SUBJECT CODE : 210241

Strictly as per Revised Syllabus of

SAVITRIBAI PHULE PUNE UNIVERSITY

Choice Based Credit System (CBCS)

S.E. (Computer) Semester - I

DISCRETE MATHEMATICS

(For IN SEM Exam - 30 Marks)

Dr. H. R. Bhapkar

M.Sc. SET, Ph.D. (Mathematics)

MIT Art, Design and Technology University's

MIT School of Engineering, Lonikalbhor, Pune

Email : drhrbhapkar@gmail.com

Mobile : 9011227141

Dr. Rajesh Nandkumar Phursule

Ph.D. (Computer Science and Engineering)

Associate Professor,

Pimpri Chinchwad College of Engineering

Nigdi, Pune.

TECHNICAL PUBLICATIONS

SINCE 1993

An Up-Thrust for Knowledge

DISCRETE MATHEMATICS

(For IN SEM Exam - 30 Marks)

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S.E. (Computer) Semester - I

First Edition : August 2020

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Published by :

TECHNICAL PUBLICATIONS

Amit Residency, Office No.1, 412, Shaniwar Peth,
Pune - 411030, M.S. INDIA, Ph.: +91-020-24495496/97
Email : sales@technicalpublications.org Website : www.technicalpublications.org

Printer :

Yogiraj Printers & Binders
Sr.No. 10/1A,
Ghule Industrial Estate, Nanded Village Road,
Tal. - Haveli, Dist. - Pune - 411041.

ISBN 978-93-332-0276-3

9 789333 202763

SPPU 19

PREFACE

The importance of **Discrete Mathematics** is well known in various engineering fields. Overwhelming response to our books on various subjects inspired us to write this book. The book is structured to cover the key aspects of the subject **Discrete Mathematics**.

The book uses plain, lucid language to explain fundamentals of this subject. The book provides logical method of explaining various complicated concepts and stepwise methods to explain the important topics. Each chapter is well supported with necessary illustrations, practical examples and solved problems. All the chapters in the book are arranged in a proper sequence that permits each topic to build upon earlier studies. All care has been taken to make students comfortable in understanding the basic concepts of the subject.

The book not only covers the entire scope of the subject but explains the philosophy of the subject. This makes the understanding of this subject more clear and makes it more interesting. The book will be very useful not only to the students but also to the subject teachers. The students have to omit nothing and possibly have to cover nothing more.

We wish to express our profound thanks to all those who helped in making this book a reality. Much needed moral support and encouragement is provided on numerous occasions by our whole family. We wish to thank the **Publisher** and the entire team of **Technical Publications** who have taken immense pain to get this book in time with quality printing.

Any suggestion for the improvement of the book will be acknowledged and well appreciated.

Authors
Dr. H. R. Bhapkar
Dr. Rajesh N. Phursule

Dedicated to the Readers of the Book

SYLLABUS

Discrete Mathematics - (210241)

Credit Scheme	Examination Scheme and Marks
03	Mid_Semester (TH) : 30 Marks

Unit I Set Theory and Logic

Introduction and significance of Discrete Mathematics, **Sets** – Naïve Set Theory (Cantorian Set Theory), Axiomatic Set Theory, Set Operations, Cardinality of set, Principle of incl usion and exclusion, **Types of Sets** - Bounded and Unbounded Sets, Diagonalization Argument, Countable and Uncountable Sets, Finite and Infinite Sets, Countably Infinite and Uncountably Infinite Sets, Power set, **Propositional Logic** - logic, Propositional Equivalences, Application of Propositional Logic - Translating English Sentences, Proof by Mathematical Induction and Strong Mathematical Induction. **(Chapters - 1, 2, 3)**

Unit II Relations and Functions

Relations and their Properties, n-ary relations and their applications, Representing relations, Closures of relations, Equivalence relations, Partial orderings, Partitions, Hasse diagram, Lattices, Chains and Anti-Chains, Transitive closure and Warshall's algorithm. **Functions** - Surjective, Injective and Bijective functions, Identity function, Partial function, Invertible function, Constant function, Inverse functions and Compositions of functions, The Pigeonhole Principle. **(Chapters - 4, 5)**

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Unit - I

1 Theory of Sets

Syllabus

Introduction and significance of Discrete Mathematics, Sets – Naïve Set Theory (Cantorian Set Theory), Axiomatic Set Theory, Set Operatio ns, Cardinality of set, Principle of inclusion and exclusion, Types of Sets - Bounded and Unbounded Sets, Diagonalization Argument, Countable and Uncountable Sets, Finite and Infinite Sets, Countably Infinite and Uncountably Infinite Sets, Power set.

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1.1 Introduction

1.2 Sets

1.3 Methods of Describing Sets

1.4 Some Special Sets

1.5 Subsets

1.6 Types of Sets

1.7 Venn Diagrams

1.8 Operations on Sets

1.9 Algebra of Set Operations

1.10 Principle of Duality Dec.-06, 08, 11, 12, 13, 14, 15, 17, 19
..... May-05, 06, 08, 14 ..... Marks 6

1.11 Cardinality of Sets

1.12 The Principle of Inclusion and Exclusion for Sets
..... Dec.-04, 05, 07, 08, 10, 13, 14, 15, 19
..... May-05, 06, 07, 08,
..... 14, 15, 17, 19, ... Marks 13

1.13 Multiset ..... Dec.-13, May-19 ..... Marks 4

1.1 Introduction

The notion of a set is a fundamental concept to all of Mathematics and every branch of mathematics can be considered as a study of sets of objects of one kind or another. A great mathematician Cantor was the founder of the theory of sets. Let us now consider the idea of a set.

1.2 Sets

A set is a collection of well defined objects. An object in the set is called a member or element of the set. The objects themselves can be almost anything. e.g. Books, numbers, cities, countries, animals, etc. In the above definition, the words set and collection for almost all practical purposes are synonymous. Elements of a set are usually denoted by lower case letters i.e. a, b, c, ... while sets are denoted by capital letters i.e.
A, B, C, ...

The symbol '$\in$' indicates the membership in a set.

If "x is an element of a set A" then we write $x \in A$.

If "x is not an element of a set A" then we write as $x \notin A$.

Examples :

1) The set of letters forming the word "MATHEMATICS"

2) The set of students in a class SE Computer Engineering

3) s = {1, 2, 3, 4, 5, 6}

4) The set of professors in SPPU university.

5) The set of all telephone numbers in the directory.

1.3 Methods of Describing Sets

The following are the most useful and common methods of describing sets.

1.3.1 Roster Method (Listing Method)

A set may be described by listing all the members of the set between a pair of braces.

In this method the order in which the elements are listed is immaterial and it is used for small sets. e.g. A = {1, 2, 3, 4, 5}, B = {a, b, c, d, e, f, g}

1.3.2 Statement Form

In this form, set is formed especially where the elements have a common characteristic.

e.g. 1) The set of all even integers

2) The set of all Prime Ministers of India

1.3.3 Set Builder Notation

It is not always possible to describe a set by the listing method or statement form.

A more compact or concise way of describing the set is to specify the common property of all elements of the set

e.g. $A = \{x \mid x \text{ is real and } x^2 > 100\}$

$B = \{x \mid x \text{ is real and } x^{20} - x^{10} + 1 = 0\}$

1.4 Some Special Sets

The following sets are important and occur frequently in our discussion.

1) Set of natural numbers = N = {1, 2, 3, ...}
2) Set of all integers = Z = {...–3, –2, –1, 0, 1, 2, 3 ....}
3) Set of all positive integers = Z^+ = {0, 1, 2, 3, ...}
4) Set of all rational numbers = $Q = \left\{\frac{p}{q} \mid p, q \in R, (p, q) = 1\ q \neq 0\right\}$
5) Set of real numbers = $\Re$
6) Set of complex numbers = $\mathbb{C}$

1.5 Subsets

A set A is said to be a subset of the set B if every element of A is also an element of B.

We also say that A is contained in B and denoted by $A \subseteq B$

If A is a subset of B i.e. $A \subseteq B$, then the set

B is called the superset of A.

The set with no elements is called an empty set or null set. It is denoted by ϕ or { }.

If $A \subseteq B$ then there are two possibilities.

i) A = B

ii) $A \neq B$, $A \subseteq B$

1.5.1 Proper Subset

A set A is called proper subset of B iff

i) $A \subseteq B$ i.e. A is a subset of B.

ii) $\exists$ at least one element in B which is not in A.

i.e. B is not subset of A.

e.g. If A = {1, 2, 3} and B = {1, 2, 3, 4, 5}

then A is a proper subset of B.

It is denoted by $A \subset B$ or $A \subset B$

1.5.2 Improper Subsets

Every set is a subset of itself and null set is a subset of every set.

Subsets A and Q are called improper subsets of A.

1.5.3 Equal sets

SPPU : May-18

Two sets A and B are said to be equal sets if $A \subseteq B$ and $B \subseteq A$. We write A = B

e.g. If $A = \{x \mid x^2 = 1\}$ and B = {1, –1} then A = B

1.6 Types of Sets

Depending upon some properties there are mainly following types of sets.

1) Universal Set :

A non empty set of which all the sets under consideration are subsets is called universal set.

It is denoted by U. Universal set is not unique.

2) Singleton Set :

A set having only one element is called a singleton set. e.g. A = {5}, B = {ϕ}

3) Finite Set :

A set is said to be finite if it has finite number of elements.

The number of elements in a set is called the cardinality of that set. It is denoted by |A| cardinality of set may be finite or infinite.

e.g. $\{1, 2, 3, 4, 5\} \Rightarrow |A| = 5$

4) Infinite Set :

A set is said to be infinite if it has infinite number of elements

e.g. N = set of natural numbers

$|N| = \infty$

5) Power Set :

The set of all subsets of a set A is called the power set of A.

The power set of A is denoted by P(A)

Hence $P(A) = \{X \mid X \subseteq A\}$

If A has n elements then P(A) has 2^n elements.

e.g.

i) $A = \{a, b\}$, $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

ii) $P(A) = \phi$ iff $A = \phi$

i.e. empty set has only subset ϕ, $\therefore$ $P(\phi) = \{\phi\}$

iii) $A = \{a, b, c\}$, $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

6) Power set of power set of A :

The power set of the power set of A is denoted by P(P(A)).

e.g. If $A = \{a, b\}$, $p(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$$P(P(A)) = \{\phi, p(A), \{\phi\}, \{a\}, \{b\}, \{a, b\}, \{\{a\}, \{b\}\} \{\phi, \{a\}\}$$
$$\{\{a\}, \{a, b\}\}, \{\phi, \{b\}\}, \{\{b\}, \{a, b\}\}$$
$$\{\phi\{a, b\}\}, \{\phi, \{a\}, \{b\}\}, \{\phi, \{a\}, \{a, b\}\}$$
$$\{\phi, \{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}\}$$

If $|A| = 2$ then $|P(A)| = 2^2$, $|P(P(A)| = 2^4 = 16$

7) A set itself can be an element of some set

Hence one should understand the difference between an element of a set and subset of a set.

e.g. If $A = \{x, y\}$, $P(A) = \{\phi, A, \{x\}, \{y\}\}$

Then

i) $x \in A$ is true. But $x \subset A$ is false as x is an element of A not subset of A

ii) $\{x\} \subset A$ is true as $\{x\}$ is a subset of A

iii) $\{x\} \in A$ is false as $\{x\}$ is not an element of A

iv) $\{x\} \in P(A)$ is true.

v) $\phi \in P(A)$, $\phi \subset P(A)$ are true.

vi) $\{x\} \in P(A)$ is true.

vii) $\{\{x\}\} \in P(A)$ is false but $\{\{x\}\} \subset P(P(A))$

viii) $\phi \subseteq \phi$, $\phi \subseteq \{\phi\}$, $\phi \in \{\phi\}$, $\{\phi\} \subseteq \{\phi\}$ are true

ix) $\phi \in \phi$ is false

x) $\{a, b\} \in \{a, b, c, \{a, b, c\}\}$ is false

xi) $\{a, b, c\} \in \{a, b, c, \{a, b, c\}\}$ is true.

xii) $\{a, b\} \subset \{a, b, c, \{a, b, c\}\}$ is true

xiii) $\{a, \phi\} \in \{\phi, \{a, \phi\}\}$

1.7 Venn Diagrams

We often use pictures in mathematics. The relation between sets can be conveniently illustrated by certain diagrams called Venn diagrams.

This representation first time used by the British Logician John Venn.

In Venn diagram

i) Universal set (U) is represented by a large rectangle.

ii) Subsets of U are represented by circles or some closed curve

iii) If $A \subseteq B$ then the circle representing A lies inside of circle representing B.

iv) If A and B are disjoint sets then circles of A and B do not have common area.

v) If A and B are not disjoint sets then circles of A and B have some common area.

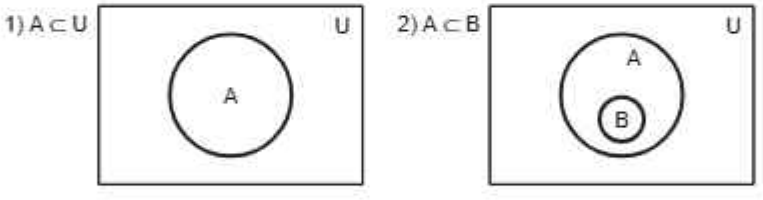


Fig. 1.7.1

1.8 Operations on Sets

To define new sets by combining the given sets, we require set operations. These operations are analogous to the algebraic operations +, –, × of numbers.

1.8.1 Union of Two Sets

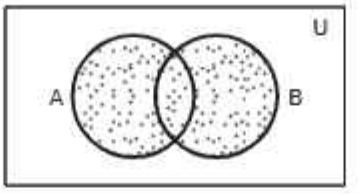
SPPU : May-18

Let A and B be two sets. The union of two sets A and B is the set of all those elements which are either in A or in B or in both sets.

If is denoted by $A \cup B$

$$\therefore \quad A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

By Venn diagram $A \cup B$ is represented as



$A \cup B =$

Fig. 1.8.1

Examples

1) $A = \{1, 2, 3, 4\}, \quad B = \{x, y, z\}$

$A \cup B = \{1, 2, 3, 4, x, y, z\}$

2) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \quad B = \{2, 3, 5, 7, 11, 13\}$

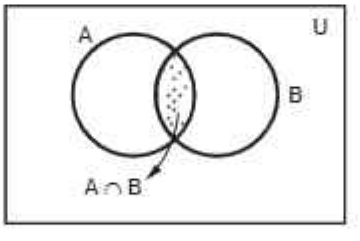
$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\}$

1.8.2 Intersection of Two Sets

The intersection of two sets A and B is the set of all elements which are in A and also in B. It is denoted by $A \cap B$ -

$\therefore \quad A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

By Venn diagram $A \cap B$ is represented as



$A \cap B :$

Fig. 1.8.2

Examples :

1) $A = \{1, 2, 3, 4, 5\}, \quad B = \{1, 4\}$

$\therefore \quad A \cap B = \{1, 4\} = B$

2) $A \cap \phi = \phi$

3) $A = \{a, b, c\}, \quad B = \{x, y\}$

$A \cap B = \phi$

1.8.3 Disjoint Sets

I) Two sets A and B are said to be disjoint sets if

$$A \cap B = \phi$$

e.g. $A = \{1, 2\}$, $B = \{x, y\}$ are disjoint sets.

1.8.4 Complement of a Set

Let A be a given set. The complement of A is denoted by $\overline{A}$ or A' and defined as

$$\overline{A} = \{x \mid x \notin A\}.$$

By Venn diagram A' is represented as

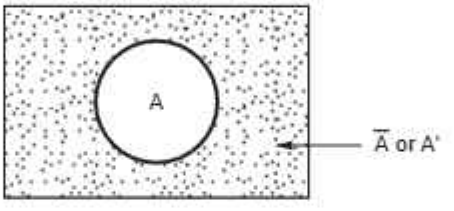


Fig. 1.8.3

Examples :

1) A = {Set of even numbers} and $\cup = IR$

then $\overline{A}$ or A' = set of odd numbers

2) $U = \{1, 2, 3, 4, 5, ... 15\}$, $A = \{1, 2, 3, 4, 5, 6, 7\}$

$\overline{A} = \{8, 9, 10, 11, 12, 13, 14, 15\}$

Properties of complement of sets

1) $\overline{U} = \phi$ and $\phi = U$

2) $\overline{(\overline{A})} = A$

3) $A \cup \overline{A} = U$, $A \cap \overline{A} = \phi$

4) $A \cup U = U$, $A \cap U = A$

1.8.5 Difference of Sets

Let A and B be two sets. The difference A-B is the set defined as

$$A - B = \{x \mid x \in A \text{ and } x \in B\}$$

It is also known as the relative complement of B in A.

Similarly,

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

By Venn diagram it is represented as

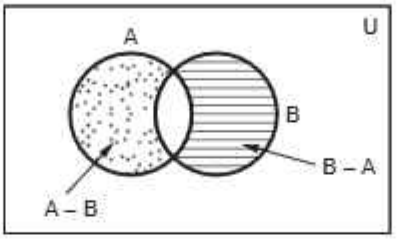


Fig. 1.8.4

Examples :

1) $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$

$\therefore$ $A - B = \{1, 2\}$, $B - A = \{6, 7\}$

2) A = Set of prime numbers

B = Set of odd integers

$A - B = \{2\}$

3) $A = \{a, b, \phi, \{a, c\}\}$

$A - \{a, b\} = \{\{a, c\}, \phi\}$

$\{a, c\} - A = \{c\}$

Properties of difference

Let A and B be two sets. Then

1) $\overline{A} = U - A$

2) $A - A = \phi$, $\phi - \phi = \phi$

3) $A - \overline{A} = A$, $\overline{A} - A = \overline{A}$, $U - \overline{\phi} = \phi$

4) $A - \phi = A$, $A - \overline{\phi} = U - A$

5) $A - B = A \cap \overline{B}$

6) $A - B = B - A$ iff $A = B$

7) $A - B = A$ iff $A \cap B = \phi$

8) $A - B = \phi$ iff $A \subseteq B$

1.8.6 Symmetric Difference of Sets

The symmetric difference of two sets A and B is denoted by $A \oplus B$ and defined as

$$A \oplus B = \{x \mid x \in A - B \text{ or } x \leftarrow B - A\}$$

i.e. $A \oplus B = (A - B) \cup (B - A)$

If is also denoted by $A \Delta B$.

By Venn diagram it is represented as

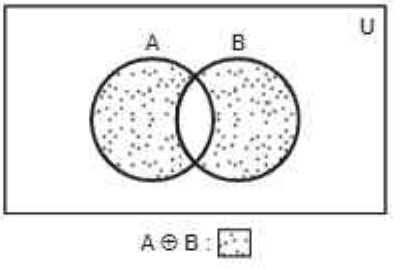


Fig. 1.8.5

Examples :

1) $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 4, 5, 6, 7, 8\}$

$A - B = \{1, 2\}$, $B - A = \{7, 8\}$

$A \oplus B = \{1, 2, 7, 8\}$

2) $A = \phi$, $B = \{x, \phi, \{\phi\}\}$

$A \oplus B = \{x, \{\phi\}\}$

3) $A = \{a\}$, $B = \{a, b\}$

$P(A) \oplus P(B) = \{\{b\}, \{a, b\}\}$

Properties of symmetric difference

1) $A \oplus A = \phi$

2) $A \oplus \phi = A$

3) $A \oplus U = U - A = \overline{A}$

4) $A \oplus \overline{A} = U$

5) $A \oplus B = (A \cup B) - (A \cap B)$

1.9 Algebra of Set Operations

The set operations obey the same rules as those of numbers such as commutativity, associativity and distributivity. In addition there are similar rules to logic such as Idempotent, Absorption De Morgan's Law's and so on.

Theorem : Let A, B, C be any sets then

1) Commutativity with respect to $\cup$ and $\cap$

a) $A \cup B = B \cup A$ b) $A \cap B = B \cap A$

2) Associativity w.r.t. $\cup$ and $\cap$

a) $A \cup (B \cup C) = (A \cup B) \cup C$

b) $A \cap (B \cap C) = (A \cap B) \cap C$

3) Distributivity

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4) Idempotent Laws

a) $A \cup A = A$ b) $A \cap A = A$

5) Absorption Laws

a) $A \cup (A \cap B) = A$ b) $A \cap (A \cup B) = A$

6) De Morgan's Laws

a) $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

7) Double Complement : $\overline{(\overline{A})} = A$

Proof : Proofs of 1), 2), 4), 5) are easy, so reader can prove these

3) To Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow (x \in A \cup B)$ and $(x \in A \cap C)$

$\Rightarrow x \in (A \cup B) \cap (A \cap C)$

Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Similarly prove that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Hence the proof

Similarly reader can prove

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

6) Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Let $x \in \overline{A \cup B} \Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \in \overline{A} \cap \overline{B}$

$\Rightarrow \overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

Similarly $\overline{A} \cap \overline{B} = \overline{A \cup B}$

Hence $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Similarly the reader can prove

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

7) $\overline{\overline{A}} = \{x \mid x \notin \overline{A}\}$

$= \{x \mid x \in A\} = A$

1.10 Principle of Duality

SPPU : Dec.-06, 08, 11, 12, 13, 14, 15, 17, 19, May-05, 06, 08, 14

The principle of duality states that any proved result involving sets and complements and operations of union and intersection gives a corresponding dual result by replacing $\cup$ by ϕ and $\cup$ by $\cap$ and vice versa.

e.g. The dual of $A \cup \overline{A} = U$ is $A \cap \overline{A} = \phi$

Examples

Example 1.10.1 *If $U = \{x \in Z' \mid -5 < x < 5\}$ and $A = \{x \in Z' \mid -2 < x < 3\}$ Where Z' = Set of integers. State the elements of the sets.*

$\overline{A}$, $\overline{A} \cap \overline{A}$, $A \cap U$, $A \cup U$, $\overline{A} \cap U$, $A \cap \overline{A}$

Solution :

$$\overline{A} = \{x \in Z' / (-5 < x < -2) \cup (3 < x < 5)\}$$

$$\overline{A} \cap \overline{A} = \overline{A}$$

$$A \cap U = A \ , \ A \cup U = U$$

$$\overline{A} \cap U = \overline{A}$$

$$\overline{A} \cap \overline{A} = \phi$$

Example 1.10.2 *If $A = \{x, y, \{x, z\}\phi\}$. Determine the following sets*

i) $A - \{x, z\}$ ii) $\{\{x, z\} - A$

iii) $A - \{\{x, y\}\}$ iv) $\{x, z\} - A$

v) $A - P(A)$ vi) $\{x\} - A$

vii) $A - \{x\}$ viii) $A - \phi$

ix) $\phi - A$ x) $\{x, y, \phi\} - A$ **SPPU : Dec.-08, Marks 4**

Solution :

i) $A - \{x, z\} = \{x, y, \phi\}$

ii) $\{\{x, z\}\} - A = \phi$

iii) $A - \{\{x, y\}\} = A$

iv) $\{x, z\} - A = \{z\}$

v) $A - p(A) = \{x, y, \{x, z\}\}$

vi) $\{x\} - A = \phi$

vii) $A - \{x\} = \{y, \{x, z\}, \phi\}$

viii) $A - \phi = \{x, y, \{x, z\}\}$

ix) $\phi - A = \phi$

x) $\{x, y, \phi\} - A = \{\{x, z\}\}$

Example 1.10.3 *If $A = \{\phi, b\}$, construct the following sets $A - \phi$, $\{\phi\} - A$, $A \cup P(A)$*

SPPU : Dec.-19, Marks 03

Solution : We have $A = \{\phi, b\}$

$\therefore \quad A - \phi = \{b\}$

$\{\phi\} - A = \phi$ and $P(A) = \{\phi, \{\phi\}, \{b\}, A\}$

$A \cup P(A) = \{\phi, \{\phi\}, \{b\}, A\}$

Example 1.10.4 *Let $U = Z'$ = Set of all integers*

A = Set of even integers

B = Set of odd integers

C = Set of prime numbers

Find $\overline{A}-B$, $\overline{B}-A$, $\overline{C}-A$, $\overline{A}-C$, $\overline{B}-C$

Solution :

$$\overline{A}-B = \overline{B}-B = \phi$$

$$\overline{B}-A = A-A = \phi$$

$$\overline{C}-A = \text{Set of non prime integers which are not even}$$

$$\overline{A}-C = B-C = \text{Set of odd integers which are not prime} = \{\pm 1, \pm 9, \pm 15, \ldots\}$$

$$\overline{B}-C = A-C = \text{Set of even integers which are not prime} = \{\pm 4, \pm 6, \pm 8, \ldots\}$$

Example 1.10.5 *Let A, B, C be three sets and $A \cap B = A \cap C$*

$\overline{A} \cap B = \overline{A} \cap C$ is it necessary that $B = C$? Justify.

Solution :

Yes, $B = C$.

Consider
$$\begin{aligned} B &= B \cap U = B \cap (A \cup \overline{A}) \\ &= (B \cap A) \cup (B \cap \overline{A}) \\ &= (A \cap B) \cup (\overline{A} \cap B) \\ &= (A \cap C) \cup (\overline{A} \cap C) \\ &= (A \cup \overline{A}) \cap C \\ &= U \cap C \end{aligned}$$

$\therefore$ $\boxed{B = C}$

Example 1.10.6 *Let A, B, C be three sets*

i) Given that $A \cup B = A \cup C$, is it necessary that $B = C$?

ii) Given that $A \cap B = A \cap C$, is it necessary that $B = C$?

Solution :

i) No,

Let $A = \{x, y, z\}$, $B = \{x\}$, $C = \{z\}$

$A \cup B = A$ and $A \cup C = A$

$\therefore$ $A \cup B = A \cup C = A$ But $B \neq C$

ii) No,

Let $A = \{x, y\}$, $B = \{y, z, w\}$, $C = \{y, p, q\}$

$A \cap B = \{y\} = B \cap C$ But $B \neq C$

Example 1.10.7 *If $A \oplus B = A \oplus C$, is $B = C$?*

Solution :

Yes, Let $x \in B \Rightarrow x \in A$ or $x \notin A$

Suppose $x \in A$ then $x \in A \cap B \Rightarrow x \notin A \oplus B$

Hence $x \notin A \oplus C \Rightarrow x \in A \cap C \Rightarrow x \in C$

i.e. if $x \in A$ then $B \subseteq C$

Suppose $x \notin A$ then $x \notin A \cap B$ so that $x \in A \oplus B$

$\Rightarrow \quad x \notin A \oplus C \Rightarrow x \notin A \cup B \Rightarrow x \in C$

Hence $B \subseteq C$

Similarly $C \subseteq B$

Hence $B = C$

Example 1.10.8 *Let A and B are two sets. If $A \subseteq B$, then prove that $P(A) \subseteq P(B)$, where $P(A)$ and $P(B)$ are power sets of A and B sets.* **SPPU : Dec.-17, Marks 6**

Solution : Let A and B be two sets.

The powr set of A is the set of all subsets of a set A.

Let P(A) be the power set of A and P(B) be the power set of B.

If A is a subset of B i.e. $A \subseteq B$ then all elements of A are in B.

If $X \subseteq P(A)$ but $A \subseteq B \Rightarrow X \in P(B)$

Thence $P(A) \subseteq P(B) \quad \forall x \in P(A)$

Example 1.10.9 *If ϕ is an empty set then find $p(\phi)$, $p(p(\phi))$ $p(p(p(\phi)))$*

Solution :

$$p(\phi) = \{\phi\}$$

$$p(p(\phi)) = \{\phi, \{\phi\}\}$$

$$p(p(p(\phi))) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$$

Example 1.10.10 *Show that* $[A \cap (B \cup \overline{A})] \cup B = B$

Solution :

$$[A \cap (B \cup \overline{A})] \cup B = [(A \cap B) \cup (A \cap \overline{A})] \cup B$$

$$= [(A \cap B) \cup \phi] \cup B$$

$$= (A \cap B) \cup B = B$$

Example 1.10.11 *Prove that* $A \cap (B - C)] \subset A - (B \cap C)$

Solution : Let $x \in A \cap (B - C)$ then

$$\Rightarrow x \in A \text{ and } x \in B - C$$

$$\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \notin C)$$

$$\Rightarrow x \in A \text{ and } x \notin (B \cap C)$$

$$\Rightarrow x \in A - (B \cap C)$$

$$\therefore \quad A \cap (B - C) \subset A - (B \cap C)$$

Example 1.10.12 *Salad is made with combination of one or more eatables, how many different salads can be prepared from onion, carrot, cabbage and cucumber?*

SPPU : Dec.-13, Marks 4

Solution : The number of different salads can be prepared from onion, carrot, cabbage and cucumber with combination of one or more eatables is $2^4 - 1 = 16 - 1 = 15$

Example 1.10.13 *Explain the concepts of countably infinite set with example.*

SPPU : Dec.-14, Marks 4

Solution : A set is said to be countable if its all elements can be labelled as 1, 2, 3, 4, ... A set is said to be countably infinite

if, i) It is countable

ii) It has infinitely many elements i.e. It's cardinality is ∞.

For example

1) The set of natural numbers {1, 2, 3, ...} is countably infinite.

2) The set of integers is countably infinite.

3) The set of real numbers is infinite but not countable.

Example 1.10.14 *Draw Venn diagram and prove the expression. Also write the dual of each of the given statements.*

i) $(A \cup B \cup C)^C = (A \cup C)^C \cap (A \cup B)^C$

ii) $(U \cap A) \cup (B \cap A) = A$ **SPPU : Dec.-11, Marks 6**

Solution : Consider the following Venn diagrams.

ii) Consider the following Venn diagrams.

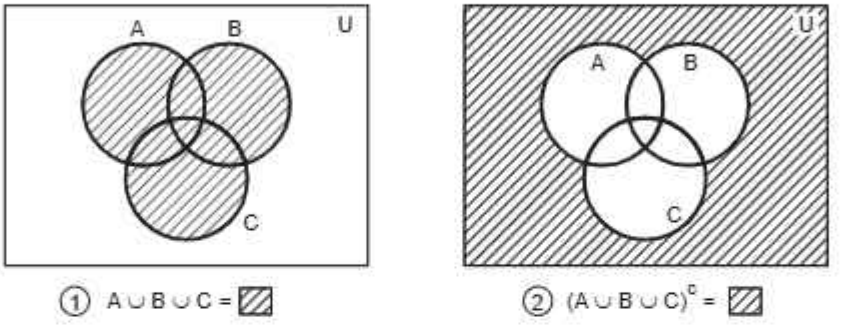


Fig. 1.10.1

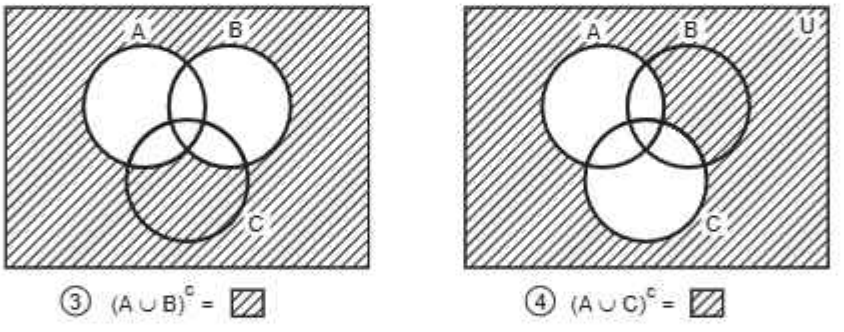


Fig. 1.10.2

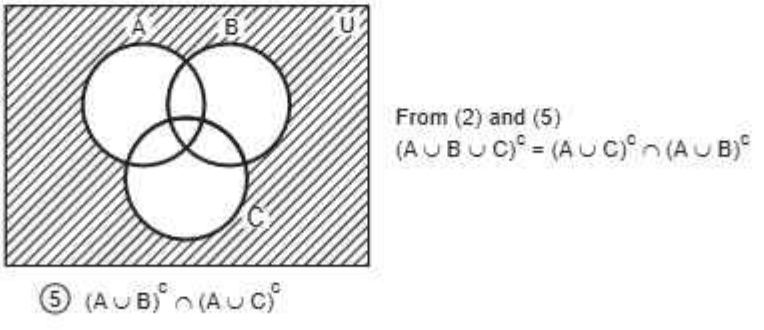


Fig. 1.10.3

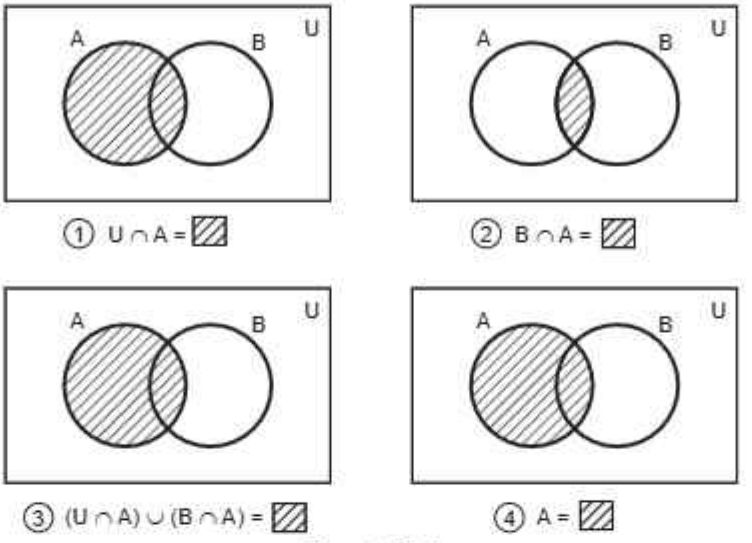


Fig. 1.10.4

From Venn diagrams (3) and (4)

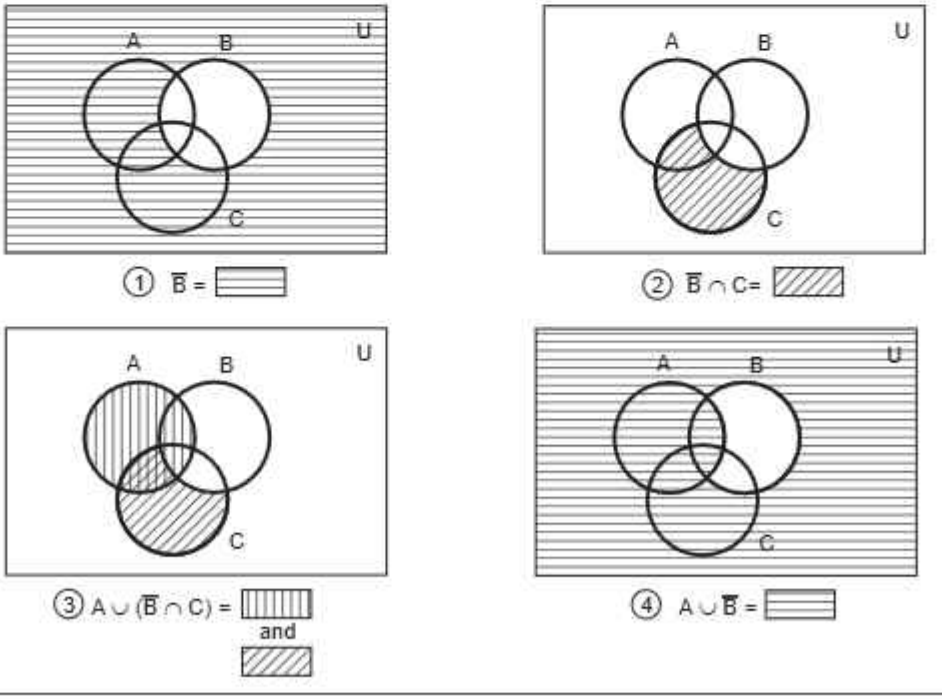
$$(U \cap A) \cup (B \cap A) = A$$

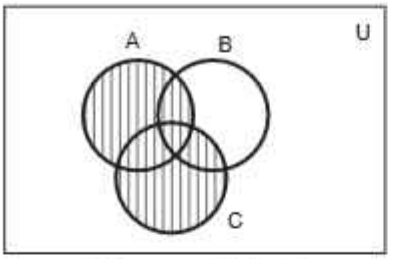
Example 1.10.15 *Using Venn diagram show that :*

$A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$ **SPPU : May-05, Marks 4**

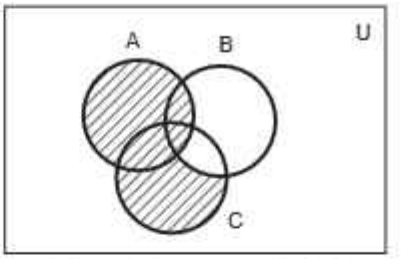
Solution : Consider the following venn diagrams

$\overline{B} = \{x / x \notin B\}$





⑤ A C =



⑥ (A $\overline{B}$) (A C) =

From ③ and ⑥,

$$A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$$

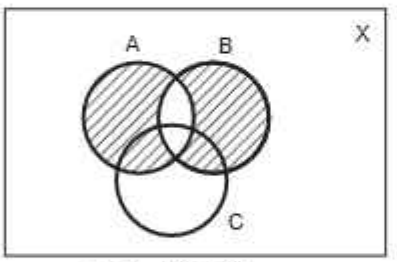
Example 1.10.16 *Using Venn diagram, prove or disprove.*

i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$ **SPPU : May-06, Marks 4**

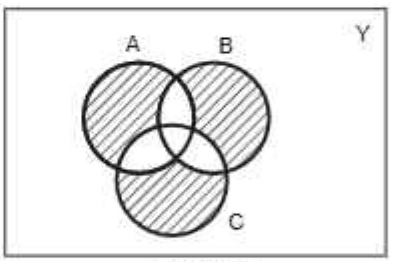
Solution : i) $A \oplus B = (A \cup B) - (A \cap B)$

$A \oplus B$: elements which are either in A or in B but not in both A and B.



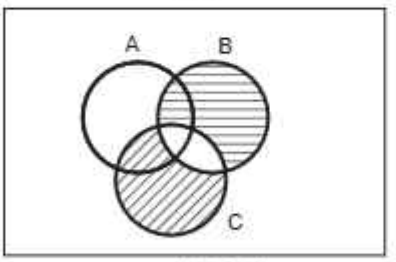
$A \oplus B =$

①



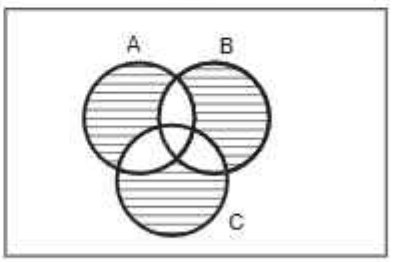
$(A \oplus B) \oplus C$

②



$B \oplus C =$

③ and



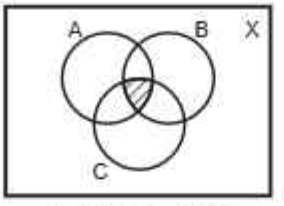
$A \oplus (B \oplus C)$

④

From ② and ④,

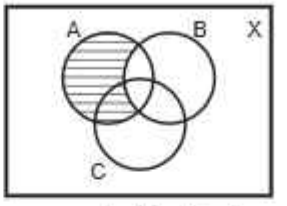
$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$



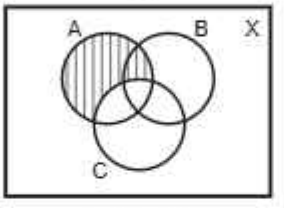
$A \cap B \cap C =$

1



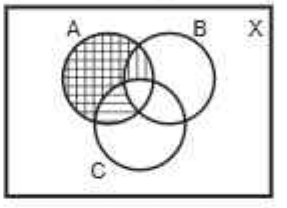
$A - B =$

2



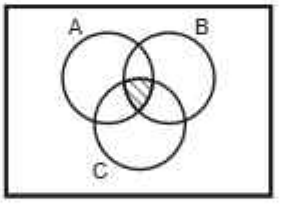
$A - C =$

3



$(A - B) \cup (A - C) =$ and

4



$A - [(A-B) \cup (A-C)] =$

5

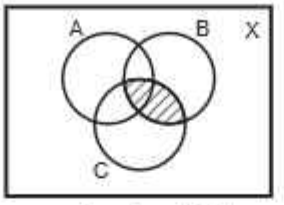
From 1 and 5

$$A \cap B \cap C = A - [(A - B) \cup (A - C)]$$

Example 1.10.17 *Using Venn diagrams show that*

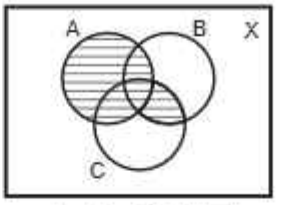
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **SPPU : May-08, Dec.-12, Marks 3**

Solution :



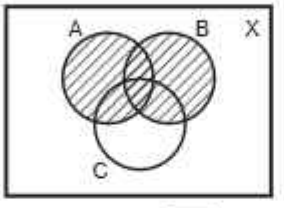
$B \cap C =$

1



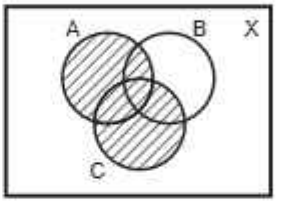
$A \cup (B \cap C) =$

2



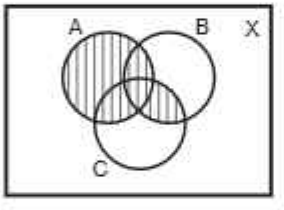
$A \cup B =$

3



$A \cup C =$

4



$(A \cup B) \cap (A \cup C) =$

5

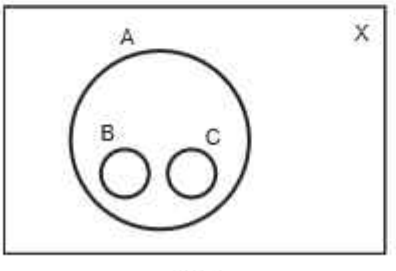
From 2 and 5,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Example 1.10.18 *Let A, B, C be sets. Under what conditions the following statements are true ?*

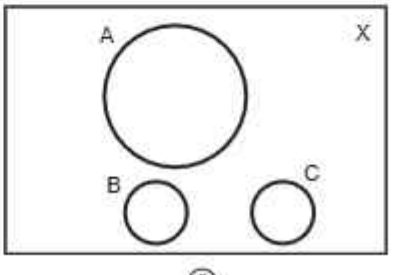
i) $(A - B) \cup (A - C) = A$ *ii)* $(A - B) \cup (A - C) = \varnothing$ **SPPU : Dec.-06, 15, Marks 3**

Solution : i) $(A - B) \cup (A - C) = A$



1

OR



2

If $B \subset A$ and $C \subset A$

then $(A - B) \cup (A - C) = A$

Or A, B, C are disjoint sets.

$$\text{i.e. } A \cap B \cap C = \varnothing$$

$$\text{Then } (A - B) \cup (A - C) = A$$

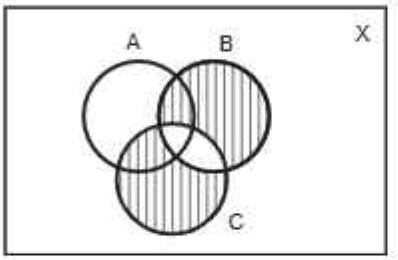
ii) $(A - B) \cup (A - C) = \varnothing$ is true.

If A is empty set i.e. $A = \varnothing$

Example 1.10.19 *Prove the following using Venn diagram.*

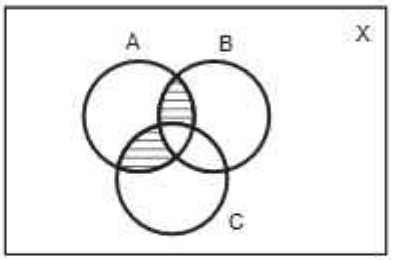
$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ **SPPU : May-08, 14, Dec.-12, Marks 3**

Solution :



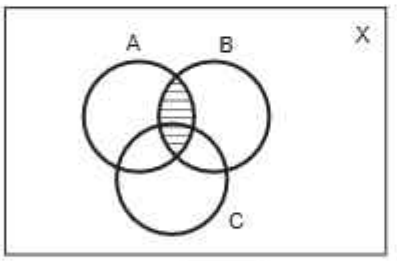
$B \oplus C =$

①



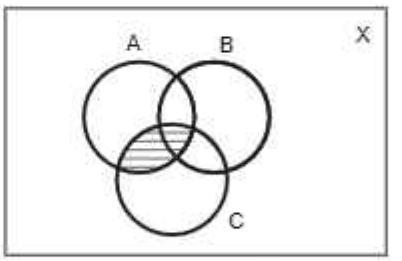
$A \cap (B \oplus C) =$

②



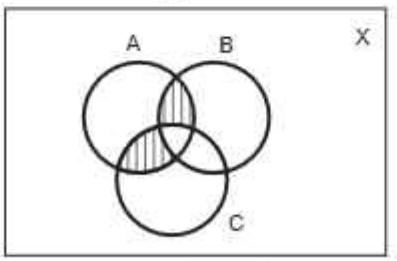
$A \cap B =$

③



$A \cap C =$

④



$(A \cap B) \oplus (A \cap C) =$

⑤

From ② and ⑤

$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

1.11 Cardinality of Sets

In the analysis of computer algorithms, we required to count the number of operations executed by various algorithms. This is necessary to estimate the cost effectiveness of a particular algorithm. Hence we require to study the cardinality of finite sets and understand related properties.

Definition :

Let A be any finite set. The number of elements in the set A, is called the cardinality of the set A. It is denoted by $|A|$ or $n(A)$.

e.g. If $A = \{x, y, z, p\}$ then $|A| = 4$

Theorem 1 : If $A = \phi$ then $|A| = 0$

Theorem 2 : If $A \subseteq B$ then $|A| \leq |B|$

Theorem 3 : (Addition Principle)

Let A and B be two finite sets which are disjoint then $|A \cup B| = |A| + |B|$

Proof : If $A = \phi$, $B = \phi$ then proof is obvious

Let us assume that $A \neq \phi$, $B \neq \phi$

Suppose $A = \{a_1, a_2, ... a_n\}$, $B = \{b_1, b_2, ... b_m\}$

$\therefore$ $A \cap B = \phi$, $|A| = n$, $|B| = m$

$\therefore$ $A \cup B = \{a_1, a_2, ... a_n, b_1, b_2, ... b_m\}$

$\therefore$ $|A \cup B| = m + n = |A| + |B|$. Hence the proof

Theorem 4 : Let $A_1, A_2, A_3, ... A_n$ be mutually disjoint finite sets then

$|A_1 \cup A_2 \cup A_3 \cup ... \cup A_n| = |A_1| + |A_2| + |A_3| + ... + |A_n|$

Theorem 5 : If A is finite set and B is any set then

$|A - B| = |A| - |A \cap B|$

Proof : By Venn diagram

$A = (A - B) \cup (A \cap B)$

$(A - B) \cap (A \cap B) = \phi$

$\therefore$ By addition principle $|A| = |A - B| + |A \cap B|$

$\Rightarrow |A - B| = |A| - |A \cap B|$

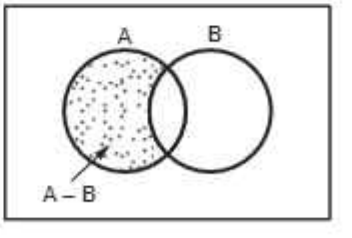


Fig. 1.11.1

1.12 The Principle of Inclusion and Exclusion for Sets

SPPU : Dec.-04, 05, 07, 08, 10, 13, 14, 15, 19
May-05, 06, 07, 08, 14, 15, 17, 19

Theorem 6 : (Principle of Inclusion - Exclusion for 2 sets)

Let A and B be finite sets then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof : By venn diagram

$$A \cup B = (A - B) \cup B$$

As A–B and B are disjoint sets.

$$|A \cup B| = |A - B| + |B|$$

$$= |A| - |A \cap B| + |B|$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

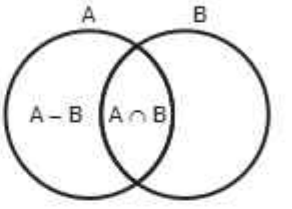


Fig. 1.12.1

Principle of Inclusion-Exclusion for three sets.

Theorem 7 : Let A, B, C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$

Proof Let $D = B \cup C$

$\therefore \quad |A \cup D| = |A| + |D| = |A \cap D|$

and $\quad |D| = |B \cup C| = |B| + |C| - |B \cap C|$

$$\therefore \quad |A \cup B \cup C| = |A \cup D| = |A| + |B| + |C| - |B \cap C| - |A \cap D|$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Theorem 8 : Let A, B, C, D be finite sets then

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D|$$

$$- |B \cap C| - |B \cap D| - |C \cap D|$$

$$= |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$

$$- |A \cap B \cap C \cap D|$$

Examples :

Example 1.12.1 *Among the integers 1 to 300 find how many are not divisible by 3, not by 5. Find also how many are divisible by 3 but not by 7.* **SPPU : Dec.-08, Marks 6**

Solution : Let A denotes the set of integers 1 to 300 divisible

by 3, B denotes the set of integers 1 to 300 divisible

by 5, C denotes the set of integers 1 to 300 divisible.

by 7 $|A| = \left[\frac{300}{3}\right] = 100$, $|B| = \left[\frac{300}{5}\right] = 60$, $|C| = \left[\frac{300}{7}\right] = 42$, $|A \cap B| = \left[\frac{300}{3 \times 5}\right] = 20$

Find $|\overline{A} \cap B|$ and $|A - C|$

We have $\overline{A} \cap \overline{B} = \overline{A \cup B} = U - (A \cup B)$

$$\therefore \quad |A \cup B| = |A| + |B| - |A \cap B|$$

$$= 100 + 60 - 20 = 140$$

$$\therefore \quad |\overline{A} \cap \overline{B}| = |U| - |A \cup B| = 300 - 140 = \mathbf{160}$$

Hence 160 integers between 1 – 300 are not divisible by 3, not by 5.

$$|A - C| = |A| - |A \cap C|, \qquad |A \cap C| = \left[\frac{300}{3 \times 7}\right] = 14$$

$$|A - C| = 100 - 14 = 86$$

Hence, 86 integers between 1 – 300 are not divisible by 3 but not by 7.

Example 1.12.2 *Out of 30 students in a college, 15 take an art course, 8 take maths course, 6 take physics course. It is know that 3 students take all courses. Show that 7 or more students take none of the courses?*

Solution : Let A be the set of students taking an art course

B be the set of students taking a maths course

C be the set of students taking a physics course

We have $|A| = 15, \qquad |B| = 8, \qquad |C| = 6,$

$$|A \cap B \cap C| = 3$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 15 + 8 + 6 - |A \cap B| - |A \cap C| - |B \cap C| + 3$$

$$= 32 - |A \cap B| - |A \cap C| - |B \cap C|$$

But $|A \cap B| \geq |A \cap B \cap C|, \qquad |B \cap C| \geq |A \cap B \cap C|$

and $|A \cap C| \geq |A \cap B \cap C|$

$\therefore \quad |A \cap B| \geq 3, \qquad |B \cap C| \geq 3, \qquad |A \cap C| \geq 3$

$\therefore \quad |A \cap B| + |A \cap C| + |B \cap C| \geq 3|A \cap B \cap C|$

$\therefore \; |A \cup B \cup C| \geq 32 - 3|A \cap B \cap C| = 32 - 3 \times 3 = 23$

Hence $|A \cup B \cup C| \geq 23$

Hence the number of students taking at least one course is ≥ 23. The students taking none of the course is $\leq 30 - 23 = 7$

Thus 7 or less students take none of the courses.

Example 1.12.3 *How many integers between 1 to 2000 are divisible by 2 or 3 or 5 or 7.*

Solution : Suppose set A denotes the number of integers between 1 to 2000 divisible by 2.

Set B is the number of integers between 1 and 2000 divisible by 3.

Set C is the number of integers between 1 and 2000 divisible by 5.

Set D is the number of integers between 1 and 2000 divisible by 7.

$$|A| = \left[\frac{2000}{2}\right] = 1000$$

$$|B| = \left[\frac{2000}{3}\right] = 666$$

$$|C| = \left[\frac{2000}{5}\right] = 400$$

$$|D| = \left[\frac{2000}{7}\right] = 285$$

$$|A \cap B| = \left[\frac{2000}{2 \times 3}\right] = 333$$

$$|A \cap C| = \left[\frac{2000}{2 \times 5}\right] = 200$$

$$|A \cap D| = \left[\frac{2000}{2 \times 7}\right] = 142$$

$$|B \cap C| = \left[\frac{2000}{3 \times 5}\right] = 133$$

$$|B \cap D| = \left[\frac{2000}{3 \times 7}\right] = 95$$

$$|C \cap D| = \left[\frac{2000}{5 \times 7}\right] = 57$$

$$|A \cap B \cap C| = \left[\frac{2000}{2 \times 3 \times 5}\right] = 66$$

$$|A \cap B \cap D| = \left[\frac{2000}{2 \times 3 \times 7}\right] = 47$$

$$|A \cap C \cap D| = \left[\frac{2000}{2 \times 5 \times 7}\right] = 28$$

$$|B \cap C \cap D| = \left[\frac{2000}{3 \times 5 \times 7}\right] = 19$$

$$|A \cap B \cap C \cap D| = \left[\frac{2000}{2 \times 3 \times 5 \times 7}\right] = 9$$

Number of elements divisible by 2 or 3 or 5 or 7 are $|A \cup B \cup C \cup D|$.

From inclusion exclusion principle

$$\begin{aligned}
|A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \\
&\quad - [|A \cap B| + |B \cap C| + |A \cap C| + |A \cap D| + |B \cap D| + |C \cap D|] \\
&\quad + [|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|] \\
&\quad - [|A \cap B \cap C \cap D|]
\end{aligned}$$

$$\begin{aligned}
\therefore \quad |A \cup B \cup C \cup D| &= 1000 + 666 + 400 + 285 - [333 + 200 + 142 + 133 + 95 + 57] \\
&\quad + [66 + 47 + 28 + 19] - 9 \\
&= 2351 - 960 + 160 - 9 \\
&= \mathbf{1542}
\end{aligned}$$

Example 1.12.4 *In the survey of 260 college students, the following data were obtained :*

64 had taken a maths course,

94 had taken a cs course,

58 had taken a business course,

28 had taken both a maths and a business course,

26 had taken both a maths and a cs course,

22 had taken both a cs and a business course,

14 had taken all types of courses.

How many students were - surveyed who had taken none of the three types of courses.

SPPU : May-17, Marks 3

Solution : Let A be the set of students, taken a maths course.

Let B be the set of students, taken a cs course

Let C be the set of students, taken a business course.

$\therefore$ We have $U = 260$, $|A| = 64$, $|B| = 94$

$|C| = 58$, $A \cap C| = 28$, $|A \cap B| = 26$

$|B \cap C| = 22$, $|A \cap B \cap C| = 14$,

We have,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 64 + 94 + 58 - 28 - 26 - 22 + 14 = 154$$

The total number of students taken none of the three types of courses

$$= |U| - |A \cup B \cup C|$$

$$= 260 - 154 = 106$$

Example 1.12.5 *Among 100 students, 32 study mathematics, 20 study physics , 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology and 30 do not study any of the three subjects.*

a) Find the number of students studying all three subjects.

b) Find the number of students studying exactly one of the three subjects.

Solution : Let A, B, C denotes the set of students studying mathematics, physics and biology respectively.

And $|X| = 100$

$|A| = 32$

$|B| = 20$

$|C| = 45$

$|A \cap C| = 15$

$|A \cap B| = 7$

$|B \cap C| = 10$

$|A' \cap B' \cap C'| = 30$

$|A' \cap B' \cap C'| = 100 - |A \cup B \cup C|$

Or $|A \cup B \cup C| = 100 - 30$

$= 70$

a) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$70 = 32 + 20 + 45 - [7 + 15 + 10] + |A \cap B \cap C|$

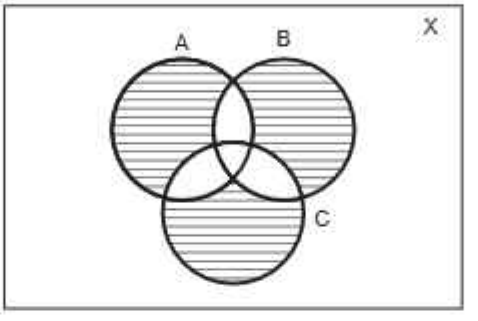
$$70 = 97 - [32] + |A \cap B \cap C|$$

$$70 - 65 = |A \cap B \cap C|$$

$$\Rightarrow |A \cap B \cap C| = 5$$

5 students study all 3 subjects.

b) Number of students studying exactly one subject.



Number of students studying only mathematics is

$$|A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$= 32 - 7 - 15 + 5$$

$$= 15$$

Number of students studying only physics is

$$|B| - |B \cap A| - |B \cap C| + |A \cap B \cap C|$$

$$= 20 - 7 - 10 + 5$$

$$= 8$$

Number of students studying only biology is

$$|C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 45 - 15 - 10 + 5$$

$$= 25$$

$\therefore$ Number of students studying exactly one subject

$$= 15 + 8 + 25 = 48$$

Example 1.12.6 *A survey was conducted among 1000 people of these 595 are Democrats, 595 wear glasses, and 550 like icecream. 395 of them are Democrats who wear glasses, 350 of them are Democrats who like icecream, and 400 of them wear glasses and like icecream 250 of them are Democrats who wear glasses and like icecream. How many of them are not Democrats do not wear glasses, and do not like icecream ? How many of them are Democrats who do not wear glasses and do not like icecream ?*

Solution : $|X| = 1000$

Let $|A|$ be the number of Democrats.

$$|A| = 595$$

$|B|$ be the number of people who wear glasses.

$$|B| = 595$$

$|C|$ be the number of people who like icecream.

$$|C| = 550$$

$$|A \cap B| = 395$$

$$|A \cap C| = 350$$

$$|B \cap C| = 400$$

$$|A \cap B \cap C| = 250$$

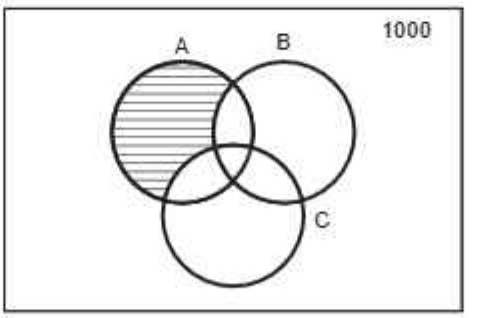
$$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |B \cap C| + |A \cap C|] + |A \cap B \cap C|$$

$$= 595 + 595 + 550 - [395 + 350 + 400] + 250$$

$$|A \cup B \cup C| = 845$$

845 people are either Democrats or wear glasses or icecream.

$\Rightarrow$ 1000 – 845 = 155 people are neither Democrats, nor wear glasses nor like icecream.

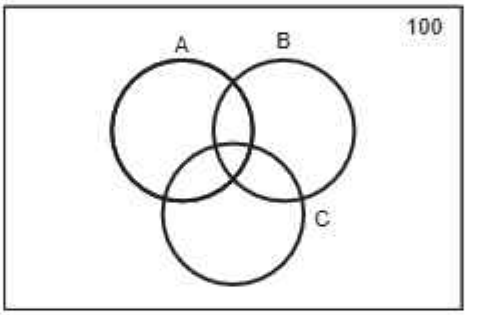


Number of people who are Democrats who do not like icecream and do not wear glasses.

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$
$$= 595 - [395 + 300] + 250$$
$$= 845 - 695$$
$$= 150$$

Example 1.12.7 *It is known that at the university 60 percent of the professors play tennis, 50 percent of them play bridge. 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog, and 40 percent play bridge and jog. If some one claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim ? Why ?* **SPPU : Dec.-13, Marks 6**

Solution :



Let A, B, C, denotes the number of professors play tennis, bridge and jog respectively.

$$|A| = 60$$
$$|B| = 50$$
$$|C| = 70$$
$$|A \cap B| = 20$$
$$|A \cap C| = 30$$
$$|B \cap C| = 40$$
$$|A \cap B \cap C| = 20$$
$$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$
$$= 60 + 50 + 70 - [20 + 30 + 40] + 20$$
$$= \mathbf{110}$$

which is not possible as $|A \cup B \cup C| \subset X$ and the number of elements in $|A \cup B \cup C|$ cannot exceed number of elements in the universal set X.

Example 1.12.8 *Among 200 students in a class, 104 students got an 'A', in first examination and 84 students got 'A' in second examination. If 68 students did not get an 'A' in either of the examination.*

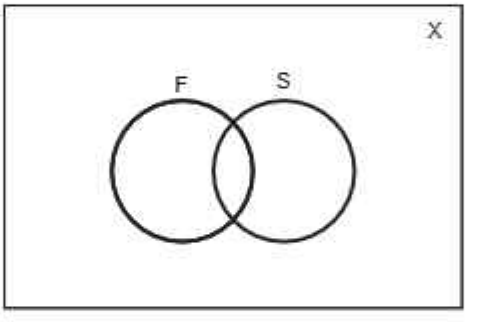
i) How many students got 'A' in both the examination.

ii) If number of students who got an 'A' in the first examination is equal to that who got an 'A' in second examination. If the total number of students who got 'A' in exactly one examination is 160 and if 16 students did not get 'A' in either examination. Determine the number of students who got 'A' in first examination, those who got 'A' in second examination and number of students who got 'A' in both examinations.

SPPU : Dec.-04, Marks 6

Solution : $X \rightarrow$ Universal set

$$|X| = 200$$



Let F denote the set of students who got an 'A' in first examination and S denote the set of students who got an 'A' in second examination.

68 students did not get A in either examination i.e.

$$|(F \cup S)'| = 68$$

$$\Rightarrow \quad |F \cup S| = |X| - 1|(F \cup S)'|$$

$$= 200 - 68$$

$$\Rightarrow \quad |F \cup S| = 132$$

i) Number of students who got A in both the examination will be $|F \cap S|$.

Now $\quad |F \cup S| = |F| + |S| - |F \cap S|$

$$132 = 104 + 84 - |F \cap S|$$

$$\therefore \quad |F \cap S| = 56$$

ii) $|F| = |S|$

The total number of students who got A in exactly one examination is $|F-S| \cup |S-F|$ i.e. $F \oplus S$

and $|F \oplus S| = 160$

16 students did not get A in either examination

$\therefore \quad |(F \cup S)'| = 16$

$\Rightarrow \quad |F \cup S| = |X| - |(F \cup S)'|$

$\Rightarrow \quad |F \cup S| = 200 - 16 = 184$

Now $\quad |F \cup S| = |F \oplus S| + |F \cap S|$

As $F \oplus S$ and $F \cap S$ are pairwise disjoint sets.

$\Rightarrow \quad 184 = 160 + |F \cap S|$

$\Rightarrow \quad |F \cap S| = 184 - 160 = 24$

$\therefore \quad |F \cup S| = |F| + |S| - |F \cap S|$

$184 = 2|F| - 24$

$2|F| = 208$

$\Rightarrow \quad |F| = 104$ and $|F| = |S|$

$\Rightarrow \quad |S| = 104$

$\therefore$ Number of students who got A in first examination

$= |F| - |F \cap S|$

$= 104 - 24 = \mathbf{80}$

Similarly number of students who got A in second examination

$= |S| - |F \cap S| = 104 - 24 = \mathbf{80}$

Example 1.12.9 *Consider a set of integers 1 to 500. Find*

i) How many of these numbers are divisible by 3 or 5 or by 11 ? **Dec.-14**

ii) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.

iii) How many are divisible by 3 or 11 but not by 5 ? **SPPU : May-05, Marks 6**

Solution : Let A denote numbers divisible by 3.

B denote numbers divisible by 5.

C denote numbers divisible by 11.

$|A|$ denotes cardinality of A similarly $|B|$ and $|C|$ denotes cardinality of B and C.

$$|A| = \left[\frac{500}{3}\right] = 166$$

$$|B| = \left[\frac{500}{5}\right] = 100$$

$$|C| = \left[\frac{500}{11}\right] = 45$$

$$|A \cap B| = \left[\frac{500}{3 \times 5}\right] = 33$$

$$|A \cap C| = \left[\frac{500}{3 \times 11}\right] = 15$$

$$|B \cap C| = \left[\frac{500}{5 \times 11}\right] = 9$$

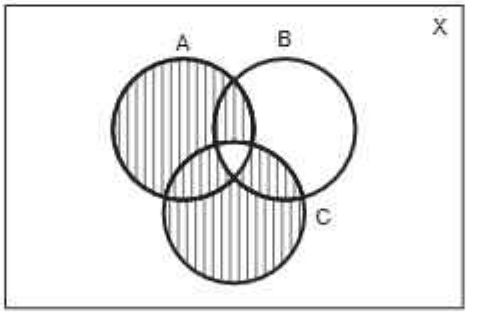
$$|A \cap B \cap C| = \left[\frac{500}{3 \times 5 \times 11}\right] = 3$$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$= 166 + 100 + 45 - [13 + 15 + 9] + 3$

$= \mathbf{257}$

ii)



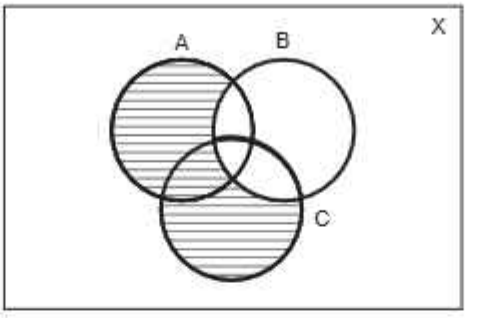
Number of integers divisible by 3 or by 11 but not by all 3, 5 and 11.

$= |A \cup C| - |A \cap B \cap C|$

$= [|A| + |C| - |A \cap C|] - |A \cap B \cap C|$

$= 166 + 45 - 15 - 3 = \mathbf{193}$

iii)



Number of integers divisible by 3 or 11 but not by 5.

$$= |A \cup B \cup C| - |B|$$

$$= 257 - 100 = \mathbf{157}$$

Example 1.12.10 *In the survey of 60 people, it was found that 25 read newsweek magazine, 26 read time, 26 read fortune. Also 9 read both newsweek and fortune, 11 read both newsweek and time, 8 read both time and fortune and 8 read no magazine at all.*

i) Find out the number of people who read all the three magazines.

ii) Fill in the correct numbers in all the regions of the Venn diagram.

iii) Determine number of people who reads exactly one magazine.

SPPU : Dec.-05, 10, 19, Marks 6

Solution : $X \rightarrow$ Universal set

$$|X| = 60$$

Let N denote the number of people who read newsweek magazine.

$$\therefore \quad |N| = 25$$

T denote the number of people who read time magazine.

$$\therefore \quad |T| = 26$$

F denote the number of people who read fortune magazine.

$$\therefore \quad |F| = 26$$

$$|N \cap T| = 11$$

$$|N \cap F| = 9$$

$$|T \cap F| = 8$$

$$|N' \cap T' \cap F'| = 8$$

Using DeMorgan's law

$$|N' \cap T' \cap F'| = |X| - |N \cup T \cup F|$$

$$8 = 60 - |N \cup T \cup F|$$

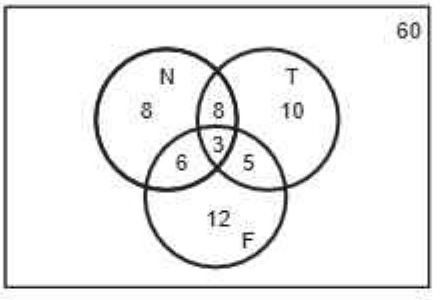
$$\Rightarrow \quad |N \cup T \cup F| = 52$$

i) $$|N \cup T \cup F| = |N| + |T| + |F| - [|N \cap T| + |N \cap F| + |T \cap F|] + |N \cap T \cap F|$$

$$\Rightarrow \quad 52 = 25 + 26 + 26 - [11 + 9 + 8] + |N \cap T \cap F|$$

$$\Rightarrow \quad |N \cap T \cap F| = 3$$

ii)



Explanation :

As $|N \cap T \cap F| = 3$

and people who read N and T both = 11 which includes people of $|N \cap T \cap F|$ also.

Hence people who read only N and T but not F will be 11 – 3, hence 8 people read only N and T similarly $|T \cap F| = 8$.

Then people read only T and F = 8 – 3 i.e. 5.

and $|N \cap F| = 9$

$\Rightarrow$ People read only N and F = 9 – 3 i.e. 6.

Also,

$|N| = 25$, which includes $|N \cap T|$, $|N \cap F|$ and $|N \cap T \cap F|$.

People who read only N $= |N| - [|N \cap T| + |N \cap F|] + |N \cap T \cap F|$

$= 25 - [11 + 9] + 3$

$\therefore$ People who read only N $= 8$ ...(1)

People who read only T $= |T| - [|N \cap T| + |T \cap F|] + |N \cap T \cap F|$

$= 26 - [11 + 8] + 3$

$\therefore$ People who read only T = 10 ... (2)

People who read only F $= |F| - [|N \cap F| + |T \cap F|] + |N \cap T \cap F|$

$= 26 - [9 + 8] + 3$

$\therefore$ People who read only F = 12 ... (3)

iii) Number of people who reads exactly one magazine

= Equation [(1) + (2) + (3)] = 8 + 10 + 12 = 30

Example 1.12.11 *Suppose that 100 out of 120 mathematics students at a college take at least one of the languages French, German and Russian. Also suppose*

65 study French, 20 study French and German

45 study German, 25 study French and Russian

42 study Russian, 15 study German and Russian

i) Find the number of students who study all the three languages.

ii) Fill in correct number of students in each region of Venn diagram.

iii) Determine the number K of students who study

a) Exactly one language.

b) Exactly two languages. **SPPU : Dec.-05, Marks 6**

Solution : X → Universal set

$|X| = 120$

Let F be the number of students who study French. G be the number of students who study German and R be the number of students who study Russian.

$|F \cup G \cup R| = 100$

$|F| = 65, \quad |F \cap G| = 20$

$|G| = 45, \quad |F \cap R| = 25$

$|R| = 42, \quad |G \cap R| = 15$

i) $|F \cup G \cup R| = |F| + |G| + |R| - [|F \cap G| + |F \cap R| + |G \cap R|] + |F \cap G \cap R|$

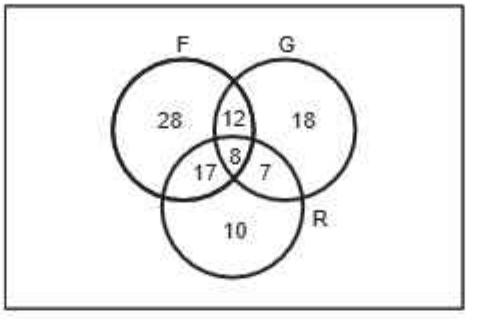
$100 = 65 + 45 + 42 - [20 + 25 + 15] + |F \cap G \cap R|$

$100 = 92 + |F \cap G \cap R|$

$\Rightarrow |F \cap G \cap R| = 8$

Hence 8 students study all three languages.

ii)



Explanation

As $|F \cap G \cap R| = 8$

Students who study F and G but not all 3 will be

$$|F \cap G| - |F \cap G \cap R|$$

$$= 20 - 8 = 12 \quad \ldots (1)$$

Students who study F and R but not all 3 will be

$$|F \cap R| - |F \cap G \cap R|$$

$$= 25 - 8 = 17 \quad \ldots (2)$$

Students who study G and R but not all 3 will be

$$|G \cap R| - |F \cap G \cap R|$$

$$= 15 - 8 = 7 \quad \ldots (3)$$

Students who study only F but not G and not R.

$$= |F| - [|F \cap G| + |F \cap R|] + |F \cap G \cap R|$$

$$= 65 - [20 + 25] + 8 = 28 \quad \ldots (4)$$

Students who study only G but not F and not R

$$= |G| - [|F \cap G| + |G \cap R|] + |F \cap G \cap R|$$

$$= 45 - [20 + 15] + 8 = 18 \quad \ldots (5)$$

Students who study only R but not F and not G.

$$= |R| - [|F \cap R| + |G \cap R|] + |F \cap G \cap R|$$

$$= 42 - [25 + 15] + 8 = 10 \quad \ldots (6)$$

iii) a) Number of students who study exactly one language

$$= \text{Equation (4)} + \text{Equation (5)} + \text{Equation (6)}$$

$$= 28 + 18 + 10 = 56$$

b) Number of students who study exactly two languages.

$$= \text{Equation (1)} + \text{Equation (2)} + \text{Equation (3)}$$

$$= 12 + 17 + 7 = 36$$

Example 1.12.12 *Out of the integers 1 to 1000.*

i) How many of them are not divisible by 3, nor by 5, nor by 7 ?

ii) How many are not divisible by 5 and 7 but divisible by 3 ?

SPPU : May-06, 08, 14, Marks 6, May-19, Marks 13

Solution : i) Let A denote numbers divisible by 3.

B denote numbers divisible by 5.

and C denote numbers divisible by 7.

$$|A| = \left[\frac{1000}{3}\right] = 333$$

$$|B| = \left[\frac{1000}{5}\right] = 200$$

$$|C| = \left[\frac{1000}{7}\right] = 142$$

$$|A \cap B| = \left[\frac{1000}{3 \times 5}\right] = 66$$

$$|A \cap C| = \left[\frac{1000}{3 \times 7}\right] = 47$$

$$|B \cap C| = \left[\frac{1000}{5 \times 7}\right] = 28$$

$$|A \cap B \cap C| = \left[\frac{1000}{3 \times 5 \times 7}\right] = 9$$

$$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 333 + 200 + 142 - [66 + 47 + 28] + 9$$

$$= 543$$

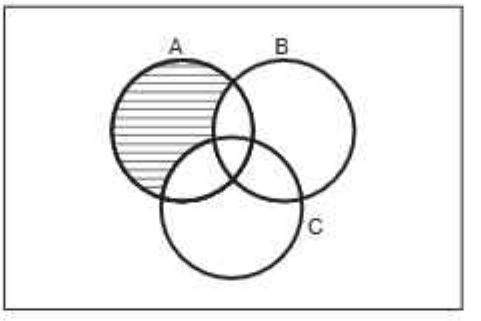
This show 543 numbers are divisible by 3 or 5 or 7. Hence numbers which are not divisible by 3, nor by 5, nor by 7.

$$= |A' \cap B' \cap C'|$$

$$= |X| - |A \cup B \cup C|$$

$$= 1000 - 543 = \mathbf{457}$$

ii)



Number of integers divisible by 3 but not by 5 and not by 7 is $|A \cap B' \cap C'|$.

$$|A \cap B' \cap C'| = |A \cap (B \cup C)'|$$

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 333 - [66 + 47] + 9 = \mathbf{229}$$

Example 1.12.13 *In the class of 55 students the number of studying different subjects are as given below : Maths 23, Physics 24, Chemistry 19, Maths + Physics 12, Maths + Chemistry 9, Physics + Chemistry 7, all three subjects 4.*

Find the number of students who have taken :

i) At least one subject ii) Exactly one subject iii) Exactly two subjects.

SPPU : May-07, Marks 6

Solution : Let X denote universal set.

Let M denote set of students studying mathematics.

P denote set of students studying physics

and C denote set of students studying chemistry.

Then $|X| = 55, |M| = 23, |P| = 24, |C| = 19, |M \cap P| = 12,$

$|M \cap C| = 9, |P \cap C| = 7$

$|M \cap P \cap C| = 4$

i) $|M \cup P \cup C| = |M| + |P| + |C| - [|M \cap P| + |M \cap C| + |P \cap C|] + |M \cap P \cap C|$

$$= 23 + 24 + 19 - [12 + 9 + 7] + 4 = \mathbf{42}$$

ii) Number of students studying mathematics but not physics and not chemistry.

$$= |M| - [|M \cap P| + |M \cap C|] + |M \cap P \cap C|$$

$$= 23 - [12 + 9] + 4 = \mathbf{6} \quad \text{... (1)}$$

Number of students studying physics but not mathematics and not chemistry.

$$= |P| - [|M \cap P| + |P \cap C|] + |M \cap P \cap C|$$

$$= 24 - [12 + 7] + 4 = \mathbf{9} \quad \text{... (2)}$$

Number of students studying chemistry but not mathematics and not physics.

$$= |C| - [|M \cap C| + |P \cap C|] + |M \cap P \cap C|$$

$$= 19 - [9 + 7] + 4 = \mathbf{7} \quad \text{... (3)}$$

Hence number of students studying exactly one subject = 6 + 9 + 7 = 22

iii) Number of students studying mathematics and physics but not chemistry

$$= |M \cap P| - |M \cap P \cap C|$$

$$= 12 - 4 = \mathbf{8} \quad \text{... (4)}$$

Number of students studying physics and chemistry but not mathematics

$$= |P \cap C| - |M \cap P \cap C|$$

$$= 7 - 4 = \mathbf{3} \quad \text{... (5)}$$

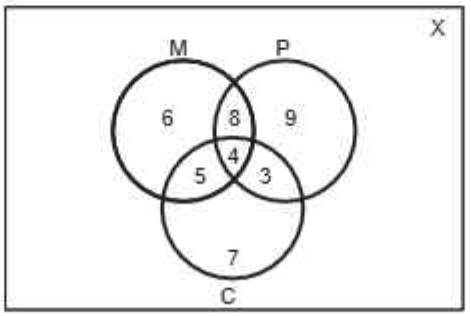
Number of students studying mathematics and chemistry but not physics

$$= |M \cap C| - |M \cap P \cap C|$$

$$= 9 - 4 = \mathbf{5} \quad \text{... (6)}$$

Hence number of students studying exactly two subjects

$$= 8 + 3 + 5 = \mathbf{16}$$



Example 1.12.14 *100 of the 120 engineering students in a college take part in atleast one of the activity group discussion, debate and quiz.*

Also : 65 participate in group discussion, 45 participate in debate, 42 participate in quiz, 20 participate in group discussion and debate 25 participate in group discussion and quiz 15 participate in debate and quiz. Find the number of students who participate in : i) All the three activities ii) Exactly one of the activities. **SPPU : Dec.-07, Marks 4**

Solution : Let X denote universal set

$$|X| = 120$$

Let A denote number of students taking part in group discussion.

B denote number of students taking part in debate

and C denote number of students taking part in quiz.

$|A \cup B \cup C| = 100, \quad |A| = 65, \quad |B| = 45, \quad |C| = 42,$

$|A \cap B| = 20, |A \cap C| = 25, \quad |B \cap C| = 15.$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$$100 = 65 + 45 + 42 - [20 + 25 + 15] + |A \cap B \cap C|$$

$\Rightarrow |A \cap B \cap C| = \mathbf{8}$

ii) Number of students taking part in group discussion but not in debate and not in quiz.

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 65 - [20 + 25] + 8 = \mathbf{28} \qquad \text{... (1)}$$

Number of students taking part in debate but not in group discussion and not in quiz.

$$= |B| - [|A \cap B| + |B \cap C|] + |A \cap B \cap C|$$

$$= 45 - [20 + 15] + 8 = \mathbf{18} \qquad \text{... (2)}$$

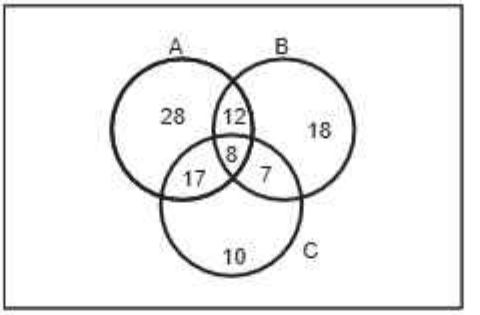
Number of students taking part in quiz but not in group discussion and not in debate.

$$= |C| - [|A \cap C| + |B \cap C| + |A \cap B \cap C|]$$

$$= 42 - [25 + 15] + 8 = \mathbf{10} \qquad \text{... (3)}$$

Hence number of students doing exactly one activity

$$= 28 + 18 + 10 = \mathbf{56}$$



Example 1.12.15 *It was found that in the first year computer science class of 80 students, 50 knew COBOL, 55 'C' and 46 PASCAL. It was also know that 37 knew 'C' and COBOL, 28 'C' and PASCAL and 25 PASCAL and COBOL. 7 students however knew none of the languages. Find*

i) How many knew all the three languages ?

ii) How many knew exactly two languages ?

iii) How many knew exactly one language ? **SPPU : May-15, Dec.-15, Marks 4**

Solution : Let A denote the set of students who know COBOL.

B denote the set of students who know 'C'.

and C denote the set of students who know PASCAL.

and X denote universal set.

Then $|X| = 80$, $|A| = 50$, $|B| = 55$, $|C| = 46$

$|A \cap B| = 37$, $|B \cap C| = 28$, $|A \cap C| = 25$

$$|A' \cap B' \cap C'| = 7$$

Hence $|(A \cup B \cup C)'| = 7$

Also $|A \cup B \cup C| = |X| - |(A \cup B \cup C)'|$

Hence $|A \cup B \cup C| = 80 - 7 = 73$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$$73 = 50 + 55 + 46 - [37 + 28 + 25] + |A \cap B \cap C|$$

$\Rightarrow$ $|A \cap B \cap C| = \mathbf{12}$

ii) Number of students who know only COBOL and 'C' but not PASCAL.

$$= |A \cap B| - |A \cap B \cap C|$$

$$= 37 - 12 = \mathbf{25} \qquad \text{... (1)}$$

Number of students who know only COBOL and PASCAL but not 'C'.

$$= |A \cap C| - |A \cap B \cap C|$$

$$= 25 - 12 = \mathbf{13} \qquad \text{... (2)}$$

Number of students who know only 'C' and PASCAL but not COBOL.

$$= |B \cap C| - |A \cap B \cap C|$$

$$= 28 - 12 = \mathbf{16} \qquad \text{... (3)}$$

Hence number of students who know exactly two languages.

$$= 25 + 13 + 16 = \mathbf{54}$$

iii) Number of students who know only COBOL but not 'C' and not PASCAL.

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 50 - [37 + 25] + 12 = \mathbf{0} \qquad \text{... (4)}$$

Number of students who know only 'C' but not COBOL and not PASCAL.

$$= |B| - [|A \cap B| + |B \cap C|] + |A \cap B \cap C|$$

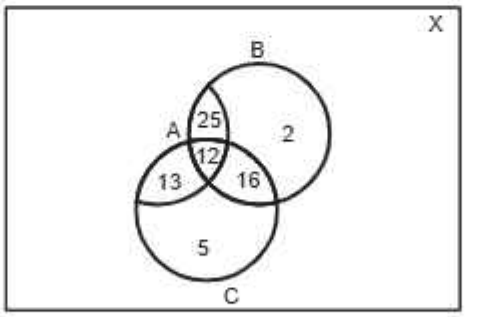
$$= 55 - [37 + 28] + 12 = \mathbf{2} \qquad \text{... (5)}$$

Number of students who know only PASCAL but not COBOL and not 'C'.

$$= |C| - [|A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 46 - [25 + 28] + 12 = \mathbf{5} \qquad \text{... (6)}$$

Hence number of students who know only one language = 0 + 2 + 5 = **7**



Example 1.12.16 *In a survey of 500 television watchers produced the following information 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball and 50 do not watch any of the three games.*

i) How many people in the survey watch all three games ?

ii) How many people watch exactly one game ? **SPPU : May-08, Marks 6**

Solution : Let F denote the set of people who watch football.

H denote the set of people who watch hockey and B denote the set of people who watch basketball.

$X \rightarrow$ Denote universal set

Then $|X| = 500, |F| = 285, |H| = 195, |B| = 115.$

$|F \cap B| = 45, |F \cap H| = 70, |H \cap B| = 50$

$$|F' \cap B' \cap H'| = 50$$

i.e. $|(F \cup B \cup H)'| = 50$

$$\Rightarrow \quad |F \cup B \cup H| = |X| - |(F \cup B \cup H)'|$$

$$= 500 - 50 = \mathbf{450}$$

i) Number of people who watch all the three games $= |F \cap H \cap B|$.

$$|F \cup H \cup B| = |F| + |H| + |B| - [|F \cap H| + |F \cap B| + |H \cap B|] + |F \cap H \cap B|$$

$$450 = 285 + 195 + 115 - [70 + 45 + 50] + |F \cap H \cap B|$$

$$|F \cap H \cap B| = 20$$

ii) Number of people who watch only football.

$$= |F| - [|F \cap B| + |F \cap H|] + |F \cap B \cap H|$$

$$= 285 - [45 + 70] + 20 = \mathbf{190} \quad \text{... (1)}$$

Number of people who watch only hockey.

$$= |H| - [|H \cap B| + |H \cap F|] + |F \cap B \cap H|$$

$$= 195 - [50 + 70] + 20 = \mathbf{95} \quad \text{... (2)}$$

Number of people who watch only basketball.

$$= |B| - [|B \cap H| + |B \cap F|] + |F \cap B \cap H|$$

$$= 115 - [50 + 45] + 20 = \mathbf{40} \quad \text{... (3)}$$

$\therefore$ Number of people who watch exactly one game.

$$= 190 + 95 + 45 = \mathbf{325}$$

1.13 Multiset

SPPU : Dec.-13, May-19

We know that the set is the collection of well defined distinct objects. But there are so many practical situations in which objects are not distinct. For example 1) The collection of alphabets in the word "Missicippi", collection = [m, i, i, i, i, s, s, c, p, p]

2) Birth month of students studying in S.E. Computer Engineering and so on.

Multiset is the generalization of set.

Definition :

A collection of objects that are not necessarily distinct is called a multiset (or Msets). To distinguish set and multiset we denote multiset by enclosing elements in a square brackets e.g. [a, a, b]

1.13.1 Multiplicity of an Element

Let S be a multiset and $x \in S$. The multiplicity of x is defined as the number of times the element x appears in the multiset S. It is denoted by $\mu(x)$.

For example i) S = [a, b, c, d, d, d, e, e]

$$\mu(a) = 1, \mu(b) = 1 \ , \ \mu(c) = 1, \ \mu(d) = 3, \ \mu(e) = 2$$

1.13.2 Equality of Multisets

Let A and B be two multisets. A and B are said to be equal multisets iff $\mu_A(x) = \mu_B(x), \ \forall x \in A \text{ or } B$

e.g. [a, b, a, b] = [a, a, b, b] but [a, a] ≠ [a]

Subset of Multisets

A multiset 'A' is said to be the multiset of B if multiplicity of each element in A is less or equal to it's multiplicity in B.

e.g. [a] ⊆ [a, a], [a, b, a, b, a] ⊆ [a, a, a, b, b, b].

1.13.3 Union and Intersection of Multisets

If A and B are multisets then $A \cup B$ and $A \cap B$ are also multisets.

The multiplicity of an element $x \in A \cup B$ is equal to the maximum of the multiplicity of x in A and in B.

The multiplicity of an element $x \in A \cap B$ is equal to the minimum of the multiplicity of x in A and in B.

Example :

A = [a, b, c, a, b, c, a, a, a], B = [a, a, b, b, b, b]

$A \cup B$ = [a, a, a, a, a, b, b, b, b, c, c]

$A \cap B$ = [a, a, b]

1.13.4 Difference of Multisets

Let A and B be two multisets. The difference A – B is a multiset such that $x \in A - B$

if $(\mu_A(x) - \mu_B(x)) \geq 1$

e.g. 1) A = [a, a, b, a], B = [a, b]

∴ A – B = [a, a]

2) A = [1, 2, 3, 4, 2, 2, 3, 3], B = [1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4]

A – B = [ ] or ϕ

1.13.5 Sum of Multisets

Let A and B be two multisets. The sum of A and B is denoted by A + B and defined as for each

$x \in A+B,\ \mu(x) = \mu_A(x) + \mu_B(x)$

e.g. A = [a, b, c, c], B = [a, a, b, b, c, c]

A + B = [a, a, a, b, b, b, c, c, c, c]

Examples :

Example 1.13.1 *Find the multiset to solve equation*

$A \cap [1, 2, 2, 3, 4] = [1, 2, 3, 4]$

$A \cup [1, 2, 2, 3] = [1, 1, 2, 2, 3, 3, 4]$

Solution : Maximum multiplicity of each element is as follows

$$\mu(1) = 2, \quad \mu(2) = 2, \quad \mu(3) = 2, \quad \mu(4) = 1$$

and minimum multiplicity of each element in A is as follows

$$\mu(1) = 1, \quad \mu(2) = 1, \quad \mu(3) = 1, \quad \mu(4) = 1$$

Therefore,

$$A = [1, 2, 3, 4] \quad \text{or} \quad A = [1, 1, 2, 2, 3, 3, 4]$$

Example 1.13.2 *Explain examples of multisets with its significance.* **SPPU : Dec.-13, Marks 4**

Solution : 1) Multisets are used to denote roots of the polynomial.

$\therefore\ x^3 + 3x^2 + 3x + 1 = 0$

Roots are –1, –1, –1

∴ A = [–1, –1, –1]

2) Multisets are used to denote prime factors of every non-negative integer.

e.g. prime factors of 72 are $2 \times 2 \times 2 \times 3 \times 3$

$\therefore$ $A = [2, 2, 2, 3, 3]$

3) Multisets are used to denote zeros or poles of analytic functions.

4) In Computer Science, multisets are applied in a variety of search and sort procedure.

Example 1.13.3 *If $P = \{a, a, a, c, d, d\}$, $Q = \{a, a, b, c, c\}$. Find union, intersection and difference of P and Q.* **SPPU : May-19, Marks 3**

Solution : We have

$$P \cup Q = \{a, a, a, b, c, c, d, d\}$$

$$P \cap Q = \{a, a, c\}$$

$$P - Q = \{a, d, d\}$$

❑❑❑

Unit - I

2 Propositional Calculus

Syllabus

Propositional Logic - logic, Propositional Equivalences, Application of Propositional Logic - Translating English Sentences.

Contents

2.1 Introduction

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2.1 Introduction

A discrete structure is defined by a set of axioms. The properties of structure are derived from the axioms as theorems. Such theorems are proved using valid rules of reasoning. The propositional calculus (Mathematical logic) is concerned with all kinds of reasoning. It has two aspects.

1) It is analytic theory of art of reasoning whose goal is systematic and codify the principles of valid reasoning.
2) It is inter-related with problems relating the foundation of mathematics.

A great mathematician Frege G. (1884-1925) developed the idea regarding a mathematical theory as applied branch of logic.

Every student of engineering should learn logic because principles of logic are valuable to problem analysis, programming, logic designing, code designing and many more.

2.2 Statements or Propositions

A statement is a declarative sentence which is either true or false but not both. The truth or falsity of a statement is called its truth value. The truth value of a true statement is denoted by 'T' and the false statement is denoted by 'F'. They are also denoted by 1 or 0.

Statements are usually denoted by A, B, C, ... or a, b, c.

Examples :

1) There are 5 days in a week
2) $2 + 5 = 7$
3) $y + 3 = 8$
4) It will rain tomorrow
5) There are 12 months in a year

Examples (2) and (5) are true statements

Example (1) is false statement

In example (3), it's truth value depends upon the value of y. If y is 5 then sentence is true and if $y \neq 5$ then sentence is false. Therefore (4) is not a statement.

In example (4), it's truth value cannot be predicted at this moment but it can be definitely determined tomorrow. Hence it is a statement.

2.3 Laws of Formal Logic

There are two laws of formal logic

2.3.1 Law of Contradiction

It states that the same statement cannot be both true and false

2.3.2 Law of Excluded Middle

It p is a statement then either p is true or p is false and there cannot be middle ground. If a student appear for the examination of discrete structure then student will be either pass or fail in that exam. There is no middle stage.

2.4 Connectives and Compound Statements SPPU : Dec.-06,18, May-05,07,08

A statement which cannot be further split into simple sentences, is called primary or primitive or atomic statements.

In day to day life, we use the words NOT, AND, OR, IF-Then, as well as, BUT, WHILE to connect two or more sentences. But these connectives are flexible in their meanings, and lead to inexact and ambiguous interpretations. However, mathematics is a very precise language and every symbol of mathematics has the unique meaning or interpretation in mathematical ocean.

Hence we take some special connectives with precise meaning to suit our purpose. Following are the common connectives with symbols or their rotations.

Sr. No.	Name of connectives	Symbol
1	Negation	$\sim$ or $\neg$
2	And (Meet)	$\wedge$
3	Or (Join)	$\vee$
4	If .... then	$\rightarrow$
5	If an only if (iff)	$\leftrightarrow$

2.4.1 Compound Statement

A statement which is formed from primary statements by using logical connectives is called a compound statement.

e.g. If p : I am studying in SE computer class

q : I am learning discrete structure subject

The compound statement is

I am studying in SE computer class and I am learning discrete structure subject.

2.4.2 Truth Table

A table showing the truth value of a statement formula is called truth table.

2.4.3 Negation

Let p be any simple statement, then the negation of p is formed by writing "It is false that" before p or introducing the word "not" at the proper place. The negation of p is also obtained by writing "p is false". The negation of p is denoted by ~ p or $\neg$ p or $\bar{p}$. If the statement p is true, then ~ p is false and if p is false then ~ p is true.

The truth table for negation of p is given below

p	~ p
T	F
F	T

Examples :

Sr. No.	Statement p	~ p
1.	p : Atharva is intelligent boy	~ p : Atharva is not intelligent boy OR It is false that Atharva is intelligent boy
2.	p : Delhi is the capital of India	~ p : Delhi is not the capital of India OR It is false that Delhi is the capital of India OR It is not the case that Delhi is the capital of India

Note : ~ p is the **unary** connective as only one statement is required to form negation.

2.4.4 Conjunction

A compound statement which is obtained by combining two primary statements by using the connective "and" is called conjunction i.e. the conjunction of two statements p and q is the statement $p \wedge q$. It is read as "p and q" or "p meet q".

The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T, otherwise $p \wedge q$ has the truth value F.

It's tabular representation is as follows :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 2.4.1

Sr. No.	*p*	*q*	*Conjunction*	$p \wedge q$
1.	*3 > 2 (T)*	*5 is prime (T)*	*3 > 2 and 5 is prime*	*T*
2.	*Delhi is in India (T)*	*3 + 30 = 30 (F)*	*Delhi is in India and 3 + 30 = 30*	*F*

3. Construct a truth table for the conjunction of n < 20 and n > 5, n ∈ N

Solution : When n < 20 and n > 5 are true then the conjunction "n < 20 and n > 5" is true.

The truth table is as follows :

p : n < 20	**q : n > 5**	**$p \wedge q$: n < 20 and n > 5 i.e. 5 < n < 20**
T (n = 15)	T (n = 15)	T (∵ 5 < 15 < 20)
T (n = 2)	F (n = 2)	F (∵ n < 5)
F (n = 30)	T (n = 30)	F (∵ n > 20)
F (n = 50)	F —	F (There does not exist any number s.t. n > 20 and n < 5)

2.4.5 Disjunction

A compound statement obtained by combining two simple statements by using the connective "or" is called the disjunction.

i.e. If p and q are simple statements then the compound statement "p or q" is the disjunction of p and q. It is denoted by $p \vee q$. $p \vee q$ is read as "p or q" or "p join q".

If p is false and q is false then $p \vee q$ is false. Otherwise $p \vee q$ is true. The truth table of $p \vee q$ is as follows :

p	q	$p \vee q$
T	T	T

T	F	F
F	T	T
F	F	F

Example 2.4.2 *1) If p : The data is wrong, q : There is an error in the program.*

Then $p \vee q$: The data is wrong or there is an error in the program.

The connective 'v' is used in the inclusive sense i.e. at least one possibility exists or even both possibilities exist.

2)

Sr. No.	p	q	Disjunction	$p \wedge q$
1.	6 < 2 (F)	5 + 3 = 4 (F)	6 > 2 or 5 + 3 = 4	F (∵ p and q both are false)
2.	Delhi is in India (T)	5.3 = 10 (F)	Delhi is in India or 5.3 = 10	T (∵ p is true)
3.	6 – 3 = 3 (T)	5 is prime (T)	6 – 3 = 3 or 5 is prime	T (∵ p and q are true)

3) Construct a truth table for disjunction of p : n is prime, q : n > 10 for $n \in N$.

Solution : The truth table for $p \vee q$ is as follows :

p : n is prime	q : n > 10	$p \vee q$: n is prime or n > 10
T (n = 13)	T (13 > 10)	T (∵ n = 13 > 10 or 13 is prime)
T (n = 5)	F (5 < 10)	T (∵ n = 5 is prime)
F (n = 12)	T (12 > 10)	T (∵ n = 12 > 10)
F (n = 6)	F (6 < 10)	F (∵ Both p and q are false)

2.4.6 Conditional Statement (If ..... then ....)

If p and q are two statements, then the statement $p \rightarrow q$ which is read as "if p then q" is called as conditional statement.

The conditional statement $p \rightarrow q$ can also be read as i) p only if q ii) p implies q iii) q if p iv) p is sufficient for q

The statement p is called the hypothesis or antecedent and q is called conclusion or consequent.

If p is true and q is false then $p \rightarrow q$ is false. Otherwise $p \rightarrow q$ is true.

The truth table of $p \rightarrow q$ is as follows :

p	q	$p \rightarrow q$
T	T	T

T	F	F
F	T	T
F	F	T

Examples :

1) Let p : Atharva is a graduate

q : 3 + 5 = 8

$\therefore$ $p \rightarrow q$: If Atharva is a graduate then 3 + 5 = 8

Let N be the set of natural numbers and Q be the set of rational numbers. We know that every natural number is a rational number.

Let p : n is a natural number

q : n is a rational number

p : $n \in N$	**q : $n \in Q$**	**$p \rightarrow q$: if $n \in N$ then $n \in Q$**
T (n = 2)	T ($n \in Q$)	T ($\because$ n = 2 is rational)
T (n = 5)	F (Impossible)	F ($\because$ Every natural number is rational)
F $\left(n=\frac{3}{2}\right)$	T $\left(\frac{3}{2} \in Q\right)$	T ($\because \exists$ number which is not natural but it is rational)
F ($n=\sqrt{2}$)	F ($\sqrt{2} \notin Q$)	T ($\because \exists$ number which is neither natural nor rational)

Note : If p then q

p only if q

q if p

p is sufficient for q

q is necessary for p

Above all are equivalent to $p \rightarrow q$

2.4.7 Biconditional (If and only if)

If p and q are two statements, then the compound statement "p if and only if q" is called a biconditional statement. It is denoted by $P \underset{\leftarrow}{\rightarrow}$ or $p \leftrightarrow q$.

The biconditional statement is also read as

"p if and only if q" or "p iff q" or "p implies and implied by q" or "p is necessary and sufficient for q".

If p and q have the same truth value then $p \leftrightarrow q$ is true. In other cases $p \leftrightarrow$ is false.

The truth table for $p \leftrightarrow q$ is as follows

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	p	T	F	F
F	F	T	T	T	T

By above table

$p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are equivalent.

Examples

1) Let $p : x = 4$ $\quad q : x + 9 = 13$

$\therefore$ $p \leftrightarrow q : x = 4$ if and only if $x + 9 = 13$

2) $5 + 6 = 11$ iff $11 + 6 = 17$

3) We know that an integer n is an even iff n is divisible by 2.

p : n is even	q : n is divisible by 2	$p \leftrightarrow q$: n is even iff n is divisible by 2
T (n = 10)	T (2 divides 10)	T ($\because$ n = 10 is even and divisible by 2)
T (n = 10)	F (Impossible)	F ($\because$ There does not exist even integer which is not divisible by 2)
F (n = 5)	T (Impossible)	F ($\because$ There does not exist odd integer which is divisible by 2)
F (n = 5)	F (5 is not divisible by 2)	T ($\because$ Integer is not even iff it is not divisible by 2)

2.4.8 Special Propositions

If $p \rightarrow q$ is a conditional statement, then

i) $q \rightarrow p$ is called its converse

ii) $\sim p \rightarrow \sim q$ is called its inverse

iii) $\sim q \rightarrow \sim p$ is called its contrapositive

The truth table of these propositions are as follows

p	q	$p \to q$	$q \to p$	$\sim p$	$\sim q$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Example 2.4.3 *Let p denote the statement, "The material is interesting'. q denote the statement, "The exercises are challenging", and r denote the statement, "The course is enjoyable".*

Write the following statements in symbolic form :

i) The material is interesting and exercises are challenging.

ii) The material is interesting means the exercises are challenging and conversely.

iii) Either the material is interesting or the exercises are not challenging but not both.

iv) If the material is not interesting and exercises are not challenging, then the course is not enjoyable.

v) The material is uninteresting, the exercises are not challenging and the course is not enjoyable. **SPPU : Dec.-06, May-08, Marks 6**

Solution :

i) $p \wedge q$

ii) $(p \to q) \wedge (q \to p)$

iii) $p \oplus \sim q$

iv) $(\sim p \wedge \sim q) \to \sim r$

v) $\sim p \wedge \sim q \wedge \sim r$

Example 2.4.4 *Express following statement in propositional form :*

i) There are many clouds in the sky but it did not rain.

ii) I will get first class if and only if I study well and score above 80 in mathematics.

iii) Computers are cheap but softwares are costly.

iv) It is very hot and humid or Ramesh is having heart problem.

v) In small restaurants the food is good and service is poor.

vi) If I finish my submission before 5.00 in the evening and it is not very hot I will go and play a game of hockey. **SPPU : May-05, Marks 6**

Solution :

i) p : There are many clouds in the sky

q : It rain

$\therefore\ p \wedge \sim q$

ii) p : I will get first class

q : I study well

r : Score above 80 in mathematics

$\therefore\ p \leftrightarrow (q \wedge r)$

iii) p : Computers are cheap

q : Softwares are costly

$\therefore\ p \wedge q$

iv) p : It is very hot

q : It is very humid

r : Ramesh is having heart problem

$\therefore\ (p \wedge q) \vee r$

v) p : In small restaurant food in good

q : Service is poor

$\therefore\ p \wedge q$

vi) p : I finish my submission before 5:00 p.m.

q : It is very hot

r : I will go

s : I will play a game of hockey

$\therefore\ (p \wedge \sim q) \rightarrow (r \wedge s)$

Example 2.4.5 *Use p : I will study discrete mathematics, q : I will go to a movie, r : I am in a good mood. Write the following in English sentence :*

a) $\sim r \rightarrow q$ *b)* $\sim q \wedge p$ *c)* $q \rightarrow \sim p$ *d)* $\sim p \rightarrow \sim r$ **SPPU : Dec.-18, Marks 04**

Solution :

a) $\sim r \rightarrow q$ means, If I am not in a good mood, then I will go to a movie

b) $\sim q \wedge p$ means, I will not go to movie and I will study discrete mathematics.

c) $q \rightarrow \sim p$ means, If I will go to a movie then I will not study discrete maths.

d) $\sim p \rightarrow \sim r$ means, If I will not study discrete mathematics then I am not in a good mood.

Example 2.4.6 *Express the contrapositive, converse and inverse forms of the following statement if 3 < b and 1 + 1 = 2, then* $\sin\frac{\pi}{3} = \frac{1}{2}$. **SPPU : May-07, Marks 6**

Solution :

$$p : 3 < b$$
$$q : 1 + 1 = 2$$
$$r : \sin\frac{\pi}{3} = \frac{1}{2}$$

Symbolic form : $p \wedge q \rightarrow r$

Contrapositive : $(\sim r \rightarrow \sim (p \wedge q))$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $\sin\frac{\pi}{3} \neq \frac{1}{2}$ then $3 \geq b$ or $1+1 \neq 2$

Converse : $r \rightarrow (p \wedge q)$

i.e. if $\sin\frac{\pi}{3} = \frac{1}{2}$ then $3 < b$ and $1 + 1 = 2$

Inverse : $\sim(p \wedge q) \rightarrow \sim r$

i.e. $(\sim p \vee \sim q) \rightarrow \sim r$

i.e. if $3 \geq b$ or $1+1 \neq 2$ then $\sin\frac{\pi}{3} \neq \frac{1}{2}$

Example 2.4.7 *Express the contrapositive, converse, inverse and negation forms of the conditional statement given below.*

'If x is rational, then x is real'. **SPPU : Dec.-06, Marks 4**

Solution :

Let p : x is rational

q : is real

Symbolic form : $p \rightarrow q$

Contrapositive : $(\sim q \rightarrow \sim p)$

If x is not real, then x is not rational

Converse : $(q \rightarrow p)$

If x is real then x is rational

Inverse - $(\sim p \rightarrow q)$

If x is not rational, then x is not real

Negation : $\sim(p \rightarrow q)$

$$\equiv \sim(p \vee \sim q)$$
$$\equiv \sim p \wedge \sim\sim q$$
$$\equiv \sim p \wedge q$$

2.5 Propositional or Statement Formula

A statement formula contains one or more simple statements and some logical connectives.

A statement formula is a string consisting of variables, parentheses and logical connective symbols.

It is called a well formed if it can be generated by the following rules :

i) A statement variable p standing alone is a well formed formula.

ii) If p is well formed formula then $\sim p$ is also well formed formula.

iii) If p and q are well formed formulas then $p \wedge q$, $p \vee q$, $p \rightarrow q$ or $q \rightarrow p$ and $p \leftrightarrow q$ are well formed formulas.

iv) A string of symbols is well formed formula iff it is obtained by finitely many applications of rules i), ii) and iii)

$\therefore$ A statement formula is not a statement and has no truth values. But if put definite statements in place of variables in a given formula we get a statement. It's truth value depends upon the truth values of variables.

2.6 Tautology

A statement formula that is true for all possible values of it's propositional variables is called a Tautology.

e.g. $p \vee \sim p$ is a tautology

2.7 Contradiction

A statement formula that is always false for all possible values of variables is called a contradiction or absurdity.

Example : $p \wedge \sim p$ is a contradiction

2.8 Contingency

SPPU : Dec.-10, 12

A statement formula which is neither tautology nor contradiction is called a contingency.

Example 2.8.1 *Prove that $p \rightarrow p$ is a tautology.*

Solution : We construct truth table for $p \rightarrow p$

p	p	$p \rightarrow p$
T	T	T
F	F	T

As $p \rightarrow p$ is always true. Hence $p \rightarrow p$ is a tautology.

Example 2.8.2 *Prove that $p \vee \sim p$ is a tautology and $\sim(p \vee \sim p)$ is a contradiction.*

Solution : Let us construct truth table for the statement $\sim(p \vee \sim p)$

p	~ p	p ∨ ~ p	~ (p ∨ ~ p)
T	F	T	F
F	T	T	F

As $p \vee \sim p$ is always true. Hence $p \vee \sim p$ is a tautology and $\sim(p \vee \sim p)$ is always false. Hence $\sim(p \vee \sim p)$ is a contradiction.

Example 2.8.3 *Show that $p \wedge \sim p$ is a contradiction and $\sim(p \wedge \sim p)$ is tautology.*

Solution : We construct truth table for $\sim(p \wedge \sim p)$

p	~ p	p ∧ ~ p	~ (p ∧ ~ p)
T	F	F	T
F	T	F	T

As $p \wedge \sim p$ is always false. Hence $p \wedge \sim p$ is a contradiction. As $\sim(p \wedge \sim p)$ is always true, Hence $\sim(p \wedge \sim p)$ is a tautology.

Example 2.8.4 *Determine whether each of the following statement formula is a tautology, contradiction or contingency.*

i) $(p \wedge q) \wedge \sim(p \vee q)$

ii) $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$

iii) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

iv) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ **SPPU : Dec.-12, Marks 6**

Solution : i) Consider the truth table

p	q	p ∧ q	p ∨ q	~ (p ∨ q)	(p ∧ q) ∧ ~ (p ∨ q)
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Hence $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction

ii) Consider the truth table

p	q	~ p	$p \to q$	$q \vee \sim p$	$(p \to q) \leftrightarrow (q \vee \sim p)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(p \to q) \leftrightarrow (q \vee \sim p)$ is always true. Hence it is a tautology.

iii) Consider the truth table

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Hence $[(p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a tautology.

iv) Consider the truth table

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence given statement formula is a tautology.

Example 2.8.5 *Show that $(p \to q) \wedge \sim q \to \sim p$ is a tautology without using truth table.*

Solution : We know that $p \to q$ is true if p is true and q is also true.

$\therefore$ We need only to show that $p \to q$ and $\sim q$ both are true imply $\sim p$ is true.

As the truth value of $\sim q$ is T, the truth value of q is F. And as $p \to q$ is true, this means that p is false ($\because F \to F$ is true)

$\therefore$ The truth value of p is T. Hence the proof.

Example 2.8.6 *Prove that $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$ is a tautology.*

SPPU : Dec.-10, Marks 4

Solution : Consider truth table

p	q	r	s	$p \to q$	$r \to s$	$p \vee r$	$(p \to q) \wedge (r \to s) \wedge (p \vee r)$ (I)	$q \vee s$	$I \to (q \vee s)$
T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	T	F	T	T
T	T	F	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	T
T	F	T	T	F	T	T	F	T	T
T	F	T	F	F	F	T	F	F	T
T	F	F	T	F	T	T	F	T	T
T	F	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T	T
F	T	F	T	T	T	F	F	T	T
F	T	F	F	T	T	F	F	T	T
F	F	T	T	T	T	T	T	T	T
F	F	T	F	T	F	T	F	F	T
F	F	F	T	T	T	F	F	T	T
F	F	F	F	T	T	F	F	F	T

Hence given statement formula is a tautology.

Example 2.8.7 *Prove by truth table $p \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$.* **SPPU : Dec.-12**

Solution : Consider the truth table

				I			II
P	**Q**	**R**	$Q \vee R$	$P \rightarrow (Q \vee R)$	$P \rightarrow Q$	$P \rightarrow R$	$(P \rightarrow Q) \vee (P \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

From truth table

$$p \rightarrow (Q \vee R) \equiv (P \rightarrow Q) \vee (P \rightarrow R)$$

2.9 Precedence Rule

The order of preference in which the connectives are applied in a formula of propositions that has no bracket is

i) $\sim$ ii) $\wedge$ iii) $\vee$ and $\oplus$ iv) $\rightarrow$ and $\leftrightarrow$

Remark :

1) '$\vee$' includes 'exclusive or' and 'inclusive or'

2) If $\vee$ and $\oplus$ are present in a statement then first apply the left most one. The same rule is applicable to $\rightarrow$ and $\leftrightarrow$

2.10 Logical Equivalence

SPPU : Dec.-12

In real life, we come across several similar things with respect to different aspects. e.g. Two cars are similar with respect to average, two students are similar with respect to F.E. marks.

Likewise in logic, we can say that two propositions are similar with respect to their truth values.

Definition : Two propositions A and B are logically equivalent iff they have the same truth value for all choices of the truth values of simple propositions involved in it.

Two propositions (formulas) are equivalent even if they have different variables. Two statement formulas P and Q are equivalent

Iff $P \leftrightarrow Q$ is a tautology.

If p is equivalent to Q then it can be represented as $P \Leftrightarrow Q$ or $P \equiv Q$.

The symbol $\Leftrightarrow$ is not a connective

Example 2.10.1 *Prove by constructing the truth table* $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$.

SPPU : Dec.-12, Marks 3

Solution : Consider truth table

p	q	r	$q \vee r$	$p \rightarrow (q \vee r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \vee (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

In the columns of $p \rightarrow (q \vee r)$ and $(p \rightarrow q) \vee (p \rightarrow r)$, truth values are same for all possible choices of truth values of p, q and r. Hence

$$p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$$

Example 2.10.2 *Prove that*

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

Solution : Consider the truth table

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$	$\sim p \vee q$	$\sim q \vee p$	$(\sim p \vee q) \wedge (\sim q \vee p)$
T	T	F	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	F	T	F
F	T	T	F	T	F	F	F	T	F	F
F	F	T	T	T	T	T	T	T	T	T

From above table

$$(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$$

2.11 Logical Identities

Sr. No.	Name of identity	Identity
1.	Idempotence of $\vee$	$p = p \vee p$
2.	Idempotence of $\wedge$	$p = p \wedge p$
3.	Commutativity of $\vee$	$p \vee q = q \vee p$
4.	Commutativity of $\wedge$	$p \wedge q = q \wedge p$
5.	Associativity of $\vee$	$p \vee (q \vee r) = (p \vee q) \vee r$
6.	Associativity of $\wedge$	$p \wedge (q \wedge r) = (p \wedge q) \wedge r$
7.	Distributivity of $\wedge$ over $\vee$	$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
8.	Distributivity of $\vee$ over $\wedge$	$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
9.	Double negation	$p = \sim(\sim p)$
10.	De Morgan's laws	$-(p \vee q) = -p \wedge -q$
11.	De Morgan's laws	$-(p \wedge p) = -p \wedge -q$
12.	Tautology	$p \vee -q =$ Tautology
13.	Contradiction	$p \wedge -p =$ Contradiction
14.	Absorption laws	$p \vee (p \wedge q) = p$
15.	Absorption laws	$p \wedge (p \vee q) = p$

Example 2.11.1 *De Morgan's laws*

i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Solution : i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Consider the truth table

p	q	~ p	~ q	p ∨ q	~ (p ∨ q)	~ p ∧ ~ q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are same for each choice of p and q

Hence $\sim(p \vee q) \equiv \sim p \wedge \sim q$

ii) $\sim(p \vee q) \equiv \sim p \vee \sim q$

Consider the table

p	q	~ p	~ q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \wedge q)$ and $(\sim p \vee \sim q)$ are same for each choice of p and q.

Hence $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$

Example 2.11.2 *Absorption laws*

i) $p \vee (p \wedge q) \equiv p$ *ii)* $p \wedge (p \vee q) \equiv p$

Solution :

i) $p \vee (p \wedge q) \equiv p$

Consider the table

p	q	$p \wedge q$	$p \vee (p \wedge q)$	p
T	T	T	T	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

From the table, in last two columns truth tables of $p \vee (p \wedge q)$ and p are same for each choice of p and q.

Hence $p \vee (p \wedge q) \equiv p$

ii) $p \wedge (p \vee q) \equiv p$

Consider the table

p	q	$p \vee q$	$p \wedge (p \vee q)$	p
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	F	F

From the table, last two columns are identical hence $p \wedge (p \vee q) \equiv p$

Example 2.11.3 *$p \to q$ and $\sim p \vee q$ are logically equivalent.*

Solution : Consider the table

p	q	~ p	~ p ∨ q	p → q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence $\sim p \vee q$ and $p \to q$ are logically equivalent.

2.12 The Duality Principle

Two statement formulas p and p* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge, \vee, T, F$ by $\vee, \wedge, F, T$ respectively.

Example 2.12.1 *Write the duals of*

i) $\sim(p \vee q)$ ii) $p \wedge (q \vee r)$ iii) $p \vee T$

Solution : The duals are i) $\sim(p \wedge q)$ ii) $p \vee (q \wedge r)$ iii) $p \wedge F$

2.13 Logical Implication

A proposition p is said to logically imply a proposition Q iff one of the following condition holds :

1) $\sim p \vee Q$ is a Tautology

2) $p \wedge Q$ is a Contradiction

3) $p \to Q$ is a Tautology

It is denoted by $p \Rightarrow Q$

e.g. 1) $(p \wedge q) \wedge \sim (q \vee p)$ is a contradiction

Hence $p \wedge q \Rightarrow q \vee p$

The relation $p \Rightarrow Q$ is reflexive, antisymmetric and transitive

Note : The symbols $\to$ & $\Rightarrow$ are not same

$\Rightarrow$ is not connective

2.14 Important Connectives

I) NAND :

The word NAND is a combination of "NOT" and "AND" where "NOT" stands for negation and "AND" stands for the conjunction. It is denoted by the symbol $\uparrow$.

