Maths - II : Assignment - 2 :

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Vector Calculus :

Verify Stoke's theorem for the vector function $F = y^2 i - (x+z) j + yz k$ and unit square $0 \le x \le 1$, $0 \le y \le 1$, $z = 0$.

Given $F = y^2 i - (x+z) j + yzk$

unit square $0 \le x \le 1$; $0 \le y \le 1$; $z = 0$.

stoke's theorem :

$$\int_c F.dr = \int_s \text{curl} F . N ds.$$

$$\text{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -(x+z) & yz \end{vmatrix}$$

$$= i(z+1) - j(0) + k(-2y-1)$$

$\because z = 0$.

$$= i - k(2y+1)$$

$N = \hat{k}$ $\quad ds = \frac{dx.dy}{1}$.

$$RHS = \int_s [i - (2y+1) k] . k . dx.dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} -(2y+1) dx\, dy = -\int_0^1 (y^2+y)\Big|_0^1 dx$$

$$= -2.$$

$LHS = \oint F.dR.$

$= \oint (y^2 i - (x+z) j + yz k)(dx\, i + dy\, j) \quad \because z=0$

$= \oint_0^1 y^2 dx - x\, dy$

when, $x=1$, $= \oint_0^1 y^2(0) - (1)dy = -1$

when $x=0$, $\oint_1^0 y^2 dx - x\, dy = 0$

when $y=0$ $\oint_0^1 y^2 dx - x\, dy = 0$

when $y=1$ $\oint_1^0 y^2 dx - x\, dy = -1$

Sum of the contributions i.e., $0-1-1+0 = -2$

$\therefore$ LHS = RHS.

It is verified.

Verify stoke's theorem for the field $F = x^2 i + 2xj + z^2 k$ on the ellipse $S = \{(x,y,z) : 4x^2+y^2 \le 4, z=0\}$.

Given $F = x^2 i + 2xj + z^2 k$.

Ellipse $S = \{(x,y,z) : 4x^2+y^2 \le 4, z=0\}$

Stokes theorem :

$\int_C F\,dR = \int curl\, F . N\, ds.$

$$curl\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 2k \qquad N = \hat{k}.$$

$ds = \frac{dx\,dy}{N \cdot k} = \frac{dx\,dy}{1}$

$RHS = \int_S (2k) k . dx\, dy$

$= 2\int_S dx\, dy \to$ [area of ellipse $= \pi ab$]

$\frac{x^2}{1} + \frac{y^2}{4} = 1$

$= 2.(2)\pi.(1)$

$= 4\pi$

$a=1 \quad b=2$

$LHS = \oint x^2 dx + 2x\,dy$

$x = \cos\theta \quad y = 2\sin\theta$

$dx = -\sin\theta\,d\theta \quad dy = 2\cos\theta\,d\theta$

$= -\int_0^{2\pi} \cos^2\theta \sin\theta\,d\theta + 2\int_0^{2\pi} 2\cos^2\theta\,d\theta$

$= 2\int_0^{2\pi} [1+\cos 2\theta]\,d\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$

$= 2[2\pi + 0] = 4\pi.$

$\therefore LHS = RHS.$

It is verified //

3) Use the divergence theorem to evaluate the surface integral $\iint F.N\,ds$ where $F = (x+y)i + z^2 j + x^2 k$ and S is the surface of the hemisphere $x^2+y^2+z^2=1$ with $z>0$ and n is outward normal to S.

Sol: $F = (x+y)i + z^2 j + x^2 k.$

$S = x^2+y^2+z^2=1 \quad$ where $z>0,$

Divergence theorem :

$\iint F.N\,ds = \int div\,F.\,dv$

$div\,F = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(x^2)$

$= 1 + 0 + 0 = 1.$

$\iint F.N\,ds = \int div\,F.\,dv = \int dv.$

$= \int dx.dy.dz$

$=$ Volume of hemisphere

$= \frac{2}{3}\pi r^3 = \frac{2\pi(1)^3}{3} = \frac{2\pi}{3}$ //

$\boxed{\therefore \iint F.N\,ds = \frac{2\pi}{3}}$

4) Verify the divergence theorem for the vector field $F = x^2 i + x^2 y j - x^2 z k$ and the surface S which is tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$

Sol: Given $F = x^2 \hat{i} + x^2 y \hat{j} + (-x^2 z)\hat{k}$.

