(Calculus)
(PART - 1)

Maths Assignment - 1 :-

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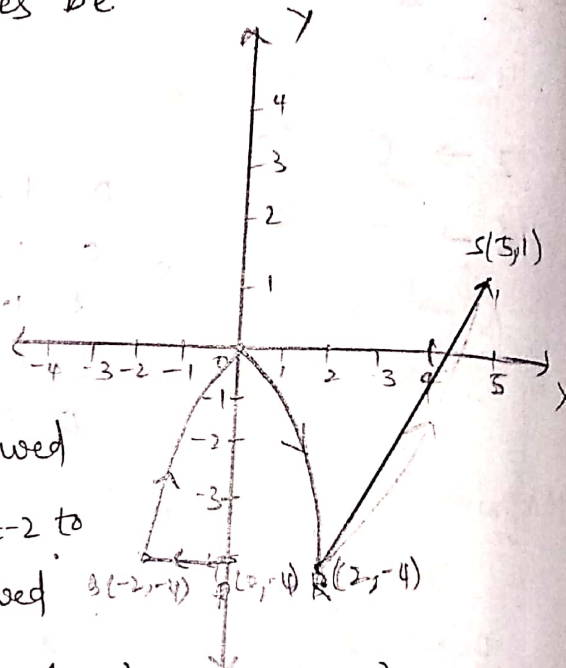
Q1) $\vec{F}(x,y) = xy\bar{i} + (1+3y)\bar{j}$;
$d\bar{r} = dx\bar{i} + dy\bar{j}$. Let the points through which the line segments passes be

P (0, -4)
Q (-2, -4)
R (2, -4)
S (5, 1)



Sol:- C is the line segment from (0,-4) to (-2,-4), followed by portion of $y=-x^2$ from $x=-2$ to $x=2$ which is in turn followed by the line segment from (2,-4) to (5,1).

→ For $y=-x^2$
$y = -4$

$y-(-4) = \frac{5}{3}(x-2)$
$5x - 3y = 22$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_P^Q \bar{F}\cdot d\bar{r} + \int_Q^R \bar{F}\cdot d\bar{r} + \int_R^S \bar{F}\cdot d\bar{r}.$$

As $\bar{r} = x\bar{i} + y\bar{j}$ $\Rightarrow dr = dx\bar{i} + dy\bar{j}$.

$$\Rightarrow \int_C xy\bar{i} + (1+3y)\bar{j}\cdot(dx\bar{i}+dy\bar{j})$$

$$= \int_P^Q xy\bar{i} + (1+3y)\bar{j}\cdot(dx\bar{i}+dy\bar{j}) + \int_Q^R xy\bar{i}+(1+3y)\bar{j}\cdot(dx\bar{i}+dy\bar{j}) + \int_S^R xy\bar{i}+(1+3y)\bar{j}\cdot(dx\bar{i}+dy\bar{j})$$

for $\vec{PQ}$, it is a line with eq $y=-4$

$dy = 0$

x varies from 0 to -2

$$= \int_0^{-2} -4x\bar{i} + (1+3(-4))\bar{j} \cdot (dx\,\bar{i} + 0\bar{j})$$

$$= \int_0^{-2} (-4x\bar{i} - 11\bar{j}) \cdot dx\,\bar{i} = -\int_0^{-2} 4x\,dx = -4\left[\frac{x^2}{2}\right]_0^{-2}$$

$$= -4\left(\frac{4}{2} - 0\right) = \underline{-8} \qquad \therefore \boxed{\int_P^Q \bar{F}\cdot d\bar{r} = -8} \rightarrow (1)$$

$\overline{QR}$, $y = -x^2$ $\quad dy = -2x\,dx$

$$\int_Q^R \bar{F}\cdot d\bar{r} = \int_Q^R xy\,\bar{i} + (1+3y)\bar{j} \cdot (dx\,\bar{i} + dy\,\bar{j})$$

$$= \int_Q^R x(-x^2)\bar{i} + (1+3(-x^2))\bar{j} \cdot (dx\,\bar{i} + dy\,\bar{j})$$

$$= \int_Q^R -x^3\bar{i} + (1-3x^2)\bar{j} \cdot (dx\,\bar{i} - 2x\,dx\,\bar{j})$$

$$= \int_Q^R -x^3\,dx - 2\left(\int_Q^R x - 3x^3\right)dx$$

$$= -2\int_{-2}^{2} x\,dx + \int_{-2}^{2} 5x^3\,dx$$

$$= -\cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_{-2}^{2} + 5\left(\frac{x^4}{4}\right)_{-2}^{2}$$

$$= -(4-4) + 5\left(\frac{2^4 - 2^4}{4}\right) = 0$$

$$\therefore \boxed{\int_Q^R \bar{F}\cdot d\bar{r} = 0} \rightarrow 2$$

$\vec{RS}$ $\rightarrow$ $5x - 3y = 22$

$3y = 5x - 22$

$dy = \frac{5}{3}dx$

$$\int_R^S \bar{F}\cdot d\bar{r} = \int_R^S xy\bar{i} + (1+3y)\bar{j}\,.\,(dx\bar{i}+dy\bar{j})$$

$$= \int_R^S x\left(\frac{5}{3}x - \frac{22}{3}\right)\bar{i} + \left(1+3\left(\frac{5x}{3}-\frac{22}{3}\right)\bar{j}\right)$$

