M-3 :-

1) Bounded and Unbounded sets.
2) Lower and Upper bounds
3) Supremum, Infimum
4) Completeness / LUB axiom

→ Upper bound :- Let S be a subset of $\mathbb{R}$. If there exists a real number m such $m \geq s$ for all $s \in S$, then m is called an upper bound of S, we say that S is bounded above.

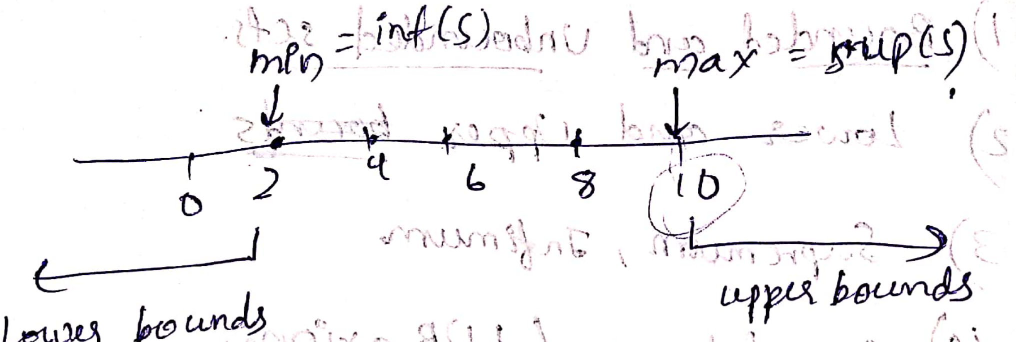
→ If $m \leq s$ for all $s \in S$, then m is a lower bound of S and S is bounded below.

→ If an upper bound m of S is a member of S, then m is called the maximum of S, we write $= m = \max S$.

→ Similarly, lower bound of S is a member of S then it is called minimum of S.

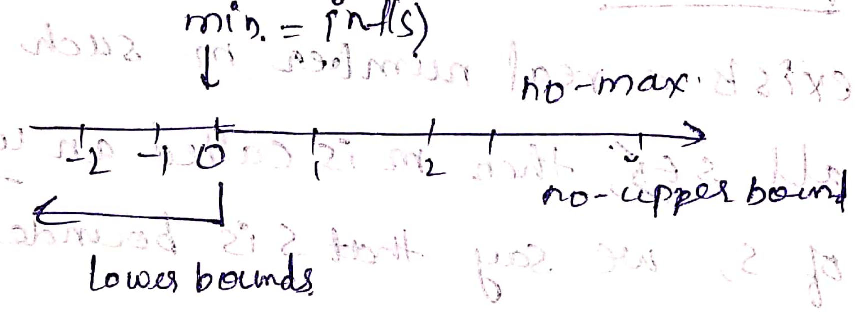
→ A set can have many upper and lower bounds but maximum and minimum are unique.

Ex:- Let $S = \{2, 4, 6, 8\}$: E-M



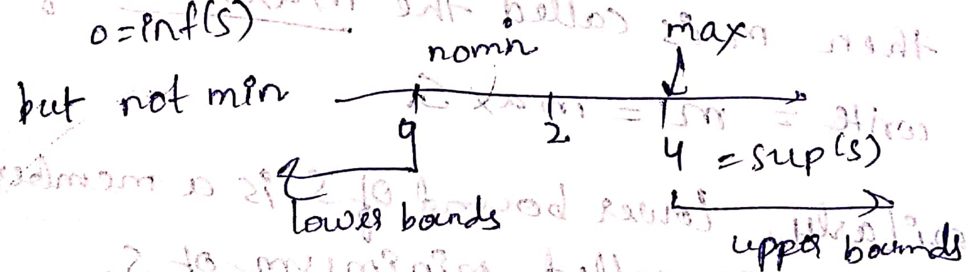
• Also, any finite set is bounded and always has a maximum and a minimum.

Ex:-



$S = [0, \infty)$

Ex:- $S = (0, 4]$



Empty set :- condition $m \geq s$ for all $s \in \phi$ is equivalent to the implication "if $s \in \phi$, then $m \geq s$". This implication is true since antecedent is false.

Likewise, ϕ is bounded by any real m.

Def 3.5 :-

S be a non empty subset of $\mathbb{R}$. If S is bounded, least upper bound of S is called its supremum and is denoted by sup S. Then, $m = \sup S$ iff

(a) $m \geq s$ for all $s \in S$, m is upper bound of S

(b) if $m' < m$, then there exists $s' \in S$ such that $s' > m'$.

