Advanced Data Structures and Algorithms

Min Cut and Karger's Algorithm

This Module

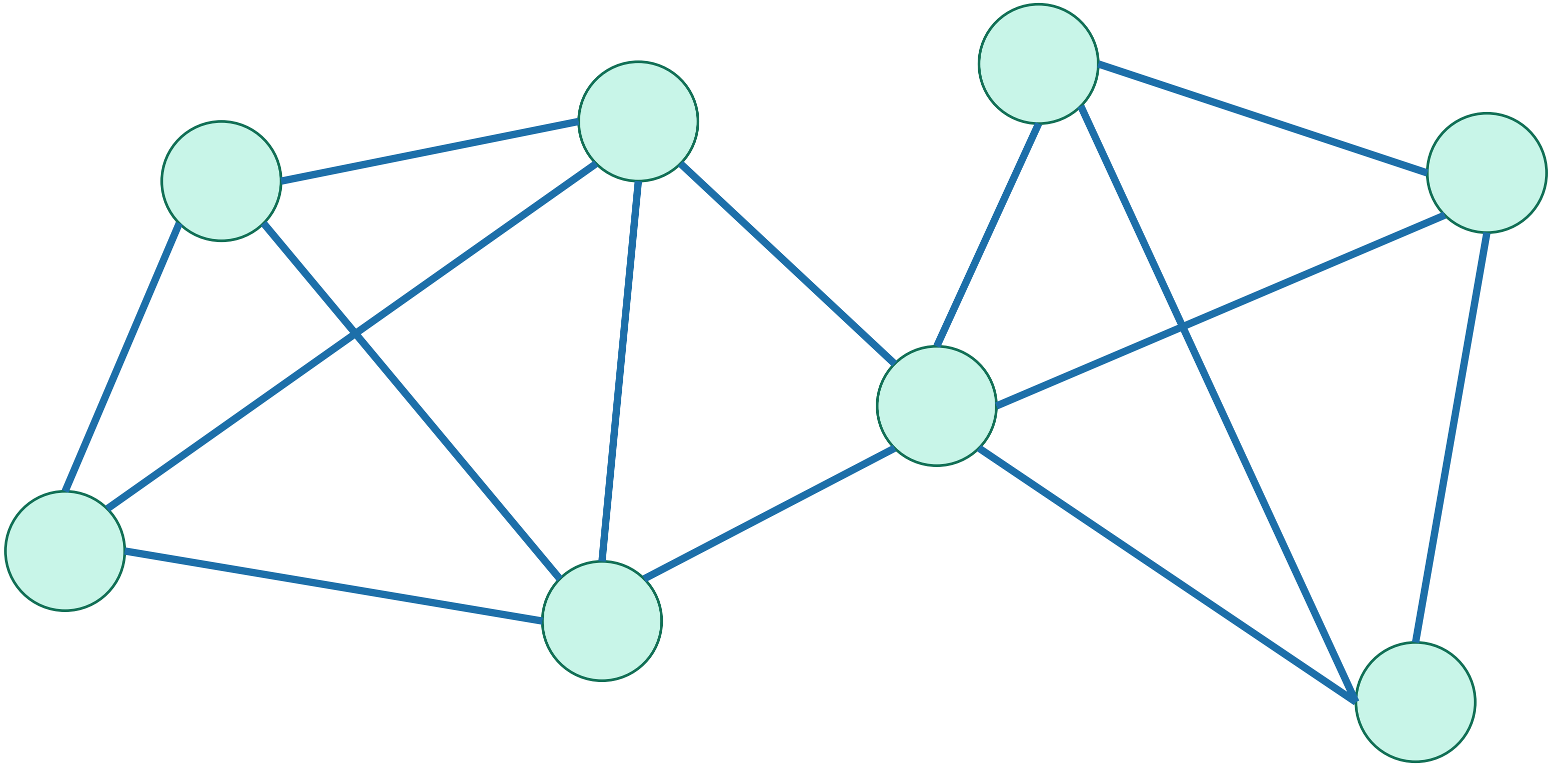
- Minimum Cuts!
  - Karger’s algorithm
  - Karger-Stein algorithm

  - Back to **randomized algorithms!**

*For today, all graphs are **undirected and unweighted.**

Recall: cuts in graphs

- A cut is a partition of the vertices into two nonempty parts.



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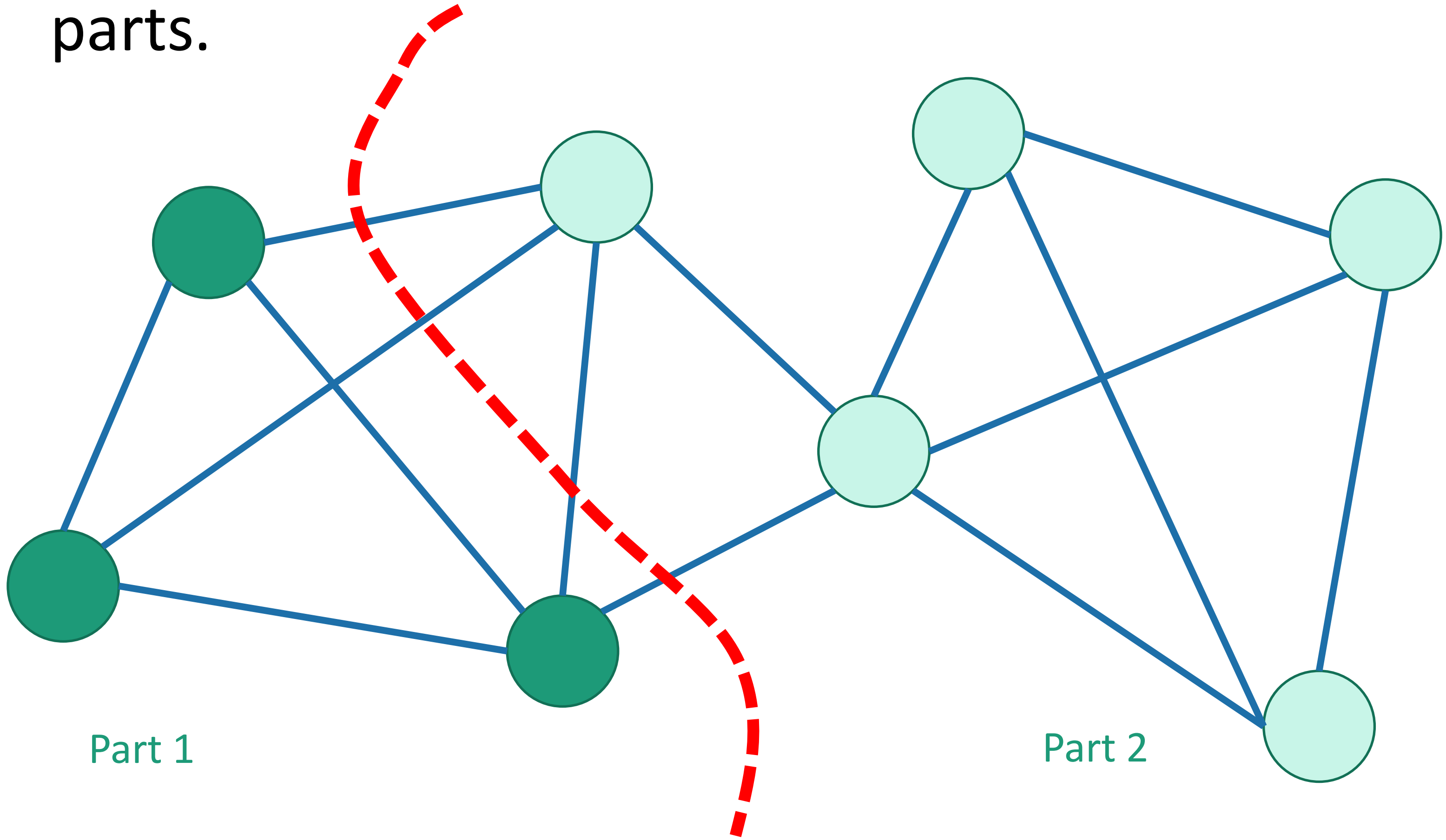
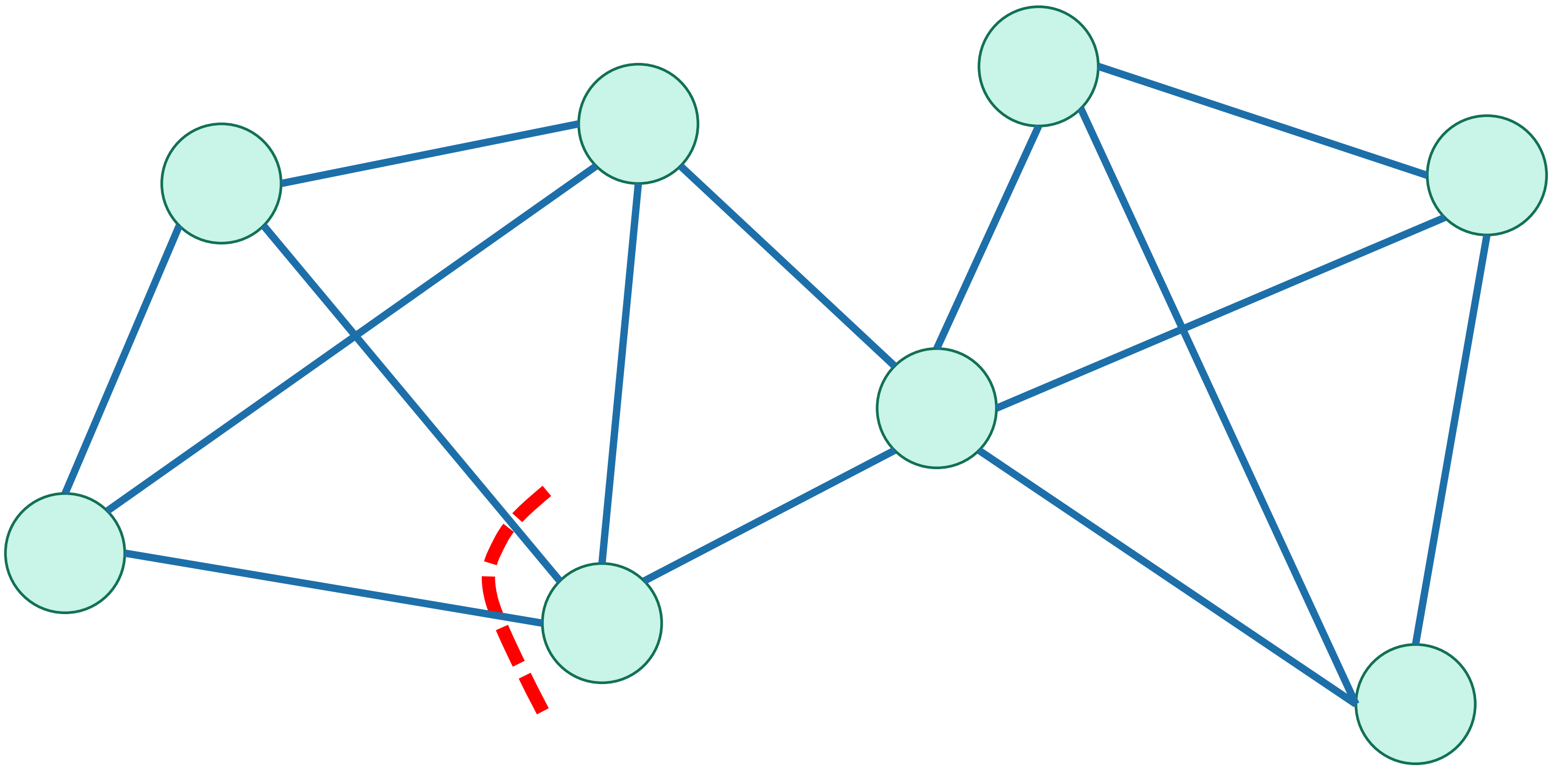


This is not a cut



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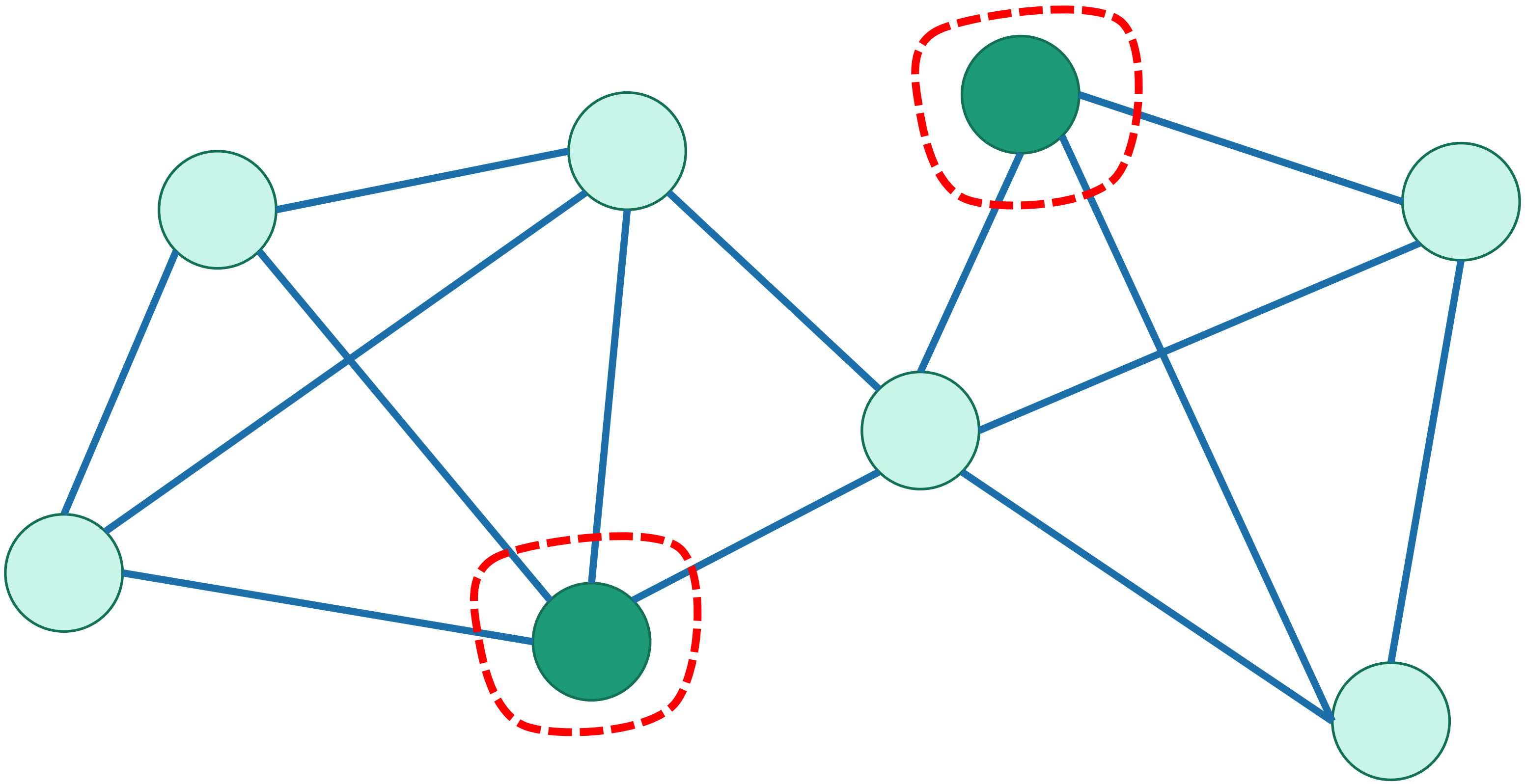
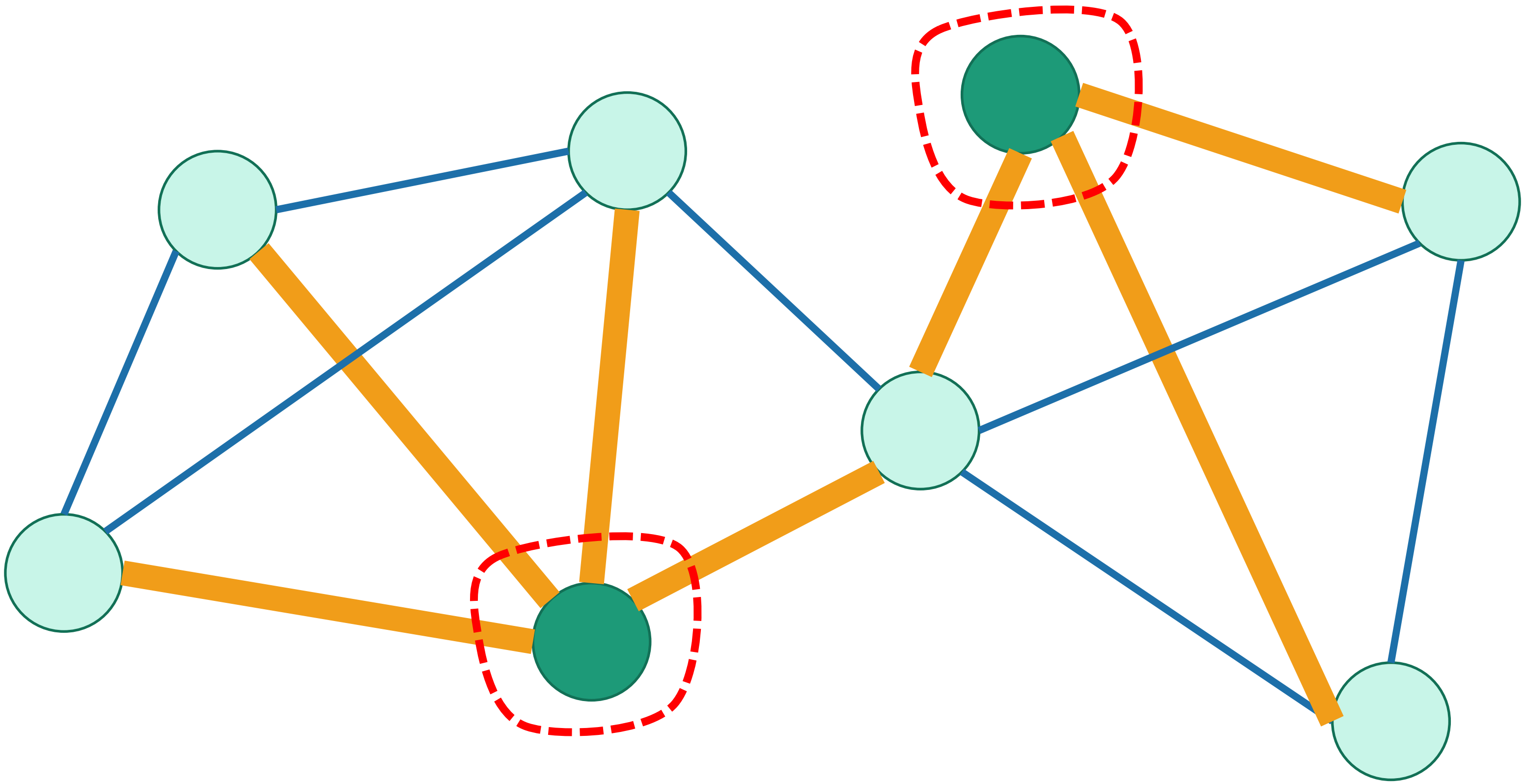


This is a cut

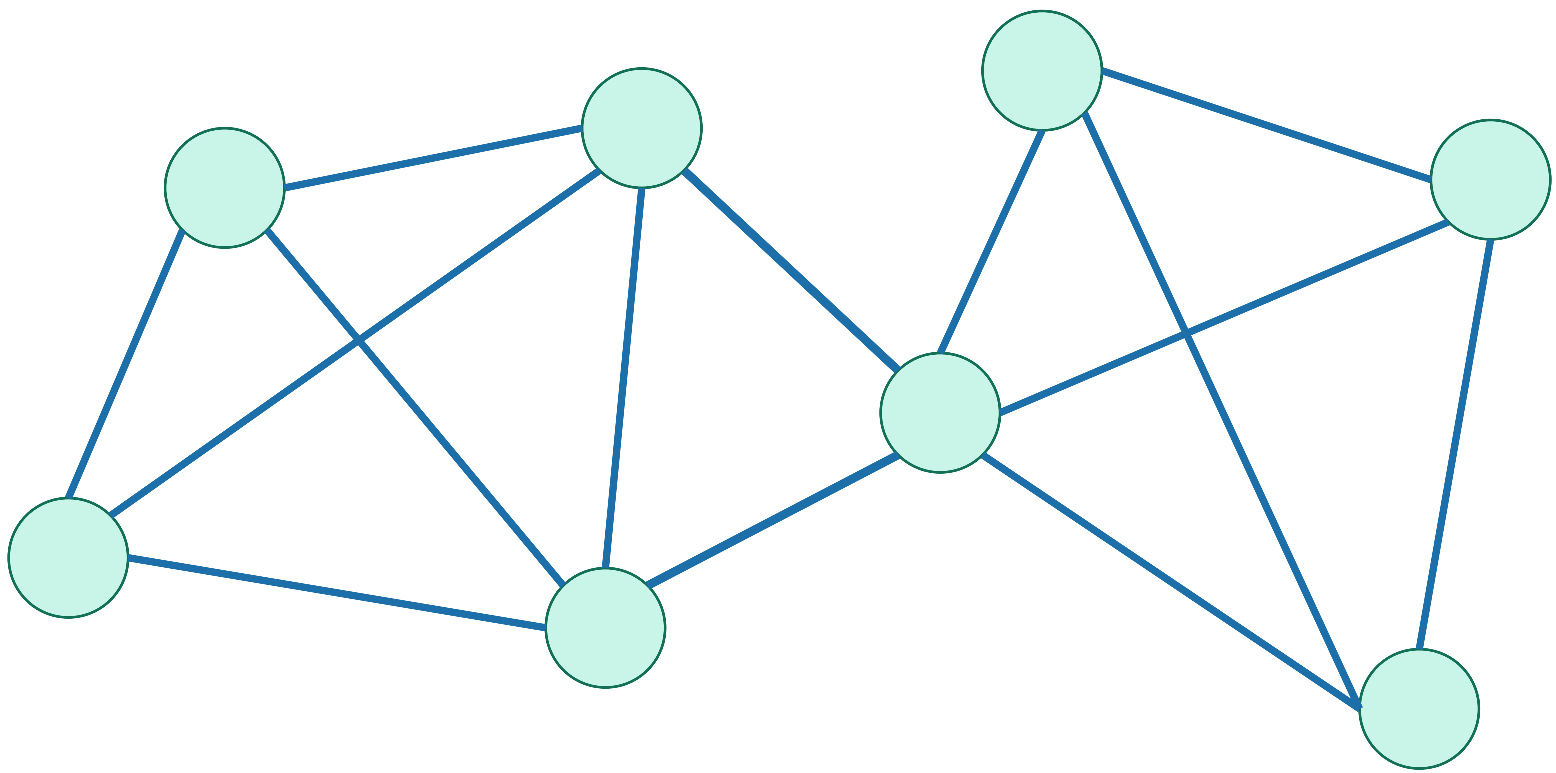
These edges **cross the cut.**

- They go from one part to the other.



A (global) minimum cut

is a cut that has the fewest edges possible crossing it.



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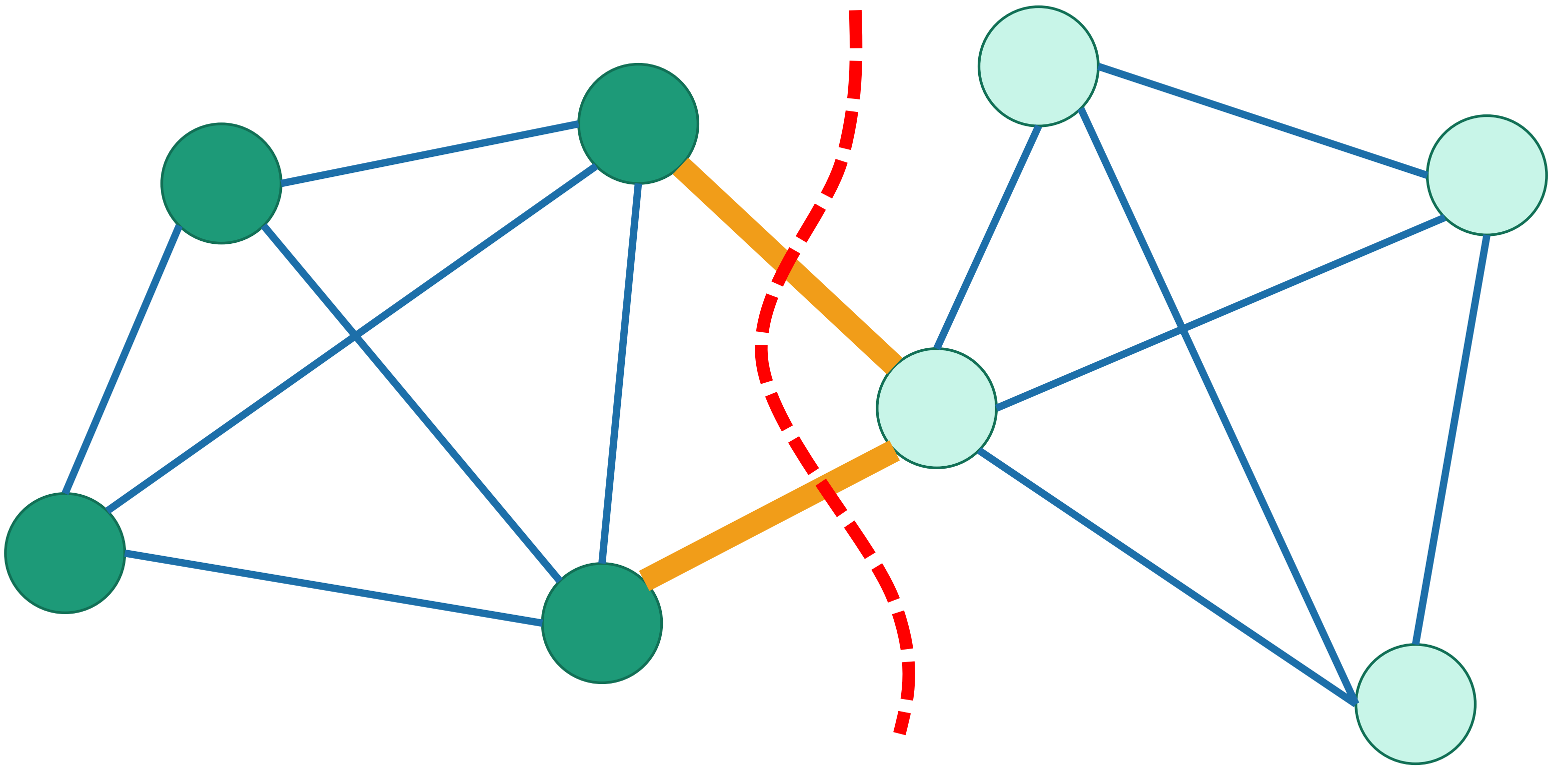

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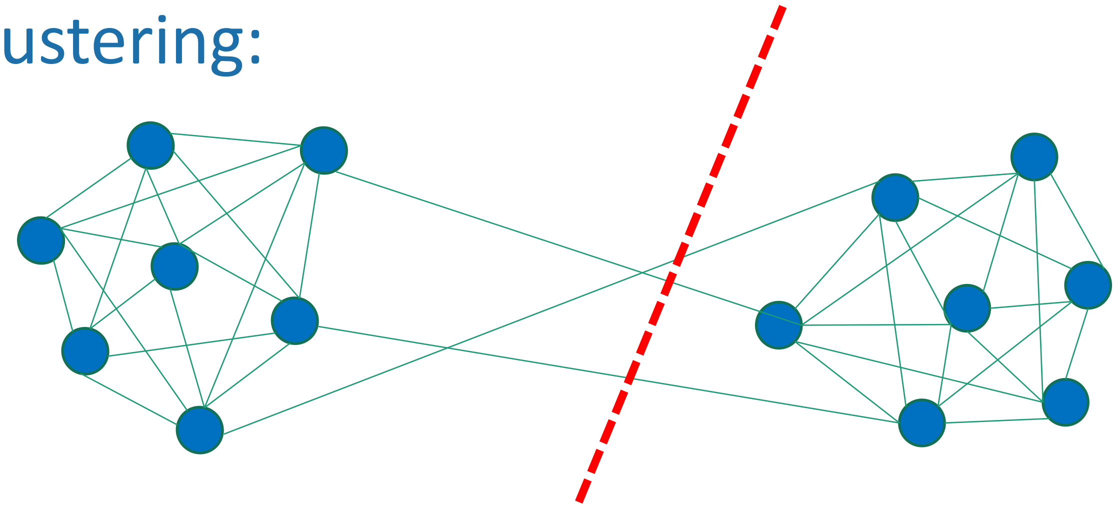

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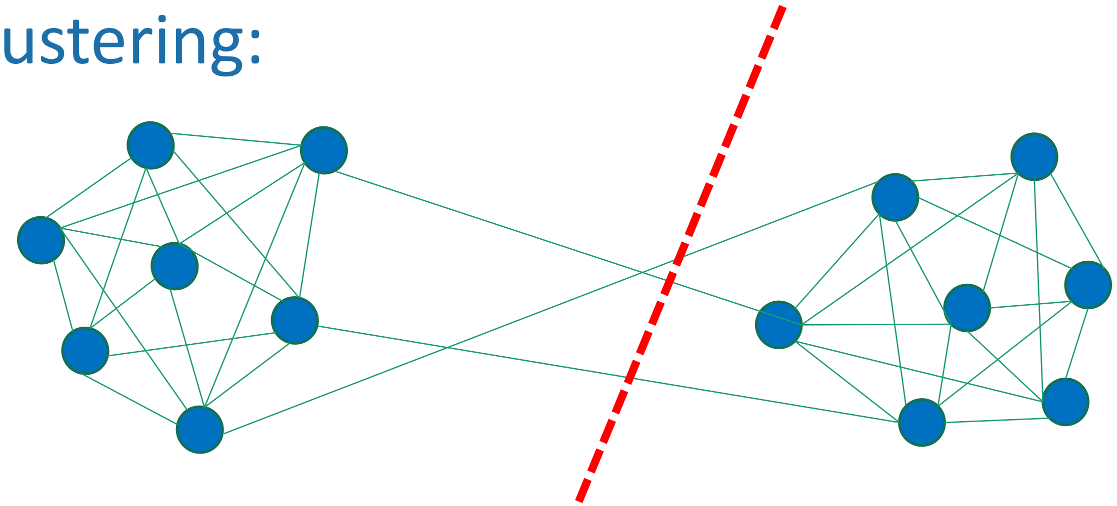
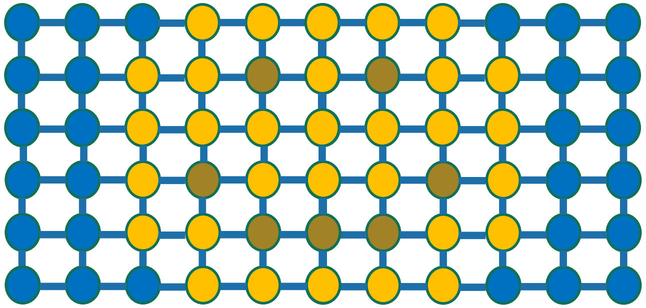


- Image Segmentation



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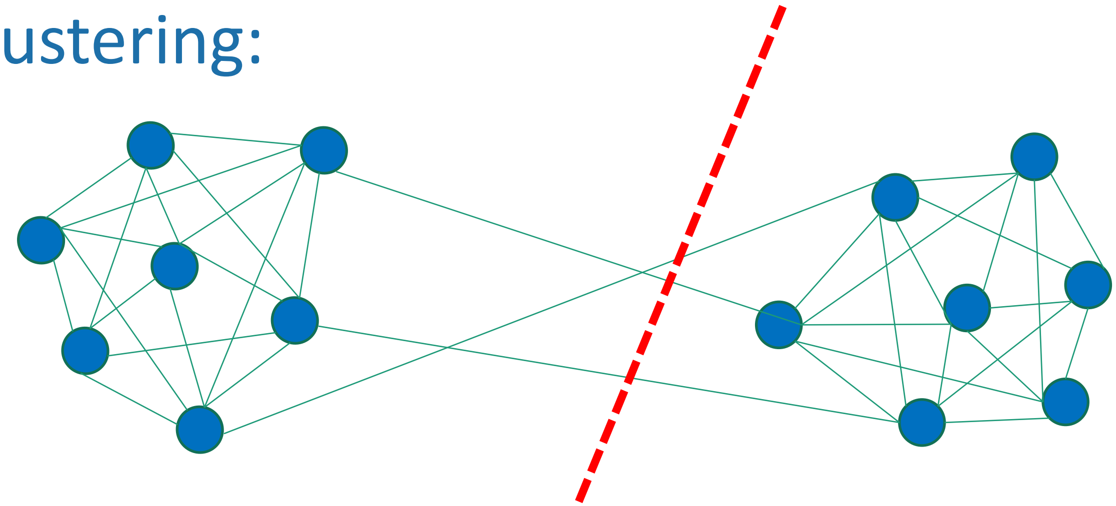

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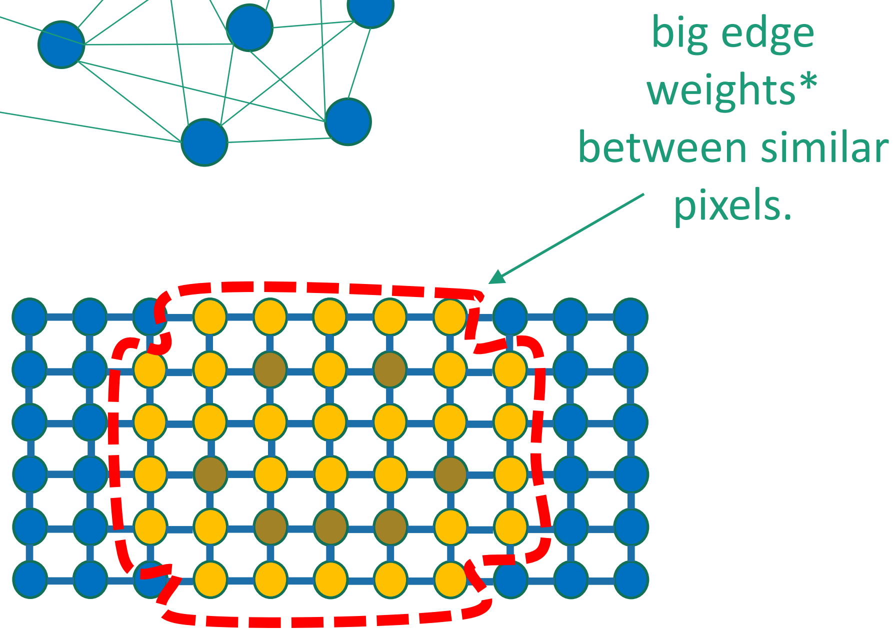

*For the rest of today edges aren't weighted; but the algorithm can be adapted to deal with edge weights.

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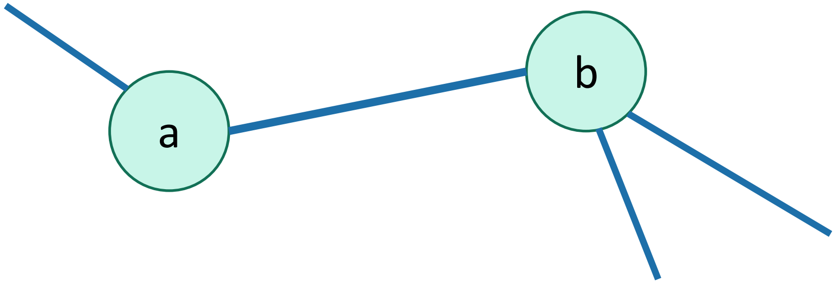
- Why would we want an algorithm that might be wrong?
  - **With high probability it won’t be wrong.**
  - Maybe the stakes are low and the cost of a deterministic algorithm is high.

Karger's algorithm



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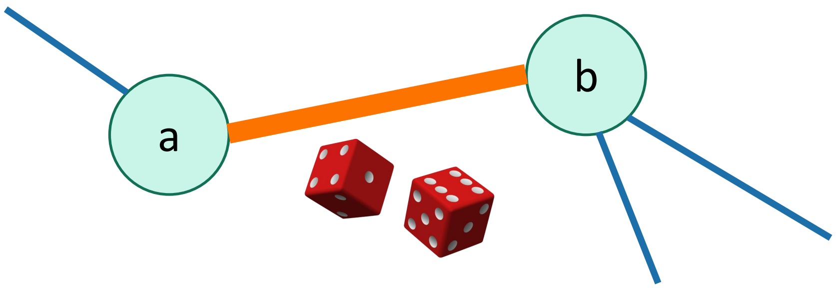

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- Pick a random edge.
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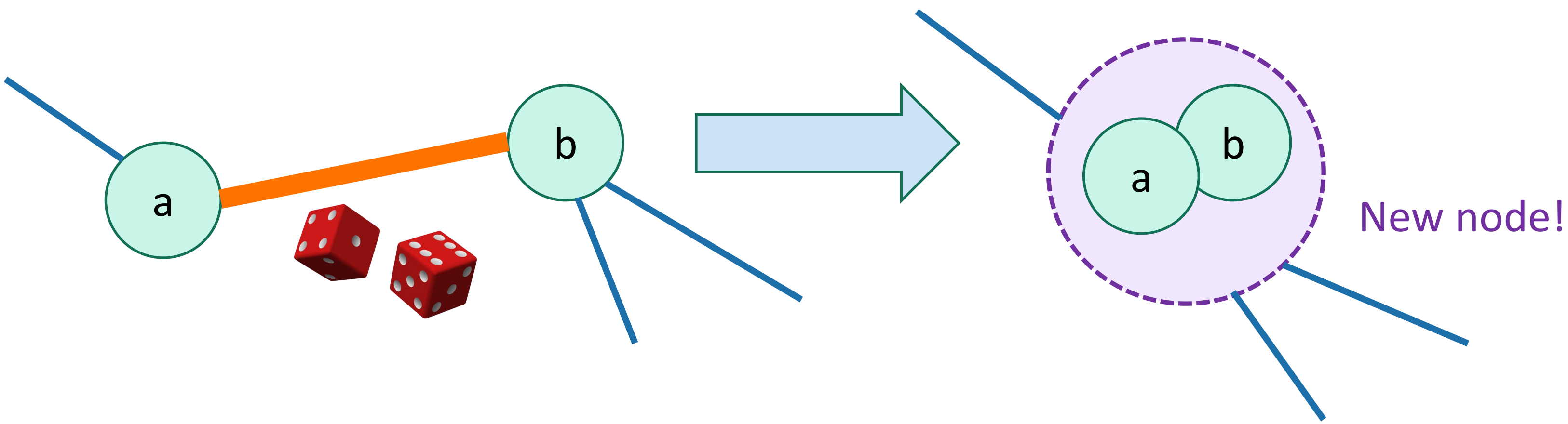

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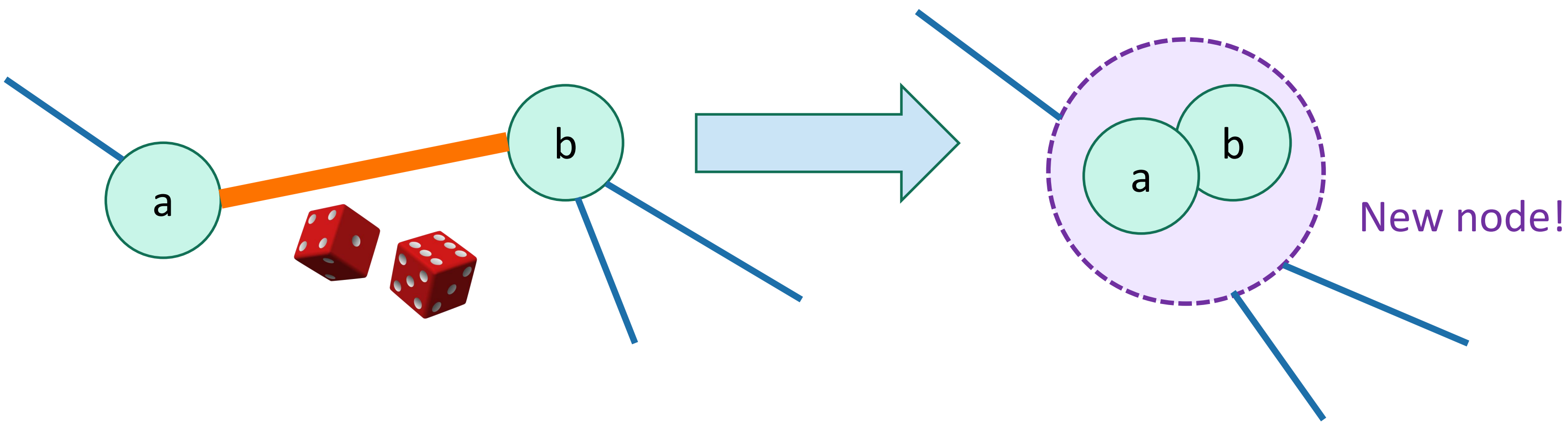

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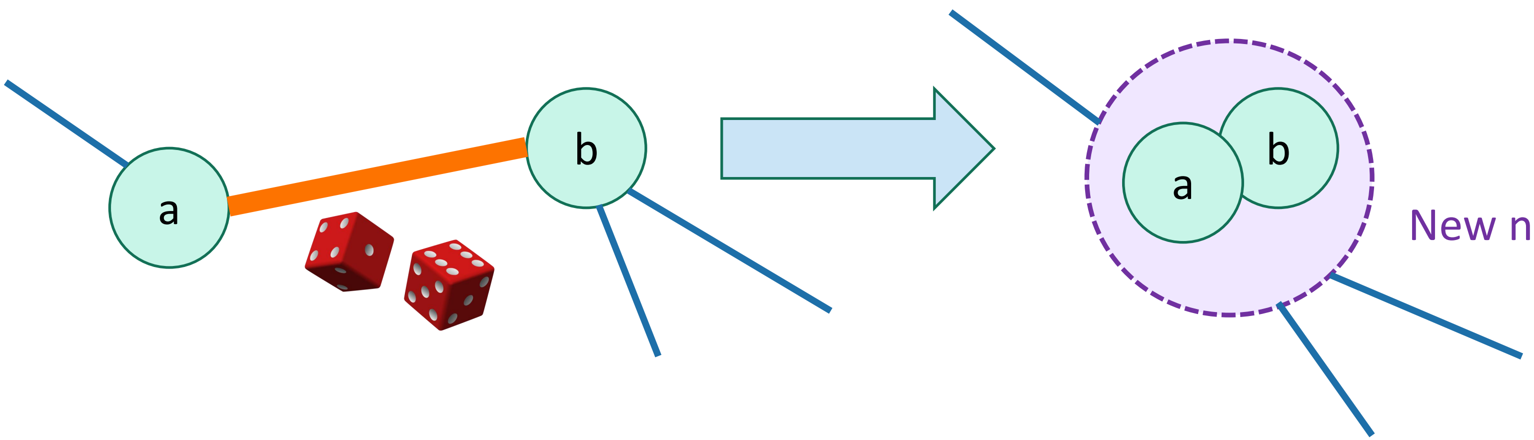
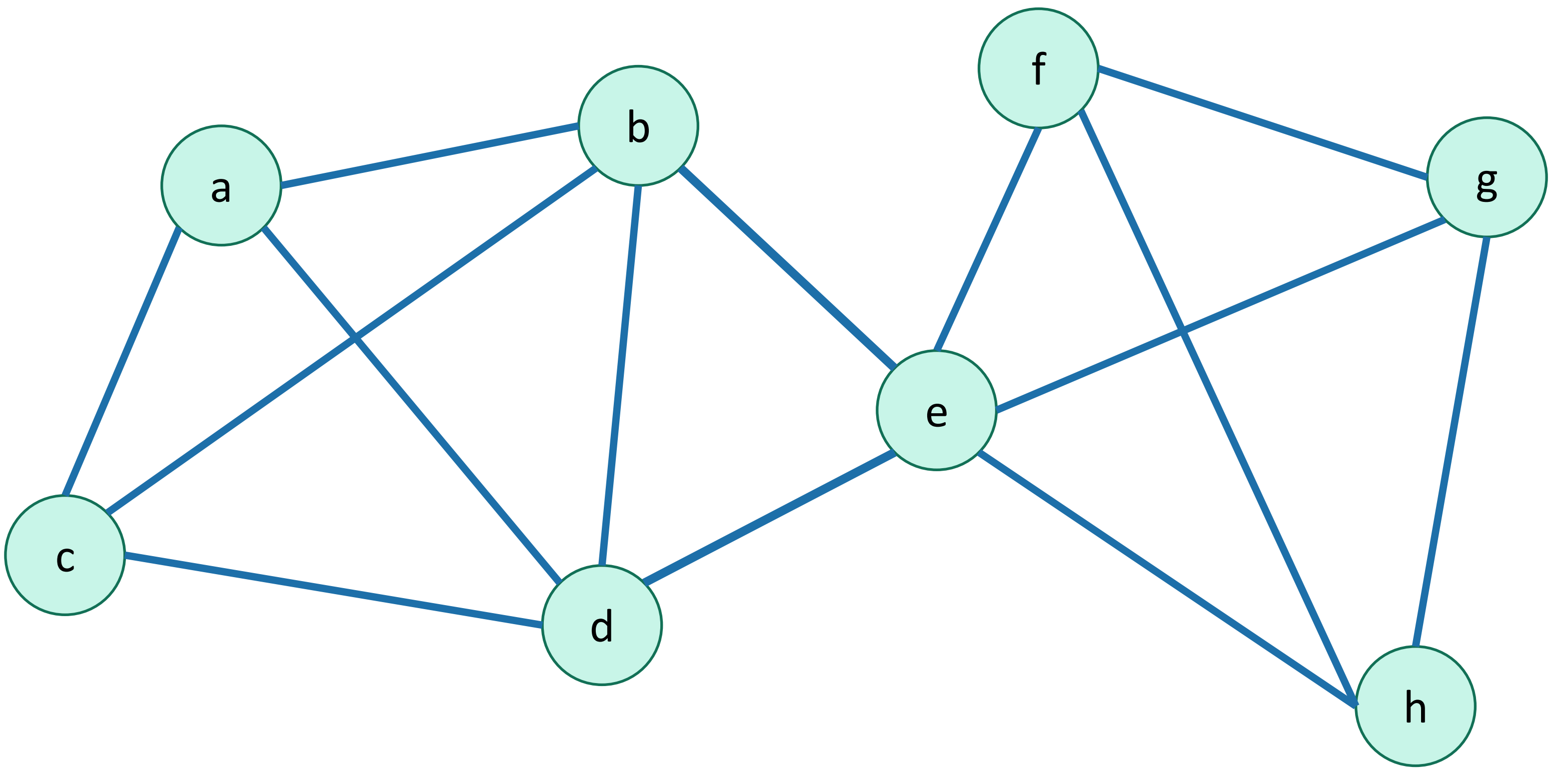


Why is this a good idea? We'll see shortly.

