Time Frequency Analysis: Short Time Fourier Transform

Fourier Transform: DFT

- Fourier analysis expands a discrete time series in terms of complex exponentials linked with different frequencies
- Essentially a Fourier analysis reveals the contribution of all frequency components in total variance (power).

N point DFT

$$X[n] = \sum_{k=0}^{k=N-1} x[k]e^{-i2\pi kn/N}$$

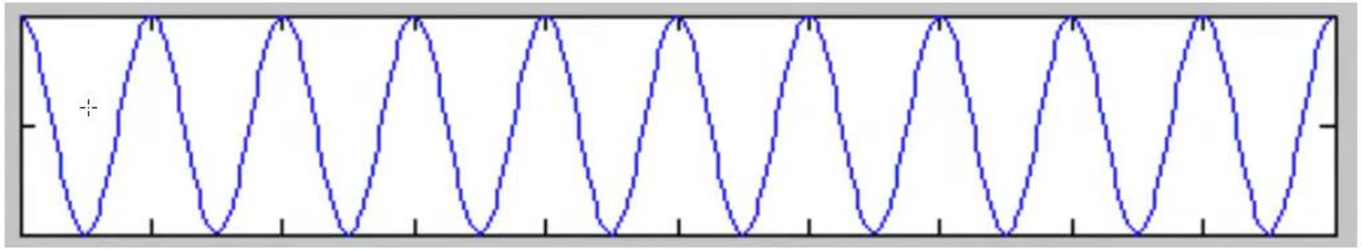
N point inverse DFT

$$x[k] = \frac{1}{N} \sum_{n=0}^{n=N-1} X[n]e^{i2\pi k/N}$$

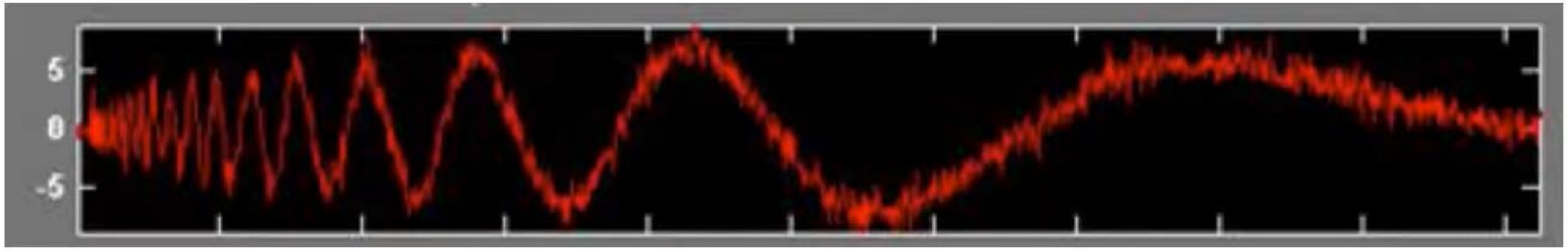
Stationarity in frequency content

- The frequency content of signal remains unchanged over time.
- All frequency components exist at all times.

Fourier Analysis is based on an indefinitely long cosine wave of a specific frequency



