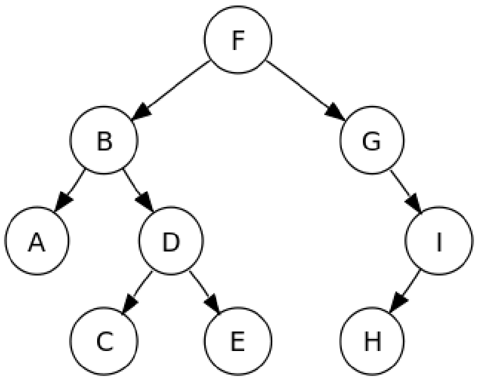
Advanced Data Structure and Algorithm

Binary Search Trees

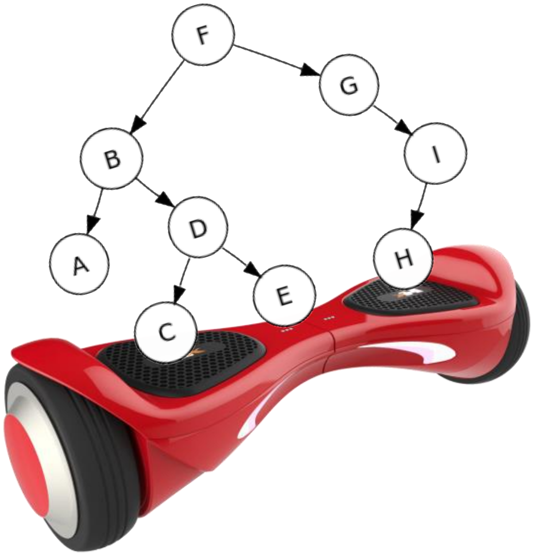
Today



- Binary search trees
  - They are better when they're balanced.

this will lead us to…

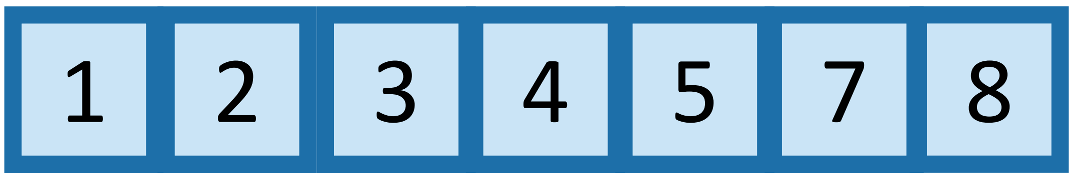
- Self-Balancing Binary Search Trees
  - **Red-Black** trees.



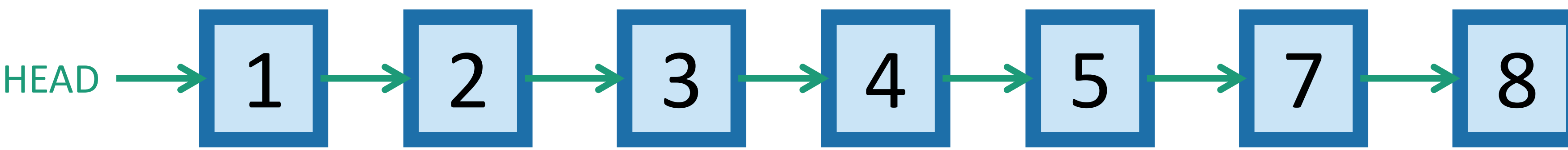
Some data structures

for storing objects like $\boxed{5}$ (aka, nodes with keys)

- (Sorted) arrays:



- (Sorted) linked lists:



- Some basic operations:
  - INSERT, DELETE, SEARCH

Sorted Arrays

1	2	3	4	5	7	8

- O(n) INSERT/DELETE:
  - First, find the relevant element (time O(log(n)) as below), and then move a bunch elements in the array:

1	2	3	4	4.5	7	8

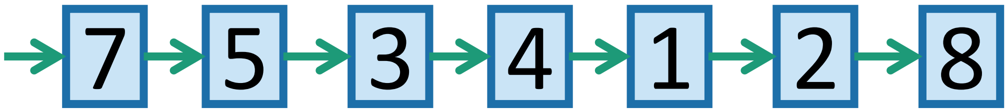
eg, insert 4.5

- O(log(n)) SEARCH:

1	2	3	4	5	7	8

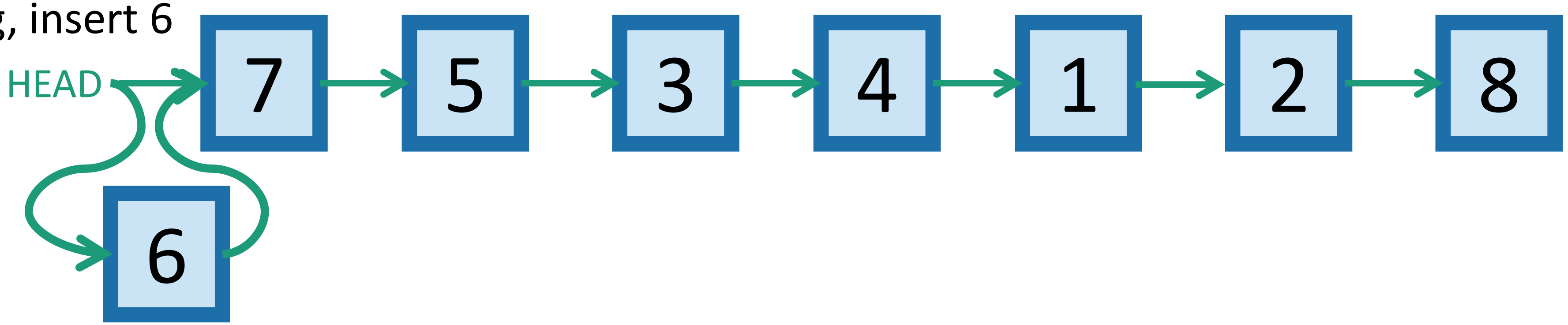
eg, Binary search to see if 3 is in A.

UNSorted linked lists

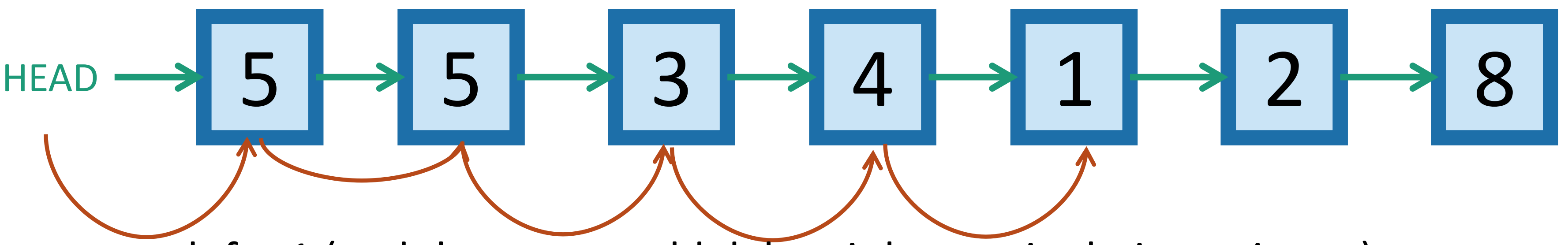


- O(1) INSERT:

eg, insert 6



- O(n) SEARCH/DELETE:



eg, search for 1 (and then you could delete it by manipulating pointers).

Motivation for Binary Search Trees

	Sorted Arrays	Linked Lists	Binary Search Trees*
Search	O(log(n))	O(n)	O(log(n))
Delete	O(n)	O(n)	O(log(n))
Insert	O(n)	O(1)	O(log(n))

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Motivation for Binary Search Trees

TODAY!

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Delete	O(n)	O(n)	O(log(n))
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For today all keys are distinct.

Binary tree terminology

Each node has two children.

The left child of 3 is 2

The right child of 3 is 4

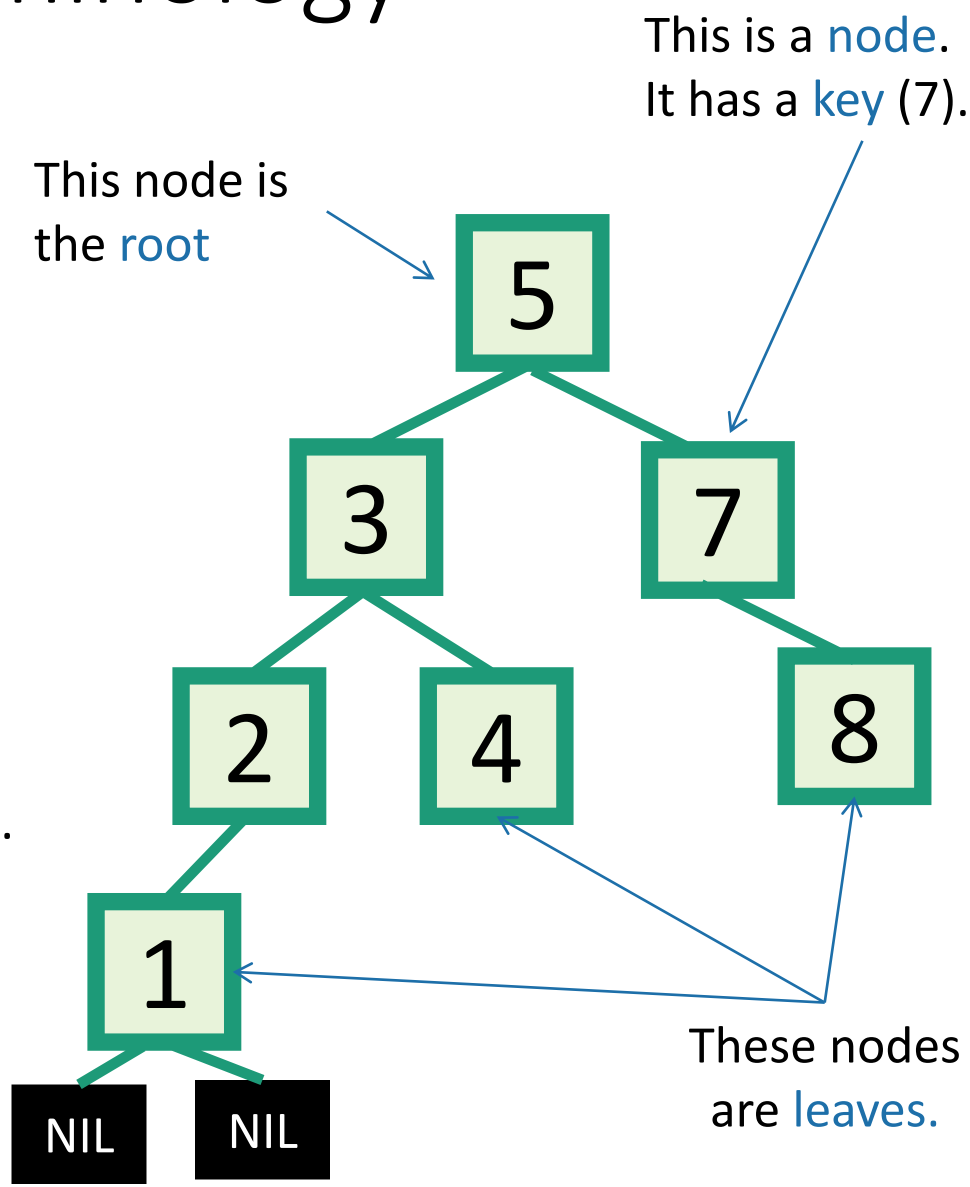
The parent of 3 is 5

2 is a descendant of 5

Each node has a pointer to its left child, right child, and parent.

