# Math H104 Homework 3

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#### Exercise 16 1

**Solution** We have the following commutative diagram:

$$Z_{1} \longleftarrow^{\phi_{1}} Y_{1} \twoheadleftarrow^{\chi_{1}} X_{1}$$

$$\downarrow^{\gamma}$$

$$Z_{2} \longleftarrow^{\phi_{2}} Y_{2} \longleftarrow^{\chi_{2}} X_{2}$$

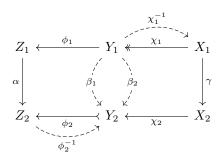
Since  $\chi_1$  is injective, it has a left inverse  $\chi_1^{-1}$  such that  $\chi_1^{-1} \circ \chi_1 = id_{X_1}$ . Similarly, since  $\phi_2$  is surjective, it has a right inverse  $\phi_2^{-1}$  such that  $\phi_2 \circ \phi_2^{-1} = id_{Z_2}$ . Now suppose there are  $\beta_1$  and  $\beta_2$  such that the diagram below commutes.

$$Z_{1} \longleftarrow^{\phi_{1}} Y_{1} \twoheadleftarrow^{\chi_{1}} X_{1}$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta_{1}} \qquad \downarrow^{\beta_{2}} \qquad \downarrow^{\gamma}$$

$$Z_{2} \longleftarrow^{\phi_{2}} Y_{2} \longleftarrow^{\chi_{2}} X_{2}$$

With the inverses added in, the diagram looks as follows:



From these diagrams we can read off the following relations:  $\alpha \circ \phi_1 = \phi_2 \circ \beta_1$  and  $\beta_2 \circ \chi_1 = \chi_2 \circ \gamma$ . Now using applying the left and right inverses gives  $\beta_1 = \phi_2^{-1} \circ \alpha \circ \phi_1$  and  $\beta_2 = \chi_2 \circ \gamma \circ \chi_1^{-1}$ . But by the commutativity of the first diagram, we have  $\alpha \circ \phi_1 \circ \chi_1 = \phi_2 \circ \chi_2 \circ \gamma$ , and applying inverses gives  $\phi_2^{-1} \circ \alpha \circ \phi_1 = \chi_2 \circ \gamma \circ \chi_1^{-1}$ . But then we have

$$\beta_1 = \phi_2^{-1} \circ \alpha \circ \phi_1 = \chi_2 \circ \gamma \circ \chi_1^{-1} = \beta_2$$

so  $\beta_1 = \beta_2 := \beta$  is unique, as desired.

#### Exercise 27 2

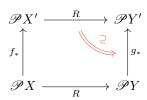
**Solution** The functions from  $\mathscr{PP}X \to \mathscr{PP}Y$  are  $f_{**}, f_{!*}, f_{*!}, f_{!!}$  and  $f^{**}$ .

# 3 Exercise 33

**Solution** The properties that are inherited by  $\phi^* \rho$  are reflexivity, transitivity, and symmetricity

### 4 Exercise 35

**Solution** We want to show the following diagram commutes:



Equivalently, we must show that for any set  $A \in \mathscr{P}X$ ,  $g_*RA \subseteq Rf_*A$ . The set  $g_*RA$  is given by

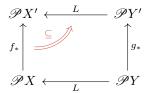
$$g_*RA = \{g(y) : \rho(x,y) \forall x \in A\}$$

and the set  $Rf_*A$  is given by

$$Rf_*A = \{y' : \rho'(f(x), y') \forall x \in A\}$$

But since (f,g) is a morphism of binary relations, we have  $(f,g)^*\rho'=\rho'\circ (f,g)$ , so if  $\rho(x,y)$  holds, then so does  $\rho'(f(x),g(y))$ . Thus, if  $y'=g(y)\in g_*RA$ , then  $y'\in Rf_*A$ , and we have  $g_*R\subseteq Rf_*$ .