If P and Q are two formulas then NAND connective is defined as

$$P \uparrow Q \leftrightarrow \sim (P \wedge Q)$$

The connective $\uparrow$ has the following equivalence

1) $P \uparrow P \leftrightarrow \sim (P \wedge P) \leftrightarrow \sim P \vee \sim P \Leftrightarrow \sim P$

$\therefore \quad P \uparrow P \equiv \sim P$

2) $(P \uparrow Q) \uparrow (P \uparrow Q) \leftrightarrow \sim (P \uparrow Q) \leftrightarrow \sim\sim (P \wedge Q) \leftrightarrow P \wedge Q$

3) $(P \uparrow P) \uparrow (Q \uparrow Q) \leftrightarrow \sim P \uparrow \sim Q \leftrightarrow \sim(\sim P \wedge \sim Q) \leftrightarrow P \vee Q$

Note : NAND connective is commutative but not associative

II) NOR :

The connective "NOR" is a combination of "NOT" and "OR" where "NOT" stands for negation and "OR" stands for the disjunction. It is denoted by the symbol $\downarrow$ and defined as

$$P \downarrow Q \leftrightarrow \sim (P \vee Q) \quad \therefore \quad P \downarrow Q \equiv \sim (P \vee Q)$$

The connective $\downarrow$ has the following equivalences

i) $P \downarrow P \leftrightarrow \sim (P \vee P) \leftrightarrow \sim P \wedge \sim P \leftrightarrow \sim P$

ii) $(P \downarrow Q) \downarrow (P \downarrow Q) \leftrightarrow \sim (P \downarrow Q) \leftrightarrow P \vee Q$

iii) $(P \downarrow P) \downarrow (Q \downarrow Q) \leftrightarrow \sim P \downarrow \sim Q \leftrightarrow \sim(\sim P \vee \sim Q) \leftrightarrow P \wedge Q$

iv) The connective $\downarrow$ is commutative but not associative

2.15 Normal Forms

SPPU : Dec.-04, 06, 08, 10, 12, 14, May-17

If a given statement formula involves n atomic variables, then we have 2 possible combinations of truth values of statements replacing the variables. The construction of the truth table involves finite number of steps but it may not be practical if number of variables are more. Therefore we reduce the given statement formula to normal form and find whether a given statement formula is a tautology or contradiction or at least satisfiable.

A formula which is a conjunction or product of the variables and their $p \wedge \sim q$ negations is called an elementary product. If p and q are statements then p, $\sim p$, $\sim p \wedge q$, $p \wedge \sim q$, $\sim p \wedge \sim q$ are some examples of elementary products or fundamental conjunctions. A formula which is a disjunction or sum of the variables and their negations is called an elementary sum or fundamental disjunctions. p, $\sim p$, $\sim p \vee q$, $p \vee \sim q$, $\sim p \vee p \vee \sim q$ are some examples of an elementary sum or fundamental disjunctions.

2.15.1 Disjunctive Normal Form (dnf)

A statement formula which consists of a disjunction ($\vee$) of fundamental conjunctions or elementary product ($\wedge$). It is abbreviated as dnf.

A disjunctive normal form of a given formula is constructed as follows :

1) Replace '$\rightarrow$' or '$\leftrightarrow$' by using logical connectives $\wedge, \vee$ & $\sim$. $p \rightarrow q \equiv \sim p \vee q$, $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$
2) Use De Morgan's laws to eliminate '~' before sums or products.
3) Apply distributive laws repeatedly and eliminate product of variables to obtain the required normal form.

Examples :

1) $(p \wedge q) \vee (q) \vee (\sim q \wedge p)$
2) $(p \wedge q \wedge r) \vee (p \wedge r) \vee (p \wedge q) \vee (p \wedge r)$
3) $(p \wedge r) \vee (p \wedge q)$
4) $(\sim p \wedge r) \vee (\sim q \wedge r) \vee (\sim r)$

All above examples are in disjunctive normal form.

2.15.2 Conjunctive Normal Form (cnf)

A statement formula which consists of a conjunction of the fundamental disjunctions ($\vee$). It is denoted as cnf.

Examples :

1) $p \wedge (p \vee q)$
2) $(p \vee q) \wedge (\sim p \vee q) \wedge (\sim q)$
3) $(p \vee q \vee r) \wedge (\sim p \vee r) \wedge (p \vee \sim q \vee r)$

All above examples are in conjunctive normal form.

2.15.3 Principal Normal Form

Let p and q be two statement variables, then $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$ are called minterms of p and q. They are also called Boolean conjunctives of p and q. The number of minterms with n variables is 2^n. None of the minterms should contain both a variable and it's negation. $\therefore p \wedge \sim p$ is not minterm.

The dual of minterm is called a maxterm.

$\therefore$ For two statement variables p and q, maxterms are $p \vee q$, $\sim p \vee q$, $p \vee \sim q$ and $\sim p \vee \sim q$.

I) Principal Disjunctive Normal Form

A statement formula which consists of a disjunction of minterms only is called the principal disjunctive normal form.

e.g.

1) $(p \vee q) \wedge (\sim p \wedge \sim q)$, $(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$ are principal cnf.
2) $p \vee (p \wedge q)$, $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim q)$, $(p \wedge q \wedge r) \vee (p \wedge q)$ are not principal dnf.

II) Principal Conjuctive Normal Form

A statement formula which consists of a conjunction of maxterms only is called the principle conjunctive normal form.

e.g.

1) $(p \vee q) \wedge (\sim p \vee q)$, $(p \vee \sim q) \wedge (\sim p \vee \sim q)$ are principal cnf.
2) $(p \vee q) \wedge (\sim p)$ is cnf but not principal cnf.

Example 2.15.1 *Obtain the conjunctive normal form and disjunctive normal form of the following formulae given below :*
i) $p \wedge (p \rightarrow q)$ *ii)* $\sim (p \vee q) \rightleftarrows (p \wedge q)$ **SPPU : Dec.-12, Marks 4**

Solution :

i)
$$\begin{aligned} p \wedge (p \rightarrow q) &\equiv p \wedge (\sim p \vee q) \sim \text{cnf} \\ &\equiv (p \wedge \sim p) \vee (p \wedge q) \\ &\equiv F \vee (p \vee q) \\ &\equiv (p \wedge q) \qquad \text{... Definition a single conjunctive} \end{aligned}$$

ii)
$$\begin{aligned} \sim (p \vee q) \rightleftarrows (p \wedge q) &\equiv (\sim\sim (p \vee q) \vee (p \wedge q)) \wedge (\sim (p \wedge q) \vee \sim (p \vee q)) \\ &\equiv ((p \vee q) \vee (p \wedge q) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q) \\ &\equiv (p \vee q) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)) \\ &\equiv (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q) \\ &\equiv (p \vee q) \wedge (\sim p \vee \sim q) \qquad \text{... cnf} \end{aligned}$$

Further

$$\begin{aligned} (p \vee q) \wedge (\sim p \vee \sim q) &\equiv ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q) \\ &\equiv (p \wedge \sim q) \vee (q \wedge \sim p) \vee ((p \wedge \sim q) \vee (q \wedge \sim q) \\ &\equiv F \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee F \\ &\equiv (p \wedge \sim p) \vee (p \wedge \sim q) \qquad \text{... Definition} \end{aligned}$$

Example 2.15.2 *Find the conjunctive normal form and disjunctive normal form for the following :*

i) $(p \vee \bar{q}) \rightarrow q$ *ii)* $p \leftrightarrow (\bar{p} \vee \bar{q})$ **SPPU : Dec.-06, 14, May-17, Marks 3**

Solution :

i) $(p \vee \bar{q}) \rightarrow q \equiv \overline{(p \vee \bar{q})} \vee q$

$\equiv (\bar{p} \wedge \bar{\bar{q}}) \vee q$

$\equiv (\bar{p} \wedge q) \vee q$ ... Definition

$(\bar{p} \wedge q) \vee q \equiv (\bar{p} \vee q) \wedge (q \vee q)$

$\equiv (\sim p \vee q) \wedge q$ ... cnf

$p \times (\bar{p} \vee \bar{q}) \equiv (\bar{p} \vee (\bar{p} \vee \bar{q})) \wedge (\overline{(\bar{p} \vee \bar{q})} \vee p)$

$\equiv (\bar{p} \vee \bar{p} \vee \bar{q}) \wedge ((\bar{\bar{p}} \wedge \bar{\bar{q}}) \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge (p \vee p) \wedge (q \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge p \wedge (q \vee p)$ ... cnf

$\equiv ((\bar{p} \wedge p) \vee (\bar{q} \wedge p)) \wedge (q \vee p)$

$\equiv (F \vee (\bar{q} \wedge p)) \wedge (q \vee p)$

$\equiv (\bar{q} \wedge p) \wedge (q \vee p)$

$\equiv (\bar{q} \wedge p \wedge q) \vee (\bar{q} \wedge p \wedge p)$

$\equiv (F \wedge p) \vee (\bar{q} \wedge p)$

$\equiv F \vee (\bar{q} \wedge p)$

$\equiv (\bar{q} \wedge p)$ ... Definition

Example 2.15.3 *Find the conjunctive and disjunctive normal forms for the following without using truth table*

i) $(p \rightarrow q) \wedge (q \rightarrow p)$ *ii)* $((p \wedge (p \rightarrow q)) \rightarrow q)$ **SPPU : Dec.-04, 10, 14, Marks 4**

Solution :

i) $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Further, using the distributive law on the above cnf we have

$((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p) = (\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p)$

$= (\sim p \wedge \sim q) \vee (q \wedge p)$ ... dnf

$(\because p \wedge \sim p \equiv q \wedge \sim q)$

ii) $(p \wedge (p \to q)) \to q \equiv \sim(p \wedge (\sim p \vee q)) \vee q$

$\equiv \sim p \vee \sim(\sim p \vee q) \vee q$

$\equiv \sim p \vee (p \wedge \sim q) \vee q$

Example 2.15.4 *Obtain cnf of each of the following*

i) $p \wedge (p \to q)$ *ii)* $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$ **SPPU : Dec.-08, Marks 4**

Solution :

i) $p \wedge (p \to q) \equiv (p \wedge (\sim p \vee q))$

$\equiv (p \wedge \sim p) \vee (p \wedge q)$

$\equiv \vee (p \wedge q)$

$\equiv (p \wedge q)$ ... cnf

ii) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q) \equiv ((q \vee p) \wedge (q \vee \sim q) \vee (\sim p \vee \sim q)$

$\equiv (q \vee p) \wedge T \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p) \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p \vee \sim p) \wedge (q \wedge p \vee \sim q)$

$\equiv (q \vee T) \wedge (p \vee q \vee \sim q)$

$\equiv T \wedge (p \vee T) \equiv T \wedge T$

$\equiv T \equiv (p \vee \sim p)$ which is the required cnf (single disjunct)

Example 2.15.5 *Find DNF of* $((p \to q) \wedge (q \to p)) \vee p$

SPPU : Dec.-14, Marks 4, May-17, Marks 3

Solution :

$((p \to q) \wedge (q \to p) \vee p \equiv [(\sim p \vee q) \wedge (\sim q \vee p)] \vee p$ $(\because p \to q \equiv - \sim p \vee q)$

$\equiv [p \vee (\sim p \vee q)] \wedge [p \vee (\sim q \vee p)]$

$\equiv [(p \vee \sim p) \vee q] \wedge [p \vee p \vee \sim p]$ $(\because p \vee p = p$, $p \vee \sim p = T)$

$\equiv (T \vee q) \wedge (p \vee \sim q)$

$\equiv T \wedge (p \vee \sim q)$

$\equiv p \vee \sim q$ which is the required DNF

Example 2.15.6 *Obtain the cnf :* $(\sim p \wedge q \wedge r) \vee (p \wedge q)$ **SPPU : Dec.-12, Marks 4**

Solution :

$(\sim p \wedge q \wedge r) \vee (p \wedge q) \equiv (p \wedge q) \vee (\sim p \wedge q \wedge r)$

$$\equiv (p \vee (\sim p \wedge q \wedge r)] \wedge [q \vee (\sim p \wedge q \wedge r)]$$

$$\equiv [(p \vee \sim p) \wedge (p \vee q)] \wedge [p \vee r)] \wedge [(q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)]$$

$$\equiv [T \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \sim p) \wedge q \wedge (q \vee r)]$$

$$\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q \qquad \text{which is the required cnf}$$

Example 2.15.7 *Find dnf by using truth table*

$(p \leftrightarrow (q \vee r)) \rightarrow \sim p$

Solution : Consider the truth table

p	q	r	~p	q ∨ r	p ↔ (q ∨ r)	[p ↔ (q ∨ r)] → ~ p
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	F	T	T	F
T	F	F	F	F	F	T ← 1
F	T	T	T	T	F	T ← 2
F	T	F	T	T	F	T ← 3
F	F	T	T	T	F	T ← 4
F	F	F	T	F	T	T ← 5

Consider only 'T' from last column and choose corresponding values of 'T' from p, q and r. For the first marked row 1 corresponding p is T, q is F and r is F. So take $p \wedge \sim q \wedge \sim r$ or $p \wedge q' \wedge r'$

For 2nd $T \rightarrow \sim p \wedge q \wedge r$, 3rd $T \rightarrow \sim p \wedge q \wedge \sim r$,

4th $T \rightarrow \sim p \wedge q \wedge r$, and 5th $T \rightarrow \sim p \wedge \sim q \wedge \sim r$

Hence the logically equivalent dnf form is

$$[p \leftrightarrow (q \vee r)] \rightarrow \sim p \equiv (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$$
$$\vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$$

Example 2.15.8 *Obtain the principal dnf of* $(\sim p \vee \sim q) \rightarrow (\sim p \wedge r)$

Solution :

$$(\sim p \vee \sim q) \rightarrow (\sim p \wedge \sim r) \equiv \sim (\sim p \vee \sim q) \vee (\sim p \wedge r)$$

$$\equiv (\sim (\sim p) \wedge \sim (\sim q)] \vee (\sim p \wedge r)$$

$$\equiv (p \wedge q) \vee (\sim p \wedge r) \qquad \text{which is dnf}$$

$$\equiv (p \wedge q \wedge (r \vee \sim r)) \vee (\sim p \wedge r \wedge (q \vee \sim q))$$

$$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q)$$

which is the required principal dnf.

Example 2.15.9 *Find the principal cnf of* $(p \wedge q) \vee (\sim p \wedge r)$

Solution :

$$\begin{aligned}
[(p \wedge q) \vee (\sim p \wedge r) &\equiv [(p \wedge q) \vee \sim p] \wedge [(p \wedge q) \vee r] \\
&\equiv (p \vee \sim p) \wedge (q \vee \sim p) \wedge (p \vee r) \wedge (q \vee r) \\
&\equiv (q \vee \sim p \vee (r \wedge \sim r) \wedge (p \vee r \vee (q \wedge \sim q)) \wedge (q \vee r \vee (p \wedge \sim p)) \\
&\equiv (q \vee \sim p \vee r) \wedge (q \vee \sim p \vee \sim r) \wedge (p \vee r \vee q) \wedge (p \vee r \vee \sim q)) \\
&\quad \wedge (q \vee r \vee p) \wedge (q \vee r \vee \sim p) \\
&\equiv (p \vee q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee \sim q \vee r)
\end{aligned}$$

which is the required principle cnf.

Example 2.15.10 *Find the principal dnf of* $\sim p \vee q$

Solution :

$$\begin{aligned}
\sim p \vee q &\equiv [\sim p \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)] \\
&\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p) \\
&\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)
\end{aligned}$$

which is the required Principle dnf.

2.16 Methods of Proof

SPPU : Dec.-09, May-06, 10

We have learned statements or propositions and their truth values. Now, we will discuss ways by which statements can be linked to form a logically valid argument.

Whenever an assertion is made, which is claimed to be true, one has to state an argument which produces the truth of the assertion. To construct a proof we need to derive new assertions from existing ones by different ways. This is also done by using valid of inference.

Valid argument : A valid argument is a finite sequence of statements $p_1, p_2, \ldots p_n$ called as premises (or assumptions or hypothesis) together with a statement C, called the conclusion such that $p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n \rightarrow C$ is a tautology.

As mathematical proof is a logical argument that verifies the truth of the theorem. There are several ways of proving a theorem which are based on one or more rules of inference.

There are following most commonly used rules of inference.

2.16.1 Law of Detachment (or Modus Ponens)

Whenever the statements p and $p \to q$ are accepted as true, then we must accept the statement q as true. This rule is represented in the following form

$$\begin{array}{l} p \to q \\ \underline{\quad p \quad} \\ \therefore q \end{array}$$

The assertions above the horizontal line are called premises OT hypothesis. And the assertion below the line is called the conclusion.

This rule constitutes a valid argument as $(p \to q) \wedge p \to q$ is a tautology.

The truth table is as follows

p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This form of valid argument is called the law of detachment as conclusion q is detached from premise $p \to q$ and p.

It is also called as the law of direct inference.

Example

If Sushma gets a first class with distinction in B.E. then she will get a good job easily.

Let p : Sushma gets a first class with distinction in B.E.

q : She will get a good job easily.

The inference rule is
$$\begin{array}{l} p \to q \\ \underline{\quad p \quad} \\ \therefore q \end{array}$$

Hence this form of argument is valid.

2.16.2 Modus Tollen (Law of Contrapositive)

Modus Tollen is a rule of denying. It can be stated as "If $p \to q$ is true and q is false, then p is false. This is represented in the following form.

$$\begin{array}{l} p \to q \\ \underline{\sim p} \\ \therefore \sim q \end{array}$$

Above argument is valid as $(p \to q) \wedge \sim q \to \sim p$ is a tautology. In above example is $b \to q$ and $\sim q$: Sushma will not get a good job easily then $\sim p$: She has not a first class with distinction in B.E.

2.16.3 Disjunctive Syllogism

This rule states that "If $p \vee q$ is true and p is false then q is true.

It is represented in the following form as

$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore \ q \end{array}$$

This argument is valid as $(p \vee q) \wedge \sim p \to q$ is a tautology.

2.16.4 Hypothetical Syllogism

It is also known as the transitive rule.

It can be stated as follows

"If $(p \to q)$ and $(q \to r)$ are true then $p \to r$ is true.

This rule is presented in the following form.

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore \ p \to r \end{array}$$

This argument is valid as $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

Example 2.16.1 *Determine whether the argument is valid or not. If I try hard and I have talent then I will become a musician. If I become a musician, then I will be happy. Therefore if I will not be happy then I did not try hard or I do not have talent.*

SPPU : May-10, Marks 4

Solution :

Let p : I try hard

q : I have talent

r : I will become musician

s : I will be happy

It's symbolic form is as follows

$$\begin{array}{l} s_1 : (p \wedge q) \rightarrow r \\ s_2 : \quad r \rightarrow s \\ \hline s_3 : \sim s \rightarrow \sim p \vee \sim q \end{array}$$

To check validity of this statement, one way is to use truth table or prove logically.

Suppose assignment is invalid. This means that for some assignment of truth values s is T, s_2 is T but s_3 is F. s_3 will have truth value F if ~ s is T and $\sim p \vee \sim q$ is F

i.e. s is F and ~ p is F and ~ q is F

i.e. s is F, and p is T and q is T

As s_2 is T, the truth values of r and s both are F. Since s, is T, r is F implies either p or q is F. This is contradiction since by assumption both p and q are true.

Hence given statement is valid.

Example 2.16.2 *Test the validity of the argument. If a person is poor, he is unhappy. If a person is unhappy, he dies young. Therefore poor person dies young.* **SPPU : Dec.-09, Marks 4**

Solution :

Let p : Person is poor

q : Person is unhappy

r : Person dies young

In symbolic form argument is

$$\begin{array}{l} s_1 : p \rightarrow q \\ s_2 : q \rightarrow r \\ \hline s : p \rightarrow r \end{array}$$

The above argument is the rule of hypothetical syllogism. Hence it is valid.

Example 2.16.3 *Determine the validity of the following argument.*

s_1 : If I like discrete structure then I will study

s_2 : Either I will study or I will fail

s : If I fail then I do not like discrete structure

Solution :

Let p : I like discrete structure

q : I will study

r : I will fail

In symbolic form

$$\begin{array}{l} s_1 : p \to q \\ s_2 : q \vee r \\ \hline s : r \to \sim p \end{array}$$

We know that for the validity of argument $s_1 \wedge s_2$ should logically imply s.

Assign the truth values T, T, T, to p, q, r respectively. Then s_1 is T and s_2 is also T

But s is $T \to F$ is F

Hence argument is invalid.

Example 2.16.4 *Determine the validity of the argument*

s_1 : *All my friends are musicians*

s_2 : *John is my friend*

s_3 : *None of my neighbours are musicians*

S : John is not my neighbour

SPPU : May-06

Solution :

Let p : All my friends are musicians

q : John is my friend

r : My neighbours are musicians

s : John is my neighbour

$\therefore$ $s_1 : p$

$s_2 : q$

$s_3 : \sim r$

In symbolic form

$$\begin{array}{c} p \\ q \\ \sim r \\ \hline \sim s \end{array}$$

As all my friends are musicians and John is my friend $\Rightarrow$ John is musician.

$\therefore$ $p \wedge q \to$ John is musician

$p \wedge q \wedge \sim r \to$ John is musician and my neighbours are not musicians

$\therefore$ $p \wedge q \wedge \sim r \to$ John is not my neighbour

$p \wedge q \wedge \sim r \to \sim s$ is true.

Therefore given argument is valid.

Example 2.16.5 *I am happy if my program runs. A necessary condition for the program to run is it should be error free. I am not happy. Therefore the program is not error free.*

Solution :

Let p : I am happy

q : My program runs

r : Program should be error free

In symbolic form

$$\begin{array}{l} q \to p \\ q \to r \\ \sim p \\ \hline \therefore \quad \sim r \end{array}$$

To check validity, consider the truth table

p	q	r	$q \to p$	$q \to r$	$\sim p$	$(q \to p) \wedge (q \to r) \wedge \sim p$ (I)	(I) $\to \sim r$
F	F	T	T	T	T	T	F

Last column is not T

$\therefore$ Given argument is not tautology.

$\therefore$ Given argument is invalid.

2.17 Quantifiers

SPPU : Dec.-08, 09,10, 11, 15, May-15

In grammar a predicate is the word in a sentence which expresses what is said about object. i.e. properties of an object or relation among objects. For example "is a good teacher", "is a clever student" are predicates. In logic predicate has a broaden role than in grammar. Predicate is presented by using a variable x in place of holder. e.g. x is a prime number.

An assertion that contains one or more variables is called a predicate. It's truth value is predicated after assigning truth values to its variables.

A predicate p containing n variables $x_1, x_2, \ldots x_n$ is called an n-place predicate and denoted by $p(x_1, x_2, x_3, \ldots x_n)$. Each variable x_i is called as argument.

The values which the variables may assume constitute a collection is called the universe of discourse or domain of discourse.

There are two types of quantifiers.

2.17.1 Universal Quantifiers

If p(x) is a predicate with x as an argument then the universal quantifier for p(x) is the statement.

"For all values of x, p(x) is true". We denote the phrase "For all" by $\forall$

$\forall$ means for all or for each or for every.

If p(x) is true for all values of x then

$\forall x\, p(x)$ is true.

For example - p(x) : $x \geq 0$, where x is any positive integer.

$\therefore$ The proposition $\forall x\, p(x)$ is true.

However, if x is an integer $\forall x\, p(x)$ is false.

2.17.2 Existential Quantifier

In some situations, we only require that there is at least one value for the predicate is true. Suppose for the predicate p(x), $\forall x\, p(x)$ is false, but there exists at least one value of x for which p(x) is true, then we say that in this proposition, x is bound by existential quantification.

The phrase "there exists an x" is called an existential quantifier.

The symbol "$\exists$" is used to denote the logical quantifier "there exists" or "there exists an x" or "there is x" or "for some x" or "for at least one x".

The existential quantifier for p(x) is denoted by $\exists x\, p(x)$.

2.17.3 Negation of Quantified Statement

Consider the statement $\forall x\, p(x)$. It's negation is "It is not the case that for all x, p(x) is true". This means that for some x = a, p(a) is not true or $\exists$ x s.t. ~ p(x) is true.

Hence the negation of $\forall x\, p(x)$ is logically equivalent to $\exists x\, [\sim p(x)]$.

Sr. No.	Statement	Negation
1.	$\forall x\, p(x)$	$\exists x\, [\sim p(x)]$
2.	$\exists x\, [\sim p(x)]$	$\forall x\, p(x)$
3.	$\forall x [\sim p(x)]$	$\exists x\, p(x)$
4.	$\exists x\, p(x)$	$\forall x [\sim p(x)]$

I) Equivalence involving quantifiers

1) Distributivity of $\exists$ over $\vee$

$$\exists x\, [p(x) \vee Q(x)] \equiv \exists x\, p(x) \vee \exists x\, Q(x)$$

$$\exists x\,[p \vee Q(x)] \equiv p \vee (\exists x\, Q(x))$$

2) Distributivity of $\forall$ over $\wedge$

$$\forall x\,[p(x) \wedge Q(x)] \equiv \forall x\, p(x) \wedge \forall x\, Q(x)$$

$$\forall x\,[p \wedge Q(x)] \equiv p \wedge (\forall x\, Q(x))$$

3) $\exists x\,[p \wedge Q(x)] \equiv p \wedge [\exists x\, Q(x)]$

4) $\forall x\,[p \vee Q(x)] \equiv p \wedge [\forall x\, Q(x)]$

5) $\sim[\exists x p(x)] \equiv \forall x\,[\sim p(x)]$

6) $\sim[\forall x\, p(x)] \equiv \exists x\,[\sim p(x)]$

7) $\forall x\, p(x) \Rightarrow \exists x\, p(x)$

8) $\forall x\, p(x) \vee \forall x\, Q(x) \Rightarrow \forall x\,(p(x) \vee Q(x)$

9) $\exists x(p(x) \wedge Q(x)) \Rightarrow \exists x\, p(x) \wedge \exists x\, Q(x)$

II) Rules of Inference for addition and deletion of quantifiers

1) Rule 1 : Universal Instantiation

$$\frac{\forall x\, p(x)}{\therefore\ p(k)}$$, k is some element of the universe

2) Rule 2 : Existential Instantiation

$$\frac{\exists x\, p(x)}{\therefore\ p(k)}$$, k is some element for which p(k) is true.

3) Rule 3 : Universal Generalization

$$\frac{px}{\forall x\, p(x)}$$

4) Rule 4 : Existential Generalization

$$\frac{p(k)}{\therefore\ \exists x\, p(x)}$$

k is some element of the universe

III)

Sr. No.	Quantifiers	Expression
1.	$\exists x \forall y\, p(x,y)$	There exists a value of x such that for all values of y, p(x, y) is true.
2.	$\forall y\, \exists x\, p(x,y)$	For each value of y, there exists x such that p(x, y) is true.
3.	$\exists x \exists y\, p(x,y)$	There exist value of x and value of y such that p(x, y) is true.
4.	$\forall x \forall y\, p(x,y)$	For all values of x and y. p(x, y) is true.

Example 2.17.1 *Represent the arguments using quantifiers and find its correctness.*

All students in this class understand logic. Ganesh is a student in this class. Therefore Ganesh understands logic. **SPPU : Dec.-11, Marks 4**

Solution :

Let C(x) : x is a student in this class

L(x) : x understands logic

In symbolic form

$$\begin{array}{ll} & \forall x\,(C(x) \rightarrow L(x)) \\ & C(a) \\ \hline \therefore & L(a) \end{array}$$

Here a means Ganesh

This is Modus Ponen

Therefore this argument is valid.

Example 2.17.2 *Let p(x) : x is even*

Q(x) : x is a prime number

R(x, y) : x + y is even

a) Using the information given above write the following sentences in symbolic form.

i) Every integer is an odd integer

ii) Every integer is even or prime

iii) The sum of any two integers is an odd integer. **SPPU : Dec.-10, Marks 4**

Solution :

i) $\forall x\,[\sim p(x)]$

ii) $\forall x\,[p(x) \vee Q(x)]$

iii) $\forall x \forall y\,[\sim R(x, y)]$

Example 2.17.3 *Using information in example 2.17.2 write an english sentence for each of the symbolic statement given below*

i) $\forall x\,(\sim Q(x))$

ii) $\exists y(\sim p(y))$

iii) $\sim[\exists x\,(p(x) \wedge Q(x))]$ **SPPU : Dec.-10, Marks 4**

Solution :

i) All integers are not prime numbers

ii) At least one integer is not even.

iii) It is not the case that there exists an integer which is even and prime.

Example 2.17.4 *Determine the validity of the following argument*

s_1 : All my friends are musician

s_2 : John is my friend

s_3 : Name of my neighbours are musician

s : John is not my neighbour.

Solution : Let the universe of discourse be the set of people.

Let F(x) : x is my friend

M(x) : x is a musician

N(x) : x is my neighbour

It's symbolic form is

$$s_1 : \forall x [F(x) \rightarrow M(x)]$$
$$s_2 : F(a) \ (a = John]$$
$$s_3 : \forall x [N(x) \rightarrow \sim M(x)]$$
$$\therefore \quad s : \sim N(a)$$

Suppose ~N(a) has value F.

∴ N(a) is T. Since s_3 is T, we must have ~M(a) is T or M(a) is F. But s_1 is T. Hence we must have F(a) to be false but this is contradiction. Hence if s is false either of s_1 or s_3 should be false. Hence argument is valid.

Example 2.17.5 *For the universe of all integers. Let*

p(x) : x > 0

Q(x) : x is even

R(x) : x is a perfect square

S(x) : x is divisible by 4

T(x) : x is divisible by 7

Write the following statement in symbolic form

i) At least one integer is even

ii) There exists a positive integer that is even

iii) If x is even then x is not divisible by 7

iv) No even integer is divisible by 7

v) There exists an even integer divisible by 7

vi) If x is even and x is perfect square then x is divisible by 4.

Solution :

i) $\exists x Q(x)$

ii) $\exists x [p(x) \wedge Q(x)]$

iii) $\forall x [Q(x) \rightarrow \sim T(x)]$

iv) $\forall x [Q(x) \rightarrow \sim T(x)]$

v) $\exists x [Q(x) \wedge T(x)]$

vi) $\forall x [Q(x) \wedge R(x) \rightarrow s(x)]$

Example 2.17.6 *Rewrite the following statements using quantifier variables and predicate symbols.*

i) All birds can fly

ii) Not all birds can fly

iii) Some men are genius

iv) Some numbers are not rational

v) There is a student who likes Maths but not Hindi

vi) Each integer is either even or odd **SPPU : Dec.-08, Marks 4**

Solution :

i) Let B(x) : x is a bird

F(x) : x can fly

Then the statement can be written as

$\forall x [B(x) \rightarrow F(x)]$

ii) $\exists x [B(x) \wedge \sim F(x)]$

iii) Let M(x) : x is a man

G(x) : x is a genius

The statement in symbolic form as $\exists x [M(x) \wedge G(x)]$

iv) Let N(x) : x is a number

R(x) : x is rational

The statement in symbolic form as $\exists x [N(x) \wedge \sim R(x)]$ or $\sim [\forall x (N(x) \rightarrow R(x)]$

v) Let S(x) : x is a student

M(x) : x likes Maths

H(x) : x likes Hindi

$\therefore$ The statement in symbolic form as $\exists x [S(x) \wedge M(x) \wedge \sim H(x)]$

vi) Let I(x) : x is an integer

E(x) : x is even

O(x) : x is odd

The statement in symbolic form as $\forall x [I(x) \rightarrow E(x) \vee O(x)]$

Example 2.17.7 *Negate each of the following statements*

i) $\forall x, |x| = x$

ii) $\exists x, x^2 = x$ **SPPU : Dec.-09, 15, May-15, Marks 4**

Solution :

i) $\exists x, |x| \neq x$ ii) $\forall x, x^2 \neq x$

Example 2.17.8 *Negate the following*

i) If there is a riot, then someone is killed.

ii) It is day light and all the people are arisen. **SPPU : May-15, Dec.-15, Marks 4**

Solution :

i) It is not the case that if there is a riot then someone is killed.

ii) It is not the case that it is day light and all the people are arisen.

OR

i) Let p : There is a riot

q : Someone is killed

Given statement is $p \rightarrow q$

Hence $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$

$\equiv$ There is a riot and someone is not killed.

ii) Let p : It is a day light

q : All the people are arisen

Given statement is $p \wedge q$

Hence $\sim (p \wedge q) = \sim p \vee \sim q$

Hence either it is not a day light or all the people are not arisen.

❑❑❑

Unit - I

3 Mathematical Induction

Syllabus

Proof by Mathematical Induction and Strong Mathematical Induction.

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3.1 Introduction

3.2 First Principle of Mathematical Induction Statement

3.3 Second Principle of Mathematical Induction Statement (Strong Mathematical Induction) ..... Dec.-04, 05, 06, 08, 10, 11,12, ... 13, 14, 15, 18, May-05, 06, 07, 08, ... 14, 15, 17, 18, 19 ..... Marks 6

3.1 Introduction

Mathematical induction is a powerful technique in applied mathematics especially in number theory, where many properties of natural numbers are proved by this method.

In day to day life, we are often required to generalise a particular pattern for the prediction purpose. The generalisation is achieved by using a statement involving a variable as natural number.

Mathematical induction is very useful technique or tool for the programmers to check whether a program statement is loop invariant or not.

There are two principles of mathematical induction :

1) First principle of mathematical induction.
2) Second principle of mathematical induction.

3.2 First Principle of Mathematical Induction Statement

Let P(n) be a statement involving a natural number $n \geq n_0$ such that,

1) If P(n) is true for $n = n_0$ where $n_0 \leq N$ and
2) Assume that P(k) is true for $k = \geq n_0$

We prove P(k+1) is also true,

Then P(n) is true for all natural numbers $n = n_0$.

Step 1 is called as the basis of induction.

Step 2 is called as the induction step.

3.3 Second Principle of Mathematical Induction Statement (Strong Mathematical Induction)

SPPU : Dec.-04, 05, 06, 10, 11, 12, 13, 14, 15, 18, May-05, 06, 07, 08, 14, 15, 17, 18, 19

Let P(n) be a statement involving a natural number $n \geq n_0$ such that,

1) If P(n) is true for $n = n_0$ where $n_0 \leq N$ and
2) Assume that P (n) is true for $n_0 < n \leq k$ i.e. $P(n_0+1)(n_0+2)$.....P(k) are true.

we prove that P(k + 1) is true,

Then P(n) is true for all natural numbers $n \geq n_0$.

Example 3.3.1 *Prove that :* $1 + 2 + 3 + 4 \ldots + n = \frac{n(n+1)}{2}$

SPPU : May-17, Dec.-18, Marks 4

Solution : Let, P(n) be the given statement

Consider the following steps,

Step 1 : Basis of induction

For $n = 1$, L.H.S. = 1

$$\text{R.H.S.} = \frac{1(1+1)}{2} = 1$$

$\therefore$ For $n = 1$, L.H.S. = R.H.S.

$\therefore$ P(1) is true.

Step 2 : Assume that P(k) is true

i.e. $1 + 2 + 3 + \ldots\ldots + k = \dfrac{k(k+1)}{2}$ ... (1)

Consider, $1 + 2 + 3 + \ldots\ldots + k + (k+1)$

$$= \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)+(k+2)}{2}$$

Hence P(k + 1) is true

$\therefore$ By the principle of mathematical induction P(n) is true for all n.

Example 3.3.2 *Prove by mathematical induction for $n \geq 1$.*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

SPPU : May-05, Marks 6

Solution : Let P(n) the given statement

1. Basis of induction

For $n_0 = 1$ L.H.S. = 1.2 = 2

$$\text{R.H.S.} = \frac{1(2)(3)}{3} = 2 \Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step

Assume that, P(k) is true

i.e. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$ ... (1)

Then we have

$$[1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1)] + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{... (Using 1)}$$

$$= (k+1)(k+2)\left[\frac{k}{3} + 1\right] = \frac{(k+1)(k+2)(k+3)}{3}$$

Hence assuming P(k) is true, P(k + 1), is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Example 3.3.3 *Show by induction that , $n \geq 1$*

$$1^2 + 3^2 + 5^2 + ..(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

SPPU : May-14, Marks 4

Solution : Let P(n) be the given statement,

1. Basis of induction :

For n = 1, L.H.S. = $1^2 = 1$,

R.H.S. = $\frac{1(1)(3)}{3} = 1$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true

2. Induction step : Assume that P(k) is true.

i.e. $$1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \ldots (1)$$

Hence

$$[1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2] + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \quad \ldots \text{(Using 1)}$$

$$= \frac{(2k+1)}{3}[2k^2 - k + 3(2k+1)]$$

$$= \frac{(2k+1)}{3}[2k^2 + 5k + 3]$$

$$= \frac{(2k+1)}{3}[2k^2 + 2k + 3k + 3]$$

$$= \frac{(2k+1)}{3}[(2k+3)(k+1)]$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

Hence assuming P(k) is true P(k + 1) is also true. Therefore by mathematical induction P(n) in true for all n ≥ 1.

Example 3.3.4 *Show that* $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+\ldots+n)^2$

SPPU : Dec.-12, May-18, Marks 4

Solution : Let P(n) be the given statement,

$$1^3+2^3+3^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2=\frac{n^2(n+1)^2}{4}$$

1. Basis of induction :

For n = 1, L.H.S. = 1,

$$\text{R.H.S.} = \frac{1(1+1)^2}{4} = 1$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $$1^3+2^3+3^3+\ldots+k^3 = \frac{k^2(k+1)^2}{4} \quad \ldots (1)$$

Then we have

$$(1^3+2^3+3^3+\ldots+k^3)+(k+1)^3 = (1+2+3+\ldots+k)^2+(k+1)^3 \quad \text{(Using 1)}$$

$$= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3$$

$$= (k+1)^2\left[\frac{k^2}{4}+k+1\right]$$

$$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k+1)^2(k+2)^2}{4}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Example 3.3.5 *Show that* $\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\ldots+\frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$

Solution : Let P(n) be the given statement,

1. Basis of induction : For n = 1

We have, $\text{L.H.S.} = \frac{1}{1.3} = \frac{1}{3}$

$\text{R.H.S.} = \frac{1(2)}{1(3)} = \frac{1}{3}$

$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$

Hence P(1) is true.

2. Induction step : Assume that P(k) is true

i.e. $$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots \frac{k^2}{(2k-1)(2k+1)} = \frac{k(k+1)}{2(2k+1)} \quad \ldots (1)$$

Then we have,

$$\left[\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{k^2}{(2k-1)(2k+1)}\right] + \frac{(k+1)^2}{[2(k+1)-1][2(k+1)+1]}$$

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} \quad \ldots \text{(Using 1)}$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{k(2k+3)+2(k+1)}{2(2k+3)}\right]$$

$$= \frac{(k+1)}{2k+1}\left[\frac{2k^2+5k+2}{2(2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k^2+4k+k+2}{2(2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k(k+2)+1(k+2)}{2(2k+3)}\right]$$

$$= \frac{(k+1)(k+2)}{2(2k+3)}$$

$$= \frac{(k+1)[(k+1)+1]}{2[2(k+1)+1]}$$

$\therefore$ P(k + 1) is true.

Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Example 3.3.6 *Show that a)* $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

b) Show that $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

c) Show that $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-1)(3n+1)}=\frac{n}{3n+1}$

SPPU : Dec.-05, Marks 6

Solution : Let P(n) be the given statement,

a) 1. Basis of induction :

For n = 1 L.H.S. $=\frac{1}{1.2}=\frac{1}{2}$

R.H.S. $=\frac{1}{1+1}=\frac{1}{2}$

L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad \ldots (1)$$

Then we have

$$\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \ldots \text{(Using 1)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}=\frac{(k+1)}{(k+1)+1}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) in true for all $n \geq 1$.

b) Let P(n) : $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

1. Basis of induction : For n = 1

$$\text{L.H.S.} = \frac{1}{1.3}=\frac{1}{3},\ \text{R.H.S.} = \frac{1}{3}$$

$$\frac{1}{3} = \frac{1}{2.1+1}$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step : Assume that P(k) is true

i.e. $$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \quad \ldots (1)$$

Then we have,

$$\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2k-1)(2k+1)}\right]+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)\,(2k+3)}$$

$$= \frac{(2k+1)\,(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

$$= \frac{k+1}{2(k+1)+1}$$

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

c) Let P(n) : $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

1. Basis of induction

For n = 1, $$\text{L.H.S.} = \frac{1}{1.4}=\frac{1}{4},\ \text{R.H.S.} = \frac{1}{4}$$

$$\frac{1}{1.4} = \frac{1}{3.1+1}$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \quad \ldots (1)$$

Then we have,

$$\left[\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\ldots+\frac{1}{(3k-2)(3k+1)}\right]+\frac{1}{(3k+1)(3k+4)} \quad \ldots \text{(Using 1)}$$

$$= \frac{k}{(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$

$$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3(k+1)+1)}$$

$$= \frac{k+1}{3(k+1)+1}$$

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Example 3.3.7 *Prove by induction for $n \geq 0$. $1+a+a^2+\ldots+a^n = \frac{1-a^{n+1}}{1-a}$.*

SPPU : Dec.-10, Marks 4

Solution : Let P(n) be the given statement,

1. Basis of induction

For n = 0, L.H.S. = 1, R.H.S. $= \frac{1-a}{1-a} = 1$

For n = 1, L.H.S. = 1 + a, R.H.S. $= \frac{1-a^2}{1-a} = 1 + a$

$\therefore$ For n = 0, 1, L.H.S. = R.H.S.

Hence P(0), P(1) are true.

2. Induction step : Assume that P(k) is true

$$\therefore \quad 1+a+a^2+\ldots+a^k = \frac{1-a^{k+1}}{1-a} \quad \ldots (1)$$

Consider,

$$1+a+a^2+....+a^k+a^{k+1} = \frac{1-a^{k+1}}{1-a}+a^{k+1} \quad \text{... (Using 1)}$$

$$= \frac{1-a^{k+1}+(1-a)a^{k+1}}{1-a}$$

$$= \frac{1-a^{k+1}+a^{k+1}-a^{k+2}}{1-a}$$

$$= \frac{1-a^{k+2}}{1-a}$$

Hence P(k + 1) is true.

Therefore by the mathematical induction P(n) is true for all $n \geq 0$.

Example 3.3.8 *Use mathematical induction to show that $n(n^2-1)$ is divisible by 24. Where n is any odd positive number.* **SPPU : Dec.-14, Marks 4**

Solution : If $n(n^2-1) = n^3-n$ is divisible by 24.

Then $n^3-n = 24\,(m)$ where m is any positive integral.

Let P(n) be the given statement,

1. Induction step : For n = 1,

$n(n^2-1) = 0$ which is divisible by 24.

For n = 3, $n\left(n^2-1\right) = 24$ which is divisible by 24.

$\therefore$ P(1) and P(3) is true.

2. Induction step :

Assume that P(k) is true.

i.e. $k(k^2-1) = k^3-k$ is divisible by 24.

$\therefore$ $k(k^2-1) = k^3-k = 24\,(m_0),\ m_0 \leftarrow z$ ... (1)

Consider

$$(k+1)\,[(k+1)^2-1] = (k+1)^3-(k+1)$$

$$= k^3+3k^2+3k+1-k-1$$

$$= k^3+3k^2+2k$$

$$= (k^3-k)+3k^2+3k \quad \text{... (Using 1)}$$

$$= 24\,m_0 + 3k\,(k+1)$$ (As k(k + 1) is multiple of 8 for k odd positive integer and $k \geq 3$)

$$= 24m_0 + 3(8m_1)$$

$$= 24\,(m_0 + m_1)$$

$$= 24\,m_2 \quad (\because m_0 + m_1 = m_2)$$

$\therefore$ P(k + 1) is true.

$\therefore$ By mathematical induction P(n) is true for all n odd positive number.

Example 3.3.9 *Show that $n^4 - 4\,n^2$ is divisible by 3 for all $n \geq 2$.* **SPPU : Dec.-15, Marks 4**

Solution : Let P(n) be the given statement,

1. Basis of induction

For $n = 2$

$$2^4 - 4\,(2^2) = 16 - 16$$

$= 0$ is divisible by 3 as 0 is divisible by every number

$\therefore$ P(2) is true.

2. Induction step : Assume that P(k) is true

i.e. $k^4 - 4k^2$ is divisible by 3

Then we have,

$$(k+1)^4 - 4(k+1)^2 = k^4 + 4k^3 + 6k^2 + 4k + 1 - 4\,(k^2 + 2k + 1)$$

$$= (k^4 - 4k^2) + 4\,(k^3 + 2k) + 6k^2 + 12k - 3$$

$k^4 - 4\,k^2$ is divisible by 3.

$k^3 + 2k$ is divisible by 3

Also $6\,k^2 + 12\,k - 3 = 3\,(2k^2 + 4k - 1)$ is divisible by 3.

Hence $(k+1)^4 - 4\,(k+1)^2$ is divisible by 3.

Hence assuming P(k) is true.

P(k + 1) in also true. Therefore P(n) is true for $n \geq 2$.

Example 3.3.10 *Prove that : $7^{2n} + 2^{3n-3} \cdot 3^{n-1}$ is divisible by 25 $\forall n$.*

SPPU : May-19, Marks 3

Solution : Let P(n) be the given statement.

1) Basis of induction : For n = 1,

$7^2 + 2^0 \cdot 3^0 = 49 + 1 = 50$, which is divisible by 25.

$\therefore$ P(1) is true.

2) Induction step :

Assume that P(k) is true.

i.e. $7^{2k} + 2^{3k-3}\, 3^{k-1} = 25\ (m)$

Consider

$$
\begin{aligned}
7^{2(k+1)} + 2^{3(k+1)-3} \cdot 3^{k+1-1} &= 7^{2k+2} + 2^{3k}\, 3^k \\
&= 49 7^{2k} + 2^{3k} \cdot 3k \\
&= 49[25m - 2^{3k-3} \cdot 3^{k-1}] + 2^{3k} \cdot 3^k \\
&= 25\ (49m) - 49\ 2^{3k-3} \cdot 3^{k-1} + 2^{3k-3} \cdot 3^{k-1}(8 \times 3) \\
&= 25\ (49m) + 2^{3k-3} \cdot 3^{k-1}\ (-49 + 24) \\
&= 25(49m) + 2^{3k-3} 3^{k-1}(-25) \\
&= 25\ [49m - 2^{3k-3} \cdot 3^{k-1}] \\
&= 25\ (P) : P \in N
\end{aligned}
$$

$\therefore$ P(k+1) is true.

$\therefore$ By mathematical induction, P(n) is true for all n.

Example 3.3.11 *Using mathematical induction, prove that*

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

SPPU : Dec.-04, Marks 6

Solution : Let P(n) be the given statement,

1. Basis of induction

For n = 1 L.H.S = 1, R.H.S. = 1

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) in true.

i.e. $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2 = (-1)^{k-1} \dfrac{k(k+1)}{2}$ ... (1)

Then we have,

$[1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2] + (-1)^k\ (k+1)^2$ ... (Using 1)

$$= (-1)^{k-1}\frac{k(k+1)}{2} + (-1)^k (k+1)^2$$

$$= (-1)^k (k+1)\left[-\frac{k}{2} + (k+1)\right]$$

$$= (-1)^k (k+1)\left[\frac{-k+2k+2}{2}\right]$$

$$= (-1)^k \frac{(k+1)(k+2)}{2}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) in true for all $n \geq 1$.

Example 3.3.12 *Prove by mathematical induction that for $n \geq 1$:*

$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1.$ **SPPU : May-08, 15, Marks 6**

Solution : Let P(n) be the given statement,

1. Basis of induction

For $n = 1$, L.H.S = 1, R.H.S. = 1

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true.

i.e. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k!$

$= (k + 1)! - 1$ ... (1)

Then we have,

$[1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k!] + (k + 1) \cdot (k + 1)!$ ... (Using 1)

$= [(k + 1)! - 1] + (k + 1) \cdot (k + 1)!$

$= (k + 1)! + (k + 1) \cdot (k + 1)! - 1$

$= (k + 1)! [k + 1 + 1] - 1$

$= (k + 2) (k + 1)! - 1$

$= (k + 2)! - 1$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for $n \geq 1$.

Example 3.3.13 *Prove that for any positive integer n the number $n^5 - n$ is divisible by 5.*

SPPU : Dec.-08, Marks 6

Solution : Let P(n) be the given statement,

1. Basis of induction :

For $n = 1$, $1^5 - 1 = 0$ is divisible by 5.

As 0 is divisible by every number.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true.

i.e. $k^5 - k$ is divisible by 5

Then we have

$$(k+1)^5 - (k+1) = (k^5 + {}^5C_1k^4 + {}^5C_2k^3 + {}^5C_3k^2 + {}^5C_4k + {}^5C_5) - (k+1)$$
$$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$
$$= (k^5 - k) + 5[k^4 + 2k^3 + 2k^2 + k]$$

$k^5 - k$ is divisible by 5.

and $5(k^4 + 2k^3 + 2k^2 + k)$ is divisible by 5.

Hence $(k+1)^5 - (k+1)$ is divisible by 5.

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for $n \geq 1$.

Example 3.3.14 *Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction for $n \geq 1$.*

SPPU : May-06, 07, Dec.-13, Marks 6

Solution : Let P(n) be the given statement,

1. Basis of induction :

For n = 1 $\quad 8^1 - 3^1 = 5$

$$= 5 \cdot 1$$

Obviously a multiple of 5.

$\therefore$ P(1) is true.

2. Induction step : Assume that, P(k) in true.

i.e. $8^k - 3^k$ is multiple of 5 say 5 r

i.e. $$8^k - 3^k = 5r \quad \text{... (1)}$$

where r is an integer

Then we have,

$$8^{k+1} - 3^{k+1} = 8^k \cdot 8 - 3^k \cdot 3$$
$$= 8^k \cdot (5+3) - 3^k \cdot 3$$
$$= 8^k \cdot 5 + (8^k \cdot 3 - 3^k \cdot 3)$$
$$= 8^k \cdot 5 + 3(8^k - 3^k)$$

Obviously $8^k \cdot 5$ is multiple of 5 and also $8^k - 3^k$ is multiple of 5.

Therefore, $8^{k+1} - 3^{k+1}$ is multiple of 5.

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Example 3.3.15 *Show that the sum of the cubes of three consecutive natural number is divisible by 9.* **SPPU : Dec.-06, Marks 6**

Solution : Let n, n + 1, n + 2 be three consecutive natural numbers.

We have to show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Let P(n) be the above statement,

1. Basis of induction : For n = 1

$$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 \quad \text{which is divisible by 9.}$$

$\therefore$ P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $k^3 + (k+1)^3 + (k+2)^3$ in divisible by 9.

Then we have,

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = [(k+1)^3 + (k+2)^3] + [k^3 + {}^3C_1 k^2(3) + {}^3C_2 k(3)^2 + {}^3C_3(3^3)]$$

$$= (k+1)^3 + (k+2)^3 + k^3 + [9k^2 + 27k + 27]$$

$$= [k^3 + (k+1)^3 + (k+2)^3] + 9[k^2 + 3k + 3]$$

$k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9 and $9(k^3 + 3k + 3)$ is divisible by 9.

$\Rightarrow (k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 9.

Hence assuming P(k) in true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Example 3.3.16 *Using mathematical induction prove that* $3 + 3.5 + 3.5^2 + \ldots + 3.5^n = \left(\frac{5^{n+1} - 1}{4}\right)$.

For non-negative number n. **SPPU : Dec.-11, Marks 6**

Solution : Cancelling 3 from the both sides of given.

Statement, We get

$$1 + 5 + 5^2 + \ldots + 5^n = \frac{5^{n+1} - 1}{5 - 1} \qquad \ldots (1)$$

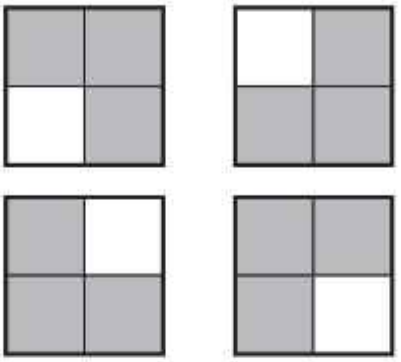
Let P(n) be the above statement.

To prove this refer example 3.3.7 for a = 5.

Example 3.3.17 *Let n be a positive integer. Show that any* $2^n \times 2^n$ *chessboard with one square removed can be covered by L-shaped pieces, where each piece covers three squares at a time.*

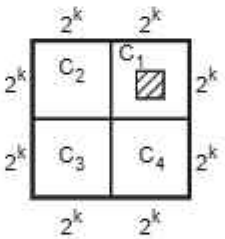
Solution : Let P(n) be the proposition that any $2^n \times 2^n$ chessboard with one square removed can be covered using L-shaped pieces.

Basis of induction : For n = 1, P(1) implies that any 2×2 chessboard with one square removed can be covered using L shaped pieces. P(1) is true, as seen below.

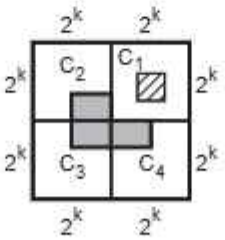


Induction step : Assume that, P(k) is true i.e. any $2^k \times 2^k$ chessboard with one square removed can be covered using L-shaped pieces.

Then, we have to show that P(k + 1) is true. For this consider, a $2^{k+1} \times 2^{k+1}$ chesshoard with one square removed. Divide the chessboard into four equal halves of size $2^k \times 2^k$, as shown below.



The square which has been removed, would have been removed from one of the four chessboards, say C_1. Then by induction hypothesis, C_1 can be covered using L-shaped pieces. Now, from each of the remaining chessboards, remove that particular piece (or tile), lying at the centre of the large chessboards.



Then by induction hypothesis, each of these $2^k \times 2^k$ chessboards with a piece (or tile) removed can be covered by the L-shaped pieces. Also the three tiles removed from the centre can be covered by one L-shaped piece. Hence the chessboard of $2^{k+1} \times 2^{k+1}$ can be covered by L-shaped pieces.

Hence proved.

Example 3.3.18 *Suppose we have unlimited stamps of two different denominations, 3 rupees and 5 rupees. We want to show that it is possible to make up exactly any postage of 8 rupees or more using stamps of these two denominations.*

Solution : For k = 8, we have one 5 rupees stamp and one 3 rupees stamp.

For k = 9, replace 5 rupees stamp by two 3 rupees stamp, similarly for k = 10, replace all 3, 3 rupees stamps by, two 5 rupees stamp an so on.

Hence let us assume that, it is possible to make up k rupees stamp using 3 rupees and 5 rupees stamps (for $k \geq 8$).

Now we have to show that it is also possible to make up (k + 1) rupees stamps using 3 rupees and 5 rupees stamps.

We examine two cases :

1) Suppose we make up stamps of k rupees using at least one 5 rupees stamp. Replacing a 5 rupees stamp by two 3 rupees stamp, we can make up k + 1 rupees stamps.

2) Suppose we make up a stamp of k rupees using 3 rupees only. Since $k \geq 8$ we must have at least 3, 3 rupees stamps. Replacing these 3, 3 rupees stamps by two five rupees stamps. We can make up stamps of k + 1 rupees.

Hence proved.

Example 3.3.19 *The king summoned the best mathematicians in the kingdom to the palace to find out how smart they were. The king told them*
"I have placed white hats on some of you and black hats on the others. You may look at, but not talk, to, one another. I will leave now and will come back every hour on the hour. Every time I return, I want those of you who have determined that you are wearing white hats to come up and tell me immediately."
As it turned out, at the nth hour every one of the n mathematician who were given white hats informed the king that she knew that she was wearing a white hat. Why ?

Solution :

1. Basis of induction : For n = 1, there is only one mathematician wearing a white hat. Since the king said that white hats were placed on some one of the mathematician (king never lie). The mathematician who saw that all other mathematicians had on black hats would realize immediately that she was wearing a white hat. Consequently she

would inform the king on the first hour. (when the king returned for the first time) that she was wearing a white hat.

Now let n = 2, i.e. two mathematicians wearing white hats.

Consider, one of these two mathematicians. She saw that one of her colleagues was wearing a white hat. She reasoned that if she were wearing a black hat, her colleague would be the only one wearing a white hat. In that case, her colleague would have figured out the situation and informed the king on the first hour. That did not happen shows that she was also wearing a white hat. Conesquently she told the king on the second hour (and so did the other mathematician with a white hat, since all the mathematicians are smart).

2. Induction step : Assume that, if there were k mathematicians wearing white hats, then they would have figured out that they were wearing white hats and informed the king so on the kth hour. Now, suppose that there were k + 1 mathematicians wearing white hats.

Every mathematician wearing a white hat saw that k of her colleagues were wearing white hats. However, that her k colleagures did not inform the king of their findings on the kth hour can only imply that there were more then k. People wearing white hats. Consequently she knew that, she must be wearing a white hat also on the $(k+1)^{th}$ hour. She (together with all other mathematicians wearing white hats) would inform the king their conclusion.

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Unit - II

4 Relations

Syllabus

Relations and their Properties, n-ary relations and their applications, Representing relations, Closures of relations, Equivalence relations, Partial orderings, Partitions, Hasse diagram, Lattices, Chains and Anti-Chains, Transitive closure and Warshall's algorithm.

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4.1 Introduction

4.2 Cartesian Product

4.3 Relation

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4.7 Types of Relations on Set ..... Dec.-10, 11, 12, 14, 18, 19, ... May-14, 17, 18, 19, ... Marks 6

4.8 Partitions of a Set ..... Dec.-11, 12, May-17, ... Marks 3

4.9 Closure of a Relation ..... Dec.-05, 07, 12, 13, 14, 15, 16 ..... May-06, 07, 08, 15, 18, ... Marks 6

4.10 Partially Ordered Set Dec.-06

4.11 Principle of Duality ..... Dec.-10, 11, 13, 14, 15, 16, ... May-14, 15, 19 ..... Marks 6

4.1 Introduction

In the chapter 1, we dealt with sets, elements, different types and general properties of sets. There may exists various relationships among elements of sets. We will study relations and functions with various properties and examples.

Relation may involves equality or inequality of elements. In mathematics the expressions like " is equal to " is similar to ", " is greater than", "is parallel to " are some relations.

4.2 Cartesian Product

Let A and B be two non empty sets. The cartesian product of A and B is denoted by $A \times B$ and defined as

$$A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$$

e.g. If $A = \{1, 2, 3, 4\}$, $B = \{a, b\}$ then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a) (3, b) (4, a) (4, b)\}$$

$$B \times A = \{(a, 1), (a, 2), (a, 3), (a, 4), (b, 1), (b, 2), (b, 3), (b, 4)\}$$

$$(a, b) \notin A \times B \text{ or } B \times A$$

and $(1, a) \neq (a, 1)$

Hence $A \times B \neq B \times A$

So the cartesian product is not commutative.

Similarly,

$$A \times B \times C = \{(a, b, c) / a \in A \text{ and } b \in B \text{ and } c \in C\}$$

In general

$$A_1 \times A_2 \times \ldots\ldots \times A_n = \{a_1, a_2, a_3 \ldots\ldots\ldots\ldots, a_n) / a_1 \in A_1 \text{ and } a_2 \in A_2, \ldots\ldots \text{ and } a_n \in A_n)\}$$

Theorem 1 : If $|A| = m$ and $|B| = n$ then $|A \times B| = |B \times A| = mn$

Proof : The proof is trivial by set theory.

Theorem 2 : If A, B, C are non empty sets then $A \subseteq B \Rightarrow A \times C \subseteq B \times C$

Proof : Let (x,y) be any element in $A \times C$, then

$\Rightarrow x \in B$ and $y \in C$

$\Rightarrow (x, y) \in B \times C$

Hence $A \times C \subseteq B \times C$

Theorem 3 : If A, B, C, are sets then

i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

iv) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof :

i) $A \times (B \cap C) = \{(x, y) / x \in A \text{ and } (y \in B \cap C)\}$

$= \{(x, y) / x \in A \text{ and } / (y \notin B \text{ and } y \in C)\}$

$= \{(x, y) / x \in A, y \in B \text{ and } y \in A, y \in C\}$

$= \{(x, y) / (x, y) \in A \times B \text{ and } (x, y) \in A \times C\}$

$= \{(x, y) / (x, y) \in (A \times B) \cap (A \times C)\}$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Similarly students can prove remaining results.

Note : If $\Re$ is the set of real numbers then

$\Re \times \Re = \Re^2 \rightarrow$ Euclidean plane

$\Re \times \Re \times \Re = \Re^3 \rightarrow$ 3D space

4.3 Relation

Let A and B be two non empty sets. A relation from A to B is any subset of $A \times B$. It is denoted by $R : A \rightarrow B$

e.g. Let $A = \{x, y, z\}$, $B = \{1, 2, 3)$

then $A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3), (z, 1), (z, 2) (z, 3)\}$

$R_1 = \{(x, 1), (y, 2), (z, 1),)\}$, $R_2 = \{(z, 3)\}$, $R_3 = \{\phi\}$

are relations from A to B

But $R_4 = \{(1, x), (2, x)\}$is not relation from A to B. R_4 is the relation from B to A.

Important Results :

1) If $(a, b) \in R$ then it is denoted by a R b.
2) If R is a relation from A to B then $R \subseteq A \times B$
3) If $R \subseteq A \times B$ then R is a relation from A to A and R is called a relation on A.
4) If R is a relation from A to B, then the set of all first elements of the ordered pairs $(a,b) \in R$ is called the domain of R. It is denoted by $D(R) = \{a/(a, b) \in R\}$.

The range of R is the set of all second co-ordinates of the ordered pairs (a,b) ∈ R. It is denoted by R (R) = {(b/(a, b) ∈ R}.

5) The null set is the subset of $A \times B$.

∴ ϕ is a relation called null relation or empty relation.

Example :

Let A = {1, 2, 3}, B = {x, y}

then R_1 = {1, x) (1, y), (3, x)} is a relation from A to B

∴ $D(R_1)$ = {1,3}

$R(R_1)$ = {x, y}

4.4 Matrix Representation of a Relation

Let $A = \{a_1, a_2, a_3 .... a_n\}$, $B = \{b_1, b_2, b_3, ... b_m\}$ and $R \subseteq A \times B$. Then the relation matrix of R is denoted by $M_R = [m_{ij}]_{n \times m}$ and defined by

$$m_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \text{ i.e. } a_i \not R b_j \\ 1 & \text{if } (a_i, b_j) \in R \text{ i.e } a_i R b_j \end{cases}$$

Example 4.4.1 *Let A = {1, 2, 3, 4} and R = {(x, y)/ x < y } then find* M_R.

Solution : R = {(1, 2) (1, 3) (2, 3) (1, 4) (2, 4) (3, 4) }

$$\therefore \quad M_R = [M_{ij}]_{4\times4} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 4.4.2 *If A = {1, 2, 3, 4, 5, 6} and a R b iff a divides b for a, b ∈ A. Find relation matrix.*

Solution : R = {(1, 2) (1, 3), (1, 4), (1, 5), (1, 6), (2, 4) (2, 6), (3, 6), (1, 1) (2, 2), (3, 3), (4, 4), (5, 5) (6, 6)}

$$\therefore \quad \text{Relation Matrix} = M_R = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times6} \end{array}$$

Example 4.4.3 *$A = \{x, y, z\}$ Find the relation matrices of the following relations.*
i) $R_1 = \{(x, x)\ (y, y)\ (z, z)\}$ ii) $R_2 = \{(x, y)\ (y, x), (y, z), (z, y)\}$ iii) $R_3 = \{(x, x)$ $(x, y), (x, z)\ (y, x)\ (y, y)\ (y, z), (z, x)\ (z, y)\ (z, z)$

Solution : (5,5) (6,6)

i) $$M_{R_1} = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \end{array}$$

ii) $$M_{R_2} = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

iii) $$M_{R_3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4.4.1 Relation Matrix Operations

We know that relation matrix is a boolean matrix as all entries are either 0 or 1

I) Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ be two relation matrices then

$A + B = [a_{ij} + b_{ij}] = [c_{ij}]_{m\times n}$

where $c_{ij} = 1$ if $a_{ij} = 1$, or $b_{ij} = 1$

$= 0$ if $a_{ij} = 0$, and $b_{ij} = 0$

II) Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times k}$ then $A \cdot B = [a_{ij}; b_{jk}] = [d_{jk}]$

where $d_{jk} = 1$ if $a_{ij} = 1$ and $b_{jk} = 1$

$= 0$ if $a_{ij} = 0$ or $b_{jk} = 0$

e.g. If $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then $$A + B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $$A \cdot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4.4.2 Properties of Relation Matrix

Let R_1 be a relation matrix from A to B and R_2 be a relation matrix from B to C then relation matrices satisfy the following properties :

i) $M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$

ii) $M_{R_1^{-1}}$ = Transpose of M_{R_1}

iii) $M_{(R_1 R_2)^{-1}} = M_{R_2^{-1}} M_{R_1^{-1}}$

4.5 Diagraphs

A relation can be represented pictorially by drawing its graph. Let A be any non empty set and R be a relation on A. R can be represented by graphically by the following procedure.

i) The elements of a set A are represented by small circles or point i.e. (∘) or ... These elements are called as vertices or nodes.

ii) If (a, b) ∈ R or a R b then vertices a and b are joined by a continuous arc with an arrow from a to b. These arcs are known as edges of the graph i.e. if aRb then

iii) If aRa then the vertex a is joined to itself by a loop around a. e.g. if aRa then

This graphical representation of the relation is called as diagraph or directed graph of R. Let us consider the following examples.

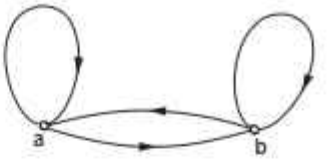
1) aRa

2) aRb

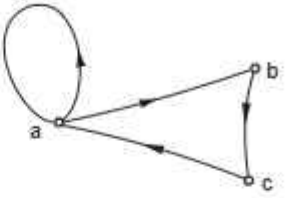
3) bRa

4) aRb ∧ bRa

5) aRa ∧ aRb ∧ bRa ∧ bRb

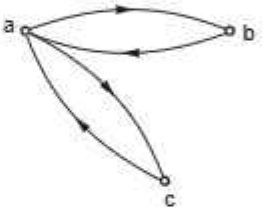


6) aRb ∧ bRc ∧ CRa ∧ aRa



7) aRa ∧ bRb ∧ cRc

8) aRb ∧ bRa ∧ aRc ∧ cRa



Examples :

Example 4.5.1 *A = {1,2,3,4,5,6} and a R_b iff a/b (a divides b). Draw diagraph of R.*

Solution : We have

$$R = \left\{ \begin{matrix} (1,1)(2,2)(3,3)(4,4)(5,5)(6,6) \\ (1,2)(1,3)(1,4)(1,5)(1,6) \\ (2,4)(2,6)(3,6) \end{matrix} \right\}$$

The diagraph of R is as follows :

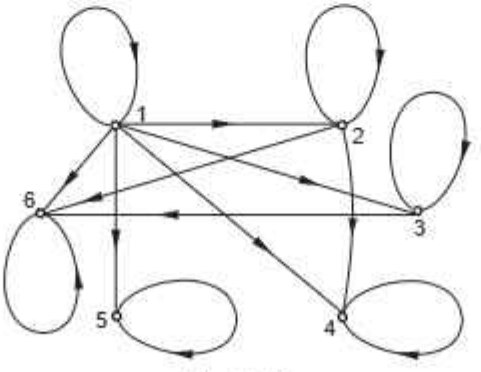


Fig. 4.5.1

Example 4.5.2 *Let A = {a, b, c, d} and* $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ *Draw diagraph of R.*

Solution : We have

R = {(a,a) (a,d) (b,b) b,d) (c,a) (c,b) (d,a)}

The digraph of R is as follows :

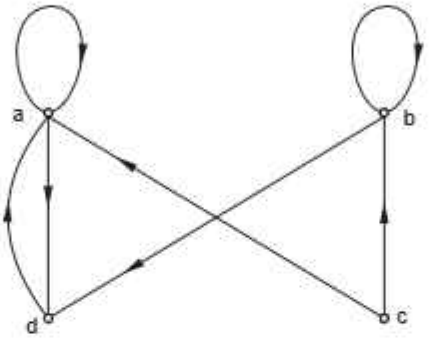


Fig. 4.5.2

Example 4.5.3 *Find the relation from the diagraph.*

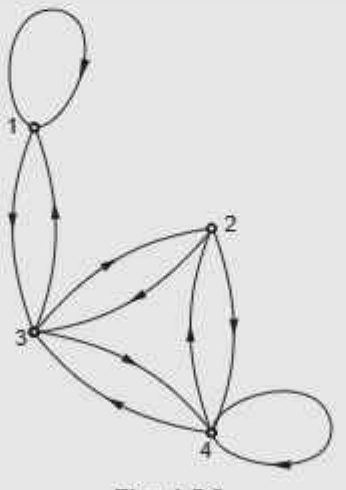


Fig. 4.5.3

Solution : From the given diagraph, the corresponding relation on A = {1, 2, 3, 4} is

R = {(1, 1) (1, 3) (3, 1) (2, 3) (3, 2), (2, 4) (4, 2) (4, 4) (3, 4) (4, 3)}

4.6 Special Types of Relations

4.6.1 Inverse Relation (OR Converse Relation)

Let R be a relation from A to B. The inverse relation of R is denoted by R^{-1} is a relation from B to A defined as

$$R^{-1} = \{(y, x) \mid y \in B, x \in A \text{ and } (x,y) \in R\}$$

i.e. if x R y then $yR^{-1}x$

It is also denoted by R^c

It is also known as the converse relation.

Example :

1) If R = {(1,2) (2,3) (3,1) (4,5) } then R^{-1} or R^c= {(2,1) (3,2) (1,3) (5,4)}

2) If $M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then $M_{R^{-1}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = [M_R]^t$

3) If the diagraph R is given by

$\therefore$ R = {(1,1) (1,2) (2,3) (4,2) (4,1) (4,3) (3,4) }

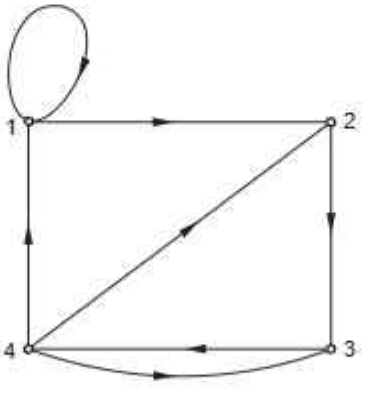


Fig. 4.6.1

then the diagraph of R^{-1} is obtained by changing the direction of arrow only which is given below.

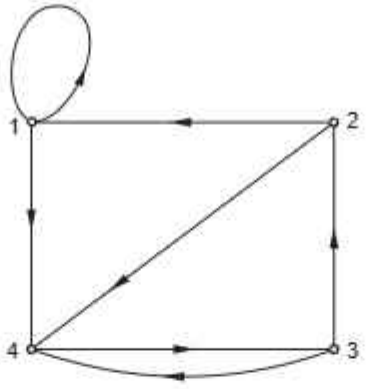


Fig. 4.6.2

Theorem 1 : If R, R_1, R_2 are relations from A to B then

i) $(R^{-1})^{-1} = R$

ii) $(R_1 \cup R_2)^{-1} = R_2^{-1} \cup R_1^{-1} = R_1^{-1} \cup R_2^{-1}$

iii) $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$

4.6.2 Complement of a Relation

Let R be a relation from A to B. The complement of R is denoted by $\overline{R}$ or $R' : A \to B$ and defined as $\overline{R} = \{(x, y) \mid (x, y) \notin R, x \in A, y \in B\}$

i.e. $x \overline{R} y$ iff $x \not R y$

Note $\overline{R}$ is the complement ary set of R w.r.t. universal domain $A \times B$.

Examples :

Example 4.6.1 *Let A = {1, 2, 3, 4}, B = { a, b},*

$A \times B$ = {(1, a) (1, b) (2, a) (2, b) (3, a) (3, b) (4, a) (4, b)},

R = {(1, a) (2, a) (3, a) (4, a)},

S = {4, a) (4, b) (3, a) (3, b)}.

Find $\overline{R}, \overline{S}, \overline{R} \cap \overline{S}, \overline{R} \cup \overline{S}$.

Solution :

$$\overline{R} = \{(1,b)\ (2,b)\ (3,b)\ (4,b)\}$$

$$\overline{S} = \{(1, a)\ (1, b)\ (2, a)\ (2, b)\}$$

$$\overline{R} \cap \overline{S} = \{(1, b)\ (2, b)\}$$

$$\overline{R} \cup \overline{S} = \{(1, a)\ (1, b)\ (2, a)\ (2, b)\ (3, b)\ (4, b)\}$$

Example 4.6.2 *If A = {a, b, c, d} and R = {(a, b) (c,d) (c,c) (d,a) (a,a) (b,b) (d,d) } is a relation on A. Draw diagraph of $\overline{R}$.*

Solution :

$$\overline{R} = \{(a,c)\,(a,d)\,(b,a)\,(b,c)\,(b,d)\,(c,a)\,(c,b)\,(d,b)\,(d,c)\}$$

∴ The diagraph of $\overline{R}$ is as follows :

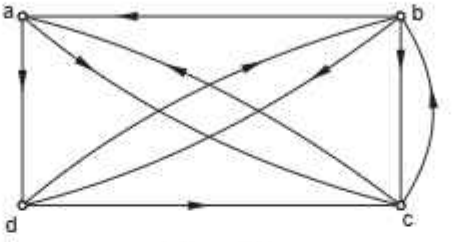


Fig. 4.6.3

Example 4.6.3 *Obtain the matrix of $\overline{R}$ if* $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution : The matrix of $\overline{R}$ is obtained from M_R be replacing 0 and 1 by 1 and 0 respectively

$$M_{\overline{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note : 1) $\overline{(R_1 \cup R_2)} = \overline{R_1} \cap \overline{R_2}$

2) $\overline{(R_1 \cap R_2)} = \overline{R_1} \cup \overline{R_2}$

4.6.3 Composite Relation

Let $R_1 : A \to B$ and $R_2 : B \to C$ be two relations.

The composition of R_1 and R_2 is denoted by

$R_1 \circ R_2$ or $R_1 R_2 : A \to C$ defined as

$R_1 \cdot R_2 = \{(x,z) \mid x R_1 y,\ y R_2 z$ i.e. $(x,y) \in R_1$ and $(y,z) \in R_2$ for $x \in A,\ z \in C\}$

Examples :

Example 4.6.4 *Let A = (x, y, z, w) and R_1 = {(x, y) (x, x) (y, z) (z, w) }, R_2 = {(x, x) (y, x) (y, y) (w, w) }, Find R_1, R_2 and diagraph of $R_1 \cdot R_2$.*

Solution :

In R_1	In R_2	$R_1 \cdot R_2$
1) (x, y)	(y, x) (y, y)	(x, x) (x, y)

i.e. $x \to y$	$y \to x$ $y \to y$	$x \to x$ $x \to y$
2) (x, x)	(x, x)	(x, x)
3) (y, z)	-	-
4) (z, w)	(w, w)	(z, w)

$\therefore \quad R_1 \cdot R_2 = \{(x, x)\}, (x, y), (z, w)\}$

The diagraph of $R_1 \cdot R_2$ is,

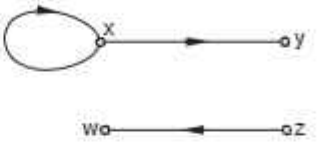


Fig. 4.6.4

Example 4.6.5 *Let $A = (1, 2, 3, 4)$. Let $R_1 = \{(x, y)/ x+y = 5\}$ and $R_2 = (x, y) / (y - x) = 1\}$ verify $(R_1 \cdot R_2)^C = R_2^C \cdot R_1^C$*

Solution : We have $A = \{1, 2, 3, 4\}$

$$R_1 = \{(1,4)\ (2,3),\ (3,2)\ (4,1)\}$$

$$R_2 = \{(1,2)\ (2,3)\ (3,4)\}$$

$$R_1 \cdot R_2 = \{(2,4)\ (3,3)\ (4,2)\}$$

$$(R_1 \cdot R_2)^C = \{(4,2)\ (3,3)\ (2,4)\}$$

$$R_1^C = \{(4,1),\ (3,2),\ (2,3),\ (1,4)\}$$

$$R_2^C = \{(2,1),\ (3,2)\ (4,3)\}$$

$$R_2^C \cdot R_1^C = \{(2,4),\ (3,3)\ (4,2)\} = (R_1 \cdot R_2)$$

Hence $(R_1 \cdot R_2)^C = R_2^C \cdot R_1^C$

4.7 Types of Relations on Set

SPPU : Dec.-10, 11, 12, 14, 18, 19, May-14, 17, 18, 19

Let R be a relation from A to A. Then we can say that R is a relation on set A.

$\therefore R \subseteq A \times A$

$A \times A$ is the universal set for R. If $R = A \times A$ then R is called the Universal Relation on A.

4.7.1 Identity Relation

Let A be any set. The relation on A is called Identity relation if

$$I_A = \{(a, a) \mid \text{for } a \in A\}$$

e.g. If A = {1, 2, 3, 4} then

R_1 = {(1, 1) (2, 2)}, R_2 = {(4, 4)} are identity relations.

The diagraph of the identity relation is the graph with loops only and it's relation matrix is the diagonal matrix with 0 and 1.

4.7.2 Reflexive Relation

1) Let R be a relation on set A. A relation R is said to be reflexive relation if $(a, a) \in R, \forall a \in A$.
2) A relation R is said to be not reflexive if $\exists$ at least one element $a \in A$ such that $(a, a) \notin R$ i.e. $a \not R\, a$.
3) A relation R is said to be irreflexive if for every $a \in A$, $(a, a) \notin R$ i.e. $a \not R\, a$

Notes :

1) If R is a reflexive relation on set A then

i) The relation matrix (M_R) is the identity matrix.

ii) Diagraph of reflexive relation will have loop for every element of A.

2) The relation matrix of irreflexive relation has all diagonal elements zero. It's diagraph is free from loops.

Examples

Example 1 :

Let A = {1,2, 3,4}. Then

i) R_1 = {(1, 1) (2, 2) (3, 3) (4, 4)} is the reflexive relation. It is known as the smallest reflexive relation on set A.

ii) R_2 = {(1, 1) (2, 2) (3, 3) (4, 4) (2, 4) (1, 2) (3, 2)} is reflexive relation.

iii) R_3 = { (1, 1) (2, 2) (3, 4)} is not reflexive relation as $3 \in A$ but $3 \not R_3 3$.

iv) R_4 = {(2, 3) (3, 4)} is irreflexive relation A as $\forall a \in A$, $a \not R_4 a$.

Example 2 :

Let A = Set of all straight lines in a plane. x R y iff x is parallel to y.

For any line $x \in A$, x is parallel to x.

$\therefore$ x Rx $\forall x \in A$

$\therefore$ R is a reflexive relation.

4.7.3 Symmetric Relation

Let R be a relation on set A. R is said to be symmetric relation if whenever a R b then b Ra for a, b $\in$ A.

Notes :

1) A relation is symmetric iff $R = R^{-1}$

2) The relation matrix of symmetric relation is a symmetric matrix.

Example : Let A = {1,2, 3, 4}. then

i) R_1 = {(1, 1) (2, 2) (3, 4) (4, 3)} is a the symmetric relation.

ii) R_2 = {(1, 1) (2, 2) (3, 3) (4, 4)} is a symmetric as well as reflexive relation.

iii) R_3 = { (1, 1) (2, 2) (3, 3) (4, 4) (3, 4)} is reflexive but not symmetric relation.

iv) R_4 = {(a, b)} $a \neq b$ is the smallest non- symmetric relation.

Example 4.7.1 *Let A = Set of all straight lines in a plane. x R y iff x || y i.e. x is parallel to y. S. T. R is a symmetric relation.*

Solution : R = {x R y | x is parallel to y}

Suppose x R y then x is parallel to y.

iff y is parallel to x.

iff y R x

i.e. whenever x R y, then y R x.

$\therefore$ R is a symmetric relation.

4.7.4 Compatible Relation

A relation R on a set A is said to be compatible relation if it is reflexive and symmetric relation. The relation matrix of a compatible relation is a symmetric matrix with the diagonal elements 1.

e.g. If A = {x, y, z} then

R = {(x, x) (y, y) (z, z) (x, y) (y, x)} is a compatible relation.

Note :

1) If R is a compatible relation on set A S.T. |A| = n then |R| $\geq$ n

4.7.4.1 Asymmetric Relation

A relation R on a set A is said to be asymmetric relation if whenever aRb then $b\not{R}a$.

Hence R is not asymmetric relation if for some a and b in A, aRb and bRa.

e.g. If A = {1, 2, 3} R = {(1, 2) (2, 3) (3, 1)} is a symmetric relation. But R_1 = {(1, 2) (2, 3) (3, 2)} is not asymmetric relation as (2, 3) and (3, 2) $\in R_1$.

4.7.5 Antisymmetric Relation

A relation R on a set A is said to be antisymmetric relation if whenever aRb and bRa then a = b. A relation R is not antisymmetric if $\exists$ a, b $\in$ A such that a $\neq$ b but aRb and bRa. The relation matrix of antisymmetric relation is never symmetric matrix.

Examples :

1) Let A = R and aRb iff $a \leq b$.

Suppose aRb and bRa then $a \leq b$ and $b \leq a$.

$\Rightarrow a = b$.

$\therefore$ R is antisymmetric relation. It is not symmetric relation.

2) Let A = {1, 2, 3} then

i) R_1 = {(1, 1) (2, 2)} is symmetric, antisymmetric relation.

ii) R_2 = {(1, 2) (2, 1)} is symmetric but not antisymmetric relation.

iii) R_3 = {(1, 1) (1, 2) (2, 2)} is antisymmetric but not asymmetric relation.

4.7.6 Transitive Relation

Let R be a relation on set A. R is said to be transitive relation if aRb and bRc, then aRc for a,b,c, in R.

$\therefore$ A relation R is not transitive if $\exists$ a, b and c in A.

Such that aRb, bRc but $a\not{R}c$

Examples :

1) Let A = R, R be the relation " $\leq$ ".

Then R is transitive as $a \leq b$, $b \leq c \Rightarrow a \leq c$

2) If A = {x, y, z w} then,

i) R_1 = {(x, y) (y, z)} is not transitive relation as x Ry, y Rz but $x \not{R} z$.

ii) $R_2 = \{(x, y)\ (y, z)\ (z, x)\ (z, y)\ (x, z)\ (y, x)\ (x, x)\ (y, y)\ (z, z)\}$ is transitive relation.

iii) $R_3 = \{(x, y)\ (z, y)\}$ is a transitive relation as there are no terms of the form x Ry and y R z,

Diagraph of transitive relation is as follows :

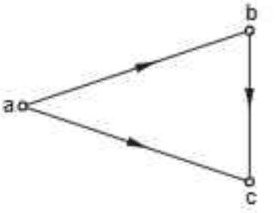


Fig. 4.7.1

4.7.7 Equivalence Relation

Let R be a relation on a set A. Then R is said to be an equivalence relation on A iff R is reflexive symmetric and transitive relations. It is denoted by 'R' or '~'.

Example :

Example 4.7.2 *Let A = R, set of all straight lines in a plane.*
$R_1 = \{x\ R_1\ y \mid x$ is parallel to $y\}$, $R_2 = \{x\ R_2\ y \mid x$ is perpendicular to $y\}$ check whether R_1 and R_2 are equivalence relations or not.

Solution : i) For relation R_1 : we know that if x, y, z are straight lines in a plane, then x is parallel to x. $\therefore R_1$ is reflexive.

ii) If $x\ R_1\ y \Rightarrow x$ is || to $y \Rightarrow y$ is parallel to x

$\Rightarrow y\ R_1\ x \Rightarrow R_1$ is symmetric.

iii) If $x\ R_1\ y,\ y\ R_1\ z \Rightarrow x \parallel y,\ y \parallel z \Rightarrow x \parallel z \Rightarrow x\ R_1\ z.$

$\Rightarrow R_1$ is transitive relation.

$\therefore R_1$ is an equivalence relation

ii) For relation R_2 :

Any line is not perpendicular to itself.

$\therefore x\ R_2 x, \forall x \in A$

$\therefore R_2$ is not reflexive.

$\therefore R_2$ is not equivalence relation.

Example 4.7.3 *If $A = \{x, y, z\}$ and $R = \{(x, x)\ (y, y)\ (z, z)\}$. It is an equivalence relation ?*

Solution : The given relation R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

4.7.8 Properties of Equivalence Relations

1) If R_1 and R_2 are equivalence relations on a set A, then $R_1 \cap R_2$ is an equivalence relation.

$\Rightarrow$ If R_1 and R_2 are reflexive, symmetric and transitive then $R_1 \cap R_2$ are also reflexive, symmetric and transitive.

$\therefore R_1 \cap R_2$ is an equivalence relation.

2) If R_1 and R_2 are equivalence relations on a set A, then it is not necessary that $R_1 \cup R_2$ is an equivalence relation.

Let $A = \{a, b, c)$

And $R_1 = \{(a, a)\ (b, b)\ (c, c)\ (a, b)\ (b, a)\}$

$R_2 = \{\ (a, a)\ (b, b)\ (c, c)\ (a, c)\ (c, a)\}$ are equivalence relation.

$R_1 \cup R_2 = \{(a, a)\ (b, b)\ (c, c)\ (a, b)\ (b, a)\ (a, c)\ (c, a)\}$

(b, a) and $(a, c) \in R_1 \cup R_2$ but $(b, c) \notin R_1 \cup R_2$.

$\therefore$ $R_1 \cup R_2$ is not an equivalence relation.

3) If R_1 and R_2 are equivalence relations then $R_1 \cup R_2$ is an equivalence relations iff $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$.

4.7.9 Equivalence Classes

Let R be an equivalence relation on a set A. The equivalence class of $a \in A$ is denoted by $[a]_R$ and defined as

$[a]_R = \{x \in A \mid x\ R\ a\}$

= The set of those elements of A which are related to a.

$\therefore$ $a \in [a]_R$

Example 1 :

Let $A = \{x, y, z\}$

$R = A \times A = \{(x, x)\ (y, y)\ (z, z)\ (x, y)\ (x, z)\ (y, z)\ (y, x)\ (z, x)\ (z, y)\}$

is an equivalence relation.

$\therefore$ The equivalence class of x is

$[x]_R = \{x, y, z\} = [y]_R = [z]_R$

All these three equivalence classes are identical.

Example 4.7.4 *If $A = z$, $R = \{(x, y) \mid x + y = even,\ x, y \in z\}$. Is R equivalence relation ? if yes, find equivalence class of 1 and 2.*

Solution : $x + x$ = even number for any $x \in z$.

$\therefore x\ Rx \Rightarrow R$ is reflexive relation.

Suppose $x\ Ry \Rightarrow x + y$ = even

$$\Rightarrow y + x = \text{even}$$

$$\Rightarrow yRx$$

$\therefore$ R is symmetric relation.

Suppose xRy and yRz $\Rightarrow x + y = \text{even}, y + z = \text{even}$

$$\Rightarrow x + y + y + z = \text{even}$$

$$\Rightarrow x + z = \text{even} - 2y = \text{even}$$

$$\Rightarrow x + z = \text{even}$$

$$\Rightarrow xRz$$

$\therefore$ R is a transitive relation.

$\therefore$ R is an equivalence relation.

$\therefore$ The equivalence class of $1 \in z$ is

$$[1]_R = \{xR_1 / x+1 = \text{even}\}$$

$$= \{xR_1 / x = \text{even} - 1 = \text{odd}\}$$

$$= \{xR_1 / x \text{ is odd}\}$$

$$[1]_R = \text{Set of all odd integers} = \{\ldots -3, -1, 1, 3, 5, \ldots\}$$

$$[2]_R = \{xR_2 \mid x+2 = \text{even}\}$$

$$= \{xR_2 \mid x = \text{even}\}$$

$$= \text{Set of all even integers}$$

$$[2]_R = \{\ldots -4, -2, 0, 2, 4 \ldots\}$$

Theorem 1 : Any two equivalence classes are either identical or disjoint.

Theorem 2 : For any $x \in A$, $x \in A$, $x \in [x]_R$

Theorem 3 : $A = \bigcup_{a \in A} [a]$

Example 4.7.5 *Which of the following are relations from A to B where A = {1, 2, 3, 4} ; B = {x, y, z}*

a) $R_1 = \{(1, x), (1, y), (1, z), (4, x)\}$

(b) $R_2 = \{(x, 1), (y, 1), (z, 1), (x, 4)\}$

(c) $R_3 = \{(1, x), (2, y), (3, z), (4, w)\}$

(d) $R_4 = \{(1, 1), (2, 2)\}$

Solution : Given that $R : A \rightarrow B$

Where A = Domain set

and B = Co-domain set

$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z), (4, x), (4, y), (4, z)\}$

(a) R is a relation from A to B because $R_1 \subset A \times B$

(b) R is not a relation from A to B as $R \not\subset A \times B$

(c) R is not a relation from A to B as $(4, w) \in R$ but $w \not\subset B$

(d) R is not a relation from A to B as $R \not\subset A \times B$

Example 4.7.6 *If $A = \{1,2,3,4\}$, $B = \{1, 4, 6, 8, 9\}$ R is a relation from A to B such that aRb iff $b = a^2$. Find domain, range, relation matrix and diagraph.* **SPPU : Dec.-18, Marks 6**

Solution : We have

$$R = \{(1, 1)\ (2, 4)\ (3, 9)\}$$

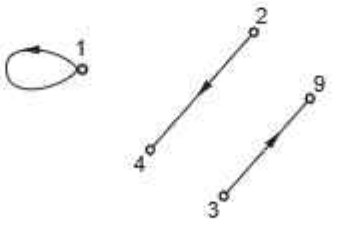
i) Domain of R is {1, 2, 3}

ii) Range of R is {1, 4, 9}

iii) Relation matrix is

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 6 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

iv) Diagraph



Example 4.7.7 *If R is a relation from A to B where $A = \{1, 2, 8\}$, $B = \{1, 2, 3, 5\}$ and aRb iff $a < b$. Find : a) R in Roster form b) Domain and Range c) Diagraph.*

SPPU : May-18, Marks 4

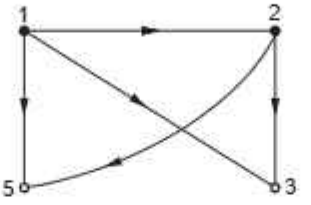
Solution : Given that A= {1, 2, 8}, B = {1, 2, 3, 5}

a) R in Roster form is R = {(1, 2) (1, 3), (1, 5), (2, 3), (2,5)}

b) Domain of R is {1, 2}

Range of R is {2, 3, 5}

c) Diagraph :



Example 4.7.8 *If R is a relation on set A = {1, 2, 3, 4, 5}, R = {(1, 1) (1, 2) (2, 1) (3, 1) (4, 1) (5, 2)}. Find domain set, codomain set, range. Draw its diagraph and find relation matrix.*

Solution : Given that A = {1, 2, 3, 4, 5}

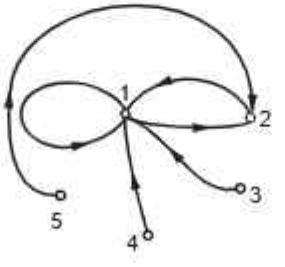
and R is a relation on set A.

Domain set = A = Co-domain set

R = {(1, 1) (1, 2) (2, 1) (3, 1) (4, 1) (5, 2)}

Range set = {1, 2} ⊂ A

Its diagraph is as follows :



The matrix form of R is

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{5\times5} \end{array}$$

Example 4.7.9 *Let A = {1, 2, 3, 4, 5} and aRb if a < b. Find i) R in Roster form ii) Domain and Range of R iii) Diagraph of R.* **SPPU : Dec.-19, Marks 3**

Solution : We have

$$A = \{1, 2, 3, 4, 5\}$$

i) By considering the given conditions,

R in Roster form is

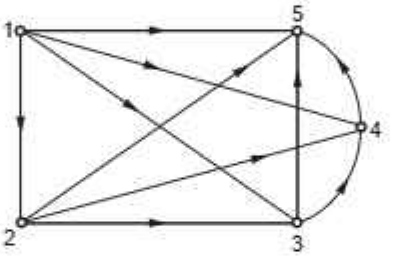
R = {(1, 2), (1, 3), (1, 4), (1, 5), (2, 3) (2, 4), (2, 5) (3, 4) (3, 5) (4, 5)}

ii) Domain of R is = {1, 2, 3, 4}

Range of R is = {2, 3, 4, 5}

iii) The vertex set of R is {1, 2, 3, 4, 5}

It's diagraph is as follows :



Example 4.7.10 *Let A = {1, 2, 3, 4, 5, 6} aRb or (a, b) ∈ R iff a is a multiple of b. Find*

(i) Range set and R (6), R (3).

(ii) Find relation matrix.

(iii) Draw diagraph

(iv) Find in and out degree of each vertex.

Solution : By considering given condition.

$$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (5, 1), (5, 5), (6, 1), (6, 2), (6, 3), (6, 6)\}$$

(i) The range set of R is {1, 2, 3, 4, 5, 6}

R (3) = {1, 3} as (3, 1) and (3, 3) ∈ R

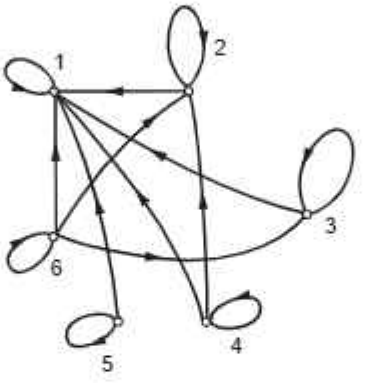
and R (6) = {1, 2, 3, 6} because (6, 1), (6, 2), (6, 3), (6, 6) ∈ R

(ii) The relational matrix is

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(iii) The diagraph of given relation is

vertex set = {1, 2, 3, 4, 5, 6}



(iv) The indegree of a vertex in diagraph is the number of edges coming towards that vertex. The outdegree of a vertex is defined as the number of edges going out from a vertex. Consider the following table.

Vertex	Indegee	Outdegree	Degree
1	6	1	7
2	3	2	5
3	2	2	4
4	3	1	4
5	2	1	3
6	4	1	5

Example 4.7.11 *Let R be a relation on set of natural numbers such that $R = \{(x, y) \mid 2x + 3y \text{ and } x, y \in N\}$. Find*

(i) The domain and codomain of R. ; (ii) R^{-1}

Solution : Given that $x, y \in N$

$\therefore$ The smallest values of x and y are 1, 1 respectively.

$\therefore \quad 2x + 3y = 2(1) + 3(1) = 5$

which is the smallest value of co-domain

$$x = 1, \quad y = 2 \Rightarrow 2x + 3y = 8$$
$$x = 2, \quad y = 1 \Rightarrow 2x + 3y = 7$$
$$x = 2, \quad y = 2 \Rightarrow 2x + 3y = 10$$
$$x = 2, \quad y = 3 \Rightarrow 2x + 3y = 13$$
$$x = 3, \quad y = 2 \Rightarrow 2x + 3y = 12$$

$\therefore$ The domain of R is = $\{x \mid x \in N\}$

and the codomain of R is = $\{y \mid y \in N\}$

(ii) The inverse relation is

$$R^{-1} = \{(y, x) / x, y \in N\}$$

Example 4.7.12 *Let A = {1, 2, 3, 4, 6} and let R be defined on A as xRy iff x | y (i.e. x divides y)*

(i) Write R as a set of ordered pairs

(ii) Find R^{-1} and describe R^{-1} in words

Solution : Given that,

$$A = \{1, 2, 3, 4, 6\} \text{ and } x\,R\,y \text{ iff x divides y.}$$

(i) $\therefore R = \{(1, 2), (1, 3), (1, 4), (1, 6), (2, 4), (2, 6), (3, 6), (1, 1),$
$(2, 2), (3, 3), (4, 4), (6, 6)\}$ ... [Here 1|2, 1|1, 2|2, ...]

(ii) The inverse relation R^{-1} is

$$R^{-1} = \{(2, 1), (3, 1), (4, 1), (6, 1), (4, 2), (6, 2), (6, 3), (1, 1), (2, 2),$$
$$(3, 3), (4, 4), (6,6)\}$$

Now R-1 is defined as : $xR^{-1}y$ iff x is divisible by y.

Example 4.7.13 *Let A = {1, 2, 3) R is the relation on A whose matrix is :*

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

show that R is transitive. **SPPU : May-17, Marks 3**

Solution : Given that : A = {1, 2, 3}

and $R = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 3)\}$

In R, $(1, 1), (1, 2) \in R \Rightarrow (1, 2) \in R$

$(1, 1), (1, 3) \in R \Rightarrow (1, 3) \in R$

$(1, 2), (2, 3) \in R \Rightarrow (1, 3) \in R$

$(1, 3), (3, 3) \in R \Rightarrow (1, 3) \in R$

$(2, 3), (3, 3) \in R \Rightarrow (2, 3) \in R$

Hence for any $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$

Thus R is a transitive relation on A.

Example 4.7.14 *Let A = Set of all students in SPPU*

xRy iff x and y belongs to same class of SPPU ∀, y ∈ A.

Is R an equivalence relation ? Find the equivalence class of Atharva in A.

Solution : Given that A = {Set of all students in SPPU}

and xRy iff x and y belong to same class of SPPU.

(i) Any student x and x belong to same class

$\therefore$ xRx , $\forall$ x $\in$ A $\Rightarrow$ R is reflexive relation.

(ii) Let xRy $\Rightarrow$ x and y belong to same class of SPPU.

$\Rightarrow$ y and x belong to same class of SPPU.

$\Rightarrow$ yRx

i.e. xRy $\Rightarrow$ yRx, $\forall$ x, y $\in$ R.

Thus R is symmetric relation.

(iii) Let xRy and yRz

$\Rightarrow$ x and y belong to same class and y and z belong to same class of SPPU

$\Rightarrow$ x, y, z belong to same class of SPPU

$\Rightarrow$ x R z

$\Rightarrow$ R is transitive relation.

Thus R is an equivalence relation.

Suppose there is one student in SPPU whose name is Atharva.

$\therefore$ The equivalence class of Atharva is the set of all students who are studying in the class of Atharva.

Example 4.7.15 *Let A = {Set of all lines in a plane} and x, y ∈ A. Define xRy iff x is perpendicular to y. Is R transitive, reflexive and symmetric ?*

Solution : Given that xRy iff x perpendicular y.

(i) There is no any line which is perpendicular to itself.

$\therefore \nexists$ any x such that xRx.

$\Rightarrow$ R is not reflexive.

(ii) Suppose xRy $\Rightarrow$ x is perpendicular to y.

$\Rightarrow$ y is perpendicular to x.

$\Rightarrow$ yRx

i.e. xRy $\Rightarrow$ yRx i.e. R is symmetric relation.

(iii) Suppose xRy and yRz

$\Rightarrow$ x is perpendicular to y and y is perpendicular to z.

$\Rightarrow$ x and z are parallel lines.

$\Rightarrow$ xRz

$\Rightarrow$ R is not transitive relation.

Example 4.7.16 *Let A = {set of lines in a plane} and x, y $\in A$. xRy iff x is parallel to y. Is R equivalence relation ? Find the equivalence class of any line 'L'.*

Solution :

(i) We know that every line is parallel to itself.

$\therefore$ xRx , $\forall$ x $\in$ A

$\Rightarrow$ R is reflexive relation.

(ii) Suppose xRy

$\Rightarrow$ x is parallel to y.

$\Rightarrow$ y is parallel to x.

$\Rightarrow$ yRx $\Rightarrow$ R is symmetric relation.

(iii) Suppose xRy and yRz.

$\Rightarrow$ x is parallel to y and y is parallel to z.

$\Rightarrow$ x is parallel to z.

$\Rightarrow$ xRz $\Rightarrow$ R is transitive relation.

Thus R is an equivalence relation.

Let L be any line in A.

Equivalence class of L = [L] = {xRL | x is parallel to L}

$$[L] = \{x \mid x \text{ is parallel to } L\}$$

$$= \text{Set of all lines which are parallel to L.}$$

Example 4.7.17 *Let $A = \{a, b, c\}$ and $M(R) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.*

Determine whether R is an equivalence relation ? Find the equivalence class of b in A.

Solution : From relation matrix, the relation R is

$$R = \{(a, a)\ (b, b)\ (c, c)\ (b, c)\ (c, b)\}$$

(i) Here $a, b, c \in A$ and $(a, a)\ (b, b)$ and $(c, c) \in R$

$\therefore$ R is reflexive relation.

(ii) As $(b, c) \in R \Rightarrow (c, b) \in R$

R is symmetric relation.

(iii) In relation R, we have,

$(b, b), (b, c) \in R \Rightarrow (b, c) \in R$

$(b, c), (c, b) \in R \Rightarrow (b, b) \in R$

$(b, c), (c, c) \in R \Rightarrow (b, c) \in R$

$(c, c), (c, b) \in R \Rightarrow (c, b) \in R$

$(c, b), (b, b) \in R \Rightarrow (c, b) \in R$

$(c, b), (b, c) \in R \Rightarrow (c, c) \in R$

So R is transitive relation. Thus R is an equivalence relation.

Now, the equivalence class of b in A is,

$$[b] = \{x \mid (x, b) \text{ or } (b, x) \in R\}$$

$$[b] = \{b, c\}$$

Example 4.7.18 *If $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\ (3, 5), (5, 3), (1, 3), (3, 1)\}$ is an equivalence relation on $X = \{1, 2, 3, 4, 5\}$. Find equivalence classes.* **SPPU : May-19, Marks 3**

Solution : If R is an equivalence relation on X and $a \in X$ then $[a] = \bar{a} = \{x \mid x \sim a\}$.

Therefore, equivalence classes are as follows

$$[1] = \bar{1} = \{1, 3\}$$

$$[2] = \bar{2} = \{2\},\ [3] = \bar{3} = \{1, 3, 5\}$$

$$[4] = \bar{4} = \{4\}$$

$$[5] = \bar{5} = \{3, 5\}$$

Example 4.7.19 *Let A be the set of integers and let R be defined as xRy iff $x \le y$. Is R equivalence relation ?*

Soltuion : Given that xRy iff $x \le y, \forall x, y \in A$

(i) We have, $x \le x, \forall x \in A$

$\Rightarrow xRx, \forall x \in A$

$\Rightarrow$ R is reflexive relation.

(ii) If $xRy \Rightarrow x \le y$

then $y \not\le x \Rightarrow y \not R x$

$\Rightarrow$ R is not symmetric relation.

$\therefore$ R is not an equivalence relation.

Example 4.7.20 *Let $A = Z^+$ the set of positive integers, and let*
$R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$
Is R symmetric, asymmetric or antisymmetric. **SPPU : May-17, Marks 3**

Ans. : Given that : $A = Z^+$ = The set of positive integers

and $R = \{(a, b)/a \text{ divides } b\}$

i) If $(a, b) \in R$ then a divides b

$\therefore \quad b = aK, \ K \in Z^+$

$\Rightarrow \quad a = \frac{1}{K} b$ but if $K \ne 1$ then $\frac{1}{K}$ is not an integer

$\therefore$ a does not divides b if

b is not multiples of K

$\therefore \quad (a, b) \in R \Rightarrow (b, a)$ may not be present in R.

$\therefore$ R is not symmetric relation.

ii) If $(a, b) \in R$ then a/b

$\therefore \quad b = aK, \ K \in Z^+$

$\Rightarrow \quad a = \frac{1}{K} b, \ \text{As } K \in Z^+$

b is not divide a

i.e. if $(a, b) \in R$ then $(b, a) \notin R$.

$\therefore$ R is asymmetric relation.

iii) As $(a, b) \in R, \ (b, a) \in R$

then $b = K_1 a, \ a = K_2 b, K_1, \ K_2 \notin Z^+$

$b = K_1, a = (K_1 K_2) b$

$\Rightarrow K_1 K_2 = 1 \Rightarrow K_1 = K_2 = 1$

$\therefore \quad a = b$

Thus R is antisymmetric relation.

Example 4.7.21 *Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9} and let N be the relation on A × A defined by (a, b) ~ (c, d) iff a + d = b + c.*

(i) Prove that ~ is an equivalence relation.

(ii) Find equivalence class of (2, 5) **SPPU : Dec.-11**

Solution : Given that (a, b) ~ (c, d) iff a + d = b + c, ∀ a, b, c, d ∈ A

(i) (a) We have a + b = b + a ⇒ (a, b) ~ (a, b)

⇒ '~' is reflexive relation.

(b) If (a, b) ~ (c, d) then a + d = b + c.

⇒ b + c = a + d ⇒ c + b = d + a

⇒ (c, d) ~ (a, b) by definition

⇒ '~' is symmetric relation.

(c) Suppose (a, b) ~ (c, d) and (c, d) ~ (e, f)

then $a + d = b + c$

and $c + f = d + e$

⇒ $a + \cancel{d} + \cancel{c} + f = b + \cancel{c} + \cancel{d} + e$

$a + f = b + e$

⇒ $(a, b) \sim (e, f)$

∴ ~ is a transitive relation.

Thus '~' is reflexive, symmetric and transitive.

⇒ '~' is an equivalence relation.

(ii) We have (2, 5) ∈ A × A

The equivalence class of (2, 5) is the set of elements of A × A which are equivalent to (2, 5)

$$\therefore \quad [(2, 5)] = \{(x, y) \mid (x, y) \sim (2, 5),\ x, y \in A\}$$

$$= \{(x, y) \mid x + 5 = y + 2\}$$

$$= \{(x, y) \mid x - y = -3 \text{ or } y = x + 3 ;\ x, y \in A\}$$

Hence, $[(2, 5)] = \{(1, 3)\ (2, 5)\ (3, 6)\ (4, 7)\ (5, 8)\ (6, 9)\}$ ... [∵ (7, 10) ∉ A × A]

Example 4.7.22 *Prove that in the set N × N, the relation R defined by (a, b) R (c, d) iff ad = bc is an equivalence relation.*

Solution : We know that relation R is an equivalence relation if R is reflexive, symmetric and transitive.

(i) We have, ab = ba

$\Rightarrow$ (a, b) R (a, b)

$\therefore$ R is reflexive relation.

(ii) Suppose (a, b) R (c, d)

$\Rightarrow$ ad = bc

$\Rightarrow$ bc = ad

$\Rightarrow$ cb = da

$\Rightarrow$ (c, d) R (a, b)

$\therefore$ R is symmetric relation.

(iii) Let (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow$ ad = bc and cf = ed

$\Rightarrow$ adcf = bced

$\Rightarrow$ af = be $\Rightarrow$ (a, b) R (e, f)

$\therefore$ R is transitive relation.

Thus, R is an equivalence relation.

Example 4.7.23 *Let $A = R \times R$ (R is set of real numbers) and define the following relation on A. (a, b) R (c, d) iff $a^2 + b^2 = c^2 + d^2$*

(i) Show that (A, R) is an equivalence relation.

(ii) Find equivalence class of (3, 2)

Solution : Given that (a, b) R (c, d) iff $a^2 + b^2 = c^2 + d^2$

(i) (a) We have, $a^2 + b^2 = a^2 + b^2$

$\Rightarrow$ (a, b) R (a, b)

R is a reflexive relation.

(b) If (a, b) R (c, d) then $a^2 + b^2 = c^2 + d^2$

$\Rightarrow$ $c^2 + d^2 = a^2 + b^2$

$\Rightarrow$ (c, d) R (a, b)

R is symmetric relation.

(c) Suppose (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow$ $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$

$\Rightarrow$ $a^2 + b^2 = e^2 + f^2$

$\Rightarrow \quad (a, b) \; R \; (e, f)$

$\therefore$ R is transitive relation.

Thus, R is an equivalence relation.

(ii) An equivalence class of (3, 2) is the set of elements of A which are equivalent to (3, 2)

$$\therefore \quad [(3, 2)] = \{(x, y) \mid (x, y) \; R \; (3, 2) \; ; \; x, y \in R\}$$

$$= \{(x, y) \mid x^2 + y^2 = 9 + 4 = 13 \; ; \; x, y \in \Re\}$$

$$[(3, 2)] = \text{The set of points on circle } x^2 + y^2 = 13.$$

Example 4.7.24 *Let R be the binary relation defined as $R = \{(a, b) \in \Re^2 / (a - b) \leq 3\}$*

(i) Determine whether R is reflexive, symmetric, transitive and antisymmetric relation.

(ii) How many binary relations are there on the finite set

Solution : Given that,

$$R = \{(a, b) \in R^2 \mid (a - b) \leq 3\}$$

(i) (a) We have, $a - a = 0 \leq 3$

$(a, a) \in R \Rightarrow$ R is reflexive.

(b) Suppose $(a, b) \in R \Rightarrow a - b \leq 3$

$$-(b - a) \geq 3$$

$$b - a \leq -3$$

$(b - a) \leq -3$ does not imply $b - a \leq 3$

e.g. $\quad a = -100 \;, \; b = 2$

$$a - b = -100 - 2 = -102 < 3$$

$$(-100, 2) \in R$$

But $\quad b - a = 2 - (-100) = 2 + 100 = 102 \nleq 3$

Thus R is not symmetric relation.

(c) If $(a, b), (b, c) \in R$

then $\quad a - b \leq 3 \;, \; b - c \leq 3$

$$a - b + b - c \leq 6$$

$$a - c \leq 6$$

$\Rightarrow a - c$ may or may not be ≤ 3.

R is not tansitive relation.

(d) Suppose aRb and bRa

$\Rightarrow \quad a - b \le 3$ and $b - a \le 3$

e.g. $\quad a = 1, \; b = 3,$

$a - b = -2 < 3$ and $b - a = 2 < 3$

but $a \ne b$.

$\therefore$ R is not antisymmetric relation.

(ii) Let n be the number of elements in a finite set A, then they are n^{n^2} binary relations on set A.

Example 4.7.25 *Let S be the set of points in a plane. Let R be a relation such that xRy iff y is within two centimeter from x. Is R equivalence relation ?*

Solution : Given that

xRy iff $|x - y| < 2, \forall x, y \in A$

(i) We have, $|x - x| = 0 < 2$

$\therefore$ $xRx, \forall \; x \in A$

$\Rightarrow$ R is reflexive relation

(ii) If $\quad xRy \Rightarrow |x - y| < 2$

$\Rightarrow |y - x| < 2$

$\Rightarrow$ yRx $\therefore$ R is symmetric relation.

(iii) Suppose xRy and yRz

$\Rightarrow \quad |x - y| < 2$ and $|y - z| < 2$

$\Rightarrow \quad |x - y| + |y - z| < 2 + 2 = 4$

$\Rightarrow \quad |x - z| < 4$

$\Rightarrow \quad |x - z| \not< 2$

$\therefore$ R is not transitive relation.

So R is not an equivalence relation.

Example 4.7.26 *Let R be a relation on the set of positive integers such that*

$R = \{(x, y) | x - y$ is an odd positive integer$\}$

Is R reflexive, symmetric, transitive, antisymmetric, an equivalence relation ?

Solution : Given that, $R = \{(x, y) \mid x - y \text{ is an odd positive integer}\}$

(i) For any positive integer x, $x - x = 0$ which is even integer.

$\therefore (x, x) \notin R$

R is not reflexive relation.

(ii) Suppose $(x, y) \in R$

then $x - y$ = Odd positive integer

$\Rightarrow$ $y - x$ = Odd negative integer

$(y, x) \notin R$

$\therefore$ R is not symmetric relation.

(iii) Suppose $(x, y), (y, z) \in R$

$\therefore$ $x - y$ = Odd positive integer = p

$y - z$ = Odd positive integer = q

$x - y + y - z = p + q$

$x - z = p + q$ which is even positive integer.

{e.g. $(5, 4) \in R$, $(4, 1) \in R$. But $(5, 1) \notin R$ as $5 - 1 = 4$}

$\therefore (x, z) \notin R$

$\therefore$ R is not transitive relation.

(iv) Suppose (x, y) and $(y, x) \in R$

then $x - y$ = Odd positive integer = p

$y - x$ = Odd positive integer = q

$\Rightarrow$ $x \geq y$ and $y \geq x$

$\Rightarrow$ $x = y$

R is antisymmetric relation.

Example 4.7.27 *Show that $R = \{(a, b) / a \equiv b \pmod{m}\}$ is an equivalence relation on z. Show that $x_1 \equiv y_1$ and $x_2 \equiv y_2$ then $x_1 + x_2 \equiv y_1 + y_2$*

Solution : Given that

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

We know that $a \equiv b \pmod{m}$ iff $m \mid (a - b)$

$$a - b = mk, \quad k \in z$$

(i) Let $a \in z$, $a - a = 0 = m(0)$

$\therefore m \mid (a - a) \Rightarrow a \equiv a \pmod{m}$

$(a, a) \in R \Rightarrow R$ is reflexive relation.

(ii) Let $(a, b) \in R$

$\Rightarrow \quad a \equiv b \pmod{m}$

$\Rightarrow \quad a - b = mk$

$\Rightarrow \quad b - a = m(-k)$

$\Rightarrow \quad m \mid b - a$

$\Rightarrow \quad b \equiv a \pmod{m}$

$\Rightarrow \quad (b, a) \in R$

$\therefore$ R is symmetric relation.

(iii) Suppose (a, b) $(b, c) \in R$

then $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$\Rightarrow \quad a - b = mk_1$ and $b - c = mk_2$

$\Rightarrow \quad a - b + b - c = mk_1 + mk_2$

$\Rightarrow \quad a - c = m(k_1 + k_2) = mk$

$\Rightarrow \quad m \mid (a - c) \Rightarrow a \equiv c \pmod{m}$

$\therefore \quad (a, c) \in R$

$\therefore$ R is transitive relation.

$\therefore$ R is reflexive, symmetric and transitive.

Thus R is an equivalence relation,

Now, $x_1 \equiv y_1 \Rightarrow x_1 \equiv y_1 \pmod{m}$

$x_2 \equiv y_2 \Rightarrow x_2 \equiv y_2 \pmod{m}$

$x_1 - y_1 = mk_1$ and $x_2 - y_2 = mk_2$

$\Rightarrow \quad x_1 - y_1 + x_2 - y_2 = mk_1 + mk_2$

$\Rightarrow \quad (x_1 + x_2) - (y_1 + y_2) = m(k_1 + k_2)$

$\Rightarrow \quad m \mid (x_1 + x_2) - (y_1 + y_2)$

$\Rightarrow \quad (x_1 + x_2) \equiv (y_1 + y_2) \pmod{m}$

$\Rightarrow \quad x_1 + x_2 \equiv y_1 + y_2 \pmod{m}$

$\Rightarrow \quad x_1 + x_2 \equiv y_1 + y_2$

Example 4.7.28 *For each of these relations on set A = {1, 2, 3, 4} decide whether it is reflexive,*
symmetric, transitive or antisymmetric.
R_1 = {(1, 1) (2, 2) (3, 3) (4, 4)}
R_2 = {(1,1)(1, 2) (2, 2) (2,1) (3, 3) (4, 4)}, R_3 = {(1, 3) (1, 4) (2, 3) (2, 4) (3, 1) (3,4,)}.

SPPU : Dec.-10

Solution : i) For R_1 :

As $\forall a \in A$, $(a, a) \in R_1$

$\therefore R_1$ is Reflexive, symmetric relation.

$\nexists$ (a, b) and $(b, c) \in R_1$ $\therefore R_1$ is transitive relation.

$\nexists$ (a, b) and $(b, a) \in R_1$ $\therefore R_1$ is antisymmetric relation.

$\therefore R_1$ reflexive, symmetric, transitive and anti-symmetric relation.

ii) For R_2 :

As $\forall a \in A$ $(a, a) \in R_2$ and $aR_2b \Rightarrow bR_1a$, for $a,b \in A$

$\therefore R_2$ reflexive and symmetric relation.

For any aR_2b and $bR_2C \Rightarrow aR_2C$

$\therefore R_2$ is transitive realtion

$\therefore R_2$ is an equivalence relation

But (1, 2) and $(2, 1) \in R_2$ and $1 \neq 2$

$\therefore R_2$ is not antisymmetric relation.

iii) For R_3 :

As $2 \in R_3$ but $(2, 2) \notin R_3$

$\therefore R_3$ is not reflexive relation.

As $(1, 4) \in R_3$ but $(4, 1) \notin R_3$

$\therefore R_3$ is not symmetric relation.

As (2, 3) and $(3, 1) \in R_3$ but $(2, 1) \notin R_3$.

$\therefore R_3$ is not transitive relation.

As (1, 3) and $(3, 1) \notin R_3$ but $1 \neq 3$.

