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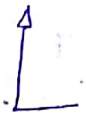
PROBABILITY.

* Probability depends on certainity.

Transmitted Signal



Received signal

 $f(t) = A\sin(\omega t)$

 $y = f(t) + n$.

$Pr(|y - f(t)| < \epsilon) \rightarrow 1$

* Received signal depends on probability.

(i) why do we need probability? ⅃

(ii) what is probability?

→ <u>classical definition</u> :- $P = \frac{n_{f}}{N}$ → (Rolling a dice)

<u>Limitation</u> :- Assume all events are equally likely.

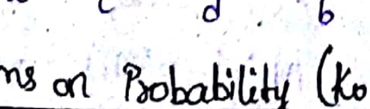
→ $P = \lim\limits_{N \to \infty} \frac{n_s}{N}$ n_s – no of success, N – Total attempts

↓

$\{H, T\} \rightarrow P \rightarrow 1/2$ (Probability converges to half).

<u>Limitation</u> :- If you flip it large no of times do not know the exact number of times.

→ If you have a water sprinkler probability of water falling in ac, cd & bd cannot be told by definitions 1 & 2.



<u>Axioms on Probability (Kolmogorov)</u>.

(Ω, F, P) Ω – sample space / sure event.

F – σ-algebra over Ω

F : set of subsets of Ω. that is a σ-algebra.

σ - algebra : closed under arbitary unions & countable interse

<u>closure</u> :- $a, b \in A$, $a+b \in A$, $\forall a, b$

A is closed under +

If $X_1, X_2 \in F$, $X_1 \cup X_2 \in F$

$X_1 \cap X_2 \in F$

$\Omega = \{H, T\}$

$F = \{\phi, \{H\}, \{T\}, \{H, T\}\}$.

$F = \phi$.

<u>Axioms</u>.

(i) $X \in F$, $(x) \geq 0$.

(ii) $P(\Omega) = 1$.

(iii) $P\left(\bigcup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} P(X_i)$, $\forall X_i; X_j \in F$ & $X_i \cap X_j = \phi$.

$P(A) + P(A^c) = 1$.

* Conditional Probability :-

P, $P(A \cup B)$, $P(A \cap B)$, $P(A^c)$.

$P(A|B) = P($A occurs given B occurs$)$

$P(A \cup B) = P($A occurs or B occurs$)$

$P(A \cap B) = P(AB) = P(A, B)$

$P(A \cap B) = P($A occurs and B occurs$)$

$P(A^c) = P($A doesn't occur$)$.

$$P(A|B) = \frac{P(A, B)}{P(B)}$$

$1. \longrightarrow P(A \cap B)$

$P(B) \rightarrow \frac{P(A \cap B)}{P(B)}$

$P(A|B) = \frac{P(AB)}{P(B)}$

$$P(B) = \sum_{\forall i} P(B|A_i) \cdot P(A_i)$$

$$P(A_i, B) = P(B|A_i)\, P(A_i).$$

$$P(A_i|B) = \frac{P(B|A_i)\, P(A_i)}{\sum_{i} P(B|A_i)\, P(A_i)}$$

$$\int_{-\infty}^{\infty} P(x,y) = P(x) \longrightarrow (\text{this is called "marginalization"})$$

$$P(AB) \xrightarrow[\text{gives}]{} P(A) \text{ or } P(B).$$

* $P(A \cup B) = P(A) + P(B) - P(AB)$

Prove using Probability axioms.

$$P(A) + P(B) - P(AB) = P(A) + P(B-A)$$

$$= P(A \cup (B-A))$$

$$= P(A \cup B).$$

Hence, proved.

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* Probability is a mathematical model to help us study physical systems in an average sense.

Ex:- If we want accurate $i(t)$ we need to keep record of 10^{23} electrons responsible in causing $i(t)$. which is practically impossible. So, we use probability.

* Types of Probability :- (4).

① Probability as intuition :- deals with judgements based on intuition.

Ex :- "She will probably marry him."

Can lead to Contradictory Behaviour.

(classical theory)

② Probability as the Ratio of Favorable to Total Outcomes :-

$$P = \frac{N_E}{N}$$

N_E – no of ways E can occur (not Experimental)

N — total ways.

* All outcomes are equally likely.

Limitations :-

(i) Cannot deal with outcomes not equally likely

(ii) Cannot handle uncountably infinite outcomes without ambiguity

③ Probability as a measure of frequency of occurrence :-
(Experimental)

Experiment is done n times.

no of times E occurs is n_E $\qquad n_E \leq n.$

$$P[E] = \lim_{n\to\infty} \frac{n_E}{n}. \qquad 0 \leq P[E] \leq 1$$

* We can estimate P(E) only from finite no of trials.
* We postulate $\frac{n_E}{n}$ approaches a limit as $n \to \infty$.

* If we flip a coin 1000 times likelihood of getting 500 heads is very small. As $n \to \infty$, getting $n/2$ heads becomes vanishingly sm

Yet we take $P(E) = 1/2$.

If $\delta > 0$ as n becomes large, δ becomes very small

$$\left| \frac{n_E}{n} - \frac{1}{2} \right| > \delta.$$

$\frac{n_E}{n}$ is ~~~~ unlikely to be $\frac{1}{2}$ but it hovers around $1/2$.

④ Probability based on Axiomatic theory :-

Mostly followed in modern T.B's.

Random exp, sample space & event ⟶ Subset of sample space following certain conditions.

Random exp: Exp in which outcomes are non deterministic.

sample space: set of all outcomes of the exp

* Sets, fields and Events :-

Set - collection of object either concrete / abstract.

Experiment - H.

set of all outcomes — Ω (sample space).

subsets of Ω - events.

Any set is subset of itself Ω is an event. certain event.

Ω < set of all outcomes. / certain event.

ntal)

$\Omega = \{a_1, a_2, \ldots, a_n\}$ — No of subsets $= 2^n$

ϕ — impossible event. this includes ϕ, empty set.

* Set Algebra.

Union/sum $\rightarrow$ $E \cup F$ / $E + F$ (Present At least in one of the sets)

if E subset of F $\rightarrow$ $E \subset F$ & $E \cup F = F$.

Intersection/set product $\rightarrow$ $E \cap F$ / EF. (common elements in E & F).

ly Small.

$E \cup E^c = \Omega$, $EE^c = \phi$.

$E - F \rightarrow$ Elements in E that are not in F. $E - F = EF^c$.

$F - E = FE^c$.

$E \oplus F = (E-F) \cup (F-E) \rightarrow$ Element in E/F but not both.

* If: $EF = \phi$ — called as disjoint sets.

* Given set E, n-partition of E consists of sets E_i, $i = 1, \ldots n$.

such that $E_i \subset E$ & $\bigcup_{i=1}^{N} E_i = E$ & $E_i E_j = \phi$ & $i \neq j$.

* Given E, F a 2-partition of F is

$$F = FE \cup FE^c.$$

* $(E \cup F)^c = E^c F^c$ $\left[\bigcup_{i=1}^{n} E_i\right]^c = \bigcap_{i=1}^{n} E_i^c$

$(EF)^c = E^c \cup F^c$ $\left[\bigcap_{i=1}^{n} E_i\right]^c = \bigcup_{i=1}^{n} E_i^c$

* The Relations Known as De-Morgan's laws.

* E, F are equal iff $E \subset F$ & $F \subset E$.

10/8/18:

$$P(A\cup B) = \overbrace{P(A-B) + P(B-A) + P(A\cap B)}^{P(A)}$$

$$= P(A) + P(B) - P(A\cap B)$$

$$= P(A\cup B).$$

Counting.

1. ① n objects placed in r locations with replacement — n^r.
② " " " " " " without " — ^nP_r.

$$\overline{n}\cdot\overline{n-1}\cdot\overline{n-2}\cdots(n-r+1)$$

$$^nP_r = n\cdot(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

③ Subpopulations — $^nC_r = \frac{n!}{(n-r)!\,r!}$

order is not important.

④ Bernoulli trials :- Repeat experiment n-times. probability of success is p. Probability of k success — $\binom{n}{k} p^k (1-p)^{n-k}$

If the limit $n\to\infty$; $p\leq 1$.

$np=\lambda$, $1-n\to\infty$.

$$\frac{n!}{n^k\cdot k!\,(n-k)!}\lambda^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

For small x

$\log(1+x)\approx x$

$\log(1-x)\approx -x$

$$\lim_{n\to\infty}\frac{n!}{n^k(n-k)} = 1.$$

$$= \frac{1}{k!}\lambda^k\, e^{(n-k)\log\left(1-\frac{\lambda}{n}\right)}$$

$$= \frac{1}{k!}\lambda^k e^{-\left(\frac{n-k}{n}\right)\lambda}$$

$$= \frac{1}{k!}\lambda^k e^{-\lambda}.$$

$\frac{n-k}{n}=1.$

$a^b = e^{b\log a}$

Probability of k success -

(*) Biased coin is flipped successively. Experiment stops when you get head in Even toss. Probability of stoppage of Experiment.

Sol) Given $p(H) = P$.

$$(1-P)\cdot P + (1-P)^3\cdot P + (1-P)^5\cdot P + \cdots \infty.$$

$$= \frac{(1-P)}{1-(1-P)^2}\cdot P = \frac{1-P}{2-P}.$$

(2) Repeat the question for odd toss.

$$P + (1-P)^2 P + (1-P)^4\cdot P + \cdots \infty.$$

$$= P\left(1 + (1-P)^2 + (1-P)^4 + \cdots\right).$$

$$1 + a + a^2 + \cdots \infty.$$

$$S_n = \frac{1}{1-a}$$

$$= P\cdot\frac{1}{1-(1-P)^2} = \frac{1}{2-P}.$$

(3) You go to a Cafe where a person supports Manchester United. what is the probability that he is located within 25 Miles of Manchester.

Given :-

(i) Probability that a man in a bar is born within 25 miles of Manchester is $1/20$.

(ii) Probability that a person born within 25 miles of Manchester & supports ManU is $7/10$.

(iii) Probability that a person not born within 25 miles of Manchester & supports Man.U is $1/10$.

$$P = \frac{1/20 \cdot 7/10}{1/20\cdot\frac{7}{10} + \frac{19}{20}\cdot 1/10} = \frac{7}{26}.$$

(4) 5 Missiles are fired against INS Vikrant Takes at least 2 to sink the carrier (all are on trajectory.)

Defense guns can destroy missiles with probability $p=0.9$. what is the probability that carrier will still be afloat after encounter.

Sol)
$$1-\sum_{i=2}^{5}\binom{5}{i}(0.1)^{i}(0.9)^{5-i} \cancel{= 1-\left({}^5C_2+{}^5C_3+{}^5C_4+{}^5C_5\right)(0.1)(0.9)^{5-i}}$$

$$=1-\sum_{i=2}^{5}\binom{5}{i}(0.1)^{i}(0.9)^{5-i}$$

(5) Emergency number is 911. 60% call police, 25% call ambulance and 15% call fire. what is the probability among 10 calls 6 will be police, 3 for ambulance, 1 for fire.

Sol)
$$\sum_{i=1}^{N} r_i = n.$$

$$P=\frac{n!\cdot(p_1)^{r_1}\cdot p_2^{r_2}\cdots p_n^{r_n}}{r_1!\cdot r_2!\cdots r_n!}$$

$$P=\binom{n}{r_1 r_2 \cdots r_n} p_1^{r_1}\cdot p_2^{r_2}\cdots p_n^{r_n}$$

$$P=\binom{10}{6,3,1}(0.6)^{6}(0.25)^{3}(0.15)^{1}.$$

(6) Exchange receives 16 calls/min. Exchange can handle 24 calls/min. Probability that exchange will saturate.

Sol) $\lambda = 16\cdot 1 = 16.$

$$P(\text{Saturation}) = \sum_{K=25}^{\infty} e^{-16}\,\frac{16^{K}}{K!}$$

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$E, F, -$ are subsets. Sometimes called algebra.

* **Sigma fields**:- Collection of subsets of a set Ω forms a field M if

(i) $\phi \in M$, $\Omega \in M$

(ii) If $E \in M$, $F \in M$ then $E \cup F \in M$ & $EF \in M$

(iii) If $E \in M$ then $E^c \in M$.

$E_1, \dots E_n$ belongs to M so do $\bigcup_{i=1}^{n} E_i \in M$ & $\bigcap_{i=1}^{n} E_i \in M$.

* Sigma(σ) field F (or) σ-algebra

F is field that is closed under any countable set of unions, intersections and combinations. Thus $E_1, \dots E_n \dots$ belong to F so do

$$\bigcup_{i=1}^{\infty} E_i \ \& \ \bigcap_{i=1}^{\infty} E_i$$

where these are defined as.

$$\bigcup_{i=1}^{\infty} E_i \triangleq \{\text{set of all elements in atleast one } E_i\}.$$

$$\bigcap_{i=1}^{\infty} E_i \triangleq \{\text{set of all elements in every } E_i\}$$

* **Events**:-

Consider experiment H.

* Ω is countable, then class of subsets of Ω make a field & each subset is an event. Every subset can be assigned a probability.

* Ω - not countable ($\mathbb{R}$), then not every subset can be given a probability. Only subsets to which probability is assigned are called event. This collection of events form a σ-field / Borel field of events.

* The three objects (Ω, F, P) called as probability space P.

P → probability measure

Axiomatic definition of Probability

Probability is a set function $P(\cdot)$ that assigns to every event $E \in F$ a number $P(E)$ called probability of E

(i) $P[E] > 0$ (ii) $P[\Omega] = 1$ (iii) $P[E \cup F] = P(E) + P(F)$ if $EF = \phi$.

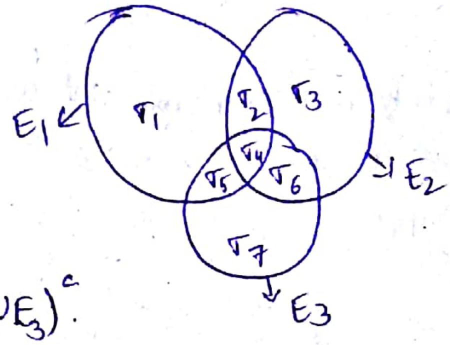
(iv) $P[\phi] = 0$ (v) $P[EF^c] = P[E] - P[EF]$ where $E \in F, F \in F$.

$$P\left[\bigcup_{i=1}^{n} E_i\right] = \sum_{i=1}^{n} P(E_i] \quad \text{if } E_iE_j = \phi \text{ all } i \neq j$$

$$P\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} P(E_i] \longrightarrow \text{union Bound.}$$

* ~~P[E∪F]~~ E_1, E_2, E_3 $\quad P\left[\bigcup_{i=1}^{3} E_i\right] = P\left(\bigcup_{i=1}^{7} \sigma_i\right) = \sum_{i=1}^{7} P[\sigma_i]$

From venn diagram we g... 7 disjoint parts (σ_i)



$$\sigma_1 = E_1E_2^cE_3^c$$
$$= E_1(E_2 \cup E_3)^c.$$

$\sigma_2 = E_1E_2E_3^c$ $\qquad \sigma_4 = E_1E_2E_3.$ $\qquad \sigma_6 = E_1^cE_2E_3.$

$\sigma_3 = E_2E_1^cE_3^c$ $\qquad \sigma_5 = E_3E_1E_2^c$ $\qquad \sigma_7 = E_1^cE_2^cE_3.$

$$P[\sigma_1] = P(E_1) - P[E_1E_2 \cup E_1E_3] \quad \cancel{P(E)-P}$$

$P[EF^c] = P(E] - P$...

~~P[σ3] = P(E2) - P[E2E1 ∪ E~~

$$P(\sigma_1] = P(E_1) - \{P(E_1E_2) + P(E_1E_3] - P(E_1E_2E_3)$$

$$P(\sigma_2] = P(E_1E_2] - P(E_1E_2E_3].$$

$$P(\sigma_4] = P(E_1E_2E_3].$$

$$P\left[\bigcup_{i=1}^{7} E_i\right] = \sum_{i=1}^{7} P(\sigma_i]. = \sum_{i=1}^{3} P(E_i] - \{P(E_1E_2] + P(E_1E_3] + P(E_2E_3]\} + P(...$$

$$= \underbrace{\sum_{i=1}^{3} P_i}_{S_1} - \underbrace{\sum_{1\leq i<j\leq 3} P_{ij}}_{S_2} + \underbrace{\sum_{1\leq i<j<k\leq 3} P_{ijk}}_{S_3}$$

$$P\left[\bigcup_{i=1}^{7} E_i\right] = S_1 - S_2 + S_3.$$

* **theorem**:- The probability P that at least one among events E... occurs in a given experiment is given by.

$$P = S_1 - S_2 + \dots \pm S_n.$$

$$S_1 = \sum_{i=1}^{n} P_i \quad , \quad S_n = \sum_{1 \le i < j < k \dots l \le n} P_{ijk\dots l}.$$

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Random Variables

why need random variables?

Probability is defined over sets. this is inconvenient

we rename the σ-algebra in terms of open subsets of $\mathbb{R}$.

(Ω, F, P) is renamed as (R, B, P).

where B is σ-algebra of semi-open intervals $(-\infty, x]$.

what kinds of renaming is accepted? $x(\mathcal{E}) : \Omega \to \mathbb{R}$.

Define a set! $x \in \mathbb{R}$. $\mathcal{E} \in \Omega$

$$E_B = \{ \mathcal{E} : x(\mathcal{E}) \in B \}. \quad E_B \subseteq F.$$

these mapping are allowed which have an inverse image in F.

*Theorem:- ~~In the real.~~

(i) $\Pr[x(\mathcal{E}) < x] \rightarrow$ Probability distribution function.

$$= F_X(x) = \Pr((-\infty, x])$$

(i) $F_X(\infty) = 1$.

(ii) $F_X(-\infty) = 0$.

(iii) $F(x)$ is monotonous increasing.

(i) Proof by contradiction.

$\Pr(\Omega)$.

Let, $e \in E \subset F$

$x(e)$: $\quad P\underset{(-\infty,\infty)}{(E)} = 1$, instead of $P(\Omega)$.

$x(E) = (-\infty, a]. \Rightarrow \Pr(\Omega) = P((-\infty, a] \cup (a, \infty)) = P((-\infty, a]) + P((a, \infty$

$Pr(\Omega) = 1+\epsilon$

$\downarrow$

Contradiction.

$P(x) = \epsilon$ $\qquad$ $Pr(\infty) = Pr([-\infty, \infty]) = 1.$

(ii) $Fx(-\infty) = 0.$

(iii) $x_1 \leq x \leq x_2$, $x_1 \leq x_2$

T.P $F(x_1) \leq F(x_2)$.

$$x \leq x_2 = (x \leq x_1) \cup (x_1 < x \leq x_2).$$

$$\Rightarrow Pr(x \leq x_2) = Pr(x \leq x_1) + Pr(x_1 < x \leq x_2)$$

$$\Rightarrow F(x_2) - F(x_1) = Pr(x_1 < x \leq x_2).$$

$Fx(x)$ is monotone increasing.

what is so special about $(-\infty, a]$?

$E_1 = X^{-1}((-\infty, a_1]) \in F$

$E_2 = X^{-1}((-\infty, a_2]) \notin F$.

$$P(E_1 \cap E_2) = P(\phi) = 0. = Pr([a_1, a_2])$$

Not consistent with axiom of Probability.

If $Fx(x)$ is differentiable.

$$fx(x) = \frac{dFx(x)}{dx}$$ probability density function f^n (p.d.f).

$Pr(x(\epsilon) \leq x)$

(i) $f_x(x) \geq 0.$

(ii) $\int_{-\infty}^{\infty} f_x(x)\,dx = 1.$

(iii) $F_x(x_2) - F_x(x_1) = \int_{x_1}^{x_2} f_x(x)\,dx.$

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* Joint, Conditional & Total Probabilities, Independence :-

C is A, B both occur , $C \triangleq AB$.

$P[C] = P[AB] \longrightarrow$ joint probability of events A & B.

can be extended for more than 1.

Conditional Probability. $P[B/A] \triangleq \frac{P[AB]}{P[A]} = \frac{n_{AB}}{n_A} = \frac{n_{AB}/n}{n_A/n}$, $P[A]>0$

B occurs given A occured.

$$P(A/B] \triangleq \frac{P[AB]}{P(B)} , \quad P(B]>0$$

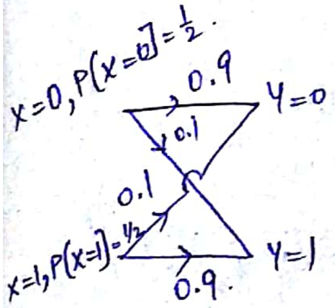
(1) In binary communication, (0,1) . Y – received symbol & X is for transmitted symbol. $\Omega = \{(x,y) : x=0 \text{ or } 1,\ y=0 \text{ or } 1\}$.

$\{x\} = \{(x,0),(x,1)\}$.

$$P[\{(x,y)=(i,j)\}] = P[\{x=i\}]\,P[\{y=j \mid x=i\}] \quad i,j=0,1.$$

$$= P[x=i,\ y=j] = P(x=i] \cap P(y=j)$$

$$\neq P(x=i] \cup P(y=j\}.$$



$P[y=1|x=1] = 0.9 = P[y=0|x=0]$.

$P(y=0|x=1] = 0.1 = P(y=1/x=0]$.

$P(x=0] = P(x=1] = 0.5$.

$P[x=0,y=0] = P[y=0|x=0]\,P(x=0] = 0.45$.

||^{ly} $P[x=0,y=1] = P[y=1|x=0]\,P[x=0] = 0.05$.

$P[x=1,y=0] = P[y=0|x=1]\,P[x=1] = 0.05$.

$P[x=1,y=1] = P[y=1|x=1]\,P[x=1] = 0.45$.

* Given E, F & $EF=\phi$ & A, all belonging to σ-field F. (Ω, F, P).

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is $P(E/A) \geq 0$ — yes.

doubt

$P[\Omega/A] = 1. = \frac{P(\Omega A)}{P(A)} = 1.$

$P[E \cup F/A] = P[E/A] + P(F/A)$ .. for $EF = \phi$.

Example, Page-19. — To be done.

* **Unconditional Probability:**

Theorem: $A_1, A_2, \ldots A_n$ are mutually exclusive events. $\bigcup_{i=1}^{n} A_i = \Omega$

A_i's are exhaustive. (No common elements.)

- B is defined over A_i's.

$$P[B] = P(B/A_1)P(A_1) + P[B/A_2]P(A_2) + \ldots$$

sometimes called as "average"/"total" probability.

$A_iA_j = \phi.\quad i \neq j \qquad B\Omega = B = B\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} BA_i$

$BA_i \subset A_i \qquad (BA_i)(BA_j) = \phi.$

$$P[B] = P\left[\bigcup_{i=1}^{n} BA_i\right] = P[BA_1] + P[BA_2] + \ldots + P[BA_n]$$

$$= P[B/A_1]P[A_1] + \ldots + P[B/A_n]P[A_n].$$

* For ex-1 on binary communication. find $P[Y=0]$ & $P[Y=1]$.

$$P[Y=0] = P[Y=0|X=0]P[X=0] + P(Y=0|X=1]P(X=1)$$

$$= (0.9)(0.5) + (0.1)(0.5)$$

$$= 0.5.$$

$\{Y=0\} \cup \{Y=1\} = \Omega \quad \therefore P[Y=1] = 1 - P[Y=0] = 0.5.$

* **Independence:**

* Two events $A, B \in F$ with $P[A] > 0$, $P[B] > 0$. are said to be independent if and only if $P[AB] = P[A]P[B]$

$$P[AB] = P(B/A)P[A] = P(A/B)\,P[B].$$

for independent events $P[A/B] = P[A]$ & $P[B/A] = P[B]$.

