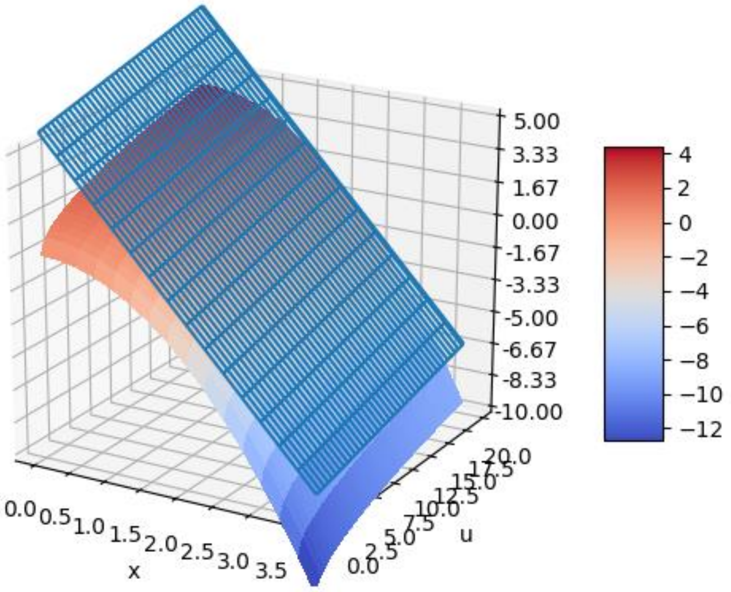
LINEARIZING DIFFERENTIAL EQUATIONS

Reducing nonlinear dynamics to linear ODEs



UNIVERSITY OF WATERLOO

LEARNING OUTCOMES

- Derive a linearized ODE for a single-input-single-output dynamic system
- Perform step and doublet tests to assess a dynamic model's response
- Compute the steady state values for a dynamic system

RESOURCES

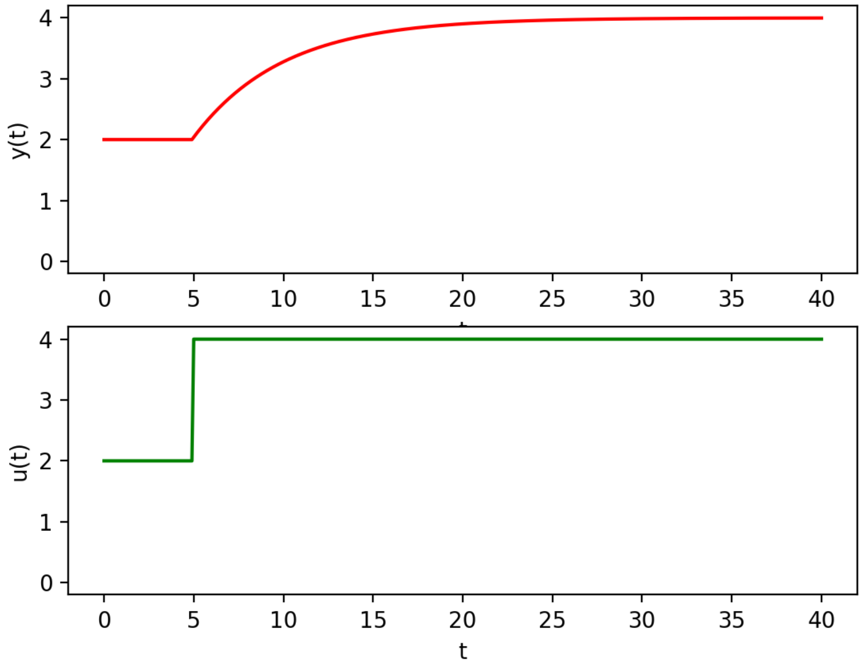
- APMonitor course
  - https://apmonitor.com/pdc/index.php/Main/ModelLinearization

STEP TESTS

- Subjects system to step change in input

$$u(t) = u_{\text{ss}} + \Delta u S(t - t_1) = \begin{cases} u_{\text{ss}} & t < t_1 \\ u_{\text{ss}} + \Delta u & t \geq t_1 \end{cases}$$

- Useful for system identification
- Limitations
  - Only deviates from steady-state on one side
  - Requires long period of off-spec operation

