Advanced Data Structures and Algorithms

The Maximum Network Flow Problem

Network Flow

Types of Networks

- Internet
- Telephone
- Cell
- Highways
- Rail
- Electrical Power
- Water
- Sewer
- Gas
- …

Maximum Flow Problem

- How can we maximize the flow in a network
  - from a source or set of sources
  - to a destination or set of destinations?

Network Flow

- Instance:
  - A Network is a directed graph G

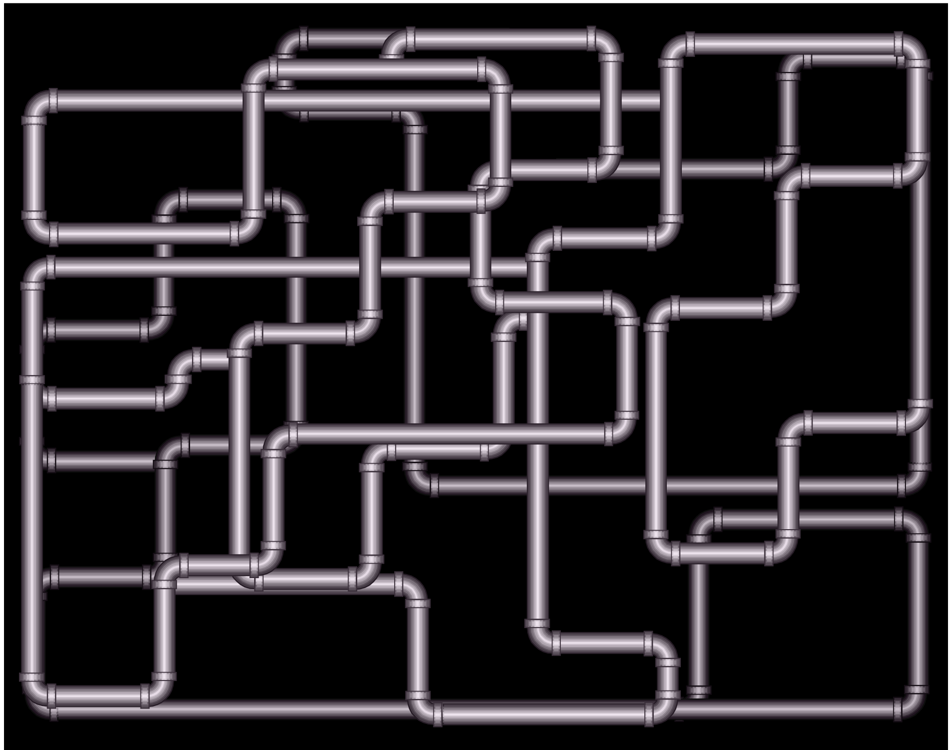

Figure courtesy of J. Edmonds

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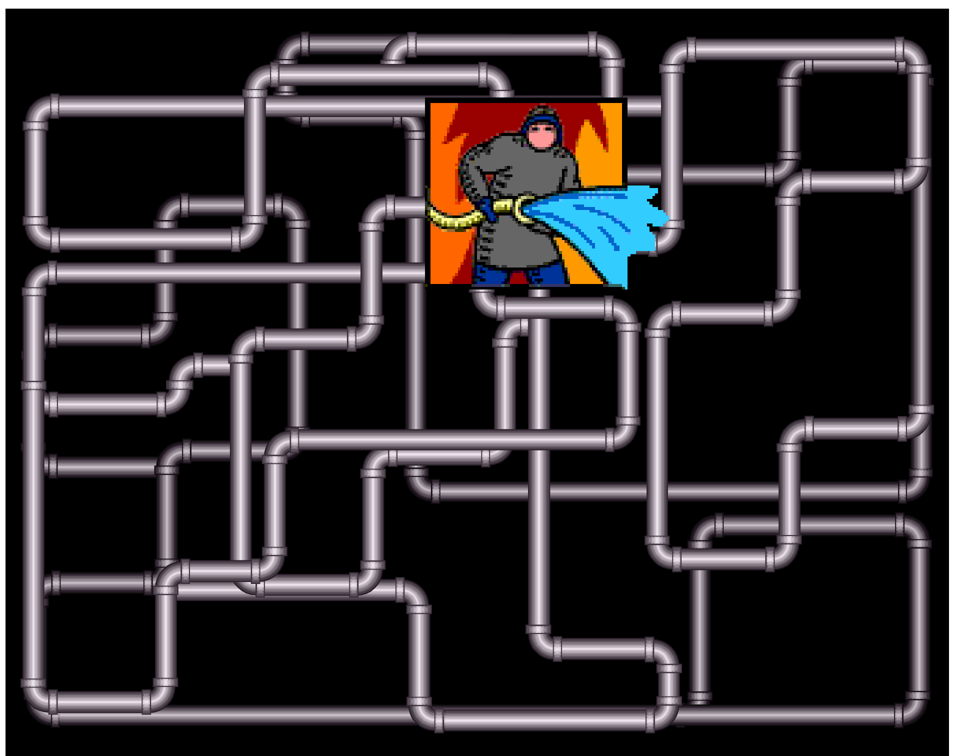

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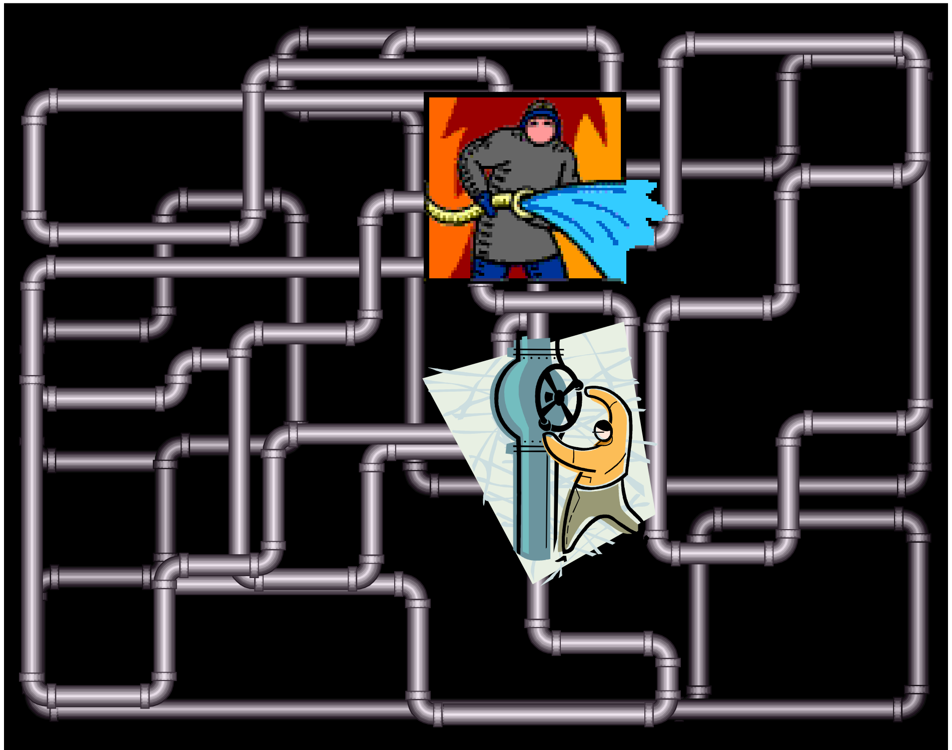

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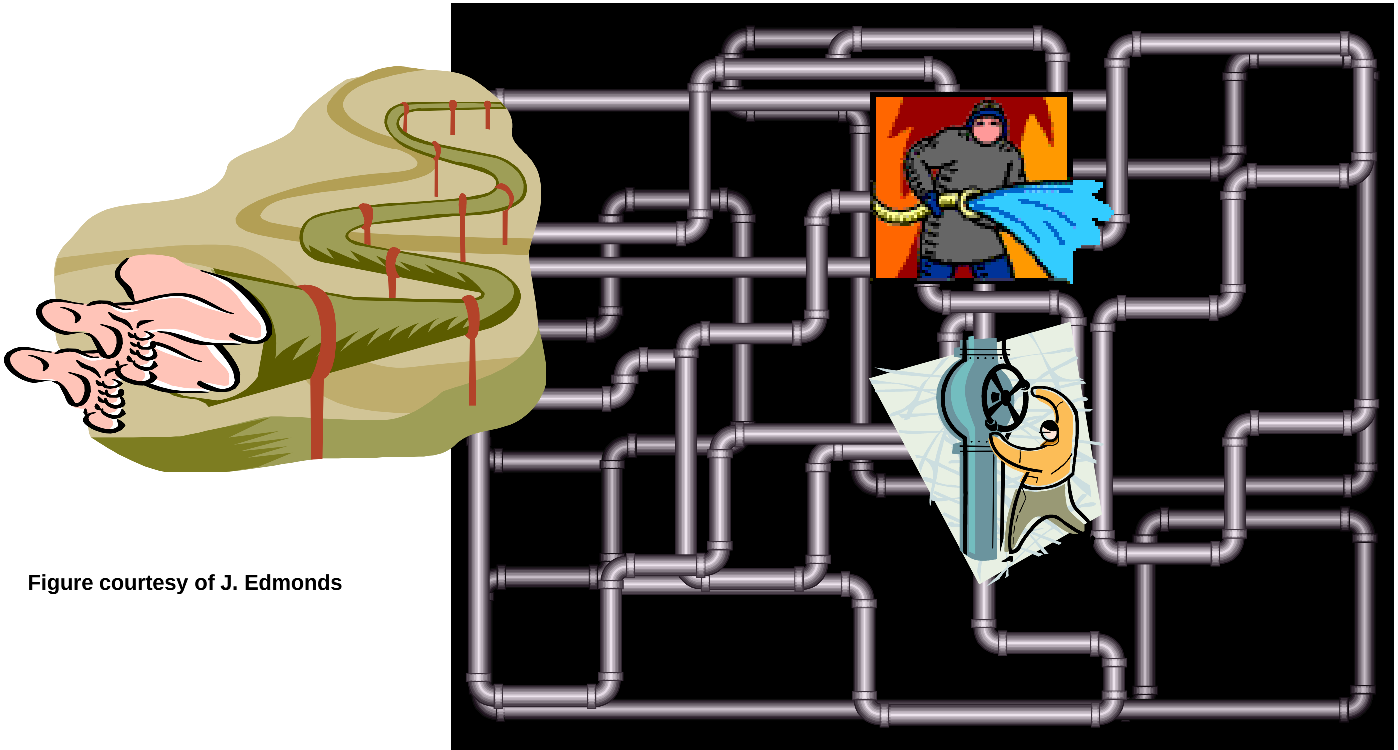

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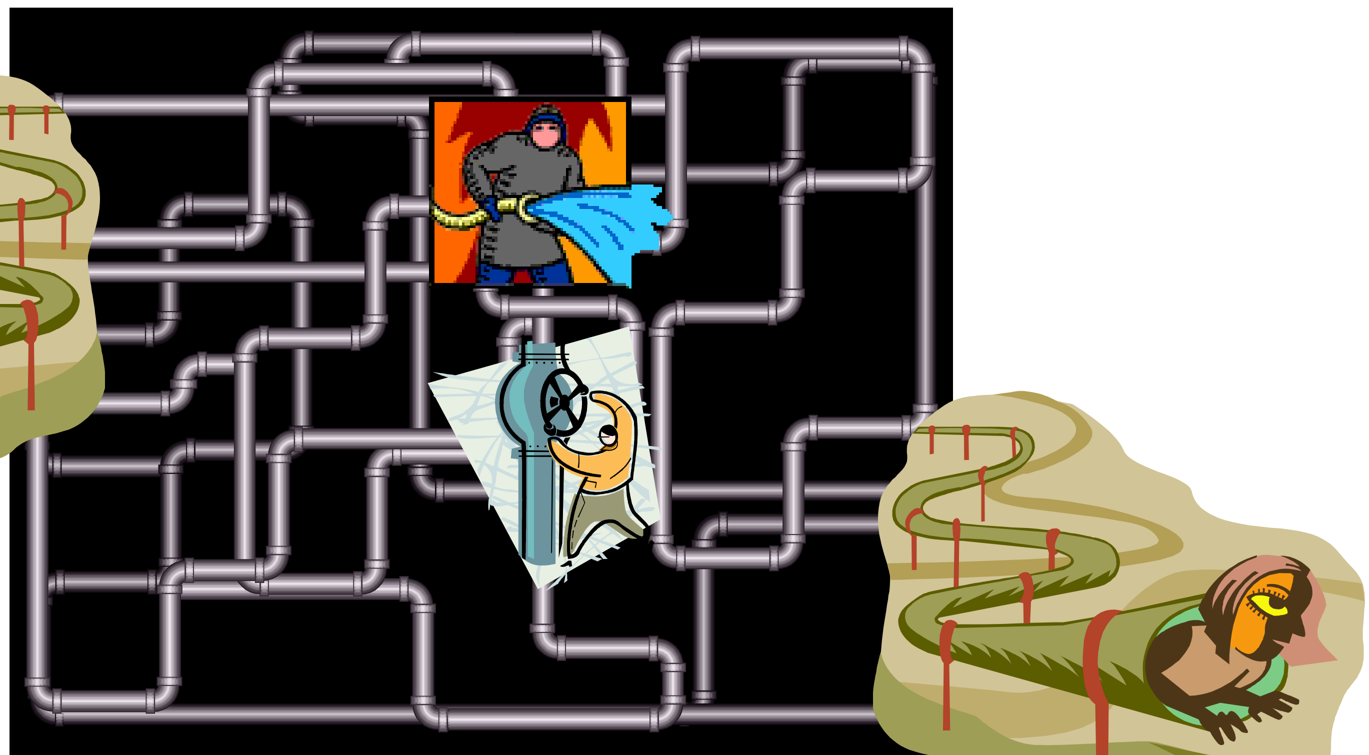

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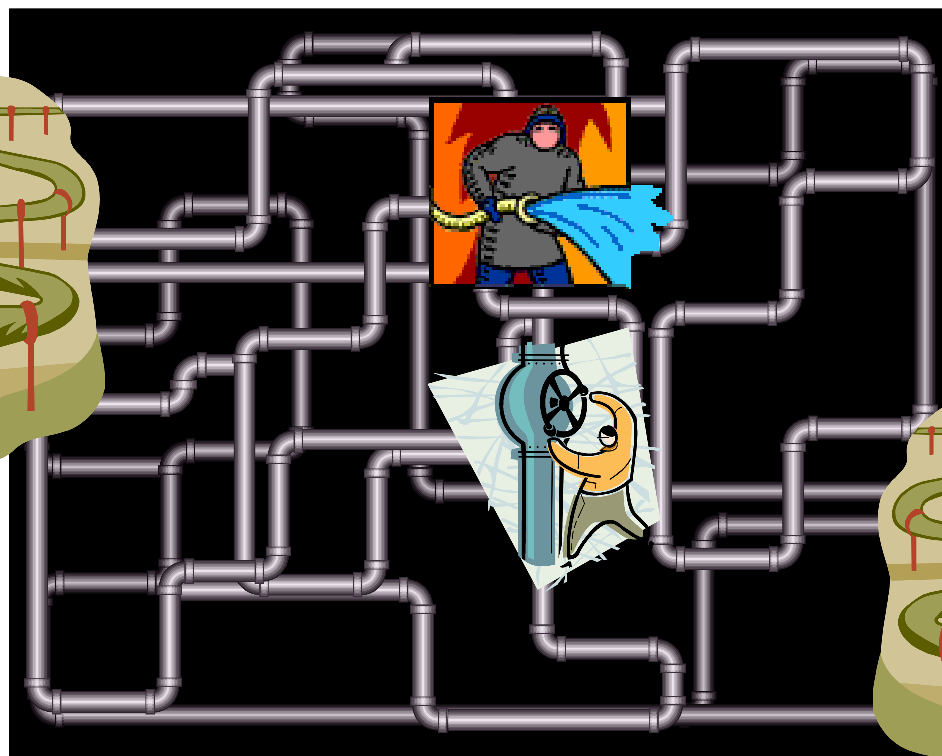
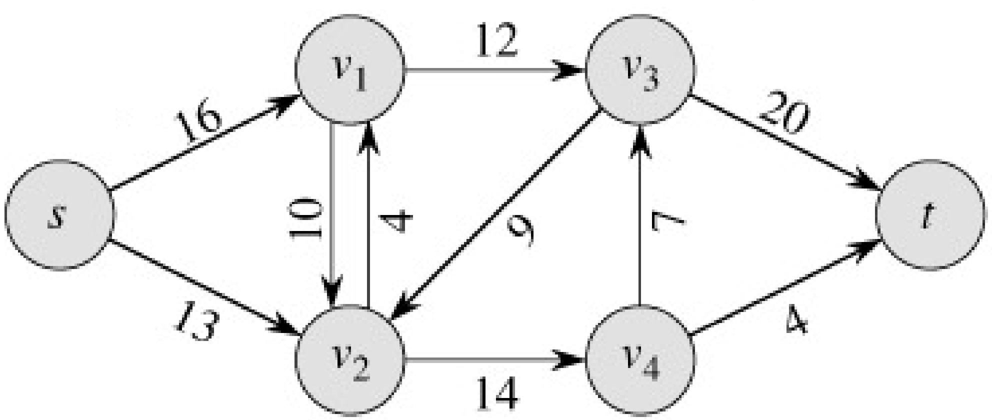


Goal:
Max Flow

Figure courtesy of J. Edmonds

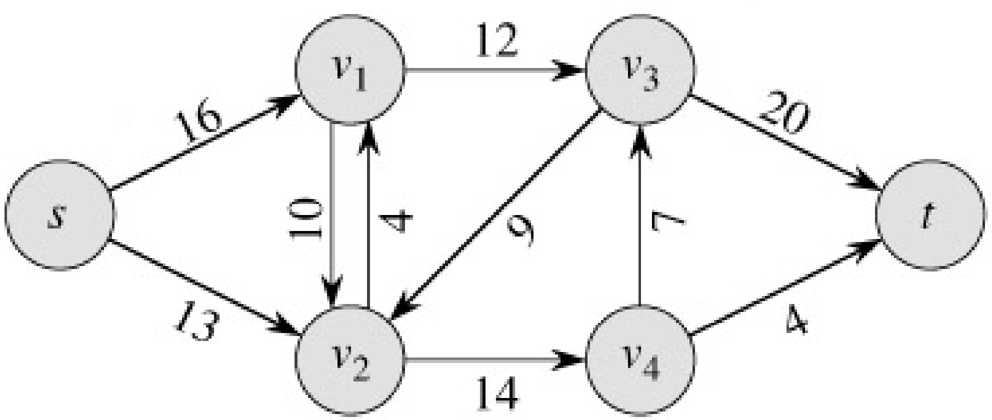
The Problem

- Use a graph to model material that flows through conduits.
- Each edge represents one conduit, and has a capacity, which is an upper bound on the flow rate, in units/time.
- Can think of edges as pipes of different sizes.
- Want to compute max rate that we can ship material from a designated source to a designated sink.



What is a Flow Network?

- Each edge (u,v) has a nonnegative capacity c(u,v).
- If (u,v) is not in E, assume c(u,v)=0.
- We have a source s, and a sink t.
- Assume that every vertex v in V is on some path from s to t.
- e.g., $c(s,v_1)=16$; $c(v_1,s)=0$; $c(v_2,v_3)=0$



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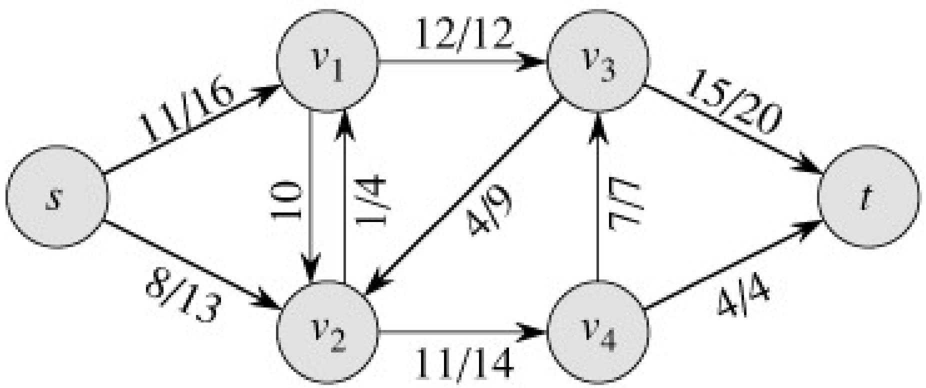
- Notes:
  - The skew symmetry condition implies that f(u,u)=0.
  - We show only the *positive* flows in the flow network.

Example of a Flow:



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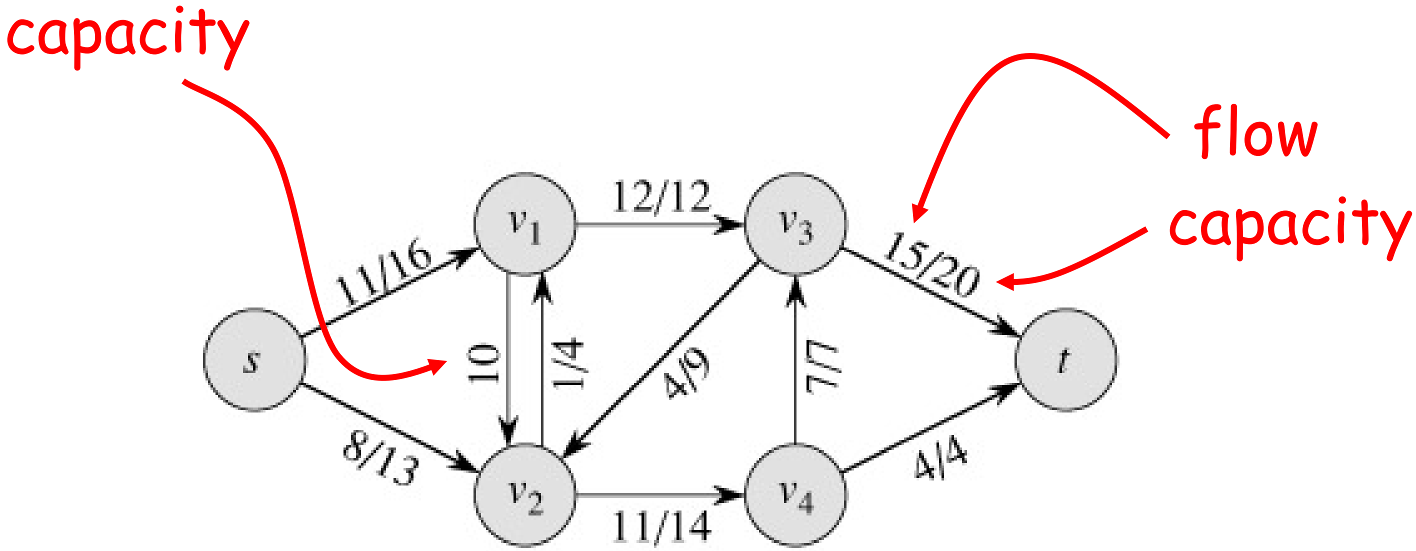

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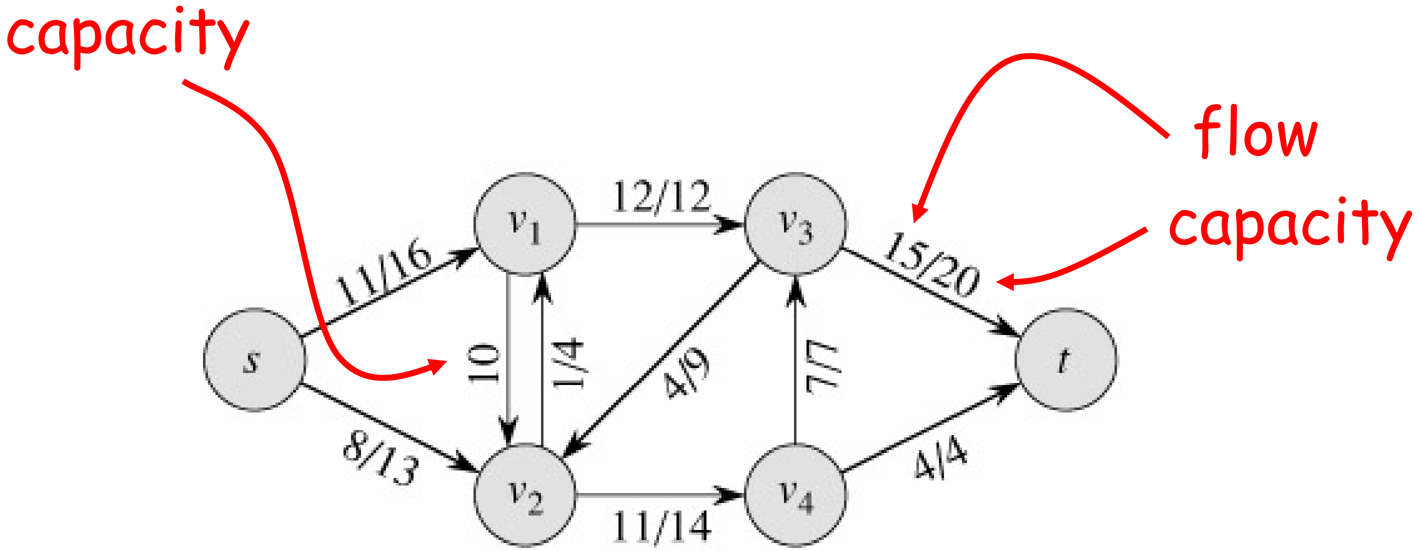

- $f(v_2, v_1) = 1$, $c(v_2, v_1) = 4$.

