Maths :- (30%)

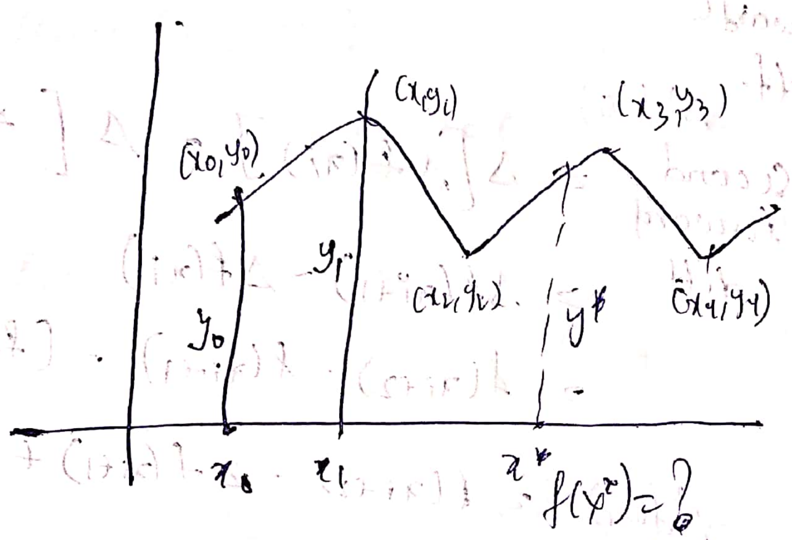
Numerical Analysis :-

Book :- Numerical Methods by SRK. Iyengar and RK. Jain

New Age International publishers.

x	$f(x)$
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

$y = f(x)$



Finite Difference operators :-

Shift operator E :-

$E(f(x_i)) = f(x_i + h) = f(x_{i+1})$

$E(f(x_0)) = f(x_0 + h) = f(x_1)$

$E(f(x_1)) = f(x_2)$

x_0	$f(x_0)$ y_0	$x_1 = x_0 + h$
x_1	y_1	$x_2 = x_1 + h$
x_2	y_2	$\vdots$
$\vdots$	$\vdots$	
x_n	y_n	$x_n = x_0 + nh$

2, 3

Interpolation with evenly spaced points

$E^2 f(x_i) = E[f(x_i)]$

$= E[f(x_{i+1})]$

$= f(x_{i+2})$

$\boxed{E^k(f(x_i)) = f(x_{i+k}) = f(x_i + hk)}$

$E^{1/2} f(x_i) = f(x_i + \frac{h}{2}) = f(x_i + \frac{h}{2})$

Forward diff operator: Δ (Del operator)

$\Delta f(x_i) = f(x_i + h) - f(x_i) = f(x_{i+1}) - f(x_i)$

First Forward diff
$$\left\{\begin{array}{l} \Delta f(x_0) = f(x_1) - f(x_0) \\ \Delta f(x_1) = f(x_2) - f(x_1) \end{array}\right.$$

Second forward diff

$\Delta^2 f(x_i)$

$= \Delta[\Delta f(x_i)] = \Delta[f(x_{i+1}) - f(x_i)]$

$= \Delta f(x_{i+1}) - \Delta f(x_i)$

$= f(x_{i+2}) - f(x_{i+1}) - (f(x_{i+1}) - f(x_i))$

$\Delta^2 f(x_i) = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)$

$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$

$= E f(x_i) - f(x_i)$

$= (E - 1) f(x_i)$

$$\boxed{\Delta f(x_i) = (E-1) f(x_i)}$$

$$\boxed{\begin{array}{l} \Delta \equiv E - 1 \\ E \equiv \Delta + 1 \end{array}}$$

$\Delta^n f(x_i) = (E-1)^n f(x_i)$

$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+n}$

$\Delta^3 f(x_i) = f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)$

Forward Difference Table:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	$f(x_0)$	$\Delta f(x_0) = f(x_1) - f(x_0)$		
			$\Delta^2 f(x_0) = \Delta f(x_1) - \Delta f(x_0)$	
				$\Delta^3 f(x_0) = \Delta^2 f(x_1) - \Delta^2 f(x_0)$
x_1	$f(x_1)$	$\Delta f(x_1) = f(x_2) - f(x_1)$		
			$\Delta^2 f(x_1) = \Delta f(x_2) - \Delta f(x_1)$	
x_2	$f(x_2)$	$\Delta f(x_2) = f(x_3) - f(x_2)$		
			$\Delta^2 f(x_2) = \Delta f(x_3) - \Delta f(x_2)$	
x_3	$f(x_3)$			

Backward Difference Operator: ∇ (Napla)

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f(x_i) - f(x_{i-1})$$

$$\nabla f(x_1) = f(x_1) - f(x_0), \quad \nabla f(x_2) = f(x_2) - f(x_1)$$

$$\nabla^2 f(x_i) = \nabla[\nabla f(x_i)] = \nabla f(x_i) - \nabla f(x_{i-1})$$

$$= f(x_i) - f(x_{i-1}) - (f(x_{i-1}) - f(x_{i-2}))$$

$$= f(x_i) - 2f(x_{i-1}) + f(x_{i-2})$$

$$\nabla^3 f(x_i) = f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})$$

$$\nabla f(x_i) = f(x_i) - f(x_{i-1}) = f(x_i) - E^{-1} f(x_i)$$

$$= (1 - E^{-1}) f(x_i)$$

$$\boxed{\begin{aligned} \nabla &\equiv (1 - E^{-1}) \\ \text{(or)} \quad E &\equiv (1-\nabla)^{-1} \end{aligned}}$$

$$\nabla^n f(x_i) = (1-E^{-1})^n f(x_i)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{x_{i-k}}$$

Backward Diff operator table :

