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THEORETICAL ARTICLE

Flow constrained minimum cost flow problem

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Abstract The paper deals with the uncapacitated minimum cost flow problems subject to additional flow constraints whether or not the sum of node capacities is zero. This is a generalization and extension of transportation problems with restrictions on total flow value. The relationship between the desired flow value and the sum of node capacities of source(s) and sink(s) gives rise to the different set of problems. Mathematical models for the various cases are formulated. For each case an equivalent standard minimum cost flow problem (MCFP) is formulated whose optimal solution provides the optimal solution to the original flow constrained problem. The paper not only extends the similar concept of transportation problem to the case of MCFP but also suggests an alternative equivalent formulation. Solving the alternative problem is computationally better in comparison to the formulation analogous to the transportation case. A broad computational comparison is carried out at the end.

Keywords Minimum cost flow problem · Network flow variant

1 Introduction

A Minimum Cost Flow Problem (MCFP) is to send flow from a set of supply nodes, through the arcs of a network, to a set of demand nodes, at the minimum total cost. The MCFP framework is particularly broad, and may be used to

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model a number of more specialized network problems, including Assignment, Transportation and Transshipment problems, the Shortest Path Problem, and the Maximum Flow problem. An MCFP has been studied extensively in literature because of its prevalent applications to a remarkably wide range of fields, including chemistry and physics, computer networking, most branches of engineering, management, manufacturing, public policy and social systems, scheduling and routing, telecommunications, and transportation.

Mathematically, it can be stated as:

$$\min_{(i,j)\in A} c_{ij}x_{ij}$$

Subject to

$$\sum_{j\in N:(i,j)\in A} x_{ij} - \sum_{j\in N:(j,i)\in A} x_{ij} = b_i,\ i \in N$$
$$l_{ij} \leqslant x_{ij} \leqslant u_{ij}\ \forall\ (i,j) \in A$$

Where N is the set of n nodes and A is the set of m arcs in the directed network $G = (N, A)$; c_{ij} is the unit flow cost on the arc $(i, j) \in A$; decision variable x_{ij} stands for the flow on the arc $(i, j) \in A$; $b_i(> 0)$ is the quantity available at the source i and $|b_j|(b_j < 0)$ is the demand at the sink j; $b_i = 0$ for transhipment nodes. In standard MCFP, it is assumed that $\sum_{i\in N} b_i = 0$. If this is not the case we can balance supply and demand by adding a dummy demand node or a dummy supply node. If $l_{ij} = 0$ and $u_{ij} = \infty$, the problem is termed as uncapacitated minimum cost network flow problem. If $S^+ \subset N$ is the set of source nodes, $S^- \subset N$ is the set of sink nodes, $S^0 \subset N$ is the set of transhipment nodes, then $N = S^+ \cup S^- \cup S^0$ and $\sum_{i\in S^+} b_i = \sum_{j\in S^-} |b_j|$. The value of feasible flow vector in the balanced case is $\sum_{i\in S^+} b_i = \sum_{j\in S^-} |b_j|$. Many efficient algorithms including the Network Simplex algorithm, Out-of-Kilter Algorithm, Dual Network Simplex Algorithm, Primal Dual Algorithms and Scaling Algorithms [2, 16, 17, 20] are available to solve an MCFP. In search of the polynomial time algorithm for the MCFP, Orlin [19] introduced a pseudopolynomial variant of the network simplex algorithm called the "premultiplier algorithm". Further, he developed a cost-scaling version of the premultiplier algorithm that solves the minimum cost flow problem in $O(\min(nm \log nC, nm^2 \log n))$ pivots. Before this a few existing polynomial algorithms for MCFP were due to Ahuja et al. [1] and Orlin et al. [18]. Klein [13] contributed one more primal method for solving an MCFP based on the observation that an optimal solution to the MCFP admits no negative cost augmenting cycle.

