M-3 :-

1) Bounded and Unbounded sets.
2) Lower and Upper bounds
3) Supremum, Infimum
4) Completeness / LUB axiom

→ Upper bound :- Let S be a subset of $\mathbb{R}$. If there exists a real number m such $m \geq s$ for all $s \in S$, then m is called an upper bound of S, we say that S is bounded above.

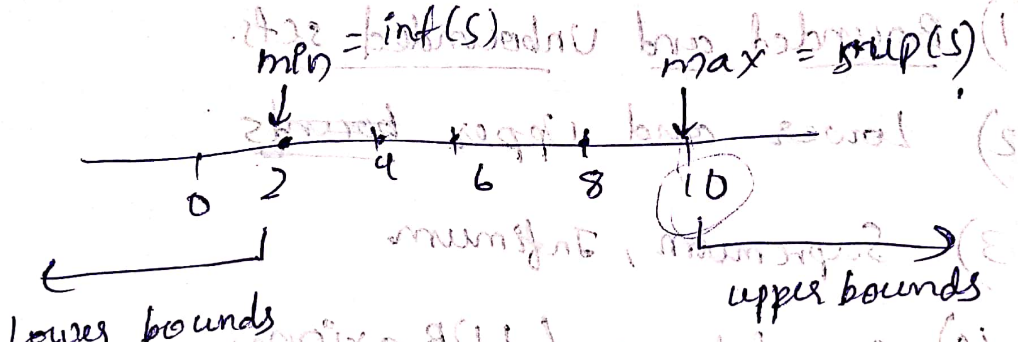
→ If $m \leq s$ for all $s \in S$, then m is a lower bound of S and S is bounded below.

→ If an upper bound m of S is a member of S, then m is called the maximum of S. we write :- $m = \max S$.

→ Similarly, lower bound of S is a member of S then it is called minimum of S.

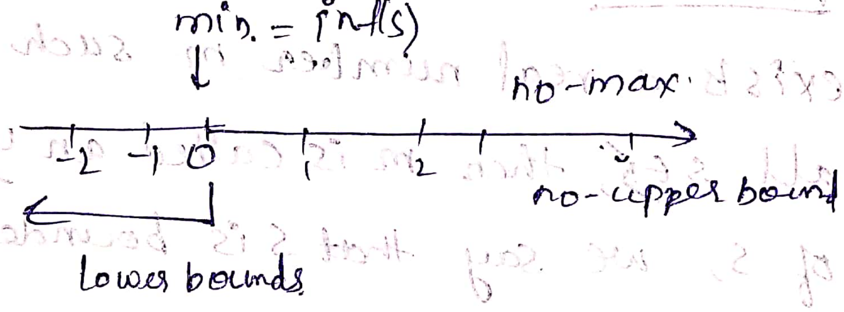
→ A set can have many upper and lower bounds but maximum and minimum are unique.

Ex:- Let $S = \{2, 4, 6, 8\}$; E-M



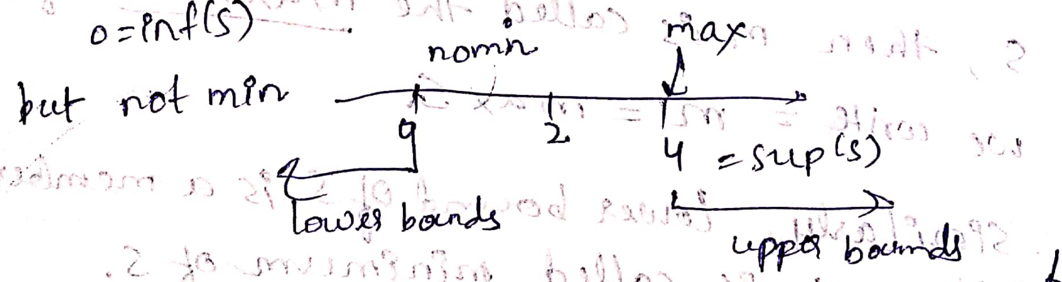
• Also, any finite set is bounded and always has a maximum and a minimum.

