DMPT-1

Probability (Syllabus)

* Basics of set theory and counting principles.
* Sample space, events, Axioms of probability, other definitions
* Conditional probability
* Baye's theorem., Independence of events
* Random variables, cumulative Distribution function, Expectation.
* Discrete random variables and their distributions.
* Properties of PDF and PMF.
* Functions of Random variables, Moment
* of Generating functions & properties.
* Conditional Distribution
* Joint & Marginal Distribution.

* Inequalities & Bounds
* Convergence & Limit theorems
* Central limit theorems & Law of large numbers

Books :-

1) An Introduction to Probability & statistics, Second edition by Rohatgi & Saleh
2) An outline of Statistical theory, Volume One by Gun Gupta & Das gupta

Basics of set Theory :-

⇒ Proper subset :- $A \subset \Omega$ $B \subset \Omega$

if $A \subseteq \Omega$, $A \neq \Omega$

then $A \subset \Omega$

⇒ Subset :- if $\omega \in A \Rightarrow \omega \in \Omega$

then $A \subseteq \Omega$

If $A = \{\omega \in \Omega : \omega \in A\}$

$A^c = \{\omega \in \Omega : \omega \notin A\}$

⇒ Union :-

$A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}$

⇒ Intersection :-

$A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}$

Ω = set of positive integers

A = Set of even ...

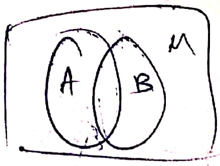
B = Set of odd ...

$\bar{A} \cup \bar{B}$

$B \cup A = \Omega$

$A \cup B = \Omega$ $\quad \bar{A} = B$

$A \cap B = \phi$ $\quad \bar{B} = A.$



$(A \cap B)^c = \Omega$

$A - B = A \cap \bar{B} = A$ only $\underline{A}$.

$A - B = \{ w \in \Omega ; w \in A, w \notin B \}$

Laws :-

→ Idempotary = $A \cup A = A \cap A = A.$

→ Commutative = $A \cup B = B \cup A$

$A \cap B = B \cap A.$

→ Associative = $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

→ Distributive $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Demorgan's law:

$\boxed{(A \cap B)^c = A^c \cup B^c}$

$A = \{1, 2, 3, 4\}$

$B = \{2, 4, 6, 8\}$

$A \cup B = \{1, 2, 3, 4, 6, 8\}$

$W = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{2, 4, 6, 8\}$

$B = \{1, 3, 5, 7\}$

$A \cup B = W$

$A \cap B = \phi$

$A = \{ w \in \Omega : w \in A$ where $A \subseteq W$

$B = \{ w \in \Omega : w \in B$ where $B \subseteq W$

$(A \cap B)^c = W - \phi$

$(A \cup B)^c = W \rightarrow ①$

$A^c = B \quad B^c = A$

$A^c \cup B^c = B \cup A = W \rightarrow ②$

$(A \cap B)^c = A^c \cup B^c$

DMPT-2

22/9/17

Experiment :- An experiment is observing something happen (or) conducting something in certain conditions which result in some outcome.

Ex:- observing weather conditions.

Types of Exp :-

1) Deterministic Exp :-

Under certain conditions of an exp is conducted, it results in a known outcome.

Ex:- throwing a ball upwards

2) Random Exp :-

Under certain conditions of an exp is conducted, its results in an unknown outcome

Ex:- 1) Tossing a coin $\{H, T\}$

2) Throwing a die. $\{1, 2, 3, 4, 5, 6\}$

3) Drawing a card from a pack of cards

4) Birth of a child. (M, F)

5) Age of death of a person (0, 120)

6) Amount of rainfall during monsoon

7) Life of a bulb $(0, \infty)$

8) Time taken to complete 100m race by an athlete $(0, 120)$ sec

Basic Terminologies :-

1) **Sample Space** :- The set of all possible outcomes of a random experiment.
Denoted by Ω, U, S.

Ex :- Drawing a card from pack of cards

$\{R, B\}$, $\{H, S, D, C\}$, $\{S_1 \ldots S_{13}\}$, $\{H_1 \ldots H_{13}\}$
$\{D_1 \ldots D_{13}\}$ $\{C_1 \ldots C_{13}\}$

Heart, Spade, Diamond, Club.

2) **Event** :- An event is any subset of a sample space.

Imp event :- ϕ.

Sure event :- Ω.

$A \cap B$

occurrence
⇒ $A \cup B$ = ~~Outcome~~ of atleast one of A and B.

$\bigcup_{i=1}^{n} A_i$

⇒ $A \cap B$ = Simultaneous occurence of A and B.

$\bigcap_{i=1}^{n} A_i$ – Simultaneous occurrence of A_1 to $\ldots A_n$

⇒ $A \cup B = \Omega$ Mutually Exhaustive. → $\bigcup_{i=1}^{n} A_i = \Omega$

$A_1 \ldots A_n$ are exhaustive

⇒ $A \cap B = \phi$ Mutually Exclusive. ⇒ $\bigcap_{i=1}^{n}$ ~~$A \cap B$~~ $= \phi$.

⇒ Pair wise Mutually Exclusive

$A_1 \ldots A_n$ $A_i \cap A_j = \phi$ $\forall i \neq j$.

Compliment of A :- A^c, $\bar{A}$

$A^c = U - A$.

First definition of probability (Classical definition / Mathematical)

↳ Given by Laplace in 1812.

Suppose a random experiment has n possible outcomes, which are mutually exclusive, exhaustive and equally likely. Let M of these outcomes be favourable to the happening of event A, then the probability of A is defined by $P(A) = \frac{M}{N}$.

Drawbacks of classical definition :-

1) It does not talk about infinite n.
2) It does not talk about the events which are not mutually exclusive, exhaustive.
3) What is the meaning of the term equally likely? (It is a contradiction)

→ Frequency definition of Probability :-

(Emperical / Statistical / Relative frequency)

↳ Given by. Von Mises.

Suppose a random experiment is conducted a large no. of times independently under identical conditions. Let an denote the no. of times the event A occurs ~~under~~ identical condⁿ in n trials, then we define probability as

$$\boxed{P(A) = \lim_{n \to \infty} \frac{a_n}{n}}$$ provided limit exists.

DMPT-3 : (Axioms of Probability)

Ω = sample space.

$\mathcal{B}$: a σ-field of subsets of Ω.

$(\Omega, \mathcal{B})$ is called a measurable space.

Let $(\Omega, \mathcal{B})$ be a measurable space. A set function $P : \mathcal{B} \rightarrow R$, is said to be a probability function if it satisfies the following three axioms :-

P_1 : $P(A) \geq 0$, $\forall A \in \mathcal{B}$ → non-negativity

P_2 : $P(\Omega) = 1$ — completeness

P_3 : For any sequence of pairwise disjoint subsets $E_i \in \mathcal{B}$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) \rightarrow \text{Countable Additivity Axiom}$$

Consequences :-

C_1 : $P(\phi) = 0$: $E_1 = \Omega$, $E_2 = E_3 \dots E_n = \phi$.

Proof :- $P_3 \Rightarrow P(\Omega) = P(\Omega) + P(\phi) \dots$

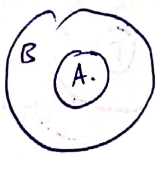
$P_2 \Rightarrow 1 = 1 + \underbrace{P(\phi) + \dots}_{\geq 0}$

$P_1 \Rightarrow P(\phi) = 0$

C_2 : $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$ $E_{n+1}, E_{n+2} = \dots = \phi$.

Proof :- P_3 $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) + \sum_{i=n+1}^{\infty} P(\phi)$ → 0.

C_3 : P is monotone



$A \subseteq B$, $P(A) \leq P(B)$

$B = A \cup (B-A)$ ≥ 0

$P(B) = P(A) + P(B-A)$ C_2. So, $P(B) \geq P(A)$

C_4: $0 \le P(A) \le 1$. $\forall A \in \mathcal{B}$

$A \subseteq \Omega$

Pf: $\Omega = A \cup (\Omega - A)$

$P(\Omega) = P(A) + P(\Omega - A) \ge 0$

P_2

$P(\Omega) = 1$ $\quad$ $P(\Omega) \ge P(A)$

$\boxed{P(A) \le 1} \rightarrow P_2$

P_1. $P(A) \ge 0$. $\quad$ So, $0 \le P(A) \le 1$.

C_5: $P(A^c) = 1 - P(A)$ $\quad$ $\Omega = A \cup A^c$

C_2: $P(\Omega) = P(A) + P(A^c)$ $\quad$ 3|1

P_1: $1 = P(A) + P(A^c)$

P_1 :- $P(A^c) = 1 - P(A)$.

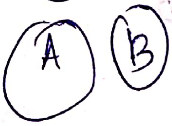
29/8/19.

DMPT-4 :-

Additive rule :- for any events $A, B \in \mathcal{B}$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

for disjoint



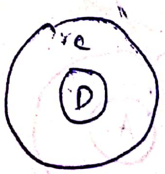
are disjoint

$P(A \cup B) = P(A) + P(B)$ $\quad$ Since for disjoint

$P(A \cup B) = P(A) + P(B) - 0$ $\quad$ $P(A \cap B) = \phi$

$P(\phi) = 0$

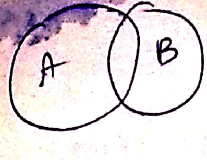
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$D \subseteq C$ $\quad$ $C = D \cup (C - D)$

$P(C) = P(D) + P(C - D)$

$\boxed{P(C-D) = P(C) - P(D)}$ So,



$A \cup B = A \cup (B - (A \cap B))$

$P(A \cup B) = P(A) + P(B - (A \cap B))$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$B - A$ diff

$B - A \cup B$ diff

$\Rightarrow P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap C) - P(A \cap B) - P(B \cap C) + P(A \cap B \cap C)$

General Additivity Rule :-

For any events $A_1 \ldots A_n \in \mathcal{B}$.

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{i<j} P(A_i \cap A_j) + \sum\sum\sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

Sub Additivity :-

$P(A \cup B) \leq P(A) + P(B)$

For $A_1, \ldots, A_n \in \mathcal{B}$, $\boxed{P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)}$ ✓

$P(A_1 \cup A_2)$

$P(A \cup B) \leq P(A) + P(B)$

$P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$

$P(A_1 \cup A_2) = P(A_1) + P(A_2)$

$P(A_1 \cup A_2 \cup A_3) \leq P(A_1) + P(A_2) + P(A_3)$.

let true $\rightarrow P(A_1 \cup A_2 \cup \ldots A_k) \leq P(A_1) + P(A_2) + \cdots P(A_k)$

for $k+1$. $\ldots A_k \cup A_{k+1}) \leq P(A_1) + P(A_2) + \ldots P(A_k) + P(A_{k+1})$

$P(A_1 \cup A_2 \cdots A_k \cup A_{k+1}) \leq P(A_1 \cup \ldots A_k) + P(A_{k+1})$

~~$P(A_1 \cup A_2 \ldots A_k \cup A_{k+1})$ $P(A_1 \cup \ldots A_k) \leq P(A_{k+1})$~~

$P($

$$P\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} P(A_i)$$

$$\rightarrow P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right)$$

$$\le P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1})$$

$$\le \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) \le \sum_{i=1}^{k+1} P(A_i)$$

$$\boxed{C_6 : P\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} P(A_i)}$$

Bonferne's Inequality :

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j}\sum P(A_i \cap A_j) \le P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

Boole's Inequality :

Let $\{A_n\}$ be a sequence of sets in R. then,

probability . $P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$

$$P(A_1 \cap A_2 \cap \dots A_\infty) \geq 1 - (P(A_1^c) + P(A_2^c) + \dots P(A_\infty^c))$$

$$P(A) = 1 - P(A^c)$$

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - P\left[\bigcap_{i=1}^{\infty} A_i\right]^c$$

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right) \quad \text{(Using Demorgan's law)}$$

$$\boxed{P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)}$$

DMPT-5 :-

29/8/18

Conditional Probability :-

2, 4, 6

$E \rightarrow$ an even number occurs $\{2,4,6\}$.

$B \rightarrow$ 2 occurs $\qquad P(B) = \frac{1}{6} \quad P\left(\frac{B}{E}\right) = \frac{P(B \cap E)}{P(E)} = \frac{1/6}{3/6} = \frac{1}{3}$

Let $(\Omega, \mathcal{B}, P)$ be a probability space and let an event belongs to $\mathcal{B}$ $(B \in \mathcal{B})$ with $P(B) > 0$. then for any $A \in \mathcal{B}$, the conditional probability of A given that B has already occured is

$$\boxed{P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}} \quad \text{for any } A \in \mathcal{B}.$$

(the conditional probability of A given that this B has already occured.)

P_1: $P(A/B) = \frac{P(A \cap B)}{P(B)} \geq 0$

P_2: $P(\Omega/B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$.

P_3: $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$

$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i/B\right) = \sum_{i=1}^{\infty} P(A_i/B)$?

$$= \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B\right)}{P(B)} = \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)} = \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)}$$

$$= \sum_{i=1}^{\infty} P(A_i/B)$$

$P(A \cap B) = P(A/B) \cdot P(B) \rightarrow P(B) \geq 0$

$= P(B/A) \cdot P(A) \rightarrow P(A) \geq 0$

$\hookrightarrow$ multiplication rule.

General multiplication rule:

Let $A_1 \cdots A_n \in \mathcal{B}$ with probability of

$P(\bigcap_{i=1}^{n} A_i) > 0$. then.

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1) P\left(\frac{A_2}{A_1}\right) P\left(\frac{A_3}{A_1 \cap A_2}\right) \cdots P\left(\frac{A_n}{\bigcap_{i=1}^{n-1} A_i}\right)$$

If $n=3$

$$P(\bigcap_{i=1}^{3} A_i) = P(A_1) P\left(\frac{A_2}{A_1}\right) P\left(\frac{A_3}{A_1 \cap A_2}\right)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \frac{P(A_1 \cap A_2)}{P(A_1)} \cdot \frac{P(A_3 \cap A_1 \cap A_2)}{P(A_1 \cap A_2)}$$

$\rightarrow$ Assuming its true

$$P(\bigcap_{i=1}^{k} A_i) = P(A_1) P\left(\frac{A_2}{A_1}\right) P\left(\frac{A_3}{A_1 \cap A_2}\right) \cdots P\left(\frac{A_k}{\bigcap_{i=1}^{k-1} A_i}\right)$$

To prove for $k+1$

$$P(\bigcap_{i=1}^{k+1} A_i) = P(A_1) P\left(\frac{A_2}{A_1}\right) P\left(\frac{A_3}{A_1 \cap A_2}\right) \cdots P(A$$

$$P(\bigcap_{i=1}^{k} A_i \bigcap_{i=k}^{k+1} A_i) = P(\bigcap_{i=1}^{k} A_i \cap A_{k+1})$$

$$= P(\bigcap_{i=1}^{k} A_i) \; P(A_{k+1})$$

Proof:- $P(\bigcap_{i=1}^{k+1} A_i) = P((A_1 \cap A_2) \bigcap_{i=3}^{k+1} A_i)$

$$= P(A_1 \cap A_2) \cdot P\left(\frac{A_3}{A_1 \cap A_2}\right) P\left(\frac{A_4}{A_1 \cap A_2 \cap A_3}\right)$$

$$= \underbrace{P(A_1) P\left(\frac{A_2}{A_1}\right)}_{\text{using multiplication rule}} P\left(\frac{A_3}{\cdot}\right) \cdots P\left(\frac{A_{k+1}}{\bigcap_{i=1}^{k} A_i}\right)$$

Theorem of total probability:

let $\{B_n\}$, $B_1, B_2, \ldots\ldots$ pair wise-disjoint sets with $B = \bigcup_{i=1}^{\infty} B_i$

then for any event A, $P(A \cap B) = \sum_{j=1}^{\infty} P(A/B_j) P(B_j)$

Corollary:

$P(B) = 1$ (or) $B = \Omega$

then $P(A) = \sum_{j=1}^{\infty} P\left(\frac{A}{B_j}\right) P(B_j)$

$$P(A \cap B) = P\left(\frac{A}{B_1}\right) P(B_1) + P\left(\frac{A}{B_2}\right) P(B_2)$$

Proof: $P(A \cap B) = P\left(A \cap \left(\bigcup_{j=1}^{\infty} B_j\right)\right)$

$$= P\left(\bigcup_{j=1}^{\infty} (A \cap B_j)\right)$$

$$= \sum_{j=1}^{\infty} P(A \cap B_j) = \sum_{j=1}^{\infty} P\left(\frac{A}{B_j}\right) P(B_j)$$

Problems:

Suppose a calculator manufactor purchases his IC's. from supplies B_1, B_2, B_3, with 40% B_1, 30% B_2, 30% B_3. Suppose 1% from B_1, 5% B_2, 10% B_3 is defective. Probability of randomly selected IC from manufactorer stock is defective.

