# Algebraic Geometry

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## 1 Introduction

Algebraic Geometry seeks to understand the connections between algebra and geometry - more specifically how algebraic properties of systems of polynomial equations affect the geometry of solutions to these polynomials. For example, the polynomial $x^2+y^2=1$ defines a circle, and is smooth, while the polynomial $y^2-x^3=0$ is not smooth. How can the algebra of these polynomials help us determine these properties?

## 2 The Main Characters

To do algebraic geometry properly, we need to develop a little bit more algebra. We start by introducing the main characters/objetcs that will be of use to us. Throughout these notes, $k$ will be a field, typically $\mathbb{R}$ or $\mathbb{C}$.

### 2.1 Affine Space

**Definition 2.1.** ***Affine Space:*** *Fix a field $k$. The set*

$$\mathbb{A}^n = k^n = \{(a_1, a_2, \ldots, a_n) : a_i \in k\}$$

*is called **n-dimensional affine space**. We denote affine space $\mathbb{A}^n$*

**Definition 2.2.** ***Polynomial:*** *A polynomial in $x_1 \ldots x_n$ over $k$ is a finite sum*

$$f = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

*with $\alpha_i \in k$. The **degree** of $f$ is given by $\deg f = \max\{\alpha_1 + \alpha_2 + \ldots + \alpha_n \ : c_\alpha \neq 0\}$*

We denote the ring of polynomials over a field $k$ by $k[x_1, x_2, \ldots, x_n]$. Polynomials define a function $f : \mathbb{A}^n \to k$ by evaluating the polynomial at points $P = (a_1, a_2, \ldots, a_n)$ in affine space.

Given a polynomial $f$, we can define interesting subsets of affine space in the following way:

**Definition 2.3.** ***Vanishing of f:*** *Given a polynomial* $f \in k[x_1, x_2 \dots, x_n]$ *define the* ***vanishing set of f*** *as*

$$V(f) = \{P \in \mathbb{A}^n : f(P) = 0\}$$

$V(f)$ *is called a* ***hypersurface***

The vanihsing of a set of polynomials can be defined in a similar way

**Definition 2.4.** ***Vanishing of a set S:*** *Given a set* $S \subset k[x_1, x_2, \dots, x_n]$*, define* $V(S)$ *by*

$$V(S) = \{P \in \mathbb{A}^n : f(P) = 0 \forall f \in S\}$$

*or equivalently*

$$V(S) = \bigcap_{f \in S} V(f)$$

Such a set $V(S)$ is called an **affine algebraic set**, and is commonly referred to as an **algebraic set**

**Example 2.1.** *Let* $S = \{f_1 = y - x^2, f_2 = y - 2\}$*. Then* $V(S) = (-\sqrt{2}, 2), (\sqrt{2}, 2)$

**Example 2.2.** *Let* $S = \{f_1 = y - x^2, f_2 = y + 1\}$*. Then depending on the field we are working in,* $V(S)$ *might differ! If our underlying field is* $\mathbb{R}$*, then* $V(S) = \emptyset$*, but if the underlying field is* $\mathbb{C}$*, then* $V(S) = \{(i, -1), (-i, 1)\}$

The above exmaple highligts the importance of clarifying the field we are working over. $\mathbb{C}$ feels "nicer" to work over, since we have "more points to work with". The next definition makes this formal

**Definition 2.5.** ***Algebraically closed fields:*** *A field* $k$ *is* ***algebraically closed*** *if every non-constant polynomial in* $k[x]$ *has a solution in* $k$*. Equivalently, every polynomial* $f$ *can be factored into linear factors such that*

$$f = \prod (x - r_1)(x - r_2) \cdots (x - r_n)$$

*with* $r_i \in k$

This explains why $\mathbb{C}$ is much nicer to work over than $\mathbb{R}$! By the Fundamental Theorem of Algebra, $\mathbb{C}$ is algebraically closed, while $\mathbb{R}$ is not algebraically closed since $x^2 + 1$ has no roots in $\mathbb{R}$

**Proposition 2.1.** *If* $S \subset \mathbb{A}^1$ *is algebraic, then* $S$ *is finite,* $S = \emptyset$*, or* $S = \mathbb{A}^1$

**Proposition 2.2.** *Here are some nice results about unions and intersections of algebraic sets:*
*Arbitrary intersections of algebraic sets are algebraic:*

$$\bigcap_{i \in \mathcal{I}} V(S_i) = V(\bigcup S_i)$$

*Finite unions of algebraic sets are algebraic:*

$$\bigcup_{i=1}^{N} S_i = V(\{\prod_{i=1}^{N} f_i : f_i \in S_i\})$$

## 2.2 Ideals

We start by defining ideals.

