Maths - II

Assignment - 3 :-

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Linear Algebra.

Q1) Let P be the set of all polynomials f(x), and let Q be the subset of P consisting of all polynomials f(x) so $f(0) = f(1) = 0$. Show that Q is a subspace of P.

Sol:- P → set of all polynomials f(x)

Q → subset of P $f(0) = f(1) = 0$.

For Q to be a subspace of P, if

(i) Additive identity: $0 \in Q$.

(ii) For any vectors $u, v \in Q$, we have $u+v \in Q$.

(iii) For any scalar c and any vector $q \in Q$. $cq \in Q$.

i) P has zero polynomial as Q.

$$Q = \{ f(x) \mid f(0) = f(1) = 0 \}.$$

0 polynomial lies in Q as $f(0) = 0$.

ii) Let g(x), h(x) belong to Q

$g(1) = g(0) = 0$ $h(0) = h(1) = 0$.

consider $k(x) = h(x) + g(x)$

$k(0) = h(0) + g(0) = 0$.

$k(1) = h(1) + g(1) = 0$.

$k(x) \in Q$.

iii) Let $h(x) \in Q$ & $r \in R$.

$K(x) = rh(x)$

$K(0) = r(h(0)) = 0.$

$K(1) = r(h(1)) = 0.$

$K(x)$ satisfies the relation of $Q \in P$.

$\therefore Q$ is a subspace of P.

2) Let $u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and let $S = \{v = R^3 \mid v \cdot u = 0\}$.

Then, S is a subspace.

Ans) $u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\quad S = \{v \in R^3 / v \cdot u = 0\}$.

Let v be $[x \ y \ z]$

$v \cdot u = 0$

$x + 2y + z = 0.$

A zero vector satisfies the above equation

$\{0\} \in S$

Let $V_1, V_2 \in S$

$V_3 = V_1 + V_2$

$V_3 = [x_1 + x_2 \quad y_1 + y_2 \quad z_1 + z_2]$

$V_3 \cdot u = 1(x + x_2) + 2(y_1 + y_2) + 1(z_1 + z_2)$

$= x_1 + 2y_1 + z_1 + x_2 + 2y_2 + z_2 = 0$

$= 0.$

$V_3 \in S.$

Hense, S is a subspace.

(or)

Given $u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

1) zero vector in S

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0.$$

2) $u + v \in W$

$S_1 = [a \; b \; c]$

$S_2 = [x \; y \; z]$

$v = S_1 + S_2 \quad [a+x \quad b+y \quad c+z]$

$$v.u = [a+x \quad b+y \quad c+z] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$= a + 2b + c + x + 2y + z$

$= S_1 . u + S_2 . u$.

Hence 2nd condition is satisfied

3) $S_1 = [a b c]$

$K \rightarrow$ constant

$P = K S_1 = [ka \; kb \; kc]$

To prove $P \in S$

$$P.u = [ka \quad kb \quad kc] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$= k(a + 2b + c)$

$\therefore P \in S$

Hence, S is a subspace.

3) Do the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}$ and $\begin{bmatrix}0\\3\\0\end{bmatrix}$

Span R^3? Prove or Disprove.

Sol: If the given vectors span R^3, then for any $x, y, z \in R$ there exists c_1, c_2, c_3, c_4 such that $(x, y, z) = c_1[1,0,1] + c_2[1,1,1] + c_3[1,-2,1] + c_4[0,3,0]$

$$= \begin{bmatrix}1&0&1\\1&1&1\\1&-2&1\\0&3&0\end{bmatrix} \xrightarrow{R_2=R_1-R_2} \begin{bmatrix}1&0&1\\0&-1&0\\1&-2&1\\0&3&0\end{bmatrix} \xrightarrow{R_3=R_1-R_3} \begin{bmatrix}1&0&1\\0&-1&0\\0&2&0\\0&3&0\end{bmatrix}$$

$$\xrightarrow{R_4=R_4+R_2} \begin{bmatrix}1&0&1\\0&-1&0\\0&3&0\\0&3&0\end{bmatrix} \xrightarrow{R_4=R_4-R_3} \begin{bmatrix}1&0&1\\0&-1&0\\0&0&0\\0&2&0\end{bmatrix}$$

$$\xrightarrow[R_4=R_4+R_2]{R_2=2R_2} \begin{bmatrix}1&0&1\\0&2&0\\0&0&0\\0&0&0\end{bmatrix}$$

No. of pivots = 2.