$$.\left(dx\bar{i}+\left(\frac{5}{3}dx\bar{j}\right)\right)$$

$$= \int_R^S \left(\frac{5}{3}x^2 - \frac{22x}{3}\right)\bar{i} + (1+5x-22)\bar{j}\,.\,\left(dx\bar{i}+\frac{5}{3}dx\bar{j}\right)$$

$$= \int_R^S \left(\frac{5}{3}x^2 - \frac{22}{3}x\right)dx + \frac{5}{3}\int_R^S (1+5x-22)\,dx$$

$$-21$$

$$\int_R^S \left(\frac{5}{3}x^2 - \frac{22}{3}x\right)dx + \int_R^S \left(\frac{25}{3}x - 35\right)dx$$

$$\int_2^5 \left(\frac{5}{3}x^2 + \cancel{2x} - 35\right)dx$$

$$= \frac{5}{3}\left(\frac{x^3}{3}\right)_2^5 + \left(\frac{x^2}{2}\right)_2^5 - 35\,(x)_2^5$$

$$= \frac{5}{3}\left(\frac{5^3-2^3}{3}\right) + \frac{25-4}{2} + 35(3)$$

$$= \frac{5}{9}(125-8) + \frac{21}{2} - 105$$

$$= 5(13) + \frac{21}{2} - 105 = -\frac{59}{2} \quad 4$$

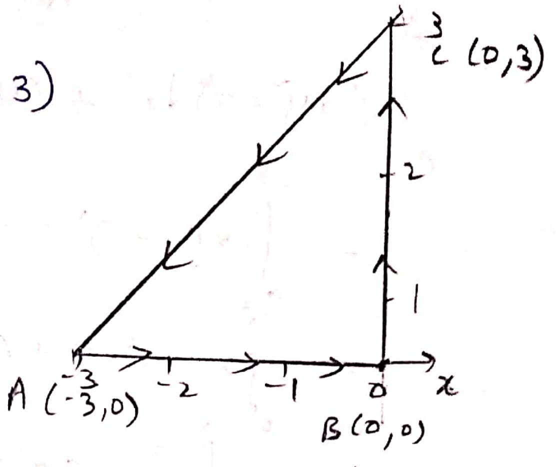
$$\boxed{\therefore \int_R^S \bar{f}\cdot d\bar{r} = -\frac{59}{2}} \rightarrow 3$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = 1 + 2 + 3 = -8 + 0 + \left(-\frac{59}{2}\right)$$

$$= \frac{-16-59}{2} = -\frac{75}{2} = -37.5$$

$$\boxed{\therefore \int_C \bar{F}\cdot d\bar{r} = -37.5}$$

2) Verify Green's theorem for $\oint_C (xy^2+x^2)\,dx + (4x-1)\,dy$ where c is shown below by (a) computing the line integral directly and (b) using Green's theorem to compute the line integral



Sol: let A (−3, 0), B(0,0), C(0,3)

(a) By computing Line integral

$$\oint_C (xy^2+x^2)\,dx + (4x-1)\,dy$$

$$= \int_A^B (xy^2+x^2)\,dx + (4x-1)\,dy + \int_B^C (xy^2+x^2)\,dx + (4x-1)\,dy + \int_C^A (xy^2+x^2)\,dx + (4x-1)\,dy$$

$\overline{AB}$: x varies from −3 to 0. eq of line is y = 0
So, dy = 0.

$$= \int_A^B (xy^2+x^2)\,dx + (4x-1)\,dy = \int_{-3}^{0} x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_{-3}^{0} = 0 - \frac{(-27)}{3} = 9$$

$$\boxed{\int_A^B \bar{F}\cdot d\bar{r} = 9} \rightarrow (1).$$

$\overline{BC}$: x = 0. | y varies from 0 to 3
dx = 0

$$= \int_B^C (xy^2+x^2)\,dx + (4x-1)\,dy = -\int_0^3 dy = -(3) = -3$$

$$\therefore \boxed{\int_B^C \bar{F}\cdot d\bar{r} = -3}$$

$\overline{CA}$: $A(-3,0)$ $C(0,3)$

$$y-3 = 1(x+0)$$

$$y = x+3$$

$$dy = dx$$

$$\int_C^A (xy^2+x^2)dx + (4x-1)dy = \int_{-3} x(y^2+x)dx + (4x-1)dy$$

$$= \int_0^{-3} (x(x+3)^2 + x^2)dx + \int_0^{-3} (4x-1)dx$$

$$= \int_0^{-3} x(x^2+9+6x) + x^2 dx + \int_0^{-3} (4x-1)dx$$

$$= \int_0^{-3} x^3+9x+6x^2 + x^2 dx + 4x-1\,dx$$

$$= \int_0^{-3+0} (x^3+7x^2+13x-1)dx$$

$$= \left(\frac{x^4}{4}\right)_0^{-3} + 7\left(\frac{x^3}{3}\right)_0^{-3} + 13\left(\frac{x^2}{2}\right)_0^{-3} - (x)_0^{-3}$$

$$= \frac{81}{4} + (-63) + \frac{117}{2} + 3$$

$$= \frac{81-252+234+12}{4} = \frac{75}{4}.$$

$$\therefore \oint (xy^2+x^2)dx + (4x-1)dy = 9-3+\frac{75}{4}$$

$$= \frac{36-12+75}{4}$$

$$= \frac{99}{4} = \underline{24.75}$$

(b) By Green's theorem.