Ex:-



$S = [0, \infty)$

Ex:- $S = (0, 4]$



Empty set:- condition $m \geq s$ for all $s \in \phi$ is equivalent to the implication "if $s \in \phi$, then $m \geq s$". This implication is true since antecedent is false.

Likewise, ϕ is bounded by an real m.

Def 3.5 :-

S be a non empty subset of R. If S is bounded, least upper bound of S is called its supremum and is denoted by sup S. Then, $m = \sup S$ iff

(a) $m \geq s$ for all $s \in S$, m is upper bound of S

(b) if $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

Nothing smaller than m is upper bound of S

S is bounded below, then the greatest lower bound of S is called its infimum and is denoted by inf S.

Bounded above :- A sequence $\{s_n\}$ is said to be bounded above if there exists a real number k such that $s_n \leq k$ $\forall n \in N$.

Tutorial :-

Q) Prove that $2 + \frac{1}{\sqrt{2}}$ is irrational.

Sol :- First let us prove $\frac{1}{\sqrt{2}}$ is irrational.

Assuming $\frac{1}{\sqrt{2}}$ is rational.

So, $\frac{1}{\sqrt{2}} = \frac{a}{b}$ where $b \neq 0$, a, b are co-primes

$b^2 = 2a^2$

b is divisible by 2

So, $b = 2k$, $4k^2 = 2a^2$

$a^2 = 2k^2$

a is also divisible by

So, it is not rational, but irrational

So, a rational + irrational is always irrational,

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(2) Greatest lower bound = Infimum

Least upper bound = Supremum

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(3) Prove that set of natural numbers ~~are~~ is order complete.

→ ~~Prove~~ ~~that~~ a every subset of natural numbers is bounded above ~~and ordered of~~ and has a supremum is called completeness.

→ Real numbers is order complete

→ Set of rational numbers is not order complete.

→ Every finite set is order complete

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(4) Set of $A = \{ \frac{(-1)^n}{n} : n$ is a natural number$\}$

(a) Find the supremum. Is this supremum a max of A? Show that A is bounded from above

(b) Show that A is bounded from below. Find infimum. Is this infimum a min of A?

$$A = \left\{ \frac{(-1)^1}{1} + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \frac{(-1)^4}{4} + \cdots \right\}$$

$-1, \; 0.5, \; -\frac{1}{3}, \; \frac{1}{4}$ (0.25)

upper bound = 0.5 = supremum.

pg No 18 ; ex : 13 | N, I are not fields

→ Every converge sequence is bounded, limit is unique

→ Triangle inequality $|A+B| \le |A|+|B|$

$$|L-K| = |L+a_k-a_k-K|$$
$$= |L-a_k+a_k-K|$$
$$\le |L-a_k|+|a_k-K|$$
$$\le |a_k-L| + |a_k-K|$$
$$\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$
$$\le \varepsilon$$

6.3) Sequence $\{a_n\}$ converges to L, & $a_n \neq 0$ for all $n \in N$, and if $L \neq 0$, sequence $\{\frac{1}{a_n}\}$ converges to $\frac{1}{L}$.

6.6)

$$|s_n - l| < \varepsilon \quad \forall n \geq m$$
$$l-\varepsilon < s_n < l+\varepsilon \quad \forall n \geq m$$
$$g = \min\{l-\varepsilon, s_1, s_2 \ldots s_{m-1}\}$$
$$G = \max(l+\varepsilon, s_1, s_2, \ldots, s_{m-1})$$

$$g \le s_n \le G \quad \forall n$$

So, Here $\{s_n\}$ is a bounded sequence

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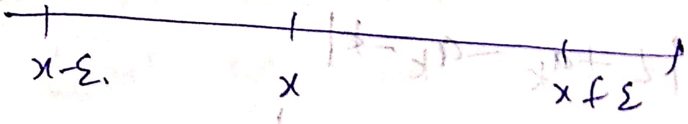
By an ε-neighbourhood of real numbers x and ε we understand the interval $(x-\varepsilon, x+\varepsilon)$



→ Limit is always a limit point, but converse need not be true.