**Definition 2.6.** ***Ideal:*** *Let $R$ be a commutative ring. An **ideal** $I \subset R$ is a subset satisfying:*

- *$I$ is closed under addition - for all $f, g \in I, f + g \in I$*
- *for any $r \in R$ and $i \in I$, $ra \in I$*

**Example 2.3.** *Let $S \subset k[x_1, \ldots, x_n]$. The **ideal generated by** $S$, denoted $\langle S\rangle$ is the set of all finite sums of the form*

$$\langle S\rangle = \sum_i h_i s_i$$

*where $h_i \in k[x_1, \ldots, x_n]$ and $s_i \in S$.*

**Proposition 2.3.** *Let $S \subset k[x_1, \ldots, x_n]$, and $I = \langle S\rangle$. Then $V(S) = V(I)$*

*Proof.* Certainly $V(I) \subset V(S)$, since $S \subset I$. Suppose $P \in V(S)$. Then $f(P) = 0$ for all $f \in I$, so $V(S) \subset V(I)$. Thus $V(I) = V(S)$ □

This tells us that every algebraic set is $V(I)$ for some ideal $I \subset k[x_1, \ldots, x_n]$.

So far, we have an operation $V(S)$ that takes a collection of polynomials and defines a subset of affine space. We can also go the other way - starting with a subset of affine space and producing a collection of polynomials.

**Definition 2.7.** *Given a subset $X \subset \mathbb{A}^n$, define*

$$I(x) = \{f \in k[x_1, \ldots, x_n] : f(P) = 0 \text{ for all } P \in X\}$$

**Lemma 2.1.** *Let $X \subset \mathbb{A}^n$. Then $I(X) \subset k[x_1, \ldots, x_n]$ is an ideal.*

*Proof.* If $f, g \in I(x)$, then for all $P \in \mathbb{A}^n$, $(f + g)(P) = f(P) + g(P) = 0$ so $f + g \in I(x)$ as well. If $h \in k[x_1, \ldots, x_n]$, then for all $P \in \mathbb{A}^n$, $(hf)(P) = h(P)f(P) = 0$ so $hf \in I(X)$ as well. Thus, $I(X)$ is an ideal.

□

**Example 2.4.** *Let $X = \{(1, 2)\} \subset \mathbb{A}^2$. Then $I(X) = \langle x - 1, y - 2\rangle$*

**Example 2.5.** *Let $X = \{(a, 0) : a \in \mathbb{Z}\} \subset \mathbb{A}^2$. Then $I(X) = \langle y\rangle$*

**Example 2.6.** $I(\emptyset) = k[x_1, \ldots, x_n]$

**Example 2.7.** $I(\mathbb{A}^n) = \langle 0 \rangle$

**Lemma 2.2.** *If $X$ is an algebraic set, then $V(I(X)) = X$*

*Proof.* $X \subset V(I(X))$, since every point in $X$ will vanish in the ideal generated by polynomials vanishing on $X$. Since $X$ is algebraic, it can be written as $V(S)$ for some $S \subset k[x_1, \ldots, x_n]$. The set $S$ must be a subset of $I(X)$, so $V(I(X)) \subset V(S) = X$, so $V(I(X)) = X$ ☐

This is a neat result! A natural question to ask after this is if we can make a similar statement for $I(V(J))$, for $J \subset k[x_1, \ldots x_n]$ an ideal. Unfortunately,— the answer is no. For example, consider the ideal $J = \langle x^2 \rangle \subset k[x]$. Then $V(J) = \{0\}$, but $I(V(J)) = \langle x \rangle \neq J$! The issue here was that one polynomial was a power of the other. The **Nullstellensatz** says this is the only possible issue we can come across in this situation. To state the Nullstellensatz, we need a few more definitions.

**Definition 2.8.** ***Radical Ideals:*** *An ideal $I$ is* ***radical*** *if $f^r \in I \implies f \in I$*

**Lemma 2.3.** *$I(X)$ is radical for $X \subset \mathbb{A}^n$*

*Proof.* Suppose $f^r \in I(X)$. Then $f^r(P) = 0$ for all $P \in X \implies f(P) = 0$ for all $P \in X$. So $f \in I(X)$ as well. ☐

**Definition 2.9.** ***Radical of an Ideal:*** *Let $I \subset R$ be an ideal. The* ***radical*** *of $I$, denoted $\sqrt{I}$ is*

$$\sqrt{I} = \{f \in R : f^n \in I \text{ for some } n\}$$

**Lemma 2.4.** *If $I \subset R$ is an ideal, then $\sqrt{I}$ is also an ideal.*

*Proof.* Let $f, g \in \sqrt{I}$. By the definition of $\sqrt{I}$, there are $n, m$ such that $f^n \in I$ and $g^m \in I$. We want to show $f + g \in \sqrt{I}$, or equivalently, $(f + g)^r \in I$ for some $r$. Suppose $r = n + m$. Then, by the binomial formula:

$$\begin{aligned} (f+g)^{n+m} &= \sum_{k=0}^{n+m} \binom{n+m}{k} f^{n+m-k} g^k \\ &= f^{n+m} + (n+m) f^{n+m-1} g + \cdots \end{aligned}$$

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