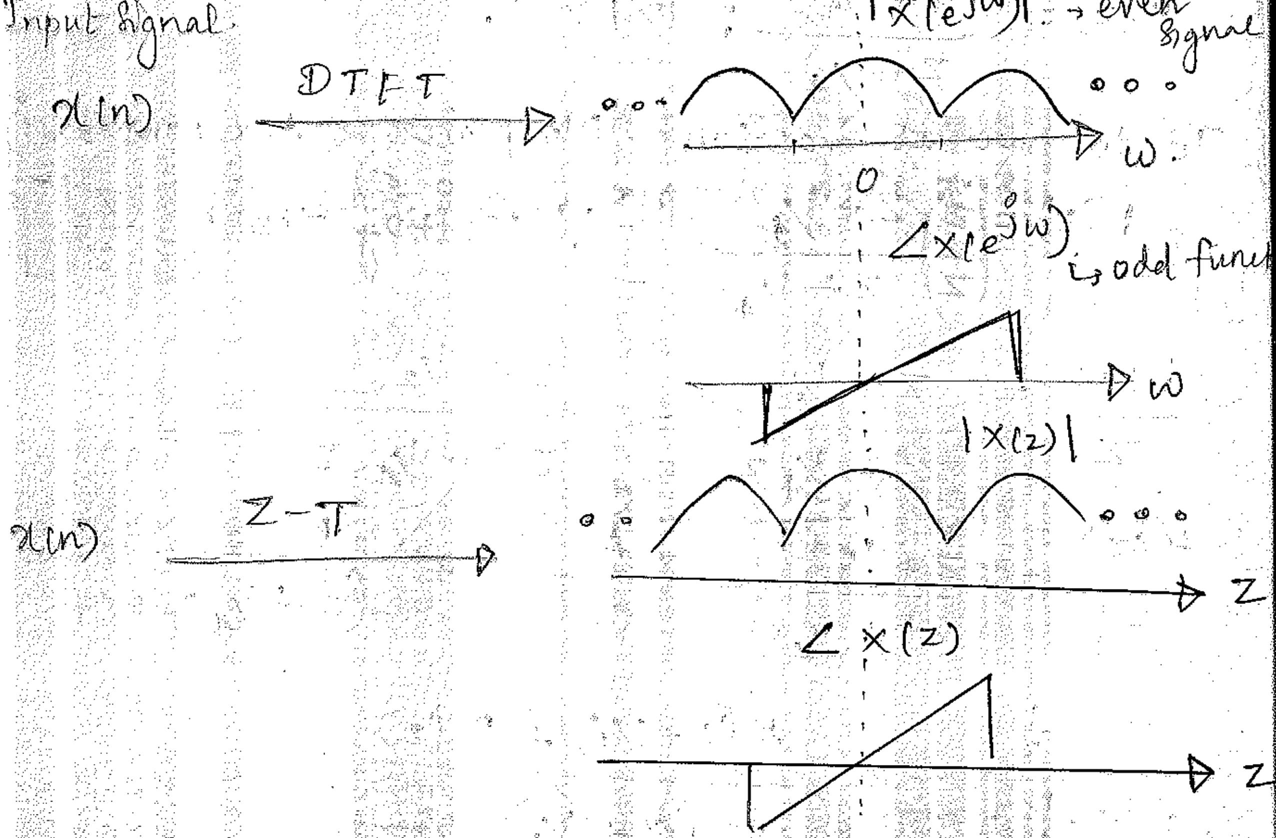
UNIT-2: FREQUENCY TRANSFORMATION



From the above both DTFT and Z-transform of the discrete-time sequence is Continuous and periodic $|X(e^{j\omega})|$ & $|X(z)|$.

Meaning, when we apply DTFT & ZT, the resultant are the frequency representation of DT signal $x(n)$, which is Continuous with respect to ω and periodic over $T = 2\pi$.

In system design, it is not good to have a Continuous representation of $x(n)$. Because, computers can process only discrete-time data. So it's better to have the Frequency domain representation as discrete function instead of continuous function.

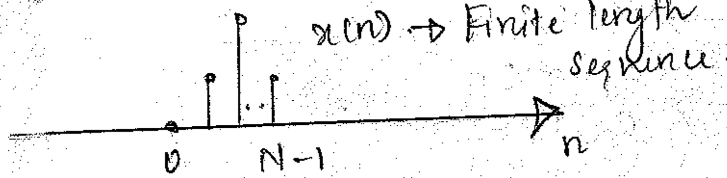
Objective is to get the Samples of DTFT (or) ZT output. how to meet this objective.