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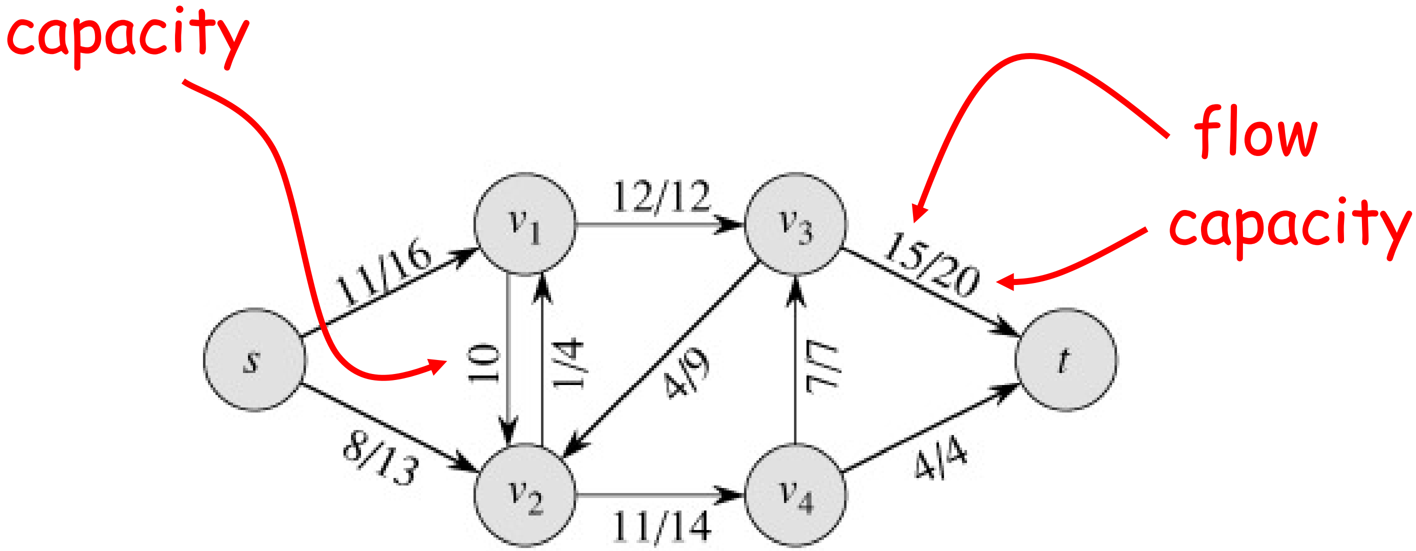

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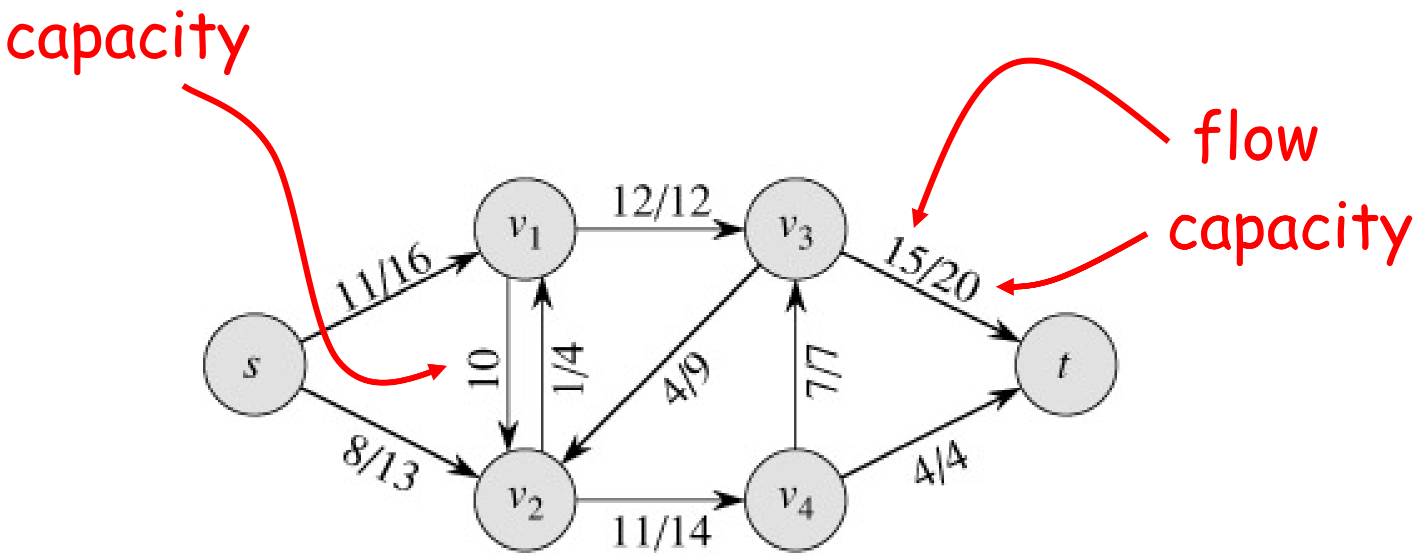

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- $f(v_1, v_2) = -1$, $c(v_1, v_2) = 10$.
- $f(v_3, s) + f(v_3, v_1) + f(v_3, v_2) + f(v_3, v_4) + f(v_3, t) =$

  $0 \quad + \quad (-12) \quad + \quad 4 \quad + \quad (-7) \quad + \quad 15 \quad = \quad 0$

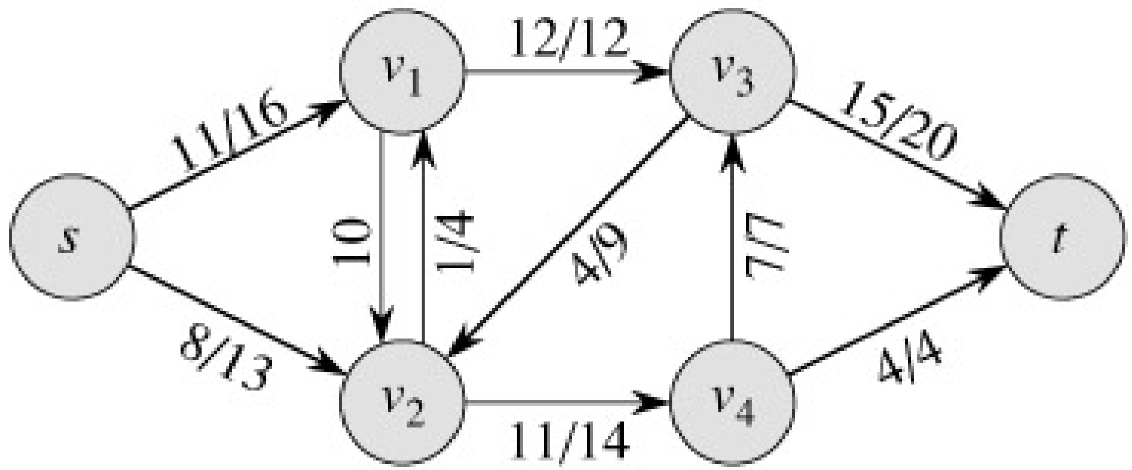
The Value of a flow

- The value of a flow is given by

$$|f| = \sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,t)$$

- This is the total flow leaving s = the total flow arriving in t.

Example:



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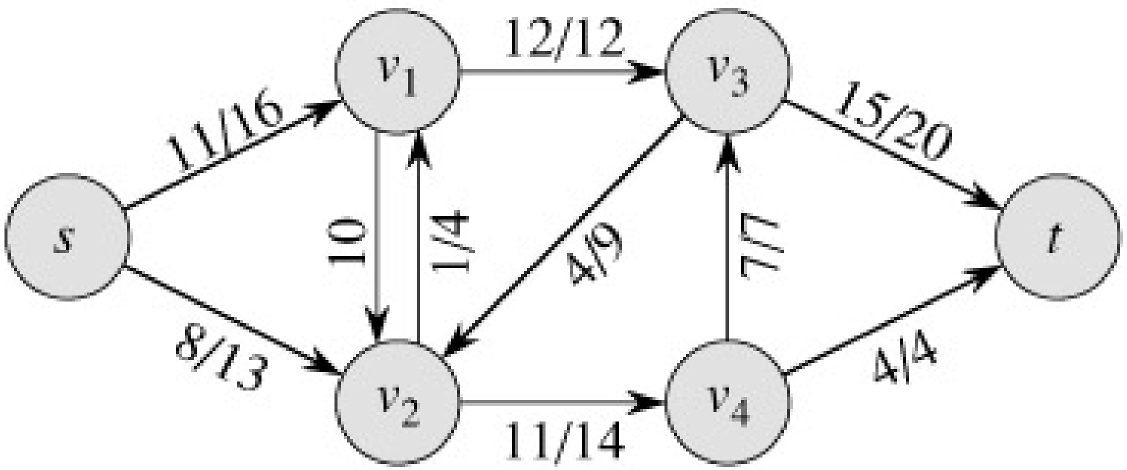

$$|f| = f(s, v_1) + f(s, v_2) + f(s, v_3) + f(s, v_4) + f(s, t) =$$
$$11 + 8 + 0 + 0 + 0 = 19$$

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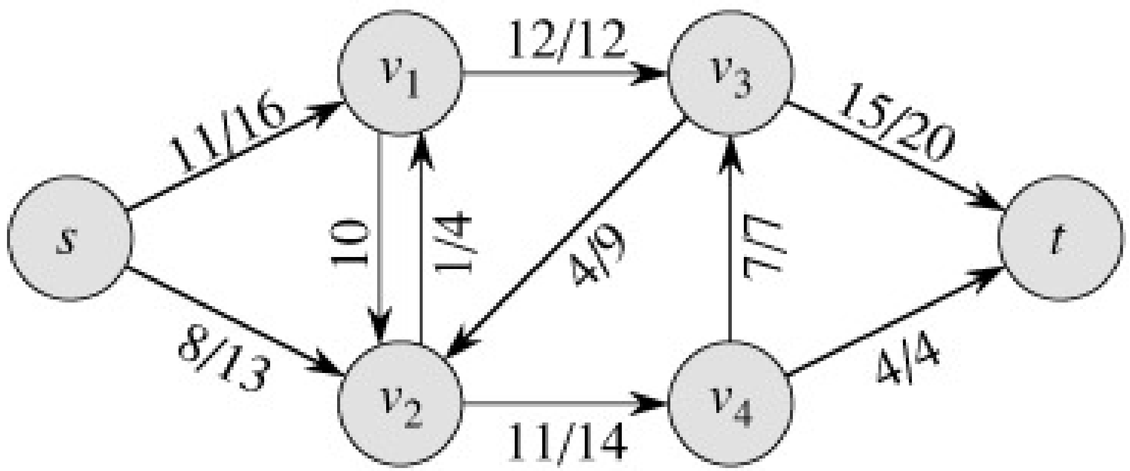
$$0 + 0 + 0 + 15 + 4 = 19$$

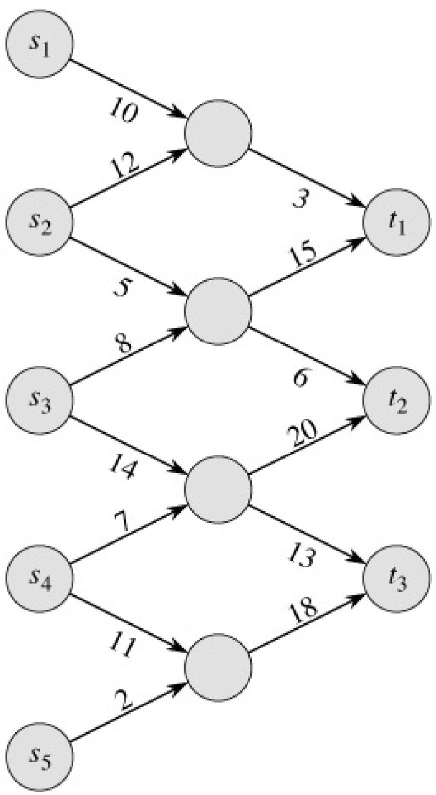
A flow in a network

- We assume that there is only flow in one direction at a time.



Multiple Sources Network

- We have several sources and several targets.
- Want to maximize the total flow from all sources to all targets.



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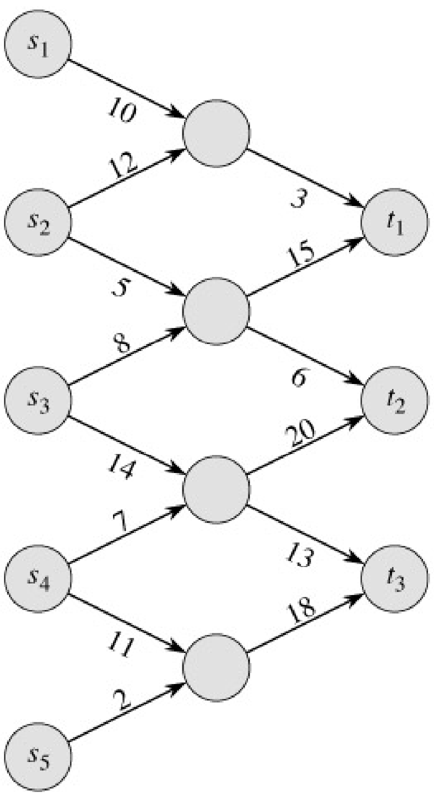

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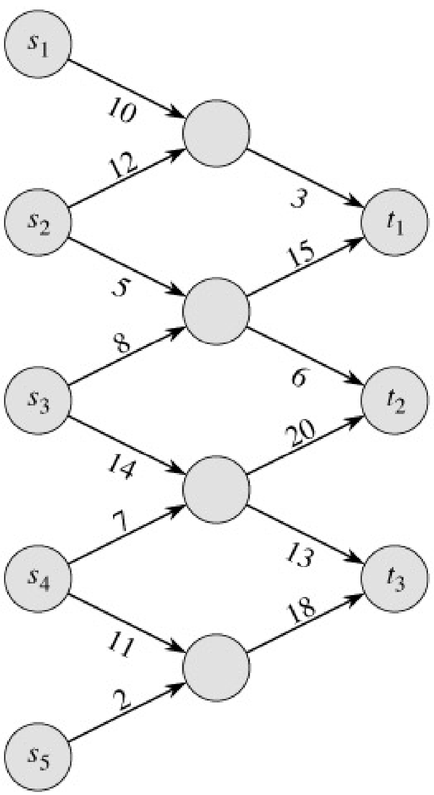

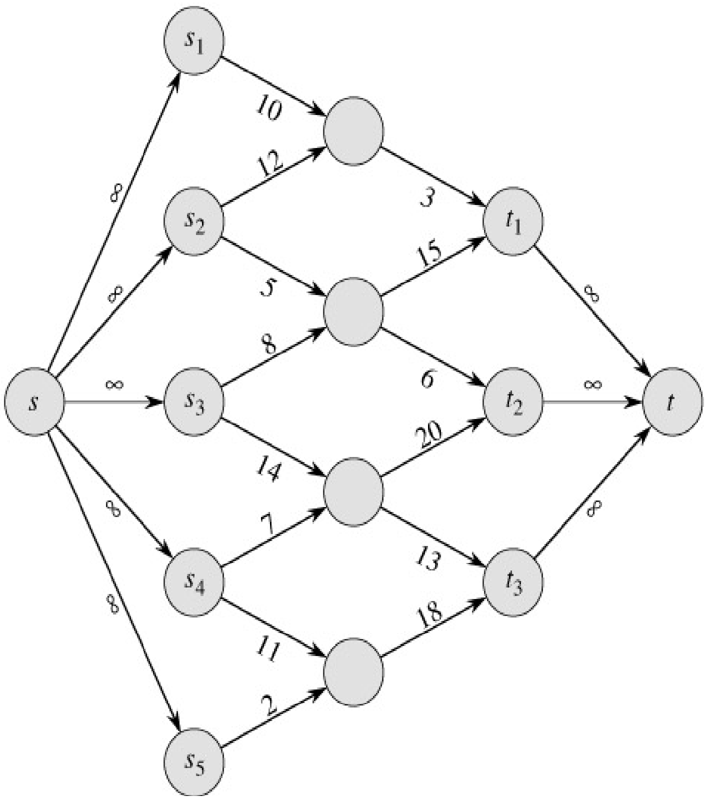

Residual Networks

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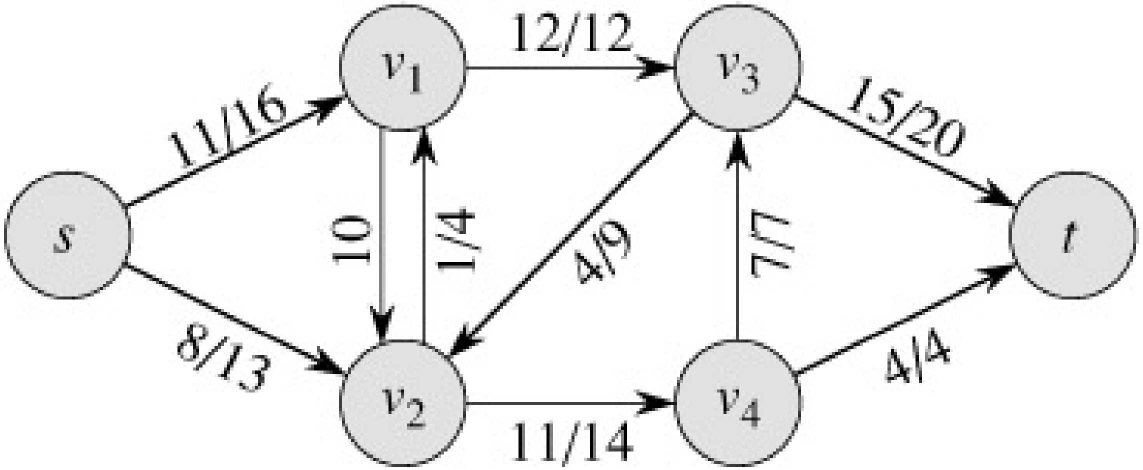
$$G_f = (V, E_f), \text{ where } E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$$

Example of Residual Network

Flow Network:



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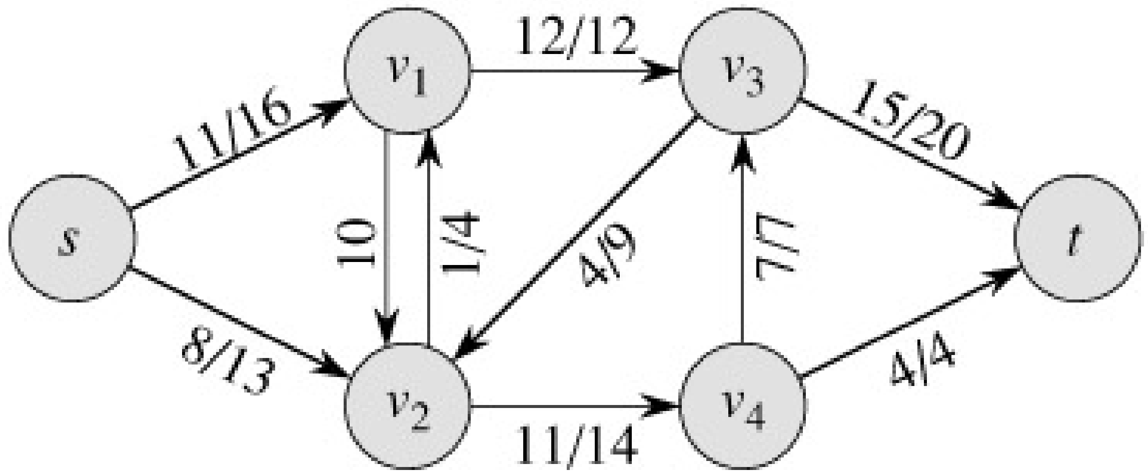

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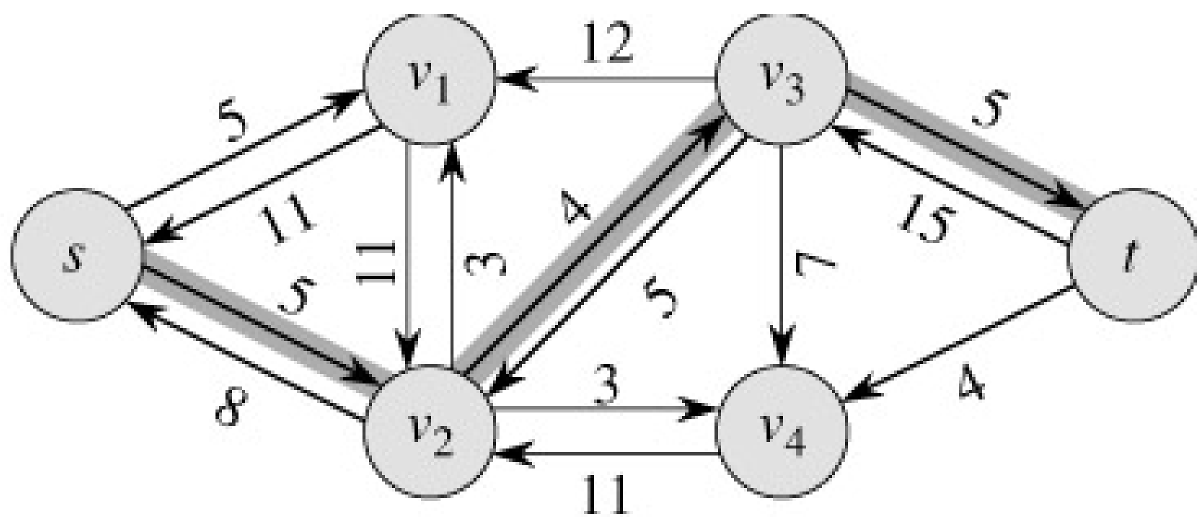

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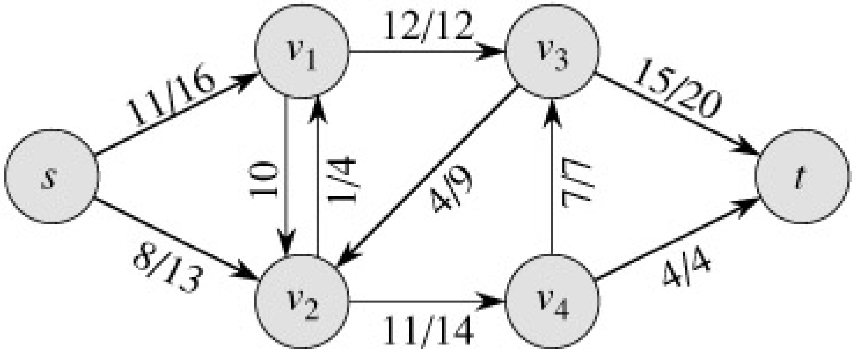
$$\boxed{c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}}$$

Augmenting Paths

Network:



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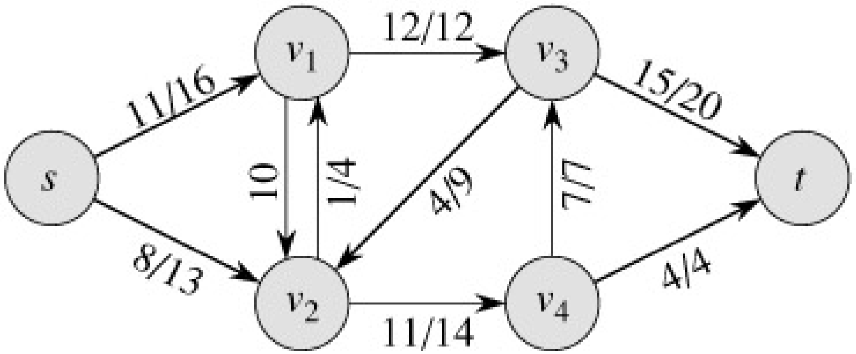

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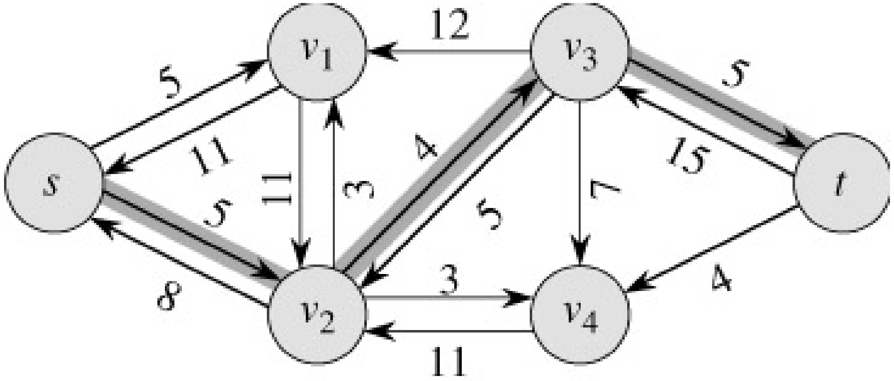

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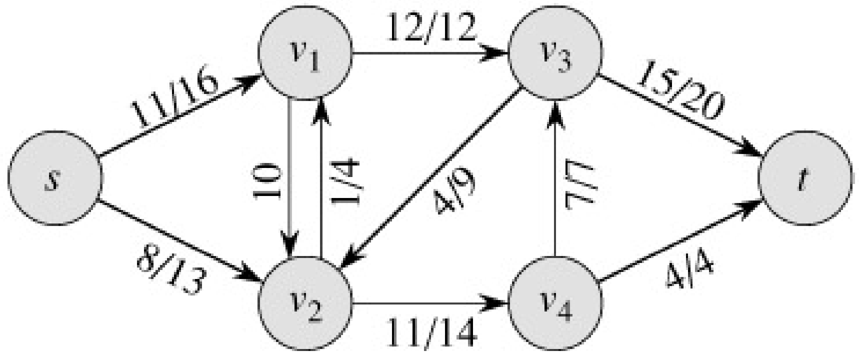

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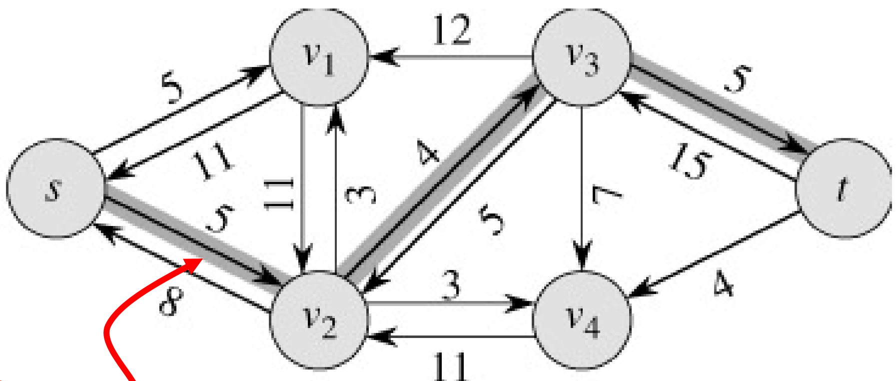

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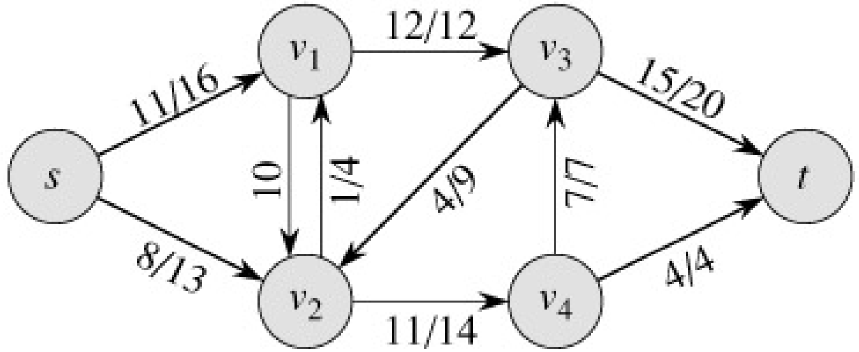

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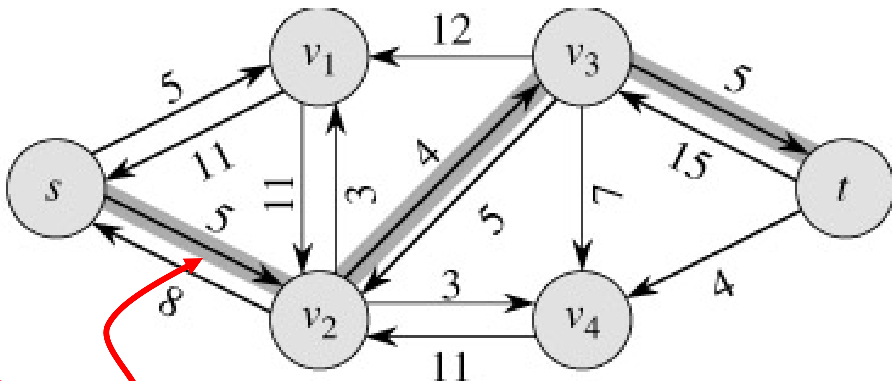


Augmenting path

The residual capacity of this augmenting path is 4.