Non-stationary frequency content

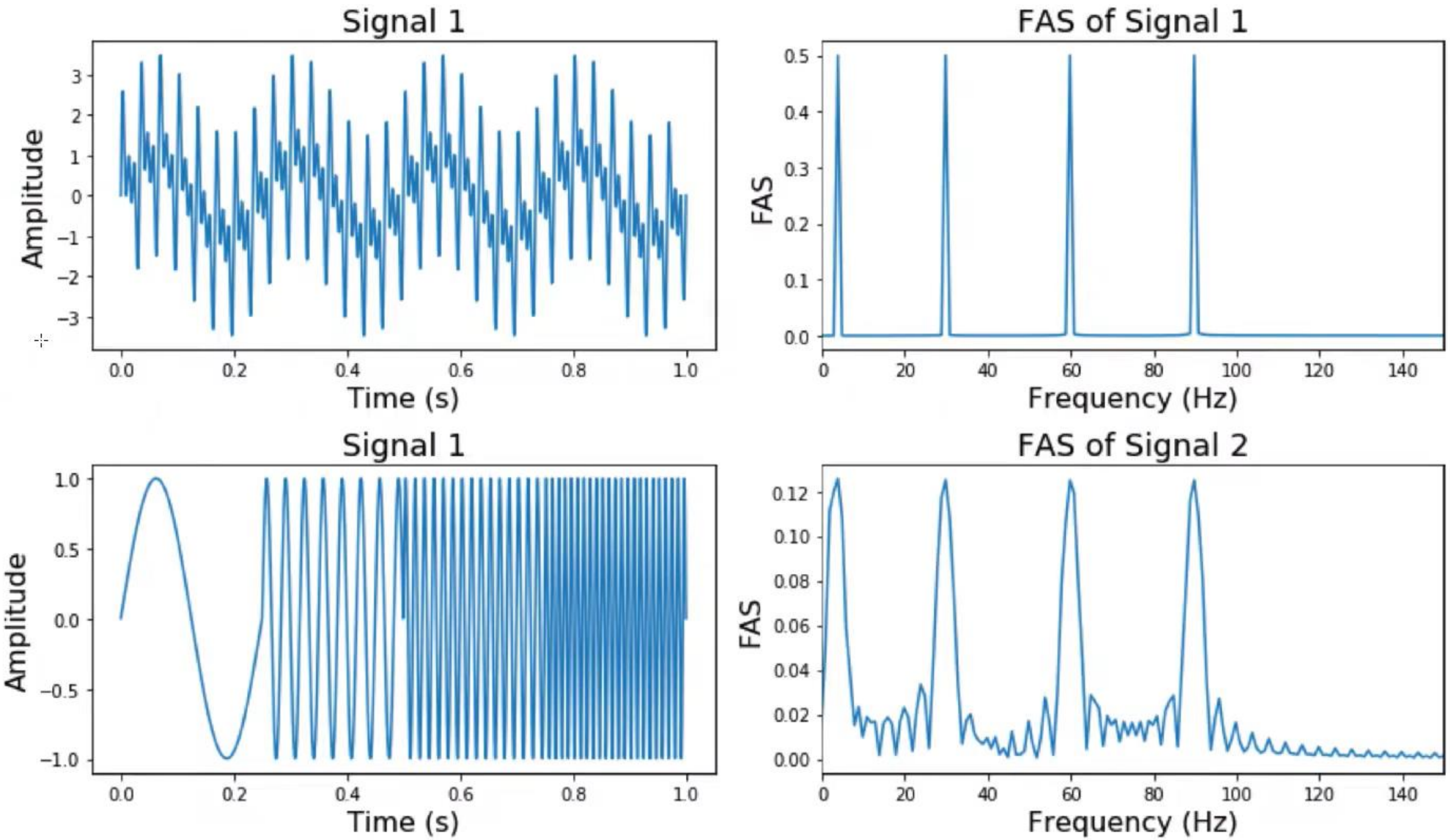


Some signals obviously have spectral characteristics that vary with time

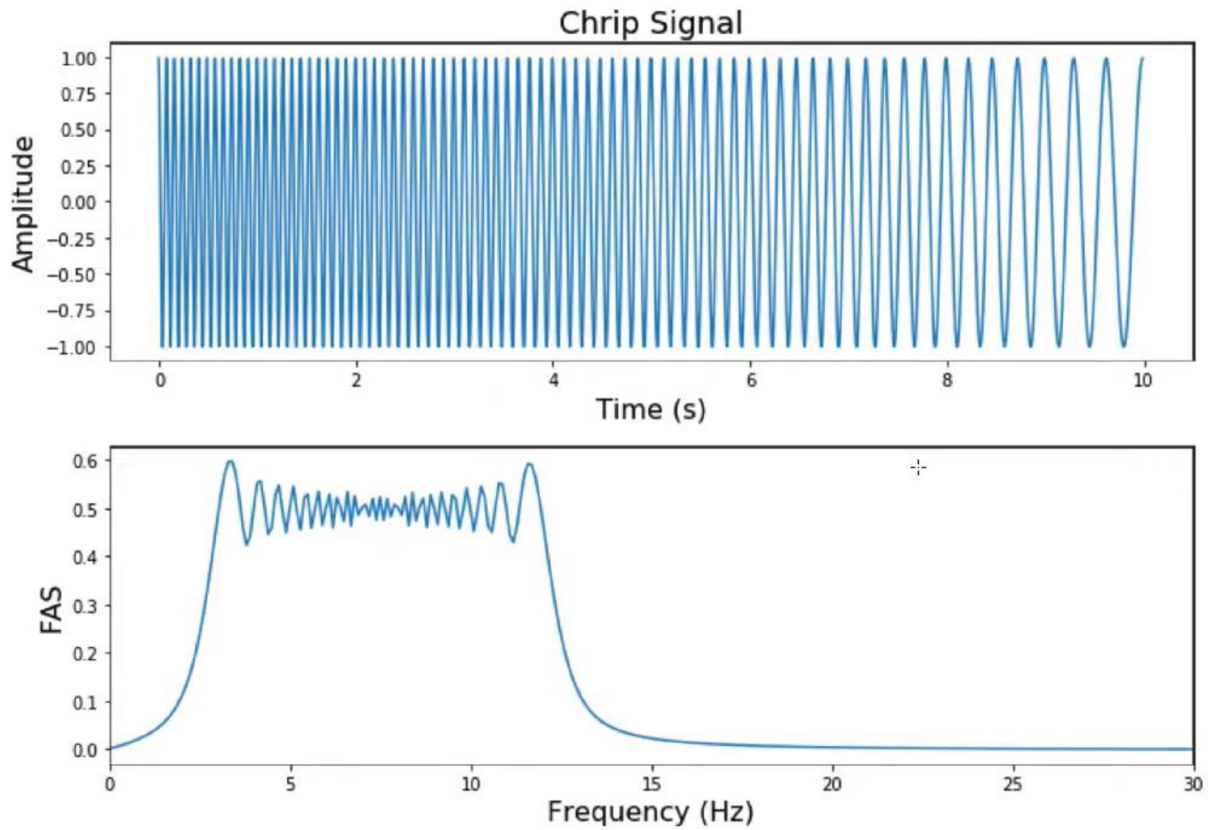
Issues with Fourier Transform

- As we have seen Fourier Transform gives complete information regarding the harmonic frequencies present in the signal and associated strength.