Karger's algorithm



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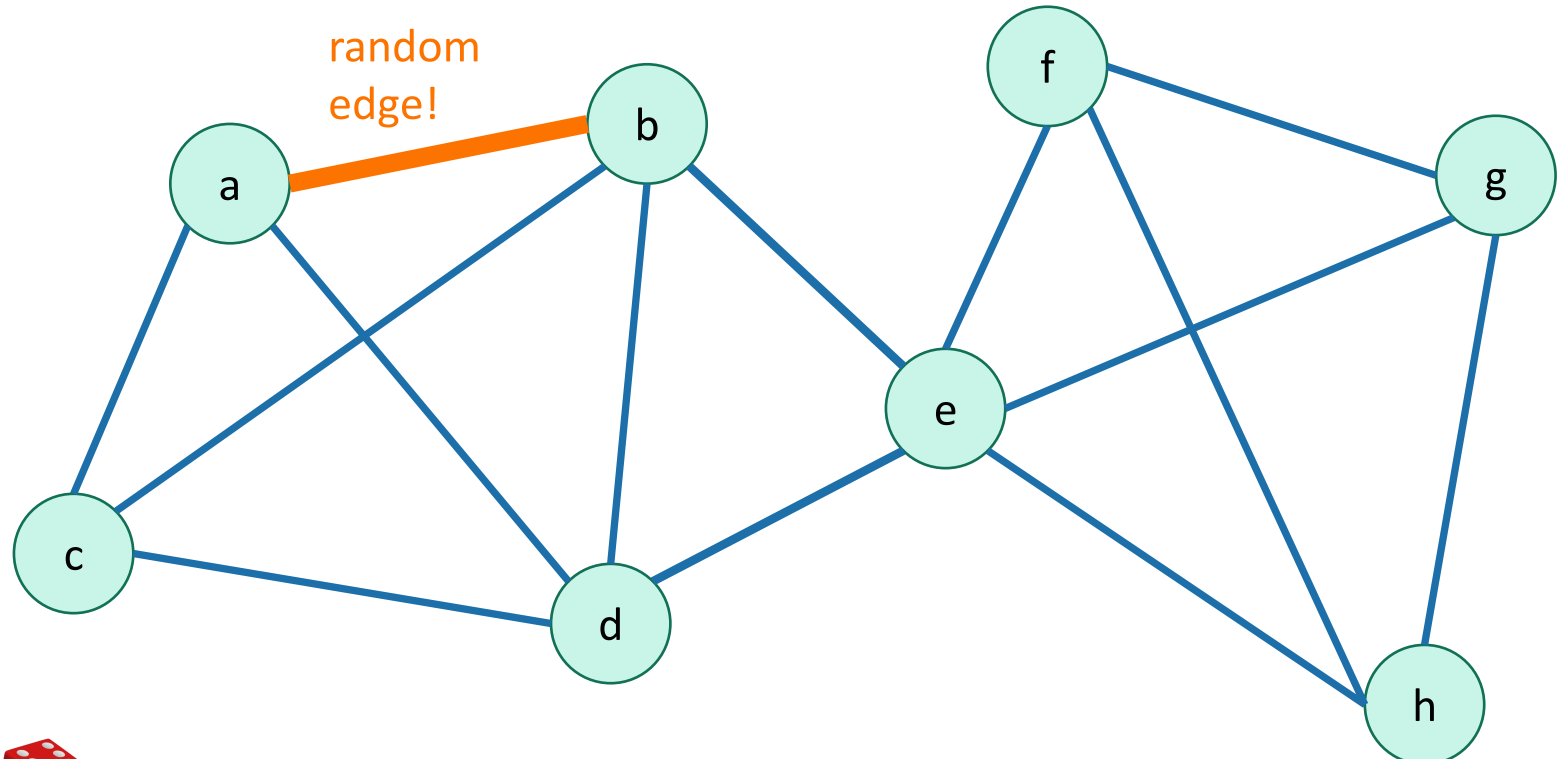

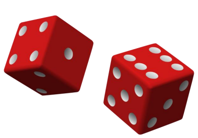

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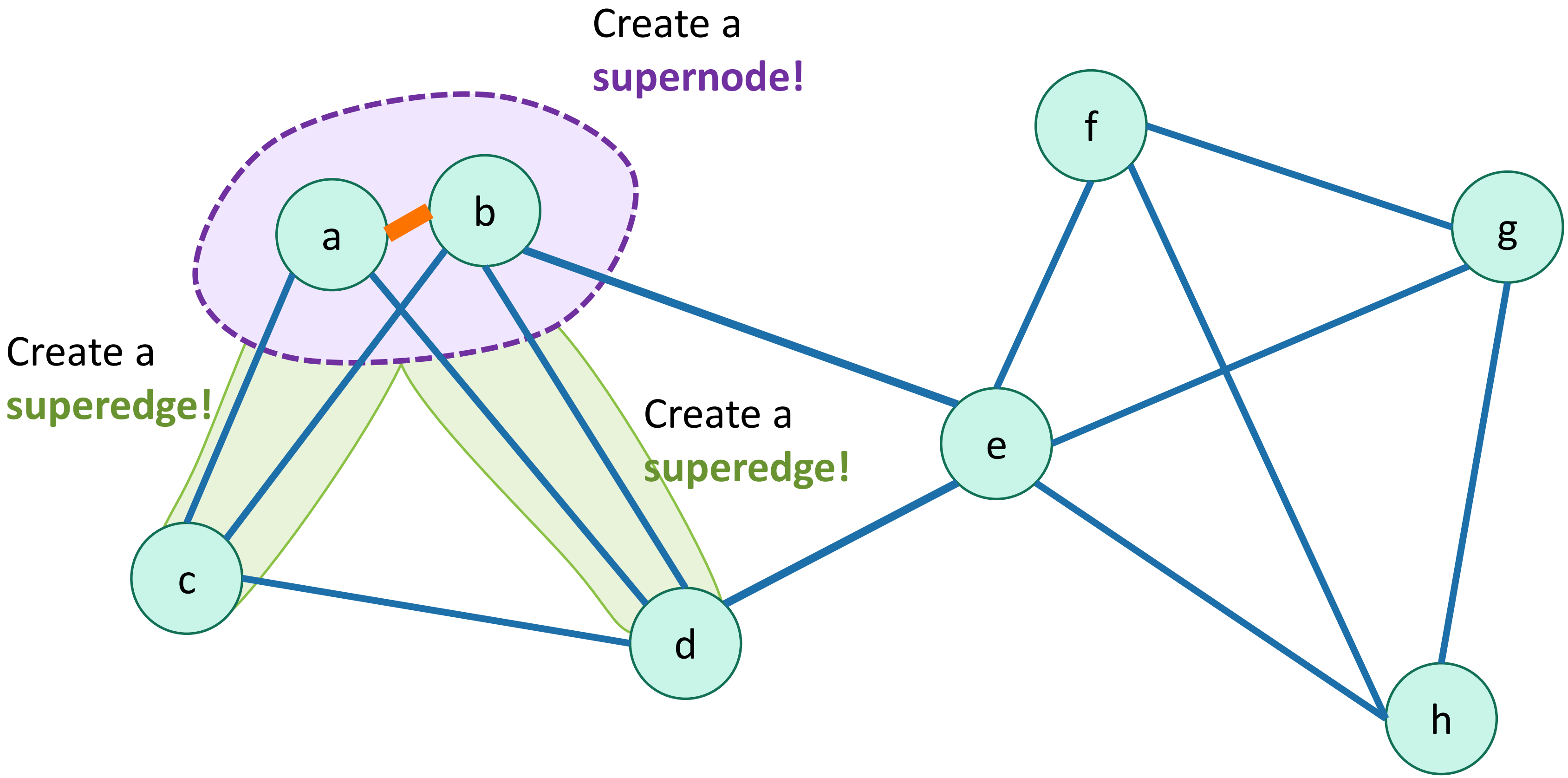

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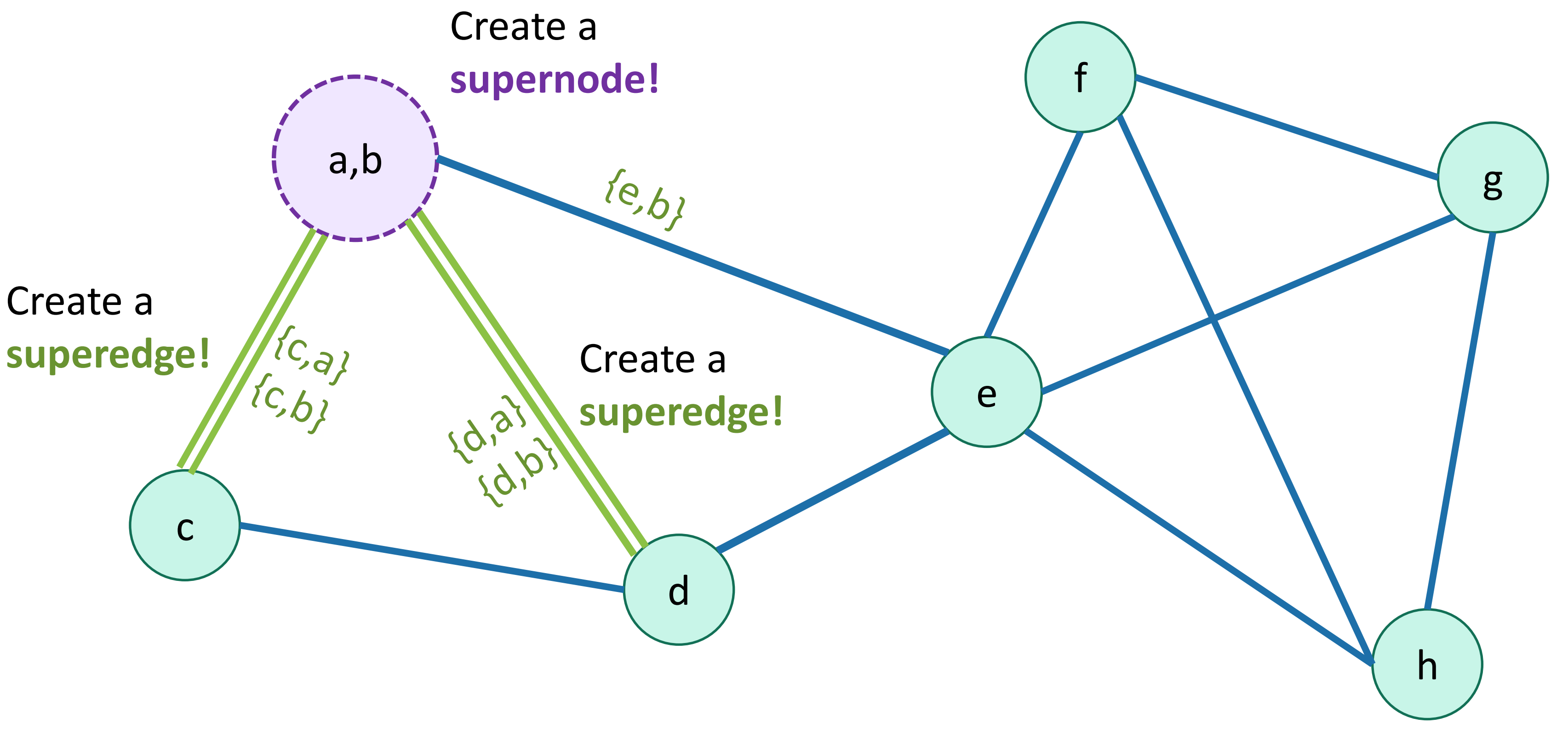

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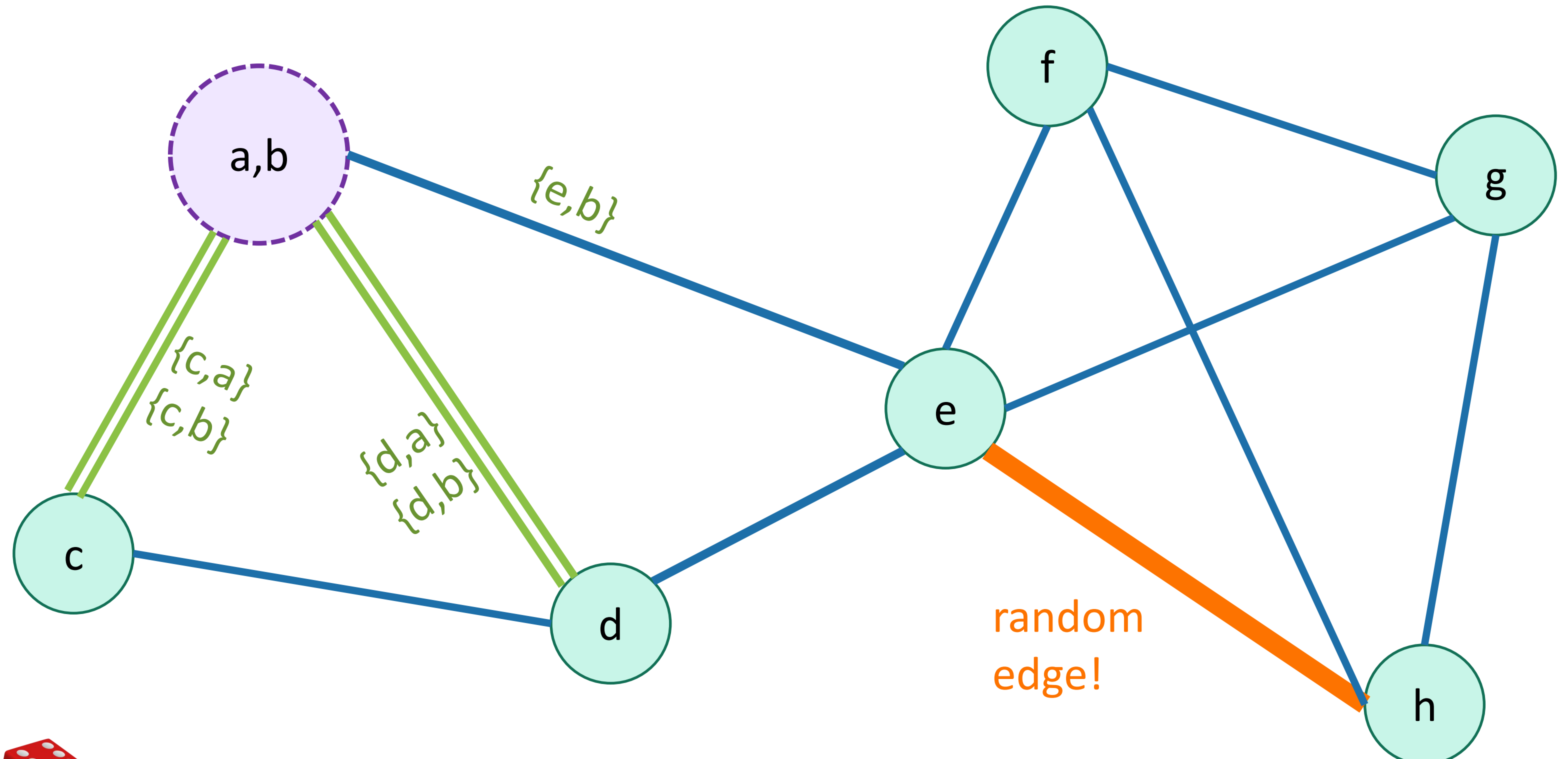

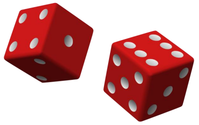

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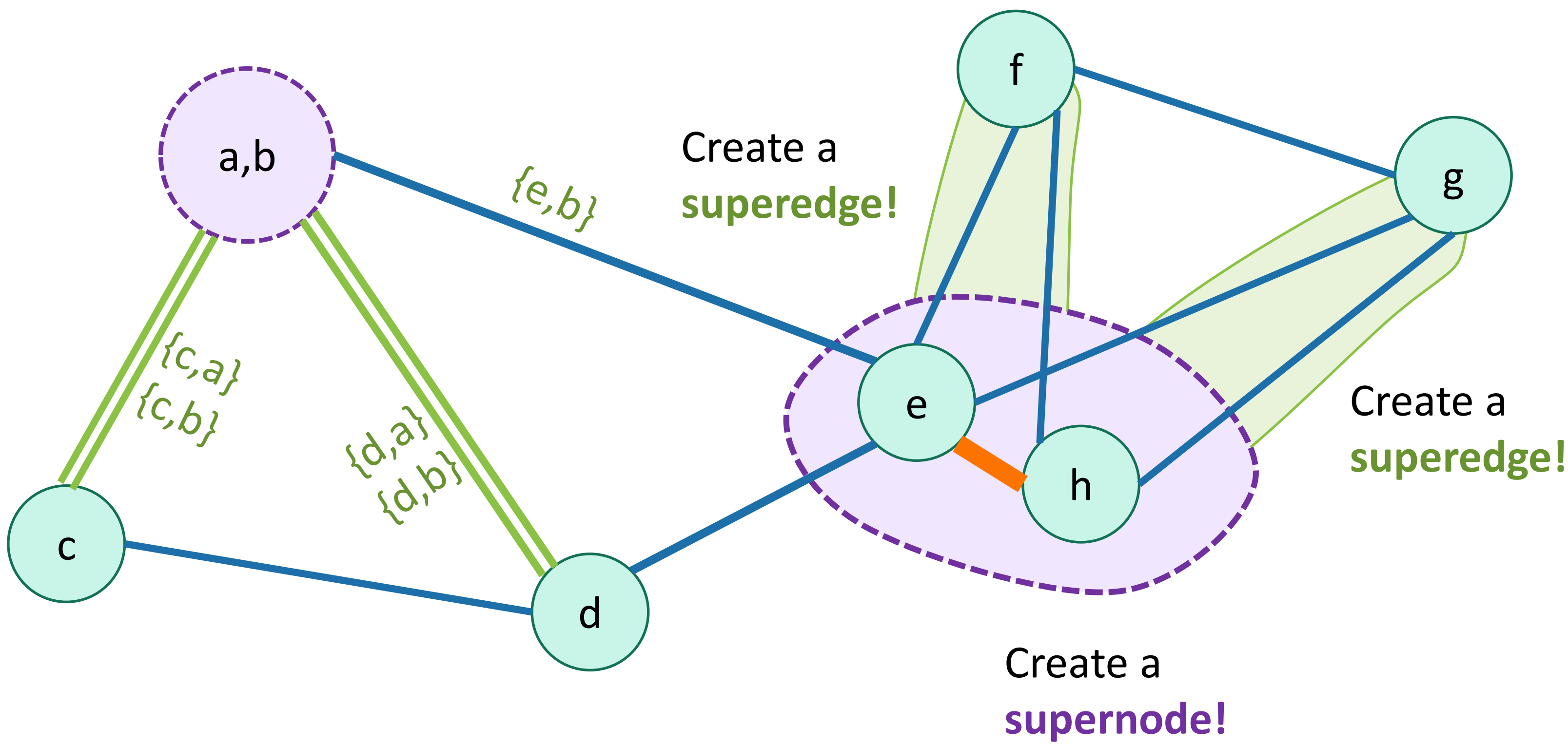

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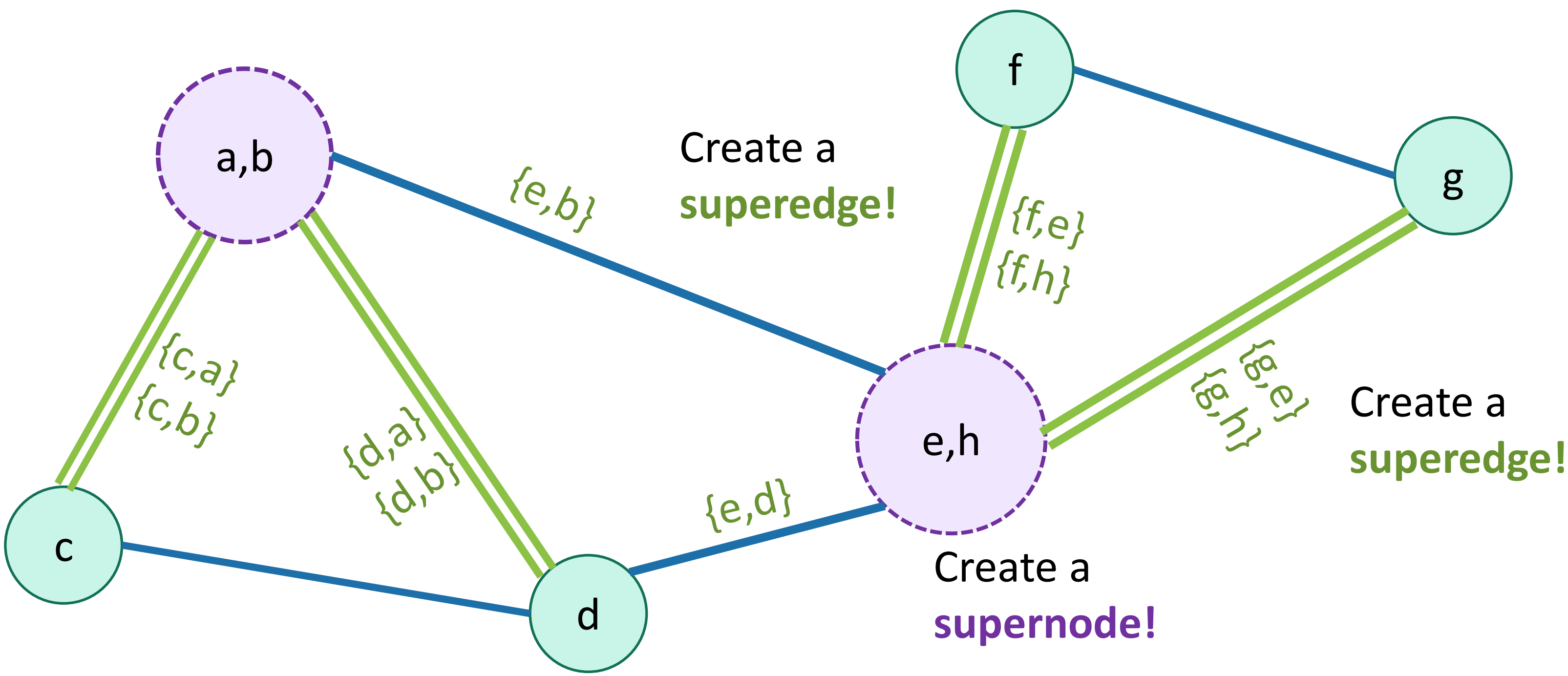

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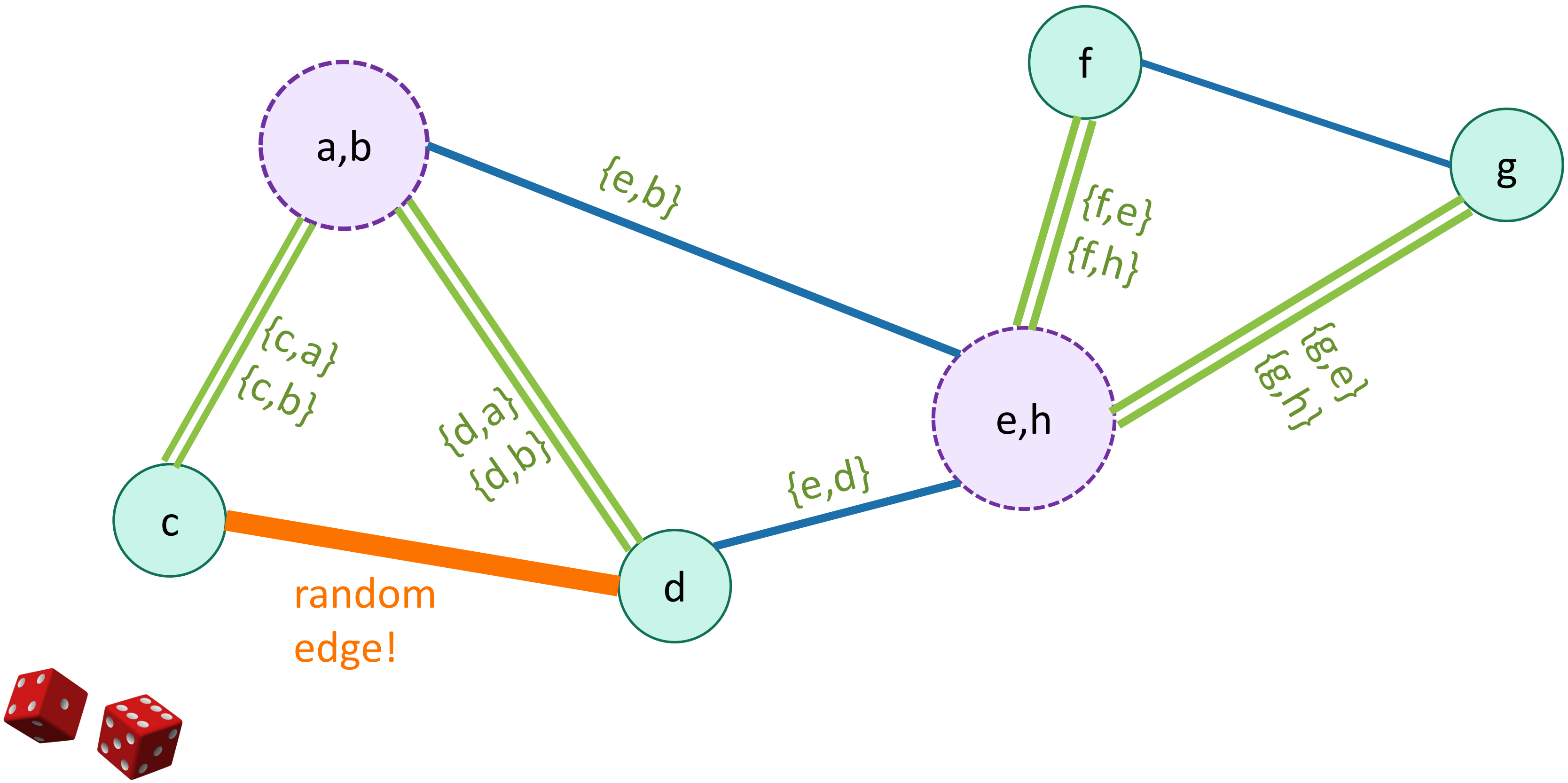

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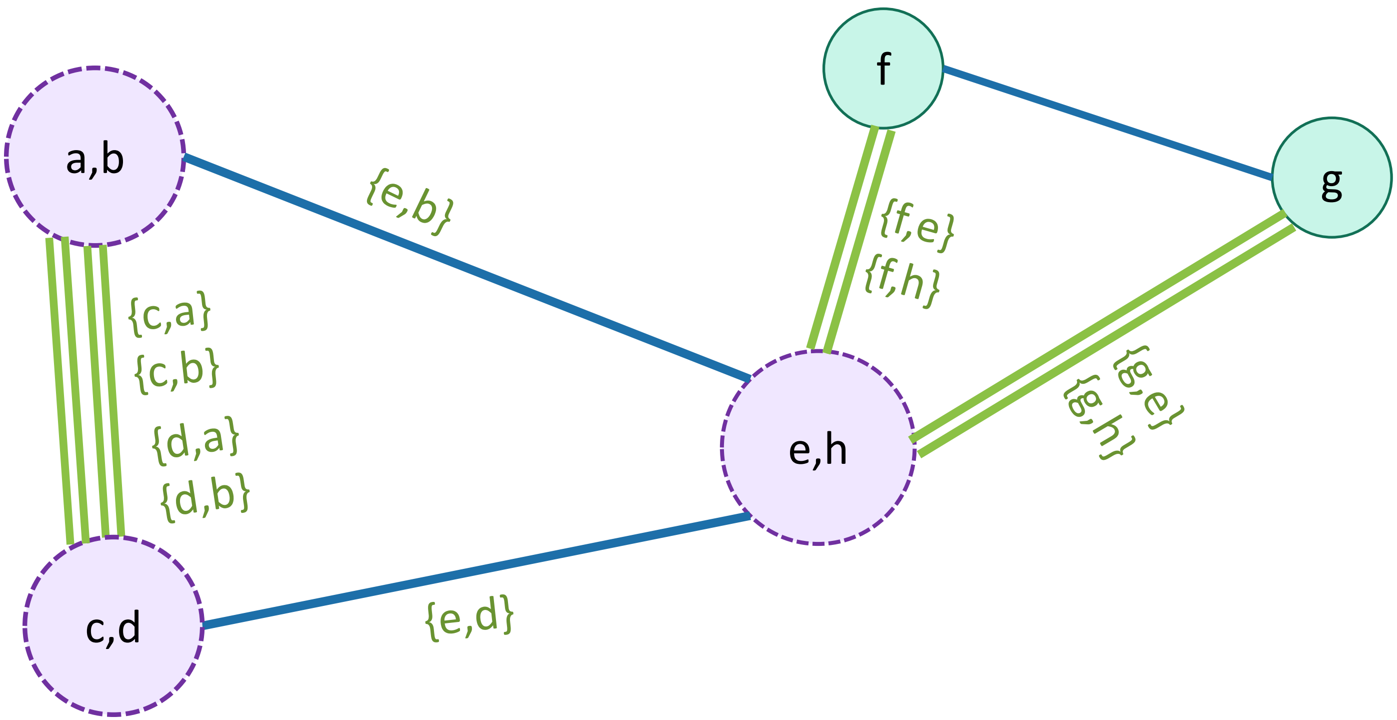

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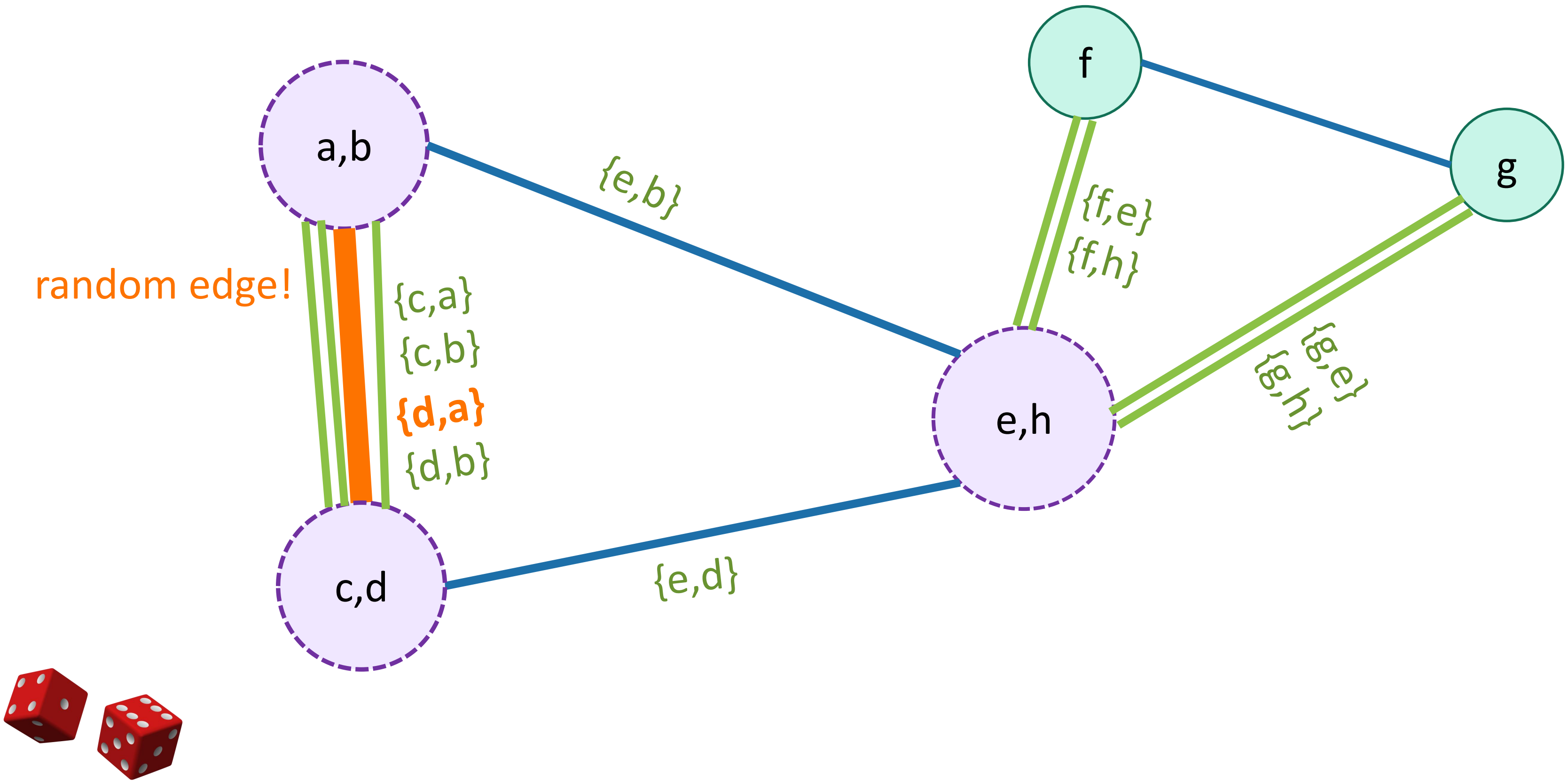

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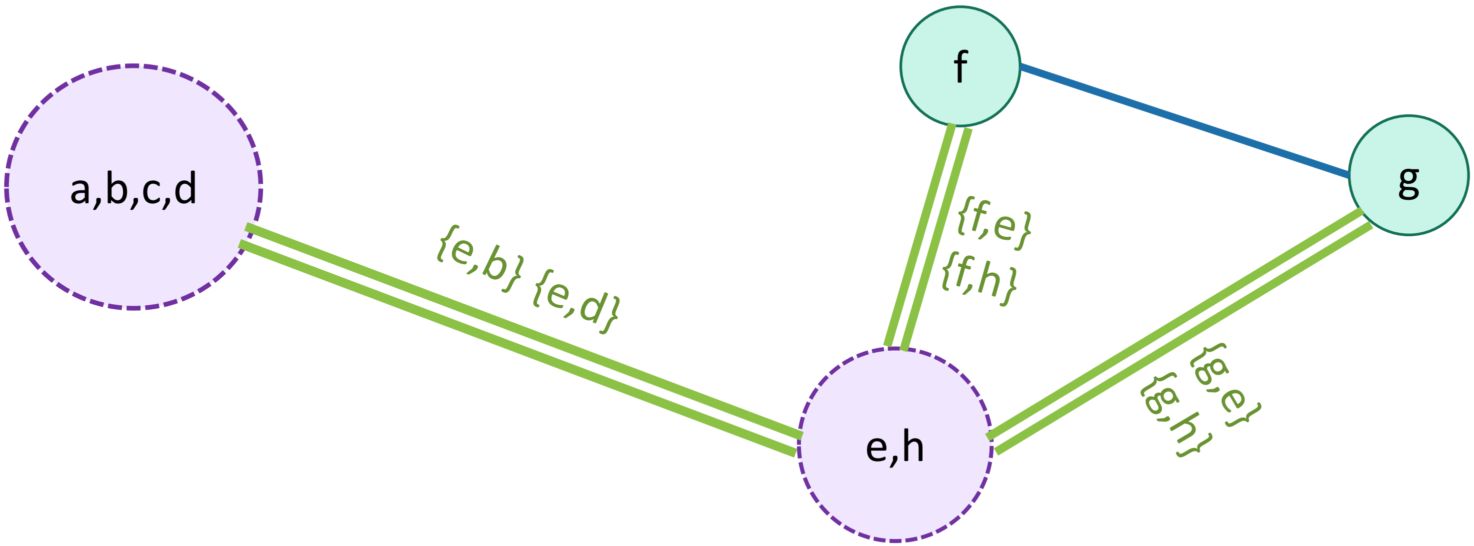

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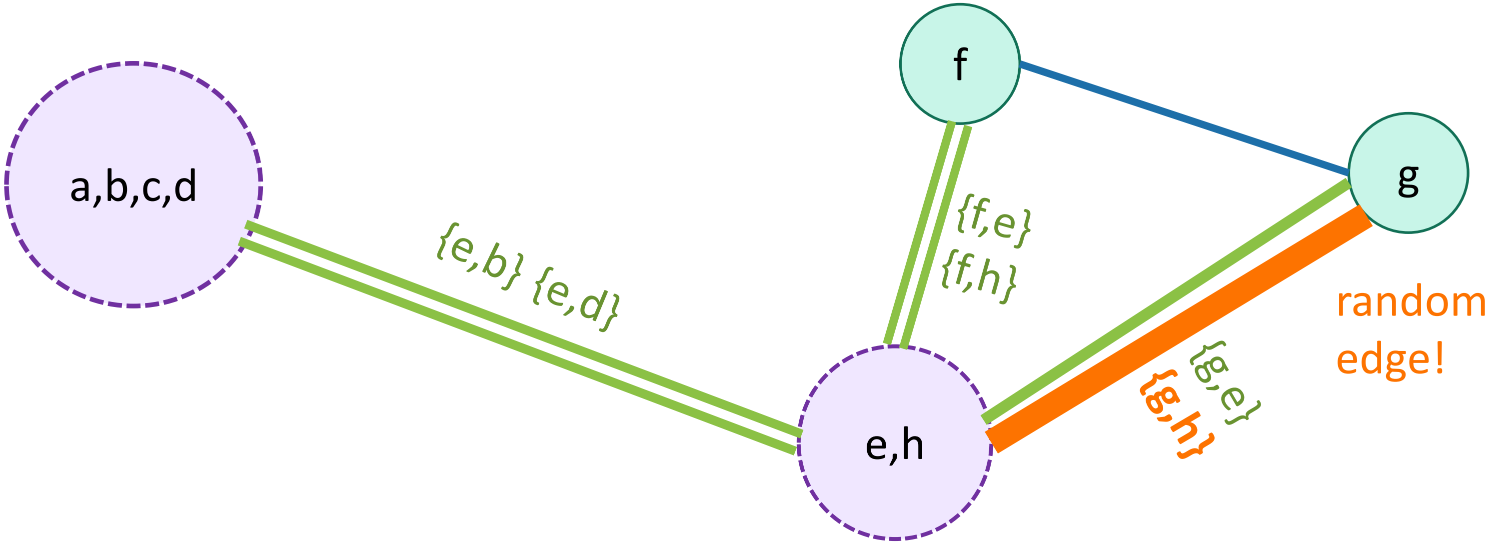

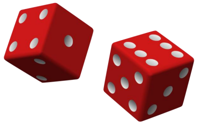

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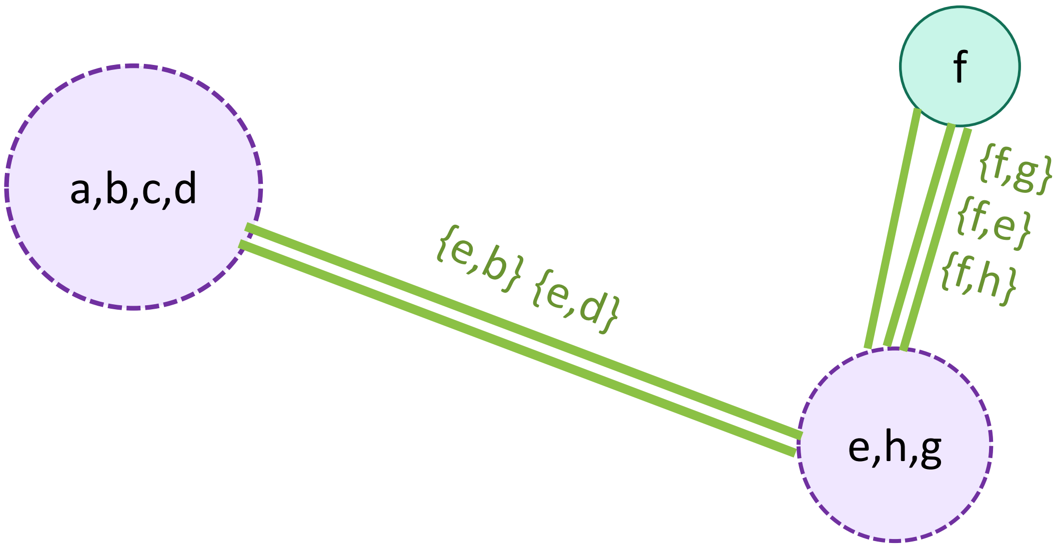

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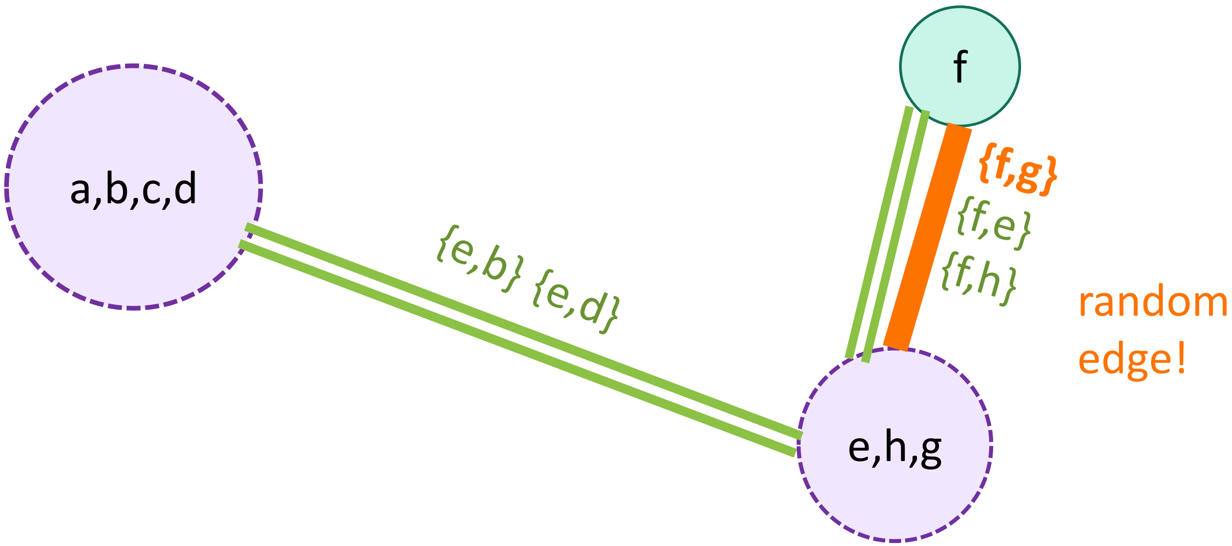

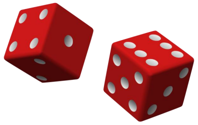

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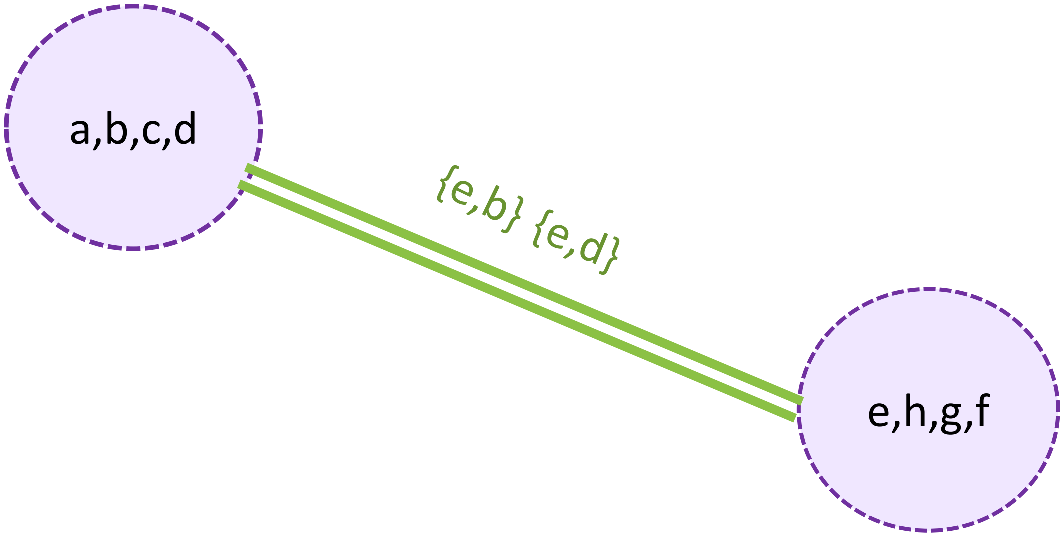

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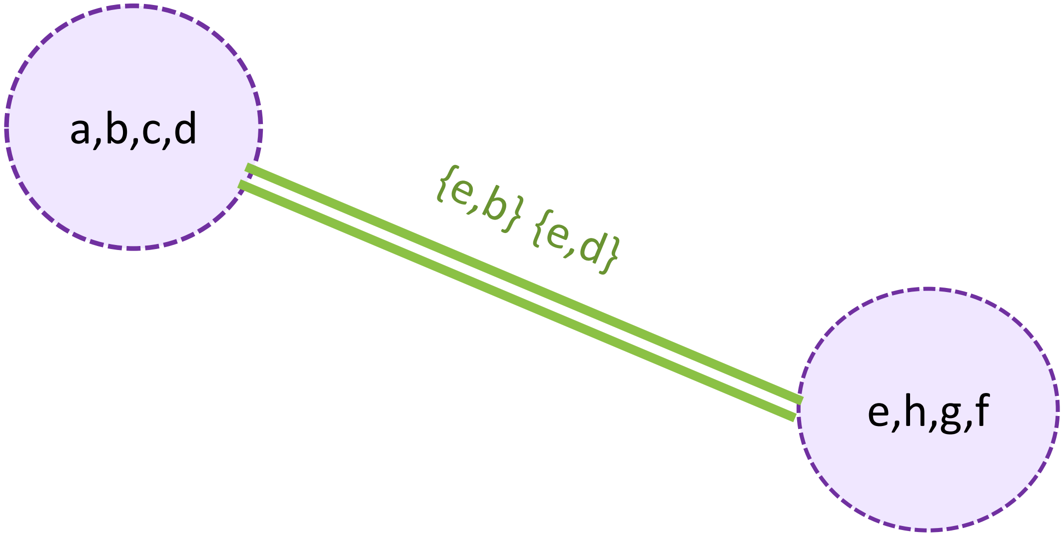
- There are only two nodes left.



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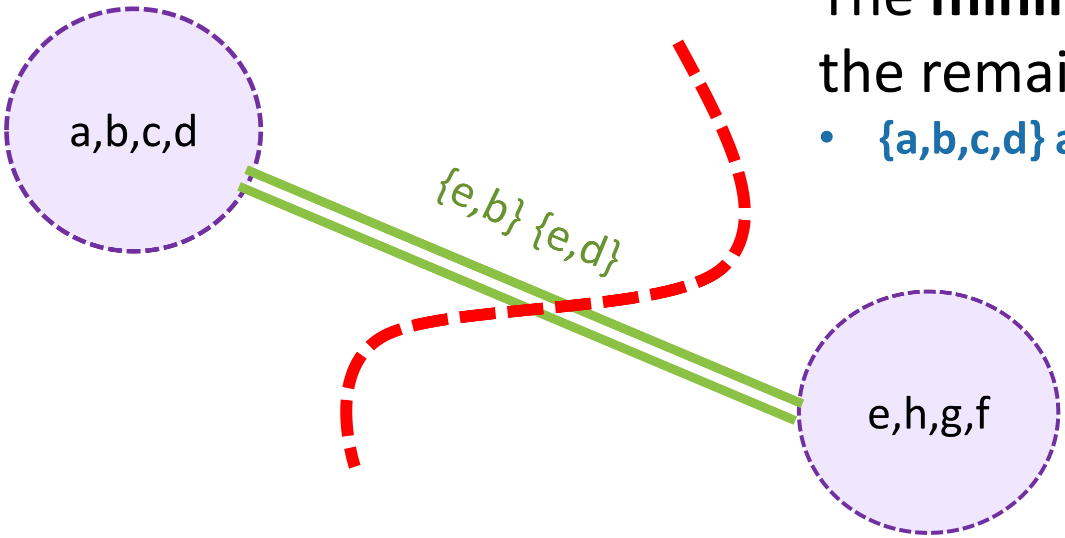

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- **{a,b,c,d} and {e,h,f,g}**

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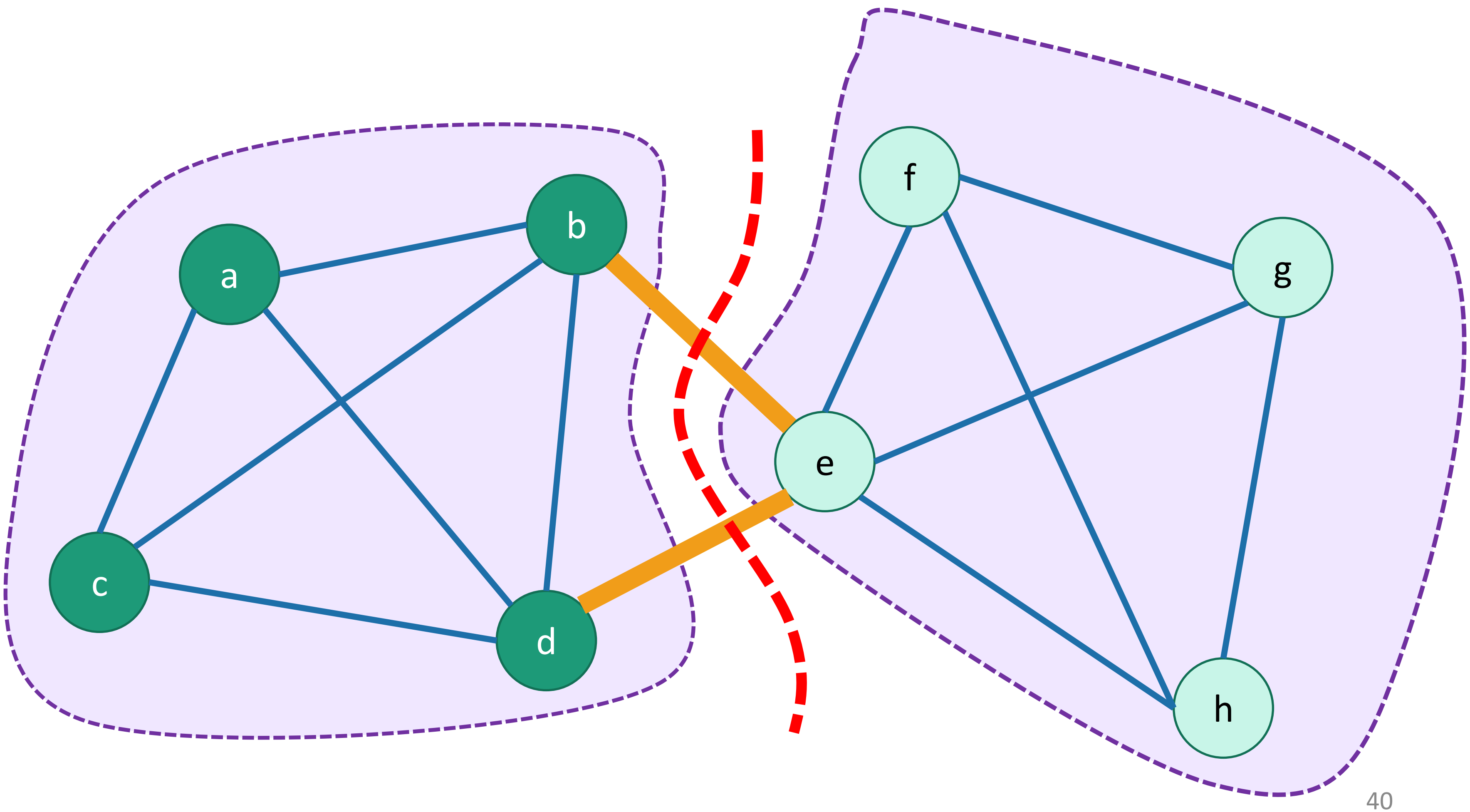

Karger’s algorithm

- Does it work?

- Is it fast?

Pseudocode

Let $\overline{\boldsymbol{u}}$ denote the SuperNode in Γ containing u
Say $E_{\overline{\boldsymbol{u}},\overline{\boldsymbol{v}}}$ is the SuperEdge between $\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}}$.

- **Karger**(G=(V,E)):
  - $\Gamma = \{$ SuperNode(v) : v in V $\}$ // one supernode for each vertex
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total runtime O(n^2)

We can do a bit better with fancy data structures, but let's go with this for now.

How do we implement this?

- Maintain a secondary “superGraph” which keeps track of superNodes and superEdges
- Running time?
  - We contract n-2 edges
    - Each time we contract an edge we get rid of a vertex, and we get rid of n – 2 vertices total.
  - Naively each contraction takes time O(n)
    - Maybe there are $\Omega(n)$ nodes in the superNodes that we are merging. (We can do better with fancy data structures).
  - So total running time O(n^2).
    - We can do $O(m \cdot \alpha(n))$ with a union-find data structure, but $O(n^2)$ is good enough for today.

Karger’s algorithm

- Does it work?

