Discrete Mathematics

(Code - 214441)

Semester III - Information Technology
(Savitribai Phule Pune University)

Strictly as per the New Credit System Syllabus (2019 Course) of Savitribai Phule Pune University w.e.f. academic year 2020-2021

Dr. Shafi K. Pathan
Ph.D. (Computer Science & Engineering)
Professor, Dept. of Computer Engineering
SKN College of Engineering
Vadagaon, Pune - 41

Rameshwar M. Indoriya
Ph.D. (Scholar), M.Tech. (CSE)
Assistant Professor,
Dept. of Computer Engineering
SKN College of Engineering
Vadagaon, Pune – 41

Lomesh K. Ahire
Ph.D. (Scholar), M.Tech. (CSE)
Assistant Professor, Dept. of Computer Engineering
SKN College of Engineering
Vadagaon, Pune - 41

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Discrete Mathematics (Code : 214441)
Dr. Shafi K. Pathan, Rameshwar M. Indoriya, Lomesh K. Ahire,
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Address For Correspondence
Tech-Neo Publications LLP
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Email : info@techneobooks.com
Website : www.techneobooks.com
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Preface

Dear students,

We are extremely happy to present the book of **"Discrete Mathematics"** for you. We have divided the subject into small chapters so that the topics can be arranged and understood properly. The topics within the chapters have been arranged in a proper sequence to ensure smooth flow of the subject.

We are thankful to Shri. Sachin Shah for the encouragement and support that they have extended to me. We are also thankful to the staff members of Tech-Neo Publications and others for their efforts to make this book as good as it is. We have jointly made every possible efforts to eliminate all the errors in this book. However if you find any, please let us know, because that will help us to improve further.

We are also thankful to our family members and friends for their patience and encouragement.

- Authors

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Syllabus...

Savitribai Phule Pune University
Second Year of Information Technology (2019 Course)
Discrete Mathematics (214441)

Teaching Scheme:	Credit	Examination Scheme:
Theory (TH) : 03 Hours/Week	03	Mid_Semester : 30 Marks
Tutorial(TUT) : 01 Hours/Week	01	End_Semester : 70 Marks
		Term Work : 25 Marks

Prerequisite Courses, if any : Basic Mathematics

Companion Course, if any : ---

Course Objectives :

1. To gain sound knowledge to formulate and solve problems with sets and propositions.
2. To understand and solve counting problems by applying elementary counting techniques to solve problems of discrete probability.
3. To understand Graph and Tree terminologies and models to be applied in real life problems.
4. To recognize types of relation, formulate and solve problems with relations and functions.
5. To understand basics of number theory and its applications.
6. To understand the various types' algebraic structures and its applications.Course Outcomes:

Course Outcomes :

On completion of the course, students will be able to –

- CO1 : Formulate and apply formal proof techniques and solve the problems with logical reasoning.
- CO2 : Analyze and evaluate the combinatorial problems by using probability theory.
- CO3 : Apply the concepts of graph theory to devise mathematical models.
- CO4 : Analyze types of relations and functions to provide solution to computational problems.
- CO5 : Identify techniques of number theory and its application.
- CO6 : Identify fundamental algebraic structures.

Course Contents

In Sem Exam 30 Marks (Unit I, II)

▶ Unit I : Sets And Propositions **(06 hrs + 2 hrs Tutorial)**

Sets : Sets, Combinations of Sets, Venn Diagram, Finite and Infinite Sets, Countable Sets, Multisets, Principle of Inclusion and Exclusion, Mathematical Induction.

Propositions : Propositions, Logical Connectives, Conditional and Bi-conditional Propositions, Logical Equivalence, Validity of Arguments by using Truth Tables, Predicates and Quantifiers, Normal forms.

Applications of Sets and Propositions. **(Refer chapters 1 and 2)**

› Mapping of Course Outcomes for Unit I CO1

▶ **Unit II : Combinatorics And Discrete Probability** **(06 hrs + 2 hrs Tutorial)**

Combinatorics : Rules of Sum and Product, Permutations, Combinations.

Discrete Probability : Discrete Probability, Conditional Probability, Bayes Theorem, Information and Mutual Information, Applications of Combinatorics and Discrete Probability. **(Refer chapter 3)**

Mapping of Course Outcomes for Unit II	CO2

End Sem Exam 70 Marks (Unit III, IV, V, VI)

▶ **Unit III : Graph Theory** **(06 hrs + 2hrs Tutorial)**

Graphs : Basic Terminologies, Multi-Graphs, Weighted Graphs, Sub Graphs, Isomorphic graphs, Complete Graphs, Regular Graphs, Bipartite Graphs, Operations on Graphs, Paths, Circuits, Hamiltonian and Eulerian graphs, Travelling Salesman Problem, Factors of Graphs, Planar Graphs, Graph Colouring.

Trees : Tree Terminologies, Rooted Trees, Path Length in Rooted Trees, Prefix Codes, Spanning Trees, Fundamental Cut Sets and Circuits, Max flow –Min Cut Theorem (Transport Network).

Applications of Graph Theory. **(Refer chapters 4 and 5)**

Mapping of Course Outcomes for Unit III	CO3

▶ **Unit IV : Relations And Functions** **(06 hrs + 2hrs Tutorial)**

Relations : Properties of Binary Relations, Closure of Relations, Warshall'sAlgorithm, Equivalence Relations, Partitions, Partial Ordering Relations, Lattices, Chains and Anti Chains.

Functions : Functions, Composition of Functions, Invertible Functions, Pigeonhole Principle, Discrete Numeric Functions.

Recurrence Relations : Recurrence Relation, Linear Recurrence Relations with Constant Coefficients, Total Solutions, Applications of Relations and Functions. **(Refer chapters 6 and 7)**

Mapping of Course Outcomes for Unit IV	CO4

▶ **Unit V : Introduction To Number Theory** **(06 hrs + 2hrs Tutorial)**

Divisibility of Integers : Properties of Divisibility, Division Algorithm, Greatest Common Divisor GCD and its Properties, Euclidean Algorithm, Extended Euclidean Algorithm, Prime Factorization Theorem, Congruence Relation, Modular Arithmetic, Euler Phi Function, Euler's Theorem, Fermat's Little Theorem, Additive and Multiplicative Inverses, Chinese Remainder Theorem. **(Refer chapter 8)**

Mapping of Course Outcomes for Unit V	CO5

▶ **Unit VI : Algebraic Structures** **(06 hrs + 2hrs Tutorial)**

Algebraic Structures : Introduction Semigroup, Monoid, Group, Abelian Group, Permutation Groups, Cosets, Normal Subgroup, Codes and Group Codes, Ring, Integral Domain, Field.

Applications of Algebraic Structures. **(Refer chapter 9)**

Mapping of Course Outcomes for Unit VI	CO6

□□□

INDEX

In Sem

End Sem

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UNIT I

Sets And Propositions

>> **Syllabus :**

- **Sets :** Sets, Combinations of Sets, Venn Diagram, Finite and Infinite Sets, Countable Sets, Multisets, Principle of Inclusion and Exclusion, Mathematical Induction.

- **Propositions :** Propositions, Logical Connectives, Conditional and Bi-conditional Propositions, Logical Equivalence, Validity of Arguments by using Truth Tables, Predicates and Quantifiers, Normal forms.

  Applications of Sets and Propositions.

Mapping of Course Outcomes for Unit I	CO1

UNIT I

CHAPTER 1 Set Theory

Syllabus

Sets : Sets, Combinations of Sets, Venn Diagram, Finite and Infinite Sets, Countable Sets, Multisets, Principle of Inclusion and Exclusion, Mathematical Induction.

A={1, 2, 3}

1.0 INTRODUCTION TO DISCRETE MATHEMATICS

- It is a branch of mathematics deals with discrete or distinct objects.
- Here, discrete means individual, separate, distinguishable or discontinuous.
- For example, the number of ceiling fans in a conference hall, will be counted as 4, 5, 6 or 7 but not as 4.5, 5.6 or 6.5 etc.
- So, the concepts of discrete mathematics based on counting.
- It focuses on whole numbers rather than floating numbers.
- For good understanding, let us take a simple example of adding up everything that exist between 0 and 7.
- In continuous mathematics, the calculation of addition will go like this,

$$\int_0^7 x \cdot dx = \left[\frac{x^2}{2}\right]_0^7 = \frac{7^2}{2} - 0 = \frac{49}{2} = 24.5$$

- In discrete mathematics, the above calculation will go like this,

$$\sum_{i=0}^{7} x_i = 0+1+2+3+4+5+6+7 = 28$$

- You can see from above calculation of adding number, the discrete mathematics is much simpler than continuous mathematics.
- That's why it is widely used in the field of computer science for programming, data structure and many more.

Significance of Discrete Mathematics in Computer Engineering

- In the recent years, discrete mathematics gain more attention due to exponential growth of computer science field.
- It can be considered as backbone for Computer Engineering.
- It also consider as the mathematical language of computer science.
- The concepts and notations used in discrete mathematics are basis for understanding and describing objects and problems in computer engineering fields, such as software development, programming languages, cryptography, algorithms complexity and automated theory proving and many more.
- In operation research, many real world idea's are derived from discrete mathematics to solve computer problems.
- Actually, set, relation, tree, probability, graph, algebraic structure, mathematical logic are important concepts from discrete mathematics which would be used in computer engineering to solve many problems.
- Mathematical logic is used in computer architecture design, in artificial intelligence for logical inferences.
- More importantly, graph theory, utilized in social networks like Facebook, Twitter for finding the mutual friends, suggested friends etc.
- Discrete mathematics play very important role in data analytics.

1.1 INTRODUCTION

- Set theory deals with the concepts of objects and their relationships.
- It is the basic and most important concepts for discrete mathematics.

Naive set Theory

- The concept of set theory was first introduced by Georg Cantor (German Mathematician) to explain the occurrence of element infinite times.
- Cantor wrote about "set of real numbers is uncountable".
- According to Naïve set theory, set is a collection of objects with similar properties.
- Here, objects are elements or members of the set.

1.2 SET

Definition : A set is an unordered collection of distinct objects.

- A set is act as container which holds its members or elements of similar properties.
- A set may contains finite or infinite number of elements.
- If set contains no element then it is null or empty set denoted by ϕ.
- Set is denoted by **capital letters** that is A, B, C ....
- The elements of set are denoted by lower case letters like x, y, a, b, c etc.
- If an element 'x' is member of set A, then it is represented as $x \in A$.
- If an elements 'x' is not member of set A, then it is represented as $x \notin A$.
- Here, the symbol '$\in$' indicate membership of set.

Examples :

i) Set of letters forming the words "computer" denoted as
   A = {c, o, m, p, u, t, e, r}
   Here, A' is name of set and all the elements of set A are included in pair of curly bracket.

ii) The set of subjects of second year Computer Engineering. S = {dm, oop, deld}

iii) The set of students of second year.

iv) The set of positive integers less than 10. z = {1,2,3,....9}

v) The set of natural numbers greater than 5. This will be infinite set but the elements in the set can be listed as 6, 7, 8, 9 ...

Note : (i) Objects in the definition of set could be anything e.g. books, pens, numbers, mobile numbers, animal, chairs, countries etc.

(ii) Set may contains unrelated elements for example A = { 2, a, ram, India, Mango }

1.3 REPRESENTATION OF SET

There are three ways to represent a particular set.

1.3.1 Roster Method

- In this method, all the elements of set described between pair of braces.
- It is generally used when elements or members of set are limited.
- The order of elements in a set is not important.

Example :

A = {6, 8, 3, 4, 5}, B = {x, y, a, b, c}

Here members of the set are written in between pair of curly braces.

1.3.2 Set Builder Method

- When the elements of set are huge, as it is not possible to write all the members or object using roster method in that condition set builder method is used.
- Elements of set can be group together by stating their common properties.
- **Example :**

  A = { x | x is natural number and x < 10 }

Explanation : Here, x is any arbitrary element which belongs to set of natural number and it's value is less than 10. So, A = {1, 2, 3, 4, 5, 6, 7, 8, 9}

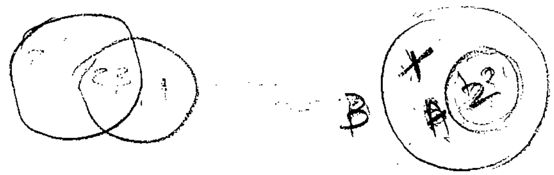
1.3.3 Statement Method

- In this method, elements of set who have common properties written in the form of statement.
- **Examples :**
  (i) The set of all books in the library.
  (ii) The set of natural numbers less than 15.
  (iii) The set of even numbers less than 8.

1.4 SOME SPECIAL SETS

- Some sets occurs frequently in the set theory, so it is important to understand those sets.
- Special symbols are used to represent them. Such symbols are as follows :

	Symbols	Meaning
i)	N	The set of natural numbers or positive integers. e.g. N = {1, 2, 3, 4, ...}
ii)	Z	The set of all integers. e.g. Z = {–, 3, –2, –1, 0, 1, 2, 3, 4, ...}
iii)	Q	The set of rational numbers e.g. $Q = \left\{ \frac{x}{y} \middle| x, y \in z \right\}$
iv)	II	The set of irrational numbers e.g. $II = \left\{ \sqrt{2}, \pi, \sqrt[3]{5}, \sqrt{5} \ldots\ldots \right\}$ $II = \{ x \mid x \in R \text{ and } x \notin Q \}$



P(A)

$A \subseteq B$ $B \supset A$ C

	Symbols	Meaning
v)	R	The set of real numbers.
vi)	σ	The set of complex numbers.
vii)	Q^+	The set of positive rational number
viii)	Z^+	The set of positive integers.

To understand the above special sets, it is important to look at below diagram of number system.

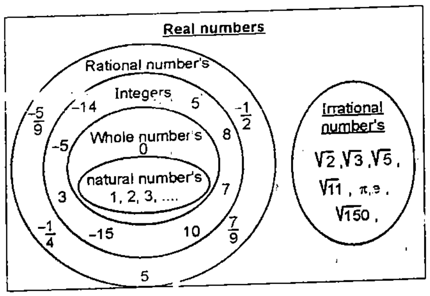


Fig. 1.4.1

Note : From the above diagram it is clear that,

$N \subseteq Z \subseteq Q \subseteq R \subseteq C$

1.5 SUBSETS

- If every element in set 'A' is also an element of a set B, that is $a \in A$ and $a \in B$. Then set 'A' is a subset of set 'B' or set 'A' is contained in set 'B'.
- It is denoted as $A \subseteq B$ or $B \supseteq A$

  Here, $A \subseteq B$ means A is a subset of B.

  $B \supseteq A$ mean B is superset of A.
- If at least one element of set A is not contained in set B. Then we can say that A is not subset of B written as $A \nsubseteq B$.
- Let's take a simple **example** to understand this concept.

  Let, $A = \{5, 6, 8\}$

  $B = \{1, 8, -3, 5, 12, 6\}$

  $C = \{15, -3, 5, 8\}$

Here, $A \subseteq B$ (All the elements of A are contained in B)

$A \nsubseteq C$ (At least one element i.e. 6 not contained in C. So A is not subset of C)

Note

(i) $A \subseteq A$ — (every set is a subset of itself, which means every element in A belongs to A)

(ii) $\phi \subseteq A$ or $\{\} \subseteq A$ — (null set or empty set is a subset of any set)

(iii) If $A \subseteq B$ and $B \subseteq A$, then $A = B$ — Here, all the elements of A present in B and all the elements of B present in A which means both A and B are equal

(iv) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ — Every elements of set A belongs to set B and every element of set B belongs to a set C. Then clearly every element of A belongs to C.

1.5.1 Proper Subset

A set 'A' is said to be proper subset of B if and only if.

(i) $A \subseteq B$ i.e. all elements of A belongs to B.

(ii) $B \nsubseteq A$ i.e. at least one element from B which is not in A.

- In other words, if $A \subseteq B$ and $A \neq B$, then we say that A is proper subset of B.
- It is denoted as $A \subset B$ or $B \supset A$.
- Indirectly, we can say that for $A \subset B$, A has some elements of set B.
- In short, proper subset is nothing but subset but not equal to

Example:

Let, $A = \{3, 6, 8\}$, $B = \{10, 3, 8, 5, 6\}$

Here, all the elements of A belongs to B which means $A \subseteq B$ and at least one element of B not belongs to A i.e. $B \nsubseteq A$. So, $A \subset B$ (A is a proper subset of B).

1.5.2 Improper Subset

- When all the elements of set A contained in set B. Then A is a improper subset of B.
- **Example :** $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 1, 2\}$

  Here, A is a improper subset of B. Because all elements of A are contained in B.

Note: ϕ or null set is improper subset of any set.

1.6 TYPES OF SET

(a) Universal set

- All sets under investigation in any application of set theory are assumed to belong to some fixed large set called as universal set.
- Denoted by U
- In other words, universal set is the set of all elements under consideration.
- All other sets are subsets of the universal set.
- It is used in particular situation.
- Example, universal set can be considered as set of students in collage, set of animals on the earth, etc.

(b) Singleton set or unit set

- When a set contains only one element then that set is singleton set.
- **Example :** $A = \{6\}$, $B = \{x \mid x \in N, x < 2\}$

(c) Finite set

- When a number of elements in a set are finite then that set is said to be finite.
- Null or void set is also finite set.
- **Example :** $A = \{5, 0, 8, 9\}$

  Here, A is finite set.

(d) Infinite set

- When a set has infinite number of elements.
- Example, consider a set of natural number, considered as infinite set.

  $|N| = \infty$

(e) Empty set or Null set

- The empty set or null set is a set that has no elements or members in it.
- It is also known as void set.
- It is denoted as ϕ or $\{\}$.
- Empty set is a finite set because $n(\phi) = 0$
- Example,

  $A = \{x \mid x \text{ is a positive integer, } x^2 = 3\}$

  Set A will contain no element.

(f) Power set

- Power set is a set of all possible subsets of a set.
- The power set of A is denoted by P(A).
- $P(A) = \{X \mid X \subseteq A\}$
- Example : Let $A = \{a, c\}$

Then

Power set of A is P(A), calculated as all possible subsets of A.

ϕ is subset of A, {a}, {c}, {a, c} are also subset of A.

So, $P(A) = \{\{\phi\}, \{a\}, \{c\}, \{a, c\}\}$

- If set A contains n elements, then power set of A, P(A) will have 2^n possible subsets.

Note: If $A = \phi$, then $P(A) = \phi$

$P(\phi) = \{\phi\}$ empty set has only subset ϕ.

(g) Disjoint sets

- Two sets are said to be disjoint, if they have no common elements.
- **Example**

  $A = \{1, 2\}$, $B = \{4, 5, 6\}$ and $C = \{5, 6, 7, 8\}$

  Here, A and B disjoint, A and C are disjoint.

  But B and C are not disjoint as B and C have some common elements i.e. 5 and 6.
- In other words, If $A \nsubseteq B$ and $B \nsubseteq A$, then A and B are disjoint sets.

(h) Overlapping sets

- Two sets are said to be overlapping, if they have at least one elements in common.
- **Example**

  $A = \{3, 6, 8\}$, $B = \{10, 5, 3\}$

  Here, A and B are overlapping sets since they have one elements i.e. 3 in common.

(i) Equal sets

- Two sets A and B said to be equal if all the elements of A belongs to B and vice versa.
- If A and B are equal sets, then it is denoted by A = B
- **Example :**

  $A = \{a, c, d\}$ $B = \{d, a, c\}$

  Here, set A and B are equal sets.

(j) Equivalent set

- Two sets A and B are said to be equivalent if both sets contains same number of elements.
- If A and B are equivalent sets, then it is denoted by $\leftrightarrow$
- **Example :**

  $A = \{a, d, f\}$, $B = \{c, l, m\}$

  Here, set A and B are equivalent sets.

Note: Equal sets are always equivalent but it is not necessary that equivalent sets are equal.

$x(x^2-6x+11)-6$ $x(x^2-6x+11$ $x \in N$ (1, 3)   AB   3, $\phi = \{\}$ $x^2+6=6$, $x^2 = 6-6$, $x^2=0$ A = {1, 2, 3}

1.6.1 Examples Based on Sets and Subsets

Ex. 1.6.1 : Which of the following sets are equal

A = {a, b, c}, B = {c, b, c, a}

C = {b, a, b, c}, D = {b, c, a, b}

☑ **Soln. :**

A = B = C = D all are equal since order and repetition do not change a set.

Ex. 1.6.2 : Which of the following sets are equal ?

$A = \{x \mid x^2 - 4x + 3 = 0\}$, $B = \{x \mid x^2 - 3x + 2 = 0\}$

$C = \{x \mid x \in N, x < 3\}$, $D = \{x \mid x \in N, x \text{ is odd}, x < 5\}$

E = {1, 2}, F = {1, 2, 1}, G = {3, 1}, H = {1, 1, 3}

☑ **Soln. :**

Here, A = D = G = H are equal and set B, C, E and F are equal.

Ex. 1.6.3 : Write down following sets.

(i) $A = \{x \mid x^2 = 4\}$

(ii) $B = \{x \mid x^2 = 9, x - 3 = 5\}$

(iii) $C = \{x \mid x^2 + 1 = 0, x \text{ is complex number}\}$

(iv) $D = \{x \mid x^2 + 1 = 0, x \text{ is real number}\}$

☑ **Soln. :**

(i) A = {2, −2} (ii) B = {ϕ}

(iii) C = {i, −i} (iv) D = {ϕ}

Ex. 1.6.4 : Find out which of the following sets are null, singleton ?

(i) $A = \{x \mid x^2 = 11, s \text{ is even integer}\}$

(ii) B = {0},

(iii) $C = \{x \mid x^2 + 6 = 6\}$

(iv) $D = \{x \mid x^2 = 7, 3x = 5\}$

☑ **Soln. :**

Sets A and D are null sets as A = ϕ, D = ϕ

Sets B and C are singleton sets as B has one element and C has one element that is {0} which satisfies given condition.

Ex. 1.6.5 : Which of the following sets are equal ?

(a) A = {1, 2, 2, 3},

(b) $B = \{x \mid x^2 - 2x + 1 = 0\}$

(c) C = {1, 2, 3}

(d) $D = \{x \mid x^3 - 6x^2 + 11x - 6 = 0\}$

(e) E = {3, 2, 1}

☑ **Soln. :** Sets A, C, D, and E are equal.

Ex. 1.6.6 : Determine whether each of the following statements to is true or false, justify your answer.

(a) ϕ ⊆ ϕ (b) ϕ ∈ ϕ

(c) ϕ ⊆ {ϕ} (d) ϕ ∈ {ϕ}

(e) {ϕ} ⊊ ϕ (f) {ϕ} ∈ ϕ

(g) {ϕ} ⊆ {ϕ} (h) {ϕ} ∈ {ϕ}

(i) {a, b} ⊆ {a, b, c, {a, b, c}} (j) {a, b} ∈ {a, b, c, {a, b, c}}

(k) {a, b} ⊆ {a, b, {{a, b}}} (l) {a, b} ∈ {a, b, {{a, b}}}

(m) {a, ϕ} ⊆ {a, {a, ϕ}} (n) {a, ϕ} ∈ {a, {a, ϕ}}

☑ **Soln. :**

(a) True, since null set is a subset of any set.

(b) False, because, element can not belong to itself.

(c) True, because null set is subset of itself.

(d) True, element belongs to set.

(e) False, null set can not be a subset of it's element.

(f) False

(g) True, null set is subset of itself.

(h) False, null set can not be the member of itself.

(i) True, all the element of {a, b} are in {a, b, c, {a, b, c}}

(j) False

(k) True

(l) False

(m) False

(n) True

Ex. 1.6.7 : If A = {ϕ, {ϕ}, {ϕ, {ϕ}}}, determine whether following statements are true or false.

Justify your answer.

(a) ϕ ∈ A (b) {ϕ} ⊆ A (c) {ϕ} ∈ A

(d) {ϕ, {ϕ}} ⊆ A (e) {{ϕ}} ∈ A

☑ **Soln. :**

(a) True, As ϕ is element of set A

(b) True, As {ϕ} is a subset of A.

(c) True, As {ϕ} is a member of A.

(d) True, As, all the elements of {ϕ {ϕ}} are in A.

(e) False As {{ϕ}} is not a member of A.

Ex. 1.6.8 : Determine whether each of the following statements is true for arbitrary sets A, B, C. Justify your answer.

(i) If A ∈ B and B ⊆ C, then A ∈ C

(ii) If A ∈ E and B ⊆ C, then A ⊆ C

(iii) If A ⊆ B and B ∈ C, then A ∈ C

(iv) If A ⊆ B and B ∈ C, then A ⊆ C

☑ **Soln. :**

(i) True, for e.g. A = {ϕ}, B = {{ϕ}}, C = {{ϕ}}
As A is contained in B, it should also belong to C since B is a subset of C.

(ii) False, As A is an element of B but not subset of B.
e.g. A = {ϕ}, B = {{ϕ}}, C = {{ϕ}}

(iii) False, e.g. A = ϕ, B = {ϕ}, C = {{C}}

(iv) False. e.g. A = {ϕ}, B = {ϕ}, C = {{ϕ}}

Ex. 1.6.9 : Let, A = {x, y, {x, y}, {{x, y}}}

Identify each of the following statements as true or false. Justify your answers.

(i) x ∈ A (ii) {x} ∈ A (iii) {x, y} ∈ A

(iv) {{x, y}} ⊆ A (v) {x, y} ⊆ A (vi) {x, {y}} ⊆ A

☑ **Soln. :**

(i) True, As x is an element of A.

(ii) False. As {x} is subset of A rather it is not an element of A.

(iii) True, As {x, y} is an element of A.

(iv) True, as {x, y} is subset of A.

(v) True, as subset containing x, y elements in A.

(vi) False, x is available in A but { y } not.

1.7 DUALITY

– Principle of duality suggest that any proved result involving sets, complements and operations of union and intersection gives a corresponding dual results by replacing U by ϕ, ∪ by ∩ and vice versa.

– From the Laws of Algebra, E is an equation of set Algebra. The dual E* of E is the equation obtained by replacing each occurrence of ∪, ∩, U and ϕ in E by ∩, ∪, ϕ and U respectively.

– If any equation E is an associative then its duality equation E* is also associative.

– **Example :**

(i) (U ∩ A) ∪ (B ∩ A) = A is (ϕ ∪ A) ∩ (B ∪ A) = A

(ii) A ∪ A = A is A ∩ A = A

(iii) (A ∪ B) = (B ∪ A) is (A ∩ B) = (B ∩ A)

1.8 SET OPERATIONS

We can do many operations on sets. Some of the important operations are listed below :

(a) Union

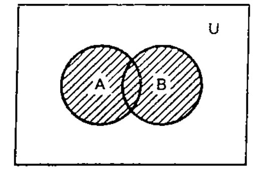
– The union of two sets A and B, denoted as A ∪ B, is the set of all elements which belongs to set A or set B.

– A ∪ B = { x | x ∈ A or x ∈ B}

– In other words, A ∪ B is the set of elements which are either in A or in B or in both sets.

– **Example 1 :** Let, A = {a, b, c, d}, B = {p, q, r, s} then
A ∪ B = {a, b, c, d, p, q, r, s}

Venn diagram



A ∪ B

Fig. 1.8.1

– **Example 2 :** Let, X = {ϕ}, Y = {b, ϕ, {ϕ}}
then, X ∪ Y = {ϕ, b, {ϕ}} = Y
since X ⊆ Y

➡ **Note :** (i) A ∪ ϕ = A (ii) A ∪ U = U
(iii) A ∪ $\bar{A}$ = U (iv) A ⊆ A ∪ B and B ⊆ A ∪ B

(b) Intersection

– The intersection of two sets A and B is the set of all the elements which belongs to both A and B.

– It is denoted by A ∩ B.

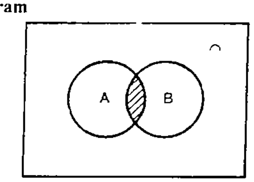
– A ∩ B = { x | x ∈ A and x ∈ B }

– **Example :**

Let, A = {a, b, c, d} B = {c, x, y, d}

A ∩ B = {c, d}

Venn diagram



A ∩ B

Fig. 1.8.2

Note : (i) $A \cap B \subseteq A$ and $A \cap B \subseteq B$
(ii) $A \cap B = \phi$, then A and B are disjoint sets.
(iii) $A \cap \phi = \phi$
(iv) $A \cap U = A$
(v) $A \cap \bar{A} = \phi$

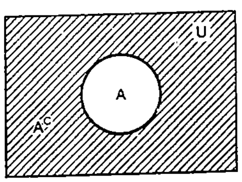
(c) Complement of set

- Let A be the given set. The complement of set A is the set of elements which belong to U but which do not belong to A.
- It is denoted by A^C or A' or $\bar{A}$.
- $A^C = \{ x \mid x \in U, x \notin A \}$

Example :

Let, $A = \{1, 2, 3, 4\}$,
$U = \{1, 2, 3, 4, 5, \ldots, 10\}$
$\bar{A} = \{5, 6, 7, 8, 9, 10\}$

Venn diagram :



A^C

Fig. 1.8.3

Properties of complement of set

(i) $U^C = \phi$ and $\phi^C = U$
(ii) $(A^C)^C = A \Rightarrow$ Law of complement
(iii) $A \cup A^C = A^C \cup A = U$
(iv) $A \cap A^C = A^C \cap A = \phi$
(v) $(A \cup B)^C = \phi$
(vi) $(A \cup B)^C = A^C \cap B^C$ (De Morgan's Law)

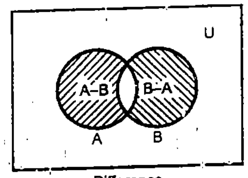
(d) Difference (Relative complement) of sets

- Let A and B are two sets. The difference of A and B denoted as $A - B$ or $A \setminus B$, is the set of elements which belongs to A but which does not belong to B.
- $A - B = \{ x \mid x \in A \text{ and } x \notin B \}$
- Similarly, $B - A = \{ x \mid x \in B \text{ and } x \notin A \}$

- **Example :**

(i) $A = \{a, b, c, d, e\}$, $B = \{f, d, a, x, y\}$,
$A - B = \{b, c, e\}$, $B - A = \{f, x, y\}$
(ii) $A = \{a, b, \phi, \{a, c\}\}$, $A - \{a, b\} = \{\{a, c\}, \phi\}$,
$\{a, c\} - A = \{c\}$

Venn diagram



Difference

Fig. 1.8.4

Properties of difference

Let, A and B be two sets then,

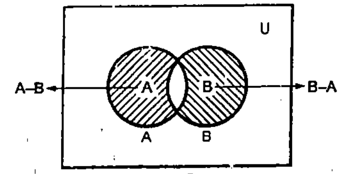
(i) $\bar{A} = U - A$ (ii) $A - A = \phi$
(iii) $\phi - \phi = \phi$ (iv) $A - \bar{A} = A, \bar{A} - A = \bar{A}$
(v) $U - \bar{\phi} = \phi$ (vi) $A - \phi = A$
(vii) $A - \bar{\phi} = U - A$ (viii) $A - B = A \cap \bar{B}$
(ix) $A - B = B - A$ iff $A = B$
(x) $A - B = A$ iff $A \cap B = \phi$ (i.e. A and B are disjoint)
(xi) $A - B = \phi$ iff $A \subseteq B$
(All the elements of A belongs to B)

(e) Symmetric difference of sets

- The symmetric difference of set A and B, denoted by $A \oplus B$, consist of those elements which belongs to A or B but not belongs to both.
- $A \oplus B = \{ x \mid x \in A - B \text{ or } x \in B - A\}$
$A \oplus B = (A - B) \cup (B - A)$
$A \oplus B = (A \cup B) - (A \cap B)$
- **Examples :**

(i) $A = \{a, b, c, d, e, f\}$, $B = \{c, d, e, f, g, h\}$
$A - B = \{a, b\}$, $B - A = \{g, h\}$
$A \oplus B = \{a, b, g, h\}$
(ii) $A = \{\phi\}$, $B = \{x, \phi, \{\phi\}\}$, $A - B = \phi$,
$B - A = \{x, \{\phi\}\}$, so, $A \oplus B = \{x, \{\phi\}\}$

Venn diagram



$A \oplus B$

Fig. 1.8.5

Properties of symmetric difference

(i) $A \oplus A = \phi$ (ii) $A \oplus \phi = A$
(iii) $A \oplus U = U - A = \bar{A}$ (iv) $A \oplus \bar{A} = U$
(v) $A \oplus B = (A \cup B) - (A \cap B)$ or $(A - B) \cup (B - A)$

1.8.1 Examples Based on Set Operations

Ex. 1.8.1 : Consider the universal set $U = \{1, 2, 3, 4, \ldots 8, 9\}$ and set $A = \{1, 2, 5, 6\}$, $B = \{2, 5, 7\}$, $C = \{1, 3, 5, 7, 9\}$

Find : (i) $A \cap B$ and $A \cap C$
(ii) A^C and C^C
(iii) $A \oplus B$ and $A \oplus C$
(iv) $A \cup B$ and $B \cup C$
(v) $A - B$ and $A - C$
(vi) $(A \cup C) - B$ and $B \oplus C - A$

Soln. :

(i) $A \cap B$ = Common elements between set A and B
$= \{2, 5\}$
$A \cap B$ = Common elements between set A and C.
$= \{1, 5\}$

(ii) $A^C = U - A$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{1, 2, 5, 6\}$
$= \{3, 4, 7, 8, 9\}$
$C^C = U - C$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{1, 3, 5, 7, 9\}$
$= \{2, 4, 6, 8\}$

(iii) $A \oplus B = (A - B) \cup (B - A)$
$= \{1, 6\} \cup \{7\} = \{1, 6, 7\}$
$A \oplus C = (A - C) \cup (C - A)$
$= \{2, 6\} \cup \{3, 7, 9\}$
$= \{2, 3, 6, 7, 9\}$

(iv) $A \cup B$ = Elements either from A or B
$= \{1, 2, 2, 5, 5, 6, 7\}$
Only distinct or unique elements are in set
So, $A \cup B = \{1, 2, 5, 6, 7\}$
$B \cup C$ = Elements either in B or C
$= \{2, 5, 7, 1, 3, 5, 7, 9\}$
= only distinct elements should be in set
$= \{1, 2, 3, 5, 7, 9\}$

(v) $A - B$ = Remove all the elements of B from A
that is $A - B = \{x \mid x \in A, x \notin B\}$
$= \{1, 6\}$
$A - C$ = Remove all the elements of C from A
$= \{2, 6\}$

(vi) $(A \cup C) - B$ = Remove all the elements of B from $(A \cup C)$
$= \{1, 3, 6, 9\}$
$B \oplus C - A$ = Remove all the elements of A from $B \oplus C$
$= \{3, 9\}$

Ex. 1.8.2 : Let, A, B, C, be subsets of U, given that
$A \cap B = A \cap C$
$\bar{A} \cap B = \bar{A} \cap C$ is it necessary that $B = C$? justify.

Soln. :

Yes, $B = C$ is necessary.

Justification : Consider $= B \cap U = B \cap (A \cup \bar{A})$
$= (B \cap A) \cup (B \cap \bar{A}) \rightarrow$ by distributive law
$= (A \cap B) \cup (\bar{A} \cap B) \rightarrow$ by commutative law
$= (A \cap C) \cup (\bar{A} \cap c) \rightarrow$ given
$= (A \cup \bar{A}) \cap C \rightarrow$ by distributive law
$B = U \cap C$
Hence, $B = C$

Ex. 1.8.3 : Let, A, B, C be three sets
(i) given that $A \cup B = A \cup C$, is it necessary that $B = C$?
(ii) given that $A \cap B = A \cap C$, is it necessary that $B = C$?

Soln. :

(i) No, There is no necessary that $B = C$.

Justification

Let, $A = \{a, b, c\}$, $B = \{a\}$, $C = \{c\}$
$A \cup B = \{a, b, c\} = A$, $A \cup C = \{a, b, c\} = A$
$A \cup B = A \cup C = A$, but, $B \neq C$

(ii) No, There is no necessary that $B = C$

Justification

Let, $A = \{a, b\}$, $B = \{b, c, d\}$, $C = \{b, l, m\}$
$A \cap B = \{b\} = A \cap C$, but $B \neq C$

Ex. 1.8.4 : For A = {a, b, {a, c}, ϕ}, determine the following sets

(i) A – { a } (ii) A – ϕ
(iii) A – { ϕ } (iv) A – {a, b}
(v) A – {a, c} (vi) A – {{a, b}}
(vii) A – {{a, c}} (viii) {a} – A
(ix) ϕ – A (x) { ϕ } – A
(xi) {a, c} – A (xii) {{a, c}} – A
(xiii) { a} – {A}

☑ **Soln. :**

(i) Removing the element a from A.
A – {a} = {b, {a, c}, ϕ}

(ii) A – ϕ = {a, b, {a, c}, ϕ} = A

(iii) A – {ϕ} = Removing the ϕ from A
= {a, b {a, c}}

(iv) A – {a, b} = Removing a, b from A
= {{a, c}, ϕ}

(v) A – {a, c} = Removing a and c from A
but, 'c' is not member of A.
= {b, {a, c}, ϕ}

(vi) A – {{a, b}} = Removing {a, b} from A,
but {a, b} is not member of A.
= {a, b, {a, c}, ϕ} = A

(vii) A – {{a, c}} = Removing {a, c} as a element from A
= {a, b, ϕ}

(viii) {a} – A = ϕ

(ix) ϕ – A = ϕ

(x) {ϕ} – A = ϕ

(xi) {a, c} – A = Removing the elements of A from {a, c}
= { c }

(xii) {{a, c}} – A = Removing the elements of A from {a, c}
= ϕ

(xiii) {a} – {A} = {a}

Ex. 1.8.5 : Let A = {ϕ, b} construct the following sets :
(i) A – ϕ (ii) {ϕ} – A (iii) A ∪ P (A)
Where P (A) is a power set.

☑ **Soln. :**

Let A = {ϕ, b} (given)

(i) A – ϕ = A
= { ϕ, b}
∵ any non-empty set relative difference with null then result will be the set itself

(ii) {ϕ} – A = ϕ
∵ null set relative difference with any non-empty set then outcome will be null i.e. ϕ.

(iii) P (A) = Number of possible subsets of A
= {ϕ, {ϕ}, {b}, {ϕ, b}}
A ∪ P (A) = {ϕ, b} ∪ {ϕ, {ϕ}, {b}, {ϕ, b}}
= {ϕ, {ϕ}, {b}, {ϕ, b}, b}

Ex. 1.8.6 : Let A, B, C be sets. Under what conditions the following statements are true ?
(i) (A – B) ∪ (A – C) = A
(ii) (A – B) ∪ (A – C) = ϕ.

☑ **Soln. :** Let, A, B, C are three sets

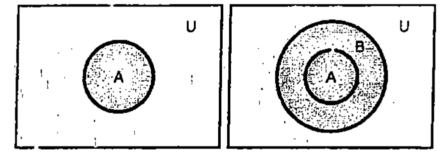
(i) **(A – B) ∪ (A – C) = A**
B and C should be null sets to get A

(ii) **(A – B) ∪ (A – C) = ϕ**
A, B, and C should be identical or equal sets to get ϕ.

1.9 VENN DIAGRAM

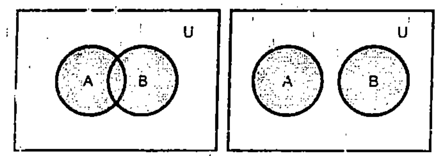
- Venn diagrams are used to show pictorial representation of relation between two sets and their operations.
- Named after the British Logician John Venn.
- In Venn diagrams
  (i) Universal set (U) is represented by rectangle.
  (ii) Subsets of universal sets are represented by circles.
  (iii) If A ⊆ B, then circle which represent A, lies inside circle represent set B.
  (iv) If set A and B are disjoint sets, then it is represented by two separated circles inside large rectangle.
  (v) If A and B are overlapping sets, then circles representing A and B have some common area.

Examples :



A ⊆ U
(A subset of universal set)

A ⊆ B
(set A is subset of set B)



A & B are overleapping

A and B are disjoint

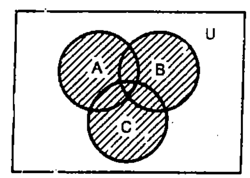
Fig. 1.9.1

1.9.1 Examples Based on Venn diagrams, Algebra of Sets and Duality

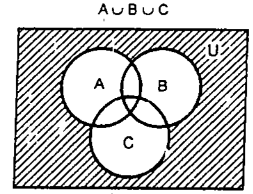
Ex. 1.9.1 : Draw the venn diagram and prove the expression. Also write the dual of each of the given statements.
(i) $(A \cup B \cup C)^C = (A \cup C)^C \cap (A \cup B)^C$
(ii) (U ∩ A) ∪ (B ∩ A) = A

☑ **Soln. :**

(i) **L.H.S = $(A \cup B \cup C)^C$**

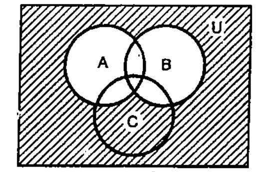


A ∪ B ∪ C

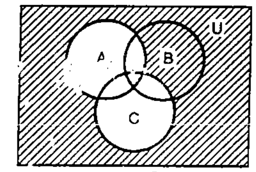


$(A \cup B \cup C)^C = U - (A \cup B \cup C)$

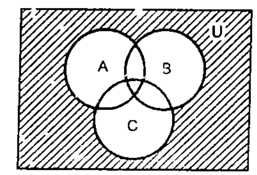
RHS = $(A \cup C)^C \cap (A \cup B)^C$



$(A \cup B)^C$



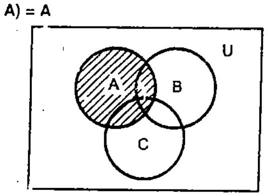
$(A \cup C)^C$



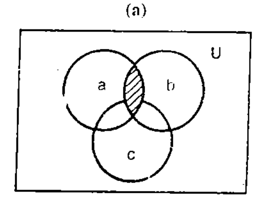
$(A \cup C)^C = \cap-(A \cup B)^C$

From L.H.S. and RHS. It is clear that
$(A \cup B \cup C)^C = (A \cup C)^C \cap (A \cup B)^C$

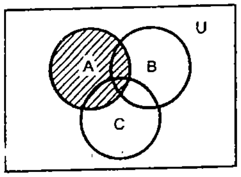
(ii) **(U ∩ A) = A**



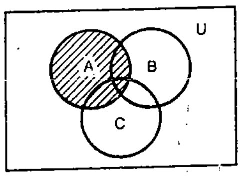
(U ∩ A) = A
(a)



(B ∩ A)
(b)



$(U \cup A) \cup (B \cap A)$
(LHS)
(c)



A (RHS)
(d)

From venn diagram (c) and (d) it is clear that

$(U \cap A) \cup (B \cap A) = A$ **...Hence proved.**

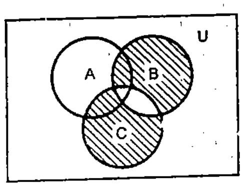
EX. 1.9.2 : Using venn diagram, prove or disprove.

(i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

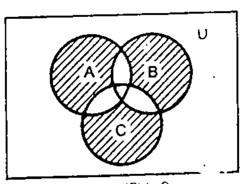
(ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$

☑ **Soln. :**

(i) **$B \oplus C$ = elements which are in B or in C, but not in both B and C.**



(i) $B \oplus C$

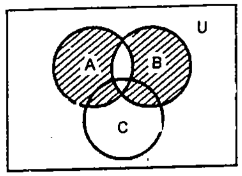


⟸ LHS

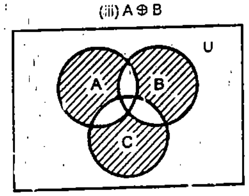
(ii) $A \oplus (B \oplus C)$

$A \oplus (B \oplus C)$ = (Elements which are either in A or $B \oplus C$, but not in both A and $B \oplus C$).

$(A \oplus B)$ = Elements which are either in A or B, but not in both A and B.



(iii) $A \oplus B$



⟸ RHS

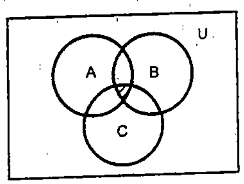
(iv) $(A \oplus B) \oplus C$

$(A \oplus B) \oplus C$ = (Elements which are either in $A \oplus B$ or in C but not in both $(A \oplus B)$ and C)

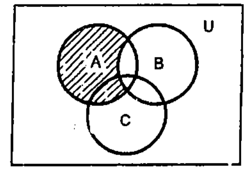
From venn diagram (ii) and (iv), it is clear that

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$ **...Hence proved.**

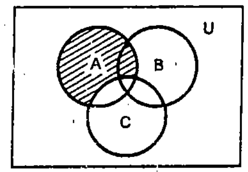
(ii) **$A \cap B \cap C$ = Elements which are in set A, B and C.**



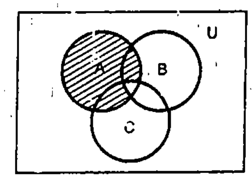
(i) $A \cap B \cap C$



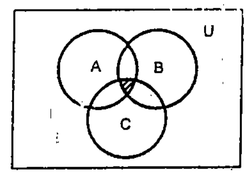
(ii) $A - B$



(iii) $A - C$



(iv) $(A - B) \cup (A - C)$



(v) $A - [(A-B) \cup (A-C)]$

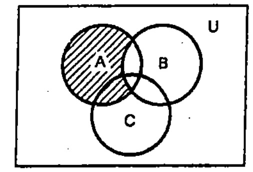
From venn diagram (i) and (v) it is observed that

$A \cap B \cap C = A - [(A - B) \cup (A - C)]$

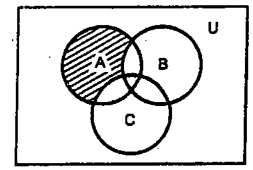
...Hence proved

Ex. 1.9.3 : Show that $(A - B) - C = A - (B \cup C)$ using venn diagram

☑ **Soln. :**

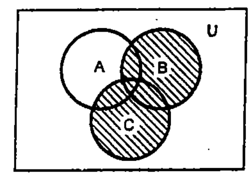


(i) $A - B$

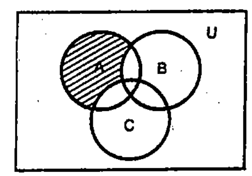


⟸ LHS

(ii) $(A - B) - C$



(iii) $B \cup C$



⟸ RHS

(iv) $A - (B \cup C)$

From, venn diagram (ii) and (iv) it is clear that

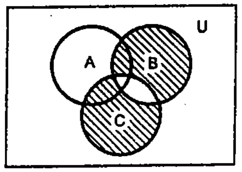
$$(A - B) - C = A - (B \cup C)$$

...Hence proved.

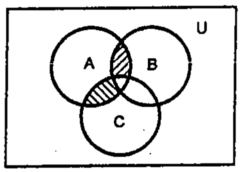
Ex. 1.9.4 : Prove the following using venn diagram

$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

☑ **Soln. :**

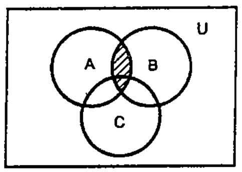


(i) $B \oplus C$

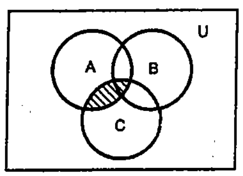


⇐ LHS

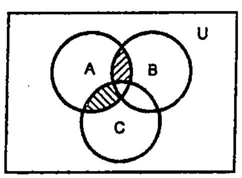
(ii) $A \cap (B \oplus C)$



(iii) $A \cap B$



(iv) $(A \cap C)$



⇐ RHS

(v) $(A \cap B) \oplus (A \cap C)$

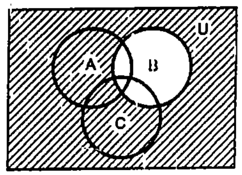
From venn diagram (ii) and (v), it is clear that

$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ ...**Hence proved.**

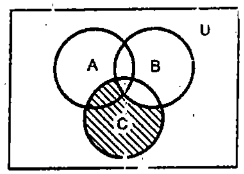
Ex. 1.9.5 : Using venn diagram show that

$A \cup (\bar{B} \cap C) = (A \cup \bar{B}) \cap (A \cup C)$

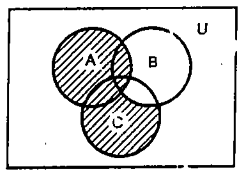
☑ **Soln. :**



(i) $\bar{B} = U - B$

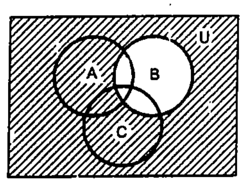


(ii) $\bar{B} \cap C$

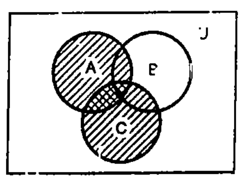


⇐ LHS

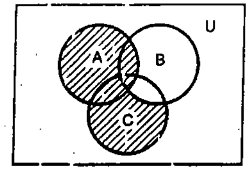
(iii) $A \cup (\bar{B} \cap C)$



(iv) $(A \cup \bar{B})$



(v) $(A \cup C)$



⇐ RHS

(vi) $(A \cup \bar{B}) \cap (A \cup C)$

From the venn diagram (iii) and (vi) it is observed that,

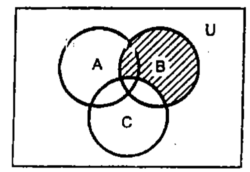
$$A \cup (\bar{B} \cap C) = (A \cup \bar{B}) \cap (A \cup C)$$

...**Hence proved.**

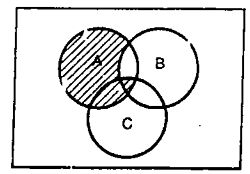
Ex. 1.9.6 : Using venn diagram,

Prove $A - (B - C) = (A - B) \cup (A \cap B \cap C)$

☑ **Soln. :**

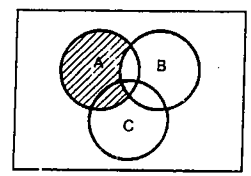


(i) $B - C$

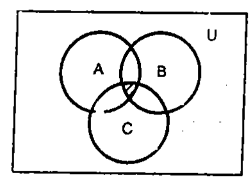


⇐ LHS

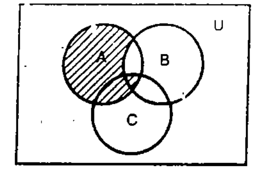
(ii) $A - (B - C)$



(iii) $A - B$



(iv) $A \cap B \cap C$



⇐ RHS

(v) $(A - B) \cup (A \cap B \cap C)$

From venn diagram (ii) and (v) it is

$A - (B - C) = (A - B) \cup (A \cap B \cap C)$ ...**Hence proved.**

1.10 ALGEBRA OF SET OPERATIONS

The algebra of sets defines the properties and Laws of sets. It includes various operations on sets. The properties and Laws are as follows :

Let, A, B, C be any set, then the operations are as follows :

(a) Idempotent Law

(i) $A \cup A = A$ (ii) $A \cap A = A$

It states that, any set union with itself, then answer will be that same set.

Example : $A = \{3, 4, 6\}$, $A \cup A = \{3, 4, 6\}$, $A \cap A = \{3, 4, 6\}$

(b) Commutative Law

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

Example :

$$A = \{1, 2, 3\}$$
$$B = \{3, 4\}$$
$$A \cup B = \{1, 2, 3, 4\}$$
$$B \cup A = \{1, 2, 3, 4\}$$
$$A \cap B = \{3\}$$
$$B \cap A = \{3\}$$

(c) Associative Law

(i) $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$

Example :

Let, $A = \{a, b, c, d\}$

$B = \{c, d, e, f\}$

$C = \{a, g, h, e\}$

$(B \cup C) = \{c, d, e, f, a, g, h\}$

$(A \cup B) = \{a, b, c, d, e, f\}$

$(A \cap B) = \{c, d\}$

$(B \cap C) = \{e\}$
$A \cup (B \cup C) = \{a, b, c, d, e, f, g, h\}$
$(A \cup B) \cup C = \{a, b, c, d, e, f, g, h\}$
$A \cap (B \cap C) = \{\phi\}$
$(A \cap B) \cap C = \{\phi\}$

(d) Distributive Law

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:
Here, union operation is distributive over intersection.

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:
Here, intersection operation is distributive over union.

(e) Identity Law

(i) $A \cup \phi = A$, $A \cup U = U$
(ii) $A \cap \phi = \phi$, $A \cap U = A$

(f) Involution Law or Double Complement Law

(i) $(\overline{\overline{A}}) = A$ or $(A^c)^c = A$

(g) Complement Law

(i) $A \cup A^C = U$, $U^C = \phi$
(ii) $A \cap A^C = \phi$, $\phi^C = U$

(h) Absorption Law

(i) $A \cup (A \cap B) = A$
(ii) $A \cap (A \cup B) = A$

Example :

Let, $A = \{3, 4, 8, 5\}$
$B = \{3, 5, 7, 6\}$
$(A \cap B) = \{3, 5\}$
$(A \cup B) = \{3, 4, 8, 5, 7, 6\}$
$A \cup (A \cap B) = \{3, 4, 8, 5\}$
$A \cap (A \cup B) = \{3, 4, 8, 5\}$

(i) De Morgan's Law

(i) $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$ (ii) $(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

1.11 CARDINALITY OF SET

- In the study of discrete mathematics, it is important to count the number of elements of the set.
- In computer engineering there is requirement of analysis of algorithm, we required to count number of operations executed by an algorithm. This is necessary to estimate the cost of effectiveness of a particular algorithm.
- Hence, it is important to study the cardinality of finite sets and understand its properties.

1.11.1 Definition

- Let, A be a finite set, then number of elements in the set A, is called the cardinality of A.
- It is denoted as $|A|$ or $n(A)$.
- **Example :** $A = \{3, 4, 8, 6\}$, then cardinality of A is $n(A)$ or $|A| = 4$

1.11.2 Properties of Cardinality of Sets

(a) If $A = \phi$ then $n(A) = 0$
(b) If $A \subseteq B$ then $n(A) \le n(B)$
(c) Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and $|A \cup B| = |A| + |B|$.

Proof :

Let, A and B are finite and $A \ne \phi$, $B \ne \phi$, $A \cap B = \phi$, $|A| = n$, $|B| = m$

$A = \{x_1, x_2, x_3, \ldots, x_n\}$
$B = \{y_1, y_2, y_3, \ldots, y_m\}$
$A \cup B = \{x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_m\}$

$|A \cup B| = m + n = |A| + |B|$. Since A and B are disjoint.

Hence proved.

The above property or theorem is **addition principle.**

(d) Let, $A_1, A_2, A_3, \ldots, A_n$ be mutually disjoint finite sets then

$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = |A_1| + |A_2| + |A_3| + \ldots + |A_n|$

(e) Let, A and B be finite sets. Then

$n(A - B) = n(A) - n(A \cap B)$

Proof :

From the venn diagram, it is observed that,

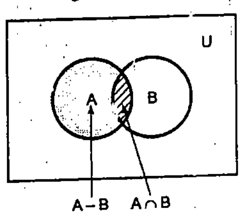


Fig. 1.11.1

$A = (A - B) \cup (A \cap B)$ because $(A - B) \cap (A \cap B) = \phi$

By addition principle, $|A| = |A - B| + |A \cap B|$

Hence, $|A - B| = |A| - |A \cap B|$

Example :

Suppose a class A has 25 students and 10 of them are taking a biology class B. Then number of students in class A which are not in class B is

$n(A - B) = n(A) - n(A \cap B) = 25 - 10 = 15.$

1.12 INCLUSION AND EXCLUSION PRINCIPLE FOR SETS

- Inclusion-exclusion is a fundamental theorem of counting.
- Let's take a simple example to understand the necessity of this principle.
- Let A and B two finite disjoint sets. The number of elements in set A and B can be calculated as,
  $|A \cup B| = |A| + |B|$
- But if the two set are not disjoint then above solution to count number of element in both sets will give incorrect result because it counts the elements of A, B and those elements which are in A and B.
- So, it is called as "over counting". To overcome the over counting, inclusion and exclusion of elements should be done.

1.12.1 Theorem for Two Sets

Let A and B are two finite sets, then number of elements of A and B as,

$|A \cup B| = |A| + |B| - |A \cap B|$

Proof :

From the following venn diagram, it is observed that,

$A \cup B = (A - B) \cup B$

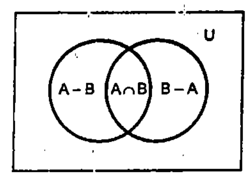


Fig. 1.12.1

According to addition principle.

$|A \cup B| = |A - B| + |B|$
$= |A| - |A \cap B| + |B|$
$(\because n(A - B) = n(A) - n(A \cap B))$

$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$

$$|A \cup B| = \underbrace{|A| + |B|}_{\text{Inclusion of elements}} - \underbrace{|A \cap B|}_{\text{Exclusion of elements}}$$

1.12.2 Inclusion-Exclusion Principle for 3-sets

Let, A, B and C are finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof :

Assume, $D = B \cup C$

$|A \cup D| = |A| + |D| - |A \cap D|$ ...(1)

$\Rightarrow |D| = |B \cup C| = |B| + |C| - |B \cap C|$ ...(2)

$\Rightarrow |A \cap D| = |A \cap (B \cup C)|$
$= |(A \cap B) \cup (A \cap C)|$
$= |A \cap B| + |A \cap C| - |A \cap B \cap C|$ ...(3)

Put the value of Equation (2) and (3) in Equation (1)

Then,

$|A \cup D| = |A| + |B| + |C| - |B \cap C| - [|A \cap B| + |A \cap C| - |A \cap B \cap C|]$

$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

1.12.3 Inclusion-Exclusion Principle for 4-sets

Let, A, B, C and D are finite sets then

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

1.12.4 Examples Based on Inclusion – Exclusion Principle

UEx. 1.12.1 (SPPU - IT, Q. 1(b), Dec. 18, 6 Marks)

100 of them 120 engineering students in a college take part in at least one of the activity group discussion, debate and quiz. 65 participate in group discussion, [illegible] participate in debate, 42 participate in quiz, [illegible] participate in [illegible] and quiz, 25 participate in group discussion and quiz, [illegible] participate in debate and quiz. Find the number of students who participate in

(i) All the three activities, (ii) Exactly one of the activities

☑ **Soln. :**

Given :

$U = 120$

$n(A)$ = Number of students participate in group discussion = 65

$n(B)$ = Number of students participate in debate = 45

$n(C)$ = Number of students participate in Quiz = 42

$n(A \cap B)$ = Number of students participate in group discussion and debate = 20

$n(A \cap C)$ = Number of students participate in group discussion and in Quiz = 25

$n(B \cap C)$ = Number of students participate in debate and Quiz = 15

$n(A \cup B \cup C)$ = The number of participate in at least one of the activity = 100

(i) The number of students participated in all the three activities is $n(A \cap B \cap C)$

According to Inclusion-Exclusion principle

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$100 = 65 + 45 + 42 - 20 - 25 - 15 + n(A \cap B \cap C)$$

$$100 - 92 = n(A \cap B \cap C)$$

$$\therefore \mathbf{n(A \cap B \cap C) = 8}$$

(ii) The number of students participate in exactly one of the activities

= (number of students participates only in group discussion)

+ (number of students participate only in debate)

+ (number of students participate only in quiz)

$$\Rightarrow [n(A \cap B^C \cap C^C) + n(A^C \cap B \cap C^C) + n(A^C \cap B^C \cap C)]$$

$$\Rightarrow [n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)]$$
$$+ [n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)]$$
$$+ [n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)]$$

$$\Rightarrow [65 - 20 - 25 + 08] + (45 - 20 - 15 + 8] + [42 - 25 - 15 + 8]$$

$$\Rightarrow (28) + (18) + (10)$$

$$\Rightarrow 56$$

The number of students participate in exactly one of the activities = 56

UEx. 1.12.2 (SPPU - IT, Q. 2(b), Dec. 18, 6 Marks)

It is known that the university 60% of the professor plays tennis, 50% of them plays football 70% cricket, 20% plays tennis and football, 30% plays tennis and cricket and 40% plays football and cricket. If someone claimed that 20% of the professor plays tennis, football and cricket would you believe that claim ? Why ?

☑ **Soln.**

Given :

$n(A)$ = Professor play tennis = 60%

$n(B)$ = Professor play football = 50%

$n(C)$ = Professor play cricket = 70%

$n(A \cap B)$ = Professor play tennis and football = 20%

$n(A \cap C)$ = Professor play tennis and cricket = 30%

$n(B \cap C)$ = Professor play football and cricket = 40%

$n(A \cap B \cap C)$ = Someone claimed that, professors play all the games = 20%

The claimed would not be accepted, because, the % of professors who does at least one of the three games as,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap c) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 60 + 50 + 70 - 20 - 30 - 40 + 20$$

$$= 110$$

Clearly, It is contradiction. It should not be exceeded 100%. **...Ans.**

UEx. 1.12.3 (SPPU - IT. : May 17, 6 Marks)

Out of a total of 130 students, 60 are wearing hats, 51 are wearing scarves, and 30 are wearing both hats and scarves. Out of 54 students who are wearing sweaters, 26 are wearing hats, 21 are wearing scarves, and 12 are wearing both hats and scarves Everyone wearing neither a hat nor a scarf is wearing gloves :

(a) How many students are wearing gloves ?

(b) How many students not wearing a sweater are wearing hats but not scarves ?

(c) How many students not wearing a sweater are wearing neither hat nor a scarf ?

☑ **Soln. :**

$U = 130$

$n(A)$ = no. of students wearing hats = 60

$n(B)$ = no. of students wearing scarf = 51

$n(C)$ = no. of students wearing sweaters = 54

$n(A \cap B)$ = no. of students wearing hats and scarf = 30

→ Every wearing neither a hat nor a scarf is wearing gloves.

According to Inclusion-Exclusion principle,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 60 + 51 - 30 = 81$$

(a) The number of students wearing gloves that is number of students wearing neither a hat nor a scarf

$$= U - n(A \cup B) = 130 - 81 = 46$$

$\because A^C = U - A$ (Complement law)

(b) The number of students not wearing a sweater are wearing hats but not scarf that is number of students wearing hats but neither sweater not scarf is

$$n(A \cap B^C \cap C^C) = n(A) - (n(A) \cap n(C \cup B))$$
$$= n(A) - n(A) \cap n(C) - n(A) \cap n(B) + n(A \cap B \cap C)$$
$$= 60 - 26 - 30 + 12 = 16$$

(c) The number of students not wearing a sweater are wearing neither hat nor a scarf that is the number of students wearing neither wearing a sweater nor hat nor scarf.

$$n(A \cup B \cup C)^C = U - n(A \cup B \cup C)$$
$$= U - n(A) - n(B) - n(C) + n(A \cap B) + n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)$$
$$= 130 - 60 - 51 - 54 + 30 + 26 + 21 - 12$$
$$= 30$$

Ex. 1.12.4 : In a DM class, every student is major in CS or math or both. The number of students having CS as major is 25, the number of students having math as major is 13 and the number of students majoring in both CS and math is 8. So, how many students in the class ?

☑ **Soln. :**

Given :

(i) Number of students having CS as major $n = (A) = 25$

(ii) $n(B)$ = number of students having math as major = 13

(iii) $n(A \cap B)$ = number of students majoring in both CS and math = 8

According to inclusion – exclusion principle

The number of students in the class is represented as $n(A \cup B) = ?$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 25 + 13 - 8$$
$$n(A \cup B) = 30$$

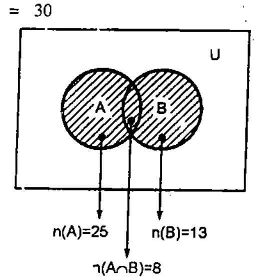


Fig. P. 1.12.4

Number of students in the class = 30 **...Ans.**

UEx. 1.12.5 (SPPU - IT, Q. 2(b), May 18, 6 Marks)

There are 2504 computer science students at a school. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these three programming languages ?

☑ **Soln. :**

U = Total number of student in the school = 2504

$n(A)$ = number of students taken java course = 1876

$n(B)$ = number of students taken linux course = 999

$n(C)$ = number of students taken C course = 345

$n(A \cap B)$ = number of students taken java and linux course = 876

$n(B \cap C)$ = number of students taken linux and C course = 231

$n(A \cap C)$ = number of students taken java and C course = 290

$n(A \cap B \cap C)$ = number of students taken all the courses = 189

$n(A^C \cap B^C \cap C^C) = n(A \cup B \cup C)^C = ?$

According to Inclusion-Exclusion principle,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$
$$= 1876 + 999 + 345 - 876 - 290 - 231 + 189$$
$$= 2012$$

We know that. According to complement law,

$$A^C = U - A$$

$$n(A \cup B \cup C)^C = U - (A \cup B \cup C)$$
$$= 2504 - 2012 = 492$$

So, 492 students have not taken any of the course.

UEx. 1.12.6 (SPPU - IT, Q. 2(a), Dec. 16, 6 Marks)

Among 130 students, 60 study mathematics, 51 study physics and 30 study both mathematics and physics. Out of 54 students studying chemistry. 26 study mathematics, 21 study physics and 12 study both mathematics and physics. All the students studying neither mathematics nor physics are studying biology :

(i) Find how many are studying biology ?

(ii) How many not studying chemistry are studying mathematics but not physics ?

(iii) How many students are studying neither mathematics nor physics nor chemistry?

Soln. :

Given

U = 130 (total number of students)

$n(A)$ = number of students study mathematics = 60

$n(B)$ = number of students study physics = 51

$n(C)$ = number of students wearing sweaters = 54

$n(A \cap B)$ = number of students study mathematics and physics = 30

→ All the students studying neither mathematics nor physics are studying biology.

(1) Number of students studying biology which means number of students study neither mathematics nor physics.

$$n(A \cup B)^C = U - (A \cup B)$$
$$= U - (n(A) + n(B) - n(A \cap B))$$
$$= 130 - (60 + 51 - 30)$$
$$= 130 - 81 = 49$$

(2) The number of students not studying chemistry are studying mathematics but not physics which means. The number of students studying mathematics but neither chemistry nor physics is,

$$n(A \cap B^C \cap C^C) = n(A) - (n(A) \cap n(B \cup C))$$
$$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 60 - 30 - 26 + 12 = 16$$

(3) The number of students studying neither mathematics nor physics nor chemistry is $(A \cup B \cup C)^C$

$$n(A \cup B \cup C)^C = U - n(A \cup B \cup C$$
$$= U - n(A) - n(B) - n(C) + n(A \cap B) + n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)$$
$$= 130 - 60 - 51 - 54 + 30 + 26 + 21 - 12$$
$$= 30$$

...Ans.

UEx. 1.12.7 (SPPU - IT, Q. 2(b), May 15, May 16, 6 Marks)

A survey of 70 high school students revealed that 35 like folk music, 15 like classical music and 5 like both. How many of the students surveyed do not like either folk or classical music ?

Soln. :

Given :

U = total number of students in the school.

$n(A)$ = number of students like folk music = 35

$n(B)$ = number of students like classical music = 15

$n(A \cap B)$ = number of students like both folk and classical music = 5

$n(A^C \cap B^C) = n(A \cup B)^C$ = number of students neither like folk nor classical music = ?

According to Inclusion – Exclusion principle

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 35 + 15 - 5$$
$$= 45$$

We know, complement law

$$A^C = U - A$$

So, $n(A \cup B)^C = U - n(A \cup B)$

$$= 70 - 45 = 25$$

The number of students neither like folk nor classical music = 25

UEx. 1.12.8 (SPPU - IT, Q. 2(a), Dec. 15, 6 Marks)

Suppose that 100 out of 120 mathematics students at a college take at least one of the languages French, German and Russian. Also suppose [illegible]

65 study French	20 study French and German
45 study German	25 study French and Russian
42 study Russian	15 study German and Russian

(i) Find the number of students who study all the three languages.

(ii) Find incorrect number of students in each region of Venn diagram,

(iii) Determine the number K of students who study

(a) Exactly one language

(b) Exactly two languages.

Soln. :

Given :

U = total number of students = 120.

$n(A)$ = number of students study French = 65

$n(B)$ = number of students study German = 45

$n(C)$ = number of students study Russian = 42

$n(A \cap B)$ = number of students study french and german = 20

$n(A \cap C)$ = number of students study French and Russian = 25

$n(B \cap C)$ = number of students study German and Russian = 15

$n(A \cup B \cup C)$ = number of students study at least one of the languages French, German and Russian = 100

(i) According to Inclusion – Exclusion principle

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$n(A \cup B \cup C) = 65 + 45 + 42 - 20 - 25 - 15$$

$$+ n(A \cap B \cap C)$$
$$100 = 92 + n(A \cap B \cap C)$$
$$n(A \cap B \cap C) = 100 - 92 = 8$$

The number of students who study all the three languages is $n(A \cap B \cap C) = 8$

(a) The number of students who study exactly one language.

= (number of students study exactly French)
+ (number of students study exactly german)
+ (number of students study exactly Russian)

$$= n(A \cap B^C \cap C^C) + n(A^C \cap B \cap C^C) + n(A^C \cap B^C \cap C)$$
$$= [n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)]$$
$$+ [n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)]$$
$$+ [n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)]$$
$$= [65 - 20 - 25 + 8] + [45 - 20 - 15 + 8] + [42 - 25 - 15 + 8]$$
$$= [28] + [18] + [10]$$
$$= 56$$

(b) The number of students who study exactly two languages.

$$\Rightarrow n(A \cap B \cap C^C) + n(A \cap B^C \cap C) + n(A^C \cap B \cap C)$$
$$\Rightarrow (n(A \cap B) - n(A \cap B \cap C)) + (n(A \cap C) - n(A \cap B \cap C)) + (n(B \cap C) - n(A \cap B \cap C))$$
$$\Rightarrow (20 - 8) + (25 - 8) + (15 - 8)$$
$$\Rightarrow (12) + (17) + (7)$$
$$\Rightarrow 36$$

The number of students study exactly two languages = 36.

UEx. 1.12.9 (SPPU - IT, Q. 1(a), Dec. 14, 6 Marks)

During a survey of the ice cream [illegible] it was found that 22 like mango, 25 like custard apple, 39 like grape, 9 like custard apple and mango, 17 like mango and grape, 20 like custard apple and grape, [illegible] and 4 like none. Then how many students were surveyed ? How many students like exactly one flavour, how many students like exactly two flavours ?

Soln. :

Given :

$n(A)$ = like mango = 22

$n(B)$ = like custard apple = 25

$n(C)$ = like grape = 39

$n(A \cap B)$ = like custard apple and mango = 9

$n(A \cap C)$ = like mango and grape = 17

$n(B \cap C)$ = like custard apple and grape = 20

$n(A \cap B \cap C)$ = like all the flavours = 6

$n(A \cup B \cup C)^C$ = like none = 4

The number of students like atleast one of the flavours is, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$$= 22 + 25 + 39 - 9 - 17 - 20 + 6$$
$$= 46$$

The number of students surveyed = 46 + 04 = 50

⇒ The number of students liked exactly one flavour

$$= n(A \cap B^C \cap C^C) + n(A^C \cap B \cap C^C) + n(A^C \cap B^C \cap C)$$
$$= (n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)) + (n(n(B) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)) + (n(C) - n(A \cap C) - (B \cap C) + n(A \cap B \cap C))$$
$$= (22 - 9 - 17 + 6) + (25 - 9 - 20 + 6) + (39 - 17 - 20 + 6)$$
$$= (2) + (2) + (8)$$
$$= 12$$

...Ans.

⇒ The number of students liked exactly two flavours

$$= n(A \cap B \cap C^C) + n(A \cap B^C \cap C) + n(A^C \cap B \cap C)$$
$$\Rightarrow (n(A \cap B) - n(A \cap B \cap C)) + (n(A \cap C) - n(A \cap B \cap C)) + (n(B \cap C) - n(A \cap B \cap C))$$
$$\Rightarrow (9 - 6) + (17 - 6) + (20 - 6)$$
$$\Rightarrow (3) + (11) + (14)$$
$$\Rightarrow 28$$

...Ans.

UEx. 1.12.10 (SPPU - IT, Q. 1(a), May 14, 6 Marks)

Explain the principle of inclusion and exclusion. [illegible]

(i) Only Japanese and French but not Russian

(ii) Only Japanese and Russian but not French.

Soln. :

Given :

U = total number of students = 120.

$n(A \cup B \cup C)$ = Number of students study atleast one of the three languages = 100

$n(A)$ = number of students study Japanese = 65

$n(B)$ = number of students study French = 45

$n(C)$ = number of students study Russian = 42

$n(A \cap B) = 20$
$n(A \cap C) = 25$
$n(B \cap C) = 15$

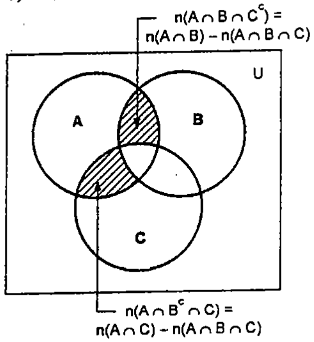


Fig. P. 1.12.10

(i) The number of students who study only Japanese and French but not Russian is $n(A \cap B \cap C^c)$

$= n(A \cap B) - n(A \cap B \cap C)$ ...(1)

According to Inclusion – Exclusion principle

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$100 = 65 + 45 + 42 - 20 - 25 + 15 + n(A \cap B \cap C)$$

$$100 = 92 + n(A \cap B \cap C)$$

$\therefore$ $\mathbf{n(A \cap B \cap C) = 100 - 92 = 08}$

Put in Equation (1)

$\mathbf{n(A \cap B \cap C^c) = 20 - 08 = 12}$

(ii) The number of students who study only Japanese and Russian but not French is $n(A \cap B^c \cap C) = n(A \cap C) - n(A \cap B \cap C)$

$= 25 - 08 = 17$ students ...Ans.

[illegible] (SPPU - IT. Q. 1(b). May 19, 6 Marks)

[illegible]
[illegible] Business India
26 read India Today
26 read Times of India
11 read both Business India and India Today
9 read both Business India and Times of India
8 read both India Today and Times of India
8 read none of these
(i) How many read all three ?
(ii) How many read exactly one ?

☑ **Soln. :**

Let, U = Total number of peoples surveyed = 60

n(A) = Number of peoples who watch Business India = 25

n(B) = number of peoples who watch India today = 26

n(C) = number of peoples who watches Times of India = 26

$n(A \cap B)$ = number of peoples watches Business India and India today = 11

$n(A \cap C)$ = number of peoples watches Business India and Times of India = 9

$n(B \cap C)$ = number of peoples watches India today and Times of India = 8

$n(A^c \cap B^c \cap C^c) = n(A \cup B \cup C)^c$ = number of peoples watches no news = 8

According to Complement Law,

The number of peoples who watches atleast one of the news $= n(A \cup B \cup C) = U - n(A \cup B \cup C)^c$

$\because A \cup U - A^c = 60 - 8 = 52$

(i) The number of peoples who watches all the three news $= n(A \cap B \cap C)$ can be calculated by inclusion – exclusion principle.

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$S_2 = 25 + 26 + 26 - 11 - 9 - 8 + n(A \cap B \cap C)$$

$$S_2 = 49 + n(A \cap B \cap C)$$

$$n(A \cap B \cap C) = 52 - 49 = 3$$

$$\mathbf{n(A \cap B \cap C) = 3}$$

(ii) The number of peoples who watch exactly one news, can be obtained by following way.

= Number of peoples who watch only business India
+ Number of peoples who watch only India today
+ Number of peoples who watch only times of India.

$= n(A \cap B^c \cap C^c) + n(A^c \cap B \cap C^c) + n(A^c \cap B^c \cap C)$

$= [n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)] + [n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)] + [n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)]$

$= [25 - 11 - 9 + 3] + [26 - 11 - 8 + 3] + [26 - 9 - 8 + 3]$

$= (8) + (10) + (12) = 30$ ...Ans.

Venn diagram

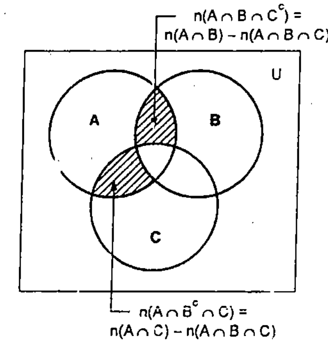


Fig. P. 1.12.11

Ex. 1.12.12 : In a class of 25 students, 12 have taken economics, 8 have taken economics but not political science.

(i) Find the number of students who have taken economics and political science.

(ii) Find the number of students those who have taken politics but not economics.

☑ **Soln. :**

Given :

n(A) = number of students have taken economics = 12.

n(B) = number of students taken political science = ?

$(A \cup B)$ = number of students in the class = 25.

$n(A \cap B^c)$ = number of students have taken economics but not political science = 08

$n(A \cap B)$ = number of students who have taken economics and political science = ?

$$\begin{Bmatrix}\text{Number of} \\ \text{students} \\ \text{taken} \\ \text{economic}\end{Bmatrix} = \begin{Bmatrix}\text{Number of} \\ \text{students taken} \\ \text{econoics but} \\ \text{not political} \\ \text{science}\end{Bmatrix} + \begin{Bmatrix}\text{Number} \\ \text{of students taken} \\ \text{both economics} \\ \text{and political} \\ \text{science}\end{Bmatrix}$$

$n(A) = n(A \cap B^c) + n(A \cap B)$

$12 = 08 + n(A \cap B)$

$\therefore 12 - 8 = n(A \cap B)$

$\mathbf{n(A \cap B) = 04}$

According to inclusion – Exclusion principle

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$25 = 12 + n(B) - 4$

$\therefore 25 = 12 - 4 + n(B)$

$25 = 8 + n(B)$

$\mathbf{n(B) = 17}$

Similarly,

$$\begin{Bmatrix}\text{Number of} \\ \text{Students taken} \\ \text{political} \\ \text{science}\end{Bmatrix} = \begin{Bmatrix}\text{Number of} \\ \text{Students} \\ \text{taken political} \\ \text{science} \\ \text{but not} \\ \text{economics}\end{Bmatrix} + \begin{Bmatrix}\text{Number of} \\ \text{Students} \\ \text{taken both} \\ \text{economics and} \\ \text{political} \\ \text{science}\end{Bmatrix}$$

$n(B) = n(A^c \cap B) + n(A \cap B)$

$17 = 4 + n(A^c \cap B)$

$17 - 4 = n(A^c \cap B)$

$\boxed{13 = n(A^c \cap B)}$

Ans. : The number of students taken political science and economics $= n(A \cap B) = 04$ students.

The number of students taken political science but no economics $= n(A^c \cap B) = 13$.

Ex. 1.12.13 : How many positive integers not exceeding 1000 are divisible by 7 or 11.

☑ **Ans. :** n(A) = Number of integers divisible by 7

$= n\left(\frac{1000}{7}\right) = 142$

n(B) = Number of integers divisible by 11

$= n\left(\frac{1000}{7}\right) = 90$

$n(A \cap B)$ = Number of integers divisible by 7 and 11

$= n\left(\frac{1000}{7 \times 11}\right) = 12$

According to inclusion – exclusion principle

The number of integers divisible by 7 or 11 not exceeding 1000 is

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 142 + 90 - 12 = 220$ ...Ans.

Ex. 1.12.14 : Out of 450 students in school, 193 students read science and 200 students read commerce, 80 students read neither. Find out how many read both.

☑ **Soln. :**

n(A) = Number of students read science = 193

n(B) = Number of students read commerce = 200

n(U) = Total number of students in the class = 450

$n(A^c \cap B^c) = n(A \cup B)^c$

= number of students read neither = 80

$n(A \cup B)$ = number of students read either science or commerce

$= U - (A \cup B)^C$

$\because A = U - A^C$ ...complement law

$= 450 - 80 = 370$

$n(A \cap B)$ = The number of students reads both science and commerce can be calculated by Inclusion – Exclusion principle

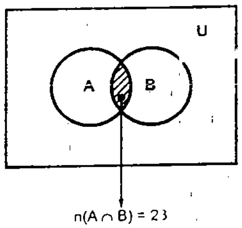


Fig. P. 1.12.4

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$370 = 193 + 200 - n(A \cap B)$$

$$370 = 393 - n(A \cap B)$$

$$n(A \cap B) = 23 \quad \text{...Ans}$$

Ex. 1.12.15 : Out of the integers 1 to 1000

(i) How many of them are not divisible by 3, nor by 5, nor by 7 ?

(ii) How many are not divisible by 5 and 7 but divisible by 3 ?

☑ **Soln. :**

$n(U)$ = Total number of integers = 1000

$n(A)$ = Number of integers divisible by 3

$= \left|\frac{1000}{3}\right| = 333$

$n(B)$ = Number of integers divisible by 5

$= \left|\frac{1000}{5}\right| = 200$

$n(C)$ = Number of integers divisible by 7

$= \left|\frac{1000}{7}\right| = 142$

$n(A \cap B)$ = Number of integers divisible by 3 and 5

$= \left|\frac{1000}{3 \times 5}\right| = 66$

$n(A \cap C)$ = Number of integers divisible by 3 and 7

$= \left|\frac{1000}{3 \times 7}\right| = 47$

$n(B \cap C)$ = Number of integers divisible by 5 and 7

$= \left|\frac{1000}{5 \times 7}\right| = 28$

$n(A \cap B \cap C)$ = Number of integers divisible by 3, 5 and 7

$= \left|\frac{1000}{3 \times 5 \times 7}\right| = 9$

(i) According to Inclusion – Exclusion principle

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 333 + 200 + 142 - 66 - 47 - 28 + 9 = 543$$

The number of integers divisible by 3, or 5 or 7 are = 543

$n(A \cup B \cup C)^C = U - (A \cup B \cup C) \quad \because A^C = U - A$

$n(A \cup B \cup C)^C = 1000 - 543 = 457$

∴ **The number of integers which are not divisible by 3, nor by 5, nor by 7 = 457.**

(ii) The number of integers divisible by 3, but not by 5 nor by 7 is $n(A \cap B^C \cap C^C)$

$$n(A \cap B^C \cap C^C) = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$

$$= 333 - 66 - 47 + 9$$

$$= 229 \quad \text{...Ans.}$$

Venn diagram

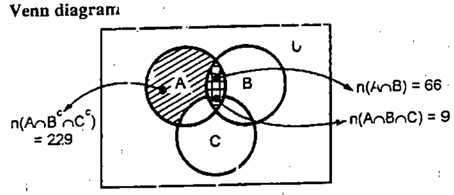


Fig. P. 1.12.15

Ex. 1.12.16 : Among the integers 1 to 500, Find how many are divisible 5, nor by 7, Find also. How many are divisible by 5, but not by 11.

☑ **Soln. :**

$n(A)$ = Number of integers divisible by 5 = $\left|\frac{500}{5}\right| = 100$

$n(B)$ = Number of integers divisible by 7 = $\left|\frac{500}{7}\right| = 71$

$n(C)$ = Number of integers divisible by 11 = $\left|\frac{500}{11}\right| = 45$

$n(A \cap B)$ = Number of integers divisible by 5 and 7

$= \left|\frac{500}{5 \times 7}\right| = 14$

$n(A \cap C)$ = Number of Integers divisible by 5 and 11

$= \left|\frac{500}{5 \times 11}\right| = 9$

$n(B \cap C)$ = Number of Integers divisible by 7 and 11

$= \left|\frac{500}{7 \times 11}\right| = 6$

According to inclusion – Exclusion Principle

$$(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= 100 + 71 - 14 = 157$$

The number of integers not divisible by 5 nor by 7 is $n(A^C \cap B^C)$ or $n(A^C \cap B)^C = U - (A \cup B)$

$$= 500 - 157 = 343 \quad \text{...Ans.}$$

The number of integers divisible by 5 but not by 11 $n(A \cap C^C)$ is $n(A - C) = n(A) - n(A \cap C)$

$$= 100 - 9 = 91 \quad \text{...Ans.}$$

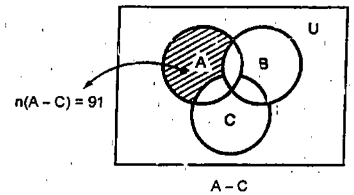


Fig. P. 1.12.16

Ex. 1.12.17 : Consider a set of integers 1 to 500.

Find :

(i) How many of these numbers are divisible by 3 or 5 or by 11.

(ii) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.

(iii) How many are divisible by 3 or 11 but not by 5 ?

☑ **Soln. :**

$n(A)$ = number of integers divisible by 3 = $\left|\frac{500}{3}\right| = 166$

$n(B)$ = Number of integers divisible by 5 = $\left|\frac{500}{5}\right| = 100$

$n(C)$ = Number of integers divisible by 11 = $\left|\frac{500}{11}\right| = 45$

$n(A \cap B)$ = Number of integers divisible by 3 and 5

$= \left|\frac{500}{3 \times 5}\right| = 33$

$n(A \cap C)$ = Number of integers divisible by 3 and 11

$= \left|\frac{500}{3 \times 11}\right| = 15$

$n(B \cap C)$ = Number of integers divisible by 5 and 11

$= \left|\frac{500}{5 \times 11}\right| = 9$

$n(A \cap B \cap C)$ = Number of integers divisible by 3, 5 and 11 $= \left|\frac{500}{3 \times 5 \times 11}\right| = 3$

(i) According to Inclusion – Exclusion principle

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 166 + 100 + 45 - 35 - 15 - 9 + 3$$

$$= 257$$

The number of integers divisible by 3 or 5 or by 11 under 500 are = 275

(ii) The number of integers divisible by 3 or by 11 but not by all 3, 5 and 11, calculated as

$= n(A \cup C) - n(A \cap B \cap C)$

$= [n(A) + n(C) - n(A \cap C)] - n(A \cap B \cap C)$

$= [166 + 45 - 15] - 3$

$= 196 - 3 = 193$

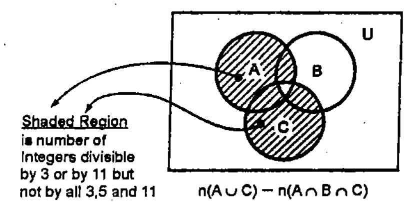


Fig. P. 1.12.17

(iii) The number of integers divisible by 3 or 11 but not by 5 = $n(A \cup B \cup C) - n(B)$

$= 257 - 100 = 157$

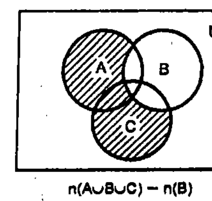


Fig. P. 1.12.17(a)

Shaded region the number of integers divisible by 3 or 11 but not by 5.

Ex. 1.12.18 : In a survey of 60 people :

25 read newsweek magazine

26 read time

26 read fortune

9 read both newsweek and fortune

11 read both newsweek and time

8 read both time and fortune

8 read no magazine at all

(i) Find the no. of people who read all the three magazines.

(ii) Find the no. of people who read exactly one magazine.

☑ **Soln. :**

Let $n(S)$ = number of peoples who read news week magazine = 25

$n(T)$ = Number of peoples who read time magazine = 26

$n(F)$ = Number of peoples who read fortune magazine = 26

$n(S \cap F)$ = Number of peoples who read both news week and fortune magazine = 9

$n(S \cap T)$ = Number of peoples who read news week and Time magazine = 11

$n(T \cap F)$ = Number of peoples who read time and Fortune magazine = 8

$n(S \cup T \cup F)^c$ = Number of peoples who read no magazine at all = 8

According to De Morgan's law

The number of peoples who read atleast one magazine that is $n(S \cup T \cup F) = U - n(S \cup T \cup F)^c$

$\because \; A = U - AC = 60 - 8 = 52$

(i) The number of peoples who read all the three magazine will be obtained by Inclusion-Exclusion principle

$$n(S \cup T \cup F) = n(S) + n(T) + n(F) - n(S \cap T) - n(S \cap F) - n(T \cap F) + n(S \cap T \cap F)$$

$$52 = 25 + 26 + 26 - 11 - 9 - 8 + n(S \cap T \cap F)$$

$$52 = 49 + n(S \cap T \cap F)$$

$\therefore$ $\mathbf{n(S \cap T \cap F) = 3}$

The number of peoples who read all the three magazines = 3 ...Ans.

(ii) The number of peoples who read exactly one magazine is obtained by following way

= (number of peoples who read only news week magazine)

+ (number of peoples who read only time magazine)

+ (number of peoples who read only fortune magazine)

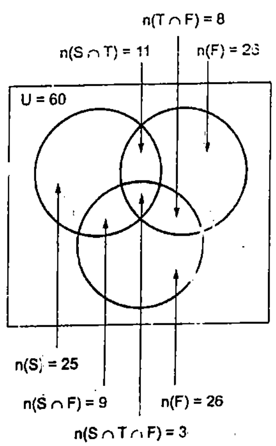
$= n(S \cap T^c \cap F^c) + n(S^c \cap T \cap F^c) + n(S^c \cap T^c \cap F)$

$= [n(S) - n(S \cap T) - n(S \cap F) + n(S \cap T \cap F)]$
$\quad + [n(T) - n(S \cap T) - n(T \cap F) + n(S \cap T \cap F)]$
$\quad + [n(F) - n(S \cap F) - n(T \cap F) + n(S \cap T \cap F)]$

$= [25 - 11 - 9 + 3] + [26 - 11 - 8 + 3]$
$\quad + [26 - 9 - 8 + 3]$

$= 8 + 10 + 12 = 30$

The number of peoples who read exactly one magazines is = 30 ...Ans.



(1A11)**Fig. P.1.12.18**

Ex. 1.12.19 : 100 sportsmen were asked whether they play cricket, football or hockey. Out of these 45 play cricket, 21 play football, 38 play hockey, 18 play cricket and hockey 9 play cricket and football, 4 play footand hockey and 23 playy none of these. Find the number of sportsmen who play :

(i) exactly one of the games (ii) exactly two of the games

☑ **Soln. :**

Let $n(U)$ = Total number of sportsmen were asked about game choice = 100

$n(C)$ = Number of sportsmen who like to play cricket = 45

$n(F)$ = Number of sportsmen like to play football = 21

$n(H)$ = Number of sportsmen like to play Hockey = 38

$n(C \cap H)$ = Number of sportsmen like to play cricket and hockey = 18

$n(C \cap F)$ = Number of sportsmen like to play cricket and football = 9

$n(F \cap H)$ = Number of sportsmen like to play football and hockey = 4

The number of sportsmen play none of the game.

$$n(C^c \cap F^c \cap H^c) = n(C \cup F \cup H)^c = 23.$$

$n(C \cup F \cup H)$ = The number of sportsmen play at least one game.

$= U - n(C \cup F \cup H)^c = 100 - 23 = 77$

Now, According to Inclusion – Exclusion principle

$$n(C \cup F \cup H) = n(C) + n(F) + n(H) - n(C \cap F) - n(C \cap H) - n(F \cap H) + n(C \cap F \cap H)$$

$$77 = 45 + 21 + 38 - 9 - 18 - 4 + n(C \cap F \cap H)$$

$n(C \cap F \cap H)$ = The number of sportsmen like to play all three games.

$= 77 - 73 = 4$

(i) The number of sportsmen who play exactly one game

= (Number of sportsmen play exactly cricket)

+ (Number of sportsmen play exactly football)

+ (Number of sportsmen play exactly hockey)

$= n(C \cap F^c \cap H^c) + n(C^c \cap F \cap H^c) + n(C^c \cap F \cap H^c)$

$= [n(C) - n(C \cap F) - n(C \cap H) + n(C \cap F \cap H)]$
$\quad + [n(F) - n(C \cap F) - n(F \cap H) + n(C \cap F \cap H)]$
$\quad + [n(H - n(C \cap H) - n(F \cap H) + n(C \cap F \cap H))]$

$= [45 - 9 - 18 + 4] + [21 - 9 - 4 + 4] + [38 - 18 - 4 + 4]$

$= \mathbf{22 + 12 + 20 = 54}$ ...Ans.

(ii) The number of sportsmen play exactly two games

= (Number of sportsmen play cricket and football)

+ (Number of sportsmen play football and hockey)

+ (Number of sportsmen play cricket and hockey)

$= n(C \cap F \cap H) + n(C^c \cap F \cap H) + n(C \cap F^c \cap H)$

$= [n(C \cap F) - n(C \cap F \cap H)] + [n(F \cap H) - n(C \cap F \cap H)] + [n(C \cap H) - n(C \cap F \cap H)]$

$= [9 - 4] + [4 - 4] + [18 - 4]$

$= \mathbf{5 + 0 + 14 = 19}$ ...Ans.

Ex. 1.12.20 : In the survey of 260 college students, the following data were obtained :

64 had taken a maths course,

94 had taken a cs course,

58 had taken a business course,

8 had taken both a maths and a business course,

26 had taken both a maths and a cs course,

22 had taken both a cs and a business course,

14 had taken all types of courses.

How many students were surveyed who had taken none of the three types of courses.

☑ **Soln. :**

Let. $n(U)$ = Total number of students surveyed

= 260

$n(M)$ = Number of students who had taken math course = 64

$n(C)$ = Number of students who had taken CS course = 94

$n(B)$ = Number of students who had taken business course = 58

$n(M \cap B)$ = Number of students taken math and business course = 8

$n(M \cap C)$ = Number of students taken math and CS course = 26

$n(C \cap B)$ = Number of students taken CS and business course = 22

$n(M \cap C \cap B)$ = Number of students taken all course = 14

According to Inclusion-Exclusion principle

$$n(M \cup C \cup B) = n(M) + n(C) + n(B) - n(M \cap C) - n(M \cap B) - n(C \cap B) + n(M \cap C \cap B)$$

$= 64 + 94 + 58 - 26 - 8 - 22 + 14$

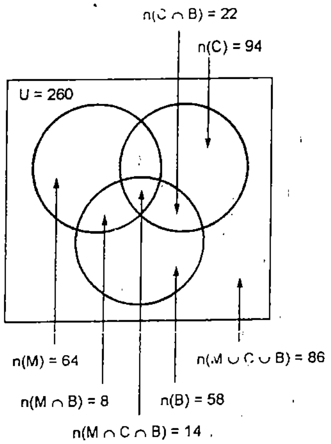
$= 174$ ...Ans.

(i) The number of students who had taken none of the three types of courses

According to De-Morgans law

$(M \cup C \cup B)^c = U - n(M \cup C \cup B)$

$= 260 - 174 = 86$ ...Ans.



(1A9)Fig. P.1.12.20

Ex. 1.12.21 : In a class of 80 students, 50 students know English, 55 know French and 46 know German language, 37 students know English and French, 28 students know French and German, 25 know English and German, 7 students know none of the languages. Find out :

(i) How many students know all the 3 languages ?

(ii) How many students know exactly 2 languages ?

(iii) How many students know only one language ?

☑ **Soln. :**

Let, $n(U)$ = Total number of students in the class = 80

$n(E)$ = Number of students know English = 50

$n(F)$ = Number of students know French = 55

$n(G)$ = Number of students know German = 46

$n(E \cap F)$ = Number of students know English and French = 37

$n(F \cap G)$ = Number of students know French and German = 28

$n(E \cap G)$ = Number of students know English and German = 25

$n(E^c \cap F^c \cap G^c)$ or $n(E \cup F \cup G)^c$ = Number of students not know all languages = 7

By using De-Morgan's law

$$n(E \cup F \cup G) = U - (E \cup F \cup G)^c$$

$$\because A = U - A^c = 80 - 7 = 73$$

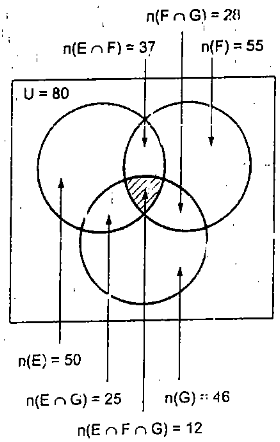
(i) The number of students know all the three languages is obtained by Inclusion-Exclusion principle

$$n(E \cup F \cup G) = n(E) + n(F) + n(G) - n(E \cap F) - n(E \cap G) - n(F \cap G) + n(E \cap F \cap G)$$

$$73 = 50 + 55 + 46 - 37 - 25 - 28 + n(E \cap F \cap G)$$

$$73 = 61 + n(E \cap F \cap G)$$

$$\mathbf{n(E \cap F \cap G) = 12} \quad \text{...Ans.}$$



(1A10)Fig. P.1.12.21

(ii) Number of students know exactly two languages

$$= n(E \cap F \cap G^c) + n(E \cap F^c \cap G) + n(E^c \cap F \cap G)$$

$$= [n(E \cap F) - n(E \cap F \cap G)] + [n(E \cap G) - n(E \cap F \cap G)] + [n(F \cap G) - n(E \cap F \cap G)]$$

$$= (37 - 12) + (25 - 12) + (28 - 12) = 25 + 13 + 16$$

$$= \mathbf{54} \quad \text{...Ans.}$$

(iii) The number of students know exactly one language

$$= n(E \cap F^c \cap G^c) + n(E^c \cap F \cap G^c) + n(E^c \cap F^c \cap G)$$

$$= [n(E) - n(E \cap F) - n(E \cap G) + n(E \cap F \cap G)] + [n(F) - n(E \cap F) - n(F \cap G) + n(E \cap F \cap G)] + [n(G) - n(E \cap G) - n(F \cap G) + n(E \cap F \cap G)]$$

$$= [50 - 37 - 25 + 12] + [55 - 37 - 28 + 12] + [46 - 25 - 28 + 12]$$

$$= (0) + (2) + (5)$$

$$= \mathbf{7} \quad \text{...Ans.}$$

Ex. 1.12.22 : Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathetmatics and biology, 7 study mathematics and physics, 10 study physics and biology and 30 do not study any of the three subjects.

(i) Find the number of students studying all three subjects.

(ii) Find the number of students studying exactly one of the three subjects.

☑ **Soln. :**

$n(U)$ = total number of students in the class = 100

$n(M)$ = Number of students study math = 32

$n(P)$ = Number of students study physics = 20

$n(B)$ = Number of students study biology = 45

$n(M \cap B)$ = Number of students study math and biology = 15

$n(M \cap P)$ = Number of students study math and physics = 7

$n(P \cap B)$ = Number of students study physics and biology = 10

$n(M^c \cap P^c \cap B^c)$ or $n(M \cup P \cup B)^c$ = number of students study none of the subjects = 30

According to De-Morgan's Law

The number of students studying at least one subject

$$n(M \cup P \cup B) = n(U) - n(M \cup P \cup B)^c$$

$$\because A = U - A^c = 100 - 30 = 70$$

(i) The number of students studying all the three subjects will be obtained by Inclusion-Exclusion principle

$$n(M \cup P \cup B) = n(M) + n(P) + n(B) - n(M \cap P) - n(M \cap B) - n(P \cap B) + n(M \cap P \cap B)$$

$$70 = 32 + 20 + 45 - 7 - 15 - 10 + n(M \cap P \cap B)$$

$$70 = 65 + n(M \cap P \cap B)$$

$$\mathbf{n(M \cap P \cap B) = 5} \quad \text{...Ans.}$$

(ii) The number of students studying exactly one subject

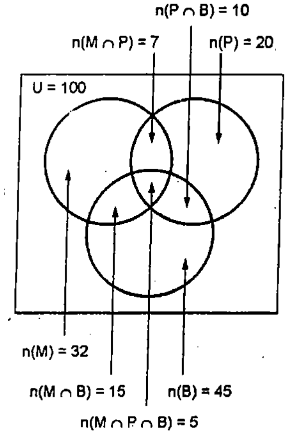
= (number of students studying only math) + (number of students studying only physics) + (number of students studying only biology)

$$= n(M \cap P^c \cap B^c) + n(M^c \cap P \cap B^c) + n(M^c \cap P^c \cap B)$$

$$= [n(M) - n(M \cap P) - n(M \cap B) + n(M \cap P \cap B)] + [n(P) - n(M \cap P) - n(P \cap B) + n(M \cap P \cap B)] + [n(B) - n(M \cap B) - n(P \cap B) + n(M \cap P \cap B)]$$

$$= [32 - 7 - 15 + 5] + [20 - 7 - 10 + 5] + [45 - 15 - 10 + 5]$$

$$= \mathbf{15 + 8 + 25 = 48} \quad \text{...Ans.}$$



(1A8)Fig. P.1.12.22

Shaded Region = Number of students studying exactly one subject = 15 + 8 + 25 = 48

Ex. 1.12.23 : Consider a set of integers 1 to 500. Find how many if these numbers are divisible by 3 or by 5 or by 11.

☑ **Soln. :**

Let, $n(U)$ = Total number of Integers = 500

$n(A)$ = Number of integers divisible by 3 $= \frac{500}{3} = 166$

$n(B)$ = Number of Integers divisible by 5 $= \frac{500}{5} = 100$

$n(C)$ = Number of Integers divisible by $= \frac{500}{11}$ = 45

$n(A \cap B)$ = Number of Integers divisible by 3 and 5 $= \frac{500}{15} = 33$

$n(A \cap C)$ = Number of Integers divisible by 3 and 11 $= \frac{500}{3 \times 11} = \frac{500}{33} = 15$

$n(B \cap C)$ = Number of integers divisible by 5 and 11. $= \frac{500}{5 \times 11} = \frac{500}{55} = 9$

$n(A \cap B \cap C)$ = Number of Integers divisible by 3, 5 and 11

$$= \frac{500}{3 \times 5 \times 11} = \frac{500}{165} = 3$$

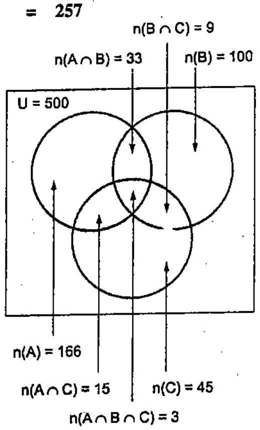
According to Inclusion-Exclusion principle,

(i) The number of integers atleast divisible by 3 or 5 or 11

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 166 + 100 + 45 - 33 - 15 - 9 + 3$$

$$= 257$$ ...Ans.



(1A7)Fig.P. 1.12.23

Ex. 1.12.24 : In the survey of 100 students. It is found that 12 students like milk only, 5 like coffee only, 8 had like tea only. 20 milk and coffee, 30 had coffee and Tea, 25 had milk and Tea, 10 students had like all the three drinks.

(i) Find, how many did not like any of the three drinks ?

(ii) How many like milk but not coffee ?

(iii) How many like tea and coffee but not milk ?

☑ **Soln. :**

Let, $n(M)$ = Number of students like milk = 12.

$n(C)$ = Number of students like coffee = 5

$n(T)$ = Number of students like Tea = 8

$n(M \cap C)$ = Number of students like milk and coffee = 20

$n(C \cap T)$ = Number of students like coffee and Tea = 30

$n(M \cap T)$ = Number of students like milk and Tea = 25

$n(M \cap C \cap T)$ = Number of students like all the drinks = 10

Find out $n(M \cup C \cup T)$ by considering all the set disjoint, initially

$$n(m \cup C \cup T) = n(M) + n(M \cap C) + n(M \cap C \cap T) + n(M \cap T) + n(C \cap T) + n(C) + n(T)$$

$$= 12 + 10 + 10 + 15 + 20 + 8 + 5 = 80$$

(i) The number of students who did not like any of the drinks from milk, coffee and Tea is

$$n(M^C \cap C^C \cap T^C) \text{ or } n(M \cup C \cup T)^C = U - n(M \cup C \cup T)$$

$$= 100 - 80$$

= 20 Students ...Ans.

(ii) The number of students like milk but not coffee will be $n(M - C)$

$$= 12 + 15 = 27$$ ...Ans.

(iii) The number of students like tea and coffee but not milk is $n(C \cap T) - n(M)$

$$n(C \cap T) - n(m) = n(C \cap T) - n(M \cap C \cap T)$$

$$= 30 - 10 = \textbf{20 Students}$$ ...Ans.

Ex. 1.12.25 : Among the integer 1 to 1000 :

(i) How many of them are not divisible by 3 nor by 5 nor by 7.

(ii) How many are not divisible by 5 and 7 but divisible by 3.

☑ **Soln. :**

Let, Total number of Integers = $n(U)$ = 1000

$n(A)$ = Number of Integers which are divisible by 3

$$= \frac{1000}{3} = 333$$

$n(B)$ = Number of Integers which are divisible by 5

$$= \frac{1000}{5} = 200$$

$n(C)$ = Number of Integers divisible by 7

$$= \frac{1000}{7} = 142$$

$n(A \cap B)$ = Number of Integers divisible by 3 and 5

$$= \frac{1000}{3 \times 5} = \frac{1000}{15} = 66$$

$n(B \cap C)$ = Number of Integers divisible by 5 and 7

$$= \frac{1000}{5 \times 7} = \frac{1000}{35} = 28$$

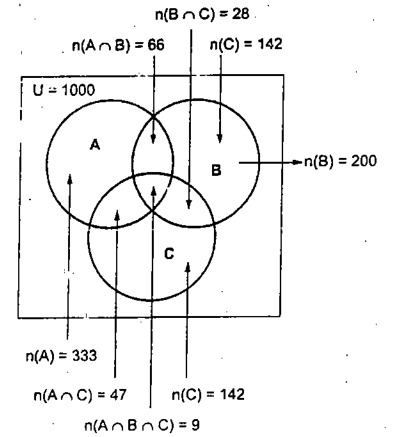
$n(A \cap C)$ = Number of Integers divisible by 3 and 7

$$= \frac{1000}{3 \times 7} = \frac{1000}{21} = 47$$

$n(A \cap B \cap C)$ = Number of Integers divisible by 3, 5 and 7

$$= \frac{1000}{105} = 9$$

Using Inclusion-Exclusion principle



(1A13)Fig. P.1.12.25

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$= (333) + (200) + (142) - (66) - (28) - (47) + 9 = 543$$

(i) According to De'Morgan's Law

$$(A^c \cap B^c \cap C^c) = (A \cup B \cup C)^c$$

The number of integers not divisible by 3 nor by 5 and nor by 7 is

$$(A \cup B \cup C)^c = U - (A \cup B \cup C)$$

$$\because A^c = U - A = 1000 - 543 = 457$$

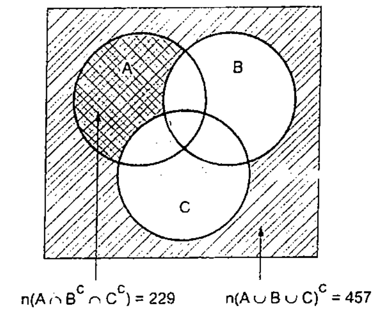
The number of Integers not divisible by 3, nor by 5 and nor by 7 = 457 ...Ans.

(ii) The number of integers not divisible by 5 and 7 but divisible by 3 written as,

$$n(A \cap B^c \cap C) = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$

$$= 333 - 66 - 47 + 9 = 229$$

The number of Integers not divisible by 5 nor by 7 but divisible by 3 = 229 ...Ans.



(1A12)Fig. P.1.12.25(a)

Ex. 1.12.26 : It was found that in the first year computer science class of 80 students, 50 knew COBOL, 55 'C' and 46 PASCAL. It was also known that 37 knew 'C and COBOL, 28 'C' and PASCAL, and 25 PASCAL and COBOL. 7 students however knew none of the language :

(i) How many knew all the three languages ?

(ii) How many knew exactly two languages ?

(iii) How many knew exactly one languages ?

☑ **Soln. :**

Let, $n(U)$ = The total number of students in the class = 80

$n(A)$ = Number of students know COBOL = 50

$n(B)$ = Number of students know C language = 55

$n(C)$ = Number of students know PASCAL = 46

$n(A \cap B)$ = Number of students know COBOL and C language = 37

$n(B \cap C)$ = Number of students know C language and PASCAL = 28

$n(A \cap C)$ = Number of students know COBOL and PASCAL = 25

$n(A^c \cap B^c \cap C^c) = n(A \cup B \cup C)^c$ = Number of students don't know any of the language = 7

The number of students know at least one language is

$$n(A \cup B \cup C) = n(U) - n(A \cup B \cup C)^c$$

$$= 80 - 7 = 73 \qquad \because A = U - A^c$$

(i) The number of students know all the three languages according to Inclusion-Exclusion principle

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$73 = 50 + 55 + 46 - 37 - 25 - 28 + n(A \cap B \cap C)$

$73 = 61 + n(A \cap B \cap C)$

$\mathbf{n(A \cap B \cap C) = 12}$ ...Ans.

(ii) The number of students know exactly two languages

= (The number of students know COBOL and C language)+(The number of students know COBOL and PASCAL) + (Number of students know C language and PASCAL)

$= (A \cap B \cap C^c) + (A \cap B^c \cap C) + n(A^c \cap B \cap C)$

$[n(A \cap B) - n(A \cap B \cap C)] + [n(A \cap C) - n(A \cap B \cap C)] + [n(B \cap C) - (A \cap B \cap C)]$

$= (37 - 12) + (25 - 12) + (28 - 12)$

$\Rightarrow 25 + 13 + 16 \Rightarrow 54$ ...Ans.

(iii) The number of students know exactly one language

$\Rightarrow$ (number of students know only COBOL)

+ (number of students know only C language)

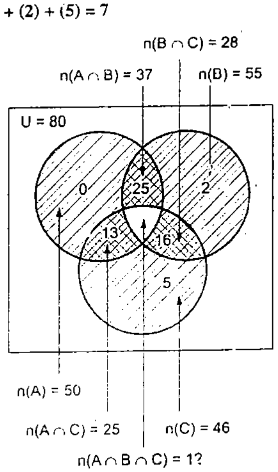
+ (number of students know only PASCAL)

$= n(A \cap B^c \cap C^c) + n(A^c \cap B \cap C^c) + n(A^c \cap B^c \cap C)$

$= [n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)] + [n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)] + [n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)]$

$= (50 - 37 - 25 + 12) + (55 - 37 - 28 + 12) + (46 - 25 - 28 + 12)$

$= (0) + (2) + (5) = 7$ ...Ans.



(1A6)Fig. P.1.12.26

1.13 MULTISET

- As we know, sets are collection of distinct objects which means set has unique elements in it.
- But there are many examples where distinct or unique object not possible, for example number of copies of same book in library, names of the students in the class, birth month of the peoples, and many more.
- Basically, multiset is generalization of a set.
- In multiset, there are multiple copies of same object.
- Multisets is also called as Msets.
- To distinguish a set and multiset, We denote multiset by enclosing elements in a square brackets.

E.g. [a, b. b, c, a].

1.13.1 Multiplicity of an Elements

- Let, A be a multiset and $x \in A$. The multiplicity of x is the number of time the element x appears in the multiset.
- In other words, multiplicity of an element in multiset is the number of time the element appears in mset.
- It is denoted by $\mu(x)$.

Example

A = [a, b, c, a, a, b]

$\mu(a) = 3, \quad \mu(b) = 2, \quad \mu(c) = 1$

1.13.2 Equality of Multisets

Equality of multisets is defined as, the number of occurrences of each element must be equal or same in the given multisets.

Example : A = [a, b, a, b], B = [a, a, b, b]

Where, multisets A and B are equal.

C = [a, b, a], Here, $A \neq C, B \neq C$

1.13.3 Union of Multisets

The union of two multisets A and B denoted as, $A \cup B$, is the multiset such that for each element $x \in A \cup B$

Where, $\mu(x) = \max \{ \mu_A(x), \mu_B(x) \}$

Example

A = [a, a, a, b, b, c, c, c]

B = [a, a, b, c, c, c, c]

$A \cup B$ = Maximum occurrence of each element of A and B

$A \cup B$ = [a, a, a, b, b, c, c, c, c]

1.13.4 Intersection of Multisets

The intersection of two multisets A and B denoted as $A \cap B$, is the multiset such that for each element $x \in A \cap B$ where.

$\mu(x) = \min \{ \mu_A(x), \mu_B(x) \}$

Example

A = [a, a, a, b, b, c, c, c]

B = [a, a, b, c, c, c, c]

$A \cap B$ = [a, a, b, c, c, c]

1.13.5 Subset of Multisets

If A is a subset of B, if multiplicity of each element of A is less than or equal to its multiplicity in B.

Examples

(i) [a, b, a, a, b] $\subseteq$ [a, a, b, b, a, b]

1.13.6 Difference of Multisets

- The difference of multisets A and B, denoted as, A – B, is a multiset such that $x \in A - B$

Where, $(\mu_A(x) - \mu_B(x)) \geq 1$

Example

(i) A = [a, a, a, b, b, c], B = [a, a, b, c]

A – B = [a, b]

(ii) A = [1, 2, 3, 4, 3, 3, 2, 2],

B = [2, 2, 2, 1, 1, 1, 3, 3, 3, 3, 4]

A – B = [ ] or ϕ

1.13.7 Sum of Multisets

- The sum of two multisets A and B, is denoted as A + B, such that, $x \in A + B$, where.

