Discrete time fourier transform represents non periodic discrete signals as a weighted integral of complex sinusoids whose frequency [illegible] continuously over [illegible]
frequency domain representation is continuous and is a periodic function of frequency with period 2π.

$x(t) \overset{F}{\rightleftharpoons} X(\omega)$

continuous | continuous

$\omega: -\infty$ to ∞

$x[n] \overset{DTFT}{\rightleftharpoons} X(e^{j\omega})$ or $X(e^{j\Omega})$

Discrete | Continuous

ω: 0 to 2π

Periodicity $\rightarrow X(e^{j(\omega+2\pi k)}) = X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad \text{Analysis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \text{Synthesis equation}$$

$\rightarrow$ $x[n] = a^n u[n]$; $0 < a < 1$

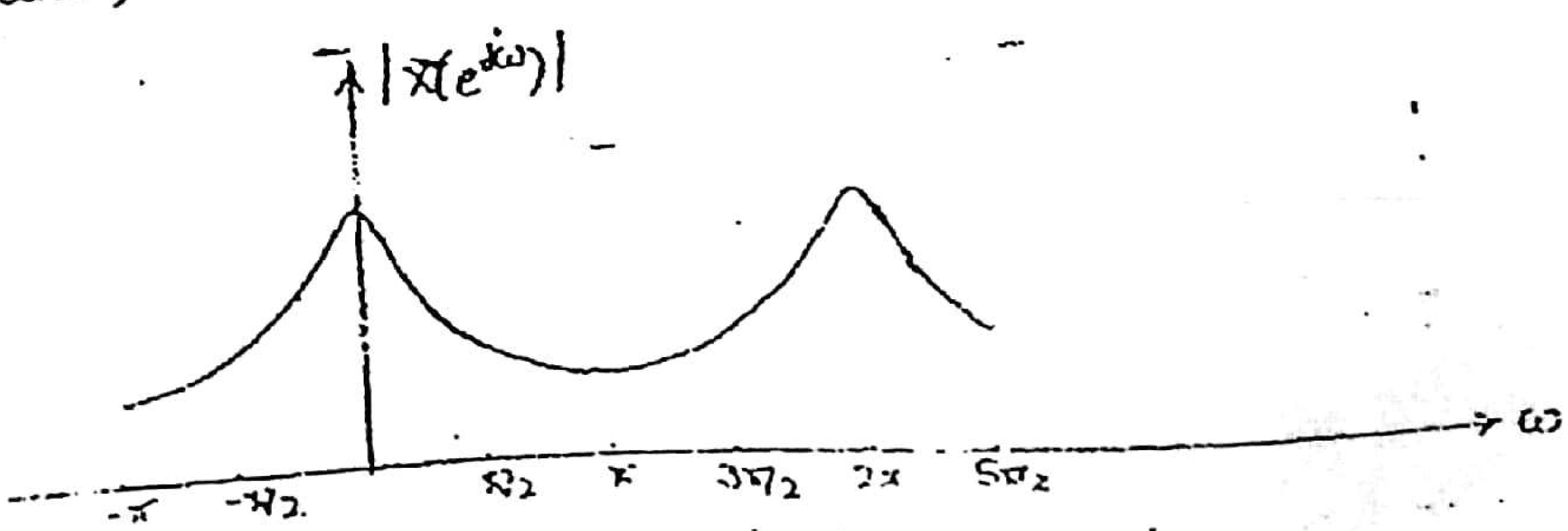
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = 1 + a e^{-j\omega} + (a e^{-j\omega})^2 + \cdots$$

$$= \frac{1}{1-ae^{-j\omega}} = \frac{1}{1-a(\cos\omega - j\sin\omega)} = \frac{1}{1-a\cos\omega + j a\sin\omega}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + a^2\sin^2\omega}} = \frac{1}{\sqrt{1-2a\cos\omega+a^2}}$$



Properties of DTFT :

(i) Linearity:

If $x_1[n] \rightleftharpoons X_1(e^{j\omega})$ and $x_2[n] \rightleftharpoons X_2(e^{j\omega})$ then

$$\alpha x_1[n] + \beta x_2[n] \rightleftharpoons \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$

(ii) Time Shifting:

If $x(n) \rightleftharpoons X(e^{j\omega})$ then $x(n-n_0) \rightleftharpoons e^{-j\omega n_0} X(e^{j\omega})$

$$F\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0)\, e^{-j\omega n}$$

put $n-n_0 = m \Rightarrow n = m+n_0$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega(m+n_0)}$$

$$= e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m}$$

$$= e^{-j\omega n_0} X(e^{j\omega})$$

(iii) Frequency Shifting:

If $x(n) \rightleftharpoons X(e^{j\omega})$ then $x(n)\, e^{j\omega_0 n} \rightleftharpoons X(e^{j(\omega-\omega_0)})$

$$F\{x(n)\, e^{j\omega_0 n}\} \rightleftharpoons \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega_0 n}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n}$$

$$= X(e^{j(\omega-\omega_0)})$$

(iv) Time Scaling:

If $x(n) \rightleftharpoons X(e^{j\omega})$ then $x(n/K) \rightleftharpoons X(e^{j\omega K})$

$$F\{x(n/K)\} = \sum_{n=-\infty}^{\infty} x(n/K)\, e^{-j\omega n}$$

put $n/K = m$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega mk}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega k m} = X(e^{j\omega k})$$