- Is it fast?
  - $O(n^2)$

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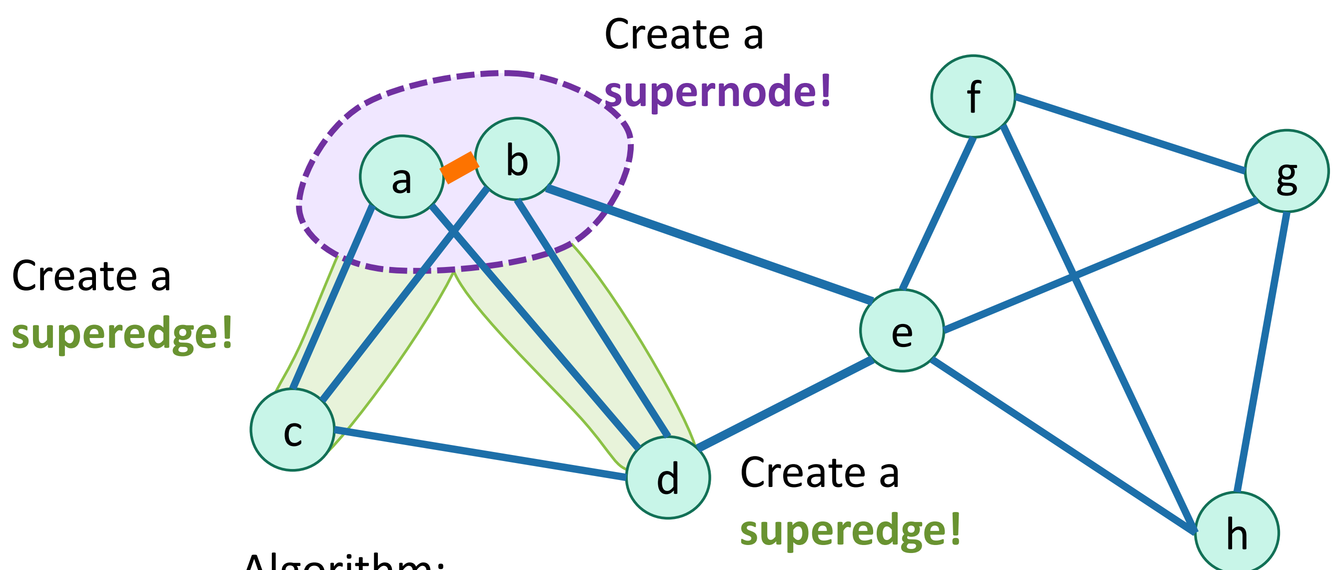
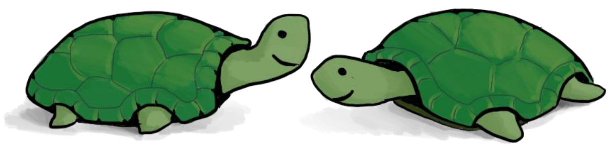


Algorithm:
- Randomly contract edges until there are only two supernodes left.

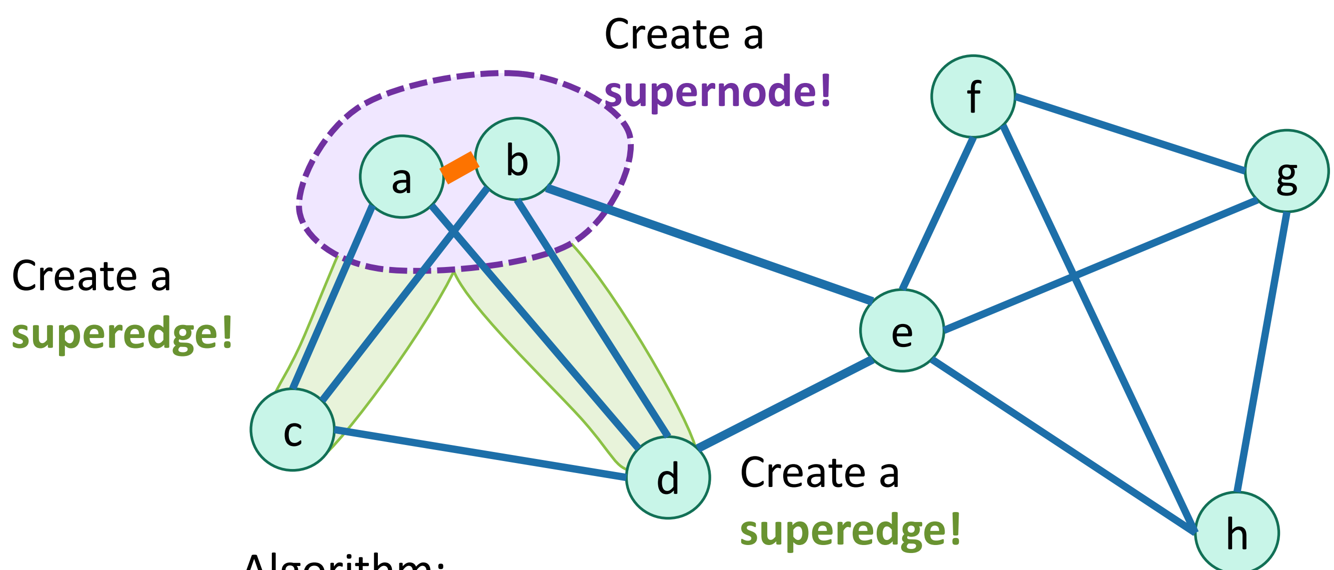
Karger’s algorithm

- Does it work?



Think-pair-share!

- Is it fast?
    - $O(n^2)$



Algorithm:

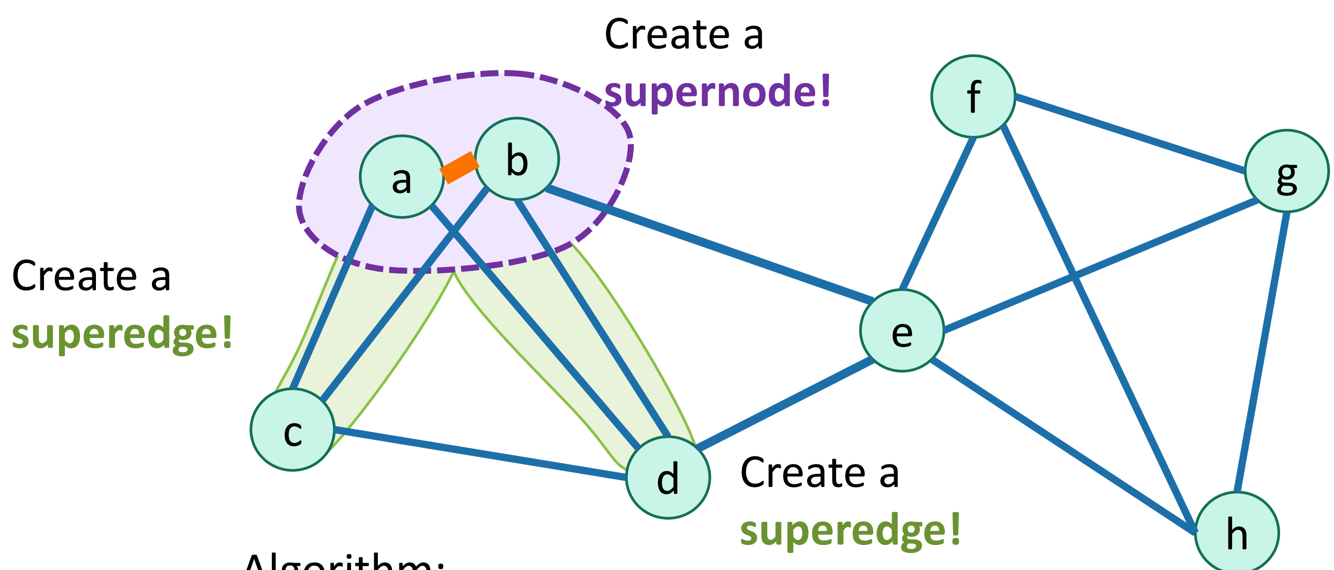
- Randomly contract edges until there are only two supernodes left.

Karger’s algorithm

- Does it work?

  No?

- Is it fast?
  - $O(n^2)$

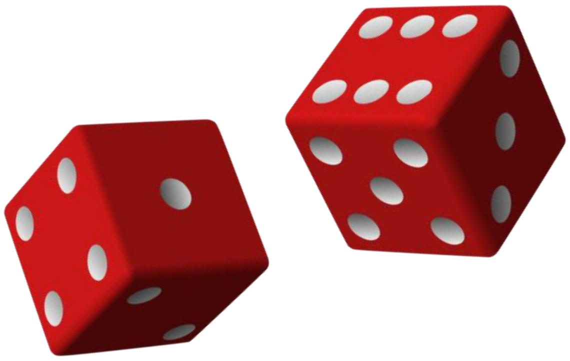


Algorithm:
- Randomly contract edges until there are only two supernodes left.

Why did that work?

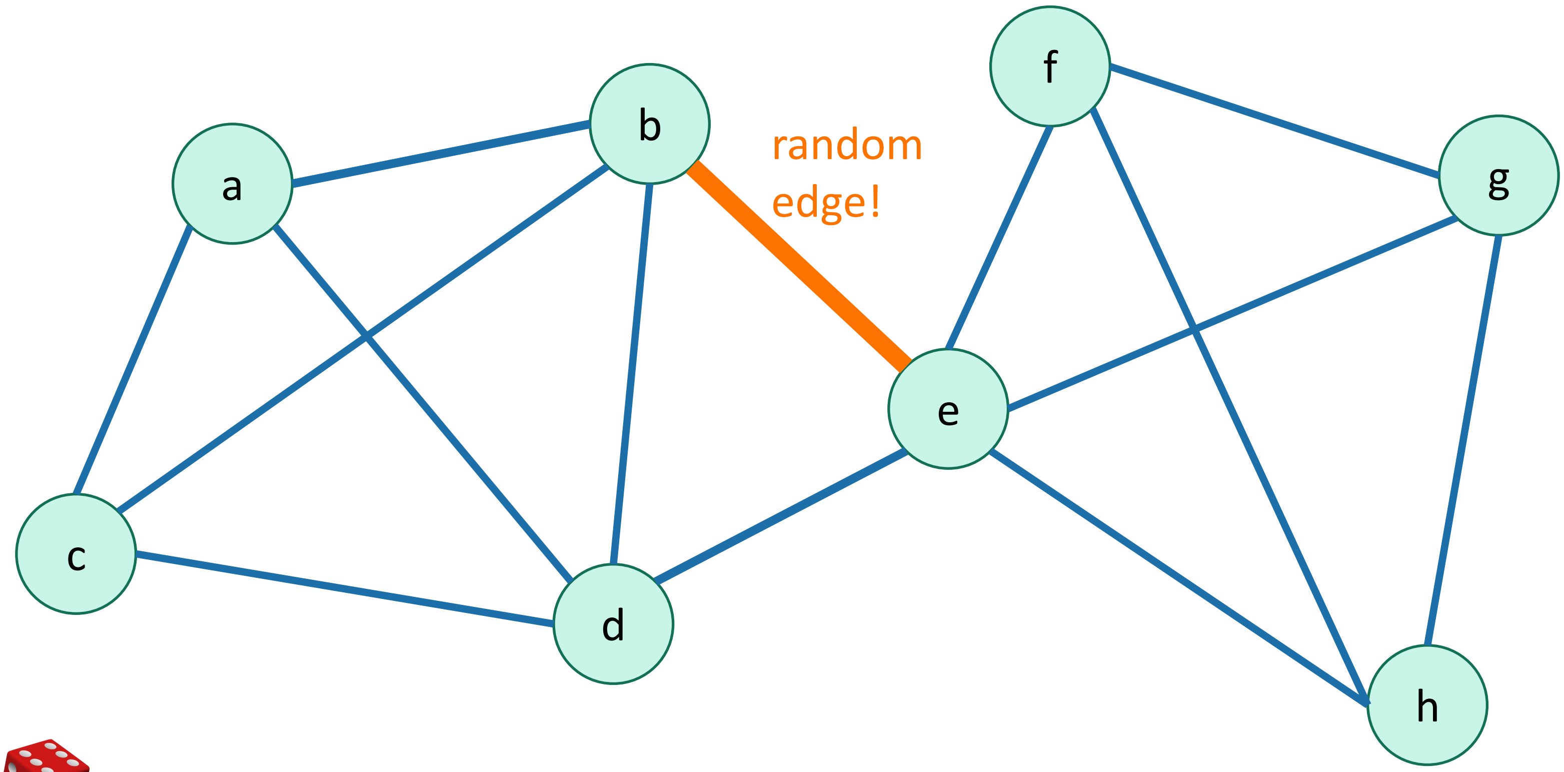
Why did that work?

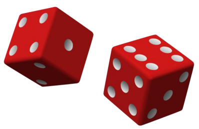
- We got really lucky!
- This could have gone wrong in so many ways.



Karger's algorithm

Say we had chosen this edge





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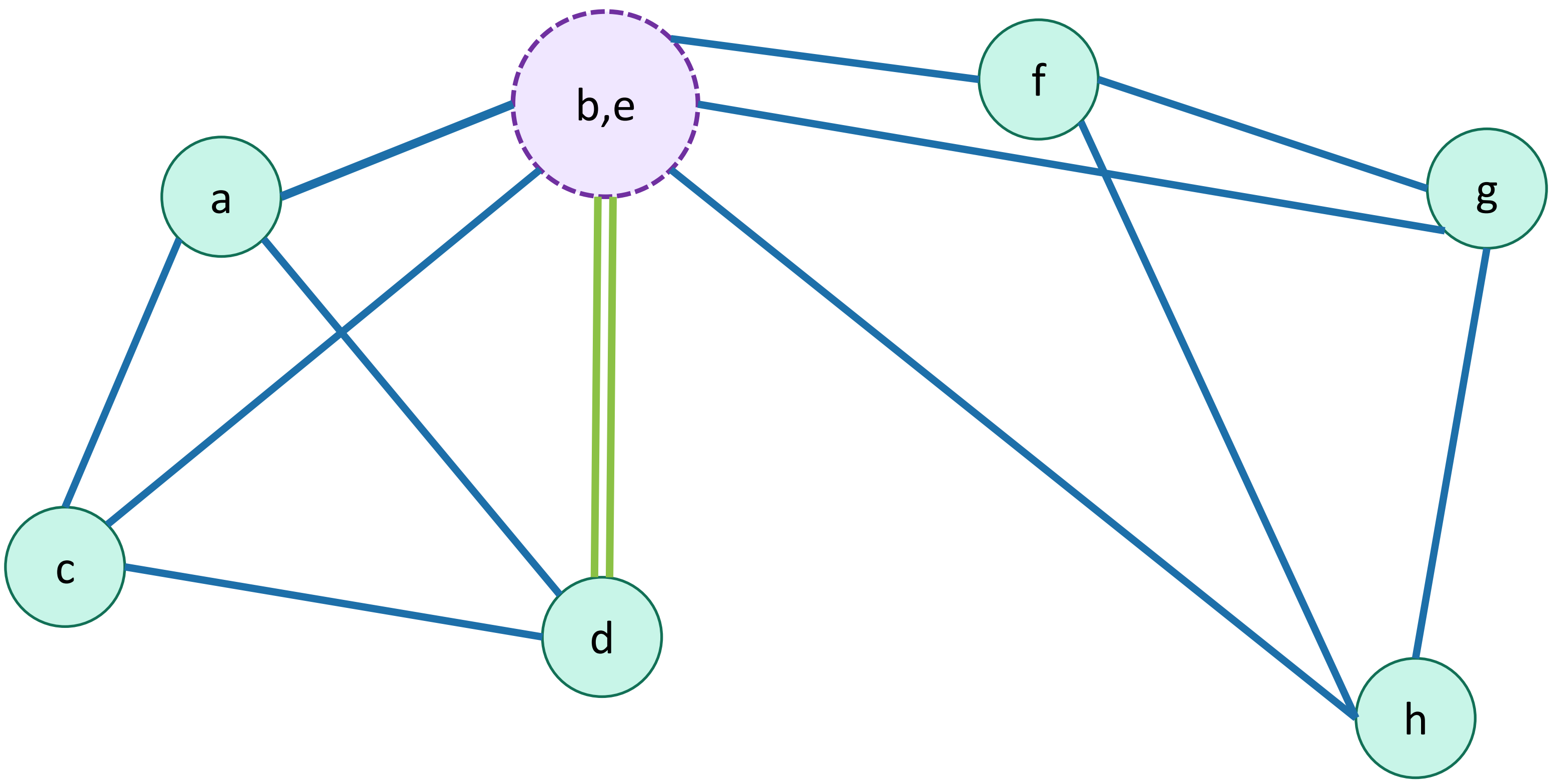
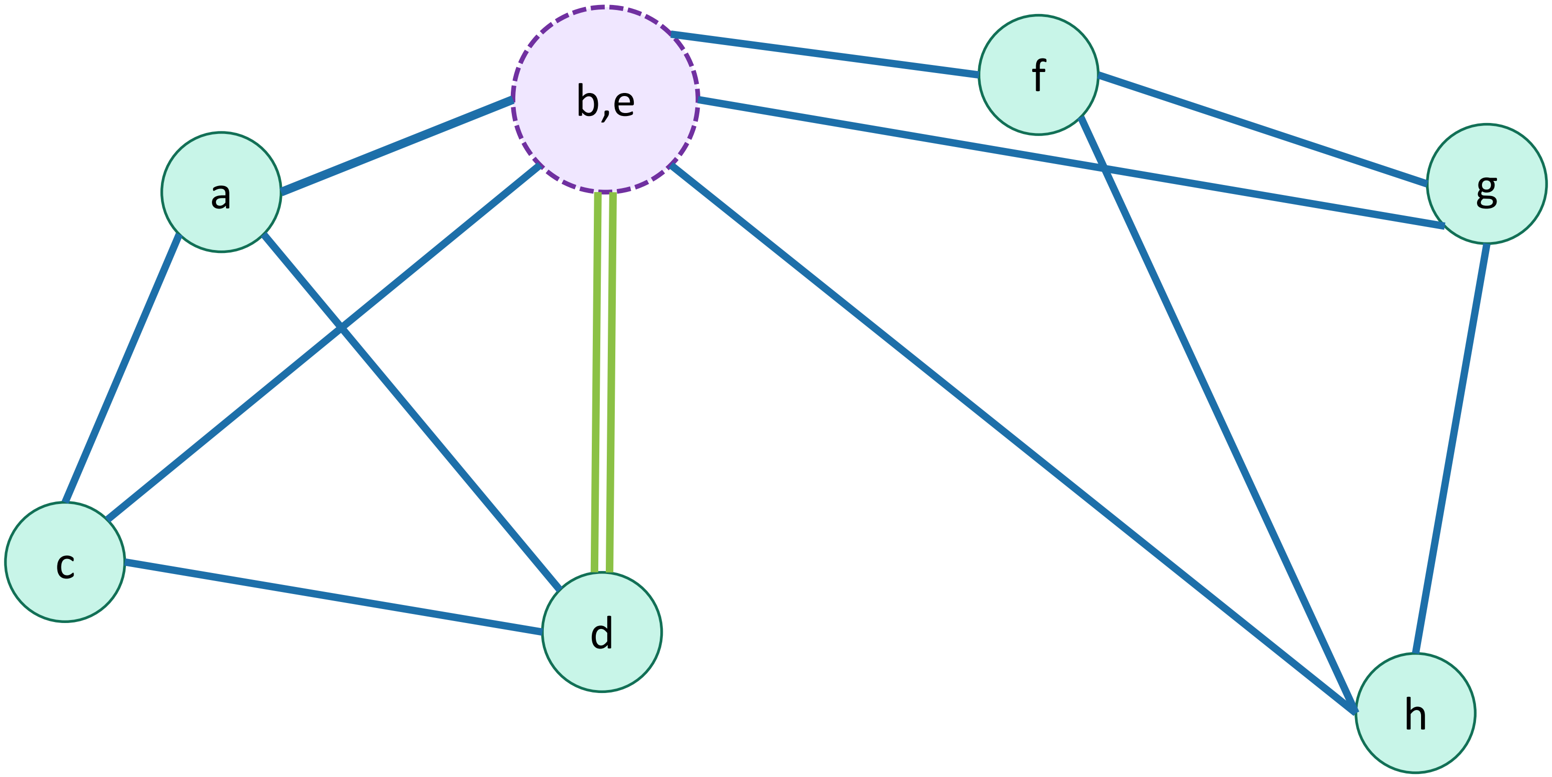


Karger's algorithm

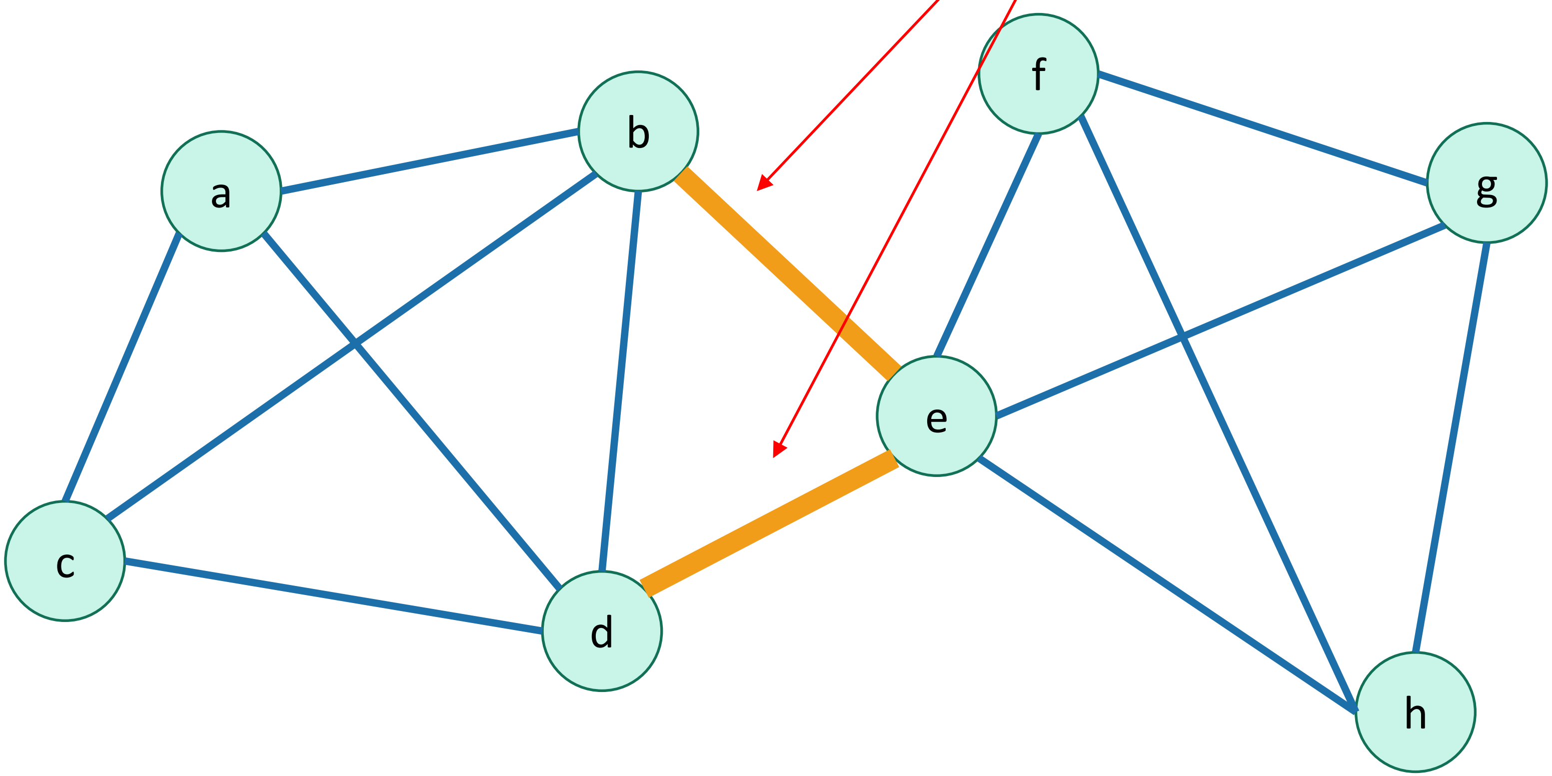
Say we had chosen this edge

Now there is **no way** we could return a cut that separates b and e.

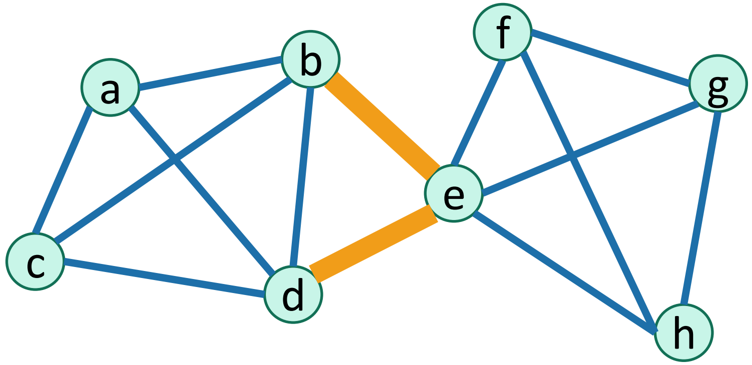


Even worse

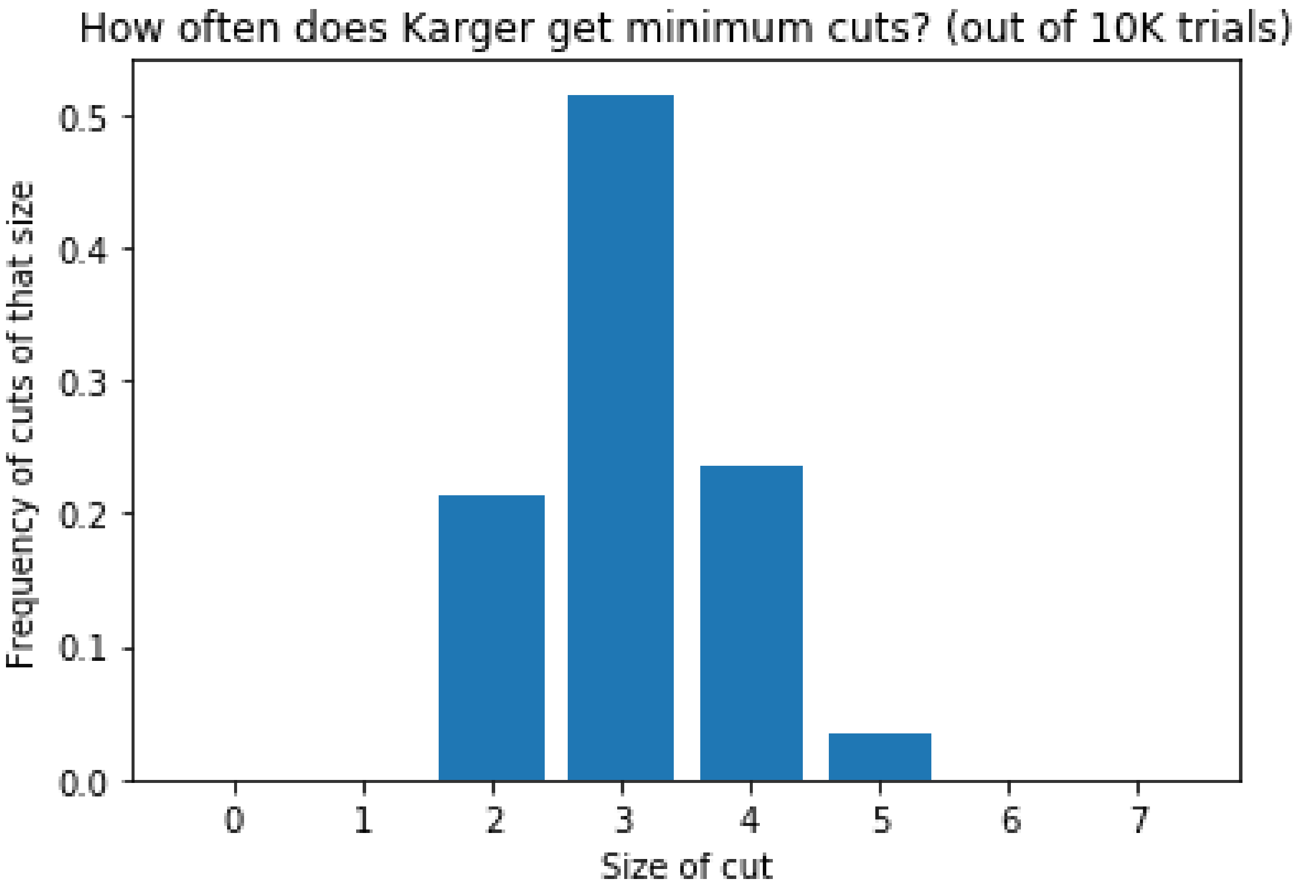
If the algorithm **EVER** chooses either of **these edges**,
it will be wrong.



How likely is that?



- For this particular graph, if we do 10,000 times:



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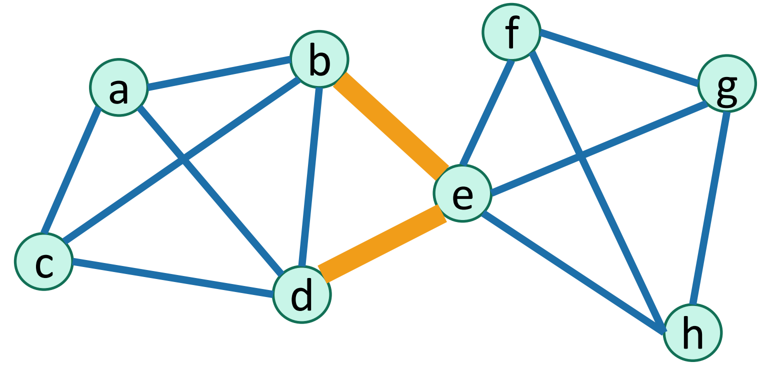

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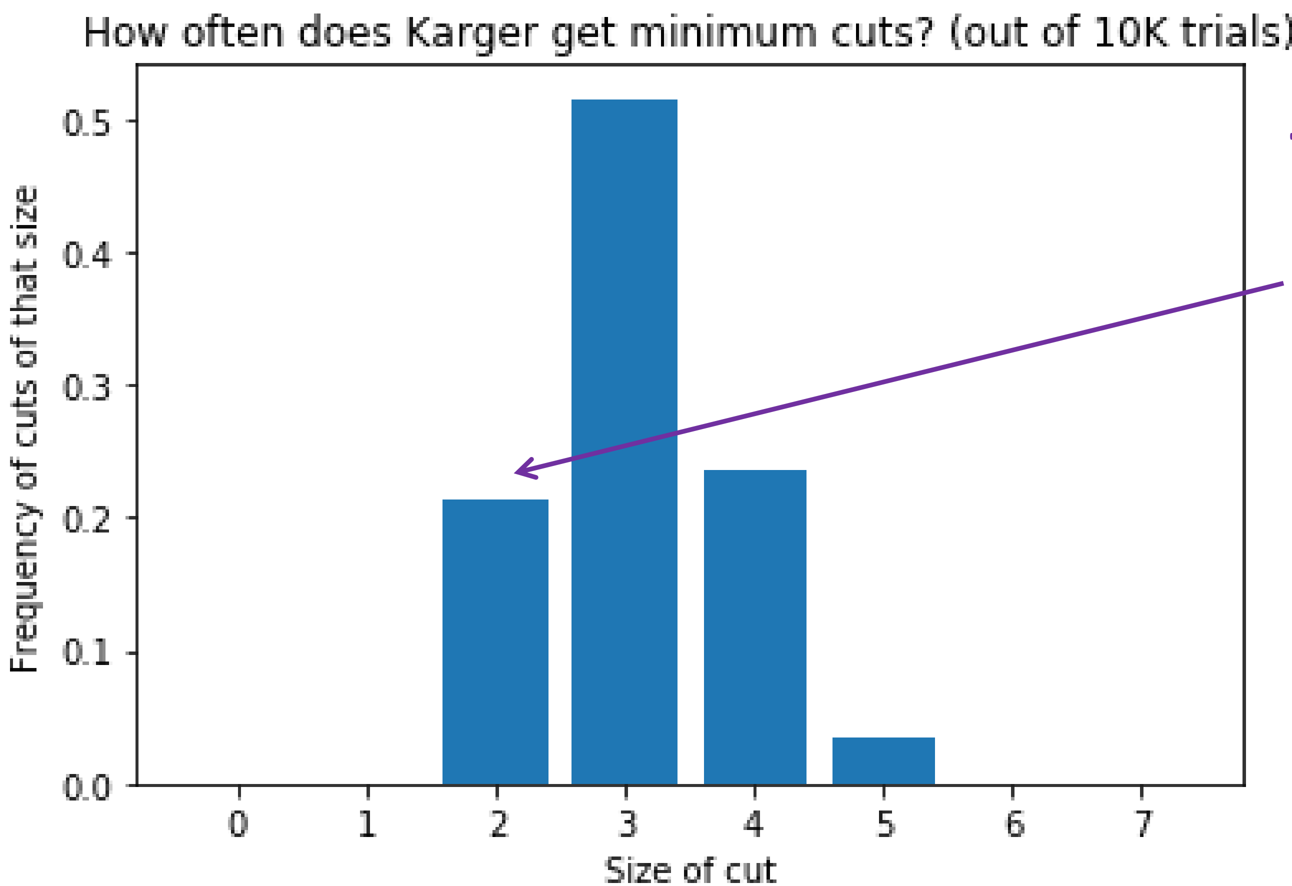
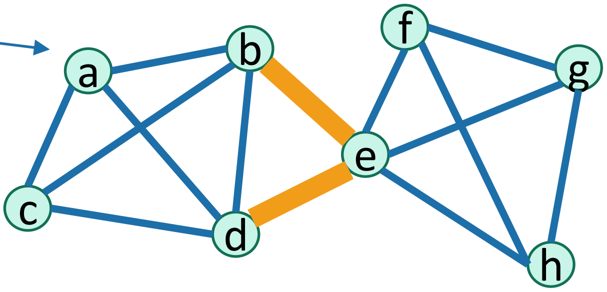


The algorithm is only correct about 20% of the time!

That doesn’t sound good

That doesn't sound good

- Too see why it's good after all, we'll do a case study of this graph.



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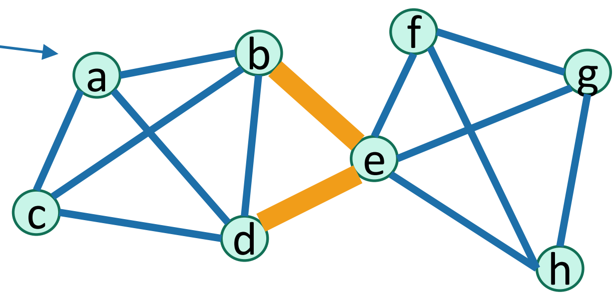

The plan:

- See that 20% chance of correctness is actually nontrivial.
- Use repetition to boost an algorithm that’s correct 20% of the time to an algorithm that’s correct 99% of the time.

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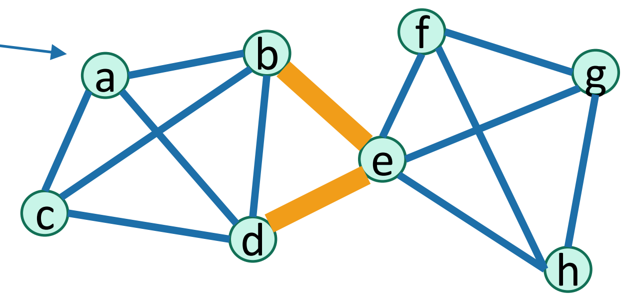

The plan:

- See that 20% chance of correctness is actually nontrivial.
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- To see the first point, let's compare Karger's algorithm to the algorithm:

Choose a completely random cut and hope that it's a minimum cut.

Uniformly random cut

- Pick a random way to split the vertices into two parts:

Uniformly random cut

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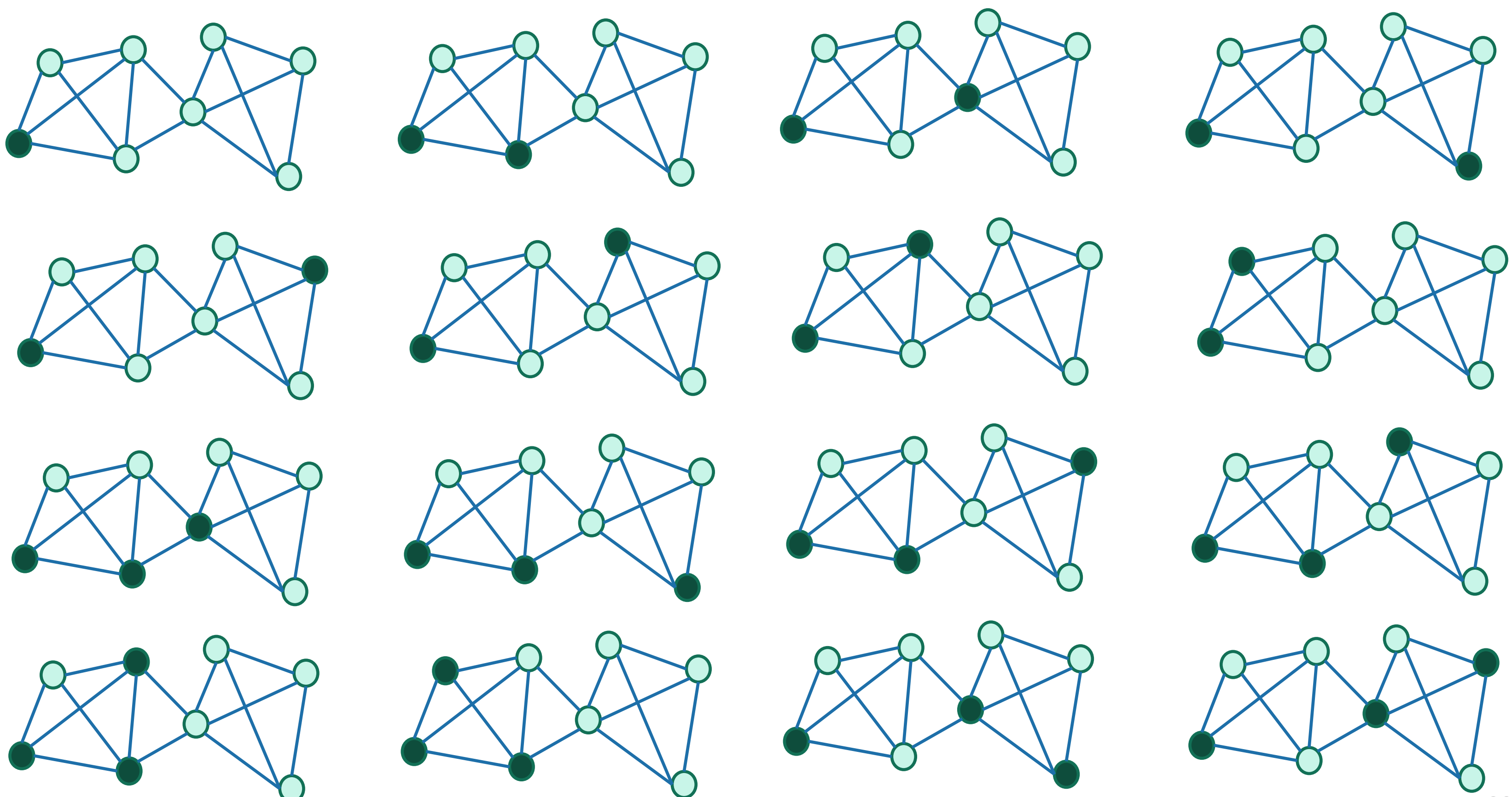

Uniformly random cut

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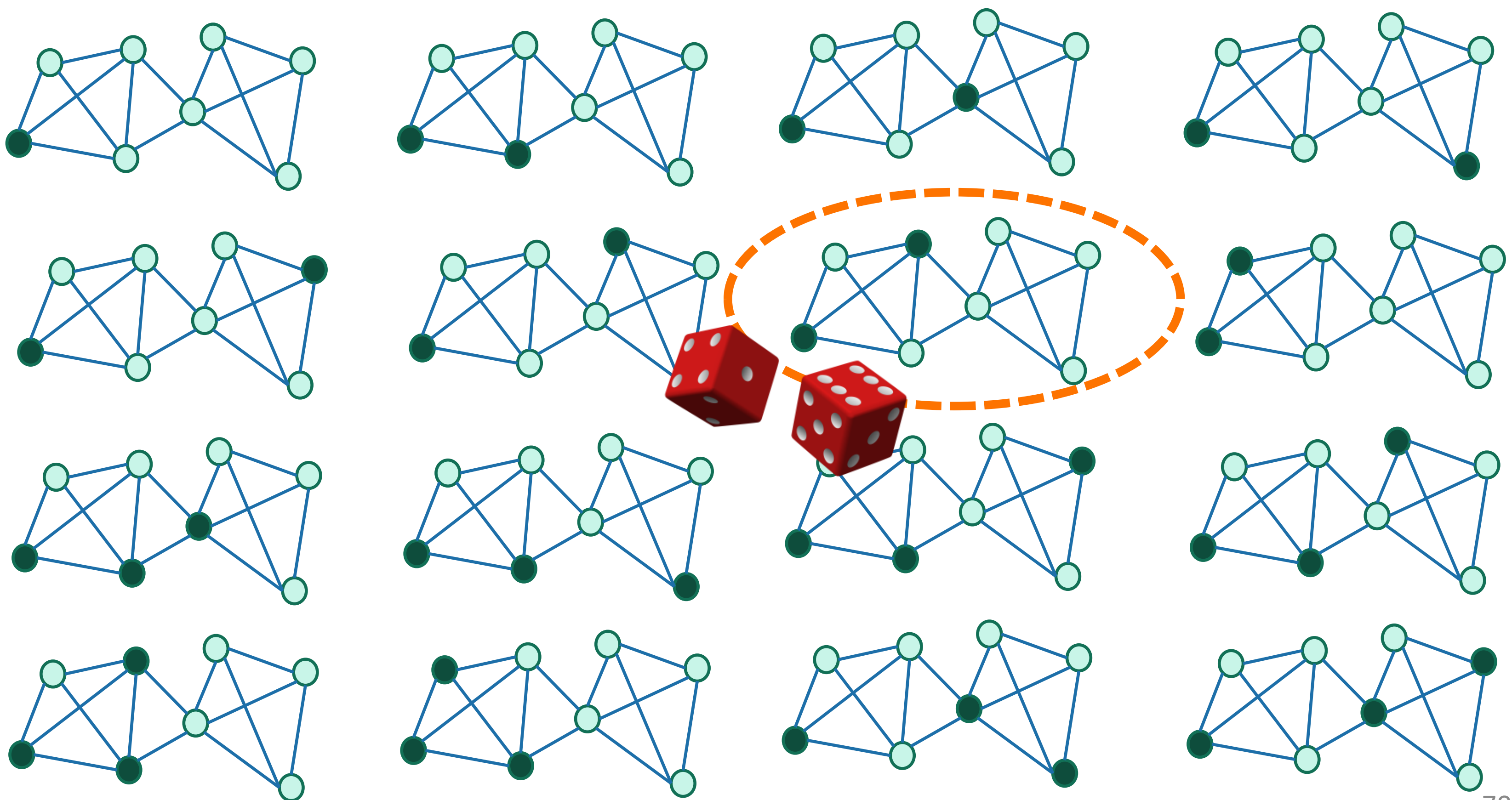


Uniformly random cut

- Pick a random way to split the vertices into two parts:

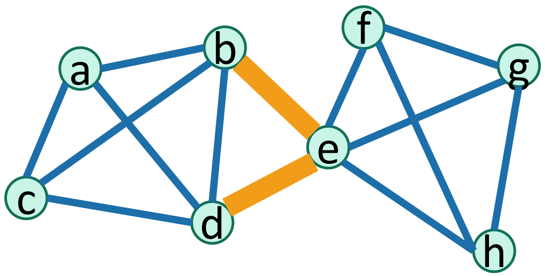
- The probability of choosing the minimum cut is*…

$$\frac{\text{number of min cuts in that graph}}{\text{number of ways to split 8 vertices in 2 parts}} = \frac{2}{2^8 - 2} \approx 0.008$$

- Aka, we get a minimum cut 0.8% of the time.

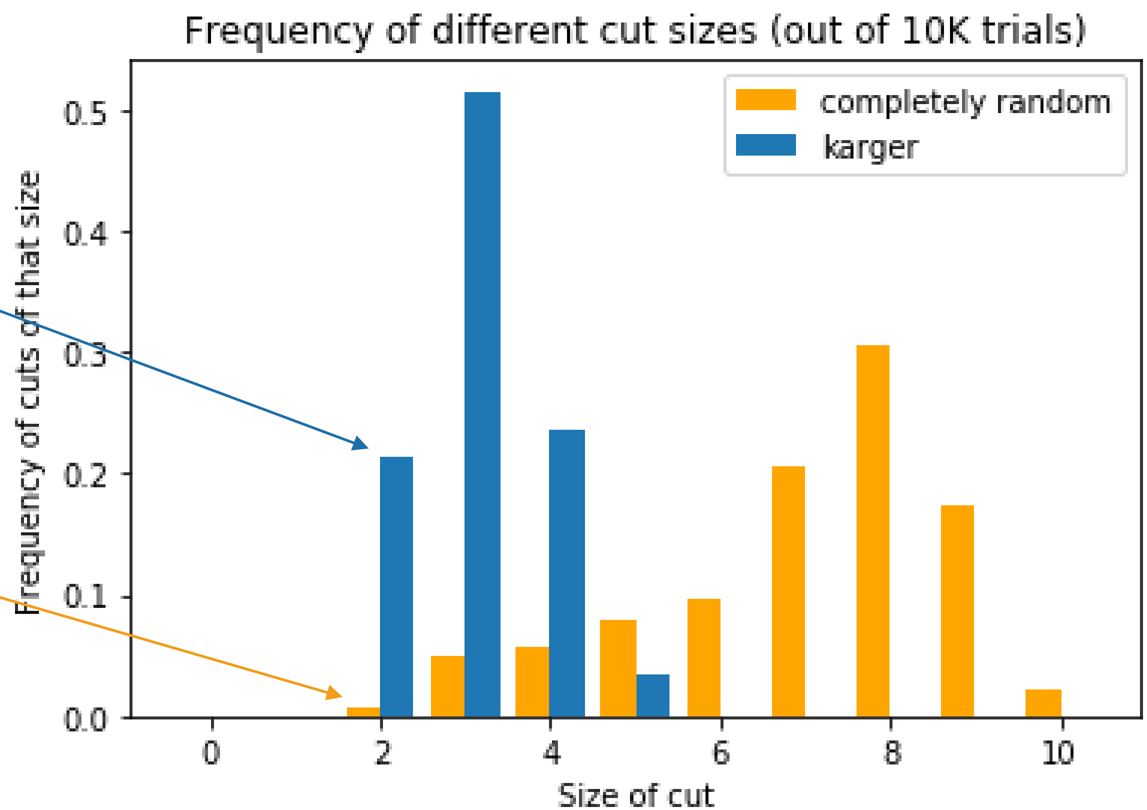
*For this example in particular

Karger is better than completely random!



Karger's alg. is correct about 20% of the time

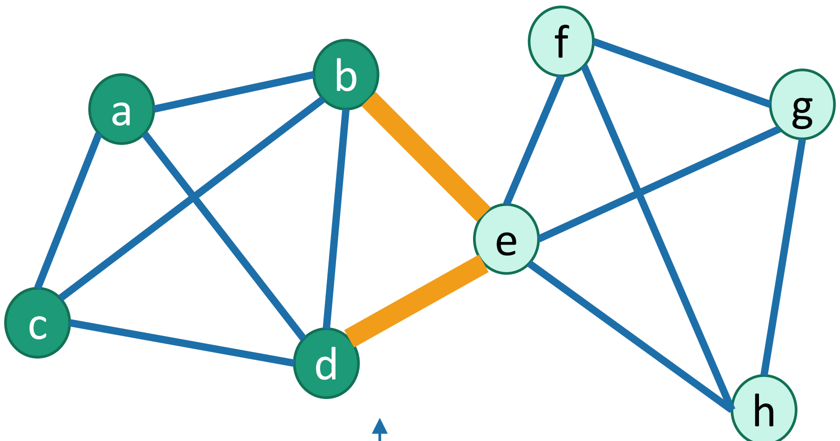
Completely random is correct about 0.8% of the time



What’s going on?

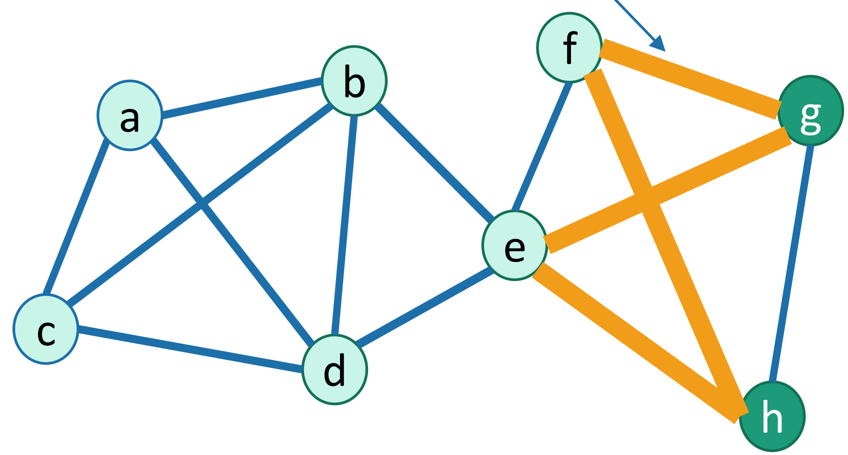
What’s going on?

- Which is more likely?



A: The algorithm never chooses either of the edges in **the minimum cut**.

B: The algorithm never chooses any of the edges in **this big cut**.