A wide range of variants of MCFP have been explored in literature. Ahuja et al. [3] in 1999 discussed the min cost equal flow problem in which a given set of arcs are bound to carry same amount of flow. Innumerable other variants like the generalized min cost flow problems, min cost circulation problems, Unsplittable min cost flow problem [21], minimum interval cost flow problem [7], fixed-charge network flow problem [4], min cost flow problem with set constraints [8] and many others [5, 14, 15] have been conferred well. In bipartite graphs pertaining to the minimum cost transportation problem, Saroj et al. [10] studied the effect of introducing constraints on the total flow when $\sum_{i\in S^+} b_i \neq \sum_{j\in S^-} |b_j|$. Many more works regarding the introduction of an additional flow constraint in various transportation problems have been discussed in literature by Saroj et al. [9, 11, 12, 22]. Redistribution of availabilities at sources and/or demands at sinks at a lower cost is studied by Anju et al. [6].

The present paper comprises of a variant of MCFP in which we discuss how to control the total flow upto a known specified level. The additional flow constraint breaks the unimodular structure of the standard MCFP and hence the standard algorithms to solve an MCFP can not be used as is to solve this variant. We develope a solution procedure to solve an MCFP with a flow constraint whether or not $\sum_{i\in S^+} b_i = \sum_{j\in S^-} |b_j|$. First, we render the extension and the generalization of the work by Saroj et al. [10] termed as 'direct extension solver' and further, we provide an alternative computationally better approach termed as 'alternate solver'.

The next section on problem formulation and solution models covers the mathematical models of various cases exhorted by the introduction of flow constraints. Further, for each of them an equivalent standard MCFP is constructed whose optimal solution provides an optimal solution for the original problem. Section 3 contains the numerical examples pertaining to each of the case. Some concluding remarks and the comparative complexity analysis are discussed in the last Section 4 followed by the references.

2 Problem formulation and solution models

Consider an uncapacitated minimum cost network flow problem. Let F be a feasible flow vector having flow value V. $V = \min\left(\sum_{i\in S^+} b_i, \sum_{j\in S^-} |b_j|\right)$ where $b_i (> 0)$ is the availability at the source node $ⓘ$ and $|b_j| (b_j < 0)$ is the demand at the jth sink node. Thus, V is the net flow originating at sources and terminating at sinks. A flow constraint is introduced which specifies a fixed flow value different from V. Let this be a positive integer $\hat{V}$. Now, there are three cases depending on how the desired flow value relates to the existing flow value V in the problem.

1. $\hat{V} < \min\left(\sum_{i\in S^+} b_i, \sum_{j\in S^-} |b_j|\right) (= V)$
2. (a) $\sum_{i\in S^+} b_i < \hat{V} < \sum_{j\in S^-} |b_j|$

   (b) $\sum_{j\in S^-} |b_j| < \hat{V} < \sum_{i\in S^+} b_i$
3. $\hat{V} > \max\left(\sum_{i\in S^+} b_i, \sum_{j\in S^-} |b_j|\right)$

2.1 Case-I: $\hat{V} < \min\left(\sum_{i\in S^+} b_i, \sum_{j\in S^-} |b_j|\right) (= V)$

Mathematical model of the problem namely (P-1) when desired flow happens to be smaller than the total supply as well as the total demand is given as:

(P-1)

$$\min\left(Z = \sum_{(i,j)\in A} c_{ij}x_{ij}\right)$$

Subject to

$$\sum_{\substack{j\in N\\(i,j)\in A}} x_{ij} - \sum_{\substack{j\in N\\(j,i)\in A}} x_{ji} \leqslant b_i, i \in S^+ \tag{2.1.1}$$

$$\geqslant b_i, i \in S^- \tag{2.1.2}$$

$$= 0, i \in S^0 \tag{2.1.3}$$

$$\sum_{\substack{i\in S^+\\(i,j)\in A}} x_{ij} \left(= \sum_{\substack{j\in S^-\\(i,j)\in A}} x_{ij}\right) = \hat{V} \tag{2.1.4}$$

$$x_{ij} \geq 0 \;\&\; \text{integers} \;\; \forall\, (i, j) \in A \tag{2.1.5}$$

In Eq. 2.1.5, integrality requirement is stated since unimodularity character might have been lost after introducing Eq. 2.1.4.

In order to solve the problem (P-1) an equivalent uncapacitated MCFP is introduced in which we have $\sum_{i\in N} b_i = 0$. This equivalent problem named as ($\overline{\text{P-1}}$) provides an optimal solution for the problem (P-1).

