External Bias and opinion clustering in Cooperative Network

Aniruddha Ghosh(Roll-231040132)

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Objective

- The objective is to study the effect of the external bias on the evolution of opinions in a cooperative network with arbitrary connectivity.
- We want to ensure the stability of final opinion states of the agents in the presence of external bias vector.

Introduction

- We consider the interactions between agents to be cooperative and it defines that the edge weights of a particular graph is positive.
- We have considered both endogenous and exogenous factors.
- Endogenous factors arise from interpersonal relationship between the agents and exogenous factors are external to the group.
- We have classified exogenous factors into two groups.(1) Uncontrollable exogenous factors like age,gender etc and (2) Controllable exogenous factors like news and advertisement etc.

Opinion Modelling

- We propose the continuous time opinion model for the ith agent like this :

$$\dot{x} = \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)) + b_i + u_i$$

- We can also write the above equation as

$$\dot{x} = -Lx + b + u$$

- L is the Laplacian matrix for underlying graph G
- b is constant bias vector,it is represented by

$$b = (b_1, b_2, ..., b_n)^T$$

- u is control input vector,it is represented by

$$u = (u_1, u_2, ..., u_n)^T$$

Effect of bias on opinion formation

- The solution of equation

$$\dot{x} = -Lx + b + u$$

becomes

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau) b d\tau$$

- We also write -L using its canonical decomposition as

$$-L = VJW^T$$

- V and W are are the matrices consisting of right and left eigen vectors of -L respectively , J is block diagonal Jordan normal form.The spectrum of -L is

$$\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

- Now the above equation looks like

$$x(t) = Ve^{Jt}W^T x_0 + V \int_0^t e^{J(t-\tau)} d\tau W^T b$$

Theorem 1

The system $\dot{x} = -Lx + b + u$ with $u = 0_n$ admits a stable solution regardless of the connectivity of the graph, if and only if the following equation holds:$\mathbf{w_i^T b{=}0} \quad \forall \boldsymbol{\sigma_i}{=}\mathbf{0}$ where $i \in \{1, 2, \ldots, n\}$. Otherwise, it is unstable. Let n_z be the number of zero eigenvalues of $-L$, then for the stable case, at steady state $\lim_{t\to\infty} x(t)$ can be given as

$$\bar{x} = V \left[{w_1}^T x_0, \ldots, w_{n_z}^T x_0, \left[-(\tilde{J}^{-1}) \tilde{W}^T b \right]_{1\times(n-n_z)} \right]^T$$

Where $\tilde{J}$ and $\tilde{W}$ are submatrices of J and W respectively, pertaining to the non zero eigenvalues of $-L$. For the unstable case, the weighted average of the opinion state evolves with time as given below.

$$ {w_i}^T x(t) = {w_i}^T (x_0 + bt) \quad \forall \sigma_i = 0, t \geq 0 $$

Proof

Now we decompose J into two parts, one with only zero eigenvalues and the other with non-zero eigenvalues of J.
So the equation $x(t) = Ve^{Jt}W^T x_0 + V\int_0^t e^{J(t-\tau)}d\tau W^T b$ can be written as

$$x(t) = Ve^{Jt}W^T x_0 + V\begin{bmatrix} T & 0 \\ 0 & (e^{Jt} - I)J^{-1} \end{bmatrix} W^T b$$

Here $T = \text{diag}(t, t, \ldots, t)$ is a diagonal matrix with the time-varying terms occurring due to the integration of the part of J corresponding to the zero eigenvalues, matrix T is a square matrix with dimension n_z.
The solution $x(t)$ becomes stable when the time-varying terms do not exist. By applying $w_i^T b = 0$ time-varying terms vanish such that

$$\bar{x} = \sum_{i=1}^{n_z} V_i W_i^T x_0 + V[0, \ldots, 0, -J^{-1}\tilde{W}^T b]^T$$

Corollary 1

For a connected undirected graph, the system $\dot{x} = -Lx + b + u$ admits a stable solution if and only if $\sum_{i=0}^{n} b_i = 0$. For $\sum_{i=0}^{n} b_i \neq 0$ the system is unstable, and the average value of the opinion states evolves with time as given below

$$\bar{x}_i(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t) = \frac{1}{n}\sum_{i=0}^{n} x_{0i} + \frac{t}{n}\sum_{i=0}^{n} b_i \quad t \geq 0$$

Proof

- We know for connected undirected graph the laplacian matrix is symmetric and has zero eigenvalue. The corresponding right and left eigenvectors are 1_n and $1_n/n$ chosen to satisfy ${v_1}^T w_1 = 1$.We can get $\sum_{i=0}^{n} b_i = 0$.
- And from ${w_i}^T x(t) = {w_i}^T (x_0 + bt) \quad \forall \sigma_i = 0, t \geq 0$ we get $\bar{x}_i(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t) = \frac{1}{n} \sum_{i=0}^{n} x_{0i} + \frac{t}{n} \sum_{i=0}^{n} b_i \quad t \geq 0$.

