Contour deformations in momentum space representations from the Landau analysis

YITP
October 26, 2023

Abstract

We study the problem of contour deformations and how to construct them for a generic amplitude in momentum space. We will start by considering systematic contour deformations in a mixed representation with both loop momenta and Feynman parameters demonstrating their relationship to the canonical Landau equations. We will then consider the Landau equations in a purely momentum space representation and demonstrate a geometric interpretation of the Landau equations. We will start from lightcone-ordered perturbation theory where the Landau equations take a particularly simple form, and return to covariant perturbation theory where we construct the most general contour deformation. Throughout, we make no assumptions about the masses of external as well as internal lines, and derive a unified framework for both cases.

Contents

1 Introduction

In recent years, there has been increasing interest in evaluating amplitudes in four dimensions using locally finite expressions [1], [2]. A loop integrand as a function of external momenta ($\{q_i\}$) as well as loop momenta ($\{k_i\}$), $I(\{k_i\}, \{q_i\})$, is said to be locally finite (or locally bounded) if there is a finite constant $C(\{k_i\})$ that is greater than the integrand everywhere in an ϵ neighborhood of the point $\{k_i\}$. For rational integrands of the form we encounter in perturbation theory, this implies that the integral exists and can be evaluated numerically.

Several obstacles arise in writing Feynman integrals in a locally finite fashion. The first is the presence of "unphysical" singularities both in covariant perturbation theory as well as in ordered perturbation theory. The second is the necessity of contour deformations for unpinched loops in any representation of perturbation. Yet another obstacle is the presence of ultraviolet singularities in all graphs. Recently, a representation of amplitudes free of "unphysical" singularities" in TOPT as well as in lightcone ordered perturbation theory (LCOPT) was proposed [7]. Another representation free of "unphysical singularities" derived directly from covariant perturbation theory [8]. This representation is closely related to that of "flow-oriented" perturbation theory [24].

In this context, using the "loop tree duality" representation [3] of amplitudes to carry out numerical integration has also found use [4], [5], [6] and the authors of Ref. [4] construct an algorithm to deform contours locally near each singular surface of the integrand. Principal value prescriptions near singularities of the integrand have also been found to be useful [9]. The need to deform contours to carry out the loop integrals for hard scattering was already recognized in formal proofs of factorization [12], [13]. However, it remains an area of active study and some interest. Previous attempts to tackle the problem involve using Feynman parametrization in a mixed representation, as well as directly studying contour deformations in momentum space Refs. [14]-[20]. Here, instead of attempting to deform a contour when a real denominator vanishes, we will deform contours and endow denominators with a positive imaginary part wherever possible. Such deformation in parameter space, with the loop momentum integrated out in Ref. [22].

In this paper, we construct a systematic set of contour deformations that one can make for any graph that contributes to an amplitude in any theory. Our representation of the amplitude will be in four-dimensional Minkowski space, and will manifestly be locally finite away from pinch surfaces. In Sec. 2 we will demonstrate our contour deformation in covariant perturbation theory using the mixed representation, where denominators in loop momentum space, combined using Feynman parameters. This proceeds in two steps, shifting loop momenta so as to shift all pinches to the origin, and deforming the shifted momenta appropriately. We will explicitly work out the shift vector in some (planar) examples. In Sec. 3 we will demonstrate how to make contour deformations in light-cone ordered perturbation theory (LCOPT), for the $\perp, +$ momentum. We will write the Landau equations for all three components in lightcone ordered perturbation theory and construct the most general deformation which admits a in a closed form expression. We will specialize to some one-loop examples and demonstrate simplifications of our formula that occur when we deform plus momenta. Here, we will be able to make contact with pinch surfaces and the Coleman-Norton [26] picture of physical propagation. We will observe that the deformation vanishes for soft lines propagating between on-shell particles, which remain pinched in all graphs. In Sec. 4, we adapt the methods presented in Sec. 3 to covariant perturbation theory constructing the most general contour deformation in covariant, momentum space representation of perturbation theory in a closed form. In Sec. 5 we will summarize our results and identify questions that remain unanswered in our analysis.

2 Contour deformations in mixed representation for covariant amplitudes

In this section, we construct a set of contour deformations for graphs in covariant perturbation theory in the mixed representation. In covariant perturbation theory, for the L loop contribution to an amplitude $\mathcal{M}(\{p_i\})$ is

$$\mathcal{M}(\{p_i\}) = \sum_G \int \prod_{i=1}^{L} \left(\frac{d^4 k_i}{(2\pi)^4} \right) \times \prod_{l=1}^{E_G} \frac{N_G\left(\{k_i\}, \{p_m\}\right)}{\left(\sum\limits_{j=1}^{L} \beta_{jl} k_j + \sum\limits_{m=1}^{N} \tilde{\beta}_l^m p_m \right)^2 - m_l^2 + i\epsilon}. \tag{2.1}$$

Here, G and E_G represent the set of Feynman graphs that contribute to the amplitude at L loops, and the number of edges in the graph G, respectively. The factor of $N_G\left(\{k_i\}, \{p_m\}\right)$ represents the numerator of the graph which is a polynomial in the momenta and is always real (up to an overall factor of i). We have also used β_{jl} to represent the incidence matrix which takes the value 1 if the loop momentum k_j flows through the line l in the forward direction, -1 if the loop momentum k_j flows through the line l in the backward direction, and zero otherwise. Here, "forward" and "backward" are relative to an arbitrary directional assignment of the loop momenta. These directions take on physical meaning in time-ordered perturbation theory as well as in LCOPT. Similarly, $\tilde{\beta}_l^m$ is the incidence matrix of the external momenta p_m on the line labeled by l. We have used N to denote the total number of external particles.

The focus of our study in this section will be the denominator structure in covariant perturbation theory, which appears in Eq. (2.1),

$$D^E = \left(\sum_{l=1}^{E} \alpha_l \left(\sum_{j=1}^{L} \beta_{jl} k_j + \sum_{m=1}^{N} \tilde{\beta}_l^m p_m \right)^2 - \alpha_l m_l^2 + i\epsilon \right)^E , \tag{2.2}$$

where we have used Feynman parametrization to combine denominators. This parametrization follows from the identity,

$$\frac{1}{\prod_{i=1}^{n+1} D_i} = \int [d\alpha_s]_{n+1} \frac{1}{\left(\sum_{i=1}^{n+1} \alpha_i D_i \right)^{n+1}},$$

$$[d\alpha_s]_{n+1} = n! \int_0^1 d\alpha_{n+1} \dots d\alpha_1 \, \delta \left(1 - \sum_{s=1}^{n+1} \alpha_s \right) . \tag{2.3}$$

To begin our analysis, we rewrite Eq. (2.2) as

$$\begin{aligned} D &= \sum_{l=1}^{E} \alpha_l \left(\sum_{j=1}^{L} \beta_{jl} k_j + \sum_{m=1}^{N} \tilde{\beta}_l^m p_m \right)^2 - \alpha_l m_l^2 + i\epsilon , \\ &= \mathbf{k}^T \mathbb{M} \left(\alpha_l, \beta_{il} \right) \mathbf{k} + 2\mathbf{k}^T \mathbf{v}(q_i, \alpha_l, \beta_{il}, \tilde{\beta}_j^m) + \Delta(p_i, m_i) + i\epsilon . \end{aligned} \tag{2.4}$$

As before, $\tilde{\beta}_j^m$ is the incidence factor of p_m on the line labeled by l. In the second equality, we have packaged loop momenta into a vector $\mathbf{k} = (k_1, k_2 \dots k_L)^T$ and defined the matrix $\mathbb{M}$ by

$$\left[\mathbb{M} \left(\alpha_l, \beta_{il} \right) \right]_{ij} = \sum_{l=1}^{E} \alpha_l \, \beta_{il} \, \beta_{jl} \, , \tag{2.5}$$

and the vector $\mathbf{v}(q_i, \alpha_l, \beta_{il}, \tilde{\beta}_l^m)$ by

$$\left[\mathbf{v}(q_i, \alpha_l, \beta_{il}, \tilde{\beta}_j^m) \right]_j = \sum_{l=1}^{E} \alpha_l \sum_{m=1}^{N} \beta_{jl} \; \tilde{\beta}_l^m \; p_m . \tag{2.6}$$