Direct Extension Solver ($\overline{\text{P-1}}$)

$$\min\left(\hat{Z} = \sum_{(i,j)\in \hat{A}} \hat{c}_{ij}\hat{x}_{ij}\right)$$

Subject to

$$\sum_{\substack{j\in\hat{N}\\(i,j)\in\hat{A}}} x_{ij} - \sum_{\substack{j\in\hat{N}\\(j,i)\in\hat{A}}} x_{ji} = \hat{b}_i, \qquad i \in \hat{S}^+ \cup \hat{S}^- \tag{2.1.6}$$

$$= 0, \qquad i \in S^0 \tag{2.1.7}$$

$$\hat{x}_{ij} \geqslant 0 \tag{2.1.8}$$

Where,

$$\hat{N} = N \cup \left\{(n+1) \in \hat{S}^+, (n+2) \in \hat{S}^-\right\}$$

$$\hat{A} = A \cup \left\{(n+1, j);\ j \in S^-\right\} \cup \left\{(i, n+2); i \in S^+\right\}$$

$$\hat{b}_i = b_i, i \in S^+, S^-$$

$$\hat{b}_{n+1} = \sum_{j\in S^-} |b_j| - \hat{V}$$

$$\hat{b}_{n+2} = -\left(\sum_{i\in S^+} b_i - \hat{V}\right)$$

$$\hat{c}_{ij} = c_{ij}, (i, j) \in A$$

$$\hat{c}_{n+1\,j} = 0,\ j \in S^-$$

$$\hat{c}_{in+2} = 0, i \in S^+$$

Lemma 1 *If $\{\hat{x}_{ij}, (i, j) \in \hat{A}\}$ is an optimal feasible flow vector of the problem $(\overline{P\text{-}1})$, then the optimal feasible flow vector for the problem $(P-1)$ is given by $x_{ij} = \hat{x}_{ij}, (i, j) \in A$ with flow value $\hat{V}$ and $(Z)_{\min} = (\hat{Z})_{\min}$.*

In this simplistic case no alternate solver is proposed.

2.2 Case-II: $\sum_{i\in S^+} b_i < \hat{V} < \sum_{j\in S^-} |b_j|$ or $\sum_{j\in S^-} |b_j| < \hat{V} < \sum_{i\in S^+} b_i$

When the desired flow value falls in between the total supply and total demand the two subcases arise, mathematical models for which are named as (P-2a) and (P-2b) respectively.

(P-2a)

$$\min\left(Z = \sum_{(i,j)\in A} c_{ij}x_{ij}\right)$$

Subject to Eqs. 2.1.2, 2.1.3, 2.1.4, 2.1.5 and

$$\sum_{\substack{j\in N\\(i,j)\in A}} x_{ij} - \sum_{\substack{j\in N\\(j,i)\in A}} x_{ji} \geqslant b_i, i \in S^+ \tag{2.2.9}$$

where $\sum_{i\in S^+} b_i < \hat{V} < \sum_{j\in S^-} |b_j|$.

In order to obtain an optimal solution of the problem (P-2a) we introduce an equivalent problem ($\overline{\text{P-2a}}$), which is formulated as:

Direct Extension Solver ($\overline{\textbf{P-2a}}$)

$$\min\left(\hat{Z} = \sum_{(i,j)\in\hat{A}} \hat{c}_{ij}\hat{x}_{ij}\right)$$

Subject to Eqs. 2.1.6, 2.1.7, 2.1.8

Where,

$$\begin{aligned}
\hat{N} &= N \cup \{(n+1), (n+2)\}\\
\hat{S}^+ &= S^+ \cup \{(n+1), (n+2)\}\\
\hat{S}^- &= S^-\\
\hat{A} &= A \cup \left\{(n+1,\ j), (n+2,\ j)\ ;\ j \in S^-\right\}\\
\hat{b}_i &= b_i, i \in S^+ \cup S^-\\
\hat{b}_{n+1} &= \hat{V} - \sum_{i\in S^+} b_i\\
\hat{b}_{n+2} &= \sum_{j\in S^-} |b_j| - \hat{V}\\
\hat{c}_{ij} &= c_{ij}, (i,\ j) \in A\\
\hat{c}_{n+1\,j} &= \min_{i\in S^+}\left(\hat{c}_{i\to j}\right),\ j \in S^-\\
\hat{c}_{n+2\,j} &= 0,\ j \in S^-
\end{aligned}$$

In the above formulation $\hat{c}_{i\to j}$ is the total cost for unit flow along the minimal cost chain from the source $i \in S^+$ to the sink $j \in S^-$.