$\therefore$ Rank of matrix = 2

Since, rank = 2, it does not span the subspace R^3.

4) Let $V = \begin{bmatrix}2\\0\\3\end{bmatrix}$ $w = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $0 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Which of the following sets are linearly dependent? Select all that apply.

a) $\{0\}$.

Ans: Given $V_1 = \begin{bmatrix}2\\0\\3\end{bmatrix}$, $w = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $0 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Let $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$av_1 = 0$ [a can be any scalar]

$\{0\}$ is linearly dependent.

b) $\{v\}$

$$a_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The only relation (solution) is $a_1 = 0$.

$\{v\}$ is linearly independent.

c) $\{w, 0\}$

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$a_1 = 0$, $a_2 \in \mathbb{R}$.

$\therefore \{w, 0\}$ is linearly dependent.

d) $\{v, 0\}$

$$a_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$a_1 = 0$, $a_2 \in \mathbb{R}$.

$\therefore \{v, 0\}$ is linearly dependent.

e) $\{v, w\}$

$$a_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

only sol.^{n} $a_1 = a_2 = 0$.

$\{v, w\}$ is linearly independent.

8) $\{v, w, 0\}$

$$a_1\begin{bmatrix}2\\0\\3\end{bmatrix} + a_2\begin{bmatrix}1\\1\\1\end{bmatrix} + a_3\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Only sol^n $a_1 = a_2 = 0$ $a_3 \in R$.

$\therefore$ $\{v, w, 0\}$ is linearly dependent.

5) Let U and W be subspace of a single vector space V such that only vector shared by both U and W is 0. let U be a nonzero vector in U and w be a nonzero vector in W. Show that $\{u, w\}$ is linearly independent?

Ans:- u, w are subspaces of V

Let $u = (\alpha_1 \alpha_2 \alpha_3)$ [non-zero vector in U]

$w = (\beta_1 \beta_2 \beta_3)$ [non-zero vector in V]

for $\{u, w\}$ to be linearly independent.

$$au + bw = 0 \quad \Rightarrow \quad a = 0 \ \& \ b = 0$$

$$a(\alpha_1, \alpha_2, \alpha_3) + b(\beta_1, \beta_2, \beta_3) = (0, 0, 0)$$

$$(a\alpha_1 + b\beta_1, a\alpha_2 + b\beta_2, a\alpha_3 + b\beta_3) = (0, 0, 0)$$

$$\alpha_1 a + b\beta_1 = 0.$$

$$a\alpha_2 + b\beta_2 = 0$$

$$a\alpha_3 + b\beta_3 = 0$$

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

Rank $= 2$. As there are no free variables

$a = 0$ & $b = 0$.

$\therefore \{u, w\}$ are linearly independent.

6) Diagonalise $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Ans:- $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Characteristic eq^n $|A - \lambda I| = 0$.

$$\begin{vmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(5-\lambda)^2 = 0$$

$$\lambda = 4, 5, 5.$$

Eigen vectors corresponding to $\lambda = 4$.

$$(A - \lambda I)x = 0.$$

$$\begin{bmatrix} 4-4 & 0 & -2 \\ 2 & 5-4 & 4 \\ 0 & 0 & 5-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

x is null space of $A-\lambda I$

$\therefore 2x_1 + x_2 + 4x_3 = 0$

$-2x_3 = 0$

set $x_3 = 0$ & $x_2 = 1$.

