Unit-4 Applications of Partial derivatives

Unit 4
556

Objectives :- After studing this chapter, the learner will be able to ;

1) Know the concept of tangent plane and linearization
2) Use the various properties of Jacobian transformation to evaluate required transformation.
3) Understand the different types of errors and find the required approximate error.
4) find the maximum and minimum values of the function of two variables.
5) Solve problems of Lagranges method of undetermined multipliers.

Jacobians: If u, v are function of independent variables x, y, i.e $u = f_1(x, y)$ $v = f_2(x, y)$ then the Jacobian J is defined as

The derivatives of the above two function u and v can be arranged in the form of determinant

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \qquad \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

If u, v, w are function of independent variables x, y, z then the Jacobian J is defined as

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

If $u = x^2 - y^2$ and $v = 2xy$ find $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$ $4x^2 + 4y^2 = 4(x^2+y^2)$

Properties of determinants:

1) If a row or column of a determinant consist of all zeros then the value of determinant is zero.

2) If any two rows or columns of a determinant are identical, then its value is zero.

3) If any two rows and columns of a determinant are proportional then its value is zero.

4) $$\begin{vmatrix} a & b & c \\ Kx & Ky & Kz \\ p & q & r \end{vmatrix} = K \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

5) $$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

2016 Ex:- If $u = x(1-y)$ and $v = xy$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.

Sol^n:- Given that $u = x(1-y)$ and $v = xy$

$\therefore \quad u = x - xy = x - v$

$\therefore \quad x = u + v$

and $\quad v = xy \Rightarrow v = (u+v)y$

$\Rightarrow \quad y = \frac{v}{u+v}$

$\therefore \quad x = u + v \longrightarrow (1)$

$\quad y = \frac{v}{u+v} \longrightarrow (2)$

$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial u} = v\left(\frac{-1}{(u+v)^2}\right) = \frac{-v}{(u+v)^2}$

$$\frac{\partial y}{\partial v} = \frac{(u+v)(1) - v(1)}{(u+v)^2} = \frac{u}{(u+v)^2}$$

We have

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \frac{-v}{(u+v)^2} & \frac{u}{(u+v)^2} \end{vmatrix} = \frac{u}{(u+v)^2} + \frac{v}{(u+v)^2}$$

$$= \frac{u+v}{(u+v)^2} = \frac{1}{u+v}$$

✗ Ex:- If $ux = yz$ and $vy = zx$, $wz = xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ 2010

Sol^n:- We have

$$u = \frac{yz}{x}, \quad v = \frac{zx}{y} \quad w = \frac{xy}{z}.$$

$$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial u}{\partial z} = \frac{y}{x},$$

$$\frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial v}{\partial y} = \frac{-zx}{y^2}, \quad \frac{\partial v}{\partial z} = \frac{x}{y},$$

$$\frac{\partial w}{\partial x} = \frac{y}{z}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}, \quad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$$

Now,

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} = -\frac{yz}{x^2}\left[\left(-\frac{zx}{y^2}\right)\left(-\frac{xy}{z^2}\right) - \left(\frac{x}{y}\right)\left(\frac{x}{z}\right)\right]$$

$$-\frac{z}{x}\left[\frac{z}{y}\left(-\frac{xy}{z^2}\right) - \left(\frac{x}{y}\right)\left(\frac{y}{z}\right)\right]$$

$$+\frac{y}{x}\left[\left(\frac{z}{y}\right)\left(\frac{x}{z}\right) - \left(-\frac{zx}{y^2}\right)\left(\frac{y}{z}\right)\right]$$

$$= -\frac{yz}{x^2}\left[\frac{\cancel{z}x^2\cancel{y}}{y^{\cancel{2}}z^{\cancel{2}}} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-x\cancel{y}\cancel{z}}{\cancel{y}z^{\cancel{2}}} - \frac{x\cancel{y}}{\cancel{y}z}\right] + \frac{y}{x}\left[\frac{\cancel{z}x}{y\cancel{z}} + \frac{\cancel{z}x\cancel{y}}{y^{\cancel{2}}\cancel{z}}\right]$$

$$= -\frac{yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[-\frac{x}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= 0 - \frac{\cancel{z}}{\cancel{x}}\left[-2\frac{\cancel{x}}{\cancel{z}}\right] + \frac{\cancel{y}}{\cancel{x}}\left[\frac{2\cancel{x}}{\cancel{y}}\right] = 2 + 2 = 4$$

