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Unitarity Constraints on power corrections and renormalons in the e^+e^- hadronic cross section

Abstract: We derive non-pertrubative unitarity bounds on the hadronic cross section for e^+e^- without committing to an underlying microscopic description. We then demonstrate a unitarity violation in the strong coupling regime if one uses naive perturbative QCD. It is necessary to add power corrections to the cross section if one is to restore unitarity in the full cross section. We therefore demonstrate a direct bound on the size of power corrections that follows from unitarity. We study connection between perturbative renormalons and power corrections, therefore finding bounds on the strength of renormalons.

Contents

1 Introduction and Motivation

Scattering processes in unitary quantum field theories with a mass gap are expected to respect the Froissart bound [1, 2], for asymptotically large centre of mass energies, which follows from the assumptions that the scattering amplitude is the boundary value of an analytic function and that the scattering amplitude is polynomially bounded for fixed physical t

$$\begin{aligned} T_{\text{phys}}(s,t) &= \lim_{\epsilon\to 0} T(s+i\epsilon,t) \\ \lim_{|s|\to\infty} |T(s,t)| &< |s|^N. \end{aligned} \tag{1.1}$$

Here, s and t are the usual Mandelstam in-variants. In particular, the derivation [3] of the Froissart bound follows from establishing that the $2 \to 2$ scattering amplitude, $T(s,t)$, is in analytic in t for $t \leq t_0 \in \mathbb{R}^+$[4], for fixed physical s. Here t_0 represents location of the s cut, $4m_\pi^2$ in a massive theory. The Froissart bound is usually written as a bound on the total cross section of pion scattering for asymptotically large energies, $s \to \infty$,

$$\sigma_T < \frac{\pi}{m_\pi^2}(\log s)^2. \tag{1.2}$$

Scattering for identical scalar particles proceeds through s, t, u channels and crossing symmetry requires an s cut for negative values of $s < t$. However, one expects a stronger, point-wise unitarity bound on processes that proceed exclusively through the s channel. This is expectation arises out of the fact that such processes have a S matrix that are entire functions of t and therefore proceed through a finite number of partial waves. Electron-positron scattering is one such process, $e^+ + e^- \to$ Hadrons. The process proceeds perturbatively through he decay of an off-shell photon into quarks which then interact strongly to hadronize. We expect it to be a point-wise, local bound because the large spin, asymptotic behaviour does not need to be bounded.

Initial states with no hadronic content, a classic example of which is e^+e^- scattering, truncated at order α_e^2 is extensively studied in QCD, and its infrared structure was elucidated in [5, 6]. Such a truncation of the electromagnetic coupling assumes that the coupling is much smaller than the strong coupling and it is therefore consistent to work to leading order in electromagnetic coupling, while attempting to work to higher and higher orders in the strong coupling. The experimentally relevant R ratio is defined by

$$R(e^+ + e^- \to \text{Hadrons}) = \frac{\sigma(e^+ + e^- \to \text{Hadrons})}{\sigma(e^+ + e^- \to \mu^+ + \mu^-)}. \tag{1.3}$$

Here, σ represents the total cross section for the process. The R-ratio is known within perturbative QCD upto four loop accuracy [7]. The R-ratio has also been probed extensively in experiments, and the cross section measurements can be found in [8].

In this paper, we study the simplest QCD processes i.e those with no initial state hadronic content using the machinery of analyticity and unitarity. It probes the vacuum of the interacting microscopic theory through the vacuum polarization tensor for the photon. One motivation to carry out such a study is to use the non perturbative tools at our disposal: analyticity and unitarity to study the often neglected power corrections to perturbation theory. Perturbation theory is understood to be the first, calculable term in a infinite series of power corrections in inverse powers of the center of mass momentum(corrections to the cross section of $O(\frac{1}{Q^n})$). Since the R-ratio is understood to be proportional to the two point function of the electromagnetic current, it is expected that through the use of Wilson's OPE [9], the two-point function may be expressed as a sum over QCD condensates, with increasing classical dimension.