$\therefore 2x_1 + 1 = 0$

$x_1 = \frac{-1}{2}$

Eigen vector corresponding to $\lambda = 4$ is $x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$

Eigen vectors corresponding to $\lambda = 5$

$$\begin{bmatrix} 4-5 & 0 & -2 \\ 2 & 5-5 & 4 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$\begin{pmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{matrix} p & f & f \end{matrix}$$

$x + 2z = 0$

$\begin{matrix} y = 1 \\ z = 0 \end{matrix} \Rightarrow x = 0 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{matrix} z = 1 \\ y = 0 \end{matrix} \Rightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Eigen vectors corresponding to $\lambda = 5$ are

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

let $S = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ formed by eigen vectors and D is diagonal matrix formed by A

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$S^{-1} \Rightarrow \left[\begin{array}{ccc|ccc} -1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

S $\quad$ I

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

I $\quad$ S^{-1}

$$\therefore S^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A = SDS^{-1}$

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

7) Given that

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 2 & 0 & 3 & 1 & 1 \\ 1 & 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix}$$

Ans: $A = \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 2 & 0 & 3 & 1 & 1 \\ 1 & 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1 \qquad R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -1 & -3 & 1 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_2 \qquad R_3 \to R_3 + R_2 \qquad R_1 \to R_1 - R_2 \qquad R_3 \leftrightarrow R_4$

$$= \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

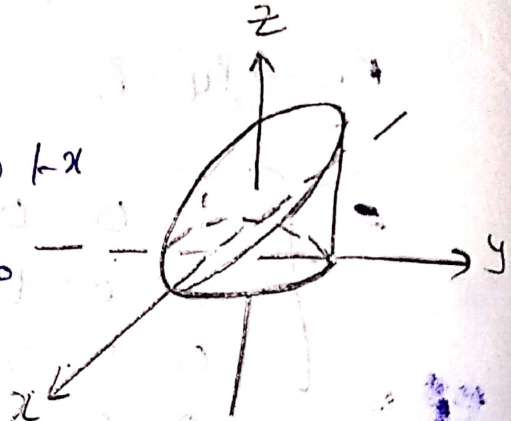
$R_1 \to R_1 + 4R_3 \,, \qquad R_2 \to R_2 - 3R_3$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Row reduced echelon form of A)

Calculus :

1) Find the volume of the solid bounded by the right circular cylinder $x^2+y^2=1$, the x-y plane, and the plane $x+z=1$.



Sol: $x^2+y^2=1$, $x+z=1$

z is varying from 0 to $1-x$

y is varying from $-\sqrt{1-x^2}$ to $+\sqrt{1-x^2}$

x is varying from -1 to 1

$$\therefore \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{1-x} dx\, dy\, dz$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x)\, dx\, dy$$

$$= \int_{-1}^{1} (2\sqrt{1-x^2})(1-x)\, dx$$

$$= 2\left[\int_{-1}^{1}\sqrt{1-x^2} - \int_{-1}^{1} x\sqrt{1-x^2}\right] \qquad \left[\int_{-1}^{1} x\sqrt{1-x^2}\, dx = 0 \text{ Odd } f^n\right]$$

$$= 2\int_{-1}^{1}\sqrt{1-x^2}\, dx$$

$$= 4\int_{0}^{1}\sqrt{1-x^2}\, dx$$

$$= 4\left[\left[\frac{x}{2}\sqrt{1-x^2}\right]_0^1 + \left[\frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1\right]$$

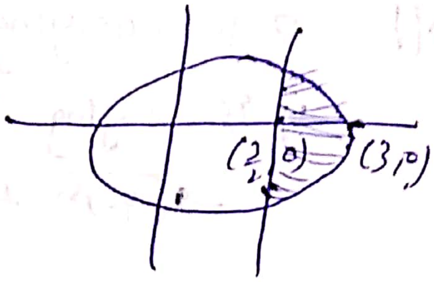
$$= 4\left[\frac{1}{2}\sin^{-1}(1)\right] = 2\sin^{-1}\left(\sin\frac{\pi}{2}\right)$$

$$= \pi$$

$\therefore$ Volume of solid bounded by right circular cylinder is $\boxed{\pi \text{ cub. units}}$

2) Use polar co-ordinates to find the area of the region inside the circle $x^2+y^2=9$ and to the right of the line $x=3/2$.