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Ex:- Given spherical polar co-ordinates,

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \text{———} \quad (1)$$

show that $\dfrac{\partial(x, y z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$

Solⁿ:- Given $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

from eqⁿ (1) we have

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \Big[\sin\theta\cos\phi\{0+\sin\theta\cos\phi\} - \cos\theta\cos\phi\{0-\cos\theta\cos\phi\} - \sin\phi\{-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi\}\Big]$$

$$= r^2\sin\theta\left[\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi\{\sin^2\theta + \cos^2\phi\}\right]$$

$$= r^2\sin\theta\left[\cos^2\phi\{\sin^2\theta+\cos^2\theta\} + \sin^2\phi\right]$$

$$= r^2\sin\theta\left[\cos^2\phi + \sin^2\phi\right] = r^2\sin\theta.$$

Jacobian of Implicit functions

Divided into two types

(1) Explicit functions

(2) Implicit functions

(1) Explicit functions :- An explicit function is a function in which one variable can be clearly (explicitly) expressed in terms of the other.

ex : $x^2 + 3x^2y + 9x^2y^3 = 1$

$x^2(1 + 3y + 9y^3) = 1$

$x = \dfrac{1}{1 + 3y + 9y^3}$ is an explicit function

(2) Implicit functions :- An implicit function is a function in which one variable can not be expressed in terms of the other.

Ex $x^2 + 2xy + y^2 = 9$

To find Jacobian of implicit functions we will use method of functions as mentioned below.

(1) Let $f_1(x, y, u, v) = 0$ and $f_2(x, y, u, v) = 0$

then $$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u,v)}}$$

(2) If $f_1(x,y,z,u,v,w)=0$

$f_2(x,y,z,u,v,w)=0$

$f_3(x,y,z,u,v,w)=0$ then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

Ex: If $u^3+v^3=x+y$, $u^2+v^2=x^3+y^3$, show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{y^2-x^2}{uv(u-v)}$$

Soln: Given, $u^3+v^3=x+y$

$\therefore f_1 = u^3+v^3-x-y$

$\therefore f_2 = u^2+v^2-x^3-y^3$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}} = \frac{N}{D} \longrightarrow (1)$$

where $N = \frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$

$N = 3y^2-(-3x^2)(-1) = 3y^2-3x^2$

$$D = \frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$D = 3(2)u.v \begin{vmatrix} u & v \\ 1 & 1 \end{vmatrix} = 6uv(u-v)$$

LHS. $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{N}{D} = \frac{3y^2 - 3x^2}{6uv(u-v)} = \frac{\cancel{3}(y^2-x^2)}{\cancel{6}_2 uv(u-v)}$$

$$= \frac{1}{2}\frac{y^2-x^2}{uv(u-v)}$$ Hence proved.

Ex: If $x+y+z = u$, $y+z = uv$, $z = uvw$, show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

Soln: from the given relations, we have

$f_1 = x+y+z-u$

$f_2 = y+z-uv$

$f_3 = z - uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} = \frac{-N}{D} \quad \text{--- (1)}$$

where $N = \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$

$\therefore \quad N = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix}$

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$\therefore \quad N = -1\left[(-u)(-uv) - 0\right] = -u^2 v$

Now, $\quad D = \dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$

24, 25, 31, 38,

41, 52, 23

$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1(1-0) - 1(0) + 1(0)$

$= 1$

$LHS = \dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \dfrac{-N}{D} = -\dfrac{(u^2 v)}{1}$ R.H.S

Hence Proved.

Partial derivatives of Implicit functions using Jacobians.

Let $f_1(x,y,u,v)=0$

$f_2(x,y,u,v)=0$

then the formula to find partial derivatives by using Jacobian can be set up by following the steps.

