Unit III

Partial Differentiation & Applications.

Function of two or more variables –

if z has one definite value for each pair of values of x & y, then z is called a function of two variables x & y. denote it by $z = f(x,y)$ or $F(x,y)$

Partial Derivatives –

① let z be a f^n of two variables x & y i.e $z = f(x,y)$

partial derivative of z w.r.t x is nothing but ordinary derivative of z w.r.t x keeping other variables constant

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \left\{ \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \right\}$$

denoted by $\frac{\partial z}{\partial x} = z_x$ or f_x

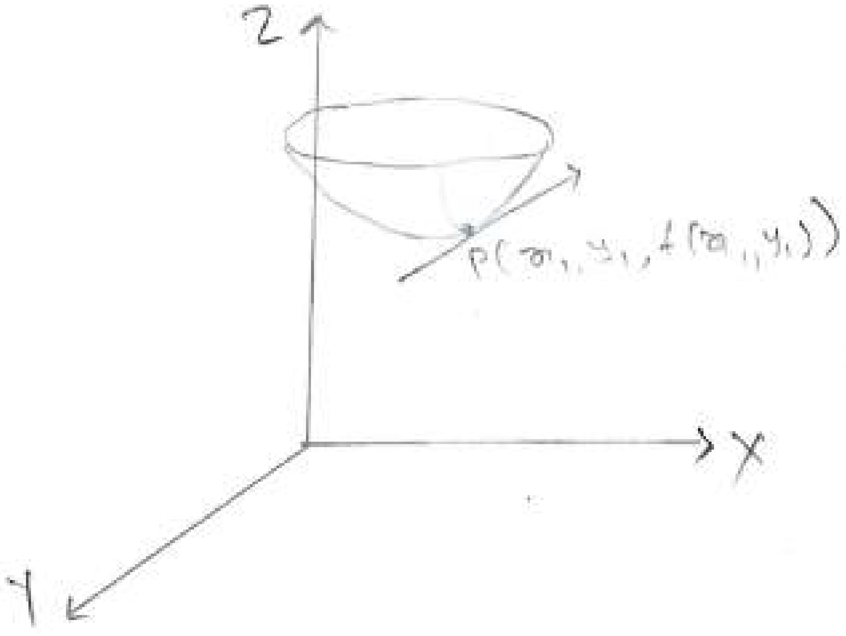
$||^{ly}$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \left\{ \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \right\} = z_y \text{ or } f_y$$

* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are called first order partial derivatives of z.

* if f is a function of n variables $x_1, \ldots, x_n$ then partial derivative of f w.r.t x_1 is ordinary derivative of f wrt x_1 keeping all other variables constant. denoted by $\frac{\partial f}{\partial x_1}$

Geometrical interpretation of z_x, z_y.



The function $z = f(x,y)$ represent a surface

$\frac{\partial z}{\partial x}$ gives the slope of the tangent drawn to the curve of intersection of the surface

$z = f(x,y)$ with a plane parallel to the zox plane

$\frac{\partial z}{\partial y}$ gives the slope of the tangent drawn to curve of intersection of the surface $z = f(x,y)$ with a plane parallel to zoy plane.

Rules of Partial Differentiation.

let u, v be two functions of two independent variables x, y then

① $$\frac{\partial}{\partial x}(u \pm v) = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial y}(u \pm v) = \frac{\partial u}{\partial y} \pm \frac{\partial v}{\partial y}$$

② $$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

③ $$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}, \quad v \neq 0$$

(4) If k is constant,

$$\frac{\partial}{\partial x}(ku) = k\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}(ku) = k\frac{\partial u}{\partial y}$$

(5) $\frac{\partial}{\partial x}(k) = 0$, $\frac{\partial}{\partial y}(k) = 0$

(6) $\frac{\partial}{\partial x}(f(y)) = 0$ ($\because$ $f(y)$ does not contain any x)

$$\frac{\partial}{\partial y}(\phi(x)) = 0$$

(7) $\frac{\partial}{\partial x}[f(x,y,z)]^n = n[f(x,y,z)]^{n-1}\frac{\partial f}{\partial x}$

$$\frac{\partial}{\partial x}\left[\frac{1}{f(x,y,z)}\right] = \frac{-1}{[f(x,y,z)]^2}\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial x}\left[\sqrt{f(x,y,z)}\right] = \frac{1}{2\sqrt{f(x,y,z)}}\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial x}[\log(f(x,y,z))] = \frac{1}{f(x,y,z)}\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial x}\left[e^{f(x,y,z)}\right] = e^{f(x,y,z)}\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial x}\left[a^{f(x,y,z)}\right] = a^{f(x,y,z)}\log a\frac{\partial f}{\partial x}$$

(8) $$\frac{\partial}{\partial x}\left[\sin f(x,y,z)\right] = \cos f(x,y,z)\cdot\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial x}\left[\cos(f(x,y,z))\right] = -\sin f(x,y,z)\cdot\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial x}\left[\tan(f(x,y,z))\right] = \sec^2 f(x,y,z)\cdot\frac{\partial f}{\partial x}$$

(9) $$\frac{\partial}{\partial x}\sin^{-1} f(x,y,z) = \frac{1}{\sqrt{1-[f(x,y,z)]^2}}\cdot\frac{\partial f}{\partial x}$$

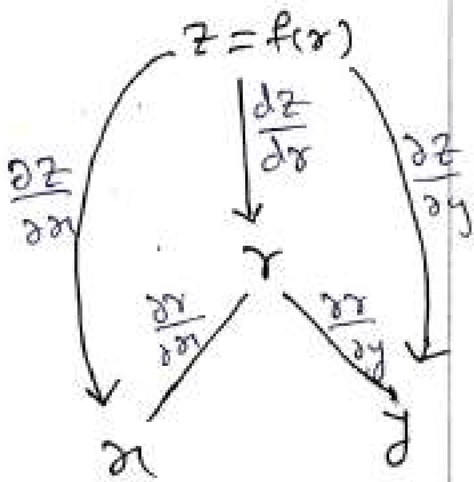
$$\frac{\partial}{\partial x}\tan^{-1} f(x,y,z) = \frac{1}{1+[f(x,y,z)]^2}\cdot\frac{\partial f}{\partial x}$$

Derivative of composite function —

if $z=f(r)$ where r is again function of two variables x & y then z becomes composite function of x & y.

$$\frac{\partial z}{\partial x} = \frac{dz}{dr}\cdot\frac{\partial r}{\partial x} = f'(r)\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{dr}\cdot\frac{\partial r}{\partial y} = f'(r)\cdot\frac{\partial r}{\partial y}$$



Partial derivative of Higher order

if $z = f(x,y)$, then the first order partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ may also be function of x & y. Hence can be differentiated again

Thus there are 4 partial derivatives of 2nd order of a fn of two variables

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z_{xx}$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial^2 z}{\partial y \partial x} = z_{yx} \\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial^2 z}{\partial x \partial y} = z_{xy} \end{aligned}\right\} \text{mixed partial derivatives.}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

Commutative property of mixed partial Derivative.

if $z = f(x,y)$ is continuous & possesses continuous partial derivatives then second order mixed partial derivative follows commutative property.

Q) if $u = \log(x^3+y^3-x^2y-xy^2)$

Prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \frac{-4}{(x+y)^2}$

$\longrightarrow$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right) \quad -(*)$$

$$u = \log(x^3+y^3-x^2y-xy^2)$$

$$= \log[x^3-x^2y+y^3-xy^2]$$

$$= \log[x^2(x-y)+y^2(y-x)]$$

$$= \log[(x^2-y^2)(x-y)]$$

$$= \log[(x-y)(x-y)(x+y)]$$

$$= \log[(x-y)^2(x+y)]$$

$$u = 2\log(x-y)+\log(x+y) \quad -(1)$$

$$\frac{\partial u}{\partial x} = \frac{2}{x-y}(1) + \frac{1}{x+y} \quad -(2)$$

$$\frac{\partial u}{\partial y} = \frac{2}{x-y}(-1) + \frac{1}{x+y} \quad -(3)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = \frac{2}{x+y} \quad -(4)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Q1) if $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$ find u_{xy}

$\longrightarrow$

$$u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

diff u wrt y

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 2y\tan^{-1}\frac{x}{y} - y^2\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial u}{\partial y} = x - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\frac{x}{y}\right]$$

$$= 1 - 2y\left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(\frac{1}{y}\right) = \frac{x^2-y^2}{x^2+y^2}$$

$\therefore$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{2}{x+y}\right)$$

$$=\frac{-2}{(x+y)^2}-\frac{2}{(x+y)^2}$$

$$=\frac{-4}{(x+y)^2}$$

Q) if $u=\log(x^3+y^3+z^3-3xyz)$

Prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2}$

$\longrightarrow$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}$$

$$v=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$$

$$u = \log(x^3+y^3+z^3-3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$\therefore v = \frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$v = \frac{3}{(x+y+z)}$$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial z} = \frac{-3}{(x+y+z)^2}$$

$$\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Q) if $v = (x^2 - y^2) f(xy)$ show that