* A, B, C are independent if $P[ABC] = P[A]P[B]P[C]$.

$$P[AB] = P[A]\cdot P[B].$$

$$P[BC] = P(B)P[C] \;\&\; P[AC] = P[A]P(C)$$

* for A_i, $i = 1, \ldots n$. $\{A_i\}$ are independent iff

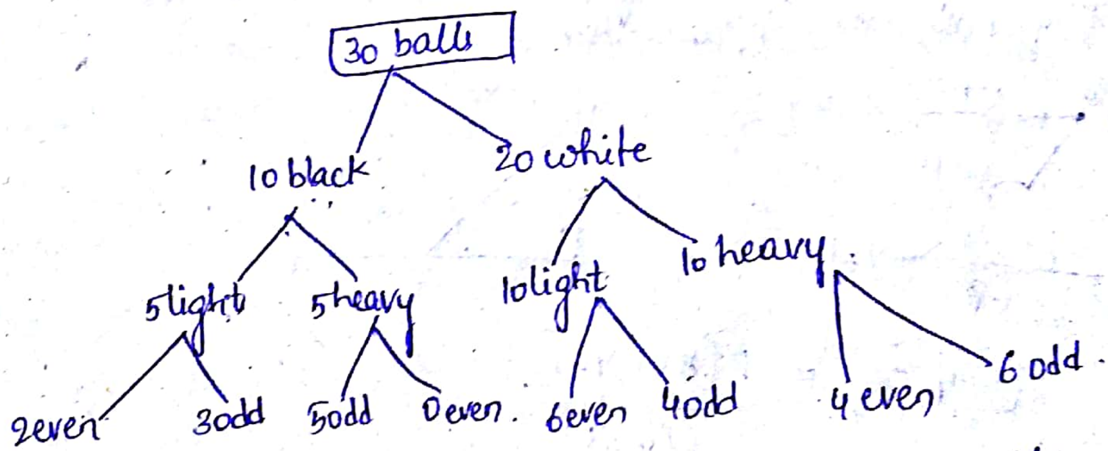
$$P[A_i A_j] = P(A_i)P(A_j)$$

$$P[A_i A_j A_k] = P[A_i]P[A_j]P[A_k] \qquad 1 \le i < j < k \ldots \le n.$$

$$\vdots$$

$$P[A_i \ldots A_n] = P[A_i] \ldots P[A_n].$$

(1.)



A — picking a black ball. B — pick light wt balls.

C — Even numbered balls.

$$P[A] = \frac{1}{3}, \quad P[B] = \frac{1}{2}, \quad P[C] = \frac{12}{30} = \frac{2}{5}.$$

$$P[ABC] = \frac{2}{30} = \frac{1}{15} \;=\; P(A)P(B)P(C) = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{5} = \frac{1}{15}.$$

$$P[AB] = \frac{5}{30} = \frac{1}{6} = P(A)P(B). \qquad P[BC] = \frac{8}{30} = \frac{4}{15} \ne P(B)P(C)$$

$$P[AC] = \frac{2}{30} = \frac{1}{15} \ne P(A)\cdot P(C).$$

$\therefore$ A, B, C are not independent events.

* **Baye's theorem** :- (biometrics, epidemiology, communication theory

Let A_i, $i=1,\ldots n$ be set of disjoint & exhaustive events defined on P. $\bigcup_{i=1}^{n} A_i = \Omega$, $A_i \cap A_j = \phi$.

B be any event defined on P with $P[B] > 0$ and $P[A_i] \neq 0$.

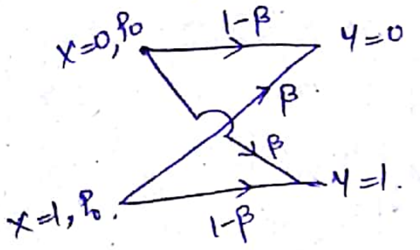
$$P[A_j|B] = \frac{P[B/A_j]\,P[A_j]}{\sum_{i=1}^{n} P[B|A_i]\,P[A_i]} \longrightarrow \frac{P[A_j B]}{P[B]}$$

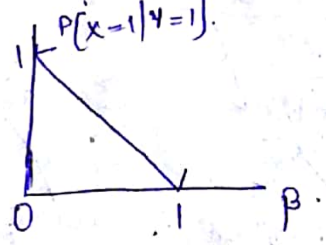
posteriori probability of A_j given B. (Measured from observations.)

$P[B/A_j]$ is called a priori probability of B given A_j

estimated from past / presupposed by experiences.

① In a communication system $P[X=0] = p_0$, $P[X=1] = 1-p_0 = p_1$. Due to noise a zero can be received as a one with probability β & a one can be received as zero with probability β. A one is observed. what is probability that one is transmitted ?





Sol)
$$P[X=1|Y=1] = \frac{P[X=1, Y=1]}{P[Y=1]}$$

$$= \frac{P[Y=1|X=1]\,P[X=1]}{P[Y=1|X=1]P[X=1] + P[Y=1|X=0]\,P[X=0]}$$

$$= \frac{p_1(1-\beta)}{p_1(1-\beta) + p_0\beta}$$

Combinatoric

$a_1, \ldots, a_n \longrightarrow a_{k_1}, a_{k_2}, \ldots a_{k_r}$ — ordered sample of size r.

i) sampling with Replacement :- $\underbrace{- - - - - - -}_{r}$ n items. n^r.

ii) " without " :- $\underbrace{- - - - -}_{r}$ n items $^nP_r = \frac{n!}{(n-r)!}$

No of subpopulations $= {}^nC_r = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$ (Binomial Coefficient)

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

*** Theorem** :- Let $r_1, \ldots r_k$ be a set of nonnegative integers such that $r_1+r_2+\ldots+r_k = n$. No of ways in which a population of n elements can be partitioned into k subpopulations of which the first contains r_1 elements, ... so forth. is $\frac{n!}{r_1!\,r_2!\ldots r_k!}$ (Multinomial coefficient.)

Order of subpopulation is essential but order within each group does not receive attention.

Balls (1,2,3,4,5) $n=5$

$r_1=3, r_2=2$

gp1 :	1,2,3	2,3,4	. . .
gp2 :	4,5	5,1	. . .

$r_1=2, r_2=3.$

gp1 :	4,5	5,1	. . .
gp2 :	1,2,3	2,3,4	. . .

$$\frac{n!}{r_1!}\cdot\frac{1}{r_2!}\cdot\frac{1}{r_3!}\ldots = \frac{n!}{r_1!(n-r_1)!}\cdot\frac{(n-r_1)!}{r_2!\,(n-r_1-r_2)!}\cdots\frac{\left(n-\sum_{j=1}^{k-1} r_j\right)!}{r_k!\underbrace{\left(n-\sum_{j=1}^{k} r_j\right)!}_{0!=1.}}$$

$$\frac{n!}{r_1!\,r_2!\ldots r_k!} = \underbrace{\binom{n}{r_1}}_{\substack{\text{choose } r_1 \text{ elem}\\ \text{from } n}}\underbrace{\binom{n-r_1}{r_2}}_{\substack{r_2 \text{ from}\\ n-r_1}}\underbrace{\binom{n-r_1-r_2}{r_3}}_{\substack{r_3 \text{ from}\\ n-r_1-r_2}}\cdots\binom{n-r_1-\ldots-r_{k-2}}{r_{k-1}}.$$

Occupancy Problems :-

This is random placement of r balls into n cells.
first ball has n choices, 2nd ball has n choices, ...

n^r possible distributions (balls are distinguishable).
less than n^r if (" " not ")

How many distinguishable distributions can be formed from r balls & n cells
n cells are represented by spaces b/w $n+1$ bars & balls by stars.

| | | | — 3 empty cells.

|*| |**| — 1 ball in cell 1
0 in 2
2 in 3.

for $r_i \geq 0$ (r_i - no of balls in ith cell). & r being total no of balls.

$$r_1 + \dots r_n = r.$$

$(r_1, r_2 \dots r_n)$ is called occupancy & r_i are occupancy numbers;
distribution is indistinguishable if corresponding occupancies are identical
n cells need $n+1$ bars. but since first & last symbols must be bars,
only $n-1$ bars & r stars can appear in any order.
no of subpopulations of size r in population of size $n-1+r$ is.

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

* No cell is Empty = No bars are adjacent.
n of the r stars should occupy the spaces but remaining $r-n$ can go anywhere. , thus $n-1$ bars, $r-n$ stars can appear in any order
No of distinct distributions = No of ways of choosing $(r-n)$ places in $(n-1)$ bars + $(r-n)$ stars or $(r-n)$ out of $n-1+r-n = r-1$ is.

$$\binom{r-1}{r-n} = \binom{r-1}{n-1}.$$

Pg-29. — Read.

Given n cells, r particles. No of distinguishable arrangements of indistinguishable identical particles is $\binom{n+r-1}{r}$ & assigned a probability of $\binom{n+r-1}{r}^{-1}$.

* 1. Exclusion principle (no two or more balls in same cell) &
2. all distinguishable arrangements satisfying (1) are equally probable
↓
$r \le n$, No of distinguishable arrangements under the hypothesis of exclusion principle is no of subpopulations of size r in a population of n elements of $\binom{n}{r}$ & probability of any state is $\binom{n}{r}^{-1}$.

Extensions & Applications :-

No of individual probability terms in the sum S_i is $\binom{n}{i}$. There are a total of n indices & in S_i, all terms have i indices.

$n=5$, $i=2$.

$S_2 = P_{12}+P_{13}+P_{14}+P_{15}+P_{23}+P_{24}+P_{25}+P_{34}+P_{35}+P_{45}$. (No repetition)

∴ No of terms in S_i is no of subpopulations of size i in a population of size n is $\binom{n}{i}$

Pg - 30 Ex → Read.
31

* Bernoulli trials - Binomial & Multinomial Probability laws :-

We throw a coin twice. Sample description is Ω_2, $\Omega_2 = \Omega \times \Omega$ where $\Omega = \{s, f\}$,

$\Omega_2 = \{ss, sf, fs, ff\} \to 2^4 = 16$ events.

$\Omega \times \Omega$ is cartesian product.

n independent trials — $\Omega_n = \underbrace{\Omega \times \ldots \Omega}_{n \text{ times}}$ & 2^n outcomes. each of which is ordered n-tuple. $\Omega_n = \{a_1, \ldots a_M\}$ $M = 2^n$.

$a_i \triangleq z_{i_1}, \ldots z_{i_n}$ $z_{ij} = s$ or f.

Each outcome z_{ij} is independent of any other outcome, the joint probability $P[z_{i_1} \ldots z_{i_n}] = P(z_{i_1}) P(z_{i_2}) \ldots P[z_{i_n}]$.

* Probability of ordered set of k successes & $n-k$ failures is $p^k q^{n-k}$.

$P[\{s\}] = p$, $P[\{f\}] = q$.

$p = P[\{H\}]$ $q = p[\{T\}]$ $P[\{HTH\}] = pqp = p^2 q$

* Exp has n bernouli trails. Ω_n has $M = 2^n$ outcomes $a_1, \ldots a_M$. where a_i is a string of n symbols, where each symbol represents a success s or failure f. Consider $A_k \triangleq \{k \text{ success in } n \text{ trials}\}$. & let a_i' denote strings with k success & $(n-k)$ failures. K denote the no of ordered arrangments with " " " " "

$$A_k = \bigcup_{i=1}^{K} \{a_i'\}.$$

Let bars be failures & stars success.

$|**|**||*$ — ntries fssfssffs.

Here $r = k$ & $(n-1)+r$ is replaced by $(n-k)+k = n$.

$$K = \binom{n}{k}$$

$\{a_i\}$ are disjoint $\{a_i\} \cap \{a_j\} = \phi$

$$P[A_k] = P\left(\bigcup_{i=1}^{K} \{a_i'\}\right) = \sum_{i=1}^{K} P[\{a_i'\}].$$

$$P[\{a_i'\}] = p^k q^{n-k} \Rightarrow P[A_k] = \binom{n}{k} p^k q^{n-k} \triangleq b(k;n,p)$$

Ex (Pg-34,35) doubt.

* Probability $B(k;n,p)$ of k or fewer successes in n tries is given by.

$$B(k;n,p) = \sum_{i=0}^{k} b(i;n,p) = \sum_{i=0}^{k} \binom{n}{i} p^i q^{n-i}$$

$B(k;n,p)$ = binomial distribution function.

Probability of atleast k successes in ntries is $\sum_{i=k}^{n} b(i;n,p) = 1 - B(k-1;n,p)$.

Probability of more than k successes but no more than j success is $\sum_{k+1}^{j} b(i;n,p)$

Multinomial Probability law :-

Generalized Bernoulli trail in which K outcomes are possible. An Elementary exp consisting of a single trail with k elementary outcomes $\xi_1, \xi_2, \ldots \xi_K$ Probability of ξ_i be p_i $\sum_{i=1}^{k} p_i = 1$.

the trial is repeated n times in which $\mathcal{E}_1$ appears r_1 times, so on $\mathcal{E}_k$ appears r_k times. Probability of this event is $p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$.

observing a prescribed order.

Here we need to count all strings of length n in which $\mathcal{E}_1$ appears r_1 times etc. In general case of n trials with r_1 outcomes of $\mathcal{E}_1$, $\mathcal{E}_2$ so on there are $\frac{n!}{r_1! r_2! \cdots r_k!}$ such strings → Multinomial Coefficient.

no of ways of selecting r_1 outcomes of $\mathcal{E}_1 = \binom{n}{r_1}$

" " " " r_2 " of $\mathcal{E}_2$ ~~an~~ within $(n-r_1)$ $= \binom{n-r_1}{r_2} \cdots$

we get $\binom{n}{r_1}\binom{n-r_1}{r_2} \cdots \binom{n-r_1-r_2-\cdots r_{k-1}}{r_k} = \frac{n!}{r_1! r_2! \cdots r_k!}$

<u>Multinomial probability law</u>:- Consider a generalized Bernoulli trail with outcomes $\mathcal{E}_1, \mathcal{E}_2, \ldots \mathcal{E}_k$ & $p[\mathcal{E}_i] = p_i$, $i=1,\ldots k$. $\sum_{i=1}^{k} p_i = 1$. Probability that in n trials $\mathcal{E}_1$ occurs r_1 times, ... is

$$\boxed{P(r; n, p) = \frac{n!}{r_1! r_2! \cdots r_k!} p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}}$$

$r = (r_1, r_2 \ldots r_k)$ $\quad p = (p_1, p_2, \ldots p_k)$. $\quad \sum_{i=1}^{k} r_i = n$.

large factorials!

$$n! \simeq (2\pi)^{1/2} n^{n+(1/2)} e^{-n}$$

*** <u>POISSON LAW</u>:-**

$b(k; n, p)$ $\quad n \gg 1$, $p \ll 1$ $\qquad$ if $n \to \infty$, $p \to 0$.

$np = a$; $q = 1-p$. $\qquad \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{a^k \cdot e^{-a}}{k!} = b(k; n, p)$.

Pg-40 Ex.

$P[k \text{ pts}] = e^{-a} \frac{a^k}{k!}$.

if $a \triangleq \lambda T$; λ = avg no of events per unit time.

T = length of the interval $(t, t+T)$.

$$P(k; t, t+T) = e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$ — λ independent of T.

$$P(k; t, t+T) = \exp\left[-\int_t^{t+T} \lambda(\xi)\, d\xi\right] \frac{1}{k!} \left(\int_t^{t+T} \lambda(\xi)\, d\xi\right)^k$$ — λ depends on T.

k events in $(x, x+\Delta x)$.

Radioactive decay — λ = avg no of emitted α particles per sec.

Water pollution monitoring — λ = avg no of bacteria per cm^3.

Size of switchboard — λ = avg no of calls phone calls/sec.

Transportation (Toll gates).

Optical receiver — λ = avg no of photons per sec per unit area.

Fiber optical transmitter receiver — λ = avg no of photoelectrons/sec.

λ = Poisson rate parameter.

$$\dim = \frac{\text{events}}{\text{unit interval}}$$

Pg – 42, 43 — Ex.

Poisson's law is based on following assumptions.

1. $P(1, t, t+\Delta t)$, of a single event occurring in $(t, t+\Delta t)$ is proportional to

$$P(1; t, t+\Delta t) \simeq \lambda(t) \cdot \Delta t \qquad \Delta t \to 0.$$

$\lambda(t)$ — poisson rate parameter.

* 2. Probability of k events in $(t, t+\Delta t)$ goes to zero $(k > 1)$.

$$P(k, t, t+\Delta t) \simeq 0 \quad \Delta t \to 0, \; k = 2, 3, \ldots$$

3. Events in nonoverlapping time intervals are statistically independent.

Pg – 44, 45, 46, 47 — Ex.

Normal Approximation to Binomial law :-

$$P[S_n = k] = \binom{n}{k} p^k q^{n-k} = b(k; n, p)$$

S_n = no of success in n Bernoulli trails, $0 \le k \le n$.

Normal distribution :- n is large.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2\right]$$

$\phi(x)$ = standard normal density.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}y^2\right] dy$$ $\Phi(x)$ – standard normal distribution.

$$\boxed{b(k;n,p) \approx \frac{1}{\sqrt{npq}} \phi\left(\frac{k-np}{\sqrt{npq}}\right)}$$ n is large

better if $npq \gg 1$. & $npq = 1.6$.

when $n \gg 1$, a, b are fixed integers.

$$\boxed{P[\alpha \leq S_n \leq \beta] \simeq \Phi\left[\frac{\beta - np + 0.5}{\sqrt{npq}}\right] - \Phi\left[\frac{\alpha - np - 0.5}{\sqrt{npq}}\right]}$$

$$\mathrm{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{y^2}{2}} dy.$$

$$\Phi(x) = \frac{1}{2} + \mathrm{erf}(x); \quad x > 0$$
$$= \frac{1}{2} - \mathrm{erf}(|x|), \quad x < 0.$$

<u>Poisson's law</u> :-

$\lambda T \gg 1$.

$$\boxed{\sum_{k=\alpha}^{\beta} e^{-\lambda T} \frac{(\lambda T)^k}{k!} = \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} \exp\left[-\frac{1}{2}y^2\right] dy.}$$

$$l_2 \triangleq \frac{\beta - \lambda T + 0.5}{\sqrt{\lambda T}} \qquad l_1 \triangleq \frac{\alpha - \lambda T - 0.5}{\sqrt{\lambda T}}.$$

$$\boxed{e^{-\lambda T} \frac{(\lambda T)^k}{k!} = \frac{1}{\sqrt{2\pi}} \int_{l_3}^{l_4} \exp\left(-\frac{1}{2}y^2\right) dy}$$

where $l_4 \triangleq \dfrac{k - \lambda T + 0.5}{\sqrt{\lambda T}}$

$l_5 \triangleq \dfrac{k - \lambda T - 0.5}{\sqrt{\lambda T}}.$

16/8/18.

Random Variables :-

We convert the outcomes of H into Numerical values. For that we need a rule/mapping from original sample space description Ω to the Real line R. Such a mapping is what a random variable is.

* Every subset of R is not an event. But these subsets are of no importance & are called as nonmeasurable. The sets of importance are [a], [a, b], (a, b], [a, b), (a, b).

* We can define more than one random variable on the same underlying sample space Ω.

* **Definition of Random Variable :-**

Consider an exp H with sample space Ω & ξ random outcomes. If to every ξ we assign a real number $X(\xi)$; we establish a correspondence rule b/w ξ & R. This rule is called a random variable.

$$E_B \subset \Omega \quad (E_B \in F).$$

The event $\{\xi : X(\xi) \le x\}$ often abbreviated as $\{X \le x\}$, will denote an event of unique importance & we need to assign probability for it.

$P[X \le x] \triangleq F_X(x)$ is called the Probability distribution function (PDF) of X.

For $F_X(x)$ to be consistent with the axiomatic defn of probability, it should satisfy the following conditions.

- For every Borel set of Numbers B, the set $\{\xi : X(\xi) \in B\}$ must correspond to an event $E_B \in F$, it must be in the domain of Probability function P(·). i.e X can be a random variable only if the inverse image of X of all Borel subsets in R, making up the field B are events.

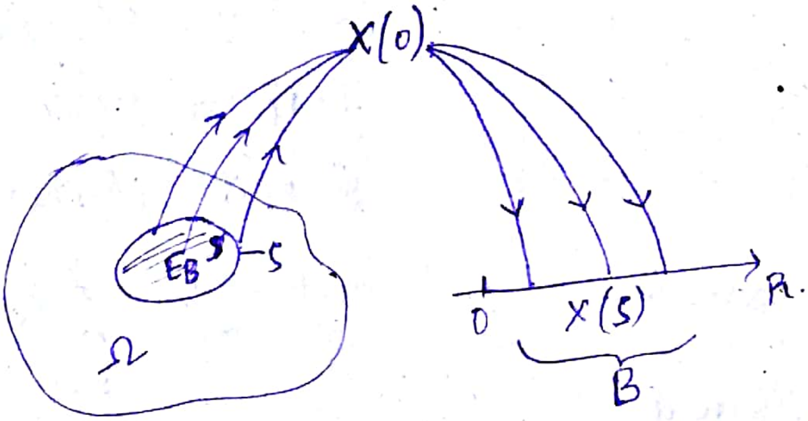
Inverse image :-

Consider an arbitary Borel set of real numbers B, the set of pts E_B in Ω for which $X(\xi)$ assumes values in B is called inverse image of B under X.

* All sets of engineering interests can be written as $(-\infty, x]$.

The event $\{\xi : X(\xi) \le x\} \in F$ gets mapped under X into $(-\infty, x] \in B$

Thus if X is random variable, the set of points $(-\infty, x]$ is an event



* Under the mapping X we have, in effect, generated a new probability space (R, B, P_X) R is Real line, B is the Borel σ-algebra of all subsets of R generated by countable unions & intersections of sets of the form $(-\infty, x]$ & P_X is set function assigning a number $P_X[A] \geq 0$ to each set $A \in B$.

* To assign desirable continuity to $F_X(x)$ at $x = \pm\infty$; events $\{X = \infty\}$ and $\{X = -\infty\}$ have probability zero.

<u>Definition</u> :- Let H be an experiment with sample description space Ω. Then the real random variable X is a function whose domain is Ω that satisfies the following:

(i) for every Borel set of numbers B, the set $E_B \triangleq \{\varsigma \in \Omega, X(\varsigma) \in B\}$ is an event & (ii) $P[X = -\infty] = P[X = +\infty] = 0$.

$F_X(x)$ is staircase func?

If X has countable set of pts — X is called discrete random variable

if Range of X is continuum — X " " continuos "

$F_X(x)$ is continuos.

Ex - doubt.

* <u>Probability distribution function</u> :- (PDF).

PDF is a function of x,.

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17/8/18

$X : \Omega \to \mathbb{R}$

$X^{-1}((-\infty, a]) \in F.$

$\xi_1 = X^{-1}((-\infty, a_1]) \in F$, some a_1

$\xi_2 = X^{-1}((-\infty, a_2]) \notin F$, some a_2.

(i) $P(\mathcal{E}) \geq 0$, $\mathcal{E} \subseteq F$

(ii) $P(\phi) = 0$, $P(\Omega) = 1$

(iii) $P\left(\bigcup_{i=1}^{\infty} \mathcal{E}_i\right) = \sum_{i=1}^{\infty} P(\mathcal{E}_i)$

$\xi_i \cap \xi_j = \phi$.

$Pr\left[X^{-1}((-\infty, a_1]) \cap X^{-1}((-\infty, a_2])\right]$

$= Pr[\xi_1 \cap \xi_2] = P[\phi] = 0.$

$Pr[(-\infty, \min(a_1, a_2)]] = \epsilon$

Contradiction

2 Possibilities.

1. F is uncountable
2. F is countable. (1-1 mapping is possible).

P.D.F.

1) $F_X(-\infty) = 0$ 2) $F_X(\infty) = 1$. 3) $x_1 \leq x \leq x_2$ $F_X(x_1) \leq F_X(x_2)$

Continuos p.d.f.