$\therefore R_3$ is not transitive relation

As (1, 3) and $(3, 1) \in R_3$ but $1 \neq 3$

$\therefore R_3$ is not antisymmetric relation

Example 4.7.29 *Consider the relation on A = { 1, 2, 3, 4, 5, 6} . R = { (i, j) | i – j| = 2}. Is R reflexive ? Is R symmetric ? Is R transitive ?*

SPPU : Dec.-14

Solution : Given that A = {1, 2, 3, 4, 5, 6}

$$R = \{(1, 3)\ (3, 1)\ (2, 4)\ (4, 2)\ (3, 5)\ (5, 3)\ (4, 6)\ (6, 4)\}$$

As $2 \in A$ but $(2, 2) \notin R \therefore R$ is not reflexive.

For any $(a, b) \in R \Rightarrow (b,a) \in R \therefore R$ is symmetric.

As $(1, 3)\ (3, 1) \in R$ but $(1, 1) \notin R \therefore R$ is not transitive.

Example 4.7.30 *Let A = { 1, 2, 3, 4, 5, 6, 7 } and R = {(x, y) | x – y is divisible by 3}. Show that R is an equivalence relation ? Draw graph of R.*

SPPU : May-14, Dec.-12, 19, Marks 6

Solution : We have R = {x, y) | x – y is divisible by 3}

We know that x – x = 0 is divisible by 3

$x\ R\ x, \forall x \in A \Rightarrow R$ is reflexive relation.

As $x\ R\ y \Rightarrow x - y$ is divisible by 3

$\Rightarrow y - x$ is also divisible by 3

$\Rightarrow y - z$ is divisible by 3

$\Rightarrow y\ R\ x$ for $x, y \in A$

$\therefore$ R is a symmetric relation.

As,

$x\ R\ y$ and $y\ R \Rightarrow x - y$ and $y - z$ are divisible by 3

$\Rightarrow (x - y) + (y - z)$ is also divisible by 3

$\Rightarrow x - z$ is divisible by 3

$\Rightarrow x\ R\ z$

$\therefore$ R is a transitive relation.

$\therefore$ R is an equivalence relation.

and $R = \{(1, 1), (2, 2)\ (3, 3)\ (4, 4)\ (5, 5)\ (6, 6)\ (7, 7)\ (1, 4)\ (4, 1)\ (1, 7)\ (7, 1)$
$(2, 5), (5, 2)\ (3, 6)\ (6, 3)\}$

Its graph is as follows

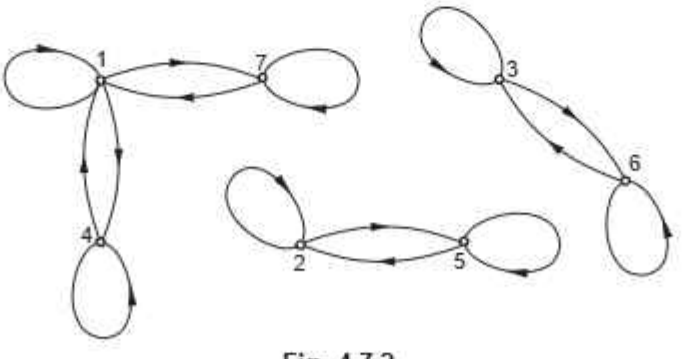


Fig. 4.7.2

4.8 Partitions of a Set

SPPU : Dec.-11, 12, May-17

We shall now discuss the concept of partitions of a set which is similar to equivalence class of set.

Definition :

Let A be any non empty set. A set

$P = \{A_1, A_2, A_3, ... A_n\}$ of non empty subsets of A is called a partition of set A if

i) $A_1 \cup A_2 \cup A_3 \cup .... \cup A_n = A = \bigcup_{i=1}^{n} A_i$

i.e. Set A is the union of the sets $A_1, A_2, ... A_n$.

ii) $A_i \cap A_j = \phi$ for $i \neq j$ i.e. All sets A_i are mutually disjoints.

The partition of a set A is denoted by π.

An element of a partition set is called a block. The rank of a partition is called as the number of blocks of that partitions. It is denoted by $r(\pi)$. For any non empty set, its partitions are not unique. There are different partitions of the same set.

e.g.

1) Let A = Z = Set of integers

A_1 = Set of even integers

A_2 = Set of odd integers

$A_3 = \{1, 2, 3, 4, ...\}$,

$A_4 = \{0\}$

$A_5 = \{-1, -2, -3, -4, ...\}$

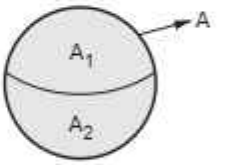


Fig. 4.8.1

$P_1 = \{A_1, A_2\}$ and $P_2 = \{A_3, A_4, A_5\}$ are different partitions of set A.

2) If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and it's subsets

$A_1 = \{1, 4, 9\}$ $A_2 = \{2, 6, 8, 10\}$, $A_3 = \{3, 5, 7\}$,

The set $P = \{A_1, A_2, A_3\}$ is such that

i) A_1, A_2, A_3 are non empty sets

ii) $A = A_1 \cup A_2 \cup A_3$

iii) $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, $A_2 \cap A_3 = \phi$

Hence $\{A_1, A_2, A_3\}$ form a partition for set A.

4.8.1 Relation Induced by a Partition

Let P be the partition of a non empty set A. We can define a relation R on a set A as

x Ry iff x and y belong to the same block of the partition P. This relation R is called as the relation induced by the partition.

e.g. i) Let $A = Z$ and $P = \{A_1, A_2\}$

where A_1 = Set of even integers and

A_2 = Set of odd integers

$\therefore$ P is a partition for $A = Z$

Define x Ry Iff x and y belong to the same partition of Z

i.e. x and y belong to either A_1 or A_2

i.e. x and y are either even or odd

This relation is an equivalence relation.

Theorem 1 : Let R be an equivalence relation on a set A then the set of equivalence classes $\{[a]_R / a \in A\}$.

Theorem 2 : Let A be any non empty set and let π be a partition of A. Then π induces an equivalence relation on set A.

Example 4.8.1 *Let $A = \{x, y, z, u, v\}$, $\pi = \{\{x, y\}, \{z\}, \{u, v\}\}$. Find the equivalence relation induced by π.*

Solution :

We have $A = \{x, y, z, u, v\}$ and Define

x Ry iff x and y belongs to the same block of the partition of A.

$\pi = \{\{x, y\}, \{z\}, \{u, v\}\}$ has 3 blocks.

The first block $\{x, y\} \rightarrow x, y \in$ same block

$\therefore$ x Rx, xRy, yRx, yRy

The second block, $\{z\} \Rightarrow zRz$

The third block, $\{u, v\} \Rightarrow uRu, uRv, vRu, vRv$

$\therefore$ The required relation is

$$R = \{(x,x)\ (x,y),\ (y,x),\ (y,y),\ (z,z),\ (u,u),\ (u,v),\ (v,u),\ (v,v)\}$$

The relation R is reflexive, symmetric and transitive

$\therefore$ R is an equivalence relation.

Example 4.8.2 *Let A = (1, 2, 3, 4) consider partition.*

P = {{1, 2, 3}, {4}}.

of A find the equivalence relation R on A determined by P. **SPPU : May-17, Marks 3**

Solution : Given that : A = {1, 2, 3, 4} and the partition P = {{1, 2, 3}, {4}}

We know that equivalence classes form a partition for the corresponding set.

P_1 = {1, 2, 3} and P_2 = {4} are two equivalence classes.

∴ In each equivalence class, every element is related to all elements of that class.

∴ Due to P_1 = (1, 1) (1, 2), (1, 3), (2, 2), (2, 3), (2, 1), (3, 3), (3, 1), (3, 2) and for P_2, (4, 4)

Hence R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)}

is an equivalence relation with respect to the given partition P.

Example 4.8.3 *Let A = {1, 2, 3, 4}, π = {{1}, {2, 3}, {4}}. Find the equivalence relation induced by π.*

Solution : Given that

$$\pi = \{\{1\}, \{2, 3\}, \{4\}\} \text{ has 3 blocks}$$

The first block, $\{1\} \Rightarrow iR_1$

The second block, $\{2, 3\} \Rightarrow 2R_2, 2R_3, 3R_2, 3R_3$

The third block, $\{4\} \Rightarrow 4R_4$

∴ The required relation is

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4)\}$$

R is reflexive symmetric and transitive

∴ R is an equivalence relation

Example 4.8.4 *Define partition of a set. Let x = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Determine whether or not each of the following is a partition of x :*

A = {{2, 4, 5, 8}, {1, 9}, {3, 6, 7}}

B = {{1, 3, 6}, {2, 8}, {5, 7, 9}} **SPPU : Dec.-11**

Solution : Please refer section 4.8 for definition.

i) For set A :

The set A has 3 blocks,

$$A_1 = \{2, 4, 5, 8\}, \quad A_2 = \{1, 9\} \quad A_3 = \{3, 6, 7\}$$

- The union of all these blocks is a set X

$$A_1 \cup A_2 \cup A_3 = X$$

- These blocks are mutually disjoints.

$\therefore$ The set A forms a partition for the set X

ii) For set B :

$\therefore$ The set B has 3 blocks.

$$B_1 = \{1, 3, 6\}, \qquad B_2 = \{2, 8\} \qquad B_3 = \{5, 7, 9\}$$

As $B_1 \cup B_2 \cup B_3 \neq X$,

B is not a partition of set X.

Example 4.8.5 *If S = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Determine whether or not each of the following is a partition of S.*

i) A = {{1, 3, 5}, {2, 6}, {4, 8, 9}}

ii) B = {{1, 3, 5}, {2, 4, 6, 8}, {7, 9}}

iii) C = {{1, 3, 5}}, {2, 4, 6, 8}, {5, 7, 9}}

iv) D = {{5}} **SPPU : Dec.-12**

Solution : Given that

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

i) A is not a partition of S because S is not the union of all blocks of A.

i.e. $S \neq A_1 \cup A_2 \cup A_3$

ii) B is the partition of S as $B_1 \cup B_2 \cup B_3 = S$ and B_1, B_2, B_3 are mutually disjoints.

iii) As blocks of set C are not disjoints. $\therefore$ The set C is not a partition of S.

iv) D = {{s}} is a partition of S, called as trivial partition.

4.8.2 Refinement of Partitions

Let π and π' be partitions of a non empty set A.

Then π' is called refinement of π if every block (or an element) of π' is contained in a block of π.

e.g. Let $A = \{1, 2, 3\}$, $\pi = \{\{1\}, \{2, 3\}\}$, $\pi' = \{\{1\}, \{2\}, \{3\}\}$

$\{1\}, \{2\}, \{3\} \notin \pi$ but $\{1\}, \{2, 3\} \in \pi'$

$\therefore \pi'$ is a refinement of π

Theorem : Let π and π' be partitions of a non empty set A and R_1, R_2 be the equivalence relations induced by π and π' respectively. Then π' refines π iff $R_2 \subseteq R_1$.

4.8.3 Product and Sum of Partitions

I) Let π_1 and π_2 be two partitions of a non empty set A. The product of π_1 and π_2 is denoted by $\pi_1 - \pi_2 = \pi$ is a partition π of A such that

1) π refines both π_1 and π_2

2) If π' refines both π_1 and π_2 then π' refines π.

II) Let π_1 and π_2 be the partitions of a non empty set A. The sum of π_1 and π_2 is denoted by $\pi = \pi_1 + \pi_2$ is a partition of A such that

1) Both π_1 and π_2 refine π

2) If π_1 and π_2 refine π' then π refines π'

4.8.4 Quotient Set

Let A be any non empty set and R be an equivalence relation on A. The set of mutually disjoint equivalence classes in which A is partitioned relatively to the equivalence relation R is called the quotient set of A for relation R. If is denoted by A/R or $\overline{A}$.

e.g. The quotient set of Z*l*, for an equivalence relation

$$R = \{(a, b)/3 \mid \{(a-b), a, b \in Z\} \text{ is}$$

$$Z/R = \{[0], [1], [2]\}.$$

Theorem 1 : Let R_1 and R_2 be an equivalence relations induced by partitions π_1 and π_2 of a non empty set A respectively, then

i) The relation $R = R_1 \cap R_2$ induces the product partition $\pi_1 \cdot \pi_2$

ii) The relation $R = (R_1 \cup R_2)^*$ {i.e. The transitive closure of $R_1 \cup R_2$} is an equivalence relation on A and induces the partition $\pi_1 + \pi_2$.

4.9 Closure of a Relation

SPPU : May-06, 07, 08, 15, 18, Dec.-05, 07, 12, 13, 14, 15, 16

Depending upon the nature of relations, there are mainly three types of closures of relations.

4.9.1 Reflexive Closure

Let R be a relation on a set which is not reflexive relation. A relation $R_1 = R \cup \Delta$ is called the reflexive closure of R if $R \cup \Delta$ is the smallest reflexive relation containing R.

If $A = \{a, b, c, d\}$ then $\Delta = \{(a, a), (b, b), (c, c), (d, d)\}$

Example 4.9.1 *Let $A = \{1, 2, 3\}$. R_1, R_2 and R_3 are relations on set A. Find the reflexive closures of R_1, R_2 and R_3.*
Where $R_1 = \{(1, 1)\ (2, 1)\}$, $R_2 = \{(1, 1)\ (2, 2), (3, 3)\}$,
$R_3 = \{(3, 1)\ (1, 3), (2, 3)\}$.

Solution :

We have $A = \{1, 2, 3\}$ $\therefore$ $\Delta = \{(1, 1), (2, 2), (3, 3)\}$

Then

i) The reflexive closure of R_1 is $R = R_1 \cup \Delta$

$\therefore$ $R = \{(1, 1)\ (2, 2)\ (3, 3)\ (2, 1)\}$

ii) The reflexive closure of R_2 is $R = R_2 \cup \Delta = R_2$

iii) The reflexive closure of R_3 is $R = R_3 \cup \Delta$

$\therefore$ $R = \{(1, 1)\ (2, 2)\ (3, 3)\ (3, 1)\ (1, 3), (2, 3)\}$

4.9.2 Symmetric Closure

Let R be a relation on a set A and R is not symmetric relation.

A relation $R_1 = R \cup R^{-1}$ is called the symmetric closure of R if $R \cup R^{-1}$ is the smallest symmetric relation containing R.

Example 4.9.2 *Find the symmetric closure of the following relations. On $A = \{1, 2, 3\}$.*

$R_1 = \{(1, 1)\ (2, 1)\}$

$R_2 = \{(1, 2)\ (2, 1)\ (3, 2)\ (2, 2)\}$

$R_3 = \{(1, 1)\ (2, 2)\ (3, 3)\}$ **SPPU : Dec.-12**

Solution : Given that

We have $A = \{1, 2, 3\}$

i) $R_1^{-1} = \{(1, 1)\ (1, 2)\}$

$\therefore$ $R = R_1 \cup R_1^{-1} = \{(1, 1)\ (1, 2)\ (2, 1)\}$ is the symmetric closure of R_1

ii) $R_2^{-1} = \{(2, 1)\ (1, 2)\ (2, 3)\ (2, 2)\}$

$\therefore$ $R = R_2 \cup R_2^{-1} = \{(1, 2)\ (2, 1)\ (3, 2)\ (2, 3)\ (2, 2)\}$ is the symmetric closure of R_2

iii) R_3 is the symmetric relation.

$\therefore$ R_3 itself is the symmetric closure.

4.9.3 Transitive Closure

Let R be a relation on a set A which is not transitive relation. The transitive closure of a relation R is the smallest transitive relation containing R. It is denoted by R^*.

The following theorem gives a procedure to find the transitive closure of a given relation.

Theorem : Let A be any non empty set and $|A| = n$. Let R be a relation on A. Then the transitive closure of R is $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Example 4.9.3 *If A = {1, 2, 3, 4, 5} and R = {(1, 2), (3, 4), (4, 5), (4, 1), (1, 1)}. Find it's transitive closure.*

Solution : Let R^* be the transitive closure of given relation R.

$$\therefore \quad R^* = R \cup R^2 \cup R^3 \cup R^4 \cup R^5$$

Now

$$R^2 = R \cdot R = \{(3, 5)\ (3, 1)\ (4, 2)\ (4, 1)\ (1, 2)\ (1, 1)\}$$

$$R^3 = R \cdot R^2 = \{(3, 2)\ (3, 1)\ (4, 2)\ (4, 1)\ (1, 1)\ (1, 2)\}$$

$$R^4 = R \cdot R^3 = \{(3, 1)\ (3, 2)\ (4, 1)\ (4, 2)\ (1, 1)\ (1, 2)\} = R^3$$

$$R^5 = R \cdot R^4 = \{(3, 1)\ (3, 2)\ (4, 1)\ (4, 2)\ (1, 1)\ (1, 2)\} = R^4 = R^3$$

$$R^* = \{(1, 2)\ (3, 4)\ (4, 5)\ (4, 1)\ (1, 1)\ (3, 5)\ (3, 1)\ (4, 2)\ (3, 2)\}$$

Which is the transitive closure of R.

4.9.4 Warshall's Algorithm to Find Transitive Closure

To find the transitive closure of a relation by computing various powers of R or product of the relation matrix is quite impractical for large relations. Warshall's algorithm gives an alternate method for finding transitive closure of R. Warshall's algorithm is practical and efficient method.

Consider the following steps to find transitive closure of the relation R on a set A.

Step 1 : We have $|A| = n$

$\therefore$ We require $W_o, W_1, W_2, \ldots W_n$. Warshall sets

$$W_o = \text{Relation Matrix of } R = M_R.$$

Step 2 : To find the transitive closure of relation R on set A, with $|A| = n$

Procedure to compute W_k from W_{k-1} is as follows

i) Copy 1 to all entries in W_k from W_{k-1}, where there is a 1 in W_{k-1}.

ii) Find the row numbers $p_1, p_2, p_3 \ldots$ for which there is 1 in column k in W_{k-1} and the column numbers $q_1, q_2, q_3 \ldots$ for which there is 1 in row k of W_{k-1}.

iii) Mark entries in W_k as 1 for (p_i, q_i). If there are not already 1.

Step 3 : Stop the procedure when W_n is obtained and it is the required transitive closure of R.

Examples

Example 4.9.4 *Find the transitive closure of R by Warshall's algorithm.*
Where A = {1, 2, 3, 4, 5, 6} and R = {(x, y)/(x − y) = 2} **SPPU : Dec.-05, 12, 13, 14**

Solution :

Step 1 : We have |A| = 6,

$$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (4, 6), (6, 4), (3, 5), (5, 3)\}$$

Thus we have to find Warshall's sets, W_o, W_1, W_2, W_3, W_4, W_5 and W_6.

The first set W_o is same as M_R. Which is shown below

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_o, we consider the first column and first row.

In a column C_1, 1 is present at R_3

In a row R_1, 1 is present at C_3

Thus add new entry in W_1, at (R_3, C_3) which is given below

$$W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the second column C_2 and second row R_2.

In C_2, 1 is present at R_4

In R_2, 1 is present at C_4

Thus add new entry in W_2 at (R_4, C_4), which is given below

$$W_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 4 : To find W_3 :

To find W_3 from W_2, we consider the third column and third row.

In C_3, 1 is present at R_1, R_3, R_5

In R_3, 1 is present at C_1, C_3, C_5

Thus add new entries in W_3 at (R_1, C_1), (R_1, C_3), (R_1, C_5)
(R_2, C_1), (R_2, C_3), (R_2, C_5) (R_3, C_1), (R_3, C_3), (R_3, C_5) which is given below

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5 : To find W_4 :

To find W_4 from W_3, we consider the fourth column and fourth row.

In C_4, 1 is present at R_2, R_4, R_6

In R_4, 1 is present at C_2, C_4, C_6

Thus add new entries in W_4 at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6) which is given below

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Step 6 : To find W_5 :

To find W_5 from W_4, we consider the 5th column and 5th row.

In C_5, 1 is present at R_1, R_3, R_5

In R_5, 1 is present at C_1, C_3, C_5

Thus add new entries in W_5 at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3), (R_5, C_5) which is given below

$$W_5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = W_4$$

Step 7 : To find W_6 :

To find W_6 from W_5 we consider the 6th column and 6th row.

In C_6, 1 is present at R_2, R_4, R_6

In R_6, 1 is present at C_2, C_4, C_6

Thus add new entries in W_6 at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6) which is given below

$$W_6 = W_5 = W_4$$

Hence W_6 is the relation matrix of R^*

$$\therefore \quad R^* = \begin{Bmatrix} (1, 1), (1, 3) (1, 5) (2, 2) (2, 4) (2, 6) (3, 1) (3, 3) (3, 5) \\ (4, 2) (4, 4) (4, 6) (5, 1) (5, 3) (5, 5) (6, 2) (6, 4) (6, 6) \end{Bmatrix}$$

Example 4.9.5 *Let R = {(a, d), (b, a), (b, d), (c, b), (c, d), (d, c)}*

Use Warshall's algorithm to find the matrix of transitive closure where A = {a, b, c, d}

SPPU : Dec.-15

Solution : Step 1 : We have

$$A = \{a, b, c, d\} \qquad \therefore \quad |A| = 4$$

$$R = \{(a, d) (b, a), (b, d) (c, b), (c, d), (d, c)\}$$

Thus we have to find Warshall's sets W_o, W_1, W_2, W_3, W_4

The first set $\quad W_o = M_R$

$$W_0 = M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_o, we consider the first column and first row.

In C_1, 1 is present at R_2

In R_1, 1 is present at C_4

Thus add new entry in W_1 at (R_2, C_4)

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2

To find W_2, from W_1, we consider the second column and second row.

In C_2, 1 is present at R_3

In R_2, 1 is present at C_1 and C_4

Thus add new entries in W_2 at (R_3, C_1), (R_3, C_4) which is given below.

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 4 : To find W_3 :

To find W_3 from W_2, we consider the 3rd column and 3rd row.

In C_3, 1 is present at R_4

In R_3, 1 is present at C_1, C_2, C_4

Thus add news entries in W_3 at (R_4, C_1), (R_4, C_2), (R_4, C_4)

Which is given below

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step 5 : To find W_4 :

To find W_4 from W_3, we consider the 4th column and 4th row.

In C_4, 1 is present at R_1, R_2, R_3, R_4

In R_4, 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_4 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_4, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)

Which is given below

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence W_4 is the relation matrix of R^*.

and $$R^* = \begin{Bmatrix} (a, a), & (a, b) & (a, c) & (a, d) & (b, a) & (b, b) & (b, c) & (b, d) \\ (c, a), & (c, b) & (c, c) & (c, d) & (d, a) & (d, b) & (d, c) & (d, d) \end{Bmatrix}$$

Example 4.9.6 *Find the transitive closure of the relation R on*

A = {1, 2, 3, 4} defined by

R = {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)}

SPPU : Dec.-07, May-15, 18, Marks 4

Solution : Step 1 :

We have $|A| = 4$, Thus we have to find

Warshall's sets, W_0, W_1, W_2, W_3, W_4

The first set $W_0 = M_R$

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and first row.

In C_1, 1 is present at R_2

In R_1, 1 is present at C_2, C_3, C_4

Thus add new entries in W_1 at (R_2, C_2), (R_2, C_3), (R_2, C_4) which is given below

$$W_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the 2nd column and 2nd row.

In C_2, 1 is present at R_1, R_2, R_3, R_4

In R_2, 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_2 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)

Which is given below

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

All entries in W_2 are 1

Hence W_2 is the relation matrix of transitive closure of R.

and $$R^* = \begin{Bmatrix} (1, 1), & (1, 2) & (1, 3) & (1, 4) & (2, 1) & (2, 2) & (2, 3) & (2, 4) \\ (3, 1), & (3, 2) & (3, 3) & (3, 4) & (4, 1) & (4, 2) & (4, 3) & (4, 4) \end{Bmatrix}$$

Example 4.9.7 *If R = {(a, b), (b, a), (b, c), (c, d), (d, a)} be a relation on the set A = {a, b, c, d}. Find the transitive closer of R using Warshall's algorithm.* **SPPU : Dec.-16, Marks 6**

Solution :

Step 1 : We have $A = \{a, b, c, d\}$

$R = \{(a, b), (b, a), (b, c), (c, d), (d, a)\}$

Thus we have to find Warshall's sets W_0, W_1, W_2, W_3, W_4.

The first set $W_0 = M_R$

$$\therefore \quad W_0 = M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1

To find W_1 from W_0, we consider the first column and first row. In C_1, 1 is present R_2 and R_4

In R_1, 1 is present at C_2

Thus add new entries in W_1 at $(R_2, C_2)(R_4, C_2)$

$$\therefore \qquad W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Step 3 : To find W_2

To find W_2 from W_1, we consider second column and second row of W_1.

In C_2, 1 is present at R_1, R_2, R_3

In R_2, 1 is present at C_1, C_2, C_3

$\therefore$ Add new entries in W_2 at (R_1, C_1), (R_1, C_2), (R_1, C_3) (R_2, C_1), (R_2, C_2), (R_2, C_3) (R_3, C_1), (R_3, C_2), (R_3, C_3)

$$\therefore \qquad W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Step 4 : To find W_3

To find W_3 from W_2, we consider the IIIrd row and IIIrd column of W_2.

In C_3, 1 is present at R_1, R_2, R_3

In R_3, 1 is present at C_1, C_2, C_3, C_4

$\therefore$ Add new entries in W_2 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_3, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4)

$$\therefore \qquad W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Step 5 : To find W_4

To find W_4 from W_3, we consider 4th row and 4th column of W_3.

In C_4, 1 is present at R_1, R_2, R_3

In R_4, 1 is present at C_1, C_2, C_3

$\therefore$ Add new entries in W_3 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_3)

$$\therefore \qquad W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Hence the W_4 is the relation metrix of R^*

$$\text{And} \qquad R^* = \begin{Bmatrix} (a,a)\ (a,b)\ (a,c)\ (a,d)\ (b,a)\ (b,b)\ (b,c) \\ (b,d)\ (c,a)\ (c,b)\ (c,c)\ (c,d)\ (d,a)\ (d,b) \end{Bmatrix}$$

Example 4.9.8 *Find the transitive closure of R by Warshall's algorithm. A = {Set of positive integers ≤ 10}.*

R = {(a, b) / a divides b} **SPPU : May-07**

Solution :

Step 1 : We have $|A| = 10$. Thus we have to find

Warshall's sets W_0, W_1, W_2, ... W_{10}

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5), (5, 10), (6, 6), (7, 7), (8, 8), (9, 9), (10, 10)\}$$

The first set $W_0 = M_R$

$$\therefore \qquad W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The relation R itself is a transitive relation on the set of positive integers. Hence $R = R^*$ and

$$W_0 = M_R \text{ is the relation matrix of } R^*$$

Example 4.9.9 *Use Warshall's Algorithm to find transitive closure of R where*

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A = \{1, 2, 3\}$$

SPPU : May-06, May-08

Solution :

Step 1 : We have $|A| = 3$. Thus we have to find Warshall's sets W_0, W_1, W_2, W_3

The first set is

$$W_0 = M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and the first row.

In C_1, 1 is present at R_1, R_3

In R_1, 1 is present at C_1 and C_3

Thus add new entries in W_1 at (R_1, C_1), (R_1, C_3), (R_3, C_1), (R_3, C_3)

Which is given below

$$W_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the 2nd column and 2nd row.

In C_2, 1 is present at R_2, R_3

In R_2, 1 is present at C_2

Thus add new entries in W_2 at (R_2, C_2), (R_3, C_2) which is given below

$$W_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = W_1$$

Step 4 : To find W_3

To find W_3 from W_2, we consider the 3rd column and 3rd row.

In C_3, 1 is present at R_1, R_3

In R_3, 1 is present at C_1, C_2, C_3

Thus add new entries in W_3 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_3) which is given below

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence W_3 is the relation matrix of R^*

and $R^* = \{(1, 1)\ (1, 2)\ (1, 3)\ (2, 2)\ (3, 1)\ (3, 2)\ (3, 3)\}$

4.10 Partially Ordered Set

SPPU : Dec.-06

A relation R on a set A is called a partially ordered relation iff R is reflexive, anti-symmetric and transitive relation.

The set A together with partially ordered relation is called a partially ordered set or POSET.

It is denoted by (A, R) or $(A, \leq)$ where $\leq$ is a partially ordered relation.

Examples :

1) $(\Re, \leq)$ $(N, \leq)$ are Posets.

where '$\leq$' is reflexive, antisymmetric and transitive relation.

2) If A = P(S) where S = (a, b, c) and for X, Y $\in$ A, Define $X \leq Y$ or XRY iff $X \subseteq Y$.

As $X \leq X \Rightarrow X \leq X$. $\therefore$ '$\leq$' is reflexive.

If $X \leq Y$, $Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$

$\therefore$ '$\leq$' is antisymmetric relation.

If $X \leq Y, Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z \Rightarrow X \leq Z$

$\Rightarrow$ '$\leq$' is transitive relation.

$\therefore$ $(P(S), \subseteq)$ or $(P(S), \leq)$ is a poset.

I) Comparable elements : Let $(A, \leq)$ be a poset. Two elements a, b in A are said to be comparable elements if $a \leq b$ or $b \leq a$. Two elements a and b of a set A are said to be non-comparable if neither $a \leq b$ nor $b \leq a$.

In above example (2),

The comparable elements are

$$\{a\} = \{a, b\}, \{b\} = \{a, b, c\}\ \{b, c\} \subseteq \{a, b, c\}$$

Non comparable elements are

$$\{a\} \not\subseteq \{b\} \quad \{a\} \not\subseteq \{a, c\}$$

II) Totally ordered set : Let A be any nonempty set. The set A is called linearly ordered set or totally ordered set if every pair of elements in A are comparable.

i.e. for any a, b $\in$ A either $a \leq b$ or $b \leq a$.

4.10.1 Hasse Diagram

It is useful tool, which completely describes the associated partially ordered relation. It is also known as ordering diagram.

A diagram of graph which is drawn by considering comparable and non-comparable elements is called Hasse diagram of that relation. Therefore while drawing Hasse diagram following points must be followed.

1) The elements of a relation R are called vertices and denoted by points.
2) All loops are omitted as relation is reflexive on poset.
3) If aRb or $a \leq b$ then join a to b by a straight line called an edge the vertex b appears above the level of vertex a. Therefore the arrows may be omitted from the edges in Hasse diagram.
4) If $a \nleq b$ and $b \nleq a$ i.e. a and b are non - comparable elements, then they lie on same level and there is no edge between a and b.
5) If $a \leq b$ and $b \leq c$ then $a \leq c$. So there is a path $a \rightarrow b \rightarrow c$. Therefore do not join a to c directly i.e. delete all edges that are implied by transitive relation.

Example 4.10.1 *Draw Hasse diagram of a poset (P(s), $\subseteq$) where S = {a, b, c}.*

Solution : P (s) = { ϕ, {a}, {b}, {c}, {a, b} {a, c} {b, c} {a, b, c}}

Now find the comparable and non comparable elements.

$\phi \subseteq$ {a}, $\phi \subseteq$ {b}, $\phi \subseteq$ {c}, $\therefore$ {a}, {b}, {c} lie above the level of ϕ.

{a} $\subseteq$ {a, b}, {b} $\subseteq$ {a, b}, {c} $\subseteq$ {a, c} $\therefore$ {a, b}, {b, c} {a,c} lies above the level of {a}, {b}, {c}.

{a, b} $\subseteq$ S, {b, c} $\subseteq$ S, {a, c} $\subseteq$ S $\therefore$ S lies above the level of {a, b}, {a, c}, {b, c}

But {a}, {b} {c} are non comparable $\therefore$ {a}, {b}, {c} lie on same level.

{a, b}, {a, c}, {b, c} are non comparable $\therefore$ lie on same level.

By considering the above observations, the Hasse diagram is as follows :

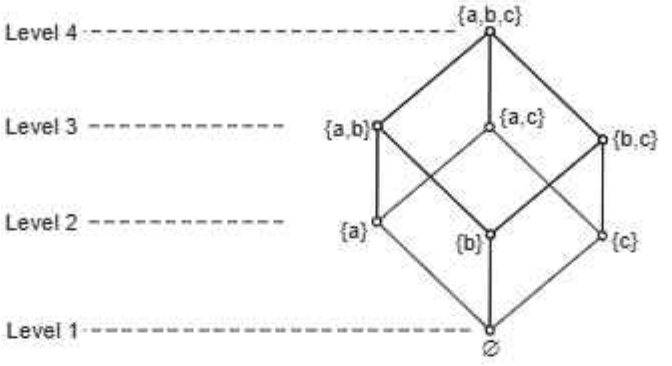


Fig. 4.10.1

4.10.2 Chains and Antichains

Let (A, ≤) be a poset. A subset of A is called a chain if every pair of elements in the subset are related.

A subset of A is called antichain if no two distinct elements in a subset are related. e.g. In above example (4.10.1)

1) The chains are $\{\phi, \{a\}, \{a, b\}, \{a, b, c\}\}$, $\{\{a\}, \{a, c\}, \{a, b, c\}\}$, $\{\{b, c\}, \{a, b, c\}\}$
2) Antichains are $\{\{a\}, \{b\}, \{c\}\}$

Note : 1) The number of elements in the chain is called the length of chain.

2) If the length of chain is n in a poset (A, ≤) then the elements in A can be partitioned into n disjoint antichains.

4.10.3 Elements of Poset

1) Let (A, ≤) be a poset. An element $a \in A$ is called a **maximal element** of A if there is no element $c \in A$ such that $a \leq c$.
2) An element $b \in A$ is called a **minimal element** of A if there is no element $c \in A$ such that $c \leq b$
3) Greatest element : An element $x \in A$ is called a **greatest element** of A if for all $a \in A$, $a \in x$. It is denoted by 1 and is called the unit element.
4) Least element : An element $y \in A$ is called a **least element** of A if for all $a \in A$, $y \leq a$.

It is denoted by 0 and is called as **zero element**.

5) Least upper bound (*l*ub) : Let (A, ≤) be a poset. For a, b, $c \in A$, an element C is called **upper bound** of a and b if $a \leq c$ and $b \leq c$ C is called as least upper bound of a and b in A if C is an upper bound a and b there is no upper bound d of a and b such that $d \leq c$. It is also known as **supremum.**

6) Greatest lower bound (glb) : Let $(A, \leq)$ be a poset. for a, b, $l \in A$, an element l is called the **lower bound** of a and b if $l \leq a$ and $l \leq b$.

An element l called the **greatest lower bound** of a and b if l is the lower bound of a and b and there is no lower bound f of a and b such that • $l \leq f$.

glb is also called as **infimum.**

Example 4.10.2 *Determine the greatest and least elements of the poset whose Hasse diagrams are shown below.*

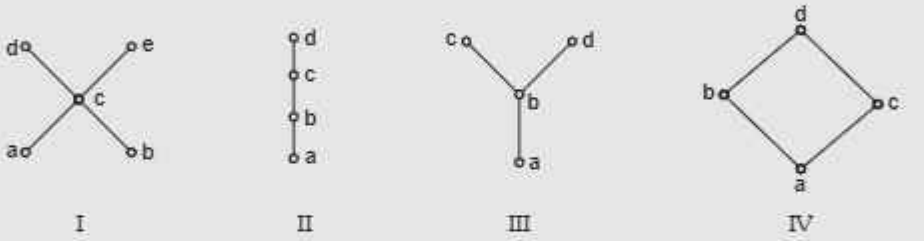


Fig. 4.10.2

Solution : The Poset shown in Fig. 4.10.2 (I) has neither greatest not least element.

The Poset shown in Fig. 4.10.2 (II), has greatest and a as least element.

The Poset shown in Fig. 4.10.2 (III), has no greatest element but a is the least element.

The Poset shown in Fig. 4.10.2 (IV), has greatest and a as least element.

Example 4.10.3 *Find glb, lwb, ub, lb, maximal, minimal, of the poset (A, R), Here aRb if a / b where. A = { 2, 3, 5, 6, 10, 15, 30, 45}*

Solution : We have A = {2, 3, 5, 6, 10, 15, 30, 45} and aRb iff a|b.

Hasse diagram is as follows :

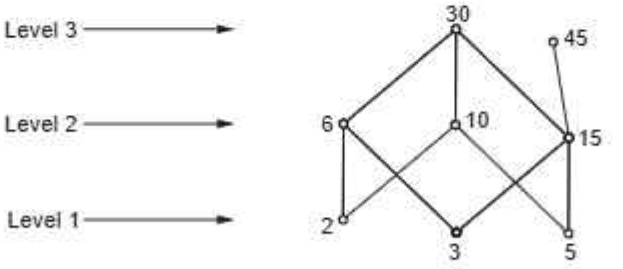


Fig. 4.10.3

1) Here 10 and 30 are upper bounds of 2 and 5, But 10 is the least upper bound of 2 and 5.

2) 5, 15, 3 are lower bounds of 30 and 45. But 15 is the greatest lower bound of 30 to 45.

3) This Poset has neither greatest element nor least element.

4) This poset has two maximal elements 45 and 30 as there is no element c such that $45 \leq C$ and $30 \leq C$.

5) This poset has three minimal elements 2, 3 and 5. because there is no element $x \in A$ such that $x \leq 2$, $x \leq 3$ and $x \leq 5$

4.10.4 Types of Lattices

A lattice is a poset in which every pair of elements has a least upper bound (lub) and a greatest lower (glb).

Let $(A, \leq)$ be a poset and $a, b \in A$ then lub of a and b is denoted by $a \vee b$. It is called the join of a and b.

i.e. $a \vee b = \text{lub}(a, b)$

The greatest lower bound of a and b is called the meet of a and b and it is denoted by $a \wedge b$

$$\therefore \quad a \wedge b = \text{glb}(a, b)$$

From the above discussion, it follows that a lattic is a mathematical structure with two binary operations $\vee$ (join) and $\wedge$ (meet). It is denoted by $[L, \vee, \wedge]$.

Examples :

Example 4.10.4 *Let $A = \{1, 2, 3\}$ $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2,3\}, \{1, 2,3\}$ show that $P(A), \subseteq)$ is a lattice*

Solution : The Hasse diagram of the poset $(P(A), \subseteq)$ is given below :

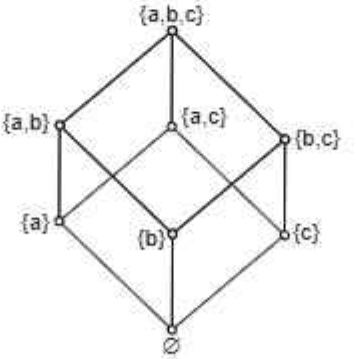


Fig. 4.10.4

Here every pair of elements of a poset has lub and glb. Hence $(P(A), \subseteq)$ is a lattice.

Example 4.10.5 *Determine which of the following posets are lattice.*

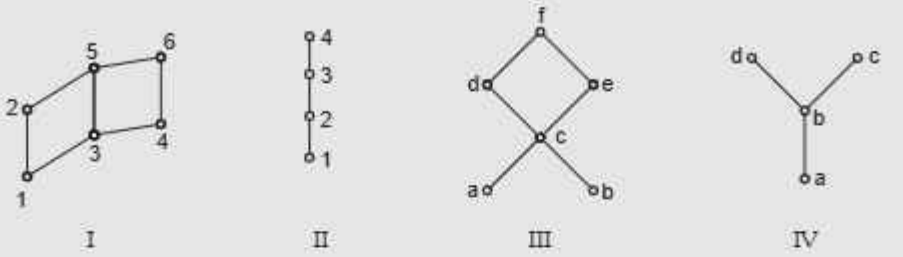


Fig. 4.10.5

Solution : I) In Fig. 4.10.5 (I), every pair of elements has lub and glb.

∴ It is a lattice.

II) In Fig. 4.10.5 (II), every pair of elements has lub and glb.

∴ It is a lattice.

II) In Fig. 4.10.5 (III), $a \wedge b$ does not exist.

∴ It is not a lattice.

IV) In Fig. 4.10.5 (IV), $c \vee d$ does not exist.

∴ It is not a lattice.

Example 4.10.6 *Let A be the set of positive factors of 15 and R be a relation on A s.t. R = {xRy| x divides y, x y ∈ A}. Draw Hasse diagram and give and* ∧ and ∨ *for lattice.*

SPPU : Dec.-06

Solution : We have A = {1, 3, 5, 15}

$$R = \{(1,1)\ (1,3)\ (1,5)\ (1,15)\ (3,15)\ (5,15)\ (15,15)\}$$

Hasse diagram of R is :

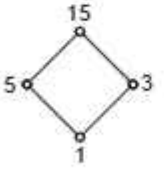


Fig. 4.10.6

Table for ∧ and ∨

∨	1	3	5	15
1	1	3	5	15
3	3	3	15	15
5	5	15	5	15
15	15	15	15	15

∧	1	3	5	15
1	1	1	1	1
3	1	3	1	3
5	1	1	5	5
15	1	3	5	15

Every pair of elements has lub and glb. ∴ It is a lattice.

4.10.5 Properties of a Lattice

Let $(L, \wedge, \vee)$ be a lattice and $a, b, c \in L$. Then L satisfies the following properties.

1) Commutative property

$$a \wedge b = b \wedge a \text{ and } a \vee b = b \vee a$$

2) Associative law

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

3) Absorption law

$$a \wedge (a \vee b) = a \text{ and } av\ (a \wedge b) = a$$

3) $a \wedge a = a$, $a \vee a = a$

4) $a \wedge b = a$ iff $a \vee b = b$

4.10.6 Types of Lattices

I) Bounded lattice : A lattice L is called a bounded lattice if it has a greatest element 1 and least element 0.

II) Sublattice : Let, $(L, \vee, \wedge)$ be a lattice. A non empty subset L_1 of L is called a sublattice of L if L_1 itself is a lattice w.r.t. the operations of L.

III) Distributive lattice : A lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any elements $a,b,c \in L$, it satisfies the following properties,

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

If the lattice does not satisfy the above properties then it is called a non distributive lattice.

Theorem : Let, $(L, \wedge, \vee)$ be a lattice with universal bounds 0 and 1 then for any $a \in L$, $a \vee 1 = 1$, $a \wedge 1 = a$, $0 \vee a = a$, $0 \wedge a = 0$.

III) Complement lattice : Let $(L, \wedge, \vee)$ be a lattice with universal bounds 0 and 1 for any $a \in L$, $b \in L$ is said to be complement of a if $a \vee b = 1$ and $a \wedge b = 0$.

A Lattice in which every element has a complement in that lattice, is called the complemented lattice.

e.g. 1) The Hasse diagram is here $0 \approx 1$ and $1 \approx 30$.

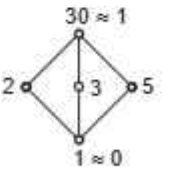


Fig. 4.10.7

i) $2 \wedge 3 = 0$ and $2 \vee 3 = 1$, $2 \wedge 5 = 0$ and $2 \vee 5 = 1$

$\therefore$ 2 has two compliments 3 and 5

Hence the complement is not unique.

4.11 Principle of Duality

SPPU : Dec.-10, 11, 13, 14, 15, 16, May-14, 15, 19

Any statement about lattice involving $\wedge, \vee \leq, \geq$ remains true if '$\wedge$' is replaced by '$\vee$', '$\vee$ by $\wedge$', '$\leq$ by $\geq$', '$\geq$ by $\leq$', '0 by 1' and '1 by 0'.

e.g. 1) $a \vee (b \wedge c) = a \wedge (b \vee c)$

2) $a \wedge (b \vee 1) = a \vee (b \wedge 0)$

Examples

Example 4.11.1 *Let A = {1, 2, 3, 4, 6, 9, 12} Let a relation R on a set A is R = {(a,b)/ a divides b ∀ a, b ∈ A}. Give list of R. Prove that it is a partial ordering relation. Draw Hasse diagram of the same. Prove or disprove it is a lattice. Give two examples of chain and antichains.* **SPPU : Dec.-11, May-19, Marks 6**

Solution : We have A = {1, 2, 3, 4, 6, 9, 12}

$$\text{and} \quad R = \left\{ \begin{matrix} (1,1), (1,2), (1,3), (1,4)\,(1,6), (1,9), (1,12), (2,2), (2,4), (2,6), (2,12)\,(3,3), (3,6), (3,9) \\ (3,12), (4,4), (4,12), (6,6), (6,12), (9,9), (12,12) \end{matrix} \right\}$$

We know that for any $a \in A$, $a \mid a$ $\therefore$ aRa

$\therefore$ R is a reflexive relation.

As $a \mid b$ and $b \mid a \Rightarrow a = b$ $\therefore$ R is antisymmetric relation.

As $a \mid b$ and $b \mid c \Rightarrow a \mid c \Rightarrow$ R is a transitive relation.

$\therefore$ R is reflexive antisymmetric and transitive

$\therefore$ (A, R) is a poset and R is a partial ordering relation.

Hasse diagram is as follows :

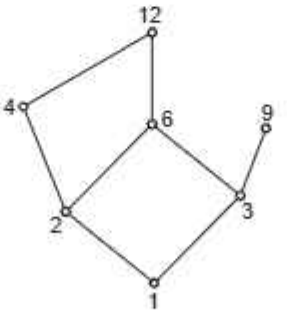


Fig. 4.11.1

In above diagram $6 \vee 9$ does not exist. $\therefore$ It is not a lattice.

i) Chains are

{1, 2, 4, 12}

{1, 2, 6, 12}

{1, 3, 6, 12}

ii) Antichains are

{6, 9} {9, 12}

{4, 9}

Example 4.11.2 *Determine whether the poset represented by each of the Hasse diagram are lattices. Justify your answer.*

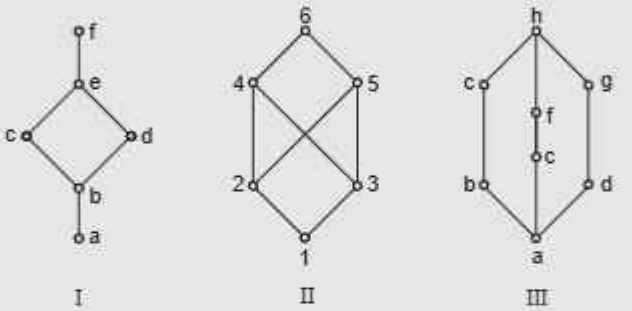


Fig. 4.11.2 **SPPU : Dec.-10**

Solution :

I) In Fig. 4.11.2 (I), every pair of element has glb and lub. ∴ It is a lattice.

II) In Fig. 4.11.2 (II), every pair of elements has lub and glb. ∴ It is a lattice.

III) In Fig. 4.11.2 (III), every pair of elements gas lub and glb. ∴ It is a lattice.

Example 4.11.3 *Show that the set of all divisors of 36 forms a lattice.* **SPPU : Dec.-14**

Solution : Let A = {1, 2, 3, 4, 6, 9,12, 18, 36 } and Let '≤' is a divisor of.

It's Hasse diagram is as follows.

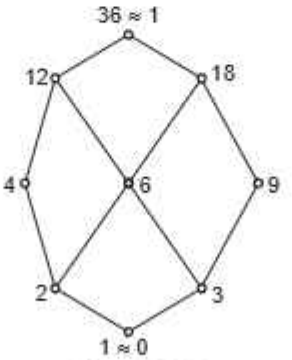


Fig. 4.11.3

The universal upper bound 1 is 36 and lower bound 0 is 1. Every pairs of elements of this poset has lub and glb.

∴ It is a lattice.

Example 4.11.4 *SLet n be a positive integer, S_n be the set of all divisors of n, Let D denote the relation of divisor. Draw the diagram of lattices for n = 24, 30, 6.* **SPPU : May-15**

Solution : Given that

i) We have S_6 = {1,2,3,6}, D is the relation of divisor.

ii) S_{24} = {1, 2, 3, 4, 6, 8, 12, 24}

iii) S_{30} = {1, 2, 3, 5, 6, 10, 15, 30}

Diagrams of Lattices are as follows.

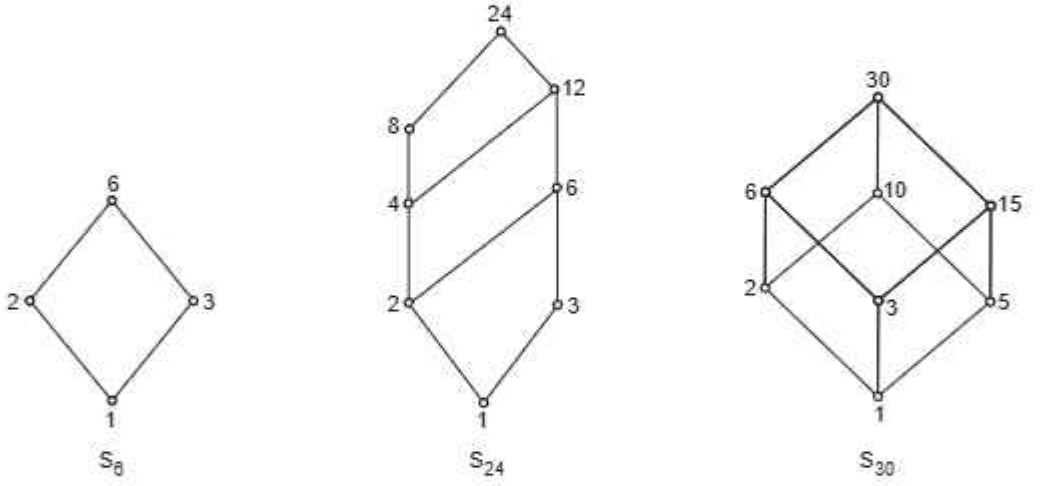


Fig. 4.11.4

Example 4.11.5 *Show that the set of all divisors of 70 forms a lattice.* **SPPU : Dec.-13, May-19, Marks 3**

Solution : Let A = {1, 2, 5, 7, 10, 14, 35, 70}

and Let '≤' is "a divisor of".

The universal upper bound 1 is 70 and the lower bound 0 is 1.

It's Hasse diagram is as follows :

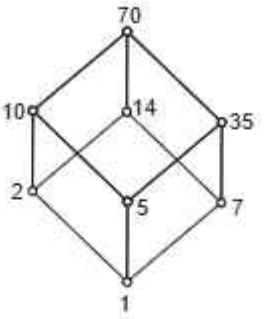


Fig. 4.11.5

Every pair of elements of A has $\wedge$ and $\vee$.

$\therefore$ It is a lattice [write table of $\wedge$ and $\vee$].

Example 4.11.6 *Let A = {1, 2, 3, 4 , 5, 6, 7, 8 ,9, 12, 18, 24} be ordered by the relation x divides y. Show that the relation is a partial ordering and draw Hasse diagram.*

SPPPU : Dec-15

Solution : We have A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24}

$$R = \{(x,y) \mid x \text{ divides } y, \text{ for } x, y \in A\}$$

$$R = \{(1, 2)\ (1, 3), (1, 4), (1, 5)\ (1, 6)\ (1, 7)\ (1, 8)\ (1, 9)\ (1, 12)\ (1, 18)\ (1, 24)$$
$$(2, 2)\ (2, 4)\ (2, 6)\ (2, 8)\ (2, 12)\ (2, 18)\ (2, 24)\ (5, 5)\ (6, 6)\ (6, 12)\ (6, 18)$$
$$(6, 24), (7, 7)\ (8, 8)\ (8, 24)\ (9, 9)\ (9, 18)\ (12, 12)\ (12, 24)\ (18, 18)\ (4, 24)\}$$

We have for any $x \in A$, $x|x \Rightarrow R$ is a reflexive for $x|y$ and $y|x \Rightarrow x = 0 \Rightarrow R$ is antisymmetric. If $x|y$ and $y|z \Rightarrow x|z$ $\therefore$ R is a transitive relation.

$\therefore$ R is a partial ordering relation, It's Hasse diagram is as follows.

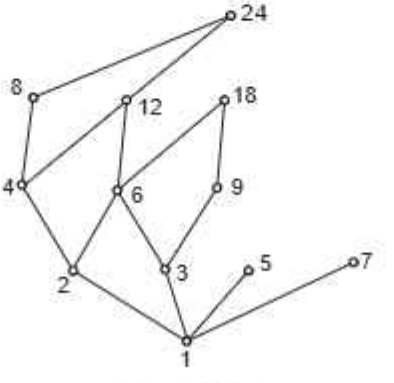


Fig. 4.11.6

Example 4.11.7 *Let x = {2, 3, 6, 12, 24, 36} and x ≤ y iff x divides y find*
i) Maximal element ii) Minimal element iii) Chain iv) Antichain v) Is Poset lattice

SPPU : May-14

Solution : We have x = {2, 3, 6, 12, 24, 36}

The relation 'R' = '≤'

∴ R = {(2, 2) (2, 6) (2, 12) (2, 24) (2, 36) (3, 3) (3, 6) (3, 12) (3, 24) (3, 36), (6, 6) (6, 12) (6, 24) (6, 36) (12, 12)(12, 24) (12, 36) (24, 24) (36, 36)}.

It's Hasse diagram is as follows .

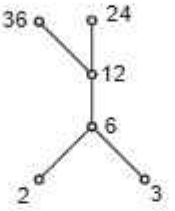


Fig. 4.11.7

i) Maximal elements are 24, 36

ii) Minimal elements are 2, 3

iii) Chain { 2, 6, 12, 24}, { 2, 6, 12, 36}, {3, 6, 12, 24}, {3, 6, 12, 36}

iv) Antichain : {2, 3} {24, 36}

v) The given poset is not a lattice as 2 ∧ 3 does not exist.

Example 4.11.8 *The following is the Hasse diagram of the poset : {(a, b, c, d, e), <} Is it a lattice ? Justify*

SPPU : Dec.-16, Marks 2

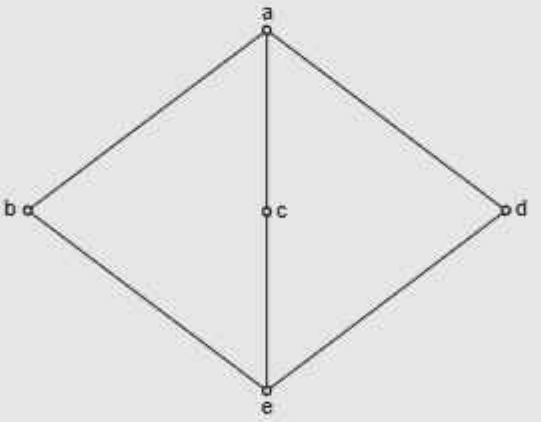


Fig. 4.11.8

Solution : Let

$$A = \{a, b, c, d, e\}$$

From the given Hasse diagram, every pair of element has glb and lub. Hence it is a lattice.

❑❑❑

Unit - II

5 Functions

Syllabus

Surjective, Injective and Bijective functions, Identity function, Partial function, Invertible function, Constant function, Inverse functions and Compositions of functions, The Pigeonhole Principle.

Contents

5.1 Introduction

5.2 Function ..... **May-07, 08, 15, Dec.-07, 12,** · Marks 4

5.3 Special Types of Functions ..... **Dec.-10, 11, 16, 17, May-18,** · Marks 6

5.4 Infinite Sets and Countability ..... **May-18, 19,** ..... Marks 4

5.5 Pigeon Hole Principle ..... **Dec.-09, 11, 12,** ..... Marks 4

5.6 Discrete Numeric Functions

5.1 Introduction

The concept of relation was defined very generally in the preceeding chapter. We shall now discuss a particular class of relations called functions. Many concepts of computer science can be conveniently stated in the language of functions.

5.2 Function

SPPU : May-07, 08, 15, Dec.-07, 12,

Let A and B be non-empty sets. A function f from A to B is denoted by $f : A \to B$ and defined as a relation from A to B such that for every $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$.

If $(a, b) \in f$ then it can be written as $f(a) = b$. Functions are also called as Mappings or Transformations.

e.g.

1) $f = \{(a, b)\ (b, c)\ (c, a)\}$ is a function on set $A = \{a, b, c\}$
2) $f = \{(a, b)\ (b, c)\ (b, a)\}$ is not a function as (b, c) and $(b, a) \in f$.
3) $f = \{(a, b)\ (b, c)\}$ is not a function on set $A = \{a, b, c\}$ as for $C \in A$ $\nexists$ any element x in A such that $(c, x) \in f$.

5.2.1 Important Definitions

I) Let $f : A \to B$ be a function such that $f(a) = b$ then $b \in B$ is called the image of $a \in A$ and a is called the pre-image of b. The element a is called argument of f.

II) Let $f : A \to B$ be a function. The set A is known as domain set of f and the set B is known as co-domain set of f.

III) The range set of a function $f : A \to B$ is denoted by R(f) and defined as

$$R(f) = \{b / b \in B \text{ and } f(a) = b \text{ for some } a \in A\}$$

In other words, range of f is the set of all images of the elements of A under f.

Examples

Example 5.2.1 *If f is a function such that $f(x) = x^2 - 1$ where $x \in R$ find the values of f(–1), f(0), f(1), f(3) and range set of f.*

Solution : We have $f(x) = x^2 - 1$

$$\therefore \quad f(-1) = (-1)^2 - 1 = 1 - 1 = 0$$

$$f(0) = 0 - 1 = -1$$

$$f(1) = 1^2 - 1 = 0$$

$$f(3) = 3^2 - 1 = 8$$

For any $x \in \Re$, $x^2 - 1 \in \Re$

$\therefore$ The range of f is $R(f) = \{x / -1 \le x < \infty\}$

Example 5.2.2 *Let $f : \Re \to \Re$ where f is defined by*

a) $f(x) = \sqrt{x}$

b) $f(x) = x^2$

c) $f(x) = \sin x \; \forall x \in \Re$

Solution :

a) We have $f(x) = \sqrt{x}$ $\quad \therefore f(4) = \sqrt{4} = \pm 2$

$\therefore$ f is not a function. Therefore we can not find range of f.

b) We have $f(x) = x^2$, x^2 is positive real number

Hence range of f is $R(f) = \{x \mid 0 \le x < \infty\}$

c) $\forall x \in \Re$, sin x lies between –1 to +1

$\therefore \quad R(f) = \{x \mid -1 \le x \le 1\}$

5.2.2 Partial Functions

Let A and B be two non empty sets. A partial function f with domain set A and codomain set B is any function from A' to B where A' CA. For any $x \in A - A'$, f(x) is not defined.

To make the distinction more clear, the function which is not partial is called as a total function.

e.g.

1)

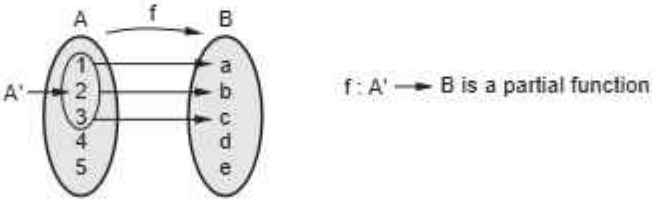


Fig. 5.2.1

2) The function $f : \Re \to \Re$ defined as $f(x) = \frac{1}{x}$ is a partial function. As it is not defined for x = 0.

3) The function $f : \Re \to \Re$ defined as $f(x) = \sqrt{x}$ is a partial function, as $\sqrt{x}$ is not defined for x < 0, in $\Re$.

5.2.3 Equality of Two Functions

Two functions $f : A \to B$ and $g : A \to B$ are said to be equal functions or identical functions iff.

$$f(x) = g(x) \ \forall \ x \in A$$

Note : Two functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \Re \to \Re$ defined as $f(x) = x, \forall x \in \mathbb{Z}$ and $g(x) = x, \forall x \in \Re$ are not identical or equal functions because their domains are not same.

5.2.4 Identity Function

Let A be any non empty set and function $f : A \to A$ is said to be the identity function if $f(x) = x, \forall x \in A$.

e.g.

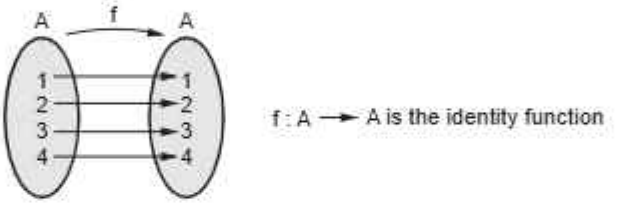


Fig. 5.2.2

5.2.5 Constant Function

A function $f : A \to B$ is said to be a constant function if $f(x) = \text{constant} = k \ ; \forall x \in A$.

The range set of a constant function consists of only one element.

5.2.6 Composite Function

Let $f : A \to B$ and $g : B \to C$ be two functions.

The composite function of f and g is denoted by gof and defined as $gof : A \to C$ is a function such that $(gof)(a) = g[f(a)] \ \forall a \in A$.

Note : gof is defined only when the range of f is a subset of the domain of g.

e.g. 1)

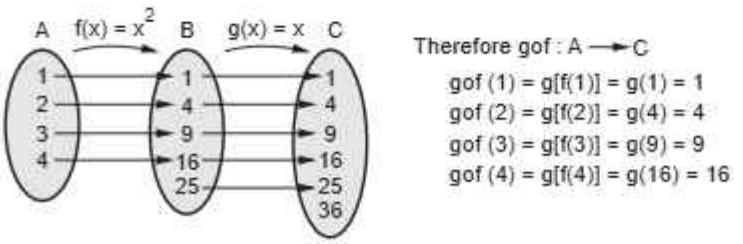


Fig. 5.2.3

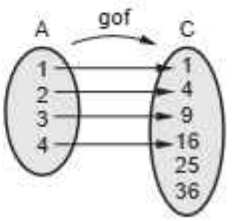


Fig. 5.2.4

Example 5.2.3 *Let $f(x) = x+2$, $g(x) = x - 2$, $h(x) = 3x$, for $x \in \Re$ Where $\Re$ is the set of real numbers*

Find i) gof ii) fog iii) fof iv) hog v) gog vi) foh vii) hof viii) fohog ix) gofoh.

SPPU : May-08, 15, Dec.-12, Marks 4

Solution : Let $x \in \Re$ be any real number.

i) $gof(x) = g[f(x)] = g[x + 2] = x + 2 - 2 = x$

ii) $fog(x) = f[g(x)] = f[x - 2] = x - 2 + 2 = x$

iii) $fof(x) = f[f(x)] = f[x + 2] = x + 2 + 2 = x + 4$

iv) $hog(x) = h[g(x)] = h[x - 2] = 3(x - 2) = 3x - 6$

v) $gog(x) = g[g(x)] = g[x - 2] = x - 2 - 2 = x - 4$

vi) $foh(x) = f[h(x)] = f[3\ x] = 3x + 2$

vii) $hof(x) = h[f(x)] = h[x + 2] = 3(x + 2) = 3x + 6$

viii) $fohog(x) = f[h(g(x))] = f[h(x - 2)]$

$= f[3(x - 2)] = f(3\ x - 6)$

$= 3x - 6 + 2 = 3\ x - 4$

ix) $gofoh(x) = g[f(h(x))] = g[f(3\ x)]$

$= g[3\ x + 2] = 3\ x + 2 - 2 = 3\ x$

Example 5.2.4 *Let functions f and g be defined by $f(x) = 2x + 1$, $g(x) = x^2 - 2$*

Find a) gof(4) and fog (4)

b) gof(a + 2) and fog(a + 2)

c) fog(5)

d) gof(a + 3)

SPPU : May-07, Dec.-07, Marks 4

Solution :

a) $gof(4) = g[f(4)] = g[2(4) + 1] = g[9] = 9^2 - 2 = 79$

$fog(4) = f[g(4)] = f(4^2 - 2) = f(14) = 2(14) + 1 = 29$

b) $gof(a + 2) = g[f(a + 2)] = g[2(a + 2) + 1] = g[2a + 5]$

$= (2a+5)^2 - 2 = 4a^2 + 20a + 23$

$fog(a + 2) = f[g(a + 2)] = f[(a+2)^2 - 2] = f[a^2 + 4a + 2]$

$= 2[(a^2 + 4a + 2] + 1 = 2a^2 + 8a + 5$

c) $fog(5) = f[g(5)] = f[25 - 2] = f(23) = 2(23) + 1 = 47$

d) $gof(a + 3) = g[f(a + 3)] = g[2(a + 3) + 1] = g[2a + 7]$

$= [2a+7]^2 - 2 = 4a^2 + 28a + 47$

5.3 Special Types of Functions

SPPU : Dec.-10, 11, 16, 17, May-18

1) Let $f : A \to B$ be a function.

i) Function f is said to be one to one (or Injective) function if distinct elements of A are mapped into distinct elements of B.

i.e. f is one to one if

$$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$$

OR $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

ii) Function f is said to be onto function if each element of B has at least one preimage in A.

OR

A function $f : A \to B$ is called onto (or surjective) function if the range set of f is equal to B.

iii) A function f is called a objective function if it is both one to one and onto.

iv) A function f is called into function if $\exists$ at least one element in B which has no preimage in A.

i.e. A function f is called into function if $R(f) \neq B$

2) Let a function $f : A \to B$ be a bijective function then $f^{-1} : B \to A$ is called the inverse mapping of f and defined as $f(b)^{-1} = a$ iff $f(a) = b$

It is also known as invertible mapping.

3) Characteristic function of a set :

Let U be a universal set and A be a subset of U.

Then the function $\psi_A : U \to \{0, 1\}$ defined by

$$\psi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

is called a characteristic function of the set A.

Examples

Example 5.3.1 *Give examples of functions of the following types by diagrams.*

a) Injective function but not surjective

b) Surjective but not injective

c) Neither injective nor surjective

d) Injective as well as surjective

e) Into function

f) Inverse function

Solution :

a) Injective but not surjective

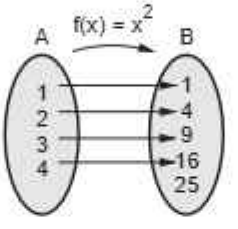


Fig. 5.3.1

b) Surjective but not injective

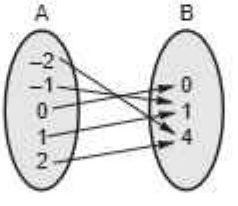


Fig. 5.3.2

c) Neither injective nor surjective

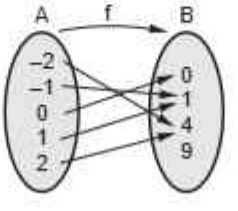


Fig. 5.3.3

d) Injective as well as surjective i.e. bijective

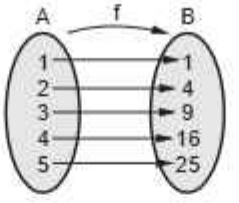


Fig. 5.3.4

e) Into function

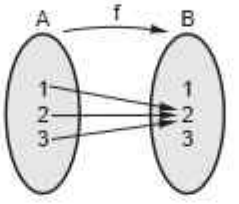


Fig. 5.3.5

f) Inverse function

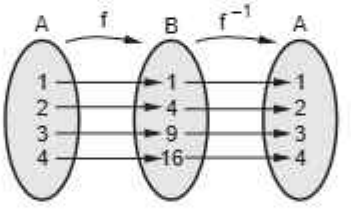


Fig. 5.3.6

Example 5.3.2 *Determine if each is a function. If yes is it injective, surjective, bijective ?*

a) Each person in the earth is assigned a number which corresponds to his age.

b) Each student is assigned a teacher.

c) Each country has assigned it's capital. **SPPU : Dec.-11, Marks 4**

Solution :

a) Every person has unique age

∴ It is a function. Two person's may have same age. ∴ It is not injective. There is no person whose age is 300 years. ∴ It is not surjective. ∴ Function is not bijective.

b) It is a function. It is not injective. It is not surjective. ∴ It is not bijective.

c) It is a function. It is injective as well as surjective. ∴ It is bijective.

Example 5.3.3 *Let A = {a, b, c, d} and B = {1, 2, 3}. Determine whether the relation R from A to B is a function. Justify. If it is function give the range :*

i) R = {(a, 1), (b, 2), (c, 1), (d, 2)}

ii) R = {(a, 1), (b, 2), (a, 2), (c, 1), (d, 2)} **SPPU : Dec.-16, Marks 4**

Solution : Given that

$$A = \{a, b, c, d\}, B = \{1, 2, 3\}$$

i) $$R = \{(a, 1), (b, 2), (c, 1), (d, 2)\}$$

Every element of the domain set A is related to the unique element of set B.

$\therefore$ R is a funtion from $A \rightarrow B$.

Range of R is {1, 2}.

ii) $$R = \{(a, 1), (b, 2), (a, 2), (c, 1) (d, 2)\}$$

Here $a \rightarrow 1$ and $a \rightarrow 2$ hence a has two images 1 and 2 in set B. i.e. a has no unique image in set B. Thus given relation R is not a function.

Example 5.3.4 *What are relations and functions. Given a relation R = {(1, 4), (2, 2), (3, 10), (4, 8), (5, 6)} and chek whether the following relations R_1, R_2, R_3 and R_4 is a function or not.*

R_1 = {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)}

R_2 = {(1, 2), (2, 4), (2, 10), (3, 8), (4, 6), (5, 4)}

R_3 = {(1, 6), (2, 2), (4, 4), (5, 10)}

R_4 = {(1, 6), (2, 2), (3, 2), (4, 4), (5, 10} **SPPU : Dec.-17, Marks 6**

Solution : Refer sections 4.3 and 5.2

Domain set = {1, 2, 3, 4, 5}

Codomain = {2, 4, 6, 8, 10}.

Given that R_1 = {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)}

All elements are mapped to 4.

$\therefore$ R_1 is a constant function.

R_2 = {(1, 2), (2, 4), (2, 10), (3, 8), (4, 6), (5, 4)}

In R_2 $2 \rightarrow 4$ and $2 \rightarrow 10$ which is not possible in function.

$\therefore$ R_2 is not a function

R_3 = {(1, 6), (2, 2), (4, 4), (5, 10)}

As $3 \in D$ but 3 has no any image in R_3

$\therefore$ R_3 is not a function.

$R_4 = \{(1, 6), (2, 2), (3, 2), (4, 4), (5, 10)\}$

Every element of a domain set has unique image in codomain set so R_4 is a function.

Example 5.3.5 *Let A = B be the set of real numbers.*

$f : A \to B$ given by $f(x) = 2x^3 - 1$

$g : B \to A$ given by $g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$

Show that f is a bijective function and g is also bijective function :

SPPU : Dec.-10, Marks 4

Solution :

1) Suppose $f(x_1) = f(x_2)$

$$\Rightarrow \quad 2x_1^3 - 1 = 2x_2^3 - 1$$

$$\Rightarrow \quad x_1^3 = x_2^3$$

$$\Rightarrow \quad x_1 = x_2$$

$\therefore$ f is injective mapping

Let $y \in B$ and $f(x) = y \Rightarrow 2x^3 - 1 = y$

$$\therefore \quad 2x^3 = 1 + y \Rightarrow x^3 = \frac{1+y}{2}$$

$$\Rightarrow \quad x = \sqrt[3]{\frac{1+y}{2}} = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} \in A = \Re \text{ for any } y \in B$$

$\therefore$ f is a surjective mapping.

Thus f is injective as well as surjective function.

Hence f is a bijective function.

We have $f(x) = y \Rightarrow x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} \Rightarrow f^{-1}(y) = x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} = g(y)$

Thus $f^{-1} = g$. We know that if f is bijective function then f^{-1} is also bijective. Hence g is bijective function.

Example 5.3.6 *Explain classification of functions with example.*

Solution : Depending upon the nature of function, there are mainly two functions.

1. **Algebraic function :** A function which consists of a finite number of terms involving powers and roots of the independent variable and four fundamental operations addition, subtraction, multiplication and division, is called an algebraic function.

There are three types of algebraic functions.

(a) Polynomial function : A function of the form

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$, where n is positive integer, $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, is called polynomial function of x in degree n.

e.g. $f(x) = x^3 - 3x^2 + 2x - 5$

(b) Rational function : A function of the form $\frac{f(x)}{g(x)}$ where f(x) and g(x) are polynomial functions and $g(x) \neq 0$ is called rational function e.g. $\frac{x^2 - 3x + 5}{x^2 + 1}$

(c) Irrational function : A function involving radicals is called irrational function.
e.g. $f(x) = x^{2/3} + 5x^2 + 1, \sqrt[2]{x+1}$

2. Transcendental function : A function which is not algebraic is called transcendental function.

e.g. $f(x) = \sin x + x^3 + 5x$

(a) Trigonometric function : The six functions sin x, cos x, tan x, sec x, cosec x, cot x, where x is in radians are called trigonometric functions.

e.g. $f(x) = \sin x + \tan x$.

(b) Inverse trigonometric function : The six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are called inverse trigonometric functions.

e.g. $f(x) = \cos^{-1} x + 5 \tan^{-1} x$

(c) Exponential function : A function of the form $f(x) = a^x$ $(a > 0)$ satisfying $a^x \cdot a^y = a^{x+y}$ and $a' = a$ is called exponential fucntion. e.g. $f(x) = 5^x$.

(d) Logarithmic function : The inverse function of the exponential function is called logarithmic function. e.g. $f(x) = \log x$.

Example 5.3.7 *If $f : A \to B$ is bijective function then f^{-1} is unique.*

Solution : Let $f\ A \to B$ is a bijective function.

Claim : Show that f^{-1} is unique.

Suppose f^{-1} is not unique, so there are two inverse functions say, g and h.

Let $x_1, x_2 \in A$, $\exists$ y in B such that

$$f^{-1}(y) = x_1 \Rightarrow g(y) = x_1 \Rightarrow f(x_1) = y$$

and $$f^{-1}(y) = x_1 \Rightarrow h(y) = x_2 \Rightarrow f(x_2) = y$$

This implies $f(x_1) = f(x_2)$, but f is 1 – 1 function.

$\therefore \quad x_1 = x_2$

$\Rightarrow \quad g(y) = h(y) \ \forall \ y$

$\Rightarrow \quad g \equiv h$ i.e. g and h are equal function.

Hence universe of f is unique.

Example 5.3.8 *If $f : A \to B$ and $g : B \to C$ are bijective functions the gof is also bijective.*

Solution : Let $f : A \to B$ and $g : B \to C$ be two bijective functions then gof $A \to C$ is a function.

(i) Let $x_1, x_2 \in A$ and suppose $gof(x_1) = gof(x_2)$

$g[f(x_1)] = g[f(x_2)]$

$\Rightarrow \quad f(x_1) = f(x_2)$ ... ($\because$ g is 1 – 1 function)

$\Rightarrow \quad x_1 = x_2$ ... ($\because$ f is 1 – 1 function)

$\Rightarrow$ gof is 1 – 1 function.

(ii) Let $z \in C$.

$\therefore \ \exists$ y in B such that $g(y) = z$ and $\exists$ x in A such that $f(x) = y$.

$\therefore \ g(y) = z \quad \Rightarrow \quad z = g(f(x)) = gof(x)$ where $x \in A$

$\Rightarrow$ gof is onto function.

Hence gof is bijective function.

Example 5.3.9 *Determine whether the function is bijective.*

$$f : I \to I \text{ such that } f(i) = \frac{i}{2} \quad \text{if } i \text{ is even}$$
$$= \frac{(i-1)}{2} \quad \text{if } i \text{ is odd}$$

Solution : Let a and b be any integers such that $a \neq b$.

There are three possibilities

Case 1 : If a and b are odd

$\therefore \quad f(a) = \dfrac{a-1}{2}, \quad f(b) = \dfrac{b-1}{2}$

$f(a) = f(b) \Rightarrow \dfrac{a-1}{2} = \dfrac{b-1}{2}$

$\Rightarrow a = b$

Case 2 : If a and b are even

then $\quad f(a) = \dfrac{a}{2}, \quad f(b) = \dfrac{a}{2}$

$$f(a) = f(b) \Rightarrow \frac{a}{2} = \frac{b}{2} \Rightarrow a = b$$

Case 3 : If a is odd and b is even

then $\quad f(a) = \dfrac{a-1}{2}$ and $f(b) = \dfrac{b}{2}$

Now, $\quad f(a) = f(b) \Rightarrow \dfrac{a-1}{2} = \dfrac{b}{2} \Rightarrow \dfrac{a}{2} - \dfrac{b}{2} = \dfrac{1}{2}$

$\Rightarrow \quad a - b = 1 \quad \therefore \quad a \neq b$

In particular $a = 7, \; b = 6$

$$f(7) = \frac{7-1}{2} = 3 \qquad f(6) = \frac{6}{2} = 3$$

$$f(7) = f(6) \text{ but } 6 \neq 7$$

Thus f is not one one function. Hence f is not bijective function.

Example 5.3.10 *Let $f(x) = ax^2 + b$ and $g(x) = cx^2 + d$, where a, b, c, d are constants. Determine for which values of constants $gof(x) = fog(x)$.*

Solution : Given that, $f(x) = ax^2 + b$ and $g(x) = cx^2 + d$

Suppose $\quad fog(x) = gof(x)$

$$f[g(x)] = g[f(x)]$$

$$f[cx^2 + d] = g[ax^2 + b]$$

$$\Rightarrow \quad a[cx^2 + d]^2 + b = c[ax^2 + b]^2 + d$$

$$\Rightarrow \quad a[c^2x^4 + 2cdx^2 + d^2] + b = c(a^2x^4 + 2abx^2 + b^2) + d$$

$$\Rightarrow \quad ac^2x^4 + 2acdx^2 + ad^2 + b = ca^2x^4 + 2cabx^2 + cb^2 + d$$

$\Rightarrow \quad$ Coefficient of x^4 = coefficient of x^4

and $\quad$ Coefficient of x^2 = coefficient of x^2

$\Rightarrow \quad ac^2 = ca^2 \Rightarrow a = c$

and $\quad 2acd = 2acb \Rightarrow b = d$

Thus $\quad a = c$ and $b = d$

Example 5.3.11 *Show that the functions $f(x) = x^3 + 1$ and $g(x) = (x-1)^{1/3}$ are converse to each other.*

Solution : We have, $g(x) = (x-1)^{1/3}$

Let $\quad y = g(x) = (x-1)^{1/3}$

$\Rightarrow \quad y^3 = x - 1$

$\Rightarrow \quad x = y^3 + 1$ ... (Which is f (x))

Hence f (x) and g (x) are converse to each other.

Example 5.3.12 *Let $f : X \to Y$ and X and Y are set of real numbers. Find f^{-1} if (i) $f(x) = x^2$; (ii) $f(x) = \frac{2x-1}{5}$*

Solution : Let $f : X \to Y$ and $X, Y \subseteq \Re$

(i) $f(x) = x^2$

Let $f(x) = y$

$x^2 = y$

$\Rightarrow \quad y = x^2$

$\Rightarrow \quad y = \pm\sqrt{x}$

Therefore, given function is not one to one. Hence f^{-1} does not exist.

(ii) Let $f(x) = y$

$\frac{2x-1}{5} = y$

$\Rightarrow \quad 2x = 5y + 1$

$x = \frac{5y+1}{2} \quad \forall \; y \in \Re$

Function f is 1 – 1 and onto.

∴ The inverse function of f is given as,

$$f^{-1}(x) = \frac{1+5y}{2}$$

Example 5.3.13 *Let X = {a, b, c}. Define $f : X \to X$ such that f = {(a, b) (b, a) (c, c)}. Find (i) f^{-1} ; (ii) f^2 ; (iii) f^3 ; (iv) f^4*

Solution : Given fucntion f is bijective function.

(i) $f^{-1} = \{(b, a) (a, b) (c, c)\}$

(ii) $f^2 = fof = \{(a, a) (b, b) (c, c)\}$

$fof(a) = f[f(a)] = f(b) = a$

$fof(b) = f[f(b)] = f(a) = b$

$fof(c) = f[f(c)] = f(c) = c$

(iii) $f^3(a) = f[f^2(a)] = f(a) = b$

$f^3(b) = f[f^2(b)] = f(b) = a$

$f^3(c) = f[f^2(c)] = f(c) = c$

$f^2 = \{(a, b), (b, a), (c, c)\}$

(iv) $f^4 = f^2 \circ f^2$

$f^4(a) = f^2[f^2(a)] = f^2(a) = a$

$f^4(b) = f^2[f^2(b)] = f^2(b) = b$

$f^4(c) = f^2[f^2(c)] = f^2(c) = c$

$\therefore \quad f^4 = \{(a, a)\ (b, b)\ (c, c)\}$

Example 5.3.14 *Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$ and f_1 and f_2 are functions from A to B.*
$f_1 = \{(1, 2), (2, 3), (3, 1)\}$
$f_2 = \{(1, 2), (2, 1), (3, 3)\}$
Compute $f_1 \circ f_2$ and $f_2 \circ f_1$ **SPPU : May-18, Marks 4**

Solution : We have $f_1 \circ f_2(x) = f_1[f_2(x)]$

$\therefore \quad f_1[f_2(1)] = f_1[2] = 3$

$f_1[f_2(2)] = f_1[1] = 2$

$f_1[f_2(3)] = f_1[3] = 1$

Now, $f_2 \circ f_1(x) = f_2[f_1(x)]$

$f_2[f_1(1)] = f_2[2] = 1$

$f_2[f_1(2)] = f_2[3] = 3$

$f_2[f_1(3)] = f_2[1] = 2$

Example 5.3.15 *Let $X = \{1, 2, 3\}$, $Y = \{p, q\}$ and $z = \{a, b\}$*
Let $f : X \to Y$ be $\{(1, p), (2, p), (3, q)\}$ and $g : Y \to Z$ be
$g = \{(p, b)\ (q, b)\}$. Find gof and show it pictorically.

Solution : Given that, $f = \{(1, p), (2, p), (3, q)\}$

$g = \{(p, b)\ (q, b)\}$

and $gof(x) = g[f(x)]$ and $gof : X \to Z$

$gof(1) = g[f(1)] = g(p) = b$

$gof(2) = g[f(2)] = g(p) = b$

$gof(3) = g[f(3)] = g(q) = b$

$gof = \{(1, b)\ (2, b)\ (3, b)\}$

It's pictorial representation is

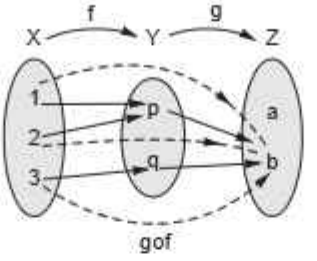


Fig. 5.3.7

Example 5.3.16 *Let R be the set of real numbers and f, g, h : $\Re \to \Re$ such that*

$f(x) = x + 2$, $g(x) = \dfrac{1}{x^2+1}$ $h(x) = 3$. *Compute*

(i) $f^{-1}\, g(x)$; *(ii)* $hf\,(gf^{-1})$ $[h\,f(x)]$

Solution : (i) Given that $f(x) = x + 2$

Let $f(x) = y = x + 2 \Rightarrow x = y - 2$

$$f^{-1}(y) = y - 2$$

$$f^{-1}[g(x)] = f^{-1}\left[\frac{1}{x^2+1}\right] = \frac{1}{x^2+1} - 2$$

$$= \frac{1-2x^2-2}{\left(x^2+1\right)} = \frac{-\left(2x^2+1\right)}{x^2+1}$$

(ii) We have $f(x) = x + 2$

$$h\,f(x) = h\,[f(x)] = h\,[x+2] = 3$$

$$g\,f^{-1}(x) = g\,[f^{-1}(x)] = g\,[x-2]$$

$$= \frac{1}{(x-2)^2+1} = \frac{1}{x^2-2x+5}$$

$$(g\,f^{-1})\,[h\,f(x)] = g\,f^{-1}(3) = \frac{1}{3^2-2(3)+5}$$

$$= \frac{1}{9-6+6} = \frac{1}{8}$$

$$\therefore \quad h\,f\,(g\,f^{-1})\,(h\,f(x)) = h\,f\left[\frac{1}{8}\right] = h\left[f\left(\frac{1}{8}\right)\right]$$

$$= h\left[\frac{1}{8}+2\right] = h\left[\frac{17}{8}\right] = 3$$

i.e. $h\,f\,(g\,f^{-1})\,(h\,f(x)) = 3$

Example 5.3.17 *Show that $f, g : N \times N \to N$ as $f(x, y) = x + y$. $g(x, y) = xy$ are onto but not one-one*

Solution : (i) Suppose $f(x_1, y_1) = f(x_2, y_2)$

$\Rightarrow \quad x_1 + y_1 = x_2 + y_2$

$\Rightarrow x_1 \neq x_2$ and $y_1 \neq y_2$

e.g. $x_1 = 5$, $x_2 = 2$, $y_1 = 4$, $y_2 = 7$.

$\therefore$ f is not one to one.

Now, $g(x_1, y_1) = g(x_2, y_2)$

$\Rightarrow \quad x_1 y_1 = x_2 y_2$

$\Rightarrow$ x_1 may or may not be equal to x_2

and y_1 may or may not be equal to y_2

$\therefore$ g is not one to one.

(ii) $f(x, y) = x + y$

$\therefore$ Every element of N can be written as the sum of two elements of N. Hence f is onto.

Now, $g(x, y) = xy$

Every element of can be written as the product of two elements of N.

$\therefore$ g is onto.

5.4 Infinite Sets and Countability

SPPU : May-18, 19

5.4.1 Infinite Set

A set A is said to be an infinite set if there exists an injective mapping (function) $f : A \to A$ such that f(A) is a proper subset of A.

If no such injective function exists, then set is finite.

Examples

1) Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 2n$, $\forall n \in N$ = Natural number set.

There range set = f(N) = {Set of positive even natural numbers} $\subset \mathbb{N}$

$\therefore$ $\mathbb{N}$ is an infinite set.

2) Define $f : \Re \to \Re$ such that $f(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$

f is an injective mapping and for $x \geq 0$, $f(x) = x + 2$

i.e. $f(x) \geq 2$ and for $x < 0$, $f(x) = x < 0$

$\therefore$ Range set = $f(\Re) = \{y \in \Re / f(x) \geq 2 \text{ or } f(x) < 0\}$

$\therefore 1 \notin f(\Re)$

Thus $f(\Re) \overset{\subset}{\neq} \Re$

Hence $\Re$ is an infinite set.

I) Properties of an infinite sets

1) If A is an infinite set the $A \times A$, P (A) are infinite sets.
2) If A and B are non empt sets and either A or B is an infinite set then $A \times B$ is an infinite set.
3) If either A or B is an infinite set then $A \cup B$ is an infinite set.
4) If $A \subseteq B$ and A is an infinite set then B is also an infinite set.

i.e. the superset of an infinite set is an infinite.

5.4.2 Countable Sets

We know that the codinality of a set is the number of elements of that set. If set is finite then, we can list elements as 1, 2, 3, ... Therefore every finite set is countable. As $|\phi| = 0$, the null set is also countable. A question remains same for an infinite sets. Let us define countable infinite sets.

Definition :

An infinite set A is said to be countable if there exists a bijection $f : \mathbb{N} \to A$

$$\therefore \qquad A = \{ f(1), f(2), f(3), \ldots\}$$

A countably infinite set is also known as a denumarable set.

i.e. If A is a denumerable set then we can least elements of A as $a_1, a_2, a_3, \ldots a_n \ldots$ or f (1), f (x) ... f (n) ... .

e.g. 1) ϕ is countable

2) $A = \{1, 2, 3, 4, \ldots 1000\}$ is countable as $|A| = 1000$.
3) As $f : \mathbb{N} \to \mathbb{N}$ defined as $f(x) = n$, $\therefore$ $\mathbb{N}$ is countable
4) The set of integers is countable

As $f : N \to Z$ such that $f(n) = \dfrac{n+1}{2}$ if $n = 1, 3, 5, \ldots$

$$= \frac{-n}{2} \text{ if } n = 0, 2, 4, 6, \ldots$$

is bijective mapping.

5) The set of rational numbers is countable.

6) The set of real numbers is not countable

7) The set of complex numbers is not countable.

8) The set of real numbers in [a, b], a < b is not countable.

9) The countable union of countable sets is countable.

Properties of countable sets :

1) A subset of a countable set is countable.

2) Let A and B be countable sets then $A \cup B$ is countable.

$\Rightarrow$ Define $f : A \cup B \to \mathbb{N}$ as $\quad f(a_i) = 2i - 1$

and $\quad f(b_i) = 2i$

f is bijective $\therefore A \cup B$ is countable

3) Prove that the set of rational numbers is countably infinite.

SPPU : May-18, 19, Marks 4

Proof : We know that the countable union of countable sets is countable. Therefore it is sufficient to prove that the set of rational numbers in [0, 1] is countable.

We have to prove that $\exists$ atleast one function f.

$f : [0, 1] \to N$ such that f is injective.

We arrange the rational numbers of the interval according to increasing denominators as

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$$

then the one to one correspondence is as follows

$0 \leftrightarrow 1$

$1 \leftrightarrow 2$

$\frac{1}{2} \leftrightarrow 3$

$\frac{1}{3} \leftrightarrow 4$

$\frac{2}{3} \leftrightarrow 5$

$\frac{1}{4} \leftrightarrow 6$ and so on

Hence set of rationals in [0, 1] is countable. Thus the set of rational numbers is countable and as the set is infinite, it is countably infinite.

4) The set of irrational numbers is uncountable.

Proof : We know that $\Re = Q \cup \overline{Q}$ where Q = set of rational numbers, $\overline{Q}$ = set of irrational numbers.

Suppose $\overline{Q}$ is countable $\Rightarrow Q \cup \overline{Q} = \Re$ is countable which is contradiction.

$\therefore$ $\overline{Q}$ is not countable i.e. uncountable.

5) The set of real numbers in (0, 1) is not countable. Assume that the set is countable. $\therefore A = \{x_1, x_2, x_3, \cdots x_n, \cdots\}$.

Proof : Any real number in (0, 1) can be written in a unique decimal without an infinite string of 9's at the end. i.e. 0.3459999 will be represented as 0.345000. Let the infinite sequence be given by,

$1 \rightarrow x_1 = 0 \cdot a_{11}\ a_{12}\ a_{13}\ \ldots$

$2 \rightarrow x_2 = 0 \cdot a_{21}\ a_{22}\ a_{23}\ \ldots$

$3 \rightarrow x_3 = 0 \cdot a_{31}\ a_{32}\ a_{33}\ \ldots$

$\vdots \quad \vdots$

$n \rightarrow x_n = 0.a_{n1}\ a_{n2}\ a_{n3}\ \ldots$

$\vdots \quad \vdots$

Construct a new number $y = 0 \cdot b_1\ b_2\ b_3\ \ldots$

Where $b_i = 0$ if $a_{ii} \neq 0$

$b_i = 1$ if $a_{ii} = 0$

Hence $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}$ ... $b_n \neq a_{nn}$

$\therefore b_i \neq a_{ii}$, $\forall$ i

$\therefore y \neq x_1$, $y \neq x_i$ $\forall$ i

Hence y is not in the list of numbers $\{x_1, x_2, \cdots x_n \cdots\}$ Thus $y \in (0, 1)$ and it is different from elements in $\{x_1, x_2, \cdots x_n \cdots\}$ which is contradiction that A is countable.

Hence A is not countable.

Thus the set of real numbers is not countable.

5.5 Pigeon Hole Principle

SPPU : Dec.-09, 11, 12

I) This principle states that if there are n + 1 pigeons and only n pigeon holes then two pigeons will share the same whole.

This principle is stated by using the analogy of the bijective mapping i.e. If A and B are any two sets such that $|A| > |B|$ then there does not exist bijective mapping from A to B.

Examples

Example 5.5.1 *If 11 shoes are selected from 10 pairs of shoes then there must be a pair and matched shoes among the selection.*

Solution : In the pigeonhole principle, 11 shoes are pigeons and the 10 pairs are the pigeon holes.

Example 5.5.2 *Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13.*

Solution : We have A = {1, 2, 3, 4, 5, ... 12}

We form the six different sets each containing 2 numbers that add uptot 13.

$A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$, $A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$, $A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$

Each of the seven numbers chosen must belong to one of these sets. As there are only six sets, by pigeonhole principle two of the chosen numbers must belong to the same set and their sum is 13.

II) The extended pigeon hole principle

If n pigeons are assigned to m pigeon holes, then one of the pigeon holes must be occupied by at least $\left[\frac{n-1}{m}\right] + 1$ pigeons. It is also known as generalized pigeon hole principle. Here $\left[\frac{n-1}{m}\right]$ is the integer division of n – 1 by m. e.g. $\left[\frac{9}{2}\right] = 4$, $\left[\frac{16}{5}\right] = 3$, $\left[\frac{8}{3}\right] = 2$.

Examples

Example 5.5.3 *Show that 7 colours are used to paint 50 bicycles, then at least 8 bicycles will be of same colour.* **SPPU : Dec.-09, 12, Marks 4**

Solution : By the extended pigeonhole principle, at least $\left[\frac{n-1}{m}\right] + 1$ pigeons will occupy one piegeonhole.

Here n = 50, m = 7 and m < n then

$$\left[\frac{50-1}{7}\right] + 1 = 7 + 1 = 8$$

Thus 8 bicycles will be of the same colour.

Example 5.5.4 *Write generalized pigeonhole principle. Use any form of pigeonhole principle to solve the given problem.*

i) Assume that there are 3 mens and 5 womens in a party show that if these people are lined up in a row at least two women will be next to each other.

ii) Find the minimum number of students in the class to be sure that three of them are born in the same month. **SPPU : Dec.-11, Marks 4**

Solution : Please refer section 5.5 (II) for definition.

i) By using analogy of pigeon hole principle, we get

3 men = pigeonholes and 5 women = pigeon

Pigeons are more than pigeon holes.

∴ At least two pigeons share the same pigeon hole i.e. at least two women in a row will be next to each other.

ii) Let h = Number of pigeons = Number of students

n = Number of pigeon holes = Number of months = 12

Given that three students in the class are born in the same month.

$$\therefore \quad \left[\frac{n-1}{m}\right] + 1 = 3$$

$$\Rightarrow \quad \frac{n-1}{12} = 3 - 1 = 2$$

$$n = 2 \times 12 + 1 = 25$$

Therefore there are 25 minimum number of students in the class.

5.6 Discrete Numeric Functions

A function whose domain is a set of natural numbers including zero and whose range set is the set of real numbers, is called a discrete numeric function or numeric function. It is also known as a sequence. If $f : \mathbb{N} \cup \{0\} \to \Re$ is a discrete numeric function then f(0, f(1), f(2), f(3), ... denote the value of function at 0, 1, 2, 3, ...

The numeric function f is written as

$$f = \{f_0, f_1, f_2, \cdots\}$$

Hereafter, to denote numeric function, we use

$$< a_r > = \{a_0, a_1, a_2, a_3 \cdots a_r, \cdots\}$$

e.g. 1) $a_r = 2^k$ if $0 \le k \le 3$

$= k$ if $k > 3$

This discrete function can be written as

$$a = \{1, 2, 4, 8, 4, 5, 6, 7, 8, \ldots\}$$

2) $$a_k = \begin{cases} 100 + r, & 0 \le r \le 5 \\ r^2 & r \ge 6 \end{cases}$$

∴ This discrete function can be written as

$$a = \{100, 101, 102, 103, 104, 105, 6^2, 7^2, 8^2, 9^2, \cdots\}$$

5.6.1 Basic Operations on Numeric Functions

Let a and b be two numeric functions. Then

i) Addition : c = a + b is also a numeric function defined as

$$c_r = a_r + b_r$$

i.e. $c_0 = a_0 + b_0, c_1 = a_1 + b_1, c_2 = a_2 + b_2$ and so on.

ii) Multiplication : d = ab is also a numeric function defined as $d_r = a_r b_r$, $\forall \in \mathbb{N} \cup \{0\}$.

iii) Scaling : Let k be any real number then b = ak is a numeric function and k is called as scalling factor.

iv) Linearity : Let p and q be any real numbers, then x = pq + qb is a numeric function where

$$x_r = p\, a_r + q\, b_r$$

v) The convolution of two numeric functions : Let a and b be two numeric functions. The convolution of two functions a and b is denoted by c = a * b and defined as

$$c_r = a_0\, b_r + a_1\, b_{r-1} + a_2\, b_{r-2} + \ldots + a_r b_0 = \sum_{n=0}^{r} a_n\, b_{r-n}$$

Example 1 :

If $$a_r = \frac{1}{3^r},\ r \ge 0 \text{ then } a = \left\{1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\right\}$$

and $$5a = \left\{5, \frac{5}{3}, \frac{5}{3^2}, \frac{5}{3^3}, \cdots\right\}$$

∴ $$5\, a_r = 5 \cdot \frac{1}{3^r}$$

Example 2 :

If $$a_r = \begin{cases} 1 & \text{if } 0 \le r \le 2 \\ 3r & \text{if } r \ge 3 \end{cases} \quad \text{and} \quad b_r = \begin{cases} 2^r + 1 & , \ 0 \le r \le 1 \\ r - 5 & , \ r \ge 2 \end{cases}$$

then $$a + b = c$$

$$\therefore \quad c_r = a_r + b_r = \begin{cases} 2^{r+1} + 2 & , \ 0 \le r \le 1 \\ 1 + (-3) & , \ r = 2 \\ 3r + r - 5 & , \ r \ge 3 \end{cases}$$

$$= \begin{cases} 2^{r+1} + 2 & , \ 0 \le r \le 1 \\ -2 & , \ r = 2 \\ 4r - 5 & , \ r \ge 3 \end{cases}$$

and $$d = ab \text{ and } d_r = a_r \, b_r = \begin{cases} 1\,(2^r + 1) & , \ 0 \le r \le 1 \\ 1\,(-3) & , \ r = 2 \\ 3r\,(r - 5) & , \ r \ge 3 \end{cases}$$

$$d_r = \begin{cases} 2^{r+1} & , \ 0 \le r \le 1 \\ -3 & , \ r = 2 \\ 3r^2 - 15r & , \ r \ge 3 \end{cases}$$

Example 3 :

If $$a_r = 3^r, \ r \ge 0$$

$$b_r = 5^r, \ r \ge 0$$

then $$c = a * b \Rightarrow c_r = \sum_{n=0}^{r} a_n \, b_{r-n} = a_0 \, b_r + a_1 \, b_{r-1} + a_2 \, b_{r-2} + \cdots + a_r \, b_0$$

5.6.2 Finite Differences of a Numeric Functions

I) Shift of a numeric function

The shift of a numeric function of a_r is denoted by $E\,[a_r]$ and defined as

$$E\,[a_r] = a_{r+1}$$

$$E^{-1}\,[a_r] = a_{r-1}$$

$$E^n\,[a_r] = a_{r+n}, \qquad E^{-n}\,[a_r] = a_{r-n}$$

Note : It is also denoted by $s^n\,[a_r] = E^n\,[a_r]$

Example 1 :

If $a_r = 5^r, r \geq 0$ then $E[a_r] = a_{r+1} = 5^{r+1}$

$E^2[a_r] = a_{r+2} = 5^{r+2}, E^{-n}[a_r] = a_{r-n} = 5^{r-n}$

Example 2 :

If $$a_r = \begin{cases} r^3 - 2r + 5 & ; \ 0 \leq r \leq 6 \\ 3 & ; \ r \geq 7 \end{cases}$$

then $$E^3[a_r] = \begin{cases} 0 & , \ 0 \leq r < 2 \\ (r+3)^3 - 2(r+3) + 5 & , \ 3 \leq r \leq 9 \\ 3 & , \ r \geq 10 \end{cases}$$

and $$E^{-2}[a_r = \begin{cases} (r-2)^3 - 2(r-2) + 5 & ; \ 0 \leq r \leq 4 \\ 3 & ; \ r \geq 5 \end{cases}$$

II) Forward difference of a numeric function

The forward difference of a numeric function a_r is denoted by Δa_r and defined as

$$\Delta a_r = a_{r+1} - a_r \ ; r \geq 0$$

Example 1 :

If $a_r = 3^r \ ; r \geq 0$ then

$$\Delta a_r = a_{r+1} - a_r = 3^{r+1} - 3^r$$

$$= (3-1)\, 3^r = 2.3^r$$

Example 2 :

If $$a_r = \begin{cases} 3 & ; \ 0 \leq r \leq 4 \\ r+2 & ; \ r \geq 5 \end{cases}$$

then $$\Delta a_r = a_{r+1} - a_r$$

$$\therefore \quad a_{r+1} = \begin{cases} 3 & ; \ 0 \leq r+1 \leq 4 \\ (r+1)+2 & ; \ r+1 \geq 5 \end{cases}$$

$$a_{r+1} = \begin{cases} 3 & ; \ 0 \leq r \leq 3 \\ r+3 & ; \ r \geq 4 \end{cases}$$

$$\therefore \quad \Delta a_r = \begin{cases} 0 & ; \ 0 \leq r \leq 3 \\ -3 + (4+3) & ; \ r = 4 \\ (r+3) - (r+2) & ; \ r \geq 5 \end{cases}$$

$$\Delta a_r = \begin{cases} 0 \; ; & 0 \le r \le 3 \\ 4 \; ; & r = 4 \\ 1 \; ; & r \ge 5 \end{cases}$$

III) Backward difference of a numeric function

The backward difference of a numeric function a_r is denoted by ∇a_r and defined as

$$\nabla a_r = a_r - a_{r-1} \; ; r \ge 1$$

Examples

Example 1 :

Let $a_r = 6^r \; ; r \ge 0$

$a_{r-1} = 6^{r-1} \; ; r - 1 \ge 0$ i.e. $r \ge 1$

$\therefore \quad \nabla a_r = a_r - a_{r-1} = 6^r - 6^{r-1} \; ; r \ge 1$

$= (6 - 1)\, 6^{r-1} \; ; r \ge 1$

$\nabla a_r = 5 \cdot 6^{r-1} \; ; r \ge 1$

Example 2 :

If $a_r = \begin{cases} 1 \; ; & 0 \le r \le 2 \\ 3^r \; ; & r \ge 3 \end{cases}$

$$a_{r-1} = \begin{cases} 1 \; ; & 0 \le r-1 \le 2 \\ 3^{r-1} \; ; & r-1 \ge 3 \end{cases}$$

$$= \begin{cases} 1 \; ; & 1 \le r \le 3 \\ 3^{r-1} \; ; & r \ge 4 \end{cases}$$

$$\nabla a_r = a_r - a_{r-1} = \begin{cases} 0 & ; r = 0 \\ 1 & ; r = 1 \\ 0 & ; r = 2 \\ 26 & ; r = 3 \\ 2 \times 3^{r-1} & ; r \ge 4 \end{cases}$$

Example 5.6.1 *Determine $a * b$ for the following numeric functions*

$$a_r = \begin{cases} 1 \; ; & 0 \le r \le 2 \\ 0 \; ; & r \ge 3 \end{cases} \quad \text{and} \quad b_r = \begin{cases} r+1 \; ; & 0 \le r \le 2 \\ 0 \; ; & r \ge 3 \end{cases}$$

Solution : By the definition of a_r, the numeric function of a is given by

$$a = \{1, 1, 1, 0, 0, 0, \ldots\} = \{a_0, a_1, a_2, a_3, \ldots\}$$

$$b = \{1, 2, 3, 0, 0, 0 \ldots\} = \{b_0, b_1, b_2, b_3, \ldots\}$$

The convolution of a and b is a numeric function c such that

$$c = a * b \quad \text{where} \quad c_r = \sum_{n=0}^{r} a_n \, b_{r-n}$$

$$\therefore \quad c_r = a_0 \, b_r + a_1 \, b_{r-1} + a_2 \, b_{r-2} + \ldots + a_r \, b_0$$

$$\therefore \quad c_0 = a_0 \, b_0 = 1 \times 1 = 1$$

$$c_1 = a_0 \, b_1 + a_1 \, b_0 = 1.2 + 1.1 = 3$$

$$c_2 = a_0 \, b_2 + a_1 \, b_1 + a_2 \, b_0 = 1.3 + 1.2 + 1.1 = 6$$

$$c_3 = a_0 \, b_3 + a_1 \, b_2 + a_2 \, b_1 + a_3 \, b_0$$

$$= 1.0 + 1.3 + 1.2 + 1.0 = 5$$

$$c_4 = a_0 \, b_4 + a_1 \, b_3 + a_2 \, b_2 + a_3 \, b_1 + a_4 \, b_0$$

$$c_4 = 0 + 0 + 1.3 + 0 + 0 = 3$$

$$c_5 = 0, \; c_6 = 0, \; c_r = 0 \; ; \; r \geq 5$$

Thus the numeric sequence of c is given by

$$c = \{1, 3, 6, 5, 3, 0, 0, 0, \ldots\}$$

$$\therefore \quad c_r = \begin{cases} 1 & ; \; r = 0 \\ 3 & ; \; r = 1 \\ 6 & ; \; r = 2 \\ 5 & ; \; r = 3 \\ 3 & ; \; r = 4 \\ 0 & ; \; r \geq 5 \end{cases}$$

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Notes

SUBJECT CODE : 210241

As per Revised Syllabus of

SAVITRIBAI PHULE PUNE UNIVERSITY

Choice Based Credit System (CBCS)

S.E. (Computer) Semester - I

DISCRETE MATHEMATICS

(For END SEM Exam - 70 Marks)

Dr. H. R. Bhapkar

M.Sc. SET, Ph.D. (Mathematics)
MIT Art, Design and Technology University's
MIT School of Engineering, Lonikalbhor, Pune
Email : drhrbhapkar@gmail.com
Mobile : 9011227141

Dr. Rajesh Nandkumar Phursule

Ph.D. (Computer Science and Engineering)
Associate Professor,
Pimpri Chinchwad College of Engineering
Nigdi, Pune.

TECHNICAL PUBLICATIONS
SINCE 1993
An Up-Thrust for Knowledge

DISCRETE MATHEMATICS

(For END SEM Exam - 70 Marks)

Subject Code : 210241

S.E. (Computer) Semester - I

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Published by :

TECHNICAL PUBLICATIONS
An Up-Thrust for Knowledge

Amit Residency, Office No.1, 412, Shaniwar Peth,
Pune - 411030, M.S. INDIA, Ph.: +91-020-24495496/97
Email : sales@technicalpublications.org Website : www.technicalpublications.org

Printer :
Yogiraj Printers & Binders
Sr.No. 10/1A,
Ghule Industrial Estate, Nanded Village Road,
Tal. - Haveli, Dist. - Pune - 411041.

ISBN 978-93-332-2155-9

9789333221559

SPPU 19

9789333221559 [1]

PREFACE

The importance of **Discrete Mathematics** is well known in various engineering fields. Overwhelming response to our books on various subjects inspired us to write this book. The book is structured to cover the key aspects of the subject **Discrete Mathematics.**

The book uses plain, lucid language to explain fundamentals of this subject. The book provides logical method of explaining various complicated concepts and stepwise methods to explain the important topics. Each chapter is well supported with necessary illustrations, practical examples and solved problems. All the chapters in the book are arranged in a proper sequence that permits each topic to build upon earlier studies. All care has been taken to make students comfortable in understanding the basic concepts of the subject.

Representative questions have been added at the end of each chapter to help the students in picking important points from that chapter.

The book not only covers the entire scope of the subject but explains the philosophy of the subject. This makes the understanding of this subject more clear and makes it more interesting. The book will be very useful not only to the students but also to the subject teachers. The students have to omit nothing and possibly have to cover nothing more.

We wish to express our profound thanks to all those who helped in making this book a reality. Much needed moral support and encouragement is provided on numerous occasions by our whole family. We wish to thank the **Publisher** and the entire team of **Technical Publications** who have taken immense pain to get this book in time with quality printing.

Any suggestion for the improvement of the book will be acknowledged and well appreciated.

Authors
Dr. H. R. Bhapkar
Dr. Rajesh N. Phursule

Dedicated to the Readers of the Book

SYLLABUS

Discrete Mathematics - (210241)

Credit Scheme	Examination Scheme and Marks
03	End_Semester (TH) : 70 Marks

Unit III Counting Principles

The Basics of Counting, rule of Sum and Product, Permutations and Combinations, Binomial Coefficients and Identities, Generalized Permutations and Combinations, Algorithms for generating Permutations and Combinations. **(Chapter-6)**

Unit IV Graph Theory

Graph Terminology and Special Types of Graphs, Representing Graphs and Graph Isomorphism, Connectivity, Euler and Hamilton Paths, the handshaking lemma, Single source shortest path- Dijkstra's Algorithm, Planar Graphs, Graph Colouring. **(Chapter-7)**

Unit V Trees

Introduction, properties of trees, Binary search tree, tree traversal, decisicon tree, prefix codes and Huffman coding, cut sets, Spanning Trees and Minimum Spanning Tree, Kruskal's and Prim's algorithms, The Max flow-Min Cut Theorem (Transport network). **(Chapter-8)**

Unit VI Algebraic Structures and Coding Theory

The structure of algebra, Algebraic Systems, Semi Groups, Monoids, Groups, Homomorphism and Normal Subgroups, and Congruence relations, Rings, Integral Domains and Fields, Coding theory, Polynomial Rings and polynomial Codes, Galois Theory -Field Theory and Group Theory. **(Chapter-9)**

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UNIT - III

6 Counting Principles

Syllabus

The Basics of Counting, rule of Sum and Product, Permutations and Combinations, Binomial Coefficients and Identities, Generalized Permutations and Combinations, Algorithms for generating Permutations and Combinations. Case Studies : Study Sudoku Solving algorithm for generation of new SUDOKU, Study Hand-shake puzzle and algorithm to solve it.

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6.6 Binomical Coefficients

6.7 Case Studies

6.8 University Questions with Answers

6.1 Introduction

Permutations and combinations deal with counting and enumeration of specified objects, patterns or designs. All techniques of the counting are very important in computer science and mathematics. In this chapter we will study basic of counting. We also develop basic ideas of permutations, combinations and discrete probability. Let us begin our study with two basic counting principles.

6.2 Basic Counting Principles

SPPU : Dec.-17

6.2.1 Sum Rule (Principle of Disjunctive Counting)

We know that, if $S = A \cup B$ and A and B are disjoint sets i.e. $A \cap B = \phi$ then

$$|S| = |A| + |B|$$

i.e. A and B are disjoint partitions of S.

Now we can extend this logic to state sum rule.

Sum rule : If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes and E_1 and E_2 are disjoint (exclusive) then there are $n_1 + n_2$ possible outcomes when E_1 or E_2 take place.

Example 6.2.1 *How many ways can we select a girl or a boy representative from the class of 30 girls and 40 boys ?*

Solution :

$\Rightarrow$ Let E_1 = Set of girls in a class

E_2 = Set of boys in a class

$\therefore$ $|E_1| = 30$, $|E_2| = 40$, E_1 and E_2 are disjoint sets.

$\therefore$ $30 + 40 = 70$ ways.

$\Rightarrow$ A girl or a boy representative can be selected by 70 different ways.

Example 6.2.2 *How many ways can we get a sum of 7 or 11 when two different dice are rolled ?*

Solution : Two dice are different, therefore the ordered pairs (a, b) and (b, a) are different when $a \neq b$.

E_1 = Set of ordered pairs in which the sum is 7

$\therefore$ $E_1 = \{(1, 6), (2, 5), (3, 4) (4, 3), (5, 2), (1, 6)\}$

There are 6 ways to get sum 7

E_2 = Set of ordered pairs whose sum is 11.

$E_2 = \{(5, 6), (6, 5)\}$

$\therefore$ There are 2 ways to get sum 11.

E_1 and E_2 are mutually exclusive. i.e. $E_1 \cap E_2 = \phi$

Therefore we can get sum 7 or 11 in 6 + 2 = 8 different ways.

Note : We can extend this principle for finite number of experiments.

6.2.2 Product Rule (The Principle of Sequential Counting)

We have, if A and B are non empty sets and $|A| = n$, $|B| = n$. then the number of elements in the cartesian product of A and B is equal to $n \times m$

i.e. $|A \times B| = n \times m$

Now extend this analogy for the product rule.

Product Rule : If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes then there are $n_1 \cdot n_2$ possible outcomes when the sequence of experiment E_1 first followed by E_2.

Example 6.2.3 *How many ways can we select a girl and a boy representatives from the class of 30 girls and 40 boys ?*

Solution : A girl representative can be selected by 30 ways.

A boy representative can be selected by 40 ways.

Therefore, a girl and a boy representative can be selected by $30 \times 40 = 1200$ different ways.

Example 6.2.4 *In how many different ways one can answer all questions of a true or false test consisting of 8 questions ?*

Solution : There are two ways of answering each question {T, F}.

Therefore by product rule, the number of ways in which all the 8 questions can be answered $= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 =$ **256** ways

6.3 Permutations

SPPU : Dec.-05, 14, May-19

An arrangement in a sequence of elements of a set is called a permutation of elements.

Depending upon the nature of arrangements, there are three types of permutations.

Type I) Permutations when all objects are distinct : A permutation of n objects taken r at a time is an arrangement of r objects out of n objects where $r \leq n$.

It is called r - permutations or r - arrangements and denoted by P (n, r) or ${}^nP_r$.

e.g. 1) Consider the three letters a, b, c. The arrangements of the letters a, b, c taken two at a time are ab, ba, ac, ca, bc, cb.

∴The number of 2 - arrangements are 6. i.e. the number of permutations of 3 symbols taken two at a time = ${}^{3}P_2 = 6$.

Therefore as discussed above, the first place in the sequence can be filled up in n - ways, the second place in (n – 1) ways, the third place in (n – 2) ways and proceeding in this manner the rth place can be filled up in n – (r – 1) = n – r + 1 ways.

$$\text{Hence } {}^{n}P_r = n\cdot(n-1)\cdot(n-2)\ldots(n-(r-1))$$

$$= \frac{n\cdot(n-1)(n-2)\ldots(n-r+1)\times(n-r)!}{(n-r)!}$$

$$\boxed{{}^{n}P_r = \frac{n!}{(n-r)!}\ ;\ 0 \le r \le n}$$

Properties : 1) $${}^{n}P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

2) $${}^{n}P_1 = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$$

$${}^{n}P_2 = \frac{n!}{(n-2)!} = \frac{n(n-1)[(n-2)!]}{(n-2)!} = n(n-1)$$

$${}^{n}P_3 = n(n-1)(n-2) \text{ and so on.}$$

3) $$0! = 1$$

Example 6.3.1 *Given that A = {1, 2, 3, 4, 5, 6}, find the number of permutations of A taken i) 2 at a time ii) 3 at a time iii) 4 at a time iv) 5 at a time v) 6 at a time.*

Solution :

We have $A = \{1, 2, 3, 4, 5, 6\}$

∴ $|A| = 6$

i) The permutation at 6 letters taken 2 at a time is ${}^{6}P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6\times5\times4!}{4!} = 30$

ii) The permutation of 6 letters taken 3 at a time is ${}^{6}P_3 = \frac{6!}{(6-3)!} = \frac{6\times5\times4\times3!}{3!} = 120$

Similarly,

iii) $${}^{6}P_4 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = 30$$

iv) $${}^{6}P_5 = \frac{6!}{(6-5)!} = \frac{6!}{1!} = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

v) $${}^{6}P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6! = 720$$

Example 6.3.2 *How many four digit numbers can be formed out of digits 1, 2, 3, ...9. if*

i) No repetition is permitted.

ii) How many of these will be greater than 3000.

Solution :

i) The number of ways of selecting 4 digits out of 9 digits is

$${}^{9}P_4 = \frac{9!}{(9-4)!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{5!} = 3024$$

ii) There is a restriction that the 4 digit numbers so formed must be greater than 3000.

Therefore the thousandth position can be filled with numbers 3, 4, 5, 6, 7, 8, 9 i.e. the thousandth place can be selected in 7 different ways.

Now out of remaining 8 digits, hundredth position can be filled in 8 different ways, Tenth place can be filled in 7 different ways and unit place can be filled in 6 different ways.

Thus the total number of 4 digit numbers greater than 3000 can be formed in

$7 \times 8 \times 7 \times 6 = 2352$ ways.

Example 6.3.3 *i) Suppose repetitions are not permittable, how many four digit numbers can be formed from six digits 1, 2, 3, 5, 7, 8 ?*

ii) How many of such numbers are less than 4000 ?

iii) How many in (i) are even ?

iv) How many in (ii) are odd ?

v) How many in (i) contain both 3 and 5.

vi) How many in (i) are divisible by 10. **SPPU : Dec.-05**

Solution :

i) Out of 6 numbers, 4 digit numbers can be formed in ${}^{6}P_4$ ways.