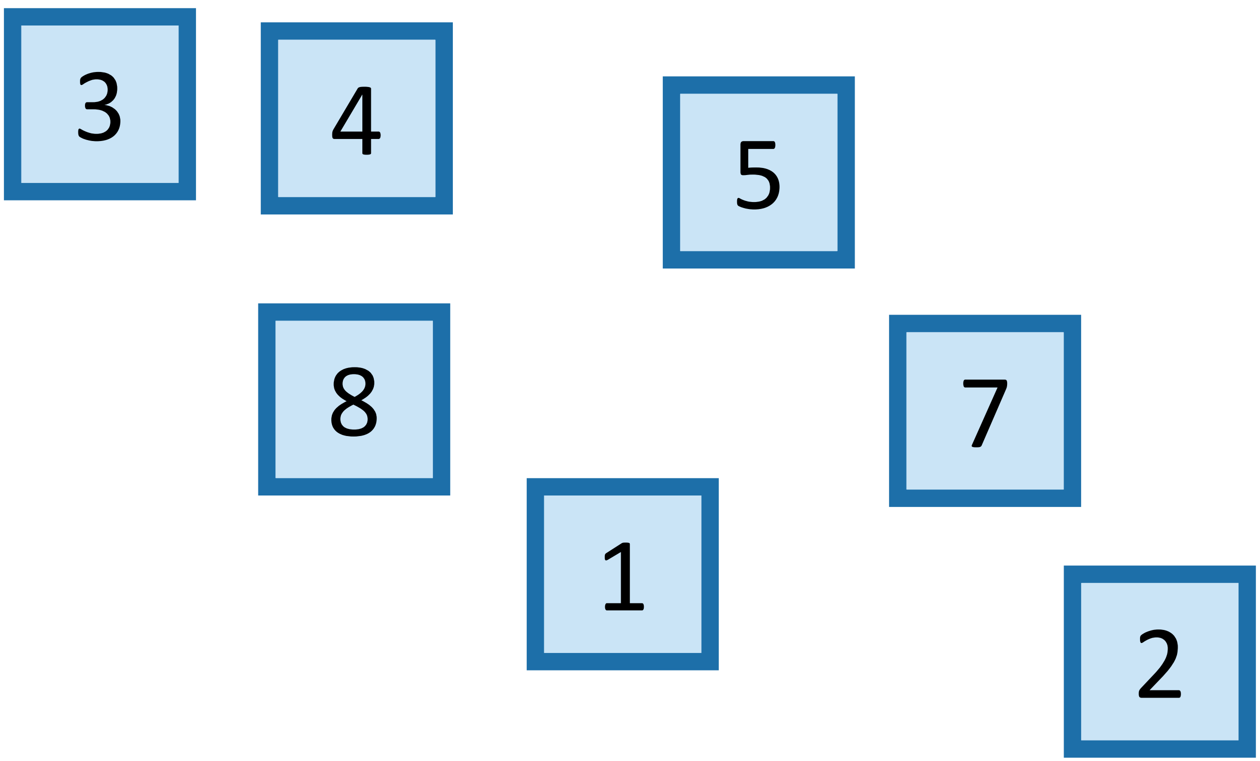
Both children of 1 are NIL.
(Not usually drawn).

The height of this tree is 3.
(Max number of edges from the root to a leaf).



Binary Search Trees

- A BST is a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:



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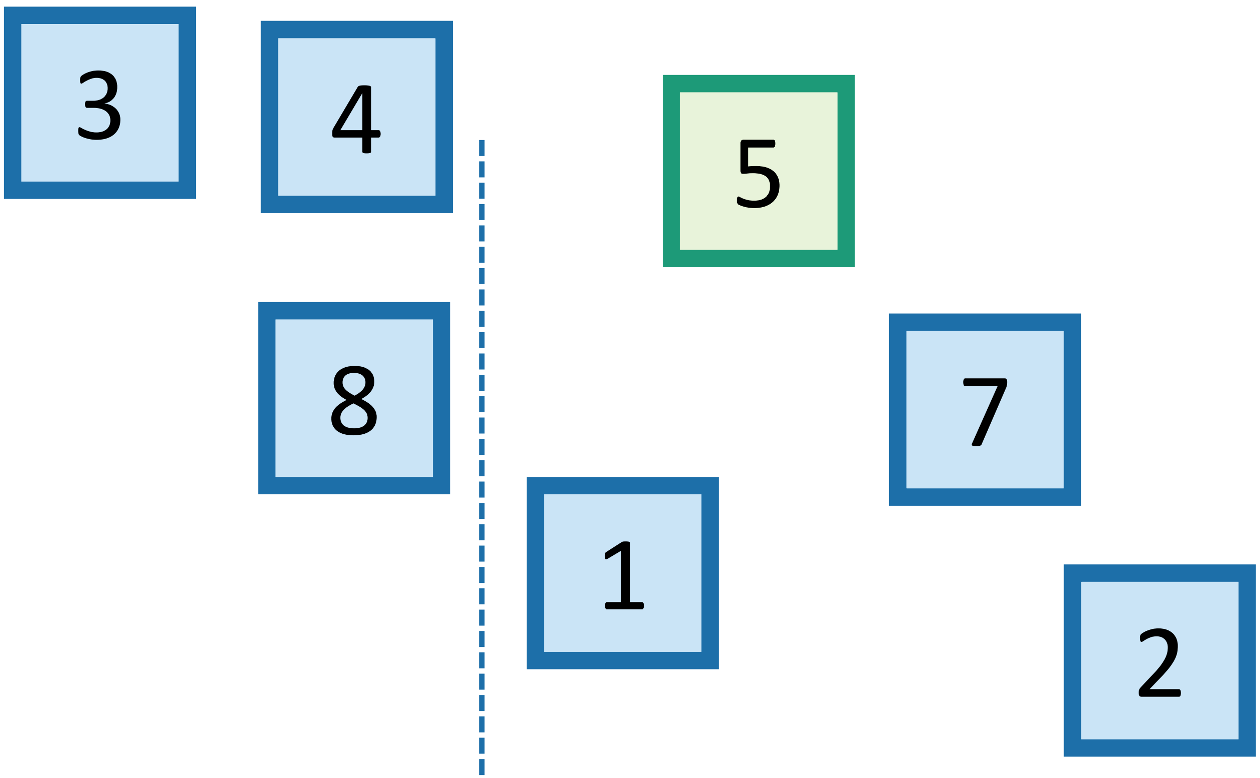

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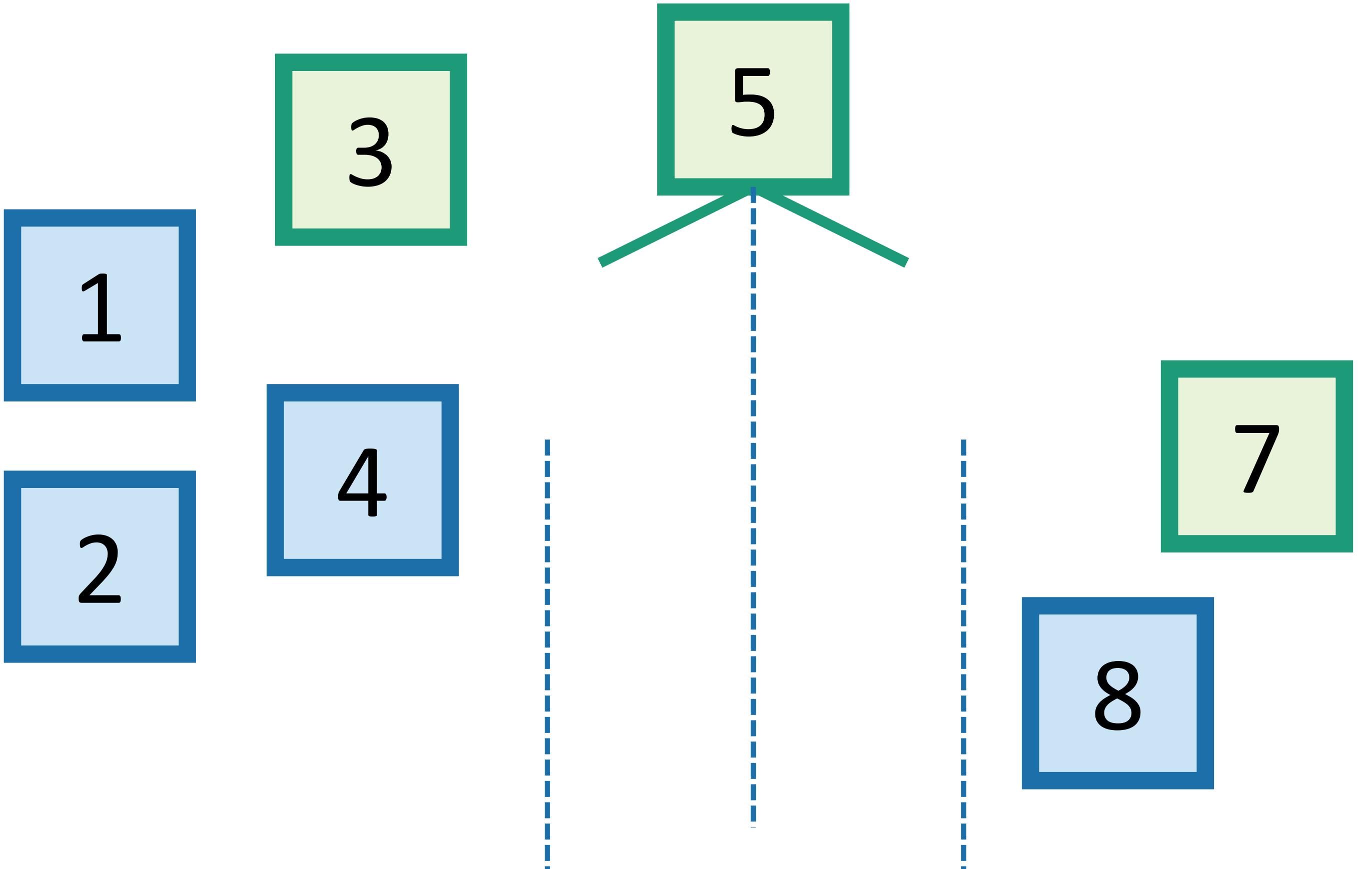