We have denoted the constant, loop momentum independent term by $\Delta(p_m, \alpha_l, \tilde{\beta}_l^m)$ defined through

$$\Delta(p_m, \alpha_l, \tilde{\beta}_l^n) = \sum_{l=1}^{E} \alpha_l \left(\sum_{i,j} \tilde{\beta}_l^j \, \tilde{\beta}_l^i \, p_i \cdot p_j - m_l^2 \right) . \tag{2.7}$$

We now observe that the matrix, $\mathbb{M}$ is a real, symmetric matrix. Applying the principal axis theorem, we conclude that it is diagonalizable by an orthogonal transformation and has real eigenvalues (this is a special case of the spectral theorem, familiar from quantum mechanics, which asserts that every Hermitian matrix is unitarily diagonalizable and has real eigenvalues). A useful feature of $\mathbb{M}$ that we will prove presently is that it is *positive definite*. This property is defined by

$$\mathbf{v}^T \mathbb{M} \mathbf{v} \geq 0 \qquad\qquad \forall v \in \mathbb{R}^L . \tag{2.8}$$

Let us start by choosing a preferred parametrization of loop momenta, applicable to any graph. To describe the first step in our algorithm in choosing a loop momentum parametrization, we identify a "large loop", any loop that contains all the external vertices. Such a loop always exists if we restrict ourselves to one-particle irreducible (1PI) graphs, but it is not unique. We note that a generic graph decomposes into a product of one-particle irreducible graphs, and we may therefore restrict our attention to 1PI graphs without loss of generality. The loop momentum parametrizing the large loop, say k is defined to be carried by the line leaving the vertex where $p_n = -\sum_{i=1}^{n-1} p_i$ enters the graph . Every subsequent line in the large loop is defined to carry "positive" loop momentum as well as "positive" external momentum i.e there are no relative negative signs between loop momenta in the parametrization. This means every line in the large loop l, satisfies $\beta_{jl} \geq 0, \tilde{\beta}_l^n \geq 0$. In particular, any connected sub-graph, G_1 with $n_1 + 1$ lines incident on the large loop, is parametrized such that a line with $-k_1 - k_2 \cdots - k_{n_1}$ entering G_1 is the earliest line to leave the large loop. Subsequent lines carry momenta k_i when entering the subgraph G_1. In this way, every line on the large loop is parametrized such that it carries loop momentum with a positive sign. As an example, the parametrization of the large loop when one connected, 1PI graph attaches to a large loop in a vacuum polarization graph is shown in Fig. 1. If

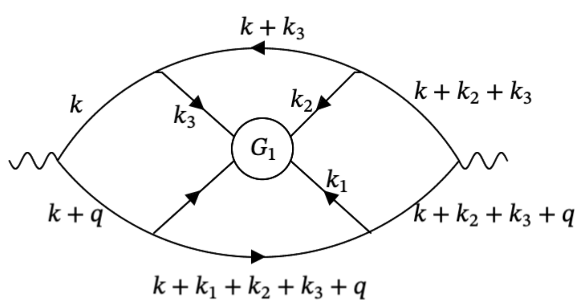

Figure 1: An example of the loop momentum parametrization we choose for an arbitrary forward scattering graph. Here, G_1 is an arbitrary, connected 1P1 graph. In general, there may be an arbitrary number of connected components, each with multiple 1PI components.

there are n connected components attached to the large loop, each with $L_1, L_2 \dots L_n$ internal loops, we have reduced the parametrization ambiguity of a L loop graph to that of n subgraphs with $L_1, L_2 \dots L_n$ loops. Notice that at this stage for each line l, we have parametrized, $\beta_{jl} \geq 0$. Here, L_1 is the set of lines in the large loop that carry external momentum while L_2 represents the set of lines that do not carry the external momentum, q.

Next, we repeat the process of finding a large loop in each of the subgraphs $G_1, G_2, \dots G_n$ (which are the connected, 1PI subgraphs that attach to the chosen large loop) and proceed to make a positive loop momentum parametrization procedure on the new set of large loops. In this way, for each line $l \in G$, we have parametrized loop momenta such that $\beta_{jl} \geq 0, \forall j$. Yet another example of our parametrization for a four-particle subgraph is shown in Fig. 2.

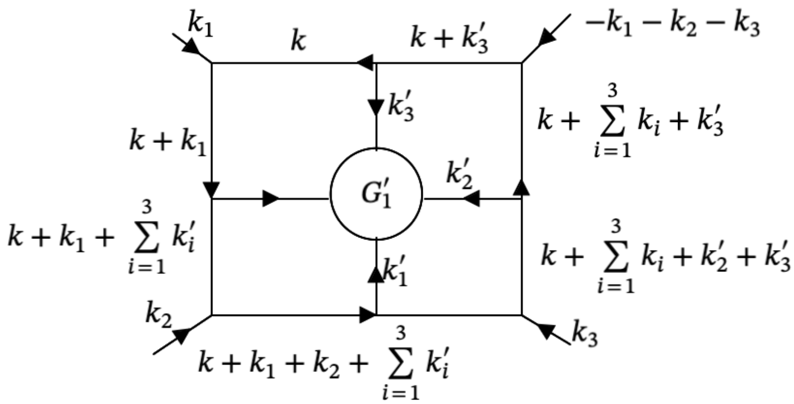

Figure 2: An example of the loop momentum parametrization we have choosen for an arbitrary four particle scattering graph. Here, G_1' is an arbitrary, connected 1P1 graph. As before, there may be an arbitrary number of connected components, each with multiple 1PI components.

Having explained our chosen parametrization of loop momenta, we now return to the Feynman-parametrized denominator of Eq. (2.4). As before, we decompose the matrix $\mathbb{M}$ into it's individual components from each line. We notice that if each $\mathbb{N}^l$ is positive definite, so is $\mathbb{M}$.

$$\begin{aligned} \mathbf{v}^T \, \mathbb{M} \, \mathbf{v} &= \sum_{l=1}^{E} \alpha_l \, \left(\mathbf{v}^T \, \mathbb{N}^l \, \mathbf{v} \geq 0 \right) , \\ &\geq 0 \end{aligned} \tag{2.9}$$

Next, we observe that for a line l, with momentum $q_l = k_{l_1} + k_{l_2} + \ldots k_{l_i} + p_{l_1} \ldots p_{l_j}$,

$$\begin{aligned} [\mathbb{N}^l]_{ab} &= 1 \, , \; a \in \{l_1, l_2 \ldots l_i\} \text{ and } b \in \{l_1, l_2 \ldots l_i\}, \\ [\mathbb{N}^l]_{ab} &= 0 \, , \; \text{otherwise} \, . \end{aligned} \tag{2.10}$$

We decompose any vector $\mathbf{v}$ in terms of the cannonical basis vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots \mathbf{e_L}$, where $\mathbf{e_i} = (0, \ldots 0, 1, 0 \ldots 0)$. Here the 1 is in the i'th position. Using this basis, the decomposition reads

$$\mathbf{v} = \sum_{i=1}^{L} v_i \; \mathbf{e_i} \, . \tag{2.11}$$

In this basis, the matrix element $\mathbf{e_j}^T \, \mathbb{N}^l \, \mathbf{e_i} = [\mathbb{N}^l]_{ji}$. Therefore, in the subspace spanned by $\{l_1, l_2, \ldots l_i\}$, the matrix $\mathbb{N}^l$ is the matrix with all entries 1, and the entries of the matrix in the orthgonal subspace, as well the entries in the matrix that map vectors inside the subspace to those outside are 0.

Evaluating the inner product of $\mathbb{N}$ with an arbitrary vector, we find

$$\begin{aligned}
\mathbf{v}^T \, \mathbb{N}^l \, \mathbf{v} &= \sum_{a=l_1}^{l_i} \left(v_a^2 + \sum_{\substack{b=l_1 \\ b \neq a}}^{l_i} v_a v_b \right) , \\
&= \sum_{a=l_1}^{l_i} v_a^2 + \sum_{a=l_1}^{l_i} \sum_{b>a} 2 v_a \, v_b \, , \\
&= \left(\sum_{a=l_1}^{l_i} v_a \right)^2 \geq 0. \qquad (2.12)
\end{aligned}$$

It follows that indeed $\mathbb{M}$ is positive definite as in Eq. (2.9). The inequality can be upgraded to a strict inequality by observing that the only way to saturate the equality $\mathbf{v}^T \, \mathbb{M} \, \mathbf{v} = 0$ is through the vanishing of all α_l's in one the loops, once again causing the reduced graph to shrink to a smaller reduced graph. We may exclude this possibility which occurs on a set of measure zero.

