Tuning the SOC gap and M point higher order Van Hove singularities in the surface bands of Sr_2RuO_4

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Loading the Library

We first load the library by executing the following:

```
In[1]:= AppendTo[$Path, StringDelete[NotebookDirectory[], "/Sr2RuO4"] <> "Package"];
<< BandUtilities`
```

Outline of the strategy

We will begin by loading and interpolating the tight-binding models for various values of θ and then separately load and interpolate through the models with nematic term added. We will then interpolate linearly between the nematicity free model and the model with the nematic term to generate a continuously tunable model.

Interpolating between the Wannier90 files

Ingredients needed for symmetrization

In[3]:=
```
ρ = {{0, -1, 0}, {1, 0, 0}, {0, 0, 1}};
(*The matrix part of the C4 rotation about one of the Ru atoms.*)
Uρ = {{0, -1, 0}, {1, 0, 0}, {0, 0, -1}};
(*Representation of C4 rotation in the d-orbital space.*)

rxy = {{0, 1, 0}, {1, 0, 0}, {0, 0, 1}}; (*x↔y reflection.*)
Urxy = {{-1, 0, 0}, {0, 1, 0}, {0, 0, -1}};
(*Representation of the x↔y reflection.*)

c0 = {1/4, 3/4, 0}; (*Location of one of the Ru atoms,
which we shall use as the centre of the C4 rotation.*)

AtomLocations =
  {{3/4, 1/4, 0}, {3/4, 1/4, 0}, {3/4, 1/4, 0}, {1/4, 3/4, 0}, {1/4, 3/4, 0}, {1/4, 3/4, 0}};
(*Location of the basis atoms in the fractional
 coordinates within the unit cell.*)

SymmetryList = {}; (*This is a multidimensional list which we shall
 construct below. Each element is a sublist corresponding to a particular
 symmetry. First element of the sublist is real space symmetry's matrix part,
second element is the translation part and the third element is
 its representation in the Hilbert space. Our point group is D4,
or D8 in some pepole's notation.*)
Do[
  AppendTo[SymmetryList,
   {MatrixPower[ρ, i], (c0 - MatrixPower[ρ, i].c0) // Simplify,
    ArrayFlatten[IdentityMatrix[2]⊗MatrixPower[Uρ, i]]}];

  AppendTo[SymmetryList,
   {MatrixPower[ρ, i].rxy, (c0 - MatrixPower[ρ, i].c0) // Simplify,
    ArrayFlatten[IdentityMatrix[2]⊗MatrixPower[Uρ, i]].
     ArrayFlatten[{{0, 1}, {1, 0}}⊗Urxy]}];
  , {i, 1, 4}];
```

Interpolation with symmetrization

We now interpolate between the Wannier90 data for 13 different rotations of the RuO octahedra: 0° through 12° to generate a continuously θ dependent Hamiltonian, whilst ensuring that the interpolated model has the right symmetry (for which we are feeding the list SymmetryList):

In[11]:=
```
FileLocn = NotebookDirectory[] <> "unrenormalized2/";

W90Data = LoadInterpolateTBMs[{FileLocn <> "Sr2RuO4_0deg.dat",
    FileLocn <> "Sr2RuO4_1deg.dat", FileLocn <> "Sr2RuO4_2deg.dat",
    FileLocn <> "Sr2RuO4_3deg.dat", FileLocn <> "Sr2RuO4_4deg.dat",
    FileLocn <> "Sr2RuO4_5deg.dat", FileLocn <> "Sr2RuO4_6deg.dat",
    FileLocn <> "Sr2RuO4_7deg.dat", FileLocn <> "Sr2RuO4_8deg.dat",
    FileLocn <> "Sr2RuO4_9deg.dat", FileLocn <> "Sr2RuO4_10deg.dat",
    FileLocn <> "Sr2RuO4_11deg.dat", FileLocn <> "Sr2RuO4_12deg.dat"},
   {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, "Symmetries" → SymmetryList,
   "OrbitalCentres" → AtomLocations, "ParameterSymbol" → "θ"];

Hamltn1[k1_, k2_, θ_] = W90Data⟦1⟧;
```

```
Invalid or no lattice vectors provided! Using unit vectors.

Tight-binding data files loaded.

Space dimension: 3

Number of orbitals: 6

Number of supercells: 49

Number of models/data-files to interpolate: 13

Number of lines in the data files:
 {741, 853, 853, 853, 845, 853, 853, 853, 853, 853, 853, 853, 853}

Interpolating between the Hamiltonians.

Valid symmetrization scheme provided! We will symmetrize the loaded Hamiltonian.

Number of supercells after symmetrization: 79
```

Phenomenological spin orbit term

We now construct the phenomenological SOC terms:

In[14]:=
```
Lx = {{0, 0, ⅈ}, {0, 0, 0}, {-ⅈ, 0, 0}};
Ly = {{0, 0, 0}, {0, 0, ⅈ}, {0, -ⅈ, 0}};
Lz = {{0, -ⅈ, 0}, {ⅈ, 0, 0}, {0, 0, 0}};

Hupup = 1/2 Lz;
Hupdown = 1/2 (Lx + ⅈ Ly);
Hdownup = 1/2 (Lx - ⅈ Ly);
Hdowndown = -1/2 Lz;
Zero2 = ConstantArray[0, {3, 3}];


HSOC = Join[Join[Hupup, Zero2, Hupdown, Zero2, 2],
   Join[Zero2, Hupup, Zero2, Hupdown, 2], Join[Hdownup, Zero2,
    Hdowndown, Zero2, 2], Join[Zero2, Hdownup, Zero2, Hdowndown, 2]];
```

Reconstructing the full interpolated Hamiltonian

The full Hamiltonian incorporating SOC with tunable SOC strength and tunable θ. A version for the $\sqrt{2} \times \sqrt{2}$ unit cell that is tailor made for fitting to the ARPES data is also defined (since the bulk has one Ru atom per unit cell, its Brillouin zone has twice the area of the default surface band BZ, which is also ‘rotated’ by 45°).

In[23]:=
```
Hamiltonian1[k1_, k2_, θ_, λ_] =
  0.5 (ArrayFlatten[IdentityMatrix[2]⊗Hamltn1[k1, k2, θ]] +
     λ HSOC + ComplexExpand[ConjugateTranspose[
       ArrayFlatten[IdentityMatrix[2]⊗Hamltn1[k1, k2, θ]] + λ HSOC]]);
```

We can check the eigenvalues at a high symmetry point to see if the symmetrisation was successful, and has eliminated the weak splitting of degeneracies at the high symmetry points

Loading and interpolating the nematic models

In[24]:=
```
FileLocn2 = NotebookDirectory[] <> "renormalized/";

W90Data2 = LoadInterpolateTBMs[
   {FileLocn2 <> "Sr2RuO4_EXP_Monolayer_1theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_1theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_2theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_3theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_4theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_5theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_6theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_7theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_8theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_9theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_10theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <> "Sr2RuO4_EXP_Monolayer_11theta_0phi1_0phi2_hr_sym_soc_nem.dat",
    FileLocn2 <>
     "Sr2RuO4_EXP_Monolayer_12theta_0phi1_0phi2_hr_sym_soc_nem.dat"},
   {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, "ParameterSymbol" → "θ"];

Hamiltonian2[k1_, k2_, θ_] = W90Data2[[1]];
```

```
Invalid or no lattice vectors provided! Using unit vectors.
Tight-binding data files loaded.
Space dimension: 3
Number of orbitals: 12
Number of supercells: 49
Number of models/data-files to interpolate: 13
Number of lines in the data files:
 {1723, 1723, 1723, 1723, 1707, 1723, 1723, 1723, 1723, 1723, 1723, 1723, 1723}
Interpolating between the Hamiltonians.
No valid symmetrization scheme
  provided! We will not symmetrize the loaded Hamiltonian.
```

Comparing the nematic and symmetric models

For the sake of plotting, let us define the Hamiltonians evaluated at $\theta = 8°$ with and without the nematic term

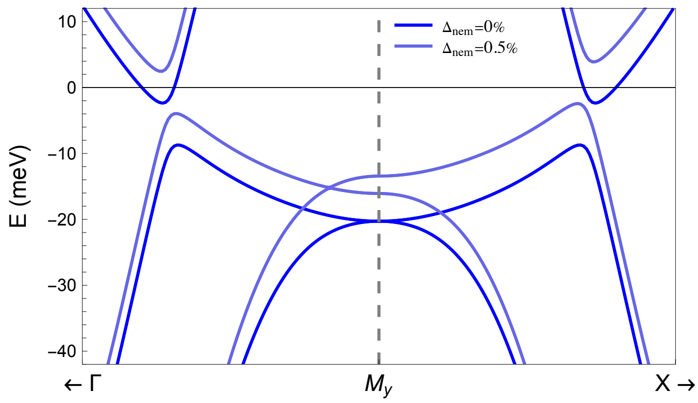
In[27]:=
```
HamTemp[k_] := 1000 × 0.2444 Hamiltonian1[k[[1]] - k[[2]], k[[1]] + k[[2]], 8, 0.175]
HamTempNem[k_] := 1000 Hamiltonian2[k[[1]] - k[[2]], k[[1]] + k[[2]], 8]
```

Using these the bands can be generated

In[29]:=
```
HSymmPts =
  {{{0, π - 3.869 × 0.36}, "← Γ"}, {{0, π}, "My"}, {{3.869 × 0.36, π}, "X →"}};
Bnd1 = GenerateBandStructure[HamTemp, HSymmPts];
Bnd2 = GenerateBandStructure[HamTempNem, HSymmPts];
```

In[32]:=
```
plt1 = PlotBandStructure[Bnd1, {-42, 12},
   "AspectRatio" → 3/5, "yLabel" → {(*Label*)"E (meV)", (*Font Size*)
     12, (*Font Color*)Black, (*Font Family*)"Arial"},
   "xLabel" → {12, Black, "Arial"}, "yTicks" → {9, Black, "Arial"},
   "LineThickness" → 0.005, "LineColorScheme" → {RGBColor[0, 0, 1]},
   "PlotKeyLegend" → {Style["Δnem=0%", FontSize → 9, FontFamily → "Times"],
     Scaled[{0.51, 1.01 - 1 / 17.}]}];

plt2 = PlotBandStructure[Bnd2, {-42, 12},
   "AspectRatio" → 3/5, "yLabel" → {(*Label*)"E (meV)", (*Font Size*)
     12, (*Font Color*)Black, (*Font Family*)"Arial"},
   "xLabel" → {12, Black, "Arial"}, "yTicks" → {9, Black, "Arial"},
   "LineThickness" → 0.005, "LineColorScheme" → {RGBColor[0.4, 0.4, 0.9]},
   "PlotKeyLegend" → {Style["Δnem=0.5%", FontSize → 9, FontFamily → "Times"],
     Scaled[{0.51, 1.01 - 2 / 17.}]}];

Show[plt1, plt2]
```

Out[34]=



The degeneracy at the M_y point has been lifted which is to be expected since the nematic term breaks the r_{xy} reflection symmetry that causes the degeneracy.

A continuously $\theta(°)$ and $\Delta_{nem}(\%)$ tuned model

We will linearly interpolate between the nematicity free model and the nematic model for $\Delta_{nem} = 0.5\%$ (which corresponds to the QPI best fit). This can be obtained as follows: $H(\boldsymbol{k}, \Delta) = H_0(\boldsymbol{k}) + \Delta \frac{(H_{0.5}(\boldsymbol{k}) - H_0(\boldsymbol{k}))}{0.5-0} = (1 - 2\,\Delta)\, H_0(\boldsymbol{k}) + 2\,\Delta\, H_{0.5}(\boldsymbol{k})$.

In[35]:=
```
HamiltonianNemTunedmeV[k1_, k2_, θ_, Δ_] =
  (1 - 2 Δ) 1000 × 0.2444 Hamiltonian1[k1 - k2, k1 + k2, θ, 0.175] +
   1000 × 2 Δ Hamiltonian2[k1 - k2, k1 + k2, θ];

HamltnTuned[k_] := HamiltonianNemTunedmeV[k⟦1⟧, k⟦2⟧, k⟦3⟧, k⟦4⟧]

HamTempNem2[k_] :=
 HamiltonianNemTunedmeV[k⟦1⟧, k⟦2⟧, 8, 1(*Nematicity set to 1%*)]
```

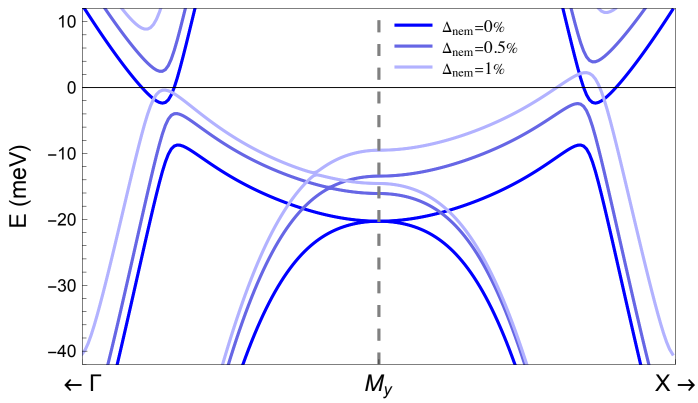
Checking the interpolation

In[38]:=
```
Bnd3 = GenerateBandStructure[HamTempNem2, HSymmPts];
```

In[39]:=
```
plt3 = PlotBandStructure[Bnd3, {-42, 12},
   "AspectRatio" → 3/5, "yLabel" → {(*Label*)"E (meV)", (*Font Size*)
     12, (*Font Color*)Black, (*Font Family*)"Arial"},
   "xLabel" → {12, Black, "Arial"}, "yTicks" → {9, Black, "Arial"},
   "LineThickness" → 0.005, "LineColorScheme" → {RGBColor[0.7, 0.7, 1]},
   "PlotKeyLegend" → {Style["Δnem=1%", FontSize → 9, FontFamily → "Times"],
     Scaled[{0.51, 1.01 - 3 / 17.}]}];
```

```
Show[plt1, plt2, plt3]
```

Out[40]=



We can see that the interpolation scheme has worked.

Iterating through Δ_{nem} and hunting for HOS

Let us outline the strategy that we shall adopt below. We iterate through a few values of Δ_{nem} including 0%, 0.5%, 1% and 2%. At each Δ_{nem}, we will try to tune the M point singularity and the SOC gap critical points to a HOS. We note down the θ_{crit} and the series expansion of the four bands.

Now the bands(points) of interest are 5&6 (M point), 7&8 (M point), 7&8 (SOC gap) and 9&10 (SOC gap). For 5&6 (M point) and 7&8 (M point), tuning to HOS is simpler. We are already at a critical point, namely the high symmetry M point. We compute det(Hes) as a function of $\delta\theta$ and set it to zero to solve for $\delta\theta$. If we repeat this a few times we get θ_{crit}. But for the SOC gap critical points, we have to adopt a different strategy since the location of the critical point is not fixed and changes with θ and Δ_{nem}.

We will use a two-pronged strategy: first we choose a range of θ to investigate, say 8° to 10°. We locate the $\boldsymbol{k}_{\text{VHS}}$ for the angles at the end points of the range and compute the Hessian determinant. If they are of opposite sign, we expect the Hessian determinant to vanish somewhere in between. We will then use binary search to locate this θ_{crit}. We have to do this separately for both the bands, and on either sides of the SOC gap (which have been made inequivalent by the nematic term).

Firstly, we have to identify the range of $\boldsymbol{k}$ in which the SOC critical points seem to exist by plotting the dispersion for Δ_{nem} = 0%, 0.5%, 1% and 2% and $\theta = 7°, 8°, \ldots, 11°$. This is needed since we need to identify a suitable range for the LocateCriticalPoint routine to work to find the actual $\boldsymbol{k}_{\text{VHS}}$.

Evolution of the band-structure

We plot a spaghetti plot showing the evolution of the band-structure with θ for different choices of Δ_{nem} (0%, 1% and 2%).