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

$f_X(\cdot)$ – probability density function.

$\left(\begin{array}{l} F_X = P_X \\ f_X = p_X \end{array} \right)$

(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x) = 1$

(iii) $x_1 \leq x \leq x_2$.

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx.$$

* Gaussian :-

1. $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ (Central limit theorem proof.)

$E[x] = \int_{-\infty}^{\infty} x f_X(x)\,dx \longrightarrow$ Expected value of X.

Variance $Var[x] = \int_{-\infty}^{\infty} (x - E(x))^2 f_X(x)\,dx$.

$E(x) = \mu$, $var[x] = \sigma^2 \Big\} \rightarrow$ After finding integrals.

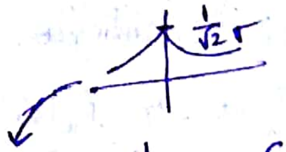
$z = x + iy$. $\quad |z| = \sqrt{x^2 + y^2}$.

Rayleigh $(\sigma > 0)$.

$$f_X(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) u(x). \qquad u(x) = 1,\ x \geq 0; \quad 0,\ x < 0.$$

Uniform Distribution:

$$f_X(x) = \frac{1}{b-a},\ a < x < b.$$
$$= 0,\ \text{else}.$$



Laplacian. $(c > 0)$.

$$f_X(x) = \frac{c}{2} e^{-c|x|}$$ $\rightarrow$ sometimes written as $f_X(x) = \frac{1}{\sqrt{2}\sigma} \exp\left[-\sqrt{2}|x|/\sigma\right]$, $\sigma > 0$.

chi-squared: $y = \sum_{i=1}^{n} x_i^2 \rightarrow$ chi-squared distribution with n-degrees of freedom.

$$f_Y(y) = \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{y^2}{2}} u(y).$$

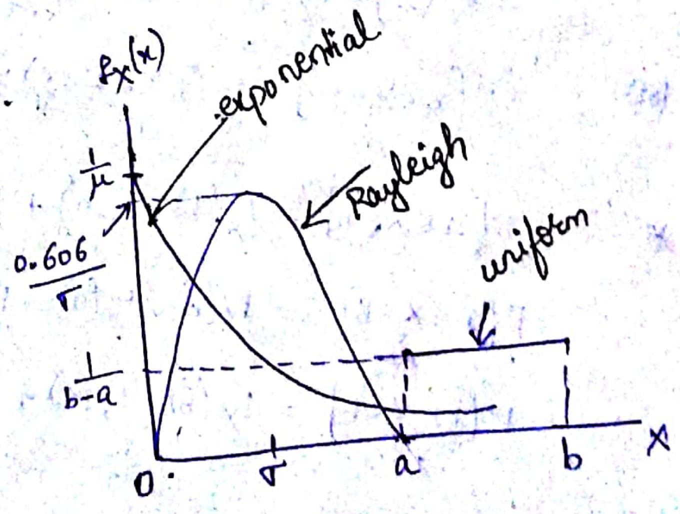
$$\Pr(a \leq x \leq b) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_a^b \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx. \qquad \frac{a-\mu}{\sigma} = a', \quad \frac{b-\mu}{\sigma} = b'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a'}^{b'} \exp\left(-\frac{x^2}{2}\right) dx. \qquad \operatorname{erf}(a) = \left(\int_0^a e^{-\frac{x^2}{2}} dx\right) \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \operatorname{erf}(b') - \operatorname{erf}(a').$$

Exponential $(\mu > 0)$:

$$f_X(x) = \frac{1}{\mu} e^{-x/\mu} u(x).$$



Probability Distribution

PDF $\rightarrow$ $F_X(x) = P\left[\{\varsigma : X(\varsigma) \le x\}\right] = P_X\left[(-\infty, x]\right]$ we merely write as $P[X \le x]$ or $P[a<X$ ... $\{X \le x\} \cup \{a<$ ...

$\{\varsigma : X(\varsigma) \le x\} \subset \Omega$ & $(-\infty, x]$ are equivalent events.

& X, Y, etc denotes random variable & x, y denotes the values they take

& * If $F_X(x)$ is discontinuous at x_0 then $F_X(x_0)$ = value of F immediately to the right of x_0 & $F_X(x_0^-)$ is value to immediate left.

* Properties of $F_X(x)$

(i) $F_X(\infty) = 1$, $F_X(-\infty) = 0$.

(ii) $x_1 \le x_2 \rightarrow F_X(x_1) \le F_X(x_2)$, $F_X(x)$ is nondecreasing function.

(iii) $F_X(x)$ is continuous from right i.e $F_X(x) = \lim\limits_{\varepsilon \to 0} F_X(x+\varepsilon)$, $\varepsilon > 0$.

Proof of (ii)

Consider $\{x_1 < X \le x_2\}$ then $[x_1, x_2]$ is non Empty & belongs to B.

$$\therefore \quad 0 \le P[x_1 < X \le x_2] \le 1.$$

but $\{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$.

& $\{X \le x_1\}\{x_1 < X \le x_2\} = \phi$.

$$\therefore \quad F_X(x_2) = F_X(x_1) + P[x_1 < X \le x_2].$$

(or) $P[x_1 < X \le x_2] = F_X(x_2) - F_X(x_1) \ge 0$ for $x_2 > x_1$.

*

$$P[a \le X \le b] = F_X(b) - F_X(a) + P[X = a]$$

$$P[a < X < b] = F_X(b) - F_X(a) - P[X = b]$$

$$P[a \le X < b] = F_X(b) - F_X(a) + P[X = a] - P[X = b]$$

Ex

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Probability Density Function (Pdf).

$F_X(x)$ is Continuos & differentiable. $\quad f_X(x) = \dfrac{dF_X(x)}{dx}$.

Properties:- If $f_X(x)$ exists.

(i) $f_X(x) \geq 0$.

(ii) $\int_{-\infty}^{\infty} f_X(\xi)\, d\xi = F_X(\infty) - F_X(-\infty) = 1$.

(iii) $F_X(x) = \int_{-\infty}^{x} f_X(\xi) d\xi = P[X \leq x]$.

(iv) $F_X(x_2) - F_X(x_1) = \int_{-\infty}^{x_2} f_X(\xi) d\xi - \int_{-\infty}^{x_1} f_X(\xi) d\xi$

$$= \int_{x_1}^{x_2} f_X(\xi) d\xi = P[x_1 < X \leq x_2].$$

Interpretation of $f_X(x)$

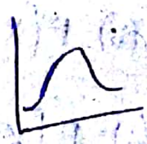
$$P[x < X \leq x + \Delta x] = F_X(x+\Delta x) - F_X(x).$$

If $F_X(x)$ is continuos in its first derivative then,

$$F_X(x+\Delta x) - F_X(x) = \int_{x}^{x+\Delta x} f(\xi) d\xi \simeq f_X(x)\Delta x \longrightarrow \text{for small } \Delta x.$$

* If $f_X(x)$ exists, then $F_X(x)$ is continuos & $\therefore P[X = x] = 0$.

Gaussian (Univariate Normal pdf).



$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \qquad \sigma^2 - \text{Variance}.$$

μ (mean), σ (standard deviation) are independent parameters.

For random variables with well defined pdf,

$$\mu = \int_{-\infty}^{\infty} x f_X(x)\, dx \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$$

For random variables that take on discrete values such as Bernoulli, Poisson etc

$$\mu = \sum_{x_i; P[X=x_i]>0} x_i P[X = x_i] \qquad \sigma^2 = \sum_{x_i; P[X=x_i]>0} (x_i-\mu)^2 P[X=x_i].$$

$\sum_{i=-\infty}^{\infty}$ $\qquad \sum_{i=-\infty}^{\infty}$

*Conversion of Gaussian pdf to standard Normal :-

We are given $X : N(\mu, \sigma^2)$ & must evaluate $P[a < X \le b]$. We have

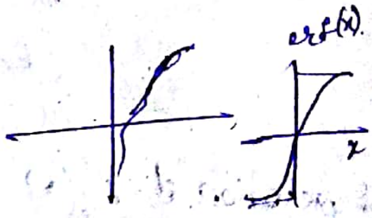
$$P[a < X \le b] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$$\beta \triangleq \frac{(x-\mu)}{\sigma}; \quad d\beta = \frac{dx}{\sigma} \qquad b' = \frac{b-\mu}{\sigma}, \quad a' = \frac{a-\mu}{\sigma}$$

$$P[a < X \le b] = \frac{1}{\sqrt{2\pi}} \int_{a'}^{b'} e^{-\frac{1}{2}x^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{b'} e^{-\frac{1}{2}x^2} dx - \frac{1}{\sqrt{2\pi}} \int_0^{a'} e^{-\frac{1}{2}x^2} dx.$$

Errorfunction $\leftarrow$ $\underline{\text{erf}(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}t^2} dt.}$



$\therefore X : N(\mu, \sigma^2)$ then.

$$P[a < X \le b] = \text{erf}\left(\frac{b-\mu}{\sigma}\right) - \text{erf}\left(\frac{a-\mu}{\sigma}\right)$$

$$\boxed{\text{erf}(-x) = -\text{erf}(x).}$$

(a) $P[X \le x] = \frac{1}{2} + \text{erf}(x).$

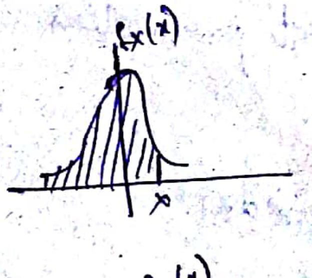
(b) $P[X > -x] = \frac{1}{2} + \text{erf}(x)$

(c) $P[X > x] = \frac{1}{2} - \text{erf}(x).$

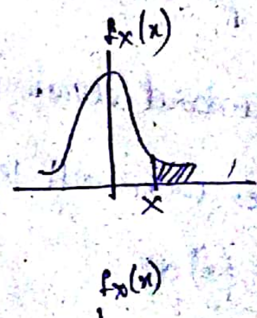
(d) $P[-x < X \le x] = 2\,\text{erf}(x).$

(e) $P[|X| > x] = 1 - 2\,\text{erf}(x).$

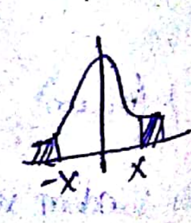
(a)



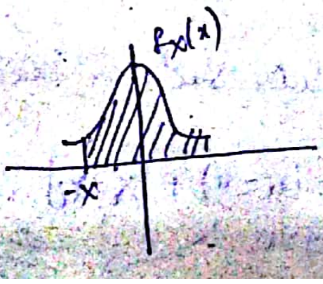
(c)



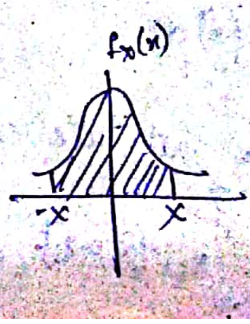
(e)



(b)



(d).



* **Gamma** $(b>0, c>0)$.

$$f_X(x) = K_\gamma x^{b-1} e^{-cx} u(x).$$

$$K_\gamma = c^b / \cancel{\Gamma(b)}\Gamma(b).$$

* **Student-t:** (n is integer).

$$f_X(x) = K_{st}\left(1+\frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}.$$

$$K_{st} = \frac{\Gamma\left(\frac{(n+1)}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi n}}.$$

/8/18.

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

Find Mean & variance of chi-squared p.d.f — P.T $E[x]=n$, $var[x]=2n$.

$$f_X(x) = \frac{x^{n/2-1}}{2^{n/2}\Gamma(n/2)} e^{-x/2} u(x).$$

$$E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx = \int_{-\infty}^{\infty} \frac{x \cdot x^{n/2-1}}{2^{n/2}\Gamma(n/2)} e^{-x/2} u(x) dx$$

$u(x) = 1, x \geq 0.$
$= 0, x < 0.$

$$\cancel{\Gamma(n/2) = \int_0^\infty x} \quad E[x] = \int_0^\infty \frac{x^{n/2} \cdot e^{-x/2} dx}{2^{n/2}\Gamma(n/2)} = 2\int_0^\infty \frac{t^{n/2} e^{-t} dt}{\Gamma(n/2)}$$

$\left(\frac{x}{2} = t\right)$

$$= \frac{2\Gamma(n/2+1)}{\Gamma(n/2)} = 2 \cdot n/2 = n.$$

$$E[x^2] \quad \cancel{var(x)} = \int_{-\infty}^{\infty} x^2 f_X(x) dx. \Rightarrow E[x^2] = \int_0^\infty \frac{x^2 \cdot x^{n/2-1}}{2^{n/2}\Gamma(n/2)} e^{-x/2} dx.$$

Ch-3. Functions of Random Variables.

First View ($Y : \Omega \to R_Y$): For every $\xi \in \Omega$, we generate a number $g[X(\xi)] \triangleq Y(\xi)$. Y is a r.v with domain Ω & range $R_Y \subset R$.

$\{\xi : Y(\xi) \leq y\}$ is equal to event $\{\xi : g[X(\xi)] \leq y\}$.

* Here, the stress is on Y as mapping from Ω to R_Y (Role of X is sup

Second (I/O view): Domain of Y is R_X & Range is R_Y.

$$X \rightarrow \boxed{g(\cdot)} \rightarrow Y.$$

$$\{\xi : Y(\xi) \leq y\} = \{\xi : g[X(\xi)] \leq y\} = \{\xi : X(\xi) \in C_y\}.$$

$$P[Y \leq y] = P[X \in C_y].$$

Ex Pg-120.

(i) $Y = aX + b$, X is cont. r.v with pdf $f_X(x)$.

$$\{aX + b \leq y\} \Rightarrow \{X \leq \tfrac{y-b}{a}\} \rightarrow C_y$$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) \quad \& \quad f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

* for $a < 0$ $\quad \{X \geq \frac{y-b}{a}\}$. $\quad$ ($\because$ $\{X < \frac{y-b}{a}\}$ & $\{X \geq \frac{y-b}{a}\}$ are disjoin union is certain

$$P\left[X < \frac{y-b}{a}\right] + P\left[X \geq \frac{y-b}{a}\right] = 1.$$

$$P\left[X \leq \frac{y-b}{a}\right] = P\left[X < \frac{y-b}{a}\right] \quad \& \quad P\left[X \geq \frac{y-b}{a}\right] = P\left[X > \frac{y-b}{a}\right].$$

$\therefore$ ~~Pe~~ $\therefore$ for $a < 0$ $\quad F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right).$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right), \quad a \neq 0.$$

Ex Pg-123 $\rightarrow$ doubt.

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General Formula of Determing the pdf of $Y=g(X)$:

pdf of conti. rv X is $f_X(x)$. & $Y \triangleq g(X)$.

$\{y < Y \leq y+dy\}$. can be written as union of disjoint elementary events $\{E_i\}$ in the Borel field generated under X. If $y=g(x)$ has n real roots $x_1, x_2, \dots x_n$, then the disjoint events have the form.

$E_i = \{x_i - |dx_i| < X \leq x_i\}$ if $g'(x_i)$ is negative or $E_i = \{x_i < X \leq x_i + |dx_i|\}$ if $g'(x_i)$ is +ve.

$P[E_i] = f_X(x_i)|dx_i|$

$$P[y < Y \leq y+dy] = f_Y(y)|dy|$$
$$= \sum_{i=1}^{n} f_X(x_i)|dx_i|.$$

$$f_Y(y) = \sum_{i=1}^{n} f_X(x_i)\left|\frac{dx_i}{dy}\right|$$

(1) $y = x^n$.

$x = y^{1/n} \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{n}|y|^{1/n - 1} = \dfrac{1}{n}|y|^{\left(\frac{1-n}{n}\right)}$

$y>0 \rightarrow f_Y(y) = \frac{1}{n}|y|^{(1-n/n)} \cdot f_X(y^{1/n})$

$y<0$. & n is odd $\rightarrow$ root is $-|y|^{1/n} \Rightarrow f_Y(y) = \frac{1}{n}|y|^{(1-n/n)} \cdot f_X(-|y|^{1/n})$.

n is even $\rightarrow$ root is $0 \Rightarrow f_Y(y) = \frac{1}{n}|y|^{(1-n/n)} f_X(0)$.

Solving problems of the type $Z=g(X,Y)$.

Find the point set C_z in the (x,y) plane such that the events $\{\zeta : Z(\zeta) \leq z\}$ & $\{\zeta : X(\zeta), Y(\zeta) \in C_z\}$.

$$\{Z \leq z\} = \{(X,Y) \in C_z\} \Rightarrow F_Z(z) = \iint_{(x,y)\in C_z} f_{XY}(x,y)\,dx\,dy.$$

*

Region $xy \leq z$

$$F_Z(z) = \int_0^{\infty}\left(\int_{-\infty}^{z/y} f_{XY}(x,y)\,dx\right)dy + \int_{-\infty}^{0}\int_{z/y}^{\infty} f_{XY}(x,y)\,dx\,dy.$$

$$G_{XY}(x,y) \triangleq \int f_{XY}(x,y)\,dx$$

$$F_Z(z) = \int_0^{\infty} [G_{XY}(z/y, y) - G_{XY}(-\infty, y)]\,dy + \int_{-\infty}^{0} [G_{XY}(\infty, y) - G_{XY}(z/y, y)]\,d$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{XY}(z/y, y)\,dy.$$

* As a special case, assume X & Y are indepent, identically distribute random variables (i.i.d) with

$$f_X(x) = f_Y(x) \triangleq \frac{\alpha/\pi}{\alpha^2 + x^2}$$

This is known as the Cauchy probability law. Bcz of independence

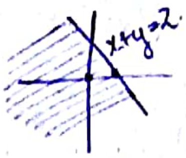
$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y).$$

$$f_Z(z) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} \frac{1}{z^2 + \alpha^2 x} \cdot \frac{1}{\alpha^2 + x}\,dx$$

$$f_Z(z) = \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{z^2 - \alpha^4} \ln \frac{z^2}{\alpha^4}.$$

Ex Pg-138 — All doubt.

* **The Sum of two independent Random Variable** : $Z = X + Y$.



$x + y \leq z$.

$$F_Z(z) = \iint_{x+y \leq z} f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\,dy.$$

$$= \int_{-\infty}^{\infty} [G_{XY}(z-y, y) - G_{XY}(-\infty, y)]\,dy.$$

$$G_{XY}(x,y) \triangleq \int f_{XY}(x,y)\,dx.$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \frac{d}{dz} [G_{XY}(z-y, y)]\,dy$$

x, y are independent

$$= \int_{-\infty}^{\infty} f_{XY}(z-y, y)\,dy = \underbrace{\int_{-\infty}^{\infty} f_X(z-y) \cdot f_Y(y)\,dy}_{\text{Convolution Integral.}}$$

OR

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\,dx.$$

If $Z = aX + bY$ $(a, b > 0)$ $\Rightarrow$ $y = \frac{z}{b} - \frac{ax}{b}$. $ax + by \leq z$

$$F_Z(z) = \iint f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} f_X\left(\int_{-\infty}^{z/a - \frac{by}{a}} f(x,y)\,dx\right)dy.$$

$$f_Z(z) = \frac{1}{a}\int_{-\infty}^{\infty} f_X\left(\frac{z}{a} - \frac{b}{a}\right) f_Y(y)\,dy.$$

(or)

$V = aX$, $W = bY$ $\Rightarrow$ $Z = V + W$.

$$f_V(v) = \frac{1}{a} f_X\left(\frac{v}{a}\right).$$

$$f_W(w) = \frac{1}{b} f_Y\left(\frac{w}{b}\right).$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_V(z-w) f_W(w)\,dw$$

$$f_Z(z) = \frac{1}{ab}\int_{-\infty}^{\infty} f_X\left(\frac{z-w}{a}\right) f_Y\left(\frac{w}{b}\right) dw.$$

* the pdf of sum of discrete Random variables is also computed by convolution. X & Y are independent.

$$P_Z(z_n) = \sum_{x_k + y_k = z_n} P[X = x_k, Y = y_j] \Rightarrow P_Z(z_n) = \sum_{x_k + y_j = z_n} P_X(x_k) P_Y(y_j)$$

$$P_Z(z_n) = \sum_{x_k} P_X(x_k) P_Y(z_n - x_k).$$

$P_Z(n) = \sum_{K} P_X(K) P_Y(n-K)$. — If z_n's & x_k's are equally spaced.

Ex

* Sum of 2 independent poisson r.v's with parameters a, b is a poisson r.v with parameter (a+b).
* Sum of i.i.d binomial r.v's with PMFs $b(k;n;p)$ is a binomial r.v with PMF $b(k;2n;p)$.
* $I_0(x) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\,d\theta$ — zero order modified bessel function of the first kind.

$f_Z(z) = \frac{z}{\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{P^2+z^2}{\sigma^2}\right)\right] I_0\left(\frac{zP}{\sigma^2}\right) u(z)$ — Rician probability density func

$I_0(0) = 1 \rightarrow$ if $P = 0$ we get Rayleigh.

$zP \gg \sigma^2$, the argument of $I_0(\cdot)$ is large, we use the approx

$$I_0(x) \approx e^x/(2\pi x)^{1/2}$$

Solving Problems of type $V=g(X,Y)$, $W=h(X,Y)$:-

$\{V\le v, W\le w\}$ & $\{(X,Y)\in C_{vw}\}$ are equal.

$$P[V\le v, W\le w] = F_{VW}(v,w) = \iint_{(x,y)\in C_{vw}} f_{XY}(x,y)\,dx\,dy.$$

$$C_{vw} = \{(x,y): g(x,y)\le v,\ h(x,y)\le w\}.$$

Obtaining f_{VW} directly from f_{XY} :-

$\{v<V\le v+dv,\ w<W\le w+dw\}$

$v=g(x,y)$ & $w=h(x,y)$.

Inverse mappings exist & are given by $x=\phi(v,w)$ & $y=\psi(v,w)$.

The Probability $P[v<V\le v+dv,\ w<W\le w+dw]$ is the probability that V & W lie in an infinitesimal rectangle of area $dv\,dw$ with vertices at (v,w), $(v+dv,w)$, $(v,w+dw)$, $(v+dv,w+dw)$. The image of this square is @ x,y system is a parallelogram.

$A(R)$ - area of rect $\qquad$ $A(P)$ - Area of pallegram

$$P[v<V\le v+dv,\ w<W\le w+dw] = \iint_R f_{VW}(\xi,\eta)\,d\xi\,d\eta = f_{VW}(v,w)A(R)$$

$$= \iint_P f_{XY}(\xi,\eta)\,d\xi\,d\eta = f_{XY}(x,y)A(P).$$

$|J|$

$$f_{VW}(v,w) = \frac{A(P)}{A(R)} f_{XY}(x,y).$$

$$|J| = \begin{vmatrix} \frac{\partial\phi}{\partial v} & \frac{\partial\phi}{\partial w} \\ \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial w} \end{vmatrix}$$

$$\boxed{f_{VW}(v,w) = \sum_{i=1}^{n} f_{XY}(x_i,y_i)\,/\,|J_i|}$$

Ex Pg - 159 — Read All

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ch-4. Expectation & Introduction of Estimation

Avg — "center of gravity" of a set.

$\hookrightarrow \mu_s = \frac{1}{N}\sum_{i=1}^{N} x_i$

If we multiply different x_i's with diff no's it is called weighted avg.

* An avg that summarises the spread/deviation from avg is the standard deviation, σ_s.

$$\sigma_s = \left[\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu_s)^2\right]^{1/2}$$

Out of N trials, x_i repeats n_i times then

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} n_i x_i = \sum_{i=1}^{N} x_i\left(\frac{n_i}{N}\right) = \sum_{i=1}^{M} x_i P[X=x_i]$$

Definition 1: Expected value of r.v X taking on values x_i with PMF $P_X(x_i)$ is

$$E[X] \triangleq \sum_i x_i P_X(x_i) = \sum_{\Omega} X(\zeta_i) P[\{\zeta_i\}]$$

Definition 2:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{\Omega} X(\zeta) P[\{d\zeta\}]$$

If $Y = g(X) \rightarrow E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy$.