	Nice	Defective
$B_1 \to$	40%	1%
$B_2 \to$	30%	5%
$B_3 \to$	30%	10%

Sol: ~~$P(\ ) = 1$~~

$$P(A) = P\left(\frac{A}{B_1}\right) P(B_1) + P\left(\frac{A}{B_2}\right) P(B_2) + P\left(\frac{A}{B_3}\right) P(B_3)$$

$$= \frac{1}{40} \times \frac{40}{100} + \frac{5}{30} \times \frac{30}{100} + \frac{10}{30} \times \frac{30}{100}$$

$\frac{1}{30} + \frac{5}{40} + \frac{10}{30}$

$\frac{11}{30} + \frac{5}{40}$

$\frac{5+10}{100} = \frac{15}{100}$

$44 + 15$, $\frac{1}{10}\left(\frac{11}{3} + \frac{5}{4}\right)$

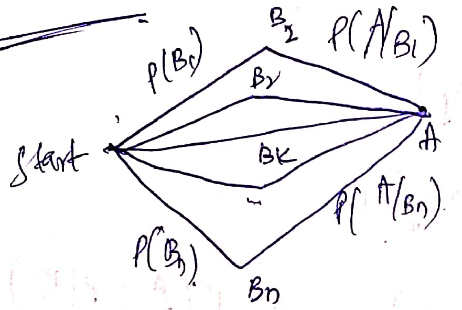
$$P(A) = P\left(\frac{A}{B_1}\right) P(B_1) + P\left(\frac{A}{B_2}\right) P(B_2) + P\left(\frac{A}{B_3}\right) P(B_3)$$

$$= \frac{1}{10\cancel{0}} \times \frac{4\cancel{0}}{100} + \frac{5}{100} \times \frac{3\cancel{0}}{10\cancel{0}} + \frac{1\cancel{0}}{100} \times \frac{3\cancel{0}}{1\cancel{0}\cancel{0}}$$

$$= \frac{4}{1000} + \frac{15}{1000} + \frac{30}{1000}$$

$$= \frac{19+30}{1000} = \frac{49}{1000} = \underline{\underline{0.049}}$$

DMPT-6:- Conditional Probability 30/8/19



$P(B_1)$, $P(B_2)$, $P(B_3)$ $\rightarrow$ prior probabilities

$P\left(\frac{B}{A_1}\right)$ = posterior probab...

Thomas Bayes $\rightarrow$ 1763-

Baye's theorem:

Let $B_1, B_2 \ldots \cancel{B_n}$ are pair-wise disjoint events with $\bigcup_{j=1}^{\infty} B_j = \Omega$ and we are given prior probabilities $P(B_j) > 0$. $j=1,2\ldots$ then for any event A, $P(A) > 0$.

$$\boxed{P\left(\frac{B_i}{A}\right) = \frac{P\left(\frac{A}{B_i}\right) P(B_i)}{\sum_{j=1}^{\infty} P(A/B_j) P(B_j)}}$$

Using ~~conditional~~ total probability

$$P(A) = \sum_{j=1}^{\infty} P\left(\frac{A}{B_j}\right) P(B_j)$$

Now, Using conditional probability

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i \cap A)}{P(A)} = \frac{P\left(\frac{A}{B_i}\right) P(B_i)}{\sum_{j=1}^{\infty} P\left(\frac{A}{B_j}\right) P(B_j)}$$

$P(A \cap B_i) = P\left(\frac{A}{B_i}\right) P(B_i)$ from multiplication rule.

$$P\left(\frac{B_1}{A}\right) = \frac{P\left(\frac{A}{B_1}\right) P(B_1)}{P\left(\frac{A}{B_1}\right) P(B_1) + P\left(\frac{A}{B_2}\right) P(B_2) + P\left(\frac{A}{B_3}\right) P(B_3)}$$

$$= \frac{\frac{4}{1000}}{\frac{49}{1000}} = \frac{4}{49}$$

$$P\left(\frac{B_2}{A}\right) = \frac{P\left(\frac{A}{B_2}\right) P(B_2)}{P\left(\frac{A}{B_1}\right) P(B_1) + P\left(\frac{A}{B_2}\right) P(B_2) + P\left(\frac{A}{B_3}\right) P(B_3)}$$

$$= \frac{\frac{15}{1000}}{\frac{49}{1000}} = \frac{15}{49}.$$

$$P\left(\frac{B_3}{A}\right) = \frac{30}{49}$$

---

prior

$P(B_1) = 40\%$.

$P(B_2) = 30\%$.

$P(B_3) = 30\%$

posterior

$P(B_1/A) = 8\%$

$P(B_2/A) = 30\%$

$P(B_3/A) = 60\%$.

A : a student gets into top IIT.

B : the student is from coaching centre 'x'

$P(B/A)$ = parents ask IIT'ians where they th

Independence of events :-

Tossing two fair coins

$\Omega = \{HH, HT, TH, TT\}$

A : Head appears in the first coin

B : Head appears in second coin

$P(A \cap B) = \frac{1}{4}$ $\quad P(A) = \frac{2}{4} = \frac{1}{2}$ $\quad P(B) = \frac{2}{4} = \frac{1}{2}$

$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{2}{4} = \frac{1}{2}$ // $\quad P(A) = \frac{1}{2}$.

$\boxed{\text{When } P(A/B) = P(A), \text{ Independent}}$

$\Rightarrow P(A \cap B) = P(A) \cap P(B)$

$= P(B).P(A/B)$

$\boxed{P(A \cap B) = P(B).P(A)}$

Definition :- Events A and B are said to be statistically independent if $P(A \cap B) = P(A).P(B)$.

1) $P(A \cap B) = P(A).P(B)$
2) $P(A \cap C) = P(A).P(C)$
3) $P(B \cap C) = P(B).P(C)$
4) $P(A \cap B \cap C) = P(A) P(B).P(C)$

General Stat: Events $A_1, A_2 \dots A_n$ are said to be independent if for any finite collection of those events, the conditions for independence are satisfied.

* Consider throwing of two die
  - A: odd number on first die
  - B: odd number on second die
  - C: odd sum.

(1,1)	~~(1,2)~~	(1,3)	~~(1,4)~~	(1,5)	~~(1,6)~~
~~(2,1)~~	(2,2)	~~(2,3)~~	(2,4)	~~(2,5)~~	(2,6)
(3,1)	~~(3,2)~~	(3,3)	~~(3,4)~~	(3,5)	~~(3,6)~~
~~(4,1)~~	(4,2)	~~(4,3)~~	(4,4)	~~(4,5)~~	(4,6)
~~(5,1)~~	~~(5,2)~~	(5,3)	~~(5,4)~~	(5,5)	~~(5,6)~~
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$P(A) = \frac{18}{36} = \frac{1}{2}$, $\quad P(C) = \frac{18}{36} = \frac{1}{2}$ // $\quad P(A/B) = \frac{P(A \cap B)}{P(B)}$

$P(B) = \frac{18}{36} = \frac{1}{2}$. $\quad = \frac{\frac{1}{4}}{\frac{1}{2}}$

$P(A/B) = P(A)$ ~~so~~, independent events.

$P(A/B) = \frac{1}{2} = P(A)$

4th one didn't satisfy so, not independent events.

2/9/19

DMPT-7 + (Practice) (Revision Tutorial)

Q1) Suppose there are n persons in a party. Assume $n \leq 365$ and no person has birthday on 29th Feb.. What is the probability that atleast two persons share the same B'day.

$=$ Total − no persons share same B'day

$$= 1 - \frac{(365!)^{n}}{(365)^{n}} = 1 - \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \ldots$$

$$= 1 - \frac{{}^{365}P_n}{(365)^n}$$

Q2) A die is rolled twice. Let all the elementary events in $\Omega = \{(i,j) : i, j = 1, 2, \ldots 6\}$ be assigned the same propability. Let A be the event that the first throw shows a number ≤ 2. and B be event that the second throw shows at least 5.

$P(A\cup B)$?

B

$P(A) = \frac{12}{36}$ $\frac{12}{36}$

$P(B)$

A →
(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

$\frac{1}{3} + \frac{1}{3}$

$\frac{4}{3/9}$

93) A coin is tossed three times. let us assign equal probability to each elementary event in Ω. Let A be the event that has atleast one head shows up in three throws. $P(A) = ?$

$\Rightarrow 1 - \frac{1}{8} = \frac{7}{8}$

HHH
HHT

---

94) ~~R~~ r distinguishable balls are to be placed in n cells. this amounts to choosing one cell for each ball. Sample space contains = ? n^r

→ r tossings with a coin. the probability that no heads will show up in r throws is ?

$\left(\frac{1}{2}\right)^r$ H T $\frac{2!}{1!\,1!}$ $2c$

→ The probability that no 6 will turn up in r throws of a die is = ? $\left(\frac{5}{6}\right)^r$

---

95) Consider a set of n elements. A sample of size r is drawn at random with replacement. then the probability that no element appears more than once is ?

$= \frac{nPr}{n^r}$ $\qquad = \frac{nCr}{n^r}$

All elements are different not identical

If identical nCr

Q6) Let Ω be the set of all permutations of n objects. Let A_i be the set of all permutation that leave the ith object unchanged.

$P\left(\bigcup_{i=1}^{n} A_i\right) = ?$ $\qquad \Omega = n!$

$= \sum_{i=1}^{n} P(A_i) - \sum\sum_{i<j} P(A_i \cap A_j) \cdots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$

$P(A_i) = \frac{(n-1)!}{n!} \times {}^nC_1 = 1$

$P(A_1 \cap A_2) = \frac{(n-2)!}{n!} \times {}^nC_2 = \frac{1}{2!}$

$P(A_1 \cap A_2 \cap A_3) = \frac{(n-3)!}{n!} \, {}^nC_3 = \frac{1}{3!}$

$\Rightarrow \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} - \cdots + \frac{1}{n!}\right)$

$= \sum_{i=1}^{n} (-1)^{i+1} \frac{1}{i!}$

Q7) Consider the random distribution of r balls in n cells. Let A_k be the event that a specified cell has exactly k balls $k = 0, 1, \ldots r$

$P(A_k) = ?$ $\qquad \frac{{}^{n+1}C_k}{n^r} = \frac{{}^nC_k \, (n-1)^{k-r}}{n^r}$

Q8) Probability that a hand of 13 cards contain 2 spades, 7 hearts, 3 diamonds and 1 club is?

${}^{13}C_2 \times {}^{13}C_7 \times {}^{13}C_3 \times {}^{13}C_1$

7 Urn Contains 5 red, 3 green, 2 blue & 4 white balls. 8 balls are selected at random without replacement. Probability that sample contains 2 red, 2 green, 1 blue and 3 white balls is?

$$= \frac{5C_2 \times 3C_2 \times 2C_1 \times 4C_3}{14C_8}$$

DMPT - 8 :-

Q1) 6 cards are drawn with replacement from an ordinary pack of cards. What is the probability that each of the 4 suits will be ~~represented~~ repeated at least once among the six cards?

$1 - \text{they will not be repeated}$

$$= 1 - \frac{13C_6}{52C_6} \times \text{------}$$

$$1 - \frac{\frac{13!}{7!\,6!}}{\frac{52!}{46!\,6!}} = 1 - \frac{\cancel{13}\times\cancel{12}\times 11\times\cancel{10}\times\cancel{9}\times\cancel{8}}{\cancel{52}\times\cancel{51}\times\cancel{50}\times 49\times\cancel{48}\times 47}$$

$52 - 6 = 46$

$$= 1 - \frac{33}{17\times 5\times 49\times 2\times 47}$$

A = all suits appear atleast one

A^c = at least one suit does not appear

$A^c = \bigcup_{i=1}^{4} B_i$ $\qquad B_1$ = Spades don't appear

B_2 = Heart

B_3 = Clubs

B_4 = Diamonds

$$P(A^c) = P\left(\bigcup_{i=1}^{4} B_i\right)$$

$$= \sum_{i=1}^{4} P(B_i) - \sum_{i<j}\sum P(B_i \cap B_j) + \sum_{i<j<k}\sum\sum P(B_i \cap B_j \cap B_k)$$

$$- P\left(\bigcap_{i=1}^{4} B_i\right)$$

$$= \left(\frac{39}{52}\right)^6 - \left(\frac{1}{2}\right)^6 + \left(\frac{1}{4}\right)^6 - 0$$

$$= \frac{3^6}{4^6} - \frac{1}{2^6} + \frac{1}{4^6}$$

$$= \sum_{i=1}^{4} P(B_i) - \sum_{i<j}^{4}\sum^{4} P(B_i \cap B_j) + \sum_{i<j<k}^{4}\sum^{4}\sum^{4} P(B_i \cap B_j \cap B_k)$$

$$- P\left(\bigcap_{i=1}^{4} B_i\right)$$

$$= 4 \times \frac{3^6}{4^6} - 6 \times \frac{1}{2^6} + 4 \times \frac{1}{4^6} - 0$$

$$= \frac{4 \times 3^6 + 4}{4^6} - \frac{3}{2^5}$$

$$\frac{4(729)+4}{} - \frac{3}{32}$$

$$\frac{4 \times 730}{4^5}$$

$$= \frac{\cancel{730}\ 365}{\cancel{1024}\ 512} - \frac{3}{32} \qquad 0.62$$

$$= \frac{365}{512} - \frac{3 \times 2^4}{32 \times 2^4}$$

$$\frac{365 - 3(16)}{512} = \frac{365-48}{512} = \frac{317}{512} = 0.38$$

$4(729+1)$

$4(730)$

$3^2 = 9$, $= 27$, $3^4 = 81$, 243, 729

$2^6 = 64$, $2^7 = 128$, $2^8 = 256$, $2^9 = 512$

$4^2 = 16$, $4^3 = 64$, $32 \times 6 = 192$, 256, 1024

$2^5 \times 2^4$

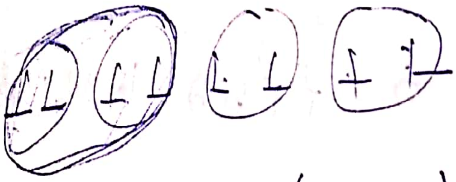
$365 - 48 = 317$

Q) 4 married couples are arrangled to be seated in a row. what is the probability that no husband is seat ~~next~~ to his wife?

$P(A) =$ No husband seat to his wife

$P(\bar{A}) =$ ~~All~~ Atleast one husbands seat to their respective wife

$=$



$$\frac{4!\,2!\,2!\,2!}{8!}$$

$$P(B_1) = \frac{2\times 7!}{8!} = \frac{1}{4}$$

$$P(B_1 \cap B_2) = \frac{6!\,2!\,2!}{8!} = \frac{\cancel{8}\times 4}{8\times 7\times \cancel{6}} = \frac{1}{14}$$

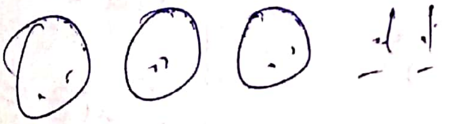
$$\frac{\cancel{4\times 3\times 2\times 1}\times 2^4}{8\times 7\times 6\times 5\times \cancel{4\times 3\times 2\times 1}}$$

$$\frac{2!}{35\times 7}$$

$$= \frac{1}{210}$$

$$1 = \frac{1}{105}$$

$$P(B_1 \cap B_2 \cap B_3) =$$



$$\frac{5!\,2!\,2!\,2!}{8!}$$

$$= \frac{5\times 2\times 2\times 2}{8\times 7\times 6} = \frac{1}{42}$$

$$= \frac{104}{105}$$

$$P(B_1 \cap B_2 \cap B_3 \cap B_4) = \frac{4!\,2^4}{8!} = \frac{2\times 2\times 2\times 2}{8\times 7\times 6\times 5} = \frac{1}{35\times 3} = \frac{1}{105}$$

$35 \times 3 = 105$

$$4\cdot\frac{1}{4} - \frac{6}{14} + \frac{4}{42} - \frac{1}{105} \quad 0.33+$$

$$+\frac{14}{42} + \frac{1}{105} \qquad 1 - \frac{6\times 3}{14\times 3} + \frac{4}{42} - \frac{1}{105} \qquad 1 - \left(1 - \frac{14}{42} - \frac{1}{105}\right)$$

$-18+4$

→ Consider all families with 2 children. and assume that B & G are equally likely

(1) If a family is choosen at random and is found to have a boy; what is probability that the other one is also a boy?

(2) If a child is ~~ran~~ choosen, and found to be a boy. What is probability that other child in that family is also a boy.

(1) Sol:

$$\frac{n(\ldots)}{n(\ldots)} \quad \frac{1}{2}$$

$\Omega = \{(b,b), (b,g), (g,b), (g,g)\}$

A = family has a boy

B = second child is also a b

(1) $P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} //$$

Mid Sem 1 - 25
mid sem 2 - 25
End sem - 40
Quiz = 10
100

(2) $\Omega = \{ b_b, b_g, g_b, g_g \}$

A = child is a boy

B = child has a brother

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} //$$

6/9/19

DMPT - 9 ?

Random Variable :-

→ A first course in Probability by Sheldon Ross (Eight Edition).

Definition:- X is a real-valued function that maps the sample space into the real line. X is called the random variable.

$X : \Omega \to \mathbb{R}$ $(-\infty, \infty)$ $X(\omega) = 7$ $\omega \in \mathbb{R}$.

X = the no. of tosses required to get the first head.

$\Omega = \{H, TH, TTH, TTTH \ldots\}$

$X = 1, 2, 3 \ldots$ $2 \le X \le 12$

$X = 2$ $\{(1,1)\}$. $P(X = x) = 0$. $x \ne 2, 3 \ldots 12$

$X = 3$ $\{(1,2), (2,1)\}$

$X = 4$ $\{(1,3)(3,1)(2,2)\}$

$\vdots$

$X = 12$ $\{(6,6)\}$

Types of random variables :-

① Discrete random variable.

A random variable that can take a countable no. of possible values is said to be discrete.