- However, in going from time domain to frequency using the Fourier transform (DFT), we loose information regarding time.
- That means Fourier transform tells us exactly which frequencies are present in a signal or time series,
- but does not tell that at which location in time these frequencies are present.

Issues with Fourier Transform: Example 1



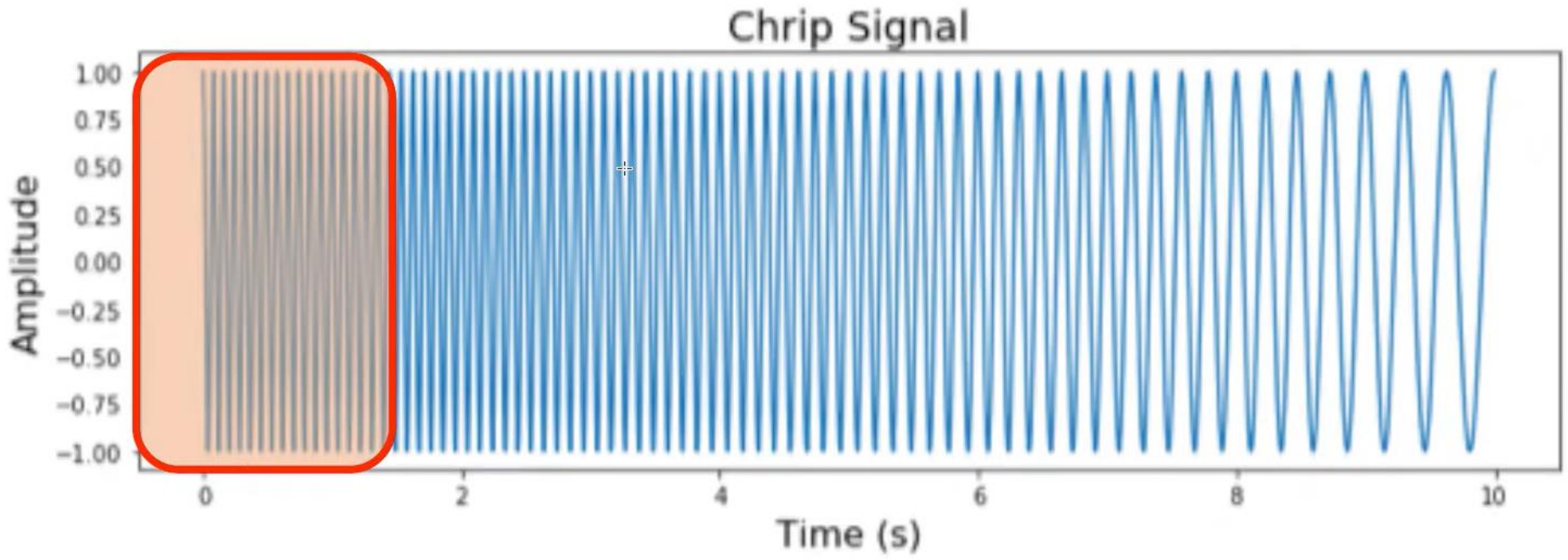
Issues with Fourier Transform: Example 2