Theorem 1: $Y = g(X)$.

$$E[Y] = \int_{-\infty}^{\infty} g(x)\cdot f_X(x)dx$$

$$\{y_j < Y \leq y_j + \Delta y_j\} = \bigcup_{k=1}^{r_j}\{x_j^{(k)} < X \leq x_j^{(k)} + \Delta x_j^{(k)}\}$$

$r_j \rightarrow$ no of real roots of eq^n $y_j - g(x) = 0$

$$y_j = g(x_j^{(1)}) = \dots = g(x_j^{(r_j)})$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy \approx \sum_{j=1}^{m} y_j P[y_j < Y \leq y_j + \Delta y_j]$$

$$= \sum_{j=1}^{m}\sum_{k=1}^{r_j} g(x_j^{(k)}) P[x_j^{(k)} < X \leq x_j^{(k)} + \Delta x_j^{(k)}].$$

$$E[Y] \approx \sum_{i=1}^{n} g(x_i) P[x_i < X \leq x_i + \Delta x]$$

$\left(\because x_j^{(k)}\right.$ are distinct, k, j can be replaced by

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \quad \text{—— (1)}$$

$$E[Y] = \sum_i g(x_i) P_X(x_i)$$

* $E\left[\sum_{i=1}^{N} g_i(x)\right] = \sum_{i=1}^{N} E[g_i(x)]$

* $Z = g(X, Y)$

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z) dz$$

$$\boxed{E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f_{XY}(x,y) dx dy}$$

Proof :-

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z) dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z f_{Z|Y}(z|y) \cdot \underbrace{f_Y(y)}_{\text{marginal pdf}} dy\, dz$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} z f_{Z|Y}(z|y) dz \right\} f_Y(y) dy \quad \text{from Eq (1)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x,y) f_Y(y) dx dy$$

$$E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dx dy$$

if $g(x,y) = X$

$$E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{XY}(x,y) dx dy = \int_{-\infty}^{\infty} \Big(\underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y) dy}_{f_X(x)} \Big) dx \cdot x$$

$$E[X] \triangleq \int_{-\infty}^{\infty} x f_X(x) dx$$

Using this we can write $E[X+Y] = E[X] + E[Y]$.

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)\,dx\,dy$$

$$E[X+Y] = E[X] + E[Y].$$

$$\boxed{E\left[\sum_N X_i\right] = \sum_N E(X_i)}$$ Independence is not required

* <u>Conditional Expectations</u>:-

<u>Definition 1</u>:

$$E[X|B] \triangleq \int_{-\infty}^{\infty} x f_{X|B}(x|B)\,dx.$$

$$E[X|B] \triangleq \sum_i x_i P_{X|B}(x_i|B).$$

<u>Definition 2</u>: X, Y - discrete r.v's with PMF $P_{XY}(x_i, y_i)$

$$E[Y|X=x_i] = \sum_j y_j P_{Y|X}(y_i|x_i). \rightarrow \frac{P_{X,Y}(x_i,y_i)}{P_X(x_i)}.$$

$$E[Y] = \sum_j y_j P_Y(y_i)$$

$$= \sum_j y_j \sum_i P_{X,Y}(x_i,y_i) = \sum_i \left[\sum_j y_j P_{Y|X}(y_i|x_i)\right] P_X(x_i).$$

$$\boxed{E[Y] = \sum_i E[Y|X=x_i]\, P_X(x_i)}$$

<u>Definition 3</u>: X, Y - cont. r.v

$$E[Y|X=x] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\,dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)\,dx\,dy.$$

$$E[Y] = \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy\right] dx$$

$$\boxed{E[Y] = \int_{-\infty}^{\infty} E[Y|X=x]\, f_X(x)\,dx}$$

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X_1, X_2 are independent.

$$P[X_1=x_1 \mid X_1+X_2=y] = \frac{P[X_1=x_1 \mid X_1+X_2=y]}{P[X_1+X_2=y]}$$

$$= \frac{P[X_1=x_1, X_2=y-x_1]}{P[X_1+X_2=y]}$$

$$= \frac{P[X_1=x_1] \cdot P[X_2=y-x_1]}{P[X_1+X_2=y]}$$

$$E[X_1 \mid X_1+X_2=y] \triangleq \sum_{x_1=0}^{y} x_1 P[X_1=x_1 \mid X_1+X_2=y].$$

* Conditional Expectation as a Random Variable :-

$Y=g(X)$.

$$E[Y] = \sum_i g(x_i) P_X(x_i) = E[g(X)]$$

$$E[Y] = \sum_i \underbrace{E[Y|X=x_i]}_{\text{number}} P_X(x_i) = E[\underbrace{E[Y|X]}_{\text{r.v (function)}}].$$

$$E[Y] = E\{E[Y|X]\}.$$

$$E[Z] = E[E[Z|X,Y]].$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z f_{Z|X,Y}(z|x,y) f_{XY}(x,y)\, dx\, dy\, dz.$$

Property :-

(i) $E[Y] = E[E[Y|X]]$

(ii) If X, Y are independent, then $E[Y|X] = E[Y]$.

$$E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy$$

$f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x) = f_Y(x) f_X(x)$ (bcoz they are independent)

$\therefore f_{Y/X}(y|x) = f_Y(y)$

$$E[Y|X=x] = \int_{-\infty}^{\infty} y f_Y(y) dy = E[Y].$$

$\therefore E[Y|X] = E[Y]$.

iii) $E[Z|X] = E[E[Z|X,Y]|X]$

$$E[Z|X=x] = \int_{-\infty}^{\infty} z f_{Z|X}(z|x) dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z f_{Z|X,Y}(z|x,y) \cdot f_{Y|X}(y|x) dz dy.$$

$$= \int_{-\infty}^{\infty} dy f_{Y|X}(y|x) \int_{-\infty}^{\infty} z f_{Z|X,Y}(z|x,y) dz.$$

$$E[Z|X] = E[E[Z|X,Y]|X=x].$$

Moments. — $\mu_X, \sigma_X^2, E[X^2]$ are called moments.

Definition 1 :- rth moment of X is defined as.

$$\xi_r = E[X^r] = \int_{-\infty}^{\infty} x^r f_X(x) dx \qquad , r=0,1,\ldots$$

$$\xi_r = \sum_i x_i^r P_X(x_i) \qquad (\xi_0 = 1 \ \& \ \xi_1 = \mu).$$

Definition 2 :- rth central moment of X is defined as

$$m_r = E[(X-\mu)^r]. \qquad r=0,1,\ldots$$

$$m_r = \sum_i (x_i-\mu)^r P_X(x_i)$$

m_2 — Variance (σ^2) — $Var[X]$

$$\sigma^2 = E[(X-\mu)^2] = E[X^2] - \mu^2$$

$\overline{X^r} = E[X^r]$. $\Rightarrow \sigma^2 = \overline{X^2} - \mu^2$.

$$\boxed{\overline{X^2} = \mu^2 + \sigma^2.}$$

$$(X-\mu)^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \mu^i X^{r-i}$$

Take Expectation on B.S:

$$\boxed{m_r = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \mu^i \xi_{r-i}}$$

Pg 195 — doubt:

$$E[|X|^r] = \int_{-\infty}^{\infty} |x|^r f_X(x)\,dx \text{ — absolute moment}$$

$$E[(X-a)^r] = \int_{-\infty}^{\infty} (x-a)^r f_X(x)\,dx \text{ — generalized moment.}$$

* Joint Moments :-

Definition 3 :- $\xi_{ij} = E[X^i Y^j]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i y^j f_{XY}(x,y)\,dx\,dy.$$

$$\xi_{ij} = \sum_{l}\sum_{m} x_l^i y_m^j P_{X,Y}(x_l, y_m).$$

Definition 4 :- $m_{ij} = E[(X-\overline{X})^i (Y-\overline{Y})^j]$. $(\overline{X} = E[X])$

$i+j \rightarrow$ Order of Moments.

$\xi_{02}, \xi_{20}, \xi_{11}, m_{02}, m_{11}, m_{20} \longrightarrow 2^{nd}$ order

$\xi_{11} \downarrow$ Correlation $\quad m_{11} \downarrow$ Covariance.

$$\rho = \frac{m_{11}}{\sqrt{m_{02}\, m_{20}}} \qquad |\rho| \leq 1$$

$$m_{11} = E[(X-\overline{X})(Y-\overline{Y})]$$
$$= E[XY] - \overline{X}\,\overline{Y}$$
$$= Cov[X,Y].$$

$$E\left[\left(\lambda(x-\bar{x})-(y-\bar{y})\right)^2\right] \geq 0.$$

$$Q(\lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\lambda(x-\bar{x})-(y-\bar{y})\right]^2 f_{XY}(x,y)\,dx\,dy \geq 0.$$

expand.

$$Q(\lambda) = \lambda^2 m_{20} + m_{02} - 2\lambda m_{11} \geq 0$$

Should have atmost one real root.

$$\left(\frac{m_{11}}{m_{20}}\right)^2 - \frac{m_{02}}{m_{20}} \leq 0 \Rightarrow m_{11}^2 \leq m_{02} m_{20} \Rightarrow |\rho| \leq 1.$$

when $m_{11}^2 = m_{02} m_{20} \Rightarrow |\rho| = 1 \rightarrow \lambda = \frac{m_{11}}{m_{20}}$

$$E\left[\left(\frac{m_{11}}{m_{20}}(x-\bar{x})-(y-\bar{y})\right)^2\right] = 0$$

$$\downarrow$$

$$y = \frac{m_{11}}{m_{20}}(x-\bar{x}) + \bar{y}.$$

When $Cov[X,Y] = 0$, then $\rho = 0$ & X, Y are called uncorrelated.

Properties of Uncorrelated R.V :-

(i) X, Y are uncorrelated

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2.$$

(ii) If X, Y are independent, they are uncorrelated.

$$Cov[X,Y] = E[XY] - E[X]E[Y]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy - \int_{-\infty}^{\infty} f_X(x)\,x\,dx \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$Cov[X,Y] = 0.$$

*** Jointly Gaussian Random Variables :-**

We say two r.v's are jointly gaussian if their joint pdf is

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\bar{x}}{\sigma_X}\right)^2 - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_X\sigma_Y} + \left(\frac{y-\bar{y}}{\sigma_Y}\right)^2\right\}\right)$$

If $\rho=0$; $f_{XY}(x,y) = f_X(x) f_Y(y)$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\bar{X}}{\sigma_X}\right)^2\right)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{1}{2}\left(\frac{y-\bar{Y}}{\sigma_Y}\right)^2\right).$$

* Two jointly Gaussian r.v's that are uncorrelated ($\rho=0$) are indepe
$f_X(x)$ & $f_Y(y)$ for jointly normal r.v's are always normal.

More than 2 Random Variables

Sum :-

$Y = X_1 + X_2 + \_\_ + X_N$

$Var(X_1+X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1,$

$E[Y] = E[X_1] + \_\_$

$\therefore Var(Y) = \sum_{i=1}^{N} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$

$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) * \ldots f_{X_n}(y)$.

$f_Y(y) = f_{X_1}(y) * f_{X_2}(y)$

$= \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(y-x)dx.$

* Moments :-

$$M_X(t) = \sum_{K=0}^{\infty} E[X^K] \cdot \frac{t^K}{K!}$$

* $M_{X_1+X_2+\_}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \ldots$

If MGF doesn't exist $\phi_X(\omega) = E[e^{j\omega X}] \rightarrow$ Characteristic function

* Random Vectors :-

$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$

X — Random Vector

$X_1, X_2, \_ X_n$ are Random Variables.

$F_X(x) = F_{X_1X_2\_X_N}(x_1, x_2, \_\_ x_n)$

$= P(X_1 \le x_1, X_2 \le x_2, \_\_ X_n \le x_n)$.

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$$f_X(x) = f_{X_1,X_2,\dots X_n}(x_1,x_2,\dots x_n).$$

$$E[X] = \begin{pmatrix} E[x_1] \\ | \\ E[x_n] \end{pmatrix}.$$

Random Matrix $= M = [X_{ij}]$.

$$E[M] = [E[X_{ij}]].$$

* $Y = AX+b. \Rightarrow E[Y] = AE[X]+b.$

* $X \to$ n-dimensional r. vector.
  $A \to m \times n$ matrix. } fixed.
  $b \to m$-dimensional vector }

* **Correlation & Covariance Matrix :-**

For a random vector X, Correlation matrix is R_X.

$$R_X = E[XX^T] = E\begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_N \\ x_2x_1 & x_2^2 & \cdots & x_2x_N \\ | & & & \\ x_nx_1 & \cdots & \cdots & x_n^2 \end{bmatrix}$$

$$\text{Covariance Matrix } (C_X) = E[(X-EX)(X-EX)^T] = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & \cdots & Cov(X_1,X_n) \\ Cov(X_2,X_1) & Var(X_2) & \cdots & \\ \vdots & & & \\ Cov(X_n,X_1) & \cdots & \cdots & Var(X_n) \end{bmatrix}$$

* $C_X = E[(X-\bar{X})(X^T-\bar{X}^T)]$

$$= E[XX^T - \bar{X}X^T - X\bar{X}^T + \bar{X}\bar{X}^T] = E[XX^T] - \bar{X}\cdot\bar{X}^T - \cancel{\bar{X}\bar{X}^T} + \cancel{\bar{X}\bar{X}^T}$$

$$C_X = R_X - \bar{X}\cdot\bar{X}^T \Rightarrow \boxed{C_X = R_X - E[X]E[X^T]}$$

(1) $Y = AX+b$

$$C_Y = E[(Y-\bar{Y})(Y-\bar{Y})^T] = E[(AX+\cancel{b}-A\bar{X}-\cancel{b})(AX+\cancel{b}-A\bar{X}-\cancel{b})^T].$$

$$= E[A(X-\bar{X})(X-\bar{X})^TA^T]$$

$$C_Y = AC_XA^T.$$

(2) $U = \begin{bmatrix} x \\ y \end{bmatrix}$. $R_u = E[xx^T] = \begin{bmatrix} E[x^2] & E[xy] \\ E[xy] & E[y^2] \end{bmatrix}$.

$$f_X(x) = \int_{-\infty}^{\infty} \left(\frac{3}{2}x^2+y\right)dy = \left[\frac{3}{2}x^2y+\frac{y^2}{2}\right]_0^1 = \left(\frac{3}{2}x^2+\frac{1}{2}\right).$$

$$f_Y(y) = \left(\frac{3}{2}\cdot\frac{x^3}{3}+y\cdot x\right)_0^1 = \left(\frac{1}{2}+y\right).$$

$$* \; E[x^2] = \int_0^1 x^2\left(\frac{3}{2}x^2+\frac{1}{2}\right)dx = \left(\frac{3}{2}\cdot\frac{x^5}{5}+\frac{1}{2}\cdot\frac{x^3}{3}\right)_0^1 = \left(\frac{3}{10}+\frac{1}{6}\right) = \left(\frac{9}{30}+\frac{5}{30}\right)$$

$$E[y^2] = \int_0^1 y^2\left(\frac{1}{2}+y\right)dy = \left(\frac{1}{2}\cdot\frac{y^3}{3}+\frac{y^4}{4}\right)_0^1 = \frac{1}{6}+\frac{1}{4} = \frac{5}{12}.$$

Properties of Covariance Matrix:-

C_X — Symmetric $\quad Cov(x_i, x_j) = Cov(x_j, x_i)$

$\downarrow$

$b^T M b \geq 0$ — $\forall b$ } M is
$b^T M b > 0$ — $b \neq 0$ } Symmetric.

$Y = b^T(X - \bar{X})$.

$E[Y^2] \geq 0 \Rightarrow E[YY^T] \geq 0$.

$b^T E[(X-\bar{X})(X-\bar{X})^T]b \geq 0$

* Eigen values ≥ 0 }
$\downarrow$ } → Positive definite (PD)
$\det(C_X) > 0$. }

$b^T C_X b \geq 0 \longrightarrow$ Positive Semidefinite

Cross Corelation matrix of X & Y. $R_{XY} = E[XY^T]$.

Cross Covariance Matrix is $C_{XY} = E[(X-\bar{X})(Y-\bar{Y})^T]$.

Functions of Random Vectors: Transformations.

$Y = G(X) \Rightarrow X = G^{-1}(Y) = H(Y)$. $\quad [H = G^{-1}]$.

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} H_1(y_1, y_2, \ldots y_n) \\ \vdots \\ H_n(y_1, \ldots y_n) \end{bmatrix}$$

$$f_Y(y) = f_X(H(y))\,|J|.$$

$$|J| = \begin{vmatrix} \frac{\partial H_1}{\partial y_1} & \frac{\partial H_1}{\partial y_2} & \cdots \\ \vdots & & \\ \frac{\partial H_n}{\partial y_1} & \frac{\partial H_n}{\partial y_2} & \cdots \end{vmatrix}$$

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$f_Y(y) =$ (1.) $Y = AX + b \Rightarrow X = A^{-1}(Y - b)$.

$J = \det(A^{-1}) = \dfrac{1}{\det(A)}$.

$$\boxed{f_Y(y) = \frac{1}{|\det(A)|} f_X\left(A^{-1}(y-b)\right)}$$

Normal (Gaussian) Random Vectors

* $Cov(X, X+Y) = \underbrace{Cov(X,X)}_{Var(X)} + Cov(X,Y)$.

$X_1, \ldots X_n$ are Jointly normal, if $a_1X_1 + \ldots a_nX_n$ is Normal r.v.

standard normal random vector $Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ z_i's are i.i.d. $z_i \sim N(0,1)$.

$f_Z(z) = f_{z_1 z_2 \ldots z_n}(z_1, \ldots z_n)$.

$= \prod_{i=1}^{n} f_{z_i}(z_i)$.

$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^{n} z_i^2\right)$.

$Z \sim N(0, I)$ — standard Gaussian Vector.

$$\boxed{f_Z(z) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} z^T z\right)}$$

* $X \sim N(\overset{\text{mean}}{m}, \overset{\text{covariance Matrix}}{C})$

$QQ^T = I$; $C = QDQ^T$

$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ 0 & \cdots & \cdots & d_{nn} \end{bmatrix}$ $D^{\frac{1}{2}} \cdot D^{\frac{1}{2}} = D \Rightarrow D^{\frac{1}{2}} = D^{\frac{1}{2}T}$

$A = QD^{\frac{1}{2}}Q^T \longrightarrow AA^T = A^TA = C$.

$X = AZ + m$.

$E[X] = AE[Z] + m$.

$= AE[Z] + m$

$= m$ //

$C_X = AC_ZA^T$. $C_Z = I$

$= AA^T$

$C_X = C$ //

$$f_X(x) = \frac{1}{|\det(A)|} f_Z\left(A^{-1}(x-m)\right).$$

$$= \frac{1}{(2\pi)^{n/2}|\det(A)|} \exp\left\{-\frac{1}{2}\left(A^{-1}(x-m)\right)^T\left(A^{-1}(x-m)\right)\right\}.$$

$$= \quad '' \quad \exp\left\{-\frac{1}{2}(x-m)^T A^{-T} A^{-1}(x-m)\right\}.$$

$$\boxed{f_X(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det C}} \exp\left\{-\frac{1}{2}(x-m)^T C^{-1}(x-m)\right\}.}$$

* **Chernoff Bounds :-**

$$P(X \geq a) \leq e^{-sa} M_X(s) \quad , s>0$$

$$P(X \leq a) \leq e^{-sa} M_X(s) \quad , s<0.$$

* $P(X \geq a) \leq \min_{s>0} e^{-sa} M_X(s).$

$P(X \leq a) \leq \min_{s<0} e^{-sa} M_X(s).$

* **Cauchy-Schwartz inequality :-**

$$|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}.$$

* **Jensen's Inequality :-**

If $g(x)$ is convex function on R_X & $E[g(x)]$ and $g(E[X])$ are finite then $E[g(x)] \geq g(E[X])$.

$g''(x) \geq 0$

* **Central limit theorem :-**

Let $X_1, X_2, \ldots X_n$ be i.i.d with $EX_i = \mu < \infty$ & Variance $(0 < \sigma^2 < \infty)$ then random variable

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \ldots + X_n - n\mu}{\sqrt{n}\,\sigma}$$ converges in distribution to standard normal random variable as n goes to infinity.

$$\lim_{n\to\infty} P(Z_n \le x) = \Phi(x). \quad \forall\, x \in R$$

$\Phi(x)$: standard normal CDF.

* Sequence of random variables is in fact a sequence of functions:-

$X_n : S \to \mathbb{R} \qquad X_n(s_i) = x_{ni}$

* Convergence
  - Convergence in probability
  - " " distribution — weakest.
  - " " mean
  - Almost sure convergence.

* **Convergence in distribution:-**

$X_1, X_2, \ldots X_n$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{D} X$ if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$. $F_X(x)$ — continuous.

* **Convergence in Probability:-**

$X_n \xrightarrow{P} X$ if $\lim_{n\to\infty} P(|X_n - X| \ge \epsilon) = 0$. $\epsilon > 0$.

* Multivariate Normal distribution:

$$f(x_1, \ldots x_k) = f(x) = \frac{1}{(2\pi)^{k/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu)} \to N(\mu, \Sigma).$$

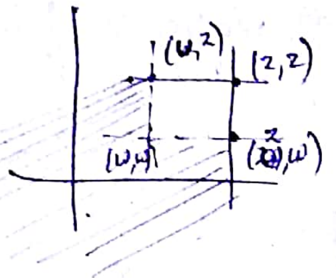
$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{1k} & \cdots & & \sigma_{kk} \end{bmatrix}$$

$$\Sigma = E[(x-\mu)(x-\mu)^T] = Cov(x_i, x_j).$$

$$|\Sigma| = \det \Sigma.$$

5/10/18.

$z = \max(x, y)$. $\quad w = \min(x, y)$.



$F_{xy}(w,z) + F_{xy}(z,w) - F_{xy}(w,w)$.

Complex Analysis.

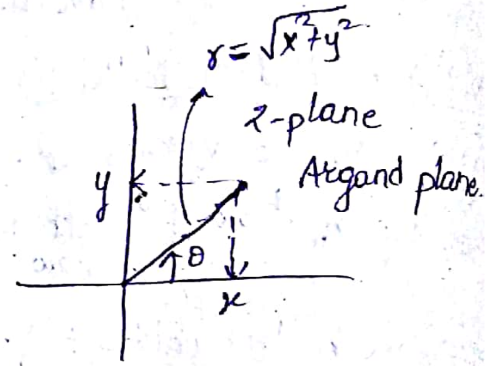
(i) $z = x+iy \quad i=\sqrt{-1}$

↓

Cartesian definition.

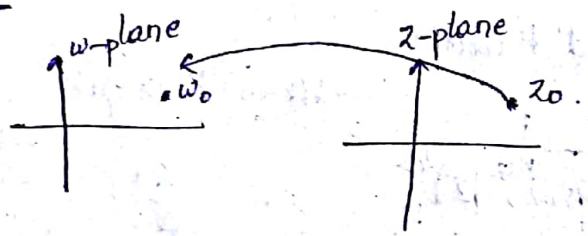
(ii) polar defⁿ : $z = r\cos\theta + i\sin\theta$.

(iii) Exponential form : $z = re^{j\theta}$.



$w = f(z) \longrightarrow$ Mapping from one Argand plane to other.

$= u+iv$.



For every $z \neq z_0$ in δ-disc around z_0 in z-plane, there exists an ϵ-disc in w-plane.

Furthermore, this also implies that the limit is independent of path.

1) $\lim_{z \to 1-i} \frac{z^2 + 4z + 3}{z+1} = 4-i$ prove that.