Nothing smaller than m is upper bound of S

S is bounded below, then the greatest lower bound of S is called its infimum and is denoted by inf S.

Bounded above :- A sequence $\{s_n\}$ is said to be bounded above if there exists a real number k such that $s_n \leq k \quad \forall n \in \mathbb{N}$.

Tutorial :-

Q) Prove that $2 + \frac{1}{\sqrt{2}}$ is irrational.

Sol :- First let us prove $\frac{1}{\sqrt{2}}$ is irrational.

Assuming $\frac{1}{\sqrt{2}}$ is rational.

So, $\frac{1}{\sqrt{2}} = \frac{a}{b}$ where $b \neq 0$, a, b are co-primes

$b^2 = 2a^2$

b is divisible by 2

So, $b = 2k$, $\quad 4k^2 = 2a^2$

$a^2 = 2k^2$

a is also divisible by

So, it is not rational, but irrational

So, a rational + irrational is always irrational.

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② Greatest lower bound = Infimum

Least upper bound = Supremum

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③ Prove that set of natural numbers ~~are~~ is order complete.

→ ~~Prove~~ ~~that~~ a every subset of natural numbers is bounded above ~~and ordered of~~ and has a supremum is called completeness.

→ Real numbers is order complete

→ Set of rational numbers is not order complete

→ Every finite set is order complete

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④ Set of $A = \{ \frac{(-1)^n}{n} : n$ is a natural number$\}$

(a) Find the supremum. Is this supremum a max of A? Show that A is bounded from above

(b) Show that A is bounded from below. Find infimum. Is this infimum a min of A?

$$A = \left\{ \frac{(-1)^1}{1} + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \frac{(-1)^4}{4} + \cdots \right\}$$

$$-1, \; 0.5, \; -\frac{1}{3}, \; \frac{1}{4}$$

0.25

upper bound = 0.5 = supremum.

pg No 18 ; ex : 13 | N, I are not fields

→ Every converge sequence is bounded, limit is unique

⇒ Triangle inequality $|A+B| \leq |A|+|B|$

$$|L-K| = |L+a_k-a_k-K|$$

$$= |L-a_k+a_k-K|$$

$$\leq |L-a_k|+|a_k-K|$$

$$\leq |a_k-L|+|a_k-K|$$

$$\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$

$$\leq \varepsilon.$$

6.3) Sequence $\{a_n\}$ converges to L, if $a_n \neq 0$ for all $n \in N$, and if $L \neq 0$, sequence $\{\frac{1}{a_n}\}$ converges to $\frac{1}{L}$.

6.6)

$$|s_n-l| < \varepsilon \quad \forall n \geq m$$

$$l-\varepsilon < s_n < l+\varepsilon \quad \forall n \geq m$$

$$g = \min\{l-\varepsilon, s_1, s_2, \ldots, s_{m-1}\}$$

$$G = \max(l+\varepsilon, s_1, s_2, \ldots, s_{m-1})$$

$$g \leq s_n \leq G \quad \forall n$$

So, Hence $\{s_n\}$ is a bounded sequence

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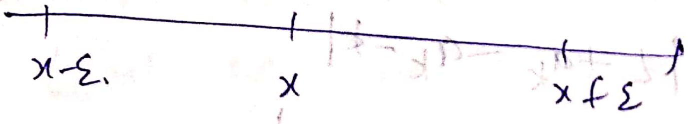
By an ε-neighbourhood of real numbers x and ε we understand the interval $(x-\varepsilon, x+\varepsilon)$



→ Limit is always a limit point, but converse need not be true.

$\{S_n\}$ to converge to L is that to each $\varepsilon > 0$ there corresponds a positive integer m such that

$$|S_n - L| < \varepsilon, \quad \forall\, n \geq m$$

Real Analysis :-

Squeeze theorem;
Statement :- If a_n converges to limit L, c_n converges to limit L, where $a_n \leq b_n \leq c_n$, then b_n also converges to limit L. for all $n > N$

Squeeze theorem :-

If $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$ and there exists an integer N such that $b_n \leq a_n \leq c_n$ for all $n \geq N$, then $\lim_{n\to\infty} a_n = L$.