$v_{xx} + v_{yy} = (x^4 - y^4) f''(xy)$

$\longrightarrow$

$v = (x^2 - y^2) f(xy)$

$$\frac{\partial v}{\partial x} = 2x f(xy) + (x^2 - y^2) f'(xy) . y$$

$$\frac{\partial^2 v}{\partial x^2} = 2f(xy) + 2x f(xy) . y$$
$$+ y\left[2x f'(xy) + (x^2 - y^2) f''(xy) . y\right]$$

$$= 2f(xy) + 4xy f'(xy) + (x^2 - y^2) f''(xy) . y^2$$

$$\frac{\partial v}{\partial y} = -2y f(xy) + (x^2 - y^2) f'(xy) . x$$

$$\frac{\partial^2 v}{\partial y^2} = -2 f(xy) - 2y f'(xy) x + x\left[-2y f'(xy)\right.$$
$$\left. + (x^2 - y^2) f''(xy) . x\right]$$

$$= -2f(xy) - 4xy f'(xy) + x^2 (x^2 - y^2) f''(xy)$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (x^2 + y^2)(x^2 - y^2) f''(xy)$$
$$= (x^4 - y^4) f''(xy)$$

Q) find value of n so that

$u = r^n(3\cos^2\theta - 1)$ satisfies the partial

differential equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = 0$$

$\longrightarrow$

$$u = r^n(3\cos^2\theta - 1) \quad \text{— (1)}$$

$$\frac{\partial u}{\partial r} = n r^{n-1}(3\cos^2\theta - 1)$$

$$\frac{\partial u}{\partial r} = n\frac{u}{r} \quad \text{— (2)}$$

$$r^2 \frac{\partial u}{\partial r} = n.ur$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}(n.ur) = nu + nr\frac{\partial u}{\partial r}$$

$$= nu + nr\left(\frac{nu}{r}\right)$$

$$= nu + n^2 u \quad \text{— (3)}$$

$$\frac{\partial u}{\partial\theta} = r^n[6\cos\theta(-\sin\theta)]$$

$$= -6r^n\cos\theta.\sin\theta$$

$$\sin\theta.\frac{\partial u}{\partial\theta} = -6r^n\cos\theta.\sin^2\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6r^n[-\sin^3\theta + 2\cos^2\theta\sin\theta]$$

$$= -6r^n\sin\theta[2\cos^2\theta - (1-\cos^2\theta)]$$

$$= -6r^n\sin\theta[3\cos^2\theta - 1]$$

$$= -6\sin\theta \cdot u$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6u$$

$$\therefore \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

$$= nu + n^2u - 6u$$

$$= (n^2+n-6)u = 0$$

$$\therefore n^2+n-6=0$$

$$\therefore n = 2, -3$$

2) $z = \tan(y+ax) + (y-ax)^{3/2}$

find the value of $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2}$

$\longrightarrow$

$$z = \tan(y+ax) + (y-ax)^{3/2}$$

$$\frac{\partial z}{\partial x} = a\sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}(-a)$$

$$\frac{\partial^2 z}{\partial x^2} = 2a^2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}a^2(y-ax)^{-1/2}$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}$$

$$a^2\frac{\partial^2 z}{\partial y^2} = 2a^2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}a^2$$

$$\therefore \frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$$

Q) if $z^3 - xz - y = 4$ find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$\longrightarrow$

$$z^3 - xz - y = 4 \quad \text{—— (1)}$$

partially diff wrt x

$$3z^2 \frac{\partial z}{\partial x} - \left(x \frac{\partial z}{\partial x} + z\right) + 0 = 0$$

$$3z^2 \frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x} - z = 0$$

$$\therefore \frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

Partially diff (1) wrt y

$$3z^2 \frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1 = 0$$

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$$

Q) if $z^3 - zx - y = 0$ prove that

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(3z^2 + x)}{(3z^2 - x)^3}$$

$\longrightarrow$

$$z^3 - zx - y = 0$$

$$3z^2 \frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1 = 0 \qquad \therefore \frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$$

$$3z^2 \frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x} - z = 0 \qquad \therefore \frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{1}{3z^2 - x}\right) = \frac{-1}{(3z^2 - x)^2}\left(6z\frac{\partial z}{\partial x} - 1\right)$$

$$= \frac{-1}{(3z^2 - x)^2}\left[\frac{6z^2}{3z^2 - x} - 1\right]$$

$$= -\frac{(3z^2 + x)}{(3z^2 - x)^3}$$

Q) find the value of n for which $u = Ae^{-gx}\sin(nt - gx)$ satisfies the partial differential equation

$\frac{\partial u}{\partial t} = m\frac{\partial^2 u}{\partial x^2}$ where A, g are const.

$\longrightarrow$

$$u = Ae^{-gx}\sin(nt - gx) \quad -(1)$$

$$\frac{\partial u}{\partial t} = Ae^{-gx}\cos(nt - gx)\cdot n \quad -(2)$$

$$\frac{\partial u}{\partial x} = -Age^{-gx}\cancel{\sin(n}\sin(nt - gx) + Ae^{-gx}\cos(nt - gx)\cdot(-g)$$

$$= -\cancel{Ag\{} \; -Age^{-gx}\left[\sin(nt - gx) + \cos(nt - gx)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = Ag^2e^{-gx}\left[\sin(nt - gx) + \cos(nt - gx)\right] - Age^{-gx}\left[\cos(nt - gx)\cdot(-g) - \sin(nt - gx)(-g)\right]$$

$$= Ag^2e^{-gx}\left[\cancel{\sin(nt - gx)} + \cos(nt - gx) + \cos(nt - gx) - \cancel{\sin(nt - gx)}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 2Ag^2 e^{-gx} \cos(nt - gx)$$

$$\therefore \frac{m\partial^2 u}{\partial x^2} = 2mg^2 A e^{-gx} \cos(nt - gx) \quad — ③$$

$$\therefore \frac{\partial u}{\partial t} = \frac{m\partial^2 u}{\partial x^2}$$

$$\Rightarrow A e^{-gx} \cos(nt - gx) \cdot n = 2mg^2 A e^{-gx} \cos(nt - gx)$$

$$\therefore n = 2mg^2$$

Q) if $u = t^n e^{-r^2/4kt}$, find n for which

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$$

$\longrightarrow$

$$u = t^n e^{-r^2/4kt} \quad — ①$$

$$\frac{\partial u}{\partial t} = n t^{n-1} \cdot e^{-r^2/4kt} + t^n \cdot e^{-r^2/4kt} \left[\frac{-r^2}{4k} \times \frac{-1}{t^2}\right]$$

$$\frac{\partial u}{\partial t} = \frac{nu}{t} + \frac{r^2}{4t^2 k} u \quad — ②$$

$$\frac{\partial u}{\partial r} = t^n e^{-r^2/4kt} \cdot \left[\frac{-1}{4kt} \times 2r\right]$$

$$= -\frac{u}{4kt} 2r$$

$$\frac{\partial u}{\partial r} = \frac{-2u}{4kt} \cdot r \quad \Rightarrow \frac{2}{r}\frac{\partial u}{\partial r} = -\frac{4u}{4kt} \quad \text{--- (3)}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{-2}{4kt}\left[r\frac{\partial u}{\partial r} + u\right]$$

$$= \frac{-2}{4kt}\left[r \cdot \left[\frac{-2u}{4kt} \cdot r\right] + u\right]$$

$$= \frac{-2}{4kt}\left[\frac{-2ur^2}{4kt} + u\right]$$

$$= \frac{4ur^2}{16k^2t^2} - \frac{2u}{4kt} \quad \text{--- (4)}$$

$$\cancel{= \frac{4ur^2}{16k^2t^2}}$$

$\therefore$

$$\frac{nu}{t} + \frac{r^2u}{4t^2k} = k\left[\frac{4ur^2}{16k^2t^2} - \frac{2u}{4kt} - \frac{4u}{4kt}\right]$$

$$\frac{nu}{t} + \frac{r^2u}{4t^2k} = \frac{ur^2}{4t^2k} - \frac{6u}{4t}$$

$$\frac{nu}{t} = -\frac{6u}{4t} \quad \Rightarrow \quad n = -2/3$$

a) Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

① $u = \tan(y+ax) - (y-ax)^{3/2}$

$\rightarrow$ $\frac{\partial u}{\partial x} = \sec^2(y+ax).a - \frac{3}{2}(y-ax)^{1/2}(-a)$

