Math 113 Homework 3

Aniruddh V.

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1 Exercise 16

Solution We have the following commutative diagram:

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\phi_1"'] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\
Z_2 & Y_2 \arrow[l, tail, "\phi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd}$$

Since χ_1 is injective, it has a left inverse χ_1^{-1} such that $\chi_1^{-1} \circ \chi_1 = id_{X_1}$. Similarly, since ϕ_2 is surjective, it has a right inverse ϕ_2^{-1} such that $\phi_2 \circ \phi_2^{-1} = id_{Z_2}$.

Now suppose there are β_1 and β_2 such that the diagram below commutes.

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\phi_1"'] \arrow[d, dashed, bend right, "\beta_1"'] \arrow[d, dashed, bend left, "\beta_2"] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\
Z_2 & Y_2 \arrow[l, tail, "\phi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd}$$

With the inverses added in, the diagram looks as follows:

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\phi_1"'] \arrow[d, dashed, bend right, "\beta_1"'] \arrow[d, dashed, bend left, "\beta_2"] \arrow[r, dashed, bend left, "\chi_1^{-1}"] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\
Z_2 \arrow[r, dashed, bend right, "\phi_2^{-1}"'] & Y_2 \arrow[l, tail, "\phi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd}$$

From these diagrams we can read off the following relations: $\alpha \circ \phi_1 = \phi_2 \circ \beta_1$ and $\beta_2 \circ \chi_1 = \chi_2 \circ \gamma$. Now using applying the left and right inverses gives $\beta_1 = \phi_2^{-1} \circ \alpha \circ \phi_1$ and $\beta_2 = \chi_2 \circ \gamma \circ \chi_1^{-1}$. But by the commutativity of the first diagram, we have $\alpha \circ \phi_1 \circ \chi_1 = \phi_2 \circ \chi_2 \circ \gamma$, and applying inverses gives $\phi_2^{-1} \circ \alpha \circ \phi_1 = \chi_2 \circ \gamma \circ \chi_1^{-1}$. But then we have

$$\beta_1 = \phi_2^{-1} \circ \alpha \circ \phi_1 = \chi_2 \circ \gamma \circ \chi_1^{-1} = \beta_2$$

so $\beta_1 = \beta_2 \coloneqq \beta$ is unique, as desired.

2 Exercise 20

Solution Suppose $f^*B \subseteq A$. Then, by definition, if $x \in X$ such that $f(x) \in B$, then $x \in A$, so $B \subseteq f_*A$. But if $x \in f_*A$, then $x \notin f_*(X \setminus A)$, so $x \in Y \setminus f_*(X \setminus A) = f_!A$. On the other hand, suppose $B \subseteq f_!A$. Then for all $x \in B$, $x \in Y \setminus f_*(X \setminus A)$, so $x \notin f_*(X \setminus A)$. Thus $x \in f_*A$ which means $f^*B \subseteq A$. Thus $f^*B \subseteq A$ if and only if $B \subseteq f_!A$

3 Exercise 23

Solution The functions $(\mathrm{id}_X)_*$, $(\mathrm{id}_X)^*$ and $(\mathrm{id}_X)_!$ are all clearly mappings from $\mathscr{P}X \to \mathscr{P}X$, so it only remains to show that the assignments of all three functions are equal to the assignments on $\mathscr{P}X$. By the definition of f_* and f^*, we obviously have $(\mathrm{id}_X)^* = \mathrm{id}_{\mathscr{P}X}$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\mathscr{P}X}$. By definition, we have $(\mathrm{id}_X)_! A = X \setminus (\mathrm{id}_X)_*(X\ A) = A$, so $(\mathrm{id}_X)_! = \mathrm{id}_{\mathscr{P}X}$

4 Exercise 28

Solution Let $f : X \to Y$ be a function, with $\mathscr{A} \subset \mathscr{P}X$ a family of subsets of X, and $\mathscr{B} \subset \mathscr{P}Y$ a family of subsets of Y. Let $y \in f_*(\cup\mathscr{A})$. Then there is $x \in \cup\mathscr{A}$ such that $f(x) = y$, so $y \in \cup f_{**}\mathscr{A}$. Thus we have $f_*(\cup\mathscr{A}) \subseteq \cup f_{**}\mathscr{A}$. Now suppose $y \in \cup f_{**}\mathscr{A}$. Then $y \in A$ for some $A \in \mathscr{A}$, so $y \in f_*(\cup\mathscr{A})$. Thus $f_*(\cup\mathscr{A}) = \cup f_{**}\mathscr{A}$. This can be represented in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{P}\mathscr{P}X \arrow[r, "f_{**}"] \arrow[d, "\cup"'] & \mathscr{P}\mathscr{P}Y \arrow[d, "\cup"] \\
\mathscr{P}X \arrow[r, "f_*"'] & \mathscr{P}Y
\end{tikzcd}$$