Ex:- Q → Determine whether $\frac{\sin n}{n}$ converges or diverges

A) $-1 \le \sin n \le 1$

$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$ for all $n > 0$

We choose $\{b_n\} = \{-\frac{1}{n}\}$ and $\{c_n\} = \{\frac{1}{n}\}$.

We ~~know~~ now have choice of $\{b_n\}$ and $\{c_n\}$, that $b_n \le a_n \le c_n$ for all n. Notice that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$. By squeeze theorem, $\lim_{n\to\infty} a_n = 0$.

In other words, $\{a_n\}$ converges to 0.

→ A neceessary and sufficient condition for convergence is it is bounded and monotonically

→ Monotically increasing bounded above sequence converges to its least upper bound.

→ Monotic increasing and not bounded above, diverges to $+\infty$.

→ Monotonic decreasing and -- -- -- below, diverges to $-\infty$.

→ A sequence is bounded and has a limit point then, it is convergent.

Comparition test :

Test I : if $\sum u_n$ & $\sum v_n$ are two series such that $u_n \leq v_n$, $\forall n$.

and $\sum v_n$ is convergent.

then $\sum u_n$ is also convergent.

Ex: $\sum u_n = \sum \frac{1}{2^n + n}$ consider $v_n = \frac{1}{2^n}$

Consider $2^n + n > 2^n \quad \forall n$

$$\frac{1}{2^n + n} < \frac{1}{2^n} \quad \forall n$$

Since $u_n < v_n \quad \forall n$, since $\sum \frac{1}{2^n}$ is convergent,

then, $\sum u_n = \sum \frac{1}{2^n + n}$ is also convergent.

$\Rightarrow$ if $\sum u_n$ & $\sum v_n$ are two series such that

$u_n \geq v_n \quad \forall n$ and $\sum v_n$ is divergent, then

$\sum u_n$ is also divergent.

Sol: $\sum u_n = \sum \frac{1}{n+1}$ Let $v_n = \frac{1}{2n}$

$$n+1 < n+n$$

$$\underbrace{\frac{1}{n+1}}_{u_n} > \underbrace{\frac{1}{2n}}_{v_n}$$

Since $u_n \geq v_n \quad \forall n$, $\frac{1}{2}\sum \frac{1}{n}$ is divergent, then

$\sum \frac{1}{n+1}$ is divergent.

P-test: The series $\sum \frac{1}{n^p}$

converges if $p > 1$

diverges if $p \leq 1$.

Limit comparission test :

if Σu_n & Σv_n are 2 series such that

$\lim_{n\to\infty} \frac{u_n}{v_n} = k \neq 0$, then both the series converge or diverge together.

Ex: $\frac{4n^2+n}{(n^7+n^3)^{1/3}}$ $\quad \lim \frac{u_n}{v_n} = k$,

$v_n = \frac{n^2}{n^{7/3}} = n^{2-\frac{7}{3}} = \frac{1}{n^{1/3}}$

$$\lim_{n\to\infty} \frac{(4n^2+n)}{(n^7+n^3)^{1/3}} \times n^{1/3}$$

$$= \frac{4n^{7/3} + n^{4/3}}{(n^7+n^3)^{1/3}} = \frac{n^{7/3}\left(4+\frac{1}{n}\right)}{\left(n^{7/3}(1+n^{-4})\right)^{1/3}}$$

$$= \frac{\cancel{n^{7/3}}\left(4+\frac{1}{n}\right)}{\cancel{n^{7/3}}\left(1+\frac{1}{n^4}\right)^{1/3}} = \frac{4+0}{1+0} = 4 \neq 0$$

Here $\lim_{n\to\infty} \frac{u_n}{v_n} = k \neq 0$ (4).

Also, Σv_n = $\lim_{n\to\infty} \frac{1}{n^{1/3}}$ diverges ($p \leq 1$)

Thus, Σu_n also diverges.

D'ALEMBERT's Ratio Test :

In a positive test series $\sum u_n$, if $\lt_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$

then, the series converges if $\lambda < 1$ & diverges if $\lambda > 1$.

