Math 143 Homework 2

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1 Problem 1

a. Solution Let $I, J \subset k[x_1, \dots, x_n]$ be ideals, and $I + J \coloneqq \{f + g : f \in I, g \in J\}$. Let $P_1, P_2 \in I + J$. Then, there are $f_1, f_2 \in I$ and $g_1, g_2 \in J$ such that $P_1 = f_1 + g_1$ and $P_2 = f_2 + g_2$. But this implies $P_1 + P_2 \in I + J$, because $P_1 + P_2 = (f_1 + f_2) + (g_1 + g_2)$, with $f_1 + f_2 \in I$ and $g_1 + g_2 \in J$ since I and J are assumed to be ideals. Next, if $r \in k[x_1, \dots, x_n]$ and $P = f + g \in I + J$, then $rP \in I + J$, since $rP = r(f + g) = rf + rg$, with $rf \in I$ and $rj \in J$, since I and J are assumed to be ideals. We have shown that if $P_1, P_2 \in I + J$, then $P_1 + P_2 \in I + J$ and if $r \in k[x_1, \dots, x_n]$, then $rP \in I + J$ for all $P \in I + J$. Thus $I + J$ is an ideal. $\square$

b. Solution Suppose a point $P \in V(I) \cap V(J)$. Then for all $f \in I$ and $g \in J$, $f(P) = g(P) = 0$, so $f(P) + g(P) = 0$ as well. Thus $(f + g)(P) = 0$, so $V(I) \cap V(J) \subseteq V(I + J)$. Alternatively, suppose $P \in V(I + J)$. Then the polynomials $f + g \in I + J$ vanish at P. One way for this to happen is if both f and g both vanish at P, that is $P \in V(I)$ and $P \in V(J)$. But then $P \in V(I) \cap V(J)$, so $V(I + J) \subseteq V(I) \cap V(J)$. But now we have $V(I) \cap V(J) \subseteq V(I + J)$ and $V(I + J) \subseteq V(I) \cap V(J)$, so $V(I) \cap V(J) = V(I + J)$ $\square$

2 Problem 2

a. Solution The set $V(y - x^2) \subset \mathbb{A}^2$ is irreducible if and only if the ideal $\langle y - x^2 \rangle \subset \mathbb{C}^2$ is prime. An ideal $R \subset I$ is prime if and only if the quotient R/I is an integral domain. To this end, consider the ring homomorphism $\phi : k[x, y] \to k[x]$ which sends $f(x, y) \mapsto f(x, x^2)$ Clearly, ϕ is surjecrtive. We claim $\ker \phi = \langle y - x^2 \rangle$. Clearly $y - x^2 \in \ker \phi$, so $\langle y - x^2 \rangle \subset \ker \phi$. If $f \in \ker \phi$, then $f(x, x^2) = 0$, so $y - x^2$ divides f and $\langle y - x^2 \rangle \supset \ker \phi$. Thus $\langle y - x^2 \rangle = \ker \phi$. Then, since ϕ is surjective, by the first isomorphism theorem, $k[x, y]/\langle y - x^2 \rangle \cong k[x]$. $k[x]$ is an integral domain, since if $f(x)g(x) = 0$, then either $f(x) = 0$ or $g(x) = 0$, since k is a field. Thus $k[x, y]/\langle y - x^2 \rangle$ is an integral domain, so $\langle y - x^2 \rangle$ is prime, and thus $V(y - x^2)$ is irreducible.

b. Solution Starting with $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3)$, factoring each polynomial gives $V\left((y^2 - x)(y^2 + x), (y^2 + x)(y - x)(y + x)\right)$. The $y^2 + x$ is common in both terms, and taking the points where $y^2 - x$, $y - x$, and $y + x$ vanish gives the set with two points $\{(1, 1), (1, -1)\}$. Then the decomposition becomes $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) = V(y^2 + x) \cup V((1, 1)) \cup V((1, -1))$, where $V(y^2 + x)$ is irreducible by part **a**.