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}$$ → Numerator variable group

$$\boxed{\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}}$$

replace $u \to x$.
in numerator only

$$\cancel{\frac{\partial y}{\partial y}} \; \frac{\partial y}{\partial u} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}$$ replace $y \to u$.

$$\boxed{\frac{\partial y}{\partial u} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,u)}}{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}}$$

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2, 4, 5, 8,
~~10~~, 22, ~~23~~, 25
27, 28, ~~30~~,
32, 38, 43
50, 53, 64,

a realation betⁿ them.

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

$$\frac{\partial y}{\partial w} = -\frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, w, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}$$

Ex: If $u+v^2 = x$, $v+w^2 = y$, $w+u^2 = z$, find $\frac{\partial u}{\partial x}$.

Solⁿ: $f_1 = u+v^2-x$

$f_2 = v+w^2-y$

$f_3 = w+u^2-z$

$$\therefore \frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}} = -\frac{N}{D} \longrightarrow (1)$$

$$N = \frac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} -1 & 2v & 0 \\ 0 & 1 & 2w \\ 0 & 0 & 1 \end{vmatrix}$$

$$N = -1(1-0) - 2v(0-0) + 0(0)$$

$$N = -1$$

and $D = \dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}$

$$D = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 2v & 0 \\ 0 & 1 & 2w \\ 2u & 0 & 1 \end{vmatrix}$$

$$D = 1[1] - 2v(0 - 4uw) + 0$$

$$= 1 + 8uvw$$

But the eq^n (1)

$$\frac{\partial u}{\partial x} = \frac{-N}{D} = \frac{-(-1)}{1+8uvw} = \frac{1}{1+8uvw}.$$

Ex: If $u+v = x^2+y^2$ and $u-v = x+2y$, find $\left(\dfrac{\partial u}{\partial x}\right)_y$

Sol^n: Given $f_1 = u+v-x^2-y^2$

$f_2 = u-v-x-2y$

and
$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\dfrac{\partial(f_1, f_2)}{\partial(x, v)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}} = -\frac{N}{D} \rightarrow (1)$$

where
$$N = \frac{\partial(f_1, f_2)}{\partial(x, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -2x & 1 \\ -1 & -1 \end{vmatrix}$$

$$N = 2x+1$$

and $D = \dfrac{\partial(f_1, f_2)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

$$D = -1-1 = -2$$

from eq^n (1).

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{N}{D} = -\frac{(2x+1)}{-2} = \frac{2x+1}{2}.$$

Ex: If $x = u+v$, $y = v^2+w^2$, $z = w^3+u^3$. Prove that

$$\frac{\partial u}{\partial x} = \frac{vw}{vw+u^2}$$

Sol^n: Given $f_1 : x-u-v$

$f_2 : y-v^2-w^2$

$f_3 : z-w^3-u^3$

$$\therefore \frac{\partial u}{\partial x} = \frac{-\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

where. $N = \dfrac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w} \end{vmatrix}$

$$N = \begin{vmatrix} 1 & -1 & 0 \\ 0 & -2v & -2w \\ 0 & 0 & -3w^2 \end{vmatrix}$$

$$= 1(6vw^2 - 0] + 1(0)$$

$$= 6vw^2$$

and $D = \dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$

$$= \begin{vmatrix} -1 & -1 & 0 \\ 0 & -2v & -2w \\ -3u^2 & 0 & -3w^2 \end{vmatrix} = -1(6vw^2 - 0) + 1(0 - 6u^2 w)$$

$$= -6vw^2 - 6wu^2$$

$$= -6w(vw + u^2)$$

from eq^n ①.

$$\frac{\partial u}{\partial x} = -\frac{N}{D} = -\frac{6vw^2}{-6w(vw+u^2)} = \frac{vw}{vw+u^2}$$

Hence proved

Ex: If $ux + vy = a$, $\dfrac{u}{x} + \dfrac{v}{y} = 1$, then by using Jacobians, Prove that $\left(\dfrac{\partial u}{\partial x}\right)_y - \left(\dfrac{\partial v}{\partial y}\right)_x = \dfrac{x^2+y^2}{y^2-x^2}$