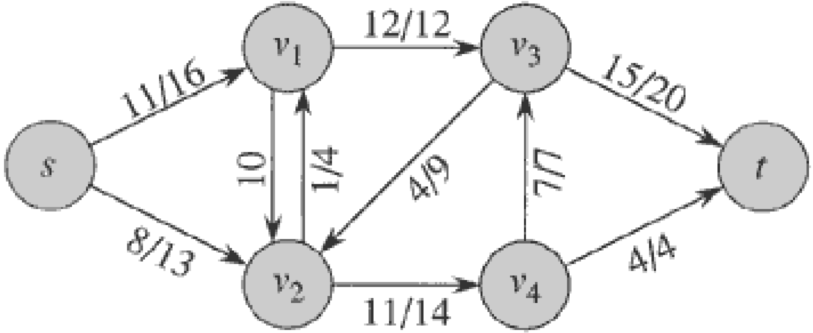
Computing Max Flow

- Classic Method:
  - Identify augmenting path
  - Increase flow along that path
  - Repeat

Ford-Fulkerson Method

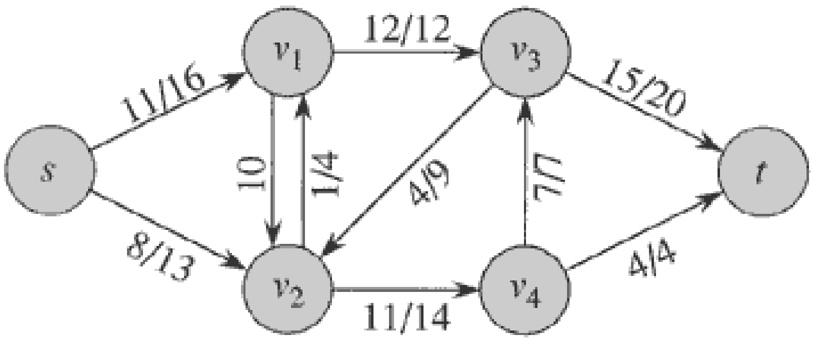
```
FORD-FULKERSON-METHOD(G, s, t)
initialize flow f to 0
while there exists an augmenting path p
      do augment flow f along p
return f
```

Example



Flow(1)

Example



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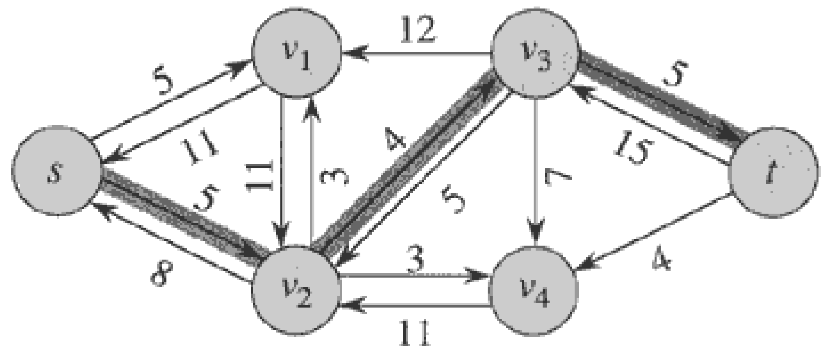

Residual(1)

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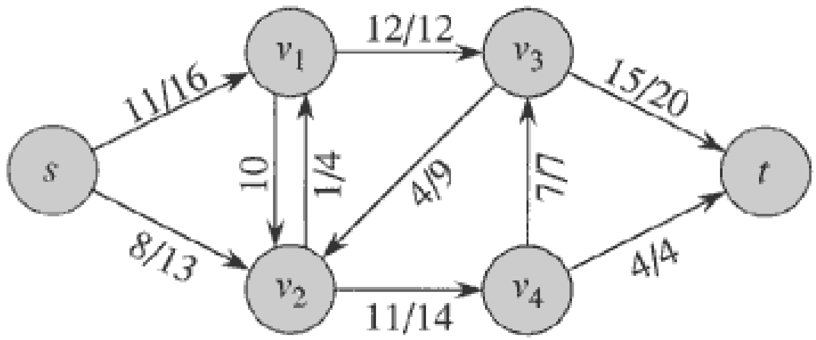

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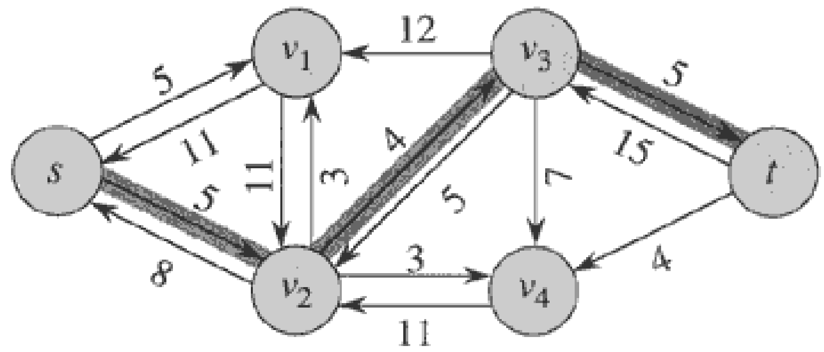

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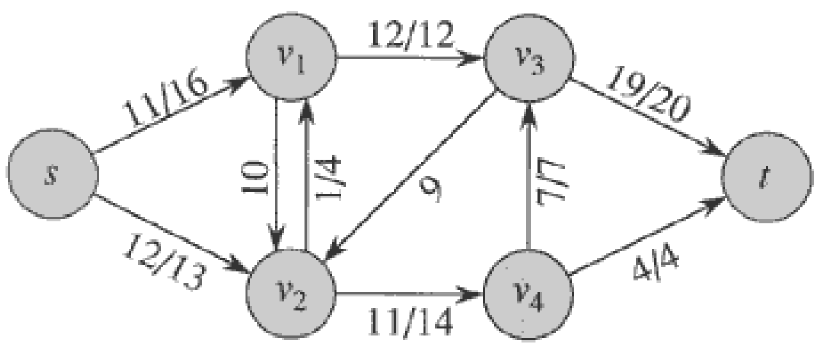
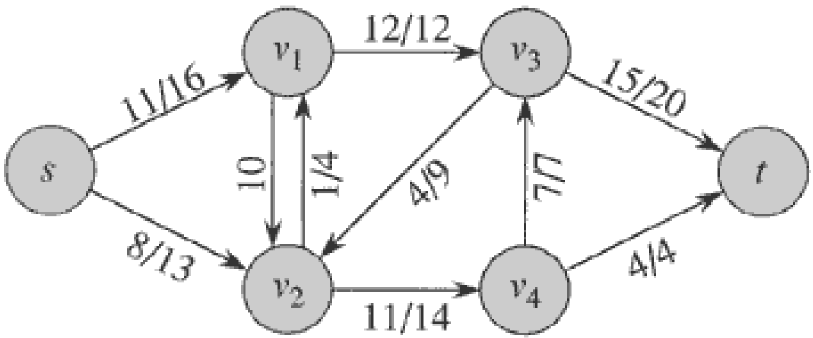


Flow(2)

Example



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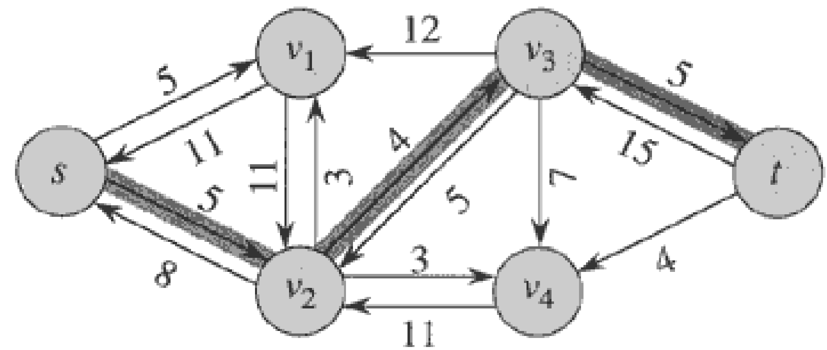

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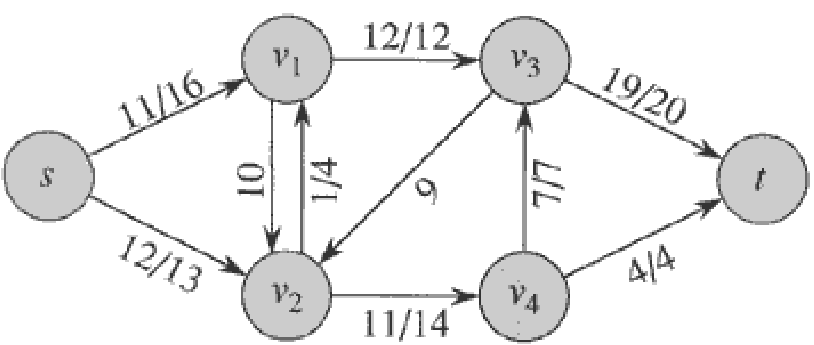

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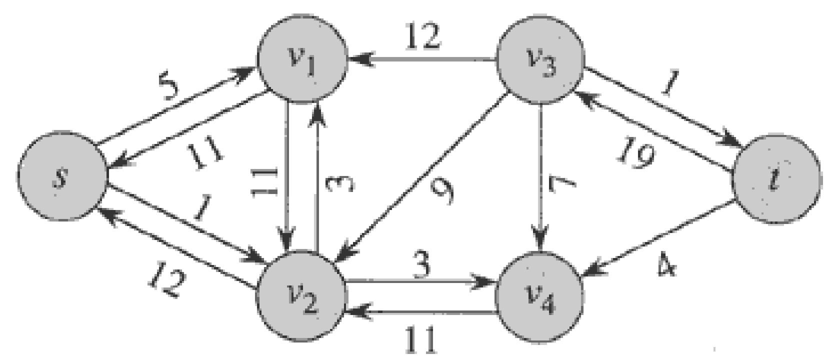

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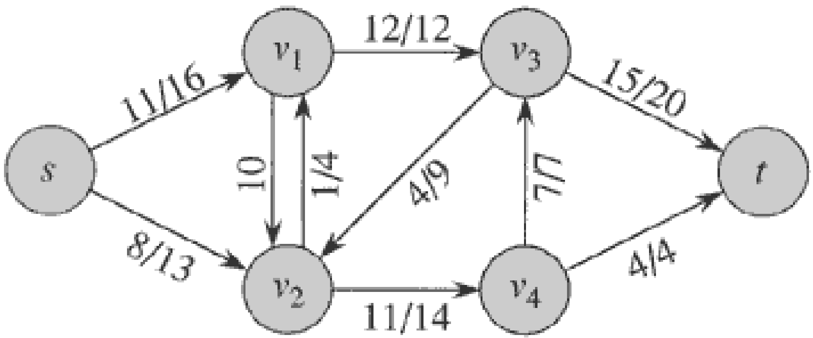

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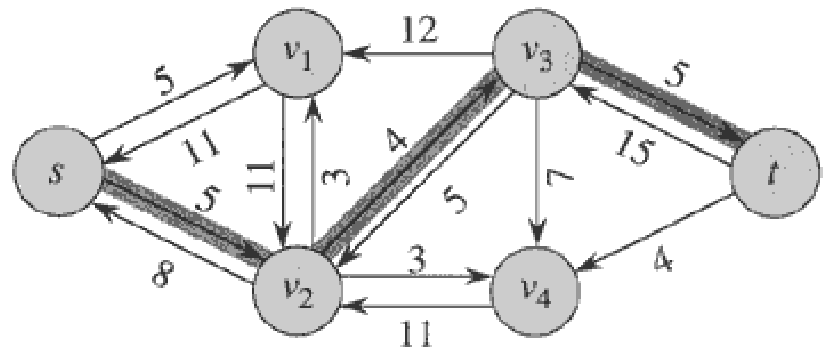
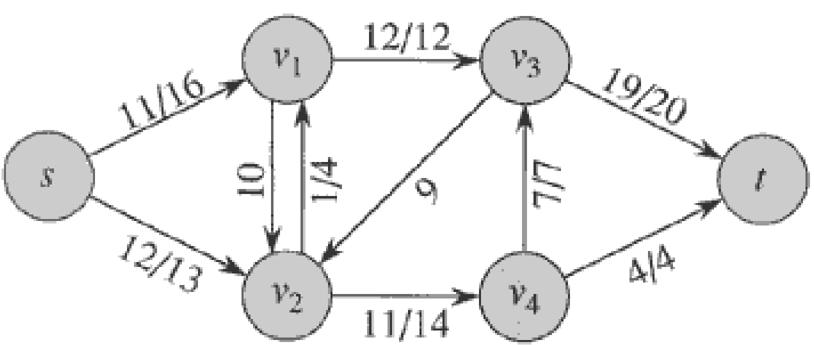
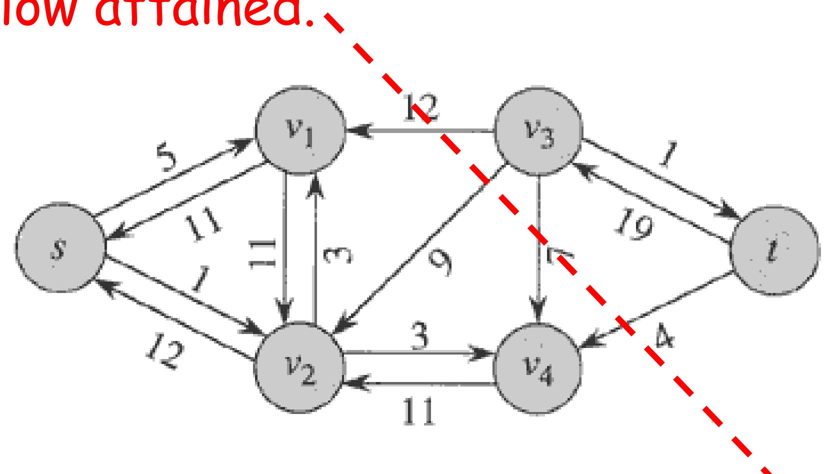


Residual(1)

No more augmenting paths → max flow attained.



Flow(2)



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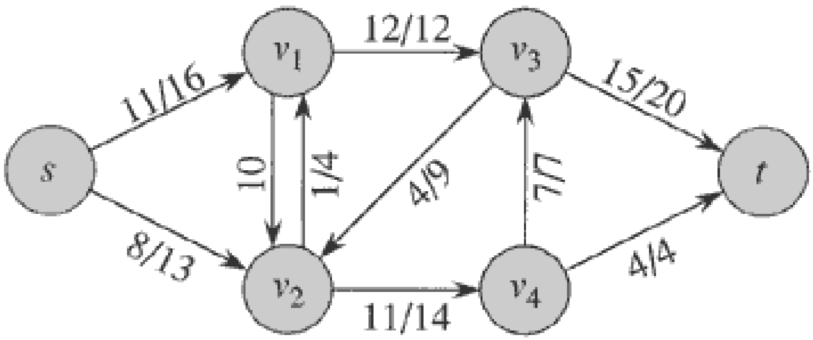

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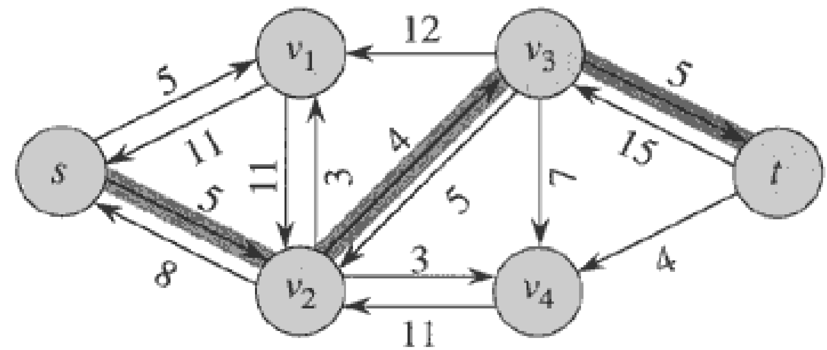

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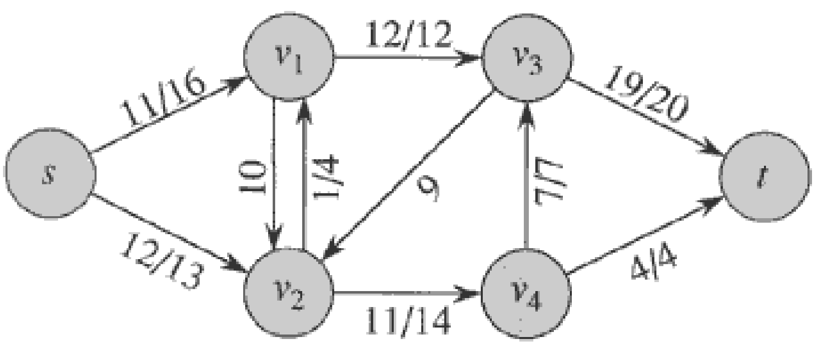

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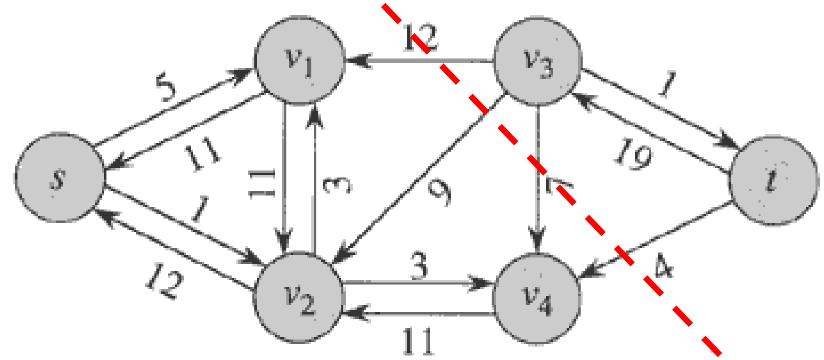
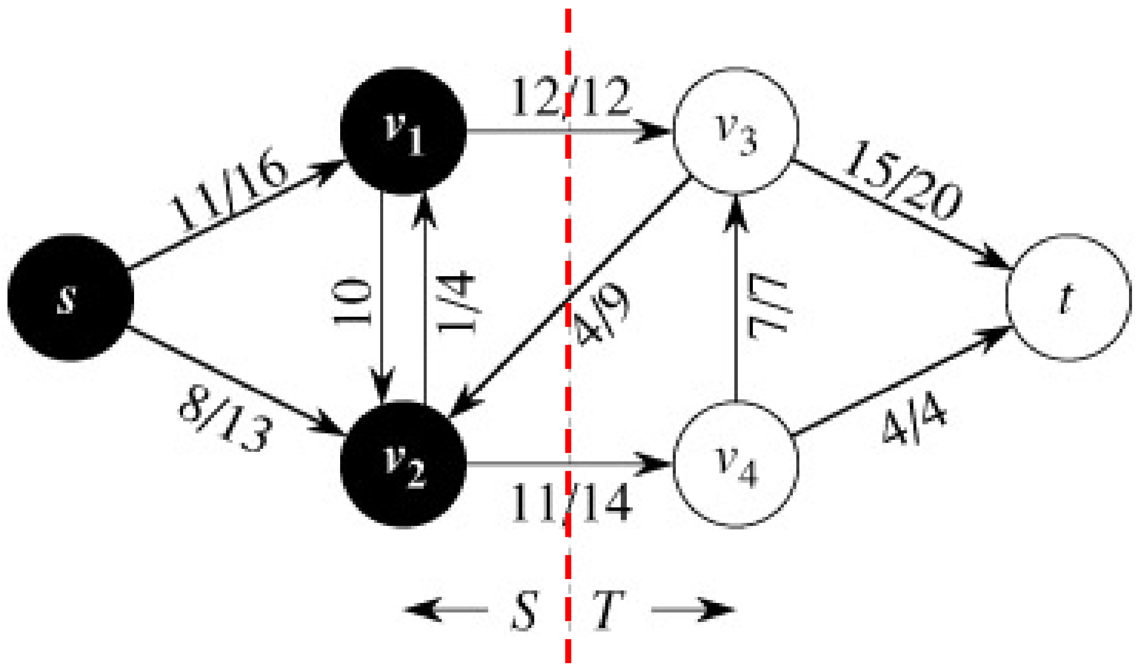


Residual(2)

Cut

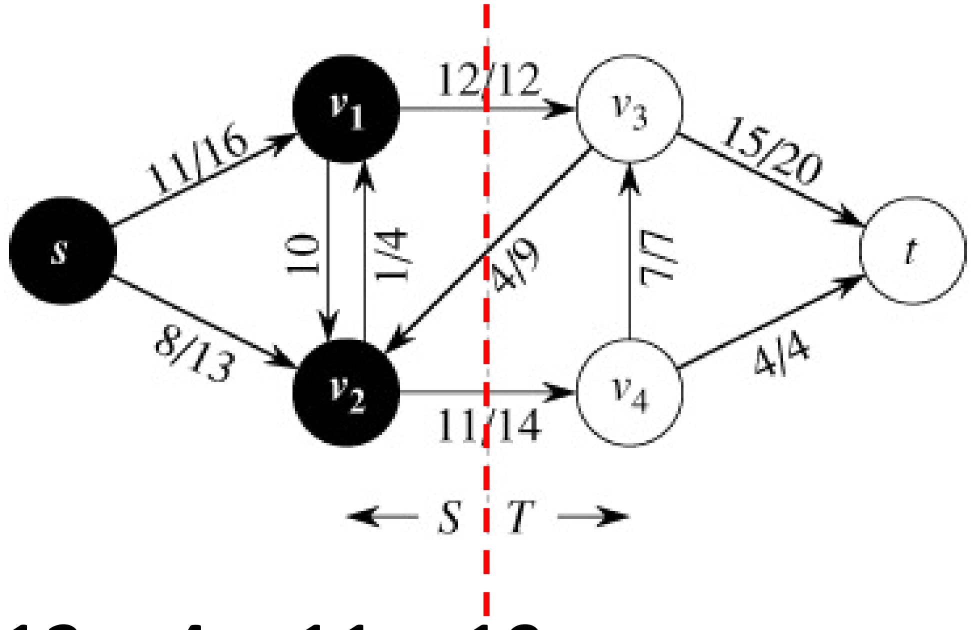
Cuts of Flow Networks

A cut (S,T) of a flow network is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$.



The Net Flow through a Cut (S,T)

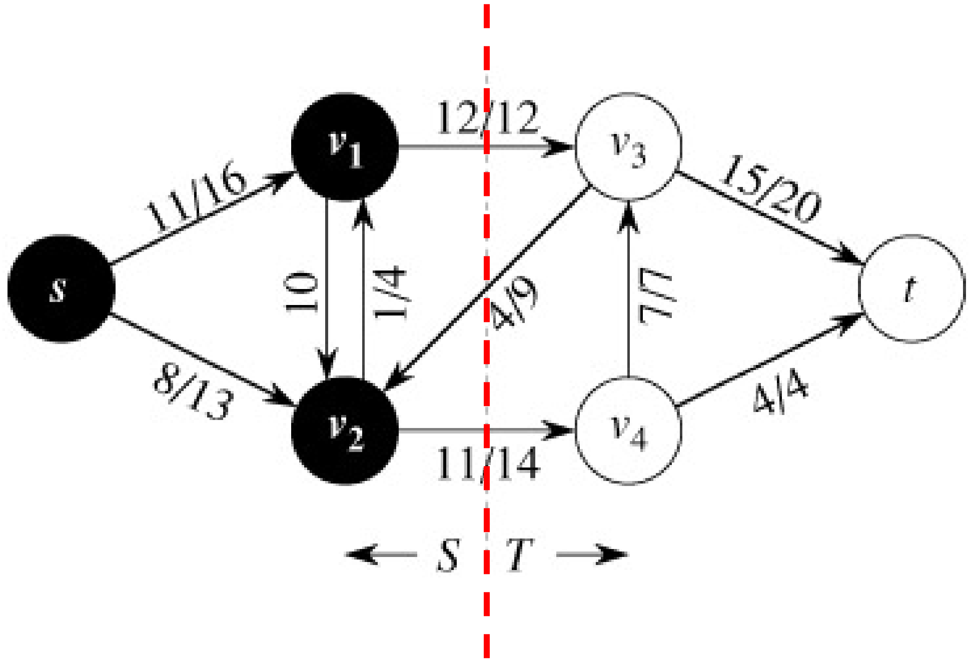
$$f(S,T) = \sum_{u \in S, v \in T} f(u,v)$$



- f(S,T) = 12 – 4 + 11 = 19

The Capacity of a Cut (S,T)

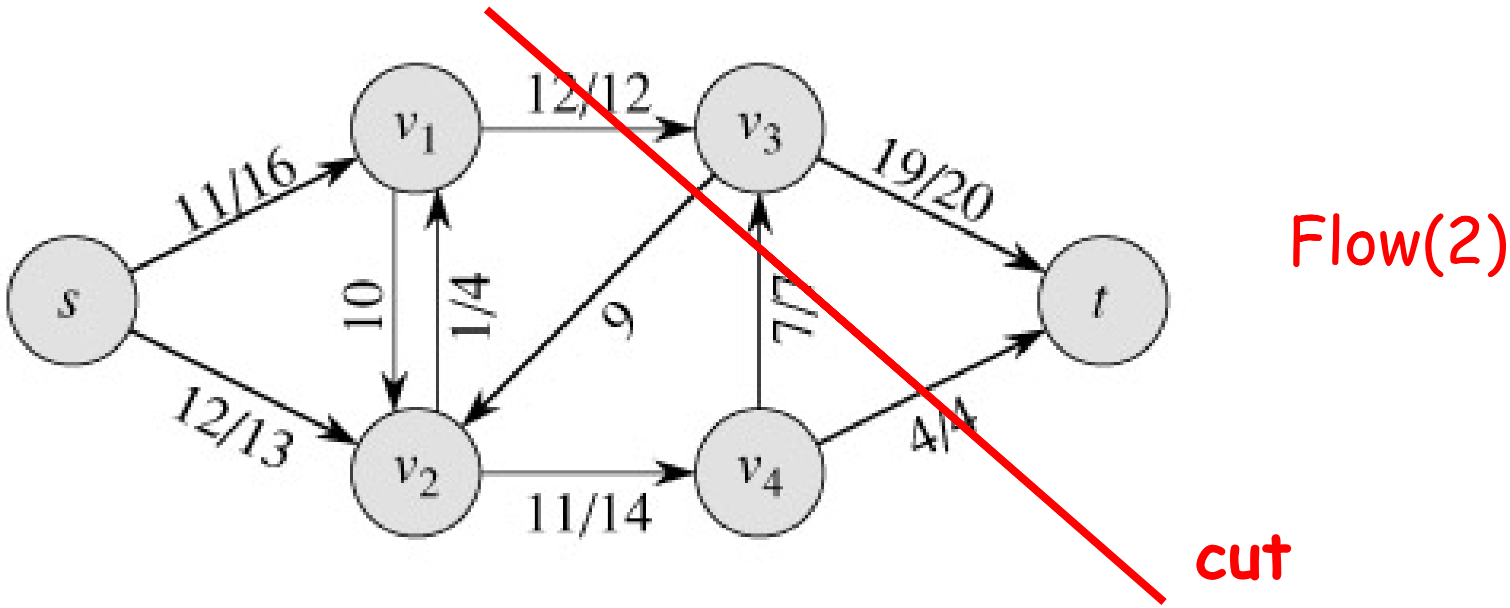
$$c(S,T) = \sum_{u \in S, v \in T} c(u,v)$$



- c(S,T)= 12+ 0 + 14 = 26

Augmenting Paths – example

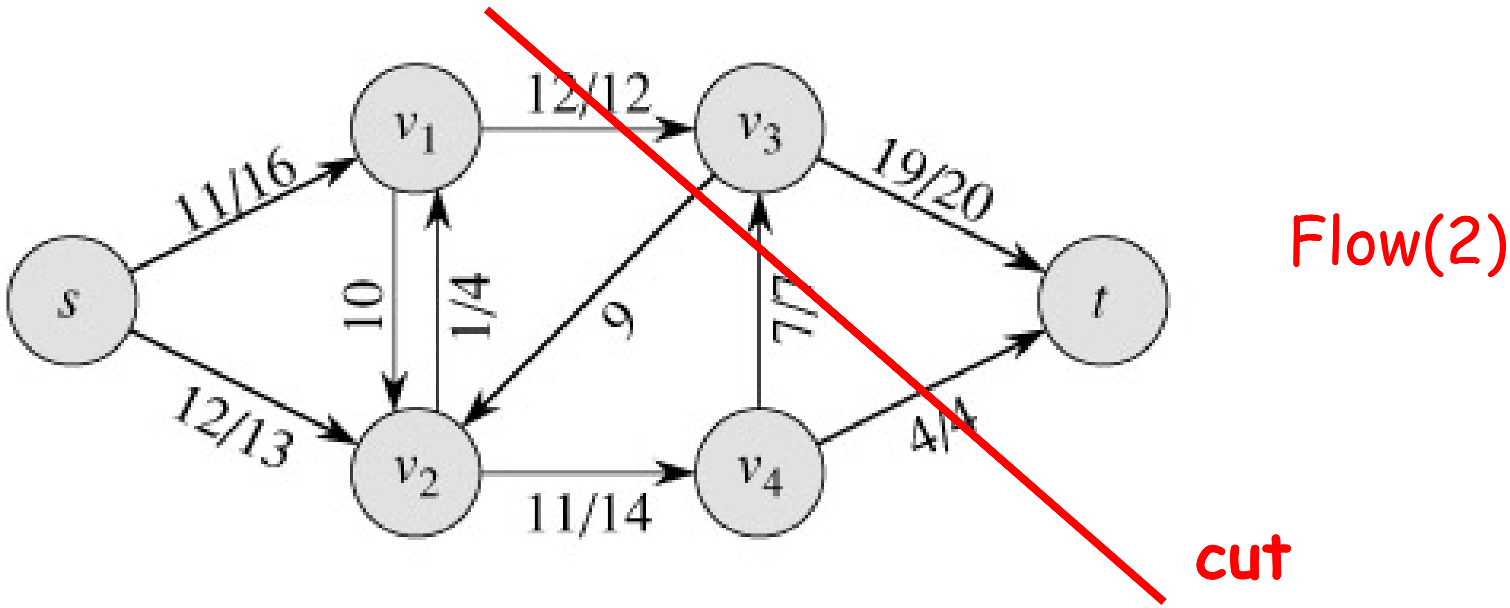
- **Capacity of the cut**
  = maximum possible flow through the cut
  = 12 + 7 + 4 = 23



- The network has a capacity of **at most** 23.

