8/8/18

DSAA.

* frequency of carrier higher than that of a signal.

$y_n = x_n + \omega_n$ / $y_n = \omega_n$

detected ✓ Target noise

$y_n = x_n + i_n + \omega_n$.

Interference (clutter).

* $x(t) = \sin(at)$.

$\forall t$ we know $x(t)$ $\therefore$ $x(t)$ is determinental.

$x(t) = \sin(at) + ~~\cot~~ \omega_t$.

for $x(t)$ we cannot give exact value, we can tell the range with the help of ω_t.

$X(t)$ → independent variable

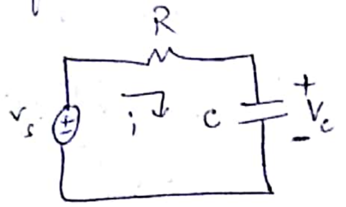
dependent (function)

9/8/18

* Continuos Time & Discrete Time Signals.

Signals can be represented in many ways, in all cases the information in the signal is contained in a pattern of variations of some form.

Ex 1 :-

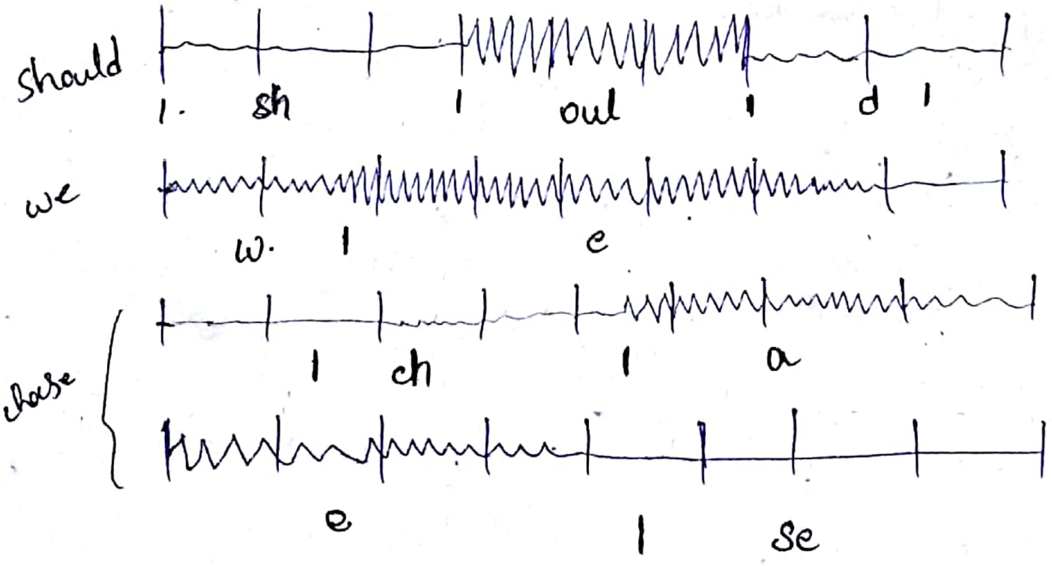


Patterns of variation over time in Source & capacitor voltages v_s & v_c is an signal.

Ex 2 :-

Car responding to applied force f from engine & to a retarding frictional force ev proportional to its velocity v

Ex 3 :- Human voice (accoustic pressure).

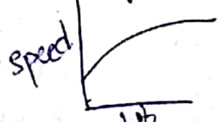


(one in our cases)

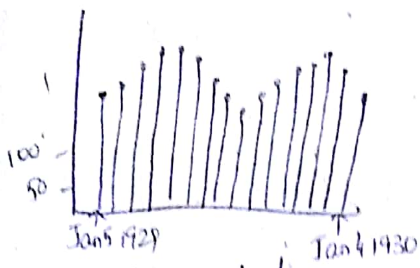
* Signal — function of one or more independent variables.

Continuos time signal → Independent variable is Continuos & thus these signals are defined for a continuum of values of an Independent variable → Any Real Number

Ex :- Speech signal as a f(t).
Atm. pressure as f(altitude).



Discrete time signals :- These are defined at discrete times & consequently, for these signals, the independent variables takes on only a discrete set of values.



Stock market

Ex: Avg Budget, crime rate, pounds of fish caught. tabulated against. family size, total population or type of fishing vessel resp.

* Continuos - time independent signals - t.
  Discrete " " " - n.

* For continuos we will enclose independent variable in ( ) → $x(t)$
  for discrete " " " " " in [ ]. → $x[n]$ → only integer values

avg budget for $2\frac{1}{2}$ family mem not possible. ← $x[n]$

Exception

* Discrete signals arises from sampling of continuos signals.
  ↓
  they represent successive samples of an underlying phenomenon for which independent variable is continuos.
  Ex: Aircraft position, velocity & heading for an autopilot & music for an audio system.

* Signal Energy & Power :-

10/8/18

Examples of Signals.

* Image $I(x,y)$ — 2D

Intensity ← I; (x,y) → spatial Coordinates.

* Air pressure $P(h)$ → altitude.

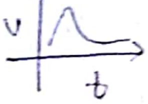
Temperature $T(h)$

Wind speed $W(h)$

h — Independent variable

t

Voltage in Electrical ckt



$v(t)$.

$v(t)$, $i(t)$

In Signals, $E = \int_{t_1}^{t_2} |x(t)|^2 dt$ — E - Energy of the signal.

$x(t)$ → Signal representation. Continuos.

Sequence/ Discrete signal. → $x[n]$.

$E = \sum_{n=0}^{N-1} |x[n]|^2$.

* Energy tells nature of signal (strong/weak)

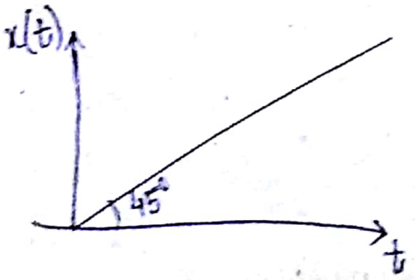
↓

Used to compare signals of same wavelength.

* <u>Infinite duration signal</u> :-

Continuos $E_\infty = \lim_{t\to\infty} \int_{-t}^{t} |x(t)|^2 dt$.

Discrete $E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2$.



$x(t) = t$, $t > 0$.

Ramp Signal.

$E_\infty = \lim_{T\to\infty} \int_0^T t^2 dt = \lim_{T\to\infty} \frac{T^3}{3} \to \infty$.

$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = \lim_{T\to\infty} \frac{T^3}{3\cdot 2T} \longrightarrow \infty$.

$$P_\infty = \lim_{t\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt \qquad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$$

continuous

$$P_\infty = \frac{E_\infty}{2T}$$

discrete

$$P_\infty = \frac{E_\infty}{2N+1}$$

* Classification of signals.

(i) Energy $\quad$ (ii) Power $\quad$ (iii) None of above

$E_\infty < \infty \qquad P_\infty < \infty$.

* $x[n] = 4$

$n \in W$.

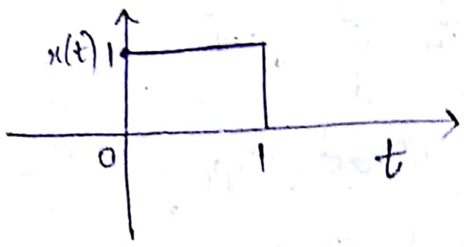
$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} 16 \longrightarrow \infty.$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} 16 = \lim_{N\to\infty} \frac{(2N+1)16}{2N+1} = 16 < \infty.$$

$\therefore x(n) = 4$ — power signal.

* Mostly power signals are Periodic signals.

*



$x(t) = 1.$ — Energy signal.

$$E = \int_0^1 1\,dt = 1.$$

$$P = \lim_{T\to\infty} \frac{1}{2T}(1) = 0.$$

* $x(t) = e^{-|t|} \quad -\infty < t < \infty$.

$e^{-t} \quad 0 < t < \infty \qquad e^{t} \quad -\infty < t < 0$.

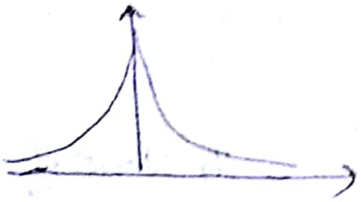
$$\cancel{E_\infty = \lim_{t\to\infty} \int_{-t}^{t} e^{2t} dt}$$

$$E_\infty = \lim_{t\to\infty} \int_0^t e^{-2t} dt + \int_{-t}^{0} e^{2t} dt$$

$$= \lim_{t\to\infty} \left(\frac{e^{-2t}}{-2}\right)_0^t + \left(\frac{e^{2t}}{2}\right)_{-t}^0 = \lim_{t\to\infty} \left(\frac{-e^{-2t}}{2} + \frac{1}{2} + \frac{1}{2} - \frac{e^{-2t}}{2}\right)$$

$E_\infty = 1$ //

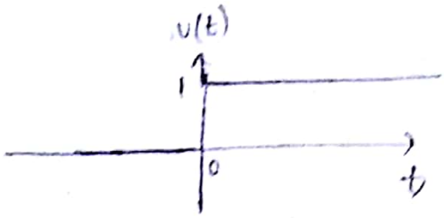
$$= \lim_{t\to\infty} \frac{1}{2}\left(e^{2t} - e^{-2t}\right)$$



$x(t) = e^{-|t|} \longrightarrow$ Energy Signal.

* Unit Step Signal :-

$$U(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

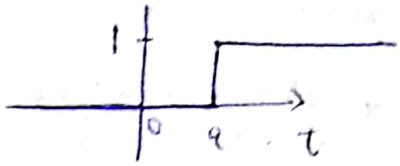


$E = \infty$, $P = \infty$

Ex:- Something that starts from Rest & continues to be in the state of Motion. (Motion of a Car.)

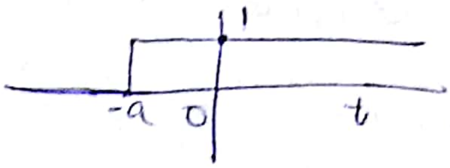
$U(t-a)$ $a>0$.

$t-a>0 \Rightarrow t>a$.



$U(t+a)$ $a>0$.

$a+t>0 \Rightarrow t>-a$



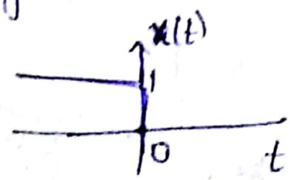
* Causal Signal → Existing in the Real World (Happens).

Non-Causal. → Existing on both sides of zero

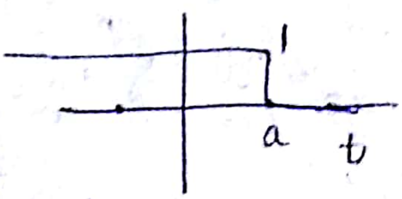
Anti-Causal → Starts at zero & goes backwards

↓

$$x(t) = U(-t) = \begin{cases} 1 & t<0 \\ 0 & t>0 \end{cases}$$



$U(a-t)$



H.W

① $x(t) = \cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$.

Compute Energy & power for $-\infty < t < \infty$.

(2) $x[n] = \cos\frac{2\pi k_0 n}{M}$ $\quad 0 \le n \le M-1$

Find Energy and Power.

(1) $x(t) = \cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$ $\quad -\infty < t < \infty$

$$E_\infty = \lim_{t\to\infty} \int_{-t}^{t} \cos^2\left(\frac{\pi t}{2}+\frac{\pi}{4}\right) dt = \frac{1}{2}\lim_{t\to\infty}\int_{-t}^{t}\left(\cos\left(\pi t+\frac{\pi}{2}\right)+1\right) dt$$

$$= \frac{1}{2}\lim_{t\to\infty}\int_{-t}^{t}(-\sin(\pi t)+1)\,dt$$

$$= \frac{1}{2}\lim_{t\to\infty}\left(+\frac{\cos(\pi t)}{\pi}+t\right)_{-t}^{t} = \frac{1}{2}\lim_{t\to\infty}\left(\frac{\cos\pi t}{\pi}+\cancel{t}+\frac{\cos\pi t}{\pi}+\cancel{t}\right)$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\lim_{t\to\infty}\underbrace{\cos\pi t}_{\text{finite}} + \frac{\cancel{2}t}{\cancel{2}} = \infty. \Rightarrow \underline{E=\infty}.$$

$$P_\infty = \frac{1}{2T}E_\infty = \frac{1}{2T}\lim_{t\to\infty}\cos\pi t + \frac{1}{4T}\lim_{t\to\infty}2t = 1/2.$$

(2) $$E_\infty = \sum_{n=0}^{M-1}\cos^2\frac{2\pi k_0 n}{M} = \cos\frac{2\pi k_0}{M}+\cos\frac{4\pi k_0}{M}+\dots+\cos\frac{(M-1)\pi k_0}{M}$$

$$= \cos a + \cos 2a + \dots + \cos(M-1)a$$

$\left(\frac{2\pi k_0}{M} = a\right)$

$$= \frac{\sin\left(\frac{(M-1)a}{2}\right)\cdot\cos\left(\frac{(M)a}{2}\right)}{\sin\left(\frac{a}{2}\right)}$$

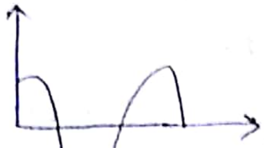
$$E = \frac{\sin\left(\frac{(M-1)\pi k_0}{M}\right)\cdot\cos(\pi k_0)}{\sin\left(\frac{\pi k_0}{M}\right)}$$

$\underline{P = 0.}$

13/8/18.

Signal Classification and Basic Signals.

(1.) $x(t) = \cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$ $\quad -\infty < t < \infty$.



$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2\left(\frac{\pi t}{2} + \frac{\pi}{4}\right) dt. \qquad T = NT_0$$

Using Periodicity $\longrightarrow$

$$P = \lim_{N\to\infty} \frac{N}{2NT_0}\int_{-T_0}^{T_0} \frac{1+\cos\left(\pi t + \frac{\pi}{2}\right)}{2}\, dt.$$

$$P = \lim_{N\to\infty} \frac{1}{2T_0}\cdot\frac{1}{2}\cdot 2T_0 + \frac{1}{2T_0}\cdot\frac{1}{2}\underbrace{\int_{-T_0}^{T_0} \cos\left(\pi t + \frac{\pi}{2}\right)}_{0}$$

$$P = \frac{1}{2}.$$

(2.) $x[n] = \cos\frac{2\pi k_0 n}{M}$ $\quad 0 \le n \le M-1$.

$$E = \sum_{n=0}^{M-1} \cos^2\frac{2\pi k_0 n}{M} = \sum_{n=0}^{M-1}\frac{1}{2} + \frac{1}{2}\sum_{n=0}^{M-1}\cos\frac{4\pi k_0 n}{M} + \frac{1}{2}\sum_{n=0}^{M-1} Re\left(e^{j\frac{4\pi k_0 n}{M}}\right.$$

$$\frac{1}{2} Re\left[\sum_{n=0}^{M-1} e^{j\frac{4\pi k_0 n}{M}}\right].$$

$$E = \frac{M}{2} + \frac{1}{2} Re\left(\sum_{n=0}^{M-1}\left(e^{j\frac{4\pi k_0}{M}}\right)^n\right).$$

$$\sum_{n=0}^{M-1} r^n = \frac{1-r^M}{1-r}$$

$$E = \frac{M}{2} + \frac{1}{2} Re\left[\frac{1 - \overbrace{e^{j4\pi k_0}}^{\to 1}}{1 - e^{-j\frac{4\pi k_0}{M}}}\right] \to 0$$

$$E = \frac{M}{2}.$$

Single channel – one source
dimensional – independent variable.

$$S_1(t) = A \sin 3\pi t$$

Ex :- Single channel & one dimensional.

* Multichannel & Multi dimensional signals → Very large array.

$$S(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \\ S_3(t) \end{bmatrix} \begin{matrix} \text{source1} \\ \text{source2} \\ \text{source3} \end{matrix} \Bigg\} \text{– 1 dimensional Multi channel.}$$

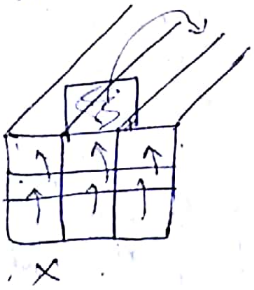
Ex :- 3 channel signal due to an Earthquake, ECG signals via 12 electrode sensors, Video games where u can connect with friends.

* Multidimensional :-

3D Signal

$I(x, y, t)$

x – cells (along columns)
y – cells (along rows)
t – time (along 3rd dim).



y

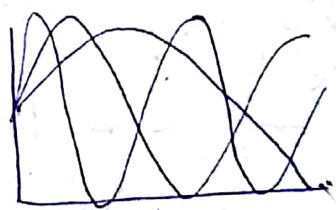
x

Each cell (x,y).
X & Y can be
(i) latitude & longitude values.
(ii) Integers representing uniform sized grid cells
(iii)

* Random Signal (weiner process)

Values change with time.

Sinusoids (deterministic).



t →

14/8/18

Signal Energy & Power

$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$

instantaneous power

$$\text{Energy} = \int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt. \qquad \text{Avg power} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt.$$

* $x(t) = \{ \quad t_1 \le t \le t_2.$ $\qquad x(n) \quad n_1 \le n \le n_2$

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt \qquad E = \sum_{n=n_1}^{n_2} |x[n]|^2.$$

$$P = \frac{E}{t_2 - t_1} \qquad P = \frac{E}{n_2 - n_1 + 1}$$

$x(t) \quad E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt.$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

$x(n) \quad E_\infty = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$$

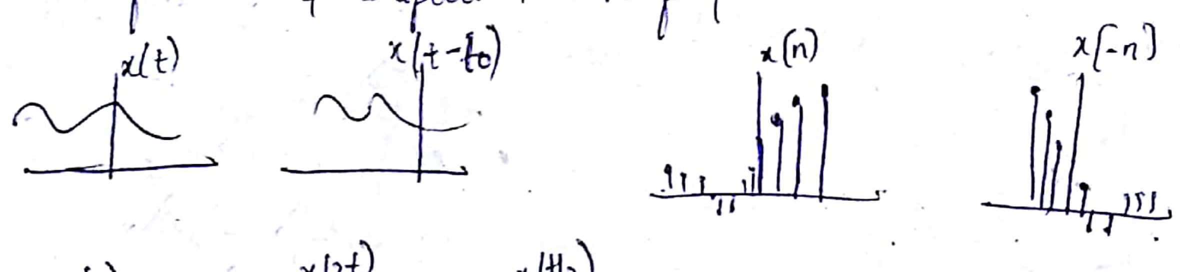
* Transformations of Independent Variable :-

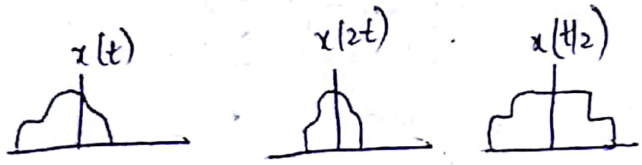
Time shift. $\quad x[n],\ x[n-n_0] \quad , \quad x(t-t_0), x(t)$

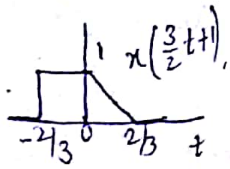
Time Reversal. $\quad x[-n], x[n]$

$x(2t),\ x(t),\ x(t/2)$

$x(\alpha t+\beta) \rightarrow$ preserves shape of $x(t)$, except that signal is linearly streched if $|\alpha|<1$, linearly compressed if $|\alpha|>1$, reversed in time if $\alpha<0$ & shifted in time if β is non zero.







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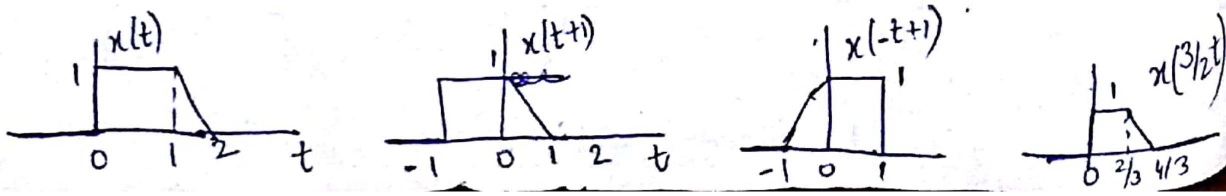
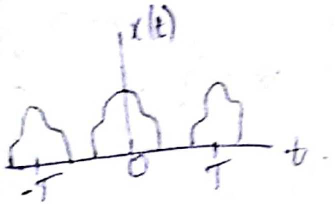


* Periodic Signals :- (aperiodic x-opposite)

$x(t) = x(t+T)$. → unchanged by time shift of T.



if $x(t)$ is periodic with period T, then

$x(t) = x(t+mT)$ m is any integer.

Smallest +ve value of T is called as fundamental period.

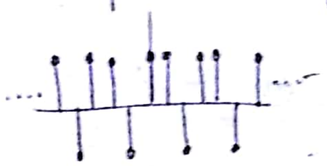
↓

Doesn't hold if $x(t)$ is a constant bcoz it is periodic for any T.

$x[n] = x[n+N]$ → unchanged by time shift of N.

Same def holds for fundamental period

↓



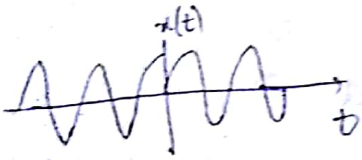
$N_0 = 3$

Ex:-

$$x(t) = \begin{cases} \cos(t) & \text{if } t<0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}$$

$\cos(t+2\pi) = \cos t$ }
$\sin(t+2\pi) = \sin t$ } periodic

But discontinuous at '0' → so non-periodic.



* Even & Odd Signals :-

Even $x(-t) = x(t)$

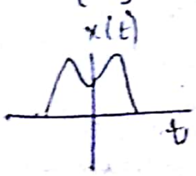
$x[-n] = x[n]$

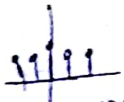
odd $x(-t) = -x(t)$

$x[-n] = -x[n]$

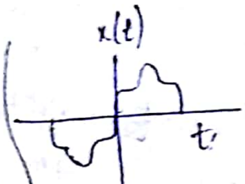
$t=0 / n=0$

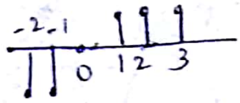
$x(t)=0 / x[n]=0$.





$$x(n) = \begin{cases} \frac{1}{2} & n<0 \\ 1 & n=0 \\ \frac{1}{2} & n>0 \end{cases}$$





$$x(n) = \begin{cases} \frac{1}{2} & n>0 \\ 0 & n=0 \\ -\frac{1}{2} & n<0 \end{cases}$$

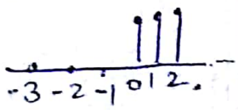
Evenpart of $x(t) = \frac{1}{2}\{x(t)+x(-t)\}$.

odd $x(t) = \frac{1}{2}\{x(t)-x(-t)\}$.

Even, odd

↑

Given signal decomposition.



$$x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n<0 \end{cases}$$

* Exponential & Sinusoidal Signals :-

$x(t) = Ce^{\alpha t}$.

C, α – complex no's

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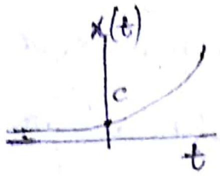
c, α are real — Real Exponential Signals.

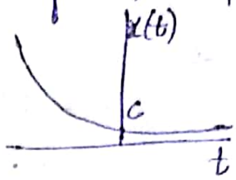
$\alpha > 0$, $x(t)$ is growing exponential

Ex: chain rxn's, atomic explosions.



$\alpha < 0$ — decaying exponential

Ex: Radioactive decay, Responses of RC, damped mechanical.



$\alpha = 0 \rightarrow$ constant.

Periodic ~~class~~ Complex Exponential & Sinusoidal Signals.

α - purely imaginary.

$$x(t) = e^{j\omega_0 t} \Rightarrow e^{j\omega_0 t} \underset{\text{Periodic}}{=} e^{j\omega_0 (t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$$

for periodicity $\Rightarrow e^{j\omega_0 T} = 1$.

If $\omega_0 = 0$, $x(t) = 1$ for any T.

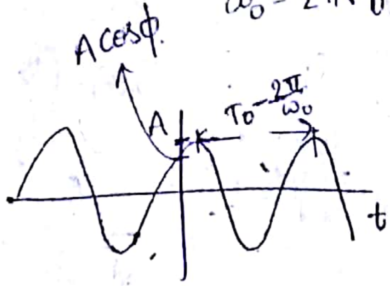
$\omega_0 \neq 0$, fundamental period T_0 is $T_0 = \frac{2\pi}{|\omega_0|}$

$e^{j\omega_0 T}$ & $e^{-j\omega_0 T}$ have same fundamental period.

A signal closely related to periodic complex exponential is sinusoidal

$$x(t) = A\cos(\omega_0 t + \phi). \qquad \phi - \text{rad}, \ \omega_0 - \text{rad/s}.$$

$\omega_0 = 2\pi f_0$. $\qquad T_0 = \frac{2\pi}{|\omega_0|}$.



Ex: Natural response of LC ckt, SHM, Acoustic pressure of a tone.

1.* By using Euler's Equation, Complex exponential can be written in terms of sinusoidal signals.

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t.$$

Same T_0.
$$A\cos(\omega_0 t + \phi) = \underbrace{\frac{A}{2}e^{j\phi}}_{} e^{j\omega_0 t} + \underbrace{\frac{A}{2}e^{-j\phi}}_{} e^{j\omega_0 t}.$$

Complex amplitudes

$A\cos(\omega_0 t+\phi) = A\,Re\{e^{j(\omega_0 t+\phi)}\}$.

$A\sin(\omega_0 t+\phi) = A\,Im\{e^{j(\omega_0 t+\phi)}\}$.

$1/T_0 \rightarrow$ fundamental frequency.

$\omega_0 = 0 \rightarrow x(t)=$ Constant, periodic for any value of T.

$\therefore$ ~~for~~ $T_0=$ undefined, fundamental frequency $= 0$.

zero rate of oscillation.

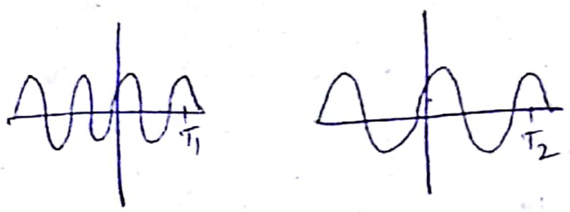
$x(t) = e^{j\omega_0 t}$

$$E_{period} = \int_0^{T_0} |e^{j\omega_0 t}|^2 dt.$$

$$= \int_0^{T_0} 1 \cdot dt = T_0.$$

$x_1(t) = \cos\omega_1 t$.

$x_2(t) = \cos\omega_2 t$.



$\omega_1 > \omega_2$ & $T_1 < T_2$

$$P_{period} = \frac{1}{T_0} E_{period} = 1$$

There are infinite no of periods as t ranges from $-\infty$ to ∞, The total energy integrated over all time is infinite. However, each period of the signal looks exactly the same. Since the avg power of signal equals 1 over each period, averaging over multiple periods gives a power 1.

$\therefore$ Complex exponential signal has finite avg power equal to.

