Math H104 Homework 1

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1 Problem 17

Solution Let ρ be a binary relation. Then, by definition, the binary relation ρ^{op} is such that $\rho(x,y) \implies \rho^{\mathrm{op}}(y,x)$. But then

$$_{\rho^{\mathrm{op}}}\langle| = \{x \in X | \rho^{\mathrm{op}}(x,y)\} = \{x \in X | \rho(y,x)\} = |\rangle_{\rho}$$

and

$$|\rangle_{\rho^{\mathrm{op}}} = \{y \in Y | \rho^{\mathrm{op}}(x,y)\} = \{y \in Y | \rho(y,x)\} = {}_{\rho}\langle|$$

2 Problem 18

Solution Consider the function $\langle| : \mathrm{Rel}(X,Y) \to \mathrm{Funct}(Y, \mathcal{P}(X))$, which takes a relation $\rho(x,y)$ and maps it to the function $f_\rho : Y \to \mathcal{P}(X)$, which sends each $y \in Y$ to subset $A_y \subset X$ defined by $A_y = \{x \in X | \rho(x,y)\}$. Then for any $f \in \mathrm{Funct}(Y, \mathcal{P}(X))$, one can define the relation $\rho_f \in \mathrm{Rel}(X,Y)$ such that $\rho_f(x,y)$ if and only if $x \in f(y)$, or x is in the subset of X defined by the image of y. This can be done for any function in $\mathrm{Funct}(Y, \mathcal{P}(X))$, so $\langle|$ is surjective.

3 Problem 19

Solution Suppose $\rho \iff \sigma$. Then $\rho(x,y)$ holds exactly when $\sigma(x,y)$ holds. Then this implies ${}_{\rho}\langle| = {}_{\sigma}\langle|$. On the other hand, suppose ${}_{\rho}\langle| = {}_{\sigma}\langle|$. Then, if $\rho(x,y)$ holds, so does $\sigma(x,y)$, so ρ and σ are equipotent. Thus $\rho \iff \sigma$ if and ${}_{\rho}\langle| = {}_{\sigma}\langle|$.

4 Problem 20

Solution Suppose $\rho \in \mathrm{Rel}_2(X)$ and $\rho \implies \leq$. This means that if $\rho(x,x')$, then $\langle x| \subseteq \langle x'|$. Clearly $\langle x| \subseteq \langle x|$, so $\rho(x,x)$ must hold, and ρ must be reflexive. Next, if $\langle x| \subseteq \langle x'|$ and $\langle x'| \subseteq \langle x''|$, then $\langle x| \subseteq \langle x''|$, so $\rho(x,x'), \rho(x',x'') \implies \rho(x,x'')$, and so ρ is transitive. Finally, ρ must be weakly antisymmetric, since if $\langle x| \subseteq \langle x'|$ and $\langle x'| \subseteq \langle x|$, then $\langle x| = \langle x'|$. Thus ρ must be an order relation.

5 Problem 21

Solution Suppose $\rho \in \mathrm{Rel}_2(X)$ and $\leq \implies \rho$. This means that if $\langle x| \subseteq \langle x'|$, then $\rho(x,x')$ holds. But $\subseteq$ is an order relation, so then ρ must also be an order relation.

6 Problem 22

Solution Suppose $\rho \in \mathrm{Rel}_2(X)$ and $\rho \implies \geq$. This means that if $\rho(x,x')$, then $|x\rangle \subseteq |x'\rangle$. Clearly $|x\rangle \subseteq |x\rangle$, so $\rho(x,x)$ must hold, and ρ must be reflexive. Next, if $|x\rangle \subseteq |x'\rangle$ and $|x'\rangle \subseteq |x''\rangle$, then $|x\rangle \subseteq |x''\rangle$, so $\rho(x,x'), \rho(x',x'') \implies \rho(x,x'')$, and so ρ is transitive. Finally, ρ must be weakly antisymmetric, since if $|x\rangle \subseteq |x'\rangle$ and $|x'\rangle \subseteq |x\rangle$, then $\langle x| = \langle x'|$. Thus ρ must be an order relation.

Similarly, if $\geq \implies \rho$, then since $\geq$ is an order relation, ρ must be an order relation as well.