$$\langle j_\mu(Q) j^\mu(0)\rangle = C_I\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) + \frac{C_{\bar{q}q}\left(\frac{Q^2}{\mu^2}\right)}{Q^4}\, m_q\langle \bar{q}q(0)\rangle + \frac{C_{F^2}\left(\frac{Q^2}{\mu^2}\right)}{Q^4}\langle \mathrm{Tr}(F_{\mu\nu}^2)\rangle + O\left(\frac{1}{Q^6}\right). \tag{1.4}$$

Further, the correlator depends only on the center of mass momentum Q, which represents the hard scale in the problem. The gluonic condensate was first introduced in [10], and condensates upto dimension 6 were used to obtain the masses of low lying resonances using the SVZ sum rules [11], [12]. These power corrections are related to IR renormalons that signal the failure of pertrubation theory to converge. In particular, IR renormalons that appear in the perturbative series for the leading OPE coefficient, are expected to be cancelled by similar renormalon poles coming from QCD condensates. This connection was made precise for the total cross section by Mueller [13] and for jet cross sections in [14]. Power corrections are also expected to restore unitarity that is naively violated by the perturbative expansion. Therefore, we can place bounds on the size of power corrections through the use of an unitarity bound.

2 Unitarity bounds and general considerations

2.1 Consequences of unitarity

Let us consider the process

$$e^-(p_1, s_1) + e^+(p_2, s_2) \longrightarrow e^-(p_3, s_1') + e^+(p_4, s_2'). \tag{2.1}$$

Eventually, we would like to set $s_1 = s_1'$ and $s_2 = s_2'$, to obtain a unitarity bound on a forward scattering like matrix element. The unitarity condition on this process is the usual one:

$$S^\dagger S = \mathbb{1}, \tag{2.2}$$

where S is the S matrix of scattering and the unitarity relationship encodes the conservation of probability. To isolate the free propagation component of the S matrix, we make the decomposition

$$S = \mathbb{1} + i\mathbb{T}. \tag{2.3}$$

Let us now consider the matrix element of $\mathbb{T}$ in the states of interest, remembering that the physical $\mathbb{T}$ matrix element is obtained by approaching the real axis of the s variable from the upper half plane.

$$\begin{aligned} &\langle e^-(p_3, s_1')e^+(p_4, s_2')|\ \mathbb{T}\ |e^-(p_1, s_1)e^+(p_2, s_2)\rangle \\ &= \lim_{\epsilon\to 0^+} T_{s_1 s_2, s_1' s_2'}(s + i\epsilon, t)(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4), \end{aligned} \tag{2.4}$$

where we have defined the usual Mandelstam invariant $s = (p_1 + p_2)^2$ and $t = (p_1 - p_3)^2$. We will now turn our attention to the matrix element with $s_1 = s_1'$ and $s_2 = s_2'$, which we will simply denote by $T_{s_1 s_2}$. We will also assume hermitian analyticity to write the relationship

$$\begin{aligned} &\langle e^-(p_3)e^+(p_4)|\ \mathbb{T}^\dagger\ |e^-(p_1)e^+(p_2)\rangle = \langle e^-(p_1)e^+(p_2)|\ \mathbb{T}\ |e^-(p_3)e^+(p_4)\rangle^* \\ &= \lim_{\epsilon\to 0^+} T_{s_1 s_2}(s - i\epsilon, t)(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4), \end{aligned} \tag{2.5}$$

where we suppressed spin indices in the states for brevity. In terms of the $\mathbb{T}$ matrix the unitarity constraint reads:

$$(\mathbb{T} - \mathbb{T}^\dagger) = i\mathbb{T}\mathbb{T}^\dagger. \tag{2.6}$$

For distinct initial and final states we may write this in terms of the matrix elements of $\mathbb{T}$ as:

$$\frac{1}{i}\langle f|\mathbb{T} - \mathbb{T}^\dagger|i\rangle = \sum_{n=2}^{\infty}\int d\mu_n \langle f|\mathbb{T}|q^{(n)}\rangle \langle q^{(n)}|\mathbb{T}^\dagger|i\rangle\,, \tag{2.7}$$

where $q^{(n)}$ is a representative point in the n particle phase space and $d\mu_n$ is the measure on the n particle phase space. For a class of final / initial states, the $\mathbb{T}\mathbb{T}^\dagger$ matrix elements satisfy a positivity constraint given by:

$$\int d\mu_n \langle f|\mathbb{T}|q^{(n)}\rangle \langle q^{(n)}|\mathbb{T}^\dagger|i\rangle \geq 0. \tag{2.8}$$

A proof of the positivty of this matrix element is supplied in Appendix A. Using this postivity constraint, we obtain a unitarity constraint in terms of the 2 particle scattering matrix alone:

$$\frac{1}{i} \langle f|\mathbb{T} - \mathbb{T}^{\dagger}|i\rangle \geq \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}} \langle f|\mathbb{T}|k_1, k_2\rangle \langle k_1, k_2|\mathbb{T}^{\dagger}|i\rangle \,. \tag{2.9}$$

At this stage, let us clarify what states appear in this inequality. We make the identification,

$$\begin{aligned} |i\rangle &= |e^-(p_1, s_1), e^+(p_2, s_2)\rangle \\ |f\rangle &= |e^-(p_3, s_1), e^+(p_4, s_2)\rangle \\ |k_1, k_2\rangle &= |e^-(k_1, s_1), e^+(k_2, s_2)\rangle \,, \end{aligned} \tag{2.10}$$

where the intermediate state spins are not summed over, but chosen to be the same as the incoming/outgoing spins. We imagine working in a theory where the QED photon couples to charged quarks in the usual way. We also assume that a truncation in the QED coupling has not yet been carried out. Let us now recast Eq.(2.9) in terms of the amplitude:

$$\mathrm{Im}[T_{s_1 s_2}(s,t)] \geq \frac{1}{2} \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) T_{s_1 s_2}(s,t') T^*_{s_1 s_2}(s,t''). \tag{2.11}$$

Next, we would like to rewrite this inequality in the partial wave basis, to obtain the standard unitarity bound on the partial wave amplitudes.

$$T_{s_1 s_2}(s,t) = \sum_{J=0}^{\infty} 16\pi(2J+1) f_J^{s_1 s_2}(s) P_J(\cos\theta), \tag{2.12}$$

Where f_J's are the partial wave amplitudes and $P_J(\cos\theta)$ are the standard Legendre polynomials. The kinematic relationship between $\cos\theta$ and t is $\cos\theta = 1 + \frac{2t}{s}$. Everywhere in this paper, we neglect the electron mass. Our normalization convention for the Legendre Polynomials is:

$$\int_{-1}^{1} dx P_J(x) P_{J'}(x) = \frac{2\delta_{JJ'}}{2J+1} \tag{2.13}$$

It is now clear that the partial wave decomposition enables us to do the phase space integrals explicitly in Eq. (2.11) since all the angular dependence is in the known Legendre Polynomials. Some more details on doing the phase space integrals can be found in Appendix B. The inequality in terms of partial wave amplitudes reads:

$$\mathrm{Im}[f_J^{s_1 s_2}(s)] \geq |f_J^{s_1 s_2}|^2. \tag{2.14}$$

Said differently, the unitarity constraint on the imaginary part is

$$1 \geq \mathrm{Im}[f_J^{s_1 s_2}] \geq 0. \tag{2.15}$$

In terms of partial waves, this is the final unitarity bound. We would now like to derive a similar bound for the spin averaged object, defined by

$$T(s,t) = \frac{1}{4} \sum_{s_1, s_2} T_{s_1 s_2}(s,t) = \frac{1}{4} \sum_{s_1, s_2, s_1', s_2'} T_{s_1 s_2, s_1' s_2'}(s,t) \delta_{s_1, s_1'} \delta_{s_2, s_2'}. \tag{2.16}$$

