Recurrences

Goal

- Learn how to design and analyze recursive algorithms
- Learn when to use or not to use recursive algorithms
- Derive & solve recurrence equations to analyze recursive algorithms

Recursion

- What is the recursive definition of *n*! ?

$$n! = \begin{cases} 1 & \text{if } n \text{ is } 0 \text{ or } 1 \\ n * ((n-1)!) & \text{otherwise} \end{cases}$$

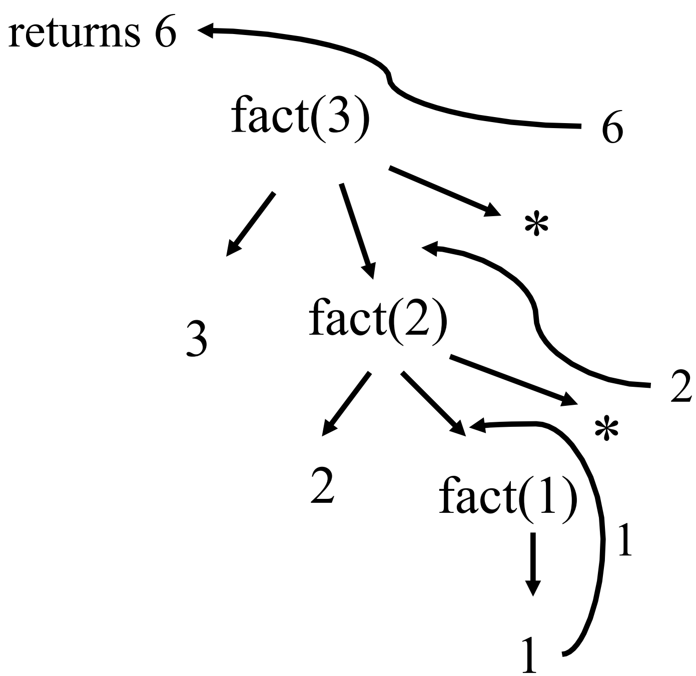
- Program

```
fact(n) {
   if (n<=1) return 1;
   else return n*fact(n-1);
}
// Note '*' is done after returning from fact(n-1)
```

Recursive algorithms

- A recursive algorithm typically contains recursive calls to the same algorithm
- In order for the recursive algorithm to terminate, it must contain code to directly solve some "base case(s)"with no recursive calls
- We use the following notation:
  - *DirectSolutionSize* is the "size" of the base case
  - *DirectSolutionCount* is the number of operations done by the "direct solution"

A Call Tree for fact(3)




```
int fact(int n)
{
    if (n<=1) return 1;
    else return n*fact(n-1);
}
```

The Run Time Environment

- When a function is called an activation records('ar') is created and pushed on the program stack.
- The activation record stores copies of local variables, pointers to other ‘ar’ in the stack and the return address.
- When a function returns the stack is popped.

Goal: Analyzing recursive algorithms

- Until now we have only analyzed (derived the count of) non-recursive algorithms.

- In order to analyze recursive algorithms, we must learn to:
    - Derive the recurrence equation from the code
    - Solve recurrence equations

Deriving a Recurrence Equation for a Recursive Algorithm

- Our goal is to compute the count (Time) $T(n)$ as a function of n, where n is the size of the problem
- We will first write a recurrence equation for $T(n)$

  For example, $T(n)=T(n-1)+1$ and $T(1)=1$

- Then we will solve the recurrence equation. What's the solution to $T(n)=T(n-1)+1$ and $T(1)=1$?

Deriving a Recurrence Equation for a Recursive Algorithm

1. Determine the “size of the problem”. The count T is a function of this *size*

2. Determine *DirectSolSize*, such that for *size* ≤ *DirectSolSize* the algorithm computes a direct solution, with the *DirectSolCount(s).*

$$T(size) = \begin{cases} DirectSolCount & size \le DirectSolSize \\ GeneralCount & \text{otherwise} \end{cases}$$

Deriving a Recurrence Equation for a Recursive Algorithm

To determine *GeneralCount*:

3. Analyze the total number of recursive calls, k, done by a single call of the algorithm and their counts, $T(n_1), \ldots, T(n_k)$ → *RecursiveCallSum* = $\sum_{i=1}^{k} T(n_i)$

4. Determine the "non recursive count" $t(size)$ done by a single call of the algorithm, i.e., the amount of work, excluding the recursive calls done by the algorithm

$$T(size) = \begin{cases} DirectSolCount & size \le DirectSolSize \\ RecursiveCallSum + t(size) & \text{otherwise} \end{cases}$$

Solving recurrence equations

- Techniques for solving recurrence equations:
  - Recursion tree: informal method
  - ***Iteration method***
  - ***Master Theorem***
- We discuss these methods with examples.

Deriving the count using the recursion tree method

- Recursion trees provide a convenient way to represent the unrolling of a recursive algorithm
- It is not a formal proof but a good technique to compute the count.
- Once the tree is generated, each node contains its "non recursive number of operations" $t(n)$ or *DirectSolutionCount*
- The count is derived by summing the "non recursive number of operations" of all the nodes in the tree
- For convenience, we usually compute the sum for all nodes at each given depth, and then sum these sums over all depths.