Corollary 2

For a digraph contains a globally reachable node(s) system admits a stable solution if and only if

$$\sum_{i \in N_G} w_{1i} b_i = 0$$

At steady state $\lim_{t \to \infty} x(t)$ becomes

$$\bar{x} = V \left[w_1^T x_0, \left[-(\tilde{J}^{-1}) \tilde{W}^T b \right]_{1 \times (n-1)} \right]^T$$

where the left eigenvector is $w_1 = [w_{11}, w_{12}, \ldots, w_{1n}]^T$ and N_G is the set of globally reachable nodes. For ${w_1}^T b \neq 0$ system becomes unstable and the weighted average of the opinion states evolves with time as

$$w_1^T x(t) = w_1^T x_0 + w_1^T bt$$

Proof

We know that L has simple zero eigenvalue and the corresponding right eigenvector is 1_n and the left eigenvector is chosen such that ${v_1}^T w_1 = 1$ and we can say $w_{1i} > 0 \forall i \in N_G$ and zero otherwise.Using these results, the proof follows in the same manner as that of Corollary 1. As we know ${w_i}^T b = 0$ from the theorem 1 we can write

$$\begin{bmatrix} w_{11} & w_{12} & \dots & w_{1n} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = 0$$

Corollary 3

For a constant bias vector b, the system admits a stable solution irrespective of the connectivity of the graph if and only if $w_i^T(b+u)=0 \quad \forall \sigma_i = 0$ Furthermore

$$w_i^T x(t) = w_i^T x(0) \quad \forall \sigma_i = 0, t \geq 0$$

Proof

This proof follows the theorem 1 where we just replacing b with $(b + u)$.
We also discuss two type of graphs that have zero as simple eigenvalues.

- For a connected undirected graph it follows $\sum_{i=1}^{n}(b_i + u_i) = 0$ for stability. Then $\sum_{i=1}^{n} x_i(t) = \sum_{i=1}^{n} x_i(0), t \geq 0$
- For a digraph containing a globally reachable node , $w_1^T(b + u) = 0$ must hold for stability.In this case we can write $w_1^T x(t) = w_1^T x(0), t \geq 0$

Remark 1

The reachable set of opinion states is defined as

$$X_R = \{x_d \in R^n | {w_i}^T x_d = {w_i}^T x_0\} \forall \sigma_i = 0$$

If the stable desired opinion states x_d belongs to reachable set X_R,then $\dot{x} = 0$ at steady state.This gives $-Lx_d + (b + u) = 0$.Then control input u to reach an opinion state x_d is given by

$$u = Lx_d - b$$

Theorem 2

For a given initial state x_0 and a constant bias vector b , the system admits any stable desired opinion state x_d, irrespective of the connectivity of the graph by designing the control input u as given below,

1. If ${w_i}^T x_d = {w_i}^T x_0$ then u is given by $u = Lx_d - b$.
2. Else u satisfies the condition, ${w_i}^T (b + u) = ({w_i}^T x_d - {w_i}^T x_0)/\bar{t}$, $\forall \sigma_i = 0$

Proof

When the desired opinion state x_d lies in the reachable set X_R,then u can be calculated using Remark 1 , which leads to condition 1 of the theorem.
When $x_d \notin X_R$ then it is impossible to reach x_d using the control input in the equation mentioned in Remark 1.To solve this problem we reach some $\tilde{x} \neq \{x_0, x_d\}$from where x_d is reachable and now we can calculate the control input which is required to reach x_d from $\tilde{x}$.
It is not possible to reach $\tilde{x}$ starting from x_0 through a stable system behaviour as ${w_i}^T\tilde{x} \neq {w_i}^T x_0$. In the presence of b and u an unstable behaviour can be modelled by the equation as,

$${w_i}^T x(t) = {w_i}^T(x_0 + (b+u)t)$$

Proof

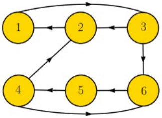
In the second stage, ${w_i}^T x_d = {w_i}^T x(\bar{t})$ holds as x_d is reachable from $x(\bar{t})$.

- When $t \leq \bar{t}$, where $\bar{t} \in (\ 0, \infty)$ is the time till which the unstable behaviour exists such that $\tilde{x} = x(\bar{t})$.
- When $t = \bar{t}$ the above equation becomes ${w_i}^T x_d = {w_i}^T (x_0 + (b + u)t)$. Rearranging this equation gives condition 2 in theorem 2 which is necessary condition the control input u must satisfy.
- When $t > \bar{t}$ the equation $u = Lx_d - b$ results in the desired behaviour.