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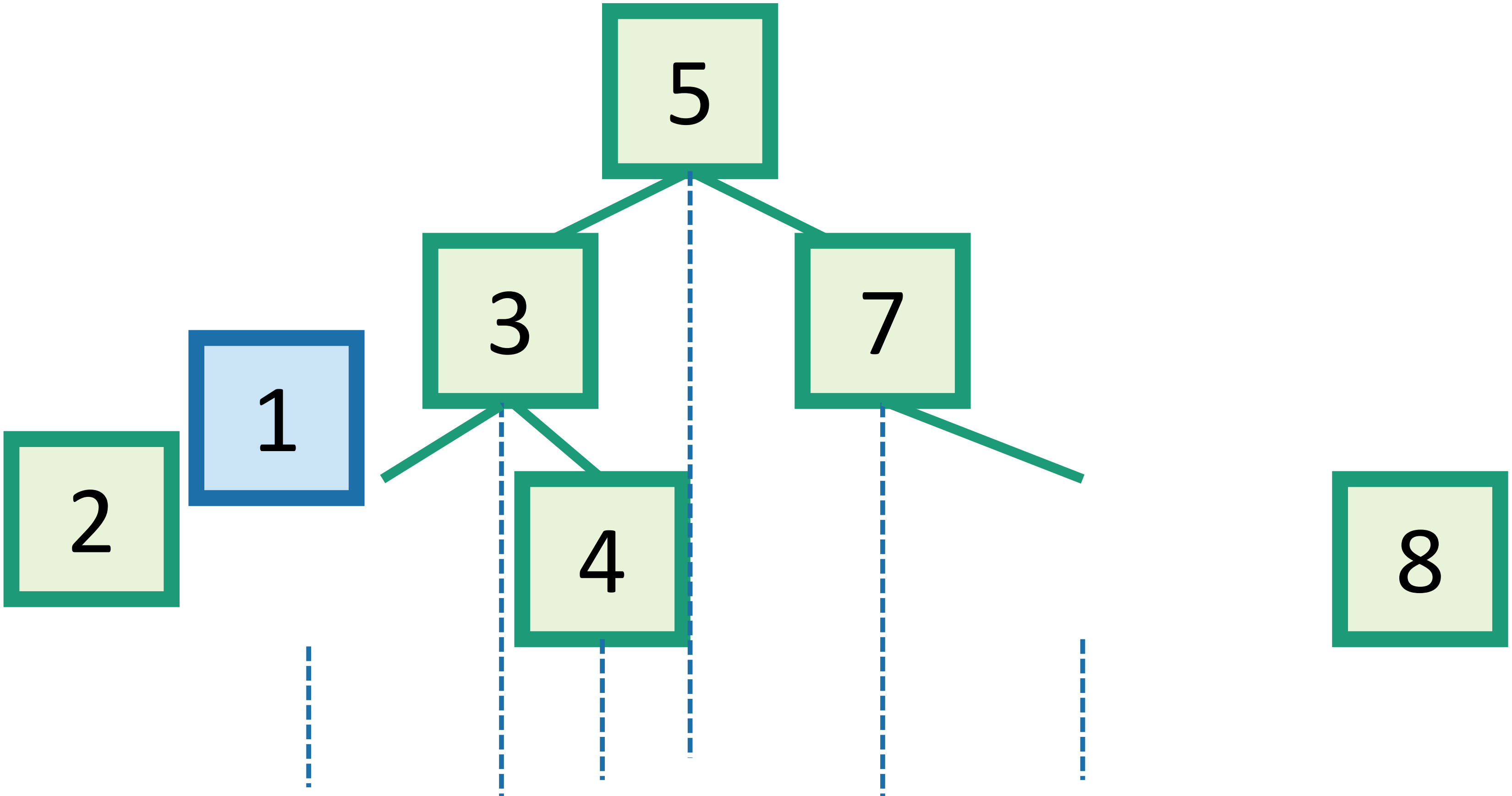

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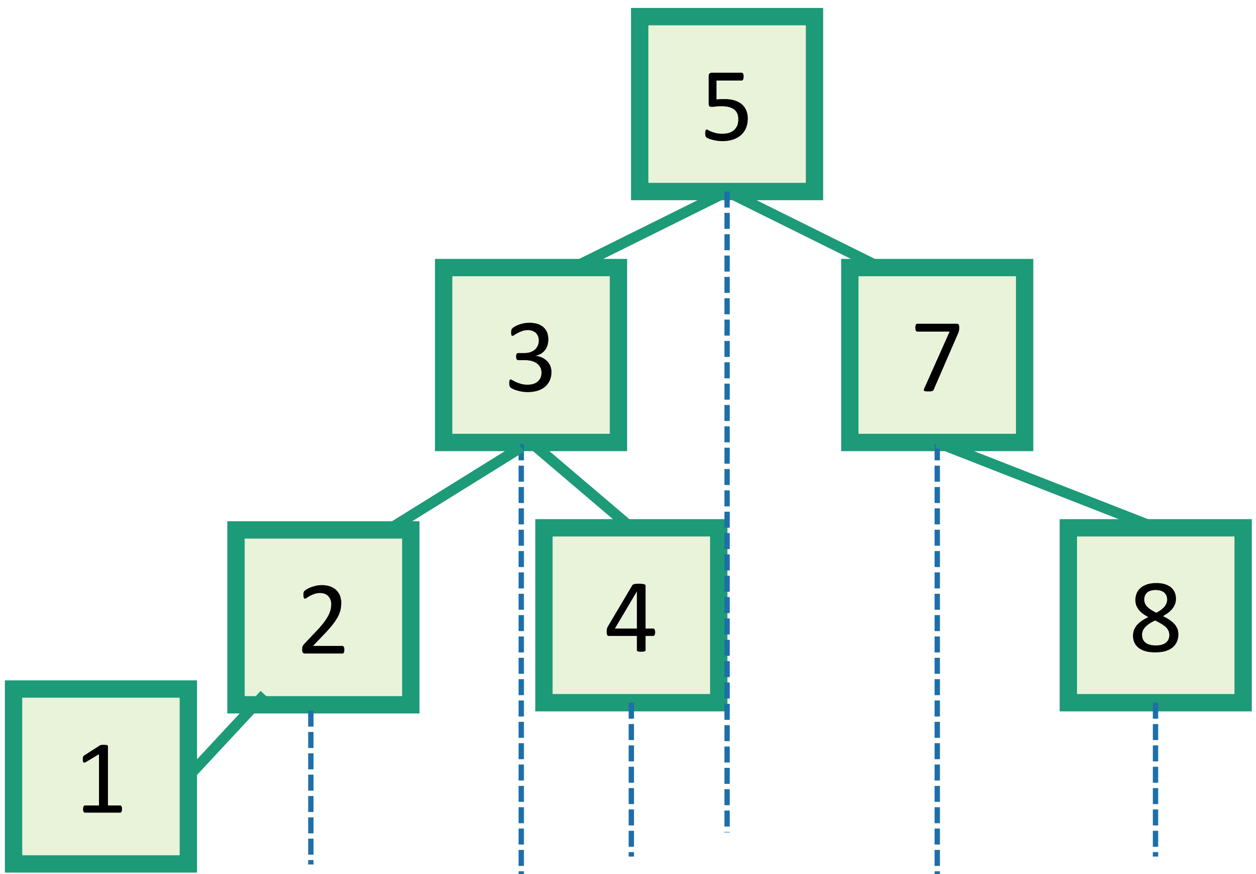
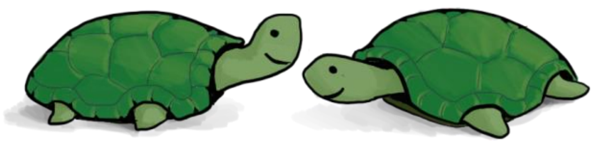


Q: Is this the only binary search tree I could possibly build with these values?

A: **No.** I made choices about which nodes to choose when. Any choices would have been fine.

Binary Search Trees

Which of these is a BST?