X is a discrete random variable

X takes the values $\{a_i\}_{i=1}^{\infty}$

1) $P(X=a_i) \geq 0 \quad i=1,2 \dots$

2) $P(X=x) = 0, \quad x \neq a_1 \dots \dots a_\infty$

3) $$\boxed{\sum_{i=1}^{\infty} P(X=a_i) = 1}$$

Probability Mass f^n (pmf)

→ Means it is defined on discrete random variable

→ It should satisfy above 3 cond^n

$P(x) / f(x)$

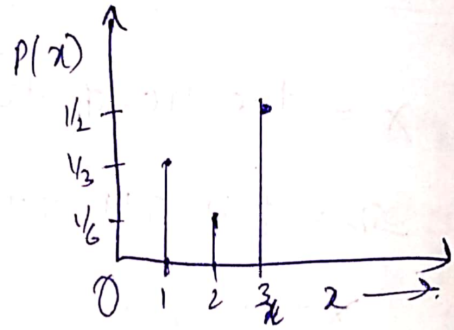
$x = 1, 2, 3 \dots$

$P(x=1) = \frac{1}{3}$

$P(x=2) = \frac{1}{6}$

$P(x=3) = \frac{1}{2}$

$P(x=\bar{x}) = 0, \forall x \neq 1,2,3.$



Define pmf for d.r.v X = sum of two no. in throwing two dice

and plot x & $P(x)$

$x = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

$P(x)$

$P(x=2) = \frac{1}{36} = \frac{1}{36}$ $\quad P(x=8) = \frac{5}{36}$

$P(x=3) = \frac{2}{36} = \frac{1}{18}$ $\quad P(x=9) = \frac{4}{36}$

$P(x=4) = \frac{3}{36} = \frac{1}{12}$ $\quad P(x=10) = \frac{3}{36}$

$P(x=5) = \frac{4}{36} = \frac{1}{9}$ $\quad P(x=11) = \frac{2}{36}$

$P(x=6) = \frac{5}{36} = \frac{5}{36}$ $\quad P(x=12) = \frac{1}{36}$

$P(x=7) = \frac{6}{36} = \frac{1}{6}$ $\quad P(x=\bar{x}) = 0 \; x \neq 2, \dots 12$

5 (1,4) (4,1) (2,3) (3,2)

(2,2)

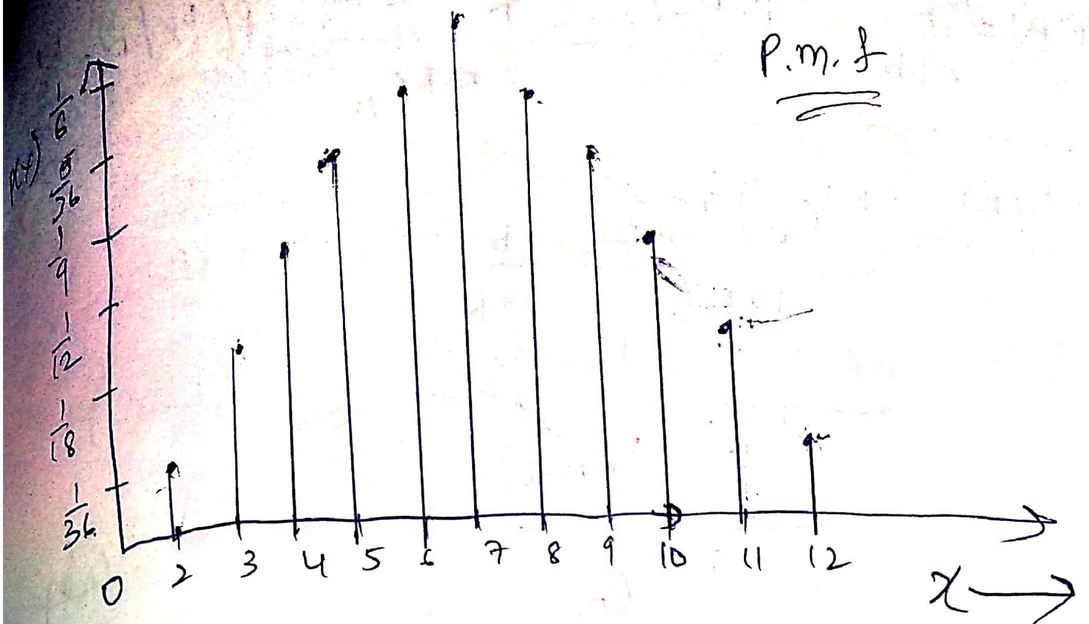
$\frac{5}{36}$ 36) 50 (0.13 −36 140

36 ×4 144

9) 10 (0.11 9 10

7) 10 (0.16 6 40

P.m.f



Tutorial :

1) A total of n balls are sequentially and randomly chosen, without replacement, from an urn containing r red and b blue balls. ($n \le r+b$). Given that k of the n balls are blue, what is the cond^n probab that the first ball choosen is blue?

$P\left(\frac{B_1}{K}\right)$

Blue 1, 2 - - - b
red b+1, - - - r+b

let B be the event that the first ball choosen is blue
& B_K be the event that the k balls choosen are blue

$P\left(\frac{B}{B_K}\right) = \frac{P(B \cap B_K)}{P(B_K)} =$ ______

$P(B) = \frac{b}{r+b}$ $\quad P(B_k) = \frac{{}^bC_k \; {}^rC_{n-k}}{{}^{r+b}C_n}$ $\quad P(B_k|B) = \frac{{}^{b-1}C_{k-1} \; {}^rC_{n-k}}{{}^{r+b-1}C_{n-1}}$

$$P(B|B_k) = \frac{\dfrac{({}^{b-1}C_{k-1})(\cancel{{}^rC_{n-k}})}{{}^{r+b-1}C_{n-1}} \cdot \dfrac{b}{r+b}}{\dfrac{({}^bC_k)(\cancel{{}^rC_{n-k}})}{{}^{r+b}C_n}}$$

$$= \frac{{}^{b-1}C_{k-1}}{{}^{r+b-1}C_{n-1}} \times \frac{{}^{r+b}C_n}{{}^bC_k} \times \frac{b}{r+b}$$

$$= \frac{\cancel{{}^{b-1}C_{k-1}}}{\cancel{{}^{r+b-1}C_{n-1}}} \; \frac{\frac{r+b}{n} \times \cancel{{}^{r+b-1}C_{n-1}}}{\frac{b}{k} \; \cancel{{}^{b-1}C_{k-1}}} \times \frac{b}{r+b}$$

$\boxed{{}^nC_r = \frac{n}{r} \, {}^{n-1}C_{r-1}}$

$$= \frac{\cancel{r+b}}{n} \times \frac{k}{\cancel{b}} \times \frac{\cancel{b}}{\cancel{r+b}} = \frac{k}{n}$$

---

2) Urn 1 contains one white and two black marbles. Urn 2 contains one black and 2 white ~~to~~ marbles, and Urn 3 contains three black and three white marbles. A die is rolled if a 1, 2 or 3 shows up, urn is selected, if a 4 shows up, urn 2 is selected. and if a 5 or 6 shows up urn 3 is selected. A marble is drawn at random from the urn selected let A be the event that the marble drawn is white

U, V, W respective denote the events that the urn selected is 1, 2, 3 find P(A), P(V/A)

Urn 1 — 1 white 2 black

Urn 2 — 2 white 1 black

Urn 3 — 3 white 3 black

$P(\text{Urn }1) = \frac{1}{2}$ $\quad P(\text{Urn }2) = \frac{1}{6}$ $\quad P(\text{Urn }3) = \frac{1}{3}$.

~~P(A) =~~

$$P\left(\frac{\text{Urn }1}{A}\right) = \frac{P\left(\frac{A}{\text{Urn}1}\right) P(\text{Urn }1)}{P\left(\frac{A}{\text{Urn}1}\right) P(\text{Urn}1) + P\left(\frac{A}{\text{Urn}2}\right) P(\text{Urn}2) + P\left(\frac{A}{\text{Urn}3}\right) P(\text{Urn}3)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{\cancel{6}_3} + \frac{1}{2} \times \frac{1}{3}}$$

P(A) =

$$= \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{9} + \frac{1}{6}} = \frac{\frac{1}{6}}{\frac{1 \times 3}{3 \times 3} + \frac{1}{9}}$$

$$\boxed{P(A) = \frac{4}{9}}$$

$$= \frac{\frac{1}{6}}{\frac{4}{9}}$$

$$= \frac{1}{\cancel{6}_2} \times \frac{\cancel{9}^3}{4}$$

$$P\left(\frac{\text{Urn }1}{A}\right) = \frac{3}{8}$$

$$P\left(\frac{\text{Urn }2}{A}\right) = \frac{\frac{1}{9}}{\frac{4}{9}} = \frac{1}{4}$$

3) An Insurance company believes that people can be divided into two classes; those who are accident prone and those who are not. The company's statistics show that, an accident prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability reduces to 0.2 for a person who is not accident prone. If we assume 30% accident prone, a what is the probability that a new policy holder will have an accident within a year of purchasing a policy.

a) $P(A) = P(A/B)\, P(B) + P(A/\bar{B})\, P(\bar{B})$

$$\frac{4}{10} \times \frac{30}{100} + \frac{2}{10} \times \frac{70}{100}$$

$$P(A) = \frac{12}{100} + \frac{14}{100} = \frac{26}{100} = 0.26.$$

b) Suppose that a new policy holder has an accident within a year of purchasing policy, what is probability that he/she is accident prone.

Sol: $P(B/A) = \frac{\cancel{12}\,6}{\cancel{26}\,13} = \frac{6}{13}$

$$= \frac{P(A/B)\, P(B)}{P(A)}$$

A: event that a new policy holder will have an accident within a year of purchasing policy.

B: person belongs to accident prone population

4) In answering a question on a multiple choice test, a student either knows the answer (or) guesses. Let p be the probability that the student knows the answer, and 1-p be the probability that the student guesses. Assume that a student who guesses at the answer will be correct with probability 1/m, where m is the number of multiple choice alternatives. What is the condⁿ probability that a know the answer to a question given that he/she answered it correctly.

$P(A) = \frac{1}{m} \times (1-p) + (1-\frac{1}{m})(p)$

$\frac{pm}{1+(m-1)p}$

$\frac{1-p}{m} + p - \frac{p}{m}$

$\frac{1}{m} - \frac{p}{m} + p - \frac{p}{m}$

$\frac{1}{m} + p - \frac{2p}{m}$

$\frac{1-2p}{m} + \frac{p}{1}$

$\frac{1-2p+pm}{m}$ //

$\boxed{\frac{1+p(m-2)}{m}}$

14 (ii)

b) I live in a locality where bulglary is common. the chance that Burglar breaks into house is 0.1 I have a dog that is highly likely to bark (say, with 0.95 probab.) if a burglar enters. However, otherwise. any dog is a quite one. If there is no burglar around, he barks with probability 0.01. I hear my dog bark. What is the chance that a burglar has entered.

$P\left(\frac{D}{B}\right)$

$$P(B/D) = \frac{P(D/B)\,P(B)}{P(D/B)\,P(B) + P(D/\bar{B})\,P(\bar{B})}$$

$$= \frac{0.95 \times 0.1}{0.95 \times 0.1 + 0.01 \times 0.9}$$

0.9
0.1
0.95
0.05
0

6) All the screws in a machine come from the same factory but it is as likely to be from A as from factory B. Two screws are inspected

1) if the first one is found to be good, what is the probability that second is also good?)

2) If first is found to be defective, what is the probability that second one is also defective?

GB GG BB BG

$$P\left(\frac{2G}{1G}\right) = \frac{P\left(\frac{1G}{2G}\right) \cdot P(2G)}{P\left(\frac{1G}{2G}\right) P(2G) + P\left(\frac{1G}{\overline{2G}}\right) P(\overline{2G})}$$

$$\frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}}$$

12/10/19 DMPT-10

$P(A)$, $A \in \mathcal{B}$

Probab. Space: $(\Omega, \mathcal{B}, P)$

$X(\omega) = x$, $x \in \mathbb{R}$.

$X : \Omega \to \mathbb{R}$.

$(\mathbb{R}, \mathcal{G}, \wp)$ $(\Omega, \mathcal{B}, P)$

$\downarrow$ σ-fa

Probability distribution.

$P(X = x)$, $\forall x \in \mathbb{R}$

$P(\omega : X(\omega) = x)$

Cumulative distribution function (c.d.f)

$$F_x(x) = P(X \leq x)$$
$$= P(\{\omega : X(\omega) \leq x\})$$

$\boxed{\begin{array}{l} x = \text{values of R.V} \\ X = \text{Denote R.V} \end{array}}$

Properties of cdf

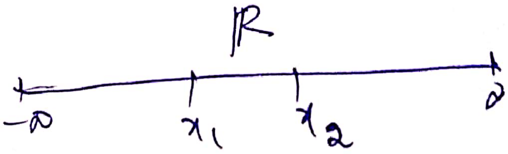
1) $\lim_{x \to -\infty} F_X(x) = 0$

2) $\lim_{x \to \infty} F_X(x) = 1$

3) $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$.

$x_1 < x_2$

$\omega_1 \leq \omega_2$



$\omega_1 : X(\omega_1) < x_1$ $\omega_2 : X(\omega_2) < x_2$

then, $\omega_1 \subseteq \omega_2$

If $\omega_1 \subseteq \omega_2$

then, $P(\omega_1) \leq P(\omega_2)$

4) $\lim_{h \to 0} F_X(x+h) = F_X(x) \to$ Right continuous

Tossing a fair coin once.

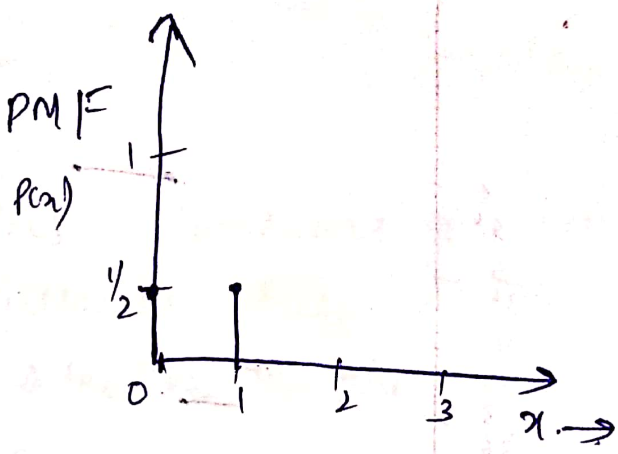
X : no. of head

$H : X = 1$

$T : X = 0$

$P(X=0) = \frac{1}{2}$

$P(X=1) = \frac{1}{2}$.



$P(X=x) = 0, \ \forall x \neq 0, 1$

$F_X(x) = P(X \le x) = 0 \quad x < 0$

$= \frac{1}{2} \quad 0 \le x < 1$

$= 1 ; \quad x \ge 1$

$P(X \le 0.99) = P(X = 0)$

$= \frac{1}{2}$.

$P_X(x_i) = P(X = x_i) \quad i = 1, \dots \infty$.

$$\boxed{F_X(x) = \sum_{\substack{x_i \le x \\ (-\infty \text{ to } x)}} P_X(x_i)}$$

$F_X(1) = P_X(0) + P_X(1)$

$= \frac{1}{2} + \frac{1}{2} = 1$

Throwing of two dice.

$F_X(x) = P(X \le x) = 0 \quad x < 2$

$X = \frac{1}{36} \quad 2 \le x < 3$

$= \frac{2}{36} \quad 3 \le x < 4$

$= \frac{3}{36} \quad 4 \le x < 5$

$= \frac{6}{36} \quad 5 \le x < 6$

$= \frac{12}{36} \quad 6 \le x < 7$

$\frac{24}{36} \quad 7 \le x < 8$

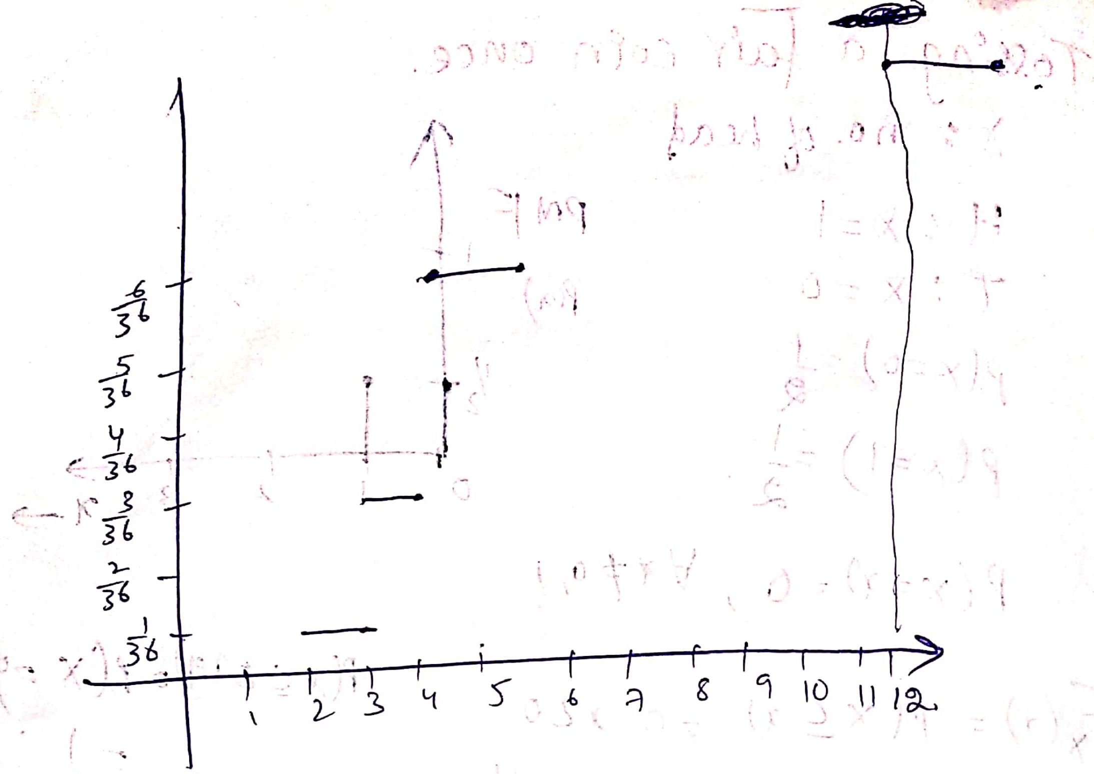
$\frac{48}{36} \quad 8 \le x < 9$

$\frac{4}{36} \quad 9 \le x < 10$

$\frac{3}{36} \quad 10 \le x < 11$

$\frac{2}{36} \quad 11 \le x < 12$

$\frac{1}{36} \quad x \ge 12$



$P(X=2) = \frac{1}{36}$

$P(X=3) = \frac{2}{36}$

$P(X=4) = \frac{3}{36}$

$P(X=5) = \frac{4}{36}$

$P(X=6) = \frac{5}{36}$

$P(X=7) = \frac{6}{36}$

$P(X=8) = \frac{5}{36}$

$P(X=9) = \frac{4}{36}$

$P(X=10) = \frac{3}{36}$

$P(X=11) = \frac{2}{36}$

$P(X=12) = \frac{1}{36}$

$$F_X(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{36} & 2 \le x < 3 \\ \frac{3}{36} & 3 \le x < 4 \\ \frac{6}{36} & 4 \le x < 5 \\ \frac{10}{36} & 5 \le x < 6 \\ \frac{15}{36} & 6 \le x < 7 \\ \frac{21}{36} & 7 \le x < 8 \\ \frac{26}{36} & 8 \le x < 9 \\ \frac{30}{36} & 9 \le x < 10 \\ \frac{33}{36} & 10 \le x < 11 \\ \frac{35}{36} & 11 \le x < 12 \\ 1 & x \ge 12 \end{cases}$$

$p_X(x_i) = P(X \le x_i) - P(X \le x_{i-1})$

$$\boxed{p_X(x_i) = F_X(x_i) - F_X(x_{i-1})}$$

* A computer store contains 10 computers of which 3 are defective. A customer buys two at random, no. of defectives in the purchase. Find PMF then CDF.