Having established that $\mathbb{M}$ is positive definite, we conclude that all it's eigenvalues are real, positive numbers with unique positive square roots. Further, we know that $\mathbb{M}$ is orthogonally diagonalizable allowing us to define a unique (upto permutation of eigenvalues) square root matrix

$$\begin{aligned}
D &= O \, \mathbb{M} \, O^T \, , \\
\sqrt{\mathbb{M}} &= O^T \, \sqrt{D} \, O, \qquad (2.13)
\end{aligned}$$

where D is a diagonal matrix and O is the orthogonal matrix that diagonalizes $\mathbb{M}$. The matrix $\sqrt{D}$ is defined to be the positive square root of the diagonal entries. It's also possible to analogously define $\sqrt{\mathbb{M}^{-1}}$ (as the inverse of the non-zero diagonal entries in D, followed by conjugation by O). It's easy to check that $\sqrt{\mathbb{M}}$ satisfies $(\sqrt{\mathbb{M}})^2 = \mathbb{M}$. Finally, we may make a shift in loop integration variables, defined by

$$\mathbf{k}' = \sqrt{\mathbb{M}} \, \mathbf{k} + \sqrt{\mathbb{M}^{-1}} \, \mathbf{v}. \qquad (2.14)$$

We notice that this shift has a non-trivial jacobian corresponding to the determinant of $\sqrt{\mathbb{M}}$, but the jacobian never vanishes (away from end points where a loop shrinks in the reduced graph). We therefore, conclude that this change of variable is real, non-singular and well defined everywhere in the region of integration. In the following sub-section, we make contact between these variables, and yet another convenient shift which enables us to make direct contact with the Landau equations.

In terms of the shift in Eq. (2.14), we may now rewrite Eq. (2.4) as

$$D = \mathbf{k'}^T \mathbf{k}' - \mathbf{v}^T \mathbb{M}^{-1} \mathbf{v} + \Delta + i\epsilon \, ,$$

$$= \sum_{i=1}^{L} k_i'^2 - \mathbf{v}^T \mathbb{M}^{-1} \mathbf{v} + \Delta + i\epsilon \tag{2.15}$$

It is now possible to make a universally well defined contour deformation to the variables $\vec{k}_i'$

$$\vec{k}_i' = \vec{l}_i e^{-i\theta}, \tag{2.16}$$

where we assume that $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\vec{l}_i$ are real integration variables.

2.1 Examples

In this section we will provide some low order examples of the shifts and deformations in the Feynman parametrized denominator, for amplitudes at one and two-loops.

The goal of this section is to make contact between the shifts in Eq. (2.14) and covariant Landau equations. We would like to demonstrate that $\mathbf{k}' = 0$, along with the vanishing of D, represents solutions of the Landau equations where we expect deformations to vanish. This explains why, despite the presence of non-trivial pinch surfaces, we are able to make a universal contour deformation.

Let us start with the simple example of the one-loop form factor. Our chosen paramterization is shown in Fig. 3.

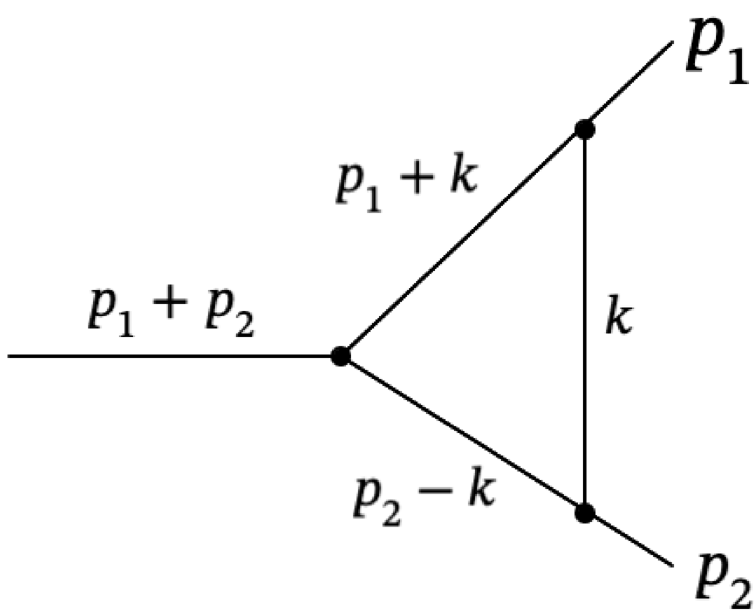

Figure 3: The one loop form factor and our momentum parametrization of the the diagram. In our notation, it is clear that overall momentum conservation implies $p_1 + p_2 = q$.

The amplitude for the one loop form factor is given by

$$F(p_1,p_2)=\int\frac{d^4k}{(2\pi)^4}\,\mathbb{N}_{\mathbb{G}}\,\frac{1}{(k^2+ie)\cdot((p_1+k)^2+i\epsilon)\cdot((p_2-k)^2+i\epsilon)}. \tag{2.17}$$

Where as before, we will focus on the denominator structure, absorbing the spin-dependent factors into the numerator $\mathbb{N}_{\mathbb{G}}$. The Feynman parametrized denominator is given by

$$D^3=\Big(k^2+2k\cdot(\alpha_2p_1-\alpha_3p_2)+i\epsilon\Big)^3. \tag{2.18}$$

From the general considerations in the previous subsection, the shifted variables are

$$k'=k+\alpha_2p_1-\alpha_3p_2, \tag{2.19}$$

which can be compared with Landau equations for the loop momentum k

$$\alpha_1k^\mu+\alpha_2(p_1+k)^\mu-\alpha_3(p_2-k)^\mu=0, \tag{2.20}$$

we see that the solution to the Landau equations, occurs precisely at ${k'}^\mu=0$. Indeed, at this point, the deformation in Eq. (2.16) vanishes.

Let us now ask what happens to a generic one-loop amplitude, with n external legs. Clearly, each graph G, is toplogically equivalent to an n-gon, and we may choose $n-1$ independent external momenta that it depends on. We choose our external momenta to be $p_1,\dots p_{n-1},p_n=-p_1\cdots-p_{n-1}$. We also use the large loop parametrization, so each momentum comes with a non-negative coefficient in each line. The generic Feynman parametrized denominator reads

$$D=k^2+2k\cdot\left(\sum_{l\in G}\sum_{j=1}^{n-1}\alpha_l\,\tilde{\beta}_l^j\,p_j\right)+\sum_{l\in G}\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\alpha_l\,\tilde{\beta}_l^j\tilde{\beta}_l^i\,p_j\cdot p_i\,-\sum_{l\in G}\alpha_lm_l^2+i\epsilon. \tag{2.21}$$

The shift vector can be read off easily and is given by

$$k'=k+\sum_{l\in G}\sum_{j=1}^{n-1}\alpha_l\,\tilde{\beta}_{jl}\,q_j \tag{2.22}$$

Once again, we may compare this to the Landau equations for the k loop,

$$\alpha_1k^\mu+\sum_{\substack{l\neq1\\ l\in G}}\alpha_l(\tilde{\beta}_l^j\,p_j+k)^\mu=0. \tag{2.23}$$

We find, yet again, that the loop is pinched if $k'=0$, where our deformation vanishes. Let us consider a two loop example. The simplest two loop graph is the two loop form factor, ladder exchange graph. Consider the parametrization in Fig. 4.