Building the recursion tree

- The initial recursion tree has a single node containing two fields:
  - The recursive call, (for example *Factorial*(n)) and
  - the corresponding count $T(n)$.
- The tree is generated by:
  - unrolling the recursion of the node depth 0,
  - then unrolling the recursion for the nodes at depth 1, etc.
- The recursion is unrolled as long as the size of the recursive call is greater than *DirectSolutionSize*

Building the recursion tree

- When the “recursion is unrolled”, each current leaf node is substituted by a subtree containing a root and a child for each recursive call done by the algorithm.
  - The root of the subtree contains the recursive call, and the corresponding “non recursive count”.
  - Each child node contains a recursive call, and its corresponding count.
- The unrolling continues, until the “size” in the recursive call is *DirectSolutionSize*
- Nodes with a call of *DirectSolutionSize*, are not “unrolled”, and their count is replaced by *DirectSolutionCount*

Divide and Conquer

- Basic idea: divide a problem into smaller portions, solve the smaller portions and combine the results (if necessary).
- Name some algorithms you already know that employ this technique.
- D&C is a **top down** approach. We often use recursion to implement D&C algorithms.
- The following is an "outline" of a divide and conquer algorithm

Divide and Conquer

- Let *size*(*I*) = *n*
- *DirectSolutionCount* = *DS*(*n*)
- *t*(*n*) = *D(n)* + *C(n) where:*
  - *D*(*n*) = instruction counts for dividing problem into subproblems
  - *C*(*n*) = instruction counts for combining solutions

$$T(n) = \begin{cases} DS(n) & \text{for } n \le DirectSolutionSize \\ \sum_{i=1}^{k} T(n_i) + D(n) + C(n) & \text{otherwise} \end{cases}$$

Divide and Conquer

- Main advantages
  - Code: simple
  - Algorithm: efficient
  - Implementation
    - Parallel computation is possible, e.g., parallel quick sort, parallel merge sort, parallel convex hull, etc.
    - Parallel algorithm is an advanced topic. If we have time, we can discuss a few representative parallel algorithms in the end of the semester

Binary search

- Assumption: The list *S*[*low*…*high*] is sorted, and x is the search key
- If the search key *x* is in the list, *x* == S[*i*], and the index *i* is returned.
- If *x* is not in the list a *NoSuchKey* is returned

Binary search

- The problem is divided into 3 subproblems
  - $x = S[mid]$, $x \in S[low,..,mid-1]$, $x \in S[mid+1,..,high]$

- The first case $x = S[mid]$ is easily solved

- The other cases
  $x \in S[low,..,mid-1]$, or $x \in S[mid+1,..,high]$ require a recursive call

- When the array is empty the search terminates with a "non-index value"

```
BinarySearch(S, x, low, high)
    if low > high then
        return NoSuchKey
    else
        mid ← floor ((low+high)/2)
        if (x == S[mid])
            return mid
        else if  (x < S[mid]) then
            return BinarySearch(S, x, low, mid-1)
        else
            return BinarySearch(S, x,  mid+1, high)
```

Worst case analysis

- A worst input (what is it?) causes the algorithm to keep searching until low>high
- Assume $2^k \leq n < 2^{k+1}$ $k = \lfloor \lg n \rfloor$
- $T(n)$: worst case number of comparisons for the call to $BS(n)$

$$\boxed{T(n) = \begin{cases} 0 \text{ for} & n = 0 \\ 1 \text{ for} & n = 1 \\ 1 + T(\lfloor n/2 \rfloor) \text{ for} & n > 1 \end{cases}}$$

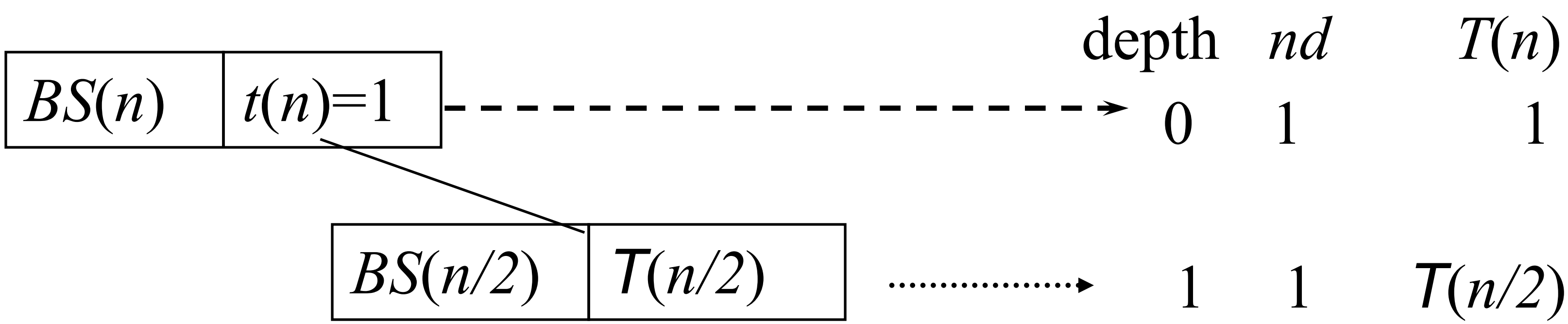
Recursion tree for BinarySearch (*BS*)

$BS(n)$	$T(n)$

- Initially, the recursive tree is a node containing the call to *BS*(*n*), and total amount of work in the worst case, *T*(n).