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 dt = 1.$$

for a complex exponential to be periodic $e^{j\omega_0 t} = 1$.

$\omega T_0 = 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$

$\omega_0 = \frac{2\pi}{T_0}$. ($\omega$ - integral multiple of ω_0).

$\phi_k(t) = e^{jk\omega_0 t}$. $\quad k=0,\ \phi_k(t) =$ constant.

$k \neq 0$, $\phi(t)$ is periodic with fundamental frequency ...

& fundamental period $= \frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$

Kth harmonic $\phi_k(t)$ is periodic with period T_0 as well as it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0.

Ex :- $x(t) = e^{j2t} + e^{j3t}$

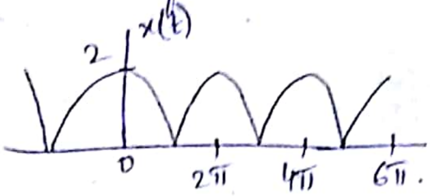
$$x(t) = e^{j2.5t}\left(e^{-j0.5t} + e^{j0.5t}\right)$$

$$x(t) = 2e^{j2.5t} \cos(0.5t).$$

$$|x(t)| = 2|\cos(0.5t)|. \qquad |e^{j2.5t}| = 1.$$

↓

full-wave rectified sinusoid.



23/8/18.

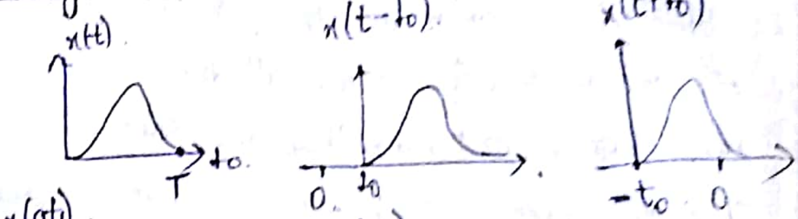
Continuous Signal

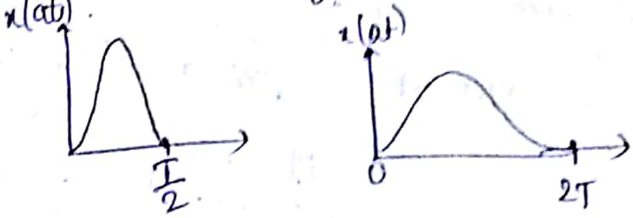
$x(t)$.

① Delay $x(t-t_0)$

② Advance $x(t+t_0)$

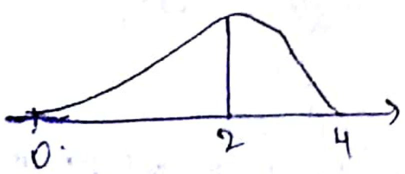
③ Scaling $x(at)$

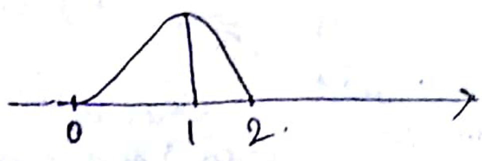




$a > 1$ (Compression). $a < 1$ (Expansion).

*. $x(t)$. $x(2t)$.



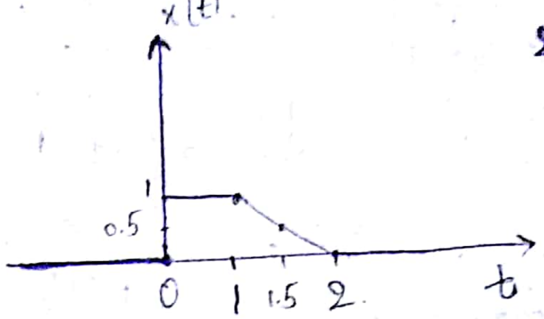


$2t = 0 \quad t = 0$

$2t = 2 \quad t = 1$

$2t = 4 \quad t = 2$

* $x(t) = \begin{cases} 0 & t<0 \\ 1 & 0<t<1 \\ 2-t & 1<t<2 \\ 0 & t>2 \end{cases}$

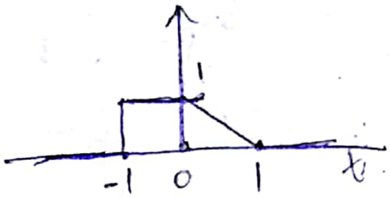


$2-t$

$2-1.5=0.5$

$2-1.1=0.9$

* $x(t+1)$



$x_1(t) = x(t+1) = \begin{cases} 0 & t<-1 \\ 1 & -1<t<0 \\ 1-t & 0<t<1 \\ 0 & t>1 \end{cases}$

* $x(at-b)$

precedence of transformations.

① Shift the signal

$v(t) = x(t-b)$

② Scale the intermediate signal.

$y(t) = v(at) = x(at-b)$

* $x_2(t) = x(-t+1)$

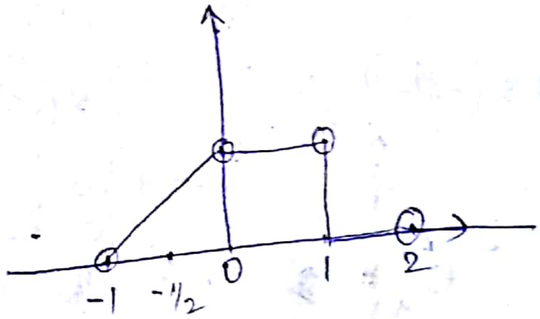
$-t+1$

$t=-1 \rightarrow t_{new} = 2$

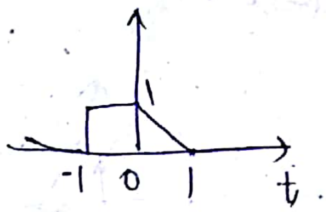
$t=0 \rightarrow t_{new} = 1$

$t=1 \rightarrow t_{new} = 0$

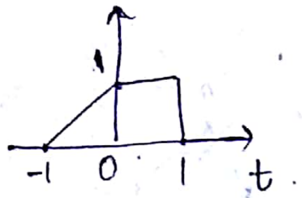
$t=2 \rightarrow t_{new} = -1$



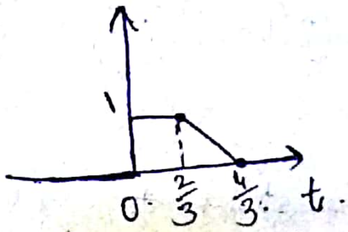
* $x(t+1)$



$x(-t+1)$



* $x\left(\frac{3}{2}t\right)$



$t=0 \quad t_{new} = 0$

$t=-1 \quad t_{new} = -\frac{3}{2}$

$t=1 \quad t_{new} = \frac{3}{2}$

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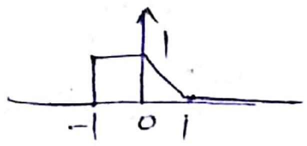
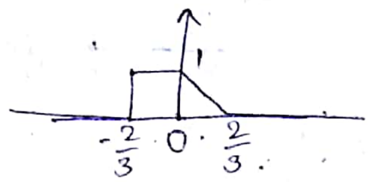
(5) plot $x\left(\frac{3t}{2}+1\right)$

$x(t+1)$.



$x\left(\frac{3}{2}t+1\right)$.



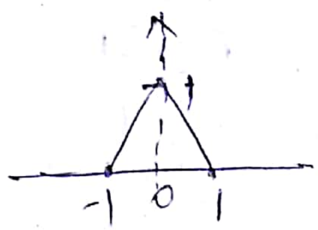
$\frac{3}{2}t = -1$. $t = -\frac{2}{3}$

$\frac{3t}{2} = -\frac{1}{2}$ $t = -\frac{1}{3}$.

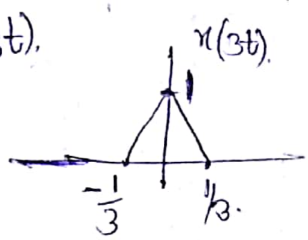
$\frac{3}{2}t = \frac{1}{2}$ $t = \frac{1}{3}$.

$\frac{3}{2}t = 1$ $t = \frac{2}{3}$.

*

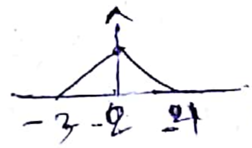


$x(3t)$.



* $x(3t+2)$.

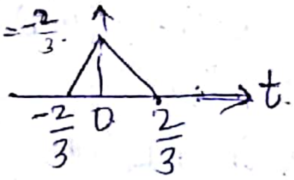
$x(t+2)$



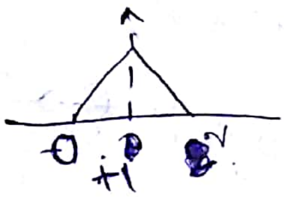
$x(3t+2)$.

$3t=-3$ $t=-1$

$3t=-2$ $t=-\frac{2}{3}$.



* $x(-2t-1)$.



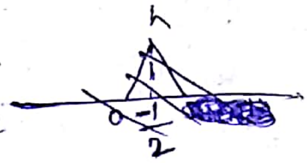
$x(-2t-1)$.

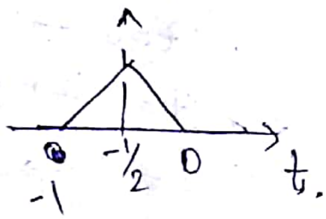
$-2t=0$

$t=0$.

$-2t=\frac{1}{2}$

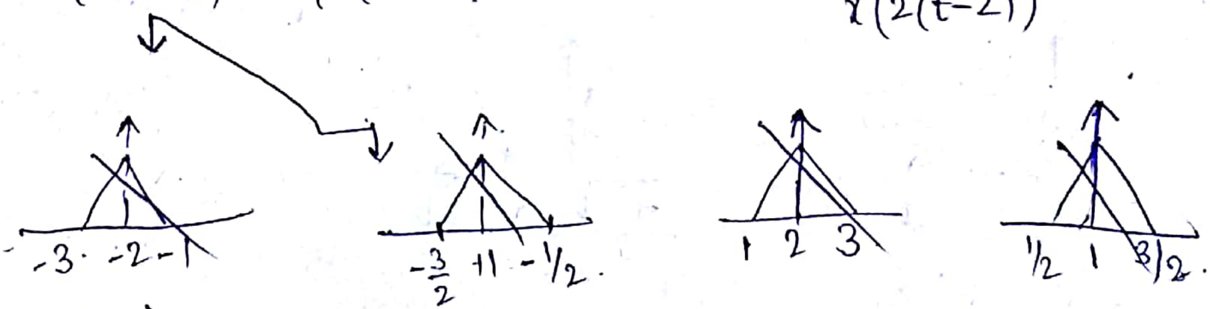
$t=1/4$.





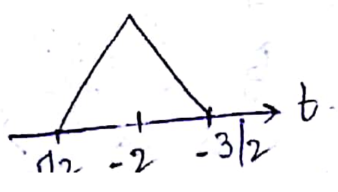
(1) $x(2(t+2))$ $x(2(t-2))$.

$x(2(t-2))$

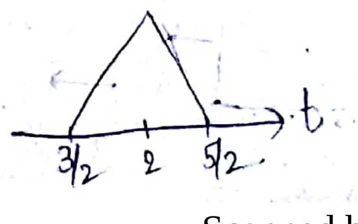


~~x(t+2)~~

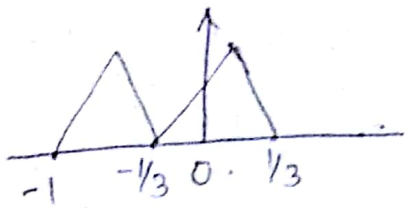
$x(2(t+2)) = x(2t+4)$.



$x(2(t-2))$



* $x(3t) + x(3t+2)$.



Matlab.

Append → $x = [x, 13]$.

if $x = [1,2,3,4]$ → row

$x = [x : [5:9]]$.

$y = x'$

$y = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ ~~y = [y ; [5 : 9]]~~

* $x = 0 : \frac{pi}{100} : 2pi$;

$y = \sin(x)$;

plot(x, y) ;

plot(x,y, ' ') → has many colours & symbols.

* ① Available

principal, In Rate

② Knowledge

Interest = principal x In Rate ; Balance = principal + Interest.

③ Display the Result

* A = n! → Time of execution (tic ; toc ; . time = toc)

(i) for loop (ii) Recursion (iii) Any Other.

* Create two vectors x, y

$z = \sum_{i=1}^{N} \text{for } x_i y_i$

x = input ('vector 1') y = input ('vector 2').

z = mydotproduct (x, y) ; disp(z)

* function dp= mydotprod (x, y)
{
dp= sum(x .*y).
}
return;

24/8/18.

Periodic Signals.

$x(t+T) = x(t)$.

Complex Exponential.

$e^{j\omega_0(t+T)} = e^{j\omega_0 t} \Rightarrow e^{j\omega_0 T} = 1.$

$\omega_0 T = 2\pi n.$

$\boxed{T = \frac{2\pi}{\omega_0} n}$

Min value of $T_0 = \frac{2\pi}{\omega_0}\ (n=1)$
↓
fundamental period.

$\omega_0 = \frac{2\pi}{T_0} \longrightarrow$ fundamental frequency.

* $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t.$

$Ae^{j(\omega_0 t+\phi)} = A\cos(\omega_0 t+\phi) + jA\sin(\omega_0 t+\phi).$

Sinat a → rational or multiple of π.

① $x(t) = e^{j2t} + e^{j3t}$

$|x(t)| = A|\cos\omega_1 t| \quad A = ?,\ \omega_1 = ?$

$x(t) = e^{j2t} + e^{j3t} = e^{j(2.5-0.5)t} + e^{j(2.5+0.5)t}.$

$= e^{j2.5t}\left(e^{-j0.5t} + e^{j0.5t}\right)$

$= 2e^{j2.5t}\cos\frac{1}{2}t.$

$|x(t)| = 2\underbrace{|e^{j2.5t}|}_{\text{always }1.}\left|\cos\frac{t}{2}\right|. \Rightarrow A = 2.\quad \omega_1 = \frac{1}{2}.$

$$x(t) = e^{jat} + e^{jbt} = e^{j\left(a+\frac{b}{2}-\frac{b}{2}\right)t} + e^{j\left(b+\frac{a}{2}-\frac{a}{2}\right)t}$$

$$= e^{j\left(\frac{a+b}{2}\right)t}\left(e^{j\frac{a-b}{2}t} + e^{-j\frac{a-b}{2}t}\right)$$

$$\boxed{|x(t)| = 2\cos\frac{a-b}{2}t}$$

* Discrete.

$$x(n+N) = x(n)$$

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \Rightarrow e^{j\omega_0 N} = 1 \Rightarrow \omega_0 N = 2\pi m$$

$m, N \in \mathbb{N}$

$$\omega_0 = 2\pi \times \frac{m}{N}$$

$$N = m\cdot\frac{2\pi}{\omega_0} \Rightarrow N_0 = \min(m)\cdot\frac{2\pi}{\omega_0}$$

natural no.

* $x[n] = e^{j\omega_0 n}$

$$e^{j(\omega_0+2\pi k)n} = e^{j\omega_0 n} \Rightarrow e^{j2\pi k n} = 1$$

Periodic in frequency.

* $e^{j(5.1)\pi n} = e^{j4\pi n} \times e^{j1.1\pi n}$

$= e^{j1.1\pi n}$

0 — 2π $-\pi$ — π

$e^{j(5.1)\pi n} = e^{j6\pi n} \times e^{-j0.9\pi n}$

$= e^{-j0.9\pi n}$

* $x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$

① Periodic? ② If yes, $N_0 = ?$

$3\pi - \frac{2\pi}{3}$

$\frac{6\pi}{3}$

$2 - \frac{3}{4}$

~~$x[n] = e^{j2\pi n} + e^{-j\frac{4\pi}{3}n} + e^{j\frac{5\pi}{4}n}$~~ $\}\to$ check.

$$\frac{2\pi}{3} = \frac{m}{N}2\pi \qquad \frac{3\pi}{4} = \frac{m}{N}2\pi$$

$$N = 3 \qquad \frac{m}{N} = \frac{3}{8} \Rightarrow N = 8$$

$m = 3$

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$N_0 = LCM(N_1, N_2) = 24.$

$n=0$	1	2	3	4	5	6	7	8	9	–	12	15	–	18	21	24
x_1 ⊙			⊙			⊙			⊙		⊙	⊙		⊙	⊙	⊙
x_2 ⊙								⊙				⊙				⊙

* $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t.$

$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\frac{\pi}{3}} = 6.$ $\qquad T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\frac{\pi}{4}} = 8.$

$LCM(T_1, T_2) = T_0 = 24.$

* $x[n] = \cos\frac{\pi}{4}n + \sin\frac{\pi n}{8} - 2\cos\frac{\pi n}{2}.$

$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\frac{\pi}{4}} = 8.$ $\qquad T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\frac{\pi}{8}} = 16.$ $\qquad T_3 = \frac{2\pi}{\omega_3} = \frac{2\pi}{\frac{\pi}{2}} = 4$

$T_0 = LCM(T_1, T_2, T_3) = 16.$

* $x(t) = \cos t + \sin\sqrt{2}t.$

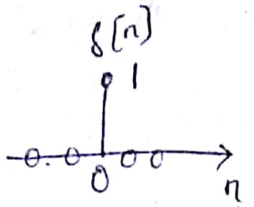
$\downarrow$

Not periodic, Irrational

28/8/18.

* Unit Impulse Sequence:-

$\delta[n] = 1, \; n = 0$
$\quad\quad\;\; 0, \; n \neq 0.$



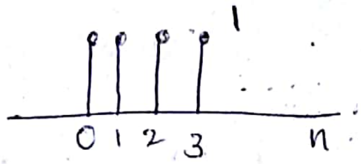
$\delta[n-2] = 1, \; n = 2.$
$\quad\quad\quad\;\; 0 \; ; \; n \neq 2.$

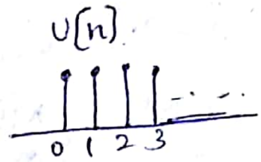
* $\delta[n-n_0] = 1, \; n = n_0.$
$\quad\quad\quad\quad 0, \; n \neq n_0.$

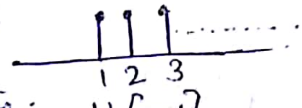
$\delta[n+n_0] = 1, \; n = -n_0.$
$\quad\quad\quad\quad 0, \; n \neq -n_0.$

* Unit Step Sequence:-

$u[n] = 1, \; n \geq 0.$
$\quad\quad\; 0, \; n < 0.$





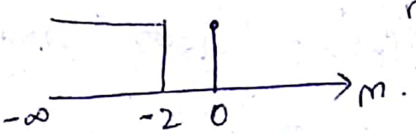


$\boxed{\delta[n] = u[n] - u[n-1]}$ → differentiation in discrete.
→ first backward difference.

$\boxed{u[n] = \sum_{m=-\infty}^{n} \delta[m].}$

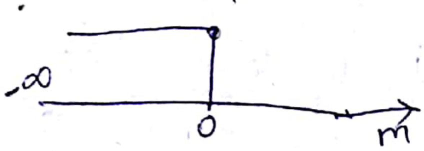
limits of summation not fixed.

$n<0$



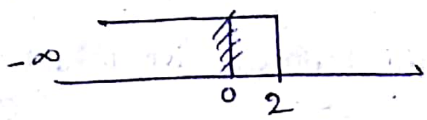
$n = -2$
$u[n] = \sum_{-\infty}^{-2} \delta[m]$
$= 0.$

$n = 0:$



$u[n] = \sum_{-\infty}^{0} \delta[m] = 0 + \ldots + 1 = 1.$

$n > 0$



$u[n] = \sum_{-\infty}^{3} \delta[m] = 0 + \ldots + 1 + 0 + 0 = 1.$

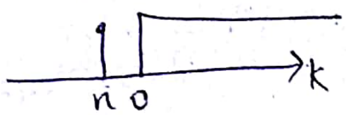
limits fixed for summation

$\boxed{u[n] = \sum_{k=0}^{\infty} \delta[n-k]}$

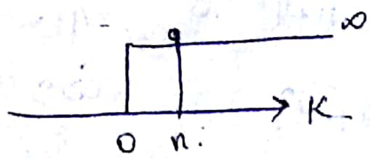
$m = n-k.$
$k = n-m.$

$m = -\infty \Rightarrow k = \infty.$
$m = n \Rightarrow k = 0.$

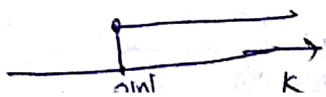
$n < 0$



$n > 0$



$n = 0$



$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$ → location. Sampling property

→ value at the location

$n_0 = 0$ $x[n]\delta[n] = x[0]\delta[n]$

* $x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k]$

Rewrite $x[n]$ as $u[Mn-n_0]$ find M, n_0 ?

*sol.

$n=0$. $x[n] = 1-(0) = 1$.

$n<0$. $x[n] = 1-(0) = 1$

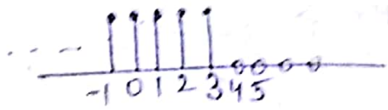
$n=1$ $x[n] = 1-0 = 1$.

$n=3$ $x[n] = 1$.

$n=4$ $x[n] = 1-1 = 0$.

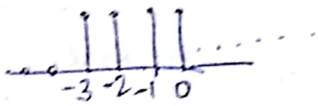
$n>4$ $x[n] = 0$.

$x[n] = 1, \; n \le 3$
$\quad\quad\;\; 0, \; n > 3$.

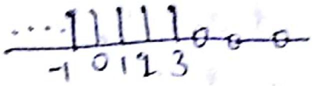


$x[n] = u[-n+3]$

$u[n+3]$



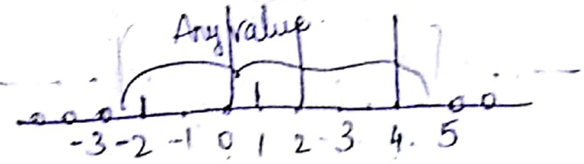
$u[-n+3]$



$M = -1$

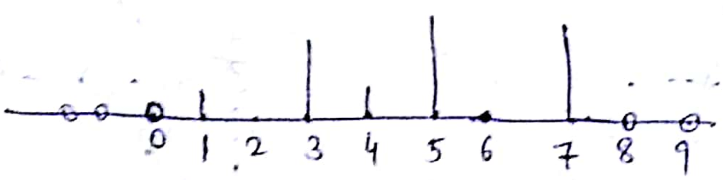
$n_0 = -3$.

* Given a $x[n] = 0$, $n<-2$ & $n>4$.



what is the interval in which $x[n-3]$ is definitely zero ?

-3 will go to 0.



$x[n-3] = 0$ $n<1$ & $n>7$.

$x[n+4] = 0$ $n<-6$ & $n>0$.

$x[-n]$ $n>2$ & $n<-4$

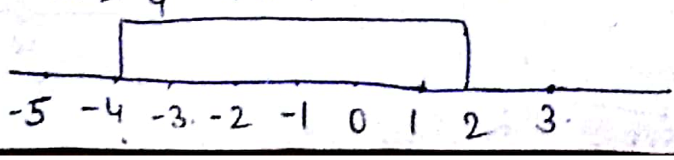
$-n<-2$

$n>2$

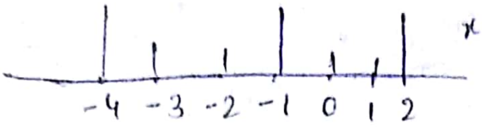
&

$-n>4$

$n<-4$.



$x[2-n]$



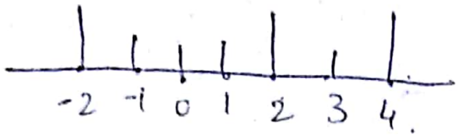
$x(n+2)$.

$2-n<-2$

$-n<-4$

$n>4$.

$x[-n+2]$.



$n<-2$ & $n>4$.

$2-n>4$.

$-n>2$

$n<-2$.

⓫ Periodicity Signals:

① Given $y_1(t)=x(2t)$; $y_2(t)=x(t/2)$ (i) if $x(t)$ is periodic is $y_1(t)$ periodic & find period (ii) if $y_1(t)$ is periodic is $x(t)$ periodic.
(iii) Same for $y_2(t)$.

② Discrete case :- Given a $x[n]$ define $\underbrace{y_1[n]=x[2n]}_{\downarrow \text{down sampling}}$ & ~~$y_2[n]=x[n/2]$~~

$y_2[n] = \begin{cases} x[n/2] & n\text{-even} \\ 0 & n\text{-odd} \end{cases}$ Upsampling.

(i) If $x[n]$ is periodic is $y_2[n]$ periodic & vice versa? (ii) same for $y_1[n]$

Answers.

① $x(t+T)=x(t)$.

$y_1(t+T_1) = x(2t+2T_1)$.

assume $2T_1=T$.

$= x(2t+T) = x(2t) = y_1(t)$.

$y_1(t+T) = y_1(t) \rightarrow$ given.

Goal :- $x(t+T_1)=x(t)$.

$x(t+T_1) = y\left(\frac{t+T_1}{2}\right)$.

$= y\left(\frac{t}{2}+\frac{T_1}{2}\right)$.

assume $\frac{T_1}{2}=T$.

$= y\left(\frac{t}{2}+T\right)$.

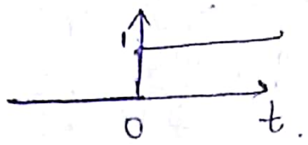
$x(t+T_1) = y\left(\frac{t}{2}\right) = x(t)$.

29/8/18

Continuos Time signals.

$u(t) = 0 \quad t<0.$

$\quad\quad\ 1 \quad t>1.$ doubt

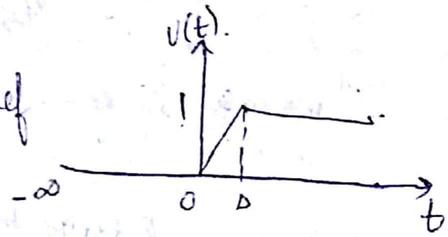


* Impulse Signal :-

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \Rightarrow \boxed{\delta(t) = \frac{d}{dt}(u(t))}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1. \qquad u[n] = \sum_{m=-\infty}^{n} \delta[m]. \qquad \tau = m\Delta\tau$$

def

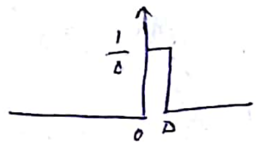


derivative $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t).$

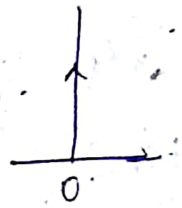
↓

↑ delta.

then $\frac{1}{\Delta}$ ↑.



when $\Delta \to 0$.



$$\delta(t) = \lim_{\Delta\to 0} \delta_\Delta(t).$$

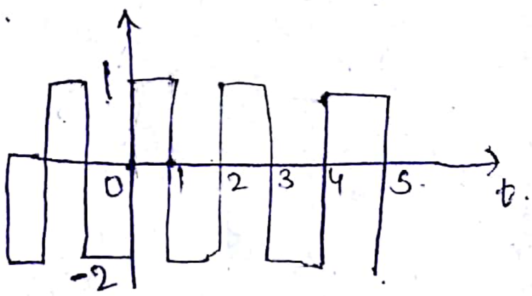
$$u(t) = \int_{-\infty}^{t} \delta(t)\,dt \quad \& \quad u(t) = \int_{0}^{\infty} \delta(t-\tau)\,dt.$$

$$x(t)\delta(t) = x(0)\delta(t).$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

$* \ x(t) = 1 \quad 0<t<1$

$\quad\quad\quad -2 \quad 1<t<2. \qquad T=2.$



$$v(t) = \frac{d}{dt} x(t)$$

(i) Plot $v(t)$.