A) $x^2+y^2=9$.



$x=3/2$

$$A=\int_{3/2}^{3}\sqrt{9-x^2}\,dx$$

$x=3\cos\theta$. | $\frac{3}{2}=3\cos\theta \quad \theta=\pi/3$

$dx=-3\sin\theta\,d\theta$. | $3=3\cos\theta \quad \theta=0$

$$A=-\int_{\pi/3}^{0}\sqrt{9-9\cos^2\theta}\,(3\sin\theta)\,d\theta$$

$$=\int_{0}^{\pi/3}3\sin\theta\cdot 3\sin\theta\,d\theta$$

$$=9\int_{0}^{\pi/3}\sin^2\theta\,d\theta$$

$$=9\int_{0}^{\pi/3}\frac{1-\cos 2\theta}{2}\,d\theta$$

$$=9\left[\left(\frac{\theta}{2}\right)_0^{\pi/3}-\left(\frac{\sin 2\theta}{4}\right)_0^{\pi/3}\right]$$

$$A=9\left[\frac{1}{2}\left(\frac{\pi}{3}-0\right)-\frac{1}{4}\left(\sin 120-0\right)\right]$$

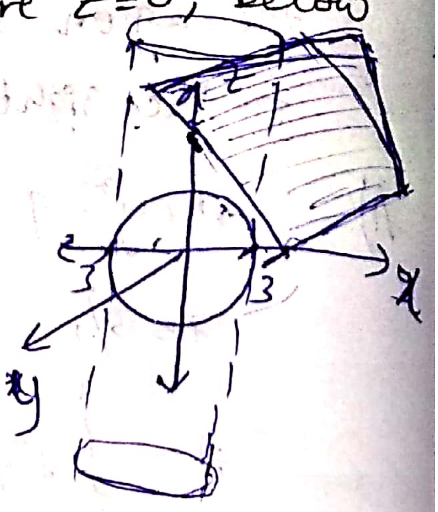
$$=9\left[\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right]=\frac{3\pi}{2}-\frac{9\sqrt{3}}{8}.$$

∴ It is area above x-axis.

Total area $=2*A=2\left(\frac{3\pi}{2}-\frac{9\sqrt{3}}{8}\right)$

$$\therefore \boxed{A=3\pi-\frac{9\sqrt{3}}{4}}$$

3) Use a triple integral to find the volume V inside $x^2+y^2=9$, and above $z=0$, below $x+z=4$.



Sol: z is varying from 0 to $4-x$.

y is varying from $-\sqrt{9-x^2}$ to $\sqrt{9-x^2}$

x is varying from -3 to 3.

$$= \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{+\sqrt{9-x^2}} \int_{0}^{4-x} dx\,dy\,dz.$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [4-x]\,dx\,dy$$

$$= \int_{-3}^{3} 2\sqrt{9-x^2}\,(4-x)\,dx.$$

$$= 2\left[\int_{-3}^{3} (\sqrt{9-x^2})4 - \int_{-3}^{3} x\sqrt{9-x^2}\right] dx$$

$$= 8\int_{-3}^{3} \sqrt{9-x^2}\,dx \qquad \left[\because \int_{-3}^{3} x\sqrt{9-x^2} = 0 \text{ odd } f^n\right]$$

$$= 16\int_{0}^{3} \sqrt{9-x^2}\,dx \qquad [\because \text{even } f^n]$$

$$= 16\left[\left[\left(\frac{x}{2}\sqrt{9-x^2}\right]_0^3 + \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3\right]$$

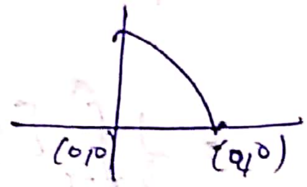
$$= 16\left[0 + \frac{9}{2}\sin^{-1}(1)\right]$$

$$= 16\left(\frac{9\pi}{4}\right) = 36\pi \text{ cu. units.}$$

$\therefore$ $\boxed{\text{~~Area~~ Volume} = 36\pi \text{ cu. units.}}$

4) Find the centre of mass of the first-quadrant part of the disk of radius a with centre at origin, if density function is y.