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(a)$
x_0	$f(x_0)$	~~$\nabla f(x_1) = f(x_1) - f(x_0)$~~		
x_1	$f(x_1)$	$\nabla f(x_1) = f(x_1) - f(x_0)$	$\nabla^2 f(x_2) = \nabla f(x_2) - \nabla f(x_1)$	$\nabla^3 f(x_3) = \Delta^2 f(x_3) - \Delta^2 f(x_2)$
x_2	$f(x_2)$	$\nabla f(x_2) = f(x_2) - f(x_1)$	$\nabla^2 f(x_3) = \nabla f(x_3) - \nabla f(x_2)$	
x_3	$f(x_3)$	$\nabla f(x_3) = f(x_3) - f(x_2)$		

$\Delta f(x_0) = \nabla f(x_1)$

$\Delta f(x_1) = \nabla f(x_2)$

$\vdots$

$\Delta^3 f(x_0) = \nabla^3 f(x_3)$

$$\boxed{\Delta^k f(x_i) = \nabla f(x_{i+k})}$$

Central - Diff operator δ def

$$\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

$$\boxed{\delta f(x_i) = f(x_{i+\frac{1}{2}}) - f(x_{i-\frac{1}{2}})}$$

$$\boxed{\delta f\left(x_i + \frac{h}{2}\right) = \delta f(x_{i+\frac{1}{2}}) = f(x_{i+1}) - f(x_i)}$$

$$\delta f\left(\frac{x_1}{2}\right) = f(x_1) - f(x_0) \quad , \quad \delta f\left(\frac{x_3}{2}\right) = f(x_2) - f(x_1)$$

$$\delta^2 f(x_i) = f(x_{i+1}) - 2 f(x_i) + f(x_{i-1})$$

$$\delta^3 f(x_i) = f(x_{i+3/2}) - 3 f(x_{i+\frac{1}{2}}) + 3 f(x_{i-\frac{1}{2}}) - f\left(x_i - \frac{3}{2}\right)$$

Even powers - nodal points
Odd power - not nodal -

$$\Rightarrow \boxed{\delta = E^{1/2} - E^{-1/2}}$$

$$\delta^n f(x_i) = ?$$

Central diff operator table :

4/11/20

R1 = $\boxed{\Delta^n f(x_i) = \nabla^n f(x_{i+n}) = \delta^n f\left(x_i + \frac{n}{2}\right)}$

$$\nabla = 1 - E^{-1} = (E-1) E^{-1} = \Delta E^{-1}$$

R1

$$\nabla^n f(x_{i+n}) = \Delta^n E^{-n} f(x_{i+n}) = \Delta^n f(x_i)$$

R2 : Let $P_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots a_n$ be a polynomial of degree n. Then $\Delta^k P_n(x) = 0 \quad k > n$

$$= a_0 n! \quad k = n$$

and $\nabla^n P_n(x) = 0 \quad k > 0$

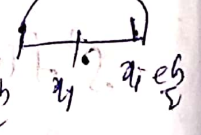
$$= a_0 n! \quad k = n$$

$$\Delta^k P_n(x) \begin{cases} = 0 & k > n \\ = a_0 n! & k = n \end{cases} \qquad \nabla^n P_n(x) \begin{cases} = 0 & k > n \\ = a_0 n! & k = n \end{cases}$$

$$\boxed{\begin{aligned} & P_2(x) = 2x^2 \\ & \Delta^3 P_2(x) = 0 \end{aligned}} \qquad \begin{aligned} & \Delta^{n+1} P_n(x) = 0 \\ & \Delta^n P_n(x) = a_0 n! \end{aligned}$$

<u>Mean operator</u>: $\mu f(x_i) = \frac{1}{2}\left[f(x_i + \frac{h}{2}) + f(x_i - \frac{h}{2})\right]$

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right].$$



$$E^{1/2} f(x_i) = f(x_i + h/2).$$

R3 (i) $\delta = \nabla(1 - \nabla)^{-1/2}$

(ii) $\mu = \left[1 + \frac{\delta^2}{4}\right]^{1/2}$

(iii) $\Delta[f(x_i)^2] = [f(x_i) + f(x_{i+1})] \times \Delta f(x_i)$

(iv) $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)\, g(x+h)}$

proof: $\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} = \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)(g(x+h))}$

$$= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x)\, g(x+h)}$$

$$= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)\, g(x+h)}$$

$E(f(x)) = f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(x) \ldots$

Taylor's series

$= f(x) + h\,Df(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \ldots$

$= \left[1 + hD + \frac{h^2}{2!} D^2 + \frac{h^3}{3!} D^3 + \ldots\right] f(x)$

$= e^{hD} f(x)$

$E = e^{hD}$

Relationship with Diff ops:

$D^2 f(x) = f''(x)$

$hD = \log E$

$hD = \log(1+\Delta)$

$hD = -\log(1-\nabla)$ $\cancel{e^{hD} f(x)}$

Newton's Forward Diff Interpolation:

$\Rightarrow F(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{1!\,h} + (x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2!\,h^2}$