S is a tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$

Divergence theorem:

$$\int F.N\,ds = \int \text{div } F.\,dR.$$

$$RHS = \int \text{div}.F.dR$$

$$\text{div } F = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(-x^2 z)$$

$$= 2x + x^2 - x^2 = 2x$$

$$= \int \text{div } F = 2\int x.\,dx.dy.dz$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} x\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^{1-x} x[1-x-y]\,dx\,dy$$

$$= 2\int_0^1 \left[xy - x^2 y - x\frac{y^2}{2}\right]_0^{1-x} dx.$$

$$= \cancel{2}\int_0^1 \left(\frac{x^3 + x - 2x^2}{\cancel{2}}\right) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^2}{2} - \frac{2x^3}{3}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{2}{3} = \frac{3}{4} - \frac{2}{3} = \frac{9-8}{12}$$

$$RHS = \frac{1}{12}.$$

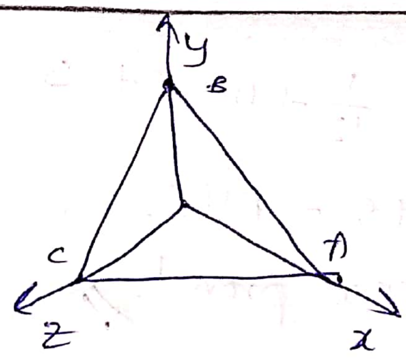
LHS = $\int F \cdot N\, ds$

$S_1 = \Delta^{le}$ on xy

$S_2 = \Delta^{le}$ on yz

$S_3 = \Delta^{le}$ on zx

$S_4 = \Delta^{le}$ ABC



for S_1 OAB outward normal is $-\hat{k}$.

$I_1 = \int x^2 z\, dx\, dy$ $\because$ $z=0$ on x-y plane

$I_1 = 0$

for S_2 OBC outward normal is $-\hat{i}$.

$I_2 = \int x^2 dy\, dz$ $\because x=0$ on y-z plane

$I_2 = 0$

Similary $I_3 = \int -x^2 y\, dx\, dz$

$I_3 = 0$ $y=0$ on x-z plane

$\therefore$ for S_4 normal of the plane $x+y+z=1$

Unit normal $= \dfrac{i+j+k}{\sqrt{3}}$

Projection on x-y plane $ds = \dfrac{dx\, dy}{N \cdot k}$

$$I_4 = \int_S (x\hat{i} + x^2 y\hat{j} - x^2 z\hat{k}) \cdot \frac{i+j+k}{\sqrt{3}} \cdot \frac{dx\, dy}{\frac{1}{\sqrt{3}}}$$

$$= \int_S (x^2 + x^2 y - x^2 z)\, dx\, dy \qquad z = 1-x-y$$

$$= \int_0^1 \int_0^{1-x} x^2 + x^2 y + x^2[1-x-y]\, dx\, dy$$

$$= \int_0^1 x^2[1-x]\, dx$$

$$= \int_0^1 [x^2 - x^3]\, dx = \left(\frac{x^3}{3}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$LHS = \frac{1}{12} + 0 + 0 + 0 \quad [\because I_1 + I_2 + I_3 + I_4]$

LHS = RHS.

Hence proved. //

LINEAR ALGEBRA :-

Q1) Diagonalize the matrix.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

by finding the eigen values of A listed in increasing order, the corresponding eigen vectors, a diagonal matrix D, and a matrix S such that $A = SDS^{-1}$.

Sol: characteristic equation $|A - \lambda I| = 0$

$$(1-\lambda)(\lambda^2 - 2\lambda + 1 - 4) = 0$$

$$(1-\lambda)(\lambda - 3)(\lambda + 1) = 0$$

eigen values are $-1, 1, 3$.