Sol: $x^2+y^2=a^2$



density f^n $\rho(x,y) = y$

$$\therefore \bar{x} = \frac{\iint_R x\,\rho(x,y)\,dx\,dy}{\iint_R \rho(x,y)\,dx\,dy}$$

$$\bar{y} = \frac{\iint_R y\,\rho(x,y)\,dx\,dy}{\iint_R \rho(x,y)\,dx\,dy}$$

$$\bar{x} = \frac{\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dx\,dy}{\int_{x=0}^{x=a}\int_{y=0}^{\sqrt{a^2-x^2}} y\,dx\,dy}$$

$$= \frac{\frac{1}{2}\int_0^a x(a^2-x^2)\,dx}{\frac{1}{2}\int_0^a (a^2-x^2)\,dx} = \frac{\cancel{\frac{1}{2}}\int_0^a (a^2x^2-x^3)\,dx}{\cancel{\frac{1}{2}}\int_0^a (a^2-x^2)\,dx}$$

$$= \frac{a^2\left[\frac{x^2}{2}\right]_0^a - \left[\frac{x^4}{4}\right]_0^a}{a^2\left[x\right]_0^a - \left[\frac{x^3}{3}\right]_0^a} = \frac{\frac{a^4}{2}-\frac{a^4}{4}}{a^3-\frac{a^3}{3}}$$

$$\bar{x} = \frac{\frac{a^4}{4}}{\frac{2a^3}{3}} = \frac{3a}{8} \; //$$

$$\boxed{\therefore \bar{x} = \frac{3a}{8}}$$

$$\bar{y} = \frac{\iint_R y.y\,dx\,dy}{\iint_R y\,dx\,dy} = \frac{\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y^2\,dx\,dy}{\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y\,dx\,dy}$$

$$= \frac{\frac{1}{3}\int_{x=0}^{a} \left[y^3\right]_0^{\sqrt{a^2-x^2}}}{\frac{1}{2}\left[a^3 - \frac{a^3}{3}\right]}$$

Consider $I = \int_{x=0}^{a} \frac{1}{3}\left[y^3\right]_0^{\sqrt{a^2-x^2}} dx$

$$= \frac{1}{3}\int_0^a (a^2-x^2)^{3/2}\,dx$$

$x = a\cos\theta$ | $\theta \to \frac{\pi}{2}$ to 0.

$dx = -a\sin\theta\,d\theta$

$$= -\frac{1}{3}\int_{\pi/2}^{0} (a^2 - a^2\cos^2\theta)^{3/2}\,a\sin\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} a^3\sin^3\theta \,.\, a\sin\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} a^4\sin^4\theta\,d\theta$$

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\times\frac{n-3}{n-2}\times\cdots\frac{1}{2}\times\frac{\pi}{2}.$$

$$\therefore I = \frac{a^4}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi a^4}{16}.$$

$$\bar{y} = \frac{\frac{\pi a^4}{16}}{\frac{1}{2}\left[\frac{2a^3}{3}\right]} = \frac{\pi a^4}{16}\times\frac{3}{a^3} = \frac{3a\pi}{16}$$

$$\therefore \boxed{\bar{y} = \frac{3a\pi}{16}}$$

$\therefore$ Centre of mass $= \left(\frac{3a}{8}, \frac{3a\pi}{16}\right)$

5) Evaluate the improper integral $\int_0^5 \frac{1}{(2-x)^{1/3}}$

Sol: It is an improper integral of type 2.

