Math H104 Homework 2

Aniruddh V.

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1 Excerise 37

Let $X_1, X_2 \in \mathcal{P}X$ with $X_1 \subseteq X_2$. Then $UX_1 = \{y \in Y | \forall x \in X_1, \rho(x,y)\}$ and $UX_2 = \{y \in Y | \forall x \in X_2, \rho(x,y)\}$. Since $X_1 \subseteq X_2$, if $x \in X_1$, then $x \in X_2$, so this implies that $UX_1 \supseteq UX_2$. Thus $U : (\mathcal{P}X, \subseteq) \to (\mathcal{P}Y, \supseteq)$ is a morphism of ordered sets.

Now, let $Y_1, Y_2 \in \mathcal{P}Y$ with $Y_1 \supseteq Y_2$. Then $LY_1 = \{x \in X | \forall y \in Y_1 \rho(x,y)\}$ and $LY_1 = \{x \in X | \forall y \in Y_1 \rho(x,y)\}$. Since $Y_1 \supseteq Y_2$, if $y \in Y_2$, then $y \in Y_1$ as well, so this implies that $UY_1 \subseteq UY_2$. Thus $L : (\mathcal{P}Y, \supseteq) \to (\mathcal{P}X, \subseteq)$ is a morphism of ordered sets.

2 Excerise 38

Suppose $A \subseteq LB$. Then for all $x \in A$, $\rho(x,y)$ holds for all $y \in B$. But this also means that if $y \in B$, $\rho(x,y)$ holds for all $x \in A$, so $B \subseteq UA$ Now suppose $B \subseteq UA$. Then if $y \in B$, the relation $\rho(x,y)$ holds for all $x \in A$. But this implies that if $x \in A$, the relation $\rho(x,y)$ holds for all $y \in B$, so then $A \subseteq LB$. Thus $A \subseteq LB$ if and only if $UA \supseteq B$

3 Excerise 39

LUA contains all x such that $\rho(x,y)$ holds for all $y \in UA$. But $y \in UA$ if and only if $\rho(x,y)$ holds for all $x \in A$, so clearly if $x \in A$, then $x \in LUA$. Thus $A \subseteq LUA$. Similarly, ULB contains all y such that $\rho(x,y)$ holds for all $x \in LB$. But $x \in LB$ if and only if $\rho(x,y)$ holds for all $y \in B$, so clearly if $y \in B$, then $y \in ULB$. Thus $ULB \supseteq B$

4 Excerise 40

Let $x \in LB$. Then for all $y \in B$, the relation $\rho(x,y)$ holds. But this means that $y \in ULB$, and hence $x \in LULB$, thus $LB \subseteq LULB$. Now let $x \in LULB$. Then for all $y \in ULB$, the relation $\rho(x,y)$ holds. But $y \in ULB$ if and only if the relation $\rho(x,y)$ holds for all $x \in LB$, thus $x \in LB$ as well, and $LULB \subseteq LB$, so $LULB = LB$

Now, let $y \in UA$. Then the relation $\rho(x,y)$ holds for all $x \in A$. But this means $x \in LUA$, and hence $y \in ULUA$, thus $UA \subseteq ULUA$ Now let $y \in ULUA$. Then for all $x \in LUA$, the relation $\rho(x,y)$ holds, But $x \in LUA$ if and only if the relation $\rho(x,y)$ holds for all $y \in UA$, thus $y \in UA$ as well, and $ULUA \subset UA$, so $ULUA = UA$

5 Excerise 41

If $A \subseteq X$, then $UA = \{Y \in \mathcal{P}X | \forall x \in A : x \in Y\} = \{Y \in \mathcal{P}X : Y \supseteq A\}$. Similarly, if $\mathcal{A} \in \mathcal{P}X$, then $L\mathcal{A} = \{x \in X | \forall Y \in \mathcal{A} : x \in Y\}$

6 Excerise 42

Let $A, B \subseteq \mathcal{P}X$. Then $UA = C \in \mathcal{P}X : A \subseteq C$ and $LB = C \in \mathcal{P}X : C \subseteq B$

7 Excerise 43

Suppose f, g is a bimorphism from ρ to ρ' and f', g' is a bimorphism from $\rho' \to \rho''$. Since f, g is a bimorphism, $(id_X, g)^*\rho \implies (f, id_y)^*\rho'$. Similarly since f', g' is a bimorphism, $(id_X, g')^*\rho' \implies (f', id_Y)^*\rho''$. But then $(id_X, g \circ g')^*\rho \to (f \circ f', id_Y)^*\rho''$ so $f \circ f', g \circ g'$ is a bimorphism from ρ to ρ''

8 Excerise 44

$$
\begin{tikzcd}
\mathcal{P}\mathcal{P}X \arrow[r, "\cap"] \arrow[d, "f_{**}"'] & \mathcal{P}X \arrow[d, "f_*"] \arrow[dl, phantom, "\subseteq"] \\
\mathcal{P}\mathcal{P}Y \arrow[r, "\cap"'] & \mathcal{P}Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}\mathcal{P}X \arrow[r, "\cup"] \arrow[d, "f_{!*}"'] & \mathcal{P}X \arrow[d, "f_!"] \arrow[dl, phantom, "\subseteq"] \\
\mathcal{P}\mathcal{P}Y \arrow[r, "\cup"'] & \mathcal{P}Y
\end{tikzcd}
$$

9 Excerise 45

Suppose f, g is a bimorphism from ρ to ρ'. This means that $(id_X, g)^*\rho \implies (f, id_y)^*\rho'$. Since $\rho^{op}(x, y) \iff \rho(y, x)$, we have $(id_Y, f)^*(\rho^{op})' \implies (g, id_X)^*\rho^{op}$ which means g, f is a bimorphism from $(\rho')^{op}$ to ρ^{op}

10 Excerise 46

Proof 1: Condition (c) of Lemme 3.1 implies $\forall x \in X|f(x)\rangle \supseteq g^*|x\rangle$. This implies that

$$\cap_{x \in X}|f(x)\rangle \supseteq \cap_{x \in X} g^*|x\rangle$$

which implies

$$U(f_*A) \supset g^*UA$$

Proof 2: Using the equivalence of statements b and b', we have $f^*LB' \supseteq L(g_*B')$. Then applying L to both sides with $B' = UA$ gives $U(f_*A) \supseteq g^*UA$

11 Excerise 47

The following conditions are equivalent:

- A pair of functions f, g is a faithful bimorphism
- $\forall B' \subseteq Y' : f^*LB' = L(g_*B')$
- $\forall A \subseteq X : U(f_*A) = g^*UA$

The second condition $\forall B' \subseteq Y' : f^*LB' = L(g_*B')$ can be expressed in the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{P}Y' \arrow[r, "g_*"] \arrow[d, "L"'] & \mathcal{P}Y \arrow[d, "L"] \\
\mathcal{P}X' \arrow[r, "f^*"'] & \mathcal{P}X
\end{tikzcd}
$$

The third condition $\forall A \subseteq X : U(f_*A) = g^*UA$ can be expressed in the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{P}X \arrow[r, "f_*"] \arrow[d, "U"'] & \mathcal{P}X' \arrow[d, "U"] \\
\mathcal{P}Y \arrow[r, "g^*"'] & \mathcal{P}Y'
\end{tikzcd}
$$