(ii) $v(t) = A_1 g(t-t_1) + A_2 g(t-t_2).$ $\quad A_1, A_2, t_1, t_2.$

$$g(t) = \sum_{k=-\infty}^{a} \delta(t-2k). \qquad k=-\infty.$$

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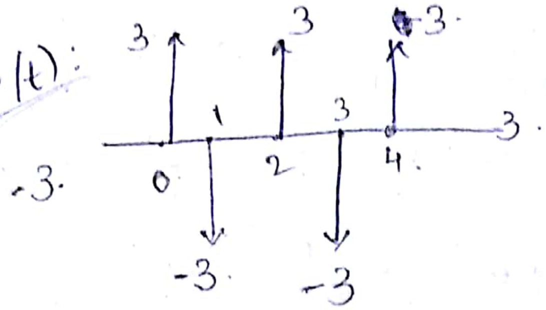
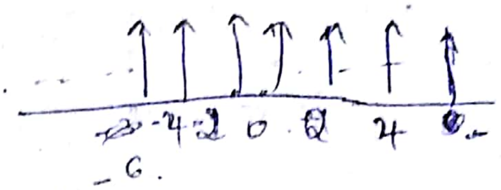
At 0 it jumps from -2 to 1 & so on.

$v(t)$:



$g(t) =$

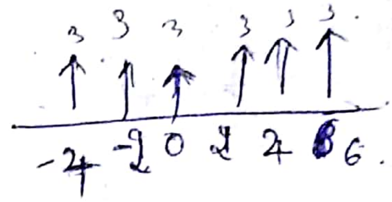


* $g(1) = \sum_{k=-\infty}^{\infty} \delta(1-2k) = 0$

$g(2) = \sum_{k=-\infty}^{\infty} \delta(2-2k)$

$v(t) = v_1(t) + v_2(t)$

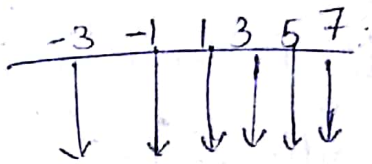
$v_1(t) =$



$v_1(t) = 3g(t)$; $A_1 = 3$, $t_1 = 0$.

$v_2(t) = -3g(t-1)$ $A_1 = -3$, $t_2 = 1$.

$v_2(t) =$



Answer :-

* $y_2[n] \begin{cases} x[n/2] & n\text{-even} \\ 0 & n\text{-odd} \end{cases}$

$x[n+N] = x[n]$.

Goal is. $y_2[n+N_2] = y_2[n]$.

$\Rightarrow y_2[n+N_2] = \begin{cases} x\left[\dfrac{n+N_2}{2}\right] & n+N_2 \text{ even} \\ 0 & n+N_2 \text{ odd} \end{cases}$

* $x\left[\frac{n}{2} + \frac{N_2}{2}\right]$.

If $\frac{N_2}{2} = N$. $N_2 = 2N$.

$x\left(\frac{n}{2} + N\right] = x\left[\frac{n}{2}\right]$. $n \to$ even.

0 n-odd.

$= y_2[n]$.

goal : $x(n+N_2] = x(n]$.

$y_2[n+N] = y_2(n]$.

$x[n+N_2] = x\left(\frac{n+N_2}{2}\right)$

$= x\left[\frac{2n+2N_2}{2}\right]$.

assume $2N_2 = N$.

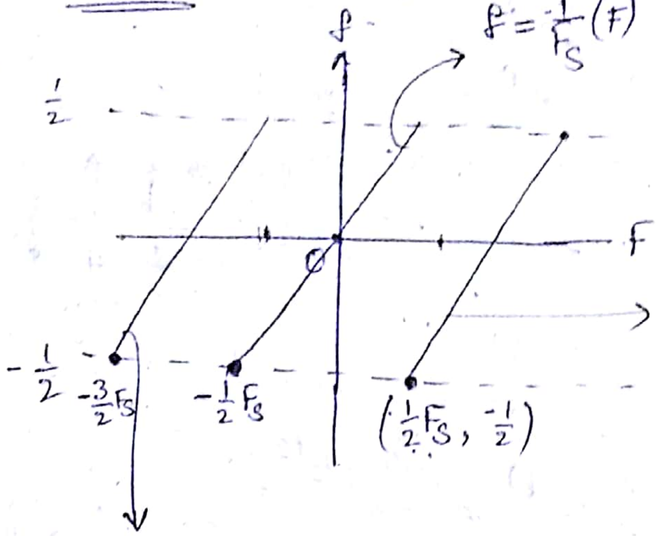
$x\left(\frac{2n+N}{2}\right) = y[2n+N]$ $2n+N$ even

$= y(2n]$.

$= x[n]$.

* Matlab

$f = \frac{1}{F_S}(F)$



$f = \left(\frac{1}{F_S}\right)F - 1$

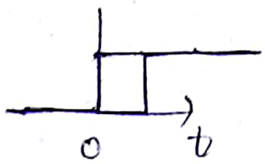
$-\frac{1}{2} = \frac{1}{2} + c \Rightarrow c = -1.$

$f = \frac{1}{F_S} \cdot F + 1$

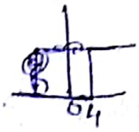
Assignment 1.

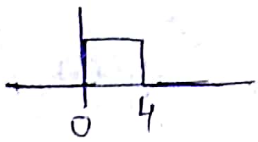
1) $x(t) = u(t) - u(t-4)$

$u(t) = 1 \quad t \geq 0$
$\quad\quad 0 \quad t < 0$



$u(t-4) = 1 \quad t \geq 4$
$\quad\quad 0 \quad t < 4.$



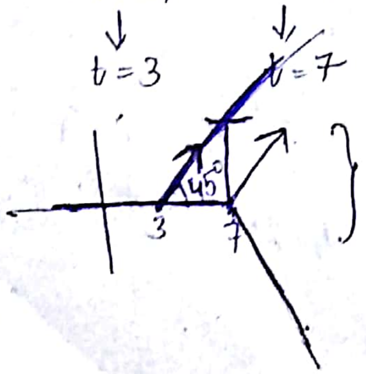


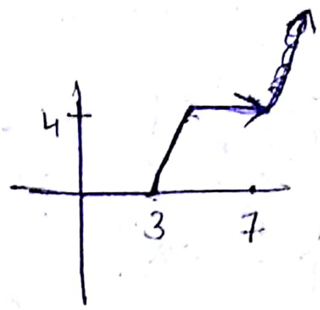
2) $r(t) = \begin{cases} t & , t \geq 0 \\ 0 & , t < 0 \end{cases}$

$r(t-3) - r(t-7).$

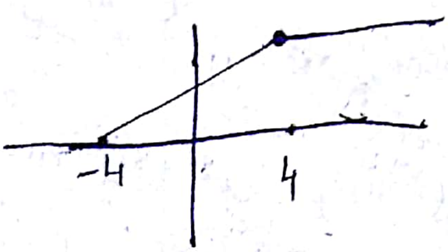
$t = 3 \quad t = 7$

$r(t-3) = \{$



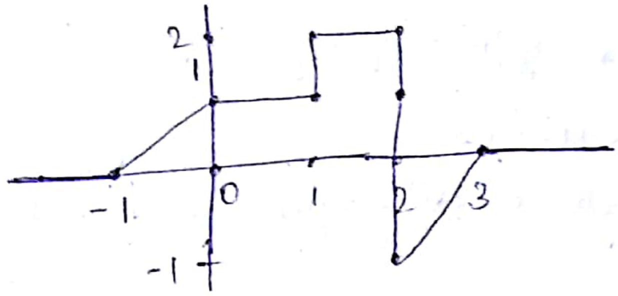


3) $r(t+4) - r(t-4).$



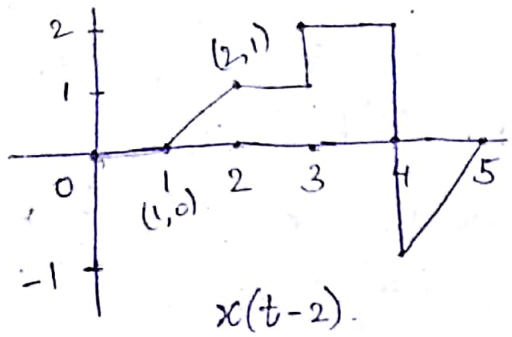
(2)

$$x(t) = \begin{cases} 0 & t \le -1 \\ 1+t & -1 \le t \le 0 \\ 1 & 0 \le t \le 1 \\ 2 & 1 \le t \le 2 \\ t-3 & 2 \le t \le 3 \\ 0 & t \ge 3 \end{cases}$$

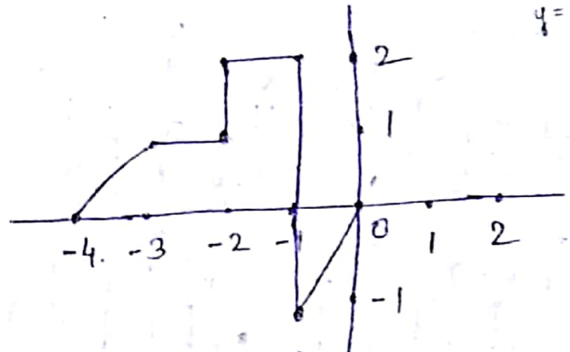


$x(t-2)$, $x(t+3)$, $x(2t-3)$, $x(-2t-1)$.

$1 = 2+c$
$c = -1$
$y = x-1$

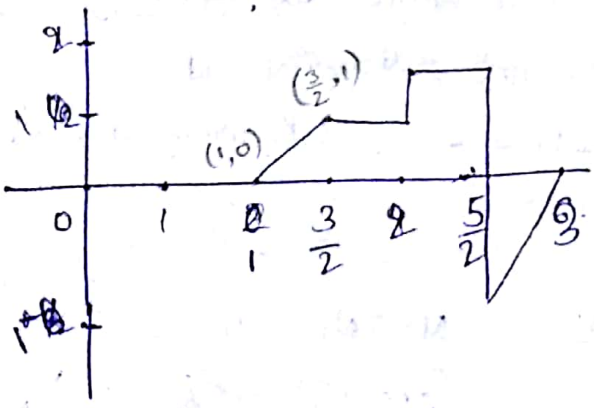


$x(t-2)$.

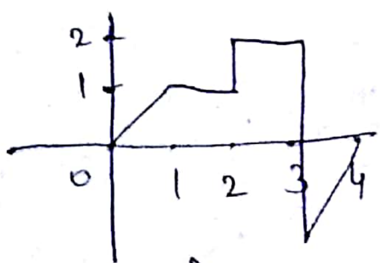


$x(t+3)$

$x(2t-3)$

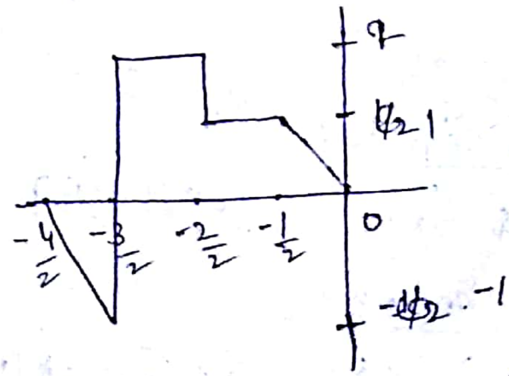


$x(-2t-1)$



$x(t-1)$.

$x(-2t-1)$.



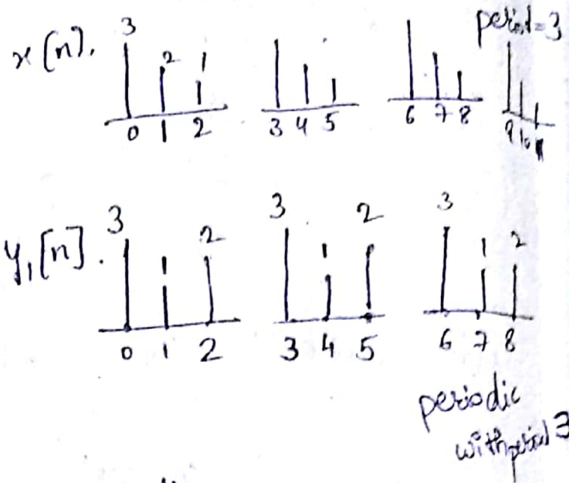
30/8/18.

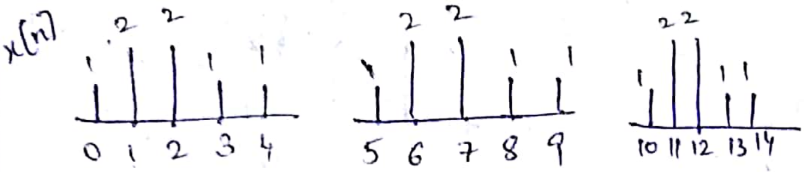
* $y_1[n] = x[2n]$.

$N \rightarrow$ odd.

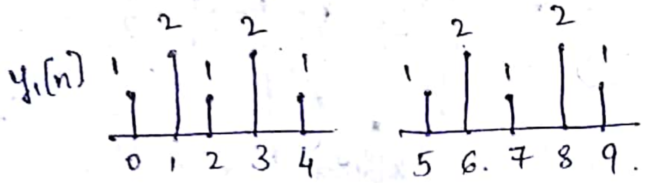
given $x[n+N] = x[n]$. is $y_1[n]$ periodic?

P.T $y_1[n+N_1] = \cancel{x[2n+2N_1]} = y[n]$.





periodic with period = 5.



periodic with period = 5.

$y_1[n+N_1] = x[2n+2N_1]$. $N_1 = N$.

$= x[2n+2N]$. (How will we know that $N_1 = N$?)

$\frac{N}{2}$ is not possible. (N-odd).

$= x[2n]$. N is odd so we can take minimum value of N_1 to be N.

$y_1[n+N_1] = y_1[n]$.

If N is even $y_1[n+N_1] = x[2n+2N_1]$ $N = 2N_1$.

$= x[2n+N]$ (since N is even)

$= x[2n] = y_1[n]$.

* **Even Signals** :-

$x(-t) = x(t)$. $x[-n] = x[n]$. Ex:- $\cos, x^2, x^4, \ldots$

* **odd signals** :-

$x(-t) = -x(t)$ $x[-n] = -x[n]$. Ex :- $\sin, x^3, \ldots$

If $x(t)$ is a signal, Even Component of signal is.

$$\boxed{Ev\{x(t)\} = \frac{x(t)+x(-t)}{2}.}$$

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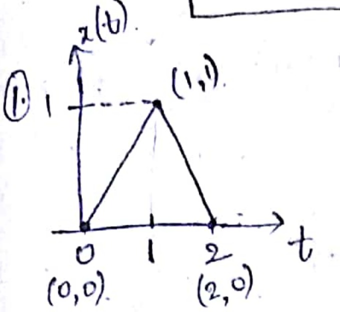
$$\boxed{Odd\{x(t)\} = \frac{x(t)-x(-t)}{2}}$$

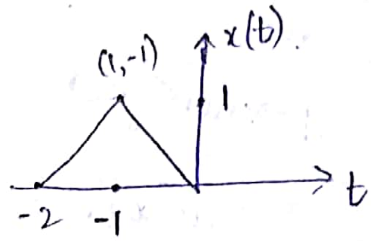
(1.)



$$x(t) = t \quad , \quad 0 \le t \le 1$$
$$= 2-t \; ; \quad 1 \le t \le 2$$

$x(-t) = -t \quad , \quad -1 \le t \le 0$

$= t-2 \; , \quad -2 \le t \le -1$

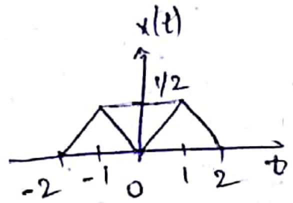


*what is the use of Even & odd Signals ?

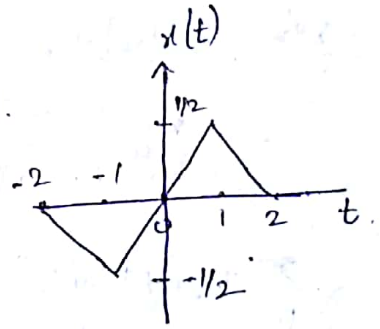
↓

When u know it is Even you need to analyse / compute for only half values, the rest follows the same thing

Even{x(t)}

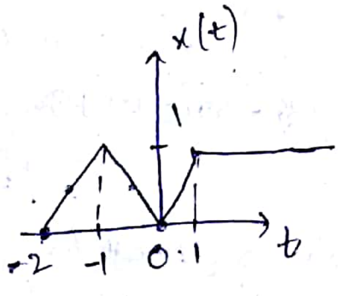


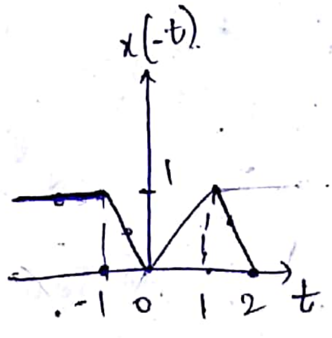
odd{x(t)}

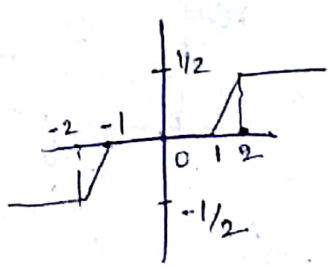


$$x(t) = Even\{x(t)\} + Odd\{x(t)\}$$

(2.)

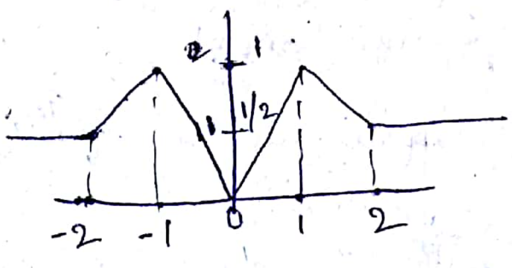


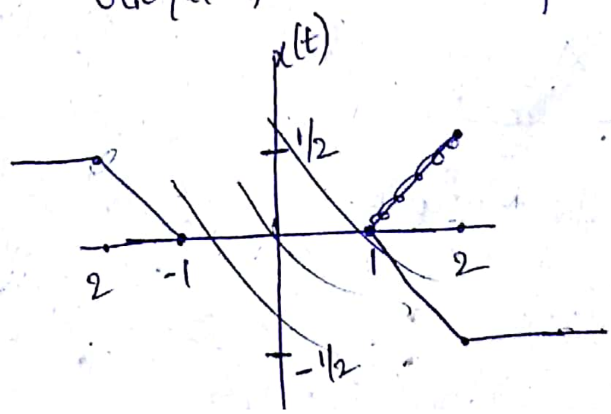




Odd{x(t)} →

Even{x(t)}





③ If $x[n]$ is an Odd Signal prove that $S = \sum_{n=-\infty}^{\infty} x[n] = 0$.

$$S = \sum_{n=-\infty}^{-1} x[n] + \overbrace{x[0]}^{0} + \sum_{n=1}^{\infty} x[n].$$

For odd $x(0) = 0$.

$n = -m$.

$$S = \sum_{m=1}^{\infty} x[-m] + \sum_{n=1}^{\infty} x[n] = -\sum_{m=1}^{\infty} \cancel{x}[m] + \sum_{n=1}^{\infty} \cancel{x}[n] = 0.$$

* $x_1[n]$ – odd, $x_2[n]$ – even.

$\boxed{x_3[n] = x_1[n]x_2[n]} \rightarrow$ odd Signal.

$$x_3[-n] = x_1[-n]x_2[-n] = -x_1[n]x_2[n] = -x_3[n].$$

$$x[n] = Ev(x[n]) + od(x[n]).$$

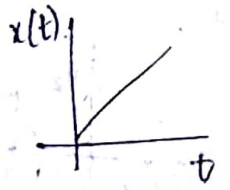
$$x[n] = x_e[n] + x_o[n].$$

$$\sum_n x^2[n] = \sum x_e^2[n] + \sum x_o^2[n].$$

$$\sum_n (x_e[n] + x_o[n])^2 = \sum_n x_e^2[n] + \sum_n x_o^2[n] + \underbrace{\sum_n 2x_e[n]x_o[n]}_{\text{odd } \& \ \sum \text{odd} = 0 \text{ from Q③}}.$$

Matlab:

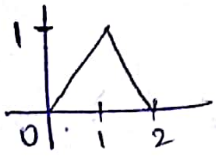
④ * Plot a Ramp Signal.



⑤. $y(t) = 3r(t+3) - 6r(t+1) + 3r(t) - 3u(t-3)$.

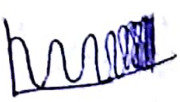
⑥ Plot e^{at} & all its variations.

⑦



Plot the derivative.

⑧ Plot a chirp Signal.



↓
(frequency changes)

⑨ Sinc. to approximate an Impulse.
↓
$\frac{\sin\theta}{\theta}$ (or) $\frac{\sin \pi\theta}{\pi\theta}$

Scanned by CamScanner

Assignment 1

(3)

1) $x_1(t) = \sin(t/2) + \sin(0.5\pi t)$

$\sin(t/2) \to 4\pi$, $\sin(0.5\pi t) \to 4$

$T = LCM(4\pi, 4)$

2) $x_2(n) = e^{\frac{j\pi n}{2}} - 1.5 e^{\frac{j\pi n}{3}}$

$\frac{\pi}{2} = \frac{m}{N} \times 2\pi$ $\qquad$ $\frac{\pi}{3} = \frac{m}{N} \times 2\pi$ $\qquad$ LCM of 4, 6 is 12

$\underline{\underline{N = 4m}}$ $\qquad$ $\underline{\underline{N = 6m}}$ $\qquad$ $\underline{\underline{N = 12}}$

$$\begin{array}{r|l} 2 & 4, 6 \\ 2 & 2, 3 \\ 3 & \end{array}$$

3) $x[n] = \cos[n/\sqrt{3}]$

irrational $\Rightarrow$ Not Periodic

4) $x[n] = \cos[\pi n/3] \cos[\pi n/4]$

$N = \frac{2\pi \times 3}{\pi} = 6$ $\qquad$ $N_0 = \frac{2\pi \times 4}{\pi} = 8$

$$\begin{array}{r|l} 2 & 6, 8 \\ & 3, 4 \end{array}$$

$\therefore$ $\underline{\underline{\text{Period} = LCM(N, N_0)}} = 24.$

31/8/18

$r(t) = t, t$

4/9/18

Systems

$SNR_{dB} = 10 \log \frac{P_{sig}}{P_{Noise}}$

* **Discrete Systems:-**

$x[n] \rightarrow$ $\boxed{\text{Discrete time system}}$ $\rightarrow y[n]$

Systems.

* Physical systems – interconnection of components, devices / subsystems
* A system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs.

* A continuous time system is a system in which continuous-time signals are applied & result in continuous-time output signals. We represent it by a notation $x(t) \rightarrow y(t)$.

$$x(t) \longrightarrow \boxed{\text{continuous time system}} \longrightarrow y(t)$$

$$x[n] \longrightarrow \boxed{\text{discrete time system}} \longrightarrow y[n]$$

Examples:-

(i) $V_0(t)$ = output

$V_i(t)$ = input

$$i(t) = \frac{V_i(t) - V_0(t)}{R}$$

$$i(t) = C\frac{dV_0(t)}{dt}$$

$$\frac{dV_0(t)}{dt} + \frac{1}{RC}V_0(t) = \frac{1}{RC}V_i(t)$$

(ii) $$\frac{dv(t)}{dt} = \frac{1}{m}\left[f(t) - \rho v(t)\right].$$

$m\rho v$ – resistance due to friction.

$$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

$$\frac{dy(t)}{dt} + a\underbrace{y(t)}_{\text{output}} = b\underbrace{x(t)}_{\text{input}}. \text{ — system}$$

(iii) $$\underbrace{y[n]}_{\substack{\text{balance at}\\ \text{the End of } n^{th} \text{ month}}} = 1.01\underbrace{y[n-1]}_{1\% \text{ interest}} + \underbrace{x[n]}_{\text{net deposit}}$$

(iv) In Ex-2 if $\frac{dv(t)}{dt}$ at $t = n\Delta$.

Resolving time into discrete intervals of length Δ.

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta}$$

$$v[n] = v(n\Delta) \text{ \& } f[n] = f(n\Delta)$$

$$v[n] - \frac{m}{(m+\rho\Delta)}v[n-1] = \frac{\Delta}{(m+\rho\Delta)}f[n].$$

5/9/18.

Linear Time Invariant Systems.

* Piece-wise linear approximation.

If the inputs to LTI can be represented as a linear combination of fundamental signals.
the output can be easily computed as a linear combination of responses to these basic signals.

Given $x[n] = \sum_k a_k \phi_k[n]$ $\qquad \phi_k[n] \rightarrow$ Basic signals.

then $y[n] = \sum_k a_k \psi_k[n]$ $\qquad \psi_k[n]$ Response to Basic signals.

$$\delta[n] \rightarrow \boxed{h[n]} \rightarrow h[n].$$ impulse Response.

$$h[n] = \sum_k \delta(k) h[n-k] = \sum_k h(k)\delta(n-k).$$

* $x(n) \longrightarrow \boxed{h[n]} \longrightarrow y[n].$
$y[n] = \underbrace{x[n]}_{I/p} * \underbrace{h[n]}_{\text{Impulse Response}}.$

Linear Convolution
↓
(*)

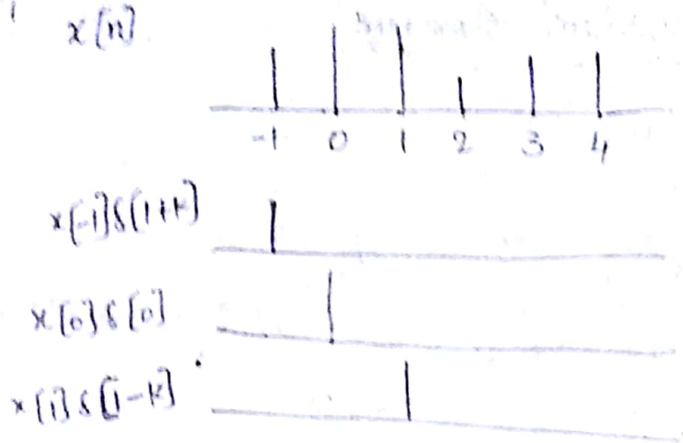
$$y[n] = \sum_k h[k]\, x[n-k] = \sum_k x[k]\, h[n-k].$$

* Impulse Response characterises the system, it is unique to the system.