Lemma 2 *If* $\{\hat{x}_{ij}, (i,\ j) \in \hat{A}\}$ *is an optimal feasible flow vector of the problem* $(\overline{P-2a})$*, then the corresponding optimal feasible flow vector for the problem* $(P-2a)$ *is*

$$\begin{aligned}
x_{ij} &= \hat{x}_{ij} + \hat{x}_{i\to j}, && \textit{if } (i,\ j) \in A \textit{ is part of the min cost chain from } i \in S^+ \textit{ to } j \in S^-.\\
& && \hat{x}_{i\to j} \textit{ being the flow along the chain } i \to j.\ (\hat{x}_{i\to j} = \hat{x}_{n+1\,j}).\\
&= \hat{x}_{ij}, && \textit{otherwise}
\end{aligned}$$

This solution will have flow value $\hat{V}$ *and* $(Z)_{\min} = (\hat{Z})_{\min}$.

As an alternative to the problem ($\overline{\text{P-2a}}$) we further introduce another balanced MCFP (alternate solver) which is also equivalent to the problem (P-2a) and requires less computation in finding its optimal solution. This problem is named as ($\overline{\overline{\text{P-2a}}}$).

Alternate Solver ($\overline{\overline{\textbf{P-2a}}}$)

$$\min\left(\hat{Z} = \sum_{(i,j)\in\hat{A}} \hat{c}_{ij}\hat{x}_{ij}\right)$$

Subject to Eqs. 2.1.6, 2.1.7, 2.1.8

Where,

$$\hat{N} = N \cup \{(n+1), (n+2)\}$$

$$\hat{S}^{+} = S^{+} \cup \{(n+1), (n+2)\}$$

$$\hat{S}^{-} = S^{-}$$

$$\hat{A} = A \cup \left\{(n+1, i), (n+2, j); i \in S^{+}, j \in S^{-}\right\}$$

$$\hat{b}_i = b_i, i \in S^{+} \cup S^{-}$$

$$\hat{b}_{n+1} = \hat{V} - \sum_{i\in S^{+}} b_i$$

$$\hat{b}_{n+2} = \sum_{j\in S^{-}} |b_j| - \hat{V}$$

$$\hat{c}_{ij} = c_{ij}, (i,j) \in A$$

$$\hat{c}_{n+1i} = 0, i \in S^{+}$$

$$\hat{c}_{n+2j} = 0, j \in S^{-}$$

Lemma 3 *If* $\{\hat{x}_{ij}, (i,j) \in \hat{A}\}$ *is an optimal feasible flow vector of the problem* ($\overline{\overline{\text{P-2a}}}$)*, then the optimal feasible flow vector for the problem (P-2) is given by* $x_{ij} = \hat{x}_{ij}, (i,j) \in A$ *with flow value* $\hat{V}$ *and* $(Z)_{\min} = (\hat{Z})_{\min}$.

The problem (P-2b) and the corresponding equivalent problems ($\overline{\text{P-2b}}$) and ($\overline{\overline{\text{P-2b}}}$) can be modeled exactly in the same manner as that of the problem (P-2a).

2.3 Case-III: $\hat{V} > \max\left(\sum_{i\in S^+} b_i, \sum_{j\in S^-} |b_j|\right)$

In this case when the total desired flow is more than the total supply as well as the total demand, we formulate the following problem named as (P-3).

