Backtracking

Sum of Subsets

and

Knapsack

Backtracking

- Two versions of backtracking algorithms
  - Solution only needs to be feasible (satisfy problem's constraints)
    - sum of subsets
  - Solution needs also to be optimal
    - knapsack

The backtracking method

- *A given* ***problem*** *has a set of constraints and may have an objective function*
- *The* ***solution*** *must be feasible and it may optimize an objective function*
- **We can represent the *solution space* for the problem using a *state space tree***
  - **The *root* of the tree represents 0 choice,**
  - **Nodes at depth 1 represent first choice**
  - **Nodes at depth 2 represent the second choice, etc.**
  - **In this tree a *path* from a root to a leaf represents a candidate solution**

Sum of subsets

- ***Problem***: *Given n positive integers $w_{1,} \ldots w_n$ and a positive integer S. Find all subsets of $w_{1,} \ldots w_n$ that sum to S.*

- **Example**:
  n=3, S=6, and w_1=2, w_2=4, w_3=6

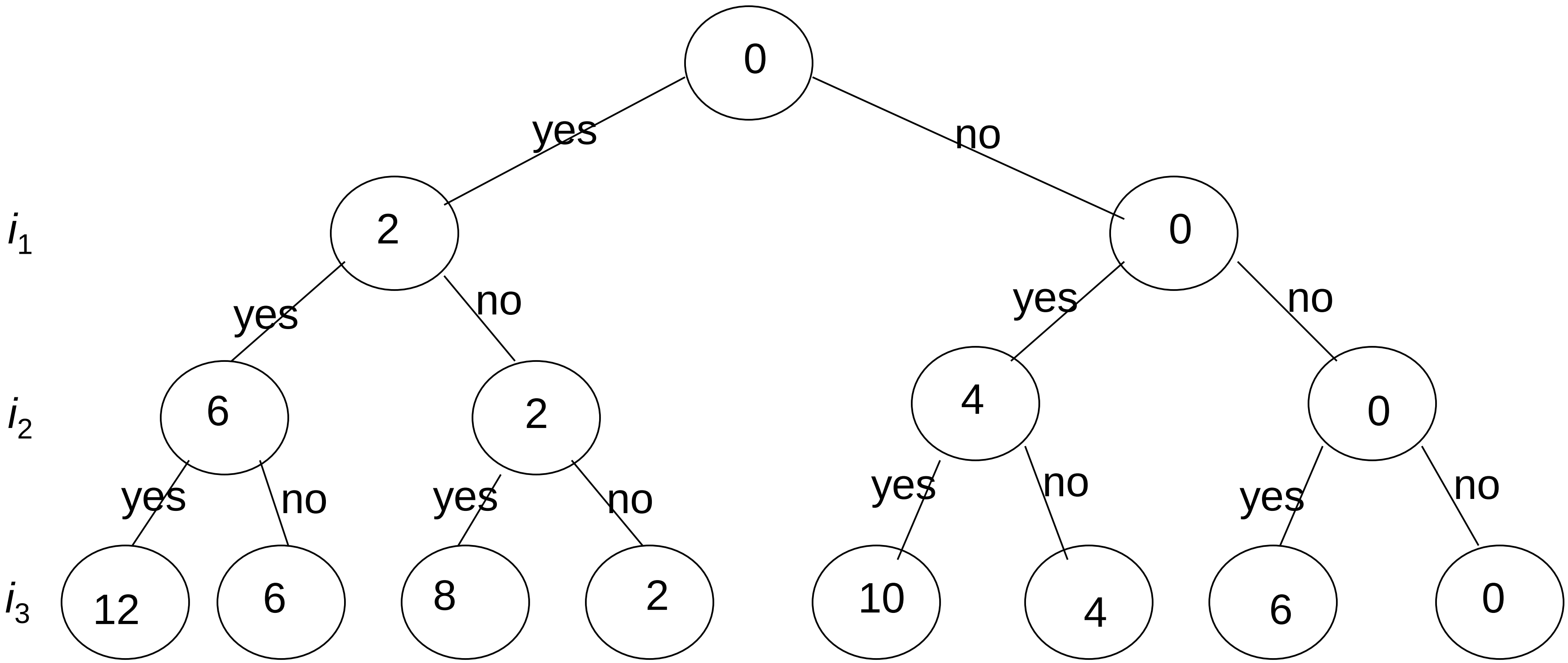
- **Solutions**:
  {2,4} and {6}

Sum of subsets

- We will assume a *binary state space tree*
- The nodes at depth 1 are for including (yes, no) item 1, the nodes at depth 2 are for item 2, etc.
- The left branch includes w_i, and the right branch excludes w_i
- The nodes contain the sum of the weights included so far

Sum of subset Problem:

State Space Tree for 3 items: $w_1 = 2$, $w_2 = 4$, $w_3 = 6$ and $S = 6$



The sum of the included integers is stored at each node.