2) $\lim_{z\to 0} \frac{z}{|z|}$ does not exist ?

3) $\lim_{z \to 0} \frac{z}{\bar{z}}$ exists/not ?

1) $\frac{1+(-1)-2i+4-4i+3}{2-i} = \frac{7-6i}{2-i} = \frac{(7-6i)(2+i)}{4+1} = \frac{14+6-5i}{5}$

$= 4-i$

3 (ii)) $\underset{x\to 0}{lt}\ \underset{y\to 0}{lt}\ \frac{x+iy}{x-iy} = 1.$ But $\underset{y\to 0}{lt}\ \underset{x\to 0}{lt}\ \frac{x+iy}{x-iy} = -1$

We are getting two different answers for same question.

$\therefore$ limit $\frac{z}{\bar{z}}$ doesn't exist.

2 (ii)) $\underset{z\to 0}{lt}\ \frac{z}{|z|} = \underset{z\to 0}{lt}\ \frac{x+iy}{\sqrt{x^2+y^2}}$ $y = mx.$

changes for diff values of m.

$= \underset{x\to 0}{lt}\ \frac{1+im}{\sqrt{1+m^2}}$ → dependent on path

$\therefore$ limit doesnot Exist.

1 (i)) $\underset{z\to 1-i}{lt}\ \frac{z^2+4z+3}{1+z} = \underset{z\to 1-i}{lt}\ \frac{z^2+3z+z+3}{z+1} = \underset{z\to 1-i}{lt}\ z+3 =$

$= \underset{\delta\to 0}{lt}\ 1+\delta-i+3 = 4-i$

9/10/18.

Analyticity.

$f'(z_0) = \underset{\Delta z\to 0}{lt}\ \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}.$

$\Delta z = \Delta x + i\Delta y.$

$f'(z_0) = \underset{\Delta x, \Delta y\to 0}{lt}\ \frac{f(z+\Delta z)-f(z)}{\Delta z}$

$= \underset{\Delta x, \Delta y\to 0}{lt}\ \frac{(u+\Delta u)+i(v+\Delta v)-(u+iv)}{\Delta x+i\Delta y}.$

Continuity.

$\underset{z\to z_0}{Lt}\ f(z) = f(z_0);$

& it should be independent of the path to z_0.

I:-

$f'(z) = \underset{\Delta x\to 0}{lt}\ \frac{\Delta u+i\Delta v}{\Delta x}.$

$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i$

II:- $\underset{\Delta y\to 0}{lt}\ \frac{\Delta u+i\Delta v}{i\Delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = f'(z)$

Compare cases I & II

$$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.}$$

CR Conditions.

necessary conditions for analyticity of $f(z) = u+iv.$

CR – Cauchy Remann

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when is it Sufficient ?

(i) CR conditions are satisfied.

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuos functions of x, y in R.

Proof:

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) \longrightarrow \text{Taylor's theorem}$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + O(\Delta x^2 + \Delta y^2)$$

$$+ i\left[v(x,y) + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right.$$

$$f(z+\Delta z) - f(z) = \left(\frac{\partial u}{\partial x}\Delta x\right) + \frac{\partial u}{\partial y}\Delta y + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)$$

$$= \left(\frac{\partial u}{\partial x}\Delta x + i^2\frac{\partial v}{\partial x}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial u}{\partial x}\Delta y\right)$$

$$= \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) = \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y}\right.$$

from CR's

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$$\boxed{\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ f'(z) &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}}$$

* $z = re^{i\theta}$

$$\frac{\partial f(z)}{\partial \theta} = f'(z) \cdot ire^{i\theta} \qquad \frac{\partial f(z)}{\partial r} = f'(z) \cdot e^{i\theta}$$

$$\Rightarrow \frac{\partial f(z)}{\partial \theta} = ir\frac{\partial f(z)}{\partial r}.$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}.}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ -r\frac{\partial v}{\partial r} &= \frac{\partial u}{\partial \theta} \end{aligned}}$$ CR Equations in Polar.

* Test. Analyticity :-

1) x^2+iy^2

2) $\sin x \cosh y + i\cos x \sinh y$

3) $xy+iy^2$

1sol) x^2+iy^2

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y}=0$

$\frac{\partial v}{\partial x}=0$, $\frac{\partial v}{\partial y} = 2y$.

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ✓

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✗

Analytic if $x=y$.

↓

along contour $x=y$.

2sol.) $\sin x \cosh y + i\cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y$. $\frac{\partial u}{\partial y} = \sin x \sinh y$.

$\frac{\partial v}{\partial x} = -\sin x \sinh y$ $\frac{\partial v}{\partial y} = \cos x \cosh y$.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓ CR satisfied

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ✓

Analytic

3sol) $xy+iy^2$

$\frac{\partial u}{\partial x} = y$ $\frac{\partial u}{\partial y} = x$

$\frac{\partial v}{\partial x} = 0$ $\frac{\partial v}{\partial y} = 2y$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ✗

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✗

∴ Analytic at Origin

4) $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$ $\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ✓

Analytic everywhere except $z=0$.

Taylors Theorem :- $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)$.

Note: (i) If a function is analytic in domain D, then u,v satisfy CR. conditions at all pts in D.

(ii) CR conditions are necessary but not sufficient for Analytic function

(iii) CR conditions are sufficient if the partial derivative are continuos.

9/10/18

Functions of a Complex Variable.

z_0 is a pt in Complex plane & z be any +ve number, then set of pts z such that $|z - z_0| < \epsilon$ is called ϵ-neighbourhood of z_0.

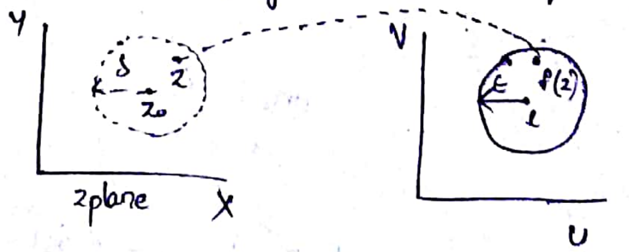
* z_0 is called interior point of a point set S if there exists a neighbourhood of z_0 lying wholly in S.

* Limit of a function of a Complex Variable.

$f(z)$ be a single valued function defined at all pts in some neighbourhood of z_0. $f(z)$ is said to have the limit l as z approaches z_0 along any path if given an arbitrary real number $\epsilon > 0$, however small there exists a real number $\delta > 0$, such that

$$|f(z) - l| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

$\lim\limits_{z \to z_0} f(z) = l \rightarrow$ for every $z \neq z_0$, in δ-disc (dotted) of $z$plane, $f(z)$ has a value lying in the ϵ-disc of w-plane.



(i) δ depends on ϵ

(ii) $z \to z_0 \Rightarrow z$ approaches z_0 along any path. limits independent of path

Questions

1) $$\lim_{z \to \infty} \frac{iz^3 + iz - 1}{(2z+3i)(z-i)^2} = \lim_{z \to \infty} \frac{i + \frac{i}{z^2} - \frac{1}{z^3}}{\left(\frac{2}{z} + \frac{3i}{z}\right)\left(1 - \frac{i}{z}\right)^2} = \frac{i}{2}.$$

2) $$\lim_{z \to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2} = \lim_{z \to 1+i} \frac{(z+i)(z-i) - 1(z+i)}{(z-1-i)(z-1+i)} = \lim_{z \to 1+i} \frac{(z+i)\cancel{(z-i-1)}}{\cancel{(z-1-i)}(z-1+i)}$$

$$= \frac{1+i+i}{\cancel{1}+i-\cancel{1}+i} = 1 - \frac{i}{2}.$$

Exercise

1) $$\lim_{z \to 0} \frac{\mathrm{Im}(z)^3}{\mathrm{Re}(z)^3} = \lim_{x \to 0, y \to 0} \frac{y^3}{x^3} \rightarrow \text{Limit doesn't exist.}$$

2) $$\lim_{z \to -i} \frac{z^2}{z+i} = \frac{1}{0} \rightarrow \text{No limit.}$$

3) $$\lim_{z \to 0} \frac{\mathrm{Re}(z)^2}{\mathrm{Im}\, z} = \frac{x^2}{y} \rightarrow \text{No limit}$$

4) $$\lim \frac{z}{\dots} = \frac{x+iy}{\dots 2xyi} \longrightarrow \text{No limit.}$$

5) $\lim_{z\to 0} \frac{Re(z)^2}{|z|} = \lim_{x\to 0, y\to 0} \frac{x^2}{\sqrt{x^2+y^2}}$ $y = mx$

$(1+i)\ (1-1$

$= \lim_{x\to 0} \frac{x^{\cancel{2}}}{\cancel{x}\sqrt{(1+m^2)}} = \lim_{x\to 0} \frac{x}{\sqrt{1+m^2}} = 0.$

6) $\lim_{z\to 1+i} \frac{2z^3}{Im(z)^2} = \lim_{z\to 1+i} \frac{2\cdot(1+i)^3}{(1)^2} = 2(1-i+3i-3) = 4(i-1).$

7) $\lim_{z\to 0} \frac{z^2+6z+3}{z^2+2z+2} = \frac{3}{2}$

*Continuity.

$|f(z)-f(z_0)| < \epsilon$ for all pts z of the domain satisfying $|z-z_0| < \delta$.

$f(z)$ is said to be Continuos at $z=z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$.

1) $f(z) = \begin{cases} \frac{z^3-iz^2+z-i}{z-i}, & z\neq i \\ 0, & z=i \end{cases} \longrightarrow \frac{z^2(z-i)+1(z-i)}{(z-i)} = z^2+1, \quad z\neq i$

$0, \quad z=i$

$\lim_{z\to i} z^2+1 = 0 = f(i) \to$ Continuos.

2) $f(z) = \lim_{z\to 0} \frac{x}{x+iy} = \lim_{x\to 0} \frac{1}{1+im} = \frac{1}{1+im}$. $\lim_{x\to 0}\lim_{y\to 0} \frac{x}{x+iy} = 1$.

limit does not exist. correct

3) $\lim_{z\to 0} \frac{y}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{y}{y\sqrt{1+m^2}} = \frac{1}{\sqrt{1+m^2}} \to$ Not Continuos.

4) $\lim_{z\to 1-i} \frac{z^2+3z+4}{z^2+i} = \lim_{z\to 1-i} \frac{\cancel{1}-\cancel{1}-2i+3-3i+4}{\cancel{1}-\cancel{1}-2i+i} = \frac{7-5i}{-i}$

Continuos.

5) $\lim_{z\to 0} \cos z$ $z = x+iy$

$\cos(x+iy) = \cos x\cosh y - \sin x \sinh y\, i$

$\lim_{z\to z_0} \cos(x+iy) = f(z_0) \Rightarrow$ Continuos//

$e^{2z} = e^{2x} + e^{2iy}$.

$\lim_{z\to z_0} f(z) = f(z_0) \rightarrow$ Continuous.

*Differentiability

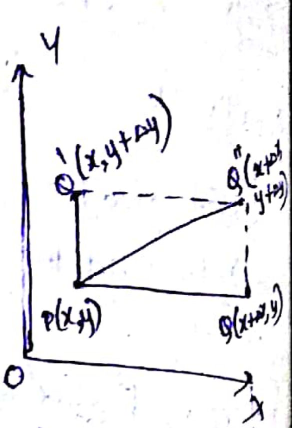
$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ provided limit exists & is independent of path along which $\Delta z \to 0$.

(1.) $f(z) = 4x+y+i(-x+4y) = u+iv$.

$u = 4x+y \quad , \quad v = -x+4y$.

$f(z+\Delta z) = 4(x+\Delta x) + (y+\Delta y) + i((-x-\Delta x)+4(y+\Delta y))$.

$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{4\Delta x + \Delta y + i(-\Delta x + 4\Delta y)}{\Delta x + i\Delta y} = \frac{\delta f}{\delta z}$.



If Q is on horizontal line $\delta y = 0$. $\delta z = \delta x$

$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\Delta x} = 4-i$.

If Q is on vertical line $\delta x = 0$ & $\delta z = \delta y$.

$\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = 4-i$

if Q is along $y = x$.

$z = x+iy = x(1+i) \Rightarrow \delta z = \delta x$.

$\therefore \frac{\delta f}{\delta z} = 4-i$

For any Q same value $\Rightarrow$ z is differentiable.

(2.) $f(z) = \begin{cases} \frac{x^3 y(y-ix)}{x^6+y^2} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

$y = mx$

$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left[\frac{x^3 y(y-ix)}{(x^6+y^2)(x+iy)}\right] = \frac{mx^4 \cdot x(m-i)}{(x^6+m^2x^2)(1+im)}$

$= \frac{m\cdot x^5 \cdot i\cancel{(-mi-1)}}{x(m^2+x^4)\cancel{(1+mi)}} = \lim_{x\to 0}\frac{-imx^3}{m^2+x^4} = 0$.

But if $y=x^3$.

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left(\frac{-ix^3 y}{x^6+y^2}\right) = \lim_{x\to 0} \frac{-ix^3(x^3)}{x^6+(x^3)^6} = \frac{-i}{2}$$

$\therefore$ Diff values at diff paths

$\therefore$ Not differentiable at $z=0$.

*theorem:- Continuity is a necessary condition but not sufficient condition for the existence of a finite derivative.

Proof:-
$$f(z_0+\Delta z) - f(z_0) = \Delta z\left\{\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right\}.$$

$$\lim_{\Delta z\to 0} f(z_0+\Delta z)-f(z_0) = \lim_{\Delta z\to 0} 0\cdot f'(z_0) \Rightarrow \lim_{\Delta z\to 0}\left(f(z_0+\Delta z)-f(z_0)\right)=0.$$

$$\lim_{z\to z_0} f(z)-f(z_0)=0 \Rightarrow \lim_{z\to z_0} f(z)=f(z_0). \rightarrow \text{continuous at } z=z_0.$$

3) $f(z)=|z|^2 = x^2+y^2 \longrightarrow$ Continuous bcoz polynomials.

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{|z+\Delta z|^2-|z|^2}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(z+\Delta z)(\bar z+\Delta\bar z)-z\bar z}{\Delta z} = \lim_{\Delta z\to 0} \frac{\bar z\Delta z + z\Delta\bar z + \Delta z\Delta\bar z}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\left\{\bar z + \Delta\bar z + \frac{z\cdot\Delta\bar z}{\Delta z}\right\}$$

$\Delta z = r(\cos\theta + i\sin\theta)$

$\Delta\bar z = r(\cos\theta - i\sin\theta)$.

$$\frac{\Delta\bar z}{\Delta z} = \frac{\cos\theta - i\sin\theta}{\cos\theta+i\sin\theta} = (\cos\theta - i\sin\theta)^2 = \cos 2\theta - i\sin 2\theta.$$

$\downarrow$ different values of θ different $\frac{\Delta\bar z}{\Delta z}$.

$\therefore$ Not differentiable.

But $z=0 \longrightarrow$ differentiable.

*Analytic function:-

$f(z)$ is said to be analytic at a pt z_0, if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0.

f(z) is analytic in a domain if it is analytic at every pt of domain. The pt at which the function is not differentiable is called a <u>singular point</u> of the function. Analytic function also known as "holomorphic", "regular", "monogenic".

* A function which is analytic everywhere is known as entire function.

Ex 13) $\sin z = \omega$.

$\omega = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y \quad \frac{\partial u}{\partial y} = \sin x \sinh y \quad \frac{\partial v}{\partial x} = -\sin x \sinh y \quad \frac{\partial v}{\partial y} = \cos x \cosh y.$

$\sin z$ is <u>Analytic</u> CR ✓

$\cosh x = \frac{e^x + e^{-x}}{2} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \cosh ix = \cos x.$

$\sinh x = \frac{e^x - e^{-x}}{2} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \cos ix = \cosh x.$

---

14) $\omega = \log z$. $\quad x = r\cos\theta,\ y = r\sin\theta$.

$\omega = \log(x+iy) = \log(r\cos\theta + ir\sin\theta) = \log r(\cos\theta + i\sin\theta)$

$= \log_e re^{i\theta} = \log_e r + i\theta = \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}.$

$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \quad \frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2} \quad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}.$

CR ✓ $\quad \therefore \log z$ is <u>Analytic</u> except when $x = y = 0$

$z = 0$. $\quad (x^2+y^2=0)$.

$\frac{d\omega}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x-iy}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}.$

15) $f(z) = z\bar{z} = x^2+y^2$.

$\frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = 2y,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 0.$

Analytic <u>only at origin</u>

$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{Lt} \frac{x+iy^2}{x+iy} \qquad y = mx$$

$$= \underset{x\to 0}{Lt} \frac{(1+m^2)\cancel{x}}{(1+im)\cancel{x}} = 0. \rightarrow \text{unique value}$$

$\therefore f(z)$ is Analytic at $z=0$.

16) $$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^3-y^3)(2x) - (x^2+y^2)(3x^2)}{(x^2+y^2)^2} = \frac{2x^4-2xy^3-3x^4-3x^2y^2}{(x^2+y^2)^2} = \frac{-x^4-2xy^3-3x^2y^2}{(x^2+y^2)^2}$$

At $z=0$.

$$\frac{\partial u}{\partial x} = \underset{h\to 0}{Lt} \frac{u(h,0)-u(0,0)}{h} = \underset{h\to 0}{Lt} \frac{\frac{h^3}{h^2}}{h} = 1.$$

$$\frac{\partial u}{\partial y} = \underset{h\to 0}{Lt} \frac{u(0,h)-u(0,0)}{h} = \underset{h\to 0}{Lt} \frac{-\frac{h^3}{h^2}}{h} = -1.$$

$$\frac{\partial v}{\partial x} = \underset{h\to 0}{Lt} \frac{v(h,0)-v(0,0)}{h} = \underset{h\to 0}{Lt} \frac{\frac{h^3}{h^2}}{h} = 1 \qquad \frac{\partial v}{\partial y} = -1.$$

CR satisfied at $z=0$.

$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{Lt} \left[\frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}\right] \rightarrow \underset{z\to 0}{Lt}$$

Let $y=x$ $\Rightarrow \underset{x\to 0}{Lt}$ $$f'(0) = \frac{\cancel{2}\cancel{x^3}\, i}{\cancel{2x^2}\cdot\cancel{x}(1+i)} = \frac{i(1-i)}{2} = \frac{1}{2}(1+i).$$

Let $y=0$. $\Rightarrow$ $f'(0) = 1+i$

$f'(0)$ is different $\Rightarrow$ $f(z)$ is Not Analytic at $z=0$.

17) $$f(z) = e^{-z^{-4}} = e^{-\frac{1}{(x+iy)^4}} = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$$

$$f(z) = e^{-\frac{1}{(x^2+y^2)^4}\left[x^4+y^4-6x^2y^2-i4xy(x^2-y^2)\right]}$$

$$f(z) = e^{\frac{-x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \cdot e^{\frac{-i4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

$$= e^{\frac{-x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \cdot \left[\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4} - i\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$$

At $z=0$.

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \frac{e^{-\frac{1}{h^4}}}{h} \cdot \cos(0) = \lim_{h\to 0}\frac{e^{-h^{-4}}}{h} = \lim_{h\to 0}\left(\frac{1}{h e^{\frac{1}{h^4}}}\right).$$

$$= \lim_{h\to 0}\left[\frac{1}{h\left[1+\frac{1}{h^4}+\frac{1}{2!h^8}+\dots\right]}\right] \qquad \boxed{e^x = 1+x+\frac{x^2}{2!}+\dots}$$

$$\frac{\partial u}{\partial x} = 0.$$

$$\text{''}^{ly} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0. \qquad \text{CR} \to \text{satisfied.}$$

$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{e^{-z^{-4}}}{z} = \lim_{r\to 0}\frac{e^{-r^{-4}} \cdot e^{-\left(e^{i\frac{\pi}{4}}\right)^{-4}}}{re^{i\frac{\pi}{4}}}$$

$$z = re^{i\frac{\pi}{4}} \qquad = \lim_{r\to 0}\frac{e^{-r^{-4}}\, e^{-\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{-4}}}{re^{i\frac{\pi}{4}}}$$

$$= \lim_{r\to 0}\frac{e^{-r^{-4}}\, e^{-\cos\pi}}{re^{i\frac{\pi}{4}}} = \lim_{r\to 0}\frac{e^{-r^{-4}}\cdot e}{re^{i\frac{\pi}{4}}} = \infty.$$

$f'(z)$ doesn't exist at $z=0$.

$\therefore$ $f(z)$ is Not Analytic at $z=0$.

18) $$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0. \qquad \text{''}^{ly} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

$$f'(0) = \lt_{z\to 0} \frac{f(z)-f(0)}{z} = \lt_{z\to 0} \frac{x^2y^5}{x^4+y^{10}} \quad y=mx$$

$$= \lt_{x\to 0} \frac{m^5 \cdot x^7 x^3}{x^4(1+x^6m^{10})} = 0.$$

But $y^5 = x^2$. $\quad f'(0) = \lt_{x\to 0} \frac{x^2 \cdot x^2}{x^4 2x} = \frac{1}{2}$

$f'(0)$ is Not Constant $\Rightarrow$ $f(z)$ is Not Analytic at $z=0$.

* CR Equations in Polar form :-

$x = r\cos\theta$

$z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$u+iv = f(z) = f(re^{i\theta})$.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot re^{i\theta} \cdot i$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) i \quad \text{or} \quad \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} i = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \quad \& \quad \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

* Derivative of W OR F(z) in polarform :-

$w = u+iv$.

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

But $$\frac{dw}{dz} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \left(-r\frac{\partial v}{\partial r} + i \cdot r\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r}$$

$$\boxed{\begin{array}{l} \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \\ w = u+iv \end{array}}$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta.$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial}{\partial r}(u+iv)\sin\theta = \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial w}{\partial r}\sin\theta$$

$$= (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}.$$

Second form of $\frac{\partial w}{\partial z}$

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial (u+iv)}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}.$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$= \frac{-i}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\left(\frac{\sin\theta}{r}\right).$$

$$= \frac{-i}{r}\frac{\partial}{\partial \theta}(u+iv)\cos\theta - \frac{\partial w}{\partial \theta}\left(\frac{\sin\theta}{r}\right) = \frac{-i}{r}\cdot\frac{\partial w}{\partial \theta}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$= \frac{-i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}.$$

$$\boxed{\begin{aligned}\frac{dw}{dz} &= (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}.\\ \frac{dw}{dz} &= \frac{-i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}.\end{aligned}}$$

$$\boxed{\frac{\partial w}{\partial \theta}\left(\frac{-i}{r}\right) = \frac{\partial w}{\partial r}}$$

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Tutorial

(1) $f(z) = \begin{cases} \frac{Re(z)}{z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$ Verify if it is Continuos at $z=0$.

Sol) $f(z) = \frac{x}{x+iy}$

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{x}{x(1+im)} = \frac{1}{1+im} \rightarrow y = mx.$$

$\therefore$ Not Continuos

(2) P.T $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$

$$\frac{\partial u}{\partial \theta} = -r \cdot \frac{\partial v}{\partial r}.$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \rightarrow \frac{\partial^2 v}{\partial r^2} = -\frac{1}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \cdot \partial r} \quad \text{— (3)}$$

$\downarrow$

(1)

$$\cancel{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}} \quad \frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta \partial r} \quad \text{— (4)}$$

$$(1) + (3) + (4) = 0.$$

(3) $f(z) = \begin{cases} \frac{x^3 y(y-ix)}{x^6+y^2} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$ $\frac{df}{dz}$ at $z=0$.

Sol) $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \frac{x^3 y(y-ix)}{x^6+y^2 \,(x+iy)} = \frac{ix^3 y\,(+(-yi)-x)}{x^6+y^2\,(x+iy)}.$$

$y = mx$ $\quad f'(z) = \lim_{z \to 0} \frac{x^3 y(-i)}{x^6+y^2} = 0.$

$y = x^3$ $\quad f'(z) = \lim_{x \to 0} \frac{-ix^3 \cdot x^3}{2 \cdot x^6} = -\frac{i}{2}$

} values depend on path.

