

4

Relations and Functions

4.1 : Relations

Important Points to Remember

1. **Cartesian Product :** Let A and B be two non empty sets. The Cartesian product of A and B is denoted by $A \times B$ and defined as
$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

(x, y) is known as order pair.

2. **Relation :** Let A and B be two non-empty sets. A relation from A to B is any subset of $A \times B$. It is denoted by $R : A \rightarrow B$. Set A is called domain set and B is called co-domain set.

The range set of relation R, is the set of elements of B that are second elements of pairs in R.

3. A relation $R : A \rightarrow A$ is called the relation on set A.

4. A relation can be represented in two forms.

(a) **Matrix form :** Let $R : A \rightarrow B$ be a relation.

$$A = \{a_1, a_2, a_3, \dots, a_n\}, \quad B = \{b_1, b_2, b_3, \dots, b_m\}.$$

Then R can be represented by $n \times m$ matrix such that

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

(b) **Digraph form :** Let R be a relation on set A. Draw small circles or dark dots for each element of A. These circles are called vertices of digraph.

Draw an edge with arrow from a_i to a_j iff $a_i R a_j$ such representation is called digraph or directed graph of R.

5. Types of Relation :

Sr. No.	Name of relation	Condition
1.	Reflexive Relation	If $(a, a) \in R$ for all $a \in A$
2.	Irreflexive	If $(a, a) \notin R$ for all $a \in A$
3.	Symmetric	Whenever $(a, b) \in R$ then $(b, a) \in R$
4.	Asymmetric	Whenever $(a, b) \in R$ then $(b, a) \notin R$
5.	Antisymmetric	Whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$
6.	Transitive relation	If $(a, b) \in R$, $(b, c) \in R$ then $(a, c) \in R$
7.	Equivalence relation	If R is reflexive, symmetric and transitive.
8.	Partial ordering relation	If r is reflexive antisymmetric and transitive.

6. Important Properties :

i) *Complement of a relation* : Let R be a relation from A to B . The complement of relation R is denoted by \bar{R} and defined as relation from A to B such that

$$\bar{R} = \{(a, b) \mid (a, b) \notin R\}$$

ii) *Inverse of relation* : Let $R : A \rightarrow B$ be a relation then the inverse relation $R^{-1} : B \rightarrow A$ is a relation such that $xR^{-1}y$ iff yRx .

iii) *Composite relation* : Let $R_1 : A \rightarrow B$ and $R_2 : B \rightarrow C$ be two relations. Then the composition of relations R_1 and R_2 is denoted by $R_1 \cdot R_2$ or $R_1 R_2$ and defined as

$$R_1 \cdot R_2 = \{(x, z) \mid xR_1y \text{ and } yR_2z\}$$

iv) *Equivalence class* : Let $R : A \rightarrow B$ be an equivalence relation. The equivalence class of an element $a \in A$ is the set of all elements of A which are related to a . It is denoted by $[a] = \{x \mid xRa ; x \in A\}$

7. Closure of Relations :

i) **Reflexive closure** : Let R be a relation on a set A and R is not reflexive relation. A relation $R_1 = R \cup \Delta$ is the reflexive closure of R if $R \cup \Delta$ is the smallest reflexive relation containing R . If $A = \{x_1, x_2, x_3, x_4\}$ then take

$$\Delta = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)\}$$

ii) **Symmetric closure** : A relation R^* is the symmetric closure of relation R if $R^* = R \cup R^{-1}$.

iii) **Transitive closure** : Let $R : A \rightarrow A$ be a relation which is not transitive. The transitive closure of a relation R is the smallest transitive relation containing R .

Note : Let A be a set with n elements and R be a relation on set A .

$$\text{Then } R^* = R \cup R^1 \cup R^2 \cup \dots \cup R^n$$

4.2 : Warshall's Algorithm to Find Transitive Closure

Important Points to Remember

To find the transitive closure of a relation by computing various powers of R or product of the relation matrix is quite impractical for large relations. Warshall's algorithm gives an alternate method for finding transitive closure of R . Warshall's algorithm is practical and efficient method.

Consider the following steps to find transitive closure of the relation R on a set A .

Step 1 : We have $|A| = n$

\therefore We require $W_0, W_1, W_2, \dots, W_n$. Warshall sets

$$W_0 = \text{Relation Matrix of } R = M_R$$

Step 2 : To find the transitive closure of relation R on set A , with $|A| = n$

Procedure to compute W_k from W_{k-1} is as follows

- Copy 1 to all entries in W_k from W_{k-1} , where there is a 1 in W_{k-1} .

- ii) Find the row numbers $p_1, p_2, p_3 \dots$ for which there is 1 in column k in W_{k-1} and the column numbers $q_1, q_2, q_3 \dots$ for which there is 1 in row k of W_{k-1} .
- iii) Mark entries in W_k as 1 for (p_i, q_i) . If there are not already 1.
- Step 3 :** Stop the procedure when W_n is obtained and it is the required transitive closure of R .

Q.1 Which of the following are relations from A to B

where $A = \{1, 2, 3, 4\}$; $B = \{x, y, z\}$

- a) $R_1 = \{(1, x), (1, y), (1, z), (4, x)\}$
 b) $R_2 = \{(x, 1), (y, 1), (z, 1), (x, 4)\}$
 c) $R_3 = \{(1, x), (2, y), (3, z), (4, w)\}$
 d) $R_4 = \{(1, 1), (2, 2)\}$

Ans. : Given that $R : A \rightarrow B$

Where A = Domain set

and B = Co-domain set

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z), (4, x), (4, y), (4, z)\}$$

- (a) R_1 is a relation from A to B because $R_1 \subset A \times B$
 (b) R_2 is not a relation from A to B as $R_2 \not\subset A \times B$
 (c) R_3 is not a relation from A to B as $(4, w) \in R$ but $w \notin B$
 (d) R_4 is not a relation from A to B as $R_4 \not\subset A \times B$

Q.2 Let $A = \{1, 2, 3, 4, 5, 6\}$ aRb or $(a, b) \in R$ iff a is a multiple of b. Find

- (i) Range set and R (6), R (3). (ii) Find relation matrix.
 (iii) Draw diagram (iv) Find in and out degree of each vertex.

Ans. : By considering given condition.

$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (5, 1), (5, 5),$

$(6, 1), (6, 2), (6, 3), (6, 6)\}$

- (i) The range set of R is $\{1, 2, 3, 4, 5, 6\}$

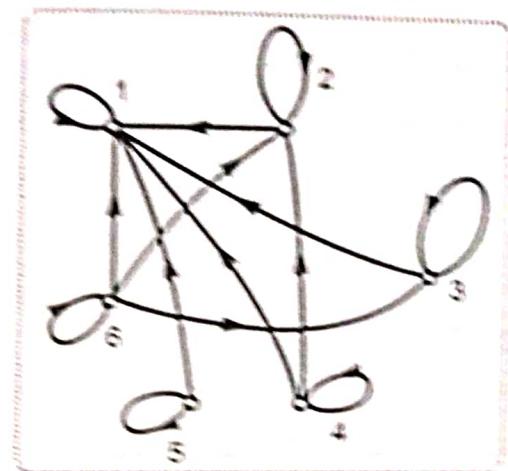
$$R(3) = \{1, 3\} \text{ as } (3, 1) \text{ and } (3, 3) \in R$$

$$\text{and } R(6) = \{1, 2, 3, 6\} \text{ because } (6, 1), (6, 2), (6, 3), (6, 6) \in R$$

- (ii) The relational matrix is,

$$M(R) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 & 0 \\ 5 & 1 & 0 & 0 & 0 & 1 \\ 6 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The diagram of given relation is
vertex set = {1, 2, 3, 4, 5, 6}



(b) The indegree of a vertex in diagraph is the number of edges coming towards that vertex. The outdegree of a vertex is defined as the number of edges going out from a vertex. Consider the following table.

Vertex	Indegree	Outdegree	Degree
1	6	1	7
2	3	2	5
3	2	2	4
4	3	1	4
5	2	1	3
6	4	1	5

- Q3 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let N be the relation on $A \times A$ defined by $(a, b) \sim (c, d)$ iff $a + d = b + c$.
- Prove that \sim is an equivalence relation.
 - Find equivalence class of $(2, 5)$

DSP [SPPU : Dec.-11]

Ans. : Given that $(a, b) \sim (c, d)$ iff $a + d = b + c, \forall a, b, c, d \in A$

(i) (a) We have $a + b = b + a \Rightarrow (a, b) \sim (a, b)$

$\Rightarrow \sim$ is reflexive relation.

(b) If $(a, b) \sim (c, d)$ then $a + d = b + c$.

$$\Rightarrow b + c = a + d \Rightarrow c + b = d + a$$

$\Rightarrow (c, d) \sim (a, b)$ by definition

$\Rightarrow \sim$ is symmetric relation.

(c) Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$

$$\text{then } a + d = b + c$$

$$\text{and } c + f = d + e$$

$$\Rightarrow a + d + c + f = b + c + d + e$$

$$a + f = b + e$$

$$\Rightarrow (a, b) \sim (e, f)$$

$\therefore \sim$ is a transitive relation.

Thus \sim is reflexive, symmetric and transitive.

$\Rightarrow \sim$ is an equivalence relation.

