Math 126 Homework 1

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1 Problem 1

Solution Suppose f_n converges to f uniformly in $C([0,1])$. Then $\|f_n - f\| \to 0$ as $n \to \infty$, so:

$$\begin{aligned}
\left|\int_0^1 f_n(x)dx - \int_0^1 f(x)dx\right| &\leq \int_0^1 |f_n(x) - f(x)|dx \\
&\leq \int_0^1 \|f_n(x) - f(x)\|dx \\
&= \|f_n(x) - f(x)\|
\end{aligned}$$

which goes to zero as $n \to \infty$, so the order of limit and integration can be swapped if $f_n \to f$ uniformly. For a counterexample when $f_n \to f$ pointwise, consider the sequence of piecewise continous functions on $[0,1]$

$$f_n = \begin{cases} n^2x & 0 \leq x \leq \frac{1}{2n} \\ -n^2x + n & \frac{1}{2n} < x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \leq 1 \end{cases}$$

Then $f_n \to 0$ pointwise on $[0,1]$. But for all n:

$$\int_0^1 f_n(x)dx = \int_0^{1/2n} n^2xdx + \int_{1/2n}^{1/n} -n^2x + ndx = \frac{1}{8}$$

But

$$\int_0^1 f(x)dx = \int_0^1 0dx = 0$$

and thus

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{8} \neq 0 = \int_0^1 f(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

and thus the order of limits and integration cannot be swapped when $f_n \to f$ pointwise.

2 Problem 2

Solution Clearly $C_0(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ is a subsapce, since if $f_1, f_2 \in C_0(\mathbb{R}^n)$ and $c_1, c_2 \in \mathbb{R}$. Then

$$\lim_{|x|\to\infty} c_1f_1 + c_2f_2 = c_1 \lim_{|x|\to\infty} f_1 + c_2 \lim_{|x|\to\infty} f_2 = 0$$

so $C_0(\mathbb{R}^n)$ is closed under linear combinations. Obviously, the zero function is in $C_0(\mathbb{R}^n)$. All that remains to show is that the uniform limit of a sequence f_n in $C_0(\mathbb{R}^n)$ is also in $C_0(\mathbb{R}^n)$.

To this end, first note that if a sequence $f_n \to f$ uniformly with f_n continuous, then f is also continuous. Thus all that remains is to show that $\lim_{|x|\to\infty} f(x) = 0$. Fix $\epsilon > 0$. Since $f_n(x) \to 0$, for each n there is a corresponding constant a_n such that if $|x| > a_n$, then $|f_n(x)| < \epsilon/2$. Let $a = \sup\{a_1, a_2, \ldots\}$.

Next, since $f_n \to f$ uniformly, there is $N \in \mathbb{N}$ such that for all $n > N$, $\|f - f_n\| < \epsilon/2$. So for all $x > a$

$$
\begin{aligned}
|f(x)| &= |f(x) - f_n(x) + f_n(x)| \\
&\leq |f(x) - f_n(x)| + |f_n(x)| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\leq \epsilon
\end{aligned}
$$

So for any ϵ, there is a constant a such that $x > a \to f(x) < \epsilon$, so

$$\lim_{n\to\infty} f(x) = 0$$

and $C_0(\mathbb{R}^n)$ is a closed subspace of $C(\mathbb{R}^n)$.

3 Problem 3

Solution Let $u : [0, T) \to \mathbb{R}^n$ be a solution of the ODE

$$
\begin{cases}
u' = F(u) \\
u(0) = u_0
\end{cases}
$$

where $F : \mathbb{R}^n \to R^n$ is a globally Lipschitz function with constant Lipschitz constant L. Then, for $t \in [0, T)$, by the Fundamental Theorem of Calculus:

$$
\begin{aligned}
u(t) - u(0) = \int_0^t u'(s)ds &= \int_0^t F(u(s))ds \\
&= \int_0^t F(u(s)) + F(u(0)) - F(u(0))ds \\
&= F(u_0)t + \int_0^t F(u(s) - F(u(0)))ds
\end{aligned}
$$

Taking the absolute value of both sides and using the Lipschitz condition gives:

$$|u(t) - u_0| \leq T|F(u_0)| + L\int_0^t |u(s) - u_0|ds$$

By Gronwall's inequality, the integral in the right hand side of the above expression is bounded, and thus u is bounded on $[0, T)$. Since T was arbitrary, the solution u is bounded for all $T < \infty$, and thus not maximal. Thus u exists for all time.