① Linearity

② Time Variance

③ Shifting. — $x[n] = \sum_K x[k]\,\delta[n-k]$



$$\delta[n] \longrightarrow \boxed{LTI} \longrightarrow h[n].$$

$$\delta[n] \longrightarrow \boxed{L} \longrightarrow h_k[n] \rightarrow$$ don't know exact structure.

$$\delta[n-1] \longrightarrow \boxed{LTI} \longrightarrow h[n-1]$$

$$\delta[n-k] \longrightarrow \boxed{LTI} \longrightarrow h[n-k]$$

---

* $$\delta[n+1] \rightarrow \boxed{h} \rightarrow h[n+1]$$
  $$\delta[n] \rightarrow \boxed{h} \rightarrow h[n].$$
  $$\delta[n-k] \rightarrow \boxed{h} \rightarrow h[n-k].$$

$$x[0]\delta[n] \longrightarrow \boxed{h} \rightarrow x[0]h[n]$$

$$x[1]\delta[n-1] \rightarrow \boxed{LTI} \rightarrow x[1]h[n-1]$$

$\downarrow$

Multiplying a constant so we get same thing on RHS.

By adding all these possible inputs.

$$x[k]\delta[n-k] \rightarrow \boxed{LTI} \rightarrow x[k]h[n-k]$$

$$\left.\begin{array}{l} x[0]\delta[n] \\ + x[1]\delta[n-1] \\ + \quad \vdots \\ + x[k]\delta[n-k]. \end{array}\right\} \rightarrow \boxed{LTI} \longrightarrow \sum_K x[k]h[n-k] = y[n].$$

$$\Rightarrow x[n] = \sum_K x[k]\delta[n-k].$$

* In a Linear System.

$$x[n] = \sum x[k]\delta[n-k].$$

$$\delta[n-k] \longrightarrow \boxed{L} \rightarrow h_k[n].$$ Since it is not a ~~translation system~~ time invariant system.

$$y[n] = \sum_K x[k]h_k[n].$$

* For Continues Systems Summation will be Replaced by Integral.

$$y(t) = \int h(\tau)x(t-\tau)d\tau.$$

$$= \int x(\tau)h(t-\tau)d\tau.$$

Commutative Property :-

$x[n] * h[n] = h[n] * x[n]$. (System is changed But output is same)

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

$n-k=m$.

$$= \sum_{m=-\infty}^{\infty} x[n-m]h[m].$$

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$.

$$= h[n] * x[n].$$

$h[n] \rightarrow \boxed{x[n]} \rightarrow y[n]$.

Distributive Property :-

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

$$x[n] * \underbrace{(h_1[n] + h_2[n])}_{g[n]} = \sum_{k=-\infty}^{\infty} x[k]g[n-k].$$

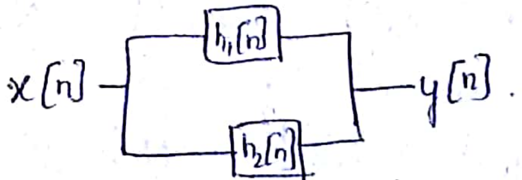
$$= \sum_{k=-\infty}^{\infty} x[k]\left(h_1[n-k] + h_2[n-k]\right).$$

$n-k=m$

$$= \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n-k].$$

$$= x[n] * h_1[n] + x[n] * h_2[n].$$

$x[n] \longrightarrow \boxed{h_1[n]+h_2[n]} \longrightarrow y$ $\equiv$



Systems in parallel add them.

Associative Property :-

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]).$$

$\sum x_1(k) g(n-k)$ $\quad g(n) = h_1(n) * h_2(n). = \sum x(k).h(n-k).$

$$\sum_k x_1(k)g(n-k) = \sum_k x_1(k)\left(h_1(n-k) * h_2(n-k)\right)$$

$$= \sum_k x_1(k) \sum_l h_1(l) h_2(n-k-l)$$

Let $n-k-l=m$

$$= \sum_k \sum_m x_1(k) h_1(n-k-m) h_2(m)$$

$$= \sum_k \sum_m x_1(k) h_1(n-m-k) h_2(m)$$

$$= \sum_m \left(x_1(n-m) * h_1(n-m)\right) h_2(m)$$

$\boxed{\text{let } g_1(n) = x_1(n) * h_1(n)}$

$$= \sum_m g_1(n-m) h_2(m)$$

$$= g_1(n) * h_2(n)$$

$$= \left(x_1(n) * h_1(n)\right) * h_2(n).$$

Hence, proved.

6/9/18.

$$x(n) * \left(h_1(n) * h_2(n)\right) = x_1(n) * \left(h_2(n) * h_1(n)\right)$$

$h_1(n) * h_2(n) = h_2(n) * h_1(n) \rightarrow$ Commutative property.

$$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n].$$

$$x[n] \rightarrow \boxed{h_2[n]} \rightarrow \boxed{h_1[n]} \rightarrow y[n].$$

* $y[n] = 4\left(x[n]\right)^2$ — Not an LTAI.

$$x[n] \rightarrow \boxed{(\ )^2} \rightarrow \boxed{4} \rightarrow y[n] = 4\left[x[n]\right]^2.$$

$$\neq$$

$$x[n] \rightarrow \boxed{4} \rightarrow \boxed{(\ )^2} \rightarrow 16\left[x[n]\right]^2$$

$x[n] \rightarrow \boxed{2} \rightarrow \boxed{(\ )^2} \rightarrow 4\left[x[n]\right]^2 \rightarrow$ But this is a different system.

* LTI System is memory less if and only iff $h[n] = 0$ for $n \neq 0$.

$h[n] = A\delta[n]$

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

$y[n] = x[n] * h[n]$

$y[n] = \sum h[k] x[n-k]$

$= \sum x[k] h[n-k]$

$= A \sum_k \delta(k) x[n-k] = A \sum_k x(k) \delta(n-k)$

$= Ax[n]$

* An LTI System is Completely characterized by its impulse response.

$h[n] = 1 \quad n = 0,1$

$0 \quad$ otherwise

$y[n] = \sum_k h(k) x[n-k]$

$y[n] = x[n] + x[n-1]$

$h[k] = 1 \quad n = 0,1$

$0 \quad$ otherwise

① $y[n] = (x[n] + x[n-1])^2$

$h[n] = (\delta[n] + \delta[n-1])^2 = 1 \quad n = 0,1$

$= 0 \quad$ otherwise

② $y[n] = \max(x[n], x[n-1])$

$h[n] = \max(\delta[n], \delta[n-1])$

$h(0) = \max(\delta[0], \delta[-1]) = 1$

$h(1) = 1$

③ $h[n] = 1 \quad n = 0,1,2$

$0 \quad$ else

$x[n] = 0.5 \quad n = 0$

$2 \quad n = 1$

$0 \quad$ else

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

$y[n] = (\cancel{x[0]h[n]} + x[1]h[n-1] \cancel{+ x[2]h[n-2]} + \ldots + x[n-2]h[2] + x[n-1]h[1] + x[n]h[0])$

$= (0 + 0 + 0 + \ldots +$

$y[n] = \sum x[k] h[n-k]$ → bcoz it has less values (generally we take which are easy to understand)

$$y[n] = \sum_{K=0}^{1} x[K]\,h[n-K]$$

$$y[n] = x[0]h[n] + x[1]h[n-1]$$

$n=-1$ $\quad y[n] = 0$

$n=0$ $\quad y[0] = (0.5)(1) + (2)(0) = 0.5$

$n=1$ $\quad y[1] = (0.5)(1) + (2)(1) = 2.5$

$n=2$ $\quad y[2] = (0.5)(1) + (2)(1) = 2.5$

$n=3$ $\quad y[3] = (0.5)(0) + (2)(1) = 2$

$n=4$ $\quad y[4] = 0$

$$y[n] = \begin{cases} 0.5 & n=0 \\ 2.5 & n=1,2 \\ 2 & n=3 \\ 0 & \text{else} \end{cases}$$

$$\{\underset{\uparrow}{0.5}\ \ 2.5\ \ 2.5\ \ 2\}$$

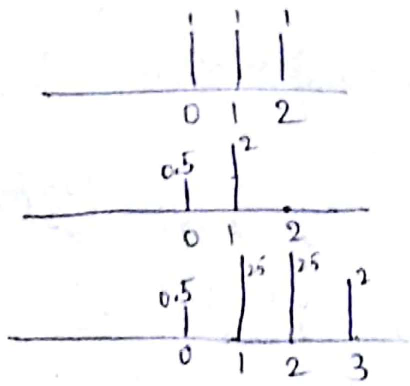
$$h[n] = \begin{cases} 1 & n=0,1,2 \\ 0 & \text{elsewhere} \end{cases} \qquad h[n] = \{\underset{\uparrow}{1}\ \ 1\ \ 1\}$$

The up arrow indicates it is zero. So $h[n]$ is 1 at 0, 1, 2.

$h[n] =$

$x[n] =$

$y[n] =$



$\text{len}\{y[n]\} = \text{len}\{x[n]\} + \text{len}\{h[n]\} - 1 \longrightarrow$ discrete.

* length is the no of non-zero values in $h[n]$, $y[n]$ etc

when you are counting length you are counting a twice so we will subtract 1

* No need to subtract 1 for a Continues signal.

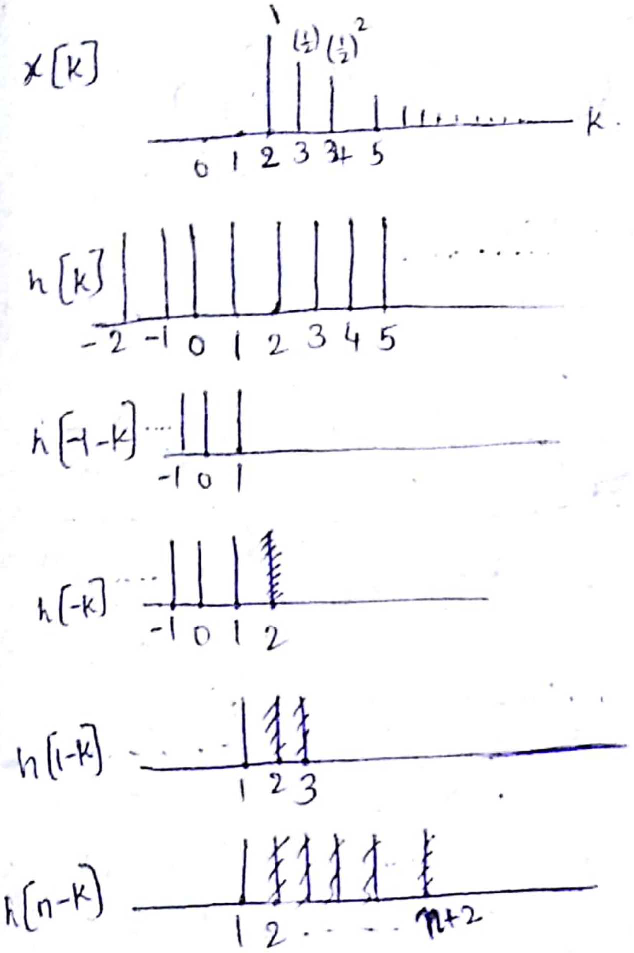
M, N, L — 3 signals

len $\{M+N-1\} \longrightarrow$ you will get new signal P.

total len = P + L − 1

(2) $x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$. $h[n] = u(n+2)$ find $y[n]$

Sol.) $y[n] = x[n] * h[n] = \sum x(k)h(n-k)$.



$n<0$ $y[n] = \sum_k x(k)\,h(n-k)$.

$h[-1-k] \rightarrow$ No overlap b/w $x[k]$ & $h[-1-k]$.

for all $n<0$ there will be no overlap.

$\boxed{n<0, \; y[n] = 0}$

$n=0$ $y[n] = \sum_k x(k)h(-k)$.

There is an overlap here.

$\boxed{n=0, y[n] = (1)(1) = 1}$

$n>0$ $n=1$

$y[1] = \sum_k x(k)h(1-k)$.

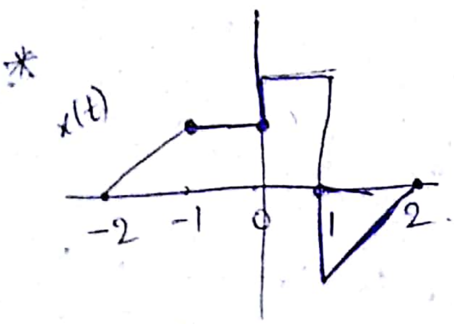
Overlap is 2 & 3.

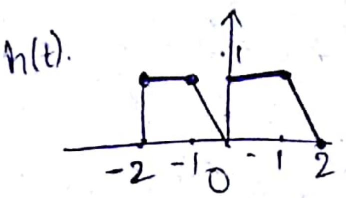
$y[n] = \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k-2} = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^n$

$\left(\begin{array}{l} k>2, \; u[k-2] > 1 \\ h[n-k] = 1. \end{array} \right)$

$\boxed{n>0, \; y[n] = 2\left(1 - \left(\frac{1}{2}\right)^n\right)}$

*





$x(t) \cdot h(t)$.

$(-2,-1) \rightarrow x(t)$

$(-1,0) \rightarrow h(t)$.

$(0,1) \rightarrow x(t)$

$(1,2) \rightarrow$

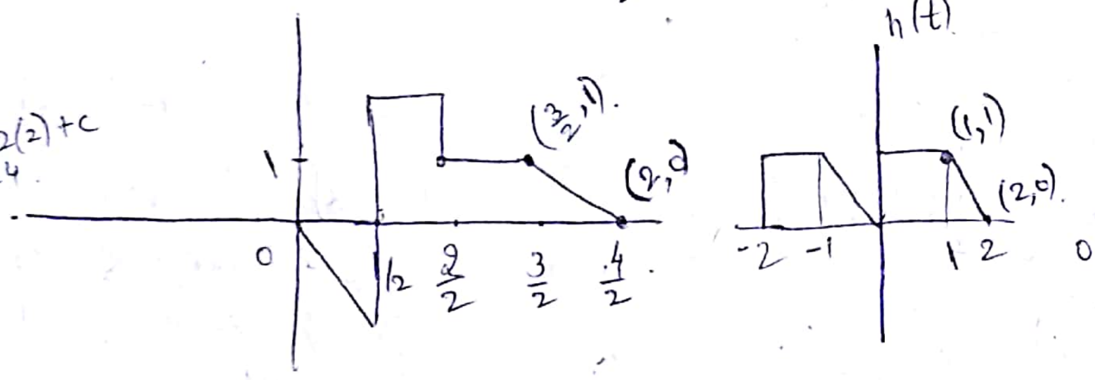
$x(2-2t)\,h(t)$.

$m = \frac{1}{\frac{3}{2}-2} = \frac{2}{-1} = -2$

y

$0 = -2(2)+c$

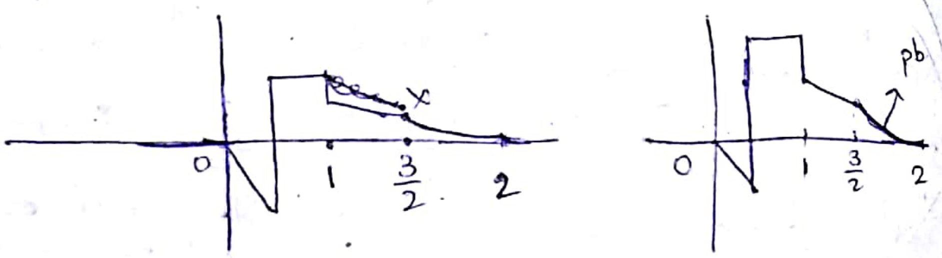
$c=4$



$0=(1)2+c$

$c=2$

$x(2-2t)h(t)$



$x(t)$

$y = -2t+4$.

$h(t) = -t+2$

$(-t+2)(-2t+4)$

$= 2t^2+8-$

11/9/18.

① $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$.

$h[n] = 2\delta[n-1] + 2\,\delta[n+1]$.

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

$x[n] = \{\underset{\uparrow}{1},2,0,-1\}$. $\quad h[n] = \{2,\underset{\uparrow}{0},2\}$.

$y[n] = x[n] * h[n]$.

$n=0 \quad y[0] = \sum_k x(k)\,h[-k] = \sum_{k=-1}^{1} h[k]\,x[-k]$.

$= (2)(2) + (0) + (2)(0) = 4$.

$\boxed{n=0\ ,\ y[0]=4.}$

$$y[1] = \cancel{(2)}(\sum_{k=-1}^{1} h[k]x[1-k]$$

$$= (2)(0) + (0) + (2)(1) = 2.$$

$$\boxed{n=1,\ y[1]=2}$$

$$y[2] = \sum_{k=-1}^{1} h[k]x[2-k]$$

$$= (2)(-1) + (0) + (2)(+2)$$

$$\boxed{y[2] = 2}$$

$$y[3] = \sum_{k=-1}^{1} h[k]x[3-k]$$

$$= (2)(0) + (0) + (2)(0)$$

$$\boxed{y[3]=0}$$

$$y[4] = \sum_{k=-1}^{1} h[k]x[4-k].$$

$$\boxed{y[4] = -2}$$

$n>4 \quad y[n]=0.$

$$y[-1] = \sum_{k=-1}^{1} h[k]x[-1-k]$$

$$= (2)(1) + (0) + (2)(0) = 2.$$

$$y[-2] = \sum_{k=-1}^{1} h[k]x[-2-k]$$

$$= (0) + (0) + (0) = 0$$

$$y(n) = \{2, \underset{\uparrow}{4}, 2, 2, 0, -2\}.$$

② $x_1[n] = [\underset{\uparrow}{0}, 1, 2, 3]$. $\quad x_2[n] = [-1, \underset{\uparrow}{-1}, 3, 4]$

Tabular Method.

$x_1 \backslash x_2$	-1	-1	3	4
0	0	0	0	0
1	-1	-1	+3	4
2	-2	-2	6	8
3	-3	-3	9	12

$$y[n] = \{\underset{\uparrow}{0}, -1, -3, -2, 7, 17, 12\}.$$

Find the sum of elements in the diagonals as drawn & that is equal to $y[n]$.

③ $x_1[n] = \{\underset{\uparrow}{0}, 2, 2, 3\}$ $\qquad x_2[n] = \{ \sin\left(\frac{3\pi n}{8}\right) \quad n=0,1,\ldots,4.$

$x_1 \backslash x_2$	0	$\sin(\frac{3\pi}{8})$	$\sin(\frac{6\pi}{8})$	$\sin(\frac{9\pi}{8})$	$\sin(\frac{12\pi}{8})$
0	0	0	0	0	0
2	0	$2\sin(\frac{3\pi}{8})$	$2\sin(\frac{6\pi}{8})$	$2\sin(\frac{9\pi}{8})$	$2\sin(\frac{12\pi}{8})$
2	0	$2\sin(\frac{3\pi}{8})$	$2\sin(\frac{6\pi}{8})$	$2\sin(\frac{9\pi}{8})$	$2\sin(\frac{12\pi}{8})$
3	0	$3\sin(\frac{3\pi}{8})$	$3\sin(\frac{6\pi}{8})$	$3\sin(\frac{6\pi}{8})$	$3\sin(\frac{12\pi}{8})$

$$y[n] = \{ \underset{\uparrow}{0}, 0, 2\sin\left(\tfrac{3\pi}{8}\right), 2\left(\sin\left(\tfrac{3\pi}{8}\right) + \sin\left(\tfrac{6\pi}{8}\right)\right), 2\left(3\sin\left(\tfrac{3\pi}{8}\right) + 2\sin\left(\tfrac{6\pi}{8}\right) + \ldots\right)$$

& so on }.

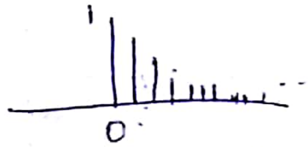
$$\underline{y[n] = x_1[n] * x_2[n]}.$$

4. $x[n] = \underbrace{\left(\tfrac{1}{2}\right)^n u[n]}_{x_1} + \underbrace{2^n u[-n]}_{x_2}$ $\qquad h[n] = u[n]$.

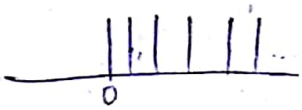
$$y[n] = x[n] * h[n].$$
$$= \underbrace{x_1[n] * h[n]}_{y_1} + \underbrace{x_2[n] * h[n]}_{y_2}.$$

$$y_1[n] = \left(\tfrac{1}{2}\right)^n u[n] * u[n].$$

$x_1[n]$.



$h[n]$.



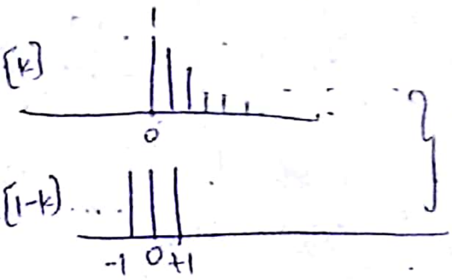
$n=0$ $\quad x[k], h[-k]$ overlap at 0

$$\boxed{n=0,\ y_1[n]=1.}$$

$n>0$, $k=1$ $\quad x[k], h[1-k]$.

$x[k]$

$h[1-k]$



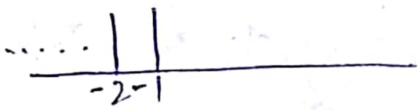
} overlap from 0 to 1

Overlap 0 to n. for any n

$$\boxed{y_1[n] = \sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^k, \ n>0}$$

$n<0$, $y_1[n] \Rightarrow h[-1-k]$.

$n=-1$



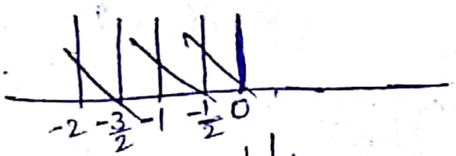
No overlap

$$\boxed{y_1[n] = 0,\ n<0.}$$

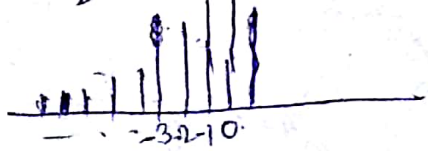
$$\boxed{y_1[n] = (2-2^{-n})\, u[n].}$$

$$y_2[n] = 2^n u[-n] * u[n].$$

$x[n]$.

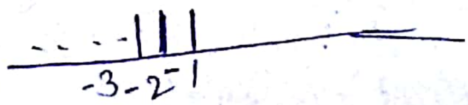


$x(n)$



$n<0$.

$h[-1-k]$



$$y_2[n] = \sum_{k=-\infty}^{n} (2)^k \qquad n<0.$$

$$k = n+m \Rightarrow \sum_{m=-\infty}^{0} 2^{m+n} = 2^n \sum_{m=-\infty}^{0} 2^m$$

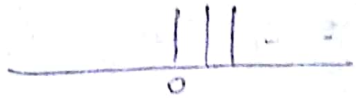
$$= 2^n \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k.$$

$$y_2[n] = 2^{n} \cdot \frac{1}{1-\frac{1}{2}} = 2^{n+1}$$

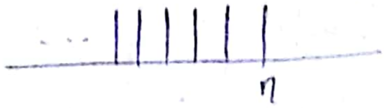
$\cancel{n=0, \; y_2[n] = 2.}$

$\cancel{n>1}$

h[k]



h[n-k]



$n>0$.

$$y_2[n] = \sum_{k=-\infty}^{0} (2)^{k} = 2.$$

$$\boxed{\begin{aligned} y_2[n] &= 2^{n+1}, \quad n<0 \\ &= 2, \quad n \geq 0 \end{aligned}}$$

$\cancel{y_2[n] = 2^{n+1} u[}$

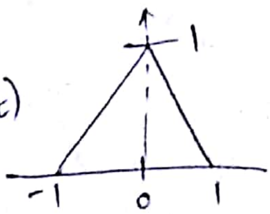
$y = y_1 + y_2$

$$y[n] = (2-2^{-n})u[n] + (2)^{n+1}u[-n-1] + 2u[n]$$

$$\boxed{y[n] = (4-2^{-n})u[n] + 2^{n+1}u[-n-1]}$$

12/9/18.

5. $\cancel{x(t)}$ h(t)

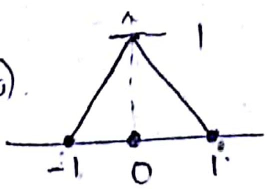


$$y(t) = \sum x(t)h(t-\tau)\,d\tau$$

$\cancel{h(t)}$ x(t)

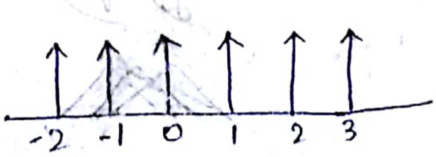
$T=1$.

$\cancel{x(t)}$ h(t)

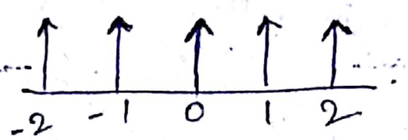


T=1

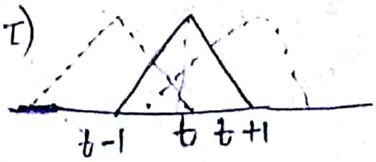
x(τ)



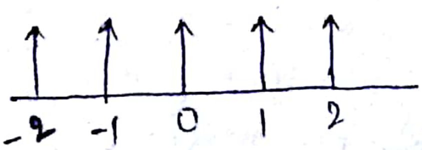
x(t) $\cancel{h(t)}$



h(t-τ)



n<0. x(-1-t)



$-1<t<0$. the triangle can move from -2 to 2 for the values of t b/w $-1<t<0$.

Overlap is at -1 & 0.

$$y(t) = \int_{t-1}^{t} \delta(\tau+1)(1+\tau-t)\,d\tau \quad \text{exists at } \tau=-1$$

$$+ \int_{t}^{t+1} \delta(\tau)(1-\tau+t)\,d\tau \quad \text{exists at } \tau=0.$$

$y = x + c$ (Here $x = \tau$), $c = 1 - t$

$\underline{y = \tau + 1 - t}$.

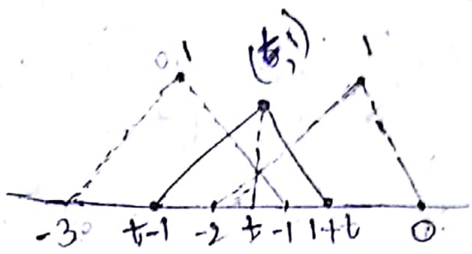
$\underline{y = -\tau + 1 + t}$.

Substitute value of $\tau = 0$ in eq's integrals

④ $y(t) = 1 - \cancel{1} - \cancel{t} + \cancel{1} + \cancel{t}$

$\underline{y(t) = 1.}$

$-2 < t < -1$.



Overlap at -2 & -1.

$m = \frac{1}{3}$

$y = \left(\frac{1}{3}\right)$

$$y(t) = \int_{t-1}^{t} \delta(\tau+2)(\tau+1-t)\,dt \rightarrow \text{exists at } -2$$

$$+ \int_{t}^{t+1} \delta(\tau+1)(1+t-\tau)\,dt \rightarrow \text{exists at } -1$$

$$y(t) = -\cancel{2} + \cancel{1} - \cancel{t} + \cancel{1} + \cancel{t} + 1 \Rightarrow \underline{y(t) = 1.}$$

⑥ $y[n] = x[n - n_0]$

(i) stable. ✓

$a < x < b$ then $a < y < b$.