$$\oint_C Pdx + Qdy = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy$$

$$\rightarrow \oint (xy^2+x^2)dx + (4x-1)dy \approx \oint Pdx + Q\,dy$$

Here $P = xy^2+x^2$ and $Q = 4x-1$

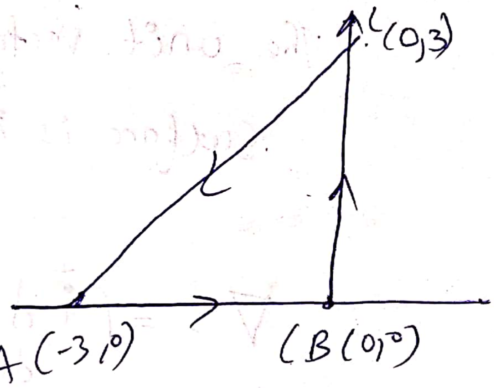
Evaluating $\oint P\,dx + Q\,dy$ by Green's theorem,

$\Rightarrow \oint (xy^2+x^2)\,dx + (4x-1)\,dy = \iint_S \left(\frac{\partial(4x-1)}{\partial x} - \frac{\partial(xy^2+x^2)}{\partial y} \right) dx\,dy$

$\frac{\partial(4x-1)}{\partial x} = 4 \quad \Big| \quad \frac{\partial(xy^2+x^2)}{\partial y} = 2xy$

$= \iint_S (4-2xy)\,dx\,dy$

$= \int_{-3}^{0} \int_{0}^{x+3} (4-2xy)\,dx\,dy$



(As x varies from -3 to 0,

y varies from 0 to $x+3$, $y = x+3 \rightarrow \overrightarrow{CA}$)

Integrating w.r.to y. $= \int_{-3}^{0} \left[4y - 2x\frac{y^2}{2} \right]_0^{x+3} dx$

$= \int_{-3}^{0} [4(x+3) - x(x+3)^2 - 0)\,dy$

$= \int_{-3}^{0} 4x + 12 - x^3 - 6x^2 - 9x)\,dx = \int_{-3}^{0} (-x^3 - 6x^2 - 5x + 12)\,dx$

Now, integrating w.r.to x, substituting limits

$= \left[-\frac{x^4}{4} - 6\frac{x^3}{3} - \frac{5x^2}{2} + 12x \right]_{-3}^{0}$

$= 0 - \left(-\frac{81}{4} + 54 - \frac{45}{2} - 36 \right)$

$= -\left(-\frac{81}{4} + 18 - \frac{45}{2} \right) = -\left(\frac{-81+72-90}{4} \right)$

$= -\left(-\frac{99}{4} \right) = \frac{99}{4}$

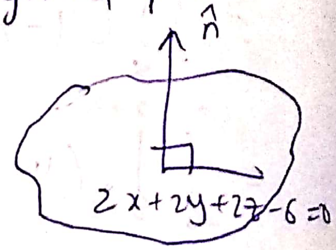
$= 24.75$

Hence, verified.

3) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$. where $\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and s is the surface of the plane $2x+y+2z=6$ in the first octant

Sol:- $S = 2x+y+2z-6=0$

The unit vector $\hat{n}$ normal to the given plane surface is $\hat{n} = \dfrac{\nabla S}{|\nabla S|}$



$$\nabla S = \left(\bar{i}\frac{\delta}{\delta x} + \bar{j}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta z}\right).\dot{S}$$

$$= \left(\bar{i}\frac{\delta}{\delta x} + \bar{j}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta z}\right)(2x+y+2z-6)$$

$$\nabla S = 2\bar{i} + \bar{j} + 2\bar{k}.$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\bar{i}+\bar{j}+2\bar{k}}{\sqrt{(2)^2+(1)^2+(2)^2}} = \frac{2\bar{i}+\bar{j}+2\bar{k}}{3}.$$

$$F.\hat{n} = \left[(x+y^2)\bar{i} + (-2x)\bar{j} + (2zy)\bar{k}\right].\ \frac{1}{3}(2\bar{i}+\bar{j}+2\bar{k}).$$

$$= \frac{2}{3}(x+y^2-x+2zy) = \frac{2}{3}(y^2+2zy)$$

The projection of $2x+y+2z=6$ on yz plane be R.

$$\Rightarrow \iint_S F.\hat{n}\,ds = \iint_R \frac{F.\hat{n}}{|\hat{n}.\bar{i}|}\,dy\,dz.$$

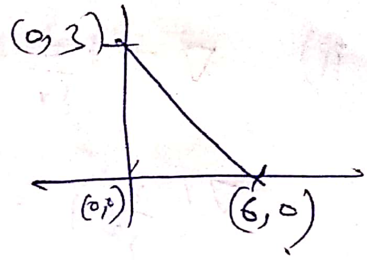
$$\hat{n}.\hat{i} = \left(\frac{2}{3}\bar{i} + \frac{\bar{j}}{3} + \frac{2\bar{k}}{3}\right)\bar{i} = \frac{2}{3}.$$

$$\Rightarrow \iint_R \frac{F.\hat{n}}{|\hat{n}.\bar{i}|}\,dy\,dz = \iint_R \frac{\frac{2}{3}(y^2+2yz)\,dy\,dz}{\frac{2}{3}}$$

As projection on y,z plane $x=0$, eq becomes

So, $y+2z=6$.