→ X: No. of defectives in purchase.

7 good 3 defective

$p_X(1) = p_X(0) = \frac{21}{45}$

$p_X(2) = \frac{3}{45}$

$E(X) = 0 \times \frac{21}{45} + 1 \times \frac{21}{45} + 2 \times \frac{3}{45}$

$= \frac{3}{5}$

PMF

$P(X=0) = \frac{7}{15}$

$P(X=1) = \frac{7}{30}$

$P(X=2) = \frac{1}{30}$

$P(X=x) = 0 \quad x \ne 0,1,2,3$

CDF

$$F_X = \begin{cases} 0 & x < 0 \\ 7/15 & 0 \le x < 1 \\ 7/15 & 0 \le x < 2 \\ 1/15 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

DMPT - 11

$P(\Omega, \mathcal{B})$

Random Variable:

$\wp$ = Probability distribution

Given a probability function $\wp$ on $(\mathbb{R}, \mathcal{B})$ there exists a f^n $F(x)$

$$\boxed{F_X(x) = P(X \leq x) = P(-\infty < X \leq x)}$$

Satisfying $\wp(-\infty, x] = F(x)$ $\forall x \in \mathbb{R}$.

It says conversely given a function F Satisfying the four properties, $\exists$ a unique probability function $\wp$ on $(\mathbb{R}, \mathcal{B})$

$\wp(-\infty, x] = F(x)$

$$\boxed{\begin{aligned} &P(a \leq X \leq b) \\ &= P(\{\omega : a \leq X(\omega) \leq b\}) \end{aligned}}$$

Continuous Random Variable:-

A Random variable (X) is said to be continuous if its cdf $F_X(x)$ is absolutely continuous f^n. i.e., $\exists$ a non-negative function $f_X(x)$ $\ni$ (such that) $\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt}$, $\forall x \in \mathbb{R}$.

this function $f_X(x)$ is called probability density function (pdf)

1) If F_X is absolutely continuous and $f_X(x)$ is cont. at point x. then $\frac{d}{dx} F_X(x) = f_X(x)$

Properties of pdf

1) $f_X(x) \geq 0 \quad \forall x \in R$

2) $\int_{-\infty}^{\infty} f_X(x) = 1$

3) $\int_a^b f(x)\,dx = P(a < X < b) = F_X(b) - F_X(a)$

$\hookrightarrow P(X \leq b) - P(X \leq a)$

X is d.r.v $\Rightarrow P_X(a) = P(X = a)$

X is c.r.v $\Rightarrow f_X(a) \neq P(X = a)$

$P(X = a) = 0$

$$f_X(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 1 \\ \frac{1}{2}, & 1 \leq x \leq 2 \\ \frac{3-x}{2}, & 2 < x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$\int \frac{3}{2}dx - \int \frac{x}{2} dx$

$\frac{3}{2}(1) - \frac{x^2}{4}$

$\frac{3}{2} - \frac{(5)}{4}$

$= \frac{1}{4}$

$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{x}{2}dx + \int_1^2 \frac{1}{2}dx + \int_2^3 \frac{3-x}{2}dx.$

$= \cancel{\frac{1}{2} \times \frac{x^2}{2}} + \frac{1}{2}(1) + \cancel{\frac{3(x)}{2}} - \cancel{\frac{x^2}{4}}$

$= \cancel{\frac{1}{2}} + \cancel{\frac{3}{4}}$

$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$

$= 1$

$$F_X(x) = \begin{cases} \int_0^x \frac{t}{2}\,dt & 0 \le x < 1 \\ \int \frac{1}{2}\,dt & 1 < x \le 2 \\ \int \frac{3-x}{2}\,dt & \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^x \frac{t}{2}\,dt & 0 \le x < 1 \\ \int_0^1 \frac{t}{2}\,dt + \int_1^x \frac{1}{2}\,dt & 1 \le x < 2 \\ \int_0^1 \frac{t}{2}\,dt + \int_1^2 \frac{1}{2}\,dt + \int_2^x \left(\frac{3-t}{2}\right)dt & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

cdf → pdf $\frac{d}{dx}$

pdf → cdf $\int$

Moments / Mathematical expectation

Let X be a discrete random variable with pmf $P_X(x_i)$, $x_i \in \mathcal{X}$. We define the expected value of X as $\boxed{E(X) = \sum_{x_i \in \mathcal{X}} x_i \times P_X(x_i)}$

Mean / Average / First order moment / around the origin

If X is a continuous random variable with pdf $f_X(x)$ $\boxed{E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx}$

(Provided sum / integration is convergent)

$X \sim f_X(x) = 1 \quad 0 \le x \le 1$
$= 0 \quad$ otherwise.

$$E(x) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

$$= \int_{-\infty}^{0} (0) \, dx + \int_{0}^{1} (x)(1) \, dx + \int_{1}^{\infty} 0 (dx)$$

$$= \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2} \; //$$

DMPT-12

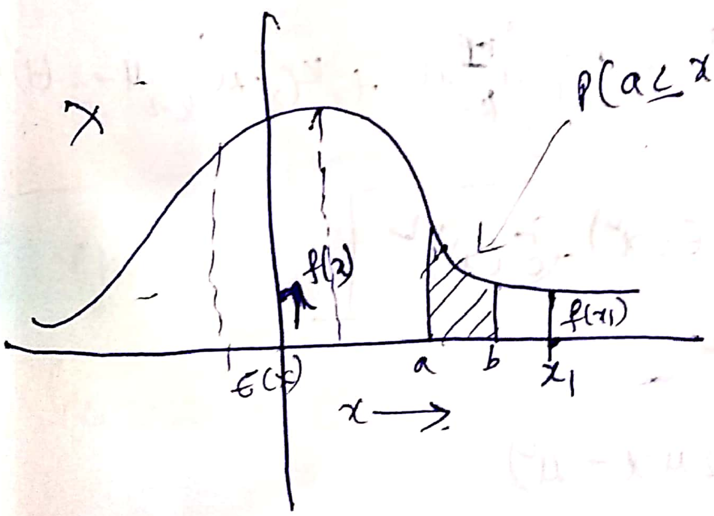
1) $E(ax+b) = a E(x) + b, \quad a \neq 0, b \in R$

$y = ax + b$

$E(Y) = aE(X) + b$

$E(c) = c.$

$$E(g(x)) = \begin{cases} \sum_{x_i \in X} g(x_i) \, P_X(x_i) & x \text{ is d.r.v} \\ \int_{-\infty}^{\infty} g(x) \cdot f(x) \, dx & x \text{ is c.r.v} \end{cases}$$



$P(a \le x \le b)$ = Area under that curve.

$f(x)$ vs x

Types of Moments

$\mu_k' = E(x^k) \rightarrow$ Around the origin
$\quad\hookrightarrow$ kth non-central moment

(μ_k prime)

$\mu_1' = E(x^1)$ $\qquad\boxed{\mu = E(x)}$

$\mu_2' = E(x^2)$

$\mu_k = E\left[(x-\mu)^k\right] \rightarrow$ around the μ (mean)
$\quad\hookrightarrow$ kth central moment

$\mu_1 = E(x-\mu) = E(x) - E(\mu)$
$\qquad = \mu - \mu = 0$

$\mu_2 = E(x-\mu)^2$

x_1
x_2 $\bar{x}$
$\vdots$ $\frac{1}{n}E(x-\bar{x})$
x_n $\leq \frac{1}{n}\Sigma x_i - \bar{x}$
$= \bar{x} - \bar{x} = 0$

Variance :- It says about the varying values of x.
It is also measure of dispersion.

$\mu_k = E(x-\mu)^k$
$= E\left(x^k - {}^kC_1 x^{k-1}\mu + {}^kC_2 x^{k-2}\mu^2 - \dots (-1)^k x^k\right)$

$= \mu_k' - {}^kC_1 \mu_{k-1}'\mu + {}^kC_2 \mu_{k-2}'\mu^2 - \dots (-1)^k \mu^k$

$\mu_2 = \boxed{\sigma^2 = E(x-\mu)^2 = E(x^2) - (E(x))^2}$

Proof: $\sigma^2 = E(x-\mu)^2$
$= E(x^2 - 2\mu x + \mu^2)$
$= E(x^2) - 2\mu E(x) + \mu^2$
$= E(x^2) - 2\mu\mu + \mu^2$
$= E(x^2) - \mu^2 = E(x^2) - (E(x))^2$

$$\mu_2 = \mu_2' - 2\mu_1'\mu + \mu^2$$
$$= \mu_2' - \mu^2$$
$$= E(x^2) - (E(x))^2$$

$\mu_k' = f(\mu_k)$ (representing μ_k' as a function of μ_k)

$$\mu_k' = E(x^k)$$
$$= E((x-\mu+\mu)^k)$$
$$= E\left((x-\mu)^k + {}^kC_1 (x-\mu)^{k-1}\mu' + {}^kC_2 (x-\mu)^{k-2}\mu^2 \dots + (\cancel{-1})^k (x-\mu)^{k}\right)$$
$$= \mu_k + {}^kC_1\, \mu_{k-1}\mu + \dots + \mu^k$$

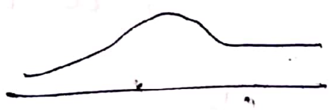
$\beta_k' = E|x|^k \rightarrow k^{th}$ absolute non-central moment

$\beta_k = E|x-\mu|^k \rightarrow k^{th}$ absolute central m

$\alpha_k = E(x(x-1)\dots(x-k+1))$

$\rightarrow k^{th}$ factorial moment

Skew distribution.



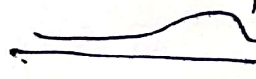
Skewness $= \dfrac{\mu_3}{\mu_2^{3/2}} = \dfrac{\mu_3}{\sigma^3}$

postive skew

$= 0$, symmetric

> 0 positively skewed

< 0 negatively skewed



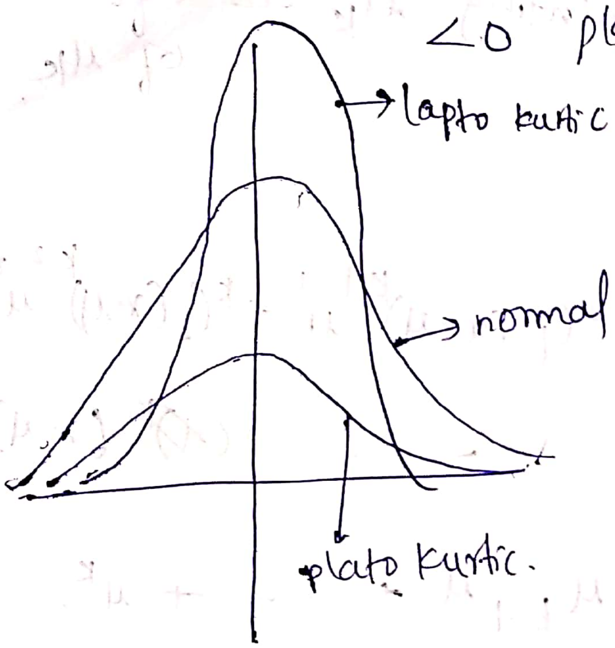
negati sk

Kurtosis :- $= \dfrac{\mu_4}{\mu_2^2} - 3$

$= 0$ normal peak

> 0 lapto kurtic

< 0 plato kurtic



Tutorial :- 18/9/19

Q. Independent trials consisting of flipping of a coin having probability 'p' of coming up heads are continuously performed until a head occurs or a total of n flips is made. Let X denote the no. of times coin is flipped. X is a random variable taking one of values of $1, \ldots, n$, then find $P\left(\bigcup_{i=1}^{n} (X=i)\right)$?

H TH TTH ... ___

p $(1-p)p$ $(1-p)^2 p$

$p(x=1) = p$

$p(x=2) = p(1-p)$

$p(x=3) = (1-p)^2 p$

$p(x=i) = (1-p)^{i-1} p$

$p(x=n-1) = (1-p)^{n-2} p$

$p(x=n) := \text{where}(TTT..H) + \text{while}(TTT..T)$

$= (1-p)^{n-1} p + (1-p)^{n-1}(1-p)$

$= (1-p)^{n-1}(p+1-p)$

$p(x=n) = (1-p)^{n-1}$

then $\sum P\left(\bigcup_{i=1}^{n}(x=i)\right) = \sum_{i=1}^{n} p(x=i)$

$= p + p(1-p) + \cdots (1-p)^{n-2} p + (1-p)^{n-1}$

$= \left(\dfrac{p\left((1-p)^{n-1}-1\right)}{(1-p)-1}\right) + (1-p)^{n-1}$

$= \dfrac{p\left((1-p)^{n-1}-1\right)}{-p} + (1-p)^{n-1}$

$= 1 - (1-p)^{n-1} + (1-p)^{n-1}$

$\sum_{i=1}^{n} P(x=i) = 1$

(Q2) Probability Mass function of a random variable is given by $P(i) = \frac{c\lambda^i}{i!}$. λ is a positive value. Find i) $P(X=0)$
ii) $P(X>2)$

$P(X=0) + P(X=1) + \dots + P(X=n) = 1$

$c\frac{\lambda^1}{1!} + \frac{c\lambda^2}{2!} + \dots + \frac{c\lambda^n}{n!}$

$c(e^{\lambda}) = 1$ $\quad$ $\underline{c = e^{-\lambda}}$

$P(X=0) = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-\lambda}}{1}$ $\quad$ $P(X=i) = \frac{e^{-\lambda}\lambda^i}{i!}$

$P(X \geq 2) = 1 - P(X \leq 2)$

$= 1 - (P(X=0) + P(X=1) + P(X=2))$

$= 1 - \left(\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}\right)$

$= 1 - e^{-\lambda}\left(1 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!}\right)$

(Q3) Let X be defined on (Ω, S, P) by $X(\omega) = c$ $\forall \omega \in \Omega$. Then find $P(X=c)$, $F(x)$ $\forall x$

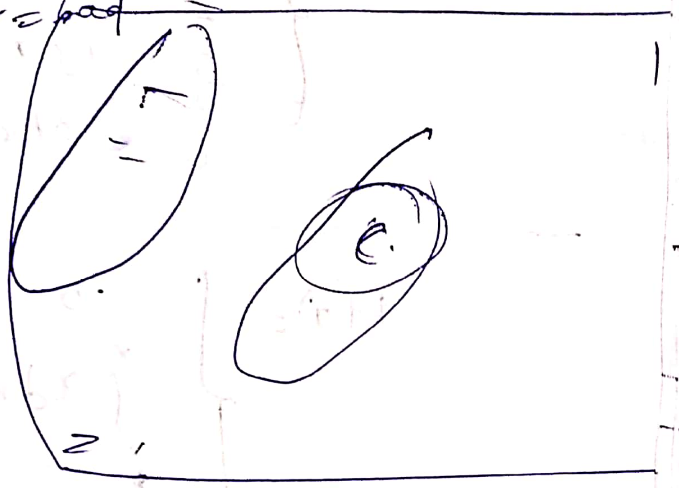
Sol: $F_X(x) = P(X \leq x)$ $\quad$ $P(X=c) = 1$

$= \int_{-\infty}^{x}$ $\quad$ $F(x) = \begin{cases} 0 & x \not\geq c \\ 1 & x \geq c \end{cases}$

Q4) A box contains good and defective items. If an item is drawn is good, we assign it to number 1, otherwise 0. Let p be the probability of drawing an item that is good. $p(x)$ and $F(x)$

(n) → good & bad

Sol:- $n \neq r \neq C_{r-1}$ $\quad r = \text{good}$, $n-r = \text{bad}$



$$p(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Q5) Let X be a random variable with DF, F given by

$$F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$$

find $f(x) = P(0.4 < X \le 0.6)$

$= F_X(0.6) - F_X(0.4)$

$= \int_{-\infty}^{0.6} 0.6 - 0.4 = 0.2$

$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}(x) = 1$

$$f_X(x) = \begin{cases} 0 & \text{otherwise} \\ 1 & 0 < x < 1 \end{cases}$$

Q6) $f(x)$ is defined as

$$f(x) = \begin{cases} x & 0 < x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$\int \frac{x^2}{2}$

$f(x)_1 = \frac{x^2}{2}$

$2(1) - \frac{1}{2} =$

$$F(x) = \begin{cases} \frac{1}{2} & 0 < x \le 1 \\ \frac{3}{2} & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$ ✗

$\frac{7}{15} + \frac{1}{6}$

$\frac{42+15}{90}$

57

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x x\,dx & 0 \le x \le 1 \\ \int_0^1 x\,dx + \int_1^x (2-x)\,dx & 1 \le x \le 2 \\ 1 & \text{or } x \ge 2 \end{cases}$$

DMPT - 13

x : r.v $\quad$ $g(x)$ is a real valued f^n.