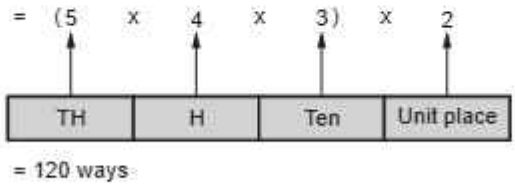
$$\therefore \quad \text{Number of ways} = \frac{6!}{2!} = 360$$

ii) The four digit numbers which are less than 4000 are the numbers in which first digit is 1, 2 or 3 i.e. 1st digit can be chosen in 3 ways, 2nd digit can be any one of the remaining 5 digits. 3rd digit can be any of the remaining 4 digit and the 4th digit is any one of the remaining 3 digits.

Hence the total number of ways = $3 \times 5 \times 4 \times 3 = 180$.

iii) Those numbers ending in 2 or 8 are even numbers. Hence the last digit (4th digit) can be chosen in 2 ways (the number 2 or 8). The first digit can be chosen in any one of the remaining 5 digits, 2nd in any of the 4 digits and 3rd in any of the 3 digits. Hence the total number of ways

iv) The numbers less than 4000 and are odd. The numbers ending with 1, 3, 5 or 7 are odd. The 4 digit numbers ending in 1 and less than 4000 should begin with eithor 2 or 3.



Then there are $2 \times 4 \times 3 \times 1 = 24$ such numbers. Similarly the number ending in 3 are 24. However the number ending with 5 or 7 are

TH UnitPlace

$\overset{1}{3} \times 4 \times 3 \times \overset{1}{2}$

= 72 ways.

Hence the total number which are odd and less than 4000 are 24 + 24 + 72 = 120

v) The digit 3 can occupy any of the 4 positions and the remaining 3 positions will be occupied by the digit 5. Hence the number of ways in which two positions are occupied by 3 and 5 will be 4×3 i.e. 12.

Now the remaining two positions will be filled by the remaining 4 numbers i.e. 1, 2, 7 and 8.

Hence out of remaining two positions one position can be occupied in 4 different ways and the remaining position will be occupied in 3 different ways.

Hence total number of 4 digit numbers in which both 3 and 5 are present = $12 \times 4 \times 3 = 144$.

vi) Not even a single number is divisible by 10 as there is no zero at unit's place.

Example 6.3.4 *A menu card in a restaurant displays four soups, five main courses, three desserts and 5 beverages. How many different menus can a customer select if,*

i) He selects one item from each group without omission.

ii) He chooses to omit the beverages, but selects one each from the other groups.

iii) He choose to omit the desserts but decides to take a beverage and one item each from the remaining groups.

Solution :

i) The customer can select the soup in 4 ways, the main course in 5 ways, the dessert in 3 ways and beverages in 5 ways.

Hence by product rule, the number of ways in which he can select one item each, without omission is $4 \times 5 \times 3 \times 5 = 300$.

ii) The number of ways in which he omit beverages = $4 \times 5 \times 3 = 60$ ways.

iii) The number of ways in which he omit desserts but he takes all other items = $4 \times 5 \times 5 = 100$ ways.

Example 6.3.5 *10 different M_1 books, 3 different M_2 books, 5 different M_3 books and 7 different D.S. books are to be arranged on a shelf. How many different arrangements are possible if*

i) The books in each subject must all be together

ii) Only M_3 books must be together.

Solution : i) M_1 books can be arranged among themselves in 10 ! ways, the M_2 books in 3 ! ways, M_3 books in 5 ! ways and D.S. books in 7 ! ways.

Hence the total number of arrangements = $4! \cdot 10! \cdot 3! \cdot 5! \cdot 7!$

ii) Consider the 5 M_3 books as a single book. Then there are 21 books which can be arranged in 2 ! ways. In each of these arrangements the M_3 books can be arranged among themselves in 5! ways.

Hence the number of arrangements in $5! \cdot 2!$

Example 6.3.6 *2 mathematics papers and 5 other papers are to be arranged at an examination. Find the total number of ways if,*

i) Mathematics papers are consecutive. ii) Mathematics papers are not consecutive.

SPPU : May-19, Marks 3

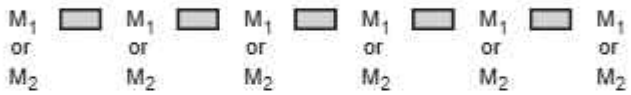
Solution : i) Both mathematics papers (M_1 and M_2) are together, consider both M_1 and M_2 as single paper.

These two papers among themselves can be arranged in 2! ways.

Now 6 papers (as M_1 and M_2 is considered as single paper) can be arranged in 6! ways.

Hence total number of arrangements = $2! \cdot 6!$

ii) If M_1 or M_2 are not consecutive than they are to be arranged between the 4 gaps or at the 2 ends.



Where ▭ denotes other papers

Hence there are 6 places where mathematics papers can be arranged. Therefore, 2 mathematics papers can be arranged in 6 places in ${}^6P_2$ ways. Five other papers can be arranged among themselves in 5! ways.

Therefore total number of arrangements

$$= 5! \cdot {}^6P_2 = 5! \cdot 6 \cdot 5$$

$$= (120) \cdot (30) = 3600$$

Example 6.3.7 *How many permutations can be made out of the letter of word "COMPUTER" ? How many of these*
i) begin with C ii) end with R iii) begin with C and end with R iv) C and R occupy the end places

Solution : There are 8 letters in the word "COMPUTER" and all are distinct

$\therefore$ The total number of permutations of these letters is 8! = 40320.

i) Permutations begin with C :

The first position can be filled in only one way i.e. C and the remaining 7 letters can be arranged in 7! ways.

$\therefore$ The total no. of permutations begining with C = $1 \times 7! = 5040$

ii) Permutations end with R :

The Last position can be filled in only one way and the remaining 7 letters can arranged in 7! ways.

$\therefore$ The total no. of permutations ending with R be = $7! \times 1 = 5040$

iii) Permutation begin with C and end with R :

The first position can be filled in only one way i.e. C and the end position also can be filled in only one way i.e. R and the remaining 6 letters can be arranged in 6! ways.

$\therefore$ The required no. of permutations = $1 \times 6! \times 1 = 720$

iv) Permutation in which C and R occupy end places :

C and R occupy end positions in 2 ! ways i.e. CR or RC and the remaining 6 letters can be arranged in 6! ways.

$\therefore$ The total no. of required permutations = $2! \times 6! = 1440$

Permutations with restrictions :

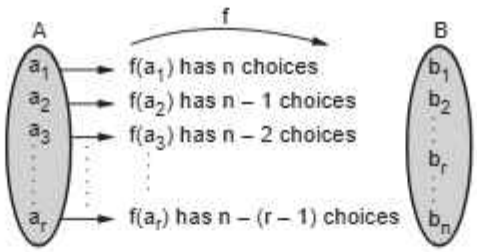
1) The number of permutations of n different objects taken r at a time in which p particular objects do not occur is ${}^{(n-p)}P_r$.

2) The number of permutations of n different objects taken r at a time in which p particular objects are present is ${}^{(n-p)}P_{r-p} \times {}^{r}P_p$.

Example 6.3.8 *Show that the number of injective functions from a set with r elements to a set with n elements is ${}^{n}P_r$; $r \le n$.*

Solution : Let A and B be two sets with $|A| = r$ and $|B| = n$.

$$A = \{a_1, a_2, \ldots a_r\} \quad \text{and} \quad B = \{b_1, b_2, b_3 \ldots b_r, \ldots b_n\}$$



Let $f : A \to B$ be an injective function

Hence by product rule, the number of injective functions from A to B is $n(n-1)(n-2)\ldots(n-r+1) = {}^{n}P_r$

Type II) Permutations when all objects are not distinct

A) If r_1 objects are of one kind

r_2 objects are of second kind

r_3 objects are of third kind

$\vdots$

r_k objects are of kth kind, where $r_1 + r_2 + \ldots r_k = n$

Then the number of permutations when all are taken at a time (i.e. $r_1, r_2, \ldots r_k$ are taken) is $\dfrac{n!}{r_1!\, r_2!\, r_3! \ldots r_k!}$

Example 6.3.9 *Find the number of permutations that can be made out of the letters i) MISSISSIPPI ii) ASSASSINATION* **SPPU : Dec.-14**

Solution : i) There are 11 letters in the word out of which S, I, P, M are distinct.

S appears 4 times

I appears 4 times

P appears 2 times

M appears 1 time

$\therefore$ The required no. of permutations $= \dfrac{11\ !}{4!\ 4!\ 2!\ 1!} = 34650$

ii) There are 13 letters of which A, I, N, S, T and O are different.

A appears 3 times

I appears 2 times

N appears 2 times

S appears 4 times

T appears 1 time

O appears 1 time

$\therefore$ The required no. of permutations $= \dfrac{13\ !}{3!\ 2!\ 2!\ 4!\ 1!\ 1!} = 10810800$

Example 6.3.10 *How many ways can the letters in the word "PIONEER" be arranged so that the two E's are always together.*

Solution : The word 'PIONEER' has two E's and remaining 5 letters are distinct. These distinct five letters can be arranged in 5 ways and for each such arrangement two E's can occupy any of the six remaining places. Hence the required no. of permutations are

$$6 \times 5! = 6! = 720.$$

Example 6.3.11 *How many seven digit numbers can be formed using digits 1, 7, 2, 7, 6, 7, 6 ?*

Solution : There are 7 digits out of which 7 is repeated 3 times, 6 is repeated twice, appears 1 time and 2 appear 1 time.

$\therefore$ The total no. of permutation $= \dfrac{7\,!}{3! \times 2! \times 1! \times 1!} = 420$

Type III) Permutations with repeated objects.

The number of different permutations of n distinct objects taken r at a time when every object is allowed to repeat any number of times is given by n^r.

Example 6.3.12 *How many 4 digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits are allowed ?*

Solution : We have 4 digits numbers.

No. of ways of filling unit's place = 4

No. of ways of filling ten's place = 4

No. of ways of filling hundred's place = 4

No. of ways of filling thousand's place = 4

Therefore the total number of 4 digits numbers = $4 \times 4 \times 4 \times 4 = 4^4 = 256$

Example 6.3.13 *How many 4 digits even numbers can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits are allowed ?*

Solution : We have 3 even numbers and 2 odd numbers. Therefore,

The no. of ways of filling unit's place = 3

The no. of ways of filling ten's place = 5

The no. of ways of filling hundred's place = 5

The no. of ways of filling thousand's place = 5

Thus, the total no. of required 4 digits numbers = $3 \times 5^3 = 3 \times 125 = 375$

Example 6.3.14 *In how many ways can 5 software projects be alloted to 6 final year students when all of five projects are not alloted to the same student.*

Solution : We have 5 projects and 6 students. Each project can be alloted in 6 ways.

Thus, the number of ways of alloting 5 projects = $6 \times 6 \times 6 \times 6 \times 6 = 6^5$

The number of ways in which all 5 projects are allotted to same student = 6.

Therefore, total number of ways to allocate 5 projects to 6 students = $6^5 - 6 = 7770$

Example 6.3.15 *A bit is either 0 or 1. A byte is a sequence of 8 bits.*
Find i) Number of bytes ii) Number of bytes that begin with 11 and end with 11.

Solution : i) Total number of byte is $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$.

ii) As the first two and last two bits are fixed i.e. 11 the remaining bits in the sequence are either 0 or 1.

$\therefore$ The required no. of total bytes = $2^4 = 16$.

Example 6.3.16 *Prove that the number of circular permutations of n different objects is $(n - 1)!$*

Solution : Let us consider that k be the number of required permutations.

For each such circular permutation of k, there are n corresponding linear permutations. We can start from every object of n objects in the circular permutation. Thus for k circular permutations, we have kn linear permutations.

Therefore $k \cdot n = n! \Rightarrow k = \frac{n!}{n} = (n-1)!$ Hence the proof.

Example 6.3.17 *How many ways can these letters A, B, C, D, E and F be arranged in a circle ?*

Solution : There are six letters. Hence the no. of ways to arrange these six letters in a circle is $(6-1)! = 5! = 120$.

Example 6.3.18 *In how many ways 10 programmers can sit on a round table to discuss the project so that project leader and a particular programmer always sit together ?*

Solution : There are 10 programmers. But project leader and particular programmer always sit together. So both become a single unit and hence there are

$(10 - 2 + 1) = 9$ remins

Thus these 9 units can be arreanged on round table in $(9-1)! = 8!$ ways.

The two programmer i.e. project leader and particular programmer can be arranged in 2! ways.

Therefore the required no. of ways $= 2! \times 8! = 80640$.

Example 6.3.19 *Determine the number of ways in which 5 software engineers and 6 electronics engineers can be sitted at a round table so that no two software engineers can sit together.* **SPPU : Dec.-16, 17, Marks 6**

Solution : There are 6 electronics engineers that can be arranged round a table in $(6-1)!$ ways. There are 5 software engineers and they are not to sit together, so there are six places for software engineers and can be placed in 6! ways as shown in Fig. 6.3.1

Therefore the required no. of ways $= (6-1)! \times 6! = 5! \times 6!$

$= 120 \times 720 = 86400$

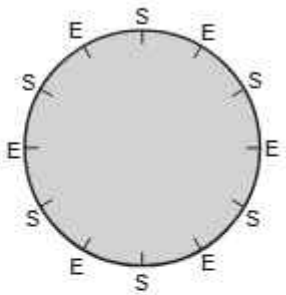


Fig. 6.3.1

6.4 Combination

SPPU : May-06, 14

A combination is a selection of some or all objects from a set of given objects where the order of the objects does not matter. In this context, we used mainly two words "Selection" and "arrangement". In selection, order of objects is immaterial i.e. selection is a set. But in arrangement, the order of objects is important it is not a set. Arrangement is a n-tupple. Arrangement is associated with permutation selection with combination.

I) Definition : The number of combinations of n different objects taken r at a time is given by ${}^nC_r$ and defined as

$$ {}^nC_r = \frac{n!}{r!\,(n-r)!} \; ; \; r \le n $$

Properties :

1) $$ {}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!} $$

2) $$ {}^nC_n = \frac{n!}{n!\,(n-n)!} = 1 $$

3) $$ {}^nC_0 = \frac{n!}{0!\,(n-0)!} = \frac{n!}{n!} = 1 $$

4) $$ {}^nC_1 = \frac{n!}{1!\,(n-1)!} = n \;,\; {}^nC_2 = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2} $$

$$ {}^nC_2 = \frac{n(n-1)(n-2)}{3!} \;,\; {}^nC_4 = \frac{n(n-1)(n-2)(n-3)}{4!} $$

Example 6.4.1 *Find the value of n if i) ${}^nC_{n-2} = 10$ and ii) ${}^{25}C_{n+2} = {}^{25}C_{2n-1}$*

Solution :

i) We have $ {}^nC_{n-2} = 10 \Rightarrow \dfrac{n!}{(n-2)!\,(n-(n-2))!} = 10$

$$ \therefore \quad \frac{n(n-1)(n-2)!}{(n-2)!\,(2!)} = 10 $$

$$ n(n-1) = 10 \times 2 = 20 $$

$$ n^2 - n - 20 = 0 $$

$$ \Rightarrow \quad (n-5)(n+4) = 0 $$

$n = 5$ or $n = -4$ as $n = -4$ is not possible

$$ \boxed{n = 5} $$

ii) we know that ${}^nC_r = {}^nC_{n-2}$

Now ${}^{25}C_{n-2} = {}^{25}C_{n-2}$

$\Rightarrow$ Either $n + 2 = 2n - 1$ or $(n + 2) + (2n - 1) = 25$

$\therefore$ either $n = 3$ or $3n = 25 - 1 = 24$

$\therefore \quad n = 3$ or $n = 8$

$\therefore \quad n = 3, 8.$

Example 6.4.2 *How many 16 bit strings are there containing exactly five 0's ?*

Solution : Each string of 16 bit has 16 digits. A 16-bit string having exactly five 0's is determined if we tell which bit are 0's. So, here order is immaterial.

$\therefore$ This can be done in ${}^{16}C_5$ ways.

$\therefore$ The total number of 16-bit strings is

$$ {}^{16}C_5 = \frac{16!}{5! \times (16-5)!} = \frac{16 \times 15 \times 14 \times 13 \times 12 \times 11!}{5 \times 4 \times 3 \times 2 \times 1 \times 11!} = 4368 $$

Example 6.4.3 *In how many ways can 30 late admitted students be assigned to three practical batches A, B, C if A can accomodate 10 students, B - 15 students and C - 5 students only ?*

Solution : The batch A can accomodate 10 students out of 30.

$\therefore$ The batch A can be assigned 10 students in ${}^{30}C_{10}$ ways.

then batch B can be assigned 15 students in ${}^{20}C_{15}$ ways.

then batch C can be assigned 5 students in ${}^{5}C_5$ ways.

Therefore by the product rule, the total no of ways of assigned students is

$$ {}^{30}c_{10} \times {}^{20}c_{15} \times {}^{5}c_5 = \frac{30!}{10!\,(20!)} \times \frac{20!}{15! \times 5!} \times 1 = \frac{30!}{10!\,15!\,5!} $$

Example 6.4.4 *How many ways can we select a software development group of 1 project leader, 15 programmers and 6 data entry operators from a group of 5 project leaders 20 programmers and 25 data entry operators.*

Solution : One project leader can be selected from 5 project leaders in ${}^{5}C_1 = 5$ ways.

15 programmers can be selected from 20 programmers in ${}^{20}C_{15}$ ways.

6 data entry operators can be selected from 25 data entry operators in ${}^{25}C_6$ ways.

Therefore the total number of ways to select the software development group is

${}^{5}C_1 \times {}^{20}C_{15} \times {}^{25}C_6 = 96101544000.$

Example 6.4.5 *From 10 programmers in how many ways can 5 be selected when i) A particular programmer is included everytime ii) A particular programmer is not included at all time.*

Solution : The number of ways to select 5 programmers from 10 is ${}^{10}C_5 = \frac{10!}{5!\,5!}$

$$= \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} = 252$$

i) When a particular programmer is included every time then the remaining 5 – 1 = 4 programmers can be selected from the remaining 10 – 1 = 9 programmers. This can be done in ${}^{9}C_4$

$$\therefore \quad {}^{9}C_4 = \frac{9!}{4!\,5!} = \frac{9\times 8\times 7\times 5}{4\times 3\times 2} = 126$$

ii) When a particular programmer is not included at all then the 5 programmers can be selected from the remaining 10 – 1 = 9 programmers.

This can be done in ${}^{9}C_5$ ways.

$$\therefore \quad {}^{9}C_5 = \frac{9!}{4!\,5!} = 126$$

Example 6.4.6 *A committee of 5 people is to be formed from a group of 4 men and 7 women. How many possible committees can be formed if at least 3 women are on the committee ?*

SPPU : May-14

Solution : If at least three women are on committee, it means committee with 3 women or 4 women or 5 women.

i) 3 women can be selected in ${}^{7}C_3$ ways.

2 men can be selected in ${}^{4}C_2$ ways.

The no. of ways this can be done is ${}^{7}C_3 \times {}^{4}C_2 = 210$ ways.

ii) 4 women and 1 man can be selected in ${}^{7}C_4$ and ${}^{4}C_1$ ways respectively.

∴The number of ways to form a committee is ${}^{7}c_4 \times {}^{4}c_1 = 140$ ways.

iii) 5 women can be selected in ${}^{7}C_5$ ways.

∴The no. of ways to form a committee is ${}^{7}C_5 = 21$ ways

Hence the total no. of ways a committee can be formed with at least 3 women is

210 + 140 + 21 = 371.

Example 6.4.7 *How many automobile license plates can be made if each plate contains two different letters followed by three different digits. Solve the problem if the first digit can not be zero.*

SPPU : May-06

Solution : The first position is a letter and can be selected from 26 letters in

$$^{26}C_1 = 26 \text{ ways.}$$

The second position is a letter and can be selected from 26 – 1 = 25 letters in

$$^{25}C_1 = 25 \text{ ways.}$$

For digits :

1) The first digit can be selected from 10 digits in $^{10}C_1 = 10$ ways.

2) The second digit can be selected from 9 digits in $^{9}C_1 = 9$ ways

3) The third digit can be selected from 8 digits in $^{8}C_1 = 8$ ways.

Therefore the total number of license plates

$$= 26 \times 25 \times 10 \times 9 \times 8 = 468000$$

Now, in license plate, the first digit can not be 0 then the first position can selected from 9 digits in $^{9}C_1$ ways.

The second digit can be zero, but one digit is already selected for the first position. Hence the second digit can be selected in $^{9}C_1$ ways. The third digit can be selected in $^{8}C_1$ ways.

Hence the total no. of required license plates are

$26 \times 25 \times 9 \times 9 \times 8 = 421200.$

Example 6.4.8 *In the discrete structure paper, there are 10 questions. In how many ways can an examiner select five questions in all the first question is compulsory.*

Solution : The first question is compulsory, so the examiner has to select 4 questions from the remaining 9 questions.

∴ The number of ways to select five questions is

$$1 \times {}^{9}C_4 = \frac{9!}{4!\,5!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2} = 126 \text{ ways.}$$

Example 6.4.9 *Determine the number of triangles that are formed by selecting points from a set of 12 points out of which 5 are colinear.*

Solution : By using 12 points, the number of triangles formed is $^{12}C_3$.

As five points are colinear i.e. lie on same line, they do not form any triangle. Thus $^{5}C_3$ triangles are lost.

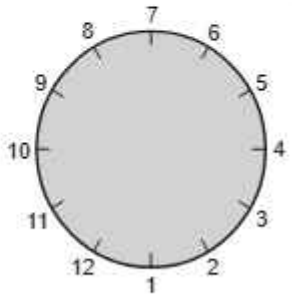
$\therefore$ The total number of triangles produced is

$$^{12}C_3 - {}^5C_3 = \frac{12!}{3! \times 9!} - \frac{5!}{3!\ 2!}$$

$$= \frac{12 \times 11 \times 10}{3 \times 2} - \frac{5 \times 4}{2} = 220 - 10 = 210$$

Example 6.4.10 *How many lines can be drawn through 12 points on a circle and line passes through exactly two points ?*

Solution : As all points on the circle are not colinear, thus no line will lost.



The total no. of lines drawn through 12 points is

$$^{12}C_2 = \frac{12\,!}{2!\ 10!} = \frac{12 \times 11}{2 \times 1} = 66$$

Example 6.4.11 *Determine the number of diagonals that can be drawn by joining the nodes of octgon.*

Solution : The number of lines that can be drawn by 2 points out of 8 points of octagon is $^8C_2 = 28$. Out of these 28 lines, 8 are the sides of the octagon.

$\therefore$ The number of diagonals = 28 – 8 = 20.

Example 6.4.12 *In a box, there are 40 floppy disks of which 4 are defective. Determine*

i) In how many ways we can select five floppy disks ?

ii) In how many ways we can select five non defective floppy disks ?

iii) In how many ways we can select five floppy disks containing exactly three defective disks ?

iv) In how many ways we can select five floppy disks containing at least 1 defective disk ?

Solution : i) There are 40 floppy disks out of which we have to select 5 floppy disks in $^{40}C_5$ ways.

$$^{40}C_5 = \frac{40\,!}{5! \times (40-5)\,!} = \frac{40 \times 39 \times 38 \times 37 \times 36}{5 \times 4 \times 3 \times 2 \times 1} = 658008$$

ii) There are 40 – 4 = 36 nondefective floppy disks out of which we have to select 5. This can be done in $^{36}C_5$ ways.

$$\therefore \quad ^{36}C_5 = \frac{36\,!}{5! \times (31)\,!} = \frac{36 \times 35 \times 34 \times 33 \times 32}{5 \times 4 \times 3 \times 2 \times 1} = 376992$$

iii) To select exactly three defective floppy disks out of 4 disks, we have $^4C_3$ ways and the remaining two floppy disks can be selected from 36 disks in $^{36}C_2$ ways.

Therefore, the required no. of ways = ${}^{4}C_3 \times {}^{36}C_2$

$$= \frac{4!}{3! \times 1!} \times \frac{36!}{2!\ 34!} = 4 \times \frac{36 \times 35}{2} = 2520$$

iv) There are 4 defective floppy disks out of which at least one must be selected. We know that the total number of ways to select 5 disks from 40 disks is ${}^{40}C_5$.

Also the number of ways to select 5 floppy disks with no defective is ${}^{36}C_5$ way.

Therefore the required no. of ways

$$= {}^{40}C_5 - {}^{36}C_5 = 658008 - 376992 = 281016$$

II) Combination with Repetitions : The number of ways to fill r slots from n categories with repetition allowed is c (n + r – 1, r) = c (n + r – 1, n – 1)

Therefore we have

$${}^{(n+r-1)}C_r = {}^{(n+r-1)}C_{n-1} = C(n + r - 1, r)$$

and C (n + r – 1, r) = The number of non-negative integers solution of

$x_1 + x_2 + \ldots + x_n = r$

= The number of ways of placing r indistinguishable balls in n numbered boxes.

= The number of binary numbers with n–1 one's and r zero's.

= C (n + r – 1 ; n – 1)

= The number of r-combinations of $\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$

Example 6.4.13 *How many 4 combinations of {1, 2, 3, 4, 5, 6} are there with unlimited repetition ?*

Solution : We have r = 4, n = 6

∴The number of 4 combinations of {1, 2, 3, 4, 5, 6} are

$$C(6 + 4 - 1, 4) = C(9, 4) = \frac{9!}{4!\ 5!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} = 126.$$

Example 6.4.14 *Find the number of 3-combinations of $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4\}$.*

Solution : We have n = 4, r = 3

∴The number of 3-combinations of the given set is

$$C(4 + 3 - 1, 3) = C(6, 3) = \frac{6!}{3!\ 3!} = \frac{6 \times 5 \times 4 \times 3!}{3 \times 2 \times 3!} = 20$$

Example 6.4.15 *The number of non negative integer solutions to $x_1 + x_2 + x_3 + x_4 = 20$*

Solution : we have r = 20, n = 4

∴The number of non negative integer solutions

$$= C(4 + 20 - 1, 20) = C(23, 20) = \frac{23!}{20! \times 3!} = \frac{23 \times 22 \times 21}{3 \times 2} = 1771$$

Example 6.4.16 *The number of ways of placing 8 similar balls in 5 numbered boxes.*

Solution : The number of ways of placing r = 8 similar balls in n = 5 boxes is

$$C(5 + 8 - 1, 8) = c(12, 8) = 495.$$

Example 6.4.17 *Find the number of binary numbers with six 1's and 4 zero's.*

Solution : The number of binary numbers with 6 one's and 4

$$\text{zero's} = C(6 + 4, 4) = C(10, 4) = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!} = 210$$

Example 6.4.18 *How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$ where each $x_i \geq 2$?*

Solution : Let $x_i = y_i + 2$ where $y_i \geq 0$.

we have $x_1 + x_2 + x_3 + x_4 + x_5 = 16$

iff $y_i + 2 + y_2 + 2 + y_3 + 2 + y_4 + 2 + y_5 + 2 = 16$

Iff $y_1 + y_2 + y_3 + y_4 + y_5 = 16 - 10 = 6$

Thus the number of integral solutions of given equation is the same as the number of integral solutions of $y_1 + y_2 + y_3 + y_4 + y_5 = 6$

There are C (5 – 1 + 6, 6) = C (10, 6) such solutions

$$\therefore \quad C(10, 6) = \frac{10!}{6! \times 4!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!} = 210$$

Example 6.4.19 *How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$.*

Solution : Let $x_1 = y_1 + 2, x_2 = y_2 + 3, x_3 = y_3 + 4, x_4 = y_4 + 2, x_5 = y_5 + 0$

$$\therefore \quad x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

$$\Rightarrow \quad y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 19$$

∴ The required number of integral solutions are

$$C(5 - 1 + 19, 19) = C(23, 19) = \frac{23 \times 22 \times 21 \times 20}{4 \times 3 \times 2 \times 1} = 8855$$

Theorem 1 : The number of integer solutions to $a_1 + a_2 + a_3 + \ldots + a_n = r$ when $a_1 \geq b_1$, $a_2 \geq b_2$, $a_3 \geq b_3$, ... $a_n \geq b_n$ is C $(n + r - 1 - b_1 - b_2 - b_3 - \ldots b_n,$ $r - b_1 - b_2 - b_3 - \ldots - b_n)$

Theorem 2 : The number of ways to select r things from n categories with b total restrictions on the r things is C $(n + r - 1 - b, r - b)$

Theorem 3 : The number of ways to select r things from n categories with atleast 1 thing from each category is C $(r - 1, r - n)$ $(\because b = n)$

6.5 Generation of Permutation and Combination

SPPU : May-18

I) Generation of permutations : Suppose we want to generate n! permutations of n distinct objects. For n = 1, 2, 3, it is simple but when n is large it is difficult to keep track of what we have written and make sure that we shall write down all permutations with no repetition or omissions.

An interesting problem is to find a systematic procedure for generating all n! permutations of a set with n distinct elements.

Suppose from the initial permutation 1, 2, 3, ... n by using the next permutation procedure repeatedly we shall obtain all the permutations of 1, 2, 3, ..., n. The last permutation is n, (n – 1), (n – 2), ... 4, 3, 2, 1.

Procedure for next permutation :

Step 1 : Given a permutation $a_1, a_2, a_3, \ldots a_n$ of 1, 2, 3, ..., n

Step 2 : Scan from right to left (L ← R). Find the first m such that $a_m < a_{m+1}$

Step 3 : $\alpha = \min \{a_k \mid k = m + 1, m + 2, m + 3, \ldots n, a_k > a_m\}$

Step 4 : The next permutation is

$a_1\ a_2\ a_3 \ldots a_{m-1}, \alpha, x, x, x \ldots$

where x, x, x ... are the remaining numbers arranged in the increasing order.

Algorithms :

I) Algorithm for generating the next Permutation in Lexicographic order :

Next permutation $[(a_1, a_2, \ldots, a_n)]$: Permutation of {1, 2, 3, ...n} not equal to (n, (n – 1) (n – 2) ... 3 2 1)]

$i := n - 1$

while $a_i > a_k$

$k := k - 1$ {a_k is the smallest integer greater than a_i to the right of a_i} interchange a_i and a_k

$r := n$

$s := j + 1$

while $r > s$

begin

interchange a_r and a_s

$r := r - 1$

$s = s + 1$

end

{This puts the tail end of permutation after the ith position in increasing order}

Example 6.5.1 *Let n = 6 and given permutation is 125364 find the next permutation.*

Solution : Step 1 : Given permutation is 125364

Step 2 : Scan from right to left and find the first m such that. $a_m < a_{m+1}$

Now start with 5th position ; As $6 > 4$, $m \neq 5$

As $3 < 6$ and position of 3 is 4th $\therefore m = 4$.

Step 3 : $\alpha = \min \{a_k \mid a_k > a_m\} = \min \{6, 4\} = 4$

Step 4 : Replace element a_m by α i.e. 3 by 4, keep previous elements as its i.e. 125 and write all remaining elements in increasing order i.e. 36.

$\therefore$ The next permutation is 125436.

Example 6.5.2 *Find the next two permutations of 125436*

Solution : i) Given that $125\underset{\uparrow}{3}6$

$\therefore \quad m = 5$ and $\alpha = \min \{6\} = 6$

$\therefore$ The next permutation is 125463.

ii) Find the next permutation of $125\underset{\uparrow}{4}63$

$m = 4, \alpha = \min \{6\} = 6$

$\therefore$ The next permutation is 125634.

Example 6.5.3 *Generate all permutations for n = 3 by next permutation method.*

SPPU : May-18, Marks 4

Solution : Consider the following table to generate all permutations for n = 3.

Sr. No.	Given permutation	Next permutation
1)	1 2 3 ↑ m = 2, α = 3	1 3 2
2)	1 3 2 ↑ m = 1, α = 2	2 1 3
3)	2 1 3 ↑ m = 2, α = 3	2 3 1
4)	2 3 1 ↑ m = 1, α = 3	3 1 2
5)	3 1 2 ↑ m = 2, α = 2	3 2 1 Last permutation.

∴The set of all permutations for n = 3 is

{(1 2 3) (1 3 2) (2 1 3) (2 3 1) (3 1 2) (3 2 1)}

II) Procedure to generate subsets of {1, 2, 3, ... n}

Let $\{a_1, a_2, a_3, \ldots a_k\}$ be a subset of size k of {1, 2, 3, ...n} with $a_1 < a_2 < a_3 < \ldots < a_k$.

The maximum possible value of a_k is n

The maximum possible value of $a_{k-1} = n - 1$

In general the maximum possible value of a_i is n – k + i. Consider the subset {1, 2, 3, ... k – 1, k} If k ≠ n, its maximum value then increase k by 1, so that the next subset {1, 2, 3, ... k – 1, k + 1} is generated. We continue this procedure till we reached to {1, 2, 3 ... (k – 1), n}. Now repeat the procedure for k – 1, if k – 1 ≠ n – 1 then increase it by 1 and continue this process with k – 1 till we reached to {1, 2, 3, ... (k – 2) (n – 1), n} Then move to (k – 2) and repeat the same process. In this manner, moving from right to left we finally reach to an element a_j such that a_j can be increased to a_{j+1} but no a_j with i > j can be increased which means that at some stage a_i is equal to its maximum value n – k + i. This procedure terminates when a_1 reaches to its maximum value.

II) Algorithm for generating the next r-combination in lexicographic order.

Next r-combination ($\{a_1, a_2, a_3, \ldots a_r\}$) : proper subset of {1, 2, 3, ...n} not equal to {n – r + 1, n – r + 2, ... n} : with $a_1 < a_2 < a_3 < \ldots a_r$)

$i := r$

while $a_i = n - r + i$

$i := i - 1$

$a_i = a_2 + 1$

for j = i + 1 to r

$a_j = a_i + j - i$

Example 6.5.4 *Generate all subsets of size 4 of {1, 2, 3, 4, 5, 6}.*

Solution : Let us begin with {1, 2, 3, 4}. We know that for any subset $\{a_1, a_2, a_3, a_4\}$ with $a_1 < a_2 < a_3 < a_4$ the maximum possible value of a_4 is 6, a_3 is 5, a_2 is 4 and a_1 is 3.

i) For $a_4 = 4$:

Hence increasing 4 by 1 we obtain a subset {1, 2, 3, 5}. Since a_4 has not still reached to 6.

∴Again increase 5 by 1, we get {1, 2, 3, 6}.

ii) For $a_3 = 3$:

The maximum value of a_3 is 5.

∴Increase a_3 successively by 1 still, we reach to 5.

∴{1, 2, 4, $\underset{\uparrow}{5}$} , {1, 2, $\underset{\uparrow}{4}$ 6} , {1, 2, 5, 6}

iii) For $a_2 = 2$: The maximum value of a_2 is 4

∴ We get (1, $\underset{\uparrow}{3}$, 4, 5}, {1, 3, 4, $\underset{\uparrow}{6}$}, {1, 3, $\underset{\uparrow}{5}$, 6}, {1, 3, $\underset{\uparrow}{4}$, 6}

iv) For $a_1 = 1$: Maximum value of a_1 is 3.

∴ We get {$\underset{\uparrow}{2}$, 3, 4, 5}, {2, 3, 4, $\underset{\uparrow}{6}$}, {2, 3, $\underset{\uparrow}{5}$, 6}, {2, 4, 5, 6}, {3, 4, 5, 6}

Thus we obtain the following 15 subsets

{ {1, 2, 3, 4} {1, 2, 3, 5} {1, 2, 3, 6}

{1, 2, 4, 5} {1, 2, 4, 6} {1, 2, 5, 6}

{1, 3, 4, 5} {1, 3, 4, 6}, {1, 3, 5, 6} {1, 4, 5, 6}

{2, 3, 4, 5}, {2, 3, 4, 6}, {2, 3, 5, 6} {2, 4, 5, 6} {3, 4, 5, 6} }

6.6 Binomical Coefficients

In the previous sections we have studied permutations and combinations of set with n elements. By using the formula of combinations we can define binomial coefficients.

Generally, Bi means two and nomial means nominee's So binomial means an experiment which has only two outcomes. e.g. i) Tossing a coin ii) Result of any examination is eitherpass or fail.

In mathematics, binomial coefficients are numbers which are coefficients in the expansion of powers of binomial expressions i.e. $(x+y)^n$

6.6.1 Binomial Theorem

Let x, y be any variables and n be a non-negative integer. Then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$= \binom{n}{n}x^n y^0 + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \ldots + \binom{n}{1}xy^{n-1} + \binom{n}{0}x^0 y^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \qquad (\because {}^n c_r = {}^n c_{n-r})$$

Examples :

1) $(x+y)^1 = x^1 + y^1 = \binom{1}{0}x + \binom{1}{1}y$

2) $(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$

3) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$$= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

$$= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$

4) $(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}x^0y^4$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

Example 6.6.1 *What is the coefficient of $x^{13}y^7$ in the expansion of $(x+y)^{20}$?*

Solution : we know that the coefficient of $x^{n-r}y^r$ in the expansion of $(x+y)^n$ is

$${}^n c_r = \binom{n}{r}$$

Here n = 20, r = 7, n − r = 13

∴ The required coefficient is

$$\binom{20}{7} = \frac{20!}{7! \times 13!} = \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times \cancel{13!}}{7 \times 6 \times 5 \times 4 \times 2 \times \cancel{13!}} = 77520$$

Example 6.6.2 *What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x-y)^{25}$?*

Solution : By the binomial theorem, the required coefficients

$$-\binom{25}{12} = \frac{-(25!)}{12!\ 13!} = -5{,}200300$$

Example 6.6.3 *What is the coefficient of $x^8 y^{12}$ in the expansion of $(2x-3y)^{20}$? Hence find the coefficient of $x^{12} y^8$*

Solution : we have

$$(2x-3y)^{20} = \sum_{r=0}^{20} \binom{20}{r} (2x)^{20-r} (-3y)^r$$

Here r = 12

∴ The coefficient of $x^8 y^{12}$ is obtained by putting r = 12 in R.H.S. of above formula.

Hence the required coefficient $= \binom{20}{12} (2)^8 (-3)^{12} = (125970)(2)^8 (-3)^{12}$

Example 6.6.4 *What is the coefficient of x^{99} in the expansion of $(x-1)^{101}$?*

Solution : The required coefficient is $= \binom{101}{99} = 5050$

Example 6.6.5 *What is the coefficient of x^{98} in the expansion of $(x-1)^{101}$?*

Solution : We have

$$(x-1)^{101} = \sum_{r=0}^{101} \binom{101}{r} x^r (-1)^{n-r}$$

∴ Put r = 98 and x = 1

∴ The required coefficient $= \binom{101}{98} (-1)^3 = -166650$

Example 6.6.6 *What is the coefficient of x^{99} in the expansion of $(2-x)^{19}$?*

Solution : We have

$$(2-x)^{19} = \sum_{r=0}^{19} \binom{19}{r} (2)^r (-x)^{19-r}$$

Put r = 10 and x = 1,

∴ The required coefficient is $= \binom{19}{10} 2^{10} (-1)^9 = -92378 \times 2^{10}$

Example 6.6.7 *What is the coefficient of $x^{101} y^{99}$ in the expansion of $(2x-3y)^{200}$?*

Solution : We have

$$(2x-3y)^{200} = \sum_{r=0}^{200} \binom{200}{r} (2x)^r (-3y)^{n-r}$$

Put r = 101 at R.H.S. and x = 1 = y

∴ The required coefficient of $x^{101} y^{99}$ is

$$= \binom{200}{101} 2^{101} (-3)^{99}$$

$$= -\binom{200}{101} 2^{101} (3)^{99}$$

Example 6.6.8 *What is the coefficient of x^{10} in the expansion of $(x+\frac{1}{x})^{100}$?*

Solution : we have

$$(x+\frac{1}{x})^{100} = \sum_{r=0}^{100} \binom{100}{r} x^r \left(\frac{1}{x}\right)^{100-r}$$

$$= \sum_{r=10}^{100} \binom{100}{r} x^{2r-100}$$

put $2r - 100 = 10 \Rightarrow 2r = 110 \Rightarrow r = 55$

∴ The coefficient of x^{10} is $\binom{100}{55}$

Example 6.6.9 *What is the coefficient of x^k in the expansion of $(x^2-\frac{1}{x})^{100}$? Hence find the coefficient of x^{51}*

Solution : we have

$$(x^2-\frac{1}{x})^{100} = \sum_{r=0}^{100} \binom{100}{r} (x^2)^r \left(-\frac{1}{x}\right)^{100-r}$$

$$= \sum_{r=0}^{100} \binom{100}{r} x^{2r} \left(-\frac{1}{x}\right)^{100-r}$$

$$= \sum_{r=0}^{100} \binom{100}{r} (-1)^{100-r} (x)^{3r-100}$$

Put $3r - 100 = k \Rightarrow r = \frac{1}{3}(k + 100)$

$$\therefore \quad (x^2 - \frac{1}{x})^{100} = \sum_{k=-100}^{200} \binom{100}{\frac{1}{3}(k+100)} (-1)^{100-\frac{1}{3}(k+100)} (x)^k$$

Thus the coefficient of x^k in $(x^2 - \frac{1}{x})^{100}$ is

$$= \binom{100}{\frac{1}{3}(k+100)} (-1)^{100-\frac{1}{3}(k+100)}$$

Example 6.6.10 *What is the coefficient of x^{20} and x^{21} in $(x^2 - \frac{1}{x})^{100}$?*

Solution : i) coefficient of x^{20} :

Put k = 20 in example (13)

$\therefore$ The required coefficient is $= \binom{100}{40} (-1)^{100-40} = \binom{100}{40}$

ii) coefficient of x^{21} :

put k = 21 in example (13), we get

$$k = 3r - 100 \Rightarrow 3r = k - 100$$

$$3r = 21 - 100 = -79$$

$$r = -\frac{79}{3} \text{ which is not possible because } r \in N \text{ (natural number)}$$

$\therefore$ The required coefficient is zero.

Example 6.6.11 *Let n be a non negative integer, then S.T.*

$\sum_{r=0}^{n} \binom{n}{r} = 2^n$ *by i) Binomial theorem ii) Combinatorial method*

Solution : i) By Binomial theorem

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n} x^o y^n$$

Put x = y = 1, we get

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

$$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

ii) Let A be a set with n elements. Then the power set of A has 2^n elements. Now find all subsets by different method.

There are subsets of A with zero elements 1, 2, 3...n elements $\{\phi\}$ is a subset of A with no element.

So there are $\binom{n}{o}$ subsets with no element (zero element) $\binom{n}{1}$ subsets with one element,

$\binom{n}{2}$ subsets with 2 elements... and $\binom{n}{n}$ subset

With n elements ∴ By addition principle, the total number of subsets are given as follows :

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

$$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

Example 6.6.12 *Let n be a positive integer then prove that* $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$

Solution : By Binomial theorem (put x = 1, y = – 1)

$$(0)^n = (1+(-1))^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r \, 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r} (-1)^r$$

Example 6.6.13 *Prove that if n and r integers with* $1 \le r \le n$ *then* $r\binom{n}{r} = n\binom{n-1}{r-1}$ *combinatorially.*

Solution : A good mathematician can prove this identity by connecting it to some real life example of set. Suppose we have a set A with n elements, count the number of ways to select a subset with r elements from A and then element of this set by two different methods.

i) The number of ways to select subsets with r elements from set A with n element is $\binom{n}{r}$ The number of ways to select an element of this set is r.

Thus the total number of elements = $r\binom{n}{r}$

ii) The number of ways to select one element from set A is n.

The number of ways to select r – 1 elements from n – 1 elements is $\binom{n-1}{r-1}$

∴ The total no. of ways = $n\binom{n-1}{r-1}$

Hence $r\binom{n}{r} = n\binom{n-1}{r-1}$

Example 6.6.14 *Let n be a non negative integer then* $\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n$

Solution : By Binomial theorem, putting x = 1, y = 2, we get

$$(1+2)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 2^r = \sum_{r=0}^{n} 2^r \binom{n}{r}$$

Hence $$\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n$$

Example 6.6.15 *Let n and r be positive integers with r ≤ n then prove that*

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

This is called as Pascal's Identity.

Solution : Let A be a set with n+1 elements. The number of ways to select subsets with r elements from A is $\binom{n+1}{r}$

Let $x \in A$ and $B = A - \{x\}$

There are two possibilities to select subset with r elements from set B with n elements.

i) The subset of A with r elements contain x with r – 1 elements of B

So there are $\binom{n}{r-1}$ subsets of A that contain x.

ii) The subset of A with r elements without x i.e. select r elements from the set $B = A - \{x\}$ with n elements.

This can be done in $\binom{n}{r}$ ways.

Thus $$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Example 6.6.16 *Let m, n and r be non negative integers with 0 ≤ r ≤ m and 0 ≤ r ≤ n Then*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

This is known as Vandermonde's Identity.

Solution : Let A and B two disjoint sets with $|A| = m$ and $|B| = n$

$\therefore \quad |A \cup B| = |A| + |B| = m+n$

The total number of ways to select r elements from $A \cup B$ is $\binom{m+n}{r}$

Alternately, to select r elements from $A \cup B$ is to select k elements from set B and r – k elements from set B is $\binom{n}{k}$. The number of ways to select r–k elements from A is $\binom{m}{r-k}$

Therefore by product rule, this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways.

Hence the total number of ways to select r elements from $A \cup B$ is also equals to

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Example 6.6.17 *Let n be a non negative integer then P.T.* $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Solution : Put m = n = r in Vandermorde's identity, we get

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 \qquad \left(\because \binom{n}{n-k} = \binom{n}{k}\right)$$

6.7 Case Studies

I) Sudoku : Sudoku puzzle is a number - placement puzzle. The objective is to fill 9×9 with digits so that each column, row and each of the nine 3×3 subgrids. Contain all the digits from 0 to 9. The puzzle developer provides a partially filled grid. Remaining grids have to fill by puzzle solver.

Here two types of algorithms are important, viz Sudoku Generator algorithm and Sudoku Solver algorithm.

1) Sudoku Generator Algorithm :

Step 1 : By using Sudoku rules and Sudoku solver algorithm, generate a full grid of numbers.

Step 2 : Now remove 1 value at a time from the grid.

Step 3 : After removing a value, apply Sudoku solver algorithm to check the grid can be solved and also count the number of solutions.

Step 4 : If resulting grid gives unique solution, goto step 2. Otherwise put the removed value again in the grid and remove other value.

Step 5 : Repeat the steps 2 to 4 several times to make the grid difficult to solve. Difficulty level of resulting grid is depends on step 2 to 4.

2) Sudoku Solver Algorithm :

Use the backtracking algorithm to find the solution -

1) The backtracking algorithm is a recursive algorithm.
2) It starts with adding a number in the grid. If the solution is not found, the algorithm backtracks and try another number.

3) Repeat step 2, until grid is full with all the rules of Sudoku puzzle.

Example :

5		4		7				
6			1	9	5			
	9	8					6	
8		9		6				3
			8		3			1
7				2				6
	6				7			
			4			6	3	5
					6		7	

Solution :

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

II) Handshake Puzzle/Problem :

Various handshaking problems are in circulation. The most common two are as follows

1) "In a room of n people, how many different hand shakes are possible."

The first person may shake hands with (n – 1) other people. The next person will shake hand (n – 2) other people, not counting the first person again. Counting like this gives total no. of handshakes are,

$$(n - 1) + (n - 2) + \ldots + 2 + 1 = \frac{n \cdot (n-1)}{2}$$

The no. of handshakes = ${}^{n}C_{2}$

$$= \frac{n!}{(n-2)!\,2!}$$

$$= \frac{n(n-1)\cdot(n-2)!}{(n-2)!\cdot 2!} = \frac{n\cdot(n-1)}{2}$$

We will see one more common example as follows :

2) There are n married couples attending a party. Each person shakes hands with every person other than his spouse. Find the total number of handshakes.

⇒ There are n couples and each couple contains 2 persons.

∴ Total no. of persons = 2n

If each person is making handshake with other person.

Total no. of handshakes = ${}^{2n}C_2$

But each married couple is not making handshake among themselves.

∴ Total no. of handshakes = ${}^{2n}C_2 - n$

$$= \frac{2n!}{(2n-2)!2!} - n$$

$$= \frac{2n(2n-1)(2n-2)!}{(2n-2)!2!} - n$$

$$= n(2n-1) - n$$

$$= 2n^2 - n - n$$

$$= 2n^2 - 2n$$

$$= 2n(n-1)$$

Example :

1) In a birthday party, every person shakehand with every other person. If there was 36 handshakes in the party, how many persons were present in the party.

Solution : Total No. of handshakes = $\frac{n(n-1)}{2}$

∴ $36 = \frac{n(n-1)}{2}$

∴ $n(n-1) = 72$

∴ $\boxed{n = 9}$

Total no. of persons in the party = 9.

6.8 University Questions with Answers

Dec. 2016

Q.1 *A bag contains 6 red and 8 green balls.*
i) If one ball is drawn at random, then what is the probability of the ball being green ?
ii) If two balls are drawn at random, then what is the probability that one is red and the other is green ? **[6]**

Ans. :

Total number of balls in the bag are 6 + 8 = 14.

i) The number of ways to draw one ball from the bag is ${}^{14}C_1 = 14$

The number of ways to draw one green ball from the bag is ${}^{8}C_1 = 8$

Hence, the required probability $= \frac{8}{14} = \frac{4}{7} = 0.57$

ii) The number of ways to draw two balls from the bag is ${}^{14}C_2 = \frac{14!}{2!\times 12!} = \frac{14\times 13}{2}$

$= 91$

The number of ways to draw one red ball is ${}^{6}C_1 = 6$

The number of ways to draw one green ball is ${}^{8}C_1 = 8$

Hence, the required probability $= \frac{8\times 6}{91} = \frac{48}{91} = 0.5274$

Q.2 *In how many ways can 6 men and 5 women be seated in a line so that no two women sit together ? In how many ways can 6 men and 5 women sit in a line so that women occupy the even places.* **(Refer example 6.3.19)** **[6]**

May 2017

Q.3 *How many words of three distinct letters can be formed from the letters of the word MAST ?* **[3]**

Ans. : Given word is "MAST"

$\therefore$ It involves 4 distinct letters.

To form words of three distinct letters,

The first letter can be selected in 4 ways.

The second letter can be selected in 3 ways

The third letter can be selected in 2 ways

$\therefore$ The total number of words are $4\times 3\times 2 = 24$

Q.4 *How many different seven-person committees can be formed each containing 3 women from an available set of 20 women and 4 men from an available set of 30 men.* **[3]**

Ans. : We have seven - person committee.

Each committee contains three women from 20 women and 4 men from 30 men.

∴ The required numbers of ways are

$$= {}^{20}C_3 \cdot {}^{30}C_4 = \frac{20!}{3!\,17!} \cdot \frac{30!}{4!\,26!}$$

$$= \frac{20\times19\times18}{3\times2} \cdot \frac{30\times29\times28\times27}{4\times3\times2\times1} = 104{,}13{,}900 \text{ ways}$$

Q.5 *How many distinguishable words that can be formed from the letters of MISSISSIPPI ?* **(Refer example 6.3.9).** **[3]**

Q.6 *Compute the number of distinct five-card hands that can be dealt from a deck of 52 cards.* **[3]**

Ans. : The number of distinct five card hands that can be dealt from a deck of 52 cards $= \binom{52}{5} = \frac{52!}{5!\,47!} = 2598960$ ways.

Dec. 2017

Q.7 *Explain the rule of sum and products with exmples.* **(Refer sections 6.2.1 and 6.2.2)** **[4]**

Q.8 *Find out how many 5-digit number greater than 30,000 can be formed from the digits 1, 2, 3, 4, 5.* **[4]**

Ans. : There is restriction that the 5 digit numbers so formed must be greater than 3000. Therefore the ten thousandth place can be filled with numbers 3, 4 or 5 only. Hence the ten thousandth place can be selected in 3 different ways. Now out of remaining digits, fill remaining places.

∴ The Thousandth's place can be filled by 5 ways.

The Hundredth place can be filled by 5 ways.

The Tenth place can be filled by 5 ways.

The unit place can be filled by 5 ways.

∴ The total number of 5 digit numbers greater than 30,000 can be formed in

$3\times5\times5\times5\times5 = 3\times5^4$ ways.

Q.9 *Find the number of permutaions which can be made with the letters of the word ENGINEERING.* **[4]**

Ans. : There are 11 letters in the word "ENGINEERING" out of which E, N, G, I, R are distinct.

E appears 3 times

N appears 3 times

I appears 2 times

G appears 2 times

R appears 1 times

$\therefore$ The required number of permutations $= \dfrac{11!}{3!\,3!\,2!\,2!\,1!} = 277200.$

May 2018

Q.10 *Write an algorithm for generating permutation of {1, 2, ...n}. Apply it for n = 3 case.* **(Refer section 6.5 and example 6.5.3)** **[4]**

Q.11 *Solve the following :*

(i) How many different car number plates are possible with 2 letters followed by 3 digits ?

(ii) How many of these number plates begin with 'MH' ? **[4]**

Ans. : i) The first position is a letter and can be selected from 26 letters in ${}^{26}C_1 = 26$ ways. The second position is a letter and can be selected from 26 letters in ${}^{26}C_1 = 26$ ways. The first digit can be selected in 10 ways. The second digit can be selected in 10 ways. The third digit can be selected in 10 ways.

Therefore the number of ways to select 3 digits is $10\times10\times10-1 = 999$

($\because$ 000 is not allowed).

Hence the required number of ways $= 26\times26\times999 = 675324$

ii) The total number of plates begin with 'MH' is $1\times1\times999 = 999$

Q.12 *In how many ways can a cricket team of eleven players be chosen out of a batch of 14 players. How many of them will :*

i) Include a particular player ii) Exclude a particular player **[4]**

Ans. : The number of ways to select 11 cricket team players from 14 is

$${}^{14}C_{11} = \frac{14\times13\times12\times11\,!}{11\,!\times3\,!} = 364.$$

i) When a particular player is included every time, then the remaining 11 – 1 = 10 players can be selected from remaining 14 – 1 = 13 players. This can be done in $^{13}C_{10} = \frac{13\times12\times11\times10!}{3!\times10!} = 286$ ways

ii) When a particular player is excluded, then select 11 players from 14 – 1 = 13 players. This can be done in $^{13}C_{11} = \frac{13\times12\times11\,!}{2\,!\times11\,!} = 78$ ways.

Dec. 2018

Q.13 *From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done ?* **[3]**

Ans. : Given that, there are at least 3 men on committee of 5 persons. It means committee will contains 3 men or 4 men or 5 men.

i) 3 men can be selected in $^{7}C_3 = \frac{7\times6\times5}{3!} = 35$ ways

2 women can be selected in $^{6}C_2 = \frac{6\times5\times4!}{2\times4!} = 15$ ways

∴The number of ways this can de done is 33 × 15 = 525

ii) The number of ways 4 men and 1 woman can be selected is

$^{7}C_4 \times {}^{6}C_1 = 35 \times 6 = 210$ days

iii) The number of ways 5 men and no woman can be selected is

$^{7}C_5 \times {}^{6}C_0 = 21 \times 1 = 21$

Hence the total number of ways a committee can be formed with at least 3 men is

525 + 210 + 21 = 756 ways

Q.14 *How many 4-letter words with or without meaning, can be formed out of the letters of the word 'LOGARITHMS', if repetition of letters is not allowed ?* **[3]**

Ans. : There are 10 letters, and all letters are distinct.

∴ The number if 4 letters words from 10 distinct letters

$$= {}^{10}P_4 = \frac{10\times9\times8\times7\times6!}{4!\times6!} = 210 \text{ ways}$$

Q.15 *If a committee has eight members :*

i) How many ways can the committee members be seated in a row ?

ii) How many ways can the committee select a president, vice-president and secretary ? **[6]**

Ans. : There are eight members in a committee

i) There are eight positions in a row

1	2	3	4	5	6	7	8

For the first position, there are 8 ways.

For the second position, there are 7 ways.

Similarly, the total number of ways the committee members can be seated in a row is 8!

$$= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

ii) The number of ways the committee can select a president is ${}^{8}C_{1} = 8$ ways

The number of ways the committee can sets a vice president and secretary respectively is 7 and 6 ways.

Hence, the required number of ways = 8 × 7 × 6 = 336.

May 2019

Q.16 *2 Mathematics papers and 5 other papers are to be arranged at an examination, find the total number of ways if, i) Mathematics papers are consecutive.* **(Refer example 6.3.6)** **[3]**

Q.17 *In the expansion of $(1+x)^6$. What is the coefficient of x^3.* **[3]**

Ans. : The coefficient of x^3 in the expansion of $(1+x^6)$ is found as follows,

We have,

$$(1+x)^6 = \sum_{r=0}^{6} \binom{6}{r} (1)^r (x)^{6-r} = \sum_{r=0}^{6} \binom{6}{r} (x)^{6-r}$$

Put, $6-r = 3 \Rightarrow r = 3$

$\therefore$ The coefficient of x^3 is,

$$\binom{6}{3} = \frac{6}{(6-3)!\cdot 3!} = \frac{6\times5\times4\times3!}{3!\times3!} = \frac{6\times5\times4}{3\times2} = 20$$

Q.18 *If the letters of the word 'REGULATIONS' be arranged at random. What is the chance that there will be exactly 4 letters between R and E ?* **[3]**

Ans. :

Total number of letters = 11

R is at first place. So, E will be at 6th place.

R is at second place. So, E will be at 7th place.

And, so on.

Finally, R is at 6th place and E will at 11th place.

So total favorable cases = 6

If R and E are interchanged

Total favorable cases = 6×2

Remaining 9 letters can be arranged by 9! Ways

Total number of ways that R and E have 4 letters between them.

$$= 6 \times 2 \times 9!$$

$$= 4354560$$

Q.19 *Use Binomial theorem to expand $(x^4 + 2)^3$.* **[3]**

Ans. : Binomial theorem states that,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$$\therefore \sum_{r=0}^{3} \frac{3!}{(3-r)! \cdot r!} (x^4)^{3-r} (2)^r$$

Expand the summation,

$$= \frac{3!}{(3-0)! \cdot 0!} (x^4)^{3-0} \cdot 2^0 + \frac{3!}{(3-1)! \cdot 1!} (x^4)^{3-1} \cdot 2^1$$

$$+ \frac{3!}{(3-2)! \cdot 2!} (x^4)^{3-2} \cdot 2^2 + \frac{3!}{(3-3)! \cdot 3!} (x^4)^{3-3} \cdot 2^3$$

$$= 1 \cdot (x^4)^3 + 3 \cdot (x^4)^2 \cdot 2 + 3 \cdot (x^4) \cdot 2^2 + 1 \cdot (x^4)^0 \cdot 2^3$$

$$= x^{12} + 6x^8 + 12x^4 + 8$$

Dec. 2019

Q.20 *The company has 10 members on its board of directors. In how many ways can they elect a president, a vice president, secretary and a treasurer ?* **[3]**

Ans. :

$$^{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = \frac{10\times9\times8\times7\times6!}{6!} = 10 \times 9 \times 8 \times 7$$

$$= 5040$$

Number of ways 10 members can elect president, vice precident a secretary and a treasures = 5040 ways.

Q.21 *Find 8th term in the expansion of $(x+y)^{13}$.* **[3]**

Ans. : Binomial theorem states that,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

In this question

x = x, y = y and n = 13

The 8th term occures when r = 8 – 1 = 7

(since sum starting with r = 0)

$\therefore 8^{th}$ term is

$$\binom{13}{7} x^{13-7} y^7 = 1716 x^6 y^7$$

Q.22 *A box contains 6 white and 5 black balls. Find number of ways 4 balls can be drawn from the box, if : (i) Two must be white (ii) All of them must have same colour.* **[3]**

Ans. : Total no. of balls = 6 + 5 = 11

i) 4 balls needs to draw, out of that 2 must be white.

i.e. 2 white and 2 black

The numbers of ways to draw 2 white balls from the box is,

$$^6C_2 = \frac{6!}{2!\times 4!} = \frac{6\times5}{2} = 15 \text{ ways}$$

The number of ways to draw 2 black balls from the box is,

$$^5C_2 = \frac{5!}{2!\times 3!} = \frac{5\times4}{2} = 10$$

$\therefore$ The no. of ways to draw 4 balls from the box = 15 × 10 = 150

ii) All of them must have same colour.

i.e. Either 4 balls are white or black.

The number of ways to draw 4 white balls from the box

$$^{6}C_{4} = \frac{6!}{4! \times 2!} = \frac{6 \times 5}{2} = 15 \text{ ways}$$

The number of ways to draw 4 black balls from box

$$^{5}C_{4} = \frac{5!}{4!} = 5 \text{ ways}$$

$\therefore$ Total no. of ways to draw 4 balls from the box with all of them must have same colour = 15 + 5 = 20 ways

Q.23 *Expand* $(3x-4)^4$ *using binomial theorem.* **[3]**

Ans. : Binomial theorem states that,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$$\therefore \sum_{r=0}^{4} \frac{4!}{(4-r)!r!} (3x)^{4-r} (-4)^r$$

Expand the summation,

$$= \frac{4!}{(4-0)!0!}(3x)^{4-0} \cdot (-4)^0 + \frac{4!}{(4-1)!1!}(3x)^{4-1} \cdot (-4)$$

$$+ \frac{4!}{(4-2)!2!}(3x)^{4-2} \cdot (-4)^2 + \frac{4!}{(4-3)!3!}(3x)^{4-3} \cdot (-4)^3 + \frac{4!}{(4-4)!4!}(3x)^{4-4}(-4)^4$$

$$= 1 \cdot (3x)^4 (-4)^0 + 4 \cdot (3x)^3 \cdot (-4) + 6 \cdot (3x)^2 \cdot (-4)^2 + 4(3x)(-4)^3 + 3 \cdot (3x)^0 \cdot (-4)^4$$

$$= 81x^4 - 432x^3 + 864x^2 - 768x + 256$$

❑❑❑

UNIT - IV

7 Graph Theory

Syllabus

Graph Terminology *and Special Types of Graphs, Representing Graphs and Graph Isomorphism, Connectivity, Euler and Hamilton Paths, the handshaking lemma, Single source shortest path-Dijkstra's Algorithm, Planar Graphs, Graph Colouring. Case Studies : Three Utility Problem, Web Graph, Google Map.*

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7.1 Introduction
7.2 Graphs
7.3 Basic Definitions **Dec.-17**
7.4 Matrix Representation of a Graph **Dec.-09, 10, May-10, 18**
7.5 Some Important Definitions **Dec.-16, May-18**
7.6 Linked Representation of Graphs
7.7 Isomorphism of Graphs ..... **Dec.-12, 18, May-18** ..... Marks 6
7.8 New Graphs from Old Graphs **May-07, 18, Dec.-09, 12**
7.9 Operations on Graphs
7.10 Paths and Circuits
7.11 Connected and Disconnected Graphs
7.12 Connectivity of Graph
7.13 Shortest Path Algorithm **May-05, 07, 14, 15, 19**
..... **Dec.-06, 07, 12, 13, 14, 15, 16, 17, 18**
.... Marks 8
7.14 Eulerian Graphs **Dec.-09, May-19**
7.15 Hamiltonian Graphs ..... **Dec.-04, 09, 10, 12, 15, May-17** .. Marks 3
7.16 Planar Graphs **May-06, Dec.-08, 09, 10, 13**
7.17 Coloring of Graphs
7.18 Web Graph
7.19 Graph Database
7.20 Google Map
7.21 Case Studies
7.22 University Questions with Answers

7.1 Introduction

The development of graph theory is very much similar to the development of probability theory. The original work of graph theory was motivated by constant efforts to understand or solve real life problems. The graph theory is a very important area of applied mathematics. The origin of graph theory started with an important and practical problem of the Konigberg bridge in 1735.

A great mathematician Euler became the father of the graph theory, when in 1736 he solved a famous unsolved problem of his days called the konigberg bridge problem. This is today called as the first problem of graph theory. He created the first graph to replicate this problem.

Mathematicians, Mobius, Gaustar Kirchhoff, De-Morgan William Hamilton, Alfred Kempe, P. K. Tait, Narsingdeo and many more contributed several result of graph theory.

The graph theory because of it's applications to many disciplines has emerged as on of the useful branch of mathematics during the cast decade. There are several reasons for the acceleration of interest in the graph theory because its applications so some areas of computer engineering, communication science, electrical and civil engineering, cryptography operations research, physics, chemistry, genetics and many more.

7.2 Graphs

A graph is an ordered pair (V(G), E(G)) where

i) V(G) is non empty finite set of elements known as vertices or nodes. V(G) is called the vertex set.

ii) E(G) is a family of unordered pairs (not necessarily distinct) of elements of v, known as edges or arc or branches of G. E(G) is known as edge set.

Graphs are so named because they can be represented diagrammatically in the plane.

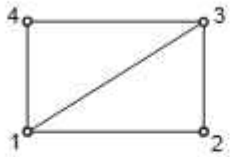
It is denoted by V(G, E).

a) Each vertex of G is represented by a point or small circle in the plane. In practical examples vertex set may be the set of states or cities or objects etc.

b) Every edge is represented by a continuous curve or straight line segment. Edges may be the route among states or cities or relation among objects etc. Diagrams of road maps, electrical circuits, chemical compounds, job scheduling family trees, all have two objects common namely vertices and edges.

Let us consider the following examples of graphs with V(G) and E(G).

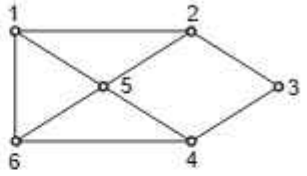
1)



$V(G_1) = \{1, 2, 3, 4\}$

$E(G_1) = \{(1, 2), (1, 3)\ (1, 4)\ (2, 3), (3, 4)\}$

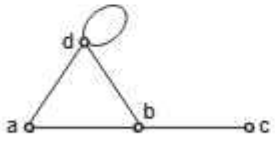
2)



$V(G_2) = \{1, 2, 3, 4, 5, 6\}$

$E(G_2) = \{(1, 2), (1, 5), (1, 6), (2,5),$
$(2, 3), (3, 4), (4, 5), (4, 6), (5, 6)\}$

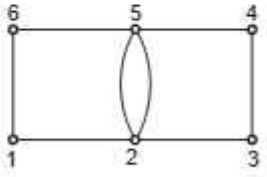
3)



$V(G_3) = \{a, b, c, d\}$

$E(G_3) = \{(a, b), (a, d)\ (b, c)$
$(b, d), (d, d)\}$

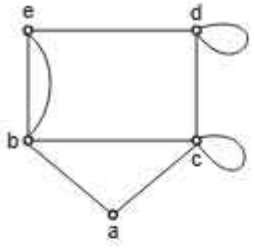
4)



$V(G_2) = \{1, 2, 3, 4, 5, 6\}$

$E(G_4) = \{(1, 2), (1, 6), (2, 3),(2,5), (2, 5),$
$(3, 4), (4, 5), (5, 6)\}$

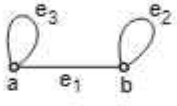
5)



$V(G_5) = \{a, b, c, d\}$

$E(G_5) = \{(a, b), (a, c), (b, e)\ (b, e), (b, c),$
$(c, c)\ (c, d)\ (d, d), (d, e)\}$

6)



$V(G_6) = \{a, b\}$

$E(G_6) = \{a, b)\ (a, a)\ (b, b) = \{e_1, e_2, e_3\}$

i) If x and y are two vertices of a graph G and unordered pair $\{x, y\} = (x, y) = e$ is an edge then we say that edge e joins x and y or e is incident to both vertices x and y.

In this case, vertices x and y are said to be incident one e.g. In example (1), e = (2, 3) ∴ e is incident at 2 and 3 and vertices 2, 3 are one incident on e = (2, 3).

ii) Two vertices x and y are said to be adjacent to each other if the pair (x, y) is an edge of G.

If e = (x, y) is an edge of G then x and y are said to be end vertices of e and we can say that e is incident at x and y.

iii) Two edges e_1 and e_2 are said to be adjacent if they have a common vertex i.e. If e_1 and e_2 are adjacent then $e_1 = \{x, y\}$ and $e_2 = \{y, z\}$.

iv) An edge joining a vertex to itself is called a loop. E.g. In example (5) there are 2 loops (c, c) and (d, d).

v) A pair of vertices of a graph is joined by two or more edges, such edges are called as multiple or parallel edges.

In example (4) (2, 5) (2, 5) are multiple edges.

7.3 Basic Definitions

SPPU : Dec.-17

1) Multigraph : A graph in which a pair of vertices is joined by two or more edges is called a multigraph or multiple graph.

i.e. A graph having multiple edges is called a multigraph. In examples (1), (2), (3), (6), graphs are not multigraphs and graphs in examples (4), (5) are multigraphs.

2) Pseudograph : A graph having loops but no multiple edges is called a Pseudograph.

Graphs in examples (3), (5) and (6) are pseudographs. A graph having only loops is called a Haary graph. For example :

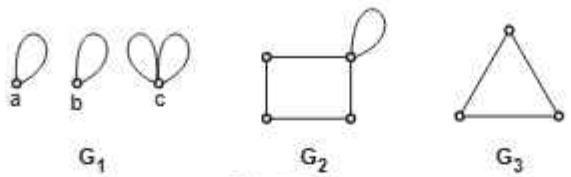
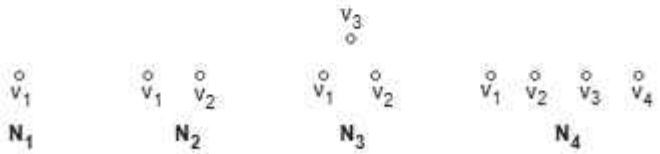


Fig. 7.3.1

Graph G is a pseudograph is well as Haary graph. Graph G_2 is Pseudograph but not Haary graph. Graph G_3 is neither Pseudo not Haary graphs.

3) Simple graph : A graph without loops and multiple edges is called a simple graph. Graphs in examples (1) and (2) are simple graphs. Graphs in examples (3), (4), (5), (6) are not simple graphs.

4) Null graph : A graph G (V, E) is said to be null graph if E is a empty set. Null graph on n vertices is denoted by N_n.



5) **Finite graph :** A graph G (V, E) in which V (x) and E (x) are finite sets is called a finite graph. Otherwise infinite graph.

6) **Directed graph :** A graph G (V, E) is said to be directed graph if the elements of E are an ordered pairs of vertices. E.g.

E = {(a, b), (b, c), (a, c)}

Here (a, c) ≠ (c, a), (c, a) ∉ E (G).

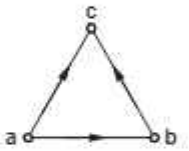


Fig. 7.3.2 Directed graph

A graph which is not directed is called Non-directed graph or graph.

7) **Weighted graph :** A graph G (V, E) in which some weight is assigned to every edge of G, is called weighted graph.

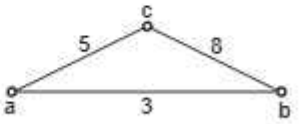


Fig. 7.3.3 Weighted graph

8) **Degree of a vertex :**

a) In a directed graph G the number of edges ending at vertex v is called the indegree of v. It is denoted by deg G^+ (v) or d^+ (v)

b) Outdegree : In a directed graph G, the number of edges beginning at vertex v is called the outdegree of v. It is denoted by deg G^- (v) or d^- (v).

c) The number of edges incident at a vertex v of a graph G with self loops counted twice is called the degree of the vertex v. It is denoted by d(v). A vertex of degree one is called pendent vertex. A vertex of degree zero is called isolated vertex. An edge incident at pendent vertex is called pendent edge.

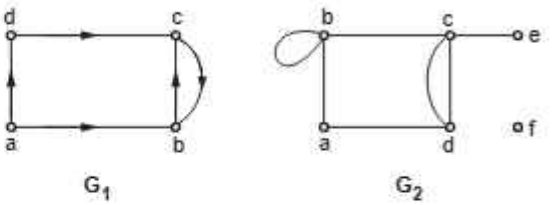


Fig. 7.3.4

e.g.

In graph G_1,

Vertices	Indegree	Outdegree
a	0	2
b	2	1
c	2	1
d	1	1

In graph G_2,

$$d(a) = 2 \ , \ d(b) = 2 + 2 = 4 \ , \ d(c) = 4 \ , \ d(e) = 1$$

$$d(d) = 3 \ , \ d(f) = 0$$

$\therefore$ f is an isolated vertex. e is a pendent vertex. An edge {c, e} is a pendent edge.

9) Order and size of graph : The number of vertices in a finite graph G is called the order of G. The number of edges in a finite graph G is called size of the graph. A graph of order n and size m is called (n, m) graph.

If G is a (p, q) graph then G has p vertices and q edges.

10) Degree sequence of a graph :

Let G be a graph with vertex set $V = \{v_1, v_2, v_3, \ldots v_n\}$ and $d_i = \deg(v_i)$ then the sequence $(d_1, d_2, d_3, \ldots d_n)$ in any order is called the degree sequence of G.

Note : 1) Vertices of G are ordered so that degree sequence is monotonically increasing.

2) Two graphs with same degree sequence are called to be degree equivalent. e.g.

$d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 2$, $d(v_5) = 5$,

$d(v_6) = 3)$, $d(v_4) = 1$

$\therefore$Its degree seq. is (4, 3, 2, 1, 5, 3)

By relabelling vertices we may write degree sequence as (1, 2, 3, 3, 4, 5).

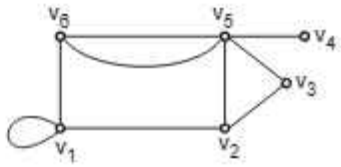


Fig. 7.3.5

Theorems :

Theorem 1 : Handshaking lemma : Let G (V, E) be any graph then $\sum_{v \in V} d(v) = 2q$ where q denotes the number of edges of G.

Proof : Let us argue by induction on q. Suppose q = 0 i.e. G has no edge i.e. E is an empty set. So $d(v) = 0, \forall v \in V$. $\therefore \sum d(v) = 2q = 0$.

Let G be a graph with $q > 0$ edges. Choose any edge $e = \{u, v\}$ of G. Consider the graph G_1 obtained from G as follows :

i) The vertex set of G_1 is same as the vertex set of G i.e. $V(G_1) = V(G) = V = p$

ii) The edges of G_1 are all edges of G except e.

In other words, G_1 is obtained from G by deleting the edge e. $\therefore$ x is a (p, q – 1) graph.

$\therefore$ By induction principle, result is true for q–1 edges

i.e. $$\sum_{x \in V} d(x) = 2(q-1) \qquad \ldots (7.3.1)$$

The degree of a vertex x other than u or v in G_1 is same as that at G. And the degree of u in G_1 is one less than the degree of u in G.

$$\text{i.e. } d_{G_1}(u) = d_G(u) - 1$$

Similarly $$d_{G_1}(v) = d_G(v) - 1$$

Hence equation (7.3.1) becomes

$$\sum_{\substack{x \in V \\ x \neq u, v}} d(x) + d_{G_1}(u) + d_{G_1}(v) = 2(q-1)$$

$$\sum_{\substack{x \in V \\ x \neq u, v}} d(x) + d_{G_1}(u) - 1 + d_G(v) - 1 = 2(q-2)$$

$\therefore$ $$\sum_{x \in V} d(x) = 2q$$ Hence the proof.

The result is so named because it implies that if several people shake hands, the total number of hands shaken must be even as two hands are involved in one handshake.

Note : If $\sum_{v \in V} d(v)$ = Odd number then there does not exist any graph with this degree sequence.

7.4 Matrix Representation of a Graph

SPPU : Dec.-09, 10, May-10, 18

We saw that graphs can be represented either set theoretically or diagrammatically. Graphs can also be represented by matrices. It is very much useful to store graphs in a computer.

7.4.1 Adjacency Matrix

Let G be a graph with n vertices and no parallel edges. The adjacency matrix of x is denoted by

$A(G) = [a_{ij}]_{n \times n}$ and defined as

$a_{ij} = 1$ if v_i and v_j are adjacent

$= 0$ if v_i and v_j are not adjacent.

Note : i) A (G) is asymmetric - binary matrix.

ii) The principal diagonal entries are all zeros if G has no loops.

iii) The ith row sum = i^{th} column sum = $d(v_i)$

e.g. 1) The adjacent matrices of the following graphs are

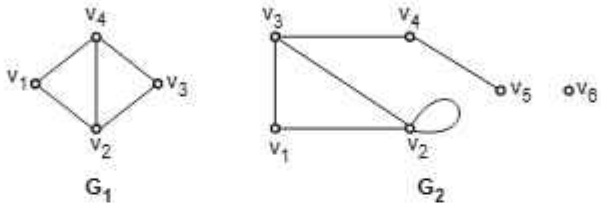


Fig. 7.4.1

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4\times 4} \end{array}, \quad A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6} \end{array}$$

The adjacency matrix for a multigraph G is a $n \times n$ Matrix $A(G) = [a_{ij}]_{n\times n}$ where

a_{ij} = Number of edges joining v_i and v_j

The adjacency matrix of the following graph is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \end{bmatrix} \end{array}$$

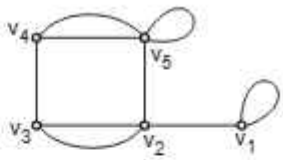


Fig. 7.4.2

7.4.2 Incidence Matrix

Let G be a graph with n vertices and m edges without self loops. The incidence matrix is denoted by X (G) or I (G) and defined as

$X(G) = [x_{ij}]_{n\times m}$ where

x_{ij} = 1 if j^{th} edge is incident on i^{th} vertex v_i.

= 0 otherwise.

X (G) is a $n \times m$ matrix whose n rows correspond to the n vertices and m columns correspond to m edges. The graph and it's incidence matrix are given below

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 7} \end{array}$$

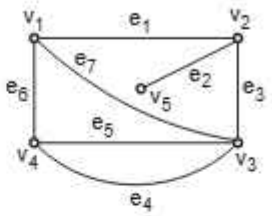


Fig. 7.4.3

Properties of Incidence Matrix

1) It contains only 0 and 1.
2) Each column in the incidence matrix has exactly two 1's appearing in that column.
3) The sum of elements in a row is equal to the degree of the corresponding vertex.
4) Two identical columns correspond to the parallel edges in graph.
5) A row with all 0's represents an isolated vertex.
6) A row with single 1 represents a pendent vertex.

The incidence matrix of a graph with loop is given as follows :

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3\times 3} \end{array}$$

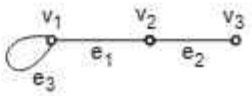


Fig. 7.4.4

7.4.3 Adjacency Matrix of a Diagraph (Directed Graph)

Let G be a directed graph with n vertices and without parallel edges. The adjacency matrix is denoted by

Where, $A(D) = [a_{ij}]_{n\times n}$

a_{ij} = 1 if there is an edge directed from v_i to v_j

= 0 otherwise

In network flow, adjacency matrix is also known as connection matrix or transition matrix. The adjancency matrix and diagraph are given below.

$$A(D) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6} \end{array}$$

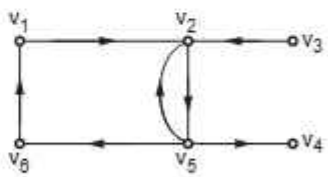


Fig. 7.4.5

7.4.4 Incidence Matrix of Diagraph

The incidence matrix of a diagraph with n vertices m edges and no self loops is a matrix.

$$X(G) = [x_{ij}]_{n \times m} \text{ where}$$

$$x_{ij} = 1 \text{ if } j^{th} \text{ edge } e_j \text{ is incident out of } i^{th} \text{ vertex } v_i.$$

$$= -1 \text{ if } j^{th} \text{ edge } e_j \text{ is incident into of vertex } v_i.$$

$$= 0 \text{ otherwise.}$$

A graph and its incidence matrix are given below :

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

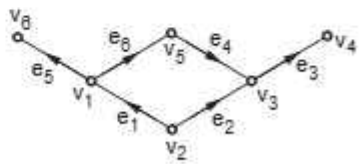


Fig. 7.4.6

Note : i) Sum of elements in each column of incidence matrix is zero.

Examples :

Example 7.4.1 *Show that in a graph the number of vertices of odd degree is even.*

Solution : Let G be a (p – q) graph.

By handshaking lemma $\sum_{v_i \in V} d(v_i) = 2q$

Now separate out vertices of even degree and odd degree

$$\therefore \quad \sum_{v_i \in V} d(v_i) = \sum_{\substack{x \in V \\ \text{even degree}}} d(x) + \sum_{\substack{y \in V \\ \text{odd degree}}} d(y) = 2q$$

$$\therefore \quad \sum_{\substack{v_i \in V \\ \text{odd degree}}} d(v_i) = 2q - \sum_{\substack{x \in V \\ \text{even degree}}} d(x) = \text{Even number}$$

$\therefore$ The sum of vertices of odd degree is even.

Hence the number of vertices of odd degree is even.

Example 7.4.2 *Show that the maximum number of edges in a simple graph with n vertices is* $\frac{n(n-1)}{2}$

Solution : Let G be a graph with n vertices m edges

∴ By handshaking lemma

$$\sum_{v \in V} d(v) = 2m \rightarrow$$

Let $x \in V$ ∴x must be adjacent to remaining $(x - 1)$ vertices ∴$d(x) = n - 1$. $\forall x \in V$

∴ Equation (1) ⇒ $(n - 1) + (n - 1) + \ldots$ n times $= 2m$

$$n(n-1) = 2m$$

$$m = \frac{n(n-1)}{2}$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$

Example 7.4.3 *How many nodes are necessary to construct a graph with exactly 8 edges in which each node is of degree 2.*

Solution : Let G be the required graph with n vertices. Each vertex is of degree 2.

By handshaking lemma

$$\sum_{i=1}^{n} d(v_i) = 2(8) = 16$$

$2 + 2 + 2 \ldots$ n times $= 16$

$$2n = 16$$

$$n = 8$$

Hence 8 vertices are required.

Example 7.4.4 *Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw to such graphs.* **SPPU : Dec.-09**

Solution : Let G be the required graph with 6 nodes and m edges.

∴ By handshaking lemma

$$\sum_{v \in G} d(v) = 2m$$

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + l(v_6) = 2m$$

$$4 + 4 + 2 + 2 + 2 + 2 = 2m$$

$$2m = 16$$

$$m = 8$$

Hence 8 edges are required.

Two such graphs are given below.

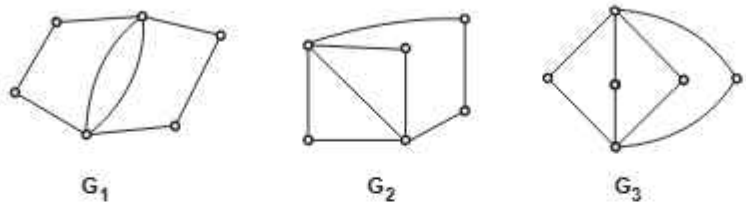


Fig. 7.4.7

Example 7.4.5 *Is it possible to construct a graph with 12 nodes such that 2 of the nodes have degree 3 and the remaining have degree 4.* **SPPU : Dec.-10**

Solution : Let G be the required graph with 12 vertices.

By handshaking lemma.

$$\sum_{v \in V(G)} d(v) = 2m$$

$$(3 + 3) + (4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4) = 2m$$

$$6 + 40 = 2m$$

$$\Rightarrow \quad m = 23$$

$\therefore$ It is possible to construct such graph.

Example 7.4.6 *Is graph exist for the degree sequence 4, 4, 3, 3, 2, 2, 1.*

Solution : Now apply handshaking lemma

$$\sum d(v) = 2m = \text{Even}$$

$$4 + 4 + 3 + 3 + 2 + 2 + 1 = \text{Even}$$

$$19 = \text{Even which is impossible}$$

$\therefore$ Such graph does not exist.

Example 7.4.7 *Let A = {1, 2, 3, 4, 5} and relation R is defined as R = {(1, 1) (2, 2), (3, 3) (4, 4) (5, 5)}. Draw graph G (A, R).*

Solution :

Fig. 7.4.8

This graph is known as Haary graph.

Example 7.4.8 *Find the adjacency matrices of the following graphs.*

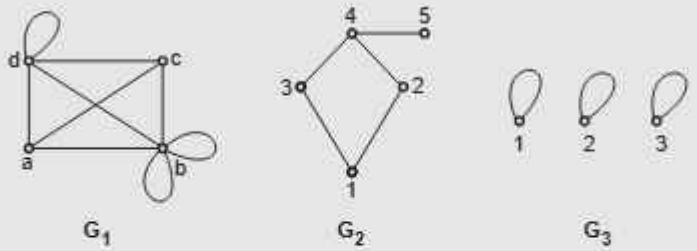


Fig. 7.4.9

Solution : The adjacency matrices for the graphs G_1, G_2, G_3 are given below :

$$A(G_1) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4} \end{array}$$

$$A(G_2) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{5\times 5}$$

$$A(G_3) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \end{array}$$

Example 7.4.9 *How many simple labelled graphs with n vertices are there ?* **SPPU : May-10**

Solution : We know that a simple graph with n vertices has maximum possible number of edges $\frac{n(n-1)}{2} = m$ (say).

To construct a simple graph with e edges and n vertices, can be done in $\binom{m}{e}$ ways.

i.e. ${}^{m}C_{e}$ ways where $m = \frac{n(n-1)}{2}$ and $0 \le e \le m$

Hence the total number of ways to construct such graphs is given by

$$\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{m}$$

$$= 2^m$$

$$= 2^{\left(\frac{n(n-1)}{2}\right)} \text{ ways.}$$

Example 7.4.10 *A man supposed to bring a tiger, a sheep a bag of cabbage across a river on a row boat. Boat is very small and he can carry one of these items on the boat at a time. Furthermore he can not leave the tiger with sheep nor the sheep with cabbage. Construct a graph to determine a possible way for the man to transport all items across the river.*

Solution : Let M = man, T = tiger, S = sheep, C = cabbage

Step 1 :

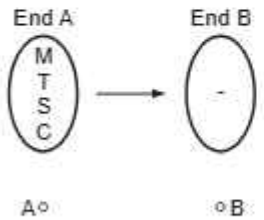


Fig. 7.4.10

Step 2 :

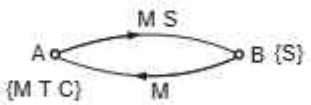


Fig. 7.4.10 (a)

Step 3 :

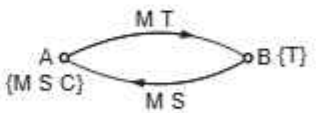


Fig. 7.4.10 (b)

Step 4 :

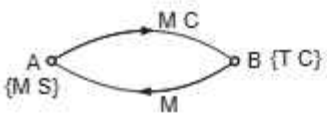


Fig. 7.4.10 (c)

Step 5 :

Fig. 7.4.10 (d)

Example 7.4.11 *Draw a non simple graph with degree sequence (1, 1, 3, 3, 3,, 4, 6, 7)*

Solution : G is non directed, therefore G permits self loops in it. The required graph is as follows :

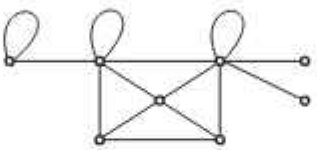


Fig. 7.4.11

Example 7.4.12 *Show that a simple graph of order 4 and size 7 does not exist.*

Solution : Let G be a simple graph with 4 vertices.

Then G has at most $\frac{n(n-1)}{2} = \frac{4 \times 3}{2} = 6$ edges.

But given that G has 7 edges which is contradiction.

$\therefore$ there can not be a simple graph with 4 vertices and 7 edges.

Example 7.4.13 *Show that every cubic graph has even number of vertices ?*

Solution : Let G be a cubic graph with n vertices.

$$\therefore \quad \sum_{v \in x} d(v) = 3P = \text{Even [by handshaking lemma]}$$

$\therefore$ 3P must be even but 3 is not even therefore P must be even.

Hence the proof.

7.5 Some Important Definitions

SPPU : Dec.-16, May-18

I) Regular Graph : A graph G is said to be r-regular graph if every vertex of G has degree r.

i) Regular graph of degree zero is called null graph.

ii) A regular graph of degree 3 is called cubic graph.

e.g.

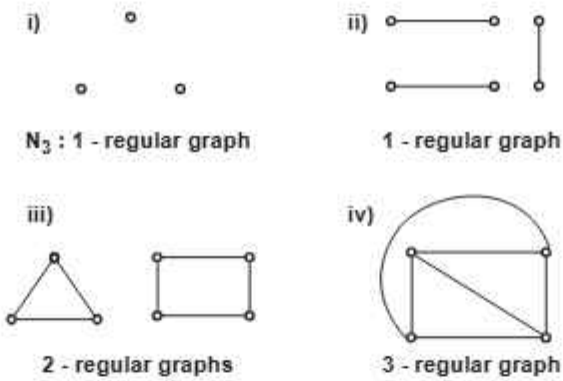


Fig. 7.5.1

II) Complete Graph : A simple graph G in which every pair of distinct vertices are adjacent is called a complete graph. If G is a complete graph on n vertices then it is denoted by K_n.

In a complete graph, there is an edge between every pair of distinct vertices.

In graph K_n, every vertex is adjacent to remaining n–1 vertices so degree of each vertex is n – 1.

Thus K_n is a (n – 1) - regular graph.

K_n has exactly $\frac{n(n-1)}{2}$ edges.

Consider the following examples :

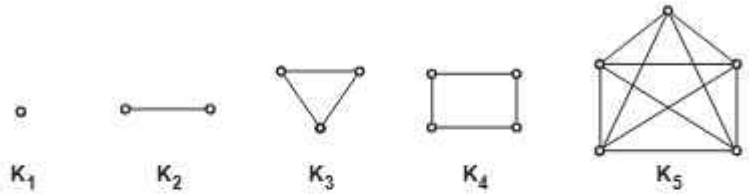


Fig. 7.5.2

III) Bipartite Graph : A graph G (v, E) is said to be bipartite graph if its vertex set can be partitioned into two disjoint subsets say v_1 and v_2 such that $v_1 \cup v_2 = v$ and $v_1 \cap v_2 = \phi$ and every edge of G joins a vertex of v_1 to a vertex of v_2.

In Bipartite graph, vertices of v_1 should not be adjacent. It is free from loops.

Following graphs are bipartite graphs

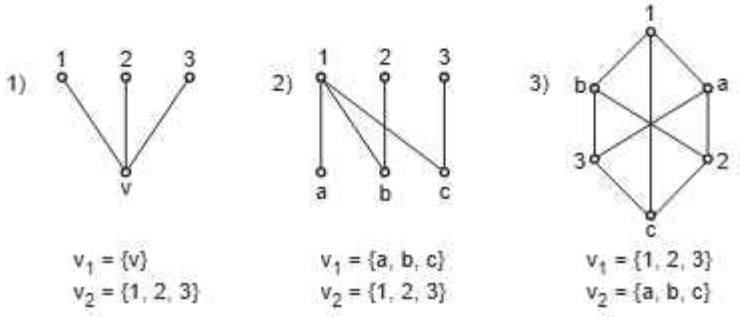


Fig. 7.5.3

IV) Complete Bipartite graph : A bipartite graph G (v, E), $v_1 \cup v_2 = v$ and $v_1 \cap v_2 = \phi$ is said to be complete Bipartite graph if each vertex of V_i is joined to every vertex of v_2 by a unique edge.

If $|v_1| = m$, $|v_2| = n$, then the complete bipartite graph $G(v_1 \cup v_2, E)$ is denoted by $K_{m,n}$

Examples :

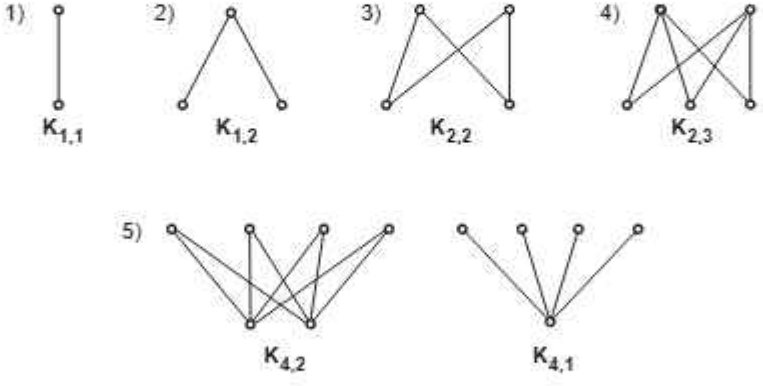


Fig. 7.5.4

The graph $K_{1,n}$ is called as star graph.

7.6 Linked Representation of Graphs

In many practical problems which involve large graph (w.r.t. vertices or edges) are solved by using computer. The link representation also is called the adjacency structure which is used for storing the graphs in computer.

In this method the graph is represented by a linear array. After assigning the vertex, in any order say 1, 2, 3, ... n, we represent each vertex k by a linear array, whose first element is k and remaining elements are the vertices that are immediate successor of K. i.e. the vertices which have a directed path of length one from K.

For example consider the following graphs :

The vertices and their adjacent vertices are as follows :

1 : 2, 3

2 : 2

3 : 2, 4

4 : 1, 5

5 : 6

6 : 4

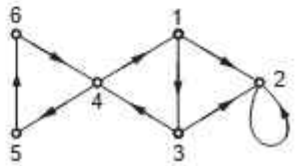


Fig. 7.6.1

This adjacency can be represented as

$$G = [1 : 2, 3 ; 2 : 2 ; 3 : 2, 4 ; 4 : 1, 5 ; 5 : 6 ; 6 : 4]$$

This is called the link presentation of the graph.

The link representation of a graph G stores the graph in the memory of computer by using its adjacency lists.

Examples :

Example 7.6.1 *Give two examples of the complete bipartite graph which are not regular.*

Solution :

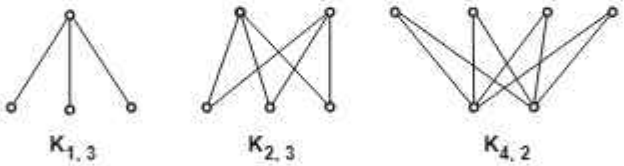


Fig. 7.6.2

$K_{1,3}$ $K_{2,3}$ $K_{4,2}$

$K_{1,3}$, $K_{2,3}$, $K_{4,2}$ all are complete bipartite graphs but not regular.

Example 7.6.2 *Is there exist any complete bipartite graph with 7 vertices and 14 edges ?*

Solution : First find all possible bipartitions of 7. They are 6 + 1, 5 + 2, 4 + 3.

We know that, if $G(v_1 \cup v_2, E)$ is a bipartite graph then the number edges in G is equal to $|v_1| \cdot |v_2|$

i.e. $|E| = |v_1| \cdot |v_2|$

Here $|E| = 14$ But $6.1 = 6$, $5.2 = 10$, $4.3 = 12$

Therefore the complete bipartite graphs with 7 vertices has 6 or 10 or 12 edges only.

Therefore any complete bipartite graph with 7 vertices and 14 edges.

Example 7.6.3 *How many edges are there in i) K_{10} ii) $K_{5,3}$ iii) $K_{5,7}$*

Solution : i) The complete graph on 10 vertices K_{10} has $\frac{10 \times 9}{2} = 45$ edges

ii) $K_{5,3}$ has $5 \cdot 3 = 15$ edges

iii) $K_{5,7}$ graph has $5 \cdot 7 = 35$ edges

7.7 Isomorphism of Graphs

SPPU : Dec.-12, 18, May-18

In real life we come across so many similar objects or figures with respect to size, shape or orientation. Similarly there are a few concepts in graph theory which deal with the similarity of graphs w.r.t. number of vertices or number of edges, number of regions and so on. Among all such similarities the most important one is an isomorphism of graphs.

Definition : Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. G_1 and G_2 are said to be isomorphic graphs if

i) There exists a bijective function $\phi : V_1 \to V_2$

ii) There exists a bijective function $\psi : E_1 \to E_2$ such that $e = (x, y)$ is an edge in G_1 iff $(\phi(x), \phi(y))$ is an edge in G_2.

The pair of functions ϕ and ψ is called an isomorphism of G_1 and G_2. It is denoted by $G_1 \cong G_2$.

Suppose two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic graphs. Then it is clear that

i) $|V_1| = |V_2|$ i.e. G_1 and G_2 must have same number of vertices.

ii) $|E_1| = |E_2|$ i.e. G_1 and G_2 must have same number of edges.

iii) G_1 and G_2 must have an equal number of vertices with the same degree.

iv) G_1 and G_2 must have an equal number of loops.

v) G_1 and G_2 must have same number of pendent.

vi) G_1 and G_2 must have same number of pendent edges.

vii) If u and v are adjacent in G_1 then the corresponding vertices in G_2 are also adjacent.

In general it is easier to prove two graphs are not isomorphic by proving that any one of the above property fails.

Consider the following example

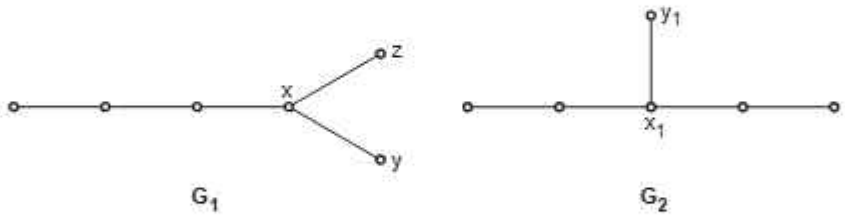


Fig. 7.7.1

These graphs have i) The same number of vertices.

ii) The same number of edges.

iii) An equal number of vertices of degree k.

These conditions are necessary for two graphs to be isomorphic but not sufficient.

In graph G_1, d (x) = 3, d (y) = 1, d (z) = 1 and x and y are adjacent to vertex x.

In graph G_2, d (x_1) = 3, d (y_1) = 1

there is only one pendent vertex adjacent to x_1. Hence adjacency is not preserved. Therefore G_1 is not isomorphic to G_2 i.e. $G_1 \ncong G_2$.

Note : Isomorphism of graphs is an equivalence relation.

Examples :

Example 7.7.1 *Draw all isomorphic graphs on vertices 2 and 3.*

Solution : i) For 3 vertices.

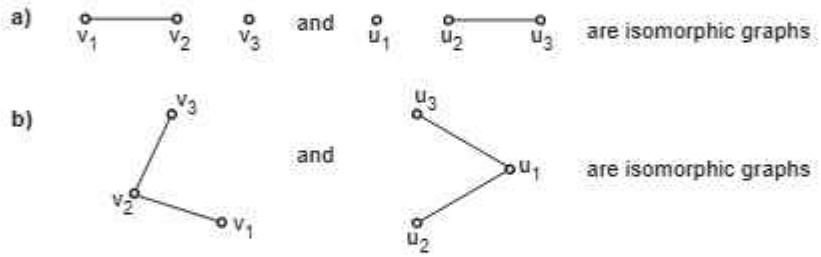


Fig. 7.7.2

ii) For two vertices.

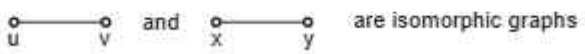


Fig. 7.7.2 (a)

Example 7.7.2 *Draw all non isomorphic graphs on 2 and 3 and 4 vertices.*

Solution : All non-isomorphic graphs on 2 vertices are

Fig. 7.7.3

All non isomorphic graphs on 3 vertices.

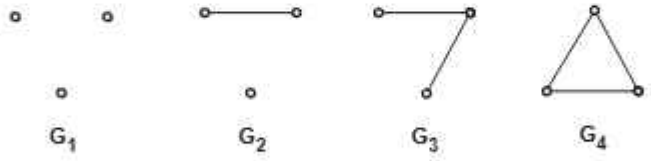


Fig. 7.7.3 (a)

All non isomorphic graphs on 4 vertices.

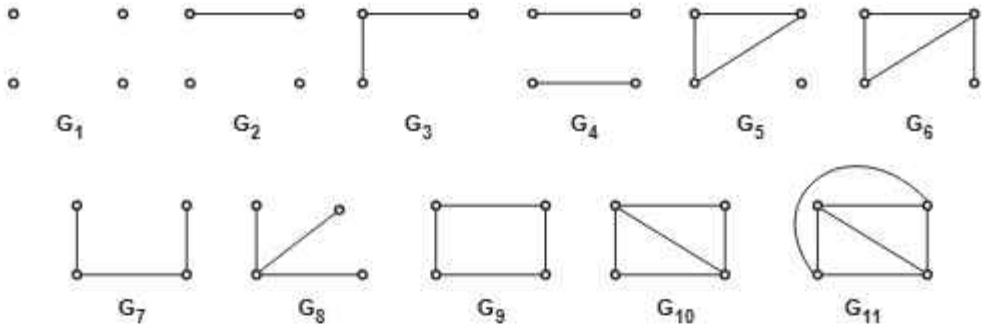


Fig. 7.7.3 (b)

Example 7.7.3 *Draw all non isomorphic graphs on 5 vertices and 5 edges.*

Solution : The following are non isomorphic graphs with 5 vertices and 5 edges.

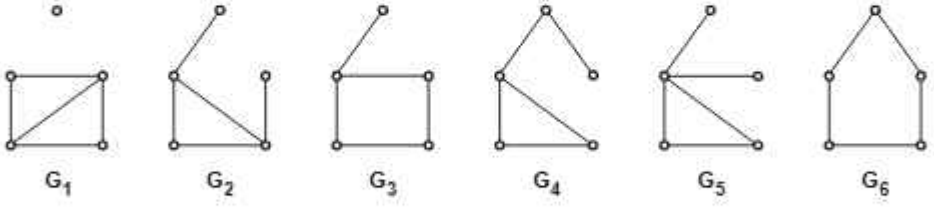


Fig. 7.7.4

Example 7.7.4 *Find whether the following pairs of graphs are isomorphic or not.*

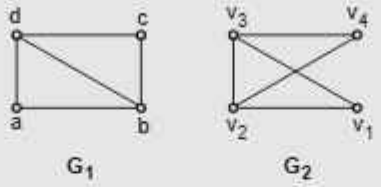


Fig. 7.7.5

Solution : i) Both the graphs have 4 vertices and 5 edges.

Both have 2 vertices of degree 3 and 2 vertices of degree 2.

$\therefore \phi : \{a, b, c, d\} \rightarrow \{v_1, v_2, v_3, v_4\}$ such that

$a \rightarrow v_1$

$c \rightarrow v_4$

$b \rightarrow v_2$

$d \rightarrow v_3$

ϕ preserves adjacency and non-adjacency of vertices.

$\psi \rightarrow E_1 \rightarrow E_2$ Is bijective.

$\therefore$ G_1 is isomorphic to Graph G_2.

ii)

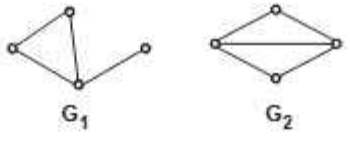


Fig. 7.7.5 (a)

As G_1 has 4 edges and G_2 has 5 edges, G_1 and G_2 are not isomorphic graphs.

iii) G_1 And G_2 are not isomorphic graphs because in G_1 vertices v_1 and v_3 of 4 degree are non adjacent while in G_2, the vertices x and y of degree 4 are adjacent.

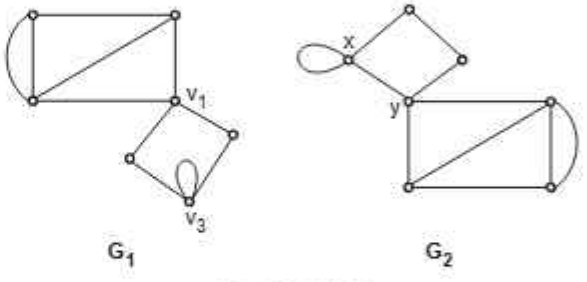


Fig. 7.7.5 (b)

Example 7.7.5 *Identify whether the given graphs are isomorphic or not* **SPPU : Dec.-12**

Solution :

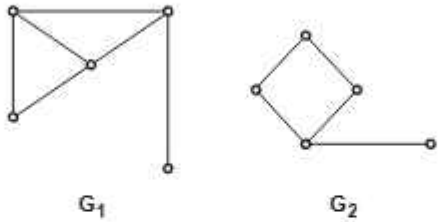


Fig. 7.7.6

In graph G_1, there are 2 vertices of degree 3. But in G_2, there is only one vertex of degree 3. So G_1 and G_2 are not isomorphic graphs.

ii)

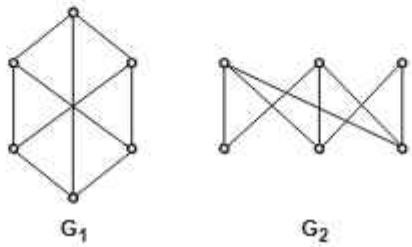


Fig. 7.7.6 (a)

Graph G_1 has 9 edges and G_2 has 8 edges.

Therefore G_1 and G_2 are not isomorphic graphs.

Example 7.7.6 *Determine whether graphs G and H are isomorphic or not. Justify your answer.* **SPPU : Dec.-18, Marks 6**

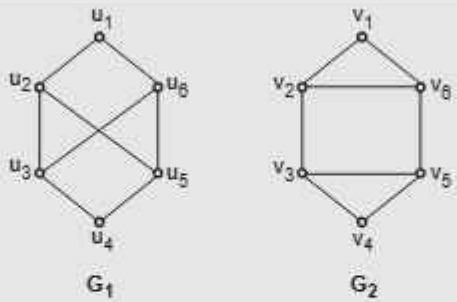


Fig. 7.7.7

Solution : G_1 And G_2 have 6 vertices, 8 edges.

Define $\phi : v(G_1) \rightarrow v(G_2)$ such that

$$u_1 \rightarrow v_1$$

$$u_2 \rightarrow v_2$$

$u_3 \rightarrow v_3$

$u_4 \rightarrow v_4$

$u_3 \rightarrow v_5$

$u_6 \rightarrow v_6$

ϕ is bijective mapping an edge $e = (u_2, u_5) \in E(G_1)$

but $\psi(e) = (\phi(u_2)), \phi(u_5)) = (v_2, v_5) \notin E(G_2)$

$\therefore$ G_1 and G_2 are not isomorphic graphs.

Example 7.7.7 *Show that the following graphs are isomorphic*

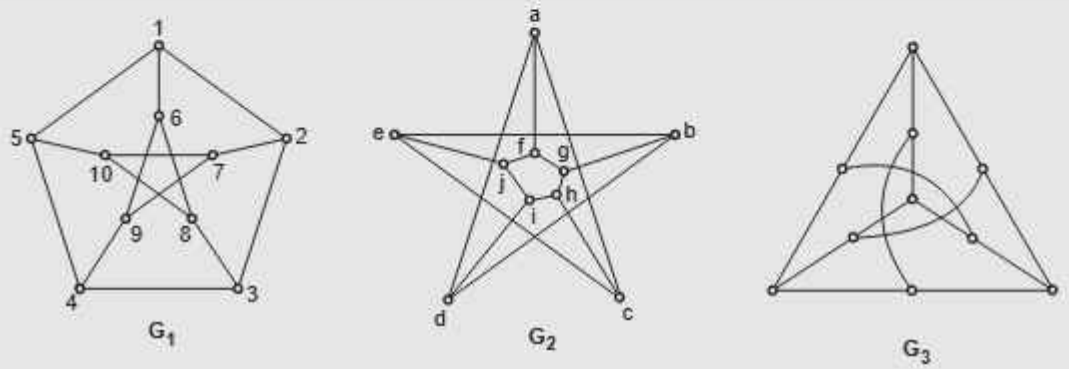


Fig. 7.7.8

Solution : All graphs G_1, G_2 and G_3 have 10 vertices and 15 edges.

All these graphs are 3-regular graphs. Also they preserve adjacency. Hence all graphs are isomorphic. Isomorphism is given by

$1 \rightarrow f$

$2 \rightarrow g$

$3 \rightarrow h$

$4 \rightarrow i$

$5 \rightarrow j$

$6 \rightarrow a$

$7 \rightarrow b$

$8 \rightarrow c$

$9 \rightarrow d$

$10 \rightarrow e$

In the similar way, we can show that G_1 and G_3 are isomorphic graphs.

Example 7.7.8 *Are the graphs isomorphic ? why ?*

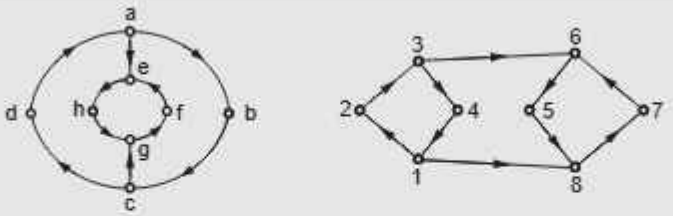


Fig. 7.7.9

Solution : Given graphs G_1 and G_2 have 8 vertices and 10 edges.

Both the graphs have 4 vertices of degree 2 and 4 vertices of degree 3. Also the adjacency is preserved.

$\phi : v(G_1) \to v(G_2)$ is defined as

$a \to 1, b \to 2, c \to 3, d \to 4, e \to 8, f \to 5, g \to 6, h \to 7,$

ϕ is bijective.

$\therefore$ G_1 and G_2 are isomorphic graphs.

7.8 New Graphs from Old Graphs

SPPU : May-07, 18, Dec.-09, 12

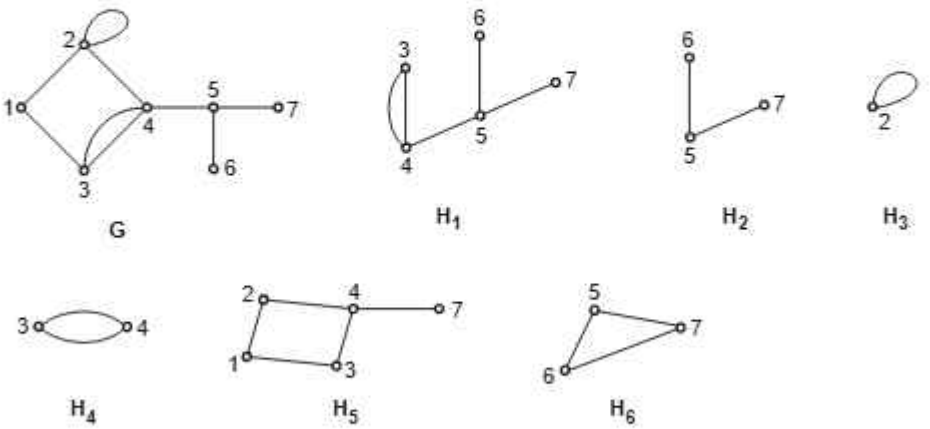
A good mathematical theory must contain sufficient number of models and examples. Moreover it must have methods to generate new objects from old ones.

In this section we derive new graphs from old graphs.

1) Subgraphs : Let G (V, E) be any graph. A graph H (V_1, E_1) is said to be subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$.

We also say that G is a supergraph of H.

e.g.

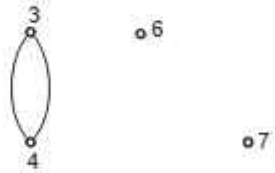


Graphs H_1, H_2, H_3 and H_4 are subgraphs of G. But graphs H_5 and H_6 are not subgraph as $(4, 7) \in E(H_5)$ but $(4, 7) \notin E(G)$ and $(6, 7) \in E(H_6)$ but $(6, 7) \notin E(G)$.

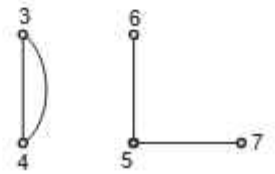
Properties :

1) Each graph is a subgraph of itself.
2) A subgraph of a subgraph of a graph G is a subgraph of G.
3) A graph G - {v} is a subgraph of G which is obtained from G by removing the vertex $v \in G$ and also the edges which are incident at v.
4) If $e \in (G)$ then G - e is a subgraph of G obtained from G by deleting the edge e.

In above example H_1 - {5} is



and H_1 - (4, 5) is given by



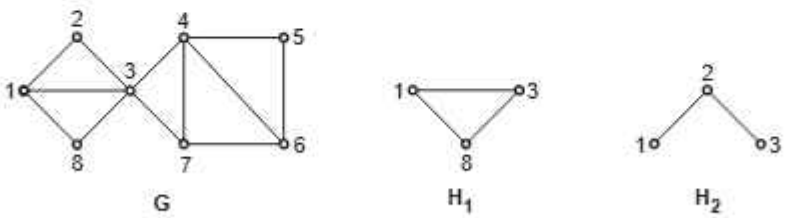
2) Edge Disjoint Subgraphs : Two subgraphs H_1 and H_2 of the graph G are said to be edge disjoint subgraphs of a graph G if there is no edge common between H_1 and H_2 i.e. $E(H_1) \cap E(H_2) = \phi$ It may have common vertex.

3) Vertex Disjoint Subgraphs : Two subgraphs H_1 and H_2 of the graph G are said to be vertex disjoint subgraphs if there is no vertex common between H_1 and H_2 i.e. $V(H_1) \cap V(H_2) = \phi$.

Note : 1) All vertex disjoint subgraphs are edge disjoint subgraphs.

4) Spanning subgraph : Let G (V, E) be any graph. A subgraph H of a graph G is said to be spanning subgraph if V (G) = V (H).

Example : Let G be the following graph :



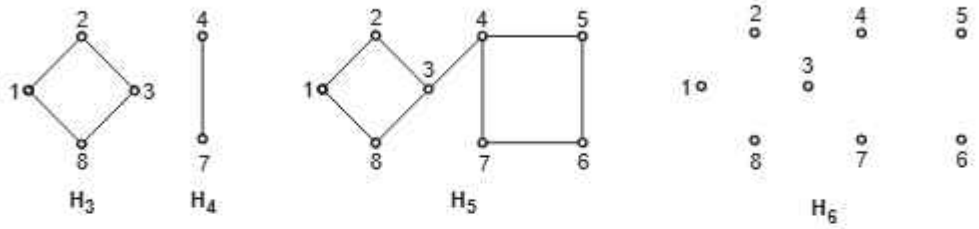


Fig. 7.8.1

Graphs H_1, H_2, ... H_6 are subgraphs of G.

H_1 and H_2 are edge disjoint subgraphs but not vertex disjoint subgraphs.

H_3 and H_4 are vertex disjoint subgraphs as well as edge disjoint subgraphs.

Subgraphs H_5 and H_6 are spanning subgraphs of G as $V(H_5) = V(H_6) = V(G)$.

5) Factors of a Graph : Let G be any graph. A k – factor of a graph G is defined to be a spanning subgraph of the graph with the degree of each of its vertex being K. i.e. K-factor is a K-regular graph.

When G has a 1-factor, say G_1, if the number of vertices are even and edges of G are point disjoint.

In particular, K_{2n+1} can not have a 1-factor but K_{2n} can have 1-factor of graph.