(P-3)

$$\min\left(Z = \sum_{(i,j)\in A} c_{ij}x_{ij}\right)$$

Subject to Eqs. 2.1.2, 2.1.3, 2.1.4, 2.1.5, 2.2.9 and

$$\sum_{\substack{j\in N\\(i,j)\in A}} x_{ij} - \sum_{\substack{j\in N\\(j,i)\in A}} x_{ji} \leqslant b_i, i \in S^- \tag{2.3.10}$$

where $\hat{V} > \max\left(\sum_{i\in S^+} b_i, \sum_{j\in S^-} |b_j|\right)$.

The Problem (P-3) is equivalent to ($\overline{\text{P-3}}$) given as under:

Direct Extension Solver ($\overline{\text{P-3}}$)

$$\min\left(\hat{Z} = \sum_{(i,j)\in \hat{A}} \hat{c}_{ij}\hat{x}_{ij}\right)$$

Subject to Eqs. 2.1.6, 2.1.7, 2.1.8

Where,

$$\begin{aligned}
\hat{N} &= N \cup \{(n+1), (n+2)\}\\
\hat{S}^+ &= S^+ \cup \{n+1\}\\
\hat{S}^- &= S^- \cup \{n+2\}\\
\hat{A} &= A \cup \left\{(n+1, j) : j \in S^-\right\} \cup \left\{(i, n+2) : i \in S^+\right\}\\
\hat{b}_i &= b_i, i \in S^+ \cup S^-\\
\hat{b}_{n+1} &= \left(\hat{V} - \sum_{i\in S^+} b_i\right)\\
\hat{b}_{n+2} &= -\left(\hat{V} - \sum_{j\in S^-} |b_j|\right)\\
\hat{c}_{ij} &= c_{ij}, (i,j) \in A\\
\hat{c}_{n+1\,j} &= \min_{i\in S^+}\left(\hat{c}_{i\to j}\right), j \in S^-\\
\hat{c}_{in+2} &= \min_{j\in S^-}\left(\hat{c}_{i\to j}\right), i \in S^+
\end{aligned}$$

Lemma 4 *If* $\{\hat{x}_{ij} : (i, j) \in \hat{A}\}$ *is an optimal feasible flow vector for* $(\overline{P\text{-}3})$*, then the corresponding optimal feasible flow vector for the problem* $(P-3)$ *is*

$$\begin{aligned} x_{ij} &= \hat{x}_{ij} + \hat{x}_{in+1} + \hat{x}_{n+1j}, && \text{if } (i, j) \in A \text{ is the part of min cost chain for source } i \in S^+ \text{ to the sink } j \in S^- \\ &= \hat{x}_{ij}, && \text{otherwise} \end{aligned}$$

Flow value will be $\hat{V}$ *and* $(Z)_{\min} = (\hat{Z})_{\min}$.

As proposed earlier in the second case above where desired flow value was lying in between the total supply and the total demand, we devise the problem $(\overline{\overline{\text{P-3}}})$, an alternative balanced MCFP equivalent to the problem (P-3), as:

Alternate Solver $(\overline{\overline{\textbf{P-3}}})$

$$\min \left(\hat{Z} = \sum_{(i,j)\in\hat{A}} \hat{c}_{ij}\hat{x}_{ij} \right)$$

Subject to Eqs. 2.1.6, 2.1.7, 2.1.8

Where,

$$\hat{N} = N \cup \{(n+1), (n+2)\}$$

$$\hat{S}^+ = S^+ \cup \{n+1\}$$

$$\hat{S}^- = S^- \cup \{n+2\}$$

$$\hat{A} = A \cup \left\{(n+1, i) : i \in S^+\right\} \cup \left\{(j, n+2) : j \in S^-\right\}$$

$$\hat{b}_i = b_i, i \in S^+ \cup S^-$$

$$\hat{b}_{n+1} = \left(\hat{V} - \sum_{i\in S^+} b_i \right)$$

$$\hat{b}_{n+2} = -\left(\hat{V} - \sum_{j\in S^-} |b_j| \right)$$

$$\hat{c}_{ij} = c_{ij}, (i, j) \in A$$

$$\hat{c}_{n+1i} = 0, i \in S^+$$

$$\hat{c}_{jn+2} = 0, j \in S^-$$

Lemma 5 *If* $\{\hat{x}_{ij}, (i, j) \in \hat{A}\}$ *is an optimal feasible flow vector of the problem* $(\overline{\overline{P\text{-}3}})$*, then the optimal feasible flow vector for the problem* $(P-3)$ *is given by* $x_{ij} = \hat{x}_{ij}, (i, j) \in A$ *with flow value* $\hat{V}$ *and* $(Z)_{\min} = (\hat{Z})_{\min}$.