We now decompose $T(s,t)$ into partial waves, with the same normalisation,

$$T(s,t) = \sum_{J=0}^{\infty} 16\pi(2J+1) f_J(s) P_J(\cos\theta). \tag{2.17}$$

It now follows from Eq. (2.15) that the imaginary part of f_J satisfies the same bound,

$$1 \geq \mathrm{Im}[f_J] \geq 0. \tag{2.18}$$

2.2 Truncation of the electromagnetic coupling

We would now like to truncate the perturbative series in the electromagnetic coupling. In doing so, we would like to separate the QED contributions to the amplitude from the QCD corrections. To this end, we imagine that the photon couples to the quarks with a coupling constant that is formally distinct from the coupling to the electrons. Lets say the coupling to the electrons is via the coupling constant e and the coupling to the quarks is through e_q. Now we truncate the electromagnetic interactions to $O(e^2 e_q^2)$. We will make a further assumption that the unitarity bound Eq. (2.18) holds order by order in perturbation theory.

An observation that we can make at this stage is that while the idea is to obtain a bound on cross section for real world QED ($e^2(m_e) = \frac{4\pi}{137}, \sum_q e_q^2 = \frac{5}{9}e^2$), any bound we obtain on the purely hadronic tensor, must hold for any consistent, weak coupling of the photon to electrons and quarks. Therefore, the bounds on the hadronic tensor itself are necessarily in-dependant of the value of e, e_q.

Let us now proceed to derive a bound on the hadronic tensor. Our strategy is to compute the imaginary part of the amplitude for e^+e^- scattering and use the bound presented in Eq. (2.18) for the partial wave amplitudes that appear in its decomposition. Tree level diagrams that occur at $O(e^2)$ have a real part but no imaginary part. At $O(e^2 e_q^2)$, the amplitude has two contributions, one in the s channel as seen in Fig. 1, and one in the t channel. However, the t channel diagram has no s channel discontinuity because it is an entire function in s. Therefore, we only need to compute the partial wave decomposition of the diagram in Fig. 1. Since its a tree level diagram, its dependence on t is trivial.

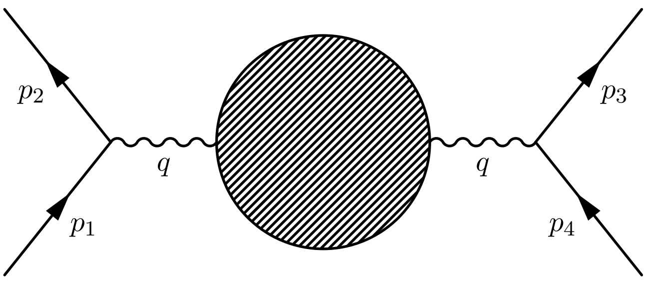

Figure 1. The $O(e^2 e_q^2)$ that is the sole contribution to the imaginary part of the scattering matrix. The shaded blob represents QCD corrections to the process at all orders.

We will parametrize our ignorance of hadronic processes through the hadronic tensor which is defined as:

$$\pi_{\mu\nu}(q^2) = \pi(q^2)(q^2\eta_{\mu\nu} - q_\mu q_\nu) = \int d^4x e^{-iq\cdot x}\langle j_\mu(x) j_\nu(0)\rangle, \tag{2.19}$$

where we have defined $q = p_1 + p_2$ and j_μ is EM current in QCD. Taking the trace, we can define the $\pi(q^2)$ through the equation:

$$\pi(q^2) = \frac{1}{3q^2}\int d^4x e^{-iq\cdot x}\langle j_\mu(x) j^\mu(0)\rangle. \tag{2.20}$$