$+ \ldots (x-x_0)(x-x_1) \ldots (x-x_n) \frac{\Delta^n f^n(x_0)}{n!\,h^n}$

$x_i = x_{i-1} + h$

x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

h

$x_i = x_{i-1} + h,$

$= i = 0(1)x$

$y = f(x)$

$x = x_0 + sh$; $x - x_i = x_0 + sh - (x_0 + ih) = (s-i)h$

$x - x_0 = sh$, $x - x_1 = (s-1)h$,

$(x - x_2) = (s-2)h$

$$f(x) = f(x_0) + s\,\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \cdots$$

$$+ \frac{s(s-1)(s-2)\cdots(s-n+1)}{n!}\Delta^n f(x_0)$$

$\boxed{\text{Alternative form}}$

$$f(x) = f(x_0 + sh) = {}^sC_0 f(x_0) + {}^sC_1 \Delta f(x_0) + {}^sC_2 \Delta^2 f(x_0)$$

$$+ \cdots {}^sC_n \Delta^n f(x_0)$$

$$s = \frac{x - x_0}{h} > 0$$

$$f(x) = f(x_0 + sh) = E^s f(x_0) = (1+\Delta)^s f(x_0)$$

$$= \left({}^sC_0 + {}^sC_1\Delta + {}^sC_2\Delta^2 + \cdots\right) f(x_0)$$

$$= {}^sC_0 f(x_0) + {}^sC_1 \Delta f(x_0) + {}^sC_2 \Delta^2 f(x_0) + \cdots {}^sC_n \Delta^n f(x_0)$$

$$+ \cdots$$

$$f(x_0 + sh) = {}^sC_0 f(x_0) + {}^sC_1 \Delta f(x_0) + {}^sC_2 \Delta^2 f(x_0) + \cdots$$

$$+ \cdots {}^sC_n \Delta^n f(x_0)$$

$$E_n(f, x) \leq \frac{(x - x_0)(x - x_1)\cdots(x - x_n)}{(n+1)!} f^{n+1}(\xi)$$

$$f^{n+1}(\xi) = f^{n+1}(x_\xi)$$

$$f(1) = f(x_1)$$

$$\rightarrow\ = {}^sC_{n+1}\, h^{n+1} f^{n+1}(\xi) \qquad 0 < \xi < n$$

→ Newton's Forward df Interpolation maintains permanance property, which means the new data added to the data table will be added to existing formula

Newton's Backward Diff Interpolation Formula

$\Rightarrow$ $$f(x) = f(x_n) + (x - x_n)\frac{1}{1!h}\nabla f(x_n) + \frac{(x-x_n)(x-x_{n-1})}{2!h^2}\nabla^2 f(x_n)$$

$$+ \cdots \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_1)}{n!\,h^n}\nabla^n f(x_n)$$

$x - x_n = sh$

$x = sh + x_n$

$s < 0$

$$f(x) = f(x_n + sh) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2}\nabla^2 f(x_0)$$

$$+ \cdots + \frac{s(s+1)\cdots(s+n-1)}{n!}\nabla^n f(x_n)$$

$$E_n(f, x) = \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_0)}{(n+1)!}f^{n+1}(\xi)$$

$0 < \xi < n$

Book $\rightarrow$ Ex 2.18

$\rightarrow$ $$E_n(f, x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!}f^{n+1}(\xi)$$

Near the starting $\rightarrow$ Use N Forward

Ending $\rightarrow$ Use N Backward

Tutorial: Construct the forward difference table.

x	-1	0	1	2
$f(x)$	-8	3	1	12

$\rightarrow$ But Backward Interpolation formula does not hold permanence

2

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
4	-8			
		11		
0	3		-13	
		-2		26
1	1		-13	
		11		
2	12			

← forward diff table

← Same for backward diff table

Ex-2 Data find $f(0.5)$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-2	15	-10	6	0
-1	5	-4	6	0
0	1	+2	6	0

Ex-3 Using newton's backward interpolation

Use Newton's backward interpolation, interpolate at $x = 1$, from the following data. $h = 0.2$

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	-1.699	-1.073	-0.375	0.443	1.429	2.631

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
0.1	-1.699	0.626	0.072	0.048	0	0
0.3	-1.073	0.698	0.120	0.048	0	
0.5	-0.375	0.818	0.168	0.048		
0.7	0.443	0.986	0.216			
0.9	1.429	1.202				
1.1	2.631					

$$f(x) = f(x_n) + (x - x_n)\frac{1}{1!\,h}\nabla f(x_n) + \frac{(x-x_n)(x-x_{n-1})}{2!\,h^2}\nabla^2 f(x_n)$$

$$+ (x-x_n)(x-x_{n-1})(x-x_{n-2})\cdot\frac{1}{3!\,h^3}\nabla^3 f(x_n)$$

$$= 2.631 + (x-1.1)\times 6.01 + 2.7(x-1.1)(x-0.9)$$

$$+ (x-1.1)(x-0.9)(x-0.7)$$

$$f(1.0) = 2.004$$

$$f(x) = P_n(x) \;\ldots\ldots\; \Delta^n f$$

$$e(x) = f(x) - P_n(x)$$

Central Diff Interpolation formula (sterling)

$$\delta y_r = y\left(x_r + \frac{h}{2}\right) - y_r\left(x_r - \frac{h}{2}\right) = y_{r+\frac{1}{2}} - y_{r-\frac{1}{2}}$$

$$\mu = \frac{E^{-1/2} + E^{1/2}}{2}$$

$$y(x_r + kh) = \left[1 + k\mu\delta + \frac{k^2}{2!}\delta^2 + \frac{k(k-1)(k+1)}{3!}\mu\delta^3\right.$$

$$\left.\frac{k^2(k^2-1)}{4!}\delta^4 + \ldots\ldots\right] y_r$$

x_0

x_1

x_2

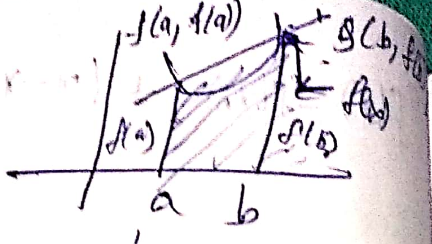
x_r

$x_r + kh$ $\quad x_{r+1}$

x_n

Numeric Integration

Trapezoidal / Trapezium Rule



$I = \int_a^b f(x)\,dx$ $\qquad \int_a^b f(x)dx$

$y = f(x) \quad a \le x \le b \qquad = \frac{b-a}{2}[f(a) + f(b)]$

$I = \int_a^b f(x)\,dx$

$= (b-a) \times f(a) + \frac{1}{2}(f(b) - f(a)) \times (b-a)$

$= (b-a)\left[f(a) + \frac{1}{2}f(b) - \frac{1}{2}f(a)\right]$

$f(x) = f(x_0) + \frac{1}{h}(x - x_0)\Delta f(x_0)$

$x_0 = a, \quad x_1 = b, \quad h = b-a.$

$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} f(x_0)\,dx + \frac{1}{h}\left(\int_{x_0}^{x_1}(x - x_0)dx\right)$