$\mu(x) = \mu_A(x) + \mu_B(x)$

Examples : A = [c, c, a, b], B = [b, b, c, a, a]

A + B = [c, c, c, b, b, b, a, a, a]

1.13.8 Examples on Multiset

Ex.1.13.1 : What is multiset. Let P and Q are two multiset defined as P = {a, a, a, c, d, d} and Q = {a, a, b, c, c}. Obtain Union, Intersection and difference of two multisets P and Q

☑ **Ans. :**

Multiset

Definition : It is a set in which objects in a set are not distinct, multiple copies of on object may occurs.

- It is generalization of set.
- It is represented by square bracket.
- E.g. The collection of alphabets in the word like "ELEVEN".

M = [E, L, E, V, E, N]

Let, P = [a, a, a, c, d, d] Q = [a, a, b, c, c]

(i) Union of multisets

It is the maximum multiplicity (number of time particular item is present) of an element in set P and in set Q.

$P \cup Q$ = [a, a, a, b, c, c, d, d]

(ii) Intersection of multisets

It is the minimum multiplicity of an element in set P and in set Q.

$P \cap Q$ = [a, a, c]

(iii) Difference of multisets

Let, P and Q are two multiset. The difference P – Q is a multiset such that for each element $x \in P - Q$

$\mu(x) = \mu_A(x) - \mu_B(x)$

P – Q = [a, d, d]

UEx. 1.13.2 (SPPU - IT : Q. 2(b), Dec. 19, 6 Marks)

Let A and B be the multisets [a, a, b, b, c, f] and [a, a, b, b, d, d] respectively. (a) $A \cup B$ (b) $A \cap B$ (c) A – B (d) B – A

☑ **Soln. :**

Let A = [a, a, b, b, c, f] , B = [a, a, b, b, d, d] be two multisets.

(a) $A \cup B$ (A Union B) :

$A \cup B$ = Maximum multiplicity of an element in A and B

= [a, a, b, b, c, d, d, f]

(b) $A \cap B$ (A Intersection B) :

$A \cap B$ = Minimum multiplicity of an element in A and B

= [a, a, b, b]

(c) A – B (A difference B)

A – B = [c, f]

(d) B – A (B difference A)

B – A = [d, d]

1.14 MATHEMATICAL INDUCTION

- It is a method of proving mathematical theorems, statements or formulae.
- In mathematical induction, induction is about **"Inductive Reasoning"**.
- It is deductive in nature.
- In mathematical induction, we first prove that the first proposition is true called as **"base of Induction"**.
- Next, we prove that if k^{th} proposition is true then $(K)^{th}$ proposition is also true known as the **"Inductive step"**.
- There are many real time examples which will show the use of mathematical induction.
  - (i) From Fig. 1.14.1 given below, it is clear that, if first the dominoes falls then all the dominoes will fall.
  - (ii) If the first teeth of zip is zipped successfully then we can successfully zip a proper zipper.
- For software developer, mathematical induction play important role in algorithm verification.
- It is used to generalized the results from any theorems or formulae's.
- There are two principles of mathematical induction.
  - (a) First principle of mathematical induction
  - (b) Second principle of mathematical induction.

Steps Involved in mathematical induction

▶ **Step I : Basis of induction**

Here, check, validity or correctness of given statement say S(n) is true for the smallest integral value of n = 1 or 2 or 3 …

▶ **Step II : Induction step**

In this step, assume given statement S(n) is true for n = k where k denotes any value of n, then it is also true that S(n) is true for n = k + 1

▶ **Step III : Conclusion**

The statement is true for all integral values of n equal to or greater than that for which it was verified in step – I.

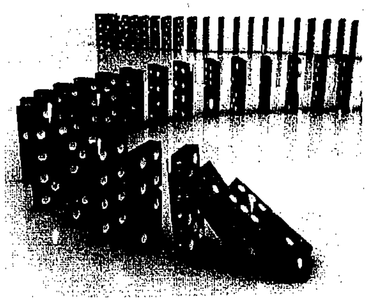


Fig. 1.14.1

1.14.1 Solved Examples

UEx. 1.14.1 (SPPU - IT, Q. 2(a), May 19, 6 Marks)

Use mathematical induction to show that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

for all n >= 1

☑ **Soln. :**

Given :

$$S(n):\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots\dots+\frac{1}{n(n+1)}=\frac{n}{n+1} \qquad \dots(1)$$

for all n >= 1

▶ **Step 1 : Basis of Induction**

Check S(n) is true for n = 1

$$\text{LHS} \Rightarrow \frac{1}{1(1+1)} = \frac{1}{2}$$

$$\text{RHS} \Rightarrow \frac{1}{1+1} = \frac{1}{2}$$

Here, LHS = RHS.Hence, S(n) is true for n = 1

▶ **Step 2 : Induction step**

Assume, S(n) is true for n = K

$$S(n):\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots\dots+\frac{1}{K(K+1)}=\frac{K}{K+1} \qquad \dots(2)$$

Now, check S(n) is true for n = K + 1

$$S(K+1):\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots\dots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+2)}=\frac{K+1}{K+2}$$

Solve the LHS of above equation.

$$S(K+1):\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+2)}$$

$$\Rightarrow \frac{K}{(K+1)}+\frac{1}{(K+1)(K+2)} \qquad \text{(From Equation (2))}$$

$$\Rightarrow \frac{K(K+1)(K+2)+(K+1)}{(K+1)(K+1)(K+2)} \Rightarrow \frac{(\cancel{K+1})[K(K+2)]+1}{(\cancel{K+1})(K+1)(K+2)}$$

$$\Rightarrow \frac{K^2+2K+1}{(K+1)(K+2)} = \frac{K^2+K+K+1}{(K+1)(K+2)}$$

$$\Rightarrow \frac{K(K+1)+(K+1)}{(K+1)(K+2)} \Rightarrow \frac{(\cancel{K+1})(K+1)}{(\cancel{K+1})(K+2)} = \frac{K+1}{K+2}$$

…Required RHS

Hence, The given S(n) is true and validated using mathematical Induction.

UEx. 1.14.2 (SPPU - IT, Q. 1(a), May 15, Dec. 16, 6 Marks)

Prove the statement is true using mathematical induction : $n^3 + 2n$ is divisible by 3 for all n >= 1.

☑ **Soln. :**

Given : S(n) : $n^3 + 2n$ is divisible by 3 for all n >= 1

▶ **Step 1 : Basis of Induction**

For n = 1

S(1) : $1^3 + 2\cdot 1 = 1 + 2 = 3$ is divisible by 3.

Hence, S(n) is true for n = 1 .

▶ **Step -2 : Induction Step**

Assume, S(n) is true for n = K, that is

S(K) = $K^3 + 2K$ is divisible by 3.

Now, Check S(n) is true for n = K + 1

$$S(K+1):(K+1)^3+2(K+1)$$

$$= K^3+3K^2+3K+1+2K+2$$

$$= (K^3+2K)+3K^2+3K+3$$

$$= (K^3+2K)+3(K^2+K+1)$$

Here, $(K^3 + 2K) \Rightarrow$ divisible by 3 (assumed)

$3(K^2 + K + 1) \Rightarrow$ divisible by 3.

Hence, S(n) is true for n = K + 1

So that,

= $n^3 + 2n$ + divisible by 3 for all n > = 1. **…Proved**

UEx. 1.14.3 (SPPU - IT, Q. 2(a), Dec. 14, 6 Marks)

State the principle of Mathematical Induction, using mathematical induction prove the following proposition :

$$P(n) = 1+4+7+\dots\dots+(3n-2)=\frac{n(3n-1)}{2}$$

☑ **Soln. :**

Given : $S(n) : 1+4+7+\dots+(3n-2)=\frac{n(3n-1)}{2}$

▶ **Step 1 : Basis of Induction**

Check S(n) is true for n = 1

S(1) : 3 – 2 = 1 ⇐ LHS

$$\frac{1(3-1)}{2}=\frac{1(2)}{2}=1 \Leftarrow \text{RHS}$$

Hence, S(n) is true for n = 1

▶ **Step 2 : Induction Step**

Assume, S(n) is true for n = K

$$S(K) = 1+4+7+\dots+(3K-2)=\frac{K(3K-1)}{2} \qquad \dots(1)$$

Now, Check S(n) is true for n = K + 1

$$S(K+1) : 1+4+7+\dots+(3K-2)+3(K+1)-2$$

$$= \frac{(K+1)(3(K+1)-1)}{2} = \frac{(K+1)(3K+2)}{2}$$

$$= \frac{K(3K-1)}{2}+3(K+1)-2 \qquad \text{...From equation (1)}$$

$$= \frac{3K^2-K+2(3K+3-2)}{2}$$

$$\Rightarrow \frac{3K^2-K+6K+6-4}{2} \Rightarrow \frac{3K^2-K+6K+2}{2}$$

$$\Rightarrow \frac{3K^2+5K+2}{2}$$

$$\Rightarrow \frac{3K^2+3K+2K+2}{2} \Rightarrow \frac{(3K+2)(K+1)}{2}$$

$$\Rightarrow \frac{(K+1)(3K+2)}{2} \quad \text{...Proved.}$$

UEx. 1.14.4 (SPPU - IT, Q. 2(A), May 17, 6 Marks)

Prove by induction that the sum of the cubes of three consecutive integers is divisible by 9.

☑ **Soln. :**

S(n) : The sum of the cubes of three consecutive integers is divisible by 9

That is

S(n) : $(n-1)^3 + n^3 + (n+1)^3$ is divisible by 9

▶ **Step 1 : Basis of Induction**

Check S(n) is true for n = 1

S(1) : $0^3 + 1 + 2^3 = 1 + 8 = 9 \Rightarrow$ divisible by 9

▶ **Step 2 : Induction Step**

Assume, S(n) is true for n = K that is

$S(K) : (K-1)^3 + k^3 + (K+1)^3$

$= 3K^3 + 6K$ is divisible by 9.

Now, Check S(n) is true for n = K + 1

$S(K+1) : (K+1-1)^3 + (K+1)^3 + (K+2)^3$

$= K^3 + (K+1)^3 + (K+2)^3$

$= 3K^3 + 9K^2 + 15K + 9$

$= 3K^3 + 9K^2 + 9K + 6K + 9$

$= 3K^3 + 6K + 9K^2 + 9K + 9$

$= \underbrace{(3K^3 + 6K)}_{\text{Already assumed that, divisible by 9}} + \underbrace{9(K^2 + K + 1)}_{\text{Divisible by 9}}$

Hence, S(n) is true for n = K + 1. ... **proved.**

UEx. 1.14.5 : (SPPU - IT, Q. 2(a), May 14, 6 Marks)

Prove by mathematics induction for n ≥ 1.

$1 \times 2 + 2 \times 3 + \ldots\ldots n(n+1) = \frac{n(n+1)(n+2)}{3}$

☑ **Soln. :**

Given : $S(n) : 1\cdot2 + 2\cdot3 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

for n > = 1

▶ **Step 1 : Basis of Induction**

Check S(n) is true for n = 1

S(1) : LHS : → 1(1 + 1) = 2

RHS: $\rightarrow \frac{1(2)(3)}{3} \Rightarrow \frac{6}{3} = 2$ Here, LHS = RHS

So that, S(n) is true for n = 1

▶ **Step 2 : Induction Step**

Assume, S(n) is true for n = K

$S(K) = 1\cdot2 + 2\cdot3 + \ldots + K(K+1) = \frac{K(K+1)(K+2)}{3}$ ...(1)

Now, Check S(n) is true for n = K + 1

$S(K+1) : 1\cdot2 + 2\cdot3 + \ldots + K(K+1) + (K+1)(K+2)$

$= \frac{(K+1)(K+2)(K+3)}{3}$

$\Rightarrow \frac{K(K+1)(K+2)}{3} + \frac{(K+1)(K+2)}{1}$ From Equation (1)

$\Rightarrow \frac{K(K+1)(K+2) + 3(K+1)(K+2)}{3}$

$\Rightarrow \frac{(K+1)(K+2)(K+3)}{3}$ ...**Proved.**

S(n) is true for n ≥ 1 by mathematical induction.

Ex. 1.14.6 : By using mathematical induction show that :

$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for all natural number values of n.

☑ **Soln. :**

Let $S(n) : 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ ...(1)

▶ **Step I : Basis of Induction**

Check S (n) is true for n = 1

LHS = 1, $RHS = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

∴ LHS = RHS

Hence, S(n) is true for n = 1

▶ **Step II : Induction step**

Assume S(n) is true for n = k

Thus we get

$S(k) : 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$ ...(2)

Now, check S(n) is true for n = k + 1 from Equation (1),

$S(k+1) : 1 + 2 + 3 + \ldots + k + k + 1 = \frac{(k+1)(k+2)}{2}$

Now, solve the LHS of above statement

$S(k+1) : \underbrace{1 + 2 + 3 + \ldots + k} + (k+1)$

$= \frac{k(k+1)}{2} + k + 1$ ∵ From Equation (2)

$= \frac{k(k+1) + 2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$ = RHS of S (k + 1)

▶ **Step III**

Conclusion : $S(n) : 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ is true using mathematical induction.

Ex. 1.14.7 : Prove by the mathematical induction

$S(n) = 1 + 3 + 6 + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$

☑ **Soln. :**

▶ **Step I : Basis of induction**

$S(n) = 1 + 3 + 6 + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$ ...(1)

For n = 1, put n = 1 in LHS and RHS of S(n)

S(1) : LHS $= \frac{1(1+1)}{2} = \frac{2}{2} = 1$,

RHS $= \frac{1(1+1)(1+2)}{6} = \frac{(2)(3)}{6} = \frac{6}{6} = 1$

LHS = RHS

Hence, S(n) is true for n = 1

▶ **Step II : Induction step**

Assume, S(n) is true for n = k

So,

$S(k) : 1 + 3 + 6 + \ldots + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$ ...(2)

Now, check S(n) is true for n = k + 1

So, put n = k + 1 in Equation (1) then,

$S(k+1) : 1 + 3 + 6 + \ldots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}$

$= \frac{(k+1)(k+2)(k+3)}{6}$

Lets start from L.H.S. of S (k + 1)

$S(k+1) : 1 + 3 + 6 + \ldots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}$

$= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$ ∵ From Equation (2)

$= \frac{2[k(k+1)(k+2)] + 6[(k+1)(k+2)]}{12}$

$= \frac{[k(k+1)(k+2)] + 3[(k+1)(k+2)]}{6}$

$= \frac{(k+1)(k+2)(k+3)}{6}$ = R.H.S of S (k + 1)

▶ **Step III : Conclusion**

This show that, if S(n) is true for n = k, then it is also S(n) is true for n = k + 1. Thus, mathematical induction is true for every value of n.

Ex. 1.14.8 : Prove by mathematical induction

$S(n) : 1 \cdot 2 + 2 \cdot 2^2 + \ldots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2$

☑ **Soln. :**

Given : $S(n) : 1\cdot2 + 2\cdot2^2 + \ldots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2$ ...(1)

▶ **Step I : Basis of induction**

Check, S(n) is true for n = 1

Put, n = 1, $L.H.S = 1\cdot2^1 = 2$

$R.H.S = (1 \cdot 1) \cdot 2^{1+1} + 2 = 0 \times 2^2 + 2 = 2$

Here, L.H.S. = R.H.S, Hence, S(n) is true for n = 1.

▶ **Step II : Induction step**

Assume, S(n) is true for n = k

Put, n = k in S(n)

$S(k) : 1\cdot2 + 2\cdot2^2 + \ldots + k \cdot 2^k = (k-1) \cdot 2^{k+1} + 2$ ...(2)

Now, check S(n) is true for n = k + 1

$S(k+1) : 1\cdot2 + 2\cdot2^2 + \ldots + k \cdot 2^k + (k+1) \cdot 2^{k+1}$

$= (k+1-1) \cdot 2^{(k+1+1)} + 2$

$S(k+1) : 1\cdot2 + 2\cdot2^2 + \ldots + k \cdot 2^k + (k+1) \cdot 2^{k+1}$

$= k \cdot 2^{(k+2)} + 2$

Taking L.H.S. of above equation,

$S(k+1) : 1\cdot2 + 2\cdot2^2 + \ldots + k \cdot 2^k + (k+1) \cdot 2^{k+1}$

$= (k-1) 2^{k+1} + 2 + (k+1) \cdot 2^{k+1}$

∵ From Equation (2)

Taking 2^{k+1} common

$= 2^{k+1} [(k - \cancel{1}) + (k + \cancel{1})] + 2$

$= 2^{k+1} [2k] + 2$

$= 2^k \cdot 2^1 \cdot 2^1 \cdot k^1 + 2 \Rightarrow (2)^{k+1+1} \cdot k + 2$

$= 2^{k+1+1} (k + 1 - 1) + 2$

$\Rightarrow (k + 1 - 1) \cdot 2^{k+1+1} + 2$

$\Rightarrow (k) \cdot 2^{k+2} + 2 = R.H.S$

▶ **Step III : Conclusion**

This shows that S(n) is true for n = k then it is also true for n = k + 1.

Thus, by using mathematical induction, S(n) is true for every integral value of n.

Ex. 1.14.9 : With the help of mathematical induction prove that, $1^2 + 3^2 + 5^2 + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

☑ **Soln. :**

Let,

$S(n) : 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ ...(1)

▶ **Step I : Basis of induction**

Check S (n) is true for n = 1

So, LHS $= (2 \cdot 1 - 1)^2 = (2-1)^2 = 1^2 = 1$

RHS $= 1 \frac{(2-1)(2+1)}{3} = \frac{(1)(3)}{3} = \frac{3}{3} = 1$,

LHS = RHS

Hence, S (n) is true for n = 1

▶ **Step II : Induction step**

Assume S (n) is true for n = k

$S(k) : 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$ ...(2)

Now, check, S (n) is true for n = k + 1

$S(k+1) : 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2(k+1)-1)$

$= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

Solve LHS of above statement

$\Rightarrow \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$ ∵ from Equation (2)

$\Rightarrow \frac{2k+1}{3}(2k^2 - k + 3(2k+1))$

$\Rightarrow \frac{2k+1}{3}(2k^2 + 5k + 3)$

$\Rightarrow \frac{2k+1}{3}((2k+3)(k+1))$

$\Rightarrow \frac{(2k+1)(2k+3)(k+1)}{3} \Rightarrow \frac{(k+1)(2k+1)(2k+3)}{3}$

$\Rightarrow \frac{(k+1)(2(k+1)-1)\cdot(2(k+1)+1)}{3}$ Required RHS

▶ **Step III :**

Conclusion : Hence S(n) is true by mathematical induction.

Ex. 1.14.10 : Show that,

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \dots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$$

using mathematical induction.

☑ **Soln. :**

$$S(n) : \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \dots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)} \quad \dots(1)$$

check, S(n) is true for n = 1

Put, n = 1 in Equation (1)

$$S(1) : \quad L.H.S = \frac{1^2}{(2\times 1-1)(2\times 1+1)} = \frac{1}{(1)(3)} = \frac{1}{3}$$

$$R.H.S = \frac{1(1+1)}{2(2\times 1+1)} = \frac{\cancel{2}}{\cancel{6}} = \frac{1}{3} \quad L.H.S. = R.H.S.$$

Hence, S(n) is true for n = 1.

▶ **Step II : Induction step**

Assume, S(n) is true for n = k.

$$S(k) : \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \dots + \frac{k^2}{(2k-1)(2k+1)} = \frac{k(k+1)}{2(2k+1)} \quad \dots(2)$$

Now, check S(n) is true for n = k + 1

$$S(k+1) : \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \dots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{[2(k+1)-1][2(k+1)+1]} = \frac{(k+1)(k+1+1)}{2(2(k+1)+1)}$$

$$S(k+1) : \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \dots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)}$$

$$S(k+1) : \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)}$$

...From Equation (2)

Take L.H.S. of S (k + 1), then solve,

$$S(k+1) : \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$= \frac{[k(k+1)]\cdot[(2k+1)(2k+3)] + [2(2k+1)(k+1)^2]}{[2(2k+1)]\cdot[(2k+1)\cdot(2k+3)]}$$

Taking $\frac{(k+1)}{(2k+1)}$ common

$$= \frac{k+1}{2k+1}\left[\frac{k(2k+3)+2(k+1)}{2(2k+3)}\right]$$

$$= \frac{k+1}{2k+1}\left[\frac{2k^2+5k+2}{2(2k+3)}\right]$$

$$= \frac{k+1}{2k+1}\left[\frac{2k^2+4k+k+2}{2(2k+3)}\right]$$

$$= \frac{k+1}{2k+1}\left[\frac{2k(k+2)+1(k+2)}{2(2k+3)}\right]$$

$$= \frac{k+1}{\cancel{2k+1}}\left[\frac{\cancel{(2k+1)}(k+2)}{2(2k+3)}\right]$$

$$= \frac{(k+1)(k+2)}{2(2k+3)} = \text{R.H.S. of } S(k+1)$$

▶ **Step III : Conclusion**

This shows that, S(n) is true for n = k then it is also true for n = k + 1.

Thus, by using mathematical induction S(n) is true for every integral value of n.

Ex. 1.14.11 : Use mathematical induction to show that $n^2 - 4n^2$ is divisible by 3 for all $n \ge 2$.

☑ **Soln. :**

Let S (n) : $n^2 - 4n^2$ is divisible by 3 for all $n \ge 2$ ...(1)

▶ **Step I : Basis of Induction**

Check S (n) is true for n = 2

S (2) : $2^2 - 4(2)^2 \Rightarrow 4 - 4 \cdot 4 = 4 - 16 = -12$ is divisible by 3 so, S (n) is true for n = 2

▶ **Step II : Induction step**

Assume, S (n) is true for n = k

S (k) : $k^2 - 4k^2$ is divisible by 3 ...(2)

Check S (k + 1) is true for n = k + 1

S (k + 1) : $(k+1)^2 - 4(k+1)^2$

$\Rightarrow (k^2 + 2k + 1) - 4(k^2 + 2k + 1)$

$\Rightarrow (k^2 + 2k + 1) - 4k^2 - 8k - 4$

$\Rightarrow (k^2 - 4k^2) - (6k + 3)$

$\Rightarrow (k^2 - 4k^2) - 3(k + 1)$

Here, $k^2 - 4k^2$ is divisible by 3 from Equation (2) and 3(k + 1) is divisible by 3.

▶ **Step III : Conclusion**

S (n) : $n^2 - 4n^2$ is divisible by 3 for all $n \ge 2$ by mathematical induction .

Ex. 1.14.12 : Use mathematical induction to show that

$$1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} n^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

☑ **Soln. :**

Let, S (n) :

$$1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} \cdot n^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2} \quad \dots(1)$$

▶ **Step I : Basis of Induction**

Check S (n) is true for n = 1

$$S(1) : LHS = (-1)^0 \cdot 1^2 = 1 \cdot 1 = 1$$

$$RHS = (-1)^0 \cdot \frac{1(1+1)}{2} = 1 \cdot \frac{2}{2} = \frac{2}{2} = 1$$

Hence, S (n) is true for n = 1

▶ **Step II : Induction step**

Assume, S (n) is true for n = k

$$S(k) : 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k-1} \cdot k^2 = (-1)^{k-1} \cdot \frac{k(k+1)}{2} \quad \dots(2)$$

Now, Check S (n) is true for n = k + 1

$$S(k+1) : 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k-1} \cdot k^2 + (-1)^{k+1-1} \cdot (k+1)^2 = \frac{(-1)^k \cdot (k+1)(k+2)}{2}$$

Solve LHS of above statement

$$\Rightarrow (-1)^{k-1} \cdot \frac{k(k+1)}{2} + (-1)^k \cdot (k+1)^2$$

...From Equation (2)

$$\Rightarrow (-1)^k (k+1)\left[\frac{-k}{2} + (k+1)\right]$$

$$\Rightarrow (-1)^k \cdot (k+1)\left[\frac{-k+2k+2}{2}\right]$$

$$\Rightarrow \frac{(-1)^k \cdot (k+1)\cdot(k+2)}{2} = \text{Required RHS}$$

▶ **Step III : Conclusion**

Hence, S (n) : $1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} \cdot n^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$ is true by mathematical induction

UEx. 1.14.13 (SPPU-IT- Q. 3(a), Dec. 19, 6 Marks)

Prove by Mathematical Induction that for $n >= 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots\dots + n \cdot n! = (n+1)! - 1$

☑ **Soln. :**

Let,

S(n) : $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! = (n+1)! - 1$ ...(1)

▶ **Step I : Basis of induction**

Check S(n) is true for n = 1

So, LHS = $1 \cdot 1! = 1$,

RHS = $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

LHS = RHS

Hence, S(n) is true for n = 1

▶ **Step II : Induction step**

Assume S(n) is true for n = k

S(k) : $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + k \cdot k! = (k+1)! - 1$ ...(2)

Now, check S(n) is true for n = k + 1

S(k + 1) : $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + k \cdot k! + (k+1)(k+1)! = (k+2)! - 1$

Solve above statement by taking LHS

$$\Rightarrow \underbrace{1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + k \cdot k!} + (k+1)(k+1)!$$

$\Rightarrow (k+1)! - 1 + (k+1)(k+1)!$

$\Rightarrow (k+2)(k+1)! - 1$

$\Rightarrow (k+2)! - 1$

$\Rightarrow$ Required RHS

▶ **Step III : Conclusion**

S(n) : $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! = (n+1)! - 1$ is true by mathematical induction

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Ex. 1.14.14 : With the help of mathematical induction prove that $8^n - 3^n$ is multiple of 5, for $n \ge 1$.

☑ **Soln. :**

Let, S (n) : $8^n - 3^n$ is multiple of 5 for $n \ge 1$

▶ **Step I : Induction Basis**

Check S (n) is true for n = 1

S (1) : $8^1 - 3^1 = 8 - 3 = 5$. Here, 5 is multiple by 5

So, S (n) is true for n = 1

▶ **Step II : Induction step**

Assume, S (n) is true for n = k that is $8^k - 3^k$ is multiple by 5

Now, check S (k + 1) is true for n = k + 1

$S(k+1) : 8^{(k+1)} - 3^{(k+1)}$

$= 8^k \cdot 8^1 - 3^k \cdot 3^1 = 8^k (5+3) - 3^k \cdot 3$

$= 8^k \cdot 5 + (8^k \cdot 3 - 3^k \cdot 3)$

$= 8^k \cdot 5 + 3 (8^k - 3^k)$

$= 5 \cdot 8^k + 3 \cdot 5^k$

$= 5 \cdot 8^k + 5^k \cdot 3$

Here, $5 \cdot 8^k$ is multiple by 5 and $5^k \cdot 3$ is multiple by 5

Therefore, $8^{k+1} - 3^{k+1}$ is multiple of 5.

▶ **Step III**

Conclusion : S (n) : $8^n - 3^n$ is multiple of 5 for $n \geq 1$

Ex. 1.14.15 : Prove : $1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

☑ **Soln. :**

Let, $S(n) : 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ ...(1)

▶ **Step I : Basis of Induction**

Check S (n) is true for n = 1

So, LHS $= 1^3 = 1$

RHS $= \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = \frac{4}{4} = 1$

LHS = RHS, Hence, S (n) is true for n = 1

▶ **Step II : Induction step**

Assume, S (n) is true for n = k

$S(k) : 1^3 + 2^3 + 3^3 + \dots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$ ...(2)

Now, prove S (n) is true for n = k + 1

$S(k+1) : 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$

Solve LHS to get required RHS

$S(k+1) : \underbrace{1^3 + 2^3 + 3^3 + \dots + k^3} + (k+1)^3$

$\Rightarrow \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$ ∵ from Equation (2)

$\Rightarrow (k+1)^2 \left[\frac{k^2}{4} + (k+1)\right]$

$\Rightarrow (k+1)^2 \left[\frac{k^2+4k+4}{4}\right] \Rightarrow (k+1)^2 \left[\frac{(k+2)^2}{4}\right]$

$\Rightarrow \left[\frac{(k+1)(k+2)}{2}\right]^2 \Rightarrow$ Required RHS

▶ **Step III : Conclusion**

So given $S(n) : 1^3 + 2^3 + 3^3 \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ is true by mathematical induction.

Ex. 1.14.16 : Show that : $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for all natural number n.

☑ **Soln. :**

Let, $S(n) : 7^{2n} + (2)^{3n-3} \cdot (3)^{n-1}$ is divisible by 25 ...(1)

▶ **Step I : Basis of Induction**

Check S (n) is true for n = 1

$S(1) : 7^2 + 2^{3-3} \cdot 3^0 \Rightarrow 49 + 1 \cdot 1 \Rightarrow 50$ divisible by 25

So, S (n) is true for n = 1

▶ **Step II : Induction step**

Assume, S (n) is true for n = k

$S(k) : 7^{2k} + 2^{3k-3} \cdot 3^{k-1}$ is divisible by 25 ...(2)

Now, check S (n) is true for n = k + 1

$S(k+1) = 7^{2(k+1)} + 2^{3(k+1)-3} \cdot 3^{(k+1)-1}$

$\Rightarrow 7^{2k+2} + 2^{3k} \cdot 3^k$

$\Rightarrow 7^2 (7^{2k}) + 2^{3k} \cdot 3^k$

$\Rightarrow 7^2 (7^{2k} + 2^{3k-3} \cdot 3^{k-1} - 2^{3k-3} \cdot 3^{k-1}) + 2^{3k} \cdot 3^k$

$\Rightarrow 49 \underbrace{(7^{2k} + 2^{3k-3} \cdot 3^{k-1})} - 49 (2^{3k-3} \cdot 3^{k-1}) + 2^{3k} \cdot 3^k$

$\Rightarrow 49 \cdot 25C - 49 (2^{3k-3} \cdot 3^{k-1}) + 2^{3k} \cdot 3^k$

∵ From Equation (2)

$\Rightarrow 49 \cdot 25C - 49 (2^{3k-3} \cdot 3^{k-1}) + 3^1 \cdot 8^1 \, 2^{3k-3} \cdot 3^{k-1}$

$\Rightarrow 49 \cdot 25C - 49 (2^{3k-3} \cdot 3^{k-1}) + 24 \cdot 2^{3k-3} \cdot 3^{k-1}$

$\Rightarrow 49 \cdot 25C - 25 (2^{3k-3} \cdot 3^{k-1})$

Here, $49 \cdot 25C \leftarrow$ divisible by 25

$25 (2^{3k-3} \cdot 3^{k-1}) \leftarrow$ divisible by 25

▶ **Step III :**

Conclusion : $S(n) \cdot 7^{2n} + 2^{3n-3} \cdot (3)^{n-1}$ is divisible by 25 by mathematical induction.

Ex. 1.14.17 : By using mathematical induction show that : $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ for all natural number values of n.

☑ **Soln. :**

Let $S(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ ...(1)

▶ **Step I : Basis of induction**

Check S (n) is true for n = 1

LHS = 1, RHS $= \frac{1(1+1)}{2} = \frac{2}{2} = 1$

∴ LHS = RHS

Hence, S(n) is true for n = 1

▶ **Step II : Induction step**

Assume S(n) is true for n = k

Thus we get

$S(k) : 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$ ...(2)

Now, check S(n) is true for n = k + 1 from Equation (1),

$S(k+1) : 1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

Now, solve the LHS of above statement

$S(k+1) : \underbrace{1 + 2 + 3 + \dots + k} + (k+1)$

$= \frac{k(k+1)}{2} + k + 1$ ∵ From Equation (2)

$= \frac{k(k+1) + 2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$ = RHS of S (k + 1)

▶ **Step III :**

Conclusion : $S(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

is true using mathematical induction.

Ex. 1.14.18 : Prove by mathematical induction :

$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ for $n \geq 1$

☑ **Soln. :**

Let,

$S(n) : \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ ...(1)

▶ **Step I : Basis of Induction**

Check S (n) is true for n = 1

$S(1) : \text{LHS} = \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{3}$

RHS $= \frac{1}{2(1)+1} = \frac{1}{3}$ LHS = RHS

Hence S (n) is true for n = 1

▶ **Step II : Induction step**

Assume S(n) is true for n = k

$S(k) : \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ ...(2)

Now, have to prove S (n) is true for n = k + 1.

$S(k+1) : \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]} = \frac{k+1}{[2(k+1)+1]}$

Solve LHS of above statement

$\Rightarrow \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]}$

$\Rightarrow \frac{k}{2k+1} + \frac{1}{[2(k+1)-1][2(k+1)+1]}$

∵ from Equation (2)

$\Rightarrow \frac{1}{2k+1} \left[k + \frac{1}{2k+3}\right]$

As ∵ $2(k+1) - 1 = (2k+1)$

∵ $2(k+1) + 1 = 2k+3$

$\Rightarrow \frac{1}{2k+1} \left[\frac{k(2k+3)+1}{2k+3}\right] \Rightarrow \frac{1}{2k+1} \left[\frac{2k^2+k+1}{2k+3}\right]$

$\Rightarrow \frac{1}{2k+1} \left[\frac{k(2k+1)+1}{2k+3}\right]$

$\Rightarrow \frac{1}{\cancel{2k+1}} \left[\frac{\cancel{(2k+1)}(k+1)}{2k+3}\right] \Rightarrow \left[\frac{k+1}{2k+3}\right]$

$\Rightarrow \frac{k+1}{2(k+1)+1} \Rightarrow$ Required RHS

▶ **Step III :**

Conclusion : Hence S(n) is true by mathematical induction.

Ex. 1.14.19 : Prove by mathematical induction, for $n \geq 0$

$S(n) : 1 + a + a^2 + \dots + a^n = \frac{1-a^{n+1}}{1-a}$

☑ **Soln. :**

Given : $S(n) : 1 + a + a^2 + \dots + a^n = \frac{1-a^{n+1}}{1-a}$ ...(1)

▶ **Step I : Basis of induction**

Check, S(n) is true for n = 0 or 1

For n = 0, L.H.S. $= a^0 = 1$

R.H.S. $= \frac{1-a}{1-a} = 1$ } L.H.S. = R.H.S.

For n = 1, L.H.S. $= 1 + a$

R.H.S. $= \frac{1-a^2}{1-a} = \frac{\cancel{(1-a)}(1+a)}{\cancel{(1-a)}}$

$= 1 + a$ ∵ L.H.S = R.H.S

Here, S(n) is true for n = 0, 1 that is S(0), S(1) are true.

▶ **Step II : Induction step**

Assume, S(n) is true for n = k (some value of n)

So,

$S(k) : 1 + a + a^2 + \dots + a^k = \frac{1-a^{k+1}}{1-a}$ ...(2)

check, S(n) is true for n = k + 1,

$S(k+1) : 1 + a + a^2 + \dots + a^k + a^{k+1} = \frac{1-a^{k+1+1}}{1-a}$

$S(k+1) : 1 + a + a^2 + \dots + a^k + a^{k+1} = \frac{1-a^{k+2}}{1-a}$

Solve above equation by taking L.H.S

$S(k+1) : 1 + a + a^2 + \ldots + a^k + a^{k+1}$

$\Rightarrow \dfrac{1-a^{k+1}}{1-a} + a^{k+1}$ ...from Equation (2)

$\Rightarrow \dfrac{1-a^{k+1} + (1-a)\cdot a^{k+1}}{1-a}$

$\Rightarrow \dfrac{(1-a^{k+1}) + a^{k+1} - a^1 \cdot a^{k+1}}{1-a}$

$\Rightarrow \dfrac{1-\cancel{a^{k+1}} + \cancel{a^{k+1}} - a^{k+1+1}}{1-a}$

$\Rightarrow \dfrac{1-a^{k+2}}{1-a} = \text{R.H.S}$

▶ **Step III : Conclusion**

This show that, S(n) is true for n = k then it is also true for n = k + 1.

Thus, by using mathematical induction, S(n) is true for every integral value of n.

Ex. 1.14.20 : Prove that for any positive integer n, the number $n^5 - n$ is divisible by 5.

☑ **Soln. :**

$S(n) : n^5 - n$ is divisible by 5

▶ **Step I : Basis of induction**

Check, S(n) is true n = 1

So, put n = 1 in S(n)

$S(1) : 0^5 - 0 = 0$ is divisible by 5

because 0 is divisible by every number

So, S(n) is true for n = 1

▶ **Step II : Induction step**

Assume S(n) is true for n = k

$S(k) : k^5 - k$ is divisible by 5

Now, check S(n) is true for n = k + 1

$S(k+1) : (k+1)^5 - (k+1)$

$\Rightarrow (k^5 + 5_{c_1} k^4 + 5_{c_2} k^3 + 5_{c_3} k^2 + 5_{c_4} k + 5_{c_5}) - (k+1)$

$\Rightarrow k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$

$\because$ C = combination

$$n_{c_r} = \frac{n!}{(n-r)! \times r!}$$

$\Rightarrow (k^5 - k) + 5\,[k^4 + 2k^3 + 2k^2 + k]$

Here, $(k^5 - k)$ is divisible by 5. (Assumed when n = k)

$5\,[k^4 + 2k^3 + 2k^2 + k]$ is also divisible by 5.

Hence, $(k+1)^5 - (k+1)$ is divisible by 5.

▶ **Step III : Conclusion**

This show that S(n) is true for n = k then it is also true for n = k + 1.

Thus, by using mathematical induction S(n) is true for every integral value of n.

Ex. 1.14.21 : Prove that $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \ldots + \dfrac{n}{2^n} = 2 - \dfrac{n+2}{2^n}$, for $n \geq 1$ by using mathematical induction

☑ **Soln. :**

$S(n) : \dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \ldots + \dfrac{n}{2^n} = 2 - \dfrac{n+2}{2^n}$ ...(1)

▶ **Step I : Basis of induction**

Check S(n) is true for n = 1

Put, n = 1 in Equation (1)

$S(1) : \dfrac{1}{2} = \text{L.H.S,}$

$\text{R.H.S} = 2 - \dfrac{1+2}{2^1} = 2 - \dfrac{3}{2} = \dfrac{4-3}{2} = \dfrac{1}{2}$

Here, L.H.S = R.H.S

Hence S(1) is true.

▶ **Step II : Induction step**

Assume, S(n) is true for n = k

$S(k) : \dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \ldots + \dfrac{k}{2^k} = 2 - \dfrac{k+2}{2^k}$ ...(2)

Now check S(n) is true for n = k + 1

$S(k+1) : \dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \ldots + \dfrac{k}{2^k} + \dfrac{k+1}{2^{k+1}} = 2 - \dfrac{k+3}{2^{k+1}}$

Solve above Equation, by taking L.H.S.

$S(k+1) : \underbrace{\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \ldots + \dfrac{k}{2^k}} + \dfrac{k+1}{2^{k+1}}$

$S(k+1) : 2 - \dfrac{k+2}{2^k} + \dfrac{k+1}{2^{k+1}}$

$\Rightarrow 2 - \left[\dfrac{k+2}{2^k} - \dfrac{k+1}{2^{k+1}}\right]$

$\Rightarrow 2 - \left[\dfrac{2^{k+1}(k+2) - 2^k(k+1)}{2^k \cdot 2^{k+1}}\right]$

$\Rightarrow 2 - \left[\dfrac{2^k \cdot 2^1 \cdot k + 2^k \cdot 2^1 \cdot 2^1 - 2^k \cdot k^1 - 2^k}{2^k \cdot 2^{k+1}}\right]$

$\Rightarrow 2 - \left[\dfrac{\cancel{2^k}(2k+4-k-1)}{\cancel{2^k} \cdot 2^{k+1}}\right]$

$\Rightarrow 2 - \left(\dfrac{k+3}{2^{k+1}}\right) \Rightarrow \text{R.H.S.}$

▶ **Step III :**

Conclusion

This, show that S(n) is true for n = k then it is also true for n = k + 1.

Thus, by using mathematical induction, S(n) is true for every integral value of n.

Ex. 1.14.22 : Show that, $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

☑ **Soln. :**

$S(n) : 1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$ ...(1)

▶ **Step I : Basis of induction**

Check, S(n) is true for n = 1

Therefore, put n = 1 in S(n)

$\text{L.H.S} \Rightarrow 1^3 = 1$

$\text{R.H.S} \Rightarrow \dfrac{1^2(1+1)^2}{4} = \dfrac{1 \times 4}{4} = 4$ here, L.H.S = R.H.S

Hence, S(n) is true for n = 1

▶ **Step II : Induction step**

Assume, S(n) is true for n = k

So, $S(k) : 1^3 + 2^3 + 3^3 + \ldots + k^3 = \dfrac{k^2(k+1)^2}{4}$ ...(2)

Now, check S(n) is true for n = k + 1.

$S(k+1) : 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$

$= \dfrac{(k+1)^2(k+1+1)^2}{4}$

$S(k+1) : 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$

$= \dfrac{(k+1)^2(k+2)^2}{4}$

Solve L.H.S of S(k + 1)

$S(k+1) : 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$

$\Rightarrow \dfrac{k^2(k+1)^2}{4} + (k+1)^3 \Rightarrow (k+1)^2 \cdot \left[\dfrac{k^2}{4} + (k+1)\right]$

$\Rightarrow (k+1)^2\left[\dfrac{k^2+4k+4}{4}\right] \Rightarrow \dfrac{(k+1)^2(k+1)^2}{4} \Rightarrow \text{R.H.S}$

▶ **Step III : Conclusion**

This show that S(n) is tru for n = k then it is also true for n = k + 1. Thus, by using mathematical induction S(n) is true for every integral value of n.

Ex. 1.14.23 : Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be covered by L-shape pieces, where each piece covers three squares at a time.

☑ **Soln. :**

Let, S(n) be the proposition that any $2^n \times 2^n$ chess board with one square removed can be covered by L-shape pieces.

▶ **Step I : Basis of induction**

Check, S(n) is true for n = 1

S(1) : Implies that any 2 × 2 chessboard with one square removed can be covered using L-shaped pieces. P(1) is true, as shown in Fig. P. 1.14.23.

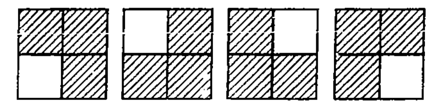


Fig. P. 1.14.23

▶ **Step II : Induction step**

Assume S(n) is true for n = k that is any $2^k \times 2^k$ chessboard with one square removed can be covered using L-shaped pieces.

Now, we have to check that S(n) is true for n = k + 1 that is S(k + 1) is true. For this consider a $2^{k+1} \times 2^{k+1}$ chessboard with one square removed. Divide the chessboard into four equal halves of size $2^k \times 2^k$ as shown in Fig. P. 1.14.23(a).

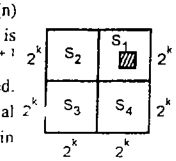


Fig. P. 1.14.23(a)

The square which has been removed, would have been removed from one of the four chessboard say S_1. Then by induction hypothesis, S_1 can be covered using L-shaped pieces. Now, from each of the remaining chessboards, remove that particular piece or tile, lying at the centre of the large chessboards.

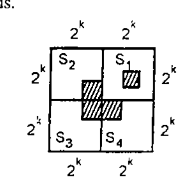


Fig. P. 1.14.23(b)

Then, by induction hypothesis each of these $2^k \times 2^k$ chessboards with a piece or tile removed can be covered by the L-shaped pieces. Also the three tiles removed from the centre can be covered by one L-shaped piece. Hence, the chessboards of $2^{k+1} \times 2^{k+1}$ can be covered by L-shaped pieces.

...Hence Proved.

Ex. 1.14.24 : Using mathematical induction show that

$$S(n) : \frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

☑ **Soln. :**

Given :

$$S(n) : \frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \quad \dots(1)$$

▶ **Step I : Basis of induction**

Check S(n) is true for n = 1

$$\text{L.H.S} \Rightarrow \frac{1}{(2-1)(2+1)} = \frac{1}{(1)(3)} = \frac{1}{3}$$

$$\text{R.H.S} \Rightarrow \frac{1}{2+1} = \frac{1}{3} \qquad \text{Here, L.H.S = R.H.S}$$

S(n) is true for n = 1

▶ **Step II : Induction step**

Assume S(n) is true for n = k

$$S(k) : \frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \quad \dots(2)$$

Now, check S(n) is true for n = k + 1

$$S(k+1) : \frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]} = \frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$$

Solve L.H.S of above equation

$$S(k+1) : \underbrace{\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \dots + \frac{1}{(2k-1)(2k+1)}} + \frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$\Rightarrow \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$\Rightarrow \frac{k(2k+1)(2k+3) + (2k+1)}{(2k+1)(2k+1)(2k+3)}$$

$$\Rightarrow \frac{\cancel{(2k+1)}[k\cdot(2k+3)+1]}{\cancel{(2k+1)}(2k+1)(2k+3)}$$

$$\Rightarrow \frac{2k^2+3k+1}{(2k+1)(2k+3)} \Rightarrow \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$

$$\Rightarrow \frac{k+1}{2k+3} \Rightarrow \text{R.H.S}$$

▶ **Step III : Conclusion**

This show that S(n) is true for n = k then it is also true for n = k + 1.

Thus, by using mathematical induction S(n) is true for every integral value of n.

Ex. 1.14.25 : Show that for any positive integers n,

$(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

☑ **Soln. :**

S(n) : $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

▶ **Step I : Basis of induction**

For n = 0, $(11)^2 + (12)^1 = 121 + 12 = 133$

Which is divisible by 133

For n = 1, $(11)^3 + (12)^3 = 1331 + 1728$

$= \mathbf{3059}$

Which is divisible by 133 as $\frac{3059}{133} = 23$

So, S(n) is true for n = 0 and 1

▶ **Step II : Induction step**

Assume, S(n) is true for n = k

S(k) : $(11)^{k+2} + (12)^{2k+1}$ is divisible by 133

now, check S(n) is true for n = k + 1

S(k + 1) : $(11)^{(k+1)+2} + (12)^{2(k+1)+1}$

$\Rightarrow (11^{k+2})\cdot 11^1 + 12^{2k+1}\cdot 12^2$

$\Rightarrow (11)\cdot(11^{k+2}) + 144.12^{2k+1}$

$\Rightarrow 11.11^{k+2} + (133-11)\cdot 12^{2k+1}$

$\Rightarrow 11.11^{k+2} + 11.12^{2k+1} + 133.12^{2k+1}$

$\Rightarrow 11(11^{k+2} + 12^{2k+1}) + 133.12^{2k+1}$

Here, $11^{k+2} + 12^{2k+1} \rightarrow$ divisible by 133

(Assumed in n = k)

$133.12^{2k+1} \rightarrow$ divisible by 133

▶ **Step III : Conclusion**

S(n) : $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

Ex. 1.14.26 : Use mathematical induction to show that : $n\cdot(n^2-1)$ is divisible by 24 where n is any odd positive number.

☑ **Soln. :**

Given : $n\cdot(n^2-1)$ is divisible by 24 where n is any odd positive number.

$\Rightarrow (n^3 - n)$ is divisible by 24

$\Rightarrow S(n) : (n^3 - n)$

▶ **Step I : Basis of induction**

Check s(n) is true for n = 1

S(1) : $1^3 - 1 = 0$, Here 0 is divisible by 24.

S(3) : $3^3 - 3^3 = 24$, 24 is divisible by 24.

▶ **Step II : Induction step**

Assume, S(n) is true for n = k.

S(k) : $k^3 - k$ is divisible by 24 ...(1)

Now check, S(n) is true for n = k + 1

S(k + 1) : $(k+1)^3 - (k+1)$

$\Rightarrow k^3 + 3k^2 + 3k + 1 - k - 1$

$\Rightarrow k^3 + 3k^2 + 2k$

$\Rightarrow (k^3 + k) + 3k^2 + 3k + 2k + k$

$\Rightarrow (k^3 - k) + 3k^2 + 3k$

$\underbrace{24\cdot C_1} + \underbrace{3k(k+1)}_{k(k+1) \text{ is multiple of 8, for } k \geq 3.}$

From Equation (1)

$= 24\cdot C_1 + 3\cdot(8\cdot C_2)$

$= 24\cdot C_1 + 24\cdot C_2$

$= 24\cdot(C_1 + C_2)$

$= 24\cdot C_3$

So, S(k + 1) is true because $24\cdot C_3$ is divisible by 24.

Hence,

▶ **Step III :**

Conclusion

S(n) : $n(n^2-1) = n^3 - n$ is divisible by 24 where n is any odd positive integer.

Ex. 1.14.27 : Show that for natural no. n :

$1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$

☑ **Soln. :**

▶ **Step I : Basis of Induction**

Let, S (n): $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$ ...(1)

Check S (n) is true for small value of n i.e. n = 1

LHS = $1^3 = 1$, RHS = $1^2 = 1$

Here, LHS = RHS So, S (n) is true for n = 1

▶ **Step II : Induction step**

Assume, S (n) is true for n = k

So, S(K) : $1^3 + 2^3 + 3^3 + \dots + k^3 = (1 + 2 + \dots + k)^2$ ...(2)

Now, we have to show that S (n) is true for n = k + 1

So, put n = k + 1 in S(n) (3)

then

S (K + 1): $1^3 + 2^3 + 3^3 + \dots + K^3 + (K+1)^3$

$= (1 + 2 + \dots + (K+1))^2$

Now, lets start from LHS of S (K + 1)

$S(K+1) = \underbrace{1^3 + 2^3 + 3^3 + \dots + K^3} + (K+1)^3$

$= (1 + 2 + \dots + K)^2 + (K+1)^3 \rightarrow$ by Equation (2)

$= \frac{K^2(K+1)^2}{4} + (K+1)^3$

$\because (1 + 2 + \dots + n^2) = \frac{n^2(n+1)^2}{4}$

$= (K+1)^2\left[\frac{K^2 + 4(K+1)}{4}\right]$

$= (K+1)^2\left[\frac{K^2 + 4K + 4}{4}\right] = \frac{(K+1)^2(K+2)^2}{4}$

$= (1 + 2 + 3 + \dots + (K+1))^2$

= RHS, S (n) is true for n = K + 1

▶ **Step III :**

Conclusion

S (n) : $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$ is true by mathematical induction.

Important Formulas

1. The number of possible subsets in power set is
   $n(P(A)) = 2^n$
   Where, n = number of elements of A
2. If $A \subseteq B$ and $B \subseteq A$ then, $A = B$
3. If $A \subseteq B$ and $B \subseteq C$ then, $A \subseteq C$
4. $A \subseteq A$, (every set is a subset of itself)
5. $\phi \subseteq A$ (null set is a subset of any set)
6. If $A \not\subseteq B$ and $B \not\subseteq A$, then, A and B are disjoint set
7. Related to union operation
   $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
   $A \cup \phi = A$

- $A \cup U = U$
- $A \cup A^c = U$
- $A \subseteq A \cup B$ and $B \subseteq A \cup B$

8. **Related to intersection**

- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- $A \cap B \subseteq A$ and $A \cap B \subseteq B$
- $A \cap B = \phi$ then A and B are disjoint sets
- $A \cap \phi = \phi$
- $A \cap U = A$
- $A \cap A^c = \phi$

9. **Related to complement of set**

- $U^c = \phi$ and $\phi^c = U$ $\boxed{A^c = U - A}$
- $(A^c)^c = A$
- $A^c = \{x \mid x \in U, x \notin A\}$
- $(A \cup B)^c = A^c \cap B^c$

10. **Related to difference of sets**

- $A - A = \phi$, $\phi - A = \phi$
- $A - B = A \cap B^c$

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$

11. **Related to symmetric difference**

- $A \oplus B = (A - B) \cup (B - A)$
- $A \oplus B = (A \cup B) - (A \cap B)$

$A \oplus B = \{x \mid x \in A - B \text{ or } x \in B - A\}$

12. **Cardinality of set**

- The number of elements in a particular set
- If $A = \phi$ then $n(A) = 0$
- Addition principle

$n(A \cup B) = n(A) + n(B)$

let A_1, A_2, A_3, A_n are finite disjoint sets then,

$$n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_n)$$

$n(A - B) = n(A) - n(A \cap B)$

13. **Inclusion – Exclusion principle**

For 2 sets : $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

For 3 sets

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - (B \cap C) + n(A \cap B \cap C)$$

For 4 sets

$N(A \cup B \cup C \cup D)$

$$= n(A) + n(B) + n(C) + n(D) - n(A \cap B) - n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D) - n(C \cap D) + n(A \cap B \cap C) + n(A \cap B \cap D) + n(A \cap C \cap D) + n(B \cap C \cap D) - n(A \cap B \cap C \cap D)$$

14. **Multisets**

Multiplicity of an Element

The number of time, element appears in multiset

Equality of Multisets

The number of occurrence of each element from the given two sets is same.

Union of Msets : $\mu(x) = \max\{\mu_A(x), \mu_B(x)\}$

Intersection of Msets : $\mu(x) = \min\{\mu_A(x), \mu_B(x)\}$

Difference of Msets : $\mu_A(x) - \mu_B(x) \geq 1$

Sum of Msets : $\mu(x) = \mu_A(x) + \mu_B(x)$

Chapter Ends ...

□□□

Note

UNIT I

CHAPTER 2

Logic and Proofs

Syllabus

Propositions : Propositions, Logical Connectives, Conditional and Bi-conditional Propositions, Logical Equivalence, Validity of Arguments by using Truth Tables, Predicates and Quantifiers, Normal forms. Applications of Sets and Propositions.

2.1 INTRODUCTION

- Logic is a fundamental concept for mathematical reasoning.
- Logic is very important to understand mathematical statements.
- The rules of logic helps us to understand con, iex mathematical statements about reasoning.
- In computer science field, it has many practical applications including designing of computing machines, artificial intelligence, computer programming, designing of algorithms and many more.
- Proofs are the base of understanding correct mathematical arguments.
- Proofs are used to prove a mathematical statement is true or not. Once it is proved true, then we call it as theorem.
- It play important rule in computer science field. In fact, to verify that computer programs produce the correct output for all possible input values, to check algorithms always produce the correct result, to implement the security of a system.
- In this chapter, we will learn about what makes up correct mathematical argument and will introduce techniques to constructs these arguments.

2.2 PROPOSITION

Definition : Proposition or statement is a declarative sentence which is either true or false but not both at the same time.

- To understand the definition, lets take some examples,
  (i) Mumbai is a capital of Maharashtra.
  (ii) Ice floats in water.
  (iii) $2 + 3 = 6$
  (iv) It will rain tomorrow.
  (v) $3 + 2 = 5$
- From the above examples, it can be understand that, example number (i), (ii) and (v) are propositions which holds TRUE VALUES. Other side, example number (iii) also a proposition which holds FALSE truth value.
- In example (iv), it's truth value can not be predicted at this moment, it will be definitely determined tomorrow or in future. So it is a statement or propositic
- E.g. (i), (ii), (iii), (iv), (v) are propositions.
- Lets take another set of examples
  (i) What is your name ?
  (ii) Where are you going ?
  (iii) $x + 3 = 8$.
- In example (iii), It's truth value depends upon the value of x. If $x = 5$, the sentence is true, if $x \neq 5$, then the sentence is false. So it is not proposition.
- While example, (i) and (ii) are not statement or proposition since it was question sentence.

2.2.1 Notations used for Proposition

- The proposition usually denoted by capital letters or small letters e.g. A, B, C, ... or a, b, c, ...
- The truth value of particular true statement is denoted by 'T' or '1'. While false statement is denoted by 'F' or '0'.
- **Examples**
  (i) P : $3 + 2 = 5$, which holds true value.
  (ii) q : $6 + 2 = 9$, which holds false value.
  (iii) x : Delhi is capital of India, which holds true value.

2.2.2 Examples Based on Proposition

Ex. 2.2.1 : Use : p : I will study discrete structure

q : I will go to a movie

r : I am in a good mood.

Write the English sentence that corresponds to each of the following :

(i) $\sim r \rightarrow q$ (ii) $\sim q \wedge p$ (iii) $q \rightarrow \sim p$ (iv) $\sim p \rightarrow \sim r$

☑ **Soln. :**

Let, p : I will study discrete structure

q : I will go to a movie

r : I am in a good mood.

(i) $\sim r \rightarrow q$

If I am not in good mood then I will go to a movie.

(ii) $\sim q \wedge p$

I will not go to movie but I will study discrete sturcture.

(iii) $q \rightarrow \sim p$

If I will go to a movie then I will not study discrete structure.

(iv) $\sim p \rightarrow \sim r$

If I will not study discrete structure then I am not in good mood.

Ex. 2.2.2 : Translate into symbolic form

(i) Atul and Ram going to movie.

(ii) He run fast while he went to ground.

(iii) Anil is rich but unhappy.

☑ **Soln. :**

(i) p : Atul going to movie.

(ii) q : Ram going to movie.

Here, Atul and Ram, both are going to movie. So, it can be written in symbolic form as :

$$p \wedge q$$

(ii) p : He run fast.

q : He went to ground.

Here, p and q are happen at the some time. So, it can be written as :

$$p \wedge q$$

(iii) p : Anil is rich

q : Anil is unhappy

It can be written as, $p \wedge q$

Ex. 2.2.3 : Using the following statements

p : Ragini is hard worker.

q : Ragini is intelligent.

Write the following statements in symbolic form.

(i) Ragini is hardworker and intelligent.

(ii) Ragini is hardworker but not intelligent.

(iii) It is false that Ragini is not hardworker or intelligent.

(iv) Ragini is hardworker or Ragini is not hard worker and intelligent.

☑ **Soln. :**

(i) $p \wedge q$

(ii) $p \wedge \sim q$

(iii) $\sim(\sim p \vee q)$

(iv) $p \vee (\sim p \wedge q)$

Ex. 2.2.4 : Using the following statements

p : The food is good.

q : The service is good.

r : The rating is three - star.

Write the following statements in symbolic form.

(i) Either the food is good, or the service is good, or both.

(ii) Either the food is good or the service is good, but not both.

(iii) The food is good while the service is poor.

(iv) It is not the case that both the food is good and rating is three star.

(v) If both the food and services are good, then the rating will be three-star.

(vi) It is not true that a three star rating always means good food and good service.

☑ **Soln. :**

(i) $p \vee q$ (ii) $(p \vee q) \wedge \sim(p \wedge q)$

(iii) $p \wedge \sim q$ (iv) $\sim(p \wedge r)$

(v) $(p \wedge q) \rightarrow r$ (vi) $\sim(r \rightarrow (p \wedge q))$

Ex. 2.2.5 : Let, p : The material is interesting.

q : The exercise are challenging.

r : The course is enjoyable.

Write the following statement in symbolic form.

(a) The material is interesting and the exercise are challenging.

(b) The material is uninteresting, the exercise are not challenging, and the course is not enjoyable.

(c) If the material is not interesting and the exercises are not challenging, then the course is not enjoyable.

(d) The material is interesting means the exercises are challenging, and conversely.

(e) Either the material is interesting or the exercises are not challenging, but not both.

☑ **Soln. :**

(a) $p \wedge q$ (b) $(\sim p) \wedge (\sim q) \wedge (\sim r)$

(c) $(\sim p \wedge \sim q) \rightarrow \sim r$ (d) $p \leftrightarrow q$

(e) $(p \vee \sim q) \wedge (\sim p \vee q)$

Ex. 2.2.6 : Let p : The weather is nice.

q : we have a picnic.

Translate the following in english and simplify if possible.

(i) $p \wedge \sim q$ (ii) $p \leftrightarrow q$

(iii) $\sim q \rightarrow \sim p$ (iv) $\sim(\sim p \vee q) \cdot \sim(p \wedge \sim q)$

☑ **Soln. :**

(i) $p \wedge \sim q$: The weather is nice but we do not have a picnic.

(ii) $p \leftrightarrow q$: If the weather is nice then we have a picnic and conversely.

(iii) $\sim q \rightarrow \sim p$: If we do not have a picnic then the weather is not nice.

(iv) $[\sim(\sim p \vee q)] \cdot [\sim(p \wedge \sim q)]$: The weather is nice but we do not have a picnic.

Ex. 2.2.7 : (a) Write a compound statement that is true whenexactly two of the three statements p, q, r are true.

(b) Write a compound statement that is true when none or one, or two of the three statements p, q, r are true.

☑ **Soln. :**

(a) $(\sim p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r)$

(b) $\sim (p \wedge q \wedge r)$

2.3 LAWS OF FORMAL LOGIC

Traditionally, there are three laws of logic.

(a) Law of contradiction

- It states that a statement can not be both true and false at a some time.
- For instance, let P is a statement, which can not be true and false at the same moment.

(b) Law of excluded middle

- Let P is a statement which is either true or false, there is no middle or third ground between true or false.
- Symbolically it is written as either P or ~ P.
- **Example :** If students are appear for examination then they either pass or fail in that exam. There is no middle true.

(c) Law of identity

- It state that, a thing identical with itself.
- That is. for all x, x is x, symbolically, $(\forall x)(x = x)$.

2.4 TYPES OF STATEMENTS / PROPOSITIONS

(a) Primitive statements

- A proposition is said to be primitive if it can not be broken down or split into simpler proposition or statement.
- It is also called as primary or atomic statements.

Examples

(i) 2 + 2 = 4 (ii) Ice floats in water

(iii) India is in Asia.

- The above statements are primitive as it can not split into simple proposition.

(b) Compound propositions statement

A proposition is said to be compound, it is formed from primary statements by using logical connectives.

Examples

(i) Roses are red and violets are blue.

(ii) Atul is smart or he studies ever night.

- From the above statements, it is clear that they are formed from primary proposition using logical connectives (like, and (∧) or (∨)).
- For instance,
  P : Roses are red.
  q : Violets are blue are combined using logical connective called and (∧).

2.5 LOGICAL CONNECTIVES

- In our day to day life, in English language, we use the words like "not", "and", "or", "but", "If-then", "while" to connect two or more statements.
- The usage of these connectives are very flexible which leads to inexact and ambiguous interpretations.
- Hence, we take some special connectives with precise meaning to suit our purpose by redefining and symbolizing them.
- Here are some special connectives with their name and notations.

Sr. No.	Name of connectives	Notation
(i)	Negation	~ or ¬
(ii)	AND	∧
(iii)	OR	∨
(iv)	If ...then	→
(v)	If and only if (iff)	↔

- We will discuss each of them in details.

2.5.1 Negation

- Let 'P' be any proposition, then the negation of 'P' can be formed by writing "It is not true that..." or "It is false that ...." before P or inserting the word "not" at proper place.
- If 'P' is true, then negation of P obtained by writing "P is false".
- The negation of P is denoted as ¬ P or ~ P read as "not P".
- The truth value of ¬ P depends upon the truth value of P as follows :

Truth table

P	~P
T	F
F	T

Example

(i) P : Delhi is capital of India.
~ P : Delhi is not capital of India.
OR
~ P : It is false that Delhi is capital of India.

(ii) P : Atul is hardworker boy.
~ P : Atul is not hardworker boy.
OR
~ P : It is false that Atul is hardworker boy.
OR
~ P : It is not the case that Atul is hardworker boy.

2.5.2 Conjunction

- Let p and q are two primary statements, the compound statement "p and q" is called as the conjunction of p and q.
- It is denoted as $p \wedge q$.
- It is read as "p and q" or "p meet q".
- The statement "$p \wedge q$" has the truth value T whenever both p and q have truth value 'T', otherwise '$p \wedge q$' has the false value 'F'.

Truth table

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Examples

(i) p : Milind is an Intelligent boy.
q : 5 is a prime number.
The conjuction of p and q written as $p \wedge q$.
$p \wedge q$: Milind is an intelligent boy and 5 is a prime number.

(ii) p : Pune is in Maharashtra.
q : DM is a second year subject.
$p \wedge q$: Pune is in Maharashtra and DM is a second year subject.

2.5.3 Disjunction

- Let p and q be the statements, then the compound statement "p or q" is formed using connective "or" (∨) is called as disjunction.
- It is denoted by "$p \vee q$".
- Disjunction of two statement ($p \vee q$) is read as "p or q" or "p join q".
- If p and q both are false, then $p \vee q$ is false. Otherwise $p \vee q$ is true.

Truth table

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Examples

(i) p : The clock is slow.
q : The time is correct.
$p \vee q$: The clock is slow or the time is correct.

(ii) p : I will read book.
q : I will go to movie.
$p \vee q$: I will read book or go to movie.

2.5.4 Conditional Statement (If ... Then)

- Let, p and q be the two primary statements, then the compound statement "If p then q" is called conditional statement or implication.
- Denoted as "$p \to q$". Here p is called the hypothesis or antecedent and q is called the consequent or conclusion.
- The conditional statement $p \to q$ can be read in many ways as

$$\left.\begin{array}{l} \text{* If p then q} \\ \text{* p only if q} \\ \text{* q if p} \\ \text{* p is sufficient for q} \\ \text{* q is necessary for p} \\ \text{* q when p} \end{array}\right\} \begin{array}{l}\text{All are same.} \\ \text{Equivalent to } p \to q\end{array}$$

If truth value of p is true and q is false then $p \to q$ is false. Otherwise $p \to q$ is True.

Truth table

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Examples

(i) p : Atul is hardworker.

q : Atul will pass the examination.

$p \rightarrow q$: If Atual is hardworker then he will pass the examination.

(ii) p : Today is Monday.

$q : 3 + 3 = 6$

$p \rightarrow q$: If today is Monday, then $3 + 3 = 6$.

Note : $p \rightarrow q$ is logically equivalent to $\sim p \vee q$

It can be cleared from below truth tables.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

- From the truth tables, it showed that,
  $p \rightarrow q \equiv (\sim p \vee q)$
- It is also important to know, $(\sim p \vee q) \neq \sim (p \vee q)$

2.5.5 Biconditional Statement (If and Only If)

- Let, p and q are two primary statements, then the compound statement "p if and only if q" is called a bi-conditional statement. It is also called as bi-implications.
- It is denoted by $p \leftrightarrow q$ or $(p \rightarrow q) \wedge (q \rightarrow p)$
- Bi-conditional statement reads in many ways as,
  - p if and only if q.
  - p iff q
  - p is necessary and sufficient for q.
  - If p then q and conversely.
- If truth values of p and q are same, then $p \leftrightarrow q$ is true. Otherwise $p \leftrightarrow q$ is false.

Truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Examples

(i) p : You can watch a movie.

q : You buy a ticket.

$p \leftrightarrow q$: You can watch a movie if and only if you buy a ticket.

(ii) p : An integer is even.

q : Integer divisible by 2.

$p \leftrightarrow q$: An integer is even if and only if it is divisible by 2.

2.5.6 Special Propositions

- There are three types proposition developed ,based on given logical implication.
- Let, $p \rightarrow q$ that is, "if p, then q" is conditional statement.
- Here
  p : It is raining
  q : Grass is wet.
  $p \rightarrow q$: "If it is raining then the grass is wet"
  Here, p is hypothesis and q is conclusion.

(a) Converse

- The converse of $p \rightarrow q$ will be $q \rightarrow p$, to form the converse of the conditional statement, interchange the Hypothesis and the conclusion.
- $q \rightarrow p$: "If the grass is wet then it is raining."

Note : A proposition may be true but have a false converse.

(b) Contra-positive

- To form the contra-positive of the conditional statement, interchange the hypothesis and conclusion and negating both.
- Contra-positive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.
- **Example**
  $\sim q \rightarrow \sim p$: If the grass is not wet then it is not raining.

Note : The contrapositive of any true proposition is also true.

(c) Inverse

- The inverse obtained by negating both hypothesis and conclusion of a conditional statement.
- If $p \rightarrow q$ is conditional statement then inverse of it will be $\sim p \rightarrow \sim q$.
- **Example :**
  If $p \rightarrow q$: If it is raining then the grass is wet.
  Inverse is,
  $\sim p \rightarrow \sim q$: "If it is not raining then the grass is not wet."

Note : Proposition may be true but it's inverse may be false.

- In short,

Statement	If p, then q
Converse	If q, then p
Inverse	If not p, then not q
Contrapositive	If not q, then not p

Note :

1. **A conditional statement and it's contra positive are equivalent.**
2. **The converse and the inverse of a conditional statement are also equivalent.**

Ex. 2.5.1 : Write contrapositive, converse and inverse forms of the following statement if $3 < b$ and $1 + 1 = 2$, then $\sin \frac{\pi}{3} = \frac{1}{2}$.

☑ **Soln. :**

Let, $p : 3 < b$

$q : 1 + 1 = 2$

$r : \sin \frac{\pi}{3} = \frac{1}{2}$

Given, if $3 < b$ and $1 + 1 = 2$, then $\sin \frac{\pi}{3} = \frac{1}{2}$

In symbolic form, $(p \wedge q) \rightarrow r$

(i) Contrapositive

$(\sim r \rightarrow \sim (p \wedge q)) \Rightarrow \sim r \rightarrow (\sim p \vee \sim q)$

That is, if $\sin \frac{\pi}{3} \neq \frac{1}{2}$ then $3 \geq b$ or $1 + 1 \neq 2$

(ii) Converse

$r \rightarrow (p \wedge q)$

That is, if $\sin \frac{\pi}{3} = \frac{1}{2}$ then $3 < b$ and $1 + 1 = 2$

(iii) Inverse

$\sim (p \wedge q) \rightarrow \sim r \Rightarrow \sim p \vee \sim q \rightarrow \sim r$

That is, if $3 \geq b$ or $1 + 1 \neq 2$ then $\sin \frac{\pi}{3} \neq \frac{1}{2}$

Ex. 2.5.2 : The converse of statements is given. Write inverse and contrapositive statements

(i) If he is considerate of others, then a man is a gentleman.

(ii) If a steel rod is stretcher, then it has been heated.

☑ **Soln. :**

Let, the converse of statements is

(i) If he considerate of others, then a man is a gentleman

Here, a : A man is gentleman

b : He is considerate of others that is $b \rightarrow a$

Inverse : $\sim b \rightarrow \sim a$ if he is not considerate of others then a men not a gentleman.

Contrapositive

$\sim a \rightarrow \sim b$

If a man is not gentleman then he is not considerate of others.

(ii) If steel rod is stretcher, then it has been heated

Here, a : Steel rod is stretchable

b : It has been heated

Converse : $b \rightarrow a$

Inverse : $\sim a \rightarrow \sim b$

If the steel rod is not stretchable then it has not been heated

Contrapositive : $\sim b \rightarrow \sim a$

If the steel rod has not been heated then it is not stretchable

Ex. 2.5.3 : Write converse, inverse and contrapositive of the following :

(i) If today is easter, then tomorrow is monday.

(ii) If a triangle is not isosceles, then it is not equilateral.

(iii) If n is prime, then n is odd or n is 2.

☑ **Soln. :**

(i) **Converse :** If tomorrow is Monday, then today is Easter.

Inverse : If today is not Easter then, tomorrow is not Monday.

Contrapositive : If tomorrow is not Monday then today is not Easter.

(ii) **Converse :** If triangle is not equilateral then it is not isosceles.

Inverse : If a triangle is isosceles then it is equilateral.

Contrapositive : If a triangle is equilateral then it is isosceles.

(iii) **Converse :** If n is odd or n is 2 then n is prime.

Inverse : If n is not odd or n is 2 then n is prime.

Contrapositive : If n is odd and n is not 2 then n is not prime.

Ex. 2.5.4 : Write the contrapositive, inverse, converse and negation of following sentence.
"If x is rational, then x is real".

☑ **Soln. :**

Let, P : x is rational, q : x is real.

In symbolic form : $p \rightarrow q$

(i) **Contrapositive**

$(\sim q \rightarrow \sim p)$

That is, "If x is not real, then x is not rational".

(ii) **Inverse**

$\sim p \rightarrow \sim q$

That is, "If x is not rational, then x is not real".

(iii) **Converse**

$q \rightarrow p$

That is, "If x is real then x is rational".

(iv) **Negation**

$\sim (p \rightarrow q) \Rightarrow \sim p (\sim p \vee q) \Rightarrow p \wedge \sim q$

That is "x is rational and not real".

2.6 TRUTH TABLES

- Truth table is mathematical table which is used to find truth value of compound statement.
- As truth table has rows and columns. The rows contain the Boolean logic true or false values while columns contains list of premises as well as conclusion.
- Truth table uses five basic operations including conjunction, disjunction, negation, conditional and bi-conditional.
- If statement or proposition contain n distinct variables then there will be 2^n possible values.

Examples

1. Conjunction ($p \wedge q$)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	T

2. Disjunction ($p \vee q$)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3. Negation

p	$\sim p$
T	F
F	T

4. Implication ($p \rightarrow q$)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

5. Bi-Implication

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

6. NOR ($p \downarrow q$)

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

7. XOR ($p \veebar q$)

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

2.7 TAUTOLOGY

- A statement formula which is always true for all possible values of it's propositional variables is called a tautology.
- e.g. $p \vee \sim p$ is tautology.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

2.8 CONTRADICTION

- A statement formula which is always false for all possible values of its propositional variables is called a contradiction.
- **Example :** $p \wedge \sim p$ is contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

2.9 CONTINGENCY

- A statement formula which is neither tautology nor contradiction is called a contingency.
- **Example :** $p \wedge q$ is contingency.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2.10 PRECEDENCE OF LOGICAL OPERATORS

- In order to get correct conclusion value from the given proposition it is necessary to know the operator precedence.
- Lets take example to understand the importance of precedence of operator.
- $(\sim p) \wedge q$ is different from $\sim (p \wedge q)$, both statement's conclusion value will be different.
- $p \wedge q \wedge r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$.
- To understand the above examples, you must have knowledge of operator priority.

Operator	Precedence
$\sim$ (negation)	1
$\wedge$ (conjunction)	2
$\vee$ (disjunction)	3
$\rightarrow$ (Implication)	4
$\leftrightarrow$ (bi-implication)	5

- From the table, it is clear that, $\sim$ (negation) operator has highest precedence while $\rightarrow$ (conditional) and bi-conditional ($\leftrightarrow$) have lowest precedence.

2.11 LOGICAL EQUIVALENCE

- In our daily life, we observed, number of similar things with respect to some aspects.
- e.g. two persons are similar with respect to their height, two students are similar to their S.E. marks.
- In the same way, two proposition or statements are similar with respect to their truth values.

> **Definition :** Two statements or propositions are said to be logically equivalent if they have same truth value.

- Two proposition are equivalent even if they have different variables.

Example

(i) p and $p \wedge p$ are logically equivalent.

p	$p \wedge p$
T	T
F	F

(ii) $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

both truth values are same.

2.12 FUNDAMENTAL LOGICAL IDENTITIES

Sr. No.	Name of Identity	Identity
1.	Idempotence of $\vee$	$p \equiv p \vee p$
2.	Idempotence of $\wedge$	$p \equiv p \wedge p$
3.	Commutativity of $\vee$	$p \vee q \equiv q \vee p$
4.	Commutativity of $\wedge$	$p \wedge q \equiv q \wedge p$
5.	Associativity of $\vee$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
6.	Associativity of $\wedge$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
7.	Distributivity of $\wedge$ over $\vee$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8.	Distributivity of $\vee$ over $\wedge$	$p \vee (q \wedge r) \equiv (p \vee r) \wedge (p \vee r)$
9.	Double negation	$p \equiv \sim(\sim p)$
10.	De-Morgan's law	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
11.	De-Morgan's law	$\sim(p \wedge q) \equiv \sim p \vee \sim q$
12.	Absorption law	$p \vee (p \wedge q) \equiv p$
13.	Absorption law	$p \wedge (p \vee q) \equiv p$
14.	Negation law of $\vee$	$p \vee \sim p = T$
15.	Negation law of $\wedge$	$p \wedge \sim p = F$
16	Identity law of $\vee$	$p \vee F = p$ $p \vee T = T$
17.	Identity law of $\wedge$	$p \wedge F = F$ $p \wedge T = p$

2.12.1 Examples Based on Truth Tables, Tautology and Logical Equivalence

Ex. 2.12.1 : Construct the truth tables for the following statements :

(i) $p \to p$ (ii) $(p \to p) \vee (p \to \sim p)$

(iii) $(p \to p) \to (p \to \sim p)$ (iv) $(p \vee \sim q) \vee \sim p$

(v) $(p \vee \sim q) \to \sim q$

(vi) $p \leftrightarrow (\sim p \vee \sim q)$

(vii) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

(viii) $(\sim q \to \sim p) \to (p \to q)$

☑ **Soln. :**

(i) $p \to p$

p	p	$p \to p$
T	T	T
F	F	T

(ii) $(p \to p) \vee (p \to \sim p)$

p	p	$\sim p$	$p \to p$	$p \to \sim p$	$(p \to p) \vee (p \to \sim p)$
T	T	F	T	F	T
F	F	T	T	T	T

(iii) $(p \to p) \to (p \to \sim p)$

p	p	$\sim p$	$p \to p$	$p \to \sim p$	$(p \to p) \to (p \to \sim p)$
T	T	F	T	F	F
F	F	T	T	T	T

(iv) $(p \vee \sim q) \vee \sim p$

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \vee \sim p$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	T	T	T

(v) $(p \vee \sim q) \to \sim q$

p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \to \sim q$
T	T	F	T	F
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

(vi) $p \leftrightarrow (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$p \leftrightarrow (\sim p \vee \sim q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	F
F	F	T	T	T	F

(vii) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

1	2	3	4	5	6	7	8	9
p	q	r	$q \to r$	$p \to (q \to r)$	$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$	$5 \to 8$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

(viii) $(\sim q \to \sim p) \to (p \to q)$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(\sim q \to \sim p) \to (p \to q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Ex. 2.12.2 : Determine whether each of the following statement formula is a tautology, contradiction or contigency.

(i) $(p \wedge q) \wedge \sim (p \vee q)$

(ii) $(p \to q) \leftrightarrow (q \vee \sim p)$

(iii) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

☑ **Soln. :**

(i) $(p \wedge q) \wedge \sim (p \vee q)$

p	q	$p \vee q$	$\sim (p \vee q)$	$p \wedge q$	$(p \wedge q) \wedge \sim (p \vee q)$
T	T	T	F	T	F
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	T	F	F

Hence, $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

(ii) $(p \to q) \leftrightarrow (q \vee \sim p)$

p	q	$\sim p$	$p \to q$	$q \vee \sim p$	$(p \to q) \leftrightarrow (q \vee \sim p)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Hence, $(p \to q) \leftrightarrow (q \vee \sim p)$ is a tautology.

(iii) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to p \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

Ex. 2.12.3 : Prove by constructing the truth table :

$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$$

☑ **Soln. :**

p	q	r	$p \to q$	$p \to r$	$q \vee r$	$p \to (q \vee r)$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	T

From the Table, $p \to (q \vee r)$ column and $(p \to q) \vee (p \to r)$ column have same truth values for all possible choices for p, q, r.

Hence, $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$ proved.

Ex. 2.12.4 : Prove $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

☑ **Soln. :**

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$\sim p$	$\sim q$	$\sim p \vee q$	$\sim q \vee p$	$(\sim p \vee q) \wedge (\sim q \vee p)$
T	T	T	T	T	T	F	F	T	T	T
T	F	F	F	T	F	F	T	F	T	F
F	T	F	T	F	F	T	F	T	F	F
F	F	T	T	T	T	T	T	T	T	T

From the Table, it is clear that, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Ex. 2.12.5 : Prove that $p \to (q \to r)$ and $(p \wedge \sim r) \to \sim q$ are logically equivalent.

☑ **Soln. :**

p	q	r	$q \to r$	$p \to (q \to r)$	$\sim r$	$p \wedge \sim r$	$\sim q$	$(p \wedge \sim r) \to \sim q$
T	T	T	T	T	F	F	F	T
T	T	F	F	F	T	T	F	F
[illegible]	[illegible]	[illegible]	[illegible]	T	F	F	T	T
T	F	F	T	T	T	T	T	T
F	T	T	T	T	F	F	F	T
F	T	F	F	T	T	F	F	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T	T

From the truth table, it is clear that,

$$p \to (q \to r) \equiv (p \wedge \sim r) \to \sim q$$

Ex. 2.12.6 : Prove that, $(p \veebar q)$ and $(p \vee q) \wedge \sim (p \wedge q)$ are logically equivalent.

☑ **Soln. :**

p	q	$p \veebar q$	$(p \wedge q)$	$\sim (p \wedge q)$	$p \vee q$	$p \vee q \wedge \sim (p \wedge q)$
T	T	F	T	F	T	F
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	F	T	F	F

From the truth table, it is observed that

$$(p \veebar q) \equiv (p \vee q) \wedge \sim (p \wedge q)$$

Here, $\veebar$ is exclusive or, or XOR

UEx. 2.12.7 (SPPU - IT - Q. 1(b), May 18, 6 Marks)

Show that each of these conditional statements is a tautology by using truth tables :

(i) $(p \wedge q) \to p$ (ii) $p \to (p \vee q)$.

☑ **Soln. :**

(i) $(p \wedge q) \to p$

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

$(p \wedge q) \to p$ is a tautology

(ii) $p \to (p \vee q)$.

p	q	$p \vee q$	$p \to p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$P \to (p \vee q)$ is a tautology

2.13 THE PRINCIPLE OF DUALITY

- Let S_1 and S_2 be the two statements are said to be dual of each other if either one can be obtained from the other by interchanging $\wedge$ (AND), $\vee$ (OR), T (TRUE) and F (FALSE) by $\vee$ (OR), $\wedge$ (AND), F (FALSE) and T (TRUE) respectively.

Example

The dual of $(p \wedge q)$ will be $(p \vee q)$.

Similarly, dual of $p \wedge F$ will be $p \vee T$.

2.14 NORMAL FORMS

- While finding, whether given statement is tautology or contradiction or contijency. We use the truth tables.
- If statement has n distinct variables then truth table will have 2^n possible combination of truth values.
- But, if number of variables are more, then constructing truth table is not convenient or impractical.
- To resolve the above mentioned issue, normal form will be the best choice to find truth values or to declare given statement tautology or contradiction or not.
- In normal form, a formula which is a conjunction (product) of the variables and their negation is said to be an fundamental product.

  e.g. $p, \sim p, \sim p \wedge q, p \wedge \sim q, \sim p \wedge \sim q$ etc.
- In the same way, a formula which is disjunction (sum) of the variables and their negation is said to be an fundamental sum.

  e.g. $p, \sim p, \sim p \vee q, p \vee q, p \vee \sim q$ etc.

2.14.1 Types of Normal Form

2.14.2 Disjunctive Normal Form (DNF)

- A statement form which consist of a disjunction ($\vee$) of fundamental conjunctions ($\wedge$) is called a disjunctive normal form.
- It is abbreviated as DNF.

Examples

(i)

$$\underbrace{(p \wedge r)}_{\text{(Conjunction)}} \underbrace{\vee}_{\text{disjunction}} \underbrace{(p \wedge q)}_{\text{Conjunction}}$$

(ii) $(p \wedge q) \vee (q) \vee (\sim q \wedge p) =$

Disjunction ($\vee$) of fundamental conjunction ($\wedge$)

(iii) $(\sim p \wedge r) \vee (\sim q \wedge r) \vee (\sim r)$

Rules to find disjunctive normal form (DNF) of given statement

(i) Replace conditional ($\to$) or bi-conditional ($\leftrightarrow$) by using logical connectives $\sim$, $\wedge$, $\vee$ etc. like

$p \to q \equiv \sim p \vee q$, $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$

(ii) Use De-Morgans law to eliminate '~' before sum or product.

(iii) Apply distributive laws repeatedly and eliminate product of variables to obtain the required normal form.

2.14.3 Conjunctive Normal Form (CNF)

- A statement form which consist of a conjunction ($\wedge$) of fundamental disjunction ($\vee$), is called conjunctive normal form.
- It is abbreviated as CNF.

Examples

(i) $p \wedge (p \vee q)$

Here p and $(p \vee q)$ are combined using conjunction. Hence it is conjunction of disjunction.

(ii)

$$\underbrace{\underbrace{(p \vee q \vee r)}_{\text{Disjunctions}} \wedge \underbrace{(\sim p \vee r)}_{\text{Disjunctions}} \wedge \underbrace{(p \vee \sim q \vee r)}_{\text{Disjunctions}}}_{\text{Conjunction}}$$

2.14.4 Problems Based on Normal Form

Ex. 2.14.1 : (i) Find DNF of : $((p \to q) \wedge (q \to p)) \vee p$.

(ii) Find CNF of : $p \leftrightarrow (\sim p \vee \sim q)$

☑ **Soln. :**

(i) Let,

$((p \to q) \wedge (q \to p)) \vee p$

$\equiv ((\sim p \vee q) \wedge (\sim q \vee p)) \vee p$ $\because a \to b \equiv \sim a \vee b$

$\equiv ((\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p)) \vee p$

$\because$ distributive law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$\equiv (\sim p \wedge \sim q) \vee (F) \vee (F) \vee (p \wedge q) \vee (p)$

$\because$ absorption law $p \vee (p \wedge q) = p$

$\equiv (\sim p \wedge \sim q) \vee (p)$

$\equiv$ Required DNF

(ii) Let,

$p \leftrightarrow (\sim p \lor \sim q)$

$\equiv (\sim p \lor (\sim p \lor \sim q)) \land (p \lor \sim(\sim p \lor \sim q))$

$\because p \leftrightarrow q \equiv (\sim p \lor q) \land (p \lor \sim q)$

$\equiv (\sim p \lor (\sim p \lor \sim q)) \land (p \lor (p \land q))$

$\equiv (\sim p \lor (\sim p \lor \sim q)) \land p$

$\equiv (\sim p \lor \sim p) \lor \sim q) \land p = (\sim p \lor \sim q) \land (p)$

$\because p \lor p = p$ absorption law $p \lor (p \land q) = p$

$\equiv$ Required CNF

Ex. 2.14.2 : Obtain the conjunctive normal form and disjunctive normal form of the following formulae given below :

(i) $p \land (p \rightarrow q)$ (ii) $\sim(p \lor q) \leftrightarrow (p \land q)$

☑ **Soln. :**

(i) $\mathbf{p \land (p \rightarrow q)}$

$\Rightarrow p \land (\sim p \lor q) \leftarrow$ CNF

$\Rightarrow (p \land \sim p) \lor (p \land q)$ $\because$ Distributive law

$\Rightarrow (F) \lor (p \land q)$ $\because (p \land \sim p) =$ Contradiction

$\Rightarrow (p \land q) \leftarrow$ DNF $\because p \lor F = p$ (Identity law)

(ii) $\mathbf{\sim(p \lor q) \leftrightarrow (p \land q)}$

$\Rightarrow [\sim\sim(p \lor q) \lor (p \land q)] \land [\sim(p \lor q) \lor \sim(p \land q)]$

$\because p \leftrightarrow q \equiv (\sim p \lor q) \land (p \lor \sim q)$

$\Rightarrow \underbrace{[(p \lor q) \lor (p \land q)]} \land [(\sim p \lor \sim q) \lor (\sim p \land \sim q)]$

$\because \sim\sim(p) = p$

$\Rightarrow [(p \lor q)] \land [(\sim p \lor \sim q \lor \sim p) \land (\sim p \lor \sim q \lor \sim q)]$

$\because$ distributive law

$\Rightarrow [(p \lor q)] \land [(\sim p \lor \sim q) \land (\sim p \lor \sim q)]$ $\because p \lor p = p$

$\Rightarrow (p \lor q) \land (\sim p \lor \sim q) \Leftarrow$ CNF $\because p \land p = p$

Further,

$[(p \lor q)] \land [(\sim p \lor \sim q)]$

$\Rightarrow [(p \lor q) \land \sim p] \lor [(p \lor q) \land \sim q]$ $\because$ distributive law

$[(p \land \sim p) \lor (q \land \sim p)] \lor [(p \land \sim q) \lor (q \land \sim q)]$

$\because$ distributive law

$\Rightarrow [(F) \lor (q \land \sim p)] \lor [(p \land \sim q) \lor (F)]$

$\because$ Negation law : $p \land \sim p = F$

$\Rightarrow [(q \land \sim p)] \lor [(p \land \sim q)]$ $\because p \lor F = p$ (Identity law)

$\Rightarrow$ DNF

Ex. 2.14.3 : Find the conjunctive and disjunctive normal forms for the following without using truth tables

(i) $(p \rightarrow q) \land (q \rightarrow p)$ (ii) $((p \land (p \rightarrow q)) \rightarrow q)$

☑ **Soln. :**

(i) $\mathbf{(p \rightarrow q) \land (q \rightarrow p)}$

$\Rightarrow (\sim p \lor q) \land (\sim q \lor p)$ $\because p \rightarrow q \equiv (\sim p \lor q)$

$\Rightarrow$ CNF

Further, $(\sim p \lor q) \land (\sim q \lor p)$

$\Rightarrow [(\sim p \lor q) \land \sim q] \lor [(\sim p \lor q) \land p]$

$\because$ distributive law $\land$ over $\lor$

$\Rightarrow [(\sim p \land \sim q)] \lor [(q \land \sim q)] \lor [(\sim p \land p) \lor (q \land p)]$

$\because$ distributive law $\land$ over $\lor$

$\Rightarrow [(\sim p \land \sim q) \lor (F)] \lor [(F) \lor (p \land q)]$

$\because$ Negation law $\sim p \land p \Rightarrow F$

$\Rightarrow (\sim p \land \sim q) \lor (p \land q)$ $\because$ Identity law $p \lor F \equiv p$

$\Rightarrow$ DNF

(ii) $\mathbf{((p \land (p \rightarrow q)) \rightarrow q)}$

$\Rightarrow \sim[(p \land (\sim p \lor q))] \lor q$ $\because p \rightarrow q = \sim p \lor q$

$\Rightarrow [\sim p \lor \sim(\sim p \lor q)] \lor q$

$\Rightarrow (\sim p) \lor (p \land \sim q) \lor (q)$

$\Rightarrow$ DNF

Further, $(\sim p) \lor (p \land \sim q) \lor (q)$

$\Rightarrow (\sim p \lor p) \land (\sim p \lor \sim q) \lor (q)$ $\because \sim p \lor p \equiv T$

$\Rightarrow (T) \land (q \lor \sim q) \lor (\sim p)$

$\Rightarrow (T) \land (T) \lor (\sim p)$

$= (T) \equiv (p \lor \sim p)$ $\Leftarrow$ CNF (Single disjunct)

Ex. 2.14.4 : Find DNF of $((p \rightarrow q) \land (q \rightarrow p)) \lor p$

☑ **Soln. :**

$((p \rightarrow q) \land (q \rightarrow p)) \lor p$

$\Rightarrow [(\sim p \lor q) \land (\sim q \lor p)] \lor p$ $\because p \rightarrow q \equiv (\sim p \lor q)$

$\Rightarrow [p \lor (\sim p \lor q)] \land [p \lor (\sim q \lor p)]$

$\because$ Distributive law $\lor$ or $\land$

$\Rightarrow [(p \lor \sim p) \lor q] \land [(p \lor p) \lor \sim q]$

$\Rightarrow [(T) \lor (q)] \land [(p) \lor (\sim q)]$ $\because p \lor \sim p = T$ and $p \lor p = p$

$\Rightarrow (T) \land (p \lor \sim q)$ $\because$ Identity law $p \lor T = T$

$\Rightarrow (p) \lor (\sim q)$ $\because$ Identity law

$\Rightarrow$ DNF $p \land T = p$

Ex. 2.14.5 : Find the conjunctive normal form and disjunctive normal form for the following :

(i) $(p \lor \sim q) \rightarrow q$ (ii) $p \leftrightarrow (\sim p \lor \sim q)$

☑ **Soln. :**

(i) $\mathbf{(p \lor \sim q) \rightarrow q}$

$\Rightarrow \sim(p \lor \sim q) \lor q$ $\because p \rightarrow q = \sim p \lor q$

$\Rightarrow (\sim p \land q) \lor q$

$\Rightarrow$ DNF

Further, $(\sim p \land q) \lor q$

$\Rightarrow (\sim p \lor q) \land (q \lor q)$ $\because$ Distributive law

$\Rightarrow (\sim p \lor q) \land (q) \Rightarrow$ CNF

(ii) $\mathbf{p \leftrightarrow (\sim p \lor \sim q)}$

$\Rightarrow [\sim p \lor (\sim p \lor \sim q)] \land [\sim(\sim p \lor \sim q) \lor p]$

$\because p \leftrightarrow q \equiv (\sim p \lor q) \land (\sim q \lor p)$

$\Rightarrow [(\sim p \lor \sim p) \lor \sim q] \land [(p \land q) \lor p]$ $\because p \lor p = p$

$\Rightarrow [(\sim p \lor \sim q)] \land [(p \lor p) \land (q \lor p)]$ $\because$ distributive law

$\Rightarrow (\sim p \lor \sim q) \land (p) \land (q \lor p) \Rightarrow$ CNF

Further, $[(\sim p \lor \sim q) \land (p)] \land (q \lor p)$

$\Rightarrow [(\sim p \land p) \lor (\sim q \land p)] \land (q \lor p)$ $\because$ distributive law

$\Rightarrow [(F) \lor (\sim q \land p)] \land (q \lor p)$ $\because p \land \sim p = F$

$\Rightarrow (\sim q \land p) \land (q \lor p)$

$\because$ Identity law

$\because p \lor F = p$

$\Rightarrow (\sim q \land p \land q) \lor (\sim q \land p \land p)$

$\Rightarrow ((F) \land p) \lor (\sim q \land p)$

$\Rightarrow (F) \lor (\sim q \land p)$ $\because$ Identity law

$\because p \land F = F$

$\Rightarrow (\sim q \land p) \leftarrow$ Single conjunct

$\Rightarrow$ DNF

Ex. 2.14.6 : Obtain CNF of $q \lor (p \land \sim q) \lor (\sim p \land \sim q)$

☑ **Soln. :** Given : $q \lor (p \land \sim q) \lor (\sim p \land \sim q)$

$\Rightarrow [(q \lor p) \land (q \lor \sim q)] \lor (\sim p \land \sim q)$

$\because$ Distributive law $\lor$ over $\land$

$\Rightarrow (q \lor p) \land (T) \lor (\sim p \land \sim q)$ $\because$ Negation law $p \lor \sim q \equiv T$

$\Rightarrow (q \lor p) \lor (\sim p \land \sim q)$ $\because$ Identity law $p \land T = p$

$\Rightarrow (q \lor p \lor \sim p) \land (q \lor p \lor \sim q)$

$\Rightarrow (q \lor (T)) \land ((T) \lor p)$

$\Rightarrow (T) \land (T)$

$\Rightarrow T \equiv (p \lor \sim p) \Leftarrow$ Single disjunct.

$\Rightarrow$ CNF

Ex. 2.14.7 : Obtain the CNF of $(\sim p \land q \land r) \lor (p \land q)$

☑ **Soln. :**

Given : $(\sim p \land q \land r) \lor (p \land q)$

$\Rightarrow [p \lor (\sim p \land q \land r)] \land [q \lor (\sim p \land q \land r)]$

$\because$ distributive law $\lor$ over $\land$

$\Rightarrow [(p \lor \sim p) \land (p \lor q) \land (p \lor r)] \land [(q \lor \sim p) \land (q \lor q) \land (q \lor r)]$

$\because$ distributive law $\lor$ over $\land$

$\Rightarrow [(T) \land (p \lor q) \land (p \lor r)] \land [(q \lor \sim p) \land (q) \land (q \lor r)]$

$\Rightarrow (p \lor q) \land (p \lor r) \land (q \lor \sim p) \land (q)$

$\because$ Absorption law $p \land (p \lor q) \equiv p$

$\Rightarrow$ CNF

Ex. 2.14.8 : Obtain principle DNF of $(\sim p \lor \sim q) \rightarrow (\sim p \land r)$

☑ **Soln. :**

Given : $(\sim p \lor \sim q) \rightarrow (\sim p \land r)$

$\Rightarrow [\sim(\sim p \lor \sim q)] \lor [(\sim p \land r)]$ $\because p \rightarrow q = \sim p \lor q$

$\Rightarrow (p \land q) \lor (\sim p \land r)$

$\Rightarrow$ DNF

$\Rightarrow$ Further, $(p \land q) \lor (\sim p \land r)$

$\Rightarrow (p \land q \land (r \lor \sim r)) \lor (\sim p \land r \land (q \lor \sim q))$

$\Rightarrow (p \land q \land r) \lor (p \land q \lor \sim r) \lor (\sim p \land r \land q)$
$\lor (\sim p \land r \land \sim q)$

$\Rightarrow$ Principle DNF

Ex. 2.14.9 : Find the disjunctive normal form

(i) $(p \rightarrow q) \land (\sim p \land q)$

☑ **Soln. :**

$(\sim p \lor q) \land (\sim p \land q)$ $\because p \rightarrow q \equiv (\sim p \lor q)$

$\Rightarrow (\sim p \land \sim p \land q) \lor (q \land \sim p \land q)$

$\Rightarrow (\sim p \land q) \lor (q \land \sim p)$ $\because p \land p = p$

$\Rightarrow$ DNF

Ex. 2.14.10 : Find the principle CNF of $(p \land q) \lor (\sim p \land r)$.

☑ **Soln. :**

Given :

$(p \land q) \lor (\sim p \land r)$

$\Rightarrow [(p \land q) \lor \sim p] \land [(p \land q) \lor r]$

$\because$ distributive law $\lor$ over $\land$

$\Rightarrow [(p \lor \sim p) \land (q \lor \sim p)] \land [(p \lor r) \land (q \lor r)]$

$\Rightarrow [(T) \land (q \lor \sim p)] \land [(p \lor r) \land (q \lor r)]$

$\Rightarrow (q \lor \sim p) \land (p \lor r) \land (q \lor r)$

$\Rightarrow \quad (q \vee \sim p \vee (r \wedge \sim r)) \wedge (p \vee r \vee (q \wedge \sim q))$

$\quad \wedge (q \vee r \vee (p \wedge \sim p))$

$\Rightarrow \quad [(q \vee \sim p \vee r) \wedge (q \vee \sim p \vee \sim r)] \wedge [(p \vee r \vee q)$

$\quad \wedge (p \vee r \vee \sim q)] \wedge [(q \vee r \vee p) \wedge (q \vee r \vee \sim p)]$

$\because$ distributive law

$\Rightarrow \quad (p \vee q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

$\quad \wedge (p \vee \sim q \vee r)$

$\Rightarrow$ Principle CNF

2.15 METHODS OF PROOF

- In mathematics, definition of any concept need not to be proved. But theorems or arguments need to be proved for their validity.
- There is requirement of drawing a conclusion from given premises or axioms for proving a theorem.
- We take certain statements, which are true and then derive other statements whose truth value is going to be established.
- The process of deriving a conclusion from a set of premises and fact or axioms by using the standard rules of inference, is called a formal proof or deduction.
- If premises (Initial collection of statements) are true, then the conclusion of the theorem is also true.
- Lets see the formal definition of valid argument.

2.15.1 Valid Argument

- Valid argument is a finite sequence of statements S_1, S_2, S_3, ... S_n called as premises together with a statement C, called a conclusion such that $S_1 \wedge S_2 \wedge S_3 \wedge \ldots \wedge S_n \rightarrow C$ is a tautology.

  e.g. $\{S_1, S_2, S_3, \ldots S_n\}$ are premises.
- If premises are true then conclusion also be true.
- That is

$$\left.\begin{matrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \end{matrix}\right\} \rightarrow \text{Set of statements}$$
$$\overline{\quad C \quad} \leftarrow \text{Conclusion}$$

2.15.2 Examples Based on Methods of Proof

Ex. 2.15.1 : Determine the validity of the argument

S_1 : All my friends are musicians.

S_2 : John is my friend.

S_3 : None of my neighbours are musicians.

S : John is not my neighbours.

☑ Soln. :

Convert above given statements or arguments into propositions.

Lets, p : All my friends are musicians.

q : John is my friend.

r : My neighbors are musicians.

s : John is my neighbour.

$\therefore \quad S_1 : p, \quad S_2 : q, \quad S_3 : \sim r$

The above mentioned arguments can be represented in symbolic form.

So,
$$\begin{array}{c} p \\ q \\ \underline{\sim r} \\ \sim S \end{array}$$

As all my friends are musicians and John is my friend which implies that John is musician.

$\therefore$ $p \wedge q \rightarrow$ John is musician.

$\therefore$ $p \wedge q \wedge \sim r \rightarrow$ John is musician and my neighbours are not musicians.

$\therefore$ $p \wedge q \wedge \sim r \rightarrow$ John is not my neighbour.

$p \wedge q \wedge \sim r \rightarrow \sim S$ is true. So the given argument is valid.

Ex. 2.15.2 : Determine whether the following is a valid argument :

If Shilpa goes to class, she is on time.

But Shilpa is late. She will therefore miss class.

☑ Soln. :

The given arguments will be written as :

p : Shilpa goes to class.

q : Shilpa is on time.

The above statements can be written in symbolic way.

$$\begin{array}{c} p \rightarrow q \\ \underline{\sim q} \\ \therefore \quad \sim p \end{array}$$

According to modus Tollens or law of contrapositive

If $(p \rightarrow q) \wedge \sim q \Rightarrow \sim p$ is a tautology.

Hence, the given argument is valid.

Ex. 2.15.3 : Determine whether the argument is valid or not. If I try hard and I have talent then I will become a musician. If I become a musician, then I will be happy. Therefore if I will not be happy then I did not try hard or I do not have talent.

☑ Soln. :

Let, p : I try hard.

q : I have talent.

r : I will become musician.

S : I will be happy.

Symbolically,

$S_1 : \ (p \wedge q) \rightarrow r$

$S_2 : \ r \rightarrow S$

$\therefore S_3 : \ \sim S \rightarrow \sim p \vee \sim q$

Consider, the argument is invalid. This means that for some assignments of truth values S_1 is T, S_2 is T but S_3 is F.

S_3 will have truth value F if ~ S is T and ~ p ∨ ~ q is F. That is S is F and ~p is F and ~ q is F that is S is F and p is T and q is T.

As, S_2 is T, the truth values of r and S both are F. Since S_1 is true, r is F which implies either p or q is F. This is contradiction since by assumption both p and q are true.

Hence, the given argument is valid.

Ex. 2.15.4 : Test the validity of the argument. If a person is poor, he is unhappy. If a person is unhappy, he dies young, therefore poor person dies young.

☑ Soln. :

Let, p : Person is poor.

q : Person is unhappy.

r : Person dies young.

The above statement written in symbolic form as :

$S_1 : \ p \rightarrow q$

$S_2 : \ q \rightarrow r$

According to hypothetical syllogism,

$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$ is a tautology.

Hence, the above given argument is valid.

Ex.2.15.5 : Determine the validity of the following argument.

S_1 : If I like discrete structure then I will study.

S_2 : Either I will study or I will fail.

S : If I will fail then I do not like discrete structure.

☑ Soln. :

Let, p : I like discrete structure.

q : I will study. r : I will fail.

In symbolic form,

$$\begin{array}{c} S_1 : p \rightarrow q \\ \underline{S_2 : q \vee r} \\ S : r \rightarrow \sim p \end{array}$$

We know that, for the validity of argument $S_1 \wedge S_2$ should logically implies S.

Assign the truth values T, T, T to p, q, r respectively.

Then S_1 is T and S_2 is also T.

But, S is T → F is F

Hence, the given argument is invalid.

Ex. 2.15.6 : If today is Monday, then there is a test in object oriented programming or in discrete mathematics. If the discrete mathematics professor is sick, there will be no test in discrete mathematics. Today is Monday and the professor of discrete mathematics is sick. Hence there will be a test in object oriented programming.

☑ Soln. : Let, convert given arguments into propositions.

p : Today is Monday.

q : There is a test in object oriented programming.

r : There is a test in discrete mathematics.

s : Discrete mathematics professor is sick.

In symbolic form,

$$\begin{array}{c} p \rightarrow (q \vee r) \\ S \rightarrow \sim r \\ \underline{p \wedge S} \\ \therefore q \end{array}$$

If $P \rightarrow (q \vee r)$, $S \rightarrow \sim r$, $P \wedge S$ are true, q is also true, because $P \wedge S$ is T which implies P is T. S is T implies ~ r is T, because r is F.

$\therefore P \rightarrow (q \vee r)$ is T implies q is T.

Hence, the argument is valid.

Ex. 2.15 7 : I am happy if my program runs. A necessary condition for the program to run is it should be error free. I am not happy. Therefore the program is not error free.

☑ Soln. :

Let, translate above arguments into simple propositions.

p : I am happy.

q : My program runs.

r : Program should be error free.

In symbolic form,

$$\begin{array}{c} q \rightarrow p \\ q \rightarrow r \\ \underline{\sim p} \\ \therefore \sim r \end{array}$$

We can check the validity of argument using truth table.

p	q	r	q→p	q→r	~p	(q→p)∧(q→r)∧~p	A→~r
F	F	T	T	T	T	Ⓐ T	F

From the values of truth table. It is observed that, in the last column, the value is F (False). So, the given argument is not tautology. Hence, it is conclude that, given argument is **not valid.**

2.16 RULES OF INFERENCE

- It is an criteria for finding the validity of an argument.
- The rules are represented in the form of statements.
- There are four types of Rules of Inference.

1. Law of Detachment (Modus ponens)

- Let, p and $p \rightarrow q$ be statements accepted as true, then we must accept the statement q as true.
- It can be represented in the following form.

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

Here,

$$\begin{array}{l} S_1 : p \rightarrow q \\ S_2 : p \\ \hline \therefore\ C : q \end{array}$$

- The statements above the horizontal line are called Premises or Hypotheses and the statement below the line is called the Conclusion. From the above discussion. It is clear that $(p \wedge (p \rightarrow q))$ is a tautology
- It can be verified using truth table as :

p	q	p → q	(p → q) ∧ p	(p → q ∧ p) → q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- **Example:** If Atul gets a first class with distinction in B.E. then he will get a good job in reputed company.

  p : Atul gets a first class with distinction in B.E.

  q : He will get a good job in reputed company.
- Here, Premises are p and $p \rightarrow q$, conclusion is q.
- Represented as,

$$\left.\begin{array}{l} p \\ p \rightarrow q \end{array}\right\} \text{Premises}$$
$$\therefore\ q \Rightarrow \text{Conclusion}$$

- Hence, this form of argument is valid.

2. Law of Contrapositive (Modus Tollen)

- This rule of inference is for denying.
- "If $p \rightarrow q$ is true and q is false, then p is false"
- It can be represented as,

$$\left.\begin{array}{l} p \rightarrow q \\ \sim q \end{array}\right\} \text{Premises}$$
$$\therefore\ \sim p \Rightarrow \text{Conclusion}$$

- The above argument is valid since $(p \rightarrow q) \wedge \sim q \rightarrow \sim p$ is tautology.
- **Example :** If Atul gets a first class with distinction then he does not get job in reputed company.
- Therefore, Atul does not get a first class with distinction.
- Let,

  p : Atul gets a first class with distinction.

  q : Atul gets a job.
- Then the inference form is,

$$\begin{array}{l} p \rightarrow q \Rightarrow \text{True} \\ \sim q \Rightarrow \text{False} \\ \hline \therefore \sim p \Rightarrow \text{False} \end{array}$$

- Hence, the above argument is valid.

3. Disjunctive syllogism

- This rule stated as **"If $p \vee q$ is true and p is false then q is true".**
- It is represented as :

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore\ q \end{array}$$

- Here, $(p \vee q) \wedge \sim p \rightarrow q$ is a tautology

  Which means $\sim p$ is true, $(p \vee q)$ is true implies q is true.

☞ Example

- "If Rajesh is going to study law or medicine and does not study law, then he will therefore study medicine".

- Let, p : Rajesh going to study law.

  q : Rajesh going to study medicine.
- The Inference form is,

$$\begin{array}{l} p \vee q \Rightarrow \text{True} \\ \sim p \Rightarrow \text{False} \\ \hline \therefore\ q \Rightarrow \text{True} \end{array}$$

- Hence, The above argument is true or valid.

4. Hypothetical Syllogism

- This form of Inference is also known as transitive rule.
- It stated as "If $(p \rightarrow q)$ and $(q \rightarrow r)$ are true then $(p \rightarrow r)$ is true".
- It is represented as :

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$$

- Here, $(p \rightarrow q) \wedge (q \rightarrow r)$ is true which implies that $(p \rightarrow r)$ is true.

☞ Example

- If Rajesh studies hard, he will obtain first class with distinction, he will get job in reputed company.
- Let, p : Rajesh studies hard.

  q : He obtains a first class with distinction.

  r : He will get a job in reputed company.
- So,

$$\begin{array}{l} p \rightarrow q \Rightarrow \text{True} \\ q \rightarrow r \Rightarrow \text{True} \\ \hline \therefore\ p \rightarrow r \Rightarrow \text{True} \end{array}$$

- Hence, the argument is valid.

2.17 PREDICATES

- In general English sentence, Predicate is the word in a sentence which express something more about object or express about properties or relation of object.
- For example, "is a good boy", "is a smart , intelligent teacher" are predicates.
- In Propositional logic, logic has broad meaning as compared to general English.
- To understand the predicates , let us have a look on some example like
  - o Dog is animal.
  - o Cat is animal.
- Here, above statements requires two different symbols to represent them. That symbols are predicates.
- "is animal" is predicate and Dog and Cat are objects or variables.

> ☐ *Definition : An assertation which contains one or more variables is called predicate.*

- It's truth value is predicated after assigning truth values to it's variables.
- Predicates represented by capital letters and objects represented by lower case letters in general.
- Symbolically, statements can be written as predicate letter followed by the name(s) of object(s) to which the predicate is applied.

☞ Example

(i) Dog is an animal.

(ii) Cat is an animal.

- Here, "is an animal" is a predicate represented by A and "Dog", "Cat" are objects represented as d and c respectively.
- Then the eg. (i) and (ii) can be written as A(d) and A(c) or it also represented as A(d,c).
- A statement must have at least one object or variable associated with the predicate.
- A predicate Z contain n variables $x_1, x_2, \ldots x_n$ is called an n - place predicate and denoted by $Z(x_1, x_2, x_3, \ldots x_n)$. Here, each variable x_i is called as argument.

☞ Example

(i) " x is a capital of Maharashtra" is denoted by Z (x).

(ii) " x is the author of discrete math book" is denoted by Z (x).

(iii) "$x + y \geq z$" is denoted by Z (x, y, z).

(iv) "x is the father of y" denoted by Z (x, y).

2.17.1 Universe of Discourse

- The values which the variables may assume constitute a collection is called Universe (domain) of discourse.
- We bind some value to the variable appearing in a predicate.
- Once, the variable is bound to some value then it becomes preposition.

Example

$Z(x) = x + 5 = 8$

- Here, Universe of discourse will be set of Integers. If we put x = 2, we get false proposition and other side, If we put , x = 3, then predicate will be true proposition.

2.18 QUANTIFIERS

- In predicate logic, quantifiers are used to express the extent to which a predicate is true over a range of elements.
- The proposition which was created using quantifiers is called quantification.
- There are two types of quantifiers.

1. Universal Quantifiers

- If P(x) is a predicate with x as an argument then the universal quantifier for P (x) is the statement which is interpreted as "For all values of x, the assertion P (x) is true".
- Here, x is said to be universally quantified.
- The phrase "for all" is denoted by $\forall$.
- $\forall$ which means "for all" or "for each" or "for every ".
- $\forall$ or "for all" is a universal quantifier.
- If P (x) is true for all values of x then $\forall x\ P(x)$ is true otherwise P (x) is false.

2. Existential Quantifiers

- Let us consider an example. "some roses are red" in which, some or at least one or there exist some roses are red not all.
- These type of predicates represented by existential quantifiers.

> **Definition :** *Let, P (x) is predicate, $\forall x\ P(x)$ is false, but there exists at least one value of x for which P (x) is true, here we can say that x is bound by existential Quantification.*

- The term "there exists an x" is called an existential quantifier.
- The symbol "$\exists$" is used to denote logical quantifier "there is at least one" or "there exists some".
- The existential quantifiers for P (x) is denoted by $\exists x \cdot P(x)$.

Example

"Some numbers are irrational" can be written as

A : x is irrational.

B : x is a number.

Then $(\exists x)\ (B(x) \wedge A(x))$

2.18.1 The Negation of Quantified Statement

- Consider the statement $\forall x.\ P(x)$. It's negation is "It is not the case that for all x, P (x) is true".
- This means that, for some x = c, P (a) is not true or $\exists$ x; such that ~ P (x) is true.
- Hence, the negation of $\forall x.\ P(x)$ is logically equivalent to $\exists x\ [\sim P(x)]$.

Example

(a) "All the professors of department present for the meeting".

- The negation of above statement will be
- "All the professors were not present for the meeting" or "some professor were not present for the meeting"
- It is denoted as $\exists x\ [\sim P(x)]$.
- The following table shows the statement and it's negation.

Statement	Negation
$\forall x\ P(x)$	$\exists x\ [\sim P(x)]$
$\exists x\ [\sim P(x)]$	$\forall x\ P(x)$
$\forall x\ [\sim P(x)]$	$\exists x\ P(x)$
$\exists x\ P(x)$	$\forall x [\sim P(x)]$

2.19 RULES OF INFERENCE FOR PREDICATE CALCULUS

During deriving the conclusion in predicate calculus, the rules of inference for statement calculus and additional rules for handling quantifiers are to be used.

(1) Rule 1 : Universal Instantiation

- The rule is to conclude that P(a) is true, where (a) is a particular member of the universe of discourse, given the premise $\forall x\ P(x)$.
- For example, If we want to conclude the statement that "Atul is brilliant" from the statement "All students are

brilliant" where Atul is a member of the universe of discourse of all students.

