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BITS Pilani
Pilani Campus

CS/IS F214
Logic in Computer Science

MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Rules for Negation

ND: Negation–Elimination Rule

$$\frac{\phi \qquad \neg\phi}{\bot}\ \neg e$$

This symbol – *read* ***bottom*** – denotes a contradiction.

This rule is also referred to as *contradiction introduction* (_|_ introduction)

ND: Contradiction-Elimination Rule

$$\frac{\perp}{\phi}\ \perp e$$

You can infer anything from a contradiction!

ND: Negation–Introduction Rule

$$\frac{\begin{array}{c}\text{Assume } \phi \\ \vdots \\ \bot\end{array}}{\neg \phi} \; \neg i$$

This rule is read as:

if you can derive a contradiction assuming φ is true then φ must be false (i.e. ¬ φ must be true)

Exercise: *Prove :*

i_am_god --> happy, i_am_god --> ¬ happy |-- ¬ i_am_god

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MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Rules for Double Negation

ND – Rules for Double negation

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg e \qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg i$$

Double Negation – Example

Exercise:

Prove the following sequent: **$p, \neg\neg(q \wedge r) \;|\!-\!- \neg\neg p \wedge r$**

	Deduction	Explanation
1	**p**	**Premise**
2	**$\neg\neg(q \wedge r)$**	**Premise**
3	**$q \wedge r$**	**$\neg\neg$e 2**
4	**r**	**$\wedge$e2 3**
5	**$\neg\neg p$**	**$\neg\neg$i 1**
6	**$\neg\neg p \wedge r$**	**$\wedge$i 3,4**

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MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Rules for Disjunction

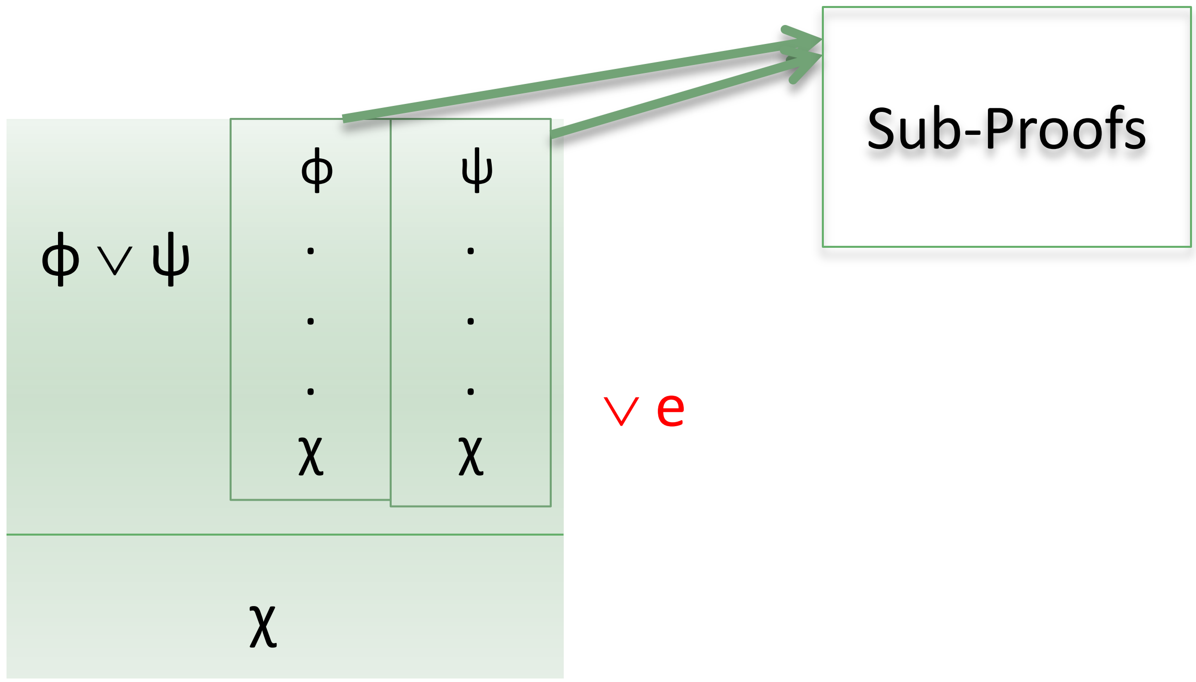
ND: Proof Rules: OR-introduction

Rules for Disjunction Introduction

$$\frac{\phi}{\phi \vee \psi} \; \vee i_1 \qquad \frac{\psi}{\phi \vee \psi} \; \vee i_2$$

