Math 143 Homework 3

Aniruddh V.

September 2023

1 Problem 1

a. Solution Let R be a ring, $I \subset R$ an ideal, and $\pi : R \to R/I$ be the canonical projection map which maps every element $r \in R$ to its coset $r + I \in R/I$. Also let $I \subseteq J \subseteq R$ be an ideal containing I. Then the image $\pi(J)$ is an ideal of R/I, since if $(a+I), (b+I) \in \pi(J)$, with $a, b \in J$, then $a+b \in J$ so $(a+I)+(b+I) = (a+b)+I = \pi(a+b) \in \pi(J)$ as well. Furthermore, if $(a+I) \in \pi(J)$ and $(r+I) \in R/I$, with $a \in J$ and $r \in R$, then $(a+I)(r+I) = (ar+I) \in \pi(J)$, since $ar \in J$ because J is an ideal. Thus $\pi(J)$ is an ideal in R/I.

Next, if K is an ideal in R/I, then $\pi^{-1}(K)$ is an ideal in R containing I. First, since K is an ideal of R/I, it contains the coset $0 + I$, and for all $a \in I$, $\pi(a) = a + I = 0 + I \in K$, so $a \in \pi^{-1}(K)$. Thus $\pi^{-1}(K) \supseteq I$. Next, if $a, b \in \pi^{-1}(K)$, then so is $a + b$, since $\pi(a+b) = ((a+b)+I) = (a+I)+(b+I) \in K$ since K is an ideal. If $r \in R$, and $x \in \pi^{-1}(K)$, then $\pi(x) \in K$, so $\pi(as) = \pi(a)\pi(s) \in K$ as well since K is an ideal. Thus $\pi^{-1}(K)$ is an ideal as well.

Now to show the bijection, it suffices to show that $\pi^{-1} \circ \pi$ and $\pi \circ \pi^{-1}$ are both the identity. Since π is surjective, $\pi(\pi^{-1}(X)) = X$ for any ideal of R/I. On the other hand, $X \subseteq \pi^{-1}(\pi(X))$ for any subset X, so certainly it holds for any ideal. Now let $a \in \pi^{-1}(\pi(X))$ for an ideal $X \subset R$. Then, Then $\pi(a) = \pi(X)$, and there is some $x \in X$ such that $\pi(a) = \pi(x)$, so $\pi(a) - \pi(x) = \pi(a - x) = 0 + I$, so $a - x \in \ker(\pi) = I$. Thus $a - x \in I \subset X$, and $x \in X$ so $a = (a - x) + x \in X$. Thus $\pi^{-1}(\pi(X)) = X$ and $\pi^{-1} \circ \pi = \pi \circ \pi^{-1} = \text{id}$. This shows there is a bijection between ideals of R/I and ideals of R containing I.

b. Solution By **a**, we have a bijective correspondence between ideals in R/I and ideals in R containing I. So all we need to check is that the image of a radical ideal under $\pi : R \to R/I$ is radical, and the pre-image of a radical ideal in R/I is a radical ideal in R. To this end, suppose J is radical. Then the quotient R/J is a reduced ring. Now consider $J' = J/I$. By the second isomorphism theorem, $(R/I)/(J/I) \cong R/J$. By assumption, R/J is a reduced ring, so then $(R/I)/(J/I)$ is reduced, and thus $J' = J/I$ is radical. On the other hand, suppose J' is radical. Since $J' = J/I$, this implies that $(R/I)/(J/I)$ is a reduced ring. But $(R/I)/(J/I) \cong R/J$, so R/J is a reduced ring as well, and thus J is radical.

c. Solution Similar to **b**, because of the bijective correspondence provided by **a**, all that needs to be shown is that the image of a maximal ideal is maximal, and that the preimage of a maximal ideal is maximal as well. To this end, suppose $J \subset R$ is a maximal ideal, and consider the corresponding ideal $J' = J/I \subset R/I$. Since an ideal is maximal J is maximal if and only if the corresponding quotient R/J is a field, we have that R/J is a field. But $R/J \cong (R/I)/(J/I)$, so this means that $J/I = J'$ is a maximal ideal of R/I as well. Finally, if $J' = J/I$ is maximal, then $(R/I)/(J/I)$ is a field. But $(R/I)/(J/I) \cong R/J$ so J is a maximal ideal of R.

Then, if L is a field, and $\phi : R \to L$ is a surjective ring homomorphism, we have $R/\ker\phi \cong L$, and since L is a field, this implies that $R/\ker\phi$ is a field, and thus $\ker\phi$ is maximal.