Augmenting Paths – example

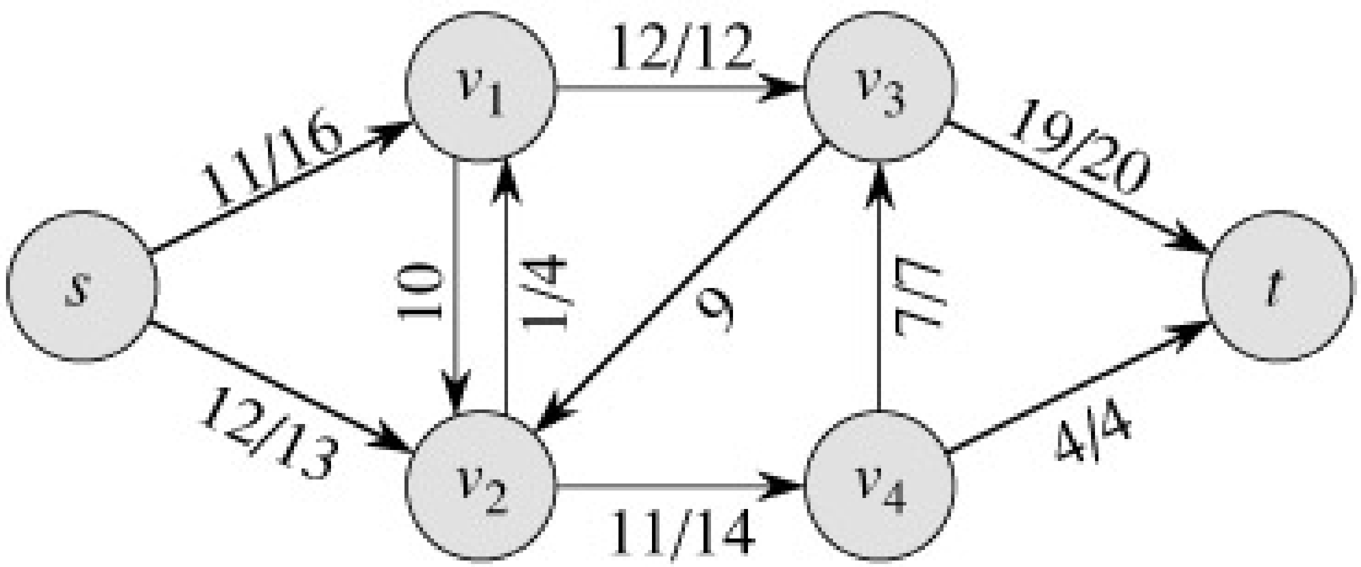
- **Capacity of the cut**
  = maximum possible flow through the cut
  = 12 + 7 + 4 = 23



- The network has a capacity of **at most** 23.
- In this case, the network **does** have a capacity of 23, because this is a **minimum cut**.

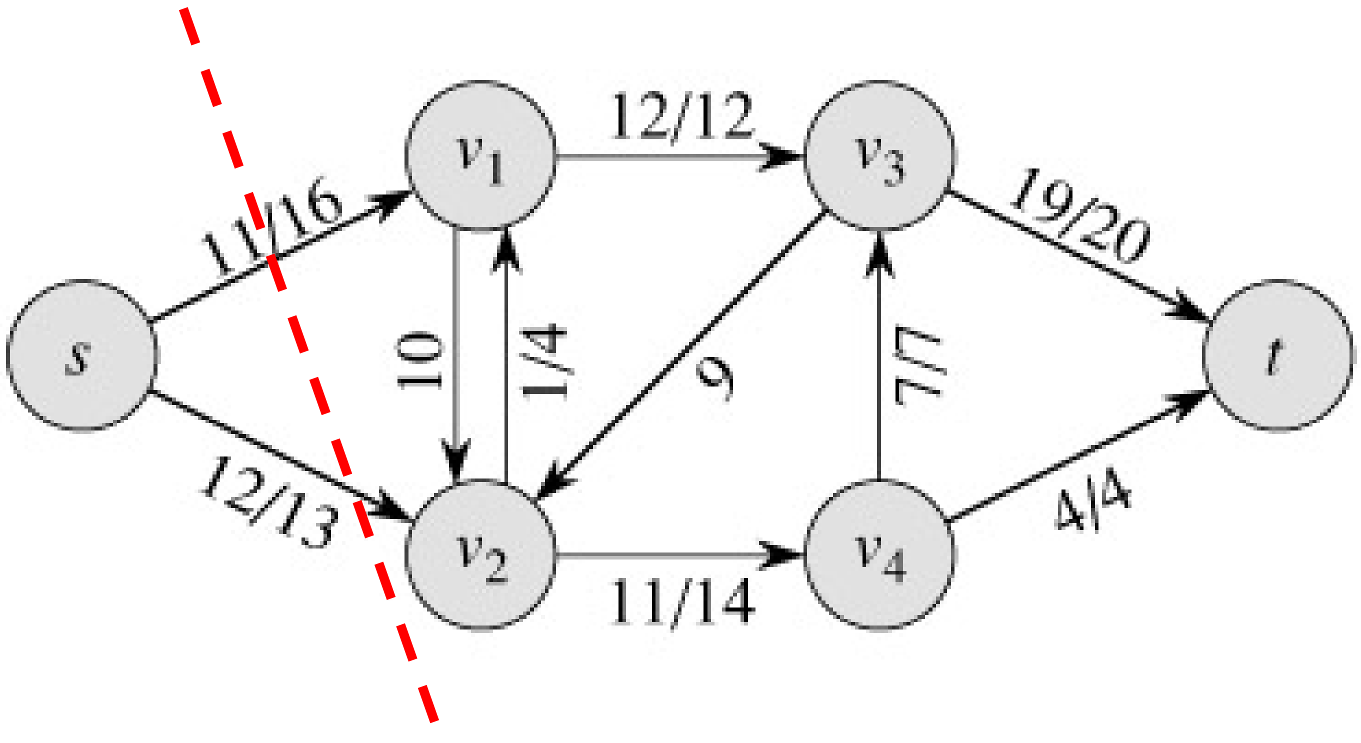
Net Flow of a Network

- The net flow across any cut is the same and equal to the flow of the network |f|.



Net Flow of a Network

- The net flow across any cut is the same and equal to the flow of the network |f|.



Net Flow of a Network

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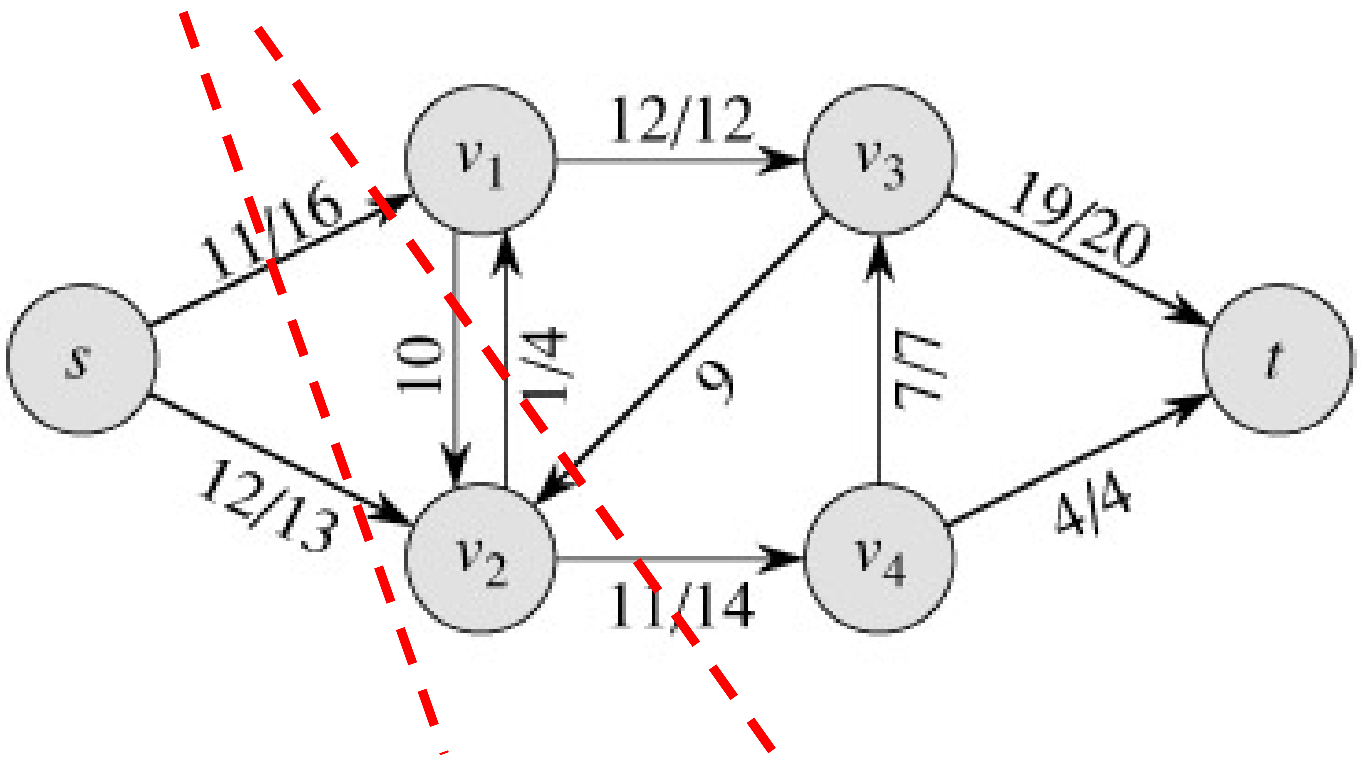

Net Flow of a Network

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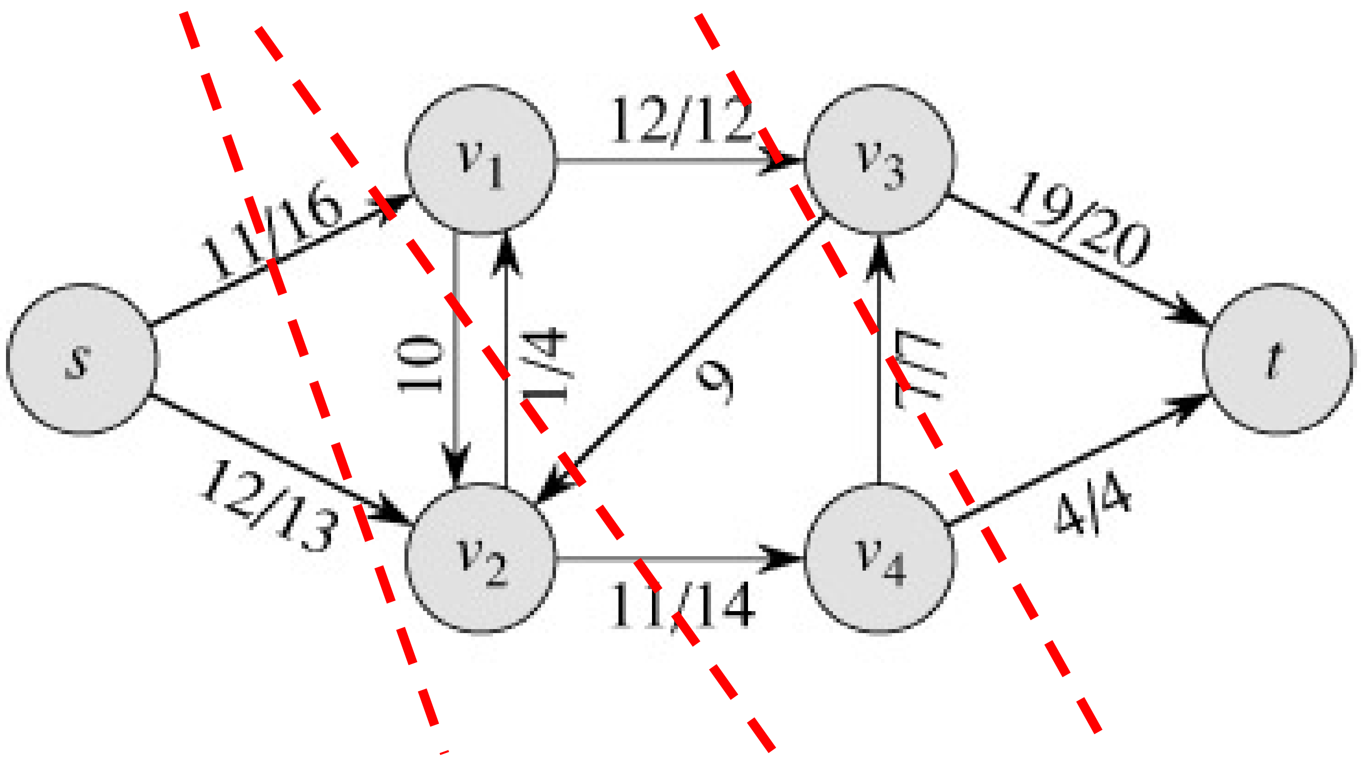
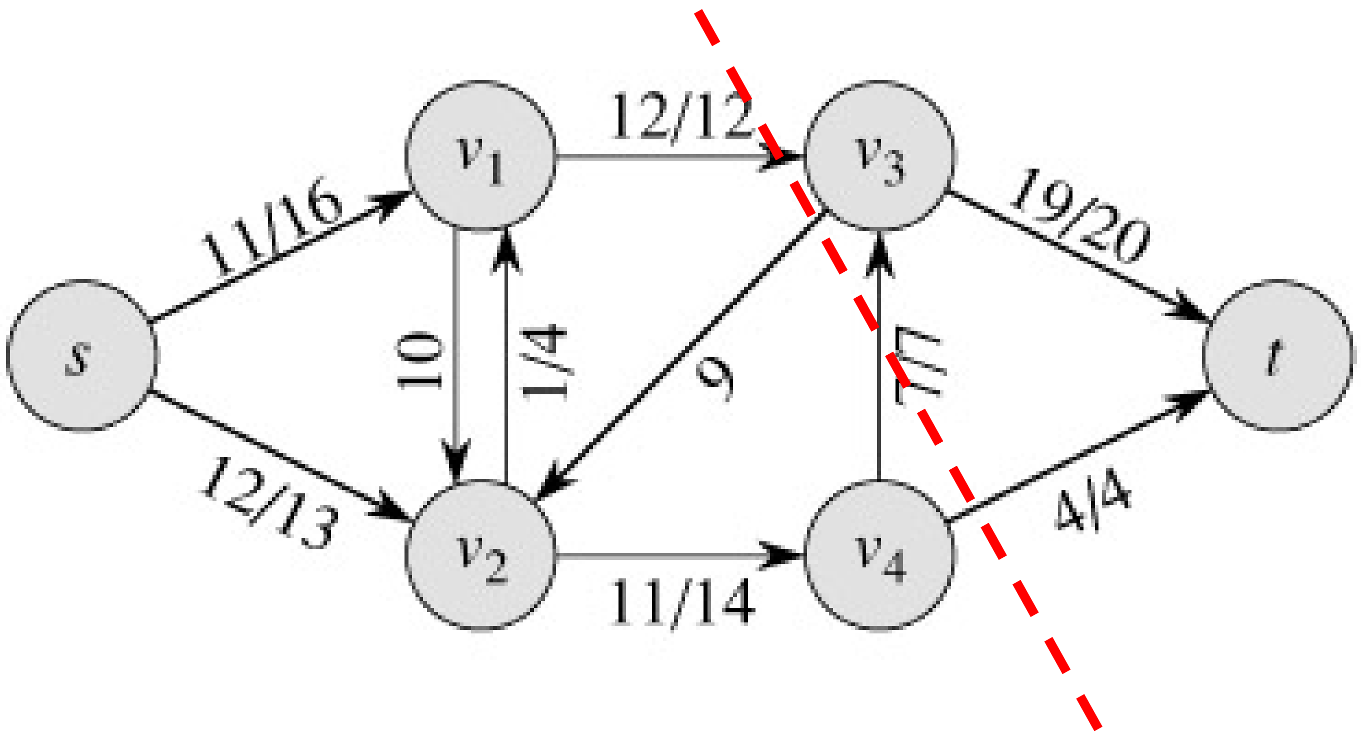


Bounding the Network Flow

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.



Max-Flow Min-Cut Theorem

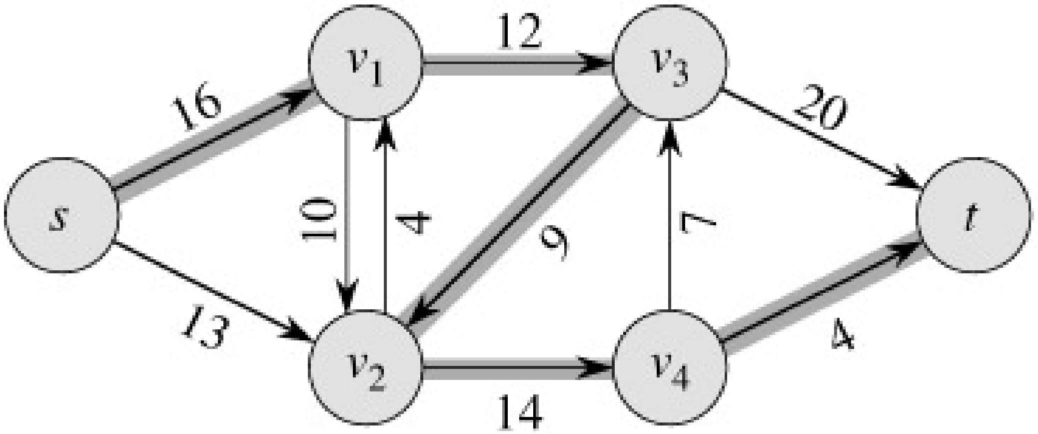
- If f is a flow in a flow network $G=(V,E)$, with source s and sink t, then the following conditions are equivalent:
  1. f is a maximum flow in G.
  2. The residual network G_f contains no augmented paths.
  3. $|f| = c(S,T)$ for some cut (S,T) (a min-cut).

The Basic Ford-Fulkerson Algorithm

FORD-FULKERSON(G, s, t)

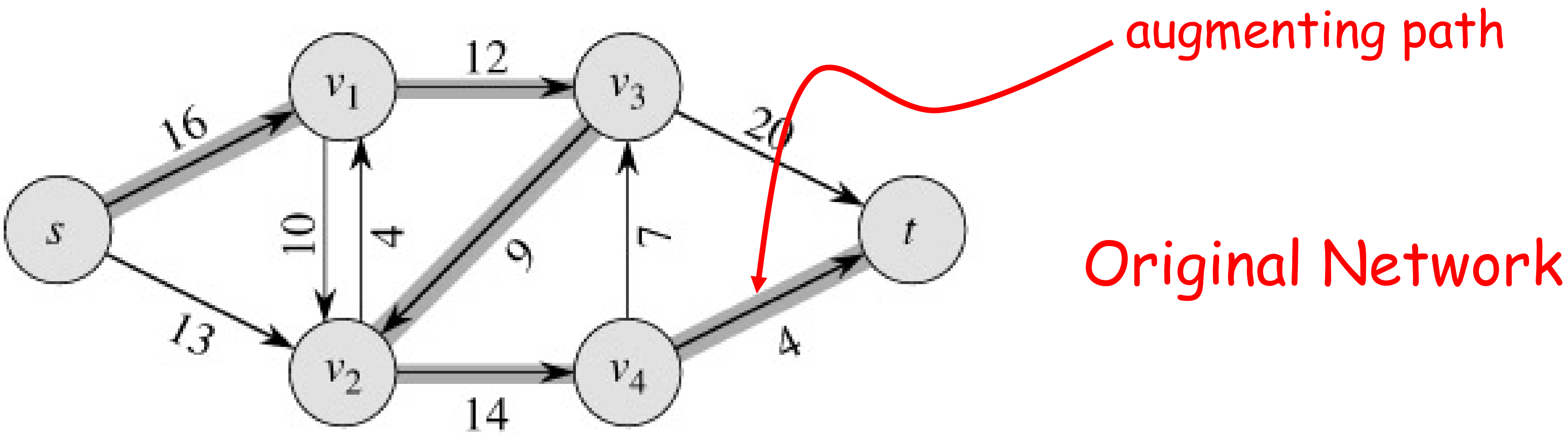
for each edge $(u, v) \in E[G]$
    do $f[u, v] \leftarrow 0$
        $f[v, u] \leftarrow 0$
while there exists a path p from s to t in the residual network G_f
    do $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
        for each edge (u, v) in p
            do $f[u, v] \leftarrow f[u, v] + c_f(p)$
                $f[v, u] \leftarrow -f[u, v]$