Soln:

$$f_1 = ux + vy - a$$

$$f_2 = \frac{u}{x} + \frac{v}{y} - 1$$

Part 1: To find $\left(\dfrac{\partial u}{\partial x}\right)_y$

$$\therefore \frac{\partial u}{\partial x} = - \frac{\dfrac{\partial(f_1,f_2)}{\partial(x,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$= - \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}}$$

$$= - \frac{\begin{vmatrix} u & y \\ \frac{-u}{x^2} & \frac{1}{y} \end{vmatrix}}{\begin{vmatrix} x & y \\ \frac{1}{x} & \frac{1}{y} \end{vmatrix}} = \frac{-\frac{u}{y} + \frac{uy}{x^2}}{\frac{x}{y} - \frac{y}{x}}$$

$$= - \frac{u\left(\frac{1}{y} + \frac{y}{x^2}\right)}{\left(\frac{x^2-y^2}{xy}\right)} = \frac{-u\left(\frac{x^2+y^2}{x^2 y}\right)}{\left(\frac{x^2-y^2}{xy}\right)}$$

$$= \frac{-u(x^2+y^2)}{x^{\cancel{2}}\cancel{y}} \times \frac{\cancel{x}\cancel{y}}{x^2-y^2} = \frac{-u}{x} \frac{(x^2+y^2)}{(x^2-y^2)} \longrightarrow (1)$$

Functional Dependence :

1) Let $u = f(x,y)$ and $v = g(x,y)$ be any two functions of x and y. The functions u and v are said to be functionally dependent if there exists a functional relation betⁿ u and v. otherwise functionally independent.
Assume that u and v are functionally dependent and the relation i.e $\phi \to u, v \to x, y$. betⁿ u and v is $\phi(u,v) = 0$.

$\longrightarrow (1)$

Thus u and v are said to be functionally dependent.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = 0, \text{ If } J = \frac{\partial(u,v)}{\partial(x,y)} \neq 0$$

then u and v are functionally independent.

2) If u, v are functions of x, y and z then u and v are functionally dependent if.

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,z)} = \frac{\partial(u,v)}{\partial(y,z)} = 0$$

3) If u, v, w are functions of x, y, z then u, v, w are functionally dependent if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

Ex :- Examine whether u, v, w are functionally dependent or not. If dependent find relation among them.

Divided

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a) $u = y+z,\ v = x+2z^2,\ w = x-4yz-2y^2$

b) $u = x+y+z,\ v = x^2+y^2+z^2,\ w = xy+yz+zx$

c) $u = x+y+z,\ v = x^2+y^2+z^2,\ w = x^3+y^3+z^3-3xyz.$

d) $u = \dfrac{x}{y-z},\ v = \dfrac{y}{z-x},\ w = \dfrac{z}{x-y}.$

Sol^n (a) Given that:

$$u = y+z,\ v = x+2z^2,\ w = x-4yz-2y^2$$

Consider.

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4z-4y & -4y \end{vmatrix}$$

$$= 0-1(-4y-4z)+1(-4z-4y) = 0$$

$$= 4y+4z-4z-4y = 0.$$

Hence u, v, w are functionally dependent.

To find Relation:

We have $u^2 = (y+z)^2 = y^2+z^2+2yz.$

$$2u^2 = 2y^2+4yz+2z^2$$

$$v-2u^2 = (x+2z^2)-(2y^2+4yz+2z^2)$$

$$= x-4yz-2y^2 = w$$

$$v-2u^2 = w$$

(b)

Given that

$u = x+y+z, \quad v = x^2+y^2+z^2, \quad w = xy+yz+zx.$

$u_x = 1, \quad u_y = 1, \quad u_z = 1, \quad v_x = 2x, \quad v_y = 2y, \quad v_z = 2z.$

$w_x = y+z, \quad w_y = x+z, \quad w_z = y+x.$

consider.

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & y+x \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & y+x \end{vmatrix}$$

Apply $R_3 + R_2$

$$= 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x+y+z)\left[1(y-z) - 1(x-z) + 1(x-y)\right]$$

$$= 2(x+y+z)\left[\cancel{y} - \cancel{z} - \cancel{x} + \cancel{z} + \cancel{x} - \cancel{y}\right] = 0$$

Hence u, v, w are functionally dependent.