$\{S_n\}$ to converge to L is that to each $\varepsilon > 0$ there corresponds a positive integer m such that

$$|S_n - L| < \varepsilon, \quad \forall\, n \geq m$$

Real Analysis :-

Squeeze theorem;

Statement :- If a_n converges to limit L, c_n converges to limit L, where $a_n \leq b_n \leq c_n$, then b_n also converges to Limit L. for all $n > N$

Squeeze theorem :-

If $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$ and there exists an integer N such that $b_n \leq a_n \leq c_n$ for all $n \geq N$, then $\lim_{n\to\infty} a_n = L$.

Ex:- Q) → Determine whether $\frac{\sin n}{n}$ converges or diverges

A) $-1 \le \sin n \le 1$

$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$ for all $n > 0$

We choose $\{b_n\} = \{-\frac{1}{n}\}$ and $\{c_n\} = \{\frac{1}{n}\}$.

We ~~know~~ now have choice of $\{b_n\}$ and $\{c_n\}$, that $b_n \le a_n \le c_n$ for all n. Notice that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$. By squeeze theorem, $\lim_{n\to\infty} a_n = 0$. In other words, $\{a_n\}$ converges to 0.

→ A neccessary and sufficient condition for convergence is it is bounded and monotonically

→ Monotically increasing bounded above sequence converges to its least upper bound.

→ Monotic increasing and not bounded above, diverges to $+\infty$.

→ Monotonic decreasing and - - - below, diverges to $-\infty$.

→ A sequence is bounded and has a limit point then, it is convergent.

Comparition test :

Test I : if $\sum u_n$ & $\sum v_n$ are two series

such that $u_n \leq v_n$, $\forall n$.

and $\sum v_n$ is convergent.

then $\sum u_n$ is also convergent.

Ex: $\sum u_n = \sum \frac{1}{2^n + n}$ consider $v_n = \frac{1}{2^n}$

Consider $2^n + n > 2^n \quad \forall n$

$$\frac{1}{2^n+n} < \frac{1}{2^n} \quad \forall n.$$

Since $u_n < v_n \quad \forall n$, since $\sum \frac{1}{2^n}$ is convergent,

then, $\sum u_n = \sum \frac{1}{2^n+n}$ is also convergent.

$\Rightarrow$ if $\sum u_n$ & $\sum v_n$ are two series such that $u_n \geq v_n \quad \forall n$ and $\sum v_n$ is divergent, then $\sum u_n$ is also divergent.

$\underline{\underline{Sol^n}}$ $\sum u_n = \sum \frac{1}{n+1}$ Let $v_n = \frac{1}{2n}$

$$n+1 < n+n$$

$$\underbrace{\frac{1}{n+1}}_{u_n} > \underbrace{\frac{1}{2n}}_{v_n}$$

Since $u_n \geq v_n \quad \forall n$, $\frac{1}{2}\sum \frac{1}{n}$ is divergent, then $\sum \frac{1}{n+1}$ is divergent.

$\underline{\underline{P\text{-test}}}$: The series $\sum \frac{1}{n^p}$

converges if $p > 1$

diverges if $p \leq 1$.

Limit comparision test :

if $\sum u_n$ & $\sum v_n$ are 2 series such that $\lim_{n\to\infty} \frac{u_n}{v_n} = k \neq 0$, then both the series converge or diverge together.