A Depth First Search solution

- **Problems can be solved using depth first search of the (implicit) state space tree**

- **Each node will save its depth and its (possibly partial) current solution**

- **DFS can check whether node v is a leaf**
  - **If it is a leaf then check if the current solution satisfies the constraints**
  - **Code can be added to find the optimal solution**

A DFS solution

- *Such a DFS algorithm will be very slow!*
  - It does not check whether a solution has been reached already in a solution state (node)
  - Neither does it check whether or not a *partial* solution can lead to a *feasible* solution
  - Is there a more efficient solution?

Backtracking

- **Definition**: We call a node *non-promising* if it cannot lead to a feasible (or optimal) solution, otherwise it is *promising*

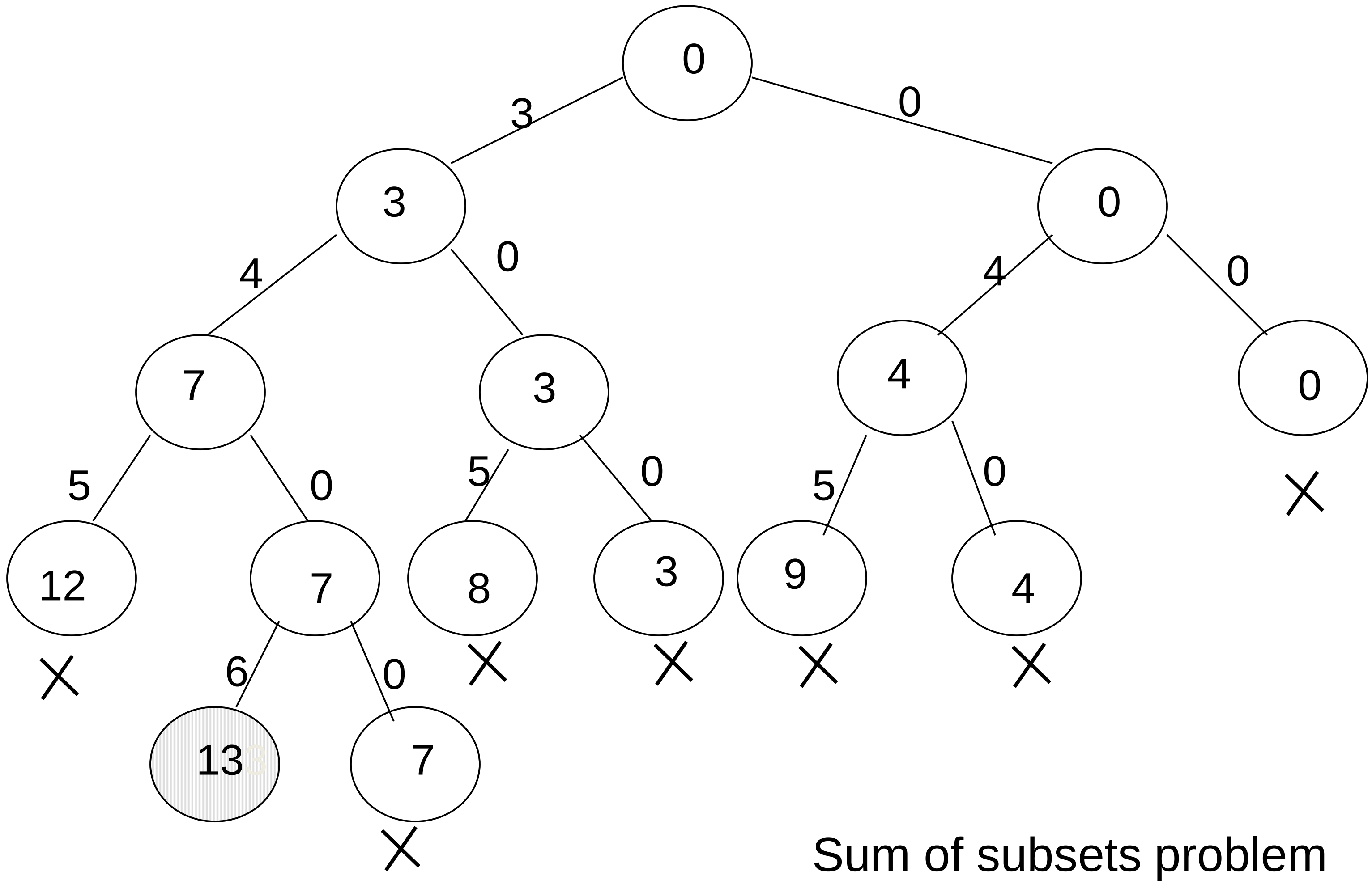
- **Main idea**:
  - Do a DFS in the state space tree
  - *Check whether each node is promising*
  - *Pruning: If the node is nonpromising, backtrack to its parent!!*