$\frac{\partial u}{\partial y \partial x} = 2\sec^2(y+ax).a.\tan(y+ax) + \frac{3a}{4}(y-ax)^{-1/2}$

$\frac{\partial u}{\partial y} = \sec^2(y+ax) - \frac{3}{2}(y-ax)^{1/2}$

$\frac{\partial u}{\partial x \partial y} = 2\sec^2(y+ax)\tan(y+ax).a + \frac{3a}{4}(y-ax)^{-1/2}$

$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

② $\tan^{-1}\left(\frac{x}{y}\right)$

$\rightarrow$ $u = \tan^{-1}\left(\frac{x}{y}\right)$

$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial y \partial x} = \frac{(x^2+y^2).1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{y^2} = \frac{-x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x \partial y} = \frac{(x^2+y^2)(-1)+x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

(3) $u = \log\left(\frac{x^2+y^2}{xy}\right)$

$\rightarrow$ $u = \log(x^2+y^2) - \log(xy)$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{y}{xy} = \frac{x^2-y^2}{x(x^2+y^2)}$$

$$\frac{\partial u}{\partial y \partial x} = \frac{x(x^2+y^2)(2x) - (x^2-y^2)[3x^2+y^2]}{x^2(x^2+y^2)^2}$$

$$= \cancel{2x^2(x^2+y^2) - 2x}$$

$$= \frac{2x^4 + 2x^2y^2 - 3x^4 - x^2y^2 + 3x^2y^2 + y^4}{x^2(x^2+y^2)^2}$$

$=$

$$u = \log(x^2+y^2) - \log(xy)$$

$$u = \log(x^2+y^2) - \log x - \log y$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x}$$

$$\frac{\partial u}{\partial y \partial x} = 2x\left[\frac{-1}{(x^2+y^2)^2} \times 2y\right]$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2y\left[\frac{-1}{(x^2+y^2)^2} \times 2x\right] = \frac{-4xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

4) $u = 3xy - y^3 + (y^2-2x)^{3/2}$

$$\rightarrow \frac{\partial u}{\partial x} = 3y + \frac{3}{2}(y^2-2x)^{1/2}(-2)$$

$$\frac{\partial u}{\partial x} = 3y - 3(y^2-2x)^{1/2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = 3 - \frac{3}{2}(y^2-2x)^{-1/2} \cdot 2y$$

$$= 3 - 3y(y^2-2x)^{-1/2}$$

$$\frac{\partial u}{\partial y} = 3x - 3y^2 + \frac{3}{2}(y^2-2x)^{1/2}(2y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = 3 + \frac{3y}{2}(y^2-2x)^{-1/2}(-2)$$

$$= 3 - 3y(y^2-2x)^{-1/2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Examples on Composite function

Q) if $u = f(r)$ where $r = \sqrt{x^2+y^2}$

Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

$$\begin{tikzcd} & u \arrow[d] & \\ & r \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

$\longrightarrow$

$$\frac{\partial u}{\partial x} = \frac{du}{dr}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{du}{dr}\frac{\partial r}{\partial y}$$

$$r = \sqrt{x^2+y^2}$$

$$r^2 = x^2+y^2$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial u}{\partial x} = \frac{du}{dr}\frac{\partial r}{\partial x} = f'(r)\cdot\frac{x}{r}$$

$$\frac{\partial u}{\partial y} = \frac{du}{dr}\frac{\partial r}{\partial y} = f'(r)\frac{y}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(f'(r)\cdot\frac{x}{r}\right)$$

$$= f'(r)\cdot\left[\frac{r. - x\frac{\partial r}{\partial x}}{r^2}\right] + \frac{x}{r}f''(r)\frac{\partial r}{\partial x}$$

$$= f'(r)\left[\frac{r - x.\frac{x}{r}}{r^2}\right] + \frac{x}{r}f''(r)\cdot\frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left[\frac{r^2 - x^2}{r^3}\right] + \frac{x^2}{r^2}f''(r)$$

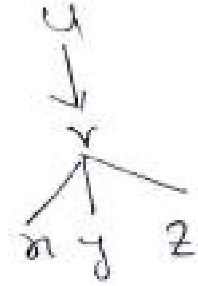
||ly $\frac{\partial^2 u}{\partial y^2} = f'(r)\left[\frac{r^2 - y^2}{r^3}\right] + \frac{y^2}{r^2}f''(r)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}\left[r^2 - (x^2+y^2)\right] + \frac{f''(r)}{r^2}(x^2+y^2)$$

$$= f''(r) + \frac{1}{r}f'(r)$$

Q) $u = f(r)$, $r = \sqrt{x^2+y^2+z^2}$ Prove that

$$u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$$



$\longrightarrow$

$$\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x}$$

$$= f'(r) \cdot \frac{x}{r}$$

$$\therefore u_{xx} = f'(r) \left\{ \frac{r \cdot 1 - x \frac{\partial r}{\partial x}}{r^2} \right\} + \frac{x}{r} f''(r) \cdot \frac{\partial r}{\partial x}$$

$$= f'(r) \left\{ \frac{r - x \cdot \frac{x}{r}}{r^2} \right\} + \frac{x^2}{r^2} f''(r)$$

$$u_{xx} = f'(r) \left\{ \frac{r^2 - x^2}{r^3} \right\} + \frac{x^2}{r^2} f''(r)$$

$$\text{||ly} \quad u_{yy} = f'(r) \left\{ \frac{r^2 - y^2}{r^3} \right\} + \frac{y^2}{r^2} f''(r)$$

$$u_{zz} = f'(r) \left\{ \frac{r^2 - z^2}{r^3} \right\} + \frac{z^2}{r^2} f''(r)$$

$$\therefore u_{xx} + u_{yy} + u_{zz} = \frac{f'(r)}{r^3} \left[3r^2 - (x^2+y^2+z^2) \right]$$

$$+ \frac{f''(r)}{r^2} (x^2+y^2+z^2)$$

$$= f''(r) + \frac{2}{r} f'(r)$$

Q) $u = \log\sqrt{x^2+y^2+z^2}$ show that

$$(x^2+y^2+z^2)(u_{xx}+u_{yy}+u_{zz}) = 1$$

$\longrightarrow$ let $x^2+y^2+z^2 = r^2$

$$\therefore u = \log r \quad -(1)$$

u
$|$
r
$x \quad y \quad z$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r} \quad -(2)$$

$$u_x = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\cdot\frac{x}{r} = \frac{x}{r^2}$$

$$u_{xx} = \frac{r^2.1 - x\,2r\frac{\partial r}{\partial x}}{r^4}$$

$$u_{xx} = \frac{r^2 - 2xr.\frac{x}{r}}{r^4} = \frac{r^2-2x^2}{r^4}$$

$\text{"}|^{\text{ly}}$ $u_{yy} = \dfrac{r^2-2y^2}{r^4}$, $u_{zz} = \dfrac{r^2-2z^2}{r^4}$

$$\therefore u_{xx}+u_{yy}+u_{zz} = \frac{3r^2-2(x^2+y^2+z^2)}{r^4}$$

$$= \frac{3r^2-2r^2}{r^4}$$

$\therefore u_{xx} + u_{yy} + u_{zz} = \frac{1}{r^2}$

$\therefore r^2(u_{xx} + u_{yy} + u_{zz}) = 1$

$\therefore (x^2+y^2+z^2)(u_{xx} + u_{yy} + u_{zz}) = 1$

Q) $u = r^m$, $r = \sqrt{x^2+y^2+z^2}$ find $u_{xx} + u_{yy} + u_{zz}$

$\longrightarrow$

$$r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$u = r^m$

$u_x = mr^{m-1} \cdot \frac{\partial r}{\partial x} = mr^{m-1} \cdot \frac{x}{r}$

$u_x = \frac{mxr^m}{r^2} = mxr^{m-2}$

$u_{xx} = m[r^{m-2} + (m-2)x^2r^{m-4}]$

$u_{xx} = mr^{(m-2)}\left[1 + \frac{x^2}{r^2}(m-2)\right]$

$u_{yy} = mr^{(m-2)}\left[1 + \frac{y^2}{r^2}(m-2)\right]$

$$u_{zz} = m r^{(m-2)}\left[1 + \frac{z^2}{r^2}(m-2)\right]$$

$\therefore u_{xx} + u_{yy} + u_{zz}$

$$= m r^{(m-2)}\left[3 + \frac{(m-2)}{r^2}(x^2+y^2+z^2)\right]$$

$$= m r^{(m-2)}\left[3 + \frac{(m-2)}{r^2} \cdot r^2\right]$$

$$= m(m+1) r^{(m-2)}$$

$$= \frac{m(m+1)}{r^2} r^m$$

$$= \frac{m(m+1)u}{(x^2+y^2+z^2)}$$

Examples on Variable to be treated as constant

<u>Notation</u> – $\left(\frac{\partial r}{\partial x}\right)_\theta$ means the partial derivative of r with respect to x treating θ constant in a relation expressing r as a function of r & θ only.

when no indication is given regarding the variable to be kept constant then by convention $\frac{\partial}{\partial x}$ always means $\left(\frac{\partial}{\partial x}\right)_y$.