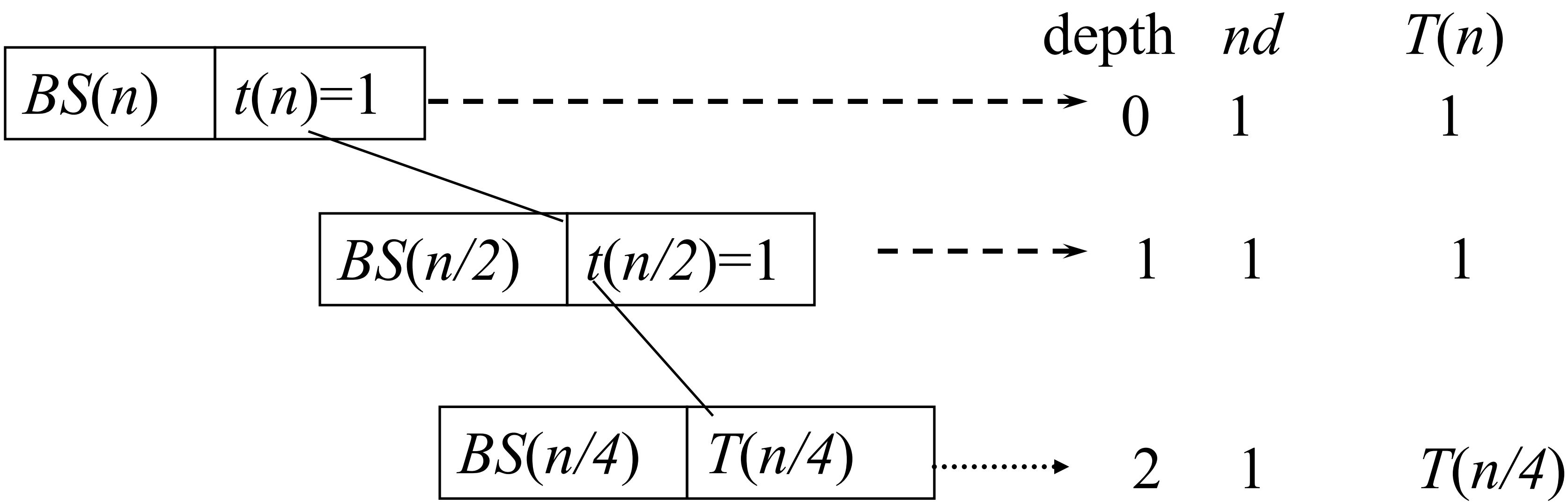
- When we unroll the computation this node is replaced with a subtree containing a root and one child:
- The root of the subtree contains the call to *BS*(*n*) , and the "nonrecursive work" for this call *t*(*n*).
- The child node contains the recursive call to *BS*(*n*/2), and the total amount of work in the worst case for this call is*T*(*n*/2).

