Math 126 Homework 2

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1 Problem 1

Solution The transport equation

$$\begin{cases} \partial_t u + 2\partial_x u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

is in canonical form, so we can proceed with the method of characteristics. Furthermore, $b(x,t) = 0$ for this problem, so the characteristics are given by $x_t = x - 2t$. Using the initial condition $u(0,x) = u_0(x)$ implies the solution is given by $u(t,x) = u_0(x-2t)$ Since $u_0(x)$ is only defined for $x \leq 0$, we must have $x - 2t \leq 0$ which implies $x \leq 2t$ and $x \leq 0$

2 Problem 2

Solution The method of characteristics gives the following equation for characteristics:

$$\frac{dy}{dx} = -\frac{y}{x}$$

which gives characteristrics of the form $x^2 + y^2 = c_1$. Then, this gives the following equation for u:

$$\frac{du}{dy} = -x$$

which gives the relation $uy - \frac{x^3}{3} = c_2$. Since u is constant along the characteristics, we have that the solution u satisfies

$$uy - \frac{x^3}{3} = F(x^2 + y^2)$$

for an aribitrary function F, Plugging in the initial condition $u(x,0) = 0$ gives

$$-\frac{x^3}{3} = F(x^2)$$

so this implies that $F(t) = -\frac{t^{3/2}}{3}$. Then, putting it all together,

$$u(x,y) = \frac{1}{3y}\left(x^3 + (x^2+y^2)^{3/2}\right)$$

This solution is defined everywhere except for the x-axis, or when $y = 0$

3 Problem 3

Solution This is Burger's equation, with initial data

$$u(t=0,x) = u_0(x) = \begin{cases} 1 & x < 0 \\ 1 - x & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

So we have the following system of characteristic ODE's

$$\begin{cases} \dot{x} = 0 & x(0) = x_0 \\ \dot{u} = 0 & u(0) = u_0(x_0) \end{cases}$$

Now all that remains is to invert the map $x_0 \to x_0 + tu_0(x_0)$, so $u = u_0(x - ut)$. When $x < 0$, $u_0(x) = 1$, so the map becomes $x_0 \to x_0 + t$, which implies $u = 1$ when $x < t$. Next, if $0 \leq x \leq 1$, then $u_0(x) = 1 - x$, so $u = 1 - (x - ut)$, so $u = \frac{x-1}{t-1}$ for $t \leq x \leq 1$. Finally, if $x > 1$, then $u_0(x) = 0$, so $u = 0$ here as well. Thus the solution $u(x, t)$ is given by:

$$\begin{cases} 1 & x < t \\ \frac{x-1}{t-1} & t \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

Thus, a Lipschitz solution only exists on the time interval $0 \leq t < 1$