& $\frac{\partial}{\partial y}$ means $\left(\frac{\partial}{\partial y}\right)_x$.

Q) $u = ax + by$ $\quad v = bx - ay$

find the value of $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u$

$\longrightarrow$

$$u = ax + by$$

$$\left(\frac{\partial u}{\partial x}\right)_y = a \quad - (1)$$

$v = bx - ay$

$ay = bx - v$

$y = \frac{b}{a}x - \frac{v}{a}$ $\quad \therefore \left(\frac{\partial y}{\partial v}\right)_x = \frac{-1}{a} \quad - (2)$

to find $\left(\frac{\partial x}{\partial u}\right)_v$ express x in terms of u & v

$$\therefore \frac{u-ax}{b} = \frac{bx-v}{a}$$

$$\Rightarrow au - a^2x = b^2x - bv$$

$$\Rightarrow x = \frac{au+bv}{a^2+b^2}$$

$$\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{a^2+b^2} \quad \text{— (3)}$$

Ily to find $\left(\frac{\partial v}{\partial y}\right)_u$ eliminate x

$$\frac{u-by}{a} = \frac{v+ay}{b}$$

$$\Rightarrow bu - b^2y = av + a^2y$$

$$\therefore v = \frac{bu-(a^2+b^2)y}{a}$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_u = -\left(\frac{a^2+b^2}{a}\right) \quad \text{— (4)}$$

$\therefore$ from (1) (2) (3) (4)

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = a\left(\frac{a}{a^2+b^2}\right)\left(\frac{-1}{a}\right)\left(-\frac{a^2+b^2}{a}\right)$$

$$= 1$$

Q) if $x = \frac{\cos\theta}{u}$; $y = \frac{\sin\theta}{u}$

evaluate $\left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial y}\right)_x$

$\longrightarrow$

$x = \frac{\cos\theta}{u} \quad \therefore \left(\frac{\partial x}{\partial u}\right)_\theta = \frac{-\cos\theta}{u^2}$

$y = \frac{\sin\theta}{u} \quad \therefore \left(\frac{\partial y}{\partial u}\right)_\theta = -\frac{\sin\theta}{u^2}$

to find $\left(\frac{\partial u}{\partial x}\right)_y$ we eliminate θ from given relations

$x^2 + y^2 = \frac{1}{u^2}$

$\therefore u^2 = \frac{1}{x^2+y^2}$

$\therefore 2u\left(\frac{\partial u}{\partial x}\right)_y = \frac{-2x}{(x^2+y^2)^2} \quad \therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{-x}{u(x^2+y^2)^2}$

$\& \; 2u\left(\frac{\partial u}{\partial y}\right)_x = \frac{-2y}{(x^2+y^2)^2} \quad \therefore \left(\frac{\partial u}{\partial y}\right)_x = \frac{-y}{u(x^2+y^2)^2}$

$\therefore$

$\left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial y}\right)_x$

$= \left(\frac{-\cos\theta}{u^2}\right)\left(\frac{-x}{u(x^2+y^2)^2}\right) + \left(-\frac{\sin\theta}{u^2}\right)\left(\frac{-y}{u(x^2+y^2)^2}\right)$

$= \frac{x\cos\theta + y\sin\theta}{u^3(x^2+y^2)^2} = \frac{1}{u^4(x^2+y^2)^2} = 1$

Q) if $x = \frac{r}{2}(e^{\theta} + e^{-\theta})$, $y = \frac{r}{2}(e^{\theta} - e^{-\theta})$

then show that $\left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}$

$\longrightarrow$

$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$

$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

$\therefore x = r\cosh\theta$, $y = r\sinh\theta$

$\left(\frac{\partial x}{\partial r}\right)_{\theta} = \cosh\theta$

$r^2 = x^2 - y^2$

$\therefore 2r\left(\frac{\partial r}{\partial x}\right)_{y} = 2x$

$\therefore \left(\frac{\partial r}{\partial x}\right)_{y} = \frac{x}{r} = \cosh\theta$

$\therefore \left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}$.

Q) if $x = r\cos\theta$, $y = r\sin\theta$

① $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}$

$\longrightarrow$ ② $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1$

③ $r_{xx}\, r_{yy} = (r_{xy})^2$

⑤ $\left(\dfrac{\partial y}{\partial r}\right)_x \cdot \left(\dfrac{\partial y}{\partial r}\right)_\theta = 1$

$\longrightarrow$

$x = r\cos\theta$, $y = r\sin\theta$

$$\begin{array}{c} r \\ x \quad y \end{array}$$

$\therefore r^2 = x^2 + y^2$

$\therefore 2r \dfrac{\partial r}{\partial x} = 2x$

$\therefore \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ — ①

$2r \dfrac{\partial r}{\partial y} = 2y$

$\therefore \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ — ②

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - x\dfrac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x.\dfrac{x}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$

(1) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}$

$= \frac{y^2}{r^3}+\frac{x^2}{r^3}$

$= \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$

$\therefore \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}$

(2) $\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2}+\frac{y^2}{r^2} = \frac{x^2+y^2}{r^2} = 1$

(3) $\frac{\partial r}{\partial x} = \frac{x}{r}$

$\therefore \frac{\partial^2 r}{\partial x \partial y} = x\left(\frac{-1}{r^2}\cdot\frac{\partial r}{\partial y}\right)$

$r_{xy} = -\frac{x}{r^2}\times\frac{y}{r} = \frac{-xy}{r^3}$

$\therefore (r_{xy})^2 = \frac{(xy)^2}{r^6}$

$r_{xx}\cdot r_{yy} = \frac{y^2}{r^3}\cdot\frac{x^2}{r^3} = \frac{(xy)^2}{r^6}$

$\therefore r_{xx} r_{yy} = (r_{xy})^2$

(5) $r^2 = x^2 + y^2$

$\therefore y^2 = r^2 - x^2$

$\therefore 2y\left(\frac{\partial y}{\partial r}\right)_x = 2r$

$\therefore \left(\frac{\partial y}{\partial r}\right)_x = \frac{r}{y} = \frac{r}{r\sin\theta} = \frac{1}{\sin\theta}$

$\partial y = r\sin\theta$

$\left(\frac{\partial y}{\partial r}\right)_\theta = \sin\theta$

$\therefore \left(\frac{\partial y}{\partial r}\right)_x \cdot \left(\frac{\partial y}{\partial r}\right)_\theta = \sin\theta \times \frac{1}{\sin\theta} = 1$

Q) if $x^2 = a\sqrt{u} + b\sqrt{v}$ &
$y^2 = a\sqrt{u} - b\sqrt{v}$ prove that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

$\longrightarrow$

$$x^2 = a\sqrt{u} + b\sqrt{v} \quad \text{— (1)}$$
$$y^2 = a\sqrt{u} - b\sqrt{v} \quad \text{— (2)}$$

diff (1) w.r.t u

$$2x\left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2\sqrt{u}}$$

$$\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{4x\sqrt{u}} \quad \text{— (3)}$$

$$\& \quad 2y\left(\frac{\partial y}{\partial v}\right)_u = \frac{-b}{2\sqrt{v}}$$

$$\therefore \left(\frac{\partial y}{\partial v}\right)_u = \frac{-b}{4y\sqrt{v}} \quad \text{— (4)}$$

Now (1) + (2)

$$x^2 + y^2 = 2a\sqrt{u}$$

$$\therefore \sqrt{u} = \frac{1}{2a}(x^2+y^2)$$

$$\therefore \frac{1}{2\sqrt{u}}\left(\frac{\partial u}{\partial x}\right)_y = \frac{2x}{2a} = \frac{x}{a}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{2x\sqrt{u}}{a}$$

① — ②

$$\sqrt{v} = \frac{1}{2b}(x^2 - y^2)$$

$$\therefore \frac{1}{2\sqrt{v}}\left(\frac{\partial v}{\partial y}\right)_x = \frac{-2y}{2b}$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_x = \frac{-2y\sqrt{v}}{b}$$

$\therefore$ ① $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v$

$$= \left(\frac{2x\sqrt{u}}{a}\right)\left(\frac{a}{4x\sqrt{u}}\right) = \frac{1}{2}$$

② $\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \left(\frac{-2y\sqrt{v}}{b}\right)\left(\frac{-b}{4y\sqrt{v}}\right) = \frac{1}{2}$