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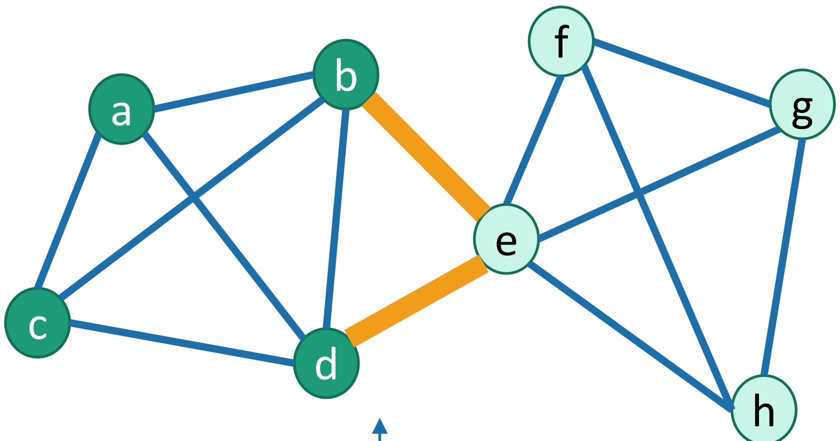

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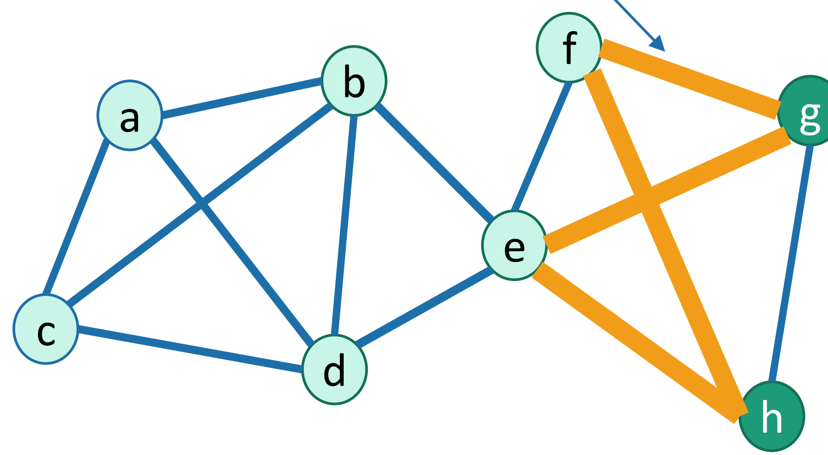
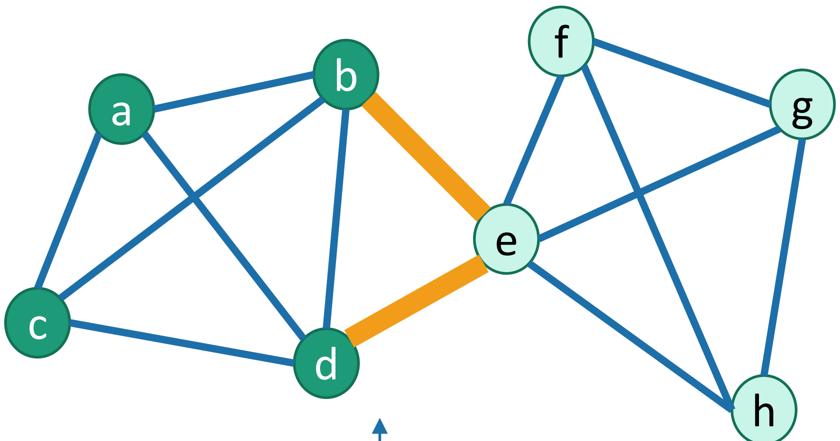


- Neither A nor B are very likely, **but A is more likely than B**.

What’s going on?

Thing 1: It’s unlikely that Karger will hit the min cut since it’s so small!

- Which is more likely?



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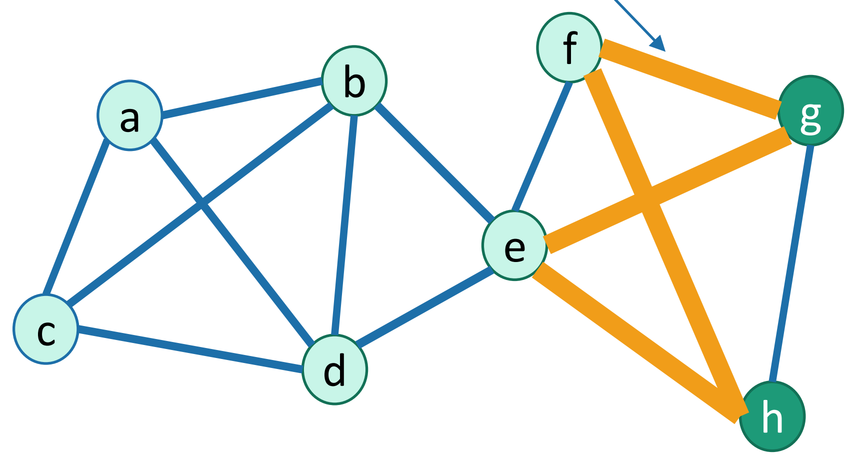

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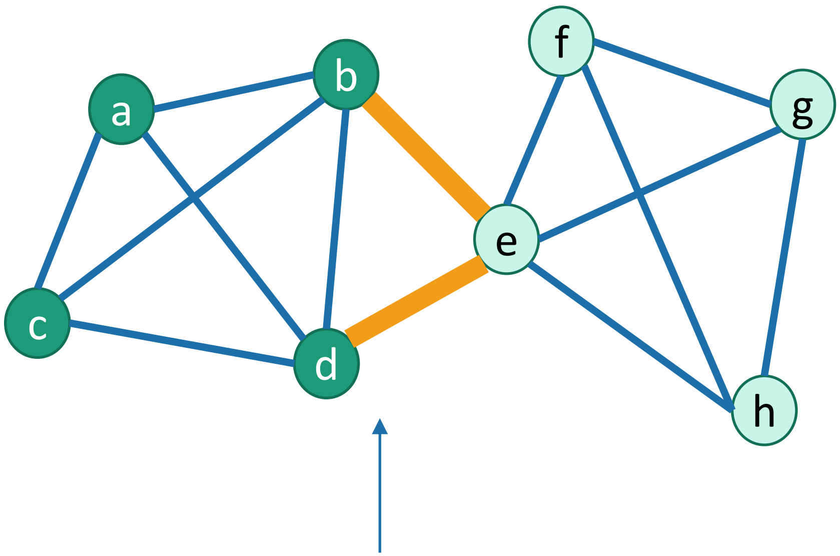
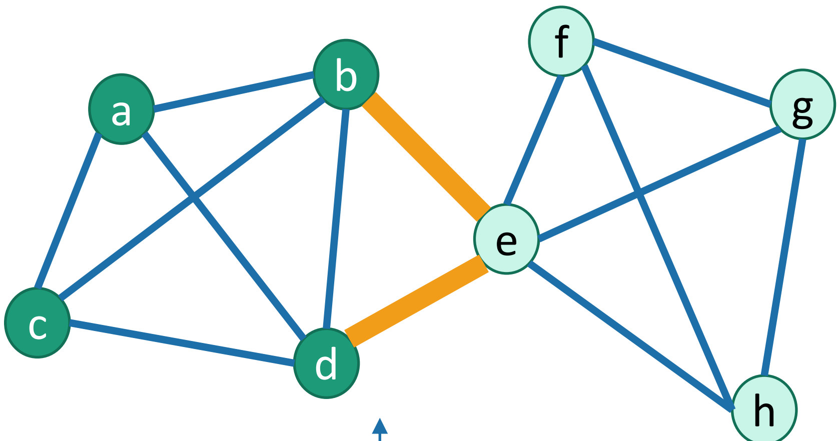


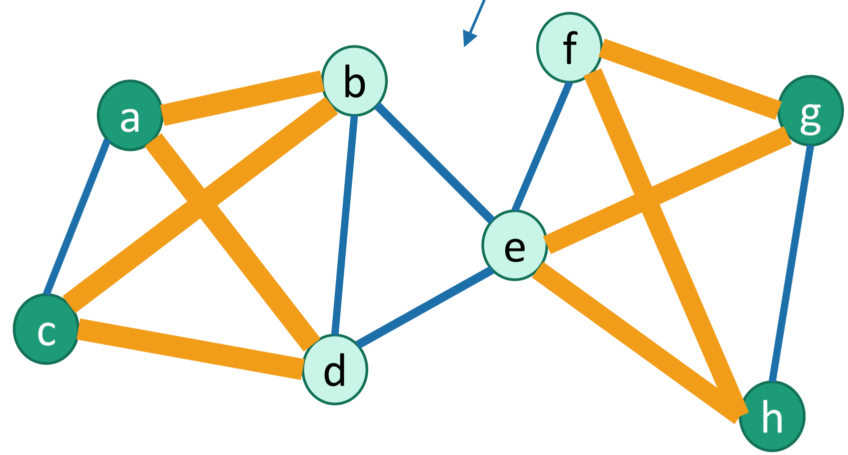
A: This cut can be returned
by Karger's algorithm.

What’s going on?



A: This cut can be returned by Karger’s algorithm.

B: This cut can’t be returned by Karger’s algorithm!
(Because how would a and g end up in the same super-node?)



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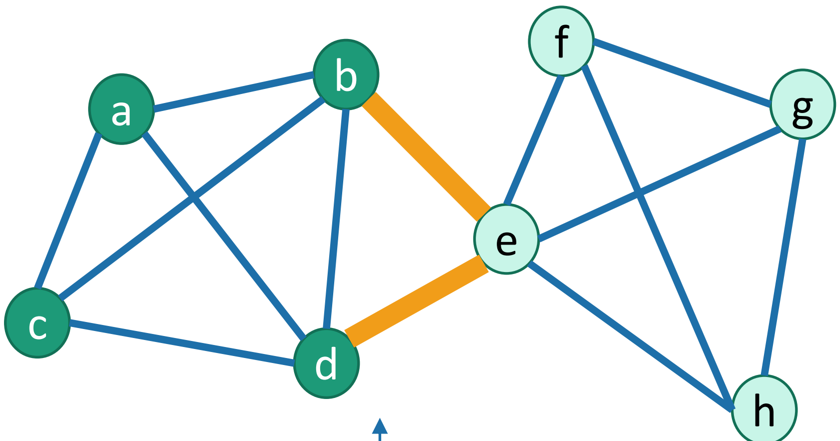

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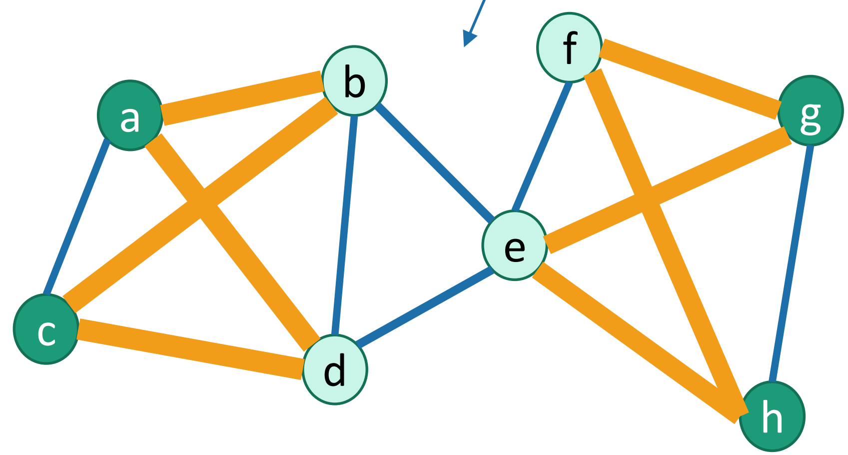
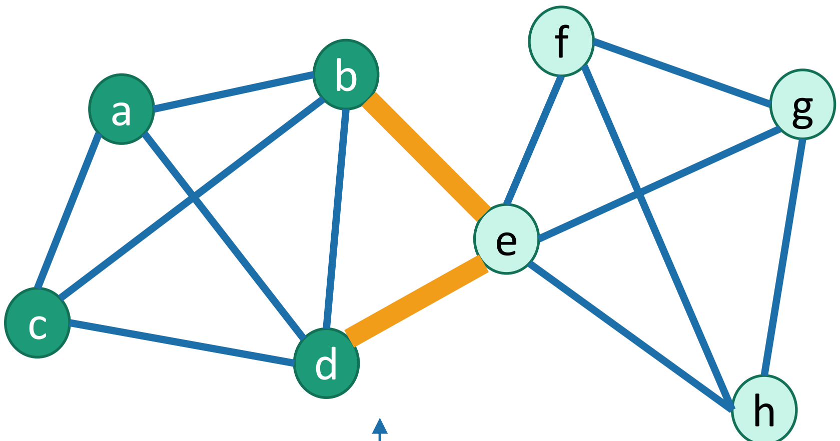


This cut actually separates the graph into three pieces, so it’s not minimal – either half of it is a smaller cut.

What’s going on?

Thing 2: By only contracting edges we are ignoring certain really-not-minimal cuts.



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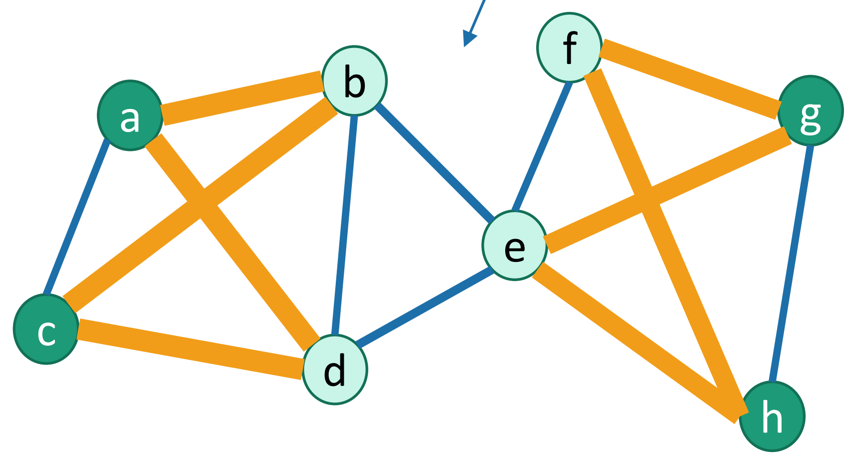

This cut actually separates the graph into three pieces, so it’s not minimal – either half of it is a smaller cut.

Why does that help?

- Okay, so it’s better than completely random…
- We’re still wrong about 80% of the time.

The plan:

- See that 20% chance of correctness is actually nontrivial.
- Use repetition to boost an algorithm that’s correct 20% of the time to an algorithm that’s correct 99% of the time.

Why does that help?

- Okay, so it's better than completely random…
- We're still wrong about 80% of the time.
- The main idea: **repeat!**
  - If I'm wrong 20% of the time, then if I repeat it a few times I'll eventually get it right.

The plan:

- See that 20% chance of correctness is actually nontrivial.
- Use repetition to boost an algorithm that's correct 20% of the time to an algorithm that's correct 99% of the time.

Thought experiment

- Suppose you have a magic button that produces one of 5 numbers, {a,b,c,d,e}, uniformly at random when you push it.
- You don't know what {a,b,c,d,e} are.
- Q: What is the minimum of a,b,c,d,e?

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4

3

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4 **3** **3**

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4 3 3 2

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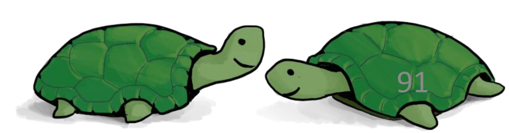
- Q: What is the minimum of a,b,c,d,e?

4 3 3 2 5 2 5

How many times do you have to push the button, in expectation, before you see the minimum value?

What is the probability that you have to push it more than 5 times? 10 times?



Pr[We push the button once and don't ever get the min] = $(1 - 0.2) = 0.8$

$$\Pr\left[\begin{array}{c}\text{We push the button}\\ \text{once and don't ever}\\ \text{get the min}\end{array}\right] = (1 - 0.2) = 0.8$$

$$E\left[\begin{array}{c}\text{Number of times}\\ \text{we push the button}\\ \text{until we get the}\\ \text{minimum value}\end{array}\right] = 1/(0.20) = 5$$

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E[Number of times we push the button until we get the minimum value] = 1/(0.20) = 5

Pr[We push the button t times and don't ever get the min] = $(1 - 0.2)^t$

Pr[We push the button 5 times and don't ever get the min] = $(1 - 0.2)^5 \approx 0.33$

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Pr[We push the button 10 times and don't ever get the min] = $(1 - 0.2)^{10} \approx 0.1$

In this context

(Lets Run Karger's!, 5 times)

In this context

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- Run Karger's! The cut size is 4!

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In this context

(Lets Run Karger’s!, 5 times)

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- Run Karger’s! The cut size is 5!

In this context

(Lets Run Karger's!, 5 times)

- Run Karger's! The cut size is 4!

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- Run Karger's! The cut size is 2! **Correct!**

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In this context

(Lets Run Karger’s!, 5 times)

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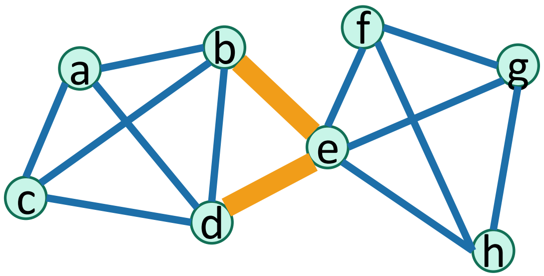
- Run Karger’s! The cut size is 3!

- Run Karger’s! The cut size is 2!   **Correct!**

- Run Karger’s! The cut size is 5!

If the success probability is about 20%, then if you run Karger’s algorithm 5 times and take the best answer you get, that will likely be correct! (with probability about 0.66)

For this particular graph



- Repeat Karger's algorithm about 5 times, and we will get a min cut with decent probability.
  - In contrast, we'd have to choose a random cut about 1/0.008 = 125 times!

The plan:

- See that 20% chance of correctness is actually nontrivial.
- Use repetition to boost an algorithm that's correct 20% of the time to an algorithm that's correct most of the time.

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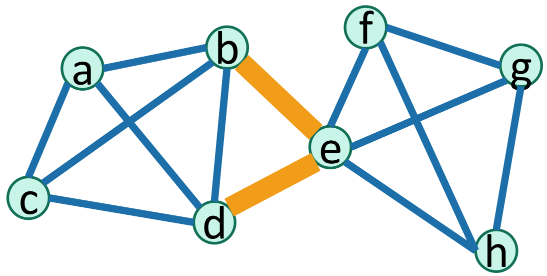

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Hang on! This "20%" figure just came from running experiments on this particular graph. What about general graphs? Can we prove something?

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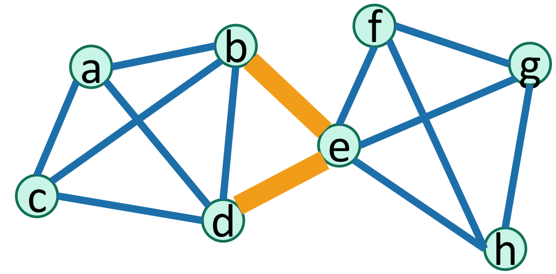

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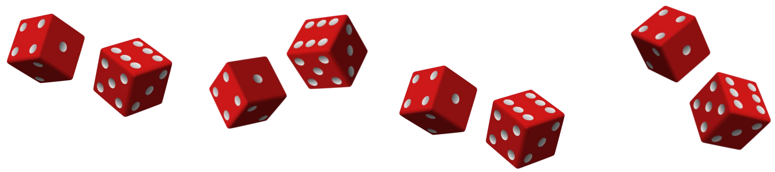
Also, we should be a bit more precise about this "about 5 times" statement.

The plan:

- ✓ See that 20% chance of correctness is actually nontrivial.
- ✓ Use repetition to boost an algorithm that's correct 20% of the time to an algorithm that's correct most of the time.

Questions

To generalize this approach to all graphs



1. What is the probability that Karger's algorithm returns a minimum cut?

2. How many times should we run Karger's algorithm to "probably" succeed?
   - Say, with probability 0.99?
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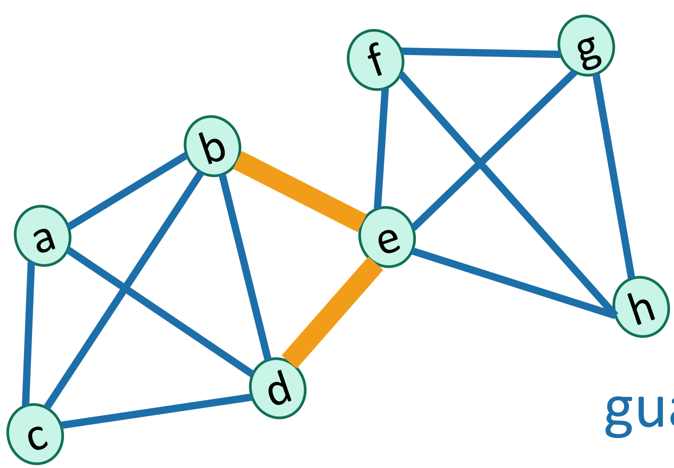
Answer to Question 1

Claim:

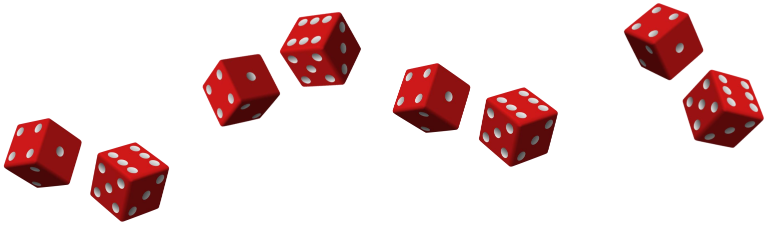
The probability that Karger's algorithm returns a minimum cut is

at least $1/\binom{n}{2}$



In this case, $1/\binom{8}{2} = 0.036$, so we are guaranteed to win at least 3.6% of the time.

Questions



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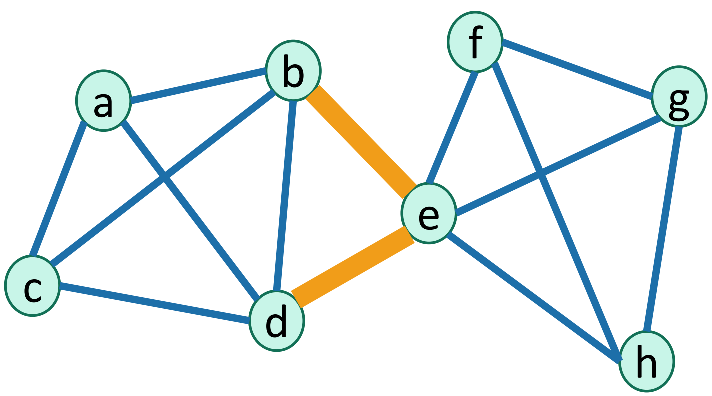
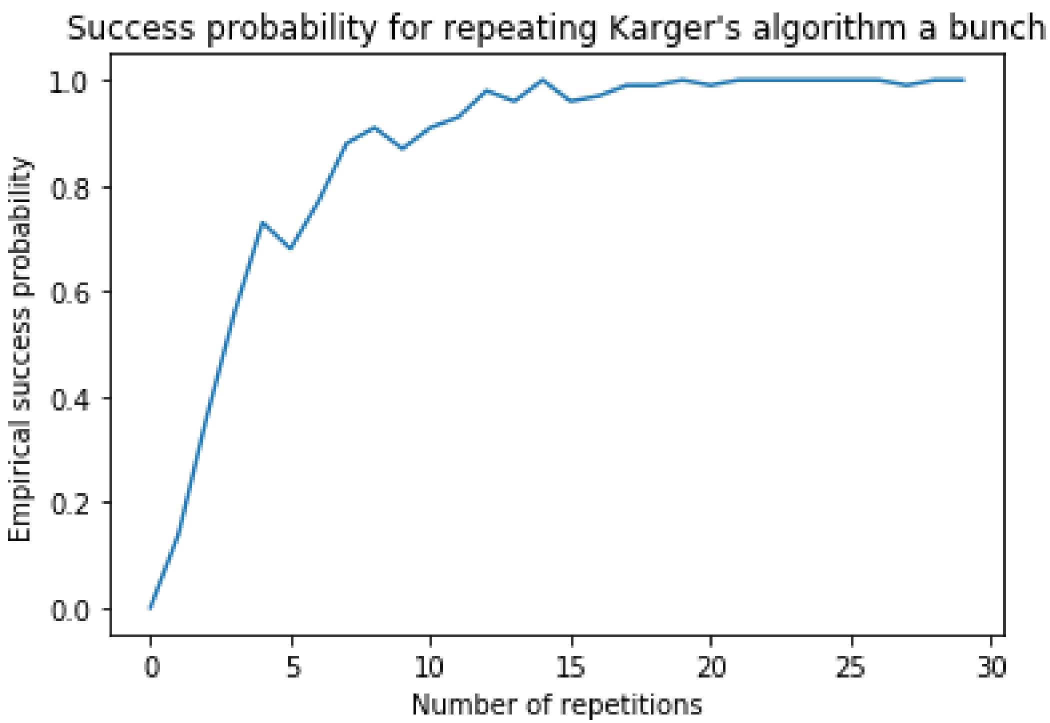
Before we prove the Claim

2. How many times should we run Karger's algorithm to succeed with probability $1 - \delta$?





A computation

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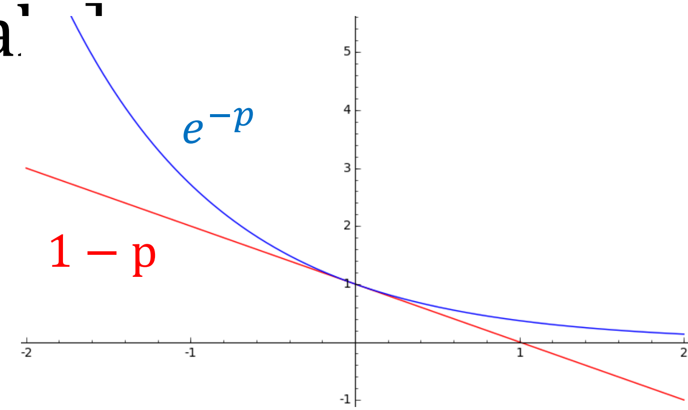

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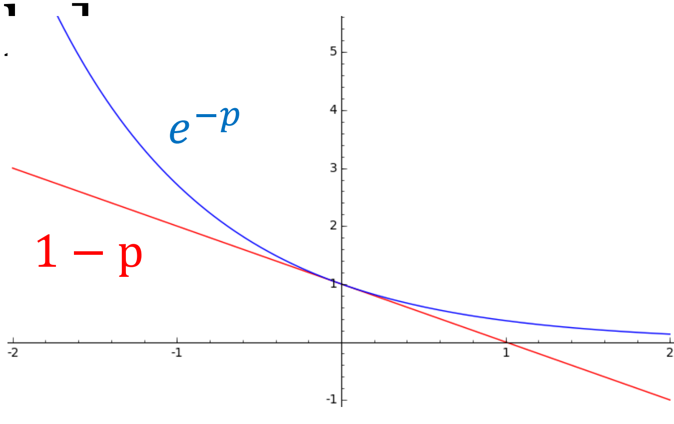

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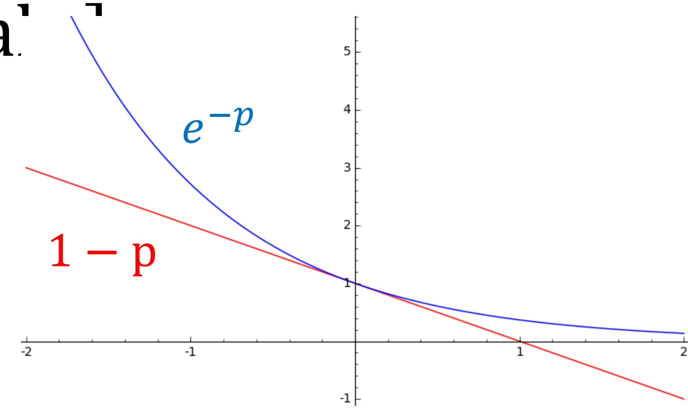

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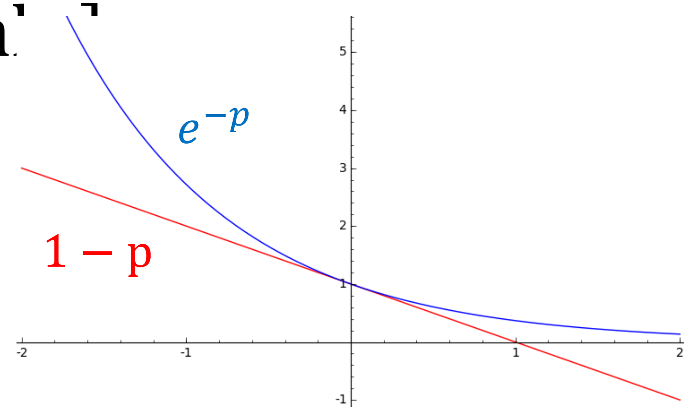

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A computation

Punchline:
If we repeat **T = $\binom{n}{2}\ln(1/\delta)$** times,
we win with probability at least $\mathbf{1-\delta}$.

- Suppose :
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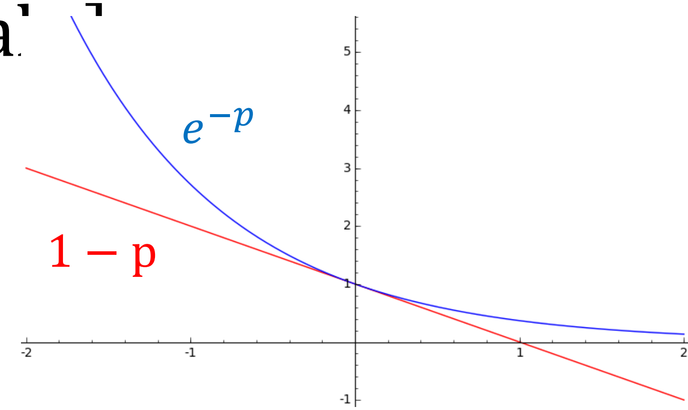

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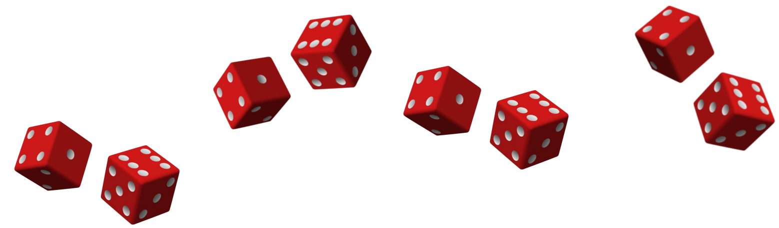
Answers



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   $\binom{n}{2} \ln\left(\frac{1}{\delta}\right)$ times.

Theorem

Assuming the claim about $1/\binom{n}{2}$ ...

- Suppose G has n vertices.

- Consider the following algorithm:
  - bestCut = None
  - **for** $\mathrm{t} = 1, \ldots, \binom{n}{2} \ln\left(\frac{1}{\delta}\right)$:
    - candidateCut ← **Karger**(G)
    - **if** candidateCut is smaller than bestCut:
      - bestCut ← candidateCut
  - **return** bestCut

- Then $\Pr[\text{ this doesn't return a min cut }] \leq \delta.$

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- $\binom{n}{2} \ln\left(\frac{1}{\delta}\right)$ repetitions, and O(n^2) per repetition.
- So, $O\left(n^2 \cdot \binom{n}{2} \ln\left(\frac{1}{\delta}\right)\right) = \mathrm{O}(\mathrm{n}^4)$ **Treating δ as constant.**

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Again we can do better with a union-find data structure.

Theorem

Assuming the claim about $1/\binom{n}{2}$ …

Suppose G has n vertices. Then [repeating Karger's algorithm] finds a min cut in G with probability at least 0.99 in time $O(n^4)$.

Now let's prove the claim…

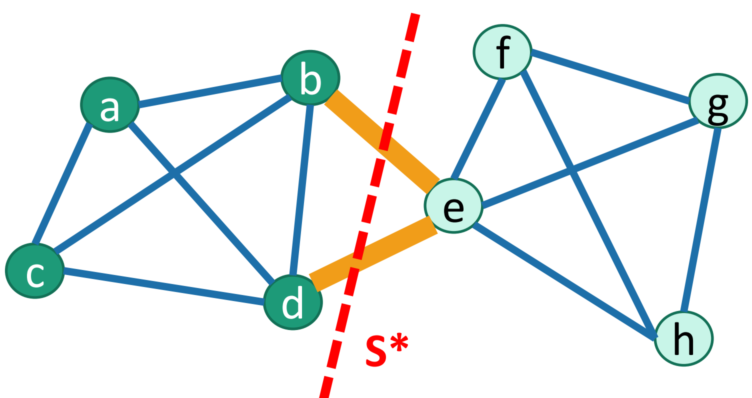
Claim

The probability that Karger's algorithm returns a minimum cut is

at least $1 / \binom{n}{2}$

Now let's prove that claim

Say that S* is a minimum cut.



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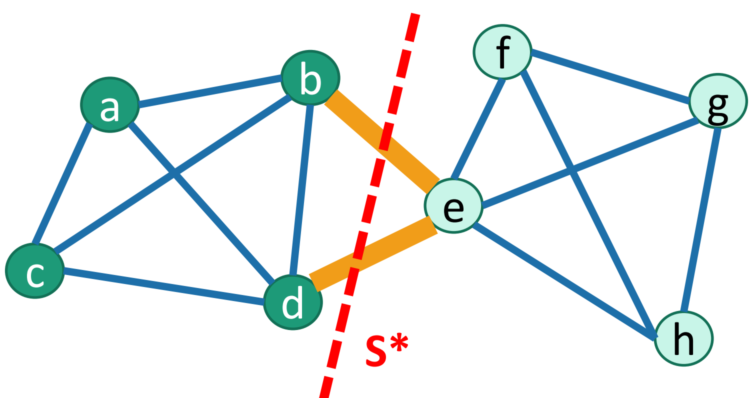

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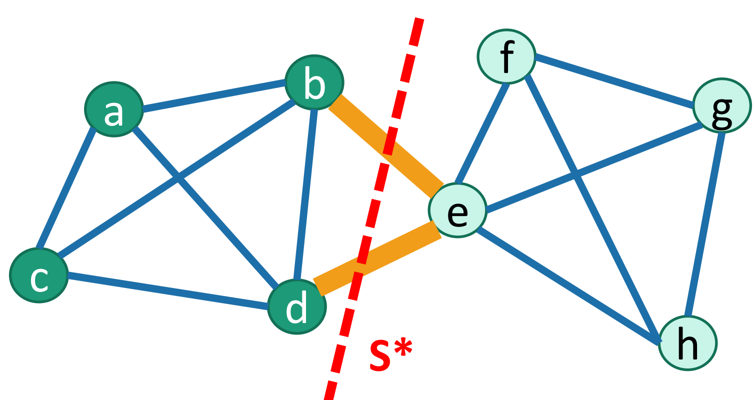

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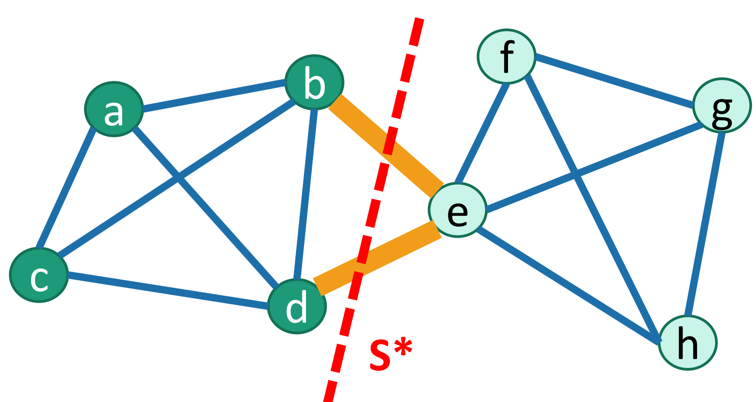

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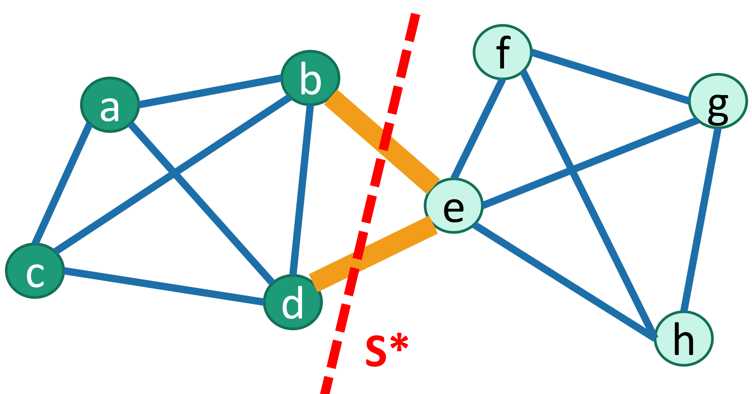
$$= \mathbf{PR}[\ e_1 \text{ doesn't cross } S^*\ ]$$
$$\times\ \mathbf{PR}[\ e_2 \text{ doesn't cross } S^* \mid e_1 \text{ doesn't cross } S^*\ ]$$
$$\ldots$$
$$\times\ \mathbf{PR}[\ e_{n-2} \text{ doesn't cross } S^* \mid e_1,\ldots,e_{n-3} \text{ don't cross } S^*\ ]$$



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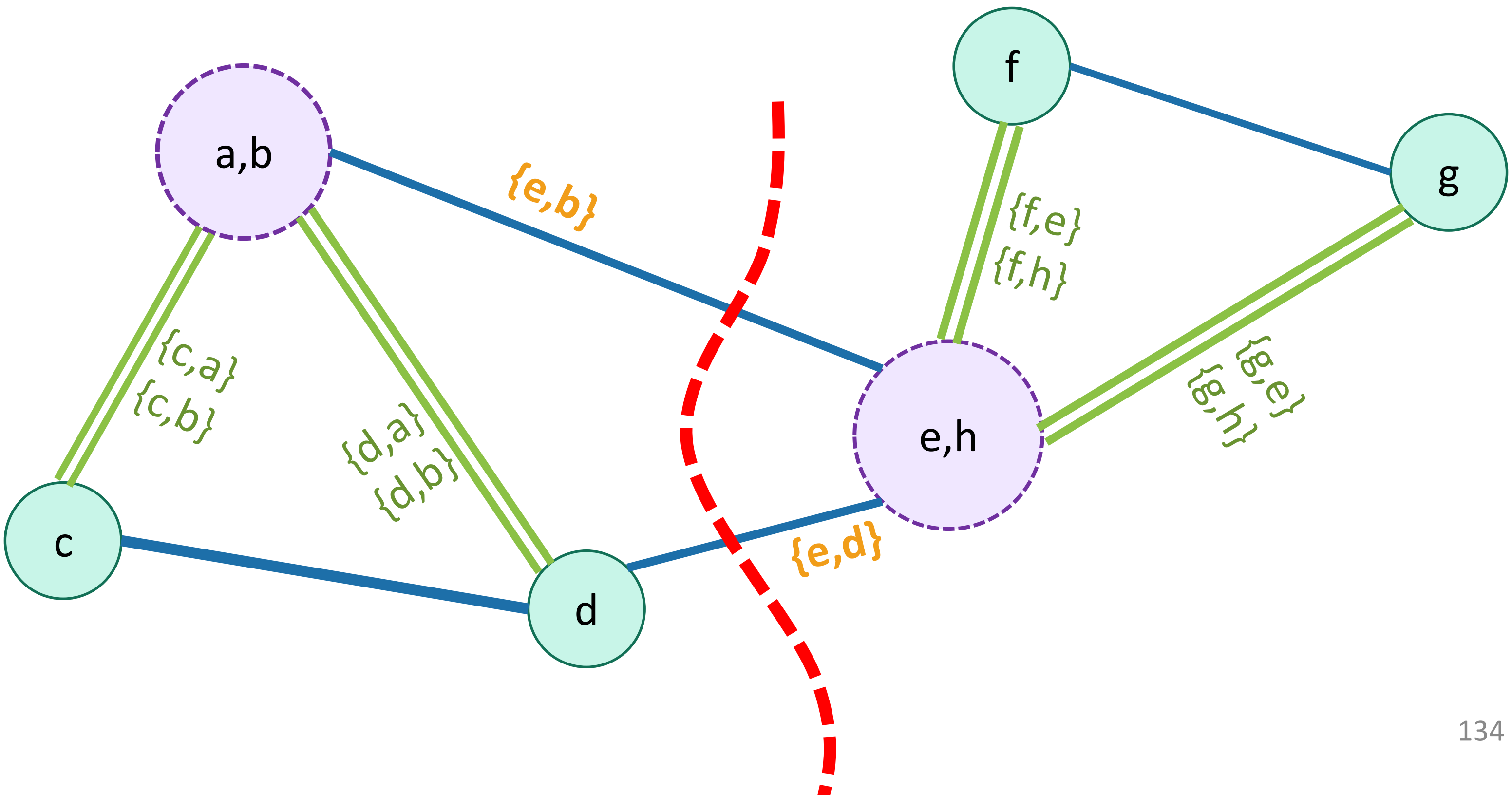
PR[e_j doesn't cross S* | $e_1,\ldots,e_{j-1}$ don't cross S*]



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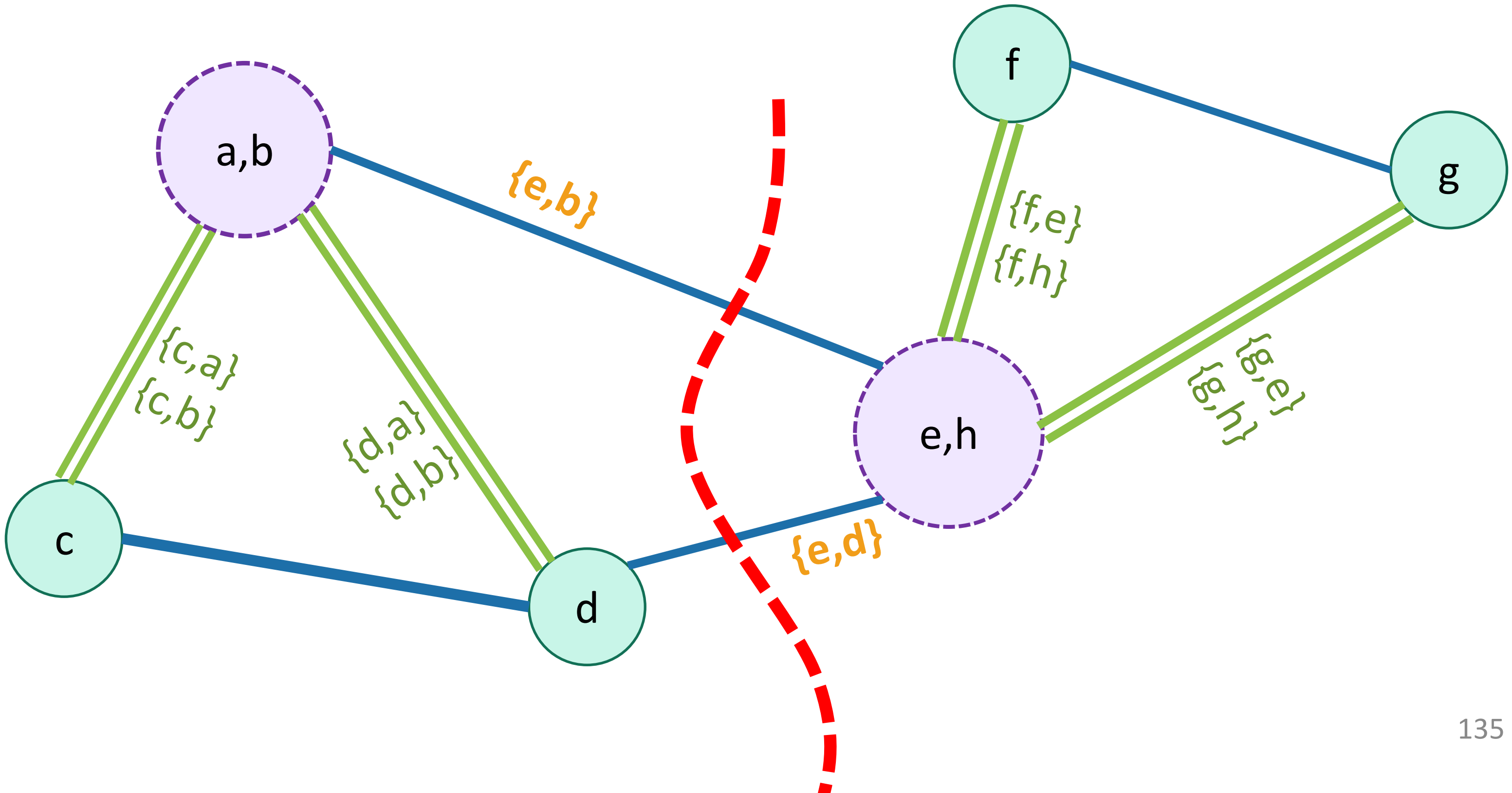

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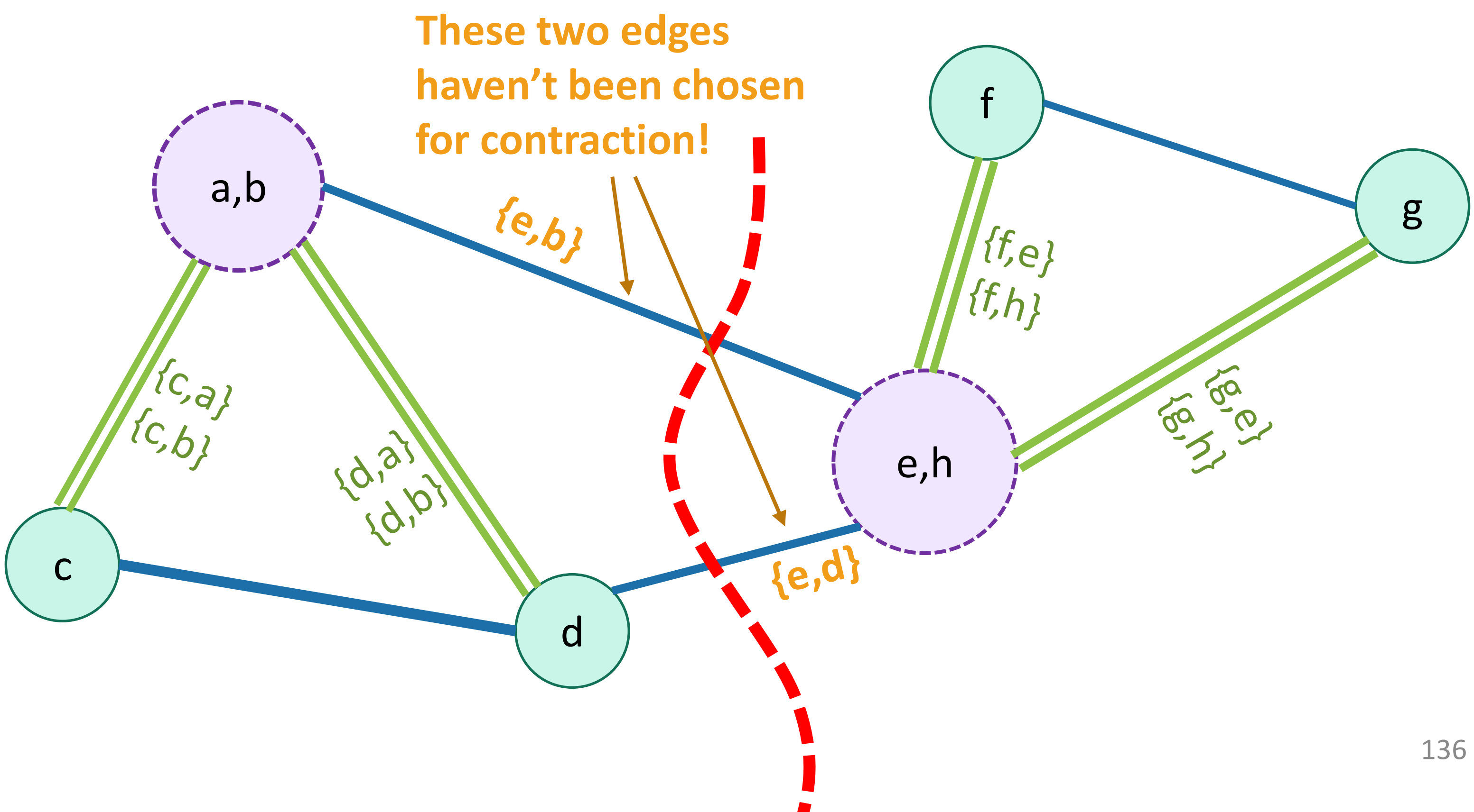

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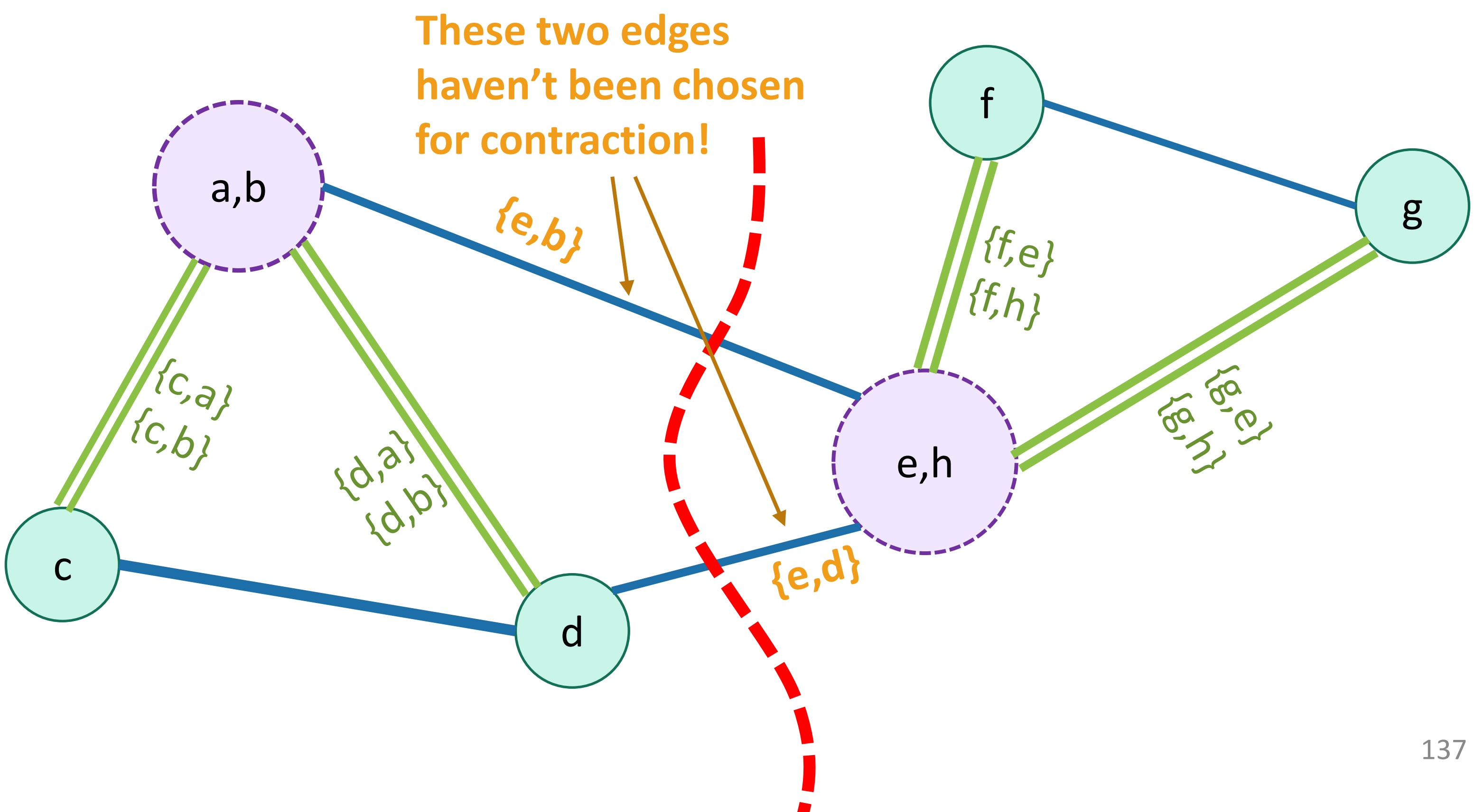

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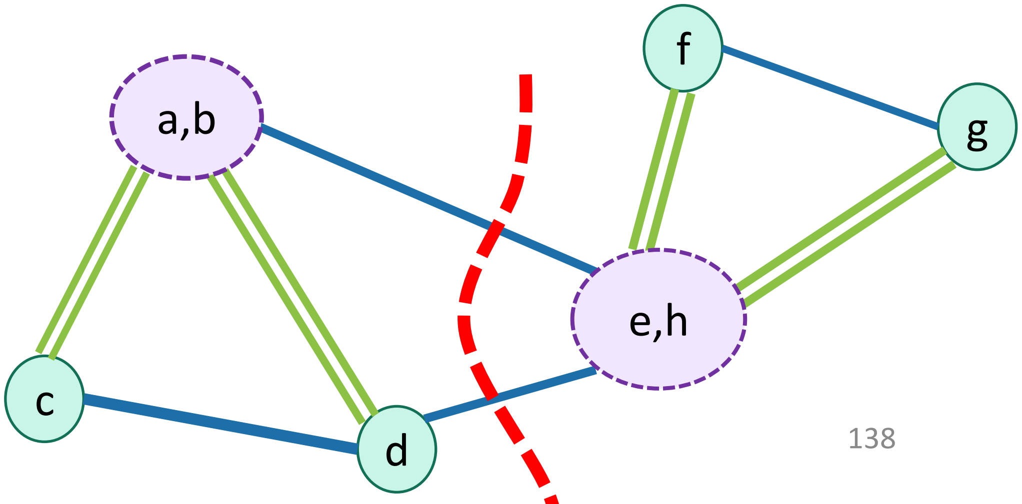

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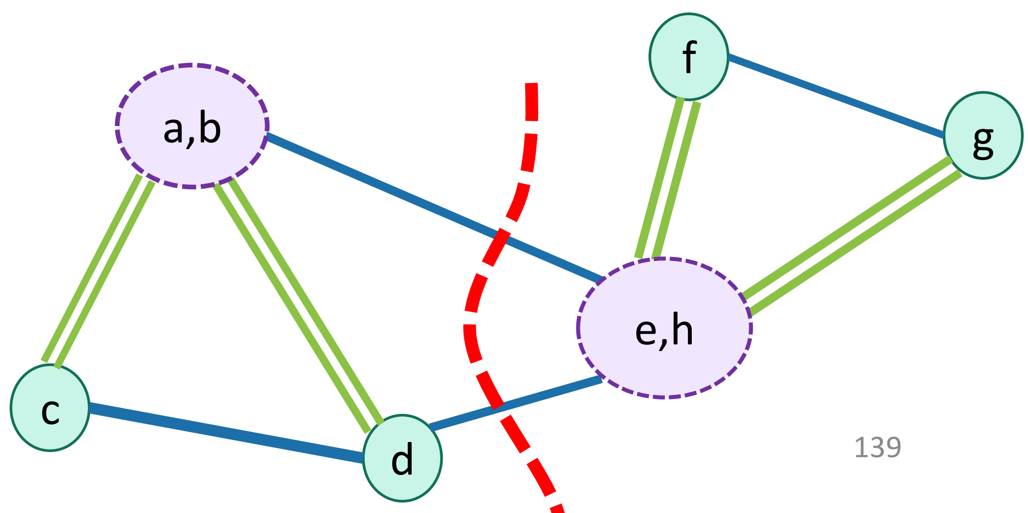
- Every supernode has at least k (original) edges coming out.
  - Otherwise we'd have a smaller cut.