Ex: $\frac{4n^2+n}{(n^7+n^3)^{1/3}}$ $\qquad$ $\lim \frac{u_n}{v_n} = k$,

$v_n = \frac{n^2}{n^{7/3}} = n^{2-\frac{7}{3}} = \frac{1}{n^{1/3}}$

$$\lim_{n\to\infty} \frac{(4n^2+n)}{(n^7+n^3)^{1/3}} \times n^{1/3}$$

$$= \frac{4n^{7/3}+n^{4/3}}{(n^7+n^3)^{1/3}} = \frac{n^{7/3}(4+\frac{1}{n})}{(n^{7/3}(1+n^{-4})^{1/3}}$$

$$= \frac{\cancel{n^{7/3}}(4+\frac{1}{n})}{\cancel{n^{7/3}}(1+\frac{1}{n^4})^{1/3}} = \frac{4+0}{1+0} = 4 \neq 0$$

Here $\lim_{n\to\infty} \frac{u_n}{v_n} = k \neq 0$ (4).

Also, $\sum v_n = \lim_{n\to\infty} \frac{1}{n^{1/3}}$ diverges ($P \leq 1$)

Thus, $\sum u_n$ also diverges.

D'ALEMBERT's Ratio Test :

In a positive test series Σu_n, if $\lt_{n\to\infty} \frac{u_{n+1}}{u_n} = k$ then, the series converges if $k<1$ & diverges if $k>1$.

Reciprocal form (or)

if $\lt_{n\to\infty} \frac{u_n}{u_{n+1}} = k$, then converges if $k>1$
diverges if $k<1$

& fails if $k=1$

Ex:- $1 + \frac{a+1}{b+1} + \frac{(a+1)(2a+1)}{(b+1)(2b+1)} + \frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}$

u_{n+1}, u_n

$\lt_{n\to\infty} \frac{u_n}{u_{n+1}} = \lt_{n\to\infty}\left(\frac{nb+1}{na+1}\right)$

$\therefore \left[\text{Reciprocal } u_{n+1} = u_n\left(\frac{na+1}{nb+1}\right) \right]$

$\lt_{n\to\infty} \frac{n\left(b+\frac{1}{n}\right)}{n\left(a+\frac{1}{n}\right)} = \frac{b}{a}$

if $\frac{b}{a}>1$ converges
i.e., $b>a$

If $\frac{a}{b}$ (or) $\frac{u_{n+1}}{u_n}$

if $\frac{b}{a}<1$ diverge
i.e., $b<a$

then $\frac{a}{b}$ converges if $\frac{a}{b}<1$. $\boxed{a<b}$

diverges if $\frac{a}{b}>1$ $\boxed{a>b}$

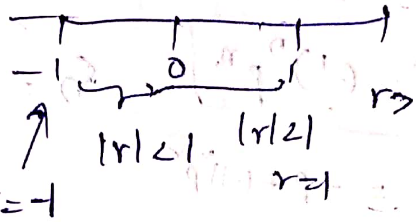
Q) <u>Geometric series</u>: Show that the series

$S_n = 1 + r + r^2 + r^3 + \dots \infty$ if i) converges to $|r| < 1$

ii) diverges if $r \geq 1$ $(r = 1 \ \& \ r > 1)$ iii) oscillates if $r \leq -1$ $(r = -1, r < -1)$

i) $S_\infty = \dfrac{1}{1-r}$

$\lim\limits_{n \to \infty} = \dfrac{1}{1-n}$



$S_n = \dfrac{1.(r^n - 1)}{r - 1}$

$\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} \dfrac{r^n - 1}{r - 1} = \dfrac{1 - r^n}{1 - r}$

$\lim\limits_{n \to \infty} \left(\dfrac{1}{1-r} - \dfrac{r^n}{1-r} \right)$, $= \cancel{0}$ if

$\lim\limits_{n \to \infty} r^n = 0$ if $|r| < 1$

$\lim\limits_{n \to \infty} \left(\dfrac{1}{1-r} \right) = \dfrac{1}{1-r}$ if $|r| < 1$

Case: 2: $\lim\limits_{n \to \infty} r^n = \infty$. $r > 1$

series diverges if $r > 1$

series becomes $1 + 1 + 1 + \dots \infty$, $S_n = n$, $\lim\limits_{n \to \infty} S_n = \infty$.

