MATH 143 Homework 1

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1 Problem 1

Solution
a. $\deg(x^5 + 15x^3y^3 + 8y^2x) = 6$
b. $\deg(x^2yz^4 + z^9 + 10xy) = 9$

2 Problem 2

Solution Let $f \in k[x, y]$ be a polynomial of degree d, and $C = V(f) \subset \mathbb{A}^{k}$. Let L be a line such that $L \not\subset C$ and $L = V(y - (ax + b))$. Then the set $L \cap C$ contains the points $(x, y) \in \mathbb{A}^2$ such that $f(x, y) = 0$ and $y = ax + b$, or alternatively the set of points $x \in \mathbb{A}^1$ such that $f(x, ax + b) = 0$. $f(x, ax + b)$ is a polynomial of degree d in $k[x]$, and so it has at most d distinct roots. Since the map $x \to ax + b$ is injective, each root of x of $f(x, ax + b)$ corresponds to exactly one point (x, y) in $L \cap C$, so the set $L \cap C$ is finite with at most d points. $\square$

3 Problem 3

Solution a. Let $S = \{(t, \sin(t))|t \in \mathbb{R}\} \subset \mathbb{A}^2$. Since $\sin(x) = \sin(x + 2n\pi)$, the intersection of the line $y = 0$ given by $L = \{(x, 0)|x \in \mathbb{R}\}$ with S gives $L \cap S = \{(0, n\pi)|n \in \mathbb{Z}\}$. L is an algebraic set, and if S were an algebraic set, then the intersection $L \cap S$ would have to be an algebraic set as well. But $L \cap S$ is an infinite set of points, and is not algebraic, so $S = \{(t, \sin(t))\}$ is **not algebraic**
b. Let $S = \{(x, y) \in \mathbb{A}^2_{\mathbb{C}}|x = 0, y \neq 0\}$. Suppose $f \in k[x, y]$ vanishes on S. Then $f(0, y) \in k[y]$ vanishes on $\{y \in \mathbb{A}^1_{\mathbb{C}}|y \neq 0\}$. But this is a proper infinite subset of A^1, and thus is not algebraic, thus $S = \{(x, y) \in \mathbb{A}^2_{\mathbb{C}}|x = 0, y \neq 0\}$ is **not algebraic**.
c. Let $S = \{(x, y) \in \mathbb{A}^2_{\mathbb{R}} : y = |x|\}$. Taking the intersection of this set with the algebraic set $A = V(y - x)$ gives $A \cap S = \{(x, y) \in \mathbb{A}^2_{\mathbb{R}} : y = x, x \geq 0\} = \{(x, x) \in \mathbb{A}^2_{\mathbb{R}} : y = x, x \geq 0\}$ which is not an algebraic set since it is a proper

infinite subset of A^1, and thus $\{(x,y) \in \mathbb{A}^2_{\mathbb{R}} : y = |x|\}$ is **not algebraic**.
d. Let $S = \{(t, t^2, t^3) \in \mathbb{A}^3_k | t \in k\}$. This set is generated by the zeroes of the polynomials $f(x,y,z) = y - x^2$ and $g(x,y,z) = z - x^3$. Thus S is **algebraic**.