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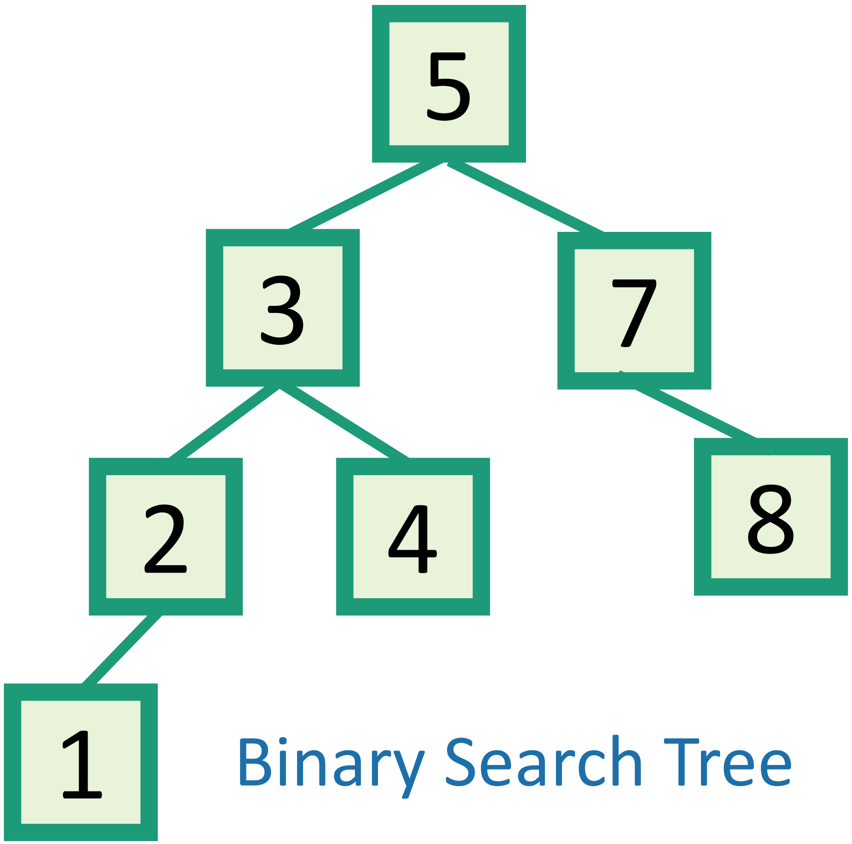
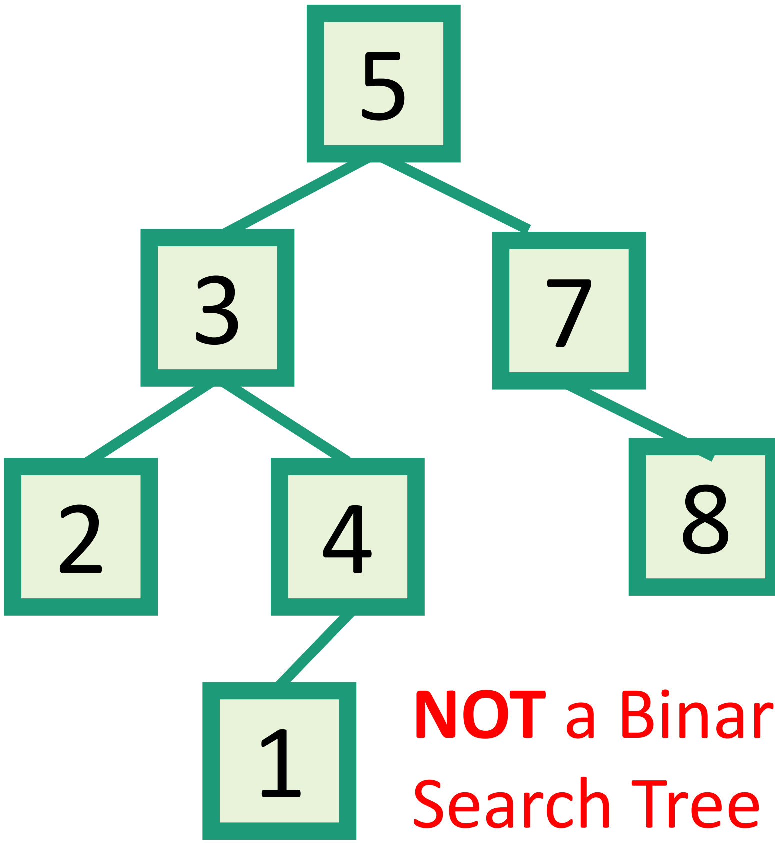


Binary Search Tree

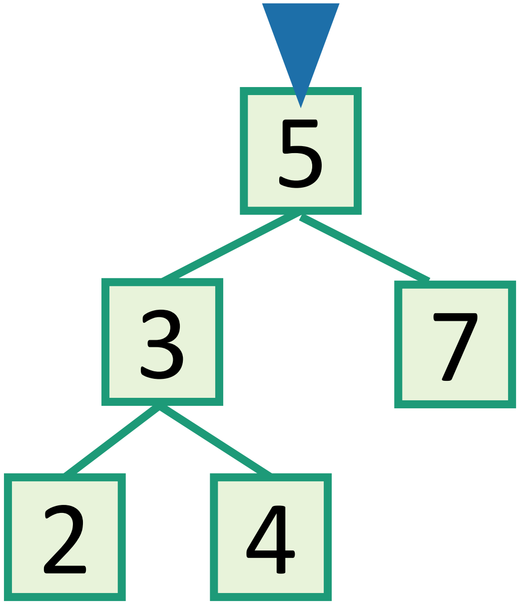


NOT a Binary Search Tree

Aside: In-Order Traversal of BSTs

- Output all the elements in sorted order!

- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal(x.left)
    - print(x.key)
    - inOrderTraversal(x.right)



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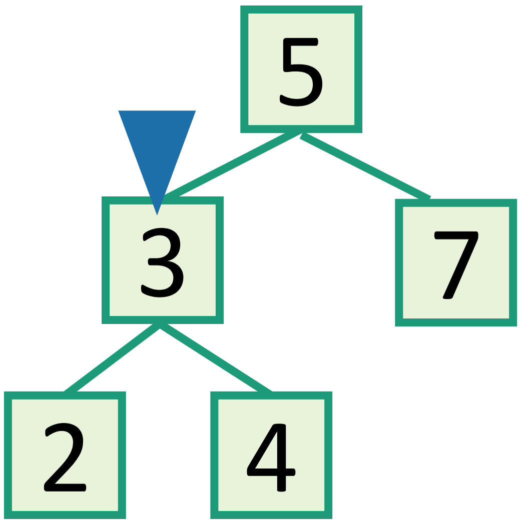

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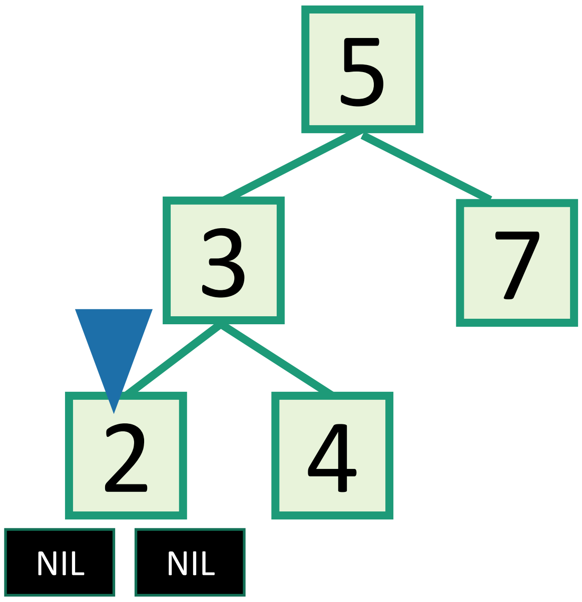

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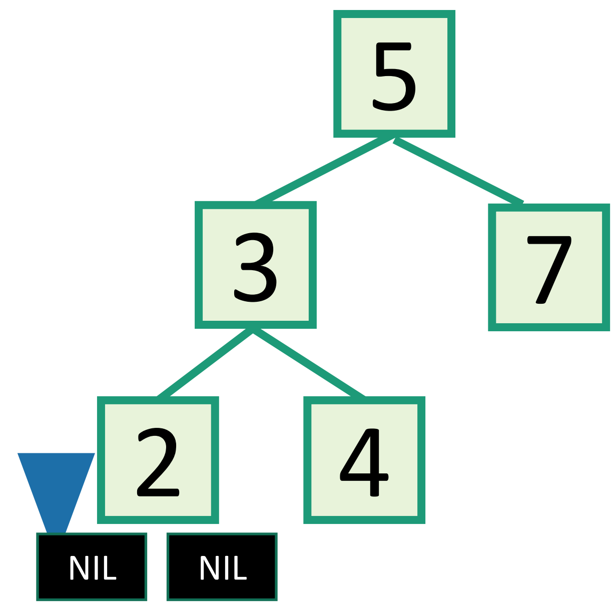

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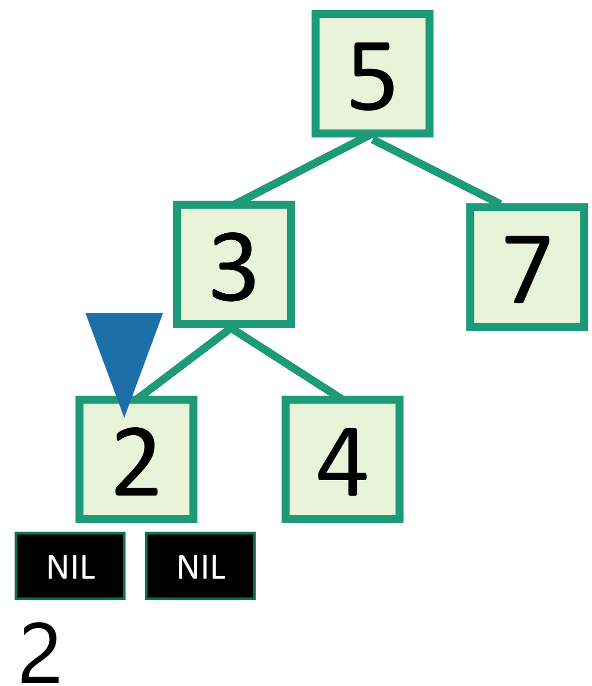

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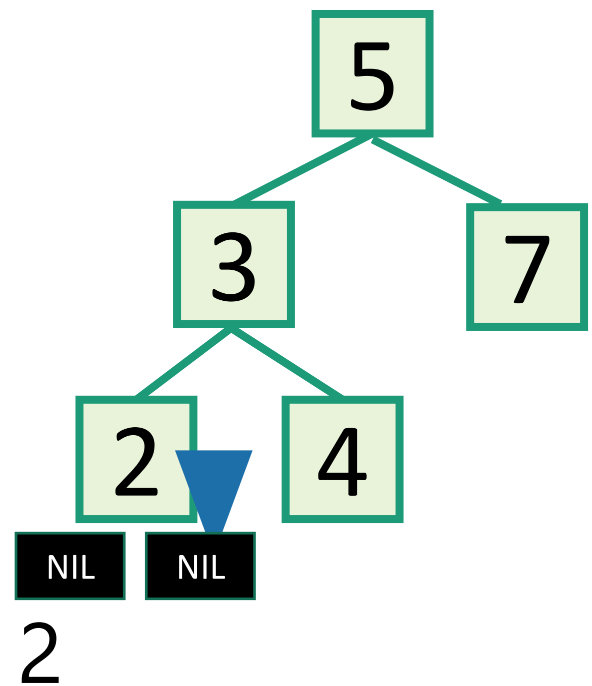

