Math 113 Homework 2

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1 Exercise 16

Let $f : X \to Y$ be a function, and let $B \subset Y$. Then $f^*(B) = \{x : f(x) \in B\}$, so the function χ_{f^*B} is given by:

$$\chi_{f^*B} = \begin{cases} 1 & f(x) \in B \\ 0 & f(x) \notin B \end{cases}$$

On the other hand, consider the function $f^*\chi_B$. This first takes the preimage of a subset $B \subset Y$, and then applies χ to that subset. This gives:

$$f^*\chi_B = \begin{cases} 1 & f(x) \in B \\ 0 & f(x) \notin B \end{cases}$$

. So $\chi_{f^*B} = f^*\chi_B$

2 Exercise 17

Let $f : X \to Y$ be a function, with $A \subset X$ and $B \subset Y$. Suppose $A \subset f^*(B)$. Then, $x \in A$ implies $f(x) \in B$. This holds for all $x \in A$, so $f_*(A) \subset B$. On the other hand, suppose $f_*(A) \subset B$. Then for all $x \in A$, $f(x) \in B$. Then for all $x \in A$, $x \in f^*(B)$, so $A \subset f^*(B)$.

3 Exercise 18

Let $x \in f^*(B^C)$. Then $f(x) \in B^C$, so $f(x) \notin B$. Then $x \notin f^*(B)$, so $x \in (f^*B)^C$. Thus $f^*(B^C) \subseteq (f^*B)^C$. On the other hand let $x \in (f^*B)^C$. Then $x \notin f^*B$, so $f(x) \notin B$. Then $f(x) \in B^C$, so $x \in f^*(B^C)$ and $(f^*B)^C \subseteq f^*(B^C)$. Therefore $f^*(B^C) = (f^*B)^C$

4 Exercise 21

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then $(g \circ f)_* : \mathcal{P}X \to \mathcal{P}Y$ takes a subset of X to its image as a subset of Z. But this can also be done by first taking a subset of X to its image under f, which is a subset of Y, and then taking the resulting subset of Y to its image under g, which results in the same subset of Z. Thus $(g \circ f)_* = g_* \circ f_*$

Next, consider $(g \circ f)^*$. This takes a subset of $C \subset Z$ and gives back the corresponding subset of $A \subset X$ that gets mapped to C under $g \circ f$. But this can also be done by finding which subset $B \subset Y$ gets mapped to C under g, and then finding the corresponding subset A that gets mapped to B under f. Thus $(g \circ f)^* = f^* \circ g^*$.

Finally, consider $(g \circ f)_!$. For a subset $A \subseteq X$This is equal to $Z \setminus (g \circ f)_*(X \setminus A)$. But by the previous part, this is equivalent to $Z\ g_*(Y \setminus B) \circ Y \setminus f_*(X \setminus A) = g_! \circ f_!$.

5 Exercise 23

Let $f : X \to Y$ be an invertible function. Then $f \circ f^{-1}$ and $f^{-1} \circ f$ are both the identity. Then the function $f^* : \mathcal{P}X \to (P)Y$ assigns every subset of X to its image under f in Y. The function $f_*^{-1} : \mathcal{P}X \to (P)Y$ assigns to every subset of $A \subset X$ the subset of $B \subset Y$ such that $f^{-1}(y) \in A$ whenever $y \in B$. But since f is invertible, this is equivalent to saying that B is the image of A under f. Thus $f^* = f_*^{-1}$

6 Exercise 25

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets, with $\mathcal{A} \subset \mathcal{B}$. Suppose $x \in \cup\mathcal{A}$. Then $x \in \cup\mathcal{B}$, since $\mathcal{A} \subset \mathcal{B}$ and thus $\cup\mathcal{A} \subset \cup\mathcal{B}$.

Now suppose $x \in \cap\mathcal{B}$. Then x is in every set $I \in \mathcal{B}$, and so it is in every set in $\mathcal{A}$, and hence in $\cap\mathcal{A}$. Thus $\cap\mathcal{A} \supset \cap\mathcal{B}$.

7 Exercise 26

The functions from $\mathcal{P}\mathcal{P}X$ to $\mathcal{P}\mathcal{P}Y$ are f_{**}, $f_{*!}$, $f_{!*}$ and $f_{!!}$.