Q) $ux+vy=0$ & $\frac{u}{x}+\frac{v}{y}=1$

then show that

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2+y^2}{y^2-x^2}$$

$\longrightarrow$

$$ux+vy=0 \quad — ①$$

$$\therefore v = -\frac{ux}{y}$$

$$\& \ \frac{u}{x}+\frac{v}{y}=1 \quad — ②$$

$$\therefore \frac{u}{x}+\frac{1}{y}\left(-\frac{ux}{y}\right)=1$$

$$\therefore u\left[\frac{1}{x}-\frac{x}{y^2}\right]=1 \quad \Rightarrow u = \frac{xy^2}{y^2-x^2}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{(y^2-x^2)(y^2)-xy^2(-2x)}{(y^2-x^2)^2}$$

$$= \frac{y^2(y^2+x^2)}{(y^2-x^2)^2} \quad — ③$$

again eqn ① becomes

$$v = -\frac{ux}{y} = -\frac{x}{y}\left(\frac{xy^2}{y^2-x^2}\right) = +\frac{yx^2}{x^2-y^2}$$

$$\therefore v = yx^2/x^2-y^2 \quad — ④$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_x = \frac{(x^2-y^2)(x^2) - 4x^2(-2y)}{(x^2-y^2)^2}$$

$$= \frac{x^2(x^2+y^2)}{(y^2-x^2)^2}$$

$$LHS = \left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x$$

$$= \frac{y^2(x^2+y^2) - x^2(x^2+y^2)}{(y^2-x^2)^2}$$

$$= \frac{x^2+y^2}{y^2-x^2}$$

$= RHS$

Q) if $x^2 = au + bv$ &

$y = au - bv$

Prove that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$

$\longrightarrow$ $x^2 = au + bv$

Diff wrt u

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a$$

$$\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$$

$y = au - bv$

diff y wrt v

$$\left(\frac{\partial y}{\partial v}\right)_u = -b$$

Now find u & v in terms of x & y

$x^2 = au + bv$ & $y = au - bv$

$\therefore x^2 + y = 2au$

$x^2 - y = 2bv$

$$\therefore u = \frac{x^2 + y}{2a}, \quad v = \frac{x^2 - y}{2b}$$

diff u wrt x

$$\frac{1}{2a} \cdot 2x = \left(\frac{\partial u}{\partial x}\right)_y$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$$

Diff v wrt y

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{-1}{2b}$$

$$LHS = \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a} \cdot \frac{a}{2x} = \frac{1}{2}$$

$$RHS = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \frac{1}{2b}(-b) = \frac{1}{2}$$

$\therefore LHS = RHS$

Q) $x = u \tan v$

$y = u \sec v$

Prove that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$

$\longrightarrow$

$x = u \tan v$

$y = u \sec v$

$y^2 - x^2 = u^2(\sec^2 v - \tan^2 v)$

$= u^2$ — ①

$\therefore -2x = 2u \left(\frac{\partial u}{\partial x}\right)_y$

$\therefore \left(\frac{\partial u}{\partial x}\right)_y = -\frac{x}{u}$ — ②

Diff ① wrt y

$2y = 2u \left(\frac{\partial u}{\partial y}\right)_x$

$\therefore \left(\frac{\partial u}{\partial y}\right)_x = \frac{y}{u}$ — ③

find v in terms of x & y

$\frac{x}{y} = \frac{u \tan v}{u \sec v} = \sin v$ — ②

$\therefore \frac{1}{y} = \cos v \left(\frac{\partial v}{\partial x}\right)_y$

$\therefore \left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{y \cos v} = \frac{\sec v}{y} = \frac{y}{uy} = \frac{1}{u}$

Diff ② w.r.t y

$$x\left(\frac{-1}{y^2}\right) = \cos v \left(\frac{\partial v}{\partial y}\right)_x$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_x = \frac{-x}{y^2 \cos v} = \frac{-x}{y^2} \sec v =$$

$$= \frac{-x}{y^2} \cdot \frac{y}{u} = \frac{-x}{uy}$$

$$LHS = \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y$$

$$= \frac{1}{u}\left(\frac{-x}{u}\right) = \frac{-x}{u^2}$$

$$RHS = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$$

$$= \frac{y}{u}\left(\frac{-x}{uy}\right) = \frac{-x}{u^2}$$

$\therefore$ LHS = RHS.

Homogeneous function –

a rational integral, algebraic function of x & y is said to be homogeneous function of n^{th} degree in x & y if the sum of the indices of x & y in each term is the same & the sum being equal to n.

An expression of the type $z = f(x, y)$ is a homo. f^n if it can be expressed as $z = y^n \phi\left(\frac{x}{y}\right)$ or

$$z = x^n f\left(\frac{y}{x}\right)$$

Homogeneous function of two variables.

a function z of two variables x & y is said to be homogeneous of degree n if it is possible to express z in the form $z = x^n f\left(\frac{y}{x}\right)$

$$z = y^n f\left(\frac{x}{y}\right)$$

e.g. $\dfrac{x^2 + y^2}{\sqrt{x} + \sqrt{y}}$ is homo. f^n of degree $3/2$

because $x^{3/2}\left[\dfrac{1 + \left(\frac{y}{x}\right)^2}{1 + \sqrt{\frac{y}{x}}}\right] = x^{3/2} f\left(\frac{y}{x}\right)$

(2) expression $4x^6 - 9x^4y^2 + 12xy^5 + y^6$

can also be put as

$$x^6\left[4 - 9\left(\frac{y}{x}\right)^2 + 12\left(\frac{y}{x}\right)^5 + \left(\frac{y}{x}\right)^6\right]$$

or $x^6 f\left(\frac{y}{x}\right)$ degree $= 6$

(3) $\sin^{-1}\left(\frac{x}{y}\right) = x^0 \sin^{-1}\left(\frac{x}{y}\right)$

(4) $\sin\left(\frac{x^2+y^2}{x+y}\right)$ is not homo.

* Eulers theorem

statement –

Eulers theorem on homogeneous functions of two variables states that if z is a homogeneous function of two variables x, y of degree n

then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

Homogeneous function of 3 variables.

Statement.-

Eulers theorem on homogeneous functions of three variables states that if u is a homogeneous function of x, y, z of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

Deduction -

if u is a homogeneous function of x, y, z of degrees n & if $u = f(v)$

then $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = \frac{nf(v)}{f'(v)}$$

Q) Determine whether the following functions are homogeneous.

(1) $\sqrt{\dfrac{x^2+y^3}{x-y}}$ it is not homogeneous function

(2) $\log\left(\dfrac{x^2+y^2}{x^2-y^2}\right)$ 3.

$$= \log\left(\frac{x^2(1+y^2/x^2)}{x^2(1-y^2/x^2)}\right)$$

$$= \log\left(\frac{1+(y/x)^2}{1-(y/x)^2}\right) = x^0\log\left(\frac{1+(y/x)^2}{1-(y/x)^2}\right)$$

homo f^n of deg 0

Q) if $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

show that $2x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = \tan u$.

$\longrightarrow$

$$u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y/x})} = x^{1/2}\phi\left(\frac{y}{x}\right)$$

$\therefore f(u) = \sin u$ is a homogeneous function of x & y degree $1/2$

$\therefore$ By Eulers theorem,

$$xu_x + yu_y = n\frac{f(u)}{f'(u)}$$

$$\therefore xu_x + yu_y = \frac{1}{2}\cdot\frac{\sin u}{\cos u}$$

$$\therefore 2xu_x + 2yu_y = \tan u.$$

Q) $u = \sin^{-1}\left[\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right]$

show that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\cdot\dfrac{\partial u}{\partial y}$

$\longrightarrow$

$$u = x^0 \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$$

$$= x^0 \sin^{-1}\left(\frac{1-\sqrt{y/x}}{1+\sqrt{y/x}}\right) = x^0 f\left(\frac{x}{y}\right)$$

$\therefore u$ is a Homogeneous function of degree 0

$\therefore$ By Eulers thm,

$$x u_x + y u_y = n = 0$$

$$\therefore x u_x + y u_y = 0$$

$$\therefore x u_x = -y u_y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{-y}{x}\frac{\partial u}{\partial y}$$

Q) if $u = \operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ show that,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right)$$

$\longrightarrow$

$$\rightarrow \operatorname{cosec} u = \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$$

$$= \sqrt{\frac{x^{1/2}(1+(y/x)^{1/2})}{x^{1/3}(1+(y/x)^{1/3})}}$$

$$= x^{1/12}\left(\frac{1+(y/x)^{1/2}}{1+(y/x)^{1/3}}\right)^{1/2}$$

$$= x^{1/12}\,\phi\left(\frac{y}{x}\right)$$

also $f(u) = \operatorname{cosec} u$

$\therefore$ By formula,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = \frac{1}{12}\,\frac{\operatorname{cosec} u}{-\operatorname{cosec} u \cdot \cot u}$$

$$= \frac{-1}{12}\tan u$$

Diff wrt x & y

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u\,\frac{\partial u}{\partial x} \quad -②$$

$$+\, x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{-1}{12}\sec^2 u \cdot \frac{\partial u}{\partial y} \quad -③$$