To find relation:

$$u^2 = (x+y+z)^2$$

$$= (x^2+y^2+z^2) + 2xy + 2yz + 2xz.$$

$$u^2 = v + 2(w)$$

$$\boxed{u^2 = v + 2w}$$

Ex: Are the following functions functionally dependent? If so, find a realation betⁿ them.

$$u = \frac{x-y}{x+z} \; ; \quad v = \frac{x+z}{y+z}$$

Solⁿ:

$$u, v \to x, y, z$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(y,z)} = \frac{\partial(u,v)}{\partial(z,x)} = 0$$

Now,

$$u = \frac{x-y}{x+z}$$

$$\frac{\partial u}{\partial x} = \frac{(x+z)(1)-(x-y)(1+0)}{(x+z)^2} = \frac{\cancel{x}+z-\cancel{x}+y}{(x+z)^2} = \frac{z+y}{(x+z)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{x+z}$$

$$\frac{\partial u}{\partial z} = \frac{-(x-y)}{(x+z)^2}\cdot(0+1) = \frac{y-x}{(x+z)^2}$$

Now

$$v = \frac{x+z}{y+z}$$

$$\frac{\partial v}{\partial x} = \frac{1}{y+z}$$

$$\frac{\partial v}{\partial y} = \frac{-(x+z)}{(y+z)^2}(1+0) = \frac{-(x+z)}{(y+z)^2}$$

$$\frac{\partial v}{\partial z} = \frac{(y+z)(0+1)-(x+z)(0+1)}{(y+z)^2} = \frac{y+\cancel{z}-x-\cancel{z}}{(y+z)^2} = \frac{y-x}{(y+z)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{z+y}{(x+z)^2} & \frac{-1}{(x+z)} \\ \frac{1}{(y+z)} & \frac{-(x+z)}{(y+z)^2} \end{vmatrix}$$

$$= \frac{-\cancel{(z+y)}}{(x+z)^2}\cdot\frac{\cancel{(x+z)}}{(y+z)^2} + \frac{1}{(x+z)}\cdot\frac{1}{(y+z)} = \frac{-1}{(x+z)(y+z)} + \frac{1}{(x+z)(y+z)} = 0$$

$|||^{ly}$ $\frac{\partial(u,v)}{\partial(y,z)} = 0$

and $\frac{\partial(u,v)}{\partial(z,x)} = 0$

$\therefore$ u and v are functionally dependent.

<u>Relation</u>;

let $u + \frac{1}{v} = \frac{x-y}{x+z} + \frac{1}{\frac{x+z}{y+z}} = \frac{x-y}{x+z} + \frac{y+z}{x+z}$

$= \frac{x - \cancel{y} + \cancel{y} + z}{x+z}$

$= \frac{x+z}{x+z} = 1$

$$\boxed{u + \frac{1}{v} = 1}$$

Ex:- Determine whether the following functions are functionally dependent, if so, find the relation betn them.

$u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

Soln: u & v are functionally dependent if $J = 0$

Now $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ ——→ (1)

let $u = \frac{x+y}{1-xy}$ $\left[\frac{u}{v} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2} \right]$

$\frac{\partial u}{\partial x} = \frac{(1-xy)(1+0) - (-y)(x+y)}{(1-xy)^2} = \frac{1 - \cancel{xy} + \cancel{yx} + y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$

$u = \frac{x+y}{1-xy}$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1)-(-x)(x+y)}{(1-xy)^2} = \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

Now $v = \tan^{-1}x + \tan^{-1}y$.

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$J = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} \begin{vmatrix} 1+y^2 & 1+x^2 \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2}\left[\frac{\cancel{1+y^2}}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{\cancel{1+x^2}}\right] = 0$$

$\therefore$ u and v are functionally dependent.

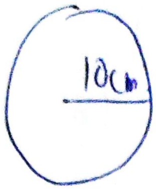
Relation: $v = \tan^{-1}x + \tan^{-1}y$.

$$v = \tan^{-1}\left[\frac{x+y}{1-xy}\right] = \tan^{-1}u.$$

$$\therefore \boxed{v = \tan^{-1}u}$$

Errors and Approximations :

Consider a circle of radius 10cm as shown in fig.