Backtracking

- The state space tree consists of the ***pruned state space tree***
- The following slide shows the pruned state space tree for the sum of subsets example
- There are only 15 nodes in the pruned state space tree
- The full state space tree has 31 nodes

A Pruned State Space Tree (find all solutions)

$w_1 = 3, w_2 = 4, w_3 = 5, w_4 = 6; \; S = 13$



Sum of subsets problem

Backtracking algorithm

```
void checknode (node v)
{
   if (promising ( v ))
      if (aSolutionAt( v ))
         write the solution
      else //expand the node
         for ( each child u of v )
            checknode ( u )
}
```

Checknode

- Checknode uses the functions:
  - *promising*(*v*) which checks that the partial solution represented by *v* can lead to the required solution
  - *aSolutionAt*(*v*) which checks whether the partial solution represented by node *v* solves the problem.

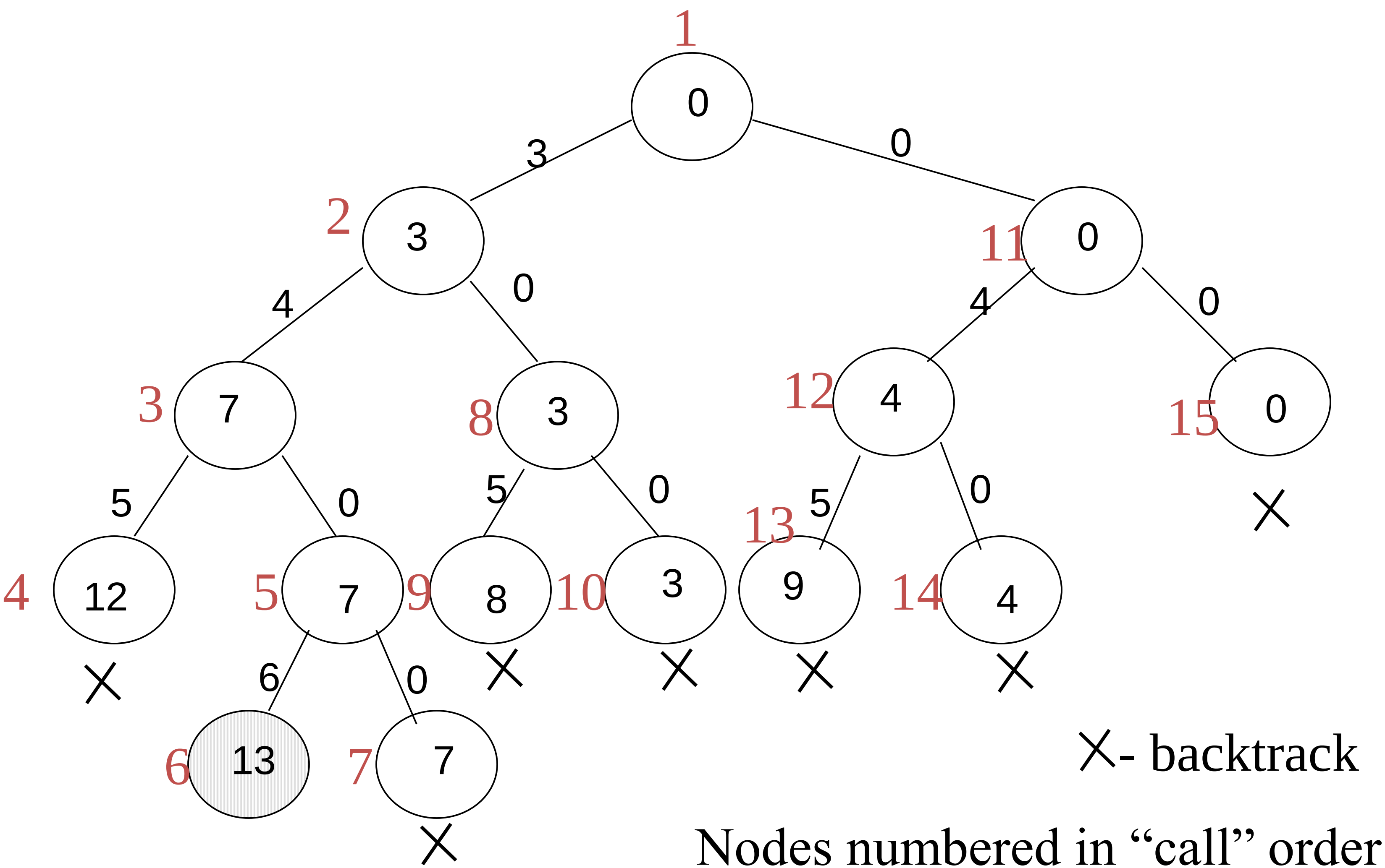
Sum of subsets – when is a node "promising"?