- The above example, represented in symbolic form as

$$\frac{\forall x\ P(x)}{\therefore P(a)}$$

- Here, (a) is some element of the universe.

(2) Rule 2 : Existential Instantiation

- This rule of inference, conclude that there exists an element (a) in the universe for which P (a) is true, if the premise $\exists x\ P(x)$ is true.
- Symbolically, it is represented as

$$\frac{\exists x\ P(x)}{\therefore P(a)}$$

Where, (a) is some element for which P (a) is true.

(3) Rule 3 : Universal Generalization

- This rule of inference is used to prove that $\forall x\ p(x)$ is true when P (a) is true for all the elements (a) belongs to universe.
- The selected element (a) must be an arbitrary element, not a specific element of the universe.
- It is represented as,

$$\frac{P(a)}{\therefore \forall x\ p(x)}$$

(4) Rule 4 : Existential Generalization

- This rule is used to conclude that, if one element (a) in the universe for which P (a) is true, then $\exists x\ P(x)$ is true.
- Symbolically, it is represented as,

$$\frac{P(a)}{\therefore \exists x\ P(x)}$$

Summary

(i) $\forall x\ P(x) \rightarrow P(a)$ (Universal Instantiation)

(ii) $\exists x\ P(x) \rightarrow P(a)$

For some element (a) (Existential Instantiation)

(iii) $P(a) \rightarrow \forall x\ P(x)$

For an arbitrary (a) (Universal generalization)

(iv) $P(a) \rightarrow \exists x\ P(x)$

For some element (a) (Existential generalization)

2.19.1 Examples Based on Predicates and Quantifiers

Ex. 2.19.1 : Over the universe of book defined propositions.

B(x): x has blue cover

M(x): x is maths book

I(x): x published in India

Translate the following

(i) $\forall x\ (M(x) \wedge I(x) \rightarrow B(x))$

(ii) **There are maths books published outside India.**

Soln. :

Let, B(x) : x has blue cover

M(x) : x is maths book

I(x) : x is published in India

(i) $\forall x\ (M(x) \wedge I(x) \rightarrow B(x))$

The maths book published in india have a blue colour.

(ii) There are maths books published outside India.

$(\exists x)\ (M(x) \wedge \sim I(x))$

Ex. 2.19.2 : Let, P(x) : x is even.

Q(x) : x is a prime number.

R(x, y) : x + y is even.

(a) Using the information given above write the following sentences in symbolic form.

(i) **Every integer is an odd integer.**

(ii) **Every integer is even or prime.**

(iii) The sum of any two integers is an odd integers.

Soln. :

(i) For every element (say x) can be represented by $\forall x$ and for odd integer, negation of P(x).

So, every integer is an odd integer can be represented as,

$\forall x\ [\sim P(x)]$

(ii) For, every integer $\Rightarrow \forall x$

For even $\Rightarrow P(x)$

For odd $\Rightarrow Q(x)$,

So, $\forall x\ [P(x) \vee Q(x)]$

(iii) The sum of any two integers is an odd integers.

$\forall x\ \forall y\ [\sim R(x, y)]$

Ex. 2.19.3 : Negate each of the statement :

(i) $\forall x, |x| = x$. (ii) $\exists x, x^2 = x$,

(iii) If there is a riot, then someone is killed.

(iii) It is day light and all the people are arisen.

☑ **Soln. :**

Let,

(i) $\forall x, |x| = x$

$\Rightarrow (\exists x)\ (|x| \neq x)$

(ii) $\exists x, x^2 = x$

$\Rightarrow \forall x\ (x^2 \neq x)$

(iii) If there is riot, then all are alive.

(iv) It is day light and some one is not arisen.

Ex. 2.19.4 : Write the following statements in symbolic form, using quantifiers.

(i) All students have taken a course in communication skills.

(ii) There is a girl student in the class who is also a sports person.

(iii) Some students are intelligent, but not hardworking.

☑ **Soln. :**

Let,

(i) P(x) : Student x has taken a course in communication skills.

The statement can be represented as,

$\boxed{\forall x\ P(x)}$

(ii) Let, P(x) : x is a student.

Q(x) : x is a girl.

R(x) : x is a sports person.

x will be student, x will be girl and x will be sports person.

Hence, the given statement can be represented as,

$\boxed{\exists x\ [P(x) \wedge Q(x) \wedge R(x)]}$

(iii) Let, p(x) : x is intelligent.

Q(x) : x is hardworking.

We know that, some x can be represented as $\exists x$

So, the given statement represented as

$\boxed{\exists x\ [\,p(x) \wedge \sim Q(x)\,]}$

Ex. 2.19.5 : Represent the arguments using quantifiers and find it's correctness. All the students in this class understand logic. Ganesh is a student in this class. Therefore Ganesh understands logic.

☑ **Soln. :**

Let, P(x) : x is a student in this class.

Q(x) : x understands logic.

In symbolic form,

$$\begin{array}{c} \forall x\ (P(x) \rightarrow Q(x)) \\ P(a) \\ \hline \therefore\ Q(a) \end{array}$$

Here, "a" holds Ganesh.

The above inference, is Modus Ponen.

That's why, this argument is valid.

Ex. 2.19.6 : Let, P(x) : x is even.

Q(x) : x is a prime number.

R(x, y) : x + y is even.

Write an English sentence for each of the symbolic statement given below :

(i) $\forall x\ [\sim Q(x)]$ (ii) $\exists y\ [\sim P(y)]$

(iii) $\sim [\exists x\ (P(x) \wedge Q(x))]$

☑ **Soln. :**

(i) All integers are not prime number.

(ii) At least one integer is not even.

(iii) It is not the case that there exists.

Ex. 2.19.7 : Rewrite the following statements using quantifiers variables and predicate symbols.

(i) All birds can fly.

(ii) Not all birds can fly.

(iii) Some men are genius.

(iv) Some numbers are not rational.

(v) There is a student who likes mathematics but not geography.

(vi) Each integer is either even or odd.

☑ **Soln. :**

(i) Let, B(x) : x is a bird.

F(x) : x can fly.

So, the given statement can be written as :

$\forall x\ [\,B(x) \rightarrow F(x)\,]$

(ii) $\sim [\,\forall x (B(x) \rightarrow F(x))\,]$ or $\exists x\ [\,B(x) \wedge \sim F(x)\,]$

(iii) Let, M(x) : x is a men.

G(x) : x is a genius.

The given statement interpreted as :

$\exists x\ [\,M(x) \wedge G(x)\,]$

(iv) Let, N(x) : x is a number.

R(x) : x is rational.

$\Rightarrow\ \sim [\,\forall x\ (N(x) \rightarrow R(x)\,]$ or $\exists x\ [\,N(x) \wedge \sim R(x)\,]$

(v) Let, S(x) : x is a student.

M(x) : x likes mathematics.

G(x) : x likes geography.

Then, $\exists x\ [\,S(x) \wedge M(x) \wedge \sim G(x)\,]$

(vi) Let, I(x) : x is an integer.

E(x) : x is even.

O(x) : x is odd.

Symbolic form

$\forall x\ [\,I(x) \rightarrow E(x) \vee O(x)\,]$

Ex. 2.19.8 : For the universe of all integers, let P(x), Q(x), R(x), S(x) and T(x) be the following statements :

P(x) : x > 0

Q(x) : x is even.

R(x) : x is a perfect square.

S(x) : x is divisible by 4.

T(x) : x is divisible by 5.

Write the following statements in symbolic form :

(i) At least one integer is even.

(ii) There exists a positive integer that is even.

(iii) If x is even, then x is not divisible by 5.

(iv) No even integer is divisible by 5.

(v) There exists an even integer divisible by 5.

(vi) If x is even and x is a perfect square, then x is divisible by 4.

☑ **Soln. :**

(i) $\exists x\ Q(x)$ (ii) $\exists x\ [\,P(x) \wedge Q(x)\,]$

(iii) $\forall x\ [\,Q(x) \rightarrow \sim T(x)\,]$ (iv) $\forall x\ [\,Q(x) \rightarrow \sim T(x)\,]$

(v) $\exists x\ [\,Q(x) \wedge T(x)\,]$ (vi) $\forall x\ [\,Q(x) \wedge R(x) \rightarrow S(x)\,]$

Ex. 2.19.9 : Transcribe the following into logical notation. Let the universe of discourse be the real numbers.

(i) For any value of x, x^2 is non-negative.

(ii) For every value of x, there is some value of y such that $x \cdot y = 1$.

(iii) There are positive values of x and y such that $x \cdot y > 0$

(iv) There is value of x such that if y is positive, then x + y is negative.

(v) For every value of x, there is some value of y such that $x - y = 1$.

☑ **Soln. :**

(i) $\forall x\ [\,x^2 \geq 0\,]$

(ii) $\forall x\ \exists y\ [\,x \cdot y = 1\,]$

(iii) $\exists x\ \exists y\ [\,(x > 0) \wedge (y > 0) \wedge (x \cdot y > 0)\,]$

(iv) $\exists x\ \forall y\ [(y > 0) \rightarrow (x + y < 0)]$

(v) $\forall x \cdot \exists y\ [\,x - y = 1\,]$

Chapter Ends ...

□□□

UNIT II

Combinatorics and Discrete Probability

>> **Syllabus :**

- **Combinatorics :** Rules of Sum and Product, Permutations, Combinations.
- **Discrete Probability :** Discrete Probability, Conditional Probability, Bayes Theorem, Information and Mutual Information, Applications of Combinatorics and Discrete Probability.

Mapping of Course Outcomes for Unit I	CO2

- Chapter 3 : Combinatorics

UNIT II

CHAPTER 3 Combinatorics

Syllabus

Combinatorics : Rules of Sum and Product, Permutations, Combinations.

Discrete Probability : Discrete Probability, Conditional Probability, Bayes Theorem, Information and Mutual Information, Applications of Combinatorics and Discrete Probability.

3.1 INTRODUCTION

- The chapter divided in two sections counting and probability.
- Counting is basic concept and we all know counting.
- Counting the object in finite set is basic need in order to compare, evaluate and predict the things.
- In computer science, counting is required to count the number of steps required to execute program, as well as to analyze the cost efficiency of algorithm.
- Counting required to determine the average and maximum lengths of [illegible] for items stored in particular data structure. The methods developed in the first section are applied to solve problems on probability, related to discrete sample space which are discussed in second section.

3.2 COMBINATIONAL PROBLEMS

- Problems involving counting are generally of the type where we have to find the number of ways to arrange some or all elements or to select some elements or their combinations.

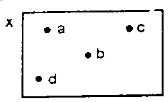


Fig. 3.2.1

- Count the elements from set X is given in Fig. 3.2.1 abcd, dcba, acdb, adcb and so on.

So from above example we understand that counting is not simple though it seems to be simple. Some more examples are :

(i) How many positive solutions of the equation $x + y + z = 100$?

(ii) How many ways to paint 4 bicycles using 4 colours so no two bicycles has same color ?

The problems stated above are interpreted as **"experiments"** or "tasks" and solutions of the problems are interpreted as **"outcomes.**

Considering experiments and their outcomes there are two rules of operations :

(1) Rule of sum : If E_1 and E_2 are two experiments having outcomes n_1 and n_2 respectively then if the two experiments E_1 and E_2 carried out separately one at a time then total number of outcomes are $(n_1 + n_2)$.

(2) Rule of product : If E_1 and E_2 are two experiments having outcomes n_1 and n_2 respectively and if the two experiments E_1 and E_2 carried out at the same time the total number of outcomes are $n_1 \cdot n_2$. e.g. If two dice rolled at same time then

Outcomes are $= 6 \times 6 = 36$ [Rule of product]

If two dice rolled exactly one at a time total outcomes $= 6 + 6 = 12$ [Rule of sum]

The problem of counting deal with two basic ideas of counting that is permutations and combinations.

3.3 PERMUTATIONS

- An arrangement of elements in a set considering the order of elements is called as permutations.
- There are three types of permutation :

Type I

Let $0 \le r \le n$. The number of ways to have an ordered sequence of n distinct elements taken 'r' at time is called as r permutations of 'n' elements and is denoted by $P(n, r)$ or ${}^nP_r$.

The first place is filled by n ways, second is filled by $(n - 1)$ ways, in this way r^{th} place is filled by $n - r + 1$ ways.

$\therefore$ ${}^nP_r$ is given as

${}^nP_r$ or $P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$

$$\boxed{P(n, r) = \frac{n!}{(n-r)!}} \quad \text{where } 0 \le r \le n.$$

3.3.1 Solved Examples - Type I

Ex. 3.3.1 : Find permutation of the set $A = \{1, 2, 3, 4\}$ taking two elements at time.

☑ **Soln.**

Here $n = 4$ as $4 = \{1, 2, 3, 4\}$ and $r = 2$.

$$\therefore {}^4P_2 = \frac{4!}{4-2!} = \frac{4!}{2!} = 3 \times 4$$

$= 12$ permutation

Ex. 3.3.2 : Four persons enter a bus in which there are 6 vacant seats. In how many ways can they take their places ?

☑ **Soln.**

The first person may seat himself in 6 ways then the second person can seat himself in 5 ways proceeding in this way, the total number of ways in which the 4 persons can seat themselves $P(6, 4) = 6 \times 5 \times 4 \times 3 = 360$ ways.

Ex. 3.3.3 : How many three digit number are formed using the digits 1, 2, 3, 4, 5, 6 ? How many of them are even ?

☑ **Soln. :**

Here $n = 1, 2, 3, 4, 5, 6 = 6$ and $r = 3$

$$\therefore {}^6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 4 \times 5 \times 6 = 120$$

$\therefore$ 120 three digit number are formed.

Among these 120 numbers the numbers ends with 2, 4, 6 are even.

So, last place is filled by 3 ways.

Total count of even number $= 5 \times 4 \times 3 = 60$ number.

$\therefore$ 60 even numbers are formed.

Ex. 3.3.4 : How many ways to arrange the letter of the word "COMPUTER" ?

☑ **Soln.**

Here $n = 8$ and $r = 8$

$${}^nP_r = {}^8P_8 = \frac{8!}{0!} = 8! \text{ ways are there.}$$

Ex. 3.3.5 : Six different mathematics books, four different discrete structures books and 3 different computer science books are to be arranged on a shelf. How many different arrangements are possible if

(i) Books in each subject must all together

(ii) Only the discrete structure must be together

☑ **Soln.**

Let Mathematics (M), Discrete Structure (DS) and Computer Science (CS)

(i) All books in each set must be all together therefore there three sets M, DS and CS are arranged by 3! ways.

M can be arranged themselves by 6! ways.

DS can be arranged themselves by 4! ways.

and CS can be arranged themselves by 3! ways.

Total number of ways $= 3!\, 6!\, 4!\, 3! = 103680$ ways.

(ii) 4 DS books can be arranged themselves by 4! ways.

Consider DS as single set there are total 10 books arranged by 10! Ways.

Therefore total arrangements $= 4!\ 10! = 87091200$ ways.

Ex. 3.3.6 : How many ways can 9 persons seated at a round table ?

☑ **Soln.**

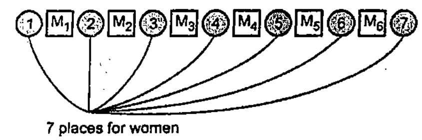
Since the persons are seated at round table they can be seat by $(9 - 1)! = 8!$ ways $= 40320$ ways.

UEx. 3.3.7 (SPPU - IT : Q. 7(b), May 15, 3 Marks)

In how many ways can 6 men and 5 women be seated in a line so that no two women sit together ?

☑ **Soln. :**

Consider 6 men as $M_1, M_2, M_3, M_4, M_5, M_6$



(1A1)**Fig. P.3.3.7 : 7 places for women**

6 mens can be arranged in themselves in 6! ways and 5 women can be seated by $7 \times 6 \times 5 \times 4 \times 3$ ways.

Total number of ways $= 6! \times 7 \times 6 \times 5 \times 4 \times 3 = 1814400$

UEx. 3.3.8 (SPPU - IT : Q. 7(a), May 16, 4 Marks)

How many auto license [illegible] alphabets followed by 4 digits if [illegible]

(i) Repetition of [illegible]

(ii) Repetition of both [illegible] allowed.

☑ **Soln. :**

License number plate made up of 3 alphabets followed by 4 digits.

There are 26 alphabets and 10 digits.

1. Total number of ways when alphabets and digits are repeated are given as,

Total number plates $= 26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10$

$= 26^3 \times 10^4$ ways $= 175,760,000$

2. If repetition of alphabets is not allowed then 3 alphabets can be selected by $26 \times 25 \times 24$ ways i.e. ${}^{26}P_3$ ways and if repetition not allowed 4 digits can be selected by ${}^{10}P_4$ i.e. $10 \times 9 \times 8 \times 7$ ways.

Total number of plates $= 26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$

$= 78,624,000$

UEx. 3.3.9 (SPPU - IT : Q. 7(b), May 16, 3 Marks)

In how many ways can five examinations be scheduled in a week so that no two examinations scheduled on same day considering Sunday as holiday ?

☑ **Soln. :**

Total number of days (n) = 6

Total number of examination (r) = 5

Number of arrangements $= {}^nP_r = {}^6P_5 = \frac{6!}{(6-1)!} = 6$

The five examination arranged themselves by 5! ways.

$\therefore$ Total number of ways of scheduling $= 6 \times 5!$

$= 6 \times 120 = 720$

Type II

The number of ways in which of the n elements can be arranged, where r_1 elements of one kind, r_2 elements are of another kind and so on till r_k elements are of another kind is given as $\frac{P(n, r)}{r_1!\, r_2! \ldots r_k!}$ where $r = r_1 + r_2 \ldots + r_k$ and $0 \le r \le n$.

3.3.2 Solved Examples - Type II

Ex. 3.3.10 : In how many ways the letters of the word "MISSISSIPPI" be arranged ?

☑ **Soln. :**

Here n = 11, r_1 = 4 (Letter I, 4 times repeated)

r_2 = 4 (Letter S, 4 times repeated)

r_3 = 2 (Letter P, 2 times repeated)

$\therefore$ Total arrangements $= \frac{P(n, r)}{4!\,4!\,2!} = \frac{11!}{4!\,4!\,2!}$

Ex. 3.3.11 : How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places.

☑ **Soln.**

The odd digits = 1, 3, 3, 1 can be arranged in their four places in $\frac{4!}{2!\,2!}$ ways.

The even digits 2, 4, 2 can be arranged in their three places in $\frac{3!}{2!}$ ways.

Hence total number of ways $\frac{4!}{2!\,2!} \cdot \frac{3!}{2!} = 6 \times 3 = 18$ ways.

Ex. 3.3.12 : In how many ways can 8 different books be divided among Sameer, Ajay and Leela if Sameer gets 4 books, Ajay and Leela gets 2 each.

☑ **Soln.**

One of the arrangements can be SALSSLSA.

Each such ordering determines distribution of books.

Hence, total number of ways in distributing the 8 books in the required manner is

$\frac{8!}{4!\,2!\,2!} = 420$ ways.

UEx. 3.3.13 (SPPU - IT : Q. 7(b), Dec. 14, 7 Marks)

Find number of arrangement that can be made out of letters

(i) ASSASSINATION (ii) GANESHPURI

☑ **Soln. :**

(i) ASSASSINATION

As in above word, total letters are 13 out of which 'A' repeated 3 times, 'S' repeated 4 times 'I' and 'N' repeated 2 times.

Therefore total number of arrangements $= \frac{13!}{3!\,4!\,2!\,2!}$

(ii) GANESHPURI

As total letters in this word are 10 and no letter is repeated.

Therefore total number of arrangements = 10!

Type III

The number of permutations of 'n' elements 'r' at a time when each element may repeated once, twice ..... upto 'r' times in any arrangement is given by n^r.

3.3.3 Solved Examples - Type III

Ex. 3.3.14 : A die rolled 3 times, find the number of faces that can appear on the top ?

☑ **Soln. :**

Every time, die is rolled the face appearing on the top can be any one of the six faces 1 to 6. So when the die is rolled 3 times.

Total number of faces appear on top $= 6^3 = 216$

Ex. 3.3.15 : How many auto license plates can be made if each is identified by 2 letters followed by 4 digits ?

☑ **Soln.**

The first two position can be occupied by $26 \times 26 = 26^2$ ways. The next four position occupied by $10 \times 10 \times 10 \times 10 = 10^4$.

Total number of auto license plates $= 26^2 \times 10^4$.

Ex. 3.3.16 : A bit is either 0 or 1. A byte is sequence of 8 bits. Find : (1) Number of bytes (2) Bytes begin with 11 and end with 11.

☑ **Soln.**

(i) Total number of bytes is $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ $= 2^8 = 256$

(ii) Since first 2 and last 2 are fixed, the remaining bits in sequence are either 0 or 1.

Hence total number of bytes $= 2 \times 2 \times 2 \times 2 = 2^4 = 16$

3.4 COMBINATIONS

Permutation is a counting method where order of elements is considered. In this section consider problem where order is not important. An arrangement of elements in a set without consideration of order is called as **Combinations.**

For $0 \le r \le n$, a selection of a set r elements from a set of n elements (distinct) is called as combinations and it is given as

$$^nC_r = C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

3.4.1 Solved Examples on Combinations

Ex. 3.4.1 : In how many ways we can select 6 balls out of 10 balls in the box ?

☑ **Soln. :**

6 balls can be selected out of 10 balls by $^{10}C_6$ ways

$= \frac{10!}{6!\,4!} = \frac{7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4} = 210$ ways

Ex. 3.4.2 : A committee of 10 peoples has to be selected from 8 boys and 6 girls, how many ways we can select committee members if.

(i) A committee consists of same number of boys and girls,

(ii) At least 3 girls are there.

☑ **Soln. :**

(i) Total number of peoples (n) = 8 + 6 = 14

A committee consists of same number of boys and girls i.e. 5 each.

Total number of ways $= {}^8C_5 \cdot {}^6C_5$

$= \frac{8!}{3!\,5!} \cdot \frac{6!}{1!\,5!} = 7 \times 8 \times 6 = 336$ ways. ...(I)

(ii) Atleast 3 girls means number of girls may be 3, 4, 5, 6

Therefore if we select

3 girls then 7 boys $= {}^6C_3 \cdot {}^8C_7$

4 girls and 6 boys $= {}^6C_4 \cdot {}^8C_6$

5 girls and 5 boys = 336 ... (from I)

6 girls and 4 boys $= {}^6C_6 \cdot {}^8C_4$

Total number of ways $= ({}^6C_3 \cdot {}^8C_7) + ({}^6C_4 \cdot {}^8C_6) + 336 + ({}^6C_6 \cdot {}^8C_4)$

$= 20 \times 8 + 15 \times 14 + 336 + 70$

$= 160 + 210 + 336 + 70$

$= 776$ ways

Ex. 3.4.3 : A fair coin is tossed 5 times. Find the number sequences in which head appears at the most 3 times.

☑ **Soln.**

As coin is tossed 5 times, we have sequence of 5 slots to be filled either by 'H' or 'T'. eg. {HHTTH, TTTHT}.

At the most 3 heads means 0, 1, 2, 3, heads.

Total number of ways $= {}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3$

$\therefore$ 26 ways

Ex. 3.4.4 : Mathematics students have to attempt six out of 10 questions in an examination in any order. How many choices have they ? How many choices do they have if they must answer at least three out of first 5 ?

☑ **Soln.**

6 question can be selected out of 10 by $^{10}C_6$ ways i.e. 210 ways.

When they have to answer at least 3 of first 5 and 3 out of second 5.

Total number of ways ${}^5C_3 \cdot {}^5C_3 + {}^5C_4 \cdot {}^5C_2 + {}^5C_5 \cdot {}^5C_1$

$= 100 + 50 + 5 = 155$ ways.

Ex. 3.4.5 : In how many ways we can distribute 10 apples among 4 children ?

☑ **Soln.**

Consider the apples as identical objects and let the children corresponds to distinct boxes. The problem can then be considered as finding the number of ways to distribute 10 identical objects in 4 distinct boxes. This can be done in $^{13}C_3 = 286$ ways.

Ex. 3.4.6 : In how many ways can 15 books be distributed among three students A, B, C so that A and B receive twice as many books as C ?

☑ **Soln.**

Let x, y, z denote the number of books of A, B, C respectively such that

$x + y + z = 15$ and $x + y = 2z$

Hence z = 5 which means that c receive 5 books. Hence 'c' can receive the number of books in one way only. Now since z = 5, we have x + y = 10. We have to find all non-negative integer solutions to this equation. The number of ways in which solution was possible is

$C(n + r - 1, n - 1) = C(2 + 10 - 1, 2 - 1) = C(11, 1) = 11$

Hence there are 11 ways in which distribution can be done.

UEx. 3.4.7 (SPPU - IT : Q. 8(b), Dec. 15, 6 Marks)

A student must answer 7 out of 10 questions in an examination.

(i) How many choices does the student have ?

(ii) How many choices does she have if she must answer the first three questions ?

(iii) How many choices does she have if she must answer at least three of the first five questions ?

☑ **Soln. :**

1. Total number of choices $= {}^{10}C_7 = \frac{10!}{3!\,7!} = \frac{8 \times 9 \times 10}{1 \times 2 \times 3}$

$= 120$ choices

2. First 3 questions are compulsory answer by ${}^{3}C_3$ ways and remaining 4 must be answered out of remaining 7 by ${}^{7}C_4$ ways.

Total number of choices $= {}^{3}C_3 \times {}^{7}C_4$

$= 3 \times \frac{7!}{3!\,4!} = 3 \times \frac{5 \times 6 \times 7}{1 \times 2 \times 3} = 105$

3. Students will have the following choices

First five	Next five
3	4
4	3
5	2

Required number of choices $= {}^{5}C_3 \times {}^{5}C_4 + {}^{5}C_4 \times {}^{5}C_3 + {}^{5}C_5 \times {}^{5}C_2$

$= \frac{5!}{2!\,3!} \times 5 + 5 \times \frac{5!}{2!\,3!} + 1 \times \frac{5!}{3!\,2!}$

$= 10 \times 5 + 5 \times 10 + 10 = 110$

UEx. 3.4.8 (SPPU - IT : Q. 7(c), May 15, 6 Marks)

What is the number of ways of choosing 4 cards from a pack of 52 playing cards ? In how many of these :

(i) Four cards are of the same suit.

(ii) Four cards belong to four different suits

(iii) Cards are of same color.

☑ **Soln. :**

4 cards can be selected out of 52 cards by

${}^{52}C_4$ i.e. $\frac{52!}{48!\,4!} = \frac{49 \times 50 \times 51 \times 52}{1 \times 2 \times 3 \times 4}$

$= 49 \times 25 \times 17 \times 13 = 270725$ ways

1. Four cards are from same suit can be drawn by ${}^{13}C_4$ ways. There are 4 suits.

$\therefore$ Total number of ways $= 4 \times {}^{13}C_4 = 4 \times \frac{13!}{9!\,4!}$

$= 4 \times \frac{10 \times 11 \times 12 \times 13}{1 \times 2 \times 3 \times 4}$

$= 4 \times 5 \times 11 \times 13 = 2860$

2. A card is drawn from a suit by ${}^{13}C_1$ ways. There are 4 suits.

Total number of ways $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$

$= 13 \times 13 \times 13 \times 13 = 28561$

3. There are 26 red cards and 26 black cards.

4 red cards can be drawn by ${}^{26}C_4$ ways. (Case-I)

4 black cards can be drawn by ${}^{26}C_4$ ways.(Case-II)

Total number of ways $= {}^{26}C_4 + {}^{26}C_4 = 2 \times {}^{26}C_4$

$= 2 \times \frac{23 \times 24 \times 25 \times 26}{1 \times 2 \times 3 \times 4}$

$= 2 \times 23 \times 25 \times 26 = 29900$ ways

UEx. 3.4.9 (SPPU - IT : Q. 8(b), May 15, 3 Marks)

In how many ways can three prizes be distributed among four winner so that no one gets more than one prize ?

☑ **Soln. :**

Prizes = 3

Winners = 4

Number of ways to distribute 3 prizes among 4 winners is ${}^{4}C_3$ i.e. 4 ways.

UEx. 3.4.10 (SPPU - IT : Q. 8(c), May 15, 6 Marks)

As 8 member team is to be formed from a group of 10 men and 15 women. In how many ways can the team be chosen if :

(i) The team must contain 4 men and 4 women.

(ii) There must be more men than women.

(iii) There must be at least two men.

☑ **Soln. :** There are 10 men and 15 women.

A team of 8 members is to be formed.

1. 4 men and 4 women can be selected as,

Number of ways $= {}^{10}C_4 \times {}^{15}C_4 = \frac{10!}{6!\,4!} \times \frac{15!}{11!\,4!}$

$= \frac{7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4} \times \frac{12 \times 13 \times 14 \times 15}{1 \times 2 \times 3 \times 4}$

$= 7 \times 3 \times 10 \times 13 \times 7 \times 15$

$= 210 \times 1365 = 286650$

2. The team can be formed with (8 men), (7 men, 1 women) (6 men, 2 women) or (5 men, 3 women)

Number ways $= {}^{10}C_8 + {}^{10}C_7 \times {}^{15}C_1 + {}^{10}C_6 \times {}^{15}C_2 + {}^{10}C_5 \times {}^{15}C_3$

$= \frac{10!}{2!\,8!} + \frac{10!}{3!\,7!} \times 15 + \frac{10!}{4!\,6!} \times \frac{15!}{13!\,2!} + \frac{10!}{5!\,5!} \times \frac{15!}{12!\,3!}$

$= \frac{9 \times 10}{1 \times 2} + \frac{8 \times 9 \times 10 \times 15}{1 \times 2 \times 3} + \frac{7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4} \times \frac{14 \times 15}{1 \times 2} + \frac{6 \times 7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4 \times 5} \times \frac{13 \times 14 \times 15}{1 \times 2 \times 3}$

$= 45 + 1800 + 22050 + 114660 = 138555$

3. Team contains at least 2 men

Case I : 2 men, 6 women $= {}^{10}C_2 \times {}^{15}C_6 = \frac{10!}{8!\,2!} \times \frac{15!}{9!\,6!}$

$= \frac{9 \times 10}{1 \times 2} \times \frac{10 \times 11 \times 12 \times 13 \times 14 \times 15}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$

$= \frac{45 \times 2 \times 11 \times 13 \times 14 \times 5}{4}$

$= 45 \times 11 \times 13 \times 7 \times 5 = 225225$

Case II : 3 men, 5 women $= {}^{10}C_3 \times {}^{15}C_5 = 120 \times 3003$

$= 360360$

Case III : 4 men, 4 women $= {}^{10}C_4 \times {}^{15}C_4 = 286650$

Case IV : 5 men, 3 women $= {}^{10}C_5 \times {}^{15}C_3 = 114660$

Case V : 6 men, 2 women $= {}^{10}C_6 \times {}^{15}C_2 = 22050$

Case VI : 7 men, 1 woman $= {}^{10}C_7 \times {}^{15}C_1 = 1800$

Case VII : 8 men $= {}^{10}C_8 = 45$

Total number of ways

$= 225225 + 360360 + 286650 + 114660 + 22050 + 1800 + 45 = 1010790$

UEx. 3.4.11 (SPPU - IT : Q. 8(a), May 16, 6 Marks)

Out of 15 employees in a software company, a group of 5 employees is to be sent for 'Linux Administration and Networking' training of one month.

(i) In how many ways can the 5 employees be selected ?

(ii) What if there are 2 employees who refuse to go together for training ?

(iii) What if there are 2 employees who want to go together i.e. either they go together or they do not go for training ?

☑ **Soln. :**

Total number of employees = 15

1. 5 employees can be selected out of 15 by ${}^{15}C_5$ by ways

$\therefore$ Total number of ways $= {}^{15}C_5 = \frac{15!}{10!\,5!}$

$= \frac{11 \times 12 \times 13 \times 14 \times 15}{1 \times 2 \times 3 \times 4 \times 5}$

$= 11 \times 13 \times 7 \times 3$

$= 3003$ ways

UEx. 3.4.12 (SPPU - IT : Q. 8(b), May 16, 6 Marks)

A committee of 5 members is to be formed from a group of 7 men and 6 women. What is the probability that

(i) At least 3 women are part of the committee

(ii) All committee members are either men or women

☑ **Soln. :**

There are 7 men and 6 women and out of 13 committee of 5 to be selected

So sample space ${}^{13}C_5$ i.e. $\frac{13!}{8!\,5!} = \frac{9 \times 10 \times 11 \times 12 \times 13}{1 \times 2 \times 3 \times 4 \times 5}$

$= 9 \times 11 \times 13 = 1287$

1. When at least 3 women are part of committee it mean 3, 4 or 5 women are the part of committee and it can be selected as,

$= {}^{6}C_3 \cdot {}^{7}C_2 + {}^{6}C_4 \cdot {}^{7}C_1 + {}^{6}C_5 \cdot {}^{7}C_0$

$= \frac{6!}{3!\,3!} \cdot \frac{7!}{5!\,2!} + \frac{6!}{2!\,4!} \cdot 7 + \frac{6!}{5!}$

$= \frac{4 \times 5 \times 6}{1 \times 2 \times 3} \times \frac{6 \times 7}{1 \times 2} + \frac{5 \times 6 \times 7}{1 \times 2} + 6$

$= 20 \times 21 + 105 + 6 = 420 + 105 + 6 = 531$

Probability $= \frac{531}{1287} = \frac{\text{Number of favorable events}}{\text{Sample space}}$

$= 0.4125$

2. Committee members either men or women.

Probability$= \frac{{}^{7}C_5 + {}^{6}C_5}{1287} = \frac{21 + 6}{1287} = \frac{27}{1287} = \mathbf{0.0209}$

3. If two employees refuse to go together then two cases are there.

Let A and B are two employees refuse to go together.

Case I : Let A will go and B will not go.

Total number of ways $= {}^{13}C_4 \cdot {}^{1}C_1$

$= \frac{10 \times 11 \times 12 \times 13}{1 \times 2 \times 3 \times 4} = 5 \times 11 \times 13$

$= 715$ ways.

Case II : Let B will go and A will not go

Then, total number of ways = 715 ways

So, total number of ways when 2 persons refuse to go together $= 715 + 715$

$= 1430$ ways

4. When 2 employees want to go together or not to go

Case I : When 2 persons go together

Number of ways $= {}^{13}C_3 \cdot {}^{2}C_2 = \frac{12 \times 13}{1 \times 2 \times 3} = 26$ ways.

Case II : When 2 persons not go

Number of ways $= {}^{13}C_5 = \frac{9 \times 10 \times 11 \times 12 \times 13}{1 \times 2 \times 3 \times 4 \times 5}$

$= 9 \times 11 \times 13 = 1287$ ways.

$\therefore$ Total number of ways $= 26 + 1287 = 1313$ ways.

UEx. 3.4.13 (SPPU - IT : Q. 8(b), Dec. 14, 6 Marks)

12 persons are made to sit around a table. Find the number of ways they can sit such that 2 specific persons are not together.

☑ **Soln. :**

There are total 12 persons.

Consider these 12 persons seated in a circle.

Therefore total number of ways $= (12-1)! = 11!$ ...(I)

Consider 2 persons as single entity then there are 11 persons altogether and they can be seated in $(11-1)! = 10!$

Thus the two specific persons can be arranged in themselves by 2! ways.

Therefore the number of ways 12 persons seated with 2 specific persons seated together are

$= 2!\ 10!$

From I, the total number of ways in which 12 persons can be seated in a round table so that 2 specific persons can not sit together is given by,

$= 11! - 2!\ 10!$

$= \mathbf{11! - 2 \times 10!}$

UEx. 3.4.14 (SPPU - IT : Q. 1(a), Dec. 18, 6 Marks)

Suppose license plate contains 2 English letters followed by 4 digits.

(i) How many different license plates can be manufactured if repetition of letters and digits are allowed ?

(ii) How many plates are possible if only the letters are repeated ?

☑ **Soln. :**

There are 26 letters and 10 digit out of which we can select the letters and digits for license plates.

(i) If repetition of letters and digits is allowed then every letter can be selected by 26 ways and every digit can be selected by 10 ways.

26 x 26 x 10 x 10 x 10 x 10

Total number of license plates

$= 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^2 \times 10^4$

(ii) If only letter repeated then we can select every letters by 26 ways and digits are not repeated hence we can select digits by, ${}^{10}P_4$ ways i.e. $7 \times 8 \times 9 \times 10$ ways.

Total number of license plates $= 26 \times 26 \times 10 \times 9 \times 8 \times 7$

UEx. 3.4.15 (SPPU - IT : Q. 2(a), Dec. 18, 6 Marks)

Out of 5 males and 6 females, a committee of 5 is to form. Find the number of ways in which it can be formed so that among the person chosen in the committee there are :

(1) Exactly a male and 2 female

(2) At least 2 male and one female.

☑ **Soln. :** There are 5 males and 6 females out of which committee of 5 is to form.

(1) Exactly a male and 2 female.

A male can be selected out of 5 males by ${}^5C_1$ i.e. 5 ways.

2 Females can be selected out of 6 females by ${}^6C_2$ i.e. $\frac{6!}{4!\ 2!}$ i.e. 15 ways.

And remaining 2 people can be selected out of 4 males and 4 females by ${}^8C_2$ ways

i.e. $\frac{8!}{6!\ 2!} = 28$ ways

Therefore total number of ways $= 5 \times 15 \times 28 = 75 \times 28$
$= 2100$ ways

(2) At least 2 males means there can be 2, 3 or 4 males of 2 males are there, then 3 females are there.

Case I : 2 males and 3 females can be selected by ${}^5C_2 \times {}^6C_3$

$= \frac{5!}{3!\ 2!} \times \frac{6!}{3!\ 3!} = 10 \times 40 = 400$ ways

Case II : 3 males and 2 female can be selected

${}^5C_3 \times {}^6C_2 = \frac{5!}{2!\ 3!} \times \frac{6!}{4!\ 2!} = 10 \times 15$
$= 150$ ways

Case III : 4 males and one female can be selected

${}^5C_4 \times {}^6C_1 = 5 \times 6 = 30$ ways

Therefore total number of ways = Case I + Case II
+ Case III
$= 400 + 150 + 30$
$= 580$ ways

UEx. 3.4.16 (SPPU - IT : Q. 2(a), Dec. 19, 6 Marks)

How many 3-digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9, which are divisible by 5 and none of the digits is repeated ?

☑ **Soln. :** Total number of digits given = 2, 3, 5, 6, 7, 9

Consider

Hun	Ten	Unit

Units placed can be filled by 6 ways i.e. ${}^6C_1$

Tens place can be filled by ${}^5C_1$ ways and

Hundreds place can be filled by 4 ways i.e. ${}^4C_1$

Total 3 digit number formed $= 6 \times 5 \times 4$ i.e. 120. out of these 120, 3 digits number formed the numbers divisible by 5 are, the numbers which ends with digit 5.

Again consider

Hun	Ten	5

So, last place can be rolled by fixed digit 5 i.e. one way and Ten's place filled by remaining 5 digits by ${}^5C_1$ and hundreds place can be filled by ${}^4C_1$ ways.

Total number divisible by $5 = {}^5C_1 \times {}^4C_1 \times 1 = 5 \times 4 \times 1$

= 20 numbers

3.5 GENERATIONS OF PERMUTATIONS AND COMBINATIONS

How to generate n! permutations ? An important but interesting problem in combinatorics is to find a systematic procedure for generating all the n! permutations of a set of n distinct elements, with no omissions or repetitions. In order to ensure that all the n! permutations are indeed generated or that there is no repetition, some kind of hierarchy or ordering needs to be introduced on permutations. One natural way to do so is to adopt the lexicographic order, which are shall now discuss :

Let $\{a_1, a_2, \ldots, a_n\}$ be a set of n distinct positive integer to be permuted. For too permutations $< a_1, a_2 \ldots a_n >$ and $< b_1, b_2 \ldots b_n >$, we shall say $< a_1, a_2 \ldots a_n >$ comes before $< b_1, b_2 \ldots b_n >$ in the lexicographic order if,

(i) for some m, $1 \le m < n$

$a_1 = b_1, a_2 = b_2 \ldots a_{m-1} = b_{m-1}$

(ii) $a_n < b_m$

For example, $< 1\ 3\ 4\ 2\ 5 >$ comes before $< 1\ 3\ 5\ 4\ 2 >$ where m = 3

On the other hand, the permutation $< 2\ 3\ 6\ 4\ 5 >$ comes after $< 2\ 3\ 5\ 4\ 6 >$. Under this ordering a permutation $< b_1, b_2 \ldots b_n >$ will immediately succeed $< a_1, a_2 \ldots a_n >$ if

(i) $a_i = b_i$, for $1 \le i \le m-1$ and $a_m < b_m$ for largest m.

(ii) b_m is the smallest element from

$\{ a_{m+1}, a_{m+2}, \ldots a_n \}$ that is larger than a_m.

(iii) $b_{m+1} < b_{m+2} < \ldots < b_m$

Consider the example, the permutation $< 1, 2, 3, 4, 5 >$ by condition (i) and (ii), m is the largest possible value for which $a_m < a_{m+1}$. In this case, obliviously m = 4, so that the permutation immediately following $< 1\ 2\ 3\ 4\ 5 >$ is $< 1\ 2\ 3\ 5\ 4 >$, m = 3. Hence by condition (ii) $b_3 = 4$ and by condition (iii) $b_4 = 3$ and $b_5 = 5$. Hence $< 1\ 2\ 4\ 3\ 5 >$ is the permutation, which comes immediately after $< 1\ 2\ 3\ 5\ 4 >$.

In this manner, all the 5! Permutations of $< 1\ 2\ 3\ 4\ 5 >$ will be generated and order in which they appear will be as follows :

$< 1\ 2\ 4\ 3\ 5 > \rightarrow < 1\ 2\ 3\ 5\ 4 > \rightarrow < 1\ 2\ 4\ 3\ 5 > \rightarrow < 1\ 2\ 4\ 5\ 3 > \rightarrow < 1\ 2\ 5\ 3\ 4 > \rightarrow < 1\ 2\ 5\ 4\ 3 > \rightarrow < 1\ 3\ 2\ 4\ 5 > \rightarrow < 1\ 3\ 2\ 5\ 4 > \rightarrow$

The procedure follow above can be succinctly stated in following steps :

1. Examine the permutation $< a_1, a_2 \ldots a_n >$ element by element from right to left and let a_m be the right most element such that $a_m < a_{m+1}$.
2. Next examine the permutation again, for the right most element such that $a_m < a_k$.
3. Interchange a_m and a_k.
4. Interchange a_{m+1} and a_n, a_{m+2} and a_{n-1}, a_{m+3} and a_{n-n} and so on.

For example, consider $< 1\ 2\ 5\ 4\ 3 >$. Here m = 2 and K = 5. Hence interchange a_2 and a_5 i.e. 2 and 5. Hence $< 1\ 2\ 5\ 4\ 3 >$ becomes $< 1\ 3\ 5\ 4\ 2 >$. Next interchange a_3 and a_5. Hence we obtain $< 1\ 3\ 2\ 4\ 5 >$ which is the immediate successor to $< 1\ 2\ 5\ 4\ 3 >$.

3.5.1 Procedure to Generate Subsets of { 1, 2, 3, ... n}

Let $\{ a_1, a_2 \ldots a_k \}$ be a subset of size k of $\{ 1, 2, \ldots n\}$ with $a_1 < a_2 < \ldots < a_k$.

Then the maximum possible value of a_{k-1} is $n - 1$ and so on. In general the maximum possible value of a_i is $n - k + i$.

Consider the subset $\{ 1, 2, \ldots k-1, k \}$. If $k \ne n$, it's maximum possible value, increase k by 1, so that the next subset $\{ 1, 2, \ldots k-1, k \}$ is generated. We continue this till the subset $\{ 1, 2, \ldots k-1, n \}$ is reached. Next repeat the procedure for $k - 1$. If $k - 1$ is not equal to it's maximum value $n - 1$, increase it by 1 and let K take all large values. Continue this procedure with $k - 1$, till $n - 1$ is reached. Then move to $k - 2$ and repeat the steps.

In this manner, moving from right to left. We finally reach the a_n, element a_j such that a_j can be increased to a_{j+1}, but no a_i with $i > j$ can be increased, which means that at some stage, a_i is equal to it's maximum value $n - k + i$.

The procedure terminates when a_1 reaches to maximum value.

Ex. 3.5.1 : Generate all the subsets of size 4 of {1,2,3,4,5,6}

☑ **Soln.**

Begin with the subset {1, 2, 3, 4}. Now for any subset { a_1, a_2, a_3, a_4 } with $a_1 < a_2 < a_3 < a_4$ maximum possible value of a_4 is 6, of a_3 is 5, of a_2 is 4 and of a_1 is 3. Hence increasing by 1 we get {1, 2, 3, 5} still a_4 didn't get maximum value, so increasing by 1 we get <1, 2, 3, 6 > same procedure we apply for a_3 we get {1, 2, 4, 5}, {1, 2, 4, 6} and {1, 2, 5, 6}. In this way if we continue we get all the 15 subsets as {1, 2, 3, 4}, {1, 2, 3, 5} {1, 2, 3, 6} {1, 2, 4, 5} {1, 2, 4, 6}, {1, 2, 5, 6}, {1, 3, 4, 5}, {1, 3, 4, 6}, {1, 3, 5, 6}, {1, 4, 5, 6}, {2, 3, 4, 5}, {2, 3, 4, 6}, {2, 3, 5, 6}, {2, 4, 5, 6} and {3, 4, 5, 6}

3.6 PROBABILITY THEORY

3.6.1 Discrete Probability

In this section we deal with probability that a particular event shall occur. Probability is numerical measure of the possibility that a certain event (mostly desirable) will take place. In other words the question of finding probability often arises in situations referred to as 'random experiments' or "trials" e.g. tossing coin, rolling a dice etc. We first define various terms of probability theory :

1. **Sample space :** Sample space is the set of all possible outcomes of experiments. It is generally denoted by "S". An outcome is also called as "Sample point".

   A sample space having finite number or countably infinite number of sample points is called as **Discrete Sample Space**.

   e.g. If we toss a coin

   Sample space (S) = 2 i.e. {T, H}

   If two coins tossed then

   Sample space (S) = 4 i.e. {TT, TH, HT, HH}

2. **Event :** A set of possible outcomes is called as an event. An event is a subset of sample space.

   e.g. In experiment of tossing pair of coins.

   A = {HH} is a event where both coins shows head on the top.

   A sample space itself is a event and called as certain event. When there is no outcome of the event then it is called as impossible event.

3. **Complement of event :** If A is an event, its set complement $\bar{A} = S - A$ is the complement event.

   If event A occurs then $\bar{A}$ will not occurs and vice versa.

   e.g. tossing a coin

   if A = {T} then $\bar{A}$ = {H}

4. **Compound event :** If A and B are two events then compound event is $A \cup B$ which is described as at least one of the events A or B occurs.

   e.g. A = {TT, HT}

   B = {HH}

   then compound event $(A \cup B)$ = {TT, HT, HH}

5. **Product event :** If A and B are events, the product event is $A \cap B$ and is described as both events A and B occur.

   e.g. A = {TT, HT}

   B = {HT, HH}

   Product event $(A \cap B)$ = {HT}

6. **Mutually exclusive events :** Events A and B are said to be mutually exclusive if the occurrence or non-occurrence of A precludes the occurrence or non-occurrence of B, that is if A occurs then B does not occurs and vice versa. i.e. $A \cap B = \phi$. In tossing coin if A = {T} and B = {H} then they are mutually exclusive.

7. **Independent events :** A and B are said to be independent events if the occurrence or non-occurrence of one does not affect the occurrence or non-occurrence of the other.

   e.g. consider the experiment of tossing a coin twice. The result of first toss does not affect the result of second loss. These two events are independent.

UEx. 3.6.1 (SPPU - IT : Q. 8(a), May 15, 4 Marks)

A basket contains 30 apples, 20 pears and 10 peaches. What is the probability that the first piece of fruit taken from the basket will be a peach.

☑ **Soln. :**

Total number of outcomes i.e. sample space

$$= 30 + 20 + 10 = 60$$

Number of favourable events = 10

$$\text{Probability} = \frac{\text{Number of favourable event}}{\text{Sample space}}$$

$$= \frac{10}{60} = \frac{1}{6}$$

UEx. 3.6.2 (SPPU - IT : Q. 7(c), May 16, 6 Marks)

Two cards are drawn at random from an ordinary deck of well shuffled 52 cards. Find the probability that

(i) First cast drawn is ace and second card drawn in face card of drawn in face card of spade.

(ii) Both are spades.

☑ **Soln. :**

$$\text{Probability (P)} = \frac{\text{Number of favorable events}}{\text{Sample space}}$$

Two cards are drawn at random out of 52 cards

$\therefore$ Sample space $= {}^{52}C_2 = 26 \times 51 = 1326$

1. An ace card drawn by ${}^4C_1$ ways i.e. 4 ways and one face card of spade drawn by ${}^3C_1$ i.e. 3 ways

$\therefore$ Number of favourable events $= 4 \times 3 = 12$

$$\text{Probability} = \frac{12}{1326} = \frac{2}{221}$$

2. Two cards of spade can be drawn by ${}^{13}C_2$ i.e. 78 ways

$$\text{Probability} = \frac{78}{1326} = \frac{13}{221} = \frac{1}{17}$$

UEx. 3.6.3 (SPPU - IT : Q. 7(a), Dec. 14, 6 Marks)

A single card is drawn from an ordinary deck S of 52 cards. Find the probability p that

(i) The card is a king.

(ii) The card is a face card (jack, queen or king).

(iii) The card is a heart.

(iv) The card is a face.

☑ **Soln. :**

$$\text{Probability (P)} = \frac{\text{Number of favorable events}}{\text{Sample space}}$$

As a single card is to be drawn the sample space will be ${}^{52}C_1$, i.e. 52.

(i) A king card can be drawn out 4 kings by ${}^4C_1$ ways i.e. 4 ways.

$$\text{Hence, Probability (P)} = \frac{4}{52} = \frac{1}{13}$$

(ii) A face card can be drawn from 12 face cards by ${}^{12}C_1$ i.e. 12 ways,

$$\text{Hence, probability (P)} = \frac{12}{52} = \frac{3}{13}$$

(iii) A heart card can be drawn out of 13 heart cards by ${}^{13}C_1$ ways i.e. 13 ways

$$\text{Probability (P)} = \frac{13}{52} = \frac{1}{4}$$

(iv) A card is a face (same as above (ii)).

UEx. 3.6.4 (SPPU - IT : Q. 1(B), May 17, 3 Marks)

Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?

☑ **Soln. :**

Tickets are numbered 1 to 20.

Let A be the event that ticket drawn is multiple of 3.

$$P(A) = \frac{\text{Numbers multiple of 3 in 1 to 20}}{20}$$

$$= \frac{6}{20} = \frac{3}{10} \quad \ldots(I)$$

Let B be the event that ticket drawn is multiple of 5.

$$P(B) = \frac{\text{Numbers multiple of 5 in 1 to 20}}{20} = \frac{4}{20}$$

$$= \frac{1}{5} \quad \ldots(II)$$

From Equations (I) and (II)

Probability that ticket drawn is multiple of 3 or 5 is $P(A \cup B)$ is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{10} + \frac{1}{5} - \frac{1}{20} = \frac{5}{10} - \frac{1}{20}$$

$$P(A \cup B) = \frac{9}{20}$$

UEx. 3.6.5 (SPPU - IT : Q. 2(B), May 17, 4 Marks)

Two cards are drawn together from [illegible] Determine the probability that one [illegible] heart.

☑ **Soln. :**

$$\text{Probability (P)} = \frac{\text{Number of favorable events}}{\text{Sample space}}$$

Two cards are drawn from 52 cards by ${}^{52}C_2$ i.e. $51 \times 26 = 1326$ ways

i.e. Sample space = 1356

One spade card is drawn by ${}^{13}C_1$ i.e. 13 ways.

One heart card is drawn by ${}^{13}C_1$ i.e. 13 ways.

Therefore

$$\text{Probability} = \frac{13 \times 13}{1326} = \frac{169}{1326}$$

UEx. 3.6.6 (SPPU - IT : Q. 2(C), May 17, 2 Marks)

Three unbiased coins are tossed. What is the probability of getting at most two heads?

☑ **Soln. :**

Three coins are tossed.

$$\text{Sample space} = \left\{ \begin{matrix} \text{HHH, TTT, TTH, THH} \\ \text{HTT, HHT, HTH, THT} \end{matrix} \right\} = 8$$

$$\begin{pmatrix} \text{Favourable event} \\ \text{at most 2 head} \end{pmatrix} = \left\{ \begin{matrix} \text{TTT, TTH, THH, HTT} \\ \text{HHT, HTH, THT} \end{matrix} \right\} = 7$$

$$\text{Probability (P)} = \frac{7}{8}$$

UEx. 3.6.7 (SPPU - IT : Q. 1(a), May 19, 6 Marks)

A single card is drawn from an ordinary deck of 52 cards. Find the probability p that :

(i) the card is a face card

(ii) the card is face card and heart

(iii) the card is face card or heart.

☑ **Soln. :**

$$\text{Probability (P)} = \frac{\text{Number of favorable events}}{\text{Sample space}}$$

For all the three events given the sample space is ${}^{52}C_1$ i.e. 52.

1. One face card can be drawn out of 12 face cards by ${}^{12}C_1$ i.e. 12 ways.

   Hence, probability $= \frac{12}{52} = \frac{3}{13}$

2. The card to be drawn is face card and heart and it can be drawn by ${}^{3}C_1$ i.e. 3 ways.

   Hence, probability $= \frac{3}{52}$

3. The card is face card or heart.

   A face card can be drawn by ${}^{12}C_1$ i.e. 12 ways and heart card can be drawn ${}^{10}C_1$ i.e. 10 ways.

   $$\left(\begin{array}{c}\text{Total number of ways that}\\ \text{card is face card or heart}\end{array}\right) = 12 + 10 = 22$$

   Hence, probability $= \frac{22}{52} = \frac{11}{26}$

UEx. 3.6.8 (SPPU - IT : Q. 1(a), Dec. 19, 6 Marks)

A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue ?

☑ **Soln. :**

$$\text{Probability (P)} = \frac{\text{Number of favorable events}}{\text{Sample space}}$$

Sample space i.e. total number of ways to select 2 balls out of total 16 balls is

$${}^{16}C_2 = \frac{16!}{14!\,2!} = \frac{15 \times 16}{1 \times 2} = 120 \text{ ways}$$

Now, one white ball can be drawn out of 6 white balls by ${}^{6}C_1$ ways i.e. 6 ways and one blue ball can be drawn out of 7 blue balls by ${}^{7}C_1$ ways i.e. 7 ways.

Total number of favourable events $= 6 \times 7 = 42$

Probability (P) $= \frac{42}{120} = \frac{7}{20}$

UEx. 3.6.9 (SPPU - IT : Q. 1(b), Dec. 19, 6 Marks)

Three cards are drawn from a well shuffled pack of 52 cards. Find the probability that they are a king a queen and a jack.

☑ **Soln. :**

$$\text{Probability (P)} = \frac{\text{Number of favorable events}}{\text{Sample space}}$$

Sample space i.e. total number of ways to draw three cards from 52 cards is ${}^{52}C_3$ ways i.e. $\frac{51!}{(52-3)!\,3!}$

$$= \frac{50 \times 51 \times 52}{1 \times 2 \times 3} = 25 \times 17 \times 52 \text{ ways}$$

Now, one king card can be drawn out of 4 king cards by ${}^{4}C_1$ i.e. 4 ways. Similarly one queen card can be drawn out of 4 queen cards by 4 ways and a jack card also drawn by 4 ways.

Three cards drawn (King, Queen and Jack) by $4 \times 4 \times 4$ i.e. 64 ways

$$\text{Probability} = \frac{64}{25 \times 17 \times 52} = \frac{16}{25 \times 17 \times 13}$$

UEx. 3.6.10 (SPPU - IT : Q. 1(b), Dec. 16, 6 Marks)

A bag contains 3 red and 5 black balls and second bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that : (i) both are red (ii) both are black (iii) 1 is red and 1 is black.

☑ **Soln. :**

Bag contains 3 red and 5 black balls and second bag contains 6 red and 4 black balls.

Let E_1 = Event of drawing red ball from first bag

E_2 = Event of drawing black ball from first bag

E_3 = Event of drawing red ball from second bag

E_4 = Event of drawing black ball from second bag

$$\therefore\ P(E_1) = \frac{3}{8},\ P(E_2) = \frac{5}{8},\ P(E_3) = \frac{6}{10},\ P(E_4) = \frac{4}{10}$$

1. Both are red

$$P(E_1 \cap E_3) = P(E_1) \cdot P(E_3) = \frac{3}{8} \times \frac{6}{10} = \frac{18}{80} = \frac{9}{40}$$

2. Both are black

$$P(E_2 \cap E_4) = P(E_2) \cdot P(E_4) = \frac{5}{8} \cdot \frac{4}{10} = \frac{20}{80} = \frac{1}{4}$$

3. One is red and one is black

$$P(E_1) \cdot P(E_4) + P(E_2) \cdot P(E_3) = \frac{3}{8} \cdot \frac{4}{10} + \frac{5}{8} \cdot \frac{6}{10} = \frac{12}{80} + \frac{30}{80} = \frac{42}{80} = \frac{21}{40}$$

3.6.2 Probability of an Event

Let A be the any event and S is it's sample space then probability (P) of event A is given as

$$P(A) = \frac{|A|\ (\text{number of sample points in A})\ [\text{Favourable event}]}{\text{Sample space}}$$

e.g. If we toss a coin and let A = {T} and B = {H} then

$P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$

Let x_i be the sample point of S then

(i) for each $x_i \in S$, $0 \le P(x_i) \le 1$

(ii) $\sum P(x_i) = 1$

Probability of sample points is also called Elementary probability.

Ex. 3.6.11 : Consider experiment of tossing a coin three times. What is the probability of getting exactly one head.

☑ **Soln. :**

Sample space $= 8 = 2^3$

Let A be the event that getting exactly one head

$$A = \{HTT, TTH, THT\} = 3$$

$$P(A) = \frac{|A|}{|S|} = \frac{3}{8}$$

Ex. 3.6.12 : If 2 cards are drawn without replacement from a pack, what is the probability that one of them is ace of heart ?

☑ **Soln.**

Let A be the event that card drawn is ace of heart sample space is 52×51. As card is drawn without replacement. For event A, since either the first card is the ace of heart or second is ace of heart therefore it's cardinality is 51×51.

$$P(A) = \frac{|A|}{|S|} = \frac{51 + 51}{52 \times 51} = \frac{2 \times 51}{52 \times 51} = \frac{2}{52}$$

$$P(A) = \frac{1}{26}$$

Ex. 3.6.13 : A student's committee consists of 2 members out of which 1 boy and 1 girl. There are 8 boys and 6 girls. Find the probability.

☑ **Soln.**

Let A be event having 1 boy and 1 girl

$|A| = 8 \times 6 = 48$

$$\text{Sample space } {}^{14}C_2 = \frac{14!}{12!\,2!} = \frac{13 \times 14}{2} = 91$$

$$\boxed{P(A) = \frac{48}{91}}$$

3.6.3 Addition Law

Let A and B be events in a finite (or countably infinite) sample space then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Let A be an event and $\bar{A}$ denotes it's complement then

$$P(\bar{A}) = 1 - P(A)$$

3.6.4 Independent Events (in terms of Probability)

Two events are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$

Ex. 3.6.14 : Two dice are rolled together. Event A denotes that sum of the numbers on top faces is even and event B denotes that there is a 4 on at least one of the top faces. Find $P(A \cup B)$ and $P(A \cap B)$.

☑ **Soln. :** Sample space $= 36 = 6 \times 6$

Let A = { (1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6) }

$|A| = 18$

$$\therefore P(A) = \frac{18}{36} = \frac{1}{2}$$

Now B = { (1, 4), (2, 4), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 4), (6, 4) }

$$P(B) = \frac{11}{36}$$

$$A \cap B = \{ (2, 4), (4, 2), (4, 4), (4, 6), (6, 4) \}$$

$$P(A \cap B) = \frac{5}{36}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{11}{36} - \frac{5}{36}$$

$$= \frac{18 + 11 - 5}{36}$$

$$\therefore P(A \cup B) = \frac{24}{36} = \frac{2}{3}$$

UEx. 3.6.15 (SPPU - IT : Q. 7(a), Dec. 15, 3 Marks)

If 2 cards are drawn from a usual deck of shuffled drawn pack of 52 cards what is the probability that 2 aces are drawn ?

☑ **Soln. :**

$$\text{Probability (P)} = \frac{\text{Number of favorable events}}{\text{Sample space}}$$

Two cards are drawn from deck of 52 cards by ${}^{52}C_2$ i.e. $\frac{52!}{50!\,2!}$ ways

$\therefore$ Sample space $= \dfrac{52!}{50!\,2!} = \dfrac{51 \times 52}{1 \times 2} = 51 \times 26$

$= 1326$

Number of favorable events $= {}^4C_2 = \dfrac{12}{2} = 6$

Probability $= \dfrac{6}{1326} = \dfrac{1}{221}$

UEx. 3.6.16 (SPPU - IT : Q. 7(b), Dec. 15, 4 Marks)

Two fair dice are rolled, what is the probability that :

(i) Sum of the faces is a perfect square

(ii) Sum of the faces is neither 5, 6 or 7.

☑ **Soln. :**

Two dice rolled

$\therefore$ Sample space $= 6 \times 6 = 36$

Number of favourable event when sum is perfect square

$= \{(2, 2)\ (1, 3)\ (3, 1)\ (3, 6)\ (6, 3)\}$

$= 5$

Probability $= \dfrac{5}{36}$

2. Number of favourable events when sum is neither 5, 6 or 7

$= \{(1, 1)\ (1, 2)\ (1, 3)\ (2, 1)\ (2, 2)\ (3, 1)\}$

$= 6$

Probability $= \dfrac{6}{36} = \dfrac{1}{6}$

UEx. 3.6.17 (SPPU - IT : Q. 8(a), Dec. 15, 7 Marks)

A bag contains 5 red, 4 white and 8 blue balls. 4 balls are drawn at random. What is the probability that there is at least one ball of each colour ?

☑ **Soln. :**

Probability (P) $= \dfrac{\text{Number of favorable events}}{\text{Sample space}}$

There are total 17 balls and 4 balls are drawn by ${}^{17}C_4$ ways.

Sample space $= {}^{17}C_4 = \dfrac{17!}{13!\,4!}$

$= \dfrac{14 \times 15 \times 16 \times 17}{1 \times 2 \times 3 \times 4}$

$= 7 \times 5 \times 4 \times 17 = 2380$

4 balls drawn contains at least one ball of each colour, It has 3 cases.

Case I : 2 red, 1 white, 1 blue

Number of favourable events $= {}^5C_2 \times {}^4C_1 \times {}^8C$

$= 10 \times 4 \times 8 = 320$

Case II : 1 red, 2 white, 1 blue

Number of favourable events $= {}^5C_1 \times {}^4C_2 \times {}^8C_1$

$= 5 \times 6 \times 8 = 240$

Case III : 1 red, 1 white, 2 blue

Number of favourable events $= {}^5C_1 \times {}^4C_1 \times {}^8C_2$

$= 5 \times 4 \times 28 = 560$

Probability $= \dfrac{320 + 240 + 560}{2380} = \dfrac{1120}{2380}$

$= \dfrac{112}{238} = \dfrac{56}{119} = \dfrac{8}{17}$

$= \dfrac{8}{17}$

UEx. 3.6.18 (SPPU - IT : Q. 7(a), May 15, 4 Marks)

A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

☑ **Soln. :**

Probability (P) $= \dfrac{\text{Number of favorable events}}{\text{Sample space}}$

Sample space $= 6 \times 2 = 12$

Number of favourable events = {(1, head), (3, head), (5, head)} = 3

Probability $= \dfrac{3}{12} = \dfrac{1}{4} = 0.25$

3.6.5 Conditional Probability

Let A be an event that has already occurred then the conditional probability of an event B with respect to that of A is given by $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

Ex. 3.6.19 : Two dice are rolled. What is the probability that the sum of the faces will not exceed 7 ? Given that at least one face shows a 4.

☑ **Soln.**

Let $A = \{(a, b) \mid a + b \le 7\}$

$A = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (6, 1)\}$

Let ~~B = {(a, b) | a = 4 or b = 4}~~

$B = \{(1, 4), (4, 1), (2, 4), (4, 2), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (6, 4)\ (4, 6)\}$

$A \cap B = \{(1, 4), (2, 4), (3, 4), (4, 1), (4, 2), (4, 3)\}$

$P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{6/36}{11/36} = \dfrac{6}{11}$

Ex. 3.6.20 : In an university 60% of the professors are men and 40% are women. It is also found that 50% of the male professors and 60% of female professors knows computer programming. Find the probability that a professor knowing programming is a women.

☑ **Soln.**

Let 'M' denotes the event that a person selected is a male and let F denote the event that the person selected is a female. Let C denote the event that the person selected knows programming.

$P(M) = 0.6,\ P(F) = 0.4,\ P(C/M) = 0.5,\ P(C/F) = 0.6$

Now to find probability P (F/C)

$P(F/C) = \dfrac{P(C \cap F)}{P(C)}$

Since $P(C/F) = 0.6$

We have $\dfrac{P(C \cap F)}{P(F)} = 0.6,\ P(C \cap F) = 0.6 \times 0.4 = 0.24$

Similarly $P(C \cap M) = 0.5 \times 0.6 = 0.30$

Now, $P(C) = P(C \cap F) + P(C \cap M)$

$= 0.24 + 0.30 = 0.54$

$P(F/C) = \dfrac{P(C \cap F)}{P(C)} = \dfrac{0.24}{0.54} = \dfrac{4}{9} = 0.44$

UEx. 3.6.21 (SPPU - IT : Q. 8(a), Dec. 14, 7 Marks)

In a certain college town, 25% of the students failed in mathematics, 15% failed in chemistry and 10% failed both in mathematics and chemistry. A student is selected at random.

(i) If he failed in chemistry, what is the probability that he failed in mathematics ?

(ii) If he failed in mathematics, what is the probability that he failed in chemistry ?

(iii) What is the probability that he failed in mathematics or chemistry ?

(iv) What is the probability that he failed neither in mathematics nor in chemistry ?

☑ **Soln. :**

Let A denotes the event that student fail in mathematics and B denote the event that student fail in chemistry.

Given that,

$P(A) = \dfrac{25}{100} = \dfrac{1}{4},\ P(B) = \dfrac{15}{100} = \dfrac{3}{20}$

and $P(A \cap B) = \dfrac{10}{100} = \dfrac{1}{10}$

(i) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/10}{3/20} = \dfrac{2}{3}$

(ii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/10}{1/4} = \dfrac{4}{10} = \dfrac{2}{5}$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \dfrac{1}{4} + \dfrac{3}{20} - \dfrac{1}{10} = \dfrac{32}{80} - \dfrac{1}{10}$

$= \dfrac{2}{5} - \dfrac{1}{10} = \dfrac{3}{10}$

(iv) $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}) = [1 - P(A)]\,[1 - P(B)]$

$= \left(1 - \dfrac{1}{4}\right) \times \left(1 - \dfrac{3}{20}\right) = \dfrac{3}{4} \times \dfrac{17}{20} = \dfrac{51}{80}$

UEx. 3.6.22 (SPPU - IT : Q. 1(C), May 17, 3 Marks)

In a box, there are 8 red, 7 blue and 6 green balls. One ball is picked up randomly. What is the probability that it is neither red nor green?

☑ **Soln.:** There are 8 red, 7 blue and 6 green balls in a box.

Let A, B, C are the events that ball picked is red, blue or green.

$P(A) = \dfrac{{}^8C_1}{21} = \dfrac{8}{21}$

$P(B) = \dfrac{{}^7C_1}{21} = \dfrac{7}{21} = \dfrac{1}{3}$

$P(C) = \dfrac{{}^6C_1}{21} = \dfrac{6}{21} = \dfrac{2}{7}$

Probability that ball is neither red nor green is

$P(\bar{A} \cap \bar{C}) = P(\bar{A}) \cdot P(\bar{C})$

$= [1 - P(A)]\,[1 - P(C)]$

$= \left[1 - \dfrac{8}{21}\right] \times \left[1 - \dfrac{2}{7}\right]$

$= \dfrac{13}{21} \cdot \dfrac{5}{7} = \dfrac{65}{147}$ ...(required probability)

UEx. 3.6.23 (SPPU - IT : Q. 2(b), Dec. 16, 6 Marks)

The probability that a contractor will get a plumbing contract is 2/3 and the probability that he will not get electric contract is 5/9. If the probability of getting at least one contract is 4/5. What is the probability that he will get both the contract ?

☑ **Soln. :**

Let A = Event that contractor get plumbing contract

B = Event that contractor get electric contract

$\therefore P(A) = \dfrac{2}{3},\ P(\bar{B}) = \dfrac{5}{9}$ and $P(A \cup B) = \dfrac{4}{5}$

$\therefore P(B) = 1 - \dfrac{5}{9}$

We have to find $P(A \cap B)$

$$\text{Consider } P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore \; P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$\therefore \; P(A \cap B) = \frac{2}{3} + \left(1 - \frac{5}{9}\right) - \frac{4}{5}$$

$$= \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{10}{9} - \frac{4}{5}$$

$$P(A \cap B) = \frac{14}{45} \quad \text{...Required probability}$$

3.6.6 Binomial Probability

Suppose an experiment is performed repeatedly, say 'n' number of times. Let A be an event with it's probability of occurrence being P. If the event. A occurs among the n-trials. We say that the trial has a success otherwise it has failure. Hence if the event A occurs r times (for $0 \le r \le n$) the trial is said to have 'r' successes. The probability that the trial has 'r' successes is given by the formula.

$$P(r) = {}^nC_r\, p^r q^{n-r}, \text{ where } q = 1 - p$$

Since, the binomial coefficients occurs in the formula, the probability is called as "Binomial Probability".

3.6.6.1 Solved Examples on Binomial Probability

Ex. 3.6.24 : An examination contains multiple choice questions so that probability of correct choice by mere guessing is 0.2. What is the probability that a student will not get more than 2 questions right out of 10, by guessing alone ?

☑ Soln.

The 10 questions are that trials. Getting a right answer by guessing is a success getting a wrong answer is a failure.

Hence, $n = 10,\ p = 0.2,\ q = 0.8,\ 0 \le r \le 2$

$$p(0 \le r \le 2) = p(0) + p(1) + p(2)$$

$$= {}^{10}C_0 (0.2)^0 (0.8)^{10} + {}^{10}C_1 (0.2)^1 (0.8)^9 + {}^{10}C_2 (0.2)^2 (0.8)^8$$

$$= 0.1073 + 0.2684 + 0.3019$$

$$= 0.68 \text{ ... (required probability)}$$

Ex. 3.6.25 : If on an average one candidate out of 10 fails in a certain exam, then find the probability that out of 5 candidates appeared for exam, at least 4 will successful.

☑ Soln.

Let A denote the event that candidate is successful in the examination.

$$p = \text{Probability of success in the examination.}$$

$$= 1 - 0.1 = 0.9$$

$$q = 0.1 \;;\; n = 5, \quad r \ge 4$$

$$p(r \ge 4) = p(4) + p(5) = {}^5C_4 (0.9)^4 (0.1) + {}^5C_5 (0.9)^5$$

$$= 0.32805 + 0.59049 = 0.91854 = 0.92$$

3.7 BAYE'S THEOREM

Many times, we have to take valide decisions on the sample space based on available sample information. E.g. consider two machines A and B producing same item. It is known that 3% of items produced by A are defective, whereas only 2% of the items produced by B are defective. If an item selected at random is found to be defective, can we determine whether it was produced by A or B ? Such types of problems can be solved by Baye's theorem. It is based on conditional probability. It has wide application is machine learning.

Let A and B are two events.

$$\text{Then} \quad P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad \text{...(3.7.1)}$$

$$\text{Also} \quad P(B/A) = \frac{P(A \cap B)}{P(A)} \qquad \text{...(3.7.2)}$$

Solving Equations (3.7.1) and (3.7.2) we get,

$$P(B/A) = \frac{P(A/B) \cdot P(B)}{P(A)} \quad \text{... (Baye's rule)}$$

3.7.1 Solved Examples on Baye's Theorem

Ex. 3.7.1 : Consider a binary communication channel over which a '0' or '1' is to be sent. Probability that 1 is sent is 0.6 and 0 sent is 0.4. Due to noise in the channel the probability that '1' is charged to '0' is 0.15 and that of '0' to '1' is 0.2. Suppose that 0 is received what is the probability that 1 was sent ?

☑ Soln.

Let A denote the event that 1 was sent and B the event that 0 was received. Hence

$$P(A/B) = \frac{P(B/A) \cdot P(B)}{P(A)}$$

$$P(B/A) = 0.15, \quad P(A) = 0.6$$

$$P(B) = (0.4)(0.8) + (0.6)(0.15) = 0.41$$

$$P(A/B) = \frac{0.15 \times 0.6}{0.41} = 0.2195$$

3.7.2 Generation of Baye's Rule

Let $A_1, A_2, \ldots A_n$ be mutually events, whose union is the sample space S. Let B is any other even in S.

$$\text{then } P(A_i/B) = \frac{P(A_i) \cdot P(B/A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(B/A_i)}$$

The probabilities $P(A_1), P(A_2) \ldots P(A_n)$, known before are called priori probabilities and $P(A_i/B)$ that are determined are called as posterior probabilities.

3.7.2.1 Solved Examples on Baye's Rule

Ex. 3.7.2 : There are 3 bags : Bag A contain 1 white, 2 red 3 green balls, Bag B contains 2 white, 3 red, 1 green and Bag C contains 3 white, 1 red and 2 green balls. Two balls are drawn from a bag chosen at random. They are found to be 1 white and 1 red. Find that two balls come from bag B.

☑ Soln.

Let A denote the event that the balls are from Bag A, B denote the event that they are from Bag B and C that they are from Bag C. Let D denote the event that the balls drawn 1 white and 1 red.

Then

$$P(B/D) = \frac{P(D/B) \cdot P(B)}{P(D/A) \cdot P(A) + P(D/B) \cdot P(B) + P(D/C) \cdot P(C)}$$

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

$$P(D/A) = \left(\frac{1}{6}\right)\left(\frac{2}{6}\right), \quad P(D/B) = \left(\frac{2}{6}\right)\left(\frac{3}{6}\right)$$

$$P(D/C) = \left(\frac{3}{6}\right)\left(\frac{1}{6}\right)$$

$$P(B/D) = \frac{\left(\frac{2}{6}\right)\left(\frac{3}{6}\right)\cdot\left(\frac{1}{3}\right)}{\frac{1}{3}\left[\left(\frac{1}{6}\right)\left(\frac{2}{6}\right) + \left(\frac{2}{6}\right)\left(\frac{3}{6}\right) + \left(\frac{3}{6}\right)\left(\frac{1}{6}\right)\right]}$$

$$P(B/D) = \frac{6}{11} \quad \text{... (required probability)}$$

Ex. 3.7.3 : Three dice were rolled. Give no two faces were same. What is the probability that there was an ace ?

☑ Soln. : Let A be the event that there was an ace and B be the event that no 2 faces are same.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{3 \cdot {}^5P_2}{6^3}$$

$$P(B) = \frac{{}^6P_3}{6^3}$$

$$P(A/B) = \frac{3 \cdot {}^5P_2}{{}^6P_3} = \frac{1}{2}$$

Ex. 3.7.4 : A room contains 10 married couples. Two people are chosen randomly from the room. What is the probability that they are husband and wife ?

☑ Soln.

Total number of ways to select 2 people from the 10 pairs is ${}^{20}C_2 = \frac{20 \times 19}{2} = 190$

Number ways to see husband wife pair is ${}^{10}C_1 = 10$

Hence required probability $= \frac{10}{190} = 0.0526$

Ex. 3.7.5 : An Urn contains 6 red balls and 4 green balls. 4 balls are selected at random from the box. What is the probability that 2 of the selected balls will be red and 2 will be green ?

☑ Soln.

The total number of ways of choosing 4 balls from a total of 10 balls is ${}^{10}C_4$ ways. The number of ways of selecting 2 red balls is ${}^6C_2$ and the number of ways of selecting 2 green balls is ${}^4C_2$ ways.

$$\therefore \text{ Required probability} = \frac{{}^6C_2 \cdot {}^4C_2}{{}^{10}C_4} = \frac{15 \times 6}{210} = \frac{3}{7}$$

Ex. 3.7.6 : A four digit number is to be formed from 1, 2, 3, 5 with no repetitions. Find the probability that the number is (i) divisible by 5 (ii) The number is odd.

☑ Soln.

Total number of ways of forming 4 digit number is 4!.

(i) The number is divisible by 5 if last digit is 5

This number can be chosen in 3! ways.

Hence the probability of getting number divisible by $5 = \frac{3!}{4!} = \frac{1}{4}$

(ii) The number is odd if the last digit 1, 3, 5.

The number of ways to select odd is $3 \cdot 3!$

Hence the probability of getting odd number $= \frac{3 \cdot 3!}{4!} = \frac{18}{24} = \frac{3}{4}$

Chapter Ends ...

□□□

Model Question Paper - I

Total No of Questions : [03] **SEAT NO. :**

[Total No. of Pages : 2]

SE/INSEM

S.E. (Information Technology) (Semester - III)

Discrete Mathematics

(2019 Pattern)

[Time: 1.00 Hour] **[Max. Marks : 30]**

Instructions to the candidates :

1. *All questions are compulsory*
2. *Neat ddiagram's must be drawn wherever necessary.*
3. *Figures to the right indicate full marks.*

Q. 1 (a) Prove the statement is true using mathematical induction : $n^3 + 2n$ is divisible by 3 for all $n > = 1$ **(5 Marks)**

(b) What is multiset ? Let A and B be the multisets [a, a, b, b, c, f] and [a, a, b, b, d, d] respectively.

Find: (i) $A \cup B$ (ii) $A \cap B$ (iii) $A - B$ (iv) $B - A$ **(5 Marks)**

OR

Q. 2(a) Show that each of the following conditional statements is a tautology by using truth tables.

(i) $(p \wedge q) \rightarrow p$ (ii) $p \rightarrow (p \vee q)$ **(5 Marks)**

(b) (i) P(x) : x is even, Q(x) : x is prime number,

R (x, y) : x + y is even. By using above write an English sentence for each of the following statement given below

1. $\forall x (\sim Q(x))$ 2. $\exists y (\sim R(y))$ 3. $\sim [x [P(x) \vee Q(x)]]$

(ii) Test the validity of the argument. If a person is poor, he is unhappy. If a person is unhappy, he dies young, therefore poor person dies young. **(5 Marks)**

Q. 3 (a) A single card is drawn from an ordinary deck of 52 cards. Find the probability P, if.

(i) the card is a face card

(ii) the card is a face card and heart

(iii) the card is a face card or heart **(5 Marks)**

(b) In how many ways can five examinations be scheduled in a week so that no two examinations scheduled on same day considering Sunday as holiday. **(5 Marks)**

OR

Q. 4 (a) Tickets number 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5 ? **(5 Marks)**

(b) A committee of 5 members is to be formed from a group of 7 men and 6 women. What is the probability that

(i) Atleast 3 women are part of the committee

(ii) All committee members are either men or women **(5 Marks)**

Q. 5 (a) Prove by mathematical induction for $n \geq 1$ **(5 Marks)**

$$1 \cdot 2 + 2 \cdot 3 + \ldots + n\,(n+1) = \frac{n\,(n+1)\,(n+2)}{3}$$

Q. 5 (b) How many 3 digits numbers can be formed from the digits 2, 3, 5, 6, 7 and 9, which are divisible by 5 and non of the digits is repeated ? **(5 Marks)**

OR

Q. 6 (a) (i) Find CNF of $q \vee (P \wedge \sim q) \vee (\sim p \wedge \sim q)$

(ii) Find DNF of $((P \rightarrow q) \wedge (q \rightarrow p)) \vee p$ **(5 Marks)**

(b) What is the number of ways of choosing 4 card from a pack of 52 playing cards ? In how many of these

(i) Four cards are of the same suit

(ii) Four cards belongs to four different suits

(iii) Cards are of same color **(5 Marks)**

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Model Question Paper - II

Total No of Questions: [04] **SEAT NO. :**

[Total No. of Pages : 2]

SE/INSEM
S.E. (Information Technology) (Semester - III)
Discrete Mathematics
(2019 Pattern)

[Time: 1.00 Hour] **[Max. Marks: 30]**

Instructions to the candidates:

1. *All questions are compulsory*
2. *Neat ddiagram's must be drawn wherever necessary.*
3. *Figures to the right indicate full marks.*

Q. 1 (a) State the principle of mathematical induction, using mathematical induction prove the following proposition $P(n) = 1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$ **(5 Marks)**

(b) A survey of 70 high school students revealed that 35 like folk music, 15 like classical music and 5 like both. How many of the students surveyed do not like either folk or classical music ? **(5 Marks)**

OR

Q. 2 (a) Using Venn diagram prove or disprove

(i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

(ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$ **(5 Marks)**

(b) Explain union, intersection, difference, symmetric difference of the sets with the help of example **(5 Marks)**

Q. 3 (a) How many auto license number plates can be created with 3-alphabets followed by 4-digits if.

(i) Repetition of both alphabets and number is allowed.

(ii) Repetition of both alphabets are numbers is not allowed. **(5 Marks)**

(b) Two fair dice are rolled, what is the probability that

(i) sum of the faces is a perfect square

(ii) sum of the faces is neither 5, 7 nor 7 **(5 Marks)**

OR

Q. 4 (a) A student must answer 7 out of 10 questions in an examination

(i) How many choices does the student have ?

(ii) How many choices does she have if she must answer the first three questions ?

(iii) How many choices does she have if she must answer at least three of the first five questions. **(5 Marks)**

(b) A basket contains 30 apples, 20 bears and 10 peaches. What is the probability that the first piece of fruit taken from the basket will be a peach. **(5 Marks)**

Q. 5 (a) Out of 15 employee in a software company, a group of 5 employee is to be sent for "linux administration and networking" training of one month. **(5 Marks)**

(i) In how many ways can the 5 employee be selected ?

(ii) What if there are 2 employees who refuse to go together for training ?

(b) Find the conjunctive and disjunctive normal forms for the following without using truth tables.

(i) $(p \rightarrow q) \wedge (q \rightarrow p)$ (ii) $((P \wedge (p \rightarrow q)) \rightarrow q$

OR

Q. 6 (a) Rewrite the following statements using quantifiers variables and predicates symbols.

(i) All birds can fly

(ii) Not all birds can fly

(iii) Some men are genius

(iv) Some numbers are not rational

(v) There is a student who likes mathematics but not geography

(vi) Each integer is either even or odd **(5 Marks)**

Q. 6 (b) Find the smallest number of people you need to choose at random so that the probability that at least two of them were both born on April 1 exceeds $\frac{1}{2}$. Assume number of days in year as 366 days. **(5 Marks)**

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Note

Note

UNIT III

Graph Theory

>> Syllabus :

- **Graphs :** Basic Terminologies, Multi-Graphs, Weighted Graphs, Sub Graphs, Isomorphic graphs Complete Graphs, Regular Graphs, Bipartite Graphs, Operations on Graphs, Paths, Circuits, Hamiltonian and Eulerian graphs, Travelling Salesman Problem, Factors of Graphs, Planar Graphs, Graph Colouring.

- **Trees :** Tree Terminologies, Rooted Trees, Path Length in Rooted Trees, Prefix Codes, Spanning Trees, Fundamental Cut Sets and Circuits, Max flow –Min Cut Theorem (Transport Network).

  Applications of Graph Theory.

Mapping of Course Outcomes for Unit III	CO3

UNIT III

CHAPTER 4 Graph Theory

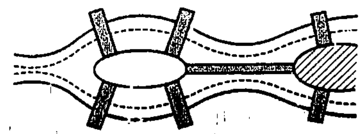
Syllabus

Basic Terminologies, Multi-Graphs, Weighted Graphs, Sub Graphs, Isomorphic graphs, Complete Graphs, Regular Graphs, Bipartite Graphs, Operations on Graphs, Paths, Circuits, Hamiltonian and Eulerian graphs, Travelling Salesman Problem, Factors of Graphs, Planar Graphs, Graph Colouring.

Previous University Paper Questions

4.1 GRAPH THEORY

- Theory of graph originated from Konigsberg Bridge problem. It says that you start from any four land area walkover the seven bridge exactly once.



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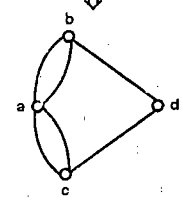


Fig. 4.1.1

- Vertices represent the land area and edge represent bridge.
- **Euler :** A mathematician, 1736 prove that the solution of this problem does not exists.
- **Graph :** A graph G is an ordered pair (V, E) where V is the set of vertices and E is the set of edges.

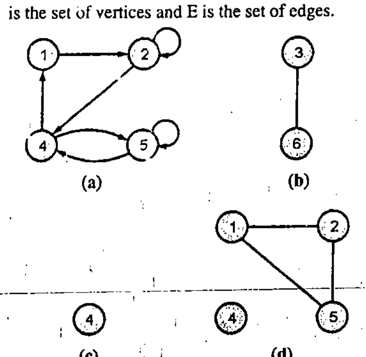


Fig. 4.1.2

GQ. Write 5 application of graph theory in the field of data analytics.

(i) Graphs are used to define the flow of computation

(ii) Graphs are used to represent networks of communication

(iii) Graphs are used to represent data organization

(iv) It is used to find shortest path in road or network.

(v) **Graph database ensure transaction – safe persistent** storing and querying of graph structure data.

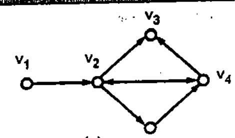
4.1.1 Basic Terminology

- **Incident :** An edge is said to be incident with the vertices, it joins.
- **Adjacent :** Two vertices are said to be adjacent, if they are joined by an edge.
- **Degree of vertices :** Number of edges incident on a particular vertex are called the degree of that vertex.
- **Indegree and outdegree :** Number of edges incident onto a vertex and number of vertex incident out of a vertex. Eg. In Fig. 4.1.2 (a) d(1) = 2 and Fig. 4.1.2 (d) d(4) = 0.
- **Loop :** If the initial vertex v_i and the terminal vertex v_j are same for an edge e_{ij}, then e_{ij} are called self loop or simply loop.
- **Parallel edge :** If there are more than one edges associated with a given pair of vertices then those edges are called parallel edges or multiple edges.
- **Isolated vertex :** A vertex is said to be isolated vertex if no edge is incident on it.
- **Pendant vertex :** A vertex with degree 1 is called a pendant vertex.

In Fig. 4.1.2 (b) vertex 3 and 6 are pendant vertex.

4.1.2 Types of Graph

UQ. Define the following terms with suitable example : (i) [illegible] Factors of graph (ii) Weighted graph (iii) Graph coloring (iv) Bipartite graph. (SPPU- Q. 4(a), Dec. 15, 6 Marks)



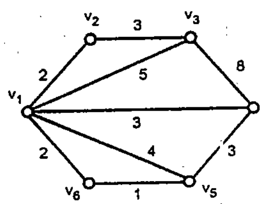


Fig. 4.1.3

☞ Directed graph

A directed graph G is defined as an ordered pair (V, E), where is V is the non empty set of vertices and E is the set of directed edges.

Eg. Fig. 4.1.3(a)

☞ Weighted graph

A graph G is said to be a weighted graph if each of the edge or vertex is assigned with some positive real number (w), called the weight.

☞ Simple graph

A graph is said to be a simple graph, if it does not contain neither the self loops nor the parallel edges.

☞ Finite and Infinite graph

A graph with finite number of edges and vertices and graph with infinite number of edges and vertices.

☞ Null graph

If the edge set of any graph G with n vertices is an empty set than the graph is called a null graph. Eg. Fig 4.1.4(a).

☞ Complete graph

A graph G is called a complete graph if every vertex in G is connected with every other vertex.

Degree of each vertex should be n – 1 and total number of edge = n (n – 1) / 2 denoted by k_n for complete graph

Eg. Fig. 4.1.4(b)

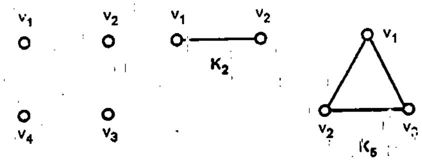


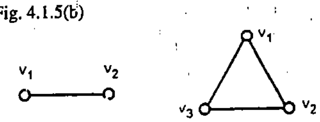
Fig. 4.1.4

☞ Regular graph

- If degree of all the vertex of a graph G is same say 'd', then it is called a regular graph. Eg. Fig. 4.1.5(a).
- Every complete graph k_n is a regular graph of degree (n – 1) but vice versa is not true.

☞ Multi graph

- A graph having self loop and parallel edges are called multi graph or multiple graph. Eg. Fig. 4.1.5(b).
- Eg. Fig. 4.1.5(b)



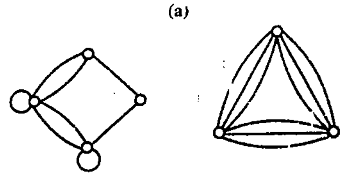


Fig. 4.1.5(b)

☞ Bipartite graph

A graph G with vertices v and edges are called bipartite graph if its vertex set can be partitioned in two sets such that – (i) $V_1 \cup V_2 = V$ (ii) $V_1 \cap V_2 = \phi$ and (iii) Each edge should join from V_1 to V_2. Eg. Fig. 4.1.6 (a).

☞ Complete bipartite graph

- If each vertex of V_1 is joined to every vertex of V_2 by an unique edge.

- It is denoted by k_{mn}. Considering m vertices in V_1 and n vertices in V_2. Eg. Fig. 4.1.6(b)

if m = n is regular graph.

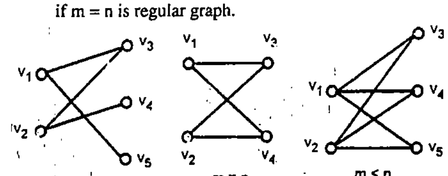


Fig. 4.1.6

Ex. 4.1.1 : Determine whether the following graphs are isomorphic to each other.

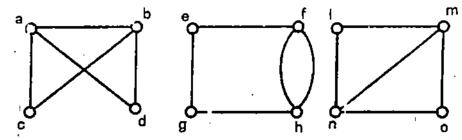


Fig. Ex. 4.1.1

☑ Soln. :

(i) Graph A, B, C all have the same number of vertices and edges.

(ii) In graph A, B, C vertices are having degrees in sequence of 3, 3, 2, 2.

(iii) In graph A and C one vertex with degree 2 have the vertices each of degree 2.

(iv) There for from above statements graph A and graph C are isomorphic to each other and graph B is not isomorphic to A and C.

GQ. Define the graph K_n and K_{mn}.

(i) A complete graph with n vertices (denoted by k_n) is a graph with n vertices in which each vertex is connected to each other.

(ii) A bipartite graph G, G = (V, E) with partition V = {V_1, V_2} is said to be complete bipartite graph if every vertex V_1 is connected to V_2. If $|V_1| = m$ and $|V_2| = n$ then complete bipartite graph is denoted by $K_{m,n}$

Ex. 4.1.2 : The graphs G and H with vertex sets V(G) and H(H), are drawn below. Determine whether or not G and H drawn below are isomorphic. If they are isomorphic, give a function g : V(G) → V(H) that define the isomorphism if they are not explain why they are not.

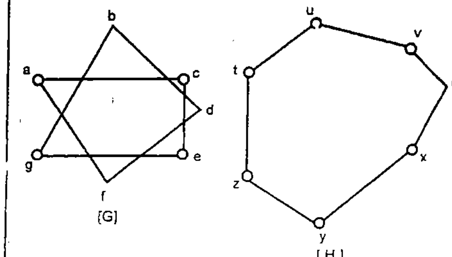


Fig. Ex. 4.1.2

☑ Soln. :

(i) In graph G and H both are having same number of vertices and edges 7 vertices and 7 edges.

(ii) In graph G and H both are having each vertex of degree 2.

(iii) In graph G and H both are having minimum length of circuit as 7.

(iv) Therefore graph G and H are isomorphic to each other.

UEx. 4.1.3 (SPPU(IT) : May 15, 6 Marks)

Determine whether the two graph are isomorphic or not explain.

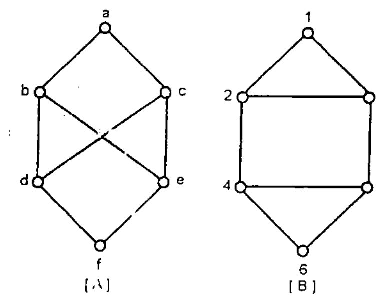


Fig. Ex. 4.1.3

☑ Soln. :

(i) In graph A and graph B both are having 6 vertices and 9 edges.

(ii) Both graph A and B are having degree sequence as 2, 3, 3, 2, 3, 3.

(iii) In graph A minimum length of any circuit in the graph is 4 and in graph B it is 3.

(iv) Therefore from above statement graph A and B are "not isomorphic".

Ex. 4.1.4 : Show that the following graphs are isomorphic.

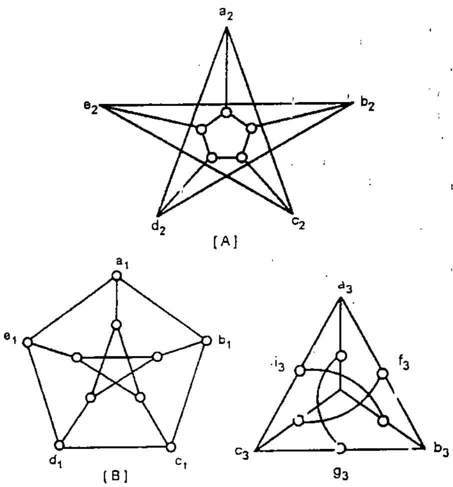


Fig. Ex. 4.1.4

☑ **Soln. :**

(i) Graph A, B, C have the same number of vertices and edges.

(ii) Degree of vertices in graph A, B, C are

$d(a_2) = d(a_1) = d(a_3) = 3$

$d(e_2) = d(e_1) = d(e_3) = 3$

$d(d_2) = d(d_1) = d(d_3) = 3$

$d(c_2) = d(c_1) = d(c_3) = 3$

$d(b_2) = d(b_1) = d(b_3) = 3$

(iii) Placement every vertices is same in all graphs

$\therefore$ Graph A, B, C are isomorphic to each other.

☞ Subgraph

If G(V, E) be any graph, then a graph G′ (V′ , E′) is said to be a subgraph of G if $E' \subseteq E$ and $V' \subseteq V$.

Ex. 4.1.5 : Define subgraph, determine whether H = H′ = (V′, E′) is a subgraph of G(V,E) shown in Fig. Ex. 4.1.5

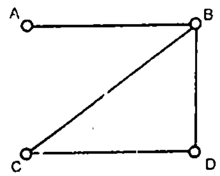


Fig. Ex. 4.1.5

☑ **Soln. :**

Subgraph is a graph whose vertices and edges are subset of another graph.

☞ Complement of a subgraph

G′ (V′, E′) with respect to graph G is another G″ (V″, E″) such that E″ = E – E′ and V″ contains only those vertices with which the edges in E″ are incident.

– **Edge disjoint subgraph :** No common edge in two subgraph.

– **Vertex disjoint subgraph :** No common vertex in two subgraph.

– **Spanning subgraph :** A subgraph which contains all the vertex of G.

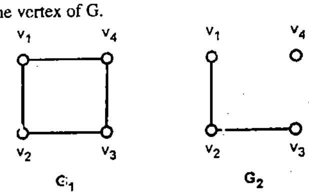


Fig. 4.1.7 : G_2 is spanning subgraph of G_1

UEx. 4.1.6 (SPPU(IT) : Q. 4(b), Dec. 19. 6 Marks)

Consider the graph given in Fig. Ex. 4.1.6, find the set V(G) of the vertice's present in G and that set E(G) of edges of G also find the degree of each vertex show that sum of degree of the vertices is twice the number of edge in graph G.

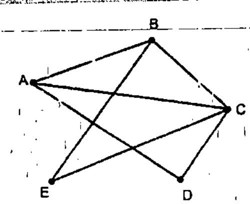


Fig. Ex. 4.1.6

☑ **Soln. :**

Here d(A) denotes the degree of vertex A

$\therefore$ d (A) = 3

d(B) = 3

d(C) = 4

d(E) = 2

d(D) = 2

$\therefore$ sum of all degree is 3 + 3 + 4 + 2 + 2 = 14

Number of edges in the graph is 7

$\therefore$ Sum of all degree = 2 × Number of edges is proved

UEx. 4.1.7 (SPPU (IT) : Q. 4(a), Dec. 15, Q. 3(b), May 19. 6 Marks)

Define the following with suitable example :

(i) Cut set (ii) Factors of graph (iii) Weighted graph

☑ **Soln. :**

(i) Cut set

Let G = (V., E) be connected graph. A subset E′ of E is called cut set of G if deletion of all the edges of E′ from G makes G disconnected.

e.g.

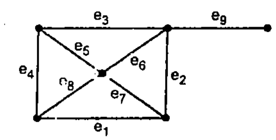


Fig. Ex. 4.1.7(a)

Cut set is $E_1 = \{e_1, e_3, e_5, e_8\}$

(ii) Factors of graph

In graph theory, a factor of a graph G is a spanning subgraph i.e. a subgraph that has same vertex set as G. A K-factor of a graph is a spanning K-regular subgraph, and k-factorization partition the edges of graph into disjoint k-factors.

(iii) Weighted graph

A graph where each edge has a numeric value called weight is called weighed graph.

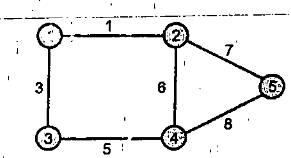


Fig. Ex. 4.1.7(b)

UEx. 4.1.8 (SPPU(IT) : Q. 3(b), May 19. 6 Marks)

Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw two such graphs.

☑ **Soln. :**

According to hand shaking lemma.

$\Sigma \deg (v) = 2 |E|$

Where V is the set of vertices and E is the set of edges.

$\therefore$ According to given statement

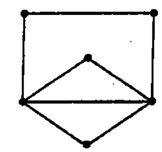
$8 + 8 = 2 |E|$

$\therefore \; E = \frac{16}{2}$

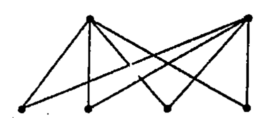
$\therefore \; E = 8$

$\therefore$ Number of edges are 8.

Graphs are



Group A



Group B

Fig. Ex. 4.1.8

UEx. 4.1.9 (SPPU(IT) : Q. 4(b)(ii), Dec. 16, 2 Marks)

Is each of the following graphs strongly connected ?

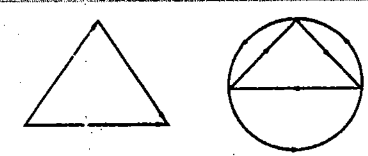


Fig. Ex. 4.1.9

☑ **Soln. :**

A directed graph is called strongly connected if there is a path in each direction between each pair of vertices of the graph.

In graph G_1

There is a path bc which is missing to make the path between each pair of vertices.

In graph G_2 we can traverse through whole graph

$\therefore$ G_1 is not strongly connected graph

G_2 is strongly connected graph.

4.1.3 Complement of Graph

A graph whose vertex set is the same as the vertex set of G and in which two vertices are adjacent iff they are not adjacent in G denoted by $\bar{G}$.

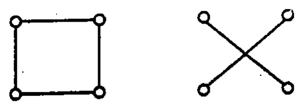


Fig. 4.1.8

- Complement of a complete graph is a null graph and vice versa.
- **Factor of a graph :** A k-factor of a graph is defined to be a spanning subgraph of the graph with the degree of each of its vertex being k.

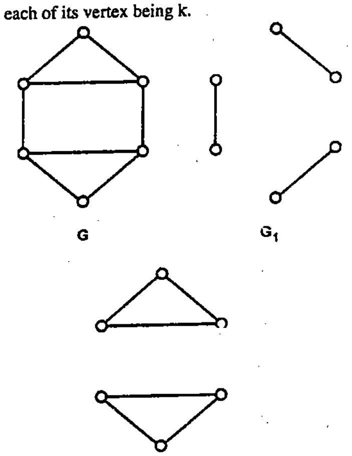


Fig. 4.1.9

G_1 and G_2 are spanning subgraph of G with k = 1 and 2 respectively.

4.1.4 Operation on Graphs

(1) Union of two graph

Where, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$

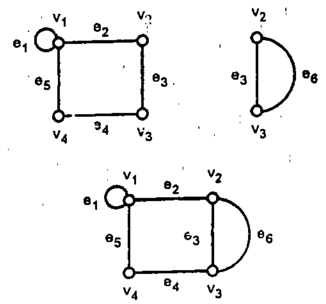


Fig. 4.1.10

then, $V_1 = \{v_1, v_2, v_3, v_4\}$

$V_2 = \{v_2, v_3\}$

$\therefore$ $V = \{v_1, v_2, v_3, v_4\}$

$E_1 = \{e_1, e_2, e_3, e_4, e_5\}$

$E_2 = \{e_3, e_6\}$

$\therefore$ $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

2. Intersection of two graphs

Where, $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$.

For above graph $V = \{V_2, V_3\}$ and $E = \{e_3\}$

Hence

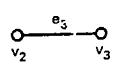


Fig. 4.1.11

3. Ring sum of two graphs

Where, $V = V_1 \cup V_2$ and of edges that either in G_1 or G_2 but not in both. Ring sum of denoted by $G_1 \oplus G_2$.

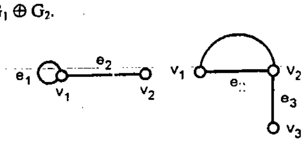


Fig. 4.1.12

For any graph G, $G_1 \oplus G_2$, then $G_1 \oplus G_2$.

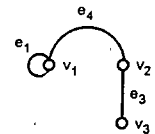


Fig. 4.1.13

4.1.5 Connectivity

1. Edge connectivity

- In a connected graph G (V, E) a cut set of edges whose removal disconnects the graph and increases the component of graph by one.

e.g.

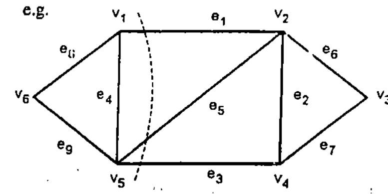


Fig. 4.1.14

For instance in Fig. 4.1.14 $\{e_1, e_5, e_3\}$ is a cut set

Whereas $\{e_1, e_4, e_8, e_9\}$ is not because its

Subset $\{e_1, e_4, e_8\}$ is cut set

- If the cutest of connected graph contains only one edge then the edge is called bridge.

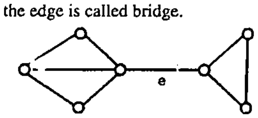


Fig. 4.1.15

e is the bridge in given graph

The number of edges in a smallest cutest of a connected simple graph G is called an edge connectivity of the graph G and it is denoted by λ (G).

$\lambda(G)$ is the smallest number of edge in G where removal disconnects the G.

For Fig. 4.1.14 $\lambda(G) = 1$.

2. Vertex connectivity

The smallest number of vertices whose removal disconnects the graph and is denoted by k(G).

Removal of v_1 and v_4 disconnect the graph. Hence k(G) = 2.

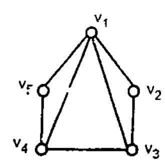
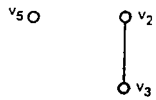


Fig. 4.1.16

A graph is said to be k-connected if its vertex connectivity is k.

The edge connectivity $\lambda(G)$ is also related to the number of edges and vertices in the given graph



i.e. $\lambda(G) \le \left[\frac{2e}{n}\right]$

Where, e = Number of edges

n = Number of vertices

- Find k(G), $\lambda(G)$ for the following graph to make disconnected the smallest cut set is 3.

Hence λ (G) = 3

- If we remove v_5, v_6, v_7 vertices then the graph will be disconnected. Hence the vertex connectivity k(G) = 3.

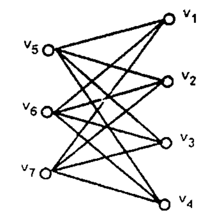


Fig. 4.1.17

- Find the edge connectivity of graph.

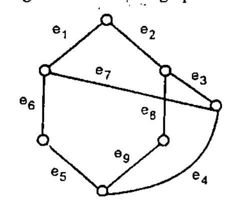


Fig. 4.1.18

In the given graph total number of edges e = 9.

by formula

$$\lambda(G) \le \left[\frac{2 \cdot e}{n}\right] \le \frac{2 \times 9}{7} \le 2$$

So Removal of e_1 and e_2 the graph becomes disconnect.

Continue from section 4.1.6

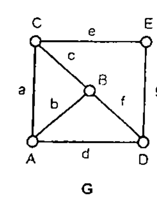


Fig. 4.1.19 : G

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix representation of graph

4.1.6 Matrix Representation of Graph

A graph can be represented by a matrix

Two ways of matrix representation

(i) Adjacency matrix

The adjacency matrix of a graph G with n vertices and no parallel edges is a symmetric binary matrix A (G) = [a_{ij}] and order $n \times n$ where

a_{ij} = 1 , if there is an edge between vertices v_i and v_j

= 0 no edge (Not adjacent)

4.1.7 Incidence Matrix

$X(G) = [x_{ij}]$ of the graph G is an $n \times e$ matrix

Where, $x_{ij} = 1$; if j^{th} edge e_j is incident on i^{th} vertex v_i

= 0 ; otherwise

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Incidence matrix representation of graph.

4.1.8 Paths and Circuits

Let G = (V, E) be any graph and let v_0 and v_n be any two vertices in V. A path of length n from v_0 to v_n is a sequence of vertices and edges of the form (v_0, e_1, v_1, e_2, ..., e_n, v_n) where each edge e_j is an edge between v_{j-1} and v_j.

The vertices v_0 and v_n are called the end vertices and remaining are called interior vertices.

- **Simple path** – No repetition of edges
- **Elementry path** – No repetition of vertices
- **Circuit** - A path becomes circuit if $v_0 = v_n$
- **Simple circuit** - No repetition of edges
- **Elementry circuit** - No repetition of vertices
- **A cyclic graph** – A group which does not contains any circuit
- **Connected graph** - If there exists a path between every pair of vertices

Otherwise the graph is disconnected.

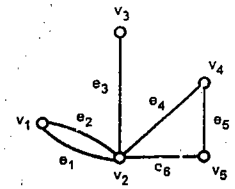


Fig. 4.1.20

Path 1 : v_1, e_2, v_2, e_4, v_4 is simple path

Path 2 : $v_4, e_4, v_2, e_2, v_1, e_1, v_2, e_3, v_3$

Path 3 : $v_3, e_3, v_2, e_2, v_1, e_1, v_2, e_3, v_3$

- Path 1 and 2 are simple path and path 3 is not.
- Path 1 is elementary path but path 2 is not

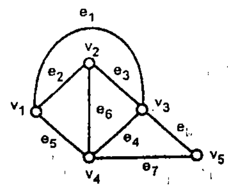


Fig. 4.1.21

Circuit 1 : $v_1, e_1, v_3, e_3, v_2, e_2, v_1$ is simple circuit and elementary circuit in G

Circuit 2 : $v_1, e_2, v_2, e_6, v_4, e_7, v_5, e_8, v_3, e_4, v_4, e_5, v_1$ is a simple circuit but not elementary as v_4 is repeated twice.

4.1.9 Eulerian Path and Eulerian Circuit

- A path is called an eulerian path if every edge of the graph G appears exactly once in the path.
- A circuit which contains every edge of the graph G exactly once is Eulerian circuit.
- **Theorem** : A graph contains an eulerian path if and only if it is connected and has either 0 or 2 vertices of odd degree.
- **Theorem** : A graph contains an eulerian circuit if and only if

It is connected and its all vertices are of even degree.

- **Theorem** : A directed graph contains an eulerian circuit if it is connected and incoming degree of every vertex is equal to outgoing degree.

Determine whether Eulerian path and Eulerian circuit exists in the following graph.

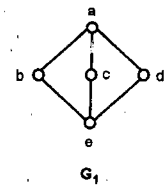


Fig. 4.1.22 : G_1

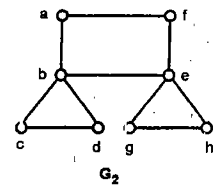


Fig. 4.1.23 : G_2

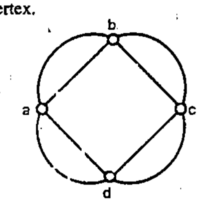
- Number of odd degree vertices in G_1 = 2. Hence G_1, contains Eulerian path.
- Degree of each vertex is odd, G_1 not contains Eulerian circuit.
- In G_2, Degree of each vertex is even. Hence contains Eulerian circuit and it contains Eulerian path as well.

Ex. 4.1.10 : Check whether the graph has an Euler circuit, Euler path, justify.

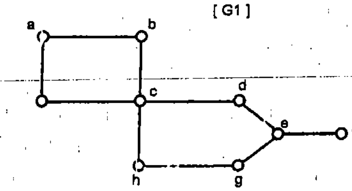
☑ **Soln. :**

Eular path in a graph is a walk through the graph which uses every edge exactly once.

Eular circuit in a graph is an Eular path starts and stops at the same vertex.



[G1]



[G2]

Fig. Ex. 4.1.6

In graph G1 every vertex has degree 4

∴ graph G1 has euler circuit.

In graph G2 every vertex e has degree 3 and F had degree 1.

∴ It has Euler path.

Ex. 4.1.11 : Under what condition $K_{m,n}$ will have eulerian circuit ?

☑ **Soln. :**

We know that a graph has Eulerian circuit if and only if all its degree are even. In Bipartite graph $K_{m,n}$ has vertices of degree m, n. So it has an Euler circuit if and only if both m and n are even.

☞ Isomorphic graph

Two graphs are said to be isomorphic if there is one to one correspondence between their vertices and their edges such that incidences and adjacency are preserved.

Following are the must have condition for Isomorphism :

(i) Same number of vertices

(ii) Same number of edges

(iii) Same number of vertices with a given degree.

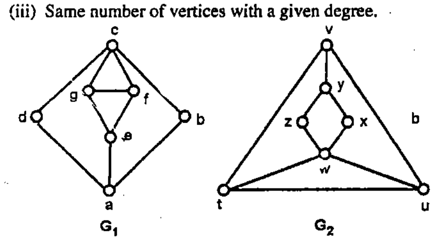


Fig. 4.1.24

$G_1 \cong G_2$ is graph G_1 and G_2 are isomorphic.

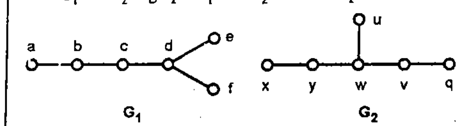


Fig. 4.1.25

$G_1 \not\cong G_2$ is graph G_1 and G_2 are not isomorphic.

Ex. 4.1.12 : Under what condition Km, n will have Eulerian circuit ?

How many colours are required to colour Km, n. ? Why ?

☑ **Soln. :**

A complete bipartite graph Km, n is connected, and each vertex has a degree m or n therefore Km, n is Eularian circuit if and only if both m and n are even.

UEx. 4.1.13 (SPPU(IT) : Q. 1(b), May 17, 6 Marks)
Define isomorphic graph, show that the following graphs G1 and G2 are isomorphic.

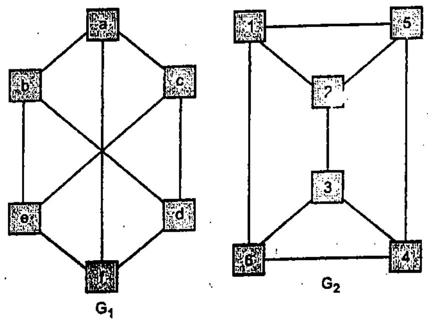


Fig. Ex. 4.1.13

☑ **Soln. :**

(i) $d(a) = 3$ $d(2) = 3$

$d(b) = 3$ $d(1) = 3$

$d(c) = 3$ $d(5) = 3$

$d(e) = 3$ $d(6) = 3$

$d(d) = 3$ $d(4) = 3$

$d(f) = 3$ $d(3) = 3$

(ii) Length of the shortest cycle in both the graph is 3.

∴ Graph G_1 and G_2 are isomorphic to each other

UEx. 4.1.14 (SPPU(IT) : Q. 4(b)(i), Dec. 16, 4 Marks)
What is complement of K_m and K_{mn} ?

☑ **Soln. :**

(i.) K_m is a complete graph, therefore the complement of a complete graph is a edgeless graph.

e.g

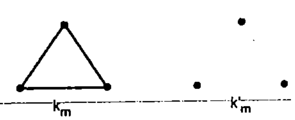


Fig. Ex. 4.1.14

(ii) K_{mn} is a bipartite graph therefore the complement of bipartite graph K_{mn} is K_m, K_n where k_m is a complete graph with m vertices and K_n is complete graph with n vertices .