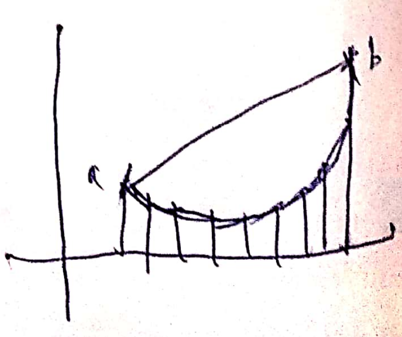
$\Delta f(x_0)$

$= (x_1 - x_0) f(x_0) + \frac{1}{2h}[f(x_1) - f(x_0)](x_1 - x_0)^2$

$\boxed{x_1 - x_0 = h}$

$= \frac{h}{2}[f(x_1) + f(x_0)]$

$= \frac{b-a}{2}[f(a) + f(b)]$



Composite Trapezoidal Rule:- $[a,b] \to N$ equal parts of length h

$h = \frac{b-a}{N}$ $\quad a = x_0,\ x_1 = x_0 + h,\ x_2 = x_1 + h = x_0 + 2h$

$\ldots x_N = x_0 + Nh = b$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_N} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \ldots \int_{x_{N-1}}^{x_N} f(x)\,dx$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \ldots f(x_{N-1}) + f(x_N) \right]$$

$$= \frac{h}{2}\left[f(x_0) + 2\left[f(x_1) + \ldots f(x_{N-1})\right] + f(x_N)\right]$$

Ex: $I = \int_0^1 \frac{dx}{1+x}$ 2, 4, 8, equal subintervals,

Find the errors.

Soln $N=2;\ h = \frac{b-a}{N} = \frac{1}{2}$ $\quad 0, 0.5, 1$

$N=4,\ h = \frac{1}{4}$ $\quad 0, 0.25, 0.5, 0.75, 1.0$

$N=8;\ h = \frac{1}{8}$ $\quad 0, 0.125, 0.25, 0.375 \ldots 1.0$

$N = 2.$

x	0	0.5	1.0
$f(x)$	1	0.6667	0.5

$I_1 = \frac{h}{2}\left[f(0) + 2f(0.5) + f(1.0)\right] = 0.708334$

$h = \frac{1}{2}$

$N=4$

x	0.	0.25	0.5	0.75	1.0
$f(x)$	1	0.8	0.666...7	0.57	0.5

$I_1 = \frac{h}{2}\left[f(0) + 2f(0.5) + f\right] \times$

$N=4 \quad I_2 = \frac{h}{2}\left[f(0) + 2\{f(0.25) + f(0.5) + f(0.75)\} + f(1.0)\right]$

$= 0.125 * \left[1 + 2[0.8 + 0.67 + 0.57] + 0.5\right]$

$= 0.692$

$I_3 = \ldots$

$$I = \int_a^b w(x) f(x) = \sum_{k=0}^{n} \lambda_k f(x_k),$$

$\lambda_k = \frac{h}{2}, \quad k = 0, 1 \ldots$

$f(a) = f(x_0)$

$f(b) = f(x_1)$

$$R(f) = \int_a^b w(x) f(x)\,dx - \sum_{k=0}^{n} \lambda_k (x_k).$$

Order of a method :

$R = 0, \ \forall$ pol^n of degree less than or equal to p.

$f(x) = 1, x, x^2, \ldots, x^p.$

$$R(x^m) = \int_a^b w(x)\, x^m\, dx - \sum \lambda_k x_k^m,$$

for $m = 0, 1, \dots p$.

$f(x) = x^{p+1}$
$$\boxed{C = \int_a^b w(x)\, x^{p+1}\, dx - \sum_{k=0}^{n} \lambda_k x_k^{p+1},}$$

C = error const.

$$R(f) = \int_a^b w(x) f(x)\, dx - \sum_{k=0}^{m} \lambda_k f(x_k)$$

$$= \frac{C}{(p+1)!} f^{p+1}$$

$$|R(f)| \le \frac{|C|}{(p+1)!} \max_{a \le x \le b} f^{p+1}(x).$$

$f(x) = 1, x$

$$R(f) = \int_a^b dx - \frac{b-a}{2} \times 2$$

$f(x) = 1$

$f(x) = x \quad R(f) = \int_a^b x\, dx - \left(\frac{b-a}{2}\right) \times (a+b)' = 0$

$f(x) = x^2 \qquad C = \int_a^b x^2\, dx - \frac{b-a}{2} (b^2 + a^2) = \frac{1}{6}(b-a)^3$

$$R(f) = \frac{C}{2!} f''(\xi) = \frac{(b-a)^3}{12}$$

$= -\frac{h^3}{12} f''(\xi)$

$|R(f)| \le \frac{h^3}{12} M_2$, $M_2 = \max_{a \le x \le b} |f''(x)|$

Composite trapezoidal rule , order = 1

$R(f) = -\frac{h^3}{12} [f''(\xi_1) + f''(\xi_2) + \dots$

$+ \dots f''(\xi_n)]$

Simpson's 1/3 Rule:- $[a,b]$ $y = f(x)$ $f(x_0)$

$h = \frac{b-a}{2}$, $x_0 = a$, $x_1 = a+h = \frac{a+b}{2}$ $f(x_1)$

$x_2 = b$ $f(x_n)$

$P(x_0, f(x_0))$, $Q(x_1, f(x_1))$, $R(x_2, f(x_2))$

$f(x) = f(x_0) + \frac{1}{h}(x-x_0)\Delta f(x_0) + \frac{1}{2h^2}(x-x_0)(x-x_1)\Delta^2 f(x_0)$