After first unrolling



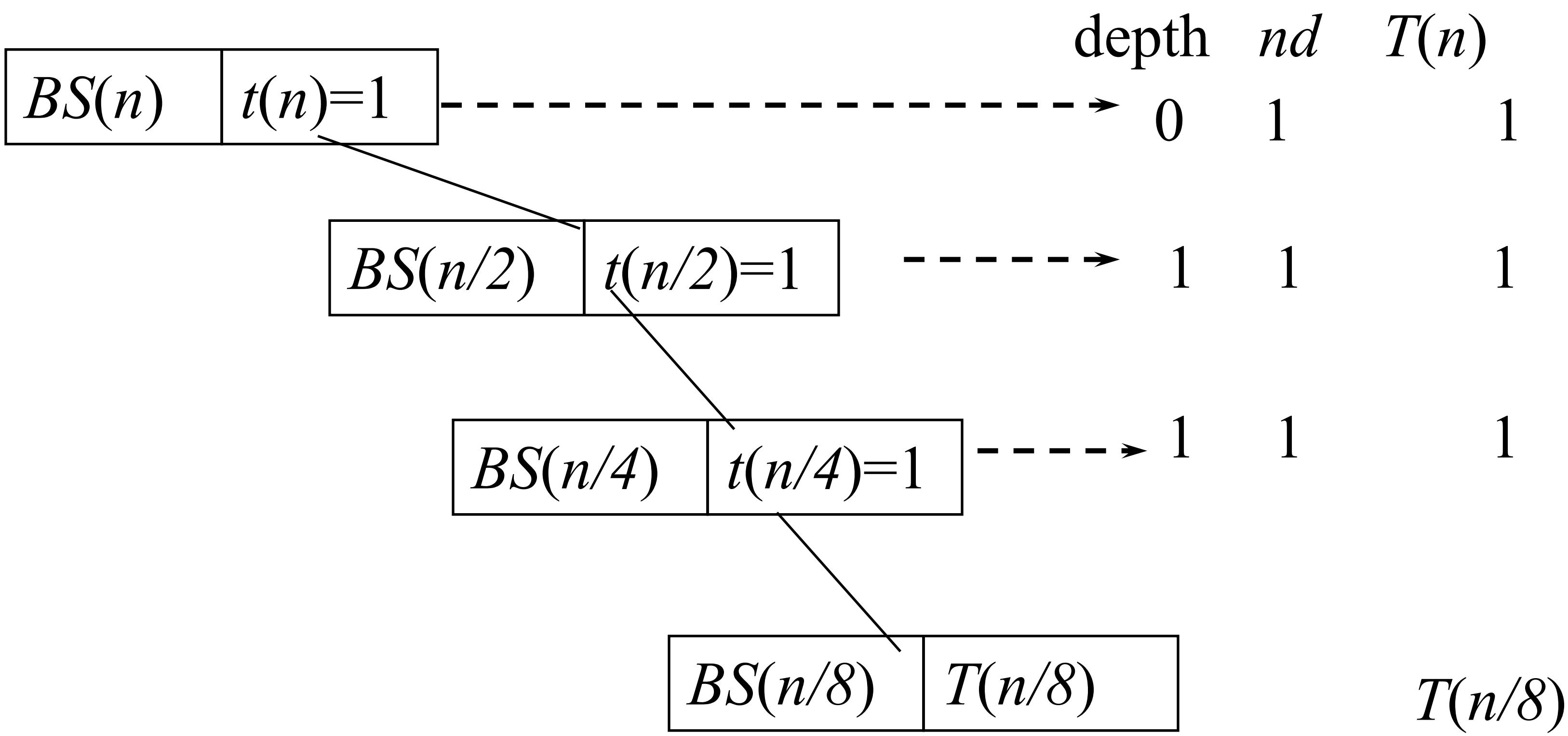
$$T(n) = \begin{cases} 0 \text{ for} & n = 0 \\ 1 \text{ for} & n = 1 \\ 1 + T(\lfloor n/2 \rfloor) \text{ for} & n > 1 \end{cases}$$

After second unrolling



$$T(n) = \begin{cases} 0 \text{ for } & n = 0 \\ 1 \text{ for } & n = 1 \\ 1 + T(\lfloor n/2 \rfloor) \text{ for } & n > 1 \end{cases}$$