Let us now write the contribution of our graph of interest to the spin averaged amplitude,

$$T(s,t) \ni \frac{1}{4}e^2\sum_q e_q^2 \sum_{s_1,s_2} \bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\bar{u}_{s_1}(p_3)\gamma^\nu v_{s_2}(p_4)\left(\frac{-\pi_{\mu\nu}}{q^4}\right), \tag{2.21}$$

where the factor of $\frac{1}{4}$ comes from the spin average. It is possible to compute the spin sum in the center of mass frame. Some details of this computation are supplied in Appendix C. The result is

$$\sum_{s_1,s_2} \bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\bar{u}_{s_1}(p_3)\gamma^\nu v_{s_2}(p_4)(q^2\eta_{\mu\nu} - q_\mu q_\nu) = -4q^4\left(1+\frac{2t}{s}\right). \tag{2.22}$$

Using this result, the amplitude takes the succinct form

$$T(s,t) \ni e^2\sum_q e_q^2 \pi(q^2) P_1(\cos\theta). \tag{2.23}$$

Therefore, the imaginary part of the amplitude is directly related to the imaginary part of the hadronic tensor. Further, the imaginary part of the hadronic tensor is also related to the imaginary part of the the partial wave amplitude f_1, since only the first Legendre polynomial appears in the first partial wave decomposition.

$$\mathrm{Im}[T(s,t)] = e^2\sum_q e_q^2 \mathrm{Im}[\pi(q^2)] P_1(\cos\theta) = 48\pi \mathrm{Im}[f_1(s)] P_1(\cos\theta). \tag{2.24}$$

Using the bound in Eq. (2.18), we obtain a bound on the the hadronic function:

$$0 \leq \mathrm{Im}[\pi(q^2)] \leq \frac{48\pi}{e^2\sum_q e_q^2}. \tag{2.25}$$

As we have argued, this bound must hold for any values of the couplings as long as perturbation theory is applicable. Since an asymptotic series continues to approximate the true function as long as successive terms are smaller, we can ask when the $O(e^2)$ tree level contribution is of the same size as the $O(e^2 e'^2)$ contribution to obtain the optimal bound.

We would now like to use analyticity to obtain a bound on full hadronic tensor, π. It is however more convenient to place a bound on the Adler D function instead. It is defined by the equation

$$D(s) = s\frac{d\pi(s)}{ds}. \tag{2.26}$$

Let us recall that since $\pi(s)$ is the hadronic contribution to a two point function, its exact analytic structure is known. In particular, we know that $\pi(s)$ has a branch cut on the real s axis corresponding to the production of multi-particle states and poles on the real axis corresponding to the single particle particle states. It has no other singularities in the complex plane. Using this, we may write a dispersion relation for the Adler function.

$$D(s) = s\int_{C_1} \frac{ds'}{2\pi i}\frac{\pi(s')}{(s'-s)^2} = s\int_{t_0}^{\infty}\frac{ds'}{\pi}\frac{\mathrm{Im}[(\pi(s')]}{(s'-s)^2}, \tag{2.27}$$

Where in the first equality, C_1 is contour wrapping around the point s and the equality follows from the definition in Eq. (2.26). The second equality follows from deforming the contour out to infinity, and dropping the contribution of the arc at infinity.Here t_0, is the starting point of the cut. A pictorial representation of the deformation is presented in Fig. 2.