$\therefore$ Not differentiable.

(4.) $f(z) = \begin{cases} \dfrac{x^2 y^5 (x+iy)}{x^4+y^{10}}, & z \neq 0 \\ = 0, & z = 0. \end{cases}$ Find if $f(z)$ is analytic/not.

Sol.)

$$\frac{\partial u}{\partial x} = \frac{3x^2y^5(x^4+y^{10}) - x^3y^5(4x^3)}{(x^4+y^{10})^4}.$$

$$\frac{\partial v}{\partial x} = \frac{2xy^6(x^4+y^{10}) - x^2y^6(4x^3)}{(x^4+y^{10})^4}$$

$$\frac{\partial u}{\partial y} = \frac{5x^3y^4(x^4+y^{10})^4 - x^3y^5(10y^9)}{(x^4+y^{10})^4}$$

$$\frac{\partial v}{\partial y} = \frac{x^2 6y^5(x^4+y^{10}) - x^2y^6(10y^9)}{(x^4+y^{10})^4}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$f'(0) = \lim_{z\to 0} \frac{x^3y^5\cancel{(x+iy)} - 0.}{(x^4+y^{10})\cancel{(x+iy)}} = \lim_{z\to 0} \frac{x^3y^5}{x^4+y^{10}}$$

$y = mx \quad f'(0) = 0 \qquad y^5 = x^2 \quad f'(0) = \frac{1}{2}.$

(5.) $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ Find Analytic function $f(z)$ by Milne-Thomson Method.

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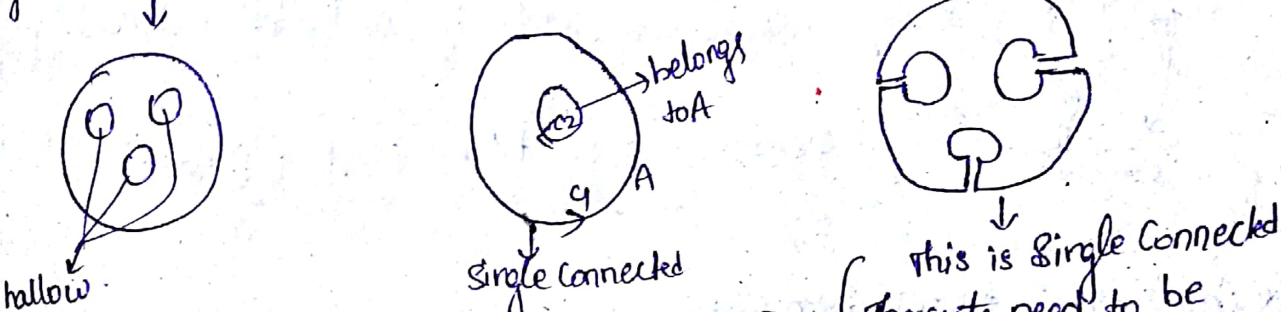
Integral Complex Analysis.

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

Connected Region :- which doesn't have non-intersection region.

*Simply Connected Region :- A Connected Region is Simply connected if all interior points of a closed curve C in region D are points of region D.

*Multiply Connected Region :- Bounded by many simply connected Regions | curves.



*A function $f(z)$ is said to be Meromorphic in R if it is analytic in Region R except at a finite no of poles.

*Multiple point :-

$f(z_1) = f(z_2) \Leftrightarrow z_1 = z_2$ then the function is single valued.

$f(z_1) = f(z_2)$ & $\exists$ z_1, z_2 such that this holds ($f(z_1) = f(z_2)$ holds)

& $z_1 \neq z_2 \Rightarrow$ Multivalued function.

*If an equation is satisfied by more than one variable in range, then the point is a Multiple point.

*Jordan Arc :- Is a continuos Arc without multiple points.

*Regular Arc :- It is a jordan arc wherein $f(z)$ is analytic $\forall$ z on the arc.

Contour is a Jordan curve consisting of a chain of Regular arcs.

*For closed contour the starting & end point are same.

Cauchy's Integral theorem for Simply Connected Regions.

$$\oint_C (u+iv)(dx+idy)$$

$$= \oint_C (udx - vdy) + i\oint_C (udy + vdx)$$

Green's theorem.

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & -v & 0 \end{vmatrix} = \iint\left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + \iint\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy = 0.$$

C_1, C_2, C_3 are out of phase

→ Simple connected if cut is infinitesimal

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \oint_{C_4} f(z)dz$$

* Cauchy's Integral Formula :-

Pole → point at which the function is not analytic.

a – pole.

$$\boxed{f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}dz}$$

$C = C_1 + C_2$. (but contribution by C_2 is very small.)

$$f(a) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)}dz.$$

$$f(a) = \frac{1}{2\pi i}\oint_{C_2}\left[\frac{f(z) - f(a) + f(a)}{(z-a)}dz.\right.$$

$z = re^{i\theta}$

$dz = ire^{i\theta}d\theta$.

$$= \frac{1}{2\pi i}\left[\int_{C_2}^{2\pi} \frac{f(\cancel{re^{i\theta}}+a) - f(a)}{re^{i\theta}} \cdot ire^{i\theta}d\theta \right.$$

$$\left. + \int_0^{2\pi} \frac{f(a)\cdot ire^{i\theta}d\theta}{re^{i\theta}}\right]$$

r is very small. $(re^{i\theta} + a) \approx a$

$$= 2 \cdot \frac{1}{2\pi i} \times \left(2\pi i\, f(a)\right)$$

$$= f(a).$$

Hence, proved.

$$\boxed{f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz.}$$

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Line Integral.

$f(z)$ — Complex function

$\int_a^b f(z)\,dz$ can be along any curve from $z=a$ to $z=b$.

$z = x+iy \qquad dz = dx + i\,dy \;,\quad dz = dx \;(y=0)\;,\quad dz = i\,dy\;(x=0)$

dz depends on path. but $\int_a^b f(z)\,dz$ is same along any regular curve.

* $\oint_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy).$

$f(z) = u(x,y) + iv(x,y)$

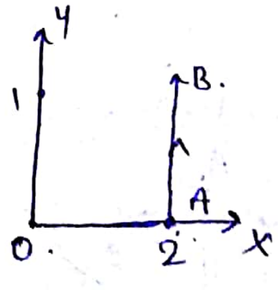
$dz = dx + i\,dy.$

$\oint_C f(z)\,dz$ — contour integral. (starting & Ending point is same.)

Q) $\int_0^{2+i} (\bar{z})^2\,dz$ — along Realaxis $z=0$ to $z=2$ & then along a line $\parallel^{el}$ to y-axis from $z=2$ to $z=2+i$.

A) $I = \int_0^{2+i} (\bar{z})^2\,dz = \int_0^{2+i} (x^2+y^2-2xyi)(dx+i\,dy) = \int_0^{2+i} (x-iy)^2 (dx+i\,dy)$

$$= \cancel{\int_0^{2+i} (x^2+y^2)\,dx - (-2xiy)\,dy + i\int_0^{2+i}}$$



$OA \rightarrow y=0,\ dy=0\ ,\ x$ varies 0 to 2.

$AB \rightarrow x=2,\ dx=0\ ,\ y$ varies 2 to $2+i$

$$I = \int_0^2 x^2\,dx + \int_2^{2+i} (2-iy)^2\,dy. = \frac{1}{3}(14+11i).$$

(2) $\int_0^{1+i}(x^2-iy)\,dz = I.$

$y=x.$
$dy=dx.$

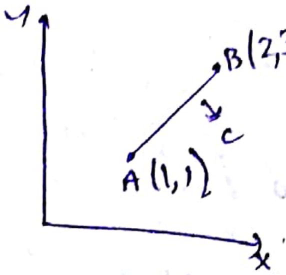
$$I=\int_0^1 x(x-i)(1+i)\,dx = (1+i)\left[\left(\frac{x^3}{3}\right)_0^1 - i\left(\frac{x^2}{2}\right)_0^1\right] = \frac{5-i}{6}.$$

$y=x^2$
$dy=2x\,dx$

$$I=\int_0^1 x^2(1-i)(1+2ix)\,dx = \frac{5+i}{6}.$$

x varies from 0 to 1

(3)



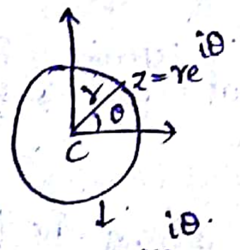
$$I=\int_C (12z^2-4iz) \qquad (z=x+iy)$$

$$y-1=\frac{3-1}{2-1}(x-1) \Rightarrow y=2x-1$$
$$dy=2dx.$$

$$I=\int_C (12x^2-12y^2+24ixy-4ix+4y)(dx+idy).$$

$$I=\int_1^2 \left(12x^2-12(2x-1)^2+24ix(2x-1)-4ix+4(2x-1)\right)(dx+2idx)$$

(4)



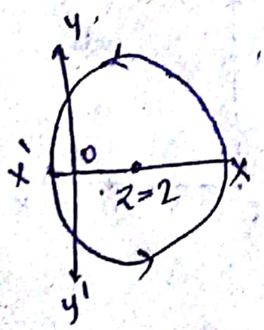
$$I=\oint_C (z-a)^n\,dz.$$

$z-a=re^{i\theta}$
$dz=ire^{i\theta}d\theta$
$\theta = 0$ to 2π

$$I=\int_0^{2\pi} r^n e^{i\theta n}\, ire^{i\theta}\,d\theta = \frac{r^{n+1}}{n+1}(1-1)=0 \quad n\neq -1$$

$$I=\int_0^{2\pi}\frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = 2\pi i$$

(5)



$|z-2|=3 \qquad z=re^{i\theta}=3e^{i\theta}$

C-upper half $\quad I=\int_C (z-z^2)\,dz = \int_0^{\pi}(3e^{i\theta}-9e^{2i\theta})\,3ie^{i\theta}d\theta$

C-lower half $\quad I=\int_{\pi}^{2\pi}(3e^{i\theta}-9e^{2i\theta})\,3ie^{i\theta}d\theta$

* **Limit of a function:** $f(z)$ is said to have a limit l at a point $z=z_0$, if for a given an arbitary chose +ve no ε, there exists a positive no δ, such that.

$$|f(z)-l|<\varepsilon \text{ for } |z-z_0|<\delta \longrightarrow \lim_{z\to z_0} f(z)=l.$$

* **Continuity:** $f(z)$ is said to be Continues at a point $z=z_0$ if let a given an arbitary +ve no ε, there exists a +ve no. δ, such that

$$|f(z)-l|<\varepsilon \text{ for } |z-z_0|\le\delta.$$

(or) $f(z)$ is continues at $z=z_0$ if $f(z_0)$ exists & $\lim_{z\to z_0} f(z)=f(z)_{z=0}$.

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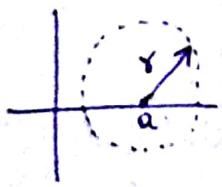
① $$\oint_C \frac{f(z)}{(z-a)}\,dz = I \qquad C:|z-a|=r.$$

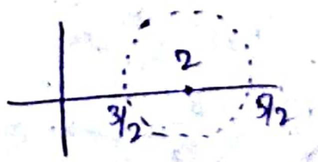
$$\oint_C \frac{dz}{z-a} = 2\pi i$$

$$I = 2\pi i f(a).$$



② $$\oint_C \frac{z}{(z^2-3z+2)}\,dz = \oint_C \frac{z\,dz}{(z-2)(z-1)} = \frac{2\cdot 2\pi i}{2-1} = 4\pi i.$$

$C:|z-2|=1/2.$



encloses only the pole 2.

③ $$\oint_C \frac{e^z}{(z+1)^2}\,dz. \qquad C:|z-1|=3.$$

$$= \frac{d}{dz} e^z \Big|_{z=-1} \cdot \frac{2\pi i}{1!} = 2\pi i e^{-1}.$$

④ $$\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz. \qquad C:|z|=1.$$

$$= \frac{2\pi i}{2!}\, \frac{d^2}{dz^2}\sin^2 z\Big|_{z=\pi/6} = \frac{\pi i}{1}\times 2\left(\frac{3}{4}-\frac{1}{4}\right) = \pi i$$

* CR → $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$\boxed{\begin{array}{c}\nabla^2 u = 0 \\ \nabla^2 v = 0\end{array}}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

↓ Harmonic functi

* If satisfies both CR's & Harmonic functions then they are Harmonic Conjugate u & v.

* $f(z) = u + iv$, $f(z)$ is Analytic.

1 u is known, find v.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy. = \underbrace{-\frac{\partial u}{\partial y}}_{M} dx + \underbrace{\frac{\partial u}{\partial x}}_{N} dy.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact Differiential Equation.}$$

1. Integrate M w.r.t x assuming y to be constant.
2. Integrate N w.r.t y removing all terms containing x.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. = \underbrace{\frac{\partial v}{\partial y}}_{M}dx - \underbrace{\frac{\partial v}{\partial x}}_{N}dy.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact Differential Equation}$$

① $u = x^2 - y^2 - 2xy - 2x + 3y$. Find v.

Sol)
$$-\frac{\partial u}{\partial y} = -(-2y - 2x + 3) = 2x + 2y - 3.$$

$$\frac{\partial u}{\partial x} = (2x - 2y - 2).$$

$$v = \int (2y + 2x - 3)\, dx + \int (2x - 2y - 2)\, dy \longrightarrow$$ should not consi the repeated term

$$= 2xy + x^2 - 3x + \cancel{2xy} - y^2 - 2y. + c.$$

* Milne Thomson Method :-

Given u/v can you find $f(z)$? · $f(z)$ is Analytic.

$f(z) = u+iv$.

$x = \frac{z+\bar{z}}{2}$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$y = \frac{z-\bar{z}}{2i}$

$= \phi_1(x,y) - i\,\phi_2(x,y)$.

$= \phi_1(z,0) - i\,\phi_2(z,0)$.

$\Rightarrow f(z) = \int [\phi_1(z,0) - \phi_2(z,0)i]\,dz + C$

$f'(z) = -\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \psi_1(x,y) + i\psi_2(x,y)$.

① $u = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2z \quad \Big| \begin{array}{l} x = z \\ y = 0. \end{array}$

$x = z, \; y = 0 \quad \Rightarrow$

$f(z) = \int 2z\,dz + C = z^2 + C$.

② $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)(2\cos 2x) + \sin 2x\, 2\sin 2x}{(\cosh 2y + \cos 2x)^2} = \frac{2 + \cosh 2y}{(\cosh 2y + \cos 2x)^2}$.

$\frac{\partial u}{\partial y} = \frac{2\sin 2x \sinh 2y + 0}{(\cosh 2y + \cos 2x)^2}$

$x = z$ & $y = 0 \longrightarrow \frac{\partial u}{\partial y} = 0$ & $\frac{\partial u}{\partial x} = \frac{2 + 2\cos 2z}{(1+\cos 2z)^2}$.

$f'(z) = \frac{2 + 2\cos 2z}{(1+\cos 2z)^2} \Rightarrow f(z) = \int \frac{dz\,(2 + 2\cos 2z)}{(1+\cos 2z)^2} + C = \int \frac{2}{2\cos^2 z}\,dz$.

$= \int 2\sec^2 z\,dz = 2\tan z + C$.

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① $u = x^2 - y^2 - 2xy - 2x + 3y$. $\quad v = ?$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \quad , \quad \frac{\partial u}{\partial y} = -2y - 2x + 3$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \underbrace{-\frac{\partial u}{\partial y}}_{\frac{\partial v}{\partial x}}dx + \underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial v}{\partial y}}dy$$

$$dv = (2x+2y-3)dx + (2x-2y-2)dy.$$

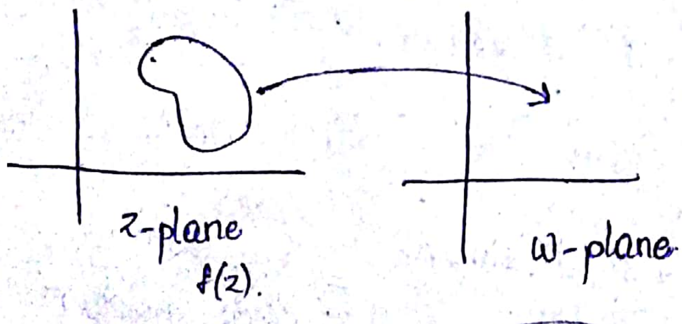
$$v = x^2 + 2xy - 3x + \psi(y).$$

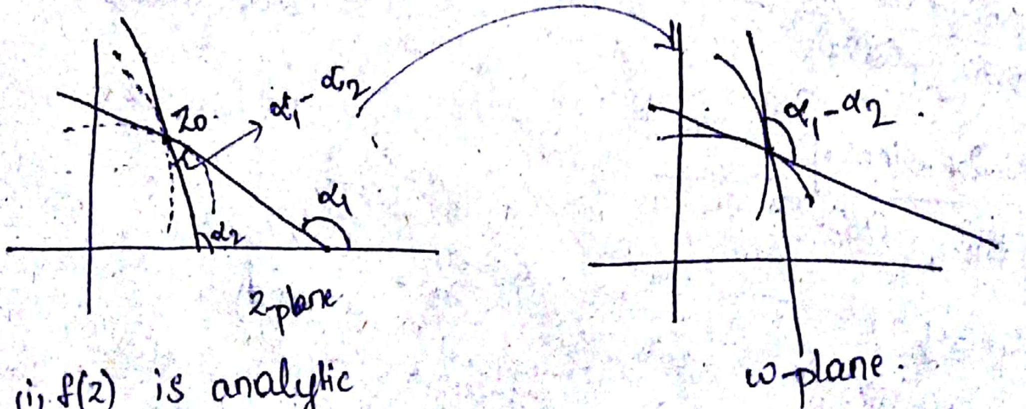
$$\frac{\partial v}{\partial y} = \cancel{2x} + \psi'(y) = \cancel{2x} - 2y - 2$$

$$\psi'(y) = -2y - 2 \Rightarrow \underline{\underline{\psi(y) = -y^2 - 2y}}.$$

$$v = \underline{x^2 + 2xy - 3x - y^2 - 2y}.$$

* <u>Conformal Mapping</u> :- Angle preserving maps.





(i) $f(z)$ is analytic

(ii) $f'(z) \neq 0$.

$$f'(z_0) = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0} = \frac{r e^{j\phi_1}}{r_1 e^{j\theta_1}} = R e^{j\lambda}$$

$R = \frac{r}{r_1}$ where $\lambda = \phi_1 - \theta_1$.

$\lambda = \beta_1 - \alpha_1$

$\lambda = \beta_2 - \alpha_2 \Rightarrow \beta_1 = \alpha_1 + \lambda$

$\beta_2 = \alpha_2 + \lambda$.

* If $f'(z) = 0$. then the mapping is not conformal.

* point where $f'(z) = 0$ is called a critical point.

2. If $f(z) = u + iv$. Angle of Rotation is $\tan^{-1}\left(\frac{v}{u}\right)$

3. $$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 = |f'(z)|^2$$

* $u = 2x^2 + y^2$, $v = \frac{y^2}{x}$, $m_1 m_2 = -1$.

$\frac{\partial u}{\partial x} = 4x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = y^2 \frac{(-1)}{x^2}$, $\frac{\partial v}{\partial y} = \frac{2y}{x}$.

$u = 2x^2 + y^2 = c$. $\qquad v = \frac{y^2}{x} = c$

$4x + 2y\frac{dy}{dx} = 0$. $\qquad \frac{2y}{x}\frac{dy}{dx} - \frac{y^2}{x^2} = 0$.

$m_1 = \frac{dy}{dx} = \frac{-2x}{y}$. $\qquad \frac{dy}{dx} = \frac{y}{2x} = m_2$.

$m_1 m_2 = -1$. $\qquad$ CR ✗

Not Analytic $\Rightarrow$ Not Conformal.

* $w = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$.

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial u}{\partial y} = -2y$. $\quad \frac{\partial v}{\partial y} = 2x$ $\quad \frac{\partial v}{\partial x} = 2y$.

C.R ✓ → Analytic → Mapping is Conformal

(i) Magnification at $z = 2 + i$

(ii) Angle of Rotation at $z = 2 + i$ → $w = 4 - 1 + 4i = 3 + 4i$

~~$\tan^{-1}(y/x)$~~ $\qquad$ angle = $\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}$

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$$\tan^{-1}\left(\frac{v}{u}\right) = \tan^{-1}\left\{\frac{2xy}{x^2-y^2}\right|_{\substack{x=2\\y=1}} = \tan^{-1}\left(\frac{4}{3}\right).$$

$$|f'(z)| = |2f(z)|. = 2|z| = 2\sqrt{(x)^2+(y)^2} = 2\sqrt{5}$$

$\downarrow$
Magnification.

27/10/18.