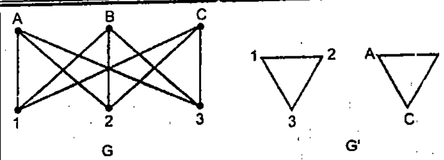


Fig. Ex. 4.1.14(a)

4.1.10 Hamilton Path and Hamilton Circuit

- A path in a connected graph G is a Hamilton path if it visit every vertex G exactly once.
- A circuit in a connected graph G, is called a Hamilton circuit if it visits vertex of G exactly once.
- **Theorem 1 :** If the sum of the degree for each pair of vertex $\geq n - 1$, then there exists a Hamilton path.
- **Theorem 2 :** If the degree of each vertex in G is $d(v) \geq n/2$, then G will contain Hamilton circuit. It is sufficient condition not necessary.
- **Example :**

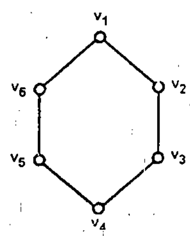


Fig. 4.1.26

In G, degree of each vertex is $\geq n/2$ where $n = 6$

Hence it has hamilton cycle

$v_1, v_2, v_3, v_4, v_5, v_6, v_7$

Ex. 4.1.15 : Determine whether the following graph has a Hamiltonian circuit or Hamiltonian path.

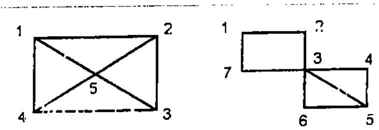


Fig. Ex. 4.1.15

☑ **Soln. :**

(i) If there exists a path in the connected graph that contains all vertices of graph, then such a path is called Hamilton path

In G_1 $1-2-3-4-5$ is a Hamilton path.

In G_2 there is no Hamilton path.

(ii) If there exists a cycle in connected graph that contain all vertices of graph, then that cycle is Hamilton circuit.

In G_1 $1-2-3-4-5-1$ is a Hamilton circuit

In G_2 there is no Hamilton circuit

Ex. 4.1.16 : Determine which of the graphs below represents Eulerian circuit, Eulerian path, Hamiltonian circuit and Hamiltonian path. Justify your answer.

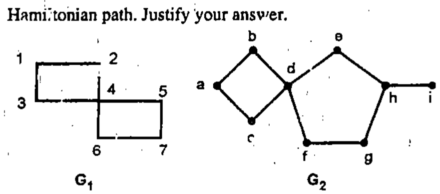


Fig. Ex. 4.1.16

☑ **Soln. :**

(i) A graph is Eular circuit if and only if degree of every vertex is even

∴ Graph G_1 and G_2 both are not Eular circuit

(ii) A graph is Eular path if and only if there are at most two vertices with odd degree. Graph G_1 has zero vertices with odd degree and G_2 has two vertice h and I with odd degree

∴ both G_1 and G_2 are Eular path

(iii) In graph G_1 we can visit every vertex exactly

∴ G_1 is a Hamilton path In G_2 we cannot visit whole graph by visiting every vertex exactly once.

∴ G_2 is not an Hamilton path.

(iv) In graph G_1 we cannot start and end with same vertex by visiting every vertex exactly at once

∴ G_1 is not an Hamilton circuit same is for graph G_2.

Ex. 4.1.17 : List and explain the necessary and sufficient conditions for Hamiltonian and Eulerian path with suitable examples.

☑ **Soln. :**

☞ Euler path

A Eular path in graph G is a simple path containing every edge of G exactly at once

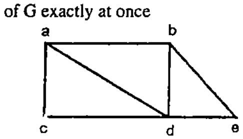


Fig. Ex. 4.1.17

☞ Hamilton path

A Hamilton path in graph G is that passes through every vertex exactly once.

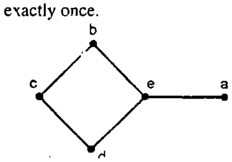


Fig. Ex. 4.1.17(a)

☞ Necessary and sufficient condition for Eular path

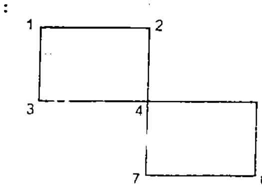
If G is connected and has exactly two vertex of odd degree, then there is a eular path in G, any eular path in G must begin at one vertex of odd degree and end at other.

☞ Necessary and sufficient condition for Hamilton path

Every vertex must have even degree

UEx. 4.1.18 (SPPU-IT : Q. 3(b), Dec. 18, 6 Marks)
Determine if the following graph (G1, G2) are having the hamiltonian circuit or path ? Justify your answer.

☑ **Soln. :**



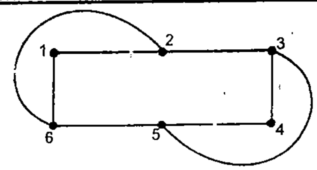


Fig. Ex. 4.1.18

(i) If there exists a path in connected graph that contains all the vertices without repeating any vertices then such path is called Hamilton path.

(ii) In graph 1 3 - 1 - 2 - 4 - 5 - 6 - 7 is the hamilton path

In graph 2 1 - 2 - 3 - 4 - 5 - 6 is a hamilton path.

(iii) If there exists a cycle in connected graph that contains all vertices of graph, then that cycle is hamilton circuit.

(iv) In graph 1 there is no Hamilton circuit.

In graph 2 1- 5 - 3 - 4 - 2 - 6 - 1 is the Hamilton circuit.

4.1.11 Handshaking Lemma

The sum of the degree of all vertices in a graph G is twice the number of edges.

$$\sum_{n=1}^{n} d(v_i) = 2 \times e$$

Where, $d(v_i)$ = degree of vertex v_i

e = Number of edges

Theorem 1

The number of vertices of odd degree in a graph is always even.

Soln. :

Let G be a graph

By handshaking lemma

$$\sum_{n=1}^{n} d(v_i) = 2 \cdot e = \text{even number}$$

No some of vertices will be odd and some of will **be even.**

i.e. $\Sigma d(v_j) + \Sigma d(v_k) = \Sigma d(v_i)$

i.e. even degree vertex + odd degree vertex

= an even number

But sum of degrees of even degree vertices is always even ie. $\Sigma d(v_k)$ = is even and $2 \cdot e$ is even number.

Hence, $\Sigma d(v_j)$ = an even number

i.e sum of odd degree vertices = even number

Ex. 4.1.19 : Show that the maximum degree of any vertex in a simple graph with n vertices is (n – 1).

Soln. :

Let G be a simple graph on n vertices, consider any vertex v of G.

Since the graph is simple, the vertex v can be adjacent to atmost (n – 1) vertices. Hence the degree of the vertex v can be at the most (n – 1). Hence maximum degree of any vertex in a simple graph with n vertices is (n – 1).

Ex. 4.1.20 : Show that the maximum number of edges in a simple graph with n vertices is n (n – 1)/2.

Soln. :

Let G be a simple graph with n vertices.

By handshaking lemma

$$\Sigma d(v_i) = 2 \times e$$

i.e. $d(v_1) + d(v_2) + d(v_3) + \ldots + d(v_n) = 2 \times e$

We know that maximum degree of each vertex in the graph G can be at the most (n – 1). Hence

$$\underbrace{(n-1) + (n-1) + \ldots\ldots + (n-1)}_{n \text{ times}} = 2 \times e$$

i.e. $n(n-1) = 2 \times e$

$e = n(n-1)/2$

Hence maximum number of edges in any simple graph with n vertices is n (n – 1) / 2.

Ex. 4.1.21 : Determine the number of edges in a graph with 6 nodes 2 of degree 4 and 4 of degree 2. Draw such graph.

Soln. :

Suppose the graph with 6 vertices has e number of edges

∴ by handshaking lemma.

$$\sum_{i=1}^{6} d(v_i) = 2e$$

i.e. $d(v_1) + d(v_2) + \ldots\ldots + d(v_6) = 2 \times e$

$4 + 4 + 2 + 2 + 2 + 2 = 2 \times e$

$16 = 2 \times e$

∴ $e = 8$

Hence number of edges = 8

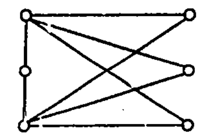


Fig. Ex. 4.1.21 : graph

Ex. 4.1.22 : Is it possible to draw a simple graph with 4 vertices and 7 edges ? Justify.

Soln. :

Let G be a simple graph with n vertices number of edges

$= n(n-1)/2$

$= 4(4-1)/2$

$= 6$

It means simple graph with 4 vertices can not have 7 edges.

Hence such graph does not exits.

Ex. 4.1.23 : Can a simple graph exist with 15 vertices each of degree five ?

Soln. :

According to handshaking lemma, in every finite undirected graph the number of vertices with odd degree is always even.

i.e. $\Sigma \deg(v) = 2(E)$

But in the above case sum of all degree

$\Sigma \deg(v) = 15 \times 5 = 75$, which is not even, which is contradiction against rule.

Ex. 4.1.24 : Consider, a graph G(V, E) where $V = \{v_1, v_2, v_3\}$ and $\deg(v_2) = 4$

(i) Does such simple graph exists ? if not why.

(ii) Does such multigraph exists ? if yes give example.

Soln. :

(i) A graph with no locps and no parallel edges are called simple graph. But in graph G(V, E) deg $(v_2) = 4$ that means it has one self loop.

∴ This graph is not simple graph.

(ii) A graph which is permitted to have multiple edges are called multigraph in graph G(V, E) deg $(v_2) = 4$ it has a self loop. It is a multigraph diagram will be

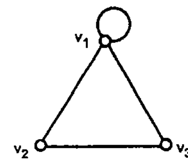
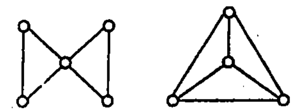


Fig. Ex. 4.1.24

4.1.12 Planar Graph

A graph G is said to be planar graph, if it can be drawn on a plane without intersecting the edges except at the common vertices.



(a) **(b)**

Fig. 4.1.27

Region

A region is characterized by the set of edges forming its boundary.

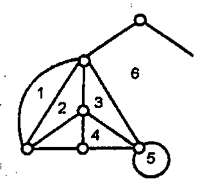


Fig. 4.1.27(c)

Number indicates region.

∴ Fig. 4.1.27(c) is having 6 region.

Ex. 4.1.25 : Show that in a connected planar graph with 6 vertices and 12 edges, each of region is bounded by 3 edges.

Soln. :

According to Eulars formula.

$v - e + r = 2$

∴ $6 - 12 + r = 2$

∴ $r = 2 + 6$

$r = 8$

By using hansharking lemma we know that

$\Sigma d(v) = 2E = 2 \times 12 = 24$

Since the number of region r = 8

∴ each region is bounded by 24/8 edges = 3 edges.

4.1.13 Euler's Formula

For any connected planar graph

$v - e + r = 2$

v = Number of vertices

e = Number of edges

r = Number of region

Proof

Let G be any connected planar graph with v vertices, e edges, and r number of regions, since G is connected two cases arise.

(i) G is tree

If G is tree then G contains $(v - 1)$ number of edges and has only one unbounded region so $r = 1$

$\therefore \quad v - e + r = v - (v - 1) + 1$

$= 2$

$\therefore \quad v - e + r = 2$ holds on tree

(ii) Suppose G is not a tree, Then G contains cycle.

We will prove the result by induction

if $e = 0$ then $v = 1$ and $r = 1$

Hence, $v - e + r = 1 - 0 + 1 = 2$

For induction step

Now from the graph G, remove the edge e which forms the cycle in G. The graph $(G - e')$ is connected and has $(e - 1)$ number of edges and $(r - 1)$ number of regions.

So by induction step

$v - (e - 1) + (r - 1) = 2$

i.e. $v - e + r = 2$

Hence proved.

Theorem :

If G (V, E) is simple connected planar graph then $e \le 3v - 6$.

Proof :

Since the graph is simple planar graph.

$\therefore$ Each region of planar graph is bounded by at least three or more edges.

Hence $e \ge 3 \cdot r$

Also each edge is included in atleast 2 regions of the planar graph G

$\therefore \quad 2e \ge 3r$ ...(1)

Now according to Eulers formula

$v - e + r = 2$

Substituting (1) in Equation

$$v - e + \frac{2e}{3} \ge 2$$

i.e. $3v - 6 \ge e$ Hence proved.

Ex. 4.1.26 : How many regions would be in plane graph with 10 vertices each of degree 3.

Soln. :

According to Euler's formula for connected planar graph with 'v' vertices, 'e' edges and 'R' Region

We have $v - e + r = 2$

$\therefore \quad r = 2 + e - v \qquad \therefore r = 2 + 15 - 10$

$r = 7$

$\therefore$ Number of regions will be 7.

4.2 SHORTEST PATH ALGORITHM

4.2.1 Dijkstra's Algorithm

Assumption

- Let G(V, E) be a simple connected graph.
- Let a and z be two vertices of this graph.
- Let L(x) denote the label of vertex x which represent the length of the shortest path from vertex a to vertex x.
- Let w_{ij} denote the weight of the edge e_{ij} (v_i, v_j)

Step 1 :

P is the set of those vertices which have the permanent labels so $P = \{ \phi \}$

$T = \{$ all the vertices of the graph $\}$

Set $L\{a\} = 0$, $L(x) = \infty$, $\forall x \in T$ and $x \ne a$

Step 2 :

Select the vertex v in T which has the smallest labels. This label will be the permanent label of v.

Set $P = P \cup \{v\}$ and $T = T - \{v\}$

If $v = z$ then L { z } is the length of shortest path from vertex a to z and stop.

Step 3 :

If $v \ne z$ then revise the label of vertices of T. The new label of a vertex x in T is given by

$L(x) = \min \{ \text{old } L(x),\ L(v) + w(v, x) \}$

Where w (x, v) is the weight of the edge joining the vertex v an x if no direct edge then $w(v, x) = \infty$.

Step 4 :

Repeat step 2 and step 3 until 2 gets the permanent labels.

Example

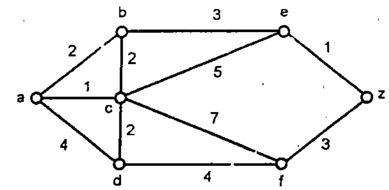


Fig. 4.2.1

1. $P = \phi$, $T = \{ a, b, c, d, e, f, z \}$

   $L(a) = 0$, $L(x) = \infty$

   $\forall x \in T, \ x \ne a$

2. $v = a$

   The permanent label of a = 0.

   $P = \{ a \}$

   $T = \{ b, c, d, e, f, z \}$

   $L(b) = \min(\infty, 0 + 2) = 2$

   $\mathbf{L(c) = \min(\infty, 0 + 1) = 1}$

   $L(d) = \min(\infty, 0 + 4) = 4$

   $L(e) = \min(\infty, 0 + \infty) = \infty$

   $L(f) = \min(\infty, 0 + \infty) = \infty$

   $L(z) = \min(\infty, 0 + \infty) = \infty$

3. $v = c$,

   The permanent label of c = 1,

   $P = \{ a, c \}$

   $T = \{ b, d, e, f, z \}$

   $\mathbf{L(b) = \min(2, 1 + 2) = 2}$

   $L(d) = \min(4, 1 + 2) = 3$

   $L(e) = \min(\infty, 1 + 5) = 6$

   $L(f) = \min(\infty, 1 + 7) = 8$

   $L(z) = \min(\infty, 1 + \infty) = \infty$

4. $v = b$,

   The permanent label of b = 2,

   $P = \{ a, c, b \}$

   $T = \{ d, e, f, z \}$

   $\mathbf{L(d) = \min(3, 2 + \infty) = 3}$

   $L(e) = \min(6, 2 + 3) = 5$

   $L(f) = \min(8, 2 + \infty) = 8$

   $L(z) = \min(\infty, 2 + \infty) = \infty$

5. $v = d$,

   The permanent label of d = 3,

   $P = \{ a, c, b, d \}$

   $T = \{ e, f, z \}$

   $\mathbf{L(e) = \min(5, 3 + \infty) = 5}$

   $L(f) = \min(8, 3 + 4) = 7$

   $L(z) = \min(\infty, 3 + \infty) = \infty$

6. $v = e$,

   The permanent label of e = 5,

   $P = \{ a, c, b, d, e \}$

   $T = \{ f, z \}$

   $L(f) = \min(7, 6 + \infty) = 7$

   $\mathbf{L(z) = \min(\infty, 5 + 1) = 6}$

7. $v = z$,

   The permanent label of z = 6,

   Hence the length of the shortest path from a to z is 6 and shortest path is a b e z.

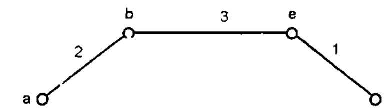


Fig. 4.2.2

Ex. 4.2.1 : Use dijkstra algorithm to find the shortest. Path between a and z.

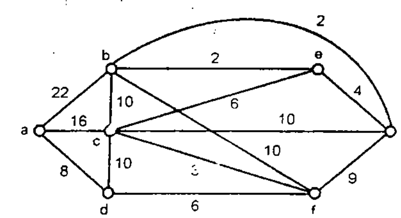


Fig. Ex. 4.2.1

Soln. :

(i) $P = \phi$, $T = \{a, b, c, d, e, f, z\}$

$L(a) = 0 \quad L(x) = \infty$

$\forall x \in T, x \ne a$

(ii) $V = a$

The permanent label of a = 0

and $P = \{a\}$, $T = \{b, c, d, e, f, z\}$

$L(b) = \min(\infty, 0 + 22) = 22$

$L(c) = \min(\infty, 0 + 16) = 16$

$\mathbf{L(d) = \min(\infty, 0 + 8) = 8}$

$L(e) = \min(\infty, \infty) = \infty$

$L(f) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, \infty) = \infty$

(iii) Taking the point with minimum distance i.e. d

$\therefore \quad V = d$,

The permanent label of d = 8

$P = \{a, d\}$, $T = \{b, c, e, f, z\}$

$L(b) = \min(22, \infty) = 22$

$L(c) = \min(16, 8 + 10) = 16$

$L(e) = \min(\infty, \infty) = \infty$

$L(f) = \min(\infty, \infty + 6) = 14$

$L(z) = \min(\infty, \infty) = \infty$

(iv) $V = f$

Permanent label of f = 14

$P = \{a, d, f\}$, $T = \{b, c, e, z\}$

$L(b) = \min(22, 8 + 6 + 10) = 22$

$L(c) = \min(16, 8 + 6 + 3) = 16$

$L(e) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, 8 + 6 + 9) = 23$

(v) $V = C$

Permanent Label of C = 16

$P = \{a, d, f, c\}$, $T = \{b, e, z\}$

$L(b) = \min(22, \infty) = 22$

$L(e) = \min(\infty, 8 + 6 + 3 + 6) = 23$

$L(z) = \min(23, 8 + 6 + 3 + 10) = 23$

(vi) $V = b$

Permanent Label of b = 22

$P = \{a, d, f, c, b\}$, $T = \{e, z\}$

$L(e) = \min(23, 8 + 6 + 3 + 10 + 2) = 23$

$L(z) = \min(23, 2 + 6 + 3 + 10 + 2) = 23$

(vii) $V = e$

Permanent Label of e = 23

$P = \{a, d, f, c, b, e\}$, $T = \{z\}$

$L(z) = \min(23, 8 + 6 + 3 + 10 + 2 + 4) = 23$

(viii) $V = z$

Permanent Label of z = 23

Hence the length of shortest path

From a to z is 23 and shortest path is a – d – f – z

OR

	a	b	d	c	e	f	z
a	$\boxed{(0, a)}$	(22,a)	(8,a)	(16,a)	(∞,a)	(∞,a)	(∞,a)
d	-	(22,a)	$\boxed{(8,a)}$	(16,a)	(∞,a)	(14,a)	(∞,a)
f	–	(21,f)	-	(16,a)	(∞,a)	$\boxed{(14,a)}$	(23,f)
b	–	$\boxed{(21,f)}$	-	(16,a)	(23,b)	–	(23,f)
c	–	–	–	$\boxed{(16,a)}$	(22,c)	–	(23,f)
e	–	–	–	–	$\boxed{(22,c)}$	–	(23,f)
z	–	–	–	–	–	–	$\boxed{(23,f)}$

Here the convention (0, a)

Distance ↓ previous point

The point denoted inside box is the point with the shortest distance from starting point.

∴ Shortest path for the given diagram with be

a – d – f – z which is 23.

Ex. 4.2.2 : Use dijkstra algorithm to find the shortest. Path.

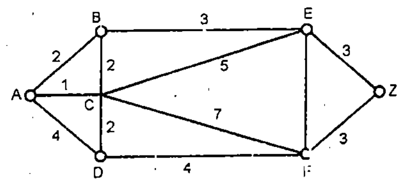


Fig. Ex. 4.2.2

☑ **Soln. :**

	A	C	B	D	E	F	Z
A	$\boxed{(0, A)}$	(1,A)	(2,A)	(4,A)	(∞,A)	(∞,A)	(∞,A)
C	-	$\boxed{(1, A)}$	(2,A)	(3,C)	(6,C)	(8,C)	(∞,A)
B	–	–	$\boxed{(2, A)}$	(3,C)	(5,B)	(8,C)	(∞,A)
D	–	–	–	$\boxed{(3, C)}$	(5,B)	(7,D)	(∞,A)
E	–	–	–	–	$\boxed{(5, B)}$	(7,D)	(8,E)
F	–	–	–	–	–	(7,D)	(8,E)
Z	–	–	–	–	–	–	$\boxed{(8, E)}$

Point (O, A) denotes O as distance from starting node and A as previous node.

Point denoted inside box are the points with shortest distance from starting point.

Ex. 4.2.3 : Apply Dijkastra's Algorithm to find the shortest path from vertex v_1 to v_5 in the graph show below in Fig. Ex. 4.2.3.

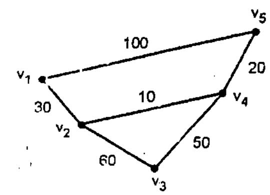


Fig. Ex. 4.2.3

☑ **Soln. :**

Apply Dijkastra's Algorithm to find the shortest path from vertex V_1 to V_5 in the grapy shown

(i) $P = \phi$, $T = \{V_1, V_2, V_3, V_4, V_5\}$

$L(V_1) = 0$, $L(x) = \infty$

$\forall x \in V_1$

(ii) $V = V_1$

The permanent label of $V_1 = 0$

and $P = \{V_1\}$, $T = \{V_2, V_3, V_4, V_5\}$

$\mathbf{L(V_2) = min(\infty, 0 + 30) = 30}$

$L(V_3) = \min(\infty, \infty) = \infty$

$L(V_4) = \min(\infty, \infty) = \infty$

$L(V_5) = \min(\infty, 0 + 100) = 100$

(iii) Taking the point with minimum distance i.e. V_2

∴ $V = V_2$

The permanent label of $V_2 = 30$

$P = \{V_1, V_2\}$, $T = \{V_3, V_4, V_5\}$

$L(V_3) = \min(\infty, 30 + 60) = 90$

$\mathbf{L(V_4) = min(\infty, 30 + 10) = 40}$

$L(V_5) = \min(100, \infty) = 100$

(iv) Taking point with minimum distance i.e. V_4

∴ $V = V_4$

$P = \{V_1, V_2, V_4\}$, $T = \{V_3, V_5\}$

$L(V_3) = \min(90, 40 + 50) = 90$

$L(V_5) = \min(100, 40 + 20) = 60$

(v) Point with minimum distance is V_5

Permanent label of $V_5 = 60$

∴ Length of shortest path from V_1 to V_5 is 60 and the path is $V_1 - V_2 - V_4 - V_5$

Ex. 4.2.4 : Use Dijkstra's algorithm to find the shortest path between a and z.

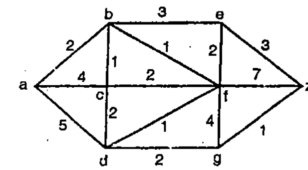


Fig. Ex. 4.2.4

☑ **Soln. :**

(i) $P = \phi$, $T = \{a, b, c, d, e, f, g, z\}$

$L(a) = 0$, $L(x) = \infty$

$\forall x \in T, x \neq a$

(ii) $V = a$

The permanant label a = 0

and $P = \{a\}$, $T = \{b, c, d, e, f, g, z\}$

$\mathbf{L(b) = min(\infty, 2) = 2}$

$L(c) = \min(\infty, 0 + 4) = 4$

$L(d) = \min(\infty, 0 + 5) = 5$

$L(e) = \min(\infty, \infty) = \infty$

$L(f) = \min(\infty, \infty) = \infty$

$L(g) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, \infty) = \infty$

(iii) Taking point with minimum distance i.e. b.

∴ $v = b$

∴ permanant label of b = 2

$P = \{a, b\}$, $T = \{c, d, e, f, g, z\}$

$L(c) = \min(4, 2 + 1) = 3$

$L(d) = \min(5, \infty) = 5$

$L(e) = \min(\infty, 2 + 3) = 5$

$L(f) = \min(\infty, 2 + 1) = 3$

$L(g) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, \infty) = \infty$

(iv) Point c and f both are having distance 3

∴ taking c

∴ $V = c$

Permanant label of c is 3

$P = \{a, b, c\}$, $T = \{d, e, f, g, z\}$

$L(d) = \min(5, 3 + 2) = 5$

$L(e) = \min(5, \infty) = 5$

$L(f) = \min(3, 3 + 2) = 3$

$L(g) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, \infty) = \infty$

(v) Taking point f

$\therefore V = f$

Permanent label of f is 3

$\therefore P = \{a, b, c, f\}, T = \{d, e, g, z\}$

$L(d) = \min(5, 3 + 1) = 4$

$L(e) = \min(5, 3 + 2) = 5$

$L(g) = \min(\infty, 3 + 4) = 7$

$L(z) = \min(\infty, 3 + 7) = 10$

(vi) Taking point with minimum length

$\therefore V = d$

$\therefore$ Permanent label of d = 4

$P = \{a, b, c, f, d\}, T = \{e, g, z\}$

$L(e) = \min(5, 8) = 5$

$L(g) = \min(7, 4 + 2) = 6$

$L(z) = \min(10, \infty) = 10$

(vii) Taking point with minimum distance

$\therefore V = e$

Permanent label of e = 5

$P = \{a, b, c, f, d, e\}, T = \{g, z\}$

$L(g) = \min(6, \infty) = 6$

$L(z) = \min(10, 5 + 3) = 8$

(viii) Taking point with min distance

$V = g$

$\therefore$ permanent label of g = 6

$\therefore P = \{a, b, c, f, d, e, g\}, T = \{z\}$

$L(z) = \min(8, 6 + 1) = 7$

(ix) $v = z$

Permanent label of Z is 7 hence length of shortest path from A to z is 7 and path is a – b – f – d – g – z

UEx. 4.2.5 (SPPU(IT) : Q. 4(b), Dec. 18, 6 Marks)

[illegible] the given graph using Dijkstra's algorithm.

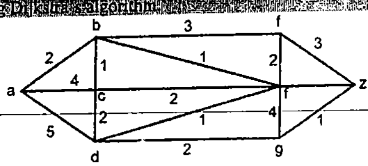


Fig. Ex. 4.2.5

☑ **Soln. :**

(i) $p = \phi, T = \{a, b, c, d, e, f, g, z\}$

$L(a) = 0, L(x) = \infty$

$\forall x \in T, x \neq a$

(ii) $v = a$

The permanent label a = 0

and $P = \{a\}, T = \{b, c, d, e, f, g, z\}$

$\mathbf{L(b) = \min(\infty, 2) = 2}$

$L(c) = \min(\infty, 0 + 4) = 4$

$L(d) = \min(\infty, 0 + 5) = 5$

$L(e) = \min(\infty, \infty) = \infty$

$L(f) = \min(\infty, \infty) = \infty$

$L(g) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, \infty) = \infty$

(iii) Taking point with minimum distance i.e. b

$\therefore v = b$

$\therefore$ Permanent label of b = 2

$p = \{a, b\}, T = \{c, d, e, f, g, z\}$

$\mathbf{L(c) = \min(4, 2 + 1) = 3}$

$L(d) = \min(5, \infty) = 5$

$L(e) = \min(\infty, 2 + 3) = 5$

$L(f) = \min(\infty, 2 + 1) = 3$

$L(g) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, \infty) = \infty$

(iv) Point c and f both are having dist. 3.

$\therefore$ taking c

$\therefore v = c$

Permanent label of c is 3

$p = \{a, b, c\}, T = \{d, e, f, g, z\}$

$L(d) = \min(5, 3 + 2) = 5$

$L(e) = \min(5, \infty) = 5$

$\mathbf{L(f) = \min(3, 3 + 2) = 5}$

$L(g) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, \infty) = \infty$

(v) Taking point f

$\therefore v = f$

Permanent label of f is 3.

$\therefore P = \{a, b, c, f\}, T = \{d, e, g, z\}$

$\mathbf{L(d) = \min(5, 3 + 1) = 4}$

$L(e) = \min(5, 3 + 2) = 5$

$L(g) = \min(\infty, 3 + 4) = 7$

$L(z) = \min(\infty, 3 + 7) = 10$

(vi) Taking point with minimum length

$\therefore v = d$

$\therefore$ permanent label of d = 4

$P = \{a, b, c, f, d\}, T = \{e, g, z\}$

$L(e) = \min(5, \infty) = 5$

$L(g) = \min(7, 4 + 2) = 6$

$L(z) = \min(10, \infty) = 10$

(vii) Taking point with minimum distance

$\therefore v = e$

Permanent label of e = 5

$P = \{a, b, c, f, d, e\}, T = \{g, z\}$

$L(g) = \min(6, \infty) = 6$

$L(z) = \min(10, 5 + 3) = 8$

(viii) Taking point with minimum distance

$v = g$

$\therefore$ permanent label of g = 6

$\therefore p = \{a, b, c, f, d, e, g\}, T = \{z\}$

$L(z) = \min(8, 6 + 1) = 7$

(ix) $v = z$

Permanent label of z is 7

Hence length of shortest path from a to z is 7 and path is

a - b - f - d - g - z

UEx. 4.2.6 (SPPU(IT) : May 19, 6 Marks)

Use dijkstra Algorithm to find the shortest path between a and z.

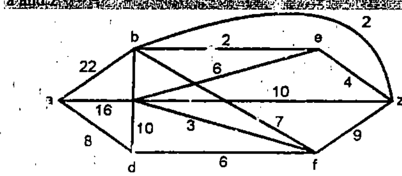


Fig. Ex. 4.2.6

☑ **Soln. :**

(i) $P = \phi, T = \{a, b, c, d, e, f, z\}$

$L(a) = 0, L(x) = \infty$

$\forall x \in T, x \neq a$

(ii) $v = a$

The permanent label of a = 0

and $p = \{a\}, T = \{b, c, d, e, f, z\}$

$L(b) = \min(\infty, 0 + 22) = 22$

$L(c) = \min(\infty, 0 + 16) = 16$

$\boxed{L(d) = \min(\infty, 0 + 8) = 8}$

$L(e) = \min(\infty, \infty) = \infty$

$L(f) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, \infty) = \infty$

(iii) Taking the point with minimum distance ie d

$\therefore v = d$

The permanent label of d = 8

$P = \{a, d\}, T = \{b, c, e, f, z\}$

$L(b) = \min(22, \infty) = 22$

$L(c) = \min(16, 8 + 10) = 16$

$L(e) = \min(\infty, \infty) = \infty$

$\mathbf{L(f) = \min(\infty, 8 + 6) = 14}$

$L(z) = \min(\infty, \infty) = \infty$

(iv) $v = f$

Permanent label of f = 14

$P = \{a, d, f\}, T = \{b, c, e, z\}$

$L(b) = \min(22, 8 + 6 + 10) = 22$

$\mathbf{L(c) = \min(16, 8 + 6 + 3) = 16}$

$L(e) = \min(\infty, \infty) = \infty$

$L(z) = \min(\infty, 8 + 6 + 9) = 23$

(v) $v = c$

Permanent label of c = 16

$P = \{a, d, f, c\}, T = \{b, e, z\}$

$\mathbf{L(b) = \min(22, \infty) = 22}$

$L(e) = \min(16, 8 + 6 + 3 + 6) = 23$

$L(z) = \min(23, 8 + 6 + 3 + 10) = 23$

(vi) $v = b$

Permanent lable of b = 22

$P = \{a, d, f, c, b\}, T = \{e, z\}$

$L(e) = \min(\infty, 8 + 6 + 3 + 10 + 2) = 23$

$L(z) = \min(23, 2 + 6 + 3 + 10 + 2) = 23$

(vii) $v = e$

Permanent label of e = 23

$P = \{a, d, f, c, b, e\}, T = \{z\}$

$L(z) = \min(23, 8 + 6 + 3 + 10 + 2 + 4) = 23$

(vii) $v = z$

Permanent label of z = 23

Hence the length of shortest path

From a to z is 23 and shortest path is a - d - f - z .

UEx. 4.2.7 (SPPU(IT) : Q. 4(b), May 15, 6 Marks)

Find the shortest path using Dijkstra's Algorithm for the given graph.

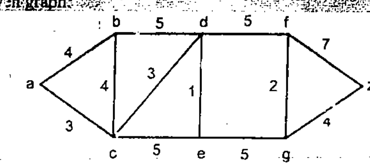


Fig. Ex. 4.2.7

☑ **Soln. :**

(i) $P = \phi, T = \{a, b, c, d, e, f, g, z\}$

$L(a) = 0, L(x) = \infty$

$\forall x \in T, x \neq a.$

(ii) P = {a}, T = {b, c, d, e, f, g, z}

L (b) = min {∞, 4} = 4

L (c) = min (∞, 3) = 3

L (d) = min (∞, ∞) = ∞

L (e) = min (∞, ∞) = ∞

L (f) = min (∞, ∞) = ∞

L (g) = min (∞, ∞) = ∞

L (z) = min (∞, ∞) = ∞

(iii) v = c

The permanent label of c = 3

P = {a, c}, T = {b, d, e, f, g, z}

L (b) = min (4, 3 + 4) = 4

L (d) = min (∞, 3 + 3) = 6

L (e) = min (∞, 3 + 5) = 8

L (f) = min (∞,∞) = ∞

L (g) = min (∞, ∞) = ∞

L (z) = min (∞, ∞) = ∞

(iv) v = b ?

The permanent label of b = 4

P = {a, c, b}, T = {d, e, f, g, z}

L (d) = min (6, 3 + 4 + 5) = 6

L (e) = min (8, ∞) = 8

L (f) = min (∞, ∞) = ∞

L (g) = min (∞, ∞) = ∞

L (z) = min (∞, ∞) = ∞

(v) v = d

Permanent label of d = 6

P = {a, c, b, d}, T = {e, f, g, z}

L (e) = min (8, 6 + 1) = 7

L (f) = min (6 + 5) = 11

L (g) = min (∞, ∞) = ∞

L (z) = min (∞, ∞) = ∞

(vi) v = e

Permanent label of e = 7

P = {a, c, b, d, e}, T = {f, g, z}

L (f) = min (11, ∞) = 11

L (g) = min (∞, 7 + 5) = 12

L (z) = min (∞, ∞) = ∞

(vii) v = f

Permanent label of f is 11

P = {a, c, b, d, e, f}. T = {g, z}

L (g) = min (12, 11 + 2) = 12

L (z) = min (∞, 11 + 7) = 18

(viii) v = g

The permanent label of g is 12

P = {a, c, b, d, e, f, g}, T = {z}

L (z) = min (18, 12 + 4) = 16

(ix) v = z

Permanent label of z is 16, hence the length of the shortest path from a to z is 16.

and path is a-c-d-e-g-z

4.2.2 Travelling Salesman Problem

Travelling salesman problem reduces to find the Hamilton circuit in the given graph with minimum weight.

Nearest – Neighbour method

- **Step 1 :** Start with arbitrary vertex. Choose the nearest vertex from it, construct the path as given step 2.
- **Step 2 :** Let v_n denotes latest vertex that was added to the path. Select the vertex v_{n+1} closet to v_n from all vertices that are not in the path and add this vertex to the path.
- **Step 3 :** Repeat step 2 until all the vertices of the graph G are included in the path.
- **Step 4 :** Lastly from a circuit by adding the edge connecting the starting vertex and the last vertex added.

The circuit obtained using the nearest – neighbor method will be the required Hamiltonian circuit.

Ex. 4.2.8 : Use nearest – neighbour method to find the Hamiltonian circuit, starting from a in the following graph. Find its weight.

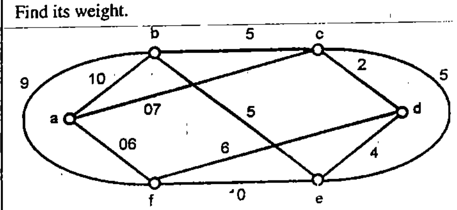


Fig. Ex. 4.2.8

☑ **Soln. :**

(i) Start from a from a nearest is f

Hence path = { a, f }

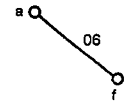


Fig. Ex. 4.2.8(a)

(ii) Path = { a, f, d }

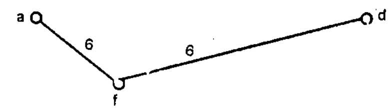


Fig. Ex. 4.2.8(b)

(iii) Path = { a, f, d, c }

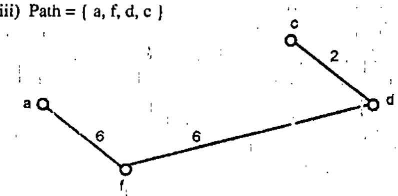


Fig. Ex. 4.2.8(c)

(iv) Path = {a, f, d, c, e }

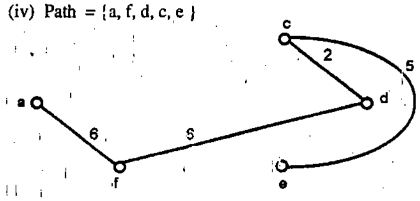


Fig. Ex. 4.2.8(d)

(v) Path = { a, f, d, c, e, b }

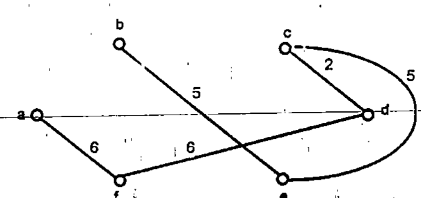


Fig. Ex. 4.2.8(e)

(vi) Hamilton circuit

= { a, f, d, c, e, b, a }

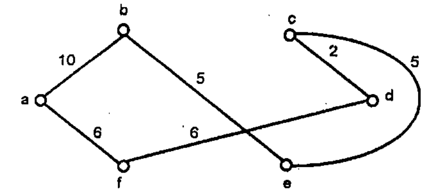


Fig. Ex. 4.2.8(f)

Total weight = 34

Ex. 4.2.9 : Use nearest neighbour method to find Hamilton circuit starting from a.

☑ **Soln. :**

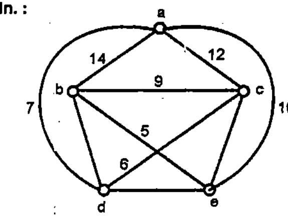


Fig. Ex. 4.2.9

(i) Starting from point a

(ii) Taking the nearest edge from point a i.e. d

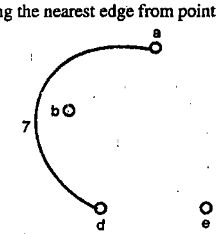


Fig. Ex. 4.2.9(a)

(iii) Now, we are at d, take the nearest edge from d i.e. c

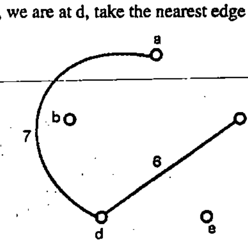


Fig. Ex. 4.2.9(b)

(iv) From, c take the nearest edge which is ce

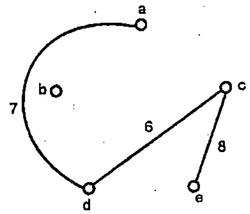


Fig. Ex. 4.2.9(c)

(v) From e check the next nearest edge which is eb.

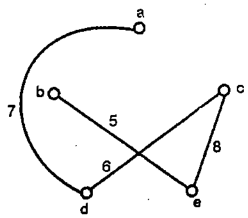


Fig. Ex. 4.2.9(d)

(vi) Hamilton circuit will be

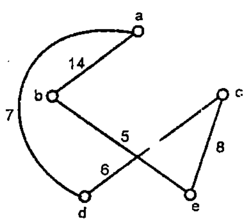


Fig. Ex. 4.2.9(e)

Weight of Hamilton circuit is 38 + 7 = 45

UEx. 4.2.10 (SPPU(IT) : Q. 3(b) (ii), May 15, 3 Marks)

Find the hamilton circuit using nearest neighbour method.

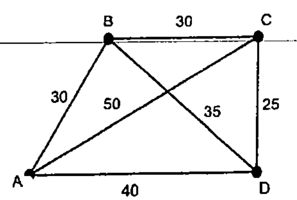
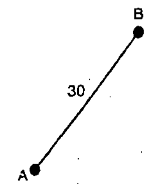


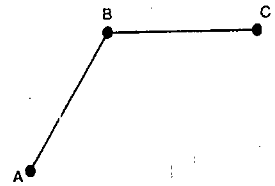
Fig. Ex. 4.2.10

☑ **Soln. :**

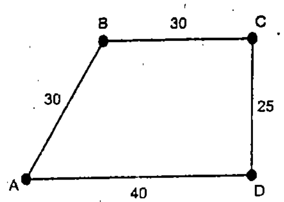
(i) Starting from point A. Taking the edge with minimum weight.



(ii) Taking nearest edge from B



(iii) Taking nearsest edge from C and complete circuit



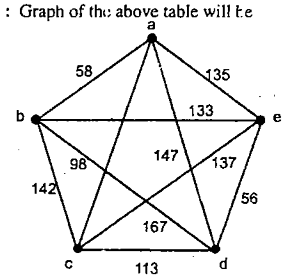
Weight of given Hamilton circuit is 30 + 30 + 40 + 25 = 125

UEx. 4.2.11 (SPPU(IT) : Q. 4(b), May 14, 6 Marks)

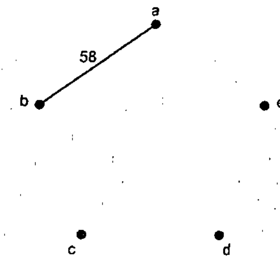
Solve the following travelling salesman problem using nearest neighborhood method.

	a	b	c	d	e
a	0	58	98	147	135
b	58	0	142	167	133
c	98	142	0	113	137
d	147	167	113	0	56
e	135	133	137	56	0

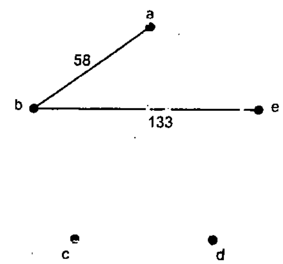
☑ **Soln. :** Graph of the above table will be



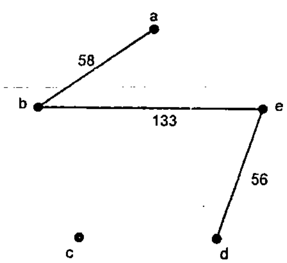
(i) Starting from point a taking the nearest edge



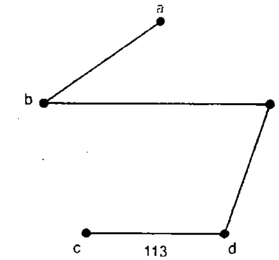
(ii)



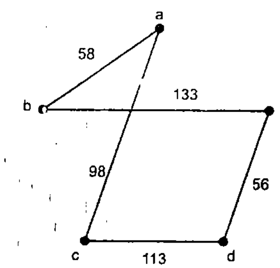
(iii)



(iv)



(v)



This is the required route using nearest neighborhood method.

Total distance = 58 + 133 + 56 + 113 + 98

= 458

4.2.3 Coloring of a Graph

One of the main way to color the vertices of G such that no two adjacent vertices have the same color it is called proper coloring.

☞ Chromatic number

The chromatic number of a graph G is the minimum number of colours required for the proper coloring of the graph G in this case X(G) = 3

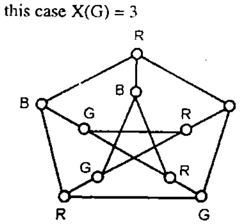


Fig. 4.2.3

R = Red , B = Blue, G = Green

From the definitions of coloring of graph, it is observed that :

1. Null graph (graph without edges) is 1-chromatic.
2. A complete graph k_n with n vertices is n-chromatic because all its vertices are adjacent. Hence a graph containing a complete graph of P vertices is at least P-chromatic.

   For example, a graph having a triangle is at least 3-chromatic.
3. A graph containing simply one circuit with number of vertices n ($n \geq 3$) is 2-chromatics, if n is even and 3-chromatics if n is odd.
4. Every 2-chromatic graph (with at least one edge) is bipartite, since coloring partitions the vertex set v into two subsets V_1 and V_2, such that no two vertices in V_1 or V_2 are adjacent.

Ex. 4.2.12 : How many colours required to colour $k_{m,n}$ why ?
(Graph G_1) or (Graph G_2)

☑ **Soln. :**

Chromatic number of any bipartite graph is 2.
e.g. $K_{2,3}$

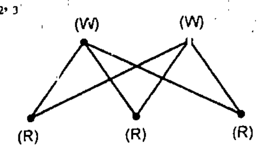


Fig. Ex. 4.2.12

Chapter Ends...

□□□

Note

UNIT III

CHAPTER 5 Trees

Syllabus

Tree Terminologies, Rooted Trees, Path Length in Rooted Trees, Prefix Codes, Spanning Trees, Fundamental Cut Sets and Circuits, Max flow –Min Cut Theorem (Transport Network).

Applications of Graph Theory.

5.1 INTRODUCTION

- In the last unit, you learned important concepts about graphs which included graph definition, representation, isomorphism and different algorithms.
- Now, in this unit, you will learn tree, basic terminologies of tree, types of tree and applications of trees.
- A tree is connected graph without a simple circuit. Trees were discovered by Kirchhoff in 1847, while investigating the electrical networks.
- In the nineteenth century, trees were used by sir Arthur cayley to count certain types of chemical compound.
- Since that time, tress were used to solve many problems of engineering. Particularly, of computer science.
- In computer science, trees are employed to solve searching, sorting of elements. Trees are also used to construct Huffman coding, which is responsible for saving cost of data transmission and storage.
- It also used to developed games, to construct networks for syntax checking and many more.

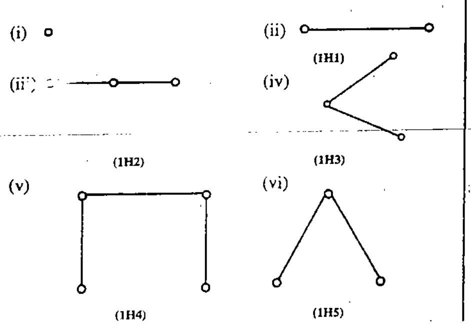
5.2 TREE

> **Definition :** *A tree is a simple connected graph without any circuit.*

A tree is a non-linear structure which has many applications, like, searching, sorting, game theory, network flow, information storage and many more.

Example 1

Following are the examples of trees with varying number of nodes (elements or vertices)



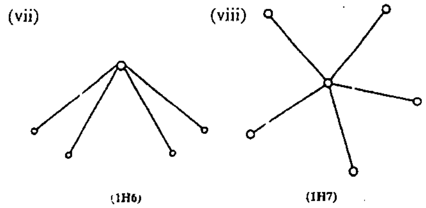
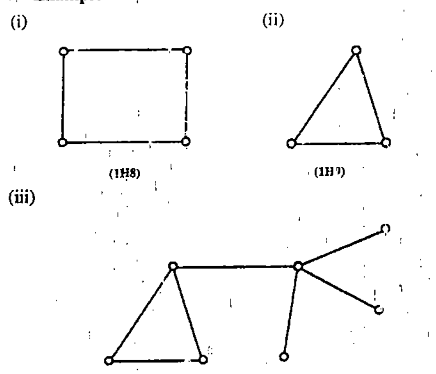


Fig. 5.2.1

- In the Fig. 5.2.1 all are trees because they are connected graph without any circuit.

Example 2



(1H10)Fig. 5.2.2

The Fig. 5.2.2 are not trees because, they contains cycles but they are connected.

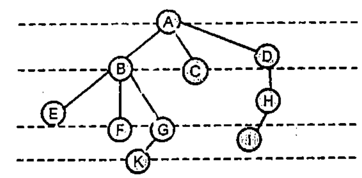
> **Note :** Edges are connection line between two nodes.

5.3 BASIC TERMINOLOGIES

> **GQ.** *Define with example : Level and height of a tree.*

- As we have understood that, tree is simple connected acyclic graph with collection of nodes called as elements. Each node of tree holds some value.
- It may have some edges which connects the different vertices or nodes.
- In general, tree has two parts called as left subtree and right subtree.

- To understand the tree concepts it is very much important to know basic terminologies associated with the trees.
- Consider a tree,



(1H11)Fig. 5.3.1

(i) Root of the tree

- A root node is the origin of the tree.
- Whole tree structure accessed from the root node. In a tree, root node is Unique (only one). Root node does not have parent node. In Fig. 5.3.1, node Ⓐ is root node.

(ii) Degree of node

- The degree of node is the number of child nodes are connected to it.
- In the Fig. 5.3.1, node Ⓐ and Ⓑ has degree 3, node Ⓓ, Ⓖ and Ⓗ has degree 1, Ⓒ, Ⓔ, Ⓕ, Ⓚ and Ⓘ has degree 0

(iii) Path of a tree

A path is nothing but, a sequence of edges between "that" nodes. There are multiple paths are exists in a single tree. In Fig. 5.3.1, A→B→G→K is a one path.

(iv) Parent node

Parent node of any node is a Immediate predecessor of that node. In Fig. 5.3.1 node Ⓑ is parent node of Ⓔ, Ⓕ and Ⓖ nodes.

(v) Child node

- Child node is immediate successor of any node.
- In Fig. 5.3.1, nodes Ⓑ, Ⓒ and Ⓓ are childs nodes of root node.

(vi) Siblings

Siblings are nodes with same parents. Example, nodes, Ⓔ,Ⓕ and Ⓖ are siblings which belongs to the parent nodeⒷ.

(vi) Leaf node

A node with no children element. In other way, a node whose degree is zero is called as leaf node.

(viii) Internal node

Internal node is a node whose degree is atleast one. Internal nodes are present between root node and leaf nodes.

e.g. nodes Ⓑ, Ⓓ, Ⓖ and Ⓗ are internal nodes.

(ix) External node

External nodes are same as like leaf node whose degree is zero.

(x) Level of tree

The level of the tree is represented by number of stages from root node to leaf node. On each stage, there are some nodes available. Generally, root node is at level 0. In Fig. 5.3.1, root node Ⓐ is at level 0, nodes Ⓑ, Ⓒ and Ⓓ at level 1 likewise, leaf nodes Ⓚ and Ⓘ are at level 3. So, level of tree is the maximum level of leaf node.

(xi) Depth of node

The depth of node is the number of edges from the node to the root node of the tree.

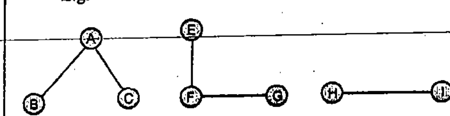
(xii) Height of a tree

The height of the tree is the number of edges on the longest downward path between the root node and a leaf node.

(xiii) Forest

Forest is a collection of disjoint trees. In other way, forest is a collection of an acyclic graph which is not connected.

E.g.



(1H12)Fig. 5.3.2 : Forest

(xiv) Degree of a tree

The maximum degree of any node in the tree is called degree of tree.

In Fig. 5.3.1, the degree of tree is 3.

5.4 PROPERTIES OF TREE

GQ. Explain the following terms : Properties of [illegible]

GQ. Prove that the number of vertices is one more than the number of edges in a tree.

(a) A graph G is a tree if and only if there exists a unique path between every distinct pair of vertices of G.

Proof

Suppose, G is a tree, then G is connected and without circuits. We know that, If graph G has circuit then It forms two or more paths between two nodes. But, G has no circuit. So it has unique path between every pair of vertices.

(b) A graph G of n vertices is a tree if and only if G is connected and has exactly (n – 1) edges

Proof

Assume, graph G is a tree. Then it is connected and acyclic graph.

We can prove graph G has (n – 1) edges when it has n-vertices by using mathematic induction.

Basis of Induction

Let, graph G has n = 1 vertex then

It has 0 edges.

Induction Step

For n = K + 1 that is n > 1

Let, e′ is on edge which act as a bridge edge or isthmus of graph G.

~~Consider, G – e′ for e′ ∈ E (G).~~

If we eliminate the e′from G, then graph G is converted into disconnected graphs say G_1 and G_2.

Here, G_1 and G_2 are connected and acyclic graph as G_1 and G_2 are subgraphs of G.

Let, G_1 has n_1 vertices and m_1 edges

G_2 has n_2 vertices and m_2 edges

According to basis of induction principle

$m_1 = n_1 - 1$ and $m_2 = n_2 - 1$

so, number of edges in graph G is shown by

$$e = (e_1 + e_2) + 1$$

$$= n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1$$

$$e = n - 1 \qquad \because (n_1 + n_2 = n)$$

Hence, from the above statement it is clear that, graph G will have n – 1 edges.

Hence proved.

5.5 CENTRE OF A TREE

To understand the centre of a tree, It is very important to get idea of eccentricity of a vertex. First, let's understand the concept of eccentricity of vertex, then understand centre of a tree.

5.5.1 Eccentricity of a Vertex

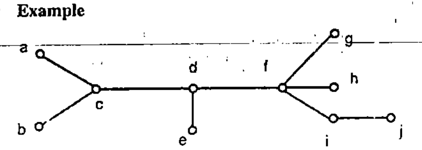
GQ. Define the following terms :
(i) Level and height of a tree
(ii) Eccentricity of a vertex

- Let, G be the connected graph and v be the vertex, ∈ V(G). The eccentricity of a V is defined as the maximum distance of v from the farthest vertex V_i in G.
- It is denoted as E (V) or e (V).
- Mathematically, Eccentricity of vertex V is
  $E(v) = \max \{d(V, V_i) \mid V_i \in V(G)\}$

5.5.2 Centre of a Graph

- A vertex in a graph G with minimum eccentricity is a centre of G.
- The minimum eccentricity of vertex is also called as radius of G. denoted as r(G).

Example



(1H13)Fig. 5.5.1

The eccentricities of all vertices as given below.

e(a) = 5, e(b) = 5 e(c) = 4, e(d) = 3.

e(e) = 4, e(f) = 3, e(g) = 4, e(h) = 4,

e(i) = 4, e(j) = 5

The minimum eccentricity is 3 for vertices d, f

Hence the vertices d and f are centres of G that is r(G) = 3.

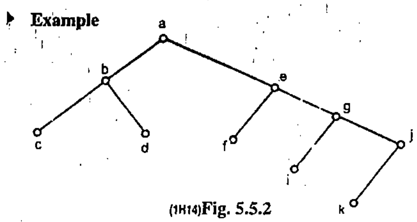
Note : In general, the eccentricity of vertex is the maximum number of edges covered to reach farthest vertex from given vertex.

Imp. Note : Every tree has either one or two centres with minimum eccentricity. In above example, There are only two vertices d and f has minimum eccentricity. Hence, the tree has 2 centres d and f.

5.5.3 Cut Vertex of a Tree

- A cut vertex is a vertex, whose removal from the Tree T, which disconnects the tree.
- In other way, A cut vertex is a vertex whose removal from tree T then tree will have two parts.
- In any tree, all the vertices except pendant vertices (End vertex or degree 1 vertex) are cut vertices.

Example



(1H14)Fig. 5.5.2

In above tree, vertices ⓐ, ⓑ, ⓔ, ⓖ and ⓙ are cut vertices and vertices ⓒ, ⓓ, f, ⓘ and ⓚ are pendant vertices.

5.6 ROOTED TREE

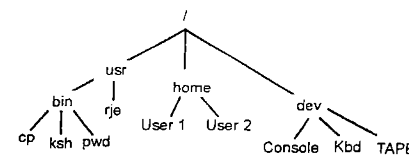
UQ. Define rooted tree.

- Normal tree can be converted into rooted tree by considering particular vertex designated as the root and specifying directions to the edges.

Definition : A rooted tree is a tree in which one vertex has aesignated as the root and every edge is directed away from the root.

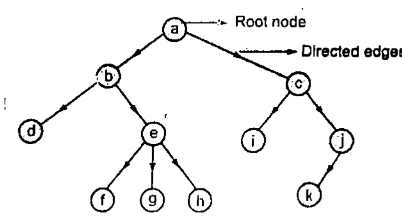
- In other way, a directed tree is said to be rooted tree if there is exactly one vertex whose incoming degree is zero (Root vertex) and all other vertices are of degree 1.
- **Root vertex :** Vertex with incoming degree zero (0).
- **External vertex or leaf vertex :** The vertex whose outgoing degree is 0.
- In a directed rooted tree, a vertex whose outgoing degree is non-zero is called a branch node or internal node.
- A rooted tree used to model the file system structure, family representation, Representing organization, use to model computer networks.

Example 1 :



(1H15)Fig. 5.6.1: Computer file system

Example 2 :



(1H1..)Fig. 5.6.2 : Rooted tree

5.6.1 M-ary tree

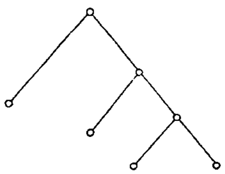
UQ. Explain the following terms : M-ary tree.

A M-ary tree is a rooted tree in which every internal node has at most m number of childs.

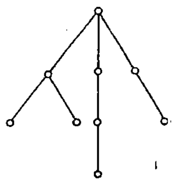
Regular m-ary tree

M-ary tree is said to be regular or full m-ary tree if every node has exactly m childs.

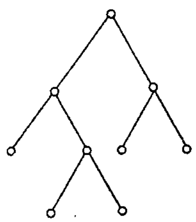
▸ **Example**



2- ary tree
(1H17)



3- ary tree
(1H18)



Full - 2- ary tree
(1H19)**Fig. 5.6.3**

☞ **Theorem :**

A regular m-ary tree with x internal nodes has mx + 1 nodes at all

Proof

- Let, T be a regular m-ary tree with n vertices. From the n vertices, there are x internal vertices.
- So, leaf nodes in T is denoted by $l = n - x$
- We know that, In given regular graph there are x internal nodes
- So, The regular m-ary tree will have m.x childs. But, Root node is not a child.
- Therefore, given tree has total (m.x + 1) number of vertices
- So, $n = m \cdot x + 1$ Hence proved

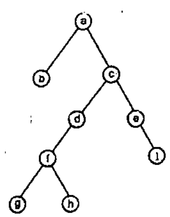
5.7 BINARY TREE

UQ. Explain binary tree, with suitable examples.

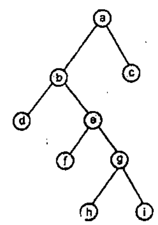
- Till now we have discussed the tree and some basic terminologies associated with it. Now, we shall discussed the another important class of tree, which has many advantages, called as binary tree.
- They are used at bigger scale in mathematics and in computer science fields.
- Lets say, for searching objects, in compression techniques, for yes or no type of coding implementations and many more.

❑ *Definition : A binary tree is a tree, in which every node or vertex has atmost 2(two) childs.*

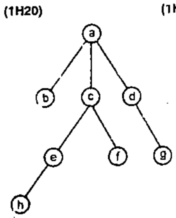
- In other sense, In binary tree each node can have zero (0), one (1) or two (2) childs but not more than 2 childs.
- **Regular Binary Tree :** It is a binary tree, in which every branch node has exactly 2 childs or zero (0) childs.
- To understand the concept In depth, lets take simple example as given below.



i) Binary tree
(1H20)



i) Full binary tree
(1H21)



iii) Not binary tree
(1H22)**Fig. 5.7.1**

➡ **Note 1 :** The maximum number of vertices in full binary tree with h levels is $1 + 2 + 2^2 + \ldots + 2^h = 2^{h+1} - 1$

➡ **Note 2 :** In a binary tree with N nodes. The minimum possible height or minimum number of levels is $[\log_2 (n + 1) - 1]$ and maximum possible height is $\frac{n-1}{2}$. Assume that the Root of the tree is at 0 level.

5.8 APPLICATIONS OF BINARY TREE

5.8.1 Binary Tree Traversal

UQ. Define the following terms with reference to the tree : Tree Traversal

(SPPU (IT) - Q. 5(a). May 16, 6 Marks)

- **Traversing :** Traversing is nothing but visiting each node in a tree, exactly once.
- Traversing is done for processing or checking or updating the value of node.
- Tree traversal is also called as tree search or walking the tree.
- The linear data structures like arrays, queue, linked list are traversed in fixed manner but binary tree as non-linear data structure traversed in many ways.
- Two basic ways to traverse a binary tree.

(a) Depth-First Traversal

- Depth-First Traversal method uses backtracking to traverse each node of tree.
- In DFS, start with the initial node of tree and then go deeper or downwards until we get leaf node.
- Then again backtracks from the leaf node towards the node which not yet explored or visited.
- There are many uses of this depth first search traversing method including for detecting cycle in a graph, for path finding to test if graph is bipartite, for solving puzzles with only one solution

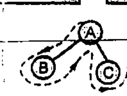
☞ **Types of depth-first traversal**

(1) Pre-order traversal

In this technique, the root node is first traverse, followed by left subtree and then the right subtree traversed.

→ Root node → Left subtree → Right subtree

▸ **Example 1**

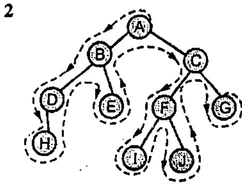


Pre-order traversal of example 1

A → B → C

(1H23)

▸ **Example 2**



(1H24)

Pre-order traversal of example 2

A → B → D → H → E → C → F → I → J → G

(1H25)

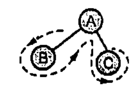
(2) In-order Traversal

In this traversal technique, the subtree processed first, then root node and lastly right subtree processed.

→ Left subtree → Root node → Right subtree

(1H25A)

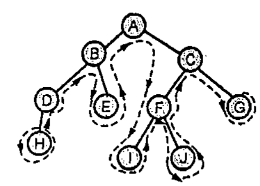
▸ **Example 1**



In-order traversal of example 1 B → A → C

(1H26)

▸ **Example 2**



(1H27)

In-order traversal of example 2

H → D → B → E → A → I → F → J → C → G

(1H28)

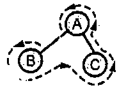
(3) Post-order traversal

In this method, left subtree processed first then right subtree and lastly root node processed.

Left-subtree → Right-subtree → Root

(1H29)

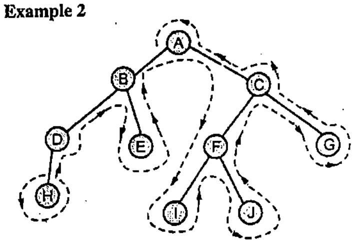
Example 1



Post order traversal → B → C → A

(1H30)

Example 2



(1H31)

Post-order traversal ⇒

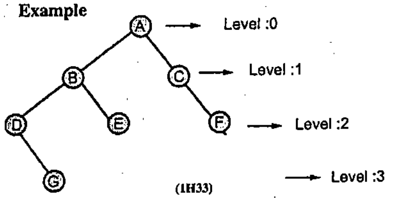
Post order traversal → H → D → E → B → I → J → F → G → C → A

(1H32)

(b) Breadth-first traversal

- In breadth first traversal method, the processing of nodes is done by level by level.
- Once all the elements of level-0 get explored, then algorithm will explored the elements from level-1
- Algorithm go, until last level elements get processed.

Example



(1H33)

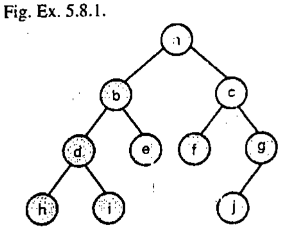
- First, element A from level : 0 processed, then go to level : 1 for B and C, then go to level : 2 for D, E and F and finally go to level 3 for G

BFS of given tree is

A → B → C → D → E → F → G

(1H34)

Ex. 5.8.1 : Find the pre, post and inorder traversal of a tree shown in Fig. Ex. 5.8.1.



1E69a

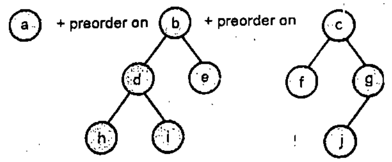
Fig. Ex. 5.8.1

☑ **Soln. :**

Given :

(1) Pre-order traversal of given tree by

In pre-order traversal : First root processed, then left subtree and lastly right subtree

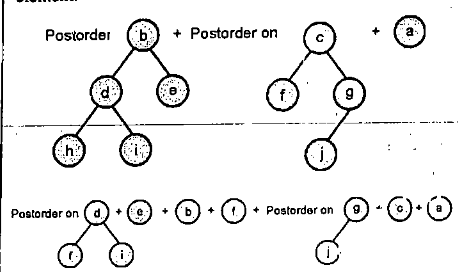


⇒ a + b + d + h + i + e + c + f + g + j

⇒ preorder traversal {a b d h i e c f g j}

(2) Postorder traversal

Traverse left subtree, then right subtree and lastly root element.



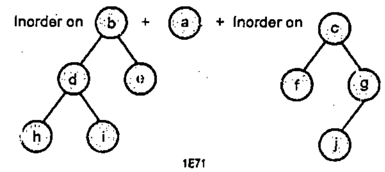
1E70

⇒ h + i + d + e + b + f + j + g + c + a

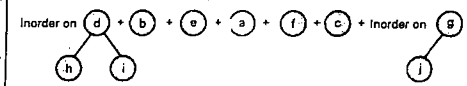
⇒ {h i d e b f j g c a}

(3) Inorder traversal

First traverse left subtree, then root element and lastly right subtree



1E71



1E72

⇒ h + d + i + b + e + a + f + c + j + g

⇒ {h d i b e a f c j g}

UEx. 5.8.2 SPPU (IT) - Q. 5(a); Dec. 19. 7 Marks

Build a binary search tree for the words banana, peach, apple, pear, coconut, mango and papaya using alphabetical order. Write sequence of visiting words in preorder and post-order traversal.

☑ **Soln. :**

Given :

- Words ⇒ Banana, Peach, Apple, Pear, Coconut, Mango and Papaya.
- Binary search tree is a kind of binary tree in which small weight item is insert into left subtree and big weight item insert into right subtree of node.

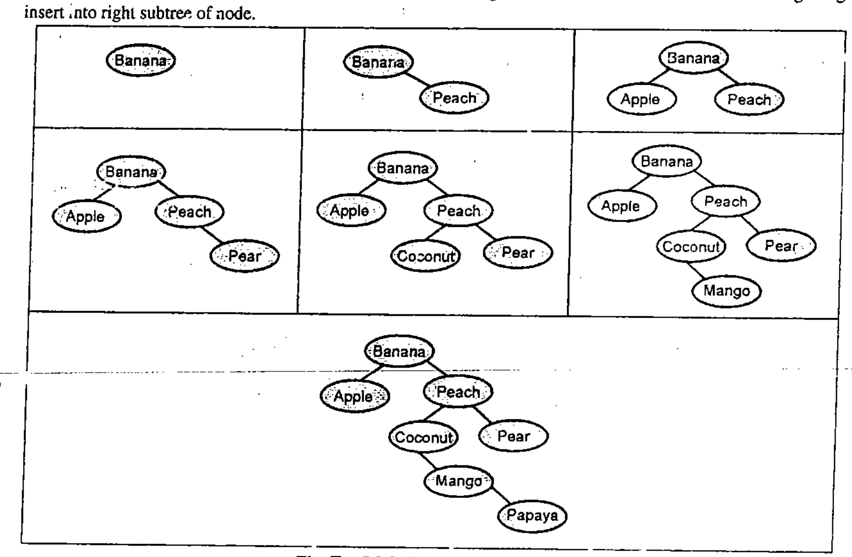
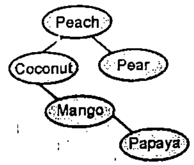


Fig. Ex. 5.8.2 : Binary search tree

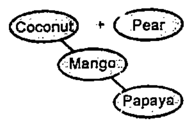
i) Pre-order Traversal

Root-node ⟶ Left subtree ⟶ Right-subtree

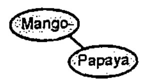
Banana + Apple + Pre-order on



Banana + Apple + Peach + Pre-order on



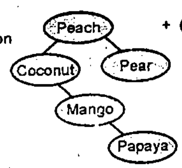
Banana + Apple + Peach + Coconut + Pre-order on



Banana + Apple + Peach + Coconut + Mango + Papaya

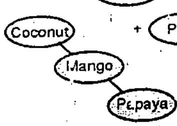
II) Post order Traversal

⇒ Apple + Post-order on



+ Banana

⇒ Apple + Post-order on



+ Pear + Peach + Banana

⇒ Apple + Post-order on

+ Coconut + Pear + Peach + Banana

⇒ Apple + Papaya + Mango + Coconut + Pear + Peach + Banana

UEx. 5.8.3 (SPPU (IT) - Q. 5(b), Dec. 19, 6 Marks)

Determine the order in which a preorder, postorder and inorder traversal visits the vertices of the given ordered rooted tree.

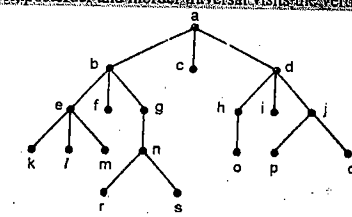
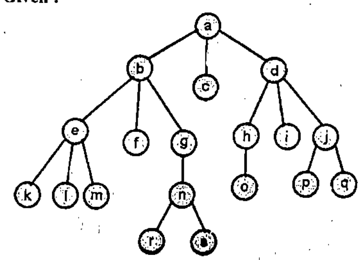


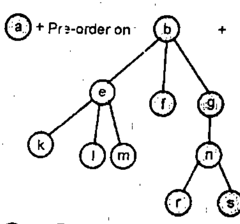
Fig. Ex. 5.8.3

☑ **Soln. : Given :**

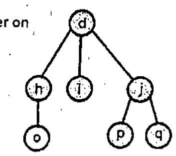


I) Preorder : Root ⟶ Left subtree ⟶ Right-subtree

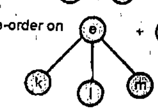
⇒ a + Pre-order on



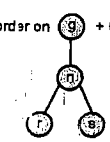
+ c + Pre-order on



⇒ a + b + Pre-order on

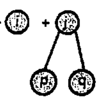


+ f + Pre-order on

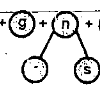


+ c + d +

+ i +



⇒ a + b + e + k + l + m + f + g +

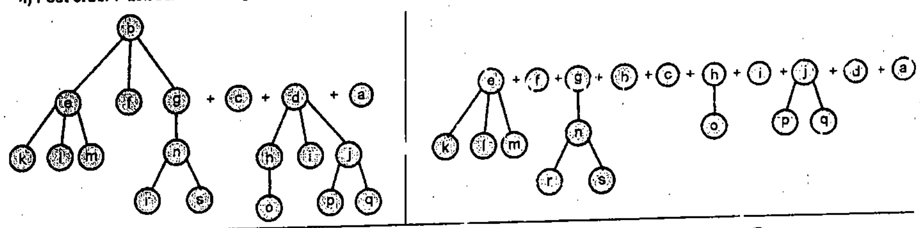


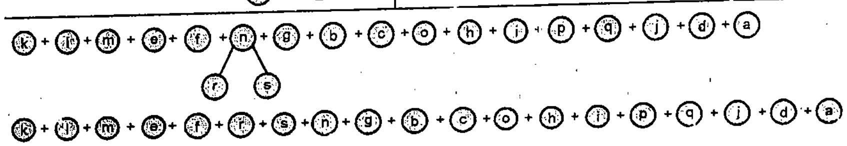
+ c + d + h + o + i + j + p + q

⇒ a + b + e + k + l + m + f + g + n + r + s + c + d + h + o + i + j + p + q

⇒ $\boxed{\text{a b e k l m g n r s c d h o i j p q}}$ Pre-order

II) Post order : Left subtree → Right subtree → Root

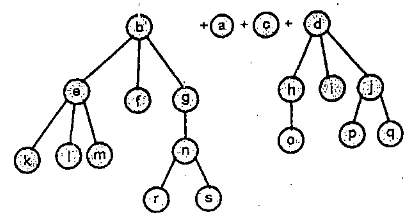


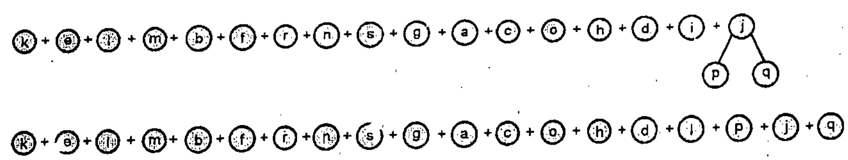


k	l	m	e	f	r	s	n	g	b	c	o	h	i	p	q	j	d	a

Post-order

III) Inorder : Left subtree → Root → Right subtree





k	e	l	m	b	f	r	n	s	g	a	c	o	h	d	i	p	j	q

Inorder

UEx. 5.8.4 (SPPU (IT) - Q. 6(a), Dec. 16, 7 Marks)

Determine the preorder, postorder and inorder traversal of the following binary tree shown in Fig. Ex. 5.19.

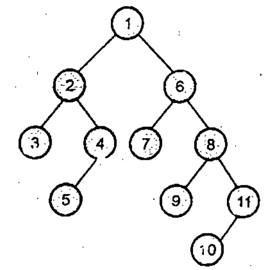
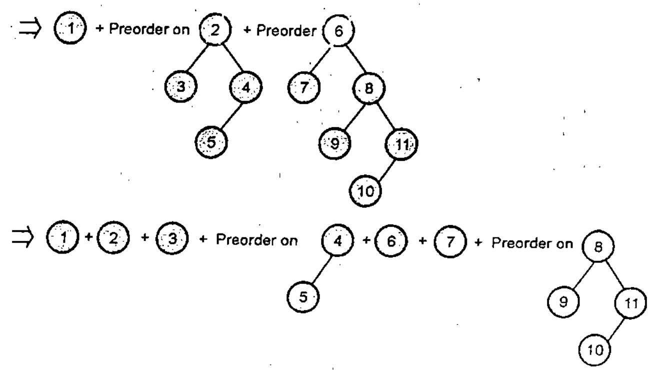


Fig. Ex. 5.8.4

☑ **Solution :**

Given :

1. **Preorder traversal**



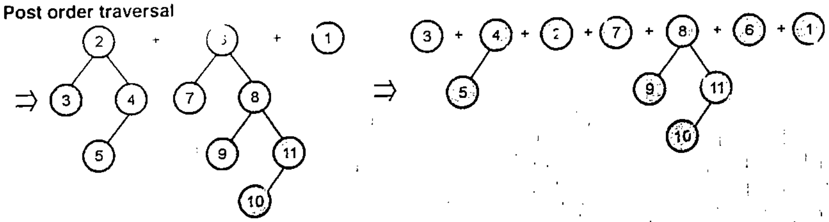
⇒ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 11 + 10

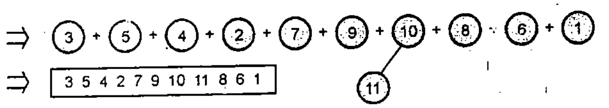
⇒

1	2	3	4	5	6	7	8	9	11	10

• NOTES •

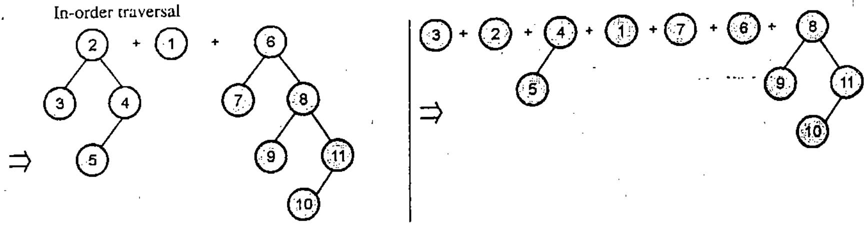
2. **Post order traversal**





$\Rightarrow$ | 3 5 4 2 7 9 10 11 8 6 1 |

3. In-order traversal



$\Rightarrow$ ③ + ② + ⑤ + ④ + ① + ⑦ + ⑥ + ⑨ + Ⓔ + ⑩ + ⑪

$\Rightarrow$ | 3 2 5 4 1 7 6 9 8 10 11 |

UEx. 5.8.5 (SPPU (IT) - Q. 5(a), Dec. 15, 6 Marks)

Construct a binary tree from given inorder and preorder traversals :

Inorder : AEBDCFGKIHJL,

Preorder : FEADBCGHIKJL

☑ **Soln. :**

Given :

Inorder : A E B D C F G K I H J L

Preorder : F E A D B C G H I K J L

From the pre-order sequence. It is clear that F is the root of the tree because in preorder sequence (Root →leftsubtree→ right subtree), Root is appear first in the sequence.

➡ **Note :** Pre order sequence gives root of tree or subtree. Inorder sequence gives left or right subtree.

Here, Ⓕ is the Root of the tree.

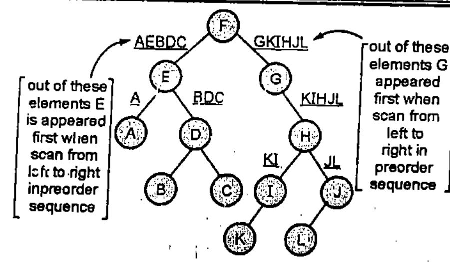


Fig. Ex. 5.8.5 : Binary tree

UEx. 5.8.6 (SPPU (IT) - Q. 6(b), May 15, 7 Marks)

Construct a binary tree with the inorder traversal as 3, 5, 6, 7, 10, 12, 13, [illegible] 12, 10, 6, 7, 13, 16, 20, 18, 23

☑ **Soln. :**

Given :

Inorder traversal

3 5 6 7 10 12 13 15 16 18 20 23

Preorder traversal

15 5 3 12 10 6 7 13 16 20 18 23

Inorder traversal

Left subtree→ Root → Right subtree

Preorder traversal

Root → Left subtree → Right subtree

From the preorder, we can get the root of the tree or subtree from the inorder, we can get left subtree and right subtree.

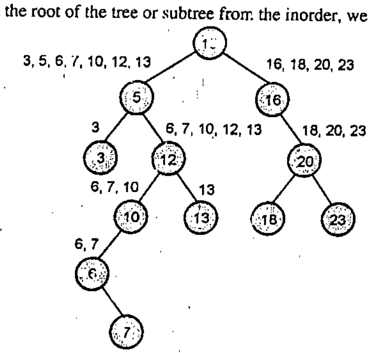


Fig. Ex. 5.8.6 : Binary tree

5.8.2 Binary Expression Tree

- Mathematical expression are in the form of operators and operands which can be expressed or represented with the help of expression tree.
- Expression tree is a kind of binary tree.

Properties of expression tree

(i) Each leaf node is an operand.

(ii) The root and internal nodes are considered as operators.

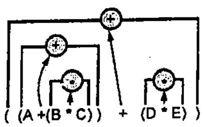
(iii) Sub trees are treated as sub expressions.

- In this study, we are going to consider only basic arithmetic operations like +, –, ×, / etc.
- A preorder traversal on the expression tree gives "prefix equivalent" of the expression.
- In the same way, post order traversal gives "postfix equivalent" of the expression.
- An inorder traversal gives "Infix equivalent" of the expression.
- **How to convert expression into binary expression tree ?**
- **Examples :** Expression A + B * C + D * E

- **Step-I :** Fully parenthesize the expression first then parenthesize the high priority operators like *, / and then low priority operators.

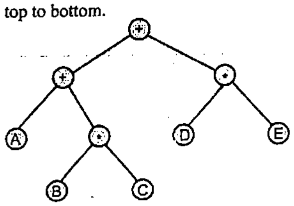
  ⇒ A + (B * C) + (D * E) ⇒ ((A + (B * C)) + (D * E))

- **Step-II :** Move the operators at the centre of group.



(1H35)

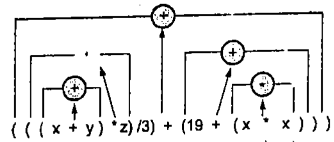
- **Step-III :** Create a binary tree by scanning operators from top to bottom.

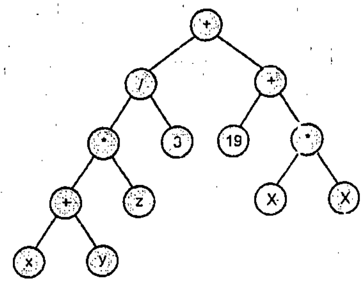


(1H36)**Fig. 5.8.1 : Binary expression tree**

Ex. 5.8.7 : Construct the labeled tree of the following algebraic expression. (((x + y) * z) / 3) + (19 + (x * x))

Soln. :

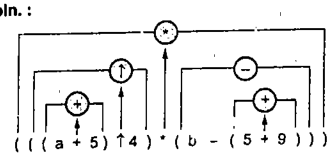




1E162**Fig. Ex. 5.8.7 : Binary expression tree**

Ex. 5.8.8 : Represent the expression ((a + 5) ↑ 4) * (b – (5 + 9)) using binary tree.

Soln. :



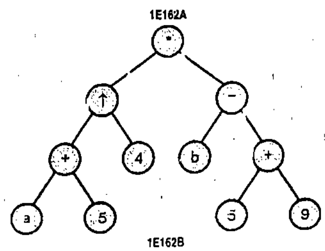


Fig. Ex. 5.8.8 : Binary expression tree

UEx. 5.8.9 (SPPU (IT) - Q. 6(a), Dec. 19, 7 Marks)

What is expression tree?

Represent the expressions (i) (x + xy) + (x / y) (ii) x + ((xy + x) / y) using binary trees. Write each of these expressions in

(a) prefix notation (b) postfix notation

Solution :

(i)

(x + xy) + x/y

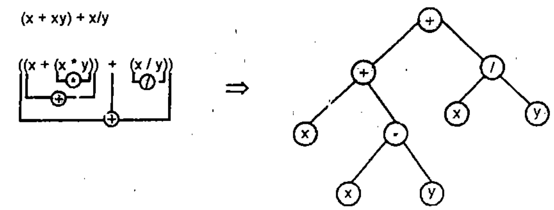
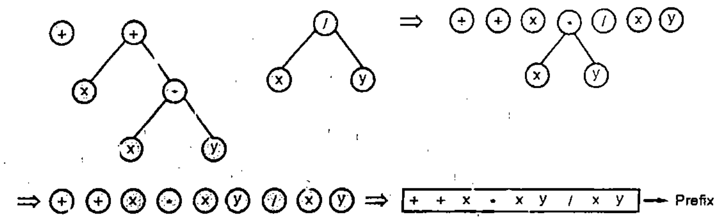
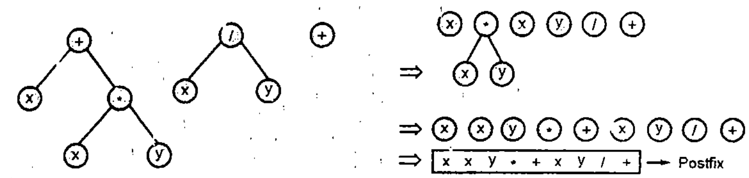


Fig. Ex. 5.8.9 : Expression tree

a) Prefix :-



b) Postfix :-



• NOTES •

(ii) x + (xy + x/y)

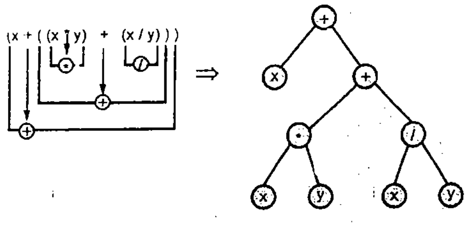
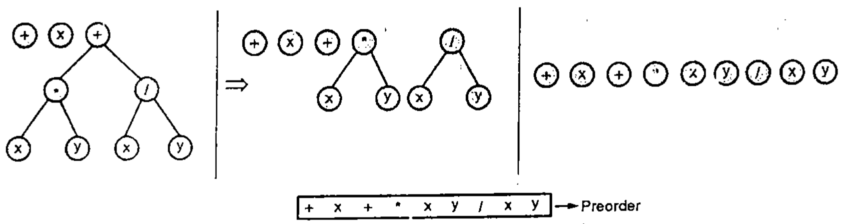
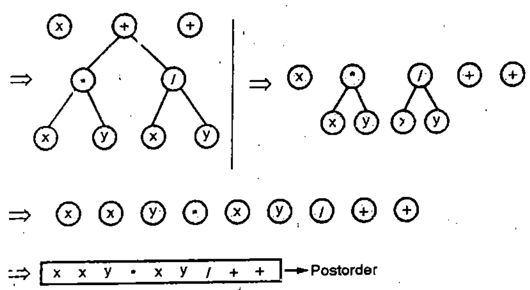


Fig. Ex. 5.8.9(a)

a) Pre order Traversal



b) Post order Traversal



UEx. 5.8.10 (SPPU (IT) - Q. 6(a), Dec. 18, 7 Marks)

Construct the labeled tree of the following algebraic expression :

$(((X + Y) * Z) / 3) + (19 + (x * x))$

Soln. :

Given :

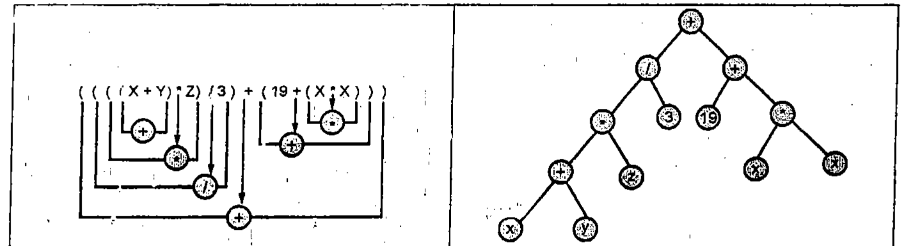


Fig. Ex. 5.8.10 : Expression tree

UEx. 5.8.11 (SPPU (IT) - Q. 6(b), May 14, 7 Marks)

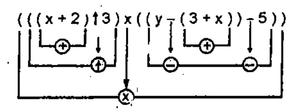
Represent the following expression in :

(i) Using a binary tree (ii) Post-fix notation (iii) Pre-fix notation

$((x + 2) \uparrow 3) \times (y - (3 + x)) - 5$

Soln. :

Given :



(i)

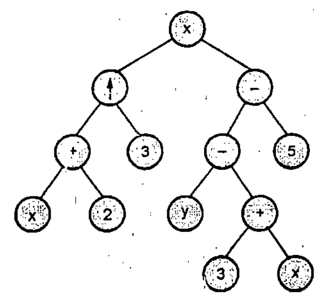
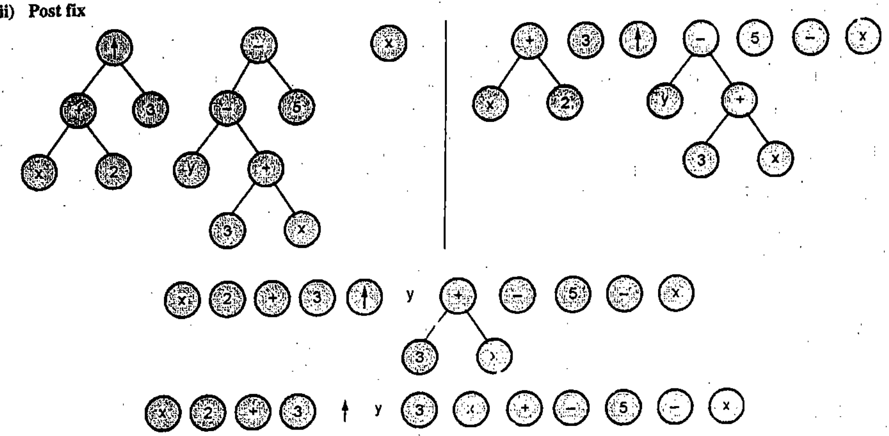
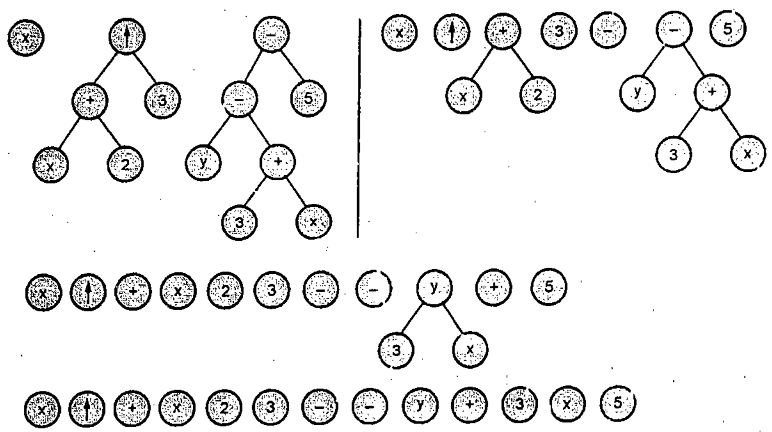


Fig. Ex. 5.8.11 : Binary tree

(ii) Post fix

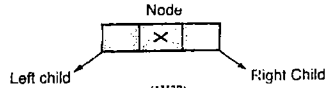


(iii) Prefix notation



5.8.3 Conversion of General Tree to Binary Tree

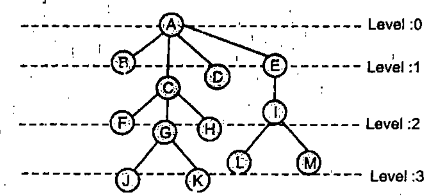
- To represent general tree in computers memory, it is difficult to represent because child per node is not limited to two.
- So, It is necessary to convert general tree to binary tree to represent in memory by using linked list.
- Linked list representation of binary tree is easy because childs per node are limited two only.



(1H37)

- Each node has exactly two links.

- First pointer of node points to it's leftmost child and other pointer points to it's next sibling.
- Lets take a simple example to understand the above procedure.



(1H38)**Fig. 5.8.2 : General tree**

For node Ⓐ

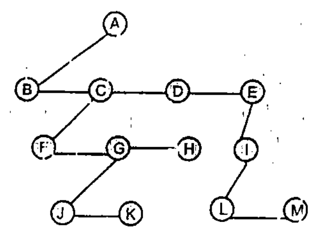
Lets start from root node, first pointer of node Ⓐ points to it's left most child i.e. Ⓑ and As the root node Ⓐ does not have sibling, then second pointer of Ⓐ is NULL.

For node Ⓑ

First pointer of Ⓑ is null since it does not have any left child and another pointer of Ⓑ points to it's siblings Ⓒ, Ⓓ and Ⓔ.

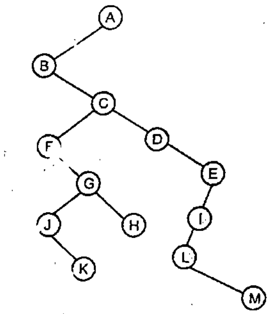
For node Ⓒ

- It's first pointer points to it's left most child that is Ⓕ and another pointer points to it's siblings Ⓓ and Ⓔ
- Follow the same procedure for all the nodes of the tree you will get following tree.



(1H39)**Fig. 5.8.3**

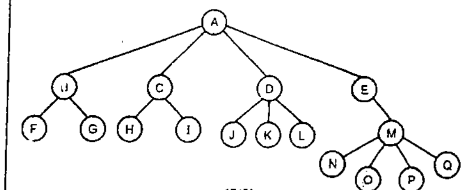
- We will get, our binary tree by rotating above tree by 45°.



(1H40)**Fig.5.8.4 : Binary tree**

> **Note :** To convert general tree into binary tree, follows Left child – Right sibling relation.

Ex. 5.8.12 : Convert the following tree into binary tree.

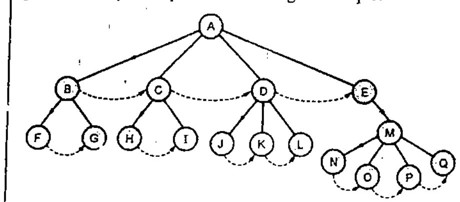


1E159

Fig. Ex. 5.8.12

☑ **Soln. :**

For converting given general tree into binary tree, we should have to use first pointers points left child and second pointer of any node points it's siblings technique.



1E160

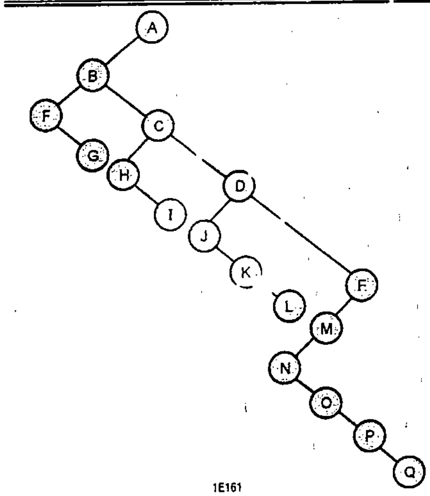


Fig. Ex. 5.8.12(a) : Binary tree

5.9 Prefix Code and Binary Search Trees

Prefix code

- Code is a sequence of bits assigned to characters.
- Information like text messages, images, computer program are represented in the form of o's and 1's.
- The code assigned to one character may or may not be the same length.
- Different types of coding system available like prefix code, infix code and postfix code.

> **Definition :** *prefix code is a code (bit sequence) assigned in such a way that the code assigned to one character is not prefix of code assigned to any other character in the set.*

- Prefix code is variable length code which means that codes assigned to characters are of different lengths.
- Consider a set of characters {a, b, c d} and code assign to it

a – 00

b – 01

c – 0

d – 1

This coding scheme leads to ambiguity because code assigned to C is prefix of codes assigned to a and b.

- So, If compressed bit stream is "0001", decompressed output may be "cccd" or "ccb" or "acd" or "ab".
- Here, original information not correctly understood.
- Lets take example of prefix code.

a – 000

b – 001

c – 01

d – 10

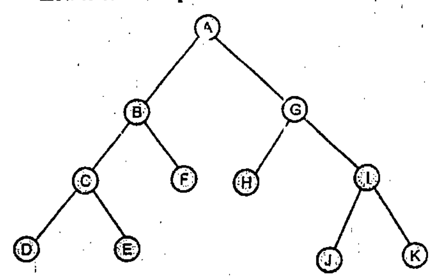
e – 11

Here, code assigned to one character is not prefix of code assigned to any other characters.

- If, "00101" code compressed at one end and decompressed at another end then actual original content will revive as "bc".

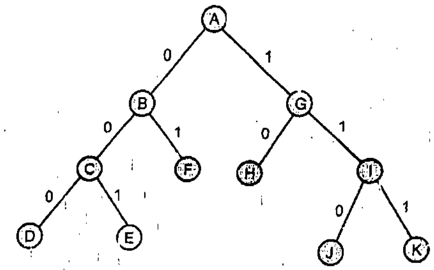
> **Note :** Prefix code is actually Prefix tree code.

- **How to construct prefix code from full binary tree ?**



(1H41)**Fig. 5.9.1 : Full binary tree**

- As we know, code is sequence of bits. Which is assigned to the characters.
- In the binary tree, assign "0" bit to left subtree and "1" bit to right subtree.
- Here, root node Ⓐ has two childs Ⓑ and Ⓖ so, assign 0 to Ⓑ and 1 to Ⓖ.
- In the same way assign 0 to Ⓒ, 1 to Ⓕ, 0 to Ⓓ, 1 to Ⓔ, 0 to Ⓗ, 1 to Ⓘ, 0 to Ⓙ and lastly 1 to Ⓚ.



(1H42)**Fig. 5.9.2**

- Therefore prefix codes of leaf node are as.

$D \Rightarrow 000$, $E \Rightarrow 001$, $F \Rightarrow 01$, $H \Rightarrow 10$

$J \Rightarrow 110$ $K = 111$

5.9.1 Optimal Tree

> **Q.** *Define the following terms with reference to tree. Optimal Binary tree.*

- Let T be any full binary tree and let, $W_1, W_2, \ldots, W_t$ be the weights of the terminal (leaf) vertices.
- Then the weight W of the binary tree is given by,

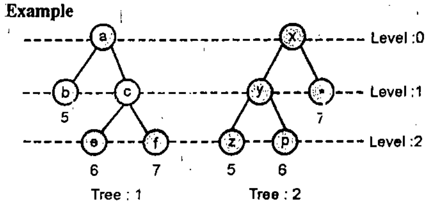
$$W(T) = \sum_{i=0}^{t} W_i l_i$$

Where l_i is the length of the path of the leaf i from the root of the tree.

Optimal tree

The full binary with it's minimum weight is called optimal tree.

Example



(1H43)**Fig. 5.9.3**

Here, 5, 6, 7 are the weights assigned to the leaf node of full binary tree.

In Tree T_1	In Tree T_2
Level of (b) i.e. l(b) = 1	Level of Z, l(z) = 2
Level of (e) i.e. l(e) = 2	Level of P, l(p) = 2
Level of (f) i.e. l(f) = 2	Level of Q, l(Q) = 1
The weight of tree T_1 is as follows W (T_1) = 5 × l(b) + 6 × l(e) + 7 × l(f) = 5 × (1) + 6 × (2) + 7 × (2) = 5 + 12 + 14 = 31	The weight of tree T_2 is as, W (T_2) = 5 × l(z) + 6 × l(P) + 7 × l(Q) = 5 × 2 + 6 × 2 + 7 × 1 = 10 + 12 + 7 = 29

Here, W (T_2) < W (T_1)

Thus, T_2 is the optimal tree for the weight 5, 6, 7.

5.9.2 Huffman Algorithm to Find an Optimal Tree

- Huffman Algorithm is developed by David A. Huffman in 1952 to develop optimal binary tree.
- The output of this algorithm is the prefix codes associated with some characters from frequency of occurrence (weight) of source symbol (character).
- Let $W_1, W_2, W_3, \ldots W_t$ be the weights of node.
- Huffman Algorithm worked in the following way

- **Step I :** Arrange the weights (Frequency of occurrence) in increasing order.
- **Step II :** First, consider the two nodes with minimum weights e.g. W_1 and W_2 make subtree whose root node is $W_1 + W_2$ with child's are W_1 and W_2.
- **Step III :** Repeat the Step-II for the remaining weights until no weights remains.
- **Step IV :** In this way, A tree is obtained which is called as optimal binary tree for given weight.

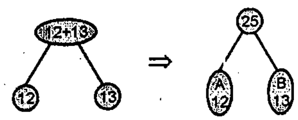
Lets take a simple example to understand the Algorithm.

Character	A	B	C	D
Weights	12	13	14	16

- **Step I :** Arrange all the characters in increasing weights.

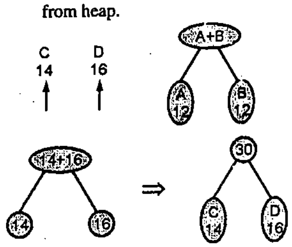
Here all weights are already sorted in increasing order.

- **Step II :** Pickup first two smallest characters. Add them and make a new subtree as



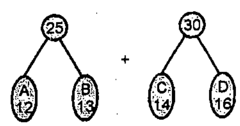
(1H44)Fig. 5.9.4

- **Step III :** Select two nodes with minimum weights from heap.

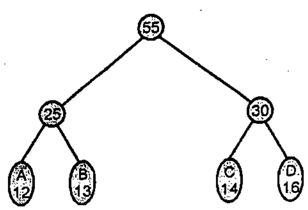


(1H45 & 1H46) Fig. 5.9.5

- **Step IV :** Select two nodes with minimum weights from heap that is,



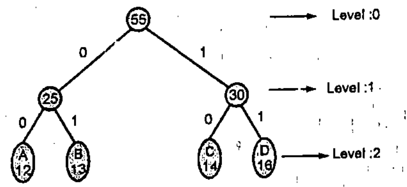
(1H47) Fig. 5.9.6



(1H48)Fig. 5.9.7

Here, min heap contain only one node that is 55.

- **Step V :** Traverse the above binary tree from root node maintain Auxiliary array. Assign 0 to left child Assign 1 to right child.



(1H49) Fig. 5.9.8 : Optimal binary tree

Characters	Prefix code
A	00
B	01
C	10
D	11

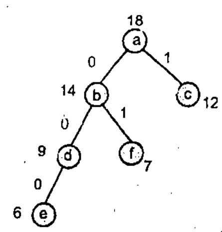
The minimum weight of the tree is,

$$W(T) = 12 \times l(A) + 13 \times l(B) + 14 \times l(C) + 16 \times l(D)$$
$$= (12 \times 2) + (13 \times 2) + (14 \times 2) + (16 \times 2)$$
$$= 24 + 26 + 28 + 32 = 110$$

5.9.3 Optimal Prefix Code

- If a binary prefix code obtained from a optimal binary tree then it is called as optimal prefix code.
- Consider a optimal tree. (Refer Fig. 5.9.9)
- The optimal prefix code for the weights e = 6, f = 7 and c = 12 is given by,

  $e \Rightarrow 000$ $f \Rightarrow 01$, $c \Rightarrow 1$



(1H50)Fig. 5.9.9

Ex. 5.9.1 : Construct the binary tree with prefix codes representing :

(i) a : 11, e : 0, t : 101, s : 100

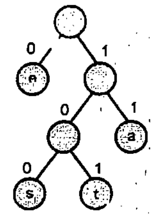
(ii) a : 1010, e : 0, t : 11, s : 1011, n : 001, i : 10001.

☑ **Soln. :**

Given :

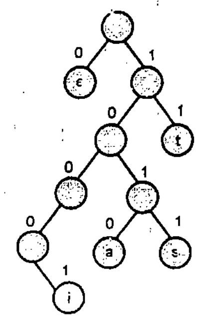
(i) a : → 11, e : 0, t : 101, s : 100

Here every character is associated with their prefix code. As we know that, in optimal binary tree, we assigned 0 to left child and 1 to right child



1E2 Fig. Ex. 5.9.1 : Binary tree

(ii) Given : a : 1010, e : 0, t : 11, s : 1011, i : 10001



(1E13) Fig. Ex. 5.9.1(a) : Binary tree

UEx. 5.9.2 (SPPU (IT) - Q. 6(a), May 19, 6 Marks)

Suppose data items A, B, C, D, E, F, G occur in the following frequencies.

Data Items	A	B	C	D	E	F	G
Weight	10	30	5	15	20	15	05

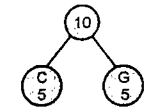
Construct a Huffman code for the data. What is the minimum weighted path length ?

☑ **Soln. :**

- **Step I :** Sort all the data items according to their weight in increasing order

Data items	C	G	A	D	F	E	B
Weights	5	5	10	15	15	20	30

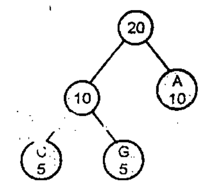
- **Step II :** Select first two data items and make small tree from them. As given below



a	d	f	e	B
10	15	15	20	30

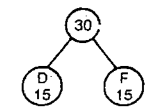
1E31

- **Step III :**



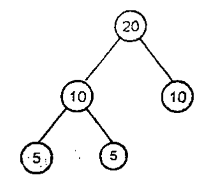
1E32

- **Step IV :** Now, select two smallest data items from D, F, E and B, D and F have smallest weights, then



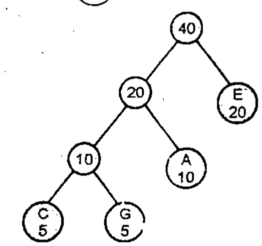
1E33

- **Step V :** Again select two smallest weights here, one is



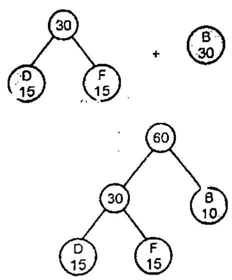
1E34

and another is E20



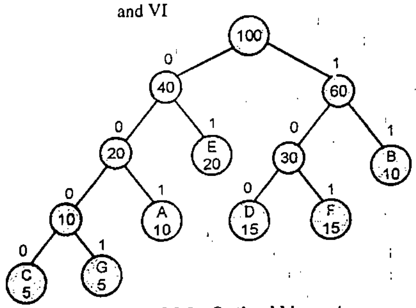
1E35

- **Step VI :** Now, select two smallest weight character and add them



1E36(a)

- **Step VII :** Lastly select two subtrees from step V and VI



1E36)**Fig. Ex. 5.9.2 : Optimal binary tree**

Here apply 0 to left child and 1 to right child

$l(A) = 3, l(B) = 2,$

$l(C) = 4, l(D) = 3,$

$l(E) = 2, l(F) = 3,$

$l(G) = 4,$

The Huffman code is as given below

05 $\Rightarrow$ 0000 or 0001

10 $\Rightarrow$ 001

15 $\Rightarrow$ 100 or 101

20 $\Rightarrow$ 01

30 $\Rightarrow$ 11

The minimum weight of the tree is

$= (3 \times 10) + (2 \times 30) + (4 \times 5) + (3 \times 15)$

$+ (2 \times 20) + (3 \times 15) + (4 \times 5)$

$= 30 + 60 + 20 + 45 + 40 + 45 + 20$

$= 260$ **...Ans.**

UEx. 5.9.3 : (SPPU (IT) - Q. 5(a). Dec. 16. 7 Marks)

A secondary storage media contains information in files with different formats. The frequency of different types of files is as follows. Exe (20), bin(75), bat(20), jpeg(85), dat(51), doc(32), sys(26), c(19), cpp(25), bmp(30), avi(24), prj (29), lst(35), zip(37). Construct the Huffman code of this.

☑ **Soln. :**

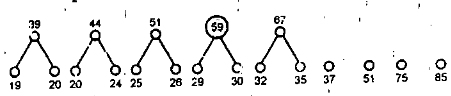
Given :

Exe	bin	bat	jpeg	dat	doc	sys	c	cpp	bmp	avi	prj	lst	zip
20	75	20	85	51	32	26	19	25	30	24	29	35	37

- **Step I :** Arrange all the items in increasing order according to their weights.

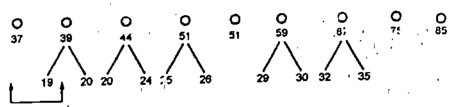
0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	20	20	24	25	26	29	30	32	35	37	51	75	85

- **Step II :** Select first two items and make a subtree.

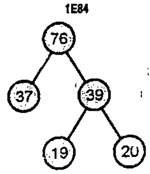


1E83

- **Step III :** Again, arrange all the items in sequence.

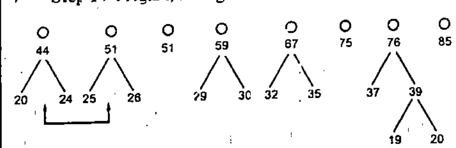


1E84

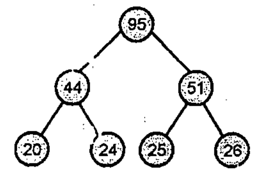


1E85

- **Step IV :** Again, arrange all the items in sequence.

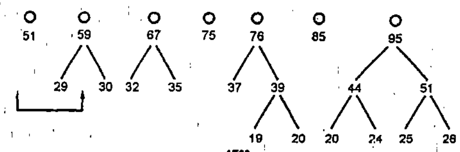


1E86

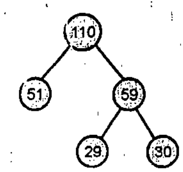


1E87

- **Step V :** Arrange all the items in sequence.

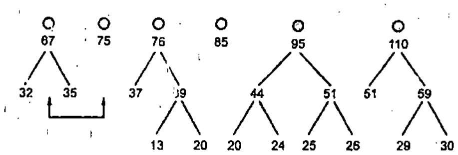


1E88

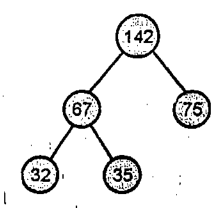


1E89

- **Step VI :** Arrange all the items in sequence.

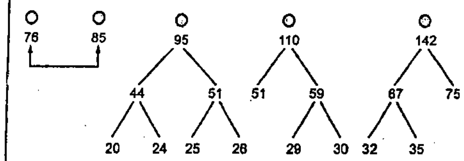


1E90

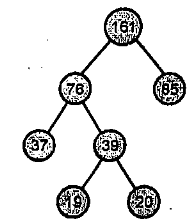


1E91

- **Step VII :** Arrange all the items in sequence.

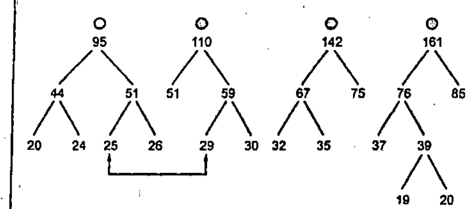


1E92

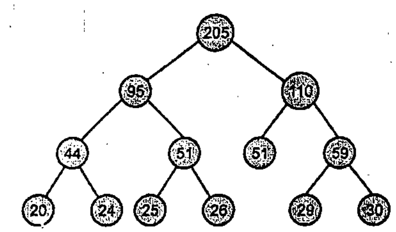


1E93

- **Step VIII :** Arrange all the items in sequence.

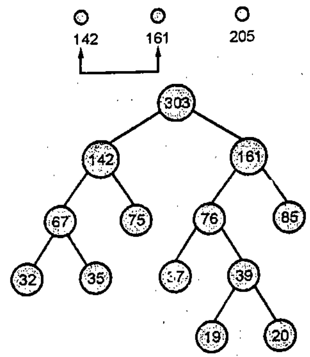


1E94



1E95

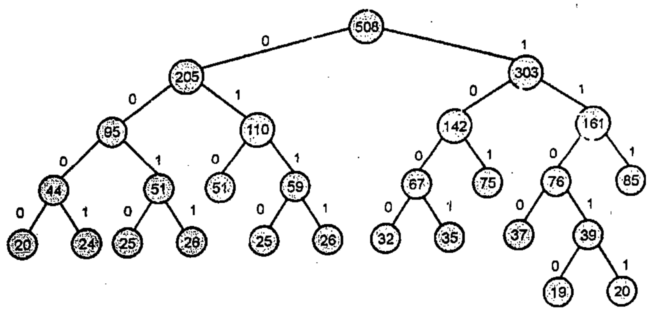
▶ **Step IX :** Arrange all the items in sequence.



1E96

▶ **Step X :** Arrange all the items in sequence. 205 303

Here, apply zero (0) to left child and one (1) to right child of optimal binary tree.



1E98 **Fig. Ex. 5.9.3 : Optional binary tree**

Characters	Binary prefix code	Characters	Binary prefix code
bat (20)	0000	C (19)	11010
uvi (24)	0001	exe (20)	11011
cpp (25)	0010	jpeg (85)	111
sys (26)	0011		
dat (51)	010		

prj (29)	0110		
bmp (30)	0111		
doc (32)	1000		
lst (35)	1001		
bin (75)	101		
zip (37)	1100		

Ex. 5.9.4 : State whether the given code is prefix code. Justify : {000, 001, 01, 10, 11}.

☑ **Soln. :**

Given : {000, 001, 01, 10, 11}

The above codes assigned to characters are prefix codes.

☞ **Justification**

– Code - 000 is not prefix to any other code in given set of code.
– In the same way, code - 001 is not a prefix of another code.
– Similarly, code - 01, 10 and 11 are not a prefix of another code.

Hence, all the codes which are included in given set, are prefix code.

UEx. 5.9.5 (SPPU (IT) - Q. 6(b), May 16, 6 Marks)

For the following sets of weights construct an optimal binary prefix code. For each weight in the set, give the corresponding code word.

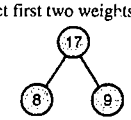
8, 9, 10, 11, 13, 15, 22.

☑ **Soln. :**

Given : weights are as

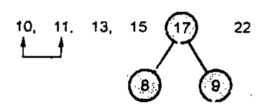
8 9 10 11 13 15 22

▶ **Step I :** All the weights are already in sequence so, select first two weights.



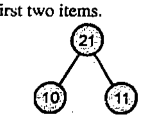
1E107

▶ **Step II :** Arrange all the items in sequence.



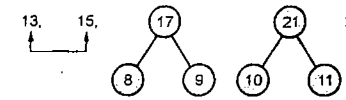
1E108

Select first two items.



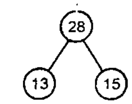
1E109

▶ **Step III :** Arrange items in sequence.



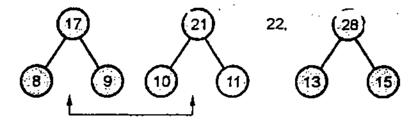
1E110

Select first two items.