- You can clearly see that such signal exhibit a non-stationary frequency content.
- That is the frequencies are not the same over the entire duration of the signal and it varies with time.

Short Time Fourier Transform

- We need a local analysis scheme for time frequency representation.
- Rather than taking the DFT of the full time series we segment the signal into narrow time intervals narrow enough to be considered stationary.
- Take the DFT of each segment.



Short-Time Fourier Transform

Steps:

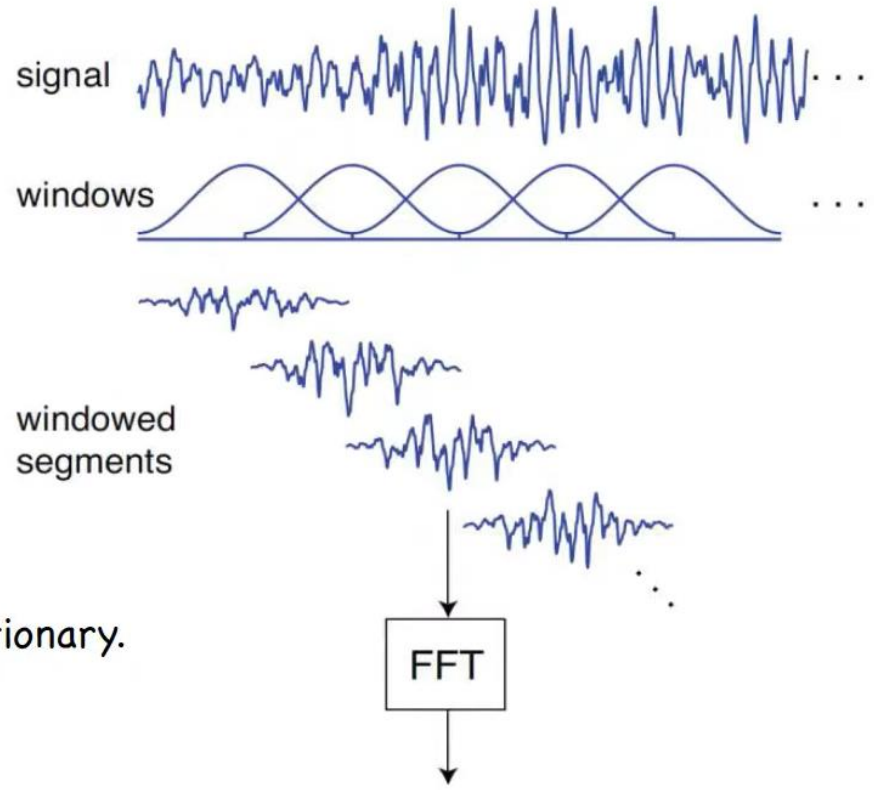
- Choose a window function of finite length. A window function is a function that is multiplied by the signal to keep a certain portion of it.
  - Rectangular window
  - Gaussian window
  - Hann window
- Place the window on top of the signal at $n = 0$
- Truncate the signal using the window
- Compute the Fourier transform of the truncated signal, and save results. For each time location we obtain a different Fourier Transform.
- Each Fourier Transform provides the spectral information of a separate time-slice of the signal providing simultaneous time and frequency information.