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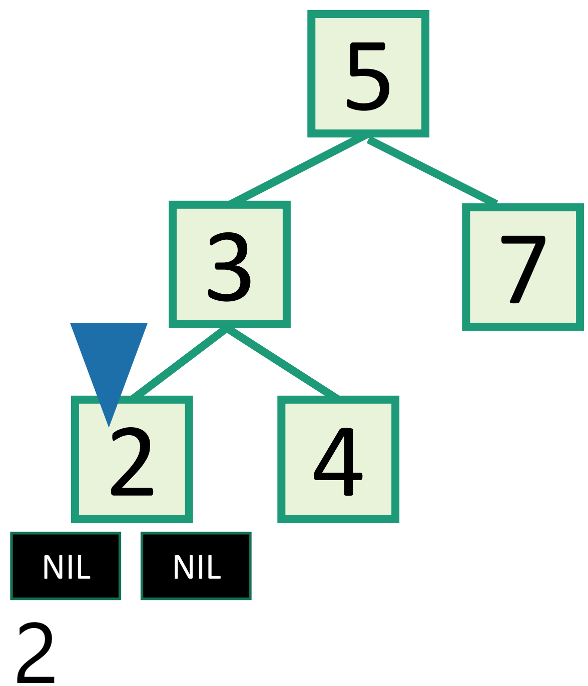

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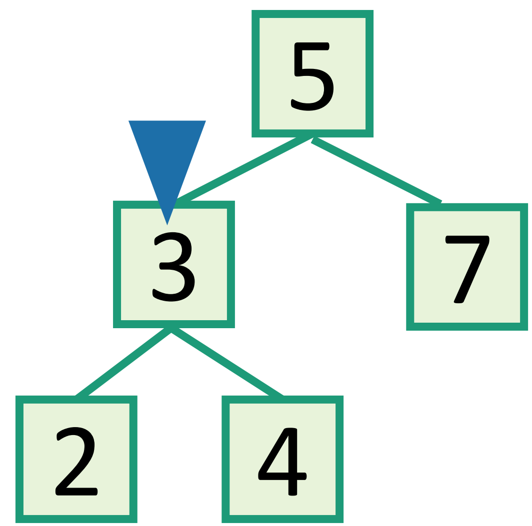

2 3

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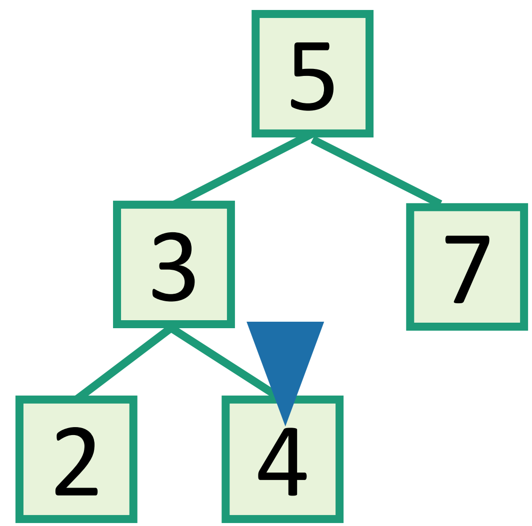

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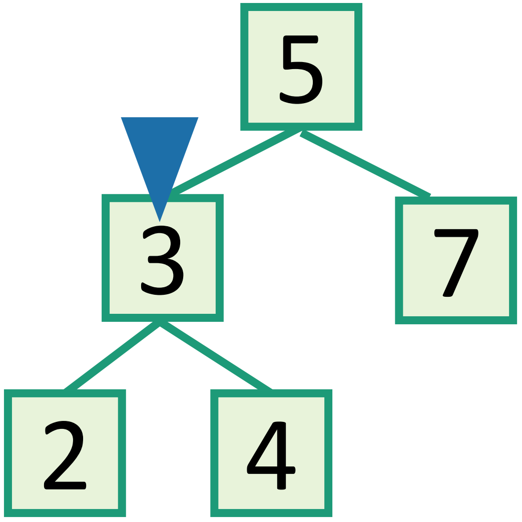

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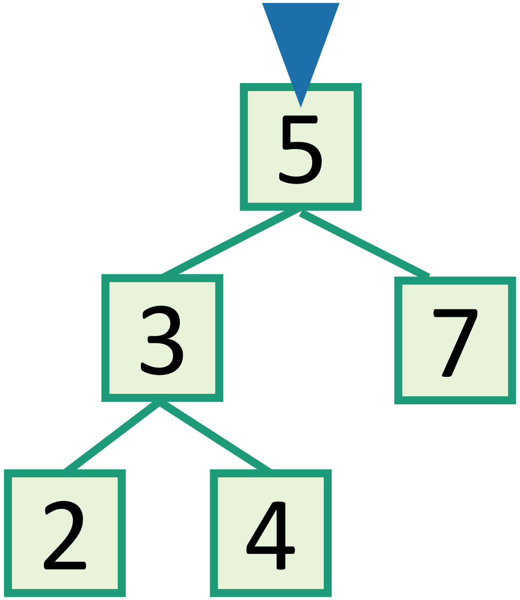

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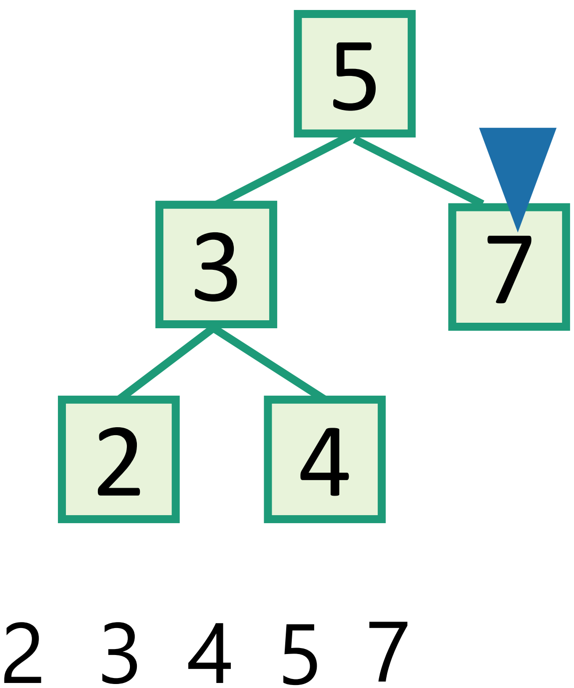

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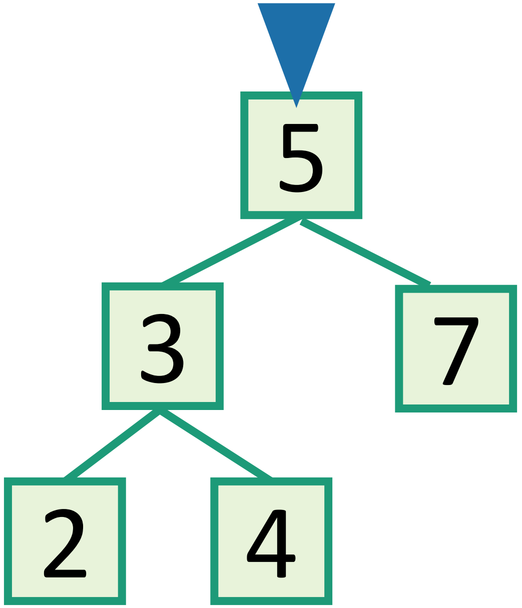


- Runs in time O(n).

2 3 4 5 7 Sorted!

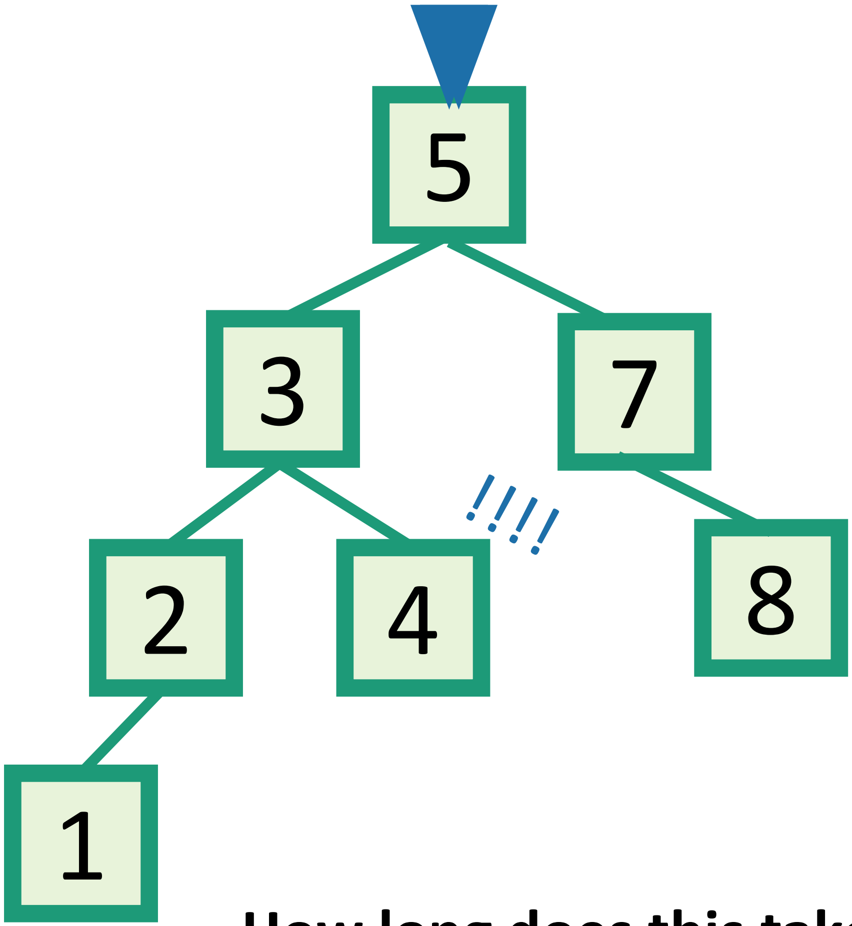
Back to the goal

Fast SEARCH/INSERT/DELETE

Can we do these?

SEARCH in a Binary Search Tree

definition by example



EXAMPLE: Search for 4.