$$I = \underbrace{\int_0^2 \frac{1}{(2-x)^{1/3}} dx}_{I_1} + \underbrace{\int_2^5 \frac{1}{(2-x)^{1/3}} dx}_{I_2}$$

$$I_1 = \int_0^2 \frac{1}{(2-x)^{1/3}} dx$$

$$= \lim_{c \to 2^-} \int_0^c \frac{1}{(2-x)^{1/3}} dx.$$

$2-x=t$
$-dx=dt$

and at $x=0$
$t=2$
at $x=c$
$t=2-c$

$$\lim_{c \to 2^-} \int_{2-c}^2 t^{-1/3} dt.$$

$$= \lim_{c \to 2^-} \left(\frac{t^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right]_{2-c}^2$$

$$= \frac{3}{2} \lim_{c \to 2^-} \left[2^{2/3} - (2-c)^{2/3}\right]$$

$$\boxed{I_1 = \frac{3}{2} \times 2^{2/3}}$$

$$I_2 = \int_2^5 \frac{1}{(2-x)^{1/3}} dx$$

$$I_2 = \lim_{c \to 2^+} \int_c^5 \frac{1}{(2-x)^{1/3}} dx$$

$2-x=t$
$-dx=dt$

at $x=c$
$t=2-c$
at $x=5$
$t=-3$

$$I_2 = \lim_{c \to 2^+} \left[- \int_{2-c}^{-3} t^{-1/3} \, dt \right]$$

$$= \lim_{c \to 2^+} \left[\int_{-3}^{2-c} t^{-1/3} \, dt \right]$$

$$= \lim_{c \to 2^+} \left[t^{2/3} \right]_{-3}^{2-c} \times \frac{3}{2}$$

$$I_2 = \frac{3}{2} \left[\lim_{c \to 2^+} \left((2-c)^{2/3} - (-3)^{2/3} \right) \right]$$

$$\boxed{I_2 = -\frac{3}{2} \times 9^{1/3}}$$

$$I = I_1 + I_2$$

$$\boxed{I = \frac{3}{2} \left[2^{2/3} - 9^{1/3} \right]}$$

6) Check the validity of Cauchy's mean value theorem for functions $f(x) = x^3$ and $g(x) = \arctan x$ (inverse of $\tan x$) on the interval $[0,1]$. Find the value of c.

Sol:- $f(x) = x^3$ and $g(x) = \tan^{-1}(x)$ in interval $[0,1]$

Cauchy's mean value Theorem

$a = 0, b = 1$

$f(0) = 0, g(0) = 0$

$f(1) = 1, g(1) = \frac{\pi}{4}$

$f'(c) = 3c^2 \quad g'(c) = \frac{1}{1+c^2}$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)},$$

$$\frac{1-0}{\frac{\pi}{4}} = \frac{3c^2}{\frac{1}{1+c^2}}$$

$$\frac{4}{\pi} = 3c^2(1+c^2)$$

$$3c^2 + 3c^4 = \frac{4}{\pi} \quad \text{put } c^2 = t$$

$$3t^2 + 3t - \frac{4}{\pi} = 0.$$

$$a = 3, \; b = 3, \; c = -\frac{4}{\pi}$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-3 \pm \sqrt{9 - 4(3)\left(-\frac{4}{\pi}\right)}}{2(3)}$$

$$t = \frac{-3 \pm \sqrt{9 + \frac{48}{\pi}}}{6}.$$

As $c^2 = t$ and t must be positive.

$$\boxed{\therefore \; c = \sqrt{\frac{-3 + \sqrt{9 + \frac{48}{\pi}}}{6}} = 0.566}$$

Also, before calculating c, we must check all the conditions of Cauchy's theorem

(i) $f(x)$ is a polynomial f^n, whereas $g(x)$ is a trigonometric f^n & both of them are continuous in $[0,1]$.

(ii) $f'(x) = 3x^2 \quad g'(x) = \frac{1}{1+x^2}$

$f(x)$ and $g(x)$ are differentiable at each point in $(0,1)$ since $f(x)$ is polynomial, $g(x)$ is trigonometric f^n..

(iii) Also $g(a) \neq g(b)$, $g'(x) \neq 0$ in (a,b).

So, all the conditions of MVT (Cauchy's) are satisfied.