Math 174 Homework 2

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1 Exercise 1.2.ii

Solution

i. Suppose $f : x \to y$ is split epic in C. Then there is an arrow $g : y \to x$ such that $fg = id_y$. Given $h : c \to y \in Mor(c, y)$, one can construct an arrow $hg : c \to x \in Mor(c, x)$ such that this diagram commutes:

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \\
& c \arrow[u, "h"'] \arrow[ul, dashed, "g\circ h=k"]
\end{tikzcd}$$

So for any $h \in Mor(c, y)$, the map $\tau : Mor(c, x) \to Mor(c, y)$, $k \mapsto f \circ k$ is surjective, since if we are given $h \in Mor(c, y)$, we can always find a map k such that $h = fk$. On the other hand, suppose $\tau : Mor(c, x) \to Mor(c, y)$ is surjective. Then for every $c \in C$ and every $h \in Mor(c, y)$, there is $k \in Mor(c, x)$ such that $fk = h$. Since this holds for every $c \in C$, taking $c = y$ gives $h = id_y : y \to y$ and $k : y \to x$ such that $fk = id_y$.

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "k"] \arrow[r, "id_y"] & y
\end{tikzcd}$$

Thus f is split epic, and f is split epic if and only if the post composition map $Mor(c, x) \to Mor(c, y)$ is surjective.

ii. Flipping the arrows in the two diagrams above gives two new diagrams:

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[dr, "k"] & y \arrow[l, bend left, "g"] \arrow[d, "k\circ g"'] \\
& c
\end{tikzcd}$$

and

$$\begin{tikzcd}
x & x \arrow[l, "id_x"'] \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"]
\end{tikzcd}$$

The first diagram shows how a given a left inverse to f or the fact that f is split monic, every arrow $k \in Mor(x, c)$ can be written as the pre-composition $f \circ gk$. On the other hand, if the pre-composition map is surjective, taking $c = x$ gives the second diagram and a left inverse for f, thus showing that f is split monic.

2 Exercise 1.2.iii

Solution

i. Suppose $f : x \to y$ and $g : y \to z$ are monic. And consider $gf : x \to z$ with $h_1, h_2 : x \to z$ as well. If $h_1gf = h_2gf$, then by associativity and and the monicity of f $h_1g = h_2g$. Then again by the monicity of g, $h_1 = h_2$, so gf is monic as well.

ii. Suppose $f : x \to y$ and $g : y \to z$ exist such that gf is monic. Then, if $h_1gf = h_2gf$, $h_1 = h_2$. But clearly if $h_1 = h_2$, then $h_1g = h_2g$, and so f must be monic as well.

i'. Applying the argument of **i** in the opposite category, and using the fact that a split monic morphism in C is split epic in C^{op} gives the desired result.

ii'. Applying the argument of **ii** in the opposite category, and using the fact that a split monic morphism in C is split epic in C^{op} gives the desired result.

3 Exercise 1.3.i

Solution A functor $F : BG \to BH$ is just a group homomorphism $\phi : G \to H$. Both BG and BH both only have one object, so they trivially must be mapped to eachother. The morphisms of BG and BH are the elements of G and H respectively, and by functoriality we have for $a, b \in Mor(BG)$, $F(ab) = F(a)F(b)$, which is the same as the homomorphism requirement $\phi(ab) = \phi(a)\phi(b)$. Similarly, since homomorphims map identities to identities, $F(e_G) = e_H$, so thus a functor between two delooping groups is just a group homomorphism between the two groups.