② $\times x$ + ③ $\times$ ④

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{-\sec^2 x}{12}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(-\frac{\sec^2 u}{12}-1\right)$$

$$=\frac{\tan u}{12}\left(\frac{\tan^2 u}{12}+\frac{13}{12}\right)$$

Q) if $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(1-4\sin^2 u)\sin 2u$$

$\longrightarrow$

$$\tan u=\frac{x^3+y^3}{x-y}=x^2\left[\frac{1+y^3/x^3}{1-y/x}\right]=x^2\phi\left(\frac{y}{x}\right)$$

= a homogeneous f^n of x,y of degree 2

$f(u)=\tan u$

By theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=2\frac{\tan u}{\sec^2 u}=2\sin u\cos u$$

$$=\sin 2u=G(u)\text{ say}$$

$$\therefore x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=G'(u)\cdot\frac{\partial u}{\partial x}\quad -①$$

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=G'(u)\cdot\frac{\partial u}{\partial y}\quad -②$$

① × x + ② × y

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=G'(u)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\cancel{= f x}$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)[G'(u)-1]$$

$$=G(u)[G'(u)-1]$$

$$=\sin 2u[2\cos 2u-1]$$

$$=\sin 2u[1-4\sin^2 u]$$

Q) $u=\left(\frac{x^3+y^3}{y\sqrt{x}}\right)+\frac{1}{x^7}\sin^{-1}\left[\frac{x^2+y^2}{x^2+2xy}\right]$

find the value of $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$

$+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ at point $(1,2)$

→

$$u = \frac{x^3+y^3}{y\sqrt{x}} + \frac{1}{x^7}\sin^{-1}\left[\frac{x^2+y^2}{x^2+2xy}\right]$$

$$u = v + w$$

$$v = \frac{x^3\left(1+(y/x)^3\right)}{x^{3/2}\left(\frac{y}{x}\right)}$$

$$= x^{3/2}\left[\frac{1+(y/x)^3}{(y/x)}\right]$$

v is homo. f^n of degree $3/2$

$$w = x^{-7}\sin^{-1}\left[\frac{1+(y/x)^2}{1+2(y/x)}\right]$$

homo f^n of degree -7

By Eulers theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{3}{2}v \quad -(1)$$

$$\therefore x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = \frac{3}{2}\left(\frac{1}{2}\right)v = \frac{3}{4}v \quad -(2)$$

also

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = -7w \quad -(3)$$

$$\therefore x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = -7(-8)w = 56w \quad -(4)$$

from ① & ③

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{3}{2}v-7w \quad \text{—⑤}$$

adding eqn ② & ④

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{3}{4}v+56w \quad \text{—⑥}$$

⑤ + ⑥

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

$$=\frac{3}{2}v+\frac{3}{4}v+56w-7w=\frac{9}{4}v+49w$$

Now at point (1,2)

$$v=\frac{9}{2},\ w=\sin^{-1}\left[\frac{5}{1+4}\right]=\frac{\pi}{2}$$

$$\therefore \frac{9}{4}v+49w=\frac{9}{4}\left(\frac{9}{2}\right)+49\frac{\pi}{2}=\frac{81}{8}+49\frac{\pi}{2}$$

$\therefore$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

$$=\frac{81}{8}+49\frac{\pi}{2}$$

Q) if $z = x^n f(y/x) + y^{-n} \phi(x/y)$

Prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$$

$\longrightarrow$

$$z = x^n f\left(\frac{y}{x}\right) + y^{-n} \phi\left(\frac{x}{y}\right) \quad — (1)$$

$$z = u + v$$

$u = x^n f(y/x)$, $v = y^{-n} \phi\left(\frac{x}{y}\right)$

$\therefore$ u, v are homogeneous f^n of x, y of degree n & $-n$

$\therefore$ By Eulers theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad — (2)$$

& $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u \quad — (3)$$

$^{||}$ly, $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -nv \quad — (4)$$

& $$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = -n(-n-1)v$$
$$= n(n+1)v \quad — (5)$$

2 + 4

$$\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)+\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=nu-nv$$

from 1 $\quad \dfrac{\partial z}{\partial x}=\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial x}$

$$\frac{\partial z}{\partial y}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nu-nv \quad -6$$

3 + 5

$$x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}\right)+2xy\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x\partial y}\right)$$

$$+y^2\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}\right)=n(n-1)u+n(n+1)v$$

$$\therefore x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n+1)v+n(n-1)u$$

-7

6 + 7

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$= n(n-1)u + n(n+1)v + nu - nv$$

$$= n^2 u + n^2 v$$

$$= n^2(u+v)$$

$$= n^2 z.$$ Hence proved

Q) if $u = \sin(\sqrt{x} + \sqrt{y})$

Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(\sqrt{x} + \sqrt{y})\cos(\sqrt{x} + \sqrt{y})$$

$\longrightarrow$

$$u = \sin(\sqrt{x} + \sqrt{y})$$

$$\frac{\partial u}{\partial x} = \frac{\cos(\sqrt{x} + \sqrt{y})}{2\sqrt{x}}$$

$$x\frac{\partial u}{\partial x} = \frac{\sqrt{x}}{2}\cos(\sqrt{x} + \sqrt{y})$$

$$\frac{\partial u}{\partial y} = \frac{\cos(\sqrt{x} + \sqrt{y})}{2\sqrt{y}}$$

$$y\frac{\partial u}{\partial y} = \frac{\sqrt{y}}{2}\cos(\sqrt{x} + \sqrt{y})$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)\cos(\sqrt{x}+\sqrt{y})$$

$$= \frac{1}{2}(\sqrt{x}+\sqrt{y})\cos(\sqrt{x}+\sqrt{y})$$

Hence proved

Q) find $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$

① $u = \tan^{-1}\left[\frac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}\right]$

$\longrightarrow$

$$u = \tan^{-1}\left[\frac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}\right]$$

$$\therefore \tan u = f(u) = \frac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{x^{3/2}\sqrt{1+(y/x)^3}}{\sqrt{x}(1+(y/x)^{1/2})}$$

$$= x\,\phi(y/x)$$

$\therefore f(u)$ is a homogeneous function in x & y of degree 1

$\therefore$ By Eulers theorem,

$$xu_x + yu_y = n\frac{f(u)}{f'(u)} = \frac{\tan u}{\sec^2 u} = \frac{1}{2}\sin 2u$$

$$= G(u)$$

$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$

$$= G(u)[G'(u) - 1]$$

$$= \frac{1}{2} \sin 2u [\tfrac{1}{2} 2\cos 2u - 1]$$

$$= \frac{1}{2} \sin 2u [\cos 2u - 1]$$

$$= -2 \sin^3 u \cos u$$

(2) $u = \sin^{-1} \sqrt{x^2+y^2}$

$\longrightarrow \sin u = f(u) = \sqrt{x^2+y^2}$

$$= x\sqrt{1+(y/x)^2}$$

$\therefore f(u)$ is homo. f^n of degree 1

$$n \frac{f(u)}{f'(u)} = 1 \cdot \frac{\sin u}{\cos u} = \tan u$$

By Eulers deduction

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = \tan u \quad = G(u)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = G(u)[G'(u) - 1]$$

$$= \tan u (\sec^2 u - 1)$$

$$= \tan^3 u$$

(8)

③ $u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$\longrightarrow$ $\sin u = f(u) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = x^{1/2}\left[\frac{1+y/x}{1+\sqrt{\frac{y}{x}}}\right]$

$= x^{1/2}\,\phi(y/x)$

f(u) is a homogeneous function of x & y of degree 1/2

By Eulers theorem

$x u_x + y u_y = n\frac{f(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u$

$= g(u)$

$\therefore x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = g(u)[g'(u) - 1]$

$= \frac{1}{2}\tan u\left[\frac{1}{2}\sec^2 u - 1\right]$

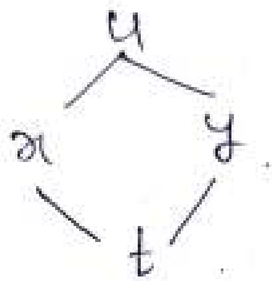
$= \frac{\sin u}{2\cos u}\left[\frac{1}{2\cos^2 u} - 1\right]$

$= \frac{\sin u[1 - 2\cos^2 u]}{2\cos u \cdot 2\cos^2 u}$

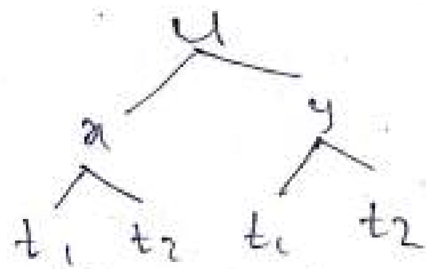
$= \frac{\sin u \cos 2u}{4\cos^3 u}.$

Composite function —

① if u be a function of two independent variables x & y where x & y are separately functions of single independent variable t then u is called composite function of t.