(iii) Oscillates if $r \le -1$.

$r = -1, \quad r < -1$

(i) $r < -1$ $\qquad S_n = \dfrac{1-r^n}{1-r}$

$\boxed{r = -p}$ $\qquad p > 1$

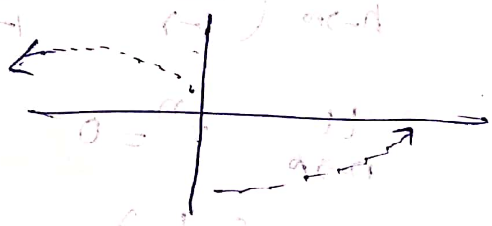
$\boxed{r^n = (-1)^n p^n}$ $\qquad S_n = \dfrac{1-(-1)^n p^n}{(1-(-p))} = \dfrac{1-(-1)^n p^n}{1+p}$

$= -p + (-1)^n p$

$\lim\limits_{n\to\infty} \dfrac{1-(-1)^n p^n}{1+p}$ $\qquad \lim\limits_{n\to\infty} p^n \to \infty \quad \because p > 1$

$S_n \to \infty$ (or) $-\infty$ depepending on n is even/odd
$\Rightarrow$ oscillates

$r = -1$, $1-1+1-1$ ------, this is oscillating



P-test: series $\sum \frac{1}{n^p}$ converges if $p > 1$
diverges if $p \le 1$

Proof: Integral test: The series $\sum \frac{1}{x^p}$ converges (or) diverges if $\int_1^\infty \frac{1}{x^p} dx$ is finite/infinite.

$\Rightarrow$ the given series $\sum \frac{1}{x^p}$, the integral is

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{m\to\infty} \int_1^m \frac{1}{x^p}\,dx = \lim_{m\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^m$$

$$= \lim_{m\to\infty} \frac{m^{1-p}-1}{1-p}$$

$\frac{1}{p-1}$ $\quad p>1$

diverges $\quad p<1$

Case II) $p=1$, $\int_1^\infty \frac{1}{x} = \lim_{n\to\infty}[\log x]_1^m$ diverges as $m\to\infty$

Cauchy's Root Test:

In a positive terms series $\sum u_n$.

Let $\lim_{n\to\infty} u_n^{1/n} = k$, now the series $\sum u_n$ converges if $k<1$ & diverges if $k>1$ & fails if $k=1$.

Problem:

$\sum (\log n)^{-2n}$

$u_n = (\log n)^{-2n}$.

$u_n^{1/n} = [\log n]^{-2n\times\frac{1}{n}}$

$= [\log n]^{-2} = \frac{1}{(\log n)^2}$

Now,

$$\lim_{n\to\infty} \frac{1}{(\log n)^2} = 0 < 1$$

Given series, converges by Cauchy's Root Test.

Absolute convergence :-

If, $u_1 + u_2 + \dots + u_n + \dots \infty$, such that

if $\sum |u_n|$ is convergent; then $\sum |u_n|$

$|u_1| + |u_2| + |u_3| + \dots |u_n| + \dots \infty$, then $\sum u_n$ is absolutely convergent.

→ if the series $\sum |u_n|$ is divergent but $\sum u_n$ is convergent,

$$|u_1| + |u_2| + \dots |u_n| + \dots \infty \text{ divergent}$$

but

$$\sum u_n = u_1 + u_2 + \dots u_n + \dots \infty \text{ convergent}$$

the series $\sum u_n$ is conditionally convergent.

Ex:- $\sum u_n = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \sum \frac{(-1)^{n+1}}{n^2}$

1) $\sum |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum \frac{1}{n^2}$

by using P-test $\sum \frac{1}{n^2}$ is convergent.

So, $\sum u_n$ is absolutely convergent.