In[41]:=
```
HSymmPtsNoLabel =
  {{{0, π - 3.869 × 0.35}, ""}, {{0, π}, ""}, {{3.869 × 0.35, π}, ""}};
AngleList = {7, 8, 9, 10, 11};

OpacityList = {0.1, 0.2, 0.35, 0.5, 1};
ColourChoice = {67/255, 49/255, 12/255}(*{45/255, 38/255, 87/255}*);
(*This for one choice of colour scheme -
 with a fixed colour and varying transparency given by the alpha channel.*)
```

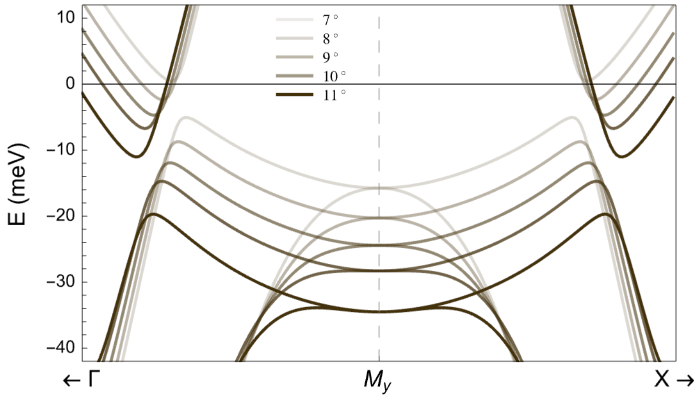
1) $\Delta_{\text{nem}} = 0 \times \%$

In[46]:=
```
BandDataList1 = First@Last@Reap[Do[
      HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], angle, 0.];

      Sow[GenerateBandStructure[HamCurr, HSymmPts]];
      , {angle, AngleList}]];
```

In[47]:=
```
BandPlotList1 = First@Last@Reap[Do[
    Sow[PlotBandStructure[BandDataList1[[i]], {-42, 12}(*meV range*),
      "AspectRatio" → 3/5, "yLabel" → {(*Label*)"E (meV)", (*Font Size*)
        12, (*Font Color*)Black, (*Font Family*)"Arial"},
      "xLabel" → {12, Black, "Arial"}, "yTicks" → {9, Black, "Arial"},
      "LineThickness" → 0.005, "LineColorScheme" → {RGBColor[ColourChoice[[1]],
         ColourChoice[[2]], ColourChoice[[3]], OpacityList[[i]]]},
      "PlotKeyLegend" → {Style[ToString[AngleList[[i]]] °, FontSize → 8,
         FontFamily → "Times"], Scaled[{0.31, 1.01 - i / 19.}]}]];
    , {i, 1, Length[AngleList]}]];
```

In[48]:=
```
SpaghettiPlot1 = Show[BandPlotList1]
```

Out[48]=

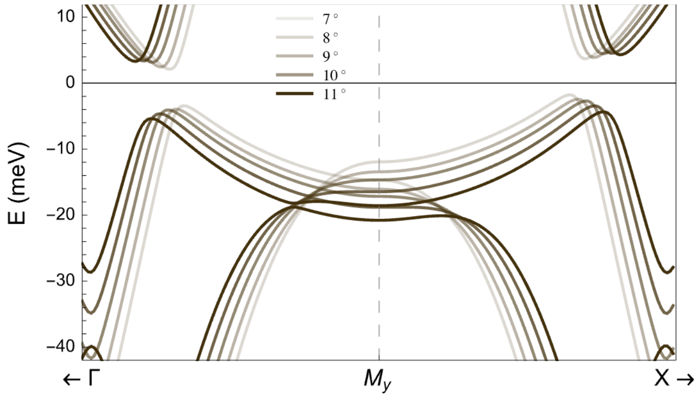


2) $\Delta_{nem} = 0.5 \times \%$

In[49]:=
```
BandDataList2 = First@Last@Reap[Do[
    HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], angle, 0.5];

    Sow[GenerateBandStructure[HamCurr, HSymmPts]];
    , {angle, AngleList}]];
```

In[50]:=
```
BandPlotList2 = First@Last@Reap[Do[
    Sow[PlotBandStructure[BandDataList2[[i]], {-42, 12}(*meV range*),
      "AspectRatio" → 3/5, "yLabel" → {(*Label*)"E (meV)", (*Font Size*)
        12, (*Font Color*)Black, (*Font Family*)"Arial"},
      "xLabel" → {12, Black, "Arial"}, "yTicks" → {9, Black, "Arial"},
      "LineThickness" → 0.005, "LineColorScheme" → {RGBColor[ColourChoice[[1]],
         ColourChoice[[2]], ColourChoice[[3]], OpacityList[[i]]]},
      "PlotKeyLegend" → {Style[ToString[AngleList[[i]]] °, FontSize → 8,
         FontFamily → "Times"], Scaled[{0.31, 1.01 - i / 19.}]}]];
    , {i, 1, Length[AngleList]}]];
```

In[51]:=
```
SpaghettiPlot2 = Show[BandPlotList2]
```

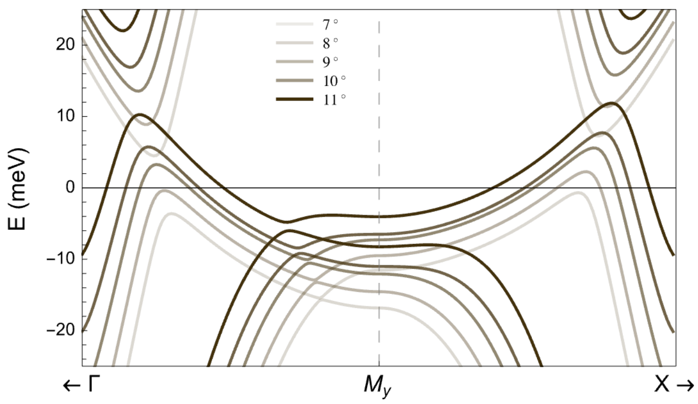
Out[51]=


3) $\Delta_{nem} = 1 \times \%$

In[52]:=
```
BandDataList3 = First@Last@Reap[Do[
        HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], angle, 1.];

        Sow[GenerateBandStructure[HamCurr, HSymmPts]];
        , {angle, AngleList}]];
```

In[53]:=
```
BandPlotList3 = First@Last@Reap[Do[
        Sow[PlotBandStructure[BandDataList3[[i]], {-25, 25}(*meV range*),
           "AspectRatio" → 3/5, "yLabel" → {(*Label*)"E (meV)", (*Font Size*)
             12, (*Font Color*)Black, (*Font Family*)"Arial"},
           "xLabel" → {12, Black, "Arial"}, "yTicks" → {9, Black, "Arial"},
           "LineThickness" → 0.005, "LineColorScheme" → {RGBColor[ColourChoice[[1]],
              ColourChoice[[2]], ColourChoice[[3]], OpacityList[[i]]]},
           "PlotKeyLegend" → {Style[ToString[AngleList[[i]]] °, FontSize → 8,
              FontFamily → "Times"], Scaled[{0.31, 1.01 - i / 19.}]}]];
        , {i, 1, Length[AngleList]}]];
```

In[54]:=
```
SpaghettiPlot3 = Show[BandPlotList3]
```

Out[54]=


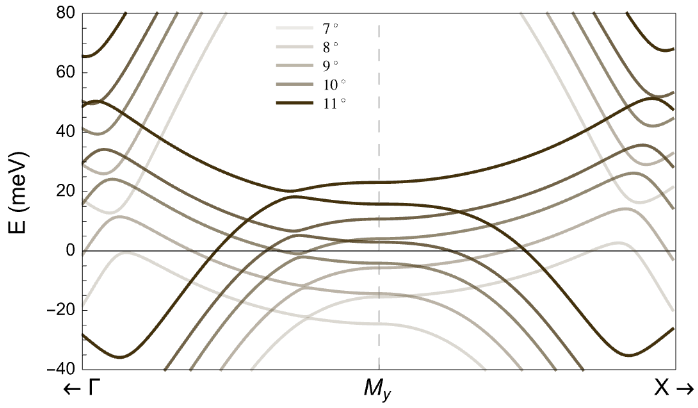
3) $\Delta_{nem} = 2 \times \%$

In[55]:=
```
BandDataList4 = First@Last@Reap[Do[
        HamCurr[k_] := HamiltonianNemTunedmeV[k⟦1⟧, k⟦2⟧, angle, 2.];

        Sow[GenerateBandStructure[HamCurr, HSymmPts]];
        , {angle, AngleList}]];
```

In[56]:=
```
BandPlotList4 = First@Last@Reap[Do[
        Sow[PlotBandStructure[BandDataList4⟦i⟧, {-40, 80} (*meV range*),
            "AspectRatio" → 3/5, "yLabel" → {(*Label*)"E (meV)", (*Font Size*)
              12, (*Font Color*)Black, (*Font Family*)"Arial"},
            "xLabel" → {12, Black, "Arial"}, "yTicks" → {9, Black, "Arial"},
            "LineThickness" → 0.005, "LineColorScheme" → {RGBColor[ColourChoice⟦1⟧,
               ColourChoice⟦2⟧, ColourChoice⟦3⟧, OpacityList⟦i⟧]},
            "PlotKeyLegend" → {Style[ToString[AngleList⟦i⟧] °, FontSize → 8,
               FontFamily → "Times"], Scaled[{0.31, 1.01 - i / 19.}]}]];
        , {i, 1, Length[AngleList]}]];
```

In[57]:=
```
SpaghettiPlot4 = Show[BandPlotList4]
```

Out[57]=



In[64]:=
```
NumberForm[{1000.3, 34.56123, 0.0034, 1.4596}, {3, 3}]
```

NumberForm : Requested number precision is lower than number of digits shown; padding with zeros.

Out[64]//NumberForm=
```
{1000.000, 34.600, 0.003, 1.460}
```

Evolution of the nature of the critical points in the SOC gap

In this section, we look at the evolution of the nature of the critical points as we change Δ_{nem} and θ, that is whether they are maxima, minima or saddles.

SOC gap upper VHS - Band 9

In[65]:=
```
Do[
 Print["Δnem=", Delt, "%:"];

 Do[

  HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θCurr, Delt];

  kVHSTrialLow = {0.6 × 3.869 × 0.36, π // N};
  kVHSTrialUp = {3.869 × 0.36, π // N};

  (*Finding true kVHS:*)
  {kVHS, CritPtFound} = LocateCriticalPoint[
    HamCurr, kVHSTrialLow, kVHSTrialUp, 9, "Messages" → False,
    "SanityChecks" → True, "MaxIterations" → 24, "Resolution" → 10^-2];

  If[CritPtFound == True,
   TaylorPoly =
    Chop[Total[TaylorExpandBand[HamCurr, kVHS, 9, 2, "Messages" → False][[1]]] /.
      {p1 → kx, p2 → ky}, 10^-2];

   Print["\t", "θ = ", θCurr, ", kcrit = ",
    NumberForm[kVHS, {4, 2}], ", Tayl = ", NumberForm[TaylorPoly, {4, 2}]];
   ,
   Print["Critical point could not be found for θ = ", θCurr];
  ]

  , {θCurr, 8, 11, 1}];

 , {Delt, {0, 0.5, 1, 2}}];
```

Δ_{nem}=0%:

θ = 8, k_{crit} = {1.02, 3.14}, Tayl = $-2.36 + 529.00\,k_x^2 - 9.48\,k_y^2$

θ = 9, k_{crit} = {1.06, 3.14}, Tayl = $-4.72 + 503.40\,k_x^2 + 4.03\,k_y^2$

θ = 10, k_{crit} = {1.10, 3.14}, Tayl = $-6.73 + 483.80\,k_x^2 + 16.24\,k_y^2$

θ = 11, k_{crit} = {1.14, 3.14}, Tayl = $-11.04 + 472.70\,k_x^2 + 26.86\,k_y^2$

Δ_{nem}=0.5%:

θ = 8, k_{crit} = {1.01, 3.14}, Tayl = $3.89 + 508.00\,k_x^2 - 10.08\,k_y^2$

θ = 9, k_{crit} = {1.05, 3.14}, Tayl = $4.46 + 485.60\,k_x^2 + 3.54\,k_y^2$

θ = 10, k_{crit} = {1.09, 3.14}, Tayl = $4.52 + 468.00\,k_x^2 + 15.86\,k_y^2$

θ = 11, k_{crit} = {1.13, 3.14}, Tayl = $4.30 + 458.10\,k_x^2 + 26.58\,k_y^2$

Δ_{nem}=1%:

θ = 8, k_{crit} = {1.07, 3.14}, Tayl = $11.38 + 416.20\,k_x^2 - 13.27\,k_y^2$

θ = 9, k_{crit} = {1.11, 3.14}, Tayl = $15.79 + 403.50\,k_x^2 - 0.81\,k_y^2$

θ = 10, k_{crit} = {1.14, 3.14}, Tayl = $18.87 + 398.80\,k_x^2 + 10.21\,k_y^2$

θ = 11, k_{crit} = {1.18, 3.14}, Tayl = $23.71 + 404.70\,k_x^2 + 19.48\,k_y^2$

Δ_{nem}=2%:

θ = 8, k_{crit} = {1.25, 3.14}, Tayl = $29.07 + 404.80\,k_x^2 - 21.41\,k_y^2$

θ = 9, k_{crit} = {1.28, 3.14}, Tayl = $42.04 + 404.60\,k_x^2 - 12.77\,k_y^2$

θ = 10, k_{crit} = {1.31, 3.14}, Tayl = $51.90 + 414.00\,k_x^2 - 5.23\,k_y^2$

θ = 11, k_{crit} = {1.34, 3.14}, Tayl = $67.46 + 432.80\,k_x^2 + 1.15\,k_y^2$

We see that all the HOS points indeed likely lie in the range [8°,11°] since the coefficient of k_y^2 changes sign from - to + in this range for all the values of Δ_{nem}. Also, **the critical point changes from an ordinary saddle to an ordinary minimum** as we increase θ at each Δ_{nem}. Now for band 8 (downward dispersing):

SOC gap lower VHS - Band 8

In[66]:=
```
Do[
 Print["Δnem=", Delt, "%:"];

 Do[

  HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θCurr, Delt];

  kVHSTrialLow = {0.5 × 3.869 × 0.35, π // N};
  kVHSTrialUp = {0.95 × 3.869 × 0.35, π // N};

  (*Finding true kVHS:*)
  {kVHS, CritPtFound} = LocateCriticalPoint[
    HamCurr, kVHSTrialLow, kVHSTrialUp, 8, "Messages" → False,
    "SanityChecks" → False, "MaxIterations" → 35, "Resolution" → 10^-2];

  If[CritPtFound == True,
   TaylorPoly =
    Chop[Total[TaylorExpandBand[HamCurr, kVHS, 8, 2, "Messages" → False][[1]] /.
      {p1 → kx, p2 → ky}, 10^-2];

   Print["\t", "θ = ", θCurr, ", kcrit = ",
    NumberForm[kVHS, {3, 2}], ", Tayl = ", NumberForm[TaylorPoly, {4, 2}]];
   ,
   Print["Critical point could not be found for θ = ", θCurr];
  ]

  , {θCurr, 8, 12, 1}];

, {Delt, {0, 0.5, 1, 1.5}}];
```

Δ_{nem}=0%:

θ = 8, k_{crit} = {0.94, 3.14}, Tayl = $-8.71 - 417.40\, k_x^2 - 12.35\, k_y^2$

θ = 9, k_{crit} = {0.98, 3.14}, Tayl = $-11.91 - 400.30\, k_x^2 + 1.12\, k_y^2$

θ = 10, k_{crit} = {1.02, 3.14}, Tayl = $-14.71 - 388.40\, k_x^2 + 13.20\, k_y^2$

θ = 11, k_{crit} = {1.06, 3.14}, Tayl = $-19.70 - 382.80\, k_x^2 + 23.35\, k_y^2$

θ = 12, k_{crit} = {1.10, 3.14}, Tayl = $-20.80 - 380.60\, k_x^2 + 30.68\, k_y^2$

Δ_{nem}=0.5%:

θ = 8, k_{crit} = {0.93, 3.14}, Tayl = $-2.43 - 395.90\, k_x^2 - 12.99\, k_y^2$

θ = 9, k_{crit} = {0.97, 3.14}, Tayl = $-2.70 - 381.50\, k_x^2 + 0.59\, k_y^2$

θ = 10, k_{crit} = {1.01, 3.14}, Tayl = $-3.43 - 371.80\, k_x^2 + 12.79\, k_y^2$

θ = 11, k_{crit} = {1.05, 3.14}, Tayl = $-4.34 - 367.70\, k_x^2 + 23.10\, k_y^2$

θ = 12, k_{crit} = {1.09, 3.14}, Tayl = $-5.34 - 366.50\, k_x^2 + 30.63\, k_y^2$

Δ_{nem}=1%:

θ = 8, k_{crit} = {0.97, 3.14}, Tayl = $2.27 - 310.20\, k_x^2 - 16.77\, k_y^2$

θ = 9, k_{crit} = {1.01, 3.14}, Tayl = $5.60 - 304.00\, k_x^2 - 4.49\, k_y^2$

θ = 10, k_{crit} = {1.05, 3.14}, Tayl = $7.73 - 304.80\, k_x^2 + 6.01\, k_y^2$

θ = 11, k_{crit} = {1.09, 3.14}, Tayl = $11.88 - 313.60\, k_x^2 + 13.99\, k_y^2$

θ = 12, k_{crit} = {1.14, 3.14}, Tayl = $12.20 - 329.70\, k_x^2 + 18.32\, k_y^2$

Δ_{nem}=1.5%:

θ = 8, k_{crit} = {1.05, 3.14}, Tayl = $7.28 - 293.30\, k_x^2 - 20.74\, k_y^2$

θ = 9, k_{crit} = {1.09, 3.14}, Tayl = $14.80 - 292.90\, k_x^2 - 10.70\, k_y^2$

θ = 10, k_{crit} = {1.13, 3.14}, Tayl = $20.46 - 301.10\, k_x^2 - 2.82\, k_y^2$

θ = 11, k_{crit} = {1.17, 3.14}, Tayl = $30.37 - 318.80\, k_x^2 + 2.29\, k_y^2$

θ = 12, k_{crit} = {1.22, 3.14}, Tayl = $32.64 - 346.60\, k_x^2 + 4.02\, k_y^2$

Here too we see that for all the Δ_{nem} the HOS likely lies in the range [8°,12°]. Also, we see that **the critical point changes from an ordinary maximum to an ordinary saddle** as we increase θ at each Δ_{nem} up to 1.5%.