Ans: Discrete Fourier Transform (DFT).

$x(n)$ DFT

Discrete Fourier Transform (DFT) :-

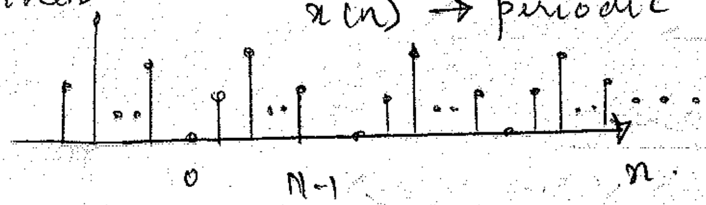
By considering a finite length sequence $x(n)$.



$x(n)$ is defined $0 \le n \le N-1$.
not defined elsewhere.

Make $x(n)$ as periodic - by repeat $x(n)$ over N period.

then $\tilde{x}(n) \rightarrow$ periodic.



$$x(n) = \hat{x}(n) \begin{cases} 0 \le n \le N-1 \\ 0 \text{ else} \end{cases}$$

WKT for any discrete, periodic function has Discrete fourier series representation exist.

$$\tilde{X}(K) \triangleq \text{DFS of } \tilde{x}(n) = \sum_{n=0}^{N-1} \tilde{x}(n).\, e^{-j\frac{2\pi n}{N}K}.$$

then DFS Co-efficients are

$$\tilde{x}(n) = \frac{1}{N}\sum_{K=0}^{N-1} \tilde{X}(K).\, e^{+j\frac{2\pi n}{N}K}.$$

Its evident that

$$\text{DFT of } x(n). = \text{DFS of } \tilde{x}(n).$$

then DFT is defined as.

$$X(K) = \sum_{n=0}^{N-1} x(n).\, e^{-j\frac{2\pi n}{N}K} = \sum_{n=0}^{N-1} x(n).\, W_N^{+nK}; \quad W_N^{nK} = e^{\left(-j\frac{2\pi}{N}\right)nK}$$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K).\, e^{j\frac{2\pi nK}{N}} = \frac{1}{N}\sum_{K=0} X(K).\, W_N^{-nK};$$

$$K = 0 \ldots N-1.$$

meaning. $X(K) = \begin{cases} \text{defined for } K = 0 \ldots N-1 \\ 0 \text{ else.} \end{cases}$

Magnitude of $X(K)$ is $|X(K)|$ → even

phase of $X(K)$ is $\angle X(K)$ → odd

$a+ib$
$|a+ib| = \sqrt{a^2+b^2}$.

Properties of DFT

(1) Time Shifting:

$$x(n) \longleftrightarrow X(K).$$

$$x(n-n_0) \longleftrightarrow e^{-j\frac{2\pi K}{N}n_0} X(K).$$

(2) frequency domain shifting.

$$x(n) \longleftrightarrow X(K).$$

$$x(n)\, e^{-j\frac{2\pi}{N}K_0 n} \longleftrightarrow X(K-K_0).$$

(3) Convolution - Linear convolution

$$x_1(n) * x_2(n) \longleftrightarrow X_1(K).\, X_2(K)$$

- Circular Convolution.

$$x_1(n) \, (N) \, x_2(n) \longleftrightarrow X_1(K).\, X_2(K)$$

(4) Conjugate:

$$x(n) \longleftrightarrow X(K).$$

$$x^*(n) \longleftrightarrow X^*(K)$$

(5) Reversal

$$x(n) \longleftrightarrow X(K)$$

$$x(-n) \longleftrightarrow X(-K).$$

(6) Parseval's Relation

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

Problems

① Compute DFT of $x(n) = a^n u(n)$; $0 \le n \le N-1$.
$\quad 0$; else.

② Compute DFT of $x(n) = \delta(n)$.

③ Compute DFT of $x(n) = \delta(n-n_0)$.

④ ✱ Compute 4 point DFT of $x(n) = \frac{1}{3}$; $0 \le n \le 2$.

(a) $x(n) = a^n u(n)$; $0 \le n \le N-1$
0 else.