Now consider the following diagram:



We want to show that for any  $B \in \mathscr{P}Y$ ,  $f_*LB \subseteq Lg*B$ . The set  $f_*LB$  is given by

$$f_*LB = \{ f(x) : \rho(x, y) \forall y \in B \}$$

and the set  $Lg_*B$  is given by

$$Lq_*B = \{x' : \rho'(x', q(y)) \forall y \in B\}$$

But (f,g) is a morphism of binary relations, so  $(f,g)^*\rho'=\rho'\circ (f,g)$ , so if  $\rho(x,y)$  holds, then so does  $\rho'(f(x),g(y))$ . Thus, if  $x'=f(x)\in f_*LB$ , then  $x'\in Lg_*B$ , and we have  $f_*L\subseteq Lg_*$ .

### 5 Exercise 39

The relations  $\leq$  and  $\geq$  are both antisymmetric transitive reflexive relations, in other words they are order relations.

# 6 Exercise 40

**Solution** Suppose  $\rho \in \text{Rel}_2(X)$  and  $\rho \Longrightarrow \leq$ . This means that if  $\rho(x, x')$ , then  $\langle x | \subseteq \langle x' |$ . Clearly  $\langle x | \subseteq \langle x |$ , so  $\rho(x, x)$  must hold, and  $\rho$  must be reflexive. Next, if  $\langle x | \subseteq \langle x' |$  and  $\langle x' | \subseteq \langle x'' |$ , then  $\langle x | \subseteq \langle x'' |$ , so  $\rho(x, x'), \rho(x', x'') \Longrightarrow \rho(x, x'')$ , and so  $\rho$  is transitive. Finally,  $\rho$  must be weakly antisymmetric, since if  $\langle x | \subseteq \langle x' |$  and  $\langle x' | \subseteq \langle x |$ , then  $\langle x | = \langle x' |$ . Thus  $\rho$  must be an order relation.

# 7 Exercise 41

**Solution** Suppose  $\rho \in \text{Rel}_2(X)$  and  $\leq \Longrightarrow \rho$ . This means that if  $\langle x | \subseteq \langle x' |$ , then  $\rho(x, x')$  holds. But  $\subseteq$  is an order relation, so then  $\rho$  must also be an order relation.

# 8 Exercise 43

**Solution** Let  $y \in \cap R_* \mathscr{A}$ . Then for all  $A \in \mathscr{A}$ , we have  $y \in R_* A$ , so clearly there is  $x \in X$  such that  $\rho(x,y)$  holds for each  $x \in A \in \mathscr{A}$  so clearly  $x \in \cup \mathscr{A}$ . Thus  $y \in R(\cup \mathscr{A})$ . On the other hand, suppose  $y \in R(\cup \mathscr{A})$ . Then for all  $x \in A \in \mathscr{A}$ , the relation  $\rho(x,y)$  holds, so for all  $A \in \mathscr{A}$ ,  $y \in R_* A$ , and thus  $y \in \cap R_* \mathscr{A}$ .

# 9 Exercise 44

a. Solution By definition, we have

$$RX = \{ y \in Y | \forall x \in X, \rho(x, y) \}$$

But by definition, if  $\rho(x,y)$  holds for all  $x \in X$ , then y is a terminal element of Y. Thus we have

$$RX = \{ y \in Y | y \text{ is a terminal element of } Y \}$$

Similarly, if  $\rho(x,y)$  holds for all  $y \in Y$ , then x is an initial element of X, so we have

$$LY = \{x \in X | x \text{ is an initial element of } X\}$$

**b.** Solution Suppose  $\xi \in X$  is a supremum of X. Then we have  $RX = |\xi\rangle$ , which means  $\xi$  is the least element of a preordered set  $(X, \geq)$ . Now suppose  $\xi$  is a smallest element of  $(X, \geq)$ . Then we have  $RX = |\xi\rangle$ , so  $\xi \in X$  is a supremum of X

Suppose  $\nu \in Y$  is a supremum of Y. Then we have  $LY = \langle \nu |$ , which means  $\nu$  is the least element of a preordered set  $(Y, \leq)$ . Now suppose  $\nu$  is a smallest element of  $(Y, \leq)$ . Then we have  $LY = \langle \nu |$ , so  $\nu \in Y$  is a supremum of Y