Evolution of the nature of the critical points at the M point

When $\Delta_{nem} = 0$, we have a pair of degenerate bands that are 90° rotated with respect to each other. However, when $\Delta_{nem} \neq 0$, the bands weakly split in energy. Since Δ_{nem} is small, we might still reasonably expect the bands to weakly preserve the $\frac{\pi}{2}$ rotation that was previously present. Nevertheless we will analyse the evolution of the Taylor expansion for both these bands separately.

Lower VHS

In[68]:=
```
Do[
  Print["Δnem=", Delt, "%:"];

  Do[

   HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θCurr, Delt];


   TaylorPoly = Chop[
     Total[TaylorExpandBand[HamCurr, {0, π // N}, 6, 2, "Messages" → False][[1]]] /.
      {p1 → kx, p2 → ky}, 10^-5];

   Print["\t", "θ = ", θCurr,
    ", Tayl = ", Chop[NumberForm[TaylorPoly, {4, 2}], 10^-2]];


   , {θCurr, 8, 12, 1}];

  , {Delt, {0.25, 0.5, 1, 1.75}}];
```

Δ_{nem}=0.25%:

θ = 8, Tayl = $-17.79 - 27.37\,k_x^2 + 16.37\,k_y^2$

θ = 9, Tayl = $-20.48 - 11.56\,k_x^2 + 15.70\,k_y^2$

θ = 10, Tayl = $-23.32 + 3.42\,k_x^2 + 15.00\,k_y^2$

θ = 11, Tayl = $-27.47 + 17.37\,k_x^2 + 14.25\,k_y^2$

θ = 12, Tayl = $-29.83 + 30.01\,k_x^2 + 11.50\,k_y^2$

Δ_{nem}=0.5%:

θ = 8, Tayl = $-16.08 - 27.77\,k_x^2 + 16.56\,k_y^2$

θ = 9, Tayl = $-17.16 - 12.08\,k_x^2 + 15.90\,k_y^2$

θ = 10, Tayl = $-18.82 + 2.79\,k_x^2 + 15.21\,k_y^2$

θ = 11, Tayl = $-20.78 + 16.61\,k_x^2 + 14.48\,k_y^2$

θ = 12, Tayl = $-22.94 + 29.09\,k_x^2 + 11.79\,k_y^2$

Δ_{nem}=1%:

θ = 8, Tayl = $-14.55 - 31.37\,k_x^2 + 17.08\,k_y^2$

θ = 9, Tayl = $-12.07 - 16.50\,k_x^2 + 16.56\,k_y^2$

θ = 10, Tayl = $-11.02 - 2.50\,k_x^2 + 16.04\,k_y^2$

θ = 11, Tayl = $-8.26 + 10.39\,k_x^2 + 15.50\,k_y^2$

θ = 12, Tayl = $-9.65 + 21.84\,k_x^2 + 13.23\,k_y^2$

Δ_{nem}=1.75%:

θ = 8, Tayl = $-14.45 - 38.72\,k_x^2 + 18.60\,k_y^2$

θ = 9, Tayl = $-6.16 - 26.19\,k_x^2 + 18.44\,k_y^2$

θ = 10, Tayl = $-0.57 - 14.58\,k_x^2 + 18.34\,k_y^2$

θ = 11, Tayl = $9.72 - 4.16\,k_x^2 + 18.30\,k_y^2$

θ = 12, Tayl = $9.88 + 4.74\,k_x^2 + 17.04\,k_y^2$

We notice that the critical point evolves from a **saddle to a minimum** as θ increases for each Δ_{nem} since the sign of the coefficient of k_x^2 changes from - to + while the coefficient of k_y^2 remains positive throughout.

Upper VHS

In[69]:=
```
Do[
  Print["Δnem=", Delt, "%:"];

  Do[

   HamCurr[k_] := HamiltonianNemTunedmeV[k〚1〛, k〚2〛, θCurr, Delt];


   TaylorPoly = Chop[
     Total[TaylorExpandBand[HamCurr, {0, π // N}, 8, 2, "Messages" → False]〚1〛] /.
      {p1 → kx, p2 → ky}, 10^-5];

   Print["\t", "θ = ", θCurr,
    ", Tayl = ", Chop[NumberForm[TaylorPoly, {4, 2}], 10^-2]];


   , {θCurr, 8, 12, 1}];

  , {Delt, {0.25, 0.5, 1, 1.75}}];
```

Δ_{nem}=0.25%:

θ = 8, Tayl = $-16.47 + 16.03\,k_x^2 - 27.76\,k_y^2$

θ = 9, Tayl = $-19.23 + 15.38\,k_x^2 - 11.94\,k_y^2$

θ = 10, Tayl = $-22.13 + 14.69\,k_x^2 + 3.08\,k_y^2$

θ = 11, Tayl = $-26.36 + 13.96\,k_x^2 + 17.05\,k_y^2$

θ = 12, Tayl = $-28.81 + 11.23\,k_x^2 + 29.73\,k_y^2$

Δ_{nem}=0.5%:

θ = 8, Tayl = $-13.44 + 15.87\,k_x^2 - 28.57\,k_y^2$

θ = 9, Tayl = $-14.66 + 15.25\,k_x^2 - 12.83\,k_y^2$

θ = 10, Tayl = $-16.46 + 14.60\,k_x^2 + 2.09\,k_y^2$

θ = 11, Tayl = $-18.58 + 13.90\,k_x^2 + 15.97\,k_y^2$

θ = 12, Tayl = $-20.91 + 11.25\,k_x^2 + 28.53\,k_y^2$

Δ_{nem}=1%:

θ = 8, Tayl = $-9.51 + 15.74\,k_x^2 - 33.11\,k_y^2$

θ = 9, Tayl = $-7.29 + 15.28\,k_x^2 - 18.13\,k_y^2$

θ = 10, Tayl = $-6.51 + 14.81\,k_x^2 - 3.98\,k_y^2$

θ = 11, Tayl = $-4.05 + 14.34\,k_x^2 + 9.05\,k_y^2$

θ = 12, Tayl = $-5.78 + 12.15\,k_x^2 + 20.67\,k_y^2$

Δ_{nem}=1.75%:

θ = 8, Tayl = $-6.53 + 16.39\,k_x^2 - 41.95\,k_y^2$

θ = 9, Tayl = $1.36 + 16.32\,k_x^2 - 29.18\,k_y^2$

θ = 10, Tayl = $6.50 + 16.30\,k_x^2 - 17.30\,k_y^2$

θ = 11, Tayl = $16.34 + 16.37\,k_x^2 - 6.57\,k_y^2$

θ = 12, Tayl = $15.99 + 15.23\,k_x^2 + 2.68\,k_y^2$

In this case as well, the critical point evolves from a **saddle to a minimum** as θ increases for each Δ_{nem} since the coefficient of the k_x^2 term remains positive while the coefficient of the k_y^2 term change sign from - to +. Furthermore, we notice that as we expected, there is only a weak breaking of the $\frac{\pi}{2}$ rotation when we compare the series expansion of the lower and upper VHS at the M point. For example the series expansion at $\Delta_{nem} = 0.25\%$ and $\theta = 9\,°$ for the pair of bands respectively reads
$-20.48 - 11.56\,k_x^2 + 15.70\,k_y^2$
$-19.23 + 15.38\,k_x^2 - 11.94\,k_y^2$
These are still “approximately” related by the transformation $(k_x, k_y) \to (-k_y, k_x)$.

Using binary search to find HOVHs in the SOC gap

Band 9, part 1

In[78]:=
```
NoIterns = 5;
CurrResln = 10^-2;
```

```
kVHSTrialLow = {0.6 × 3.869 × 0.36, π // N};
kVHSTrialUp = {3.869 × 0.36, π // N};

HosListBand9 = {};
SetSharedVariable[HosListBand9];

ParallelDo[
 HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], k[[3]], Delt];

 NoTry = 0;
 HOSFound = False;
 HOSPossible = True;
 θmax = 11;
 θmin = 8;

 (*Print["Binary search initiated for Δnem = ",Delt];*)

 While[NoTry ≤ 14 && HOSFound == False && HOSPossible == True,
  HamCurrUp[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmax, Delt];
  HamCurrDown[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmin, Delt];

  (*We compute the Hessian determinant for θmax*)
  (*Finding true kVHS:*)
  {kVHSmax, CritPtFound} = LocateCriticalPoint[
    HamCurrUp, kVHSTrialLow, kVHSTrialUp, 9, "Messages" → False,
    "SanityChecks" → True, "MaxIterations" → 24, "Resolution" → 10^-2];

  TaylorPolyMax = TaylorExpandBand[HamCurrUp, kVHSmax, 9, 2, "Messages" → False];

  If[CritPtFound == False,
   HOSPossible = False;
  ];

  TaylorPolyMax = Chop[TaylorPolyMax, CurrResln];

  (*Print["kcrit,max = ",kVHSmax,TaylorPolyMax[[1]]];*)

  HesDetMax = (D[TaylorPolyMax[[1, 3]], {p1, 2}] × D[TaylorPolyMax[[1, 3]], {p2, 2}] -
       (D[TaylorPolyMax[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

  (*We now compute the Hessian determinant for θmin*)
  (*Finding true kVHS:*)
  {kVHSmin, CritPtFound} = LocateCriticalPoint[
    HamCurrDown, kVHSTrialLow, kVHSTrialUp, 9, "Messages" → False,
    "SanityChecks" → True, "MaxIterations" → 24, "Resolution" → 10^-2];
```

```
TaylorPolyMin =
 TaylorExpandBand[HamCurrDown, kVHSmin, 9, 2, "Messages" → False];

If[CritPtFound == False,
 HOSPossible = False;
];

TaylorPolyMin = Chop[TaylorPolyMin, CurrResln];

(*Print["k_{crit,max} = ",kVHSmax,TaylorPolyMax[[1]]];*)

HesDetMin = (D[TaylorPolyMin[[1, 3]], {p1, 2}] × D[TaylorPolyMin[[1, 3]], {p2, 2}] -
     (D[TaylorPolyMin[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

(*Now we begin the binary search if the
 Hessian determinants computed above are opposite in sign*)
If[Sign[HesDetMax] == -Sign[HesDetMin] && HOSFound == False && HOSPossible == True,

 θmid = 0.5 (θmax + θmin);
 HamCurrMid[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmid, Delt];

 (*Finding true kVHS1:*)
 {kVHSmid, CritPtFound} = LocateCriticalPoint[
   HamCurrMid, kVHSTrialLow, kVHSTrialUp, 9, "Messages" → False,
   "SanityChecks" → True, "MaxIterations" → 24, "Resolution" → 10^-2];

 TaylorPolyMid =
  TaylorExpandBand[HamCurrMid, kVHSmid, 9, 2, "Messages" → False];

 If[CritPtFound == False,
  HOSPossible = False;
 ];

 TaylorPolyMid = Chop[TaylorPolyMid, CurrResln];

 (*Print["k_{crit,max} = ",kVHSmax,TaylorPolyMax[[1]]];*)

 HesDetMid = (D[TaylorPolyMid[[1, 3]], {p1, 2}] × D[TaylorPolyMid[[1, 3]], {p2, 2}] -
      (D[TaylorPolyMid[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 If[CritPtFound == False,
  HOSPossible = False;
 ];

 TaylorPolyMid = Chop[TaylorPolyMid, CurrResln];
```

```
HesDet = (D[TaylorPolyMid[[1, 3]], {p1, 2}] × D[TaylorPolyMid[[1, 3]], {p2, 2}] -
     (D[TaylorPolyMid[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

If[Chop[HesDet, 1.] == 0,
 HOSFound = True;
 ,
 If[Sign[HesDet] == Sign[HesDetMax],
  θmax = θmid;
  ,
  If[Sign[HesDet] == Sign[HesDetMin],
   θmin = θmid;
  ]
 ]
];

NoTry++;
,
If[HOSFound == False,
 Print[Style["No HOS in the range!", Red]];
 HOSPossible = False;
]
];
];

If[HOSFound == True,
 AppendTo[HosListBand9, {Delt, θmid, NoTry, kVHSmid}];
 ,
 Print[Style["No HOS in the range!", Red]];
];
, {Delt, 0, 2, 0.25}]
```

Since the list was constructed with a parallelized Do loop, we need to sort it

In[86]:=
```
HosListBand9 = SortBy[HosListBand9, First];
```

We are now in a position to examine the series expansion of the HOVHS for the critical tuning situations obtained above

```
In[168]:= Do[
  HamCurrHOS[k_] := HamiltonianNemTunedmeV[k⟦1⟧, k⟦2⟧, hosdata⟦2⟧, hosdata⟦1⟧];
  TaylorPolyHOS =
   TaylorExpandBand[HamCurrHOS, hosdata⟦4⟧, 9, 4, "Messages" → False];

  Print["Δnem = ", hosdata⟦1⟧,
   ". Binary search converged after ", hosdata⟦3⟧, " iterations."];
  Print["\t", "θcrit=", NumberForm[hosdata⟦2⟧, {4, 2}], ", Tayl = ",
   NumberForm[Chop[Total[TaylorPolyHOS⟦1⟧] /. {p1 → kx, p2 → ky}, 10^-2], {6, 2}]];
  DiagnoseHOS[Chop[Total[TaylorPolyHOS⟦1⟧], 10^-2]];
  Print[
   "*****************************************************************"]
  , {hosdata, HosListBand9}]
```

Δ_{nem} = 0.. Binary search converged after 11 iterations.

θ_{crit}=8.70, Tayl =
$-4.09 + 510.83\, k_x^2 - 4716.23\, k_x^3 + 23497.90\, k_x^4 - 116.26\, k_x\, k_y^2 + 1997.04\, k_x^2\, k_y^2 - 73.75\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.25. Binary search converged after 7 iterations.

θ_{crit}=8.68, Tayl =
$-0.22 + 524.97\, k_x^2 - 5211.00\, k_x^3 + 29212.00\, k_x^4 + 0.05\, k_x^3\, k_y - 118.23\, k_x\, k_y^2 + 2195.08\, k_x^2\, k_y^2 - 74.55\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.5. Binary search converged after 11 iterations.

θ_{crit}=8.73, Tayl =
$4.23 + 491.34\, k_x^2 - 4475.94\, k_x^3 + 22621.60\, k_x^4 + 0.08\, k_x^3\, k_y - 111.84\, k_x\, k_y^2 + 1914.84\, k_x^2\, k_y^2 - 74.38\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.75. Binary search converged after 11 iterations.

θ_{crit}=8.86, Tayl =
$9.57 + 440.23\, k_x^2 - 3323.17\, k_x^3 + 12723.50\, k_x^4 + 0.04\, k_x^3\, k_y - 104.19\, k_x\, k_y^2 + 1462.58\, k_x^2\, k_y^2 - 72.74\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.. Binary search converged after 11 iterations.