(ii) We have $(2, 5) \in A \times A$

The equivalence class of $(2, 5)$ is the set of elements of $A \times A$ which are equivalent to $(2, 5)$

$$\begin{aligned} \therefore [(2, 5)] &= \{(x, y) \mid (x, y) \sim (2, 5), x, y \in A\} \\ &= \{(x, y) \mid x + 5 = y + 2\} \\ &= \{(x, y) \mid x - y = -3 \text{ or } y = x + 3; x, y \in A\} \end{aligned}$$

$$\text{Hence, } [(2, 5)] = \{(1, 3) (2, 5) (3, 6) (4, 7) (5, 8) (6, 9)\}$$

$\dots [\because (7, 10) \notin A \times A]$

Q.4 Show that $R = \{(a, b) / a \equiv b \pmod{m}\}$ is an equivalence relation on \mathbb{Z} . Show that $x_1 \equiv y_1$ and $x_2 \equiv y_2$. then $x_1 + x_2 \equiv y_1 + y_2$

Ans. : Given that,

$$R = \{(a, b) | a \equiv b \pmod{m}\}$$

We know that $a \equiv b \pmod{m}$ iff $m | (a - b)$

$$a - b = mk, k \in \mathbb{Z}$$

(i) Let $a \in \mathbb{Z}$, $a - a = 0 = m(0)$

$$\therefore m|(a - a) \Rightarrow a \equiv a \pmod{m}$$

$(a, a) \in R \Rightarrow R$ is reflexive relation.

(ii) Let $(a, b) \in R$

$$\Rightarrow a \equiv b \pmod{m}$$

$$\Rightarrow a - b = mk$$

$$\Rightarrow b - a = m(-k)$$

$$\Rightarrow m | b - a$$

$$\Rightarrow b \equiv a \pmod{m}$$

$$\Rightarrow (b, a) \in R$$

$\therefore R$ is symmetric relation.

(iii) Suppose $(a, b), (b, c) \in R$

then $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$$\Rightarrow a - b = mk_1 \text{ and } b - c = mk_2$$

$$\Rightarrow a - b + b - c = mk_1 + mk_2$$

$$\Rightarrow a - c = m(k_1 + k_2) = mk$$

$$\Rightarrow m|(a - c) \Rightarrow a \equiv c \pmod{m}$$

$\therefore (a, c) \in R$

$\therefore R$ is transitive relation.

$\therefore R$ is reflexive, symmetric and transitive.

Thus R is an equivalence relation,

$$\text{Now, } x_1 \equiv y_1 \Rightarrow x_1 \equiv y_1 \pmod{m}$$

$$x_2 \equiv y_2 \Rightarrow x_2 \equiv y_2 \pmod{m}$$

$$x_1 - y_1 = mk_1 \text{ and } x_2 - y_2 = mk_2$$

$$\Rightarrow x_1 - y_1 + x_2 - y_2 = mk_1 + mk_2$$

$$\Rightarrow (x_1 + x_2) - (y_1 + y_2) = m(k_1 + k_2)$$

$$\Rightarrow m|(x_1 + x_2) - (y_1 + y_2)$$

$$\Rightarrow (x_1 + x_2) \equiv (y_1 + y_2) \pmod{m}$$

$$\Rightarrow x_1 + x_2 \equiv y_1 + y_2 \pmod{m}$$

$$\Rightarrow x_1 + x_2 = y_1 + y_2$$

Q.5 For each of these relations on set $A = \{1, 2, 3, 4\}$ decide whether it is reflexive, symmetric, transitive or antisymmetric.

$$R_1 = \{(1, 1) (2, 2) (3, 3) (4, 4)\}$$

$$R_2 = \{(1, 1)(1, 2) (2, 2) (2, 1) (3, 3) (4, 4)\}, R_3 = \{(1, 3) (1, 4) (2, 3) (2, 4) (3, 1) (3, 4)\}.$$

[SPPU : Dec. 10]

Ans. : i) For R_1 :

As $\forall a \in A. (a, a) \in R_1$

$\therefore R_1$ is Reflexive, symmetric relation.

$\exists (a, b)$ and $(b, c) \in R_1 \therefore R_1$ is transitive relation.

$\exists (a, b)$ and $(b, a) \in R_1 \therefore R_1$ is antisymmetric relation.

$\therefore R_1$ reflexive, symmetric, transitive and anti-symmetric relation.

ii) For R_2 :

As $\forall a \in A (a, a) \in R_2$ and $aR_2 b \Rightarrow bR_1 a$, for $a, b \in A$

$\therefore R_2$ reflexive and symmetric relation.

For any $aR_2 b$ and $bR_2 c \Rightarrow aR_2 c$

$\therefore R_2$ is transitive relation

$\therefore R_2$ is an equivalence relation

But $(1, 2)$ and $(2, 1) \in R_2$ and $1 \neq 2$

$\therefore R_2$ is not antisymmetric relation.

iii) For R_3 :

As $2 \in R_3$ but $(2, 2) \notin R_3$

$\therefore R_3$ is not reflexive relation.

As $(1, 4) \in R_3$ but $(4, 1) \notin R_3$

$\therefore R_3$ is not symmetric relation.

As $(2, 3)$ and $(3, 1) \in R_3$ but $(2, 1) \notin R_3$.

$\therefore R_3$ is not transitive relation.

As $(1, 3)$ and $(3, 1) \in R_3$ but $1 \neq 3$.

$\therefore R_3$ is not transitive relation

As $(1, 3)$ and $(3, 1) \in R_3$ but $1 \neq 3$

$\therefore R_3$ is not antisymmetric relation

Q.6 Consider the relation on $A = \{1, 2, 3, 4, 5, 6\}$.

$R = \{(i, j) \mid i - j = 2\}$. Is R reflexive? Is R symmetric? Is R transitive?

[SPPU : Dec.-14]

Ans.: Given that $A = \{1, 2, 3, 4, 5, 6\}$

$$R = \{(1, 3) (3, 1) (2, 4) (4, 2) (3, 5) (5, 3) (4, 6) (6, 4)\}$$

As $2 \in A$ but $(2, 2) \notin R \therefore R$ is not reflexive.

For any $(a, b) \in R \Rightarrow (b, a) \in R \therefore R$ is symmetric.

As $(1, 3) (3, 1) \in R$ but $(1, 1) \notin R \therefore R$ is not transitive.

Q.7 Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $R = \{(x, y) \mid x - y \text{ is divisible by } 3\}$. Show that R is an equivalence relation? Draw graph of R .

[SPPU : May-14, Dec.-12]

Ans.: We have $R = \{(x, y) \mid x - y \text{ is divisible by } 3\}$

We know that $x - x = 0$ is divisible by 3

$\forall x, \forall x \in A \Rightarrow R$ is reflexive relation.

As $x R y \Rightarrow x - y$ is divisible by 3

$$\Rightarrow y - x \text{ is also divisible by } 3$$

$$\Rightarrow y - z \text{ is divisible by } 3$$

$$\Rightarrow y R x \text{ for } x, y \in A$$

$\therefore R$ is a symmetric relation.

As,

$x R y$ and $y R \Rightarrow x - y$ and $y - z$ are divisible by 3

$$\Rightarrow (x - y) + (y - z) \text{ is also divisible by } 3$$

$$\Rightarrow x - z \text{ is divisible by } 3$$

$$\Rightarrow x R z$$

$\therefore R$ is a transitive relation.

$\therefore R$ is an equivalence relation.

and

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (1, 4), (4, 1), (1, 7), (7, 1), (2, 5), (5, 2), (3, 6), (6, 3)\}$$

Its graph is as follows

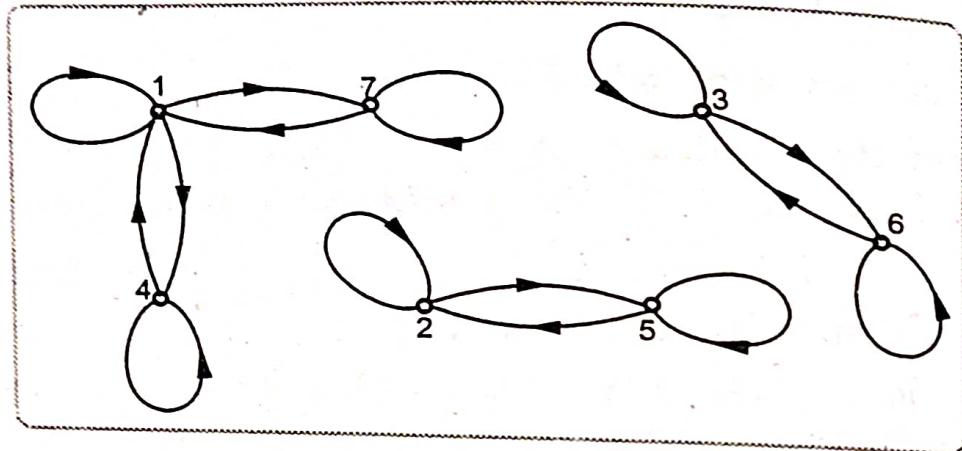


Fig. Q.7.1

Q.8 Define partition of a set. Let $x = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Determine whether or not each of the following is a partition of x :

$$A = \{\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}\}$$

$$B = \{\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}\}$$

[SPPU : Dec.-11]

Ans. : Please refer section 5.8 for definition.

i) For set A :

The set A has 3 blocks,

$$A_1 = \{2, 4, 5, 8\}, \quad A_2 = \{1, 9\} \quad A_3 = \{3, 6, 7\}$$

- The union of all these blocks is a set X

$$A_1 \cup A_2 \cup A_3 = X$$

- These blocks are mutually disjoint.

∴ The set A forms a partition for the set X

ii) For set B :

∴ The set B has 3 blocks.

$$B_1 = \{1, 3, 6\}, \quad B_2 = \{2, 8\} \quad B_3 = \{5, 7, 9\}$$

$$\text{As } B_1 \cup B_2 \cup B_3 \neq X,$$

B is not a partition of set X.

Q.9 If S = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Determine whether or not each of the following is a partition of S.

- i) $A = \{\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}\}$
 ii) $B = \{\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}\}$
 iii) $C = \{\{1, 3, 5\}\}, \{2, 4, 6, 8\}, \{5, 7, 9\}\}$
 iv) $D = \{\{5\}\}$

[SPPU : Dec.-12]

Ans. : Given that

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- i) A is not a partition of S because S is not the union of all blocks of A .
 i.e. $S \neq A_1 \cup A_2 \cup A_3$
 ii) B is the partition of S as $B_1 \cup B_2 \cup B_3 = S$ and B_1, B_2, B_3 are mutually disjoints.
 iii) As blocks of set C are not disjoints. \therefore The set C is not a partition of S .
 iv) $D = \{\{s\}\}$ is a partition of S , called as trivial partition.