After third unrolling

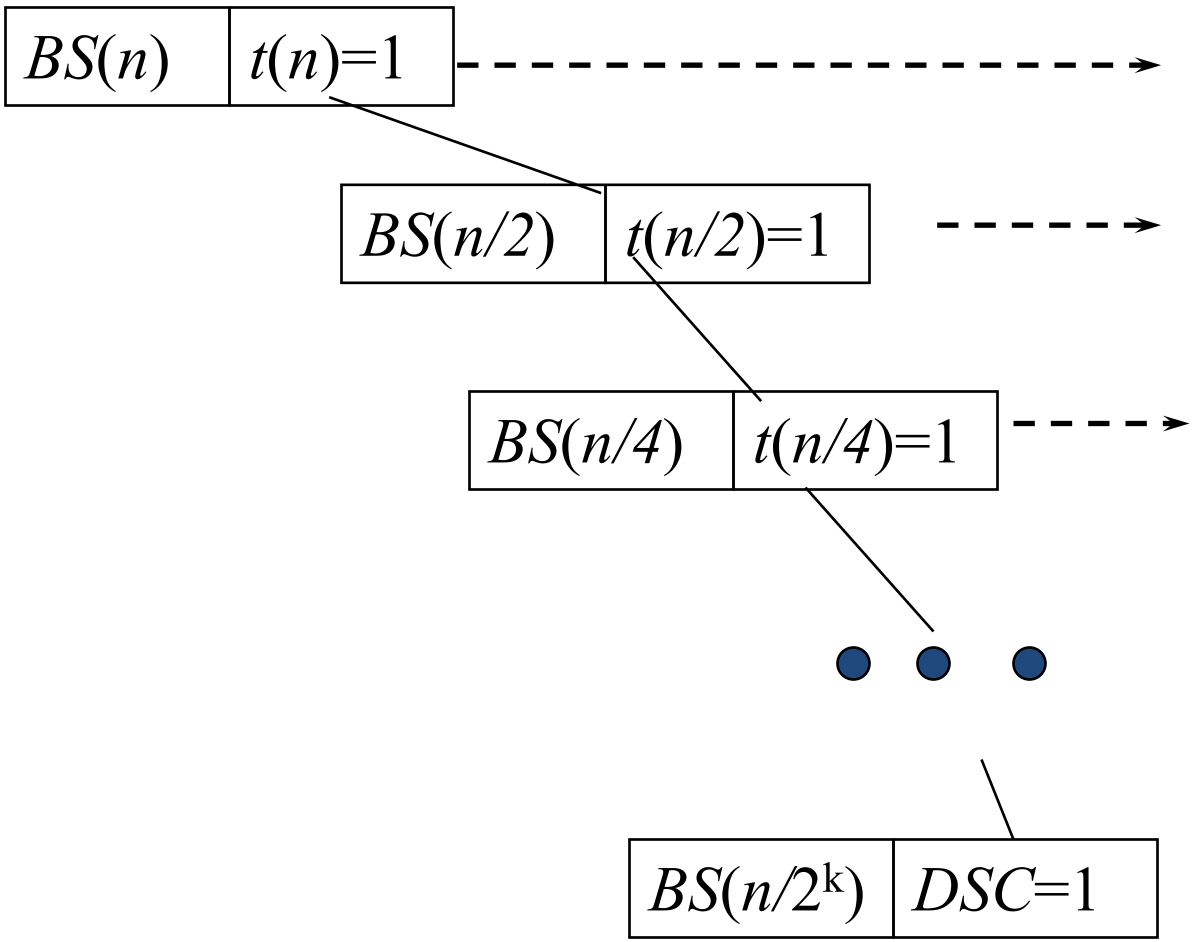


For *BinarySearch, DirectSolutionSize* = 0 or 1
and *DirectSolutionCount* = 0 for 0 and 1 for 1

Terminating the unrolling

- Let $2^k \le n < 2^{k+1}$
- $k = \lfloor \lg n \rfloor$
- When a node has a call to $BS(n/2^k)$, (or to $BS(n/2^{k+1})$):
  - The size of the list is *DirectSolutionSize* since $\lfloor n/2^k \rfloor = 1$, (or $\lfloor n/2^{k+1} \rfloor = 0$)
  - In this case the unrolling terminates, and the node is a leaf containing *DirectSolutionCount* (*DSC*) = 1, (or 0)

The recursion tree



depth	*nd*	*T*(*n*)
0	1	1
1	1	1
2	1	1
k	1	1

$$T(n)=k+1=\lfloor \lg n \rfloor +1$$

Iteration for binary search

$$
\begin{aligned}
W(n) &= 1 + W(\lfloor n/2 \rfloor) \\
&= 1 + (1 + W(\lfloor \lfloor n/2 \rfloor / 2 \rfloor)) = 2 + W(\lfloor n/4 \rfloor) \\
&= 2 + (1 + W(\lfloor n/8 \rfloor)) = 3 + W(\lfloor n/8 \rfloor) \\
&\;\ldots \\
&= k + W(\lfloor n/2^k \rfloor) = k + W(1) = k + 1 \\
&= \lfloor \lg n \rfloor + 1 \in \Theta(\lg n)
\end{aligned}
$$

Merge Sort

Input: S of size n.

Output: a permutation of S, such that if $i > j$ then $S[\,i\,] \geq S[\,j\,]$

Divide: If S has at least 2 elements, divide it into S_1 *and* S_2. S_1 contains the the first $\lceil n/2 \rceil$ elements of S. S_2 has the last $\lfloor n/2 \rfloor$ elements of S.

Recursion: Recursively sort S_1 and S_2.

Conquer: Merge sorted S_1 and S_2 into S.