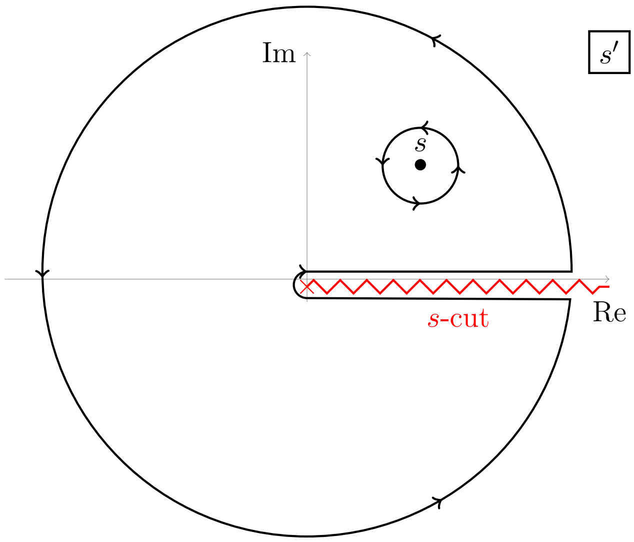

Figure 2. The two contours on which the integral is equal. The first contour is deformed into the second. Analyticity relates the imaginary part of the hadronic function $\pi(s)$ to the full Adler function $D(s)$.

Using the unitarity bound Eq. (2.25), let us now obtain a bound on the Adler function $D(s)$. We utilize the identity $|\int dz f(z)|\leq \int dz|f(z)|$ to write:

$$|D(s)|\leq |s|\int_{t_0}^{\infty}\frac{ds'}{\pi}\frac{\mathrm{Im}[(\pi(s')]}{|s'-s|^2}. \tag{2.28}$$

It is then possible to utilize the bound on the imaginary part of the hadronic tensor, and extend the integral down to 0

$$|D(s)|\leq \frac{48}{e^2\sum_q e_q^2}\int_0^{\infty} ds' \frac{|s|}{|s'-s|^2}. \tag{2.29}$$

Finally, we carry out the s' integral to obtain a bound on the Adler function:

$$|D(s)| \leq \frac{48}{e^2 \sum_q e_q^2} \frac{\pi - \theta}{\sin\theta}, \tag{2.30}$$

where θ is the argument of s, i.e. $s = |s|e^{i\theta}$. Its easy to check that the bound is both positive definite everywhere on the complex s plane and is a bound on the combined real and imaginary parts of the Adler function. Notice that the bound is finite (and equal to $\frac{48}{e^2 \sum_q e_q^2}$) on the negative real axis, in the region of euclidean momenta, while being unbounded close to the positive real axis, due to the presence of a possible branch cut due to the production of multi-particle states in QCD.

3 Power corrections and the unitarity bound

In this section, we will supply a unitarity bound on the size of power corrections using the bound on the Adler function in Eq. (2.30). Power corrections and a clean separation between perturbative and non-perturbative contributions to the cross section were discussed in [13]. Let us first review this how this separation is achieved and the connection between perturbative predictions and the OPE.

3.1 A review of power corrections to the total cross section

A Proof of the positivity constraint used to obtain the unitarity bound

In this Appendix, we will prove the positivity of the matrix element in the scattering of identical particles. We emphasize that the positivity constraint is not respected if the particles are not identical. Let us recall that the constraint was

$$\int d\mu_n \langle f|\mathbb{T}|q^{(n)}\rangle \langle q^{(n)}|\mathbb{T}^\dagger|i\rangle \geq 0. \tag{A.1}$$

Let us prove this relationship in the center of mass frame, without loss of generality. In doing so, we assume that the S matrix, and the T matrix, both commute with Lorentz transformations. In the COM frame, we have

$$\begin{aligned} |i\rangle &= |e^-(\vec{p_1}, s_1), e^+(-\vec{p_1}, s_2)\rangle \\ |f\rangle &= |e^-(\vec{p_3}, s_1), e^+(-\vec{p_3}, s_2)\rangle \\ &. \end{aligned} \tag{A.2}$$

If R is a rotation matrix, such that $R \cdot p_3 = p_1$, there exists an Unitary operator $U[R]$, which is the image of R under the representation of the Lorentz group on the Hilbert space. $U[R]$ satisfies:

$$U[R]\,|e^-(\vec{p_3}, s_1), e^+(-\vec{p_3}, s_2)\rangle = |e^-(\vec{p_1}, s_1), e^+(-\vec{p_1}, s_2)\rangle\,. \tag{A.3}$$

