Separable Temporal Exponential Random Graph Models
STERGM

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Papers

- Krivitsky, P. N., Handcock, M. S. (2014). A Separable Model for Dynamic Networks.
- Hanneke, S., Xing, E.P. (2010). Discrete Temporal Models of Social Networks.

Introduction

Networks are used to represent relational phenomena.There is a need for realistic and tractable statistical models for these networks, especially when the phenomena evolves over time.

Goal is to develop a family of statistical models (Dynamic Network Models) for social network evolution over time, which represents an extension of Exponential Random Graph Models (ERGMs).

Introduction

Exponential Random Graph Model (ERGM):

$$P(Y = y|X) = \frac{\exp\left(\theta' g(y, X)\right)}{k(\theta, y, X)}$$

where Y is the random variable for the state of the network (with realization y), $g(y, X)$ is the vector of model statistics for network y, X is the attributes of the nodes, θ is the vector of coefficients for those statistics, and $k(\theta, y, X)$ is the normalizing constant.

This can be re-expressed in terms of the conditional log-odds of a single actor pair:

$$\text{logit}\left(Y_{ij} = 1|y_{ij}^c\right) = \theta' \delta\left(y_{ij}\right)$$

The variable $\delta\left(y_{ij}\right)$ equals $g\left(y_{ij}^+\right) - g\left(y_{ij}^-\right)$, where y_{ij}^+ is defined as y_{ij}^c along with y_{ij} set to 1, and y_{ij}^- is defined as y_{ij}^c along with y_{ij} set to 0.

Temporal ERGM (TERGM)

TERGMs represent tie dynamics only, and are typically formulated in discrete time.

Model Definition:
Suppose we have n actors, Y^t is the network at time t

$$\mathcal{P}\left(Y^2, Y^3, \ldots, Y^t | Y^1\right) = \mathcal{P}\left(Y^t | Y^{t-1}\right) \mathcal{P}\left(Y^{t-1} | Y^{t-2}\right) \cdots \mathcal{P}\left(Y^2 | Y^1\right)$$

Markov assumption, Y^t is the network at time t then we might make the assumption that Y^t is independent of $Y^1, \ldots, Y^{t-2}$ given Y^{t-1}.

TERGM

$Y^t | Y^{t-1}$ admits to an ERGM representation:

$$P\left(Y^t | Y^{t-1}, \theta\right) = \frac{1}{Z\left(\theta, Y^{t-1}\right)} \exp\left\{\theta' \Phi\left(Y^t, Y^{t-1}\right)\right\}$$

where $\Phi : \mathbb{R}_{n\times n} \times \mathbb{R}_{n\times n} \to \mathbb{R}^k$

They are essentially stepwise ERGM in time.

This can be generalised to k^{th} order dependence.

$$\mathrm{P}\left(Y^t | Y^{t-1}, \ldots, Y^{t-k}; \theta\right) = \frac{\exp\left(\theta' \Phi\left(Y^t, Y^{t-1}, \ldots, y^{t-k}\right)\right)}{Z\left(\theta, Y^{t-1}, \ldots, Y^{t-k}\right)}$$

TERGM

Define the following statistics, which represent density, stability, reciprocity, and transitivity, respectively.

$$\Phi_D\left(Y^t, Y^{t-1}\right) = \frac{1}{(n-1)}\sum_{ij} Y_{ij}^t$$
$$\Phi_S\left(Y^t, Y^{t-1}\right) = \frac{1}{(n-1)}\sum_{ij}\left[Y_{ij}^t Y_{ij}^{t-1} + \left(1 - Y_{ij}^t\right)\left(1 - Y_{ij}^{t-1}\right)\right]$$
$$\Phi_R\left(Y^t, Y^{t-1}\right) = n\left[\sum_{ij} Y_{ji}^t Y_{ij}^{t-1}\right] / \left[\sum_{ij} Y_{ij}^{t-1}\right]$$
$$\Phi_T\left(Y^t, Y^{t-1}\right) = n\left[\sum_{ijk} Y_{ik}^t Y_{ij}^{t-1} Y_{jk}^{t-1}\right] / \left[\sum_{ijk} Y_{ij}^{t-1} Y_{jk}^{t-1}\right]$$

The transition probability for this temporal network model can then be written as follows:

$$\mathcal{P}\left(Y^t | Y^{t-1}, \boldsymbol{\theta}\right) = \frac{1}{Z\left(\boldsymbol{\theta}, Y^{t-1}\right)} \exp\left\{\sum_{j\in\{D,S,R,T\}} \theta_j \Phi_j\left(Y^t, Y^{t-1}\right)\right\}$$

Separable Temporal ERGM (STERGM)

Intuitively, those social processes and factors that result in ties being formed are not the same as those that result in ties being dissolved,i.e., the probability a relationship ending during a particular time interval would not, in general, be a perfect reflection the differences in the probability of it forming during such a time interval.

Therefore, STERGM posit two models:
1. One for tie formation
2. One for tie dissolution.