Reciprocal form (or)

if $\lt_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda$, then converges if $\lambda > 1$
diverges if $\lambda < 1$

& fails if $\lambda = 1$

Ex:

$$1 + \frac{a+1}{b+1} + \frac{(a+1)(2a+1)}{(b+1)(2b+1)} + \frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}$$

$$\lt_{n\to\infty} \frac{u_n}{u_{n+1}} = \lt_{n\to\infty} \left(\frac{nb+1}{na+1}\right)$$

$\therefore$ [Reciprocal $u_{n+1} = u_n \left(\frac{na+1}{nb+1}\right)$]

$$\lt_{n\to\infty} \frac{n\left(b+\frac{1}{n}\right)}{n\left(a+\frac{1}{n}\right)} = \frac{b}{a}$$

if $\frac{b}{a} > 1$ converges i.e., $b > a$

if $\frac{b}{a} < 1$ diverge i.e., $b < a$

If $\frac{a}{b}$ (or) $\frac{u_{n+1}}{u_n}$

then $\frac{a}{b}$ converges if $\frac{a}{b} < 1$. $\boxed{a < b}$

diverges if $\frac{a}{b} > 1$ $\boxed{a > b}$

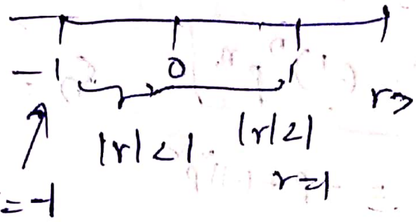
Q) <u>Geometric Series</u>: Show that the series

$S_n = 1 + r + r^2 + r^3 + \dots \infty$ if i) converges to $|r| < 1$

ii) diverges if $r \geq 1$ ($r = 1$ & $r > 1$)   iii) oscillates if $r \leq -1$ ($r = -1$, $r < -1$)

i) $S_\infty = \dfrac{1}{1-r}$

$\lim_{n\to\infty} = \dfrac{1}{1-r}$



$$S_n = 1.\frac{(r^n-1)}{r-1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^n-1}{r-1} = \frac{1-r^n}{1-r}$$

$$\lim_{n\to\infty}\left(\frac{1}{1-r} - \frac{r^n}{1-r}\right),$$

$$\lim_{n\to\infty} r^n = 0 \quad \text{if } |r| < 1$$

$$\lim_{n\to\infty}\left(\frac{1}{1-r}\right) = \frac{1}{1-r} \quad \text{if } |r| < 1$$

Case: 2: $\lim_{n\to\infty} r^n = \infty$ $r > 1$

series diverges if $r > 1$

Series becomes $1+1+1+\dots\infty$, $S_n = n$, $\lim_{n\to\infty} S_n = \infty$

iii) Oscillates if $r \le -1$.

$r = -1$, $r < -1$

i. $r < -1$ $\qquad S_n = \frac{1-r^n}{1-r}$

$\boxed{r = -p}$ $\qquad p > 1$ $\qquad -1 \quad 0$

$\boxed{r^n = (-1)^n p^n}$ $\qquad S_n = \frac{1-(-1)^n p^n}{(1-(-p))} = \frac{1-(-1)^n p^n}{1+p}$

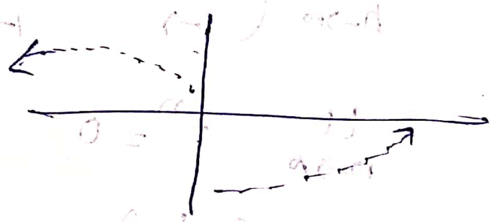
$= -p + (-1)^n p$

$\lim_{n \to \infty} \frac{1-(-1)^n p^n}{1+p}$ $\qquad \lim_{n \to \infty} p^n \to \infty$ $\because p \ge 1$

$S_n \to \infty$ (or) $-\infty$ depending on n is even/odd

$\Rightarrow$ oscillates

$r = -1$, $1 - 1 + 1 - 1$ ------, this is oscillating



P-test: series $\sum \frac{1}{n^p}$ converges if $p > 1$

diverges if $p \le 1$

Proof: Integral test: The series $\sum \frac{1}{x^p}$ converges (or) diverges if $\int_1^\infty \frac{1}{x^p} dx$ is finite/infinite.

$\Rightarrow$ the given series $\sum \frac{1}{x^p}$, the integral is

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{m\to\infty} \int_1^m \frac{1}{x^p}\,dx = \lim_{m\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^m$$

$$= \lim_{m\to\infty} \frac{m^{1-p}-1}{1-p}$$

$$\frac{1}{p-1} \qquad p>1$$

$$\text{diverges} \qquad p<1$$

Case II) $p=1$, $\int_1^\infty \frac{1}{x} = \lim_{n\to\infty} [\log x]_1^m$ diverges as $m\to\infty$

Cauchy's Root Test :

In a positive terms series $\sum u_n$.