$0 \le y \le 6 \quad , \quad 0 \le z \le \frac{6-y}{2}$



$$\int_{y=0}^{6}\int_{z=0}^{\frac{6-y}{2}} (y^2+2yz)\,dy\,dz$$

$$= \int_0^6 y^2[z]_0^{\frac{6-y}{2}} + 2y\left[\frac{z^2}{2}\right]_0^{\frac{6-y}{2}}$$

$$= \int_0^6 \left[zy\,(y+z)\right]_0^{\frac{6-y}{2}} dy = \int_0^6 \left[\left(\frac{6-y}{2}\right).y.\left(y+\frac{6-y}{2}\right)\right] dy$$

$$= \int_0^6 \frac{1}{4}(36y-y^3)\,dy = \frac{1}{4}\left[36\left[\frac{y^2}{2}\right]_0^6 - \left[\frac{y^4}{4}\right]_0^6\right]$$

$$= \frac{1}{4}\left[18[36-0] - 324\right] = \frac{1}{4}(628-324)$$

$$= \frac{324}{4} = \underline{81}$$

$$\boxed{\therefore \iint_S \bar{F}.\hat{n}\,ds = 81}$$

4) If $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and $r = |\vec{r}| \neq 0$ show that.

(a) (i) $\nabla\left(\frac{1}{r^2}\right) = -\frac{2\vec{r}}{r^4}$, (ii) $\nabla.\left(\vec{r}/r^2\right) = \frac{1}{r^2}$.

(i) $\nabla = \bar{i}\frac{\delta}{\delta x} + \bar{j}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta z}$

$$\nabla\left(\frac{1}{r^2}\right) = \left(\bar{i}\frac{\delta}{\delta x} + \bar{j}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta z}\right)\left(\frac{1}{x^2+y^2+z^2}\right)$$

$$= \bar{i}\frac{\delta}{\delta x}\left(\frac{1}{x^2+y^2+z^2}\right) + \bar{j}\frac{\delta}{\delta y}\left(\frac{1}{x^2+y^2+z^2}\right) + \bar{k}\frac{\delta}{\delta z}\left(\frac{1}{x^2+y^2+z^2}\right)$$

$$\nabla\left(\frac{1}{r^2}\right) = \frac{(-2x)\bar{i}}{(x^2+y^2+z^2)^2} + \frac{(-2y)\bar{j}}{(x^2+y^2+z^2)^2} + \frac{(-2z)\bar{k}}{(x^2+y^2+z^2)^2}$$

$$\nabla\left(\frac{1}{r^2}\right) = -\frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)^2} = \frac{-2\bar{r}}{r^4} \;//$$

ii) $\nabla = \bar{i}\frac{\delta}{\delta x} + \bar{j}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta z}$

$$\nabla\cdot\left(\frac{\bar{r}}{r^2}\right) = \left(\bar{i}\frac{\delta}{\delta x} + \bar{j}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta z}\right)\cdot\left(\frac{x\bar{i} + y\bar{j} + z\bar{k}}{x^2+y^2+z^2}\right)$$

$$= \frac{\delta}{\delta x}\left(\frac{x}{x^2+y^2+z^2}\right) + \frac{\delta}{\delta y}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\delta}{\delta z}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$= \frac{-2x(x) + (x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} + \frac{-2y(y) + (x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$+ \frac{-2z(z) + (x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$= \left(-\cancel{x^2} + y^2 + z^2 - \cancel{y^2} + x^2 + \cancel{z^2} - \cancel{z^2} + \cancel{x^2} + \cancel{y^2}\right) / (x^2+y^2+z^2)^2$$

$$= \frac{(x^2+y^2+\cancel{z^2})}{(x^2+y^2+z^2)^{\cancel{2}}} = \frac{1}{(x^2+y^2+z^2)} = \frac{1}{r^2}$$

$$\therefore \nabla\cdot\left(\frac{\bar{r}}{r^2}\right) = \frac{1}{r^2} \text{ //}$$

(b) (i) $\nabla\cdot(r^n\cdot\bar{r}) = (n+3)\, r^n$ (ii) $\nabla\times(r^n\cdot\bar{r}) = \bar{0}$.

$$\left(\bar{i}\frac{d}{dx} + \bar{j}\frac{d}{dy} + \bar{k}\frac{d}{dz}\right)\left(\sqrt{x^2+y^2+z^2}\right)^n (x\bar{i} + y\bar{j} + z\bar{k})$$

$$= \frac{d}{dx}\left(x(x^2+y^2+z^2)^{n/2}\right) + \frac{d}{dy}\left(y(x^2+y^2+z^2)^{n/2}\right)$$

$$+ \frac{d}{dz}\left(z(x^2+y^2+z^2)^{n/2}\right)$$

$$= x\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x) + (x^2+y^2+z^2)^{n/2}(1)$$

$$+ \frac{n}{2}y(x^2+y^2+z^2)^{\frac{n}{2}-1}(2y) + (x^2+y^2+z^2)^{n/2}$$

$$+ \frac{n}{2}z(x^2+y^2+z^2)^{n/2-1}(2z) + (x^2+y^2+z^2)^{n/2}$$

$$= (x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot n \cdot (x^2+y^2+z^2) + 3(x^2+y^2+z^2)^{n/2}$$

$$= n \cdot (x^2+y^2+z^2)^{n/2} + 3(x^2+y^2+z^2)^{n/2}$$

$$= (n+3)(x^2+y^2+z^2)^{n/2}$$

$$\boxed{\therefore \nabla \cdot (r^n \bar{r}) = (n+3)\, r^n.}$$

(ii) $\nabla \times (r^n \bar{r}) = \bar{0}$

$$\nabla \times (r^n \bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2+y^2+z^2)^{n/2} & y(x^2+y^2+z^2)^{n/2} & z(x^2+y^2+z^2)^{n/2} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}\left(z(x^2+y^2+z^2)^{n/2}\right) - \frac{\partial}{\partial z}\left(y(x^2+y^2+z^2)^{n/2}\right)\right)$$