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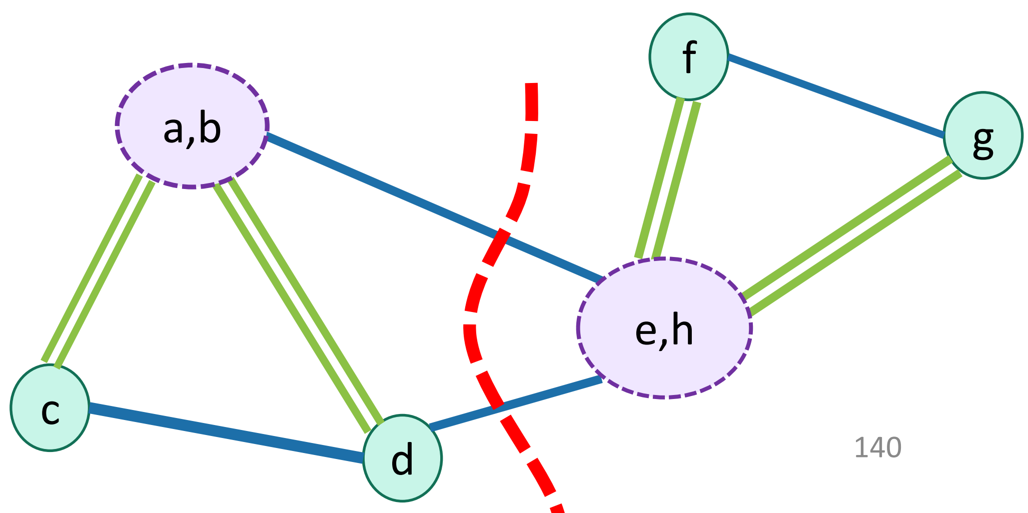
- Thus, there are at least (n-j+1)k/2 edges total.
  - because there are n - j + 1 supernodes left, each with k edges.



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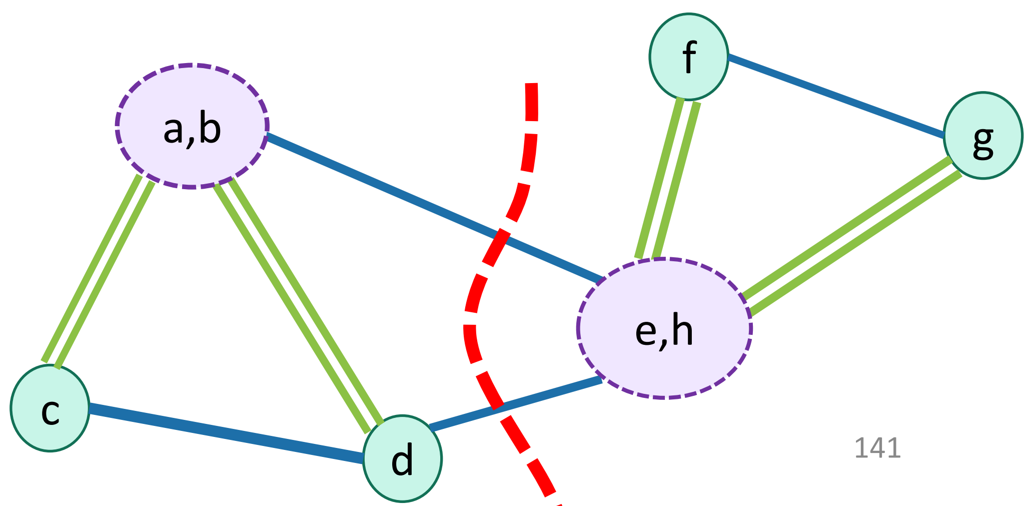
So the probability that we choose one of the k edges crossing S* at step j is at most:

$$\frac{\boldsymbol{k}}{\left(\frac{(\boldsymbol{n}-\boldsymbol{j}+\boldsymbol{1})\boldsymbol{k}}{\boldsymbol{2}}\right)} = \frac{\boldsymbol{2}}{\boldsymbol{n}-\boldsymbol{j}+\boldsymbol{1}}$$



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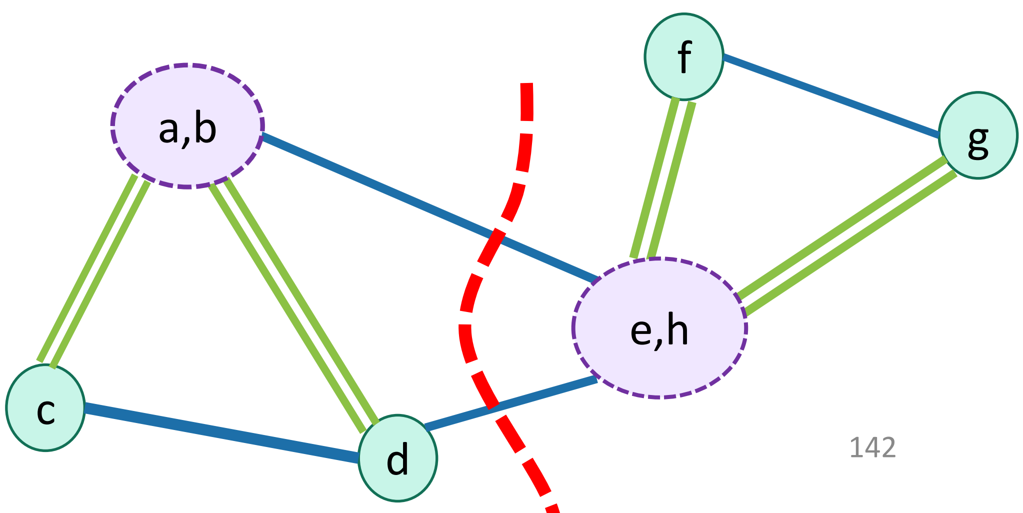

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- So the probability that we choose one of the k edges crossing S* at step j is at most:

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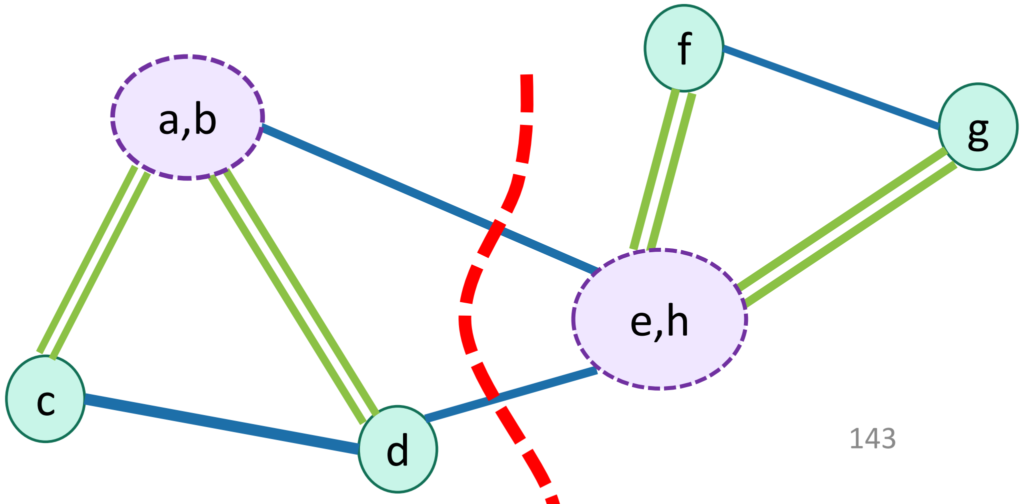
- The probability we **don't** choose one of the k edges is at least:

$$1 - \frac{2}{n-j+1} = \frac{n-j-1}{n-j+1}$$



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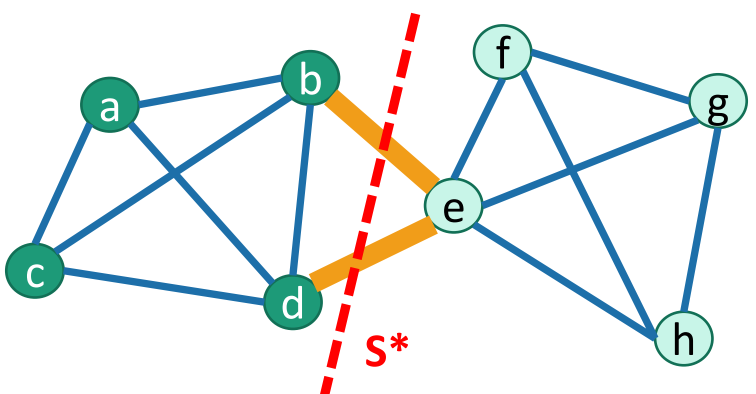

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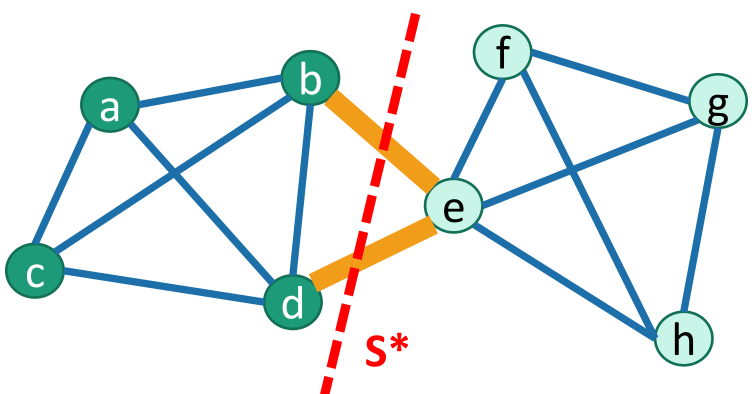

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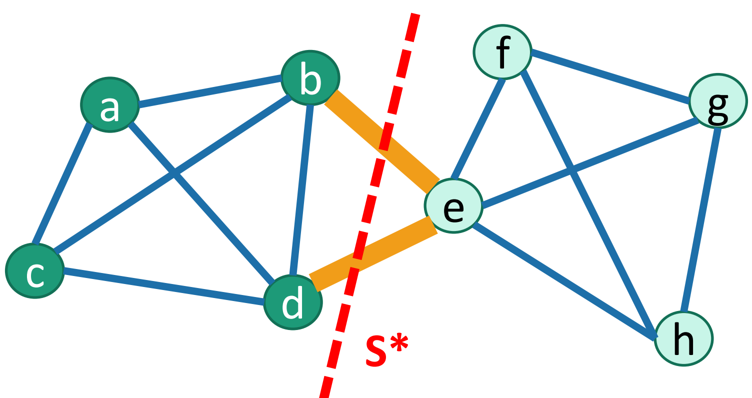

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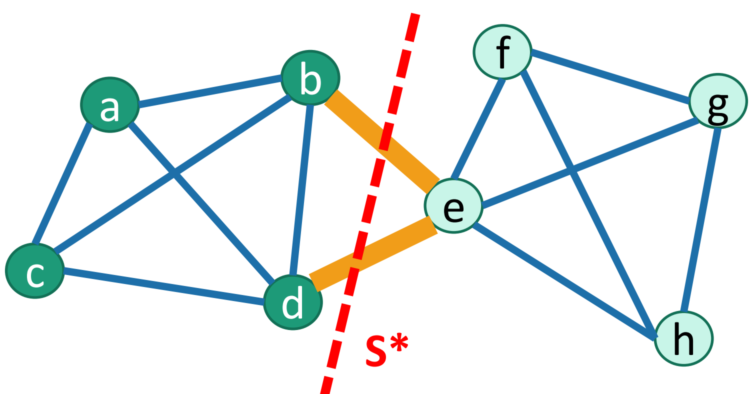

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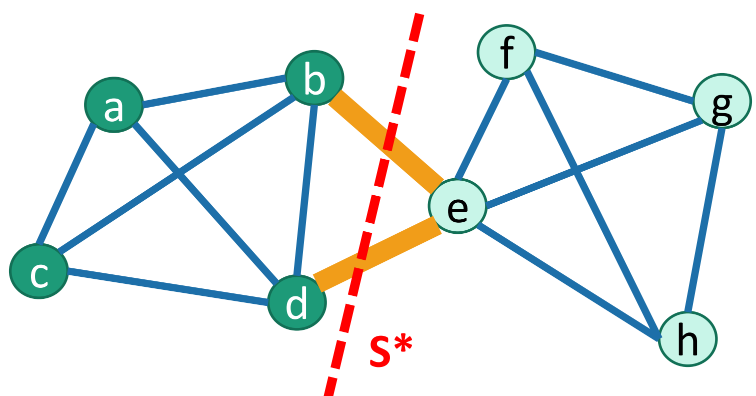

Now let's prove that claim

Say that S* is a minimum cut.

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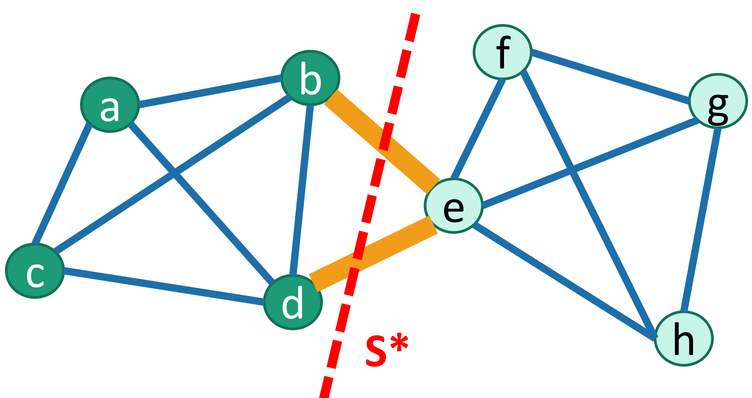

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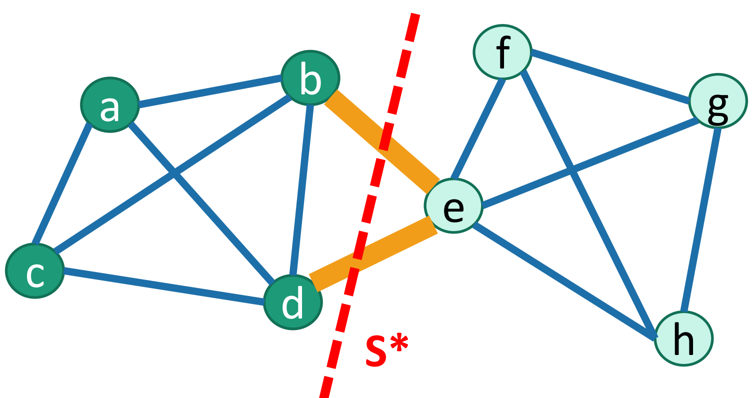
$$= \left(\frac{2}{n(n-1)}\right)$$



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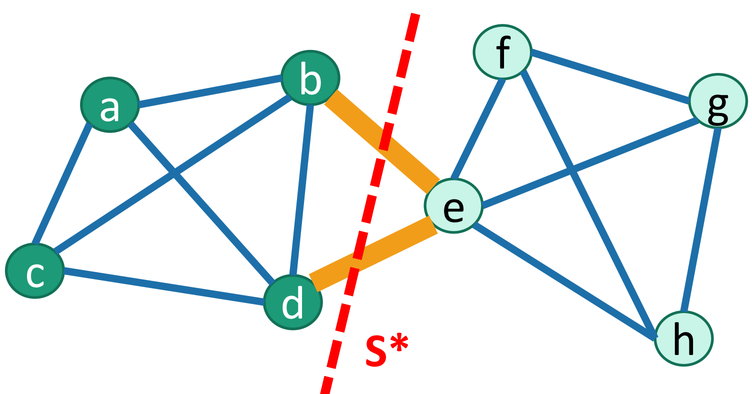

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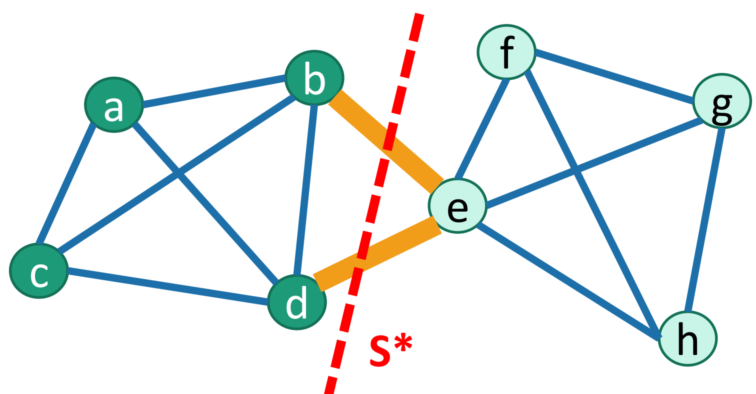
CLAIM PROVED



Theorem

Assuming the claim about $1/\binom{n}{2}$ …

Suppose G has n vertices. Then [repeating Karger's algorithm] finds a min cut in G with probability at least 0.99 in time $O(n^4)$.

That proves this Theorem!

What have we learned?

- If we randomly contract edges:
  - It's unlikely that we'll end up with a min cut.
  - But it's not **TOO** unlikely
  - By repeating, we likely will find a min cut.

Here I chose $\delta = 0.01$ just for concreteness.

- Repeating this process:
  - Finds a **global min cut in time O(n^4), with probability 0.99**.
  - We can run a bit faster if we use a **union-find** data structure.

More generally

- If we have a Monte-Carlo algorithm[1] with a small success probability,
- and we can check how good a solution is,
- Then we can **boost** the success probability by repeating it a bunch and taking the best solution.

[1]In computing, a Monte Carlo algorithm is a randomized algorithm whose output may be incorrect with a certain (typically small) probability.

Can we do better?

- Repeating $O(n^2)$ times is pretty expensive.
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  - $O(\ n^2\log^2(n)\ )$ runtime for the same success probability.
  - **Warning!** This is a tricky algorithm! We'll sketch the approach here: the important part is the high-level idea, not the details of the computations.

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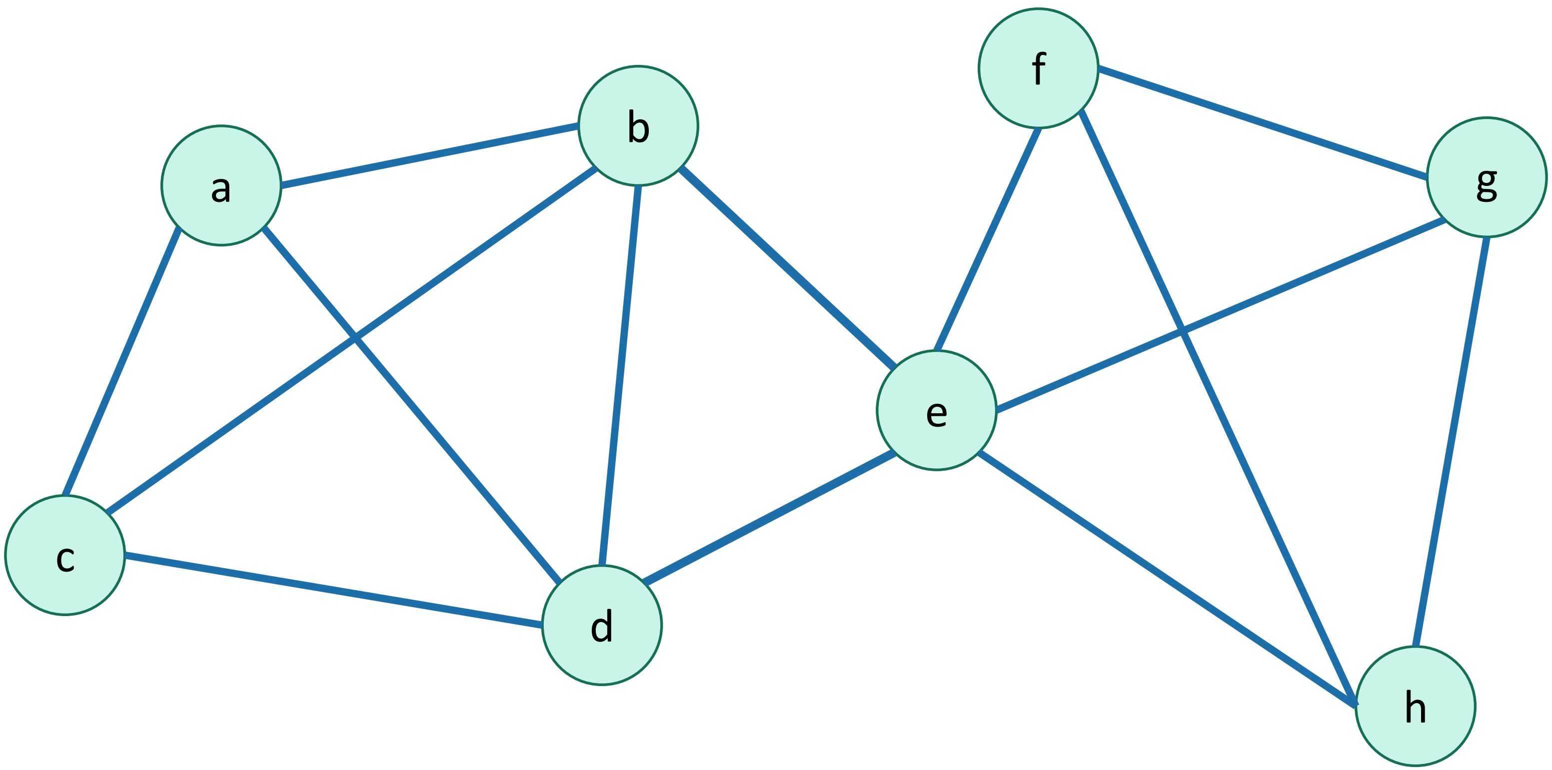
To see how we might save on repetitions, let's run through Karger's algorithm again.

Karger’s algorithm

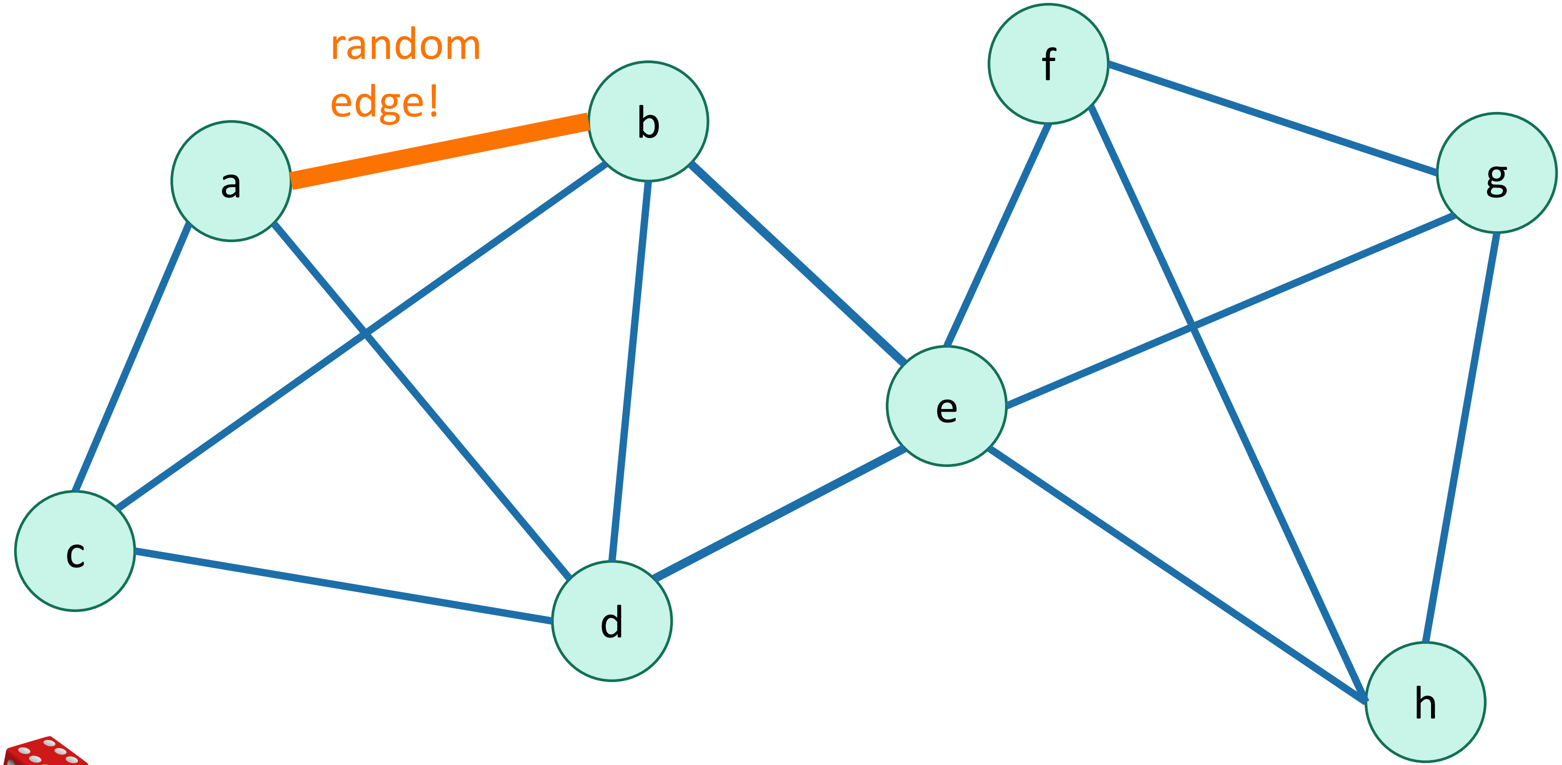


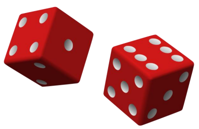
Karger's algorithm

Probability that we didn't mess up:

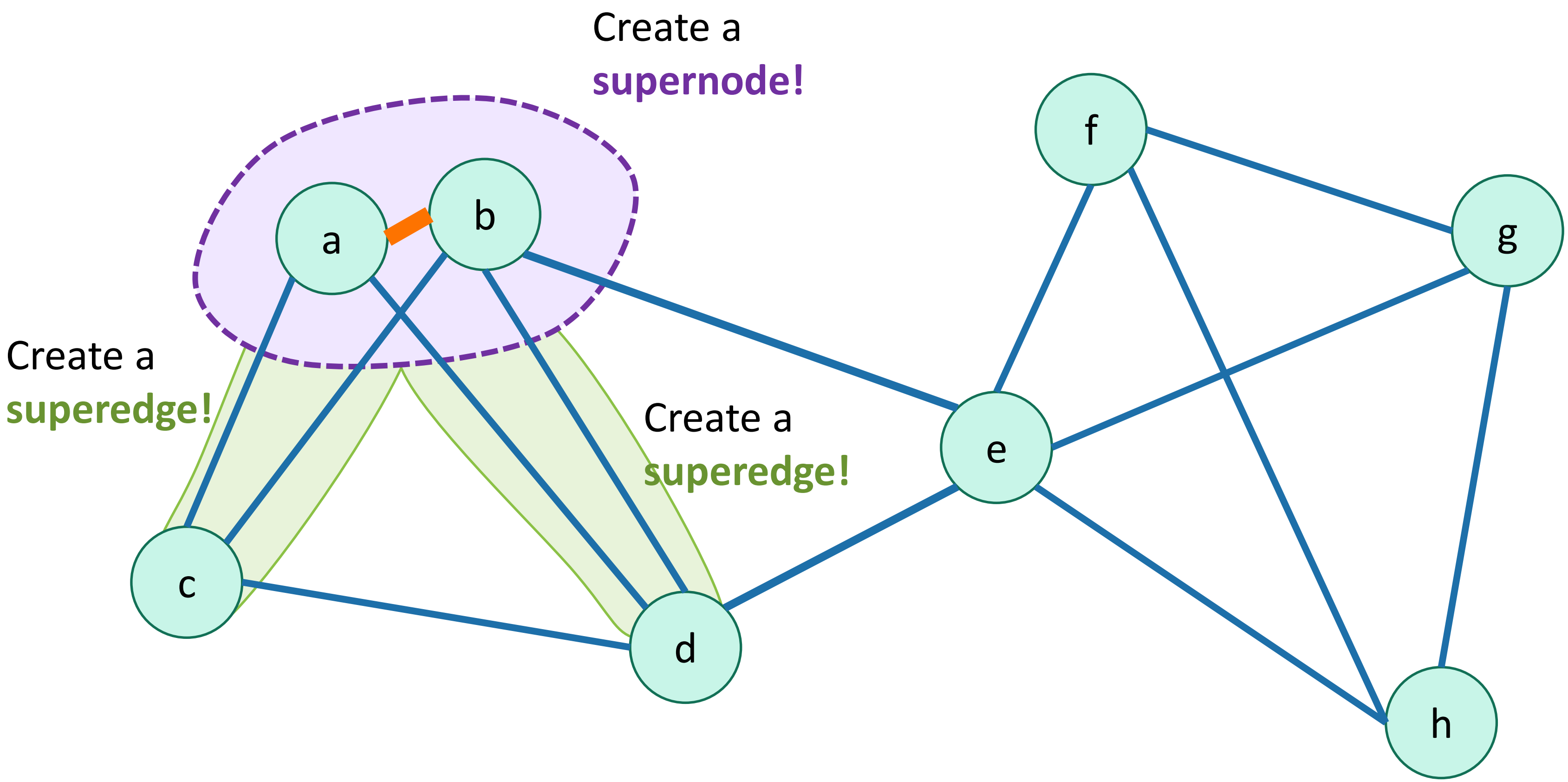
12/14

There are 14 edges, 12 of which are good to contract.

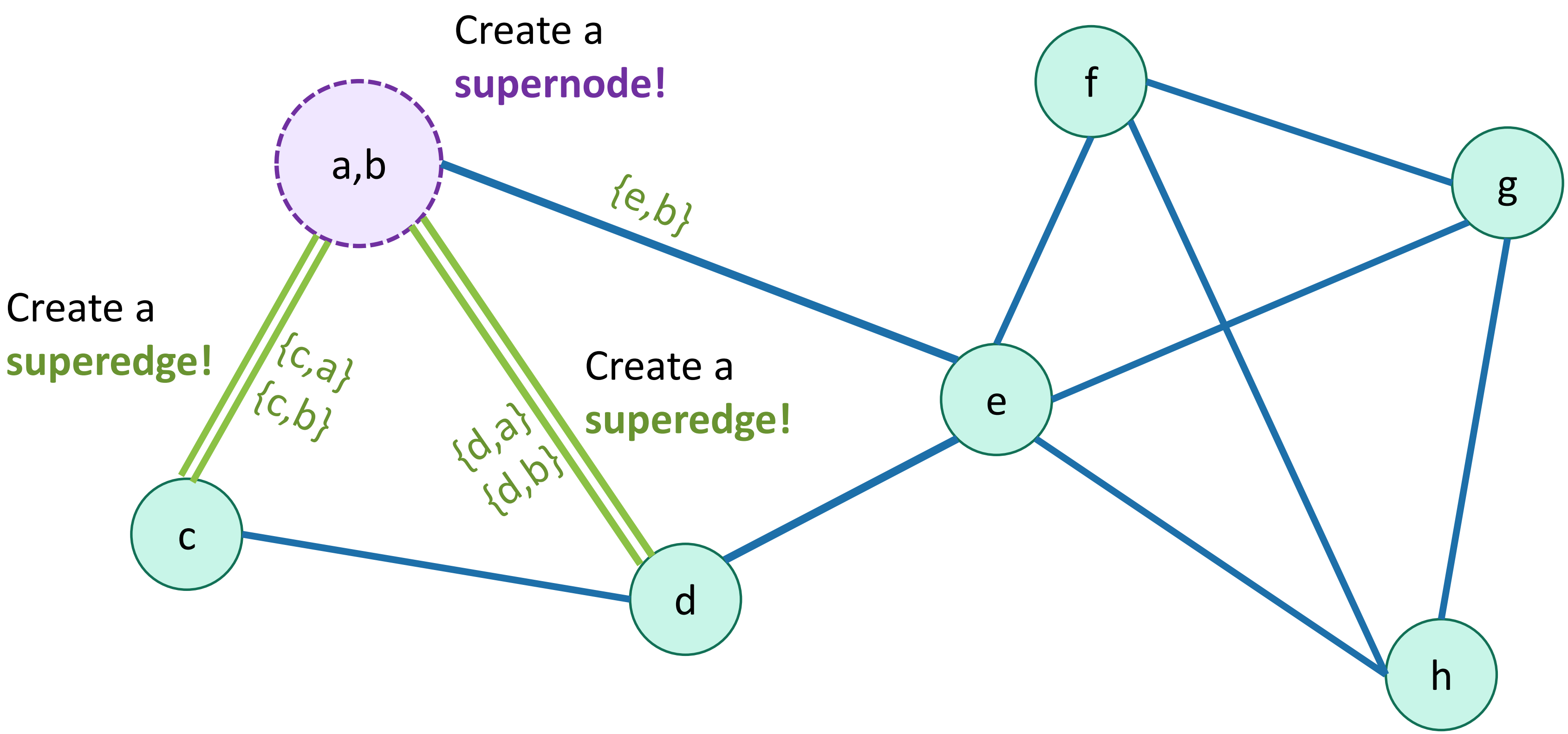




Karger’s algorithm



Karger’s algorithm

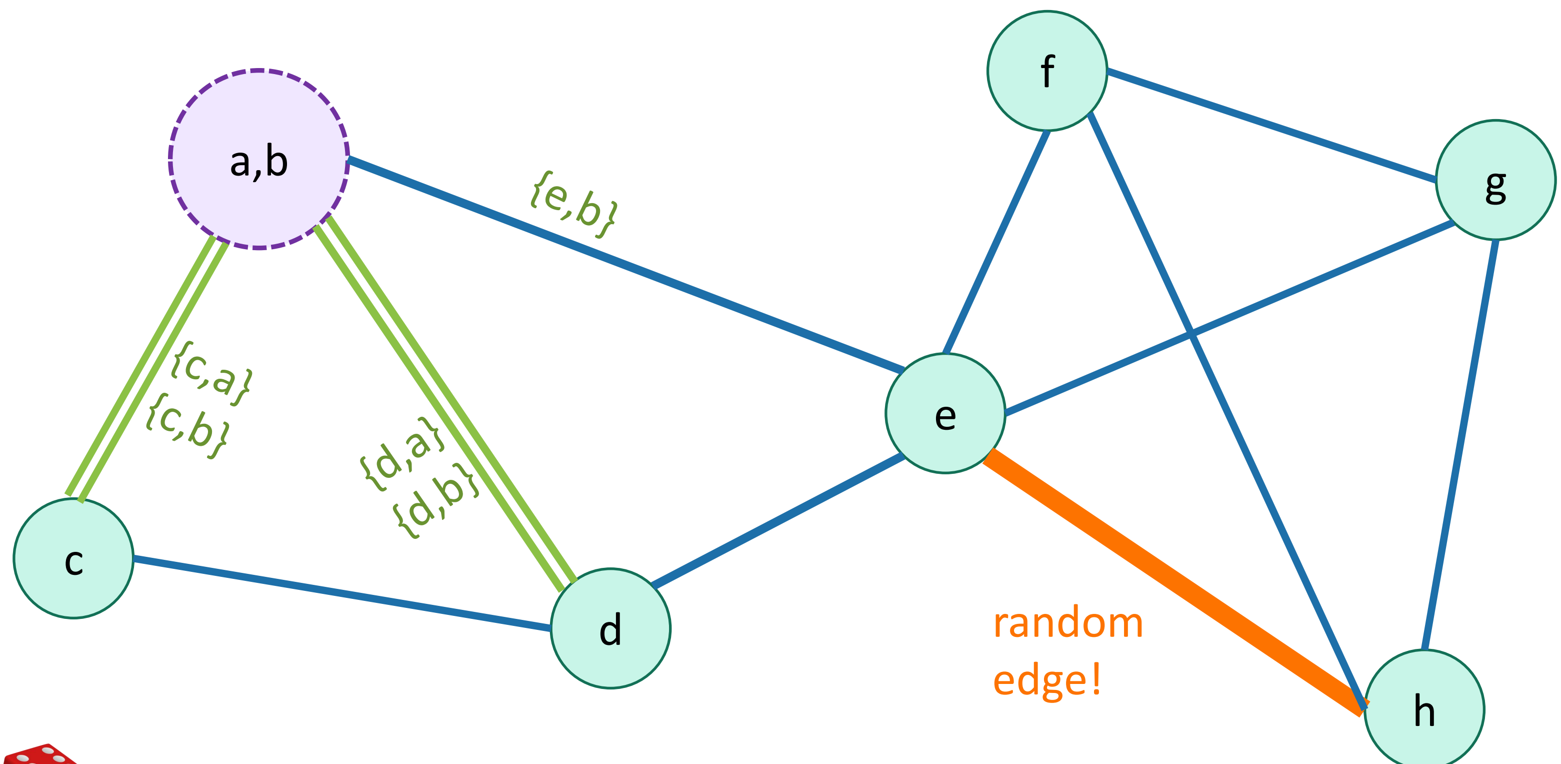


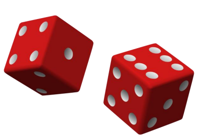
Karger's algorithm

Probability that we didn't mess up:

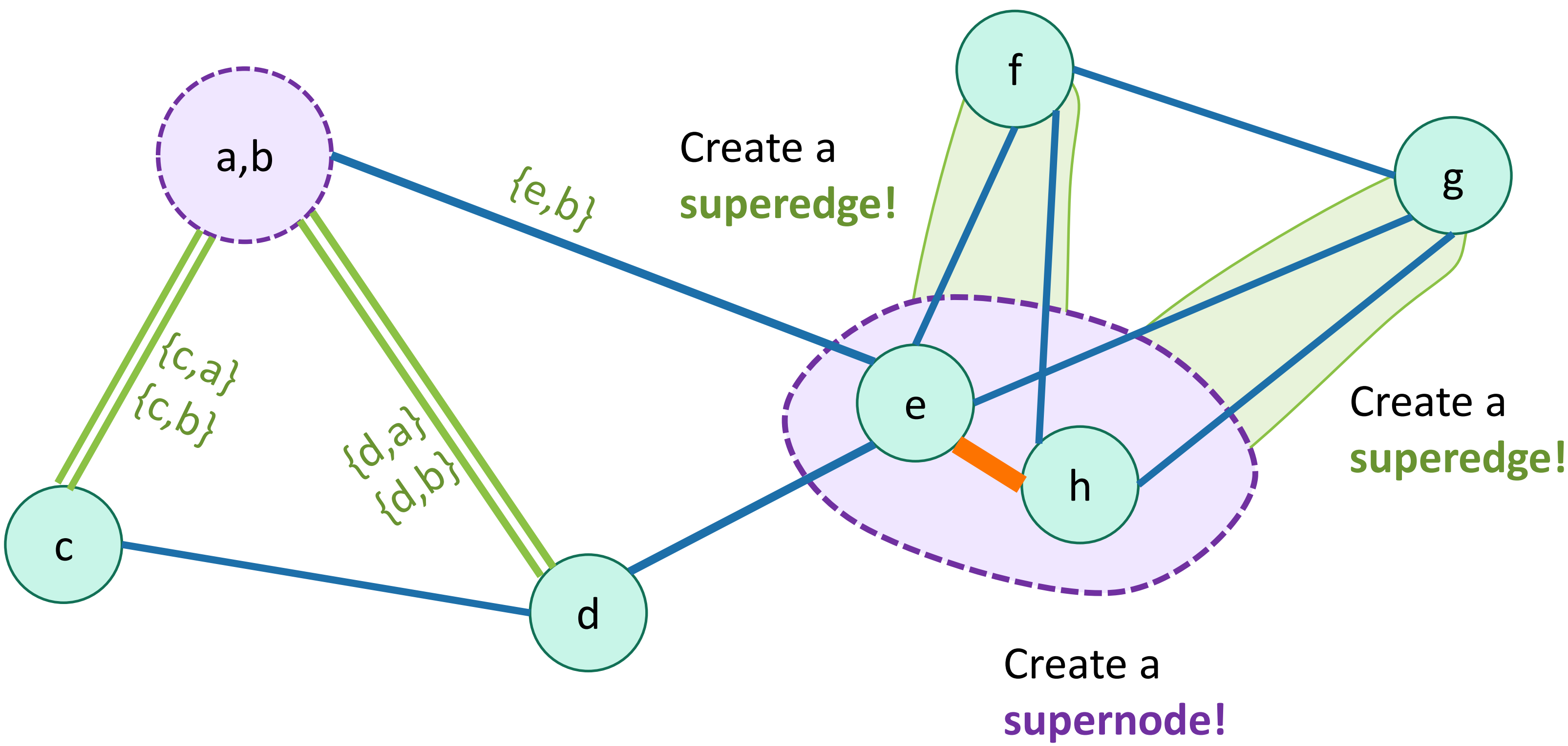
11/13

Now there are only 13 edges, since the edge between a and b disappeared.

