MUSA 174 Homework 1

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1 Problem 1.1.i

Solution Let C be a category, and let $f : x \to y$, $g : y \to x$ and $h :$ be a morphisms such that $gf = 1_x$ and $fh = 1_y$. Then, $g = h$, since

$$g = g1_y = g(fh) = (gf)h = 1_x h = h$$

Then f is an isomorphism as well, since there is a morphism $f^{-1} = g = h$ such that $f^{-1}f = 1_x$ and $ff^{-1} = 1_y$.

2 Problem 1.1.iii.i

Solution Fixing a category C and an object $c \in C$, define a new category c/C such that:

- Objects in c/C are morphisms in C whose domain is c, or morphisms in C of the form $f : c \to x$
- A morphism in c/C with domain $f : c \to x$ and codomain $g : c \to y$ is another morphism $h : x \to y$ in C such that $g = hf$, or so that the following diagram commutes:

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"] & & y \end{tikzcd}$$

- Identity elements for each object $f : c \to x$ will be the corresponding identity morphism $1_x : x \to x$ in C
- Given three objects $o_1 : c \to x_1$, $o_2 : c \to x_2$, and $o_3 : c \to x_3$, with morphisms $m_1 : x_1 \to x_2$ and $m_2 : x_2 \to x_3$, define the composition

$m_2 \circ m_1$ so that the following diagram commutes.

$$\begin{tikzcd}
& c \arrow[dl, "o_1"'] \arrow[d, "o_2"'] \arrow[dr, "o_3"] & \\
x_1 \arrow[r, "m_1"'] \arrow[rr, bend right, "m_2 \circ m_1"'] & x_2 \arrow[r, "m_2"'] & x_3
\end{tikzcd}$$

By the first diagram, $o_2 = m_1 o_1$ and $o_3 = m_2 o_2$. Combining these gives $o_3 = m_2 m_1 o_1$ or $o_3 = (m_2 \circ m_1)o_1$

- Since all morphisms are some type of mapping, they are assosciative as well

Thus c/C meets all the requirements to be a category.