(v) Time Reversal:

If $x(n) \rightleftharpoons X(e^{j\omega})$ then

$$x(-n) \rightleftharpoons X(e^{-j\omega})$$

(vi) Conjugate:

If $x[n] \rightleftharpoons X(e^{j\omega})$ then $x^*[n] \rightleftharpoons X^*(e^{-j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

Replace 'ω' by '$-\omega$'

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{j\omega n}$$

Conjugating both sides

$$X^*(e^{-j\omega}) = \sum_{n=-\infty}^{+\infty} x^*[n]\, e^{-j\omega n}$$

(vii) Frequency Differentiation:

If $x[n] \rightleftharpoons X(e^{j\omega})$ then $n\,x[n] \rightleftharpoons j\frac{d}{d\omega}X(e^{j\omega})$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Differentiating both sides w.r.t. 'ω'.

$$\frac{d}{d\omega}X(e^{j\omega}) = \frac{1}{j}\sum_{n=-\infty}^{\infty} \{n\,x[n]\}\, e^{-j\omega n}$$

$$\therefore\; n\,x[n] \rightleftharpoons j\frac{d}{d\omega}X(e^{j\omega})$$

(viii) Convolution in n-domain:

If $x_1[n] \rightleftharpoons X_1(e^{j\omega})$ and $x_2[n] \rightleftharpoons X_2(e^{j\omega})$ then

$$x_1[n] * x_2[n] \rightleftharpoons X_1(e^{j\omega})\,X_2(e^{j\omega})$$

$$F\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty} \{x_1[n] * x_2[n]\}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_1[m]\, x_2[n-m]\, e^{-j\omega n}$$

put $n-m=p$

$$= \sum_{p=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_1[m]\, x_2[p]\, e^{-j\omega(p+m)}$$

$$= \sum_{m=-\infty}^{\infty} x_1[m]\, e^{-j\omega m} \sum_{p=-\infty}^{\infty} x_2[p]\, e^{-j\omega p}$$

$$= X_1(e^{j\omega})\, X_2(e^{j\omega})$$

$\frac{1}{2\pi} [X_1(e^{j\omega}) \circledast X_2(e^{j\omega})]$

(x) **Periodicity:**

If $x[n] \rightleftharpoons X(e^{j\omega})$ then $X\{e^{j(\omega+2\pi k)}\} = X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Replace 'ω' by '$\omega+2\pi k$'

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \cdot e^{-j2\pi k n}$$

$$= X(e^{j\omega})$$

(xi) **First Difference:**

If $x[n] \rightleftharpoons X(e^{j\omega})$ then $x[n] - x[n-1] \rightleftharpoons [1 - e^{-j\omega}]\, X(e^{j\omega})$

(xii) **Parseval's Theorem:**

If $x[n] \rightleftharpoons X(e^{j\omega})$ then $\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{energy}} = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} \underbrace{|X(e^{j\omega})|^2}_{\text{power}}\, d\omega$

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x^*[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X^*(e^{j\omega})\, e^{-j\omega n}\, d\omega \longrightarrow (1)$$

$$\text{LHS} = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X^*(e^{j\omega})\, e^{-j\omega n}\, d\omega$$

Adjusting Summation and Integration

$$= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X^*(e^{j\omega}) \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, d\omega = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X^*(e^{j\omega})\, X(e^{j\omega})$$

$$= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |X(e^{j\omega})|^2\, d\omega$$

Problems:-

① $x(n) = a^n u(n)$; $0 \le a < 1$

$\rightarrow$ $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$\left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad a<1 \right]$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n = \frac{1}{1 - a e^{-j\omega}}.$$

② $x(n) = \left(\frac{1}{2}\right)^{n-1} u(n-1)$

$\rightarrow$ WkT $$a^n u(n) \longleftrightarrow \frac{1}{1 - a e^{-j\omega}}$$

then
$$x(n-n_0) \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

$$\left(\frac{1}{2}\right)^n u(n) \longleftrightarrow \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

NOTES

$$\left(\frac{1}{2}\right)^{n-1} u(n-1) \longleftrightarrow e^{-j\omega(1)} \times \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}.$$

(3) $x(n) = \left(\frac{1}{4}\right)^n u(n+2)$

$\longrightarrow$

$$x(n) \longleftrightarrow X(e^{j\omega})$$

$$\left(\frac{1}{4}\right)^n u(n+2) = \left(\frac{1}{4}\right)^{n+2-2} u(n+2) = \left(\frac{1}{4}\right)^{-2} \cdot \left(\frac{1}{4}\right)^{n+2} u(n+2)$$

$$= 16 \left(\frac{1}{4}\right)^{n+2} u(n+2)$$

$$16 \times \left(\frac{1}{4}\right)^n u(n) = 16 \times \frac{1}{1 - \left(\frac{1}{4}\right) e^{-j\omega}}$$

$$16 \times \left(\frac{1}{4}\right)^{n+2} u(n+2) = 16 \times e^{j\omega 2} \left[\frac{1}{1 - \frac{1}{4} e^{-j\omega}}\right]$$

$$X(e^{j\omega}) = \frac{16 e^{j\omega 2}}{1 - \frac{1}{4} e^{-j\omega}}$$

NOTES