Math H104 Homework 3

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1 Exercise 16

Solution We have the following commutative diagram:

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\phi_1"'] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\
Z_2 & Y_2 \arrow[l, tail, "\phi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd}$$

Since χ_1 is injective, it has a left inverse χ_1^{-1} such that $\chi_1^{-1} \circ \chi_1 = id_{X_1}$. Similarly, since ϕ_2 is surjective, it has a right inverse ϕ_2^{-1} such that $\phi_2 \circ \phi_2^{-1} = id_{Z_2}$.

Now suppose there are β_1 and β_2 such that the diagram below commutes.

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\phi_1"'] \arrow[d, dashed, bend right, "\beta_1"'] \arrow[d, dashed, bend left, "\beta_2"] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\
Z_2 & Y_2 \arrow[l, tail, "\phi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd}$$

With the inverses added in, the diagram looks as follows:

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\phi_1"'] \arrow[d, dashed, bend right, "\beta_1"'] \arrow[d, dashed, bend left, "\beta_2"] \arrow[r, dashed, bend left, "\chi_1^{-1}"] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\
Z_2 \arrow[r, dashed, bend right, "\phi_2^{-1}"'] & Y_2 \arrow[l, tail, "\phi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd}$$

From these diagrams we can read off the following relations: $\alpha \circ \phi_1 = \phi_2 \circ \beta_1$ and $\beta_2 \circ \chi_1 = \chi_2 \circ \gamma$. Now using applying the left and right inverses gives $\beta_1 = \phi_2^{-1} \circ \alpha \circ \phi_1$ and $\beta_2 = \chi_2 \circ \gamma \circ \chi_1^{-1}$. But by the commutativity of the first diagram, we have $\alpha \circ \phi_1 \circ \chi_1 = \phi_2 \circ \chi_2 \circ \gamma$, and applying inverses gives $\phi_2^{-1} \circ \alpha \circ \phi_1 = \chi_2 \circ \gamma \circ \chi_1^{-1}$. But then we have

$$\beta_1 = \phi_2^{-1} \circ \alpha \circ \phi_1 = \chi_2 \circ \gamma \circ \chi_1^{-1} = \beta_2$$

so $\beta_1 = \beta_2 \coloneqq \beta$ is unique, as desired.

2 Exercise 27

Solution The functions from $\mathscr{P}\mathscr{P}X \to \mathscr{P}\mathscr{P}Y$ are $f_{**}, f_{!*}, f_{*!}, f_{!!}$ and f^{**}.

3 Exercise 33

Solution The properties that are inherited by $\phi^*\rho$ are reflexivity, transitivity, and symmetricity

4 Exercise 35

Solution We want to show the following diagram commutes:

$$\begin{tikzcd}
\mathscr{P}X' \arrow[r, "R"] & \mathscr{P}Y' \\
\mathscr{P}X \arrow[u, "f_*"] \arrow[r, "R"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g_*"']
\end{tikzcd}$$

Equivalently, we must show that for any set $A \in \mathscr{P}X$, $g_*RA \subseteq Rf_*A$. The set g_*RA is given by

$$g_*RA = \{g(y) : \rho(x, y) \forall x \in A\}$$

and the set Rf_*A is given by

$$Rf_*A = \{y' : \rho'(f(x), y') \forall x \in A\}$$

But since (f, g) is a morphism of binary relations, we have $(f, g)^*\rho' = \rho' \circ (f, g)$, so if $\rho(x, y)$ holds, then so does $\rho'(f(x), g(y))$. Thus, if $y' = g(y) \in g_*RA$, then $y' \in Rf_*A$, and we have $g_*R \subseteq Rf_*$.