θ_{crit}=9.07, Tayl =
$16.11 + 402.83\, k_x^2 - 2464.46\, k_x^3 + 6242.84\, k_x^4 - 0.02\, k_x^3\, k_y - 103.28\, k_x\, k_y^2 + 1117.78\, k_x^2\, k_y^2 - 68.83\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.25. Binary search converged after 11 iterations.

θ_{crit}=9.36, Tayl = $24.22 + 385.65\, k_x^2 - 1930.96\, k_x^3 + 2457.90\, k_x^4 + 0.01\, k_x^2\, k_y - 0.09\, k_x^3\, k_y - 110.85\, k_x\, k_y^2 + 892.90\, k_x^2\, k_y^2 - 62.33\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.5. Binary search converged after 11 iterations.

θ_{crit}=9.74, Tayl = $33.82 + 386.60\, k_x^2 - 1607.31\, k_x^3 - 124.87\, k_x^4 + 0.01\, k_x^2\, k_y - 0.14\, k_x^3\, k_y - 126.96\, k_x\, k_y^2 + 742.99\, k_x^2\, k_y^2 - 0.01\, k_x\, k_y^3 - 53.18\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.75. Binary search converged after 9 iterations.

θ_{crit}=10.26, Tayl = $44.42 + 402.43\, k_x^2 - 1405.95\, k_x^3 - 2426.43\, k_x^4 - 0.01\, k_x^2\, k_y - 0.03\, k_x^3\, k_y - 151.17\, k_x\, k_y^2 + 623.55\, k_x^2\, k_y^2 - 0.01\, k_x\, k_y^3 - 41.76\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 2.. Binary search converged after 9 iterations.

θ_{crit}=10.85, Tayl =
$62.25 + 429.31\, k_x^2 - 1253.64\, k_x^3 - 5084.61\, k_x^4 + 0.09\, k_x^3\, k_y - 183.41\, k_x\, k_y^2 + 515.60\, k_x^2\, k_y^2 - 28.19\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

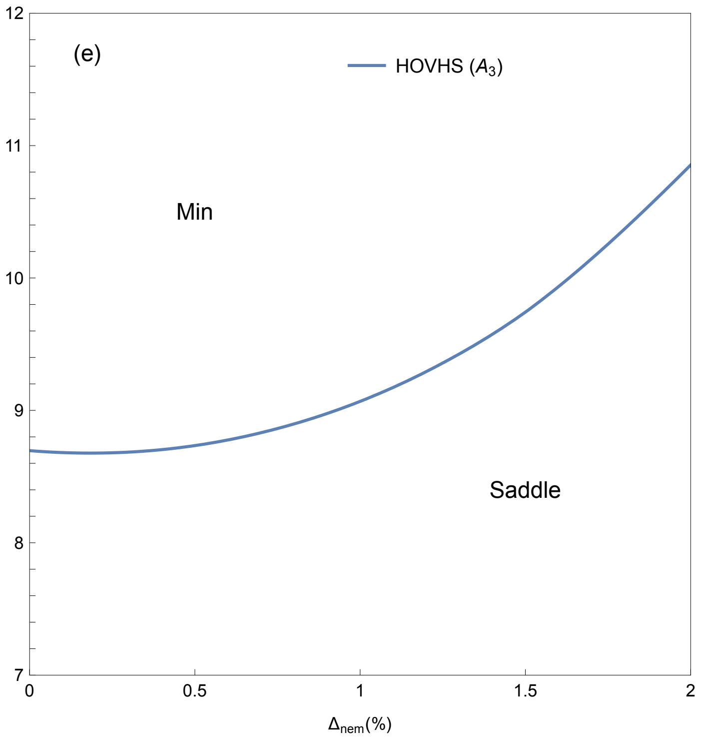
****

In[95]:= **HosListBand9⟦All, 1 ;; 2⟧**

Out[95]= {{0., 8.6958}, {0.25, 8.67969}, {0.5, 8.73389}, {0.75, 8.86279}, {1., 9.06787}, {1.25, 9.35791}, {1.5, 9.7417}, {1.75, 10.2559}, {2., 10.8535}}

In[96]:=
```
FitFunc1 = Interpolation[HosListBand9[[All, 1 ;; 2]]];
PlotBand9 = ListPlot[HosListBand9[[All, 1 ;; 2]]];
PanelE = Show[Plot[FitFunc1[x], {x, 0, 2},
   FrameLabel → {"Δnem(%)", (*"θ(°)"*)None}, PlotLegends → Placed[
     {Style["HOVHS (A3)", FontFamily → "Arial", FontSize → 10]}, {0.6, 0.92}],
   PlotRange → {{0, 2}, {7, 12}}, Frame → True, AspectRatio → 1,
   FrameTicks → {{Automatic, None}, {{0, 0.5, 1, 1.5, 2}, None}}], Graphics[
   Text[Style["Saddle", FontFamily → "Arial", FontSize → 12], {1.5, 8.4}]],
  Graphics[Text[Style["Min", FontFamily → "Arial", FontSize → 12], {0.5, 10.5}]],
  Graphics[Text[Style["(e)", FontFamily → "Arial", FontSize → 12], {0.175, 11.7}]]
  (*,PlotBand9*)]
```

Out[98]=



Band 8, part 1

In[107]:=
```
NoIterns = 5;
CurrResln = 10^-2;
kVHSTrialLow = {0.5 × 3.869 × 0.35, π // N};
kVHSTrialUp = {0.95 × 3.869 × 0.35, π // N};

HosListBand8 = {};
SetSharedVariable[HosListBand8];

ParallelDo[
 HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], k[[3]], Delt];

 NoTry = 0;
 HOSFound = False;
 HOSPossible = True;
```

```
θmax = 12;
θmin = 8;

(*Print["Binary search initiated for Δnem = ",Delt];*)

While[NoTry ≤ 18 && HOSFound == False && HOSPossible == True,
 HamCurrUp[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmax, Delt];
 HamCurrDown[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmin, Delt];

 (*We compute the Hessian determinant for θmax*)
 (*Finding true kVHS:*)
 {kVHSmax, CritPtFound} = LocateCriticalPoint[
   HamCurrUp, kVHSTrialLow, kVHSTrialUp, 8, "Messages" → False,
   "SanityChecks" → True, "MaxIterations" → 28, "Resolution" → 10^-2];

 TaylorPolyMax = TaylorExpandBand[HamCurrUp, kVHSmax, 8, 2, "Messages" → False];
 TaylorPolyMax = Chop[TaylorPolyMax, CurrResln];

 (*Print["kcrit,max = ",kVHSmax,TaylorPolyMax[[1]]];*)

 HesDetMax = (D[TaylorPolyMax[[1, 3]], {p1, 2}] × D[TaylorPolyMax[[1, 3]], {p2, 2}] -
      (D[TaylorPolyMax[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 If[CritPtFound == False,
  HOSPossible = False;
  (*Print["Critical point not found for θmax at Δnem = ",Delt,"."];*)
 ];


 (*We now compute the Hessian determinant for θmin*)
 (*Finding true kVHS:*)
 {kVHSmin, CritPtFound} = LocateCriticalPoint[
   HamCurrDown, kVHSTrialLow, kVHSTrialUp, 8, "Messages" → False,
   "SanityChecks" → True, "MaxIterations" → 28, "Resolution" → 10^-2];

 TaylorPolyMin =
  TaylorExpandBand[HamCurrDown, kVHSmin, 8, 2, "Messages" → False];
 TaylorPolyMin = Chop[TaylorPolyMin, CurrResln];

 (*Print["kcrit,max = ",kVHSmax,TaylorPolyMax[[1]]];*)

 HesDetMin = (D[TaylorPolyMin[[1, 3]], {p1, 2}] × D[TaylorPolyMin[[1, 3]], {p2, 2}] -
      (D[TaylorPolyMin[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 If[CritPtFound == False,
  HOSPossible = False;
```

```
 (*Print["Critical point not found for θmin at Δnem = ",Delt,"."];*)
];

(*Now we begin the binary search if the
 Hessian determinants computed above are opposite in sign*)
If[Sign[HesDetMax] == -Sign[HesDetMin] && HOSFound == False && HOSPossible == True,

 θmid = 0.5 (θmax + θmin);
 HamCurrMid[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmid, Delt];

 (*Finding true kVHS1:*)
 {kVHSmid, CritPtFound} = LocateCriticalPoint[
   HamCurrMid, kVHSTrialLow, kVHSTrialUp, 8, "Messages" → False,
   "SanityChecks" → True, "MaxIterations" → 28, "Resolution" → 10^-2];

 TaylorPolyMid =
  TaylorExpandBand[HamCurrMid, kVHSmid, 8, 2, "Messages" → False];
 TaylorPolyMid = Chop[TaylorPolyMid, CurrResln];

 (*Print["kcrit,max = ",kVHSmax,TaylorPolyMax[[1]]];*)

 HesDetMid = (D[TaylorPolyMid[[1, 3]], {p1, 2}] × D[TaylorPolyMid[[1, 3]], {p2, 2}] -
      (D[TaylorPolyMid[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 If[CritPtFound == False,
  HOSPossible = False;
  (*Print["Critical point not found for θmid at Δnem = ",Delt,"."];*)
 ];


 If[Chop[HesDetMid, 1.] == 0,
  HOSFound = True;
  ,
  If[Sign[HesDetMid] == Sign[HesDetMax],
   θmax = θmid;
   ,
   If[Sign[HesDetMid] == Sign[HesDetMin],
    θmin = θmid;
   ]
  ]
 ];

 NoTry++;
 ,
 If[HOSFound == False,
  Print[Style["No HOS in the range!", Red]];
```

```
        HOSPossible = False;
       ]
      ];
     ];

    If[HOSFound == True,
     AppendTo[HosListBand8, {Delt, θmid, NoTry, kVHSmid}];
     ,
     Print[Style["No HOS in the range!", Red]];
    ];
    , {Delt, 0, 1.5, 0.25}]
```

In[128]:=
```
HosListBand8 = SortBy[HosListBand8, First];
```

In[167]:=
```
Do[
 HamCurrHOS[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], hosdata[[2]], hosdata[[1]]];
 TaylorPolyHOS =
  TaylorExpandBand[HamCurrHOS, hosdata[[4]], 8, 4, "Messages" → False];

 Print["Δnem = ", hosdata[[1]],
  ". Binary search converged after ", hosdata[[3]], " iterations."];
 Print["\t", "θcrit=", NumberForm[hosdata[[2]], {4, 2}], ", Tayl = ",
  NumberForm[Chop[Total[TaylorPolyHOS[[1]]] /. {p1 → kx, p2 → ky}, 10^-2], {6, 2}]];
 DiagnoseHOS[Chop[Total[TaylorPolyHOS[[1]]], 10^-2]];
 Print[
  "******************************************************************"]
 , {hosdata, HosListBand8}]
```

Δ_{nem} = 0.. Binary search converged after 9 iterations.

θ_{crit}=8.91, Tayl =
$-11.66 - 401.64\, k_x^2 - 4209.66\, k_x^3 - 25725.60\, k_x^4 + 117.43\, k_x\, k_y^2 + 1457.44\, k_x^2\, k_y^2 - 86.96\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.25. Binary search converged after 11 iterations.

θ_{crit}=8.88, Tayl =
$-6.94 - 414.45\, k_x^2 - 4642.84\, k_x^3 - 31225.00\, k_x^4 + 121.33\, k_x\, k_y^2 + 1605.98\, k_x^2\, k_y^2 - 88.42\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.5. Binary search converged after 11 iterations.

θ_{crit}=8.96, Tayl =
$-2.70 - 382.06\, k_x^2 - 3966.23\, k_x^3 - 24498.20\, k_x^4 + 113.62\, k_x\, k_y^2 + 1381.22\, k_x^2\, k_y^2 - 87.38\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 0.75. Binary search converged after 5 iterations.

θ_{crit}=9.13, Tayl =
$1.66 - 334.73\, k_x^2 - 2939.67\, k_x^3 - 14578.00\, k_x^4 + 101.17\, k_x\, k_y^2 + 1040.56\, k_x^2\, k_y^2 - 84.05\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.. Binary search converged after 10 iterations.

θ_{crit}=9.39, Tayl =
$6.87 - 302.99\, k_x^2 - 2204.97\, k_x^3 - 8045.38\, k_x^4 - 0.02\, k_x^3\, k_y + 91.00\, k_x\, k_y^2 + 807.63\, k_x^2\, k_y^2 - 78.27\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.25. Binary search converged after 10 iterations.

θ_{crit}=9.80, Tayl =
$13.44 - 294.24\, k_x^2 - 1773.94\, k_x^3 - 3998.47\, k_x^4 - 0.09\, k_x^3\, k_y + 83.39\, k_x\, k_y^2 + 688.22\, k_x^2\, k_y^2 - 69.12\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.5. Binary search converged after 6 iterations.

θ_{crit}=10.56, Tayl = $22.82 - 310.42\, k_x^2 - 1508.70\, k_x^3 - 360.33\, k_x^4 - 0.01\, k_x\, k_y - 0.01\, k_x^2\, k_y + 0.14\, k_x^3\, k_y + 71.60\, k_x\, k_y^2 + 648.09\, k_x^2\, k_y^2 + 0.02\, k_x\, k_y^3 - 54.10\, k_y^4$

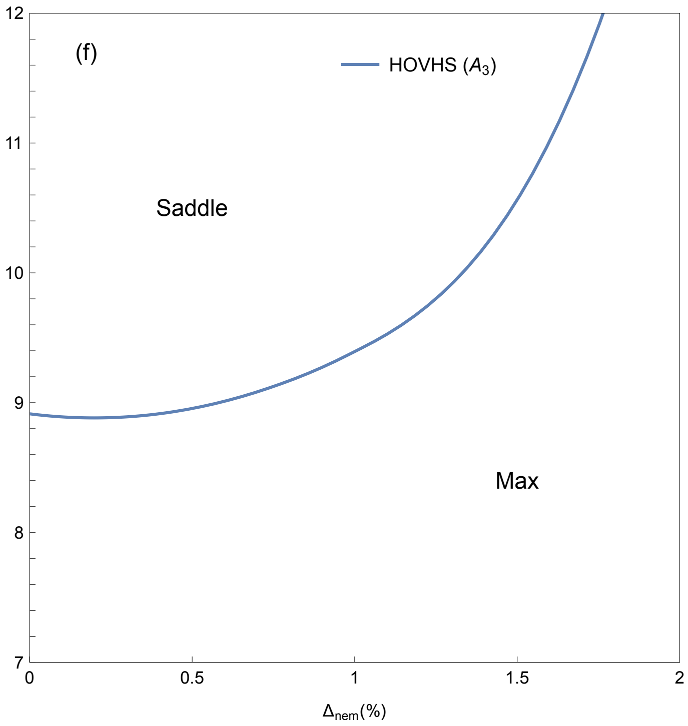
Polynomial has corank 1. It is 2-determinate. Its determinacy is either 2 or 1 and its codimension is 1. Comparing with the known list of singularities:

Point singularity has a finite determinacy but cannot identified by name!