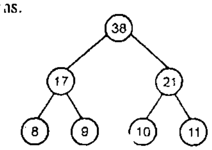
1E111

▶ **Step IV :** Arrange items in sequence.



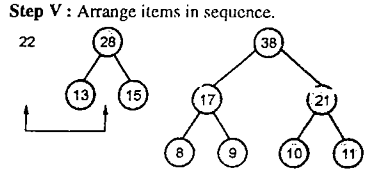
1E112

Select two items.



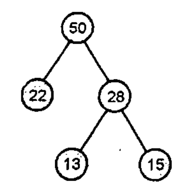
1E113

▶ **Step V :** Arrange items in sequence.



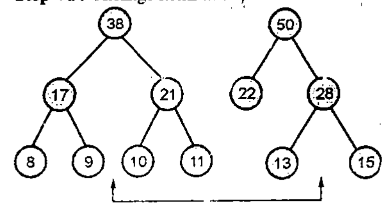
1E114

Select first two items.



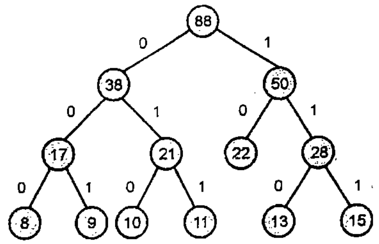
1E115

Step VI : Arrange items in sequence.



1E116

Add two items (38) and (50)



1E117 **Fig. Ex. 5.9.5 : Optimal binary tree or Huffman tree**

As we know, apply zero (0) to left childs and one (1) to right childs.

Binary prefix code

9 → 001
10 → 010
11 → 011
22 → 10
13 → 110
15 → 111

UEx. 5.9.6 (SPPU (IT) - Q. 5(a). Dec. 18, 7 Marks)

For the following set of weights, construct optimal binary prefix code :

α	5
β	6
γ	6
δ	11
ε	20

☑ **Soln. :**

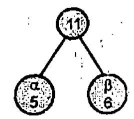
Given :

Characters	α	β	γ	δ	∈
Weights	5	6	6	11	20

Step I : Sort the weights in increasing order.

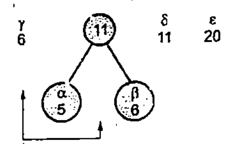
α β γ δ ∈
5 6 6 11 20

Select first two weights or frequency, which are indirectly smallest weights and make a subtree of $\alpha + \beta$. In the following way



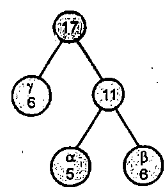
1E137

Step II : Again, arrange all the items in sequence according to their weights.



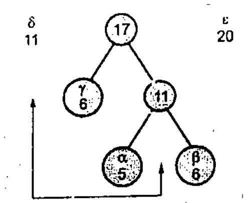
1E138

Select, first two smallest weights and make a subtree of them.

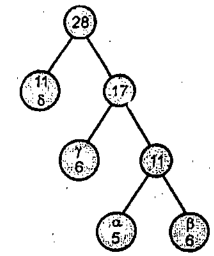


1E139

Step III : Again, arrange all the weights in sequence and select first two smallest weights.

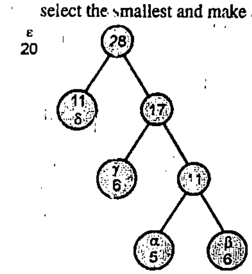


1E140

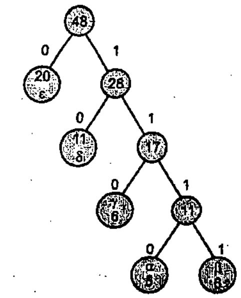


1E141

Step IV : Again, arrange all the items in sequence and select the smallest and make a subtree.



1E142



1E143 **Fig. Ex. 5.9.6 : Optimal binary tree**

Here, apply zero (0) to left childs and 1 (one) to right childs.

The Huffman code for given items :

α (5) → 1110
β (6) → 1111
γ (6) → 110
δ (11) → 10
∈ (20) → 0 ...Ans.

UEx. 5.9.7 (SPPU (IT) - Q. 6(b). Dec. 19, 6 Marks)

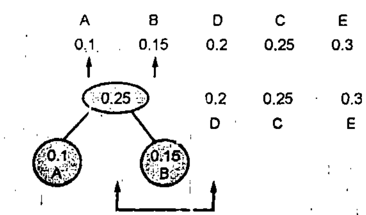
For the following set of characters, construct Huffman code. Find average bit length of the code.

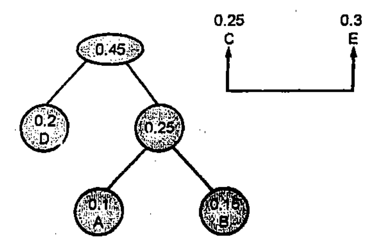
Character	A	B	C	D	E
Frequency	0.1	0.15	0.25	0.2	0.3

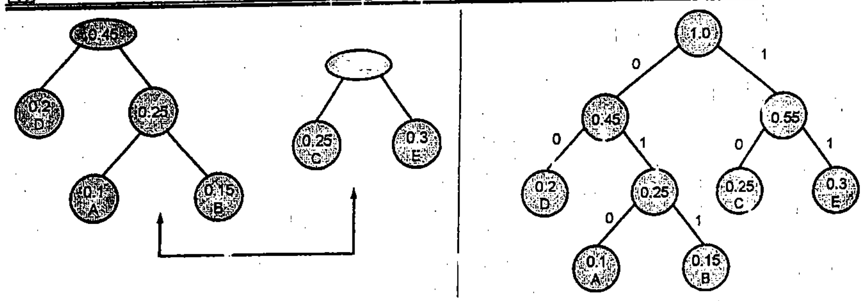
☑ **Solution :**

Given :

Arrange all the characters in ascending order.







Assign 0 to left child and 1 to right child.

$A(0.1) \Rightarrow 010$, $B(0.15) \Rightarrow 011$, $C(0.25) \Rightarrow 10$, $D(0.2) \Rightarrow 00$, $E(0.3) \Rightarrow 11$

Total number of bits = Freq(A) * Code length of (A) + Freq(B) * Code length (B) + Freq(C) * Code length (C) + Freq(D) * Code length (D) + Freq(E) * Code length (E)

$= (0.1 * 3) + (0.15 * 3) + (0.25 * 2) + (0.2 * 2) + (0.3 * 2)$

$= (0.3) + (0.45) + (0.5) + (0.4) + (0.6) = 2.25$

$$\text{Average code length} = \frac{\text{Total number of bits}}{\Sigma \text{ Frequency}} = \frac{2.25}{0.1 + 0.15 + 0.25 + 0.2 + 0.3}$$

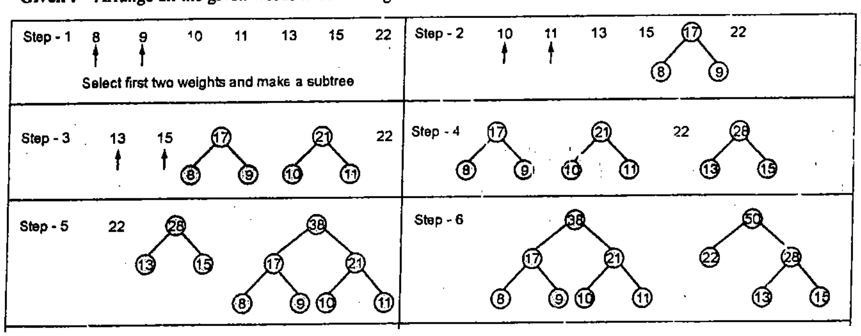
$$= \frac{2.25}{1} = 2.25$$

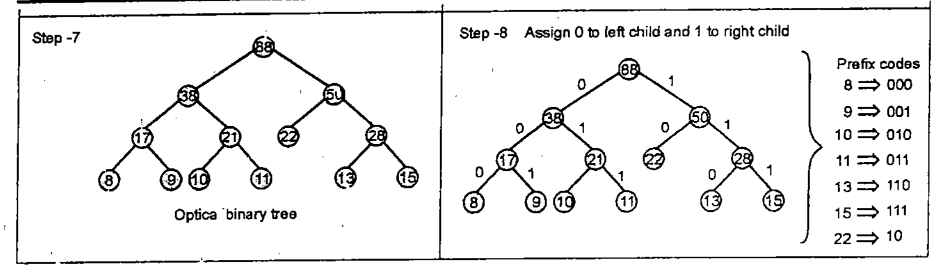
UEx. 5.9.8 (SPPU (IT) - Q. 5(a), May 19, 6 Marks)

Construct an optimal binary tree for the set of weights as {15, 22, 9, 11, 10, 13, 8}. Find the weight of an optimal tree. Also assign the prefix codes and write the code words.

☑ **Solution :**

Given : Arrange all the given words in ascending order.





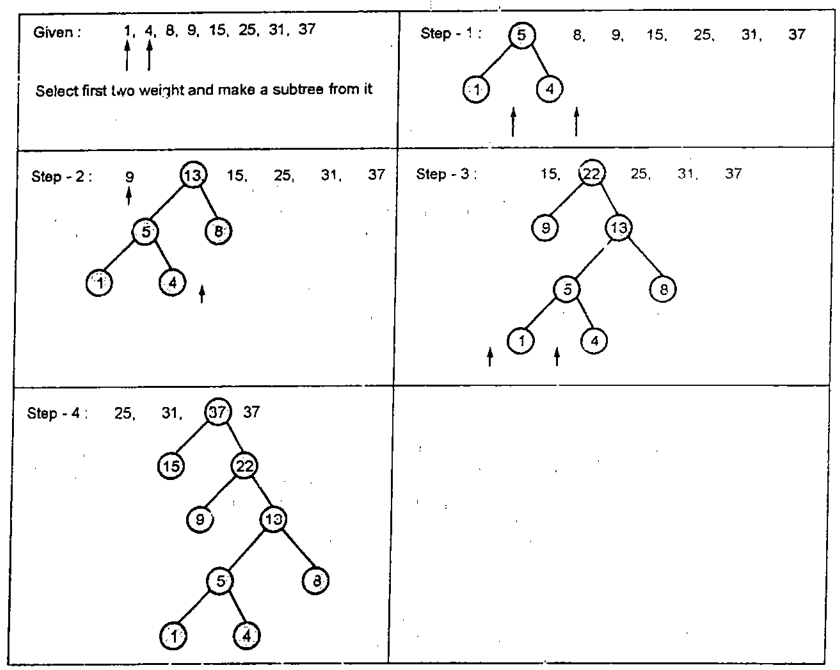
Weight of the optimal binary tree $= (8 \times 3) + (9 \times 3) + (10 \times 3) + (11 \times 3) + (13 \times 3) + (15 \times 3) + (22 \times 2)$

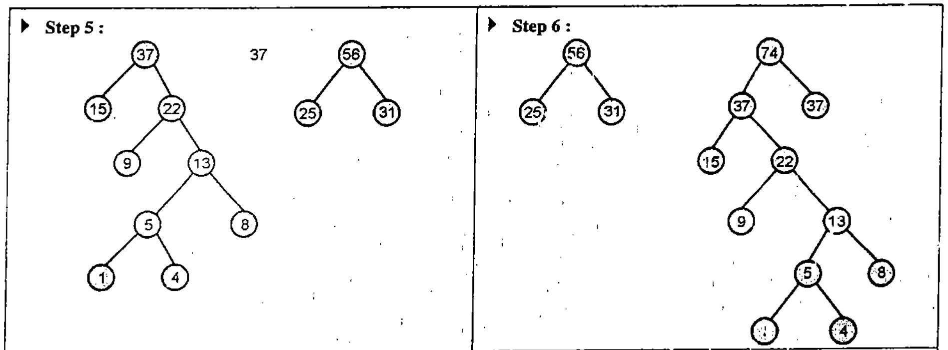
$= (24) + (27) + (30) + (33) + (39) + (45) + (44) = 242$ ...Ans.

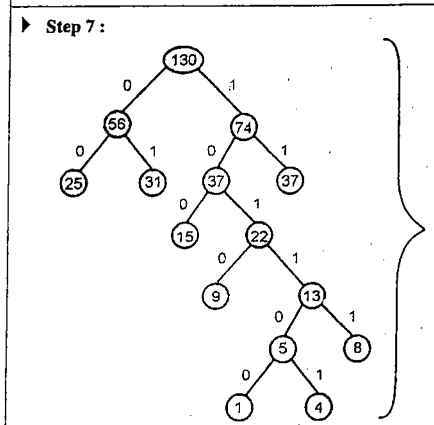
UEx. 5.9.9 (SPPU (IT) - Q. 5(B), May 17, 6 Marks)

Define optimal tree. For the following set of weights, construct optimal binary prefix code. For each weight in the set, give corresponding prefix code : 1, 4, 8, 9, 15, 25, 31, 37.

☑ **Solution :**







Optimal binary pre-fix code

01 ⇒ 101100
04 ⇒ 101101
08 ⇒ 10111
09 ⇒ 1010
15 ⇒ 100
25 ⇒ 00
31 ⇒ 01
37 ⇒ 11

Fig. Ex. 5.9.9 : Optimal Binary Tree

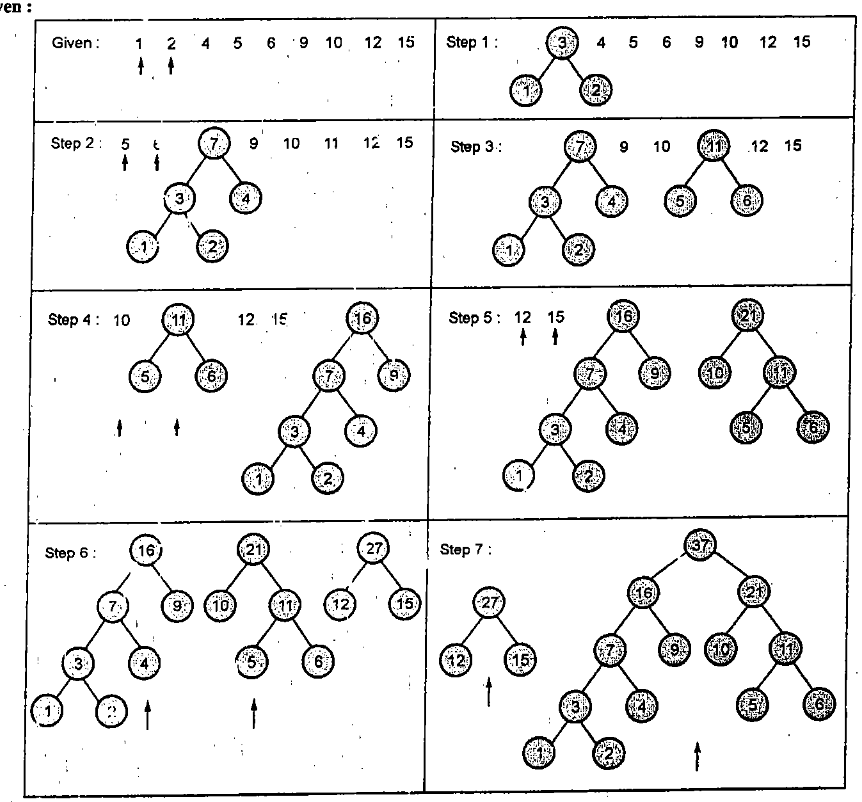
• NOTES •

UEx. 5.9.10 (SPPU (IT) - Q. 6(b). Dec. 15, 7 Marks)

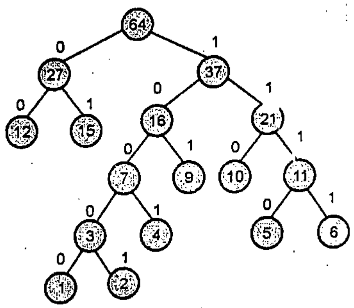
For the following set of weight construct the optimal binary prefix tree. For each of the weight in the set give the corresponding prefix code : 1, 2, 4, 5, 6, 9, 10, 12, 15.

☑ **Soln. :**

Given :



Step 8 :



Binary prefix code

$1 \Rightarrow 10000$

$2 \Rightarrow 10001$

$4 \Rightarrow 1001$

$5 \Rightarrow 1110$

$6 \Rightarrow 1111$

$9 \Rightarrow 101$

$10 \Rightarrow 110$

$12 \Rightarrow 00$

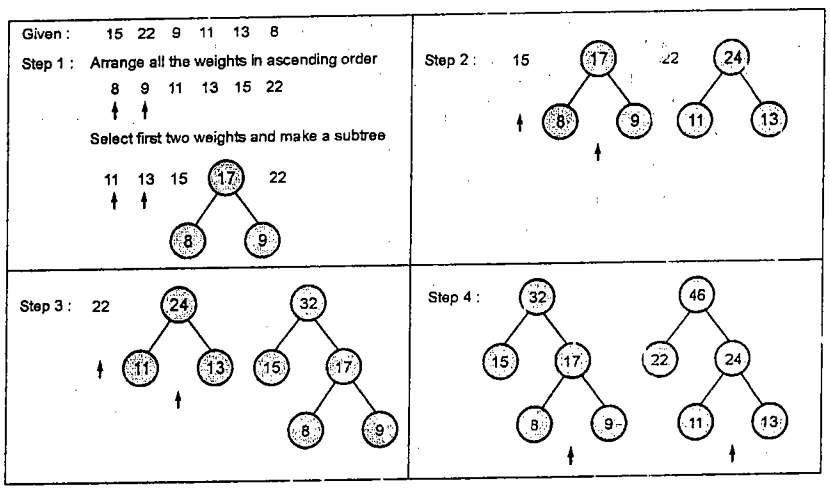
$15 \Rightarrow 01$

Fig. Ex. 5.28 : Optimal binary tree

UEx. 5.9.11 (SPPU (IT) - Q. 5(a), May 15, 6 Marks)

Construct an optimal binary tree for the set of weights as (15, 22, 9, 11, 13, 8). Find the weight of an optimal tree. Also assign the prefix codes and write the code words.

☑ **Solution :**



Step 5 :

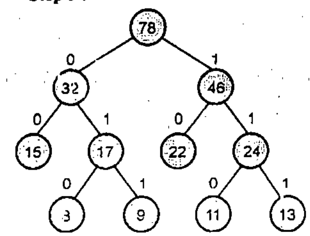


Fig. Ex. 5.9.11 : Optimal binary tree

Prefix code

$8 \Rightarrow 010$

$9 \Rightarrow 011$

$11 \Rightarrow 110$

$13 \Rightarrow 111$

$15 \Rightarrow 00$

$22 \Rightarrow 10$

Weight of optimal binary tree

$= (8 \times 3) + (9 \times 3) + (11 \times 3) + (13 \times 3) + (15 \times 2) + (22 \times 2)$

$= (24) + (27) + (33) + (39) + (30) + (44)$

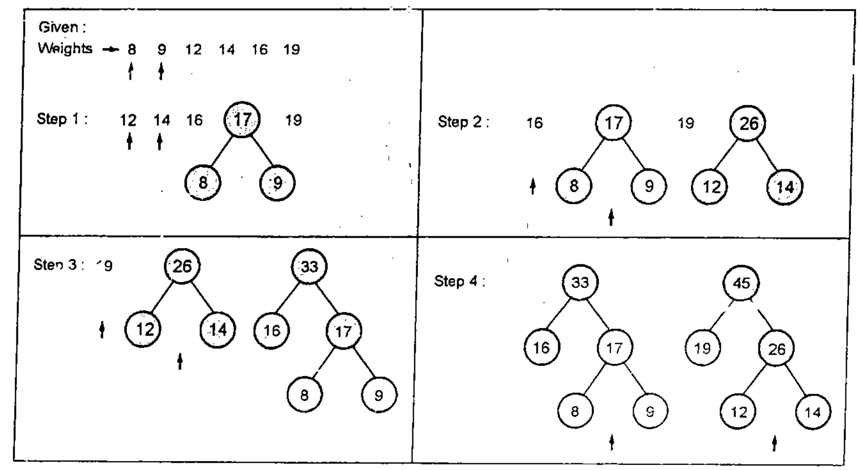
$= 197$ ...Ans.

UEx. 5.9.12 (SPPU (IT) - Q. 6(a), Dec. 14, 7 Marks)

Define optimal tree. For the following set of weights, construct optimal binary prefix code. For each weight in the set, give corresponding code words – 8, 9, 12, 14, 16, 19.

☑ **Solution :**

Optimal tree



Step 5 :

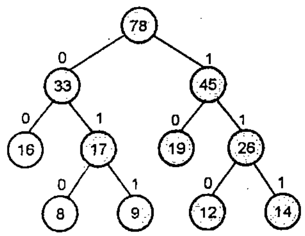


Fig. Ex. 5.9.12 : Optimal binary tree

Prefix codes

$8 \Rightarrow 010$

$9 \Rightarrow 011$

$12 \Rightarrow 110$

$14 \Rightarrow 111$

$16 \Rightarrow 00$

$19 \Rightarrow 10$

UEx. 5.9.13 (SPPU (IT) - Q. 5(b), Dec. 14. 7 Marks)

Use Huffman coding to encode the following symbols.

Symbol	Frequency
A	0.08
B	0.10
C	0.12
D	0.15
E	0.20
F	0.35

☑ **Soln. :**

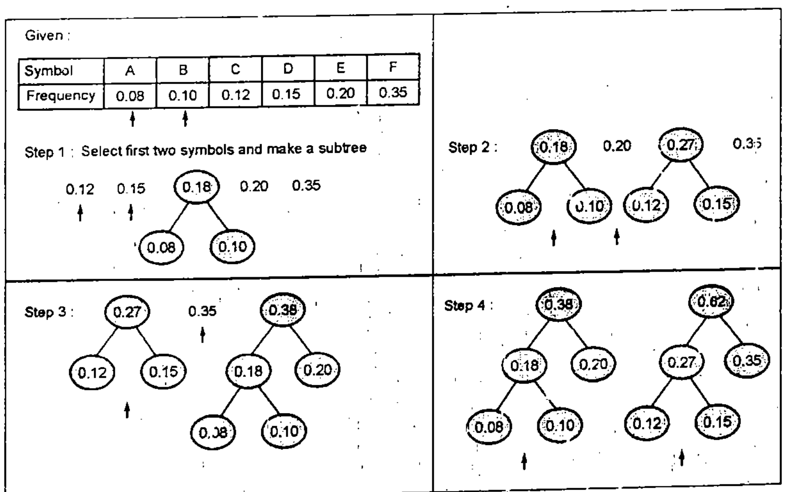


Fig. Ex. 5.9.13 : Optimal binary tree

Prefix codes

$A(0.08) \Rightarrow 000$

$B(0.10) \Rightarrow 001$

$C(0.12) \Rightarrow 100$

$D(0.15) \Rightarrow 101$

$E(0.20) \Rightarrow 01$

$F(0.35) \Rightarrow 11$

5.9.4 Binary Search Tree

GQ. Define with example : Binary search tree.

- Binary search tree is an important application of binary tree where we can search, delete, insert nodes easily.
- Binary search tree contains sorted data arranged in tree like structure.
- Binary search tree can be used in databases, syntax parsing and file compression and many more.
- Binary search tree has single root node which contains data as well as references to left and right nodes.
- Each left and right nodes can also have data and references to it's own left and right nodes respectively. In this way, tree get recursively formed.

Properties of binary search tree

(i) The left subtree of any particular node contains only nodes with key lesser than the node's key (Information weight).

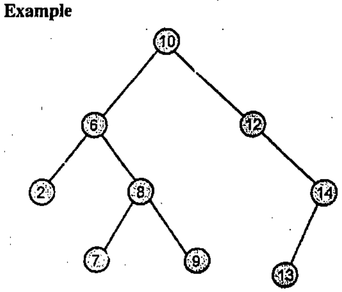
(ii) The right subtree of a node contains any nodes with keys greater than the node's key.

(iii) The left and right subtree must itself have to be binary search tree.

(iv) Each node of BST must unique.

In binary search tree. we can apply different traversal algorithms like Preorder, Inorder or Postorder.

Example



(1H50)**Fig. 5.9.10 : Binary search tree**

- By looking at tree, it is clearly observed that each left child have less weight than the right child of node.
- **Insertion of new node to BST :** A new node is always inserted at leaf.

Steps

(i) Start from the root.

(ii) Compare the inserting node or element with root node. If inserting node is less than the root node, then go to left subtree else goto right subtree.

(iii) In this way, carry out some procedure to get leaf node.

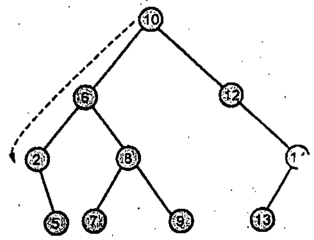
(iv) After getting leaf node, insert new node at left side (If less than leaf node) or otherwise insert at right side.

Example

- Suppose , if we want to insert node ⑤ then, node ⑤.

  First compare with root node ⑩.
- Here, 5 is less than 10, then we will go to left subtree of root node.
- In the same way, we iterate procedure till we get leaf node, lastly node ② is the leaf node where we will insert our new node.

– But it will be placed to right side of ② because ⑤ > ②.



(1H51)Fig. 5.9.11 : Binary search tree with new element

Note : Inorder traversal of binary search tree always produces sorted output.

Ex. 5.9.14 : Construct a binary search tree : J, R, D, G, W, E, M, H, P, A, F, Q

☑ **Soln. :**

Given :

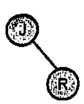
J, R, D, G, W, E, M, H, P, A, F, Q

As we know that, in binary search tree (BST), all nodes in the left subtree are less than the root node and all the nodes in the right sub tree are greater than the root note.

Same procedure is recursively applied to each node

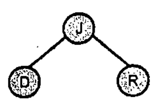
▶ **Insert (J) :**

▶ **Insert (R) :**



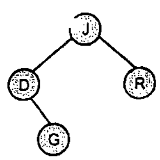
1E2

▶ **Insert (D) :**



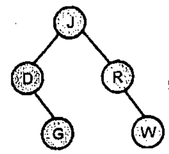
1E2 1E3

▶ **Insert (G) :**



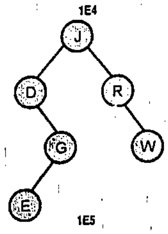
1E3

▶ **Insert (W) :**



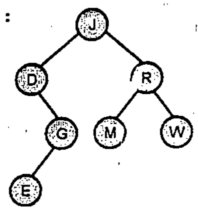
1E4

▶ **Insert (E) :**



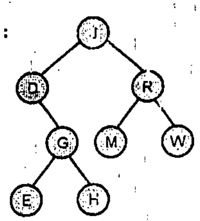
1E5

▶ **Insert (M) :**



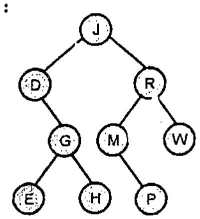
1E6

▶ **Insert (H) :**



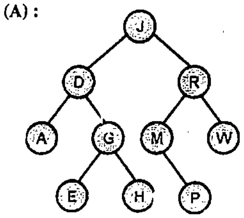
1E7

▶ **Insert (P) :**



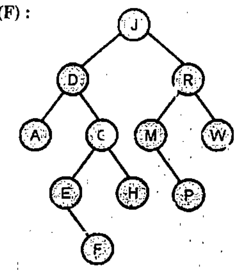
1E8

▶ **Insert (A) :**



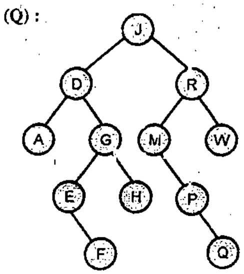
1E9

▶ **Insert (F) :**



1E10

▶ **Insert (Q) :**



1E11

5.10 SPANING TREES

GQ. Explain the following terms : Spanning trees.

– Spanning tree is developed from the graph.

– The spanning tree of graph G is a subset of G that covers all of it's vertices using optimum or minimum number of edges.

– In spanning tree, number of vertices of graph G are equal to number of vertices of spanning tree that is,

$$V(G) = V(T)$$

☞ Properties of a spanning tree

(i) It can not be disconnected since it covers all the vertices.

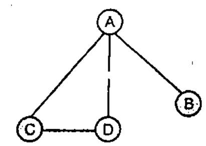
(ii) Spanning tree can not have any cycle and consist of (n – 1) edges.

(iii) A single graph can have more than one spanning tree.

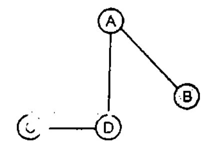
(iv) In complete graph, maximum number of spanning tree = n^{n-2}.

▶ **Example**

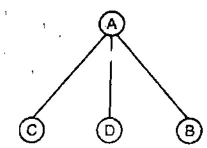
Let G be the graph and T_1, T_2 and T_3 are spanning trees developed from graph G.



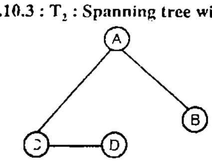
(1H52)Fig. 5.10.1 : Graph G which contain cycle



(1H53)Fig. 5.10.2 : T_1 : Spanning tree which covers all the vertices



(1H54)Fig. 5.10.3 : T_2 : Spanning tree with all vertices



(1H55)Fig. 5.10.4 : Spanning tree

Here T_1, T_2 and T_3 are different possible spanning trees developed.

5.10.1 Minimum Spanning Tree

- Let G be a weighted, connected undirected graph and T_1 be a spanning tree of G is called minimum spanning tree. If its weight is minimum than other spanning trees of graph G.
- Here, weight of the tree = The sum of the weights of its edges.

Applications of minimum spanning tree

(i) Designing of networks of telephone lines railway lines, roads, electric lines, gas pipelines, etc.
(ii) Travelling salesman problem.
(iii) Cluster analysis.
(iv) To design protocols to avoid network cycles.
(v) Real time face tracking and verification.

Note : Minimum spanning tree may or may not be unique because there may be two edges with same weights.

5.10.2 Prim's Algorithm

- Prim's Algorithm is used to find the minimum spanning tree of graph G.
- This algorithm was developed or designed by Robert clay prim in 1957.
- To apply prim's algorithm, the given graph must be weighted connected and undirected.
- Prim's algorithm start with picking up any arbitrary vertex from the graph and put it into empty spanning tree
- Then minimum weight edge connected to this vertex is selected and added to the spanning tree. Now spanning tree is consisting of two vertices.
- Now, select and add the edge with the minimum weight that has one end in an already selected vertex (Part of spanning tree) and other end in an unselected vertex.
- This process have to be repeated until the spanning tree contains an the vertices.

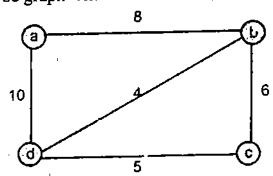
Note : The number of edges in minimum tree are $n - 1$ where n = number of vertices.

Let G (V, E) be a connected, weighted graph.

- **Step I :** Select any arbitrary vertex V_o in the graph G.
  Set, MST = $\{V_o, \phi\}$ ← Initially MST have only one vertex, there is no edge selected.
- **Step II :** Find the edge $e_i = (V_o, V_i)$ in set of Edges (E) such that one end of edge is $V_0 \in$ MST and other end in non-selected region of graph and it's weight is minimum.
  So, MST = $\{\{V_o, V_1\}, \{e_i\}\}$
- **Step III :** Choose next edge $e_x = \{V_x, V_j\}$ in such a way that it's one end $V_x \in$ MST and other end belongs to unselected area of graph. The weight of the edge e_x should be minimum or small as possible. Now include vertex V_j and e_x to MST.
- **Step IV :** Repeat the Step-III, until MST (Minimum Spanning Tree) contains all the vertices of graph G. The set MST will give the minimum spanning tree of graph G.

Example

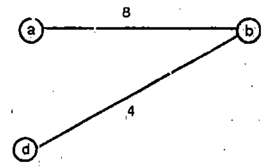
Let G be graph with four vertices a, b, c, d.



(1H56)Fig. 5.10.5

- **Step I :** Select any arbitrary vertex of graph G as a starting of MST.
  MST = $\{\{a\}, \phi\}$
- **Step II :** Vertex "a" is adjacent to vertices ⓑ and ⓓ
  $a \xrightarrow{8} b$, $a \xrightarrow{10} d$
  Here, minimum weight is $a \xrightarrow{8} b$
  Therefore, include it to MST.
  MST = $\{(a, b), (e_1)\}$ ⓐ $\overset{8}{\text{——}}$ ⓑ
- **Step III :** From MST, Vertex ⓐ is adjacent to ⓑ and ⓓ.
- Vertex ⓑ is adjacent to ⓐ, ⓒ and ⓓ vertices but vertices ⓐ and ⓑ are already part of MST.
- Here minimum weight edge is, b $\overset{4}{\text{——}}$ d.

- So, include vertex ⓓ in MST.
  MST = $\{\{a, b, d\} \{e_1, e_2\}\}$



(1H57)Fig. 5.10.6

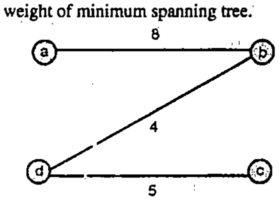
- **Step IV :** From MST, Vertex ⓓ is adjacent to ⓐ, ⓑ and ⓒ.

Here minimum weight edge is ⓓ $\overset{5}{\text{——}}$ ⓒ.

So include vertex ⓒ in MST.

MST = $\{\{a, b, d, c\}, \{e_1, e_2, e_3\}\}$

The weight of minimum spanning tree.



(1H58)Fig. 5.10.7 : Minimum spanning tree

W (MST) = W (a – b) + W (b – d) + W (d – c)
= 8 + 4 + 5 = 17 unit.

Notes : (i) MST weight is the sum of weights of its all edges.
(ii) If any graph contains any self loop and parallel edges, then remove it while creating MST.

Ex. 5.10.1 : Give the stepwise construction of minimum spanning tree using Prim's Algorithm for the following graph shown in Fig. Ex. 5.10.1. Obtain the total cost of minimum spanning tree.

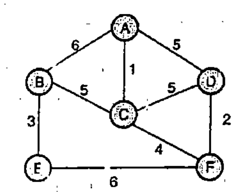


Fig. Ex. 5.10.1

☑ **Soln. :**

Initially, minimum spanning tree has no vertex i.e. empty

MST = { }

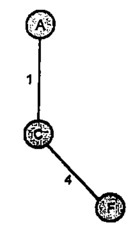
- **Step 1 :** Select any arbitrary vertex. Let select vertex (A). MST = {A, ϕ} vertex (A) is adjacent (B), (C) and (D).

The minimum weight Edge Ⓐ – Ⓒ. Add vertex Ⓒ into MST

So MST = $\{\{A, C\}, \{e_1\}\}$

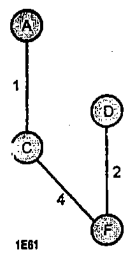
1E59

- **Step II :** Vertex Ⓐ is adjacent to Ⓑ and Ⓓ.
  Vertex Ⓒ is adjacent to Ⓓ, Ⓑ and Ⓕ. The minimum weight edge is Ⓒ → Ⓕ, so include it in MST.
  MST = $\{\{A, C, F\} \{e_1, e_2\}\}$



1E60

- **Step III :** Vertex A is adjacent to Ⓑ and Ⓓ, C is adjacent to Ⓑ and Ⓓ and vertex Ⓕ is adjacent to Ⓔ and Ⓓ. the minimum weight edge is F → D. So, include vertex Ⓓ in MST
  MST = $\{\{A, C, F, D\}, \{e_1, e_2, e_3\}\}$

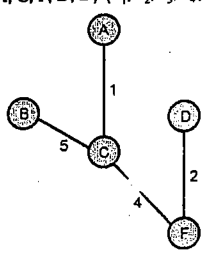


1E61

- **Step IV :** Vertex A is adjacent to B, C is adjacent to B, F is adjacent to E

The minimum weight edge is C → B. so include vertex B into MST.

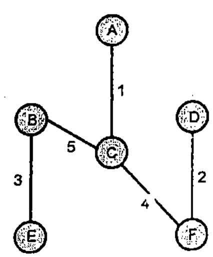
MST – {{A, C, F, D, B} {e_1, e_2, e_3, e_4}}



1E62

- **Step V :** Vertex B is adjacent to E, F is adjacent to E. But, the minimum weight edge is B → E so include vertex E into MST

MST = {{A, C, F, D, B, E} {e_1, e_2, e_3, e_4, e_5}}



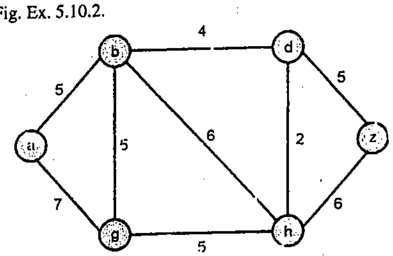
1E63

The weight of the minimum spanning tree is

W (T) = w (AC) + w (CF) + w (DF) + w (CB) + w (BE)

⇒ W (T) = 15 ...Ans.

Ex. 5.10.2 : Use Prim's Algorithm to find the minimum spanning tree for the connected weighted graph G as shown in Fig. Ex. 5.10.2.



1E64

Fig. Ex. 5.10.2

Soln. :

Given :

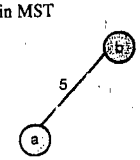
Initially, Minimum Spanning Tree (MST) is empty i.e. MST = {ϕ}

- **Step I :** Select any arbitrary vertex and include it into MST.

MST = {{a}, {ϕ}} → No edge a

- **Step II :** Vertex a is adjacent to b and g. the minimum weight edge is a → b. So, include vertex b in MST

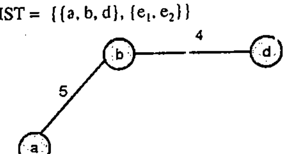
MST = {{a, b}, {e_1}}



1E65

- **Step III :** Vertex a is adjacent to g, b is adjacent to g, d and h. The minimum weight edge is b → d so include d in MST

MST = {{a, b, d}, {e_1, e_2}}

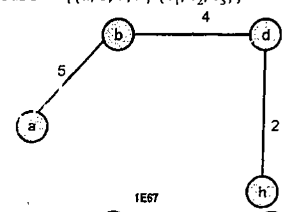


1E66

- **Step IV :** Vertex d is adjacent to z, h. The b is adjacent to h and g and vertex a is adjacent to g.

The minimum weight edge is d → h. so include h in MST

MST = {{a, b, d, h} {e_1, e_2, e_3}}

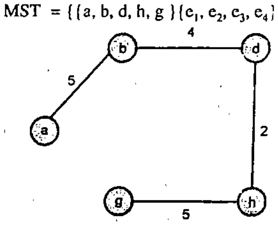


1E67

- **Step V :** Vertex h is adjacent to g and z vertex d is adjacent to z

The minimum weight edge is g → h. so include g in MST.

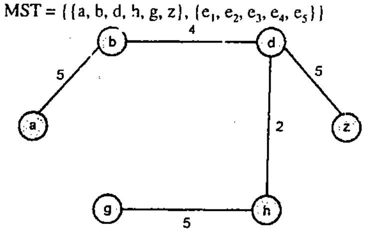
MST = {{a, b, d, h, g }{e_1, e_2, e_3, e_4}}



1E68

- **Step VI :** Vertex d is adjacent to z. h is adjacent to z. The minimum weight edge is d → h. so include z in MST.

MST = {{a, b, d, h, g, z}, {e_1, e_2, e_3, e_4, e_5}}



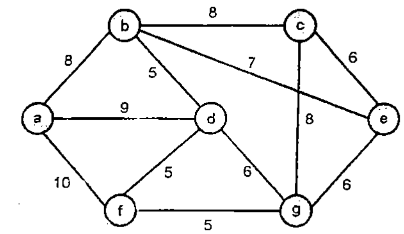
(1E69) **Fig. Ex. 5.10.2(a) : Minimum spanning tree**

The weight of minimum spanning tree is as

W (T) = w (ab) + w (bd) + w (dh) + w(gh) + w (dz)

= 5 + 4 + 2 + 5 + 5 = 21 unit ...Ans.

Ex. 5.10.3 : Find the minimum spanning tree for given graph using Prim's algorithm.



(1E130) **Fig. Ex. 5.10.3**

Soln. :

Given :

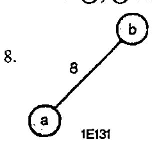
- **Step I :** Select any vertex. Lets select vertex a.

MST = {{a}, {ϕ}} a

- **Step II : Vertex a is adjacent to vertex b, d and f.**

The minimum weight edge is a → b = 8.

MST = {{a, b}, {e_1}}



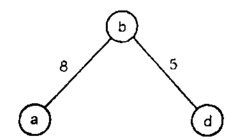
1E131

- **Step III :** Vertex a is adjacent to f and d. vertex b is adjacent to c, d and e.

The minimum weight edge is b → d = 5.

Select vertex d

MST = {{a, b, d}, {e_1, e_2}}

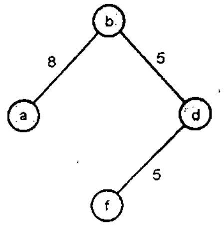


1E132

- **Step IV :** Vertex a is adjacent to f and vertex b is adjacent to c, e. Vertex d is adjacent to f and g.

Among all these edges, the minimum weight edge is ⓓ → ⓕ = 5. So that, include ⓕ in MST.

MST = {{a, b, d, f}, {e_1, e_2, e_3}}



1E133

- **Step V :** Vertex ⓐ is adjacent to ⓑ, ⓓ and ⓕ but they all are in MST.

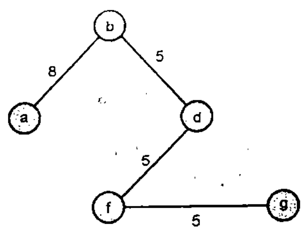
Vertex ⓓ is adjacent to ⓒ and ⓔ.

Vertex ⓓ is adjacent to ⓖ. Vertex ⓕ is adjacent to ⓖ.

Among all these edges, minimum weight edge = fg = 5

So, include ⓖ in MST.

MST = {{a, b, d, f, g }, {e_1, e_2, e_3, e_4}}



1E134

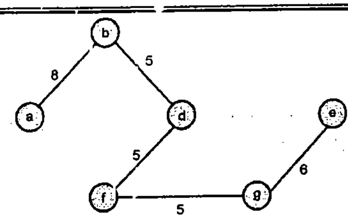
- **Step VI :** Vertex ⓖ is adjacent to ⓒ and ⓔ.

Vertex ⓑ is adjacent to ⓒ and ⓔ.

Among these edges, the minimum weight edge is ⓖ → ⓔ = 6

So, include ⓖ vertex ⓔ in MST.

MST = {{a, b, d, f, g, e }, {e_1, e_2, e_3, e_4, e_5}}



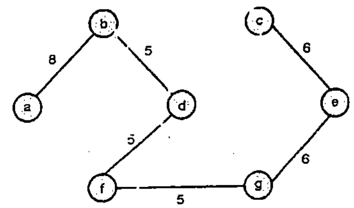
1E135

- **Step VII :** Now, only one vertex is remaining to include in MST. The vertex ⓒ is adjacent to ⓑ, ⓔ, ⓖ.

The minimum weight edge is ec = 6

So, include ⓒ in MST.

MST = {{a, b, d, f, g, e, c}, {e_1, e_2, e_3, e_4, e_5, e_6}}



1E136

Fig. Ex. 5.10.3 (a) : Minimum spanning tree

The weight or cost of minimum spanning tree is

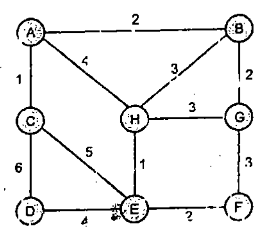
W (T) = W (ab) + W (bd) + W (df) + W (fg) + W (ge) + W (ec)

= 8 + 5 + 5 + 5 + 6 + 6

= 35 units ...Ans.

UEx. 5.10.4 (SPPU (IT) - Q. 5(b). Dec. 18, May 19, 7 Marks)

Give the stepwise construction of minimum spanning tree using Prim's algorithm for the following graph. Obtain the total cost of minimum spanning tree.



1E99 **Fig. Ex. 5.10.4**

☑ **Soln. :**

Initially, minimum spanning tree is empty.

So that MST = { φ }

- **Step I :** Select any arbitrary vertex. Lets select vertex Ⓐ

MST = {{A}, {φ}}

↑ set of edges

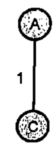
Set of vertices

Ⓐ

- **Step II : Vertex Ⓐ is adjacent to Ⓑ, Ⓒ and Ⓗ.**

The minimum weight edge is A → C. So include vertex Ⓒ in MST.

MST = {{A, C}, {e_1}}



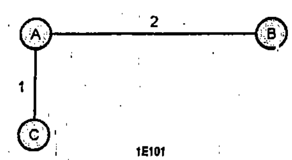
1E100

- **Step III :** Vertex Ⓐ is adjacent to Ⓑ, and Ⓗ, vertex C is adjacent to Ⓓ and Ⓗ.

The minimum weight edge is A → B

So, include Ⓑ in MST.

MST = {{A, C, B}, {e_1, e_2}}



1E101

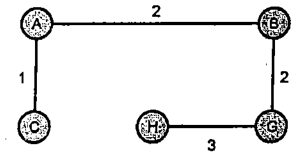
- **Step IV :**

MST = {{A, C, B, G}, {e_1, e_2, e_3}}

1E102

- **Step V :**

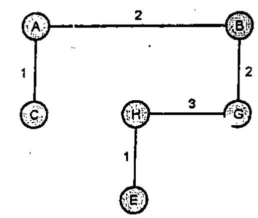
MST = {{A, C, B, G, H} {e_1, e_2, e_3, e_4}}



1E103

- **Step VI :**

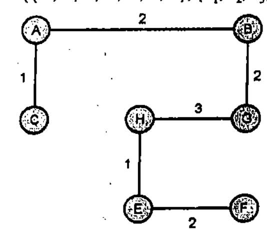
MST = {{A, C, B, G, H, E}, {e_1, e_2, e_3, e_4, e_5}}



1E104

- **Step VII :**

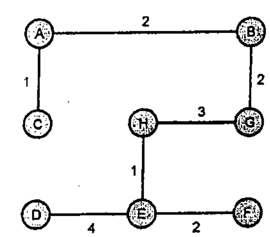
MST = {{A, C, B, G, H, E, F}, {e_1, e_2, e_3, e_4, e_5, e_6}}



1E105

- **Step VIII :**

MST = {{A, C, B, G, H, E, F, D}, {e_1, e_2, e_3, e_4, e_5, e_6, e_7}}



1E106 **Fig. Ex. 5.10.4 (a) : Minimum spanning tree**

Here, all the vertices get visited with minimum number of edges with less weight. So, algorithm stoped here .

The cost of spanning tree is also called as weight of the spanning tree.

$W(T) = W(AC) + W(AB) + W(BG) + W(HG) + W(HE) + W(ED) + W(EF)$

$= 1 + 2 + 2 + 3 + 1 + 4 + 2$

$= \mathbf{15}$ **unit** ...Ans.

UEx. 5.10.5 (SPPU (IT) - Q. 5(b), Dec. 16, 6 Marks)

Find minimum spanning tree for the graph given below using Prim's algorithm.

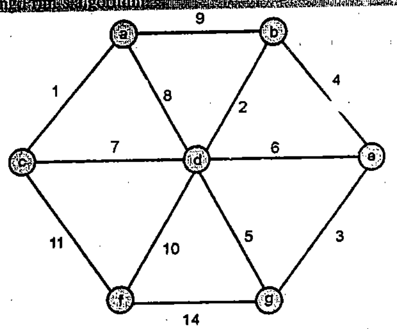


Fig. Ex. 5.10.5

☑ **Solution :**

Given :

Initially minimum spanning tree is empty.

So, MST = { ϕ }

▸ **Step 1 :**

Select any arbitrary vertex from the graph. Lets select ⓐ

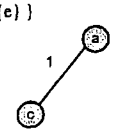
MST = { {a}, {ϕ} } ⓐ

▸ **Step 2 :**

Vertex ⓐ is adjacent to vertex ⓑ, ⓒ and ⓓ.

The minimum weight edge is ⓐ —1— ⓒ So add a—c into MST.

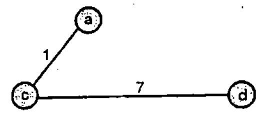
MST = { {a, c}, {e} }



▸ **Step 3 :**

Vertex ⓐ is adjacent to vertex ⓑ and ⓓ. Vertex ⓒ is adjacent to ⓓ and ⓕ minimum weight edge is c-d.

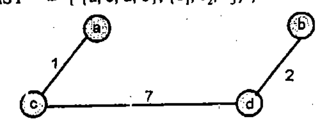
MST = { {a, c, d}, $\{e_1, e_2\}$ }



▸ **Step 4 :**

Vertex ⓐ is adjacent to ⓑ, c is adjacent to ⓕ and ⓓ is adjacent to ⓑ, ⓔ and ⓖ. The minimum weight edge is ⓓ —— ⓑ.

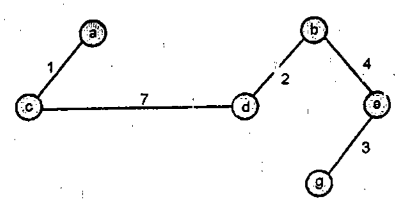
MST = { {a, c, d, b}, $\{e_1, e_2, e_3\}$ }



▸ **Step 5 :**

Vertex ⓐ is adjacent to ⓑ, ⓒ is adjacent to ⓕ. ⓓ is adjacent to ⓔ and ⓖ. ⓑ is adjacent to ⓔ. The minimum weight edge is ⓑ – ⓔ.

MST = { {a, c, d, b, e}, $\{e_1, e_2, e_3, e_4\}$ }



▸ **Step 6 :**

Vertex ⓔ is adjacent to ⓖ. ⓓ is adjacent to ⓕ. The minimum weight edge is ⓔ – ⓖ.

MST = { {a, c, d, b, e, g} $\{e_1, e_2, e_3, e_4, e_5\}$ }

▸ **Step 7 :**

Vertex ⓓ is adjacent to ⓕ. ⓖ is adjacent to ⓕ. The minimum weight edge is ⓓ – ⓕ.

MST = { {a, c, d, b, e, g, f}, $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ }

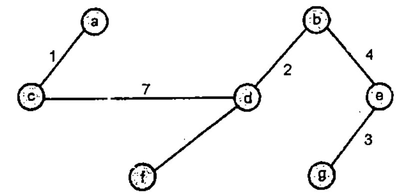


Fig. Ex. 5.10.5(b) : Minimum spanning tree

UEx. 5.10.6 (SPPU (IT) - Q. 6(a), Dec. 15, 6 Marks)

Determine the minimum spanning tree using Prims algorithm for the following graph.

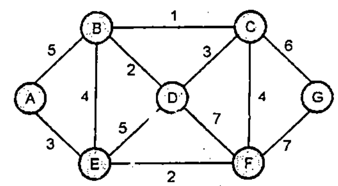


Fig. Ex. 5.10.6 (a)

☑ **Solution : Given :**

▸ **Step 1 :**

Initially minimum spanning tree is empty. Select any arbitrary vertex from the graph. Lets select vertex Ⓐ. Include it in MST.

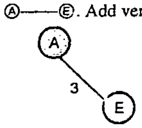
MST = { {A}, {ϕ} } Ⓐ

▸ **Step 2 :**

Vertex Ⓐ is adjacent to Ⓑ and Ⓔ.

The minimum weight edge is Ⓐ——Ⓔ. Add vertex Ⓔ in MST.

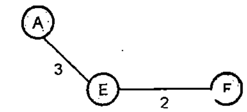
MST = { {A, E}, $\{e_1\}$ }

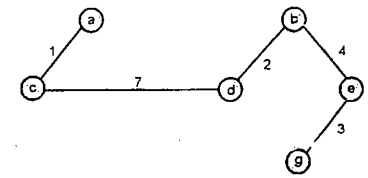


▸ **Step 3 :**

Vertex A is adjacent to Ⓑ, Ⓔ is adjacent to F, D and B. The minimum weight edge is Ⓔ – Ⓕ include F in MST.

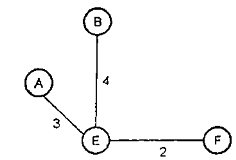
MST = { {A, E, F}, $\{e_1, e_2\}$ }





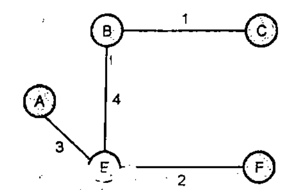
▸ **Step 4 :**

MST = { {A, E, F, B}, $\{e_1, e_2, e_3\}$ }



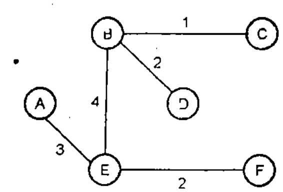
▸ **Step 5 :**

MST = { {A, E, F, B, C}, $\{e_1, e_2, e_3, e_4\}$ }



▸ **Step 6 :**

MST = { {A, E, F, B, C, D}, $\{e_1, e_2, e_3, e_4, e_5\}$ }



▸ **Step 7 :**

MST = { {A, E, F, B, C, D, G}, {e_1, e_2, e_3, e_4, e_5, e_6} }

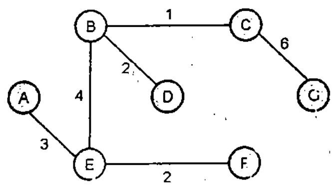


Fig. Ex. 5.10.6(b) : Minimum spanning tree

Weight of minimum spanning tree

$= 3 + 2 + 4 + 2 + 1 + 6$

= 18 unit ...Ans.

UEx. 5.10.7 (SPPU (IT) - Q. 4(a), May 15, 6 Marks)

Find minimum spanning tree and its minimum weight using Prim's algorithm.

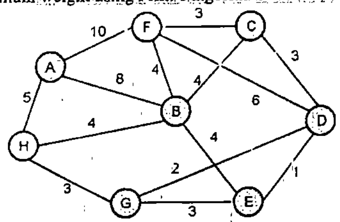


Fig. Ex. 5.10.7

☑ **Solution :**

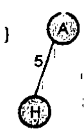
Given :

▸ **Step 1 :**

MST = { {A}, {ϕ} }

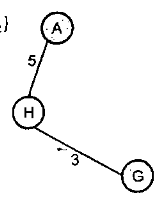
▸ **Step 2 :**

MST = { {A, H}, {e_1} }



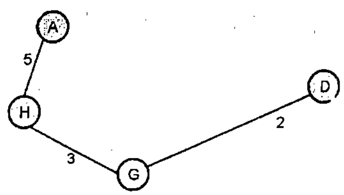
▸ **Step 3 :**

MST = { {A, H,G}, {e_1, e_2}



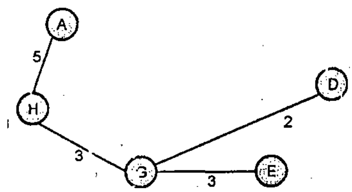
▸ **Step 4 :**

MST = { {A, H, G, D}, {e_1, e_2, e_3} }



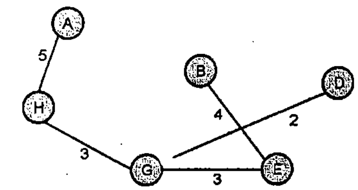
▸ **Step 5 :**

MST = { {A, H, G, D, E}, {e_1, e_2, e_3, e_4} }



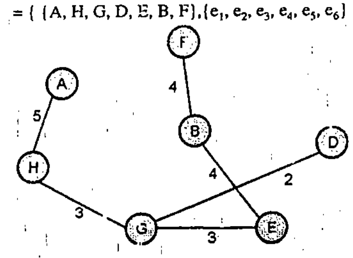
▸ **Step 6 :**

MST = { {A, H, G, D, E, B}, {e_1, e_2, e_3, e_4, e_5} }



▸ **Step 7 :**

MST = { {A, H, G, D, E, B, F},{e_1, e_2, e_3, e_4, e_5, e_6} }



▸ **Step 8 :**

MST = { {A, H, G, D, E, B, F, C}, {e_1, e_2, e_3, e_4, e_5, e_6, e_7} }

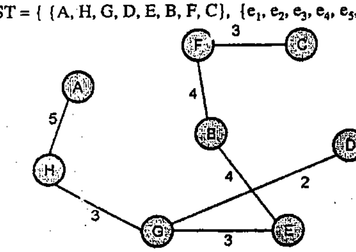


Fig. Ex. 5.10.7(a) : Minimum spanning tree

Weight of minimum spanning tree

W(T) = 5 + 3 + 3 + 4 + 2 + 4 + 3

= 24 Unit ...Ans.

UEx. 5.10.8 (SPPU (IT) - Q. 6(a), May 14, 6 Marks)

Find the MST for the graph given below using Prim's algorithm.

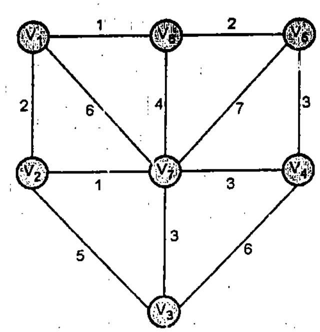


Fig. Ex. 5.10.8 (a)

Soln. :

Step 1 :
Select any arbitrary vertex. Lets select V_1

MST = $\{ \{v_1\}, \{\phi\} \}$ V_1

Step 2 :

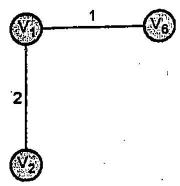
V_1 is adjacent to V_2, V_6 and V_7

The minimum weight edge is V_1 — V_6

MST = $\{ \{v_1, v_6\}, \{e_1\} \}$ $V_1 \overset{1}{—} V_6$

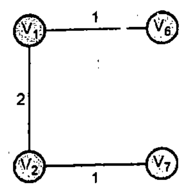
Step 3 :

MST = $\{ \{v_1, v_6, v_2\}, \{e_1, e_2\} \}$



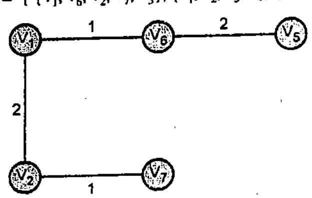
Step 4 :

MST = $\{ \{v_1, v_6, v_2, v_7\}, \{e_1, e_2, e_3\} \}$



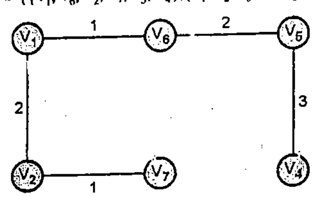
Step 5 :

MST = $\{ \{v_1, v_6, v_2, v_7, v_5\}, \{e_1, e_2, e_3, e_4\} \}$



Step 6 :

MST = $\{\{v_1, v_6, v_2, v_7, v_5, v_4\}, \{e_1, e_2, e_3, e_4, e_5\}\}$



Step 7 :

MST = $\{ \{v_1, v_6, v_2, v_7, v_5, v_4, v_3\}, \{e_1, e_2, e_3, e_4, e_5, e_6\} \}$

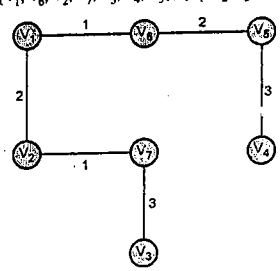


Fig. Ex. 5.10.8(b) : Minimum spanning tree

Weight of the minimum spanning tree.

$$W(T) = 1 + 2 + 2 + 1 + 3 + 3$$
$$= 12 \text{ Unit}$$

5.10.3 Kruskal Algorithm

GQ. Explain the following : Kruskal's Algorithm.

- Kruskal Algorithm is also used to find minimum spanning tree of given graph.
- It is developed by Joseph Kruskal in 1956.
- It find minimum spanning tree of graph by selecting minimum weight edges. It avoids those edges which creates cycle in MST.

Working of Kruskal Algorithm

Let G (V, E) be a weighted, connected graph.

Step I : Pickup an edge e_i of G with minimum weight.

Step II : If edges $e_1, e_2, \ldots, e_n$ have been selected then pick up an edge e_{n+1} such that,

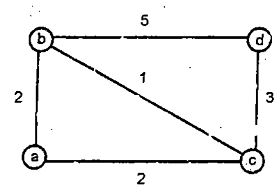
(i) $e_{n+1} \neq c_i$ for i = 1, 2, 3, ...n.

(ii) The edges $e_1, e_2, e_3, \ldots e_n, e_{n+1}$ do not form circuit or cycle.

Step III : Stop when, all the vertices get covered.

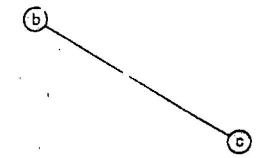
Example

Let G be a graph.



(1H59)**Fig. 5.10.8**

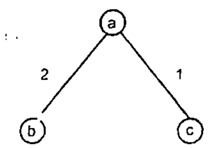
Step I : Select an edge with minimum weight among others. $b \overset{1}{—} c$ has minimum weight.



(1H60)**Fig. 5.10.9**

Step II : Select other edge whose weight is minimum other than b — c.

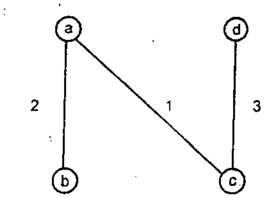
Here there are two edge with weight is 2 a — b = 2 and a — c = 2 select any one of them. We are selecting a — b = 2.



(1H61)**Fig. 5.10.10**

Step III : Select a — c = 2 since it has minimum weight but it creates a circuit in MST. So avoid it. Moving further, select minimum weight edge of graph. cd = 3 has minimum weight. Select it.

Required Hast

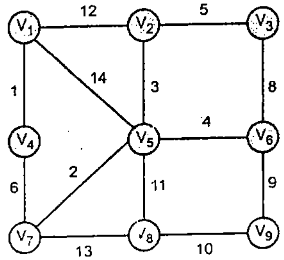


(1H62)**Fig. 5.10.11 : Minimum spanning tree**

The weight of minimum spanning tree

$$= W(a — b) + W(a — c) + W(c — d)$$
$$= 2 + 1 + 3 = 6 \text{ Unit}$$

Ex. 5.10.9 : Give the stepwise construction of minimum spanning tree using Kruskal's algorithm for the following graph. Obtain the total cost of minimum spanning tree.



(1E14A) **Fig. Ex. 5.10.9**

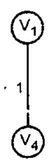
☑ **Soln. :**

In Kruskal algorithm, we have to select minimum weight edge first. Arrange all the edges in increasing order and select one by one and avoid those edge which will make circuit in tree

edges	V_1-V_4	V_5-V_7	V_2-V_5	V_5-V_6	V_2-V_3	V_4-V_7	V_3-V_6	V_6-V_9	V_8-V_9
weight	1	2	3	4	5	6	8	9	10

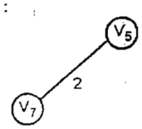
edges	V_5-V_8	V_1-V_2	V_7-V_8	V_1-V_5
Weight	11	12	13	14

- Select V_1-V_4 :



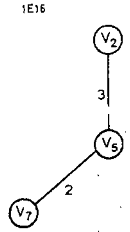
1E16

- Select V_5-V_7 :



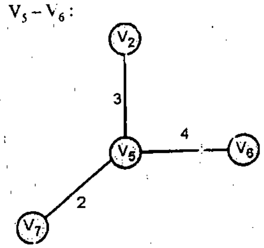
1E16

- Select V_2-V_5 :



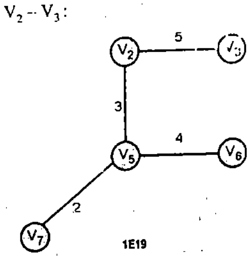
1E17

- Select V_5-V_6 :



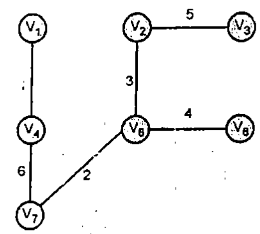
1E18

- Select V_2-V_3 :



1E19

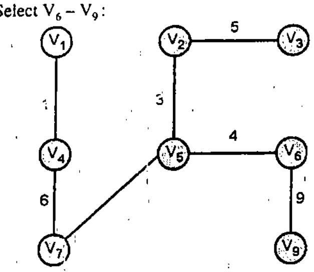
- Select V_4-V_7 :



1E20

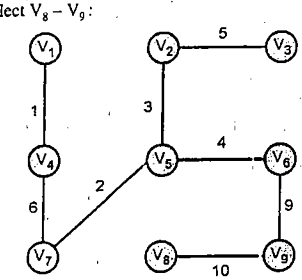
When we select V_3-V_6. It will make an circuit in a spanning tree, so, neglect it. Proceed to next edge

- Select V_6-V_9 :



1E21

- Select V_8-V_9 :



1E22

- Select V_5-V_8 :

It will create a circuit in spanning tree, so neglect it.

- Select V_1-V_2 :

It will also create a circuit in spanning tree, so neglect it.

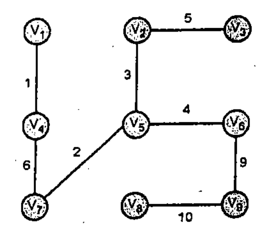
- Select V_7-V_8 :

It will also create a circuit, so neglect it.

- Select V_1-V_5 :

It also creates a circuit, so neglect it.

– Hence the minimum spanning tree is as



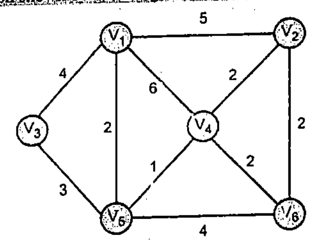
1E23 **Fig. Ex. 5.10.9(a): Minimum spanning tree**

Cost of minimum spanning tree

$$= w(V_1-V_4)+w(V_4-V_7)+w(V_2-V_5)$$
$$+ w(V_2-V_3)+w(V_5-V_6)+w(V_6-V_9)$$
$$+ w(V_8-V_9)$$
$$= 1+6+2+3+5+4+9+10$$
$$= 40$$

• NOTES •

UEx. 5.10.10 (SPPU (IT) - Q. 5(a), May 14, 6 Marks)
Find minimum spanning tree for the graph shown below using Kruskal's algorithm.



(1E46A) **Fig. Ex. 5.10.10**

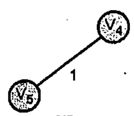
Soln. :

- **Step I :** Arrange all the edges according to their weights in increasing order in edge table

Edges	V_4-V_5	V_1-V_5	V_2-V_4	V_2-V_6	V_4-V_6	V_3-V_5	V_1-V_3	V_5-V_6	V_1-V_2	V_1-V_4
Weights	1	2	2	2	2	3	4	4	5	6

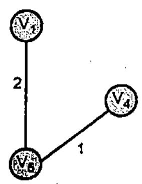
- **Step II :** Select each edge one by one and make a spanning tree. Those edges which will creates a circuit in tree neglect that edge.

- Select V_4-V_5 :



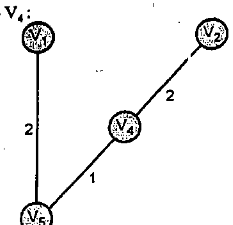
1E47

- Select V_1-V_5 :



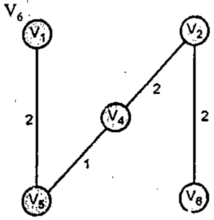
1E48

- Select V_2-V_4 :



1E49

- Select V_2-V_6 :

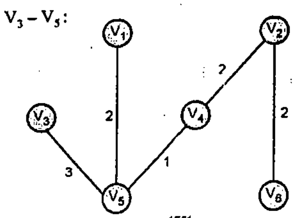


1E50

- Select V_4-V_6:

While selecting $V_4 \rightarrow V_6$. This edge creates an circuit in spanning tree. So neglect it.

- Select V_3-V_5 :



1E51

- Select V_1-V_3 :

The edge $V_1 \rightarrow V_3$ makes a circuit in spanning tree. So neglect it.

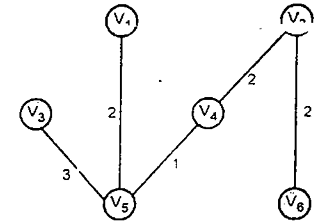
- Select V_1-V_3 :

It will also creates a circuit in spanning tree. So, neglect it.

V_1-V_2 edge also creates an circuit, so neglect it.

V_1-V_4 edge also creates an circuit, so neglect it.

Required minimum spanning tree is given below :



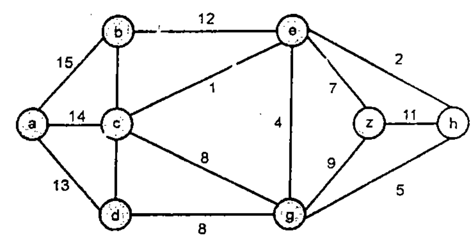
1E52

The weight of the minimum spanning tree is

$$W(T) = 3+2+1+2+2$$
$$= 10 \text{ units}$$

...**Ans.**

Ex. 5.10.11 : Obtain the minimum spanning tree using Kruskal algorithm for the following graph. Obtain the total cost of minimum spanning tree.



(1E73A) **Fig. Ex. 5.10.11**

Soln. :

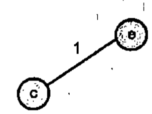
- **Step I :** Arrange all the edges in ascending order according to their weights is edge table

Edge table

Edges	c – e	e – h	c – d	e – g	g – h	e – f	c – g	d – g	f – g	b – c	f – h	a – d	a – c	a – b	b – e
Weights	1	2	3	4	5	6	7	8	9	10	11	13	14	15	12

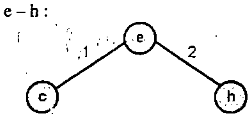
- **Step II :**

- Select c – e :



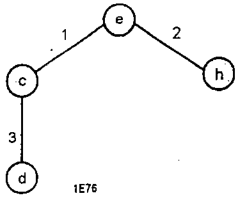
1E74

- Select e – h :



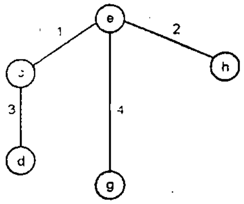
1E75

- Select c – d :

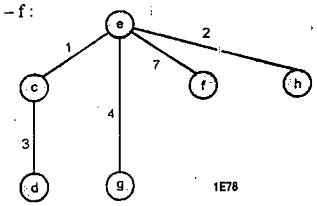


1E76

- Select e – g :

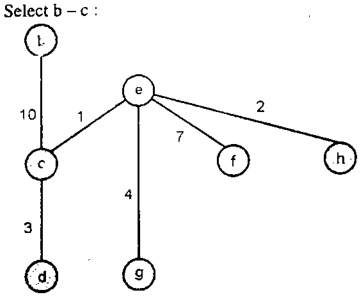


- Select g – h :
  By selecting edge "g – h". it will create on circuit in spanning tree so neglect it and proceed further for next edge.
- Select e – f :



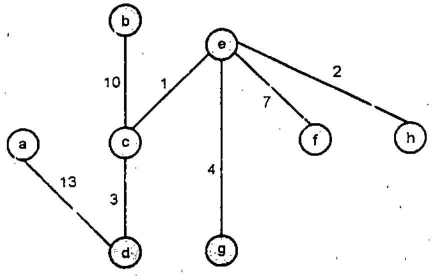
1E78

- Select c – g : edge c – g circuit. So neglect it.
- Select d – g :
  Edge d – g creates a circuit in spanning tree so neglect it.
- Select b – c :



1E79

- Select f – h :
  Neglect, it creates circuit in ST.
- Select b – g :
  Neglect it, it creates circuit in ST.
- Select a – d :



1E80 **Fig. Ex. 5.10.11(a) : Minimum spanning tree**

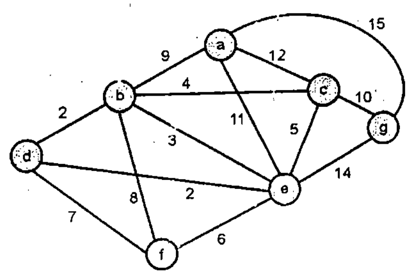
As all the vertices get covered. We will stop here. Cost **of minimum spanning tree is**

$$C(T) = 10 + 3 + 1 + 4 + 7 + 2 + 13$$

$$= 40 \quad \text{...Ans.}$$

Cost is nothing but weight of minimum spanning tree.

Ex. 5.10.12 : Give the stepwise construction of minimum spanning tree using Prim's algorithm for the graph in Fig. Ex. 5.10.12. Obtain the total cost of minimum spanning tree.



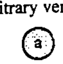
(1E118) **Fig. Ex. 5.10.12**

☑ **Soln. :**

Initially, minimum spanning tree = { φ }

- **Step I :** Select any arbitrary vertex. Let's select ⓐ.

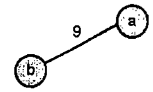
  MST = {{a}, {φ}}



1E119

- **Step II**

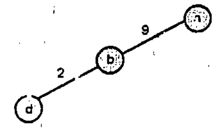
  MST = {{a, b}, {e_1}}



1E120

- **Step III :**

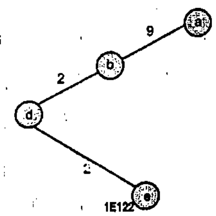
  MST = {{a, b, d}, {e_1, e_2}}



1E121

- **Step IV :**

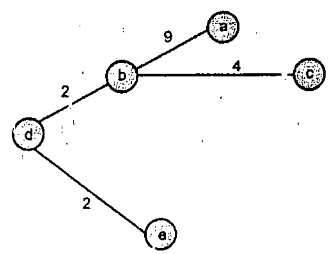
  MST = {{a, b, d, e}, {e_1, e_2, e_3}}



1E122

- **Step V :**

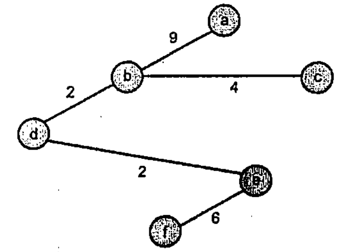
  MST = {{a, b, d, e, c}, {e_1, e_2, e_3, e_4}}



1E123

- **Step VI :**

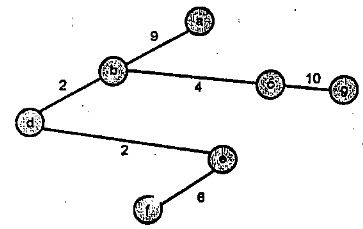
  MST = {{a, b, d, e, c, f}, {e_1, e_2, e_3, e_4, e_5}}



1E124

- **Step VII :**

  MST = {{a, b, d, e, c, f, g}, {e_1, e_2, e_3, e_4, e_5}}



(1E125) **Fig. Ex. 5.10.12(a) : Minimum spanning tree**

The cost (weight) of the minimum spanning tree is as :

$$W(T) = 9 + 2 + 2 + 6 + 4 + 10 = 33 \text{ Unit.}$$

Ex. 5.10.13 : Find the minimum spanning tree of the given Fig. Ex. 5.10.13 using Kruskal's algorithm.

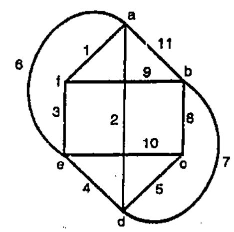


Fig. Ex. 5.10.13

☑ **Soln. :**

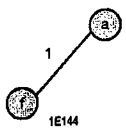
- **Step I :** Arrange all the edges in ascending order according to their weights in edge-table.

Edge table

Edges	a-f	a-d	f-e	e-d	d-c	a-e	b-d	b-c	f-b	e-c	a-b
Weights	1	2	3	4	5	6	7	8	9	10	11

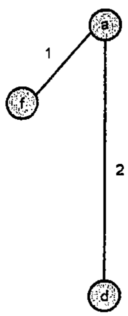
- **Step II :** Select edges one by one and keep developing spanning tree. Those edges who creates on circuit in spanning tree, neglect them and proceed to next edge.

Select : a – f



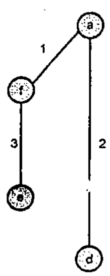
1E144

Select : a – d



1E145

Select : f – e

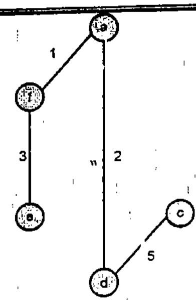


1E146

Select : e – d :

If we select edge e-d, it will creates on circuit in spanning tree so, neglect this edge and proceed further.

Select : d – c

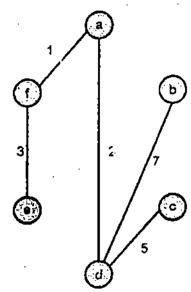


1E147

Select : a – e

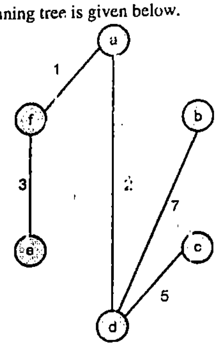
Edge a – e creates an circuit so, neglect it.

Select : b – d



1E148

All the vertices get included in minimum spanning tree, with minimum number of edges. So that , the required minimum spanning tree is given below.



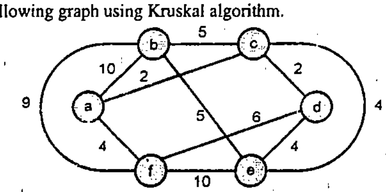
1E149

Fig. Ex. 5.10.13(a) : Minimum spanning tree

The weight (cost) of minimum spanning tree is

$$W(T) = W(af) + W(ad) + W(fe) + W(db) + W(dc)$$
$$= 1 + 2 + 3 + 7 + 5$$
$$= 18 \text{ units} \qquad \text{...Ans.}$$

Ex. 5.10.14 : Find the minimum spanning tree for the following graph using Kruskal algorithm.



1E162

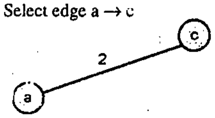
Fig. Ex. 5.10.14

Soln. :

- **Step I :** Arrange all the edges in ascending order in edge table.

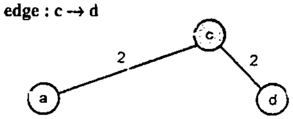
Edges	a-c	c-d	a-f	d-e	e-c	b-c	b-e	d-f	b-f	a-b	f-e
Weights	2	2	4	4	4	5	5	6	9	10	10

- **Step II :** Select edge a → c



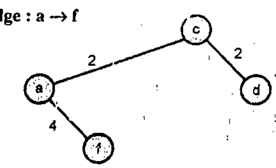
1E163

Select edge : c → d



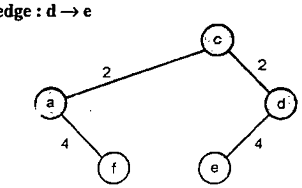
1E164(a)

Select edge : a → f



1E165

Select edge : d → e

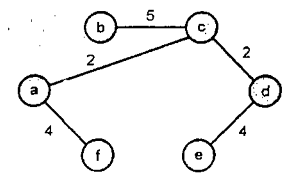


1E166

Select edge : e → c

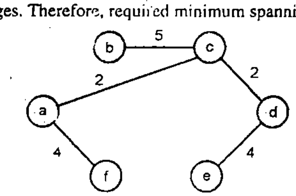
Edge e → c create an circuit in spanning tree. So, neglect it.

Select edge : b → c



1E167

All the vertices get covered with minimum number of edges. Therefore, required minimum spanning tree is



1E168

The weight or cost of minimum spanning tree

$$W(T) = W(ac) + W(cd) + W(de) + W(af) + W(bc)$$
$$= 2 + 2 + 4 + 4 + 5 = 17 \text{ units} \qquad \text{...Ans.}$$

UEx. 5.10.15 (SPPU (IT) - Q. 5(A), May 17, 7 Marks)

Find the minimum spanning tree and weight of it for the given graph using Kruskal's algorithm.

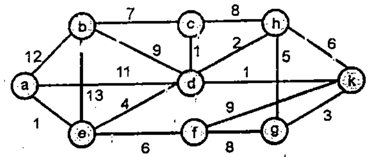


Fig. Ex. 5.10.15

☑ **Soln. :**

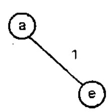
Given :

Arrange all the edges in ascending order according to their weights in the edge table-Edge table

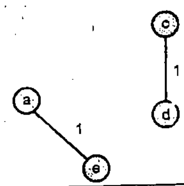
Edge	a-e	c-d	d-k	d-h	g-k	d-e	h-g	h-k	e-f	b-c	c-h	f-g	b-d	f-k	a-d	a-b	b-e
Weight	01	01	01	02	3	4	5	6	6	7	8	8	9	09	11	12	13

Select the edges one by one and make a spanning tree until all vertices get covered. Avoid those edges, who will create a cycle in a tree.

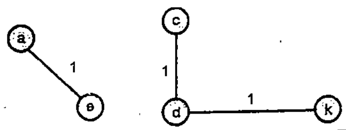
Select a-e



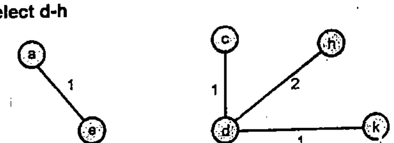
Select cd



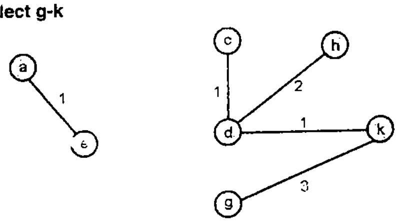
Select d-k



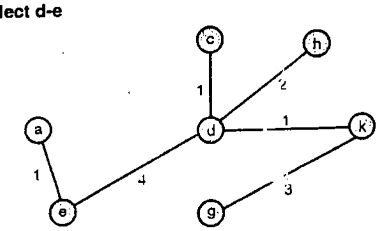
Select d-h



Select g-k



Select d-e



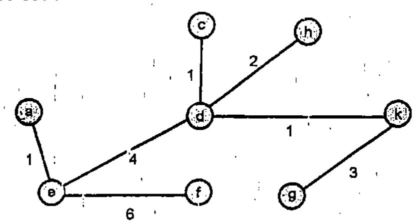
Select h-g

Neglect it. It makes a circuit.

Select h-k

It makes a circuit in spanning tree. So neglect.

Select e-f



Select bc

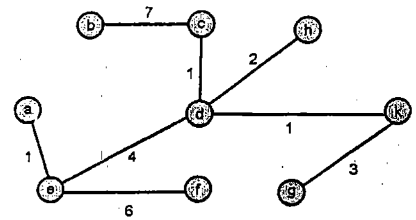


Fig. Ex. 5.10.15(a) : Minimum spanning tree

Weight of MST

W(T) = 01 + 01 + 01 + 02 + 03 + 04 + 06 + 07

= **25 unit** ...**Ans.**

UEx. 5.10.16 (SPPU (IT) - Q. 6(b), Dec. 16, 6 Marks)

Find minimum spanning tree for the graph given in Fig. Ex. 5.10.16 using Kruskal's algorithm.

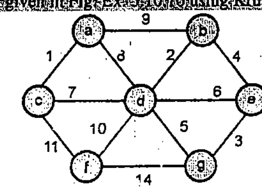


Fig. Ex. 5.10.16 (a)

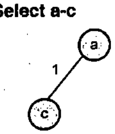
☑ **Solution : Given :**

Arrange all the edges in ascending order in edge table.

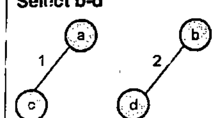
Edges	a-c	b-d	e-g	b-e	d-g	d-e	c-d	a-d	a-b	d-f	c-f	f-g
Weight	01	02	03	04	05	06	07	08	09	10	11	14

Select edges one by one and make a spanning tree. Edge which creates a cycle in S.T., neglect that edge.

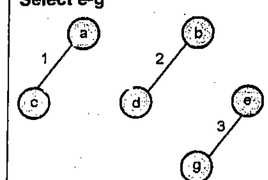
Select a-c



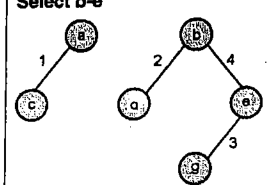
Select b-d



Select e-g



Select b-e



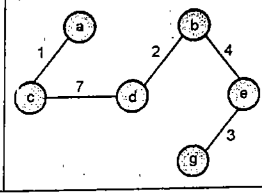
Select d-g

Neglect edge d-g. It creates a loop in S.T.

Select d-e

Neglect edge d-e. It creates loop in S.T.

Select c-d



Select a-d

Edge a-d creates a loop in S.T. so neglect it.

Select a-b

Neglect it.

It creates a loop in S.T.

Select d-f

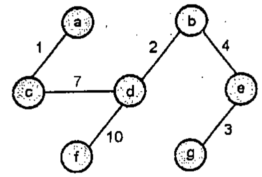


Fig. Ex. 5.10.16(b) : Minimum spanning tree

...Ans.

UEx. 5.10.17 (SPPU (IT) - Q. 6(a), May 16, 6 Marks)

Determine the minimum spanning tree using Kruskal's algorithm for the following graph :

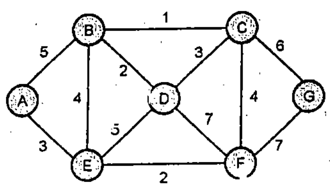


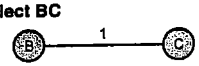
Fig. Ex. 5.10.17

☑ **Solution :**

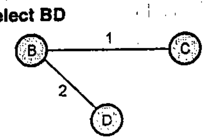
Given : Arrange all the edges in ascending order according to their weights in edge table.

Edges	BC	BD	EF	CD	AE	BE	CF	AB	DE	CG	DF	FG
Weight	01	02	02	03	03	04	04	05	05	06	07	07

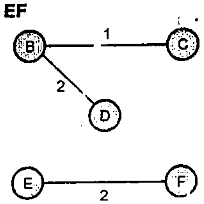
Select BC



Select BD



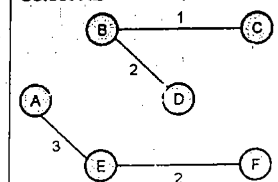
Select EF



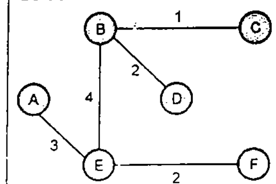
Select CD

Neglect CD. It creates a cycle in S.T.

Select AE



Select BE



Select CF

Neglect CF. It creates a cycle in spanning tree.

Select AB

Neglect AB. It creates a cycle in spanning tree.

Select DE

Neglect DE. It creates a cycle in spanning tree.

Select CG

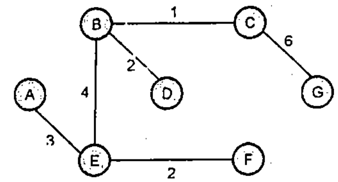


Fig. Ex. 5.10.17(a) : Minimum spanning tree

Weight of minimum panning tree = 3 + 2 + 4 + 2 + 1 + 6
= 18

UEx. 5.10.18 (SPPU (IT) - Q. 5(b), May 15, 7 Marks)

Find the minimum spanning tree and weight of it for the given graph using Krushal's algorithm.

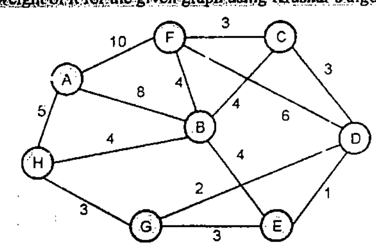


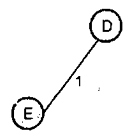
Fig. Ex. 5.10.18

☑ **Solution :**

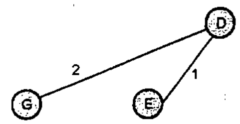
Edge table

Edges	DE	DG	EG	GH	CF	CD	BH	BF	BC	BE	AH	DF	AB	AF
Weight	01	02	03	03	03	03	04	04	04	04	05	06	08	10

Select DE



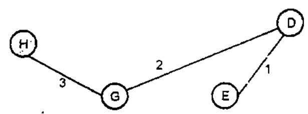
Select DG



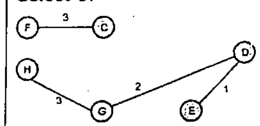
Select EG

Neglect EG. Because EG creates a circuit in spanning tree

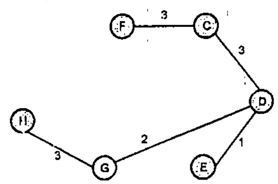
Select GH



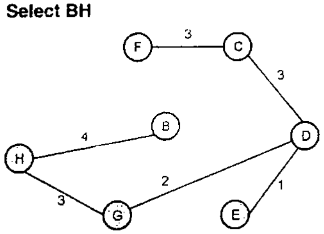
Select CF



Select CD



Select BH



Select BF

Neglect it. It creates a circuit in spanning tree. We neglected edges BC and BE. Since they both were creating circuit in spanning tree.

Now, Select AH

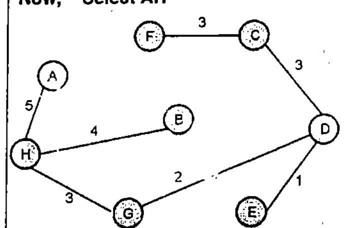


Fig. Ex. 5.10.18(a) : Minimum spanning tree

Weight of minimum spanning tree

$$W(T) = 5+4+3+2+1+3+3$$

$$= 21 \text{ Unit}$$

...Ans.

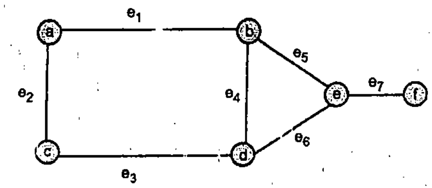
5.11 FUNDAMENTAL CIRCUITS AND CUTSETS

GQ. Explain fundamental system of cutsets with suitable examples.

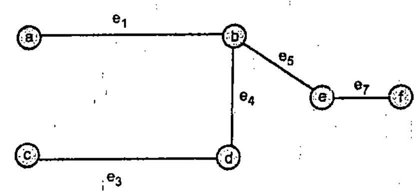
- Let, G be a connected graph and T be a spanning tree of G.
- **Branch :** An edge associated with the spanning tree is called as "branch".
- **Chord :** An edge associated with the graph G is called "chord".
- **Fundamental circuit :** When a circuit is formed by adding chord to spanning tree i.e. (T + e), then it is called as fundamental circuit.
- A fundamental circuit of connected graph G is always associated with spanning tree (T) of graph G.

Note : Different spanning trees of graph may have different fundamental circuits.

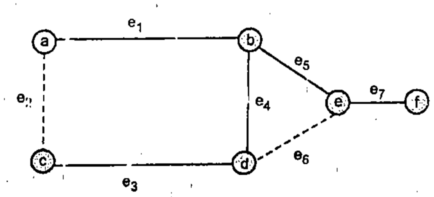
Example



(1H64)Fig. 5.11.1 : Graph G



(1H65)Fig. 5.11.2 : Spanning tree of G



(1H66)Fig. 5.11.3 : Spanning tree with fundamental circuit

- Vertices of graph G, V (G) = { a, b, c, d, e, f }
- Edges of graph G, E(G) = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
- Branches of spanning tree $E_T = \{e_1, e_3, e_4, e_5, e_7\}$
- Chords of spanning tree = $E - E_T = \{e_2, e_6\}$
- As we know that when chord is added to spanning tree, it makes circuit,

Sr. No.	Chords	Fundamental circuit
(i)	e_2	$\{e_1, e_4, e_3, e_2\}$
(ii)	e_6	$\{e_4, e_5, e_6\}$

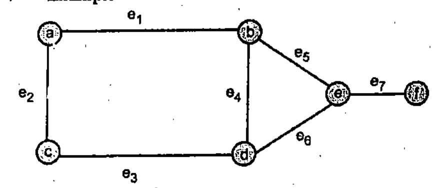
☞ Fundamental cutsets

- We know that, in a spanning tree, every edge is a bridge or isthmus. If we cut any of the edge, the spanning tree will splits into two trees like T_1 and T_2.

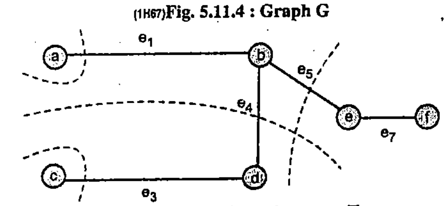
Definition : Cut set is a set of minimal edges in a graph G such that when broken, the whole graph breaks into exactly two parts.

- Fundamental cutset of a graph is always respect to a spanning tree of the graph.
- In spanning tree, removal or cutting of any edge, breaks the spanning tree into exactly two trees or two set of vertices.
- The number of fundamental cutsets of G with respect to spanning tree "T" is the number of branches (edges in T) of T.

Example



(1H67)Fig. 5.11.4 : Graph G



(1H68)Fig. 5.11.5 : Spanning tree : T

- Branches of T are : e_1, e_3, e_4, e_5 and e_7.
- Fundamental cutsets of graph G with respect to spanning tree "T" are as given below in Table 5.11.1.

Table 5.11.1

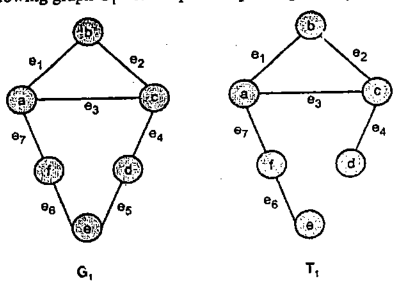
Branches of T	Fundamental cutsets
e_1	$\{e_1, e_2\}$
e_3	$\{e_2, e_3\}$
e_4	$\{e_2, e_4, e_6\}$
e_5	$\{e_5, e_6\}$
e_7	$\{e_7\}$

- In the Table 5.11.1, if we cut branch e_1 of T then, it will split whole tree into two parts, one part consist of single vertex and another part contains vertices b, c, d, e, f. While cutting edge e_1, chord e_2 of tree T also get cut. Hence fundamentals cutest of $e_1 = \{e_1, e_2\}$.

– In this same way, we have to cut all the branches. In such a way that whole tree get splits into exactly two subtrees.

> **Note :** (a) In a connected graph with n number of vertices, the spanning tree will have (n – 1) edges and there are (n – 1) fundamental cutsets

Ex. 5.11.1 : Find fundamental cutsets and circuits of the following graph G_1 with respect to spanning tree T_1.



(1E29A) **Fig. Ex. 5.11.1**

☑ **Soln. :**

(i) Fundamental circuit is associated with chords of T_1

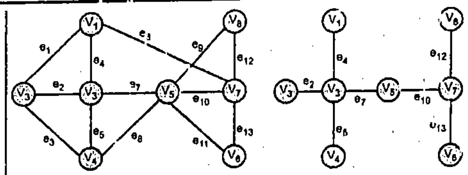
Chords	Corresponding fundamental circuit
e_3	$\{e_1, e_2, e_3\}$
e_5	$\{e_4, e_5, e_6, e_1, e_2\}$

(ii) Fundamental cutest is associated with branches of T_1

Braches	Corresponding fundamental cutest
e_1	$\{e_1, e_3, e_5\}$
e_2	$\{e_2, e_3, e_5\}$
e_4	$\{e_4, e_5\}$
e_6	$\{e_5, e_6\}$
e_7	$\{e_5, e_7\}$

UEx. 5.11.2 (SPPU (IT) - Q. 5(b), Dec. 14, 7 Marks)

Find the fundamental system of cutset for the graph G shown below with respect to the spanning tree T.



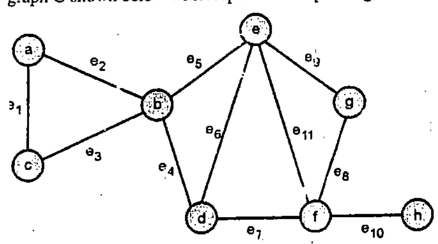
(11E82AB) **Fig. Ex. 5.11.2**

☑ **Soln. :**

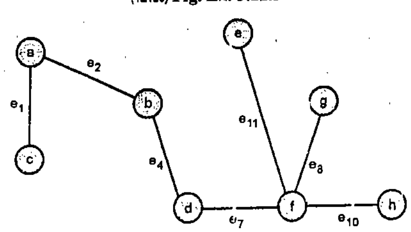
Fundamental cutsets are always associated with branches of spanning tree. There are 7 branches (edges) $\{e_2, e_4, e_5, e_7, e_{10}, e_{12}, e_{13}\}$ so that there will be 7fundomental cutsets.

Branches	Fundamental cutsets
e_2	$\{e_1, e_2, e_3\}$
e_4	$\{e_1, e_4, e_6\}$
e_5	$\{e_3, e_5, e_6\}$
e_7	$\{e_6, e7, e_8\}$
e_{10}	$\{e_6, e_9, e_{10}, e_{11}\}$
e_{12}	$\{e_9, e_{12}\}$
e_{13}	$\{e_{11}, e_{13}\}$

Ex. 5.11.3 : Find the fundamental system of cut-set for the graph G shown below with respect to the spanning tree T.



(1E126) **Fig. Ex. 5.11.3**



(1E127) **Fig. Ex. 5.11.3 (a) : Spanning tree (T)**

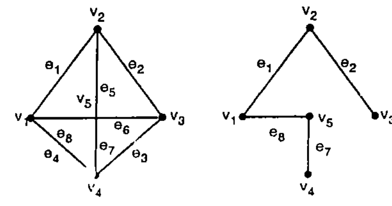
☑ **Soln. :**

Fundamental cutsets are corresponds to branches (edges) of spanning tree. Here, T has 7 edges. Then it has 7 corresponding cutsets.

Branches	Fundamental cutsets		Branches	Fundamental cutsets
e_1	$\{e_1, e_3\}$		e_{11}	$\{e_9, e_{11}, e_6, e_5\}$
e_2	$\{e_2, e_3\}$		e_8	$\{e_8, e_9\}$
e_4	$\{e_4, e_6\}$		e_{10}	$\{e_{10}\}$
e_7	$\{e_7, e_6, e_5\}$			

UEx. 5.11.4 (SPPU (IT) - Q. 6(b), Dec. 18, 6 Marks)

Find out fundamental cut sets of the following graph with respect to given spanning tree.



(a) (b)

Fig. Ex. 5.11.4

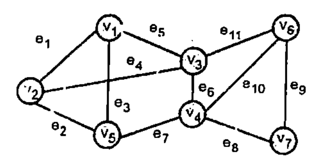
☑ **Soln. :**

The fundamental cutsets are associated with number of branches (edges) in spanning tree. Here, there are 4 branches of tree. Then there will be 4 cutsets.

Branches	Fundamental cutsets
e_1	$\{e_1, e_5, e_6, e_3\}$
e_2	$\{e_2, e_6, e_3\}$
e_7	$\{e_3, e_4, e_7\}$
e_8	$\{e_4, e_8, e_5, e_6, e_3\}$

UEx. 5.11.5 (SPPU (IT) - Q. 6(B), May 17, 6 Marks)

Find the fundamental system of cut set for the graph G shown below with respect to the spanning tree T.



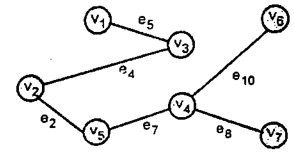


Fig. Ex. 5.11.5

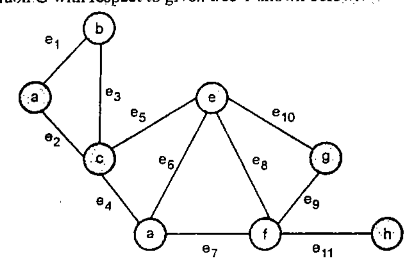
☑ **Solution :**

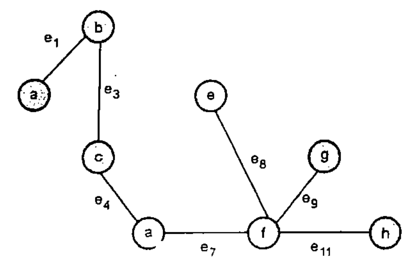
Given :

Branches	Fundamental Cutsets
e_2	$\{e_2, e_3, e_6, e_{11},\}$
e_4	$\{e_1, e_3, e_4, e_6, e_{11}\}$
e_5	$\{e_1, e_3, e_5\}$
e_7	$\{e_7, e_6, e_{11}\}$
e_8	$\{e_8, e_9\}$
e_{10}	$\{e_9, e_{10}, e_{11}\}$

UEx. 5.11.6 (SPPU (IT) - Q. 5(b), May 16, 6 Marks)

Define fundamental circuit. Find fundamental circuit of graph G with respect to given tree T shown below.





☑ **Soln. :**

Fundamental circuit

Given :

Chords		Fundamental Circuits
e_2	⟶	$\{e_1, e_2, e_7\}$
e_5	⟶	$\{e_4, e_5, e_8, e_7\}$
e_6	⟶	$\{e_6, e_8, e_7\}$
e_{10}	⟶	$\{e_8, e_9, e_{10}\}$

5.12 NETWORK FLOWS

GQ. Explain any two of the following : Transport network.

- In daily life we regularly deal with railway lines, telephone line, gas line, water line and so on, here it is necessary to understand the maximum flow from one station to other in any type of network.
- These type of networks which we see every day can be represented by weighted directed graph where each station is represented by vertex and the pipes or the lines (carry gas or water or oil) is represented by edges.

Basic terminologies

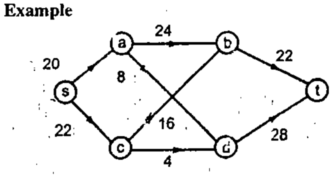
1. **Source :** Source is a vertex from which flows start that is, source does not have incoming edge. In whole network or graph, there is only one source which is generally represented by S.
2. **Sink :** Sink is a vertex to which all flows comes in that is, sink does not have outgoing edge. There is one and only one sink node in the network graph. Generally, it is represented by t.
3. **Residual capacity :** It is the original capacity of edge or pipe minus (–) the flow of that edge.
   e.g. let, say, ab is an edge whose maximum capacity is 10 and current flow of ab is 6, then residual capacity of ab will be 10 – 6 = 4.
4. **Augmented path :** Augmented path is a path of network graph that we selected for flowing material into it.

Transport network

Transport network is an weighted directed, connected graph which must be satisfied the following conditions.

- There must be one and only one source node or station.
- There must be one and only one sink node.
- It must be without loop.
- Every edge or pipe or line must have non-negative real number weight is known as capacity of the edge.

▸ **Example**



(1H69)**Fig. 5.12.1 : Transport network**

In transport network,

S ⇒ Source node, t ⇒ sink node

Capacity of s – a = C_{sa} = 20

Capacity of s – c = C_{sc} = 22

Capacity of a – b = C_{ab} = 24

Capacity of c – d = C_{cd} = 4

Capacity of a – d = C_{ad} = 8

Capacity of b – t = C_{bt} = 22

Capacity of d – t = C_{dt} = 28

Some important points

(i) Consider a vertex ⓧ in the network then flow into the vertex ⓧ = Flow away from the vertex x.

(ii) Flow coming from the source vertex ⓢ = Flow into the sink vertex ⓣ.

(iii) The actual flow through the edge or pipe must be less than or equal to (≤) capacity of the pipe.

e.g. consider a pipe or edge a – b whose capacity is 24, then flow must be

$\therefore \ 0 \le f_{ab} \le C_{ab}$ for all a, b, ∈ V.

(iv) A given edge (x, y) is said to be saturated if flow of xy is equal to capacity of xy that is,

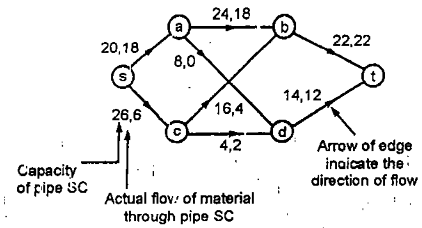
$f_{xy} = C_{xy}$

(v) A given edge (x, y) is said to be unsaturated if flow of xy is less than capacity of xy that is,

$f_{xy} < C_{xy}$

(vi) Flow through the network can be represented as (s, t) – flow. A label on edge or pipe has two parameters first is capacity of the pipe and second is actual flow through the pipe.

▸ **Example**



(1H71)**Fig. 5.12.2 : Flow of material through pipe SC**

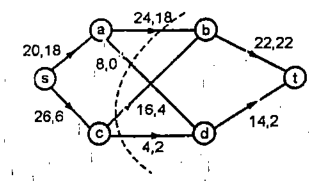
Theorem

The value of any flow in a transport network is less than or equal to the capacity of any cut in the network.

The max flow min cut theorem

- **Theorem :** The maximum amount of flow passing from the source (S) node to the sink (S) node is equal to the total weight of the edges in the minimum cut (the smallest total weight of the edges, if removed, will separates the source from the sink).
- Maximum flow in the network = Minimum cut.
- If we cut the edges, then there will be no flow from the source to sink.

▸ **Example**



(1H70)**Fig. 5.12.3 : Transport network**

- If we cut the edges, in order to separate the source from destination.
- Lets, cuts the edges ab, ad, cb and cd, the whole network divided into two parts as

S = {s, a, c}

T = {t, b, d}

and there will be now flow from source to sink.

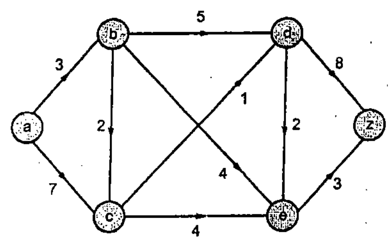
Capacity of cut

Capacity of ab + Capacity of ad + Capacity of cb + Capacity of cd

$= C_{ab} + C_{ad} + C_{cb} + C_{cd}$

$= 24 + 8 + 16 + 4 = 52$

Ex. 5.12.1 : Using the labelling procedure to find maximum flow in the transport network in the following figure. Determine the corresponding minimum cut.



(1E24A) **Fig. Ex. 5.12.1**

☑ **Soln. :**

▸ **Step I :** Select any arbitrary path of given network. Lets select ⓐ → ⓒ → ⓔ → ⓩ

Initially, flow on selected path is zero. Then apply unit flow, till we get any saturated path.

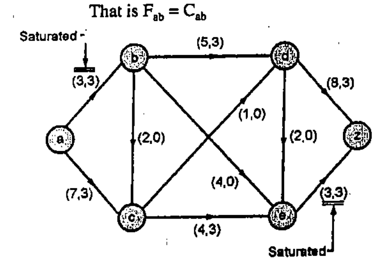
On the unit flow 3, edge e – z get saturated first. Flow of ⓔ → ⓩ

Here,

The flow of path

a → c → e → z = 3 unit.

▸ **Step II :** Select another path say "abdz". When we apply unit flow on this path and incrementing one at a time, then edge "ab" get saturated, when flow reached to 3 units. That is $F_{ab} = C_{ab}$



1E26

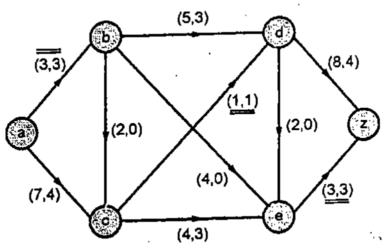
Here, the flow of path

a → b → d → z = 3 unit

▸ **Step III :** Now select another unsaturated path, where (flow and capacity) and apply unit flow we get saturated path

Path ⇒ $a \to c \to d \to z$

The edge "cd" get saturated when unit flow reached to "1" units flow of cd = capacity of cd



1E27

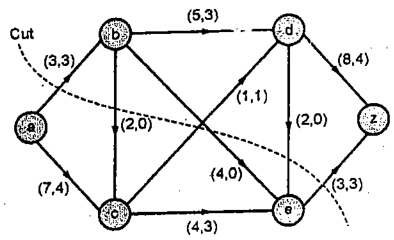
Here, The flow of path, $a \to c \to d \to z$

flow = 1 unit

Note : There is no unsaturated path is remaining to apply unit flow, so we will stop here.

According to max flow – min cut theorem

Maximum flow in the network = minimum cut



1E28

Maximum flow in the network

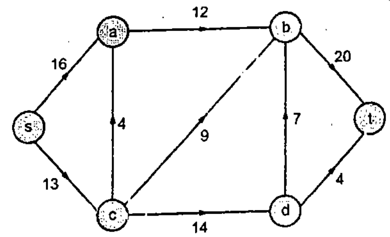
$= F_{ez} + F_{ab} + F_{cd} = 3 + 3 + 1 = 7$ units

Capacity of cut $= C_{ab} + C_{cd} + C_{ez}$

$= 3 + 1 + 3 = 7$ units

Minimum cut = 7

Ex. 5.12.2 : Using the labelling procedure to find maximum flow in the transport network in the following figure. Determine the corresponding minimum cut.

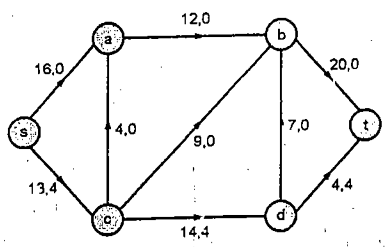


(1E37A) **Fig. Ex. 5.12.2**

☑ **Soln. :**

- **Step I :** Select any arbitrary path of network and apply unit flow so, here we have selected $s \to c \to d \to t$ path

After applying unit flow one at a time the edge "dt" get saturated at unit flow = 4 that is flow of dt ⇒ $f_{dt} = C_{dt}$



1E38

Here, the flow of path

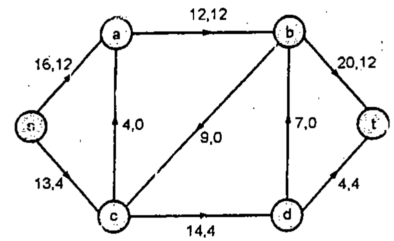
$S \to C \to d \to t = 4$ units

- **Step II :** Select "$s \to a \to b \to t$" path and apply unit flow. Edge "ab" get saturated first as unit flow = 12 that is flow of ab = capacity of ab ⇒ $F_{ab} = C_{ab}$

Here the flow of path

$s \to a \to b \to t$

= 12 units



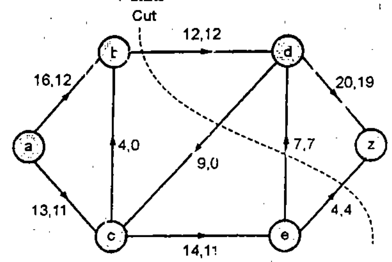
1E39

- **Step III :** Select "$s \to c \to d \to b \to t$" path to apply unit flow. Edge "db" get saturated at unit flow = 7 units which means flow of db = capacity of db

Here flow of path

$s \to c \to d \to b \to t$

= 7 units



1E40

⇒ The maximum flow of network is given below

⇒ Flow of "dt" + Flow of "ab" + flow of "db"

⇒ $F_{dt} + F_{ab} + F_{db}$ ⇒ 4 + 12 + 7 = 23 units

⇒ The capacity of minimum cut

= capacity of (ab) + capacity of (db) + capacity of (dt)

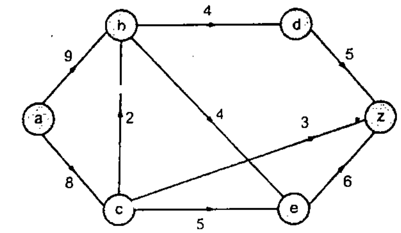
= 12 + 7 + 4 = 23 ...Ans.

According to max. flow – min. cut theorem

Maximum flow = minimum cut

23 = 23

Ex. 5.12.3 : Find maximum flow in the transport network using labelling procedure. Determine the corresponding min cut :



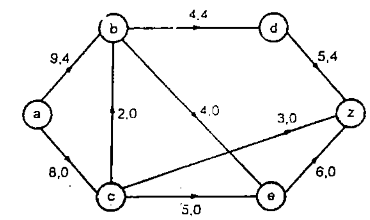
(1E41A) **Fig. Ex. 5.12.3**

☑ **Soln. :**

- **Step 1 :** Select any arbitrary path and apply unit flow. We have selected "$a \to b \to d \to z$". the edge "bd" get saturated at unit flow 4

Here the flow of the path

$a \to b \to d \to z = 4$ units

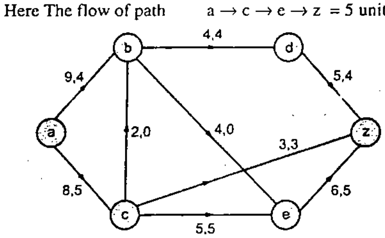


(1E42) **Fig. Ex. 5.12.3(a)**

- **Step 2 :** Select "$a \to b \to d \to z$" path and apply unit flow, Edge "ce" get saturated at unit flow = 5.