Merge Sort Example

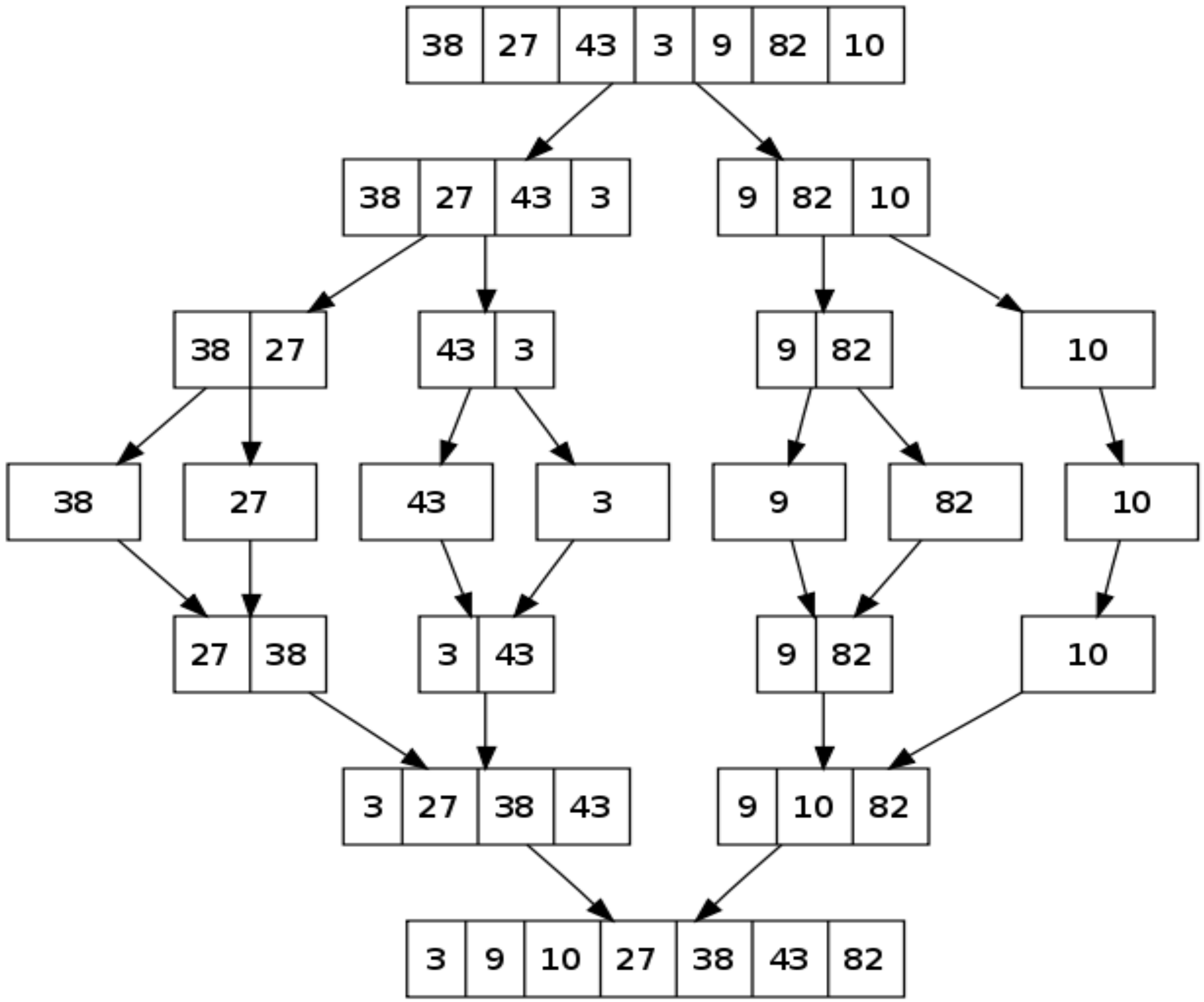


Image source: http://en.wikipedia.org/wiki/File:Merge_sort_algorithm_diagram.svg

Deriving a recurrence equation for Merge Sort

```
Sort(S)
    if (n ≥ 2)
            // Divide S into S1 and S2
            Sort(S1)              // recursion
            Sort(S2)              // recursion
            Merge( S1, S2, S )    // conquer
```

DirectSolutionSize is n < 2

DirectSolutionCount is $\theta(1)$

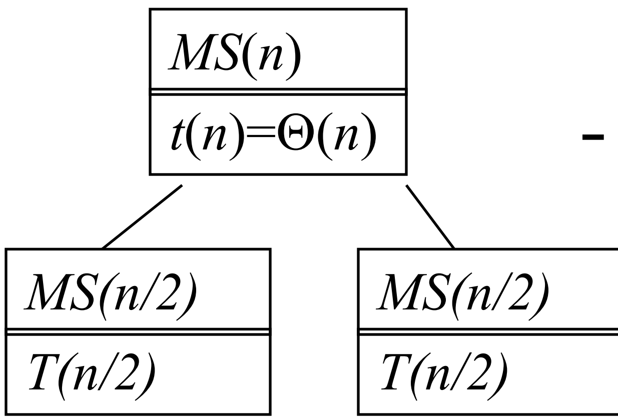
RecursiveCallSum is $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$

The non-recursive count $t(n) = \theta(n)$ ➔ Merge

Recurrence Equation (cont'd)

- The cost of division is O(1) and *merge* is $\Theta(n)$. So, the total cost for dividing and merging is $\Theta(n)$.
- The recurrence relation for the run time of MergeSort is:
  $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n)$.
  $T(0) = T(1) = \Theta(1)$
- The solution is $T(n) = \Theta(n \lg n)$

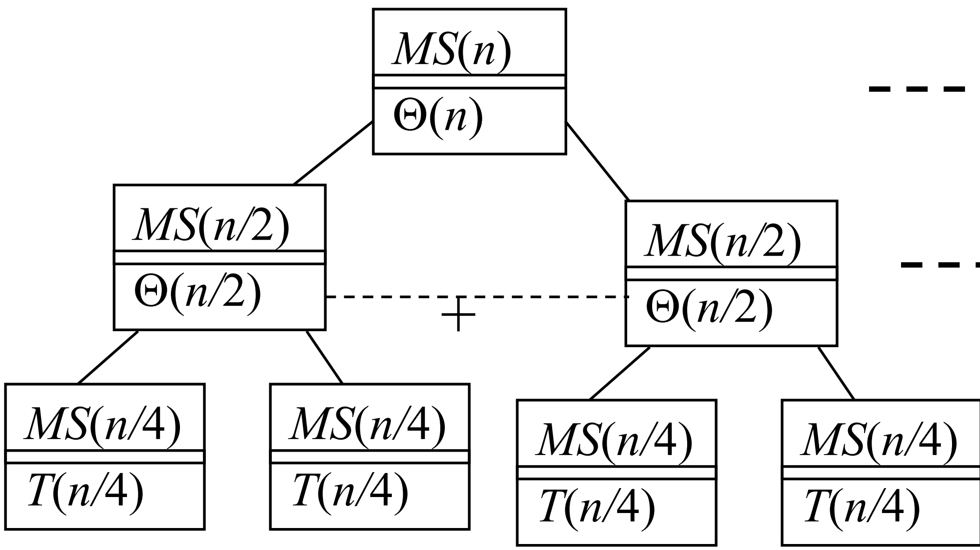
After first unrolling of mergeSort



depth	nd	$T(n)$
0	1	$\Theta(n)$
		$2T(n/2)$

$$T(n) = \begin{cases} 1 \ \text{for} \quad \mathrm{n} \le 1 \\ 2T(n/2) + \theta(\mathrm{n}) \quad \text{for} \quad \mathrm{n} > 1 \end{cases}$$