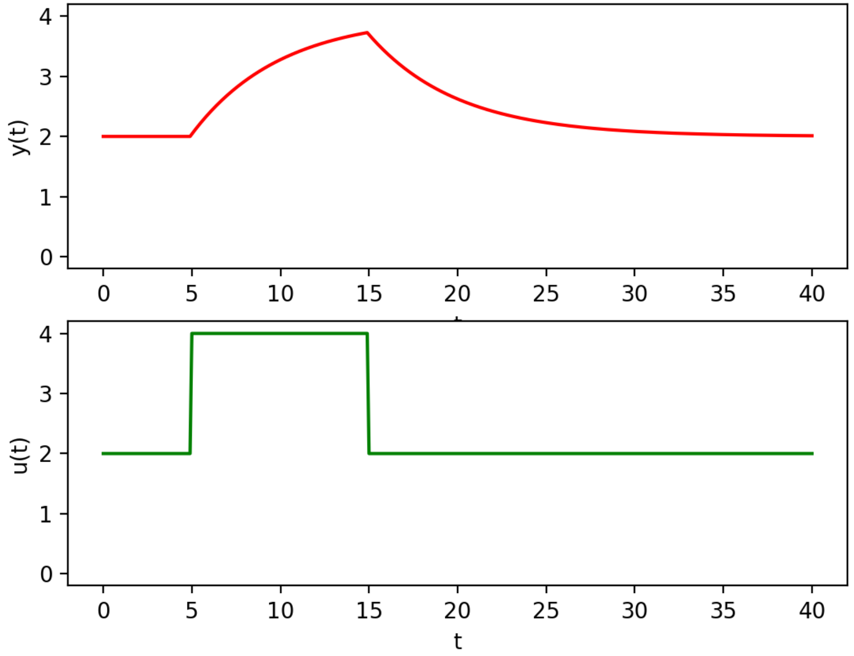


PULSE TESTS

- Subjects system to step change in input

$$u(t) = \begin{cases} u_{\mathrm{ss}} & t < t_1 \\ u_{\mathrm{ss}} + \Delta u & t_1 \le t < t_2 \\ u_{\mathrm{ss}} & t \ge t_2 \end{cases}$$

- Only has brief deviation from steady-state
- Limitations
  - Only deviates from steady-state on one side

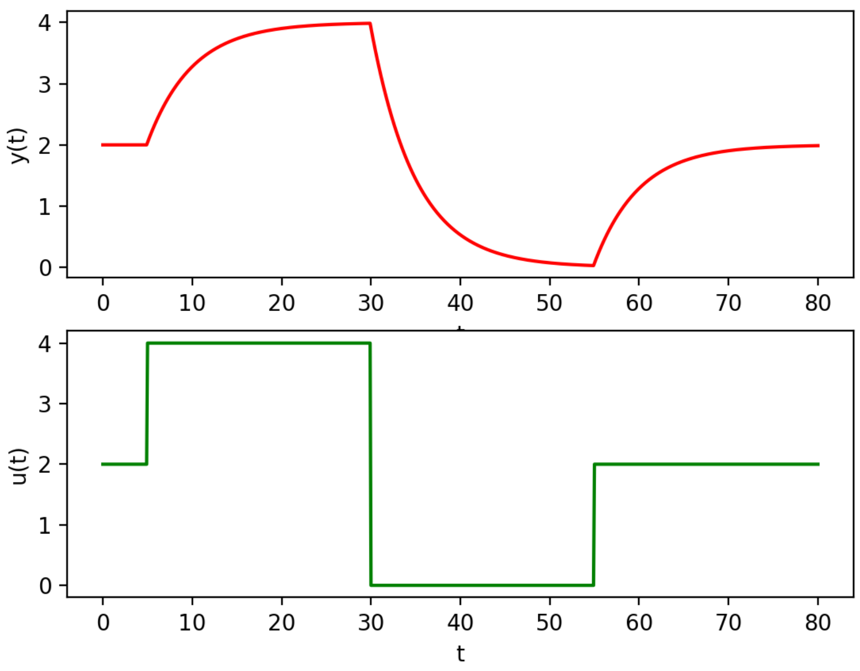


DOUBLET TESTS

- Subject system to two pulses in opposite directions

$$u(t) = \begin{cases} u_{\text{ss}} & t < t_1 \\ u_{\text{ss}} + \Delta u & t_1 \leq t < t_2 \\ u_{\text{ss}} - \Delta u & t_2 \leq t < t_3 \\ u_{\text{ss}} & t \geq t_3 \end{cases}$$

- Tests both sides of steady-state
- Short deviation from steady-state
- Common in practice
- More difficult system identification