Area of circle $= A = \pi r^2$

$= \pi (10)^2$

$= 100\pi \text{ cm}^2$.

Now, there can be a human error while calculating the radius of circle, it can be positive or negative. lets say, radius calculated is 10.1cm. instead of 10cm.

$\therefore A = \pi r^2 = \pi (10.1)^2$

$= 102.01\pi \text{ cm}^2$.

$\therefore$ for $(10.1 - 10)$ cm $= 0.1$ cm error in radius, there will corresponding $(102.01\pi - 100\pi) = 2.01\pi \text{cm}^2$ error in area.

And this is called as error.

Note:- ① If dr is the actual error made in measurement of radius 'r', then the dA is the approximate error introduced in area 'A'.

$\frac{100 dr}{r}$ and $\frac{100 dA}{A}$ are known as % error in radius and area respectively.

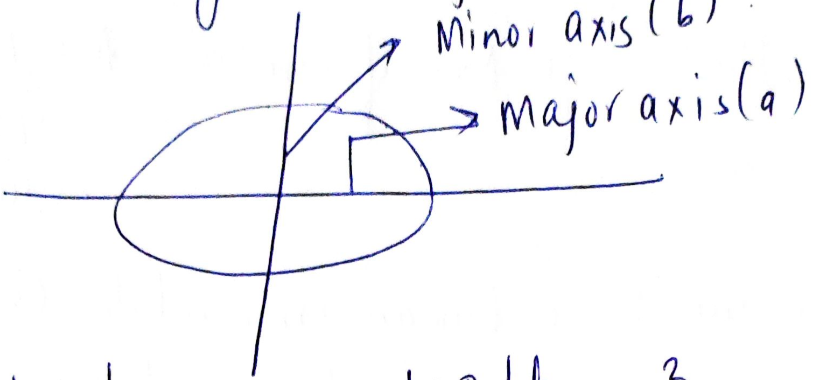
(i) dx, dy, dz are known as actual errors in x, y, z, resp.

(ii) $\frac{dx}{x}$, $\frac{dy}{y}$, $\frac{dz}{z}$ are known as relative error in x, y, z, respectively.

(iii) $\frac{100dx}{x}$, $\frac{100dy}{y}$, $\frac{100dz}{z}$ are known as Percentage errors in x, y, z, resp.

Ex: find the Percentaage error in the area of an ellipse when an error of 2% and 3% is made in measuring its major and minor axis

Soln



Given

$$\frac{100\,da}{a} = 2, \quad \frac{100\,db}{b} = 3$$

Area of the ellipse is given by,

$$A = \pi ab$$

Taking log on both sides

$$\log A = \log(\pi ab) = \log\pi + \log a + \log b$$

$$\left[\because \log(abc) = \log a + \log b + \log c\right]$$

Differentiation

$$\frac{1}{A}\,dA = 0 + \frac{1}{a}\,da + \frac{1}{b}\,db$$

Multiplying by 100

$$\frac{100\,dA}{A} = \frac{100\,da}{a} + \frac{100\,db}{b} = 2 + 3 = 5$$

$\therefore$ % error in area $A = 5\%$

Ex: In calculating the volume of a right circular cone, errors of 2% and 1% are made in measuring the height and radius of base resp.

Find the error in calculated volume.

Solⁿ: Let h = height

Given r = radius of base

$$\frac{100\,dh}{h} = 2; \quad \frac{100\,dr}{r} = 1$$

Volume V of right circular cone is given by,

$$V = \frac{1}{3}\pi r^2 h$$

Taking log on both sides,

$$\log V = \log\left[\frac{1}{3}\pi r^2 h\right]$$

$$\log V = \log\frac{1}{3} + \log\pi + \log r^2 + \log h$$

[By $\log m^n = n\log m$]

$$\log V = \log\frac{1}{3} + \log\pi + 2\log r + \log h$$

Differentiating,

$$\frac{1}{V}dV = 0 + 0 + 2\frac{1}{r}dr + \frac{1}{h}dh$$

Multiplying by 100

$$\frac{100\,dV}{V} = 2\frac{100\,dr}{r} + \frac{100\,dh}{h}$$

$$\therefore \frac{100\,dV}{V} = 2(1) + 2 = 4$$

$\therefore$ % error in volume $(V) = 4\%$

Ex: Find the possible percentage error in calculating parallel resistance 'r' of the three resistances r_1, r_2, r_3 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ if r_1, r_2, r_3 are each in error by plus 1.2%.