Both models are ERGMs, and they are "separable" in the sense that they assume formation is independent of dissolution within time step, and Markov dependent between steps. This allows the factors that influence formation to be different than those that influence dissolution.

STERGM

The expression below says that the number of ties present in the cross-section (prevalence) is a function of the rate at which they form (incidence) and the rate at which they dissolve (1/duration).

$$\text{prevalence} = \text{incidence} \times \text{duration}$$

ERGMs are models for the prevalence of ties. STERGMs are models that explicitly represent the incidence and duration processes.

STERGM

Model:

$$P\left(Y^{+}|Y^{t-1},\theta^{+}\right)=\frac{1}{Z\left(\theta^{+},Y^{t-1}\right)}\exp\left\{\theta^{+}\Phi\left(Y^{+},Y^{t-1}\right)\right\}$$

$$P\left(Y^{-}|Y^{t-1},\theta^{-}\right)=\frac{1}{Z\left(\theta^{-},Y^{t-1}\right)}\exp\left\{\theta^{-}\Phi\left(Y^{-},Y^{t-1}\right)\right\}$$

Re-expressed in terms of conditional log-odds:

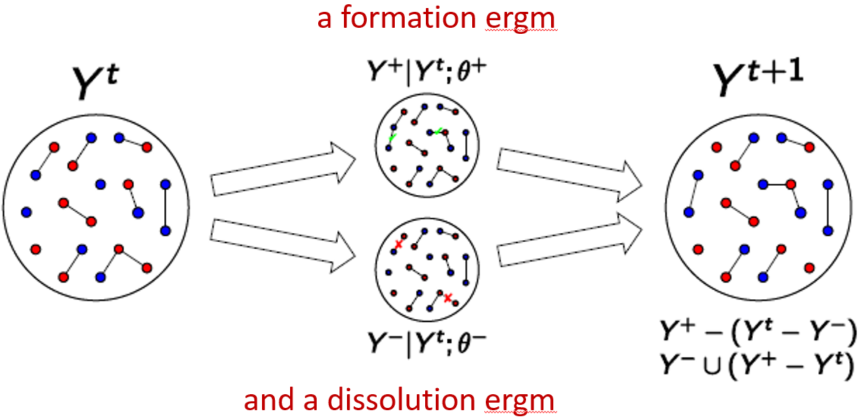
$$\ln\frac{P\left(Y_{ij,t}=1|y_{ij}^{c},Y_{ij,t-1}=0\right)}{P\left(Y_{ij,t}=0|y_{ij}^{c},Y_{ij,t-1}=0\right)}=\theta^{+}\delta\left(\Phi^{+}\left(y_{ij}\right)\right)$$

$$\ln\frac{P\left(Y_{ij,t}=1|y_{ij}^{c},Y_{ij,t-1}=1\right)}{P\left(Y_{ij,t}=0|y_{ij}^{c},Y_{ij,t-1}=1\right)}=\theta^{-}\delta\left(\Phi^{-}\left(y_{ij}\right)\right)$$

These parallel the traditional cross-sectional ERGM. We have simply:

- added a time index to the tie values
- defined two θ and Φ vectors instead of just one.
- added in a new conditional.

STERGM


a formation ergm
Y^t
$Y^+|Y^t;\theta^+$
Y^{t+1}
$Y^-|Y^t;\theta^-$
$Y^+ - (Y^t - Y^-)$
$Y^- \cup (Y^+ - Y^t)$
and a dissolution ergm

Package

STATNET:

Author: Mark S. Handcock, David R. Hunter, Carter T. Butts, Steven M. Goodreau, Pavel N. Krivitsky, Skye Bender-deMoll, Martina Morris

- Requires R 3.5 or newer to work
- ERGM, TERGM-STERGM
- networkDynamic: extends network with functionality to store information about about evolution of a network over time, defining a networkDynamic object class.
- tsna: is a collection of extensions to sna that provide descriptive summary statistics for temporal networks
- ndtv: Visualizing dynamic networks
- The package has a well documented github page.

Example (Sampson monastery data)