$Y = ax + b$ $\quad$ Then $g(x)$ is a r.v

$$E(ax+b) = aE(x) + b, \quad a \ne 0, b \in \mathbb{R}$$

$$\mathrm{Var}(\underbrace{ax+b}_{Y}) = E(Y - E(Y))^2$$

$$= E(ax + b - (aE(x) + b))^2$$

$\mathrm{var}(x) = E(x - E(x))^2$

$$= E(ax - aE(x))^2$$

$$= a^2 E(x - E(x))^2$$

$$\mathrm{var}(ax+b) = a^2\,\mathrm{var}(x)$$

$var(c) = E(c-c)^2 = 0$

$E(c) = c$

$Y = g(X) = X^2$

pmf of Y

X r.v

$P(X=-2) = \frac{1}{5}$ $P(X=-1) = \frac{1}{6}$ $P(X=0) = \frac{1}{5}$

$P(X=1) = \frac{1}{15}$, $P(X=2) = \frac{11}{30}$

$P(X=x) = 0 \ \forall x \neq 0, \pm 1, \pm 2.$

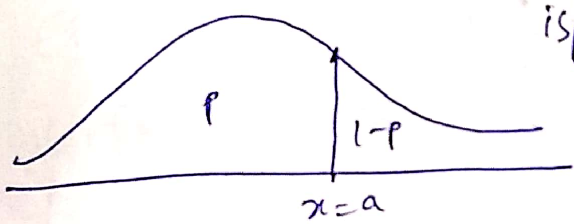
$Y = 0, 1, 4.$

$P(Y=0) = \frac{1}{5}$

$P(Y=1) = \frac{1}{6} + \frac{1}{15} = \frac{5+2}{30} = \frac{7}{30}$

$P(Y=4) = \frac{1\times 6}{5\times 6} + \frac{11}{30} = \frac{17}{30}$

Quantile:-



The point left of which prob is p and right of which prob is 1-p, it is called Quantile

Q_p : p^{th} quantile.

$P(X \le Q_p) = p$

$P(X \le Q_{1/2}) = 1/2.$

Types of Quantiles

1) Quartile:

Divide the area into four equal parts

$Q_{1/4}$ $Q_{1/2}$ $Q_{3/4}$

2) Decile.

Divide the area into ten equal parts

$Q_{1/10}$ ...... $Q_{9/10}$

3) percentiles:

$X \to Q_{1/100}$ ... $Q_{99/100}$.

Area into 100 equal parts.

To find Q_p for a discrete ran. var.

→ A number Q_p satisfying this $P(X \le Q_p) \ge p$ and $P(X \geq Q_p) \ge 1-p$ where $0<p<1$, is called p^{th} quartile (or) a quartile of p.

→ If F is absolutely continuously c.d.f $F(Q_p) = p$. i.e., a unique quantile.

$$f_X(x) = \frac{1}{\pi} \times \frac{1}{1+x^2}$$

$$F_X(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{dx}{1+x^2} = \left(\tan^{-1}(x)\right)_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= \frac{1}{\pi}\left(\tan^{-1}x + \frac{\pi}{2}\right)$$

$$E(x) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2x\, dx}{1+x^2} \qquad 1+x^2 = t, \quad 2x\,dx = dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dt}{t}$$

$$= \frac{1}{2\pi}\left(\log(1+x^2)\right)_{-\infty}^{\infty} = \text{un-defined}$$

$$F_X(0) = \frac{1}{2}$$

$Q_{1/2}$ $M = 0$ $\quad Q_{1/4} = -1$, $\quad Q_{3/4} = 1$.

8) for discrete case

$p(x=-2) = P(x=0)$ ~~$P(x=0) = \frac{1}{4}$~~ $P(x=1) = \frac{1}{3}$

$= \frac{1}{4}$ $P(x=2) = \frac{1}{6}$

$E(x) = -2 \times \frac{1}{4} + 1 \times \frac{1}{3} + 2 \times \frac{1}{6}$ $\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$

$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

$P(x \leq 0) = \frac{1}{2}$

$M \in [0,1]$ $P(x \leq 0.1) = \frac{1}{2}$

$\boxed{p = \frac{1}{2}}$ $P(x \geq 1) = \frac{1}{2}$

$f_X(x) = 1 \quad 0 \leq x \leq 1$

$= 0 \quad$, otherwise

$E(x^2) = \int_{-\infty}^{x} x^2 f_x(x)\, dx$

$E(x) = \frac{1}{2}$ $= \frac{x^3}{3}$

$Var(x) = E(x^2) - (E(x))^2$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

MGF :- (Moment Generating Function)

let X be a random variable, the function $\boxed{M_X(t) = E(e^{tx})}$ is called moment generating function of random variable x, provided the function exists around $t=0$.

$$f_X(x) = \frac{1}{\pi} \times \frac{1}{1+x^2}, \quad -\infty < x < \infty$$

$$E(e^{tx}) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{tx} \times \frac{dx}{1+x^2} \, dx$$

does not exist except for !..

$$f_X(x) = \begin{cases} \frac{1}{2} e^{-x/2} & x > 0 \\ 0 & , x \le 0 \end{cases}$$

$$M_X(t) = E(e^{tx}) = \frac{1}{2} \int_{0}^{\infty} e^{-x/2} dx = e^t(-2)dt$$

$$= -\cancel{2} \int_0^{\infty} e^t \, dt$$

$$= \frac{1}{2} \int_{-\infty}^{0} 0 \, dx = 0$$

$\cancel{e^{-x/2} = t}$ $\quad -\frac{x}{2} = t$

$-\frac{x}{2} = t \quad -\frac{0}{2} = t$

$-\frac{dx}{2} = dt$

$$E(e^{tx}) = \frac{1}{1-2t} \quad t < \frac{1}{2}$$

DMPT - 14

→ MGF uniquely determines a c.d.f and Conversely if MGF exits, it is unique.

→ If the MGF $M_x(t)$ exists $\forall |t| < t_0$. the derivatives of all orders exists at $t=0$ and can be evaluated under integral sign (or) summation sign (discrete) (continu).

$$\left. \frac{d^k(M_x t)}{dt^k} \right|_{t=0} = \mu_k' = E(x^k)$$

$\frac{d}{dt} E(e^{tx}) = E\left(\frac{d}{dt} e^{tx}\right) = E(x e^{tx})$

at $t=0$ $E(x) = \mu_1'$ $\longrightarrow$ First order moment

at $t=0$, $\frac{d^2}{dt^2} E(e^{tx})\Big|_{t=0} = E(x^2) = \mu_2'$

$E(e^{tx})$

$E\left(1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \dots\right)$

$= 1 + \frac{t}{1!} E(x) + \frac{t^2 E(x^2)}{2!} + \dots$

$= 1 + \frac{t}{1!} \mu_1' + \frac{t^2 \mu_2'}{2!} + \dots$

$X \sim \exp(\lambda) \rightarrow \underline{\lambda > 0}$

$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \lambda > 0 \\ 0, & \text{otherwise.} \end{cases}$

$-\lambda x = t$

$-\lambda dx = dt$

$M_X(t) = \int \lambda e^{-\lambda x}$

$= E(e^{tx}) = \int_0^\infty \lambda e^{-\lambda x} dx . e^{tx}$

$= \cancel{\int -dt\, e^t} = -\int_0^\infty e^t dt$

$= -e^t$

$= \lambda \int_0^\infty e^{-\lambda x} . e^{tx} dx$

$= \lambda \int_0^\infty e^{(t-\lambda)x} dx = \lambda \left(\frac{e^{(t-\lambda)x}}{t-\lambda}\right)_0^\infty = \frac{\lambda}{\lambda - t}$

$= \lambda \left(- \frac{(e^{-\lambda}}{t-\lambda}\right) = \frac{(\lambda)}{t-\lambda}$

$E(x) = M_x'(t)\Big|_{t=0}$ $\frac{\lambda}{\lambda-t}$, $t<\lambda$

$= \frac{\lambda}{(\lambda-t)^2}$

$E(x^2) = M_x''(t) = \frac{2\lambda}{(\lambda-t)^3}$

$= \frac{1}{\lambda}$

$E(x^2)\Big|_{t=0} = \frac{2}{\lambda^2}$

Discrete Distributions:

Discrete Uniform Dist^n

$x \to 1, 2 \ldots N$

$P(x=i) = \frac{1}{N}$ $i=1 \ldots N$

$= 0$, otherwise

$E(x) = \sum_{i=1}^{N} i \times \frac{1}{N} = \frac{1}{N} \times \frac{N(N+1)}{2} = \frac{N+1}{2}$.

$V = E(x^2) - (E(x))^2$

$E(x^2) = 1 \times \frac{1}{N} + 4 \times \frac{1}{N} + 9 \times \frac{1}{N}$

$= \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4}$

$= \frac{1}{N}(1 + 2^2 + 3^2 + \ldots)$

$= \frac{1}{N} \frac{(N)(N+1)(2N+1)}{6}$

$\boxed{V = \frac{N^2-1}{12}}$

$$M_X(t) = E(e^{tX}) = \frac{1}{N}\sum_{j=1}^{N} e^{tj}$$

$$= \frac{1}{N}\left(e^t + e^{2t} + e^{3t} + \dots e^{nt}\right)$$

$$M_X(t) = \begin{cases} \dfrac{e^t\left((e^t)^N - 1\right)}{N\,(e^t - 1)}, & t \neq 0 \\ = 1 & t = 0 \end{cases}$$

Degenerate distribution:

$p(X = c) = 1$

$= 0$, otherwise.

$E(X) = c$

$\mu_k' = E(X^k) = c^k$; $k = 1, \dots$

Bernoulli Distⁿ:

A Bernoulli trial is an experiment with two possible outcomes, success p, failure $1-p$

$P_X(0) = 1-p \quad P_X(1) = p \quad 0 < p < 1$

$P_X(x) = 0$ otherwise

$E(X) = p(1) = p$

Variance $(X) = E(X^2) - (E(X))^2$

$= p - p^2 = p(1-p) = pq$

$q = 1-p$

$\mu_k' = E(X^k) = p \quad k = 1, 2 \ldots$

$M_X(t) = E(e^{tX}) = \int (1-p)e^{t(0)} + pe^{t}$

$= p(X=0)e^{tx} + P(X=1)e^{tx}$

$$\boxed{M_X(t) = (1-p) + pe^t}$$

DMPT-15

Binomial distribution :-

$X \to$ no. of successes in n trials.

$X \to 0, 1, 2, \ldots, n.$ $\quad X \sim Bin(n, p)$

$P(X=x) = {}^nC_x (1-p)^{n-x} p^x \quad x = 0, 1, 2 \ldots n$

Consider n independent and identical bernoullie trials, with probability of success in each trial as p, then,

Let $X \to$ no. of successes in n trials,

Then, It follows binomial distr with pmf.

$$P(X=x) = \begin{cases} \boxed{{}^nC_x (1-p)^{n-x} p^x} & x = 0, 1, 2 \ldots n \\ = 0 & , \text{otherwise} \end{cases}$$

$$\sum_{x=0}^{n} P_X(x) = \sum_{x=0}^{n} {}^nC_x p^x (1-p)^{n-x} = \underline{(q+p)^n}$$

$$= (p + 1 - p)^n$$

$$= 1^n = 1$$

$$\mu_1' = E(x) = \sum_{x=1}^{n} x \frac{n!}{x(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

$\underline{t = x-1}$

$$= np \sum_{t=0}^{n} \frac{(n-1)!}{t!(n-t-1)!} p^t (1-p)^{n-t-1}$$

$$= np \, \xi \, {}^{n-1}C_t \cdot p^t (1-p)^{n-t-1}$$

$$= np\,(1-p+p)^{n-1}$$

$$\boxed{E(x) = np = \mu_1'} = np\,(1) = np$$

$$Var(x) = E(x^2) - (E(x))^2 = n(n-1)p^2 + np - (np)^2$$

$$E(x(x-1)) = n(n-1)p^2 \qquad = n(n-1)p^2 + np - n^2p^2$$

$$E(x^2 - x) = n(n-1)p^2 \qquad = (n^2-n)p^2 + np - n^2p^2$$

$$\qquad = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$E(x^2) - E(x) = n(n-1)p^2 \qquad = np(1-p)$$

$$E(x^2) = n(n-1)p^2 + E(x) \qquad \boxed{Var(x) = npq} = \sigma^2$$

$$\boxed{E(x^2) = n(n-1)p^2 + np}$$

$$B_1 = \frac{\mu_3}{\sigma^3} = \frac{1-2p}{(npq)^{1/2}}$$

$$B_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{1-6pq}{npq}$$

Geometric Dist^n.

Suppose independent Bernoullian Trials are conducted till a success is achieved. let X is no. of trials needed for the first success.

$q \quad qr \quad q^2 r$

$\frac{q}{1-r}$

$X \to 1, 2, 3 \ldots \infty$

$$P(X=x) = \begin{cases} q^{x-1} p & x = 1, 2 \ldots \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{\sum_{x=1}^{\infty} P(X=x) = 1}$$

$$= \sum_{x=1}^{\infty} q^{x-1} p$$

$$= p(q^0 + q^1 + q^2 + \ldots)$$

$$= \frac{p}{1-q}, \quad 0 \le q < 1.$$

$q^0 p +$

$p(q^0 + 2q + 3q^2$

$p(1 + 2q + 3q^2 + \ldots$

$p\left(\frac{1}{p^2}\right) = \frac{1}{p}$

$$E(X) = \sum_{x=1}^{\infty} x\, q^{x-1} p$$

$$= \frac{1}{p}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$=$$

$$\boxed{Var(X) = \frac{q}{p^2}}$$

$M_X(t)$ for geometric dist

$$M_X(t) = E_X(e^{tX})$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

$$= \frac{pe^t}{1-qe^t}, \quad 0 < qe^t < 1$$

$e^t < \frac{1}{q}$

$1 + qe^t + (qe)^t$

$t < -\log q$

$Bin(n, p)$

$$M_X(t) = E(e^{tX})$$

$$= \sum_{j=0}^{n} ({}^nC_j)\, p^j q^{n-j} \times e^{tj}$$

$$= \sum_{j=0}^{n} ({}^nC_j)\,(pe^t)^j (1-p)^{n-j}$$

$$= (1-p+pe^t)^n$$

$$\boxed{M_X(t) = (q+pe^t)^n}$$

$$P(X>m) = \sum_{j=m+1}^{\infty} q^{j-1} p$$

$$= p\,(q^m + q^{m+1} + \dots)$$

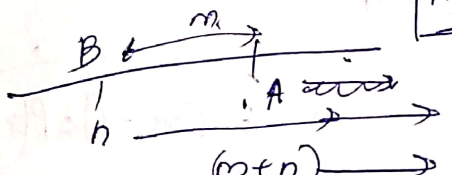
$$= \frac{p\,q^m}{(1-q)}$$

$$\boxed{P(X>m) = q^m}$$

memoryless property

$$P(\underbrace{X > m+n}_{A} \,/\, \underbrace{X>n}_{B}) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{q^{m+n}}{q^n} = q^m$$

$\boxed{A \subseteq B}$



(-ve) Binom-

Inverse Binomial distⁿ

Consider independent Bernoulli trials under identical conditions till r^{th} success is achieved.

Total = r

$X \to r, r+1, \dots$

$$P(X=x) = {}^{x-1}C_{r-1}\, q^{x-r} p^r \qquad x = r, r+1, r+2, \dots$$

$$= 0 \qquad \text{otherwise.}$$

$$\boxed{E(X) = \frac{r}{p}} \quad \boxed{V(X) = \frac{rq}{p^2}}$$

Tutorial

1) There is a dinner party where n men check their hats. the hats are mixed up during dinner, so that afterwards each man recieves a random hat. In particular, each man gets his hat with $P = \frac{1}{n}$, what is expected no of men that get their own hat.

Sol: let G be no. of men receives their own hat.

$$G_i = \begin{cases} 1 & \text{if person receives his own hat} \\ 0 & \text{otherwise.} \end{cases}$$

$$G = G_1 + G_2 + \dots + G_n$$

$$E(G) = E(G_1 + G_2 + \dots + G_n).$$

$$E(G) = E(G_1) + E(G_2) + \dots E(G_n) \qquad \boxed{E(x_i = 1) = P(x = i)}$$

$$E(G) = \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}$$

$$E(G) = 1$$

which means on-average each person gets his own hat.

---

2) Game A: We win \$2 with proba 2/3 and lose \$1 with prob 1/3

Game B: We win \$1002 with pr 2/3 and loose \$2001 with prob 1/3. which game is better financially?

$$E(A) = 2 \times \frac{2}{3} - 1 \times \frac{1}{3} = \frac{4}{3} - \frac{1}{3} = 1$$

$$E(B) = 1002 \times 2/3 - 2001 \times \frac{1}{3} = \frac{2004 - 2001}{3} = 1$$

$(x) = \mathcal{E}(x^2) - (\mathcal{E}(A))$

$\mathcal{E}(A^2) = 4 \times \frac{2}{3} - 1 \times \frac{1}{3} = \frac{7}{3}$

$V(A) = \frac{7}{3} - 1 = \frac{4}{3}$,

Higher variance, Higher risk

~~$\mathcal{E}(B^2) =$~~

variance $(A - \mathcal{E}(A)) = \begin{cases} 1 & \text{prob } 2/3 \\ -2 & \text{prob } 1/3 \end{cases}$

$(A - \mathcal{E}(A))^2 = \begin{cases} 1 & \text{prob } 2/3 \\ 4 & \text{prob } 1/2 \end{cases}$

$\mathcal{E}(A - \mathcal{E}(A))^2 = 1 \times \frac{2}{3} + 4 \times \frac{1}{3} = \frac{6}{3} = 2$

$\mathcal{E}(B) = 1$

$(B - \mathcal{E}(B)) = \begin{cases} 1001 & 2/3 \\ -2002 & 1/3 \end{cases}$

$(B - \mathcal{E}(B))^2 = \begin{cases} 1002001 & 2/3 \\ 4008004 & 1/3 \end{cases}$

$var(B) = 1002001 \times \frac{2}{3} + 4008004 \times \frac{1}{3} = 2004002$

3) In a card game, each card has a point value. Cards 2-9 has 5 points, 10 and face cards 10 points, A has 15 points; if deck is shuffled, what is expected total point value of top 3 cards.