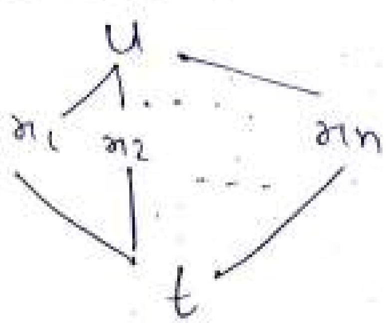
② $u = f(x,y)$, $x = \phi_1(t_1, t_2)$
$y = \phi_2(t_1, t_2)$



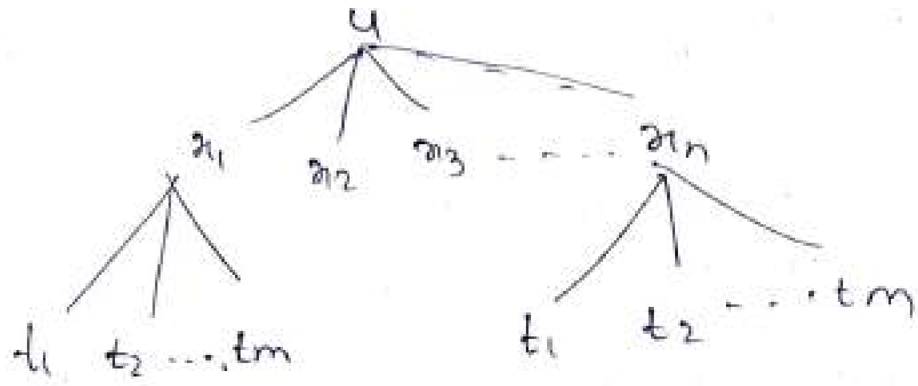
then u is called composite f^n of two independent variables t_1, t_2.

$u = x^2 + y^2$, $x = t_1 + t_2$, $y = t_1 - t_2$

③ if $u = f(x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are functions of single independent variable t then u is called composite f^n of single variable t

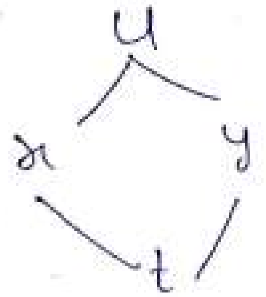


(5) $u = f(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are functions of m independent variables $t_1, t_2, t_3, \ldots t_m$ then u is called the composite function of several independent variables.



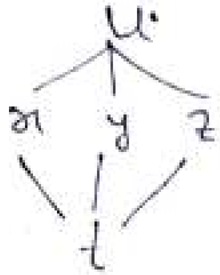
* Total Derivative

(1) $u = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$ then u can be expressed as function of single variable t where u possesses continuous derivatives with respect to x & y & x, y possess derivatives w.r.t t then



$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

(2) u be a composite function of t given by the relation $u = \phi(x,y,z)$, $x = \phi(t)$, $y = \psi(t)$, $z = \xi(t)$



total derivative –

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

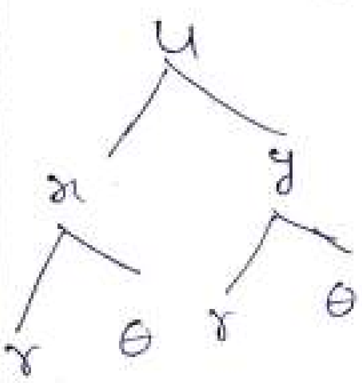
total differential

$$du = u_x\,dx + u_y\,dy + u_z\,dz$$

llly if $u = f(x_1, \ldots, x_n)$

then $$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt}$$

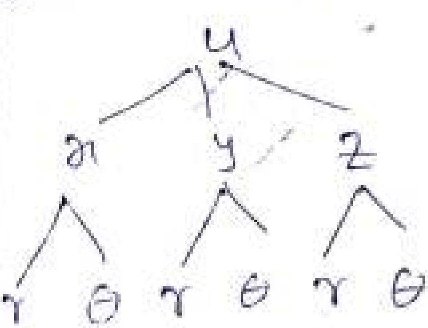
(3) $u = f(x,y)$, $x = f_1(r,\theta)$, $y = f_2(r,\theta)$



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

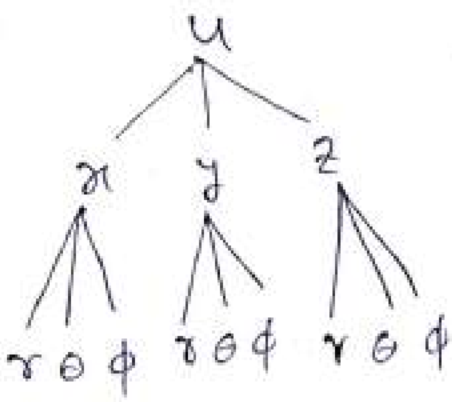
(4) $u = f(x,y,z)$, $x = f_1(r,\theta)$, $y = f_2(r,\theta)$, $z = f_3(r,\theta)$



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

(5) $u = f(x, y, z)$, $x = f_1(r, \theta, \phi)$, $y = f_2(r, \theta, \phi)$, $z = f_3(r, \theta, \phi)$



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \theta}$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \phi}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Put $t = x$ & $t = y$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dy} = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{dx}{dy}$$

* Differentiation of implicit function -

let $z = f(x, y) = 0$

then $\frac{dz}{dx} = 0$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\partial z/\partial x}{\partial z/\partial y}$$

let $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

$$\therefore \frac{dy}{dx} = \frac{-p}{q}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{q^2r - 2pqs + p^2t}{q^3}\right] \quad \therefore q^3\frac{d^2y}{dx^2} = \begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix}$$

Q) $z = \cancel{f(x,y)}\ f(u,v)$ & u, v are homogeneous functions of degree n in x & y then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

$\longrightarrow$

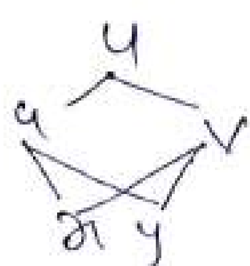
Given that u & v are homogeneous f^n of degree n

By Eulers theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{—} (1)$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv \quad \text{—} (2)$$

treating z as a function of u & v & u, v are again functions of x & y, z becomes composite function of x & y ie



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad - (3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad - (4)$$

$x \times (3) + y \times (4)$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$+ \frac{\partial z}{\partial v}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial z}{\partial u}((nu)) + \frac{\partial z}{\partial v}(nv)$$

$$= n\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

Q) if $x = r\cos\theta$, $y = r\sin\theta$, r, θ are functions of t prove that $x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}$

→

Here x & y are composite functions of t

$$\therefore \frac{dx}{dt} = \frac{\partial x}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\cdot\frac{d\theta}{dt}$$

$$= \cos\theta\cdot\frac{dr}{dt} - r\sin\theta\cdot\frac{d\theta}{dt}$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial y}{\partial \theta} \cdot \frac{d\theta}{dt}$$

$$\frac{dy}{dt} = \sin\theta \frac{dr}{dt} + r\cos\theta \cdot \frac{d\theta}{dt}$$

$$\therefore\ x\frac{dy}{dt} - y\frac{dx}{dt} = r\cos\theta\left(\sin\theta \frac{dr}{dt} + r\cos\theta \frac{d\theta}{dt}\right)$$

$$- r\sin\theta\left(\cos\theta \cdot \frac{dr}{dt} - r\sin\theta \frac{d\theta}{dt}\right)$$

$$= r^2(\cos^2\theta + \sin^2\theta)\frac{d\theta}{dt}$$

$$= r^2\frac{d\theta}{dt}$$

Q) if $z = f(u,v)$, $u = x^2 - 2xy - y^2$ & $v = y$ show that

$$(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = (x-y)\frac{\partial z}{\partial v}$$

→ also prove that the equation

$(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$ is equivalent to

$\frac{\partial z}{\partial v} = 0$ Hence show that

$z = \phi(x^2 - y^2 - 2xy)$, ϕ is arbitrary fn.

$$\begin{tikzcd} & & & z \arrow[dll, dash] \arrow[drr, dash] & & & \\ & u \arrow[dl, dash] \arrow[dr, dash] & & & & v \arrow[dl, dash] \arrow[dr, dash] & \\ x & & y & & x & & y \end{tikzcd}$$

z is a composite function of x & y

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(2x-2y) + \frac{\partial z}{\partial v}(0) = 2(x-y)\frac{\partial z}{\partial u} \quad \text{--- (1)}$$

$$(x+y)\frac{\partial z}{\partial x} = 2(x^2-y^2)\frac{\partial z}{\partial u}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2x-2y) + \frac{\partial z}{\partial v}$$

$$(x-y)\frac{\partial z}{\partial y} = -2(x^2-y^2)\frac{\partial z}{\partial u} + (x-y)\frac{\partial z}{\partial v} \quad \text{--- (2)}$$

