Math 143 Homework 4

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1 Problem 1

a. Solution Let R be an integral domain, and let the inclusion map $\iota : R \to \text{Frac}(R)$ be the map that sends $a \mapsto a/1$. This is injective, since if $\iota(a) = \iota(b)$, then $a/1 = b/1$ and thus $a = b$.

b. Solution Let K be a field, and $\phi : R \to K$ be a ring homomorphism. Consider the map $\beta : \text{Frac}(R) \to K$, which sends $a/b \mapsto \phi(a)\phi(b)^{-1}$. Then, if we first send $a \mapsto a/1$, then send $a/1 \mapsto \phi(a)\phi(1)^{-1}$. But $phi(1)$ must equal 1 in a ring homomorphism and so we have $\beta(a/1) = \phi(a)$ which implies that $\phi = \beta \circ \iota$

2 Problem 2

a. Solution A ring homomorphism is injective if and only if its kernel is $\{0\}$. Furthermore, the kernel of a homomorphism is an ideal. Since the only ideals of a field k are $\{0\}$ and k itself, there are only two possibilities for the kernel of the homomorphism, and thus two possibilities for the homomorphism itself. When $\ker \varphi = k$, then φ is the zero map, and when $\ker \varphi = \{0\}$, φ is injective.

b. Solution No, there cannot be such a map. Notice that since $k[y]$ is a domain, $k(y)$ is a field, and thus we have a surjective map from a ring to a field, say $\phi : k[x_1, \ldots, x_n] \to k(y)$. Since ϕ is surjective, we have $k[x_1, \ldots, x_n]/\ker \phi \cong k(y)$. But if a ring mod an ideal is a field, that ideal must be maximal, and Weak Nullstellensatz 3 tells us that if m is maximal, then $k[x_1, \ldots, x_n]/m$ is a finite extension of k. But this is a contradiction, since $k(y)$ is not a finite extension. Thus, no such map can exist.

3 Problem 3

a. Solution Let k be a field, and let $f(x) \in k[x]$ be a polynomial of degree $n > 0$. Consider the map $\phi : k[x] \to k[x]/\langle f \rangle$. To show $S = \{\phi(1), \phi(x), \ldots \phi(x^{n-1})\}$ is a basis for $k[x]/\langle f \rangle$ over k, it suffices to show that S is linearly independent, and S spans $k[x]/\langle f \rangle$. To this end, suppose $a(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} = 0$ in $k[x]/\langle f \rangle$. This means that $f|a$, which implies $a_i = 0$ since any nonzero multiple of f must have degree greater than f. Thus S is linearly independent. Now, let $b(x) \in k[x]$. By the Euclidian algorithm, we can write $b(x) = f(x)q(x) + r(x)$, with $\deg r < \deg f$. But then by definition, $\phi(r) = \phi(f)$, and $\phi(r) \in \text{span } S$, so $\phi(g) \in \text{span } S$ and S spans $k[x]/\langle f \rangle$. Thus S is a basis for $k[x]/\langle f \rangle$.

b. Solution In part **a**, we took the quotient by a degree d polynomial to obtain a d-dimensional k vector space. Now we have a collection of d^2 monomials, with a degree d monomial in x and a degree d monomial in y, so the quotient $k[x, y]/I$ is a d^2 dimensional vector space.

4 Problem 4

Solution First suppose we only have two points P_1 and P_2. Then a polynomial f such that $f(P_1) = 1$ and $f(P_2) = 0$ is given by:

$$f(x) = \frac{x - P_2}{P_1 - P_2}$$

Similarly, we have

$$g(x) = \frac{x - P_1}{P_2 - P_1}$$

satisfies $g(P_1) = 0$ and $g(P_2) = 1$ Now suppose we have F_{ij} such that $F_{ij}(P_i) = 0$ and $F_{ij}(P_j) = 1$. Then the product $\prod_{i \neq j}$ is 0 for all P_i when $i \neq j$ and 1 on P_j. Thus the final polynomial is:

$$F_{ij} = \prod_{i \neq j} \frac{x - P_i}{P_j - P_i}$$

5 Problem 5

a. Solution Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, and $Z \subset \mathbb{A}^r$ be algebraic sets, and $\phi : X \to Y$ and $\psi : Y \to Z$ be polynomial maps. Then for all $P \in X$, $P' \in Y$, there are polynomials $f_1, \dots, f_m$, $g_1, \dots, g_r$ such that $\phi(P) = (f_1(P), \dots, f_m(P))$ and $\psi(P') = (g_1(P'), \dots g_r(P'))$ Then $\psi \circ \phi$ is the map that sends $P \mapsto (g_1(f_1(P), \dots f_m(P)), \dots, g_r(f_1(P), \dots f_m(P)))$. Since each g_i and f_j are polynomials, the compositions $g_i(f_j)$ is also a polynomial, so $\psi \circ \phi$ is also a polynomial map.

b. Solution The pullback map $(\psi \circ \phi)^* : \Gamma(Z) \to \Gamma(X)$ which sends $g \mapsto g \circ \psi \circ \phi$ for $g \in \Gamma(Z)$. But this is the same as first starting with the pullback map $\psi^* : \Gamma(Z) \to \Gamma(y)$, which sends $g \mapsto g \circ \psi$, and then applying the pullback $\phi^* : \Gamma(Y) \to \Gamma(X)$ which maps $g \circ \psi \mapsto g \circ \psi \circ \phi$. Thus $(\psi \circ \phi)^* = \phi^* \circ \psi^*$