Sol $\mathcal{E}(5) = 5 \times \frac{8}{13}$

$P(A) = \frac{1}{13}$, $\frac{4}{52}$

$\mathcal{E}(10) = 10 \times \frac{4}{13}$

$$E(X_1) = 5 \times \frac{8}{13} + 10 \times \frac{4}{13} + 15 \times \frac{1}{13} = \frac{95}{13}$$

$$\begin{array}{r} 95 \\ \times\ 3 \\ \hline 285 \end{array}$$

$X = X_1 + X_2 + X_3$.

$$E(X) = E(X_1 + X_2 + X_3) = 3 \times \frac{95}{13} = \frac{285}{13}$$

$E(3X) = 3\, E(X)$

---

4) Let T be the time for first success in bernoulli's trials process. We take same space $\Omega = 1, 2, 3 \ldots$ and assign geometric distribution $m(j) = P(T=j) = q^{j-1}p$. find $E(T)$

$$\boxed{\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}}$$

$$E(T) = \sum_{t=1}^{\infty} t \cdot q^{t-1} p$$

$$= p \sum_{t=1}^{\infty} q^{t-1} t$$

$$= p \times \frac{1}{p^2} = \frac{1}{p}$$

$$\boxed{\text{Sum AGP} = \frac{a}{1-r} + \frac{dr}{(1-r)^2}}$$

$$\frac{1}{1-q} + \frac{(1)(\cancel{1-q})q}{(1-q)^2}$$

$$\cancel{\frac{1}{1-q} + \frac{p}{q^2}}$$

$$\frac{(1-q) + (1-q)}{(1-q)^2}$$

$$= \frac{(1-q)}{(1-q)^2} + \frac{(1)(q)}{(1-q)^2}$$

$$= \frac{(1-q)}{(1-q)^2} =$$

$$\frac{1}{(1-q)^2} = \frac{1}{p^2}$$

5) When looking at person's eye colour, it turns out 1% of people of the world has green eyes. Consider a group of 20 people.

1) Find probability that none of them has green eyes $P(x=0) = {}^{20}C_0 \left(\frac{1}{100}\right)^0 \left(\frac{99}{100}\right)^{20}$

2) Nine of have green eyes $= P(x=9) = {}^{20}C_9 \left(\frac{1}{100}\right)^9 \left(\frac{99}{100}\right)^{n}$

3) At most three have green eyes

$P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$\underline{P(x \geq 3)} = \cancel{1 - P(x \leq 3)}$

4) At least 4 green eyes

$P(x \geq 4) = 1 - P(x < 4)$

$= 1 - (P(x=0) + P(x=1) + P(x=2) + P(x=3))$

6) According to center for disease control [CDC] about 1 in 88 children in US have been diagnosed with autism. Suppose you have 10 children

1) None has autism

2) Seven has autism

3) At most two have autism

4) At least five have autism

7) Two players A and B play a coin tossing game. A gives B one dollar if a head shows turns up otherwise. B pays A one doller if tail shows up. The probability that head is shown is p. Find expected gain of A.

Sol. A $\xleftarrow{+1}$ B if p

A $\xrightarrow{-1}$ B if $1-p$

$E(A) = 1(p) - 1(1-p)$

$= 2p - 1$

8)

8) A fair die is rolled n times. Find the probability of obtaining atleast one 6? find minimum value of n for which probability of getting atleast one six is greater than (or) equal to 1/2.

$p = \frac{1}{6}$

$p(x \geq 6)$?

$p(x \geq 6) \geq \frac{1}{2}$

$\left(\frac{5}{6}\right)^{n-1} \geq \left(\frac{1}{2}\right)$ ~~x~~

Conditional Expectation, Tutorial

Markov's Inequality: If X is a random variable that takes only non negative values, then for any value $a>0$,

$$\boxed{P(X \geq a) \leq \frac{E(x)}{a}}$$

Discrete: X takes values $x_1 < x_2 < x_3 \cdots x_j < x_{j+1} \cdots$ (Ascending order) let $x_j = a$

Consider $E(x) = \sum_{i=1}^{\infty} x_i P(x_i)$

$$= \sum_{i=1}^{j-1} x_i P(x_i) + \sum_{i=j}^{\infty} x_i P(x_i)$$

$$E(x) \geq \sum_{i=j}^{\infty} x_i P(x_i)$$

$$E(x) \geq a \sum_{i=j}^{\infty} P(x_i)$$

$$\frac{E(x)}{a} \geq P(x \geq a)$$

Continuous

$$E(x) = \int_{-\infty}^{\infty} x f_x(x)\, dx.$$

but takes only positive values

$$= \int_{0}^{\infty} x f_x(x)\, dx.$$

$$E(x) = \int_0^a x f_x(x) dx + \int_a^{\infty} x f_x(x) dx,$$

$$E(x) \geq \int_a^{\infty} x f_x(x) dx$$

$$\geq a \int_a^{\infty} f_x(x) dx$$

$$E(x) \geq a P(X \geq a)$$

$$\therefore \boxed{P(x \geq a) \leq \frac{E(x)}{a}}$$

Chebeshev's Inequality:- If x is a random variable with finite mean μ and variance then for any value $k>0$, $\boxed{P\{|x-\mu|\geq k\} \leq \frac{\sigma^2}{k^2}}$

Proof:- Since $(x-\mu)^2$ is non-negative, by applying Markov's inequality with $a=k^2$:

$$P\{(x-\mu)^2 \geq k^2\} \leq \frac{\sigma^2}{k^2}$$

Since, $(x-\mu)^2 \geq k^2$, if $|x-\mu| \geq k$, we get

$$P\{|x-\mu| \geq k\} = \frac{\sigma^2}{k^2} \qquad E(x-\mu)^2 = \sigma^2$$

1) Suppose that it is known that, the number of items produced in a factory during a week is a r.v with mean 50.

(a) What can be said about prob that this week production will exceed 75.

Sol: $E(x) = 50$. $a = 75$ From Markov's inequ

$$P(x \geq 75) \leq \frac{\cancel{50}\,2}{\cancel{75}\,3}$$

(b) if the variance is 25, what can be said about prob, that this week production is between 40 & 60.

$$P\{|x-50| \geq 10\} \leq \frac{25}{10^2} = \frac{1}{4} \qquad \frac{25\times 25}{75\times 75}$$

$$1 - P\{|x-50| \geq 10\} \geq 1 - \frac{1}{4}.$$

$$P\{|x-50| < 10\} \geq 3/4.$$

$$P\{40 < x < 60\} \geq \frac{3}{4} = 0.75$$

$x - 50 > -10$

$x > 40$

$x - 50 > 10$

$x > 60.$

2) If X is a uniformly distributed over the interval $(0,10)$, then $P\{|x-5| > 4\}$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E(x) = \int_0^{10} \frac{1}{(10)} x\,dx = \frac{1}{10} \cdot \left[\frac{x^2}{2}\right]_0^{10} = \frac{1}{10}\left(\frac{100}{2}\right) = 5$$

$$var(x) = E(x^2) - (E(x))^2$$

$$= \frac{1}{10}\int_0^{10} x^2\,dx - 5^2 = \frac{1}{10}\left(\frac{x^3}{3}\right)_0^{10} - 25$$

$$= \frac{1}{10}\left(\frac{1000}{3}\right) - 25 = \frac{100}{3} - 25 = \frac{25}{3}.$$

$$P\{|x-5| > 4\} \le \frac{25}{3(16)} = 0.52$$

$$P\{|x-\mu| > k\} \le P\{|x-\mu| \ge k\} \le \sigma^2/k^2.$$

$$1 - P\{|x-5| \le 4\} = 1 - P\{-4 < x-5 < 4\}$$

$$= 1 - P\{1 < x < 9\}$$

$$= 1 - \frac{1}{10}\int_1^9 dx = 1 - \frac{1}{10}(8)$$

$$= 1 - 0.8$$

$$= 0.2$$

Theorem: if $var(x) = 0$, then, $P\{x = E(x)\} = 1$

by Chebeshev's inequality. for any $n \geq 1$.

$$P\{|x-\mu| > \frac{1}{n}\} < \frac{\sigma^2}{1/n^2} = n^2\sigma^2 \leq 0$$

which means equal to zero

$$P\{|x-\mu| > \frac{1}{n}\} = 0$$

letting $n \to \infty$, by continuity property of probability

$$\lim_{n\to\infty} P\{|x-\mu| > \frac{1}{n}\} = 0.$$

$$P\{\lim_{n\to\infty} |x-\mu| > \frac{1}{n}\} = 0$$

$$P\{|x-\mu| > 0\} = 0$$

$$P\{x \neq \mu\} = 0$$

Compliment $\quad P\{x = \mu\} = 1$

$$\therefore \boxed{P\{x = E(x)\} = 1}$$

Theorem: Weak law of large numbers:-

Let $X_1, X_2, X \ldots$ be a sequence of identically distributed random variables, each having finite mean $E(x_i) = \mu$, then for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1 + X_2 - - + X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0 \text{ as } n \to \infty.$$

$$X = \frac{X_1 + X_2 + X_3 \cdots X_n}{n}$$

$$E(\bar{x}) = \frac{1}{n} E(x_1 + x_2 + \dots + x_n)$$

$$= \frac{1}{n} \left(E(x_1) + E(x_2) + \dots E(x_n) \right)$$

$$= \frac{1}{\cancel{n}} (\mu \times \cancel{n}) = \mu.$$

$$V(\bar{x}) = var\left(\frac{x_1 + x_2 + \dots x_n}{n} \right)$$

$$= \frac{1}{n^2} var(x_1 + x_2 + \dots + x_n)$$

$$= \frac{1}{n^2} \left(var(x_1) + var(x_2) + \dots var(x_n) \right)$$

$$= \frac{1}{n^{\cancel{2}}} \times \sigma^2 \cancel{n} = \frac{\sigma^2}{n}$$

By applying Chebeshev's inequality

$$P \left\{ \left| \frac{x_1 + x_2 + \dots x_n}{n} - \mu \right| \geq \varepsilon \right\} \leq \left(\frac{\sigma^2}{n \varepsilon^2} \right.$$

$$\lim_{n \to \infty} P \left\{ \left| \frac{x_1 + x_2 + \dots x_n}{n} - \mu \right| \geq \varepsilon \right\} = 0.$$

Tutorial

1) Calculate $var(x)$ if x represents the outcome when a fair die is rolled.

2) The following gambling game, known as wheel of fortune as follows. A player bets on one of the numbers between 1 to 6. three dice are rolled and if the number bet by the player appear i times $i = 1, 2, 3$. then, the player wins i units. If the numbers bet by player does not

appear he looses 1 unit. Is the game fair?

$\left(E(x) = ?\right)$

Tutorial 8 — 9/10/19

1) A die is rolled until a 1 occurs what is the prob that we have to roll the die

a) once. What is the resulting. Geometric dist

b) twice

c) thrice.

$P(X=x) = q^{x-1} \cdot p$

$p = \frac{1}{6}$

$q = \frac{5}{6}$

Sol: a)

a) $P(x=1) = \left(\frac{5}{6}\right)^0 \left(\frac{1}{6}\right)$

b) $P(X=2) = \left(\frac{5}{6}\right)^1 \left(\frac{1}{6}\right)$

c) $P(x=3) = \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)$

General $P(x=x) = \left(\frac{5}{6}\right)^{x-1} \left(\frac{1}{6}\right)$

---

2) 2 coins are tossed until HH occurs 3 times. What is probability that what is probability that are necessary

a) exactly 5 tosses of the 2 coins are necessary

b) $P(x<6)$

c) $E(X) = ?$ $var(x) = ?$

Sol: let X be no. of tosses of 2 coins

$\{HH, TH, HT, TT\}$ $\frac{3}{4}$, $\frac{1}{4}$

$P(X=x) = {}^{x-1}C_{r-1}\, q^{x-r} p^r$ $r=3$

$p = \frac{1}{4}$

$q = \frac{3}{4}$.

$P(x=x) = {}^{x-1}C_2\, q^{x-3} p^3$.

$P(x=5) = {}^4C_2\, q^2 p^3$

$= {}^4C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^3$.

$$P(x<6) = \cancel{P(x=0)} + \cancel{P(x=1)} + \cancel{P(x=2)} + \dots P(x=5)$$

no meaning

$^{x-1}C_{r-1}$

$x-1 \geq r-1$

$$= P(x=3) + P(x=4) + P(x=5)$$

$$= {}^{2}C_{2}\left(\tfrac{1}{4}\right)^{3}\left(\tfrac{3}{4}\right)^{0} + {}^{3}C_{2}\left(\tfrac{1}{4}\right)^{3}\left(\tfrac{3}{4}\right)^{1}$$

$$= \left(1 + \frac{9}{4} + \frac{3 \times 9}{8}\right)\left(\frac{1}{4}\right)^{3} + {}^{4}C_{2}\left(\tfrac{1}{4}\right)^{3}\left(\tfrac{3}{4}\right)^{2}$$

$$= \frac{13 \cdot 2}{4 \cdot 2} + \frac{27}{8} = \frac{53}{8} \times \frac{1}{4^3}$$

$E(x) = \dfrac{r}{p}$

$= \dfrac{3(4)}{1}$

$\boxed{E(x) = 12}$

$V(x) = \dfrac{rq}{p^2}$

$V(x) = \dfrac{(3)\left(\frac{3}{4}\right)}{\frac{1}{16}}$

$= \dfrac{9}{4} \times \dfrac{16}{1}$

$\boxed{V(x) = 36}$

---

3) During a bad economy, a graduating ECE student goes to career fair booths in the technology sector and his/her likelihood of receiving an off-campus interview invitation after a career booth depends on how well he/she did in ECE 313 specifically, an A in 313 results in prob $p = 0.95$ of obtaining an invitation, whereas as C in 313 results in $p = 0.15$.

a) Give a pmf for random variable Y which denotes the no. of career fair booths visits a student make before his/her first invitation.

b) $E(Y) = ?$ for A & C

c) Assuming that each Student visits 5 booth during a typical career fair, find prob that an A in 313 will not get an off-campus interview invitation. Similarly, find prob that C will get an invitation during a typical career fair.

a) $P(Y=k) = q^{k-1} p$

$P(Y=k) = (1-p)^{k-1} p$

b) $E(Y) = \frac{1}{p}$ for both A & C

$E(Y) = \begin{cases} \frac{1}{0.95} & \text{for A} \\ \frac{1}{0.15} & \text{for C} \end{cases}$

c) $P(Y \leq k) = (1-p)^4 (p)$

$= (0.05)^4 (0.95)$ for A.

$P(Y=k) = 1 - P(Y=k)$

$= 0.05 - (0.95)(0.05)^4$

c) (i) $(1-p)^5 = (1-0.95)^5$

$P(X=x) = q^{x-1} p$

(ii) $1 -$ C will not invitation

$= 1 - (1-p)^5 = \{1 - (1-0.15)^4$

4) Suppose that 105 passengers hold reservations for a 100-passenger flight. The no. of passengers will show up at the gate can be modeled as a binomial random variable X with parameters (105, 0.9)

a) On average, how many passengers show at gate

$E(x) = np = 105 (0.9) = 94.5 = 94 \cdot / -$

b) $P(X \leq 100) = 1 - P(x > 100)$

$= 1 - (P(X = 101) + P(X = 102) + P(X = 103) + P(X = 104) + P(X = 105))$

$= 1 - {}^{105}C_{101} (0.9)^{101} (0.1)^4 - {}^{105}C_{102} (0.9)^{102} (0.1)^3$

$- {}^{105}C_{103} (0.9)^{103} (0.1)^2 - {}^{105}C_{104} (0.9)^{104} (0.1)^1$

$- {}^{105}C_{105} (0.9)^{105} (0.1)^0$

c) Explain, why the no. of no-shows can be modeled as binomial random variable (105, 0.1)

⇒ If X is binomial r.v, then, r.v $Y = n - X$ with parameters $(n, 1-p)$

(Success is no show)

Tutorials (9/10/19)

5) An airline knows that 5% of people making reservations do not turn up for the flight, so it sells 52 tickets for a 50 seat flight. what is prob that every passenger who turn up will get a ticket.?

Sol: $P(x \le 50) = 1 - P(x > 50)$

$= 1 - (P(X=51) + P(X=52))$

$q = \frac{5}{100}$

$p = \frac{95}{100}$

$= 1 - {}^{52}C_{51}\left(\frac{95}{100}\right)^{51}\left(\frac{5}{100}\right)^{1} + {}^{52}C_{52}\left(\frac{95}{100}\right)^{52}\left(\frac{5}{100}\right)^{0}$

6) Suppose independent tests are conducted on monkeys to develop a vaccine. If the prob. of success is 1/3. in each trial, what is prob that atleast 5 trials are needed to get first success.?

$p = 1/3$

$q = 2/3.$

$P(x \ge 5) = \sum_{n=5}^{\infty} \left(\frac{2}{3}\right)^{n-1}\left(\frac{1}{3}\right)$

$\boxed{P(X \ge k) = q^{k-1}}$

$= \frac{1}{3}\left(\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 + \cdots\right)$

$= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right)$

$= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{1-\frac{2}{3}}\right)$

$= \left(\frac{\cancel{3}}{\cancel{3}}\right)\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)^4$

7) An urn contains N white and M black balls. Balls are randomly selected one at a time, until a black one is obtained. If we assume that each ball selected is replaced before the next one is drawn, what is probab that

a) Exactly n draws are made $= \left(\frac{N}{M+N}\right)^{n-1}\left(\frac{M}{M+N}\right)$

b) at least k draws are made.

$$\sum_{n=k}^{\infty}\left(\frac{N}{M+N}\right)^{n-1}\left(\frac{M}{M+N}\right)$$

$$= \left(\frac{M}{M+N}\right)\left(\left(\frac{N}{M+N}\right)^{k-1}+\left(\frac{N}{M+N}\right)^{k}+\dots\right)$$

$$= \left(\frac{M}{M+N}\right)\left(\frac{N}{M+N}\right)^{k-1}\left(1+\left(\frac{N}{M+N}\right)+\dots\right)$$

$$= \left(\frac{M}{M+N}\right)\left(\frac{N}{M+N}\right)^{k-1}\left(\frac{1}{1-\frac{N}{M+N}}\right)$$

$$= \cancel{\left(\frac{M}{M+N}\right)}\left(\frac{N}{M+N}\right)^{k-1}\times\cancel{\frac{M+N}{M}}$$

$$= \left(\frac{N}{M+N}\right)^{k-1}$$

---

8) Find the expected value and variance of no. of times one must throw a die until the outcome (1) has occurred 4 times.