Corresponding eigen vectors $(A - \lambda I)x = 0$

$\rightarrow \lambda = -1$

$$A - \lambda I = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\rightarrow \lambda = 3$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow \lambda = 1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} +1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

2) Identify which of the following matrices have two linearly independent eigenvectors.

Sol:

① $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A - \lambda I = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$

$\det(A - \lambda I) = 0 \qquad (1-\lambda)^2 = 0 \qquad | \quad V = (1,1)$

$\lambda = 1$

$\therefore$ It has only one eigenvector.

② $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow B - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (1-\lambda)^2 = 0 \qquad | \quad V = (1,0)$

$\lambda = 1$

$\therefore$ It has only one eigen vector

③ $C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow (C - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$

$\det(C - \lambda I) = 0 \qquad (1-\lambda)(2-\lambda) = 0$

$\lambda = 1 \;\&\; \lambda = 2 \qquad | \quad V_1 = (1,0)$

$\therefore$ It has two independent eigen vectors $\qquad | \quad V_2 = (1,1)$

4) $D = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow D - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}$

$\det(D - \lambda I) = 0 \qquad (1-\lambda)(2-\lambda) - 2 = 0$

$\cancel{2} - \lambda - 2\lambda + \lambda^2 - \cancel{2} = 0$

$\lambda^2 - 3\lambda = 0$

$\lambda(\lambda - 3) = 0.$

$\lambda = 0, \lambda = 3$

∴ It has two independent eigen vectors

$v_1 = (-1, 1), \quad v_2 = (1/2, 1)$

5) $E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow (E - \lambda I) = \begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}$

$\det(E - \lambda I) = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0$

∴ It has only one eigen vector $v = (0,1)$.

6) $F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow F - \lambda I = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}$

$\det(F - \lambda I) = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0$

~~$v = (0,1)$~~ ∴ It has infinite eigen vectors.

7) $G = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow G - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & -\lambda \end{bmatrix}$

$\lambda(\lambda - 1) - 4 = 0$

$\lambda^2 - \lambda - 4 = 0$

∴ It has two similar eigen values

8) $H = \begin{bmatrix} 3 & 0 \\ 1 & -3 \end{bmatrix} \Rightarrow H - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 1 & -3-\lambda \end{bmatrix}$

$\det(H - \lambda I) = (3-\lambda)(-3-\lambda) \qquad \lambda = 3, \lambda = -3$

$\therefore$ It has 2 independent eigen vectors

$\lambda = 3$, $\lambda = -3$; $V_1 = (6, 1)$, $V_2 = (0, 1)$

$\therefore$ The matrix C, D and H have 2 independent eigen vectors.

3) Find an orthogonal matrix R that diagonalizes

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Sol: Given matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

A is a symmetric matrix.

Now, $A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0$ $\quad (1-\lambda)^2(2-\lambda) - 1(2-\lambda) = 0$

$(2-\lambda)(1+\lambda^2-2\lambda-1) = 0$

$(\lambda-2)(\lambda)(\lambda-2) = 0$

$\lambda = 2, \lambda = 0.$

Now, $\lambda = 0$.

$$A - \lambda I = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[\text{elimination}]{\text{By}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z = 0$, $y = 1$ and $x = 1$

$V_1 = (1, 1, 0)$

Now, $\lambda = 2$,

$$A - \lambda I = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Elimin--}} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We, get two vectors, they are

$v_1 = (-1, 1, 0)$ and $v_2 = (0, 0, 1)$

Now, to get unit vectors, we divide by its magnitude

$$v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad v_2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$v_3 = (0, 0, 1)$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{+1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = R$$

$$\text{As } Q^T = Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = R^{-1} \& R^T$$

$\therefore$ the matrix $R = Q$

4) Suppose that A is a 3×3 matrix with eigen values $\lambda_1 = -1$, $\lambda_2 = 0$ and $\lambda_3 = 1$, and corresponding eigenvectors

$$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

a) Find the matrix A.