Short Time Fourier Transform: The basic idea

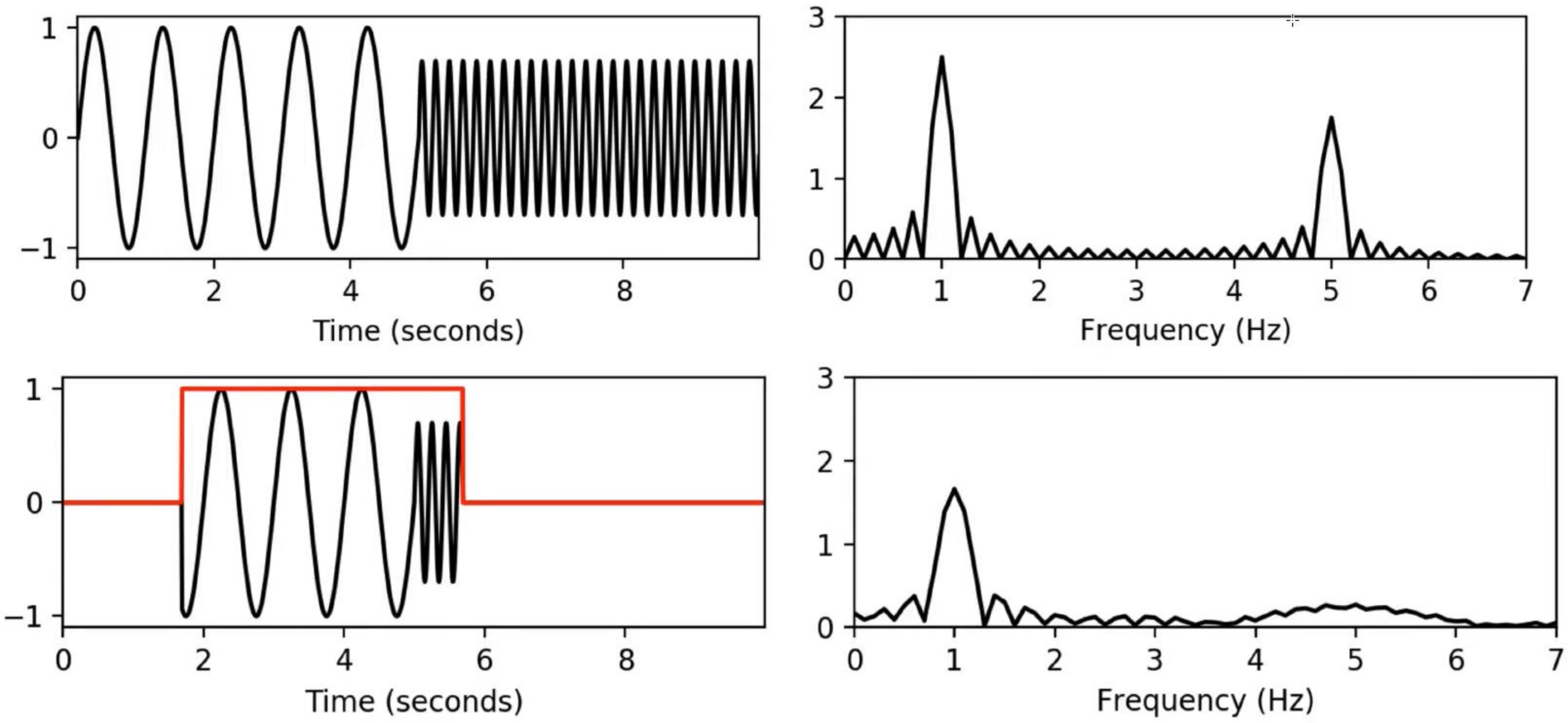
- Then slide the window towards the right inclemently till the window reaches the last index of the signal.
- Finally we can plot the different Fourier transforms at each time step next to each other.

Dennis Gabor (1946) first used STFT.

The segment of the signal within the windows is assumed stationary.