Let us analyze the graph in Fig. 4 by considering it's Landau singularities. The Landau equations are

$$\alpha_1(k_1+p_1)^\mu+\alpha_6(k_1-k_2)^\mu-\alpha_2(p_2-k_1)^\mu\;\;=\;\;0,$$

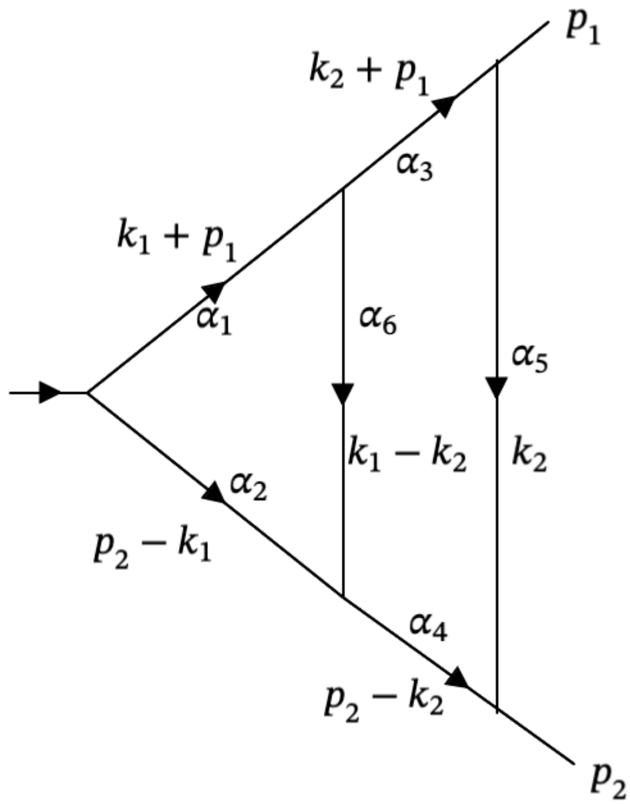

Figure 4: The two loop form factor and our momentum parametrization of the the diagram. We have also indicated an arbitrary labelling of the lines.

$$\alpha_3(k_2+p_1)^\mu - \alpha_6(k_1-k_2)^\mu - \alpha_4(p_2-k_2)^\mu + \alpha_5 k_2^\mu \;=\; 0. \tag{2.24}$$

At two loops, it becomes physically transparent to make a shift given by

$$\begin{aligned}\tilde{k_1} &= \alpha_1(k_1+p_1)^\mu + \alpha_6(k_1-k_2)^\mu - \alpha_2(p_2-k_1)^\mu,\\ \tilde{k_2} &= \alpha_3(k_2+p_1)^\mu - \alpha_6(k_1-k_2)^\mu - \alpha_4(p_2-k_2)^\mu + \alpha_5 k_2^\mu.\end{aligned} \tag{2.25}$$

We observe that the shift in Eq. (2.25) is made so as to move the location of the pinch surfaces to $\tilde{k}_{1,2} = 0$. It is a homogenous, linear transformation of the shift made in Eq. (2.14) and therefore, the shift in Eq. (2.14) also occurs at $k' = 0$.

The Feynman parametrized denominator of this graph is given by the expression

$$\begin{aligned}D &= \alpha_1(k_1+p_1)^2 + \alpha_2(p_2-k_1)^2 + \alpha_3(k_2+p_1)^2 + \alpha_4(p_2-k_2)^2\\ &\quad +\alpha_5 k_2^2 + \alpha_6(k_1-k_2)^2 - \sum_l m_l^2\alpha_l + i\epsilon.\end{aligned} \tag{2.26}$$

In order to recast this equation in matrix form, we find explicit expressions for the matrix $\mathbb{M}$ defined in Eq. (2.5), vector structure defined in Eq. (2.6) and the k

independent piece defined in Eq. (2.7), which take the form

$$\begin{aligned}
\mathbb{M} &= \begin{pmatrix} \alpha_1+\alpha_2+\alpha_6 & -\alpha_6 \\ -\alpha_6 & \alpha_3+\alpha_4+\alpha_5+\alpha_6 \end{pmatrix} \\
\mathbf{v} &= \begin{pmatrix} \alpha_1 p_1 - \alpha_2 p_2 \\ \alpha_3 p_1 - \alpha_4 p_2 \end{pmatrix} \\
\Delta &= (\alpha_1+\alpha_3)p_1^2 + (\alpha_2+\alpha_4)p_2^2 - \sum_l m_l^2 \alpha_l + i\epsilon. \qquad (2.27)
\end{aligned}$$

We now write this denominator in terms of the $\tilde{k}_1, \tilde{k}_2$, using the $\mathbb{M}$ matrix

$$\begin{aligned}
D &= \mathbf{k}^T \mathbb{M}\,(\alpha_l)\,\mathbf{k} + 2\mathbf{k}^T \mathbf{v}(p_1, p_2, \alpha) + \Delta + i\epsilon \\
&= \tilde{\mathbf{k}}^T \mathbb{M}^{-1} \tilde{\mathbf{k}} - \mathbf{v}^T \mathbb{M}^{-1} \mathbf{v} + \Delta + i\epsilon. \qquad (2.28)
\end{aligned}$$

Here, we have the symbol $\mathbb{M}^{-1}$ to represent the inverse matrix

$$\mathbb{M}^{-1} = \frac{1}{\det \mathbb{M}} \begin{pmatrix} \alpha_3+\alpha_4+\alpha_5+\alpha_6 & \alpha_6 \\ \alpha_6 & \alpha_1+\alpha_2+\alpha_6 \end{pmatrix}. \qquad (2.29)$$

Finally, we may use the positive definiteness of $\mathbb{M}$ to conclude that $\mathbb{M}^{-1}$ is also positive definite and that

$$\sum_{i,j\in\{1,2\}} \vec{\tilde{k}}_i \, \mathbb{M}^{-1}_{ij} \, \vec{\tilde{k}}_i \geq 0. \qquad (2.30)$$

We may now make the deformation $\vec{\tilde{k}}_i = \vec{l}_i e^{-\frac{\theta}{2}}$ which is a universal deformation that vanishes near at a pinch surface. We observe that the relationship in Eq. (2.28) holds for any graph if we define $\tilde{\mathbf{k}} = \frac{1}{2}\frac{\partial D}{\partial \mathbf{k}}$, and the vanishing of $\tilde{\mathbf{k}}$ implies that the Landau equations are satisfied. Further, we can verify the relationship between $\tilde{k}$ and k' defined in Eq. (2.14) is through a linear, homogeneous, transformation given by $\mathbf{k}' = \sqrt{\mathbb{M}^{-1}}\tilde{\mathbf{k}}$.

3 The geometric Landau analysis in LCOPT

In this section and the next, we will not require that the denominator is Feynman parametrized in order to construct a deformation. To make contact with the Landau equations as we did in the previous section, we will need to construct the Landau equations in loop momentum space. The covariant version of the Landau equations in momentum space were already derived in [27]. Here, we continue the analysis into LCOPT and TOPT from a geometric perspective.

To begin, let us recall the Feynman rules in LCOPT. In what follows, we use a definition of lightcone variables which follows from the relationship $k^2 = 2k^+k^- - \vec{k}_\perp^2$. In LCOPT all vertices are ordered in all possible ways to indicate increasing lightcone "time" and summing over lightcone orders is necessary to recover the original Feynman graph. Ordering vertices on the lightcone allows us to carry out one of the lightcone momentum integrals using the residue theorem. If we choose to carryout the k^- integrals in momentum space, vertices are ordered by increasing x^+. A "cut" or "state" is defined to live between two consecutive vertices and consists of all lines that flow from the left of the cut to the right of the cut. The associated rules are:

- For each loop the plus, transverse integrals remain. The measure of the integral is $\prod_{i=1}^{L} \frac{d^2 k_{i,\perp}}{(2\pi)^2} \frac{dk_i^+}{2\pi}$
- For each line with momentum l in the direction of increasing x^+, we assign the factor $\frac{\theta(l^+)}{2l^+}$. Therefore, there is an overall factor of $\prod_{i=1}^{E} \frac{\theta(l_i^+)}{2l_i^+}$.
- For each state s, we associate a denominator of the form $\frac{i}{\sum_{\alpha<s} P_\alpha^- - \sum_{j\in s} \frac{\vec{p}_{j\perp}^2}{2p_j^+} + i\epsilon}$ where P_α represents the external $(-)$ component of the momentum to have entered the state prior to s.
- The numerator factor of the Feynman graph $N_G(\{k_i^\mu\})$, is evaluated using the on-shell value of l^-, for each line $l(k,p)$ i.e for a lightcone order labeled by λ, $N_\lambda = N_G\left(\left\{\frac{\vec{l}_{i,\perp}^2}{2l_i^+}, l_i^+, \vec{l}_{i,\perp}\right\}\right)$.

