# Math 174 Homework 3

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## 1 Excercise 1.3.iii

**Solution** Consider a category C, with four objects a,b,c and d, with morphisms  $f:a\to b$  and  $g:c\to d$ , and a category D with objects x,y,z, and morphims  $x\to y,\ y\to z$ , and  $z\to x$ . Consider the functor  $F:C\to D$  which maps  $F(a)=x,\ F(b)=F(c)=y,$  and F(d)=z. Since categories are closed under composition of morphisms, and there is no morphism  $a\to d$ , the image of F cannot be a category.

## 2 Exercise 1.3.iv

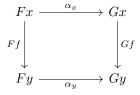
Solution Let C be a locally samll catgeory. Let the functor  $Mor(c,-): C \to \text{Set}$  have the following assignments. Given an object  $x \in C$ ,  $x \mapsto Mor(c,x)$ . Next, given a morphism  $f: x \to y$  in C, Mor(c,-) acting on the morphism gives a new map  $f_*: Mor(c,x) \to Mor(c,y)$ . We now check the Functoriality axioms. First,  $F(id_c) = id_{F(c)}$ . So given the identity morphism  $id_x: x \to x$ , Mor(c,-) maps  $id_x$  to the map  $Mor(c,x) \to Mor(c,x)$ . That is  $id_x \mapsto id_{Mor(c,x)}$ , and the first axiom of functoriality is satisfied. Next, given two morphisms  $a: x \to y$  and  $b: y \to z$  in C, we want to show that F(ab) = F(a)F(b). Indeed, ab is a morphism from  $x \to z$ , and so F(ab) must be a morphism from  $Mor(c,x) \to Mor(c,z)$ . But F(a)F(b) is the morphism  $Mor(c,x) \to Mor(c,y) \to Mor(c,z)$ , which when composed together gives the same as F(ab). Thus F(ab) = F(a)F(b), and so the axioms of functoriality are satisfied for Mor(c,-)

Now consider the contravariant functor Mor(-,c). This behaves similarly to Mor(c,-), except now a morphism  $f: x \to y$  in C is mapped to a morphism  $f^*: Mor(y,c) \to Mor(y,c)$ . The idenity law of functoriality remains the same as for Mor(c,-), now the only thing left to be checked is F(gf) = F(f)F(g) for morphisms  $f: x \to y$  and  $g: y \to z$ . But gf is a morphism from  $x \to z$  and since F(f)F(g) also maps  $z \to x$ , we have equality.

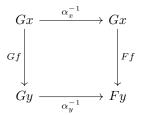
Thus Mor(c, -) and Mor(-, c) are both functors

## 3 Exercise 1.4.i

**Solution** Let  $\alpha: C \Longrightarrow D$ . Then for all morphisms  $f: x \to y$  in C, the following diagram commutes



Further,  $\alpha_x$  and  $\alpha_y$  are isomorphisms. This tells us that  $Gf \circ \alpha_x = \alpha_y \circ Ff$ . But since  $\alpha$  is an isomorphism, it has an inverse for x and y, so we also have  $\alpha_y^{-1} \circ Gf = Ff \circ \alpha_x$ . So we have the following commutative diagram as well.



This shows  $\alpha^{-1}: G \implies F$  is a natural isomorphism.