Example

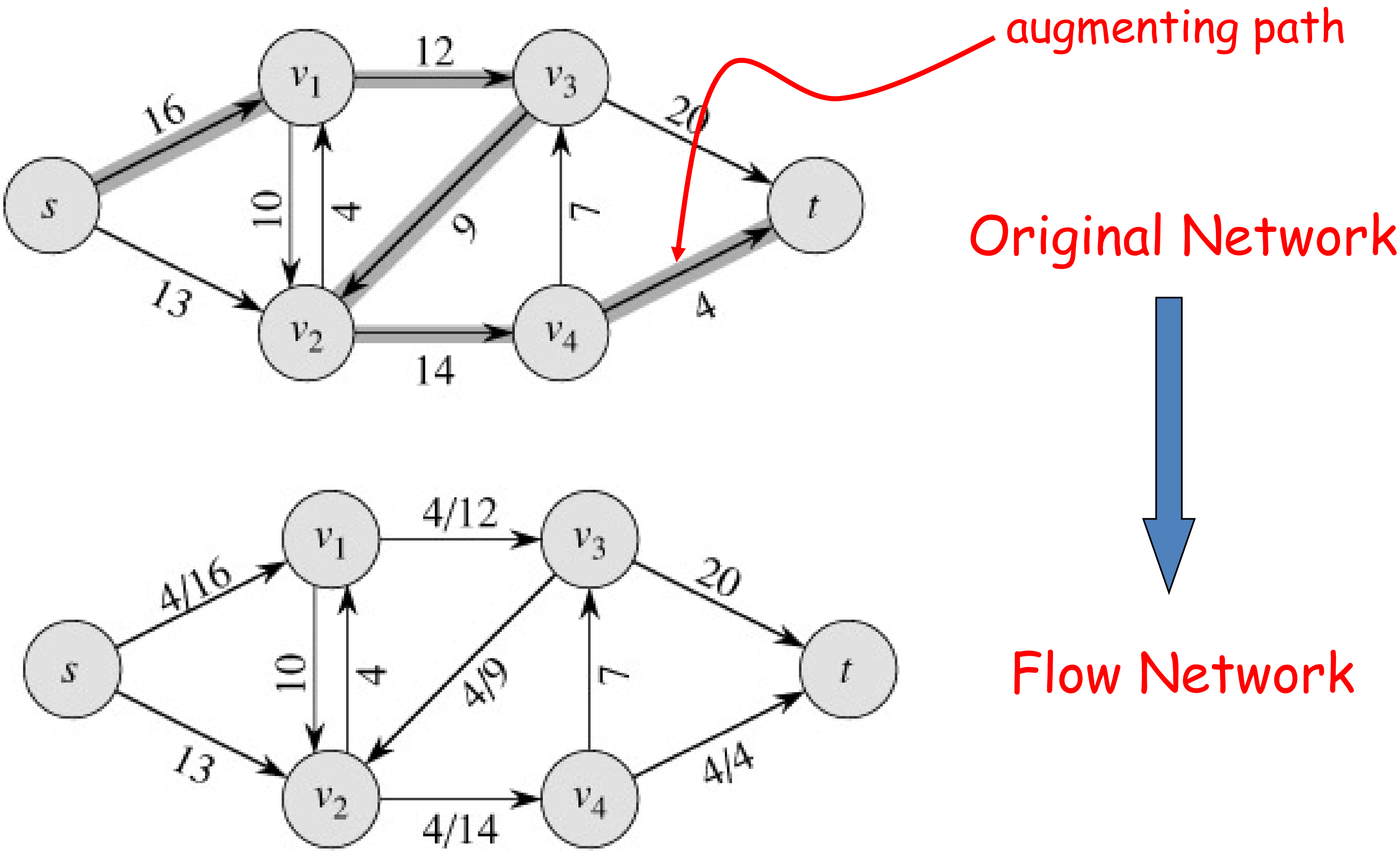


Original Network

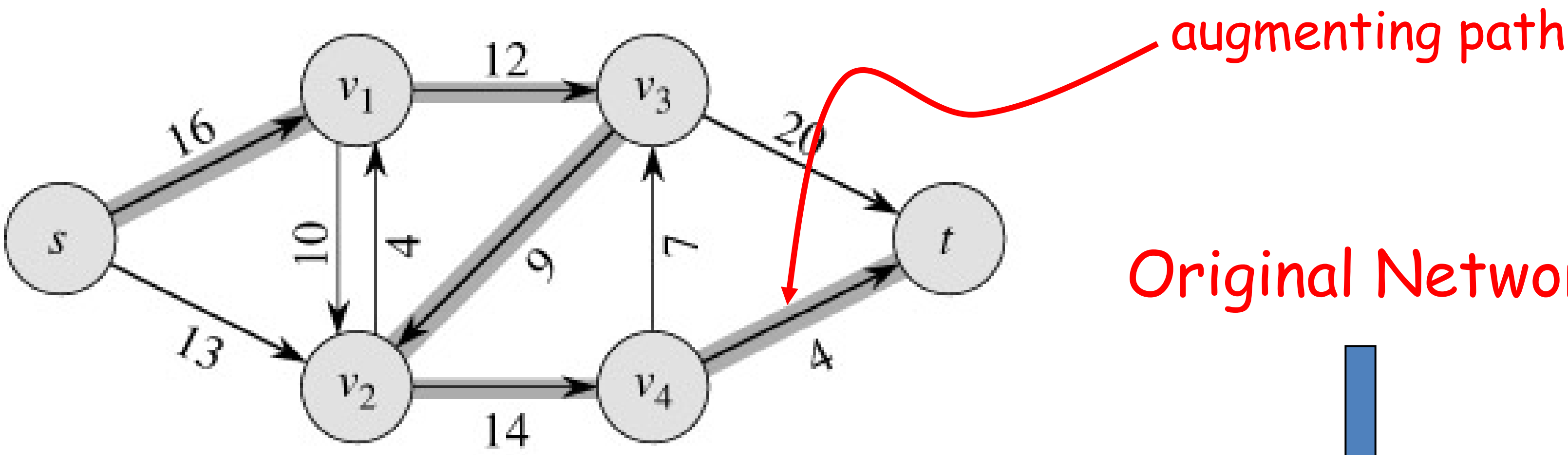
Example


augmenting path
16
12
20
10
4
9
7
13
14
4
s
t
v_1
v_2
v_3
v_4
Original Network

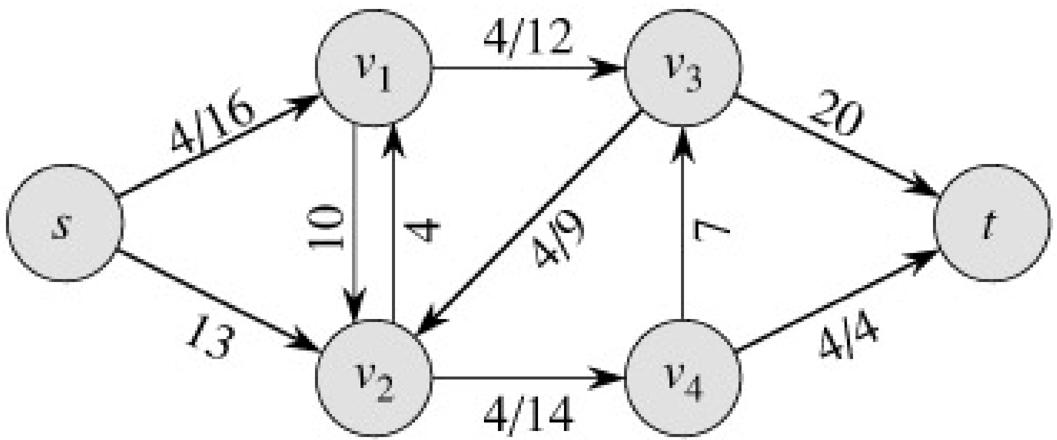
Example


augmenting path
Original Network
Flow Network
s
v_1
v_2
v_3
v_4
t
16
12
20
10
4
9
7
13
14
4
4/16
4/12
20
10
4
4/9
7
13
4/14
4/4

Example



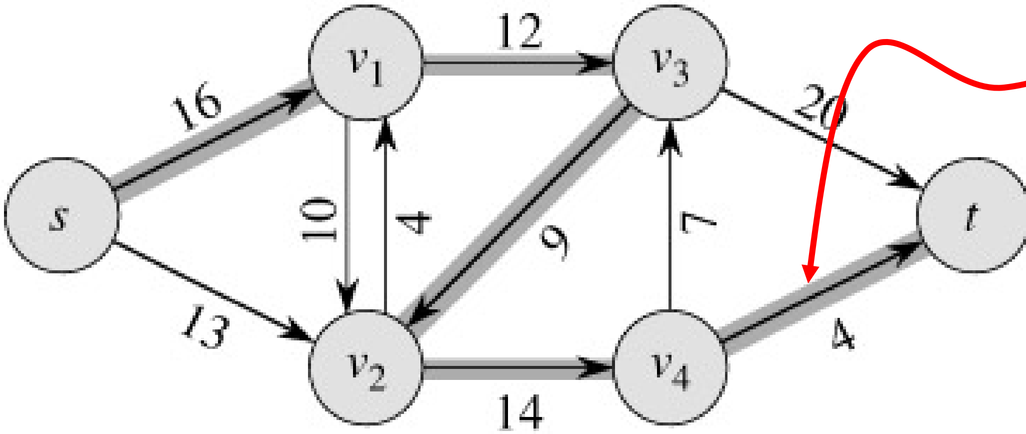
Original Network



Flow Network

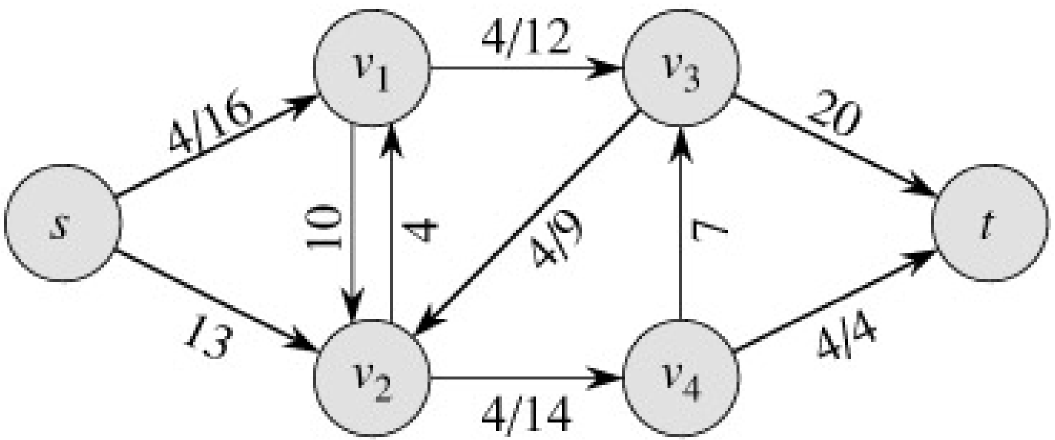
Resulting Flow = ?

Example



augmenting path

Original Network

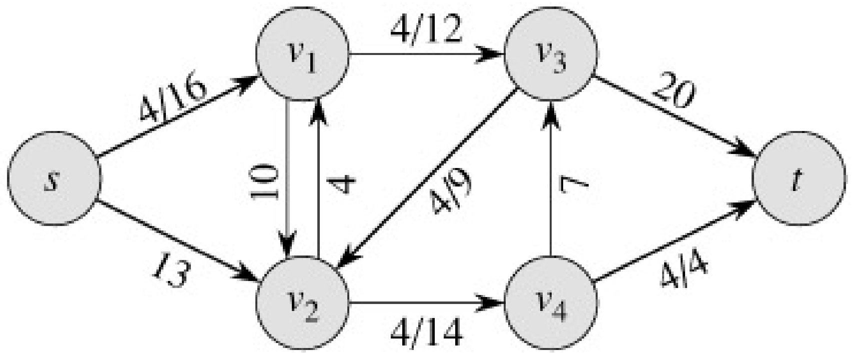


Flow Network

Resulting Flow = 4

Example

Flow Network

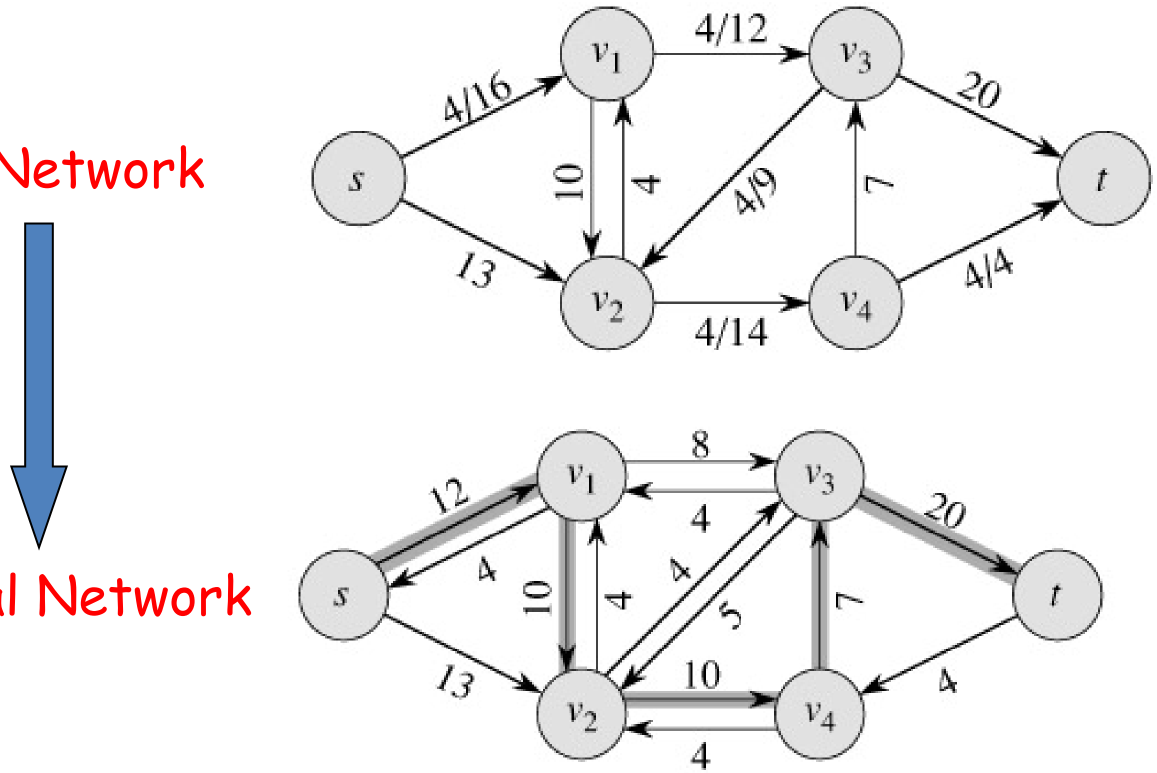


Resulting Flow = 4

Example

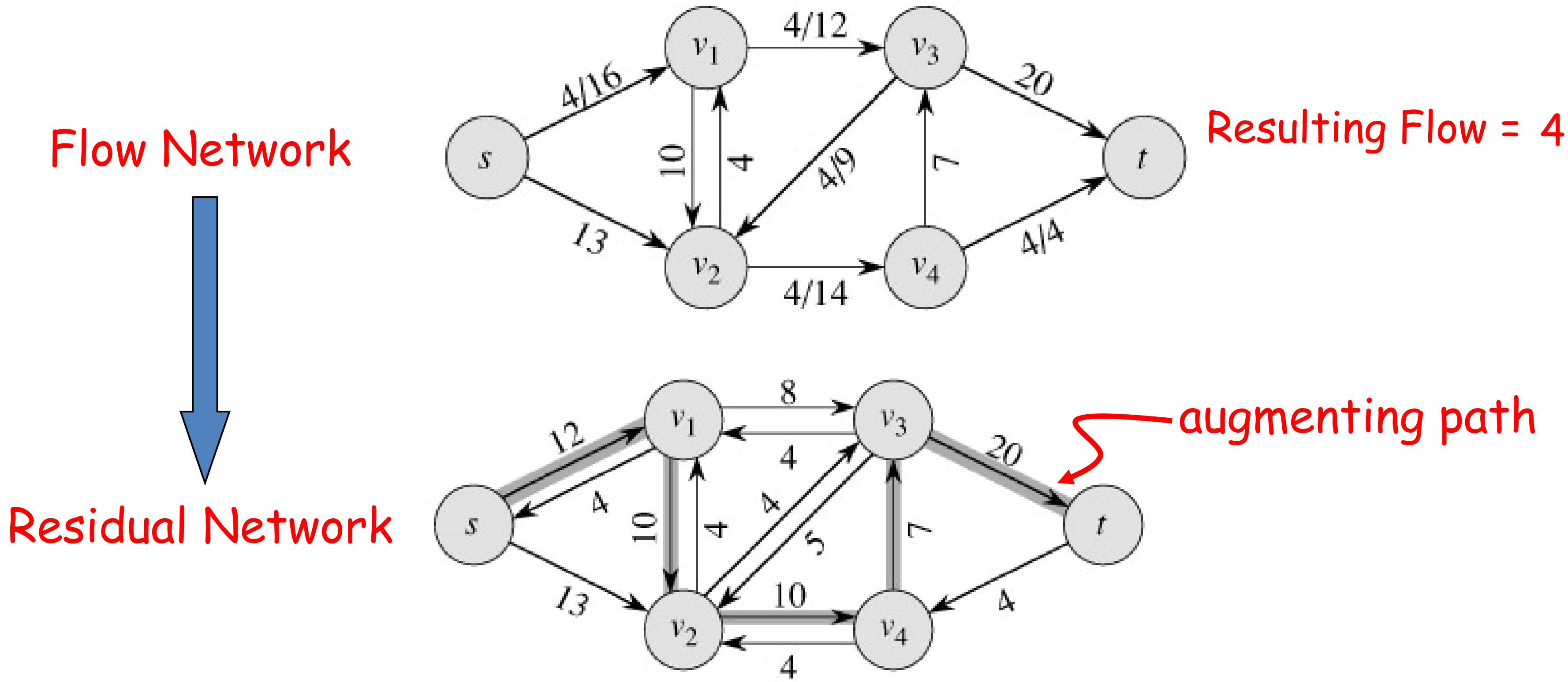
Flow Network

Resulting Flow = 4



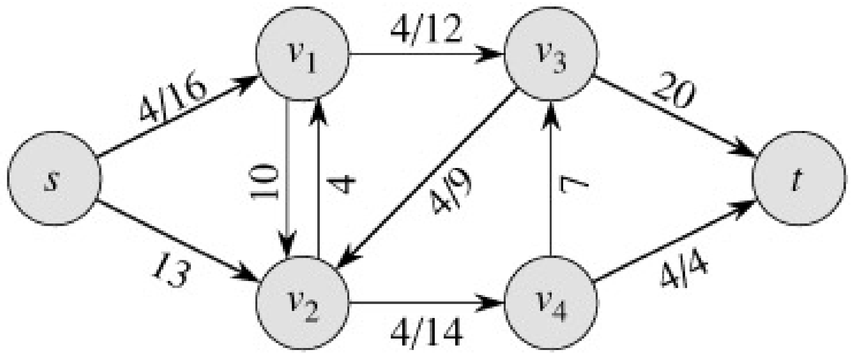
Residual Network

Example


Flow Network
Residual Network
s
v1
v2
v3
v4
t
4/16
13
10
4
4/12
4/9
7
4/14
20
4/4
Resulting Flow = 4
12
4
8
5
10
augmenting path

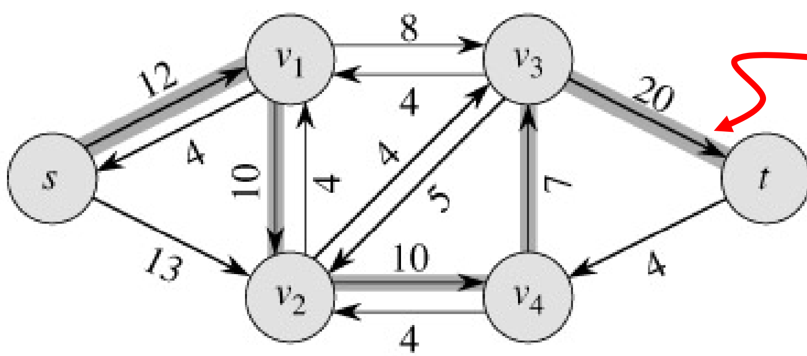
Example

Flow Network



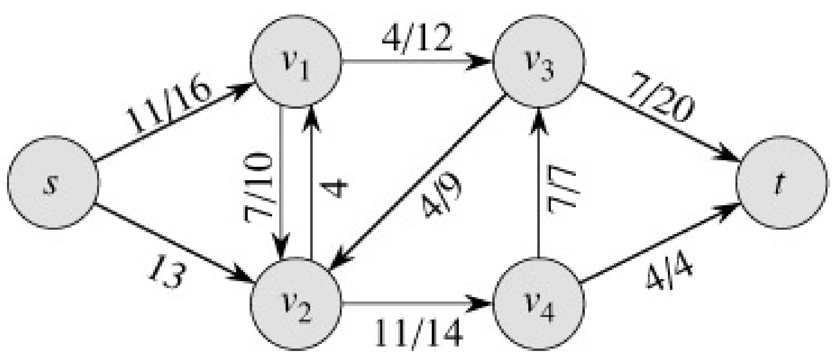
Resulting Flow = 4

Residual Network



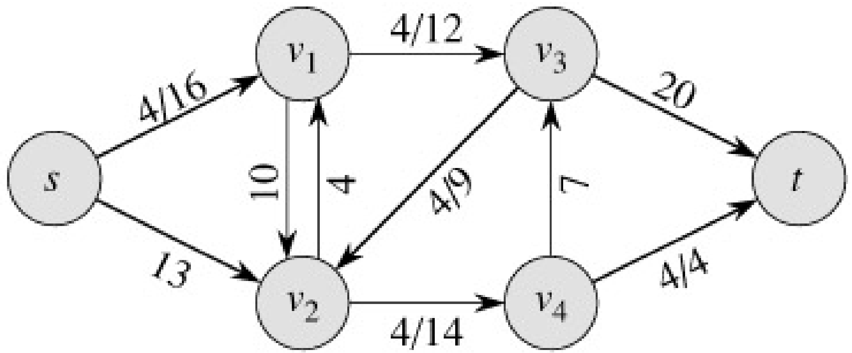
augmenting path

Flow Network



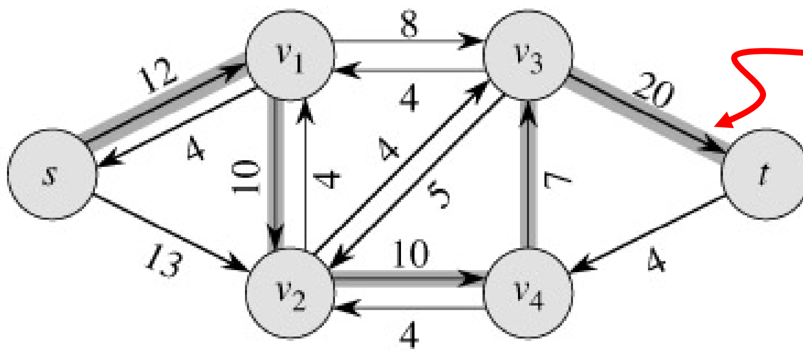
Example

Flow Network



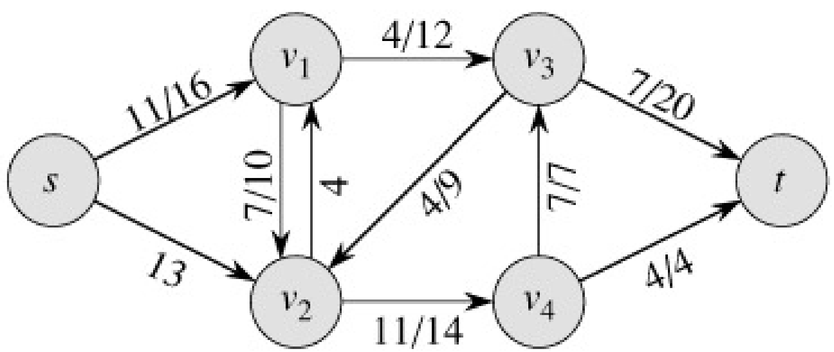
Resulting Flow = 4

Residual Network



augmenting path

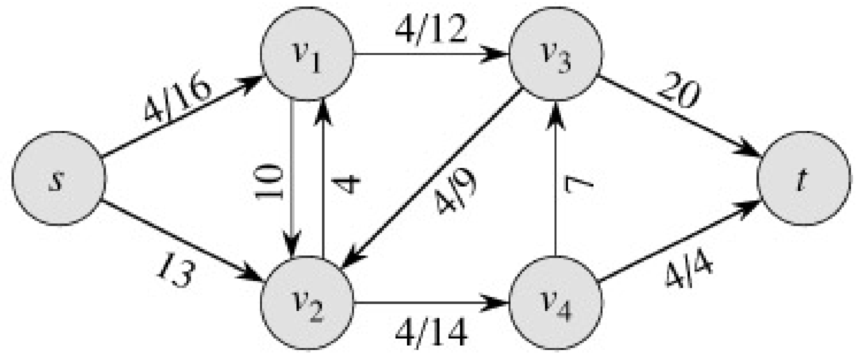
Flow Network



Resulting Flow = ?