- *Sort* w_i*'s* in *<u>non-descending</u>* order

- Consider a node at depth i
  - *weightSoFar* = weight of a node (sum of numbers included in the partial solution that the current node represents)
  - *totalPossibleLeft* = weight of the remaining items i+1 to n (for a node at depth i)
  - A node at depth *i* is *non-promising*
    if (weightSoFar + totalPossibleLeft < S)
    or (weightSoFar + w[i+1] > S)

A Pruned State Space Tree

$w_1 = 3$, $w_2 = 4$, $w_3 = 5$, $w_4 = 6$; $S = 13$



```
sumOfSubsets ( i, weightSoFar, totalPossibleLeft )
    1) if (promising ( i ))                          //may lead to solution
   2)      then if ( weightSoFar == S )
    3)           then print include[ 1 ] to include[ i ]      //found solution
                       return
    4)     else   //expand the node when weightSoFar < S
    5)           include [ i + 1 ] = "yes"                   //try including
    6)           sumOfSubsets ( i + 1, weightSoFar + w[i + 1],
                        totalPossibleLeft - w[i + 1] )
    7)            include [ i + 1 ] = "no"                     //try excluding
    8)            sumOfSubsets ( i + 1, weightSoFar , totalPossibleLeft –
                 w[i + 1] )
     9) return // nonpromising
```

Initial call: sumOfSubsets(0, 0, $\sum_{i=1}^{n} w_i$)

```
boolean promising (i )
    1) return ( weightSoFar + totalPossibleLeft ≥ S)  and
            ( weightSoFar == S  or weightSoFar +  w[i + 1] ≤ S )
```

Prints all solutions!

Backtracking for ***optimization*** problems

Backtracking for optimization problems

- For optimization, compute:
  - *best*: *value of the best solution achieved so far*
  - *value*(*v*): value of the solution at node *v*
  - *Modify promising(v)*

- *Best* is initialized to a value that is equal to a candidate solution or worse than any possible solution

- *Best* is updated to *value*(*v*) if the solution at *v* is "better"
  - "better" means:
    - larger for maximization
    - smaller for minimization

Modifying promising

- A node is *promising if*
  - *it is feasible and can lead to a feasible solution and*
  - *there is a chance that a better solution than the (current) best can be achieved by expanding it*

- *Bound on the best solution that can be achieved by expanding the node is computed*

- *If the bound > best for maximization (bound < best for minimization), the node is promising*

- *Otherwise, it is non-promising*

Modifying promising for Maximization Problems

- **For a *maximization* problem the bound is an *upper bound***
  - **The largest possible solution that can be achieved by expanding the node is smaller than or equal to the *upper bound***
- **If *upper bound > best* so far, a better solution may be found by expanding the node and the feasible node is *promising***

Modifying promising for Minimization Problems

- *For <u>minimization</u>, the bound is a <u>lower bound</u>*
  - *The smallest possible solution that can be achieved by expanding the node is larger than or equal to the lower bound*

- *If lower bound < best, a better solution may be found and the feasible node is promising*

Template for backtracking in the case of optimization problems

```
Procedure checknode (node v )
{
    if ( value(v) is better than best )
            best = value(v);
    if (promising (v) )
            for (each child u of v)
                    checknode (u);
}
```

- *best* is the best value so far
- *value*(*v*) is the value of the solution at node v

0-1 Knapsack problem

- Solve 0-1 knapsack problem via backtracking
- *How to compute the upper bound?*
  - *Use the <u>optimal greedy algorithm for the fractional</u> knapsack just to compute the upper bound*
  - *<u>Just to check whether node v is promising or not</u>*
  - *Could be higher than actual benefit*