$\int_a^b f(x)dx = \int_{x_1}^{x_2} f(x)dx = (x_2 - x_0) f(x) + \frac{1}{h}\left[\frac{1}{2}(x-x_0)^2\right]_{x_1}^{x_2} \Delta f(x_0)$

$+ I_1$

$\therefore \int_a^b f(x)dx = 2h f(x_0) + 2h \Delta f(x_0) + I_1$

$$I_1 = \frac{1}{2h^2}\left[\frac{x^3}{3} - (x_0+x_1)\frac{x^2}{2} + x_0 x_1 x\right]_{x_0}^{x_2} \Delta^2 f(x_0)$$

$$= \frac{h}{3} \Delta^3 f(x_0)$$

$$\int_a^b f(x)dx = 2hf(x_0) + 2h\,\Delta f(x_0) + \frac{h}{3}\Delta^2 f(x_0)$$

$$= \frac{h}{3}\left[6f(x_0) + 6\left(f(x_1) - f(x_0)\right] + \right.$$

$$\left.\{ f(x_0) - 2f(x_1) + f(x_2) \}\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right], \quad h = \frac{x_2 - x_0}{2}$$

Error : Integrates exactly pol^n of degree ≤ 3.

$R(f) = 0$ for $f(x) = 1, x, x^2, x^3$.

$f(x) = x^2$, $R(f) = \int_a^b x^2 dx - \frac{b-a}{6}\left[a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right]$

$$= \frac{b^3 - a^3}{3} - \frac{(b-a)}{3}\left[a^2 + ab + b^2\right]$$

$$= 0$$

$f(x) = x^3$, $R(f) = \int_a^b x^3 dx - \frac{(b-a)}{6}\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right]$

$$= \frac{1}{4}(b^4 - a^4) - \frac{(b-a)}{4}\left[a^3 + a^2 b + ab^2 + b^3\right]$$

$$= 0$$

Order of Simpson's $\frac{1}{3}$ rd Rule is 3.

$f(x) = x^4$

$$C = \int_a^b x^4 dx - \frac{(b-a)}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right]$$

$$= -\frac{(b-a)^5}{120}.$$

$a \le \xi \le b$

$$\boxed{R(f) = \frac{C}{4!} f^4(\xi)} \boxed{= -\frac{(b-a)^5}{2880} f^4(\xi)}$$

$$|R(f)| \le \frac{(b-a)^5}{2880} M_4 \qquad M_4 = \max_{a \le x \le b} |f^4(x)|.$$

Composite Simpson's $\frac{1}{3}$ rule :- 3 x_0, x_1, x_2

$[a, b]$ $\boxed{2N}$ $h = \frac{b-a}{2N}$.

$a = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ...

$x_{2N} = x_0 + 2Nh = b$

$$\int_a^b f(x)dx = \int_{x_0}^{x_{2N}} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx$$

$$+ \int_{x_4}^{x_6} f(x)dx + \cdots + \int_{x_{2N-2}}^{x_{2N}} f(x)dx$$

$$= \frac{h}{3}\left[f(x_0) + 4\left[f(x_1) + f(x_3) + \dots f(x_{N-1}) \right] + 2\{ f(x_2) + f(x_4) + \dots + f(x_{N-2}) \} + f(x_{2N}) \right]$$

$$R(f) = \frac{-h^5}{90}\left[f^4(\xi_1) + f^4(\xi_2) + \dots f^4(\xi_n) \right]$$

$$x_0 < \xi_1 < x_2 ,\ x_2 < \xi_2 < x_4 ; \dots$$

$$|R(f)| \le \frac{(b-a)^5}{2880 N^4} M_4 \qquad M_4 = \max_{a \le x \le b} |f^4(x)|$$

Composite simpson's rule is also of order 3

Ex: $I = \int_0^1 \frac{dx}{1+x}$ 2, 4, 8 errors?

Sol: $2N = 2, 4, 8.$ $N = 1, 2, 4$..

$N = 1$, $h = \frac{b-a}{2N} = \frac{1}{2}$, 0, 0.5, 1.0

$N = 2$ $h = \frac{b-a}{2N} = \frac{1}{4}$ 0, 0.25, 0.5, 0.75, 1.0

$2N = 2$

x	0	0.5	1
$f(x)$	1.0	0.667	0.5

$$I_1 = \frac{h}{3}\left[f(0) + 4f(0.5) + f(1.0) \right]$$

$$= \frac{1}{6}\left[1 + 4 \times 0.667 + 0.5 \right] = 0.6944$$

$2N=4$ x 0 0.25 0.5 0.75 1

$f(x) \rightarrow$ 1.0 0.8 0.667 0.57 0.5

$$I_2 = \frac{h}{3}\left[f(0) + 4\{f(0.25) + f(0.75)\}\right.$$

$$\left. + 2f(0.5) + f(1.0)\right)$$

$= 0.693254$

$2N=8$. $I_3 = 0.693155$ //

$I = \ln 2 = 0.693$

$2N=2$, Error $|I_1 - I| = 0.001297$. Error is decreasing

$2N=4$ $|I - I_2| = 0.00107$

$2N=8$, $|I - I_3| = 0.000008$.