→ $$X(K) = \sum_{n=0}^{N-1} x(n).\, e^{-j\frac{2\pi nK}{N}}$$

$$= \sum_{n=0}^{N-1} a^n u(n).\, e^{-j\frac{2\pi}{N}nK}$$

$$= \sum_{n=0}^{N-1} a^n.\, e^{-j\frac{2\pi}{N}nK}$$

$$= \sum_{n=0}^{N-1} \left(a.\, e^{-j\frac{2\pi}{N}K}\right)^n$$

WKT $$\sum_{n=0}^{N-1} a^n = \begin{cases} \dfrac{1-a^N}{1-a}; & a \neq 1 \\ N; & a = 1 \end{cases}$$

then
$$X(K) = \frac{1-\left(ae^{-j\frac{2\pi}{N}K}\right)^N}{1-ae^{-j\frac{2\pi}{N}K}}$$

$k = 0 \ldots N-1$.

(b) $x(n) = \delta(n)$

→ WKT $\delta(n) = \begin{cases} 1; & n=0 \\ 0; & n \neq 0 \end{cases}$

$$X(K) = \sum_{n=0}^{N-1} x(n).\, e^{-j\frac{2\pi}{N}nK}$$

$$= 1.\, e^0 \Rightarrow 1$$

$X(K) = 1$

$K = 0 \ldots N-1$.

(c) $x(n) = \delta(n-n_0)$

Apply time shifting property

$x(n-n_0) = e^{-j\frac{2\pi}{N}Kn_0}\, X(K)$.

$x(n) = \delta(n-n_0)$

$X(K) = e^{-j\frac{2\pi K}{N}n_0}\ 1$

(4) $x(n) = \frac{1}{3}$; $0 \le n \le 2$

→ 4 point DFT means $N=4$.

$$X(K) = \sum_{n=0}^{N-1} x(n).\, e^{-j\frac{2\pi n}{N}k}$$

$K = 0 \ldots N-1$.

then 4 point DFT

$$X(K) = \sum_{n=0}^{4-1} x(n).\, e^{-j\frac{\cancel{2}\pi n}{\cancel{4}_2}K}$$

$k = 0 \ldots 4-1$

$$X(K) = \sum_{n=0}^{3} x(n).\, e^{-j\frac{\pi}{2}nK}$$

$k = 0, 1, 2, 3$.

Given
$x(n) = \{ \underset{\uparrow\, n=0}{1/3},\ \underset{n=1}{1/3},\ \underset{n=2}{1/3} \}$

but 4 point DFT holds for $n=0,\ n=1,\ n=2,\ n=3$.

So $x(n) = \{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \}$

$$X(K) = \sum_{n=0}^{3} x(n).\, e^{-j\frac{\pi}{2}nK}$$

$$X(K) = x(0).\, e^{-j\frac{\pi}{2}(0)K} + x(1)e^{-j\frac{\pi}{2}K} + x(2).\, e^{-j\frac{\pi}{2}(2)K} + x(3).\, e^{-j\frac{\pi}{2}(3)K}$$

$$= \frac{1}{3} + \frac{1}{3}.\, e^{-j\pi/2\, K} + \frac{1}{3}e^{-j\pi K} + 0.\, e^{-j\frac{3\pi}{2}}$$

$$X(K) = \frac{1}{3} + \frac{1}{3}e^{-j\frac{\pi}{2}K} + \frac{1}{3}e^{-j\pi K} + 0.$$

$K=0$

$X(0) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

$K=1$

$X(1) = \frac{1}{3} + \frac{1}{3} e^{-j\pi/2} + \frac{1}{3} e^{-j\pi}$

$= \frac{1}{3} + \frac{1}{3}\left[\cancel{\cos\left(-\frac{\pi}{2}\right)} + j\sin\left(-\frac{\pi}{2}\right)\right]$

$+ \frac{1}{3}\left[\cos(-\pi) + \cancel{j\sin(-\pi)}\right]$

$= \cancel{\frac{1}{3}} - \frac{j}{3} - \cancel{\frac{1}{3}} = -j/3$

$X(1) = -j/3$

$K=2$

$X(2) = \frac{1}{3} + \frac{1}{3} e^{-j\pi} + \frac{1}{3} e^{-j2\pi} = \frac{1}{3}$