Notation for knapsack

- We use *maxprofit* to denote *best (so far)*
- *profit*(v) to denote *value*(v)

The state space tree for knapsack

- Each *node v* will include 3 values:
  - *profit* (v) = sum of profits of all items included in the knapsack (on a path from root to *v*)
  - *weight* (v)= the sum of the weights of all items included in the knapsack (on a path from root to *v*)
  - *upperBound(v)* is greater than or equal to the maximum benefit that can be found by expanding the whole subtree of the state space tree with root *v.*
- The nodes are numbered in the order of expansion

Promising nodes for 0/1 knapsack

- *Node v is promising if weight(v) < C and upperBound(v) > maxprofit*
- *Otherwise, it is not promising*

- *Note that when weight(v) = C or upperbound(v) = maxprofit the node is non-promising*
  - *Further expansion of v is impossible or won’t increase the total profit*

Main idea for upper bound

- ***Main idea****: KWF (knapsack with fraction) is used to compute upper bound*
- ***Theorem****: The optimal profit for 0/1 knapsack $\leq$ optimal profit for KWF*
  - Clearly the optimal solution to 0/1 knapsack is a possible solution to *KWF*. So the optimal profit of *KWF* is greater than or equal to that of 0/1 knapsack

Computing the upper bound for 0/1 knapsack

- Given node *v* at depth *i,*

  UpperBound(v) =

  KWF4(i+1, weight(v), profit(v), w, p, C, n)

  where w and p are arrays of weights and profits

- *To use KWF4, sort the items in non-ascending p_i / w_i order before applying the backtracking algorithm*

KWF4(i, weight, profit, w, p, C, n)

```
bound = profit

for j = i to n
   x[j] = 0  //initialize variables to 0

while (weight < C && i <= n)  {     //not "full" and more items
    if weight + w[i] <= C                //room for next item
         x[i]=1                          //item i is added to knapsack
         weight = weight + w[i]; bound = bound +p[i];
   else
        x[i]=(C - weight)/w[i]  //fraction of  i added to knapsack
        weight = C; bound = bound + p[i]*x[i]
   i=i+1                          // next item
}

return bound
```

- KWF4 is in O(n) (if items are sorted before applying backtracking)

Pseudo code

- **The arrays *w*, *p*, *include* and *bestset* have size n+1.**
- **Location 0 is not used**
- ***include* contains the current solution**
- ***bestset* the best solution so far**

Knapsack

```
num = 0;    //number of items considered
maxprofit = 0;
knapsack(0, 0, 0);
cout << maxprofit;
for (i = 1; i <= num; i++)
    cout << bestset[i]; //the best solution
```

- maxprofit is initialized to $0, which is the worst profit that can be achieved with positive p_i

knapsack(i, profit, weight)

```
if ( weight <= C && profit > maxprofit ) {
      // save better solution
       maxprofit = profit
       num = i; bestset = include;
   }
if promising(i) {
      include[i + 1] = " yes"
      knapsack(i+1, profit + p[i+1], weight + w[i+1])
      include[i+1] = "no"
      knapsack(i+1,profit,weight)
   }
```

Promising(i)