Ex

2) $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \sum \frac{(-1)^{n+1}}{n}$

$\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum \frac{1}{n}$

by using P-test $\sum \frac{1}{n}$ is divergent.

$\sum u_n$ is → convergent.

$\sum |u_n|$ is divergent.

therefore, $\sum u_n = \sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

<u>Note</u> :- Every absolutely convergent series is convergent but every convergent series need not be absolutely convergent.

$\sum u_n = \sum \frac{(-1)^{n+1}}{n}$ Convergent but $\sum |u_n| = \sum \frac{1}{n}$ is divergent.

$\sum u_n$ $\quad \sum |u_n|$ $\quad ((-1)-1)^{n+1}$

Absolute convergence Abs. conver $\leftarrow$ convergent

<u>Leibnitz's Test</u> :-

An alternating series $u_1 - u_2 + u_3 - \cdots u_n - \infty$ <u>Converges</u> if i) $u_{n+1} < u_n$ numerically each term is numerically less than preceding term i.e., i) $u_{n+1} < u_n \ \forall n$

ii) $\lim_{n \to \infty} u_n = 0$.

$$\sum \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} \cdots \quad \Rightarrow u_{n+1} = \frac{1}{n+1}$$

$$u_n = \frac{1}{n}.$$

$$n+1 > n$$

$$\frac{1}{n+1} < \frac{1}{n} \quad \forall n.,$$

(i) $u_{n+1} < u_n$.

(ii) $\lim_{n \to \infty} \frac{1}{n} = 0$.

So, the above alternating series u_n converges.

Ex:
$$\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} + \dots$$

$$u_n = \frac{(-1)^{n+1}}{\log(n+1)}$$

$$u_n = \frac{1}{\log(n+1)} = \frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} \dots$$

$$u_{n+1} = \frac{1}{\log(n+2)} = \frac{1}{\log 3} + \frac{1}{\log 4} + \frac{1}{\log 5}$$

$\frac{0.9}{\log 2} < \frac{0.9}{\log 3}$

i) $\log 2 < \log 3$

$\log(n+1) < \log(n+2)$

$$\frac{1}{\log(n+1)} > \frac{1}{\log(n+2)}$$

$u_n > u_{n+1}$

ii) $\lim_{n\to\infty} u_n = 0$

So, the series converges.

Just Ex: $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$

$e^{(\text{base}-1)\text{power}}$ $\left[\because \frac{\text{base}}{\text{power}} = 1^{\infty}\right]$

$* (-1)^{n+1}$

$$\frac{e^{(-1-1)(n+1)}}{n} = \frac{e^{-2(n+1)}}{n} = \frac{e^{-\infty}}{\infty} = \frac{1}{\infty}$$

$= 0$

Power Series:

[8/9/20]

A series of the form $a_0 + a_1x + a_2x^2 + \dots a_nx^n \dots \infty$ where a's are independent of x is called Power series.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{a_{n+1} x^{n+1}}{a_n x^n} = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) x$$

<u>Abs case:</u>

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right|$$

if $|x| < 1$

$5x < 1$

$x < \frac{1}{5}$

$$= \lim_{n\to\infty} \left|\left(\frac{a_{n+1}}{a_n}\right) x\right|$$

$$\lim_{n\to\infty} = \left(\frac{a_{n+1}}{a_n}\right) = l.$$

$$= |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= |x|\, l$$

Series converges if $|x| l < 1 \Rightarrow |x| < \frac{1}{l}$

Interval of convergence $\rightarrow$ $\boxed{-\frac{1}{l} < x < \frac{1}{l}}$

<u>Ex:</u> $$\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3}$$

$$u_n = \frac{1}{n(1-x)^n} \qquad u_{n+1} = \frac{1}{(n+1)(1-x)^{n+1}}$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{1}{(n+1)\cancel{(1-x)^n}(1-x)} \cdot n\cancel{(1-x)^n}\right|$$

$$\therefore \lim_{n\to\infty} = \left|\frac{n}{(n+1)(1-x)}\right| \Rightarrow \cancel{\left|\frac{n}{n+1}\right| \lim_{n\to\infty} \left|\frac{1}{1-x}\right|}$$

$$\Rightarrow \frac{1}{|1-x|} \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \frac{1}{|1-x|} \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}$$