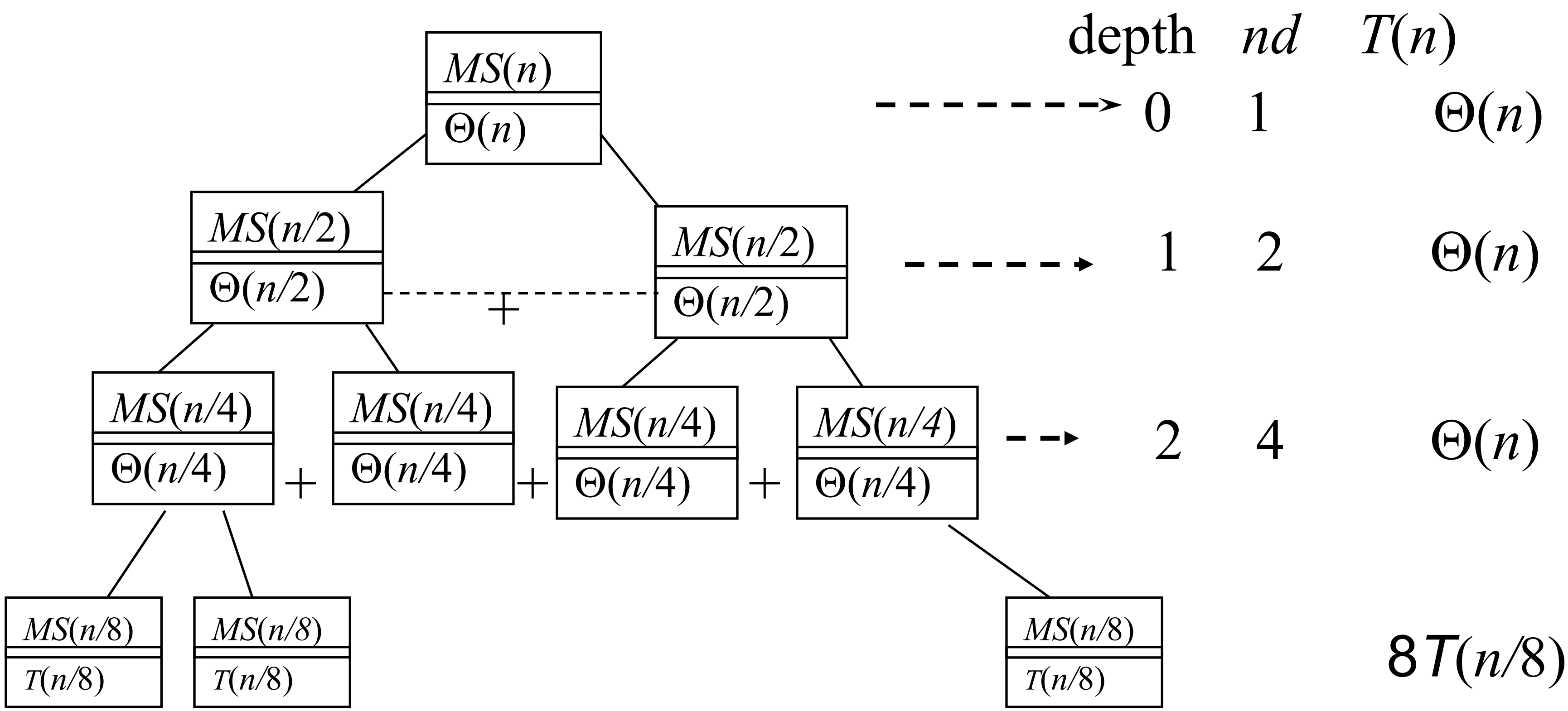
After second unrolling



depth	*nd*	*T*(*n*)
0	1	$\Theta(n)$
1	2	$\Theta(n)$
		$4T(n/4)$

$$T(n) = \begin{cases} 1 \text{ for } n \le 1 \\ 2T(n/2) + \theta(n) \quad \text{for } \quad n > 1 \end{cases}$$