(ii) Causal.

$n_0 \geq 0$ — causal. $n_0 < 0$ — not causal.

(iii) Linear ✓

$x_1 \rightarrow y_1$

$x_2 \rightarrow y_2$

$x_3 = ax_1 + bx_2$

$$y = ax_1(n-n_0) + bx_2(n-n_0)$$

$$\underline{y = ay_1 + by_2.}$$

(iv) Time Invariance. ✓

$y_1 = x(n - n_0 - k)$ $\underline{y(n-k) = x(n-k-n_0) = y_1}$.

(7) $y[n] = ax[n] + b$.

(i) stable ✓

$v_1 a < x < b v_2$; $a v_1 + b < y < a v_2 + b$. $\sum_n |h(n)| < \infty$.

(ii) Causal

$n < 0$, $y[n] = 0$. ✗

$x[n] = 0$, y $n < 0$.

(iii) Linear ✗

$x_1 \to y_1$, $x_2 \to y_2$ $x_3 = \overset{u}{a} x_1 + \overset{v}{b} x_2$.

$x_3 = y_3 = u x_1 + v x_2 + b$.

$y_3 = u y_1 + v y_2$

(iv) Time Invariance. ✓

$y_1 = \overset{a}{x}(n-k) + b$.

$y(n-k) = ax(n-k) + b$.

(8) $y[n] = x[-n]$

(i) stable ✓

$a < x < b$ & $-a < y < -b$.

(ii) Causal.

$n \geq 0 \to$ Causal $n < 0 \to$ Non Causal.

(iii) Linear ✓

$x_3 = a x_1 + b x_2$.

$y_3 = a y_1 + b y_2$

(iv) Time Invariance ✗

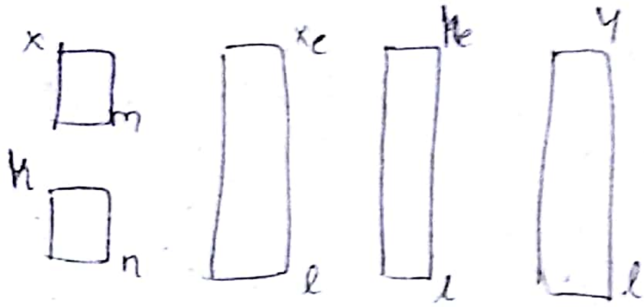
$y_1 = x(-n-k)$ $y(n-k) = x(-n+k)$.

*

MATLAB.

Linear Convolution

$y = h_e * x_e$



$x_e(1:m) = x$

$h_e(1:n) = y.$

for $l = 1$ to L.

$y(l) = 0.$

for $k = 1$ to l

$y(l) = y(l) + h(k)x(l-k+1)$

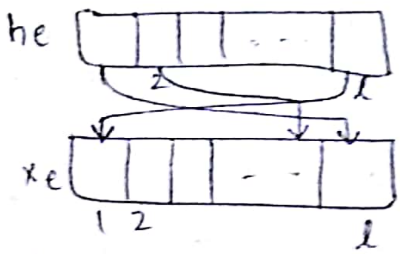
end

end

* $y = \text{Conv}(x,h).$

Reverse a vector is fliplr(x).

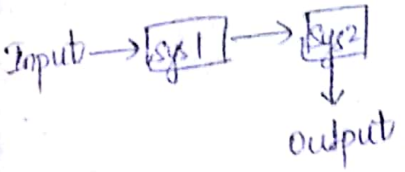
$y(l) = x_e(l)h_e(1) + h_e(2)x_e(l-1) + h_e(3)x_e(l-2) + \ldots + h_e(l)x_e(1)$



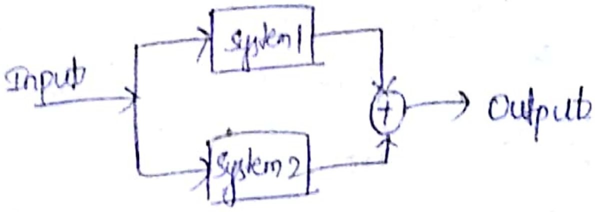
18/9/18.

Interconnection of Systems.

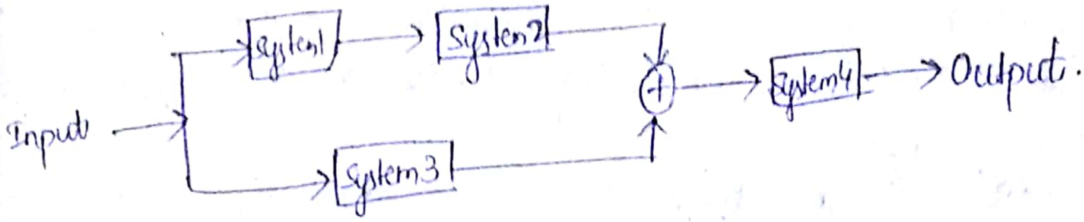
(i) Series/Cascade



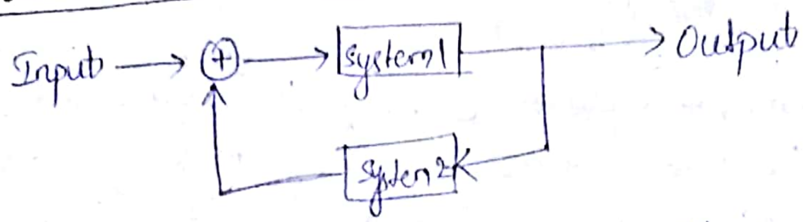
(ii) Parallel



(iii) Series parallel.



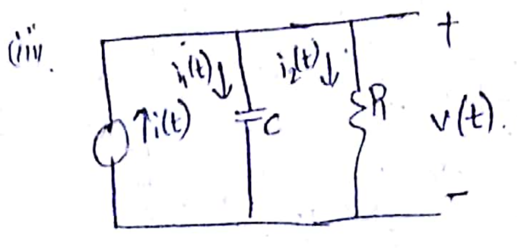
(iv) Feedback interconnection.

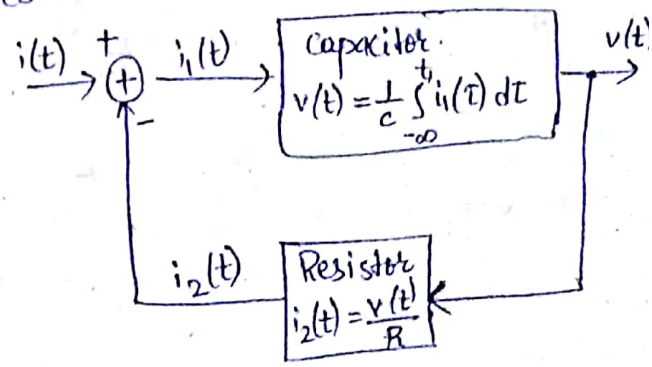


Ex (i) A cruise control system on an automobile senses the vehicle's velocity & adjusts the fuel flow in order to keep the speed at the desired level.

(ii) Digitally controlled aircraft in which differences b/w actual & desired speed, heading or altitude are fed back through the autopilot in order to correct these discrepancies.

(iii)





Simple electric ckt.

*Systems With & Without Memory:-

Memoryless → Dependent only on the input at the same time.

$$y[n] = (2x[n] - x^2[n])^2.$$

Ex $y(t) = R(x(t))$ $\quad$ $y(t)$ = Voltage & $x(t)$ = current.

$y(t) = x(t)$ & $y[n] = x[n]$.

Scanned by CamScanner

* An Example of discrete-time system with memory is an accumulator or summer.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Delay → $y[n] = x[n-1]$.

Capacitor → $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$.

* Memory corresponds to presence of a mechanism in the system that retains or stores information about input values at times other than current time.

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n].$$

$$y[n] = y[n-1] + x[n].$$

In many physical systems, memory is directly associated with storage of Energy. In discrete-time systems implemented with Computers or digital microprocessors, memory is typically directly associated with storage registers than retain values b/w clock pulses.

* **Invertibility & Inverse Systems :-**

If distinct inputs lead to distinct outputs.

$$x[n] \longrightarrow \boxed{\text{System}} \xrightarrow{y[n]} \boxed{\text{Inverse System}} \longrightarrow w[n] = x[n]$$

let $y(t) = 2x(t)$.

Inverse of this is

$$w(t) = \frac{1}{2}y(t).$$

$y(t) = x^2(t)$ is Non-invertible

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$w[n] = y[n] - y[n-1]$ (Invertible)

$y[n] = 0$ is Non-invertible.

* System that produce '0' output for any input & in case where we cannot determine the sign of the input from knowledge of the output are Non-invertible systems.

Ex:- Encoding. $\text{Msg} \longrightarrow \boxed{\text{sys}} \xrightarrow[\text{Msg}]{\text{Encoded}} \boxed{\text{Inverse Sys}} \xrightarrow{} \text{Msg}$

* Causality :-

If output at any time depends only on values of input at the present time & the past. Such system is often referred to as being nonanticipative, as the system output doesnot anticipate future values of the input.

Ex:- RC ckt, motion of an automobile.

All Memoryless systems are causal, since output responds to only current value of input.

$y[n] = x[n] - x[n+1]$; $y(t) = x(t+1)$ X Both are not causal.

* NonCausal Averaging system is $y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$.

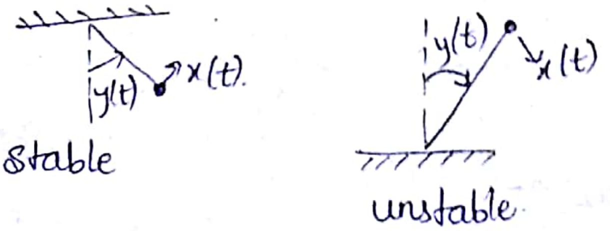
(i) $y[n] = x[-n]$.

if $n = -4$, then $y[-4] = x[4]$ — depends on future value, so Not causal.

(ii) $y(t) = x(t)\cos(t+1)$ — causal. (Memoryless.)

* Stability :-

Stable System is one in which small inputs leads to responses that do not diverge.



Stability of physical systems generally results from the presence of mechanisms that dissipate energy. [If the input to a stable system is bounded then the output must also be bounded & therefore cannot diverge.]

Ex:- $\frac{\rho v}{m} = \frac{1}{m} \cdot F \Rightarrow$ $v = \frac{F}{\rho}$ → bound for velocity is F is applied.

$y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n]$ ⟶ unstable (No Bounds)

$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$ ⟶ stable (bounded)

(ii) $y(t) = t\,x(t)$

if $x(t) = 1$

$y(t) = t$ — unbounded.

$\therefore$ Unstable

(Go by Counter Example).

(iii) $y(t) = e^{x(t)}$.

$|x(t)| < B$.

$-B < x(t) < B$.

$e^{-B} < |y(t)| < e^{B}$.

Bounded $\Rightarrow$ stable.

* Time Invariance :

A System is time invariant if the behavior & characteristics of the system are fixed over time. A System is time Invariant if a time shift in the input signal results in an identical time shift in the output signal.

if $x[n] \to y[n]$ then $x[n-n_0] \to y[n-n_0]$.

(i) $y(t) = \sin[x(t)]$.

$\sin[x(t-t_0)] = y(t-t_0)$

$\therefore$ Time Invariant.

(ii) $y[n] = n x[n]$.

$(n-n_0)\,x[n-n_0] = n x[n-n_0] - n_0 x[n-n_0]$

$\neq y[n-n_0]$

$\therefore$ Time Varying

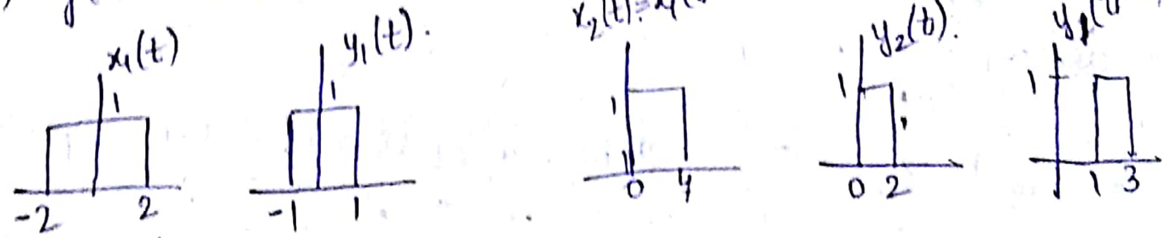
Let $x_1[n] = \delta[n]$ $(\delta[n] = 0)$.

$y_1[n] = n\,\delta[n] = 0$.

$x_2[n] = \delta[n-1]$ $\Rightarrow$ $y_2[n] = n\,\delta[n-1] = \delta[n-1]$.

$x_2[n]$ is shifted version of $x_1[n]$ while $y_2[n]$ is not shifted version of $y_1[n]$.

(iii) $y(t) = x(2t)$.



$y_2(t) \neq y_1(t-2)$

$\therefore$ Not Time Invariant.

* **Linearity** :- (Superposition).

If an input consists of the weighed sum of several signals then the output is the superposition - that is, the weighed sum - of the responses of the system to each of these signals.

* Response to $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$. — additive property.
  Response to $ax_1(t)$ is $ay_1(t)$. — Scaling / homogeneity property.

If $x[n] = \sum_k a_k x_k[n]$ then $y[n] = \sum_k a_k y_k[n]$ $\quad K=1,2,3,\ldots$

$\downarrow$

Superposition property.

* An input which is zero for all time results in an output which is zero for all time. $x[n] \to y[n]$. from homogeneity.

$$0 = 0\cdot x[n] \xrightarrow{} 0\cdot y[n] = 0.$$

(i) $y(t)=x^2(t)$.

$x_1(t) \to y_1(t) = x_1^2(t)$ $\qquad x_3(t) \to y_3(t) = x_3^2(t)$.

$x_2(t) \to y_2(t) = x_2^2(t)$ $\qquad = \left(ax_1(t)+bx_2(t)\right)^2$.

$y_3(t) = \underline{\ } ay_1(t)+by_2(t)+2ab\,x_1(t)x_2(t)$

$\neq$

$\therefore$ Non-linear.

(ii) $y[n] = Re\{x[n]\}$. — additive

$x_1[n] = r[n]+js[n]$. $\longrightarrow y_1[n] = r[n]$.

$x_2[n] = jx_1[n]$ $\longrightarrow y_2[n] = -s[n]$ but $y_2[n] \neq j\,y_1[n]$.

$\therefore$ Not linear.

$$x(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \oplus \longrightarrow y(t). \qquad \oplus \xleftarrow{} y_0(t)\ \text{(from above)}$$

Incrementally linear System.

$\Updownarrow$

$y_0(t) = 3\ ek \longrightarrow$ zero-input Response

* The Difference between the responses to any two inputs to an incrementally linear system is linear function of the difference between the two inputs. $\quad y[n] = 2x[n]+3$.

$y_1[n] - y_2[n] = 2x_1[n] + \cancel{3} - 2x_2[n] - \cancel{3} \ = 2\{x_1[n] - x_2[n]\}$ //

Linear Time Invariance (LTI).

Sifting Property:- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$.

$$y[n] = \sum_{K=-\infty}^{\infty} x[k]\, h_k[n] \quad \text{— linear.}$$

For LTI $h_k[n] = h_0[n-k]$.

$$\boxed{y[n] = \sum_{K=-\infty}^{\infty} x[k]\, h[n-k].} \quad \text{— LTI.}$$

$\downarrow$

Convolution Sum / Superposition Sum.

19/9/18.

Accumulator & First difference

$$y[n] = \sum_{m=-\infty}^{n} x[m].$$

$h[n] \longrightarrow$ impulse Response.

$$h[n] = \sum_{m=-\infty}^{n} \delta[m] = u[n].$$

Given $h[n] = u[n]$.

$$y[n] = \sum_{K} h[k]\, x[n-k] = \sum_{K} u[k]\, x[n-k].$$

$$y[n] = \sum_{k=0}^{\infty} x[n-k].$$

$n-K = m.$

$m = n$

$m = \infty$.

$\downarrow$

$$y[n] = \sum_{m=-\infty}^{n} x[m].$$

$$y[n] = \sum_{m=-\infty}^{n} x[m] = \sum_{m=-\infty}^{n-1} x[m] + x[n].$$

$$y[n] = y[n-1] + x[n].$$

$$w[n] = y[n] - y[n-1].$$

$$h_1[n] = \delta[n] - \delta[n-1].$$

$$x[n] \longrightarrow \boxed{h[n]} \xrightarrow{y[n]} \boxed{h_1[n]} \longrightarrow w[n]$$

$$w[n] \equiv x[n].$$

$$x[n] \longrightarrow \boxed{h * h_1} \longrightarrow w[n]$$

$$x[n] \longrightarrow \boxed{\delta[n]} \longrightarrow w[n]$$

$$\sum_k x[k]\delta[n-k] = x[n].$$

$$h[n] * h_1[n]$$
$$= u[n] * (\delta[n] - \delta[n-1])$$
$$= u[n] - u[n-1]$$
$$= \delta[n].$$

* Causality

LTI Systems.

$$h[n] = 0 \quad , n < 0.$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

$$n-k = m$$
$$m = n$$
$$m = -\infty.$$

$$= \sum_{k=-\infty}^{n} h[n-m]x[m]$$

* $x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

$$|x[n]| < B, \qquad |y[n]| < B' < \infty.$$

$$y[n] = \sum x(k)h[n-k] = \sum h(k)x[n-k]$$

$$|y[n]| = \left|\sum_k h(k)x(n-k)\right| \leq \sum_k |h(k)\cdot x(n-k)|$$

$$|y[n]| \leq \sum_k |h(k)x[n-k]|$$

$$= \sum_k |h(k)||x[n-k]| \quad \&$$

$$|y[n]| \leq \sum_k |h(k)|\cdot B. \Rightarrow |y[n]| < B'$$

$$\sum_k |h(k)| < D < \infty.$$

↳ Absolutely summable.

$\int h(t)\, dt < \infty$.

$\downarrow$

Absolutely Integrable.

(i) $h[n] = \left(\frac{-1}{2}\right)^n u[n] + (1.01)^n u[n-1]$. Causal?

$h[n]=0$, $n<0$. ✓ — Causal

stability? $\longrightarrow$ $\sum_n |h[n]| = \sum_n \left|\left(\frac{-1}{2}\right)^n\right.$ ~~and~~ ~~and~~.

$$= \sum_n \left|\left(\frac{-1}{2}\right)^n u[n] + (1.01)^n u[n-1]\right|.$$

$$= \sum_n \left|\left(-\frac{1}{2}\right)^n u[n]\right| + \sum_n \left|(1.01)^n u[n-1]\right|$$

$$= \sum_{n=0}^{\infty} 2^{-n} + \sum_{n=1}^{\infty} (1.01)^{n} \longrightarrow \text{diverges.}$$

(ii) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[1-n]$.

Non-Causal $\rightarrow$ $h[n] \neq 0$, $n<0$.

stability. $\rightarrow$ $\sum_n |h[n]| = \sum_n \left|\left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[1-n]\right|$

$$= \sum_{n=-\infty}^{-1} \left|(1.01)^n\right| + \sum_{n=0}^{1} \left|\left(-\frac{1}{2}\right)^n + (1.01)^n\right|$$

$$+ \sum_{n=2}^{\infty} \left(\frac{-1}{2}\right)^n$$

$$= \sum_{n=-\infty}^{0} (1.01)^n - 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 1 - \frac{1}{2} + 2 + \left|-\frac{1}{2} + 1.01\right|$$

m.

$$= \sum_{m=0}^{\infty} \left(\frac{1}{1.01}\right)^m - 1 + \left(-1 - 1\tfrac{1}{2}\right) + 2 \left| -\frac{1}{2} + 1.01 \right|$$

$$= \frac{1}{1 - \frac{1}{1.01}} - 1 + 2 - 1.5 + 0.51 + 2.$$

* Unit Step Response :-

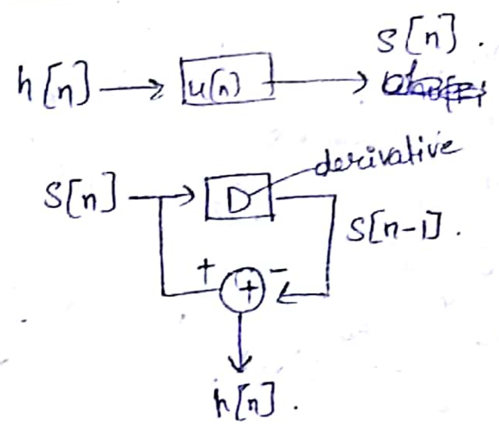
$u[n] \rightarrow \boxed{h[n]} \rightarrow s[n]$.

$\hookrightarrow$ Unit step response.

$$s[n] = u[n] * h[n] = h[n] * \overset{u}{\delta}[n].$$

$$s[n] = \sum_{k=0}^{\infty} h(n-k) = \sum_{m=0}^{n} h[m].$$

$$s[n] = s[n-1] + h[n].$$



* $$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau.$$

$$h(t) = \frac{d}{dt} s(t).$$

20/9/18.

(1.) LTI System. $\longrightarrow$ Causal

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

System eq^n $\quad y[n] = \frac{1}{4} y[n-1] + x[n]$.

Find $y[n]$ for $x[n] = \delta[n-1]$.

Sol.)

$y[0] = x[0]$.

$y[-1] = 0$.

$$y[1] = \frac{1}{4}x[0] + x[1].$$

$$y[2] = \frac{1}{4}\left(\frac{1}{4}x[0] + x[1]\right) + x[2].$$

$\vdots$

$$y[n] = \left(\frac{1}{4}\right)^n x[0] + \left(\frac{1}{4}\right)^{n-1} x[1] + \dots\dots + \left(\frac{1}{4}\right) x[n-1] + x[n].$$

$$y[n] = \sum_{k=0}^{n} \left(\frac{1}{4}\right)^{n-k} x[k].$$

$$h[n] = \sum_{k=0}^{n} \left(\frac{1}{4}\right)^{n-k} \delta(k) = \left(\frac{1}{4}\right)^n, \quad n \geq 0.$$

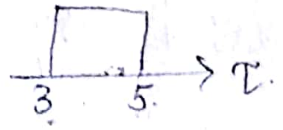
$$h[n] = \left(\frac{1}{4}\right)^n u[n]. \Rightarrow h[n-1] = \left(\frac{1}{4}\right)^{n-1} u[n-1].$$

② $x(t) = u(t-3) - u(t-5)$

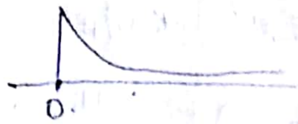
$h(t) = e^{-3t}u(t)$

$y(t) = ?$

$x(\tau)$



$h(\tau)$



$y(t) = 0 \quad , \quad t<0$

$y(t) = 0 \quad , \quad 0<t\leq 3$

$y(t) = \int_0^{t-3} h(\tau)d\tau \quad , \quad 3<t\leq 5$

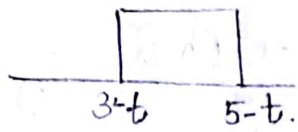
$$y(t) = \int_0^{t-3} e^{-3\tau}u(\tau)d\tau$$

$$= -\frac{1}{3}\left(e^{-3\tau}\right)_0^{t-3}$$

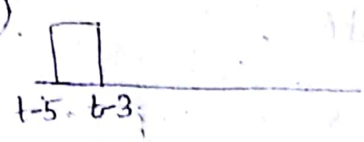
(i $y(t) = -\frac{1}{3}\left(e^{9-3t}-1\right)$.

$t<0$

$x(\tau+t)$

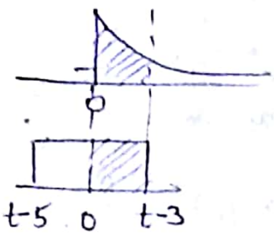


$x(-\tau+t)$

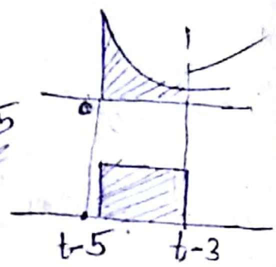


$y(t)$

$3<t\leq5$



$t>5$



$t>5, \quad y(t) = \int_{t-5}^{t-3} e^{-3\tau}u(\tau)d\tau$

$$y(t) = -\frac{1}{3}\left(e^{-3\tau}\right)_{t-5}^{t-3} \Rightarrow y(t) = \frac{1}{3}\left(e^{15-3t} - e^{9-3t}\right)$$

$$y(t) = \begin{cases} 0 & , \ t\leq 3 \\ \frac{1}{3}\left(1-e^{9-3t}\right) & , \ 3\leq t\leq 5 \\ \frac{(1-e^{-6})}{3}e^{-3(t-5)} & , \ t\geq 5 \end{cases}$$

③ Replace $x(t)$ with $x'(t)$ & send that through the linear system & find the output $g(t)$.

$x'(t) = \delta(t-3) - \delta(t-5) \quad , \quad h(t) = e^{-3t}u(t)$.

$g(t) = h(t) * x'(t)$.

$$g(t) = e^{-3t}u(t) * \delta(t-3) - e^{-3t}u(t) * \delta(t-5)$$

$$g(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)$$

$$y'(t) = \begin{cases} 0 & , t < 3 \\ e^{-3(t-3)} & , 3 \leq t < 5 \\ (1-e^{-6})e^{-3(t-5)} & , t \geq 5 \end{cases}$$

$$\downarrow$$

$$\boxed{g(t) = y'(t)} \qquad y'(t) = e^{-3(t-3)}\left(u(t-3) - u(t-5)\right) - (1 - e^{-6})e^{-3(t-5)}u(t-5)$$

4. $x[n] = \alpha^n u[n] \quad ; \quad \alpha = \frac{1}{2}$

$y[n] = \left(\frac{1}{2}\right)^n\left(u[n+2] - u[n-2]\right)$. Find $h[n]$.

Sol) $$y[n] = x[n+2]4 - \frac{x[n-2]}{4}$$

$$h[n] = 4\delta[n+2] - \frac{1}{4}\delta[n-2]$$

5. $y(t) = x(t) * h(t)$ LTI.

$y_1(t) = x'(t) * h(t)$ ∴ Have to prove that $y_1(t) = y'(t)$.

Sol) (i) $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \Rightarrow y'(t) = \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)d\tau$.

$$y'(t) = \int_{-\infty}^{\infty} h(\tau)\, x'(t-\tau)\, d\tau$$

$$y'(t) = x'(t) * h(t) = y_1(t).$$

(ii) $x'(t) = \lim_{k \to 0} \frac{x(t) - x(t-k)}{k}$

$$y_1(t) = x'(t) * h(t) = \lim_{k \to 0} \frac{x(t)*h(t) - x(t-k)*h(t)}{k}$$

$y_1(t) = \lim_{k \to 0} \frac{y(t) - y(t-k)}{k}$ → LTI input is delayed by k, So output should be delayed by k.

$y_1(t) = y'(t)$.

Hence, proved.

Tutorial

① Determine whether the following LTI systems with the given impulse responses are stable (or) Not.