Soln: Given:

$$\frac{100\,dr_1}{r_1} = 1.2; \quad \frac{100\,dr_2}{r_2} = 1.2, \quad \frac{100\,dr_3}{r_3} = 1.2.$$

and
$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \longrightarrow (1).$$

Differentiating,

$$-\frac{1}{r^2}\,dr = -\frac{1}{r_1^2}\,dr_1 - \frac{1}{r_2^2}\,dr_2 - \frac{1}{r_3^2}\,dr_3 \quad \left(\because \frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}\right)$$

$$-\frac{1}{r}\cdot\frac{dr}{r} = -\left[\frac{1}{r_1}\frac{dr_1}{r_1} + \frac{1}{r_2}\frac{dr_2}{r_2} + \frac{1}{r_3}\frac{dr_3}{r_3}\right]$$

Multiplying by 100 on both sides.

$$\frac{1}{r}\frac{100\,dr}{r} = \frac{1}{r_1}\frac{100\,dr_1}{r_1} + \frac{1}{r_2}\frac{100\,dr_2}{r_2} + \frac{1}{r_3}\frac{100\,dr_3}{r_3}\Bigg]$$

$$\therefore \frac{1}{r}\frac{100\,dr}{r} = \frac{1}{r_1}(1.2) + \frac{1}{r_2}(1.2) + \frac{1}{r_3}(1.2)$$

$$\therefore \frac{1}{r}\cdot\frac{100\,dr}{r} = \left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right](1.2)$$

But from eqn (1). $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

So, $\frac{1}{\cancel{2}} \cdot \frac{100 ds}{s} = \frac{1}{\cancel{2}}(1.2) = 1.2)$

$\therefore$ % Error in s is 1.2%

Ex: The focal length of a mirror is found from the formula $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$. find the percentage error in f if u and v are both in error by p% each.

Soln: $\frac{100\,du}{u} = p$ and $\frac{100\,dv}{v} = p$ (Given)

and $\frac{1}{v} - \frac{1}{u} = \frac{2}{f} \longrightarrow (1)$.

Differentiating

$$-\frac{1}{v^2}dv + \frac{1}{u^2}du = -\frac{2}{f^2}df$$

$$\cancel{+}\left(\frac{1}{v}\cdot\frac{dv}{v} - \frac{1}{u}\cdot\frac{du}{u}\right) = \cancel{+}\frac{2}{f}\cdot\frac{df}{f}$$

Multiplying by 100

$$\frac{1}{v}\cdot\frac{100\,dv}{v} - \frac{1}{u}\frac{100\,du}{u} = \frac{2}{f}\cdot\frac{100\,df}{f}$$

$$\frac{1}{v}(p) - \frac{1}{u}(p) = \frac{2}{f}\cdot\frac{100\,df}{f} \quad \ldots \text{(Given)}$$

$$p\left(\frac{1}{v} - \frac{1}{u}\right) = \frac{2}{f}\cdot\frac{100\,df}{f}$$

But from eqn (1). $p\left(\frac{\cancel{2}}{\cancel{f}}\right) = \frac{\cancel{2}}{\cancel{f}}\cdot\frac{100\,df}{f} \Rightarrow p = \frac{100\,df}{f}$

% error in f is p%

Ex:- If $x = u(1-v)$ and $y = uv$, then Prove that $JJ' = 1$.

Soln: Given $x = u(1-v)$ and $y = uv$

Part 1: To find J

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + uv$$

$$= u - uv + uv = u = J$$

Part 2: To find J' (Reciprocal of J)

Given $x = u(1-v)$ and $y = uv$

$$\frac{1}{J} = J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{1}{u}$$

$$JJ' = 1 = u \cdot \frac{1}{u} = 1$$ Hence Proved.

Note: For J', we will always follow the method of functions as explained in this example.