****

In[130]:=
```
FitFunc2 = Interpolation[HosListBand8[[All, 1 ;; 2]]];
PlotBand8 = ListPlot[HosListBand8[[All, 1 ;; 2]]];
PanelF = Show[Plot[FitFunc2[x], {x, 0, 2},
   FrameLabel → {"Δnem(%)", (*"θ(°)"*)None}, PlotLegends → Placed[
     {Style["HOVHS (A3)", FontFamily → "Arial", FontSize → 10]}, {0.6, 0.92}],
   PlotRange → {{0, 2}, {7, 12}}, Frame → True, AspectRatio → 1,
   FrameTicks → {{Automatic, None}, {{0, 0.5, 1, 1.5, 2}, None}}], Graphics[
   Text[Style["Max", FontFamily → "Arial", FontSize → 12], {1.5, 8.4}]], Graphics[
   Text[Style["Saddle", FontFamily → "Arial", FontSize → 12], {0.5, 10.5}]],
  Graphics[Text[Style["(f)", FontFamily → "Arial", FontSize → 12], {0.175, 11.7}]]
  (*,PlotBand8*)]
```

Out[132]=



Band 9, part 2

In[118]:=
```
NoIterns = 5;
CurrResln = 10^-2;
kVHSTrialLow = {0, π - 3.869 × 0.36};
kVHSTrialUp = {0, π - 0.6 × 3.869 × 0.36};

HosListBand9p2 = {};
SetSharedVariable[HosListBand9p2];

ParallelDo[
 HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], k[[3]], Delt];

 NoTry = 0;
 HOSFound = False;
 HOSPossible = True;
```

```
θmax = 11;
θmin = 8;

(*Print["Binary search initiated for Δnem = ",Delt];*)

While[NoTry ≤ 14 && HOSFound == False && HOSPossible == True,
 HamCurrUp[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmax, Delt];
 HamCurrDown[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmin, Delt];

 (*We compute the Hessian determinant for θmax*)
 (*Finding true kVHS:*)
 {kVHSmax, CritPtFound} = LocateCriticalPoint[
   HamCurrUp, kVHSTrialLow, kVHSTrialUp, 9, "Messages" → False,
   "SanityChecks" → True, "MaxIterations" → 24, "Resolution" → 10^-2];

 TaylorPolyMax = TaylorExpandBand[HamCurrUp, kVHSmax, 9, 2, "Messages" → False];

 If[CritPtFound == False,
  HOSPossible = False;
 ];

 TaylorPolyMax = Chop[TaylorPolyMax, CurrResln];

 (*Print["kcrit,max = ",kVHSmax,TaylorPolyMax[[1]]];*)

 HesDetMax = (D[TaylorPolyMax[[1, 3]], {p1, 2}] × D[TaylorPolyMax[[1, 3]], {p2, 2}] -
      (D[TaylorPolyMax[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 (*We now compute the Hessian determinant for θmin*)
 (*Finding true kVHS:*)
 {kVHSmin, CritPtFound} = LocateCriticalPoint[
   HamCurrDown, kVHSTrialLow, kVHSTrialUp, 9, "Messages" → False,
   "SanityChecks" → True, "MaxIterations" → 24, "Resolution" → 10^-2];

 TaylorPolyMin =
  TaylorExpandBand[HamCurrDown, kVHSmin, 9, 2, "Messages" → False];

 If[CritPtFound == False,
  HOSPossible = False;
 ];

 TaylorPolyMin = Chop[TaylorPolyMin, CurrResln];

 (*Print["kcrit,max = ",kVHSmax,TaylorPolyMax[[1]]];*)

 HesDetMin = (D[TaylorPolyMin[[1, 3]], {p1, 2}] × D[TaylorPolyMin[[1, 3]], {p2, 2}] -
```

```
    (D[TaylorPolyMin⟦1, 3⟧, p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

(*Now we begin the binary search if the
 Hessian determinants computed above are opposite in sign*)
If[Sign[HesDetMax] == -Sign[HesDetMin] && HOSFound == False && HOSPossible == True,

 θmid = 0.5 (θmax + θmin);
 HamCurrMid[k_] := HamiltonianNemTunedmeV[k⟦1⟧, k⟦2⟧, θmid, Delt];

 (*Finding true kVHS1:*)
 {kVHSmid, CritPtFound} = LocateCriticalPoint[
   HamCurrMid, kVHSTrialLow, kVHSTrialUp, 9, "Messages" → False,
   "SanityChecks" → True, "MaxIterations" → 24, "Resolution" → 10^-2];

 TaylorPolyMid =
  TaylorExpandBand[HamCurrMid, kVHSmid, 9, 2, "Messages" → False];

 If[CritPtFound == False,
  HOSPossible = False;
 ];

 TaylorPolyMid = Chop[TaylorPolyMid, CurrResln];

 (*Print["k_crit,max  = ",kVHSmax,TaylorPolyMax⟦1⟧];*)

 HesDetMid = (D[TaylorPolyMid⟦1, 3⟧, {p1, 2}] × D[TaylorPolyMid⟦1, 3⟧, {p2, 2}] -
      (D[TaylorPolyMid⟦1, 3⟧, p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 If[CritPtFound == False,
  HOSPossible = False;
 ];

 TaylorPolyMid = Chop[TaylorPolyMid, CurrResln];

 HesDet = (D[TaylorPolyMid⟦1, 3⟧, {p1, 2}] × D[TaylorPolyMid⟦1, 3⟧, {p2, 2}] -
      (D[TaylorPolyMid⟦1, 3⟧, p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 If[Chop[HesDet, 1.] == 0,
  HOSFound = True;
  ,
  If[Sign[HesDet] == Sign[HesDetMax],
   θmax = θmid;
   ,
   If[Sign[HesDet] == Sign[HesDetMin],
    θmin = θmid;
   ]
```

```
      ]
     ];

     NoTry++;
     ,
     If[HOSFound == False,
      Print[Style["No HOS in the range!", Red]];
      HOSPossible = False;
     ]
    ];
   ];

   If[HOSFound == True,
    AppendTo[HosListBand9p2, {Delt, θmid, NoTry, kVHSmid}];
    ,
    Print[Style["No HOS in the range!", Red]];
   ];
   , {Delt, 0, 2, 0.25}]
```

In[126]:=
```
HosListBand9p2 = SortBy[HosListBand9p2, First];
```

In[166]:=
```
Do[
 HamCurrHOS[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], hosdata[[2]], hosdata[[1]]];
 TaylorPolyHOS =
  TaylorExpandBand[HamCurrHOS, hosdata[[4]], 9, 4, "Messages" → False];

 Print["Δnem = ", hosdata[[1]],
  ". Binary search converged after ", hosdata[[3]], " iterations."];
 Print["\t", "θcrit=", NumberForm[hosdata[[2]], {4, 2}], ", Tayl = ",
  NumberForm[Chop[Total[TaylorPolyHOS[[1]]] /. {p1 → kx, p2 → ky}, 10^-2], {6, 2}]];
 DiagnoseHOS[Chop[Total[TaylorPolyHOS[[1]]], 10^-2]];
 Print[
  "****************************************************************"]
 , {hosdata, HosListBand9p2}]
```

Δ_{nem} = 0.. Binary search converged after 11 iterations.

θ_{crit}=8.70, Tayl =

$$-4.09 - 73.75\, k_x^4 + 116.26\, k_x^2\, k_y + 510.83\, k_y^2 + 1997.04\, k_x^2\, k_y^2 + 4716.23\, k_y^3 + 23497.90\, k_y^4$$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.25. Binary search converged after 11 iterations.

θ_{crit}=8.64, Tayl =

$$-0.88 - 73.87\, k_x^4 + 123.15\, k_x^2\, k_y + 546.36\, k_y^2 + 2294.10\, k_x^2\, k_y^2 + 5496.38\, k_y^3 - 0.06\, k_x\, k_y^3 + 30376.60\, k_y^4$$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3
and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.5. Binary search converged after 5 iterations.

θ_{crit}=8.66, Tayl =

$2.83 - 73.08\,k_x^4 + 120.97\,k_x^2\,k_y + 531.17\,k_y^2 + 2084.18\,k_x^2\,k_y^2 + 4968.16\,k_y^3 - 0.10\,k_x\,k_y^3 + 24333.20\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3
and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.75. Binary search converged after 10 iterations.

θ_{crit}=8.75, Tayl =

$7.37 - 70.86\,k_x^4 + 116.16\,k_x^2\,k_y + 491.11\,k_y^2 + 1644.82\,k_x^2\,k_y^2 + 3853.39\,k_y^3 - 0.08\,k_x\,k_y^3 + 13891.20\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3
and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.. Binary search converged after 9 iterations.

θ_{crit}=8.92, Tayl =

$13.15 - 66.45\,k_x^4 + 117.07\,k_x^2\,k_y + 460.17\,k_y^2 + 1281.90\,k_x^2\,k_y^2 + 2955.17\,k_y^3 - 0.02\,k_x\,k_y^3 + 6573.65\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3
and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.25. Binary search converged after 10 iterations.

θ_{crit}=9.18, Tayl =

$20.68 - 59.43\,k_x^4 + 126.62\,k_x^2\,k_y + 448.43\,k_y^2 + 1034.66\,k_x^2\,k_y^2 + 2371.82\,k_y^3 + 0.06\,k_x\,k_y^3 + 1964.09\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3
and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.5. Binary search converged after 11 iterations.

θ_{crit}=9.54, Tayl = $30.18 - 49.76\,k_x^4 + 145.12\,k_x^2\,k_y +$
$455.23\,k_y^2 + 0.01\,k_x\,k_y^2 + 860.97\,k_x^2\,k_y^2 + 2001.61\,k_y^3 + 0.16\,k_x\,k_y^3 - 1502.14\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3
and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.75. Binary search converged after 8 iterations.

θ_{crit}=10.03, Tayl = $41.06 - 37.73 k_x^4 + 172.51 k_x^2 k_y + 0.02 k_x^3 k_y + 478.47 k_y^2 + 0.01 k_x k_y^2 + 713.88 k_x^2 k_y^2 + 1757.47 k_y^3 + 0.18 k_x k_y^3 - 4928.18 k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 2.. Binary search converged after 6 iterations.

θ_{crit}=10.67, Tayl = $55.76 - 23.62 k_x^4 + 209.31 k_x^2 k_y + 0.01 k_x^3 k_y + 516.06 k_y^2 + 554.24 k_x^2 k_y^2 + 1576.22 k_y^3 - 0.08 k_x k_y^3 - 9202.16 k_y^4$

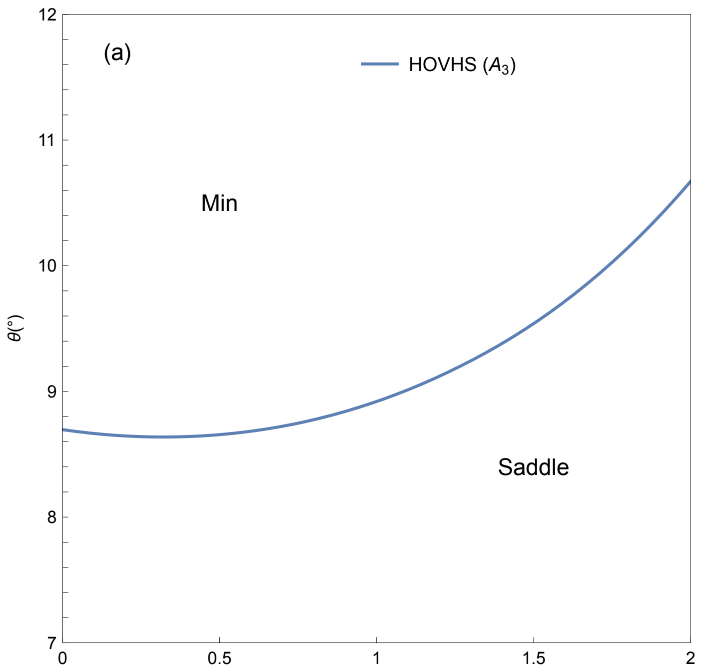
Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

In[146]:=
```
FitFunc3 = Interpolation[HosListBand9p2[[All, 1 ;; 2]]];
PlotBand9p2 = ListPlot[HosListBand9p2[[All, 1 ;; 2]]];
PanelA = Show[(*PlotBand9,*)Plot[FitFunc3[x], {x, 0, 2},
   FrameLabel → {(*"Δnem(%)"*)None, "θ(°)"}, PlotLegends → Placed[
     {Style["HOVHS (A3)", FontFamily → "Arial", FontSize → 10]}, {0.6, 0.92}],
   PlotRange → {{0, 2}, {7, 12}}, Frame → True, AspectRatio → 1,
   FrameTicks → {{Automatic, None}, {{0, 0.5, 1, 1.5, 2}, None}}], Graphics[
   Text[Style["Saddle", FontFamily → "Arial", FontSize → 12], {1.5, 8.4}]],
  Graphics[Text[Style["Min", FontFamily → "Arial", FontSize → 12], {0.5, 10.5}]],
  Graphics[
   Text[Style["(a)", FontFamily → "Arial", FontSize → 12], {0.175, 11.7}]]]
```

Out[148]=



Band 8, part 2

In[133]:=
```
NoIterns = 5;
CurrResln = 10^-2;
```

```
kVHSTrialLow = {0, π - 0.95 × 3.869 × 0.35};
kVHSTrialUp = {0, π - 0.5 × 3.869 × 0.35};

HosListBand8p2 = {};
SetSharedVariable[HosListBand8p2];

ParallelDo[
 HamCurr[k_] := HamiltonianNemTunedmeV[k〚1〛, k〚2〛, k〚3〛, Delt];

 NoTry = 0;
 HOSFound = False;
 HOSPossible = True;
 θmax = 12;
 θmin = 8;

 (*Print["Binary search initiated for Δnem = ",Delt];*)

 While[NoTry ≤ 18 && HOSFound == False && HOSPossible == True,
  HamCurrUp[k_] := HamiltonianNemTunedmeV[k〚1〛, k〚2〛, θmax, Delt];
  HamCurrDown[k_] := HamiltonianNemTunedmeV[k〚1〛, k〚2〛, θmin, Delt];

  (*We compute the Hessian determinant for θmax*)
  (*Finding true kVHS:*)
  {kVHSmax, CritPtFound} = LocateCriticalPoint[
    HamCurrUp, kVHSTrialLow, kVHSTrialUp, 8, "Messages" → False,
    "SanityChecks" → True, "MaxIterations" → 28, "Resolution" → 10^-2];

  TaylorPolyMax = TaylorExpandBand[HamCurrUp, kVHSmax, 8, 2, "Messages" → False];
  TaylorPolyMax = Chop[TaylorPolyMax, CurrResln];

  (*Print["kcrit,max = ",kVHSmax,TaylorPolyMax〚1〛];*)

  HesDetMax = (D[TaylorPolyMax〚1, 3〛, {p1, 2}] × D[TaylorPolyMax〚1, 3〛, {p2, 2}] -
       (D[TaylorPolyMax〚1, 3〛, p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

  If[CritPtFound == False,
   HOSPossible = False;
   (*Print["Critical point not found for θmax at Δnem = ",Delt,"."];*)
  ];


  (*We now compute the Hessian determinant for θmin*)
  (*Finding true kVHS:*)
  {kVHSmin, CritPtFound} = LocateCriticalPoint[
    HamCurrDown, kVHSTrialLow, kVHSTrialUp, 8, "Messages" → False,
    "SanityChecks" → True, "MaxIterations" → 28, "Resolution" → 10^-2];
```

```
TaylorPolyMin =
 TaylorExpandBand[HamCurrDown, kVHSmin, 8, 2, "Messages" → False];
TaylorPolyMin = Chop[TaylorPolyMin, CurrResln];

(*Print["kcrit,max = ",kVHSmax,TaylorPolyMax[[1]]];*)

HesDetMin = (D[TaylorPolyMin[[1, 3]], {p1, 2}] × D[TaylorPolyMin[[1, 3]], {p2, 2}] -
      (D[TaylorPolyMin[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

If[CritPtFound == False,
 HOSPossible = False;
 (*Print["Critical point not found for θmin at Δnem = ",Delt,"."];*)
];

(*Now we begin the binary search if the
 Hessian determinants computed above are opposite in sign*)
If[Sign[HesDetMax] == -Sign[HesDetMin] && HOSFound == False && HOSPossible == True,

 θmid = 0.5 (θmax + θmin);
 HamCurrMid[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], θmid, Delt];

 (*Finding true kVHS1:*)
 {kVHSmid, CritPtFound} = LocateCriticalPoint[
   HamCurrMid, kVHSTrialLow, kVHSTrialUp, 8, "Messages" → False,
   "SanityChecks" → True, "MaxIterations" → 28, "Resolution" → 10^-2];

 TaylorPolyMid =
  TaylorExpandBand[HamCurrMid, kVHSmid, 8, 2, "Messages" → False];
 TaylorPolyMid = Chop[TaylorPolyMid, CurrResln];

 (*Print["kcrit,max = ",kVHSmax,TaylorPolyMax[[1]]];*)

 HesDetMid = (D[TaylorPolyMid[[1, 3]], {p1, 2}] × D[TaylorPolyMid[[1, 3]], {p2, 2}] -
      (D[TaylorPolyMid[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 If[CritPtFound == False,
  HOSPossible = False;
  (*Print["Critical point not found for θmid at Δnem = ",Delt,"."];*)
 ];


 If[Chop[HesDetMid, 1.] == 0,
  HOSFound = True;
  ,
  If[Sign[HesDetMid] == Sign[HesDetMax],
```

```
      θmax = θmid;
      ,
      If[Sign[HesDetMid] == Sign[HesDetMin],
       θmin = θmid;
      ]
     ]
    ];

    NoTry++;
    ,
    If[HOSFound == False,
     Print[Style["No HOS in the range!", Red]];
     HOSPossible = False;
    ]
   ];
  ];

  If[HOSFound == True,
   AppendTo[HosListBand8p2, {Delt, θmid, NoTry, kVHSmid}];
   ,
   Print[Style["No HOS in the range!", Red]];
  ];
  , {Delt, Append[Table[i, {i, 0, 1.5, 0.25}], 1.6]}]
```

In[140]:=
```
HosListBand8p2 = SortBy[HosListBand8p2, First];
```

In[165]:=
```
Do[
 HamCurrHOS[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], hosdata[[2]], hosdata[[1]]];
 TaylorPolyHOS =
  TaylorExpandBand[HamCurrHOS, hosdata[[4]], 8, 4, "Messages" → False];

 Print["Δnem = ", hosdata[[1]],
  ". Binary search converged after ", hosdata[[3]], " iterations."];
 Print["\t", "θcrit=", NumberForm[hosdata[[2]], {4, 2}], ", Tayl = ",
  NumberForm[Chop[Total[TaylorPolyHOS[[1]]] /. {p1 → kx, p2 → ky}, 10^-2], {6, 2}]];
 DiagnoseHOS[Chop[Total[TaylorPolyHOS[[1]]], 10^-2]];
 Print[
  "*****************************************************************"]
 , {hosdata, HosListBand8p2}]
```

Δ_{nem} = 0.. Binary search converged after 9 iterations.