Let $\lim_{n\to\infty} u_n^{1/n} = k$, now the series $\sum u_n$ converges if $k<1$ & diverges if $k>1$ & fails if $k=1$.

Problem :

$\sum (\log n)^{-2n}$

$u_n = (\log n)^{-2n}$.

$u_n^{1/n} = [\log n]^{-2n \times \frac{1}{n}}$

$= [\log n]^{-2} = \frac{1}{(\log n)^2}$

Now,

$$\lim_{n\to\infty} \frac{1}{(\log n)^2} = 0 < 1$$

Given series, converges by Cauchy's Root Test.

Absolute convergence :-

If, $u_1 + u_2 + \dots + u_n + \dots \infty$; such that

if $\Sigma |u_n|$ is convergent; then $\Sigma |u_n|$

$|u_1| + |u_2| + |u_3| + \dots |u_n| + \dots \infty$, then Σu_n is absolutely convergent.

→ if the series $\Sigma |u_n|$ is divergent but Σu_n is convergent;

$$|u_1| + |u_2| + \dots |u_n| + \dots \infty \text{ divergent}$$

but

$$\Sigma u_n = u_1 + u_2 + \dots u_n + \dots \infty \rightarrow \text{convergent}$$

the series Σu_n is conditionally convergent.

Ex:- $\Sigma u_n = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \Sigma \frac{(-1)^{n+1}}{n^2}$

1) $\Sigma |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \Sigma \frac{1}{n^2}$

by using P-test $\Sigma \frac{1}{n^2}$ is convergent.

So, Σu_n is absolutely convergent.

Ex

2) $\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \Sigma \frac{(-1)^{n+1}}{n}$

$\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \Sigma \frac{1}{n}$

by using P-test $\Sigma \frac{1}{n}$ is divergent.

Σu_n is ---→ convergent.

$\sum |u_n|$ is divergent.

therefore, $\sum u_n = \sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

<u>Note</u> :- Every absolutely convergent series is convergent but every convergent series need not be absolutely convergent.

$\sum u_n = \sum \frac{(-1)^{n+1}}{n}$ convergent but $\sum |u_n| = \sum \frac{1}{n}$ is divergent.

$\sum u_n$	$\sum \lvert u_n \rvert$	$(-1)^{n+1}$
Absolute convergence	Abs. conver ← convergent	

<u>Leibnitz's Test</u> :-

An alternating series $u_1 - u_2 + u_3 - \cdots u_n - \infty$

<u>Converges</u> if i) $u_{n+1} < u_n$ numerically

each term is numerically less than preceding term i.e., i) $u_{n+1} < u_n$ $\forall n$

ii) $\lim_{n \to \infty} u_n = 0$

$\sum \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} \cdots\cdots$ $\Rightarrow u_{n+1} = \frac{1}{n+1}$

$u_n = \frac{1}{n}$.

$n+1 > n$

$\frac{1}{n+1} < \frac{1}{n}$ $\forall n$,

(i) $u_{n+1} < u_n$.

(ii) $\lim_{n \to \infty} \frac{1}{n} = 0$.

So, the above alternating series u_n converges.

Ex:

$$\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} + \cdots$$

$$u_n = \frac{(-1)^{n+1}}{\log(n+1)}$$

$$u_n = \frac{1}{\log(n+1)} = \frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} \cdots$$

$$u_{n+1} = \frac{1}{\log(n+2)} = \frac{1}{\log 3} + \frac{1}{\log 4} + \frac{1}{\log 5}$$

$\frac{0.9}{\log 2} < \frac{0.9}{\log 3}$

i) $\log 2 < \log 3$

$\log(n+1) < \log(n+2)$

$\frac{1}{\log(n+1)} > \frac{1}{\log(n+2)}$

$u_n > u_{n+1}$

ii) $\lim_{n\to\infty} u_n = 0$

So, the series converges.