Q.10 Find the symmetric closure of the following relations.

On $A = \{1, 2, 3\}$.

$$\begin{aligned} R_1 &= \{(1, 1) (2, 1)\} & R_2 &= \{(1, 2) (2, 1) (3, 2) (2, 2)\} \\ R_3 &= \{(1, 1) (2, 2) (3, 3)\} \end{aligned}$$

[SPPU : Dec.-12]

Ans. :

We have $A = \{1, 2, 3\}$

$$i) R_1^{-1} = \{(1, 1) (1, 2)\}$$

$$\therefore R = R_1 \cup R_1^{-1} = \{(1, 1) (1, 2) (2, 1)\}$$

is the symmetric closure of R_1

$$ii) R_2^{-1} = \{(2, 1) (1, 2) (2, 3) (2, 2)\}$$

$$\therefore R = R_2 \cup R_2^{-1} = \{(1, 2) (2, 1) (3, 2) (2, 3) (2, 2)\}$$

is the symmetric closure of R_2

iii) R_3 is the symmetric relation.

$\therefore R_3$ itself is the symmetric closure.

Q.11 Find the transitive closure of R by Warshall's algorithm.

Where $A = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(x, y) / (x - y) = 2\}$

[SPPU : Dec.-05, 12, 13, 14, 16]

Ans. : Step 1 : We have $|A| = 6$,

$$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (4, 6), (6, 4), (3, 5), (5, 3)\}$$

Thus we have to find Warshall's sets,

$W_0, W_1, W_2, W_3, W_4, W_5$ and W_6 .

The first set W_0 is same as M_R . Which is shown below

	1	2	3	4	5	6
1	0	0	1	0	0	0
2	0	0	0	0	1	0
3	1	0	0	0	1	0
4	0	1	0	0	0	1
5	0	0	1	0	0	0
6	0	0	0	1	0	0

$$W_0 = M_R =$$

Step 2 : To find W_1 :

To find W_1 from W_0 , we consider the first column and first row.

In a column C_1 , 1 is present at R_3

In a row R_1 , 1 is present at C_3

Thus add new entry in W_1 , at (R_3, C_3) which is given below

0	0	1	0	0	0
0	0	0	1	0	0
1	0	1	0	1	0
0	1	0	0	0	1
0	0	1	0	0	0
0	0	0	1	0	0

Step 3 : To find W_2 :

To find W_2 from W_1 , we consider the second column C_2 and second row R_2 .

In C_2 , 1 is present at R_4

In R_2 , 1 is present at C_4

Thus add new entry in W_2 at (R_4, C_4) , which is given below

0	0	1	0	0	0
0	0	0	1	0	0
1	0	1	0	1	0
0	1	0	1	0	1
0	0	1	0	0	0
0	0	0	1	0	0

Step 4 : To find W_3 :

To find W_3 from W_2 , we consider the third column and third row.

In C_3 , 1 is present at R_1, R_3, R_5

In R_3 , 1 is present at C_1, C_3, C_5

Thus add new entries in W_3 at
 $(R_1, C_1), (R_1, C_3), (R_1, C_5)$
 $(R_2, C_1), (R_2, C_3), (R_2, C_5)$
 $(R_3, C_1), (R_3, C_3), (R_3, C_5)$ which
 is given below

Step 5 : To find W_4 :

To find W_4 from W_3 , we consider
 the fourth column and fourth row.

In C_4 , 1 is present at R_2, R_4, R_6

In R_4 , 1 is present at C_2, C_4, C_6

Thus add new entries in W_4 at
 $(R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2),$
 $(R_6, C_4), (R_6, C_6)$ which is given below

Step 6 : To find W_5 :

To find W_5 from W_4 , we consider the 5th column and 5th row.

In C_5 , 1 is present at R_1, R_3, R_5

In R_5 , 1 is present at C_1, C_3, C_5

Thus add new entries in W_5 at
 $(R_1, C_1), (R_1, C_3), (R_1, C_5),$
 $(R_3, C_1), (R_3, C_3), (R_3, C_5),$
 $(R_5, C_1), (R_5, C_3), (R_5, C_5)$
 which is given below

Step 7 : To find W_6 :

To find W_6 from W_5 we consider the 6th column and 6th row.

In C_6 , 1 is present at R_2, R_4, R_6

In R_6 , 1 is present at C_2, C_4, C_6

Thus add new entries in W_6 at $(R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2),$
 $(R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6)$ which is given below

$$W_6 = W_5 = W_4$$

Hence W_6 is the relation matrix of R^*

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$W_5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = W_4$$

$$\therefore R^* = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\}$$

Q.12 Let $R = \{(a, d), (b, a), (b, d), (c, b), (c, d), (d, c)\}$

Use Warshall's algorithm to find the matrix of transitive closure where $A = \{a, b, c, d\}$

[SPPU : Dec.-15]

Ans. : Step 1 : We have

$$A = \{a, b, c, d\}$$

∴

$$|A| = 4$$

$$R = \{(a, d), (b, a), (b, d), (c, b), (c, d), (d, c)\}$$

Thus we have to find Warshall's sets

$$W_0, W_1, W_2, W_3, W_4$$

$$\text{The first set } W_0 = M_R$$

Step 2 : To find W_1 :

To find W_1 from W_0 , we consider the first column and first row.

In C_1 , 1 is present at R_2

In R_1 , 1 is present at C_4

Thus add new entry in W_1 at (R_2, C_4)

	a	b	c	d
a	0	0	0	1
b	1	0	0	1
c	0	1	0	1
d	0	0	1	0

W ₁ =	[0	0	0	1
		1	0	0	1
		0	1	0	1
		0	0	1	0

Step 3 : To find W_2

To find W_2 , from W_1 , we consider the second column and second row.

In C_2 , 1 is present at R_3

In R_2 , 1 is present at C_1 and C_4

Thus add new entries in W_2 at (R_3, C_1) , (R_3, C_4) which is given below.

$\therefore W_2 =$	[0	0	0	1
		1	0	0	1
		1	1	0	1
		0	0	1	0

Step 4 : To find W_3 :

To find W_3 from W_2 , we consider the 3rd column and 3rd row.

In C_3 , 1 is present at R_4

In R_3 , 1 is present at C_1, C_2, C_4

Thus add new entries in W_3 at (R_4, C_1) , (R_4, C_2) , (R_4, C_4)

Which is given below

$W_3 =$	[0	0	0	1
		1	0	0	1
		1	1	0	1
		1	1	1	1

Step 5 : To find W_4 :

To find W_4 from W_3 , we consider the 4th column and 4th row.

In C_4 , 1 is present at R_1, R_2, R_3, R_4

In R_4 , 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_4 at $(R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_4, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)$

Which is given below

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence W_4 is the relation matrix of R^* :

and $R^* = \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d), (d, a), (d, b), (d, c), (d, d)\}$

Q.13 Find the transitive closure of the relation R on

$A = \{1, 2, 3, 4\}$ defined by

$R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)\}$

[SPPU : Dec.-07, May-15]

Ans. : Step 1 :

We have $|A| = 4$, Thus we have to find

Warshall's sets, W_0, W_1, W_2, W_3, W_4

The first set $W_0 = M_R$

Step 2 : To find W_1 :

To find W_1 from W_0 , we consider the first column and first row.

In C_1 , 1 is present at R_2

In R_1 , 1 is present at C_2, C_3, C_4

Thus add new entries in W_1 at $(R_2, C_2), (R_2, C_3), (R_2, C_4)$ which is given below

$$\therefore W_0 = M_R = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1 , we consider the 2nd column and 2nd row.

In C_2 , 1 is present at R_1, R_2, R_3, R_4

In R_2 , 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_2 at $(R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)$

Which is given below

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

All entries in W_2 are 1

Hence W_2 is the relation matrix of transitive closure of R .

$$\text{and } R^* = \{(1, 1), (1, 2) (1, 3) (1, 4) (2, 1) (2, 2) (2, 3) (2, 4) \\ (3, 1), (3, 2) (3, 3) (3, 4) (4, 1) (4, 2) (4, 3) (4, 4)\}$$

Q.14 Use Warshall's Algorithm to find transitive closure of R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A = \{1, 2, 3\}$$

[SPPU : May-06, May-08]

Solution : **Step 1 :** We have $|A| = 3$. Thus we have to find Warshall's sets W_0, W_1, W_2, W_3

The first set is

Step 2 : To find W_1 :

To find W_1 from W_0 , we consider the first column and the first row.

In C_1 , 1 is present at R_1, R_3

In R_1 , 1 is present at C_1 and C_3

Thus add new entries in W_1 at $(R_1, C_1), (R_1, C_3), (R_3, C_1), (R_3, C_3)$

Which is given below

$$W_0 = M_R = \boxed{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

$$W_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1 , we consider the 2nd column and 2nd row.

In C_2 , 1 is present at R_2, R_3

In R_2 , 1 is present at C_2 .

Thus add new entries in W_2 at $(R_2, C_2), (R_3, C_2)$ which is given below

$$W_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = W_1$$

Step 4 : To find W_3

To find W_3 from W_2 , we consider the 3rd column and 3rd row.

In C_3 , 1 is present at R_1, R_3

In R_3 , 1 is present at C_1, C_2, C_3

Thus add new entries in W_3 at $(R_1, C_1), (R_1, C_2), (R_1, C_3), (R_3, C_1)$
 $(R_3, C_2), (R_3, C_3)$ which is given below

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence W_3 is the relation matrix of R^*

and $R^* = \{(1, 1) (1, 2) (1, 3) (2, 2) (3, 1) (3, 2) (3, 3)\}$

4.3 : Partially Ordered Set

Important Points to Remember

A relation R on a set A is called a partially ordered relation iff R is reflexive, anti-symmetric and transitive relation.
The set A together with partially ordered relation is called a partially ordered set or POSET.