Combined with the assertion that $U[R]$ commutes with $\mathbb{T}$, we may rewrite left hand side of Eq. (A.1) as:

$$\int d\mu_n \langle i|\mathbb{T}\; U[R]|q^{(n)}\rangle \langle q^{(n)}|\mathbb{T}^\dagger|i\rangle\,. \tag{A.4}$$

Let us now make the definition

$$\mathbb{O}_n = \int d\mu_n U[R] \, |q^{(n)}\rangle \, \langle q^{(n)}| \, . \tag{A.5}$$

With this definition, we can prove our claim if we can show that $\mathbb{O}_n$ is a positive semi definite matrix, since the left hand side of Eq. (A.1) is $\langle i|\mathbb{T}\mathbb{O}_n\mathbb{T}^\dagger|i\rangle$. To do this, we will need to argue that $\langle\psi|\mathbb{O}_n|\psi\rangle \geq 0$. Since, $\mathbb{O}_n$ acts asymptotic states and the action of $\mathbb{O}_n$ on any m particle state $(n \neq m)$ is zero, it is sufficient to consider the n particle subspace.

Within the n particle subspace,$\langle\psi|\mathbb{O}_n|\psi\rangle$ is either 0 or $\int d\mu_n |\langle q^{(n)}|q^{'(n)}\rangle\,|^2 \geq 0$. This proves the claim.

B Doing the two particle phase space integrals

In this Appendix, we supply some details on doing the two particle phase space integral. We follow the conventions of [15]. The two particle phase space integral we encounter is

$$\frac{1}{2}\int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) T(s,t') T^*(s,t''). \tag{B.1}$$

Here, $s = (p_1+p_2)^2$, $t' = (p_1 - k_1)^2$ and $t'' = (k_1 - p_3)^2$. First, its possible to the use the delta functions to do the $\vec{k_2}$ integrals, setting it to $\vec{k_2} = \vec{p_1} + \vec{p_2} - \vec{k_1}$. Next, in the center of mass frame, $\vec{p_1} + \vec{p_2} = 0$ and $s = (p_1^0 + p_2^0)^2$. In this frame, the energy delta function is reduces to $\delta(\sqrt{s} - 2|\vec{k_1}|)$, allowing us to carry out the $|\vec{k_1}|$ integral. The integrals that remain are:

$$\frac{1}{16(2\pi)^2}\int d^2\Omega_2 T(s,t')T(s,t''). \tag{B.2}$$

Next, the angular integrals admit a rewriting in terms of the kernel $K(z,z',z'')$ as

$$\int d^2\Omega_2 = \int_{-1}^{1} dz' \int_{-1}^{1} dz'' K(z,z',z''). \tag{B.3}$$

Here $z = \cos(\theta(p_1,p_3))$, $z' = \cos(\theta(p_1,k_1))$ and $z' = \cos(\theta(k_1,p_3))$. The kernel $K(z,z',z'')$ has the explicit form

$$K(z,z',z'') = 2\sqrt{1-z^2}\,\delta(1 - z^2 - z^{'2} - z^{''2} + 2zz'z''). \tag{B.4}$$

It admits a partial wave decomposition given by:

$$K(z,z',z'') = \sum_{J=0}^{\infty} \pi(2J+1) P_J(z) P_J(z') P_J(z''). \tag{B.5}$$

We can combine this with the partial wave decomposition of the amplitudes to rewrite equation Eq. (B.2) as:

$$\begin{aligned}\sum_{J,J',J''} \frac{1}{16(2\pi)^2} \int dz' dz'' \pi(2J+1) P_J(z) P_J(z') P_J(z'') \\ \times 16\pi(2J'+1) f_{J'} P_{J'}(z') \times 16\pi(2J''+1) f^*_{J''} P_{J''}(z'').\end{aligned} \tag{B.6}$$

We can now use the orthogonality of the Legendre Polynomials to write this as:

$$\sum_J 16\pi(2J+1)|f_J|^2 P_J(z). \tag{B.7}$$

Having carried out the phase space integrals, the unitarity relation in Eq. (2.14) is manifest.