Given an arbitrary graph in LCOPT, we start our analysis by asking when the momentum in the $\perp$ directions are pinched. This question is the two dimensional analog of the situation encountered in four dimensional covariant perturbation theory. Since momentum denominators are also quadratic in covariant perturbation theory, we expect that the Landau equations for the transverse momenta alone, to take a similar form. Suppose we are studying an L-loop graph, with integration variables given by transverse momenta $\vec{k}_{1\perp}, \vec{k}_{2\perp} \dots \vec{k}_{L\perp}$ and plus momenta $k_1^+ \dots \kappa_L^+$. The general amplitude in any lightcone order $P(V)$, which represents a permuation of the vertex set V, is given by

$$\begin{aligned} \mathcal{M}_G(\{p_i\}) &= \sum_{P(V)} \int \prod_{i=1}^{L} \frac{d^2 k_{i,\perp}}{(2\pi)^2} \frac{dk_i^+}{2\pi} \prod_{i=1}^{E} \frac{\theta(l_i^+)}{2l_i^+} \\ &\quad \times \prod_{s=1}^{V-1} \frac{i}{\sum_{\alpha<s} P_\alpha^- - \sum_{j\in s} \frac{\vec{l}_{j\perp}^2}{2l_j^+} + i\epsilon} N_G\left(\left\{\frac{\vec{l}_{i,\perp}^2}{2l_i^+}, l_i^+, \vec{l}_{i,\perp}\right\}\right). \end{aligned} \tag{3.1}$$

The starting point of our analysis is to write the most general contour deformation possible for the set of integration variables at hand

$$\begin{aligned} \vec{k}_{i\perp} &= \vec{k}_{R,i\perp} + i\lambda_i\, \vec{n}_{i\perp}(\vec{k}_{R,j\perp}) \\ k_i^+ &= k_{R,i}^+ + i\delta_i\, n_i^+(k_{R,i}^+). \end{aligned} \tag{3.2}$$

Here, $\vec{k}_{R,i\perp}$ represents a real integration variable which parametrizes the contour of integration and λ_i, δ_i are small parameters that can be continuously varied and control the size of the deformation. Let us list some properties of $\vec{n}_\perp, n^+$ which we require for any acceptable contour deformation

- The deformation vectors $\vec{n}_\perp, n^+$ are continuous functions of each the integration variables. This ensures that the contour is continuously connected to the real contour of integration.
- The deformation vectors $\vec{n}_\perp, n^+$ are piecewise differentiable functions of the integration variables. This ensures that the real integration variables represent a good parametrization of the contour and the jacobian of the transformation is single valued and finite almost everywhere.
- The deformation vectors $\vec{n}_\perp, n^+$ vanishe near a pinch and always add to the $i\epsilon$ in any state. This ensures that the deformation vectors do not cross a pole for any value of λ_i, δ_i.
- The deformation n_i^+ vanishes near the endpoints of integration for k_i^+. This restriction applies to the plus component exclusively because $+$ momentum integrals have support over bounded sets in LCOPT.

After implementing the contour deformation, the integrand takes the form

$$\begin{aligned}\mathcal{M}_G(\{p_i\}) &= \sum_{P(V)} \int \prod_{i=1}^{L} \frac{d^2 k_{R,i\perp}}{(2\pi)^2} \left| \frac{\partial \vec{k}_{i\perp}}{\partial \vec{k}_{R,j\perp}} \right| \frac{dk^+_{R,i}}{2\pi} \left| \frac{\partial k_i}{\partial k^+_{R,j}} \right| \prod_{i=1}^{E} \frac{\theta(l^+_{R,i})}{2l_i^+} \\ &\quad \times \prod_{s=1}^{V-1} \frac{i}{\sum\limits_{\alpha<s} P_\alpha^- - \sum\limits_{j\in s} \frac{\vec{l}^2_{j\perp}}{2l_j^+} + i\epsilon} N_G\left(\left\{\frac{\vec{l}^2_{i,\perp}}{2l_i^+}, l_i^+, \vec{l}_{i,\perp}\right\}\right). \quad (3.3)\end{aligned}$$

We would now like to define the set of first derivative vectors for the transverse momentum

$$\begin{aligned}S_{j\perp} &= \left\{\vec{v}_{i,j} \,\middle|\, \vec{v}_{i,j} = \frac{\partial c_i}{\partial \vec{k}_{j\perp}} \neq 0,\ i \in [1, V-1]\right\} \\ c_i\left(\{\vec{k}_{m\perp}, k_m^+\}\right) &= \sum_{\alpha<i} P_\alpha^- - \sum_{j\in i} \frac{\vec{l}^2_{j\perp}(\{\vec{k}_{m\perp}\})}{2l_j^+(\{k_m^+\})} + i\epsilon\,. \quad (3.4)\end{aligned}$$

We emphasize that in Eq. (3.4), the derivative is defined with respect to the argument of c_i, and not it's real or imaginary parts. We can also construct an analogous set of derivatives with respect to the plus momenta, by

$$S_j^+ = \left\{v^+_{i,j} \,\middle|\, v^+_{i,j} = \frac{\partial c_i}{\partial k_j^+} \neq 0,\ i \in [1, V-1]\right\}, \quad (3.5)$$

where c_i's appearing in this equation are those defined in Eq. (3.4). We now observe that the LCOPT states c_i, as well as their imaginary parts admit an expansion around the real values of the loop momenta, given by

$$\text{Im}(c_i\left(\{\vec{k}_{m\perp}, k_m^+\}\right) = \epsilon + \sum_{j=1}^{L} \lambda_j \vec{n}_{j,\perp} \cdot \vec{v}_{i,j}\Big|_{k_m=k_{R,m}} + \sum_{j=1}^{L} \delta_j n_j^+ v_{i,j}^+\Big|_{k_m=k_{R,m}} + O(\delta^2\lambda) \tag{3.6}$$

Before proceeding, let us discuss some features of Eq. (3.6). We note that only odd derivatives of c_i's occur in the expansion since we are expanding the imaginary part. Further, higher powers of λ_j alone (e.g. λ_j^3) do not occur since the denominator is quadratic in the perp momenta. We also observe that all higher powers of δ occur in a taylor series expansion. We contrast this with the situation in covariant perturbation theory with quadratic and linear (in the case of Wilson lines) denominators, where the expansion truncates. In what follows, we assume the $\delta_j n_j^+$ is sufficiently small to allow us to consistently drop higher order terms. We will derive the Landau equations in LCOPT presently, which are only contour trapping conditions for small deformations. For larger deformations, we will have to keep non-linear terms in our expansion and derive the "non-linear" Landau equations. The sufficiency of the Landau equations for arbitrary deformations was demonstrated in covariant perturbation theory in Ref. [27]. The considerations in this section also apply to time ordered perturbation theory (TOPT), assuming small deformations of the three momenta.

Let us now concretely state what constitues the condition on the contour deformations to prohibit crossing a pole. The deformation set $\vec{n}_{m\perp}, n_m^+$ constitute an "allowed" deformation if

$$\begin{gathered}\nexists\, \lambda_1, \delta_1 \geq 0, \lambda_2, \delta_2 \geq 0 \ldots \lambda_L, \delta_L \geq 0 \;: \\ \sum_{j=1}^{L} \lambda_j \vec{n}_{j,\perp} \cdot \vec{v}_{i,j}\Big|_{k_m=k_{R,m}} + \sum_{j=1}^{L} \delta_j n_j^+ v_{i,j}^+\Big|_{k_m=k_{R,m}} < 0,\ \forall c_i.\end{gathered} \tag{3.7}$$

It is clear that the condition in Eq. (3.7) is satisfied if and only if

$$n_j^+ v_{i,j}^+\Big|_{k_m=k_{R,m}} > 0\ \forall i \in [1, V-1] \text{ and } \forall j \in [1, L], \tag{3.8}$$

$$\vec{n}_{j,\perp} \cdot \vec{v}_{i,j}\Big|_{k_m=k_{R,m}} > 0\ \forall i \in [1, V-1] \text{ and } \forall j \in [1, L]. \tag{3.9}$$

If no such n_j^+ exists, we say the $k_{R,j}^+$ contour is trapped and n_j^+ must identically be 0. Similarly, if no such $\vec{n}_{j\perp}$ exists, we say the $\vec{k}_{R,j\perp}$ contour is trapped and $\vec{n}_{j\perp}$ must identically be 0. For any loop momentum k_j^+, we see that such an n_j^+ exists if and only if all the $v_{i,j}^+\Big|_{k_m=k_{R,m}}$'s have the same sign. Therefore, the condition for a pinch in the k_j^+ contour at leading order in the Landau analysis is

$$\exists \beta_1, \beta_2 \ldots \beta_{|S_j^+|} \geq 0, \text{ and } \exists i : \beta_i > 0 : \sum_{i=1}^{|S_j^+|} \beta_i\, v_{j_i,j}^+\Big|_{k_m=k_{R,m}} = 0 \tag{3.10}$$