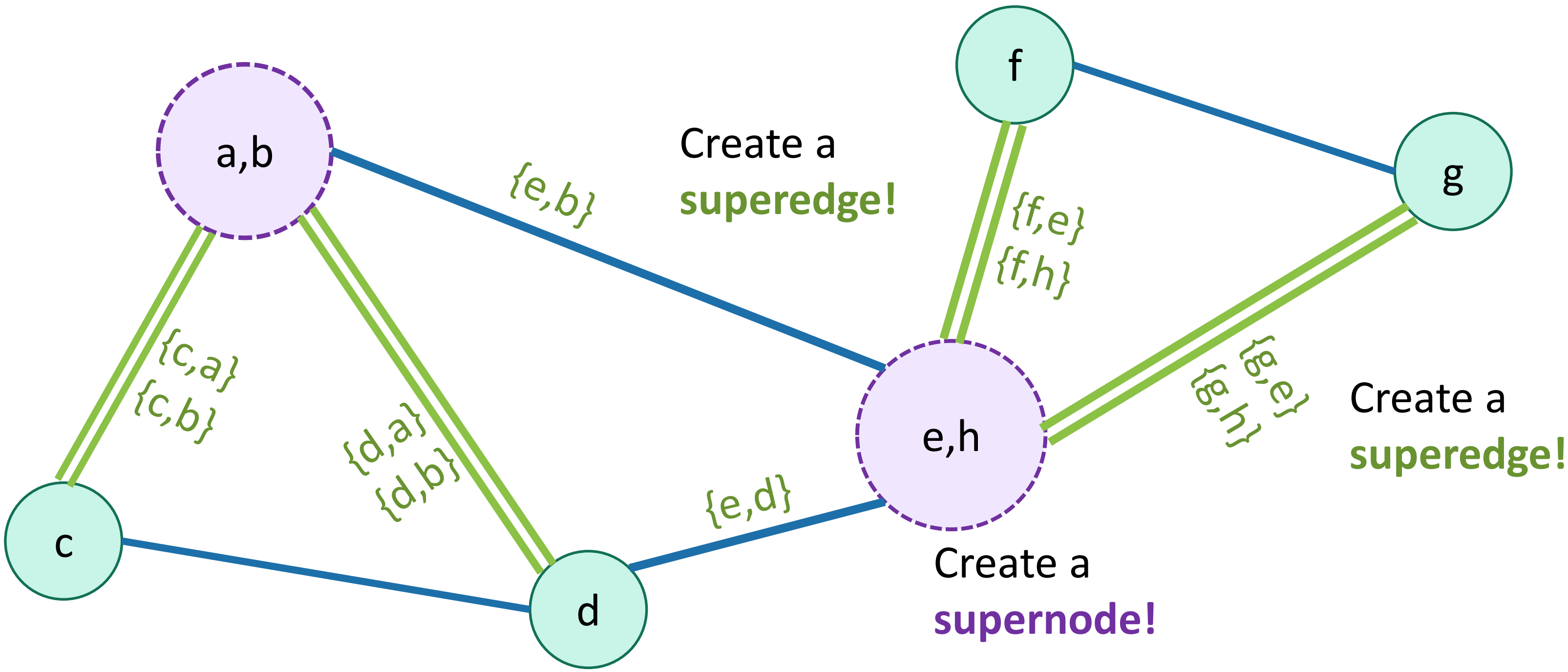




Karger's algorithm



Karger's algorithm

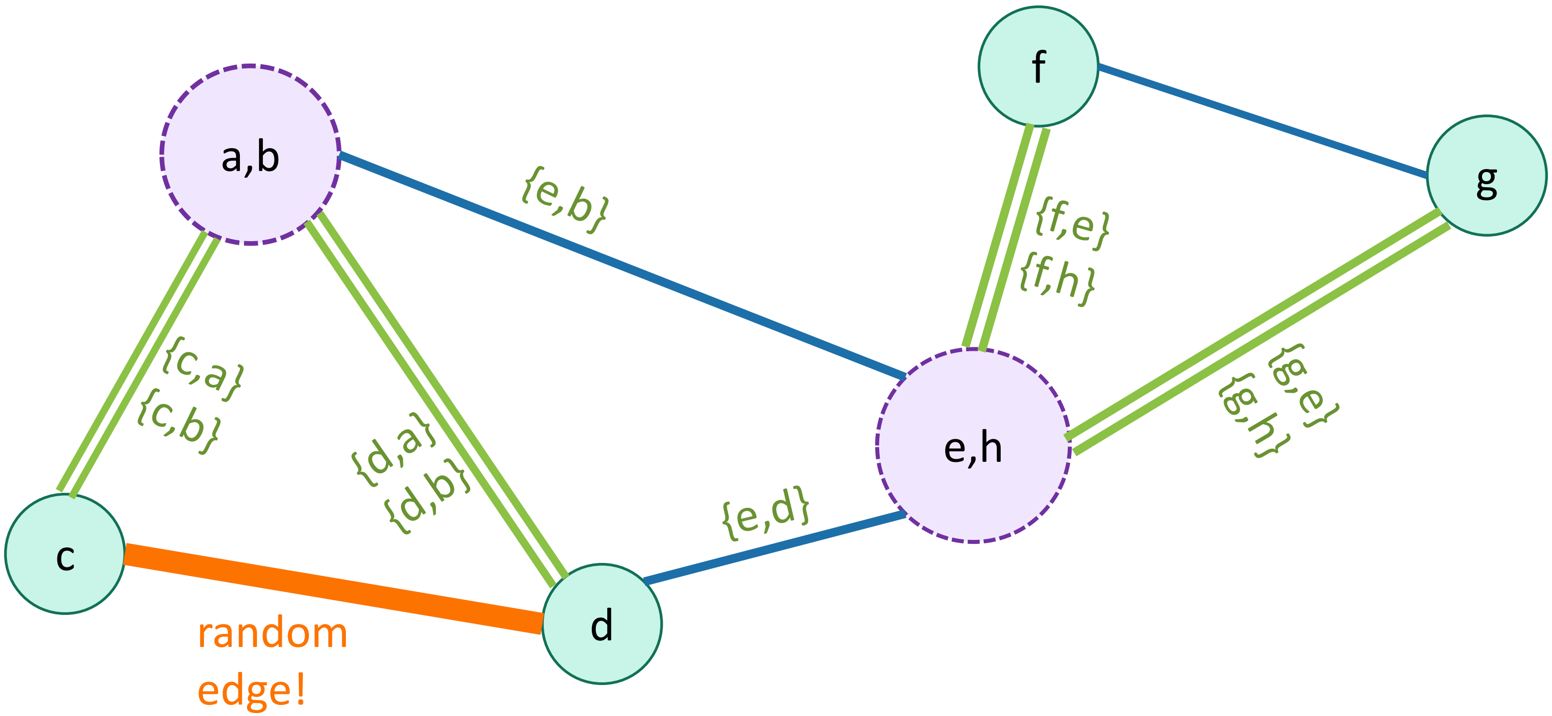


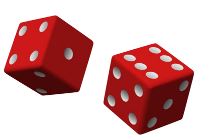
Karger's algorithm

Probability that we didn't mess up:

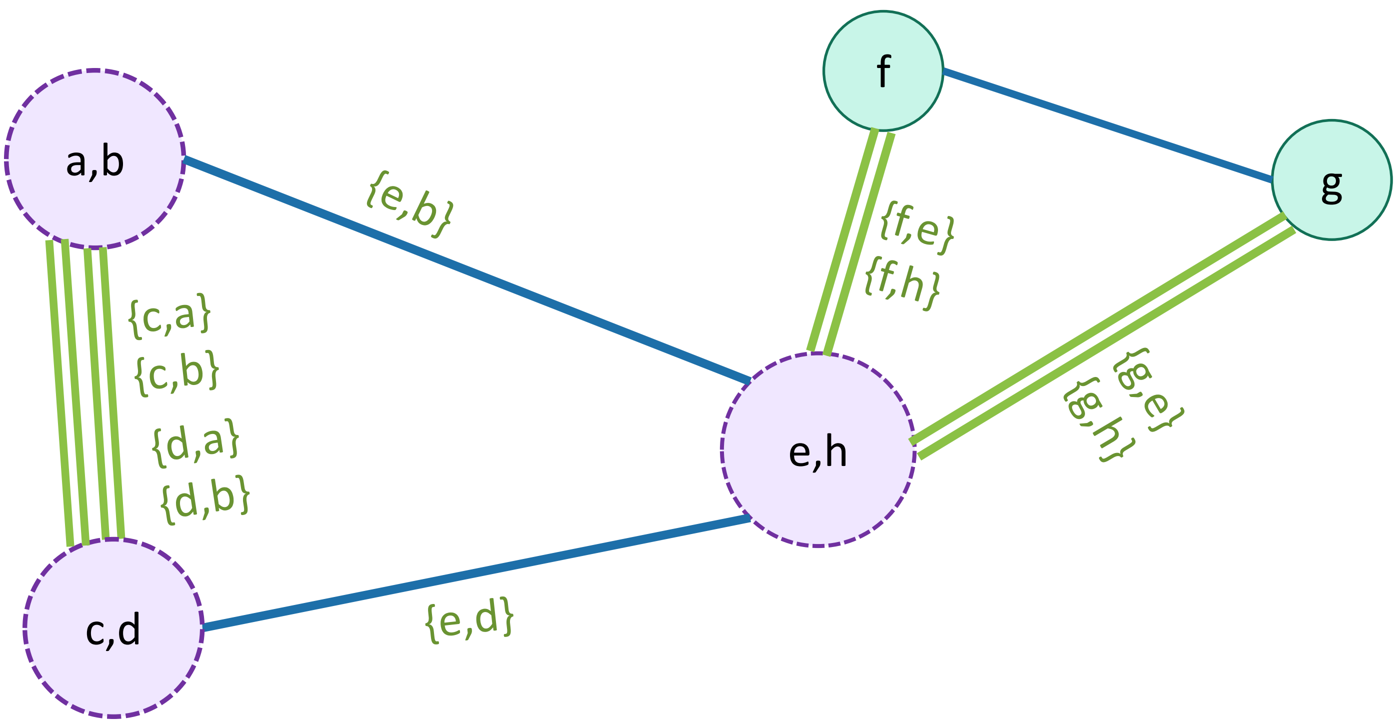
10/12

Now there are only 12 edges, since the edge between e and h disappeared.





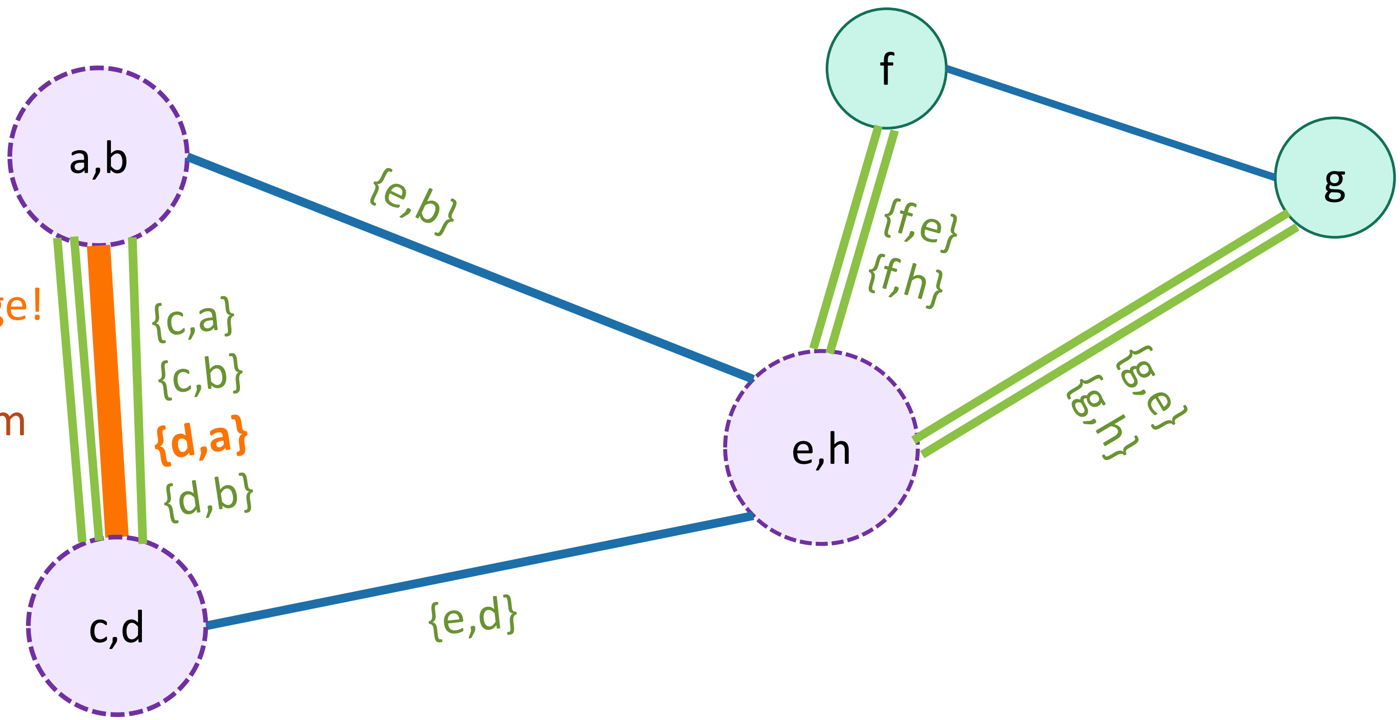
Karger’s algorithm



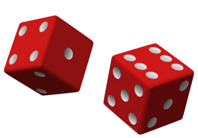
Karger’s algorithm

Probability that we didn’t mess up:

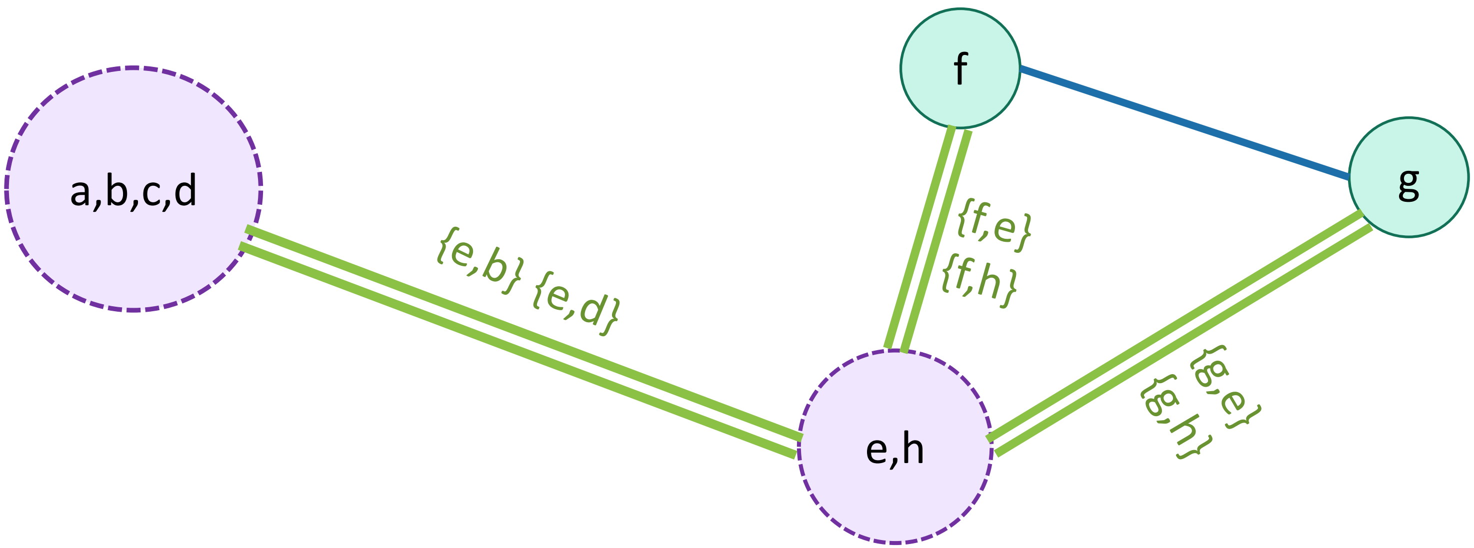
9/11



random edge!
(We pick at random from the original edges).



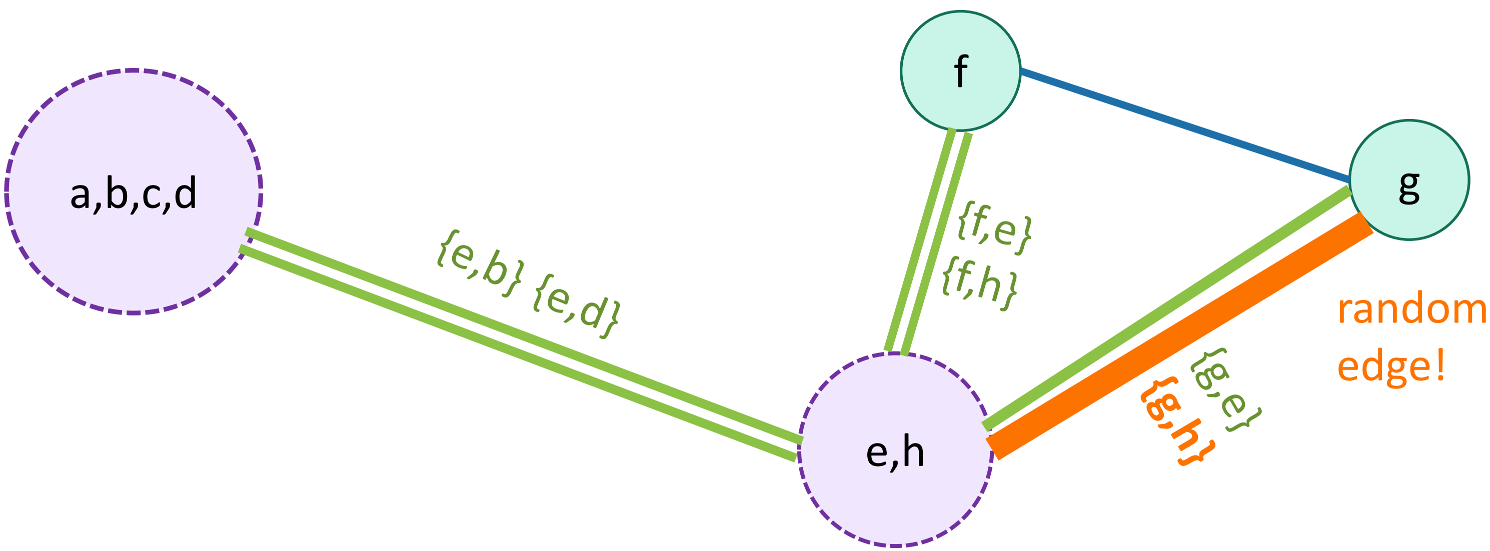
Karger's algorithm

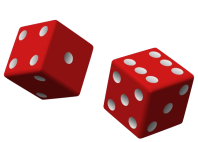


Karger's algorithm

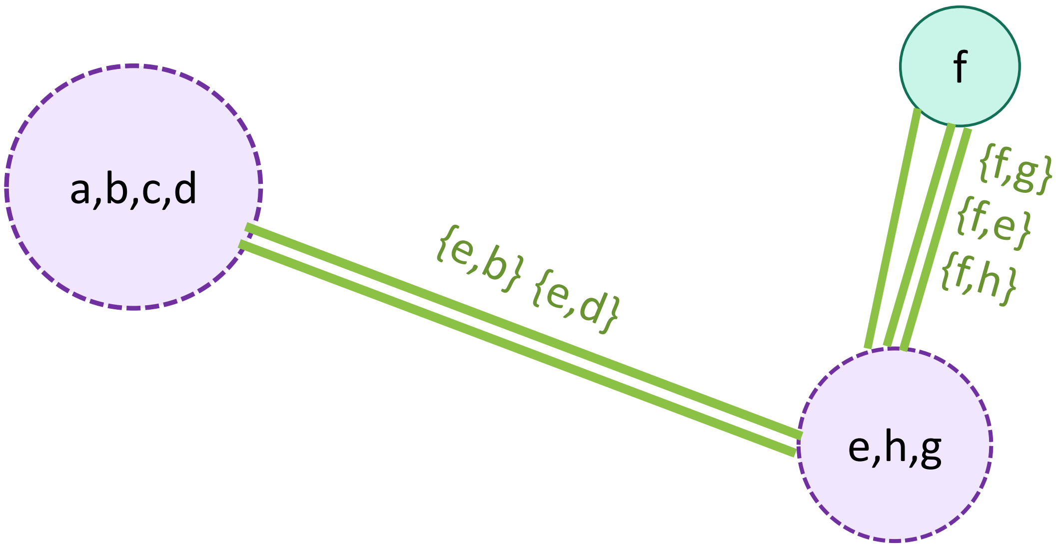
Probability that we didn't mess up:

5/7





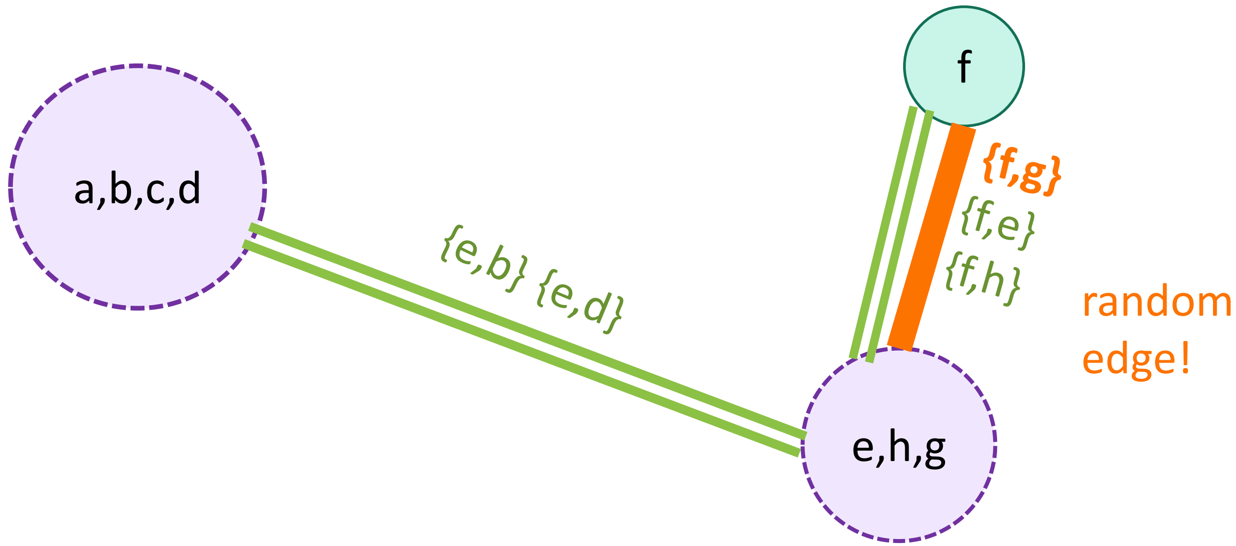
Karger's algorithm



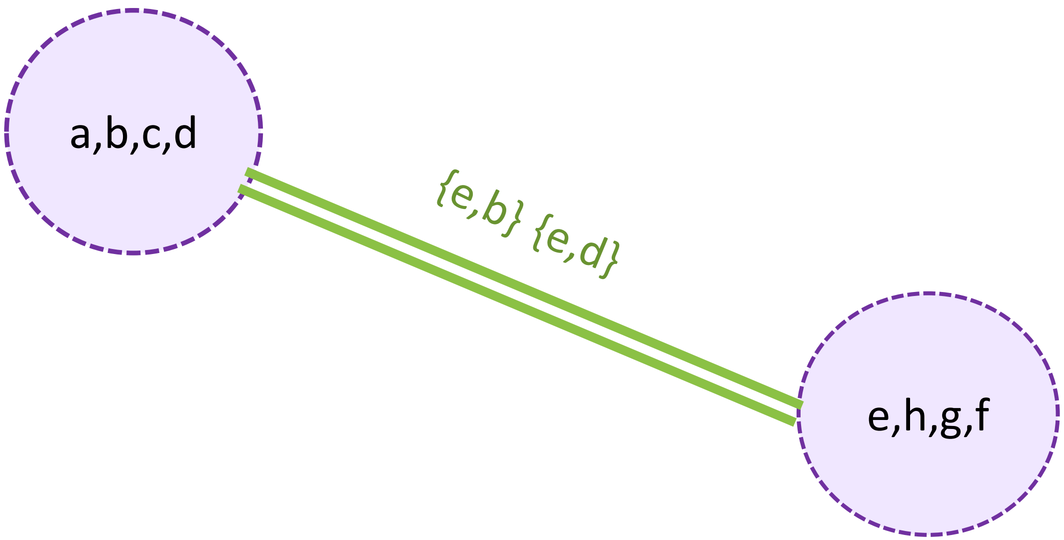
Karger's algorithm

Probability that we didn't mess up:

3/5



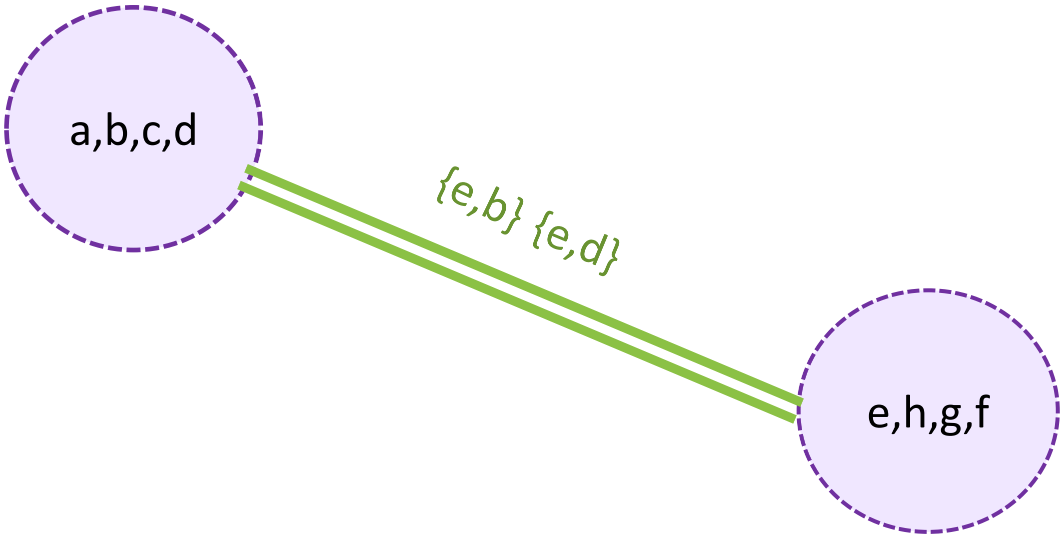
Karger's algorithm



Karger's algorithm

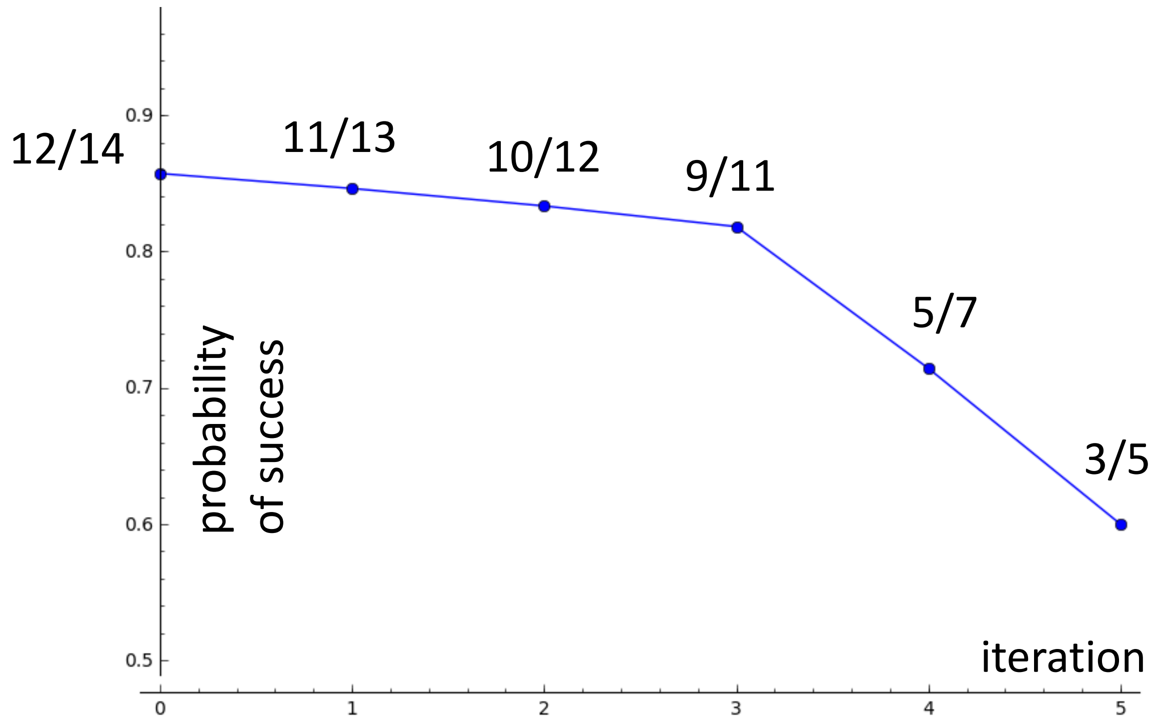
Now stop!

- There are only two nodes left.



Probability of not messing up

- At the beginning, it's pretty likely we'll be fine.
- The probability that we mess up gets worse and worse over time.



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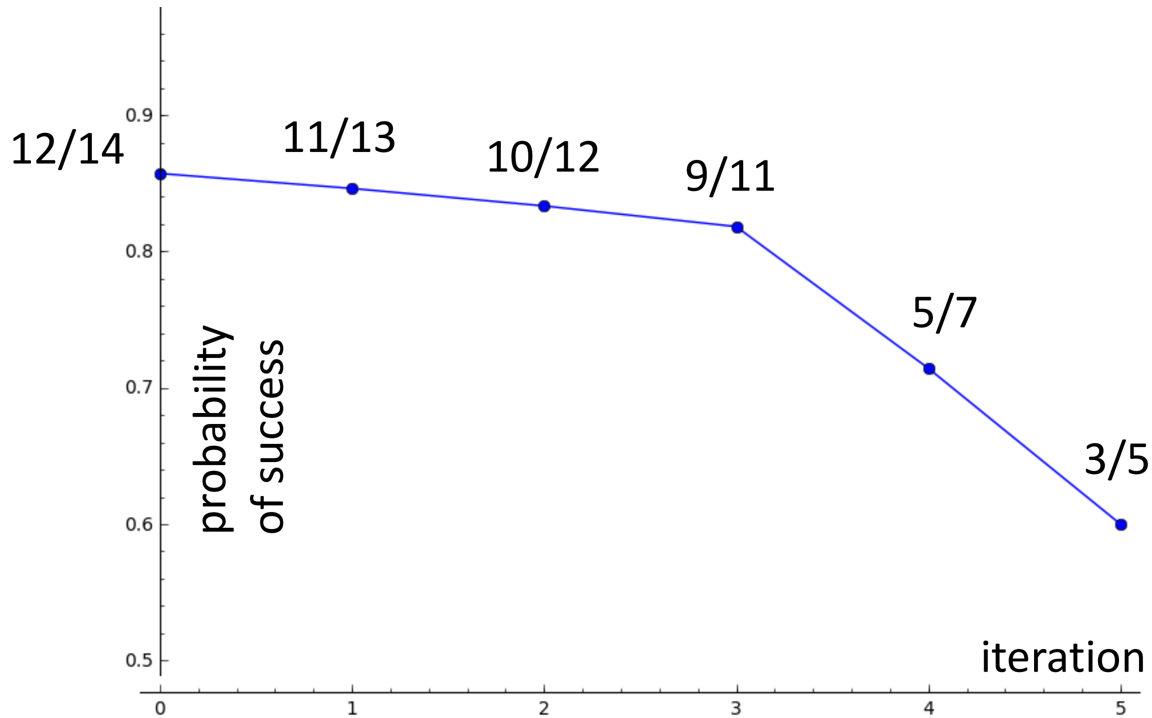
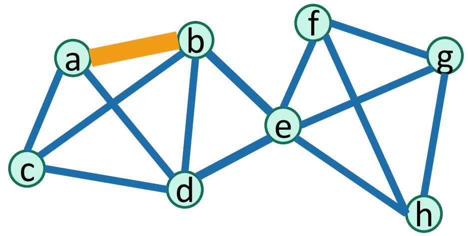


Moral:

Repeating the stuff from the beginning of the algorithm is **wasteful**!

Instead...



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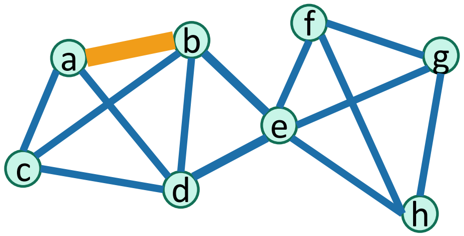
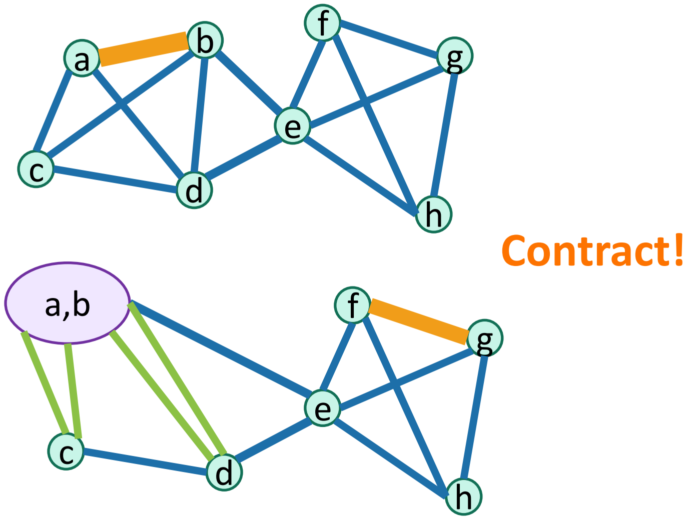


Contract!

Instead…



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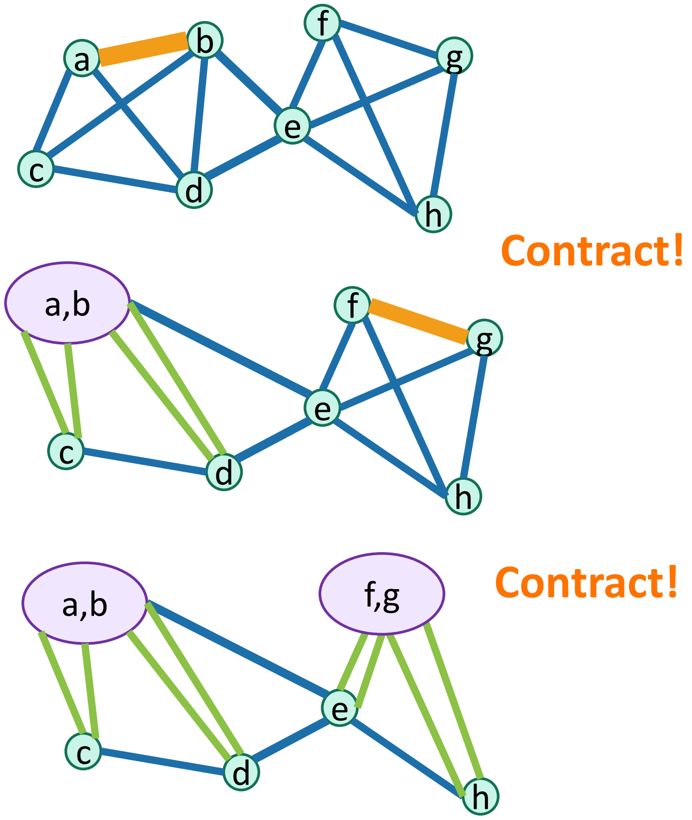

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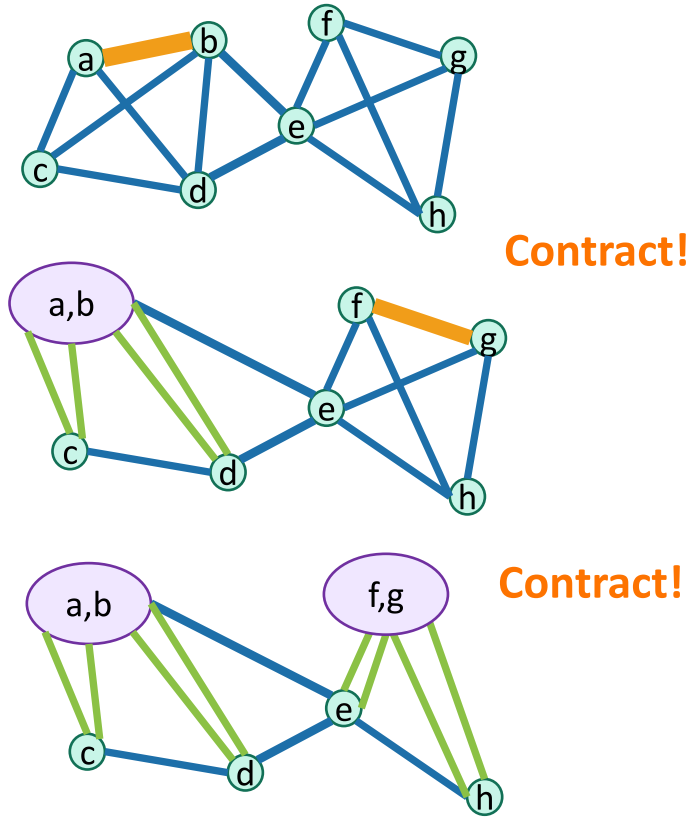
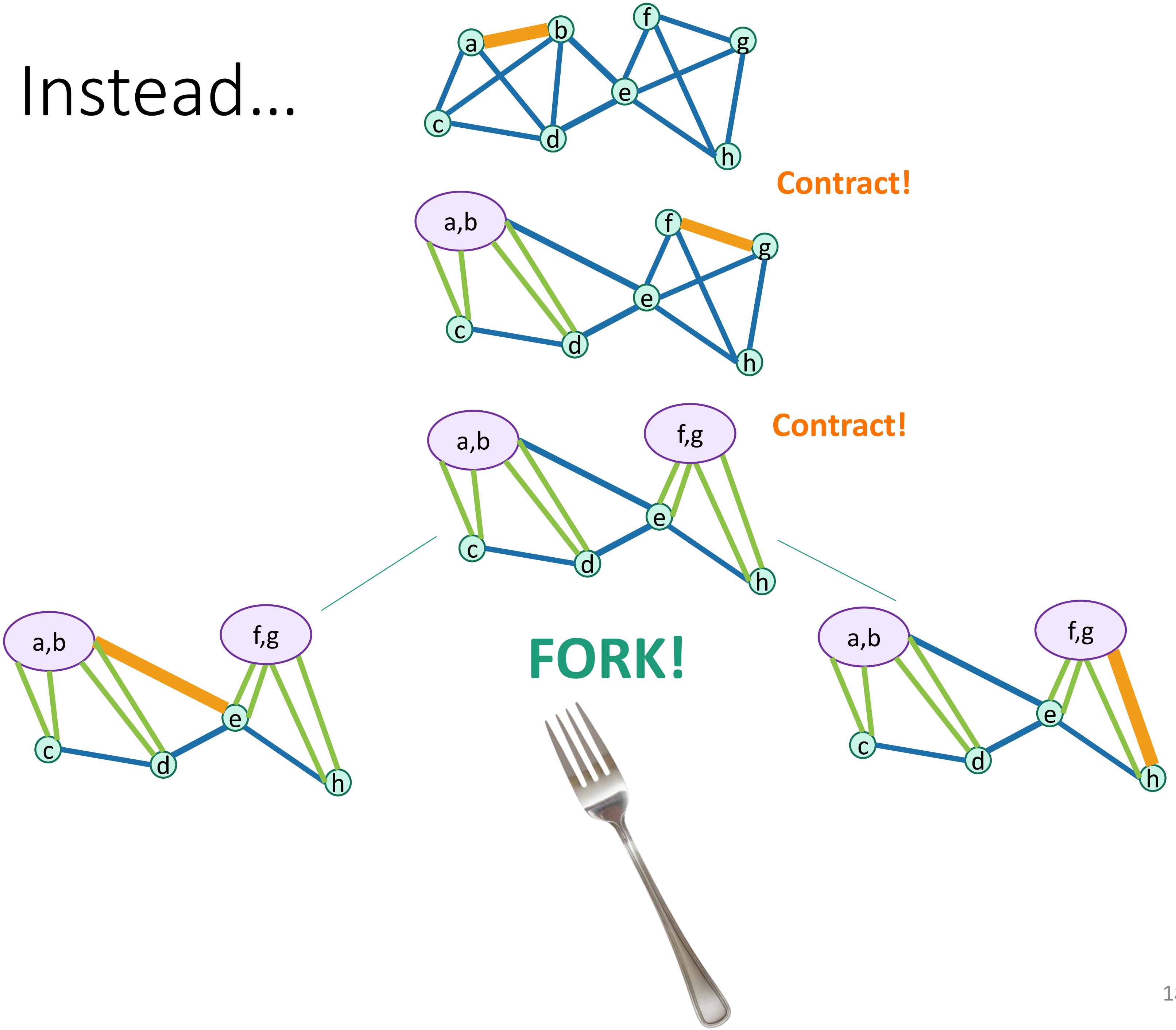


FORK!

Instead…



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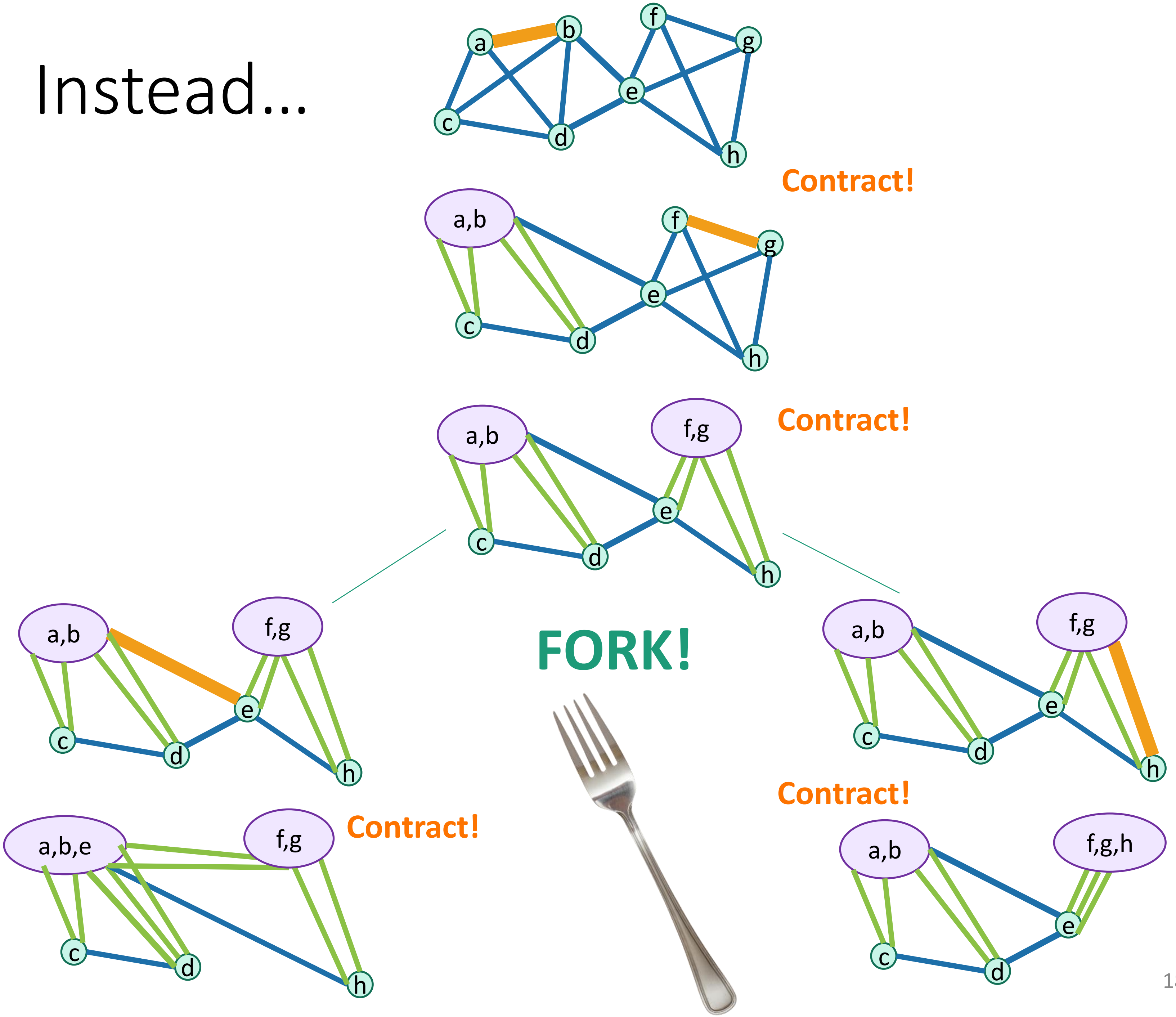
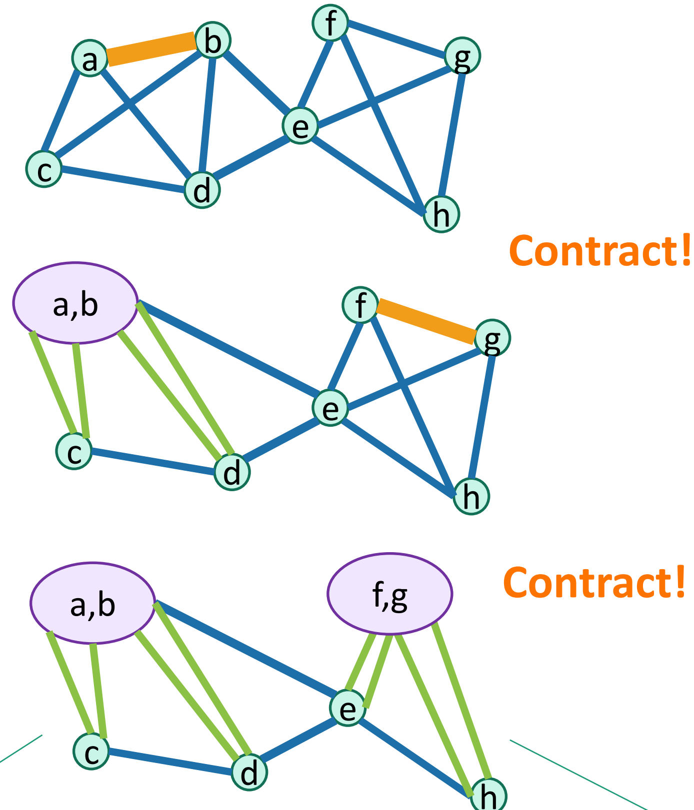
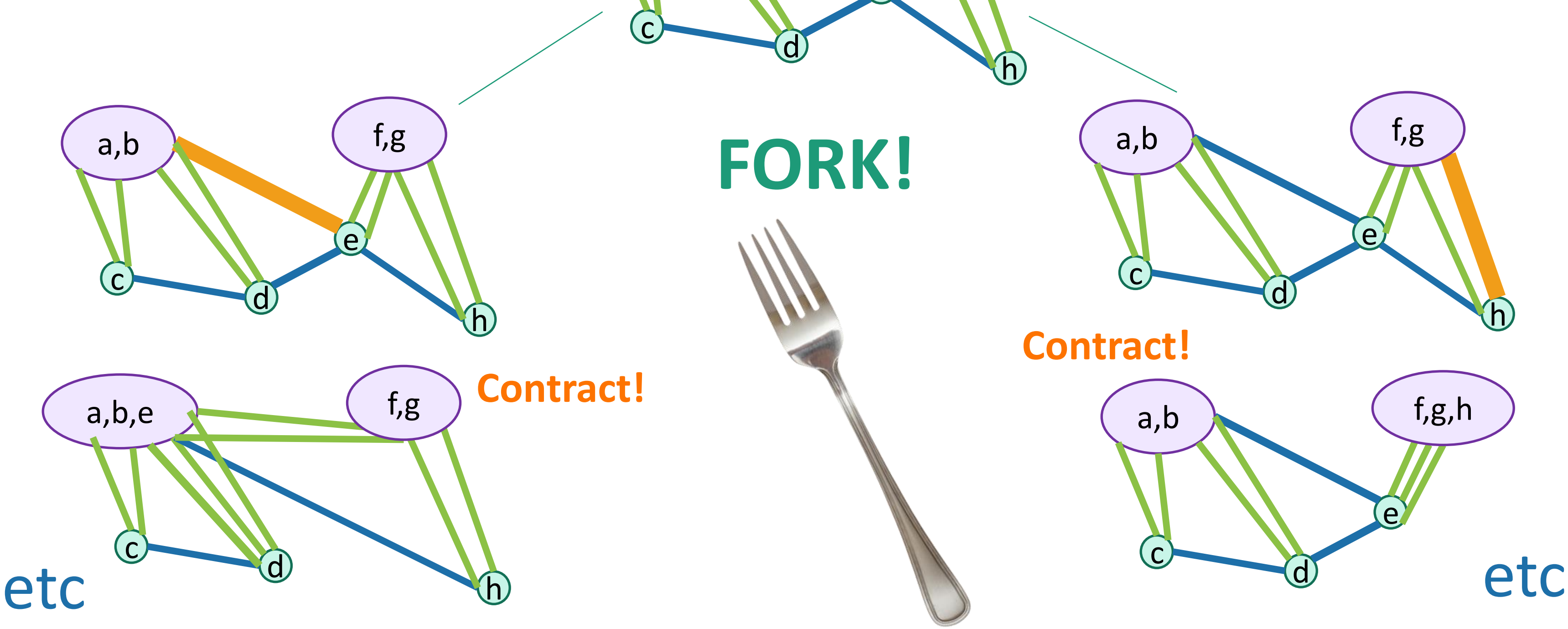


a
b
c
d
e
f
g
h
Contract!
a,b
Contract!
f,g
FORK!
a,b,e
Contract!
f,g,h
Contract!

