Design and Analysis of Algorithms

Math review

The goal of this document is to review some of the math needed for this course. It also contains links to web sites that provide added information on some of the topic. It contains also additional formulas and procedures.

Proofs and Mathematical Induction

http://www.purplemath.com/modules/inductn.htm

http://zimmer.csufresno.edu/~larryc/proofs/proofs.html

http://www.teachers.ash.org.au/mikemath/mathsc/structureproof/

Logarithms

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^y = y \log_a x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$x^{\log_b y} = y^{\log_b x}$$

$$\lg x = \log_2 x$$

$$\ln x = \log_e x$$

$$\log^k n = (\log n)^k$$

$$\lg \lg n = \lg(\lg n)$$

http://www.purplemath.com/modules/logs.htm

Other

$$m \bmod n = m - n\left\lfloor \frac{m}{n} \right\rfloor$$

Simple Series

1. **Arithmetic series:** The first n values are: a, $a + d$, $a + 2d$, …, $a + (n - 1)d$ where a is an initial value and d is a fixed increment. The sum of the first n values is: $\sum_{i=0}^{n-1} a + id = \frac{(2a + (n-1)d)n}{2}$.

Let $a_1 = a$, $a_n = a + (n-1)d$ the sum is also $\frac{(a_1 + a_n)n}{2}$.

2. **Geometric series:** The first n values are: $a, ar, ar^2, \ldots, ar^{n-1}$ where a is an initial value and $r \neq 1$ is a fixed multiplier. The sum of the first n values is: $\sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}$.

3. When $-1 < r < 1$ and $r \neq 1$ the sum of the infinite geometric progression converges to: $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$.

4. **Harmonic series:** The first n values are: 1, 1/2, 1/3,…,1/n. The sum of the first n values is: $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n$ satisfies $\ln(n+1) < H_n \leq 1 + \ln n$.

[Arithmetic and geometric progressions]

L'Hopital's Rule:

If f(x) and g(x) are both differentiable with derivatives f(x) and g(x), respectively, and if $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$ then

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = \lim_{x \to \infty} \frac{g'(x)}{f'(x)}$$

whenever the limit on the right exists

Review of limits

$$\lim_{x \to c} f(x) = l \text{ iff } \begin{cases} \text{for each } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \text{if } 0 < |x - c| < \delta, \text{ then } |f(x) - l| < \varepsilon. \end{cases}$$

If $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g(x) = m$ then:

a) $\lim_{x \to c} [f(x) + g(x)] = l + m$

b) $\lim_{x \to c} af(x) = al$ for any real a

c) $\lim_{x \to c} [f(x)g(x)] = lm$

d) If $m \neq 0$ then $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$ and $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{l}{m}$

Review of derivatives

$$p(x) = x^n \text{ then } \quad p'(x) = nx^{n-1} \text{ for } n \neq 0$$

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$$

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}(f(g)) = f'(g(x))g'(x)$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{f(x)}) = e^{f(x)} f'(x)$$

$$\frac{d}{dx}(p^x) = p^x \ln p$$

$$\frac{d}{dx}[p^{g(x)}] = p^{g(x)} g'(x) \ln p$$

$$\frac{d}{dx}(\log_p x) = \frac{1}{x \ln p}$$

$$\frac{d}{dx}(\log_p g(x)) = \frac{g'(x)}{g(x) \ln p}$$

Master Theorem

Version 1

$$
T(n) = \begin{cases}
\Theta(n^{\log_b a}) & f(n) \in O(n^{\log_b a - \varepsilon}), \varepsilon > 0 \qquad (1) \\
\Theta(n^{\log_b a} \lg^{k+1} n) & f(n) \in \Theta(n^{\log_b a} \lg^k n), k \geq 0 \qquad (2) \\
\Theta(f(n)) & f(n) \in \Omega(n^{\log_b a + \varepsilon}), \varepsilon > 0 \text{ and} \qquad (3) \\
 & af(n/b) \leq cf(n) \text{ for some} \\
 & c < 1 \text{ and} \\
 & \text{all sufficiently large } n
\end{cases}
$$

Version 2

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \dfrac{f(n)}{n^{\log_b a}} \in O(n^{-\varepsilon}), \varepsilon > 0 \quad (1) \\ \Theta(n^{\log_b a} \lg^{k+1} n) & \dfrac{f(n)}{n^{\log_b a}} \in \Theta(\lg^k n), k \geq 0 \quad (2) \\ \Theta(f(n)) & \dfrac{f(n)}{n^{\log_b a}} \in \Omega(n^{\varepsilon}), \varepsilon > 0 \text{ and} \quad (3) \\ & af(n/b) \leq cf(n) \text{ for some} \\ & c < 1 \text{ and} \\ & \text{all sufficiently large } n \end{cases}$$

Using limits to compare asymptotic growth:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} c \neq 0 \;\; c \neq \infty & f(n) = \Theta(g(n)) \quad * \\ 0 & f(n) = O(g(n)) \\ \infty & f(n) = \Omega(g(n)) \end{cases}$$

* Θ implies both O and Ω

Changing a linear programming problem to standard form

1. Inequality with $\geq$ action: (multiply inequality by –1)

2. Equality = action: (transform to one inequality with $\geq$, and one with $\leq$)

3. Objective function is minimum action: (Multiply objective function by –1 and replace min by max)

4. x is unrestricted action: (Replace x in problem by $y - z$, and add $y \geq 0$, $z \geq 0$)

5. $x \leq c$ for $c < 0$ action: (Replace x in problem by $-y + c$, and add $y \geq 0$)