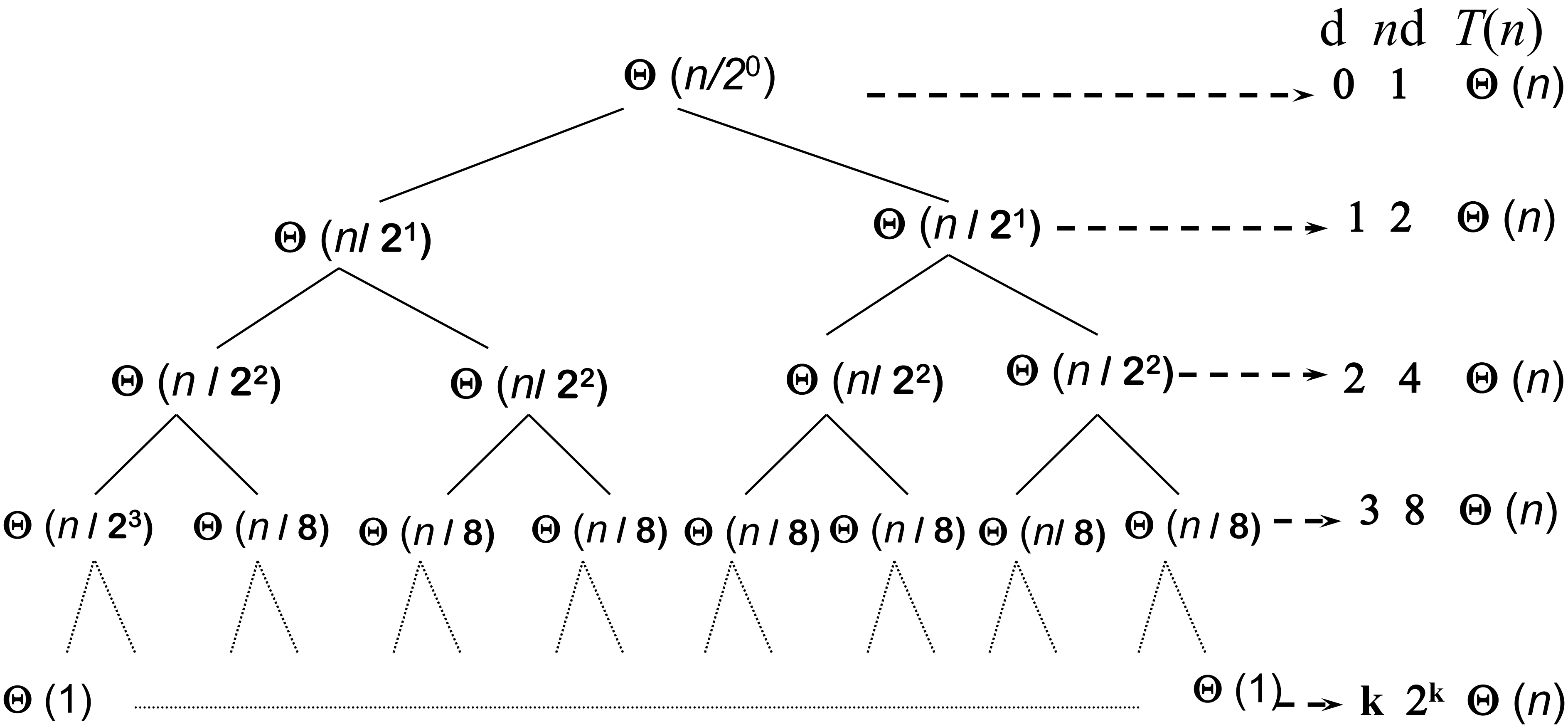
After third unrolling



Terminating the unrolling

- For simplicity let $n = 2^k$
- $\lg n = k$
- When a node has a call to *MergeSort*($n/2^k$):
  - The size of the list to merge sort is *DirectSolutionSize* since $n/2^k = 1$
  - In this case the unrolling terminates, and the node is a leaf containing *DirectSolutionCount* = $\Theta(1)$

The recursion tree



$$T(n) = (k+1)\,\Theta(n) = \Theta(n \lg n)$$

Master Theorem to Solve General Recurrence Equations

- Suppose that:
  - $T(n) = aT(n/b) + cn^k$
  - $T(1) = d$

  where $b \geq 2$ and $k \geq 0$ are constant integers and a, c, d are constants such that $a > 0$, $b > 0$, and $d \geq 0$. Then,
  - $T(n) = \Theta(n^k)$ if $a < b^k$
  - $T(n) = \Theta(n^k \lg n)$ if $a = b^k$
  - $T(n) = \Theta(n^{\log_b a})$ if $a > b^k$

Master method examples

- Case 1:
  - $T(n) = 8T(n/4) + 5n^2$ for $n>1$, n is a a power of 4
  - $T(1) = 3$

→ $a=8$, $b=4$, $k=2$

→ As $a < b^k$ (i.e., $8 < 4^2$), $T(n) = \Theta(n^2)$

- Case 2:
  - $T(n) = 8T(n/2) + 5n^3$ for $n>64$, n is a power of 2
  - $T(64) = 200$

→As $a = b^k$ (i.e., $8 = 2^3$), $T(n) = \Theta(n^3 \lg n)$

- Case 3:
  - T(n) = 9T(n/3) + 5n for n > 1, n is a power of 3
  - T(1) = 7

→a = 9, b = 3, k = 1

→Since a > b^k , $T(n) = \Theta(n^{\log_3 9}) = \Theta(n^2)$