$P(x=x) = {}^{x-1}C_{r-1}\, q^{x-r}\, p^{r}$

$= {}^{x-1}C_{3} \left(\frac{5}{6}\right)^{x-4} \left(\frac{1}{6}\right)^{4}$

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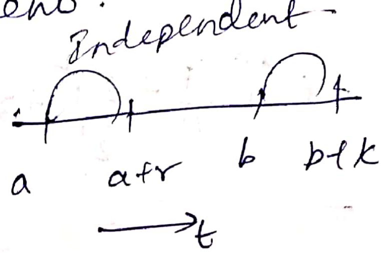
Poisson process

observations / occurences / happenings over a time/space

Assumptions:-

① No. of outcomes (or) occurences during disjoint time intervals are independent.



② Probability of a single occurence during a small time interval is proportional to length of the interval.

③ Probability of more than one occurence during a small time interval is negligible.

$\lambda \to$ no. of occures in an interval of length t.

$$P_n(t) = P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

(2) $\to P_1(h) = P(X(h) = 1) = \lambda h$

h = Small time interval

→ The distribution of occurences in a fixed time interval during a poisson process is called poisson distribution.

Poisson distribution:-

$\boxed{\lambda \geq 0}$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2 \ldots$$

λ = Rate con

$$= 0, \quad \text{elsewhere.}$$

$$\sum_{x=-\infty}^{\infty} P_X(x_i) = 1$$

$$\mu_1' = E(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x x}{x(x-1)!}$$

$$= e^{-\lambda}\left(\frac{\lambda}{1!} + \frac{2\lambda^2}{2!} + \frac{3\lambda^3}{3!} \ldots\right) \quad e^{-\lambda} \frac{\lambda^0}{}$$

$$\mu_1' = E(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$x - 1 = y$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = \sum_{y=0} e^{-\lambda} \frac{\lambda^y}{y!} \times \lambda = \lambda$$

$$\mu_2' = E(X^2) = E(X(X-1)) + E(X) = \lambda^2 + \lambda$$

$$E(X(X-1)) = \lambda^2$$

$$\mu_2 = var(X) = \mu_2' - (\mu_1')^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\alpha_k = E(x(x-1)\dots(x-k+1)) = \lambda^k$$

$$= \boxed{\sigma^2 = \lambda}$$

$$\mu_3' = \cancel{\lambda^3}\ \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_3 = \cancel{\lambda}\ \lambda$$

$$\beta_1 = \frac{\mu_3}{\sigma^3} = \frac{\lambda}{\lambda^{3/2}} = \frac{1}{\sqrt{\lambda}} > 0.$$

$$\beta_2 = \frac{\mu_4}{\sigma^4} - 3 = \frac{\mu_4}{\lambda^2} - 3 = \frac{1}{\lambda} > 0$$

$$\mu_4' = \cancel{\lambda^4}\ \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda.$$

$$\mu_4 = 3\lambda^2 + \lambda$$

$a^{lm}\ b^m$

$\mu_3 = ?$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} (\lambda e^t)^x \frac{e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

→ Let X follows ~~Fol~~ binomial $\sim (n, p)$

let $n \to \infty$, $p \to 0$.

Such that $np = \lambda$. then, this distribution

$$f_X(x) \to \frac{e^{-\lambda}\lambda^x}{x!}$$

Proof:- $P(X=x) = {}^nC_x\, p^x q^{n-x}$. $\qquad np = \lambda$, $p = \frac{\lambda}{n}$

$M_X(t) = (q + pe^t)^n$ → for Binomial

$= (1 - p + pe^t)^n$

$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n} + \frac{\lambda e^t}{n}\right)^n$

$= \lim_{n\to\infty} \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n \qquad = 1^\infty$

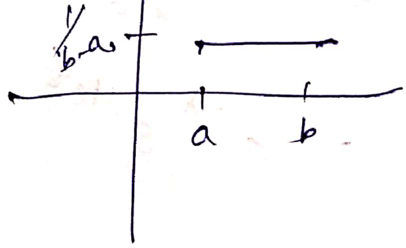
$e^{\left(1 + \frac{\lambda(e^t - 1)}{n} - 1\right)\cancel{\infty}}$ so $e^{(\text{base}-1)p}$

$M_X(t) = e^{\lambda(e^t - 1)}$ → for poisson

Continuous distribution

$f_X(x) = k \cdot \frac{1}{b-a} \quad a \le x \le b$ $\qquad P(a \le x \le b) = \int_a^b f_X(x)\,dx$.

$\quad 0 \quad$ elsewhere.



$\int_a^b k\,dx = 1$

$\boxed{k = \frac{1}{b-a}}$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_a^x \frac{dx}{b-a}$$

$$= \frac{x-a}{b-a}.$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_a^b \frac{x\,dx}{b-a} = \frac{1}{b-a} \times \left(\frac{x^2}{2}\right)_a^b = \frac{b^2-a^2}{(b-a)(2)} = \frac{a+b}{2} \;//$$

$$V(x) = \int_a^b \frac{x^2\,dx}{b-a} = \left(\frac{x^3}{3}\right)_a^b \left(\frac{1}{b-a}\right) = \frac{b^3-a^3}{(b-a)(3)}$$

$$= \frac{(\cancel{b-a})(b^2+a^2+ab)}{(3)\cancel{(b-a)}}$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{a^2+b^2-ab}{3} - \left(\frac{a+b}{2}\right)^2 \qquad E(x^2) = \frac{a^2+b^2+ab}{3}$$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a^2+b^2+2ab}{4}\right)$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{b} e^{tx} f_X(x)\,dx$$

$$= \int_a^b e^{tx} \times \frac{dx}{b-a} = \frac{1}{b-a}\left(\frac{e^{tx}}{t}\right)_a^b = \frac{e^{bt}-e^{at}}{(b-a)t}$$

$$= \frac{e^{bt}-e^{at}}{(b-a)t}$$

$$M_X(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

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Let us consider a poisson process with rate λ

Let T be the time of the first occurence.

$$P(T > t) = P(X(t) = 0) = e^{-\lambda t} \quad t > 0$$
$$= 1 \quad , t \leq 0.$$

$$\boxed{P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}} \quad n \geq 0.$$

$$\text{cdf} = F_T(t) = P(T \leq t) = 1 - P(T > t) = 0 \quad , t \leq 0$$
$$1 - e^{-\lambda t} \quad , t > 0$$

$$\boxed{f_T(t) = \begin{matrix} 0 & , t \leq 0 \\ \lambda \cdot 0 - \cancel{e^{-\lambda t}} \cancel{\frac{}{-\lambda}} & t \geq 0 \end{matrix}}$$

Exponential distribution

(or)

(-ve) exponential dist

$$\text{pdf} = f_T(t) = \begin{cases} 0 & t \leq 0 \\ \lambda e^{-\lambda t} & t > 0 \end{cases}$$

$$\mu_k' = E(x^k) = \int_0^\infty t^k \lambda e^{-\lambda t} dt = \frac{\lambda \sqrt{k+1}}{\lambda^{k+1}} \quad ?$$

$$= \frac{k!}{\lambda^k}$$

Memoryless prope

$$P(T > a) = e^{-\lambda a}$$

$$\rightarrow P(\underbrace{T > a+b}_{A} / \underbrace{T \geq b}_{B}) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda(b)}} = e^{-\lambda a} //$$

$$\Gamma(x) = \int t^{x-1} e^{-t}\, dt$$

$$\Gamma(x+1) = x\sqrt{x}$$

$$\sqrt{(x+1)} = x!$$

$$\mu_1' = E(x) = \frac{1}{\lambda}$$

$$\mu_2' = \frac{2}{\lambda^2} = E(x^2)$$

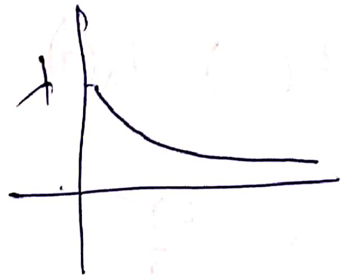
$$\mu_3' = \frac{6}{\lambda^3}$$

$$\mu_4' = \frac{24}{\lambda^4}$$

$$Var(x) = \mu_2 = E(x^2) - (E(x))^2 = \frac{1}{\lambda^2}.$$

$\mu_3, \mu_4, \beta_1, \beta_2$.

$$\mu_3 = \frac{2}{\lambda^3} \qquad \mu_4 = \frac{9}{\lambda^4}$$



$$\beta_1 = \frac{\frac{2}{\lambda^3}}{\frac{1}{\lambda^3}} = 2 > 0 \qquad \beta_2 = 6 > 0.$$

$$\beta_2 = \frac{\frac{9}{\lambda^4}}{\frac{1}{\lambda^4}} - 3 = 6,$$

$$M_x(t) = E(e^{tx})$$

Normal Distribution:

$$X \sim N(\mu, \sigma^2)$$

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$\frac{x-\mu}{\sigma} = z$ $\quad$ $\frac{dx}{\sigma} = dz$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \cancel{\sigma}\, dz$$

$(\sigma)(1) - (x-\mu)(0)$

$\frac{\sigma}{\sigma^2} = \frac{1}{\sigma}$

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

$F_X(-x) = F_X(x)$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{+\frac{z^2}{2}} \cancel{dz}\, dz$$

$t = +\frac{z^2}{2}$

$$\left(= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{t} \frac{\cancel{dt}}{+z} \right) \times$$

$dt = -\frac{\cancel{2} z\, \cancel{dz}}{\cancel{2}}$

$$= \frac{\cancel{2}}{\sqrt{\cancel{2}\pi}} \int_0^{\infty} e^{-t} \frac{dt}{\cancel{\sqrt{2}}\sqrt{t}}$$

$z^2 = 2t$

$z = \sqrt{2t}$

$dz = \frac{\sqrt{2}\,dt}{\sqrt{2}\sqrt{2}\,t^{1/2}}$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t}\,dt}{\sqrt{t}} \qquad \left[\Gamma(x) = \int t^{x-1} e^{-t}\right.$$

$dz = \frac{dt}{\sqrt{2t}}$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} t^{-1/2} e^{-t}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \times \sqrt{\frac{\pi}{2}} \rightarrow \sqrt{\pi} \quad = 1$$

Gamma

$$\mu_k' = E\left(\frac{x-\mu}{\sigma}\right)^k = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^k \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$z = \frac{x-\mu}{\sigma}$

$$= \int_{-\infty}^{\infty} z^k \times \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

If $k = 2m+1$, $E\left(\frac{x-\mu}{\sigma}\right)^k = 0$

$k = 2m$, $E\left(\frac{x-\mu}{\sigma}\right)^k = (2m-1)(2m-3)\ldots 5.3.1$

$k=1$, for $E\left(\frac{x-\mu}{\sigma}\right)^k = E\left(\frac{x-\mu}{\sigma}\right) = \frac{E(x)-\mu}{\sigma} = 0$

$E(x) - \mu = 0$

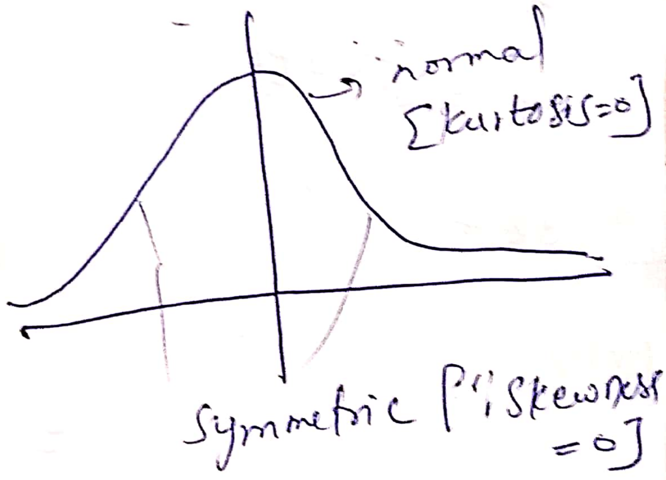
$E\left(\frac{x-\mu}{\sigma}\right)^2 = 1$ [∵ $2m = 2$, $m = 1$ $\left(E\left(\frac{x-\mu}{\sigma}\right)^k = (2m-1)\ldots 5.3.1\right)$

$E(x-\mu)^2 = \sigma^2$

$\boxed{Var(x) = E(x - E(x))^2 = \sigma^2} = 1$

$E\left(\frac{x-\mu}{\sigma}\right)^4 = 3.$

$\mu_4 = E(x-\mu)^4 = 3\sigma^4$



$\mu_3 = 0$ $\quad$ $\mu_4 = 3\sigma^4$

$\beta_1 = 0$ $\quad$ $\beta_2 = \frac{\mu_4}{\sigma^4} - 3$

$= 0$

Mean = Median = Mode = μ.

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$z = \frac{x-\mu}{\sigma}$

$dz =$

$$= \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma z t} e^{-\frac{z^2}{2}} dz.$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma z t - \frac{z^2}{2}} dz.$$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma z t + \sigma^2 t^2)} dz$$

$(z-\sigma t)^2 = m$

$z - \sigma t = \sqrt{m}$

$z = \sqrt{m} + \sigma t$

$dz =$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

$$= \cancel{e^{\mu t} + \frac{1}{2}}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(z-\sigma t)^2} dz}{\sqrt{2\pi}}$$

$\to 1$

$$\boxed{M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

$Z \sim N(\sigma t, 1)$

$X \sim N(\mu, \sigma^2)$

$Y = ax + b \qquad a \neq 0, \; b \in R.$

$M_Y(t) = E(e^{tY})$

$= E(e^{t(ax+b)})$

$= E(e^{a(x + bt})$

$= E(e^{atx} \cdot e^{bt})$

$= e^{bt} E(e^{(at)x})$

$= e^{bt} \underline{M_x(at)} \longrightarrow M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$= e^{bt} \times e^{\mu a t + \frac{\sigma^2 a^2 t^2}{2}}$

$M_Y(t) = e^{t(a\mu + b) + \frac{t^2}{2}(\sigma^2 a^2)}$

$Y \sim N(a\mu + b, a^2\sigma^2)$

$Y = ax + b$

$E(Y) = aE(x) + b$

$\boxed{E(Y) = a\mu + b}$

$V(Y) = a^2 V(x)$

$\boxed{V(Y) = a^2\sigma^2}$

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$X \sim N(\mu, \sigma^2)$

$Y \sim N(a\mu + b, a^2\sigma^2)$

Linearity property of Normal dist?

$X \sim N(\mu, \sigma^2)$

$Z = \frac{x - \mu}{\sigma} \qquad a = \frac{1}{\sigma}, \; b = -\frac{\mu}{\sigma}.$

$Z \sim N(0, 1)$

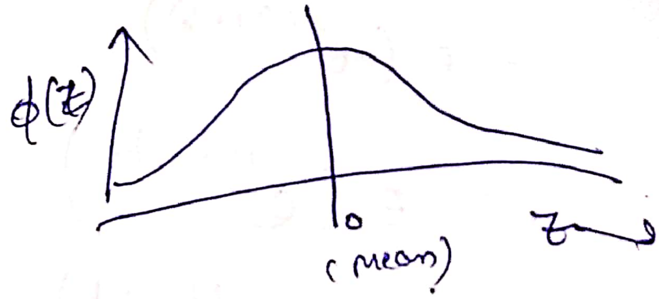
Standard normal random variable.

$z \sim N(0,1)$

Standard Normal random variable.

Pdf of S. N.V. (standard Normal ra varia)

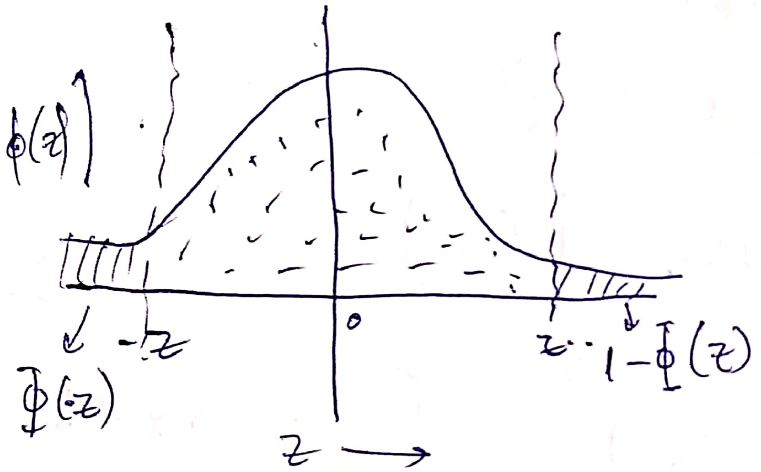
$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



Cdf of z

$$F_Z(z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

So, $\Phi(-z) = 1 - \Phi(z)$



$$\boxed{\Phi(z) + \Phi(-z) = 1}$$

If $z=0$ $\Phi(0) = 1/2$

$P(a < X < b)$

$P(a-\mu < x-\mu < b-\mu)$

$$P\left(\frac{a-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$

$$P\left(\frac{a-\mu}{\sigma} < z < \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$\Phi\left(\frac{b-\mu}{\sigma}\right) = \int_{-\infty}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz.$$

Joint dist^n

$X : \Omega \to \mathbb{R}$.