It is denoted by (A, R) or (A, \leq) where \leq is a partially ordered relation.

Examples :

1) (\mathbb{N}, \leq) (\mathbb{N} , \leq) are Posets.

where ' \leq ' is reflexive, antisymmetric and transitive relation.

2) If $A = P(S)$ where $S = \{a, b, c\}$ and for $X, Y \in A$, Define $X \leq Y$ or $X R Y$ iff $X \subseteq Y$.

As $X \leq X \Rightarrow X \subseteq X$. $\therefore \leq$ is reflexive.

If $X \leq Y, Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z \Rightarrow X \leq Z$

$\therefore \leq$ is antisymmetric relation.

If $X \leq Y, Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z \Rightarrow X \leq Z$

$\Rightarrow \leq$ is transitive relation.

$\therefore (P(S), \subseteq)$ or $(P(S), \leq)$ is a poset.

I) **Comparable elements** : Let (A, \leq) be a poset. Two elements a, b in A are said to be comparable elements if $a \leq b$ or $b \leq a$. Two elements a and b of a set A are said to be non-comparable if neither $a \leq b$ nor $b \leq a$.

In above example (2),

The comparable elements are

$$\{a\} = \{a, b\}, \{b\} = \{a, b, c\}, \{b, c\} \subseteq \{a, b, c\}$$

Non comparable elements are

$$\{a\} \not\subseteq \{b\}, \{a\} \not\subseteq \{c\}$$

II) **Totally ordered set** : Let A be any nonempty set. The set A is called linearly ordered set or totally ordered set if every pair of elements in A are comparable.

i.e. for any $a, b \in A$ either $a \leq b$ or $b \leq a$.

4.4 : Hasse Diagram

Important Points to Remember

It is useful tool, which completely describes the associated partially ordered relation. It is also known as ordering diagram.

A diagram of graph which is drawn by considering comparable and non-comparable elements is called Hasse diagram of that relation.

Therefore while drawing Hasse diagram following points must be followed.

- 1) The elements of a relation R are called vertices and denoted by points.
- 2) All loops are omitted as relation is reflexive on poset.
- 3) If aRb or $a \leq b$ then join a to b by a straight line called an edge the vertex b appears above the level of vertex a . Therefore the arrows may be omitted from the edges in Hasse diagram.
- 4) If $a \leq b$ and $b \leq a$ i.e. a and b are non-comparable elements, then they lie on same level and there is no edge between a and b .
- 5) If $a \leq b$ and $b \leq c$ then $a \leq c$. So there is a path $a \rightarrow b \rightarrow c$. Therefore do not join a to c directly i.e. delete all edges that are implied by transitive relation.

Q.15 Draw Hasse diagram of a poset $(P(s), \subseteq)$ where

$$S = \{a, b, c\}.$$

$$\text{Ans. : } P(s) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Now find the comparable and non comparable elements.

$$\emptyset \subseteq \{a\}, \emptyset \subseteq \{b\}, \emptyset \subseteq \{c\}, \therefore \{a\}, \{b\}, \{c\} \text{ lie above the level of } \emptyset$$

$$\{a\} \subseteq \{a, b\}, \{b\} \subseteq \{a, b\}, \{c\} \subseteq \{a, c\} \quad \therefore \{a, b\}, \{b, c\}, \{a, c\} \text{ lies above the level of } \{a\}, \{b\}, \{c\}.$$

$$\{a, b\} \subseteq S, \{b, c\} \subseteq S, \{a, c\} \subseteq S \quad \therefore S \text{ lies above the level of } \{a, b\}, \{a, c\}, \{b, c\}$$

But $\{a\}, \{b\}, \{c\}$ are non comparable $\therefore \{a\}, \{b\}, \{c\}$ lie on same level.

$\{a, b\}, \{a, c\}, \{b, c\}$ are non comparable \therefore lie on same level.

By considering the above observations, the Hasse diagram is as follows :

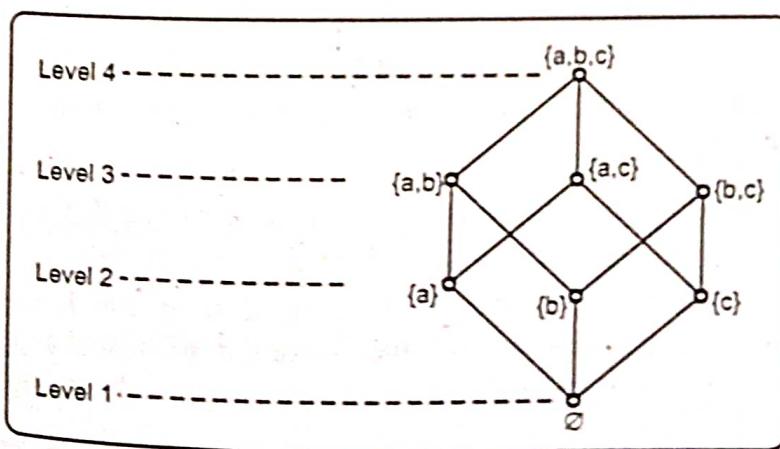


Fig. Q.15.1

4.5 : Chains , Antichains and Elements of Poset

Important Points to Remember

Let (A, \leq) be a poset. A subset of A is called a chain if every pair of elements in the subset are related.

A subset of A is called antichain if no two distinct elements in a subset are related.

1) The chains are

$\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}, \{\{a\}, \{a, c\}, \{a, b, c\}\}, \{\{b, c\}, \{a, b, c\}\}$

2) Antichains are $\{\{a\}, \{b\}, \{c\}\}$

Note : 1) The number of elements in the chain is called the length of chain.

2) If the length of chain is n in a poset (A, \leq) then the elements in A can be partitioned into n disjoint antichains.

3) Let (A, \leq) be a poset. An element $a \in A$ is called a maximal element of A if there is no element $c \in A$ such that $a \leq c$.

4) An element $b \in A$ is called a minimal element of A if there is no element $c \in A$ such that $c \leq b$

5) Greatest element : An element $x \in A$ is called a greatest element of A if for all $a \in A$, $a \leq x$. It is denoted by 1 and is called the unit element.

6) Least element : An element $y \in A$ is called a least element of A if for all $a \in A$, $y \leq a$.

It is denoted by 0 and is called as zero element.

7) Least upper bound (lub) : Let (A, \leq) be a poset. For $a, b, c \in A$, an element C is called upper bound of a and b if $a \leq c$ and $b \leq c$. C is called as least upper bound of a and b in A if C is an upper bound of a and b there is no upper bound d of a and b such that $d \leq c$. It is also known as supremum.

8) Greatest lower bound (glb) : Let (A, \leq) be a poset. for $a, b, l \in A$, an element l is called the lower bound of a and b if $l \leq a$ and $l \leq b$. An element l called the greatest lower bound of a and b if l is the lower bound of a and b and there is no lower bound f of a and b such that $l \leq f$.

glb is also called as infimum.

4.6 : Types of Lattices

Important Points to Remember

A lattice is a poset in which every pair of elements has a least upper bound (lub) and a greatest lower bound (glb).

Let (A, \leq) be a poset and $a, b \in A$ then lub of a and b is denoted by $a \vee b$. It is called the join of a and b .

$$\text{i.e. } a \vee b = \text{lub}(a, b)$$

The greatest lower bound of a and b is called the meet of a and b and it is denoted by $a \wedge b$

$$a \wedge b = \text{glb}(a, b)$$

\therefore From the above discussion, it follows that a lattice is a mathematical structure with two binary operations \vee (join) and \wedge (meet). It is denoted by (L, \vee, \wedge) .

Properties of a Lattice

Let (L, \wedge, \vee) be a lattice and $a, b, c \in L$. Then L satisfies the following properties.

1) Commutative property

$$a \wedge b = b \wedge a \text{ and } a \vee b = b \vee a$$

2) Associative law

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

3) Absorption law

$$a \wedge (a \vee b) = a \text{ and } a \vee (a \wedge b) = a$$

$$a \wedge a = a, a \vee a = a$$

$$a \wedge b = a \text{ iff } a \vee b = b$$

Types of Lattices

I) **Bounded lattice** : A lattice L is called a bounded lattice if it has a greatest element 1 and least element 0.

II) **Sublattice** : Let, (L, \vee, \wedge) be a lattice. A non empty subset L_1 of L is called a sublattice of L if L_1 itself is a lattice w.r.t. the operations of L .

III) **Distributive lattice** : A lattice (L, \vee, \wedge) is called a distributive lattice if for any elements $a, b, c \in L$, it satisfies the following properties,

$$\text{i) } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$\text{ii) } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

If the lattice does not satisfy the above properties then it is called a non distributive lattice.

Theorem : Let, (L, \wedge, \vee) be a lattice with universal bounds 0 and 1 then for any $a \in L$, $a \vee 1 = 1$, $a \wedge 1 = a$, $0 \vee a = a$, $0 \wedge a = 0$.

III) Complement lattice : Let (L, \wedge, \vee) be a lattice with universal bounds 0 and 1 for any $a \in L$, $b \in L$ is said to be complement of a if $a \vee b = 1$ and

$$a \wedge b = 0.$$

A Lattice in which every element has a complement in that lattice, is called the complemented lattice.

e.g. 1) The Hasse diagram is here $0 \approx 1$ and $1 \approx 30$.

$$\text{i) } 2 \wedge 3 = 0 \text{ and } 2 \vee 3 = 1, 2 \wedge 5 = 0 \text{ and } 2 \vee 5 = 1$$

$\therefore 2$ has two compliments 3 and 5

Hence the complement is not unique.

Principle of Duality

Any statement about lattice involving \wedge, \vee, \leq, \geq remains true if ' \wedge ' is replaced by ' \vee' ', ' \vee ' by ' \wedge' ', ' \leq ' by ' \geq ', ' \geq ' by ' \leq ', '0' by '1' and '1' by '0'.

$$\text{e.g. 1) } a \vee (b \wedge c) = a \wedge (b \vee c)$$

$$\text{2) } a \wedge (b \vee 1) = a \vee (b \wedge 0)$$

Q.16 Let A be the set of positive factors of 15 and R be a relation on A s.t. $R = \{xRy \mid x \text{ divides } y, x, y \in A\}$. Draw Hasse diagram and give and \wedge and \vee for lattice.