ND: Proof Rules: OR Elimination

Rule for Disjunction Elimination



Proofs using rules for Disjunction – Example 1

- Recall from Boolean algebra, the distribution rule
  - $p \wedge (q \vee r)$ "is equivalent to" $(p \wedge q) \vee (p \wedge r)$
- *To prove that the two formulas are equivalent*
  - we must *prove that one of them can be derived from the other* and *vice versa*

- For this example:
  - We will prove
    - $\mathbf{p \wedge (q \vee r) \mid\text{--} (p \wedge q) \vee (p \wedge r)}$
  - and this will be an exercise for you:
    - $\mathbf{(p \wedge q) \vee (p \wedge r) \mid\text{--} p \wedge (q \vee r)}$

Proofs using rules for Disjunction – Example 1

- Prove:
  - $p \wedge (q \vee r)$ |-- $(p \wedge q) \vee (p \wedge r)$

	Deduction	Explanation	
	$p \wedge (q \vee r)$	**Premise**	
?	$q \vee r$		
	q	**Assumption**	Sub-proof
	…		
?	$(p \wedge q) \vee (p \wedge r)$		
	r	**Assumption**	Sub-proof
	…		
2	$(p \wedge q) \vee (p \wedge r)$		
1	$(p \wedge q) \vee (p \wedge r)$	**$\vee$e ?-2, ?-?,**	

Proofs using rules for Disjunction – Example 1

- Prove:
  - $p \wedge (q \vee r) \quad |\text{--} \quad (p \wedge q) \vee (p \wedge r)$

	Deduction	Explanation	
	$p \wedge (q \vee r)$	Premise	
	p		
?	$q \vee r$		
	q	Assumption	
	…		
5	$(p \wedge q) \vee (p \wedge r)$		
4	r	Assumption	
3	$p \wedge r$	$\wedge i$?, 4	Sub-proof
2	$(p \wedge q) \vee (p \wedge r)$	$\vee i_2$ 3	
1	$(p \wedge q) \vee (p \wedge r)$	$\vee e$ 4-2, ?-5,	

Proofs using rules for Disjunction – Example 1

- Prove:
  - $p \wedge (q \vee r) \quad |\text{--} \quad (p \wedge q) \vee (p \wedge r)$

	Deduction	Explanation	
10	**$p \wedge (q \vee r)$**	**Premise**	
9	**p**	**$\wedge e_1$ 10**	
8	**q|r**	**$\wedge e_2$ 10**	
7	**q**	**Assumption**	Sub-proof
6	**$p \wedge q$**	**$\wedge i$ 9,7**	
5	**$(p \wedge q) \vee (p \wedge r)$**	**$\vee i_1$ 6**	
4	**r**	**Assumption**	
3	**$p \wedge r$**	**$\wedge i$ 9, 4**	
2	**$(p \wedge q) \vee (p \wedge r)$**	**$\vee i_2$ 3**	
1	**$(p \wedge q) \vee (p \wedge r)$**	**$\vee e$ 4-2, 7-5,**	

Proofs using rules for Disjunction - Example

- Consider the following program fragment in C:
  - if (x>y) { m = x; }
  - else /* x <= y */ { m = y; }
- *Prove the post-condition* (i.e. condition after execution)
  - m *holds the maximum of the two values* x *and* y
- How would the proof proceed?

1 ϕ is ____________
2 ψ is _____________
3 $\phi \vee \psi$ is true.
4 χ is ______________
5 Now, apply disjunction elimination.

- **Exercise:** *Prove:*
  - **$\neg$rains|wet_road |-- rains --> wet_road**

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MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Derived Rules

Modus Tollens

$$\frac{\phi \dashrightarrow \psi \qquad \neg \psi}{\neg \phi}$$

MT (modus tollens)

What is the relation between this and modus ponens?

How do you derive it?

Proof by Contradiction

$$\dfrac{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}{\phi}\ \text{PBC}$$

- One can infer anything from a contradiction.
- But in this case the contradiction resulted from an assumption i.e. $\neg\phi$.
    - Therefore it is meaningful to infer ϕ that the assumption led to the contradiction
        - Implicit meta-assumption: that the proof is sound!
- In fact one must infer ϕ to eliminate the assumption $\neg\phi$.
    - Why?