θ_{crit}=8.91, Tayl =
$-11.66 - 86.96\, k_x^4 - 117.43\, k_x^2\, k_y - 401.64\, k_y^2 + 1457.44\, k_x^2\, k_y^2 + 4209.66\, k_y^3 - 25725.60\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 0.25. Binary search converged after 12 iterations.

θ_{crit}=8.84, Tayl =
$-7.58 - 88.03\, k_x^4 - 125.54\, k_x^2\, k_y - 435.67\, k_y^2 + 1696.36\, k_x^2\, k_y^2 + 4929.44\, k_y^3 + 0.01\, k_x\, k_y^3 - 32820.80\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 0.5. Binary search converged after 11 iterations.

θ_{crit}=8.87, Tayl =
$-4.08 - 86.54\, k_x^4 - 121.31\, k_x^2\, k_y - 421.81\, k_y^2 + 1536.27\, k_x^2\, k_y^2 + 4462.10\, k_y^3 + 0.01\, k_x\, k_y^3 - 26931.40\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 0.75. Binary search converged after 10 iterations.

θ_{crit}=9.00, Tayl =
$-0.46 - 82.65\, k_x^4 - 110.74\, k_x^2\, k_y - 385.56\, k_y^2 + 1209.01\, k_x^2\, k_y^2 + 3471.89\, k_y^3 - 16446.60\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.. Binary search converged after 10 iterations.

θ_{crit}=9.24, Tayl =
$4.16 - 76.28\, k_x^4 - 101.51\, k_x^2\, k_y - 360.68\, k_y^2 + 969.41\, k_x^2\, k_y^2 + 2694.18\, k_y^3 + 0.02\, k_x\, k_y^3 - 8873.05\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.25. Binary search converged after 10 iterations.

θ_{crit}=9.61, Tayl =
$10.64 - 66.64\, k_x^4 - 94.74\, k_x^2\, k_y - 356.79\, k_y^2 + 843.98\, k_x^2\, k_y^2 + 2193.99\, k_y^3 + 0.12\, k_x\, k_y^3 - 3645.34\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.5. Binary search converged after 9 iterations.

θ_{crit}=10.40, Tayl = $19.95 - 50.80\, k_x^4 - 81.88\, k_x^2\, k_y -$
$0.01\, k_x^3\, k_y - 378.00\, k_y^2 + 801.88\, k_x^2\, k_y^2 + 1832.04\, k_y^3 - 0.08\, k_x\, k_y^3 + 1691.74\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

```
**********************************************************************
```

Δ_{nem} = 1.6. Binary search converged after 10 iterations.

θ_{crit}=10.93, Tayl =

$31.06 - 42.02\, k_x^4 - 78.97\, k_x^2\, k_y - 397.35\, k_y^2 + 854.97\, k_x^2\, k_y^2 + 1649.95\, k_y^3 - 0.08\, k_x\, k_y^3 + 5058.27\, k_y^4$

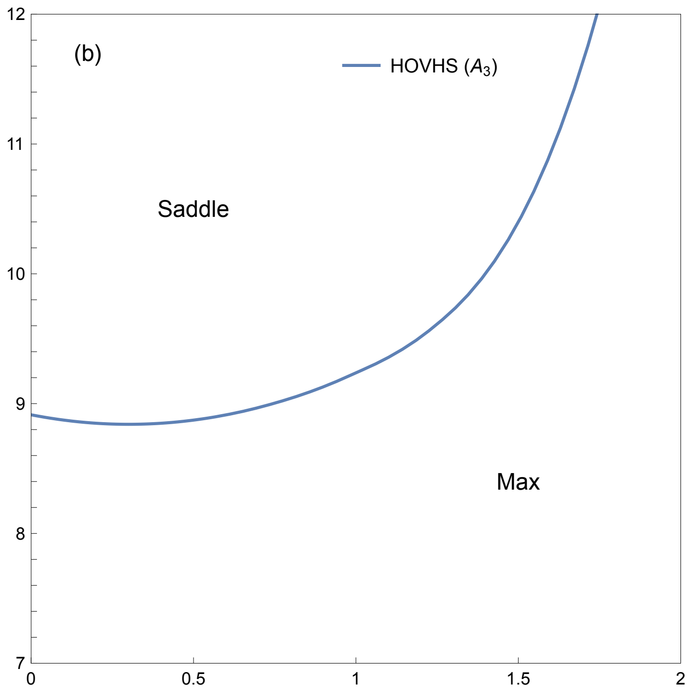
Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

```
**********************************************************************
```

In[143]:=
```
FitFunc4 = Interpolation[HosListBand8p2⟦All, 1 ;; 2⟧];
PlotBand8p2 = ListPlot[HosListBand8p2⟦All, 1 ;; 2⟧];
PanelB = Show[(*PlotBand9,*)Plot[FitFunc4[x], {x, 0, 2},
   FrameLabel → {(*"Δnem(%)"*)None, (*"θ(°)"*)None}, PlotLegends → Placed[
     {Style["HOVHS (A3)", FontFamily → "Arial", FontSize → 10]}, {0.6, 0.92}],
   PlotRange → {{0, 2}, {7, 12}}, Frame → True, AspectRatio → 1,
   FrameTicks → {{Automatic, None}, {{0, 0.5, 1, 1.5, 2}, None}}],
  Graphics[Text[Style["Max", FontFamily → "Arial", FontSize → 12], {1.5, 8.4}]],
  Graphics[Text[Style["Saddle", FontFamily → "Arial", FontSize → 12],
    {0.5, 10.5}]], Graphics[
   Text[Style["(b)", FontFamily → "Arial", FontSize → 12], {0.175, 11.7}]]]
```

Out[145]=



Finding HOVHs at the M point

Upper VHS - Bands 7 & 8

In[158]:=
```
HosListBand8M = {};
SetSharedVariable[HosListBand8M];
```

```
ParallelDo[
 HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], k[[3]], Delt];

 θCurr = 10.5;
 CurrResln = 10^-2;
 HOSFound = False;
 NoTry = 1;

 While[NoTry ≤ 4 && HOSFound == False,
  TaylorPoly =
   Chop[TaylorExpandBand[HamCurr(*The Hamiltonian*), {0, π // N, θCurr}
     (*the (k,t) point at which we are going to series expand.*), 8
     (*Band to series expand*), 2(*Degree of Taylor polynomial*),
     "Dimension" → 2(*k-space dimension*), "TuningDegree" → 2
     (*Degree of tuning parameter terms*),
     "TuningSymbol" → "δθ"(*Symbols for the tuning parameters*),
     "Resolution" → 10^-4(*Resolution below which we shall kill terms*),
     "SampleDirection" → {0.333, 0.779, 0.1335}
     (*A sample k-vector that will be used for diagonalising symbolic
       matrices. This is not needed in principle since we will
       generate a random one in the routine if this is not provided*),
     "Messages" → False(*YesMessage*)], 10^-6];

  HesDet = (D[TaylorPoly[[1, 3]], {p1, 2}] × D[TaylorPoly[[1, 3]], {p2, 2}] -
       (D[TaylorPoly[[1, 3]], p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

  If[Chop[ReplaceAll[HesDet, {δθ → 0}], CurrResln] == 0,
   HOSFound = True;
  ];

  δθSolns = Solve[HesDet == 0, δθ, Reals];

  MinSoln = 1;
  Do[
   If[Abs[ReplaceAll[δθ, δθSolns[[l]]]] < Abs[ReplaceAll[δθ, δθSolns[[MinSoln]]]],
    MinSoln = l;
   ]
   , {l, 1, Length[δθSolns]}];

  θCurr = θCurr + δθ /. δθSolns[[MinSoln]];
 ];

 If[HOSFound == True,
  TaylorPoly =
   Chop[TaylorExpandBand[HamCurr(*The Hamiltonian*), {0, π // N, θCurr}
```

```
        (*the (k,t) point at which we are going to series expand.*), 8
        (*Band to series expand*), 4(*Degree of Taylor polynomial*),
        "Dimension" → 2(*k-space dimension*), "TuningDegree" → 0
        (*Degree of tuning parameter terms*),
        "TuningSymbol" → "δθ"(*Symbols for the tuning parameters*),
        "Resolution" → 10^-4(*Resolution below which we shall kill terms*),
        "SampleDirection" → {0.333, 0.779, 0.1335}
        (*A sample k-vector that will be used for diagonalising symbolic
          matrices. This is not needed in principle since we will
          generate a random one in the routine if this is not provided*),
        "Messages" → False(*YesMessage*)], 10^-6];

    AppendTo[HosListBand8M, {Delt, θCurr, NoTry, Total[
        Chop[ReplaceAll[TaylorPoly[[1]], {p1 → kx, p2 → ky, δθ → 0}], CurrResln]]}];
    ,
    Print[Style["No HOS found in the range!", Red]];
   ];

  , {Delt, 0, 1.75, 0.25}]
```

In[161]:=
```
HosListBand8M = SortBy[HosListBand8M, First];
```

In[164]:=
```
Do[
  Print["Δnem = ", hosdata[[1]],
   ". Search converged after ", hosdata[[3]], " iterations."];
  Print["\t" <> "θcrit = ", NumberForm[hosdata[[2]], {4, 2}],
   ", Tayl = ", NumberForm[hosdata[[4]], {3, 2}]];
  DiagnoseHOS[Chop[hosdata[[4]], 10^-1]];
  Print[
   "*******************************************************************"]

  , {hosdata, HosListBand8M}]
```

Δ_{nem} = 0.. Search converged after 1 iterations.

θ_{crit} = 9.84, Tayl = $-27.70 - 98.80\, k_x^4 + 15.20\, k_y^2 + 4.52\, k_x^2 k_y^2 - 8.08\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.25. Search converged after 1 iterations.

θ_{crit} = 9.79, Tayl = $-21.40 + 14.90\, k_x^2 - 6.01\, k_x^4 + 1.98\, k_x^2 k_y^2 - 98.60\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.5. Search converged after 1 iterations.

θ_{crit} = 9.85, Tayl = $-16.10 + 14.70\,k_x^2 - 6.16\,k_x^4 + 2.65\,k_x^2\,k_y^2 - 98.80\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 0.75. Search converged after 1 iterations.

θ_{crit} = 10.03, Tayl = $-11.30 + 14.60\,k_x^2 - 5.04\,k_x^4 + 1.32\,k_x^2\,k_y^2 - 97.60\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.. Search converged after 1 iterations.

θ_{crit} = 10.30, Tayl = $-6.93 + 14.60\,k_x^2 - 2.98\,k_x^4 - 0.01\,k_x^3\,k_y - 1.37\,k_x^2\,k_y^2 - 95.10\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.25. Search converged after 1 iterations.

θ_{crit} = 10.68, Tayl = $-1.12 + 14.80\,k_x^2 - 0.23\,k_x^4 - 0.02\,k_x^3\,k_y - 5.33\,k_x^2\,k_y^2 - 91.10\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.5. Search converged after 1 iterations.

θ_{crit} = 11.09, Tayl = $12.10 + 15.70\,k_x^2 + 2.54\,k_x^4 + 0.01\,k_x^3\,k_y - 10.30\,k_x^2\,k_y^2 - 84.20\,k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

$\triangle_{nem}$ = 1.75. Search converged after 1 iterations.

θ_{crit} = 11.44, Tayl =
$33.90 + 17.20\,k_x^2 + 5.17\,k_x^4 - 0.03\,k_x\,k_y + 0.08\,k_x^3\,k_y - 15.60\,k_x^2\,k_y^2 - 74.30\,k_y^4$

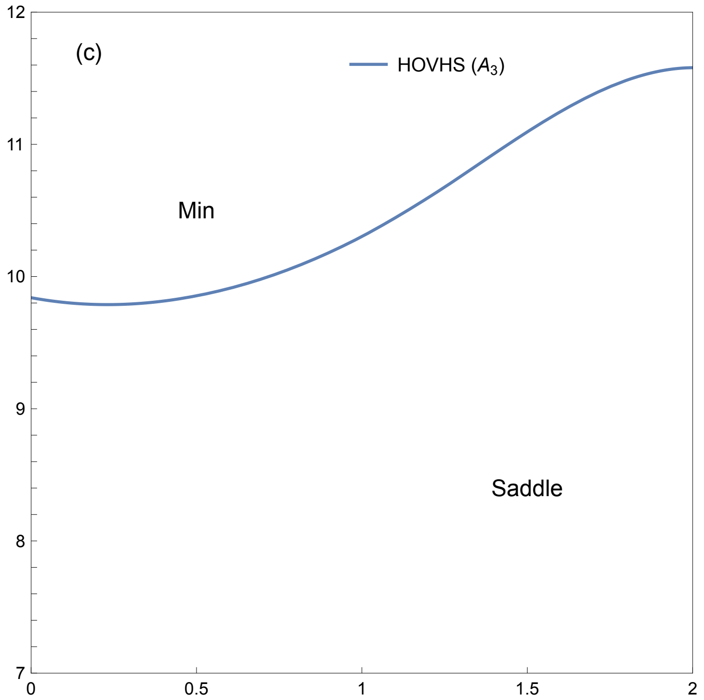
Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

In[177]:=
```
FitFunc5 = Interpolation[HosListBand8M[[All, 1 ;; 2]]];
PlotBand8p3 =
  ListPlot[HosListBand8M[[All, 1 ;; 2]], AxesLabel → {"Δnem (%)", "θ (°)"},
   PlotLegends → Placed[{"HOVHs (A3)"}, {0.5, 0.965}]];
PanelC = Show[(*PlotBand9,*)Plot[FitFunc5[x], {x, 0, 2},
   FrameLabel → {(*"Δnem (%)","θ (°)"*)None, None}, PlotLegends → Placed[
     {Style["HOVHS (A3)", FontFamily → "Arial", FontSize → 10]}, {0.6, 0.92}],
   PlotRange → {{0, 2}, {7, 12}}, Frame → True, AspectRatio → 1,
   FrameTicks → {{Automatic, None}, {{0, 0.5, 1, 1.5, 2}, None}}], Graphics[
   Text[Style["Saddle", FontFamily → "Arial", FontSize → 12], {1.5, 8.4}]],
  Graphics[Text[Style["Min", FontFamily → "Arial", FontSize → 12], {0.5, 10.5}]],
  Graphics[
   Text[Style["(c)", FontFamily → "Arial", FontSize → 12], {0.175, 11.7}]]]
```