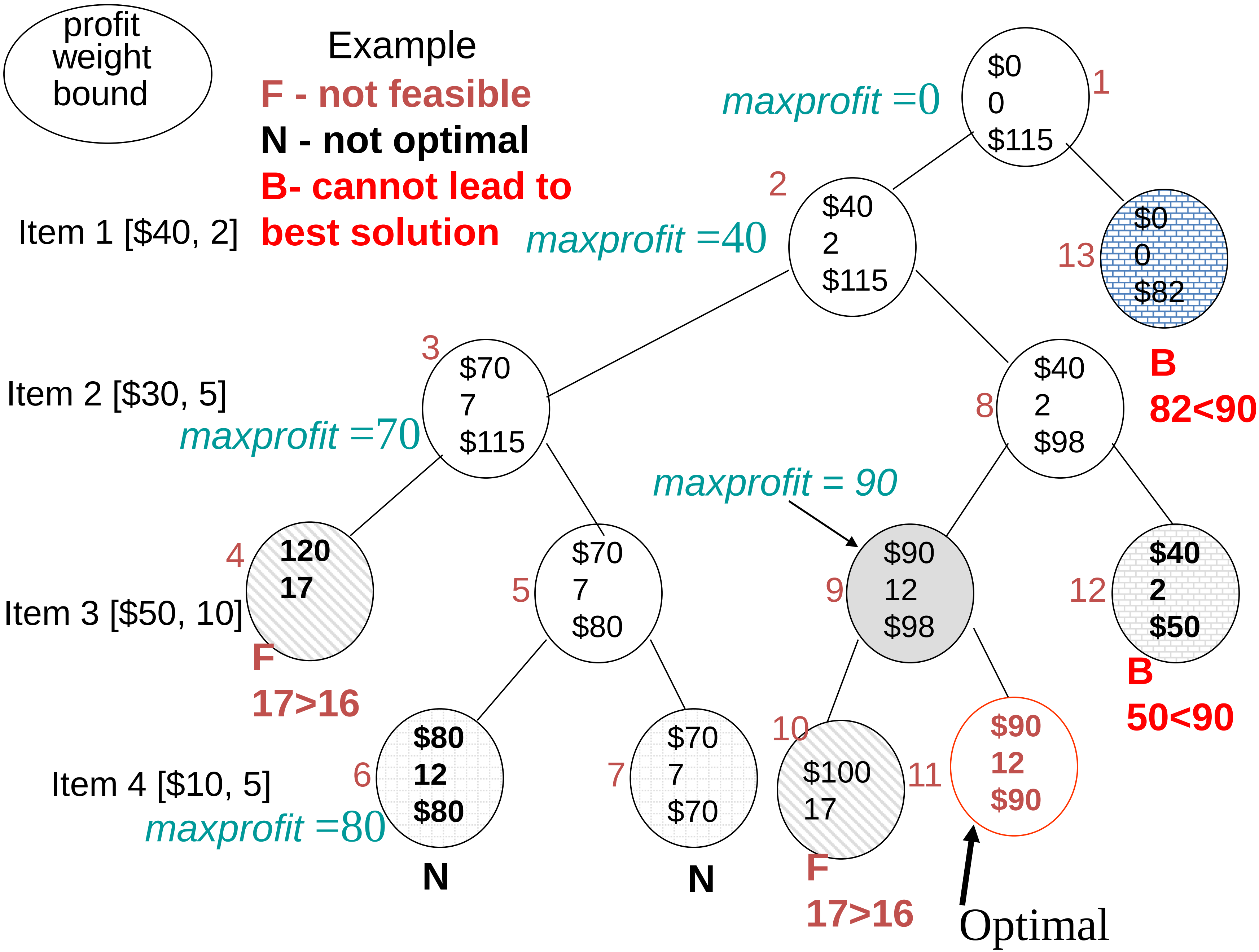
```
promising(i)
{
   //Cannot get a solution by expanding node i
   if weight >= C return false

   //Compute upper bound
   bound = KWF4(i+1, weight, profit, w, p, C, n)

   return (bound > maxprofit)
}
```

Example

- Suppose $n = 4$, C = 16, and we have the following:

i	p_i	w_i	p_i / w_i
1	\$40	2	\$20
2	\$30	5	\$6
3	\$50	10	\$5
5	\$10	5	\$2

- Note the items should be in the *correct order* needed by *KWF (largest profit/weight first)*


profit
weight
bound
Example
F - not feasible
N - not optimal
B- cannot lead to
best solution
Item 1 [$40, 2]
Item 2 [$30, 5]
Item 3 [$50, 10]
Item 4 [$10, 5]
maxprofit =0
1
$0
0
$115
2
$40
2
$115
maxprofit =40
13
$0
0
$82
B
82<90
3
$70
7
$115
maxprofit =70
8
$40
2
$98
maxprofit = 90
4
120
17
F
17>16
5
$70
7
$80
9
$90
12
$98
12
$40
2
$50
B
50<90
6
$80
12
$80
maxprofit =80
N
7
$70
7
$70
N
10
$100
17
F
17>16
11
$90
12
$90
Optimal

The calculation for node 1

maxprofit = \$0 ($n = 4$, $C = 16$)

Node 1

a) *profit* = \$ 0

weight = 0

b) $bound = profit + p_1 + p_2 + (\boldsymbol{C} - 7) * p_3 / w_3$

= \$0 + \$40 + \$30 + (16 -7) X \$50/10 =\$115

c) 1 is promising because its weight =0 < ***C*** = 16 and its bound \$115 > 0 (*maxprofit).*

The calculation for node 2

Item 1 with profit \$40 and weight 2 is included

maxprofit = \$40

a) *profit* = \$40

weight = 2

b) $bound = profit + p_2 + (\boldsymbol{C} - 7) \text{ X } p_3 / w_3$

= \$40 + \$30 + (16 -7) X \$50/10 =\$115

c) 2 is promising because its weight =2 < ***C*** = 16 and its bound \$115 > \$40 the value of *maxprofit.*

The calculation for node 13

Item 1 with profit \$40 and weight 2 is not included

At this point maxprofit=\$90 and is not changed

a) *profit* = \$0
weight = 0

b) $bound = profit + p_2 + p_3 + (\boldsymbol{C} - 15) \times p_4 / w_4$
$= \$0 + \$30 + \$50 + (16 - 15) \times \$10/5 = \$82$

c) 13 is nonpromising because its bound \$82 < \$90 the value of *maxprofit.*

Worst-case time complexity

Check number of nodes:

$$1+2+2^2+2^3+...+2^n=2^{n+1}-1$$

Time complexity:

$$\theta(2^n)$$

When does it happen?