Given $x = u(1-v)$ and $y = uv$

$$\therefore f_1 = x - u(1-v) = x - u + uv$$

$$f_2 = y - uv = y - uv$$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1, f_2)}{\partial(x,y)}}{\frac{\partial(f_1, f_2)}{\partial(u,v)}} = \frac{N}{D}$$

$$\therefore J' = \frac{N}{D} \longrightarrow (1)$$

$$N = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

and

$$D = \frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -1+v & u \\ -v & -u \end{vmatrix}$$

$$= -(1+v)(-u) - u(-v)$$

$$= u + \cancel{uv} + \cancel{uv} = u$$

from eqn (1)

$$\therefore J' = \frac{N}{D} = \frac{1}{u} =$$

Now LHS $= JJ' = u \cdot \frac{1}{u} = 1 =$ R.HS

Hence proved.

Ex: If $x = uv$ and $y = \frac{u+v}{u-v}$, prove that, $\frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}$

Soln: Given $x = uv$ and $y = \frac{u+v}{u-v}$.

$f_1 : x - uv$

$$y(u-v) = u+v$$

$$yu - yv = u+v$$

$$f_2 : yu - yv - u - v$$

Now $\frac{\partial(u,v)}{\partial(x,y)} = (-1)^{1} \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$

$\frac{\partial(u,v)}{\partial(x,y)} = \frac{N}{D}$

where $N = \frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & u-v \end{vmatrix}$

$N = u-v$

and $D = \frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -v & -u \\ y-1 & -y-1 \end{vmatrix}$

$D = (-v)(-y-1) - (-u)(y-1)$

$D = vy + v + uy - u$

Now $LHS = \frac{\partial(u,v)}{\partial(x,y)} = \frac{N}{D} = \frac{u-v}{vy+v+uy-u}$

But $y = \frac{u+v}{u-v}$ (Given)

$= \frac{u-v}{v\left(\frac{u+v}{u-v}\right) + v + u\left(\frac{u+v}{u-v}\right) - u}$

$= \frac{u-v}{\frac{v(u+v)+v(u-v)+u(u+v)-u(u-v)}{u-v}}$

$$= \frac{(u-v)^2}{uv+\cancel{v^2}+uv-\cancel{v^2}+\cancel{u^2}+uv-\cancel{u^2}+uv}$$

$$= \frac{(u-v)^2}{4uv} = \text{R.H.S} \quad \text{Hence proved}$$

→ If $x = v^2+w^2$; $y = w^2+u^2$, $z = u^2+v^2$, then Prove that $JJ' = 1$

Sol^n: To find J

Given: $x = v^2+w^2$, $y = w^2+u^2$, $z = u^2+v^2$

$$\therefore J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

Taking common, $2u$ from 1st col^n

$2v$ from 2nd column

$2w$ from 3rd column.

$$\therefore J = (2u)(2v)(2w) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 8uvw\left(0 - 1(a-1) + 1(1)\right) = 8uvw(2)$$

$$J = 16uvw$$

To find J'

Given $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$

$$f_1 = x - v^2 - w^2$$

$$f_2 = y - w^2 - u^2$$

$$f_3 = z - u^2 - v^2$$

$$\therefore \quad J' = \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$J' = \frac{-N}{D}$$

when $N = \dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$

$$N = \begin{vmatrix} f_{1x} & f_{1y} & f_{1z} \\ f_{2x} & f_{2y} & f_{2z} \\ f_{3x} & f_{3y} & f_{3z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$N = 1$$

& $D = \dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} f_{1u} & f_{1v} & f_{1w} \\ f_{2u} & f_{2v} & f_{2w} \\ f_{3u} & f_{3v} & f_{3w} \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2v & -2w \\ -2u & 0 & -2w \\ -2u & -2v & 0 \end{vmatrix} = (-2u)(-2v)(-2w) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -8uvw\left[0 - 1(0-1) + 1(1)\right]$$

$$= -8uvw(2) = -16uvw$$

But from eqn ①.

$$J' = -\frac{N}{D} = \frac{-1}{-16uvw} = \frac{1}{16uvw}.$$

$$\text{LHS } JJ' = 16uvw \times \frac{1}{16uvw} = 1$$ Hence Proved.