Out[179]=



Lower VHS Bands 5 & 6

In[169]:=
```
HosListBand6M = {};
SetSharedVariable[HosListBand6M];

ParallelDo[
 HamCurr[k_] := HamiltonianNemTunedmeV[k[[1]], k[[2]], k[[3]], Delt];

 θCurr = 10.5;
 CurrResln = 10^-2;
 HOSFound = False;
 NoTry = 1;

 While[NoTry ≤ 4 && HOSFound == False,
```

```
 TaylorPoly =
  Chop[TaylorExpandBand[HamCurr(*The Hamiltonian*), {0, π // N, θCurr}
    (*the (k,t) point at which we are going to series expand.*), 6
    (*Band to series expand*), 2(*Degree of Taylor polynomial*),
    "Dimension" → 2(*k-space dimension*), "TuningDegree" → 2
    (*Degree of tuning parameter terms*),
    "TuningSymbol" → "δθ"(*Symbols for the tuning parameters*),
    "Resolution" → 10^-4(*Resolution below which we shall kill terms*),
    "SampleDirection" → {0.333, 0.779, 0.1335}
    (*A sample k-vector that will be used for diagonalising symbolic
      matrices. This is not needed in principle since we will
      generate a random one in the routine if this is not provided*),
    "Messages" → False(*YesMessage*)], 10^-6];

 HesDet = (D[TaylorPoly⟦1, 3⟧, {p1, 2}] × D[TaylorPoly⟦1, 3⟧, {p2, 2}] -
      (D[TaylorPoly⟦1, 3⟧, p1, p2])^2) /. {p1 → 0, p2 → 0} // Simplify;

 If[Chop[ReplaceAll[HesDet, {δθ → 0}], CurrResln] == 0,
  HOSFound = True;
 ];

 δθSolns = Solve[HesDet == 0, δθ, Reals];

 MinSoln = 1;
 Do[
  If[Abs[ReplaceAll[δθ, δθSolns⟦l⟧]] < Abs[ReplaceAll[δθ, δθSolns⟦MinSoln⟧]],
   MinSoln = l;
  ]
  , {l, 1, Length[δθSolns]}];

 θCurr = θCurr + δθ /. δθSolns⟦MinSoln⟧;
];

If[HOSFound == True,
 TaylorPoly =
  Chop[TaylorExpandBand[HamCurr(*The Hamiltonian*), {0, π // N, θCurr}
    (*the (k,t) point at which we are going to series expand.*), 6
    (*Band to series expand*), 4(*Degree of Taylor polynomial*),
    "Dimension" → 2(*k-space dimension*), "TuningDegree" → 0
    (*Degree of tuning parameter terms*),
    "TuningSymbol" → "δθ"(*Symbols for the tuning parameters*),
    "Resolution" → 10^-4(*Resolution below which we shall kill terms*),
    "SampleDirection" → {0.333, 0.779, 0.1335}
    (*A sample k-vector that will be used for diagonalising symbolic
      matrices. This is not needed in principle since we will
      generate a random one in the routine if this is not provided*),
```

```
        "Messages" → False(*YesMessage*)], 10^-6];

      AppendTo[HosListBand6M, {Delt, θCurr, NoTry, Total[
          Chop[ReplaceAll[TaylorPoly[[1]], {p1 → kx, p2 → ky, δθ → 0}], CurrResln]]}];
      ,
      Print[Style["No HOS found in the range!", Red]];
     ];

    , {Delt, 0, 1.75, 0.25}]
```

In[172]:=
```
HosListBand6M = SortBy[HosListBand6M, First];
```

In[173]:=
```
Do[
  Print["Δnem = ", hosdata[[1]],
   ". Search converged after ", hosdata[[3]], " iterations."];
  Print["\t" <> "θcrit = ", NumberForm[hosdata[[2]], {4, 2}],
   ", Tayl = ", NumberForm[hosdata[[4]], {3, 2}]];
  DiagnoseHOS[Chop[hosdata[[4]], 10^-1]];
  Print[
   "*****************************************************************"]

  , {hosdata, HosListBand6M}];
```

Δ_{nem} = 0.. Search converged after 1 iterations.

θ_{crit} = 9.84, Tayl = $-27.70 - 98.80\, k_x^4 + 15.20\, k_y^2 + 4.52\, k_x^2\, k_y^2 - 8.08\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.25. Search converged after 1 iterations.

θ_{crit} = 9.76, Tayl = $-22.50 - 101.00\, k_x^4 + 15.20\, k_y^2 + 7.55\, k_x^2\, k_y^2 - 8.87\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.5. Search converged after 1 iterations.

θ_{crit} = 9.81, Tayl = $-18.30 - 100.00\, k_x^4 + 15.40\, k_y^2 + 6.40\, k_x^2\, k_y^2 - 8.20\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 0.75. Search converged after 1 iterations.

θ_{crit} = 9.95, Tayl = $-14.70 - 99.50\, k_x^4 + 15.60\, k_y^2 + 6.24\, k_x^2\, k_y^2 - 7.77\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.. Search converged after 1 iterations.

θ_{crit} = 10.19, Tayl =
$-11.20 - 98.20\, k_x^4 + 0.02\, k_x^3\, k_y + 15.90\, k_y^2 + 6.43\, k_x^2\, k_y^2 - 0.01\, k_x\, k_y^3 - 7.26\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.25. Search converged after 1 iterations.

θ_{crit} = 10.53, Tayl =
$-7.09 - 96.30\, k_x^4 + 0.03\, k_x^3\, k_y + 16.30\, k_y^2 + 7.02\, k_x^2\, k_y^2 - 0.02\, k_x\, k_y^3 - 6.81\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.5. Search converged after 1 iterations.

θ_{crit} = 10.93, Tayl = $2.20 - 93.30\, k_x^4 + 0.01\, k_x^3\, k_y + 17.20\, k_y^2 + 8.54\, k_x^2\, k_y^2 - 7.13\, k_y^4$

Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

Δ_{nem} = 1.75. Search converged after 1 iterations.

θ_{crit} = 11.30, Tayl =
$21.30 - 88.10\, k_x^4 - 0.07\, k_x^3\, k_y + 11.60\, k_x^2\, k_y^2 + 0.02\, k_x\, k_y^3 - 8.72\, k_y^4 + k_y\, (0.02\, k_x + 18.80\, k_y)$

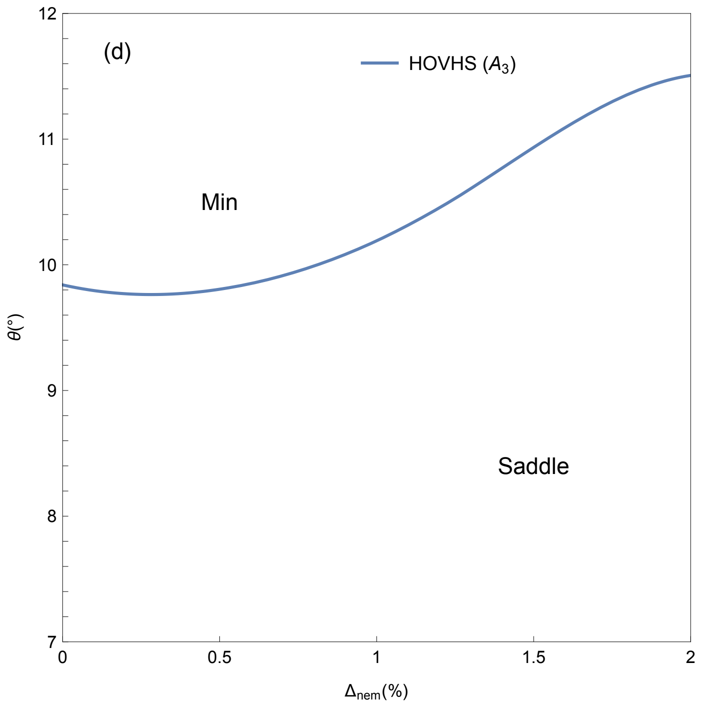
Polynomial has corank 1. It is 4-determinate. Its determinacy is either 4 or 3 and its codimension is 2. Comparing with the known list of singularities:

Polynomial has a cusp (A_3) catastrophe.

****

In[174]:=
```
FitFunc6 = Interpolation[HosListBand6M[[All, 1 ;; 2]]];
PlotBand6 = ListPlot[HosListBand6M[[All, 1 ;; 2]], AxesLabel → {"Δnem (%)", "θ(°)"},
   PlotLegends → Placed[{"HOVHs  (A3)"}, {0.5, 0.965}]];
PanelD = Show[(*PlotBand9,*)Plot[FitFunc6[x],
   {x, 0, 2}, FrameLabel → {"Δnem (%)", "θ(°)"}, PlotLegends → Placed[
     {Style["HOVHS  (A3)", FontFamily → "Arial", FontSize → 10]}, {0.6, 0.92}],
   PlotRange → {{0, 2}, {7, 12}}, Frame → True, AspectRatio → 1,
   FrameTicks → {{Automatic, None}, {{0, 0.5, 1, 1.5, 2}, None}}], Graphics[
   Text[Style["Saddle", FontFamily → "Arial", FontSize → 12], {1.5, 8.4}]],
  Graphics[Text[Style["Min", FontFamily → "Arial", FontSize → 12], {0.5, 10.5}]],
  Graphics[
   Text[Style["(d)", FontFamily → "Arial", FontSize → 12], {0.175, 11.7}]]]
```

Out[176]=



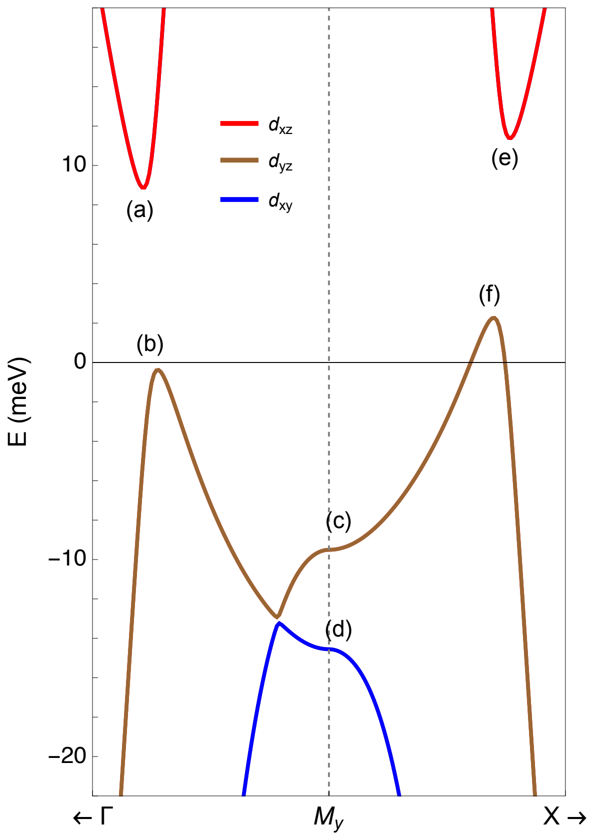
Assembling the results

Labeling the critical points of interest

In[180]:=
```
Bnd4 = GenerateBandStructure[HamTempNem2, HSymmPts, "OrbitalProjection" → True,
   "OrbitalGrouping" → {{1, 4, 7, 10}, {2, 5, 8, 11}, {3, 6, 9, 12}}];
```

In[181]:=
```
PltCritPts =
  PlotBandStructure[Bnd3, (*{-0.042,0.012}*){-22, 18}, AspectRatio → 5/3,
   "yLabel" → {(*Label*)"E (meV)", (*Font Size*)13, (*Font Color*)
     Black, (*Font Family*)"Arial"}, "xLabel" → {13, Black, "Arial"},
   "yTicks" → {12, Black, "Helvetica"}, "LineThickness" → 0.008,
   "LineColorScheme" → {Red, Brown, Blue}, (*Dividing Lines*)"DividingLines" →
    {(*Dashing[..]*)0.01, (*Color*)Gray, (*Thickness*)0.004},
   (*,Legend*)"PlotKeyLegend" → {{"dxz", "dyz", "dxy"}, {0.25, 0.8}}];
PanelCritPts = Show[PltCritPts,
  Graphics[Text[Style["(a)", FontFamily → "Arial", FontSize → 12], {0.28, 7.8}]],
  Graphics[Text[Style["(b)", FontFamily → "Arial", FontSize → 12], {0.34, 1}]],
  Graphics[Text[Style["(c)", FontFamily → "Arial", FontSize → 12], {1.45, -8}]],
  Graphics[Text[Style["(d)", FontFamily → "Arial", FontSize → 12],
    {1.45, -13.5}]], Graphics[
   Text[Style["(e)", FontFamily → "Arial", FontSize → 12], {2.43, 10.42}]],
  Graphics[Text[Style["(f)", FontFamily → "Arial", FontSize → 12], {2.34, 3.5}]]]
```

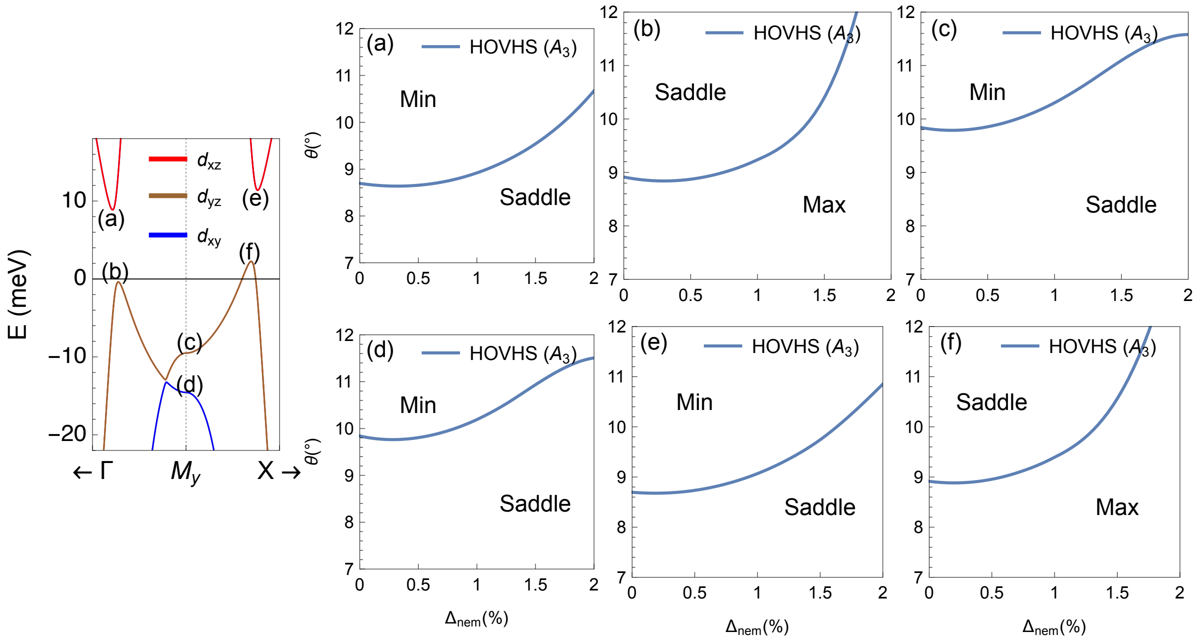
Out[182]=



Phase diagrams for the critical points

In[183]:=
```
Show[GraphicsGrid[{{PanelCritPts, PanelA, PanelB, PanelC},
    {SpanFromAbove, PanelD, PanelE, PanelF}},
  Spacings → {Scaled[0.], Scaled[0.02]}], ImageSize → Full]
```

Out[183]=



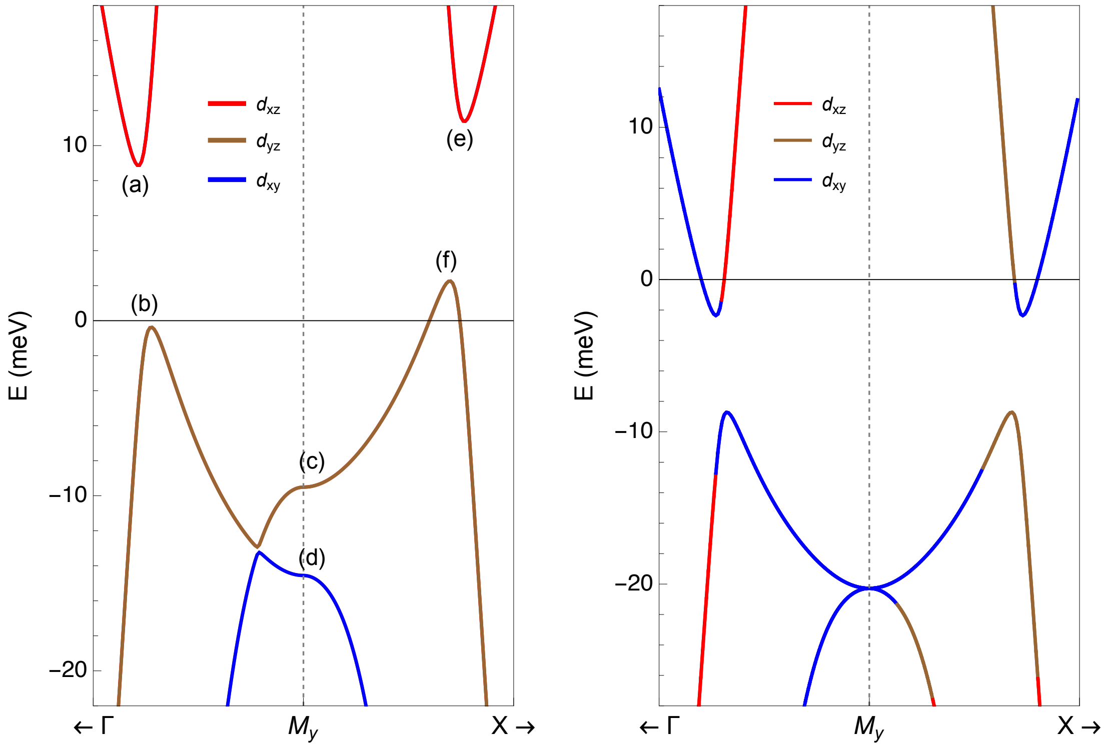
We can also examine the orbital-projected band structure in the absence of nematicity and compare with the nematic model