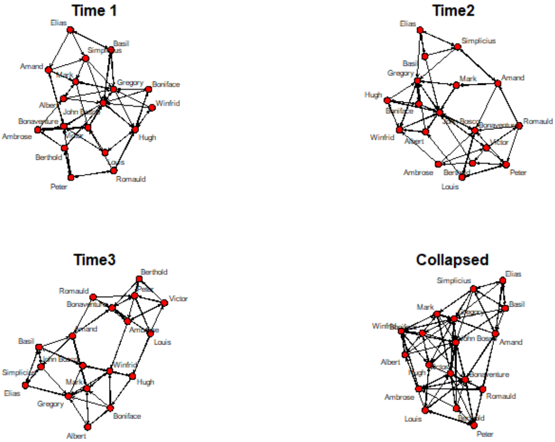


Figure: Sampson monastery dataset

TimePrism using scatterplot3d

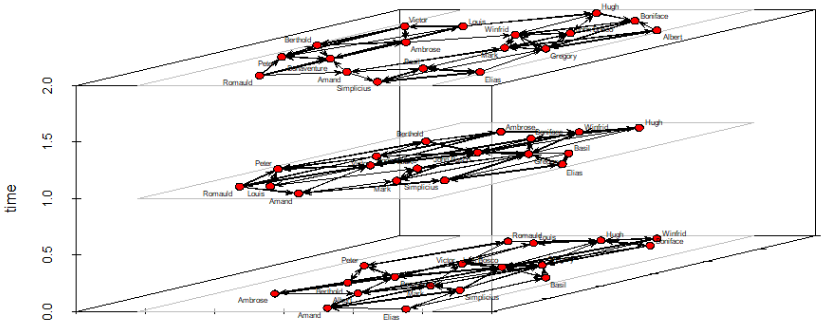


Figure: Sampson monastery dataset

Summary statistics

```
> tSnaStats(sampdyn,"degree") # Changes in degree centrality
Time Series:
Start = 0
End = 3
Frequency = 1
  John Bosco Gregory Basil Peter Bonaventure Berthold Mark Victor Ambrose
0         12      10     5     6           8        4    7      7       5
1         11      12     4     8          10        5    6      5       4
2          7       9     7     7           9        5    8      5       7
3         NA      NA    NA    NA          NA       NA   NA     NA      NA
  Romauld Louis Winfrid Amand Hugh Boniface Albert Elias Simplicius
0       4     5       4     5   10        4      5     4          5
1       4     5       7     5    6        6      5     5          6
2       4     5       9     5    5        5      4     5          6
3      NA    NA      NA    NA   NA       NA     NA    NA         NA
> ## ------------------------------------------------------------------------
> tErgmStats(sampdyn, "~ edges+triangle") # Notice the increase in triangles
Time Series:
Start = 0
End = 3
Frequency = 1
  edges triangle
0    55       31
1    57       56
2    56       62
3     0        0
```

STERGM

```
stergm (nw,
        formation,
        dissolution,
        estimate,
        times=NULL,
        offset.coef.form=NULL,
        offset.coef.diss=NULL,
        targets=NULL,
...)
```

samp.fit = stergm(samplist, formation = edges+mutual+cyclicalties+ transitiveties, dissolution = *,estimate = "CMLE",times = c(1:3))

Result

```
================================
Summary of formation model fit
================================

Formula:   ~edges + mutual + cyclicalties + transitiveties

Iterations:  2 out of 20

Monte Carlo MLE Results:
               Estimate Std. Error MCMC % z value Pr(>|z|)
edges           -3.4768     0.3386      0 -10.267   <1e-04 **
mutual           2.0493     0.4029      0   5.086   <1e-04 **
cyclicalties    -0.1427     0.2042      0  -0.699   0.4845
transitiveties   0.4003     0.2410      0   1.661   0.0967 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

     Null Deviance: 693.1  on 500  degrees of freedom
 Residual Deviance: 240.2  on 496  degrees of freedom

AIC: 248.2    BIC: 265    (Smaller is better.)
```

```
================================
Summary of dissolution model fit
================================

Formula:   ~edges + mutual + cyclicalties + transitiveties

Iterations:  2 out of 20

Monte Carlo MLE Results:
               Estimate Std. Error MCMC % z value Pr(>|z|)
edges            0.2141     0.3027      0   0.707    0.479
mutual           0.7943     0.5210      0   1.525    0.127
cyclicalties    -0.1929     0.2510      0  -0.769    0.442
transitiveties   0.5049     0.2856      0   1.768    0.077 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

     Null Deviance: 155.3  on 112  degrees of freedom
 Residual Deviance: 136.8  on 108  degrees of freedom

AIC: 144.8    BIC: 155.7    (Smaller is better.)
```

Result

Data	Sampson monastery	Windsurfer	Marriage
Graph type	directed	undirected	undirected
Nodes	18	95	16
Timepoints	3	31	25
Formation	E+M+C+T	E+G	E+G
Dissolution	E+M+C+T	E+G	E
AIC (BIC)-F	248.4 (265.3)	6580 (6600)	174.3 (185.9)
AIC (BIC)-D	144.7 (155.6)	401 (410.5)	262.5 (266.3)

where, E=edges, M=mutual, C=cyclicalties, T=transitiveties, G=gwesp(0, fixed=T)

Conclusion and Summary

- The methods builds on the foundations of exponential-family random graph models for cross-sectional networks and inherits the flexibility and interpretability of these models.
- It is important to emphasize that STERGMs jointly model the formation and dissolution of ties. While the two processes are modeled as conditionally independent within a time step, they are modeled as dependent over time.
- Ease of specification, tractability of the model, and substantial improvement in interpretability.
- Existing methods developed for ERGMs are readily adapted to apply to the these models as well.
- The these framework allows a number of extensions to the model like multiple relations in the network, actor attributes, relation attributes, longer-range Markov dependencies, change in number of actors/actor attributes, or a host of other possibilities. Also can deal with missing data. It is possible to incorporate the network size adjustment into these dynamic models.

Thank you

Any questions?