$X(2) = \frac{1}{3}$

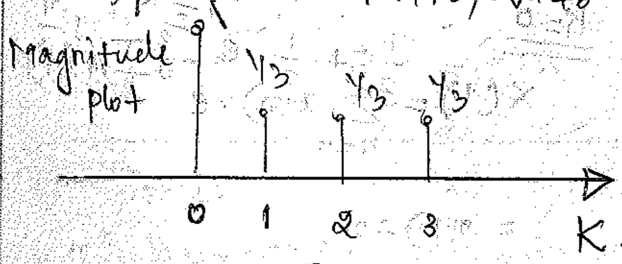
$K=3$

$X(3) = \frac{1}{3} + \frac{1}{3} e^{-j3\pi/2} + \frac{1}{3} e^{-j3\pi} = j/3$

$X(3) = j/3$

$X(K) = \left\{1, -j/3, \frac{1}{3}, j/3\right\}$

$|X(K)|$ $\qquad |a+ib| = \sqrt{a^2+b^2}$



$|X(0)| = \sqrt{1^2} = 1$

$|X(1)| = \sqrt{0^2 + \left(-\frac{1}{3}\right)^2} = 1/3$

$|X(2)| = \sqrt{0^2 + \left(\frac{1}{3}\right)^2} = 1/3$

$|X(3)| = \sqrt{0^2 + \left(\frac{1}{3}\right)^2} = 1/3$

$\angle X(K)$ $\qquad \angle a+ib = \tan^{-1}(b/a)$

$\angle X(0) = \tan^{-1}(0/1) = 0$

$\angle X(1) = \tan^{-1}(-\infty)$ $\qquad \left[\tan^{-1}(-\theta) = -\tan^{-1}(\theta)\right]$

$= -\tan^{-1}(\infty)$

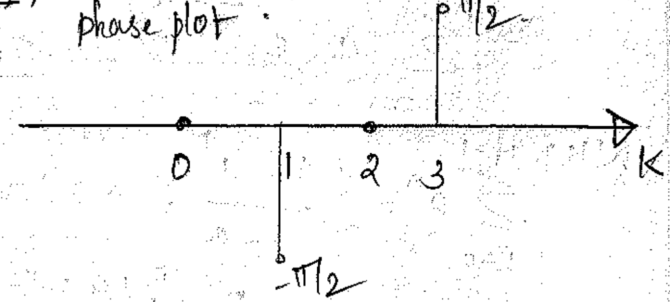
$= -\tan^{-1}(\infty) = -\pi/2$ $\qquad \left[\tan^{-1}\infty = \ldots\right]$

$\angle X(1) = -\pi/2$

$\angle X(2) = \tan^{-1}\left(\frac{0}{1/3}\right) = 0$

$\angle X(3) = \tan^{-1}\left(\frac{1/3}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$\angle X(K)$
phase plot



Short cut method → 4 point DFT

For 4 point DFT

$X(K) = W_4\, x(n)$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

$X(K) = W_4\, x(n)$

$$\underbrace{\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}}_{X(K)} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}}_{W_4} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}}_{x(n)}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1/3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+0 \\ \cancel{\frac{1}{3}} - j/3 - \cancel{\frac{1}{3}} + 0 \\ \frac{1}{3} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} + 0 \\ \cancel{\frac{1}{3}} + j/3 - \cancel{\frac{1}{3}} + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -j/3 \\ 1/3 \\ j/3 \end{bmatrix}$$

$$X(K) = \{1,\ -j/3,\ \tfrac{1}{3},\ +j/3\}.$$

Problems: [4 point DFT]

(1) Compute DFT of sequence.

$x(n) = \{0, 1, 2\}$.

(2) Compute DFT of sequence.

$x(n) = \{1, 1, 1, 1\}$

(3) Compute DFT of sequence.

$x(n) = \{1, -3, 5, -6\}$

→

(a) $x(n) = \{0, 1, 2\}$

→ Apply 4 point DFT (S.C method)

$$X(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$X(K) = \{3,\ -2-j,\ 1,\ -2+j\}.$$

(b) $x(n) = \{1, 1, 1, 1\}$.

→ By applying 4 point DFT (S.C) method.

$$X(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & +1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X(K) = \{4, 0, 0, 0\}.$$

(c) $x(n) = \{1, -3, 5, -6\}$.