Ans. : We have $A = \{1, 3, 5, 15\}$

[SPPU : Dec.-06]

$$R = \{(1,1) (1,3) (1,5) (1, 15) (3,15) (5,15) (15,15)\}$$

Hasse diagram of R is :

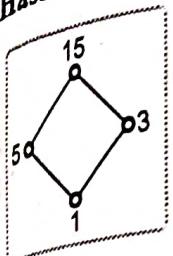


Fig. Q.16.1

Table for \wedge and \vee

\vee	1	3	5	15	\wedge	1	3	5	15
1	1	3	5	15	1	1	1	1	1
3	3	3	15	15	3	1	3	1	3
5	5	15	5	15	5	1	1	5	5
15	15	15	15	15	15	1	3	5	15

Every pair of elements has lub and glb. \therefore It is a lattice.

Q.17 Let $A = \{1, 2, 3, 4, 6, 9, 12\}$ Let a relation R on a set A is $R = \{(a,b) / a \text{ divides } b \ \forall a, b \in A\}$. Give list of R . Prove that it is a partial ordering relation. Draw Hasse diagram of the same. Prove or disprove it is a lattice.

[SPPU : Dec.-11]

Ans. : We have $A = \{1, 2, 3, 4, 6, 9, 12\}$

and $R = \left\{ (1,1), (1,2), (1,3), (1,4), (1,6), (1,9), (1,12), (2,2), (2,4), (2,6), (2,12), (3,3), (3,6), (3,9), (3,12), (4,4), (4,12), (6,6), (6,12), (9,9), (12,12) \right\}$

We know that for any $a \in A$, $a | a \therefore aRa$

$\therefore R$ is a reflexive relation.

As $a | b$ and $b | a \Rightarrow a = b \therefore R$ is antisymmetric relation.

As $a | b$ and $b | c \Rightarrow a | c \Rightarrow R$ is a transitive relation.

$\therefore R$ is reflexive antisymmetric and transitive

$\therefore (A, R)$ is a poset and R is a partial ordering relation.

Hasse diagram is as follows :

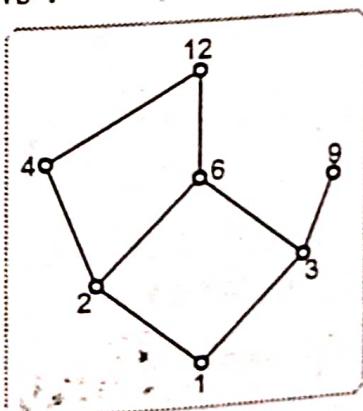


Fig. Q.17.1

In above diagram $6 \vee 9$ does not exist. \therefore It is not a lattice.

Q.18 Determine whether the poset represented by each of the Hasse diagram are lattices. Justify your answer.

[SPPU : Dec.-10]

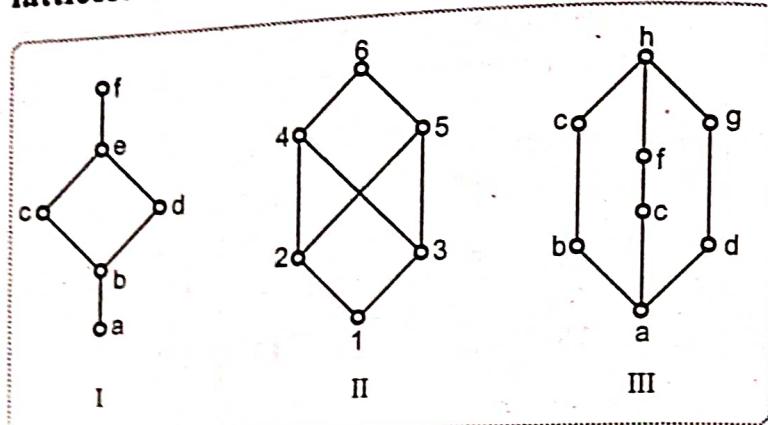


Fig. Q.18.1

Ans. : I) In Fig. Q.18.1 (I), every pair of element has glb and lub. \therefore It is a lattice.

II) In Fig. Q.18.1 (II), every pair of elements has lub and glb. \therefore It is a lattice.

III) In Fig. Q.18.1 (III), every pair of elements has lub and glb. \therefore It is a lattice.

Q.19 Show that the set of all divisors of 36 forms a lattice.

[SPPU : Dec.-14]

Ans. : Let $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ and Let ' \leq ' is a divisor of. Its Hasse diagram is as follows.

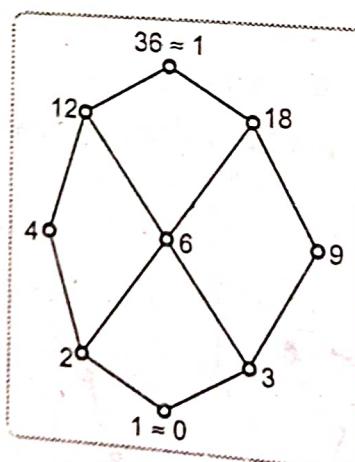


Fig. Q.19.1

The universal upper bound 1 is 36 and lower bound 0 is 1. Every pairs of elements of this poset has lub and glb.
 \therefore It is a lattice.

Q.20 Let n be a positive integer, S_n be the set of all divisors of n , let D denote the relation of divisor. Draw the diagram of lattices for $n = 24, 30, 6$.

☞ [SPPU : May-15]

Ans. : Given that

We have $S_6 = \{1, 2, 3, 6\}$, D is the relation of divisor.

- $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$
- $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Diagrams of Lattices are as follows.

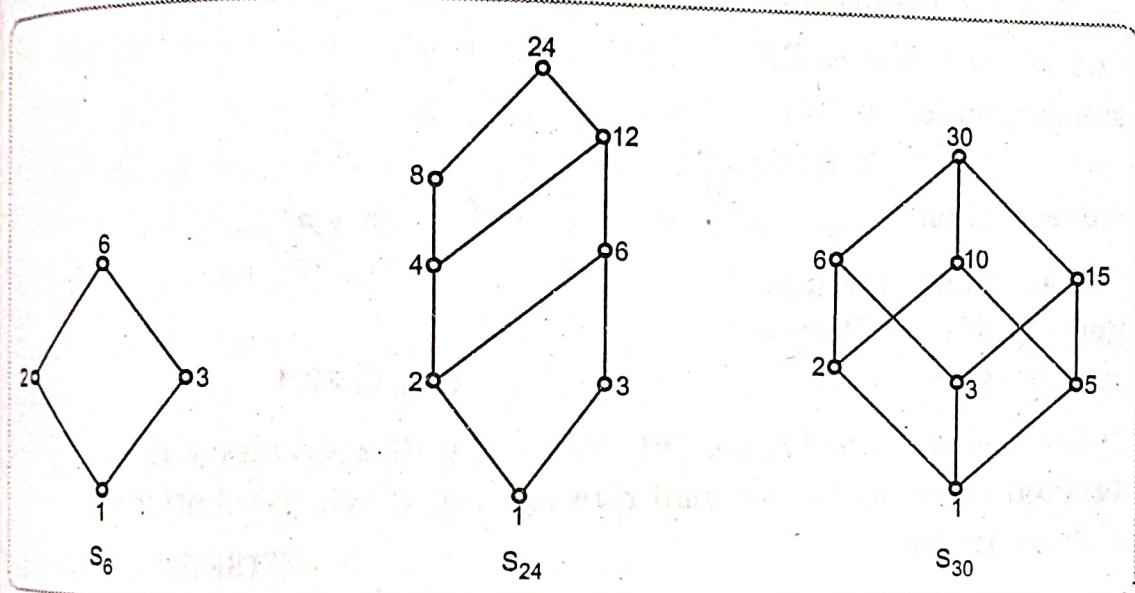


Fig. Q.20.1

Q.21 Show that the set of all divisors of 70 forms a lattice.

☞ [SPPU : Dec.-13]

Ans. : Let $A = \{1, 2, 5, 7, 10, 14, 35, 70\}$

and Let ' \leq ' is "a divisor of".

The universal upper bound 1 is 70 and the lower bound 0 is 1.

Its Hasse diagram is as follows :

Every pair of elements of A has \wedge and \vee .

\therefore It is a lattice [write table of \wedge and \vee].

Q.22 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$ be ordered by the relation x divides y . Show that the relation is a partial ordering and draw Hasse diagram.

☞ [SPPU : Dec-15]

Ans. : We have $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$

$$R = \{(x, y) \mid x \text{ divides } y, \text{ for } x, y \in A\}$$

$$\begin{aligned} R = & \{(1, 2) (1, 3), (1, 4), (1, 5) (1, 6) (1, 7) (1, 8) (1, 9) \\ & (1, 12) (1, 18) (1, 24) \\ & (2, 2) (2, 4) (2, 6) (2, 8) (2, 12) (2, 18) (2, 24) (5, 5) \\ & (6, 6) (6, 12) (6, 18) \\ & (6, 24), (7, 7) (8, 8) (8, 24) (9, 9) (9, 18) (12, 12) \\ & (12, 24) (18, 18) (4, 24)\} \end{aligned}$$

We have for any $x \in A$,
 $x|x \Rightarrow R$ is a reflexive for
 $x|y$ and $y|x \Rightarrow x = y \Rightarrow R$
is antisymmetric. If $x|y$
and $y|z \Rightarrow x|z \therefore R$ is a
transitive relation.

$\therefore R$ is a partial ordering
relation, It's Hasse
diagram is as follows.