$$-\bar{j}\left(\frac{\partial}{\partial x}\left(z(x^2+y^2+z^2)^{n/2}\right) - \frac{\partial}{\partial z}\left(x(x^2+y^2+z^2)^{n/2}\right)\right)$$

$$+\bar{k}\left(\frac{\partial}{\partial x}\left(y(x^2+y^2+z^2)^{n/2}\right) - \frac{\partial}{\partial y}\left(x(x^2+y^2+z^2)^{n/2}\right)\right)$$

$$= \bar{i}\left(\frac{n}{\cancel{2}} z(x^2+y^2+z^2)^{\frac{n}{2}-1}(\cancel{2}y) - \frac{n}{\cancel{2}} y(x^2+y^2+z^2)^{\frac{n}{2}-1}(\cancel{2}z)\right)$$

$$-\bar{j}\left(\frac{n}{\cancel{2}} z(x^2+y^2+z^2)^{\frac{n}{2}-1}(\cancel{2}x) - \frac{n}{\cancel{2}} x(x^2+y^2+z^2)^{\frac{n}{2}-1}(\cancel{2}z)\right)$$

$$+\bar{k}\left(\frac{n}{\cancel{2}} y(x^2+y^2+z^2)^{\frac{n}{2}-1}(\cancel{2}x) - \frac{n}{\cancel{2}} x(x^2+y^2+z^2)^{\frac{n}{2}-1}(\cancel{2}y)\right)$$

$$= \bar{i}(0) - \bar{j}(0) + \bar{k}(0) = \vec{0}.$$

$$\therefore \nabla \times (r^n \bar{r}) = \vec{0}.$$

(c) $\nabla\left[\nabla\cdot\frac{\bar{r}}{r}\right]$

$$\nabla\cdot\frac{\bar{r}}{r} = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x\bar{i}+y\bar{j}+z\bar{k}}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\left(-\frac{1}{2}(x)(2x)+(x^2+y^2+z^2)+\left(-\frac{1}{2}y\right)(2y)+(x^2+y^2+z^2)+\frac{1}{2}z(2z)+(x^2+y^2+z^2)\right)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{-(x^2+y^2+z^2)+3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$\therefore \nabla\cdot\frac{\bar{r}}{r} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{(x^2+y^2+z^2)^{1/2}}$$

$$\nabla\left(\nabla\cdot\frac{\bar{r}}{r}\right) = 2\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$= 2\left(i\frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)^{1/2}}\right)+j\frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2+z^2)^{1/2}}\right)+k\frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2+z^2)^{1/2}}\right)\right)$$

$$= 2\left[i\left(\frac{-\frac{1}{2}(2x)}{(x^2+y^2+z^2)^{3/2}}\right)+j\left(\frac{-\frac{1}{2}(2y)}{(x^2+y^2+z^2)^{3/2}}\right)+\bar{k}\left(\frac{-\frac{1}{2}(2z)}{(x^2+y^2+z^2)^{3/2}}\right)\right]$$

$$= -2\left[\frac{xi+yj+zk}{(x^2+y^2+z^2)^{3/2}}\right] = -\frac{2\bar{r}}{r^3}.$$

$$\therefore \nabla\cdot\left[\nabla\cdot\frac{\bar{r}}{r}\right] = \frac{-2\bar{r}}{r^3}$$

LINEAR ALGEBRA. (PART - II)

(i) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$

The matrix $H = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$

is 4×4 matrix. Use Gauss-Jordan elimination to compute $K = H^{-1}$. then K_{44} is (exactly) ______.

Now, create a new matrix H' by replacing each entry in H by its approximation to 3 decimal places. (For example $\frac{1}{6}$ by 0.167). Use Gauss Jordan elimination to find the inverse of K' of H'. then K'_{44} is ______.

Sol: Let the given matrix is A, Inverse of $A = B$

We know that $A^{-1}A = I$

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/4 & 1/3 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

First rows of A^{-1} also remains as it is.

first row of $A^{-1} = [1\ 0\ 0\ 0]$

Let 2^{nd} row of A^{-1} be $[a\ b\ c\ d]$

then
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & b & c & d \\ \vdots & - & - & - \\ \vdots & - & \cdot & \cdot \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$a + \frac{b}{4} + \frac{c}{3} + \frac{d}{2} = 0$$

$$b + \frac{c}{3} + \frac{d}{2} = 1$$

$$c + \frac{d}{2} = 0$$

$$d = 0$$

$\therefore$ $d = 0$, $c = 0$, $b = 1$ & $a = -\frac{1}{4}$

Thus, 2nd of A^{-1} $\begin{bmatrix} -\frac{1}{4} & 1 & 0 & 0 \end{bmatrix}$

Similarly, 3rd row of $A^{-1} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \end{bmatrix}$

4th row of $A^{-1} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}$

Combining everything $B = A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ -1/4 & -1/3 & 1 & 0 \\ -1/4 & -1/3 & -1/2 & 1 \end{bmatrix}$