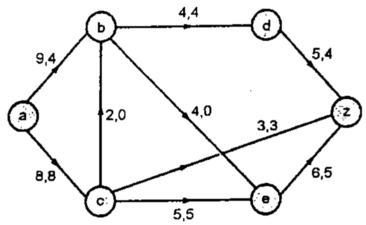
$F_{ce} = C_{ce}$

Here The flow of path $a \to c \to e \to z = 5$ units



(1E43) **Fig. Ex. 5.12.3(b)**

▸ **Step 3 :** Select "a – c – z" path and apply unit flow over it. Edge "ac" and "cz" get saturated at unit flow = 3



(1E44) **Fig. Ex. 5.12.3(c)**

Here,

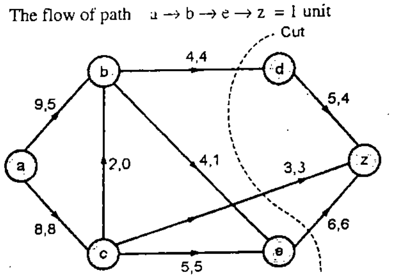
The flow of path

$a \to c \to z = 3$ units

▸ **Step 4 :** Select "$a \to b \to e \to z$" path and apply unit flow. Edge "ez" get saturated at unit flow = 1

Here

The flow of path $a \to b \to e \to z = 1$ unit



(1E45) **Fig. Ex. 5.12.3(d)**

The maximum flow through the network is as :

$\Rightarrow$ flow of bd + flow of ce + flow of cz + flow of ez

$\Rightarrow$ 4 + 5 + 3 + 1

$\Rightarrow$ 13 units

The minimum cut = capacity of bd + capacity of cz + capacity of ez

= 4 + 3 + 6 = 13 units

According max. flow – min cut theorem

Maximum flow = minimum cut

13 = 13

Ex. 5.12.4 : Determine the maximum flow in the transport network shown in Fig. Ex. 5.12.4 using labelling procedure. Determine the corresponding min. cut.

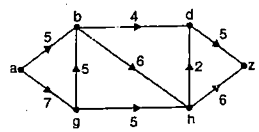
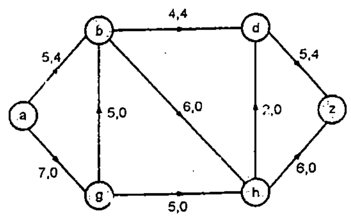


Fig. Ex. 5.12.4

☑ **Soln. :**

▸ **Step 1 :** Select any arbitrary path and keep applying unit flow. Here we have selected path "$a \to b \to d \to z$" edge "bd" get saturated at unit flow 4.



1E54

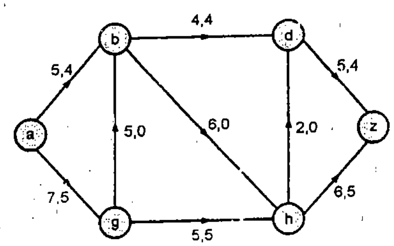
Here flow of path

$a \to b \to d \to z = 4$ units

▸ **Step 2 :** Select path $a \to g \to h \to z$. Edge "gh" is saturated at unit flow 5.

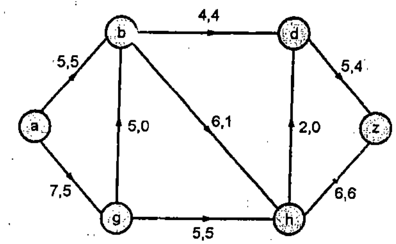
The flow of path

$a \to g \to h \to z = 5$ units



1E55

▸ **Step 3 :** Select path "$a \to b \to h \to z$". Edge "ab" and "hz" get saturated at unit flow 1.



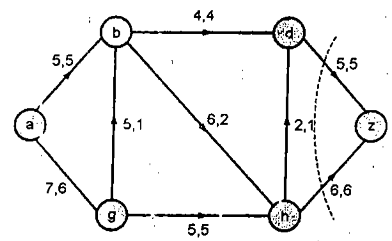
1E56

The flow of path

$a \to b \to h \to z = 1$ unit

▸ **Step 4 :** Select path $a \to g \to b \to h \to d \to z$.

Edge "dz" get saturated at unit flow 1



1E57

The flow of path

$a \to g \to b \to h \to d \to z = 1$ unit

The maximum flow of the network is

$\Rightarrow$ Flow of "bd" + flow of "gh" + flow of "ab" + flow of "dz"

$= F_{bd} + F_{gh} + F_{ab} + F_{dz}$

$= 4 + 5 + 1 + 1 = 11$ units ...**Ans.**

Minimum cut, which can separate source (a) and sink (z) so that, there will be no flow from d to z.

Cut the edges dz and hz

Capacity of cut $= C_{dz} + C_{hz} = 5 + 6 = 11$ unit ...**Ans.**

According to max. flow – min. cut theorem

The maximum flow through the network = minimum cut

11 = 11

Ex. 5.12.5 : Find the maximum flow of the transport network given in the Fig. Ex. 5.12.5.

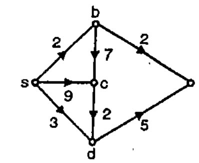


Fig. Ex. 5.12.5

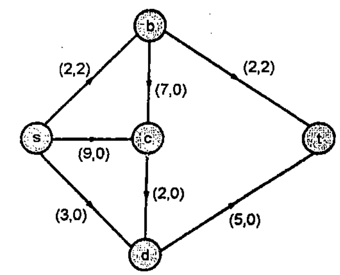
☑ **Soln. :**

▸ **Step I :** Select any arbitrary path. Apply unit flow.

Lets select "$s \to b \to t$" path.

Initially, start applying unit flow on selected path, till we get any saturated edge.

At unit flow 2, edge $s \to b$ and $b \to t$ get saturated, that is $f_{sb} = C_{sb}$ and $f_{bt} = C_{bt}$.

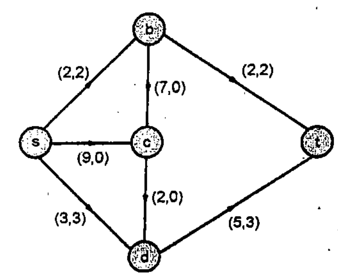


1E151

The flow of the path, $s \to b \to t$ is 2

▸ **Step II :** Select "$s \to d \to t$" path

The edge "sd" get saturated at unit flow 3.

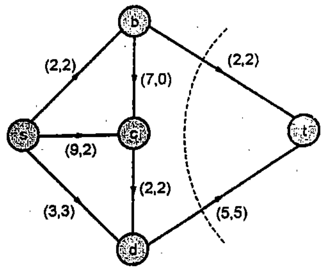


1E152

The flow of path $s \to d \to t$ is 3.

Step III : Select "s → c → d → t" path.

The edge cd get saturated at unit flow = 2.



1E153

Here, the flow of the path

$s \to c \to d \to t$ is 2.

The maximum flow of given network is

⇒ Flow of $(s \to b \to t)$ + flow of $(s \to d \to t)$ + flow of $(s \to c \to d \to t)$

$\Rightarrow F_{sb} + F_{sd} + F_{cd}$

$\Rightarrow 2 + 3 + 2 =$ **7 units** ...Ans.

The minimum cut is

Capacity of C_{bt} + Capacity of $C_{dt} = 2 + 5 = 7$ units

According to maximum flow – Minimum cut theorem

Maximum flow in the network = Minimum cut

$7 = 7$

UEx. 5.12.6 (SPPU (IT) - Q. 5(a), Dec. 14. 7 Marks)

Find the Maximum flow of the given Transport network in Fig. Ex. 5.12.6.

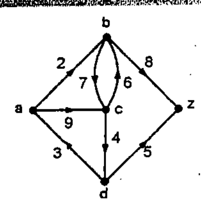
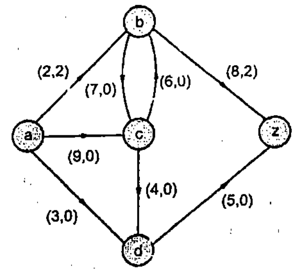


Fig. Ex. 5.12.6

☑ **Soln. :**

Step I : Select any arbitrary path from the given network. So, lets select $a \to b \to z$

Apply unit flow. Edge $a \to b$ gets saturated when unit flow reached to two (2) units.



1E155

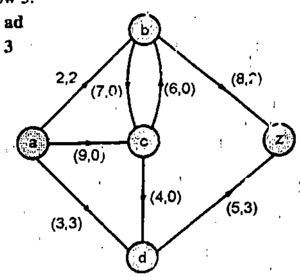
arc = ab

Flow = 2

Step II : Select "a → d → z" path, and apply unit flow on it. Edge $a \to d$ get saturated at unit flow 3.

Arc = ad

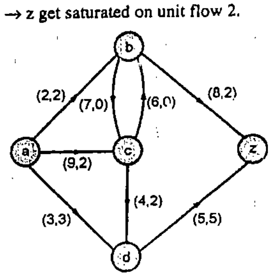
Flow = 3



1E156

Step III : Select another path "a → c → d → z" and apply unit flow on it.

Edge $d \to z$ get saturated on unit flow 2.



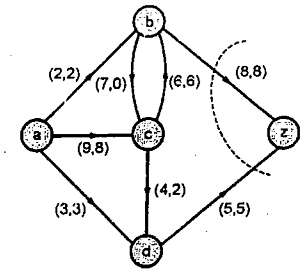
1E157

Arc = dz ; Flow = 2

Step IV : Select "a → c → b → z" and apply unit flow on it.

Edge bc and bz get saturated at unit flow 6.

arc = bc , Flow = 6



1E158

There is no path remaining, which is unsaturated so, stop here. The maximum flow in the network = Flow of (ab) + flow of (ad) + flow of (dz) + flow of (bc)

$= 2 + 3 + 2 + 6 = 13$ unit

Minimum cut which separates the source and sink. In order to stop the flow between source and sink.

Minimum cut $= C_{bz} + C_{dz} = 8 + 5 = 13$ unit

UEx. 5.12.7 (SPPU (IT) - Q. 6(A), May 17, 7 Marks)

Find the maximum flow for the following transport network.

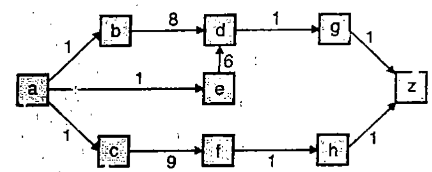
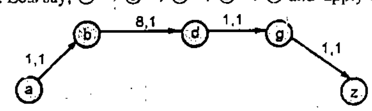


Fig. Ex. 5.12.7

☑ **Solution : Given :**

Step 1 :

Select any arbitrary path. Lets say, ⓐ → ⓑ → ⓓ → ⓖ → ⓩ and apply unit flow on it.



The edges a → b , d → g & g → z get

Saturated at unit flow = 1

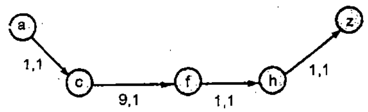
That is, $F_{ab} = C_{ab} = 1$ unit

Step 2 :

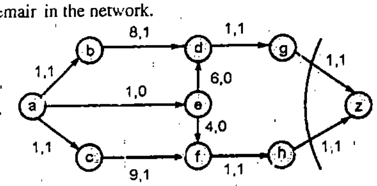
Select other unsaturated path. Lets take a → c → f → h → z

The edges a → c , f → h & h → z

get saturated at unit flow = 1



There is no unsaturated path remair in the network.



Maximum flow through network = $F_{ab} + F_{ac}$ = 1 + 1 = 2 unit

Capacity of cut = $C_{gz} + C_{hz}$ = 1 + 1 = 2 unit

According to maximum flow - minimum cut theorem.

Maximum flow = Minimum cut

$\boxed{2 = 2}$

UEx. 5.12.8 (SPPU (IT) - Q. 5(b). Dec. 15. 7 Marks)

Find the maximum flow in the following transport network :

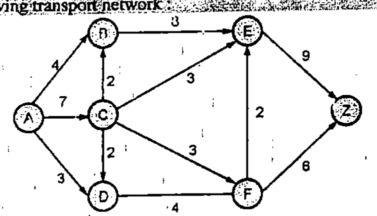


Fig. Ex. 5.12.8

☑ **Solution : Given :**

▸ **Step 1 :**

Select any arbitrary path from the network and apply unit flow onto it until path get saturated.

select A → B → E → Z

Here, edge A—B get saturated at unit flow = 4 that is

$\boxed{F_{AB} = C_{AB} = 4 \text{ unit}}$

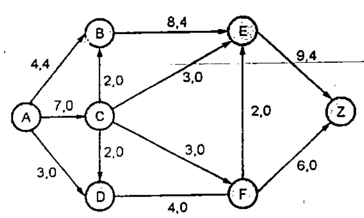


Fig. Ex. 5.12.8(a)

▸ **Step 2 :**

select A → D → F → Z

Here, edge A—D get saturated at unit flow = 3

i.e. $\boxed{F_{AD} = C_{AD} = 3}$

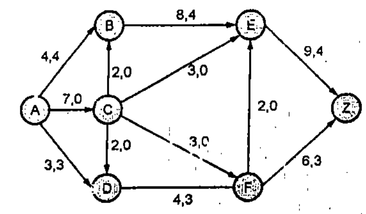


Fig. Ex. 5.12.8(b)

▸ **Step 3 :**

select A → C → B → E → Z

Edge C—B get saturated at unit flow = 2

i.e. $\boxed{F_{CB} = C_{CB} = 2 \text{ unit}}$

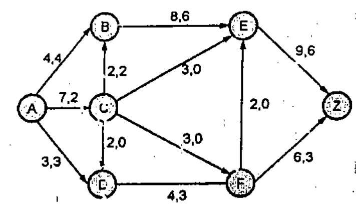


Fig. Ex. 5.12.8(c)

▸ **Step 4 :**

select A → C → E → Z

Here, edge C—E get saturated at unit flow = 3

i.e. $\boxed{F_{CE} = C_{CE} = 3}$

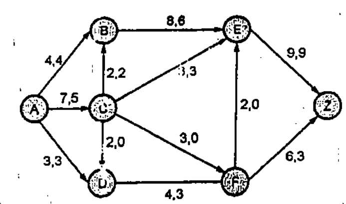


Fig. Ex. 5.12.8(d)

Step 5 :

select A → C → F → Z

Here, edge A—C get saturated at unit flow = 2

i.e. $\boxed{F_{AC} = C_{AC} = 2 \text{ unit}}$

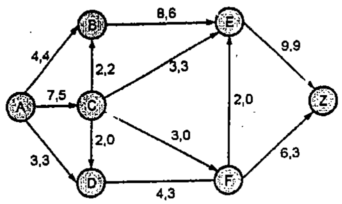


Fig. Ex. 5.12.8(e)

The maximum flow in the network is

⇒ Flow of AB + flow of AD + Flow of CB + Flow of CE + Flow of AC

⇒ $4 + 3 + 2 + 3 + 2 = 14$ Unit

According to max.flow – min. cut theorem.

Capacity of cut = Capacity of (AB + AC + AD) = 4 + 7 + 3 = 14 unit

Maximum flow = Minimum cut
14 = 14

Chapter Ends...

□□□

Note

UNIT IV

Relations And Functions

>> Syllabus :

- **Relations :** Properties of Binary Relations, Closure of Relations, Warshall'sAlgorithm, Equivalence Relations, Partitions, Partial Ordering Relations, Lattices, Chains and Anti Chains.
- **Functions :** Functions, Composition of Functions, Invertible Functions, Pigeonhole Principle, Discrete Numeric Functions.
- **Recurrence Relations :** Recurrence Relation, Linear Recurrence Relations with Constant Coefficients, Total Solutions, Applications of Relations and Functions.

Mapping of Course Outcomes for Unit IV	CO4

- **Chapter 6 : Relations**
- **Chapter 7 : Functions**

UNIT IV

CHAPTER 6 Relations

Syllabus

Relations : Properties of Binary Relations, Closure of Relations, Warshall's Algorithm, Equivalence Relations, Partitions, Partial Ordering Relations, Lattices, Chains and Anti Chains.

Recurrence Relations: Recurrence Relation, Linear Recurrence Relations with Constant Coefficients, Total Solutions, Applications of Relations and Functions.

6.1 INTRODUCTION

- In chapter 1, we have understood, set is a collection of related objects. We might say that two or more objects are related if they have common properties.
- Lets take on example to understand it. Let, X be the set, which contain names of students.
- Here, we might say two students are related if the first letters of their names are same.
- Also, consider the set of integers {1, 2, 3, ... 10}. We might say that four integers from the set are related if they are divisible by 2.
- Thus, integers 2, 4, 6, 8 and 10 are related because they followed or satisfied some certain condition.
- In this chapter, we will study the relations among discrete objects.

6.2 CARTESIAN PRODUCT

- Let, A and B are two non-empty sets. The Cartesian product of set A and B is denoted as $A \times B$, is the set of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$. In other way,

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

Example

Let, $A = \{1, 2, 3\}$, $B = \{a, b\}$ then.

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

- Here, $(1, a) \neq (a, 1)$ because every pair is ordered.
  Hence, $A \times B \neq B \times A$
- Thus, the Cartesian product is not commutative. In the same way, Cartesian product of three sets A, B, C will be as,

$$A \times B \times C = \{ (a, b, c) \mid a \in A, b \in B \text{ and } c \in C \}$$

- In particular,

$$A_1 \times A_2 \times A_3 \times \ldots \times A_n = \{ (a_1, a_2, a_3, \ldots a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots \text{ and } a_n \in A_n \}$$

6.2.1 Properties of Cartesian Product

1. If $|A| = m$ and $|B| = n$ then
   $|A \times B| = |B \times A| = m \cdot n$
2. Let, A, B, C are non-empty sets then
   $A \subseteq B \Rightarrow A \times C \subseteq B \times C$
3. If A, B, C are sets then
   $\rightarrow A \times (B \cap C) = (A \times B) \cap (A \times C)$
   $\rightarrow (A \cap B) \times C = (A \times C) \cap (B \times C)$
   $\rightarrow A \times (B \cup C) = (A \times B) \cup (A \times C)$
   $\rightarrow (A \cup B) \times C = (A \times C) \cup (B \times C)$

6.3 RELATION

- Let, A and B be two non-empty sets. A binary relation from A to B is a subset of $A \times B$ that is some elements in A are related to some of the elements in B.
- It is denoted by $R : A \rightarrow B$.
- Here, if R is a binary relation from A to B and if the ordered pair (a, b) is in R, then we would say that, the element a is related to the element b.
- Relations are nothing but set of ordered pairs.
- A Relation is defined on one or more sets.

Example,

$A = \{ x, y, z \}$, $B = \{ a, b, c \}$

Then $A \times B = \{ (x, a), (x, b), (x, c), (y, a), (y, b), (y, c), (z, a), (z, b), (z, c) \}$

Here, $R_1 = \{ (x, a), (y, b), (z, c) \}$

$R_2 = \{ (z, c) \}$

$R_3 = \{ \phi \}$

$R_4 = \{ (a, x), (b, z), (c, x) \}$

- In the above three relations, some elements in A are related to some elements in B, with respect to certain condition. So, $R : A \rightarrow B$.
- But, $R_4 = \{ (a, x), (b, z), (c, x) \}$ is not a relation from A to B. R_4 is the relation from B to A.
- If x relates with a, then it is x Ra. If x not relates with a, then it is $x \not R a$.

6.3.1 Important Deduction about Relation

(i) If relation $R = \phi$, then R is said to be void or empty relation. Which is subset of $A \times B$.

(ii) If relation, $R = A_1 \times A_2 \times \ldots \times A_n$, then R is said to be universal relation.

(iii) If $(a, b) \in R$ then it is denoted aRb which means element 'a' is related to b.

(iv) If R is a relation from A to B, then $R \subseteq A \times B$.

(v) If $R \subseteq A \times A$ then R is a relation from A to A. Which means R is called a relation on A.

6.3.2 Domain of Relation

- Let, A and B are two non-empty sets. R is the relation from A to B, then the **set of all first elements** of the ordered pairs $(a, b) \in R$ is called the domain of R.
- It is denoted by $D(R) = \{ a \mid (a, b) \in R \}$

Example

Let, $A = \{ 1, 2 \}$, $B = \{ x, y \}$

and relation $R = \{ (1, x), (1, y), (2, x) \}$

Domain of $R \Rightarrow D(R) = \{ 1, 2 \}$

6.3.3 Range of Relation

- Let, A and B be two non-empty sets. R is the relation from A to B, then the set of all second elements of the ordered pairs $(a, b) \in R$ is called the range of R.
- It is denoted by $R(R) = \{ b \mid (a, b) \in R \}$

Example

Let, $A = \{ 1, 2 \}$, $B = \{ x, y \}$

and relation $R = \{ (1, x), (1, y), (2, x) \}$

Range of $R \Rightarrow R(R) = \{ x, y \}$

Ex. 6.3.1 : Let, $A = \{1, 2, 3, 4, 5\}$

Define the following relation R on A aRb if and only if $a < b$.

Find : (i) R in roster form

(ii) Domain and range of R

(iii) Diagraph of R

Soln. :

Given, $A : \{1, 2, 3, 4, 5\}$

Let R be a relation defined on A with condition as (a, b) such that $(a < b)$, a R b.

So $R = \{ (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5) \}$

(i) Roster form of relation R

$R = \{ (a, b) \mid a < b \text{ where } a, b \in A \}$

(ii) Domain of R = Dom (R) = all first elements from the ordered pairs of R.

Dom (R) = $\{1, 2, 3, 4\}$

Range of R = Ran (R) = all second elements from the ordered pair of R.

$= \{2, 3, 4, 5\}$

(iii) Diagraph of R :

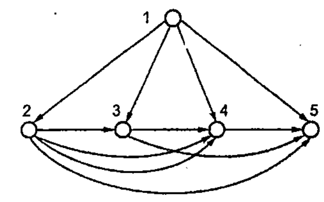


Fig. P. 6.3.1

6.4 REPRESENTATION OF RELATION

6.4.1 Matrix Representation of Relation

- For a relation R, a matrix is formed, called as relation matrix and is denoted by M_R.
- In the relation matrix, we place 1 in the i^{th} row and j^{th} column if the element $a_i \in A$ is related to the element $b_j \in B$, otherwise 0 is placed.
- Mathematically, let, $A = \{ a_1, a_2, a_3, \ldots, a_n \}$
  $B = \{ b_1, b_2, b_3, \ldots b_m \}$ and relation $R \subseteq A \times B$ then, relation matrix $M_R = [M_{ij}]$ can be defined as

$$M_R = [M_{ij}] = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$$

Example

Let, $A = \{ 1, 2, 3, 4 \}$, $B = \{ a, b, c, d \}$

and relation $R = \{(1, a), (1, d), (2, b), (3, c), (4, d)\}$

- Then, relation matrix,

$$M_R = [M_{ij}]_{4 \times 4} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- Here, in M_R, set A represents rows and set B represents columns.

6.4.2 Directed Graph Representation of Relation

Another way to represent relation is through directed graph. But this representation used only when a finite set relates with itself.

☞ Example

A = { a, b, c, d }

- R is the relation from A to A i.e. $R : A \rightarrow A$
  R = { (a, a), (a, c), (b, d), (c, d), (d, a), (d, c), (a, b) }

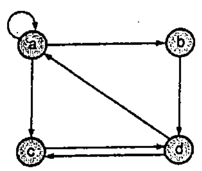


Fig. 6.4.1

- Here, each node in the directed graph is the element of set.

6.4.3 Tabular form Representation

- In the table form, relation from A to B i.e. $R : A \rightarrow B$ represented in the form of table where the rows of the table correspond to the elements in A and the columns of the table corresponds to the elements in B
- And check mark in a particular cell means the element in the row containing the cell is related to the element in the column containing the cell.

☞ Example

- Let, A and B are finite sets and R is the relation from A to B.
  A = { 1, 2, 3, 4 }, B = { a, b, c, d }
  R = { (1, c), (2, b), (3, d), (4, b), (4, d) }
- Then, relation R represented in tabular form as :

	a	b	c	d
1			✓	
2		✓		
3				✓
4		✓		✓

- Here, in the table we can replace ✓ check mark with 1's and other cells with 0's.

6.4.4 Arrow Diagram Representation

- In the arrow diagram representation, draw two ellipses for the sets A and B.
- Write down the elements of A in first ellipse and the elements of B in second ellipse in column-wise.
- Then, draw an arrow from the first ellipse to the second ellipse if a is related b where $a \in A$ and $b \in B$.

☞ Example

Let, A = { 1, 2, 3, 4 }, B = { a, b, c, d }

- R is the relation from A to B i.e. $R : A \rightarrow B$.
  R = {(1, a), (2, a), (3, b), (3, d), (4, c), (4, d)}

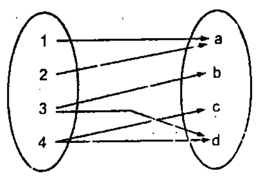


Fig. 6.4.2

☞ Properties of relation matrix

Let, R_1 be a relation from A to B and R_2 be a relation from B to C, then the relation matrix posses following properties.

1. $M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$
2. $M_{R_1^{-1}}$ or $M_{R_1^c}$ = Transpose of M_{R_1}
3. $M_{(R_1 \cdot R_2)^{-1}}$ or $M_{(R_1 \cdot R_2)^c} = M_{R_2^c \cdot R_1^c} = M_{R_2^c} \cdot M_{R_1^c}$

6.4.5 Examples Based on Ration and It's Representation

Ex. 6.4.1 : Let A = {1, 2, 3, 4, 5, 6} and R = { (a, b) | a divides b }

List all the ordered pairs and find relation matrix of R.

☑ **Soln. :**

R = { (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6) }

The relation matrix of R is M_R as given below

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Ex. 6.4.2 : Represent each of these relations on A = { 1, 2, 3} with matrix

(i) R_1 = { (1, 1), (1, 2), (1, 3) }

(ii) R_2 = { (1, 2), (2, 1), (2, 2), (3, 3) }

(iii) R_3 = { (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3) }

(iv) R_4 = { (1, 3), (3, 1) }

☑ **Soln. :**

(i)

$$M_{R1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

(ii)

$$M_{R2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

(iii)

$$M_{R3} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

(iv)

$$M_{R4} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Ex. 6.4.3 : If A = { 2 }, B = { x, y }, C = { c, d }, then find $A \times B \times C$, A^2, $B^2 \times A$, C^3.

☑ **Soln. :**

(i) $A \times B \times C$ = {(2, x, c), (2, y, c), (2, x, d), (2, y, d) }

(ii) $A^2 = A \times A$ = { (2, 2) }

(iii) B^2 = { (x, x) (x, y), (y, y), (y, x) }

$B^2 \times A$ = { ((x, x), 2), ((x, y), 2), ((y, y), 2), ((y, x), 2) }

(iv) $C^3 = C \times C \times C$ = { (c, c, c), (c, c, d), (c, d, d), (c, d, c), (d, c, c), (d, d, c), (d, c, d), (d, d, d) }

Ex. 6.4.4 : If A = { a, b, c }, B = { b, c } then find $(A \times B) \cap (B \times A)$

☑ **Soln. :**

$A \times B$ = { (a, b), (a, c), (b, b), (b, c), (c, b), (c, c) }

$B \times A$ = { (b, a), (b, b), (b, c), (c, a), (c, b), (c, c) }

$(A \times B) \cap (B \times A)$ = **{ (b, b), (b, c), (c, b), (c, c) }**

Ex. 6.4.5 : If A = { 2, 3, 4, 6, 8 } B = {2, 3, 4, 6, 8} xRy iff $x + y \leq 10$. Find it's relation matrix.

☑ **Soln. :**

$A \times B$ = { (2, 2), (2, 3), (2, 4), (2, 6), (2, 8), (3, 2), (3, 3), (3, 4), (3, 6), (3, 8), (4, 2), (4, 3), (4, 6), (4, 8), (6, 2), (6, 3), (6, 4), (6, 6), (6, 8), (8, 2), (8, 3), (8, 4), (8, 6), (8, 8) }

R = { (2, 2), (2, 3), (2, 4), (2, 6), (2, 8), (3, 2), (3, 3), (3, 4), (3, 6), (4, 2), (4, 3), (4, 6), (6, 2), (6, 3), (6, 4), (8, 2) }

$$M_R = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 6 \\ 8 \end{array}\begin{array}{c} \begin{array}{ccccc} 2 & 3 & 4 & 6 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ex. 6.4.6 : Let, A = {a, b, c, d} and

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ Draw diagraph of R.

☑ **Soln. :**

Relation "R" in the form of ordered pairs, can be drawn from it's relation matrix.

~~R = { (a, a), (a, d), (b, b), (b, d), (c, a), (c, b), (d, a) }~~

The diagraph of R is as follows :

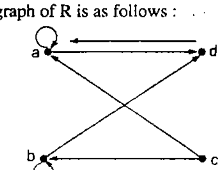


Fig. P. 6.4.6

Ex. 6.4.7 : $A = \{1, 2, 3, 4, 5, 6\}$, $R = \{(a,b) \mid a \text{ divides } b\}$

Draw diagraph of R.

Soln. :

$$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}$$

The diagraph of R is as follows

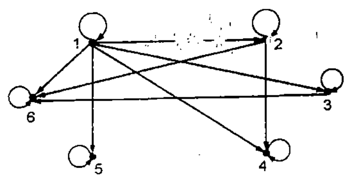


Fig. P. 6.4.7

Ex. 6.4.8 : Find the Relation and it's Relation matrix from the diagram.

Soln. :

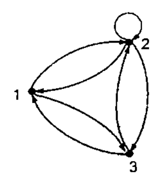


Fig. P. 6.4.8

$$R = \{(1,2), (1,3), (2,3), (3,2), (3,1), (2,1), (2,2)\}$$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

6.5 FUNDAMENTAL CONCEPTS OF RELATION

6.5.1 Inverse or Converse of Relation

- Let, R be the relation from A to B i.e. $R : A \rightarrow B$, the inverse of relation R is denoted by R^{-1} is a relation from B to A.
- It is also known as converse of relation denoted as R^C.
- Mathematically,

$$R^{-1} \text{ or } R^C = \{(y, x) \mid y \in B, x \in A \text{ and } (x, y) \in R\}$$

- If, element 'a' relates with 'b' that is aRb then it's inverse is bRa.

Example

Let, $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$

R is the relation from A to B which is

$$R = \{(1, c), (3, b), (2, d), (4, a), (4, d)\}$$

- The converse or inverse of relation R is R^C or R^{-1} is

$$R^C \text{ or } R^{-1} = \{(c, 1) (b, 3), (d, 2), (a, 4), (d, 4)\}$$

Note : R^{-1} or $R^C \subseteq B \times A$, if $R \subseteq A \times B$.

Important deduction

1. $(R^C)^C = R$
2. $(R \cup S)^C = R^C \cup S^C$
3. $(R \cap S)^C = R^C \cap S^C$

6.5.2 Complement of Relation

- Let, R be the relation from A to B. The complement of R is denoted by $\bar{R}$ or $R' : A \rightarrow B$.
- Mathematically,

$$\bar{R} \text{ or } R' = \{(x, y) \mid (x, y) \notin R \text{ but } x \in A, y \in B\}$$

- That is $x \bar{R} y$ if and only if $x \not{R} y$

Imp. Note : $\bar{R}$ or R' is a complementary set of R with respect to $A \times B$ (Universal Relation) that is

$$\bar{R} = (A \times B) - R$$

Example

Let, $A = \{1, 2, 3\}$, $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$R = \{(2, a), (3, b)\}$

$\bar{R}$ or $R' = (A \times B) - R = \{(1, a), (1, b), (2, b), (3, a)\}$

Important deduction

- Let, R and S be two relations.

1. $\overline{R \cup S} = \bar{R} \cap \bar{S}$
2. $\overline{R \cap S} = \bar{R} \cup \bar{S}$

6.5.3 Composition of Relation

- Composition of relation formed from already available sequence of relations.
- It plays an important role in data structure, execution of program.

Definition : Let, A, B and C are three sets. Let R be a relation from A to B and S be a relation from B to C that is $R : A \rightarrow B$ and $S : B \rightarrow C$, then composite relation from A to C denoted by R o S or RS : $A \rightarrow C$.

Mathematically,

$$R \circ S = \{(a, c) \mid aRb, bSc \text{ i.e. } (a, b) \in R \text{ and } (b, c) \in S \text{ for } a \in A, c \in C\}$$

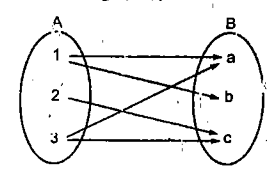
Example,

Let, $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $C = \{p, q, r\}$

Let, $R : A \rightarrow B$, $S : B \rightarrow C$

$R = \{(1, a) (1, b), (2, c), (3, a), (3, c)\}$

$S = \{(a, p), (a, r), (b, p), (b, q), (c, p), (c, q), (c, r)\}$



R : A → B

Fig. 6.5.1(a)

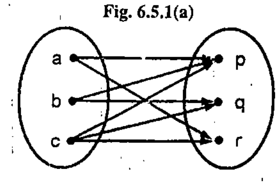


Fig. 6.5.1(b)

- The composition of R and S is R o S or RS.

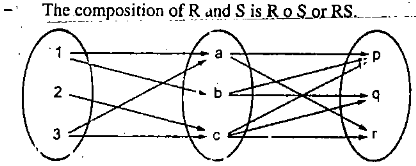


Fig. 6.5.1(c) : RoS

$$RoS = \{(1, p), (1, r), (1, q), (2, p), (2, q), (2, r), (3, p), (3, q), (3, r)\}$$

Important deduction

1. Let R_1, R_2 and R_3 be relations from A to B, B to C and C to D. Then $(R_1R_2) R_3 = R_1 (R_2R_3)$.
2. Let, R_1 be relation from A to B and R_2 be relation from B to C. Then $(R_1 \cdot R_2)^C = R_2^C R_1^C$.
3. Let, R be a relation from set A to itself. Then, composition of relation R with itself is RoR sometimes denoted as R^2. In the same way.

$$R^3 = R^2 o R = R o R o R$$

Thus, R^n is defined for all positive n.

6.5.4 Examples Based on Types of Relation

Ex. 6.5.1 : Let, $A = \{1, 2, 3\}$, $B = \{a, b\}$

$R = \{(1, 0), (2, b), (3, a)\}$, $S = \{(2, a), (3, b)\}$

Find $\bar{R}, \bar{S}, \bar{R} \cap \bar{S}, \bar{R} \cup \bar{S}$.

Soln. : From set A and B

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$R = \{(1, a) (2, b), (3, a)\}$

$S = \{(2, a), (3, b)\}$

(i) $\bar{R} = (A \times B) - R = \{(1, b) (2, a), (3, b)\}$

$\bar{R}$ = complement of R

(ii) $\bar{S} = (A \times B) - S = \{(1, a) (1, b), (2, b), (3, a)\}$

$\bar{S}$ = complement of S

(iii) $\bar{R} \cap \bar{S} = \{(1, b)\}$

(iv) $\bar{R} \cup \bar{S} = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

Ex. 6.5.2 : Obtain the matrix of $\bar{R}$ if

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Soln. :

The relation matrix of $\bar{R}$ is obtained by replacing 0 by 1 and 1 by 0 of M_R.

$$M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex. 6.5.3 : Let, $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

$R = \{(1, a), (2, a), (3, c)\}$

Find (i) R^C (ii) $D(R^C)$ (iii) Range of R^C

☑ **Soln. :** $R = \{(1, a), (2, a), (3, c)\}$

(i) The converse of R is $R^C = \{(a, 1), (a, 2), (c, 3)\}$

(ii) The domain of R^C is $D(R^C) = \{a, c\}$

(iii) The range of $R^C = \{1, 2, 3\}$

Ex. 6.5.4 : Let, $A = \{3, 4, 5, 6, 8\}$

R and S be the relations on A such that

$R = \{(x, y) \mid x = y + 1 \text{ or } y = 2x\}$

$S = \{(x, y) \mid x \text{ divides } y\}$ find $(R \cap S)^C$.

☑ **Soln. :**

$R = \{(4, 3), (5, 4), (6, 5), (3, 6)\}$

$S = \{(3, 3), (3, 6), (4, 4), (4, 8), (5, 5), (6, 6), (8, 8)\}$

$R \cap S = \{(3, 6)\}$

The converse of $(R \cap S)$ is $(R \cap S)^C = \{(6, 3)\}$

Ex. 6.5.5 : Let, $A = \{1, 2, 3, 4\}$,

$R_1 = \{(1, 2) (1, 1), (2, 3), (3, 4)\}$

$R_2 = \{(1, 1), (2, 1), (2, 2), (4, 4)\}$

Find (i) $R_1 \cdot R_2$ (ii) $R_2 \cdot R_1$ (iii) R_2^2 (iv) R_2^3

(v) diagraph of $R_1 \cdot R_2$.

☑ **Soln. :**

R_1	R_2	$R_1 \cdot R_2$
(1, 2)	(2, 1) (2, 2)	(1, 1) (1, 2)
(1, 1)	(1, 1)	(1, 1)
(2, 3)	–	–
(3, 4)	(4, 4)	(3, 4)

R_2	R_1	$R_2 \cdot R_1$
(1, 1)	(1, 2) (1, 1)	(1, 2) (1, 1)
(2, 1)	(1, 2) (1, 1)	(2, 2) (2, 1)
(2, 2)	(2, 3)	(2, 3)
(4, 4)	–	–

(i) $\mathbf{R_1 \cdot R_2 = \{(1, 1), (1, 2), (3, 4)\}}$

(ii) $\mathbf{R_2 \cdot R_1 = \{(1, 2), (1, 1), (2, 2), (2, 1), (2, 3)\}}$

(iii)

R_2	R_2	$R_2 \cdot R_2 = R_2^2$
(1, 1)	(1, 1)	(1, 1)
(2, 1)	(1, 1)	(2, 1)
(2, 2)	(2, 2) (2, 1)	(2, 1) (2, 2)
(4, 4)	(4, 4)	(4, 4)

$$R_2^2 = (1, 1), (2, 1) (2, 2), (4, 4)$$

(iv) $R_2^3 = R_2 \cdot R_2^2$

R_2	R_2^2	R_2^3
(1, 1)	(1, 1)	(1, 1)
(2, 1)	(1, 1)	(2, 1)
(2, 2)	(2, 2) (2, 1)	(2, 2) (2, 1)
(4, 4)	(4, 4)	(4, 4)

$$R_2^3 = \{(1, 1), (2, 1) (2, 2), (4, 4)\}$$

(v)

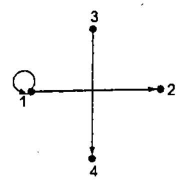


Fig. P. 6.5.5

Ex. 6.5.6 : Let, $A = \{1, 2, 3, 4\}$, $R_1 = \{(x, y) \mid x + y = 5\}$

$R_2 = \{(x, y) \mid (y - x) = 1\}$ verify $(R_1 \cdot R_2)^C = R_2^C \cdot R_1^C$

☑ **Soln. :**

The relations R_1 and R_2 which satisfies above conditions

$$R_1 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$$

$$R_2 = \{(1, 2), (2, 3), (3, 4)\}$$

$$R_1 \cdot R_2 = \{(2, 4), (3, 3), (4, 2)\}$$

$$(R_1 \cdot R_2)^C = \{(4, 2), (3, 3), (2, 4)\} \quad ...(1)$$

The converse or inverse of R_1 is R_1^C

$$R_1^C = \{(4,1), (3, 2), (2, 3), (1, 4)\}$$

The converse or inverse of R_2 is R_2^C

$$R_2^C = \{(2, 1), (3, 2), (4, 3)\}$$

The composition of R_2^C and R_1^C is

$$R_2^C \cdot R_1^C = \{(2, 4), (3, 3), (4, 2)\} \quad ...(2)$$

From statement (1) and (2) it is clear that

$$(R_1 \cdot R_2)^C = R_2^C \cdot R^C$$

6.6 PROPERTIES OF RELATIONS

- Let, R be relation from A to A. Here, we can say that R is a relation on set A, which is subset of $A \times A$ that is $R \subseteq A \times A$.
- $A \times A$ = universal set for relation R.
- If $R = A \times A$ them R is said to be universal relation.

6.6.1 Reflexive Relation

- Let, R be a relation on set A. R is said to be a reflexive relation if $(a, a) \in R$ for every a in A.
- R is not reflexive relation if for some element $a \in A$, $a \not R a$ or $(a, a) \notin R$.
- In a reflexive relation, every element in A is related to itself.
- In general, let, A be the set of positive integers and R be a relation defined on A such that (a, b) is in R if and only if a divides b.
- Here, every integer is always divides itself, so R is reflexive relation.

Example

Let, $A = \{1, 2, 3, 4\}$

(i) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (2, 3)\}$

Here, R_1 is reflexive relation. Since every element in A is related to itself.

(ii) $R_2 = \{(1, 2), (1, 1), (3, 4), (2, 2), (3, 3)\}$

Here, R_2 is not reflexive relation. Since $4 \in A$ but $(4, 4) \in R_2$.

Representation of reflexive relation

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

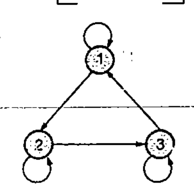


Fig. 6.6.1

- In matrix representation of relation, a binary relation on a set is reflexive if and only if all the cells or elements on the main diagonal of matrix contains 1's.
- In diagraph representation of relation, if each vertex contains a loop, then the corresponding relation is a reflexive relation.

6.6.2 Irreflexive Relation

A relation R is said to be irreflexive if for every element $a \in A$, $(a, a) \notin R$ i.e. $a \not R a$

Example

Let, $A = \{1, 2, 3, 4\}$

(i) $R_1 = \{(2, 3) (3, 4)\}$

- Here, R_1 is irreflexive relation since (1, 1), (2, 2) (3, 3) and $(4, 4) \notin R_1$. In general, for all $a \in A$, $a \not R a$.

(ii) $R_2 = \{(1, 1), (2, 3), (3, 3), (3, 4)\}$

- Here, R_2 is not irreflexive since, (1. 1) and $(3, 3) \in R_2$ also, R_2 is not reflexive since (2, 2) and $(4, 4) \notin R_2$.

> **Note :**
>
> (a) If R is a reflexive relation on set A, then,
>
> (i) The relation matrix (M_R) is the identity matrix. (diagonal elements are 1's).
>
> (ii) Diagraph of reflexive relation will have loop for every element of A.
>
> (b) The relation matrix of irreflexive relation has all diagonal elements are zero. It's diagraph is free from the loops.

6.6.3 Symmetric Relation

Let, R be a relation on set A. R is said to be symmetric relation if $(a, b) \in R$ which implies that $(b, a) \in R$ for $a, b \in A$.

Example

Let, $A = \{1, 2\}$ then

(i) $R_1 = \{(1, 1), (1, 2), (2, 1)\}$

Here, R_1 is symmetric relation, since $(a, b) \in R$ and $(b, a) \in R$.

(ii) $R_2 = \{(1, 1) (2, 2)\}$

Here, R_2 is reflexive as well as symmetric relation since, $(a, a) \in R$, $(a, b) \in R$ and $(b, a) \in R$.

(iii) $R_3 = \{(1, 1), (2, 2), (1, 2)\}$

Here, R_3 is reflexive relation but not symmetric because, every element of A is relates to itself but a relates to b, but b not relates to a that is, $(a, b) \in R_3$ but $(b, a) \notin R_3$.

Note :

(i) If relation $R = R^{-1}$, then given relation is symmetric relation.

(ii) The relation matrix of symmetric relation is a symmetric matrix.

Matrix and diagraph representation of symmetric relation

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

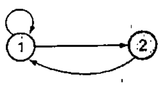


Fig. 6.6.2

6.6.4 Asymmetric Relation

- A relation R on a set A is said to be asymmetric if (a, b) ∈ R i.e. aRb (a relates with b) then (a, b) ∉ R i.e. a$\not R$ b (a does not relates with b).
- Hence, R is not asymmetric relation if for some a, b ∈ A, (a, b) ∈ R and (b, a) ∈ R.

Example

(i) Let A = { 1, 2, 3 }, R_1 = { (1, 2), (2, 3), (1, 3) }

- Here, R_1 is asymmetric relation because (1, 2) ∈ R_1 (2 1) ∉ R_1, (2, 3) ∈ R_1 and (3, 2) ∉ R_1, (1, 3) ∈ R_1 and (3, 1) ∉ R_1.

(ii) R_2 = { (1, 2), (2, 3), (3, 2) }

- Here, R_2 is not asymmetric relation because (2, 3) ∈ R and (3, 2) ∈ R.

6.6.5 Anti-symmetric Relation

- Let, R be a relation on set A. R is said to be an anti-symmetric relation if aRb and bRa then there should be a = b.
- In other words, R is said to be anti-symmetric relation if (a, b) in R implies that (b, a) is not in R unless a = b.
- If a relates with b and b relates with a, then there must be a = b.
- A Relation R is not anti-symmetric if some elements a, b ∈ A, where (a, b) in R as well as (b, a) in R, but a ≠ b.
- The relation matrix of antisymmetric relation is never symmetric matrix.

Example

Let, A = { 1, 2, 3 }, then

(i) R_1 = { (1, 1), (3, 3) }

- Here, R_1 is symmetric and anti-symmetric relation.

(ii) R_2 = { (1, 3) (3, 1) }

- Here, R_2 is symmetric but not antisymmetric relation as (1, 3) and (3, 1) is in R, but 1 ≠ 3.

(iii) R_3 = { (1, 1), (1, 2), (2, 2) }

- Here, R_3 is anti-symmetric relation because (a, b) ∈ R, (b, a) ∈ R and a = b.
- R_3 is not asymmetric because (a, b) ∈ R and (b, a) ∈ R but for asymmetric this condition must not be there.

6.6.6 Transitive Relation

- Let R be a binary relation on A. R is said to be a transitive relation if (a, b) in R and (b, c) in R then there must be (a, c) in R.
- A relation R is not transitive if there exist some elements a, b and c in A such that (a, b) in R, (b, c) in R but (a, c) not in R. i.e. if aRb, bRc but a$\not R$ c, then relation R is not transitive.
- If aRb, bRc and their exist aRc, then R is transitive relation.

Example

1. '≤' (less than equal to relation) is transitive relation. If a ≤ b, b ≤ c then a ≤ c so R is transitive.
2. Let, A = { a, b, c, d }

   (i) R_1 = { (a, b), (b, c) }

   - R_1 is not transitive relation. Since, (a, b) in R, (b, c) in R present but (a, c) not in R.

   (ii) R_2 = { (a, b), (b, c), (c, a), (c, b), (a, c), (b, a), (a, a), (b, b), (c, c) }

   - R_2 is a transitive relation because there is (a, b) ∈ R_2 and (b, c) ∈ R_2, then (a, c) ∈ R_2 is available.

   (iii) R_3 = { (a, b), (c, b) } is transitive relation because there is no term in the form of (a, b) ∈ R_3 and (b, c) ∈ R_3.

Graphical representation

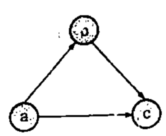


Fig. 6.6.3

Note : The relation ≤, ⊆ and / are transitive because

(i) If a ≤ b, b ≤ c then a ≤ c

(ii) If a ⊆ b, b ⊆ c then a ⊆ c

(iii) If a/b, b/c then a/c

Ex. 6.6.1 : Let A = {1, 2, 3} R is the relation on A whose matrix is :

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

show that R is transitive.

Soln. : Let, A = {1, 2, 3}

R be a relation defined on A. M_R is relation matrix.

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

R = {(1, 1), (1, 2), (1, 3), (2, 3), (3, 3)}

For R to be transitive relation, R must be

(x, y) ∈ R and (y, z) ∈ R then (x, z) ∈ R.

Here, R is transitive relation since,

→ (1, 1) ∈ R and (1, 2) ∈ R then (1, 2) ∈ R

→ (1, 2) ∈ R and (2, 3) ∈ R then (1, 3) ∈ R

→ (1, 3) ∈ R and (3, 3) ∈ R then (1, 3) ∈ R

→ (2, 3) ∈ R and (3, 3) ∈ R then (2, 3) ∈ R

Ex. 6.6.2 : Let A = Z^+ the set of positive integers, and let R = {(a, b) ∈ A × A | a divides b}

Is R symmetric, asymmetric or antisymmetric.

Soln. :

Let, A = z^+ i.e. set of positive integers.

R = {(a, b) ∈ A × A | a divides b}

⇒ Suppose (a, b) ∈ R then $\frac{b}{a}$ ∈ R but $\frac{a}{b}$ ∉ R.

That is (a, b) ∈ R but (b, a) ∉ R

Hence, R is not symmetric or asymmetric.

⇒ $\frac{a}{b}$ ∈ R and $\frac{b}{a}$ ∈ R true when a = b that is

$\frac{a}{b}$ is treated as $\frac{a}{a}$ = true for all a.

Hence, Relation R is antisymmetric.

Ex. 6.6.3 : Consider the following relation on {1, 2, 3, 4, 5, 6} : R = {(i, j) : | i − j | = 2}. Is R transitive ? Is R reflexive ? Is R symmetric ?

Soln. :

Let, A = {1, 2, 3, 4, 5, 6}

R = {(i, j) such that | i − j | = 2}

R = {(1, 3), (2, 4), (3, 5), (4, 6), (3, 1), (4, 2), (5, 3), (6, 4)}

(i) Relation R is not reflexive, since for every i ∈ A, (i, i) ∉ R or i $\not R$ i

In other way, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6) ∉ R.

(ii) Relation R is symmetric, since for any (i, j) ∈ R there is (j, i) ∈ R. that is (1, 3) ∈ R and (3, 1) ∈ R

(iii) R is not transitive, because for any (i, j) ∈ R and (j, k) ∈ R, then (i, k) ∉ R

that is, (1, 3) and (3, 1) belongs to R but (1, 1) does not belongs to R.

6.7 EQUIVALENCE RELATION

Let, R be a relation on set A. Then R is said to be equivalence relation if and only if,

(i) R must be reflexive i.e. for all a ∈ A, aRa.

(ii) R must be symmetric i.e. aRa, then there must be bRa.

(iii) R must be transitive i.e. if aRb and bRc then there must be aRc.

Example

Let, A = { 1, 2, 3 }

R = { (1, 1), (2, 2), (2, 3), (3, 2), (3, 3) }

- Here R is reflexive because (1, 1), (2, 2), (3, 3) ∈ R. R is symmetric because for all (a, b) ∈ A, there is aRb and bRa.
- R is transitive because, for all (a, b, c) ∈ A there is aRb, bRc then aRc.

☞ Diagraph Representation

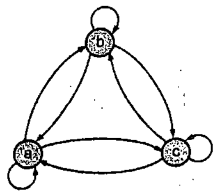


Fig. 6.7.1

From the diagraph, it is clear that, every vertex has a loop. If there is an arc from a to b, then there must be an arc from b to a. If there is an arc from a to b and b to c, then there must be an arc from a to c.

> **Note :**
> (i) The equality i.e. "=" relation is an equivalence relation that is a = a for every $a \in A$ (Reflexive). If a = b then b = a (symmetric). If a = b, b = c then a = c (transitive).
> (ii) Let L be the set of lines and let T be the set of triangles in the plane then
> The relation "is parallel to or identical to" is an equivalence relation on L.
> The relations of congruence and similarity are equivalence relation on T.
> (iii) The relation ⊆ is not an equivalence relation because $A \subseteq B \neq B \subseteq A$ (not symmetric).
> (iv) Let m be a fixed positive integer two integers x and y are said to be congruent modulo m, written as $x \equiv y$ (mod m) if m divides x − y and m = 4 then $11 \equiv 3$ (mod 4) and $22 \equiv 6$ (mod 4) so "congruence modulo m" is an equivalence relation because 4 divides 11 − 3 = 8 and 4 divides 22 − 6 = 16.
> (v) If R_1 and R_2 are equivalence relation on set A then $R_1 \cap R_2$ is an equivalence relation but $R_1 \cup R_2$ will not be equivalence relation.

UEx. 6.7.1 : Let A = {1, 2, 3, 4} consider partition.
P = {{1, 2, 3,}, {4} of A. Find the equivalence relation R on A determined by P.

☑ **Soln. :**

Let A = {1, 2, 3, 4}

Partition P = {{1, 2, 3}, {4}} of A.

There are two blocks in partition P.

$Block_1 = \{1, 2, 3\}$, $Block_2 = \{4\}$

We can define relation for 1, 2 and 3 belongs to the same block of the partition of A.

– Relation associated with block1 is :
1R1, 1R2, 1R3, 2R1, 2R2, 2R3, 3R1, 3R2, 3R3.
– Relation associated with block 2 is : 4R4.

The required relation is

R = {(1, 1) (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)}

The relation R is reflexive since all elements of $A \in R$, symmetric since $(a, b) \in R$ then $(b, a) \in R$, transitive,

Since $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Hence R is an equivalence relation.

⏭ 6.8 EQUIVALENCE CLASSES

– Let, R be an equivalence relation on set A. The equivalence class of $a \in A$ is denoted as $[a]_R$ and defined as,

$[a]_R$ = { x∈A | xRa }
= The grouping of those elements of A which are related to a.
= $[a]_R \neq \phi$ because $a \in [a]_R$

☞ Example

Let, A = { a, b, c }
R = A × A = { (a, a), (b, b), (c, c), (a, b), (a, c), (b, c), (b, a), (c, a), (c, b) }
is an equivalence relation.

– The equivalence class of a is $[a]_R$ = { a, b, c } which contain elements with them a is related.
– Similarly,
$[b]_R$ = {a, b, c} ⇒ Here, b is related to itself, to a and to c
$[c]_R$ = {a, b, c} ⇒ Here, c is related to itself, to a and to b

So, $[a]_R = \{a, b, c\} = [b]_R = [c]_R$.

– All these three equivalence are identical.

☞ Rank of relation

The rank of R is the number of distinct equivalence classes of R.

☞ Properties of equivalence classes

(i) Any two equivalence classes are either identical or disjoint that is for all a, b ∈ A, either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \phi$.
(ii) For any x∈ A, x∈ $[x]_R$
(iii) $A = \bigcup_{a \in A} [a]_R$

UEx. 6.8.1 (Q. 1(a), May 16, 6 Marks)
What is an equivalent relation ? Identify whether the given relation is an equivalent relation where A = {1, 2, 3, 4 }, relation R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)}. Also find the equivalent classes.

☑ **Soln. :**

Given : A = {1, 2, 3, 4}

R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)}

For a relation to be equivalence, It should satisfies reflexive, symmetric and transitive properties.

(i) Is R Reflexive ?
Yes. Since, every element of A is related to itself. That is xRx, x∀x∈ A.

(ii) Is R symmetric ?
No, because (4, 1) ∈R and (1, 4) ∉ R.

(iii) Is R transitive ?
Yes. Since, xRy ∈R and yRz∈R then xRz∈R
Relation R does not satisfied the symmetric property.
Hence, R is not Equivalence relation.

Equivalence classes

$[a]_R$ = {x∈A | xRa}
$[1]_R$ = {1, 2, 3, 4}
$[2]_R$ = {1, 2}
$[3]_R$ = {1, 3}
$[4]_R$ = {4}

⏭ 6.9 PARTITIONS

The concept of partitions is closely related to equivalence class.

> ☐ **Definition :** Let, S be a non-empty set. A partition of S is a subdivision of S into non-overlapping, nonempty subsets.

– Mathematically, S = $\{A_1, A_2, A_3, \dots A_n\}$
– Here, S is a collection of disjoint, non-empty partitions (subsets).
– Set S is the union of all partitions (subsets)
i.e. $S = A_1 \cup A_2 \cup A_3 \cup \dots A_n = \bigcup_{i=1}^{n} A_i$
– All the subsets (partitions) are mutually disjoint
i.e. $A_i \cap A_j = \phi$ for $i \neq j$
– Each a (element) in S belongs to one of the A_i.
– The partition of a set denoted by π.
– An element of partition set is called a block.
– The rank of partition is called as the number of blocks of that partition denoted by r(π).

> ➡ **Note :** For any non-empty set, it's partitions are not be unique. There are different partitions of the same set.

Fig. 6.9.1 shows, the partitions of non-empty set S.

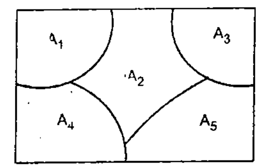


Fig. 6.9.1 : Venn diagram of partition of set S

☞ Example

Let, S = {1, 2, 3, 4, 5, 6, 7, 8, 9}

The subsets of S are as follows :

(i) A_1 = {1, 3, 5}, (ii) A_2 = {2, 6},
(iii) A_3 = {4, 8, 9}, (iv) A_4 = {2, 4, 6, 8},
(v) A_5 = {5, 7, 9}, (vi) A_6 = {7, 9}

Now

(a) S = [{1, 3, 5}, {2, 6}, {4, 8, 9}]
= $\{A_1, A_2, A_3\}$
= This is not a partition of S because 7' in S does not belong to any of the subsets.

(b) S = [{1, 3, 5}, {2, 4, 6, 8}, {5, 7, 9}]
= $\{A_1, A_4, A_5\}$
= This is not partition of S because A_1 and A_5 are not disjoint (5 is common element in A_1 and A_5)

(c) S = [{1, 3, 5}, {2, 4, 6, 8}, {7, 9}]
= $\{A_1, A_4, A_6\}$

= This is a partition of set S, since

$A_1 \cup A_4 \cup A_6 = S$ and

$A_1 \cap A_4 \cap A_6 = \phi$

6.9.1 Examples Based on Properties of Relations and Partitions

Ex. 6.9.1 : A = {1, 2, 3, 4}, define relations on A which have properties

(i) Reflexive, transitive but not symmetric.

(ii) Symmetric but neither reflexive nor transitive.

(iii) Reflexive, symmetric and transitive.

☑ **Soln. :** Given that, A = { 1, 2, 3, 4 }

$A \times A$ = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 3), (3, 1), (3, 2), (3, 4), (4, 4), (4, 1), (4, 2), (4, 3) }

(i) For R_1 to be reflexive, there should be for all a ∈ A, aR_1a. For R_1 to be transitive if aR_1b and bR_1c then there should be aR_1c.

R_1 = { (1, 1), (2, 2), (3, 3), (4, 4), (1, 2) (2, 3), (1, 3) }

So, R_1 is reflexive and transitive but not symmetric.

(ii) For R_2 to be symmetric, if aRb then bRa.

R_2 = { (1, 2), (2, 1), (2, 3), (3, 2) }

Here R_2 is symmetric neither reflexive nor transitive.

(iii) R_3 = { (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3), (3, 2) }

R_3 is Reflexive symmetric as well as transitive.

Ex. 6.9.2 : Let R be the relation defined on the set of natural number N as

R = { (x, y) | x ∈ N, y ∈ N, 2x + y = 41 }

Find the domain and range of the relation R. Also verify whether R is reflexive, symmetric and transitive.

☑ **Soln. :**

Given : x ∈ N, y ∈ N and 2x + y = 41

Relation R which satisfies above condition is as

R = { (1, 39), (2, 37), (3, 35), (4, 33), (5, 31), (6, 29), (7, 27), (8, 25), (9, 23), (10, 21), (11, 19), (12, 17), (13, 15), (14, 13), (15,11), (16, 9), (17, 7), (18, 5), (19, 3),(20, 1) } }

Domain of R = DOM (R) = {1, 2, 3, 4, 5, 6, 7, ... , 20}

Range of R = Ran (R) = {39, 37, 35, ...,1}

(i) R is not reflexive , As (1, 1) (2, 2) does not belongs to R that is (x, x) ∉ R

(ii) R is not symmetric As (1, 39) ∈ R but (39, 1) ∉ R that is (x, y) ∈ R but (y, x) ∉ R.

(iii) R is not transitive As (11, 19) ∈ R, (19.3) ∈ R but (11, 3) ∉ R, that is (x, y) ∈ R and (y, z) ∈ R but (x, z) ∉ R.

Ex. 6.9.3 : Let A = { set of all straight lines in a plane } x relates with y iff x is parallel to y. Show that R is a symmetric relation.

☑ **Soln. :**

Let, R = { xRy | x is parallel to y }

– If we consider, xRy when x is parallel to y and y is parallel to x.

– If the same way, yRx, when y is parallel to x and x is parallel to y.

Thus, whenever x relates with y, then t y relates with x.

Hence, Relation R is symmetric relation.

Ex. 6.9.4 : Show that the relation R is the set A of all the books in a library of a college given by

R = {(x, y) | x and y have same number of pages } is an equivalence relation.

☑ **Soln. :**

Given,

A = {set of all the books in a library of a college }

R = { (x, y) | x and y have some number of pages }

(i) **Is R reflexive ? :** wkt x ∈ A,

Putting y = x, we will have x book and have same number of pages that is (x, x) ∈ R or xRx for all x ∈ A.

Hence **R is Reflexive.**

(ii) **Is R symmetric ? :** Let, x, y ∈ A and (x, y) ∈ R

If x book and y book have same number of pages then it is clear that y and x have same number of pages therefore (y, x) ∈ R.

Hence, **R is symmetric.**

(iii) Let, x, y, z ∈ A (x, y) ∈ R that is x and y have same number of pages (y, z) ∈ R that is y and z have same number of pages.

If (x, y) ∈ R and (y, z) ∈ R then it is clear that (x, z) also belongs to R. Which means x and z book have same number of pages.

Hence, R is transitive.

Here, R satisfies reflexive, symmetric and transitive properties.

Hence R is equivalence relation.

Ex. 6.9.5 : Let A = { set of all straight lines in a plane }

R_1 = { xR_1 y | x is parallel to y }

R_2 = {xR_2y | x is perpendicular to y } check whether R_1 and R_2 are equivalence relations or not.

☑ **Soln. :**

(a) For relation R_1 :

(i) Let, R_1 = { x R_1 y | x is parallel to y }

x, y, z are straight lines in a plane.

x is parallel to x that is every line is parallel to itself so, xR_1 x.

Hence, R_1 is reflexive.

(ii) If x is parallel to y i.e. (x, y) ∈ R_1 then y is also parallel to x i.e.,(y, x) ∈ R_1.

Hence R_1 is symmetric.

(iii) If x is parallel to y, y is parallel z then it is clear that, is also parallel to z. (x, y) ∈ R_1 and (y, z) ∈ R_1 then (x, z) ∈ R_1

Hence, R_1 is transitive relation.

R_1 satisfies reflexive, symmetric and transitive properties so that R_1 is equivalence relation.

(b) For relation R_2 :

x is a line, which never perpendicular to itself so, (x, x) ∉ R_2 or x $\not R_2$ x for all x ∈ A.

Hence, R_2 is not reflexive.

Indirectly, R_2 is not equivalence Relation.

Ex. 6.9.6 : Let A = {x, y, z} and $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Determine whether R is an equivalence Relation ? Find the equivalence class of y in A.

☑ **Soln. :**

Given : Relation matrix of R is = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

R = { (x, x), (y, y), (y, z), (z, y), (z, z) }

(i) **Is R Reflexive ?**

For x, y, z ∈ A and (x, x), (y, y) and (z, z) ∈ R

Hence, R is Reflexive.

(ii) **Is R symmetric ?**

(y, z), (z, y) ∈ R that is if (x, y) ∈ R then (y, x) ∈ R

So that, R is symmetric Relation.

(iii) **Is R transitive ?**

In relation R, we have

(y, y) ∈ R and (y, z) ∈ R ⇒ (y, z) ∈ R

(y, z) ∈ R and (z, y) ∈ R ⇒ (y, y) ∈ R

(z, y) ∈ R and (y, y) ∈ R ⇒ (z, y) ∈ R

(z, z) ∈ R and (z, y) ∈ R ⇒ (z, y) ∈ R

So, R is transitive relation.

Thus, R satisfied reflexive, symmetric and transitive properties. Hence R is equivalence relation.

Ex. 6.9.7 : Let A = {Set of positive Integers} R be a relation R = { (x, y) | x – y is an odd positive integer}. Is R reflexive, symmetric, transitive an equivalence relation ?

☑ **Soln. :**

Given : R = { (x, y) | x – y is an odd positive integer }

(i) **Is R Reflexive ?**

For any positive integer, say x, x – x = 0

Which is even integer, that is (x, x) ∉ R

R is not reflexive relation

(ii) **If we consider (x, y) ∈ R**

Which means, x – y = odd positive integer

y – x = odd negative integer ∉ R

that is (y, x) ∉ R

∴ R is not symmetric relation.

(iii) **Again, if we consider (x, y) ∈ R and (y, z) ∈ R**

Which means that,

x – y = odd positive integer = a

y – z = odd positive integer = b

(x + y) + (y + z) = a + b

x – z = a + b (odd + odd = even)

= Even positive integer

(lets take simple e.g. to understand,

~~(8, 5) ∈ R As 8 – 5 = 3 (odd positive integer)~~

(5, 2) ∈ R As 5 – 2 = 3 (odd positive integer)

But (8, 2) ∉ R As 8 – 2 = 6 (even positive integer)

Hence, (x, z) ∉ R.

∴ R is not transitive relation.

(iv) R is not Equivalence relation since R not satisfied reflexive, symmetric and transitive properties.

Ex. 6.9.8 : Let,

A = { set of points in the plane } R be a relation

R = { (x, y) | y is within 2 centimeter from x }

Is R equivalence Relation ?

Soln. :

Given that, A = { Set of points in the plane }

R = { x, y) such that $(y - x) \leq 2$ for all $x, y \in A$ }

(i) Is R Reflexive ?

Suppose x = y then $x - x = 0 \leq 2$

(e.g. x = 4, then y = 4, $x - x = 4 - 4 = 0 \leq 2$)

Therefore $(x, x) \in R$ for all $x \in A$

So, R is Reflexive Relation.

(ii) Is R symmetric ?

Suppose $(x, y) \in R$ then $(y - x) \leq 2$

In the same way,. $(y, x) \in R$ then $(x - y) \leq 2$

(e.g. x = 3, y = 4, $y - x = 4 - 3 = 1 < 2$, x Ry

$x - y = 3 - 4 = -1 < 2$ that is yRx)

Therefore, $(x, y) \in R$ and $(y, x) \in R$

Hence, R is symmetric relation

(iii) Suppose $(x, y) \in R$ and $(y, z) \in R$

That is $(y - x) \leq 2$ and $(z - y) \leq 2$

Let, $z - x = (y - x) + (z - y)$

$z - x = 2 + 2$

$z - x \leq 4$ that is $\boxed{z - x \nleq 2}$

Which means, **$(x, z) \notin R$**

Therefore, R is not transitive Relation.

Hence, Relation R is not Equivalence Relation.

Ex. 6.9.9 : Define partition of a set.

Let X = { 1, 2, 3, 4, 5, 6, 7, 8, 9 }

Determine whether or not each of the following is a partition of x

A = { {2, 4, 5, 8}, {1, 9}, {3, 6, 7} }

B = { {1, 3, 6}, {2, 8}, {5, 7, 9 } }

Soln. :

Given that, X = {1, 2, 3, 4, 5, 6, 7, 8, 9}

(i) Definition of partition of a set : Please Refer theory.

(ii) For Set A

The set A has 3 blocks

$A_1 = \{2, 4, 5, 8\}$, $A_2 = \{1, 9\}$, $A_3 = \{3, 6, 7\}$

⇒ $A_1 \cup A_2 \cup A_3 = X$

⇒ A_1, A_2 and A_3 are mutually disjoints.

Hence, set A is a partition of X.

(iii) For set B

The set B has 3 blocks

$B_1 = \{1, 3, 6\}$, $B_2 = \{2, 8\}$, $B_3 = \{5, 7, 9\}$

⇒ $B_1 \cup B_2 \cup B_3 \neq X$

⇒ Element 4 is not included in B.

⇒ Hence, set B is not partition of X.

Ex. 6.9.10 : If S = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

Determine whether or not each of the following is a partition of S.

(i) A = {{1, 3, 5}, {2, 6}, {4, 8, 9} }

(ii) B = {{1, 3, 5}, {2, 4, 6, 8}, {7, 9} }

(iii) C = {{1, 3, 5}, {2, 4, 6, 8}, {5, 7, 9} }

(iv) D = { {S} }

Soln. :

Given that, S = {1, 2, 3, 4, 5, 6, 7, 8, 9}

(i) A has three blocks

$A_1 = \{1, 3, 5\}$, $A_2 = \{2, 6\}$, $A_3 = \{4, 8, 9\}$

$A_1 \cup A_2 \cup A_3 \neq S$

Element 7 is not in A. So, A is not partition of S.

(ii) B has three blocks

$B_1 = \{1, 3, 5\}$, $B_2 = \{2, 4, 6, 8\}$, $B_3 = \{7, 9\}$

$B_1 \cup B_2 \cup B_3 = S$ and B_1, B_2 and B_3 are mutually disjoints.

(iii) Set C is not partition of S because

Blocks of set C are not unique as element 5 is common in two blocks.

(iv) Set D is a partition of S because set S itself a block of Set D. It is called as trivial partition.

Ex. 6.9.11 : Let X = {1, 2, … n}

R = {(x, y) | x − y}

is divisible by 3. Show that R is equivalence relation. Draw the diagraph for R where n = 7.

Soln. :

Here, n = 7, so set X = {1, 2, 3, 4, 5, 6, 7}

R is a relation on X = {(x, y) | x − y is divisible by 3}

From the given condition for Relation, let we find ordered pairs of relation which satisfies the condition.

If x = 1, y = 1 then 1 − 1 = 0 is divisible by 3.

If x = 1, y = 4 then 1 − 4 = − 3 is divisible by 3.

If x = 1, y = 7 then 1 − 7 = − 6 is divisible by 3

Likewise, Relation R obtained

R = { (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (1, 4), (4, 1), (1, 7), (7, 1), (2, 5), (5, 2), (3, 6), (6, 3)}

For, R to be equivalence relation, it must be reflexive, symmetric and transitive.

(i) Reflexive property : From the relation R, it is observed that, every element x relates to itself i.e. x R x.

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)

So, R is Reflexive relation.

(ii) Symmetric property : From relation R it is seen that

x R y ⇒ x − y is divisible by 3.

⇒ y − x is also divisible by 3.

⇒ i.e. y R x for $x, y \in X$

So, R is a symmetric relation.

(iii) Transitive property : Again from relation R, it has seen that,

x R y and y R z ⇒ x − y and y − z are divisible by 3.

⇒ (x − y) + (y − z) is also divisible by 3.

⇒ So, x − z is divisible by 3.

⇒ that is, x R z.

So, R is a Transitive relation.

Relation, R satisfies reflexive, symmetric and transitive property. Hence R is equivalence relation.

Diagraph of R

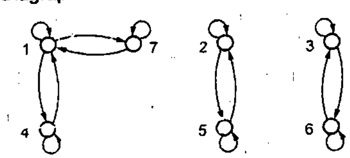


Fig. P. 6.9.11

Ex. 6.9.12 : Relation on {1, 2, 3, 4, 5}. If relation is defined as {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (3, 5), (5, 3), (1, 3), (3, 1)}

Find the equivalence classes.

Soln. :

Given : A = {1, 2, 3, 4, 5}

R = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (3, 5), (5, 3), (1, 3), (3, 1)}

Equivalence classes of the elements of set A are :

$[a]_R$ = { x ∈ A | xRa}

$[1]_R$ = {1, 5, 3} = {1, 3, 5}

$[2]_R$ = {2}

$[3]_R$ = {3, 1} = {1, 3}

$[4]_R$ = {4}

$[5]_R$ = {5, 1, 3} = {1, 3, 5}

Hence, $[1]_R = [5]_R$

6.10 CLOSURE OF A RELATION

GQ. Define the closure of Relation. Discuss about the following closure properties with examples :

1. Reflexive closure 2. Symmetric closure

3. Transitive closure.

Definition : Let, A be a set. R be a relation on set A. A closure of relation R is always with respect to some property "P" of relations. The P-closure of R is defined as the smallest relation on A containing R and possessing the property P.

There are mainly three types of closures of relations.

6.10.1 Reflexive Closure

- Let R be a relation on set A, which is not reflexive relation. A relation $R_1 = R \cup \Delta$ is called the reflexive closure of R if $R \cup \Delta$ is the smallest reflexive relation containing R.
- Here, Δ is the minimum number of ordered pair which satisfy the property P (Reflexive Property)

Example

Let, A = {a, b, c, d}

R = {(a, a), (a, b), (c, d), (c, c)}

- Here R is not reflexive relation.

$R_1 = R \cup \Delta$ is the smallest relation containing R which satisfies reflexive property.

Δ = {(b, b), (d, d)} ⇒ minimum number of ordered . Pair, need to add in R to make R as reflexive relation.

i.e. Δ = { (a, a) | a ∈ A }

- So, the reflexive closure of R is

$$R_1 = R \cup \Delta = \{(a, a)\ (b, b), (c, c), (d, d), (a, b), (c, d)\}$$

6.10.2 Symmetric Closure

- Let, R be a relation on a set A, which is not symmetric relation.
- A relation $R_1 = R \cup R^{-1}$ is called the symmetric closure of R if $R \cup R^{-1}$ is the smallest symmetric relation containing R

Example

Let, A = { a, b, c }

R = { (a, a) (b, a) }

- Here, R is not symmetric relation. To make it symmetric relation, we need to add minimum number of ordered pairs.

i.e. $(b, a) \in R$ So, the symmetric closure of R is

$$R_1 = R \cup R^{-1}$$

$$R_1 = \{(a, a), (b, a), (a, b)\}$$

- Here, (a, a), (b, a) $\in R$ and $(a, b) \in R^{-1}$

6.10.3 Transitive Closure

- Let, R be a relation on set A, which is not transitive relation. The transitive closure of a relation R is the smallest transitive relation containing R.
- It is denoted by R*
- We know that, the composition of relation is $R^2 = R \circ R$, $R^4 = R^2 \circ R^2$, $R^5 = R \circ R^4$.

Finding transitive closure

- Let, A be a non-empty set and | A | = n and R be a relation on A.
- The transitive closure of R is R*.

$$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$$

Example

Let, A = {a, b, c}, R be a relation which is not transitive

R = {(a, b), (b, c), (c, c)}

- The transitive closure of R is R* calculated as

$$R^* = R \cup R^2 \cup R^3,$$

- Here, n = 3 (number of elements of A)

$$R^2 = R \cdot R = \{(a, c), (b, c), (c, c)\}$$

$$R^3 = R^2 \cdot R = \{(a, c), (b, c), (3, 3)\}$$

$$R^* = \{(a, b), (b, c), (c, c), (a, c)\}$$

- This method is critical to find transitive closure.
- So, matrix method is utilized, which we will discuss in next section.

6.10.4 Warshall's Algorithm

- To find transitive closure of relation by computing many powers of R or using product of relation matrix is somewhat impractical for relations on large set.
- Warshall's algorithm comes with alternative method for finding transitive closure of relation.
- It is practical and efficient method.

Steps to find transitive closure of relation

Step I :

Let, | A | = n

- ∴ Here we require W_n warshall sets from W_0, W_1, W_2 ... to W_n
- W_0 is the relation matrix of R = M_R.

Step II :

- We have set A with number of elements = n. Here, to compute Warshall set W_k from W_{k-1} as
  - (i) Copy 1 to all entries in W_k from W_{k-1}.
  - (ii) Find the row numbers $r_1, r_2, r_3, \ldots$ for which there is 1 is corresponding column in W_{k-1} and find the column numbers $c_1, c_2, c_3, \ldots$ for which there is 1 in corresponding row in W_{k-1}.
  - (iii) Mark the entries in W_k as 1 for (r_i, c_j), if there are not already 1.

Step III :

Repeat Step II until we get Warshall set W_n and it is the required transitive closure of R.

UEx. 6.10.1 (SPPU(IT) - Q. 4(a), May 18, 6 Marks)

Use Warshall's algorithm to find transitive closure of the following relation on the set.

{ 1, 2, 3, 4 }, R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }

☑ **Soln. :**

Given : A = {1, 2, 3, 4}

Relation R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

The relation matrix of R is as

$$M_R = W_0 = \begin{array}{cc} & \begin{array}{cccc}1&2&3&4\end{array}\\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}\end{array}$$

Step 1 : Find W_1 from W_0. Consider 1st Row and 1st column of W_0.

In $C_1 \Rightarrow$ 1 is not present at any of the Rows.

So, W_1 will be same as W_0

$$W_1 = \begin{array}{cc} & \begin{array}{cccc}1&2&3&4\end{array}\\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}\end{array}$$

Step 2 : Find W_2. Consider 2nd row and 2nd column of W_1

In $C_2 \Rightarrow$ 1 is present at R_1

In $R_2 \Rightarrow$ 1 is present at C_3 and C_4

Make new entries at (R_1, C_3) and (R_1, C_4) in W_2

$$W_2 = \begin{array}{cc} & \begin{array}{cccc}1&2&3&4\end{array}\\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}\end{array}$$

Step 3 : Find W_3 consider 3rd row and 3rd column of W_2

In $C_3 \Rightarrow$ 1 is present at R_1 and R_2

In $R_3 \Rightarrow$ 1 is present at C_4

Make new entries at (R_1, C_4), (R_2, C_4) in W_3.

$$W_3 = \begin{array}{cc} & \begin{array}{cccc}1&2&3&4\end{array}\\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}\end{array}$$

Step 4 : To Find W_4 consider 4th row and 4th column

In $C_4 \Rightarrow$ 1 is present at R_1, R_2 and R_3

In $R_4 \Rightarrow$ 1 is not present at any of the columns.

So, W_4 will be same as W_3

$$W_4 = \begin{array}{cc} & \begin{array}{cccc}1&2&3&4\end{array}\\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}\end{array}$$

W_4 is the transitive closure of relation R

R* = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)} ...Ans.

UEx. 6.10.2 (SPPU(IT) - Q. 4(a), May 18, 6 Marks)

Find the transitive closure of R by Warshall's algorithm where A = {1, 2, 3, 4, 5, 6} and R = {(x, y) | x – y | = 2} and draw it digraph.

☑ **Soln. :**

Given :

A = {1, 2, 3, 4, 5, 6} and R = {(x, y) | x – y | = 2}

The relation R which satisfies the given condition.

R = {(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4)}

The relation matrix of R is as

$$M_R = W_0 = \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\1&0&0&0&1&0\\0&1&0&0&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\end{bmatrix}\end{array}$$

Step 1 : Find W_1 consider 1st row and first column of W_0.

$\left.\begin{array}{l}\text{In } C_1 \Rightarrow 1 \text{ is present at } R_3\\ \text{In } R_1 \Rightarrow 1 \text{ is present at } C_3\end{array}\right\}$ make new entry at (R_3, C_3) in W_1

$$W_1 = \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\1&0&1&0&1&0\\0&1&0&0&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\end{bmatrix}\end{array}$$

Step 2 : Find W_2. Consider 2nd row and 2nd column

In $C_2 \Rightarrow$ 1 is present at R_4

In $R_2 \Rightarrow$ 1 is present at C_4

Make new entry at (R_4, C_4) is W_2.

$$W_2 = \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\end{bmatrix}\end{array}$$

Step 3 : Find W_3. Consider 3rd row and 3rd column.

In $C_3 \Rightarrow$ 1 is present at R_1, R_3 and R_5

In $R_3 \Rightarrow$ 1 is present at C_1, C_3 and C_5

Make new entries at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3), (R_5, C_5) in W_3.

$$W_3 = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

▶ **Step 4 : Find W_4. Consider 4^{th} row and 4^{th} column.**

In $C_4 \Rightarrow 1$ is present at R_2, R_4 and R_6.

In $R_4 \Rightarrow 1$ is present at C_2, C_4 and C_6. Make new entries at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_4) and (R_6, C_6) in W_4.

$$W_4 = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

▶ **Step 5 : Find W_5. Consider 5^{th} row and 5^{th} column.**

In $C_5 \Rightarrow 1$ is present at R_1, R_3 and R_5.

In $R_5 \Rightarrow 1$ is present at C_1, C_3 and C_5 make new entries at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3) and (R_5, C_5) is W_5.

$$W_5 = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

▶ **Step 6 : Find W_6. Consider 6^{th} row and 6^{th} column.**

In $C_6 \Rightarrow 1$ is present at R_2, R_4 and R_6.

In $R_6 \Rightarrow 1$ is present at C_2, C_4 and C_6 make new entries at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4) and (R_4, C_6) is W_6.

$$W_6 = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

W_6 is the transitive closure of Relation R.

$R^* = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\}$ ...Ans.

UEx. 6.10.3 (SPPU(IT) - Q. 2(b), Dec. 15, 6 Marks)

Let A = [illegible] and let R be a relation on A whose relation [illegible]

Find transitive closure using Warshall's method.

☑ **Soln. :**

Given : $A = \{a, b, c, d\}$

$R = \{(b, a), (c, b), (a, d), (b, d), (c, d)\}$

Relation matrix of R is

$$M_R = W_0 = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

▶ **Step 1 : Find W_1 consider 1^{st} row and 1^{st} column of W_0.**

$\left.\begin{array}{l} \text{In } C_1 \Rightarrow 1 \text{ is present at } R_2 \\ \text{In } R_1 \Rightarrow 1 \text{ is present at } C_4 \end{array}\right\}$ make new entry at (R_2, C_4) in W_2

$$W_1 = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

▶ **Step 2 : Find W_2 consider 2^{nd} row and 2^{nd} column.**

In $C_2 \Rightarrow 1$ is present at R_3

In $R_2 \Rightarrow 1$ is present at C_1 and C_4

make new entries at (R_3, C_1), (R_3, C_4) in W:

$$W_2 = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

▶ **Step 3 : Find W_3 consider 3^{rd} row and 3^{rd} column.**

In $C_3 \Rightarrow 1$ is not present at any of the row. So there will be no new entries in W_3.

So, W_3 will be same as W_2.

$$W_3 = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

▶ **Step 4 : Find W_4. Consider 4^{th} row and 4^{th} column.**

In $C_4 \Rightarrow 1$ is present at R_1, R_2, R_3

In $R_4 \Rightarrow 1$ is not present at any of the columns.

So, $W_4 = W_3$

$$W_4 = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

W_4 is the transitive closure of relation R

$R^* = \{(a, d), (b, a), (b, d), (c, a), (c, b), (c, d)\}$ ...Ans.

UEx. 6.10.4 (SPPU(IT) - Q. 1(b), May 15, 6 Marks)

Find the transitive closure by using Warshall's algorithm for the given relation as :

$R = \{(1, 1), (1, 4), (2, 1), (2, 2), (3, 3) (4, 4)$

☑ **Soln. :**

Given :

$A = \{1, 2, 3, 4\}$

$R = \{(1, 1), (1, 4), (2, 1), (2, 2), (3, 3), (4, 4)\}$

The relation matrix of R

$$M_R = W_0 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

▶ **Step 1 : Find W_1 consider 1^{st} row and 1^{st} column of W_0.**

In $C_1 \Rightarrow 1$ is present at R_1 and R_2

In $R_1 \Rightarrow 1$ is present at C_1 and C_4 make new entries at (R_1, C_1), (R_1, C_4) and (R_2, C_1), (R_2, C_4) in W_1.

$$W_1 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

▶ **Step 2 : Find W_2 consider 2^{nd} row and 2^{nd} column of W_1.**

In $C_2 \Rightarrow 1$ is present at R_2 and

In $R_2 \Rightarrow 1$ is present at C_1, C_2 and C_4

Make new entries at (R_2, C_1), (R_2, C_2) and (R_2, C_1) in W_2.

$$W_2 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

▶ **Step 3 : Find W_3 consider 3^{rd} row and 3^{rd} column of W_2.**

In $C_3 \Rightarrow 1$ is present at R_3

In $R_3 \Rightarrow 1$ is present at C_3

So, make new entry at (R_3, C_3) in W_3.

$$W_3 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

▶ **Step 4 : Find W_4 consider 4^{th} row and 4^{th} column of W_3.**

In $C_4 \Rightarrow 1$ is present at R_1, R_2 and R_4

In $R_4 \Rightarrow 1$ is present at C_4

Make new entries at (R_1, C_4), (R_2, C_4) and (R_4, C_4) in W_4.

$$W_4 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

W_4 is the transitive closure of relation R

$R^* = \{(1, 1), (1, 4), (2, 1), (2, 2), (2, 4), (3, 3), (4, 4)\}$ ...Ans.

UEx. 6.10.5 (SPPU(IT) - Q. 2(b), Dec. 15, 6 Marks)

Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 1), (1, 2), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (4, 4)\}$.

Find Transitive closure of R using Warshall's Algorithm.

☑ **Soln. :**

Given : $A = \{1, 2, 3, 4\}$

$R = \{(1, 1), (1, 2), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (4, 4)\}$

Relation matrix of R is

$$M_R = W_0 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

▶ **Step 1 : Find W_1 . Consider 1^{st} row and 1^{st} column of W_0.**

In $C_1 \Rightarrow 1$ is present at R_1 and R_3

In $R_1 \Rightarrow 1$ is present at C_1 . C_2 and C_4 make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_4) (R_3, C_1), (R_3, C_2) and (R_3, C_4) in W_1.

$$W_1 = \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

▶ **Step 2 : Find W_2 consider 2^{nd} row and 2^{nd} column of W_1.**

In $C_2 \Rightarrow 1$ is present at R_1, R_3 and R_4.

In $R_2 \Rightarrow 1$ is present at C_4

Make new entries at (R_1, C_4), (R_3, C_4) and (R_4, C_4) in W_2.

$$W_2 = \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

▶ **Step 3 : Find W_3 consider 3^{rd} row and column.**

In $C_3 \Rightarrow 1$ is present at R_4

In $R_3 \Rightarrow 1$ is present at C_1, C_2 and C_4

make new entries at (R_4, C_1), (R_4, C_2) and (R_4, C_4) in W_3.

$$W_3 = \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

▶ **Step 4 : Find W_4 consider 4^{th} row and 4^{th} column of W_3.**

In $C_4 \Rightarrow 1$ is present at R_1, R_2, R_3, R_4

In $R_4 \Rightarrow 1$ is present at C_1, C_2, C_3, C_4

Thus, Make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1) (R_3, C_2), (R_3, C_3), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4) in W_4.

$$W_4 = \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

W_4 is the transitive closure of relation R

$R^* = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

...Ans.

UEx. 6.10.6 (SPPU(IT) - Q. 2(b), May 14, 6 Marks)

Let $R = \{(a, b), (b, a), (b, c), (c, d), (d, a)\}$ over the set $A = \{a, b, c, d\}$.

Find transitive closure using Warshall's algorithm.

☑ **Soln. :**

Given : $A = \{a, b, c, d\}$.

$R = \{(a, b), (b, a), (b, c), (c, d), (d, a)\}$

The relation matrix of R is

$$M_R = W_0 = \begin{array}{cc} & \begin{matrix}a & b & c & d\end{matrix} \\ \begin{matrix}a\\b\\c\\d\end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

▶ **Step 1 : Find W_1 consider 1^{st} row and 1^{st} column of W_0.**

In $C_1 \Rightarrow 1$ is present at R_2 and R_4

In $R_1 \Rightarrow 1$ is present at C_2

make new entries at (R_2, C_2) and (R_4, C_2) in W_1.

$$W_1 = \begin{array}{cc} & \begin{matrix}a & b & c & d\end{matrix} \\ \begin{matrix}a\\b\\c\\d\end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

▶ **Step 2 : Find W_2 consider 2^{nd} row and 2^{nd} column of W_1.**

In $C_2 \Rightarrow 1$ is present at R_1 and R_2.

In $R_2 \Rightarrow 1$ is present at C_1, C_2 and C_3

Make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_2, C_1), (R_2, C_2) and (R_2, C_3) in W_2.

$$W_2 = \begin{array}{cc} & \begin{matrix}a & b & c & d\end{matrix} \\ \begin{matrix}a\\b\\c\\d\end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

▶ **Step 3 : Find W_3 consider 3^{rd} row and 3^{rd} column of W_2.**

In $C_3 \Rightarrow 1$ is present at R_1, R_2 and R_4

In $R_3 \Rightarrow 1$ is present at C_4

So, make new entries at (R_1, C_4), (R_2, C_4) and (R_4, C_4) in W_3.

$$W_3 = \begin{array}{cc} & \begin{matrix}a & b & c & d\end{matrix} \\ \begin{matrix}a\\b\\c\\d\end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

▶ **Step 4 : Find W_4 consider 4^{th} row and 4^{th} column of W_3.**

In $C_4 \Rightarrow 1$ is present at R_1, R_2, R_3 and R_4

In $R_4 \Rightarrow 1$ is present at C_1, C_3 and C_4

make new entries at (R_1, C_1), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_3), (R_3, C_4) (R_4, C_1), (R_4, C_3), (R_4, C_4) in W_4.

$$W_4 = \begin{array}{cc} & \begin{matrix}a & b & c & d\end{matrix} \\ \begin{matrix}a\\b\\c\\d\end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

W_4 is the transitive closure of relation R

$R^* = \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, c), (c, d), (d, a), (d, c), (d, d)\}$

...Ans.

UEx. 6.10.7 (SPPU(IT) - Q. 3(a), Dec. 18, 6 Marks)

Find the transitive closure of the relation R on $A = \{1, 2, 3, 4\}$ defined by

$R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 2), (4, 3)\}$

☑ **Soln. :**

Given :

$A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4) (4, 2), (4, 3)\}$.

The relation matrix of given relation as

$$M_R = W_0 = \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

▶ **Step 1 : Find W_1, consider 1^{st} Row and 1^{st} column of W_0**

In $C_1 \Rightarrow 1$ is present at R_2

In $R_1 \Rightarrow 1$ is present at C_2, C_3 and C_4

Make new entries at (R_2, C_2), (R_2, C_3) and (R_2, C_4) in W_1

$$W_1 = \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

▶ **Step 2 : Find W_2, consider 2^{nd} Row and 2^{nd} column of W_1**

In $C_2 \Rightarrow 1$ is present at R_1, R_2, R_3 and R_4

In $R_2 \Rightarrow 1$ is present at C_1, C_2, C_3 and C_4

Make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4) in W_2.

$$W_2 = \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

From the W_2, It is clear that W_2 is matrix representation of universal relation and we know that universal relation satisfied reflexive, symmetric and transitive properties.

So, W_2 is the transitive closure of relation R

$R^* = \{(R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)\}$ ...Ans.

UEx. 6.10.8 (SPPU(IT) - Q. 3(a), May 19, 6 Marks)

Let $A = \{1, 2, 3, 4, 5\}$ and R be the relation on A such that $R = \{(1, 1), (1, 4), (2, 2), (3, 4), (3, 5), (4, 1), (5, 2), (5, 5)\}$. Find the transitive closure of R by Warshall's algorithm.

☑ **Soln. :**

Given :

$A = \{1, 2, 3, 4, 5\}$ and R be the relation such that

$R = \{(1, 1), (1, 4), (2, 2), (3, 4), (3, 5), (4, 1), (5, 2), (5, 5)\}$

Set a has 5 elements so that we need to find 5 Warshall sets $(w_1, w_2, w_3, w_4, w_5)$

Represent the given relation in the matrix form and that will be $M_R = W_0$.

$$M_R = W_0 = \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4 & 5\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

▶ **Step 1 : Find W_1**

To find W_1 from W_0, consider the 1^{st} Row and first column of W_0.

In C_1, 1 is present at R_1 and R_4

In R_1, 1 is present at C_1 and C_4

Thus, Add new entry in W_1 at (R_1, C_1), (R_1, C_4), (R_4, C_1) and (R_4, C_4)

$$W_1 = \begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}1&0&0&1&0\\0&1&0&0&0\\0&0&0&1&1\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}\end{array}$$

▶ **Step 2 : Find W_2**

To find W_2, consider 2^{nd} column and 2^{nd} row of W_1

In $C_2 \Rightarrow$ 1 is present at R_2 } so, make new entry at (R_2, C_2)

In $R_2 \Rightarrow$ 1 is present at C_2 } in W_2

$$W_2 = \begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}1&0&0&1&0\\0&1&0&0&0\\0&0&0&1&1\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}\end{array}$$

▶ **Step 3 : Find W_3. Consider third column and third Row of W_2.**

In $C_3 \Rightarrow$ 1 is not present at any of the rows.

So, W_3 will be same as W_2.

$$W_3 = \begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}1&0&0&1&0\\0&1&0&0&0\\0&0&0&1&1\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}\end{array}$$

▶ **Step 4 : Find W_4. Consider 4^{th} row and 4^{th} column of W_3.**

In $C_4 \Rightarrow$ 1 is present at R_1, R_3 and R_4 at the rows.

In $R_4 \Rightarrow$ 1 is present at C_1 and C_4

So, make new entries at (R_1, C_1), (R_1, C_4), (R_3, C_1), (R_3, C_4), (R_4, C_1) and (R_4, C_4) in W_4.

$$W_4 = \begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}1&0&0&1&0\\0&1&0&0&0\\1&0&0&1&1\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}\end{array}$$

▶ **Step 5 : To find W_5, consider 5^{th} row and 5^{th} column of W_4.**

In $C_5 \Rightarrow$ 1 is present at R_3 and R_5 at the rows.

In $R_5 \Rightarrow$ 1 is present at C_2 and C_4

Thus, make new entries at (R_3, C_2), (R_3, C_4), (R_5, C_2) and (R_5, C_4) is W_5.

$$W_5 = \begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}1&0&0&1&0\\0&1&0&0&0\\1&1&0&1&1\\1&0&0&1&0\\0&1&0&1&1\end{bmatrix}\end{array}$$

W_5 is the transitive relation matrix of R*

Transitive closure of R is

R* = {(1, 1), (1, 4) (2, 2), (3, 1), (3, 2), (3, 4), (3, 5), (4, 1), (4, 4), (5, 2), (5, 4), (5, 5)}

6.10.5 Example Based on Closure of Relations

Ex. 6.10.9 : Let, A = {a, b, c } and Relation R be defined on A as follows

R = { (a, a), (b, c), (a, b) }. Then, write minimum number of ordered pairs to be added in R to make R Reflexive and Transitive.

☑ **Soln. :**

(i) **For Reflexitive**

The minimum number of ordered pairs are as

Δ = (b, b), (c, c)

(ii) **For Transitive**

The minimum number of ordered pairs to make R as transitive = (a, c)

Ex. 6.10.10 : R = { (1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3) } on A = {1, 2, 3, 4 }. Find Reflexive and symmetric closure of relation R.

☑ **Soln. :**

A = {1, 2, 3, 4 }

R = { (1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3) }

(i) **For Reflexivity**

Δ = { (2, 2), (3, 3), (4, 4) }

The Reflexive closure of R is $R_1 = R \cup \Delta$

R_1 = { (1, 3), (2, 4), (3, 1), (3, 3), (4, 3), (1,1), (2,2) (3, 3), (4, 4) }

(ii) **For symmetric**

The inverse of R is R^{-1}

= {(1, 1), (3, 1), (4, 2), (1, 3) (3, 3), (3, 4) }

The symmetric closure of R is $R \cup R^{-1}$.

= { (1, 1), (1, 3), (3, 1), (2, 4), (4, 2), (3, 3), (4, 3), (3, 4) }

Ex. 6.10.11 : Let, A = { a, b, c } R_1, R_2 and R_3 are relations on set A. Find the Reflexive closure of R_1, R_2 and R_3 where

(i) R_1 = { (a, a), (b, a) }

(ii) R_2 = { (a, a), (b, b), (c, c) }

(iii) R_3 = { (c, a), (a, c), (b, c) }

☑ **Soln. :**

(i) Δ = { (b, b), (c, c) } are minimum number of ordered pair to make relation R_1 Reflexive.

So, Reflexive closure of R_1 is $R = R_1 \cup \Delta$

R = { (a, a), (b, a), (b, b), (c, c) }

(ii) $\Delta = \phi$

Reflexive closure of R_2 is $R = R_2 \cup \Delta$

R = { (a, a), (b, b), (c, c) }

(iii) Δ = { (a, a), (b, b), (c, c) } Δ are the minimum number of pairs which make relation R_3 Reflexive.

So, Reflexive closure of R_3 is $R = R_3 \cup \Delta$.

R = { (a, a), (c, a), (a, c), (b, c), (b, b), (c, c) }

Ex. 6.10.12 : Find the symmetric closure of the following Relations on A = { x, y, z }

R_1 = { (x, x), (y, x) } , R_2 = { (x, y), (y, x), (z, y), (y, y) }

R_3 = { (x, x), (y, y), (z, z) }

☑ **Soln. :**

Given, A = { x, y, z }

(i) The inverse Relation of R_1 is R_1^{-1} = { (x, x), (x, y) }

The symmetric closure of R_1 is R

$R = R_1 \cup R_1^{-1}$ = { (x, x), (x, y), (y, x) }

(ii) The inverse relation of R_2 is R_2^{-1}

R_2^{-1} = { (y, x), (x, y), (y, z), (y, y) }

The symmetric closure of R_2 is R which is

$R = R_2 \cup R_2^{-1}$ = { (x, y), (y, x), (3, 2), (2, 3), (2, 2)}

(iii) The Inverse of R_3 is R_3^{-1} = { (x, x), (y, y), (z, z)

The symmetric closure of R_3 is R

$R = R_3 \cup R_3^{-1}$ = { (x, x), (y, y), (z, z) }

Ex. 6.10.13 : Let Relation R = {(a, a), (a, b), (b, c), (c, c) } on A = {a, b, c}. Find :

(i) Reflexive closure

(ii) Symmetric closure

(iii) Transitive closure of R

☑ **Soln. :**

Given : R = { (a, a), (a, b), (b, c), (c, c) }

A = { a, b, c }

(i) Reflexive closure of R is obtained by adding minimum number of diagonal pairs of A × A to R which are not currently in R.

Δ = { (b, b) }

Reflexive closure of R is $R_1 = R \cup \Delta$

R_1 = { (a, a), (a, b), (b, c), (b, b), (c, c) }

(ii) The symmetric closure of R is obtained by adding all the pairs in R^{-1} to R which are not currently in R symmetric closure of R is $R \cup R^{-1}$.

= { (a, a), (a, b), (b, a), (b, [illegible]

(iii) Transitive closure of relation R is obtained by taking the union of R with R^2 (R · R) and R^3 (R · R · R).

Since set A has three elements.

R^2 = R · R = { (a, a), (a, b), (a, c), (b, c), (c, c) }

$R^3 = R \cdot R^2$ or R · R · R

= { (a, a), (a, b), (a, c), (b, c), (c, c) }

Hence, Transitive closure of R is R*.

$R^* = R \cup R^2 \cup R^3$

= { (a, a), (a, b), (a, c), ([illegible]), (c, c) }

Ex. 6.10.14 : Let R = {(1, 4), (2, 1), (2, 5), (2, 4), (4, 3), (5, 3),(3, 2)} on the set A = {1, 2, 3, 4, 5}. Use Warshall's algorithm to find transitivie closure of R.

☑ **Soln. :**

Let, A = {1, 2, 3, 4, 5} and R be a relation on R

R = {(1,4), (2,1), (2,5), (2,4), (4,3), (5,3), (3,2)}

Here, we have to find W_1, W_2, W_3, W_4, W_5 warshall sets since A has 5 elements.

First, represent the given relation in matrix form, that is M_R, which is also W_0.

$$M_R = W_0 = \begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}0&0&0&1&0\\1&0&0&1&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&0&0\end{bmatrix}\end{array}$$

▶ **Step I : Find W_1**

To find W_1 from W_0, consider the 1^{st} column and first row of W_0.

$$W_1 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 \end{array}$$

In C_1, 1 is present at R_2.

In R_1, 1 is present at C_4.

Thus, add new entry in W_1 at (R_2, C_4) but 1 is already present at (R_2, C_4) in W_0. So, keep original entry as it is in W_1.

▶ Step II : Find W_2

To find W_2 from W_1, consider the 2[nd] column and 2[nd] row of W_1.

In $C_2 \Rightarrow 1$ is present at R_3.

In $R_2 \Rightarrow 1$ is present at C_1, C_4 and C_5.

Thus, make a new entry at (R_3, C_1), (R_3, C_4) and (R_3, C_5) in W_2.

$$W_2 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 \end{array}$$

▶ Step III : Find W_3

To find W_3 from W_2, consider 3[rd] column and 3[rd] row.

In $C_3 \Rightarrow 1$ is present at R_4 and R_5.

In $R_3 \Rightarrow 1$ is present at C_1, C_2, C_4 and C_5.

So, make new entry at (R_4, C_1), (R_4, C_2), (R_4, C_4), (R_4, C_5), (R_5, C_1), (R_5, C_2), (R_5, C_4) and (R_5, C_5) in W_3.

$$W_3 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 & 1 & 1 \end{array}$$

▶ Step IV : Find W_4

To find W_4 from W_3, consider 4[th] column and 4[th] row.

In $C_4 \Rightarrow 1$ is present at R_1, R_2, R_3, R_4 and R_5.

In $R_4 \Rightarrow 1$ is present at C_1, C_2, C_3, C_4 and C_5.

So make new entry at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_1, C_5), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_2, C_5), (R_3, C_1), (R_3, C_2), (R_3, C_1), (R_3, C_4), (R_3, C_5), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4), (R_4, C_5), (R_5, C_1), (R_5, C_2), (R_5, C_3), (R_5, C_4), (R_5, C_5) in W_4.

$$W_4 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 & 1 & 1 \end{array}$$

If we observed, W_4, it is matrix representation of universal relation. $(A \times A)$.

Hence, W_4 is act as relation matrix of R^*.

So, transitive closure of R is

$R^* = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)\}$

Ex. 6.10.15 : Compute the transitive closure of given diagraph using Warshall's algorithm.

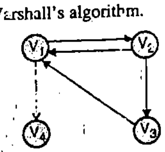


Fig. P. 6.10.15

☑ Soln. :

Refer Fig. P. 6.10.15,

$A = \{V_1, V_2, V_3, V_4\}$

$n(A) = 4$

The relation $R = \{(V_1, V_2), (V_1, V_4), (V_2, V_1), (V_3, V_1), (V_2, V_3)\}$

Thus, for transitive closure, we required to find W_1, W_2, W_3 and W_4 warshall sets.

Represent the given relation in matrix form.

$$M_R = W_0 = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 1 & 0 & 1 \\ V_2 & 1 & 0 & 1 & 0 \\ V_3 & 1 & 0 & 0 & 0 \\ V_4 & 0 & 0 & 0 & 0 \end{array}$$

▶ Step I : Find W_1

To find W_1, from W_0 consider 1[st] column and 1[st] row.

In $C_1 \Rightarrow 1$ is present at R_2 and R_3.

In $R_1 \Rightarrow 1$ is present at C_2 and C_4.

So, make new entry at (R_2, C_2), (R_2, C_4), (R_3, C_2), (R_3, C_4) in W_1.

$$W_1 = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 1 & 0 & 1 \\ V_2 & 1 & 1 & 1 & 1 \\ V_3 & 1 & 1 & 0 & 1 \\ V_4 & 0 & 0 & 0 & 0 \end{array}$$

▶ Step II : Find W_2

To find W_2 from W_1 consider 2[nd] column and 2[nd] row.

In $C_2 \Rightarrow 1$ is present at R_1, R_2 and R_3.

In $R_2 \Rightarrow 1$ is present at C_1, C_2, C_3 and C_4.

So, make a new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3) and (R_3, C_4) in W_2.

$$W_2 = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 1 & 1 & 1 & 1 \\ V_2 & 1 & 1 & 1 & 1 \\ V_3 & 1 & 1 & 1 & 1 \\ V_4 & 0 & 0 & 0 & 0 \end{array}$$

▶ Step III : Find W_3

To find W_3, consider 3[rd] column and 3[rd] row of W_2.

In $C_3 \Rightarrow 1$ is present at R_1, R_2 and R_3.

In $R_3 \Rightarrow 1$ is present at C_1, C_2, C_3 and C_4.

So, make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4) in W_3.

$$W_3 = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 1 & 1 & 1 & 1 \\ V_2 & 1 & 1 & 1 & 1 \\ V_3 & 1 & 1 & 1 & 1 \\ V_4 & 0 & 0 & 0 & 0 \end{array}$$

Here, $W_2 = W_3$

It is observed that, if we find next Warshall set W_4, that will also be same as W_2 and W_3.

So,

$$W_4 = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 1 & 1 & 1 & 1 \\ V_2 & 1 & 1 & 1 & 1 \\ V_3 & 1 & 1 & 1 & 1 \\ V_4 & 0 & 0 & 0 & 0 \end{array}$$

Hence, here, W_4 is transitive closure of R is

$R^* = \{(V_1, V_1), (V_1, V_2), (V_1, V_3), (V_1, V_4), (V_2, V_1), (V_2, V_2), (V_2, V_3), (V_2, V_4), (V_3, V_1), (V_3, V_2), (V_3, V_3), (V_3, V_4)\}$

Ex. 6.10.16 : Find transitive closure using Warshall algorithm :

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

☑ Soln. :

Let, $A = \{a, b, c, d\}$ $\quad n(A) = 4$

$R = \{(a, b), (b, a), (b, c), (c, d)\}$

The relation matrix of R is

$$W_0 = M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$$

Here, size of A = 4, so, we required warshall sets W_1, W_2, W_3 and W_4 to find transitive closure.

▶ Step I : Find W_1

To find W_1, consider 1[st] column and 1[st] row of W_0.

In $\text{column}_1 \Rightarrow 1$ is present at R_2.

In $\text{row}_1 \Rightarrow 1$ is present at C_2.

So, make new entry at (R_2, C_2) in W_1.

$$W_1 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 1 & 1 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$$

Step II : Find W_2

To find W_2, consider 2^{nd} column and 2^{nd} row of W_1.

In $column_2 \Rightarrow 1$ is present at R_1 and R_2.

In $row_2 \Rightarrow 1$ is present at C_1, C_2 and C_3.

So, make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_2, C_1), (R_2, C_2) and (R_2, C_3).

$$W_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step III : Find W_3

To find W_3, consider 3^{rd} column and 3^{rd} row of W_2.

In $column_3 \Rightarrow 1$ is present at R_1 and R_2.

In $row_3 \Rightarrow 1$ is present at C_4.

So, make new entries at (R_1, C_4) and (R_2, C_4)

$$W_3 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step IV : Find W_4

To find W_4, consider 4^{th} column and 4^{th} row of W_3.

In $column_4 \Rightarrow 1$ is present at R_1, R_2 and R_3.

In $row_4 \Rightarrow$ no 1 is present.

So, W_3 is the relation matrix of transitive closure of R.

$R^* = \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, d)\}$

Here, in R^*, (a, d), (b, d) and (c, d) have to be neglected because $(a, d) \in R$ $(d, b) \notin R$.

Ex. 6.10.17 : If, R = {(a, b), (b, a), (b, c), (c, d), (d, a)} be a relation on the set A = {a, b, c, d}. Find the transitive closer of R using Warshall's algorithm.

☑ **Soln. :**

Let, R = {(a, b), (b, a), (b, c), (c, d), (d, a)} be a relation on the set A = {a, b, c, d}.

Here, n (A) = 4. So will required W_1, W_2, W_3, W_4 Warshall sets to get transitive closure of R.

Find relation matrix of given relation and which is equal to W_0.

$$M_R = W_0 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step I : Find W_1

To find W_1, consider 1^{st} row and 1^{st} column of W_0.

In $column_1 \Rightarrow 1$ is present at R_2 and R_4.

In $row_1 \Rightarrow 1$ is present at C_2 only.

So, make new entries in W_1 at (R_2, C_2), (R_4, C_2)

$$W_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Step II : Find W_2

To find W_2, consider 2^{nd} row and 2^{nd} column of W_1.

In $column_2 \Rightarrow 1$ is present at R_1, R_2, and R_3.

In $row_2 \Rightarrow 1$ is present at C_1, C_2 and C_3.

So, make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_3) in W_2.

$$W_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Step III : Find W_3

To find W_3, consider, 3^{rd} column and 3^{rd} row of W_2.

In $column_3 \Rightarrow 1$ is present at R_1, R_2 and R_3.

In $row_3 \Rightarrow 1$ is present at C_1, C_2, C_3 and C_4.

So, make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3) and (R_3, C_4) in W_3.

$$W_3 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Step IV : Find W_4

To find W_4, consider 4^{th} column and 4^{th} row of W_3.

In $column_4 \Rightarrow 1$ is present at R_1, R_2 and R_3.

In $row_4 \Rightarrow 1$ is present at C_1 and C_2.

So, make new entries at (R_1, C_1), (R_1, C_2), (R_2, C_1), (R_2, C_2), (R_3, C_1) and (R_3, C_2), But already 1 is present at these locations.

Thus $W_4 = W_3$.

$$W_4 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

W_4 is the transitive closure of relation R.

$R^* = \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d), (d, a), (d, b)\}$

Ex. 6.10.18 : Use Warshall's algorithm to compute the transitive closure of $R \cup S$ for the relations R and S defined on A = {1, 2, 3, 4} described as :

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad M_s = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

☑ **Soln. :**

Let, R and S be the relations defined on set

A = {1, 2, 3, 4}

Relation matrix of R and S as given below.

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$M_{RUS} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array} = W_0$$

In n (A) = 4, so we will required, W_1, W_2, W_3 and W_4 Warshall sets to get transitive closure of RUS.

Step I : Find W_1

To find W_1 from W_0, consider 1^{st} column and 1^{st} row.

In $column_1 \Rightarrow 1$ is present at R^1.

In $row_1 \Rightarrow 1$ is present at C_1, C_2 and C_4.

So, make new entries at (R_1, C_1), (R_1, C_2) & (R_1, C_4) in W_1.

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Step II : Find W_2

To find W_2 from W_1, consider 2^{nd} row and 2^{nd} column.

In $column_2 \Rightarrow 1$ is present at R_1, R_2, R_3 and R_4.

In $row_2 \Rightarrow 1$ is present at C_2.

So, make new entries at (R_1, C_2), (R_2, C_2), (R_3, C_2) and (R_4, C_2).

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Here, $W_0 = W_1 = W_2$

It is clear that,

The transitive closure of $R \cup S$ is $R \cup S$ itself.

Thus, transitive closure of $R \cup S$ is

$(R \cup S)^* = \{(1, 1), (1, 2), (1, 4), (2, 2), (3, 2), (3, 3), (4,2), (4, 3), (4, 4)\}$

Ex. 6.10.19 : Let, R = {(a, d), (b, a), (b, d), (c, b), (c, d), (d, c)}. Use Warshall's algorithm to find the matrix of transitive closure where A = {a, b, c, d}.

☑ **Soln. :**

Let, R = {(a, d), (b, a), (b, d), (c, b), (c, d), (d, c)}

be a relation defined on set A = {a, b, c, d}

Here, n (A) = 4 elements. So, we will required W_1, W_2, W_3 and W_4 Warshall sets to get transitive closure of R

The relation matrix of R is

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = W_0$$

▸ Step I : Find W_1

To find W_1 from W_0, consider 1^{st} row and 1^{st} column of W_0.

In $column_1 \Rightarrow 1$ is present at R_2.

In $row_1 \Rightarrow 1$ is present at C_4.

So, make new entry at (R_2, C_4) in W_1, but 1 is already present at (R_2, C_4). Thus, $W_1 = W_0$.

$$W_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

▸ Step II : Find W_2

To find W_2 from W_1, consider 2^{nd} column and 2^{nd} row of W_1.

In $column_2 \Rightarrow 1$ is present at R_3.

In $row_2 \Rightarrow 1$ is present at C_1 and C_4.

So, make new entries at (R_3, C_1) and (R_3, C_4) in W_2.

$$W_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

▸ Step III : Find W_3

To find W_3 from W_2, consider 3^{rd} column and 3^{rd} row of W_2.

In $column_3 \Rightarrow 1$ is present at R_4.

In $row_3 \Rightarrow 1$ is present at C_1, C_2 and C_4.

Thus, add new entries at (R_4, C_1), (R_4, C_2), (R_4, C_4) in W_3.

$$W_3 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

▸ Step IV : Find W_4

To find W_4 from W_3, consider 4^{th} column and 4^{th} row of W_3.

In $column_4 \Rightarrow 1$ is present at R_1, R_2, R_3 and R_4.

In $row_4 \Rightarrow 1$ is present at C_1, C_2, C_3 and C_4.

Thus, make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4) in W_4.

$$W_4 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Hence, W_4 is the relation matrix of transitive closure of R.

$R^* = \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d), (d, a), (d, b), (d, c), (d, d)\}$

Ex. 6.10.20 : Find the transitive closure of the relation R on :

A = {1, 2, 3, 4} defined by

R = {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)}

☑ Soln. :

Let, A = {1, 2, 3, 4}

R = {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)} be a relation on set A.

Here, n (A) = 4. Therefore we will required W_1, W_2, W_3 and W_4 Warshall sets to transitive closure of R.

The relation matrix of R is

$$W_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = W_0$$

▸ Step I : Find W_1

To find W_1 from W_0, consider 1^{st} row and 1^{st} column of W_0.

In $column_1 \Rightarrow 1$ is present at R_2.

In $row_1 \Rightarrow 1$ is present at C_2, C_3 and C_4.

Thus, make new entries at (R_2, C_2), (R_2, C_3), (R_2, C_4) in W_1.

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

▸ Step II : Find W_2

To find W_2 from W_1, consider 2^{nd} column and 2^{nd} row of W_2.

In $column_2 \Rightarrow 1$ is present at R_1, R_2, R_3 and R_4.

In $row_2 \Rightarrow 1$ is present at C_1, C_2, C_3 and C_4.

Thus, make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4).

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

In W_2, every cell contain 1 in it. So no need to find W_3 and W_4.

The relation matrix W_2 is the transitive closure of R.