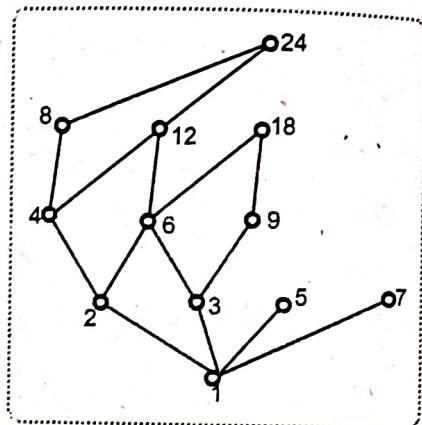


Fig. Q.22.1

Q.23 Let $x = \{2, 3, 6, 12, 24, 36\}$ and $x \leq y$ iff x divides y find
i) Maximal element ii) Minimal element iii) Chain iv) Antichain
v) Is Poset lattice

[SPPU : May-14]

Ans. : We have $x = \{2, 3, 6, 12, 24, 36\}$

The relation ' R ' = ' \leq '

$\therefore R = \{(2, 2) (2, 6) (2, 12) (2, 24) (2, 36) (3, 3) (3, 6) (3, 12) (3, 24)$
 $(3, 36), (6, 6) (6, 12) (6, 24) (6, 36) (12, 12) (12, 24) (12, 36) (24, 24)$
 $(36, 36)\}.$

It's Hasse diagram is as follows .

- i) Maximal elements are 24, 36
- ii) Minimal elements are 2, 3
- iii) Chain { 2, 6, 12, 24}, { 2, 6, 12, 36},
 $\{3, 6, 12, 24\}, \{3, 6, 12, 36\}$
- iv) Antichain : {2, 3} {24, 36}
- v) The given poset is not a lattice as $2 \wedge 3$ does not exist.

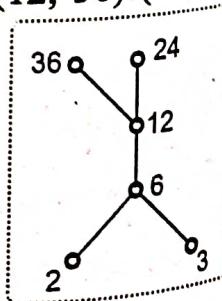


Fig. Q.23.1

4.7 : Functions

Important Points to Remember

1. Function : Let A and B be two non empty sets. A function from A to B, denoted as $f : A \rightarrow B$, is a relation from A to B such that for every element $a \in A$, there exists a unique element $b \in B$ such that $(a, b) \in f$ or $f(a) = b$.

where A = Domain set,

 B = Codomain set

$$\begin{aligned}\text{Range set} &= \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\} \\ &= \{\text{Set of all image points}\}\end{aligned}$$

2. Equality of functions : two functions f and g from A to B are said to be equal iff $f(a) = g(a) \forall a \in A$. we write as $f \equiv g$.

3. Types of functions :

(a) *Identity function* : Let A be any non empty set. A function f from $A \rightarrow A$ is said to be the Identity function iff $f(a) = a, \forall a \in A$.

(b) *One to one function* : (Injective function) : A function $f : A \rightarrow B$ is said to be one to one function if for all elements $x, y \in A$ such that $f(x) = f(y) \Rightarrow x = y$.

(c) *Into function* : A function $f : A \rightarrow B$ is said to be into function if range set of f is a proper subset of B i.e. \exists at least one element in B which has no pre-image in A.

(d) *Onto function (Surjective function)* : A function $f : A \rightarrow B$ is said to be onto function if range set of f is equal to set B i.e. every element in B has pre-image in A.

(e) *Bijective function* : A function which is one to one (injective) as well as onto (surjective) is called bijective function.

(f) *Inverse function* : Let A, B any non-empty sets and $f : A \rightarrow B$ is a bijection function. The inverse function $f^{-1} : B \rightarrow A$ is a function defined as

$$f^{-1}(b) = a \text{ iff } f(a) = b \text{ for } a \in A \text{ and } b \in B$$

(g) **Composite function** : Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions. The composition of f and g is denoted by gof and $gof : A \rightarrow C$ such that

$$(gof)(x) = g[f(x)], \forall x \in A.$$

Q.24 Explain classification of functions with example.

Ans. : Depending upon the nature of function, there are mainly two functions.

1. Algebraic function : A function which consists of a finite number of terms involving powers and roots of the independent variable and four fundamental operations addition, subtraction, multiplication and division, is called an algebraic function.

There are three types of algebraic functions.

a) **Polynomial function** : A function of the form

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n, \text{ where } n \text{ is positive integer,}$$

$a_0, a_1, a_2, \dots, a_n$ are real numbers and $a_0 \neq 0$, is called polynomial function of x in degree n .

$$\text{e.g. } f(x) = x^3 - 3x^2 + 2x - 5$$

b) **Rational function** : A function of the form $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions and $g(x) \neq 0$ is called rational function e.g. $\frac{x^2 - 3x + 5}{x^2 + 1}$

c) **Irrational function** : A function involving radicals is called irrational function. e.g. $f(x) = x^{2/3} + 5x^2 + 1, \sqrt[3]{x+1}$.

2. Transcendental function : A function which is not algebraic is called transcendental function.

$$\text{e.g. } f(x) = \sin x + x^3 + 5x$$

a) **Trigonometric function** : The six functions $\sin x, \cos x, \tan x, \sec x, \operatorname{cosec} x, \cot x$, where x is in radians are called trigonometric functions.

$$\text{e.g. } f(x) = \sin x + \tan x.$$

- b) *Inverse trigonometric function* : The six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are called inverse trigonometric functions.
e.g. $f(x) = \cos^{-1} x + 5 \tan^{-1} x$
- c) *Exponential function* : A function of the form $f(x) = a^x$ ($a > 0$) satisfying $a^x \cdot a^y = a^{x+y}$ and $a' = a$ is called exponential function. e.g. $f(x) = 5^x$.
- d) *Logarithmic function* : The inverse function of the exponential function is called logarithmic function. e.g. $f(x) = \log x$.

Q.25 Let $f(x) = x+2$, $g(x) = x - 2$, $h(x) = 3x$, for $x \in \mathbb{R}$ Where \mathbb{R} is the set of real numbers. Find i) gof ii) fog iii) fof iv) hog v) gog vi) foh vii) hof viii) fohog ix) gofoh . [SPPU : May-08, 15, Dec.-12, Marks 4]

Ans. : Let $x \in \mathbb{R}$ be any real number.

$$\begin{aligned} \text{i)} \quad \text{gof}(x) &= g[f(x)] = g[x+2] = x+2-2=x \\ \text{ii)} \quad \text{fog}(x) &= f[g(x)] = f[x-2] = x-2+2=x \\ \text{iii)} \quad \text{fof}(x) &= f[f(x)] = f[x+2] = x+2+2=x+4 \\ \text{iv)} \quad \text{hog}(x) &= h[g(x)] = h[x-2] = 3(x-2) = 3x-6 \\ \text{v)} \quad \text{gog}(x) &= g[g(x)] = g[x-2] = x-2-2=x-4 \\ \text{vi)} \quad \text{foh}(x) &= f[h(x)] = f[3x] = 3x+2 \\ \text{vii)} \quad \text{hof}(x) &= h[f(x)] = h[x+2] = 3(x+2) = 3x+6 \\ \text{viii)} \quad \text{fohog}(x) &= f[h(g(x))] = f[h(x-2)] = f[3(x-2)] = f(3x-6) \\ &= 3x-6+2=3x-4 \\ \text{ix)} \quad \text{gofoh}(x) &= g[f(h(x))] = g[f(3x)] \\ &= g[3x+2] = 3x+2-2=3x \end{aligned}$$

Q.26 Let functions f and g be defined by $f(x) = 2x + 1$, $g(x) = x^2 - 2$
Find a) $\text{gof}(4)$ and $\text{fog}(4)$ b) $\text{gof}(a+2)$ and $\text{fog}(a+2)$
c) $\text{fog}(5)$ d) $\text{gof}(a+3)$ [SPPU : May-07, Dec.-07, Marks 4]

Ans. : a) $\text{gof}(4) = g[f(4)] = g[2(4)+1] = g[9] = 9^2 - 2 = 79$
 $\text{fog}(4) = f[g(4)] = f(4^2 - 2) = f(14) = 2(14) + 1 = 29$

b) $\text{gof}(a+2) = g[f(a+2)] = g[2(a+2)+1] = g[2a+5]$
 $= (2a+5)^2 - 2 = 4a^2 + 20a + 23$

$$\begin{aligned} \text{fog}(a+2) &= f[g(a+2)] = f[(a+2)^2 - 2] = f[a^2 + 4a + 2] \\ &= 2[(a^2 + 4a + 2) + 1] = 2a^2 + 8a + 5 \end{aligned}$$

c) $\text{fog}(5) = f[g(5)] = f[25 - 2] = f(23) = 2(23) + 1 = 47$

d) $\text{gof}(a+3) = g[f(a+3)] = g[2(a+3) + 1] = g[2a + 7]$
 $= [2a + 7]^2 - 2 = 4a^2 + 28a + 47$

Q.27 Determine if each is a function. If yes is it injective, surjective, bijective ?

- a) Each person in the earth is assigned a number which corresponds to his age.
- b) Each student is assigned a teacher.
- c) Each country has assigned its capital.  [SPPU : Dec.-11, Marks 4]

Ans. : a) Every person has unique age

∴ It is a function. Two person's may have same age. ∴ It is not injective. There is no person whose age is 300 years. ∴ It is not surjective. ∴ Function is not bijective.

b) It is a function. It is not injective. It is not surjective. ∴ It is not bijective.

c) It is a function. It is injective as well as surjective. ∴ It is bijective.

Q.28 Let $A = B$ be the set of real numbers.

$$f : A \rightarrow B \text{ given by } f(x) = 2x^3 - 1$$

$$g : B \rightarrow A \text{ given by } g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$$

Show that f is a bijective function and g is also bijective function :

Ans. : 1) Suppose $f(x_1) = f(x_2)$

 [SPPU : Dec.-10, Marks 4]

$$2x_1^3 - 1 = 2x_2^3 - 1$$

$$\Rightarrow x_1^3 = x_2^3$$

$$\Rightarrow x_1 = x_2$$

∴ f is injective mapping

Let $y \in B$ and $f(x) = y \Rightarrow 2x^3 - 1 = y$

$$\therefore 2x^3 = 1+y \Rightarrow x^3 = \frac{1+y}{2}$$

$$x = \sqrt[3]{\frac{1+y}{2}} = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} \in A = \mathbb{R} \text{ for any } y \in B$$

\Rightarrow $\therefore f$ is a surjective mapping.