$f'(z) = u+iv.$

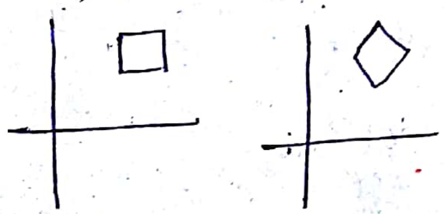
$\angle$ of Rotation by a Conformal Map $= \angle f'(z)\big|_{z=z_0}$

$$f(z_0) = \lim_{z_1\to z_0} \frac{w_1-w_0}{z_1-z_0}.$$

1. Translation :- $c = a+ib$

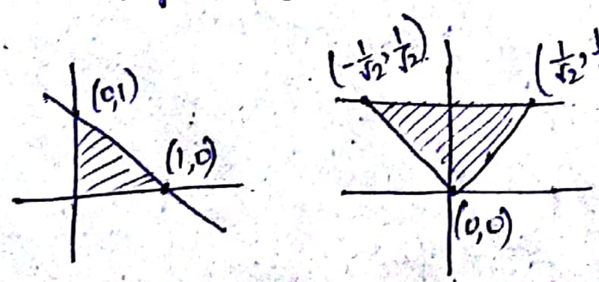
$y = z+c.$
$= x+iy+a+ib.$
$= (x+a)+i(y+b).$

z-plane $\rightarrow (x,y)$.
w-plane $\rightarrow (x+a, y+b)$



$$w = ze^{i\theta} = (x+iy)(\cos\theta+i\sin\theta).$$
$$= (x\cos\theta - y\sin\theta) + i(y\cos\theta + x\sin\theta).$$

* $x=0, y=0, x+y=1.$



$w = ze^{j\pi/4}.$

2. Magnification :-

$w = cz, \ c \in \mathbb{R}$

$z = re^{i\theta}.$

$w = c(r\cos\theta + ir\sin\theta).$

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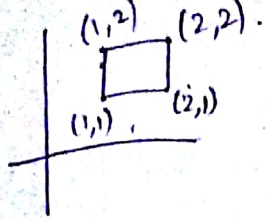
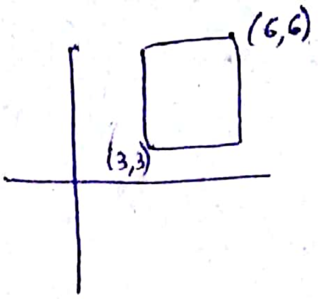
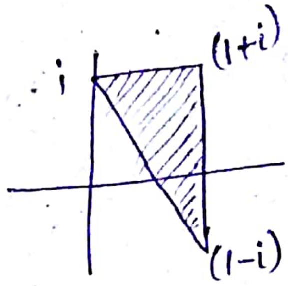




c = Real



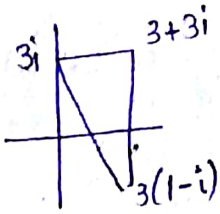
$c \in \mathbb{C}$ $\quad c = a+ib$

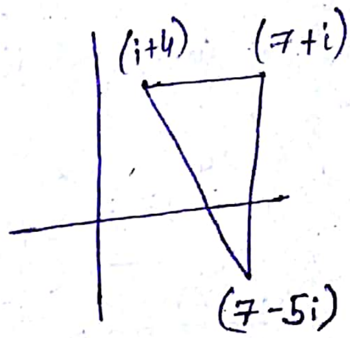
$c = ae^{i\phi}$

$w = 3z+4-2i$

$u+iv = (a+ib)(x+iy)$

$= (ax-by)+i(bx+ay)$





$$\left.\begin{aligned} u &= ax-by \\ v &= bx+ay \end{aligned}\right\} \longrightarrow y = \frac{av-bu}{a^2+b^2}$$

$$x = \frac{au+bv}{a^2+b^2}$$

$$* \; w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = u+iv \quad \text{—①}$$

$$Z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2} = x+iy \quad \text{—②}$$

th. 1 circle maps to other with reflection. c.m.

$$x^2+y^2+2gx+2fy+c=0.$$

$$\Rightarrow \frac{1}{(u^2+v^2)} + \frac{2gu}{(u^2+v^2)} - \frac{2fv}{(u^2+v^2)} + c = 0.$$

$$* w = \frac{1}{z}, \quad y-x+1=0.$$

From eq ① & ② $\quad x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2}$

$$\frac{v}{u^2+v^2} - \frac{u}{u^2+v^2} + 1 = 0 \Rightarrow u^2+v^2+v-u+1=0.$$

$$\left(u-\tfrac{1}{2}\right)^2 + \left(v-\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{\sqrt{2}}\right)^2$$

* Bilinear Transformation :-

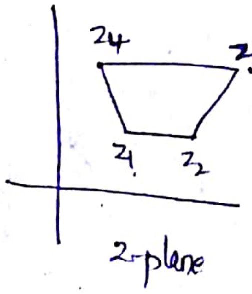
$$w = \frac{az+b}{cz+d} \quad , \quad ad-bc \neq 0.$$

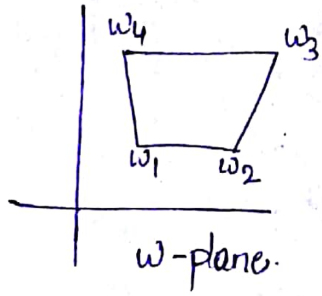
$$\Rightarrow wcz + wd = az + b.$$

$$(wc-a)z = b - wd.$$

$$z = \frac{b-wd}{wc-a} \quad \text{except } w = \frac{a}{c}.$$

*** theorem :-** w_1, w_2, w_3, w_4 be images of z_1, z_2, z_3, z_4 under BLT & they form vertices of a closed Region.





$$\frac{(w_1-w_2)(w_4-w_3)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_2)(z_4-z_3)}{(z_1-z_4)(z_2-z_3)} \quad — ①$$

$$w_1 = \frac{az_1+b}{cz_1+d}$$

$$w_2 = \frac{az_2+b}{cz_2+d}$$

$$w_1 - w_2 = \frac{(ad-bc)(z_1-z_2)}{\underbrace{(cz_1+d)(cz_2+d)}_{F(z_1,z_2)}}$$

$$w_4 - w_3 = \frac{(ad-bc)(z_4-z_3)}{\underbrace{(cz_4+d)(cz_3+d)}_{F(z_4,z_3)}}.$$

Eq ①

$$L.H.S = \frac{\dfrac{(z_1-z_2)(z_4-z_3)}{\cancel{F(z_1,z_2)\,F(z_4,z_3)}}}{\dfrac{(z_1-z_4)(z_2-z_3)}{\cancel{F(z_1,z_4)\,F(z_2,z_3)}}} = \frac{(z_1-z_2)(z_4-z_3)}{(z_1-z_4)(z_2-z_3)} = R.H.S.$$

Hence, proved.

* Show that the transformation $w = \frac{3-z}{z-2}$ maps the circle with centre $(5/2, 0)$ & radius $1/2$ in z-plane to imaginary axis on w-plane & the interior to right half of w-plane.

Sol)

$$w = \frac{\frac{1}{2}+\frac{5}{2}-z}{z-\frac{5}{2}+\frac{1}{2}} \qquad -z+\frac{5}{2} = re^{i\phi}$$

$$w = \frac{\frac{1}{2}+re^{i\phi}}{\frac{1}{2}-re^{i\phi}} = \frac{\left(\frac{1}{2}+re^{i\phi}\right)\left(\frac{1}{2}-re^{-i\phi}\right)}{\left(\frac{1}{2}-re^{i\phi}\right)\left(\frac{1}{2}-re^{-i\phi}\right)} = \frac{\left(\frac{1}{4}-r^2\right)+ir\sin\phi}{\frac{1}{4}+r^2+r\cos\phi}$$

If $r = 1/2$ then w will be ~~on~~ in imaginary axis.

$r < 1/2$ then it will be inside the circle.

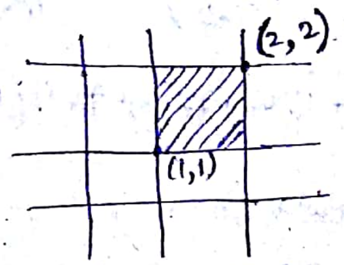
Th

* Family of circles & lines under BLT is closed.

$$w = \frac{az+b}{cz+d} = K \cdot \frac{1}{cz+d} + \frac{a}{c} \qquad K = \frac{-ad+bc}{c}$$

(i) Verify that lines & circles are closed w.r.t translation, reflection, scaling & Rotation.

$$w = z^2 = (x+iy)^2 = \underbrace{x^2-y^2}_{u} + \underbrace{2xy}_{v}i$$



$y = c,\ x = c$

(i) $y = c$

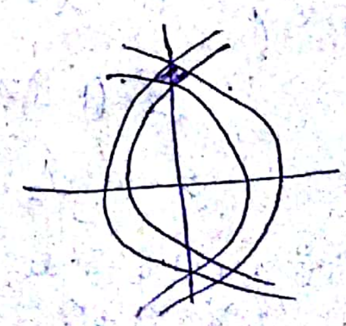
$x^2 = u + c^2 \Rightarrow x = \pm\sqrt{u+c^2}$

$v = 2c \pm \sqrt{u+c^2} \Rightarrow v^2 = 4c^2(u+c^2)$

(ii) $x = c$

$y = \pm\sqrt{c^2-u^2}$ $\qquad v = 2c \pm \sqrt{c^2-u} \Rightarrow v^2 - 4c^2 = c^2 - u$

$v^2 = -4c^2(u-c^2)$



31/10/18.

Tutorial.

① Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ ~~or~~ to the straight line $4x+3=0$.

Sol)
$$w = \cancel{2z}\ \frac{2(x+iy)+3}{x+iy-4} = \frac{(2x+2iy+3)((x-4)-iy)}{((x-4)+iy)((x-4)-iy)}$$

$$= \frac{2x^2-8x+\cancel{2xyi}-8iy+3x-12-\cancel{2xyi}+2y^2-3yi}{(x-4)^2+y^2}$$

$$w = \frac{2x^2-\cancel{8}5x+2y^2-12+i(-\cancel{8y}5y)}{(x-4)^2+y^2}$$

$z = \frac{4w+3}{w-2}$, given w. $z = x+iy$.

Eq^n of circle, $x^2+y^2-4x=0$ can be written as $z\bar{z}-2(z+\bar{z})=0$.

$$\Rightarrow \left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right) - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

$$\Rightarrow 16w\bar{w}+12w+12\bar{w}+9-2(4w\bar{w}+3\bar{w}-8w-6+4w\bar{w}+3w-8\bar{w}-6)$$

$$\Rightarrow 22(w+\bar{w})+33=0$$

$w = u+iv$.

$$\Rightarrow 22(2u)+33=0.$$

$$4u+3=0 \longrightarrow \text{St. line.}$$

② Find the bilinear transformation where fixed points are 1 and 2.

$w = \frac{az+b}{cz+d}$ — (1) [Hint: fixed pts are given by $w=z$]

Sol) points are given by $z = \frac{az+b}{cz+d}$ $\Rightarrow cz^2+dz = az+b$

$$cz^2+(d-a)z+b=0.$$

$$z^2-\frac{(a-d)}{c}z-\frac{b}{c}=0 \text{ — (2).}$$

Fixed points are 1 & 2 so.

$$(z-1)(z-2)=0 \Rightarrow z^2-3z+2=0 \text{ — (3).}$$

Comparing (2) & (3).

$\frac{a-d}{c} = 3$. $-\frac{b}{c} = 2 \Rightarrow b = -2c$

$a-d = 3c$.   from (1) $w = \frac{az-2c}{cz+a-3c}$ Taking $a=1, c=-1$

$\underline{d = a-3c}$ $\underline{w = \frac{z+2}{4-z}}$ (can take any a & c values.)

③ If a is any Real +ve number, show that the transformation $w = \frac{z-a}{z+a}$ transforms conformally the plane $x>0$ to the unit circle $|w|<1$. what is the transform of arg w = constant in z-plane?

Sol) We have $w = \frac{z-a}{z+a}$; $|w|<1 \Rightarrow \left|\frac{z-a}{z+a}\right|<1$ Let $z = x+iy$.

$\Rightarrow |z-a| < |z+a|$.

$(x-a)^2 + \cancel{y^2} < (x+a)^2 + \cancel{y^2}$

$\cancel{x^2} + \cancel{a^2} - 2ax < \cancel{x^2} + \cancel{a^2} + 2ax \Rightarrow \underline{x>0}$. (a is +ve).

• $\arg w =$ constant. $w = \frac{z-a}{z+a}$

$Re^{i\phi} = \frac{z-a}{z+a}$. $w = Re^{i\phi} \Rightarrow |w| = R$ & $\arg w = \phi$.

$\log R + i\phi = \log(z-a) - \log(z+a)$.

$\log R + i\phi = \log((x-a)+iy) - \log((x+a)+iy)$.

$= \log(r_1 e^{i\theta_1}) - \log(r_2 e^{i\theta_2})$.

$= \log r_1 + i\theta_1 - \log r_2 - i\theta_2$.

| $z = x+iy = re^{i\theta}$. then $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Compare LHS and RHS. Imaginary parts $\phi = \theta_1 - \theta_2$.

$\phi = \tan^{-1}\left(\frac{y}{x-a}\right) - \tan^{-1}\left(\frac{y}{x+a}\right)$.

$\phi = \tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right) = \alpha$.

As, $\arg w$ = constant = α (Let) $[\phi = \alpha]$.

$\boxed{x^2+y^2-a^2 = 2ay\cot\alpha}$ —— z-plane.

4) If $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$, find $f(z)$.

Milne ~~theorem~~. thomson method.

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u is given, $f(z) = \phi_1(z,0) - i\phi_2(z,0)$.

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)\, 2\cos 2x - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2(1)}{(\cosh 2y + \cos 2x)^2} = \phi_1(x,y)$$

For $x = z,\ y = 0$, $\phi_1(z,0) = \frac{2\cos 2z + 2}{(1+\cos 2z)^2}$

$$\frac{\partial v}{\partial y} = \frac{-\sin 2x(2\sinh 2y)}{(\cosh 2y + \cos 2x)^2} - \phi_2(x,y).$$

At $x = z,\ y = 0,\ \phi_2(z,0) = 0.$

$$f(z) = \int \frac{2\cos 2z + 2}{(1+\cos 2z)^2}\, dz = 2\int \frac{1}{1+\cos 2z}\, dz = 2\int \frac{1}{\cos^2 2z}\, dz.$$

$$f(z) = \tan z + c$$

2/11/18.

$\omega = z^n, \quad z = re^{i\theta}$.

$\omega = r^n e^{in\theta} \Rightarrow |\omega| = r^n, \quad \angle\omega = n\theta$.

$\omega = z + \frac{1}{z}$

$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$

$= (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$.

$\frac{u}{r+\frac{1}{r}} = x, \quad \frac{v}{r - \frac{1}{r}} = y$.

$x^2 + y^2 = c^2$

$\Rightarrow \frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = c^2$.

z-plane	w-plane
circle	ellipse.
circle $r=1$	
line $\theta = \theta_0$	Hyperbola.

$u = r\cos\theta(r + \frac{1}{r})\Big|_{r=1} = 2\cos\theta$.

$\frac{u^2}{\cos^2\theta_0} - \frac{v^2}{\sin^2\theta_0} = (r+\frac{1}{r})^2 - (r - \frac{1}{r})^2 = 1$.

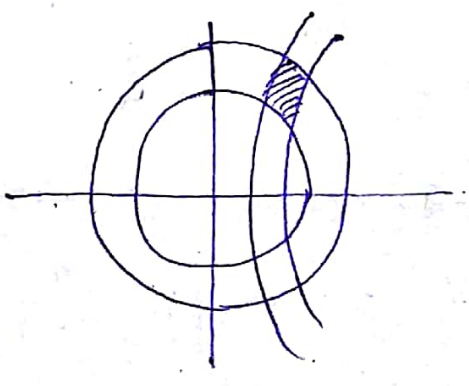
* $\omega = e^z = e^{x+iy} = e^{r(\cos\theta + i\sin\theta)}$.

z-plane	w-plane
$x = c$	$R = e^c$ R-radius
$x = 0$	$R = 1$
$y \in [0, \pi]$.	Upper ½ w-plane
$y \in [-\pi, 0]$.	Lower ½ w-plane
$y = c + 2\pi$	whole plane.

$\omega = \cosh z$.

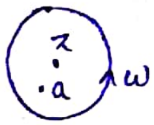
$z = x + iy \quad \omega = \cosh(x+iy)$

$\omega = \underbrace{\cosh x\cos y}_{x} + i\underbrace{\sinh x\sin y}_{v}$



$y = a$ maps to $\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = 1$

$x = a$ maps to $\frac{u^2}{\cosh^2 a} + \frac{v^2}{\sinh^2 a} = 1$.



$$\left|\frac{z-a}{w-a}\right|<1$$

RoC → Region of Convergence

$$\frac{1}{w-z}=\frac{1}{w-a+a-z}=\frac{1}{w-a\left[1-\frac{z-a}{w-a}\right]}$$

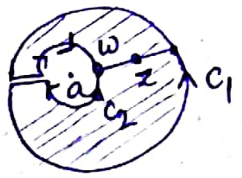
$$=\frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\frac{(z-a)^2}{(w-a)^2}+\dots+\infty\right]$$

$$\oint\frac{f(w)}{(w-z)}dz=\frac{f(w)}{(w-a)}dw+(z-a)\oint_C\frac{f(w)}{(w-a)^2}dw+(z-a)^2\oint_C\frac{f(w)}{(w-a)^3}dw+\dots$$

$$\oint\frac{f(w)}{(w-z)}dz=\underline{f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)}+\dots\infty.$$

↓

Taylor Series (Assumption is Analyticity).



$$\left|\frac{z-w}{a-w}\right|<1.$$

$$f(z)=\frac{1}{2\pi i}\oint_C\frac{f(w)}{(w-z)}dw$$

$$=\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-z)}dw-\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{(w-z)}dw.$$

$$f(z)=\underbrace{f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\dots+\infty}_{C_1}+\underbrace{\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}}_{C_2}.$$

$$w-z=-[z-w]=-[z-a+a-w]$$

$$=-(z-a)\left[1-\frac{w-a}{z-a}\right].$$

$$\frac{f(w)}{w-z}=\frac{-f(w)}{(z-a)}\left[1+\frac{(w-a)}{(z-a)}+\frac{(w-a)^2}{(z-a)^2}+\dots+\infty\right]$$

$$\Rightarrow -\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z} = \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \dots \infty.$$

$$b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw.$$

$$(1-\alpha t)^{-\beta} = \sum_{n=0}^{\infty}\frac{(\beta)_m}{m!}(\alpha t)^m. \qquad (\beta)_m = \frac{\Gamma(\beta+m)}{\Gamma(\beta)}.$$

* Lawrence Series.

* $f(z) = \frac{1}{(z-1)(z-2)}$, $1 \leq |z| \leq 2$. $\longrightarrow$ Lawrence Series $\rightarrow$ Anular

$z > 1$

$z^{-1} < 1$.

$$= \frac{(z-1)-(z-2)}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{z^{-1}}{1-z^{-1}}$$

$$= -z^{-1}(1+z^{-1}+z^{-2}+\dots) - \frac{1}{z(1-\frac{z}{2})}$$

$$f(z) = -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\infty\right].$$

22/11/18.

* Singular point :- Point at which $f(z)$ ceases to be analytic, say $z=a$.

* Isolated singularity :- Considering an ϵ-neighbourhood of a, there $\exists$ no other singularity.

* If there are poles in ϵ-neighbourhood, then it is a non-isolated singularity.

* Residue :-

$$f(z) = \sum_{n=0}^{\infty} a_m (z-a)^m + \sum_{m=1}^{\infty} b_m (z-a)^{-m}$$

$$a_m = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{m+1}} dz.$$

$$b_m = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{-m+1}} dz.$$

In particular, $b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz.$ b_1 is called Residue.

(i) $\underset{f(z)}{\text{Res}} [z=a] = \underset{z\to a}{\text{Lt}} (z-a) f(z). \longrightarrow$ single order pole.

$$(z-a) f(z) = \sum_{m=0}^{\infty} a_m (z-a)^{m+1} + \sum_{m=1}^{\infty} b_m (z-a)^{-m+1}.$$

$\because$ Pole is of order 1 $\quad b_1 \neq 0,\ b_2 = b_3 = \dots = 0.$

$$\Rightarrow \underset{z\to a}{\text{lt}} (z-a) f(z) = b_1.$$

(ii) Pole of order n.

$$(z-a)^n f(z) = \sum_{m=0}^{\infty} a_m (z-a)^{m+n} + \sum_{m=1}^{n} b_m (z-a)^{n-m+1}$$

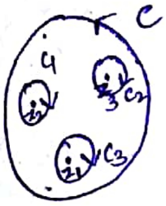
$$\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right] = b_1.$$

* $$\underset{\frac{\phi(z)}{\psi(z)}}{\text{Res}} [z=a]. = \underset{z\to a}{\text{lt}} \frac{(z-a)\phi(z)}{\psi(z)}.$$

$$= \lim_{z\to a} \frac{(z-a)\,\phi(z)}{\psi(a)+(z-a)\psi'(a)+\frac{(z-a)^2}{2}\psi''(a)+\dots+\infty}$$

$$= \frac{\phi(a)}{\psi'(a)}.$$

*Residue theorem :-



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz.$$

$$= 2\pi i\left(\operatorname{Res}_{f(z)}[z=z_1] + \operatorname{Res}_{f(z)}[z=z_2] + \operatorname{Res}_{f(z)}[z=z_3]\right).$$

① $\dfrac{1}{2\pi i}\oint \dfrac{e^{zt}\,dz}{(z^2+2z+2)z^2}$

$z=-1\pm i$, order $=1$

$z=0$, order $=2$.

$$\frac{d}{dz}\left.\frac{e^{zt}}{z^2+2z+2}\right|_{z=0} = \left.\frac{e^{zt}\,\cancel{t}(z^2+2z+2)-e^{zt}(2z+2)}{(z^2+2z+2)^2}\right|_{z=0}$$

$$\operatorname{Res}[z=0] = \frac{2t-2}{4}$$

$$\operatorname{Res}[z=0] = \frac{t-1}{2} \quad \text{—①}$$

$$\operatorname{Res}[z=-1+i] = \left.\frac{e^{zt}}{z^2(z+1+i)}\right|_{z=-1+i}$$

$$= \frac{e^{t(-1+i)}}{(1-1-2i)\cdot 2i} = \frac{e^{(-1+i)t}}{4} \quad \text{—②}$$

$$\operatorname{Res}[z=-1-i] = \left.\frac{e^{zt}}{z^2(z+1-i)}\right|_{z=-1-i} = \frac{e^{-(1+i)t}}{+4} \quad \text{—③}.$$

$$\int_C f(z)\,dz = \left(\frac{t-1}{2}\right) + \frac{e^{-t}\cos t}{2} \qquad [\text{Add ①+②+③}]$$

② $\displaystyle\oint_C \frac{e^{2z}+z^2}{(z-1)^5}\,dz.$

$f(z) = e^{2z}+z^2 \Rightarrow$

$f'(z) = 2e^{2z}+2z.$

$f''(z) = 4e^{2z}+2$

$f'''(z) = 8e^{2z}$

$f''''(z) = 16e^{2z}.$

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$$= 2\pi i \times \frac{1}{\frac{24}{3}} \times 16 e^{2z}\Big|_{z=1} = \frac{4\pi i e^2}{3}.$$

(3) $f(z) = \frac{1}{z^2 \sinh z} = \frac{2}{z^2(e^z - e^{-z})}$.

$$= \frac{1}{z^2\left(z + \frac{z^3}{6} + \frac{z^5}{120} + \dots + \infty\right)} = \frac{1}{z^3\left(1 + \frac{z^2}{6} + \frac{z^4}{120} + \dots + \infty\right)}$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} - \frac{z^4}{120} + \dots \infty\right]$$

$\text{Res}[z=0] = -1/6.$

(4) $I = \oint \frac{\sin z}{z^6}\, dz$. Contour is $|z| = 2$.

$f(z) = \sin z \quad f'(z) = \cos z \quad f''(z) = -\sin z \quad f'''(z) = -\cos z \quad f''''(z) = \sin z$.

$f'''''(z) = \cos z$.

$$I = \frac{2\pi i}{120} \cos z\Big|_{z=0} \Rightarrow I = \frac{\pi i}{60}.$$

* $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \text{Res}[z=n] = \sum_{n=-\infty}^{\infty} f(n).$

$$\underbrace{-f(z)\cot(\pi z)\pi}_{f_1(z)}$$

$$\text{Res}_{f_1(z)}[z=n] = \lim_{z\to n} -\pi(z-n) f(z) \cot(\pi(z-n) + n\pi)$$

$$= \lim_{z\to n} -\pi(z-n) f(z) \frac{\cos(\pi(z-n) + n\pi)}{\sin(\pi(z-n) + n\pi)}$$

$\text{Res}_{f_1(z)}[z=n] = f(n).$

(5) $\sum_{n=-\infty}^{\infty} \frac{1}{(3n-1)^2} = \frac{4\pi^2}{27}.$

$$f(z) = \frac{1}{(3z-1)^2} = \frac{1}{9(z-1/3)^2}$$

$$f_1(z) = \frac{-\pi \cot \pi z}{9(z-1/3)^2}.$$

$$= \underset{z=1/3}{\text{Res}}\, f_1(z).$$

$$= \frac{d}{dz}\frac{\cancel{(z-1/3)^2}}{9}\cdot\frac{-\pi\cot(\pi z)}{\cancel{(z-1/3)^2}}\Bigg|_{z=1/3} = \frac{\pi^2}{9}\text{cosec}^2(\pi z)\Big|_{z=1/3}.$$

$$= \frac{\pi^2}{9}\times\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4\pi^2}{27}.$$

3/11/18.

