Math 174 Homework 3

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1 Excercise 1.3.iii

Solution Consider a category C, with four objects a, b, c and d, with morphisms $f : a \to b$ and $g : c \to d$, and a category D with objects x, y, z, and morphims $x \to y$, $y \to z$, and $z \to x$. Consider the functor $F : C \to D$ which maps $F(a) = x$, $F(b) = F(c) = y$, and $F(d) = z$. Since categories are closed under composiition of morphisms, and there is no morphism $a \to d$, the image of F cannot be a category.

2 Exercise 1.3.iv

Solution Let C be a locally samll catgeory. Let the functor $Mor(c, -) : C \to \text{Set}$ have the following assignments. Given an object $x \in C$, $x \mapsto Mor(c, x)$. Next, given a morphism $f : x \to y$ in C, $Mor(c, -)$ acting on the morphism gives a new map $f_* : Mor(c, x) \to Mor(c, y)$. We now check the Functoriality axioms. First, $F(id_c) = id_{F(c)}$. So given the identity morphism $id_x : x \to x$, $Mor(c, -)$ maps id_x to the map $Mor(c, x) \to Mor(c, x)$. That is $id_x \mapsto id_{Mor(c,x)}$, and the first axiom of functoriality is satisfied. Next, given two morphsims $a : x \to y$ and $b : y \to z$ in C, we want to show that $F(ab) = F(a)F(b)$. Indeed, ab is a morphism from $x \to z$, and so $F(ab)$ must be a morphism from $Mor(c, x) \to Mor(c, z)$. But $F(a)F(b)$ is the morphism $Mor(c, x) \to Mor(c, y) \to Mor(c, z)$, which when composed together gives the same as $F(ab)$. Thus $F(ab) = F(a)F(b)$, and so the axioms of functoriality are satisfied for $Mor(c, -)$

Now consider the contravariant functor $Mor(-, c)$. This behaves similarly to $Mor(c, -)$, except now a morphism $f : x \to y$ in C is mapped to a morphism $f^* : Mor(y, c) \to Mor(y.c)$. The idenity law of functoriality remains the same as for $Mor(c, -)$, now the only thing left to be checked is $F(gf) = F(f)F(g)$ for morphisms $f : x \to y$ and $g : y \to z$. But gf is a morphism from $x \to z$ and since $F(f)F(g)$ also maps $z \to x$, we have equality.

Thus $Mor(c, -)$ and $Mor(-, c)$ are both functors

3 Exercise 1.4.i

Solution Let $\alpha : C \implies D$. Then for all morphisms $f : x \to y$ in C, the following diagram commutes

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\
Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}$$

Further, α_x and α_y are isomorphisms. This tells us that $Gf \circ \alpha_x = \alpha_y \circ Ff$. But since α is an isomorphism, it has an inverse for x and y, so we also have $\alpha_y^{-1} \circ Gf = Ff \circ \alpha_x$. So we have the following commutative diagram as well.

$$\begin{tikzcd}
Gx \arrow[r, "\alpha_x^{-1}"] \arrow[d, "Gf"'] & Gx \arrow[d, "Ff"] \\
Gy \arrow[r, "\alpha_y^{-1}"'] & Fy
\end{tikzcd}$$

This shows $\alpha^{-1} : G \implies F$ is a natural isomorphism.