Conclusion

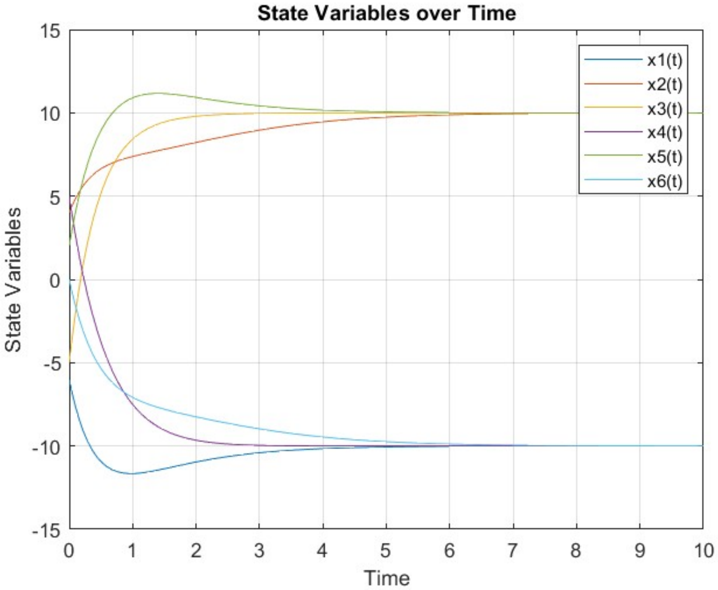
1. We study how the external bias factors effects on the evolution of opinions in a cooperative network in arbitrary connectivity.
2. In the presence of constant bias b the conditions has been provided which can ensure the stability of final opinion states of the agents.
3. We will design a control input u to extend the reachable set of opinion states to R^n.
4. In future we also develop a nonlinear model to better understanding a real world scenario.

Graph and the model



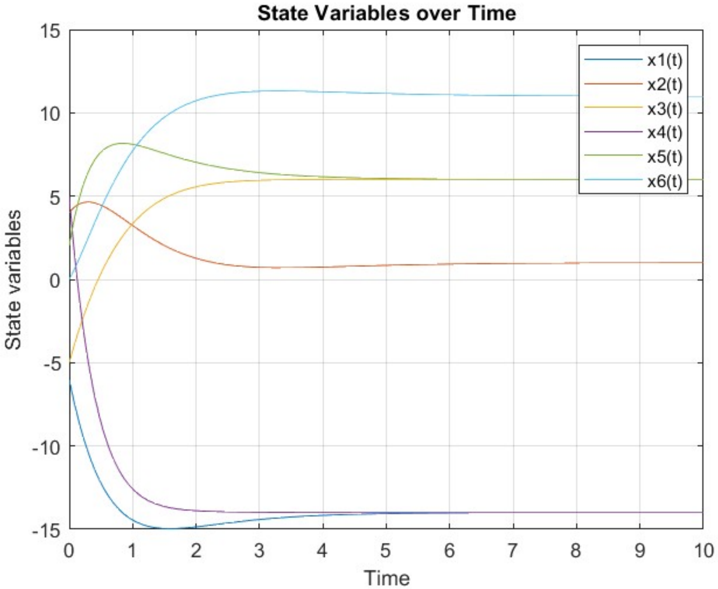
- For the above graph I have simulated the model and getting two graphs which are shown in the next slides.
- For the 1st graph $u = 1_n$, $x(0) = [-6.4, -5.5, 2, 0]^T$, $b = [-20, 20, 20, -20, 20, -20]^T$, L we have calculated by using the formula, $L = D_{\text{out}} - A$
- For the 2nd graph $u = [0, 5, -20, -20, 0, 25]^T$ and have $x(0)$, b as same as the previous one.
- In 1st graph we are getting $x_f = [-10, 10, 10, -10, 10, -10]^T$ and from the result we can say polarization of opinion is happening in the group of agents.
- In 2nd graph we are getting $x_f = [-14, 1.6, -14, 6, 11]^T$ and from the result we can say clustering is happening .

Simulation



State Variables over Time
15
10
5
0
-5
-10
-15
0
1
2
3
4
5
6
7
8
9
10
Time
State Variables
x1(t)
x2(t)
x3(t)
x4(t)
x5(t)
x6(t)

Simulation



State Variables over Time
State variables
Time
x1(t)
x2(t)
x3(t)
x4(t)
x5(t)
x6(t)
15
10
5
0
-5
-10
-15
0
1
2
3
4
5
6
7
8
9
10