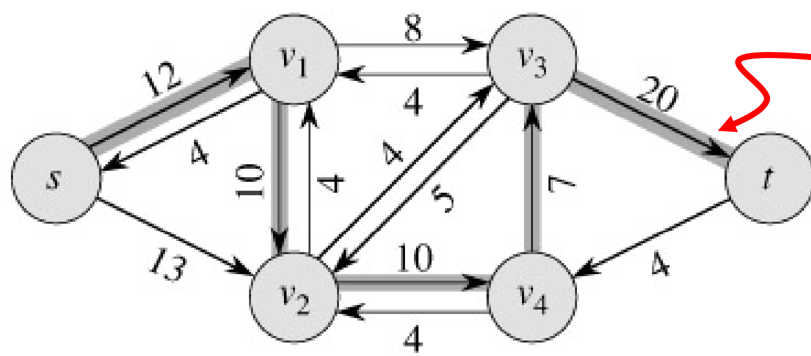
Example

Flow Network



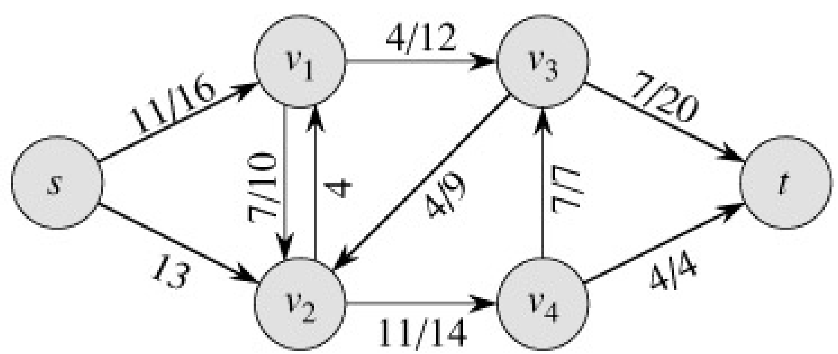
Resulting Flow = 4

Residual Network



augmenting path

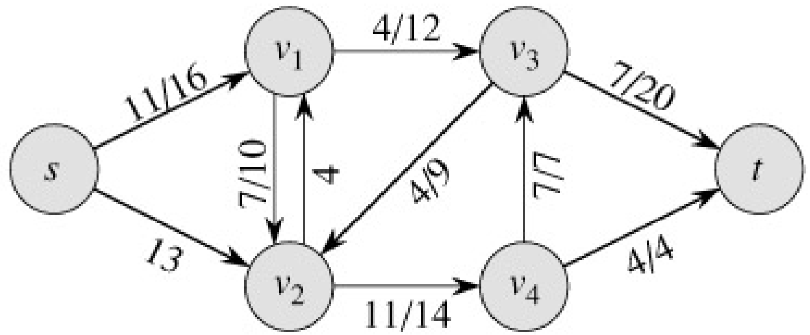
Flow Network



Resulting Flow = 11

Example

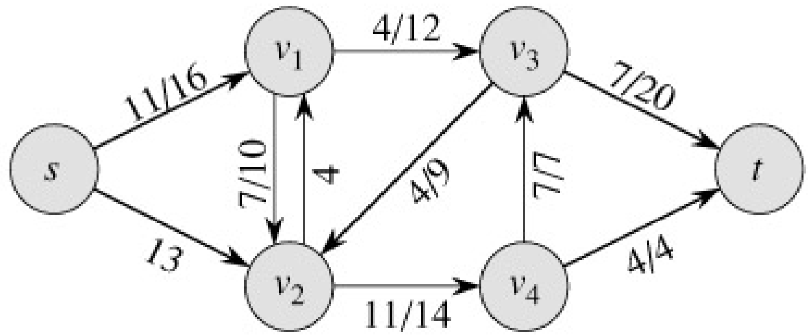
Flow Network



Resulting Flow = 11

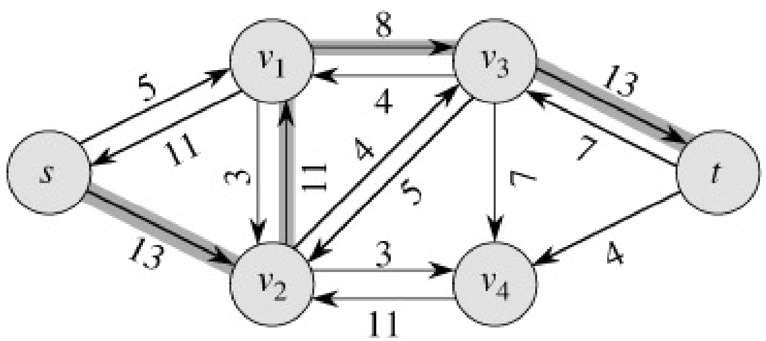
Example

Flow Network



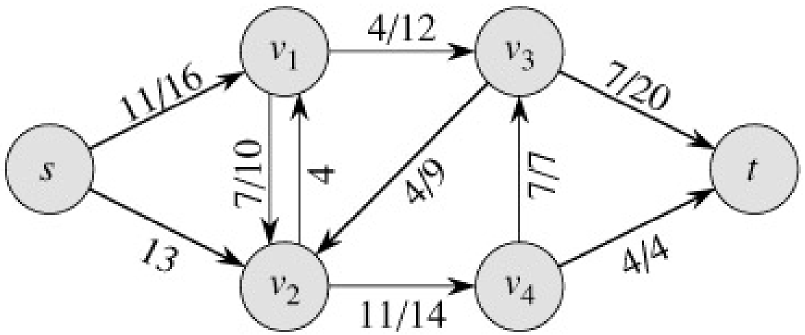
Resulting Flow = 11

Residual Network



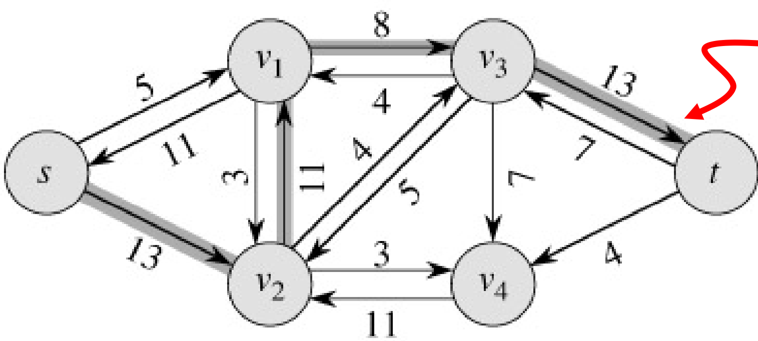
Example

Flow Network



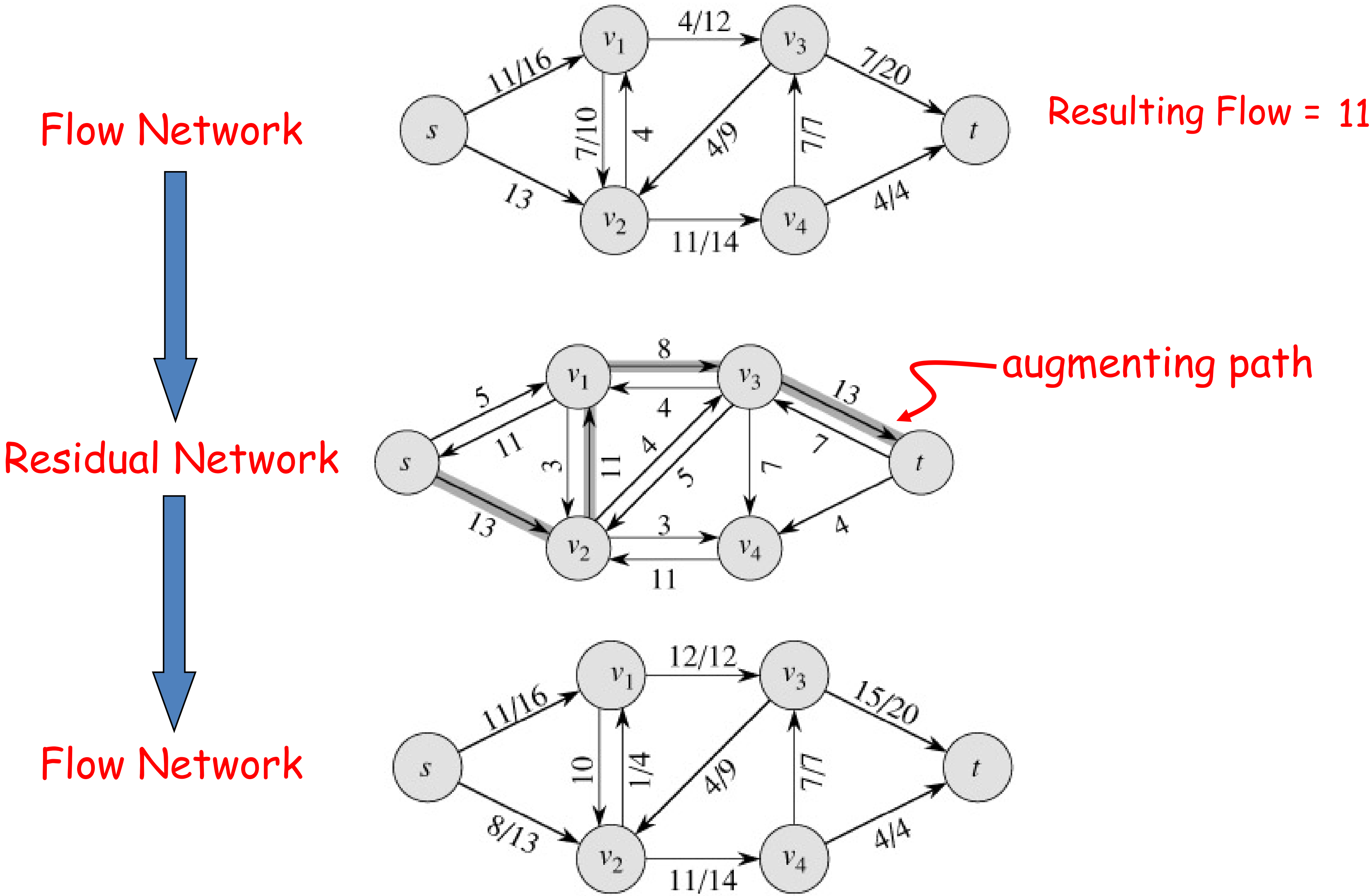
Resulting Flow = 11

Residual Network



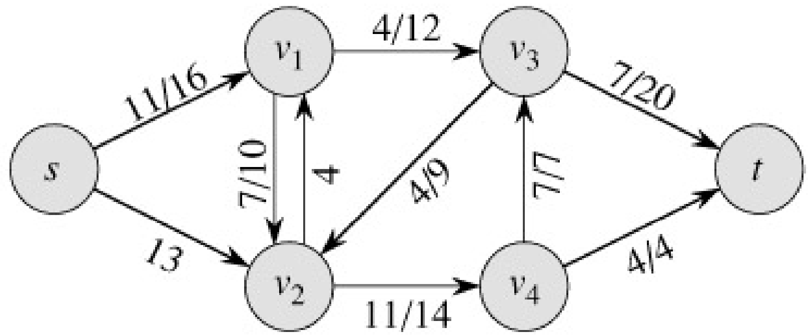
augmenting path

Example


Flow Network
Residual Network
Flow Network
Resulting Flow = 11
augmenting path
s
t
v_1
v_2
v_3
v_4
11/16
13
7/10
4
4/12
4/9
7/7
7/20
4/4
11/14
5
11
8
4
3
11
4
5
7
13
7
4
3
11
13
11/16
8/13
10
1/4
12/12
4/9
7/7
15/20
4/4
11/14

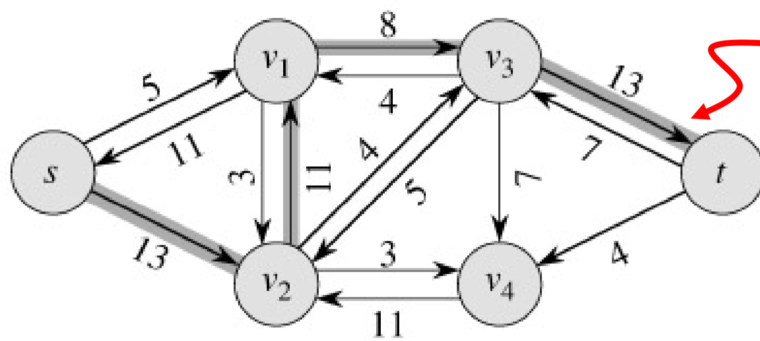
Example

Flow Network



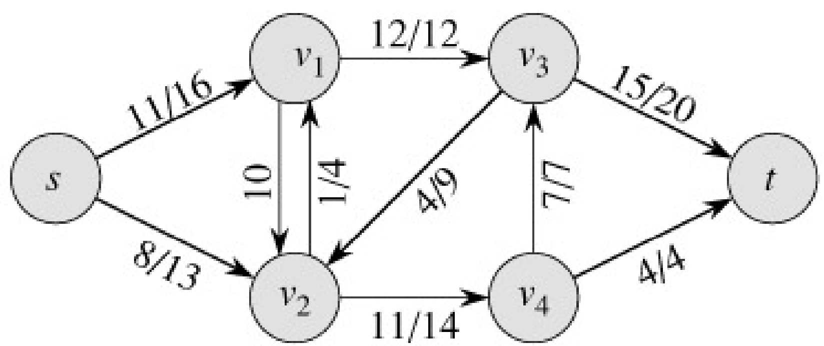
Resulting Flow = 11

Residual Network



augmenting path

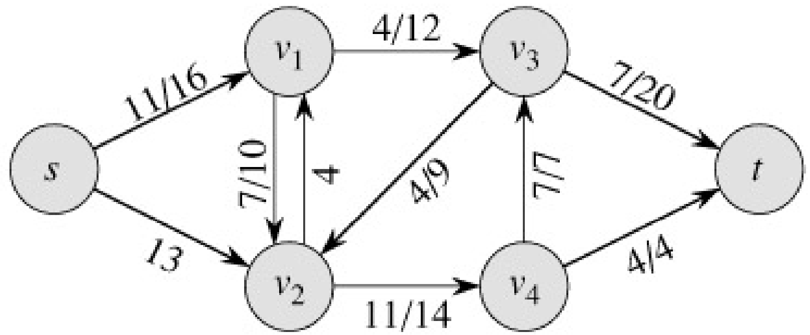
Flow Network



Resulting Flow = ?

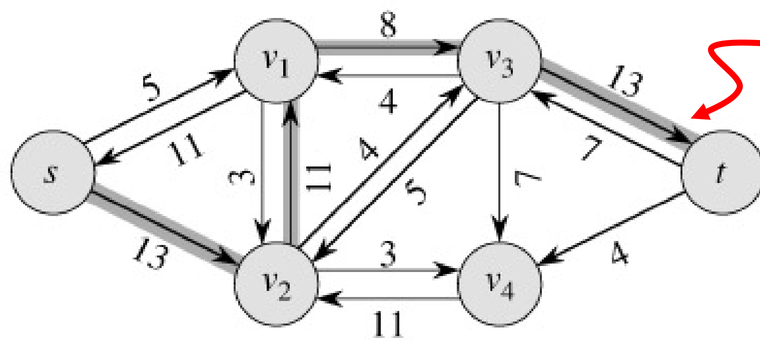
Example

Flow Network



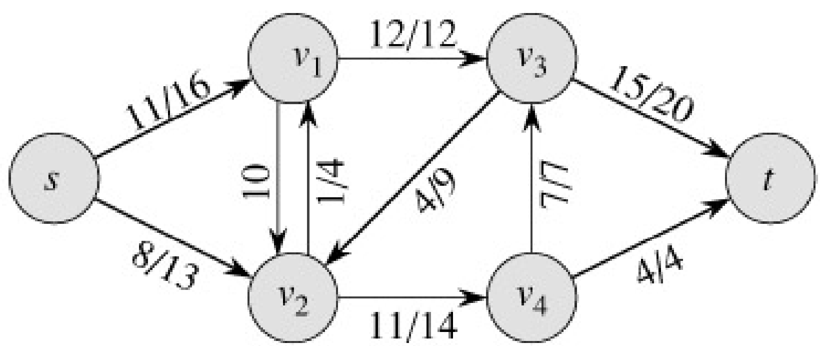
Resulting Flow = 11

Residual Network



augmenting path

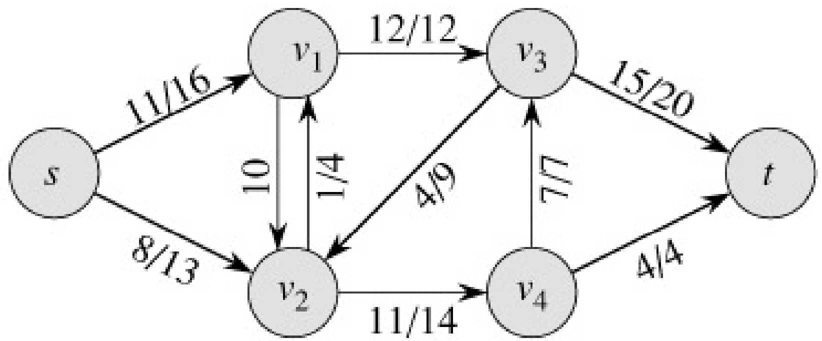
Flow Network



Resulting Flow = 19

Example

Flow Network

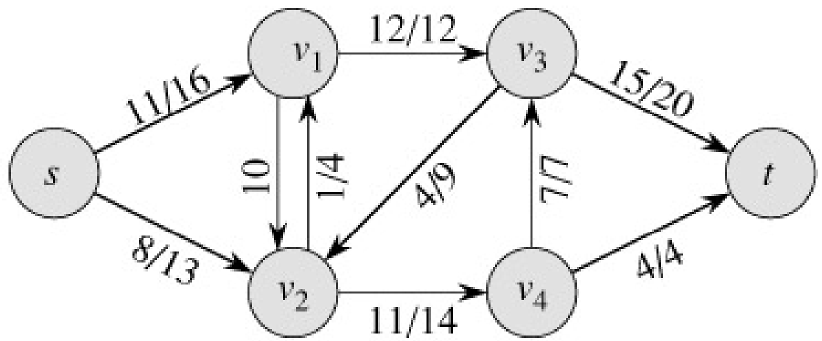


Resulting Flow = 19

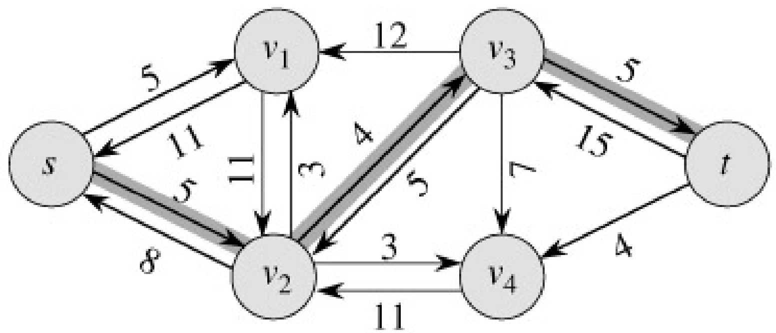
Example

Flow Network

Resulting Flow = 19

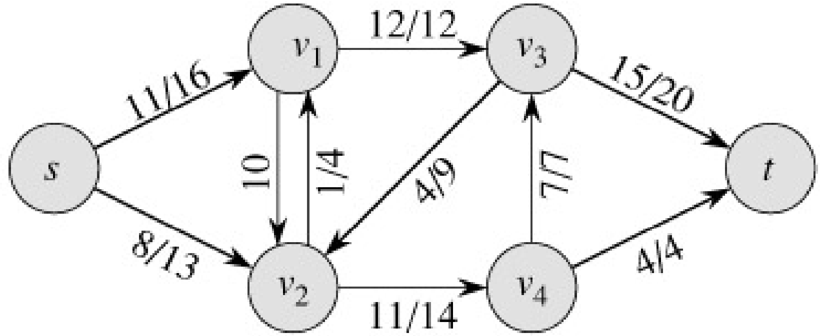


Residual Network



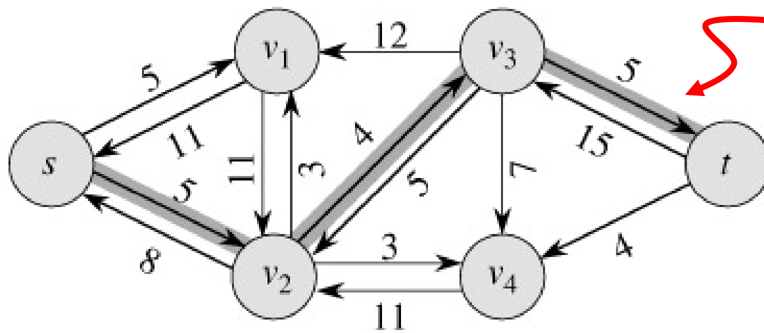
Example

Flow Network



Resulting Flow = 19

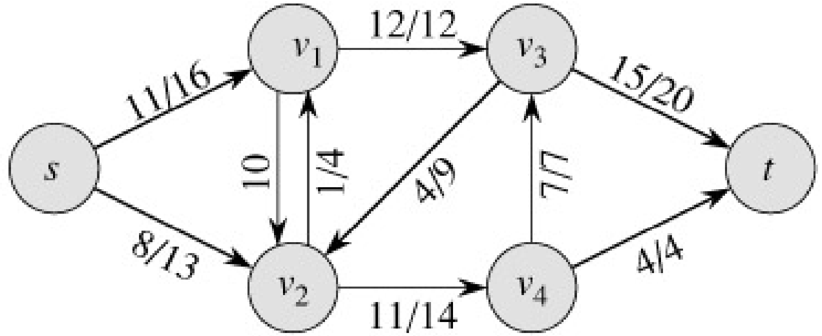
Residual Network



augmenting path

Example

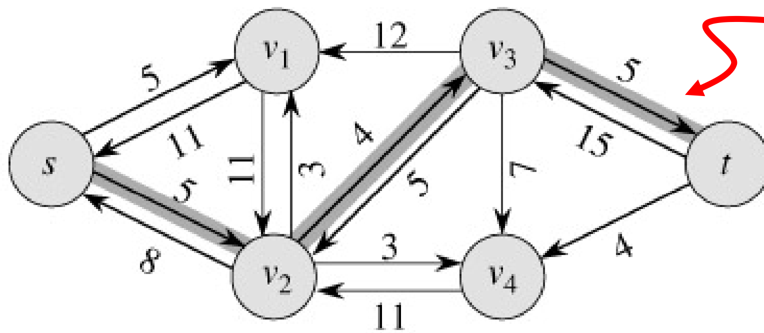
Flow Network



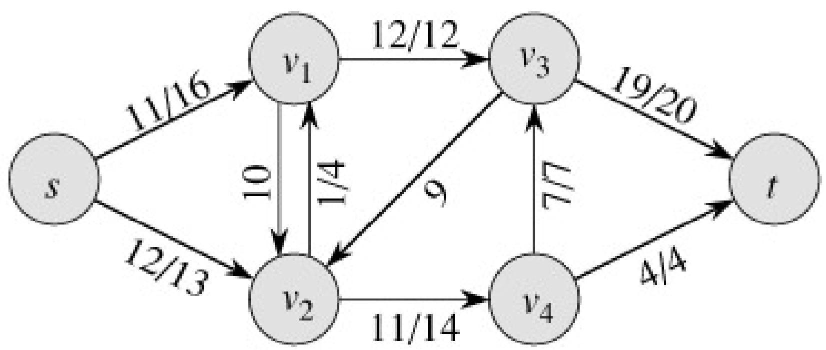
Resulting Flow = 19

Residual Network

augmenting path

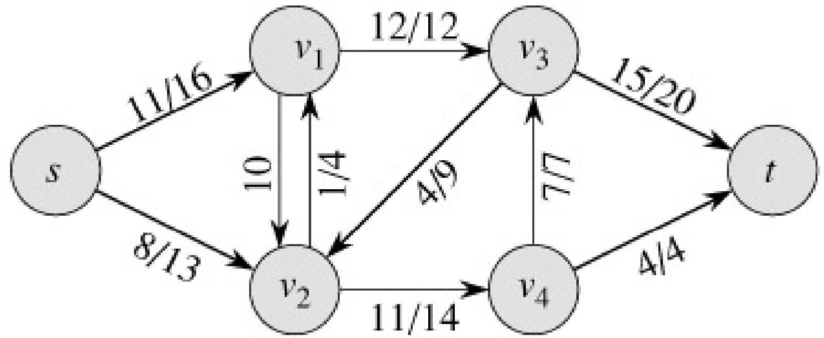


Flow Network



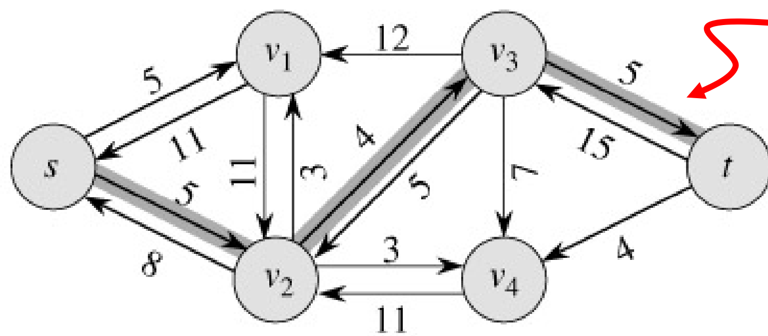
Example

Flow Network



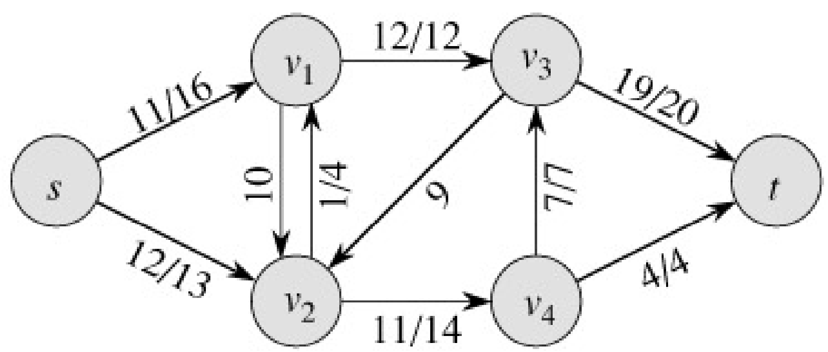
Resulting Flow = 19

Residual Network



augmenting path

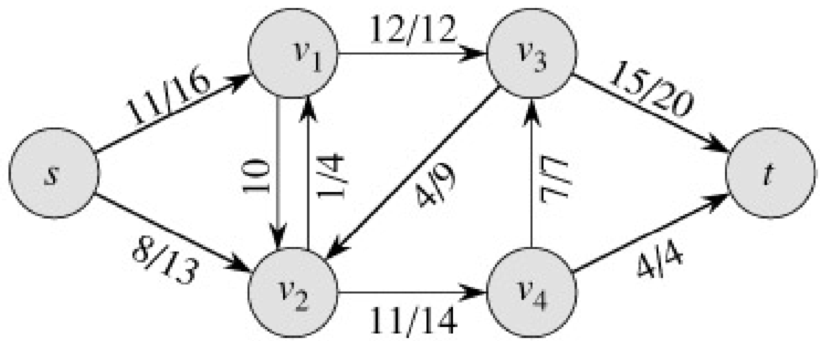
Flow Network



Resulting Flow = ?

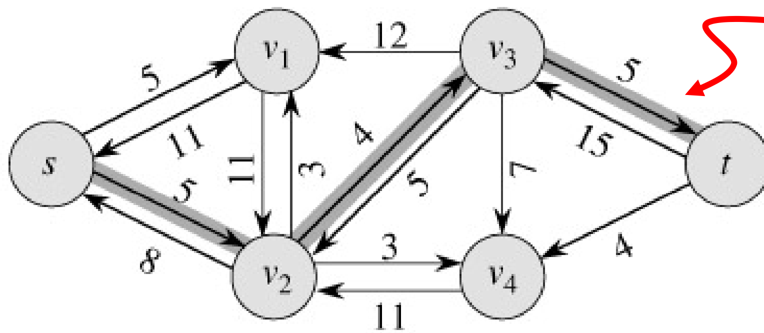
Example

Flow Network



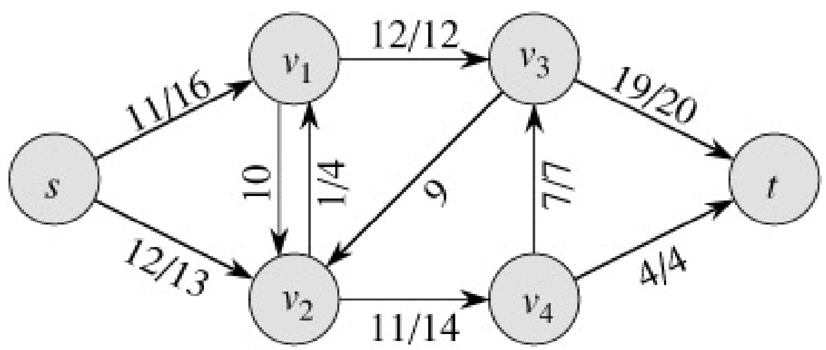
Resulting Flow = 19

Residual Network



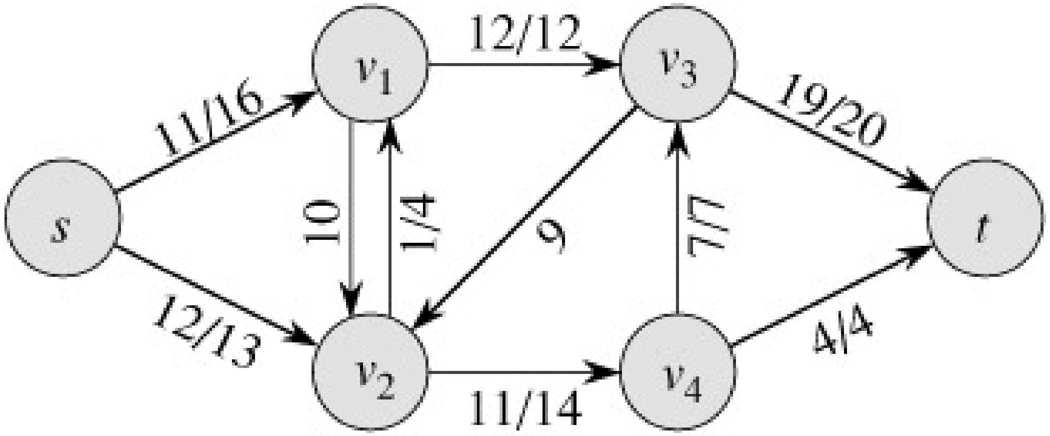
augmenting path

Flow Network



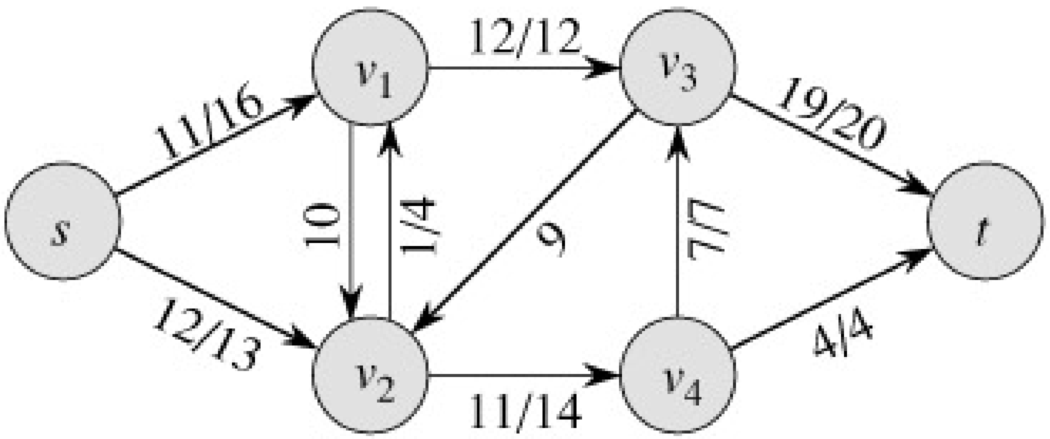
Resulting Flow = 23

Example

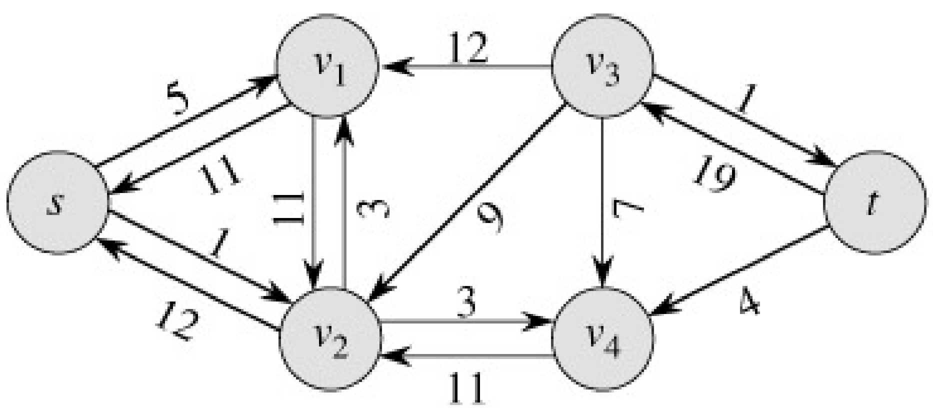


Resulting Flow = 23

Example

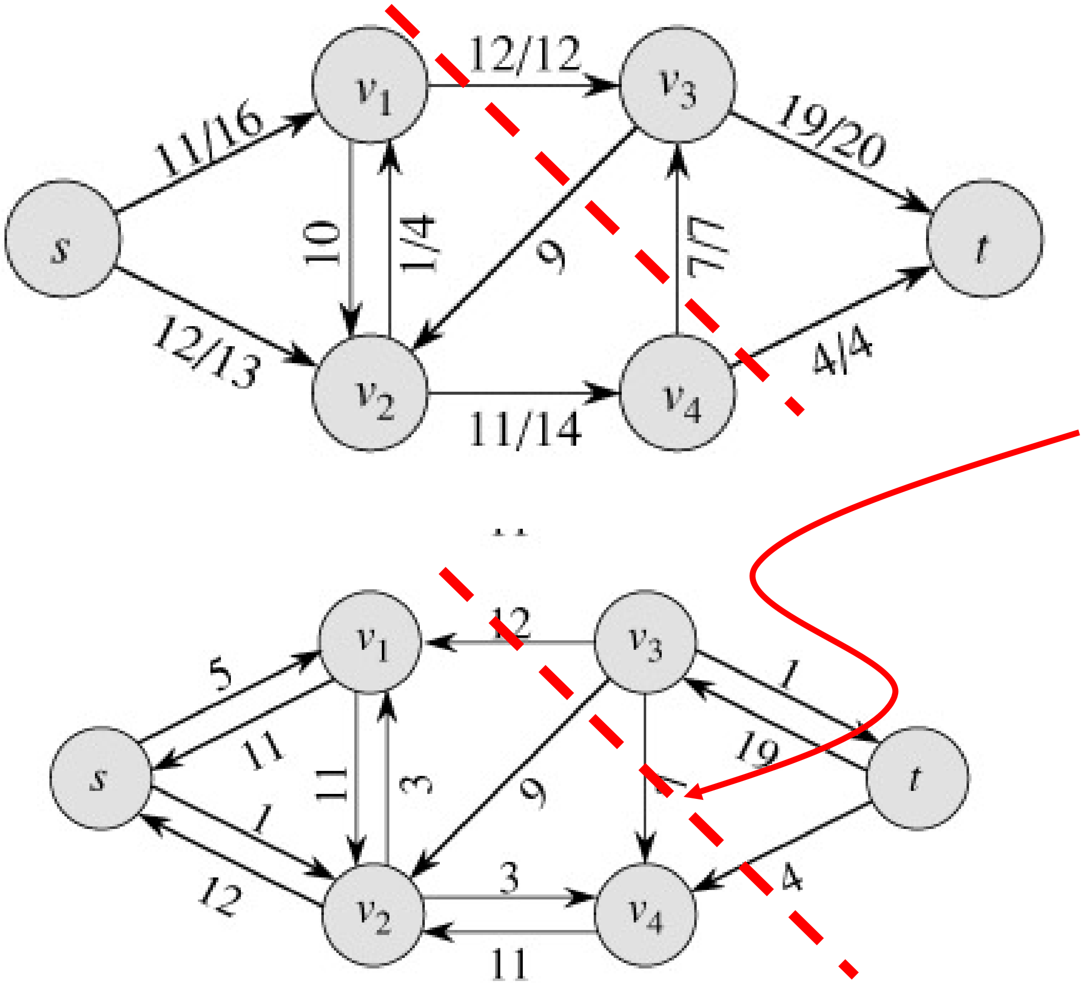


Resulting
Flow = 23



Residual Network

Example



Resulting
Flow = 23

No augmenting path:

Maxflow=23

Residual Network

Analysis

FORD-FULKERSON(G, s, t)

for each edge $(u, v) \in E[G]$
    do $f[u, v] \leftarrow 0$
        $f[v, u] \leftarrow 0$
while there exists a path p from s to t in the residual network G_f
    do $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
        for each edge (u, v) in p
            do $f[u, v] \leftarrow f[u, v] + c_f(p)$
                $f[v, u] \leftarrow -f[u, v]$

Analysis

```
FORD-FULKERSON(G, s, t)
for each edge (u, v) ∈ E[G]
        do f[u, v] ← 0
           f[v, u] ← 0
while there exists a path p from s to t in the residual network G_f
        do c_f(p) ← min {c_f(u, v) : (u, v) is in p}
           for each edge (u, v) in p
                do f[u, v] ← f[u, v] + c_f(p)
                   f[v, u] ← −f[u, v]
```

O(E)

Analysis

FORD-FULKERSON(G, s, t)

for each edge $(u, v) \in E[G]$
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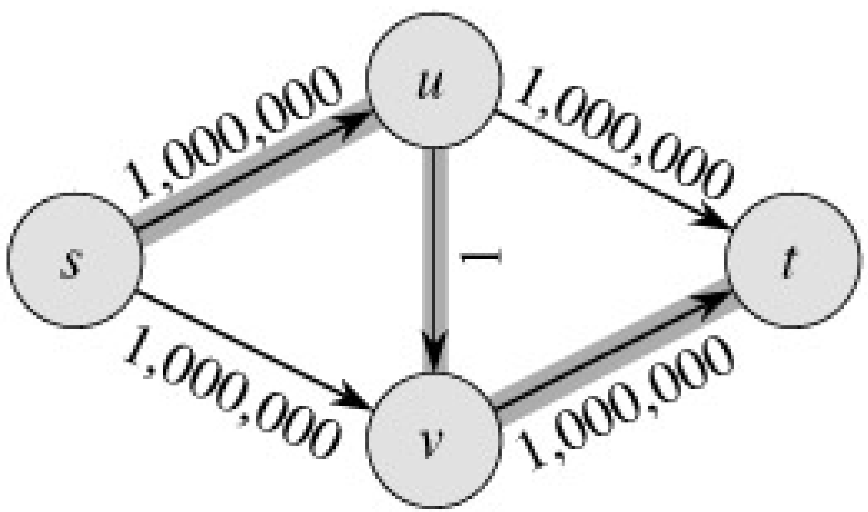
- Note that this running time is **not polynomial** in input size. It depends on $|f^*|$, which is not a function of $|V|$ or $|E|$.
- If capacities are rational, can scale them to integers.
- If irrational, FORD-FULKERSON might never terminate!

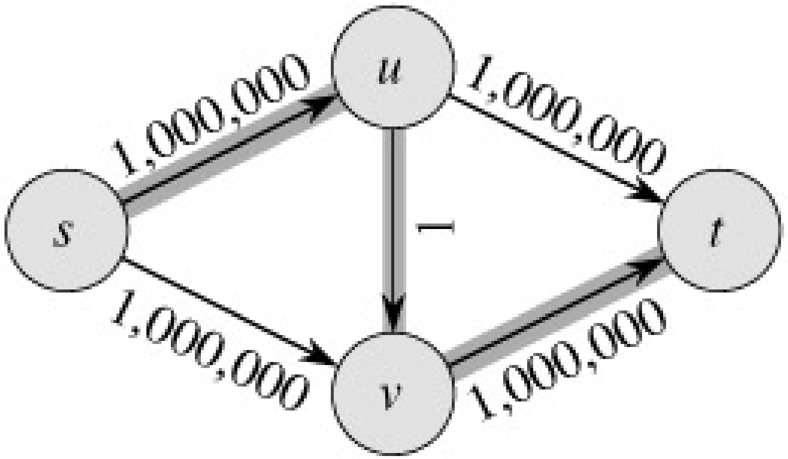
The Basic Ford-Fulkerson Algorithm

- With time O (E |f*|), the algorithm is **not** polynomial.
- This problem is real: Ford-Fulkerson may perform very badly if we are unlucky:

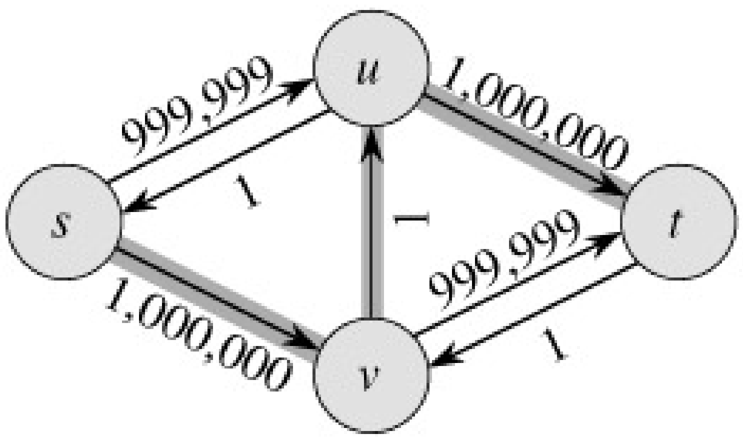


|f*|=2,000,000

Run Ford-Fulkerson on this example



Augmenting Path



Residual Network

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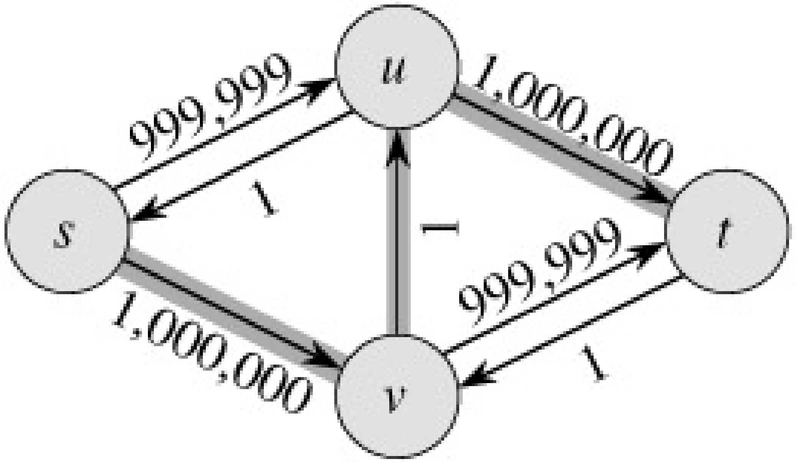

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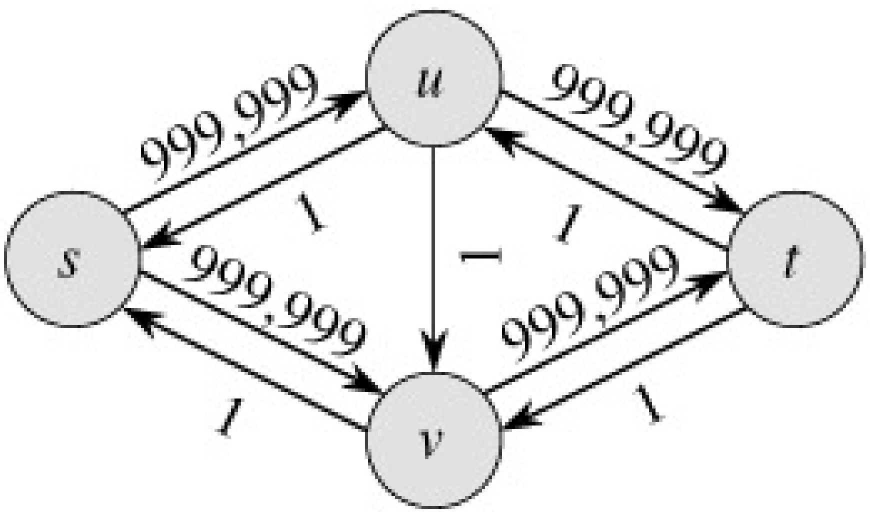

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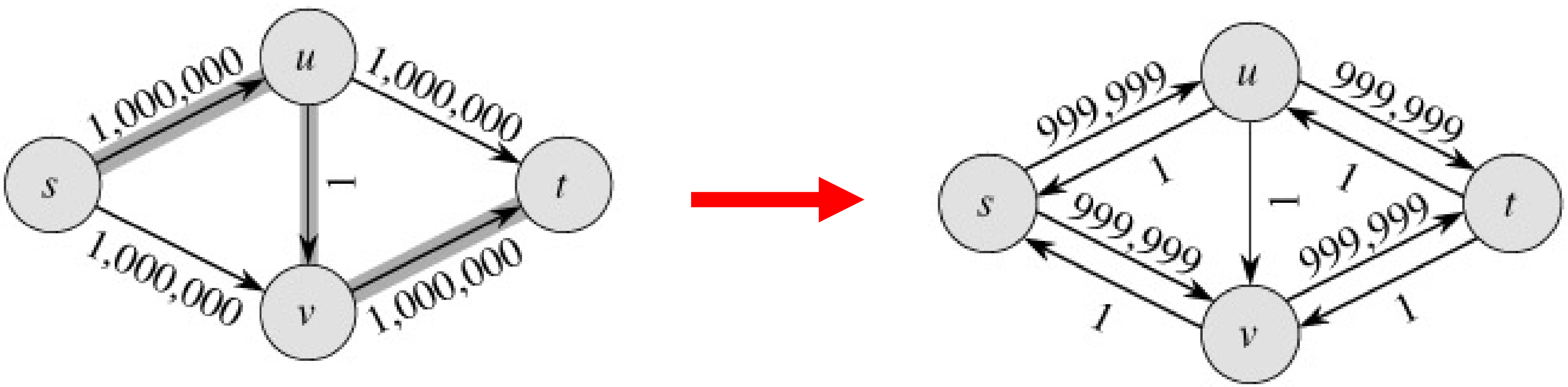

- Repeat 999,999 more times…
- Can we do better than this?

The Edmonds-Karp Algorithm

- A small fix to the Ford-Fulkerson algorithm makes it work in polynomial time.

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- Select the augmenting path using **breadth-first search** on residual network.
- The augmenting path p is the shortest path from s to t in the residual network (treating all edge weights as 1).
- Runs in time $O(V E^2)$.

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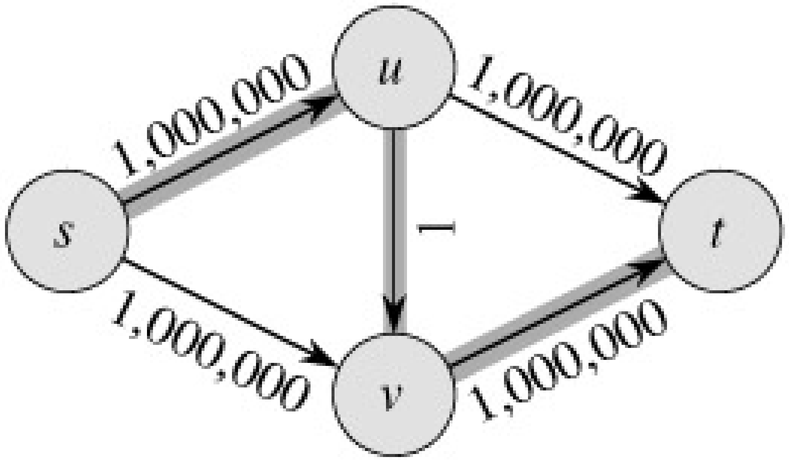
The Edmonds-Karp Algorithm - example



- The Edmonds-Karp algorithm halts in only 2 iterations on this graph.

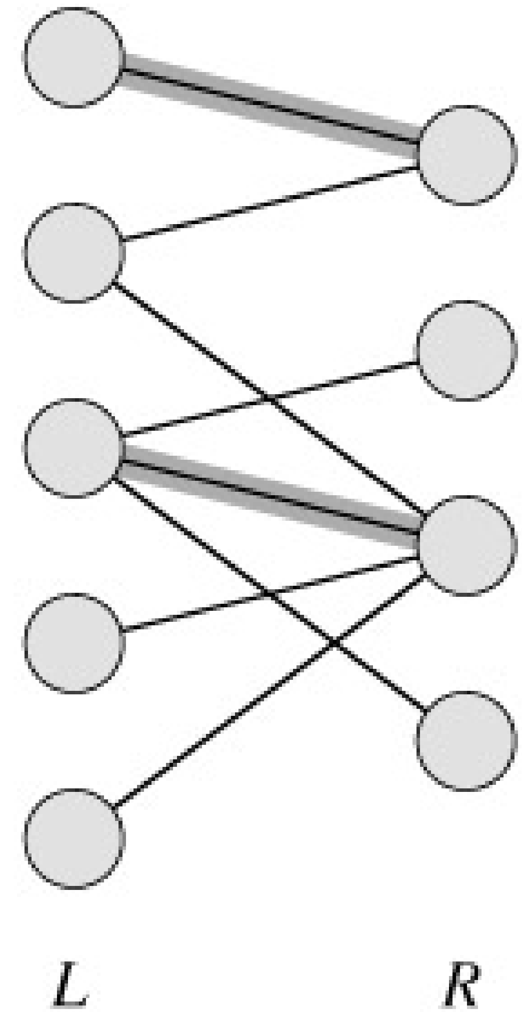
Further Improvements

- Push-relabel algorithm – $O(V^2 E)$.
- The relabel-to-front algorithm – $O(V^3)$.
- (We will not cover these)

An Application of Max Flow:

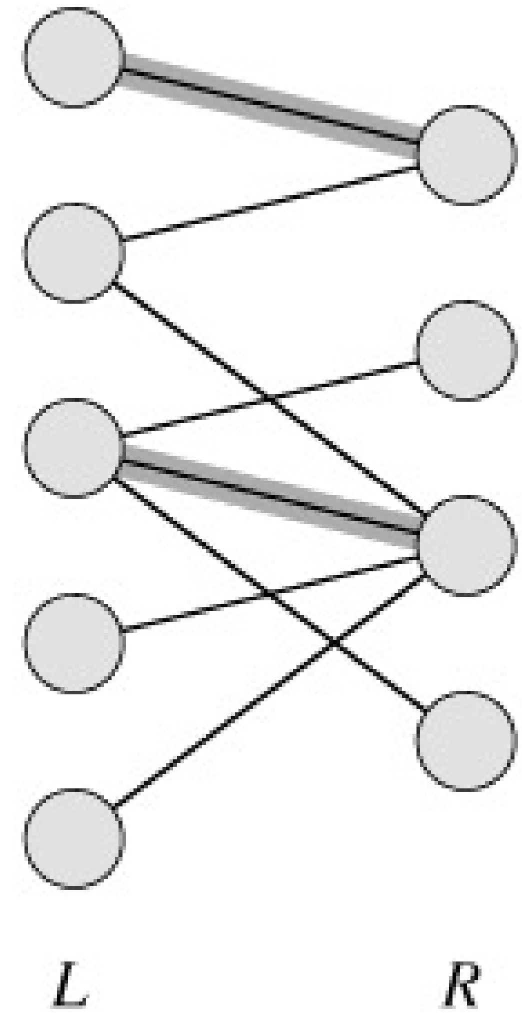
Maximum Bipartite Matching

Maximum Bipartite Matching



Maximum Bipartite Matching

- A bipartite graph is a graph $G=(V,E)$ in which V can be divided into two parts L and R such that every edge in E is between a vertex in L and a vertex in R.



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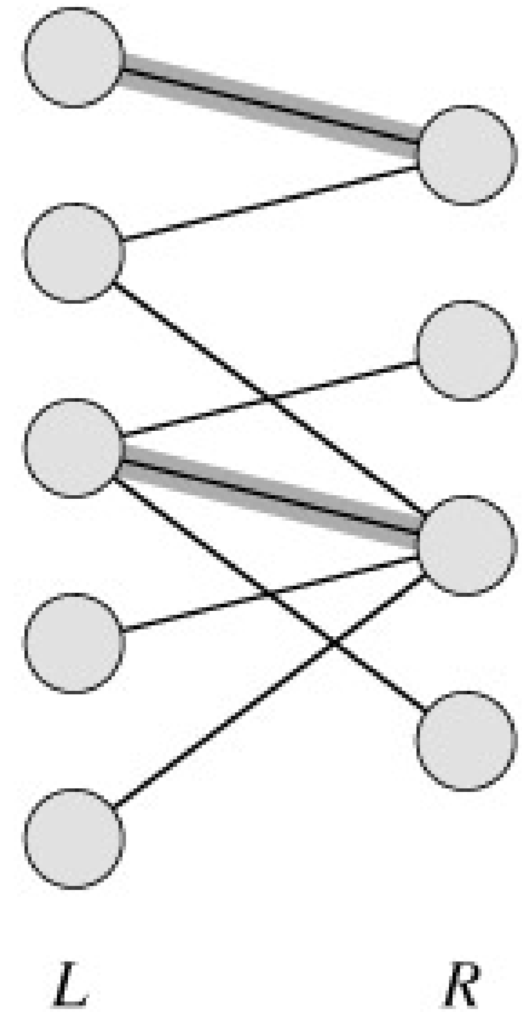
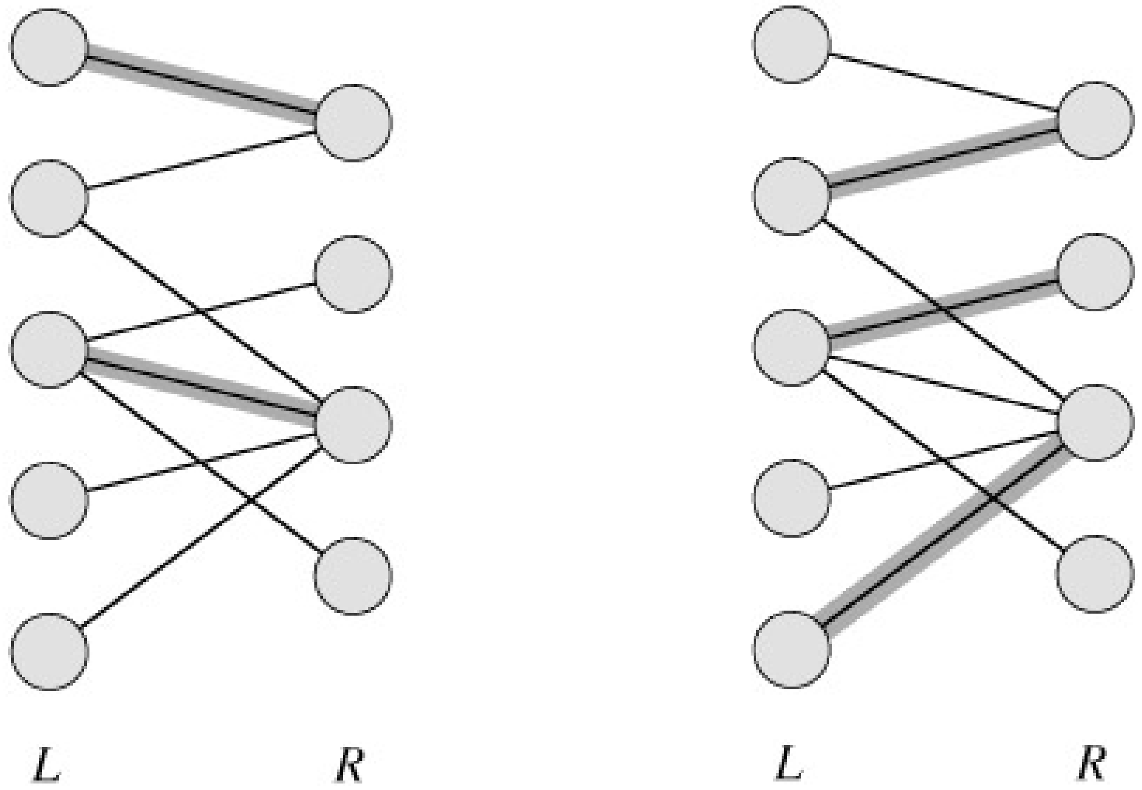
- A bipartite graph is a graph $G=(V,E)$ in which V can be divided into two parts L and R such that every edge in E is between a vertex in L and a vertex in R.

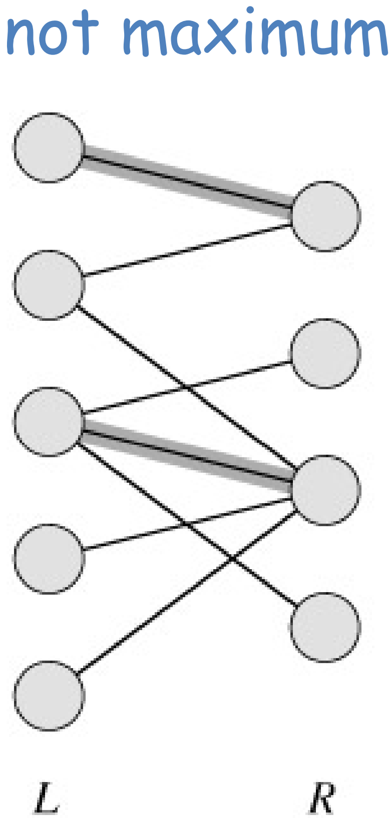
- e.g. vertices in L represent skilled workers and vertices in R represent jobs. An edge connects workers to jobs they can perform.

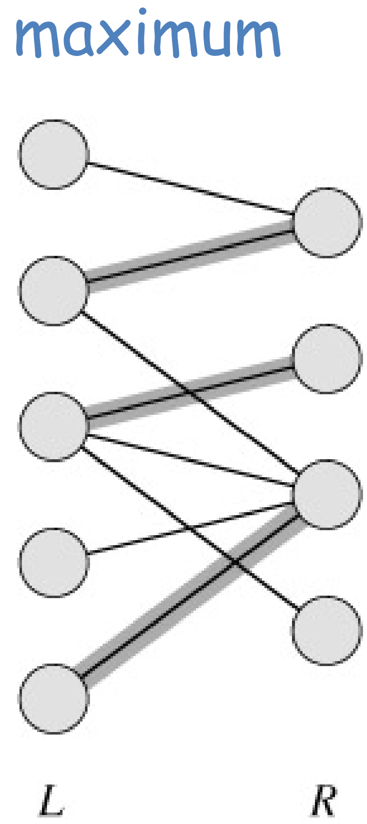


- A matching in a graph is a subset M of E, such that for all vertices v in V, at most one edge of M is incident on v.



- A **maximum matching** is a matching of maximum cardinality (maximum number of edges).

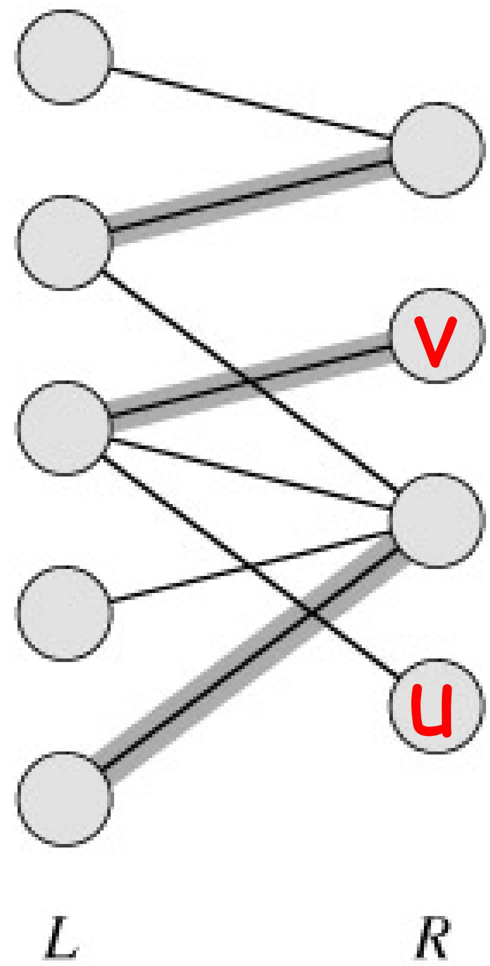




A Maximum Matching

- No matching of cardinality 4, because only one of v and u can be matched.