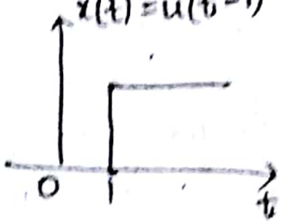
(i) $h(n) = a^n u[n]$.

$$\sum_n |h[n]| = \sum_n |a^n u[n]| = \sum_{n=0}^{\infty} |a^n| \quad \text{(Sum of G.P)}$$

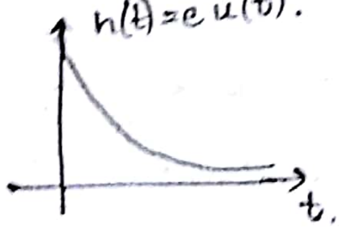
if $|a|<1 \rightarrow \sum_n |h[n]| = \frac{1}{1-a} < \infty$. — Stable

If $|a|>1 \rightarrow \sum_{n=0}^{\infty} a^n = \infty \longrightarrow$ unstable.

② $x(t) = u(t-1)$



$h(t) = e^{-t}u(t)$.



Find $x(t) * h(t)$.

$$= \int_a^b h(\tau)\, x(t-\tau)\, d\tau$$

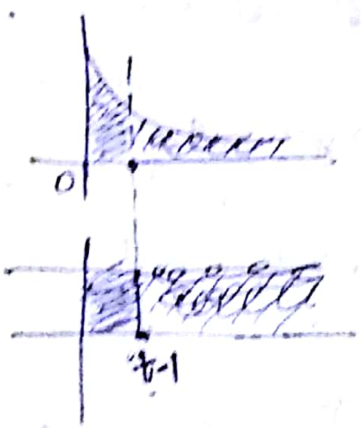
Sol)

h(τ)

t>1

0<t<1

x(t-τ)



$y(t) = 0 \quad , t<0$

$y(t) = 0 \quad , 0<t\leq 1.$

$$y(t) = \int_{t-1}^{\infty} e^{-t}\, dt = -\left(e^{-t}\right)_{t-1}^{\infty}$$

$$= -\left(0 - e^{1-t}\right)$$

$$y(t) = \int_0^{t-1} e^{-t}\, dt = -\left(e^{-t}\right)_0^{t-1}$$

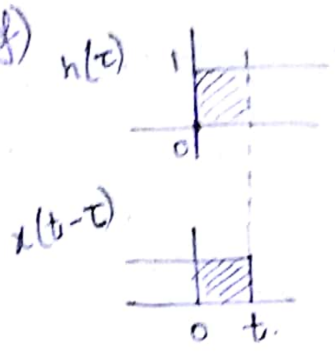
$$= -\left(e^{1-t} - 1\right)$$

$$y(t) = \begin{cases} 0 , & t \leq 1 \\ e^{1-t} , & t>1 \end{cases}$$

$$y(t) = \begin{cases} 0, & t \leq 1 \\ 1 - e^{-t}, & t > 1. \end{cases}$$

(3) $x(t) = u(t)$, $h(t) = u(t)$.

Sol)



$y(t) = 0$, $t < 0$.

$y(t) = \int_0^t u(t)\,dt = \int_0^t dt = t.$, $t \geq 0.$

3/10/18.

Fourier Analysis.

Motivation.

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$y(t) = \int x(\tau)h(t-\tau)\,d\tau = \int h(\tau)x(t-\tau)\,d\tau$

$X(s) = \int_0^\infty x(t)e^{-st}\,dt. \longrightarrow$ Laplace transform.

s - Complex function.

$[s = r + j\sigma]$.

If $x(t) = e^{st}$.

$\downarrow$

$y(t) = \int h(\tau)e^{s(t-\tau)}\,d\tau$

$= e^{st}\int h(\tau)e^{-s\tau}\,d\tau$

$\boxed{y(t) = H(s)\,e^{st}.} \longrightarrow$ Eigen function — if you give an input the output will be same as the input with just change in Amplitude.

$H(s)$ $\downarrow$ independent of input.

* $x(t) = a_1e^{s_1t} + a_2e^{s_2t} + a_3e^{s_3t}$

$y(t) = a_1(H(s_1))e^{s_1t} + a_2(H(s_2))e^{s_2t} + a_3H(s_3)e^{s_3t}.$

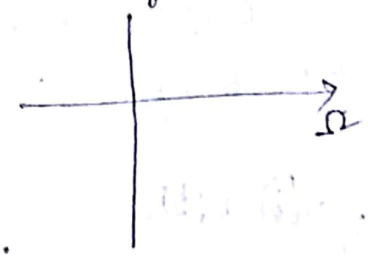
* $x(t) = \sum_k a_k e^{s_kt}.$

$\hookrightarrow y(t) = \sum_k a_k H(s_k)e^{s_kt}.$

* $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$.

$a_0 = 1, \quad a_1 = a_{-1} = \frac{1}{4} \quad a_2 = a_{-2} = \frac{1}{2}$

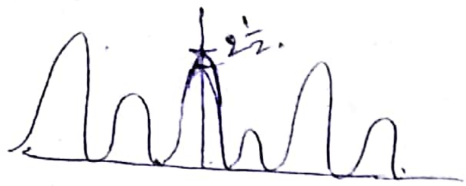
$a_3 = a_{-3} = \frac{1}{3}$.



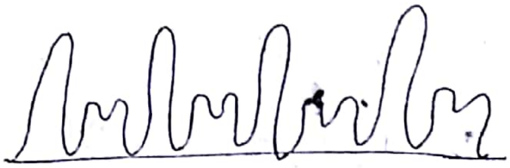
$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$x(t) = \underset{x_0}{1} + \underset{x_1}{\frac{1}{2}\cos 2\pi t} + \underset{x_2}{\cos 4\pi t} + \underset{x_3}{\frac{2}{3}\cos 6\pi t}.$$

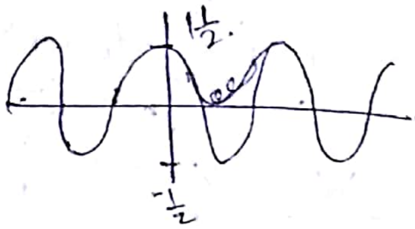
$x_0 + x_1 + x_2$.



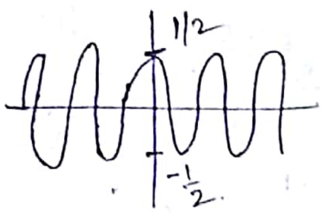
$x_0 + x_1 + x_2 + x_3$.



$x_1 + x_0$.



x_2.



x_3

*__Definition :-__

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$. — synthesis eq^n. $\quad \omega_0 = 2\pi f_0$. ↳ fundamental frequency in Radians.

* Fourier ⟶ applied for Continuous periodic signal.

$x(t) = x(t+T)$.

• __Assumption ①.__

$x(t)$ — purely real.

→ $x(t) = x^*(t)$

$$x^*(t) = x(t) = \left[\sum_k a_k e^{jk\omega_0 t}\right]^*$$

$$= \sum_k a_k^* \cdot e^{-j\omega_0 t \cdot k}$$

$K=-l$ $\quad \sum_{K=-\infty}^{\infty} a_{-l}^{*} e^{j l \omega_0 t}$

$$= \sum_{K=-\infty}^{\infty} a_{-l}^{*} e^{j l \omega_0 t}.$$

$\boxed{a_K = a_{-K}^{*} \text{ or } a_K^{*} = a_{-K}}$ $\longrightarrow$ Only Real signals.

$\downarrow$

fourier coefficients.

* $$x(t) = a_0 + \sum_{K=1}^{\infty} a_K e^{jk\omega_0 t} + \sum_{K=-\infty}^{-1} a_K e^{jk\omega_0 t}.$$

$$= a_0 + \sum_{K=1}^{\infty} a_K e^{jk\omega_0 t} + \sum_{l=1}^{\infty} a_{-K} e^{-jl\omega_0 t}.$$

$$= a_0 + \sum_{K=1}^{\infty} \left[a_K e^{jk\omega_0 t} + a_K^{*} e^{-jk\omega_0 t} \right].$$

$$= a_0 + 2 \sum_{K=1}^{\infty} \mathrm{Re}\{ a_K e^{jk\omega_0 t} \}.$$

$\cancel{a_K = A_K e^{j\theta_K}}$ $\quad a_K = A_K e^{j\theta_K}.$

$$x(t) = a_0 + 2 \sum_{K=1}^{\infty} \mathrm{Re}\{ A_K e^{(jk\omega_0 t + j\theta_K)} \}.$$

$$x(t) = a_0 + 2 \sum_{K=1}^{\infty} A_K \cos(k\omega_0 t + \theta_K).$$

$$= a_0 + 2 \sum_{K=1}^{\infty} A_K \cos\theta_K \cdot \cos k\omega_0 t - A_K \sin\theta_K \sin k\omega_0 t.$$

Trignometric fourier series.

$$x(t) = a_0 + 2 \sum_{K=1}^{\infty} B_K \cos k\omega_0 t - 2 \sum_{K=1}^{\infty} C_K \sin k\omega_0 t.$$

$$a_K = B_K + j C_K.$$

$B_K = a_K \cos\theta_K$

$C_K = a_K \sin\theta_K.$

Transfer function : $H(s)$.

$$x(t) \longrightarrow \boxed{H(s).} \longrightarrow y(t).$$

$$H(s) = \frac{Y(s)}{X(s)}. \qquad Y(s) = X(s) \cdot H(s).$$

Causal & stable —— Work in Real world.

Causality.

(i) $h(t) = 0 \quad t < 0$.

(ii) $H(s) \rightarrow$ poles only in LHS plane (left half of s-plane)

$\downarrow$ leads to

$\int |h(t)| dt < \infty$.

4/10/18.

$$H(s) = \frac{Y(s)}{X(s)} \longrightarrow \text{Transfer function.}$$

$H(S)$ has poles only on L.H.S of S-plane.

$$H(s) = \frac{A(s)}{B(s)} = \frac{\prod_{n=1}^{M} (s - z_n)}{\prod_{n=1}^{N} (s - P_k)}.$$

z_ns are zeroes

P_ks are poles.

$$H(s) = \frac{A(s)}{B(s)} = \sum_{K} \frac{C_k}{s - P_K}.$$

$$h(t) = L^{-1}[H(s)].$$

$$L[u(t)] = \frac{1}{s}$$

$$\int_0^\infty e^{-st} dt = \frac{e^{-st}}{-s} \Big|_0^\infty = \frac{1}{s}.$$

$$L\{x(t-a)\} = e^{as} \cdot X(s).$$

$$L^{-1}\left(\frac{1}{s - P_K}\right) = e^{P_K t}.$$

$$\int_0^\infty e^{P_k t} e^{-st} dt = \frac{1}{s - P_k}$$

$$H(s) = \sum_K \frac{c_K}{s - P_K}$$ Poles on L.H.S $\Rightarrow P_K < 0$.

$$h(t) = \sum_K c_K e^{P_K t}$$ $re(P_K) < 0$.

$$\int |h(t)| dt = \int \left| \sum_K c_K e^{P_K t} \right| dt \leq \int |c_1| |e^{P_1 t}| dt + \int |c_2| |e^{P_2 t}| dt + \ldots$$

$$\int |h(t)| dt < \infty \text{ iff } P_K < 0 \Rightarrow \text{Poles on L.H.S}$$

if $P_K > 0 \longrightarrow$ then sum diverges.

* **Trignometric Integrals:-**

$\rightarrow \int_{T_0} \sin \omega_0 t \, dt = 0.$

$\rightarrow \int_{T_0} \cos \omega_0 t \, dt = 0.$

$\rightarrow \int_{T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt = \frac{1}{2} \int_{T_0} \cos(m+n)\omega_0 t \, dt + \frac{1}{2} \int_{T_0} \cos(n-m)\omega_0 t \, dt.$

$= 0 \quad , n \neq m.$

$= \frac{T_0}{2} \quad , n = m \; (n = -m) \rightarrow$ if we consider -ve numbers.

$$\int_{T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt = \frac{T_0}{2} \delta(n-m).$$

$\rightarrow \int_{T_0} \cos n\omega_0 t \sin m\omega_0 t \, dt = \cancel{\frac{T_0}{2} \delta(n-m)} \; 0$

$\rightarrow \int_{T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \frac{T_0}{2} \delta(n-m). \rightarrow$ ~~[illegible]~~

$\rightarrow \int_{T_0} e^{-jm\omega_0 t} dt = T_0 \delta(m)$ — $m = 0 \;\; T_0$; $m \neq 0 \;\; 0$.

$$e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow H(k\omega_0) e^{jk\omega_0 t}.$$

$x(t) = \sum a_k e^{jk\omega_0 t}$ $\quad x(t)$ is periodic

$$y(t) = \sum b_k e^{jk\omega_0 t}.$$

$$b_k = a_k \cdot H(k\omega_0).$$

$$x(t) = \sum_k a_k e^{jk\omega_0 t} \quad \text{— (1)}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt, \quad \text{— Fourier coefficients.}$$

$$a_0 = \frac{1}{T}\int_T x(t)\, dt. \quad \text{— (2)}$$

$$x(t)\cdot e^{-jl\omega_0 t} = \sum_k a_k e^{jk\omega_0 t}\cdot e^{-jl\omega_0 t}.$$

$$\int_{T_0} x(t) e^{-jl\omega_0 t} = \int_{T_0} \sum_k a_k e^{-j(l-k)\omega_0 t} dt.$$

$$= \sum_k a_k \int_{T_0} e^{-j(l-k)\omega_0 t} dt.$$

$$= \sum_k a_k T_0 \cdot \delta(l-k).$$

$$= T_0 \cdot a_l.$$

$$\boxed{a_l = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jl\omega_0 t}\, dt.}$$

(1) Given $x(t) = A\cos(\omega_0 t + \phi)$. $a_k = ?$

$$x(t) = A\cos\omega_0 t\cos\phi - A\sin\omega_0 t\sin\phi.$$

$$x(t) = A\cos\phi \cdot \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} - A\sin\phi \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

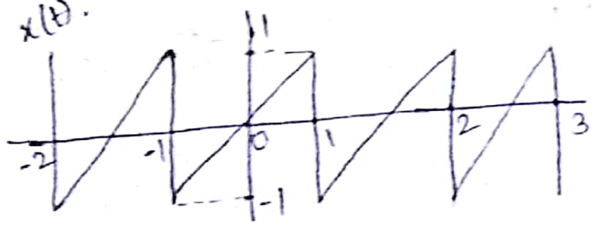
$$x(t) = \tfrac{1}{2}(A\cos\phi + jA\sin\phi) e^{j\omega_0 t} + (A\cos\phi - jA\sin\phi)e^{-j\omega_0 t}\cdot\tfrac{1}{2}$$

$$a_0 = 0, \quad a_1 = \tfrac{1}{2}Ae^{j\phi}, \quad a_{-1} = \tfrac{1}{2}Ae^{-j\phi}.$$

$\phi = 0$, $a_1 = \frac{1}{2}A = a_{-1}$

$\phi = \frac{\pi}{2}$, $a_1 = \frac{j}{2}A$, $a_{-1} = \frac{-j}{2}A$

② Given a triangular series



Tutorial

① Derive the Trignometric Fourier Series & Coefficients & also Express Exponential Fourier Series in terms of Trignometric Fourier Series.

Sol)

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + \sum_{k=1}^{\infty} b_k \sin k\omega_0 t.$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = \int_{t_0}^{t_0+T} a_0\,dt + \sum_{k=1}^{\infty} \int_{t_0}^{t_0+T} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)\,dt.$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt.$$

$$\int_{t_0}^{t_0+T} x(t)\cdot\cos l\omega_0 t = \underbrace{\int_{t_0}^{t_0+T} a_0 \cos l\omega_0 t\,dt}_{0} + \sum_{k=1}^{\infty} \int_{t_0}^{t_0+T} a_k \cos k\omega_0 t \cdot \cos l\omega_0 t + \sum_{k=1}^{\infty} \underbrace{\int_{t_0}^{t_0+T} b_k \sin k\omega_0 t \cos l\omega_0 t\,dt}_{0}$$

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cdot\cos l\omega_0 t\,dt.$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cdot\sin l\omega_0 t\,dt.$$

$$C_k = \frac{1}{T}\int_0^T x(t).e^{-j\omega_0 kt}\, dt.$$

$$= \frac{1}{T}\int_0^T x(t)\left[\cos k\omega_0 t - j\sin k\omega_0 t\right] dt.$$

$$C_k = \frac{a_k}{2} - j\frac{b_k}{2}.$$

$$C_k = \frac{a_k - b_k j}{2}$$

(2) $x(t) = 3\sin(4\omega_0 t)$. Exponential fourier series Representation.

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt.$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} \sin(4\omega_0 t)dt.3 = \frac{3}{T}\times -\frac{1}{4\omega_0}\left(\cos(4\omega_0 t)\right)_{t_0}^{t_0+T}.$$

$$a_0 \cancel{=} \frac{-3\cos(4\omega_0 T)}{4\omega_0 T}. \quad a_0 = 0.$$

$$a_k = \frac{2\times3}{T}\int_T \cos l\omega_0 t \sin(4\omega_0 t)\, dt. = \frac{6}{T}\left(0\right) \Rightarrow a_k = 0.$$

$$b_k = \frac{2\times3}{T}\int_T \sin l\omega_0 t \sin(4\omega_0 t)dt = 3\delta(l-4).$$

$$C_k = -\frac{3\delta(l-4)j}{2}$$

(3) $x[n] = \cos\left[\frac{\pi n^2}{8}\right] \rightarrow$ Periodicity?

$n = n + N.$

$$Re\left\{e^{j\left(\frac{n^2\pi}{8}\right)}\right\} = Re\left\{e^{\frac{j(n+N)^2\pi}{8}}\right\}.$$

$$e^{\frac{j\pi n^2}{8}} = e^{\frac{j\pi n^2}{8}} \cdot e^{\frac{j\pi N^2}{8} + j\pi 2Nn}$$

$$e^{j\frac{(N^2+2N)\pi}{8}} = 1 \Rightarrow \frac{\pi}{8}(N^2+2Nn) = 2\pi.$$

$N^2 + 2Nn = 16. \Rightarrow \cancel{N^2=16}.$

N & n are independent.

$N \cdot n = 0.$

$N^2 = 16$

$N = 4K.$

For $K=1$ ✗.

For $K=2 \rightarrow N=8$ – periodicity.

4/10/18.

Chapter-3.

*Response of LTI Systems to Complex Exponentials:-

Continuous time: $e^{st} \longrightarrow H(s)e^{st}.$

discrete time: $z^n \rightarrow H(z) \cdot z^n.$

* A signal for which the system output is a (possibly complex) constant times the input is referred to as an eigenfunction of the system, & the amplitude factor is referred to as the system's eigenvalue.

→ Complex Exponentials are Eigenfunctions of LTI Systems.

$x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)} d\tau.$$

$$y(t) = e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau}_{\text{converges.}}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau$$

↓

Eigen value

$$\boxed{y(t) = H(s) e^{st}}$$

→ Complex Exponential Sequences are Eigenfunctions

$x[n] = z^n.$

$$y[n] = \sum_{K=-\infty}^{\infty} h(K)x[n-k] = z^n \underbrace{\sum_{K=-\infty}^{\infty} h[k] \cdot z^{-k}}_{\text{Converges.}}$$

$$\boxed{y[n] = H(z) \cdot z^n}$$

$$H(z) = \sum_{K=-\infty}^{\infty} h[K] \cdot z^{-K}.$$

↓

Eigen Value for a specified value of z.

$$x(t) = \sum_k a_k e^{s_k t} \Rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$
$$x[n] = \sum_k a_k z_k^n \Rightarrow y[n] = \sum_k a_k H(z_k) z_k^n.$$

1) $y(t) = x(t-3)$. $x(t) = e^{j2t}$.

$$\Rightarrow y(t) = e^{j2t} e^{-6j}$$

$$H(s) = \int_{-\infty}^{\infty} \delta(t-3) e^{-s\tau} d\tau = e^{-3s}.$$

$H(2j) = e^{-6j}$

$h(t) = \delta(t-3)$.

2) $y(t) = \cos(4(t-3)) + \cos(7(t-3))$. for $x(t) = \cos 4t + \cos 7t$.

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} + \frac{1}{2} e^{j7t} + \frac{1}{2} e^{-j7t}.$$

$$y(t) = \frac{1}{2} e^{-j12} \cdot e^{j4t} + \frac{1}{2} e^{j12} e^{-j4t} + \frac{1}{2} e^{-j21} e^{j7t} + \frac{1}{2} e^{j21} e^{-7jt}.$$

$$y(t) = \frac{1}{2} e^{4j(4t-3)} + \frac{1}{2} e^{-j(4t-3)4} + \frac{1}{2} e^{j7(t-3)} + \frac{1}{2} e^{-j7(t-3)}$$

$$y(t) = \cos(4(t-3)) + \cos(7(t-3)).$$

*Linear Combination of Harmonically Related Complex Exponentials

$x(t) = \cos \omega_0 t$ & $x(t) = e^{j\omega_0 t}$ $\omega_0 = \frac{2\pi}{T}$.

$\phi_k(t) = e^{jk\omega_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}$. $k = 0, \pm 1, \pm 2, \ldots\ldots$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}. \rightarrow \text{periodic } (T).$$

Term for $k=0$ – Const.

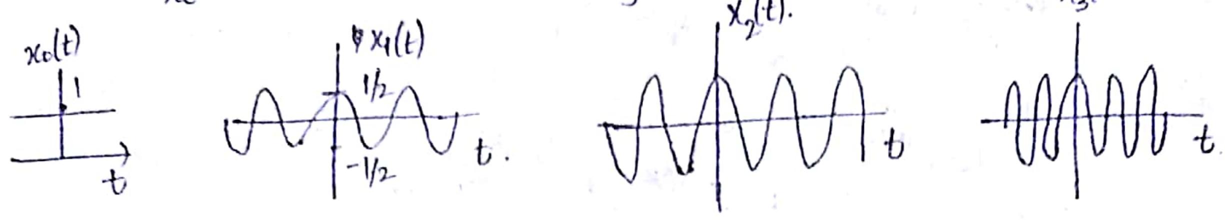
$k=-1$ & $k=1$ → have fundamental frequency equal to ω_0 & are collectively referred as fundamental components or the first harmonic components.

$k=-2$ & $k=2$ → periodic with half the period.

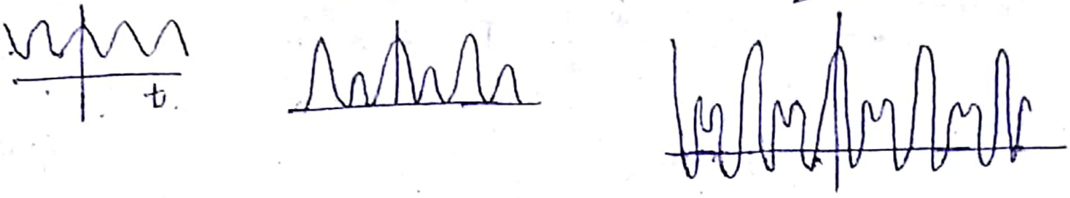
$k=-N$ & $k=N$ → Nth harmonic Components.

$$\boxed{x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}} \longrightarrow \text{Fourier Series.} \quad \text{---(1)}$$

$$x(t) = \underset{\downarrow\, x_0}{1} + \underset{\downarrow\, x_1}{\tfrac{1}{2}\cos 2\pi t} + \underset{\downarrow\, x_2}{\cos 4\pi t} + \underset{\downarrow\, x_3}{\tfrac{2}{3}\cos 6\pi t}.$$



$x_0 + x_1$. $\quad x_0 + x_1 + x_2$. $\quad x_0 + x_1 + x_2 + x_3$.



Since, $x^*(t) = x(t)$. — $x(t)$ is Real..

$$x(t) = \sum_{K=-\infty}^{\infty} a_K^* e^{-jK\omega_0 t} \longrightarrow \text{Replacing K by -K we get } x(t) = \sum_{K=-\infty} a_{-K}^* e^{jK\omega_0 t}$$

$$\boxed{a_K^* = a_{-K}} \quad \text{or} \quad \boxed{a_K = a_{-K}^*}$$

$$(1) \Rightarrow x(t) = a_0 + \sum_{K=1}^{\infty} \left\{ a_K e^{jK\omega_0 t} + a_{-K} e^{-jK\omega_0 t} \right\} = a_0 + \sum_{K=1}^{\infty} \left(a_K e^{jK\omega_0 t} + a_K^* e^{-jK\omega_0 t} \right)$$

Complex Conjugates of Each other.

$$x(t) = a_0 + \sum_{K=1}^{\infty} 2\,\mathrm{Re}\left\{ a_K e^{jK\omega_0 t} \right\}.$$

$$a_K = A_K e^{j\theta_K} \longrightarrow \text{Polar form.}$$

$$x(t) = a_0 + \sum_{K=1}^{\infty} 2\,\mathrm{Re}\left\{ A_K e^{j(K\omega_0 t + \theta_K)} \right\}.$$

$$x(t) = a_0 + 2\sum_{K=1}^{\infty} A_K \cos(K\omega_0 t + \theta_K).$$

$$a_K = B_K + jC_K.$$

$$x(t) = a_0 + 2\sum_{K=1}^{\infty} \left[B_K \cos K\omega_0 t - C_K \sin K\omega_0 t \right].$$

$\longrightarrow$ Fourier Series.

* Determination of the Fourier Series Representation of a Continuous-time periodic Signal.

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}} \longrightarrow \text{Synthesis Equation.}$$

$$x(t).e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}.$$

$T = \frac{2\pi}{\omega_0}$.

$$\int_0^T x(t).e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt.$$

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos(k-n)\omega_0 t \, dt + j\underbrace{\int_0^T \sin(k-n)\omega_0 t \, dt}_{0}.$$

$$= T.\delta(k-n).$$

Analysis Equation. $\longleftarrow$ $\boxed{a_n = \frac{1}{T}\int_0^T x(t).e^{-jn\omega_0 t}.}$ $\longrightarrow$ Same Answer over any interval of length T.

$\{a_k\} \rightarrow$ Fourier Series Coefficients / spectral coefficients of $x(t)$

*these Complex Coefficients measure the portion of the signal $x(t)$ that is at each harmonic of the fundamental component.

$a_0 \rightarrow$ dc / constant component of $x(t)$.

$$a_0 = \frac{1}{T}\int_T x(t)\,dt. \longrightarrow \text{Avg value}$$

(1) $\sin\omega_0 t$.

a. $x(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}.$

$a_k = 0. \quad k \neq \pm 1.$

$a_1 = \frac{1}{2j} \quad a_{-1} = -\frac{1}{2j}$

(2) $x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$

$$x(t) = 1 + \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) + \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + \frac{1}{2}\left[e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{j(2\omega_0 t + \frac{\pi}{4})}\right]$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}$$

$a_0 = 1.$ $\quad a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j$ $\quad a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j$

$a_2 = \frac{1}{2} e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1+j)$ $\quad a_{-2} = \frac{1}{2} e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1-j).$

$a_k = 0 \,, \; |k| > 2.$

3.

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$ fundamental period T

$$-\frac{T}{2} \leq t < \frac{T}{2}$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}. \quad ; \quad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = \frac{-1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \; k \neq 0.$$

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \; k \neq 0.$$

* Convergence of the Fourier Series :-

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$

$e_N(t) \rightarrow$ approximation error

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$

$$E_N = \int_T |e_N(t)|^2 dt. \longrightarrow$$ Energy in the Error over 1 period.