Instead…

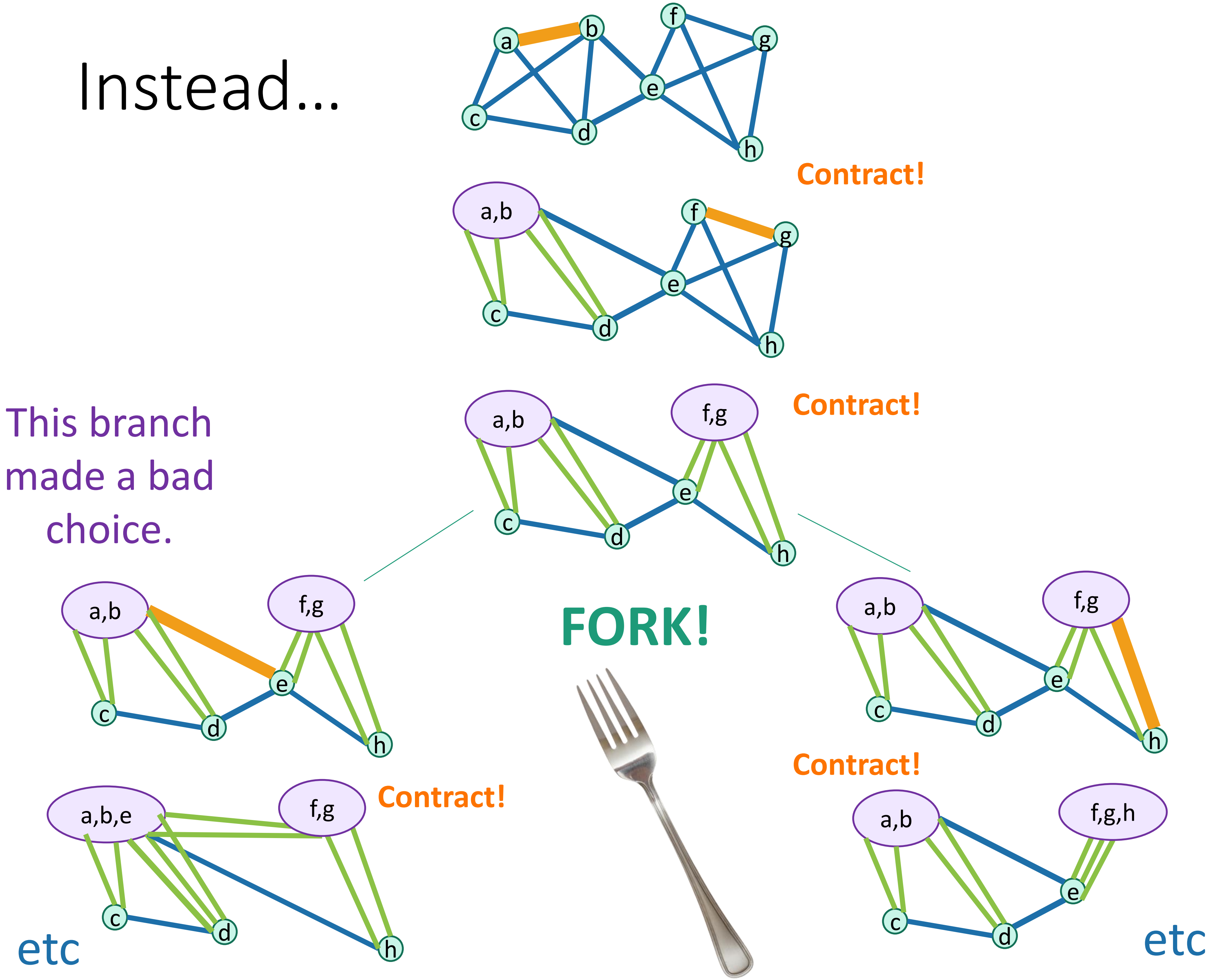


a
b
c
d
e
f
g
h
Contract!
a,b
Contract!
f,g

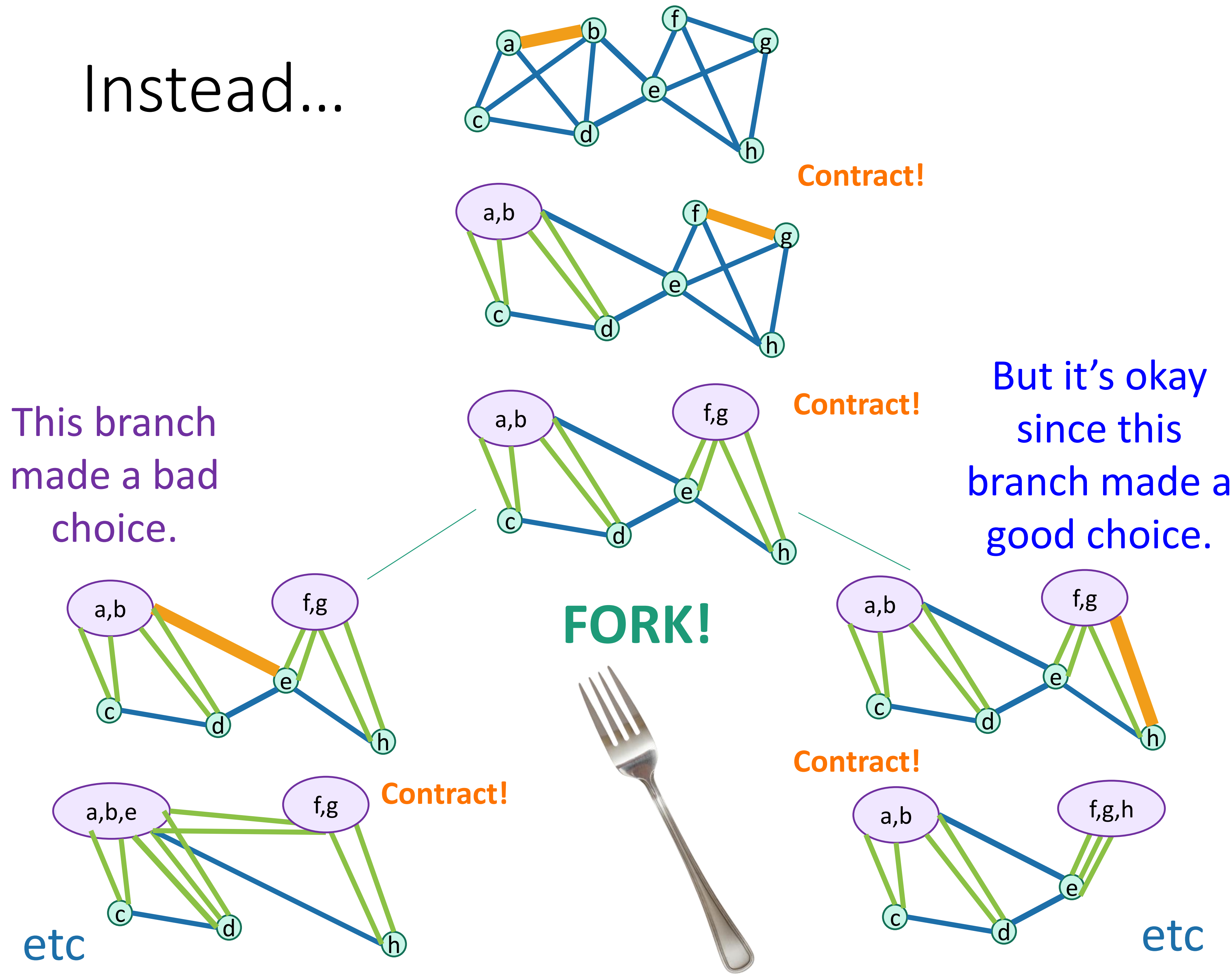


FORK!
a,b
f,g
a,b,e
Contract!
f,g,h
Contract!
etc
etc

Instead…



a
b
c
d
e
f
g
h
Contract!
a,b
Contract!
f,g
This branch made a bad choice.
FORK!
a,b,e
Contract!
f,g,h
Contract!
etc
etc



Instead…
Contract!
Contract!
This branch made a bad choice.
But it's okay since this branch made a good choice.
FORK!
Contract!
Contract!
etc
etc
a
b
c
d
e
f
g
h
a,b
f,g
a,b,e
f,g,h

In words

- Run Karger's algorithm on G for a bit.
  - Until there are $\frac{\mathrm{n}}{\sqrt{2}}$ supernodes left.

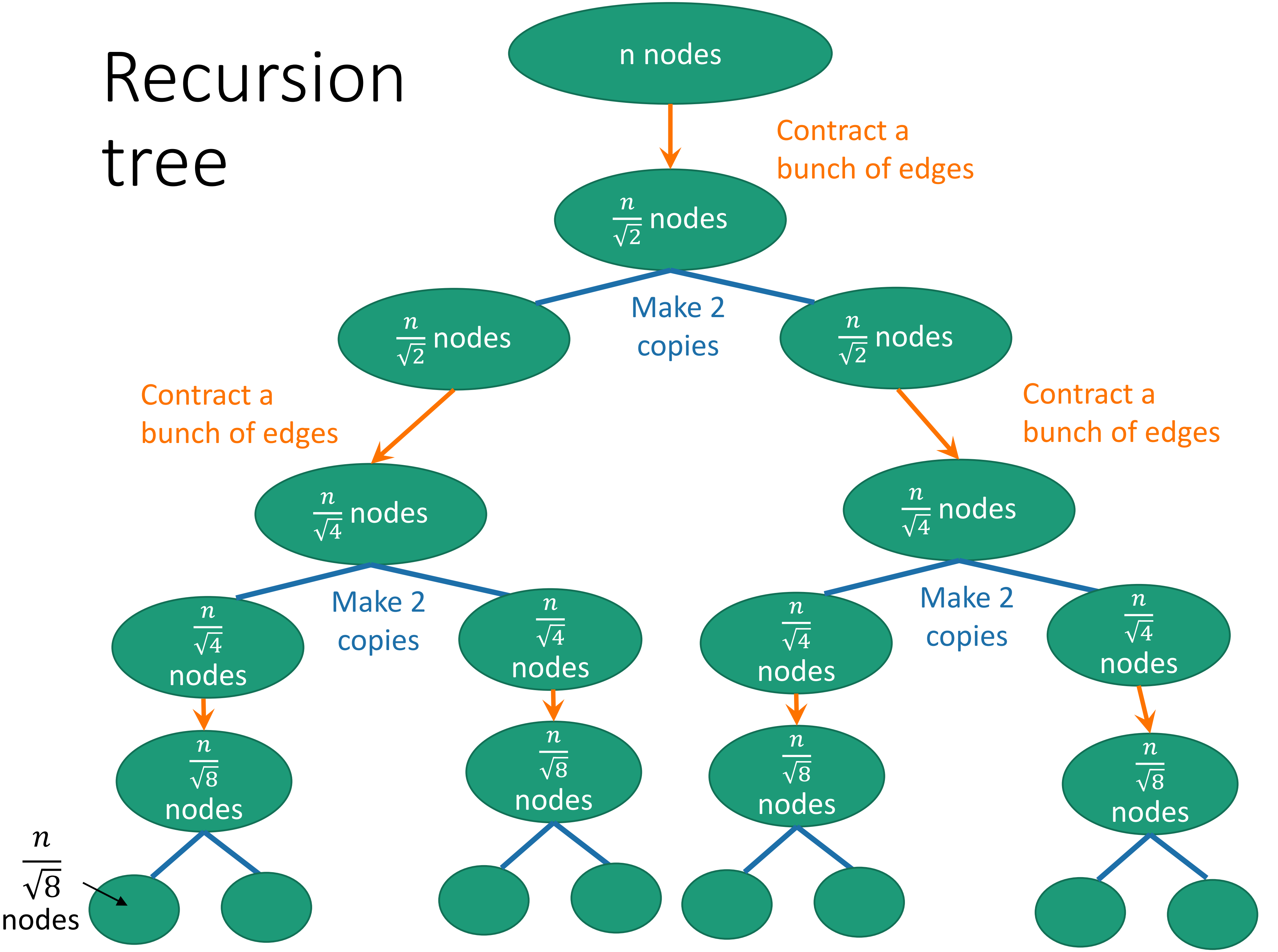
Why $\frac{\mathrm{n}}{\sqrt{2}}$? We'll see later.

- Then split into two independent copies, G_1 and G_2
- Run Karger's algorithm on each of those for a bit.
  - Until there are $\frac{\left(\frac{\mathrm{n}}{\sqrt{2}}\right)}{\sqrt{2}} = \frac{n}{2}$ supernodes left in each.
- Then split each of those into two independent copies…

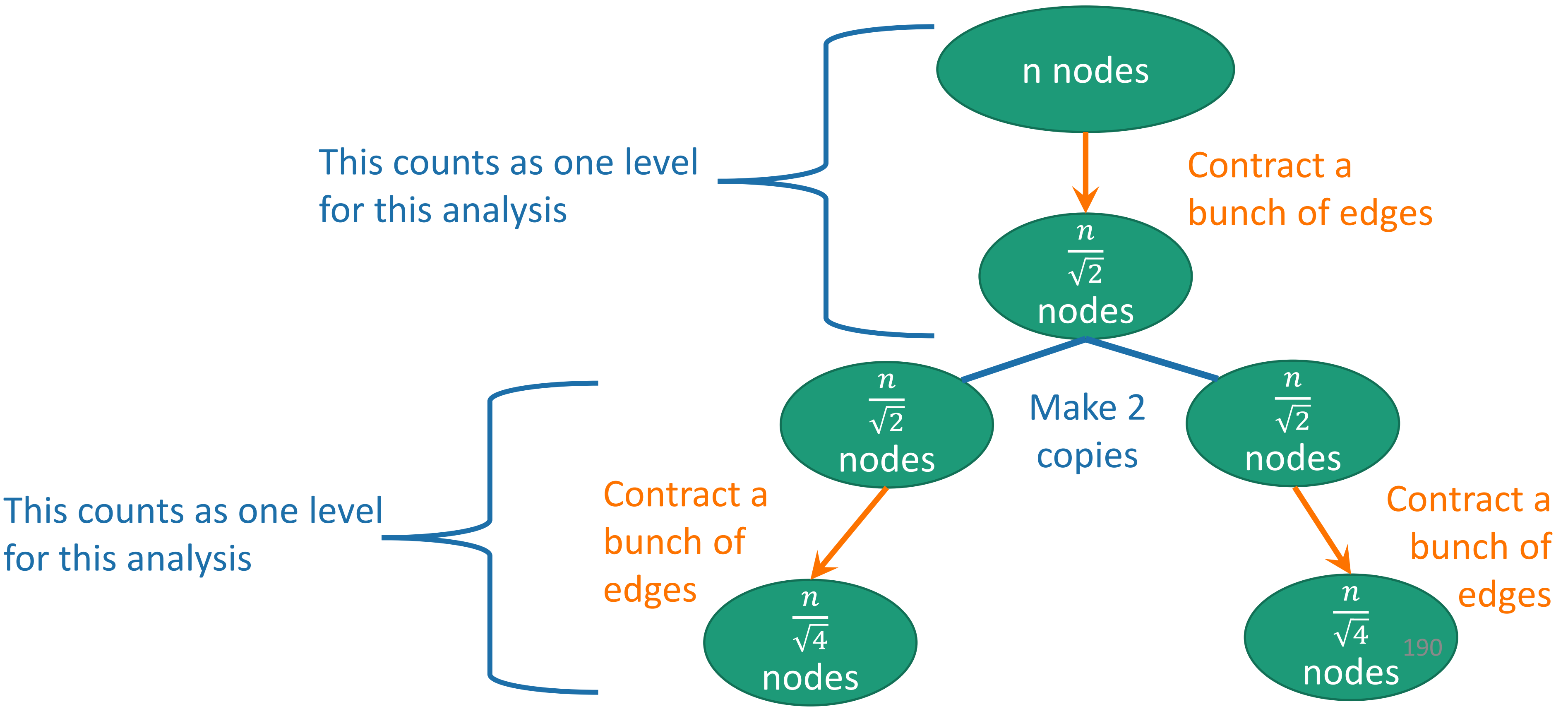
In pseudocode

- **KargerStein**(G = (V,E)):
  - n ← |V|
  - if n < 4:
    - find a min-cut by brute force \\ time O(1)
  - Run Karger's algorithm on G with independent repetitions until $\left\lfloor \frac{n}{\sqrt{2}} \right\rfloor$ nodes remain.
  - G_1, G_2 ← copies of what's left of G
  - S_1 = **KargerStein**(G_1)
  - S_2 = **KargerStein**(G_2)
  - **return** whichever of S_1, S_2 is the smaller cut.

Recursion tree

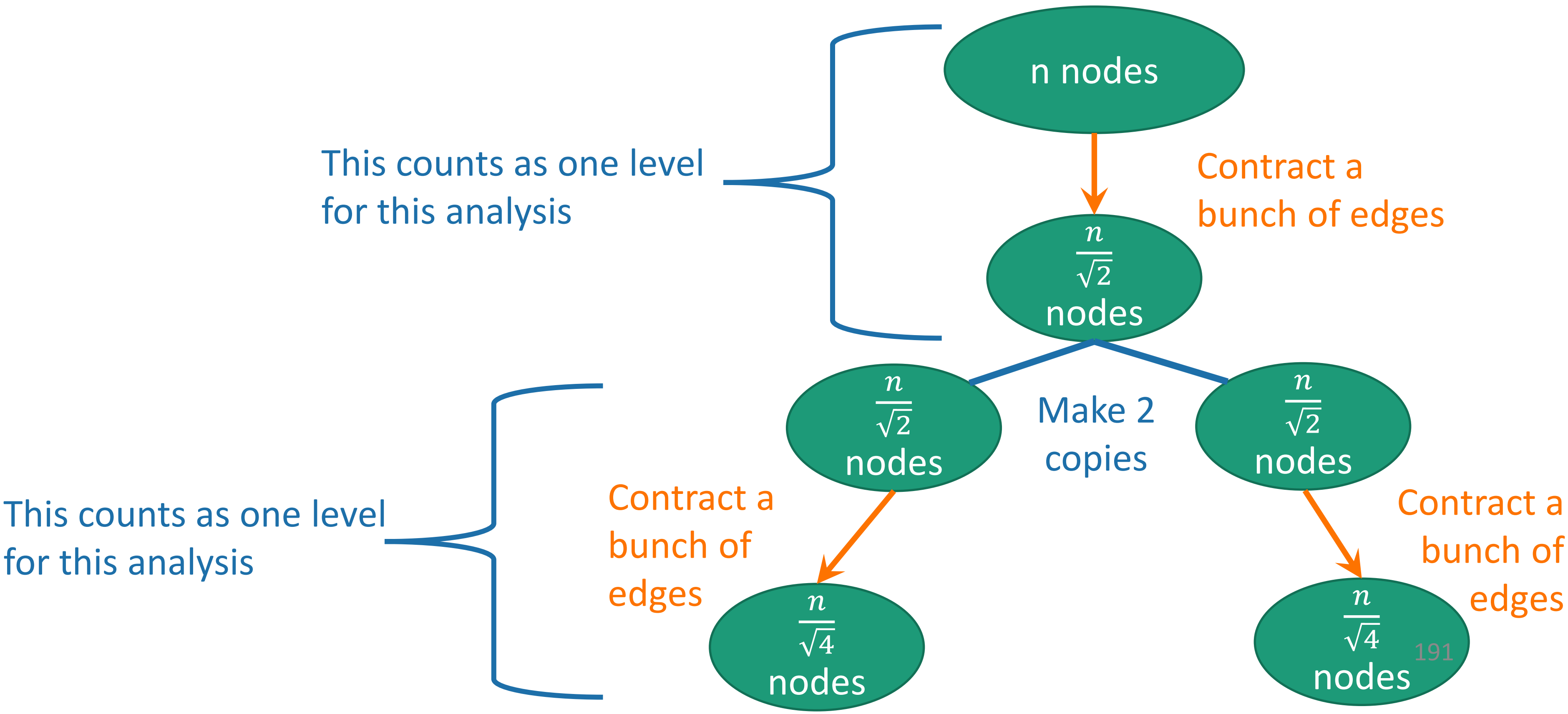


Recursion tree



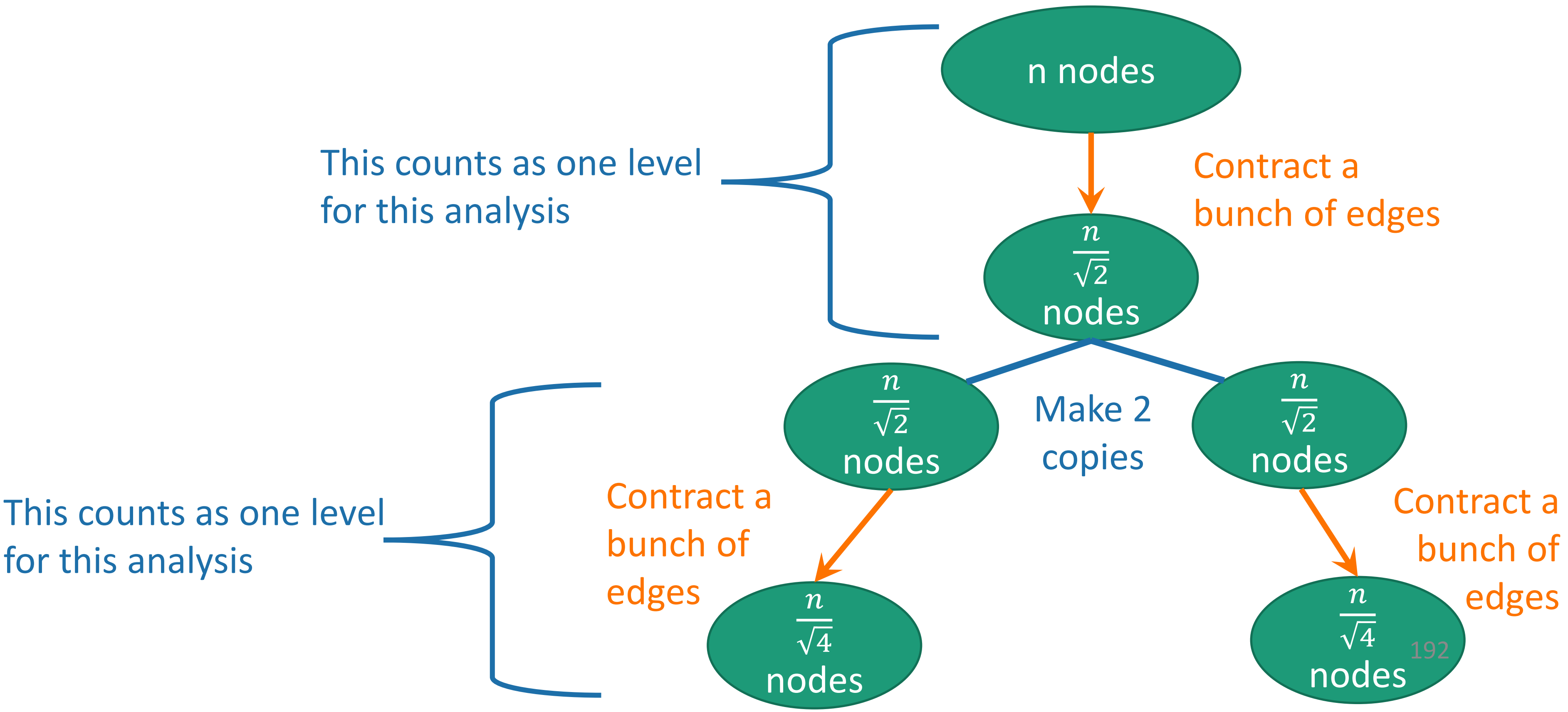
Recursion tree

- depth is $\log_{\sqrt{2}}(n) = \frac{\log(n)}{\log(\sqrt{2})} = 2\log(n)$



Recursion tree

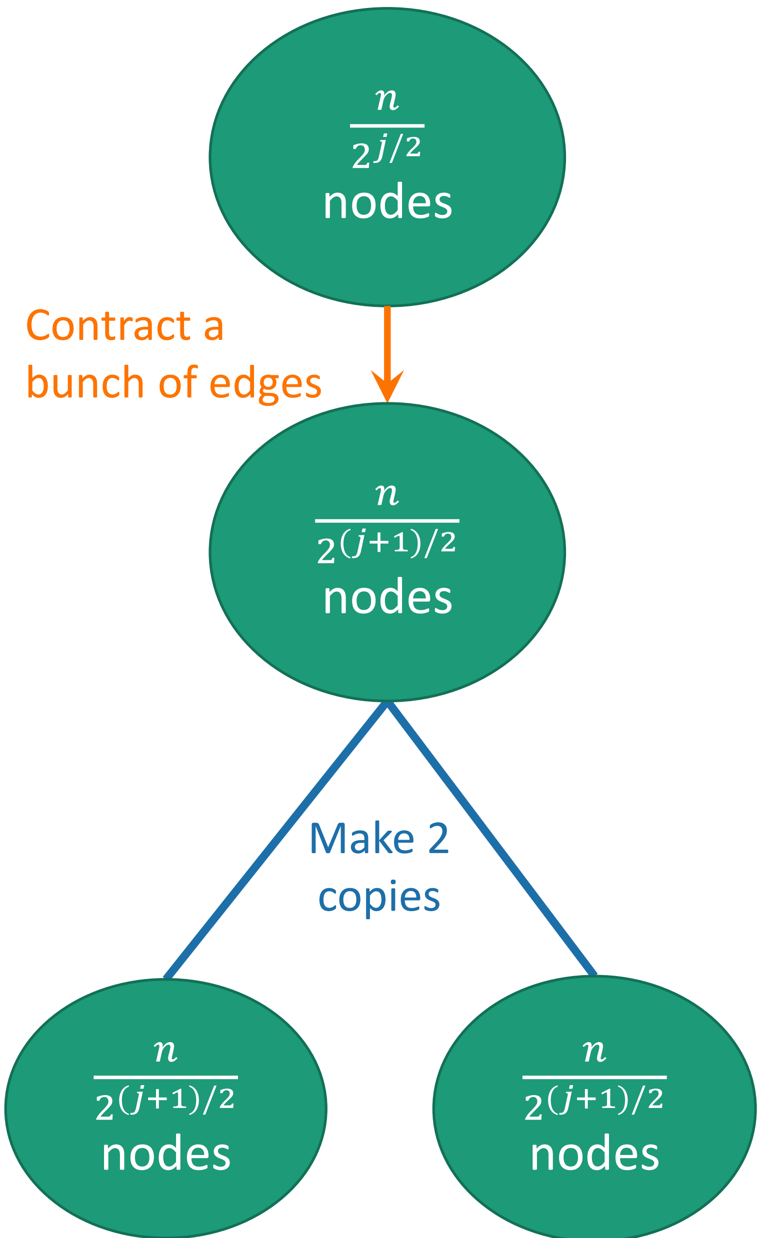
- depth is $\log_{\sqrt{2}}(n) = \frac{\log(n)}{\log(\sqrt{2})} = 2\log(n)$
- number of leaves is $2^{2\log(n)} = n^2$



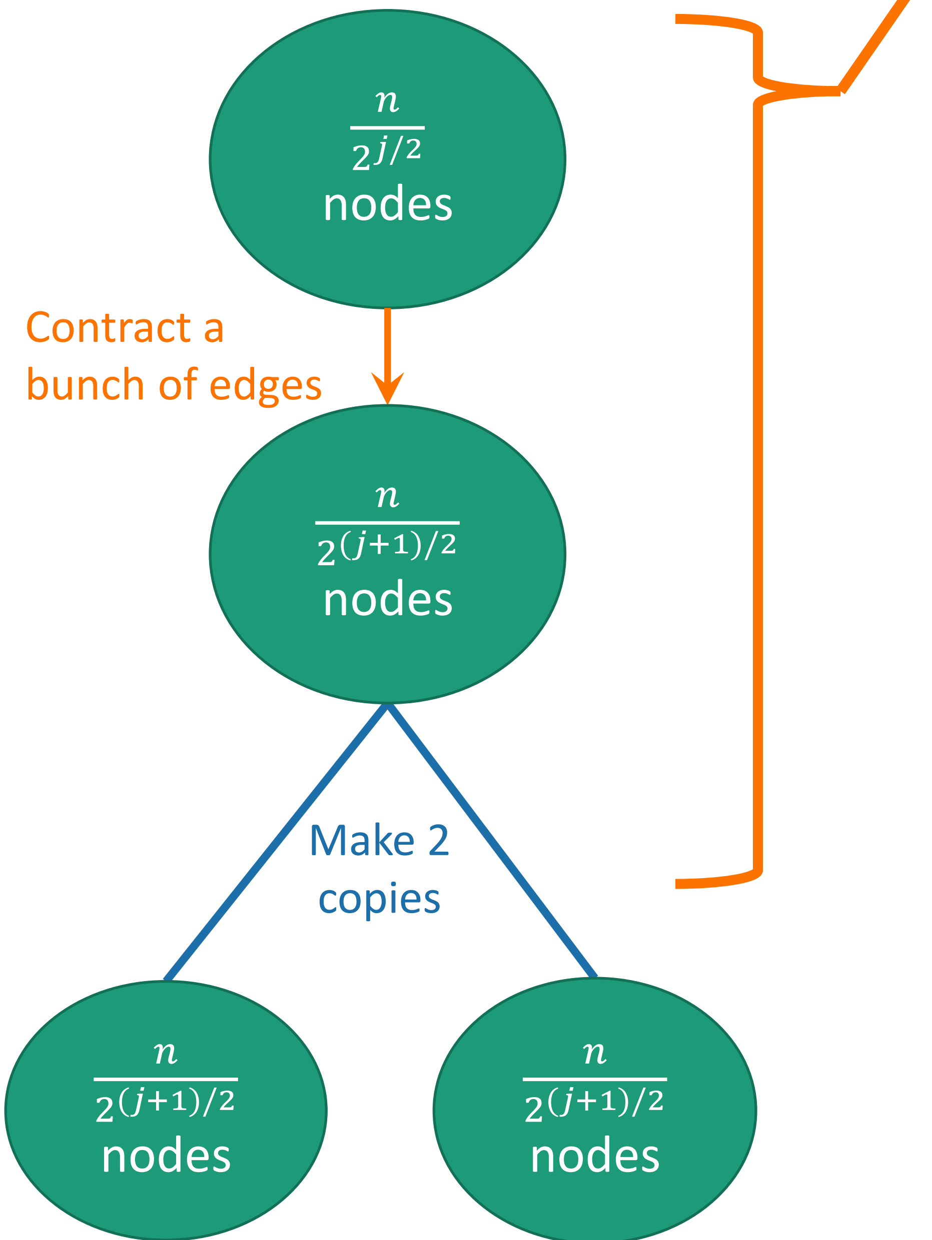
Two questions

- Does this work?
- Is it fast?

At the jth level

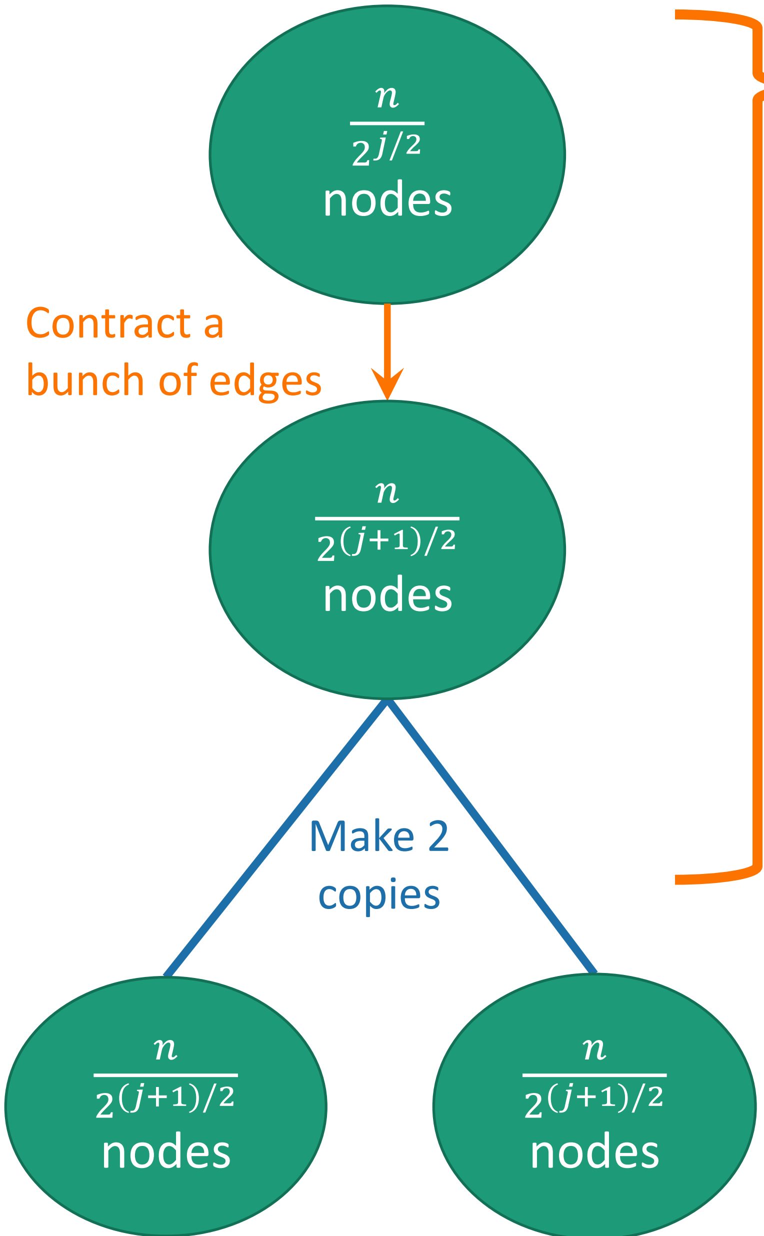


At the jth level



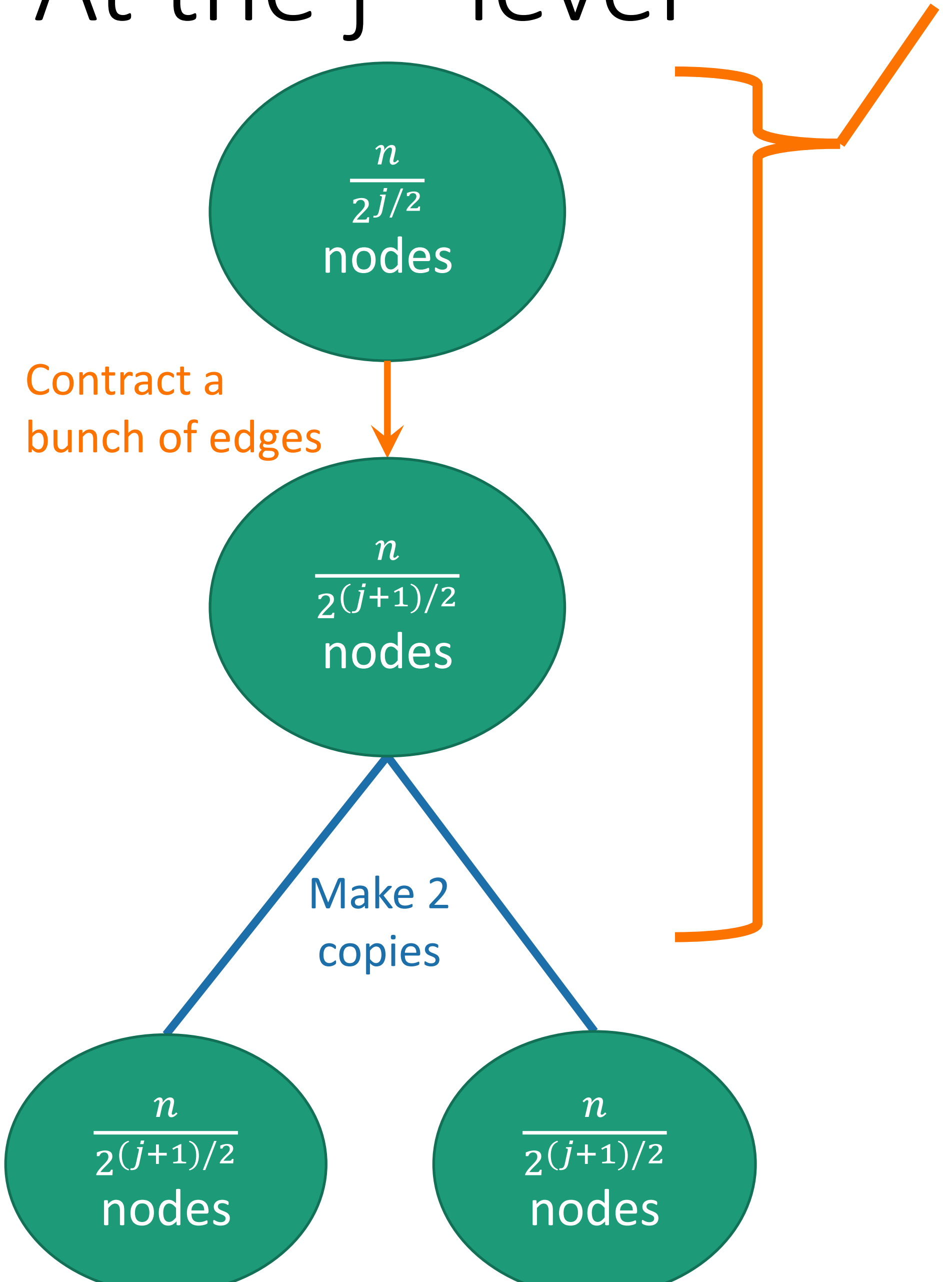
- The amount of work per level is the amount of work needed to reduce the number of nodes by a factor of $\sqrt{2}$.

At the j^{th} level



- The amount of work per level is the amount of work needed to reduce the number of nodes by a factor of $\sqrt{2}$.
- That's at most $O(n^2)$.
  - since that's the time it takes to run Karger's algorithm once, cutting down the number of supernodes to **two**.

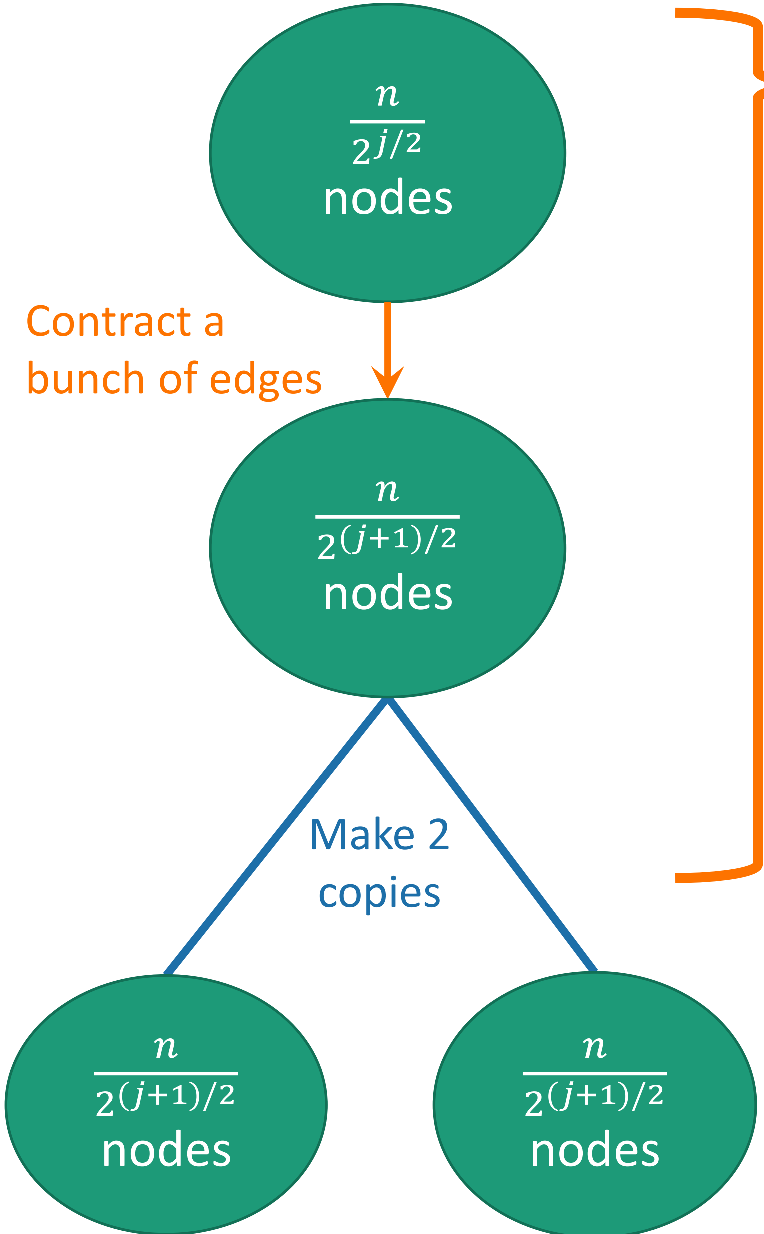
At the j^{th} level



- The amount of work per level is the amount of work needed to reduce the number of nodes by a factor of $\sqrt{2}$.
- That's at most $O(n^2)$.
  - since that's the time it takes to run Karger's algorithm once, cutting down the number of supernodes to **two**.
- Our recurrence relation is…

$$T(n) = 2T(n/\sqrt{2}) + O(n^2)$$

At the j^th level



- The amount of work per level is the amount of work needed to reduce the number of nodes by a factor of $\sqrt{2}$.
- That's at most $O(n^2)$.
  - since that's the time it takes to run Karger's algorithm once, cutting down the number of supernodes to **two**.
- Our recurrence relation is…

$$T(n) = 2T(n/\sqrt{2}) + O(n^2)$$

- The Master Theorem says…

$$T(n) = O(n^2 \log(n))$$

Jedi Master Yoda

Two questions

- Does this work?

- Is it fast?
  - Yes, $O(n^2\log(n))$.

Why $n/\sqrt{2}$?

Suppose we contract n – t edges, until there are t supernodes remaining.

- Suppose the first n-t edges that we choose are

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  × **PR**[e_2 doesn't cross S* | e_1 doesn't cross S*]

  …

  × **PR**[e_{n-t} doesn't cross S* | $e_1,\ldots,e_{n-t-1}$ don't cross S*]

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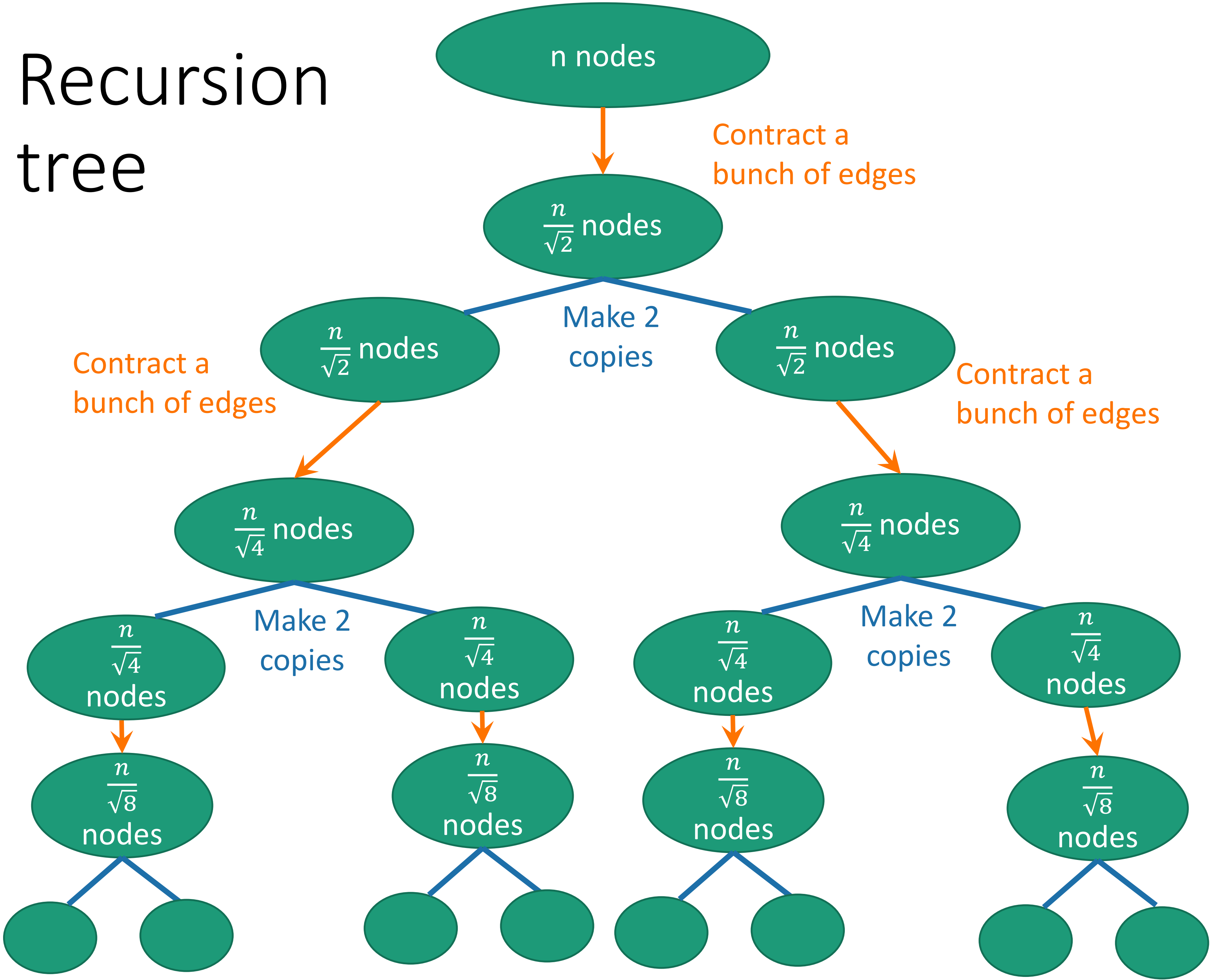
Choose $\boldsymbol{t = n/\sqrt{2}}$

$$= \frac{\frac{n}{\sqrt{2}}\cdot\left(\frac{n}{\sqrt{2}}-1\right)}{n\cdot(n-1)} \approx \mathbf{\frac{1}{2}}$$

when n is large

Recursion tree



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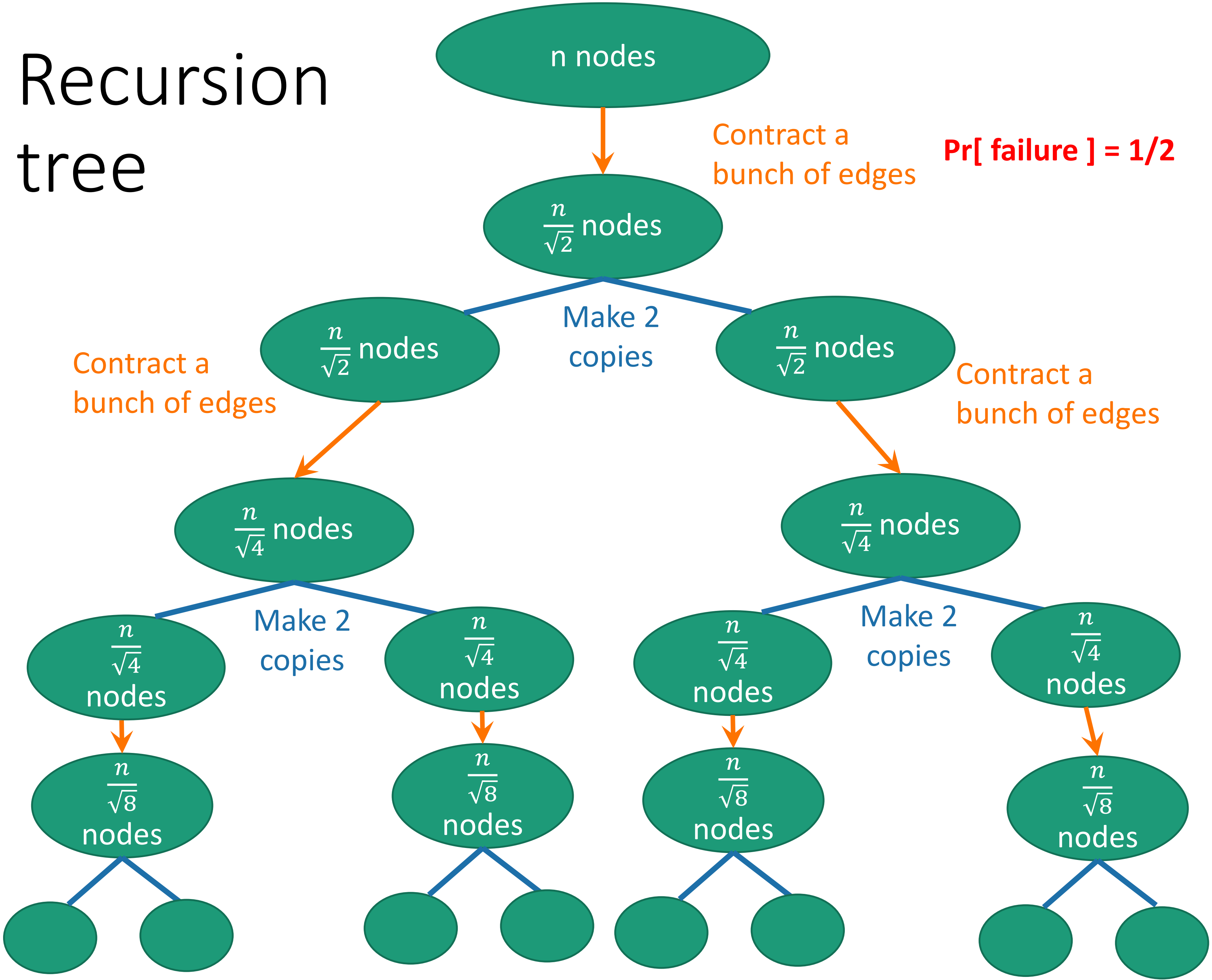

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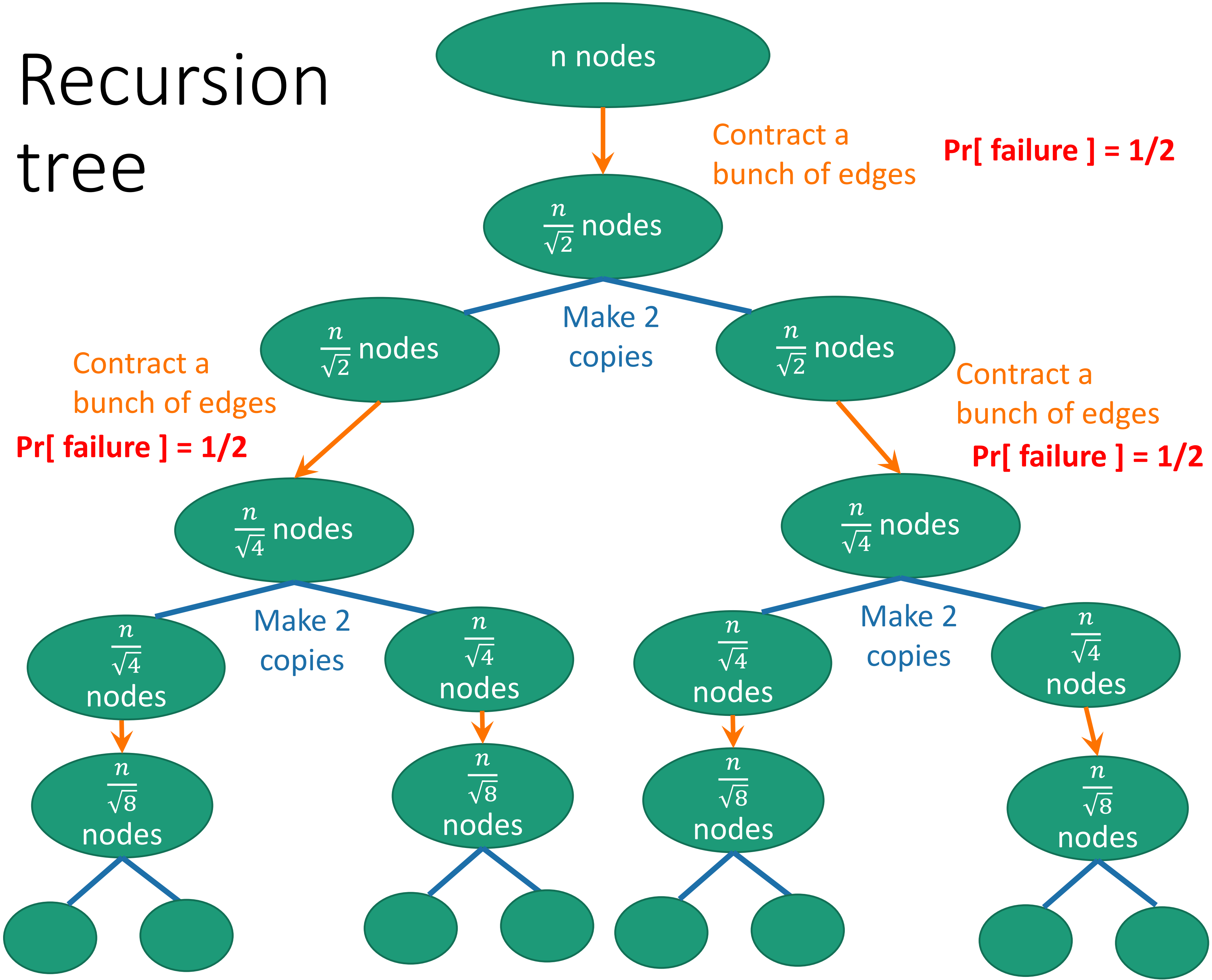