EXAMPLE: Search for 4.5

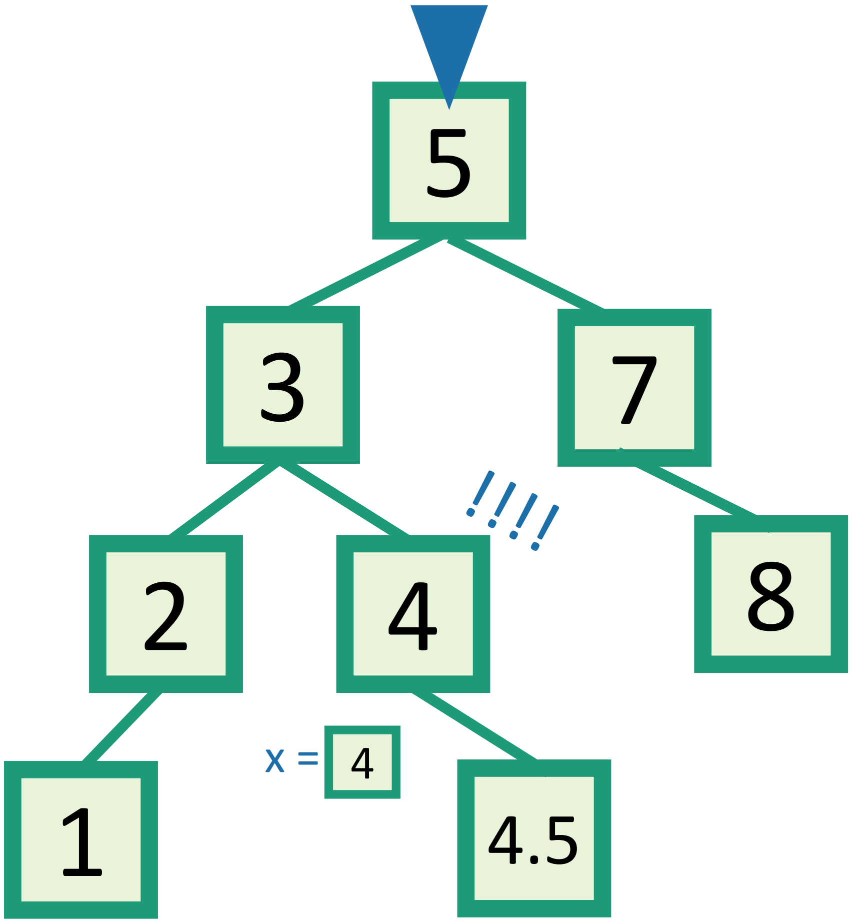
- It turns out it will be convenient to **return 4** in this case
- (that is, **return** the last node before we went off the tree)

Write pseudocode (or actual code) to implement this!

How long does this take?

O(length of longest path) = O(height)

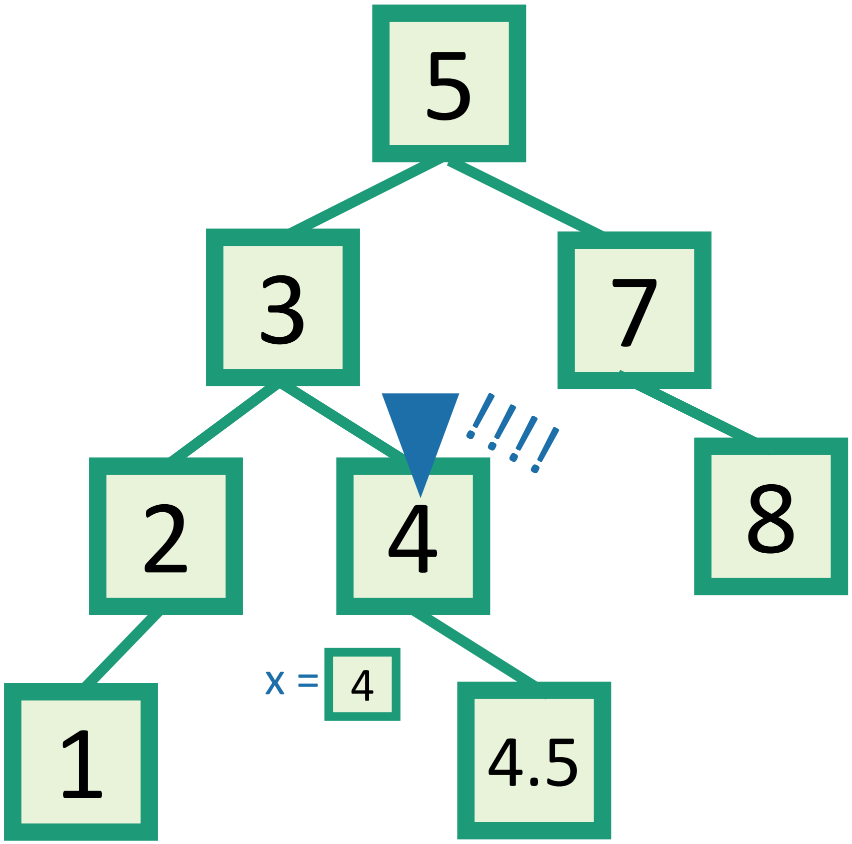
INSERT in a Binary Search Tree



EXAMPLE: Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - **Insert** a new node with desired key at x…

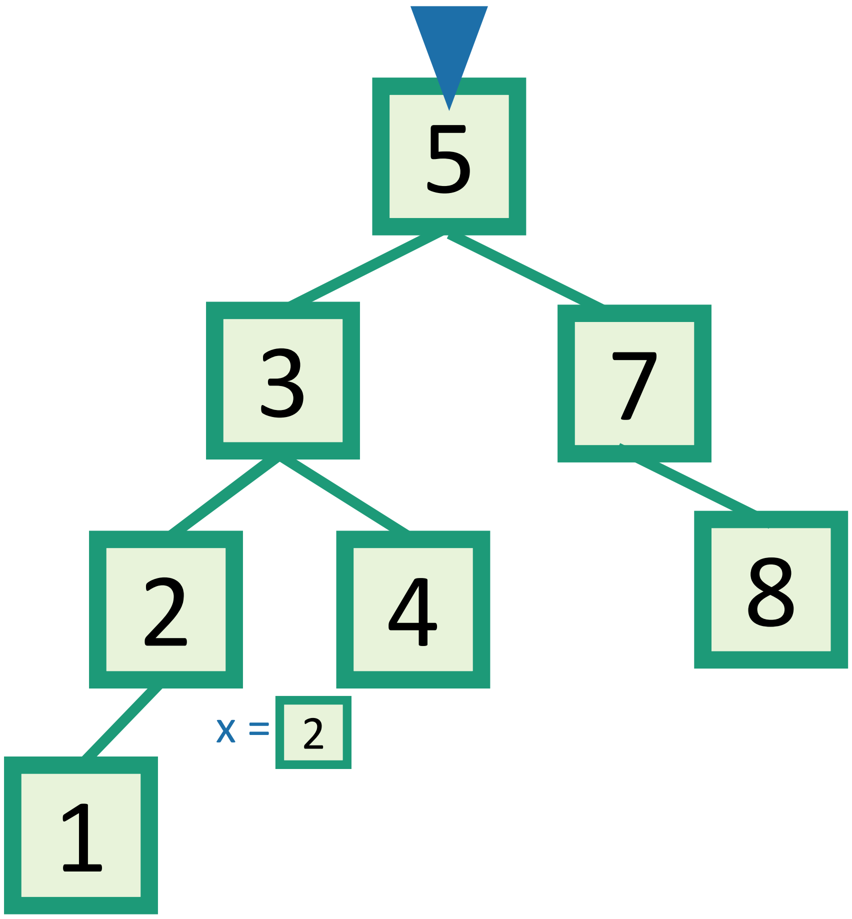
INSERT in a Binary Search Tree



EXAMPLE: Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - **if** key > x.key:
    - Make a new node with the correct key, and put it as the right child of x.
  - **if** key < x.key:
    - Make a new node with the correct key, and put it as the left child of x.
  - **if** x.key == key:
    - **return**

DELETE in a Binary Search Tree



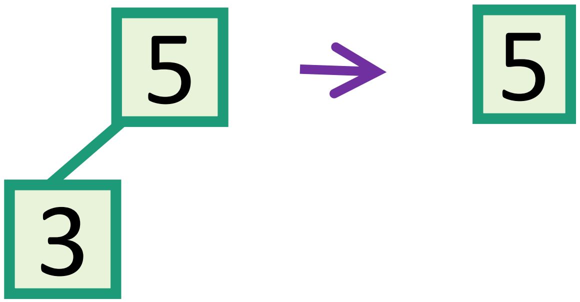
EXAMPLE: Delete 2

- DELETE(key):
  - x = SEARCH(key)
  - **if** x.key == key:
    - ....delete x....

DELETE in a Binary Search Tree

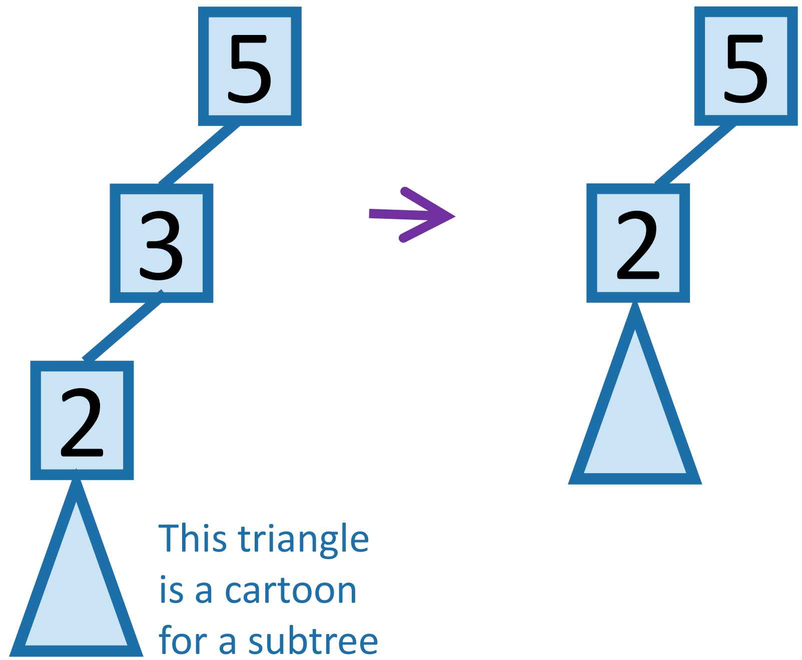
several cases (by example)

say we want to delete 3



Case 1: if 3 is a leaf, just delete it.