The particular choice for coefficients that minimize the energy in the error is $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt.$

$x(t)$ has a Fourier Series Representation, then the limit of E_N as $N \to \infty$ is '0'

* No Convergence difficulties for large classes of periodic Signals. For Ex, every Continuos periodic signal has a Fourier series representation for which the energy E_N in approximation Error approaches 0 as $N \to \infty$. This is also true for many discontinuos signals.

* One class of Signals representable by Fourier Series is those which have finite Energy over a single period i.e signals for which

$$\int_T |x(t)|^2 dt < \infty . \Rightarrow \text{Coefficient } a_k \text{ are finite.}$$

$$x_N(t) = \sum_{K=-N}^{N} a_K e^{jk\omega_0 t}$$

$x_N(t)$ – approximation of $x(t)$ obtained by using these coefficients for $|K| \leq N$.

$$e(t) = x(t) - \sum_{K=-\infty}^{\infty} a_K e^{jk\omega_0 t}.$$

$$\Downarrow$$

$$\int_T |e(t)|^2 dt = 0.$$

doesnot imply that $x(t)$ & $\sum_{K=-\infty}^{\infty} a_K e^{jk\omega_0 t}$ are equal.

<u>Dirichlet Conditions</u> :-

$x(t)$ is equal to its Fourierseries representation, except at isolated values of t for which $x(t)$ is discontinuos. At these values the infinite series $\sum_{K=-\infty}^{\infty} a_K e^{jk\omega_0 t}$ converges to the values on either side of discontinuity.

<u>Condition 1</u> :- Over any period, $x(t)$ must be absolutely integrable; that is

$$\int_T |x(t)| dt < \infty .$$

$$|a_K| \leq \frac{1}{T} \int_T |x(t) e^{-jk\omega_0 t}| dt = \frac{1}{T} \int_T |x(t)| dt.$$

So if $\int_T |x(t)| dt < \infty$ then $|a_K| < \infty$.

$x(t) = \frac{1}{t} \quad 0 < t \leq 1 \rightarrow$ violates condition: 1

$x(t)$ is periodic with period 1.

<u>Condition 2</u> :- In any finite interval of time, $x(t)$ is of bounded variation that is, there are no more than a finite no of maxima & minima during any single period of the signal.

$x(t) = \sin\left(\frac{2\pi}{t}\right)$; $0 < t < 1$ → Meets Cond 1 but not Cond 2.

Periodic at $T=1$ $\int_0^1 |x(t)|\,dt < 1$.

this function has, however an infinite # of maxima & minima in interval.

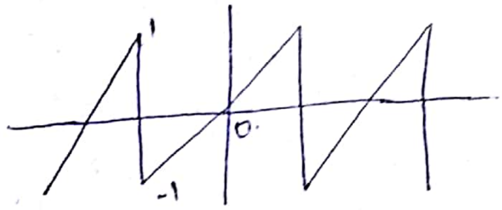
Condition 3 :- In any finite interval of time, there are only a finite number of discontinuities. Futhermore, each of these discontinuities is finite.

10/10/18.

$x(t) = t \quad -1 < t < 1$.

$T = 2$

$a_0 = \frac{1}{T}\int_T x(t)\,dt = 0$.
↓
odd function.



$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,dt$.

$= \frac{1}{2}\int_{-1}^{1} t e^{-jk\omega_0 t}\,dt$.

$T = \frac{2\pi}{\omega_0} = 2 \Rightarrow \omega_0 = \pi$.

$a_k = \frac{1}{2}\left[\frac{t e^{-jk\omega_0 t}}{-jk\omega_0}\Big|_{-1}^{1} - \int_{-1}^{1}\frac{e^{-jk\omega_0 t}}{-jk\omega_0}dt\right]$

$a_k = \frac{1}{2}\left[\frac{e^{-jk\omega_0} + e^{jk\omega_0}}{-jk\omega_0}\right] - \frac{1}{2}\left(\frac{e^{-jk\omega_0 t}}{(j\omega_0 k)^2}\right)\Big|_{-1}^{1}$

$a_k = \frac{e^{-jk\omega_0}}{-jk\omega_0} - \cancel{\frac{1 \cdot 2 e^{-}}{2(jk\omega_0)^2}}$

$a_k = \frac{j\cos k\omega_0}{k\omega_0} + \frac{j\sin k\omega_0}{k^2\omega_0^2}$

$a_k = \frac{j\cos k\pi}{k\pi} + \cancel{\frac{j\sin k\pi}{k^2\pi^2}} \to 0$.

$a_k = \frac{j\cos k\pi}{k\pi} = \frac{j(-1)^k}{k\pi}$.

$x(t) = \sum_{k=-\infty}^{\infty} \frac{j(-1)^k}{k\pi} e^{jk\pi t}$.

$B_K = \text{real}(a_K) = 0.$

$C_K = \text{imag}(a_K) = \dfrac{(-1)^k}{K\pi}.$

$$x(t) = 2\sum_{K=1}^{\infty} \frac{(-1)^k}{K\pi} \sin K\pi t.$$

K=1. K=-1.

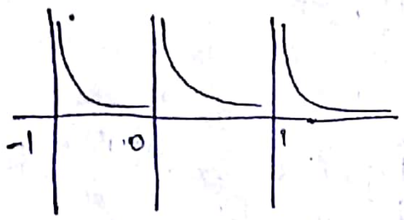
$$\frac{j(-1)}{1.\pi}(\cancel{\cos\pi t} + j\sin\pi t) + \frac{j(-1)^1}{-\pi}(\cancel{\cos\pi t} - j\sin\pi t).$$

* All Cos will get cancelled & sin will survive.

* Dirichlet Conditions :- To know where to apply Fourier series.

$x(t) = 1 \rightarrow$ satisfy Condition 1.

$x(t) = \frac{1}{t}$ $x_n(t) = \frac{1}{t-n}.$

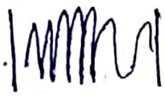


$$\int_0^1 \frac{1}{t}\,dt = (\log t)_0^1 = \infty.$$

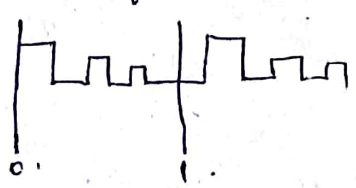
$\therefore$ Doesn't satisfy Condition 1.

Condition 2 :- No of maxima & minima must be finite

$\sin\omega t$ 

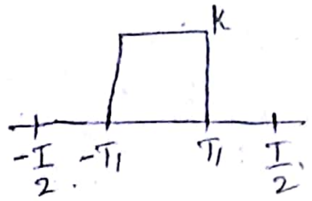
$\sin(2\pi \frac{1}{t})$  $\rightarrow$ cannot have uniform interval. Miss many values if interval is large & cannot calculate if interval is small.

Condition 3 :- In any finite Interval no. of discontinuities must be finite



Infinite discontinuities.

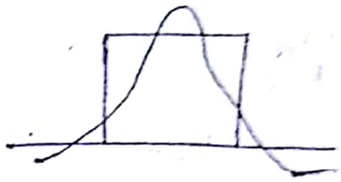
* Gibbs Phenomenon :-



$$a_K = \frac{A}{T}\int_{-T_1}^{T} e^{-jK\omega_0 t}\,dt.$$

$$x_N(t) = \sum_{K=-N}^{N} a_K e^{jK\omega_0 t}$$

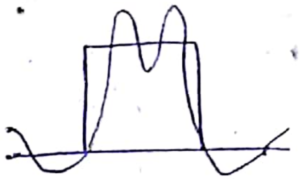
$$x_1(t) = a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}.$$



$N = 1.$

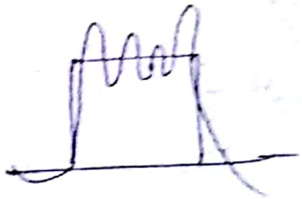
$\rightarrow$ 1peak.

$N = 3. \rightarrow$ 3 Maxima & Minima

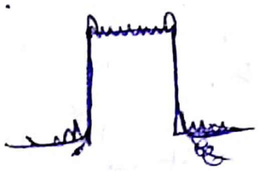


$N = 7$

7 Maxima & Minima



$N = 79.$



Still it is uneven but ~~the~~ it is approximated in the Middle to the Square.

$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}.$$

$$x_N(t) = \sum_{K=-N}^{N} a_K e^{jK\omega_0 t}. \rightarrow \int_T |x(t) - x_N(t)|^2 dt. = e_N \cdot T$$

Convergence :-

$\hookrightarrow \lim_{N\to\infty} e_N \rightarrow 0.$

* The implication is that the truncated Fourier series approximation $x_N(t)$ of a discontinues signal $x(t)$ will in general exhibit high-frequency ripples & overshoot $x(t)$ near the discontinuities

* Properties of Continuos Time Fourier Series :-

1) Linearity :-

periodic $\begin{cases} x(t) \longleftrightarrow a_K \\ y(t) \longleftrightarrow b_K. \end{cases}$ $\rightarrow$ fourier coefficients.

$\rightarrow$ should be periodic

$z(t) = Ax(t) + By(t).$

$z(t) \longleftrightarrow Aa_K + Bb_K. \rightarrow$ irrespective of individual periods

$$c_K = \frac{1}{T}\int_T z(t)e^{-jK\omega_0 t}dt = \frac{A}{T}\int x(t)e^{-jK\omega_0 t}dt + \frac{B}{T}\int y(t)e^{-jK\omega_0 t}dt$$

$$c_K = Aa_K + Bb_K.$$

2) Time shifting :-

$x(t) \leftrightarrow a_K$.

$x(t-t_0) \leftrightarrow e^{-jK\frac{2\pi}{T}t_0} a_K$.

$y(t) = x(t-t_0)$.

$b_K = \frac{1}{T}\int_T x(t-t_0)e^{-jK\omega_0 t}dt$.

$t' = t - t_0$

$b_K = \frac{1}{T}\int_T x(t')e^{-jK\omega_0(t'+t_0)}dt'$

$b_K = e^{-jK\omega_0 t_0} \cdot a_K$.

* frequency shifting :-

$e^{jM\omega_0 t}x(t) \leftrightarrow a_{K-M}$.

11/10/18.

3) Scaling :-

$x(t) \leftrightarrow a_K$. → period $= \frac{T}{\alpha}$ & $f = \alpha\omega_0$

$x(\alpha t) \leftrightarrow a_K$. → fourier coefficients dont change but frequency changes.

$b_K = \frac{1}{T'}\int_{T'} x(\alpha t)e^{-jK\omega_0' t}dt = \cancel{\frac{\alpha}{T}}\int$ $\quad \alpha t = t'$.

$t - 0 \to t'$

$t' - 0 \to \frac{T'}{\alpha} = T$.

$T' = \frac{T}{\alpha}$; $\omega_0' = \omega_0 \alpha$

$b_K = \frac{\alpha}{T}\int_T x(t')e^{-jK\omega_0 t'}\frac{dt'}{\alpha}$.

$x(\alpha t) = \sum_{K=-\infty}^{\infty} a_K e^{jK(\alpha\omega_0)t}$

$b_K = a_K$.

4) Time Reversal :-

$x(t) \leftrightarrow a_K$.

$x(-t) \leftrightarrow a_{-K}$

$b_K = \frac{1}{T}\int_T x(-t)e^{-jK\omega_0 t}dt$.

$t = -t'$

$x(t) = x(-t) \Rightarrow a_{-K} = a_K$ } even.

$= \frac{1}{T}\int_T x(t')e^{-j(-K)\omega_0(+t')}dt'$

$t' = 0$

$t' = -T$

$x(t) = -x(-t) \Rightarrow a_{-K} = -a_K$

↳ odd

$b_K = a_{-K}$.

(-) in dt' will be used to Reverse the integral limits.

5) Multiplication.

$x(t) \leftrightarrow a_K$

$y(t) \leftrightarrow b_K$

$x(t)y(t) \leftrightarrow a_K * b_K = \sum_{l=-\infty}^{\infty} a_l b_{K-l}$

$$C_K = \frac{1}{T}\int_T x(t)y(t)e^{-jK\omega_0 t}\,dt = \frac{1}{T}\int_T \sum_l a_l e^{jl\omega_0 t}\sum_m b_m e^{jm\omega_0 t} e^{-jK\omega_0 t}\,dt.$$

$$= \frac{1}{T}\sum_l \sum_m a_l b_m \int_T e^{-j(K-l-m)\omega_0 t}\,dt.$$

$$= \frac{1}{\cancel{T}}\sum_l a_l \sum_m b_m \cancel{T}\cdot\delta(K-l-m).$$

$$C_K = \sum_l a_l b_{K-l}$$

$x(t)$ is Real & odd.
$a_K \rightarrow$ purely imag & odd.

6) Conjugation :-

$x(t) \longleftrightarrow a_K$.

$x^*(t) \longleftrightarrow a^*_{-K}$

$x(t)$ is Real.

$x(t) = x^*(t)$ $\quad |a_K| = |a_{-K}|$.

$a_{-K} = a_K^*$

Conjugate symmetric

$x(t)$ is Real & Even.

$a_K = a_{-K}$, $a^*_K = a_{-K}$.

$a_K = a_K^*$

7) Parseval's theorem :-

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{K=-\infty}^{\infty} |a_K|^2$$

$|a_K|^2 \rightarrow$ Power Spectral Density.

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T \left|\sum_l a_l e^{jl\omega_0 t}\right|^2 dt.$$

$$= \frac{1}{T}\int_T \sum_l a_l e^{jl\omega_0 t}\sum_K a_K^* e^{-jK\omega_0 t}\,dt.$$

$$= \frac{1}{T}\sum_l a_l \sum_K a_K^* \int_T e^{-j(l+K)\omega_0 t}\,dt.$$

$$= \frac{1}{\cancel{T}}\sum_l a_l \sum_K a_K^* \cdot \cancel{T}\cdot\delta(l+K) = \sum_l a_l \cdot a_l^*.$$

$$\frac{1}{T}\int |x(t)|^2 dt = \sum_{l=-\infty}^{\infty} |a_l|^2$$

Statement :- ~~Total Avg~~ power in a periodic signal equals the sum of avg. powers in all of its harmonic components

* Tutorial Problems :-

1. $x_2(t) = x_1(1-t) + x_1(t-1)$.

(i) $\omega_1 \rightarrow$ freq of $x_1(t)$, $\omega_2 \rightarrow$ freq of $x_2(t)$.

$\omega_2 = f(\omega_1)$

Relation b/w ω_2 & ω_1.

(iii) $x_2(t) \leftrightarrow b_K$, $b_K = g(a_K)$ find b_K?

(2) Verify if Signals are Real or not?

(i) $x_1(t) = \sum_{K=0}^{100} \left(\frac{1}{2}\right)^K e^{jK\frac{2\pi}{50}t}$

$x_2(t) = \sum_{K=-100}^{100} \cos(K\pi)\, e^{jK\frac{2\pi}{50}t}$.

$x_3(t) = \sum_{K=-100}^{100} j\sin\frac{K\pi}{2}\, e^{jK\frac{2\pi}{50}t}$.

(3) Given $x(t) \leftrightarrow a_K$ what is the fourier coeffient of $g(t) = \frac{d}{dt}x(t) \leftrightarrow b_K$?

(4) Find a signal that satisfies the following Conditions.

(i) $x(t)$ is Real & Odd.

(ii) $x(t)$ is periodic $T=2$

(iii) $x(t) \leftrightarrow a_K$ & $|a_K| = 0$ for $|K|>1$.

(iv) $\frac{1}{2}\int_0^2 |x(t)|^2 dt = 1$.

* Fourier Transform and Fourier Series :-

$\tilde{x}(t) \rightarrow$ periodic Signal.

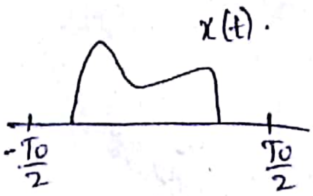
$\tilde{x}(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}$.

$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$x(t) \rightarrow$ aperiodic

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$.

$\tilde{x}(t) \rightarrow$ periodic version of $x(t)$.

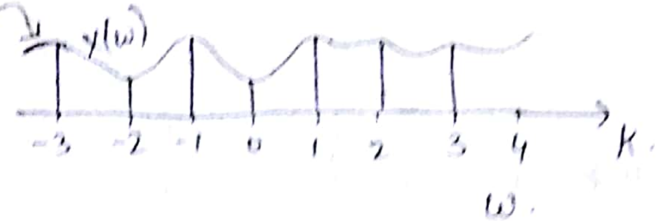


$x(t) = \tilde{x}(t) \rightarrow$ Only for fundamental Interval.
Outside this $x(t) = 0$.

$\tilde{x}(t) = x(t) \quad |t| < \frac{T_0}{2}$

$$a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

Envelope → y(w)



↓
Continuous Signal
↓
x(w)

$\omega_k = k\omega_0$.

If you sample from $x(\omega)$ we will get $x(k\omega_0)$.

$$x(k\omega_0) = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

$$\boxed{x(k\omega_0) = T_0 \cdot a_k}$$

$$\tilde{x}(t) = \sum_k a_k e^{jk\omega_0 t} = \sum \frac{x(k\omega_0)}{T_0} e^{jk\omega_0 t}$$

Take $T_0 \rightarrow \infty$. $\quad T_0 = \frac{2\pi}{\omega_0}$

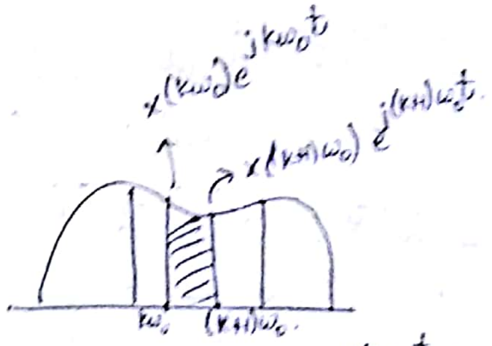
then $\tilde{x}(t) = x(t)$.

$$x(t) = \tilde{x}(t) = \sum_k \frac{\omega_0}{2\pi} x(k\omega_0) \cdot e^{jk\omega_0 t}$$

$$x(t) = \int \frac{1}{2\pi} d\omega \cdot x(\omega)e^{j\omega t}$$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)e^{j\omega t}d\omega}$$

→ inverse fourier transform Eqn



area $= \omega_0 \cdot x(k\omega_0)e^{jk\omega_0 t}$.

If the width is infinitesimal then ω_0 will become $d\omega$.

area $= d\omega \cdot x(\omega)e^{j\omega t}$.

* Fourier Transform can be Applied to both Periodic & Aperiodic Signals.

$X(j\omega)$ — Fourier Transform / Fourier integral of $x(t)$.

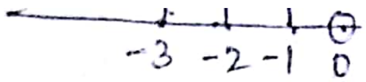
* Fourier Transform also satisfies Convergence & Dirichlet Conditions.

$$\frac{dx(t)}{dt} \longleftrightarrow jk\omega_0 a_k$$

$$\int_{-\infty}^{t} x(t)dt \longleftrightarrow \left(\frac{1}{jk\omega_0}\right)a_k$$

$x_e(t) \longleftrightarrow Re\{a_k\}$

$x_o(t) \longleftrightarrow j\,Im\{a_k\}$.



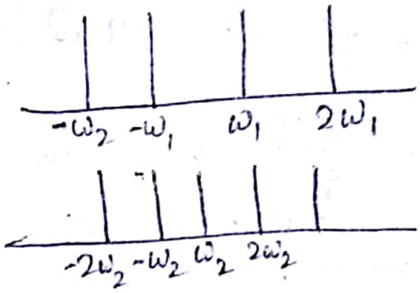
$y[-1], y[-2], y[-3]$.

24/10/18

Tutorial Answers.

(1.) $x_2(t) = x_1(1-t) + x_1(t-1)$.

$\omega_2 = \omega_1$.



$x_1(1-t) = y(t)$. $\quad x_1(t+T_1) = x_1(t)$.

$y(t+T_1) = y(t)$.

$x_1(1-t+T_1) = x_1(1-t)$.

$x_1(t) \leftrightarrow a_K$.

$$b_K = \frac{1}{T_1}\int_{T_1} x_2(t)\, e^{-jK\omega_1 t}\, dt.$$

$$b_K = \frac{1}{T_1}\int_{T_1} x_1(1-t)\, e^{-jK\omega_1 t}\, dt + \frac{1}{T_1}\int_{T_1} x_1(t-1)\, e^{-jK\omega_1 t}\, dt.$$

$t \to 0 \qquad 1-t \to 1$

$t \to T_1 \qquad 1-t \to 1-T_1$

$1-t = t'$ | $t-1 = t''$

$-dt = dt'$ | $dt = dt''$

$\to -1$

$\to T_1 - 1$

$$b_K = \frac{1}{T_1}\int_{T_1} x_1(t')\, e^{-jK\omega_1(1-t')}\, dt + \frac{1}{T_1}\int_{T_1} x_1(t'')\, e^{-jK\omega_1(t''+1)}\, dt''$$

$$= \frac{1}{T_1}\int_{T_1} x_1(t')\, e^{-jK\omega_1(-t')} \cdot e^{-jK\omega_1}\, dt' + \frac{1}{T_1}\int_{T_1} x_1(t'')\, e^{-jK\omega_1 t''}\, e^{-jK\omega_1}\, dt$$

$$\boxed{b_K = e^{-jK\omega_1} a_{-K} + e^{-jK\omega_1} a_K}$$

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(2.) $x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t}$

real signals $\leftarrow$ $a_k^* = a_k$. bcoz $a_{-k} = 0$.

$\therefore x_1(t)$ is Not Real Signal.

- $x_2(t) = \sum_{k=-100}^{100} \underbrace{\cos(k\pi)}_{\text{even}} e^{jk\frac{2\pi}{50}t}$.

$a_k^* = a_{-k} \longrightarrow$ Real Signal.

This is Even signal.

$x_2(-t) = \sum_{k=-100}^{100} \cos(k\pi) e^{-jk\frac{2\pi}{50}t}$.

- $x_3(t) = \sum_{k=-100}^{100} j\sin\frac{k\pi}{2} e^{jk\frac{2\pi}{50}t}$

$a_k^* = -j\sin\frac{k\pi}{2}$.

$a_{-k} = j\sin\left(-\frac{k\pi}{2}\right)$

$\therefore$ Real function.

(3.) $x(t) \leftrightarrow a_k$. $T=2$, $a_0 = 2$.

$g(t) = \frac{d}{dt}x(t)$.

$g(t) \longleftrightarrow b_k$.

~~$b_k = \frac{1}{T}\int_T \frac{d}{dt}x(t) e^{-jk\omega_0 t} dt$.~~

$x(t) = \sum_k a_k e^{jk\omega_0 t}$.

$g(t) = \sum_k a_k e^{jk\omega_0 t} \cdot (jk\omega_0) = \sum \underbrace{jk\omega_0 a_k}_{b_k} \cdot e^{jk\omega_0 t}$

(4.) $x(t) = A\sin\omega_1 t$.

$\omega_1 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

~~Ae~~ $\frac{1}{2}\int_0^2 |x(t)|^2 dt = 1 \Rightarrow \frac{1}{2}\int_0^2 A^2\sin^2\omega_1 t\, dt = 1 \Rightarrow \frac{A^2}{2}\int_0^2 \sin^2\pi t\, dt = 1$

$= \frac{A^2}{2}\int_0^2 \underbrace{\frac{1-\cos 2\pi t}{2} dt}_{1} = 1$

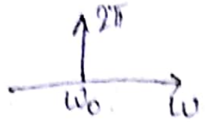
$\Rightarrow$ ~~$\frac{A^2}{2} \cdot \frac{2}{2}$~~ $\left(\frac{T}{2} = \frac{2}{2} = 1\right)$

$\frac{A^2}{2} = 1 \Rightarrow A = \sqrt{2}$.

$x(t) = e^{j\omega_0 t}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \qquad \int_{-\infty}^{\infty} \left| e^{j\omega_0 t} \cdot e^{-j\omega t} \right| dt \longrightarrow \infty.$$

$$X(\omega) = 2\pi\,\delta(\omega - \omega_0).$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega.$$

$$= \frac{2\pi}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega = e^{j\omega_0 t}.$$

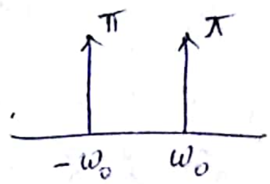
$$\boxed{e^{j\omega_0 t} \rightleftharpoons 2\pi\,\delta(\omega - \omega_0).}$$

* Applications in. Communication Engineering.
* spectral lines.

* $\sin\omega_0 t = x(t) = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$F[x(t)] = \frac{F(e^{j\omega_0 t}) - F(e^{-j\omega_0 t})}{2j} = \frac{2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)}{2j}$$

$$X(\omega) = -j\pi\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right].$$



$$\boxed{\sin\omega_0 t \rightleftharpoons -j\pi\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]}$$

$\angle X(\omega) = \tan^{-1}\left(\frac{-\pi}{0}\right) = \frac{-\pi}{2} \quad \omega = -\omega_0$

phase

$\frac{\pi}{2} \quad \omega = \omega_0.$

* $x(t) = \cos\omega_0 t = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

$$F(x(t)) = \frac{2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)}{2}$$

$$X(\omega) = \pi\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right].$$

$\angle X(\omega) = \tan^{-1}(0) = 0.$

$$\boxed{\cos\omega_0 t \rightleftharpoons \pi\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right]}$$

25/10/18

(1) $x(t) = e^{-at} u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{a+j\omega}$$

$|X(\omega)| \rightarrow$ Magnitude spectrum

$\angle X(\omega) \rightarrow$ phase spectrum

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$\downarrow$

phase of Num − phase of denominator

(2) $x(t) = e^{-a|t|}$ $\quad a>0$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at-j\omega t} dt + \int_{0}^{\infty} e^{-at-j\omega t} dt$$

$$= \left.\frac{e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} + \frac{1}{a+j\omega} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$X(\omega) = \frac{2a}{a^2+\omega^2}$$

phase = 0 (bcoz No img part). $\quad |X(\omega)| = \frac{2a}{a^2+\omega^2}$

(3) $x(t) = e^{-2(t-1)} u(t-1)$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-2(t-1)} e^{-j\omega t} dt\, u(t-1) = \int_{1}^{\infty} e^{-2t-j\omega t} \cdot e^{2} dt = e^{2} \left.\frac{e^{(-2-j\omega)t}}{(-2-j\omega)}\right|_{1}^{\infty}$$

$$= e^{2}\left(0 + \frac{e^{-2-j\omega}}{2+j\omega}\right)$$

$$X(\omega) = \frac{1}{e^{j\omega}(2+j\omega)}$$

$$|X(\omega)| = \frac{1}{\sqrt{4+\omega^2}}$$

$$\angle X(\omega) = -\omega - \tan^{-1}\left(\frac{\omega}{2}\right)$$

Matlab Code for Q2:-

```
omegavec = -10*pi:pi/100:10*pi;
a=1;
mag-spec = 1./sqrt(a^2 + omegavec.^2);
phase-spec = -atan(omegavec/a);
subplot(211)
plot(omegavec, abs(mag-spec))
subplot(212).
plot(omegavec, phase-spec);
```

4) $x(t) = e^{-2|t-1|}$ $a>0$.

$$X(\omega) = \int_{-\infty}^{\infty} e^{-2|t-1|} e^{-j\omega t}\, dt = \int_{-\infty}^{1} e^{2(t-1)} e^{-j\omega t}\, dt + \int_{1}^{\infty} e^{-2(t-1)-j\omega t}\, dt.$$

$$= \left.\frac{e^{(2-j\omega)t}\, e^{-2}}{(2-j\omega)}\right|_{-\infty}^{1} + \frac{e^{-j\omega}}{2+j\omega}$$

$$X(\omega) = \frac{4e^{-j\omega}}{4+\omega^2}$$

$$|X(\omega)| = \frac{4}{4+\omega^2} \qquad \angle X(\omega) = -\omega.$$

* phase of $e^{-j\omega}$ is $-\omega$.