Sol^n of Eq^ns : $\left\| \begin{matrix} 3x + 5y = 2 \\ 2x + 7y = 5 \end{matrix} \right\|$ $f(x) = 0$

$f(x) = \ln(x) = a_0 x^n + a_1 x^{n-1}$

$+ \dots a_{n-1} x + a_n = 0$

$3x^2 + 7x + 2 = 0 \rightarrow$ algebraic eq^n or polynomial eq^ns

$\cos x - xe^x = 0$

$\tan^{-1} x + \log 3x = 0$

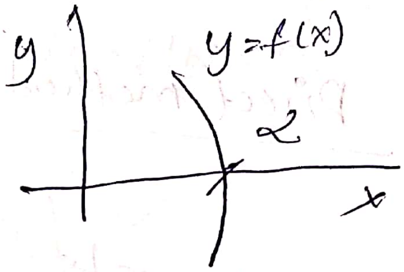
$\downarrow$

transendental eq^ns

$[a, b] \rightarrow f(x) = 0$ $f(x)$ is continuous on $[a,b]$

Root / zero : α for which $f(\alpha)=0$ is called a root of eq^n $f(x)=0$ (or) zero of $f(x)$

Simple root : α is a simple root of $f(x)=0$, if $f(\alpha)=0$ and $f'(\alpha) \neq 0$



$f(x) = (x-\alpha)\, g(x)$, $g(\alpha) \neq 0$.

$f(x) = (x-\alpha)\, g(x)$

$f(\alpha) = 0$ ✓

$f(x) = x^3 + x - 2 = 0$

$f(x) = (x-1)(x^2+x+2)$

$= (x-1)(g(x))$

$g(x) = (x-\alpha)\, h(x)$

$f(x) = (x-\alpha)^2 h(x)$ ✗

$f(1) = 0$, $g(1) \neq 0$.

$f'(1) \neq 0$.

then , $x=1$ is simple root

multiple root : α is a multiple root of multiplicity m, of $f(x)=0$ if $f(\alpha)=0$, $f'(\alpha)=0$, ... $f^{(m-1)}(\alpha)=0$, and $f^{(m)}(\alpha) \neq 0$.

$f(x) = (x-\alpha)^m g(x)$, $g(\alpha) \neq 0$.

Ex : $f(x) = x^3 - 3x^2 + 4 = 0$.

$f(x) = \underbrace{(x-2)^2 (x+1)}_{g(x)}$

2 is a multiple root of $f(x)=0$ with multiplicity 2.

$f(2)=0$, $f'(2)=0$, $f''(2) \neq 0$.

A polynomial eq^n of degree n has exactly n roots, real (or) complex, simple (or) multiple.

Direct method :- for $x^2-2x+1=0$.

$ax^2+bx+c=0$

Exact roots. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Iterative method :- $f(x)=0$ x

$x_{k+1} = \phi(x_k)$ x_0 x_1 x_2

$\boxed{\phi = \frac{x_1+x_0}{2}}$ Ex

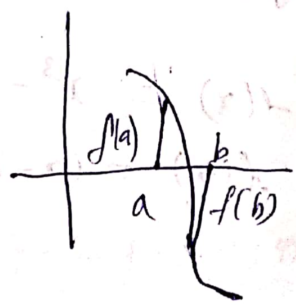
$|f(x_k)| \le \epsilon$

$|x_{k+1}-x_k| \le \epsilon$.

(1) Location of Roots Iterative method.

Method of Bisection :-

Bolzano's theorem :- If the function $f(x)$ is continuous in closed interval $[a,b]$ and if $f(a)$ and $f(b)$ are of opposite signs., then there exits atleast one real root of $f(x)=0$ between a and b.



Bisection: let $f(x)=0$ has a root in $[a,b]$,

$f(x)$ is cont in $[a,b]$, $f(a)$ & $f(b)$ are of opp signs

$f(a)f(b)<0$.

Let

$x_1 = \frac{a+b}{2}$, If $f(x_1)=0$

If $f(a).f(x_1)<0 \Rightarrow x \in [a, x_1]$ ✓

else $f(b)f(x_1)<0 \Rightarrow x \in [x_1, b]$ ✗

$a_1 = a$

$b_1 = x_1$

$x_2 = \frac{a_1+b_1}{2}$ $\quad f(x_2)=0$

$f(a_1)f(x_2)<0 \to [a_1, x_2]$

or $f(b_1).f(x_2)<0 \to [x_2, b_1]$

Ex: Find by bisection, a real positive root of $2x - \log_{10}(x) = 7$

$f(x) = 2x - \log_{10}(x) - 7 = 0$

$f(1) = -5$

$f(2) = -3.3$

$f(3) = -1.477$

$f(4) = 0.3979$

$f(3).f(4)<0$.

$\alpha \in [3,4]$

x	a_n (-ve)	b_n (+ve)	$x_{n+1} = \frac{a_n+b_n}{2}$	$f(x_{n+1})$
0	3	4	3.5	-0.5441
1	3.5	4	3.75	-0.0740
2	3.75	4	3.875	0.16177
3	3.75	3.875	3.8125	0.0438
4	3.75	3.8125	3.7813	-0.0151
⋮				
9	3.7891	3.7910	3.7910	

$\alpha = 3.79$

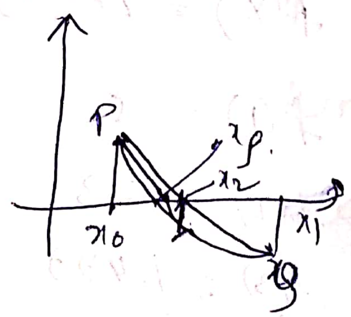
<u>Regular False Method</u> : (Improvement of Bisection method)

[Linear interpolation method, Chord method, method of false position].