3 Numerical illustrations

3.1 Example Problem (P-1)

Consider the following flow constrained MCFP with two sources ①, ② and two sinks ③, ④. The desired flow value $\hat{V} = 15$ which is less than both supply $(b_1 + b_2 = 22)$ and demand $(|b_3| + |b_4| = 20)$ (Fig. 1).

The equivalent problem $(\overline{\text{P-1}})$ with a dummy source ⑥ and a dummy sink ⑤ is formulated whose optimal basic feasible solution (OBFS) is shown in Fig. 2.

The equivalent OBFS pertaining to the original problem is read as shown in Fig. 3. Note that the flow value is restricted to 15 units and the optimal cost is 63.

3.2 Example Problem (P-2a)

Now, consider another flow constrained MCFP (Fig. 4) with two sources ①, ② and two sinks ③, ④. Desired flow value is $\hat{V} = 28$ lying between the supply $(b_1 + b_2 = 22)$ and demand $(|b_3| + |b_4| = 33)$. The equivalent problem $(\overline{\text{P-2a}})$ with two dummy sources ⑤ and ⑥ is formulated and its optimal flow vector is shown in Fig. 5.

The equivalent OBFS pertaining to original problem is read as shown in Fig. 6.

The alternate equivalent problem $(\overline{\overline{\text{P-2a}}})$ with two dummy sources ⑤ and ⑥ is formulated. Its optimal solution is provided in Fig. 7. The OBFS of the original problem read from the OBFS of the equivalent problem is shown in Fig. 8.

Note that, the flow is constrained to the value 28, i.e. the source ① is now overproducing 6 units and sink ④ is not receiving 5 units of commodity.

3.3 Example Problem (P-3)

Now, consider another flow constrained MCFP (Fig. 9) with two sources ①, ② and two sinks ③, ④. Desired flow value $\hat{V} = 29$ is higher than both the supply $(b_1 + b_2 = 24)$ and demand $(|b_3| + |b_4| = 22)$.

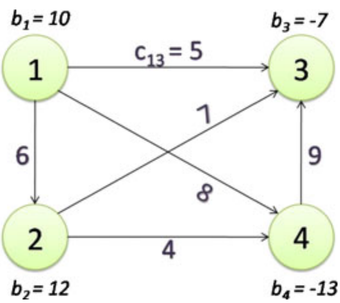


Fig. 1 Flow constrained MCFP (P-1)

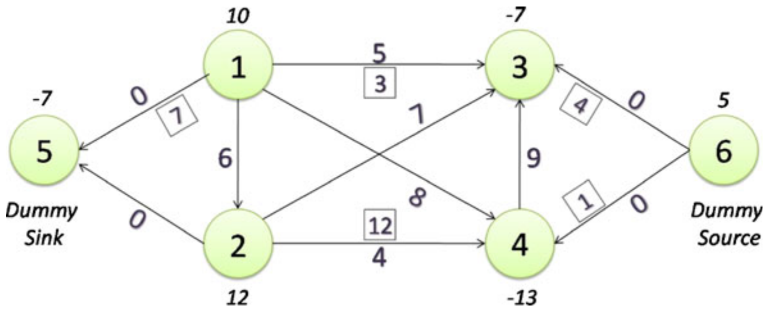


Fig. 2 OBFS—($\overline{\text{P-1}}$)

Fig. 3 OBFS - (P-1)

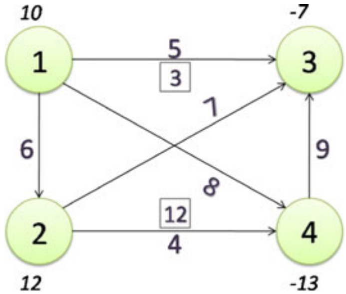
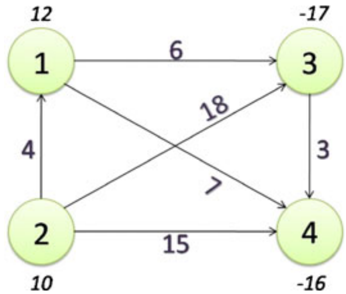