(1) + (2)

$$(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = (x-y)\frac{\partial z}{\partial v}$$

if $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0 \Rightarrow (x-y)\frac{\partial z}{\partial v} = 0$

$\therefore \frac{\partial z}{\partial v} = 0 \Rightarrow$ z is function of only u

$\therefore z = \phi(u) = \phi(x^2-2xy-y^2)$

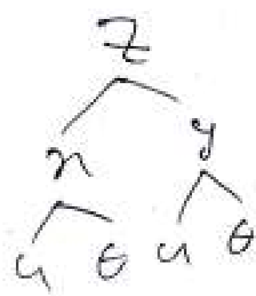
(1) $x = \frac{\cos\theta}{u}$, $y = \frac{\sin\theta}{u}$ then show that

$$u\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \theta} = (y-x)\frac{\partial z}{\partial x} - (y+x)\frac{\partial z}{\partial y}.$$

$\longrightarrow$

treating z as a function of ~~u & v where~~ x, y where x, y are function of u, θ

$\therefore$ z is composite function of u, θ



$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \left(-\frac{\cos\theta}{u^2}\right)\frac{\partial z}{\partial x} + \left(-\frac{\sin\theta}{u^2}\right)\frac{\partial z}{\partial y}$$

$$\therefore u\frac{\partial z}{\partial u} = -\frac{\cos\theta}{u}\frac{\partial z}{\partial x} - \frac{\sin\theta}{u}\frac{\partial z}{\partial y}$$

$$= -x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}\left(-\frac{\sin\theta}{u}\right) + \frac{\partial z}{\partial y}\left(\frac{\cos\theta}{u}\right)$$

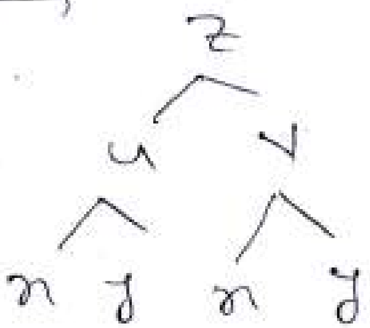
$$= -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

$$\therefore u\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \theta} = (-x+y)\frac{\partial z}{\partial x} - (y+x)\frac{\partial z}{\partial y}$$

Q) $u = x^2 - y^2$, $v = 2xy$, $z = f(u, v)$

Show that $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 2\sqrt{u^2+v^2}\,\dfrac{\partial z}{\partial u}$

$\longrightarrow$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= 2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= -2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}$$

$$x\frac{\partial z}{\partial x} = 2x^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v}$$

$$y\frac{\partial z}{\partial y} = -2y^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v}$$

$$\therefore x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2+y^2)\frac{\partial z}{\partial u}$$

$u^2 = (x^2-y^2)^2$, $v^2 = 4x^2y^2$

$$\therefore u^2+v^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$= (x^2+y^2)^2$$

$\therefore x^2+y^2 = \sqrt{u^2+v^2}$

$$\therefore x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2+v^2}\,\frac{\partial z}{\partial u}$$

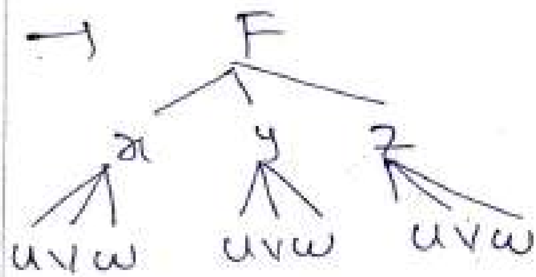
Q) if $x = u+v+w$, $y = uv+vw+wu$

$z = uvw$. & F is a function of x, y, z

show that,

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

$\rightarrow$



F is a composite function of u, v, w.

$$\therefore \frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{\partial F}{\partial x} + (v+w)\frac{\partial F}{\partial y} + vw\frac{\partial F}{\partial z}$$

$$\therefore u\frac{\partial F}{\partial u} = u\frac{\partial F}{\partial x} + (uv+uw)\frac{\partial F}{\partial y} + uvw\frac{\partial F}{\partial z} \quad -(1)$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{\partial F}{\partial x} + (u+w)\frac{\partial F}{\partial y} + uw\frac{\partial F}{\partial z}$$

$$v\frac{\partial F}{\partial v} = v\frac{\partial F}{\partial x} + (uv+vw)\frac{\partial F}{\partial y} + uvw\frac{\partial F}{\partial z} \quad -(2)$$

$$\frac{\partial F}{\partial w} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial w}$$

$$= \frac{\partial F}{\partial x} + (u+v)\frac{\partial F}{\partial y} + uv\frac{\partial F}{\partial z}$$

$$\therefore w\frac{\partial F}{\partial w} = w\frac{\partial F}{\partial x} + (uw+vw)\frac{\partial F}{\partial y} + uvw\frac{\partial F}{\partial x} \quad -(3)$$

(1) + (2) + (3)

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = (u+v+w)\frac{\partial F}{\partial x} + 2(uv+vw+uw)\frac{\partial F}{\partial y}$$

$$+ 3uvw\frac{\partial F}{\partial z} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

Q) if $u = f(x-y, y-z, z-x)$

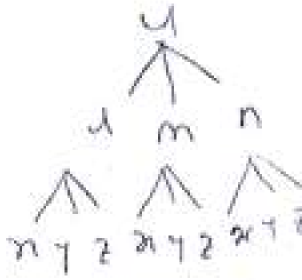
then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

$\longrightarrow$

$u = f(x-y, y-z, z-x)$

let $l = x-y$, $m = y-z$, $n = z-x$

then $u = f(l, m, n)$



u is a composite f^n of x, y, z

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n} \quad \text{—} (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= -\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m} \quad \text{—} (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= -\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n} \quad \text{—} (3)$$

(1) + (2) + (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

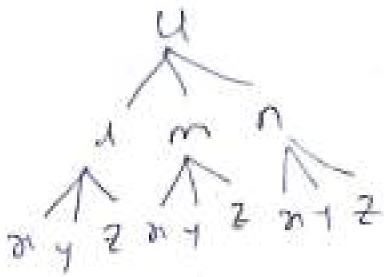
Q) $u = f(x^2-y^2, y^2-z^2, z^2-x^2)$ prove that

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

$\rightarrow$ let $l = x^2-y^2$

$m = y^2-z^2$

$n = z^2-x^2$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial x}$$

$$= 2x\frac{\partial u}{\partial l} - 2x\frac{\partial n}{\partial x}$$

$$\therefore \frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial n}{\partial x} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial y}$$

$$= -2y\frac{\partial u}{\partial l} + 2y\frac{\partial u}{\partial m}$$

$$\therefore \frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial z}$$

$$= -2z\frac{\partial u}{\partial m} + 2z\frac{\partial u}{\partial n}$$

$$\therefore \frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n} \quad \text{—— (3)}$$

(1) + (2) + (3)

$$\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=0$$

(Q) ~~$V=f(e^{x-y}, e^{y-z}, e^{z-x})$~~

$V=f(e^{x-y}, e^{y-z}, e^{z-x})$

show that $\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial V}{\partial z}=0$

$\longrightarrow$ $l=e^{x-y}$, $m=e^{y-z}$, $n=e^{z-x}$.

$$\frac{\partial V}{\partial x}=\frac{\partial V}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial V}{\partial m}\cdot\frac{\partial m}{\partial x}+\frac{\partial V}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$=e^{x-y}\frac{\partial V}{\partial l}-e^{z-x}\frac{\partial V}{\partial n} \quad -(1)$$

$$\frac{\partial V}{\partial y}=\frac{\partial V}{\partial l}\frac{\partial l}{\partial y}+\frac{\partial V}{\partial m}\frac{\partial m}{\partial y}+\frac{\partial V}{\partial n}\frac{\partial n}{\partial y}$$

$$=-e^{x-y}\frac{\partial V}{\partial l}+e^{y-z}\frac{\partial V}{\partial m} \quad -(2)$$

$$\frac{\partial V}{\partial z}=\frac{\partial V}{\partial l}\frac{\partial l}{\partial z}+\frac{\partial V}{\partial m}\frac{\partial m}{\partial z}+\frac{\partial V}{\partial n}\frac{\partial n}{\partial z}$$

$$=-e^{y-z}\frac{\partial V}{\partial m}+e^{z-x}\frac{\partial V}{\partial n} \quad -(3)$$

$$\therefore \frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial V}{\partial z}=0$$