In[184]:=
```
BndTemp = GenerateBandStructure[HamTemp, HSymmPts, "OrbitalProjection" → True,
    "OrbitalGrouping" → {{1, 4, 7, 10}, {2, 5, 8, 11}, {3, 6, 9, 12}}];
```

In[185]:=
```
PltTemp =
  PlotBandStructure[BndTemp, (*{-0.042,0.012}*){-28, 18}, AspectRatio → 5/3,
   "yLabel" → {(*Label*)"E (meV)", (*Font Size*)13, (*Font Color*)
     Black, (*Font Family*)"Arial"}, "xLabel" → {13, Black, "Arial"},
   "yTicks" → {12, Black, "Helvetica"}, "LineThickness" → 0.008,
   "LineColorScheme" → {Red, Brown, Blue}, (*Dividing Lines*)"DividingLines" →
    {(*Dashing[..]*)0.01, (*Color*)Gray, (*Thickness*)0.004},
   (*,Legend*)"PlotKeyLegend" → {{"dxz", "dyz", "dxy"}, {0.25, 0.8}}];
```

In[186]:=
```
GraphicsRow[{PanelCritPts, PltTemp}]
```

Out[186]=



The dominant orbitals seem to have changed. (Note that we have computed the total contribution of each type of orbital across both the Ru atoms and both spin species).

Phase diagrams

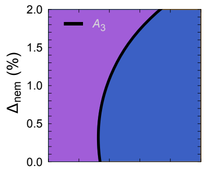
We will now generate the phase diagram for the SOC gap singularities that is included in the manuscript

In[187]:=
```
θmin = 7;
θmax = 12(*9.6*);

cm = 72 / 2.54;

TicsFontSize = 6;
AxesFontSize = 8;
RegionFontColour = LightGray;
FigLength = 3.5 cm;
LeftPadding = 0.5 cm;
RightPadding = 0.2 cm;
BottomPadding = 0.5 cm;
TopPadding = 0.15 cm;
xTicsLabeled = {{7, "7"}, {8, "8"}, {9, "9"}, {10, "10"}, {11, "11"}, {12, "12"}};
xTics = {7, 8, 9, 10, 11, 12};
yTics = {0, 0.5, 1, 1.5, 2};
```

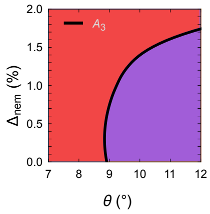
```
VHsColour = {(*125*)161/255, (*9*)93/255, (*184*)216/255, 1(*0.65*)};
MinColour = {(*9*)59/255, (*73*)96/255, (*184*)196/255, 1(*0.65*)};
MaxColour = {(*207*)242/255, (*17*)70/255, (*51*)70/255, 1(*0.65*)};

PanA = ParametricPlot[
   {{v FitFunc3[u] + θmin (1 - v), u}, {(1 - v) FitFunc3[u] + θmax v, u}},
   {u, 0, 2}, {v, 0, 1}, PlotRange → {{θmin, θmax}, {0, 2}}, Frame → True,
   PlotStyle → {RGBColor[VHsColour], RGBColor[MinColour]}, AspectRatio → 1,
   (*FrameTicks→{True,True},*)(*FrameLabel→{None,Style["Δnem (%)",
      FontFamily→"Arial",FontSize→AxesFontSize,FontColor→Black]},*)
   FrameTicksStyle → {Directive[Black, FontFamily → "Arial", TicsFontSize],
     {Directive[Black, FontOpacity → 0, FontSize → 0],
      Directive[Black, FontOpacity → 0, FontSize → 0]}}, FrameTicks →
    {True, {xTics, xTics}}, ImagePadding → {{LeftPadding, RightPadding},
     {0.6 TopPadding, TopPadding}}, ImageSize → FigLength];

PanAp2 = ParametricPlot[{FitFunc3[u], u}, {u, 0, 4}, Axes → False, Frame → False,
   PlotStyle → {Black, Thickness[0.018]}, PlotRange → {{θmin, θmax}, {0, 2}},
   PlotLegends → Placed[LineLegend[{Style["A3", FontFamily → "Arial",
         FontSize → TicsFontSize, FontColor → RegionFontColour]},
      LegendMarkerSize → {10, 1.3}], {{0.25, 0.92}, {0.5, 0.5}}]];

Labeled[Show[PanA, PanAp2], {Style["Δnem (%)",
   FontFamily → "Arial", FontSize → AxesFontSize, FontColor → Black], ""},
 {Left, Bottom}, Spacings → {0, 0}, RotateLabel → True]


PanB = ParametricPlot[
   {{v FitFunc4[u] + θmin (1 - v), u}, {(1 - v) FitFunc4[u] + θmax v, u}},
   {u, 0, 2}, {v, 0, 1}, PlotRange → {{θmin, θmax}, {0, 2}}, Frame → True,
   PlotStyle → {RGBColor[MaxColour], RGBColor[VHsColour]}, AspectRatio → 1,
   FrameTicksStyle → {Directive[Black, FontFamily → "Arial", TicsFontSize],
     {Directive[Black, FontFamily → "Arial", TicsFontSize],
      Directive[Black, FontOpacity → 0, FontSize → 0]}},
   FrameTicks → {True, {xTicsLabeled, xTics}}, ImagePadding →
    {{LeftPadding, RightPadding}, {BottomPadding, 0.6 TopPadding}},
   ImageSize → FigLength];

PanBp2 = ParametricPlot[{FitFunc4[u], u}, {u, 0, 2},
   Axes → False, Frame → False, PlotStyle → {Black, Thickness[0.018]},
   PlotLegends → Placed[LineLegend[{Style["A3", FontFamily → "Arial",
         FontSize → TicsFontSize, FontColor → RegionFontColour]},
      LegendMarkerSize → {10, 1.3}], {{0.25, 0.92}, {0.5, 0.5}}]];
```

```
Labeled[Show[PanB, PanBp2],
 {Style["Δnem (%)", FontFamily → "Arial", FontSize → AxesFontSize,
   FontColor → Black], Style["θ (°)", FontFamily → "Arial",
   FontSize → AxesFontSize, FontColor → Black]},
 {Left, Bottom}, Spacings → {0, 0}, RotateLabel → True]
```

Out[206]=

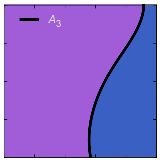


Out[209]=

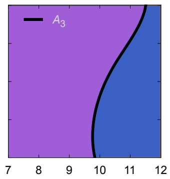


In[210]:=
```
PanC = ParametricPlot[
    {{v FitFunc5[u] + θmin (1 - v), u}, {(1 - v) FitFunc5[u] + θmax v, u}},
    {u, 0, 2}, {v, 0, 1}, PlotRange → {{θmin, θmax}, {0, 2}}, Frame → True,
    PlotStyle → {RGBColor[VHsColour], RGBColor[MinColour]}, AspectRatio → 1,
    (*FrameTicks→{True,True},*)(*FrameLabel→{None,Style["Δnem (%)",
       FontFamily→"Arial",FontSize→AxesFontSize,FontColor→Black]},*)
    FrameTicksStyle → {{Directive[Black, FontOpacity → 0, FontSize → 0],
       Directive[Black, FontOpacity → 0, FontSize → 0]},
      {Directive[Black, FontOpacity → 0, FontSize → 0],
       Directive[Black, FontOpacity → 0, FontSize → 0]}},
    FrameTicks → {{yTics, yTics}, {xTics, xTics}},
    ImagePadding → {{LeftPadding, RightPadding}, {0.6 TopPadding, TopPadding}},
    ImageSize → FigLength];

PanCp2 = ParametricPlot[{FitFunc5[u], u}, {u, 0, 4}, Axes → False, Frame → False,
    PlotStyle → {Black, Thickness[0.018]}, PlotRange → {{θmin, θmax}, {0, 2}},
    PlotLegends → Placed[LineLegend[{Style["A3", FontFamily → "Arial",
          FontSize → TicsFontSize, FontColor → RegionFontColour]},
       LegendMarkerSize → {10, 1.3}], {{0.25, 0.92}, {0.5, 0.5}}]];

Show[PanC, PanCp2]
(*Labeled[Show[PanA,PanAp2],
 {Style["Δnem (%)",FontFamily→"Arial",FontSize→AxesFontSize,FontColor→Black],
  ""},{Left,Bottom},Spacings→{0, 0},RotateLabel→True]*)


PanD = ParametricPlot[
    {{v FitFunc6[u] + θmin (1 - v), u}, {(1 - v) FitFunc6[u] + θmax v, u}},
    {u, 0, 2}, {v, 0, 1}, PlotRange → {{θmin, θmax}, {0, 2}}, Frame → True,
    PlotStyle → {RGBColor[VHsColour], RGBColor[MinColour]}, AspectRatio → 1,
    FrameTicksStyle → {{Directive[Black, FontOpacity → 0, FontSize → 0],
       Directive[Black, FontOpacity → 0, FontSize → 0]},
      {Directive[Black, FontFamily → "Arial", TicsFontSize],
       Directive[Black, FontOpacity → 0, FontSize → 0]}},
    FrameTicks → {{yTics, yTics}, {xTicsLabeled, xTics}},
    ImagePadding → {{LeftPadding, RightPadding}, {BottomPadding, 0.5 TopPadding}},
    ImageSize → FigLength];

PanDp2 = ParametricPlot[{FitFunc6[u], u}, {u, 0, 2},
    Axes → False, Frame → False, PlotStyle → {Black, Thickness[0.018]},
    PlotLegends → Placed[LineLegend[{Style["A3", FontFamily → "Arial",
          FontSize → TicsFontSize, FontColor → RegionFontColour]},
       LegendMarkerSize → {10, 1.3}], {{0.25, 0.92}, {0.5, 0.5}}]];

Show[PanD, PanDp2]
```

Out[212]=



Out[215]=



In[221]:=
```
PhaseDiagramFinal =
  GraphicsGrid[{{Labeled[Show[PanA, PanAp2, Graphics[Rotate[Text[
          Style["Saddle", FontFamily → "Arial", FontColor → RegionFontColour,
           FontSize → TicsFontSize], {8, 1}], 90 Degree]], Graphics[
        Text[Style["Minimum", FontFamily → "Arial", FontColor → RegionFontColour,
          FontSize → TicsFontSize], {10.5, 1.0}]], ListPlot[{{9, 0.5}},
        PlotMarkers → {▲, 2.5}, PlotStyle → {LightGray, Thickness[0.02]}],
       Graphics[Text[Style["QPI", FontFamily → "Arial", FontColor →
           RegionFontColour, FontSize → (TicsFontSize - 1)], {9.45, 0.5}]],
       ListPlot[{{8, 0.1}}, PlotMarkers → {"↓", 8}, PlotStyle → {LightGray}],
       Graphics[Text[Style["ARPES", FontFamily → "Arial", FontColor →
           RegionFontColour, FontSize → (TicsFontSize - 1)], {7.9, 0.25}]]],
      {Style["Δnem (%)", FontFamily → "Arial", FontSize → AxesFontSize,
        FontColor → Black]}, {Left},
      Spacings → {0.1, 0}, RotateLabel → True, ImageSize → FigLength]
     ,
     Labeled[Show[PanC, PanCp2,
       Graphics[Rotate[Text[Style["Saddle", FontFamily → "Arial", FontColor →
            RegionFontColour, FontSize → TicsFontSize], {8, 1}], 0 Degree]],
       Graphics[Rotate[Text[Style["Minimum", FontFamily → "Arial",
           FontColor → RegionFontColour, FontSize → TicsFontSize], {11.2, 1.0}],
         90 Degree]], ListPlot[{{9, 0.5}}, PlotMarkers → {▲, 2.5},
        PlotStyle → {LightGray, Thickness[0.02]}], Graphics[
        Text[Style["QPI", FontFamily → "Arial", FontColor → RegionFontColour,
          FontSize → (TicsFontSize - 1)], {9.45, 0.5}]],
       ListPlot[{{8, 0.1}}, PlotMarkers → {"↓", 8}, PlotStyle → {LightGray}],
       Graphics[Text[Style["ARPES", FontFamily → "Arial", FontColor →
           RegionFontColour, FontSize → (TicsFontSize - 1)], {7.9, 0.25}]],
       ImageSize → FigLength], {Style["", FontSize → 0]}, {Right},
      Spacings → {0.1, 0}, RotateLabel → True]
    }, {Labeled[Show[PanB, PanBp2,
       Graphics[Rotate[Text[Style["Maximum", FontFamily → "Arial", FontColor →
            RegionFontColour, FontSize → TicsFontSize], {8, 1}], 90 Degree]],
       Graphics[Text[Style["Saddle", FontFamily → "Arial",
          FontColor → RegionFontColour, FontSize → TicsFontSize], {10.5, 1.0}]],
       ListPlot[{{9, 0.5}}, PlotMarkers → {▲, 2.5},
        PlotStyle → {LightGray, Thickness[0.02]}],
```

```
      Graphics[Text[Style["QPI", FontFamily → "Arial", FontColor →
          RegionFontColour, FontSize → (TicsFontSize - 1)], {9.45, 0.5}]],
      ListPlot[{{8, 0.1}}, PlotMarkers → {"↓", 8}, PlotStyle → {LightGray}],
      Graphics[Text[Style["ARPES", FontFamily → "Arial", FontColor →
          RegionFontColour, FontSize → (TicsFontSize - 1)], {7.9, 0.25}]]],
     {Style["Δnem (%)", FontFamily → "Arial", FontSize → AxesFontSize,
       FontColor → Black], Style["θ (°)", FontFamily → "Arial",
       FontSize → AxesFontSize, FontColor → Black]},
     {Left, Bottom}, Spacings → {0.1, 0}, RotateLabel → True],
    Labeled[Show[PanD, PanDp2,
      Graphics[Rotate[Text[Style["Saddle", FontFamily → "Arial", FontColor →
           RegionFontColour, FontSize → TicsFontSize], {8, 1}], 0 Degree]],
      Graphics[Rotate[Text[Style["Minimum", FontFamily → "Arial",
          FontColor → RegionFontColour, FontSize → TicsFontSize], {11.2, 1.0}],
        90 Degree]], ListPlot[{{9, 0.5}}, PlotMarkers → {▲, 2.5},
       PlotStyle → {LightGray, Thickness[0.02]}], Graphics[
       Text[Style["QPI", FontFamily → "Arial", FontColor → RegionFontColour,
         FontSize → (TicsFontSize - 1)], {9.45, 0.5}]],
      ListPlot[{{8, 0.1}}, PlotMarkers → {"↓", 8}, PlotStyle → {LightGray}],
      Graphics[Text[Style["ARPES", FontFamily → "Arial", FontColor →
          RegionFontColour, FontSize → (TicsFontSize - 1)], {7.9, 0.25}]]],
     {Style["θ (°)", FontFamily → "Arial", FontSize → AxesFontSize,
       FontColor → Black], Style["", FontSize → 0]}, {Bottom, Right},
     Spacings → {0.1, 0}, RotateLabel → {True, True}]}}, Spacings → {0, -8}]

(*Export[NotebookDirectory[]<>"Phase_horiz.pdf",
 Show[PhaseDiagramFinal,ImageSize→4.5cm]]*)
```

Out[221]=