Therefore $R^* = W_2$

$R^* = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

Ex. 6.10.21 : Let R be the relation on the set A = {a, b, c, d, e, f} and R = {(a, c), (b, d), (c, a), (c, e), (d, b), (d, f), (e, c), (f, d)}. Find the transitive closure of R using Warshall's algorithm.

☑ Soln. :

Let, A = {a, b, c, d, e, f} , n (A) = 6

R = {(a, c), (b, d), (c, a), (c, e), (d, b), (d, f), (e, c), (f, d)}

As, A has 6 elements, then we will required W_1, W_2, W_3, W_4, W_5, W_6 Warshall sets to get transitive closure.

The relation matrix of R is given below.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} = W_0$$

▸ Step I : Find W_1

To find W_1 from W_0, consider 1^{st} column and 1^{st} row of W_0.

In $column_1 \Rightarrow 1$ is present at R_3.

In $row_1 \Rightarrow 1$ is present at C_3.

Thus, make new entry at (R_3, C_3) in W_1.

$$W_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

▸ Step II : Find W_2

To find W_2 from W_1, consider 2^{nd} column and 2^{nd} row of W_1.

In $column_2 \Rightarrow 1$ is present at R_4.

In $row_2 \Rightarrow$ is present at C_4

So, make new entry at (R_4, C_4) in W_2.

$$W_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

▸ Step III : Find W_3

To find W_3 from W_2, consider 3^{rd} column and 3^{rd} row of W_2.

In $column_3 \Rightarrow 1$ is present at R_1 and R_3 and R_5.

In row$_3$ $\Rightarrow$ 1 is present at C_1, C_3 and C_5.

Thus, make new entries at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3), (R_5, C_5) in W_3.

$$W_3 = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 1 & 0 & 1 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 1 & 0 & 0 \\ c & 1 & 0 & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 1 & 0 & 1 \\ e & 1 & 0 & 1 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

▶ **Step IV : Find W_4**

To find W_4 from W_3, consider 4[th] column and 4[th] row of W_3.

In column$_4$ $\Rightarrow$ 1 is present at R_2, R_4 and R_6.

In row$_4$ $\Rightarrow$ 1 is present at C_2, C_4 and C_6.

Thus, make new entries at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4) and (R_6, C_6).

$$W_4 = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 1 & 0 & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 1 & 0 & 1 \\ c & 1 & 0 & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 1 & 0 & 1 \\ e & 1 & 0 & 1 & 0 & 1 & 0 \\ f & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

▶ **Step V : Find W_5**

To find W_5 from W_4, consider 5[th] column and 5[th] row of W_4.

In column 5[th] $\Rightarrow$ 1 is present at R_1, R_3 and R_5

In row 5[th] $\Rightarrow$ 1 is present at C_1, C_3 and C_5

Make new entries at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3), (R_5, C_5) in W_5.

$$W_5 = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 1 & 0 & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 1 & 0 & 1 \\ c & 1 & 0 & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 1 & 0 & 1 \\ e & 1 & 0 & 1 & 0 & 1 & 0 \\ f & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

▶ **Step VI : Find W_6**

To find W_6 from W_5, consider 6[th] column and 6[th] row of W_5.

In column$_6$ $\Rightarrow$ 1 is present at R_2, R_4 and R_6.

In row$_6$ $\Rightarrow$ 1 is present at C_2, C_4 and C_6.

Thus, make new entries at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6) in W_6.

$$W_6 = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 1 & 0 & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 1 & 0 & 1 \\ c & 1 & 0 & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 1 & 0 & 1 \\ e & 1 & 0 & 1 & 0 & 1 & 0 \\ f & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

Relation matrix W_6 is the transitive closure R^* of R.

$R^* = \{(a, a), (a, c), (a, e), (b, b), (b, d), (b, f), (c, a), (c, c), (c, e), (d, b), (d, d), (d, f), (e, a), (e, c), (e, e), (f, b), (f, d), (f, f)\}$.

Ex. 6.10.22 : Find the transitive closure by using Warshall's algorithm :

$A = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(x, y) \mid |x - y| = 2\}$.

☑ **Soln. :**

Let, $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(x, y) \mid |x - y| = 2\}$

Here, n (A) = 6, so that, we required to find W_1, W_2, W_3, W_4, W_5, W_6 Warshall sets to get transitive closure.

The relation matrix of R is given below

$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (4, 6), (6, 4), (3, 5), (5, 3)\}$

$$W_0 = M_R = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

▶ **Step I : Find W_1**

To find W_1 from W_0, consider 1[st] column and 1[st] row of W_0.

In column$_1$ $\Rightarrow$ 1 is present at R_3.

In row$_1$ $\Rightarrow$ 1 is present at C_3.

Make new entry at (R_3, C_3) in W_1.

$$W_1 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

▶ **Step II : Find W_2**

To find W_2 from W_1, consider 2[nd] column and 2[nd] row of W_1.

In column$_2$ $\Rightarrow$ 1 is present at R_4.

In row$_2$ $\Rightarrow$ 1 is present at C_4.

Thus, make new entry at (R_4, C_4) in W_2.

$$W_2 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

▶ **Step III : Find W_3**

To find W_3 from W_2, consider 3[rd] row and 3[rd] column of W_2.

In column$_3$ $\Rightarrow$ 1 is present R_1, R_3 and R_5

In row$_3$ $\Rightarrow$ 1 is present at C_1, C_3 and C_5

Thus, make new entries at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3) and (R_5, C_5) in W_3.

$$W_3 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

▶ **Step IV : Find W_4**

To find W_4 from W_3, consider 4[th] column and 4[th] row of W_3.

In column$_4$ $\Rightarrow$ 1 present at R_2, R_4 and R_6.

In row$_4$ $\Rightarrow$ 1 is present at C_2, C_4 and C_6.

Thus, make new entries at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4) and (R_6, C_6) in W_4.

$$W_4 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 \\ 6 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

▶ **Step V : Find W_5**

To find W_5 from W_4, consider 5[th] column and 5[th] row of W_4.

In column$_5$ $\Rightarrow$ 1 is present at R_1, R_3 and R_5.

In row$_5$ $\Rightarrow$ 1 is present at C_1, C_3 and C_5.

Thus, make new entries at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3) and (R_5, C_5) in W_5.

$$W_5 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 \\ 6 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

▶ **Step VI : Find W_6**

To find W_6 from W_5, consider 6[th] column and 6[th] row of W_5.

In column$_6$ $\Rightarrow$ 1 is present at R_2, R_4 and R_6.

In row$_6$ $\Rightarrow$ 1 is present at C_2, C_4 and C_6.

Thus, make new entries at (R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4) and (R_6, C_6) in W_6.

$$W_6 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 \\ 6 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

W_6 is the relation matrix of transitive closure of R.

Hence,

R* = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)}

6.11 PARTIAL ORDERED SET

- If we want to compare the elements of set, then order relation to be considered.
- Order relation is a transitive relation.

(a) Partial order relation

A binary relation R on non-empty set A is said to be partial ordered relation iff R :

(i) Reflexive i.e. for every $a \in A$, aRa.

(ii) Anti-symmetric i.e. if aRb and bRa, then a = b.

(iii) Transitive i.e. if aRb and bRc, then aRc.

(b) Partial ordered set

- The ordered pair (A, R) is called a partially ordered set or POSET.
- It is also denoted as (A, ≤) here, "≤" is the most familiar relation. Which is "less than equal to" relation.

(c) Example

(i) (R, ≤), (N, ≤) are POSETS

- Here, R and N are sets of real numbers and natural numbers respectively.
- Whereas, "≤" is reflexive, anti-symmetric and transitive relation i.e. partially ordered relation.

(d) Comparable elements

- Let, a and b are two elements of a POSET (A, ≤), then a and b are said to be comparable elements if either a ≤ b or b ≤ a.
- Two elements a and b are non-comparable if neither a ≤ b nor b ≤ a
- Here, "≤" is partial ordered relation which may be any kind of relations like ≤, ≥, |, <, >.

Example

Let, (Z^+, |) is a POSET.

- where Z^+ is the set of all positive integers and | is the divides relation.
- Let, (3, 7, 9, 10) $\in Z^+$ here 3 and 9 are comparable because 3/9 i.e. 3 divides 9. But ∵ and 10 are not comparable because 7 \ 10 and 10 \ 7 that is 7 does not divides 10 or 10 does not divides 7.

(e) Total ordered set

- Let, A be a non-empty set.
- The set A is said to be total ordered set if every element in A is comparable to every other element with respect to any relation on A.

Example

- Let, (A, ≤) is a POSET.

  Where, A = {1, 2, 3, 4, 5}

- Here, A is a total ordered set because every element is comparable with respect to '≤' relation.

6.11.1 Hasse Diagram

- A diagram that is used to describe partial order relation associated with POSET is called as Hasse diagram or ordering diagram.
- Hasse diagram is drawn with an implied upward orientation.
- It is similar to diagraph representation of relation.
- Hasse diagram is drawn by considering comparable and non-comparable elements of POSET.

Procedure to draw Hasse diagram

(i) The elements of POSET represented by dots (·).

(ii) All loops must be removed as relation is reflexive on POSET.

(iii) If two elements p and q are related, then these two elements represented by dots and joined by line or edge.

(iv) If two elements p and q are non-comparable elements, then they lie on the same revel and there is no edge between p and q.

(v) If two elements p and q and p < q in the POSET, then join p and q by edge but point corresponding to p appears lower than points corresponding to q in the diagram.

(vi) If p ≤ q and q ≤ r then p ≤ r. So there is a path p to q and q to r. So, do not join p to r directly. So, delete all the edges that are implied by transitive relation.

Example

- Let, (A, R) be POSET, where

  A = {1, 2, 3, 4, 6, 12}

- And relation R is divisible / relation, then Hasse diagram of given POSET is

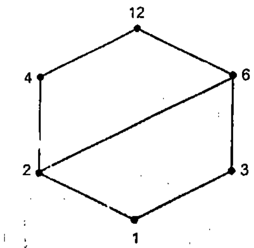


Fig. 6.11.1 : Hasse diagram of POSET

Explanation

(i) Here, in Hasse diagram, all the elements are represented in the form of dots.

(ii) All comparable elements are joined by edges.

(iii) There is no self loop. So no reflexive.

(iv) The comparable elements are as

A = { (1 < 2), (1 < 3), (1 < 4), (1 < 6), (1 < 12), (2 < 4), (2 < 6), (2 < 12), (3 < 6), (3 < 12), (4 < 12), (6 < 12) }

(v) The element "1" is divides all the elements and smallest. Hence, it is placed at the bottom in the diagram.

(vi) Elements 2 and 3 are not divisible to each other hence they are not comparable. So placed at same level, but joined by 1 (both divisible by 1).

(vii) Element 4 and 6 are not divisible. So not comparable not joined by edge directly, but 4 is divisible by 2 and 1 while 6 is divisible by 3, 2 and 1. Hence, 4 is joined by 2 and 6 is joined by 3 and 2.

(viii) 12 is divisible by all other elements. Hence joined by 4 and 6.

6.11.2 Chains and Anti-chains

(a) Chain

Definition : Let (A, ≤) be a POSET. A subset of A is said to be a chain if every pair of elements in the subset is related.

- In any chain, with a finite number of elements, {x_1, x_2, …. x_k} is a chain, there should have sequence like $x_1 \le x_2 \le x_3 \le \ldots \le x_k$.
- Here, x_1 is an element which is less than (related to) every element in the chain and there is an element x_2 which is related to every other element in that chain except x_1.
- In the same way, relation exists in the chain.
- **Length of chain :** The number of elements in the chain is called as the length of the chain.

Note : If set A itself is a chain, then the POSET (A, ≤) is called a totally ordered set or linearly ordered set.

(b) Anti-chain

Definition : Let (A, ≤) be a POSET. [illegible] A is said to be [illegible] elements in a subset [illegible]

Example

- Let, (P(S), ⊆) be a POSET, where S = {a, b, c} P(S) = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}} P(S) is a power set of S.
- The Hasse diagram of (P(S), ⊆) is as,

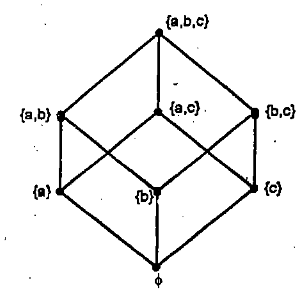


Fig. 6.11.2

From the Hasse diagram,

(i) The chains are

c_1 = { ϕ, {a}, {a, b}, {a, b, c} }

c_2 = { {a}, {a, c}, {a, b, c} }

c_3 = { {b, c}, {a, b, c} }

- In the chain c_1, all the elements are related or comparable likewise in chain c_2 all the elements are related.
- The chains c_1, c_2 and c_3 are subsets of POSET (P(S), ⊆).

(ii) The anti-chains are

$c_1 = \{\{a\}, \{b\}, \{c\}\}$,

$c_2 = \{\{a, b\}, \{a, c\}, \{b, c\}\}$,

In the anti-chain c_1, all the elements are non-comparable or not related.

6.11.3 Elements of POSET

(a) Maximal Element

- Let, (A, ≤) be a POSET. An element a∈A is said to be an maximal element of set A **if there is no element c∈A such that a ≤ c.** It is not unique in POSET.
- In other words, maximal elements are those which are not succeeded by another element.

(b) Minimal Element

- Let, (A, ≤) be a POSET. An element b∈A is said to be an minimal element of set A **if there is no element** c∈A such that c ≤ b. It is not unique in POSET.
- In other words, minimal elements are those which are not preceded by another element.

(c) Upper Bound

Let, a, b be the elements in a POSET (A, ≤). An element c is said to be on upper bound of a and b iff a ≤ c and b ≤ c.

(d) Least Upper Bound (LUB)

- An element c is said to be a least upper bound (lub) of a and b elements. If there is no upper bound of a and b and if there is no other upper bound d of a and b such that d ≤ c.
- It is also known as supremum or join denoted as "∨".
- There is only one least upper bound in POSET. It is unique.

(e) Lower Bound

- Let, a, b be the elements in a POSET (A, ≤). An element "e" is said to be an lower bound of a and b iff e ≤ a and e ≤ b.

(f) Greatest Lower Bound (GLB)

- An element e is said to be greatest lower bound (glb) of a and b iff there is no other lower bound "f" of a and b such that e ≤ f.
- It is also known as infimum or meet.
- It is denoted as "∧".
- There is any only greatest lower bound in POSET. It is unique.

Example

Following diagrams showing Hasse diagram of POSET.

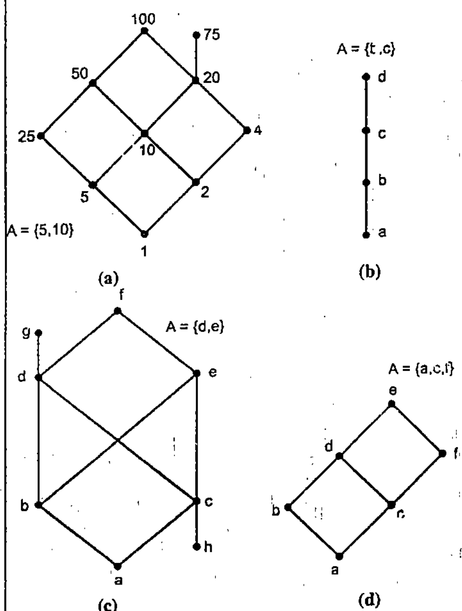


Fig. 6.11.3

(a) Maximal Elements

- **In, Fig. 6.11.3(a) :** 100 and 75 are maximal elements in POSET.
- **In Fig. 6.11.3(b) :** d is the maximal element as it succeeding all elements in the POSET.
- **In Fig. 6.11.3(c) :** f and g are maximal elements in POSET as they are succeeding all the elements.
- **In Fig. 6.11.3(d) :** e is the maximal element. Since e is succeeding all the elements in the POSET.

(b) Minimal Elements

- **In Fig. 6.11.3(a) :** 1 is the minimal element which is preceding all the elements.
- **In Fig. 6.11.3(b) :** a is the minimal element.
- **In Fig. 6.11.3(c) :** a and h are minimal elements. Since, they are preceding all other elements in POSET.
- **In Fig. 6.11.3(d) :** a is minimal element.

(c) Upper Bound (UB)

- **In Fig. 6.11.3(a) :** The upper bounds of A = {5, 10} are 10, 20, 50, 75 and 100.Here, 10 is upper bound of itself.
- **In Fig. 6.11.3(b) :** The upper bounds of {b, c} are c and d. Here, c is upper bound of itself.
- **In Fig. 6.11.3(c) :** The upper bounds of {d, e} is f. Here, e or d is not upper bound of itself because d and e are not related to each other as well as g is not UB because d is related to g but e is not related to g.
- **In Fig. 6.11.3(d) :** The upper bounds of {a, c, f} are e and f. Here, f is upper bound of itself or in other words, a and c are related to f.

(d) Lower Bound (LB)

- **In Fig. 6.11.3(a) :** Lower bounds of {5, 10} are 5 and 1 Here, 5 is lower of itself or 5 is related to 10.
- **In Fig. 6.11.3(b) :** Lower bounds of {b, c} are b and a.
- **In Fig. 6.11.3(c) :** Lower bounds of {d, e} are b, c, a and h.
- **In Fig. 6.11.3(d) :** Lower bounds of {a, c, f} is a.

(e) Least Upper Bound (LUB)

- **In Fig. 6.11.3(a) :** Least upper bound is not available for {5, 10}.
- **In Fig. 6.11.3(b) :** Least upper bound of {b, c} is c.
- **In Fig. 6.11.3(c) :** Least upper bound of {d, e} is not available since g and f are not related and e is not related to g
- **If Fig. 6.11.3(d) :** Least upper bound of {a, c, f} is f. Since a, c and f are related to f.

(f) Greatest Lower Bound (GLB)

- **In Fig. 6.11.3(a) :** The greatest lower bound of {5, 10} is 5.
- **In Fig. 6.11.3(b) :** The greatest lower bound of {b, c} is b.
- **In Fig. 6.11.3(c) :** The greatest lower bound of {d, e} is φ. Since b and c are not related to a and h are not related.
- **In Fig. 6.11.3(d) :** The greatest lower bound of {a, c, f} is a. Since a is related to a, c and f.

Ex. 6.11.1 : Let, A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24} be ordered by the relation x divides y. Show that the relation is partial ordering and draw the Hasse diagram.

Soln. :

Let, A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24}

R be a relation on set A.

R = {(x, y) | x divides y where x, y ∈ A }

R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (1, 12), (1, 18), (1, 24), (2, 2), (2, 4), (2, 6), (2, 8), (2, 12), (2, 18), (2, 24), (3, 3), (3, 6), (3, 9), (3, 12), (3, 18), (3, 24), (4, 8), (4, 12), (4, 24), (5, 5), (6, 6), (6, 12), (6, 18), (6, 24), (7, 7), (8, 8), (8, 24), (9, 9), (9, 18), (12, 12), (12, 24), (18, 18), (24, 24)}

We have, for any x, y, z ∈ A.

→ (x divides x) ∈ R or x R x ⇒ so, R is reflexive.

→ (x | y and y | x) ∈ R for x = y ⇒ So, R is anti-symmetric.

→ (x | y) and (y | z) then x | z which is belongs to R.

So, R is transitive.

R satisfies reflexive, anti-symmetric and transitive properties. Therefore R is partial order relation.

Hasse diagram of POSET (A, /)

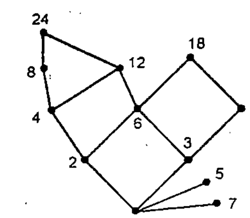


Fig. P. 6.11.1

6.12 LATTICE

> **Definition :** *A lattice is a POSET (L, ≤) in which every pair of elements has a Least Upper Bound (LUB) or join and a Greatest Lower Bound (GLB) or meet.*

Join semi lattice

- In a POSET (L, ≤) if least upper bound or join for every ordered pair of elements is available then POSET is said to be join semi lattice.
- Which means in a POSET, only join is present.

☞ Meet semi lattice

In a POSET (L, ≤) if for every pair of elements has a greatest lower bound or meet is said to be meet semi lattice.

☞ Join

- Let (L, ≤) be a POSET. If a, b ∈ L then least upper bound (*lub*) of a and b is denoted as "**a ∨ b**".
- It is called as "join" of a and b.

  i.e. a ∨ b = *lub* (a, b) ← Join

☞ Meet

- Let, (L, ≤) be a POSET. If a, b ∈ L then greatest lower bound (*glb*) of a and b is denoted as "**a ∧ b**".
- It is called as "meet" of a and b.

  i.e. a ∧ b = *glb* (a, b)
- As we know, Lattice is a mathematical structure with two binary operations join (∨) and meet (∧).
- So, Lattice can be denoted as (L, ∨, ∧)

☞ Example

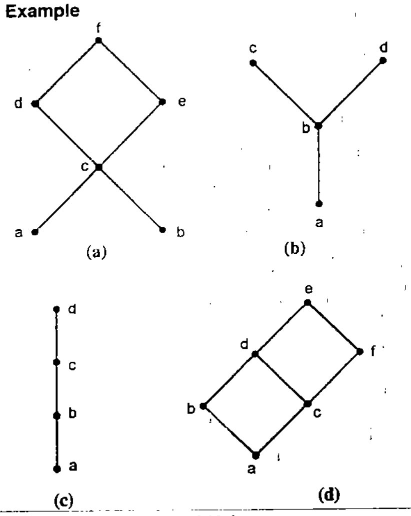


Fig. 6.12.1

- **In Fig. 6.12.1(a)** : For every pair of elements, has least upper bound (join) but there is no greatest lower bound (meet) for every pair of elements. Hence, POSET in Fig. 6.12.1(a) is join semi Lattice.
- **In Fig. 6.12.1(b)** : For every pair of elements, has greatest lower bound (meet) but there is no least upper bound (join) for every pair of elements. Hence, POSET in Fig. 6.12.1(b) is meet semi Lattice.
- **In Fig. 6.12.1(c)** : For every pair of elements, has meet and join. Hence, POSET in Fig. 6.12.1(c) is a Lattice structure.
- **In Fig. 6.12.1(d)** : For every pair of elements, has join and meet. Hence, POSET in Fig. 6.12.1(d) is a Lattice structure.

6.12.1 Properties of Lattice

- Let, (L, ∨, ∧) be a lattice and a, b, c, ∈ L.
- Lattice satisfies the following properties.
- Here ∨ = join and ∧ = meet.

1. Idempotent Property

a ∨ a = a

a ∧ a = a

2. Commutative Property

a ∧ b = b ∧ a

a ∨ b = b ∨ a

3. Associative Property

a ∨ (b ∧ c) = (a ∨ b) ∧ (a ∨ c)

a ∧ (b ∨ c) = (a ∧ b) ∨ (a ∧ c)

4. Absorption Property

a ∧ (a ∨ b) = a

a ∨ (a ∧ b) = a

5. a meet b = a iff a join b = b

i.e. a ∧ b = a iff a ∨ b = b

6.12.2 Types of Lattice

1. Bounded Lattice

- A Lattice L is said to be bounded Lattice if it has greatest element (when maximal element unique) and least element (when minimal element is unique).
- For understanding, greatest element symbolize by I and least element is by O.

2. Sub Lattice

Let, L be a Lattice. A non-empty subset L_1 of L is said to be a sublattice if L_1 itself is a Lattice with respect to operation of L.

3. Complement Lattice

- Let (L, ∧, ∨, O, I) be a bounded Lattice with least element "O" and greatest element "I" and for each element x∈L, the element x′∈L is a complement of x, which satisfies x ∧ x′ = O and x ∨ x′ = I, is said to be complement Lattice.
- For sake of understanding, Lattice "L" is said to be complemented if every element ∀a∈L must have at least one complement.
- As we know, in the set theory, A ∪ A^C = ∪ and A ∩ A^C = ϕ.
- In the same way. In lattice, a ∨ a^C = I (greatest element) and a ∧ a^C = O (least element).
- The complement of I is O and O complement is I. Lets understand the complement of element from following Lattice structure.

☞ Example

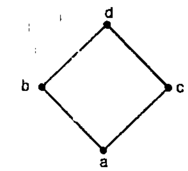


Fig. 6.12.2

Here, a^c = d and d^c = a

a = Least element = O

d = Greatest element = I

- If we consider, complement of b is c and complement of c is b then we must get greatest and least element, in the following way

  b ∨ c = d (Greatest element = I)

  b ∧ c = a (Least element = O)
- Hence, complement of b is c and c complement is b.
- For every element in L, complement is available.
- Hence, Lattice shown in Fig. 6.12.2, is complement Lattice.

4. Distributive Lattice

Let, (L, ∨, ∧) be a Lattice which said to be distributive Lattice if for any element a, b, c ∈ L which must satisfies following properties.

(i) a ∨ (b ∧ c) = (a ∨ b) ∧ (a ∨ c)

(ii) a ∧ (b ∨ c) = (a ∧ b) ∨ (a ∧ c)

Another important properties of distributive Lattice.

(a) a ∧ y = a ∧ x and a ∨ y = a ∨ x together implies that x = y.

(b) If a ∧ x = O and a ∨ x = I, Here O and I are least and greatest element of a lattice a and x are said to be complementary pair.

Theorem

Let, (L, ≤) be a Lattice with the universal bounds O and I then for any a ∈ L.

→ a ∨ I = I

→ a ∧ I = a

→ O ∨ a = a

→ O ∧ a = O

6.12.3 Principle of Duality

- Any statement about lattice involving the join and meet operation and the relations like ≤ (less than equal to), ≥ (greater than equal to) remains true if ∧ (meet) is replaced by ∨ (join), "≤" is replaced by "≥", "O" is replaced by I and "I" is replaced by "O".
- Hence, a ∨ a = a then a ∧ a = a is also true.

  Likewise

  (i) a ∨ (b ∧ C) = a ∧ (b ∨ C) is true.

  (ii) a ∧ (b ∨ I) = a ∨ (b ∧ O) is true.

  (iii) a ∧ b ≤ a = a ∨ b ≥ a is true.

6.12.4 Examples based on POSET and Lattice

Ex. 6.12.1 : Let, A = { 1, 2, 3, 4, 6, 12 } and R = { (x, y) | x divides y }

Show that, R is a partial order Relation and draw its Hasse diagram.

☑ **Soln. :**

Given, A = {1, 2, 3, 4, 6, 12 }

R = { (1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 2), (2, 4), (2, 6), (2, 12), (3, 3), (3, 6), (3, 12), (4, 4), (4, 12), (6, 6), (6, 12), (12, 12) }

(i) Is R Reflexive ? :

R is Reflexive since, for all elements, (x, x) ∈ R

(ii) Is R antisymmetric ? :

R is anti-symmetric since, x divides y and y divides x when x = y that is (1, 1), (2, 2), (3, 3) (4, 4), (6, 6) and (12, 12) ∈ R

(iii) Is R Transitive ? :

Yes, R is transitive, since for every (x, y) ∈ R and (y, z) ∈ R, there is (x, z) ∈ R.

(iv) Hasse diagram of R :

⇒ 1 is Related to all elements.

⇒ 2 and 3 are non-comparable therefore, they are on s..... level.

⇒ 4 and 6 are non-comparable.

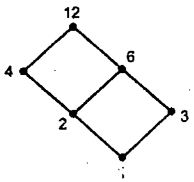


Fig. P. 6.12.1

Ex. 6.12.2 : Let, A = {1, 2, 3, 4 }

Relation R = { (1, 1), (1, 2), (2, 2), (2, 4), (1, 3)
(3, 3), (3, 4), (1, 4), (4, 4) }

Then show that R is partial order and draw it's Hasse diagram.

☑ **Soln. :**

Given : A = { 1, 2, 3, 4}

R = { (1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4) }

(i) Is R Reflexive ?

Every element of A is Related to itself, that is ∀ a ∈ A there is (a, a) ∈ R.
(e.g. (1, 1), (2, 2), (3, 3) and (4, 4) ∈ R)

(ii) Is R anti-symmetric ?

Yes, R is Anti-symmetric because (1, 2) ∈ R but (2, 1) ∉ R, (2, 4) ∈ R but (4, 2) ∉ R, (1, 3) ∈ R but (3, 1) ∉ R.
For other ordered pairs have (a, b) ∈ R and (b, a) ∈ R When a = b.

(iii) Is R Transitive ?

Yes, Relation R is transitive because for every ordered pair if (a, b) ∈ R and (b, c) ∈ R then (a, c) ∈ R.
Hence, Relation R is partial ordered Relation As it satisfied. Reflexive Anti-symmetric and Transitive properties.

(iv) Hasse diagram of Relation R

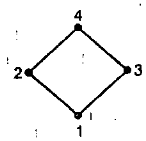


Fig. P.6.12.2

Ex. 6.12.3 : Let, A = {x, y, z} show that (P(A), ⊆) is a POSET and draw it's Hasse diagram.

☑ **Soln. :**

Given that, A = {x, y, z}

Power set of A is P(A)

P(A) = {ϕ, { x }, { y }, { z }, {x, y},{x z},{ y, z} {x,y,z }

⇒ ϕ is subset of all the elements of P(A). So it will lie below all the elements in Hasse diagram.

⇒ ϕ is related to all the elements because ϕ is a subset of all the sets.

⇒ { x }, { y } and { z } are non-comparable elements so that they will lie on same level.

⇒ { x } ⊆ { x, y }, { x } ⊆ { x, z}, { x } ⊆ { x, y, z }
{ y } ⊆ { x, y }, { y } ⊆ { y, z }, { y } ⊆ { x, y, z }
{ z } ⊆ { x, z}, { z } ⊆ { y, z }, { z} ⊆ { x, y, z }

⇒ {x, y}, {x, z} and {y, z} are non-comparable elements so that they will lie on same level above { x }, {y } { z }.

☞ **Hasse diagram**

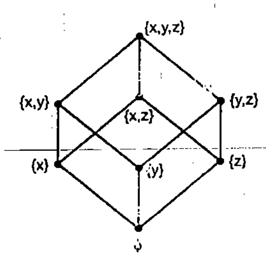
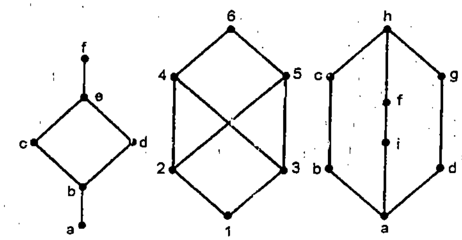


Fig. P. 6.12.3

Ex. 6.12.4 : Determine whether the POSET represented by each of the Hasse diagram are lattice. Justify your answer.



(1) (2) (3)

Fig. P. 6.12.4

☑ **Soln. :**

1. In Fig. P. 6.12.4(1) Hasse diagram, for every pair of elements, has greatest lower bound (meet) and Least upper bound (join).
   So that Fig. P. 6.12.4 (1) Hasse diagram is a Lattice.
2. In Fig. P. 6.12.4(2) Hasse diagram, Least upper bound of 2 and 3 is 4 and 5. LUB should be unique as well as greatest lower bound of 4 and 5 is 2 and 3. GLB should be unique. Hence the given Hasse diagram is not Lattice.
3. In Fig. P. 6.12.4(3) Hasse diagram, for every pair of elements, has least upper bund (join) and greatest lower bound (meet). Hence given diagram is Lattice.

Ex. 6.12.5 : Let, A ={x, y, z}, Power set of A is P(A) = { ϕ, {X }, {y}, {z}, {x, y}, {x, z}, {y, z},{x, y, z} }
Show that (P(A), ⊆) POSET is a lattice.

☑ **Soln. :**

Given that, A = { x, y, z }

The Hasse diagram of (P(A), ⊆) is as follows.

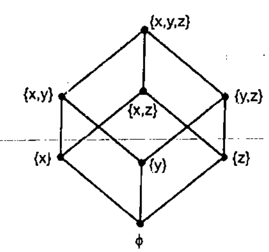


Fig. P. 6.12.5

For every pair of elements of a POSET has joint (∨) and meet (∧) that is For,

(i)	{x} ∨ { ϕ} = {x},	{x} ∧ {ϕ} = {ϕ}
(ii)	{x} ∨ {y} = {x, y},	{x} ∧ {y} = {ϕ}
(iii)	{x, y} ∨ {x, z} = {x, y, z},	{x, y} ∧ {x, z} = { x }
(iv)	{y} ∨ {z} = {y, z},	{y} ∧ {z} = ϕ
(v)	{x} ∨ {z} = {x, z} ,	{x} ∧ {z} = ϕ
(vi)	{x, y} ∨ {y, z} = {x, y, z},	{ (x, y} ∧ {y, z} = {y}

So for every pair of elements there has join and meet. Hence, Hasse diagram of POSET (P(A), ⊆) is a Lattice.

Ex. 6.12.6 : Let, A = {2, 3, 5, 6, 10, 15, 30, 45}

R = { (x, y) | x divides y }

Find

(i) Maximal and minimal elements.
(ii) Maximum (greatest) and minimum (least) element.
(iii) Upper Bound and least upper bound of {6, 10,15}
(iv) Lower bound and greatest lower bound of {6, 10,1 5}
(v) IS POSET (A, /) Lattice or not ?

☑ **Soln. :**

Given that A = {2, 3, 5, 6, 10, 15, 30, 45}

R = { (2, 2), (2, 6), (2, 10), (2, 30), (3, 3), (3, 6), (3, 15), (3, 30), (3, 45), (5, 5), (5, 10), (5, 15), (5, 30), (5, 45), (6, 6), (6, 30), (10, 10),(10, 30), (15, 15), (15, 30), (15, 45), (30, 30), (45, 45) }

☞ **Hasse diagram**

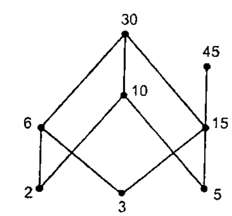


Fig. P. 6.12.6

(i) Maximal Element : 30 and 45 are maximal elements as they are not related to any other elements in a POSET.

– **Minimal Element :** 2, 3 and 5 are minimal elements, as no element is Related with them.

(ii) Maximum Element (greatest) : no maximum element, as 30 and 45 are not related.
Minimum Element (least) : no minimum element, as 2, 3 and 5 are not related.
(There is no single top, single bottom in diagram)

(iii) Upper Bound of {6, 10, 15} is 30

Least Upper Bound of {6, 10, 15} is 30.

(iv) Lower Bound of {6, 10, 15} is ϕ (no lower Bound)

Greatest lower bound of {6, 10, 15} is also ϕ.

Ex. 6.12.7 : Let A = {1, 2, 3, 4, 6, 9, 12} let aRb if a divided b. show that R is POSET, draw Hasse diagram Prove or disprove if it is a lattice.

☑ **Soln. :**

Given : A : {1, 2, 3, 4, 6, 9, 12}

Condition for relation is "a divides b"

So,

Relation R = {(1, 1), (1, 2), (1, 3), (1,4), (1,6), (1, 9),
(1, 12), (2,2), (2, 4), (2, 6), (2,12), (3, 3),
(3,6), (3, 9), (3, 12), (4, 4), (4, 12), (6, 6)
(6, 12), (9, 9), (12, 12)}

For relation R to be POSET, R should satisfies reflexive, anti-symmetric and transitive properties.

(i) **Reflexive property** : From the relation R, it is observed that, every element of A is related to itself that is. aRa or (a, a) ∈ R. Hence, R is reflexive.

(ii) **Anti-symmetric** : For R to be anti-symmetric, if a R b and b R a then a = b.

R is anti-symmetric since. a divides b and b divides a ⇒ a = b.

(iii) **Transitive property** : For R to be transitive, if a R b, and b R c then a R c.

By observing R, it is clear that R is transitive relation.

Hence, R is a POSET.

☞ Hasse diagram of POSET

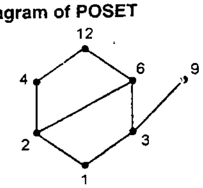


Fig. P. 6.12.7

For POSET to be lattice, POSET should have least upper bound (∨) and greatest lower bound (∧) for every ordered pair of values.

For pair 9 and 12, there is no least upper bound or join that is 9 ∨ 12 = not defined.

Hence, POSET is not lattice.

Ex. 6.12.8 : Show that the set of all divisors of 70 for divisibility relation forms a lattice.

☑ **Soln. :**

The divisors of 70 are as

A : {1, 2, 5, 7, 10, 14, 35, 70}

Let, (A, ≤) be a POSET where "≤" is "divisor of" relation on A.

The Hasse diagram of given POSET is as,

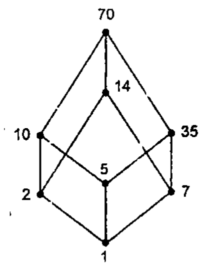


Fig. P. 6.12.8

Here, every pair of elements of A has least upper bound (join) and greatest lower bound (meet).

Hence, it is a lattice.

Ex. 6.12.9 : The following is the Hasse diagram of the poset {(a, b, c, d, e), <}. Is it a lattice? Justify.

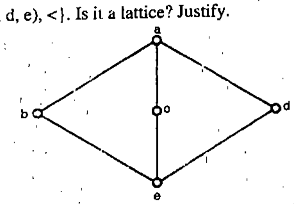


Fig. P. 6.12.9

☑ **Soln. :**

Let, {(a, b, c, d, e), < } be a POSET.

Given Hasse diagram.

From the diagram, it is clear that, the given POSET system is lattice because for every pair of elements, join (∨) and meet (∧) is available.

Lets see in details.

→ e ∨ d = d, e ∧ d = e
→ e ∨ c = c, e ∧ c = e
→ e ∨ b = b, e ∧ b = e
→ b ∨ c = a, b ∧ c = e

→ c ∨ d = a, c ∧ d = e
→ b ∨ a = a, b ∧ a = b
→ d ∨ a = a, d ∧ a = d
→ c ∨ a = a, c ∧ a = c

Hence, Hasse diagram for given POSET is lattice.

Ex. 6.12.10 : Let n be a positive integer S_n, be the set of all divisors of *n*. Let D denote the relation of 'division'. Draw the diagrams of lattices for :

(i) n = 24 (ii) n = 30 (iii) n = 6.

☑ **Soln. :**

Let. S_n = {set of all divisors of n}

D = relation of divisor on S_n

(i) n = 24

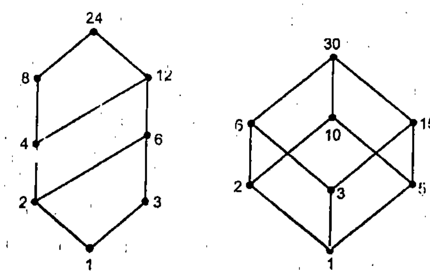
S_{24} = {1, 2, 3, 4, 6, 8, 12, 24}

(ii) n = 30

S_{30} = {1, 2, 3, 5, 6, 10, 15, 30}

(iii) n = 6

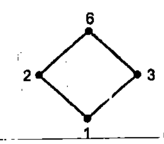
S_6 = {1, 2, 3, 6}



Lattice for n = 24 Lattice for n = 30

(i) (II)

Fig. P. 6.12.10



Lattice for n = 6

Fig. P. 6.12.10(III)

Ex. 6.12.11 : Show that the set of all divisors of 36 forms a lattice.

☑ **Soln. :**

Let, A = {set of all divisors of 36}

A = {1, 2, 3, 4, 6, 9, 12, 18, 36}

R be a relation **"a is a divisor of b"** on A.

R = {(1, 2), (1, 3), (1, 4), (1, 6), (1, 9), (1, 12), (1, 18), (1, 36), (2, 2), (2, 4), (2, 6), (2, 12), (2, 18), (2, 36), (3, 3), (3, 6), (3, 9), (3, 12), (3, 18), (3, 36), (4, 4), (4, 12), (4, 36), (6, 6), (6, 12), (6, 18), (6, 36), (9, 9), (9, 18), (9, 36), (12, 12), (12, 36), (18, 18), (18, 36), (36, 36)}

The Hasse diagram of POSET (A, R) is

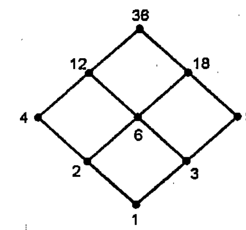


Fig. P. 6.12.11

From the diagram it is clear that, every pair of elements has least upper bound (join) and greatest lower bound (meet). Hence, (A, R) POSET is a lattice.

Ex. 6.12.12 : Let x = {12, 3, 6, 12, 24, 36} x < y if x divides y

Find :

1. Maximal element 2. Minimal element
3. Chain 4. Antichain
5. Is poset lattice ?

☑ **Soln. :**

Let, x = {2, 3, 6, 12, 24, 36}

R = {(x, y) such that x ≤ y if x divides y}

R = {(2, 6), (2, 12), (2, 24), (2, 36), (3, 6), (3, 12), (3, 24), (3, 36), (6, 12), (6, 24), (6, 36), (12, 24), (12, 36)}

The Hasse diagram OR POSET (A, R) is given below

(i) **Maximal elements :** There are two maximal elements th .. 24 and 36.

(ii) **Minimal elements :** 2 and 3 are minimal elements.

(iii) **Chain :** {2, 6, 12, 24}, {3, 6, 12, 36}

(iv) **Antichain :** {2, 3}, {24, 36}

(v) The given POSET is not a lattice because,

→ for 2 and 3 pair, there is no greatest lower bound.

→ for 24 and 36 pair, there is no least upper bound.

That is for (2, 3) no meet or $2 \wedge 3$ = ?

for (24, 36) no join or $(24 \vee 36)$ = ?

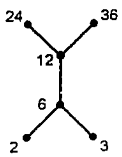


Fig. P. 6.12.12

UEx. 6.12.13 (SPPU(IT) - Q. 4(a), Dec. 18, 6 Marks)
Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24} be ordered by [illegible] Show that the relation is partial ordered and draw Hasse diagram.

☑ **Soln. :**

Given : A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24 }

The relation R, which satisfies the condition.

R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (1, 12), (1, 18), (1, 24), (2, 2), (2, 4), (2, 6), (2, 8), (2, 12), (2, 18), (2, 24), (3, 3), (3, 6), (3, 9), (3, 12), (3, 18), (3, 24), (4, 4), (4, 8), (4, 12), (4, 24), (5, 5), (6, 6), (6, 12), (6, 18), (6, 24), (7, 7), (8, 8), (8, 24) (9, 9) (9, 18), (12, 12), (12, 24), (24, 24), (18, 18) }

1. **Is R Reflexive ?**

Yes because, every element of A is related to itself that is $\forall$ x $\in$ A.

2. **Is R anti-symmetric ?**

Yes because xRy and yRx such that x = y

3. **Is R transitive ?**

Yes because xRy and yRz such that xRz

Hence, R is a partial order Relation.

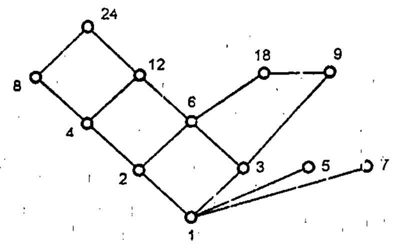


Fig. P. 6.12.13 : Hasse diagram

UEx. 6.12.14 (SPPU(IT) - Q. 3(b), May 18, 6 Marks)
Draw the graph and its equivalent Hasse diagram for divisibility on the set : {1, 2, 3, 6, 12, 24, 36, 48 }.

☑ **Soln. :**

Given :

A = {1, 2, 3, 6, 12, 24, 36, 48}

The relation R which satisfies the divisibility property is as

R = {(1, 1), (1, 2), (1, 3), (1, 6), (1, 12), (1, 24), (1, 36), (1, 48), (2, 2), (2, 6), (2, 12), (2, 24), (2, 36), (2, 48), (3, 3), (3, 6), (3, 12), (3, 24), (3, 36), (3, 48), (6, 6), (6, 12), (6, 24), (6, 36), (6, 48), (12, 12), (12, 24), (12, 36), (12, 48), (24, 24), (24, 28), (36, 36), (48, 48)}

1. **The graphical representation of R**

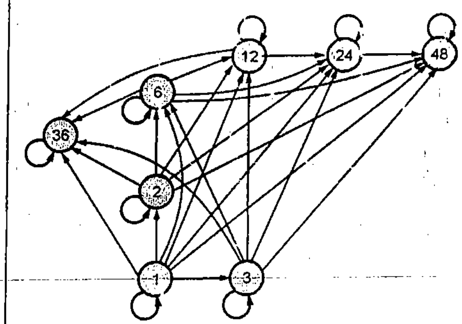


Fig. P. 6.12.14 (a)

By removing all the loops and the edges which shows transitivity.

The Hasse diagram is shown below.

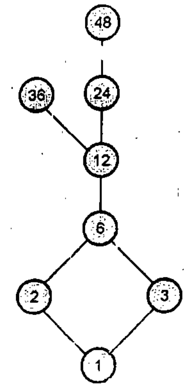


Fig. P. 6.12.14 (b) : Hasse diagram

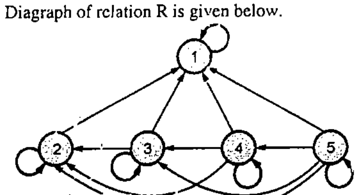
UEx. 6.12.15 (SPPU(IT) - Q. 1(b), Dec. 15, 6 Marks)
Draw the Hasse diagram of relation R on A.
Let A = {1, 2, 3, 4, 5} and [illegible]
R = {(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)}.

☑ **Soln. :**

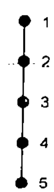
Given : A = {1, 2, 3, 4, 5}

R = {(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)}.

Diagraph of relation R is given below.



Remove self loop and transitive edges. From above graph.



Hasse diagram

6.13 RECURRENCE RELATION

Introduction

Consider the following instructions for generating a sequence.

(i) Start with 1

(ii) Add 4 to get the next term.

If we list the terms of sequence, we get

1, 5, 9, 13, 17, ...(1)

Here, first term is 1 because of instruction 1. The second terms is 5 because of instruction 2 (which says to add 4 to 1, to get the next term 5.)

In the same way, we get the sequence

By following the instructions 1 and 2.

If we denote the sequence (1) in the form of a_1, a_2, a_3, ...we can write the instruction(i)

as $a_1 = 1$ ...(2)

and instruction (ii) as $a_n = a_{n-1} + 4$; $n \geq 1$ ...(3)

Taking n = 1 in equation (3).

$a_2 = a_1 + 4 = 1 + 4 = 5$

similarly, taking n = 3 in equation (3).

$a_3 = a_2 + 4 \Rightarrow a_3 = 5 + 4 = 9$

Equations (2) and (3) are same as instructions (i) and (ii) respectively. Equation (3) is an example of recurrence relation.

> ☐ ***Definition :** A recurrence relation is an equation which is defined in the form of sequence where next term is formed by taking previous terms.*

In other words, it is a sequence where n^{th} value obtained in terms of its predecessors.

For generating recurrence relation. Initial or startup values are called as initial condition is required.

☞ **Example**

(a) Fibonacci numbers or series

Fibonacci series or numbers are defined using the recurrence relation.

$$f_n = f_{n-1} + f_{n-2}$$

with values

$$f_0 = 0$$
$$f_1 = 1$$

In particular,

$$f_2 = f_1 + f_0$$
$$f_3 = f_2 + f_1$$
$$f_4 = f_3 + f_2$$

we can obtain the sequence of Fibonacci series which start with

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,....

(b) Consider an sequence

0!, 1!, 2!, 3!, ...

Represented as recurrence relation

$a_n = n\, a_{n-1}$ where $(n \geq 1)$

conversely, given this relation and the initial condition $a_0 = 1$, one can recover the entire sequence by iteration.

$$\begin{aligned} a_n &= n\,[(n-1)a_{n-2}] \\ &= n\,(n-1)\cdot[(n-2)a_{n-3}] \\ &= \ldots\ldots \\ &= n\,(n-1)\cdot(n-2)\cdot(n-3)\ldots 3.2.1 \\ &= n! \end{aligned}$$

Homogeneous linear Recurrence relations

Definition : *A linear recurrence relation of degree k with constant coefficients is a recurrence relation of the form.*

$$a_n = c_1\cdot a_{n-1} + c_2\, a_{n-2} + \ldots + c_K\, a_{n-K}$$

where $c_1, c_2, c_K \in R$ (Real no.) and $c_K \neq 0$

Here, linear refers to the fact that $a_{n-1}, a_{n-2}, \ldots a_{n-K}$ Present in separate terms and to the first power.

- Homogeneous refers to the degree of each terms is the same (no constant term is there.)
- Constant coefficients $c_1, c_2, \ldots c_K$ are fixed real numbers that do not depend on n.
- Degree k is the expression for a_n contains the previous k terms $a_{n-1}, a_{n-2}, \ldots a_{n-K}$.

Example

Consider the recurrence relation

$x_n + 2x_{n-1} - 15\, x_{n-2} = 0$

for, $n \geq 2$, where $x_0 = 0$ and $x_1 = 1$

Let

$$G(z) = x_0 + x_1 z + x_2 z^2 + x_3 Z^3 + \ldots$$

Be the generating function of the sequence $x_0, x_1, x_2, \ldots$

After multiplying by Z and Z^2 we find

$$\begin{aligned} ZG(z) &= x_0 Z + x_1 Z^2 + x_2 Z^3 + \ldots \\ Z^2 G(z) &= x_0 Z^2 + x_1 Z^3 + \ldots \end{aligned}$$

We get the expression

$G(z) + 2ZG(z) - 15z^2 G(z)$

Because of the recurrence relation most of the terms cancel and so

$$G(z)\,[1 + 2z - 15z^2] = x_0 + (x_1 + 2x_0)z$$

Using the initial values $x_0 = 0$ and $x_1 = 1$ and dividing by $(1 + 2Z - 15z^2)$ reads to the closed form

$$G(z) = \frac{z}{1 + 2Z - 15Z^2}$$

Now, we have, $1 + 2Z - 15z^2 = (1 - 3z)(1 + 5z)$ and ready to find A and B such that

$$\frac{z}{1 + 2z - 15z^2} = \frac{A}{1 - 3z} + \frac{A}{1 + 5z}$$

By solving above statement, we can get

$A = \frac{1}{8}$ and $B = -\frac{1}{8}$

Thus,

$$G(z) = \frac{1}{8}\left[\frac{1}{1-3z}\right] - \frac{1}{8}\left[\frac{1}{1+5z}\right]$$

Using the formula for $(1 - z)^{-1}$, replacing z by 3z for the first term and z by $-5z$ for the second term, we obtain

$$\begin{aligned} G(z) = {} & \frac{1}{8}\,[1 + 3Z + 3^2\cdot z^2 + 3^3\cdot z^3 + \ldots] \\ & - \frac{1}{8}\,[1 - 5Z + 5^2\cdot z^2 - 5^3 z^3 + \ldots] \end{aligned}$$

The coefficient of z^n is G(z) is x_n and from the expression for G(z) just found we have the solution

$$\boxed{x_n = \frac{1}{8}3^n - \frac{1}{8}(-5)^n} \quad \leftarrow \text{Recurrence relation}$$

From this example, It is clear that the solution to a homogeneous recurrence relation is a combination of n^{th} powers.

We can find our example by doing following steps.

- **Step 1 :** Find the value of λ such that $x_n = \lambda^n$ is a solution to the recurrence relation.
- **Step 2 :** Find out what combinations of the solutions found in step (1) satisfy the initial conditions

Here is how this approach works for the out previous example.

$x_n + 2x_{n-1} - 15x_{n-2} = 0$

for $n \geq 2$ with initial conditions $x_0 = 0$ and $x_1 = 1$ substitute $x_n = \lambda^n$ in to recurrence.

After cancellation we see that λ satisfies

$\lambda^2 + 2\lambda - 15 = 0$

i.e. $(\lambda - 3)(\lambda + 5) = 0$, so $\lambda = 3$ and $\lambda = -5$

The next step is to put $x_n = A(3)^n + B(-5)^n$ and find the values of A and B which satisfies the initial conditions.

i.e.

$$\begin{aligned} A + B &= 0 \\ 3A - 5B &= 1 \end{aligned}$$

Therefore, $A = \frac{1}{8}$ and $B = -\frac{1}{8}$

Thus, $x_n = \frac{1}{8}[(3)^n] - \frac{1}{8}[(-5)^n]$

(a) First order linear recurrence relations

Consider an recurrence relation.

$$x_n = a\cdot x_{n-1}$$

If, $G(z) = x_0 + x_1\cdot z + x_2\cdot z^2 + x_3\cdot z^3 + \ldots$

Then,

$$Z\,G(z) = x_0\cdot Z + x_1 z^2 + x_2 z^3 + \ldots$$

and therefore

$(1 - az)\,G(z) = x_0$,

Thus the generating function for G(z) is $x_0\,(1 - az)^{-1}$

and $x_n = x_0\cdot a^n$

We can prove it by mathematical induction.

For n = 1, we have, $x_1 = ax_0$ by assumption

On the other side, if $n \geq 1$ and $x_n = a^n\cdot x_0$

$$\begin{aligned} \text{Then, } x_{n+1} &= ax_n = a\cdot a^n\cdot x_0 \\ &= a^{n+1}\cdot x_0 \end{aligned}$$

(b) Second order linear recurrence relations

$$x_n = ax_{n-1} + b\cdot x_{n-2}, \quad \text{for } n \geq 2$$

If $x'_0, x'_1, x'_2 \ldots$ and $x''_0, x''_1, x''_2 \ldots$ are two sequences which satisfies the recurrence relation and if A and B are constants then.

$x_n = A\cdot x'_n + B\,x''_n$ also satisfies it

Thus linear combination $(Ax'_n + Bx''_n)$ of the two solutions (x'_n) and (x''_n) is also a solution.

General solution to second order Recurrence relations

We introduce the generating function G(z) of $x_0, x_1, x_2 \ldots$

$$\begin{aligned} G(z) &= x_0 + x_1\cdot z + x_2 z^2 + x_3 z^3 + \ldots \\ z\cdot G(z) &= x_0 z + x_1 z^2 + x_2 z^3 + \ldots \\ z^2\cdot G(z) &= x_0 z^2 + x_1 z^3 + \ldots \end{aligned}$$

Therefore,

$G(z) - azG(z) - bz^2 G(z) = x_0 + (x_1 - ax_0)\,z$

We obtain the closed form

$$G(z) = \frac{x_0 + (x_1 - ax_0)\,z}{1 - az - bz^2}$$

Factorize, $1 - az - oz^2$

$(1 - az - bz^2) = (1 - \lambda_1 z)(1 - \lambda_2 z)$

For some numbers λ_1 and λ_2.

Find partial fraction expansion for G(z)

$$G(z) = \frac{A}{1 - \lambda_1 z} + \frac{B}{1 - \lambda_2 z}$$

For some constants A and B, where as if $\lambda_1 = \lambda_2$, it will be of the form

$$G(z) = \frac{A}{1 - \lambda_1 z} + \frac{B}{(1 - \lambda_1 z)^2}$$

- **Case I**

The coefficient of z^n in G(z) is

$$x_n = A\cdot\lambda_1^n + B\,\lambda_2^n$$

- **Case II**

The coefficient of z^n is

$$\begin{aligned} x_n &= A\cdot\lambda_1^n + B\,(n+1)\,\lambda_1^n \\ &= (A + B)\,\lambda_1^n + n\cdot B\lambda_1^n \end{aligned}$$

In order to find the values of A and B, we required to know initial conditions x_0 and x_1.

The characteristic equation

λ_1 and λ_2 are the roots of the characteristic equation.

$\lambda^2 - a\lambda - b = 0$

Theorem

If $1 - az - bz^2 = (1 - \lambda_1 z)(1 - \lambda_2 z)$. then the general solution to the recurrence relation.

$x_n = ax_{n-1} + bx_{n-2}$, for $n \geq 2$ is either

$$\begin{aligned} x_n &= A\cdot\lambda_1^n + B\cdot\lambda_2^n \quad \text{or} \\ x_n &= c\cdot\lambda_1^n + n\cdot D\cdot\lambda_1^n \end{aligned}$$

$\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2$ Respectively

Similarly, for k^{th} order linear recurrence relation.

$x_n = A_1\cdot\lambda_1^n + A_2\cdot\lambda_2^n + \ldots + A_k\,\lambda_k^n$ is the solution for all distinct roots, $\lambda_1, \lambda_2, \ldots \lambda_K$.

if all $\lambda_1 = \lambda_2 = \ldots = \lambda_K = \lambda$

Then, $x_n = [A_1 n^k + A_2 n^{K-1} + A_3\cdot n^{K-3} + \ldots + A_K]\,\lambda^n$

6.13.1 Solved Examples based on the Recursion Relation

UEx. 6.13.1 (SPPU(IT) - Q. 4(a), Dec. 16, 6 Marks)

What is recurrence relation ? Solve the following recurrence relation : $a_n - 4a_{n-1} + 4a_{n-2} = 0$ given that $a_0 =$ [illegible] and $a_1 =$ [illegible]

Soln. :

The characteristic equation is

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$, $\lambda = 2,2$

So, that, The solution of the given recurrence relation is,

$$a_r = (A + B \cdot r)2^r \quad \ldots(1)$$

Here, A and B are constants

To find A and B, putting r = 0 in equation (1)

$$a_0 = (A + 0)\, 2^0$$

$$a_0 = A$$

$$\therefore A = 1$$

Also put r = 1 in equation (1)

$$a_1 = (A + B.1)2^1$$

$$6 = (A + B)2$$

$$3 = A + B$$

$$3 = 1 + B$$

$$\therefore A = 1$$

$$3 - 1 = B$$

$$B = 2$$

Put the value of A and B in equation (1), so

The homogeneous solution of given recurrence relation is

$$a_r = (1 + 2 \cdot r) \cdot 2^r$$

UEx. 6.13.2 (SPPU(IT) - Q. 2(a), May 15, 6 Marks)

Solve the following recurrence relation:

$a_n - 7a_{n-1} + 10a_{n-2} = 0$, $a_0 = 0$, $a_1 = 3$

☑ **Soln. :**

The characteristic equation is

$$1 \cdot \lambda^2 - 7\lambda + 10 = 10$$

$$(\lambda - 5)(\lambda - 2) = 0$$

$$\lambda = 5 \text{ and } \lambda = 2$$

So, the solution of the given recurrence relation is

$$a_n = A \cdot (5)^n + B(2^n) \quad \ldots(1)$$

Here, A and B are constants

To get the value of A and B, put n = 0 in equation (1)

$$a_0 = A + B \Rightarrow 0 = A + B \quad \ldots(2)$$

also put n = 1 in equations (1)

$$a_1 = a \cdot 5^1 + B \cdot 2^1$$

$$\Rightarrow a_1 = 5A + 2B$$

$$\Rightarrow 3 = 5A + 2B \quad \ldots(3)$$

From equation (2) and equation (3)

$$A + B = 0 \quad \ldots(2) \text{ multiply by 2}$$

$$5A + 2B = 3 \quad \ldots(3)$$

$$2A + \cancel{2B} = 0 \quad \ldots(2)$$

$$5A + \cancel{2B} = 3 \quad \ldots(3)$$

$$-3A = -3,$$

A = 1 put in equation (2) then **B = –1**

Hence the homogeneous solution of the given recurrence relations is $a_n = 5^n - 2^n$.

UEx. 6.13.3 (SPPU(IT) - Q. 4(a), May 19, 6 Marks)

Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial condition $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$.

☑ **Soln. :**

The characteristic equation of the given recurrence relations is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6$$

$(\lambda - 1)(\lambda - 2)(\lambda - 3)$, so, $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$

These are the characteristic roots so that, The solutions to this recurrence relation are of the form.

$$a_n = A \cdot 1^n + B \cdot 2^n + C \cdot 3^n \quad \ldots(1)$$

To find the constants A, B and C use initial condition's put n = 0 in equations (1)

$$a_0 = A + B + C \quad \ldots(2)$$

put, n = 1 in equation (1)

$$a_1 = A + 2B + 3C \quad \ldots(3)$$

likewise, put n = 2, in equation (1)

$$a_2 = A + 4B + 9c \quad \ldots(4)$$

Solve these simultaneous equations (2), (3) and (4)

We get, A = 1, B = – 1, and C = 2

Hence, the unique solution to given recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n \quad \ldots\text{Ans.}$$

Ex. 6.13.4 : Find the solution of the recurrence relation. $a_n = a_{n-1} + 2a_{n-2}$ with initial condition $a_0 = 2$ and $a_1 = 7$

☑ **Soln. :**

Given : $a_n = a_{n-1} + 2 \cdot a_{n-2}$

The characteristic equation of the recurrence relation is

$$\lambda^2 - \lambda - 2 = 0$$

By solving above equation, we get the roots as **λ = 2** and **λ = – 1**

A solution to the recurrence relation is

$$a_n = A \cdot 2^n + B \cdot (-1)^n \quad \ldots(1)$$

By considering initial conditions,

$$a_0 = 2 = A \cdot 2^0 + B(-1)^0 = A + B \quad \ldots(2)$$

$$a_1 = 7 = A \cdot 2^1 + B(-1)^1 = A \cdot 2 + B(-1) \quad \ldots(3)$$

Solving these simultaneous equations (2) and (3) we get A = 3 and B = – 1

Hence, the solution to the given recurrence relation is

$$a_n = 3 \cdot 2^n - (-1)^n \Rightarrow a_n = 3\, 2^n + 1^n \quad \ldots\text{Ans.}$$

Ex. 6.13.5 : Find the solution of the recurrence relation.

$a_n = 6a_{n-1} - 9a_{n-2}$ with initial condition $a_0 = 1$ and $a_1 = 6$.

☑ **Soln. :**

Given : $a_n = 6a_{n-1} - 9a_{n-2}$

The characteristic equation of the given relation of degree 3i

$$\lambda^2 - 6\lambda + 9 = 0$$

The root of the above equation is **λ = 3**

The solution to the given relation will be,

$$a_n = A \cdot 3^n + B \cdot n \cdot 3^n$$

For some constants A and B.

Using initial conditions

$$a_0 = 1 = A \cdot 3^0 + B \cdot (0) \cdot 3^0 \Rightarrow A + 0 \Rightarrow A \quad \ldots(1)$$

$$a_1 = 6 = A \cdot 3^1 + B \cdot (1) \cdot 3^1 \Rightarrow 3A + 3B \quad \ldots(2)$$

Solving equations (1) and (2) we get

A = 1 and B = 1

The solution to the recurrence relation is

$$a_n = 1 \cdot 3^n + 1 \cdot n \cdot 3^n$$

$$a_n = 3^n + n \cdot 3^n \quad \ldots\text{Ans.}$$

Ex. 6.13.6 : Find out the solution of the recurrence relation. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial condition $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

☑ **Soln. :**

The characteristic equation of the given recurrence relation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above equation is $(\lambda + 1)^3 = \lambda = -1$

The solution of the recurrence relation is

$$a_n = A_{1,0}(-1)^n + A_{1,1}\, n(-1)^n + A_{1,2}\, n^2(-1)^n$$

To find constants $A_{1,0}$, $A_{1,1}$ and $A_{1,2}$, use initial conditions ,

$$a_0 = 1 = A_{1,0}(-1)^0 + A_{1,1}(0)(-1)^0 + A_{1,2}(0)^0(-1)^0$$

$$= A_{1,0}$$

$$a_1 = -2 = A_{1,0}(-1)^1 + A_{1,1}(1)(-1)^1 + A_{1,2}(1)^1(-1)^1$$

$$= -A_{1,0} - A_{1,1} - A_{1,2}$$

$$a_2 = -1 = A_{1,0}(-1)^2 + A_{1,1}(2)(-1)^2 + A_{1,2}(2)^2(-1)^2$$

$$= A_{1,0} + 2A_{1,1} - 4A_{1,2}$$

By solving above three simultaneous equations, we get

$$A_{1,0} = 1 , \; A_{1,1} = 3 , \; A_{1,2} = -2$$

Hence the solution of the given recurrence relation is,

$$a_n = 1 \cdot (-1)^n + 3 \cdot n(-1)^n - 2 \cdot n^2(-1)^n$$

$$a_n = (1 + 3n - 2n^2)(-1)^n \quad \ldots\ldots\ldots$$

Chapter Ends ...

□□□

UNIT IV

CHAPTER 7 Functions

Syllabus

Functions – Functions, Composition of Functions, Invertible Functions. Pigeonhole Principle, Discrete Numeric Functions.

7.1 INTRODUCTION

- In the previous chapter, we have studied relation which tells about the relationships between elements of one or many sets.
- In this chapter we will learn, special type of relation called as function.
- Functions are like a assignment of elements to other. We have to take care while assignment of elements.
- Consider an instance, that each student in a second year class is assigned a letter grade from the set {A, B, C, D, E}.
- The assignments of grades to name of the student is depicted in following Fig. 7.1.1.

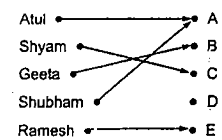


Fig. 7.1.1 : Assignment of grades to second year class

- In the Fig. 7.1.1, grade A is for Atul and Shubham, B for Geeta, C for Shyam, E for Ramesh. However, grade D is not assigned to any of the student.
- Function is a kind of input-output relation.
- Function play an important role in computer science and in mathematics.
- Many of the technical concepts of computer science are friendly stated by the use of functions.
- If we want to find the time required by computer to solve problems, then function is used.
- In this chapter, we will discuses the concept of infinite sets, types of functions, pigeonhole principle.

7.2 FUNCTION

❑ *Definition 1 : Let, A and B be non-empty sets. A function from set A to set B is denoted as $f : A \to B$ (read as f : a function from A into B) or (f maps A to B) and defined as assignment of exactly one element of B to each element of A.*

- A function is also relation from A to B which is subset of $A \times B$. i.e $f \subseteq A \times B$.
- A function $f : A \to B$ is a relation from A to B such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.
- In other way, function is a relation in which no two ordered pair have same first element.

❑ *Definition 2 : A function f which maps the elements of A into B if every element of set A has one & only one image in set B.*

- Functions are sometimes also called as "**mappings**" or "**transformations**".

❑ *Definition 3 : A function is a relation which maps the each element of first set into unique or distinct element of another set.*

Example 1 :

Let, f_1 be a function which maps set A to B.

Where, $A = \{a, b, c\}$ $B = \{d, e, f\}$

$f_1 = \{(a, e), (b, d), (c, e)\}$

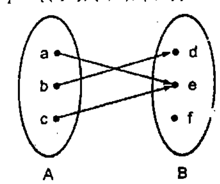


Fig. 7.2.1 : Function

- Here, f_1 is a function because, no two ordered pairs in f_1 have same first element and assignment of exactly one element of B to each element of A.

Example 2 :

$f_2 : A \to B$ $A = \{a, b, c\}$, $B = \{1, 2, 3\}$

$f = \{(a, 1), (a, 3), (b, 2), (c, 3)\}$

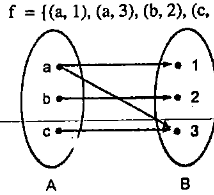


Fig. 7.2.2 : $F_2 : A \to B$

Here, f_2 is **not** a function, as two elements 1 and 3 assigned to the same element $a \in A$.

In other words, two order pairs have same first element.

Example 3 :

$f_3 : A \rightarrow B$ where, $A = \{a, b, c\}$, $B = \{1, 2, 3\}$

$f_3 : \{(a, 1), (b, 2)\}$

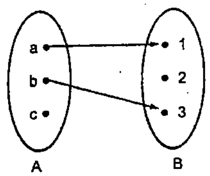


Fig. 7.2.3

Here, f_3 is not a function. Because no element of B is assigned to the element $c \in A$.

7.2.1 Basic Terminologies

(1) **Image :** Let $f : A \rightarrow B$ be a function and $(a, b) \in f$ such that $f(a) = b$ then $b \in B$ is called the image of element $a \in A$.

(2) **Pre-image :** Let, $f : A \rightarrow B$ be a function and $(a, b) \in f$ such that $f(a) = b$ then element $a \in A$ is called pre-image of element $b \in B$.

(3) **Domain of function :** Let, $f : A \rightarrow B$ be a function. The set A is known as domain set of function f.

(4) **Co-domain of function :** The set B is known as co-domain set of function f.

(5) **Range of function :** Let, $f : A \rightarrow B$ be a function. The range of f is the set of all images of elements of A. That is set of elements of B known as range of function.

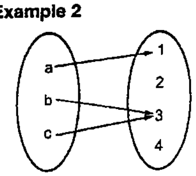
Example 1

Let, f be a function on set $A = \{1, 2, 3, 4\}$

$B = \{a, b, c, d\}$ and $f = \{(1, a), (2, b), (3, c), (4, d)\}$

- Here, "a" is a image of 1, "b" is image of 2, "C" is a image of 3 and "d" is a image of 4.
- Similarly, "1" is pre-image of a, "2" is pre-image of b, "3" is pre-image of C and "4" is pre-image of d.
- Domain : Set of elements of A is domain so, domain of function is {1, 2, 3, 4}
- Co-domain : B = {a. b, c, d}
- Range of function f is {a, b, c, d}

Example 2



domain = { a,b,c }
Co-domain = { 1,2,3,4 }
Range = { 1,3, }

Fig. 7.2.4

7.2.2 Properties of Function

(1) **Addition :** Let, f_1 and f_2 be functions from A to B, then $f_1 + f_2$ is defined as

$$f_1 + f_2 (x) = f_1(x) + f_2 (x)$$

(2) **Multiplication :** Let f_1 and f_2 be functions from A to B, then $f_1 \cdot f_2$ are defined as

$$f_1 \cdot f_2(x) = f_1(x) \cdot f_2(x)$$

7.2.3 Types of Function

(1) **Partial functions :** Let, A and B be two non-empty sets. A partial function $f : A' \rightarrow B$ for some subset A' of A. where $A' \subset A$ i.e. A' is a proper subset of A. For any element $x \in A - A'$, the value of f(x) is said to be undefined.

Example :

Let, $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e\}$

F be a function from A to B i.e. $f : A' \rightarrow B$

Here, f is a partial function which maps elements of A' (proper subset of A) to elements of B.

$A - A' = \{4, 5\}$ are undefined.

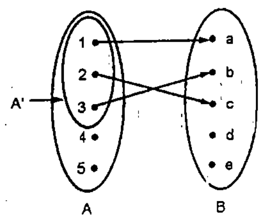


Fig. 7.2.5 : $f : A' \rightarrow B$

- The function which is not partial is called as total function.

(2) **Identity function :** Let, A be a non-empty set and f be a function $f : A \rightarrow A$ is said to be the identity function if each element of set A has an image on itself that is $f(a) = a$, $\forall a \in A$. It is denoted by I.

Example : Let, $A = \{a, b, c, d\}$, $f : A \rightarrow A$

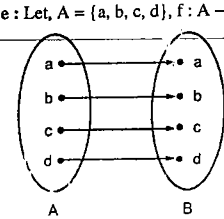


Fig. 7.2.6 : $f : A \rightarrow A$ (Identity function)

(3) **Equality of two functions :** Let, f be a function from A to B ($f : A \rightarrow B$) and g be a function from A to B ($g : A \rightarrow B$) are equal when they have same domain, have the same co-domain and map elements of their common domain to the same elements in their common co-domain.

Two functions are inequal when there is a change in either the domain or the co-domain of a function.

Example : Let, $f = \{(3, x), (4, y), (5, z)\}$, $g = \{(3, x), (4, y), (5, z)\}$

Here, f and g are equal functions

(4) **Constant function**

A function $f : A \rightarrow B$ is said to be constant function, if every element of A has the same image in B.

Thus, f be a function $f : A \rightarrow B$, $f(x) = C$ for $\forall x \in A$ then $c \in B$ is a constant function.

Example : Let, $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$

$f = \{(1, b), (2, b), (3, b), (4, b)\}$

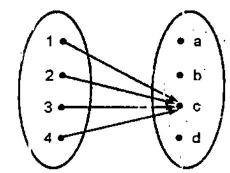


Fig. 7.2.7 : $f : A \rightarrow B$

Here, f is a constant function since, every domain element has same image that is "C".

(5) **Composite function**

- A composite function is a function whose mapping is depends upon another function.
- Composite function is created when one function is substituted into another function.

> **Definition :** *Let, g be a function from the set A to B. Let, f be a function from set B to C. The composition of functions f and g, denoted by fog(x) or f (g(x)) or fg(x) (read as "f of g of x").*

- To find fog(x), we have to apply the function g to x to obtain g(x) and then we apply the function f to the result g(x) to obtain fog(x) = f(g(x)).

> **Note :** Composition of fog is defined only when the range of g is a subset of the domain of f.

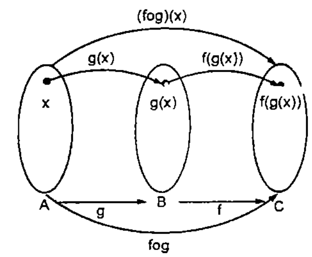


Fig. 7.2.8 : Composition of the functions

Example

Let, $f(x) = 3x + 2$ and $g(x) = x + 5$

The composition of f and g is fog(x) or f(g(x)) is :

$$f(g(x)) = f(x + 5) = 3(x + 5) + 2$$
$$= 3x + 15 + 2 = 3x + 17$$

The composition g and f is g o f(x) or g (f(x))

$$g(f(x)) = g(3x + 2) = (3x + 2) + 5 = 3x + 7$$

Ex. 7.2.1 : Let, $X = \{1, 2, 3\}$ and $Y = \{4, 5\}$. Find whether the following subsets of $X \times Y$ are functions from X to Y or not.

(i) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$

(ii) $g = \{(1, 4), (2, 4), (3, 4)\}$

(iii) $h = \{(1, 4), (2, 5), (3, 5)\}$

(iv) $k = \{(1, 4), (2, 5)\}$

☑ **Soln. :**

(i) f is not a function because two ordered pairs have same first element.

(ii) g is a function because each element of set X is uniquely mapped into elements of set Y.

(iii) Clearly "h" is a function since each element of set X is exactly mapped into elements of set Y.

(iv) k is not a function since element 3 of set X is not mapped into elements of set Y.

Ex. 7.2.2 : Let, f and g are Function defined by

$f(x) = 2x + 1$, $g(x) = x^2 - 2$. Find

(a) g o f(4) and fog(4) (b) g o f (a + 2) and fog (a + 2)

(c) fog (5) (d) g o f (a + 3)

☑ **Soln. :**

(a)

$$g \circ f(4) = g(f(4)) = g(2(4) + 1) = g(9)$$
$$= 9^2 - 2 = 81 - 2 = 79$$
$$f \circ g(4) = f(g(4)) = f(4^2 - 2) = f(14)$$

$= 2(14) + 1 = 28 + 1 = 29$

(b) $g o f (a+2) = g(f(a+2)) = g(2(a+1)+1)$
$= g(2a+5) = (2a+5)^2 - 2$
$= 4a^2 + 20a + 23$

$f o g (a+2) = f(g(a+2)) = f((a+2)^2 - 2)$
$= f(a^2 + 4a + 2)$
$= 2(a^2 + 4a + 2) + 1$
$= 2a^2 + 8a + 4 + 1$
$= 2a^2 + 8a + 5$

(c) $f o g (5) = f(g(5)) = f(5^2 - 2)$
$= f(25 - 2) = f(23)$
$= 2(23) + 1 = 46 + 1 = 47$

(d) $g o f (a+3) = f(g(a+3)) = f(2(a+3)+1)$
$= f(2a+6+1) = f(2a+7)$
$= (2a+7)^2 - 2$
$= 4a^2 + 28a + 47$

Ex. 7.2.3 : Let $A = \{1, 2, 3\}$ and f_1 and f_2 are functions from A to B given by $f_1 = \{(1, 2), (2, 3), (3, 1)\}$ and $f_2 = \{(1, 2) (2, 1), (3, 3)\}$

Compute $f_1 o f_2$ and $f_2 o f_1$.

☑ **Soln. :**

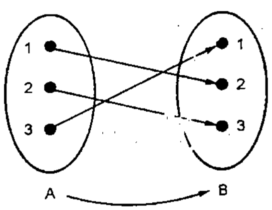
Given : $A = \{1, 2, 3\}$

f_1 and f_2 be the functions from A to B.

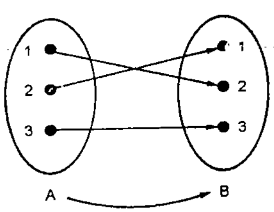
$f_1 = \{(1, 2), (2, 3), (3, 1)\}$

and $f_2 = \{(1, 2), (2, 1), (3, 3)\}$

(i) $f_1 o f_2$

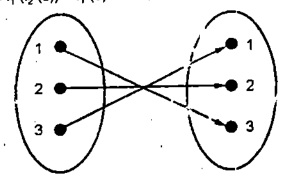


f_1



f_2

$$\left.\begin{aligned} f_1 o f_2 (1) &= f_1 (f_2 (1)) = f_1 (2) = 3 \\ f_1 o f_2 (2) &= f_1 (f_2 (2)) = f_1 (1) = 2 \\ f_1 o f_2 (3) &= f_1 (f_2 (3)) = f_1 (3) = 1 \end{aligned}\right\} f_1 o f_2 = \{(1, 3), (2, 2), (3, 1)\}$$



$f_1 o f_2$

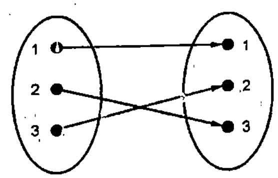
(ii) $f_2 o f_1$

$f_2 o f_1 (1) = f_2 (f_1 (1)) = f_2 (2) = 1$

$f_2 o f_1 (2) = f_2 (f_1 (2)) = f_2 (3) = 3$

$f_2 o f_1 (3) = f_2 (f_1 (3)) = f_2 (1) = 2$

$f_2 o f_1 = \{(1, 1), (2, 3), (3, 2)\}$



$f_2 o f_1$

Ex. 7.2.4 : Let, $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$.

Determine whether the relation R from A to B is a function. Justify. If it is function give the range :

(i) $R = \{(a, 1), (b, 2), (c, 1), (d, 2)\}$,

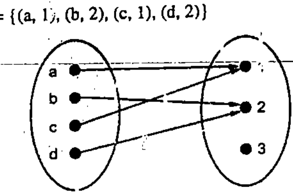
(ii) $R = \{(a, 1), (b, 2), (a, 2), (c, 1), (d, 2)\}$.

☑ **Soln. :**

Let, $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$

Let, R be a relation from A to B.

(i) $R = \{(a, 1), (b, 2), (c, 1), (d, 2)\}$

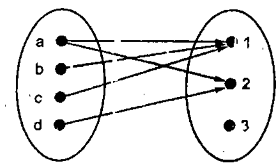


Yes, R is a function from A to B.

Because, each and every element of set A exactly mapped into distinct element of set B.

Range of function $= \{1, 2\}$

(ii) $R = \{(a, 1), (b, 2), (a, 2), (c, 1), (d, 2)\}$



Here, R is not a function, because, element a from set A is not uniquely mapped into element of B.

In other way, two ordered pairs have same first element.

Ex. 7.2.5 : Let $f(x) = x + 2$, $g(x) = x - 2$

$h(x) = 3x$ for xR where R is the set of real number,

find :

(i) g o f (ii) f o g (iii) f o f (iv) h o g

(v) g o g (vi) f o h (vii) h o f (viii) f o h o g

☑ **Soln. :**

Let, $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$

$x \in R$ where R is the set of real numbers.

(i) $g o f (x) = g (f (x)) = g (x + 2) = (x + 2) - 2 = x$

(ii) $f o g (x) = f (g (x)) = f (x - 2) = (x - 2) + 2 = x$

(iii) $f o f (x) = f (f (x)) = f (x + 2) = (x + 2) + 2 = x + 4$

(iv) $h o g (x) = h (g (x)) = h (x - 2) = 3 (x - 2) = 3x - 6$

(v) $g o g (x) = g (g (x)) = g (x - 2) = (x - 2) - 2 = x - 4$

(vi) $f o h (x) = f (h (x)) = f (3x) = (3x) + 2 = 3x + 2$

(vii) $h o f (x) = h (f (x)) = h (x + 2) = 3 (x + 2) = 3x + 6$

(viii) $f o h o g (x) = f (h (g (x))) = f (h (x - 2)) = f (3 (x - 2))$
$= f (3x - 6) = (3x - 6) + 2 = 3x - 4$

UEx. 7.2.6 (Q. 4(A), May 17, 6 Marks)

Functions f, g and h are defined on a set X = {1, 2, 3} as

f = {(1, 3), (2, 1), (3, 2)}, g = {(1, 2), (2, 3), (3, 1)},

h = {(1, 2), (2, 1), (3, 3)}

(i) Find fog and gof. Are they equal?

(ii) Find fogoh and fohog.

☑ **Soln. :**

Given : $X = \{1, 2, 3\}$

Function

$f = \{(1, 3), (2, 1), (3, 2)\}$,

$g = \{(1, 2), (2, 3), (3, 1)\}$;

$h = \{(1, 2), (2, 1), (3, 3)\}$.

(i)

$$\left.\begin{aligned} fog (1) &= f(g(1)) = f(2) = 1 \\ fog (2) &= f(g (2)) = f(3) = 2 \\ fog (3) &= f(g (3)) = f(1) = 3 \end{aligned}\right\} fog(x) = \{(1,1),(2,2) (3,3)\}$$

$$\left.\begin{aligned} gof (1) &= g(f(1)) = g(3) = 1 \\ gof (2) &= g(f (2)) = g(1) = 2 \\ gof (3) &= g(f (3)) = g(2) = 3 \end{aligned}\right\} gof (x) = \{(1,1),(2,2) (3,3)\}$$

fog and gof are equals.

(ii) $fogoh (1) = f(g(h(1))) = f (g(2)) = f(3) = 2$

$fogoh (2) = f(g(h(2))) = f (g(1)) = f(2) = 1$

$fogoh (3) = f(g(h(3))) = f (g(3)) = f(1) = 3$

$\boxed{fogoh(x) = \{(1, 2), (2, 1), (3, 3)\}}$

$fohog (1) = f(h(g(1))) = f (h(2)) = f(1) = 3$

$fohog (2) = f(h(g(2))) = f (h(3)) = f(3) = 2$

$fohog (3) = f(h(g(3))) = f (h(1)) = f(2) = 1$

$\boxed{fohog(x) = \{(1, 3), (2, 2), (3, 1)\}}$

7.3 SPECIAL TYPE OF FUNCTIONS

There are mainly three types of special functions.

(a) one to one function (Injective)

- A function f : A → B is said to be one to one function if for each element of A there is a distinct element of B. It is also known as injective.
- f is defined as f : A → B such that $f(a_1) = f(a_2)$ if $a_1 \in A$ and $a_2 \in B$.
- In general, one to one functions maps distinct element of set A into distinct elements of B.

☞ **Example**

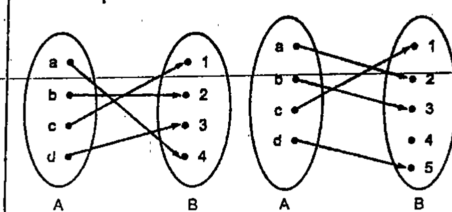


Fig. 7.3.1

Note : For sake of understanding, In injective function set B element can not mapped to many A elements.

(b) onto function (surjective)

- A function $f : A \rightarrow B$ is said to be onto function if each element of B has at least one pre-image in A.
- In onto function, there is no requirement of distinct mapping, but each and every element of B mapped to at least one element of A.
- It is also known as surjective function.

Example

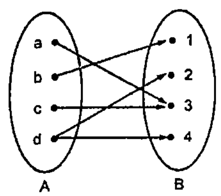


Fig. 7.3.2 : $f : A \rightarrow B$ (surjective function)

(c) One to one correspondence function (Bijective)

- A function $f : A \rightarrow B$ is said to be one to one correspondence if the function f is both one to one and onto function.
- Which means, in one to one correspondence function, each element of A distinctly mapped into distinct element of B. and every element of B has a pre-image in A.

Example

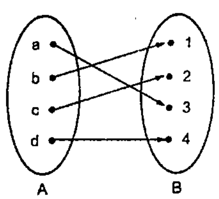


Fig. 7.3.3 : $f : A \rightarrow B$ (Bijective function)

(d) Inverse function

- Let, $f : A \rightarrow B$ be a function from A to B. f be a bijective function then $f^{-1} : B \rightarrow A$ is called the inverse function (mapping) of f and defined as $f(b^{-1}) = a$ iff $f(a) = b$
- If function f is not one to one correspondence i.e. bijective function, then we can not define an inverse function of f.
- In inverse function, $f^{-1} \neq \frac{1}{f}$
- A Bijective function is called invertible because we can define an inverse of the function.

Example

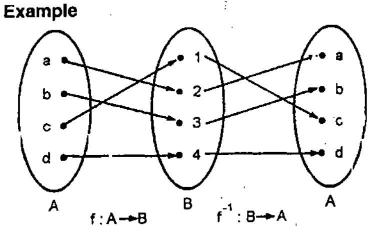


Fig. 7.3.4

(e) Strictly increasing and strictly decreasing function

A function $f : A \rightarrow B$ is strictly increasing if $f(x) < f(y)$ while a function is strictly decreasing if $f(x) > f(y)$.

(f) Increasing and decreasing function

A function $f : A \rightarrow B$ is increasing if $f(x) \leq f(y)$ and a function is decreasing if $f(x) \geq f(y)$.

Properties of composition of function

(i) $fog(x) \neq gof(x)$

(ii) $f^{-1}of = f^{-1}(f(a)) = f^{-1}(b) = a$

(iii) $f \circ f^{-1} = f(f^{-1}(b)) = f(a) = b$

(iv) $(f^{-1})^{-1} = f$

(v) $(fog)^{-1} = g^{-1} \circ f^{-1}$

(vi) If f and g both are one to one function (injective) then fog is also one to one function.

(vii) If f and g both are onto function (surjective) then fog also onto function.

(viii) If f and fog are onto function, then g does not mean to onto function.

(ix) If f and fog are one to one function, then g is also one to one function.

Important deduction about functions

(i) A one to one function never assigns the same value to two different domain elements.

(ii) For onto function, range and co-domain are same.

(iii) If a function is not bijective, then inverse function of f can not be defined.

(iv) A function is one to one if it is strictly increasing or strictly decreasing.

Some examples to understand functions and it's types

(a) One to one, but not onto

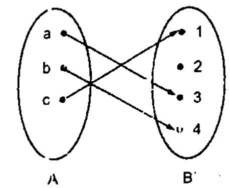


Fig. 7.3.5

(b) Onto, but not one to one

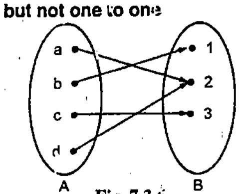


Fig. 7.3.6

(c) One to one and onto

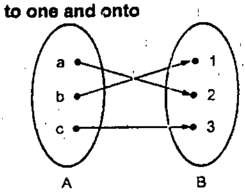


Fig. 7.3.7

(d) Neither one to one nor onto

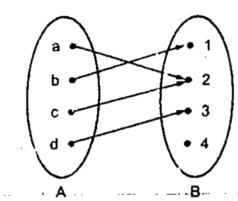


Fig. 7.3.8

(e) Not a function

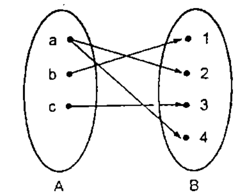


Fig. 7.3.9

(f) Function

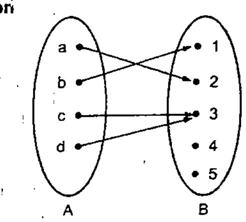


Fig. 7.3.10

Ex. 7.3.1 : What are relations and functions. Given a Relation R = {(1, 4), (2, 2), (3, 10), (4, 8)} (5, 6)} and check whether the following relations R_1, R_2, R_3 & R_4 is a function or not.

$R_1 = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)\}$

$R_2 = \{(1, 2), (2, 4), (2, 10), (3, 8), (4, 6), (5, 4)\}$

$R_3 = \{(1, 6), (2, 2), (4, 4), (5, 10)\}$

$R_4 = \{(1, 6), (2, 2), (3, 2), (4, 4), (5, 10)\}$

☑ **Soln. :**

For relations and functions, Please see theory

Given : R = {(1, 4), (2, 2), (3, 10), (4, 8), (5, 6)}

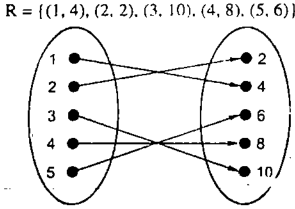


Fig. P. 7.3.1(a)

Here, Relation R is a function. Since, each element of domain is mapped to co-domain.

(i) $R_1 = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4)\}$

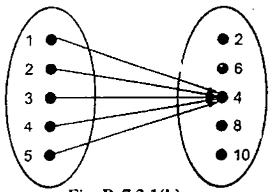


Fig. P. 7.3.1(b)

R_1 is a function and it is constant function since, every element of domain is uniquely mapped into co-domain.

(ii) $R_2 = \{(1, 2), (2, 4), (2, 10), (3, 8), (4, 6), (5, 4)\}$

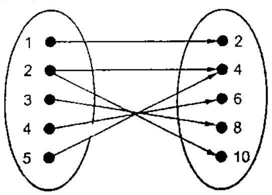


Fig. P. 7.3.1(c)

Here, R_2 is not a function, since, element 2 is not uniquely mapped into co-domain.

In other way, R_2 is not a function because two ordered pairs have same first element.

(iii) $R_3 = \{(1, 6), (2, 2), (4, 4), (5, 10)\}$

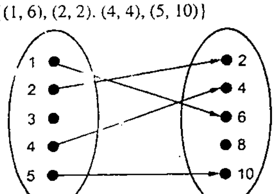


Fig. P. 7.3.1(d)

R_3 is not a function because 3 is not mapped into any of the element of co-domain.

(iv) $R_4 = \{(1, 6), (2, 2), (3, 2), (4, 4), (5, 10)\}$

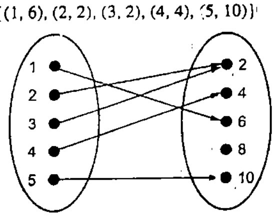


Fig. P. 7.3.1(e)

It is a function, since every element of domain is mapped into co-domain.

Ex. 7.3.2 : Give examples of functions of the following types by diagrams

(a) Injective function but not surjective.

(b) Surjective but not Injective.

(c) Neither Injective nor surjective.

(d) Injective as well as surjective

(e) Into Function.

(f) Inverse Function.

☑ **Soln. :**

(a) Injective but not surjective

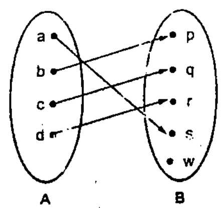


Fig. P. 7.3.2

(b) Surjective but not Injective

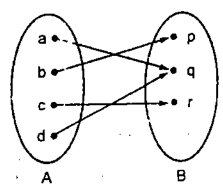


Fig. P. 7.3.2(a)

(c) Neither Injective nor Surjective

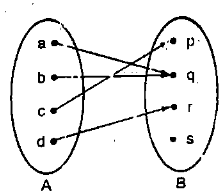


Fig. P. 7.3.2(b)

(d) Injective as well as Surjective

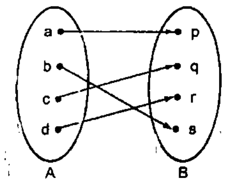


Fig. P. 7.3.2(c)

(e) Into Function :

A Function $f : A \rightarrow B$ is an Into function if there exists an element in B having no preimage in A.

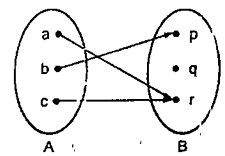


Fig. P. 7.3.2(d)

(f) Inverse function

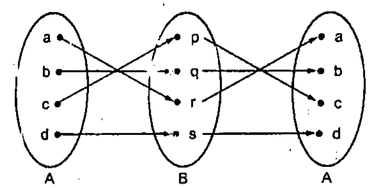


Fig. P. 7.3.2(e)

Ex. 7.3.3 : Let, $A = \{a, b, c, d, e\}$, $g : A \rightarrow A$ be a function as shown below

Find g o g, g o(g o g). Determine whether each is one-to-one or onto Function

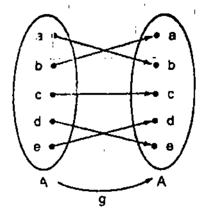


Fig. P. 7.3.3

☑ **Soln. :**

(i) gog

$$g(g(a)) = g(b) = a,$$
$$g(g(b)) = g(a) = b$$
$$g(g(c)) = g(c) = c$$
$$g(g(d)) = g(e) = d$$
$$g(g(e)) = g(d) = e$$

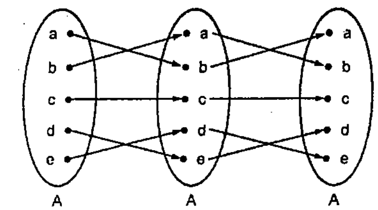


Fig. P. 7.3.3(a) : g o g (one to one and onto function)

(ii) go (gog)

$$g(g \circ g(a)) = g(a) = b$$
$$g(g \circ g(b)) = g(b) = a$$
$$g(g \circ g(c)) = g(c) = c$$
$$g(g \circ g(d)) = g(d) = e$$
$$g(g \circ g(e)) = g(e) = d$$

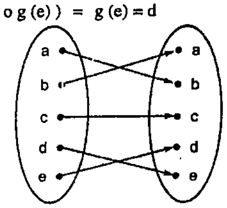


Fig. P. 7.3.3(b) : g o (g o g) (one to one and onto function)

Ex. 7.3.4 : Determine if each is a function. If yes, is it injective, surjective, bijective ?

(i) Each person on the earth is assigned a number which corresponds to his age.

(ii) Each student is assigned a teacher.

(iii) Each country has assigned its capital.

(iv) To each country assign the number of people living in the country.

(v) To each book written by only one author, assign the author.

(vi) To each country having prime minister, assign the prime minister.

☑ **Soln. :**

(i) It is a function. Since each person has unique age.

- It is not injective function, because two person's may have same age.
- There is no person whose age is 250 years. Hence Function will not be surjective. It is not bijective function.

(ii) It is a function, each student is not exactly mapped to teacher. So, it is not injective function, as many students may be assigned to the same teacher. It is surjective function as each teacher will have some student assigned to him. It is not bijective.

(iii) It is a function : It is injective function as every country has unique capital. Function is surjective as each capital city has mapped to it's country. Hence function is bijective.

(iv) It is a function : It is a Injective function as every country has different population.

- It is not surjective function, as each number represent population may not have a valid count of the people in the country. It is not bijective.

(v) It is a function. It is not injective as one author can have many books. It is a surjective function, as every author has written a book. It is not bijective.

(vi) It is a function

- It is both Injective and Surjective function as each country has unique prime minister and each prime minister is associated with a country.
- Hence it is bijective function.

Ex. 7.3.5 : Consider $f : R \to R$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

☑ **Soln. :**

Given that, $f : R \to R$

$$\underset{\text{domain}}{R} \to \underset{\text{co-domain}}{R}$$

$$f(x) = 4x + 3 \qquad \ldots(1)$$

We know that, a function f is invertible iff f is one to one and onto function. Now check f is one to one and onto.

(i) Is f one – one ?

Let, x_1 and $x_2 \in$ domain R such that $f(x_1) = f(x_2)$

$\therefore$ By Equation (1),

$$4x_1 + 3 = 4x_2 + 3 \Rightarrow 4x_1 = 4x_2$$

Dividing by 4 to both sides, we get $x_1 = x_2$

Hence, **f is one to one function**

(ii) **Is f onto ?**

Let, $y \in$ co-domain R

Let, $y = f(x) = 4x + 3 \qquad \ldots$by (1)

Here, find x in terms of y

$$y = 4x + 3 \Rightarrow y - 3 = 4x \Rightarrow \boxed{x = \frac{y-3}{4}} \qquad \ldots(2)$$

From Equation (1),

$$f(x) = 4x + 3 = 4\left(\frac{y-3}{4}\right) + 3$$

$$\Rightarrow \quad y - 3 + 3 = y$$

that is every y is f(x). Which means every co-domain element is a range. Which means, every y is get mapped into $x \in R$ (domain). Hence f is onto function.

So, f is both one to one and onto function, that is f is bijective function.

We know that, every bijective function has inverse. Hence f^{-1} is exists.

$$\Rightarrow f(x) = y \Rightarrow f^{-1}(y) = x \text{ that is } \boxed{f^{-1}(y) = \frac{y-3}{4}}$$

…by Equation (2)

Ex. 7.3.6 : Let, set A = B be the set of Real numbers.

$f : A \to B$ given by $f(x) = 2x^3 - 1$.

$g : B \to A$ given by $g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$

Show that, f is a bijective function and g is also bijective function.

☑ **Soln. :**

(a) Given that, $f : A \to B$

A = domain ; B = co-domain

$$f(x) = 2x^3 - 1 \qquad \ldots(1)$$

We know that, for a function f is bijective function iff f is one to one (Injective) and onto (surjective).

Now, check

(i) **Is f Injective ? :**

Let x_1 and $x_2 \in A$ such that $f(x_1) = f(x_2)$

$\therefore$ By Equation (1),

$$\Rightarrow 2x_1^3 - 1 = 2x_2^3 - 1$$

$$\Rightarrow 2x_1^3 = 2x_2^3$$

$$\Rightarrow x_1^3 = x_2^3$$

$\Rightarrow$ Taking cube root on both the sides, then

$$\Rightarrow x_1 = x_2$$

Hence, f is Injective function.

(ii) **Is f surjective ?**

Let, $y \in B$ and $f(x) = 2x^3 - 1 = y$

By Equation (1),

Here, find x in terms of y

$$\therefore \quad 2x^3 = 1 + y \Rightarrow x^3 = \frac{1+y}{2} \Rightarrow x = \sqrt[3]{\frac{1+y}{2}}$$

$$\Rightarrow x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}} \Rightarrow x = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} \in A \text{ For any } y \in B \qquad \ldots(2)$$

From Equation (1), $f(x) = 2x^3 - 1$

Here, Put x value in terms of y

$$f(x) = 2\left(\sqrt[3]{\frac{y}{2} + \frac{1}{2}}\right)^3 - 1 \Rightarrow 2\left(\frac{y}{2} + \frac{1}{2}\right) - 1$$

$$\Rightarrow \left(\frac{2y}{2} + \frac{2}{2}\right) - 1 \Rightarrow y$$

$$\therefore \boxed{f(x) = y}$$

From the above statement. It is clear that, every $y \in B$ is having f(x). In other way, every co-domain element is a range which is every y is get mapped into $x \in A$.

$\therefore$ f is surjective.

Thus, function f is Injective and Surjective mapping.

Hence, function f is Bijective function.

(b) Now, to check g is bijective function,

We have, $f(x) = y \Rightarrow$ The inverse of y will be

$$x \Rightarrow f^{-1}(y) = x \text{ that is } f^{-1}(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} = g(y)$$

Thus, $f^{-1} = g$. We know that, if f is bijective function then f^{-1} is also bijective.

Hence, function g is also bijective.

Ex. 7.3.7 : Show that : If $f : P \to Q$ is bijective function then f^{-1} is unique.

☑ **Soln. :**

Let, $f : P \to Q$ is bijective function.

We will prove the statement by contradiction.

Assume, f^{-1} is not unique, then there will be two Inverse functions, say g and h.

Let, $x_1, x_2 \in p$, for some y in Q such that,

We know that, $f(x) = y$. Then inverse of f is as

$$f^{-1}(y) = x_1 \Rightarrow g(y) = x_1 \text{ then } f(x_1) = y$$

and $\quad f^{-1}(y) = x_2 \Rightarrow h(y) = x_2$ then $f(x_2) = y$

From the above statements. It is clear that,

$f(x_1) = f(x_2)$, but f is Injective (given)

So that, $\quad x_1 = x_2$ then $g(y) = h(y)$ for all y

$\Rightarrow$ $g \equiv h$ that is g and h are two inverse function of f and they are equal.

Hence, It is clear that, Inverse of function $f : P \to Q$ is unique.

7.4 FINITE SET AND IT'S CARDINALITY

- In chapter 1, we have already seen finite set and it's cardinality.
- Finite set is either null (void) set or it's elements can be listed.
- In finite set, we can count the elements associated with it.
- The number of elements in the finite set can be denoted as n (A) where A is finite set. It is also called as order or cardinal number of a set A.
- In short, finite set has finite number of elements.

e.g. $A = \{1, 2, 3, 4, 5 \ldots 50\}$ $B = \{a, b, c, d, \ldots, z\}$ Here, A and B are finite sets

There are some theorems associated with finite set are as given below.

(1) Theorem 1

Let A and B be finite set and suppose there is bijection from A to B. Then n(A) = n(B).

(2) Theorem 2

If A and B be finite sets with same cardinality.

$f : A \to B$ be a function then f is injective iff f is surjective.

7.4.1 Pigeonhole Principle

- As we know, two sets are equal or their cardinalities are same when there is a bijection function $f : A \to B$ exists. Which means, $n(A) = n(B)$ when $f : A \to B$ where f is bijective.

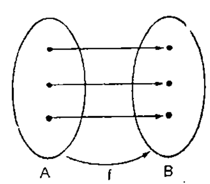


Fig. 7.4.1

- But, if n(A) > n(B), then there will be no bijection exist from A to B. This type of situation can be expressed by one famous principle known as "Pigeon-hole principle".
- The pigeonhole principle is simple and beautiful concept to apply when sets have different cardinalities
- Lets maps the concept of pigeons to sets with unequal cardinalities.
- Consider, a set A of pigeons and set B of pigeonholes. If all the pigeons flies into a pigeonholes and there are more pigeons than holes, then one of the pigeonholes have contain more than one pigeon.
- **Principle :** If there are n +1 pigeons and only n pigeons holes, then two pigeons will share the same whole.
- Consider there are 'n' pigeons and f(n) are pigeonholes, then it can be represented as

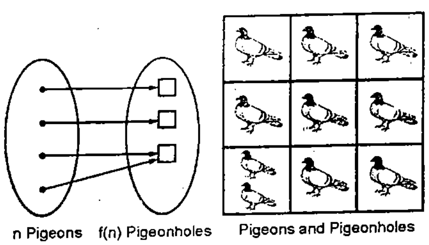


Fig. 7.4.2

7.4.2 Advanced Pigeonhole Principle

- Let f : A → B be a function from A to B.

$$n(A) = n,\ n(B) : m, \text{ where } n > m$$

then,

$$K = \left[\frac{n-1}{m}\right] + 1$$

There exists k elements $a_1, a_2, \ldots a_k \in A$ such that $f(a_1) = f(a_2) \ldots = f(a_k)$

In other way, if n pigeons are occupied to m pigeon holes, then one of the pigeonholes must be occupied by at least $\left[\frac{n-1}{m}\right] + 1$ pigeons.

Ex. 7.4.1 : Find the minimum number of students in the class to be sure that three of them are born in the same month.

☑ **Soln. :**

Here, n = Number of pigeons = number of students = ?

m = Number of pigeon holes = Number of months

= 12

We know that, three students from the class are born in the same month.

∴ According to extended pigeonhole principle at least $\left[\frac{n-1}{m}\right] + 1$ students born in the same month,

$$\Rightarrow \left[\frac{n-1}{m}\right] + 1 = 3 \Rightarrow \left[\frac{n-1}{12}\right] = 3 - 1$$

$$\Rightarrow n = (2 \times 12) + 1 = 25$$

$$\Rightarrow \mathbf{n = 25}$$

There are 25 minimum number of students in the class.

Ex. 7.4.2 : Prove that among 5000 peoples, there are four who are born at exactly the same minute.

☑ **Soln. :**

Let, n = number of peoples (number of pigeons) = 5000

m = number of minutes (number of pigeon holes)

$= 24 \times 60 = 1440$

Here, n > m that is peoples are more than Hours.

Apply, Extended pigeonhole principle,

$$K = \left[\frac{n-1}{m}\right] + 1$$

$$= \left[\frac{5000-1}{24 \times 60}\right] + 1 = 3 + 1 = 4 \text{ peoples}$$

Hence, there are at least 4 peoples who born on the same day or same minute.

Ex. 7.4.3 : Suppose, 20 students in a class appear at an entrance examination. Prove that there exists two among them whose seat numbers differ by a multiple of 19.

☑ **Soln. :**

Let, n = number of students = (number of pigeons) = 20

m = number of seat numbers = 19

= number of pigeon holes.

$= 24 \times 60 = 1440$

n > m, According to Extended pigeonhole principle.

There is atleast $\left[\frac{n-1}{m}\right] + 1$ students whose seat numbers differ by multiple of 19.

$$K = \left[\frac{n-1}{m}\right] + 1$$

$$= \left[\frac{20-1}{19}\right] + 1 = \left(\frac{19}{19}\right) + 1 = 1 + 1$$

= 2 students.

Therefore, 2 students whose seat number differ by multiple of 19.

Ex. 7.4.4 : Show that 7 colours are used to paint 50 bicycles, then at least 8 bicycles of same colour.

☑ **Soln. :**

Given : 7 colours are similar to pigeonholes

50 bicycles are similar to pigeons.

So, n = 50 and m = 7 pigeons are more than pigeonholes.

n > m, that is number of pigeons are more than pigeonholes.

According to Extended pigeonhole principle, at least $\left[\frac{n-1}{m}\right] + 1$ pigeons (bicycles) have same pigeonhole (colour).

$$\Rightarrow \left[\frac{50-1}{7}\right] + 1 \Rightarrow \frac{49}{7} + 1 = 7 + 1 = 8$$

Hence, it is clear that, 8 bicycles will have same colour.

...proved

Ex. 7.4.5 : Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13.

☑ **Soln. :**

We have, A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }

We can form the six different sets, each containing 2 numbers that add upto 13.

$A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$

$A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$

$A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$

Each of the seven numbers chosen must belong to one of these sets. As there are only six sets, by pigeonhole principle two of the chosen numbers must belongs to the same set and their sum is 13.

Ex. 7.4.6 : Assume that there are 3 mens and 5 womens in a party. Show that if these people are lined up in a row, at least two women will be next to each other.

☑ **Soln. :**

m = men (pigeonholes) = 3 mens

n = women (pigeons) = 5 womens

that is, pigeons are more than pigeonholes.

According to pigeonhole principle, when pigeons are more than pigeonholes, then at least two pigeons will share same pigeonhole.

Which means, at least two women in a row will be next to each other. **...proved.**

7.5 INFINITE SETS AND COUNTABILITY

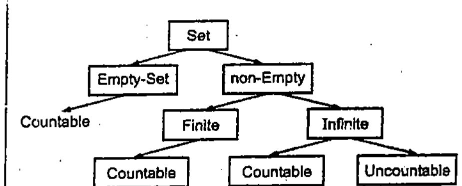


Fig. 7.5.1

7.5.1 Infinite Set

Definition : [illegible] set if there exists an [illegible] mapping [illegible] f : A → A such that f(A) [illegible]

Examples ;

(i) Set of all positive integers ⇒ Z^+ ⇒ Infinite set

(ii) Set of all whole numbers ⇒ W ⇒ infinite set

(iii) Set of all natural numbers ⇒ N ⇒ Infinite set

Properties of infinite set

(i) If A be a infinite set then Cartesiann product (A × A) and power set (P(A)) are also infinite sets.

(ii) If A and B are non-empty set and either A or B is an infinite set then A × B is an infinite set.

(iii) If either A or B is an infinite set then $A \cup B$ is an infinite set.

(iv) If $A \subseteq B$ and A is an infinite set then B is also an infinite set.

7.5.2 Countable Set

- As, we know empty set is a countable set because $n(\phi) = 0$.
- In finite set we can list the element, so finite set is countable.
- But, infinite set can be countable and uncountable.

Countably Infinite

- An infinite set A is said to be countable if there exists a bijection mapping : between natural number set (N) and set A i.e. $f : N \to A$.
- In other words, we can maps the elements of set "A" to the natural numbers.

Here,

$a_1 = f(1)$

$a_2 = f(2)$

:

:

$a_n = f(n)$

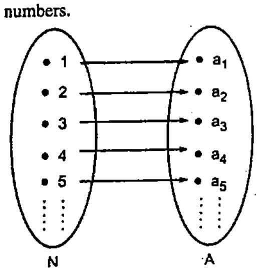


Fig. 7.5.2

- In set A and N, there are infinite elements, but their bijection mapping is possible.
- A countably infinite set is also known as a denumerable set.

Example :

(i) A = {1, 2, 3, 4,...5000} is countable.

(ii) The set of rational numbers is countable.

(iii) The set of real numbers is uncountable.

(iv) The set of real numbers in [a, b], a < b is uncountable.

Properties of countable set

(i) A subset of a countable set is countable.

(ii) Let, A and B be countable sets then $A \cup B$ is countable.

(iii) Set of rational number is countable.

(iv) The set of irrational number is uncountable.

(1) Theorem : The set of rational numbers is countable.

Proof : Georg cantor state that, if there is an one to one correspondence between two sets then they have same size.

Rational number

- Every positive rational number is the quotient $\frac{P}{q}$ of two positive integers.

$$= \left\{\frac{P}{q} \mid P \text{ and } q \in N, N > 0\right\}$$

- The rational number can be mapped to positive integers in following way,

Cantor diagonalization argument

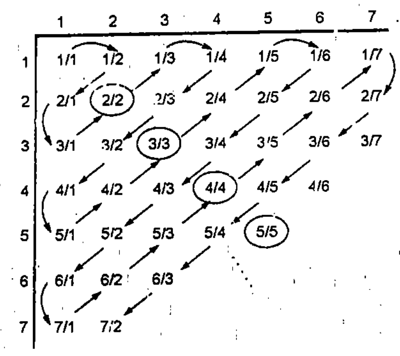


Fig. 7.5.3

- From the above arrangement of positive integers in tabular form, which represents diagonal. If we expand the diagonals from first row to the first column by omitting circled elements. We get the enumeration

$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{5}{1}, \frac{4}{2}, \frac{2}{4}, \ldots$

- From the cantor diagonalization argument, it is clear that, we can map the elements of rational number set into positive integers.

Hence, set of rational numbers is countable.

(2) The set of real numbers is an uncountable set

Proof

- To prove the theorem, we first assume that the set of real numbers is countable and arrive at a contradiction.
- The subset of all real numbers that falls between 0 and 1 would also be countable (since any subset of countable set is also countable).

- According to assumptions, we can arrange the elements of real numbers between 0 and 1 as an sequence. Each sequence being represented as a unique decimal as

$$a_1 = 0.a_{11}\, a_{12}\, a_{13} \ldots$$
$$a_2 = 0.\, a_{21}\, a_{22}\, a_{23} \ldots$$
$$a_3 = 0.\, a_{31}\, a_{32}\, a_{33} \ldots$$
$$\vdots$$
$$a_k = 0.\, a_{k1}\, a_{k2}\, a_{k3} \ldots$$
$$\vdots$$

Here, $a_1, a_2, a_3 \ldots$ are real numbers between 0 and 1.

So, every number between 0 and 1 must lie in the sequence.

Define,
$b_1 = 2$ if $a_{11} = 1$ otherwise
$b_1 = 1$ i.e. if $a_{11} \neq 1$
$b_2 = 2$ if $a_{22} = 1$
$b_2 = 1$ if $a_{22} \neq 1$

In the same way,

$$b_i = \begin{cases} 2 & \text{if } a_{ii} = 1 \\ 1 & \text{if } a_{ii} \neq 1 \end{cases} \quad \text{for all i}$$

So we can get cantor diagonlization

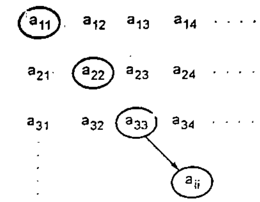


Fig. 7.5.4

Here, initially, we have made b_{ii} differ from diagonal element of array i.e. a_{ii}.

Thus. b_i is always different than a_{ii} now consider

$$b = 0.\, b_1\, b_2\, b_3\, b_4 \ldots$$

This forms a number which is greater than 0 and less than 1. That is $b \in (0, 1)$

So, it must be a_i for some i

but $b \neq a_i$; (as b differs from a_1 in the first decimal place).

b differs from a_2 in the second decimal place

:

b is differs from a_i in the i^{th} decimal place.

So, our assumption is false (contradiction) to the fact that real numbers between (0, 1) interval is countable.

Conclusion : The set of real numbers is an uncountable set.

Chapter Ends...

□□□

UNIT

V

Introduction To Number Theory

>> **Syllabus :**

- **Divisibility of Integers :** Properties of Divisibility, Division Algorithm Greatest Common Divisor GCD and its Properties, Euclidean Algorithm, Extended Euclidean Algorithm, Prime Factorization Theorem, Congruence Relation, Modular Arithmetic, Euler Phi Function, Euler's Theorem, Fermat's Little Theorem, Additive and Multiplicative Inverses, Chinese Remainder Theorem..

Mapping of Course Outcomes for Unit V	CO5

- **Chapter 8 : Introduction To Number Theory**

UNIT V

CHAPTER 8 Introduction to Number Theory

Syllabus

Divisibility of Integers : Properties of Divisibility, Division Algorithm, Greatest Common Divisor GCD and its Properties, Euclidean Algorithm, Extended Euclidean Algorithm, Prime Factorization Theorem, Congruence Relation, Modular Arithmetic, Euler Phi Function, Euler's Theorem, Fermat's Little Theorem, Additive and Multiplicative Inverses, Chinese Remainder Theorem.