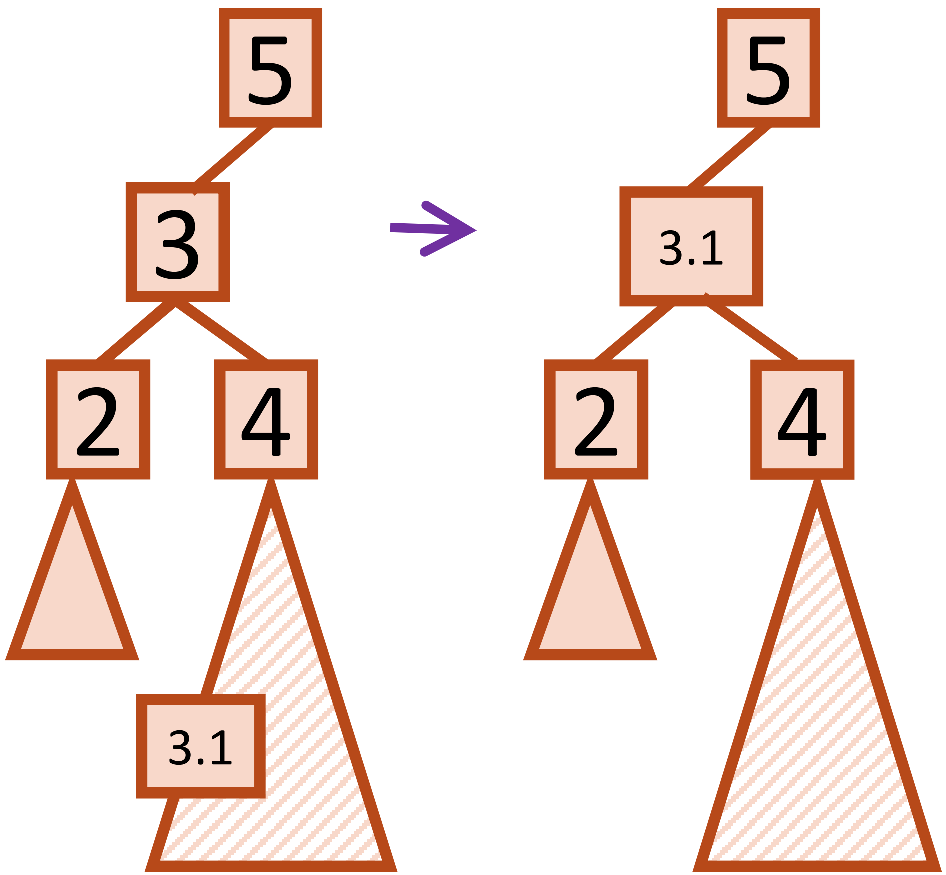
Write pseudocode for all of these!



Case 2: if 3 has just one child, move that up.

DELETE in a Binary Search Tree

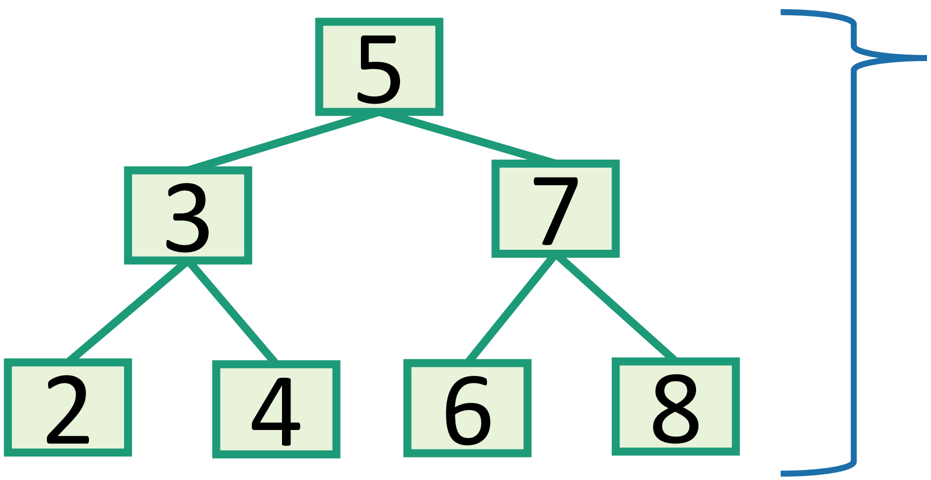
Case 3: if 3 has two children,
replace 3 with it's immediate successor.
(aka, next biggest thing after 3)



- Does this maintain the BST property?
  - Yes.
- How do we find the immediate successor?
  - SEARCH for 3 in the subtree under 3.right
- How do we remove it when we find it?
  - If [3.1] has 0 or 1 children, do one of the previous cases.
- What if [3.1] has two children?
  - It doesn't. (can not have two children)

How long do these operations take?

- SEARCH is the big one.
  - Everything else just calls SEARCH and then does some small O(1)-time operation.

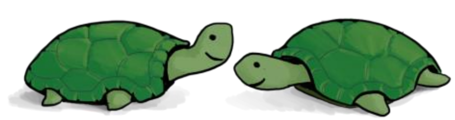


Time = O(height of tree)

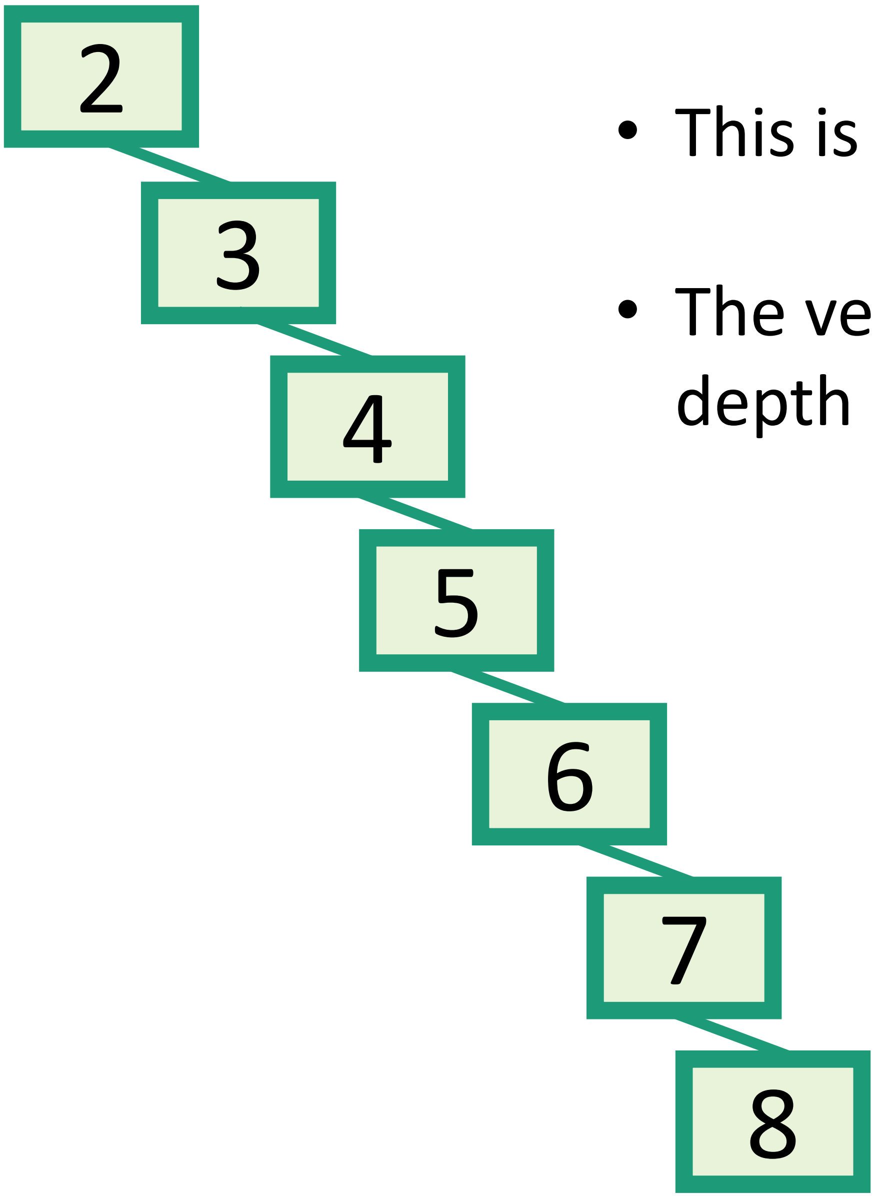
Trees have depth O(log(n)). **Done!**

Wait a second…

How long does search take?



Search might take time O(n).



- This is a valid binary search tree.
- The version with n nodes has depth n, **not** O(log(n)).

What to do?

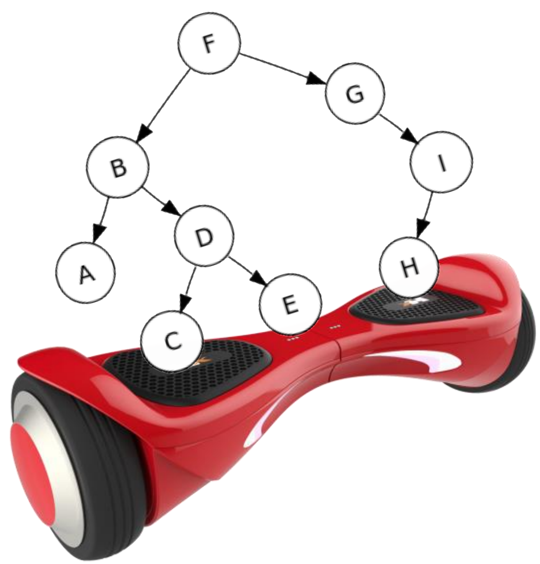
How often is "every so often" in the worst case? It's actually pretty often!

- Goal: Fast SEARCH/INSERT/DELETE
- All these things take time O(height)
- And the height might be big!!! ☹

- Idea 0:
    - Keep track of how deep the tree is getting.
    - If it gets too tall, re-do everything from scratch.
        - At least $\Omega(n)$ every so often....