LINEARIZING NONLINEAR FUNCTIONS

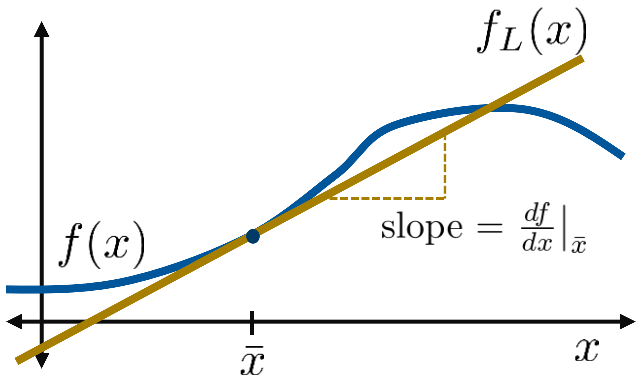
- Nonlinear functions can be expensive to simulate and/or difficult to analyze
- Represent nonlinear function $f(x)$ via Taylor series expansion:

$$f(x) = f(\bar{x}) + \frac{1}{1!}\frac{df}{dx}\bigg|_{\bar{x}}(x-\bar{x}) + \cancel{\frac{1}{2!}\frac{d^2f}{dx^2}\bigg|_{\bar{x}}(x-\bar{x})^2} + \cancel{\frac{1}{3!}\frac{d^3f}{dx^3}\bigg|_{\bar{x}}(x-\bar{x})^3} + \dots$$

- Truncate all 2nd order and higher terms to get linear approximation

$$f(x) \approx f_L(x) := f(\bar{x}) + \frac{df}{dx}\bigg|_{\bar{x}}(x-\bar{x})$$

- Error is proportional to truncated terms



LINEARIZING DIFFERENTIAL EQUATIONS

- Consider single-input single-output (SISO) ODE w/ output $y(t)$ and input $u(t)$

$$\frac{dy}{dt} = f(y, u)$$

- Linearize the RHS using truncated Taylor series

$$\frac{dy}{dt} = f(y, u) \approx f(\bar{y}, \bar{u}) + \left.\frac{\partial f}{\partial y}\right|_{\bar{y}, \bar{u}} (y - \bar{y}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y}, \bar{u}} (u - \bar{u})$$

- At steady-state w/ values y_{ss}, u_{ss} and $\frac{dy}{dt} = 0$ we have $f(y_{ss}, u_{ss}) = 0$
  - Let $\bar{y} = y_{ss}, \bar{u} = u_{ss}$

$$\frac{dy}{dt} = f(y, u) \approx \left.\frac{\partial f}{\partial y}\right|_{y_{ss}, u_{ss}} (y - y_{ss}) + \left.\frac{\partial f}{\partial u}\right|_{y_{ss}, u_{ss}} (u - u_{ss})$$

LINEARIZING DIFFERENTIAL EQUATIONS

- Define deviation variables

$$y' = y - y_{ss} \qquad u' = u - u_{ss}$$

$$\frac{dy'}{dt} = \frac{d(y - y_{ss})}{dt} = \frac{dy}{dt} - \cancel{\frac{dy_{ss}}{dt}} \qquad \frac{du'}{dt} = \frac{d(u - u_{ss})}{dt} = \frac{du}{dt} - \cancel{\frac{du_{ss}}{dt}}$$

- Simplify linearized ODE

$$\frac{dy'}{dt} = \alpha y' + \beta u' \qquad \alpha = \left.\frac{\partial f}{\partial y}\right|_{y_{ss},u_{ss}} \qquad \beta = \left.\frac{\partial f}{\partial u}\right|_{y_{ss},u_{ss}}$$

- Add disturbance variable $d(t)$ to the model

$$\boxed{\frac{dy'}{dt} = \alpha y' + \beta u' + \gamma d' \qquad \alpha = \left.\frac{\partial f}{\partial y}\right|_{y_{ss},u_{ss},d_{ss}} \qquad \beta = \left.\frac{\partial f}{\partial u}\right|_{y_{ss},u_{ss},d_{ss}} \qquad \gamma = \left.\frac{\partial f}{\partial d}\right|_{y_{ss},u_{ss},d_{ss}}}$$

EXAMPLE

- Linearize the following ODE with $u(t) = 16$

$$\frac{dy}{dt} = -y^2 + \sqrt{u}$$

a) Derive general linearized ODE

b) Determine steady-state values and simply the linearized ODE

c) Simulate linear and nonlinear ODEs with doublet test