b) Compute the matrix A^{20}.

a) $$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

$\therefore A = SDS^T$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

$$\boxed{\therefore A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ -4 & -4 & 1 \end{bmatrix}}$$

b) $A^{20} = (SDS^{-1})^{20} = \underbrace{(SDS^{-1}) \cdots\cdots (SDS^{-1})}_{20 \text{times}}$

$$= SD^{20}S^{-1}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} (-1)^{20} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1^{20} \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ //

5) Proof or counterexample. Here A is an $n\times n$ real symmetric matrix. If A is positive definite, then it is invertible.

Sol:- Given, A is a $n\times n$ real symmetric matrix

If A is positive definite

$x^TAx > 0$ for all $x \neq 0$.

Hence, $Ax \neq 0$ for all $x \neq 0$.

i.e, All the rows and columns are linearly Independent.

[$\because \det|A| \neq 0$]

i.e, A is a full rank matrix.

And when a matrix is full rank, it is invertible.

$\therefore$ When A is positive definite, it is invertible

[OR]

Given, A is an $n \times n$ real symmetric matrix

Now, let us consider A as positive definite

as it ~~is real~~ symmetric ~~matrix~~

→ Condition for positive definite is all eigen values i.e, λ greater than ($\lambda > 0$) zero.

Let's take $n=2$ $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

For a matrix all $\lambda > 0$, its pivots should be greater than zero.

$\therefore$ pivots of A are 'a' and '$ac - b^2$'

→ We know that $a > 0$ and $ac - b^2 > 0$. As per given statement, we can say that $\det|A| > 0$. which means it does not have any free column.

→ So, given $A_{n \times n}$ is positive definite, then A is invertible.

6) Show that if x is an eigen vector of A, then so is kx for any $k \neq 0$.

Sol) Characteristic eq is $(A - \lambda I)x = 0$

Also, $Ax = \lambda x$.

Now, multiplying each side with k.

$$A(kx) = \lambda(kx)$$

as kx also satisfies this condition

$$Ax' = \lambda x'$$

Here, kx is a eigen vector of (A)

$\therefore$ Hence, ~~vector~~ proved.

7) Show that if λ is an eigen value of A, then $k\lambda$ is an eigen value of kA $(k \neq 0)$

Sol) Consider $A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}_{n \times n}$.

eigen values are $a_1 \cdots a_n$ $\lambda = a_1$.

$kA = \begin{bmatrix} ka_1 & & & \\ & ka_2 & & \\ & & \ddots & \\ & & & ka_n \end{bmatrix}_{n \times n}$

eigen values are $ka_1, ka_2 \cdots ka_n$ $k\lambda = ka_1$

$\therefore$ Hence, proved.

8) Diagonalise $A = \begin{pmatrix} 1 & 1 \\ 0 & 1/2 \end{pmatrix}$ and use the diagonalisation to calculate A^{12}.

Sol: Characteristic equation is $|(A - \lambda I)| = 0$

$$\lambda = 1, \lambda = 1/2$$

eigen values are $1, \frac{1}{2}$

for $\lambda = 1$ $\begin{pmatrix} 0 & 1 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$x_2 = 0, \ x_1 = n$ $\begin{pmatrix} n \\ 0 \end{pmatrix}$

$\lambda = \frac{1}{2}$ $\begin{pmatrix} 1/2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 = -2x_1$ $m \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$ $S = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ $S^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

$A^{12} = S D^{12} S^{-1}$

$= \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2^{12}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} \frac{-1}{2^{11}} & 1 \\ \frac{1}{2^{12}} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-1}{2^{11}} + 2 \\ 0 & \frac{1}{2^{12}} \end{pmatrix}$

$A^{12} = \begin{pmatrix} 1 & \frac{2^{12}-1}{2^{11}} \\ 0 & \frac{1}{2^{12}} \end{pmatrix}$

4) Diagonalise the matrices below. If possible, orthogonally diagonalize

Sol: Given Matrix $A = \begin{bmatrix} 10 & 8 & 4 \\ 8 & 10 & 4 \\ 4 & 4 & 4 \end{bmatrix}$

As, A is symmetric, it is possible to diagonalise orthogonally.