- Turns out that's not a great idea. Instead we turn to…

Self-Balancing
Binary Search Trees

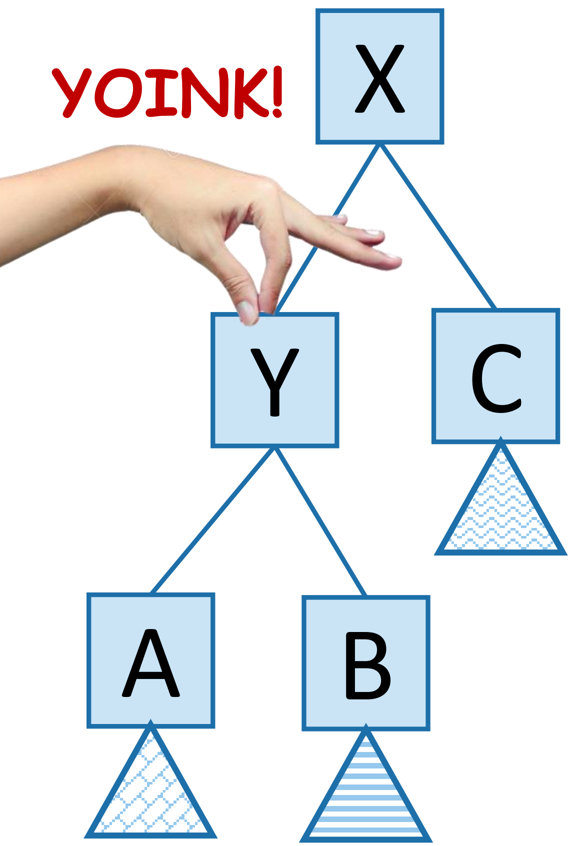


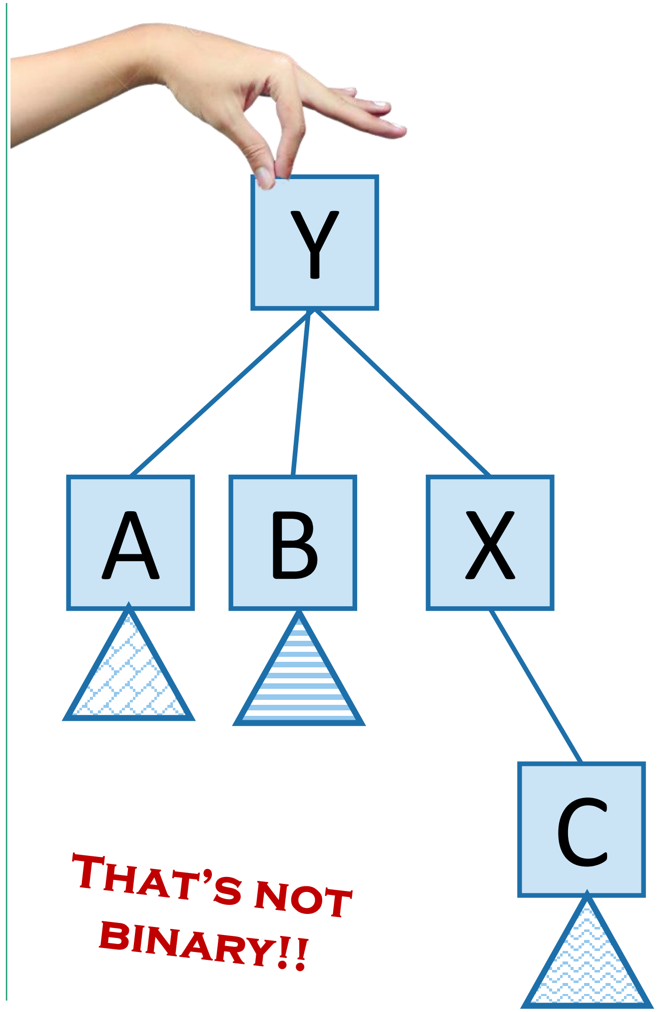
Idea 1: Rotations

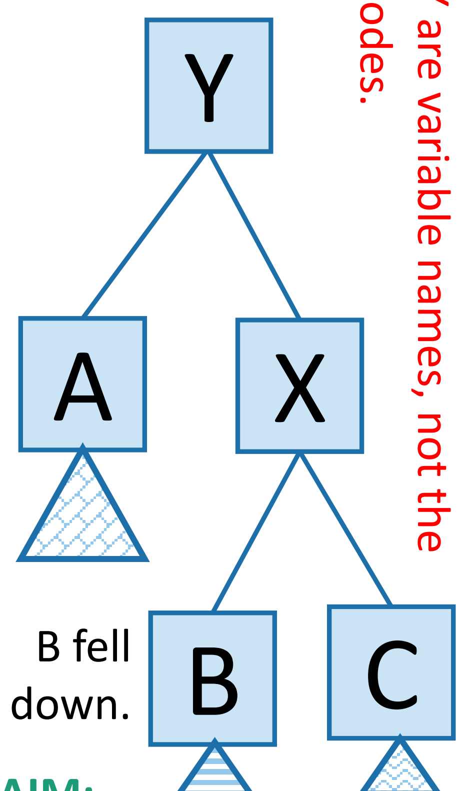
No matter what lives underneath A,B,C, **this takes time O(1).** (Why?)

- Maintain Binary Search Tree (BST) property, while moving stuff around.

Note: A, B, C, X, Y are variable names, not the contents of the nodes.

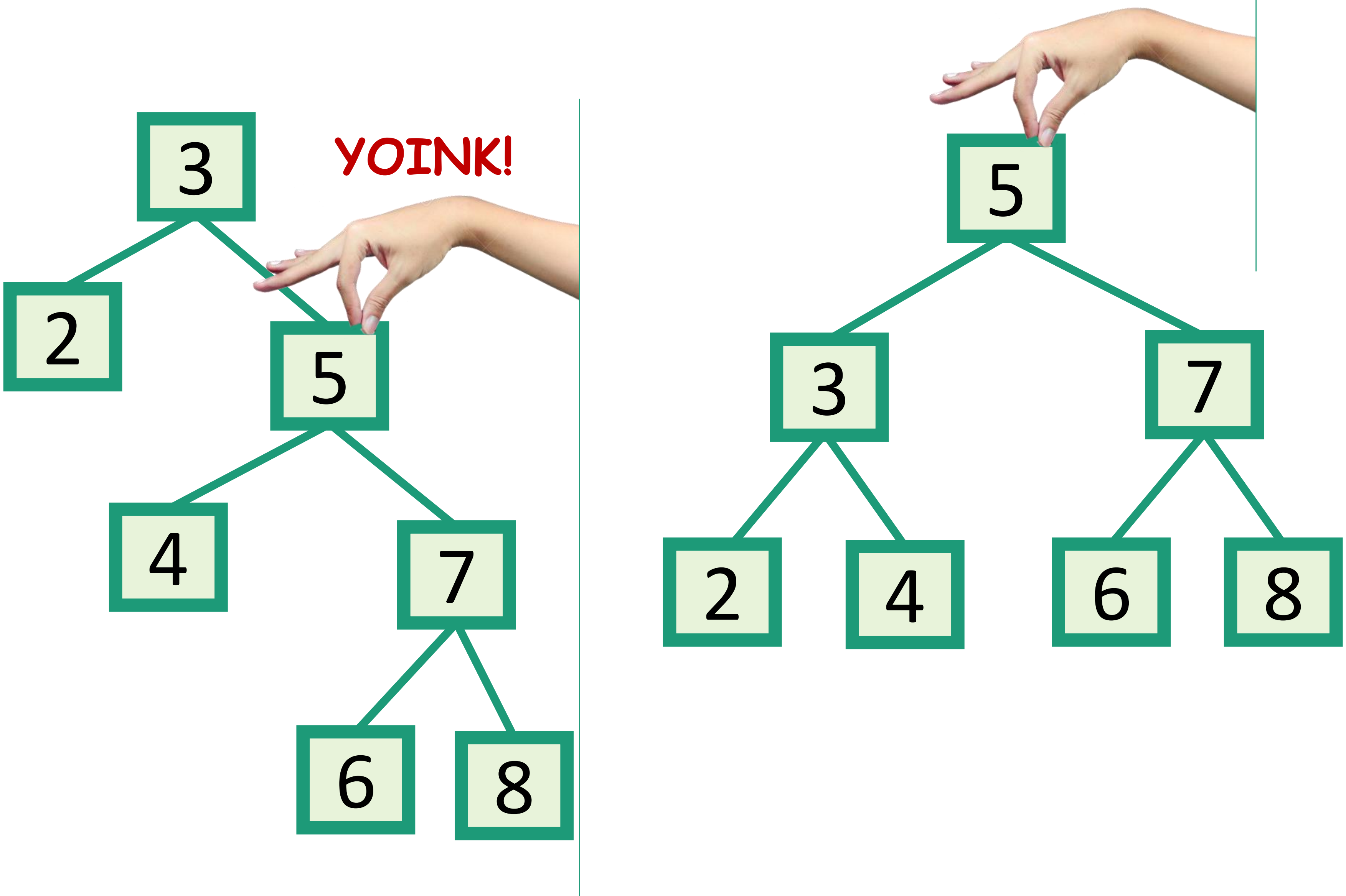






CLAIM:
this still has BST property.

This seems helpful



Strategy?

- Whenever something seems unbalanced, do rotations until it's okay again.

This is pretty vague.

What do we mean by "seems unbalanced"?

What's "okay"?

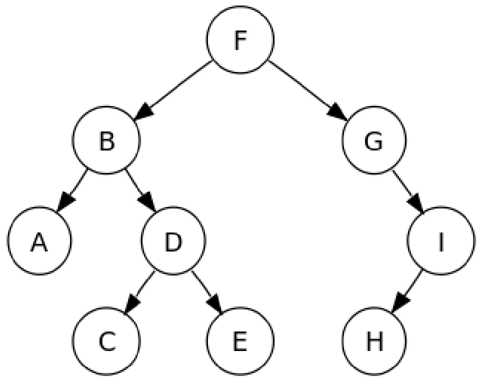
Idea 2: have some proxy for balance

- Maintaining perfect balance is too hard.
- Instead, come up with some proxy for balance:
  - If the tree satisfies [SOME PROPERTY], then it's pretty balanced.
  - We can maintain [SOME PROPERTY] using rotations.

There are actually several ways to do this, but we'll see:

1. AVL Tree (Covered in DSA)
2. Multiway-Search Tree (2-4 Tree)
3. Red-Black Tree

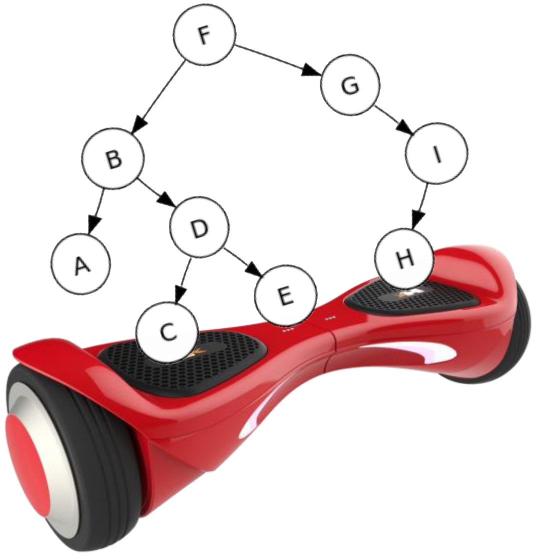
Today



- Begin a brief foray into data structures!
- Binary search trees
  - They are better when they’re balanced.

this will lead us to…

- Self-Balancing Binary Search Trees
  - AVL Tree
  - Multiway-Search Tree
  - Red-Black Tree



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