By Gauss Jordan

$$\begin{array}{c} \begin{array}{cc} A & I \end{array} \\ \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$R_2 \to R_2 - \frac{R_1}{4}$, $R_3 \to R_3 - \frac{R_1}{3}$, $R_4 \to R_4 - \frac{R_1}{2}$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 1/4 & 1 & 0 & -1/3 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 1 & -1/2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - \frac{R_2}{3}$, $R_4 \to R_4 - \frac{R_2}{3}$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & -1/3 & 1 & 0 \\ 0 & 0 & 1/2 & 1 & -3/8 & -1/2 & 0 & 1 \end{array}\right]$$

$$R_4 \to R_4 - \frac{R_3}{2}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & -1/3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1/4 & -1/3 & -1/2 & 1 \end{array}\right]$$

$$\quad I \qquad\qquad A^{-1}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ -1/4 & -1/3 & 1 & 0 \\ -1/4 & -1/3 & -1/2 & 1 \end{bmatrix}.$$

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

Now, to find $K = H^{-1}$, we have to perform Gauss Jordan method.

$$\left[\begin{array}{cccc|cccc} 1 & 1/2 & 1/3 & 1/4 & 1 & 0 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 1/5 & 0 & 1 & 0 & 0 \\ 1/3 & 1/4 & 1/5 & 1/6 & 0 & 0 & 1 & 0 \\ 1/4 & 1/5 & 1/6 & 1/7 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - \frac{R_1}{2}, \quad R_3 \to R_3 - \frac{R_1}{2}, \quad R_4 \to R_4 - \frac{R_1}{2}$$

$$\left[\begin{array}{cccc|cccc} 1 & 1/2 & 1/3 & 1/4 & 1 & 0 & 0 & 0 \\ 0 & 1/12 & 1/12 & \frac{3}{40} & -1/2 & 1 & 0 & 0 \\ 0 & 1/12 & 4/45 & 1/12 & -1/3 & 0 & 1 & 0 \\ 0 & 3/40 & 1/12 & 9/12 & -1/4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to 12R_2$

$$\left[\begin{array}{cccc|cccc} 1 & 1/2 & 1/3 & 1/4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 9/10 & -6 & 12 & 0 & 0 \\ 0 & 1/12 & \frac{4}{45} & 1/12 & -1/3 & 0 & 1 & 0 \\ 0 & 3/40 & 1/12 & 9/112 & -1/4 & 0 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1 - \frac{R_2}{2}$; $R_3 \to R_3 - \frac{R_2}{12}$; $R_4 \to R_4 - \frac{3R_2}{40}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1/6 & -1/5 & 4 & -6 & 0 & 0 \\ 0 & 1 & 1 & 9/10 & -6 & 12 & 0 & 0 \\ 0 & 0 & 1/180 & 1/120 & 1/6 & -1 & +1 & 0 \\ 0 & 0 & 1/120 & 9/700 & 1/5 & -9/10 & 0 & 1 \end{array}\right]$$

$R_3 \to 180 R_3$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1/6 & -1/5 & 4 & -6 & 0 & 0 \\ 0 & 1 & 1 & 9/10 & -6 & 12 & 0 & 0 \\ 0 & 1 & 1 & 3/2 & 30 & -180 & 180 & 0 \\ 0 & 0 & 1/120 & 9/700 & 1/5 & -9/10 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_3$, $R_1 \to R_1 + \frac{R_3}{6}$, $R_4 \to R_4 - \frac{R_3}{120}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1/20 & 9 & -36 & 30 & 0 \\ 0 & 1 & 0 & -3/5 & -36 & 192 & -180 & 0 \\ 0 & 0 & 1 & 3/2 & 30 & -180 & 180 & 0 \\ 0 & 0 & 0 & 1/2800 & -1/20 & 3/5 & -3/2 & 1 \end{array}\right]$$

$R_4 \to R_4 \times 2800$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1/20 & 9 & -36 & 30 & 0 \\ 0 & 1 & 0 & -3/5 & -36 & 192 & -180 & 0 \\ 0 & 0 & 1 & 3/2 & 30 & -180 & 180 & 0 \\ 0 & 0 & 0 & 1 & -140 & 1680 & -4200 & 2800 \end{array}\right)$$

$R_1 \to R_1 - \frac{R_4}{20}$, $R_2 \to R_2 + \frac{3R_4}{5}$, $R_3 \to R_3 - \frac{3}{2} R_4$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 16 & -120 & 240 & -140 \\ 0 & 1 & 0 & 0 & -120 & 1200 & -2700 & 1680 \\ 0 & 0 & 1 & 0 & 240 & -2700 & 6480 & -4200 \\ 0 & 0 & 0 & 1 & -140 & 1680 & -4200 & 2800 \end{array}\right)$$

$$\boxed{\therefore K_{4\times4} = \begin{bmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{bmatrix}}$$

$$H' = \begin{bmatrix} 1 & 0.5 & 0.333 & 0.25 \\ 0.5 & 0.333 & 0.25 & 0.2 \\ 0.333 & 0.25 & 0.2 & 0.169 \\ 0.25 & 0.2 & 0.167 & 0.142 \end{bmatrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0.5 & 0.333 & 0.25 & 1 & 0 & 0 & 0 \\ 0.5 & 0.333 & 0.25 & 0.2 & 0 & 1 & 0 & 0 \\ 0.333 & 0.25 & 0.2 & 0.167 & 0 & 0 & 1 & 0 \\ 0.25 & 0.2 & 0.167 & 0.142 & 0 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \to R_2 - 0.5R_1$; $R_3 \to R_3 - 0.33 R_1$; $R_4 \to R_4 - 0.25 R_1$

$$\left[\begin{array}{cccc|cccc} 1 & 0.500 & 0.333 & 0.250 & 1 & 0 & 0 & 0 \\ 0 & 0.083 & 0.0835 & 0.075 & -0.5 & 1 & 0 & 0 \\ 0 & 0.0835 & 0.0891 & 0.0837 & -0.333 & 0 & 1 & 0 \\ 0 & 0.075 & 0.0837 & 0.0803 & -0.25 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to 0.83R_1 - 0.5R_2$; $R_3 \to 0.083R_3 - 0.0835 R_2$