$f(x) = 0$

(x_{k-1}, x_k)

$f_{k-1} \cdot f_k < 0$. $\boxed{f(x_k) = f_k}$



$P(x_{k-1}, f_{k-1}), Q(x_k, f_k)$ $f(x) = 0$

(x_{k-1}, x_k) Eq of PQ : $\frac{y - f_k}{f_{k-1} - f_k} = \frac{x - x_k}{x_{k-1} - x_k}$

point at x-axis $y = 0$

$$x = x_k - \left(\frac{x_{k-1} - x_k}{f_{k-1} - f_k}\right) f_k$$

$$= \frac{x_k f_{k-1} - \cancel{x_k f_k} - x_{k-1} f_k + \cancel{x_k f_k}}{f_{k-1} - f_k}$$

$$x = \frac{x_k f_{k-1} - x_{k-1} f_k}{f_{k-1} - f_k} \quad //$$

$$\boxed{x_{k+1} = x_k - \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right) f_k}$$

$$x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}} \qquad k = 1, 2, \ldots$$

(x_0, x_1) $\quad x_2 = \dfrac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$

$f(x_0).f(x_2) < 0$

$f(x_1).f(x_2) < 0.$

Alternatively,

$(x_{k-1}, x_k) \to f(x) = 0.$

$P(x_{k-1}, f_{k-1})$, $Q(x_k, f_k)$

$f(x) = ax + b$ $\qquad$ x-axis $y = 0$

$x = -\frac{b}{a}$

$f_{k-1} = a x_{k-1} + b$ $\qquad a = \dfrac{f_k - f_{k-1}}{x_k - x_{k-1}}$

$f_k = a x_k + b$

$b = f_k - a x_k$

$x_{k+1} = -\dfrac{b}{a}$

$= \dfrac{-(f_k - a x_k)}{a} = x_k - \dfrac{f_k}{a}$

$$x_{k+1} = x_k - \frac{f_k(x_k - x_{k-1})}{(f_k - f_{k-1})}$$

Ex: Locate the intervals which contain the positive real roots of the eq $x^3 - 3x + 1 = 0$. Obtain the roots correct to 3 decimal places

x	0	1	2	3
$f(x)$	1	-1	3	19

$f(x) > 0$ $\forall x > 2$

$(0,1) = x_0 = 0, \quad x_1 = 1, \quad f_0 = f(x_0) = f(0) = 1$

$f_1 = f(x_1) = -1$

$x_2 = \dfrac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \dfrac{0 - (1)(1)}{-1 + 1} = \dfrac{+1}{+2} = 0.5$

$f(x_2) = f(0.5) = +0.375 \qquad (0, 0.375)$

$f(0)\, f(0.5) < 0 \;;\; (\overset{x_0}{0}, \overset{x_2}{0.5})$

$$x_3 = \frac{x_0 f_2 - f_0 x_2}{f_2 - f_0} = 0.3634$$

$f(0.363) = -0.04283$

$f(0)\, f(0.36364) < 0 \quad (0, 0.36364)$

$$x_4 = \frac{x_0 f_3 - f_0 x_3}{f_3 - f_0} = 0.34870$$

$f(x_4) = -0.0370.$

$f(0)\, f(0.34870) < 0 \quad (0, 0.34870)$

$$x_5 = \frac{x_0 f_4 - f_0 x_4}{f_4 - f_0} = 0.34741$$

$f(x_5) = -0.00030.$

$f(0)\, f(0.34741) < 0 \quad (0, 0.\underline{347}41)$

$$x_6 = \frac{x_0 f_5 - f_0 x_5}{f_5 - f_0} = 0.\underline{347}306$$

$f(x_6) = \ldots \text{———} \ldots$

So, root has computed correctly upto 3 places

$|x_6 - x_5| = 0.001$

$x = x_6 = 0.347306$

Now, root between (1,2)

$x_0 = 1$, $x_1 = 2$ $f_0 = -1$, $f_1 = -3$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = 1.25 \quad f(x_2) = -0.796875$$

$\Rightarrow$

$f(1) \cdot f(1.25) < 0$

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = 1.407407$$

$f(x_3) = -0.43447$

$\Rightarrow$

$f(x_3) \cdot f(1.407) < 0$

⋮

$\Rightarrow$

$x_9 = 1.531729$

$x_8 = 1.531116$

$x_{10} = 1.531958$

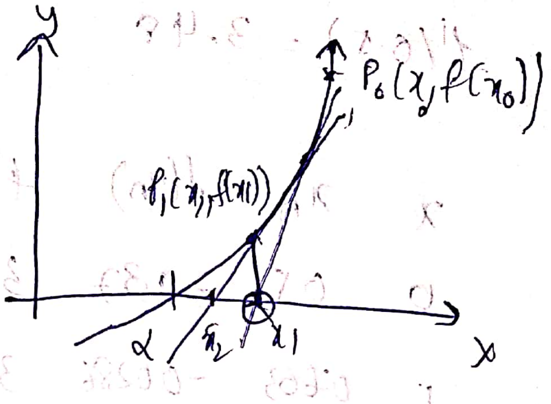
Newton Raphson Method :

$f(x)=0,\ x_0,\ P(x_0, f_0),\ f_0 = f(x_0)$

$y - f(x_0) = (x - x_0)\, f'(x_0)$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},\ f'(x_0) \neq 0$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},\ f'(x_1) \neq 0$



$x = \alpha$ $f(x) = 0$

x_0.

$x_k \to f(x) = 0$ $f(x_k + \Delta x) = 0$

$f(x_k) + \Delta x \cdot f'(x_k) + \frac{(\Delta x)^2}{2!} f''(x_k) + \cdots = 0$

$(x_0) \neq (\alpha)$ far way

diverges

$f(x_k) + \Delta x f'(x_k) = 0$

$x \in (a, b)$
$\alpha \in (a, b)$ } then converges

(or) $\Delta x = -\frac{f(x_k)}{f'(x_k)}$

$f'(x) = 0$ if failure

$f'(x) \neq 0,\ k = 0, 1, 2$

$\boxed{x_{k+1} = x_k + \Delta x = x_k - \frac{f(x_k)}{f'(x_k)}}$

Ex: $f(x) = 3x - \cos x - 1 = 0$; $f(0) = -2$, $f(0.5) = 1.5 \cdots$

$= -0.37$

$f(0.7) = 0.34$.