Fig. 4 Flow constrained MCFP (P-2a)



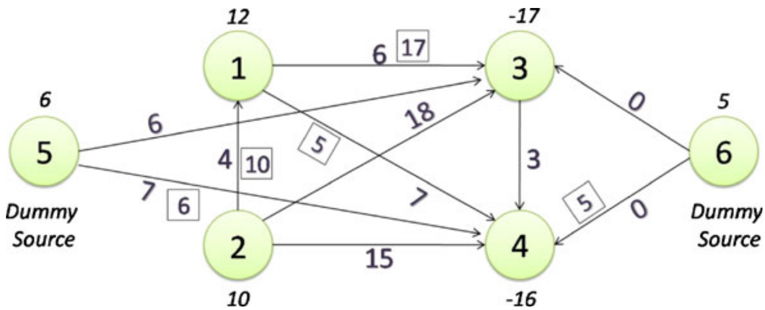
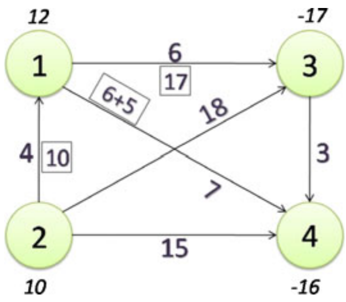


Fig. 5 OBFS—($\overline{\text{P-2a}}$)

Fig. 6 OBFS—(P-2a)



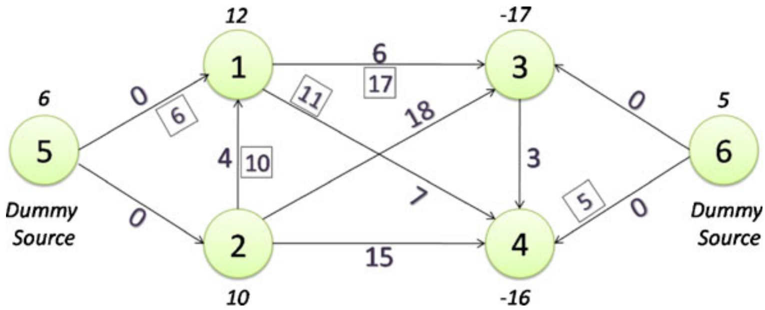


Fig. 7 OBFS—($\overline{\overline{\text{P-2a}}}$)

Fig. 8 OBFS—(P-2a)

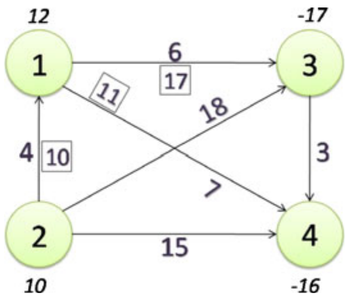
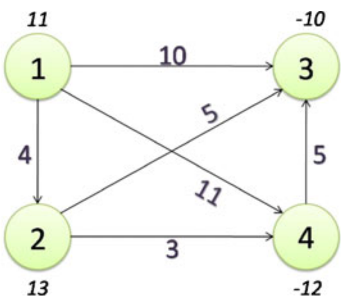


Fig. 9 Flow constrained MCFP (P-3)



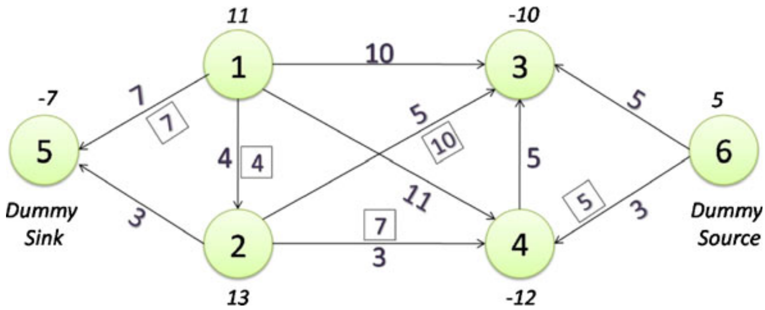
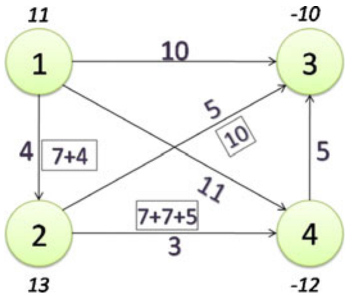