Thus f is injective as well as surjective function.

Hence f is a bijective function.

$$\text{We have } f(x) = y \Rightarrow x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} \Rightarrow f^{-1}(y) = x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} = g(y)$$

Thus $f^{-1} = g$. We know that if f is bijective function then f^{-1} is also bijective. Hence g is bijective function.

Q.29 If $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijective functions the gof is also bijective.

Ans.: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two bijective functions then $gof : A \rightarrow C$ is a function.

(i) Let $x_1, x_2 \in A$ and suppose $gof(x_1) = gof(x_2)$

$$g[f(x_1)] = g[f(x_2)]$$

... ($\because g$ is 1 - 1 function)

$$\Rightarrow f(x_1) = f(x_2)$$

... ($\because f$ is 1 - 1 function)

$$\Rightarrow x_1 = x_2$$

$\Rightarrow gof$ is 1 - 1 function.

(ii) Let $z \in C$.

$\therefore \exists y \text{ in } B \text{ such that } g(y) = z \text{ and } \exists x \text{ in } A \text{ such that } f(x) = y$.

$\therefore g(y) = z \Rightarrow z = g(f(x)) = gof(x) \text{ where } x \in A$

$\Rightarrow gof$ is onto function.

Hence gof is bijective function.

Q.30 Let $f : X \rightarrow Y$ and X and Y are set of real numbers. Find f^{-1} if

$$(i) f(x) = x^2 ; (ii) f(x) = \frac{2x-1}{5}$$

Ans.: Let $f : X \rightarrow Y$ and $X, Y \subseteq \mathbb{R}$

$$(i) \quad f(x) = x^2$$

$$\text{Let } f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow y = x^2$$

$$\Rightarrow y = \pm \sqrt{x}$$

Therefore, given function is not one to one. Hence f^{-1} does not exist.

(ii) Let $f(x) = y$

$$\frac{2x-1}{5} = y$$

$$\Rightarrow 2x = 5y + 1$$

$$x = \frac{5y+1}{2} \quad \forall y \in \mathbb{R}$$

Function f is $1 - 1$ and onto.

\therefore The inverse function of f is given as,

$$f^{-1}(x) = \frac{1+5x}{2}$$

4.8 : Pigeon Hole Principle

Important Points to Remember

I) This principle states that if there are $n + 1$ pigeons and only n pigeon holes then two pigeons will share the same hole.

This principle is stated by using the analogy of the bijective mapping i.e. If A and B are any two sets such that $|A| > |B|$ then there does not exist bijective mapping from A to B .

II) The extended pigeon hole principle

If n pigeons are assigned to m pigeon holes, then one of the pigeon holes must be occupied by at least $\left\lceil \frac{n-1}{m} \right\rceil + 1$ pigeons. It is also known as

generalized pigeon hole principle. Here $\left\lceil \frac{n-1}{m} \right\rceil$ is the integer division of $n-1$ by m . e.g. $\left\lceil \frac{9}{2} \right\rceil = 4$, $\left\lceil \frac{16}{5} \right\rceil = 3$, $\left\lceil \frac{8}{3} \right\rceil = 2$.

Q.31 If 11 shoes are selected from 10 pairs of shoes then there must be a pair and matched shoes among the selection.

Ans. : In the pigeonhole principle, 11 shoes are pigeons and the 10 pairs are the pigeon holes.

Q.32 Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13.

Ans. : We have $A = \{1, 2, 3, 4, 5, \dots, 12\}$

We form the six different sets each containing 2 numbers that add upto 13.

$$A_1 = \{1, 12\}, A_2 = \{2, 11\}, A_3 = \{3, 10\}, A_4 = \{4, 9\}, A_5 = \{5, 8\}, \\ A_6 = \{6, 7\}$$

Each of the seven numbers chosen must belong to one of these sets. As there are only six sets, by pigeonhole principle two of the chosen numbers must belong to the same set and their sum is 13.

Q.33 Show that 7 colours are used to paint 50 bicycles, then at least 8 bicycles will be of same colour.  [SPPU : Dec.-09, 12, Marks 4]

Ans. : By the extended pigeonhole principle, at least $\left[\frac{n-1}{m} \right] + 1$ pigeons will occupy one pigeonhole.

Here $n = 50$, $m = 7$ and $m < n$ then

$$\left[\frac{50-1}{7} \right] + 1 = 7 + 1 = 8$$

Thus 8 bicycles will be of the same colour.

Q.34 Write generalized pigeonhole principle. Use any form of pigeonhole principle to solve the given problem.

i) Assume that there are 3 mens and 5 womens in a party show that if these people are lined up in a row at least two women will be next to each other.

ii) Find the minimum number of students in the class to be sure that three of them are born in the same month.

 [SPPU : Dec.-11, Marks 4]

Ans. :

This principle states that if there are $n + 1$ pigeons and only n pigeon holes then two pigeons will share the same whole.

This principle is stated by using the analogy of the bijective mapping i.e. If A and B are any two sets such that $|A| > |B|$ then there does not exist bijective mapping from A to B .

i) By using analogy of pigeon hole principle, we get

3 men = pigeonholes and 5 women = pigeons

Pigeons are more than pigeon holes.

∴ At least two pigeons share the same pigeon hole i.e. at least two women in a row will be next to each other.

ii) Let $h = \text{Number of pigeons} = \text{Number of students}$
 $n = \text{Number of pigeon holes} = \text{Number of months} = 12$

Given that three students in the class are born in the same month.

$$\therefore \left[\frac{n-1}{m} \right] + 1 = 3$$

$$\Rightarrow \frac{n-1}{12} = 3 - 1 = 2$$

$$n = 2 \times 12 + 1 = 25$$

Therefore there are 25 minimum number of students in the class.

4.9 : Generating Function

Important Points to Remember

Let $a = \{a_0, a_1, a_2, \dots, a_r, \dots\}$ be a numeric function.

An infinite series

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_r z^r + \dots$$

is called the generating function of the sequence or numeric function a .

Note : The sum and product of two generating functions is a generating function.

I) Types of generating functions

Depending upon the nature of numeric sequence, there are two types of generating functions.

1) Ordinary generating function :

An infinite sum $\sum_{r=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + \dots + a_r z^r + \dots$ is

called the ordinary generating function of the numeric function $\{a_r\}_{r=0}^{\infty}$.

2) Exponential generating function :

An infinite sum $\sum_{r=0}^{\infty} a_r \frac{z^r}{r!} = a_0 + a_1 z + a_2 \frac{z^2}{2!} + a_3 \frac{z^3}{3!} + \dots + a_r \frac{z^r}{r!}$

+ ... is called an exponential generating function of $\{a_r\}_{r=0}^{\infty}$.

III) Generating functions for some numeric functions

Sr. No.	Numeric function (a_r)	Generating function
1	a^r	$\frac{1}{1 - az}$
2	$k a^r$	$\frac{k}{1 - az}$
3	r	$\frac{z}{(1 - z)^2}$
4	$r + 1$	$\frac{1}{(1 - z)^2}$
5	$(r + 1) a^r$	$\frac{1}{(1 - az)^2}$
6	$r a^r$	$\frac{az}{(1 - az)^2}$
7	$\frac{1}{r!}$	e^z
8	$n C_r ; 0 \leq r \leq n$ $0 ; r > n$	$(1 + z)^n$

Q.35 Find the generating function for $a_r = \begin{cases} 0 & ; r = \text{odd} \\ 2^{r+1} & ; r = \text{even} \end{cases}$

Ans. : The numeric function is

$$\{a_r\}_{r=0}^{\infty} = \{2, 0, 2^3, 0, 2^5, 0, 2^7, \dots\}$$

$$\therefore a_r = 2^r + (-2)^r ; r \geq 0$$

∴ It's generating function is

$$A(z) = \frac{1}{1-2z} + \frac{1}{1+2z} = \frac{2}{1-4z^2}$$

Q.36 Find the numeric function for $A(z) = \frac{2}{1-4z^2}$.

[SPPU : Dec.-05, 07, Marks 4]

functions

Ans. : We have $A(z) = \frac{2}{1-4z^2} = \frac{2}{(1+2z)(1-2z)}$

$$= \frac{A}{1-2z} + \frac{B}{1+2z}$$

$$= \frac{1}{1-2z} + \frac{2}{1+2z}$$

$$A(z) = \sum_{r=0}^{\infty} (2z)^r + \sum_{r=0}^{\infty} (-2z)^r$$

Thus numeric function for $A(z)$ is

$$a_r = 2^r + (-2)^r ; r \geq 0$$

4.10 : Recurrence Relation

I) In the previous section, we have studied about the generating functions and its discrete numeric functions or sequences. In that computations, a_r is given in terms of r only. But there so many problems in which previous are used to define next terms i.e. to define a_r we use a_{r-1} , a_{r-2} and so on.

A relation in which previous terms are used to define next terms, is called the **recurrence relation**. A recurrence relation is also called as difference equation.

e.g. i) $a_r = a_{r-1} + a_{r-2} ; r \geq 2$

ii) $a_r = a_{r-1} + 5$ are recurrence relations

II) A recurrence relation of the form

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \dots + c_k a_{r-k} = f(r) \text{ for } r \geq k \quad \dots (i)$$

Where $c_0, c_1, c_2, \dots, c_k$ are constants, is called a linear recurrence relation with constant coefficients of order k provided c_0 and c_k are non zero.

e.g. $a_r - 3a_{r-1} = 2r^2$

is the first order linear recurrence relation with coefficients.

$$a_r + a_{r-1} + 5a_{r-3} = 0$$

is the third order linear recurrence relation with coefficients.