$\lim_{n\to\infty} = \left|\frac{1}{1-x}\right|$

$-1 < 1-x < 1$

$-1-1 < x-x-x < 1-1$

$-2 < -x$

Converges if $\left|\frac{1}{1-x}\right| < 1.$

$|1-x| > 1$

$1-x > 1 \quad \& \quad 1-x < -1$

$\cancel{2 > x}, \qquad x > 2$

$x < 0 \quad \&$

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for $x = 0$ $\quad \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots + \sum \frac{1}{n}$ so $p = 1$

the series diverges

$x = 2 \Rightarrow -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots$

$= \sum \frac{(-1)^n}{n} \rightarrow$ convergent by Lebinitz's test

So, the series converges for $x < 0$ & $x > 2$.

Convergence of Exp Series:

Ex: $1 + \frac{x}{1} + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots \frac{1}{n!}x^n + \ldots \infty$

$u_n = \frac{1}{n!}x^n \qquad u_{n+1} = \frac{1}{(n+1)!}x^{n+1}$

$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{(n+1)!}x^{n+1} \cdot \frac{n!}{x^n} \quad \left|\frac{n!}{(n+1)!}\right| x$

$= \frac{n!}{(n+1)\,n!} x = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0$

Series is convergent for all values of x.

Find the interval of convergence for

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \frac{(-1)^n x^n}{n} + \dots \infty.$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = (-1)$$

Converges for $|x| < 1$
diverges for $|x| > 1$
Converges for $x = 1$
diverges for $x = -1$

Ex: $\sum a_n = \sum \frac{n^n}{3^{1+2n}}$ using Root test

$\Rightarrow$

Ex: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 \dots$

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{x^{n+1}}{x^n} = |x|$$

$$\lim_{n\to\infty} |x| < 1 \qquad -1 < x < 1.$$

Sequence of f^n $\{ f_n(x) \}_{n=1}^{\infty}$

$f_n(x) = x + n$, $n \in N$, $x \in [0,1]$

$x = \frac{1}{2}$, $f_1(x), f_2(x) \dots$

$f_1(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$ $f_2(x) = 2 + \frac{1}{2} = \frac{5}{2}$

at $x = \frac{1}{2}$, $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$

$\downarrow$

$\Rightarrow$ This sequence converges, so, $f_n(x)$ is convergent at $x = x_0$

(1) Converges at a point; $f_n(x)$ is convergent, at $x = x_0$.

(2) Point wise convergence $f_n(x) = \frac{x}{n}$:

$x = 0.4 \Rightarrow f(n)\, f(0.4) = \frac{0.4}{n}$ $\quad a_n = \frac{0.4}{n}$.

$\{a_n\} = \frac{0.4}{1}, \frac{0.4}{2}, \frac{0.4}{3}, \ldots\ldots$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{0.4}{n} = 0$

Suppose $\{f_n(x)\}$ is a sequence of f^n's, we say that $f_n(x)$ is point wise, convergent to $f(x)$ if $\lim_{n\to\infty} f_n(x) = f(x)$, for each x in that interval.

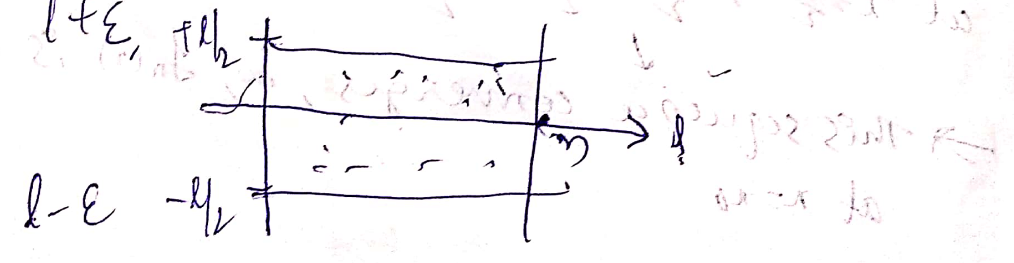
$\{f_1(x), f_2(x), \ldots\} \to f(x)$.