Example 1)

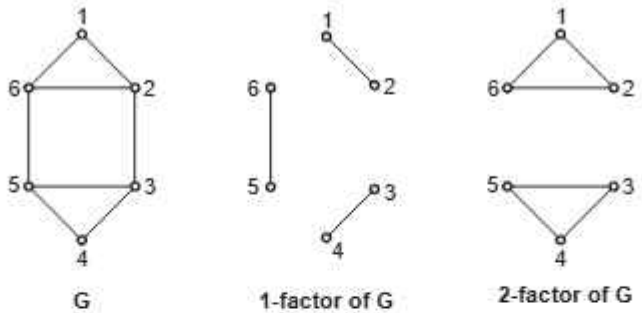


Fig. 7.8.2

Example 2)

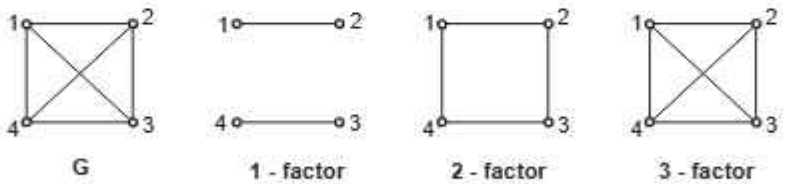
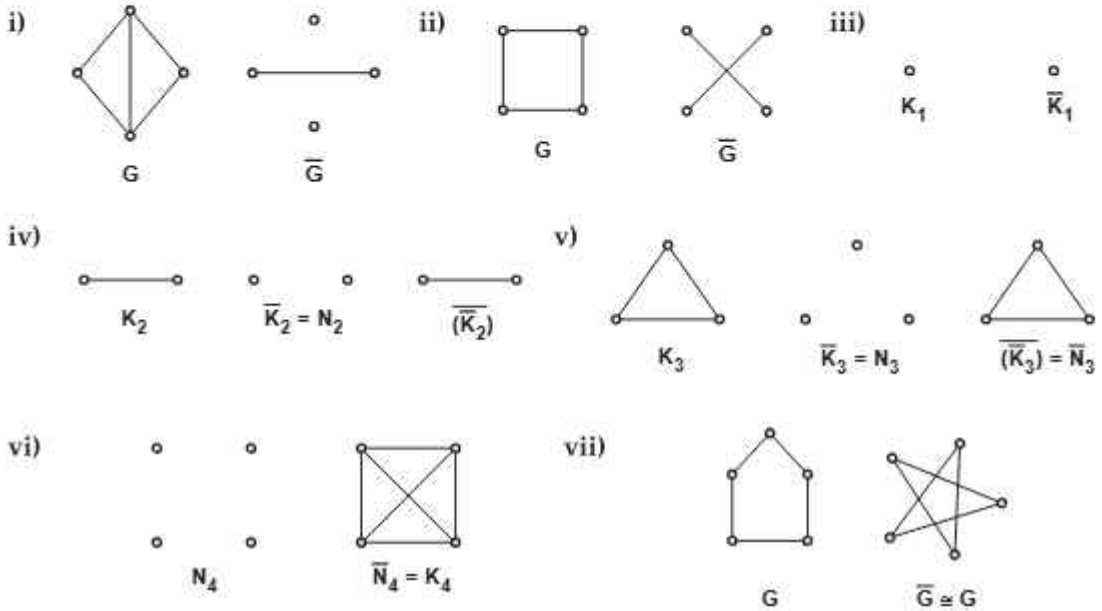


Fig. 7.8.3

6) Complement of a Graph : Let G be a simple graph. The complement of G is denoted by $\overline{G}$ is the graph whose vertex set is the same as the vertex set of G and in which two vertices are adjacent if and only if they are not adjacent in G.

A graph is said to be self complementary graph if it is isomorphic to its complement.

e.g.



G is isomorphic to $\overline{G}$. ∴ G is self complementary graph.

Note :

1) For any graph G, $\overline{(\overline{G})} = G$

2) The complement of the null graph on n vertices is the complete graph K_n on n vertices and vice versa.

3) K_1 is self complementary graph.

Examples :

Example 7.8.1 *For the following graphs, determine whether H (V', E') is a subgraph of G where*

i) V' = {A, B, C}, E' = {(A, B), (A, F)}

ii) V' = {B, C, D}, E' = {(B, C), (B, D)}

iii) V' = {A, B, C, D}, E' = {(A, C)}

SPPU : May-07, Dec.09

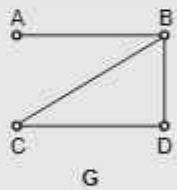


Fig. 7.8.4

Solution : i) H is not a subgraph of G because F ∈ V (H)

but F ∉ V (G), so V (H) ⊄ V (G)

ii) Here $V' \subset V(G)$, $E' \subset E(G)$, so $H(V', E')$ is a subgraph of G.

iii) Here $V' \subset v(G)$, but $E' \not\subset E(G)$ Therefore $H(V', E')$ is not a subgraph of G.

Example 7.8.2 *Find the 1-factor, 2-factor graph of the following graphs :*

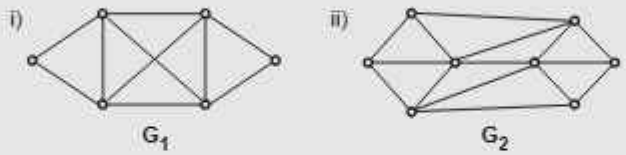


Fig. 7.8.5

Solution : i) A) 1-factor graphs :

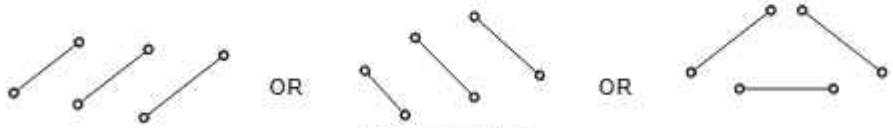


Fig. 7.8.5 (a)

B) 2-factor graphs :

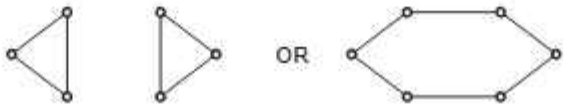


Fig. 7.8.5 (b)

ii) A) 1-factor graphs :

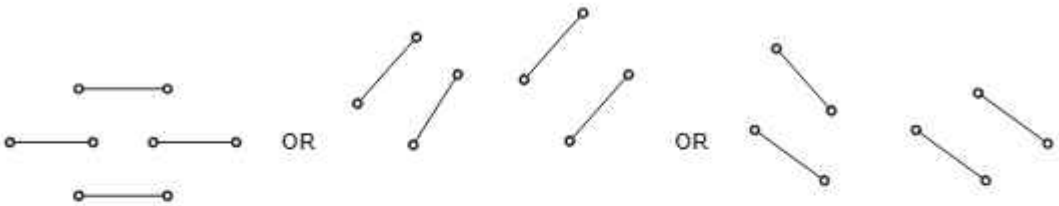


Fig. 7.8.5 (c)

B) 2-factor graphs :

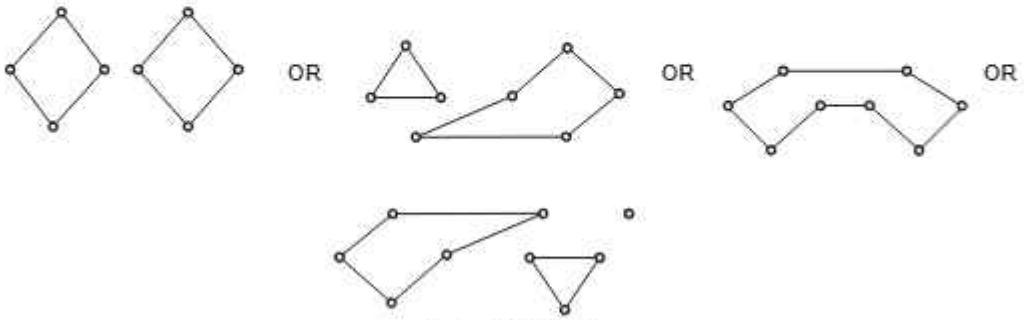


Fig. 7.8.5 (d)

Example 7.8.3 *Find the complement of the following graph. Is it self complementary.*

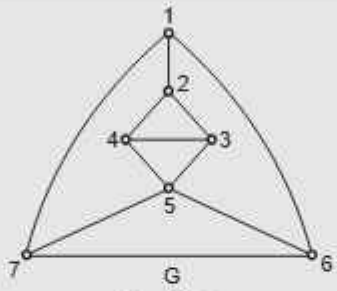


Fig. 7.8.6

Solution : The complement of G is as follows :

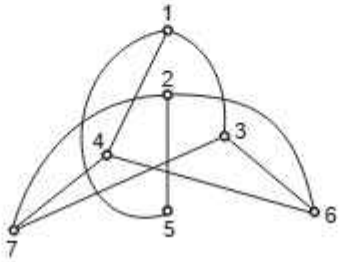


Fig. 7.8.6 (a)

It is not self complementary because G has a vertex of degree 4 but $\overline{G}$ do not have any vertex of degree 4.

Example 7.8.4 *Draw all self complementary graphs on 5 vertices.* **SPPU : Dec.-12**

Solution : The following graphs are self complementary graphs on 5 vertices.

Here $\overline{G_1} = G_2$ and $\overline{G_2} = G_1$

$\therefore G_1$ as well as G_2 are self complementary graphs.

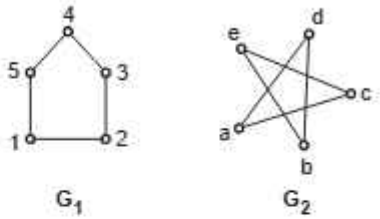


Fig. 7.8.7

Example 7.8.5 *Draw the complement of the following graphs.*

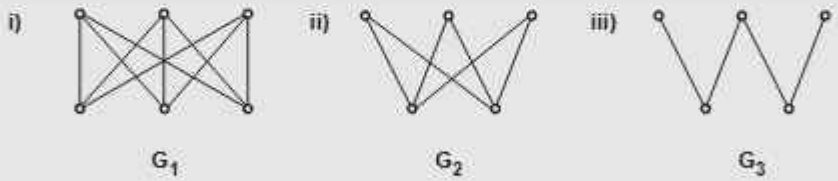


Fig. 7.8.8

Solution : The complements of given graphs are as follows :

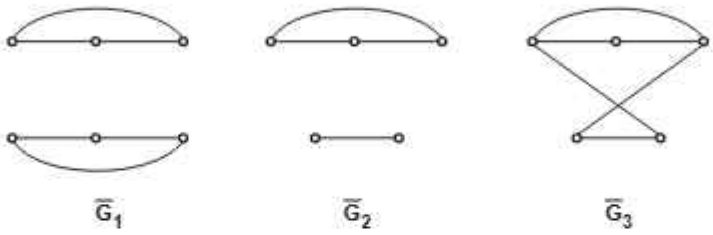


Fig. 7.8.8 (a)

Note : The complement of the complete bipartite graph $K_{m,n}$ gives the complete graphs K_m and K_n and vice versa.

Example 7.8.6 *Find the complement of the following graphs :*

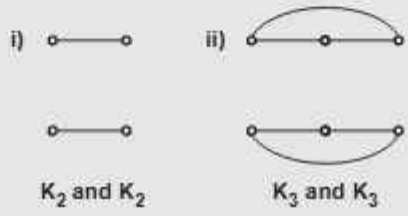


Fig. 7.8.9

Solution :

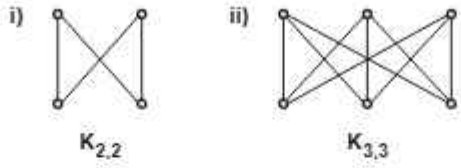


Fig. 7.8.9 (a)

7.9 Operations on Graphs

In this section, we define some standard operations of graphs like intersection, union, Ringsum etc.

A) Intersection of Two Graphs : The intersection of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is a graph G (V, E) whose vertex set is $V = V_1 \cap V_2$ and edge set is $E = E_1 \cap E_2$. The intersection of G_1 and G_2 is denoted by $G_1 \cap G_2$.

e.g.

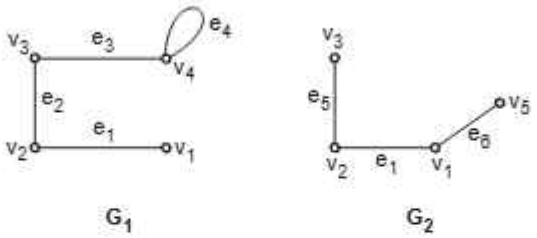


Fig. 7.9.1

$V_1 = \{v_1, v_2, v_3, v_4\}$ $V_2 = \{v_1, v_2, v_3, v_5\}$

$E_1 = \{e_1, e_2, e_3, e_4\}$ $E_2 = \{e_1, e_5, e_6\}$

Therefore $G = G_1 \cap G_2$ (v, E) where

$V = V_1 \cap V_2 = \{v_1, v_2, v_3\}$, $E = E_1 \cap E_2 = \{e_1\}$

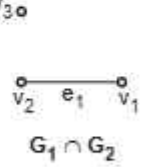


Fig. 7.9.2

B) Union of Two Graphs : Let G_1 (V_1, E_1), G_2 (V_2, E_2) be two graphs. The union of G_1 and G_2 is denoted by $G_1 \cup G_2 = G$ (v, E) and it is a graph whose vertex set is $V = V_1 \cup V_2$ and Edge set is $E = E_1 \cup E_2$

Consider the graphs G_1 and G_2 as shown in above example :

The union of G_1 and G_2 is given by G (v, E)

where $V = V_1 \cup V_2 = \{v_1, v_2, v_3, v_4, v_5\}$

$V = V_1 \cup V_2 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

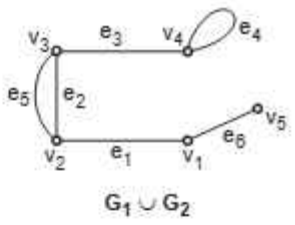


Fig. 7.9.3

Note : Both graphs G_1 and G_2 are subgraphs of $G_1 \cup G_2$.

C) The Ring Sum of Two Graphs : The ring sum of two graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) is denoted by $G = G_1 \oplus G_2$ (V, E) whose vertex set is $V = V_1 \cup V_2$ and the edge set consists of those edges which are either in E_1 or in E_2 but not in both i.e. $E = (E_1 \cup E_2) - (E_1 \cap E_2)$.

The ring sum of above graphs G_1 and G_2 is given by $G (V, E) = G_1 \oplus G_2$

$V = \{v_1, v_2, v_3, v_4, v_5\} = V_1 \cup V_2$

$E = (E_1 \cup E_2) - (E_1 \cap E_2) = \{e_2, e_3, e_4, e_5, e_6\}$

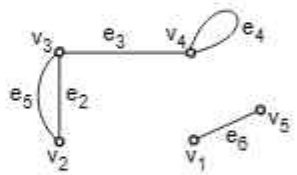


Fig. 7.9.4

D) Sum of Two Graphs : The sum of two vertex disjoint graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) is denoted by $G_1 + G_2 = G$ (V, E) is defined as the graph whose vertex set is V $(G_1 \cup G_2)$ and consisting of edges which are in G_1 or G_2 together with the edges

obtained by joining each vertex of G_1 to each vertex of G_2. Thus $G_1 + G_2$ is nothing but the graph $G_1 \cup G_2$ in which each vertex of G_1 is joined to each vertex of G_2 by an edge.

e.g. If

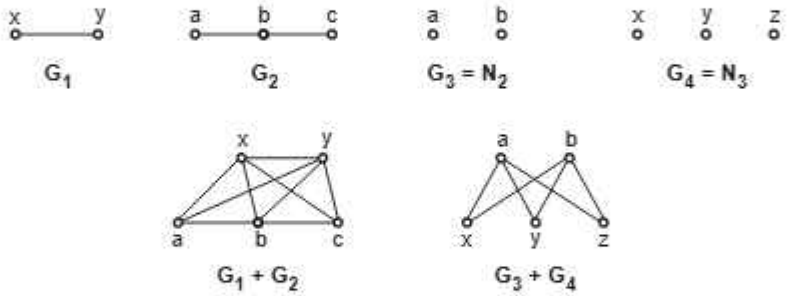


Fig. 7.9.5

Note : The sum $N_m + N_n$ of null graphs is nothing but the complete bipartite graph $K_{m,n}$.

E) Product of Two Graphs : Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two vertex disjoint graphs then the product of G_1 and G_2 is denoted by $G_1 \times G_2 = G(V, E)$ is a graph whose vertex set is $V = V_1 \times V_2$ and two edges (x_1, x_2) and (y_1, y_2) are adjacent if $x_1 = y_1$ and x_2 is adjacent to y_2 in G_2 or $x_2 = y_2$ and x_1 is adjacent to y_1 in G_1.

e.g. If

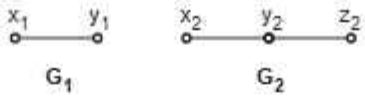


Fig. 7.9.6

Then $G_1 \times G_2$ is given below :

Fig. 7.9.7

F) Decomposition : A graph G is said to have been decomposed into two subgraphs H and K if $H \cup K = G$ and $H \cap K$ = Null graph i.e. each edge of G occurs either in H or in K but not in both. But vertices may occur in both. In this context isolated vertices are not considered.

e.g. The decomposition of G into H and K is given below :

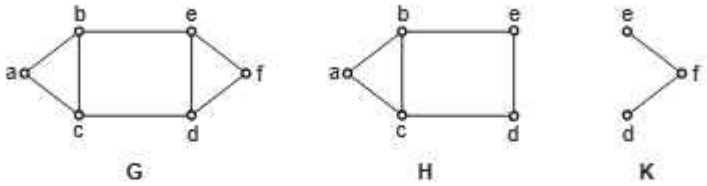


Fig. 7.9.8

G) Fusion of vertices : A pair of vertices a and b in a graph G are said to be fused if a and b are replaced by a single new vertex say c such that every edge that was incident on either a or b or both is incident on the new vertex c. The fusion of two vertices do not change the number of edges but reduced number of vertices by 1.

e.g.

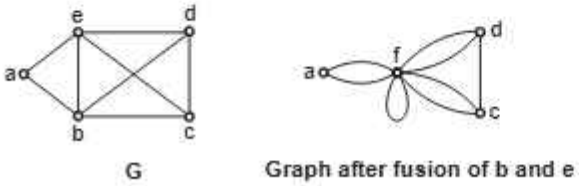


Fig. 7.9.9

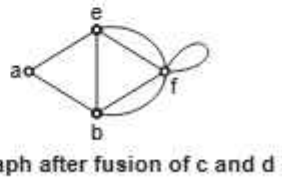


Fig. 7.9.10

7.10 Paths and Circuits

1) Path : An alternating sequence of vertices and edges $v_0 - e_1 - e_2 - e_3 - \ldots v_{n-1} - e_n - v_n$ beginning are ending with vertices in which each edge is incident with the two vertices immediately preceding if and following it is called a path.

The vertices v_o and v_n are called terminal vertices and $v_1, v_2, \ldots v_{n-1}$ are called its interior vertices.

e.g. Let G be the following graph.

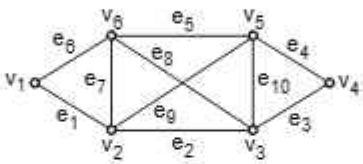


Fig. 7.10.1

Following are some examples of path

i) $v_1 - e_1 - v_2 - v_3$

ii) $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_2 - v_2$

iii) $v_6 - e_5 - v_5 - e_{10} - v_3 - e_8 - v_6$

iv) $v_1 - e_6 - v_6$

There are so many paths between every distinct pair of vertices of given graph G. Depending upon the nature of terminal vertices, there are two types at path.

A path in which terminal vertices are equal is called a **closed path**. A closed path is known as circuit. A path in which terminal vertices are distinct, is called an **open path**.

In above examples, paths in (i) and (iv) are open paths and (ii) and (iii) are closed paths.

7.10.1 Simple Path

A path in a graph G is said to be a simple path if the edges do not repeat in the path. Vertices may be repeated.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5$ is a simple path.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_7 - v_6$ is a simple path in which v_3 is repeated.

iii) $v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_8 - v_6$ is a simple path in which v_3 is repeated.

iv) $v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_3 - v_4$ is not a simple path as an edge e_3 is repeated.

7.10.2 Elementary Path

A path in a graph G is said to be elementary path if vertices do not repeat in the path. Every elementary path is a simple path.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3$ is an elementary path.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_8 - e_6 - e_7 - v_2$ is not an elementary path. But it is simple path.

7.10.3 Simple and Elementary Circuits

A closed path is known as circuit.

A simple path which is closed is called a simple circuit of graph.

In other words, A circuit in a graph G is said to be simple circuit if all edges of a circuit are distinct.

A circuit in a graph G is said to be elementary circuit if all vertices of a circuit are distinct except the terminal vertices i.e. the first and last vertices. The number of edges in any circuit (or path) is called the length of the circuit (or path).

In above graph G.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_1 - v_1$ is a circuit with e_1 repeated twice and v_2 is also repeated twice.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_7 - v_6 - e_6 - v_1$ is a simple circuit but not elementary circuit as v_2 is repeated.

iii) $v_1 - e_1 - v_2 - e_7 - v_6 - e_6 - v_1$ is an elementary circuit.

7.11 Connected and Disconnected Graphs

A graph G is said to be connected graph if there exists a path between every pair of vertices. A graph which is not connected is called the disconnected graph.

A disconnected graph consists of two or more parts called components or blocks if each of which is connected and there is no path between two vertices if they belong to different components.

A connected graph has only one component, e.g.

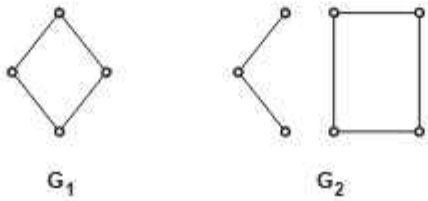


Fig. 7.11.1

Graph G_1 is connected and G_2 is disconnected graph with two components.

i) A directed graph is said to be connected if the undirected graph derived from it by ignoring the directions of the edges, is connected.

ii) A directed graph is said to be strongly connected if for every two vertice x and y in the graph there is a path from x to y as well as y to x.

iii) A directed graph (or diagraph) is said to be weakly connected if it is not strongly connected and its underlying graph is connected.

iv) A directed graph which is neither strongly connected nor weakly connected is called as disconnected digraph.

Consider the following examples :

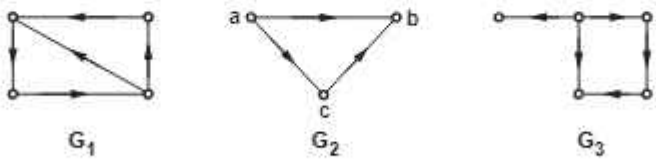


Fig. 7.11.2

Graph G_1 is strongly connected. In graph G_2 there is no path from b to a. $\therefore G_2$ is not strongly connected. G_2 is weakly connected. G_3 is also weakly connected.

7.12 Connectivity of Graph

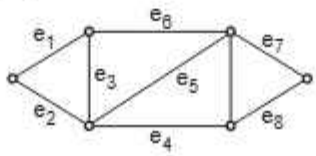
In this section, we study the measure, in some, sense, of the connectedness of a graph. For this, we must know how many edges or vertices are to be removed so that the number of components increases.

7.12.1 Edge Connectivity

A set of edges of a connected graph G whose removal disconnects G is called a disconnecting set of G. A cutset is defined as a minimal disconnecting set i.e. A minimal set of edges whose removal from G gives a disconnected graph is called a cutset.

If a cutset of a graph contains only one edge, then that edge is called as an isthmus or bridge. The number of edges in the smallest cutset of G is called the edge connectivity of G. It is denoted by λ (G).

e.g. Consider the following graph G.



G

Fig. 7.12.1

Cutsets of G are as follows :

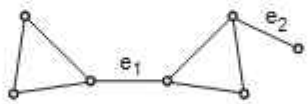
i) $\{e_4, e_5, e_6\}$,

ii) $\{e_1, e_3, e_6\}$,

iii) $\{e_1, e_2\}$,

A set $\{e_1, e_2, e_3\}$ is not a cutset because its subset $\{e_1, e_2\}$ is a cutset. The edge connectivity of graph G is 2. i.e. λ (G) = 2.

Consider the following graph G_1.



G_1

Fig. 7.12.2

Graph G_1 has edge connectivity 1 as $G_1 - \{e_1\}$ is a disconnected graph. $\therefore e_1$ is an isthmus or Bridge.

$G - e_2$ is also disconnected graph. $\therefore e_2$ is also isthmus.

7.12.2 Vertex Connectivity

The vertex connectivity k(G) of a simple connected graph G is defined as the smallest number of vertices whose removal disconnects the graph.

In graph G, the sets $\{v_2, v_5, v_4\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_2, v_3\}$ disconnect graph G. The smallest set is $\{v_2, v_3\}$ $\therefore$ k (G) = 2.

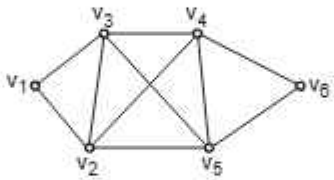


Fig. 7.12.3

1) A graph G is said to be k-connected if its vertex connectivity is k.
2) A graph G is said to be seperable graph if its vertex connectivity is one.
3) A vertex v of a connected graph G is said to be cut vertex if G – {v} is a disconnected graph.
4) $k(G) \le \lambda(G) \le \delta$

i.e. vertex connectivity ≤ edge connectivity ≤ minimum degree of a vertex in G and

$$\lambda(G) \le \left[\frac{2e}{n}\right]$$

where e = number of edges in G.

n = no. of vertices in G.

7.13 Shortest Path Algorithm

SPPU : May-05, 07, 14, 15, 19, Dec.-06, 07, 12, 13, 14, 15, 16, 17, 18

Suppose there is associated to each edge e of a graph a real number w (e). w (e) is called the weight of e. A weighted graph is a graph in which each edge has a weight. The weight of graph G is the sum of weight of all edges of G. Weighted graph has many applications in communication networks. Given a railway network connecting several cities, determine a shortest route between two cities.

We consider the weighted graph where the vertices are the towns, rail roads are the edges and the weight represent the distance between directly linked cities. Therefore weights are non negative integers. The problem is to find a path of minimum weight connectivity two given cities. Of course, this is possible theoretically. One has to list all paths, find their weights and select minimum one. But for large networks (large number of vertices and edges) this may not be efficient. So we required different method to take such problems. The algorithm was found by Dijkstra (1959) and is known as Dijkstra's algorithm.

A) Dijkstra's algorithm to find the shortest path from the vertex a to vertex z of a graph G. Let G (v, E) be a simple graph and a, z ∈ V.

Suppose L(x) is the label of the vertex which represents the length of the shortest path from the vertex a. W_{ij} = weight of an edge $e_{ij} = (v_i, v_j)$

Consider the following steps :

Step 1 : Let P be the set of those vertices which have permanent labels and T = set of all vertices of G.

Set L (a) = 0, L (x) = ∞ ; ∀ x ∈ T and x ≠ a

P = ϕ and T = v.

Step 2 : Select the vertex v in T which has the smallest label. This label is called the permanent label of v. Also set P as P ∪ {v} and T as T - {v}.

If v = z then L (z) is the length of the shortest path from the vertex a to z and stop the procedure.

Step 3 : If v ≠ z, then revise the labels of the vertices of T. i.e. The vertices which do not have permanent labels.

The new label of x in T is given by

$$L(x) = \min \{\text{old } L(x), L(v) + w(v, x)\}$$

where w (v, x) is the weight of the edge joining v and x. If there is no edge joining v and x then take w (v, x) = ∞.

Step 4 : Repeat the steps 2 and 3 until z gets the permanent label.

Examples :

Example 7.13.1 *Use Dijkstra's algorithm to find the shortest path between a and z.*

SPPU : May-05, 14, 19, Marks 8, Dec.-06, Marks 6

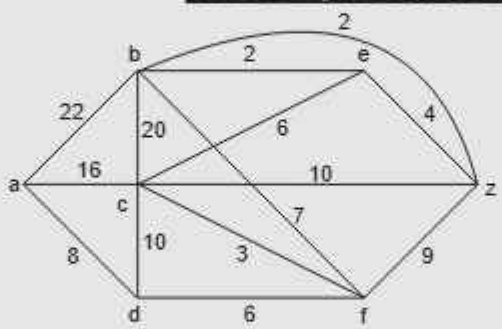


Fig. 7.13.1

Solution :

Step 1 : P = ∅ , T = {a, b, c, d, e, f, z}

$L\{a\} = 0$

$L\{x\} = \infty, \quad \forall\ x \in T, \quad x \neq a$

Step 2 : $v = a$, the permanent label of a is 0

$P = \{a\}, \quad T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 22\} = 22$

$L\{c\} = \min\{\infty, 0 + 16\} = 16$

$L\{d\} = \min\{\infty, 0 + 8\} = 8$

$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$

$\therefore \quad L\{d\} = 8$ is the minimum label.

Step 3 : $v = d$, the permanent label of d is 8.

$P = \{a, d\}, \quad T = \{b, c, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(d) + w(d, b)\}$

$= \min\{22, 8 + \infty\} = 22$

$L\{c\} = \min\{16, 8 + 10\} = 16$

$L\{e\} = \min\{\infty, 8 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 8 + 6\} = 14$

$L\{z\} = \min\{\infty, 8 + \infty\} = \infty$

$\therefore \quad L\{f\} = 14$ is the minimum label.

Step 4 : $v = f$, the permanent label of f is 14.

$P = \{a, d, f\}, \quad T = \{b, c, e, z\}$

$L\{b\} = \min\{\text{old } L(b), L(f) + w(b, f)\}$

$= \min\{22, 14 + 7\} = 21$

$L\{c\} = \min\{16, 14 + 3\} = 16$

$L\{e\} = \min\{\infty, 14 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 14 + 9\} = 23$

$\therefore \quad L\{c\} = 16$ is the minimum label.

Step 5 : $v = c$, the permanent lable of c is 16.

$P = \{a, d, f, c\}, \quad T = \{b, e, z\}$

$L\{b\} = \min\{\text{old } L(b), L(f) + w(f, b)\}$

$= \min\{21, 16 + 20\} = 21$

$L\{e\} = \min\{\infty, 16 + 6\} = 22$

$L\{z\} = \min\{23, 16 + 10\} = 23$

$\therefore \quad L\{b\} = 2$ is the minimum label.

Step 6 : $v = b$, the permanent label of b is 21.

$P = \{a, d, f, c, b\}, \quad T = \{e, z\}$

$L\{e\} = \min\{\text{old } L(e), L(b) + w(e, b)\}$

$= \min\{22, 21 + 2\} = 22$

$L\{z\} = \min\{23, 21 + 2\} = 23$

$\therefore \quad L\{e\} = 22$ is the minimum label.

Step 7 : $v = e$, the permanent label of e is 22.

$P = \{a, d, f, c, b, e\}, \quad T = \{z\}$

$L(z) = \min\{\text{old } L(z), L(e) + w(e, z)\}$

$= \min\{23, 22 + 4\} = 23$ which is the minimum label

Step 8 : $v = z$, the permanent label of z is 23.

Hence the length of the shortest path from a to z is 23.

The shortest path is adfz or adfbz.

Fig. 7.13.2 (a)

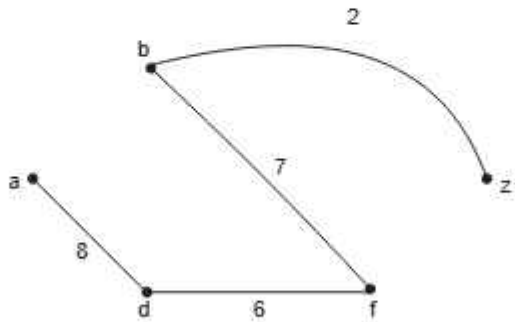


Fig. 7.13.2 (b)

Example 7.13.2 *Find the shortest path from a-z in the given graph using Dijkstra's algorithm.*

SPPU : May-07, Dec.-07, 15, 18, Marks 6

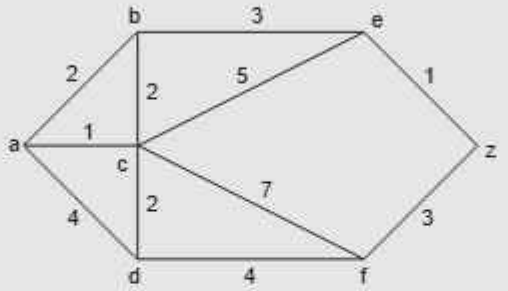


Fig. 7.13.3

Solution :

Step 1 : Set $P = \varnothing$, $T = \{a, b, c, d, e, f, z\}$

$L\{a\} = 0$

$L\{x\} = \infty, \ \forall x \in T, \ x \neq a$

Step 2 : $v = a$, the permanent label of a is 0.

$P = \{a\}, \ T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 2\} = 2$

$L\{c\} = \min\{\infty, 0 + 1\} = 1$

$L\{d\} = \min\{\infty, 0 + 4\} = 4$

$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$

$\therefore$ $L\{c\} = 1$ is the minimum label.

Step 3 : $v = c$, the permanent label of c is 1.

$P = \{a, c\}, \ T = \{b, d, e, f, z\}$

$L\{b\} = \min\{2, 1 + 2\} = 2$

$L\{d\} = \min\{4, 1 + 2\} = 3$

$L\{e\} = \min\{\infty, 1 + 5\} = 6$

$L\{f\} = \min\{\infty, 1 + 7\} = 8$

$L\{z\} = \min\{\infty, 1 + \infty\} = \infty$

$\therefore \quad L\{b\} = 2$ is the minimum label.

Step 4 : $v = b$, the permanent label of b is 2.

$P = \{a, c, b\}, \quad T = \{d, e, f, z\}$

$L\{d\} = \min\{3, 2 + \infty\} = 3$

$L\{e\} = \min\{6, 2 + 3\} = 5$

$L\{f\} = \min\{8, 2 + \infty\} = 8$

$L\{z\} = \min\{\infty, 2 + \infty\} = \infty$

$\therefore \quad L\{d\} = 3$ is the minimum label.

Step 5 : $v = d$, the permanent label of d is 3.

$P = \{a, c, b, d\}, \quad T = \{e, f, z\}$

$L\{e\} = \min\{5, 3 + \infty\} = 5$

$L\{f\} = \min\{8, 3 + 4\} = 7$

$L\{z\} = \min\{\infty, 3 + \infty\} = \infty$

$\therefore \quad L\{e\} = 5$ is the minimum label.

Step 6 : $v = e$, the permanent label of e is 5.

$P = \{a, c, b, d, e\}, \quad T = \{f, z\}$

$L\{f\} = \min\{7, 5 + \infty\} = 7$

$L\{z\} = \min\{\infty, 5 + 1\} = 6$

$\therefore \quad L\{z\} = 6$ is the minimum label.

Step 7 : $v = z$, the permanent label of z is 6.

Hence the length of shortest path from a to z is 6.

The shortest path is a b e z.

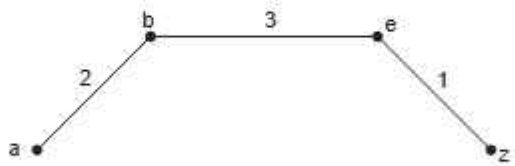


Fig. 7.13.3 (a)

Example 7.13.3 *Find the shortest path between a-z for the given graph : using Dijkstra's algorithm.*

SPPU : Dec.-12, 13, 14, 16, May-15, Marks 8

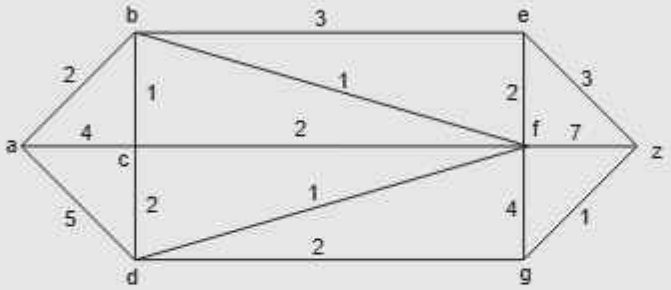


Fig. 7.13.4

Solution :

Step 1 : Set $P = \varnothing$, $T = \{a, b, c, d, e, f, g, z\}$

$L\{a\} = 0$

$L\{x\} = \infty, \quad \forall\ x \in T, \quad x \neq a.$

Step 2 : $v = a$, the permanent label of a is 0.

$P = \{a\}, \quad T = \{b, c, d, e, f, g, z\}$

$L\{b\} = \min\{\text{old } L(b),\ L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 2\} = 2$

$L\{c\} = \min\{\infty, 0 + 4\} = 4$

$L\{d\} = \min\{\infty, 0 + 5\} = 5$

$L\{e\} = L\{f\} = L\{g\} = L\{z\} = \infty$

$\therefore \quad L\{b\} = 2$ is the minimum label. The permanent label of b is 2.

Step 3 : $v = b$

$P = \{a, b\}, \quad T = \{c, d, e, f, g, z\}$

$L\{c\} = \min\{L(c), L(b) + w\{b, c\}\}$

$= \min\{4, 2 + 1\} = 3$

$L\{d\} = \min\{5, 2 + \infty\} = 5$

$L\{e\} = \min\{\infty, 2 + 3\} = 5$

$L\{f\} = \min\{\infty, 2 + 1\} = 3$

$L\{g\} = L\{z\} = \infty$

$\therefore \quad L\{c\} = L\{f\} = 3$ are the minimum labels.

Step 4 : $v = c$ or f

Let $v = f$, permanent label of f is 3.

$P = \{a, b, f\}, \quad T = \{c, d, e, g, z\}$

$L\{c\} = \min\{3, 3 + 2\} = 3$

$L\{d\} = \min\{5, 3 + 1\} = 4$

$L\{e\} = \min\{5, 3 + 2\} = 5$

$L\{g\} = \min\{\infty, 3 + 4\} = 7$

$L\{z\} = \min\{\infty, 3 + 7\} = 10$

$\therefore \quad L\{c\} = 3$ is the minimum label.

Step 5 : $v = c$, permanent label of $c = 3$.

$P = \{a, b, f, c\}, \quad T = \{d, e, g, z\}$

$L\{d\} = \min\{4, 3 + 2\} = 4$

$L\{e\} = \min\{5, 3 + \infty\} = 5$

$L\{g\} = \min\{7, 3 + \infty\} = 7$

$L\{z\} = \min\{10, 3 + \infty\} = 10$

$\therefore \quad L\{d\} = 4$ is the minimum label.

Step 6 : $v = d$, permanent label of $d = 4$.

$P = \{a, b, f, c, d\}, \quad T = \{e, g, z\}$

$L\{e\} = \min\{5, 4 + \infty\} = 5$

$L\{g\} = \min\{7, 4 + 2\} = 6$

$L\{z\} = \min\{10, 4 + \infty\} = 10$

$\therefore \quad L\{e\} = 5$ is the minimum label.

Step 7 : $v = e$, permanent label of e is 5.

$P = \{a, b, f, c, d, e\}, \quad T = \{g, z\}$

$L\{g\} = \min\{6, 5 + \infty\} = 6$

$L\{z\} = \min\{10, 5 + 3\} = 8$

$\therefore \quad L\{g\} = 6$ is the minimum label.

Step 8 : $v = g$, permanent label of g is 6.

$P = \{a, b, f, c, d, e, g\}, \quad T = \{z\}$

L {z} = min {8, 6 + 1} = 7 which is the minimum label.

Step 9 : v = z, permanent label of z is 7.

Hence the length of shortest path from a to z is 7.

The shortest path is a b f d g z.

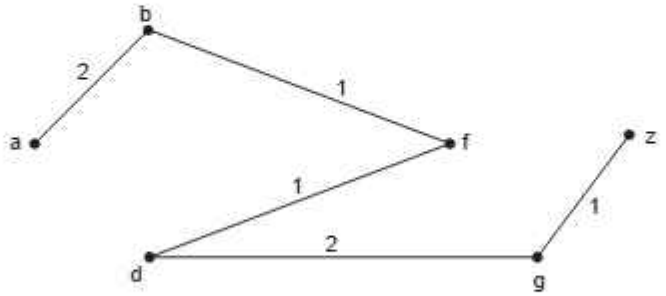


Fig. 7.13.4 (a)

7.14 Eulerian Graphs

SPPU : Dec.-09, May-19

7.14.1 Konigsberg's Seven Bridges

The city of Konigsberg, the capital of old East Prussia was founded by the tentonic knights in 1254. The river pregel flowed through the city which included an island called Kneiphof. There were seven bridges on this river connecting the island and various parts of the city as shown below :

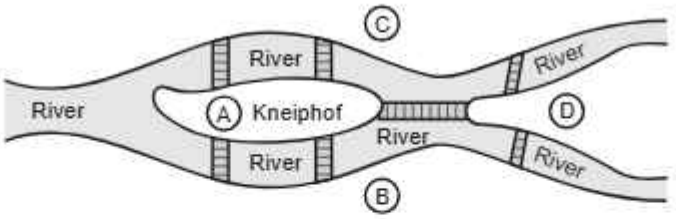


Fig. 7.14.1

The seven bridges problem of Konigsberg :

"Is it possible to cross the seven bridges in continuous walk without recrossing any of them ?"

On 26 August 1735, a swiss mathematician Leonhard Euler presented a paper which described the solution of this problem. Euler concluded that such a continuous walk over the bridges was impossible. This paper of Euler is considered as the origin of graph theory.

Euler represented the arrangement of the river and its bridges by means of a graph in which the land areas were the vertices and bridges were the edges. In this section we shall study the general problem of traversing the vertices and edges of a graph.

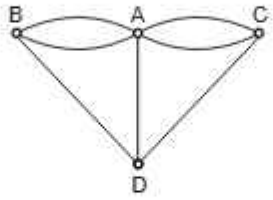


Fig. 7.14.2

In this section we shall study the general problem of traversing the vertices and edges of a graph.

7.14.2 Eulerian Path and Circuit

A path is called an Eulerian path if every edge of graph G appears exactly once in the path.

A circuit of a graph which contains every edge of graph exactly once is called the Eulerian circuit.

A graph which has an Eulerian circuit is called as Eulerian graph.

The problem of find Eulerian path is the same as the problem of drawing a network without lifting the pencil off the paper and without retracing an edge.

Consider the following graphs :

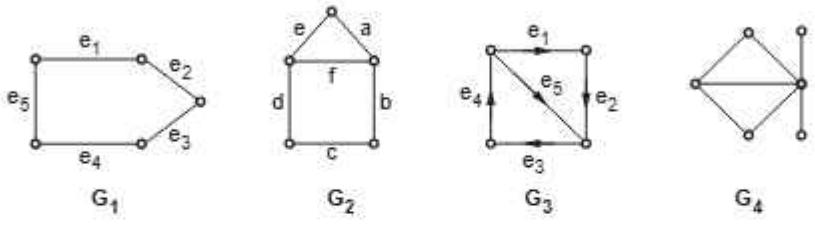


Fig. 7.14.3

In graph G_1, Eulerian circuit is $e - e_2 - e_3 - e_4 - e_5 - e_1$

$\therefore G_1$ is an Eulerian graph.

In graph G_2, Eulerian circuit does not exist.

$\therefore G_2$ is not Eulerian graph.

In graph G_3, Eulerian path is $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4 \rightarrow e_4 \rightarrow e_5$ but G_3 does not have any Eulerian circuit.

$\therefore G_3$ is not Eulerian graph.

G_4 is also not an Eulerian graph.

The existence of Eulerian paths and circuits in a graph depends upon the degree of vertices.

Theorem 1 : An undirected graph possesses an Eulerian path iff it is connected and has either zero or two vertices of odd degree.

Theorem 2 : An undirected graph possesses an Eulerian circuit iff it is connected and its vertices are all of even degree.

Theorem 3 : A directed graph possesses an Eulerian circuit iff it is connected and incoming degree of every vertex is equal to its outgoing degree.

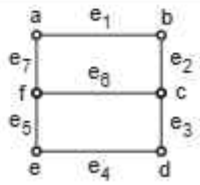
Examples :

Example 7.14.1 *Draw a graph which has Eulerian path but not Eulerian circuit.*

Solution : This graph G has no Eulerian circuit because d (c) = 3 in G.

Eulerian path is

$c - e_3 - d - e_4 - e - e_5 - f - e_6 - e - e_2 - b - e_1 - a - e_7 - f$



G

Fig. 7.14.4

Example 7.14.2 *For what values of n does K_n, the complete graph on n vertices have an Eulerian circuit ? an Eulerian path ?*

Solution : In the complete graph on n vertices (K_n), vertex is adjacent to remaining (n–1) vertices. So degree of each vertex is n–1.

If n is an odd, the degree of each vertex will be an even. Hence by theorem 1, Eulerian circuit exists and Graph is Eulerian.

For an Eulerian path the graph should have either zero or exactly two vertices of odd degree. It is possible only in K_2.

Fig. 7.14.5

Hence Eulerian path exists for K_2 only.

Example 7.14.3 *Find under what conditions $K_{m,n}$ the complete bipartite graph will have an Eulerian circuit*

SPPU : Dec.-09, May-19, Marks 3

Solution : In $K_{m,n}$ consider the following cases.

Case 1 : m = n and both m and n are even :

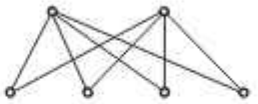
In this case, degree of each vertex is even, Hence by theorem 1, $K_{m,n}$ will have an Eulerian circuit. For example $K_{1,2}$ and $K_{4,4}$

$K_{2,2}$ $K_{4,4}$

Fig. 7.14.6

Case 2 : If m = n and m, n are odd :

In this case degree of each vertex is odd. Hence Eulerian circuit will not exist.



$K_{2,4}$

Fig. 7.14.7

Case 3 : If m ≠ n but m and n are even :

In this case, degree of each vertex is even. So there exists an Eulerian circuit.

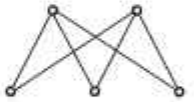
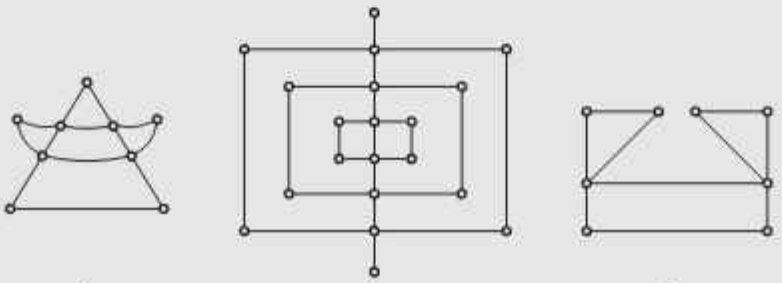


Fig. 7.14.8

Case 4 : If m ≠ n and either m is odd or n is odd or both are odd : then graph will have vertices of odd degree. Hence Eulerian circuit does not exist. e.g. K_2, 3.

Example 7.14.4 *Which of the following graphs possess Eulerian circuits or paths ?*



G_1 G_2 G_3

Fig. 7.14.9

Solution : In graph G_1, each vertex is of even degree, hence it possesses an Eulerian circuit.

In graph G_2 there are exactly 2 vertices of odd degree. Hence G_2 possesses an Eulerian path.

In graph G_3, each vertex is of even degree. Hence G_3 has an Eulerian circuit.

7.15 Hamiltonian Graphs

SPPU : Dec.-04, 09, 10, 12, 15, May-17

In this section, we introduce a class of graphs which possess a strinking similarity to Eulerian graphs.

We will now define Hamiltonian path and circuits of the connected graph.

A path in a connected graph G is called a Hamiltonian path if it contains every vertex of G exactly once.

A circuit in a connected graph G is called a Hamiltonian circuit if it contains every vertex of G exactly once.

A graph which has a Hamiltonian circuit is called a Hamiltonian Graph.

Consider the following graphs :

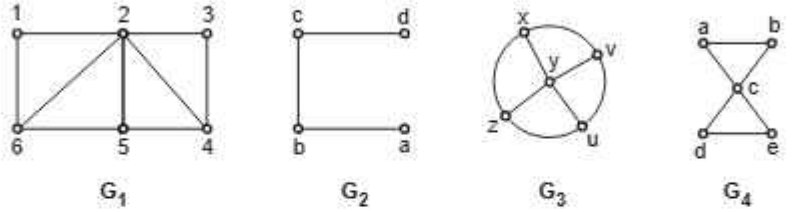


Fig. 7.15.1

In graph G_1, Hamiltonian circuit is 1-2-3-4-5-6-1

$\therefore G_1$ is a Hamiltonian graph.

In graph G_2, Hamiltonian path is a-b-c-d but Hamiltonian circuit does not exist.

$\therefore G_2$ is not Hamiltonian graph.

In graph G_3, Hamiltonian circuit is x-y-z-u-v-x

$\therefore G_3$ is Hamiltonian graph but it is not Eulerian.

In graph G_4, Hamiltonian path is a-b-c-d-e but Hamiltonian circuit does not exist.

$\therefore G_4$ is not Hamiltonian but Eulerian graph.

Remark : Hamiltonian graphs were named after W.R. Hamilton (in 1856) who described a mathematical game on the dodecahedron as given in Fig. 7.15.2

In this vertices are represented various cities and one is required to make a smallest spanning cycle.

Thus, dodecahedron graph is a Hamiltonian graph.

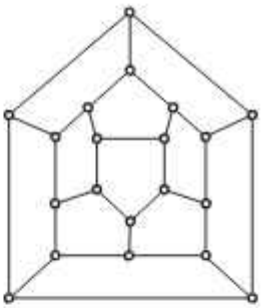


Fig. 7.15.2

Important theorems :

Theorem 1 : Let G be a simple connected graph on n vertices. If the sum of the degree of each pair of vertices in G is (n – 1) or large then there exists a Hamiltonian path in G.

Consider the following graphs :

In graph G_1, n = 5, degree sum of every pair of vertices is 4 or more. Hence ∃ a Hamiltonian path in G_1 which is a-b-c-d-e.

In graph G_2 n = 6, degree of every pair of vertices is 4 which is not equal to n – 1 = 5.

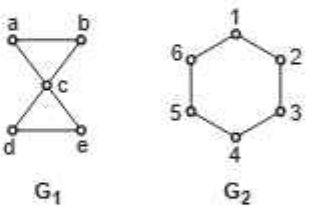


Fig. 7.15.3

But ∃ a Hamiltonian path 1-2-3-4-5-6. Therefore the condition in theorem 1 is a sufficient but not a necessary condition for existence of Hamiltonian path.

Theorem 2 : If G (v, E) is a simple connected graph on n vertices and $d(v) = \frac{n}{2}$; $\forall v \in V$ then G will contain a Hamiltonian circuit.

This condition is sufficient condition but not necessary.

Theorem 3 : Let G (v, E) be a connected simple graph. If G has a Hamiltonian circuit then for every proper non empty subsets of v, the components in the graph G-S is less than or equal to the number of vertices in S.

Theorem 4 : A Hamiltonian graph contains no cut vertices and hence is 2-connected.

Example 7.15.1 *Show that the complete bipartite graph $K_{m,n}$ is Hamiltonian for $m = n$ and for $m \neq n$, $K_{m,n}$ is not Hamiltonian graph.*

Solution : In a complete bipartite graph $K_{m,n}$ for m = n i.e. $K_{n,n}$, degree of each vertex is n.

Therefore $d(v) \geq \frac{n}{2}$ for all $v \in v(K_{n,n})$

By theorem 2, G contains a Hamiltonian circuit.

Hence $K_{n,n}$ is a Hamiltonian graph.

If $m \neq n$, Let V_1 and V_2 be the partitions of the vertex set of $k_{m,n}$ where $|v_1| = m$ and $|v_2| = n$. Without loss of generality assume that $m < n$.

The graph $K_{m,n} - V_1$ is a null graph on n vertices.

Hence it is a disconnected graph with n components.

Therefore the number of components in $K_{m,n} - V_1 = n$ which is greater than the number of vertices in V_1

Hence by theorem 3, $K_{m,n}$ does not contain a Hamiltonian circuit when $m \neq n$.

Example 7.15.2 *Show that the complete graph K_n $(n \geq 3)$ is a Hamiltonian graph. What is the length of that circuit ? How many circuits exist in K_n ?* **SPPU : Dec.-09**

Solution : The complete graph K_n has n vertices, $n \geq 3$ and degree of each vertex is $n - 1$. As $n \geq 3$.

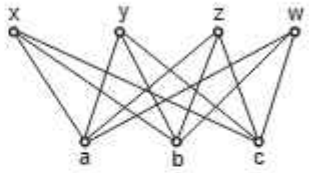
$$d(v) = n - 1 \geq \frac{n}{2} ; \forall v \in V(K_n).$$

Therefore by theorem 2, K_n has a Hamiltonian circuit. Hence K_n is a Hamiltonian graph.

Hamiltonian circuit contains all vertices of graph and length of circuit is the number of vertices present it. Hence in K_n, the length of the Hamiltonian circuit is n and there are $\frac{(n-1)!}{2}$ Hamiltonian circuits in K_n.

Example 7.15.3 *Find the Hamiltonian path and circuit in $K_{4,3}$?* **SPPU : Dec.-04**

Solution : The complete bipartite graph $K_{4,3}$ is given by



$K_{4,3}$

Fig. 7.15.4

In $K_{4,3}$, $4 \neq 3$ Hence it does not contain Hamiltonian circuit. Here degree of each vertex is either 3 or 4.

$\therefore$For x, y any two vertices in $K_{4,3}$ $d(x) + d(y) = 7.1 = 6$

Hence by theorem 1, the graph $K_{4,3}$ has a Hamiltonian path. It is given by x a y b z c w.

Example 7.15.4 *Give an example of the following graphs*

a) Eulerian but not Hamiltonian.

b) Hamiltonian but not Eulerian.

c) Eulerian as well as Hamiltonian.

d) Neither Eulerian nor Hamiltonian.

Solution : a) Eulerian but not Hamiltonian graph.

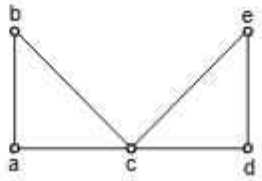


Fig. 7.15.5

Eulerian circuit : a b c d e c a

No Hamiltonian circuit.

b) Hamiltonian but not Eulerian

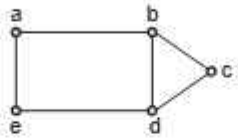


Fig. 7.15.5 (a)

Hamiltonian circuit : abcdea

No Eulerian circuit because d (b) = 3.

c) Eulerian and Hamiltonian graph.

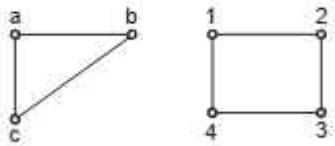


Fig. 7.15.5 (b)

Hamiltonian circuit : a-b-c-a, 1-2-3-4-1

Eulerian circuit : a-b-c-a, 1-2-3-4-1

d) Neither Eulerian nor Hamiltonian

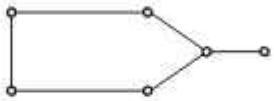


Fig. 7.15.5 (c)

No Hamiltonian circuit and no eulerian circuit.

Example 7.15.5 *Determine, if the following graphs are having the Hamiltonian circuit or path. Justify your answer.* **SPPU : Dec.-12, May-17, Marks 3**

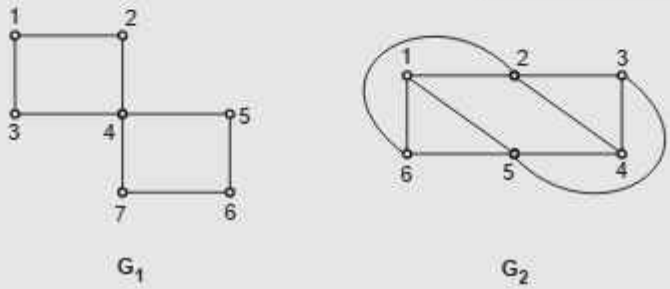


Fig. 7.15.6

Solution : In graph G_1, there are n = 7 vertices.

d (4) = 4 and all remaining vertices is 2.

So to draw Hamiltonian circuit, we have to visit vertex 4 twice. Which is not possible in Hamiltonian path. G_1 has no Hamiltonian circuit. Hamiltonian path is 1-2-3-4-5-6-7.

In graph G_2, there are 6 vertices and degree of each vertex is 3 or 4.

If we consider two vertices of lowest degree then also their sum is 6 which is equal to the number of vertices. So there exists a Hamiltonian path in G_2. ∴ Path is 1-2-3-4-5-6.

In graph G_2, $d(x) = \frac{6}{2} = 3$; $x \in V(G_2)$

∴By theorem 2, ∃ a Hamiltonian circuit

∴Hamiltonian circuit is 1-2-3-4-5-6-1

Example 7.15.6 *Which of the following have a Euler circuit or path or Hamiltonian cycle ? Write the path or circuit* **SPPU : Dec.-10**

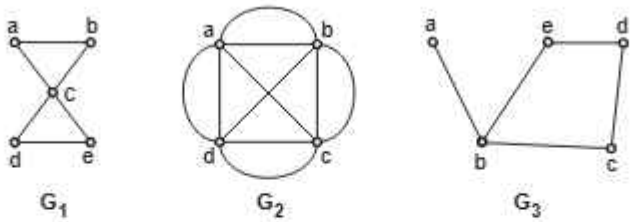


Fig. 7.15.7

Solution : In graph G_1, degree of each vertex is an even so ∃ an Eulerian circuit which is a-b-c-d-e-c-a.

In graph G, there are 5 vertices and degree sum of every pair of vertices is 4 or greater than 4. Hence there exists a Hamiltonian path in G_1 which is given by a-b-c-d-e. But there is no any Hamiltonian circuit as vertex c is a vertex. In graph G_2, degree of each vertex is 5 which is odd integer, so there is no Eulerian path in G_2, degree of each vertex is $5 > \frac{4}{2}$. Hence there exixts a Hamiltonian circuit which is given by a-b-c-d-a.

In G_3, Eulerian path is a-b-c-d-e-b No Eulerian circuit as d (a) = 1. Hamiltonian path is a-b-c-d-e.

No Hamiltonian cyle because b is a cut vertex.

7.15.1 The Travelling Salesman Problem (TSP)

A salesman is required to travel a number of cities during a trip. Given the distance among cities, in what order should he travel so that he travels as minimum distance as possible ? This is known as Travelling Salesman Problem (TSP).

In terms of graph theory, the TSP is to a Hamiltonian circuit with the smallest weight. In the case of K_n the problem can be solved theoretically by listing all the possible Hamiltonian circuits and select one which has least weight. But this method is highly impractical for the large graphs. In fact no efficient algorithm is there to solve TSP. It is therefore desirable to obtain a reasonably good but not an optimal solution.

One possible approach is to first find a Hamiltonian cycle and search for other Hamiltonian cycles of lesser weight. The simple method is as follows :

Fig. 7.15.8

Let C be the Hamiltonian circuit of a graph G.

Let further uv and xy be two no-adjacent edges of C such that the vertices u, v, x and y occur in that order in C.

If ux and vy are edges such that v (u x) + w (y v) < w (uv) + w (yx) then replace the edges uv and xy in (by ux and vy). The new cycle C' would still be Hamiltonian cycle and w (C') < w (C). This process can be continued until one gets a reasonably good Hamiltonian cycle.

7.15.2 Nearest Neighbour Method

In this method, we start with any arbitrary vertex and find the vertex which is nearest to it. Continuing this way and coming back to the starting vertex by travelling through all the vertices exactly once, we will get Hamiltonian cycle or circuit.

Consider the following steps to find Hamiltonian cycle by this method.

Step 1 : Start with any arbitrary vertex say v_1, choose the vertex closest to v_1 to form an initial path of one edge. Construct this path by selecting different vertices as described in step 2.

Step 2 : Let v_n be the latest vertex that was added to the path. Select the vertex v_{n+1} closest to v_n from all vertices that are not in the path and add this vertex to the path. Select those vertices which will not form a circuit in this stage.

Step 3 : Repeat step (2) till all the vertices of G are included in the path.

Step 4 : Lastly form a circuit by adding the edge connecting to v_1 and the last added vertex.

The circuit obtained using the nearest neighbour method will be the required Hamiltonian circuit.

Note : If we start with an arbitrary vertex in TSP then we may or may not minimum Hamiltonian circuit, But if we start with a vertex whose incident edge has the minimum weight in graph then we will get minimum Hamiltonian circuit as compared with arbitary starting vertex. For more details see example (2).

Examples :

Example 7.15.7 *Use nearest neighbour method to find the Hamiltonian circuit starting from a in the following graph. Find its weight.* **SPPU : Dec.-15**

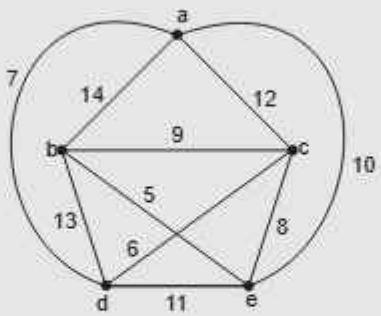


Fig. 7.15.9

Solution :

Step 1 : Let a be the starting vertex. Vertex a is adjacent to b, c, d, e. But minimum path is {a, d} which is the initial path.

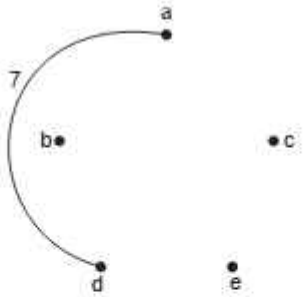


Fig. 7.15.9 (a)

Step 2 : There are three vertices adjacent to d. but closest one is C ∴The path is {a, d, c}.

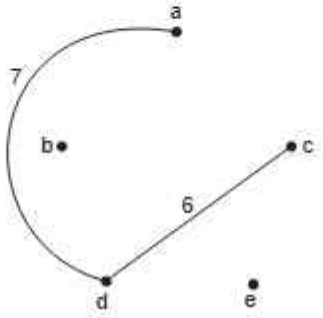


Fig. 7.15.9 (b)

Step 3 : There are 4 vertices adjacent to c. but closest is e ∴ The path is {a, d, c, e}.

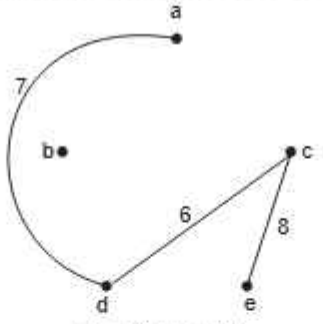


Fig. 7.15.9 (c)

Step 4 : There are 4 vertices adjacent to e. but closest is b ∴The path is {a, d, c, e, b}.

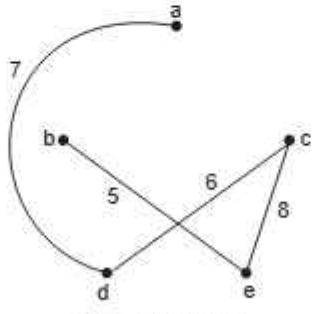


Fig. 7.15.9 (d)

Step 5 : Here are all vertices are corered so to complete Hamiltonian an circuit there should be a path from b to a.

∴ Hamiltonian circuit is {a, d, c, e, b, a}

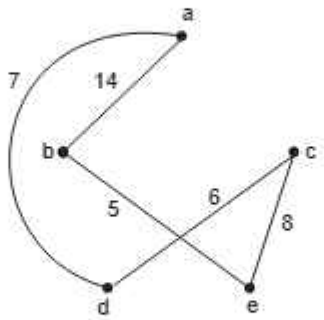


Fig. 7.15.9 (e)

Weight of the Hamiltonian circuit = 40.

Example 7.15.8 *For the following figure solve the following options :*

a) Use nearest neighbourhood method to find out Hamiltonian circuit for the graph in the following Fig. 7.15.10 starting at vertex a.

b) Repeat the part (a), starting at vertex 'd' instead.

c) Determine the minimum Hamiltonian circuit for the graph in the following Fig. 7.15.10

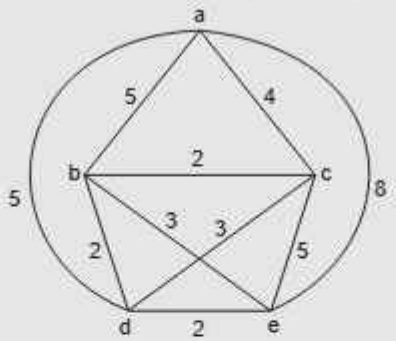


Fig. 7.15.10

Solution : a) Start with vertex a. There are 4 adjacent vertices b, c, d and e. The closest vertex is c (as length of the edge (a, c) = 4, which is minimum).

Step 1 : Initial path = {a, c}

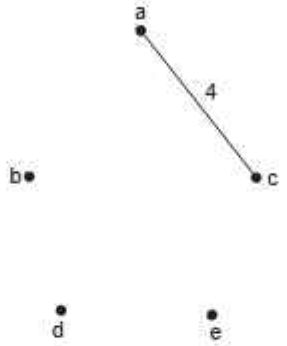


Fig. 7.15.10 (a)

Step 2 : There are three vertices adjacent to c, namely b, d, e (except a). The closest is b

Path = {a, c, b}

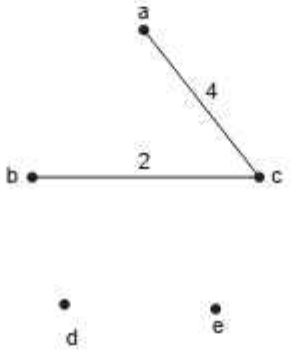


Fig. 7.15.10 (b)

Step 3 : There are two vertices adjacent to b, namely d and e. The closest is d

Path = {a, c, b, d}

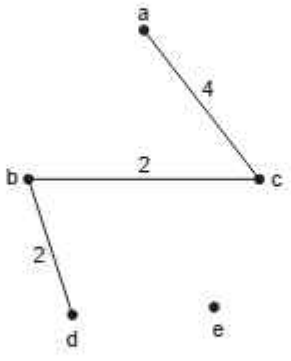


Fig. 7.15.10 (c)

Step 4 : path = {a, c, b, d, e}

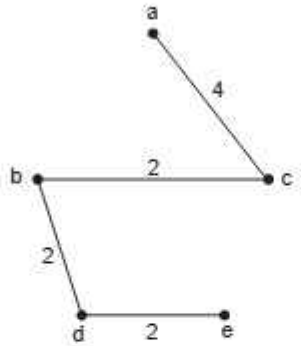


Fig. 7.15.10 (d)

Step 5 : Since all the vertices are traversed, to complete the Hamiltonian circuit, there should be a path from e to a.

Hamiltonian circuit = {a, c, b, d, e, a}

The weight of Hamiltonian circuit = 18

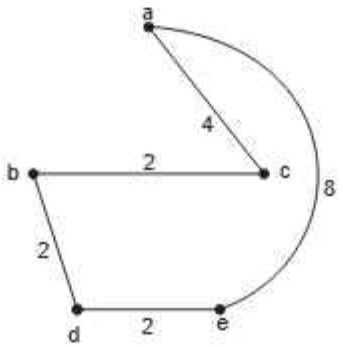


Fig. 7.15.10 (e)

(b) To find the Hamiltonian circuit starting from d, consider the vertex d. Since both the vertices b and e are nearest to d as both are at distance 2 from d, hence we can choose any vertex.

Let e be the vertex.

Step 1 : Initial path = {d, e}

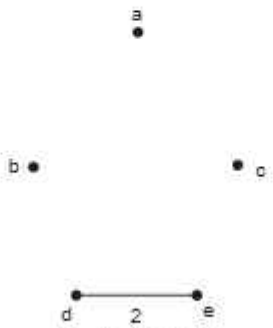


Fig. 7.15.10 (f)

Step 2 : a, b, c are adjacent to e but b is the nearest neighbour of e.

Path = {d, e, b}

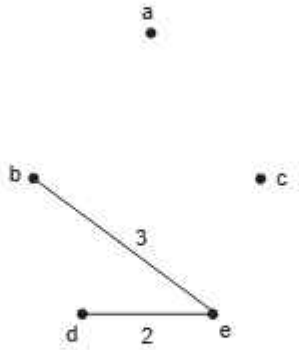


Fig. 7.15.10 (g)

Step 3 : Path = {d, e, b, c}

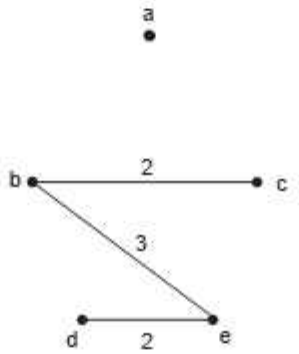


Fig. 7.15.10 (h)

Step 4 : Path = {d, e, b, c, a}

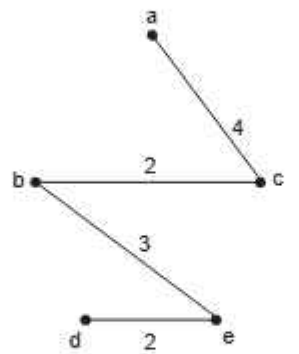


Fig. 7.15.10 (i)

Step 5 : Since all the vertices are traversed, to complete the Hamiltonian circuit, there should be a path from a to d.

Hamiltonian = {d, e, b, c, a, d}

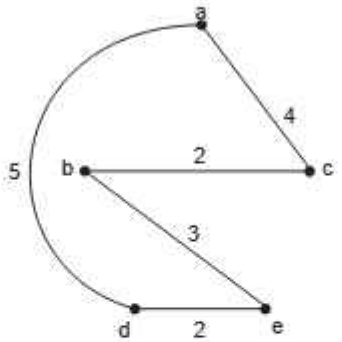


Fig. 7.15.10 (j)

The weight of the Hamiltonian circuit is 16.

C) The minimum weight of Hamilton circuit is 16.

7.16 Planar Graphs

SPPU : May-06, Dec.-08, 09, 10, 13

In this section we will study drawing of graphs without crossing edges on surfaces and deal with some important properties of such graphs.

Definition : A graph is said to be planar graph if it can be drawn on a plane such that no edges intersect or cross in a point other than their end vertices.

A graph G is said to be non-planar if it is not possible to draw graph be without crossing.

Examples :

1) Following graphs i.e. Planar graphs :

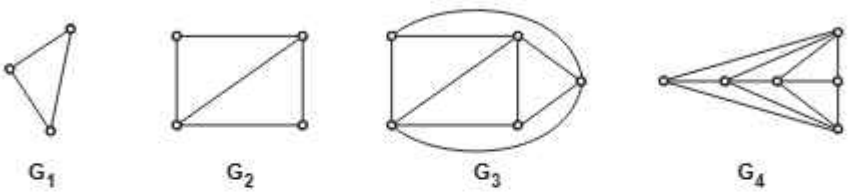


Fig. 7.16.1

2)

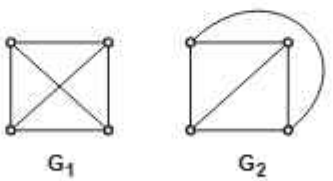


Fig. 7.16.2

Graph G_1 looks like non-planar graph out if we redraw graph we get graph G_2 which is planar.

7.16.1 Regions

A plane representation of a graph divides the plane into parts or regions. They are also known as faces or windows or meshes.

A region or face is characterised by the set edges forming it's boundary.

A region is said to be finite if it's area is finite. A region is said to be infinite or unbounded if it's area is infinite. Every planar graph has an infinite region.

Consider the graph given below :

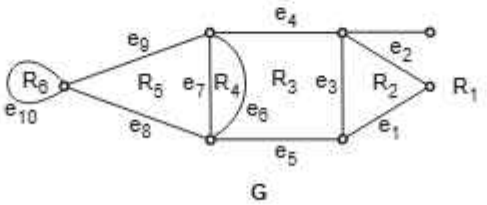


Fig. 7.16.3

The graph G has 6 regions, 7 vertices and 11 edges. Region R_1 is an infinite region known as exterior region. We have

$$R_2 = \{e_1, e_2, e_3\} = \text{Region bounded by } e_1, e_2, e_3$$

$$R_3 = \{e_3, e_4, e_5, e_6\}$$

$$R_4 = \{e_6, e_7\}, \quad R_6 = \{e_{10}\}.$$

It is observed that $n = 7$, $e = 11$, $r = 6$

$$\therefore \quad n + r - 2 = 7 + 6 - 2 = 11 = e$$

Now let us define Euler's formula.

7.16.2 Euler's Formula

Statement : For any connected planar graph G, with v number of vertices, e number of edges and r number of regions

$$\boxed{v - e + r = 2}$$

or $v + r - 2 = e$

Proof : Let G be a connected planar graph with v vertices, e edges and r regions. We shall prove the theorem by induction on e.

Step 1 : For $e = 0$, we get $v = r = 1$. Thus

$$v - e + r = 1 - 0 + 1 = 2$$

Hence result is true for $e = 0$

Step 2 : Let $e \geq 1$. Assume that the result is true for all connected planar graphs with less than e edges. Let G be a graph with v vertices, e edges and r regions

Step 3 : Case 1 : If G has a pendent vertex say x then $G - \{x\}$ is a connected graph with $v - 1$ vertices, $e - 1$ edges and r regions.

So by induction hypothesis

$$(v - 1) - (e - 1) + r = 2$$

$$v - e + r = 2$$

Case 2 : If G has no pendent vertex the G is a connected graph with circuit. Let e_1 be the edge of a circuit in G. Then $G - \{e_1\}$ is a connected graph with v vertices, $e - 1$ edges and $r - 1$ regions {If we remove edge from a circuit, then it reduces region by 1}.

By induction hypothesis

$$v - (e - 1) + (r - 1) = 2$$

$$\Rightarrow \quad v - e + r = 2$$

Thus by the principle of mathematical induction the result is true for all e.

Corollary 1 : If G (V, E) is a simple connected planar graph with v vertices and e edges then $e \leq 3v - 6$

Proof : Give that, G is a simple planar graph, so each region of G is bounded by three or more edges.

If G has r number of regions then the total number of edges in G is $e \geq 3r$.

Also each edge of G is included in exactly two regions of G. therefore $2e \geq 3r$

$$\Rightarrow \quad \frac{2e}{3} \geq r$$

Substitute these values in Euler's theorem, we get

$$v - e + r = 2$$

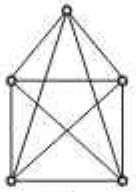
$$v - e + \frac{2e}{3} \geq 2$$

$$3v - e \geq 6$$

$$e \leq 3v - 6 \text{ Hence the proof.}$$

Corollary 2 : Prove that, K_5 (the complete graph on 5 vertices) is not planar.

Proof : The complete graph on 5 vertices K_5 is given below :



K_5

Fig. 7.16.4

K_5 has 5 vertices and 10 edges. i.e. v = 5 and e = 10.

Now $3v - 6 = 15 - 6 = 9$

By corollary 1, $e \leq 3v - 6$

$10 \leq 9$ which is impossible.

Therefore K_5 is not planar graph.

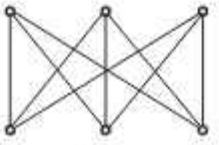
K_5 is the smallest planar graph with respect to number of vertices.

Consider the graph $K_{3,3}$

Here $v = 6, e = 9,$

$3v - 6 = 18 - 6 = 12 > 9 = e$

i.e. $e \leq 3v - 6$



$K_{3,3}$

Fig. 7.16.5

But $K_{3,3}$ is not a planar graph.

$\therefore$ The graph $K_{3,3}$ is the smallest non planar graph with respect to number of edges.

The graph K_5 is called the Kuratowski's first graph and $K_{3,3}$ is called the Kuratowski's second graph.

In 1930, Kuratowski gave a necessary and sufficient condition for a graph to be planar.

Kuratowski's Theorem : A graph G is a planar if G does not contain any subgraph that is isomorphic to within vertices of degree two to either K_5 or $K_{3,3}$

Two graphs are said to be isomorphic to within vertices of degree two if they are isomorphic or they can be reduced to isomorphic graphs by repeated insertion of vertices of degree 2 or by merging the edges which have exactly one common vertex of degree 2.

For example the following graph are isomorphic to within vertices of degree 2. (Homeomorphic)

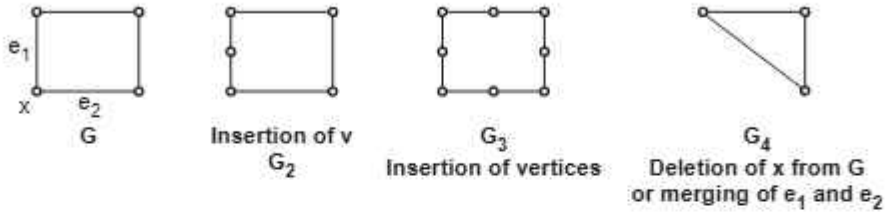


Fig. 7.16.6

Examples :

Example 7.16.1 *Every planar graph with at least 3 vertices contains a vertex of degree at the most 5.*

Solution : Without loss of generality assume that G is a plannar connected graph with v vertices and e edges.

If $v \leq 6$ there is nothing to prove.

Let $v > 6$ and there is no vertex of degree ≤ 5 then $2e = \sum_{x_i \in v(G)} d(x_i) \geq 6\,v$

i.e. $6\,v \leq 2e$ but $e \leq 3v - 6$

$\Rightarrow \quad 6\,v \leq 2e \leq 6v - 2$

$6\,v \leq 6v - 12$ which is impossible.

Hence there is a vertex of degree at the most 5.

Example 7.16.2 *Draw a planar representation of graphs given below if possible.*

SPPU : May-06, Dec.-09

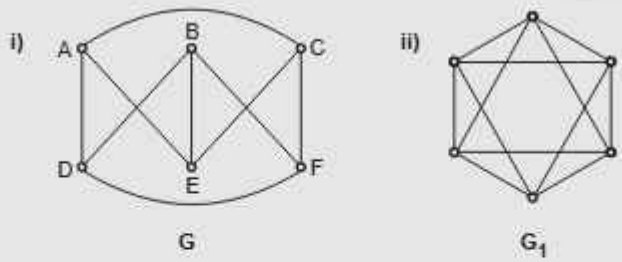


Fig. 7.16.7

Solution : The planar representation of G_1 and G_2 is as follows :

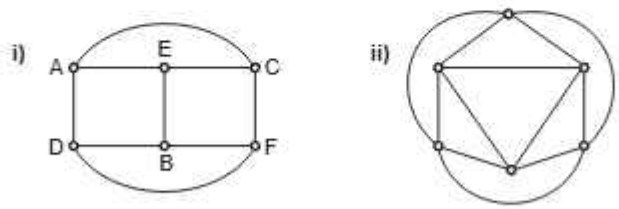
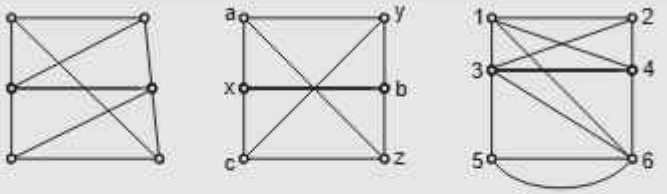


Fig. 7.16.7 (a)

Example 7.16.3 *Identify whether the graphs are planar or not Justify ?* **SPPU : Dec.-08**



(i) (ii) (iii)

Fig. 7.16.8

Solution : i)

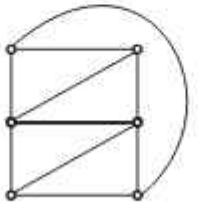


Fig. 7.16.8 (a)

Given graph is planar graph.

ii) Given graph is isomorphic to $K_{3,3}$

$\therefore$ Given graph is not planar.

iii)

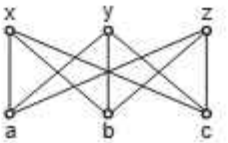


Fig. 7.16.8 (b)

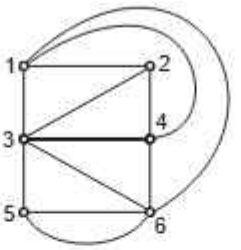


Fig. 7.16.8 (c)

$\therefore$Given graph is planar.

Example 7.16.4 *Without drawing graph, prove that G is non planar graph.*

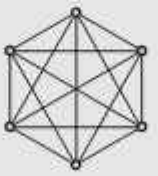


Fig. 7.16.9

Solution : Given graph is a complete graph on 6 vertices G has $\frac{n(n-1)}{2} = \frac{6\times 5}{2} = 15$ edges.

$\therefore$ v = 6 and e = 15

$$3v - 6 = 3 \times 6 - 6 = 12$$

We know that for a planar graph $e \leq 3v - 6$

Here $15 \leq 3v - 6 = 12$ which is impossible.

Hence G is not planar graph.

Example 7.16.5 *Show that in a connected planar graph with 6 vertices, 12 edges each of region is bounded by 3 edges.* **SPPU : Dec.-10, 13**

Solution : According to Eulers theorem for planar graphs.

$$v - e + r = 2$$

Here $v = 6, e = 12$

$6 - 12 + r = 2 \Rightarrow r = 8$

We know that, each edge contributed twice in a regions we have 12 edges.

So $12 \times 2 = 24$ edges are distributed among 8 regions.

$\Rightarrow \quad \frac{24}{8} = 3$ edges for each region.

So each region is bounded by 3 edges.

Example 7.16.6 *Prove that $K_{3,3}$ is not planar graph.*

Solution : $K_{3,3}$ has 6 vertices and 9 edges. Suppose $K_{3,3}$ is planar, then the boundary of each region has at least 4 edges because it is bipartite and contains no triangles. Each edge lies on boundary of two regions.

Therefore, $2e \geq \sum_{i=1}^{r}$ (the number of edges in the ith region)

$$2e \geq 4r$$

$$2e \geq 4(2 + e - v)$$

$\Rightarrow \quad e \leq 2v - 4$

But $\quad e = 9$ and $v = 6$

$\therefore \quad 9 \leq 12 - 4 = 8$ which is impossible.

Hence $K_{3,3}$ is not planar graph.

7.17 Coloring of Graphs

The coloring of all vertices of a connected graph such that adjacent vertices have different colors is called a proper coloring or vertex coloring or simply a coloring of graphs.

A graph G is said to be properly colored graph if each vertex of G is colored according to a proper coloring.

e.g. 1) Consider the following graphs with proper coloring

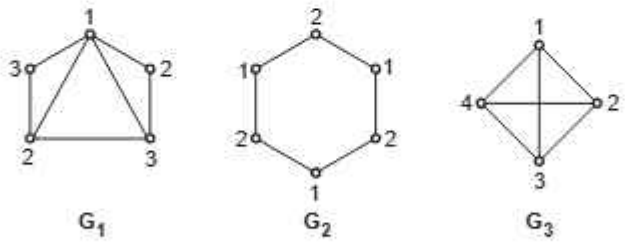


Fig. 7.17.1

7.17.1 Chromatic Number of Graph

The chromatic number of a graph G is denoted by X(G) and defined as the minimum number of colors required to color the vertices of G so that the adjacent vertices get different colors.

A graph G is said to be K-colorable if all vertices of G can be properly colored using at most K different colors. Obviously, a K-colorable graph is K+1 colorable.

If G is k-colorable then X (G) ≤ K.

e.g. In above example (1) X(G) = 3, $X(G_2)$ = 2, $G(G_3)$ = 4. If G is any graph with x(G) = K then the addition or deletion of loops or multiple edges do not change the chromatic number of that graph. Thus hereafter for a coloring of problem we consider only simple connected graphs.

In general, proper coloring of graph is very plain or simple but a proper coloring with minimum number of colors is not an easy task. We list some observation of coloring of graphs as follows :

i) Null graph is 1 - chromatic.

ii) The complete graph on n vertices is n-chromatic.

iii) Bipartite graph is 2-chromatic.

iv) Cycle graph with even edges is 2-chromatic and odd edges is 3-chromatic.

v) If graph is r-chromatic then it has r distinct partitions and that graph is called r-partite graph.

7.17.2 Chromatic Polynomial

We have studied the properly coloring of graph in many different ways using a sufficiently large number of colors.

The chromatic polynomial of a graph is denoted by $P_n(\lambda)$ and defined as the number of ways of properly coloring of graph using λ or fewer colors

e.g. 1) The chromatic polynomial of the complete graph K_1 is $P_1(\lambda) = \lambda$.

2) The chromatic polynomial of the complete graph K_n on n vertices is

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$$

7.17.3 Coloring of Planar Graph

A map or atlas is a plane representation of a connected planar graph. Two regions of a planar graph G are said to be adjacent if they have an edge common.

The coloring of a planar graph or map means an assignment of a color to each region of a planar graph G such that adjacent regions have different colors.

A planar graph is - n colorable if minimum n different colors are required to color graph G.

Theorem 1 : (Four Color Theorem)

Every planar graph is 4 - colorable.

Initially it was a conjecture, but in 1979 Appel and Haken proved this. That's why this conjecture became theorem.

7.17.4 Open Problem of Coloring

A lot of research is done in the coloring of planar graphs, particularly coloring of vertices, or edges or regions of a planar graph.

The following open problem is stated by Dr. H. R. Bhapkar and proved partially first time.

Open Problem :

How many minimum colors will be required to color planar graph such that

i) Adjacent vertices have different colors

ii) Incident edges have different colors.

iii) Adjacent regions have different colors.

iv) A region, boundary edges and boundary vertices of that region have different colors.

This type of coloring is known as perfect coloring of G and denoted by PC (G).

We list some observations of perfect coloring of planar graph as follows :

i) If G is a null graph then PC (G) = 2

ii) If G is a chain graph when n vertices then,

$$pc(G) = \Delta(G) + 2$$

where $\Delta(G)$ = Highest degree of a vertex in G.

7.18 Web Graph

A directed graph whose nodes correspond to static pages on the web and whose arcs correspond to links among these pages, is called web graph.

There are severel reasons for the developement of this graph, Few of them are as follows :

i) Designing crawl strategies on the web.

ii) Understanding of the sociology of content creation on the web.

iii) Analyzing the behavior of web algorithms that make use of link information.

iv) Predicting the evolution of web structures such as bipartite cores and webrings and developing better algorithms for organizing and discovering them.

v) Predicting the emergence of very important new phenomena in the web graph.

We detail a number of experiments on a web crawl of approximately 200 million pages and 1.5 billion links. So the scale of this experiment is very large.

7.19 Graph Database

In computing, a graph database is database that uses graph structures for semantic queries with nodes, edges and properties to represent and store data. A key concept of the system is the graph (or edges or relationships) which directly relates data items in the store.

Alternately, Graph databases are based on graph theory graph databases employ nodes, edges and properties.

i) Nodes : Nodes representes entities such as people businesses, accounts or any other item you might want to keep track of. They are roughly the equivalent of the record, relations, the documents in a document database.

ii) Edges are known as relationships are the lines that connect nodes to other nodes, they represent the relationship among them. Edges are the key concepts in graph databse, representing an abstraction that is not directly implemented in other systems.

iii) Properties are pertinent information that relate to nodes.

Examples :

1) The set at all the users who's phone number contains the area code "141".

2) The set of all trees in the world with similar properties.

7.20 Google Map

- A google map is a big giant graph with nodes and edges to find out fastest and shortest route from source to destination.
- A graph theory can be applied to find fastest and shortest route in the following way.

1) A shortest path from source to destination - Dijkstra's Algorithm.

- We consider the weighted graph where the vertices are the towns, cities (sources) and edges are the roads. The weight between two nodes is the distance between two cities. Therefore weights are non-negative. The problem is find a path of minimum weight connecting two given cities. One has to list all paths, find their weights and select minimum one. But for a large network, this is not efficient. Dijkstra's algorithm is used to find out the shortest path from the vertex 'a' to the vertex 'z' of the graph. A detailed explanation of Dijkstra's algorithm is in section 7.13.

2) The Busiest Interconnection : The degree of vertex of directed graph is used to find out busiest interconnection. Each vertex in a directed graph and in degree and out degree. So such type of node (vertex) can be avoided during travel.

3) Find the nearest location :

Fig. 7.20.1

7.21 Case Studies

1) **Three Utility Problem :** It is a classical mathematical puzzle. It is also called as three cottages problem or water, gas and electricity problem. In 1917, Henry Dudeney states this problem is 'water, gas and electricity'. This problem is stated as follows :

- Suppose that there are three cottages H_1, H_2 and H_3 and each needs to be connected to each of the three utilities Water (W), Gas (G) and Electricity (E). Is there any way to make nine connections without any crossover of lines ?
- Consider the following graph :

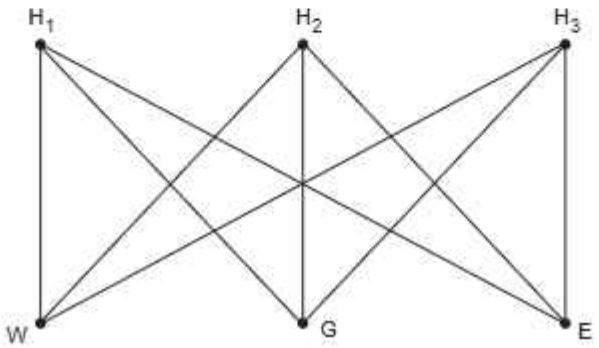


Fig. 7.21.1

- The above graph is complete bipartite graph $K_{3,3}$. Three, utility problem is a question whether this $K_{3,3}$ graph is planar. But $K_{3,3}$ is not a planar graph. In other words, there is no way to make all nine connections without line crossover. Some attempts are as follows :

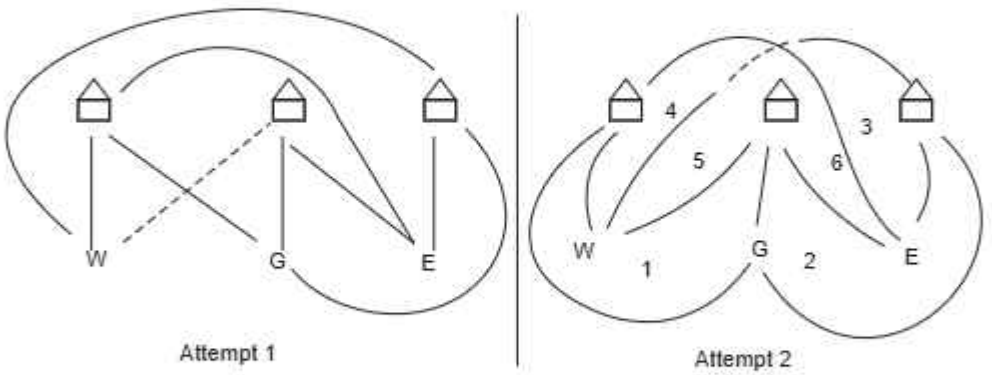


Fig. 7.21.2

- There is at least two lines crosses each other.

Proof : Eular's formula exists for planar graph as

$$F + V = E + 2 \text{ where}$$

$$V = \text{No. of vertices}$$

$$F = \text{No. of faces}$$

$$E = \text{No. of edges}$$

In above graph,

$$V = 6, E = 9, F = 4$$

$$\therefore \quad 4 + 6 \neq 9 + 2$$

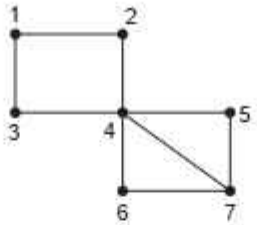
- Hence above graph is not planer. Hence there is no way to make all nine connection without line crossover.

2) Web graph : Refer section 7.18.

7.22 University Questions with Answers

Dec. 2016

Q.1 *Determine which of the graphs below represents Eulerian circuit, Eulerian path, Hamiltonian circuit and Hamiltonian path. Justify your answer.* **[4]**



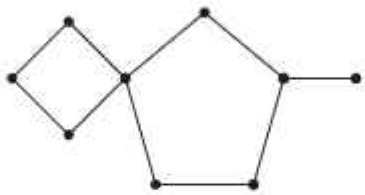


Fig. 7.22.1

Ans. : We know that,

i) An undirected graph possesses an Eulerian path iff it is connected and has either zero or two vertices of odd degree.

ii) An undirected graph possesses an Eulerian circuit iff it is connected and its vertices are all at even degree.

For graph (1)

Eulerian path : 4 - 3 - 1 - 2 - 4 - 5 - 7 - 4 - 6 - 7

Eulerian circuit does not exist.

Hamiltonian path and circuit do not exists.

For graph (2)

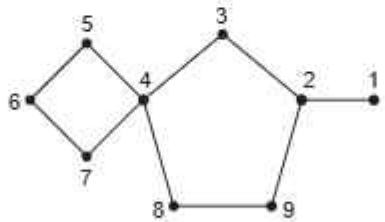


Fig. 7.22.2

Eulerian path : 1 - 2 - 3 - 4 - 5 - 6 - 7 - 4 - 8 - 9 - 2

Eulerian circuit does not exist.

Hamiltonian path and circuit do not exists.

Q.2 *Define the graph K_n and K_{mn}.* **(Refer section 7.5 (ii) and (iv))** **[2]**

Q.3 *Use Dijkstra's algorithm to find the shortest path between a and z.* **(Refer example 7.13.3)** **[6]**

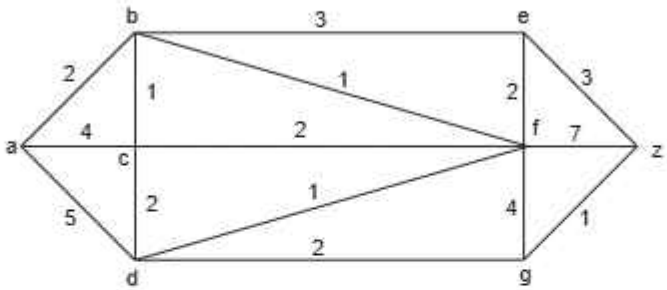
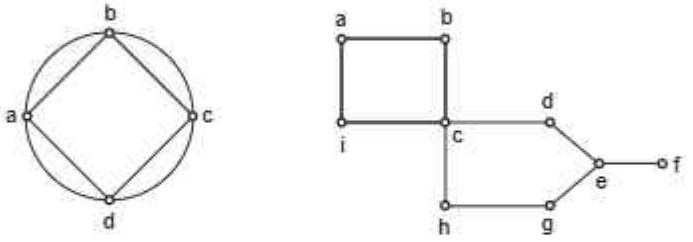


Fig. 7.22.3

May 2017

Q.4 *Check whether the graph has an Euler circuit, Euler path, Justify :* **[3]**



(Graph G_1) (Graph G_2)

Fig. 7.22.4

Ans. : For graph G_1 :

Enter circuit : a-b-c-d-a-b-c-d-a

Enter path : a-b-c-d-a-b-c-d-a

For graph G_2 :

Enter circuit does not exist

Enter path : f-e-g-h-c-i-a-b-c-d-e

Q.5 *How many colours required to colour $k_{m,n}$ why ?* **[3]**

Ans. : In $k_{m,n}$ graph, $V(k_{m,n}) = V_1 \cup V_2$

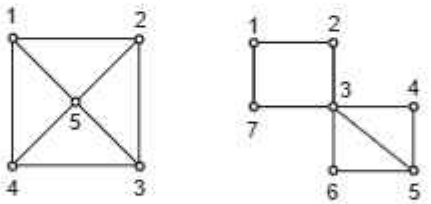
V_1 and V_2 are the partitions of $V(k_{m,n})$

$\therefore$ Vertices in V_1 are not adjacent to each other.

So assign color 1 to all vertices of V_1 and color 2 to all vertices of V_2

$\therefore$ Every $k_{m,n}$ is two colorable graph.

Q.6 *Determine whether the following graph has Hamiltonian circuit of Hamiltonian path.* **(Refer example 7.15.5)** **[3]**



(Graph 1) (Graph 2)

Fig. 7.22.5

Q.7 *Write 5 applications of graph theory in the field of data analytics.* **[3]**

Ans. : Applications of graph theory in the field of data analytics are given below.

1) Project Evaluation Review Technique (PERT).
2) Critical Path Method (CPM)
3) Game theory
4) Operations research
5) Combinatorial problems.

Dec 2017

Q.8 *Explain the directed and undireted grpah with suitable example.* **(Refer section 7.3(6))** **[4]**

Q.9 *Explain the Dijkstra's algorithm in details.* **(Refer section 7.13)** **[4]**

Q.10 *Define subgraph.*
Determine whether $H = H' = (V', E')$ is a subgraph of $G(V, E)$ shown in Fig. 7.22.6 :
i) $V' = \{A, B, F\}$
$E' = \{(A, B), (A, F)\}$
ii) $V' = \{B, C, D\}$
$E' = \{(B, C), (B, D)\}$ **[4]**

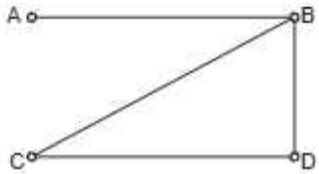


Fig. 7.22.6

Ans. : Refer section 7.8 (1).

Given that

$V = \{A, B, C, D\}$, $E = \{(A, B), (B, C), (B, D), (C, D)\}$

i) We have $V' = \{A, B, F\}$

As $F \in V'$ but $F \notin V$

$\therefore$ H is not a subgraph of G.

We have $V' = \{B, C, D\}$ $\therefore V' \subseteq V$

and $E' \{(B, C), (B, D)\}$ $\therefore E' \subset E$

$\therefore$ H is a subgraph of G.

May 2018

Q.11 *Consider a graph $G(V, E)$ where $V = \{v_1, v_2, v_3\}$ and deg $(v_2) = 4$:*
i) Does such simples graph exists ? If not, why ?
ii) Does such a multigraph exists ? If yes, give example. **[4]**

Ans. : Given that $V = \{v_1, v_2, v_3\}$, deg $(v_2) = 4$

i) G_e has 3 vertices and degree of v_2 is 4.

V_2 has either a loop or multiple edges.

$\therefore$ G_e is not a simple graph.

ii) Multigraph with given conditions exists. Examples of such graphs are given below -

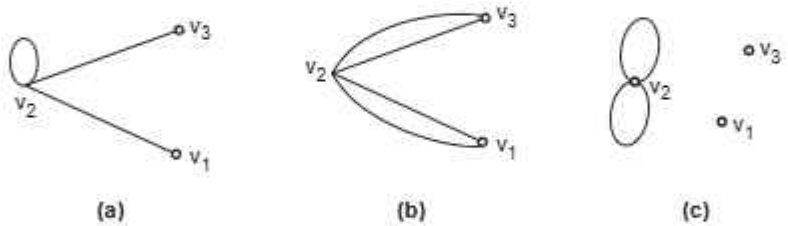


Fig. 7.22.7

Q.12 *Explain the following in brief :*

i) Subgraphs and spanning subgraph **(Refer section 7.8 (1) and (4))**

ii) Isomorphic graph **(Refer section 7.7)**

iii) Bipartite graph **(Refer section 7.5 (iii))**

iv) Adjacency matrix and incidence matrix of undirected graph.

(Refer sections 7.4.1 and 7.4.2) **[4]**

Q.13 *Apply Dijkastra's algorithm to find the shortest path from vertex v_1 to v_5 in the graph shown below in Fig. 7.22.8.* **[4]**

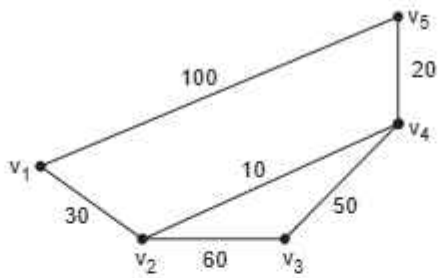


Fig. 7.22.8 Weighted graph

Ans. : Step 1 : $P = \phi$ and $T = \{v_1, v_2, v_3, v_4, v_5\}$

$L\{v_1\} = 0$, $L\{x\} = \infty$, $\forall$ $x \in T$, $x \neq v_1$

Step 2 : $v = v_1$, the permanent lable of v_1 is 0.

$P\{v_1\}$, $T = \{v_2, v_3, v_4, v_5\}$

$$L\{v_2\} = \min\{\text{old } L(v_2), L(v_1) + w(v_1, v_2)\}$$

$$= \min\{\infty, 0 + 30\} = 30$$

$$L\{v_3\} = \min\{\infty, 0 + \infty\} = \infty$$

$L\{v_4\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{v_5\} = \min\{\infty, 0 + 100\} = 100$

$L\{v_2\} = 30$ is the minimum label.

Step 3 : $v = v_2$, the permanent label of v_2 is 30

$P = \{v_1, v_2\}$; $T = \{v_3, v_4, v_5\}$

$L\{v_3\} = \min\{\text{old } L(v_3), L(v_2) + w(v_2, v_3)\}$

$= \min\{\infty, 30 + 60\} = 90$

$L\{v_4\} = \min\{\infty, 30 + 10\} = 40$

$L\{v_5\} = \min\{\infty, 30 + \infty\} = \infty$

$L\{v_4\} = 40$ is the minimum label

Step 4 : $v = v_4$ the permanent label of v_4 is 40.

$P = \{v_1, v_2, v_4\}$; $T = \{v_3, v_5\}$

$L\{v_3\} = \min\{\infty, 40+50\} = 90$

$L\{v_5\} = \min\{\infty, 40+20\} = 60$

$L\{v_5\}$ is the minimum label.

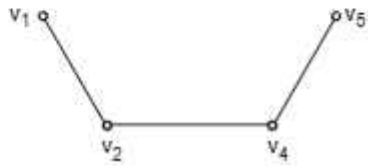


Fig. 7.22.9

Step 5 : $v = v_5$ the permanent label of v_5 is 60.

Hence the length of the shortest path from v_1 to v_5 is 60. The shortest path is $v_1 - v_2 - v_4 - v_5$.

Dec. 2018

Q.14 *Determine whether the two graphs are isomorphic or not. Explain.* **(Refer example 7.7.6)** **[6]**

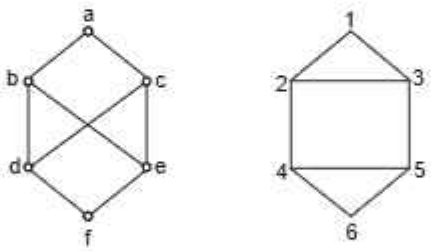


Fig. 7.22.10

Q.15 *Use Dijkstra's algorithm to find the shortest path between A and Z in figure :* **(Refer example 7.13.2)** **[6]**

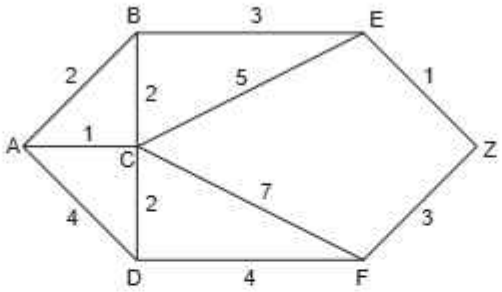


Fig. 7.22.11

May 2019

Q.16 *Use dijkstra's algorithm to find the shortest path between a and z.* **(Refer example 7.13.1)** **[6]**

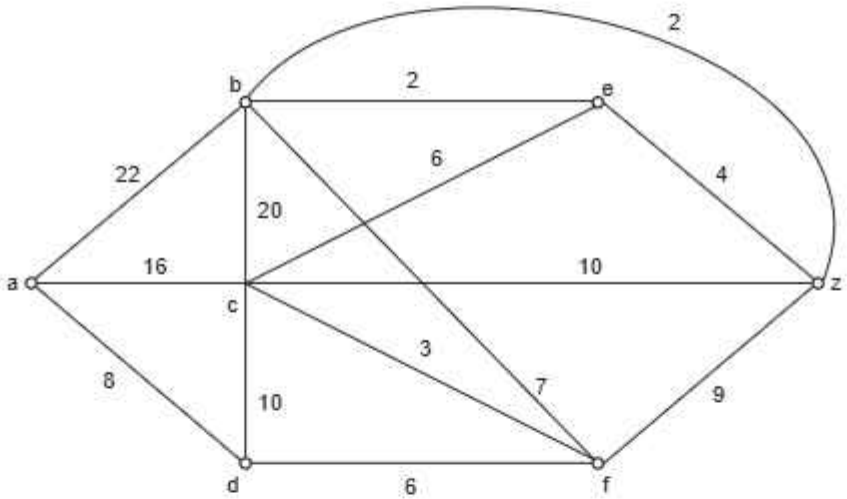


Fig. 7.22.12

Q.17 *Under what condition K_{mn} will have eulerian circuit ?* **(Refer example 7.14.3)** **[3]**

Q.18 *The graphs G and H with vertex sets V(G) and V(H), are drawn below. Determine whether or not G and H drawn below are isomorphic. If they are isomorphic, given a function g : V(G) –> V(H) that defines the isomorphism. If they are not explain why they are not.* **[3]**

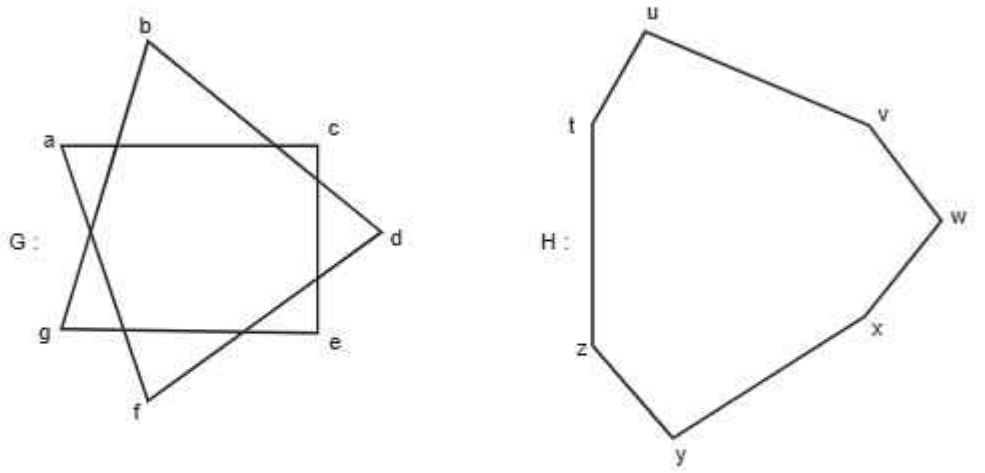


Fig. 7.22.13

Ans. : Number of vertices in both graphs = 7

Number of edges in both graphs = 7

Both graph have 7 vertices of degree 2

$\phi : \{a, b, c, d, e, f, g\} \rightarrow \{t, u, v, w, x, y, z\}$

Such that,

$a \rightarrow u, b \rightarrow x, c \rightarrow t, d \rightarrow w, e \rightarrow z, f \rightarrow v, g \rightarrow y$

ϕ Preserves adjacency and non-adjacency of vertices

$\Psi \rightarrow E_1 \rightarrow E_2$ is bijective

Hence, both graphs G and H are isomorphic.

Dec. 2019

Q.19 *Can a simple graph exist with 15 vertices, each of degree five ?* **[3]**

Ans. : By the handshaking Lemma

$$2e = \sum_{u \in v} \deg(u) \text{ where}$$

$$e = \text{No. of edges and } v = \text{No. of vertices}$$

It suggests that sum of degrees of all the vertices is even.

But in given example

$$\sum_{u \in v} \deg(u) = 15 \times 5 = 75 \text{ which is odd}$$

Hence simple graph never exists with 15 vertices, each of degree 5.

Q.20 *For which values of n, m are the following graph regular :*
(i) K_n **[3]**

Ans. : A regular graph is a graph where each vertex has same degree

1) K_n - It is a complete graph

Complete graph is regular graph for any n.

Q.21 *Determine whether the following graphs are isomorphic to each other.* **[3]**

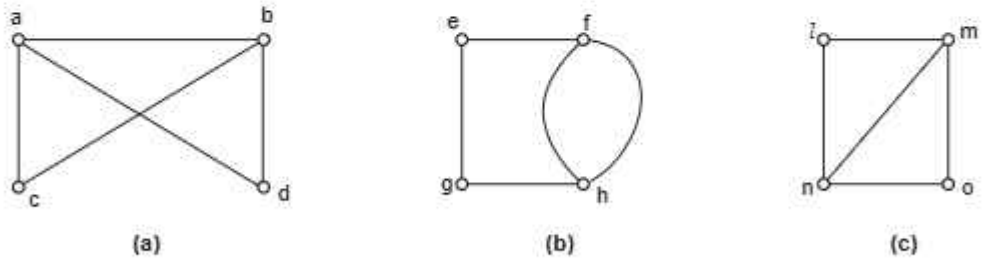


Fig. 7.22.14

Ans. :

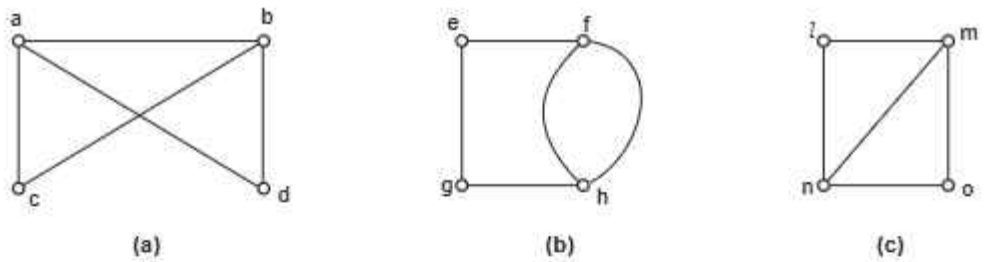


Fig. 7.22.15

All graphs have 4 vertices and 5 edges.

All graphs have 2 vertices of degree 3 and 2 vertices of degree 2.

i) $\therefore \phi : \{a, b, c, d\} \rightarrow \{m, n, l, o\}$ such that

$a \rightarrow m$

$b \rightarrow n$

$c \rightarrow l$

$d \rightarrow o$

ϕ preservers adjancy and non-adjancy of vertices for graph a and graph c.

$\psi \rightarrow E_1 \rightarrow E_2$ is bijective

ii) In graph a, vertex having degree 3 is connected to all vertices i.e. b with degree 3, C with with degree 2 and d with degree 2. But in graph B, f and h have degree 3 but not connected to all vertices.

Hence it is not preserving adjancy and non adjancy of vertices for graph a and graph b.

Hence graph a and graph C are isomophic graph. Graph B is not isomorphic to them hence A, B, C are not isomorphic.

Q.22 *How many regions would there be in a plane graph with 10 vertices each of degree 3.* **[3]**

Ans. : $V = 10$

According to handshake Lemma

$$2e = \sum_{u \in v} \deg(u)$$

$$2e = 10 \times 3$$

$$2e = 30$$

$$\therefore \quad e = 15$$

Total no. of edges = 15

According to Euler's formula

$$V - e + r = 2$$

$$\therefore \quad 10 - 15 + r = 2$$

$$\therefore \quad r = 7$$

$$\boxed{\text{Total no. of regions} = 07}$$

❑❑❑

Notes

UNIT - V

8 Trees

Syllabus

Introduction, Properties of trees, Binary search tree, Tree traversal, Decision tree, Prefix codes and Huffman coding, Cut sets, Spanning trees and minimum spanning tree, Kruskal's and prim's algorithms, The max flow-min cut theorem (Transport network). Case studies : Algebraic Expression Tree, Tic-Tac-Toe Geme Tree

Contents

8.1 Introduction

8.2 Tree **Dec.-16, May-17,18**

8.3 Centre of a Tree **Dec.-18**

8.4 Rooted Tree and Binary Tree..... **Dec.-09,16,18,19, May-18,19**

8.5 Binary Tree **Dec.-09,16, May-10,12,17**

8.6 Prefix Code and Binary Search Trees **Dec.-12,14,15,16,18,**
..... **May-08, 15,17,19**

8.7 Spanning Trees **Dec.-11,13,14,15,16,17,18,**
..... **May-05,14,15,18,19**

8.8 Fundamental Circuits and Cutsets **Dec.-07,12,14,15,16, May-07,15**

8.9 Network Flows **Dec.-12,13,14,16,17,18,**
..... **May-07,14,17, April-11**

8.10 Game Tree **Dec.-17, May-18**

8.11 Case Studies

8.12 University Questions with Answers

8.1 Introduction

In this chapter we will study one of the simplest types of the connected graphs known as trees. This class of graphs has wide applications and has been the subject of study of many outstanding scientiests of different fields. Trees are discovered by Kircnnoff in 1847, while investigating the electrical networks. Sir Arthur Cayley used trees to study and enumerate isomers of saturated hydrocarbons.

The important applications of tree include searching, sorting, syntax checking and database managements. The tree is one of the most non-linear structures used for algorithm development in computer science.

8.2 Tree

SPPU : Dec.-16, May-17,18

A tree is a connected graph without any circuit i.e. tree is a connected acyclic graph. The collection or set of an acyclic graphs. (not necessarily connected) is called a forest.

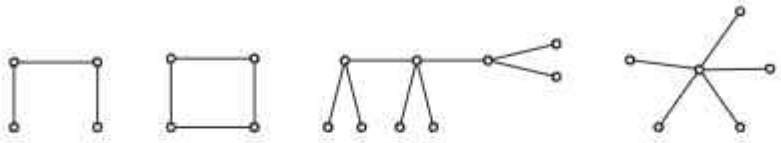
Examples :

Example 1

(i) (ii) (iii)

All these three graphs are trees. There are unique tree on one vertex, 2 vertices and 3 vertices.

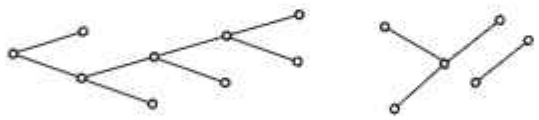
Example 2



(i) (ii) (iii) (iv)

Graphs (i), (iii) and (iv) are trees but (ii) is not a tree as it has a cycle.

Example 3 : Consider the following graph G.

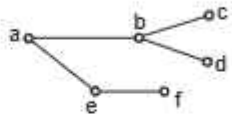


G

This graph G is not a tree because it is not connected graph. But G has a connected components i.e. G has 3 components which are trees. Therefore G is a forest.

A) A vertex of degree 1 in a tree is called a **leaf** or a **terminal node.** A vertex of a degree greater than one is called a **branch node** or **internal node.**

e.g. In a tree



c, d and e are leaves or terminal nodes and a, b, c are branch nodes.

B) Some properties of tree are obvious

i) Every edge of a tree is an isthmus. Conversely if every edge in a connected graph G is an isthmus then G is a tree.

ii) Every tree is a simple graph because loops and multiple edges field cycles.

iii) Every tree is a bipartite graph with $n \geq 2$ vertices.

Theorem 1 : A graph G is a tree iff there exists a unique path between every distinct pair of vertices of G.

Proof : Suppose G is a tree. So G is a connected and without circuits. We know that a circuit forms two or more paths. But G has no circuit so it has unique path between every pair of vertices.

Conversely assume that there is a unique path between every pair of vertices in G. This implies that G is a connected and G has no cycles. i.e. G is a connected acyclic graph.

Therefore G is a tree.

Theorem 2 : A graph G on n vertices is a tree iff G is connected and has exactly $n - 1$ edges.

Proof : Suppose graph G is a tree. Therefore is a connected and acyclic graph.

It is sufficient to prove that G has $n - 1$ edges only.

We can prove this by induction principle on **n**.

For $n = 1$, the result is obvious

Let $n > 1$, consider $G - e'$ for any $e' \in E(G)$

As G is a tree, e' is an isthmus of G.

$\therefore G - e'$ is a disconnected graph with two components say G_1 and G_2.

Now G_1 and G_2 are connected and a cyclic graphs as G_1 and G_2 are subgraph of G.

Let G_1 has n_1 vertices and m_1 edges and G_2 has n_2 vertices and m_2 edges

$\therefore$ By induction principle

$$m_1 = n_1 - 1 \qquad \text{and} \qquad m_2 = n_2 - 1$$

Therefore the number of edges in G is given by

$$e = (e_1 + e_2) + 1 = n_1 - 1 + n_2 - 1 + 1$$
$$= n_1 + n_2 - 1$$
$$e = n - 1 \qquad (\because\ n = n_1 + n_2)$$

Hence G has n – 1 edges only.

Conversely assume that G is connected graph and has exactly n – 1 edges.

Claim : Prove that G is a tree.

It is suffices to prove that G is noncyclic graph.

Suppose G contains a cycle C.

Let P denote the number of vertices in C

$\therefore$ The number of edges in C = P

As G is connected graph, the remaining n – P vertices must be connected to vertices in C. To connect each vertex of G which is not in C, we required n – P edges as each edge of G can connect only one vertex to the vertices in C.

Hence the total number of edges in G is given by

$$e = (n - P) + P = n \Rightarrow e = n \text{ which is contradiction}$$

$\therefore$ G has n – 1 edges only.

Thus G is a tree.

Theorem 3 : Let G be a graph with n vertices and m edges. If any of the following is true then all are true.

i) G is a tree.

ii) G is connected and m = n – 1

iii) G is acyclic graph and m = n – 1

iv) Every edge of G is an isthmus and G is connected.

v) There is exactly one path between every pair of vertices in G.

Theorem 4 : A non trivial tree contains at least two vertices of degree 1. (i.e. pendent vertices)

Proof : Let G be a tree with n vertices, so G is connected.

$\therefore \quad d(x) \geq 1 \ ; \ \forall x \in V(G)$

By handshaking lemma

$$\sum d(x) = 2 \times \text{Number of edges in } G = 2(n-1)$$

$x \in V(G)$

Suppose there is no vertex of degree 1.

Then $\quad 2n \le \sum_{x \in V(G)} d(x) = 2n - 2$

i.e. $2n \le 2n-2 \Rightarrow n \le n-1$ which is impossible.

Thus there is at least one vertex of degree 1.