8.1 DIVISORS

- When working with an integer, it is natural to try to factor the integer and thereby write it as a product. For this reason, numbers that cannot be factored themselves such as 2 are particularly important.
- To initiate our study of integers and their divisors, we start with the consideration of whether or not an integer is multiple of 2.

8.1.1 Divisibility

- An integer being even is simply a special case or of one integer divisible by another

> **Definition :** *Given integers n and d, we say that d divides n, written as $\frac{d}{n}$, if n = dK for some integer K. In this case we also say that n is divisible by d, that n is multiple of d, that d is a divisor of n, and that d is a factor of n when n is not divisible by d, we write $\frac{d}{n}$.*

Remark

n is even if and only if $\frac{2}{n}$

Example 8.1.1

Let a and m be integers with $m \geq 1$, show that $\frac{a}{a^m}$

Proof

- Observe that $a^m = a.a^{m-1}$ and since $m - 1 \geq 0$, $a^{m-1} \in Z$
- From the algebra of real numbers, we know that if a < b and b < c then a < c. That property is called as the transisitivity of the lees than relation.
- We have an analogous property of divisibility among integers.

Example 8.1.2

Let a, b and c be integers shown that if a|b and b|c then a|c.

Proof

- Suppose a|b and b|c
- So b = ak and c = bl for some k, $l \in z$
- Observe that c = bl = aKl = a (Kl)
- Since K$l \in$ Z, we have established that a|c

Remark

- The divisors of a positive integers cannot exceeds that integers.

Theorem 8.1.1 : Let a,b, $\in$ z with b > 0 if a|b then a $\leq$ b

Proof

- Suppose a|b
- So b = ak for some K $\in$ z
- **Case 1 :** a $\leq$ 0

  We have a $\leq$ 0 < b, and the conclusion a $\leq$ b is immediate
- **Case 2 :** a > 0
- Since aK = b > 0, it must be that K > 0 too (otherwise aK < 0)
- Since K is an integer 1 $\leq$ K
- Multiplication by (the positive value) 'a' gives that $a = a \cdot 1 \leq a \cdot K = b$
- In both cases, we conclude that a $\leq$ b assume b > 0, (necessary) for example if b = – 6 then each of the divisors a $\in$ {– 3, – 2, – 1, 1, 2, 3, 6} satisfies a|b and a > b only if an integer b is positive will none of its divisors be larger.

8.1.2 Primes

The most important divisors of an integer are those that cannot be factored any further.

> **Definition :** *An integer p is said to be prime if P > 1 and the only positive divisor of p are 1 and P.*

8.1.3 Composite

- An integer n > 1 that is not prime is called as composite.
- E.g. primes – 2, 3, 5, 7, 11 and so on.
- Composite – 4 = 2 · 2, 6= 3 · 2, 10 = 5 · 2 and so on.

Example 8.1.3

Show that 2 is only even prime.

Proof

- Let p be an even prime.
- So p = 2K for some K $\in$ z

- In fact, since $p > 0$ and $2 > 0$, it must be that $K > 0$
- Since p is prime and K is a positive divisor of P. we must have either K = l or K = P.
- Since $p = 2p$ is impossible. it has to be that K = 1 Therefore p = 2 K = 2
- The term composite is defined as the negation of the term prime. However, it is often useful to have a direct characterization of composites.
- An integer $n > 1$ is composite if and only if $\exists$ r, s $\in$ Z such that $r > 1$, $S > 1$ and $rs = n$

8.1.4 Greatest Common Divisor

- A number c is called as common divisor of the numbers (integers) m and n if c|m and c|n.
- That is, c is a divisor of both m and n for example. 10 is divisor of both 40 and 60, 20 is common divisor of 40 and 60 and is largest one.

> **Definition :** *Given integers m and n not both zero, their greatest common divisor, denoted gcd (m, n) is the unique integer d such that*
> *(i) $d > 0$*
> *(ii) $d|m$ and $d|n$ and*
> *(iii) $\forall$ $c \in Z+$, if $c|m$ and $c|n$, then $c \leq d$*

Remark

1. gcd (0, 0) is undefined, since every positive integer is a divisor of 0. Hence there cannot be a greatest one.
2. It is possible for two integers to have no positive divisors in common other than 1, which devides all integers.

- **Relatively prime :** Two integers m and n said to be relatively prime if gcd (m, n) = 1

Example 8.1.4

(i) gcd (18, 30) = 6
As 6 is maximum in all common divisors of 18 and 30 i.e. {1, 2, 3, 6}

(ii) 14 and 9 are relatively prime, as gcd (14, 9) = 1

8.1.5 Properties of Greatest Common Divisor

Property 1 :

If one can factor each of two positive integers m and n as a product of powers of primes then gcd can quickly be determined from those factorization

e.g.

(1) gcd (81989600, 231254595000) = 2928200
We have the factorization
$81989600 = 2^5 \cdot 5^2 \cdot 7 . 11^4$ and
$231254595000 = 2^3 \cdot 3^5 \cdot 5^4 \cdot 11^4 \cdot 13$

- We see that 2, 5, and 11 are the prime numbers factors common to both integers.
If follows that
- gcd (81989600, 231254595000) $= 2^3 \cdot 5^2 \cdot 11^4$
= 2928200
- Here the exponents on the primes 2, 5 and 11 are chosen to be the smaller of the two exponents in the factorization of 81989600 and 231254595000

(2) gcd (26159679, 644436) = 3159
From the factorization
$26159679 = 3^5 \cdot 7^2 \cdot 13^3$ and
$6444361 = 2^2 \cdot 3^6 \cdot 13 \cdot 17$
We get
gcd (26159679, 644436) $= 3^5 \cdot 13 = 3159$

Remark

- Disadvantage of this method is that it is extremely difficult to factor the large numbers.

Property 2 :

For given any positive integer K, gcd (K, 0) = K

Proof

- Observe that K is positive and that K|K and K|0
- Hence, conditions (i) and (ii) are satisfied if c $\in$ Z + and c|K and c|o, then in particular c|K.
- We have $c \leq K$. Thus condition (iii) is satisfied. We conclude that K = gcd (K, 0)
Property 3 : Given any integer $K \neq 0$, gcd (K, 0) = |K|
Property 4 : Let p and n be integers with p prime then
$$\text{gcd}(p, n) = \begin{cases} p \text{ if } p/n \\ 1 \text{ if } p\ n \end{cases}$$

- **Case I :** $p \setminus n$
- Since p is prime, p is positive
- Certainly p|p and p|n.
- So suppose that c is a positive integer such that c|p and c|n.
- Since c|p. if follows from that $c \leq p$
- Therefore p = gcd (p, n)
- **Case II :** p not \ n
- It is clear that 1 is positive and that 1|p and 1|n

- So suppose that c is positive integer such that c|p and c|n.
- Since c|p and p is prime, it follows that c = 1 Or
- Since c|n and p|n, in this case it must be that c = l.
- In particular, $C \leq 1$.
- Therefore 1 = gcd (p, n)
- **Property 5 :** let n be any integer. Then n and n + 1 are relatively prime.

Proof

- We shall show that gcd (n, n + 1) = 1
- certainly, 1 > 0, 1|n and 1| n + 1
- So suppose c $\in$ z+ and c|n and c | n + 1
- So n = cK and n + 1 = c*l* for some K, L $\in$ z
- Hence, L = (n + 1) – n = c*l* – cK = c (l – K)
- That is, c|1 and therefore $c \leq 1$
- It follows that gcd (n, s + 1) = 1

Well ordering principal for integers.

- Each non – empty subset of the non negative integers has a smallest element

Example 8.1.5

- Let S = {S : S = 15 x + 25 y, where x, y $\in$ z and 15 x + 25 y > 0). Obviously S is a non-empty subset of non-negative integers.
- Hence the well ordering principal guarantees that S has a smallest element.
- However, it is not hard to see directly in this case that the smallest element of S is 5 first, 5 $\in$ S, since 5 = 15(2) + 25 (– 1). Second any element of 5 can be expended as
S = 15 x + 25 y = 5 (3x + 5y)
- A multiple of S, since none of 1, 2, 3, 4 is divisible by 5, no element smaller than 5.

8.2 DIVISION ALGORITHM

- Given any integer n and any positive integer d, there exists unique integers q and r such that

(i) n = dq + r and (ii) $0 \leq r < d$

Theorem

- Let q is the quotient and r is the remainder upon division of n by d we also write.
- q = n div d and r = n mod d.

Examples

1. If n = 124 and d = 9 then, q = 13 and r = 7 that is, 124 = 9(13) + 7
So 124 div 9 = 13 and 124 mod 9 = 7
2. If n = 60 and d = 5 then q = 12 and r = 0

Ex. 8.2.1 : Show that for any integer n, $n^2 - 2$ is not divisible by 5.

Soln. :

- Let n be any integer
- By the division algorithm, n must take one of the forms.
5K, 5 K + 1, 5 K + 2, 5 K + 3, 5 K + 4
- For some integer K. That is dividing n by 5 leaves a remainder of 0, 1, 2, 3 or 4
- Here, K is being used to represent the quotient.
- We consider each possible form for n
- **Case 0 :** n = 5 K for some K $\in$ Z.

Observe that

$$\begin{aligned} n^2 - 2 &= 25 K^2 - 2 \\ &= 25 K^2 - 5 + 3 \\ &= 5 (5K^2 - 1) + 3 \end{aligned}$$

That is, $(n^2 - 2)$ mod 5 = 3 and not 0

- **Case 1 :** n = 5K + 1 for some K $\in$ Z

Observe that

$$\begin{aligned} n^2 - 1 &= 25K^2 + 10 K + 1 - 2 \\ &= 25 K^2 + 10 K - 5 + 4 \\ &= 5 (5 K^2 + 2K - 1) + 4 \end{aligned}$$

That is $(n^2 - 2)$ mod 5 = 4 and not 0

- **Case 2 :** n = 5 K + 2 for some K $\in$ Z

Observe that

$$\begin{aligned} n^2 - 2 &= 25 K^2 + 20 K + 4 - 2 \\ &= 25 K^2 + 20 K + 2 \\ &= 5 (5K^2 + 4K) + 2 \end{aligned}$$

That is $(n^2 - 2)$ mod 5 = 2 and not 0

- **Case 3 :** n = 5 K + 3 for some K $\in$ Z

Observe that

$$\begin{aligned} n^2 - 2 &= 25 K^2 + 30K + 9 - 2 \\ &= 25 K^2 + 30 K + 5 + 2 \\ &= 5 (5K^2 + 6K + 1) + 2 \end{aligned}$$

That is $(n^2 - 2)$ mod 5 = 2 and not 0

- **Case 4 :** n = 5 K + 4 for some K $\in$ Z

Observe that

$$\begin{aligned} n^2 - 2 &= 25 K^2 + 40 K + 16 - 2 \\ &= 25 K^2 + 40 K + 10 + 4 \\ &= 5 (5K^2 + 8K + 2) + 4 \end{aligned}$$

That is, $(n^2 - 2) \bmod 5 = 4$ and not 0

On each case, we see that $5 \nmid (n^2 - 2)$

Ex. 8.2.2 : Show that $\forall\ n \in Z$, $n^2 + n - 1$ is odd.

☑ **Soln. :**

- Let $n \in z$
- By the division algorithm, n must take either the form 2K or 2K + 1, for some $K \in Z$ (That is n is either even or odd)

▸ **Case 0 :** $n = 2K$ for some $K \in Z$

Observe that

$$\begin{aligned} n^2 + n - 1 &= 4K^2 + 2K - 1 \\ &= 4K^2 + 2K - 2 + 1 \\ &= 2(2K^2 + K - 1) + 1 \end{aligned}$$

Thus $(n^2 + n - 1) \bmod 2 = 1$ and not 0 (That is $n^2 + n - 1$ is odd)

▸ **Case I :** $n = 2K + 1$ for some $K \in z$

Observe that

$$\begin{aligned} n^2 + n - 1 &= 4K^2 + 4K + 1 + 2K + 1 - 1 \\ &= 4K^2 + 6K + 1 \\ &= 2(2K^2 + 3K) + 1 \end{aligned}$$

Thus $(n^2 + n + 1) \bmod 2 = 1$ and not 0

That is, $n^3 + n + 1$ is odd

In both cases, $n^2 + n - 1$ is odd.

8.3 EUCLID'S ALGORITHM

One of the greatest early contributors to number theory (and to mathematics) was the greek mathematician Euclid of Alexandria although his famous work elements. is perhaps better known for it's contribution to geometry, it also includes important theorems in number theory for example, it contains the devar argument used in our proof that there are infinitely many primes.

This section focuses on two important result i.e. algorithm useful to find greatest common divisors and more and second is a technical lemma that is fundamental to results on factoring

Before we discuss Euclid's Algorithm for finding gcd's we should be aware of very useful form in which greatest common divisors can be expressed.

Theorem 8.3.1

Expressing the GCD as linear combination given integers m and n not both zero, there exists integers x and y such that

$$\gcd(m, n) = mx + ny$$

▶ **Proof**

- Let $S = \{t : t = mu + nv$. For some $u, v \in Z$ where $+ nv > 0\}$ since $m.\ m + n \cdot n > 0$, S is non-empty. The well ordering principle gives us a smallest element d of S. so $d = mx + ny$ for some $x, y \in Z$ and $d > 0$ our aim is to show that $d = \gcd(m, n)$. we first show that $d | m$.
- By the division algorithm, there are integers q and r such that $m = dq + r$ and $0 \le r < d$.

  Observe that.
- $r = m - dq = m - (mx + ny)q = m(1 - qx) + x(-qy)$ it is were the case that $r > 0$ then $r \in s$ and r is smaller than d, the smallest element of S.
- Hence it must be that $r = 0$ That is $m = dq$ and $d | m$. A similar argument shows that $d | n$.
- It remains to show that d is the greatest among common divisors. So suppose that c is an integers such that $c | m$ and $c | n$. Hence $m = ac$ and $n = bc$ for some $a, b \in Z$. It follows that

$$d = mx + ny = acx + bcy = (ax + by)c$$

so $c | d$ as $c \le d$ therefore $\gcd(m, n) = d = mx + ny$

Corollary 8.3.1

Two integers m and n are relatively prime if and only if there exists integers x and y such tht $mx + ny = 1$.

Example 8.3.1

1. Observe that $\gcd(18, 30) = 6$ and $6 = 18(-3) + 30(2)$
2. Observe that 14 and 9 are relatively prime and $14(2) + 9(-3) = 1$

8.3.1 Euclid's Algorithm

- An algorithm is finite list of simple instructions leading to a desired result.
- There does exists a method, known as Euclid's Algorithm, for finding not only greatest common divisors but also the integers x and y.
- However, to see how this algorithm works, we initially focus our attention on the computation of the greatest common divisor.
- First observe that since $\gcd(m, n) = \gcd(|m|, |n|)$ it suffices to have a method for computing. gcd (m, n) When m and n are both non-negative.
- The critical result exploited by Euclid is an application of the division algorithm.

Theorem 8.3.2 GCD Reduction

Let n and m be integers such that $n \ge m > 0$ write $n = mq + r$, where $q, r \in Z$ with $0 \le r < m$ then $\gcd(n, m) = \gcd(m, r)$

▶ **Proof**

- The fact that $\gcd(n, m) = \gcd(m, r)$ is an immediate consequence of the fact, which we now prove, that the pairs (n, m) and (m, r) have the same set of common divisors.
- Suppose that 'c' is a common divisor of n and m. That is $n = ck$ and $m = cl$ for some integer K and l. Observe that

  $$r = n - mq = ck - clq = c(K - lq)$$
- Thus $c | r$ as well $c | m$. That is, c is a common divisor of m and r.
- Suppose that c is a common divisor of m and r. That is $m = ck$ and $r = cl$ for some integers K and l. observe that.

  $n = mq + r = ckq + cl = c(Kq + l)$
- Thus $c | n$ as well as $c | m$. That is c is a common divisor of n and m.
- The point of this theorem is that it allows us to exchange a calculation or the gcd or the pair (n, m) for a calculation of the gcd of the smaller pair (m, r) By a sequence of such reductions we can hope to reduce a general problem to a much smaller problem for which the solution is easy in particular, these reduction will eventually result in a pair in which one of the integers is 0.
- At that point the result from previous example completes the calculation.
- If K is positive integer then $\gcd(K, 0) = K$.

Ex. 8.3.2 : Compute gcd (48, 18)

☑ **Soln. :**

We repeatedly theorem of gcd reduction

$$\begin{aligned} \gcd(48, 18) &= \gcd(8, 12) && \text{since } 48 = (18)2 + 12 \\ &= \gcd(12, 6) && \text{since } 18 = (12)1 + 6 \\ &= \gcd(6, 0) && \text{since } 12 = (6)2 + 0 \\ &= 6 \end{aligned}$$

8.3.2 Algorithm

Euclid's Algorithm for finding gcd (n, m)

Let $n, m \in Z+$, with $n \ge m$

Algorithm

```
While m > 0
\began
Let r = n mod m
Let n = m
Let m = r
\end
Return n
```

Ex. 8.3.3 : Find gcd (630, 96) and write in the form $630x + 96y$ for $x, y \in z$.

☑ **Soln. :**

The initial steps are the same as those used in previous example.

$$\begin{aligned} \gcd(630, 96) &= \gcd(96, 54) && \text{since } 630 = (96)6 + 54 \\ &= \gcd(54, 42) && \text{since } 96 = (54)1 + 42 \\ &= \gcd(42, 12) && \text{since } 54 = (42)1 + 12 \\ &= \gcd(12, 6) && \text{since } 42 = (12)3 + 6 \\ &= \gcd(6, 0) && \text{since } 12 = (6)2 + 0 \end{aligned}$$

$\gcd(630, 96) = 6$

- With some additional work, these calculations can be used to find x and y.
- Note that the greatest common divisor 6 is the last non-zero remainder occurring in our calculation.
- Down to that least non-zero remainder we have,

$$\begin{aligned} 54 &= 630 - (96)6 \\ 42 &= 96 - (54)1 \\ 12 &= 54 - (42)1 \\ 6 &= 42 - (12)3 \end{aligned}$$

Substituting backward from the last row, we get

$$\begin{aligned} 6 &= 42 - (12)3 \\ &= 42 - (54 - 1(42)3) - (54)(-3) + (42)(4) \\ &= 54(-3) + (96 - 1(54)(4)) \\ &= (96)(4) + (54)(-7) \\ &= (96)(4) + (630 - 6(96)(-7)) \\ &= 630(-7) + (96)(46) \end{aligned}$$

That is, $\gcd(630, 96) = 6 = 630x + 96y$ for $x = -7$ and $y = 46$

Ex. 8.3.4 : Find gcd (102, 30) and write in the form $102x + 30y$ for $x, y \in Z$.

☑ **Soln. :**

$$\begin{aligned} \gcd(102, 30) &= \gcd(30, 12) \text{ since } 102 = (30)3 + 12 \\ \text{so } 12 &= 102 - (30)3 \\ &= \gcd(12, 6) \text{ since } 30 = (12)2 + 6 \\ \text{So } 6 &= 30 - (12)2 \\ &= \gcd(6, 0) \text{ since } 12 = (6)2 + 0 \\ &= 6 \end{aligned}$$

Therefore

$$6 = 30 - (12)2$$
$$= 30 - (102 - 3(30))2$$
$$= (102)(-2) + (30)(7)$$

- That is gcd (102, 30) = 6 = 102 x + 30 y for x = – 2 and v = 7
- The procedure used in example 2 and 3 to find x and y so that d = gcd (m n) = mx + ny called as extended Euclidean Algorithm.

8.4 EUCLID'S LEMMA

Let m, n and c be integers. If clmn and gcd (c, m) = 1 the cln

Proof

Suppose clmn and gcd (c, m) = 1, so mn = ck for some $K \in Z$ by corollary, there are $x, y \in z$ such that cx + my = 1. Observe that

$$n = (cx + my)\, n = cnx + mny = cnx + cky = c\,(nx + ky)$$

Therefore cln

Note :

1. Let m, n and p be integers with p prime of p | mn then p | m or p | n.
2. Let m_1 m_2 .....m_n and p be integers with p prime if $P|(m_1, m_2 .....m_n)$ then p\m1 for $1 \le i \le n$.
3. Let m_1, n and p be integers with n > 0 and p prime if $p|m^n$, then $p|^m$.

Ex. 8.4.1 : Let a, b and n be integers with n > 0 show that if gcd (a, b) = 1 then gcd (a, b^n) = 1.

Soln. :

Suppose gcd (a, b) = 1, Let d = gcd (a, b^n), suppose towards a contradiction, that d > 1, we have a prime P such that pld. so pla and plb^n but plb, so Plgcd (a, b). That is Pl1, which is impossible since p is prime. Hence d = 1.

8.4.1 Prime Factorization Theorem

- The fundamental theorem of arithmetic states that every natural number greater than 1 can be written as product of prime numbers and that up to rearrangement of the factors, this product is unique.
- This is called as prime factorization of the number.
- e.g. 36 can be written as 6 × 6, 4 × 9 or 3 × 12 or 2 × 18. But there is unity one wary to write it as a product where all the factor are primes.

$$36 = 2 \times 2 \times 3 \times 3$$

- This is prime factorization of 36, often written with exponent as

$$36 = 2^2 \times 3^2$$

- The prime factorization can be found using factor tree start by finding two factors when multiplied together, give the number keep splitting each branch of the tree into a pair of factors until all the branches terminate in prime numbers.
- Consider factor tree of 1386.

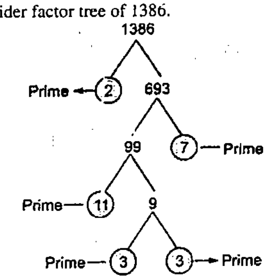


Fig. 8.4.1

- Prime factorization of 1386 = 2 × 3 × 3 × 7 × 11

8.5 MODULAR ARITHMETIC

- Many things are numbered cyclically for example, four hours after 10 o'clock, it is 2 o' clock, not 14 o'clock. Seven hours before 3 o' clock, it is 8 o' clock, twenty eight hours after 1' o' clock, it is 5 o' clock.
- In specifying time of day, we equate 10 + 4 with 2, we equate 3 – 7 with 8.
- These equivalences hold because the differences (10 + 4) – 2, (3 – 7) – 8 and (1 + 28 – 5) respectively, are divisible by 12 in the same way two dates full on the same day of the week if and only if the number of days by which they differ is divisible 7.
- These types of calculations are sometimes called clock arithmetic. The more formal term is 'Modular arithmetic'.

8.5.1 Definition – Congruent Modules

Given integers a, b and n with n > 1, we say that a is congruent to b modulo n, written as, a = b modulo $n \equiv b \pmod n$ if n l (a – b).

e.g.

$$14 \equiv 2 \pmod{12} \text{ since } 12 \mid (14 - 2)$$
$$-4 = 2 \pmod{12} \text{ since } 12 \mid (-4 - 8)$$

8.5.2 Congruence Relation

An equivalence relation is called as congruence relation

Let a, b and n be integers with n > 1

(a) $a \equiv a \pmod n$

(b) if $a \equiv b \pmod n$, Then $b \equiv a \pmod n$

(c) if $a \equiv b \pmod n$ and $b \equiv c \pmod n$ then $a \equiv c \pmod n$

Arithmetic properties of congruence relation

Let a_1, a_2, b_1, and n be integers with n > 1.

If $a_1 \equiv a_2 \pmod n$ and $b_1 \equiv b_2 \pmod n$ then

(1) $a_1 + b_1 \equiv a_2 + b_2 \pmod n$, and

(2) $a_1 b_1 \equiv a_2 b_2 \pmod n$

e.g.

- Note that 4 = – 6 (mod 10) and $22 \equiv 2$ (mod) As promised by above property, 4 + 22 = – 6 + 2 (mod 10) and 4 × 22 = – 6 × 22 (mod 10)
- **Corollary :** Let m, a_1, a_2 and n be integers with n > 1

If $a_1 \equiv a_2 \pmod n$ then

1. $ma_1 = ma_2 \pmod n$ and
2. If $m \ge 0$, then $a_1^m \equiv a_2^m \pmod n$

e.g.

- Note that $3 \equiv -2 \pmod 5$ by the corollary we also have $4(3) \equiv 4(-2) \pmod 5$ and $3^4 \equiv (-2)^4 \pmod 5$

Theorem 8.5.1

- Given integers n and d with d > 1
- n mod d ≡ n (mod d) and moreover, n mod d is the unique integer in {0, 1, ...d – 1} congruent to n modulo n.

Proof

- Let r ≡ n mod d.
- That is r is the remainder upon division of n by d guaranteed by the division algorithm.
- By letting Q = n div d, we have n = dq + r and $0 \le r < d$. So d (– q) = r – r. since dl(r – n), in the notation of congruence, we have $r \equiv n \pmod d$, That is, n mod d ≡ n (mod d)
- To establish our uniqueness assertion, suppose $r' \in \{0, 1, ... d - 1\}$ and $r' \equiv n \pmod d$, since d | (r′ – n), there is some $q' \in z$ such that d(– q′) ≡ r′ – n. that is, n = dq′ + r′.
- since $0 \le r'$ d the uniqueness assertion in the division algorithm tells us that r′ = r = n mod d.

Example 8.5.1

Compute (5162387 + 83645) mod 10

- Note that the value of a positive integer mod 10 is its one digit.
- Since $5162387 \equiv 7 \pmod{10}$ and $83645 \equiv 5 \pmod{10}$
- We get $5162387 + 83645 = 5 + 7 \equiv 12 \equiv 2 \pmod{10}$
- That is, we can conclude that (5162387 + 83645) mod 10 = 2 without having to compute 5162387 + 83645 = 5246032

Example 8.5.2

Compute (34861 + 571028) mod 5

- Note that it is easy to determine the value of an integer mod 5 based on its ones digit since $34861 \equiv 1 \pmod 5$ and $571028 \equiv 8 \equiv 3 \pmod 5$
- We get $34861 + 571028 \equiv 1 + 3 = 4 \pmod 5$ that is , we can conclude that (34861 + 571028) mod 5 = 4
- Without having to compute 34861 + 571028 = 605889

Example 8.5.3

Compute 3^{35} mod 10

- Note that $3^4 \equiv 81 \equiv 1 \pmod{10}$
- So $3^{32} = (3^4)^8 \equiv 1^8 \equiv 1 \pmod{10}$
- Hence, $3^{35} = 3^{32} \times 3^3 = 1(27) = 27 \equiv 7 \pmod{10}$
- We conclude that 3^{35} mod 10 = 7 and we did not need to compute the value of 3^{35} = 50031545098999707

Example 8.5.4

Compute 2^{13} mod 209

- Note that $2^8 = 256 \equiv 47 \pmod{209}$ and $2^5 = 32 = 32 \pmod{209}$
- Hence $2^{13} = 2^8 \cdot 2^5 = 47 \times 32 = 1504 = 41 \pmod{209}$
- we conclude that,
- $2^{13} \bmod (209) = 41$

Example 8.5.5

Compute 41^7 mod 209

- Note that $41^2 = 1681 = 9 \pmod{209}$
- Hence $41^6 = (41^2)^3 = 9^3 = 729 = 102 \pmod{209}$
- Therefore, $41^7 = 41^6 (41) = 102 (41) = 4182 = 2 \pmod{209}$
- We conclude that $41^7 \bmod 209 = 2$

8.5.3 Modular Cancellation Rule

- Let a, b_1, b_2 and n be integers with $n > 1$ suppose that $ab_1 \equiv ab_2 \pmod n$ and gcd $(a, n) = 1$ then $b_1 \equiv b_2 \pmod n$

Proof

- Since $n | a (b_1 - b_2)$ and gcd $(n, a) = 1$, it follows from Euclid's theorem that $n | (b_1 - b_2)$. That is $b_1 \equiv b_2 \pmod n$
- Cancelling a from both sides of the congruences $ab_1 = ab_2 \pmod n$ is effectively dividing both sides by a from another point of view, we are multiplying both sides by an inverse of a

8.5.4 Additive and Multiplication Inverse

- Given a, $n \in Z$ with $n > 1$ then additive inverse of a modulo n is an integer c such that $a + c = 0 \pmod n$ and multiplicative inverse of a modulo n is an integer c such that $ac = 1 \pmod n$.
- **Theorem :** Given $n \in Z$ with $n > 1$, an integer a has a multiplicative inverse module n if and only If gcd $(a, b) = 1$.

Proof

Observe that the following statement are equivalent.

1. $\exists c \in Z$ such that $ac \equiv 1 \pmod n$
2. $\exists$ c, $d \in Z$ such that $n (-d) = ac - 1$
3. $\exists$ c, $d \in Z$ such that $ac + d = 1$

8.6 EULER'S THEOREM

- If a and n are coprime positive integers where $n \geq 2$ in z + and $a \in Z$ that $(a, n) = 1$

  then

  $a^{\phi(n)} = a \bmod n$
- Where $\phi(n)$ is the number of invertible integers modulo n.
- When m = p is prime, all non-zero integers modulo p are invertible, so $\psi(p) = p - 1$ and Euler's theorem becomes farmat's little theorem which we discussed in next section

How to compute ϕ (n) ?

- Consider n = 12, to count the number of invertible integers modulo 12 write down a set of representatives for integers modulo 12, such as 1,2,3,4,5,6,7,8,9,10,11,12
- The numbers here that are invertible modulo are 1, 5, 7, 11 so $\psi(12) = 4$. Euler's theorem for n = 12 says $a^4 = 1 \bmod 12$ when $(a, 12) = 1$.
- Being invertible modulo n is the same as being relatively prime to n (that is we can solve $ax \equiv 1 \bmod n$ for x exactly when $(a, m) = 1$) so we can describe $\psi(n)$ concretely as

  $\phi(n) = |\{a : 1 \leq a \leq n, (a, n) = 1\}|$
- Here is a small table of values derived from this formula.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\psi(n)$	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8

- It seems $\phi(n)$ is even for m > 2 to prove this observe that when a mod n is invertible so is – a mod n. so using standard representatives modulo n, invertible numbers modulo n come in pairs as (a, m, – a).
- This pair of different numbers mod m, since if $a \equiv m - a \bmod m$, for $0 < a < n$ then $a = n - a$. So $a = n/2$ and n is even. But $(n/2, n) = n/2$ and $n/2 > 1$ when n > 2, so $n/2$ mod n is not invertible. Thus if n > 2 the invertible numbers modulo n come in pairs {a, n – a mod n}, so ϕ (n) is even.

8.6.1 Euler's phi ϕ Function

- The Euler's phi (ϕ) function on positive integers defined by
- $\phi(n)$ = (the number of integers in {1, 2, ... n – 1} which are relatively prime to n.

  For example.
- $\phi(24)$ = 8, because there are eight positive integer less then 2y which are relatively prime to 25

  i.e. 1, 5, 7, 11, 13, 17, 19, 23
- on the other hand $\phi(11) = 10$, because all of the number in {1, ...10} are relatively prime to 11.

Remarks

1. If p is prime, $\phi(p) = p - 1$
2. If p is prime and $n \geq 1$ the $\phi(p^n) = p^n - p^{n-1}$.
3. $\phi(n)$ counts the elements in {1, 2, 3, ... n – 1} which are invertible mod n.

Example

1. Reduce 37^{103} (mod 40) to a number in the range {0, 1, ...39}

   Euler's theorem says that

   $37^{16} = 1 \pmod{40}$ so,

   $37^{103} = 37^{96} \cdot 37^7 = (37^{16})^6 \cdot 37^1 = 1.13 \pmod{40} = 13$
2. Solve $15x = 7 \pmod{32}$
- Note that $(15, 32) = 1$ and $\phi(32) = 16$ therefore $15^{16} = 1$ (Mod 32). Multiply the equation by 15^{15}

  $x = 7.15^{15} \pmod{32}$

Now,

$7.15^{15} = 105.15^{14} = 105 (15^2)^7 = 105.225^7 = 9.1^7$

$= 9 \pmod{32}$

so $\quad$ **x = 9 (mod 32)**

8.7 FARMAT'S LITTLE THEOREM

- If p is a prime, $a \in Z$ and $p \nmid a$ then $a^{p-1} \equiv 1 \pmod p$

Proof

- Since no pair of integers from the lest

  0.1, 2, ... P – 1 ...(8.7.1)

  is congruent modulo p, it follows from the modular cancellation law that no pair of integers from the list

  a.1, a.1, a.2, ... a (P – 1) ...(8.7.2)

  is congruent modulo n consequently, there must be a way to pair off the entries in Equation (8.7.1) with the entries in Equation (8.7.2) so that each pair is congruent modulo p certainly, 0 pairs with a.0.

  Since the list 1, 2, ... p – 1 gets paired, in some order, with the lest a.1, a.2, ...a (p – 1), if follows from the given theorem of asthmatic properties of congruence that the products of these lists are congruent modulo p. that is $(p - 1)! \equiv a^{p-1} (p - 1)! \pmod p$. since p is prime, gcd $(p, (p - 1)!) = 1$. the modular cancellation rule then tells us that $1 \equiv a^{p-1} \pmod p$.

Example 8.7.1

1. Compute 7^{2222} mod 13.

   Farmat's little theorem tells us that $7^{12} = 1$ (mod 13 thus

   $7^{2222} = (7^{12})^{185} 7^2 = 1^{185} 49 = 10 \pmod{13}$

   We conclude that

   $7^{2222} \bmod 13 = 10$

Remark

1. If p is prime then for all $a \in Z$, $a^p = a \pmod p$
2. Let $n \in Z$ with $n > 1$. If there exist $a \in Z$ such that $a^n \not\equiv a \pmod n$, then n is not prime

Example 8.7.2

Consider n = 221 observe that

$3^5 = 243 = 22 \pmod{221}$ and $22^4 = 234256 \equiv -4 \pmod{221}$

Thus,

$3^{220} = (3^5)^{44} = 22^{44} \equiv (22^4)^{11} = (-4)^{11} = -4194304$

$\equiv 55 \pmod{221}$

Since $3^{221} = 165 \not\equiv 3 \pmod{221}$

- Therefore from above remark (2) we conclude that 221 Is not prime
- If an integer n satisfies $a^n \equiv a \pmod n$ for a particular value of a, the this is not enough to conclude that n is prime for example $n = 91 = 7.13$ is certainly not prime despite the fact that $3^{91} = 3 \pmod{91}$.
- A more useful computation in this case is the observation that $2^{91} \equiv 37 \neq 2 \pmod{91}$

8.8 CHINESE REMAINDER THEOREM

Find a number x such that have remainders of 1 when divided by 3, 2 when divided by 5 and 3 when divided by 7 i.e.

1. $x = 1 \bmod 3$
2. $x = 2 \bmod 5$
3. $x = 3 \bmod 7$

Integers can be represented by their residues modulo a set of pair wise relatively prime moduli, for example. In Z_{10}, integer 8 can be represented by the residues of the 2 relatively prime factors of 10 (2 and 5) as a tuple (0, 3).

Let $M = m_1 \times m_2 \times m_3 \times \ldots \times m_K$, where m_i s are pair wise relatively prime, (i.e. gcd $(m_i, m_j) = 1$

$1 \leq i \neq j \leq k$

8.8.1 Assertion

I ... $\leftrightarrow (a_1, a_2, \ldots a_k)$ where $A \in Z_m$, $a_i \in Z_{mi}$ and $a_i = A$ mod mi for $1 \leq i \leq k$.

(a) One to one correspondance (bisections) between Z_m and the cartesion Product $Z_{m1} \times Z_{m2} \times Z_{mk})$

(b) For every integer A such that $0 \leq A \leq M$, there is unique K-tuple $(a_1, a_2, \ldots a_k)$ with $0 \leq a_i < m_i$

(c) For every such k-tuple $(a_1, a_2, \ldots a_k)$ there is a unique A in Z_m.

(d) Transformation from A to $(a_1, a_2, \ldots a_k)$ is unique.

(e) Computing A from $(a_1, a_2, \ldots a_k)$ is done as follows :

1. Let $M_i = M/m_i$ for $1 \leq i \leq K$ i i,e, $M_i = m_1 \times m_2 \times \ldots \times m_{j-1} \times m_K$.
2. Note that $M_i \equiv 0$ (mod m_j) for all $j \neq i$.
3. Let $C_j = M_j \times (M_j^{-1} \bmod m_j)$ for $1 \leq j \leq K$.
4. Then $A \equiv (a_1 c_1 + a_2 c_2 + \ldots + a_k c_k) \bmod M_1$
5. $a_i = A$ mod mi, since $c_j = M_j \equiv 0$ (mod m_i)

If $j \neq i$ and $c_i \equiv 1$ (mod m_i)

- Operations performed on the elements of Z_m can equivalently performed on the corresponding K-tuple by performing the operation independently in each co-ordinate position.

Example

$A \leftrightarrow (a_1, a_2, \ldots a_k)$, $B \leftrightarrow (b_1, b_2, \ldots\ldots b_k)$

$(A + B) \bmod M \leftrightarrow ((a_1 + b_1) \bmod m_1, \ldots (a_k + b_k) \bmod m_k)$

$(A - B) \bmod M \leftrightarrow ((a_1 - b_1) \bmod m_1, \ldots (a_k - b_k) \bmod m_k)$

$(A \times B) \bmod M \leftrightarrow ((a_1 \times b_1) \bmod m_1, \ldots (a_k \times b_k) \bmod m_k)$

- Chinese remainder theorem provides a way to manipulate (potentially large) numbers mod M in terms of tuple of small number

8.8.2 Theorem – Chinese Reminder Theorem

Suppose gcd (m, n) = 1, given a_1 and b_1, there exists exactly one solution x (mod mn) to the simultaneous congruence under certain conditions

$x = a$ (mod m); $x = b$ (mod n)

Proof

There exists integers s, t such that $ms + nt = 1$

Then $ms = 1$ (mod n) and $nt = 1$ (mod m).

Let $x \equiv bms + ant$. Then $x \equiv ant \equiv a$ (mod m) and

$x \equiv bms \equiv b$ (mod n) as desired

Suppose x, is another solution. Then $x = x_1$ (mod m) and $x = x_1$ (mod n), so $x - x_1$ is a multiple of both m and n.

Lemma 8.8.1

Let m, n be integers with gcd (m, n) = 1, if an integer c is a multiple of both m and n, then c is a multiple of mn.

Proof

Let $c = mk = nl$ write $ms + nt = 1$ with integers s, t multiply by c to obtain

$c = cms + cnt = mnls + mnkt = mn\,(ls + kt)$

To finish the proof of the theorem let

$c = x - x_1$, in the lemma to find that $x - x_1$ is multiple of mn.

Therefore $x \equiv x_1$ (mod mn)

This means that any two solutions x to the system of congruence's are congruent most mn, as claimed.

Examples 8.8.1

Solve $x \equiv 3$ (mod 7), $x \equiv 5$ (mod 15)

$\Rightarrow x \equiv 80$ (mod 105) note $105 = 7.15$. since $80 \equiv$? (mod 7) and $80 \equiv 5$ (mod 15), 80 is a solution.

The theorem guarantees that such a solution exists, and says that it is uniquely determined mod the product Mn, which is 105 in the present example.

How to solve

- One way, which works with small numbers M and n is to list the numbers congruent to be (mod n) until you find one that is congruent to a (mod M).
- For example, the numbers congruent to 5 x = 101 (mod 210) (mod 15) are 5, 20, 35, 50, 65, 80, 95, ... mod 7, there are 5, 6, 0, 1, 2, 3, 4, ... since we want 3 (mod 7), we choose 80.
- For slightly larger numbers m and n making a list would be inefficient however, a similar idea works. The numbers congruent to b (mod n) are of the form b + nk with K an integer, so we need to solve $b + nK \equiv a$ (mod m). This is the same as.

  $nK \equiv a - b$ (mod m)
- Since gcd (m, n) = 1 by assumption, there is a multiplicative inverse i for n (mod m) multiplication by 1 gives.
- $K \equiv (a - b)\, i$ (mod M).
- Substituting back into $x = b + nk$, then reducing mod mn, gives the answer.

Examples 8.8.2

Solve $x \equiv 7$ (mod 12345), $x \equiv 3$ (mod 11111)

First we know from our calculations in section that the inverse of 11111 (mod 12345) is $i = 2471$.

There $K = 2471\,(7 - 3) \equiv 9884$ (mod 12345)

This yields $x = 3 + 11111 \equiv 9884 \equiv 109821127$ (mod 11111.12345)

In a Chinese remainder theorem, let n = 210 and let $n_1 = 5$, $n_2 = 6$, $n_3 = 7$.

Compute F^{-1} (3, 5, 2), i.e. given $x_1 = 3$, $x_2 = 5$, $x_3 = 3$, compute x.

$N_1 = n_2 \times n_3 = 42$

$N_2 = n_1 \times n_3 = 35$

$N_3 = n_1 \times n_2 = 30$

$v_1 \equiv (N_1)^{-1} \equiv 42^{-1} \equiv 2^{-1} \equiv 3$ (mod 5)

$v_2 \equiv (N_2)^{-1} \equiv 35^{-1} \equiv 5^{-1} \equiv 5$ (mod 6)

$v_3 \equiv (N_3)^{-1} \equiv 30^{-1} \equiv 2^{-1} \equiv 4$ (mod 7)

$x \equiv a_1 v_1 N_1 + a_2 v_2 N_2 + a_{133} N_3$

$\equiv 126 + 875 + 360$

$\equiv 1361$

Chapter Ends ...

□□□

UNIT VI

Algebraic Structures

>> Syllabus :

- **Algebraic Structures :** Introduction Semigroup, Monoid, Group, Abelian Group, Permutation Groups, Cosets, Normal Subgroup, Codes and Group Codes, Ring, Integral Domain, Field.
- Applications of Algebraic Structures.

Mapping of Course Outcomes for Unit VI	CO6

- **Chapter 9 : Algebraic Structures**

UNIT VI

CHAPTER 9 Algebraic Structures

Syllabus

Algebraic Structures : Introduction Semigroup, Monoid, Group, Abelian Group, Permutation Groups, Cosets, Normal Subgroup, Codes and Group Codes, Ring, Integral Domain, Field.
Applications of Algebraic Structures.

Introduction

In this chapter, we study the additional structure with set, called as algebraic system induced by one or more binary operations on the elements of set. Algebraic system obey a set of rules which are similar to the rules of addition and multiplication of numbers in elementary algebra.

In first section we study a general algebraic system and it's properties such as semigroup, group, ring and fields. In other section we discuss the application of group in coding theory where techniques are developed for detecting and correcting errors in transmitted data.

9.1 ALGEBRAIC SYSTEM

UQ. Define : (i) Algebraic System

(SPPU(IT) - Q. 7(a), Dec. 18, 2 Marks)

An n-ary operation on a non-empty set A is a function $f : A^n \rightarrow A$, A^n being the product set of A.

Algebraic system is defined as, it is an ordered pair (A, *), where A is the set of elements and '*' is set of m-ary operations such that for $\forall$ **a, b ∈ A, a * b ∈ A**

Note that, here '*' is not multiplication if the set of m-ary operations like (A, +), (A, +, *) etc.

e.g.

(1) Let E = {0, 2, 4, ...} then set E along with binary operation '+' (addition) is an algebraic system i.e. (E, +) is algebraic system.

(2) Set of integers Z with operations '+' and '×' is the algebraic system i.e. (Z , +, ×)

9.2 PROPERTIES OF BINARY OPERATION

(1) **Commutative property :** A binary operation '*' is called as commutative if a * b = b * a, for all elements a, b ∈ A.

e.g.

(1) Binary operation of addition on set of integers is commutative, but subtraction operation is not commutative.

(2) Binary operation multiplication on the set of integers is commutative.

(2) **Associative property :** A binary operation * is called as associative if a * (b * c) = (a * b) * c, for all a, b, c, ∈ A

e.g.

(1) Binary operation addition is associative on set of integers but operation subtraction is not associative on set of integers.

(2) The binary operation multiplication is associative on the set of integers.

(3) **Idempotent property :** A binary operation * on set A is called as idempotent if it satisfy the property a * a = a for all a ∈ A.

e.g. consider lattice L with two operations meet (∧) and join (∨). Then ∧ and ∨ are the binary operations such that for all a ∈ A

a ∧ a = a

and a ∨ a = a

therefore both ∧ and ∨ operations satisfy the idempotent property.

(4) **Identity Element :** An element e ∈ A is called as identity element for the operation *, if

a * e = e * a = a, for all a ∈ A

(5) **Distributive property :** For a, b, c ∈ A and the operation * and + , the operation '*' distributes over '+', if a * (b + c) = (a * b) + (a * c)

Similarly operation + distribution over * if

a + (b * c) = (a + b) * (a + c)

(6) **Cancellation law :** For a, b, c ∈ A and operation * satisfies the cancellation law (property) then

a * b = a * c ⇒ b = c

(7) **Diverse element :** For each element a ∈ A, there exists an element b ∈ A such that

a * b = e, where e is identity element

9.3 SEMI-GROUP

UQ. Define : Semigroup,

Also show that <Z6, + > is abelian group.

(SPPU(IT) - Q. 4(a), May 14, May 16, 2 Marks

Q. 8(b), Dec. 16, 2 Marks)

An algebraic system (A, *) with a binary operation * on A is said to be a semigroup if * is associative.

a * (b * c) = (a * b) * c for all a, b, c ∈ A

e.g.

(1) If R is set of real numbers then (R, +) and (R, ×) is semigroup. This is also commutative therefore it is also called commutative semigroup.

(2) Z is set of integers and (Z, +) is commutative semigroup.

(3) For a non-empty set A, (P (A), U) is a commutative semigroup (P(A)) is a power set of A)

(4) (Z, –) is not semigroup as subtraction operation is not associative.

9.4 MONOID

UQ. *Define the following terms with suitable example : Monoid.*

SPPU(IT) - Q. 4(a), May 14, 2 Marks
Q. 7(b), Dec. 16, May 19, Dec. 19, 2 Marks
Q. 7(a), Dec. 18, 2 Marks

A semigroup (A, *) with an identity element is called as monoid.

Example

(1) (z, +) is a monoid where 0 is the identity element.

$7 + 0 = 0 + 7 = 7$

(2) (z, x) is a monoid where 1 is the identity element

$7 \times 1 = 1 \times 7 = 7$

9.5 GROUP

UQ. *Define the following terms with suitable example : Group.*

SPPU(IT) - Q. 4(a), May 14, 2 Marks
Q. 7(b), May 16, Dec. 16, Dec. 19, 2 Marks

An algebraic system (A, *) is said to be a group if it has following properties.

(1) Associative

i.e. $a * (b * c) = (a * b) * c$, for all a, b, c, ∈ A

[∴ semigroup]

(2) It has identity element [monoid]

$a * e = e * a = a$ for all a ∈ A

e is identity element

(3) It has inverse element

For each a ∈ A there exists b ∈ A such that

$a * b = b * a = e$

where e is identity element.

9.6 ABELIAN GROUP

UQ. *Define the following : (c) Abelian group.*

SPPU(IT) - Q. 7(b), May 19, Dec. 19
Q. 7(a), May 16, Dec. 18, 2 Marks

UQ. *Define : (iv) Abelian group.*
Also show that <Z6, + > is abelian group.

(SPPU(IT) - Q. 8(b), Dec. 16, 2 Marks)

An algebraic system (A, *) is called as abelian group if it is group and if it satisfies the commutative property such that

$a * b = b * a$, for all a, b ∈ A

Algebraic system (A, *)

(1) Associative	→ semigroup
(2) Identity element	→ Monoid
(3) Inverse Element	→ Group
(4) Commutative	→ Abelian group

9.7 SUBGROUP

UQ. *Define the following terms with suitable example : Subgroup.*

SPPU(IT) - Q. 7(b), Dec. 16, 2 Marks

Let (G, *) be any group then (H, *) is called as subgroup of group (G, *) if the following condition are satisfied.

(1) H is subset of G

(2) e ∈ H where e is identity element of (G, *)

(3) For any a ∈ H, a^{-1} ∈ H

(4) For a, b ∈ H, a * b ∈ H

Note that (H, *) itself a group with same identity element as that of (G, *)

All properties of G are inherited by H.

({e}, *) is a trival subgroup

9.8 SOLVED EXAMPLES

Ex. 9.8.1 : Consider the binary relation * defined on the set A = {a, b, c, d} by the following table.

*	A	b	c	D
a	A	c	b	D
b	d	a	b	C
c	C	d	a	A
d	D	b	a	C

Is * commutative ? Associative ?

☑ **Soln. :**

* is not commutative as

$a * c = b$ and $c * a = c$

* is not associative as

$a * (b * c) = a * b = c$ {∵ b * c = b}

but,

$(a * b) * c = c * c = a$

∴ $a * (b * c) \neq (a * b) * c$

Hence * is neither commutative nor associative

Ex. 9.8.2 : Consider binary operation * on set A, where A is set of rational numbers defined by

$a * b = a + b - ab, \quad \forall a, b \in A$

Find whether * is associative

☑ **Soln. :**

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc \\ (a * b) * c &= (a + b - ab) * c \\ &= a + b - ab + c - (a + b - ab)c \\ &= a + b - ab + c - ac - bc + abc \\ &= a + b + c - ab - ac - bc + abc \end{aligned}$$

Hence $a * (b * c) = (a * b) * c$, * is associative

Ex. 9.8.3 : For group G prove the following :

(1) The identity element is unique.

(2) Every element a in G has unique inverse a^{-1}

(3) ab = ac implies b = c

☑ **Soln. :**

(1) Consider (G, *) be a group and it has two identity elements e and e_1

Now $e * e_1 = e$ and $e * e_1 = e_1$ ∴ $e = e_1$

Therefore, the identity element is unique.

(2) Let us assume that an element x and G has two inverse values x^{-1} and y

Now $x = x^{-1} = e$ and $x * y = e$

$x^{-1} * (x * y) = x^{-1} * e = x^{-1}$

and

$(x^{-1} * x) * y = e * y = y$

For a group, associativity holds

∴ $x^{-1} * (x * y) = (x^{-1} * x) * y$ or $x^{-1} = y$

∴ It shows that, each 'a' in G has unique inverse

(3) Now, we have to prove that

$a * b = a * c$, implies b = c

Consider $a^{-1} * (a * b)$

i.e. $a^{-1} * (a * b) = a^{-1} * (a * c)$

or $(a^{-1} * a) * b = (a^{-1} * a) * c$

or $e * b = e * c$

or $b = c$

Hence, if a * b = a * c, then **b = c** ...{Proved}

Ex. 9.8.4 : Let A = {a, b}, which of the following tables defines a semigroup on A ? Which defines a monoide on A?

(i)

*	a	b
a	a	b
b	a	a

(ii)

*	a	b
a	a	b
b	b	b

☑ **Soln. :**

For the table (i)

$b * (b * b) = b * a = a$

and $(b * b) * b = a * b = b$

Associativity property not satisfied for table (i) but associativity property is satisfied by table (ii) as

$(b * b) * b = b * b = b$

and $b * (b * b) = b * b = b$

So, table (i) does not define semigroup but table (ii) defines semigroup

For table (ii) has an identity element a as

$a * b = b$

$b * a = b$

$a * a = a$

Table (ii) defines monoide but table (i) does not define monoide.

Ex. 9.8.5 : Le G be a group for fixed element of G

Let : G_x {a ∈ G : ax = xa}

Shot that G_x is subgroup of G, ∀ x ∈ G

☑ **Soln. :**

Let e be the identity element of G

∴ $ex = xe = x$ as $x \in G$

∴ e ∈ G_x and e is also the identity element of G_x

Let us see whether G_x is closed. Let a, b ∈ G_x

Then we have to prove that for every a, b ∈ G_x

ab ∈ G_x

$xa = ax$ and $bx = xb$

$(ab)x = a(bx) = a(xb)$

$= (ax)b = (xa)b = x(ab)$

∴ ab ∈ G_x is a subgroup of G.

Ex. 9.8.6 : ~~Consider group (Z, +). Let H = {3n : n ∈ Z}~~ show that H is a subgroup of Z.

☑ **Soln. :**

First we have to show that it is closed under +

3x and 3y ∈ H

∴ $3x + 3y = 3(x + y) \in H$

3x ∈ H then it's inverse – 3x also belongs to H.

∴ H is subgroup of (Z, +)

UEx. 9.8.7 (SPPU(IT) - Q. 7(a), Dec. 19, 7 Marks)

Let G = {even, odd} and binary operation ⊕ be define as :

⊕	even	odd
even	even	odd
odd	odd	even

Show that (G, ⊕) is a group.

☑ **Soln. :** G = {Even, odd}

1. Closure property :

   For ∀ even, odd ∈ G

   Even ⊕ odd · ∈ G

2. Semi group :

   ∀ Even, odd ∈ G

   Even ⊕ (Even ⊕ Even) = Even ⊕ Even = Even

   And (Even ⊕ Even) ⊕ Even = Even ⊕ Even = Even

   From associative property is satisfied.

   ∴ G is semigroup.

3. **Monoid :** The element even ∈ G is the identity element.

   as Even ⊕ Even = Even

   Even ⊕ Odd = Odd ⊕ Even = odd

   ∴ e = Even

4. **Inverse :** For element x ∈ G there is inverse

Element	Inverse
Even	Even
Odd	Odd

Therefore (G, ⊕) satisfies all the properties required to be a group.

∴ (G, ⊕) is a group.

UEx. 9.8.8 (SPPU(IT) - Q. 8(a), Dec. 19, 7 Marks)

Show that (G, +) is an abelian group

where G = (0, 1, 2, 3, 4, 5, 6, 7)

☑ **Soln. :** Refer solved examples 9.9.1.

UEx. 9.8.9 (SPPU(IT) - Q. 7(a), May 19, 7 Marks)

Consider the group (Z, +).

Prove that (Z, *) is an abelian group where * is binary operation defined by a*b = a + b + 1 for all a, b ∈ Z.

☑ **Soln. :** (Z, +) is a group.

We have,

$$a * b = a + b + 1,\ \forall a, b \in Z \qquad \ldots(I)$$

(i) **Closure property**

Refer topic Section 9.18

For a, b ∈ z ⇒ a + b + 1 ∈ z ⇒ a * b ∈ z

∴ Z is closed with respect to *.

(ii) **Associative :** Let a, b, c, ∈ Z

$$(a * b) * c = (a + b + 1) * c \qquad \text{(From I)}$$

$$= a + b + 1 + c + 1 = a + b + c + 2 \qquad \ldots(II)$$

Now, $a * (b * c) = a * (b + c + 1)$

$$= a + b + c + 1 + 1$$

$$= a + b + c + 2 \qquad \ldots(III)$$

From II and III

∴ * is associative on Z.

(iii) **Existence of identity**

Let e be the identity element in z.

∴ For any a ∈ Z, a * e = e * a = a

⇒ a + e + 1 = a

⇒ e + 1 = a

⇒ e = – 1 is the identity element.

(iv) **Existence of inverse**

Let a ∈ Z, suppose b ∈ Z is the inverse of a in z.

∴ a * b = e

a + b + 1 = e

a + b + 1 = – 1

a + b = – 2

b = – a – 2 ∈ Z

∴ The inverse exists for all a ∈ z.

Therefore (z, *) is a group.

Consider,

a * b = a + b + 1

= b + a + 1

= b * a . ∀ a, b ∈ Z

∴ * is commutative in z.

∴ (Z, *) is a abelian group.

UEx. 9.8.10 (SPPU(IT) - Q. 7(b), Dec. 18, 7 Marks)

Let R = {0°, 60°, 120°, 180°, 240°, 300°}

and * = binary operation, so that for a and b in R, a * b is overall angular rotation corresponding to successive rotations by a and then by b. Show that (R, *) is a Group.

☑ **Soln. :**

R = {0°, 60°, 120°, 180°, 240°, 300°}

1. **Closure property :** It is operation which is overall angular rotation corresponding to successive rotations.

   ∀ a, b ∈ R, a * b ∈ R

   e.g. 0° 60° ∈ R

   It satisfies closure property.

2. **Associative property :** Let a, b, c ∈ R

   The associative property is satisfied ∀ a, b, c ∈ R as

   0° * (60° * 180°) = (0° * 60°) * 180°

3. **Existence of identity**

   The element 0° ∈ R is the identity element as,

   0° * e = e * 0° = e

4. **Existence of inverse :** For every element a ∈ R there exists a^{-1} (Inverse of a) ∀ a ∈ R. Therefore existence of inverse element is in R.

   ∴ (R, *) is a group.

UEx. 9.8.11 (SPPU(IT) - Q. 8(a), Dec. 18, 6 Marks)

Show that $(G, +_8)$ is an abelian group where :

G = {0, 1, 2, 3, 4, 5, 6, 7}

☑ **Soln. :** Refer Solved Ex. 9.9.1.

UEx. 9.8.12 (SPPU(IT) - Q. 7(a), May 18, 7 Marks)

What is abelian group ? Show that $(Z_6, +)$ is an Abelian Group ?

☑ **Soln. :** For abelian group → Refer Section 9.6.

Let $Z_6 = \{0, 1, 2, 3, 4, 5\}$

1. **Closure property :** For all a ∀ Z_6

   $a +_6 a \in Z_6$

   ∴ Closure property satisfied.

$+6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

2. **Associative property**

   Let a, b, c ∈ Z_6

   Then ∀ a, b, c ∈ Z_6

   $(a +_6 b) +_6 C = a +_6 (b +_6 C)$ is satisfied.

3. **Existence of identity :** 0 is the identity element of $(Z_6, +_6)$

4. **Existence of inverse :** From the table of '$+_6$' we observe that ∀ a ∈ Z_6 there exists.

   it's inverse a^{-1} as,

   $1 +_6 3 = 4 = 3 +_6 1$

5. **Commutative :** From the table of '$+_6$' we can observe that,

   For every a, b ∈ Z_6

   $a +_6 b = b +_6 a$

   e.g. $1 +_6 3 = 3 +_6 1 = 4$

   Hence $(Z_6, +_6)$ is an abelian group.

UEx. 9.8.13 (SPPU(IT) - Q. 8(a), May 18, 7 Marks)

Let R = {0^2, 60^2, 120^2, 180^2, 240^2, 300^2} and * = binary operation, so that for a and b in R, a * b is overall angular rotation corresponding to successive rotations by a and then by b. Show that (R, *) is a Group.

☑ **Soln. :** Refer Example 9.8.10.

UEx. 9.8.14 (SPPU(IT) - Q. 8(b), May 18, 6 Marks)

Let G = {even, odd} and binary operation ⊕ be defined as :

⊕	even	odd
even	even	odd
odd	odd	even

Show that (G, ⊕) is a group

☑ **Soln. :** Refer Example 9.8.7

UEx. 9.8.15 (SPPU(IT) - Q. 7(A), May 17, 7 Marks)

Let Q_1 be the set of all rational numbers other than 1. Show that with operation * defined on the set Q_1 by (a * b = a + b – ab) is an Abelian group.

☑ **Soln. :**

We have a * b = a + b – ab. ∀ a, b ∈ Q_1.

(i) **Closure property :** Let a, b ∈ Q_1, a ≠ 1, b ≠ 1

∴ ab ≠ 1.

∴ a * b = a + b – ab ≠ 1 and a * b ∈ Q_1

Q_1 is closed with respect to *.

(ii) **Associatively :** Let a, b, c ∈ Q_1

$$(a * b) * c = (a + b - ab) * c$$

$$= a + b - ab + c - (a + b - ab) * c$$

$$= a + b - ab + c - ac - bc + abc \qquad \ldots(1)$$

$$a * (b * c) = a * (b + c - bc)$$

$$= a + b + c - bc$$

$$= a * (b + c - bc)$$

$$= a + b + c - bc - ab - ac + abc \qquad \ldots(2)$$

From (1) and (2).

(a * b) * c = a * (b * c)

∴ * is associative.

(iii) **Existence of identity :**

Let e be the identity element in Q_1

a * e = a

⇒ a + e = ae = q

⇒ e – ae = 0

⇒ e (1 – a) = 0

$$e = 0$$

$\therefore$ 0 is the identity element.

(iv) Existence of inverse element : Let $a \in Q_1$, $a \neq 1$

Suppose $b \in Q$ is the inverse of a

$$a * b = e$$

$$\Rightarrow \quad a + b - ab = 0$$

$$\Rightarrow \quad a + b(1 - a) = 0$$

$$b(1 - a) = -a$$

$$\Rightarrow \quad b = \frac{-a}{1-a}$$

$$= \frac{a}{a-1} \neq 1 \text{ and } b \in Q_1$$

$\therefore$ The inverse exists for all $a \in Q_1$

Therefore $(Q_1, *)$ is a group.

Also, for $a, b \in Q_1$

$$a * b = a + b - ab$$

$$= b + a - ba$$

$$a * b = b * a$$

$\therefore$ $(Q_1, *)$ is abelian group.

UEx. 9.8.16 (SPPU(IT) - May 16, 4 Marks)

Determine whether the set together with binary operation is a group. If it is group, determine if it is abelian, specify the identity and inverse.

☑ **Soln. :**

1. Not a group as 3 has no inverse. Inverse of 3 is $\frac{1}{3}$ which is not an integer.
2. e = 0 is the identity. Associativity property is satisfied. It is a group with the inverse of dement x as – x.

   It is all abelian group as a + b = b + a, for all a, b ∈ set of rational numbers.

UEx. 9.8.17 (SPPU(IT) - Q. 3(a), Dec. 14, 6 Marks)

Consider an algebraic system (G, *), where G is the set of all non-zero real number and * is a binary operation defined by a * b = ab/4. Show that (G, *) is an abelian group.

☑ **Soln. :**

1. Closure properly : Let $a, b \in G$

$$a * b = \frac{ab}{4} \in G \text{ as } ab \neq 0.$$

2. Associatively : Let $a, b, c \in G$

$$\text{Consider } a * (b * c) = a * \frac{bc}{4} = \frac{a(bc)}{8} = \frac{abc}{8}$$

$$\text{And } (a * b) * c = \frac{ab}{4} * c = \frac{(ab)c}{8} = \frac{abc}{8}$$

$\therefore$ * is associative in G.

3. Existence of identity : Let $a \in G$ and there exists e such that,

$$a * e = \frac{ae}{4} = a$$

$$\Rightarrow \quad ae = 4a$$

$$e = 4$$

$\therefore$ 4 is the identity element in G.

4. Existence of inverse :

Let $a \in G$ and $b \in G$ such that $a * b = e = 4$

$$\Rightarrow \quad \frac{ab}{4} = 4$$

$$ab = 16$$

$$b = \frac{16}{a}$$

$\therefore$ The inverse of a is $\frac{4}{a}$, $\forall a \in G$.

5. Commutativity

$$a * b = \frac{ab}{4}$$

$$\text{and } b * a = \frac{ba}{4} = \frac{ab}{4}$$

$\therefore$ * is commutative.

Therefore (G, *) is an abelian group.

9.9 ADDITION AND MULTIPLICATION MODULO

(1) Addition modulo

Let z be the set of integers and n be any number then addition modulo of z is denoted by $(z, +n)$ and it is given as

Consider $x, y \in Z$ then

Let $n \in Z$ be an integer

Let Z_n be the set of integers modulo n.

$$Z_n = \{0, 1, \ldots, n-1\}$$

The operation of addition modulo n is defined on Z_n as :

x + ny equals to remainder after x + y has been divided by n.

(2) Multiplication modulo

Let $n \in Z$ be an integer

Let Z_n be the set of integers modulo n

$$Z_n = \{0, 1, \ldots n-1\}$$

The operation of multiplication modulo n is defined on Z_n as :

x × ny equals the remainder after xy has been divided by n.

Let $\quad Z = \{1, 2\}$ and $n = 2$

then

	$(Z, +_2)$	1	2	
Addition	1	0	1	As (1 + 1) mod 2 = 0
modulo	2	1	0	(1 + 2) mod 2 = 1
				and so on

	$(Z, \times_2)$	1	2	
Addition	1	1	0	As (1 × 1) mod 2 = 1
modulo	2	0	0	(1 × 2) mod 2 = 0
				and so on

9.9.1 Examples Based on Addition and Multiplication Modulo

Ex. 9.9.1 : Let G = {0, 1, 2, 3, 4, 5, 6, 7} and operation '$+_8$' addition modulo 8 than show that $(G, +_8)$ is an abelian groups

☑ **Soln. :** Addition table for $+_8$

$+_8$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

From above table,

The element 0 is identity element as $0 +_8 x = x$

Where $x \in G$

Associativity property is satisfied.

For every element $x \in G$ there exist inverse x^{-1}

if $\quad x = 0$ then $x^{-1} = 0$ as $0 +_8 0 = 0$

$\quad x = 3$ then $x^{-1} = 5$ as $3 +_8 5 = 0$

$\quad x = 6$ then $x^{-1} = 2$ as $6 +_8 2 = 0$

and $\quad a +_8 b = b +_8 a$ for all $a, b \in G$

as $\quad 3 +_8 4 = 4 +_8 3 = 7$ and $3, 4 \in G$

Hence it is commutative

Therefore $(G, +_8)$ is an abelian group

Ex. 9.9.2 : Let Z = {0, 1, 2, 3, 4, 5} *prove that multiplication* modulo (z, x_6) is an abelian group.

☑ **Soln. :**

Multiplication table

x_6	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

From the above table 1 is an identity element

as $x \times 1 = 1 \times x = x$; for all $x \in z$

For each element it is observed that no inverse element exists there (z, x_6) is not a group and therefore it is not a abelian group.

Ex. 9.9.3 : Is (G, x_6) an abelian group ?

where G = {1, 2, 3, 4, 5, 6}

☑ **Soln. :**

Here $\quad 2, 3 \in G$ but $2 \times_6 3 = 0 \notin G$

$\therefore$ x_6 is not binary operation in G

$\therefore$ x_6 is not closed in G.

Hence (G, x_6) is not group

Thus (G, x_6) is not abelian group

Ex. 9.9.4 : Show that (G, x_7) is an abelian group,

where G = {1, 2, 3, 4, 5, 6}

☑ **Soln. :** Consider following table

x_7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(1) Closure property

Every element of table belongs to G

i.e. $a\, x_7\, b \in G$ for all $a, b \in G$

$\therefore$ x_7 is closed in G.

(2) Associativity

For any $a, b, c \in G$

$a\, x_7 (b\, x_7\, c) = (a\, x_7\, b)\, x_7\, C$

$\therefore$ x_7 is associative

(3) Existence of identity element

From the above table, we see 1 is the identity element as

$a \times_7 1 = 1 \times_7 a = a$, for all $a \in G$

(4) Existence or inverse

From table

As $1 \times_7 1 = 1$	1 is inverse of 1
As $2 \times_7 4 = 4 \times_7 2 = 1$	2 is inverse of 4 and vice versa.
As $3 \times_7 5 = 5 \times_7 3 = 1$	3 is inverse of 5 and vice versa
As $6 \times_7 6 = 1$	6 is inverse of 6

Hence inverse exists for all $a \in G$

(5) Commutative property

From the table for all $a, b \in G$

$a \times_7 b = b \times_7 a$

$\therefore$ $\times_7$ is commutative in G

$\therefore$ Therefore $(G, \times_7)$ is an abelian group

UEx. 9.9.5 (SPPU(IT) - Q. 8(b), Dec. 19, 6 Marks)

Prove that $G = \{0, 1, 2, 3, 4, 5\} = Z_6$ is an abelian group of order 6 with respect to addition modulo 6.

☑ **Soln. :**

$G = \{0, 1, 2, 3, 4, 5\} = Z_6$

☞ Table of addition

$+6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

From above table

1. $\forall a, b \in Z_6$

   $a +_6 b \in Z_6$, closure properly is satisfied.

2. Associative property is also satisfied as,

   $1 +_6 (3 +_6 4) = 1 +_6 1 = 2$

   $(1 +_6 3) +_6 4 = 4 +_6 4 = 2$

3. The element 0 is the identity element as

   $0 +_6 x = x, \forall x \in G.$

4. For every element $x \in G$ there exists inverse element x^{-1} as $x = 0$ then $x^{-1} = 0$ as $0 +_6 0 = 0$ also as $x = 3$ then $x^{-1} = 3$ as $3 +_6 3 = 0$.

5. $\forall a, b \in G, a +_6 b = b +_6 a$

   $\therefore$ Therefore it is commutative as $3 +_6 4 = 1 = 4 +_6 3$

   $\therefore$ $(Z_6, +_6)$ is an abelian group.

UEx. 9.9.6 (SPPU(IT) - Q. 8(a), May 19, 7 Marks)

Let $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Let R is a relation under the operations addition modulo 7 and multiplication modulo 7. Does this system form a ring ? Is it a commutative ring ?

☑ **Soln. :** Refer solved Ex. 9.9.1.

UEx. 9.9.7 (SPPU(IT) - Q. 8(b), Dec. 18, 7 Marks)

Show that (F, +, *) is a field where F is set of all rational numbers and + and * are ordinary additional and multiplication operators ?

☑ **Soln. :** To show (F, +) is an abelian group.

1. **Closure property :** For all $a, b \in R$ (Rational)

   $a + b \in R$ is satisfied.

2. **Associative :** For all $a, b, c, \in R$

   Associatively i.e. $a + (b + c) = (a + b) + c$ is satisfied.

3. **Identity element :** As R is the let of rational

   Let's 0 is the identity element.

   $\forall a \in R, a + 0 = 0 + a = a$

4. **Existence of inverse :** For $\forall a \in R$ there exist

   $a^{-1} \in R$ which inverse of a

5. **For all a, b, ∈ R**

   $a + b = b + a$

   Therefore (f, +) is also commutative.

   $\therefore$ (F, t) is abelian group.

   Now (F, *) is also satisfying associative

   Proper for $\forall a, b, c \leftarrow R$

   $a * (b * c) = (a * b) * C$

   And (f, *) is also commutative as,

   $a * b = b * a, \forall a, b \in R$

   Hence (F, +, *) is commutative ring.

   As R is the set of rational number, every non zero element in R has it's multiplicative inverse as,

   $a * a^{-1} = a^{-1} * a = a, \forall a \in R$

   Therefore, (F, +, *) is a field.

UEx. 9.9.8 (SPPU(IT) - Q. 7(B), May 17, 6 Marks)

Let I be the set of all integers. For each of the following determine whether * is an associative operation or not :

(1) $a * b = \max(a, b)$ (2) $a * b = \min(x + 2, b)$

(3) $a * b = a - 2b$ (4) $a * b = \max(2a - b, 2b - a)$.

☑ **Soln. :**

1. $a * b = \max(a, b)$

   Let $a, b, c \in I$, consider $b * c$

   $b * c = \max(b, c)$

   $\therefore b * c = b$ if $b \geq c$

   $= c$ if $c \geq b$

   Or let us suppose $(b, c) = b$ so that,

   $b * c = b$

   Then $a * (b * c) = \max(a, b)$

   $= a$, if $a \geq b$

   $= b$, if $b \geq a$

   or

   if $a \geq b$ then $a * (b * c) = a$

   Then $a * b = \max(a, b) = a$

   Hence $(a * b) * c = \max(a, c)$

   $= a$ since $a \geq b$ and $b \geq c$

   Hence if $a \geq b \geq c$ then $(a * b) * c = a * (b * c)$

   Similarly one can prove $(a * b) * c = a * (b * c)$

   For other cases.

   Therefore * is associative.

2. $a * b = \min(a + 2, b)$

   * is not associative since

   Let us take $4, 3, 1 \in I$

   $4 * (3 * 1) = 4 * 3 = 4$ while

   $(4 * 3) * 1 = 3 * 1 = 3$

   $\therefore 4 * (3 * 1) \neq (4 * 3) * 1$

3. $a * b = a - 2b$

   * is not associative since

   Let us take $4, 3, 1 \in I$

   $4 * (3 * 1) = 4 * 1 = 2$

   $(4 * 3) * 1 = -2 * 1 = -4$

   $\therefore 4 * (3 * 1) \neq (4 * 3) * 1$

4. $a * b = \max(2a - b, 2b - a)$

Let us take $4, 3, 1 \in I$

$4 * (3 * 1) = 4 * 5 = 3$ ...(I)

$(4 * 3) * 1 = 5 * 1 = 9$ ...(II)

From (I) and II,

$4 * (3 * 1) \neq (4 * 3) * 1$

$\therefore$ It is not associative.

UEx. 9.9.9 (SPPU(IT) - May 16, 6 Marks)

(i) The set of odd integers are closed under operation of multiplication.

(ii) Q, the set of all rational numbers closed under operation of addition.

☑ **Soln. :** Refer example no. 9.9.7

9.10 PERMUTATION GROUP

(I) Let $s = \{1, 2, 3, \ldots, n\}$ be a finite set with n distinct elements. If $R : s \to s$ is a bijective function then R is called a permutation of degree n.

Let $R(a_1) = b_1, R(a_2) = b_2, f(a_3) = b_3 \ldots + (a_n) = b_4$

Then the permutation is denoted by

$$f = \begin{pmatrix} a_1 & a_2 \ldots & a_n \\ b_1 & b_2 \ldots & b_n \end{pmatrix} \begin{array}{l} \longrightarrow \text{Elements in domain} \\ \longrightarrow \text{Elements in co-domain} \end{array}$$

(II) The permutation corresponding to the identity function is called as identity permutation

$$I = \begin{pmatrix} a_1 & a_2 & a_3 \ldots & a_n \\ b_1 & b_2 & b_3 \ldots & b_n \end{pmatrix}$$

(III) If

$$f_1 = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}, f_2 = \begin{pmatrix} b_1 & b_2 & \ldots & b_n \\ c_1 & c_2 & \ldots & c_n \end{pmatrix}$$

then product of two permutations

$$f_1 f_2 = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ c_1 & c_2 & c_3 & \ldots & c_n \end{pmatrix}$$

9.10.1 Example Based on Permutation Group

Ex. 9.10.1 : Show that the set of all permutations on $s = \{1, 2, 3\}$ forms a group w.r.t. permutation multiplication.

☑ **Soln. :**

Let

$P_3 = \{f/f$ is a permutation degree 3$\}$

$$P_3 = \begin{pmatrix} (123) & (123) & (123) & (123) & (123) & (123) \\ (123) & (132) & (213) & (231) & (312) & (321) \end{pmatrix}$$

$\quad\quad I \quad f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5$

Consider following table

O	I	f_1	f_2	f_3	f_4	f_5
I	I	f_1	f_2	f_3	f_4	f_5
f_1	f_1	I	f_3	f_2	f_5	f_4
f_2	f_2	f_4	I	f_5	f_1	f_3
f_3	f_3	f_5	f_1	f_4	I	f_2
f_4	f_4	f_2	f_5	I	f_3	f_1
f_5	f_5	f_3	f_4	f_1	f_2	I

(1) Closure property : Each element or table belongs to P_3

e.g. for any $f, g \in P_3$, $Reg \in P_3$

$\therefore$ P is called w.r.t. composition of permutation

(2) Associativity

For any f, g, h $\in P_3$

f o (g o h) = (fog) oh

Hence P_3 is associative

(3) Existence of identity

By observing, first row and first column of table.

For any $R \in P_3$, $\exists\, I \in P_3$

Such that f o I = I o f = f

$\therefore$ I is the identity permutation

(4) Existence of the inverse

From table, identify the identity element I.

As $\quad IoI = I \Rightarrow I^{-1} = I$

$f_1 o f_1 = I \Rightarrow f_1^{-1} = f_1$

$f_2 o f_2 = I \Rightarrow f_2^{-1} = f_2$

$f_3 o f_4 = I = f_4 o f_3 = f_3^{-1} = f_4$ and $f_4^{-1} = f_3$

$f_5 o f_5 = I = f_5^{-1} = f_5$

$\therefore$ Every element of P has inverse in P_3

Hence, $(P_3, \cdot)$ is a group

But, $\quad f_1 o f_2 \neq f_3$ and $f_2 o f_1 = f_4$

$\Rightarrow$ $f_1 o f_2 \neq f_2 o f_1 \Rightarrow$ It is not commutative

$\Rightarrow$ $(P_3, 0)$ is not abelian group

$\Rightarrow$ $(P_3, 0)$ is the smallest non-abelian group

9.11 COMPLEXES OF A GROUP

Let (G, *) be a group then any non-empty subset of group G is called a complex of G e.g.

$H_1 = \{1, 2, 3, 4, 5\}$, $\quad H_2 = (1, 2, 3, \ldots..)$, $\quad H_3 = z$

are complexes of group (R, +)

9.12 COSET

Let (C, *) be a group and H be any subgroup of G.

Let a $\in$ G be any element, then the set

H * a = {(h * a} | $\forall$ h $\in$ H} is called a right coset of H in G and

a * H = {(a * h} | $\forall$h$\in$H} is called a left coset of H in G

> **Note**
> (1) H * a and a * H are subsets of G
> (2) $\because$ (G, *) is an abelian group then H * a = a * H in G

e.g.

(1) Let (Z, +) is a group and

H = {.... – 10, – 5, 0, 5, 10, ...} is a subgroup of G = z

$\therefore$ For $1 \in Z$, H + 1 = {.... – 9, – 4, 1, 6, 11, ...}

$3 \in z$, H + 3 = {.... – 7, – 2, 3, 8, 13, ...}

$5 \in z$, H + 5 = {.... – 5, 0, 5, 10, ...} = H

are right cosets of H in G

9.13 ORDER OF AN ELEMENT OF A GROUP

Let (G, $\cdot$) be a group. The smallest positive integer is called the order of an element a $\in$ G, if

$a^n = e$ (identity element in G)

It is denoted by $0(a) = \eta$

If no such positive number exists, then are say that 'a' is of infinite order or zero order.

> **Note**
> (1) For the order of the group is the number of distinct element in G.
> (2) The order of the identity element is 1 i.e. 0(e) = 1
> (3) In a group G, $0(a) = 0(a^{-1})$, $\forall a \in G$
> (4) In a group G, $0(a) \leq 0(G)$

Examples

(1) G = {1, – 1, i, – i} is a multiplicative group 1 = identity element in G and 0(G) = y

0 (1) = 1

0 (– 1) = 2 $\quad$ as $\quad (-1)^2 = 1$

0 (– i) = 4 $\quad$ as $\quad (i)^4 = 1$

0 (– i) = 4 $\quad$ as $\quad (i)^4 = 1$

(2) (z, +) is a group of infinite order 'o' is the identity element 0 (0) = 1 we have 5 $\in$ z and 5^n = 5 + 5 + ... + n times = ns $\neq$ 0

$\therefore$ o (5) = infinity or zero

9.14 CYCLIC GROUP

A group G is called as cyclic group if there exists at least one element a $\in$ G such that every element x $\in$ G can be written as $x = a^m$ where m is some integer.

The element a is called the generator of G and denoted as G = <a>

Example

Fourth roots of unity form a cyclic group with respect to multiplication.

G = {1, – 1, i, – i}

We have, $(i)^1 = i$, $(i)^2 = -1$, $(i)^3 = -i$, $(i)^4 = 1$

$\therefore$ i is the generation of G $\Rightarrow$ G is cyclic group moreover, $(-i)^1 = -i$, $(-i)^2 = -1$, $(-i)^3 = i$, $(-i)^4 = 1$

$\therefore$ – i is also generator of G

$\therefore$ $\quad$ G = <i> = <– i>

9.15 NORMAL SUBGROUPS

A subgroup H of group (G, *) is said to be a normal subgroup of G if for all g $\in$ G and for all h $\in$ H

$g * h * g^{-1} \in H$ $\quad$ (i.e. $g h g^{-1} \in H$)

Every group G possesses at least two normal subgroups namely {e} and G. These groups are called as improper normal subgroups.

Simple group

A group G is said to be simple group if it has only two normal subgroups {e} and G.

> **Note**
> (1) Every subgroup of an abelian group is normal.
> (2) The intersection of normal subgroups is a normal subgroup

Examples : (z, t) is an abelian group

$\therefore$ (2z, +) and (3z, +) are normal subgroup of (z, +)

9.16 QUOTIENT GROUPS

Let (G, *) be a group and N be a normal subgroup of G. Let G / N be the collection of all cosets of N in G.

G / N = {N * a / a $\in$ G}

(G / N, *) is called as quotient group or factor group.

9.16.1 Examples Based on Quotient Groups

Ex. 9.16.1 : Give an example of a finite abelian group which is not cyclic.

☑ **Soln. :** Let G be the set of the four real matrices.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

It can be easily observed that G is an abelian group with respect to multiplication of matrices.

The identity element of this group is the identity matrix I.

Let us find the order of each element of G.

We have, 0 (I) = I

Also

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad (\because 0\,(A) = 2)$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad (\because 0\,(B) = 2)$$

$$C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$(\because 0\,(C) = 2)$

Now, G is a group of order 4 and G contains no element of order 4. Therefore G is not a cyclic group. Hence G is finite abelian group which is not cyclic.

Ex. 9.16.2 : Show that the multiplicative group {1, – 1} is a subgroup of multiplicative group {1, – 1, i, – i}

☑ **Soln. :**

Let $\quad$ H = {1, – 1} and G = {1, – 1, i, – i},

H $\in$ G consider composition table

x	1	– 1
1	1	– 1
– 1	– 1	1

(1) Closure property : All elements of table belongs to H. Hence it is closed w.r.t multiplication

(2) Multiplication of complex numbers is associative

(3) Existence of identity : 1 $\in$ H, is identity element.

(4) Existence of inverse : We have $(1)^{-1} = 1$ and $(-1)^{-1} = -1$

Therefore, (H, x) is subgroup of group (G, x)

Ex. 9.16.3 : Show that the set (1, w, w^2) is a cyclic group of order 3 with respect to multiplication, w is the cube root of unity.

☑ **Soln. :**

Let $\quad$ G = {1, w, w^2} and $w^3 = 1$

We have, $\quad 1 = 1^1 = w^3 = (w^2)^3$

$w = w^1 = (w^2)^2$

$w^2 = w^2 = (w^2)^1$

Thus every element of G can be expressed as in the powers of w or w^2. Therefore G is the cyclic group and it's generators are w and w^2.

9.17 HOMOMORPHISM OF GROUPS

Let $(G_1, *)$ and $(G_2, 0)$ be two groups a function f : $(G_1, *) \to (G_2, 0)$ is said to be homomorphism if f(a * b) = f(a) 0 f(b) for all a, b $\in$ G, i.e. a * b in $G_1 \to$ f(a) of (b) in G_2

A homomorphism from G to itself is called as endomorphism.

Properties of group homomorphism.

Let $f : G_1 \to G_2$ be group homomorphism and $(G_1, *)$ and $(G_2, *)$ are groups then

(i) $f(e_1) = f(e_2)$

(ii) $f(a^{-1}) = [f(a)]^{-1}$

Proof

(i) Let $a \in G_1$ and $f(a) = G_1$ and e_2 is the identity element in G_2

$\therefore$ $f(a) \circ e_2 = f(a)$

$= f(a * e_1)$

$f(a) \circ e_2 = f(a) \, 0 \, f(e_1)$

$f(e_1) = e_2$

(2) Let $a \in G$, then $a^{-1} \in G$

and $e_2 = f(e_1)$

$= f(a * a^{-1})$

$e_2 = f(a) \, 0 \, f(a^{-1})$

$\Rightarrow$ $f(a^{-1}) = [f(a)]^{-1}$

9.18 ISOMORPHISM OF GROUPS

UQ. *What is homomorphism and automorphism in an algebraic system ? Explain by giving example of each.*

(SPPU(IT) - Q. 8(b), May 19, 7 Marks)

UQ. *Define each of the following :*

[illegible]

Also show that <Z6, +> is abelian group.

(SPPU - Q. 8(b), Dec. 16, 2 Marks)

Let $(G_1, *)$ and $(G_2, 0)$ be two groups. A function $f : (G_1, *) \to (G_2, 0)$ is said to be isomorphism.

If (i) f is homomorphism from $G_1 \to G_2$

(ii) f is bijective function

of $f : G_1 \to G_2$ is an isomorphism of groups then G_1 and G_2 are called as isomorphic groups and denoted by $G_1 \cong G_2$

An isomorphism from G to itself is called as automorphism of group G.

9.18.1 Examples Based on Isomorphism of Groups

Ex. 9.18.1 : Let G be a group with identity e show that a function $f : G \to G$ defined by $f(a) = e \ \forall a \in G$ is a homomorphism (endomorphism)

☑ **Soln. :**

We have $f : G \to G$ and $f(a) \ e, \forall a \in G$

Let, $a, b \in G \Rightarrow f(a), f(b) \in G$

$\therefore$ $f(a * b) = e$

$= e * e$

$= f(a) * f(b)$

$\therefore$ f is homomorphism

Ex. 9.18.2 : Let $(G_1, *_1)$, $(G_2, *_2)$ and $(G_3, *_3)$ be semigroups and $f : G_1 \to G_2$ and $g : G_2 \to G_3$ be homomorphism. Prove that g o f is homomorphism from $G_1 \to G_3$

☑ **Soln. :**

Let $a, b \in G_1$ then

$(g \circ f)(a *_1 b) = g(f(a *_1 b)) = g(f(a) *_2 f(b))$

$= g(f(a)) *_3 (f(b)) = (g \circ f)(a) *_3 (g \circ f)(b)$

$\therefore$ g o f is homomorphism from G_1 to G_3

Ex. 9.18.3 : Let G be the group of real numbers under addition and G' be the group of positive real numbers under multiplication.

Let $f : G \to G'$ be defined as $f(x) = e^x$

Show that f is an isomorphism.

☑ **Soln. :**

$f(x + y) = e^{x+y} = e^x \times e^y$

$= f(x) \times f(y)$

$\therefore$ The mapping $f : G \to G'$ is a homomorphism

Since the mapping $f(x) = e^x$ is one-one onto we can conclude that

$f : G \to G'$ is an isomorphism.

Ex. 9.18.4 : Let G be a group. Show that the function $f : G \to G$ defined by $f(a) = a^2$ is a homomorphism iff G is abelian.

☑ **Soln. :**

Let G be abelian and $a, b \in G$ then

$f(ab) = (ab)^2 = abab = aabb = a^2b^2 = f(a) \cdot f(b)$

Hence if G is abelian then f is a homomorphism.

Conversely if f is a homomorphism then,

$f(ab) = f(a) \cdot f(b)$

or $(ab)^2 = a^2b^2$

or $abab = aabb = ba = ab$

$\therefore$ If f is homomorphism then G is abelian.

9.19 RINGS, INTEGRAL DOMAIN AND FIELDS

UQ. *Define : (iv) Isomorphic group*

(SPPU(IT) - May 16, 2 Marks)

UQ. *Define : (ii) Isomorphism of group;*

Also show that <Z6, +> is abelian group.

(SPPU(IT) - Q. 8(b), Dec. 16, 2 Marks)

In this section we discuss the algebraic structures with two binary operation such as rings, integral domain and fields.

9.19.1 Rings

Let R be a non empty set equipped with two binary operations called addition and multiplication and denoted by '+' and '·' respectively.

An algebraic structure $(R, +, \cdot)$ is called as ring if it satisfies following axioms.

(1) $(R, +)$ is an abelian group i.e.

(i) Closure property : for $a, b \in R, a + b \in R$

(ii) Associativity : for $a, b, c \in R$, $a + (b + c) = (a + b) + c$

(iii) Existence of identity : for any $a \in R, \exists e \in R$ Such that $a + e = e + a = a$

Then, e is called as additive identity element

(iv) **Existence of Inverse** : for each $a \in R \ \exists (-a) \in R$ such that

$a + (-a) = (-a) + a = 0$

$-a$ is called as additive inverse of a

(v) **Commutative property** : For $a, b \in R$, $a + b = b + a$

(2) $(R, \cdot)$ is a semigroup i.e.

(i) Closure property : $\forall a, b \in R, a \cdot b \in R$

(ii) Associativity : $\forall a, b, c \in R, a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(3) Multiplication distributes over addition $\forall a, b, c \in R$

(i) $a \cdot (b + c) = a \cdot b + a \cdot c$ ... Right distribution law

(ii) $(a + b) \cdot c = a \cdot c + b \cdot c$ ....Left distribution law

9.19.2 Commutative Ring

A ring $(R, +, \cdot)$ is called as commutative ring if $\forall a, b \in R, a \cdot b = b \cdot a$.

9.19.3 Ring With Unity

A ring $(R, +, \cdot)$ is called as ring with unity if $\forall a \in R$, $\exists 1 \in R$, such that $a \cdot 1 = 1 \cdot a = a$

e.g.

(1) $(z, +, \cdot)$ is commutative ring with unity.

(2) $(2z, +, \cdot)$ is commutative ring without unity when $2z$ = set of even integers.

9.19.4 Properties of A Ring

If $(R, +, \cdot)$ is a ring with identity 0 and unit element 1 then following are true for all $a, b, c \in R$

(i) $a \cdot 0 = 0 \cdot a = 0$

(ii) $a \cdot (-b) = (-a)(b) = -(a \cdot b)$

(iii) $(-a) \cdot (-b) = a \cdot b$

(iv) unit element is unique

9.19.5 Subring

Let $(R, +, \cdot)$ be a ring. A non empty subset s of R is called as subring of R if $(s, +, \cdot)$ is a ring e.g. $(z, +, \cdot)$ is a subring $(R, +, \cdot)$

9.19.6 Zero Devisors

Let $(R, +, \cdot)$ be a commutative ring. An element $a \neq 0$ in R is called as zero divisor.

if $\exists b \neq 0$ in R such that $a \cdot b = 0$

A ring $(R, +, \cdot)$ is said to be without zero divisors.

if $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$, $\forall a, b \in R$

e.g.

1. $\overline{2}$ is a zero divisor in $(Z_4, +, \cdot)$ as $\overline{2} \cdot \overline{2} = \overline{4} = 0$

2. $[(M_{2\times2}, R), +, \cdot]$ is a ring with zero divisor.

$$\text{as } A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

but $A \neq 0$ and $B \neq 0$

3. $(Z, +, \cdot)$ is a ring without zero divisors.

i.e. $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$.

9.19.7 Integral Domain

UQ. *Define and illustrate with example : Integral Domain.*

(SPPU(IT) - Q. 3(a), May 14, 2 Marks)

– A commutative ring with zero divisors is called as integral domain.

e.g.

1. $(R, +, \cdot)$, $(Z, +, \cdot)$ are integral domains.

2. $(Z_4, +, \cdot)$ is ring with zero divisors.

$\therefore$ It is not integrate domain.

9.19.8 Field

UQ. Define and illustrate with example : Fields (SPPU(IT) - Q. 3(a), May 14, 2 Marks)

- A commentator ring with unity in which every non zero element possesses their multiplicative inverse, is called as field.
- A field is an integral domain.
  1. (R, + , ·), (Q, +, ·) are fields.
  2. (Z, +, ·) is integral domain but not field.

9.19.9 Ring Homomorphism

- Let (R, +, *) and (S, +, *') be two rings.
- A function $\phi : R \to S$ is called a ring homomorphism.
- For any a, b, ∈ R
  (i) $\phi(a + b) = \phi(a) + \phi(b)$
  (ii) $\phi(a \wedge b) = \phi(a) *' \phi(b)$
- Of ϕ is bijective then it is called as ring isomorphism.
- The kernel of ring homomorphism is defined as the set $\{a \in R \mid \phi(x) - 0'\}$
- It is denoted by Ker (ϕ) or Ker ϕ

9.19.10 Examples Based on Rings, Integral Domain and Fields

Ex. 9.19.1 : Show that $S = \{a + b\sqrt{2} ; a, b \in Z\}$, for operation +, * is an integral domain.

☑ **Soln. :**

1. $(a + b\sqrt{2}) \cdot 1 = a + b\sqrt{2} = 1 \cdot (a + b\sqrt{2})$
   ∴ 1 is the identity element.
2. $(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)$
   $\sqrt{2} = (c + d\sqrt{2}) * (a + b\sqrt{2})$
   ∴ (s, *) is commutative.

∴ S is a commutative ring with unity 1 to prove that S does not have divisors of zero we proceed with,

$$(a + b\sqrt{2}) * (c + d\sqrt{2}) = 0$$

$$\text{or } (ac + bd) + (ad + bc)\sqrt{2} = 0$$

For the above quaton to be true.

$$(ac + bd) = 0 \text{ and } ad + bc = 0$$

$$ac + bd = 0$$

1. a and b are zero. 2. a and d are zero.
3. c and b are zero. 4. c and d are zero.

$$ac + bc = 0$$

1. a and b are zero.
2. a and c are zero.
3. d and b are zero.
4. d and c are zero.

Now we conclude that either a and b are zero or c and d are zero.

∴ Either $a + b\sqrt{2}$ is zero or $c + d\sqrt{2}$ is zero.

$$\therefore (a + b\sqrt{2}) * (c + d\sqrt{2}) = 0$$

$$\Rightarrow (a + b\sqrt{2}) = 0$$

$$\text{Or } (c + d\sqrt{2}) = 0$$

The given ring 'S' is a commutative ring without divisors of zero and hence it is an integral domain.

Ex. 9.19.2 : Let $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Let R be the relation under operation addition modulo 7 and multiplication modulo 7. Does this system form a ring ? is it commutative ring ?

☑ **Soln. :**

Let ⊕ : Stands for addition modulo 7.

⊗ : Stands for multination modulo 7.

☞ Table for ⊕

⊕	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

☞ Table for ⊗

⊗	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

1. $(Z_8, \oplus)$ is an abelian group as,
   $a \oplus b = b \oplus a$ for all $a, b \in Z_8$

2. $(Z_8, \otimes)$ is a semigroup. It can be verified from the product table that,
   $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ for all $a, b, c, \in Z_8$
3. The operation ⊗ is distributive over ⊕.
   It may be noted that the remainder of a · (b + c) will be same as the remainder of ab + ac when divided by any element.
4. The operation ⊗ is commutative as it can be verified from the table of ⊗.
   $a \otimes b = b \otimes a$ for every $a, b, \in Z_8$.
   ∴ System forms a ring and it is commutative ring.

Ex. 9.19.3 : Prove that (R, +, *) is a ring with zero divisors, where R is 2 × 2 matrix and + and * are usual addition and multiplication operative.

☑ **Soln. :**

R = {a set of 2 × 2 matrices}

First we show that (R, +) is an abelian group

(i) Closure : Since the sum of 2 × 2 matrices is also a 2 × 2 matrix closure is satisfied.

(ii) Associative : If A, B and C are 2 × 2 matrices then, $(A_{2\times2} + B_{2\times2}) + C_{2\times2} = A_{2\times2} + (B_{2\times2} + C_{2\times2})$ always exists.

(iii) Identity : The identity element $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$ belongs to R and $A + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A$ identity is satisfied.

(iv) Inverse : The inverse of a 2 × 2 matrix say A is always – A which is also 2 × 2 matrix.

(v) Commutative : The sum of 2 matrices is always commutative.

∴ (R, +) is an abelian group. (R, ·) is also a semigroup with respect to multiplication also multiplication is distributive with respect to addition. Hence (R, +, ·) is a ring.

Now, Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ be the two nonzero elements of this ring.

Then we have $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus the product of two non-zero elements or the ring is equal to the zero element of ring.

∴ AB is ring with zero divisors.

Ex. 9.19.4 : Prove that the set E or all even integers is a commutative ring with respect to usual addition and multiplication, but it has no unit element.

☑ **Soln. :**

Let 2z be the set of all even integers, then (i) for any two element 2m, 2n ∈ 2z, m, n ← z.

$$2m + 2n = 2(m + n) \in 2z \text{ and}$$

$$2m \cdot 2n = 2\,mn \in 2z \text{ and}$$

$$2m \cdot 2n = 2\,mn \in 2z$$

Since m + n ∈ z and 2mn ∈ z

(ii) Since (Z, +) (Z, ·) are associative and commutative 2z ∈ z so (2z, +) and (2z, ·) are also associative and commutative.

(iii) For any 2m ∈ 2z – 2m = (– m) ∈ 2z ∈ z, so "·" is distributive on "+" over 2z.

(iv) 0 ∈ 2z such that a + 0 = 0 + a, ∀ a ∈ 2z

Hence, (2z, +, ·) is a commutative ring but it is not a ring with unity as it does not contain I ∈ E, (As it is the set of all even integers).

Ex. 9.19.5 : Consider a ring (R, +, *) defined by a * a = a. Determine whether ring is commutative or not ?

☑ **Soln. :**

Let a, b ∈ R

$$\begin{aligned} \text{and } (a + b)^2 &= a + b \\ (a + b)(a + b) &= a + b \\ a^2 + ab + ba + b^2 &= a + b \\ a + ab + ba + b &= a + b \\ (a + b) + (ab + ba) &\ a + b \\ ab + ba &= 0 \\ a + b &= 0 \\ a + b &= a + a \\ \therefore\ b &= a \\ \therefore\ ab &= ba \end{aligned}$$

∴ R is commutative ring.

UEx. 9.19.6 (SPPU(IT) - Q. 8(A), May 17, 7 Marks)

Let Z_n be the set of integers {0, 1, 2, ..., n – 1}. Let ⊕ be a binary operation on Z_n such that :

$$a \oplus b = \begin{cases} a + b & \text{if } a + b < n \\ a + b - n & \text{if } a + b \geq n \end{cases}$$

Let ⊙ be a binary operation on Z_n such that [illegible]

a ⊙ b = the remainder of ab divided by n

Show that $(Z_n, \oplus, \odot)$ is Ring.

☑ **Soln. :**

Let us consider $(Z_n, \oplus)$ take Z_8.

1. **Closure :**
   $\forall\ a, b \in Z_n$

$$a \oplus b = \begin{cases} a+b \in Z_n \text{ for } a+b<n \\ a+b-n \in Z_n \text{ for } a+b \geq n \end{cases}$$

Therefore Z_n is closed under $\oplus$

2. **Associative :** Let a, b, c $\in Z_n$

Let us take 4, 3, 1 $\in Z_8$

$(4 \oplus 3) \oplus 1 = 7 \oplus 1 = 0$ ...(I)

$4 \oplus (3 \oplus 1) = 4 \oplus 4 = 0$ ...(II)

From (I) and (II),

$\therefore$ $\oplus$ is associative in Z_n.

3. **Existence of identity :** 0 is the identity element in Z_n.
4. **Existence of inverse :** For all $a \in Z_n$ there exists a^{-1} element which inverse of a.

$\therefore$ $(Z_n, \oplus)$ is a group.

Now consider (1) operation.

(Zn, $\odot$)

1. $\odot$ is associative as,

Let 4, 3, 1 $\in Z_8$

$4 \odot (3 \odot 1) = 4 \odot 3 = 4$ ...(I)

$(4 \odot 3) \odot 1 = 4 \odot 1 = 4$ ...(II)

From (I) and (II),

$\therefore$ $\odot$ is associative with Z_n.

2. $\boxdot$ is observed that,

$\odot$ is distributive ove $\oplus$.

As for a, b, c $\in Z_n$.

$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$

$\therefore$ $(Z_n, \oplus, \odot)$ is a ring.

9.20 GROUP CODES

Codes are used for representation of digits and alphabets. They are very sensitive to transmission errors that may occur because of equipment Relive or noise in the transmission channel.

9.20.1 Error Detecting Codes

UQ. Define the following terms with suitable examples. Ring SPPU(IT) - Q. 3(a), May 14, Q. 7(b), May 16, Dec. 16, 2 Marks)

- Let us consider a 4 bit BCD codes given by {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001} of an error occurs in the least significant digit of 0010, the code word 0011 results, and since it is a valid code word, it is incorrectly interpreted by receiver.
- If a code possesses the properly that the occurrence of single error transforms a valid code word into a invalid code word, it is said to be a single-error detecting code.
- The error detection in BCD code is accomplished by parity check. The basic idea in the parity check is to add an extra bit to each code word or a given code, so as to make the numbers of 1's in each code word either odd or even. Even parity mechanism is used in the code of table as follows :

Decimal digit	Even parity BCD Code	Parity
0	0000	0
1	0001	1
2	0010	1
3	0011	0
4	0100	1
5	0101	0
6	0110	0
7	0111	1
8	1000	1
9	1001	0

- In the above table, number of is in a code word is even. Now if a single error occurs, a valid code gets transformed into a invalid one. Thus the detection of single error becomes taught forward.
- The distance between two code words is the number of digits that must change in one word so that the other word results. The distance between 1010 and 0100 is 3. The 2 code words differ in 3 bit positions as shown as follows :

1	0	1	0
0	1	0	0

Distance is 3

9.20.2 Hamming Distance

- The distance between two codes words is also known as hamming distance. The minimum distance of a code is the minimum of the hamming distances between all pairs of code words in the code.

Example :

Let the 3 code words.

x = 1001, y = 0100 and z = 1000 are given

- Now, the hamming distance between x and y represented as H (x, y) = 3.

- Similarly, H (x, z) = 1, H (y, z) = 2

Distance of above code = min (3, 1, 2) = 1

9.20.3 Examples Based on Hamming Distance

Ex. 9.20.1 : Find hamming distance between code words of C = {(0000), (0101), (1011), (0111)}

Rewrite the message by adding even parity check bit.

☑ **Soln. :**

☞ **Hamming distance**

Sr. No.	Code (m)	Code (n)	Hamming distance H (m, n)
1.	0000	0101	2
2	0000	1011	3
3	0000	0111	3
4	0101	1011	3
5	0101	0111	1
6	1011	0111	2

☞ **Even parity code**

Code word	Parity check
0000	0
0101	0
1011	1
0111	1

Ex. 9.20.2 : Find hamming distance between x and y

(i) a (x) = 110110, y = 000101

(ii) b (x) = 001100, y = 010110

☑ **Soln. :**

(i)

x = 110110

y = 000101

x x xx → Places at bits differ

Hamming distance = 4

(ii)

x = 001100

y = 010110

x x x ← Places at bits differ

Hamming distance = 3.

Ex. 9.20.3 : Find the minimum distance of the following (2, 5) encoding function e.

e (00) = 00000

e (10) = 00111

e (01) = 01110

e (11) = 11111

☑ **Soln. :**

d (00000, 00111) = 3

d (00000, 01110) = 3

d (00000, 11111) = 5

d (00111, 01110) = 2

d (00111, 11111) = 2

d (01110, 11111) = 2

$\therefore$ Minimum distance of code = Minimum of all distances = 2

Ex. 9.20.4 : Find the minimum distance of an encoding function e : $B^2 \to B^5$ given as : e (2, 5)

e (0, 0) = 00000 e (0, 1) = 10011

e (1, 0) = 01110 e (1, 1) = 11111

☑ **Soln. :**

d (00000, 10011) = 3

d (00000, 01110) = 3

d (00000, 11111) = 5

d (10011, 01110) = 4

d (10011, 11111) = 2

d (01110, 11111) = 2

$\therefore$ Minimum distance of code = Minimum distance of all 2.

UEx. 9.20.5 SPPU(IT) - Q. 7(b), May 18, 6 Marks

Find the hamming distance between code words of : C = {(0000), (0101), (1011), (0111)}.

Rewrite the message by adding even parity check bit and odd parity check bit.

☑ **Soln. :**

C = {(0000), (0101), (1011), (0111)}

Refer Solution from Section 9.21.3 Solved Ex. 9.21.1.

9.20.4 Error Correcting Code

- A code is called as an error correcting code if the correct code word always be found from the erroneous word in an error correcting code, an erroneous word can be uniquely associated with only one of the valid code words.

- A code can correct all combinations of n or fewer errors if and only if the distance of the code is at least 2n + 1.

9.20.5 Group Code

Definition :

A group code is a group (G, ⊕) with following properties :

1. *Identity element e = 000, ..., 000.*
2. *Each element has it's own inverse.*
3. *⊕ denotes the addition modulo 2 on (0, 1). It is same as exclusive (OR).*

⊕	0	1
0	0	1
1	1	0

9.20.6 Hamming Error Correction Code

- Let us consider a code word length n, in which there are k parity bits and m information bits.
- To each code word, k parity checking bits are added. These parity bits are denoted as, $P_1, P_2, \ldots P_k$.
- The number k must be large enough to describe the location of any of the m + k possible k must satisfy the inequality.

$$2^k \geq m + k + 1$$

For example, if m is 4 then k must be 3.

So that $2^3 \geq 4 + 3 + 1$.

- Now, we will see how the hamming code is constructed for m = 4 and k = 3 parity bits are constructed in such way that in case of any error, these parity bits can give the position of the error.
- The following table gives the seven error positions and the corresponding values of position number.

Error position	Position number		
0 (no error)	C_3	C_2	C_1
0 (no error)	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

- Parity bit P_1 is selected to establish even parity in position 1, 3, 5, 7.
- Parity bit P_2 is selected to establish even parity in position 2, 3, 6, 7.
- Parity bit P_3 is selected to establish even parity in positions 4, 5, 6, 7.
- Hamming error correction code for BCD is shown in following table.

Decimal position digit	1	2	3	4	5	6	7
	P_1	P_2	m_1	P_3	m_2	m_3	m_7
	x_5	x_6	x_1	x_7	x_2	x_3	x_4
0	0	0	0	0	0	0	0
1	1	1	0	1	0	0	1
2	0	1	0	1	0	1	0
3	1	0	0	0	0	1	1
4	1	0	0	1	1	0	0
5	0	1	0	0	1	0	1
6	1	1	0	0	1	1	0
7	0	0	0	1	1	1	1
8	1	1	1	0	0	0	0
9	0	0	1	1	0	0	1

As shown in above table,

1. x_5 (i.e. P_1) = $x \oplus x_2 \oplus x_4 = [x_1 + x_2 + x_4]$ mod 2
2. $x_6 = x_1 \oplus x_3 \oplus x_4 = [x_1 + x_3 + x_4]$ mod 2
3. $x_7 = x_2 \oplus x_3 \oplus x_4 = [x_2 + x_0 + x_4]$ mod 2

It may also noted that,

1. $[x_1 + x_2 + x_4 + x_5]$ mod 2 = 0
2. $[x_1 + x_3 + x_4 + x_6$ mod 2 = 0
3. $[x_2 + x_3 + x_4 + x_7]$ mod 2 = 0

- This table is called as parity check matrix. The 3 operations, given are represented using a coefficient matrix. The coefficient matrix H is given as below.

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \leftarrow \; x_1 + x_2 + x_4 + x_5 = 0 \\ \leftarrow \; x_1 + x_3 + x_4 + x_6 = 0 \\ \leftarrow \; x_2 + x_3 + x_4 + x_7 = 0 \end{array}$$

- The coefficient matrix (Parity check matrix) can be used to generate code words.
- The minimum weight of the non-zero code words in a group code is equal to it's minimum distance, weight of code word is equal to number of 1's in it.

9.20.7 Examples Based on Group Code

Ex. 9.20.6 : Find the number of codes generated by the given check matrix H. Also find all code words.

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Soln. :

The given matrix, represents the following set of operations :

$$\left. \begin{array}{l} x_1 + x_2 + x_4 = 0 \\ x_2 + x_3 + x_5 = 0 \\ x_1 + x_3 + x_6 = 0 \end{array} \right\} \text{mod } 2$$

These equations can be written as,

$$\left. \begin{array}{l} x_4 = x_1 + x_2 \\ x_5 = x_2 + x_3 \\ x_6 = x_1 + x_3 \end{array} \right\} \text{all mod } 2$$

A code word consists of 3 message bits (x_1, x_2, x_3) and 3 parity bits (x_4, x_5, x_6).

x_4 can be found by exclusive or of x_1 and x_2

x_5 can be found by exclusive or of x_2 and x_3

x_6 can be found by exclusive or of x_1 and x_3

The number of code generated by the matrix = 2^m where m is the number of data bits : $2^m = 2^3 = 8$ code words are given as follows :

Data bits			Parity bits		
x_1	x_2	x_3	x_4	x_5	x_6
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	1	1	0
0	1	1	1	0	1
1	0	0	1	0	1
1	0	1	1	1	0
1	1	0	0	1	1
1	1	1	0	0	0

∴ The required code

= {(000000), (001011), (010110), (011101), (100101), (101110), (110011), (111000)}

Ex. 9.20.7 : A parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find the minimum distance of the code generated by H. How many errors it can detect and correct.

Soln. :

The parity check matrix H generates a code word of weight q if and only if there exists a set of q columns of H such that their k-tuple sum is zero. The minimum weight of the non-zero code words in a group code is equal to it's minimum distance.

Let us take the modulo 2 addition of the first three columns.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

∴ The maximum number of columns that have zero sum is 3.

∴ Minimum distance of the code = 3.

As the minimum distance of the code is 3, it can detect all 2 or less errors. This code can correct all single errors.

UEx. 9.20.8 (SPPU(IT) - May 17, 6 Marks)

Consider the (2, 7) encoding function :

e (00) = 0000000, e(01) = 1010101,

e(10) = 0111110, e(11) = 0110110

Soln. :

d (0000000, 1010101) = 4

d (0000000, 0111110) = 5

d (0000000, 0110110) = 4

d (1010101, 0111110) = 5

d (1010101, 0110110) = 4

d (0111110, 0110110) = 1

Minimum distance of code i.e.

hamming distance of code = 1

The code can detect 0 errors.

UEx. 9.20.9 (SPPU(IT) - Q. 4(a), Dec. 14, 6 Marks)

Find the degree of f(x) [illegible]

polynomials are on the integer (mod [illegible]

addition and multiplication.

You have f(x) = 2x + 4x [illegible], g(x) = 2 + 6x + 4x [illegible]

☑ **Soln. :**

$$[f(x) + g(x)] = (2x + 4x^2) + (2 + 6x + 4x^2)$$
$$= 2 + 8x + 8x^2$$
$$= 2 \pmod 8$$

Hence degree of $f(x) + g(x) = 0$

$$[f(x) \cdot g(x)] = (2x + 4x^2) \cdot (2 + 6x + 4x^2)$$
$$= 4x + 20x^2 + 32x^3 + 16x^4$$
$$= 4x + 4x^2 + 0x^3 + 0x^4 \pmod 8$$
$$= 4x + 4x^2 \pmod 8$$

Hence degree of $[f(x) \cdot g(x)] = 2$.

9.21 CYCLIC CODE

- A code of length n is called a cyclic code if
  1. Every code word is of the same length n.
  2. If $q_0, q_1, \ldots q_{n-2}\, q_{n-1}$ is a code word then the sequence $q_{n-1}\, q_0\, q_1 \ldots q_{n-1}$ is also a code word.
- For example, if 111000 is a code word then 011100, 001110, 000111, 100011, 110001 will also be codewords.
- Every binary sequence $q_0\, q_1 \ldots q_{n-2}\, q_{n-1}$ can be associated with the polynomial.

$$f(x) = q_0 + q_1 \times q_2\, x^2 + Q_2\, x^3 + \ldots + Q_{n-2}\, x^{n-2} + Q_{n-1}\, x^{n-1} \in Z^n [x]$$

- If f(x) is a polynomial then x f (x) will correspond to a code word in the cyclic code.

$$x \cdot f(x) = Q_0 x + Q_1 x^2 + \ldots + Q_{n-2}\, x^{n-1} + q_{n-1}$$
$$= Q_{n-1} + Q_0\, x + Q_1\, x^2 + Q_2\, x^3 + \ldots + a_{n-1}\, x^{n-1}$$

This corresponds to the code word

$Q_{n-1} + q_0\, q_1\, q_2 \ldots q_{n-1}$

9.21.1 Example based on Cyclic Code

Ex. 9.21.1 : Find the roots of

$f(t) = 2t^4 - 11t^3 + 33t^2 - 19t - 6s$

Given that $t = 2 + 3i$ is one roo

☑ **Soln. :**

$2 + 3i$ is a root of f (t). The conjugate $2 - 3i$ will also be a root of f(t).

The polynomial f (t) will be divisible by,

$$(t - (2 + 3i)(t - (2 - 3i))) = t^2 - t(2 + 3i + 2 - 3i) + (2 + 3i)(2 - 3i)$$
$$= t^2 - 4t + 13$$

Dividing $2t^4 - 11t^3 + 33t^2 - 19t - 65$ by $t^2 - 4t + 13$

$2t^2 - 3t - 5$

$$\begin{array}{r|l} t^2 - 4t + 13 & 2t^4 - 11t^3 + 33t^2 - 19t - 65 \\ & 2t^4 - 8t^3 + 26t^2 \\ \hline & \quad -3t^3 + 7t^2 - 19t - 65 \\ & \quad -3t^3 + 12t^2 - 39t \\ \hline & \qquad -5t^2 + 20t - 65 \\ & \qquad -5t^2 + 20t - 65 \\ \hline & \qquad\qquad 0 \end{array}$$

Now, $2t^2 - 3t - 5 = 2t^2 + 2t - 5t - 5$
$$= 2t + (t + 1) - 5(t + 1)$$
$$= (2t - 5)(t + 1)$$

Hence, the roots of (t) are, 1, 5/2, 2 + 3i and 2 – 3i.

...*Chapter Ends*

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Note

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Note