$R_4 \to 0.083 R_4 - 0.075 R_2.$

$$\left[\begin{array}{cccc|cccc} 0.083 & 0 & -0.01411 & -0.0167 & 0.333 & -0.5 & 0 & 0 \\ 0 & 0.083 & 0.0835 & 0.075 & -0.5 & 1 & 0 & 0 \\ 0 & 0 & 4.239\times10^{-4} & 6.887\times10^{-4} & 0.01411 & -0.083 & 0.083 & 0 \\ 0 & 0 & 6.887\times10^{-4} & 1.056\times10^{-4} & 0.0167 & -0.075 & 0 & 0.08 \end{array}\right]$$

$R_1 \to 4.23963\times10^{-4} R_1 + 0.01411 R_3$

$R_2 \to 4.23963\times10^{-4} R_2 - 0.0835 R_3$

$R_4 \to 4.23963\times10^{-4} R_4 - 0.68875\times10^{-4} R_3$

$$\left[\begin{array}{cccc} 3.5188929\times10^{-5} & 0 & 0 & 2.61757\times10^{-5} \\ 0 & 3.5188929\times10^{-5} & 0 & -2.57134\times10^{-5} \\ 0 & 0 & 4.2396\times10^{-4} & 6.8875\times10^{-4} \\ 0 & 0 & 0 & -2.6459\times10^{-8} \end{array}\right.$$

$$\left|\begin{array}{cccc} 3.403\times10^{-4} & -1.30925\times10^{-3} & 1.712\times10^{-3} & 0 \\ -1.39025\times10^{-3} & 7.3962\times10^{-3} & -6.93\times10^{-3} & 0 \\ 0.014111 & -0.0835 & +0.083 & 0 \\ -2.6175\times10^{-6} & 2.57113\times10^{-5} & -5.7166\times10^{-5} & 3.51\times10^{-5} \end{array}\right]$$

$R_1 \to -2.6459\times10^{-8} R_1 - 2.61751\times10^{-6} R_4$

$R_2 \to -2.6459\times10^{-8} R_1 + 2.57134\times10^{-5} R_4$

$R_3 \to -2.645\times10^{-8} R_3 - 6.8875\times10^{-4} R_4$

$$\begin{bmatrix} -9.3106\,9943\times10^{-13} & 0 & 0 & 0 \\ 0 & -9.3106\times10^{-13} & 0 & 0 \\ 0 & 0 & -1.12179\times10^{-11} & 0 \\ 0 & 0 & 0 & -2.6459653\times10^{-8} \end{bmatrix}$$

$$\left[\begin{array}{cccc} 2.64\times10^{-12} & -3.052N175\times10^{-11} & -1.18\times10^{-10} & -9.21\times10^{-10} \\ -3.052\times10^{-9} & 4.6\times10^{-10} & -1.28\times10^{-9} & 9.04\times10^{-10} \\ 1.429478\times10^{-9} & 3.08\times10^{-8} & -1.55\times10^{-8} & -2.4236\times10^{-8} \\ -2.6117571\times10^{-6} & 2.57\times10^{-5} & -5.71\times10^{-5} & 3.518\times10^{-5} \end{array}\right]$$

$R_1 \to \dfrac{R_1}{-9.31\times10^{-13}}$ $\quad R_2 \to \dfrac{R_2}{-9.31\times10^{-13}}$ $\quad R_3 \to \dfrac{R_3}{-1.12179\times10^{-11}}$ $\quad R_4 \to \dfrac{R_4}{-2.64\times10^{-8}}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2.8418 & 32.7886 & -127.4306 & 98.9286 \\ 0 & 1 & 0 & 0 & 32.7806 & -499.9384 & 1381.8081 & -971.8142 \\ 0 & 0 & 1 & 0 & -127.4283 & 1381.7830 & -3314.0835 & 2160.5064 \\ 0 & 0 & 0 & 1 & 98.9268 & -971.7965 & 2160.5064 & -1329.9089 \end{array}\right]$$

$\therefore K'_{4\times4} = \uparrow$

2) Prove that rank A^TA = rank A for any $A \in M_{n\times n}$.

Soln: $A_{m\times n}$, $A^T_{n\times m}$

$A^TA \to n\times n$

let $x \in N(A)$ where $N(A)$ is null space of A.

So, $Ax = 0$.

$\therefore A^TA\,x = 0$ [$\because$ multiplying by A^T on both sides]

$\therefore x \in N(A^TA)$

Hence, $N(A) \subseteq N(A^TA) \to ①$

$A^TAx = 0$.

$A^T A x = 0.$

$x^T A^T A. x = 0$ [$\because$ Multiplying both sides by x^T]

$(Ax)^T . Ax = 0$

$A \quad \overline{x}$

$Ax = 0$

$x \in N(A)$

Hence, $N(A^TA) \subseteq N(A) \rightarrow$ (2).

from (1) & (2);

$N(A^TA) = N(A)$

$\dim(N(A^TA)) = \dim(N(A))$

$\text{rank}(A^TA) = \text{rank}(A)$

Hence, proved.

3) It is given that $\{u, v, w\}$ be a linearly independent set in a vector space V. Use the definition of linear independence to give a careful proof that the set $\{u+v, u+w, v+w\}$ is linearly independent in V.

Soln: As, u, v, w are linearly independent.