We will call Eq. (3.10) the Landau equation for the plus momenta. If all the derivative vectors $v^+_{i,j}$ all have the same sign, they cannot add to zero with positive coefficients and if atleast one has a different sign, we can find a set of β_i's such that they add to zero. It is now straightforward to construct a contour deformation in the absence of a pinch

$$\begin{aligned} n^+_j \;\; &= \;\; \left(\text{Min}[v^+_{j1,j}v^+_{j2,j},\, v^+_{j1,j}v^+_{j3,j}\, \ldots\, v^+_{j1,j}v^+_{j_{|S^+_j|},j}] - \text{Min}[v^+_{j1,j}v^+_{j2,j},\, v^+_{j1,j}v^+_{j3,j}\, \ldots\, v^+_{j1,j}v^+_{|S^+_j|,j},\, 0]\right) \\ &\quad \times f_j(\vec{k}_{R,m\perp}, k^+_{R,m}) \left(\frac{k^+_{j,u} - k^+_{R,j}}{p^+_m}\right)\left(\frac{k^+_{R,j} - k^+_{j,l}}{p^+_m}\right), \end{aligned} \tag{3.11}$$

where p_m+ is the largest external plus momentum in the graph and $k^+_{j,u}$,$k^+_{j,l}$ represent the upper and lower limits of integration for the momentum k^+_j. The function f_j is any positive function, continuous in each of arguments. We note that the difference of minimum functions is positive when all the $v^+_{i,j}$ have the same sign and zero otherwise. In Eq. (3.11), all $v^+_{i,j}$ are evaluated at real values of the loop momenta. We henceforth drop the dependence of $v^+_{i,j}, \vec{v}_{i,j}$ on it's arguments, and it is implicit that they are evaluated at real values of the loop momenta. The space of all continuous functions f_j scan the space of all allowed contour deformations. We would like to emphasize the size of f_j is constrained by the requirement that higher order terms do not change the sign of the deformation.

Let us now repeat the forgoing analysis for the transverse momenta. The equivalent formalism is in one dimension higher (since there are now two components in the transverse momenta). To develop the tools necessary to write a contour deformation, we make a few more definitions.

We first define the convex cone of the set of vector in $S_{j\perp}$ to be

$$C(S_{j\perp}) = \left\{\alpha_{j_1}\vec{v}_{j1,j} + \alpha_{j_2}\vec{v}_{j2,j} \ldots + \alpha_{j_{|S_{j\perp}|},j}\vec{v}_{j_{|S_{j\perp}|},j} \,\middle|\, \alpha_i \geq 0 \text{ and } \exists i : \alpha_i > 0\right\}. \tag{3.12}$$

With this definition of the convex cone, we notice that the zero vector lies in the closure of the convex cone, i.e. there exist a sequence of vectors in the cone which approaches the zero vector. We conclude that either $\vec{0}$ lies inside the cone or it lies on the boundary of the cone. Upon inspecting Eq. (3.9), we notice that a contour deformation of the transverse momenta is possible only if there exists a line through the origin, with normal $\vec{n}_{j\perp}$, such that all the vectors lie on one side of the line, i.e. $\vec{n}_{j\perp}.\vec{v}_{i,j} > 0\ \forall i$, the cone lies entirely on one side of the line.

If the zero vector lies inside the cone clearly such a line does not exist, since every line through the origin necessarily intersects the cone at more than one point, therefore diving the cone into two pieces. If the zero vector lies on the boundary, however, we appeal to the supporting hyperplane theorem [28], to conclude that there exists a hyperplane (a line in this case) through the boundary point $\vec{0}$, such that the entire convex set lies on one side of the plane. This situation is depicted in Fig. 5.

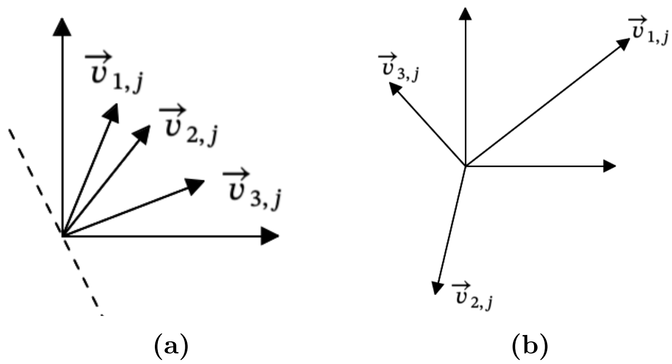

Figure 5: (a) A situation where a supporting hyperplane exists. The dotted line represents a supporting hyperplane, and the convex set lies entirely on side of the dotted line. The convex cone is the region between the vectors $\vec{v}_{1,j}$ and $\vec{v}_{3,j}$. (b) A situation where a supporting hyperplane does not exist. The entire plane is the convex cone of the vectors, and line through origin divides the plane into two disjoint parts.

If any of the vectors $\vec{v}_{i,j}$, are zero when evaluated at the real value of the loop momenta, zero automatically lies inside the cone and a deformation of the $\vec{k}_{j\perp}$ contour is not possible. If line carrying the loop momentum us soft in the transverse directions, the transverse momenta cannot be deformed.

The Landau equations, when satisfied represent a trapping of the contours of integration onto the real axis, are equivalent to the null vector being inside the cone.

$$\exists \alpha_1, \alpha_2 \dots \alpha_{|S_j^+|} \geq 0, \text{ and } \exists i : \alpha_i > 0 : \sum_{i=1}^{|S_{j\perp}|} \alpha_i \, \vec{v}_{j_i,j}\Big|_{k_m = k_{R,m}} = 0. \tag{3.13}$$

Having derived contour trapping conditions for the plus and transverse momenta, we conclude that Eqs. (3.13), (3.10) are the Landau equations for small deformations.

Finally, we construct a contour deformation, or a separating hyperplane, when it exists. Given the vector $\vec{v}_{i,j}$, the set of vectors with a positive inner product with $\vec{v}_{i,j}$, is the convex half plane generated by $\vec{v}_{i,j}^{\perp,1}$, $\vec{v}_{i,j}^{\perp,2} = -\vec{v}_{i,j}^{\perp,1}$, where $\vec{v}_{i,j}^{\perp,1}$ is a vector orthogonal to $\vec{v}_{i,j}$, given by

$$\vec{v}_{i,j}^{\perp,1} = \begin{pmatrix} v_{i,j}^{(2)} \\ -v_{i,j}^{(1)}. \end{pmatrix} \tag{3.14}$$

It is easy to check that indeed $\vec{v}_{i,j}^{\perp,1} \cdot \vec{v}_{i,j} = 0 = \vec{v}_{i,j}^{\perp,2} \cdot \vec{v}_{i,j}$. It is now straightforward to construct the intersection of these half spaces. The most general contour deformation is convex cone generated by the vectors $\vec{v}_{i,j}^{\perp,1}, \vec{v}_{i,j}^{\perp,2}$ which satisfy $\vec{v}_{i,j}^{\perp,1} \cdot \vec{v}_{i',j}, \vec{v}_{i,j}^{\perp,2} \cdot \vec{v}_{i',j} > 0$ for all $i' \neq i$. This conclusion follows from the fact that the intersection of two convex cones is always bounded by lines that define the cones.

$$\begin{aligned}\vec{n}_{j\perp} \quad = \quad & \sum_{\substack{l=1,2 \\ \vec{v}_{i,j}\in S_{j\perp}}} \Bigg(\mathrm{Min}[\vec{v}^{\perp,l}_{i,j}\cdot\vec{v}_{j_1,j},\, \vec{v}^{\perp,l}_{i,j}\cdot\vec{v}_{j_1,j},\ldots\vec{v}^{\perp,l}_{i,j}\cdot\hat{\vec{v}}_{i,j}\ldots\,\vec{v}^{\perp,l}_{i,j}\cdot\vec{v}_{j_{|S_{j\perp}|},j}] \\ & - \mathrm{Min}[\vec{v}^{\perp,l}_{i,j}\cdot\vec{v}_{j_1,j},\, \vec{v}^{\perp,l}_{i,j}\cdot\vec{v}_{j_1,j},\ldots\vec{v}^{\perp,l}_{i,j}\cdot\hat{\vec{v}}_{i,j}\ldots\,\vec{v}^{\perp,l}_{i,j}\cdot\vec{v}_{j_{|S_{j\perp}|},j},0]\Bigg) \\ & \times f_{ij}(\vec{k}_{R,m\perp},k^{+}_{R,m})\,, \end{aligned} \tag{3.15}$$

where as before f_{ij} are positive, continuous functions of it's arguments. Here $\hat{\vec{v}}_{i,j}$ indicates that the term with $\vec{v}_{i,j}$ does not occur in the list.