The basic idea



Discrete Short-Time Fourier Transform

$$X(n,\omega) = \sum_{m=-\infty}^{\infty} x[m]w[n-m]e^{-j\omega n}$$

With a Gaussian window this was also called Gabor Transform

Since we are working with discrete time and discrete frequency signals.

$$X(n,k) = X(n,\omega)\Big|_{\omega=\frac{2\pi}{N}k}$$

$$S(n,k) = \log|X(n,k)|^2$$

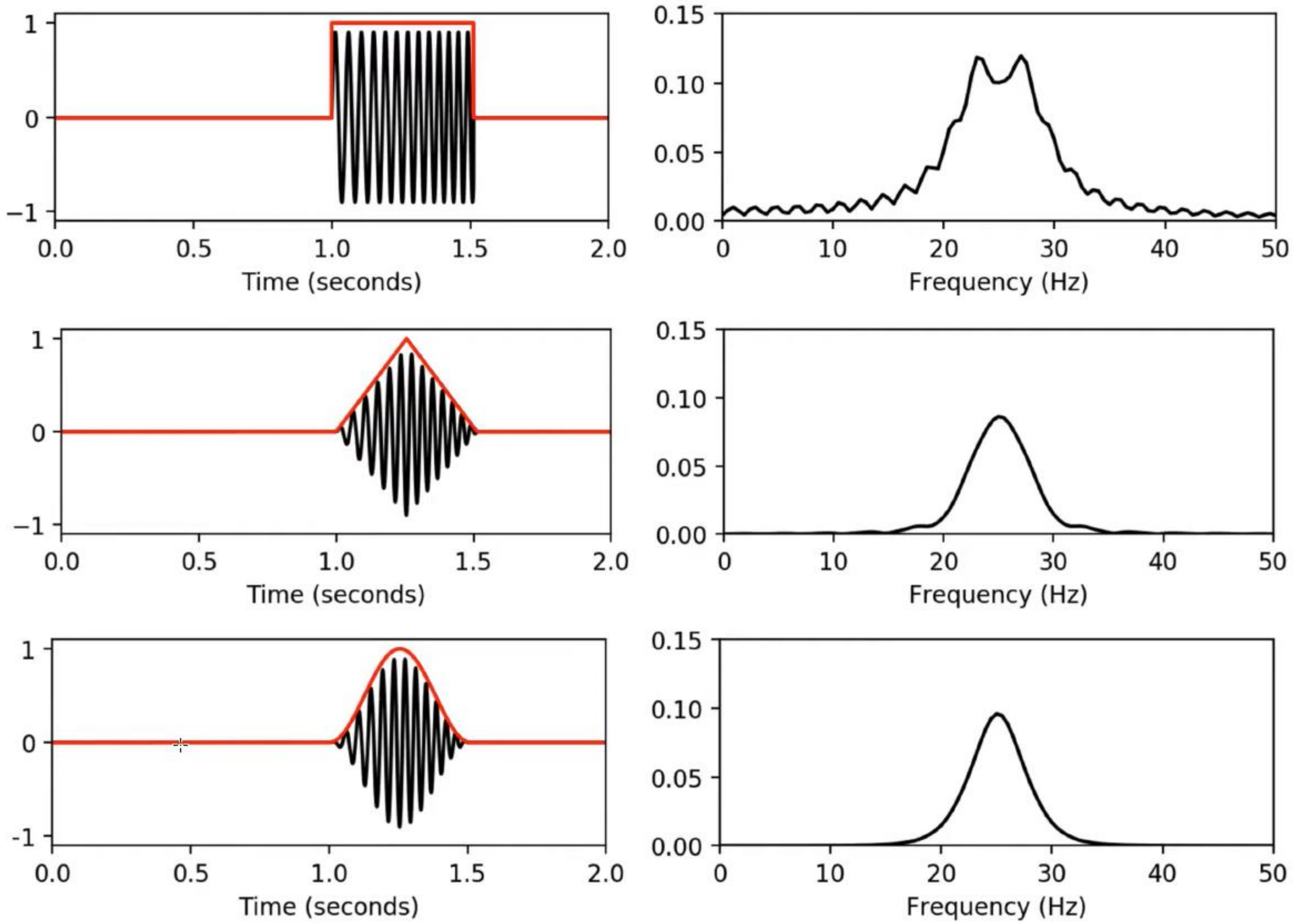
Spectrogram

Effect of window shape

Rectangular window

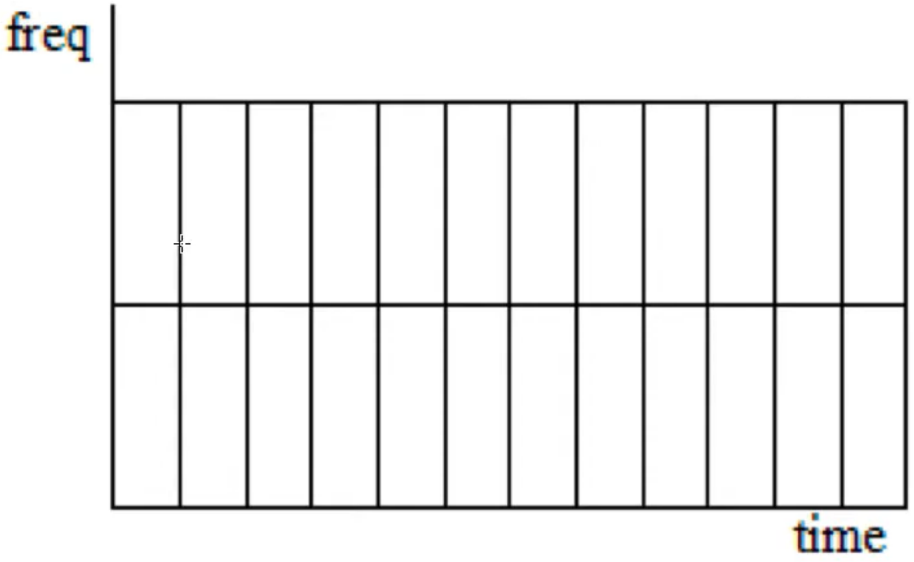
Triangular window

Hann Window

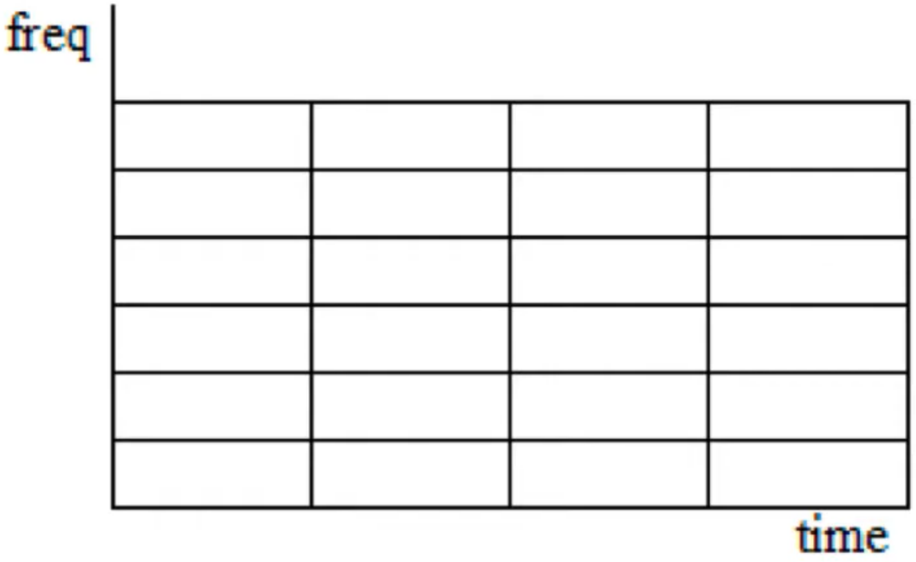


Time and Frequency Resolution

We can not have equally good resolution in time and frequency simultaneously.



Good time resolution, poor frequency resolution



Good frequency resolution, poor time resolution

Time and Frequency Resolution

Tine and frequency resolution

$$\Delta T = N \times T_s$$

$$T_s = \frac{\Delta T}{N}$$

$$f_{samp} = \frac{1}{T_s} = \frac{N}{\Delta T}$$

$$f_{Nyq} = 1/2(f_{samp}) = \frac{N}{2\,\Delta T}$$

- If you want to increase frequency resolution then you would like to have more frequency samples inside a smaller frequency range.
- That means reducing f_{Nyq} but keeping N constant, but this increase the ΔT (length of time window). That means you will loose time resolution.

STFT: our initial example