Now consider the following diagram:

$$\begin{tikzcd}
\mathscr{P}X' & \mathscr{P}Y' \arrow[l, "L"'] \\
\mathscr{P}X \arrow[u, "f_*"] \arrow[ur, phantom, "\subseteq"] & \mathscr{P}Y \arrow[l, "L"] \arrow[u, "g_*"']
\end{tikzcd}$$

We want to show that for any $B \in \mathscr{P}Y$, $f_*LB \subseteq Lg * B$. The set f_*LB is given by

$$f_*LB = \{f(x) : \rho(x, y) \forall y \in B\}$$

and the set Lg_*B is given by

$$Lg_*B = \{x' : \rho'(x', g(y)) \forall y \in B\}$$

But (f, g) is a morphism of binary relations, so $(f, g)^*\rho' = \rho' \circ (f, g)$, so if $\rho(x, y)$ holds, then so does $\rho'(f(x), g(y))$. Thus, if $x' = f(x) \in f_*LB$, then $x' \in Lg_*B$, and we have $f_*L \subseteq Lg_*$.

5 Exercise 39

The relations $\leq$ and $\geq$ are both antisymmetric transitive reflexive relations, in other words they are order relations. z

6 Exercise 40

Solution Suppose $\rho \in \mathrm{Rel}_2(X)$ and $\rho \implies \leq$. This means that if $\rho(x, x')$, then $\langle x| \subseteq \langle x'|$. Clearly $\langle x| \subseteq \langle x|$, so $\rho(x, x)$ must hold, and ρ must be reflexive. Next, if $\langle x| \subseteq \langle x'|$ and $\langle x'| \subseteq \langle x''|$, then $\langle x| \subseteq \langle x''|$, so $\rho(x, x'), \rho(x', x'') \implies \rho(x, x'')$, and so ρ is transitive. Finally, ρ must be weakly antisymmetric, since if $\langle x| \subseteq \langle x'|$ and $\langle x'| \subseteq \langle x|$, then $\langle x| = \langle x'|$. Thus ρ must be an order relation.

7 Exercise 41

Solution Suppose $\rho \in \mathrm{Rel}_2(X)$ and $\leq \implies \rho$. This means that if $\langle x| \subseteq \langle x'|$, then $\rho(x, x')$ holds. But $\subseteq$ is an order relation, so then ρ must also be an order relation.

8 Exercise 43

Solution Let $y \in \cap R_* \mathscr{A}$. Then for all $A \in \mathscr{A}$, we have $y \in R_* A$, so clearly there is $x \in X$ such that $\rho(x, y)$ holds for each $x \in A \in \mathcal{A}$ so clearly $x \in \cup \mathscr{A}$. Thus $y \in R(\cup \mathscr{A})$. On the other hand, suppose $y \in R(\cup \mathscr{A})$. Then for all $x \in A \in \mathscr{A}$, the relation $\rho(x, y)$ holds, so for all $A \in \mathscr{A}$, $y \in R_* A$, and thus $y \in \cap R_* \mathscr{A}$.

9 Exercise 44

a. Solution By definition, we have

$$RX = \{y \in Y | \forall x \in X, \rho(x, y)\}$$

But by definition, if $\rho(x, y)$ holds for all $x \in X$, then y is a terminal element of Y. Thus we have

$$RX = \{y \in Y |\ y \text{ is a terminal element of } Y\ \}$$

Similarly, if $\rho(x, y)$ holds for all $y \in Y$, then x is an initial element of X, so we have

$$LY = \{x \in X |\ x \text{ is an initial element of } X\}$$

b. Solution Suppose $\xi \in X$ is a supremum of X. Then we have $RX = |\xi\rangle$, which means ξ is the least element of a preordered set $(X, \geq)$. Now suppose ξ is a smallest element of $(X, \geq)$. Then we have $RX = |\xi\rangle$, so $\xi \in X$ is a supremum of X

Suppose $\nu \in Y$ is a supremum of Y. Then we have $LY = \langle\nu|$, which means ν is the least element of a preordered set $(Y, \leq)$. Now suppose ν is a smallest element of $(Y, \leq)$. Then we have $LY = \langle\nu|$, so $\nu \in Y$ is a supremum of Y