We close this section by making some brief remarks about these deformations. We first note that we have constructed the most general deformation in momentum space i.e. the basis of deformations in momentum space. There may very well be vectors in the space of deformations that are more convenient than others. While the deformations in Eq. (3.8) are restricted by the requirement that they are not dominated by higher order terms in the expansion, the deformations in Eq. (3.15) are arbitrary and are not necessarily small.

3.1 Simplifications in plus deformations and examples

In this section, we discuss two simple one-loop examples, where we can explicitly write down a deformation in terms of a single unknown plus that parametrizes the plus momentum deformation. In these examples, we will be able to show that this deformation alone is sufficient to ensure that the denominators are non singular everywhere except at a pinch singular point. Consequently in this section, we will show how to construct plus momentum deformations for any graph in the large loop parametrization introduced in Sec. 2. We emphasize that here, the deformation, while still parametrized by a single arbitrary function, does not require an auxiliary evaluation minimum functions introduced in Eq. (3.11), but is only valid when the deformation is normalized to a numer small enough that the $O(\lambda^2\delta)$ term in Eq. (3.6) is subleading. Alternatively, one can imagine only deforming the plus momenta, while leaving the transverse momentum undeformed. In this situation, there are no higher order restrictions.

As a first example, consider the one loop form factor in the lightcone order shown in Fig. 6. The denominator of this graph is given by the two states that live between the three vertices of the graph.

$$D = \left(q^- - \frac{(\vec{q}-\vec{k})^2_\perp}{2(q-k)^+} - \frac{\vec{k}^2_\perp}{2k^+} + i\epsilon\right)\left(p_1^- - \frac{(\vec{p_1}-\vec{k})^2_\perp}{2(p_1-k)^+} - \frac{\vec{k}^2_\perp}{2k^+} + i\epsilon\right), \tag{3.16}$$

where we have dropped masses of the lines for notational clarity. Next, we introduce variables that are convenient parametrizations of the transverse momenta $\bar{l}_\perp = \frac{\vec{l}_\perp}{l^+}$, the

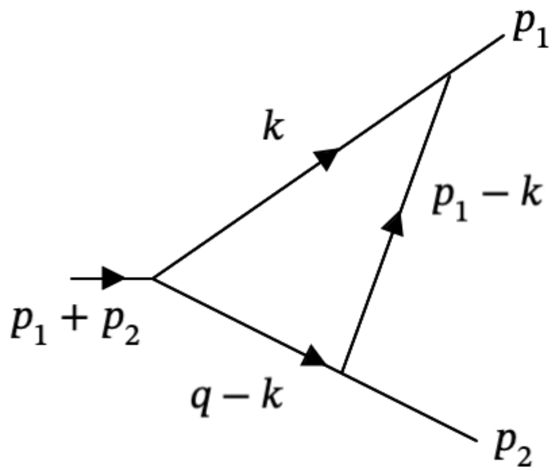

Figure 6: One lightcone order for the one loop form factor. Here, we have drawn arrows on the lines to indicate the flow x^+ time. We have also used the shorthand $q = p_1 + p_2$.

analog of velocities of lines in LCOPT. In terms of these variables, Eq. (3.16) takes the form

$$D = \left(-(\bar{q}_\perp - \bar{k}_\perp)^2 \frac{q^+ k^+}{2(q-k)^+} + \frac{Q^2}{q^+} + i\epsilon\right)\left(-(\bar{p_1}_\perp - \bar{k}_\perp)^2 \frac{{p_1}^+ k^+}{2(p_1-k)^+}\right), \tag{3.17}$$

where we have dropped the $i\epsilon$ in the second term to emphasize that it is a real denominator i.e. it never vanishes in a theory where the intermediate lines are massive and vanishes only near the collinear pinch when the intermediate lines are massless. Here, we have assumed that p_1, p_2 are massless and on-shell. We can immediately construct a contour deformation for k^+, given by

$$n^+ = -f(\vec{k_\perp}, k^+) \times \frac{k^+(p_1^+ - k^+)}{(q^+)^2}, \tag{3.18}$$

where as before, f is any positive continuous function of it's arguments. Further, we notice that all odd derivatives of $\frac{k^+}{(q-k)^+}$ are positive and therefore, there are no constraints coming from higher-order terms.

As a next example consider the two-to-two scattering graph in LCOPT, shown in Fig. 7. There are three denominators in this process, and they are given by

$$\begin{aligned} D \;&=\; \left({p_1}^- - \frac{(\vec{p_1} - \vec{k})_\perp^2}{2(p_1-k)^+} - \frac{\vec{k}_\perp^2}{2k^+} + i\epsilon\right)\left(p_1^- + p_2^- - \frac{(\vec{p_1} + \vec{p_2} - \vec{k})_\perp^2}{2(p_1+p_2-k)^+} - \frac{\vec{k}_\perp^2}{2k^+} + i\epsilon\right) \\ &\qquad \times \left({p_4}^- - \frac{(\vec{p_4} - \vec{k})_\perp^2}{2(p_4-k)^+} - \frac{\vec{k}_\perp^2}{2k^+} + i\epsilon\right). \end{aligned} \tag{3.19}$$

Once again, we have set the masses to zero for notational clarity. Using a change in the transverse momentum variables, it is possible to rewrite these in a form similar to

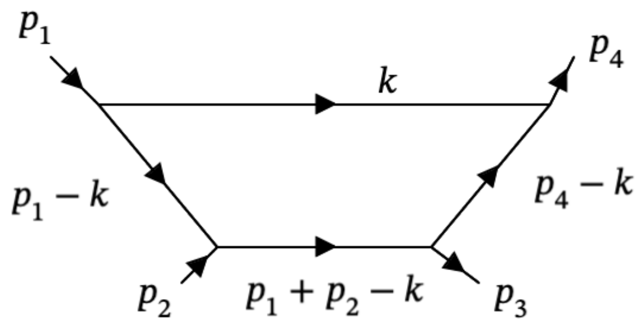

Figure 7: One lightcone order for the two to two scattering graph. As before, we have drawn arrows on the lines to indicate the flow x^+ time. Here, p_1^+, p_2^+ are taken to be incoming and p_3^+, p_4^+ are outgoing.

Eq. (3.17) giving

$$\begin{aligned} D &= \left(-(\bar{p_1}_\perp - \bar{k}_\perp)^2 \frac{{p_1}^+ k^+}{2(p_1-k)^+} + i\epsilon\right)\left(-(\bar{p_4}_\perp - \bar{k}_\perp)^2 \frac{{p_4}^+ k^+}{2(p_4-k)^+} + i\epsilon\right) \\ &\times \left(-(p_1 \bar{+} p_{2\perp} - \bar{k}_\perp)^2 \frac{p_1+{p_2}^+ k^+}{2(p_1+p_2-k)^+} + (\bar{p_1}_\perp - \bar{p_2}_\perp)^2 \frac{p_1^+ p_2^+}{2(p1+p_2)^+} + i\epsilon\right) \end{aligned} \tag{3.20}$$

Inspecting the form of the denominator in Eq. (3.20), we once again infer that the plus momentum admits a simple deformation given by

$$n^+ = -f(\vec{k_\perp}, k^+) \times \frac{k^+(\mathrm{Min}[p_1^+ + p_4^+] - k^+)}{(p_1^+ + p_2^+)^2}, \tag{3.21}$$

Given the simplicity of the deformations in Eqs. (3.21), (3.18), in comparison to the deformation in Eq. (3.11), we might ask if this simplicity is a feature of all graphs at any loop order.