$\therefore$ now, $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 10-\lambda & 8 & 4 \\ 8 & 10-\lambda & 4 \\ 4 & 4 & 4-\lambda \end{vmatrix} = 0$$

$(10-\lambda)\left[(10-\lambda)(4-\lambda) - 16\right] - 8\left[8(4-\lambda) - 16\right]$
$+ 4\,(32 - 4(10-\lambda)] = 0.$

$\Rightarrow (10-\lambda)^2(4-\lambda) - 160 + 16\lambda - 8\left[32 - 8\lambda - 16\right]$
$+ 4\left[32 - 40 + 4\lambda\right] = 0$

$= (10-\lambda)^2(4-\lambda) - 160 + 16\lambda - 128 + 64\lambda - 32 + 16\lambda = 0$

$= (10-\lambda)^2(4-\lambda) - 320 + 96\lambda = 0$

$= (100 + \lambda^2 - 20\lambda)(4-\lambda) + 96\lambda - 320 = 0$

$= 400 + 4\lambda^2 - 80\lambda - 100\lambda - \lambda^3 + 20\lambda^2 + 96\lambda - 320 = 0$

$= -\lambda^3 + 24\lambda^2 - 84\lambda + 80 = 0$

$= \lambda^3 - 24\lambda^2 + 84\lambda - 80 = 0$

$\therefore \lambda = 2, \quad \lambda = 2 \quad \& \ \lambda = 20.$

For $\lambda = 2$, Eigen vectors are $v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 20$, Eigen vector is $v_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$\therefore S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $\quad A^T A = 5$

$\therefore B = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 1 \\ -2/5 \end{bmatrix}$ $\quad A^T b = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 4$

$A^T C = 0 \quad B^T C = 0 \quad C = c$

$V_1 = A = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $\quad \therefore \|V_1\| = \sqrt{5}$

$V_2 = B = \begin{bmatrix} -4/5 \\ 1 \\ -2/5 \end{bmatrix}$ $\quad \therefore \|V_2\| = \frac{3}{\sqrt{5}}$

$V_3 = c = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ $\quad \therefore \|V_3\| = 3$

$$\therefore \boxed{R = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{1}{3} \end{bmatrix}}$$

$|R| = 1$

Here $R^{-1} = R^T$.

ii) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$(A - \lambda I) = 0$ $\quad \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$ $\quad (1-\lambda)^2 - 1 = 0$

$\lambda(\lambda - 2) = 0$

$\lambda = 0, \lambda = 2$

for $\lambda = 0$

$Ax = 0$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix}1\\0\end{bmatrix}x_1 + \begin{bmatrix}1\\0\end{bmatrix}x_2 = \begin{bmatrix}0\\0\end{bmatrix} \quad , \; x_2 = 1 , \; x_1 = -1$$

Eigen vector $= \begin{bmatrix}-1\\1\end{bmatrix}$.

for $\lambda = 2$

$$\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix} \rightarrow \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1\\0\end{bmatrix}x_1 + \begin{bmatrix}1\\0\end{bmatrix}x_2 = \begin{bmatrix}0\\0\end{bmatrix}$$ here x_2 is free variable

$x_2 = 1$

$x_1 = 1$

Eigen vectors $= \begin{bmatrix}1\\1\end{bmatrix}$

$D =$ diagonal matrix $= \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}$

$\boxed{A = SDS^{-1}}$

$S = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}$, $S^{-1} = \begin{bmatrix}-1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$.

Now, after normalization

$x_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$, $x_2 = \begin{bmatrix}1\\1\end{bmatrix} \times \frac{1}{\sqrt{2}}$

$$\boxed{P = \begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 1/\sqrt{2}\end{bmatrix}}$$

10) Compute the projection of the vector $v=(1,1,0)$ onto the plane $x+y-z=0$.