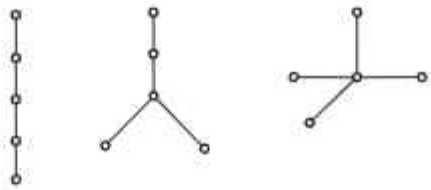
By a similar argument there is one more vertex of degree 1.

Hence every tree has at least two pendent vertices.

Example 8.2.1 *Draw all non isomorphic trees on 4 and 5 vertices.*

Solution : a) Non isomorphic trees on 4 vertices

b) Non isomorphic trees on 5 vertices



Example 8.2.2 *Is it possible to draw a tree with five vertices having degrees 1, 1, 2, 2, 2, 4 ?*

Solution : According to given degrees graph G has 6 vertices and total degree of graph is 1 + 1 + 2 + 2 + 2 + 4 = 12. So G has 6 edges as each edge contributes degree two.

Therefore G is a graph with 6 vertices and 6 edges which is contradiction to the fact that a tree on 6 vertices has 6 – 1 = 5 edges.

Hence the tree with given degree sequence of vertices does not exist.

Example 8.2.3 *Under what conditions trees are the complete bipartite graphs.*

Solution : Suppose T is a tree which is the complete bipartite graph.

Let $\quad T = K_{m,n}$

$\therefore$ T has m + n vertices and $(m+n-1)$ edges.

But $K_{m,n}$ has $m \cdot n$ number of edges.

Therefore $\quad mn = m + n - 1$

$$mn - m - n + 1 = 0$$

$$(m - 1)(n - 1) = 0$$

$\Rightarrow$ $m = 1$ or $n = 1$

i.e. $m = 1$ and any n or $n = 1$ and any m

Hence $K_{1,n}$ and $K_{m,1}$ are the only complete bipartite graphs. These are known as star graphs.

Thus T is a star graph.

Example 8.2.4 *a) Is it possible to draw a tree with 10 vertices which has vertices either of degree 1 or 3 ?*

If possible draw tree. Is it possible to draw same type of tree with 13 vertices ?

b) For which values of n (number of vertices), such type of tree exist ?

Solution : Given that tree T has 10 vertices so it must have 9 edges.

Let x and y be the number of vertices of degree 1 and 3 in T respectively.

$$\therefore \quad x + y = 10 \qquad \text{... (1)}$$

By handshaking lemma

$$\sum_{V \in V(T)} d(v) = 2(\text{Number of edges in T})$$

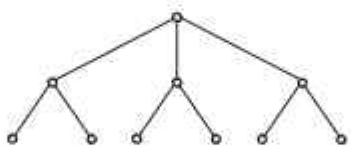
$$x + 3y = 2 \times 9 = 18$$

$$x + 3y = 18 \qquad \text{... (2)}$$

Solving equations (1) and (2) we get

$$y = 4 \quad \text{and} \quad x = 6$$

Therefore there are 6 vertices of degree 1 and 4 vertices of degree 3. Such type of graph is given below.



Now consider a tree with 13 vertices and 12 edges.

By using similar theory, we get

$$x + y = 13 \qquad \text{... (3)}$$

$$x + 3y = 24 \qquad \text{... (4)}$$

Solving (3) and (4), we get $1y = 11 \Rightarrow y = \frac{11}{2}$ and $x = \frac{15}{2}$ which is impossible.

Therefore it is not possible to draw such type of tree.

b) Let T be a tree with n vertices.

Let x and y be the number of vertices of degree 1 and 3 in T respectively. T has n – 1 edges.

Therefore $x + y = n$ ... (5)

By handshaking lemma $x + 3y = 2(n - 1)$

$$x + 3y = 2n - 2 \quad \text{... (6)}$$

Equation (6) – Equation (5) $\Rightarrow 2y = 2n - 2 - n$

$$2y = n - 2$$

$$\Rightarrow \quad y = \frac{n-2}{2} = \frac{n}{2} - 1$$

$$\Rightarrow \quad x = n - y = n - \frac{n}{2} + 1 = \frac{n}{2} + 1$$

i.e. $x = \frac{n}{2} + 1$ and $y = \frac{n}{2} - 1$. These should be non negative integers.

Case 1 : If n is even then $x = \frac{n}{2}$ is odd

and $\frac{n}{2} + 1 = x$ is an even integer

$\frac{n}{2} - 1 = y$ is an even integer

Thus the tree exists with required condition on even number of vertices.

$T_2, T_4, T_6, T_8 \ldots T_{2n}$ are such type of trees.

Case 2 : If n is odd then $\frac{n}{2}$ is not an integer there x and y are not an integer.

Hence the required tree does not exist on odd number of vertices.

8.3 Centre of a Tree

SPPU : Dec.-18

We know that the distance between two vertices in a connected graph i.e. is a length of the shortest path between that vertices. Before defining centre of a tree first find the distance between vertices of a tree which is defined as follows :

8.3.1 Eccentricity of a Vertex

Let G be a connected graph G and $v \in V(G)$. The eccentricity of a vertex v is denoted by E(v) or e(v) and defined as the distance from v to the vertex farthest from v in G.

i.e. $E(v)$ or $e(v) = \max\{d(v, v_i)/\forall v_i \in v(G)\}$

8.3.2 Centre of a Graph

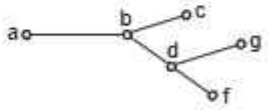
A vertex in a graph G with minimum eccentricity is called a centre of G and it's eccentricity is called as radius of G. It is denoted by r(G).

Consider the following examples

Example 1 :

$e(a) = 3, \quad e(b) = 2, \quad e(c) = 3$

$e(g) = 3, \quad e(d) = 2, \quad e(f) = 3$



G

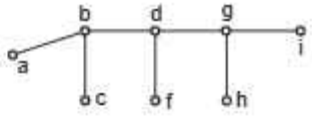
The smallest eccentricity is 2 for vertices b and d

$\therefore$ b and d are centre of G and $r(G) = 2$

Example 2

$e(a) = 4, \quad e(b) = 3, \quad e(c) = 4, \quad e(d) = 2,$

$e(f) = 3, \quad e(h) = 4, \quad e(i) = 4$



The minimum is 2 $\therefore$ Centre is at d and $r(G) = 2$

It is observed that the graph has more than one centre.

Theorem 1 : Prove that every tree has either one or two centres.

Proof : Let T be a tree. Then e(v) for a vertex v is at a vertex farthest from v. As T is a tree, it is attained at a pendent vertex.

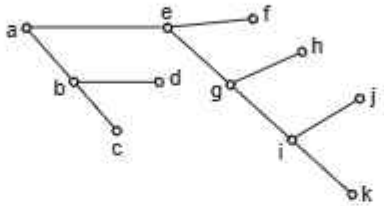
We know that every non trivial tree has at least two pendent vertices. Now, delete all pendent vertices from G, we get new graph G′ which is also a tree. Moreover as one edge at each pendent vertex is removed, the eccentricity of any vertex will be reduced by one. Therefore centres of G will still remain centres of G′. We continue the process with G′ until we arrive at K_2 or K_1. K_2 has two centres and K_1 has one centre. Hence the proof.

8.3.3 Cut Vertex of a Tree

We know that the vertex v whose removal from a connected graph G, disconnects the graph is called as cut vertex of G.

In any tree all vertices except pendent vertices are cut vertices.

e.g. Consider the following tree



G

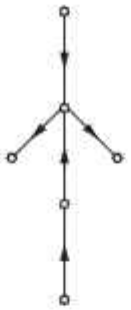
In graph G, a, b, c, g, i are cut vertices and c, d, f, h, j and k are not cut vertices.

8.4 Rooted Tree and Binary Tree

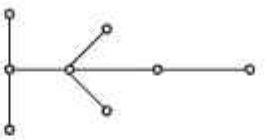
SPPU : Dec.-09, 16, 18, 19, May-18, 19

A connected acyclic, directed graph is called a **directed tree.** In other words, a directed graph is said to be a directed tree if it will become a tree when the directions of the edges are ignored.

e.g.



Directed tree



Non directed tree or tree

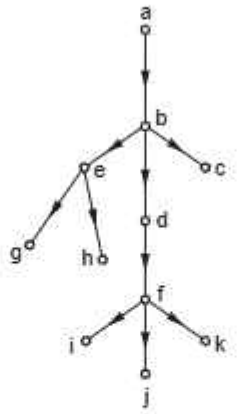
8.4.1 Rooted Tree

A directed tree is called a **rooted tree** if there is exactly one vertex whose incoming degree is zero and the incoming degrees of all other vertices are one.

The vertex with incoming degree 0 is called **the root** of the rooted tree. The vertex whose outgoing degree is zero is called **leaf** or **terminal node.**

A vertex whose outgoing and incoming degrees are non zero is called a **branch note** or an **internal node.** Consider the following example.

Given graph is a rooted tree a is the root of tree. Vertices b, e, d, f are called branch nodes. The vertices c, h, g, i, j and k are leaves.



Rooted tree

8.4.2 Level and Height of a Tree

A vertex v in a rooted tree is said to be at **level** n if there is a path of length n from the root to the vertex v.

The **height** of the tree is the maximum of the levels of its vertices.

Example

In the given graph G .... a is a root

Vertices b, c, d are at level 1

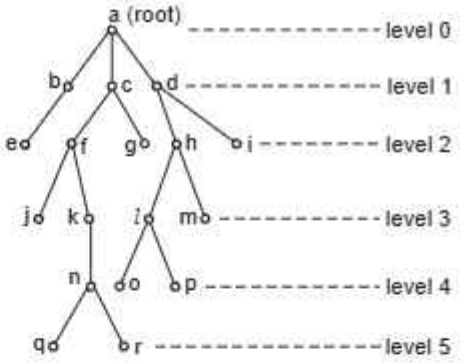
Vertices e, f, g, h and i are at level 2

Vertices j, k, l and m are at level 3

Vertices n, o and p are at level 4

Vertices q and r are at level 5

The maximum level is 5. Therefore the height of tree is 5.

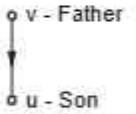


G

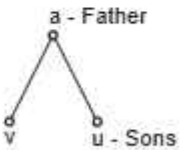
Rules for rooted tree : In a rooted tree

i) If the level of a vertex u is greater than the level of vertex v then u is below v.

ii) If vertex u is below vertex v and there is an edge from v to u then u is said to be son of v (or child of v) and v is said to be the father of u (or parent of u) i.e.



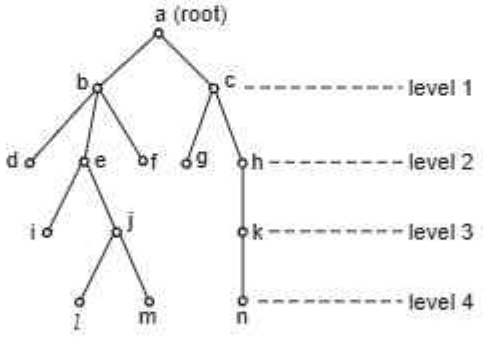
iii) Two vertices u and v are said to be brothers if they are the sons of the same vertex.



iv) A leaf is a vertex without children.

v) If $p = \{a, v_1, v_2, v_3, \ldots v_{n-1}, b\}$ is a path from a to b then b is called as descendent of a and a is called as ancestor of b.

Consider the following example



From above figure i) a is a root of the tree.

ii) b and c lie at level 1. ∴ b and c are sons of root a i.e. a is father of b and c and b and c are brothers.

iii) b has three sons d, e, f

∴ b is a father of d, e, f

c has 2 sons g and h

iv) e has two sons i and j

h has only one son k

v) i has no son j has two sons *l* and m. So k and *l* are brothers. k has one son n. n has no brother as n is a leaf.

l, m, n are known as descendent of a and a is ancestor of d, e, f, g, h, i, j, k, *l*, m, n and so on.

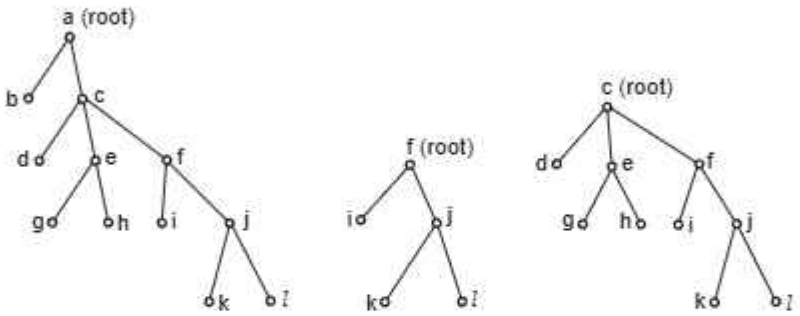
8.4.3 Subtrees

Let T be a rooted tree and $x \in V(T)$.

A vertex x together will all its descendants is called the **subtree** of T rooted at x.

The subtree corresponding to the root node is the **entire tree.** The subtree corresponding to any other node is called a **proper subtree.**

Example



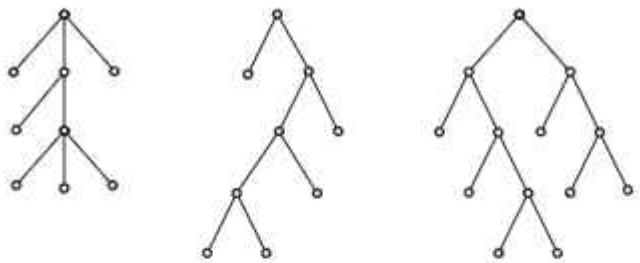
Tree Subtree with root at f Subtree with root at c

8.4.4 M-ary Trees

A rooted tree in which every interior node has at most m sons is called an **m-ary tree.**

A m-ary tree is said to be **regular m-ary tree** or full m-ary tree if every branch node has exactly m sons.

Consider the following examples.



3 - ary tree 2 - ary tree Full 2 - ary tree

Theorem 1 : A regular m-ary tree with p interior nodes has mp + 1 nodes at all.

SPPU : Dec.-09

Proof : Let T be a regular m-ary tree with n vertices. Out of n vertices there are p interior vertices or branch nodes.

Therefore there are t = n – p number of sons or leaves in T.

But given graph is regular and p interior nodes.

So the regular m-ary tree will have mp sons.

But root is not a son.

Therefore give tree has total (mp + 1) number of vertices

Hence $\boxed{n = mp + 1}$

8.5 Binary Tree

SPPU : Dec.-09, 16, May-10, 12, 17

So far we have discussed the tree and its properties. Now, we shall study about a special class of rooted tree known as binary trees. Binary trees play an important role in decision making. They are extensively used in the study of computer search methods, binary identification problems and coding theory. Binary tree is a particular case of m-ary tree.

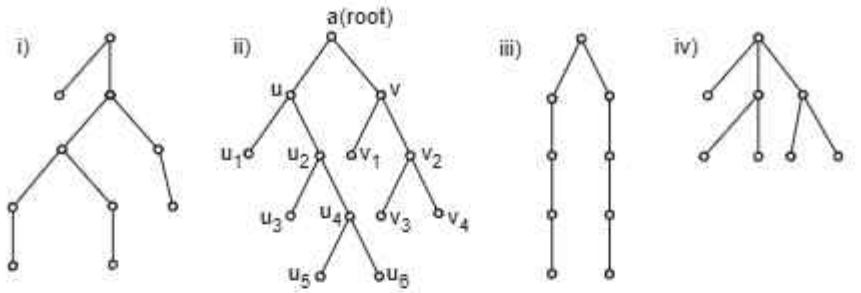
Definition :

An **m**-ary tree is known as **binary tree** if every branch node has at most 2 sons.

In other words, a tree in which there is exactly one vertex of degree 2 and each of the remaining vertices of degree or three, is called a binary tree.

A binary tree is called as **regular binary tree** or **full binary tree** if every branch node has exactly 2 sons or zero son.

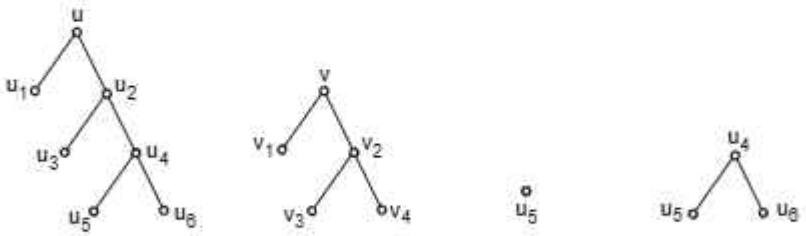
Consider the following examples



Binary tree | Full binary tree | Binary tree | Not binary tree

In binary trees, instead of referring the first or second subtree of a branch node, we use to the left subtree or right subtree of the node.

In above example (ii), we have



Left subtree of a | Right subtree of a | Left subtree of u_4 | Right subtree of u_2

Note : The maximum number of vertices in a full binary tree with h levels is $1+2+2^2+ \ldots 2^h$.

$\therefore$ This is in geometric progression.

Let $$S_{h+1} = 1+2+2^2+ \ldots 2^h \quad \ldots (8.5.1)$$

$$2S_{h+1} = 2+2^2+2^3+ \ldots 2^h+2^{h+1} \quad \ldots (8.5.2)$$

$\therefore$ Equation (8.5.2) – Equation (8.5.1) $\Rightarrow$

$$\boxed{S_{h+1} = 2^{h+1}-1}$$

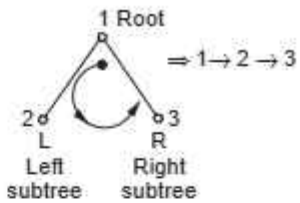
Theorem : The minimum height of a binary tree on n vertices is $[\text{Log}_2(n+1)-1]$ and maximum possible height is $\frac{n-1}{2}$. Assume that the root of T is at 0 level.

8.5.1 Binary Tree Traversal

Traversing means visiting or processing all the nodes of a tree. A binary tree traversal is the visiting of each node of a tree only once according to some sequence.

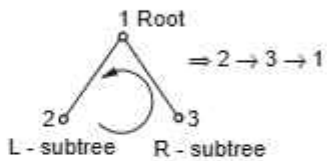
There are two types of traversing binary trees.

1) Depth-first traversal : In this method, the processing proceeds from the root or the most distant descendent of the first child. There are three types of Depth-First Traversal.



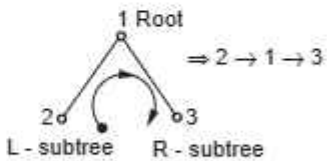
A) Pre order traversal : In this traversal, the root node is traversal first, followed by the left subtree and then the right subtree as shown below.

B) Post order traversal : If processes first the left subtree then the right subtree and then at the last root of the tree as shown below.

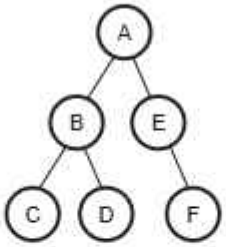


C) In order traversal : It processes the left subtree first then the root and at the last right subtree.

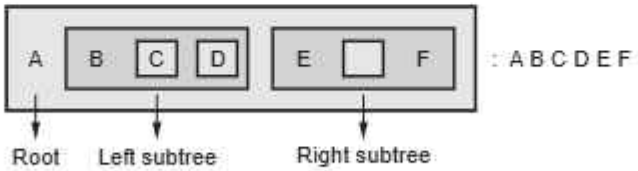
The prefix "in" means root is processed in between the subtrees. It is shown as below.



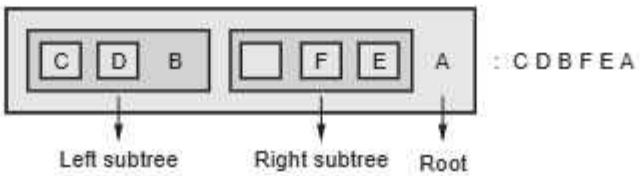
Consider the following binary tree.



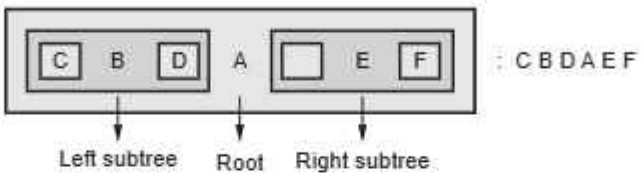
i) Pre order traversal



ii) Post order traversal

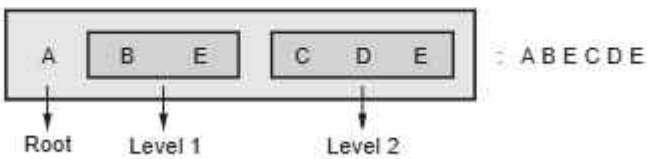


iii) In order traversal



2) Breadth first traversal : In this traversal, the processing proceeds horizontally from the root of all of its children, then to its childre's children and so on until all the nodes have been processed. That is first write node of zero level, then all nodes of level 1 then level 2 and so on.

The breadth first process for the above binary is shown below.



8.5.2 Binary Expression Tree

An algebraic expression can be conveniently expressed by its expression tree. An expression tree is a binary tree with the following properties.

i) Each leaf node is an operand

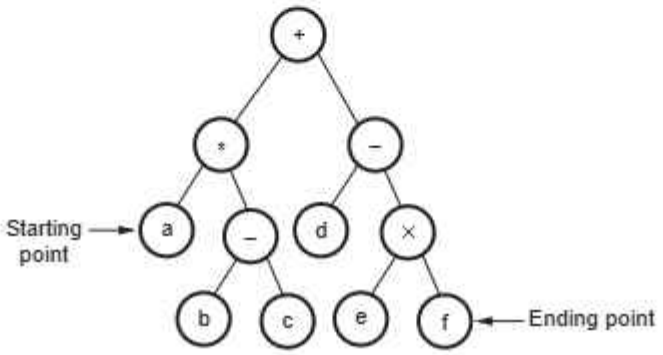
ii) The root and internal nodes are operators

iii)Subtrees are subexpressions, with the root being an operator.

In this context, we consider only standard arithmetic operations +, –, ×, /, .

For example

i)



Binary tree

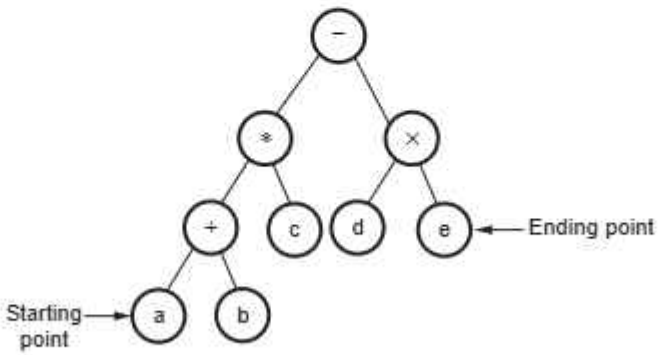
The expression of given binary tree is

$[a*(b-c)] + [d - (e \times f)]$

2) The binary tree of the expression

$[a+b)*c] - [d \times e]$

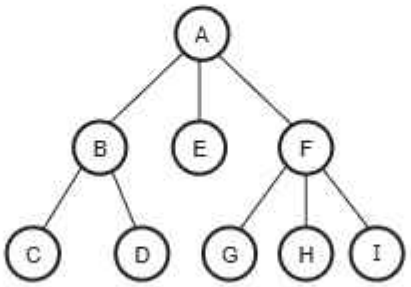
is given below



8.5.3 Conversion of General Tree to Binary Tree

In general life we come across so many applications in which general trees are used. It is considerably easier to represent binary trees in programs than it is to represent general trees.

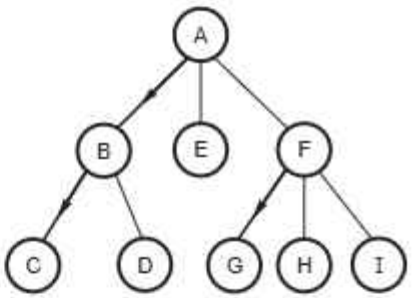
Now, let us explain the method to convert general tree to binary tree with the help of the following example.

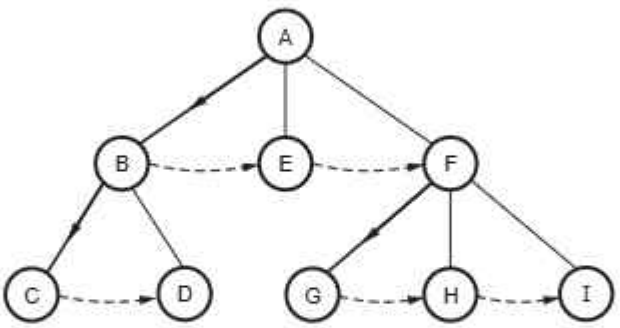


Consider the following steps

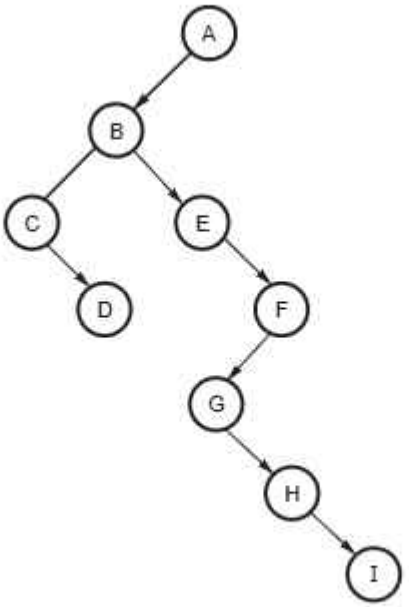
Step 1) To convert it into binary tree, we first identify the branch from the parent to its first or left most child. These branches from each parent become left pointers in the binary tree.

Step 2) Connect sibling, starting with the left most or first child, using a branch for each sibling to its right sibling. They are the right pointers in the binary tree.



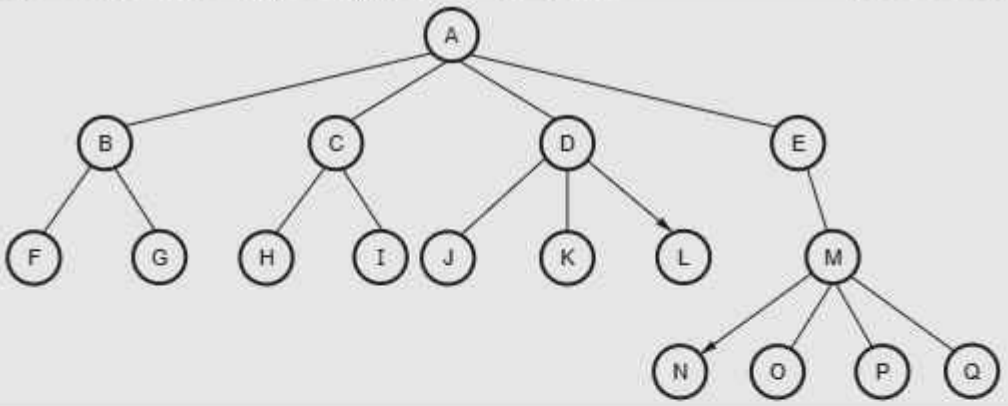


Step 3) Now remove all unneeded branches from the parent to its children. Therefore remove $A \rightarrow E$, $B \rightarrow D$, $A \rightarrow F$, $F \rightarrow H$, $F \rightarrow I$ we get the following required binary tree.

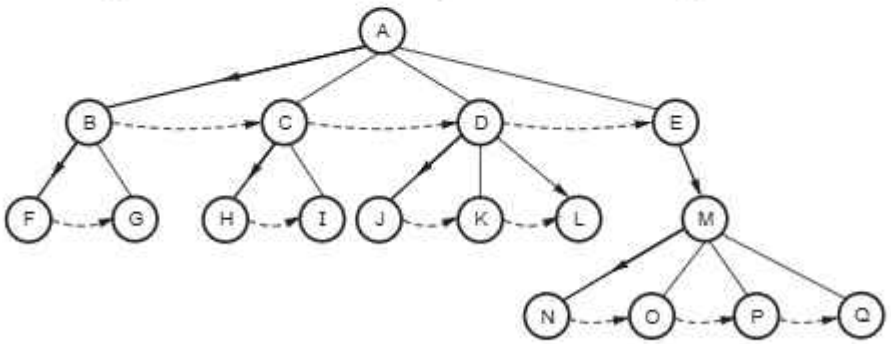


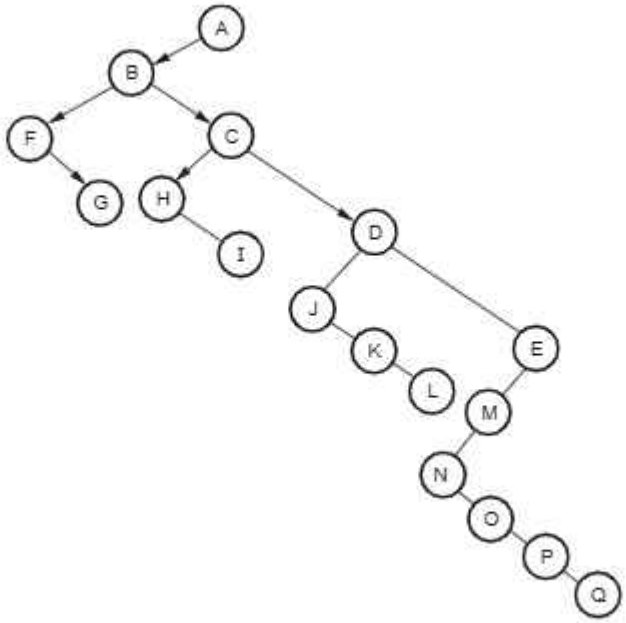
Resulting binary tree

Example 8.5.1 *Convert the following tree into binary tree.* **SPPU : Dec.-09**



Solution : The steps involve to convert the given tree into binary tree are as follows

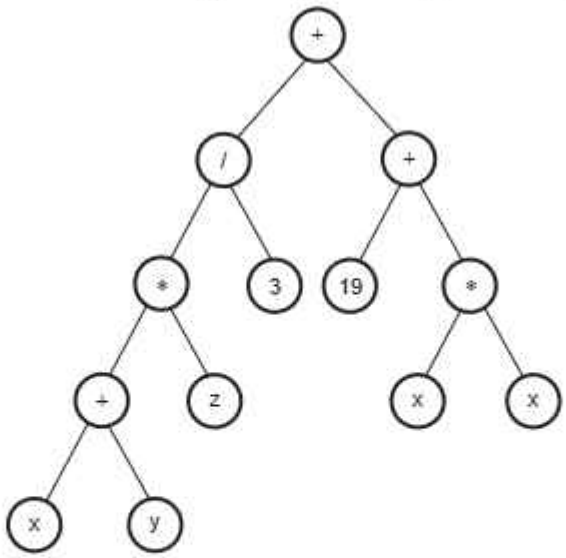




Binary tree

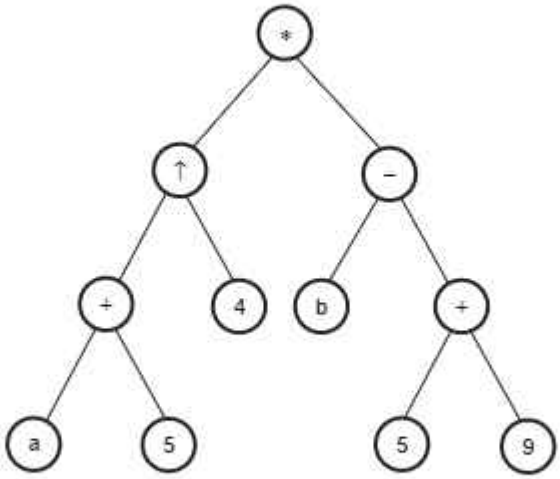
Example 8.5.2 *Construct the labeled tree of the following algebraic expression* $(((x+y)*z)/3)+(19+(x*x))$ **SPPU : May-10**

Solution : The expression tree for the given algebraic expression is given below :

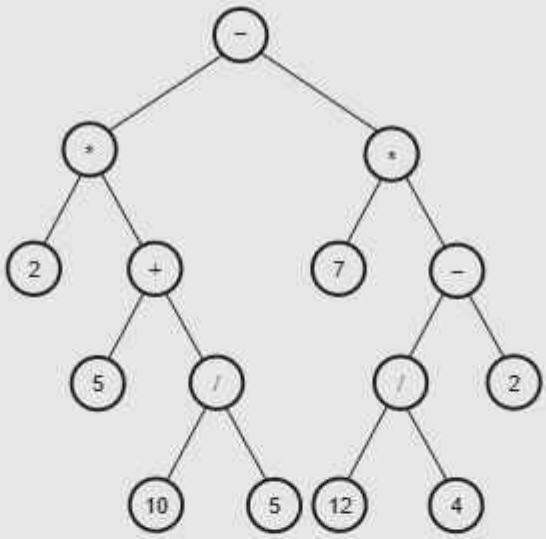


Example 8.5.3 *Represent the expression* $((a+5)\uparrow 4)*(b-(5+9))$ *using a binary tree.* **SPPU : May-12**

Solution : The binary tree for the given expression is as follows :



Example 8.5.4 *Write and evaluate the expression tree shown below.* **SPPU : Dec.-09**

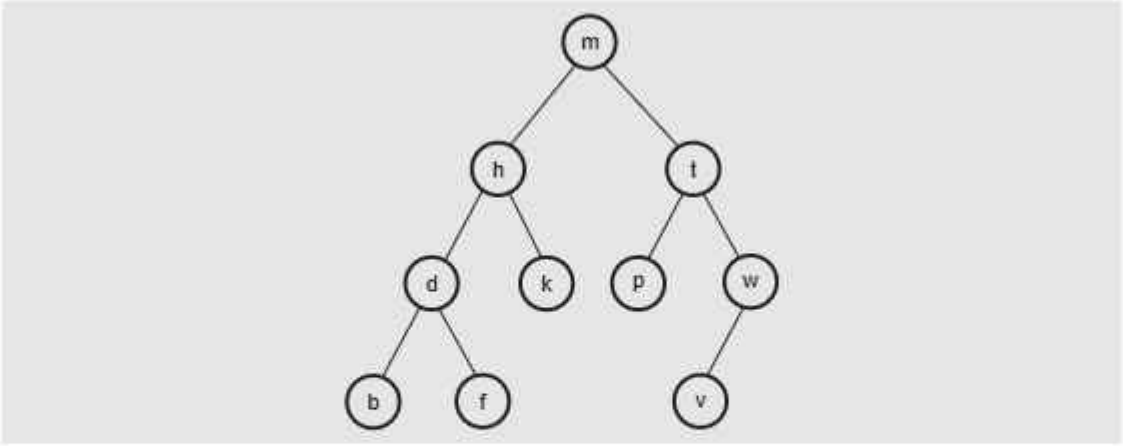


Solution : The algebraic expression of the given binary tree is

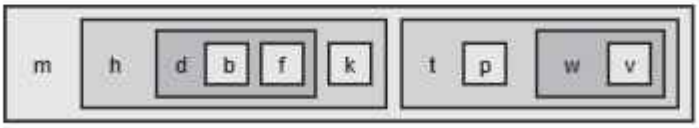
$(2*((5)+(10/5))) - ((7)*(((12/4)-(2))))$

It's value is $2*(5+2)-(7*(3-1)) = 14 - 7 = \mathbf{14}$

Example 8.5.5 *Find the preorder, postorder and inorder traversal of the following tree.* **SPPU : May-10**



Solution : i) Preorder traversal

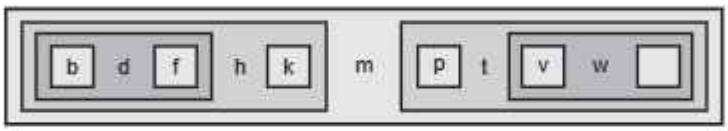


i.e. m b d b f k t p w v

ii) Postorder traversal

i.e. b f d k h p v w t m

iii) Inorder traversal



i.e. b d f h k m p t v w

Example 8.5.6 *How many leaves does a full binary tree with h levels have ?*

Solution : A full binary tree with h levels has 2^h leaves.

Example 8.5.7 *How many internal vertices does a full binary tree with h levels have ?*

Solution : The total number of vertices in full binary tree with h levels

$$= 2^{h+1} - 1$$

Out of which 2^h are leaves.

Hence internal vertices $= (2^{h+1}-1)-2^h$

$$= 2^h(2-1)-1$$

$$= 2^h-1$$

Example 8.5.8 *How many vertices will a full ternary tree with h levels have ?*

Solution : The number of vertices in a full ternary tree with h levels is

$$S = 1+3+3^2+3^3+\ldots+3^h$$

$$S = \frac{3^{h+1}-1}{3-1} = \frac{3^{h+1}-1}{2}$$

Hence the total number of vertices $= \dfrac{3^{h+1}-1}{2}$

Example 8.5.9 *What is the total number of nodes in a full binary tree with 20 leaves ?*

Solution : Let n be the number of nodes in a full binary tree.

Then $n = mP + 1$

$m = 2$

$\Rightarrow$ $n = 2P + 1$ Where P is the number of branch nodes.

The number of nodes in a tree is the sum of branch nodes and leaves.

Hence $n = P + 20 \Rightarrow P = n - 20$

$\Rightarrow$ $n = 2n - 40 + 1 = 2n - 39$

$2n - n = 39$

$n = 39$

Hence the total number of nodes in a full binary tree with 20 leaves are 39.

Example 8.5.10 *Can be have a ternary tree with exactly 20 nodes ?*

Solution :

We have $n = mP + 1$

$20 = 3P + 1$

$3P = 20 - 1 = 19$

$P = \dfrac{19}{3}$ Which is impossible

Therefore such tree does not exist.

Example 8.5.11 *Show that the number of vertices in a binary tree is odd.*

Solution : Let T be a binary tree with n vertices. T contains exactly one vertex of degree 2 and the remaining vertices of T are of degree one or three. Therefore the number of vertices of odd degree are n – 1. But for any graph the number of vertices of odd degree must be even. So n – 1 must be even.

Hence n is odd.

Example 8.5.12 *If T is a binary tree on n vertices then the number of pendent vertices in T is* $\frac{n+1}{2}$.

Solution : Let T be a binary tree with n vertices. Out of these vertices P vertices are pendent vertices and one vertex is of degree 2.

Therefore remaining all vertices are of degree 3.

Thus there are n – P – 1 vertices are of degree 3.

The number of edges in T is n – 1.

∴ Total degree of T is 2(n–1)

Thus $2 + P \times 1 + 3(n - P - 1) = 2(n - 1) = \text{Total degree}$

$$P + 3n - 3P - 1 = 2n - 2$$

$$n + 1 = 2P$$

$$\boxed{P = \frac{n+1}{2}}$$

Hence the number of pendent vertices are $\frac{n+1}{2}$.

Example 8.5.13 *Find the maximum of possible height of a binary tree with 13 vertices and draw graph.*

Solution :

We have $n = 13$

The maximum possible height of the binary tree is

$$\frac{n-1}{2} = \frac{13-1}{2} = 6$$

The required graph as shown in Fig. 8.5.1

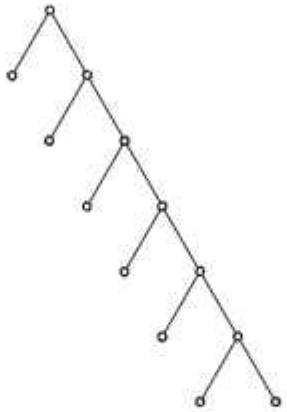


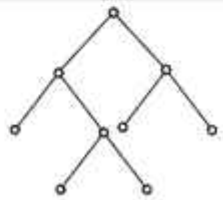
Fig. 8.5.1

Example 8.5.14 *Find the minimum height of the binary tree with 9 vertices.*

Solution : The minimum height of a binary tree

$$= [\log_2(n+1)-1] = \log_2(g+1)-1 = 3$$

It's graph is given below



8.6 Prefix Code and Binary Search Trees

SPPU : Dec.-12, 14, 15, 16, 18, May-08, 15, 17, 19

A **code** is simply an organized way of representing all information using a optimal set of symbols.

We know that modern computers rely on a binary code system consisting only of 0 and 1. Text messages, images, computer programs all such types of information is stored in computer by using coded form as a sequence of 1's and 0's.

The codes of the word may or may not be same length. So it is very difficult to convert codes into words. Therefore the codes should have some special characteristics so that easily we can convert that codes into unique meaningful words. Let us define prefix code.

Definition

A set of sequences is said to be a **prefix code** if no sequences in the set is a prefix of another sequence in the set.

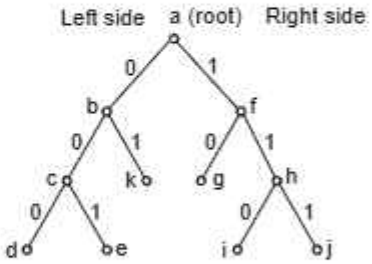
For example the set {000, 001, 01, 10, 11} is a prefix code as no sequence of symbols is present at the beginning of another sequences. All these sequence are distinct.

The set {1, 00, 000, 0001} is not a prefix code because the sequence 00 is the prefix of the sequences 000 and 0001.

A question comes in everyone's mind "How to construct prefix code to a full binary trees?"

We can solve this question by adding some flavours and binary codes to a full binary trees.

For a given full binary tree, we label the two branches incident from each internal node with 0 and 1. For the left branch we assign 0 and for the right branch we assign 1 of every rooted tree or subtree. Consider the following example of full binary tree.



In given figure root a has 2 sons. Left son is ab and right is af, so assign 0 to ab and 1 to af. Now b has two sons. Assign 0 to left son be and 1 to right son bk. Similarly assign 0 or 1 to every edge of a tree.

Now assign to each leaf, a sequence of 0's and 1's which is the sequence of labels of the edges in the path from the root to the leaf.

For example, d is a leaf and the path a to d is a – b – c – d and their respective labels are 0 – 0 – 0 so the prefix code of d is 000.

For leaf e, path is a – b – c – e and labels 0 – 0 – 1

∴ The prefix code of e is 001

Thus the prefix code of above tree is {000, 001, 01, 10, 110, 111}.

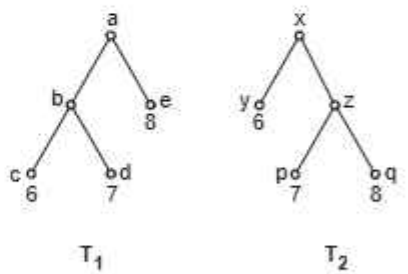
8.6.1 Optimal Tree

Let T be any full binary tree and $W_1, W_2, W_3, \ldots W_t$ be the weights of the leaves (terminal vertices) then the weight W of the full binary tree is given by

$$W(T) = \sum_{i=0}^{t} W_i l_i$$

Where $l_i = l(i)$ is the length of the path of the leaf i from the root of the tree. The full binary tree is called an optimal tree if its weight is minimum.

For example, suppose 6, 7, 8 are the weights of the leaves in a full binary tree as given below.



In T_1, $l(c) = 2$, $l(d) = 2$, $l(e) = 1$

∴ The weight of T is given by

$$W(T_1) = 6 \times l(c) + 7 \times l(d) + 8 \times l(e)$$

$$= 6 \times 2 + 7 \times 2 + 8 \times 1$$

$$= 12 + 14 + 8 = 34$$

In T_2, $l(y) = 1$, $l(p) = 2$, $l(q) = 2$

$$\therefore \quad W(T_2) = 6 \times 1 + 7 \times 2 + 8 \times 2$$

$$= 6 + 14 + 16 = 36$$

Hence $W(T_1) < W(T_2)$

Thus T_1 is the optimal tree for the weights 6, 7, 8.

8.6.2 Huffman Algorithm to Find an Optimal Tree

Let $W_1, W_2, W_3, \ldots W_t$ be the weights of the leaves and it is required to construct an optimal binary tree.

The following steps of an algorithm gives the required optimal binary tree.

Step 1 : Arrange the weights in increasing order

Step 2 : Consider two leaves with the minimum weights W_1 and W_2. Replace these two leaves and their father by a leaf. Assign weight $W_1 + W_2$ to this new leaf.

Step 3 : Repeat the step 2 for the weights $W_1, W_2, W_3, \ldots W_t$ until no weight remains.

Step 4 : The tree obtained in this way is an optimal tree for given weights and stop.

8.6.3 Optimal Prefix Code

A binary prefix code obtained from a optimal tree is called an optimal prefix code.

8.6.4 Binary Search Trees

In computer science, a binary search tree (BST) is a binary tree which has the following properties.

i) Each node has a value (weight)

ii) A total order is defined on these values.

iii) The left subtree of a node contains only values less than the nodes values.

iv) The right subtree of a node contains only values greater than or equal to the nodes values.

BST is used to construct more abstract data structures such as sets, multisets and associated arrays.

If a BST allows repeated values then it represents a multiset.

For multisets, we used everything in the right subtree is either > or = to the value of node.

For non repeated sets (All elements are distinct)

We used > for right subtree and < for left subtree

Let us formulate precisely the problem of searching for an item an ordered list.

i) Let $k_1, k_2, \ldots k_n$ be the n items in an ordered list which are known as keys.

ii) Assume that $k_1 < k_2 < k_3 < \ldots < k_n$

iii) Given an item x, our problem is to search the keys and determine whether x is equal to one of the keys or x falls between keys k_1 and k_{i+1} for some i.

iv) A search procedure consists of a sequence of comparisons between x and the keys k_i where each comparison gives.

i) $x = k_i$ or ii) $x > k_i$ iii) $x < k_i$

We explain now how we can describe a search procedure using a binary tree representation.

We define a search tree for the keys $k_1, k_2, \ldots k_n$ to be a binary tree with n branch nodes and n + 1 leaves. The branch nodes are labeled as $k_1, k_2, \ldots k_n$ and the leaves are labeled $k_0, k_1, k_2, \ldots k_n$ such that for the branch node with the label k_i, its left subtree contains only vertices with labels k_j for $j > i$ and its right subtree contains only vertices with labels k_j for $j \geq i$

Starting with the root of a search tree, we compare a given item x with label of root k_i.

If $x = k_i$ then search is completed.

If $x < k_i$, then we compare x with left son of the root.

If $x > k_i$ then we compare x with right son of the root.

Such comparison continues till, we find a match for x or till a leaf is reached. If a leaf labeled k_j is reached, it means that x is larger than the key k_j but less than the key k_{j+1}. If leaf labeled k_0 is reached it means that $x < k_1$.

If leaf labeled k_n is reached it means that $x > k_n$.

Example 8.6.1 *For the following set of weights construct optimal binary prefix code.*

$\alpha - 5$

$\beta - 6$

$\gamma - 6$

$\delta - 11$

$\epsilon - 20$

SPPU : Dec.-14

Solution : Here we use Huffman's coding method.

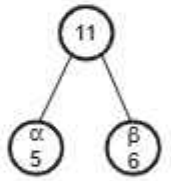
Step 1 : Sort the weights or frequencies of the given letters in increasing order in a queue.

α	β	γ	δ	∈
5	6	6	11	20
↑	↑			

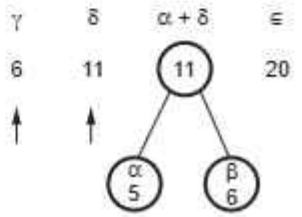
{↑ indicates the first 2 smallest weights}

Consider the two symbols with lowest weights say α, β. The root of the first subtree has a weight $5 + 6 = 11$.

As $6 < 11$, it can not be placed at the beginning. It has to be placed an appropriate position means after δ. The first subtree is given below



Step 2 : Again rewrite sequence of weights in increasing order by replacing and β by new subtree weight as

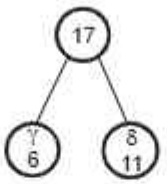


The first two smallest weights are 6 and 11.

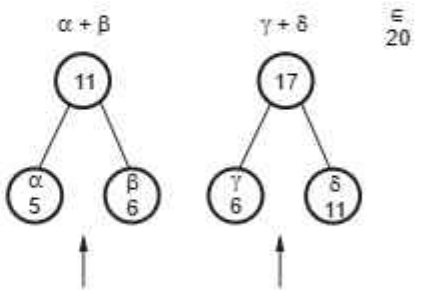
∴ The root of new subtree has a weight

$$6 + 11 = 17$$

The new subtree is as follows



Step 3 : Rewrite sequence of weight in increasing order as

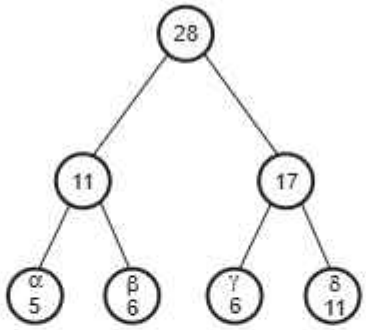


The first two smallest weights are 11 and 17.

∴ The root of new subtree has a weight

$$11 + 17 = 28$$

The new subtree is as follows

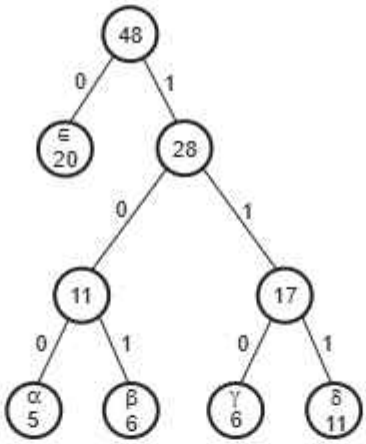


Step 4 : Rewrite the sequence of weights in increasing order

$$\begin{array}{cc} \in & \\ 20 & 28 \\ \uparrow & \uparrow \end{array}$$

The root of new subtree is 20 + 28 = 48

∴ The new subtree is as follows



This is the optimal binary tree.

Symbols leaves	α	β	γ	δ	∈
Binary prefix code	100	101	110	111	0

The weight of the optimal tree is

$$W = 20 \times 1 + 5 \times 3 + 6 \times 3 + 6 \times 3 + 11 \times 3$$

$$= 20 + 5 + 18 + 18 + 33 = \mathbf{104}$$

Note : All readers are requested to understand example 8.6.1 properly and then see next examples. Hereafter we have given solutions in shortforms.

Example 8.6.2 *Suppose data items A, B, C, D, E, F, G occur in the following frequencies.*

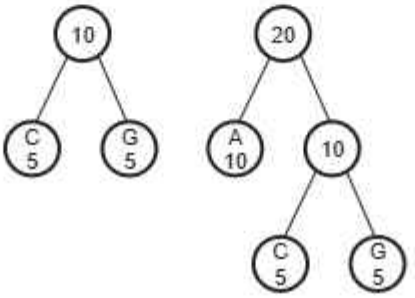
Data Items	A	B	C	D	E	F	G
Weight	10	30	5	15	20	15	05

Construct a Huffman code for the data.
What is the minimum weighted path length. **SPPU : May-08, 19, Dec.-14**

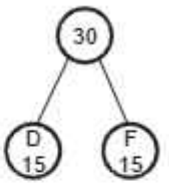
Solution : Consider the following steps

Step 1 : Sequence in increasing order is as follows

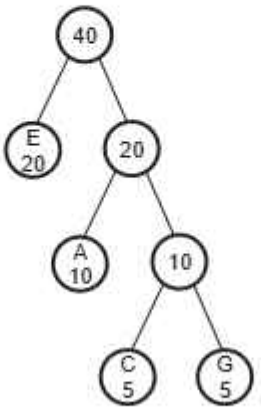
C D A D F E B
5 5 10 15 15 20 30
↑ ↑



Step 2 : Sequence : D F F
15, 15, 20, 20, 30
↑ ↑



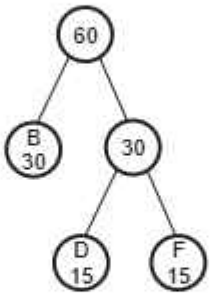
Step 3 : Sequence : E B
20, 20, 30, 30
↑ ↑



Step 4 : Sequence : B

30, 30, 40

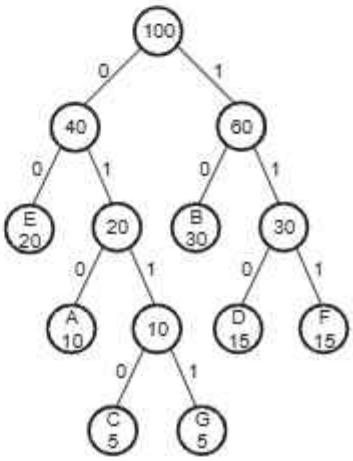
↑ ↑



Step 5 : Sequence : 40, 60

↑ ↑

Items	Binary prefix code
5	0111 or 0110
10	010
15	110 or 111
20	00
30	11



The minimum weight path length for the vertices as follows

$A \rightarrow 3,\ B \rightarrow 2,\ C \rightarrow 4,\ D \rightarrow 3,\ E \rightarrow 2,\ F \rightarrow 3,\ G \rightarrow 4$

∴ The minimum weight of tree is

$$W = 10 \times 3 + 30 \times 2 + 5 \times 4 + 15 \times 3 + 20 \times 2 + 15 \times 3 + 5 \times 4 = \mathbf{260}$$

Example 8.6.3 *For the following set of weights, construct optimal binary prefix code. For each weight in the set, give the corresponding code words.*
8, 9, 12, 14, 16, 19.

Solution :

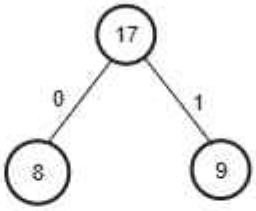
Step 1 : Increasing order of weights :

8, 9, 12, 14, 16, 19

↑ ↑

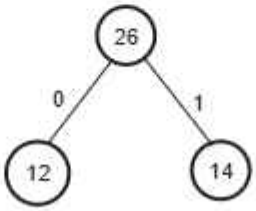
(↑ : First two smallest weights)

The first sub tree is as follows

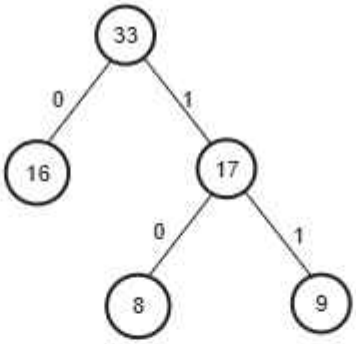


Step 2 : Rewrite sequence of weights in increasing order

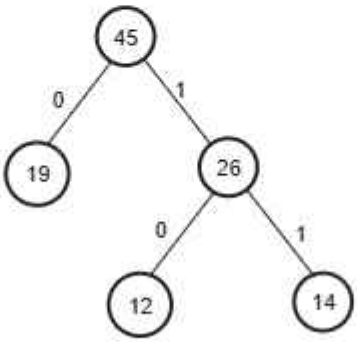
12, 14, 16, 17, 19
↑ ↑



Step 3 : Sequence : 16, 17, 19, 26,
↑ ↑

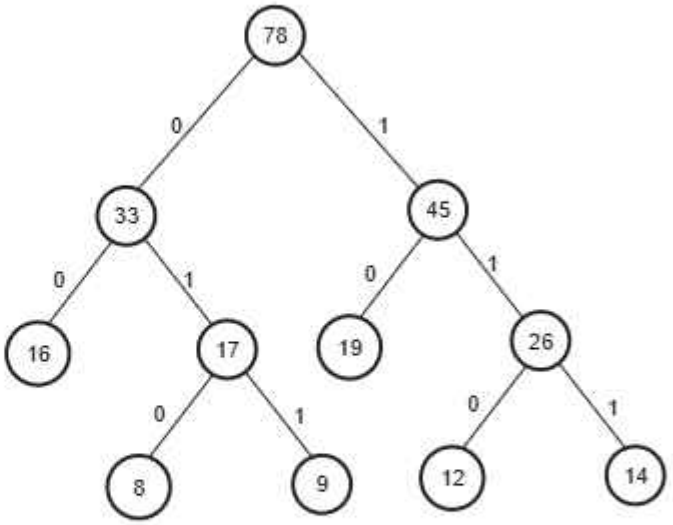


Step 4 : Sequence : 19, 26, 33,
↑ ↑



Step 5 : Sequence : 33, 45
↑ ↑

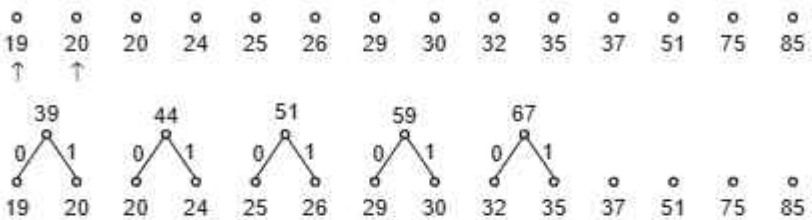
Symbols	Binary Prifix Codes
16	00
8	010
9	011
19	10
12	110
14	111



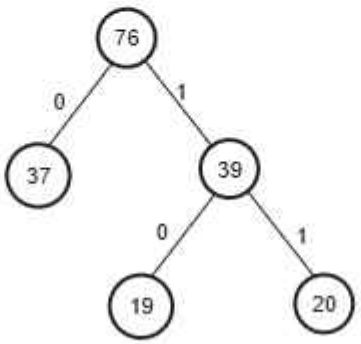
Example 8.6.4 *A secondary storage media contains information in files with different formats. The frequency of different types of files is as follows.*
Exe (20), bin (75), bat (20), jpeg (85), dat (51), doc (32), sys (26), c (19), cpp (25), bmp (30), avi (24), prj (29), 1st (35), zip (37).
Construct the Huffman code for this. **SPPU : May-15, Dec.-15**

Solution : Consider the following steps

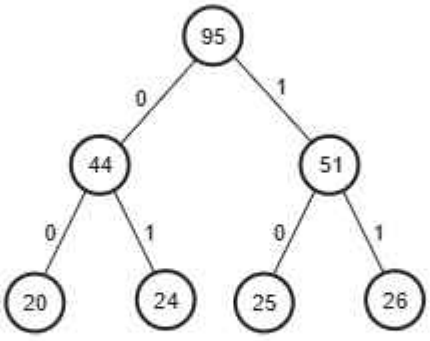
Step 1 : Sequence :



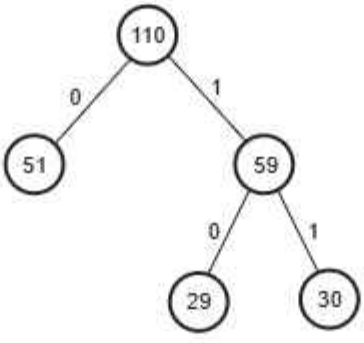
Step 2 : Sequence : 37, 39, 44, 51, 51, 59, 67, 75, 85
↑ ↑



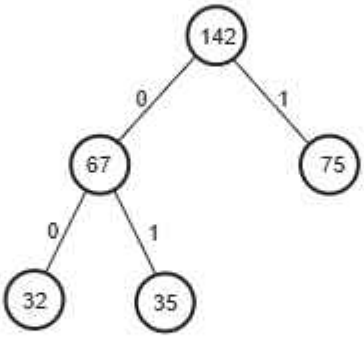
Step 3 : Sequence : 44, 51, 51, 59, 67, 75, 76, 85
↑ ↑



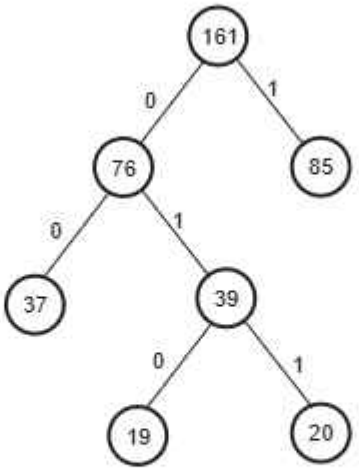
Step 4 : Sequence : 51, 59, 67, 75, 76, 85, 95
↑ ↑



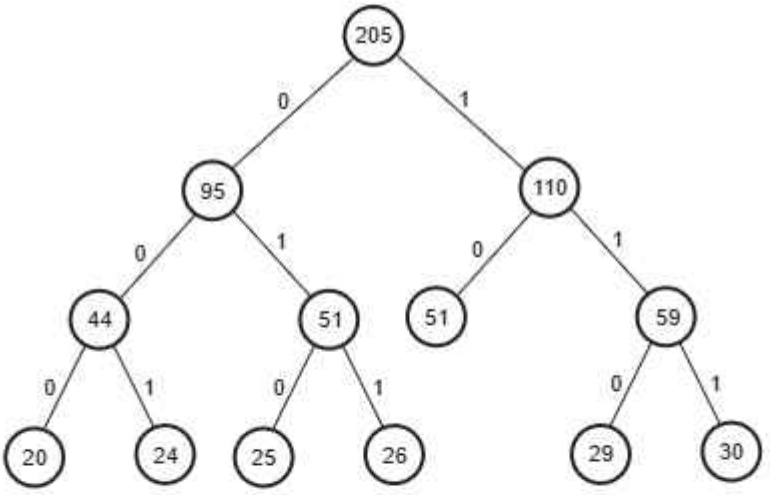
Step 5 : Sequence : 67, 75, 76, 85, 95, 100
↑ ↑



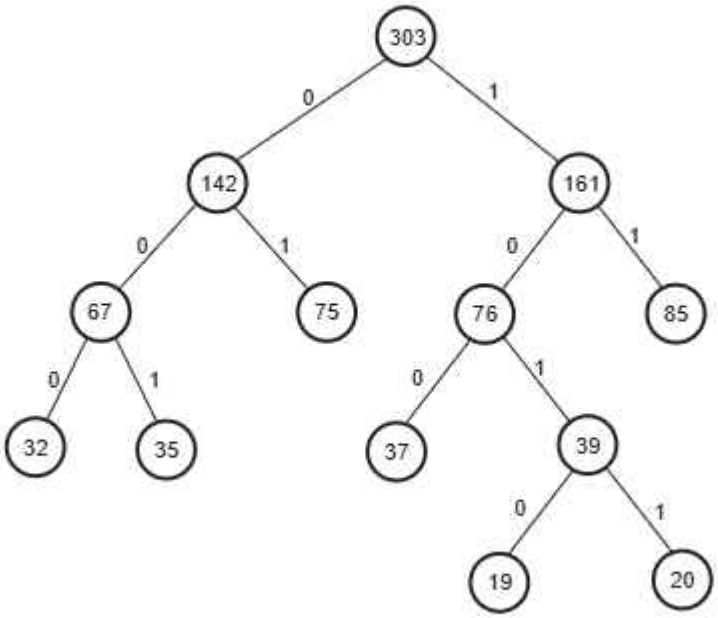
Step 6 : Sequence : 76, 85, 95, 110, 142,
↑ ↑



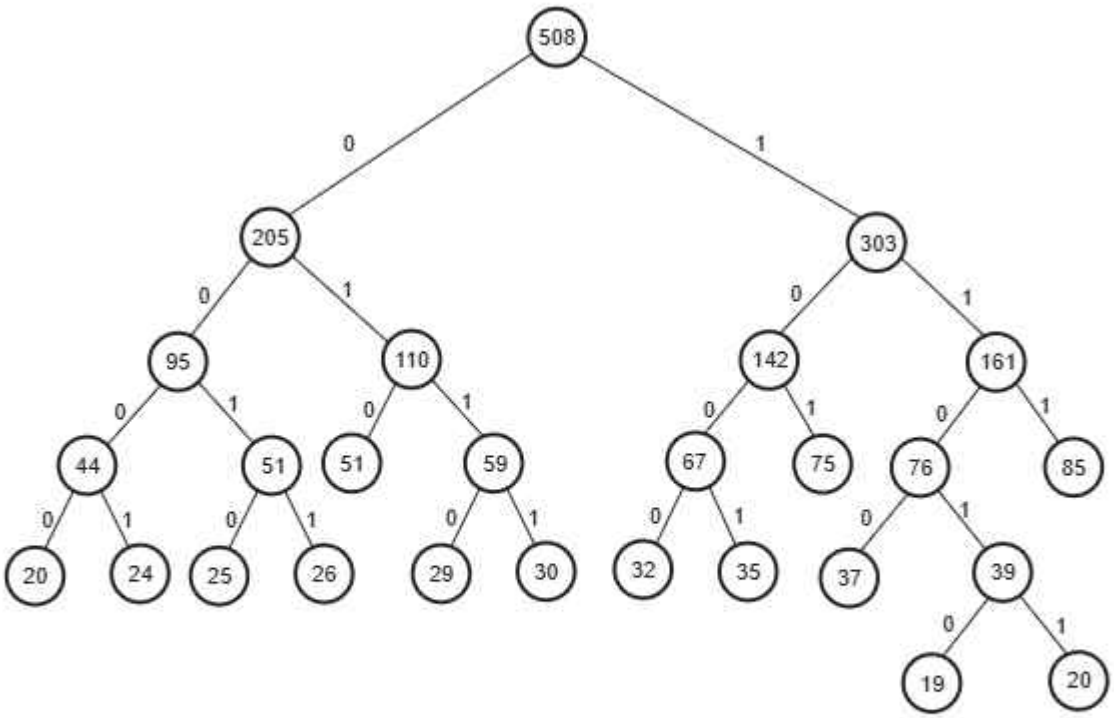
Step 7 : Sequence : 95, 110, 142, 161,
↑ ↑



Step 8 : Sequence : 142, 161, 205
↑ ↑



Step 9 : Sequence : 205, 303,
↑ ↑



Numbers	Binary prifix codes
20	0000
24	0001
25	0010
26	0011
51	010
29	0110
30	0111
32	1000
35	1001
75	101
37	1100
19	11010
20	11011
85	111

Example 8.6.5 *For the following sets of weights, construct an optimal binary prefix code. For each weight in the set, give the corresponding code word*

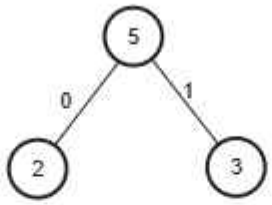
i) 2, 3, 5, 7, 9, 13

ii) 8, 9, 10, 11, 13, 15, 22 **SPPU : Dec.-12**

Solution : Consider the following steps.

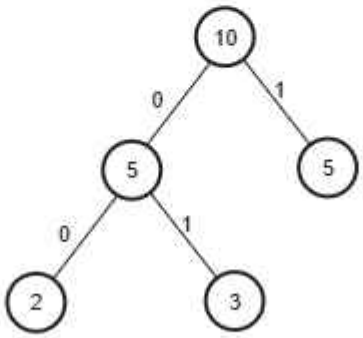
i) Step 1 : Sequence : 2, 3, 5, 7, 9, 13

↑ ↑



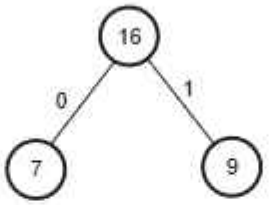
Step 2 : Sequence : 5, 5, 7, 9, 13,

↑ ↑



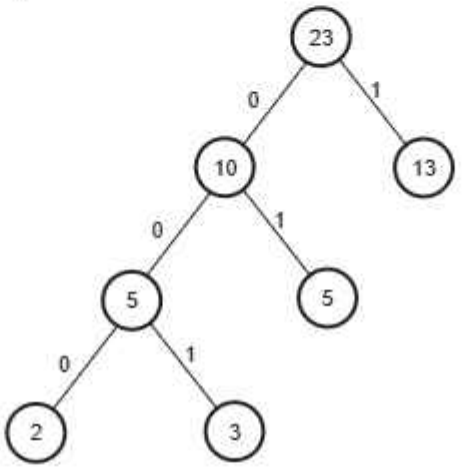
Step 3 : Sequence : 7, 9, 10, 13

↑ ↑



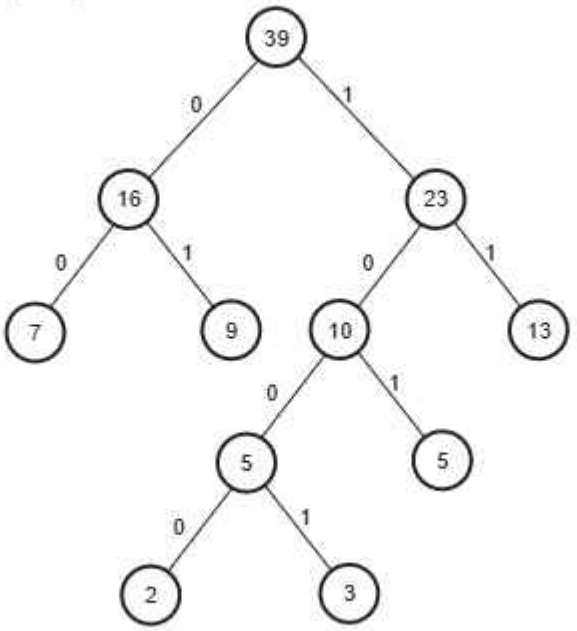
Step 4 : Sequence : 10, 13, 16,

↑ ↑



Step 5 : Sequence : 16, 23

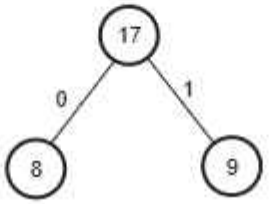
↑ ↑



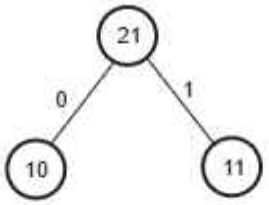
Symbols	Binary prefix code
7	00
9	01
2	1000
3	1001
5	101
13	11

ii) Consider the following steps

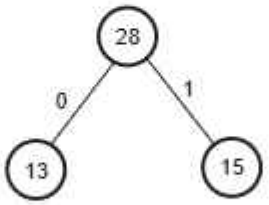
Step 1 : Sequence : 8, 9, 10, 11, 13, 15, 22
↑ ↑



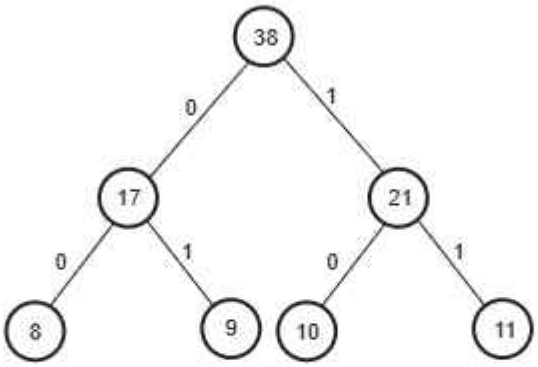
Step 2 : Sequence : 10, 11, 13, 15, 17, 22
↑ ↑



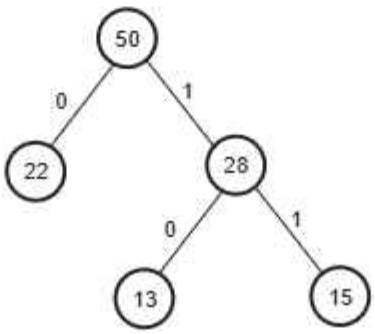
Step 3 : Sequence : 13, 15, 17, 21, 22,
↑ ↑



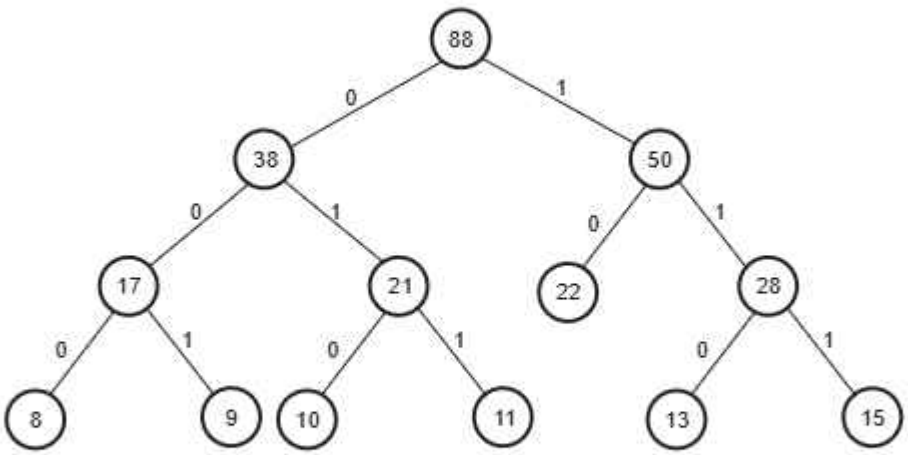
Step 4 : Sequence : 17, 21, 22, 28,
↑ ↑



Step 5 : Sequence : 22, 28, 38
↑ ↑



Step 6 : Sequence : 38, 50
↑ ↑

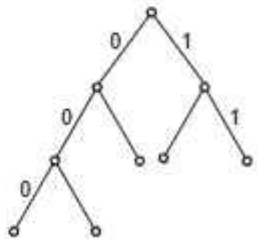


Symbols	Binary prefix code
8	000
9	001
10	010
11	011
22	10
13	110
15	111

Example 8.6.6 *State whether the given set is a prefix code. Justify {000, 001, 01, 10, 11}*

SPPU : May-08, Dec.-16

Solution : The given set which contains 5 elements will be a prefix code if we can construct a full binary tree with five leaves.



Let n be the number of vertices and P be the number of interior vertices in a full binary tree with five leaves.

Then $n = P + 5 \Rightarrow P = n - 5$ ... (1)

and $n = 2P + 1 = 2(n - 5) + 1$

$\Rightarrow 2n - 2P = -9$ ... (2)

Solving (1) and (2) we get n = 9 and P = 4.

Therefore we have a full binary tree with 9 vertices and five leaves. Such binary tree is shown in the figure.

8.7 Spanning Trees

SPPU : Dec.-11,13,14,15,16,17,18, May-05,14,15,18,19

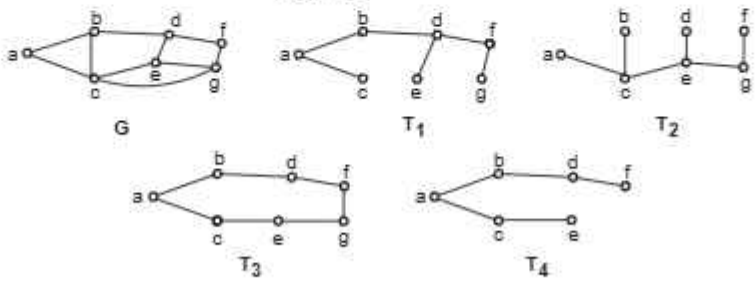
We know that a subgraph H of a graph G is a **Spanning subgraph** of G if V(G) = V(H).

Now define spanning subgraph by using tree.

Definition :

A spanning subgraph T of a connected graph G is said to be **spanning tree** of G if T is a tree. In other words, A subgraph T of a graph G is said to be spanning tree if V(T) = V(G).

For example consider the following graph G.



Trees T_1 and T_2 are spanning trees of graph G.

T_3 is not a tree so it is not spanning tree of G.

T_4 is a tree but $g \notin V(T_4)$, so T_4 is not spanning tree.

8.7.1 Minimum Spanning Tree

Let G be a weighted connected graph. A spanning tree of G is called a **Minimum Spanning tree** if it's weight is minimum.

Minimum spanning trees are using in constructing or designing a network of roads, railway lines, TV cable line, Telephone lines, Electric lines etc. for connecting number of cities or objects.

The connector problem is to design a network that connect all required stations such that the total cost of construction is least.

There are several spanning trees.

The minimum spanning tree may or may not be unique. There are any realistic situations in which we have two paths with exactly same weights.

There are two algorithms to find minimum spanning trees.

8.7.2 Prim's Algorithm

Let G(V, E) be a connected weighted graph.

To construct minimum spanning tree of G consider the following steps.

Step 1 : Select any vertex v_o in graph G.

Set $T = \{v_o, \phi\}$

Step 2 : Find edge $e_i = (v_o, v_i)$ in E such that its one end vertex is $v_o \in T$ and its weight is minimum.

$\therefore$New set $T = \{\{v_0, v_1\}, \{e_i\}\}$

Step 3 : Choose next edge $e_k = (v_k, v_j)$ in such a way that it's one end vertex $v_k \in T$ and other vertex $v_j \notin T$ and weight of e_k is as small as possible. Again join vertex v_j and edge e_k to T.

Step 4 : Repeat the step 3 until T contains all the vertices of G. The set T will give the minimum spanning tree of the graph G.

8.7.3 Kruskal Algorithm

In 1956, Kruskal developed an algorithm to find a minimum spanning tree of a given weighted connected graph.

Let G(V, E) be a weighted connect graph.

Consider the following steps.

Step 1 : Pick up an edge e_i of G such that its weight $W(e_i)$ is minimum.

(If there are more edges of the minimum weight then select all those edges which do not form a circuit).

Step 2 : If edges $e_1, e_2, \ldots e_n$ have been chosen then pick an edge e_{n+1} such that

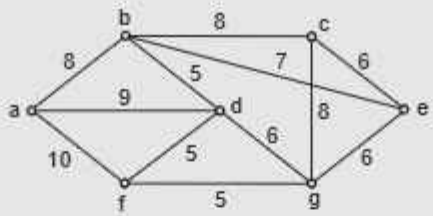
i) $e_{n+1} \neq e_i$ for $i = 1, 2, \ldots n$

ii) The edges $e_1, e_2, \ldots e_n, e_{n+1}$ do not form a circuit.

iii) $W(e_{n+1})$ is as small as possible subject to condition (ii).

Step 3 : Stop when step two cannot be implemented.

Example 8.7.1 *Find the minimum spanning tree for the graph given in the following figure using Prim's algorithm.* **SPPU : Dec.-14**



Solution : Consider the following steps for the construction of the minimum spanning tree.

Step 1 : Select $a \in G$ as a starting vertex

$\therefore \quad T = \{\{a\}, \phi\}$

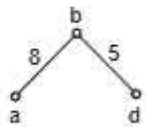
Step 2 : Vertex a is adjacent to vertices b, d and f.
Among these edges minimum weight is $\{a, b\} = 8$

$\therefore \quad T = \{\{a, b\}, \{e_1\}\}$

Step 3 : Vertex a is adjacent to f, d.

Vertex b is adjacent to c, d, e

The minimum weight is of an edge $\{b, d\} = 5$



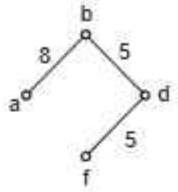
$\therefore \quad T = \{\{a, b, d\}, \{e_1, e_2\}\}$

Step 4 : Vertex a is adjacent to f and af = 10

Vertex b is adjacent to c, e and bc = 8, be = 7

Vertex d is adjacent to f and g and df = 5, dg = 6

Among all these weights minimum is 5 = df



$\therefore \quad T = \{\{a, b, d, f\}\{e_1, e_2, e_3\}\}$

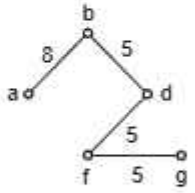
Step 5 : Vertex is adjacent to b, d, f but all are in T.

Vertex b is adjacent to c and e

Vertex d is adjacent to g.

Vertex f is adjacent to g

Among all these edges minimum weight is of fg.

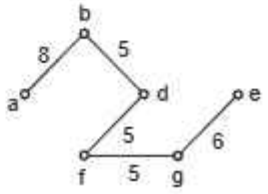


$\therefore \quad T = \{\{a, b, d, f, g\}\{e_1, e_2, e_3, e_4\}\}$

Step 6 : Vertex g is adjacent to c and e

Vertex b is adjacent to c and e

Among these edges the minimum weight is ge = 6

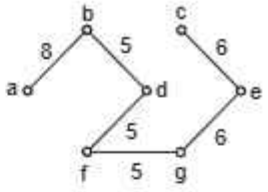


$\therefore \quad T = \{\{a, b, d, f, g, e\}\{e_1, e_2, e_3, e_4, e_5\}\}$

Step 7 : Now only one vertex is remaining.

The vertex C is adjacent to b, e, g.

The minimum weight is ec = 6

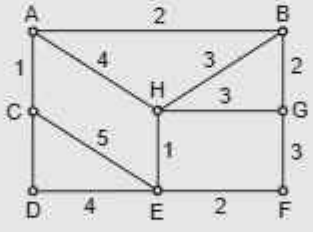


$\therefore \quad T = \{\{a, b, d, f, g, e, c\}\{e_1, e_2, e_3, e_4, e_5, e_6\}\}$

The graph obtained is the minimum spanning tree of weight

$= 8 + 5 + 5 + 5 + 6 + 6 = 35$

Example 8.7.2 *Use Prim's algorithm to find minimum spanning tree. Take A as starting vertex.* **SPPU : Dec.-11, 16**



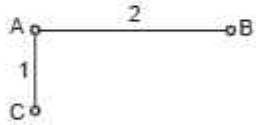
Solution : Consider the following steps for the construction of the minimum spanning tree.

Step 1 : Vertex A is starting vertex

$$T = \{\{A\}, \phi\}$$

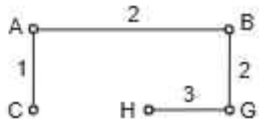
Step 2 : $T = \{\{A, C\}, e_1\}$

Step 3 : $T = \{\{A, C, B\}, \{e_1, e_2\}\}$

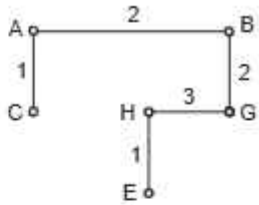


Step 4 : $T = \{\{A, C, B, G\}, e_1, e_2, e_3\}$

Step 5 : $T = \{\{A, C, B, G, H\}, e_1, e_2, e_3, e_4\}$



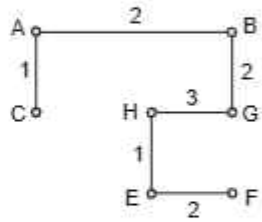
Step 6 : $T = \{\{A, C, B, G, H, E\}, \{e_1, e_2, e_3, e_4, e_5\}\}$



Step 7 : $T = \{\{A, C, B, G, H, E, F\}, \{e_1, e_2, e_3, e_4, e_5, e_6\}\}$

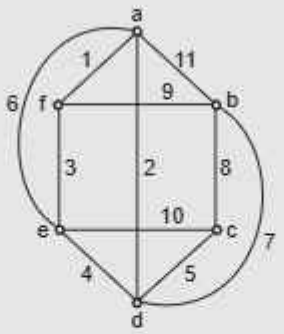
The graph obtained is the minimum spanning tree of weight

$$= 1 + 2 + 2 + 3 + 1 + 2 = 11$$



Example 8.7.3 *Determine minimum spanning tree for the given graph by Prism's algorithm.*

SPPU : May-14, 18

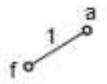


Solution : Consider the following steps for the construction of the minimum spanning tree.

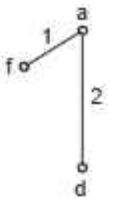
Step 1 : Starting with vertex a

$$T = \{\{a\}, \phi\}$$

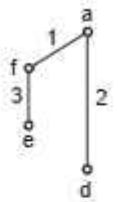
Step 2 : $T = \{\{a, f\}, e_1\}$



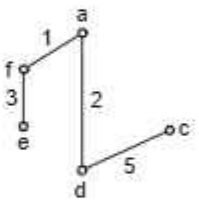
Step 3 : $T = \{\{a, f, d\}, \{e_1, e_2\}\}$



Step 4 : $T = \{\{a, f, d, e\}, \{e_1, e_2, e_3\}\}$



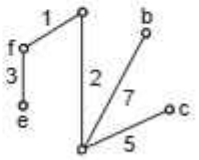
Step 5 : $T = \{\{a, f, d, e, c\}, \{e_1, e_2, e_3, e_4\}\}$



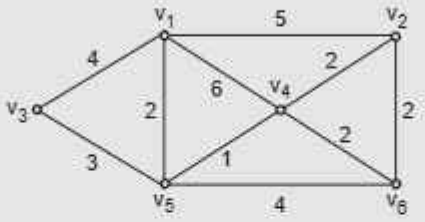
Step 6 : $T = \{\{a, f, d, e, c, b\}, \{e_1, e_2, e_3, e_4, e_5\}\}$

The graph obtained is the minimum spanning tree of weight

$$= 1 + 2 + 1 + 5 + 7 = 18$$

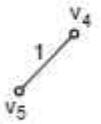


Example 8.7.4 *Find the minimum spanning tree for graph given below by Kruskal's algorithm.* **SPPU : Dec.-13**

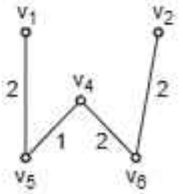


Solution : Consider the following steps for the construction of the minimum spanning tree.

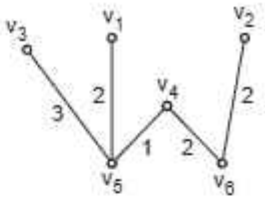
Step 1 : The minimum weight in given graph is 1, so select an edge $\{v_4, v_5\}$



Step 2 : The minimum weight is 2 in remaining graph. There are four edges of weight 2. These edges form a circuit so select edges which do not form a circuit with selected edge. We select three edges $\{v_5, v_1\}$, $\{v_4, v_6\}$, $\{v_6, v_2\}$



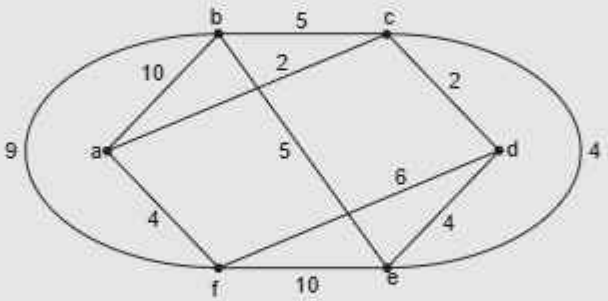
Step 3 : The minimum weight is 3 in the remaining graph so select $\{v_5, v_3\}$



The graph obtained is the minimum spanning tree of weight

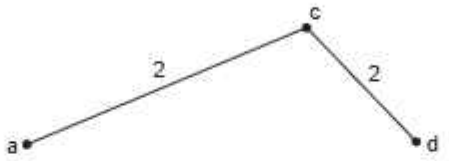
$$= 3 + 2 + 1 + 2 + 2 = \mathbf{10}$$

Example 8.7.5 *Use the Kruskal's algorithm to find the minimum spanning tree for the graph shown.* **SPPU : May-05**

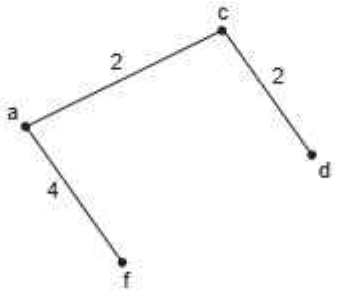


Solution : Consider the following steps for the construction of the minimum spanning tree.

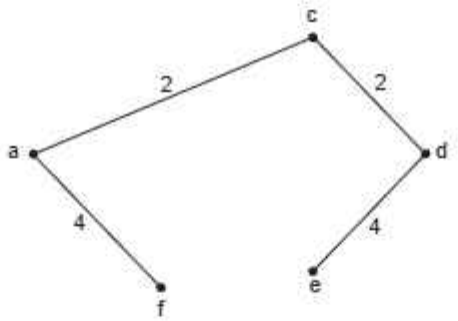
Step 1 : The minimum weight is 2 associated with two edges. These edges do not form a circuit. So select these to edges.



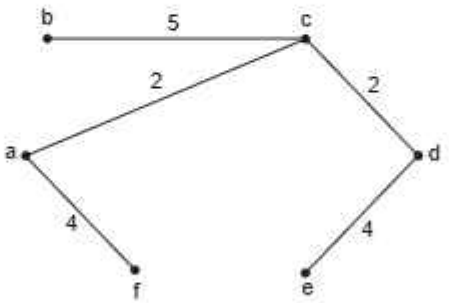
Step 2 : Minimum weight = 4



Step 3 : Maximum weight = 4

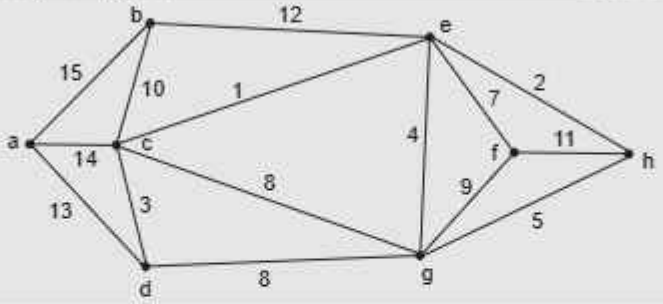


Step 4 : Minimum weight = 4 associated with {c, e} which forms a circuit so select next minimum weight = 5



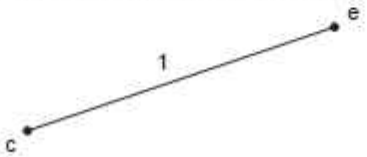
The graph is the minimum spanning tree and its total weight is 17.

Example 8.7.6 *Obtain the minimum spanning tree for the following graph. Obtain the total cost of minimum spanning tree.* **SPPU : May-15, Dec.-15**

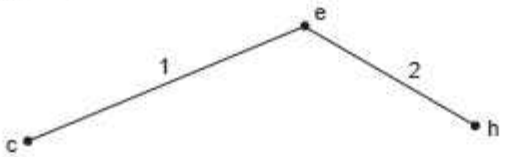


Solution : Using Kruskal algorithm, the minimum spanning tree is obtained as follows :

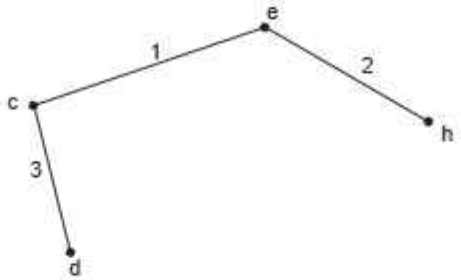
Step 1 : Minimum weight in a given graph is 1 associated with edge {a, e}



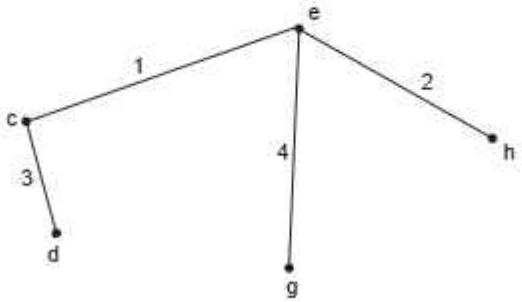
Step 2 : Minimum weight = 2



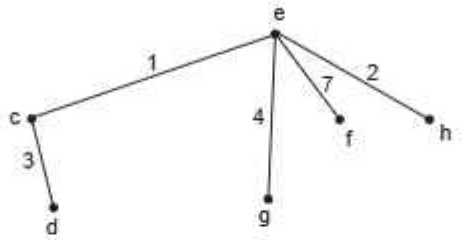
Step 3 : Minimum weight = 3



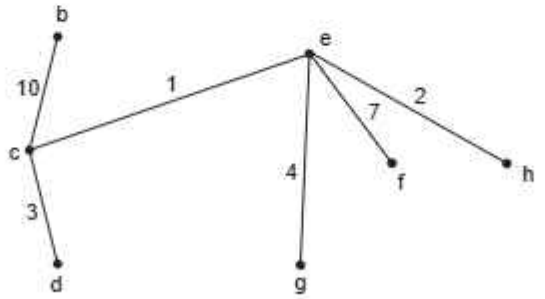
Step 4 : Minimum weight = 4



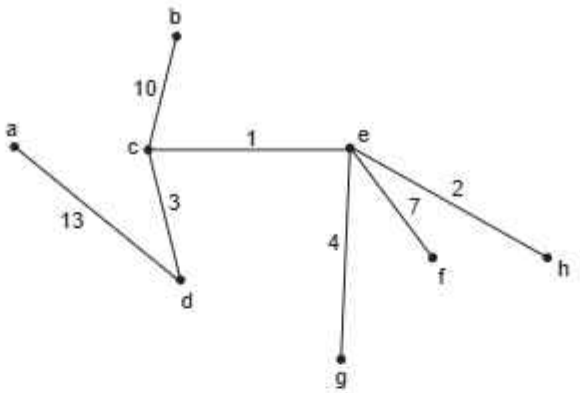
Step 5 : Minimum weight = 7



Step 6 : Minimum weight = 10



Step 7 : Minimum weight = 13



The obtained graph is the minimum spanning tree of the given graph. Its total cost is 40.

8.8 Fundamental Circuits and Cutsets

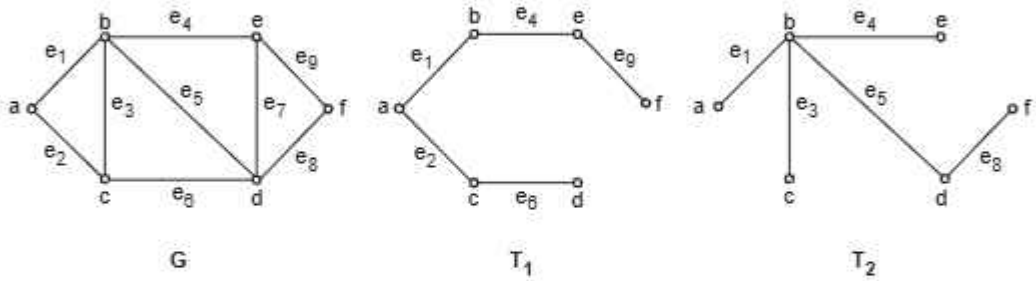
SPPU : Dec.-07, 12, 14, 15, 16, May-07, 15

Let G be a connected graph and T be a spanning tree of G. An edge of a tree is called a **branch.** An edge of G which is not in T is called chord of T. T + e contains a unique circuit called fundamental circuit of G with respect to T.

A fundamental circuit of a connected graph G is always with respect to a spanning tree of G.

Therefore the different spanning trees will have different fundamental circuits.

Consider the following graph G with two spanning trees T_1 and T_2.



In graph G, vertex set V(G) = {a, b, c, d, e, f}

Edge set E(G) = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$

A) For Spanning Tree T_1 :

Branches of T_1 are e_1, e_2, e_4, e_6, e_9

Chords of T_1 are e_3, e_5, e_4, e_8

Consider the following table of chords and corresponding fundamental circuits.

Chords	Corresponding Fundamental Circuits
e_3	$\{e_1, e_2, e_3\}$
e_5	$\{e_1, e_2, e_6, e_5\}$
e_7	$\{e_1, e_2, e_4, e_6, e_7\}$
e_8	$\{e_1, e_2, e_4, e_6, e_9, e_8\}$

B) For Spanning Tree T_2

Branches of T_2 are e_1, e_3, e_5, e_4, e_8

Chords of T_2 are e_2, e_4, e_6, e_7, e_9

Consider the following table of chords and corresponding fundamental circuits.

Chords	Corresponding Fundamental Circuits
e_2	$\{e_1, e_3, e_2\}$
e_6	$\{e_3, e_5, e_6\}$
e_7	$\{e_4, e_5, e_7\}$
e_9	$\{e_4, e_5, e_8, e_9\}$

Fundamental Cutsets

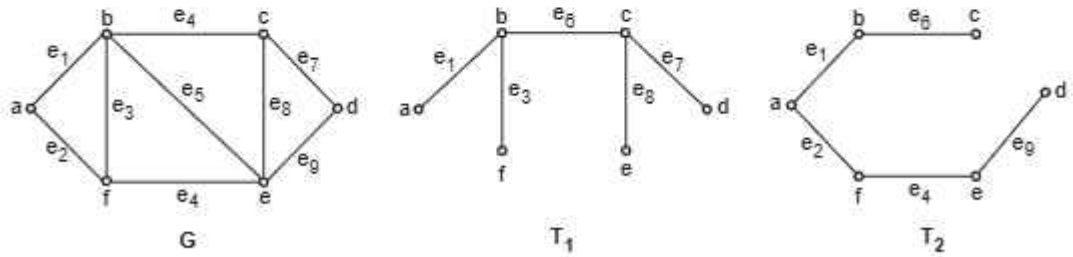
Let T be k spanning tree of a connected graph G. Since every edge e of a tree is an isthmus or bridge, T-e splits into two components say T_1 and T_2. But
$V(T) = V(T_1) \cup V(T_2) = V(G)$

The set E of edges of G which join a vertex in $V(T_1)$ to a vertex in $V(T_2)$ is a cutset of G.

A cutset of G obtained in this manner is called a fundamental cutset of G with respect to T. To each edge of edge there is a fundamental cutset and every fundamental cutset is obtained in this way. Thus the number of fundamental cutsets of G w.r.t. T is the number of branches of T.

Theorem : Let G be a connected graph with n vertices then its spanning tree has $n-1$ edges and there are $n-1$ fundamental cutsets only.

e.g. Consider the following and its spanning tree.



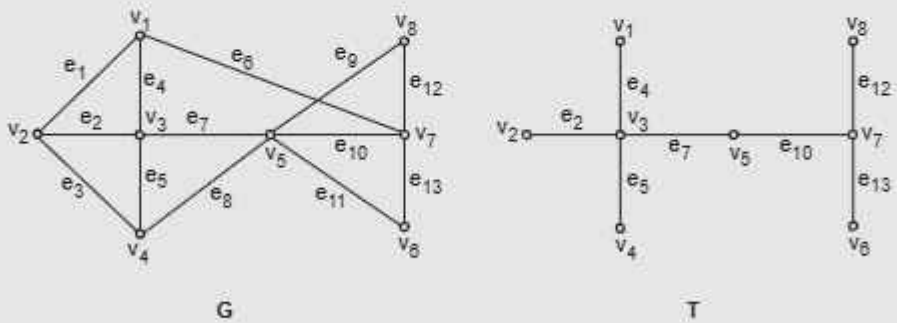
A) For Spanning Tree T_1

Branches of T_1 are e_1, e_3, e_6, e_7, e_8

Consider the following table

Branches T_1	Corresponding Fundamental Cutset
e_1	$\{e_1, e_2\}$
e_3	$\{e_3, e_2, e_4\}$
e_6	$\{e_6, e_5, e_4\}$
e_7	$\{e_7, e_9\}$
e_8	$\{e_8, e_4, e_9, e_5\}$

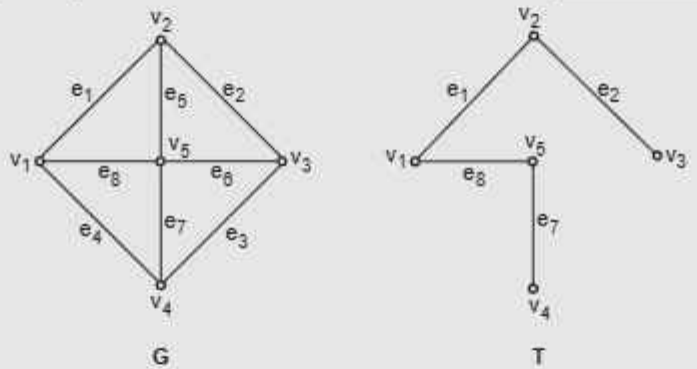
Example 8.8.1 *Find the fundamental system of cutset for the graph G shown below w.r.t. the spanning tree T.* **SPPU : May-15, Dec.-12, 15**



Solution : The spanning tree T has 7 branches $\{e_2, e_4, e_5, e_7, e_{10}, e_{12}, e_{13}\}$. Therefore are seven fundamental cutsets of G w.r.t. T which are given below :

Branch	Fundamental Cutset
e_2	$\{e_2, e_1, e_3\}$
e_4	$\{e_4, e_1, e_6\}$
e_5	$\{e_5, e_3, e_8\}$
e_7	$\{e_7, e_6, e_8\}$
e_{10}	$\{e_{10}, e_6, e_9, e_{11}\}$
e_{12}	$\{e_{12}, e_9\}$
e_{13}	$\{e_{13}, e_{11}\}$

Example 8.8.2 *Find the fundamental cutsets and fundamental circuits of the following graph w.r.t. given spanning tree.* **SPPU : May-07, Dec.-07, 14**



Solution : Here the spanning tree has 4 branches e_1, e_2, e_7, e_8. Therefore there are 4 fundamental cutsets corresponding to each branch of T which are given below.

Branch	Corresponding Fundamental Cutset
e_1	$\{e_1, e_5, e_6, e_3\}$
e_2	$\{e_2, e_6, e_3\}$
e_7	$\{e_7, e_3, e_4\}$
e_8	$\{e_8, e_4, e_5, e_6, e_3\}$

The chords of T are e_3, e_4, e_5, e_6

Therefore there are 4 fundamental circuits corresponding to each chord of T which are given below :

Chord	Corresponding Fundamental Cutset
e_3	$\{e_1, e_2, e_7, e_8, e_3\}$
e_4	$\{e_4, e_7, e_8\}$
e_5	$\{e_5, e_1, e_8\}$
e_6	$\{e_6, e_1, e_2, e_8\}$

8.9 Network Flows

SPPU : Dec.-12, 13, 14, 16, 17, 18, May-07, 14, 17, April-11

In a network of telephone lines, highways, railroads, pipelines of oils (gas or water) and so on, it is essential to know the maximum rate of flow that is possible from one station to another in the network. This type of network is represented by a weighted directed graph in which the vertices are stations and edges are lines through which the given commodity (water, gas, oils, etc.) flows

Transport Network

A weighted directed connected graph is said to be a **transport network** if the following conditions are satisfied.

i) It is without loops.

ii) There is one and only one vertex in the graph that has no incoming edge. It is known as **source.**

iii) There is one and only one vertex in the graph that has no outgoing edge. It is known as **sink.**

iv) The weight of each edge is a non negative real number. It is known as capacity of that edge. The capacity of the (i, j) edge is denoted by W(i, j).

A flow ϕ, in a transport network is an assignment of a non negative number $\phi(i, j)$ to each edge (i, j) such that

i) $$\phi(i, j) \leq W(i, j) \quad \text{for each } (i, j)$$

i.e. the amount of material to be transported through a route (edge) cannot greater than the capacity of that route.

ii) $$\sum_{\text{all } i} \phi(i, j) = \sum_{\text{all } k} \phi(i, j) \quad \text{for each vertex j except the source a and sink z.}$$

i.e. (Incoming material flow = Outgoing material flow)

The value of flow is given by

$$\phi_v = \sum_{\text{all } i} \phi(a, i) - \sum_{\text{all } k} \phi(k, z)$$

For a given flow an edge (i, j) is said to be **saturated** if $\phi(i, j) = W(i, j)$

A edge (i, j) is said to be unsaturated if ϕ (i, j) < W(i, j)

8.9.1 Maximum Flow

The maximum flow in a transport network is a flow that achieves the largest possible value.

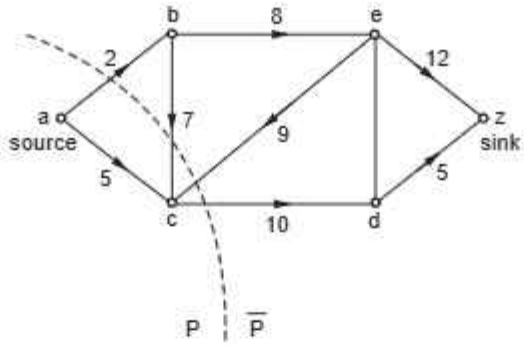
Cut : A cut in a transport network is a cutset of the undirected graph, obtained from the transport network by ignoring the direction of the edges that separate the source from sink.

The notation $(P, \overline{P})$ is used to denote a cut that divides the vertices into two subsets P and $\overline{P}$ where the subset P contains the source and the subset $\overline{P}$ contains the sink.

The capacity of a cut is denoted by $w(P, \overline{P})$ and defined as the sum of the capacities of those edges incident from the vertices in P to the vertices in $\overline{P}$.

i.e. $$w(P, \overline{P}) = \sum_{i \in p \text{ and } j \in \overline{P}} w(i, j)$$

Consider the following example.



The dashed line in the above figure identifies a cutset that separates subset of vertices {a, c} = P from the subset of vertices $\overline{P}$ = {b, d, e, z}

The capacity of this cut is

$$w(P, \overline{P}) = w(a, b) + w(c, d)$$
$$= 2 + 10 = 12$$

Theorem 1 : The value of any flow in a transport network is less than or equal to the capacity of any cut in the network.

Theorem 2 : For any cut $(P, \overline{P})$, the value of a flow equal to the sum of flows in the edges from the vertices in P to the vertices in $\overline{P}$ minus the sum of flows in the edges from the vertices in $\overline{P}$ to the vertices in P.

Theorem 3 : In a given transport network G, the maximum value of a flow from source a to sink z is equal to minimum value of the capacities of all the cuts in G that separate the source a from the sink z.

e.g. : Consider transport network in the given Fig. 8.9.1 below

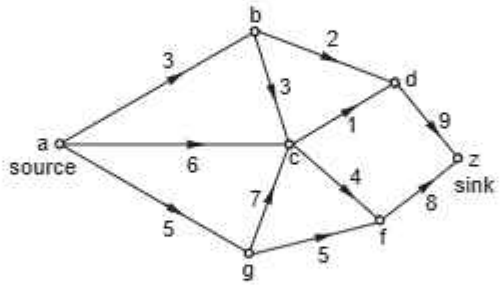


Fig. 8.9.1

Vertex a is a source and z is a sink.

The vertex set of some such cut are listed in the following table.

Vertex set P	Capacity $w(p, \bar{p})$
{a}	14(3 + 6 + 5)
{a, b}	16(5 + 6 + 3 + 2)
{a, c}	13(3 + 5 + 1 + 4)
{a, d}	23(3 + 6 + 5 + 9)
{a, f}	22(3 + 6 + 5 + 8)
{a, b, c}	15(5 + 2 + 1 + 4 + 3)
{a, b, d}	23(6 + 5 + 3 + 9)
{a, b, c, g}	12(2 + 1 + 4 + 5)

8.9.2 Constructing a Maximum Flow in a Transport Network

For any cut $(P, \bar{P})$ the values of a flow in a transport network equal the sum of flows in the edges from the vertices in P to the vertices in $\bar{P}$ minus the sum of flows in the edges from the vertices in $\bar{P}$ to the vertices in P.

Whenever we can construct a flow ϕ the value of which is equal to the capacity of some cut, we can be certain that ϕ is a maximum flow.

8.9.3 Labelling Procedure for Finding Maximum Flow in the Network

1) The source a is labelled $(-, \infty)$. It means that (out from nowhere) the source can supply can infinite amount of material to the other vertices.

2) A vertex b that is adjacent from a is labelled $(a^+, \Delta b)$, where Δb is equal to $w(a, b) - \phi(a, b)$,

if $w(a, b) > \phi(a, b)$;

i.e. $$\Delta b = w(a, b) - \phi(a, b) \quad [\text{if } w(a, b) > \phi(a, b)]$$

The vertex is not labelled if $w(a, b) = \phi(a, b)$

3) Scan and label all the remaining vertices adjacent to a. Also scan and label all the vertices adjacent to labelled vertices.

Suppose vertex q is adjacent to labelled vertex b, then q is labelled as $(b^+, \Delta q)$

where $$\Delta q = \min\{\Delta b, [w(b, q) - \phi(b, q)]\}$$

if $w(b, q) > \phi(b, q)$

Vertex q is not labelled if $w(b, q) = \phi(b, q)$

We can also label vertex q as $(\bar{b}, \Delta q)$ where

$$\Delta q = \min[\Delta b, \phi(q, b)] \quad \text{if} \quad \phi(q, b) > 0$$

4) Repeat step 3 till we reach to sink z.

5) If we repeat this labelling procedure, two cases shall arise while labelling the sink z.

Case I :

i) Sink z is labelled, say with a label $(y^+, \Delta z)$ [z is never labelled $(y^-, \Delta z)$]

ii) Vertex y can be labeled as $(q^+, \Delta y)$ or $(q^-, \Delta y)$ for some adjacent vertex q.

iii) If y is labelled $(q^+, \Delta y)$ (q^+ means increase in flow) then we increase the flow in edge (q, y) to $\phi(q, y)$ to Δz. Similarly for the label $(q^-, \Delta y)$ we decrease the flow in edge (q, y) to $\phi(q, y) - \Delta y$.

iv) This process is continued back to source a till the value of flow is increased by amount Δz.

v) Again start the labelling procedure to further increase value of flow in the network.

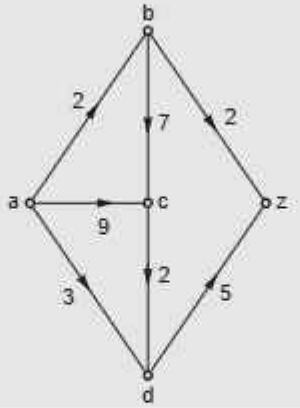
Case II :

i) If the sink z is not labelled, then denote all labelled vertices as P and all unlabelled vertices as $\bar{P}$.

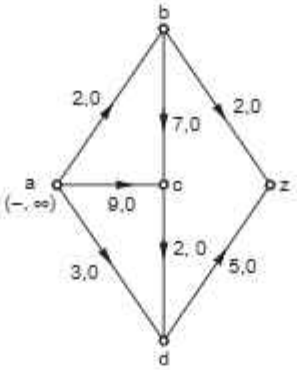
ii) The fact that sink z is not labeled means flow each edge directed from vertices of P to vertices of $\bar{P}$. is equal to capacity of cut $(P, \bar{P})$ is thus maximum flow.

Example 8.9.1 *Determine the maximal flow in the following transport network.*

SPPU : Dec.-12, May-14



Solution : Step 1 : Assign the flow zero to each edge and the label $(-, \infty)$ to the source a.



Step 2 : The vertices b, c, d are adjacent to the source a.

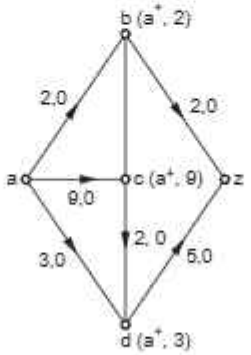
Therefore, we label the vertices b, c, d.

For the vertex b

$$w(a, b) - \phi(a, b) = 2$$

i.e. $\Delta b = 2$

Similarly $\Delta c = 9$ and $\Delta d = 3$



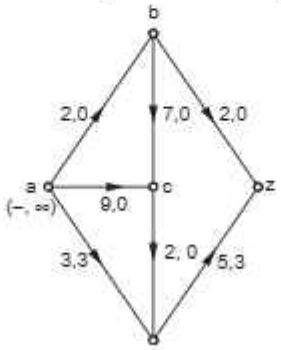
Now sink z is adjacent to both vertices b and d, so we arbitrarily choose vertex d and label.

sin z as $(d^+, 5)$ as

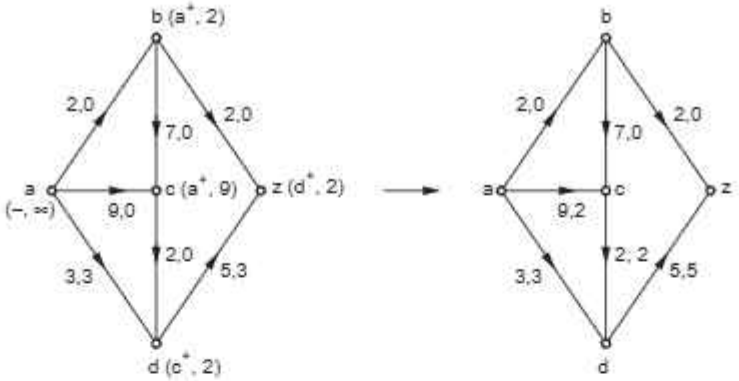
$$w(d, z) - \phi(d, z) = 5 - 0 = 5$$

and $$\Delta z = \min\{\Delta d, \{w(d, z) - \phi(d, z)\}$$
$$= \min\{3, 5\} = 3$$

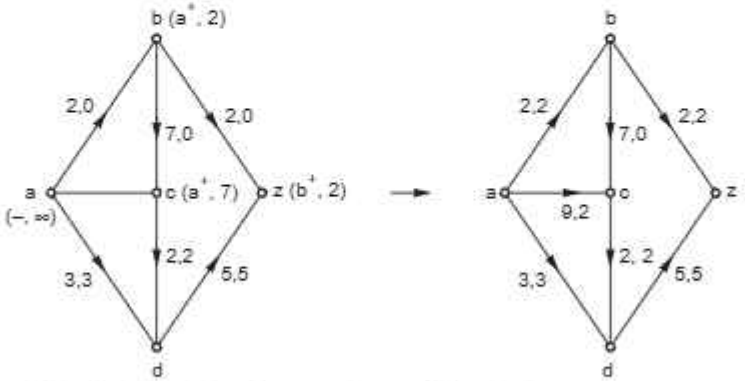
According to label of sink, we now adjust flow of edge (d, z) and edge (a, d)



Step 3 :

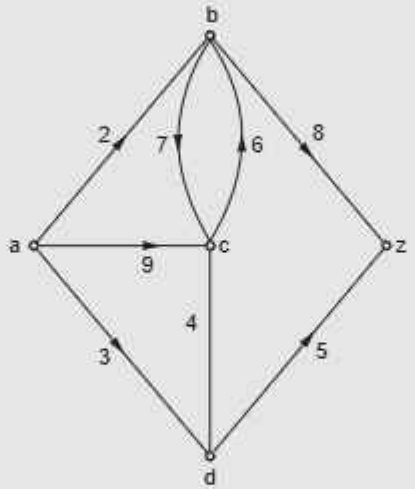


Step 4 :

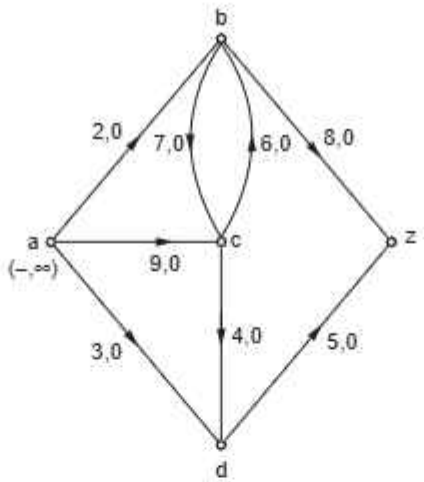


From the step 4, it is clear that the maximum flow is 7.

Example 8.9.2 *Determine the maximal flow in the flowing transport network.* **SPPU : Dec.-13**



Solution : Step 1 : Assign the flow zero to each edge and the label $(-, \infty)$ to the source a.

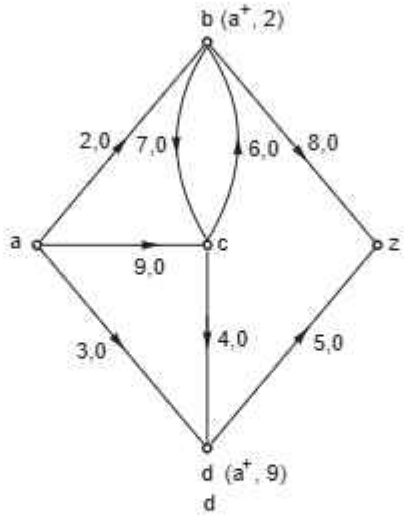


Step 2 : The vertices b, c, d are adjacent to the source a. Therefore, we label the vertices b, c, d for the vertex b.