$f'(x) = 3 + \sin x$ $x_0 = 0.5$

$f'(0.5) = 3.48$

x	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h$
0	0.5	-0.37	3.48	0.1063	0.603
1	0.603	-0.0286	3.56983	0.00801	0.6071
2	0.607	-0.000023	3.5704	0.0000064	0.607101
3	0.60710606	- - -	- - -		

Error of approximation: $\varepsilon_k = x_k - \alpha$, $k = 0, 1, 2, \ldots$

$|\varepsilon_{k+1}| \to 0$, as $k \to \infty$.

order p or has the rate of convergence p if p is the largest +ve real number.

for which $\exists$ a finite const $c \neq 0$. s.t.

$$|\varepsilon_{k+1}| \leq c|\varepsilon_k|^p$$

Bisection → linear convergence rate

False position → Linear

Faster than Bisection

$$x_{k+1} = \frac{x_0 f_k - x_k f_0}{f_k - f_0}$$

Substitute,

$x_k = \varepsilon_k + \alpha$

$x_{k+1} = \varepsilon_{k+1} + \alpha$

$x_0 = \varepsilon_0 + \alpha$

$f(\alpha) = 0$

$$\boxed{\varepsilon_{k+1} = c\,\varepsilon_0\,\varepsilon_k}$$

$$c = \frac{f''(\alpha)}{2f'(\alpha)}$$

$|\varepsilon_{k+1}| = |x_{k+1} - \alpha| = \frac{1}{2} \times |x_k - \alpha|$

Bisection: $|\varepsilon_{k+1}| = \frac{1}{2} \alpha |\varepsilon_k| \to p = 1$, linear

Newton Raphson:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad f'(x_k) \neq 0.$$

$x_{k+1} = \varepsilon_{k+1} + \alpha \qquad x_k = \varepsilon_k + \alpha$

$$\varepsilon_{k+1} = C\,\varepsilon_k^2, \quad C = \frac{f''(\alpha)}{2f'(\alpha)}$$

$\boxed{p = 2}$

(P) $\varepsilon_k = 0.1$

$C(0.1)^2$

→ Newton's method is faster than Bisection and Regular falsi method.

→ Newton's method fails when $f'(x) = 0$ at any point in interval and x_0, α are very far away.

21/11/2020

(3) $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n-1}x_{n-1} + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots a_{2n-1}x_{n-1} + a_{2n}x_n = b_2$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn-1}x_{n-1} + a_{nn}x_n = b_n$

$a_{ii} \neq 0$

$$\Rightarrow x_1 = \frac{b_1}{a_{11}} - 0x_1 - \frac{a_{12}}{a_{11}}x_2 - \dots - \frac{a_{1n-1}}{\boxed{a_{11}}}x_{n-1} - \frac{a_{1n}}{a_{11}}x_n \quad (4)$$

$$\Rightarrow x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - 0x_2 - \dots - \frac{a_{2n-1}}{a_{22}}x_{n-1} - \frac{a_{2n}}{a_{22}}x_n$$

(1) Solving linear Eqtn

$3x(2x+3y)=5$ || $x=1, y=1$

$2x(3x+5y)=8$

Direct method: Gauss - elimination method,

$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2$ $\quad$ $\det A \neq 0.$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n = b_n$

Gauss - Seidel method

(1) Initial guess: $(x_1^{(0)}, x_2^{(0)}, \ldots x_n^{(0)})$

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} \ldots a_{1n-1}x_{n-1}^{(0)} - a_{1n}x_n^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)} \ldots - a_{2n-1}x_{n-1}^{(0)} - a_{2n}x_n^{(0)}\right]$$

$\ldots\ldots\ldots$

$$x_n^{(1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(0)} - a_{n2}x_2^{(0)} \ldots - a_{nn-1}x_{n-1}^{(0)}\right]$$

---

(2) Iterative method: Jacobi's method:

Gauss, Seidal method:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{n2} \end{bmatrix}_{n \times n} \qquad |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

Unique sol^n:

Ex:- $x_1 + x_2 + 4x_3 = 9$

$8x_1 - 3x_2 + 2x_3 = 20$

$4x_1 - 11x_2 - x_3 = 33$

$\Rightarrow$ Diagonally row-dominant eq^{n}s:

$8x_1 - 3x_2 + 2x_3 = 20$

$4x_1 - 11x_1 - x_3 = 33$

$x_1 + x_2 + 4x_3 = 9$

$x_1 = \frac{1}{8}(20 + 3x_2 - 2x_3)$

$x_2 = \frac{1}{11}(33 - 4x_1 + x_3)$

$x_3 = \frac{1}{4}(9 - x_1 - x_3)$

$x_1^{(0)} = 1, \; x_2^{(0)} = 1, \; x_3^{(0)} = 0$

$x_1^{(1)} = 3.057$ $\quad x_1^{(1)} = \frac{1}{8}(20 + 3\times1 - 2\times0) = 2.875$

$x_2^{(2)} = 2.114$ $\quad x_2^{(1)} = \frac{1}{11}(33 - 4\times1 + 0) = 2.636$

$x_3^{(2)} = 0.872$ $\quad x_3^{(1)} = \frac{1}{4}(9 - 1 - 1) = 1.75.$

$x_1^{(6)} = 2.997$

$x_2^{(6)} = 2.004$

$x_3^{(6)} = 1.002$

$x_1 = 3, x_2 = 2, x_3 = 1$