- In the workers-jobs example a max-matching provides work for as many people as possible.



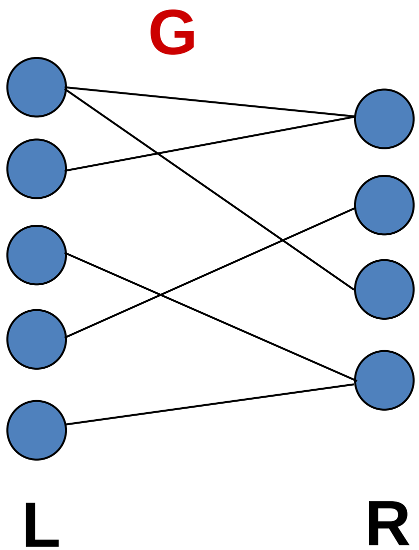
Solving the Maximum Bipartite Matching Problem

- Reduce the maximum bipartite matching problem on graph **G** to the max-flow problem on a corresponding flow network **G'**.

- Solve using Ford-Fulkerson method.

Corresponding Flow Network

- To form the corresponding flow network **G'** of the bipartite graph **G**:



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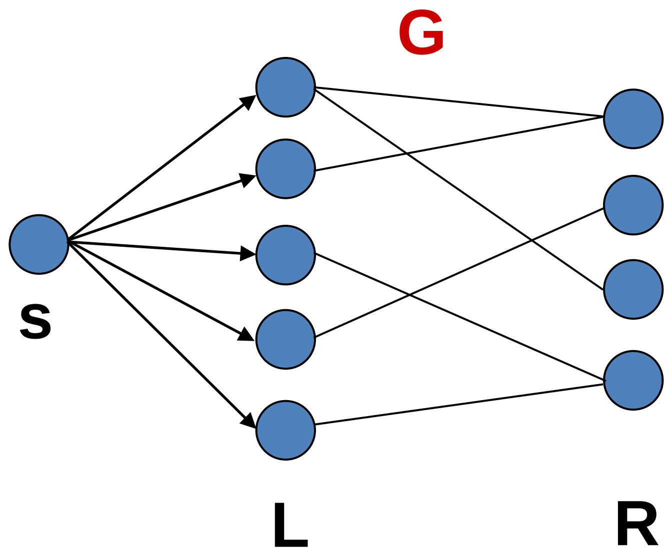
- To form the corresponding flow network **G'** of the bipartite graph **G**:
  - Add a source vertex s and edges from s to L.



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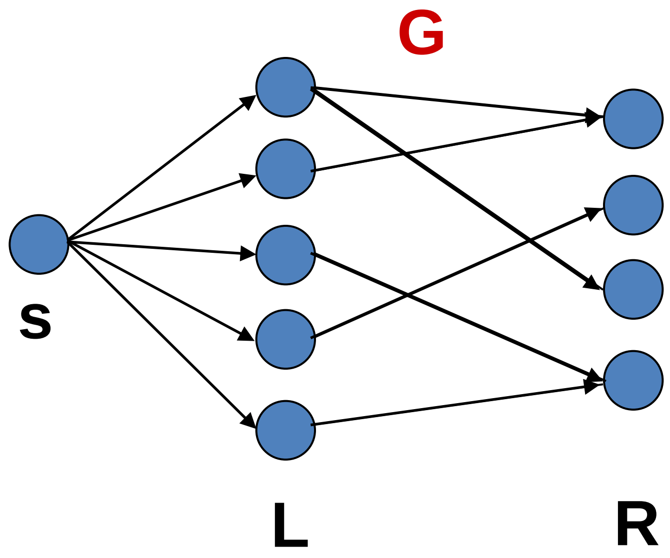

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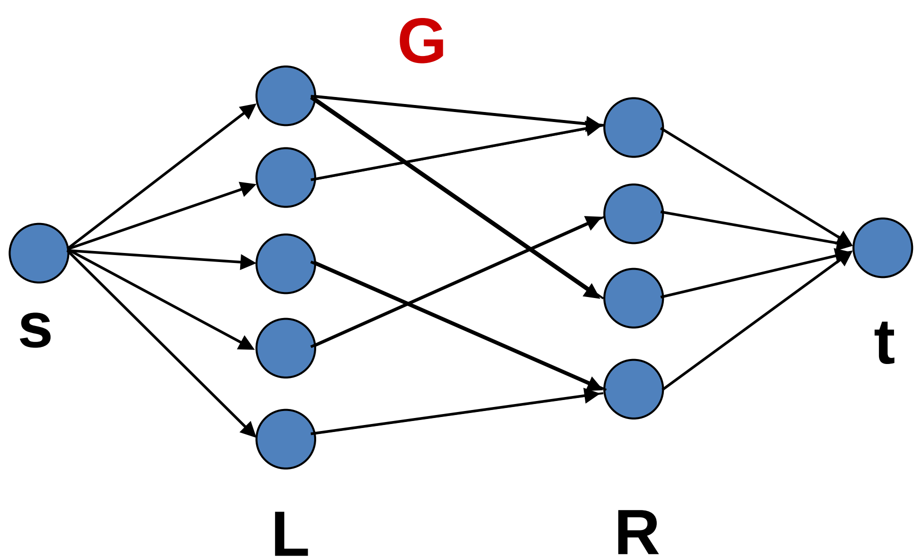

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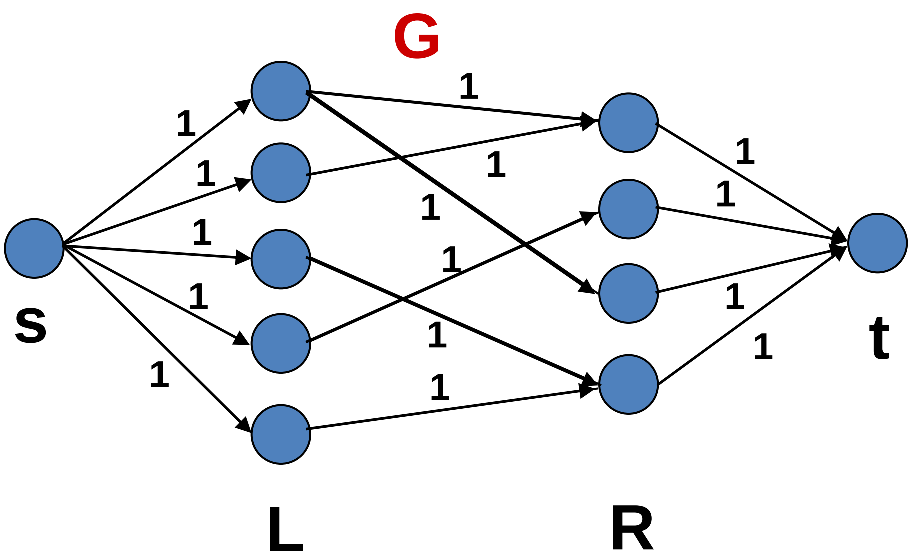

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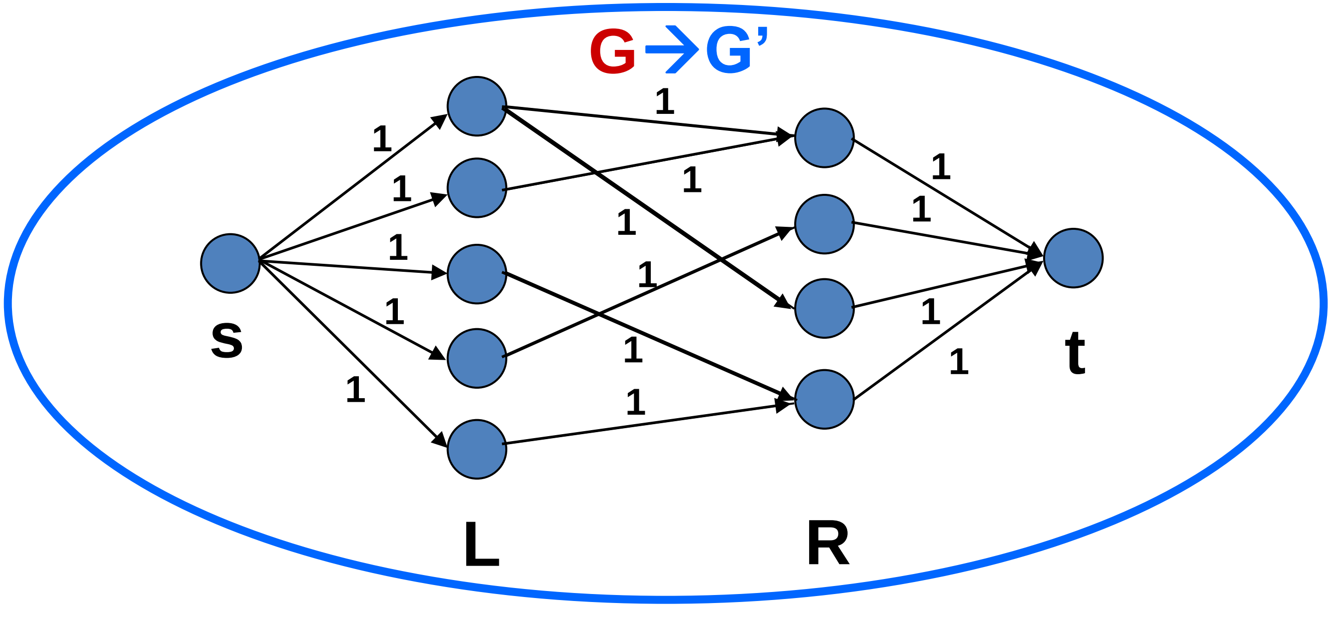

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  - Add a source vertex s and edges from s to L.
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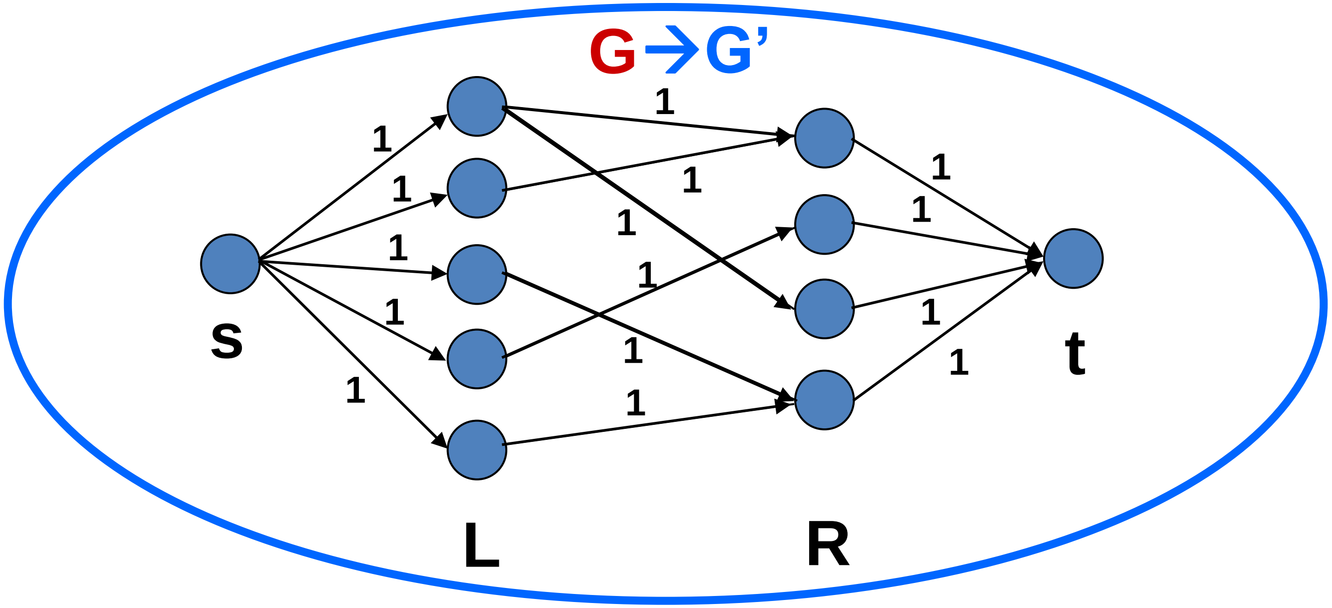
  - Assign a capacity of 1 to all edges.
- Claim: max-flow in **G'** corresponds to a max-bipartite-matching on **G**.



Solving Bipartite Matching as Max Flow

Let $G = (V, E)$ be a bipartite graph with vertex partition $V = L \cup R$.

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Conversely if f is an integer-valued flow in G',

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Thus $\max|M|=\max(\text{integer }|f|)$

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- Problem: we haven't shown that the max flow f(u,v) is necessarily integer-valued.

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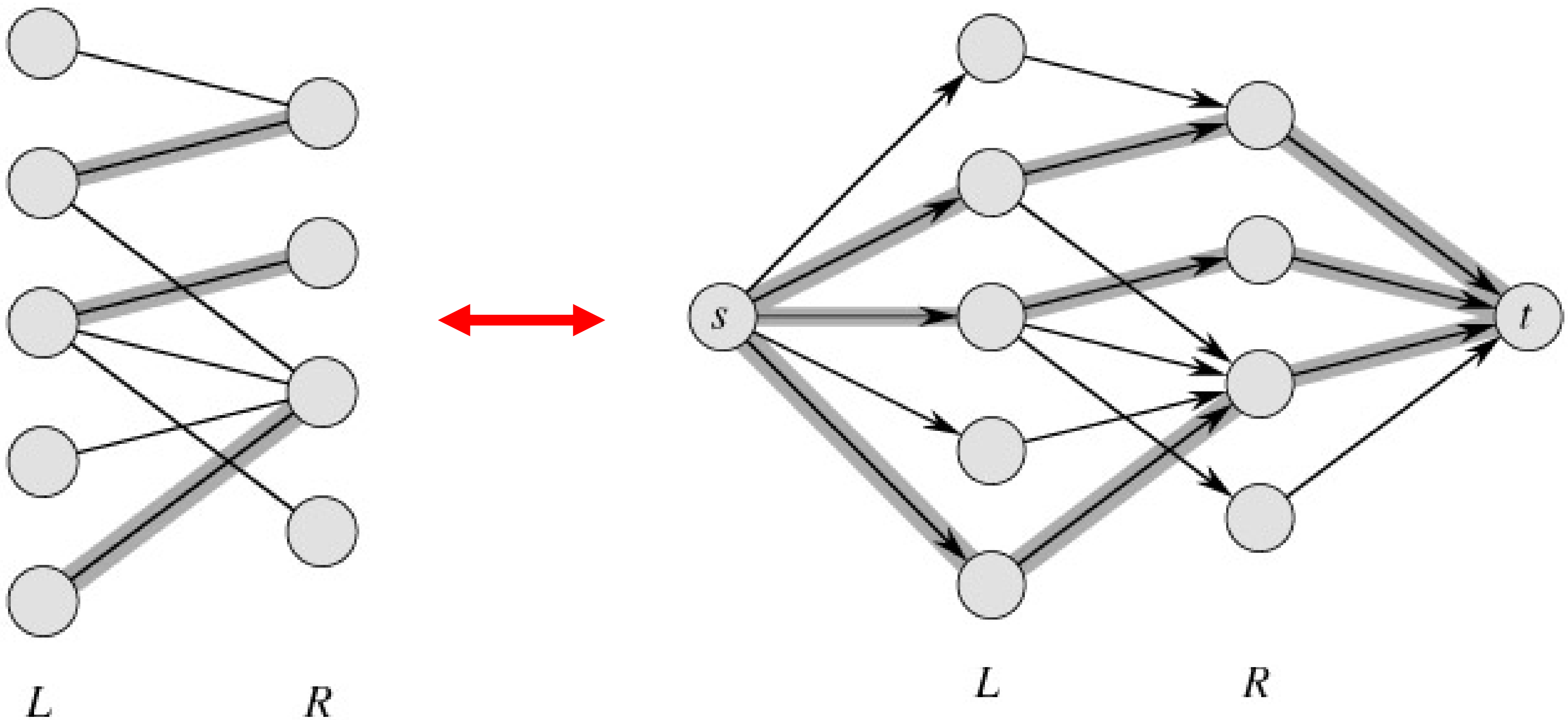
- So $\max|M| = \max |f|$

Example



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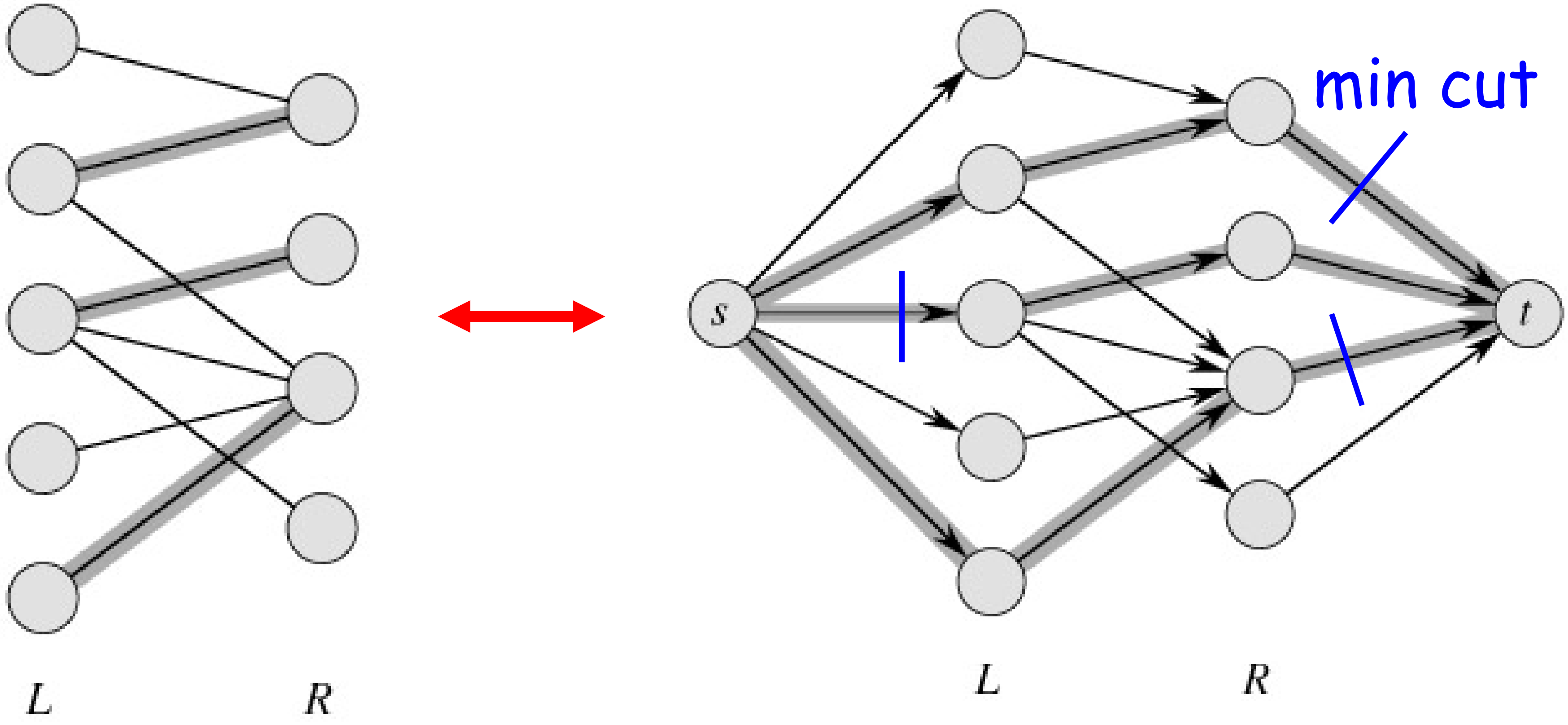

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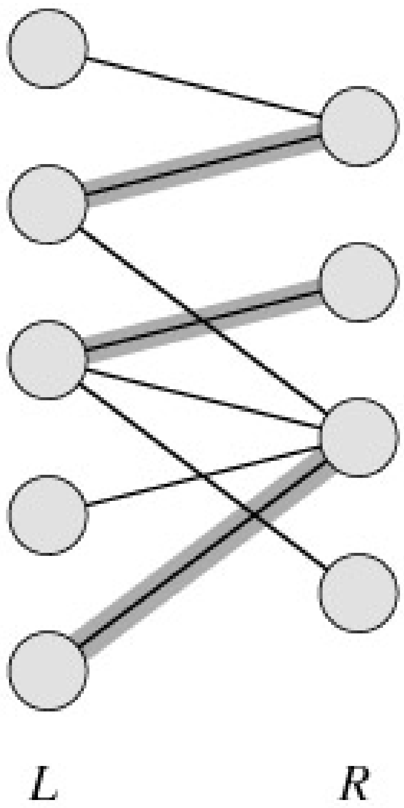

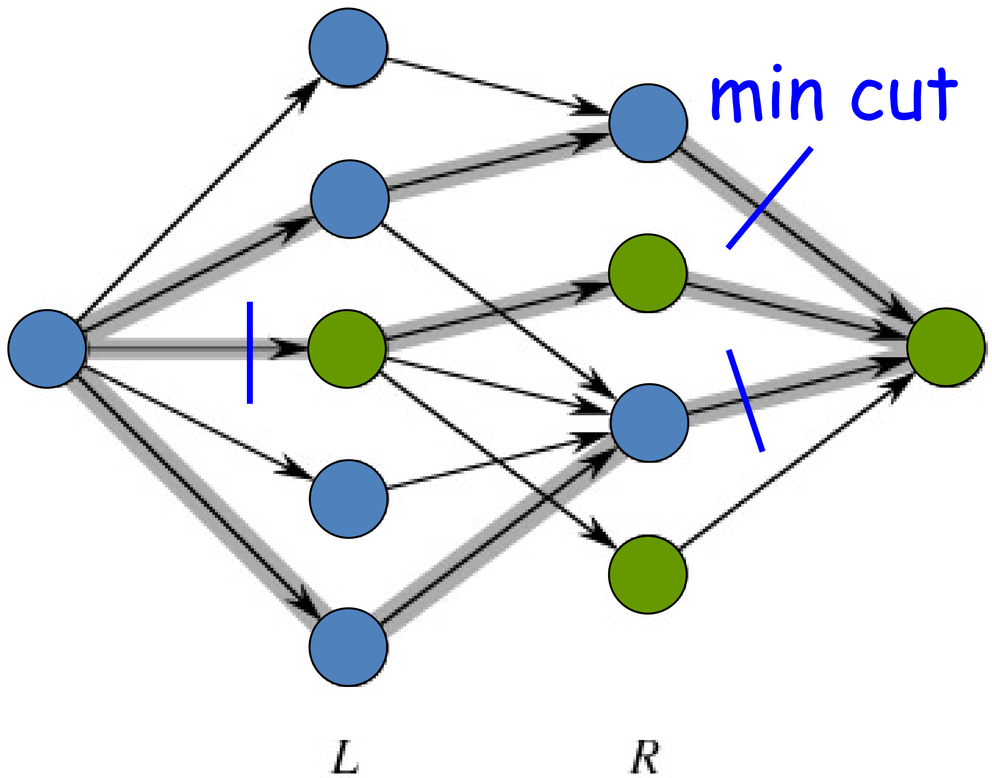

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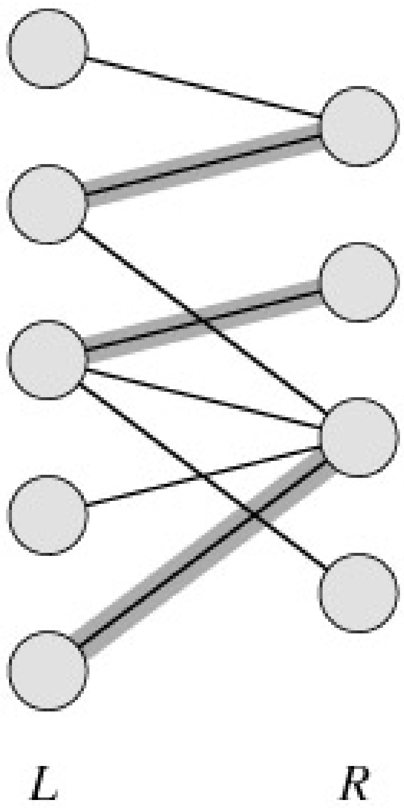

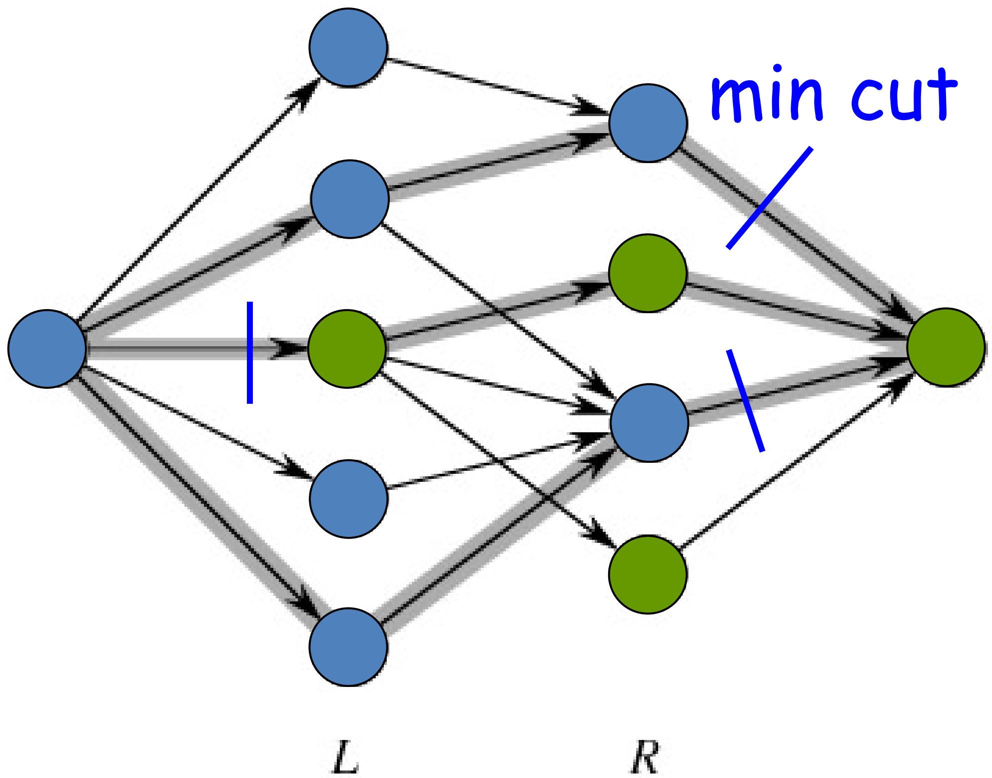

|M| = 3 ←→ max flow =|f|= 3

Conclusion

- Network flow algorithms allow us to find the maximum bipartite matching fairly easily.
- Similar techniques are applicable in other combinatorial design problems.

Example

- In a department there are n courses and m instructors.
- Every instructor has a list of courses he or she can teach.
- Every instructor can teach at most 3 courses during a year.

- The goal: find an allocation of courses to the instructors subject to these constraints.