III) A recurrence relation (i) is called **homogeneous recurrence relation** if $f(r) = 0$.

$$\text{i.e. } c_0 a_r + c_1 a_{r-1} + \cdots + c_k a_{r-k} = 0$$

IV) The **characteristic equation** of the k^{th} order homogeneous relation is $c_0 \alpha^k + c_1 \alpha^{k-1} + c_2 \alpha^{k-2} + \cdots + c_k = 0$ which is the k^{th} degree polynomial in α .

The roots of this equation are called the characteristic roots.

A solution of a recurrence relation which is obtained from homogeneous recurrence relation is called the homogeneous solution. It is denoted by $a_r^{(h)}$.

The homogeneous solution depends upon the nature of characteristic roots. The following table gives the details about homogeneous solution and roots of characteristic equations.

Sr. No.	Nature of roots	Homogeneous solution $a_r^{(h)}$
1.	Distinct real roots i) α_1, α_2 ii) $\alpha_1, \alpha_2 \dots \alpha_n$	$a_r^{(h)} = A_1 \alpha_1^r + A_2 \alpha_2^r$ $a_r^{(h)} = A_1 \alpha_1^r + A_2 \alpha_2^r + \dots + A_n \alpha_n^r$
2.	Repeated real roots i) α_1, α_1 ii) $\alpha_1, \alpha_1, \alpha_1$	$a_r^{(h)} = (A_1 + A_2 r) \alpha_1^r$ $a_r^{(h)} = (A_1 + A_2 r + A_3 r^2) \alpha_1^r$
3.	Distinct complex roots $\alpha + i\beta, \alpha - i\beta$	$a_r^{(h)} = r_1^r (A \cos r\theta + B \sin r\theta)$ where $r_1 = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1} \left(\frac{\beta}{\alpha} \right)$

V) Particular solution

The solution which satisfies the linear recurrence relation with $f(r)$ on right side is called the particular solution. It is denoted by $a_r^{(p)}$. There is no general method to determine the particular solution. It depends on the nature of $f(r)$. Consider the following table of particular solutions for given right hand side function $f(r)$.

Sr. No.	$f(r)$ (Right hand side function)	Form of particular solution
1.	$f(r) = d = \text{constant}$	$a_r^{(p)} = p$
2.	$d_0 + d_1 r$	$p_0 + p_1 r$
3.	$d_0 + d_1 r + d_2 r^2 + \dots + d_n r^n$	$p_0 + p_1 r + p_2 r^2 + \dots + p_n r^n$
4.	$d b^r$ if b is not characteristic root	$p b^r$
5.	$d b^r$ if b is the root of characteristic equation with multiplicity m	$p r^m b^r$

VI) Total solution

The total solution or general solution of a recurrence relation is the sum of homogeneous solution and particular solution.

i.e. $a_r = a_r^{(h)} + a_r^{(p)}$

Q.37 Solve the following recurrence relation :

$$a_r - 3a_{r-1} = 2, r \geq 1, a_0 = 1$$

[SPPU : May-07, Marks 4]

Ans. : Step 1 : The characteristic equation is

$$(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 3$$

Which is real and distinct characteristic root
Hence homogeneous solution is

$$a_r^{(h)} = A(3)^r$$

Particular solution is of the type P (constant)

$$\text{Hence } a_r = P$$

$$\text{Therefore } a_{r-1} = P$$

Substituting the value of a_r and a_{r-1} in the given recurrence relation.

$$P - 3P = 2$$

$$\Rightarrow -2P = 2$$

$$\Rightarrow P = -1$$

$$\text{Hence } a_r = A(3)^r - 1$$

Using initial condition $a_0 = 1$

$$1 = A - 1 \Rightarrow A = 2$$

Hence $a_n = 2(3^n) - 1$

Q.38 Find the homogeneous solution for the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \text{ with } a_0 = 2, a_1 = 5, a_2 = 15$$

[SPPU : Dec.-10, 12, Marks 6]

Ans. : We have

$$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0 \quad \dots (\text{Q.38.1})$$

Step 1 : The characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$$

$\alpha = 1$ is a trial root

\therefore By synthetic division method

1	1	-6	11	-6
		1	-5	6
		1	-5	6
			0	

$$\therefore (\alpha - 1)(\alpha^2 - 5\alpha + 6) = 0$$

$$(\alpha - 1)(\alpha - 2)(\alpha - 3) = 0$$

$\alpha = 1, 2, 3$ are real and distinct roots

\therefore Its homogeneous solution is

$$a_n^{(h)} = A_1 1^n + A_2 2^n + A_3 3^n \quad \dots (\text{Q.38.2})$$

Step 2 : But given that $a_0 = 2, a_1 = 5, a_2 = 15$

$$\therefore a_0 = A_1 + A_2 + A_3 \Rightarrow 2 = A_1 + A_2 + A_3 \quad \dots (\text{Q.38.3})$$

$$a_1 = A_1 + 2A_2 + 3A_3 \Rightarrow 5 = A_1 + 2A_2 + 3A_3 \quad \dots (\text{Q.38.4})$$

$$a_2 = A_1 + 4A_2 + 9A_3 \Rightarrow 15 = A_1 + 4A_2 + 9A_3 \quad \dots (\text{Q.38.5})$$

equation (Q.38.4) + equation (Q.38.3) and equation (Q.38.5) - equation (Q.38.3)

$$\Rightarrow 3 = A_2 + 2A_3 \quad \dots (\text{Q.38.6})$$

$$\text{and } 13 = 3A_2 + 8A_3 \quad \dots (\text{Q.38.7})$$

equation (Q.38.7) - 3 × equation (Q.38.6)

$$4 = 0 + 2 A_3 \Rightarrow$$

$$\boxed{A_3 = 2}$$

$$\text{equation (Q.38.6)} \Rightarrow A_2$$

$$= 3 - 2 A_3 = 3 - 4 = -1$$

$$\text{equation (Q.38.3)} \Rightarrow$$

$$A_1 = 2 - A_2 - A_3 = 2 - (-1) - 2 = 1$$

$$\text{Hence } a_n^{(h)} = 1 + (-1) 2^n + 2 (3)^n$$

Q.39 Solve $a_r - 4a_{r-1} + 4a_{r-2} = 0$ given that $a_0 = 0$ and $a_1 = 6$.

[SPPU : Dec.-16]

Ans. : We have

$$a_r - 4a_{r-1} + 4a_{r-2} = 0$$

... (Q.39.1)

Step 1 : The characteristic equation is

$$\alpha^2 - 4\alpha + 4 = 0$$

$$(\alpha - 2)^2 = 0$$

$\alpha = 2, 2$ which are real and repeated.

∴ The homogeneous solution is

$$a_r = (A_1 + A_2 r) 2^r$$

... (Q.39.2)

Step 2 : Given that, $a_0 = 0$ and $a_1 = 6$

∴ Put $r = 0$ in equation (Q.39.2), we get

$$a_0 = (A_1 + 0) 2^0 \Rightarrow A_1 = 0$$

∴ Put $r = 1$ in equation (Q.39.2), we get

$$a_1 = (A_1 + A_2) 2^1$$

$$6 = (0 + A_2) 2$$

$$\Rightarrow A_2 = 3 \text{ and } a_r^{(p)} = 0$$

∴ Equation (Q.39.2) becomes,

$$a_r = 3r 2^r$$

Q.40 Solve the recurrence relation :

$$a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 2^n, n \geq 3 \quad a_0 = 0, a_1 = 0, a_2 = 2$$

[SPPU : May-05, Marks 6]

Ans. : Step 1 : The characteristic equation is

$$\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$$

$$\Rightarrow (\lambda + 2)^3 = 0$$

Therefore $\lambda = -2, -2, -2$ which are real and repeated roots

Hence the homogeneous solution is

$$a_n^{(h)} = (A_1 n^2 + A_2 n + A_3) (-2)^n$$

Step 2 : The particular solution will be of the form $P(2^n)$. Substituting the given recurrence relation,

we get,

$$P(2^n) + 6P(2^{n-1}) + 12P(2^{n-2}) + 8P(2^{n-3}) = 2^n$$

$$\Rightarrow 2^n P \left[1 + \frac{6}{2} + \frac{12}{4} + \frac{8}{8} \right] = 2^n$$

$$\text{Or } P[1 + 3 + 3 + 1] = 1 \Rightarrow P = \frac{1}{8}$$

$$\therefore a_n^{(P)} = \frac{1}{8}(2^n) = 2^{n-3}$$

Step 3 : Total solution is

$$a_n = a_n^{(h)} + a_n^{(P)} = (A_1 n^2 + A_2 n + A_3) (-2)^n + 2^{n-3}$$

Using the initial conditions

$$a_0 = 0, a_1 = 0, a_2 = 2$$

$$0 = A_3 + \frac{1}{8}$$

$$\Rightarrow A_3 = -\frac{1}{8} \quad \dots (Q.40.1)$$

$$0 = (A_1 + A_2 + A_3)(-2)^1 + 2^{1-3}$$

$$A_1 + A_2 + A_3 = \frac{1}{8} \quad \dots (Q.40.2)$$

Also $2 = (A_1 \cdot 4 + A_2 \cdot 2 + A_3) (-2)^2 + 2^{2-3}$

$$2 = (4A_1 + 2A_2 + A_3)(4) + \frac{1}{2}$$

$$\frac{3}{2} = 4(4A_1 + 2A_2 + A_3)$$

$$\Rightarrow 4A_1 + 2A_2 + A_3 = \frac{3}{8} \quad \dots (\text{Q.40.3})$$

From equations (Q.40.1), (Q.40.2) and (Q.40.3) $A_1 = 0$, $A_2 = \frac{1}{4}$,
 $A_3 = -\frac{1}{8}$

$$a_n = \left(\frac{1}{4}n - \frac{1}{8} \right) (-2)^n + 2^{n-3}$$

END... ↗