Ex: $f_n(x) = \frac{n}{1+nx}$ $\quad f: (0,1) \to R$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{1+nx} = \lim_{n\to\infty} \frac{n}{n(\frac{1}{n}+x)} = \frac{1}{x} = f(x)$

$\qquad \frac{1}{n} \to 0.$

Let $\{a_n\}$ be a sequence, we can say that sequence is convergent to l, if for each $\epsilon > 0$, $\exists m$ such that $|a_n - l| < \epsilon, \forall n \geq m$



Let $\{f_n(x)\}$ be a sequence of functions $\{f_n(x)\}$ is convergent to $f(x)$ or f, if for each $\varepsilon > 0$, $\exists\, m$ such that $|f_n(x) - f| < \varepsilon$, $\forall n \geq m$ for each x in given interval.

Point wise convergence for a sequence of f^n.

Ex: $f_n(x) = \frac{n}{1+nx}$ $\quad f(x) = \frac{1}{x}$

for $\varepsilon > 0$, $|f_n(x) - f(x)| < \varepsilon$, for each $x \in (0,1]$

$$\left|\frac{n}{1+nx} - \frac{1}{x}\right| < \varepsilon \Rightarrow \left|\frac{nx - 1 - nx}{(1+nx)x}\right| < \varepsilon$$

$$\frac{1}{x(1+nx)} < \varepsilon \Rightarrow (1+nx)x > \varepsilon$$

$$\Rightarrow (1+nx) > \frac{\varepsilon}{x} \quad nx > \frac{\varepsilon}{x} - 1 \quad n > \frac{\varepsilon}{x^2} - \frac{1}{x} = m$$

2) Uniform convergence

$\{f_n(x)\}$ is said to uniformly convergent to $f(x)$ if for $\varepsilon > 0$, $\exists\, m$ such that $|f_n(x) - f(x)| \varepsilon$, $\forall n \geq m$, & for all x in that interval.

$f_n(x) = \frac{2nx}{1+n^4x^2}$ $\quad f(x) = 0$

$$\left|\frac{2nx}{1+n^4x^2} - 0\right| < \varepsilon \Rightarrow$$

$1 + n^4x^2 \geq 2n^2x$

$$\frac{2nx}{1+n^4x^2} \leq \frac{2nx}{2n^2x}$$

$$|f_n(x) - 0| \leq \frac{1}{n}.$$

$\rightarrow$ $f_n(x)$, if $\exists$ a_n such that $|f_n(x)-f| < a_n$

& $\lim_{n\to\infty} a_n = 0$ then $F_n(x)$ is uniformly

convergent to f. $|f_n(x) - 0| < \frac{1}{n}$.

$\rightarrow$ $f_n \to f$ pointwise iff $\forall \varepsilon > 0$, $\forall x \in E$, $\exists N \in \mathbb{Z}^+$ s.t.

$\forall n > N$ $|f_n(x) - f(x)| < \varepsilon$. [N depends on ε and x]

$\rightarrow$ $f_n \to f$ uniformly iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{Z}^+$, st

$\forall n > N$, $\forall x \in E$ $|f_n(x) - f(x)| < \epsilon$

[N depends on ε]

Ex: $f_n(x) = x^n$ on $[0,1]$

if $x = 0$ $f_n(0) = 0^n = 0$

if $x = 1$, $f_n(1) = 1^n = 1 \to 1$

If $0 < x < 1$, $f^n(x) = x^n = 0$

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$