$w(a, b) - \phi(a, b) = 2$

i.e. $\Delta b = 2,$

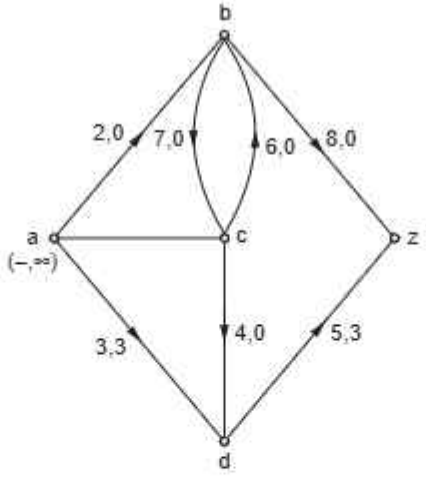
Similarly $\Delta c = 9,$ $\Delta d = 3$



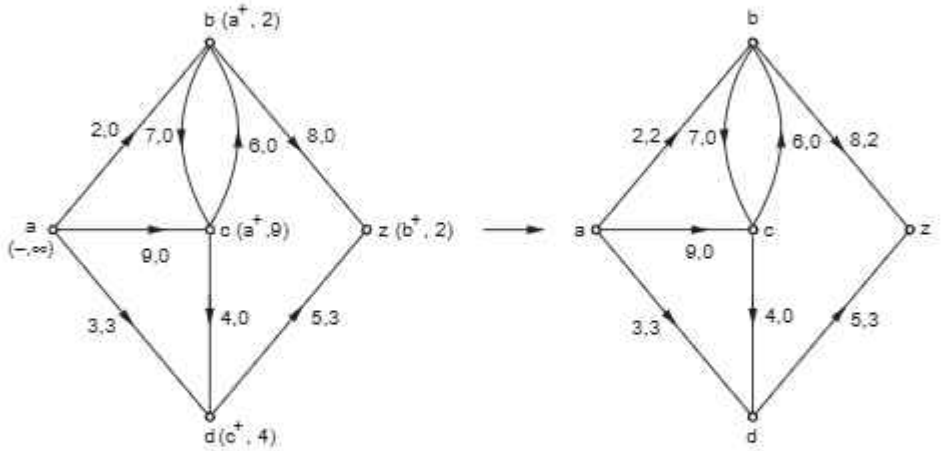
Now sink z is adjacent to both vertices b and d, so we can choose any vertex b or d for labelling of sink z. Let us choose the vertex d. The label of sink z is $(d^+, 5)$ as $w(d, z) - \phi(d, z) = 5 - 0 = 5$

and $\Delta z = \min\{\Delta d, w(d, z) - \phi(d, z)\} = \min\{3, 5\} = 3$

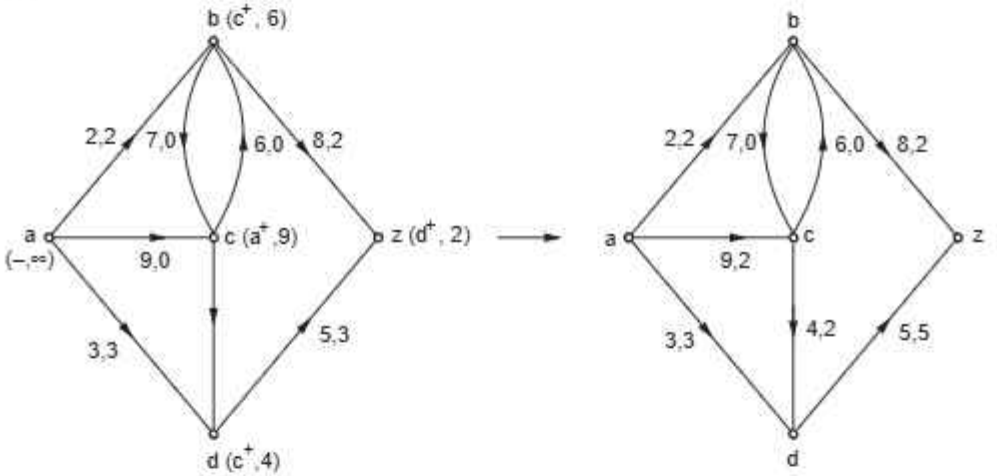
According to label of sink, we now adjust flow of edge (d, z) and edge (a, d)



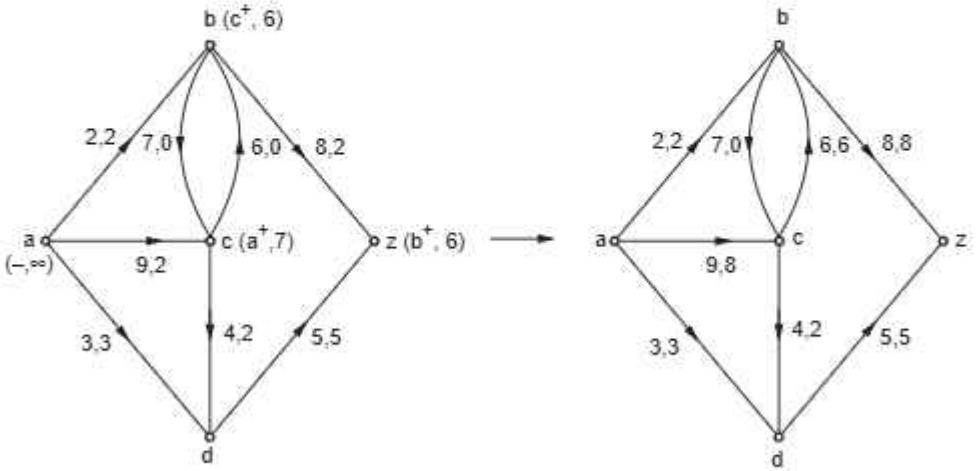
Step 3 :



Step 4 :



Step 5 :



Example 8.9.3 *Use labelling procedure to find a maximum flow in the transport network shown in the following Fig. 8.9.2. Determine the corresponding minimum cut.*

SPPU : May-07, 17, Dec.-14, 16, 18

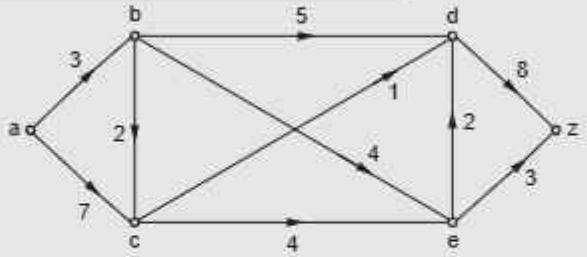
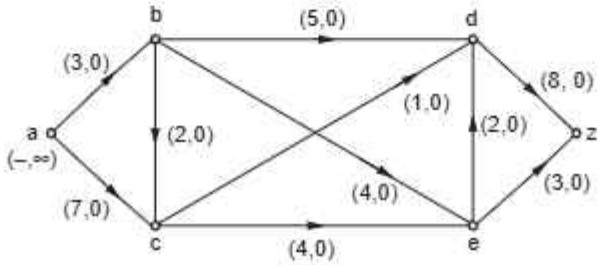


Fig. 8.9.2

Solution : To find the maximum flow in the given transport network, assign the flow zero to each edge and label $(-, \infty)$ to the source a.

Step 1 :



Step 2 : Since the vertices b and c are adjacent to the source a, therefore label the vertices b and c as $(a^+, \Delta b)$ and $(a^+, \Delta c)$ respectively.

Where $\Delta b = ((a, b) - f(a, b) = 3$

$\Delta c = ((a, b) - f(a, c) = 7$

$\therefore$ Label of b is $(a^+, 3)$ and label of c is $(a^+, 7)$

Now the vertices d and e are adjacent to labeled vertices b $(a^+, 3)$ and $(a^+, 7)$ therefore we label the vertices d and e as $(b^+, \cdot d)$ and $(c^+, \cdot e)$ where

$$\Delta d = \min(\Delta b, c(b, d) - f(b, d))$$
$$= \min(3, 5, -0)$$
$$= 3$$

Similarly $\Delta e = \min(\Delta c, c(c, e) - f(c, e))$

$$= \min(7, 4, -0)$$
$$= 4$$

$\therefore$ Label of d is $(b^+, 3)$ and label of e is $(c^+, 4)$

Now the sink z is adjacent to both the vertices d and e, we can choose any vertex d or e to label the sink z.

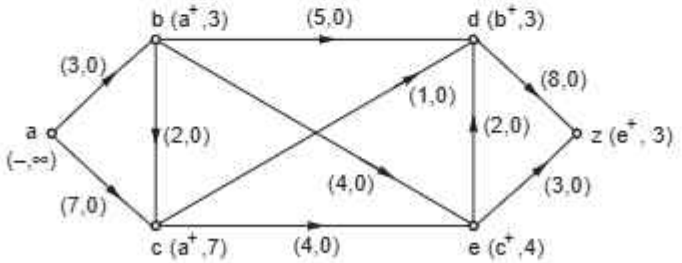
Let us choose the vertex e, which is labeled as $(c^+, 4)$

The label of z will be $(e^+, \Delta z)$ where

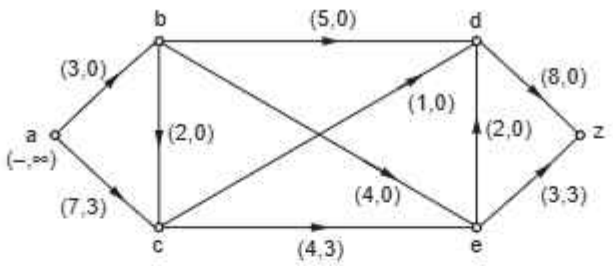
$$\Delta z = \min(4, 3, -0)$$
$$= 3$$

Hence label of z is $(e^+, 3)$

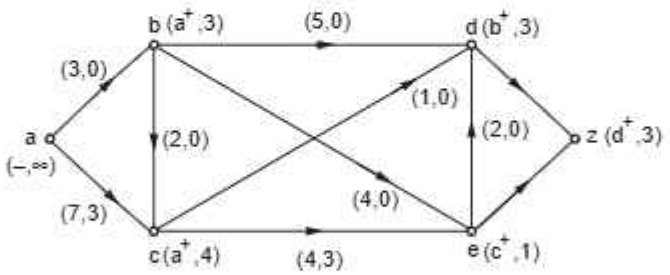
The labels of vertices can be shown as follows :



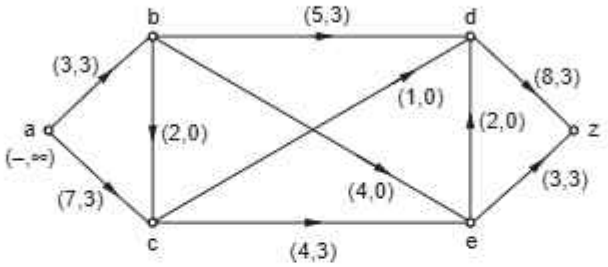
According to the label of the sink z, adjust the flow in the edges (e, z), (c, e) and (a, c)



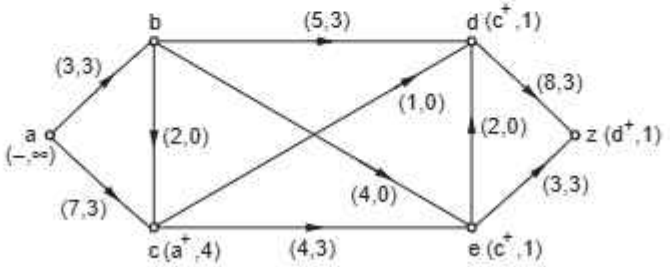
Repeat the step 2 in each pass, the new value of the flow is obtained.



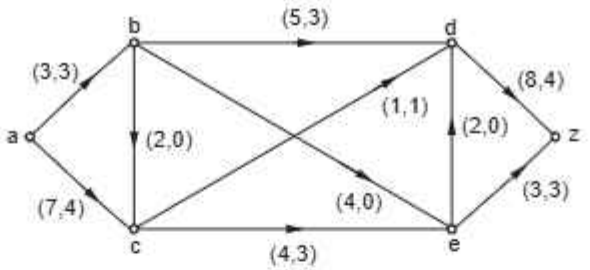
After adjusting the flow we have

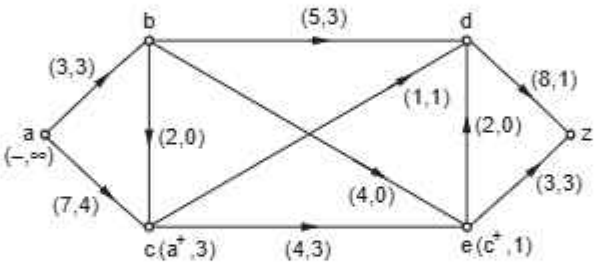


Step 4 :



Adjust the flow according to sink z label.





Since the vertex z (sink) cannot be labelled, we have to stop. The vertices b, d and z cannot be labelled because either the edges approaching to vertices are saturated or the edges in the opposite direction have flow zero. At this stage, we have minimum cut $(P, \overline{P})$.

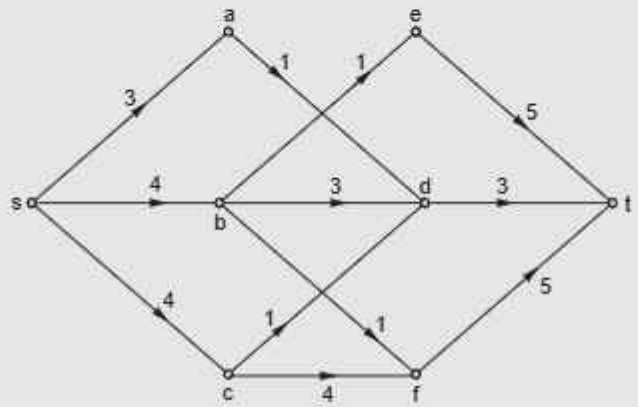
P is the set of labelled vertices P = {a, c, e}. $\overline{P}$ is the set of unlabeled vertices, $\overline{P}$ = {b, d, z}

The edges in cut P $\overline{P}$ {(a, b), (c, d), (e, z)} capacity of the cut is c(a, b) + c(c, d) + c(e, z) = 3 + 1 + 3 = 7.

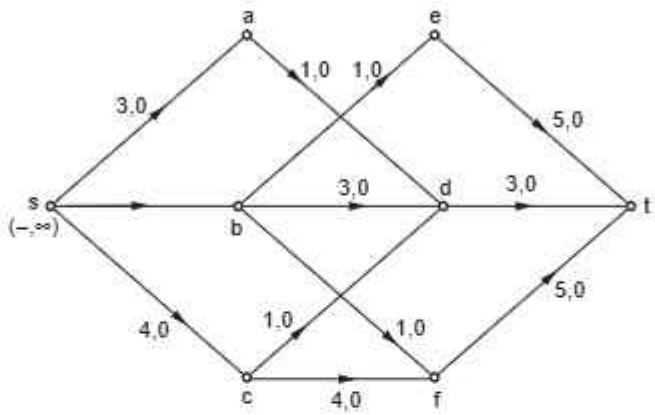
Hence the maximum flow in the given transport network 7.

Example 8.9.4 *Find the maximum flow in the following network using labeling algorithm.*

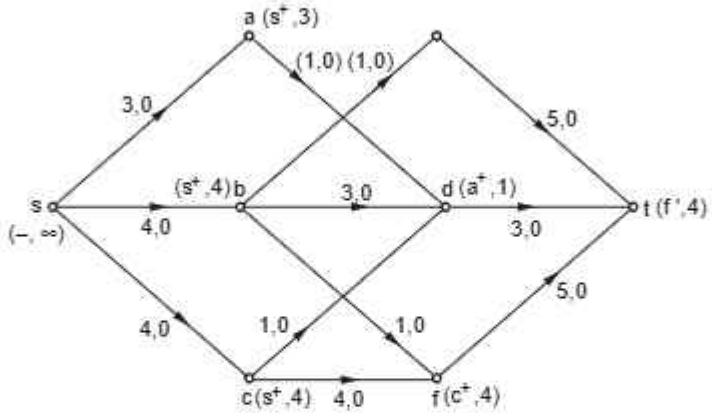
SPPU : April-11

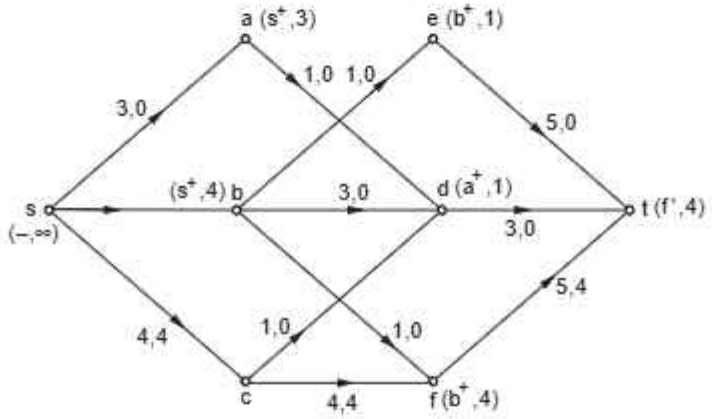


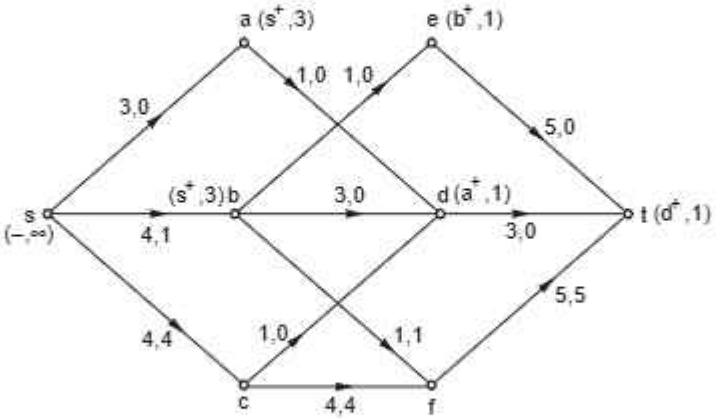
Solution : To find the maximum flow in the given transport network, assign the flow zero to each edge and label (–, ∞) to the sources.

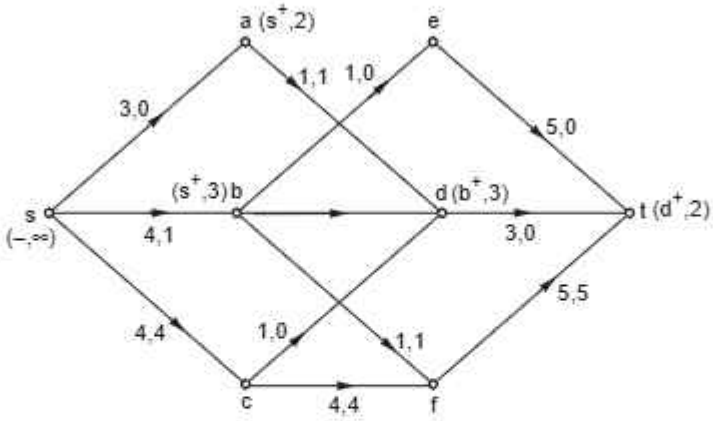


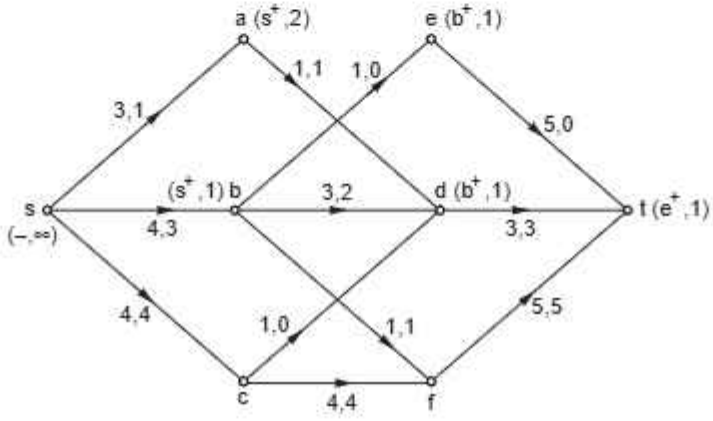
Using labelling procedure the maximum flow can be obtained using following steps.

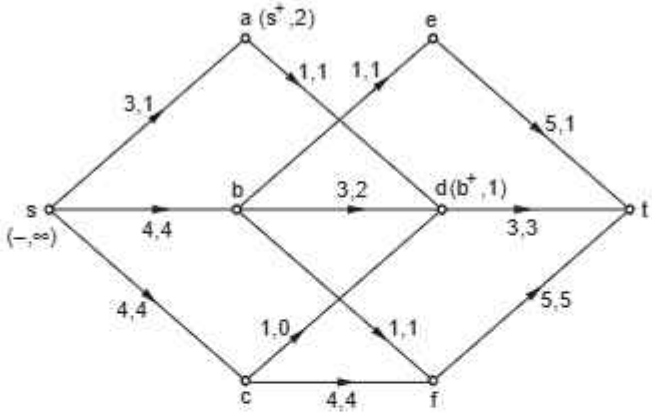

a (s+,3)
(1,0) (1,0)
3,0
5,0
s
(–, ∞)
(s+,4) b
3,0
d (a+,1)
t (f',4)
4,0
3,0
5,0
4,0
1,0
1,0
c (s+,4)
4,0
f (c+,4)


a (s+,3)
e (b+,1)
1,0 1,0
3,0
5,0
s
(–,∞)
(s+,4) b
3,0
d (a+,1)
t (f',4)
3,0
5,4
4,4
1,0
1,0
c
4,4
f (b+,4)


a (s+,3)
e (b+,1)
1,0 1,0
3,0
5,0
s
(–,∞)
(s+,3) b
3,0
d (a+,1)
t (d+,1)
4,1
3,0
5,5
4,4
1,0
1,1
c
4,4
f







Since t cannot be labelled we will stop the procedure minimum cut from the set.

$$P = \{s, a\} \text{ to } \overline{P} = \{b, c, d, e, f, t\} \text{ is } \{(s, c), (s, b), (q, d)\}$$

The capacity of the cut is $c\{s, c\} + c\{s, b\} + c\{q, d\}$

$$= 4 + 4 + 1 = 9$$

$\Rightarrow$ Maximum flow in the network is 9.

8.10 Game Tree

SPPU : Dec.-17, May18

A directed graph whose nodes are positions in a game and edges are moves, is called a game tree.

The complete game tree for a game is the game tree starting at the initial position and containing all possible moves from each position. The complete tree is the same tree as that obtained from the extensive form game representation.

We look at using trees for game planning, in particular the problems of searching game trees. We classify all games into following three types.

1) Single player path finding problems :

For examples :

a) Travelling salesman problem

b) Sliding puzzle

c) Hamiltonian paths

d) Rubik's cude

2) Two player games :

For examples :

a) Chess

b) Badminton

c) Tennis

d) Checkers

3) Constraint satisfication problems :

For examples :

a) Sudoku

b) Eight queen problem

c) Mathematical puzzles

d) Four queen problem

Each game consists of a problem space, an initial state and a set of goal states.

A problem space is a mathematical abstraction in a form of a tree :

- The root represents current state
- Nodes represent states of the game
- Edges represents moves

- Leaves represent final states (Win, loss or draw)

For example, in the 8-puzzle game

- Nodes : The different permentations of the tiles.
- Edges : Moving the black file up, down, right or left.

Minimax :

We consider game with two players in which one players gains are the result of another players losses so called zero sum games.

The minimax algorithm is a specialized search algorithm which returns the optimal sequence of moves for a player in an zero sum game.

In the game tree that result from the algorithm, each level represents a move by either of two players, say A and B player. The minimax algorithm explores the entire game tree using a depth-first search.

At each node in the tree where A-player has to move, A-player would like to play the move that miximizes the payoff. Thus, A-player will assign the maximum score amongst the children to the node where max makes a move. Similarly, B-player will minimize the pay off to A-player. The maximum and minimum scores are taken at alternating levels of the tree, Since A and B alternate turns. The minimax algorithm computes the minimx decision for the leaves of the game tree and than backs up through the tree to give the final value to the current state.

8.11 Case Studies

1) Algebraic expression tree : Refer section 8.5.2.
2) Tic - Tac - Toe game tree : A game free is a directed graph whose nodes are positions in a game and edges are move Tic - Tac - Toe is a paper - pencil game for two players, X and O, who take turns making the spaces in a 3×3 grid. The player who succeeds in placing three of their marks in a horizontal, vertical or diagonal row wins the game.

O	O	X
X	X	X
X	O	O

Minimax algorithm : Min-Max is a decision making algorithm which uses decision theory, game theory, statistics and philosophy to calculate the optimal move. It is a two-player game. The mechanism evaluates minimum loss and maximum profit.

In the Tic - Tac - Toe game, a player tries to ensure two cases :

i) Maximize a player's own chances of win

ii) Minimize opponent's chances of win.

Maximize profit : The profit can be maximized by either fork or win.

Fork : Initially player will create opportunity where he or she can win in two ways.

Win : If there are two same 'X' or 'O' in a row, then play the third to get three in a row.

Minimize loss : The loss can be minimized by a block.

Block : If two 'X' or 'O' of the opponent are in a row then block it or else block opponents fork.

8.12 University Questions with Answers

Dec. 2016

Q.1 *Find maximum flow in the transport network using labeling procedure. Determine the corresponding min cut.* **(Refer similar example 8.9.3)** [7]

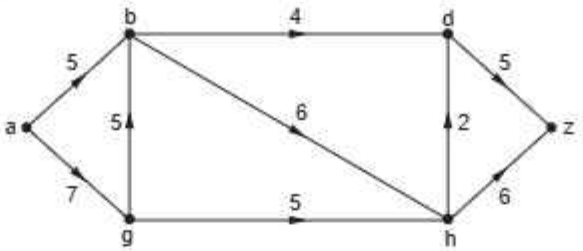


Fig. 8.12.1

Q.2 *Define :*
i) Forest **(Refer section 8.2)** *ii) Height of a tree* **(Refer section 8.4.2)**
iii) Ordered tree **(Refer section 8.4)**
iv) Properties of tree **(Refer theorem 1 to 4 statements only of section 8.2)** [4]

Q.3 *State whether the given code is prefix code. Justify :*
(000, 001, 01, 10, 11) **(Refer example 8.6.6)** [2]

Q.4 *Give the stepwise construction of minimum spanning tree using Prim's algorithm for the following graph. Obtain the total cost of minimum spanning tree.*
(Refer example 8.7.2) [6]

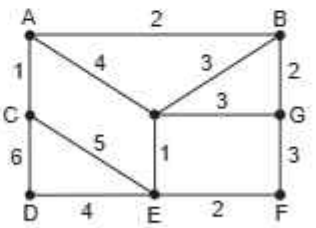


Fig. 8.12.2

Q.5 *Explain fundamental system of cutsets with suitable examples.*
(Refer section 8.8) [4]

Q.6 *Explain binary tree, binary search tree and ordered tree with suitable examples.*
(Refer section 8.6) [3]

May 2017

Q.7 *Use labeling procedure of find a maximum flow in the transport network given in the following figure. determine the corresponding minimum cut.*
(Refer example 8.9.3) [7]

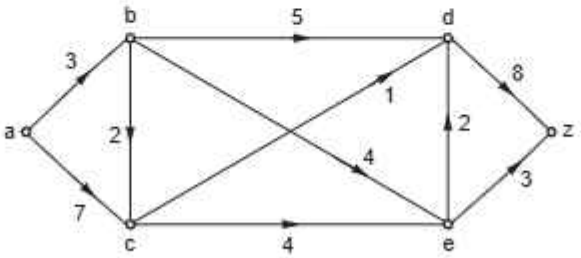


Fig. 8.12.3

Q.8 *Explain the following :*
1) Difference between binary tree and binary search tree.
2) Rooted free 3) Cut-sets. **(Refer sections 8.5 and 8.6.4)** [6]

Q.9 *Find minimum spanning tree for given graph using Krustkal's algorithm.* [6]

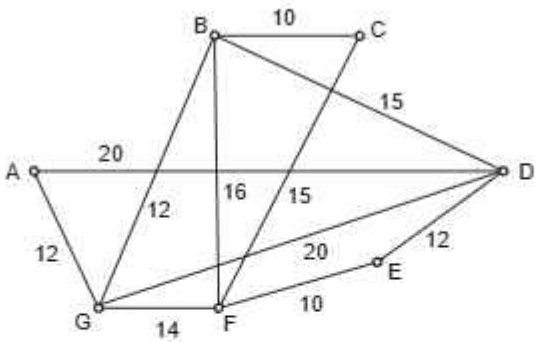
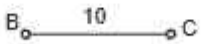


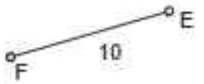
Fig. 8.12.4

Ans. :

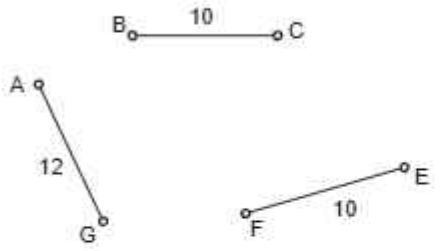
Step 1 :

Step 2 :

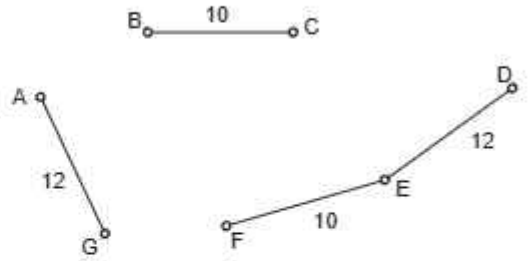




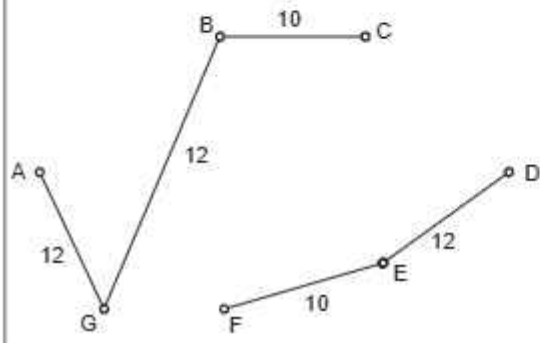
Step 3 :



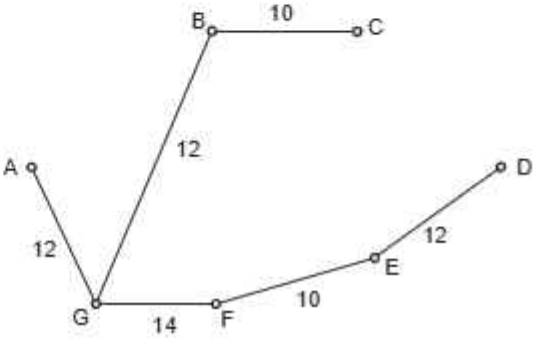
Step 4 :



Step 5 :



Step 6 :



Weight of MS = 10+10+10+12+12+14 = 68

Q.10 *Explain the following terms :*
i) Application of cutset in computer engineering domain.
ii) Prefix code construction using Huffman coding. **(Refer section 8.6.2)**
iii) Properties of trees. **(Refer section 8.2)** [7]

Ans. : **i)** Applications of cutset

- transport network
- Map coloring
- Ad-hoc network
- Network security
- Fault tolerant computing system.

Dec. 2017

Q.11 *Use Prim's algorithm to find he minimum spanning tree for the connected weighted graph G as shown in Fig. 8.12.5.* [7]

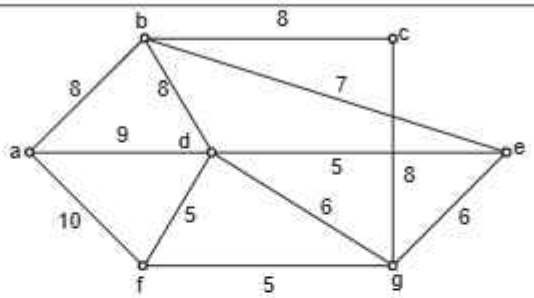


Fig. 8.12.5 Connected weighted graph

Ans. : Consider the following steps for the construction of the minimum spanning tree by prim's algorithm.

Step 1 : Vertex a is starting vertex

$T = \{\{a\}, \phi\}$

Fig. 8.12.6

Step 2 : $T = \{\{a, b\}, e_1\}\}$

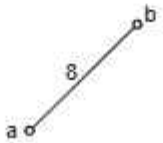


Fig. 8.12.7

Step 3 : $T = \{\{a, b, e\}, \{e_1\ e_2\}\}$

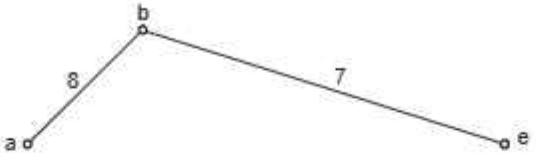


Fig. 8.12.8

Step 4 : $T = \{\{a, b, e, d\}, \{e_1\ e_2, e_3\}\}$

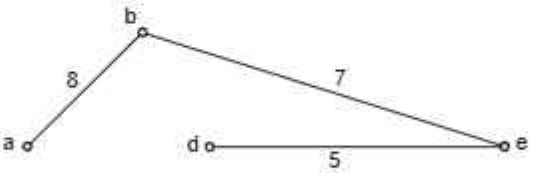


Fig. 8.12.9

Step 5 : $T = \{\{a, b, e, d, f\}, \{e_1\ e_2, e_3, e_4\}\}$

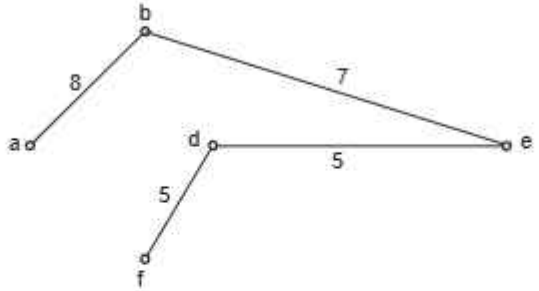


Fig. 8.12.10

Step 6 : $T = \{\{a, b, e, d, f, g\}, \{e_1\ e_2, e_3, e_4, e_4, e_5\}\}$

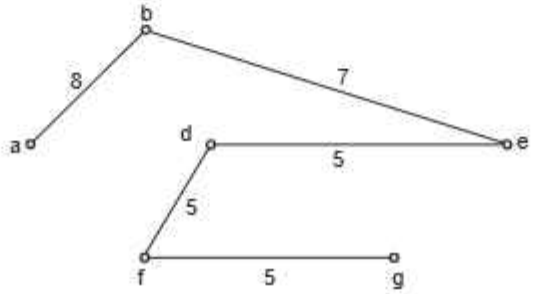


Fig. 8.12.11

Step 7 : $T = \{\{a, b, e, d, f, g, c\}, \{e_1, e_2, e_3, e_4, e_4, e_5, e_6\}\}$

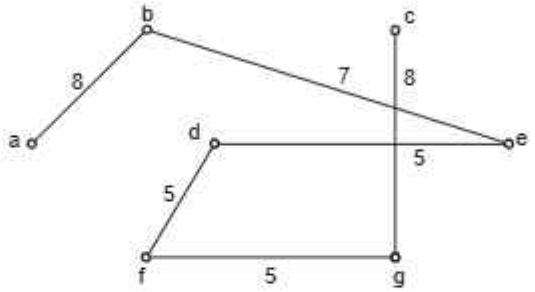


Fig. 8.12.12

The above grpah is the minimum spanning tree and it's total weight is $8 + 7 + 5 + 5 + 5 + 8 = 38$.

Q.12 *Explain any two of the following :*
i) Min-max tree case study ii) Transport network iii) Decision free. [6]

Ans. : i) Refer section 8.10

ii) Refer section 8.9.

iii) **Decision tree :** A decision tree is a decision support tool that uses a tree of decisions and their possible consequences, including chance event, outcomes, resource cost and utility. It is one way to display an algorithm that only contains conditional control statements.

Decision trees are used in decision analysis to help identify a strategy most likely to reach goal, but are also popular tool in machine learning.

Q.13 *Explain the Kruskal's algorithm in detail.* **(Refer section 8.7.3)** [7]

Q.14 *Find the pre, post and inorder traversal of a tree shown in Fig. 8.12.13.* [6]

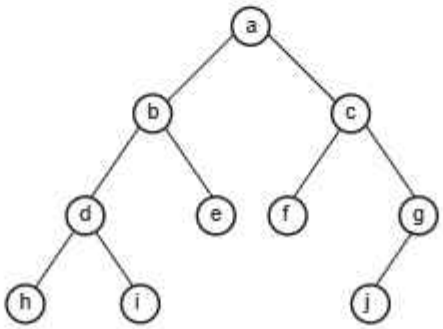


Fig. 8.12.13 Tree

Ans. : Preorder :a, b d h i e c f g j

Inorder : h d i b e a f c j g

Postorder : h i d e b f j g c a

May 2018

Q.15 *Determine the maximum flow in the transport network shown in Fig. 8.12.14 using labelling procedure. Determine the corresponding min. cut.* [7]

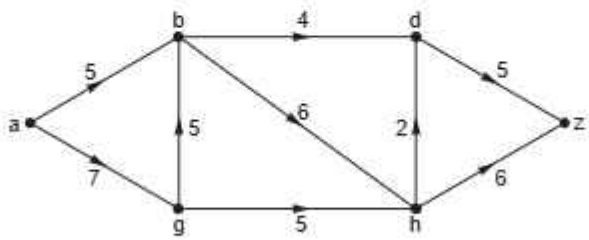


Fig. 8.12.14

Ans. : The maximum flow in the transport network using labelling procedure can be obtained using following steps.

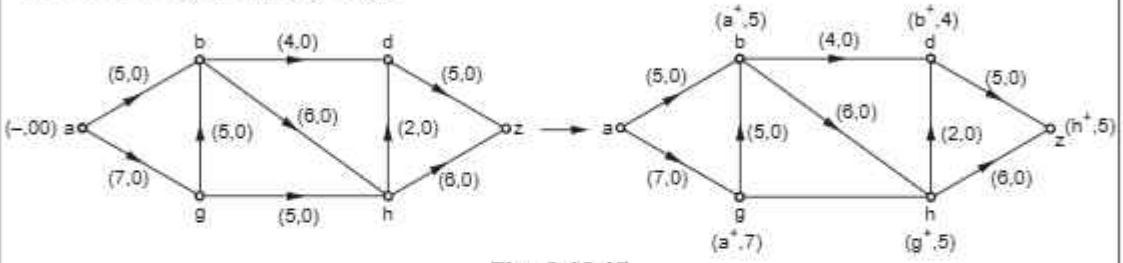


Fig. 8.12.15

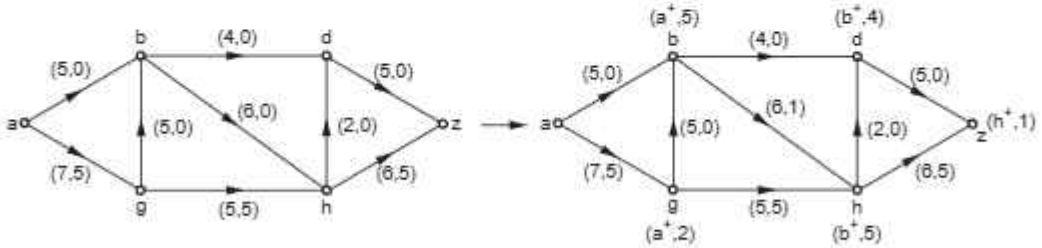


Fig. 8.12.16

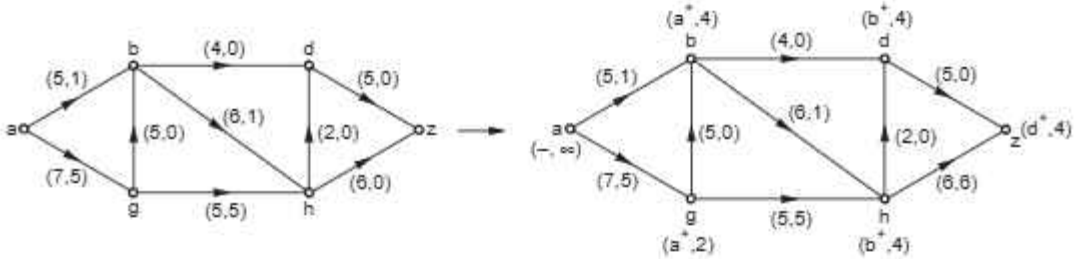


Fig. 8.12.17

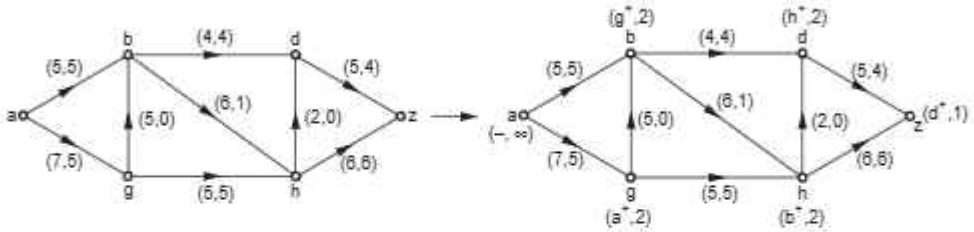


Fig. 8.12.18

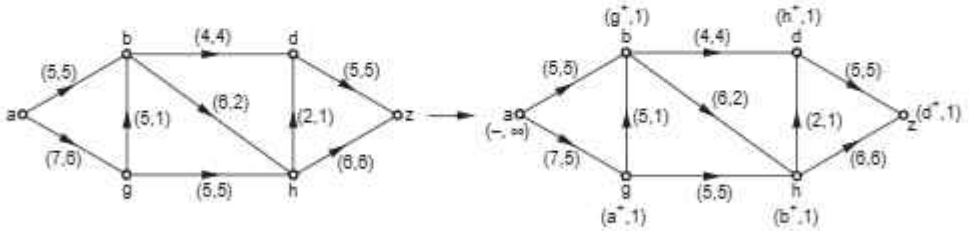


Fig. 8.12.19

Since z can not be labeled so we have to stop.

The maximum cut is $\overline{PP}$ where P is the set of labeled vertices and $\overline{P}$ is the set of unlabelled vertices.

$$P = \{a, b, g, d, h\}, \quad \overline{P} = \{z\}$$

Edges is cut $\overline{PP} = \{(d, z), (h, z)\}$

Capacity of the maximum cut = C (d, z) + C (h, z) = 5 + 6 = 11

The maximum flow in transport work is 11.

Q.16 *Explain the following terms :*
i) Spanning trees **(Refer section 8.7)**
ii) Properties of trees **(Refer section 8.2, Theorem 1 to 4)**
iii) M-ary tree **(Refer section 8.4.4)** [6]

Q.17 *Give the stepwise construction of minimum spanning tree using Prim's algorithm for the following graph shown in Fig. 8.12.20. Obtain the total cost of minimum spanning tree.* [7]

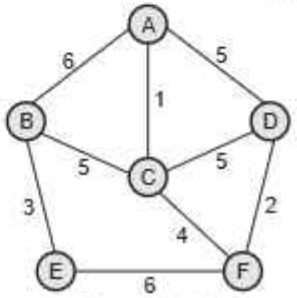


Fig. 8.12.20

Ans. : Consider the following steps for the construction of the minimum spanning tree.

Step 1 : Select A ∈ V(G) as a starting vertex.

$$T = \{\{A\}, \phi\}$$

Step 2 : Vertex A is adjacent to vertices B, C and D

Among all these edges minimum weight is

W(A, C) = 1

$$\therefore \quad T = \{[A, C], [(A, C)\}$$

Fig. 8.12.21

Step 3 : Vertex A is adjacent to B and D.

C is adjacent to B, D, F.

W{A, B} = 6, W{A, D} = 5, W(C, B) = 5, W(C, D) = 5, W(C, F) = 4.

Among these minimum weight is W(C, F) = 4.

$$\therefore \quad T = \{\{A, C, F\} \{(A, C), (C, F)\}\}$$

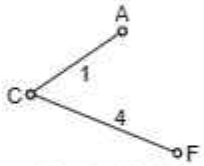


Fig. 8.12.22

Step 4 : Vertex A is adjacent to B and D.

C is adjacent to B and D.

F is adjacent to D and E.

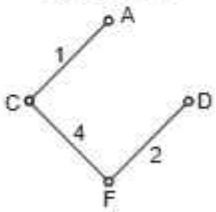


Fig. 8.12.23

Among all these minimum weight W{F, D} = 2.

$\therefore$ T = {{A, C, F, D}, {(A, C), (C, F), (F, D)}}

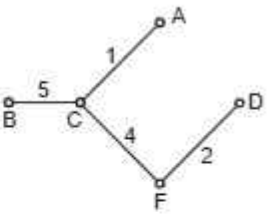


Fig. 8.12.24

Step 5 : Vertex A is adjacent to B.

C is adjacent to B.

F is adjacent to E.

$\therefore$ W{A, B} = 6, W(C, B) = 5, W(F, E) = 6.

$\therefore$ W (C, B) = 5 is minimum

T = {{A, C, F, D, B}, {(A, C), (C, F), (F, D), (C, B)}}

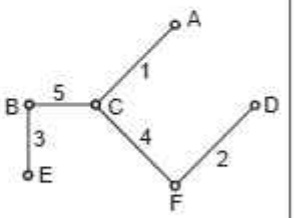


Fig. 8.12.25

Step 6 : Vertex B is adjacent to E, W(B, E) = 3

Vertex F is adjacent to E, W(F, E) = 6

Minimum is W (B, E) = 3.

T = {{A, C, F, D, B, E,}, {(A, C), (C, F), (F, D), (C, B), (B, E)}}

The graph obtained is the minimum spanning tree of weight = 1 + 5 + 3 + 4 + 2 = 15.

Q.18 *Explain the following : i) Game tree* **(Refer section 8.10)**
ii) Kruskal's algorithm **(Refer example 8.7.3)** [6]

Dec. 2018

Q.19 *Find maximum flow in the transport network using labeling procedure. Determine the corresponding min cut :* **(Refer similar example 8.9.3)** [7]

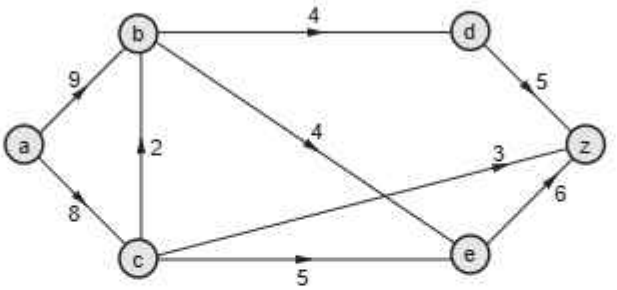


Fig. 8.12.26

Q.20 *Define the following terms :*
(i) Level and height of a tree **(Refer section 8.4.2)**
(ii) Cut points **(Refer section 8.3.3)**
(iii) Eccentricity of a vertex **(Refer section 8.3.1)** [6]

Q.21 *Find minimum spanning tree for the graph shown below using Kruskal's algorithm.* **(Refer section 8.7.4)** [7]

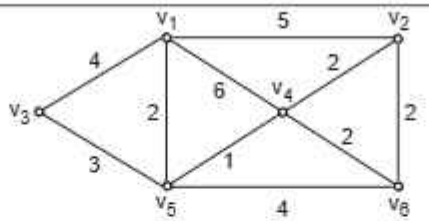


Fig. 8.12.27

Q.22 *Suppose data items A, B, C, D, E, F, G occur in the following frequencies respectively 10, 30, 5, 15, 20, 15, 5. Construct a Huffman code for the data. What is the minimum weighted path length ?* **(Refer section 8.6.2)** [6]

May 2019

Q.23 *Suppose data items A,B,C,D,E,F,G occurs in the following frequencies.*

Data items	*A*	*B*	*C*	*D*	*E*	*F*	*G*
Weight	*10*	*30*	*5*	*15*	*20*	*15*	*05*

Construct a Huffman code for the data.
What is the minimum weighted path length. **(Refer example 8.6.2)** [6]

Q.24 *Using the labelling procedure to find maximum flow in the transport network in the following figure. Determine the corresponding minimum cut.* [7]

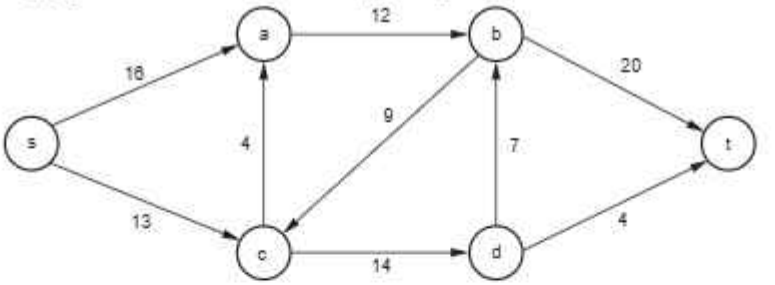


Fig. 8.12.28

Ans. : Assign the flow zero to each edge and $(-, \infty)$ to the source s.

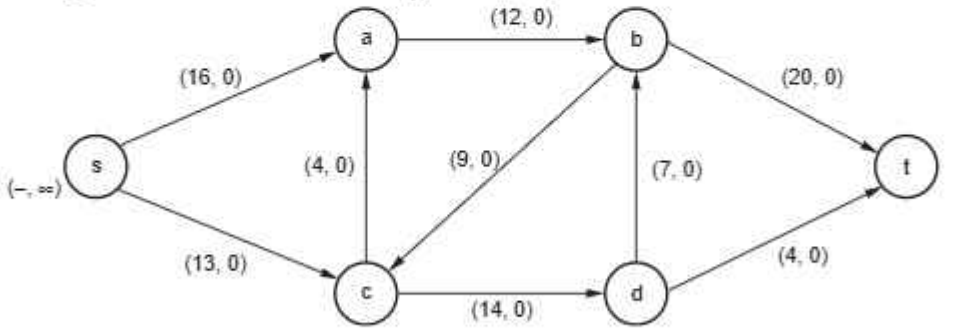


Fig. 8.12.29

Step 2 :

Since, vertices a and c are adjacent to source s. Therefore, label the vertices a and c as $(s^+, \Delta a)$ and $(s^+, \Delta c)$ respectively.

Where,

$$\Delta a = ((s-a) - f(s,a)) = 16$$

$$\Delta c = ((s-c) - f(s,c)) = 13$$

Label of a is $(s^+, 16)$ and label of c is $(s^+, 13)$

Now vertices b and d are adjacent to labelled vertices $(s^+, 16)$ and $(s^+, 13)$. Therefore, label the vertices b and d as $(a^+, \Delta b)$ and $(c^+, \Delta d)$.

Where,

$$\Delta b = \min(\Delta a, c(a,b) - f(a,b))$$

$$= \min(16, 12-0)$$

$$= 12$$

Similarly, $\Delta d = 13$

Lable b is $(a^+, 12)$ and d is $(c^+, 13)$

Now t is the sink and adjacent to b and d , we can choose any vertex b or d to label the sink t.

Lets, choose vertex d, which is labelled as $(c^+, 13)$. The label of t will be $(d^+, \Delta t)$ where

$$\Delta t = \min(13, 4-0) = 4$$

Hence, label of t is $(d^+, 4)$. It is shown in the following diagram.

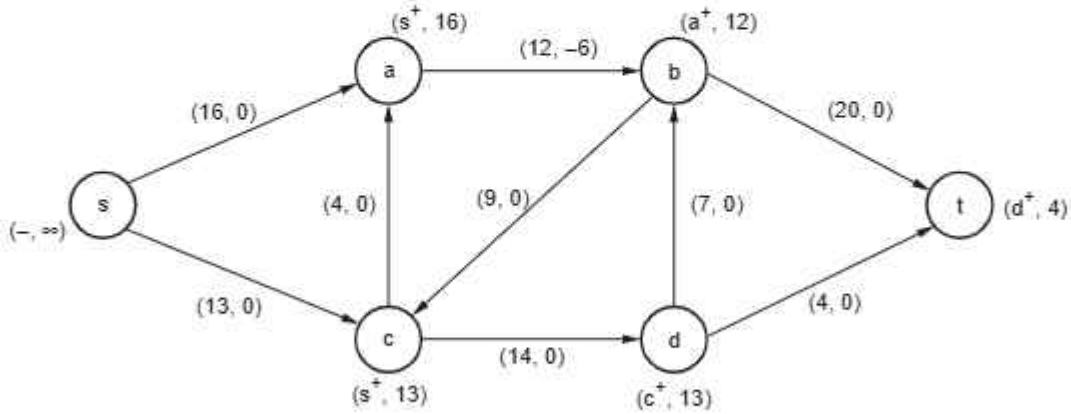


Fig. 8.12.30

Now adjust the flow in the edges

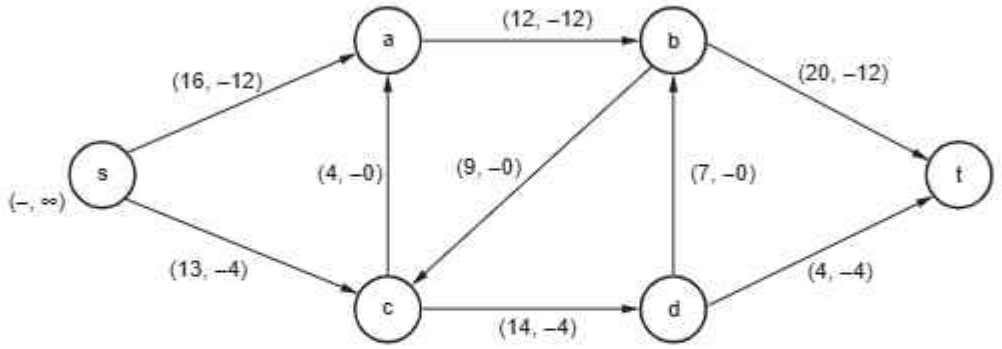


Fig. 8.12.30 (a)

Now adjust the flow in the edges

s–c, c–d, d–b and label the nodes

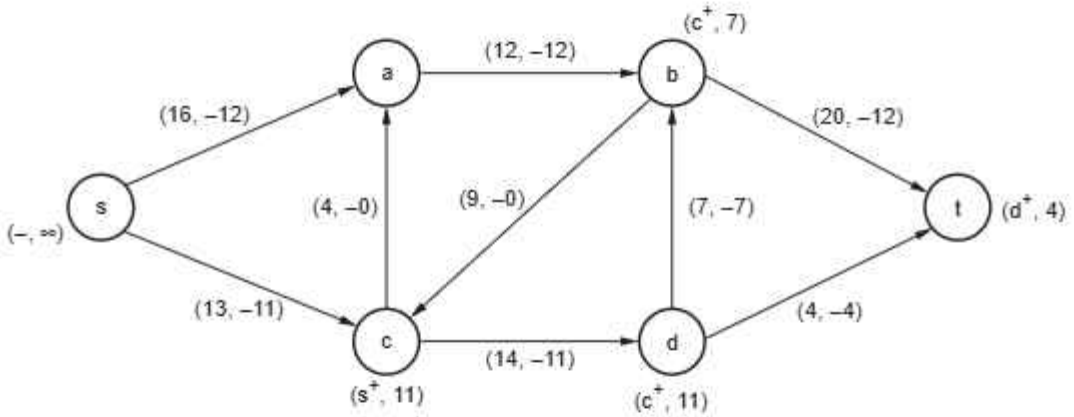


Fig. 8.12.30 (b)

Now adjust the flow in the edges b–t

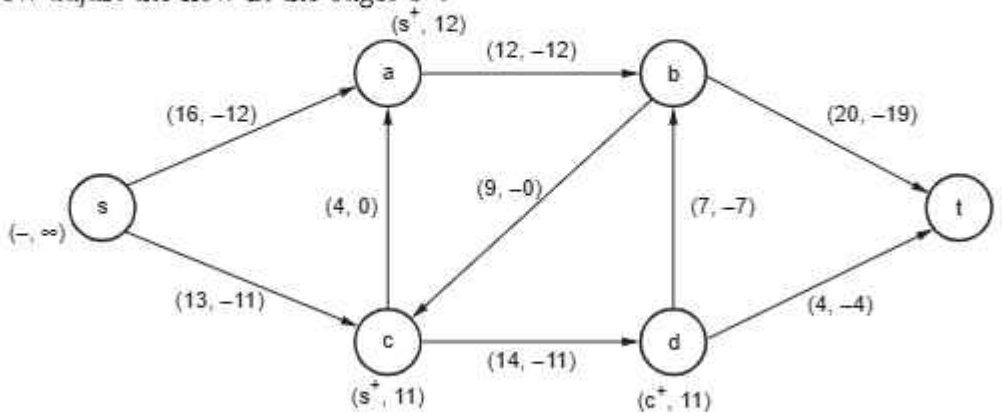


Fig. 8.12.30 (c)

We have to stop here.

P is set of labelled vertices

{s,a,c,d} and unlabelled vertices {b,t}

Capacity of cut is c(a + b) + (d, b) + c(d, t) = 12 + 7 + 4 = 23

Hence, maximum flow in the network = 23.

Q.25 *Give the stepwise construction of minimum spanning tree using Prime algorithm for the following graph. Obtain the total cost of minimum spanning tree.* [7]

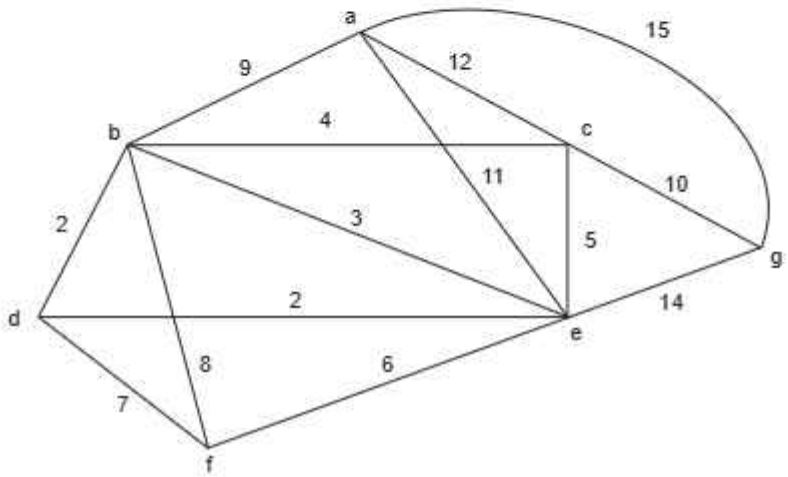


Fig. 8.12.31

Ans. :

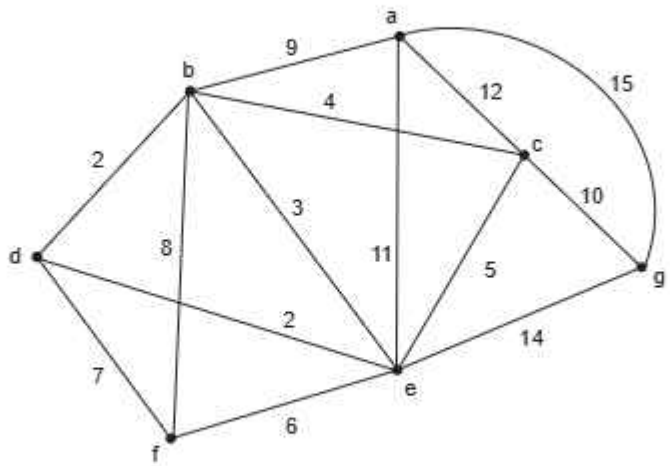


Fig. 8.12.31 (a)

i) Select $a \in G$ as a starting vertex

$$T = \{\{G\}, \phi\}$$

Fig. 8.12.31 (b)

ii) Select edge having minimum weight and one end is connected to a is {a,b} = 9.

$$T = \{\{a, b\}, e_1\}$$

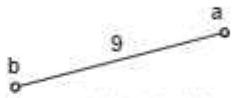


Fig. 8.12.31 (c)

iii) Select edge {b,d} = 2

$$T = \{\{a, b, d\}, \{e_1, e_2\}\}$$

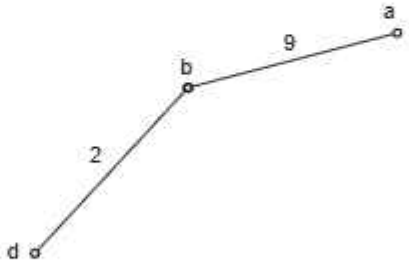


Fig. 8.12.31 (d)

iv) Select edge {d,e} = 2

$$T = \{\{a, b, d, e\}, \{e_1, e_2, e_3\}\}$$

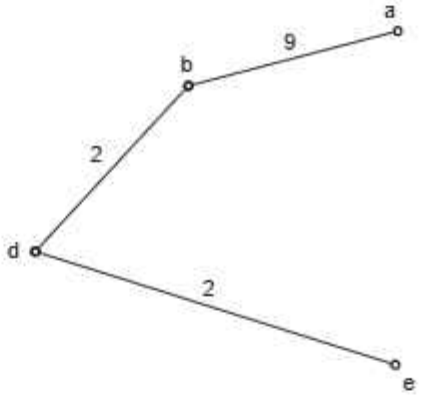


Fig. 8.12.31 (e)

v) Discard edge {b,e} = 3, since it creates cycle.

Select edge {b,c} = 4.

$$T = \{\{a, b, d, e, c\}, \{e_1, e_2, e_3, e_4\}\}$$

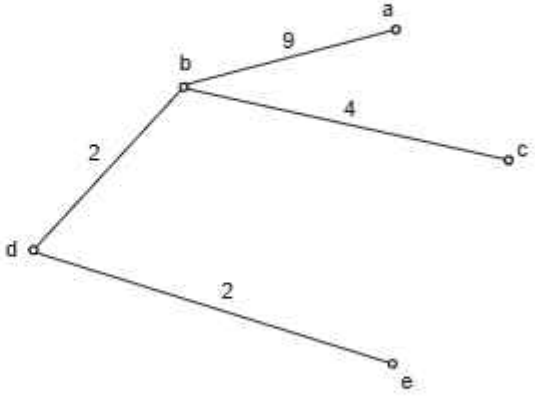


Fig. 8.12.31 (f)

vi) Discard edge {c,e} = 5, since it creates cycle. Select edge {e,f} = 6.

$$T = \{\{a, b, d, e, c, f\}, \{e_1, e_2, e_3, e_4, e_5\}\}$$

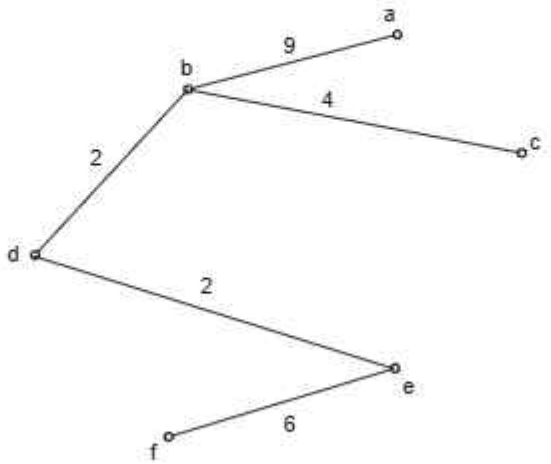


Fig. 8.12.31 (g)

vii) Discard edge {f,d} = – 7, {f,b} = 8, since it creates cycle. Select edge {c,g} = 10.

$$T = \{\{a, b, d, e, c, f, g\}, \{e_1, e_2, e_3, e_4, e_5, e_6\}\}$$

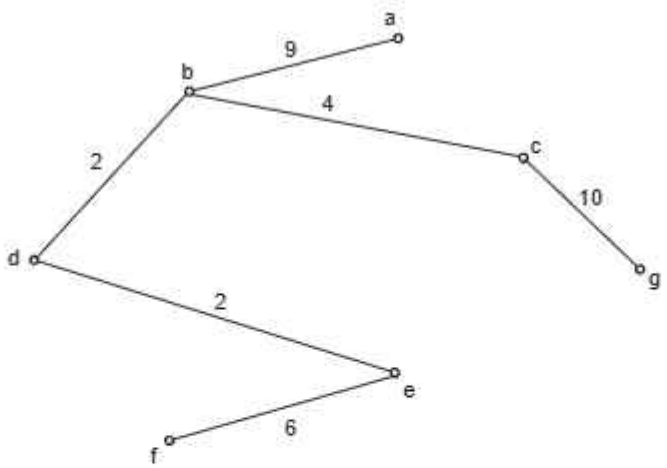


Fig. 8.12.31 (h)

This is a minimum spanning tree and cost of MST = 2+2+4+6+9+10 = 33

Q.26 *Define with example.*

i) Level and height of tree **(Refer section 8.4.2)**

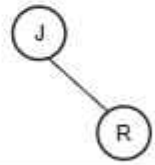
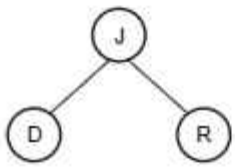
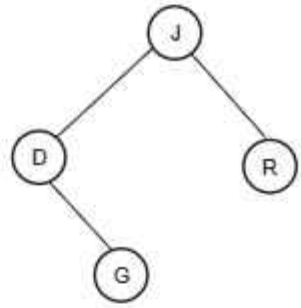
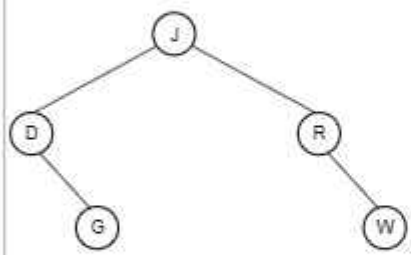
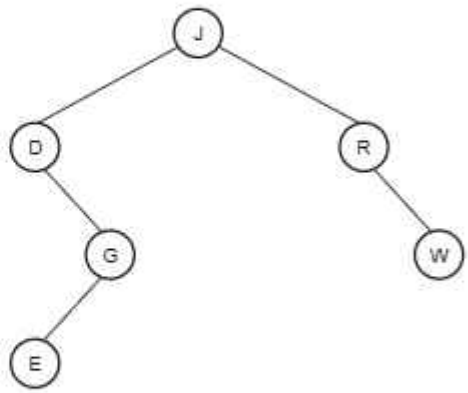
ii) Binary search tree **(Refer section 8.6.4)**

iii) Spanning tree. **(Refer section 8.7)** [6]

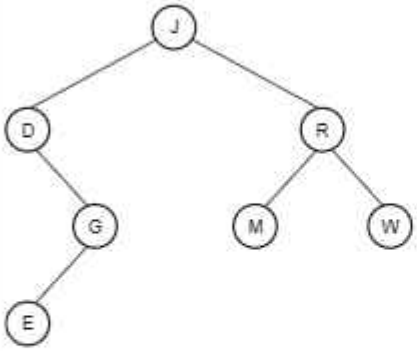
Dec. 2019

Q.27 *Construct a binary search tree : J, R, D, G, W, E, M, H, P, A, F, Q.* [4]

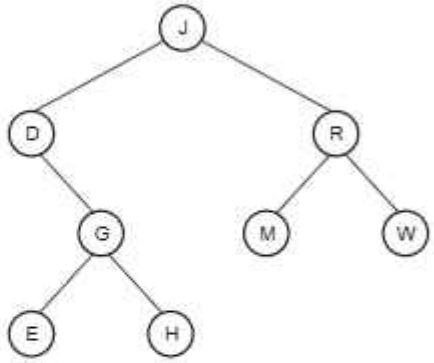
Ans. : Given

i)	J	ii)	R 
iii)	D 	iv)	G 
v)	W 	vi)	E 

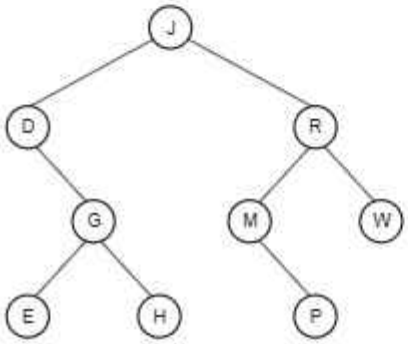
vii) M



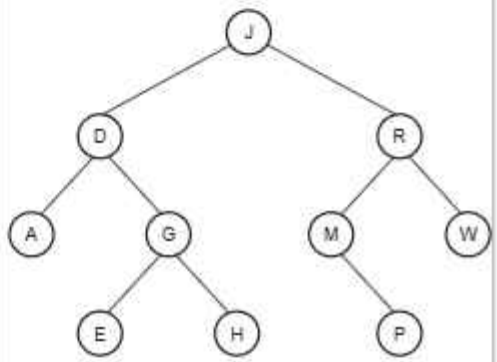
viii) H



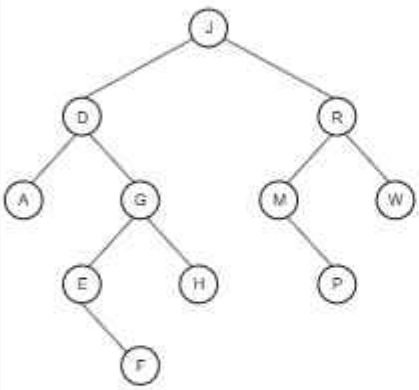
ix) P



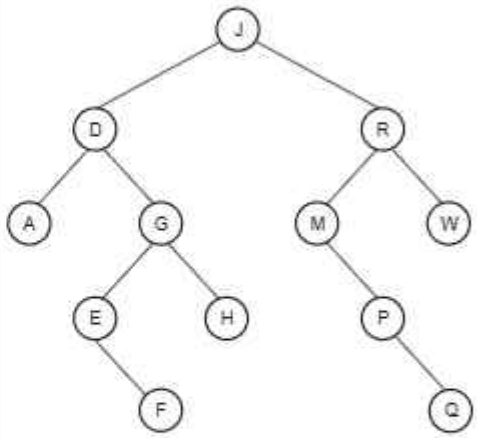
x) A



xi) F



xii) Q

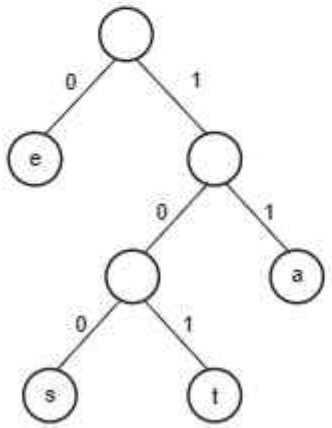


This is binary search tree

Q.28 *Construct the binary tree with prefix codes representing :*
(i) a : 11, e : 0, t : 101, s : 100
(ii) a : 1010, e : 0, t : 11, s : 1011, n : 1001, i : 10001 **[4]**

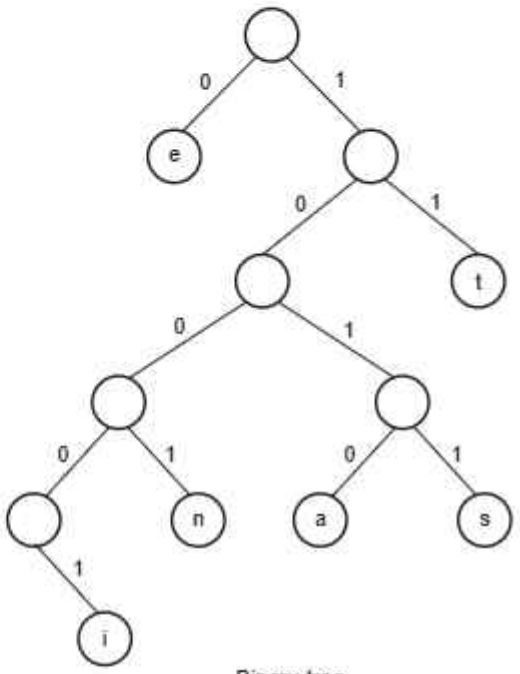
Ans. :

i)



Binary tree

ii)



Binary tree

Q.29 *Give the stepwise construction of minimum spanning tree using Kruskal's algorithm for the following graph. Obtain the total cost of minimum spanning tree.* **[5]**

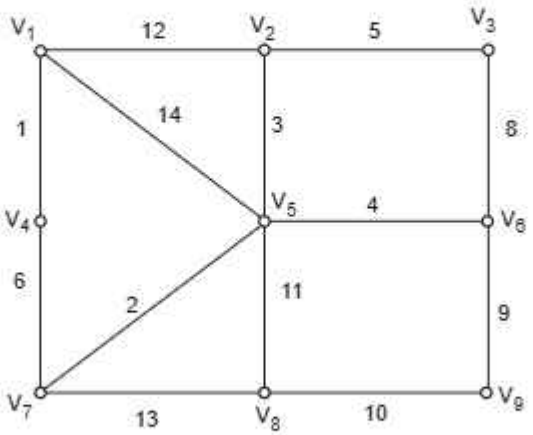


Fig. 8.12.32

Ans. :

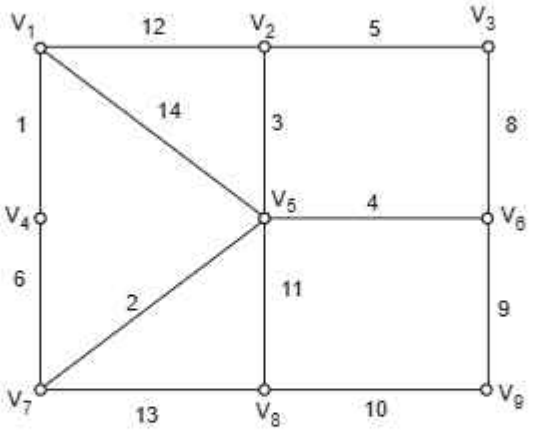
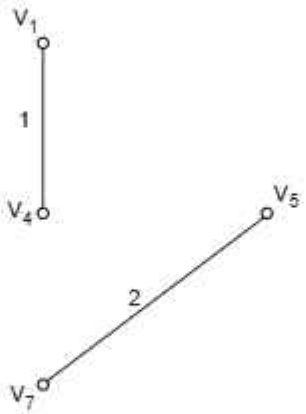


Fig. 8.12.32 (a)

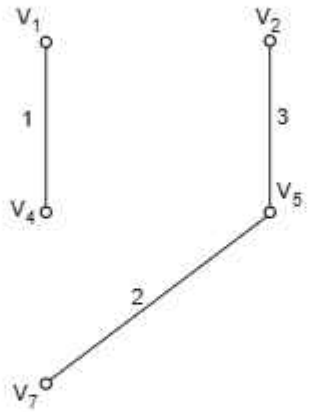
Using Kruskal's algorithm minimum spanning tree is obtained as follows.

Step 1 : Select the edge $(V_1 - V_4)$ with minimum weight = 1

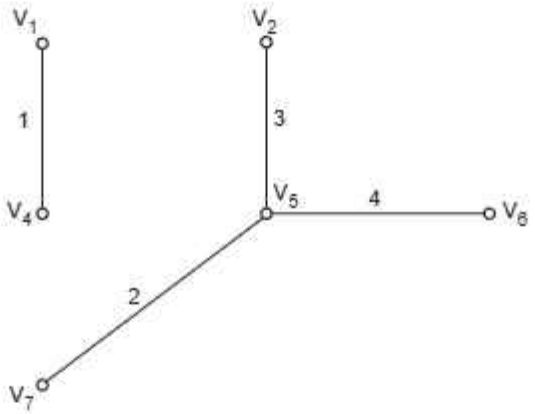
Step 2 : Select edge $(V_5 - V_7)$ weight = 2



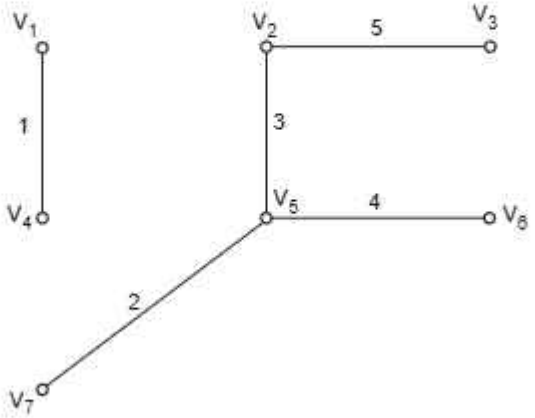
Step 3 : Select edge (V_2 - V_5) weight = 3



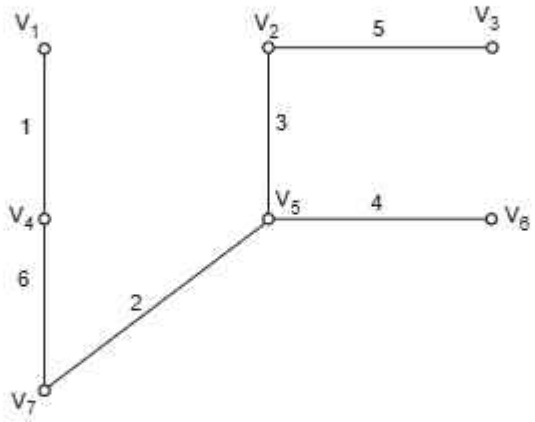
Step 4 : Select edge (V_5 - V_6) weight = 4



Step 5 : Select edge (V_2 - V_3) weight = 5

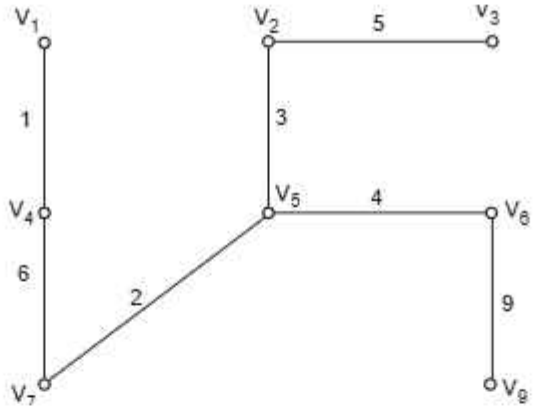


Step 6 : Select edge (V_4 - V_7) weight = 6

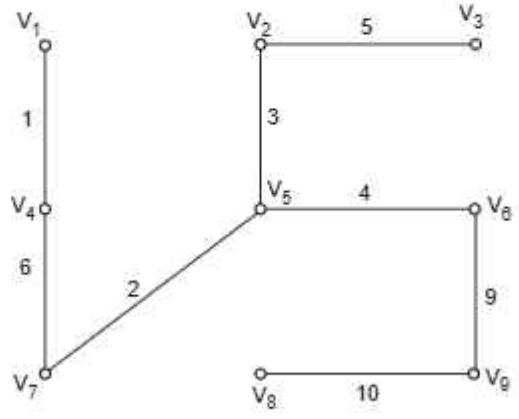


Step 7 : Select edge (V_3 - V_6) with weight = 8. But it will form the cycle so discard it.

Step 8 : Select edge (V_6 - V_9) with weight = 9



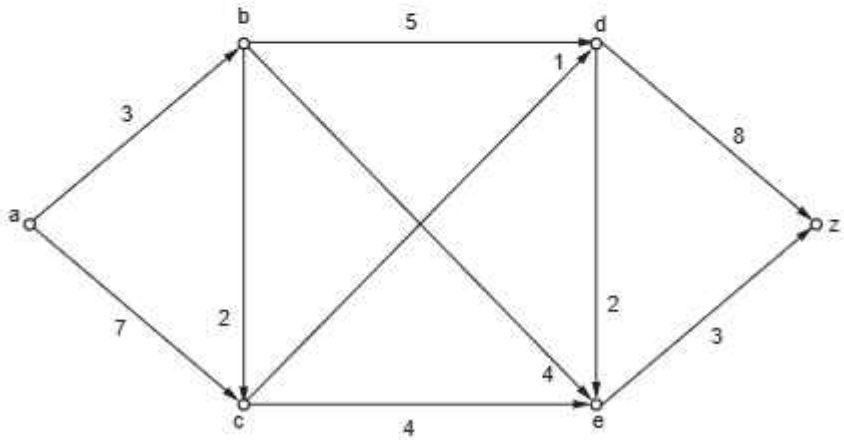
Step 9 : Select edge (V_8 - V_9) with weight = 10



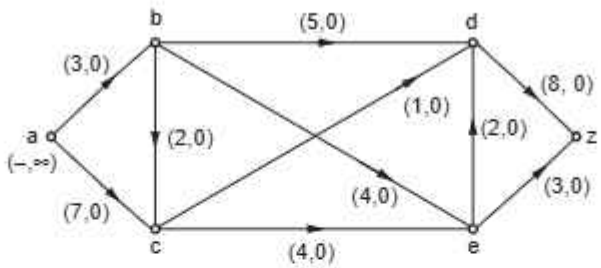
The obtained grpah is minimum spanning free of the given graph. The total cost is

$(1 + 2 + 3 + 4 + 5 + 6 + 9 + 10) = 40$

Q.30 *Using the labelling procedure to find maximum flow in the transport network in the following figure. Determine the corresponding minimum cut.* [7]



Ans. : Step 1 : To find the maximum flow in the given transport network, assign the flow zero to each edge and label (–, ∞) to the source a.



Step 2 : Since the vertices b and c are adjacent to the source a, therefore label the vertices b and c as $(a^+, \Delta b)$ and $(a^+, \Delta c)$ respectively.

Where $\Delta b = ((a, b) - f(a, b) = 3$

$\Delta c = ((a, b) - f(a, c) = 7$

$\therefore$ Label of b is $(a^+, 3)$ and label of c is $(a^+, 7)$

Now the vertices d and e are adjacent to labeled vertices b $(a^+, 3)$ and $(a^+, 7)$ therefore we label the vertices d and e as $(b^+, \cdot d)$ and $(c^+, \cdot e)$ where

$$\Delta d = \min(\Delta b, c(b, d) - f(b, d))$$
$$= \min(3, 5, -0)$$
$$= 3$$

Similarly $\Delta e = \min(\Delta c, c(c, e) - f(c, e))$
$$= \min(7, 4, -0)$$
$$= 4$$

$\therefore$ Label of d is $(b^+, 3)$ and label of e is $(c^+, 4)$

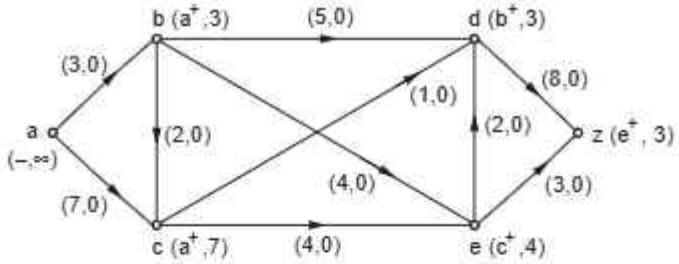
Now the sink z is adjacent to both the vertices d and e, we can choose any vertex d or e to label the sink z.

Let us choose the vertex e, which is labeled as $(c^+, 4)$

The label of z will be $(e^+, \Delta z)$ where

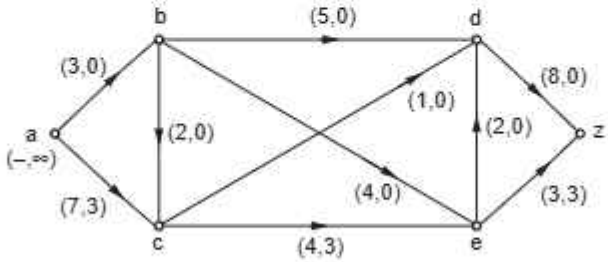
$$\Delta z = \min(4, 3, -0)$$
$$= 3$$

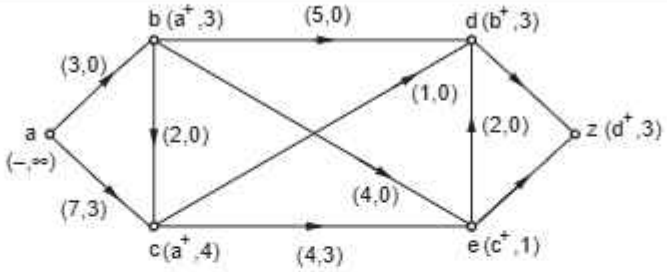
Hence label of z is $(e^+, 3)$



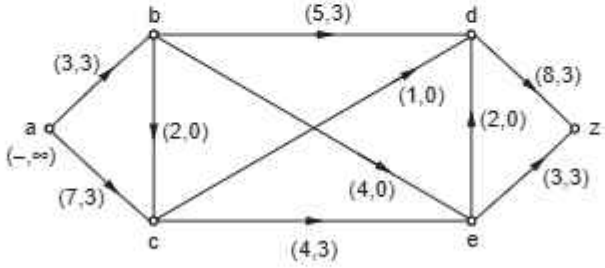
The labels of vertices can be shown as follows :

According to the label of the sink z, adjust the flow in the edges (e, z), (c, e) and (a, c)

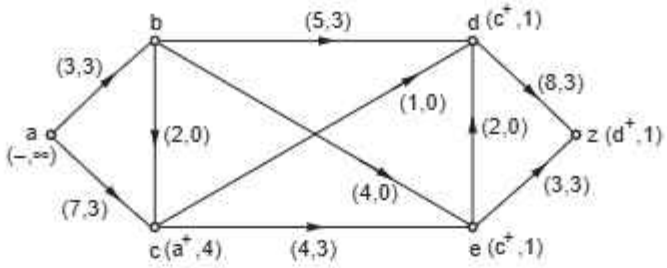




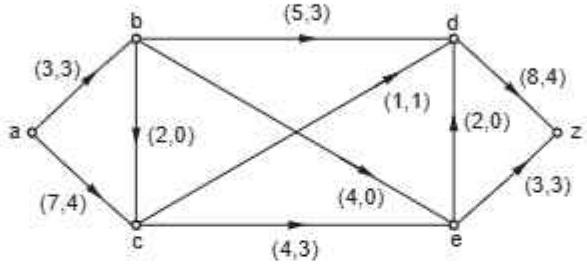
Repeat the step 2 in each pass, the new value of the flow is obtained.

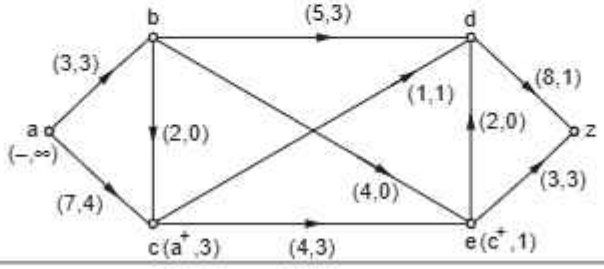


After adjusting the flow we have



Step 4 :





Adjust the flow according to sink z label.

Since the vertex z (sink) cannot be labelled, we have to stop. The vertices b, d and z cannot be labelled because either the edges approaching to vertices are saturated or the edges in the opposite direction have flow zero. At this stage, we have minimum cut $(P, \overline{P})$.

P is the set of labelled vertices P = {a, c, e}. $\overline{P}$ is the set of unlabeled vertices, $\overline{P}$ = {b, d, z}

The edges in cut $P\,\overline{P}$ {(a, b), (c, d), (e, z)} capacity of the cut is c(a, b) + c(c, d) + c(e, z) = 3 + 1 + 3 = 7.

Hence the maximum flow in the given transport network 7.

Q.31 *Find fundamental cutsets and circuits of the following graph G_1 with respect to spanning tree T_1* **[4]**

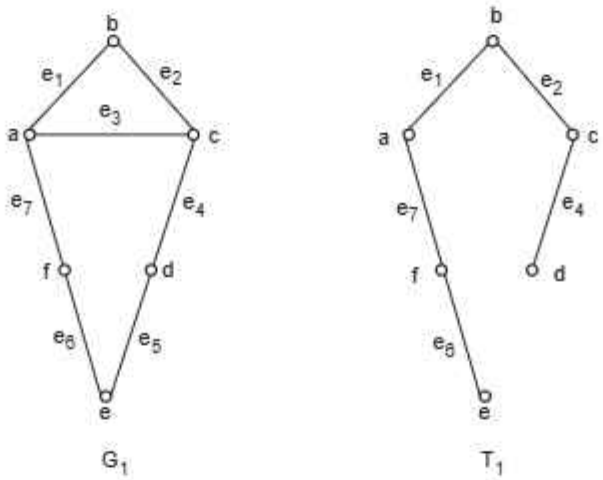


Fig. 8.12.33

Ans. :

Given spanning tree has 5 branches. These are e_1, e_2, e_4, e_6 and e_7

Chords of the T_1 are e_3, e_5

Fundamental circuits :

Consider the following table of chords and corresponding fundamental circuits

Chords	Corresponding fundamental circuits
e_3	$\{e_1, e_2, e_3\}$
e_5	$\{e_1, e_2, e_4, e_5, e_6, e_7\}$

Fundamental cut sets

Branches of T_1 are e_1, e_2, e_4, e_6, e_7

Branches	Corresponding fundamental cutset
e_1	$\{e_1, e_3, e_5\}$
e_2	$\{e_2, e_3, e_5\}$
e_4	$\{e_4, e_5\}$
e_4	$\{e_4, e_5\}$
e_7	$\{e_5, e_7\}$

Q.32 *Define with example : Level and height of a tree.* **(Refer section 8.4.2)** [2]

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UNIT - VI

9 Algebraic Structures and Coding Theory

Syllabus

The structure of algebra, Algebraic Systems, Semi Groups, Monoids, Groups, Homomorphism and Normal Subgroups, and Congruence relations, Rings, Integral Domains and Fields, Coding theory, Polynomial Rings and polynomial Codes, Galois Theory -Field Theory and Group Theory. Case studies : Cryptography used in world war - II.

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9.1 Introduction

9.2 Algebraic Structures ..... **Dec.-05, May-17,19** ..... Marks 3

9.3 Special Algebraic Structures ..... **Dec.-06, 13,17, May-18,19** · Marks 4

9.4 Group **Dec.-08, 09, 10, 12,17,18**
..... **May-14,17,19** ..... Marks 6

9.5 Modulo m **Dec.-11, May-08**

9.6 Permutation Group

9.7 Complexes and Subgroups ..... **May-18** ..... Marks 6

9.8 Cosets

9.9 Order of an Element of a Group

9.10 Cyclic Group ..... **Dec.-18** ..... Marks 6

9.11 Normal Subgroups ..... **Dec.-16** ..... Marks 5

9.12 Quotient Groups ..... **Dec.-14. 15, May-15,17** .... Marks 3

9.13 Homomorphism of Groups **Dec.-11, 12, 14, 15, May-15,17**

9.14 Isomophism of Groups **Dec.-10, 11, 12, 13,16,**
..... **May-08,10,17,19** ..... Marks 4

9.15 Rings, Integral Domains and Fields **Dec.-08, 10, 11, 12, 13, 14,16,17,18**
..... **May-14,17,18,19**

9.16 Group Codes **Dec.-06, 11, 12, 15,18,**
..... **May-18** ....., Marks 5

9.17 Case Studies

9.18 University Questions with Answers

9.1 Introduction

In this chapter, we shall deal with sets with additional structures, induced by one or more binary operations on the elements of the set. We will study some algebraic structures, groups, rings integral domains and fields.

9.2 Algebraic Structures

SPPU : Dec.-05, May-17,19

9.2.1 Binary Operation

I) Let A be any non empty set. A function $f : A \times A \to A$ is called the binary operation on the set A.

'$*$' is a binary operation on the set A iff $a * b \in A \ \forall \ a, b \in A$ and $a * b$ is unique.

II) An n-ary operation on a set $A \neq \phi$, is a function $f : A \times A \times \ldots \times A$ (n times) $\to A$ i.e. $f : A^n \to A$

The n-ary operation is defined for each n-tuple $(a_1, a_2, \ldots a_n) \in A$ for $a_2 \in A$

If n = 1 then f is called unary operation

If n = 2 then f is called binary operation

If n = 3 then f is called ternary operation

Examples

1) A function $f : \Re \to \Re$ such that $f(x) = x \ ; \forall x \in \Re$ then f is a unary operation.
2) A function $f : \Re \times \Re \to R$ such that $f(x, y) = x + y$ then f is a binary operation.
3) A function $f : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that $f(x, y, z) = x + y + z$. then f is a ternary operation.
4) If $A = \Re$ (or $\mathbb{C}$)

where $\Re$ = Real number set

$\mathbb{C}$ = Complex number set

then '+' '–' and ' × ' are binary operations as

$x \pm y$ and $x \cdot y \in \Re$ (or $\mathbb{C}$) ; $\forall$ x, y $\in \Re$ (or $\mathbb{C}$)

5) For any $x, y \in N = \{1, 2, 3, 4, \ldots\}$, $x + y, x \times y \in N$

$\therefore$ '+' and '×' are binary operations on N

But '–' and ÷ are not binary operations on N

as $2 - 5 = -3 \notin N$ and $\frac{2}{3} \notin N$

6) Union, intersection and difference are binary operations in P(A), for any non empty set A.

9.2.2 Properties of Binary Operations

I) Commutative Property

A binary operation '$*$' on A is said to be commutative if $a * b = b * a$, for all $a, b \in A$.

e.g. $x + y = y + x$ and $x \times y = y \times x$ for all $x, y \in \Re$

$\therefore$ '+' and '×' are commutative binary operations on $\Re$.

But '–' is not commutative on $\Re$.

II) Associative Property :

A binary operation '$*$' on set A is said to be associative if $a*(b*c) = (a*b)*c$; $\forall$ a, b, c $\in$ A

e.g. '+' and '×' are associative on the set of real numbers '–' is not associative on $\Re$.

III)Idempotent

A binary operation '$*$' on set A is said to be idempotent if $a * a = a$; for all $a \in A$

e.g.

1) 1 is idempotent element in $\Re$ w.r.t. binary operation 'X'.
2) 0 is idempotent element in $\Re$ w.r.t. '+'.
3) 1 is not idempotent element in $\Re$ w.r.t. '+'.

Examples

Example 9.2.1 *For each of the following, determine whether $*$ is a binary operation.*

i) R is the set of real numbers and $a * b = ab$

ii) z^+ is the set of positive integers and $a * b = a/b$.

iii) On z^+ where $a * b = a - b$.

iv) On R, where $a * b = \min\{a, b\}$

v) On R, where $a * b = a \times |b|$

vi) On z, where $a * b = a^b$.

Solution :

i) Yes, since $f : R^2 \to R$ defined as $f(a, b) = ab$ is a function, with $ab \in R$.

ii) No, since $(a, b) \in z^+ \times z^+$ does not imply that $a * b = a/b \in z$

$(1, 2) \in z^+ \times z^+$, but $1/2 \notin z^+$

iii) No, since $(1, 2) \in z^+ \times z^+$ but

iv) Yes, since, $*$ is a function with $\min\{a, b\} \in R$

v) Yes, since $*$ is a function with $a \times |b| \in R$

vi) No, since $2 * (-1) = 2^{-1} = \frac{1}{2} \notin z$

Example 9.2.2 *Let $(A, *)$ be algebraic system where $*$ is a binary operation such that for any $a, b \in A$, $a * b = a$.*

i) Show that $*$ is an associative operation.

ii) Can $*$ ever be a commutative operation ?

Solution :

i) $$a * (b * c) = a * b = a \cdot (a * b) * c$$
$$= a * c = a$$

Hence $*$ is associative

ii) $*$ is commutative only if $a * b = b * a$ i.e. $a = b$, for all $a, b \in A$. This is possible if A is the singleton set $\{a\}$ and $a * a = a$ i.e. $*$ is an idempotent operation on A.

Example 9.2.3 *Determine whether or not following operations on the set of integers $\mathbb{Z}$ are associative.*

i) Division ii) Exponentiation **SPPU : Dec.-05, Marks 3**

Solution :

i) Division on the set of integers is not associative as

$$(a/b)/c \neq a/(b/c)$$

ii) Exponentiation on the set of integers is not associative as

$$(a^b)^c \neq a^{(b^c)}$$

Example 9.2.4 *Consider the binary operation $*$ defined on the set $A = \{a, b, c, d\}$ by the following table.*

*	A	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Find i) $c * d$ and $d * c$ ii) $b * d$ and $d * b$

iii) $a * (b * c)$ and $(a * b) * c$ iv) Is $*$ commutative, associative ?

Solution :

i) $c * d = a$ $\quad$ $d * c = a$

ii) $b * d = c$ $\quad$ $d * b = b$

iii) $b * c = b$ $\quad$ $a * (b * c) = a * b = c$

$a * b = c$ $\quad$ Hence $(a * b) * c = c * c = a$

iv) $*$ is not commutative, since $b * d \neq d * b$

$*$ is also not associative, since $a * (b * c) \neq (a * b) * c$

9.3 Special Algebraic Structures

SPPU : Dec.-06, 13,17, May-18,19

9.3.1 Groupoid

A non empty set k with binary operation '$*$' is called groupoid if the binary operation '$*$' satisfies $\forall a, b \in G$, $a * b \in G$

In other words, every algebraic structure is groupoid.

e.g. $(\Re, +)$, $(\Re, -)$, $(\mathbb{Z}, +)$, $(\mathbb{Z}, \times)$ are groupoids.

9.3.2 Semi Group

A non empty set G with binary operation '$*$' is called a semigroup if it satisfies the following properties.

$$a * (b * c) = (a * b) * c \; ; \forall a, b, c \in G$$

i.e. '$*$' is associative in G

A semigroup is said to be commutative if $*$ is commutative.

e.g. i) $(\Re, +)$, $(N, +)$, $(\mathbb{Z}, +)$ are commutative semigroups.

ii) $(\Re, \times)$, $(\mathbb{Z}, \times)$ are commutative semigroups.

iii) $(\mathbb{Z}, -)$ is not a semigroup as '$-$' is not associative.

9.3.3 Monoid

SPPU : Dec.-13

Let G be a non empty set and $*$ be a binary operation on G. $(G, *)$ is called monoid if.

i) Associative property : $a * (b * c) = (a * b) * c \; ; \forall a, b, c \in G$

ii) Existence of identity : $\exists$ an element $e \in G$ such that $e * a = a * e = a \; ; \forall a \in G$

The element 'e' is called the identity element.

e.g. 1) $(\Re, +)$ is a monoid as $a + b \in \Re$, $\forall a, b \in \Re$

i.e. $(\Re, +)$ is closed

and $a + (b + c) = (a + b) + c$

and $0 + a = a + 0 = a, \quad \forall\ a \in \Re$

0 is the identity element in $\Re$ w.r.t. '+'.

Example 2

$(\mathbb{C}, +), (\mathbb{C}, \times), (\mathbb{Z}, +), (\mathbb{Z}, \times), (\mathbb{Q}, +)$ are monoids.

Example 3

$(\mathbb{N}, \times)$ is a monoid but $(\mathbb{N}, +)$ is not monoid as $0 \notin \mathbb{N}$

Example 9.3.1 *Show that the algebraic system (A, +) is a monoid, where A is the set of integers and '+' is a binary operation giving addition of two integers.*

SPPU : Dec.-06, Marks 4

Solution : Let A be the set of all integers and '+' defined on A.

i) Closure property : $a + b \in A$; $\forall\ a, b, \in A$ as $a + b$ is an integer.

ii) Associative property :

$$a + (b + c) = (a + b) + c \ ; \forall\ a, b, c \in A$$

iii) Existence of identity element :

For any $a \in A$, $\exists\ 0 \in A$ such that

$$a + 0 = 0 + a = a$$

Therefore (A, +) is a monoid.

9.4 Group

SPPU : Dec.-08, 09, 10, 12,17,18, May-14,17,19, Marks 4

Let G be a non empty set equipped with a binary operation '$*$'. $(G, *)$ is called a group if it satisfies the following postulates or axioms.

i) Associativity : $a * (b * c) = (a * b) * c$; $\forall\ a, b, c \in G$

ii) Existence of the identity : For any $a \in G$, $\exists\ e \in G$

s.t. $\quad a * e = e * a = a;$

An element e is called the identity element in $(G, *)$

iii) Existence of the inverse :

For all $a \in G$, $\exists\ b \in G$ such that

$$a * b = b * a = e$$

Then b is called the inverse of a in $(G, *)$

$\therefore \quad b = a^{-1}$

9.4.1 Abelian Group or Commutative Group

A group (G, *) is called an Abelian group if

$$a * b = b * a \quad ; \quad \forall\, a, b \in G.$$

i.e. * is commutative in (G, *)

9.4.2 Properties of Group

SPPU : Dec.10, Marks 2

I) The identity element in a group is unique.

Suppose e_1 and e_2 are two identity elements in group G.

We have, $e_1 e_2 = e_1$; if e_2 is identity element in G.

and $e_1 e_2 = e_2$; If e_1 is identity element in G.

$\Rightarrow \quad e_1 e_2 = e_1 = e_2$. Hence the identity element in group G is unique.

II) The inverse of each element in group G is unique. **SPPU : Dec.-10, Marks 4**

Let a be any element of a group G and let e be identity element in group G.

Suppose b and c are two inverses of a in G.

$\therefore \quad ba = ab = e$ and $ac = ca = e$

We have, $b = be = b\,(ac)$

$b = (ba)\,c$

$b = ec$

$b = c$

Hence, the inverse of each element is unique.

III) The inverse of an inverse of the element is the original element. i.e. If the inverse of a is a^{-1} then $(a^{-1})^{-1} = a$.

Let $e \in G$ be the identity element of the group G.

Let $a \in G$.

We have, $a^{-1}a = e$

$[(a^{-1})^{-1}\,(a^{-1}]a = (a^{-1})^{-1}\,e$ ... (Multiplying by $(a^{-1})^{-1}$)

$[(a^{-1})^{-1}\,(a^{-1}]\,a = (a^{-1})^{-1}$ ... ($\because$ Associativity of e identity)

$e\,a = (a^{-1})^{-1}$

$\Rightarrow \quad a = (a^{-1})^{-1}$ Hence, the proof

IV) Prove that the inverse of the product of two elements of a group G is the product of the inverses taken in reverse order. i.e. $(ab)^{-1} = b^{-1} a^{-1}, \forall a ; b \in G$.

Let a^{-1} and b^{-1} be the inverses of a and b in a group G respectively. Let e be the identity in G. Then $a^{-1} a = a a^{-1} = e$ and $b^{-1} b = b^{-1} b = e$

Consider, $(ab)(b^{-1} a^{-1}) = a(bb^{-1}) a^{-1}$ (∵ Associativity)

$= a(e) a^{-1}$ $(\because bb^{-1} = e)$

$= (ae) a^{-1}$

$= aa^{-1} = e$ ... (1)

Similarly, $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1} a) b$

$= b^{-1}(e) b$

$= b^{-1} b = e$ ... (2)

From equation (1) and equation (2), $(ab)(b^{-1}a^{-1}) = (b^{-1} a^{-1})(ab)$

$b^{-1} a^{-1}$ is the inverse of ab

$$(ab)^{-1} = b^{-1} a^{-1}$$

V) Prove that the cancellation laws hold in a group.i.e. If $a, b, c \in G$ then $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$.

Let a be any element in G and e be the identity element of a group G.

Now we have, $ab = ac$

Permultiplying by a^{-1} we get,

$a^{-1}(ab) = a^{-1}(ac)$

$(a^{-1} a) b = (a^{-1} a) c$ ... (∵ Associativity)

$eb = ec$ ... $(\because aa^{-1} = e)$

$b = c$

Similarly, $ba = ca$

$\Rightarrow$ $(ba) a^{-1} = (ca) a^{-1} \Rightarrow be = ce$

$\Rightarrow$ $b = c$ Hence, the proof

VI)If a, b are any elements of a group G then equation $ax = b$ and $ya = b$ have unique solutions in G.

Let $a \in G$

$\therefore \exists a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$

$a, a^{-1} \in G$ and $b \in G \Rightarrow a^{-1}b \in G$.

Now substituting $a^{-1}b$ for x in the L.H.S. given equation, we get,

$$ax = a(a^{-1}b) = (aa^{-1})b = eb = b$$

Thus, $x = a^{-1}b$ is the solution of $ax = b$

Let us suppose that x_1 and x_2 are two solutions of $ax = b$.

$\therefore \quad ax_1 = b$ and $ax_2 = b$

$\Rightarrow \quad b = ax_1 = ax_2 \Rightarrow x_1 = x_2$

Hence solution is unique.

Similarly prove for $ya = b$.

Examples

Example 9.4.1 *Let (A, x) be a monoid such that for every $x \in A$, $x * x = e$ where e is the identity element. Show that (A, *) is an abelian group.*

Solution : Given that, (A, *) is a monoid. Therefore it satisfies closure property, associativity and the existence of identity.

We have $x * x = e, \ \forall x \in A$

$$x = x^{-1}, \ \forall x \in A$$

Thus inverse exists for all $x \in A$.

$\therefore$ (A, *) is a group.

Consider,

$(a * b) * (b * a)$; $a, b, \in A = a * (b * b) * a$ ... ($\because$ * is asssociative)

$= a * e * a$ ... ($\because b * b = e$)

$= a * a = e$

and $(b * a) * (a * b) = b * (a * a) * b$

$= b * e * b = b * b = e$

Thus $b * a$ is the inverse of $a * b$

But $x = x^{-1}, \forall x$ in A

$b * a = a * b$

Thus, (A, *) is an abelien group.

Example 9.4.2 *Show that the set $G = \{a + b\sqrt{2} \ / \ a, b \in Q\}$ is a group with respect to addition.*

Solution : Given that, $G = \{a + b\sqrt{2} \ / \ a, b \in Q\}$

(i) Closure property : Let $x, y \in G$.

$$x = a + b\sqrt{2}, \ y = c + d\sqrt{2} \ a, b, c, d \in Q.$$

Consider, $x + y = (a + b\sqrt{2})(c + d\sqrt{2})$

$$= (a + c) + (b + d)\sqrt{2}, \qquad a + c \text{ and } b + d \in Q$$

$\Rightarrow \quad x + y \in G$

$\therefore$ G is closed w.r.t. +

(ii) Associativity : All elements of G are real numbers and associativity holds in R.

$\therefore$ Associativity holds in G.

(iii) Existence of the idenity : We have, $\theta \in Q$

$$0 + 0\sqrt{2} = 0 \in G$$

Consider

$$(0 + 0\sqrt{2}) + (a + b\sqrt{2}) = (0 + a)(0 + b)\sqrt{2} = a + b\sqrt{2}$$

and $(a + b\sqrt{2}) + (0 + 0\sqrt{2}) = a + b\sqrt{2}$

$\therefore$ 0 is the additive identity element in G.

(iv) Existence of the inverse : We have a, b and $-a, -b \in Q$.

$\therefore \quad a + b\sqrt{2}$ and $-a - b\sqrt{2}$ G.

Consider, $(a + b\sqrt{2}) + (-a - b\sqrt{2}) = (a - a) + (b - b)\sqrt{2}$

$$= 0 + 0\sqrt{2} = 0$$

and $\quad (-a - b\sqrt{2}) + (a + b\sqrt{2}) = (-a + a) + (-b + b)\sqrt{2}$

$$= 0 + 0\sqrt{2} = 0$$

$\therefore -a - b\sqrt{2}$ is the inverse of $a + b\sqrt{2}$. Hence G is a group with respect to addition.

Example 9.4.3 *If set Q_1 of all rational numbers other than 1 with $a * b = a + b - ab$. Show that (G, *) is a group.* **SPPU : Dec.-09,18, Marks 4**

Solution : We have, $a * b = a + b - ab, \forall a, b \in G$.

(i) Closure property : Let $a, b \in Q_1$, $a \neq 1$, $b \neq 1 \therefore ab \neq 1$.

$\therefore \quad a * b = a + b - ab \neq 1$ and $a * b \in Q_1$

Q_1 is closed w.r.t. *

(ii) Associativity : Let $a, b, c \in Q_1$.

$$(a * b) * c = (a + b - ab) * c$$
$$= (a + b - ab + c - (a + b - ab) c$$
$$= a + b - ab + c - ac - bc - abc \quad \ldots (1)$$
$$a * (b * c) = a * (b + c - bc)$$
$$= a + b + c - bc$$
$$= a (b + c - bc)$$
$$= a + b + c - bc - ab - ac + abc \quad \ldots (2)$$

$\therefore$ From equation (1) and equation (2)

$$(a * b) * c = a * (b * c)$$

$\therefore$ * is associative

(iii) Existence of the identity : Let e be the identity element in Q_1

$$a * e = a$$
$$\Rightarrow \quad a + e = ae = a$$
$$\Rightarrow \quad e - ae = 0$$
$$\Rightarrow \quad e (1 - a) = 0 \quad \text{(But } a \neq 1)$$
$$\Rightarrow \quad e = 0$$

$\therefore$ 0 is the identity element.

(iv) Existence of the inverse : Let $a \in Q_1$, $a \neq 1$.

Suppose $b \in Q$, is the inverse of a.

$$\therefore \quad a * b = e$$
$$\Rightarrow \quad a + b - ab = 0$$
$$\Rightarrow a + b (1 - a) = 0$$
$$b (1 - a) = -a$$
$$\Rightarrow \quad b = \frac{-a}{1-a} = \frac{a}{a-1} \neq 1 \text{ and } b \in Q_1.$$

$\therefore$ The inverse exist for all a in Q_1.

Thus, $(Q_1, *)$ is a group.

Example 9.4.4 *If $S = \{(a, b) \mid a \neq 0, a, b, \in \Re\}$ and $(a, b) * (c, d) = (ac, bc + d)$ then show that G is a group but not abelian group w.r.t.**

Solution : (i) Closure property : Let $(a, b)\ (c, d) \in S$; $a \neq 0$; $c \neq 0$

$\therefore\ ac \neq 0.$

$(a, b) * (c, d) = (ac, bc + d) \in S$

(ii) Associativity : Let (a, b), (c, d) and $(e, f) \in S$

Consider

$$[(a, b) * (c, d)] * (e, f) = [ac, bc + d) * (e, f)$$

$$= ([ac]\ e, [bc + d]\ e + f)$$

$$= (ace, bce + de + f) \quad \text{... (1)}$$

and $(a, b) * [(c, d) * (e, f)] = (a, b) * [ce, de + f]$

$$= [ace, b\ (ce) + de + f]$$

$$= (ace, bce + de + f) \quad \text{... (2)}$$

From equation (1) and equation (2) * is associative operation.

(iii) Existence of the identity : Let $(a, b) \in S$ and $(x, y) \in S$, $x \neq 0$

Consider, $(a, b) * (x, y) = (a, b)$

$$(ax, bx + y) = (a, b)$$

$\Rightarrow \quad ax = a$ and $bx + y = b$

$\Rightarrow \quad x = 1$ and $b + y = b$

$\Rightarrow \quad y = 0$

Similarly, $(x, y) * (a, b) = (a, b)$

$(1, 0)$ is the identity element in S.

(iv) Existence of the inverse : Let (a, b) and $(c, d) \in S$.

$$(a, b) * (c, d) = (1, 0)$$

$$(ac, bc + d) = (1, 0)$$

$\Rightarrow \quad ac = 1$

and $\quad bc + d = 0$

$\therefore \quad c = \dfrac{1}{a}$

and $\quad d = -bc = -\dfrac{b}{a}$ and $c \neq 0$

Thus $\left(\frac{1}{a}, -\frac{b}{a}\right)$ is the inverse of (a, b) in S.

Thus (S, *) is a group.

Consider, $(a, b) * (c, d) = (ac, bc + d)$

and $(c, d) * (a, b) = (ca, da + b)$

$\Rightarrow$ $(a, b) * (c, d) \neq (c, d) * (a, b)$

Thus, (S, *) is not an abelian group.

Example 9.4.5 *If* $G = \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \Big/ a \text{ is non zero real number}\right\}$ *show that G is an abelian group w.r.t. matrix multiplication.*

Solution : **(i) Closure property :** Let $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$ be any two elements of G where a, b are non-zero real numbers.

$$AB = \begin{bmatrix} ab & 0 \\ 0 & 0 \end{bmatrix}, \text{ ab is non zero real number.}$$

$\therefore$ $AB \in G$

$\therefore$ G is closed w.r.t. multiplication.

(ii) Associativity : We know that matrix multiplication is associative.

(iii) Existence of the identity : Let $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in G$

Consider $AE = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = A$

and $EA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = A$

$\therefore$ $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the identity element w.r.t. matrix multiplication.

(iv) Existence of the inverse :

Let $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \in G$

Consider $AB = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} ab & 0 \\ 0 & 0 \end{bmatrix}, \ ab \neq 0$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad ab = 1 \Rightarrow b = \frac{1}{a}$

$$B = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{bmatrix} \in G$$

And $\quad AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ B is the inverse of A in G.

(v) Commutativity :

$$AB = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ba & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad AB = BA$

Hence G is an abelian group.

Example 9.4.6 *If* $G = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix} \Big/ x \text{ is non zero real number} \right\}$. *Show that G is an abelian group w.r.t. matrix multiplication.*

Solution : (i) Closure property :

Let $\quad A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ and $B = \begin{bmatrix} y & y \\ y & y \end{bmatrix} \in G$

then $\quad AB = \begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} y & y \\ y & y \end{bmatrix} = \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} \in G$

(ii) Associativity : We know that any matrix multiplication is associative.

(iii) Existence of the identity : Let $E = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$ such that

$AE = A \Rightarrow$

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2xe & 2xe \\ 2xe & 2xe \end{bmatrix} = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$$

$$\Rightarrow \quad 2xe = e$$

$$\Rightarrow \quad 2x = 1$$

$$\Rightarrow \quad x = \frac{1}{2}$$

$$\therefore \quad E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ is the identity element of G.}$$

(iv) Existence of the inverse : Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \in G$ and

$B = \begin{bmatrix} y & y \\ y & y \end{bmatrix} \in G$ such that $AB = E$

$$\Rightarrow \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \quad 2xy = \frac{1}{2}$$

$$\Rightarrow \quad xy = \frac{1}{4}$$

$$\Rightarrow \quad y = \frac{1}{4x}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix}$$

(v) Commutative property :

We have, $AB = BA = \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} \in G$

Thus G is an abelian group w.r.t. matrix multiplication.

Example 9.4.7 *If* $G = \left\{ \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \Big/ \alpha \text{ is non zero real number} \right\}$, *then G is a group w.r.t. matrix multiplication.*

Solution : (i) Closure property :

Let A and $B \in G$

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \text{ and } B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$AB = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \in G$$

(ii) Matrix multiplication is associative.

(iii) Existence of the identity : As $0 \in R$

$$E = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AE = EA = E$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is the idenity element in G.}$$

(iv) Existence of the inverse :

Let $A \in G$ and $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$B = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \in G$$

Consider $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BA$

$\Rightarrow$ B is the inverse of A in G.

Thus G is a group w.r.t. matrix multiplication.

Example 9.4.8 *Let G be the set of all non-zero real numbers and let $a * b = \frac{ab}{2}$. Show that (G, *) is an abelian group.* **SPPU : Dec.-08, May-17, Marks 6**

Solution : (i) Closure property : Let a, b ∈ G.

$$a * b = \frac{ab}{2} \in G \text{ as } ab \neq 0$$

(ii) Associativity : Let a, b, c ∈ G

Consider $a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{4} = \frac{abc}{4}$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{(ab)c}{4} = \frac{abc}{4}$$

⇒ * is associative in G.

(iii) Existence of the identity : Let a ∈ G and ∃ e such that

$$a * e = \frac{ae}{2} = a$$

$$\Rightarrow \quad ae = 2a$$

$$\Rightarrow \quad e = 2$$

∴ 2 is the identity element in G.

(iv) Existence of the inverse : Let a ∈ G and b ∈ G such that a * b = e = 2.

$$\Rightarrow \quad \frac{ab}{2} = 2$$

$$\Rightarrow \quad ab = 4$$

$$\Rightarrow \quad b = \frac{4}{a}$$

∴ The inverse of a is $\frac{4}{a}$, ∀ a ∈ G.

(v) Commutativity : Let a, b ∈ G

$$a * b = \frac{ab}{2}$$

and $b * a = \frac{ba}{2} = \frac{ab}{2}$

⇒ * is commutative

Thus, (G, *) is an abelian group.

Example 9.4.9 *Show that the four fourth roots of unity namely 1, –1, i, –i form a group w.r.t. multiplication.*

Solution : Let G = { 1, –1, i, –i } . Consider the following table.

	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	– i	–1	1
–i	–i	i	1	–1

(i) Closure property : All elements of table belongs to G.

∴ G is closed w.r.t. multiplication.

(ii) Associativity : We know that multiplication of complex numbers is associative.

(iii) Existence of the identity : From table

$$(1)\,(1) = 1 \qquad\qquad (-1)\,1 = -1$$

$$(i)\,1 = i \qquad \text{and} \qquad (-i)\,1 = -i$$

∴ 1 is the identity element in G.

(iv) Existence of the inverse : From table,

$$1^{-1} = 1,$$

$$(-1)^{-1} = 1,$$

$$(i)^{-1} = -i,$$

and $$(-i)^{-1} = i$$

∴ The inverse exists for all elements of G. Thus, 'G' is a group.

Example 9.4.10 *Show that the set G = {1, w, w^2} where w is the cube root of unity is a group with respect to multiplication.* **SPPU : Dec.-12, Marks 6**

Solution : Consider the following composition table of G.

	1	w	w^2
1	1	w	w^2
w	w	w^2	1
w^2	w^2	1	w

(i) Closure property : From table all elements belong to G.

$\therefore$ G is closed w.r.t. multiplication.

(ii) Associativity : As all elements of G are complex numbers and multiplication of complex numbers is associative.

$\therefore$ (G, x) is associative.