To see that this is indeed the case, we adopt the large loop parametrization described in Sec. 2. Recall, that the key feature of the large loop parametrization was that for a line j, carrying momentum l_j, expanded in terms of the incidence matrices $l_j = \eta_j^m k_m + \tilde{\eta}_j^m p_m$, the incidence matrices were positive definite, i.e. $\eta_j^m, \tilde{\eta}_j^m \in \{0, 1\}$. Say a state C_s appears in the denominator D. Then for every state $C_s \in D$, we have

$$\begin{aligned} C_s &= \sum_{\alpha<s} P_\alpha^- - \sum_{j\in s} \frac{\vec{l}_{j\perp}^2}{2l^+} + i\epsilon \\ &= \sum_{\alpha<s} P_\alpha^- - \sum_{j\in s} \frac{(\eta_j^m \vec{k}_{m\perp} + \tilde{\eta}_j^m \vec{p}_{m\perp})^2}{2|l^+|^2} \left(\eta_j^m (k_{R,m}^+ - i\delta_m n_m^+) + \tilde{\eta}_j^m p_m^+\right) + i\epsilon, \end{aligned} \tag{3.22}$$

where in the second line, we have expanded l_j and multiplied and divided by $(l^+)^*$. The positivity of η_j^m ensures that the coefficient of n_m^+ is positive everywhere in phase space. There, the deformation in this parametrization reads

$$n_j^+ \quad = \quad f_j(\vec{k}_{R,m\perp}, k^+_{R,m}) \left(\frac{k^+_{j,u} - k^+_{R,j}}{p^+_m}\right) \left(\frac{k^+_{R,j} - k^+_{j,l}}{p^+_m}\right), \tag{3.23}$$

where as before p_m^+ is a largest external plus momenta. Once again, we notice that despite the possibility of higher order correction to the deformation, the situation does not occur in LCOPT (if the $O(\lambda^2\delta)$ deformation is also positive). It is also easy to see that internal lines that massive do not change situation since the state denominator is obtained by the replacement $\frac{\vec{l}^2_{j\perp}}{2l^+} \to \frac{\vec{l}^2_{j\perp}}{2l^+} + \frac{m_j^2}{2l^+}$, as long as the m^2 parameters are positive.

4 Geometric Landau analysis in covariant perturbation theory and contour deformations

In this section, we briefly summarize the result of adapting the analysis of Sec. 3 to covariant perturbation theory, to construct a basis of deformations in momentum space. While the arguments are unchanged, the analysis in covariant perturbation theory differs in some details.

5 Conclusions

In this paper, we have constructed contour deformations in the mixed representation, with both Feynman parameters and loop momenta, as well as in loop momentum space. These contour deformations were systematic, in that there exists an algorithmic procedure to find them and they admit closed from expressions. In one-loop examples, we constructed a contour deformation that takes a particularly simple form and endows each state with a positive imaginary imaginary part. We have identified a framework where one might deduce the most general contour deformation that can be made in momentum space. However, such contour deformations were still parametrized by arbitrary positive functions. One would like to answer the question of which choice of functions are numerically and analytically tractable. Further, one would like to entirely eliminate the intermediate problem of solving the geometric convex cone at each point in phase space and give a direct prescription for the contour deformation.

References

[1] C. Anastasiou, R. Haindl, G. Sterman, Z. Yang and M. Zeng, "Locally finite two-loop amplitudes for off-shell multi-photon production in electron-positron anni-

hilation," JHEP **04**, 222 (2021) doi:10.1007/JHEP04(2021)222 [arXiv:2008.12293 [hep-ph]].

[2] C. Anastasiou and G. Sterman, "Locally finite two-loop QCD amplitudes from IR universality for electroweak production," JHEP **05**, 242 (2023) doi:10.1007/JHEP05(2023)242 [arXiv:2212.12162 [hep-ph]].

[3] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J. C. Winter, "From loops to trees by-passing Feynman's theorem," JHEP **09**, 065 (2008) doi:10.1088/1126-6708/2008/09/065 [arXiv:0804.3170 [hep-ph]].

[4] Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni and B. Ruijl, "Numerical Loop-Tree Duality: contour deformation and subtraction", JHEP **04** (2020) 096, [1912.09291].

[5] Z. Capatti, V. Hirschi, D. Kermanschah and B. Ruijl, "Loop-Tree Duality for Multiloop Numerical Integration", Phys. Rev. Lett. **123** (2019) 151602, [1906.06138].

[6] Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni and B. Ruijl, "Manifestly Causal Loop-Tree Duality", 2009.05509

[7] G. Sterman, A. Venkata, "Four-dimensional infrared safety in time-ordered perturbation theory " [arXiv:2309.13023[hep-ph]]

[8] Z. Capatti, "Exposing the threshold structure of loop integrals," Phys. Rev. D **107**, no.5, L051902 (2023) doi:10.1103/PhysRevD.107.L051902 [arXiv:2211.09653 [hep-th]].

[9] D. Kermanschah, "Numerical integration of loop integrals in momentum space," doi:10.3929/ethz-b-000602637

[10] G. F. Sterman, "Mass Divergences in Annihilation Processes. 2. Cancellation of Divergences in Cut Vacuum Polarization Diagrams," Phys. Rev. D **17**, 2789 (1978) doi:10.1103/PhysRevD.17.2789

[11] G. F. Sterman, "Zero Mass Limit for a Class of Jet Related Cross-sections," Phys. Rev. D **19**, 3135 (1979) doi:10.1103/PhysRevD.19.3135

[12] G. T. Bodwin, "Factorization of the Drell-Yan Cross-Section in Perturbation Theory," Phys. Rev. D **31**, 2616 (1985) [erratum: Phys. Rev. D **34**, 3932 (1986)] doi:10.1103/PhysRevD.34.3932

[13] J. C. Collins, D. E. Soper and G. F. Sterman, "Factorization for Short Distance Hadron - Hadron Scattering," Nucl. Phys. B **261**, 104-142 (1985) doi:10.1016/0550-3213(85)90565-6

[14] S. Becker and S. Weinzierl, "Direct contour deformation with arbitrary masses in the loop," Phys. Rev. D **86**, 074009 (2012) doi:10.1103/PhysRevD.86.074009 [arXiv:1208.4088 [hep-ph]].

[15] S. Becker, C. Reuschle and S. Weinzierl, "Numerical NLO QCD calculations," JHEP **12**, 013 (2010) doi:10.1007/JHEP12(2010)013 [arXiv:1010.4187 [hep-ph]].

[16] Z. Nagy and D. E. Soper, "General subtraction method for numerical calculation of one loop QCD matrix elements," JHEP **09**, 055 (2003) doi:10.1088/1126-6708/2003/09/055 [arXiv:hep-ph/0308127 [hep-ph]].

[17] W. Gong, Z. Nagy and D. E. Soper, "Direct numerical integration of one-loop Feynman diagrams for N-photon amplitudes," Phys. Rev. D **79**, 033005 (2009) doi:10.1103/PhysRevD.79.033005 [arXiv:0812.3686 [hep-ph]].

[18] M. Assadsolimani, S. Becker and S. Weinzierl, "A Simple formula for the infrared singular part of the integrand of one-loop QCD amplitudes," Phys. Rev. D **81**, 094002 (2010) doi:10.1103/PhysRevD.81.094002 [arXiv:0912.1680 [hep-ph]].

[19] C. Anastasiou, S. Beerli and A. Daleo, "Evaluating multi-loop Feynman diagrams with infrared and threshold singularities numerically," JHEP **05**, 071 (2007) doi:10.1088/1126-6708/2007/05/071 [arXiv:hep-ph/0703282 [hep-ph]].

[20] D. E. Soper, "Choosing integration points for QCD calculations by numerical integration," Phys. Rev. D **64**, 034018 (2001) doi:10.1103/PhysRevD.64.034018 [arXiv:hep-ph/0103262 [hep-ph]].

[21] G. F. Sterman, "An Introduction to quantum field theory," Cambridge University Press, 1993, ISBN 978-0-521-31132-8

[22] M. Borinsky, H. J. Munch and F. Tellander, "Tropical Feynman integration in the Minkowski regime," Comput. Phys. Commun. **292**, 108874 (2023) doi:10.1016/j.cpc.2023.108874 [arXiv:2302.08955 [hep-ph]].

[23] G. F. Sterman, "Partons, factorization and resummation, TASI 95," [arXiv:hep-ph/9606312 [hep-ph]].

[24] M. Borinsky, Z. Capatti, E. Laenen and A. Salas-Bernárdez, "Flow-oriented perturbation theory," JHEP **01**, 172 (2023) doi:10.1007/JHEP01(2023)172 [arXiv:2210.05532 [hep-th]].

[25] P. T. Komiske, E. M. Metodiev and J. Thaler, "The Hidden Geometry of Particle Collisions," JHEP **07**, 006 (2020) doi:10.1007/JHEP07(2020)006 [arXiv:2004.04159 [hep-ph]].

[26] S. Coleman and R. E. Norton, "Singularities in the physical region," Nuovo Cim. **38**, 438-442 (1965) doi:10.1007/BF02750472

[27] J. Collins, "A new and complete proof of the Landau condition for pinch singularities of Feynman graphs and other integrals," [arXiv:2007.04085 [hep-ph]].

[28] S. Boyd, and L. Vandenberghe, "Convex optimization", Cambridge Univ. Press, UK. (2004).