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{\forall k}\underset{z=z_k}{\text{Res}}\,[f_1(z)].$$

$f_1(z) = f(z)\cdot\pi\cos(\pi z)$. where z is poles of $f(z)$.

$$\lim_{z\to n}\,(z-n)\,\pi\cot\pi z\, f(z) = f(n).$$

$$\oint_R f_1(z)\,dz = 0.$$

$$\Rightarrow \sum_{n=-N}^{N} f(n) + \sum_{\forall k}\text{Res}\,[f_1(z)].$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} f(n) = -\underset{z=z_k}{\text{Res}}\,[f_1(z)].$$

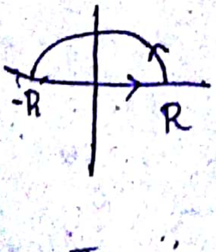
① $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}$ $\qquad f(z) = \dfrac{1}{z^2+a^2}$

$z_k\text{'s} = \pm ai.$

$$\underset{z=ia}{\text{Res}}\left[\frac{\pi\cot\pi z}{(z+ia)(z-ia)}\right] + \left[\underset{z=-ia}{\text{Res}}\,\frac{\pi\cot\pi z}{(z+ia)(z-ia)}\right]$$

$$= \frac{-\pi\cot\pi ia}{2ai} = \frac{\pi}{2a}\coth(\pi a).$$

* Finding Real Definite Integrals.



$$\int_{-\infty}^{\infty}\frac{dx}{x^2+1} = \int_{-\infty}^{\infty}\frac{dx}{(x+i)(x-i)} = 2\pi i\times\frac{1}{2i}$$

$$= \pi\cdot\tan^{-1}x\Big|_{-\infty}^{\infty} = \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) = \pi.$$

$$\int_0^{\pi}\frac{\cos mx}{x^2+1}\,dx = \text{Re}\left[\int_0^{\infty}\frac{e^{imx}}{x^2+1}\,dx\right]\cdot 2\pi i$$

$$= Re\left[\frac{e^{imx}}{x+i}\Big|_{x=i}\right] \cdot 2\pi i$$

$$= \frac{e^{-m}}{2i} \cdot 2\pi i = -\pi e^{-im} = \frac{\pi \cdot e^{im}}{\cdot}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = \sum_{K} \left[\operatorname{cosec} \pi z f(z)\right].$$

$$= \lim_{z\to n} \frac{f(z)}{\sin\pi(z-n)} \cdot \pi(z-n)(1)^n$$

$$= (-1)^n f(n).$$

$$\sum_{n=-N}^{N} (-1)^n f(n) + Res_{z=z_k}$$

6/11/18.

*theorem:- $f(x) = \sum_{K=0}^{\infty} \frac{\phi(K)}{K!}(-x)^K$

$$\int_0^\infty x^{s-1} f(x)\,dx = \nu(s)\,\phi(-s).$$

$$Res_{z=-n}\left[\nu(z)\right]. = \lim_{z\to -n} (z+n)\,\nu(z).$$

$$= \lim_{z\to -n} (z+n) \cdot \frac{\nu(z+n+1)}{z(z+1)\cdots(z+n+1)} = \frac{(-1)^n}{n!}$$

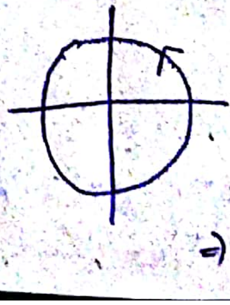
$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \nu(s)\,\phi(-s)\,x^{-s}\,ds!$$

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s!} x^s \phi(s), = \sum_{s=0}^{\infty} \frac{(-x)^s}{\lfloor s} \phi(s).$$

$$\int_0^\infty x^{s-1} f(x) = \nu(s)\,\phi(-s).$$

① $\int_0^{2\pi} \frac{4\,d\theta}{5+4\sin\theta} = \oint_c \frac{4\,dz}{iz\left[5+\frac{4}{2i}(z-1/z)\right]}$

$$= \oint_c \frac{4\,dz}{5iz+2(z^2-1)}$$



$e^{i\theta} = z.$

$\Rightarrow ie^{i\theta}\,d\theta = dz$

$\Rightarrow d\theta = \frac{dz}{iz}$

$$= \oint_C \frac{4\left[z(z^2-1) - 5iz\right] dz}{4(z^2-1)^2 + 25z^2} = \oint_C \frac{4\left[z(z^2-1)-5iz\right] dz}{4(z^4+1-2z^2)+25z^2}$$

$$= \oint_C \frac{4\left[z(z^2-1)-5iz\right]}{4z^4+4+17z^2} dz = \oint_C \frac{2(z^2-1)-5iz}{(z+i/2)(z-i/2)(z^2+4)} dz.$$

$$\text{Re}_{z=i/2}\left[\cdot\right] = 2(-1/4-1) - 5i\cdot\frac{i}{2} = 0.$$

$$\text{Re}_{z=-i/2}\left[\cdot\right] = \frac{2(-1/4-1)+5i\cdot i/2}{-i\cdot 15/4} = \frac{\cancel{8}\,1}{i\cdot 3/4} = \frac{4}{3i}$$

sum of Residue $= \left(0+\frac{4}{3i}\right)\cdot 2\pi i = \frac{8\pi}{3}.$

(2) $$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \oint_C \frac{dz}{iz\left(2+\frac{1}{2}(z+1/z)\right)}$$

$e^{i\theta} = z.$
$\Rightarrow ie^{i\theta}d\theta = dz.$

$$= \oint_C \frac{dz}{i\left(2z+\frac{1}{2}(z^2+1)\right)} = \oint_C \frac{2dz}{(4z+z^2+1)i}$$

z^2+4z+1

$\frac{-4\pm\sqrt{16-4}}{2}$

$\frac{-4\pm 2\sqrt{3}}{2}$

$-2\pm\sqrt{3}.$

$$= \oint_C \frac{2dz}{i(z+2+\sqrt{3})(z+2-\sqrt{3})}.$$

outside the contour (i.e unit circle).

$$\text{Re}_{z=-2+\sqrt{3}}[\cdot] = \frac{2}{i(-\cancel{2}+\sqrt{3}+\cancel{2}+\sqrt{3})} = \frac{\cancel{2\pi}}{\cancel{2\sqrt{3}}}\frac{1}{i\sqrt{3}}.$$

Ans $= 2\pi \times \frac{1}{i\sqrt{3}} = \frac{2\pi}{i\sqrt{3}}.$

(3) $$\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta \, d\theta}{(3+2\cos\theta)}.$$

$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta.$

$$= \int_0^{2\pi} \frac{(1+2\cos\theta)^n e^{in\theta} d\theta}{(3+2\cos\theta)} = \oint_C \frac{(1+(z+1/z))^n z^n \, dz}{(3+z+1/z)\; iz} = \oint_C \frac{(z^2+z+1)^n \, dz}{(z^2+3z+1)i}$$

$$\cancel{\text{Res}_{z=\frac{-3+\sqrt{5}}{2}}[\cdot] = \frac{}{i\left(z-\frac{-3+\sqrt{5}}{2}\right)\left(z-\frac{(3-\sqrt{5})}{2}\right)}}$$

$$z = \frac{-3\pm\sqrt{5}}{2}$$

$$\text{Re}_{\frac{-3+\sqrt5}{2}}[\cdot] = \frac{(3-\sqrt5)^n}{i\left(\frac{-3+\sqrt5+3+\sqrt5}{2}\right)}$$

$$\left.(z^2+3z+1)\right|_{z=\frac{-3+\sqrt5}{2}}$$

$$= \left(\frac{-3+\sqrt5}{2}\right)^2 + \left(\frac{-3+\sqrt5}{2}\right) + \frac{4}{4}$$

$$= 3-\sqrt5.$$

$$\text{Ans} = \frac{2\pi \cancel{i}(3-\sqrt5)^n}{\cancel{i}\sqrt5} = \frac{2\pi}{\sqrt5}(3-\sqrt5)^n.$$

9/11/18.

① Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ if $a>|b|$.

$z=e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$d\theta = \frac{dz}{iz}$.

$$= \int_C \frac{dz}{iz\left(a+\frac{b}{2i}(z-1/z)\right)}$$

$$= \frac{1}{i}\int_C \frac{dz}{az+\frac{bz^2}{2i}-\frac{b}{2i}} = \frac{1}{i}\int_C \frac{dz}{\frac{b}{2i}\left(z^2+az\left(\frac{2i}{b}\right)-1\right)}$$

$$= \frac{2}{b}\int_C \frac{dz}{\left(z^2+\frac{2azi}{b}-1\right)} = \frac{2}{b}\int_C \frac{dz}{(z-\alpha)(z-\beta)}$$

$\alpha+\beta = \frac{-2ai}{b}$

$\alpha\beta = -1.$

$$= 2\pi i \times \frac{2}{b} \cdot \lim_{z\to\alpha} \frac{\cancel{(z-\alpha)}}{\cancel{(z-\alpha)}(z-\beta)} = \frac{2}{b} \times \frac{2\pi i}{(\alpha-\beta)}$$

$$= \frac{2}{b} \times \frac{2\pi i}{\sqrt{(\alpha+\beta)^2-4\alpha\beta}} = \frac{2}{b} \times \frac{2\pi i}{\sqrt{\left(\frac{-2ai}{b}\right)^2-4(-1)}} = \frac{2}{b} \times \frac{b\times 2\pi i}{2\sqrt{b^2-a^2}}$$

$$= \frac{2\pi i}{\sqrt{b^2-a^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

② Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta} d\theta = \int \frac{1-\cos2\theta}{2(5-4\cos\theta)} d\theta$

$d\theta = \frac{dz}{iz}$

$$= \text{Re}\left(\int_C \frac{1-\cos2\theta - i\sin2\theta}{2(5-4\cos\theta)} \frac{dz}{iz}\right.$$

$$= \text{Re}\left(\int_C \frac{1-e^{i2\theta}}{2\left(5-4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)\right)} \frac{dz}{iz}.\right.$$

$$= Re\left(\int_c \frac{1-z^2}{10-4z-\frac{4}{z}}\cdot\frac{dz}{iz}\right) = Re\left(\frac{1}{i}\int_c \frac{1-z^2}{10z-4z^2-4}\,dz\right)$$

$$= Re\left(\frac{1}{4i}\int_c \frac{-1+z^2}{z^2-\frac{5}{2}z+1}\,dz\right)$$

$$\frac{\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2}$$

$$= Re\left(\frac{1}{4i}\int_c \frac{z^2-1}{(z-\frac{1}{2})(z-2)}\,dz\right)$$

$$\frac{\frac{5}{2}\pm\frac{3}{2}}{2} \quad \frac{5\pm3}{4}$$

$$\frac{2}{4}, \frac{8}{4} \quad 2, \frac{1}{2}.$$

$$= Re\left(\frac{1}{\cancel{4i}_2}\times \cancel{2\pi i}\times \lim_{z\to 1/2} \frac{(z^2-1)\cancel{(z-1/2)}}{\cancel{(z-1/2)}(z-2)}\right).$$

$$= Re\left(\frac{\pi}{2}\times\frac{(\frac{1}{4}-1)}{(\frac{1}{2}-2)}\right) = Re\left(\frac{\pi}{\cancel{2}}\times\frac{(-\cancel{3})\times\cancel{2}}{4\times(\cancel{3})}\right)$$

$$= \frac{\pi}{4}.$$

③ Evaluate $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2}$, $a^2<1$

$$d\theta = \frac{dz}{iz}.$$

$$\int_c \frac{Re\{e^{i2\theta}\}\,d\theta}{1-\frac{2a}{2}(e^{i\theta}+e^{-i\theta})+a^2} = \int_c \frac{Re\{z^2\}\,dz}{iz(1-a(z+1/z)+a^2)}.$$

$$= Re\frac{1}{i}\int_c \frac{z^2\,dz}{z-az^2-a+\cancel{z}\,a^2z} = Re\cdot\frac{1}{i}\int_c \frac{z^2dz}{-a\left(z^2-\left(\frac{a^2+1}{a}\right)z+1\right)}$$

$$= Re\times\frac{1}{i(-a)}\int_c \frac{z^2dz}{z^2-\left(\frac{a^2+1}{a}\right)z+1}.$$

$$\cancel{\frac{\frac{a^2+1}{a}\pm\sqrt{\left(\frac{a^2+1}{a}\right)^2-4}}{2}}$$

$$= Re\,\frac{1}{i}\int_c \frac{dz\,z^2}{z(1-az)-a(1-az)}.$$

$$\frac{a^2+1}{a}\pm\sqrt{}$$

$$= Re\int_c \frac{-iz^2\,dz}{(z-a)(1-az)} = Re\int_c \frac{z^2\,dz}{(z-a)(az-1)}.$$

$$= Re\left[\lim_{z\to a}\frac{\cancel{(z-a)}\,iz^2}{\cancel{(z-a)}(az-1)}\right]\times 2\pi i = Re\left[\frac{(-1)\times z^2}{(az-1)}\right]_{z=a}$$

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$$= Re\left[\frac{(-1)(a)^2}{(a^2-1)}\right] \times 2\pi$$

$$= \frac{2\pi a^2}{1-a^2}.$$

13/11/18.

Practice Problems :-

1.) $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$

$\cancel{f(z) = \frac{1}{z^4}.}$ $\qquad$ $\cancel{f_1(z) = \frac{-\pi\cot\pi z}{(z)^4}.}$

$\cancel{\text{Res}\, f_1(z)\Big|_{z=0} = \frac{1}{3!} \times \frac{d^3}{dz^3}\left(z^4 \times \frac{-\pi\cot\pi z}{z^4}\right)}$

$\cancel{= \frac{-\pi}{6} \times \left(\frac{d^2}{dz^2}\left(-\pi\,\text{cosec}^2\pi z\right)\right).}$

$\cancel{= \frac{\pi^2}{6} \times \frac{d}{dz}\left(-\pi \times 2\,\text{cosec}\,\pi z \times\right.}$

2.) $\sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6}.$

$$f(z) = \frac{1}{z^2} \times \pi\cot\pi z = \frac{1}{z^2} \times \pi \cdot \frac{\cos\pi z}{\sin\pi z}$$

$$= \frac{\frac{1}{z^2} \times \pi \times \left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} + \cdots\right)}{\underbrace{\left(\pi z - \frac{\pi^3 z^3}{6} + \frac{\pi^5 z^5}{120} + \cdots + \infty\right)}_{\times}}$$

$(1-x)^{-1} = 1 + x + x^2 + \cdots + \infty.$

$$= \frac{1}{z^3}\left(1 + \frac{\pi^2 z^2}{6} - \frac{\pi^5 z^5}{120} + \cdots + \infty\right)\left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} + \cdots\right)$$

$$= \left(-1/2 + 1/6\right)\pi^2 = \frac{\pi^2}{3}.$$

$\sum_{n=-\infty}^{\infty} 1/n^2 = \frac{\pi^2}{3} \Rightarrow \sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6}.$ $\qquad$ $f(n) = 1/n^2,\ n \neq 0.$ $\quad$ $f(n) = 0,\ n = 0.$

1) $\sum_{n=1}^{\infty} 1/n^4 = \frac{\pi^4}{90}$.

$$f_1(z) = \frac{1}{z^4} \times \pi \cot \pi z = \frac{1}{z^4} \cdot \frac{\pi \cos \pi z}{\sin \pi z}$$

$$= \frac{\pi \cdot \left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} + \dots \infty\right)}{z^4 \left(\pi z - \frac{\pi^3 z^3}{6} + \frac{\pi^5 z^5}{120} + \dots\right)}$$

$$= \frac{1}{z^5} \left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} + \dots\right)\left(1 + \frac{\pi^2 z^2}{6} - \frac{\pi^4 z^4}{120} + \dots\right).$$

$$= \frac{1}{z^5}\left(1 + \frac{\pi^2 z^2}{6} - \frac{\pi^4 z^4}{120} - \frac{\pi^2 z^2}{2} + \dots\right).$$

Compare arguments of z^4.

$$= \left(\frac{1}{24} - \frac{1}{12} - \frac{1}{120} + \frac{1}{36}\right) \pi^4 \longrightarrow$$ when taking inverse U square it i.e x^2 is third term $\Rightarrow \left(\frac{\pi^2 z^2}{6}\right)^2$.

$$= \frac{-\pi^4}{45}.$$

$$\sum_{n=-\infty}^{\infty} 1/n^4 = +\frac{\pi^4}{45} \Rightarrow \sum_{n=1}^{\infty} 1/n^4 = \frac{\pi^4}{90}.$$

Probability.

(Ω, F, P).

$\downarrow$

F-algebra of subsets of Ω.

$\downarrow$

closed under complements, arbitrary unions, countable intersections

(i) $P(\mathcal{E}) \geq 0$; $\mathcal{E} \subseteq \Omega$ (F)

(ii) $P(\phi) = 0$

(iii) $P\left(\bigcup_{i=1}^{n} \mathcal{E}_i\right) = \sum_{i=1}^{n} P(\mathcal{E}_i)$; $\mathcal{E}_i \cap \mathcal{E}_j = \phi$.

* $X: \mathcal{E} \longrightarrow (-\infty, x]$.

$Pr\left((-\infty, x]\right) = F_X(x)$. $\qquad f_X(x) = \frac{dF_X(x)}{dx}$.

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$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$y = g(x) \qquad P_Y(y) = \frac{P_X(g^{-1}(y))}{g'(x)\big|_{x=g^{-1}(y)}}$$

$$X : \mathcal{E} \to (-\infty, x_1] \times (-\infty, x_2] \times \cdots (-\infty, x_N].$$

$$F_X(x_1, x_2, \ldots x_N).$$

$$f_X(x_1, x_2, \ldots x_N) = \frac{\partial^N F_X(x_1, x_2, \ldots x_N)}{\partial x_1 \partial x_2 \ldots \partial x_N}.$$

$$\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_N} f_X(x_1, x_2, \ldots x_N)\,dx = F_X(x_1, x_2, \ldots x_N).$$

* $y_1 = g_1(\underline{x})$, $y_2 = g_2(\underline{x})$ — — $y_n = g_n(\underline{x})$

$$P(\underline{y}) = \frac{P(\underline{x})}{\|J\|}$$

$$J = \text{mod.}\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_N} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_N} \\ & & & \vdots \\ \frac{\partial y_N}{\partial x_1} & \cdots & \cdots & \frac{\partial y_N}{\partial x_N} \end{vmatrix}$$

*We take Jacobian of only differentiable functions.

*__Last Question of Mid1.__

$$y = \mathrm{Tr}(\underline{x}^T\underline{x}) + \log|X|.$$

$$\sum_i \sum_j x_{ij}^2 \qquad 2\underline{X} + X^{-1}$$

$$\Delta X^{-1}\left(\log|X + \Delta X|\right) - \log|X|.$$

$$\frac{P_X(\underline{x})}{\|2\underline{X} + \underline{X}^{-1}\|} = P(y).$$

20/11/18.

* Orthogonal Curves:

Two curves are said to be orthogonal to each other, when they intersect at right angle at each of their points of intersection.

$f(z) = u+iv$. $u(x,y)=c_1$ & $v(x,y)=c_2$. u & v form an orthogonal system.

$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy = 0$. $\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1$. $\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$.

$$\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2.$$

$$m_1m_2 = \left(\frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(\frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right) = \left(\frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(\frac{-\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}\right) = -1 \rightarrow [\text{C.R Conditions}]$$

$m_1m_2 = -1 \Rightarrow$ u & v are orthogonal.

* Harmonic Function:

Any function which satisfies the Laplace Eq^n is known as Harmonic function.

Theorem: If $f(z)=u+iv$ is an analytic function, then u, v are both harmonic functions.

Proof: $f(z)=u+iv$ is analytic $\Rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$\frac{\partial^2 u}{\partial x^2} = +\frac{\partial^2 v}{\partial y \partial x} \quad \text{—④} \qquad \& \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}. \quad \text{—③}$$

③+④ $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$. $\quad$ $\parallel^{ly}$ $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0$ $\Big\} \rightarrow$ Laplace Equation.

$\therefore$ u & v are harmonic conjugates.

Such functions u, v are called conjugate harmonic functions if $u+iv$ is also analytic function.

* Calculating Harmonic Conjugate:

$u(x,y) = 2x(1-y) \longrightarrow$ V is harmonic conjugate.

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = -\frac{\partial U}{\partial y}dx + \frac{\partial U}{\partial x}dy \longrightarrow CR$$

$$= -(-2x)dx + (2-2y)dy + C.$$

$$= 2x\,dx + (2dy - 2y\,dy) + C$$

$$V = x^2 + 2y - y^2 + C.$$

* For u, v to be harmonic conjugates they need to satisfy CR condition and Laplace Equation.

Ex21:- $u(x,y), v(x,y)$ – harmonic functions, P.T $\left[\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]$ is an analytic function of $z = x+iy$.

Sol.) u, v – harmonic $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow (1)$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \rightarrow (2)$

$$F(z) = R + iS = \left(\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right)$$

$$\frac{\partial R}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} \qquad \frac{\partial S}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial R}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial S}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial R}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \longrightarrow \text{(putting values from (2))}$$

$$\Rightarrow \frac{\partial R}{\partial x} = \frac{\partial S}{\partial y} \qquad \text{||}^{ly} \quad \frac{\partial R}{\partial y} = -\frac{\partial S}{\partial x}$$

∴ CR conditions satisfied, hence function is analytic.

* Method to find Conjugate function :-

Case I :- $f(z) = u + iv$, u is known. to find v, conjugate function.

Method :- $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy.$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy. \Rightarrow v = -\int \overbrace{\frac{\partial u}{\partial y}}^{M} dx + \int \overbrace{\frac{\partial u}{\partial x}}^{N} dy$$

$$v = -\int M dx + \int N dy$$

$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$ $\quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

Since, u is conjugate $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \underbrace{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

satisfies condition for integration
So you can integrate M & N.

Case II :- v is given, find u.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \Rightarrow u = \int \frac{\partial v}{\partial y}dx - \int \frac{\partial v}{\partial x}dy.$$

Ex 30) Deduce $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

① — $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

② — $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Sol) ① $\Rightarrow$

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\cdot\frac{\partial^2 v}{\partial r \partial \theta} \quad \text{— ③}$$

② $\Rightarrow \frac{\partial^2 u}{\partial \theta^2} = -r\cdot\frac{\partial^2 v}{\partial \theta \partial r}$ — ④.

③ $+ \frac{1}{r}$① $+ \frac{1}{r^2}$④ — gives result.

* $f'(z) = \frac{r}{z}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$ ~~or r [...]~~

Sol)
$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = r\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}\right)$$

$$= r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)$$

$$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\right)$$

$$= x f'(z) + iy f'(z) = (x+iy)f'(z) = zf'(z)$$

$$\Rightarrow f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) //$$

22/11/18.

Milne thomson Method.

① u given

$$\phi_1(x,y) = \frac{\partial u}{\partial x} \quad \phi_2(x,y) = \frac{\partial u}{\partial y}$$

$$f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\} dz + c$$

② v given

$$\psi_2(x,y) = \frac{\partial v}{\partial x}, \quad \psi_1(x,y) = \frac{\partial v}{\partial y}.$$

$$f(z) = \int \{\psi_1(z,0) - i\psi_2(z,0)\} dz + c$$

③ u−v given ⇑

$F(z)$ → $U = u-v$ — given
→ $V = u+v$.

$$f(z) = \frac{F(z)}{1+i}$$

④ u+v given. ⇑

$F(z)$ → $U = u-v$
→ $V = u+v$ — given

$$f(z) = \frac{F(z)}{1+i}$$

* $$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right).$$

* Line Integral :-

$\oint_C f(z)\,dz$ — contour integral (initial & final pt of integration coincide such that c is a closed curve.)

$$dz = dx + idy.$$

$$\oint_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int(v\,dx + u\,dy)$$

* Cauchy's Integral theorem :-

If a function $f(z)$ is analytic & its derivative $f'(z)$ continuous at all points inside & on a simple closed curve c, then $\int_C f(z)dz = 0$.

* If $f(z)$ is analytic in the Region R b/w two simple closed curves c_1 & c_2 then. $\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$.

* Cauchy Integral formula :-

If $f(z)$ is analytic within and on a closed curve C, and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_z \frac{f(z)}{z-a}\,dz$$

$$\boxed{f^{n}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}}$$

$$f'(a) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^2}.$$

* Transformation

For every pt (x,y) in z-plane, the relation $w=f(z)$ defines a corresponding point (u,v) in w-plane. We call this "transformation or mapping into w-plane". z_0 maps to w_0 (w_0 – image of z_0)