(iii) Existence of the identity : From the composition table

$$1\ (1) = 1\ ,\ 1\ (w) = 1\ ,\ 1\ (w^2) = w^2$$

$\therefore$ 1 is multiplicative identity in G.

(iv) Existence of the inverse : From table, the inverses of 1, w, w^2 are 1, w^2 , w respectively. Thus (G, x) is a group.

Example 9.4.11 *Prove that G = {0, 1, 2, 3, 4, 5} = $\mathbb{Z}_6$ is an abelian group of order 6. w.r.t. addition modulo 6.*

Solution : Let us consider the composition table

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(i) Closure property : From the composition table all elements belong to $\mathbb{Z}_6$.

$\therefore$ $\mathbb{Z}_6$ is closed w.r.t. $+_6$

(ii) Associativity : Let a, b, c $\in \mathbb{Z}_6$

$a +_6 (b +_6 c)$ = The smallest non zero remainder when a + (b + c) is divided by 6.

= The smallest non zero remainder when (a + b) + c is divided by 6.

$= (a +_6 b) +_6 c$

$\therefore$ $+_6$ is associative.

(iii) Existence of the identity : We have $0 \in \mathbb{Z}_6$

If $a \in \mathbb{Z}_6$ then $a +_6 0 = a = 0 +_6 a$

$\therefore$ 0 is the additive identity.

(iv) Existence of the inverse : From the composition table, the inverses 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 respectively.

(v) Commutativity : Let $a, b \in \mathbb{Z}_6$

$$a +_6 b = b +_6 a \qquad \text{... } (\because \text{ From table})$$

Thus $(\mathbb{Z}_6, +_6)$ is an abelian group.

Example 9.4.12 *Let (A, *) be a group. Show that (A, *) is abelian group iff $a^2 * b^2 = (a * b)^2$* **SPPU : Dec.-12, Marks 4**

Solution : Let (A, *) be an abelian group.

$$a * b = b * a$$

Consider,
$$\begin{aligned} a^2 * b^2 &= (a * a) * (b * b) \\ &= a * (a * b) * b \qquad (* \text{ is associative}) \\ &= a * (b * a) * b \\ &= (a * b) * (a * b) \\ a^2 * b^2 &= (a * b)^2 \end{aligned}$$

Conversely suppose

$$a^2 * b^2 = (a * b)^2 \qquad (1)$$

$$\text{L.H.S.} = a^2 * b^2 = (a * a) * (b * b) = a * (a * b) * b$$

$$\text{R.H.S.} = (a * b)^2 = (a * b) * (a * b) = a * (b * a) * b$$

Equation (1) $\Rightarrow$

$$a * (a * b) * b = a * (b * a) * b$$

By cancellation laws,

$$a * b = b * a$$

$\Rightarrow$ (A, *) is an abelian group.

Examples

Example 9.4.13 *Show that the set of all idempotents in a commutative monoid S is a submonoid of S.* **SPPU : Dec.-12, Marks 4**

Solution : We know that an element $x \in S$ is called an idempotent if $x * x = x$. Let T be the set of all idempotents in S.

i) For any $x, y \in T$, Consider

$$(x * y) * (x * y) = ((x * y) * x) * y$$

$$= (y * (x * x)) * y$$
$$= (y * x) * y$$
$$= (x * y) * y$$
$$= x * (y * y)$$
$$= x * y$$

$\therefore$ $x * y \in T$ for all $x, y \in T$

T is closed w.r.t. '$*$'

ii) Associativity, Let $x, y, z \in T \subseteq S$

$\therefore$ $x * (y * z) = (x * y) * z$ in T

$*$ is associative in T

iii) Existence of the identity :

Let e be the identity element in S

As $e * e = e \quad \therefore \quad e \in T$

$\therefore$ e is the identity element in T.

$\therefore$ T is a monoid

Thus T is a submonoid of S.

Example 9.4.14 *Prove that the set $\mathbb{Z}$ of all integers with binary operation $*$ defined by $a * b = a + b + 1$ such that $\forall\ a, b \in \mathbb{Z}$ is an abelian group :* **SPPU : May-14, Marks 4**

Solution :

We have $a * b = a + b + 1 \ , \ \forall\ a, b \in \mathbb{Z}$

i) Closure property

For $a, b \in \mathbb{Z} \Rightarrow a + b + 1 \in \mathbb{Z} \Rightarrow a * b \in \mathbb{Z}$

$\therefore$ $\mathbb{Z}$ is closed w.r.t. $*$

ii) Associative : Let $a, b, c \in \mathbb{Z}$

$$(a * b) * c = (a + b + 1) * c$$
$$= a + b + 1 + c + 1 = a + b + c + 2 \qquad \ldots (1)$$

and $a * (b * c) = a * (b + c + 1)$
$$= a + b + c + 1 + 1 = a + b + c + 2 \qquad \ldots (2)$$

From equation (1) and equation (2) $(a * b) * c = a * (b * c)$

$\therefore$ $*$ is associative in $\mathbb{Z}$

iii) Existence of the identity :

Let e be the identity in $\mathbb{Z}$

$\therefore$ For any $a \in \mathbb{Z}$, $a * e = e * a = a$

$$\Rightarrow a + e + 1 = a$$

$$\Rightarrow e + 1 = a$$

$$\Rightarrow e = -1 \text{ is the identity element in } \mathbb{Z}$$

iv) Existence of the inverse :

Let $a \in \mathbb{Z}$. Suppose $b \in \mathbb{Z}$ is the inverse of a in $\mathbb{Z}$

$$\therefore \quad a * b = e$$

$$a + b + 1 = -1$$

$$a + b = -2$$

$$b = -a - 2 \in \mathbb{Z}$$

$\therefore$ The inverse exists for all $a \in \mathbb{Z}$

Thus $(\mathbb{Z}, *)$ is a group.

Now consider

$$a * b = a + b + 1 \qquad (\because a + b = b + a)$$

$$= b + a + 1$$

$$= b * a \ , \ \forall \ a, b \in \mathbb{Z}$$

$\therefore$ $*$ is commutative in $\mathbb{Z}$

$\therefore$ $(\mathbb{Z}, *)$ is an abelian group.

9.5 Modulo m

SPPU : Dec.-11, May-08

I) Let and b are any integers and m is a fixed positive integer then the addition modulo m denoted by $a +_m b$ and defined as $a +_m b = r \ ; \ 0 \le r \le m$

Where r is the least non negative remainder when a + b is divided by m.

e.g. $5 +_3 9 = 2$ as $5 + 9 = 14$ and $14 = 3 \times 4 + 2$

$15 +_5 25 = 0$ as $15 + 25 = 40$ and $40 = 5 \times 8 + 0$

II) Let a and b be any integers and m is a fixed positive integer. Then the multiplication modulo m is denoted by $a \times_m b$ and defined as

$$a \times_m b = r \ ; \ 0 \le r \le m$$

Where r is the least non negative remainder when $a \times b$ is divided by m.

e.g. $3 \times_4 5 = 3$ as $3 \times 5 = 15$ and $15 = 4 \times 3 + 3$

$9 \times_6 4 = 0$ as $9 \times 4 = 36$ and $36 = 6 \times 6 + 0$

III) If a and be are any two integers such that a – b is divisible by a fixed positive integer m, is called "a congruent to b modulo m".

It is denoted by $a \equiv b \pmod{m}$

Note :

1) If $a \equiv b \pmod{m}$ iff a – b is divisible by m

i.e. $a - b = pm$; for some integer p.

$a = pm + b$

2) If $a \equiv b \pmod{m}$ then $b \equiv a \pmod{m}$

3) For any $a \in \mathbb{Z}$, $a \equiv a \pmod{m}$

e.g.

i) $5 \mid (12-2) \Rightarrow 12 \equiv 2 \pmod{5}$

$5 \mid (12-2) \Rightarrow 2 \equiv 12 \pmod{5}$

ii) $8 \mid (30-6) \Rightarrow 30 \equiv 6 \pmod{8}$

iii) $5 \mid (9-9) \Rightarrow 9 \equiv 9 \pmod{5}$

Examples

Example 9.5.1 *Show that $(G, +_8)$ is an abelian group where $G = \{0, 1, 2, 3, 4, 5, 6, 7\}$*

Solution : Consider the following table for $+_8$

$+_8$	0	1	2	3	4	5	6	7
0	$\boxed{0}$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	$\boxed{0}$
2	2	3	4	5	6	7	$\boxed{0}$	1
3	3	4	5	6	7	$\boxed{0}$	1	2
4	4	5	6	7	$\boxed{0}$	1	2	3
5	5	6	7	$\boxed{0}$	1	2	3	4
6	6	7	$\boxed{0}$	1	2	3	4	5
7	7	$\boxed{0}$	1	2	3	4	5	6

1) Closure property : Every element of the table belongs to G

i.e. $a +_8 b \in G$; $\forall$ a, b $\in$ G

$\therefore$ $+_8$ is closed in G.

2) Associativity : For any a, b, c $\in$ G

$$a +_8 (b +_8 c) = (a +_8 b) +_8 c$$

$\therefore$ $+_8$ is associative in G.

3) Existence of the identity

By observing first row and first column,

For any $a \in G$, $\exists\, 0 \in G$ such that

$$a +_8 0 = 0 +_8 a = a$$

Hence 0 is the identity element for '$+_8$' in G

4) Existence of the inverse

Identify the identity element 0 from table.

$\therefore$ By table ; As $0 +_8 0 = 0$; 0 is the inverse of 0

As $1 +_8 7 = 0$; 1 is the inverse of 7

As $7 +_8 1 = 0$; 7 is the inverse of 1

As $2 +_8 6 = 0$; 2 is the inverse of 6

As $6 +_8 2 = 0$; 6 is the inverse of 2

As $3 +_8 5 = 0$; 5 is the inverse of 3

As $5 +_8 3 = 0$; 3 is the inverse of 4

As $4 +_8 4 = 0$; 4 is the inverse of 4

Hence, inverse exists in G for all $a \in G$

5) Commutative property : From table, $\forall$ a, b $\in$ G

$$a +_8 b = b +_8 a$$

$\therefore$ '$+_8$' is commutative in G.

Hence, (G, $+_8$) is an abelian group.

Example 9.5.2 *Show that (G, $\times_7$) is an abelian group, where G = {1, 2, 3, 4, 5, 6}*

Solution : Consider the following table

$\times_7$	1	2	3	4	5	6
1	$\boxed{1}$	2	3	4	5	6
2	2	4	6	$\boxed{1}$	3	5
3	3	6	2	5	$\boxed{1}$	4
4	4	$\boxed{1}$	5	2	6	3
5	5	3	$\boxed{1}$	6	4	2
6	6	5	4	3	2	$\boxed{1}$

i) Closure Property

Every element of table belongs to G

i.e. $a \times_7 b \in G$ for all $a, b \in G$

$\therefore$ $\times_7$ is closed in G.

ii) Associativity

For any $a, b, c \in G$

$a \times_7 (b \times_7 c) = (a \times_7 b) \times_7 c$

$\therefore$ $\times_7$ is associative in G.

iii) Existence of the Indentity

By observing the first row and the first column,

For any $a \in G$, $\exists$ $1 \in G$ such that

$$a \times_7 1 = 1 \times_7 a = a$$

$\therefore$ 1 is the identity element in G.

iv) Existence of the Inverse

Identify the identity elements from table

$\therefore$ By table,

As $1 \times_7 1 = 1$, 1 is the inverse of 1

As $2 \times_7 4 = 4 \times_7 2 = 1$, $\therefore$ 2 is the inverse of 4 and 4 is the inverse of 2

As $3 \times_7 5 = 5 \times_7 3 = 1$, $\therefore$ 3 is the inverse of 5 and 5 is the inverse of 3

As $6 \times_7 6 = 1$, $\therefore$ 6 is the inverse of 6

Hence, inverse exists in G for all $a \in G$

v) Commutative Property

From table, for all $a, b \in G$

$$a \times_7 b = b \times_7 a$$

$\therefore$ $\times_7$ is commutative in G.

Hence $(G, \times_7)$ is an abelian group.

Example 9.5.3 *Is $(G, \times_6)$ an abelian group? Where $G = \{1, 2, 3, 4, 5\}$*

Solution : Here $2, 3 \in G$ but $2 \times_6 3 = 0 \notin G$

$\therefore$ $\times_6$ is not binary operation in G

i.e. $\times_6$ is not closed in G.

Hence $(G, \times_6)$ is not a group.

Thus $(G, \times_6)$ is not a abelian group.

Example 9.5.4 *Show that $(z_6, +)$ is an abelian group.* **SPPU : May-08**

Solution : Let z_6 be the set of residue classes modul o6.

$\therefore$ $z_6 = \{[0], [1], [2], [3], [4], [5]\}$

Or $z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

Consider the following table

+	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\boxed{\bar{0}}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\boxed{\bar{0}}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\boxed{\bar{0}}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\boxed{\bar{0}}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{5}$	$\boxed{\bar{0}}$	$\bar{1}$	$\bar{2}$	$\bar{1}$
$\bar{5}$	$\bar{5}$	$\boxed{\bar{0}}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$

i) Closure Property

Every element of a table belongs to z_6

i.e. $\bar{a} + \bar{b} \in z_6 \quad \forall\, \bar{a}, \bar{b} \in z_6$

$\therefore$ + is closed in z_6

ii) Associativity

By table, for any $\bar{a}, \bar{b}, \bar{c} \in z_6$

$$\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$$

$\therefore$ '+' is associative in z_6

iii) Existence of the Identity

By observing the first row and the first column,

For any $\bar{a} \in z_6$, $\exists\ \bar{0} \in z_6$ such that

$$\bar{a} + \bar{0} = \bar{0} + \bar{a} = \bar{a}$$

$\therefore$ $\bar{0}$ is the identity element in z_6.

iv) Existence of the Inverse

From table, identify the identity elements

As $\bar{0} + \bar{0} = \bar{0}$, $\bar{0}$ is the inverse of $\bar{0}$

As $\bar{1} + \bar{5} = \bar{5} + \bar{1} = \bar{0}$, $\bar{1}$ is the inverse of $\bar{5}$ and $\bar{5}$ is the inverse of $\bar{1}$

As $\bar{2} + \bar{4} = \bar{4} + \bar{2} = \bar{0}$, $\bar{2}$ is the inverse of $\bar{4}$ and $\bar{4}$ is the inverse of $\bar{2}$

As $\bar{3} + \bar{3} = \bar{0}$, $\bar{3}$ is the inverse of $\bar{3}$

$\therefore$ Every element of z_6 has inverse in z_6.

v) Commutative Property

From table, For all $\bar{a}, \bar{b} \in z_6$

$$\bar{a} + \bar{b} = \bar{b} + \bar{a}$$

$\therefore$ '+' is commutative in z_6.

Hence $(z_6, +)$ is an abelian group.

Example 9.5.5 *Let $z = \{0, 1, 2, 3, 4, \ldots (n-1)\}$ and '$\Diamond$' be a binary operation such that $a \Diamond b$ = remainder of abelian when divided by n. Construct a table for $n = 4$, Is $(z_4, \Diamond)$ is groupoid, monoid, semigroup and abelian group ?* **SPPU : Dec.-11**

Solution :

We have $z_4 = \{0, 1, 2, 3\}$

Table of z_4 is

$\Diamond$	0	1	2	3
0	$\boxed{0}$	0	0	0
1	0	1	2	3
2	0	2	$\boxed{0}$	2
3	0	3	2	1

i) From table, for any $a, b \in z_4$, $a \Diamond b \in z_4$

$\therefore$ $(z_4, \Diamond)$ is a groupoid and $\Diamond$ is a binary operation in z_4 i.e. $\Diamond$ is closed in z_4.

ii) Semigroup : By table for any a, b, c, $\in z_4$

$$a \Diamond (b \Diamond c) = (a \Diamond b) \Diamond c$$

$\therefore$ $\Diamond$ is associative in z_4

OR Let $a \Diamond b = r$ and $b \Diamond c = t$

$\therefore$ $ab = 4p + r$ and $bc = 4l + t$

$\therefore$ $(a \Diamond b) \Diamond c = r \Diamond c = s$ where $rc = 4q + s$

$a \Diamond (b \Diamond c) = a \Diamond t = k$ where $at = 4m + k$

Prove that $s = k$

$$a(bc) = 4\,al + at = 4\,al + 4m + k \quad \ldots (1)$$

$$(ab)c = (4p + r)c = 4pc + rc = 4pc + 4q + s \quad \ldots (2)$$

Equations (1) and equation (2) are equal

$$\Rightarrow 4\,al + 4m + k = 4pc + 4q + s$$

$$\Rightarrow k = s$$

Hence $(a \Diamond b) \Diamond c = a \Diamond (b \Diamond c)$

Thus $(z_4, \Diamond)$ is a semigroup

iii) Monoid

By observing the first row and the first column of table

For any $a \in z_4$, $\exists$ $0 \in z_4$ such that

$$a + 0 = 0 + a = a$$

$\therefore$ 0 is the identity element in z_4

Thus from (i), (ii) and (iii), $(z_4, +)$ is a monoid.

iv) Existence of the inverse

From table, identify the identity elements.

x is the inverse of 3 in z_4

$\therefore$ $x \Diamond 3 = 0$ and $x \Diamond 3 = 4$ if $3x = 4m + r$

$\therefore$ $3x - 4m = 0$

$\Rightarrow$ $3x = 4m$

$\Rightarrow$ $x = \dfrac{4m}{3}$ which is an integer

For $m = 1, 2,$ x is not an integer

For $m = 3,$ $x = 4$ which is an integer

But $x = 4 \notin z_4$

Thus 3 does not have inverse in z_4

$\therefore$ $(z_4, \lozenge)$ is not a group

Hence $(z_4, \lozenge)$ is not an abelian group.

9.6 Permutation Group

I) Let $S = \{1, 2, 3, ..., n\}$ be a finite set with n distinct elements. If $f : S \to S$ is a bijective function then f is called a permutation of degree n.

Let $f(a_1) = b_1, \quad f(a_2) = b_2, \quad f(a_3) = b_3, \quad .... f(a_n) = b_4$

Then the permutation is denoted by

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & .... & a_n \\ b_1 & b_2 & b_3 & .... & b_n \end{pmatrix} \begin{matrix} \to \text{Elements in Domain} \\ \to \text{Elements in Co-domain} \end{matrix}$$

II) The permutation corresponding to the identity function is called the identity permutation

$$I = \begin{pmatrix} a_1 & a_2 & a_3 & .... & a_n \\ b_1 & b_2 & b_3 & .... & b_n \end{pmatrix}$$

III) If $f_1 = \begin{pmatrix} a_1 & a_2 & a_3 & .... & a_n \\ b_1 & b_2 & b_3 & .... & b_n \end{pmatrix}$ and $f_2 = \begin{pmatrix} b_1 & b_2 & b_3 & .... & b_n \\ c_1 & c_2 & c_3 & .... & c_n \end{pmatrix}$

Then the product of two permutations is given by

$$f_1 f_2 = \begin{pmatrix} a_1 & a_2 & a_3 & .... & a_n \\ c_1 & c_2 & c_3 & .... & c_n \end{pmatrix}$$

(To write product use $a_1 \to b_1 \to c_1 \Rightarrow a_1 \to c_1$ and so on)

Example 9.6.1 *Show that the set of all permutations on S = {1, 2, 3} forms a group w.r.t. permutation multiplication.*

Solution :

Let $P_3 = \{f / f \text{ is a permutation degree } 3\}$

$$P_3 = \left\{ \begin{matrix} (123) & (123) & (123) & (123) & (123) & (123) \\ (123) & (132) & (213) & (231) & (312) & (321) \end{matrix} \right\}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ I & f_1 & f_2 & f_3 & f_4 & f_5 \end{matrix}$$

Consider the following table

0	I	f_1	f_2	f_3	f_4	f_5
I	I	f_1	f_2	f_3	f_4	f_5
f_1	f_1	I	f_3	f_2	f_5	f_4
f_2	f_2	f_4	I	f_5	f_1	f_3
f_3	f_3	f_5	f_1	f_4	I	f_2
f_4	f_4	f_2	f_5	I	f_3	f_1
f_5	f_5	f_3	f_4	f_1	f_2	I

i) Closure Property : Each element of a table belongs to P_3

e.g. for any $f, g \in P_3$, $fog \in P_3$

$\therefore$ P is closed w.r.t. composition of permutations.

ii) Associativity

For any $f, g, h \in P_3$

$$f o (g o h) = (fog) \cdot h$$

Hence it is associative

iii) Existence of the Identity Element

By observing, first row and the first column of table

For any $f \in P_3$, $\exists\ I \in P_3$

such that $foI = Iof = f$

$\therefore$ I is the identity permutation.

iv) Existence of the Inverse

From table, identify the identity element I,

As $I o I = I \Rightarrow I^{-1} = I$

$$f_1 o f_1 = I \Rightarrow f_1^{-1} = f_1$$

$$f_2 o f_2 = I \Rightarrow f_2^{-1} = f_2$$

$$f_3 o f_4 = I = f_4 o f_3 \Rightarrow f_3^{-1} = f_4 \text{ and } f_4^{-1} = f_3$$

$$f_5 o f_5 = I \Rightarrow f_5^{-1} = f_5$$

$\therefore$ Every element of P has inverse in P_3.

Hence $(P_3, \cdot)$ is a group.

But $f_1 \circ f_2 \neq f_3$ and $f_2 \circ f_1 = f_4$

$\Rightarrow$ $f_1 \circ f_2 \neq f_2 \circ f_1 \Rightarrow$ It is not commutative.

$\Rightarrow$ $(P_3, \circ)$ is not abelian group.

$(P_3, \circ)$ is the smallest non abelian group.

9.7 Complexes and Subgroups

SPPU : May-18

Depending upon the nature of subset of a group there are two types of subsets, which are given below.

9.7.1 Complex of a Group

Let $(G, *)$ be a group. Any non empty subset of a group G is called a complex of the group.

e.g. $H_1 = \{1, 2, 3, 4, 5\}$, $H_2 = \{1, 2, 3, ...\}$, $H_3 = \mathbb{Z}$

are complexes of a group $(\Re, +)$

9.7.2 Subgroup

Let $(G, *)$ be a group. A non empty subset H of a group G, is said to be subgroup of G if $(H, *)$ itself is a group.

Examples

1) $(\mathbb{Z}, +)$ is a subgroup of $(\Re, +)$
2) $(\Re, +)$ is a subgroup of $(\mathbb{C}, +)$
3) $(N, +)$ is not a subgroup of $(\mathbb{Z}, +)$
4) $(\Re_0, \cdot)$ is a subgroup of $(\mathbb{C}_0, \cdot)$
5) $(\{1\}, \cdot)$ is a subgroup of $(Q_0, \cdot)$, $(\Re_0, \cdot)$, $(\mathbb{C}, \cdot)$
6) $(\Re, +)$ is a subgroup of $(\Re, +)$

Note :

1) $\{e\}$ and $\{G\}$ are subgroups of group G : These subgroups are called improper subgroups.
2) Subgroups which are not improper are called proper subgroups.

9.7.3 Properties of Subgroup

Theorem I) : Prove that the identity of a subgroup is the same as that of the group.

Let H be a subgroup of the group G.

Let e and e' be the identities of G and H respectively.

Now, $a \in H \Rightarrow e' a = a$ ... [$\because$ e' is identity of H]

also $a \in H \Rightarrow a \in G \Rightarrow ea = a$ [$\because$ e is identity of G]

$$e'a = ea$$

$\therefore$ in G we have,

$\Rightarrow$ $e' = e$ ... [by right cancellation law in G]

Theorem II) : Prove that the inverse of any element of a subgroup is the same as the inverse of the same regarded as an element of the group.

Let H be a subgroup of the group G.

Let e be the identity of G as well as of H.

Let $a \in H$, suppose b is the inverse of a in H and c is the inverse of a in G. Then we have,

$$ba = e$$

and $$ca = e$$

$\therefore$ in G we have, $ba = ca \Rightarrow b = c$

Theorem III) : Prove that the order of any element of a subgroup is the same as the order of the element regarded as a member of the group.

Ans. : Let H be a subgroup of the group G.

Let $a \in H$, But $a \in G$ and $a^n = e$ in G.

$a^n = e$ in H also. Hence the proof.

Theorem IV) : A necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup is that $a \in H$, $b \quad H \Rightarrow ab^{-1} \in H$ where b^{-1} is the inverse of b in G.

Suppose H is a subgroup of G.

Let $a \in H$, $b \in H$. Now each element of H must possess inverse because H itself is a group.

$$b \in H \Rightarrow b^{-1} \in H$$

Further H must be closed with respect to multiplication i.e. the composition in G.

$\therefore\ a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H$

(i) The condition is sufficient. Now it is given that $a \in H, b \in H \Rightarrow ab^{-1} \in H$. We have to prove that H is a subgroup of G.

(ii) Existence of identity :

We have $a \in H, a \in H \Rightarrow aa^{-1} \in H$ ... By given condition

$\Rightarrow e \in H$

Thus the identity e is an element of H.

(iii) Existence of inverse :

Let a be any element of H. Then by the given condition, we have,

$e \in H, a \in H \Rightarrow ea^{-1} \in H \Rightarrow a^{-1} \in H.$

Thus each element of H possesses inverse.

(iv) Closure property : Let $a, b \in H$. Then as shown above $b \in H \Rightarrow b^{-1} \in H$. Therefore applying the given condition we have,

$a \in H, b^{-1} \in H \Rightarrow a\,(b^{-1})^{-1} \in H \Rightarrow ab \in H.$

(v) Associativity : The elements of H are also the elements of G. The composition in G is associative. Therefore, it must also be associative in H.

Hence H itself is a group for the composition in G. Therefore, H is a subgroup of G.

Theorem V) : If H_1 and H_2 are two subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G.

Let H_1 and H_2 be any two subgroups of G. Then $H_1 \cap H_2 \neq \phi$ Since at least the identity element e is common to both H_1 and H_2.

In order to prove that $H_1 \cap H_2$ is a subgroup it is sufficient to prove that

$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1}\ H_1 \cap H_2$

Now, $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$a \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

But H_1, H_2 are subgroups.

$\therefore\ a \in H_1, b \in H_1, \Rightarrow ab^{-1} \in H_1$

$a \in H_2, b \in H_2 \Rightarrow ab^{-1} \in H_2$

Finally $ab^{-1} \in H_1, ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Thus, we have shown that, $a \in H_1 \cap H_2$, $b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$ Hence $H_1 \cap H_2$ is a subgroup of G.

Theorem VI) : Show that the union of two subgroups is a subgroup if and only if one is contained in the other.

Suppose H_1 and H_2 are two subgroups of a group G.

Let $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Then $H_1 \cup H_2 = H_2$ or H_1.

But H_1, H_2 are subgroups and therefore, $H_1 \cup H_2$ is also a subgroup. Conversly suppose $H_1 \cup H2$ is a subgroup. To prove that $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Let us assume that H_1 is not a subset of H_2 and H_2 is also not a subset of H_1.

Now H_1 is not a subset of $H_2 \Rightarrow \exists\, a \in H_1$ and $a \notin H_2$ ... (1)

and H_2 is not a subset of $H_1 \Rightarrow \exists\, b \in H_2$ and $B \notin H_1$ ... (2)

From equation (1) and (2).

We have, $a \in H_1 \cup H_2$ and $b \in H_1 \cup H_2$

Since $H_1 \cup H_2$ is a subgroup, therefore $ab = c$ (say) is also an element of $H_1 \cup H_2$.

But $ab = c \in H_1 \cup H_2 \Rightarrow ab = c \in H_1$ or H_2

Suppose $ab = c \in H_1$

Then $b = a^{-1} c \in H_1$, ... [$\because H_1$ is a subgroup. $\therefore a \in H_1 \Rightarrow a^{-1} \in H_1$]

But from (2), we have $b \in H_1$. Thus we get the contradiction.

Again suppose $ab = c \in H_2$

Then, $a = cb^{-1} \in H_2$ ... [$\because H_2$ is a subgroup, therefore $b \in H_2, \Rightarrow b^{-1} \in H_2$]

But from (1), we have $a \notin H_2$. Thus here also we get a contradiction.

Hence either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Example 9.7.1 *Is union of two subgroups is a subgroup? If not, give example.*

Solution : The union of two subgroups is not necessarily a subgroup.

Let $H_1 = \{...., -4, -2, 0, 2, 4, 6, ...\}$

$H_2 = \{...., -6, -3, 0, 3, 6, ...\}$ are subgroup of $(\mathbb{Z}, +)$

Now $H_1 \cup H_2 = \{-6, -4, -3, -2, 0, 2, 3, 4, 6, ...\}$

$2, 3 \in H_1 \cup H_2$ but $2 + 3 = 5 \notin H_1 \cup H_2$

$\therefore$ + is not binary operation on $H_1 \cup H_2$

$\Rightarrow$ $H_1 \cup H_2$ is not a subgroup of G.

9.8 Cosets

Let (G, $*$) be a group and H be any subgroup of G.

Let $a \in G$ be any element, then the set

$$H * a = \{h * a / \forall h \in H\} \text{ is called a right coset of H in G.}$$

and $$a * H = \{a * h / \forall h \in H\} \text{ is called a left coset of H in G.}$$

Note :

1) $H * a$ and $a * H$ are subsets of G.
2) If (G, $*$) is an abelian group then $H * a = a * H$ in G.

e.g. 1) Let ($\mathbb{Z}$, +) is a group and

$$H = \{...., -10, -5, 0, 5, 10, ...\} \text{ is a subgroup of } G = \mathbb{Z}$$

$\therefore$ For $1 \in \mathbb{Z}$, $H+1 = \{...., -9, -4, 1, 6, 11, ...\}$

$3 \in \mathbb{Z}$, $H+3 = \{...., -7, -2, 3, 8, 13, ...\}$

$5 \in \mathbb{Z}$, $H+5 = \{...., -5, 0, 5, 10, ...\} = H$

are right cosets of H in G.

9.9 Order of an Element of a Group

Let (G, $\cdot$) be a group. The smallest positive integer is called the order of an element $a \in G$ if

$$a^n = e(\text{identity element in G})$$

If is denoted by $o(a) = n$.

If no such positive number exists, then we say that a is of infinite order or zero order.

Note :

1) For the order of the group is the number of distinct elements in G.
2) The order of the identity element is 1 i.e. $o(e) = 1$
3) In a group G, $o(a) = o(a^{-1})$; $\forall a \in G$
4) In a group G, $o(a) \leq o(G)$

Examples

1) $G = \{1, -1, i, -i\}$ is a multiplicative group

1 = Identity element in G and $o(G) = 4$

$o(1) = 1$

$o(-1) = 2$ as $(-1)^2 = 1$

$o(-i) = 4$ as $(i)^4 = 1$

$o(-i) = 4$ as $(-i)^4 = 1$

2) $(\mathbb{Z}, +)$ is a group of infinite order

o is the identity element, $o(O) = 1$

We have $5 \in \mathbb{Z}$ and $5^n = 5 + 5 + \ldots$ n times $= n5 \neq 0$

$\therefore o(5)$ = infinity or zero

9.10 Cyclic Group

SPPU : Dec.-18

A group G is called a cyclic group if $\exists$ at least one element $a \in G$ such that every element $x \in G$ can be written as $x = a^m$ where m is some integer.

The element a is called the generator of G and denoted by $G = \langle a \rangle$

Example 1 : Four fourth roots of unity form a cyclic group with respect to multiplication.

$$G = \{1, -1, i, -i\}$$

We have $(i)^1 = i, \quad (i)^2 = -1, \quad (i)^3 = -i, \quad (i)^4 = 1$

$\therefore$ i is the generator of $G \Rightarrow G$ is a cyclic group.

Moreover, $(-i)^1 = -i, \quad (-i)^2 = -1, \quad (-i)^3 = i, \quad (-i)^4 = 1$

$\therefore$ $-i$ is also generator of G

$\therefore \quad G = \langle i \rangle = \langle -i \rangle$

9.11 Normal Subgroups

SPPU : Dec.-16

A subgroup H of a group $(G, *)$ is said to be a normal subgroup of G if for all $g \in G$ and for all $h \in H$

$g * h * g^{-1} \in H$ (We may write $g\,h\,g^{-1} \in H$)

Every group G possesses at least two normal subgroups namely {e} and G. These groups are called improper normal subgroups.

Simple Group : A group G is said to be simple group if it has only two normal subgroups, {e} and G.

Notes :

1) Every subgroup of an abelian group is normal.
2) The intersection of normal subgroups is a normal subgroup.

Example 1 : $(\mathbb{Z}, +)$ is an abelian group.

$\therefore$ $(2\mathbb{Z}, +)$, $(3\mathbb{Z}, +)$ are normal subgroups of $(\mathbb{Z}, +)$

9.12 Quotient Groups

SPPU : Dec.-14. 15, May-15,17

Let $(G, *)$ be a group and N be a normal subgroup of G. Let G/N be the collection of all cosets of N in G.

$$G/N = \{N * a / a \in G\}$$

$(G/N, *)$ is called the quotient group or factor group.

Theorem 1) Prove that $(G/N, *)$ is a group. **SPPU : Dec.-15, May-15**

Solution : Let $(G, *)$ be a group and N is the normal subgroup at G

$$\therefore \quad G/N = \{N * a / \forall a \in G\}$$

Theorem :

1) Closure Property : Let $a, b \in G$, $\therefore N * a, N * b \in G/N$

$$\begin{aligned}(N * a) * (N * b) &= N * (a * N) * b && (\because N \text{ is normal})\\ &= N * (N * a) * b\\ &= (N * N) * (a * b)\\ &= N * c && (\because N * N = N \text{ and } a * b = c \in G)\end{aligned}$$

$\therefore$ $(N * a) * (N * b) \in G/N$

$\therefore$ G/N is closed w.r.t. $*$

2) Associativity : Let $a, b, c \in G$

and $N * a$, $N * b$ and $N * c \in G/N$

$$\begin{aligned}\therefore \quad (N * a) * [(N * b) * (N * c)] &= N * a * [N * (b * c)] && \text{by (1)}\\ &= N * a * (b * c) && \text{by (1)}\end{aligned}$$

$$= N * (a * b) * c) \qquad (* \text{ is associative in } G)$$

$$= (N * (a * b)) * (N * c)$$

$$= [(N * a) * (N * b)] * (N * c)$$

$$= [(N * a) * (N * b)] * (N * c)$$

Thus $*$ is associative in G/N

3) Existence of the identity :

We have $\quad N = N * e \in G/N$ and for any $N * a \in G/N$

$$(N * a) * (N * e) = N * (a * e) \quad \text{(by (1))}$$

$$= N * a$$

$\therefore \quad N * e = N$ is the identity element in G/N.

4) Existence of the inverse : Let $a \in G$, $N * a \in G/N$

$\therefore \exists\, a^{-1} \in G$ and $N * a^{-1} \in G/N$ such that

$$(N * a) * (N * a^{-1}) = N * (a * a^{-1}) = N * e = N$$

Hence $N * a^{-1}$ is the inverse of $N * a$ in G/N

Thus (G/N, $*$) is a group, known as quotient group.

Theorem 1 :

Lagranges theorem : The order of each subgroup of a finite group is a divisor of the order of the group.

Ans. : Let G be a group of finite order n. Let H be a subgroup of G and let O (H) = m. Suppose $h_1, h_2 \ldots h_m$ are the m members of H.

Let $a \in G$. Then Ha is a right coset of H in G and we have

$$Ha = \{h_1 a, h_2 a, \ldots\ldots, h_m a\}$$

Ha has m distinct members, since $h_i a = h_j a \Rightarrow h_i = h_j$.

Therefore each right coset of H in G has m distinct members. Any two distinct right cosets of H in G are disjoint i.e. they have no element in common. Since G is a finite group, the number of distinct right cosets of H in G will be finite, say equal to K. The union of these K distinct right cosets of H in G is equal to G.

Thus, if $Ha_1, Ha_2, \ldots\ldots Ha_k$ are the K distinct right cosets of H in G, then

$$G = Ha_1 \cup Ha_2 \cup \ldots\ldots\ldots\ldots \cup Ha_k$$

$\Rightarrow$ The number of elements in G = The number of elements in Ha_1 + The number of elements in Ha_2 + ... + The number of elements in Ha_k

... [$\because$ two distinct right cosets are mutually disjoint]

$$\Rightarrow \quad O(G) = Km \Rightarrow n = Km$$

$$\Rightarrow \quad K = \frac{n}{m} \Rightarrow m \text{ is a divisor of } n$$

$\Rightarrow$ O (H) is a divisor of O (G).

Theorem 2 :

The order of every element of a finite group is a divisor of the order of the group.

Suppose G is a finite group of order n.

Let $a \in G$ and O (a) = m. Prove that m is a divisor of n.

Let $H = \{ \ldots\ldots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \ldots\ldots\}$ be the subset of G consisting of all integral powers of a.

We know that H is a subgroup of G.

We have to show that H contains only m distinct elements and that they are a, $a, a^2, a^3, \ldots\ldots, a^m = e\ a^0$

Let $1 \le r \le m$, $1 \le s \le m$, and $r > s$.

then $a^r = a^s \Rightarrow a^r a^{-s} = a^s a^{-s} \Rightarrow a^{r-s} \Rightarrow a^0 \Rightarrow a^{r-s} = e$

Thus there exists a positive integer $r - s$ less than m such that $a^{r-s} = e$. m is the least positive integer such that $a^m = e$.

$$\therefore \quad a^r \neq a^s$$

$\therefore$ $a, a^2, a^3, \ldots\ldots, a^m = a^0 = e$ are all distinct elements of H.

Now suppose at is any element of H where t is any integer.

By division algorithm,

We have, $t = mp + q$ ... [p and q are some integers, $0 \le q < m$]

We have, $a^t = a^{mp+q}$

$$= a^{mp} a^q$$

$$= \left(a^m\right)^p a^q = a^q \qquad \ldots (0 \le q < m)$$

$\therefore$ a^q is one of the m element $a, a^2, \ldots\ldots, a^m = a^0$

Hence, H has only m distinct elements. Thus order of H is m.

By Lagrange's theorem m is a divisor of n.

Theorem 3 :

If G is a finite group of order n and $a \in G$, then $a^n = e$.

In a finite group, the order of each element is finite. Let O (a) = m. The subset H of G consisting of all integral powers of a is a subgroup of G and the order of H is m.

By Lagrange's theorem, m is a divisor of n.

Let $k = \frac{n}{m}$ then $n = mk$

Now, $a^n = a^{mk} = (a^m)^k = e^k$ ... $O(a) = m \Rightarrow am = e$

$= e$

Theorem 4 :

Every cyclic group is an abelian group. **SPPU : Dec.-14, Marks 3**

Let G = {a} be a cyclic group generated by a.

Let x, y be any two elements of G. Then there exist integers r and s such that $x = a^1$, $y = a^s$.

Now, $xy = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = yx$

Thus, we have, $xy = yx \ \forall \ x, y \in G$.

Therefore G is abelian.

Theorem 5 :

If a is a generator of a cyclic group G, then a^{-1} is also a generator of G.

Let G = {a} be a cyclic group generated by a.

Let a^1 be any element of G, where r is some integer.

as $a^r = (a^{-1})^{-r}$

$\therefore$ – r is also some integer.

$\therefore$ Each element of G, is generated by a^{-1}

Thus, a^{-1} is also a generator of G.

Theorem 6 :

Every group of order 3 is cyclic.

OR

Every group of prime order is cyclic.

Suppose G is a finite group whose order is a prime number P, then to prove that G is a cyclic group. As an integer P is said to be a prime number if $P \neq 0$, $P \neq \pm 1$, and if the only divisors of P are $\pm 1, \pm P$.

$\because$ G is a group of prime order, therefore G must contain at least 2 elements.

As 2 is the least positive prime integer.

There must exist an element $a \in G$ such that $a \neq t$, the identity element e.

Since a is not the identity element, therefore O (a) is definitely ≥ 2.

Let O (a) = m, If H is the cyclic subgroup of G generated by a then

$$O(H) = O(a) = m.$$

By Lagrange's theorem m must be divisor of P.

But P is prime and $m \geq 2$. Hence m = P

$\therefore$ H = G. Since H is cyclic. Therefore G is cyclic and a is a generator of G.

7) Every subgroup of a cyclic group is cyclic.

Suppose G = {a} is a cyclic group generated by a. If H = G or {e}, then obviously H is cyclic. So let H be a proper subgroup of G. The elements of H are integral powers of a. If $a^s \in H$, then the inverse of a^s.

i.e. $a^{-s} \in H$.

$\therefore$ H contains elements which are positive as well as negative integral powers of a.

Let m be the least positive integer such that $a^m \in H$.

Then we shall prove that $H = \{a^m\}$

i.e. H is cyclic and is generated by a^m.

Let a^1 be any arbitrary element H.

By division algorithm,

there exist integers q and r such that $t = mq + r$, $0 \leq r < m$.

Now, $a^m \in H \Rightarrow (a^m)^q \in H$ ... by closure property

$$\Rightarrow a^{mq} \in H \Rightarrow (a^{mq})^{-1} \in H$$

$$\Rightarrow a^{-mq} \in H$$

Also, $a^t \in H \;\; a^{-mq} \in H \Rightarrow a^t q^{-mq} \in H$

$$\Rightarrow a^{t-mq} \in H$$

$$\Rightarrow a^r \in H.$$

Now m is the least positive integer such that $a^m \in H$ and $0 \leq r < m$.

Therefore r must be equal to 0. Hence t = mq.

$$\therefore \quad a^t = a^{mq} = (a^m)^q$$

Thus every element $a^t \in H$ is of the form $(a^m)\,q$.

Therefore H is cyclic and a^m is a generator of H.

Examples :

Example 9.12.1 *Give an example of a finite abelian group which is not cyclic.*

Solution : Let G be the set of the four real matrices.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

It can be easily seen that G is an abelian group with respect to multiplication of matrices.

The identity element of this group is the identity matrix I.

Let us find the order of each element of G.

We have, $O(I) = I$

Also, $$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad [\because O(A) = 2]$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad [\because O(B) = 2]$$

$$C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad [\because O(C) = 2]$$

Now G is a group of order 4 and G contains no element of order 4.

Therefore G is not a cyclic group. Hence G is a finite abelian group which is not cyclic.

Example 9.12.2 *Show that the multiplicative group {1, –1} is a subgroup of the multiplicative group {1, –1, i, –i}*

Solution : Let $H = \{1, -1\}$ and $G = \{1, -1, i, -i\}$, $H \subset G$.

Consider the composition table

X	1	– 1
1	1	– 1
– 1	– 1	1

(i) Closure property : All elements of a table belong to H. Hence it is closed w.r.t. multiplication.

(ii) Multiplication of complex numbers is associative.

(iii) Existence of the identity : $1 \in H$, is the identity element.

(iv) Existence of the inverse : We have, $(1)^{-1} = 1$ and $(-1)^{-1} = -1$.

Therefore, (H, X) is a subgroup of a group (G, X).

Example 9.12.3 *Define the subgroup of a group. Let (G, 0) be a group. Let H = {a | a ∈ G and aob = boa for all b in G}. Show that H is normal subgroup of G.*

Solution : Let (G, x) be a group. A non empty subset H of a group G is said to be a subgroup of G if (H, *) itself is a group.

Given that, $H = \{a \mid a \in G \text{ and } a \circ b = b \circ a \; ; \forall b \in G\}$

Let $a, b \in H \Rightarrow a \circ x = x \circ a$ and $b \circ x = x \circ b, \; \forall x \in G.$

$$\Rightarrow (b \circ x)^{-1} = (x \circ b)^{-1}$$

$$\Rightarrow x^{-1} \circ b^{-1} = b^{-1} \circ x^{1} \qquad \text{... (1)}$$

$$\Rightarrow b^{-1} \in H.$$

Now, $(a \circ b^{-1}) \circ x = a \circ (b^{-1} \circ x)$ [∵ o is associative]

$= a \circ (x \circ b^{-1})$ [∵ use (1) or $b^{-1} \in H$]

$= (a \circ x) \circ b^{-1}$ [∵ $a \in H$]

$= (x \circ a) \circ b^{-1}$

$= x \circ (a \circ b^{-1})$

$\Rightarrow \quad a \circ b^{-1} \in H$

Therefore H is a subgroup of group G.

Let $h \in H$ and $g \in G$ and any x in G.

Consider,

$(g \circ h \circ g^{-1}) \circ x = (g \circ g^{-1} \circ h) \circ x$ [∵ $h \in H$]

$= (e \circ h) \circ x = h \circ x$

$= x \circ h$ (∵ $h \in H$)

$= x \circ (h \circ g \circ g^{-1})$

$= x \circ (g \circ h \circ g^{-1})$ (∵ $h \in H$)

$\Rightarrow \quad g \circ h \circ g^{-1} \in H$ for any $g \in G$

∴ H is a normal subgroup of G.

Example 9.12.4 *Show that the set {1, w, w^2} is a cyclic group of order 3 with respect to multiplication, w is the cube root of unity.*

Solution : Let $G = \{1, w, w^2\}$ and $w^3 = 1$

We have, $1 = 1^1 = w^3 = (w^2)^3$

$w = w^1 = (w^2)^2$

$w^2 = w^2 = (w^2)^1$

Thus every element of G can be expressed in the powers of w or w^2. Therefore G is cyclic group and its generators are w and w^2 .

Example 9.12.5 *Let $G = \{1, -1, i, -i\}$ where $i = -1$ and multiplication is the binary operation in G.*

(I) Determine whether G is an abelian group.

(II) If G is a cyclic group then find its generators.

(III) Find the order of each element of G.

Solution :

(I) Let $G = \{1, -1, i, -i\}$.

The composition table of G with respect to multiplication is given below.

X	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	– i	–1	1
–i	–i	i	1	–1

(i) Closure property : All elements of the table belong to G. Therefore G is closed w.r.t. X.

(ii) Associativity : The multiplication of complex numbers is associative. $\therefore$ (G, X) is associative.

(iii) Existence of the identity : $1 \in G$ is the identity element.

(iv) Existence of the inverse : From table

$$(1)^{-1} = 1,\ (i)^{-1} = -i,\ (-1)^{-1} = -1,\ (-i)^{-1} = i$$

(v) Commutativity : The composition table is symmetric about main diagonal.

$\therefore$ (G, X) is commutative.

Thus (G, X) is an abelian group.

(II) We have, $(i)^4 = 1$, $(i)^1 = i$, $(i)^2 = -1$, $(i)^3 = -i$

$$(-i)^4 = 1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = -i$$

Thus every element of G can be expressed as i^n or $(-i)^n$

$\therefore$ (G, X) is a cyclic group with i and – i generators.

(III) We have, $1^1 = 1$, $(-1)^2 = 1$, $(i)^4 = 1$ and $(-i)^4 = 1$.

$\therefore$ The orders of 1, –1, i, –i are 1, 2, 4, 4 respectively.

Example 9.12.6 *Show that the four permutations I, (a b), (c d), (a b) (c d) on four symbols a, b, c, d form a finite abelian group with respect to the multiplication.*

Solution : Let $f_1 = I = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$

$$f_2 = (a\ \ b) = \begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}$$

$$f_3 = (c\ \ d) = \begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix}$$

$$f_4 = (a\ b)\ (c\ \ d) = \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}$$

Let $G = \{f_1, f_2, f_3, f_4\}$

Consider the multiplication table

X	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1

(i) Closure property : All elements of table belong to G. (G, x) is closed w.r.t. x.

(ii) Multiplication of permutations is associative.

(iii) $f_1 = I$ is the identity element in G.

(iv) In G, $f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$ and $f_4^{-1} = f_4$

v) Multiplication table is symmetric about main diagonal $\therefore$ (G, x) is commutative.

Hence (G, x) is an ablein group of order 4.

9.13 Homomorphism of Groups

SPPU : Dec.-11, 12, 14, 15, May-15,17

Let $(G_1, *)$ and (G_2, o) be two groups. A function

$f : (G_1, *) \to (G_2, o)$ is said to be homomorphism.

If $f(a * b) = f(a)\, o\, f(b)$ for all $a, b \in G_1$.

i.e. $a * b$ in $G_1 \to f(a)\, o\, f(b)$ in G_2

A homomorphism from G to itself is called as endomorphism.

Properties of Group Homomorphism.

Let $f : G_1 \to G_2$ be group homomorphism and $((G_1, *)$ and $(G_2 * o)$ are groups then

i) $f(e_1) = (e_2)$

ii) $f(a^{-1}) = [f(a)]^{-1}$

Proof : i) Let $a \in G_1$ and $f(a) \in G_1$ and e_2 is the identity element in G_2.

$$\therefore \quad f(a) \circ e_2 = f(a)$$
$$= f(a * e_1)$$
$$f(a) \circ e_2 = f(a) \circ f(e_1)$$
$$\Rightarrow f(e_1) = e_2$$

2) Let $a \in G_1$ then $a^{-1} \in G$

$$\text{and} \quad e_2 = f(e_1)$$
$$= f(a * a^{-1}) \qquad \text{(f is homomorphism)}$$
$$e_2 = f(a) \circ f(a^{-1})$$
$$\Rightarrow \quad f(a^{-1}) = [f(a)]^{-1}$$

9.14 Isomorphism of Groups

SPPU : Dec.-10,11,12,13,16, May-08,10,17,19

Let $(G_1, *)$ and $(G_2, \circ)$ be two groups. A function

$f : (G_1, *) \to (G_2, \circ)$ is said to be isomorphism.

If i) f is a homomorphism from $G_1 \to G_2$

ii) f is bijective function.

If $f : G_1 \to G_2$ is an isomorphism of groups then G_1 and G_2 are called as isomorphic groups and denoted by $G_1 \cong G_2$.

An isomorphism from G to itself is called as automorphism of group G.

Examples :

Example 9.14.1 *Let G be a group with identity e show that a function $f : G \to G$ defined by $f(a) \forall = e, a \in G$ is a homomorphism (Endomorphism).* **SPPU : May-08,10,19, Marks 3**

Solution : We have $f : G \to G$ and $f(a) = e, \forall a \in G$.

$$\text{Let} \quad a, b \in G \Rightarrow f(a), f(b) \in G$$
$$\therefore \quad f(a * b) = e \qquad \text{(as } a * b \in G\text{)}$$
$$= e * e$$
$$= f(a) * f(b)$$

$\therefore$ f is a homomorphism.

Example 9.14.2 *Explain homomorphism and antomorphism of groups with examples.*

SPPU : Dec.-11, 12, 13, Marks 3

Solution : Refer 9.12 and 9.13 for definition.

e.g. 1) The homomorphism $f : (\mathbb{Z}, +), (\mathbb{Z}, +)$ such that

$$f(n) = -n \text{ is an antomorphism of group.}$$

2) The homomorphism $f : (R_o, o) \to (R_o, o)$ such that

$$f(x) = x; \ \forall x \in \Re_o \text{ is an antomorphism of groups.}$$

Example 9.14.3 *Let $\Re$ be the additive group of real numbers and $\Re$ + be the multiplicative group of positive real numbers. Prove that the mapping*
$f : (R, +) \to (\Re_+, X)$ defined by $f(x) = e^x$, $\forall x \in \Re$ is an isomorphism of $\Re$ onto $\Re_+$

SPPU : Dec.-10, Marks 4

Solution : If x is any real number, then e^x is always positive real number and e^x is unique. Therefore, $f : \Re \to R_+$ is a function such that $f(x) = e^x$.

Let $x_1, x_2 \in \Re$ then $f(x_1) = f(x_2)$

$$\Rightarrow \quad e^{x1} = e^{x2} \Rightarrow x_1 = x_2$$

$\therefore$ f is one to one mapping.

For any $y \in \Re_+$ then $\log y \in \Re$ such that

$$f(\log y) = e^{\log y} = y$$

$\therefore$ f is onto.

Now for any $x_1, x_2 \in \Re$

Consider, $f(x_1 + x_2) = e^{x1+x2}$

$$= e^{x1 \times x2}$$

$$= f(x_1) \times f(x_2)$$

$\therefore$ f preserves compositions in $\Re$ and $\Re$+

$\therefore$ f is an isomorphism of $\Re$ onto $\Re$+.

Hence $\Re \cong \Re_+$

Example 9.14.4 *Show the, the additive group of integers.*
$G = \{...., -3, -2, -1, 0, 1, 2, 3, ...\}$ is isomorphic to the additive group
$G' = \{...., -3m, -2m, -1m, 0, 1m, 2m, 3m, ...\}$
where m is any fixed integer not equal to zero.

Solution : If $x \in G$, then obviously $mx \in G'$

Let $f : G \to G'$ be defined by $f(x) = mx \ \forall \ x \in G$

(i) f is one to one. Let $x_1, x_2 \in G$.

Then $f(x_1) = f(x_2)$

$\Rightarrow \quad mx_1 = mx_2$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ f is one to one.

(ii) f is onto : Suppose y is any element of G'.

Then obviously $\frac{y}{m} \in G$. Also $f\left(\frac{y}{m}\right) = m\left(\frac{y}{m}\right) = y$

Thus $y \in G' \Rightarrow$ that there exists $\frac{y}{m} \in G$ such that $f\left(\frac{y}{m}\right) = y$.

$\therefore$ Each element of G' is the f image of some element of G. Hence f is onto. Again if x_1 and x_2 are any two elements of G, then

$$f(x_1 + x_2) = m(x_1 + x_2) \quad \text{... by definition of f.}$$

$$= mx_1 + mx_2 \quad \text{... by distributive law for integers}$$

$$= f(x_1) + f(x_2)$$

Thus f preserves compositions in G and G'. Therefore, f is an isomorphic mapping of G onto G'.

Hence G is isomorphic to G'.

9.15 Rings, Integral Domains and Fields

SPPU : Dec.-08, 10, 11, 12, 13, 14,16,17,18, May-14,17,18,19

In previous sections, we have discussed an algebraic structures with a single binary operation. Now, we will study an algebraic structures with two binary operations such as rings, integral domains and fields.

9.15.1 Rings

SPPU : Dec.-13, May-14

Let R be a non empty set equipped with two binary operations called addition and multiplication and denoted by '+' and '·' respectively.

An algebraic structure (R,+, ·) is called a ring if it satisfies following axioms.

1) (R, +) is an abelian group i.e.

i) Closure property : for $a, b \in R$, $a + b \in R$

ii) Associativity : for $a, b, c \in R$, $a + (b + c) = (a + b) + c$

iii) Existence of the identity : For any $a \in R$, $\exists\, 0 \in R$ s.t, $a + 0 = 0 + a = a$.

$\therefore$ 0 is called as the additive identity element of ring.

iv) Existence of the inverse : for each $a \in R$, $\exists -a \in R$

Such that $a + (-a) = -a + a = 0$

$-a$ is called the additive inverse of a

v) Commutative property : For $a, b \in R$

$$a + b = b + a$$

2) $(R, \cdot)$ is semigroup i.e.

i) Closure property : $\forall a, b \in R$, $a \cdot b \in R$

ii) Associativity : for $a, b, c \in R$,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

3) Multiplication distributes over addition $\forall a, b, c \in R$

i) $a \cdot (b+c) = a \cdot b + a \cdot c$ (Right distributive law)

ii) $(a+b) \cdot c = a \cdot c + b \cdot c$ (Left distributive law)

9.15.2 Commutative Ring

A ring $(R, +, \cdot)$ is said to be commutative ring if $\forall\, b \in R$, $a \cdot b = b \cdot a$

9.15.3 Ring with Unity

A ring $(R, +, \cdot)$ is said to be ring with unity if $\forall\, a \in R$, $\exists\, 1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$.

Examples :

1) $(\mathbb{Z}, +, \cdot)$ is a commutative ring with unity.
2) $(2\mathbb{Z}, +, \cdot)$ is a commutative ring without unity where $2\mathbb{Z}$ = set of even integers.
3) The set of $n \times n$ matrices over real numbers with respect to usual matrix addition and multiplication is a non commutative ring with unity.

9.15.4 Properties of a Ring

If $(R, +, \cdot)$ is a ring with identity 0 and unit element 1 then following are true for all $a, b, c \in R$.

i) $a \cdot 0 = 0 \cdot a = 0$

ii) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

iii) $(-a) \cdot (-b) = a \cdot b$

iv) Unit element is unique

9.15.5 Subring

Let $(R, +, \cdot)$ be a ring. A non empty subset S of R is said to be subring of R if $(S,+, \cdot)$ is a ring. e.g. $(\mathbb{Z},+, \cdot)$ is a subring of $(\Re, +, \cdot)$.

9.15.6 Zero Divisors

Let $(R, +, \cdot)$ be a commutative ring. An element $a \neq 0$ in R is said to be zero divisor if $\exists\, b \neq 0$ in R such that $a \cdot b = 0$.

A ring $(R, +, \cdot)$ is said to be without zero divisors.

if $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0, \forall\, a, b \in R$.

e.g. 1) $\overline{2}$ is a zero divisor in $(\mathbb{Z}_4,+,\cdot)$ as $\overline{2} \cdot \overline{2} = \overline{4} = 0$

2) $(M_{2\times2}\, \Re), +, \cdot 1)$ is a ring with zero divisors.

as $$A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \text{ but } A \neq 0 \text{ and } B \neq 0$$

3) $(\mathbb{Z}, +, \cdot)$ is a ring without zero divisors i.e. $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$.

9.15.7 Integral Domain

SPPU : Dec.-13, 11, 10, May-14

A commutative ring with zero divisors is called an integral domain.

e.g.1) $(\Re,+, \cdot)$, $(\mathbb{Z}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are integral domains.

2) $(\mathbb{Z}_4, +, \cdot)$ is a ring with zero divisors

$\therefore$ It is not integral domain.

9.15.8 Field

SPPU : Dec.-13, 12, 11, 10, May-14

A commutative ring with unity in which every non zero element possesses their mulplicative inverse, is called as field.

A field is an integral domain.

e.g. 1) $(\Re,+, \cdot)$, $(\mathbb{C}, +, \cdot)$, $(Q,+, \cdot)$ are fields.

2) $(\mathbb{Z}, +, \cdot)$ is integral domain but not field.

9.15.9 Ring Homomorphism

Let $(R, +, *)$ and $(S, +', \xi *')$ be two rings.

A function $\phi : R \to S$ is called a ring homomorphism

If for any $a, b \in R$

i) $\phi(a+b) = \phi(a)+\phi(b)$

ii) $\phi(a*b) = \phi(a)*' \phi(b)$

If ϕ is bijective then it is called as a ring isomorphism.

The kernal of ring homomorphism is defined as the set $\{a \in R \mid \phi(x)=0'\}$.

It is denoted by ker (ϕ) or ker ϕ.

Examples :

Example 9.15.1 *Let* $R = \left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in z\right\}$ *f is the mapping that takes* $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ *to* $a - b$.

i) Show that f is *a homomorphism. ii) Find Kernel of* f. **SPPU : Dec.-08**

Solution :

$$f = \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} + \begin{bmatrix} c & d \\ d & c \end{bmatrix}\right) = f\left(\begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix}\right)$$

$$= (a+c)-(b+d)$$

$$= (a-b)+(c-d)$$

$$= f\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) + f\left(\begin{bmatrix} c & d \\ d & c \end{bmatrix}\right)$$

Also $f\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) \cdot \left(\begin{bmatrix} c & d \\ d & c \end{bmatrix}\right) = f\begin{bmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{bmatrix}$

$$= (ac+bd)-(ad+bc)$$

$$= (ac-bc)+(bd-ad)$$

$$= (a-b)(c-d)$$

$$= f\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) \cdot f\left(\begin{bmatrix} c & d \\ d & c \end{bmatrix}\right)$$

ii) $\text{Ker } f = \left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} : a-b=0\right\}$ i.e. $\left\{\begin{bmatrix} a & a \\ a & a \end{bmatrix} ; a \in z\right\}$

Example 9.15.2 *Show that* $S = \left\{a+b\sqrt{2} ; a, b \in z\right\}$ *for the operations +, × is an integral domain but not a field.* **SPPU : Dec.-14,16, May-17,19**

Solution : We have,

$$(a+b\sqrt{2})+(c+d\sqrt{2}) = (a+c)+(b+d)\sqrt{2}$$

$$(a+b\sqrt{2})+(c+d\sqrt{2} = (ac + 2bd) + (bc + ad)\sqrt{2}$$

Clearly S is commutative ring with unit element 1.

We have to prove S is an integral domain.

Let $\left(a+b\sqrt{2}\right)\left(c+d\sqrt{2}\right) = 0$

$\therefore$ $ac + 2bd = 0$ ... (1)

and $bc + ad = 0$ ...(2)

Suppose $a = 0$; then $bd = bc = 0$

either $b = 0$ or both $d = c = 0$

Hence, if $a = 0$, $a+b\sqrt{2} = 0$

or $c+d\sqrt{2} = 0$

Assume $a \neq 0$. Multiplying equation (1) by d

we have, $acd + 2bd^2 = 0$ ... (3)

From equation (2)

$$ad = -bc$$

Hence substituting this value in equation (3)

We have, $-bc^2 + 2bd^2 = 0$

$\Rightarrow b(2d^2 - c^2) = 0$

$\therefore b = 0$ or $c^2 = 2d^2$, i.e. $c = \sqrt{2}\, d$

Since c is an integer, $c^2 = 2d^2$ is true only if $c = d = 0$

Hence if $c^2 \neq 2d^2$, $b = 0$. But $b = 0$ implies $a = 0$

Hence, in any case either $a + b\sqrt{2} = 0$ or $c + d\sqrt{2} = 0$

Hence, S is an integral domain.

To show that S is not a field consider the element $2+\sqrt{2}$. It's multiplicative inverse does not exit in S, for $\left(2+\sqrt{2}\right)\left(c+d\sqrt{2}\right) = 1$

$$\Rightarrow \quad 2c + 2d = 1 \Rightarrow c + d = \frac{1}{2}$$

Absurd, since $c, d \in z$.

Example 9.15.3 *If R is a ring such that $a^2 = a, \forall a \in R$ then prove that $a + a = 0$, $\forall a \in R$. R is a commutative ring.*

Solution : (i) Let $b = -a$ is the inverse of a,

Now prove that, $a = b$

Consider, $a - b = (a - b)(a - b)$

$$\therefore \quad a - b = a^2 - ba - ab + b^2$$

$$= a - ba - ab + b$$

$$= (a + b) - ba - ab$$

$$\therefore \quad a - b = -ba - ab = -(ba + ab) \quad \text{... (1)}$$

Now, $a + b = (a + b)\ (a + b)$

$$= a^2 + ab + ba + b^2$$

$$= a + ab + ba + b$$

$$= a + b + (ab + ba)$$

$$= ab + ba \quad \text{... (2)}$$

But, $a + b = 0$

$$\Rightarrow \quad ab + ba = 0$$

$\therefore$ Equation (1) $\Rightarrow a - b = 0$

$$\Rightarrow \quad a = b$$

Example 9.15.4 *Consider a ring (R, +, *) defined by a * a = a. Determine whether ring is commutative or not ?*

Solution : Let $a, b \in R$

and $(a + b)^2 = a + b$

$$(a + b)(a + b) = a + b$$

$$a^2 + ab + ba + b^2 = a + b$$

$$a + ab + ba + b = a + b \quad \text{... } (\because aa = a)$$

$$(a + b) + (ab + ba) = a + b$$

$$ab + ba = 0$$

$$\Rightarrow \quad a + b = 0$$

$$\Rightarrow \quad a + b = a + a$$

$$\Rightarrow \quad b = a$$

$\Rightarrow \qquad ab = ba$

$\therefore$ R is commutative ring.

Example 9.15.5 *Prove that the set E of all even integers is a commutative ring with respect to usual addition and multiplication, but it has no unit element.*

Solution : Let 2z, be the set of all even integers, then

(i) For any two elements $2m, 2n \in 2z$, $m, n \in z$;

$$2m + 2n = 2(m + n) \in 2z \text{ and}$$

$$2m \cdot 2n = (2mn) \in 2z \text{ and}$$

$$2m \cdot 2n = (2mn) \in 2z,$$

Since $m + n \in z$ and $2mn \in z$.

(ii) Since $(z, +)$ $(z, \cdot)$ are associative and commutative, $2z \in z$, so $(2z, +)$ and $(2z, \cdot)$ are also associative and commutative.

(iii) $0 \in 2z$ such that $a + 0 = 0 + a = a \ \forall \ a \in 2z$.

(iv) For any $2m \in 2z - 2m = (-m) \in 2z \in z$, so " $\cdot$ " is distributive on " + " over 2z.

Hence $(2z, +, \cdot)$ is a commutative ring but it is not a ring with unity as it does not contain $1 \in E$. (Being the set of all even integers).

Example 9.15.6 *Prove that (R, +, *) is a ring with zero divisors, where R is 2 × 2 matrix and + and * are usual addition and multiplication operations.*

Solution : R = { a set of 2 × 2 matrices }

First we show that (R, t) is an abelian group.

(i) Closure : Since the sum of two 2 × 2 matrices is also a 2 × 2 matrix closure is satisfied.

(ii) Associative : If A, B and C are 2 × 2 matrices then

$$(A_{2\times2} + B_{2\times2}) + C_{2\times2} = A_{2\times2} + (B_{2\times2} + C_{2\times2}) \text{ always exists.}$$

(iii) Identity : The identity element $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$ belong to R and

$$A + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A. \text{ Identity is satisfied.}$$

(iv) Inverse : The inverse of a 2 × 2 matrix say A is always – A which is also 2 × 2 matrix.

(v) Commutative : The sum of two matrices is always commutative.

$\therefore$ (R, +) is an abelian group. (R, ·) is also a semigroup w.r.t. multiplication and also multiplication is distributive with respect to addition. Hence (R, ·, +) is a ring.

Now, Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ be two non zero elements of this ring.

Then, we have $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus the product of two non-zero elements of the ring is equal to the zero element of the ring.

$\therefore$ AB is a ring with zero divisors.

Example 9.15.7 *Write out the operation table for $[z_2^2, +_2, x_2]$ Is z_2^2 a ring ? Is an integral domain ? Is a field ? Explain.*

Solution : The operation tables are as follows :

We have, $z_2 = \{0, 1\}$

+	0	1
0	0	1
1	1	1

×	0	1
0	0	0
1	0	1

Since $(z_2, +_2, x_2)$ satisfies the following properties :

(i) Closure axiom : All the entries in both the tables belong to z_2. Hence closure is satisfied.

(ii) Commutative : In both the tables all the entries about the main diagonal are same, therefore comutativity is satisfied.

(iii) Associative law : The associative law for addition and multiplication are also satisfied.

(iv) Identity : Here 0 is the additive identity and 1 is the multiplicative identity. Identity property is satisfied.

(v) Inverse : Inverse exists in both the tables. The additive inverse of 0, 1 are 1, 0 and the multiplicative inverse of non-zero element of z_2 is 1.

(vi) Multiplication : It is distributive over addition. Hence, $(z_2, +_2, x_2)$ is a ring as well as field. Since we know that every field is an integral domain, therefore it is also an integral domain.

Example 9.15.8 *Let $z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Let R is a relation under the operations addition modulo 7 and multiplication modulo 7. Does this system form a ring ? It is a commutative ring ?* **SPPU : Dec.-11**

Solution : Consider the following tables,

Table 1 :

$\times_7$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	0
2	0	2	4	6	1	3	5	0
3	0	3	6	2	5	1	4	0
4	0	4	1	5	2	6	3	0
5	0	5	3	1	6	4	2	0
6	0	6	5	4	3	2	1	0
7	0	0	0	0	0	0	0	0

Table 2 :

$+_7$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	0
1	1	2	3	4	5	6	0	1
2	2	3	4	5	6	0	1	2
3	3	4	5	6	0	1	2	3
4	4	5	6	0	1	2	3	4
5	5	6	0	1	2	3	4	5
6	6	0	1	2	3	4	5	6
7	0	1	2	3	4	5	6	0

I) S.T. $(z_8, +_7)$ is an abelian group.

i) From table 2, all element are in z_8 $\therefore$ z_8 is closed w.r.t. $+_7$

ii) Associativity : for all a, b c $\in z_8$.

$a +_7 (b +_7 c) = (a +_7 b) +_7 c$

iii) By observing the first row of table 2. 0 is the additive identity in z_8.

9.15.10 Polynomial Ring

Let R be an arbitary ring and x an indeterminate. The set of all polynomials f (x)

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

Where the a_i' s are elements of the ring R and only a finite number of them are equal to zero is called R[x].

R[x] from a ring with repect to addition and multiplication of polynomials and hence R[x] is known as polynomial ring.

Note : 1) If $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{i=0}^{\infty} b_i x^i$

and if f(x) = g(x) then $a_i = b_i$ ∀ non negative integer i.

Thus, two polynomials are equal iff their corresponding coefficients are equal.

2) $f(x) + g(x) = c_0 x^0 + c_1 x^1 + \ldots$

$$= \sum_{i=0}^{\infty} C_i x^i$$

Where $c_i = a_i + b_i$, $\forall_i$ thus in order to add two polynomials we should add the coefficients of like power of x

3) $f(x)\, g(x) = d_0 x^0 + d_1 x^1 + \ldots d_n x^n$

Where, $c_i = d_0 x^0 + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0$

Thus $$d_i = \sum_{i+j=n} a_i b_j$$

Examples :

Example 9.15.9 *Find the deg of [f(x)+g(x)], f (x) g(x) h(x) h(x) h(x) where the polynomials are on the integer (mod 8) and operation are addition and multiplication and* $f(x) = 2x + 4x^2$, $g(x) = 2 + 6x + 4x^2$, $h(x) = 2 + 4x$.

Solution : $f(x) + g(x) = (2x + 4x^2) + [2 + 6x + 4x^2]$

$$= 2 + 8x + 8x^2$$

$$= 2 \pmod 8$$

Hence deg f(x) + g(x) = 0

$$f(x)\, g(x) = (2x + 4x^2) \cdot (2 + 6x + 4x^2)$$

$$= 0 + (2 \cdot 2 + 0 \cdot 6) + (2 \cdot 4 + 6 \cdot 2 + 4 \cdot 0)x^2 + (2 \cdot 4 + 4 \cdot 6)x^3 + (4 \cdot 4)x^4$$

$$= 4x + 20x^2 + 32x^3 + 16x^4$$

$$= 4x + 4x^2 + 0x^3 + 0x^4 \pmod 8$$

$$= 4x + 4x^2 \pmod 8$$

Hence deg $f(x) + g(x) = 2$

Also $h(x)\,h(x)\,h(x) = (2 + 4x)\,[(2 + 4x)\,(2 + 4x)]$

$$= (2 + 4x)\,[2 \cdot 2 + (2 \cdot 4 + 4 \cdot 2)x + 4 \cdot 4x^2]$$

$$= (2 + 4x)\,(4 + 16x + 16\,x^2)$$

[Now $4 + 16x + 16x^2 = 4$ (mods 8)]

$$= (2+4x) \cdot 4$$

$$= (2 \cdot 4) + 4 \cdot 4\,x$$

$$= 8 + 16\,x$$

$$= 0 \pmod 8$$

Hence deg $[h(x)\,h(x)\,h(x)] = 0$.

9.16 Group Codes

SPPU : Dec.-06, 11, 12, 15,18, May-18

In communication, a code is a rule for converting a data (information or message) into another form or representation, not necessarily of the same type. In communication and information processing, encoding is the process by which data from a source is converted into some symbols to be communicated. Decoding is the reverse process of encoding. In decoding process we convert these code symbols back into information or data which is understandable to receiver.

For a example, in the Chapter 2 of In sem (Propositional Calculus), we convert the information (statement) into mathematical symbol (quantifiers) and vice versa. This is the process of encoding and decoding.

One of the reason for coding to enable communication in places where ordinary spoken or written language in difficult to understand.

The coding problem is that of representing distinct messages or statements by distinct sequence of letters from a given symbols e.g. alphabets, mathematical symbols or binary alphabets {0,1}.

In this contest, we assume that symbols are only binary alphabets {0,1}.

- A sequence of letters from an alphabets is also called as word.
- Code is the collection of words which are used to represent messages. A word in code is also known as code word.
- A block code is consisting of words of the same length.
- The number of bits (digits) in the binary word is called the length of binary word.

One of the reason for choosing a block code to represent set of messages is its ability to correct errors. Suppose some code word is transmitted from its origin to its destination. In the course of transmission due to some problem, some 1's are received as 0's the and some 0's are received as 1's. Due to this error the received word is different from the transmitted word.

Therefore to find the exact error and correct it is essential to get correct message. It is known as error detection and correction.

From above discussion, it is clear that the error detection and correction both depend upon the words in the set of possible transmitted words being sufficiently different from one another. The codes with error detecting and correcting properties have a mathematical structure known as group structure we now present this group structure of codes with some definitions.

9.16.1 Group Code

Let S_n be the set of all binary words of length n. Let $\oplus$ be a binary operation on S_n such that for all $x, y \in S_n$ where $x = (x_1, x_2 \ldots x_n)$, $y = (y_1, y_2, \ldots y_n)$, $x \oplus y = (x_1 \oplus y_1, x_2 \oplus y_2, \ldots x_n \oplus y_n)$

Where the operation $\oplus$ denotes the addition modulo 2 on {0, 1} and is given by the following table.

$\oplus$	0	1
0	0	1
1	1	0

The algebraic structure $(S_n, \oplus)$ forms a group in which n tuple of 0's (0, 0, 0, ..., 0) is the identity and each element is its own inverse. In general, any code which is a group under the operation $\oplus$ is called a group code. Group code was first introduced by Hamming and it is very useful in binary encoding techniques.

e.g. if $x = (1\ 0\ 1\ 0\ 1)$ and $y = (0\ 1\ 1\ 0\ 0)$

then $x \oplus y = (1\ 1\ 0\ 0\ 1)$

9.16.2 Hamming Distance

SPPU : Dec.-11, Dec.-15

Let x be a word is S_n. The weight of x is denoted by w(x) and defined as

$$w(x) = \text{Number of one's in } x$$

e.g. $w(0\ 0\ 1\ 0\ 1) = 2$, $w(0\ 0\ 0) = 0$, $w = (1\ 1\ 1\ 1) = 4$

Let $x = (x_1, x_2, \ldots x_n)$, $y = (y_1, y_2, \ldots y_n)$ be any two elements in $(S_n, \oplus)$. The Hamming distance between x and y is denoted by d(x,y) and defined as

$d(x, y)$ = The number of co-ordinates at which x_i and y_i are different.

e.g. If $x = (1\ 0\ 1\ 1\ 0\ 1)$

$y = (0\ 1\ 1\ 1\ 1\ 0)$

$\downarrow\downarrow \quad \downarrow\downarrow$

different

$\therefore \quad d(x, y) = 4$, as x and y have first second, fifth, sixth positions different)

Now $x \oplus y = (1\ 1\ 0\ 0\ 1\ 1)$

$w(x \oplus y) = 4 = d(x, y)$

Thus for any $x, y \in S_n$, $d(x, y) = w(x \oplus y)$

Properties of Hamming Distance :

Let $x, y, z \in (S_n, \oplus)$ then

i) $d(x, y) \geq 0$

ii) $d(x, y) = 0$ iff $x = y$

iii) $d(x, y) = d(y, x)$

iv) $d(x, z) \leq d(x, y) + d(y, z)$

The minimum distance of a code is the minimum of all the distances between district pair of code words.

e.g. Let $x\ (1\ 1\ 0\ 1\ 1\ 0)$, $y = (0\ 0\ 1\ 1\ 1\ 1)$, $z = (1\ 0\ 1\ 0\ 1\ 0)$

$\therefore \quad d(x, y) = 4$, $d(y, z) = 3$, $d(x, z) = 3$

Out of 4, 3, 3, minimum is 3.

Therefore the minimum distance between the words x, y, z is 3

By using the weight and minimum distance, a combination of errors can be detected and corrected.

Theorem 1) The minimum weight of the non zero code words in a group code is equal to its minimum distance.

Theorem 2) A code can detect all combinations of k or fewer errors iff the minimum distance between any two code words is at least k +1,

Theorem 3) A code can correct all combinations of k or fewer errors iff the minimum distance between any two code words is at least 2k+1.

For example, Let a code containing words of length 3

$c = \{(0, 0, 0), (1, 1, 1)\}$, c contains only two words.

The minimum weight of non zero code word is 3.

$\therefore$ By theorem 1, the minimum distance is also 3.

By theorem 2, this code can detect any combination of two errors or one error. And by theorem 3, this code can correct only one error.

9.16.3 Generation of Codes By Using Parity Checks

In 1950, Hamming developed the first complete error detecting and error correcting encoding procedure. This procedure has been frequently used in computer systems.

Hamming constructed the codes, called Hamming codes by introducing redundant digit called parity digits. In a message, that is n digits long, m digits ($m < n$) are used to represent the information part of the message and the remaining $k = n - m$ digits are used for the detection and correction of errors. These K digits are called parity checks.

Hamming single error detecting codes can be described as follows.

i) The actual message is contained in the first (n–1) digits of a code word of length n and the last digit position is set to 0 or 1 so as to make the entire message contain an even numbers of 1's. such an encoding procedure is called a even parity check.

ii) Odd parity check can be used by making the entire message containing an odd no. of 1's. e.g. In even parity check {00, 10, 11} becomes $\{\underset{\uparrow}{000}, \underset{\uparrow}{101}, \underset{\uparrow}{110}\}$ In odd parity check { 00, 01, 10, 11} becomes $\{\underset{\uparrow}{001}, \underset{\uparrow}{010}, \underset{\uparrow}{100}, \underset{\uparrow}{111}\}$.

The code words of length n in which information is contained m digits ($m < n$) and remaining $k = n - m$ digits are parity checks, can be generated by using $k \times n$ matrix H. This matrix is called parity check matrix, where elements are from set {0, 1} A single error correcting code of length n generated by H with k parity check is given by

$$2^k \geq n + 1 = (k + m) + 1$$

$$\therefore \quad m \leq 2^k - k - 1$$

The number of code words generated by H is $2^m = 2^{n-k}$ and the code generated in this way called the Hamming code.

Theorem 4 : Let H be a parity check matrix which consists of k rows and n columns. Then the set of words $x = (x_1, x_2, x_3, \ldots, x_n)$ which belongs to the following set.

$C = \{ x \mid xH^t = 0 \pmod 2\}$ is a group code under the operation $\oplus$ (addition modulo 2) where H^t = Transpose of matrix H.

Theorem 5 : The parity check matrix H generates a code word of weight q iff $\exists$ a set of columns of H such that their k - tuple sum is zero.

For the parity check matrix H of order $k \times n$.

Where n = Length of code word

k = Number of parity check bits

The information digits will be $m = n - k$. The number of code words generated is $2^m = 2^{n-k}$. Consider the following steps for the generation of code words.

Step 1 : Find the system of equations from $x \cdot H^t = 0$.

Step 2 : Find the values of parity check bits interms of information digits from the equations obtained in step 1.

Step 3 : Give values 0 or 1 to information digits and calculate the values of parity check bits according to step 2. The binary words obtained in this way will be the required code words generated by H.

To find the number of errors defected and corrected first find the columns of H whose sum (addition modulo 2) is zero. The number of theses columns is equal to the minimum weight of the code and which is equal to the minimum distance of the code.

Examples :

Example 9.16.1 *Find the minimum distance of an encoding function $e : B^2 \rightarrow B^5$ given as .*

e (0 0) = 00000, e (0 1) = 1 0 0 1 1, e (1 0) = 0 1 1 1 0, e (1 1) = 1 1 1 1 1.

SPPU : Dec.-06

Solution :

d [e (0 0), e (0 1)] = 3

d [e (0 0), e (1 1)] = 5

d [e (0 0), e (0 1)] = 3

d [e (0 0), e (1 0)] = 3

d [e (0 1), e (1 0)] = 4

d [e (0 1), e (1 1)] = 2

d [e (1 0), e (1 1)] = 2

The minimum along all distances is 2.

Thus the minimum distance of an encoding is 2.

Example 9.16.2 *What is Hamming function (distance) ? Find the distance between the code words of C = { (0 0 0 0), (0 1 0 1), (1 0 1 1), (0 1 1 1)} Rewrite the message by adding even parity check bit and parity check bit.* **SPPU : Dec.-11, Marks 5**

Solution : Refer 9.16.2 for the definition.

We have d [e (0 0 0 0), e (0 1 0 1)] = 2

d [e (0 0 0 0), e (1 0 1 1)] = 3

d [e (0 0 0 0), e (0 1 1 1)] = 3

d [e (0 1 0 1), e (1 0 1 1)] = 3

d [e (0 1 0 1), e (0 1 1 1)] = 1

d [e (1 0 1 1), e (0 1 1 1)] = 2

Even parity check bit of c is {(0 0 0 0 0), (0 1 0 1 0), (1 0 1 1 1), (0 1 1 1 1)}

Odd parity check bit of c is {(0 0 0 0 1), (0 1 0 1 1), (1 0 1 1 0), (0 1 1 1 0)}

Example 9.16.3 *Given the parity check matrix* $H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ *find the minimum distance of code generated by H. How many errors can it detect and correct.*

SPPU : Dec.-15, Marks 5

Solution : From the given matrix H, select minimum number of column whose sum is zero.

$\therefore$ Consider columns $h_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $h_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $h_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

The sum of these columns

$$h_1 \oplus h_2 \oplus h_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the minimum weight of the code is 3 which is equal to its minimum distance.

We know that the code can detect all combinations of k or fever errors iff the minimum distance between any two code words is at least k +1.

Here minimum distance is 3 = k+1 $\Rightarrow$ k = 2.

$\therefore$ This code can detect 2 or less errors.

Also it can correct k errors if the minimum distance is 2k +1.

$\therefore \quad 2k + 1 = 3 \Rightarrow k = 1$

This code can correct only one error.

Therefore this is a single error correcting code.

Example 9.16.4 *Find the number of code words generated by the parity check matrix H given by* $H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ *find all the code words generated.*

SPPU : Dec.-12, Marks 5

Solution : The parity check matrix is

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{3\times 6}$$

H is of order 3×6. Hence the length of the code word is 6 in which last three digits are parity check bits. The information digits are

$$M = n - k = 6 - 3 = 3$$

$\therefore$ The matrix H will generate. $2^m = 8$ code words.

They are the solution of $x \cdot H^t = 0$.

$$[x_1\, x_2\, x_3\, x_4\, x_5, x_6] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 0]$$

$$\Rightarrow \begin{aligned} x_1 + x_2 + \quad + x_4 + \quad &= 0 \\ x_2 + x_3 + \quad + x_5 + \quad &= 0 \\ x_1 + \quad + x_3 + \quad + x_6 &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} x_4 &= x_1 + x_2 \\ x_5 &= \quad x_2 + x_3 \\ x_6 &= x_1 + \quad + x_3 \end{aligned}$$

By giving different combinations of 0 and 1 to x_1, x_2, x_3, we get the following code words.

x_1	x_2	x_3	x_4	x_5	x_6
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	1	1	0
0	1	1	1	0	1
1	0	0	1	0	1
1	0	1	1	1	0
1	1	0	0	1	1
1	1	1	0	0	0

Hence the code is

C { (0 0 0 0 0) (0 0 1 0 1 1) (0 1 0 1 1 0) (0 1 1 1 0 1) (1 0 0 1 0 1)
(1 0 1 1 1 0) (1 1 0 0 1 1) (1 1 1 0 0 0)}

9.16.4 Decoding

For group codes, there is an efficient way to determine the transmitted word corresponding to the word received according to minimum distance criteria. Basic group theory is involved in this process.

Let (C, ⊕) be a group code. Suppose the codewords transmitted contain n binary digits in which the first n digits contain the information part and the last k digits are parity check bits.

Let $(S_n, \oplus)$ be a set of all n-tuples, where (C, ⊕) is a subgroup of $(S_n, \oplus)$. Suppose the codeword $x \in C$ being transmitted through the noisy channel and the received word is y. Let e be the error n the received word y where e consists of one's in digit position in which an error occurs.

Thus $x \oplus e = y$, where $y \in S_n$

Since the operation ⊕ is addition modulo 2 therefore $x = e \oplus y$. This shows that the transmitted codeword x is closed to y in terms of Hamming distance i.e. the error e should be of least weight.

To find error with least weight, we use the concept of cosets of subgroup C of S_n. We know that set of all cosets of C in S_n forms a partition in S_n.

Consider the element $y \in S_n$, clearly y is in the left coset $y \oplus C$ among all the elements $y \oplus C$ we can determine the element 'a' with least weight.

Such an element is known as coset leader. Thus for each coset, we have a coset leader. For the coset of C itself, 0 is the coset leader, since C has 2^m elements, the number of cosets will be

$$r = 2^{n-m}$$

Hence all together the coset leaders are $0, a_1, a_2, \ldots, a_{r-1}$ if a_1 is coset leader, then elements of coset $a_1 \oplus C$ are given as $a_1 \oplus x$ where $x \in C$. Further the element of C which is closed to $a_1 \oplus x$. Thus, if the received word is $a_1 \oplus x$, the trammitted code word is x.

Consider the following steps for constructing a decoding table.

Step i : List the codewords of C, starting with 0 as the first elements.

Step ii : Choose any word $a_1 \in S_n$ not in the first row of minimum weight. List the coset $a_1 \oplus C$ as the second row by putting a_1 under 0 and $a_1 \oplus x$ under x for each $x \in C$

Step iii : From these words of S_n, not in first and second row, choose a_2 of minimum weight and list the coset $a_2 \oplus C$ as in step 2 to get third row.

Step iv : Continue in this way untill all the cosets are listed and every element of S_n appears exactly once.

The decoding table has the form

0	x	x^1	x^n	...
a_1	$a_1 \oplus x$	$a_1 \oplus x^1$	$a_1 \oplus x^n$	...
a_2	$a_2 \oplus x$	$a_2 \oplus x^1$	$a_2 \oplus x^n$	...
⋮	⋮	⋮	⋮	...

In the table, the left most column denotes the coset leader and the top row denotes the codewords generated. A received word 'y' can be denoted by first finding y in a row of the decoding table. Suppose y is in $(i+1)^{th}$ row of the decoding table. Then the decoded word is given by $x = a_i \oplus y$ where a_i is the coset leader of $(i+1)^{th}$ row. Also if no error has occurred in transmission, then the coset leader will be 0.

9.16.5 Cyclic Code

Let C be a linear code over a finite field F of **block length** n. C is called a cyclic code, if for every codeword $C = (c_1, c_2 \ldots, c_n)$ from C, the word $(c_n, c_1, \ldots, c_{n-1})$ in F^n obtained by a cyclic right shift of components is also codeword. If C is the smallest

cyclic code containing c, or, in other words C is the linear code generated by c and all codeword's obtained by cyclic shifts of its components.

I) Algebraic Structure

Cyclic codes can be linked to ideals in certain rings.

Consider the polynomial code over G f (2) = {0, 1} with n = 5, m = 2 and generator polynomial g (x) = $x^2 + x + 1$. This code consists of the following code words,

$0, x^2+x+1, x^3+x^2+x, \quad x^3+1, x^4+x^3+x^2, x^4+x^3+x+1, x^4+x, x^4+x^2+1$

equivalently expressed as strings of binary digits of the codewords are,

00000, 00111, 01110, 01001, 11100, 11011, 10010, 10101

A block code is called a cyclic code if for every codeword $(a_0\, a_1\, a_2 \ldots a_{n-2}\, a_{n-1})$ the sequence $(a_{n-1}\, a_0\, a_1\, a_2 \ldots a_{n-2})$ is also a codeword.

In other words, for a cyclic code, a cyclic shift of a codeword is also a codeword.

To describe cyclic codes,

let (F, +, ·) be a field of integer modulo 2.

Now the binary sequence $a_0\, a_1\, a_2 \ldots a_{n-1}$ can be expressed as a polynomial.

$a_0 + a_1 x + \ldots a_{n-1} x^{n-1}$ in the polynomial ring (F [x], +, ·)

Let $(F_n[x], \oplus, \odot)$ be the ring of polynomials modulo $1 + x^n$.

Let I be an ideal or $(F_n[x], \oplus, \odot)$

Now to show that polynomial in I constitute a cyclic code.

As (I, ⊕) form an abelian group, the polynomial in I constitute group codes.

Let f (x) ∈ I

then ∀ x ∈ F [x], f(x) ∈ I

$x \odot f(x) \in I$ [by definition of ideals ∀ r ∈ R, s ∈ S, r s ∈ S also s r ∈ s where R is ring and S is subring of R]

Let $f(x) = a_0 + a_1 + \ldots + a_{n-1} X^{n-1}$

then $x \odot f(x) = \text{remainder of} \left(\dfrac{a_0 x + a_1 x^2 + \ldots + a_{n-1} x^n}{1 + x^n} \right)$

$$= \frac{a_0 x + a_1 x^2 + \ldots + a_{n-1} + a_{n-1}(1 + x^n)}{1 + x^n}$$

$$= a_{n-1} + a_0 x + a_1 x^2 + \ldots a_{n-2} x^{n-1}$$

Therefore polynomial in I constitute a cyclic code.

On the other hand, let I denote a set of polynomials corresponding to the codeword in a cyclic code.

(I, ⊕) is an abelian group.

If f (x) is in I.

x ⊙ f (x) is in I.

Hence x ⊙ (x ⊙ f (x)) = x^2 ⊙ f (x) is in I.

and this way

x^i ⊙ f (x) is in I.

and b_j x^j ⊙ f (x) is in I.

Because the polynomials in I form a cyclic code, it follows that B (x) ⊙ f (x) is in I for B (x) in F_n [x].

9.17 Case Studies

Case Study : Cryptography used in a world-war II -

- Cryptography is a technique of securing information and communication by converting it to non-readle format through the use of code. This information can understand the person for whome the information is intended. Thus, preventing unauthorized access to information.
- Cryptography contains two phases encryption and decryption. In encryption original message (plain text) is converted into a non-readable format (cipher text). It is done at the sender's end. In decryption, again cipher text is converted into plain text. It is done at receiver's end. There are different algorithms used for cryptography some wellknown algorithms are RSA, DES, AES, etc. There are two types of algorithms public key cryptography and private key cryptography. RSA belongs to public key cryptography and DES, AES belongs to private key cryptography.
- One of the most famous example is Engima machine used by German Government and military during word-war II. The standard military Engima used there 26-point wired metal and black-plastic rotors selecting from a set of 5 to 8. Each rotor was a cylinder with a large, movable, notched wheel on one end with an alphabet or numbers around its circumference. One face of the cylinder had 26 spring-loaded copper pin protruding from it and the other face had 26 flush copper contacts. Inside each cylinder was a wired 'maze' connecting the contacts to the pins.

- Engima used 3 rotors, each one wired differently and a reflector. When a letter was input, the rightmost wheel would turn one space forward and electrical pulse would rotate through each wheel in turn.
- It uses a monoalphabetic substitution cipher, rotation cipher and rotating substitution cipher. A alphabet encypts to many letter. Repeated letters will not encrypt the same way in different positions.

9.18 University Questions with Answers

Dec. - 2016

Q.1 *Consider binary relation * defined on the set :*
A = {a, b, c, d} by the following table. Fill the empty cell. **[2]**

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b				
c				

Ans. : Given that,

$$A = \{a, b, c, d\}$$

Given table :

x	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b				
c				

a and c ε A but a * c = e which is not in A

∴ The relation is not binary relation on A.

Hence this equation is wrong.

Q.2 *Prove that :*
$((a+b\sqrt{2}),+,*)$ where a, b ε R is integral domain. **(Refer example 9.15.2)** **[6]**

Q.3 *Define normal subgroup and rings with example.*
(Refer sections 9.11 and 9.15.1) **[5]**

Q.4 *Let R = {0°, 60°, 120°, 180°, 240°, 300°} and * is a binary operation so that a and b in R, a * b is overall angular rotation corresponding to successive rotations by a and then b. Show that (R, *) is a group.* **[6]**

Ans. : Given that,

$$R = \{0°, 60°, 120°, 180°, 240°, 300°\}$$

Consider the composition table,

*	0°	60	120	180	240	300
0	0	60	120	180	240	300
60	60	120	180	240	300	0
120	120	180	140	300	0	60
180	180	240	300	0	60	120
240	240	300	0	60	120	180
300	300	0	60	120	180	240

i) Every element of table is present in R

∴ R is closed w.r.t. *

ii) Rotation of objects is associative

∴ * is associative in R.

iii) From the composition table, 0° is the identity element in R.

iv) From table, the inverses of 0, 60, 120, 180, 240, 300 are 0, 300, 240, 180, 120, 60 respectively.

Thus (R, *) is a group.

Q.5 *Let A = {0, 1}. Is A closed under : i) Multiplication ii) Addition* **[3]**

Ans. : Given that

A = {0, 1} consider the following tables

×	0	1
0	0	0
1	0	1

+	0	1
0	0	1
1	1	2

By multiplication table, 'A' is closed w.r.t. ×.

By addition table, 'A' is not closed w.r.t. + as $1 + 1 = 2 \notin A$.

Q.6 *Explain isomorphism and automorphism.* **(Refer section 9.14)** **[4]**

May - 2017

Q.7 *Prove that : $(a+b\sqrt{2}, +, x)$ where $a, b \in R$ is integral domain.* **(Refer example 9.15.2)** **[6]**

Q.8 *Explain isomorphism and homomorphism of two semigroups.* **(Refer sections 9.13 and 9.14)** **[3]**

Q.9 *Prove that every cyclic group is an abelian group.* **(Refer section 9.12 theorem 4)** **[4]**

Q.10 *Let G be set of all non-zero real numbers and let : $a * b = \frac{ab}{2}$. Show that (G, *) is an abelian group.* **(Refer example 9.4.8)** **[6]**

Q.11 *Explain Galois theory.* **[3]**

Ans. : • Galois used permutating graphs to describe how the various roots of a given polynomial equations are related to each other.

- The central idea of Galois therory is to consider permutations of roots such that any algebraic equation satisfied by the roots is still satisfied after the roots have been permuted.
- The permutations together form a permutation group, also called Galois group of polynomial.

Q.12 *Explain properties of binary operations.* **(Refer section 9.2.2)** **[4]**

Dec. - 2017

Q.13 *Consider the set A = {1, 3, 5, 7, 9,.....} i.e. a set of odd positive integers. Determine whether A is closed under : i) Addiion ii) Multiplieation* **[4]**

Ans. : Given that A = {1, 3, 5, 7, 9, ...}

i) **Addition :** We know that the addition of two odd numbers is an even number

i.e. $x, y \in A \Rightarrow x + y$ = even number $\notin A$.

In particular, $1, 3 \in A$ but $1 + 3 = 4 \notin A$

Addition is not closed on A.

ii) **Multiplication :** We know that the multiplication of two odd numbers is an odd number

$\therefore$ For any $x, y \in A$, $x \times y$ is an odd number

$\therefore$ $x \times y \in A$

$\therefore$ Multiplication is closed on A.

Q.14 *Check whether the algebraic system (A, *) whose table is given below is a SEMI group.* **[4]**

*	a	b	c
a	a	b	c
b	a	c	b
c	a	b	c

Ans. : An algebric structure (A *) is a semi group if a * (b * c) = (a * c) * c

$\forall$ a, b, c, $\in$ A

i.e. * is associative on A.

Consider

(a * c) * b = c * b = b

a * (c * b) = a * b = b

Now, (b * a) * c = a * c = c

b * (a * c) = b * c = b

$\therefore$ (b * a) * c $\neq$ b * (a * c) $\Rightarrow$ * is not associative

$\therefore$ (A *) is not a semigroup

Q.15 *Discuss in brief about the Galois theory - Field theory and group theory* **(Refer Q.11).** **[5]**

Q.16 *Define algebraic system. Explain the steps to identify the following with suitable example.*

i) Monoid **(Refer section 9.3.3)**

ii) Abelian group **(Refer section 9.4.1)**

iii) Ring. **(Refer section 9.15.1)** **[13]**

May - 2018

Q.17 *Let R = {0, 60, 120, 180, 240, 300} and * = binary operation so that for a and b in R, a * b is overall angular rotation corresponding to successive rotation by a and by b. Show (R, *) is a group.* **(Refer Q.4)** **[7]**

Q.18 *Explain the following terms with examples :*

i) Ring **(Refer section 9.15.1)** *ii) Integral domain* **(Refer section 9.15.7)**

iii) Field **(Refer 9.15.8)** **[6]**

OR

Q.19 *Show that (I, $\oplus$, $\square$) is a commutative ring with identity where $\oplus$ and $\square$ are defined as : $a \oplus b = a + b - 1$ and $a \square b = a + b$* **[7]**

Ans. : We have (I, ⊕, □)

We have, $a \oplus b = a + b - 1$ and $a \square b = a + b$

I) (R, ⊕) is an abelian group

i) Closure property : for any $a, b \in I$

$$a \oplus b = a + b - 1 \in I$$

ii) Associativity : for any $a, b, c \in I$

$$(a \oplus b) \oplus c = (a + b - 1) \oplus c = a + b - 1 + c - 1 = a + b + c - 2$$

$$a \oplus (b \oplus c) = a \oplus (b + c - 1) = a + b + c - 1 - 1 = a + b + c - 2$$

$$\therefore (a \oplus b) \oplus c = a \oplus (b \oplus c)$$

iii) Existence of the Identity :

for any $a \in I, \exists 1 \in I$ S.t.

$a \oplus 1 = a + 1 - 1 = a = 1 \oplus a$

∴ 1 is called the identify element.

iv) Existence of the inverse : For any $a \in I$

$\exists 2 - a \in I$ such that $a \oplus (2 - a) = a + 2 - a - 1 = 1$

∴ $2 - a$ is the inverse of a

v) Commutative property : For any $a, b \in I$

$a \oplus b = 4 + b - 1 = b + a - 1 = b \oplus a$

II) (I, □) is semigroup

i) Closure property : $a \square b = a + b \in I$

ii) Associative property : $(a \square b) \square c = a \square (b \square c) \forall a, b, c \in I$

III) Multiplication distributes over addition :

$a \square (b \oplus c) = a \square [b + c - 1] = a + b + c - 1$

$(a \square b) \oplus (a \square c) = (a + b) \oplus (a + c) = a + b + a + c - 1$

$a \square (b \oplus c) \neq (a \square b) \oplus (a \square c)$

Multiplication is not distribute over addition.

∴ (I, ⊕ , □) is not a ring, Hence. it is not commutative ring.

Q.20 *Explain the following terms :*

i) Monoids **(Refer section 9.3.3)**

ii) Sub-group **(Refer section 9.7.2)**

iii) Group codes **(Refer section 9.16)** **[6]**

Dec. - 2018

Q.21 *Let $Z_n = \{0, 1, 2, ..., n-1\}$. In Z_{13} what is the order of 3, 6 and 8.* **[3]**

Ans. : We have $Z_{12} = \{0, 1, 2, 3, ..., 11\}$

In Z_{12}, $+_{12}$ is the binary operation.

n is the order of x in Z_{12} if nx = 0

We have,

$$4(3) = 0 \therefore 0(3) = 4$$

$$2(6) = 0 = 12, \quad \therefore \quad 0(6) = 2$$

$$3(8) = 24 = 0, \quad \therefore \quad 0(8) = 3$$

Q.22 *Let (Q, *) is an Algebraic system. * is a binary operation defined as $a * b = a + b - ab$ $\forall$ a, b $\in$ Q. Determine whether (Q, *) is a group.*
(Refer similar example 9.4.3) **[4]**

Q.23 *Define :*
i) Rings **(Refer section 9.15.1)**
ii) Integral domain **(Refer section 9.15.7)**
iii) Field **(Refer section 9.15.8)** **[6]**

Q.24 *Let $Z_n = \{0, 1, 2, ..., n-1\}$. Let * be a binary operation such that a * b = remainder of (a + b) divided by n. Construct a table for n = 4. Is (Z_4, *) a monoid, semigroup group and abelian group.* **[7]**

Ans. : Let $Z_4 = \{0, 1, 2, 3\}$

In, Z_4, a * b = Remainder of (a + b) divided consider the following table

*	0	1	2	3
0	$\boxed{0}$	1	2	3
1	1	2	3	$\boxed{0}$
2	2	3	$\boxed{0}$	1
3	3	$\boxed{0}$	1	2

i) All elements of table are in Z_4 $\therefore$ * is closed on Z_4

ii) As * is defined in terms of addition

$\therefore$ * is associative on Z_4

iii) 0 is the identity element w.r.t *

iv) The inverse of 0 is 0 $\therefore$ $0^{-1} = 0$

$$(1)^{-1} = 3, (2)^{-1} = 2, (3)^{-1} = 1$$

$\therefore$ Every element in Z_4 has inverse in Z_4 w.r.t. *

v) Table is symmetric $\therefore$ a * b = b * a

Hence (Z, *) is a monoid, semigroup, group and abelian group.

Q.25 *Define :*
(i) Group code **(Refer section 9.16)**
(ii) Galois theory **(Refer Q.11)**
(iii) Cyclic group **(Refer section 9.10)** **[6]**

May - 2019

Q.26 *What is Monoid ? Show that the algebraic structure (A,+) is a monoid, where A is set of integers and + is a binary operation giving addition of two integers.*
(Refer section 9.3.3 and example 9.3.1) **[3]**

Q.27 *Define the following terms.*
i) Ring **(Refer section 9.15.1)**
ii) Field **(Refer section 9.15.8)**
iii) Integral domain. **(Refer section 9.15.7)** **[3]**

Q.28 *Show that* $R = \{a + b\sqrt{2} \; ; b \in I\}$ *for the operation* $+, \cdot$ *is integral domain but not a field.*
(Refer example 9.15.2) **[7]**

Q.29 *Let A = {0,1}. Is a closed under*
1) Multiplication 2) Addition **(Refer Q.5)** **[4]**

Q.30 *Define*
1) Properties of binary operations
2) Ring with unity **(Refer section 9.15.1)** **[4]**

Q.31 *Let R = {0,60,120,180,240,300} and * = binary operation so that for a and b in R a * b is overall angular rotation corresponding to successive rotations by a and by b show (R, *) is a group.* **(Refer Q.4)** **[5]**

Q.32 *Write properties of binary operations.* **(Refer section 9.2.2)** **[5]**

Q.33 *Prove that the set 2 of all integers with binary operation * defined by :* **[5]**
$a * b = a + b + 1 \quad \forall a, b \in 2$ *is an abelian group.* **(Refer section 9.4.14)**

Q.34 *Let A {0, 1}. Is A closed under : 1) Multiplication 2) Addition.* **[3]**

Ans. : 1) Multiplication

*	0	1
0	0	0
1	0	1

Every element of table belongs A. i.e. $0, 1 \in A$

Hence A is closed w.r.t multiplication

2) Addition

+	0	1
0	0	1
1	1	2

Here $2 \notin A$. Hence A is not closed w.r.t addition.

Q.35 *$f : G \to G$, G is group with identity 'e' such that $f(a) \in e$ for all $a \in G$ prove that function f is homomorphism.* **(Refer example 9.14.1)** **[5]**

Q.36 *In the set R of real number. Decide whether the following composition is associative $a, b, c \in R$: (1) $a*b = a + 2b$ (2) $a* b = a$.* **[4]**

Ans. : Associate property : A binary operation '*' on set A is said to be associative if $a * (b * c) = (a * b) * c \ \forall \ a, b, c \in A$

1) $a * b = a + 2b$

$$a * (b * c) = a * (b + 2c)$$
$$= a + 2 (b + 2c)$$
$$= a + 2b + 4c$$
$$(a * b) * c = (a + 2b) * c$$
$$= (a + 2b) + 2c$$
$$= a + 2b + 2c$$
$$a * (b * c) \neq (a * b) * c$$

Hence * is not associative

2) $a * b = a$

$$a * (b * c) = a * b = a$$

$(a * b) * c = a * c = a$

Hence * is associative

Q.37 *Prove that the set : A = {0, 2, 4, 6, 8} with t_{10} and x_{10} operation i.e. R = {A, t_{10}, x_{10}} is a ring.* **[4]**

Ans. :

$+_{10}$ = Addition modulo 10

$*_{10}$ = Multiplication modulo 10

Consider the table for $+_{10}$

I) $(A, +_{10})$

i) From Table 9.18.1, all elements are in A.

i.e. A is closed with respect to $+_{10}$

ii) Associativity : For all a, b, c ∈ A

$a +_{10} (b +_{10} c) = (a +_{10} b) +_{10} c$

iii) 0 is the identity element

iv) Each element of A has inverse element

i.e. 0 is the inverse of 0, 2 is the inverse of 8, 4 is the inverse of 6.

v) Commutativity : For all a, b ∈ A

$a +_{10} b = b +_{10} a$

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

Table 9.18.1

$*_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

Table 9.18.2

Hence $(A, +_{10})$ is an abelian group

II) $(A, +_{10})$

i) From Table 9.18.2, all elements are in A

i.e. A is closed w.r.t $*_{10}$

ii) Associativity

For all a, b, c $\in$ A

$a *_{10} (b *_{10} c) = (a *_{10} b) *_{10} c$

Hence $(A, *_{10})$ is a semigroup

$\therefore (A, +_{10}, *_{10})$ is a ring.

❑❑❑

SOLVED MODEL QUESTION PAPER

(As Per 2019 Pattern)

Discrete Mathematics

S.E. (Comp) Sem - I (End Sem)

Time : 2 $\frac{1}{2}$ Hours] **[Total Marks : 70**

Instructions to the candidates :

1) **Answer Q.1 or Q.2, Q.3 or Q.4, Q.5 or Q.6, Q.7 or Q.8.**
2) **Figures to the right indicate full marks.**
3) **Neat diagrams must be drawn wherever necessary**
4) **Assume suitable data if necessary.**

Q.1 a) *How many permutations can be made out of the letter of word "COMPUTER"? How many of these : i) begin with C ii) end with R iii) begin with C and end with R iv) C and R occupy the end places.* **(Refer example 6.3.7)** **[8]**

b) *From 10 programmers in how many ways can 5 be selected when i) A particular programmer is included every time ii) A particular programmer is not included at all time.* **(Refer example 6.4.5)** **[6]**

c) *In the expression of $(1+x)^6$. What is the coefficient of x^3.*
(Refer Q.17 of section 6.8) **[4]**

OR

Q.2 a) *If a committee has eight members :*
i) How many ways can committee members be seated in a row ?
ii) How many ways can the committee select a president, vice-president and secretary ? **(Refer Q.15 of section 6.8)** **[6]**

b) *Solve the following :*
i) How many different car number plates are possible with 2 letters followed by 3 digits ? ii) How many of these number plates begin with 'MH' ?
(Refer Q.11 of section 6.8) **[6]**

c) *i) Find 8th term in the expression $(x+y)^{13}$.* **(Refer Q.21 of section 6.8)**
ii) Expand $(3x-4)^4$ using binomial theorem. **(Refer Q.23 of section 6.8)** **[6]**

Q.3 a) *Determine whether graphs G and H are isomorphic or not. Justify your answer.* **(Refer example 7.7.6)** **[6]**

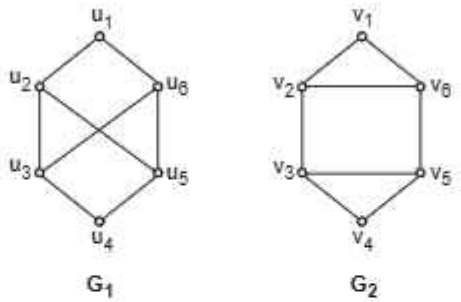


Fig. 1

b) *Use Dijkstra's algorithm to find the shortest path between a and z.*
(Refer example 7.13.1) **[6]**

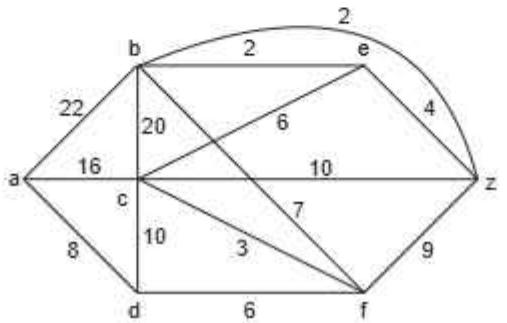


Fig. 2

c) *Determine which of the graphs below represents Eulerian circuit, Eulerian path, Hamiltonian circuit and Hamiltonian path. Justify your answer.*
(Refer Q.1 of section 7.22) **[5]**

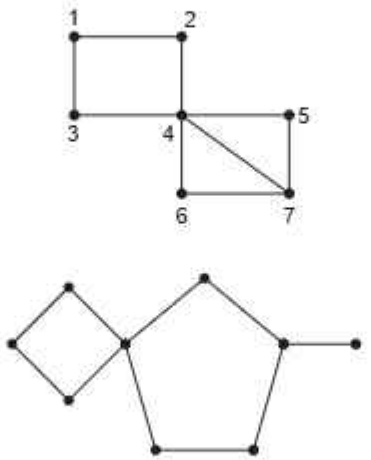


Fig. 3

OR

Q.4 a) *Use dijkstra's algorithm to find the shortest path between a and z.* **(Refer Q.16 of section 7.22)** **[6]**

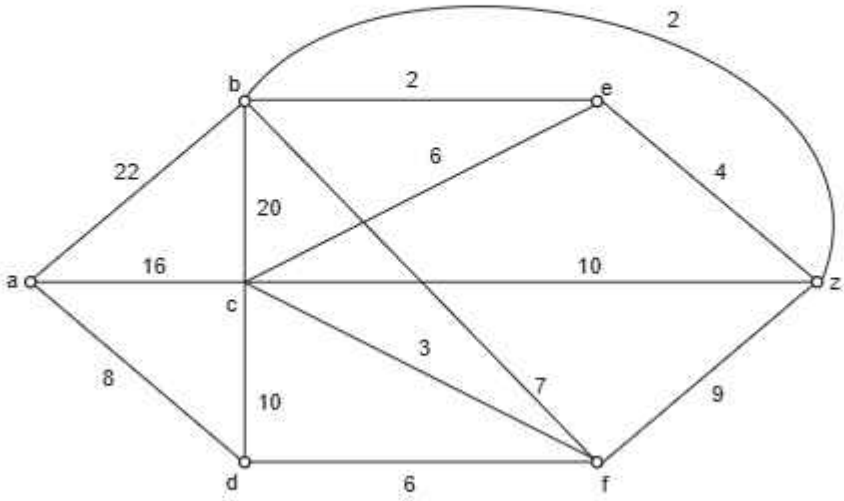
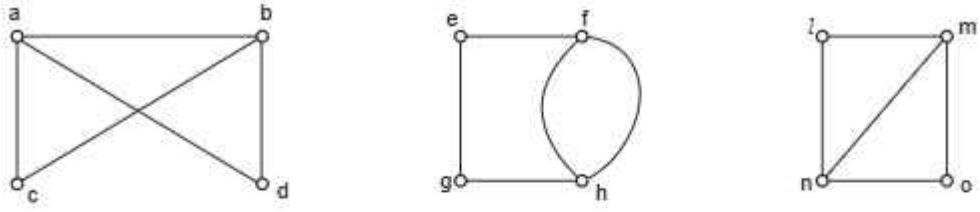


Fig. 4

b) *Determine whether the following graphs are isomorphic to each other.* **(Refer Q.21 of section 7.22)** **[6]**



(a) (b) (c)

Fig. 5

c) *i) Define the graph K_n, K_{mn}.* **(Refer section 7.5 (ii) and (iv))**

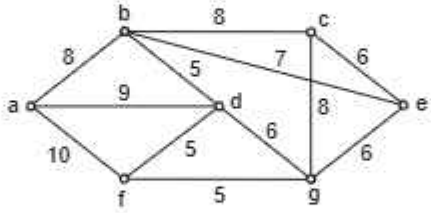
ii) How many colors required to color K_{mn} ? **(Refer Q.5 of section 7.22)** **[5]**

Q.5 a) *Suppose data items A, B, C, D, E, F, G occur in the following frequencies.*

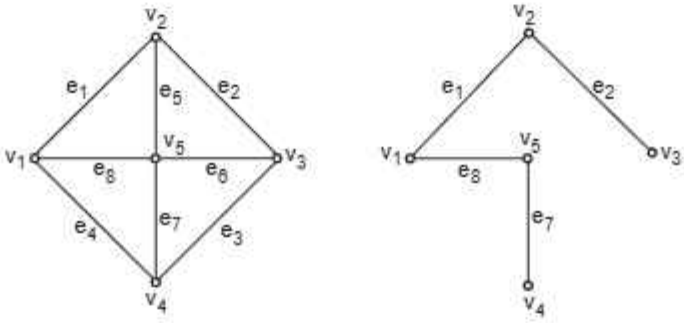
Data Items	A	B	C	D	E	F	G
Weight	10	30	5	15	20	15	05

Construct a Huffman code for the data.
What is the minimum weighted path length.
(Refer example 8.6.2) **[6]**

b) *Find the minimum spanning tree for the graph given in the following figure using Prim's algorithm.* **(Refer example 8.7.1)** **[6]**



c) *Find the fundamental cutsets and fundamental circuits of the following graph w.r.t. given spanning tree.* **(Refer example 8.8.2)** **[6]**



OR

Q.6 a) *Use labelling procedure to find a maximum flow in the transport network shown in the following Fig. 6. Determine the corresponding minimum cut.*
(Refer example 8.9.3) **[6]**

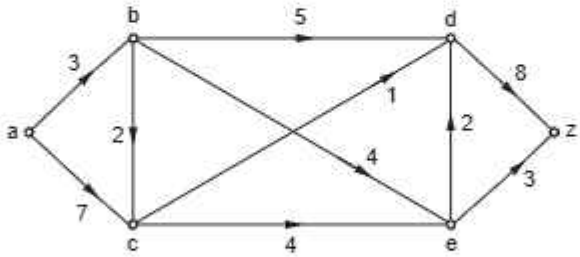
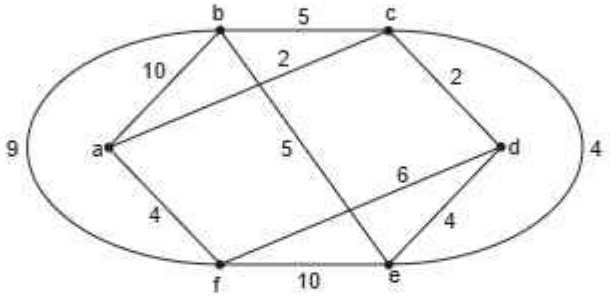


Fig. 6

b) *Use the Kruskal's algorithm to find the minimum spanning tree for the graph shown.* **(Refer example 8.7.5)** **[6]**



c) *Find the pre, post and inorder traversal of a tree shown in Fig. 7.*
(Refer Q.14 of section 8.12) **[6]**

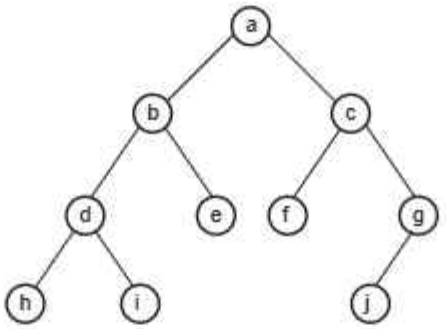


Fig. 7 Tree

Q.7 a) *Let R = {0°, 60°, 120°, 180°, 240°, 300°} and * is a binary operation so that a and b in R, a * b is overall angular rotation corresponding to successive rotations by a and then b. Show that (R, *) is a group.* **(Refer Q.4 of section 9.18)** **[6]**

b) *Prove that :* $(a+b\sqrt{2}, +, x)$ *where a, b ∈ R is integral domain.*
(Refer Q.7 of section 9.18) **[5]**

c) *Explain the following terms :*
i) Monoids ii) Sub-group iii) Group codes **(Refer Q.20 of section 9.12)** **[6]**

OR

Q.8 a) *Show that (I, ⊕, ⊓) is a commutative ring with identity where ⊕ and ⊓ are defined as :* $a \oplus b = a + b - 1$ *and* $a \sqcap b = a + b$.
(Refer Q.19 of section 9.18) **[6]**

b) *Find the number of code words generated by the parity check matrix H given by* $H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ *find all the code words generated.*

(Refer example 9.16.4) **[6]**

c) *Define :*

i) Rings ii) Integral domain iii) Field **(Refer Q.23 of section 9.18)** **[5]**

❑❑❑