2 Problem 2

a. Let $J \subset k[x_1, \dots x_n]$ be a radical ideal. Then $\sqrt{J} = J$. Consider a polynomial $f \in J$. Then since $\emptyset \neq I \neq k[x_1, \dots, x_n]$, f vanishes on finitely many points $P_1, \dots P_m \in \mathbb{A}^n$. Consider the set

$$V(J) = P \in \mathbb{A}^n : \exists f \in J : f(P) = 0$$

Then $V(J)$ is the union of the singleton sets X_n containing only a single point in affine space. But since $I(A \cup B) = I(A) \cap I(B)$, we have

$$I(V(J)) = \cap_i I(X_i)$$

Each X_i contains just a single point, so by the Second Weak Nullstellensatz, every $I(X_i)$ is maximal. Furthermore, since J is a radical ideal, we have $I(V(J)) = \sqrt{J} = J$, so putting it all together

$$J = \cap_i I(X_i)$$

Clearly J is contained in each X_i, so a radical ideal of $k[x_1, \dots, x_n]$ is equal to the intersection of all of the maximal ideals containing it.

On the other hand, suppose $J \subset k[x_1, \dots, x_n]$ is the intersection of all of the maximal ideals containing it. Since every maximal ideal is the ideal of a point in affine space, J contains all polynomials that vanish on a set of points P_n in affine space. Now suppose $f^r \in J$. This means $f^r(P) = 0$, but $f^r(P) = f(P)^r$, so $f(P) = 0$ and thus $f \in J$ as well. Thus J is radical, and an ideal $J \subset k[x_1, \dots, x_n]$ is radical if and only if it is equal to the intersection of all of the maximal ideals containing it.

3 Problem 3

Solution The second equation $xz - x$ implies that either $x = 0$ or $z = 1$. If $x = 0$, then the first equation becomes $yz = 0$, which implies $y = 0$ or $z - 0$. These correspond to the two sets $S_1 = V(x, y)$, which is the z axis, or $S_2 = V(x, z)$, which is the y axis. If $z = 1$, then the first equation becomes $x^2 - y$, which corresponds to the set $S_3 = V(z - 1, x^2 - y)$.

To see that S_1, S_2 and S_3, it suffices to show that their ideals are prime. The corresponding ideals are $I_1 = \langle y, z \rangle, I_2 = \langle x, z \rangle$ and $I_3 = \langle z - 1, y^2 - x \rangle$. These are all prime, since $\mathbb{C}[x, y, z]/\langle y, z \cong \mathbb{C}[x]$, $\mathbb{C}[x, y, z]/\langle x, z \cong \mathbb{C}[y]$, and $\mathbb{C}[x, y, z]/\langle z - 1, x^2 - y \cong \mathbb{C}[x]$, which are all domains. Therefore, $V(x^2 - yz, xz - x) = S - 1 \cup S_2 \cup S_3$, and each S_i is irreducible.

4 Problem 4

a. Solution Let k be a field, and L be an extension. Let $v \in L$ such that v is algebraic over k. Then there is a polynomial $f \in k[x]$ such that $f(v) = 0$. In other words, we have $v^n + a_1 v^{n-1} + \cdots + a_n = 0$. But this implies $v(v^{n-1} + \cdots) = -a_n$. Since k is a field, $-a_n$ has a multiplicative inverse, so $v[-a_n^{-1}(v^{n-1} + \cdots)] = 1$. Thus any algebraic element v has a multiplicative inverse, and the set of all algebraic elements in a field extension is itself a field.

b. Solution R is already a subring of L, so it suffices to show that every $x \neq 0 \in R$ has a multiplicative inverse. Since L is a finite extension of k, it is also a finite dimensional vector space over k, say of dimension n. Then the elements $1, v, \dots, v^n$ with $v \in R$ are linearly dependent, so there are $a_i \in k$ such that $v^n + a_1 v^{n-1} + \cdots + a_n = 0$. But this is equivalent to saying $v(v^{n-1} + \cdots) = -a_n$. Since k is a field, $-a_n$ has a multiplicative inverse, so $v[-a_n^{-1}(v^{n-1} + \cdots)] = 1$ Thus v has a multiplicative inverse, and R is a field.

5 Problem 5

Solution Suppose $k \subset L$ is an algebraic extension, and $L \subset L'$ is also an algebraic extension. Let $a \in L'$. Then a is algebraic over L so there is a polynomial $f(x) \in L[x]$ such that $f(a) = 0$. Suppose $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n$. Each c_i is in L and is thus algebraic over k. Consider the finite (and hence algebraic) extension $M = k(c_0, c_1 \dots, c_n)$. Now f is a polynomial in $M[x]$ and $f(a) = 0$, so a is algebraic over M. Now consider $M(a)$, which is a finite extension of k, so a is algebraic over k, and since a was arbitrary, L' is a finite extension of k as well.