3 Problem 3

a. Solution Let $I \subset R$ be an ideal, with $a^n, b^m \in I$. Then, by the binomial formula:

$$(a + b)^{n+m} = \sum_{k=1}^{n+m} \binom{n+m}{k} a^{n+m-k} b^k$$

Expanding this gives

$$(a + b)^{n+m} = a^{n+m} + (n + m)a^{n+m-1}b + \cdots \binom{+n + m}{m} a^n b^m + \cdots + (n + m)ab^{n+m-1} + b^{n+m}$$

Factoring gives

$$(a + b)^{n+m} = a^n \underbrace{(a^m + \cdots + b^m)}_{r_1 \in R} + b^m \underbrace{\left(\binom{n+m}{m+1} a^{n-1}b + \cdots + b^n\right)}_{r_2 \in R}$$

Since I is an ideal, $r_1a^n \in I$ and $r_2b^m \in I$, and thus $r_1a^n + r_2b^m = (a+b)^{n+m} \in I$ as well.
b. Solution Let $a, b \in \sqrt{I}$. Then, there are n, m such that $a^n \in I$ and $b^m \in I$. By part **a**, $(a+b)^{n+m} \in I$ as well, and since there is t such that $(a+b)^t \in I$, $a+b \in \sqrt{I}$. Next, if $r \in R$ and $a \in \sqrt{I}$, there is some n such that $(ra)^n = r^n a^n \in I$ since I is an ideal, so $ra \in \sqrt{I}$. Thus $\sqrt{I}$ is an ideal.
c. Solution Suppose $f^r \in \sqrt{I}$. Then there exists s such that $(f^r)^s \in I$. But $(f^r)^s = f^{rs}$, and since there is $n = rs$ such that $f^n \in I$, $f \in \sqrt{I}$. So $f^r \in \sqrt{I} \implies f \in \sqrt{I}$ and thus $\sqrt{I}$ is radical.
d. Solution Let $I \subset R$ be a prime ideal. Then if $ab \in I$, either $a \in I$ or $b \in I$. Now suppose $a^n \in I$. Since $a^n = a \cdot a \cdots a \cdot a$, I being a prime ideal implies $a \in I$. Thus I is radical as well.

4 Problem 4

a. Solution Let X, Y be algebraic sets, and let $p \in I(X \cup Y)$. Then p vanishes on all of X and Y, so certainly p vanishes on X and Y individually. This means $p \in I(X)$ and $p \in I(Y)$, so $p \in I(X) \cap I(Y)$ and $I(x \cup Y) \subseteq I(X) \cap I(Y)$. Now let $p \in I(X) \cap I(Y)$. Then p vanishes on X and Y, so p vanishes on $X \cup Y$ as well, and $p \in I(X \cup Y)$. Thus $I(X \cup Y) = I(X) \cap I(Y)$
b. Solution This does not hold in general. Take $X = V(y)$, $Y = V(y + x^2)$. Then $I(X) + I(Y) = \langle y \rangle + \langle y + x^2 \rangle = \langle y, x^2$. But the intersection $X \cap Y$ contains only the origin, and thus $I(X \cap Y) = \langle y, x \rangle$. So in general $I(X \cap Y) \neq I(X) + I(Y)$

5 Problem 5

Solution Let R be a ring such that every ascending chain of ideals $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$ is finite. For the sake of contradiction, suppose I is an ideal of R that is infinitely generated. Then there is an infinite set G that generates I, that is $I = \langle G \rangle$. Suppose the elements in G are $g_1, g_2, \ldots$. Then the ascending chain of ideals $\langle g_1 \rangle \subsetneq \langle g_1, g_2 \rangle \subsetneq \langle g_1, g_2, g_3 \rangle \subsetneq \cdots$ is an infinite ascending chain of ideals, contradicting the initial assumption that every ascending chain of ideals is finite. Thus, if R is a ring in which every ascending chain of ideals is finite, then every ideal must be finitely generated, and hence R is Noetherian.