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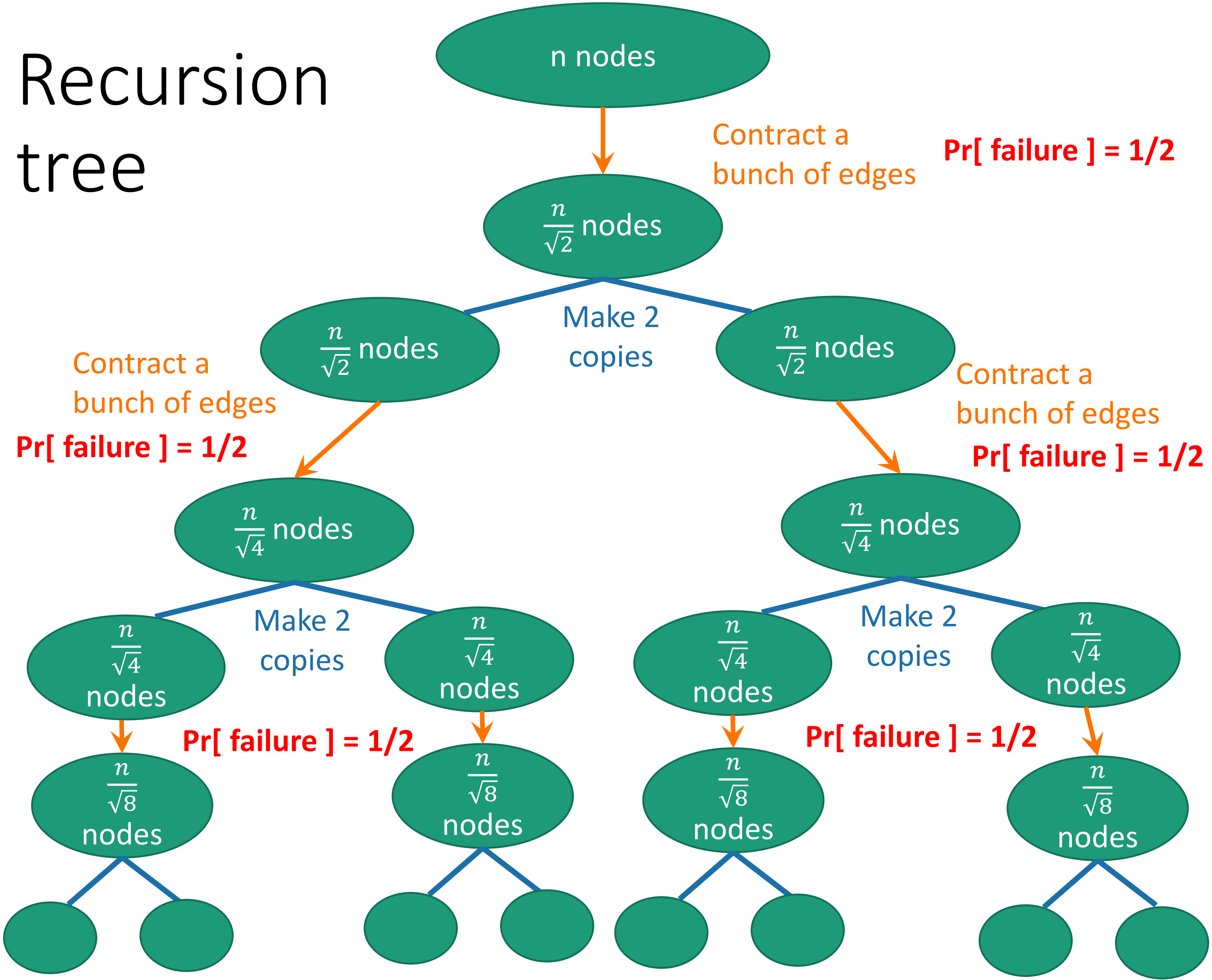

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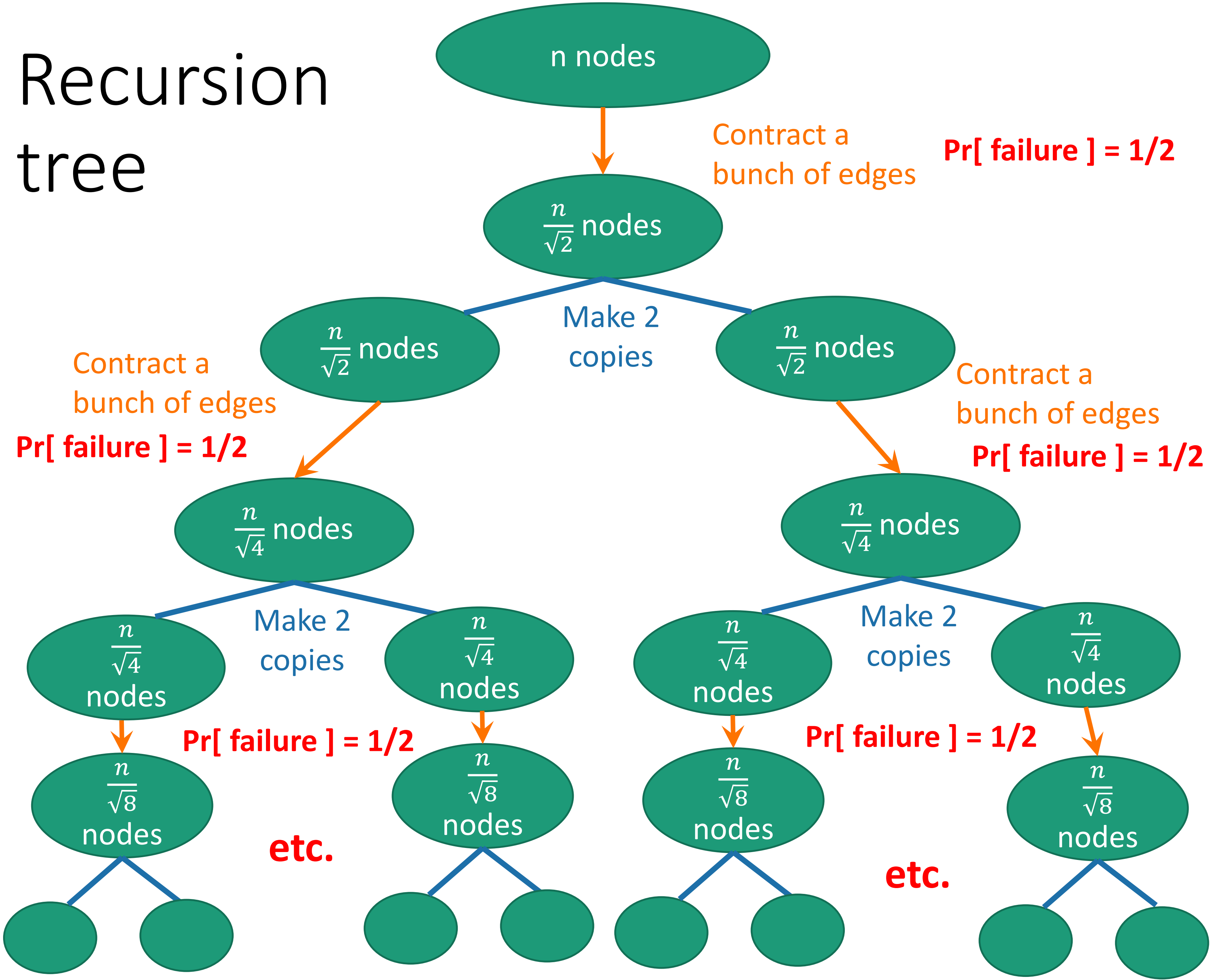
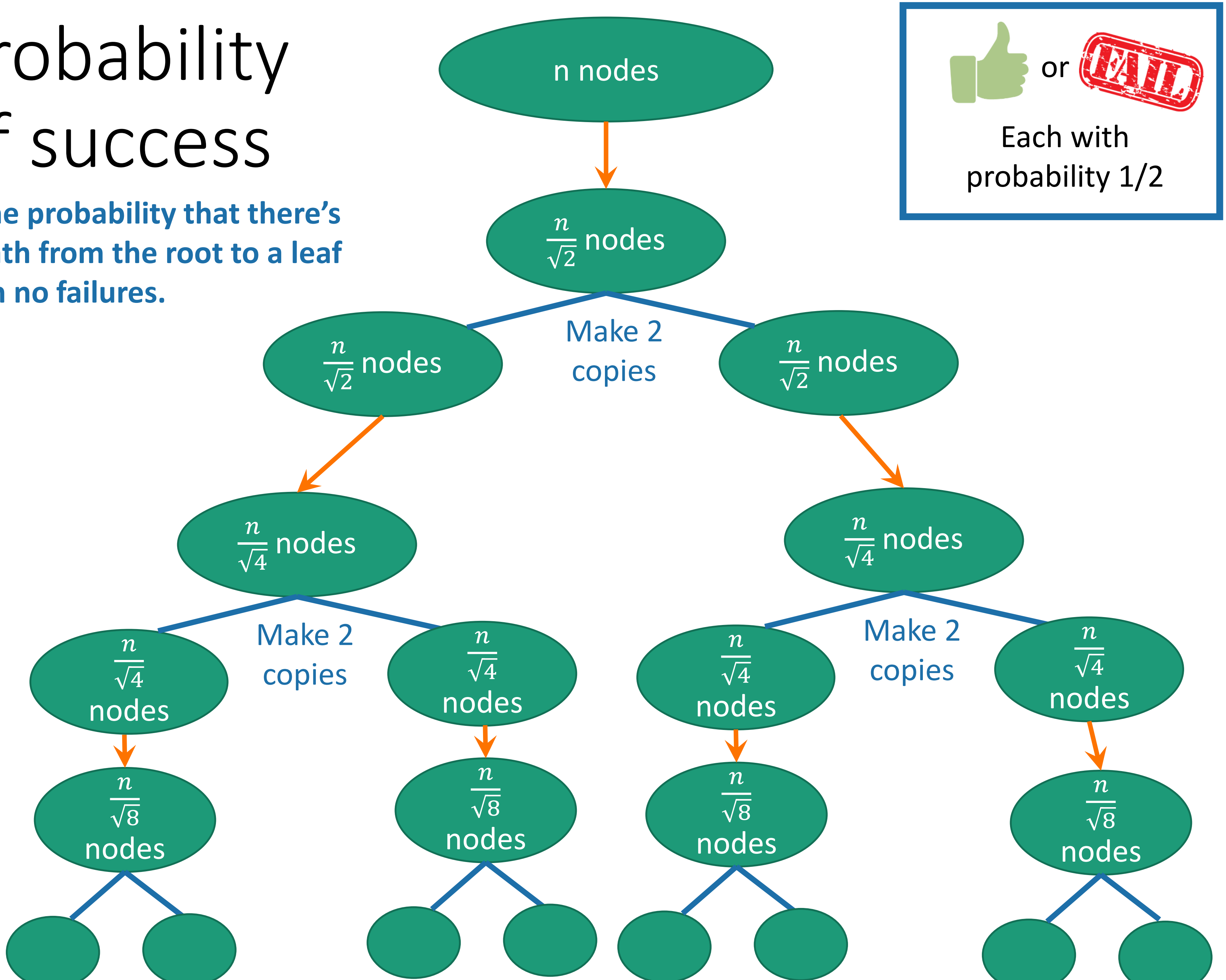


Probability of success

Is the probability that there's a path from the root to a leaf with no failures.



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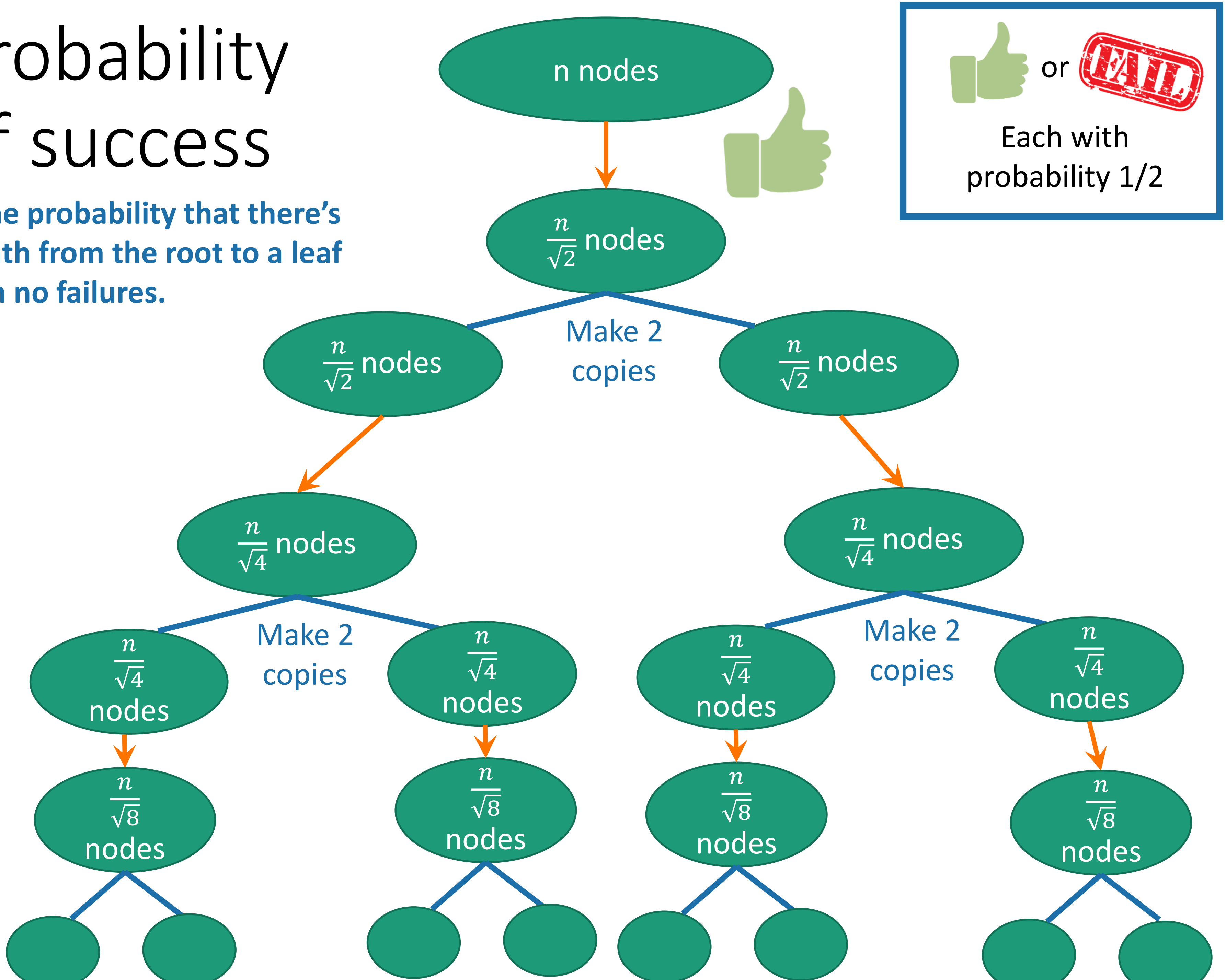

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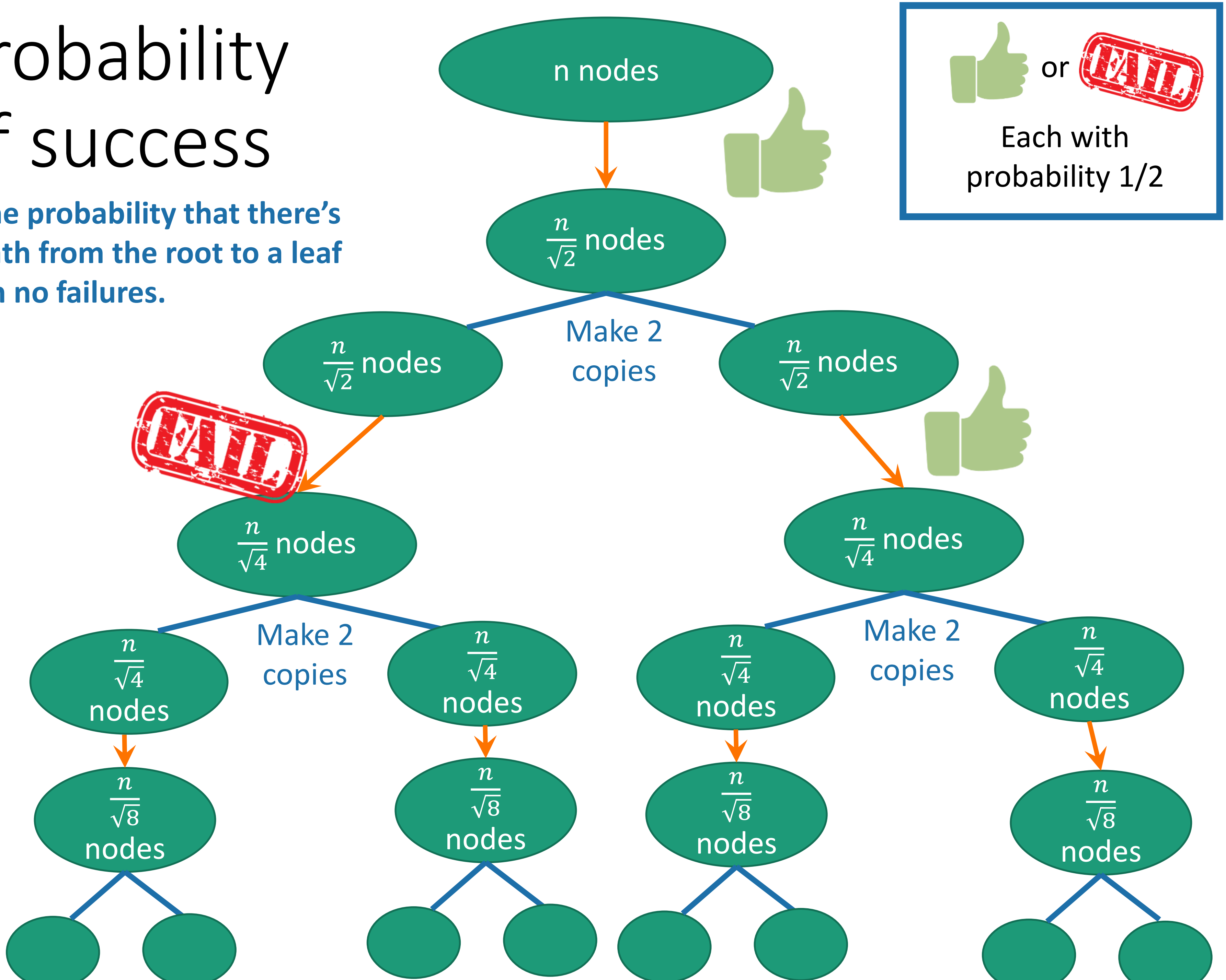

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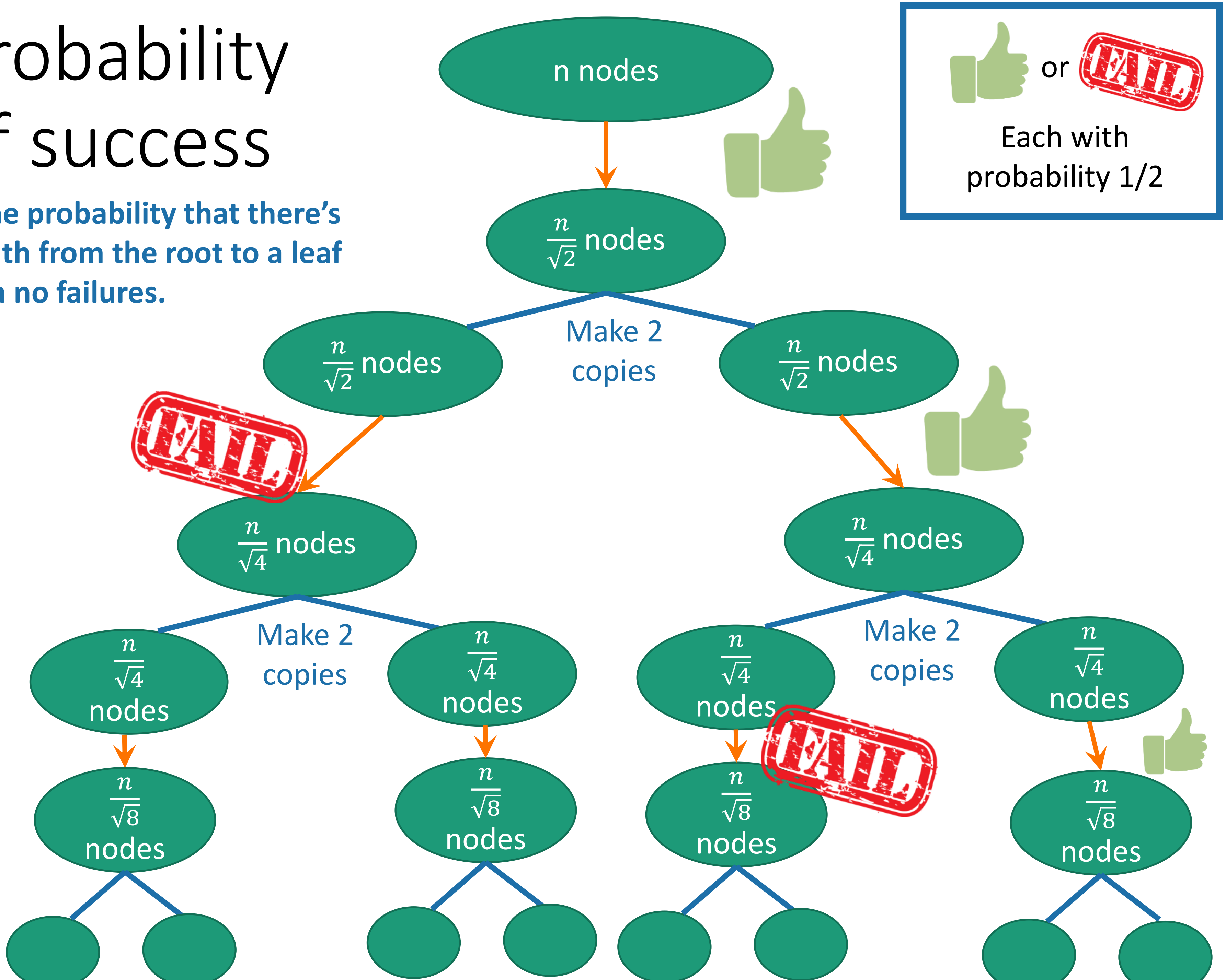

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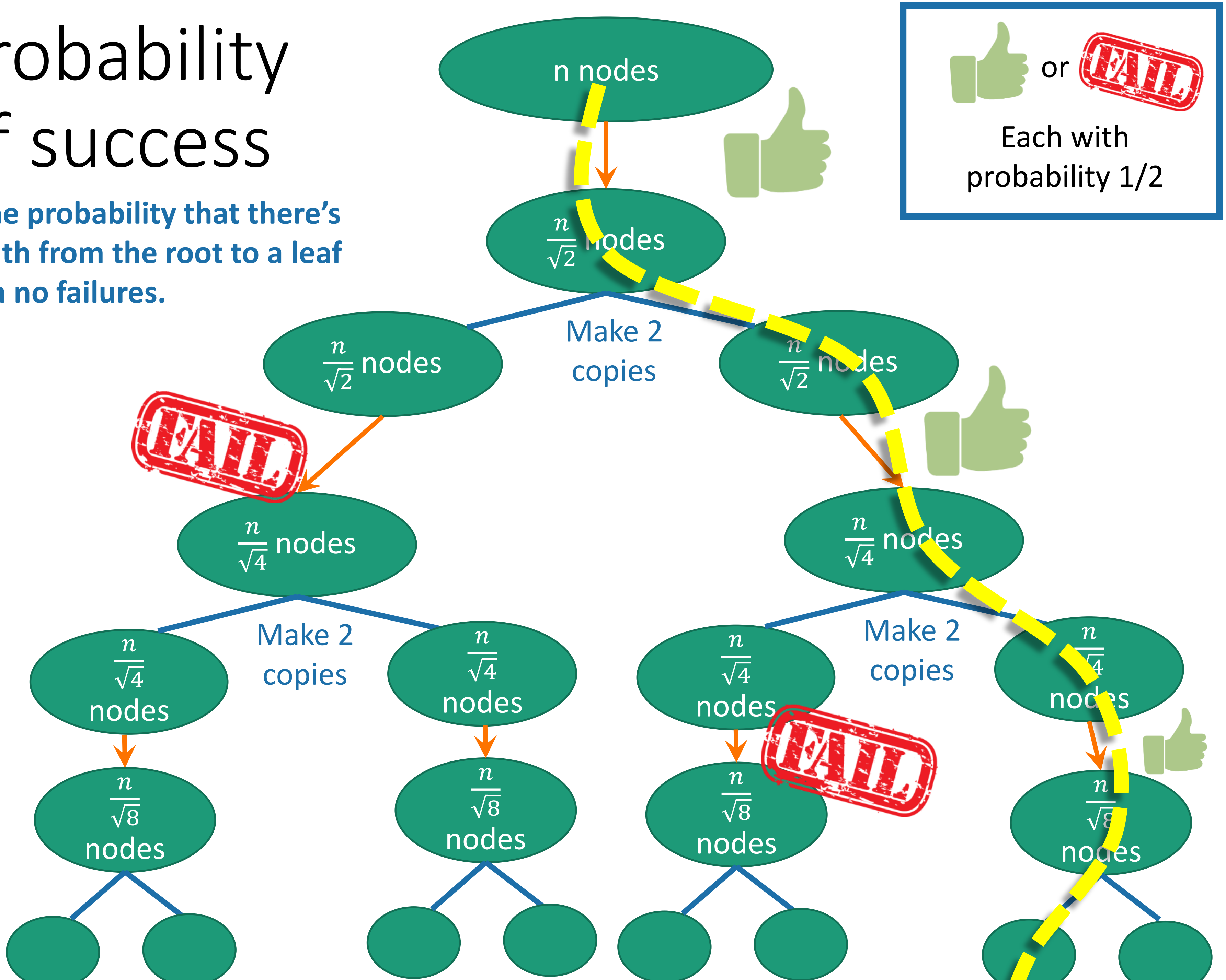


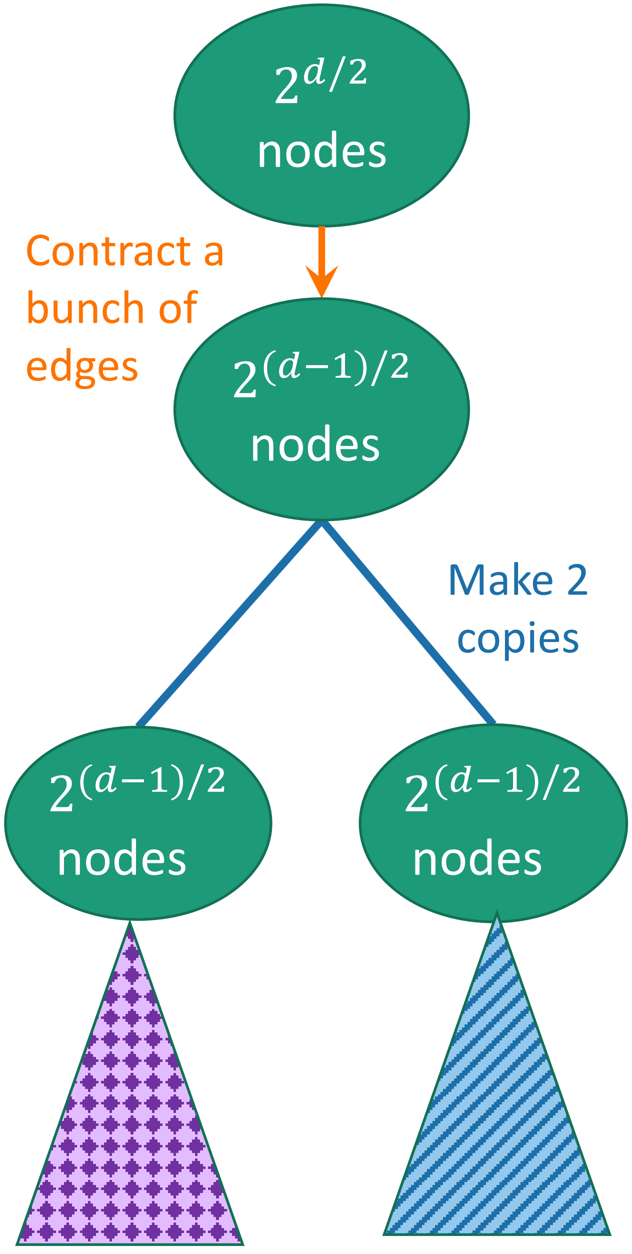
The problem we need to analyze

- Let T be binary tree of depth 2log(n)

- Each node of T succeeds or fails independently with probability 1/2

- What is the probability that there's a path from the root to any leaf that's entirely successful?

Analysis



$2^{d/2}$
nodes
Contract a
bunch of
edges
$2^{(d-1)/2}$
nodes
Make 2
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$2^{(d-1)/2}$
nodes
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nodes

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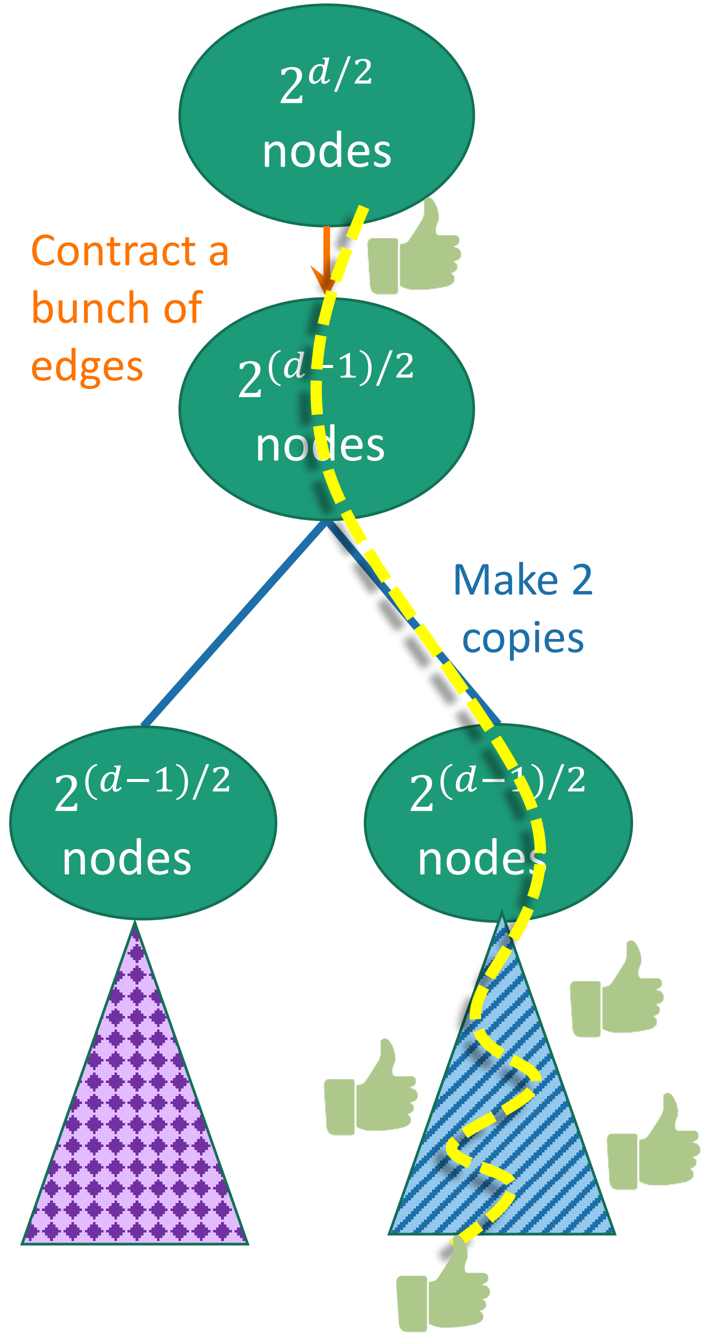

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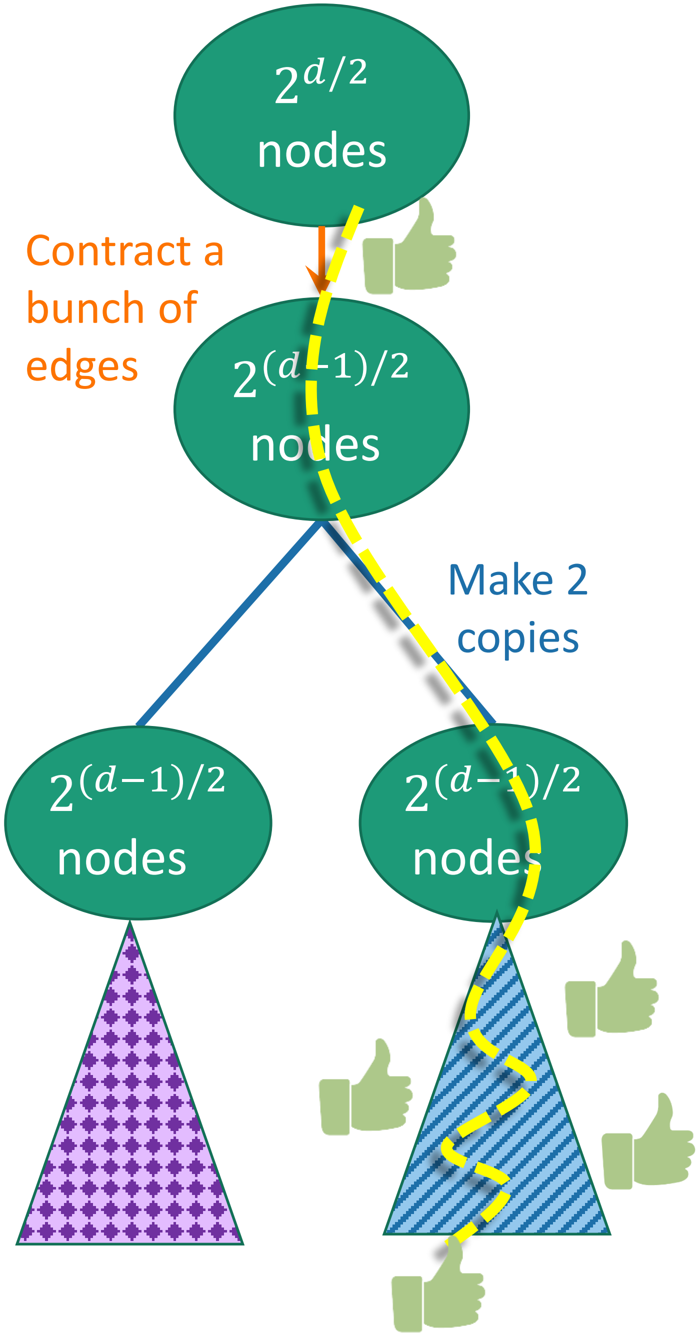

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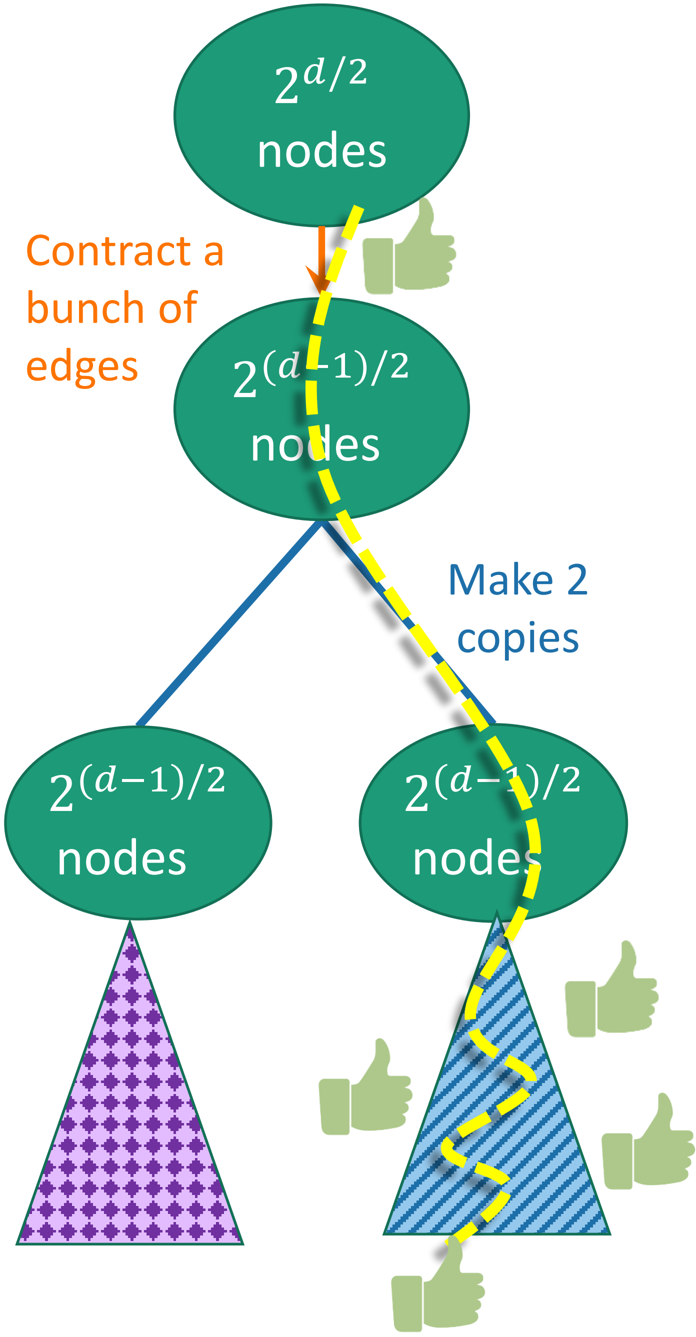

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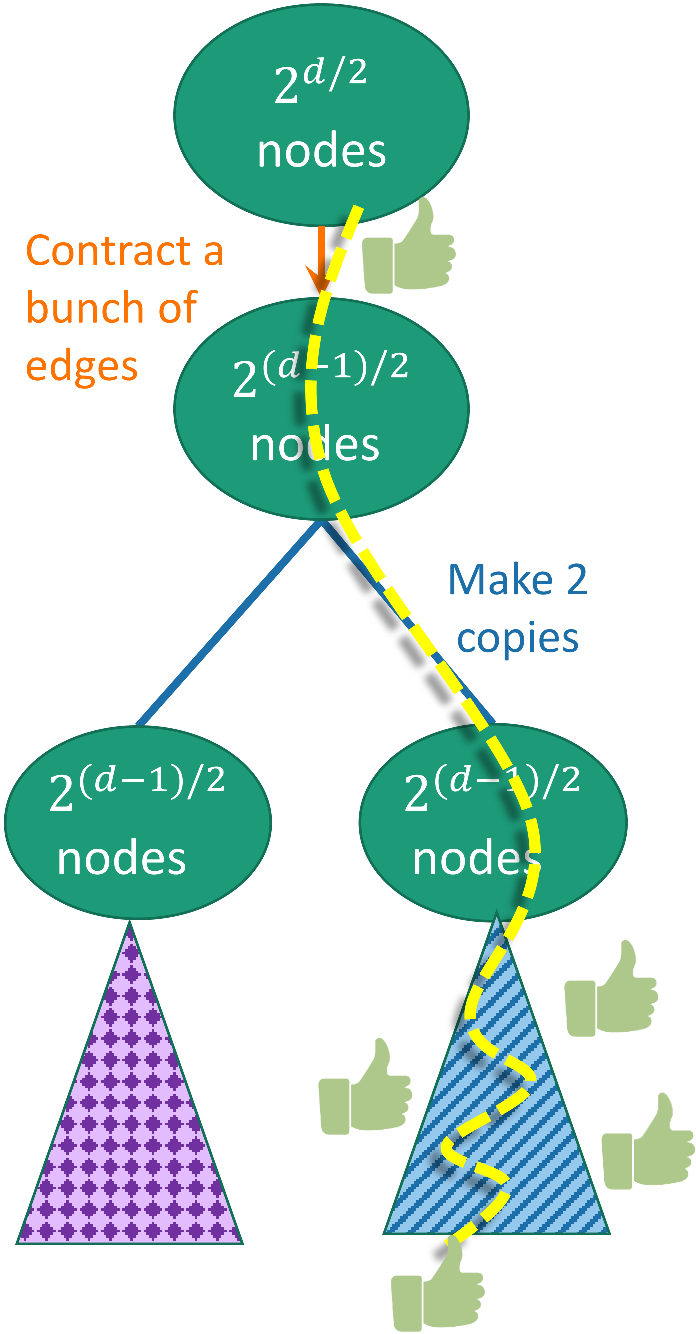

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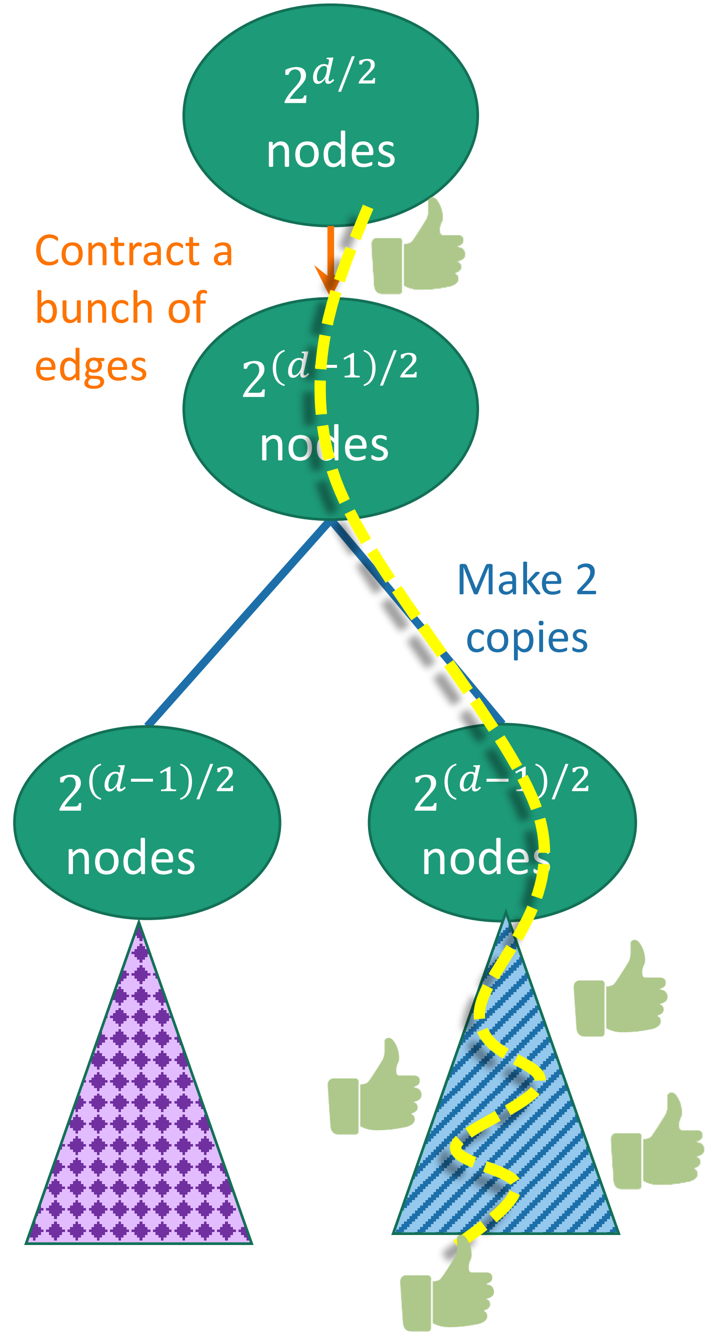
$$= \frac{1}{2} \cdot \left(p_{d-1} + p_{d-1} - p_{d-1}^2 \right)$$



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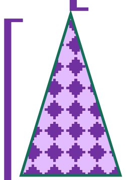
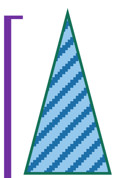
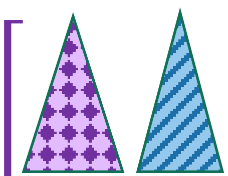
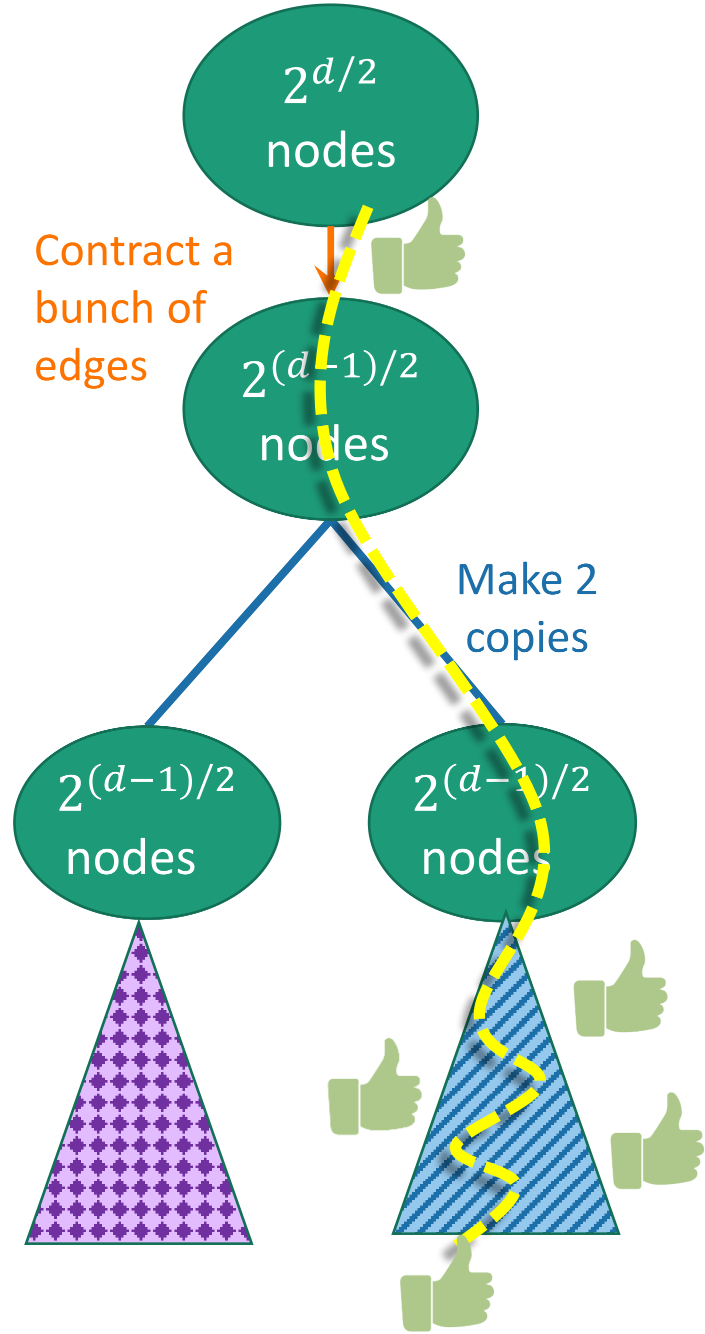
$= \frac{1}{2} \cdot \Big(\Pr\Big[$  wins $\Big] + \Pr\Big[$  wins $\Big]$

$- \Pr\Big[$  both win $\Big] \Big)$

$$= \frac{1}{2} \cdot \left(p_{d-1} + p_{d-1} - p_{d-1}^2 \right)$$

$$= p_{d-1} - \frac{1}{2} \cdot p_{d-1}^2$$



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- We are really good at those.
- In this case, the answer is:
  - **Claim**: for all d, $p_d \geq \frac{1}{d+1}$

Recurrence relation

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- $p_0 = 1$

- **Claim**: for all d, $p_d \geq \frac{1}{d+1}$
- **Proof**: induction on d.
  - **Base case**: $1 \geq 1$. **YEP.**
  - **Inductive step**: say d > 0.
    - Suppose that $p_{d-1} \geq \frac{1}{d}$.
    - $p_d = p_{d-1} - \frac{1}{2} \cdot p_{d-1}^2$
    - $\geq \frac{1}{d} - \frac{1}{2} \cdot \frac{1}{d^2}$
    - $\geq \frac{1}{d} - \frac{1}{d(d+1)}$
    - $= \frac{1}{d+1}$

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- aka,

Pr[Karger-Stein is successful] = $\Omega\left(\frac{1}{\log(n)}\right)$

Altogether now

- We can do the same trick as before to amplify the success probability.
  - Run Karger-Stein $O\left(\log(n) \cdot \log\left(\frac{1}{\delta}\right)\right)$ times to achieve success probability $1 - \delta$.

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- Each iteration takes time $O(n^2 \log(n))$
  - That's what we proved before.
- Choosing $\delta = 0.01$ as before, the total runtime is

$$O(n^2 \log(n) \cdot \log(n)) = O(n^2 \log^2(n))$$

Much better than $O(n^4)$!

What have we learned?

- Just repeating Karger's algorithm isn't the best use of repetition.
  - We're probably going to be correct near the beginning.
- Instead, Karger-Stein repeats when it counts.
  - If we wait until there are $\frac{n}{\sqrt{2}}$ nodes left, the probability that we fail is close to ½.
- This lets us (probably) find a global minimum cut in an undirected graph in time **O(n^2 $\log^2$(n)).**
  - Notice that we can't do better than n^2 in a dense graph (we need to look at all the edges), so this is pretty good.

Recap

- Some algorithms:
  - Karger's algorithm for global min-cut
  - Improvement: Karger-Stein
- Some concepts:
  - Monte Carlo algorithms:
    - Might be wrong, are always fast.
  - We can boost their success probability with repetition.
  - Sometimes we can do this repetition very cleverly.

Acknowledgement

- Stanford University