C Computing the fermionic spin sum

In this Appendix, we supply some details on carrying out the fermionic spin sum. The spin sum we encounter is

$$\sum_{s_1,s_2} \bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\bar{u}_{s_1}(p_3)\gamma^\nu v_{s_2}(p_4) \tag{C.1}$$

This spin sum requires us to compute $\sum_{s_1} u_{s_1}(p_1)\bar{u}_{s_1}(p_3)$ and $\sum_{s_2} v_{s_2}(p_4)\bar{v}_{s_2}(p_2)$. We would like to describe how these sums are to be computed. In the Dirac basis, boosted spinors are related to those in the rest frame by:

$$\frac{\not{p}+m}{\sqrt{(E+m)2m}}u(\vec{0}) = u(\vec{p}). \tag{C.2}$$

This relationship can be verified using the explicit form of the Gamma matrices and solutions to the Dirac equation. Using the relationship $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, it's easy to see that a similar relationship holds for $\bar{u}$ as well.

$$\bar{u}(\vec{0})\frac{\not{p}+m}{\sqrt{(E+m)2m}} = \bar{u}(\vec{p}). \tag{C.3}$$

The spin sum in the rest frame is simple to carry out and is given by:

$$\sum_{s_1} u_{s_1}(p_1)\bar{u}_{s_1}(p_3) = \frac{(\not{p}_1+m)(\gamma^0+\mathbb{I})(\not{p}_3+m)}{2\sqrt{(E_1+m)(E_3+m)}} \xrightarrow[m=0]{} \frac{(\not{p}_1)(\gamma^0+\mathbb{I})(\not{p}_3)}{2\sqrt{E_1E_3}}. \tag{C.4}$$

Analogously, one can check that the spinors v also satisfy a boost relationship in the Dirac basis

$$\frac{-\not{p}+m}{\sqrt{(E+m)2m}}v(\vec{0}) = v(\vec{p}). \tag{C.5}$$

This allows us to carry out the other spin sum

$$\sum_{s_2} v_{s_2}(p_4)\bar{v}_{s_2}(p_2) = \frac{(-\not{p}_4+m)(\gamma^0-\mathbb{I})(-\not{p}_2+m)}{2\sqrt{(E_2+m)(E_4+m)}} \xrightarrow[m=0]{} \frac{(\not{p}_4)(\gamma^0-\mathbb{I})(\not{p}_2)}{2\sqrt{E_2E_4}}. \tag{C.6}$$

Therefore, the spin sum in Eq. (C.1) reduces to

$$\mathrm{Tr}[\gamma^\mu\not{p}_1(\gamma^0+\mathbb{I})\not{p}_3\gamma^\nu\not{p}_4(\gamma^0-\mathbb{I})\not{p}_2]\frac{1}{4\sqrt{E_1E_2E_3E_4}}. \tag{C.7}$$

In the center of mass frame, the denominator is s, while the numerator is $\mathrm{Tr}[\gamma^\mu\not{p}_1\gamma^0\not{p}_3\gamma^\nu\not{p}_4\gamma^0\not{p}_2] - \mathrm{Tr}[\gamma^\mu\not{p}_1\not{p}_3\gamma^\nu\not{p}_4\not{p}_2]$. The cross terms identically vanish since they contain an odd number of γ matrices. The second term is readily calculated, while the first has additional γ^0's.

However, in the center of mass frame, we can utilize identity $\gamma^0\not{p}_3 = \not{p}_4\gamma^0$ and the fact that the photon polarization tensor $\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$ is diagonal and spatial to evaluate the trace only in the spatial components. This reduces the first term to $-\mathrm{Tr}[\gamma^i\not{p}_1\not{p}_4\gamma^j\not{p}_3\not{p}_2]$, which is again readily evaluated using standard formulae. This yields the result in Eq. (2.22).

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