Fig. 10 OBFS—($\overline{\text{P-3}}$)

Fig. 11 OBFS—(P-3)



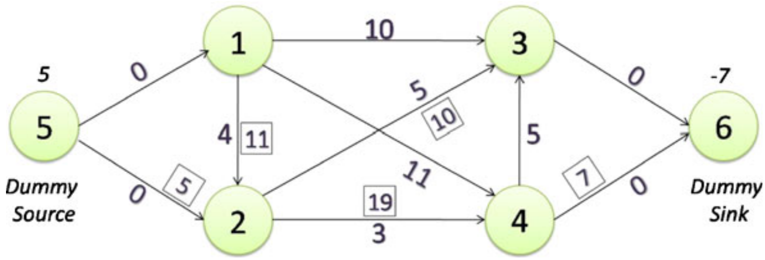
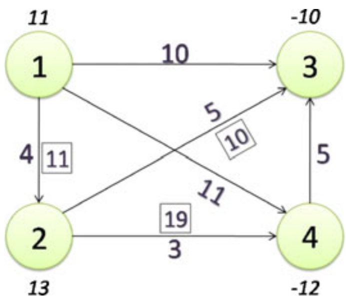


Fig. 12 OBFS—($\overline{\overline{\text{P-3}}}$)

Fig. 13 OBFS—(P-3)



An OBFS of the equivalent problem ($\overline{\text{P-3}}$) with a dummy source ⑥ and a dummy sink ⑤ is shown in Fig. 10.

The equivalent OBFS pertaining to original problem is read as shown in Fig. 11.

An optimal solution of the alternate equivalent problem ($\overline{\overline{\text{P-3}}}$) with a dummy source ⑥ and a dummy sink ⑤ is shown in Fig. 12.

The equivalent OBFS pertaining to original problem is read as shown in Fig. 13.

Note that, the flow is enhanced to the value 29.

4 Concluding remarks

Owing to the widespread applications of the minimum cost network flow problems and its variants, we have covered a variant of MCFP which is existing and well appreciated in the class of transportation problems. To conclude, the discussed problem is the generalization of the flow constrained transportation problems to the uncapacitated min cost flow problems. The proposed approach discusses simpler solution methodology in comparison to the direct extension to the generalized case. A comparison in terms of the computational complexity of both the approaches, pertaining to each of the case discussed in the paper, is carried out and presented in Table 1 for a flow constrained MCFP with n nodes (n_1 sources & n_2 sinks) and m arcs.

Thus, to summarize an alternative suggested extension is not only simpler to understand and implement but is faster as well except the case of the problem (P-1). In case of the problem (P-1) both work equally well.

MCFPs with finite arc capacities can be dealt in the similar manner with the same ease. As a future work in this direction, one can think of the same idea extended to the case of multi commodity minimal cost network flow problems.

Table 1 Complexity analysis

Problem	Complexity (direct extension solver)	Complexity (alternative solver)
(P-1)	$O_M(n+2, m+n_1+n_2)$	$O_M(n+2, m+n_1+n_2)$
(P-2a)	$O_M(n+2, m+2n_2)+$ $O_S(n_1, n_2)+$ $O(n^2+mn\log(n))$	$O_M(n+2, m+n_1+n_2)$
(P-3)	$O_M(n+2, m+n_1+n_2)+$ $O_S(n_1, n_2)+$ $O(n^2+mn\log(n))$	$O_M(n+2, m+n_1+n_2)$

$O_M(n, m)$ denotes complexity to solve a MCFP with n nodes and m arcs
$O_S(n_1, n_2)$ denotes complexity to find all source-sink shortest paths between n_1 sources and n_2 sinks

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