→ By applying 4 point DFT S-C method

$$X(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \\ -6 \end{bmatrix}$$

$$X(K) = \{-3,\ -4-3j,\ 15,\ -4+3j\}.$$

Problems. [4 point Inverse DFT]

(1) Compute IDFT of $X(K)$

$X(K) = \{-3, -4-3j, 15, -4+3j\}$

→ Short Cut Method - 4 point IDFT

$$x(n) = \frac{1}{N}\left[W_N^*\right] X(K).$$

4 point IDFT

$$x(n) = \frac{1}{4}\left[W_4^*\right] X(K)$$

$$W_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$$

$$x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} \begin{bmatrix} -3 \\ -4-3j \\ 15 \\ -4+3j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -3 - 4 - \cancel{3j} + 15 - 4 + \cancel{3j} \\ -\cancel{3} - \cancel{4j} + \cancel{3} - 15 + \cancel{4j} + 3 \\ -3 + 4 + \cancel{3j} + 15 + 4 - \cancel{3j} \\ -3 + \cancel{4j} - 3 - 15 - \cancel{4j} - 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \\ 20 \\ -24 \end{bmatrix}$$

$x(n) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ ∴ $x(n) = \{1, -3, 5, -6\}$

(b) Compute IDFT of $X(K)$

$X(K) = \{3, -2-j, 1, -2+j\}$.

$\rightarrow$ 4 point IDFT S.C method

$$x(n) = \frac{1}{N}\left[W_N^*\right] . X(K)$$

$$= \frac{1}{4} W_4^* X(K).$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}\begin{bmatrix} 3 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 0 \\ 4 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

$x(n) = \{0, 1, 2, 0\}$.

8 point IDFT

Short-cut method

$$x(n) = \frac{1}{N}\left[W_N^*\right]\left[X(K)\right]$$

$$x(n) = \frac{1}{8}\left[W_8^*\right]\left[X(K)\right]$$

W_8^* = Complex Conjugate of W_8

8-point DFT

8 point DFT Short-Cut method.

$$X(K) = \underset{8\times 8}{\left[W_8\right]}\ \underset{8\times 1}{\left[x(n)\right]}$$

$$W_8 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt2}-j/\sqrt2 & -j & -\frac{1}{\sqrt2}-j/\sqrt2 & -1 & -\frac{1}{\sqrt2}+j/\sqrt2 & +j & \frac{1}{\sqrt2}+j/\sqrt2 \\ 1 & -j & -1 & j & +1 & -j & -1 & +j \\ 1 & -\frac{1}{\sqrt2}-j/\sqrt2 & j & \frac{1}{\sqrt2}-j/\sqrt2 & -1 & \frac{1}{\sqrt2}+j/\sqrt2 & -j & -\frac{1}{\sqrt2}+j/\sqrt2 \\ 1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\ 1 & -\frac{1}{\sqrt2}+j/\sqrt2 & -j & \frac{1}{\sqrt2}+j/\sqrt2 & -1 & \frac{1}{\sqrt2}-j/\sqrt2 & +j & -\frac{1}{\sqrt2}-j/\sqrt2 \\ 1 & +j & -1 & -j & +1 & +j & -1 & -j \end{bmatrix} \end{array}$$

(2) First 5 points of 8 point DFT of real valued sequence are $\{0.25, -j0.03018, 0, 0, 0.125-j0.0518\}$ determine the remaining three points of the DFT.

→

$$\boxed{X(K) = X^*(N-K)} \rightarrow (1)$$

$X(0) = 0.25$, $X(1) = -j0.03018$, $X(2) = 0$, $X(3) = 0$, $X(4) = 0.125-j0.0518$
$X(5) = ?$, $X(6) = ?$, $X(7) = ?$

By using (1) $X(K) = X^*(N-K)$; $N=8$

put $K=5$ $X(5) = X^*(8-5) = X^*(3) = 0$.

$X(5) = 0$.

put $K=6$ $X(6) = X^*(8-6) = X^*(2) = 0$.

$X(6) = 0$.

put $K=7$ $X(7) = X^*(8-7) = X^*(1) = [-j0.03018]^*$
$= j0.03018$

$X(7) = j0.03018$

$X(5)=0$; $X(6)=0$; $X(7) = j0.03018$.

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(3). Find 5 points of 8 point DFT of real valued sequence are $\{28, -4+j9.656, -4+4j, -4+j1.656, -4\}$ determine the remaining 3 points of the DFT.

→ $X(0) = 28$, $X(1) = -4+j9.656$; $X(2) = -4+4j$; $X(3) = -4+j1.656$.
$X(4) = -4$

By using $X(K) = X^*(N-K)$.

$X(5) = X^*(3) = (-4+j1.656)^* = -4-j1.656$

$X(6) = X^*(2) = (-4+4j)^* = -4-4j$

$X(7) = X^*(1) = (-4+j9.656)^* = -4-j9.656$.

$X(5) = -4-j1.656$; $X(6) = -4-4j$; $X(7) = -4-j9.656$.