Given n items, suppose that

- $W=n$
- $P_i = 1$, $w_i = 1$ for $1 \le i \le n-1$
- $P_n = n$, $w_n = n$

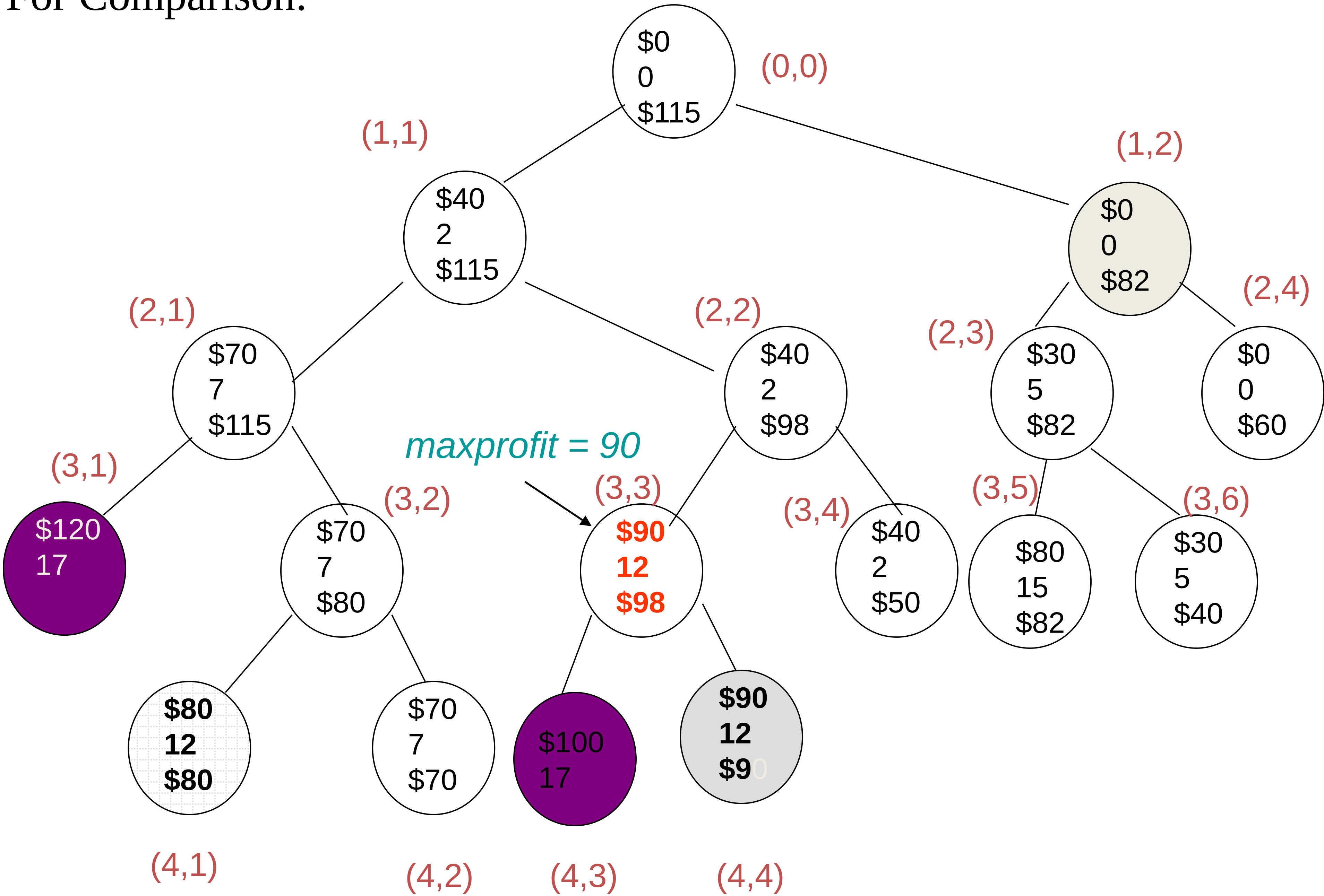
Branch-and-Bound

Knapsack

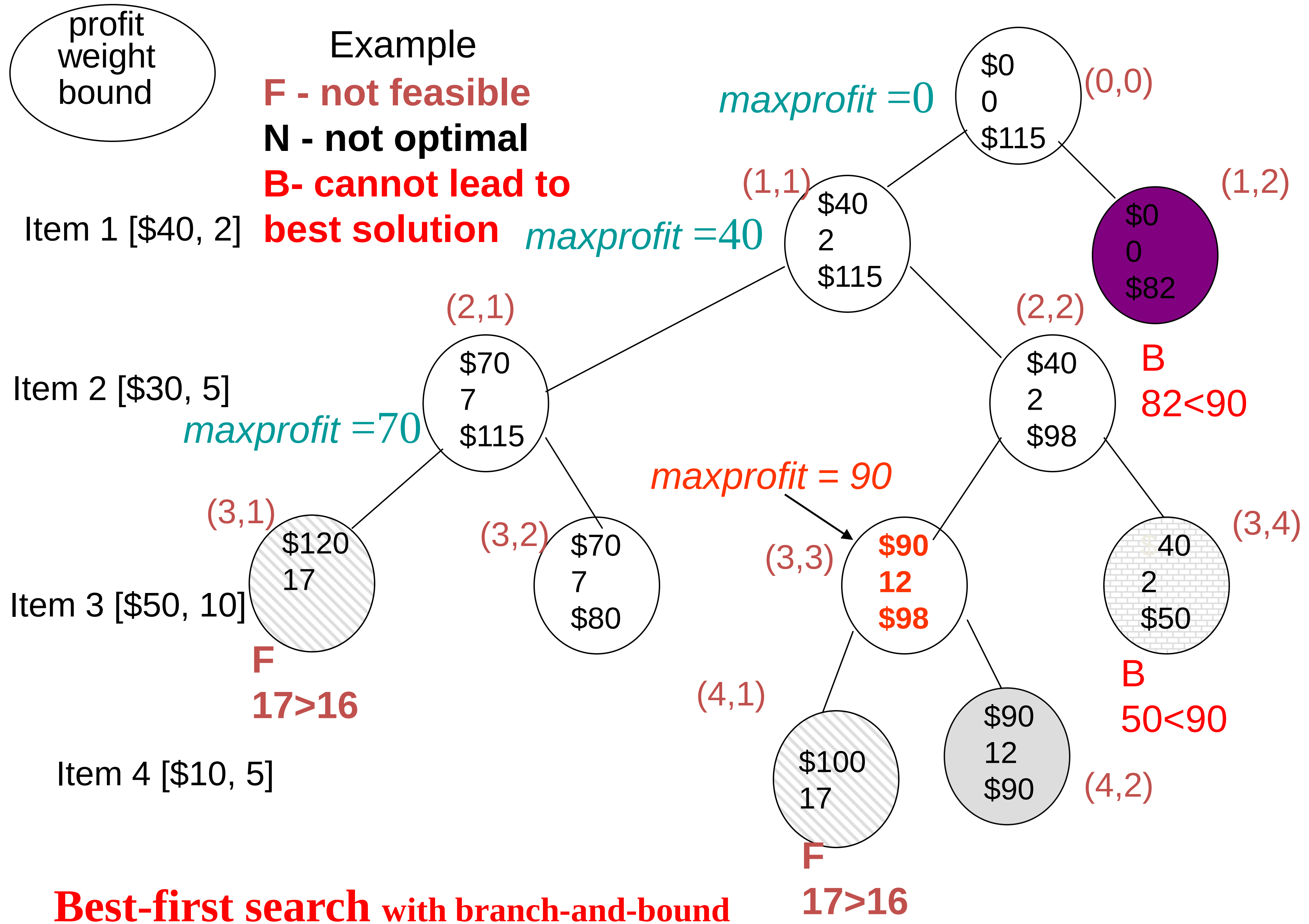
Characteristics

- Use strategy similar to breadth-first-search with some modification
- Visit all the children of a given node to look at all the promising, unexpanded nodes and expand beyond the one with the best bound (e.g., greatest bound)
- Exponential-time in the worst case (same as backtracking algorithm), but could be very efficient for many large instances.

For Comparison:



Breath-first search with branch-and-bound pruning



Best-first search **with branch-and-bound**

(Expand the unexplored node with the greatest bound)