Sol: Given vector $V=(1,1,0)$ and plane is $S=x+y-z=0$

$\therefore$ Normal to plane is $\nabla S = i+j-k = \vec{n}$.

$\vec{v} = i+j$.

Now, we have to find projection of v onto S.

$\therefore$ Projection (onto plane) $= \vec{v} -$ Projection (onto normal of plane) $\rightarrow$ ①.

So, Projection onto normal of plane is given by

$$\vec{P_1} = \frac{\vec{v}.\vec{n}}{|\vec{n}|^2}.\vec{n}$$

$$= \frac{(i+j).(i+j-k)}{3} \times \vec{n}$$

$$= \frac{2}{3}(i+j-k)$$

Now, projection on plane is given by

$\vec{p} = \vec{v} - \vec{P_1}$ from eq ①.

$$\vec{p} = (i+j) - \frac{2}{3}i - \frac{2}{3}j + \frac{2}{3}k$$

$$\therefore \boxed{\vec{p} = \frac{1}{3}\bar{i} + \frac{1}{3}\bar{j} + \frac{2}{3}\bar{k}}$$

Projection of vector $(1,1,0)$ on plane $x+y-z=0$

is $\begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$.

11) Compute the projection matrix Q for the subspace W of R^4 spanned by the vectors $(1,2,0,0)$ and $(1,0,1,1)$

Sol:
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad A^TA = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$$

$$Q = A(A^TA^{-1})^{-1}A^T = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{3}{14} & \frac{-1}{14} \\ -1/14 & \frac{5}{14} \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{14} & \frac{4}{14} \\ \frac{6}{14} & \frac{-2}{14} \\ \frac{-1}{14} & \frac{5}{14} \\ \frac{-1}{14} & \frac{5}{14} \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Projection matrix Q

$$\boxed{\therefore Q = \frac{1}{14}\begin{bmatrix} 6 & 4 & 4 & 4 \\ 4 & 12 & -2 & -2 \\ 4 & -2 & 5 & 5 \\ 4 & -2 & 5 & 5 \end{bmatrix}}$$

12) Compute the Cholesky factorizations of the following matrices.

(a) $H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ce & d^2+e^2+f^2 \end{bmatrix}$

let $L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ $\boxed{H = L \times L^T}$

By simplifying, we get $a^2 = 2$ $a = \sqrt{2}$

$ab = 1$ $\boxed{b = \frac{1}{\sqrt{2}}}$

$ad = 0$ $\boxed{d = 0}$

$b^2 + c^2 = 2$

$\frac{1}{2} + c^2 = 2$

$c^2 = \frac{3}{2}$ $\boxed{c = \frac{\sqrt{3}}{\sqrt{2}}}$

$bd + ce = 1$

$0 + e\left(\frac{\sqrt{3}}{\sqrt{2}}\right) = 1$ $\boxed{e = \frac{\sqrt{2}}{\sqrt{3}}}$

$d^2 + e^2 + f^2 = 2$

$0 + \frac{2}{3} + f^2 = 2$

$f = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{1}{2} & \sqrt{3/2} & 0 \\ 0 & \sqrt{2/3} & \frac{2}{\sqrt{3}} \end{bmatrix}$

b) Here, $H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ce & d^2+e^2+f^2 \end{bmatrix}$

here also $L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$

→ Here, the matrix is not symmetric positive definite, so, cholesky factorization is not possible

(or)

we can check this,

$a^2 = 2$ → $a = \sqrt{2}$

$ab = 1$ $b = \frac{1}{\sqrt{2}}$

$b^2 + c^2 = 2$ $c = \sqrt{\frac{3}{2}}$

$ad = 0$ $\boxed{d = 0}$

$ce = 1$

$e = \sqrt{\frac{2}{3}}$

$e^2 + f^2 = -2$

Factorization is not possible is sum of squares cant be negative