$P(a<x<b)$ $\quad P(5'<x<6')$

$P(a<x<b, c<Y<d) = P(5'<x<6, 56<Y<70)$

$$= \int_a^b \int_c^d f_{x,y}(x,y)\, dx\, dy.$$

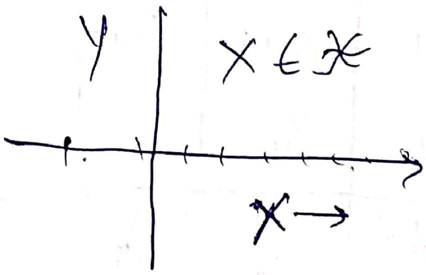
$$= \sum_{x_i=a}^{b} \sum_{y_j=c}^{d} P_{x,y}(x_i, y_i)$$

$X :: \Omega \to \mathbb{R}^p$

$x = (X_1, X_2, X_3, X_4, X_5, X_6) \to$ random vector.

Bivariate dist^n

$\boxed{X, Y} \to$ discrete.



$P_{X,Y}(x,y)$

$(X,Y) \in \mathcal{X} \times Y$. (Discrete)

The pmf of (X,Y) satisfies.

(i) $P(X = x_i, Y = y_i) = P_{x,y}(x_i, y_i)$

(ii) $P_{x,y}(x_i, y_i) \geq 0 \;\; \forall (x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$..

(iii) $\sum\sum_{\mathcal{X} \times \mathcal{Y}} P_{x,y}(x_i, y_i) = 1$,

Q) Suppose a car showroom has 10 cars, of a brand out of which 5 are good, 2 of are having defective transmission, 3 are having defecting steering problem. Now, 2 cars are selected at random,

(0,1,2). X denotes no. of cars with DT.

(0,1,2) - Y → no. of cars with D.S.

G → 5
DT → 2
DS → 3

$$P_{X,Y}(0,0) = \frac{\frac{5!}{3!\,2!}}{\frac{10!}{8!\,2!}} = \frac{5 \times 4}{10 \times 9} = \frac{2}{9}$$!!

$$P_{X,Y}(0,1) = \frac{5C_1 \times 3C_1}{10C_2} = \frac{5 \times 3 \times 2}{10 \times 9} = \frac{1}{3}$$!!

X \ Y	0	1	2	$P_X(x)$
0	2/9	1/3	3/45	$\frac{28}{45}$
1	10/45	6/45	0	$\frac{16}{45}$
2	1/45	0	0	$\frac{1}{45}$
$P_Y(y)$	$\frac{21}{45}$	$\frac{21}{45}$	$\frac{3}{45}$	

$\frac{10}{45} + 15 + 3$

$P_{X,Y}(x,y)$ $P(X \le 1, Y \le 1) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=1, Y=0) + P(X=1, Y=1)$

$= \frac{41}{45}$!!

P.d.f. of X

$P_X(0) = \frac{28}{45}$

$P_X(1) = \frac{16}{45}$

$P_X(2) = \frac{1}{45}$

Marginal distⁿ

$$P_X(x_i) = \sum_{y_i \in \mathcal{Y}} P_{X,Y}(x_i, y_i)$$

$$P_X(y_i) = \sum_{x_i \in \mathcal{X}} P_{X,Y}(x_i, y_i)$$

Conditional distⁿ

$$P_{X/Y=x_i}(x_i) = \frac{P_{X,Y}(x_i, y_i)}{P_Y(y_i)} \quad x_i \in \mathcal{X} \quad \left(\because P(A/B) = \frac{P(A \cap B)}{P(B)}\right)$$

$$P_{Y/X=x_i}(y_i) = \frac{P_{X,Y}(x_i, y_i)}{P_X(x_i)}, \quad y_i \in \mathcal{Y}$$

$\frac{10}{45}$ $\frac{28}{45}$ 5 14

$$P_{Y/X=0}(0) = \frac{P_{X,Y}(0,0)}{P_X(0)} = \frac{2/9}{\frac{28}{45}} = \frac{10}{28} = \frac{5}{14}.$$

$$P_{Y/X=0}(1) = \frac{P_{X,Y}(0,1)}{P_X(0)} = \frac{\frac{15}{45}}{\frac{28}{45}} = \frac{15}{28}$$

$$P_{Y/X=0}(2) = \frac{P_{X,Y}(0,2)}{P_X(0)} = \frac{\frac{3}{45}}{\frac{28}{45}} = \frac{3}{28}$$ //

~~Main~~ (Continuous case)

let (X, Y) be both continuous. $f_{(X,Y)}(x,y)$

Properties

1) $f_{X,Y}(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$.
2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1.$
3) $\int_a^b \int_c^d f_{X,Y}(x,y)\,dx\,dy = P(c < x < d,\ a < Y < b).$

Marginal distⁿ

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \quad , x_i \in X$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \quad , y_i \in Y$$

16/10/19

Tutorial :

Q1) The lifetime T(years) of an electronic component is a continuous random variable with probability density function given by:

$$f(t) = e^{-t} \quad t \geq 0.$$

a) find the lifetime L which a typical component is 60% certain to exceed.

(b) If five components are sold to a manufacturer find probability that at least one of them will have life time less than 2 years

$$E(x) = \int_0^{\infty} t e^{-t} dt$$

a) $P(T > L) = 0.6$

$\ln e^{-L} = \ln 0.6$

$-L = \log 0.6$

$L = -\log(0.6)$ $\quad \log \frac{10}{6}$

$L = \ln\left(\frac{1}{0.6}\right)$

$= \ln\left(\frac{5}{3}\right) = 2.303\,(0.22) = 0.51$

(b) P(atleast of one of them will have time < 2 years

$= 1 - P(\text{Lifetime more than } 2)$

$= 1 - P((T > 2))$

$= 1 - (1 - p)^5$

$= 1 - (0.6)^5 = \underline{0.92}$

Q.) Suppose that crowd size at home games for a football club follows a Normal distn with mean 26000 and standard dev 5000. what percentage of crowd are between 31000 & 36000.

$P(31000 < X < 36000)$

$P(10000 < X - 10000 < 15000)$

$P\left(\frac{10000}{5000} < \frac{X - 10000}{5000} < \frac{15000}{5000}\right)$

$P(2 < Z < 3) = P(3 < Z$

$= \Phi(3) - \Phi(2)$

$=$

$P(31000 < X < 36000)$

$= \Phi(2) - \Phi(1)$

$= 0.9772 - 0.8413$

$= 0.1359$

Q) 3 let $X \sim N(3, 4)$

find $P(2 \leq x \leq 5) = ?$

$\mu = 3$

$\sigma = 2$

$$\frac{2-3}{2} \leq \frac{x-3}{2} \leq \frac{5-3}{2}$$

$$P\left(\frac{-1}{2} \leq \cancel{z} \leq 1\right)$$

$$= \Phi(1) - \Phi\left(\frac{-1}{2}\right)$$

$$= 0.8413 - (0.3085)$$

$$= 0.5328$$

$1 - 0.6915$

Q) 4. Assume that the time required for a distance runner to run a mile is a normal random variable with parameters $\mu = (4$ min, 1 sec$)$ and $\sigma = 2$ sec.. What is probability that this athelete will run a mile in less than 4 min! in more than 3 min 55 sec?

60
×4

Sol: $\mu = 241$ sec

$\sigma = 2$ sec.

$P(X < 4$ min$)$

$P(X < 240)$

4×60

$= P(X < 240)$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{240-241}{2}\right) = \Phi\left(\frac{-1}{2}\right) = 0.3085$$

$P(X > 235) = 1 - P(X < 235)$

$= 1 - P\left(\frac{X-\mu}{\sigma} < \frac{235-241}{2}\right)$

$= 1 - \Phi(-3)$

$= 1 - 0.0013$

$P(X > 235) = 0.9987$

180
55
235

Q5) Assume that the high price of a commodity tomorrow is a normal random variable with $\mu = 10$, $\sigma = \frac{1}{4}$. What is the shortest interval that had probability 0.95 of including tommorrow's high price for this commodity?

$\frac{a-\mu}{\sigma}$

$\mu = 10 \quad \sigma = \frac{1}{4} \qquad P = (0.95)$

$P\left(\cancel{\frac{10-a}{\frac{1}{4}}} < z <\right)$

$P\left(\frac{a-10}{\frac{1}{4}} < z < \frac{b-10}{\frac{1}{4}}\right)$

$0.95 = \Phi(4(b-10)) - \Phi(4(a-10))$

$a = \frac{1}{6} \quad b = \frac{-\mu}{6}$

$a = \frac{4}{1} \quad b = -104$

$a = 4 \quad b = -40$

24

let $a = 10 - a$ $b = 10 + a$,

$$P(\overset{a}{\overline{10-a}} < X < \overset{b}{\overline{10+a}}) = 0.95$$

$$\cancel{P\left(\frac{a}{1/4} < \bar{X}_5 < \frac{a}{1/4}\right) = 0.95}$$

$$P\left(\frac{10-a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{10+a-\mu}{\sigma}\right)$$

$$= P[-4a < Z < 4a)$$

$$= \phi(x < 4a) - \phi(x > 4a)$$

$$= P(x < 4a) - (1 - P(x < 4a)$$

$$0.95 = 2P(x < 4a) - 1$$

$$0.95 + 1 = 2\phi(4a)$$

$$\frac{1.95}{2} = \phi(4a)$$

$\phi(4a) = \phi(1.96)$

$$0.975 = \phi(4a)$$

$4a = 1.96$

$$\frac{1.96}{4} = a$$

$$\underline{0.49 = a}$$

Actual interval

$A = 10 - 0.49$ $B = 10 + 0.49$

DMPT - 20 :-

17/10/19

Independence

X, Y are independent if

$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y), \quad \forall x, y \in \mathbb{R}^2.$$

1) X, Y are discrete

we call X and Y independent if

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y) \; ; \forall (x,y) \in \mathcal{X} \times \mathcal{Y}.$$

2) X, Y are continuous, we call them independent if

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \forall (x,y) \in \mathbb{R}^2.$$

Ex:
$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 < x < 1, \; 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_0^1 (1)\, dy \qquad = \int_0^1 (1)\, dx$$

$$f_X(x) = 1 \qquad f_Y(y) = 1.$$

$$\boxed{\begin{aligned} f_{X,Y}(x,y) &= f_X(x) \cdot f_Y(y) \\ 1 &= 1 \cdot 1 \; \checkmark \\ \text{LHS} &= \text{RHS} \end{aligned}}$$

$X = 0, 1$

$Y = 0, 1$

$X \backslash Y$	0	1	$P_X(x)$
0	1/4	1/4	1/2
1	1/4	1/4	1/2
$P_Y(y)$	1/2	1/2	1

$P_X(0) = 1/2 \quad P_X(1) = 1/2$

$P_Y(0) = 1/2 \quad P_Y(1) = 1/2$

Since $P_X(0) = \frac{P_X(0,0)}{P_Y(0)}$

$= \frac{1/4}{1/2} =$

Moments :-

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy.$$

Product Moments :-

$\mu'_{r,s} = E(X^r Y^s) \rightarrow (r,s)^{th}$ non-central product mom

$r = 1, s = 1$

$\mu'_{1,1} = E(XY) = (1,1)^{th}$ non - - - -

$\mu_{r,s} = E((X - \mu_X)^r (Y - \mu_Y)^s) \rightarrow (r,s)^{th}$ central mom

$\mu_{1,1} = E((X - \mu_X)(Y - \mu_Y)) \rightarrow$ Covariance (X, Y)

$Covar(X,Y) = E(XY - X\mu_Y - \mu_X Y + \mu_X \mu_Y)$

$= E(XY) - \mu_X \mu_Y - \cancel{\mu_X \mu_Y} + \cancel{\mu_X \mu_Y}$

$Covar(X,Y) = E(XY) - E(X)E(Y)$

$\sigma_x^2 = var(Y) = E(x-\mu_x)(x-\mu_y)$

$\sigma_{x,Y} = \text{Co-var}(X,Y) = E(x-\mu_x)(Y-\mu_y)$

says, how X, Y are co-varying with respect to mean

$\frac{\sigma_{X,Y}}{\sigma_X \sigma_Y} = \rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$ $\quad \sigma_X = +\sqrt{\sigma_X^2}$

$-1 \leq \rho_{X,Y} \leq 1$ $\quad \boxed{Y = aX + b}$

Co-effecient of co-relation between X and Y

$\rho_{(X,Y)} = 1$ perfectly linearlo co-relation between X and Y.

$\rho_{(X,Y)} = -1$ perfectly negative linear co-re

$(\times \rho_{X,Y} = 0$, co-var$(X) = 0$, X, Y are independent ?X

But, Theorem:

If X, Y are independent $\rho_{X,Y} = 0$, covar$(X) = 0$.

converse (not true).

$P_{X,Y}(0,-1) = P_X(0) P_Y(-1)$

$0 \neq \frac{1}{3} \times \frac{1}{3}$

X \ Y	-1	0	1	$P_X(x)$
0	0	1/3	0	1/3
1	1/3	0	1/3	2/3
$P_Y(Y)$	1/3	1/3	1/3	1

$E(XY) = 0(-1)(0) + 0.0.\frac{1}{3} + \ldots$

$= 0$

$E(X) = 2/3 \quad E(Y) = 0$.

$COV(X,Y) = 0 - 0 .. = 0$

$\rho_{X,Y} = 0$

Theorem: X, Y are independent $Cov(X,Y) = 0$

then $E[g(X) \cdot h(Y)] = E[g(X)]\,E[h(Y)]$.

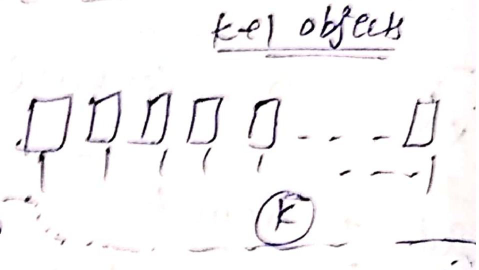
$g(X) = X^4$, $h(Y) = Y^5$

$E(X^4 Y^5) = E(X^4)\,E[Y^5]$

Discrete Maths :-

PHP (Pigeon Hole Principle):

If k is a positive integer, $k+1$ (or) more objects are placed into k boxes, then there is atleast one box containing two (or) more of the objects.



Generalised function:

If N objects are placed into k boxes, then there is atleast one box containing

$\lceil N/k \rceil \rightarrow$ ceil objects.

$\downarrow$

$\lfloor N/k \rfloor$: floor

Prob: 1, 11, 111, 1111, 11111, ...

atleast one number in above series will be divisible by 7777.

ceil $\lceil \ \rceil = [x] + 1 \rightarrow$ step

$\lfloor \ \rfloor$ floor $= [x] \rightarrow$ step

A) 1, 11, 111, 1111, - - - -

1, 2, ---, 7776 → remainders possible.

1111 . . . . . 1 = a × 7777 + d

1111 . . 1 = b × 7777 + d

lets take finite 1s . 1, 11, 111, - - - , 111 . . . 1

numbers 7778

1111 0000 = (a-b) 7777 × 10^k

There exists one remainder, atleast which repeats

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DMPT-21

2) Given m integers $a_1, a_2, \ldots, a_m$. there exists k, l

$1 \le k \le l \le m$ Such that $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.

A) Remainders = 1, 2, 3, - - - m-1

$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \ldots + a_m$

$a_1 + a_2 + \ldots + a_k + \ldots + a_l = ma + d$

$a_1 + a_2 + \ldots + a_k = mb + d$

$a_{k+1} + a_{k+2} + \ldots + a_l = (a-b)m$

So, divisible by m.

Books Reference for DM:

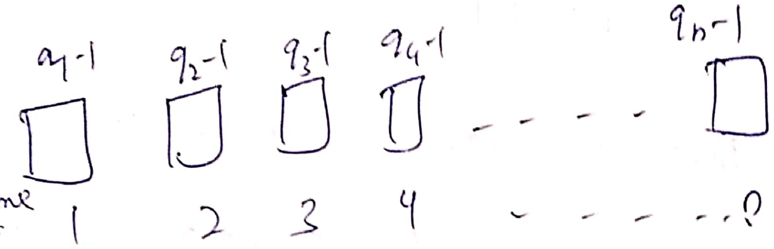
by Keneth Rosen

Strong form of PHP:-

let $q_1\ q_2 \ldots\ldots q_n$ be positive integers, if $q_1+q_2+\ldots+q_n-n+1$ objects are put into n boxes, then either the first box contains atleast q_1 objects or second box contain (q_2 objects) (or) third box (q_3 objects) ...... nth box contain q_n objects

A)

We assume



but atleast one is present which contains more than q_n-1 as there are q_n-n+1.

Product rule:-

Q) Suppose that a procedure into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways of doing the second task, there are $n_1 n_2$ ways to do the procedure.

Q) The chairs of an auditorium are to be labelled with an upper case English letter followed by an positive integer not exceeding 100. What is largest no. of chairs that can be labelled separately?

26×100

Sum Rule :-

Q2) If a task can be done in either 1 of n_1 ways (or) 1 of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways. then there are $n_1 + n_2$ ways to do the task.

Q) In how many ways, can you choose 2 books of different languages among 5 books in latin, 7 books in latin, 10 books in French?

Sol :-

L G $\quad 5 \times 7 = 35$

G F $\quad 7 \times 10 = 70$

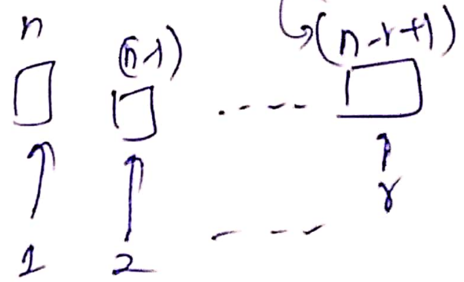
L, F $\quad 5 \times 10 = 50$

$= 120 + 35 = 155$

$${}^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-r\ldots$$

select ${}^{n-1}C_{r-1}$

not select ${}^{n-1}C_r$



$${}^nP_r = {}^nC_r \times {}^rP_r$$

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

To prove: $(k!)!$ is divisible by $(k!)^{(k-1)!}$