* Properties.

1) Linearity.

$x_1(t) \rightleftharpoons X_1(\omega)$

$x_2(t) \rightleftharpoons X_2(\omega)$

$a x_1(t) + b x_2(t) \rightleftharpoons a X_1(\omega) + b X_2(\omega)$.

$x(t) = a x_1(t) + b x_2(t)$

$$F\{x(t)\} = F\{ax_1(t) + bx_2(t)\} = \int (ax_1(t) + bx_2(t))\, e^{-j\omega t}\, dt$$

$$= aX_1(\omega) + bX_2(\omega)$$

2) Symmetry:

$X(-\omega) = (X(\omega))^*$ $\quad x(t)$ is Real.

$$X(\omega) = \int x(t)\, e^{-j\omega t}\, dt \qquad X(-\omega) = \int x(t)\, e^{-j(-\omega)t}\, dt$$

$$= \left(\int x(t)\, e^{-j\omega t}\, dt\right)^*$$

$$X(-\omega) = (X(\omega))^*$$

$$Re[X(-\omega)] + jIm[X(-\omega)] = Re[X^*(\omega)] + j\, Im[X^*(\omega)]$$

$$\left.\begin{aligned} Re[X(-\omega)] &= Re[X^*(\omega)] = Re[X(\omega)] \\ Im[X(-\omega)] &= Im[X^*(\omega)] = -Im[X(\omega)] \end{aligned}\right\} \; x(t) \text{ is Real}$$

If $x(t)$ is Real & Even.

$$X(\omega) = \int x(t) e^{-j\omega t} dt = \int x(-t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t') e^{j\omega t'} dt' = X(-\omega).$$

$\therefore$ No Imaginary part for Real & Even Signal.

3) Time Shifting:

$x(t - t_0) \rightleftharpoons$

$$\int x(t - t_0) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t') e^{-j\omega t' - j\omega t_0} dt' = e^{-j\omega t_0} X(\omega)$$

$x(t) \rightleftharpoons X(\omega)$

$$\frac{dx(t)}{dt} \rightleftharpoons j\omega X(\omega) \qquad x(t) = \frac{1}{2\pi} \int X(\omega)\, e^{j\omega t}\, d\omega.$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int X(\omega) (j\omega) e^{j\omega t} dt.$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \rightleftharpoons 2\pi X(0)\delta(\omega) + \frac{X(\omega)}{j\omega}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega\tau}\,d\omega$$

$$\int_{-\infty}^{t} x(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{t}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega\tau}\,d\omega\,d\tau$$

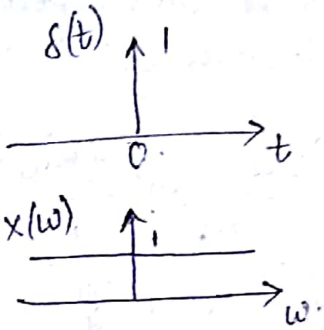
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\int_{-\infty}^{t} e^{j\omega\tau}\,d\tau\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X(\omega)}{j\omega}\,e^{j\omega t}\,d\omega.$$

Ex :- $x(\tau) = \delta(\tau)$

$$X(\omega) = \int \delta(t)\,e^{-j\omega\tau}\,d\tau = 1$$

$$g(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t).$$



30/10/18.

Examples. T.B.

$\omega_0 = \frac{2\pi}{T}$ $\quad T = 4$.

① $g(t) = x(t-1) - 1/2$

$$e^{-jk\omega_0(1)}a_k \qquad c_k = \begin{cases} 0, & k \neq 0 \\ -\frac{1}{2}, & k = 0 \end{cases}$$

$$g(t) \rightarrow d_k = \begin{cases} a_0 - \frac{1}{2} & k = 0 \\ a_k e^{-jk\omega_0} & k \neq 0 \end{cases}$$

* Fourier Series & LTI Systems :-

$x(t) = e^{st}$ $\quad y(t) = H(s)e^{st}$ $\quad H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau.$

$x[n] = z^n$ $\quad y[n] = H(z)z^n$ $\quad H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$

$z = e^{j\omega}$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}.$$

$$H(j\omega) = \int h(\tau)e^{-j\omega t}\,d\tau.$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)\,e^{jk\omega_0 t}.$$

3/10/18.

Discrete time Fourier Series.

Applies to periodic signals.

$x[n+N] = x[n] \quad \forall n. \quad , N \rightarrow +\text{ve integer}.$

$x[n+mN] = x[n].$

need not be fundamental frequency.

synthesis $\boxed{x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n]}$ ① $\Rightarrow \phi_k[n] = e^{jk\frac{2\pi}{N}n}$

Basis function

$\phi_k[n+N] = \phi_k[n].$ — periodic in time and.

$\phi_{k+N}[n] = \phi_k[n]$ — periodic in frequency

$k = \langle N\rangle \quad (m \text{ to } m+N-1).$

Values of k can be taken from any interval of length N bcoz it is periodic in frequency.

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\frac{2\pi}{N}n}.$$

① $\Rightarrow$

$$\sum_{n=\langle N\rangle} x[n] \cdot e^{-jr\frac{2\pi n}{N}} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n - jr\frac{2\pi n}{N}}$$

$$= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} \big(\underbrace{e^{j(k-r)\frac{2\pi}{N}}}_{\alpha}\big)^n.$$

if $(k-r) \neq mN.$

$$\sum_{n=0}^{N-1} (\alpha)^n = \frac{1-\alpha^N}{1-\alpha} = \frac{1-1}{1-\alpha} = 0.$$

$$\sum_{n=0}^{N-1} (\alpha)^n = N\delta(k-r-mN).$$

$$= N \cdot \sum_{k=\langle N\rangle} a_k \cdot \delta(k-r-mN).$$

$k \in (0, N-1).$

survive only at $k=r$.

$$= N \cdot a_r.$$

Analysis. $\boxed{a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\frac{2\pi n}{N}}}$ $\rightarrow a_k$ is also periodic signal.

$$a_{k+sN} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j(k+sN)\frac{2\pi n}{N}} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\frac{2\pi n}{N}}$$

periodic $\boxed{a_{k+sN} = a_k}$ $\rightarrow$ periodic in frequency.

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi n}{N}}$$

theorem :- $\sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi n}{N}} = \sum_{k=m}^{m+N-1} a_k e^{jk\frac{2\pi n}{N}}$

$0 \quad N-1$

$m \quad m+N-1$

$$R.H.S = \underbrace{\sum_{k=m}^{N-1} a_k e^{j\frac{2\pi n}{N}}}_{y} + \sum_{k=N}^{m+N-1} a_k e^{j\frac{2\pi kn}{N}}$$

$k - N = l.$

$l = 0$ to $m-1$

$$= \sum_{k=m}^{N-1} y + \sum_{l=0}^{m-1} \underbrace{a_{l+N}}_{a_l} \underbrace{e^{j\frac{2\pi(l+N)}{N}n}}_{\frac{2\pi ln}{N}}$$

$$= \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi n}{N}}$$

R.H.S = L.H.S.

*Discrete case is like sampled version of Continuos case.

① $x[n] = \sin \omega_0 n.$ $\quad \omega_0 = \frac{2\pi}{N} = \frac{2\pi}{5} \; (N=5)$

$$a_k = \frac{1}{5} \sum_{n=-2}^{2} \sin\frac{2\pi}{5}(n) \cdot e^{-jk\frac{2\pi n}{N}}$$

$$x[n] = \frac{e^{j\frac{2\pi}{5}n} - e^{-j\frac{2\pi}{5}n}}{2j}$$

$a_0 = 0$, $a_1 = -\frac{1}{2}j$

$[\underset{\uparrow}{0}, -\frac{1}{2}j, 0, 0, \frac{1}{2}j]$

$a_4 = a_{-1}.$

② $\omega_0 = \frac{2\pi}{5} \times 3.$

$$x[n] = \frac{e^{j\frac{2\pi 3}{5}n} - e^{-j\frac{2\pi}{5}.3n}}{2j}$$

$a_0 = 0$, $a_1 = 0$, $a_2 = a_3$, $a_3 = \frac{1}{2j}$, $a_{-3} = -\frac{1}{2j}$, $a_4 = 0.$

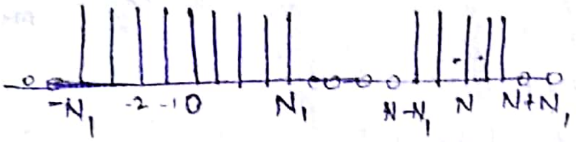
∴ a_k for discrete are also known as spectral Coefficients of $x[n]$.

7/11/18.

(1.) $x[n] = 1 \quad -N_1 \le n \le N_1. \quad N_1 < N$

$\quad\quad\; 0 \quad$ elsewhere $\quad$ N-time period.

$x[n+N] = x[n]$. Find fourier Coefficients.



$$a_k = \frac{1}{N}\sum_{n=\langle n\rangle} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} (1)e^{-j\frac{2\pi}{N}kn} \quad \text{—(1)}$$

$$a_k = \frac{1}{N}\left(e^{-j\frac{2\pi k}{N}(-N_1)} + e^{-j\frac{2\pi k}{N}(-N_1+1)} + \dots + 1 + \dots + e^{-j\frac{2\pi k}{N}(N_1-1)} + e^{-j\frac{2\pi k}{N}(N_1)}\right)$$

$$a_k = \frac{e^{-j\frac{2\pi k}{N}}}{N}\left(e^{-N_1} + e^{-N_1+1} + \dots + e^{0} + \dots + e^{N_1}\right).$$

$$(1) \Rightarrow a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-j\frac{2\pi}{N}k(m-N_1)} \qquad \left[\begin{array}{l} m = n+N_1 \\ m = 0. \\ m = 2N_1 \end{array}\right]$$

$$= \frac{1}{N}\, e^{j\frac{2\pi}{N}kN_1}\sum_{m=0}^{2N_1} e^{-j\frac{2\pi}{N}km}.$$

$$= \frac{1}{N}e^{j\frac{2\pi kN_1}{N}}\left(\frac{1 - e^{-j\frac{2\pi}{N}k(2N_1+1)}}{1 - e^{-j\frac{2\pi}{N}k}}\right)$$

$$= \frac{e^{j\frac{2\pi}{N}k(N_1+\frac{1}{2}-\frac{1}{2})} - e^{-j\frac{2\pi}{N}k(N_1+\frac{1}{2}+\frac{1}{2})}}{1 - e^{-j\frac{2\pi}{N}k}} \times \frac{1}{N}.$$

$$= \frac{e^{-j\frac{2\pi}{N}\frac{k}{2}}\left[e^{j\theta} - e^{-j\theta}\right]}{N\left(1 - e^{-j\frac{2\pi}{N}k}\right)}. \qquad \theta = \frac{2\pi}{N}k\left(N_1+\tfrac{1}{2}\right).$$

$$\phi = \frac{\pi}{N}k.$$

$$= \frac{\cancel{e^{-j\frac{\pi}{N}k}} \times 2j\sin\theta}{N \times \cancel{e^{-j\frac{\pi}{N}k}}\left(e^{j\frac{\pi}{N}k} - e^{-j\frac{\pi}{N}k}\right)} \Rightarrow \boxed{\frac{1}{N}\cdot\frac{\sin\theta}{\sin\phi} = a_k.}$$

$$a_0 = \frac{2N_1+1}{N}$$

* **Linearity:** $Ax[n] + By[n] \rightleftharpoons Aa_k + Bb_k$
* **Time shifting:** $x[n-n_0] \rightleftharpoons a_k e^{-jk\frac{2\pi}{N}n_0}$
* **Frequency shifting:** $e^{j\frac{2\pi}{N}n.M} x[n] \rightleftharpoons a_{k-M}$
* **Conjugate:** $x^*[n] \rightleftharpoons a^*_{-k}$
* **Time Reversal** $x[-n] \rightleftharpoons a_{-k}$

$x[n]y[n] \rightleftharpoons \cancel{a_k * b_k} = \sum_{l=\langle N\rangle} a_l b_{k-l}$ — Multiplication.

$e^{j\omega_0 t}$ $\qquad\qquad$ $e^{j\omega_0 n}$

Any real number (ω_0). $\qquad$ rational multiple of π $(a \cdot 2\pi = \omega_0)$

Periodic: Any Rational ω_0 or multiple of π. $\qquad$ Periodic

* **Discrete Time Fourier Transform:- (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

need not be periodic. (aperiodic)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$\omega = k\omega_0 \Rightarrow X(k\omega_0) = \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$ $\qquad$ $a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}(n) e^{-jk\omega_0 N}$

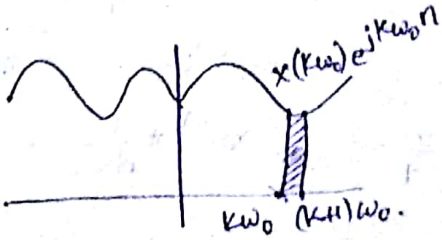
periodic

$\tilde{x}[n] = x[n] \rightarrow n \in (0, N-1)$. $\qquad$ $x[n]$ is Repeated to form $\tilde{x}[n]$.

$$\boxed{X(k\omega_0) = N \cdot a_k}$$

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} \frac{1}{N} X(k\omega_0) e^{jk\omega_0 n} \qquad \left(\omega_0 = \frac{2\pi}{N}\right)$$

$$= \frac{\omega_0}{2\pi} \sum_{k=\langle N\rangle} X(k\omega_0) e^{jk\omega_0 n}$$



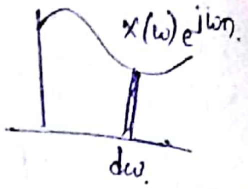
If you make rectangle infinitesimal,

$\omega \to K\omega_0$.

$K = 0 \quad , \omega = 0.$

$K = N-1 \quad , \omega = \frac{2\pi}{N}(N-1)$

$\omega = 2\pi$



[N & N-1 are nearly equal].

$$\boxed{\tilde{x}[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}\,d\omega}$$

(1) $x[n] = 1 \quad , \forall n.$

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}\,d\omega.$$

$$\boxed{1 \rightleftharpoons 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)}$$

$$x[n] = \frac{1}{2\pi} \times 2\pi \int_0^{2\pi} \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)e^{j\omega n}\,d\omega.$$

$$= \sum_{l=-\infty}^{\infty} \int_{-\pi}^{\pi} \delta(\omega - 2\pi l)e^{j\omega n}\,d\omega. = \sum_{l=-\infty}^{\infty} \delta(l).$$

$\delta(0)$

$x[n] = 1.$

survives only for $l = 0$, for remaining value it will be zero.

(2) Find FT of $(n+1)a^n u[n] \to X_1(\omega)$

Sol)

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}. \qquad |a| < 1.$$

$$X(\omega) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$\frac{d}{d\omega} X'(\omega) = -j\sum_{n=0}^{\infty} n a^n \cdot e^{-j\omega n} \Rightarrow \sum_{n=0}^{\infty} n a^n e^{-j\omega n} = \frac{X'(\omega)}{-j}$$

$$X_1(\omega) = X(\omega) + \frac{d}{d\omega}X(\omega)j = \frac{1}{1 - ae^{-j\omega}} + \frac{+1(+ae^{-j\omega})(+\cancel{j})}{(1 - ae^{-j\omega})^2} \cdot \cancel{j}$$

$$X_1(\omega) = \frac{1-ae^{-j\omega}+ae^{-j\omega}}{(1-ae^{-j\omega})^2} = \frac{1+ae^{-j\omega}(1-1)}{(1-ae^{-j\omega})^2} = \frac{1}{(1-ae^{-j\omega})^2}.$$

6/11/18.

* FSDT.

Fourier series Representation of a discrete signal is finite. therefore, no issues of Convergence i.e No Gibbs phenomenon. In fact, there is no convergence issues with the discrete signals bcoz of the fact that $x[n]$ is completely specified by a finite no of N of parameters, namely the values of the sequence over one period.

*Properties.

$x[n], y[n]$ — period $= N$. $\omega_0 = \frac{2\pi}{N}$.

Linearity. $Ax[n]+By[n] \longleftrightarrow Aa_k + Bb_k$

Time shifting. $x[n-n_0] \leftrightarrow a_k e^{-jk(2\pi/N)n_0}$.

Frequency shifting $e^{jM(2\pi/N)n}x[n] \longleftrightarrow a_{k-M}$.

Conjugation $x^*[n] \longleftrightarrow a^*_{-k}$

Time Reversal. $x[-n] \longleftrightarrow a_{-k}$

Time Scaling. $x_m[n] = \begin{cases} x[n/m] & n \text{ multiple of } m \\ 0 & n \neq km \end{cases} \longleftrightarrow \frac{1}{m} a_k.$ (viewed as periodic with period mN)

Periodic Convolution: $\sum_{r=\langle N\rangle} x[r]y[n-r] \longleftrightarrow Na_k b_k$

Multiplication. $x[n]y[n] \longleftrightarrow \sum_{l=\langle N\rangle} a_l b_{k-l}$.

First Difference $x[n]-x[n-1] \longleftrightarrow \left(1-e^{-jk(2\pi/N)}\right)a_k$.

Running Sum $\sum_{k=-\infty}^{n} x[k]$ (finite valued & periodic only if $a_0=0$) $\longleftrightarrow \left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$

Conjugate Symmetry for Real signals. $x[n]$ real $\longleftrightarrow \begin{cases} a_k = a^*_{-k} \\ Re\{a_k\} = Re\{a_{-k}\}. \\ Im\{a_k\} = -Im\{a_{-k}\}. \\ |a_k| = |a_{-k}| \\ \measuredangle a_k = -\measuredangle a_{-k}. \end{cases}$

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Real & Even Signals. $x[n] \longleftrightarrow a_K$ – Real & even

Real & Odd signals. $x[n] \longleftrightarrow a_K$ – purely imag & odd.

Even - Odd Decomposition of Real Signals.

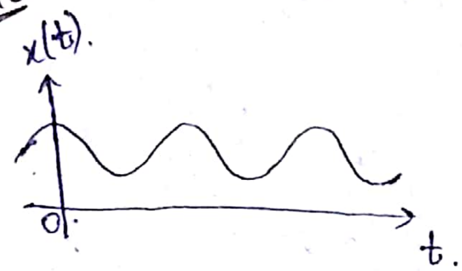
$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $\longleftrightarrow$ $Re\{a_k\}$

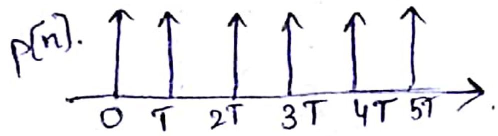
$x_o[n] = Od\{x[n]\}$ $[x[n]$ real$]$. $\longleftrightarrow$ $j Im\{a_K\}$

*Parseval's Relation for Periodic Signals $\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_K|^2$

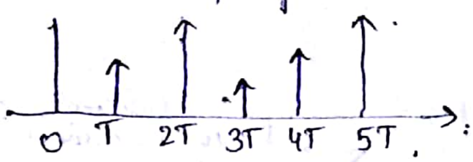
$\downarrow$ Avg power in one period. $\downarrow$ Avg power in Kth harmonic component.

8/11/18.





T → sampling interval.



$x(nT) = x(t)\, p(nT)$.

Sampling Theorem :- Sampling Rate is greater than or equal to twice the maximum frequency. Only then we can construct $x(t)$ from $x[n]$.

$$P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = p(t)\, x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)\, x(t).$$

$(x_p(t) = x(nt) \rightarrow$ by applying sifting property$)$.

*If it is Multiplication in time domain, it will be convolution in frequency domain.

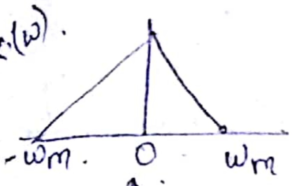
$$x_p(\omega) = P(\omega) * X(\omega) \quad P(\omega) = 2\pi \sum \delta(\omega - k\omega_s). \qquad \omega_s = \frac{2\pi}{T}$$

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * 2\pi \sum_{K} \delta(\omega - K\omega_s)$$

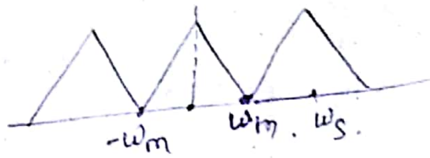
$$= \sum_{K} \int x(K) \cdot \delta(\omega - k\omega_s - K)\, dK$$

$$= \sum_{K} X(\omega - K\omega_s)$$

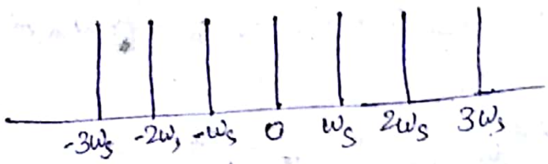
$X(\omega)$



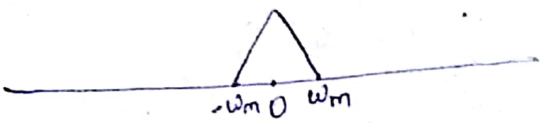
$\omega_s > 2\omega_m$., so that there won't be any overlap.

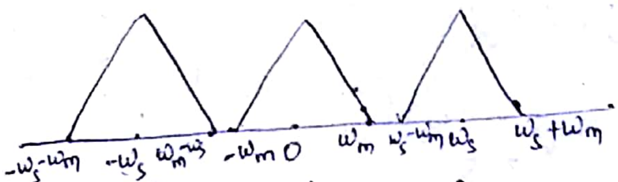


$P(\omega)$.

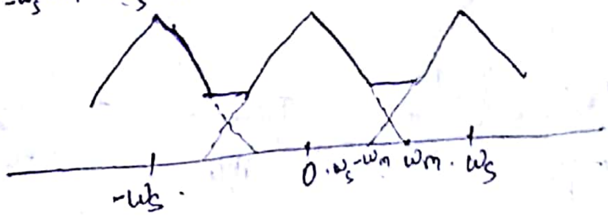


$X(\omega)$.

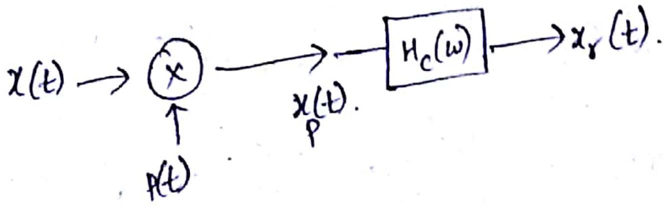




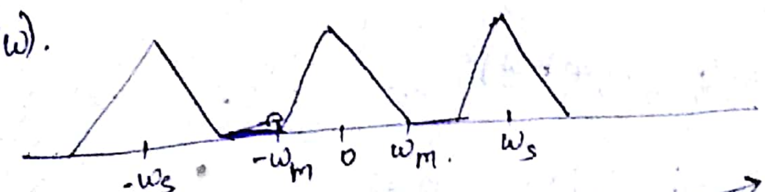
$\omega_s \geq 2\omega_m$. ✓



$\omega_s < 2\omega_m$. X.
Leads to Aliasing.



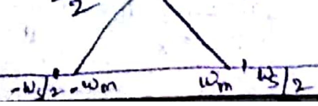
$X_p(\omega)$.



$H(\omega)$.

→ ideal low pass filter.

$X_r(\omega)$.



$\rightarrow (F^{-1}) \rightarrow x_r(t)$

Sampling

(1) $x(\omega)$ is band limited. $\omega_m \rightarrow$ finite value

(2) $\omega_s > 2\omega_m$.

(3) $\omega_s = \frac{2\pi}{T}$, $F_s = \frac{\omega_s}{2\pi}$

$x[n]$. Goal : Recover $x(t)$. $x_p(t) \longrightarrow X_p(\omega)$.

(4) $\hat{X}_r(\omega) = X_p(\omega)\, H(\omega)$.

$\omega_m < \omega_c < \omega_s - \omega_m$. $\omega_c \rightarrow$ cutoff frequency of lowpass filter

$$H(\omega) = \begin{cases} T & -\omega_c < \omega < \omega_c \\ 0 & \text{elsewhere} \end{cases}$$

$h(t) = ?$

$h(t) = \mathcal{F}^{-1}\{H(\omega)\}$. — inverse fourier transform

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{+j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} T e^{+j\omega t}\, d\omega.$$

$$= \frac{T}{2\pi}\left(\frac{+e^{+j\omega t}}{jt}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{+T}{2\pi j t}\left[e^{+j\omega_c t} - e^{-j\omega_c t}\right)$$

$$h(t) = \frac{T\left(e^{j\omega_c t} - e^{-j\omega_c t}\right)}{2\pi j t} = \frac{T \sin(\omega_c t)}{\pi t} \rightarrow \text{Sinc}$$

$$h(t) = \frac{\omega_c T}{\pi}\,\text{sinc}(\omega_c t).$$

$$x_r(t) = x_p(t) * h(t).$$

$$= \sum_n x(nT)\,\delta(t-nT) * h(t).$$

$$= \sum_n x(nT)\int \delta(\tau - nT)\, h(t-\tau)\, d\tau.$$

$$= \sum_n x(nT)\, h(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\cdot\frac{T}{\pi}\,\frac{\sin\omega_c(t-nT)}{(t-nT)}.$$

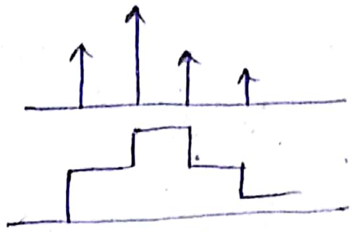
$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\frac{T}{\pi}\,\frac{\sin\omega_c(t-nT)}{(t-nT)}}$$ $\longrightarrow$ Exact Reconstruction.

* Zero-Order Hold.

$$x(t) \longrightarrow \underset{\uparrow \atop p(t)}{\otimes} \xrightarrow{x_p(t)} \boxed{H_0(\omega)} \xrightarrow[x_0(t)]{} \boxed{H_r(\omega)} \longrightarrow r(t).$$

Sample & hold.

$$h_0(t) = \begin{cases} 1 & 0<t<T \\ 0 & \text{else.} \end{cases}$$



Sample & hold.

$$x_0(t) = h_0(t) * x_p(t).$$

$$H_0(\omega) = e^{-j\omega\frac{T}{2}}\,\frac{2\sin(\omega T/2)}{\omega}.$$

$$H(\omega) = H_0(\omega)H_r(\omega). \Rightarrow H_r(\omega) = \frac{H(\omega)}{H_0(\omega)}.$$

* Interpolation.