$\therefore c_1 u + c_2 v + c_3 w = 0.$ if $c_i = 0, i = 1, 2, 3.$

we have to prove

$\{u+v, v+w, u+w\}$ are linearly independent.

Let's assume these vectors are linearly independent.

let, $a(u+v) + b(v+w) + c(u+w) = 0$

$(a+c)u + (a+b)v + (b+c)w = 0$

Since, u, v, w are linearly independent,

$a+c = 0, \quad a+b = 0, \quad b+c = 0.$

$a = -c, \quad b = c \quad 2c = 0 \quad \boxed{c = 0}.$

$\therefore \boxed{a = 0} \quad \boxed{b = 0}$

i.e., $a(u+v) + b(v+w) + c(u+w) = 0$, is true is

$a=0,\ b=0,\ c=0.$

$\therefore$ They satisfy the condition of linearly Independent Vectors, $\therefore u+v,\ v+w,\ u+w$. are also linearly independent.

Hence, proved.

(4) The system of equations.

$$\begin{cases} 2y + 3z = 7 \\ x + y - z = -2 \\ -x + y - 5z = 0. \end{cases}$$

is solved by applying Gauss-Jordan reduction to the augumented coeffecient matrix.

$$A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & -1 & -2 \\ -1 & 1 & -5 & 0 \end{bmatrix}.$$ Give the names of the elementary 3×3 matrices $x_1 \cdots x_r$

which implement the following reduction

$A \xrightarrow{R_1 \to R_2} B$

Sol:

$$\underset{E}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & -1 & -2 \\ -1 & 1 & -5 & 0 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 & 4 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ -1 & 1 & 5 & 0 \end{bmatrix}}$$

& $x_1 = E_1$

$$\boxed{x_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

(ii) $B \xrightarrow{R_3 \to R_3 + R_1} C$

$$\underset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ -1 & 1 & -5 & 0 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & -6 & -2 \end{bmatrix}}$$

& $x_2 = E_2$

$$\boxed{x_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$$

(iii) $C \xrightarrow{R_3 \to R_3 - R_2} D$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & -6 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & -9 & -9 \end{bmatrix}$$

$\therefore \quad x_3 = E_3$

$$\boxed{x_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$$

(iv) $D \xrightarrow{R_3 \to -R_3} E$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{9} \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & -9 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x_4 = E_4$

$$\boxed{x_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/9 \end{bmatrix}}$$

(v) $E \xrightarrow{R_2 \to R_2 - 3R_3} F$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}}_{E_5} \underbrace{\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{F}$$

$x_5 = E_5$

$$\boxed{x_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}}$$

(vi) $F \xrightarrow{R_2 \to R_2/2} G$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_6} \underbrace{\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{F} = \underbrace{\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{G}$$

$X_6 = E_6$

$$\boxed{X_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

(vii) $G \xrightarrow{R_1 \to R_1 + R_3} H$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_7} \underbrace{\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{G} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{H}$$

$X_7 = E_7$

$$\boxed{X_7 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

(viii) $H \xrightarrow{R_1 \to R_1 - R_2} I$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_8} \underbrace{\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{H} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{I}$$

$X_8 = E_8$

$$\boxed{X_8 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

5) Prove that the vectors $(1,1,0)$, $(1,2,3)$ and $(2,-1,5)$ form a basis for $\mathbb{R}^3$.

Sol:

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad w = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

let, $$c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-\left(\frac{c_1 + c_2}{2}\right) = c_3 \rightarrow (1)$$

$$c_1 + 2c_2 = +c_3 \rightarrow (2)$$

$$3c_2 = -5c_3 \rightarrow (3)$$

from (1) and (2)

$$\frac{-3c_1}{5} = c_2 \rightarrow (4)$$

Sustituting 4 in (1),

$$c_1 = -5c_3 \rightarrow (5)$$

from (3) $\frac{-3c_2}{5} = c_3$ and (4)

$$c_1 = \frac{25c_3}{9} \rightarrow (6)$$

But 5 & 6 are possible if $c_1 = 0$.

If $c_1 = 0$ then $c_3 = 0$, from 5 & 6 itself.

From (1)

if $c_1 = c_3 = 0$ then $c_2 = 0$

and if $c_i = 0$; $i = 1, 2, 3$, Hence the vectors are linearly independent as per the definition

And if these vectors are linearly independent in R^3, then it means that it spans R^3.

thus, u, v, w are

(i) linearly independent

(ii) spans R^3.

Therefore, u, v, w form a basis in R^3

[∵ Clear Explaination,

$$c_1 + c_2 = -2c_3 \rightarrow (1)$$

$$c_1 + 2c_2 = c_3 \rightarrow (2)$$

$$3c_2 = -5c_3 \rightarrow (3)$$

Subtract (1) - (2), we get $-c_2 = -3c_3$

$$\boxed{c_2 = 3c_3} \rightarrow (4)$$

Now, multiply eq (4) with 3. $3c_2 = 9c_3 \rightarrow (5)$

Solve eq (5) and eq (3).

$$\begin{aligned} \cancel{3c_2} - 9c_3 &= 0 \\ \cancel{3c_2} + 5c_3 &= 0 \\ - \quad - \quad & \\ \hline -14c_3 &= 0 \end{aligned}$$

$$\boxed{c_3 = 0}$$

Substituting $c_3 = 0$ in eq (3), we get $\boxed{c_2 = 0}$.

Then, substituting it in eq (1), we get $c_1 = 0$.

Therefore, $c_1, c_2, c_3 = 0$.