Now suppose $x \in f^*(\cup\mathscr{B})$. Then there is $y \in \cup\mathscr{B}$ such that $f(x) = y$. But $y \in B \in \mathscr{B}$ we also have $x \in \cup {f^*}_*\mathscr{B}$. Thus we have $f^*(\cup\mathscr{B}) \subseteq \cup {f^*}_*\mathscr{B}$. On the other hand, suppose $x \in \cup {f^*}_*\mathscr{B}$. Then $x \in B$ for some $B \in \mathscr{B}$, so we also have $x \in f^*(\cup\mathscr{B})$. Thus $f^*(\cup\mathscr{B}) = \cup {f^*}_*\mathscr{B}$. This can be represented in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{P}\mathscr{P}Y \arrow[r, "{f^*}_*"] \arrow[d, "\cup"'] & \mathscr{P}\mathscr{P}X \arrow[d, "\cup"] \\
\mathscr{P}Y \arrow[r, "f^*"'] & \mathscr{P}X
\end{tikzcd}$$

5 Exercise 29

Solution Let $x \in f^*(\cap\mathscr{B})$. Then for all $B \in \mathscr{B}$, there is $y \in B \in \mathscr{B}$ such that $y = f(x)$. But then $x \in {f^*}_*\mathscr{B}$, so $x \in \cap {f^*}_*\mathscr{B}$. Thus $f^*(\cap\mathscr{B}) \subseteq \cap {f^*}_*\mathscr{B}$ Now suppose $x \in \cap {f^*}_*\mathscr{B}$. Then there is $y = f(x)$ such that $y \in \cap\mathscr{B}$, so $x \in f^*(\cap\mathscr{B})$. Thus $f^*(\cap\mathscr{B}) = \cap {f^*}_*\mathscr{B}$. This is represented in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{P}\mathscr{P}Y \arrow[r, "{f^*}_*"] \arrow[d, "\cap"'] & \mathscr{P}\mathscr{P}X \arrow[d, "\cap"] \\
\mathscr{P}Y \arrow[r, "f^*"'] & \mathscr{P}X
\end{tikzcd}$$

Next, let $y \in f_!(\cap\mathscr{A})$. Then for all $A \in \mathscr{A}$, $y \in f_!A$, so $y \in \cap f_{!*}\mathscr{A}$. Thus $f_!(\cap\mathscr{A}) \subseteq \cap f_{!*}\mathscr{A}$. Now, let $y \in f_{!*}\mathscr{A}$. Then for all $A \in \mathscr{A}$, $y \in f_!A$. Thus $y \in f_!(\cap\mathscr{A})$, and $f_!(\cap\mathscr{A}) = \cap f_{!*}\mathscr{A}$. This can be represented in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{P}\mathscr{P}X \arrow[r, "f_{!*}"] \arrow[d, "\cap"'] & \mathscr{P}\mathscr{P}Y \arrow[d, "\cap"] \\
\mathscr{P}X \arrow[r, "f_!"'] & \mathscr{P}Y
\end{tikzcd}$$

6 Exercise 30

Solution Let $y \in f_*(\cap \mathscr{A})$. Then for all $A \in \mathscr{A}$, $y \in f_* A$, so $y \in \cap f_{**}\mathscr{A}$. Thus $f_*(\cap \mathscr{A}) \subseteq \cap f_{**}\mathscr{A}$.

Next, let $y \in f_!(\cup \mathscr{A})$. Then there is an $A \in \mathscr{A}$ such that $y \in f_! A$, so $y \in \cup f_{!*}\mathscr{A}$. Thus $f_!(\cap \mathscr{A}) \subseteq \cap f_{!*}\mathscr{A}$.

7 Exercise 31

Solution Let $(A, \cdot)$ be a binary algebraic structure with an associative binary operation. Let $\lambda_a : b \mapsto ab$ be the left multiplication function. Then we have

$$\lambda_{ab} = \lambda_{ab}(x) = (ab)x = a(bx) = \lambda_a(\lambda_b(x)) = \lambda_a \circ \lambda_b$$

so left multiplication is a homomorphism of binary algebraic structures. On the other hand, suppose left multiplication was a homomorphism of binary algebraic structures. Then this implies $\lambda_{ab} = \lambda_a \circ \lambda_b$. But this implies $\lambda_{ab}(x) = \lambda_a \circ \lambda_b(x)$, or $(ab)x = a(bx)$. So the binary algebraic structure is associative. Thus, a binary associative structure is associative if and only if left mutiplication is a homomorphism.