Just Ex: $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$

$e^{(\text{base}-1)\text{power}}$ $\left[\because \text{base}^{\text{power}} = 1^{\infty}\right]$

$* (-1)^{n+1}$ $\quad \frac{e^{(-1-1)(n+1)}}{n} = \frac{e^{-2(n+1)}}{n} = \frac{e^{-\infty}}{\infty} = \frac{1}{\infty}$

$= 0$

Power Series:

[8/9/20]

A series of the form $a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n \cdots \infty$

where a's are independent of x is called

Power Series.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{a_{n+1}x^{n+1}}{a_n x^n} = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) x$$

Abs case:

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|$$

if $|x|<1$

$5x<1$

$x<\frac{1}{5}$

$$= \lim_{n\to\infty} \left|\left(\frac{a_{n+1}}{a_n}\right)x\right|$$

$$\lim_{n\to\infty} = \left(\frac{a_{n+1}}{a_n}\right) = l.$$

$$= |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= |x|\, l$$

Series converges if $|x|l<1 \Rightarrow |x|<\frac{1}{l}$

Interval of convergence $\rightarrow$ $\boxed{-\frac{1}{l}<x<\frac{1}{l}}$

Ex: $\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3}$

$u_n = \frac{1}{n(1-x)^n}$ $\quad u_{n+1} = \frac{1}{(n+1)(1-x)^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{1}{(n+1)\cancel{(1-x)^n}(1-x)} \cdot n\cancel{(1-x)^n}\right|$$

$$\therefore \lim_{n\to\infty} = \left|\frac{n}{(n+1)(1-x)}\right| \Rightarrow \cancel{\left|\frac{n}{n+1}\right| \lim_{n\to\infty}\left|\frac{1}{1-x}\right|}$$

$$\Rightarrow \frac{1}{|1-x|} \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \frac{1}{|1-x|} \lim_{n\to\infty} \frac{\cancel{n}}{n(1+\frac{1}{n})}$$

$\lim_{n\to\infty} = \left|\frac{1}{1-x}\right|$

$-1 < 1-x < 1$

$-1-1 < 1-x-1 < 1-1$

$-2 < -x$

Converges if $\left|\frac{1}{1-x}\right| < 1$.

$|1-x| > 1$

$1-x > 1$ & $1-x < -1$

$\cancel{2 > x}$, $x > 2$

$x < 0$ &

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for $x=0$ $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ $\cdots + \sum \frac{1}{n}$ so $p=1$

the series diverges

$x=2 \Rightarrow -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\cdots$

$= \sum \frac{(-1)^n}{n} \to$ convergent by Lebinitz's test

So, the series converges for $x \leq 0$ & $x \geq 2$.

Convergence of Exp Series:

Ex: $1+\frac{x}{1}+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\frac{1}{n!}x^n + \to \infty$

$u_n = \frac{1}{n!}x^n$ $\quad u_{n+1} = \frac{1}{(n+1)!}x^{n+1}$

$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{(n+1)!}x^{n+1} \cdot \frac{n!}{x^n}$ $\quad \left|\frac{n!}{(n+1)!}\right| x$

$= \frac{n!}{(n+1)n!}x = |x| \lim_{n\to\infty}\frac{1}{n+1} = 0$

Series is convergent for all values of x.

Find the interval of convergence for

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \frac{(-1)^n x^n}{n} + \cdots \infty.$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = (-1)$$

Converges for $|x| < 1$
diverges for $|x| > 1$
Converges for $x = 1$
diverges for $x = -1$

Ex: $\sum a_n = \sum \frac{n^n}{3^{1+2n}}$ using Root test

$\Rightarrow$

Ex: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 \cdots$

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{x^{n+1}}{x^n} = |x|$$

$$\lim_{n\to\infty} |x| < 1 \qquad \underline{-1 < x < 1}.$$

<u>Sequence of f^n</u> $\{f_n(x)\}_{n=1}^{\infty}$

$f_n(x) = x + n$, $n \in N$, $x \in [0, 1]$

$x = \frac{1}{2}$, $f_1(x), f_2(x) \cdots$

$f_1(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$ $\quad f_2(x) = 2 + \frac{1}{2} = \frac{5}{2}$

at $x = \frac{1}{2}$, $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots$

$\Rightarrow$ This sequence converges, so, $f_n(x)$ is convergent at $x = x_0$

(1) Converges at a point; $f_n(x)$ is convergent, at $x = x_0$.

(2) Point wise convergence $f_n(x) = \frac{x}{n}$;

$x = 0.4 \Rightarrow f(n)\, f(0.4) = \frac{0.4}{n}$ $\quad a_n = \frac{0.4}{n}$.

$\{a_n\} = \frac{0.4}{1}, \frac{0.4}{2}, \frac{0.4}{3}, \ldots\ldots$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{0.4}{n} = 0.$

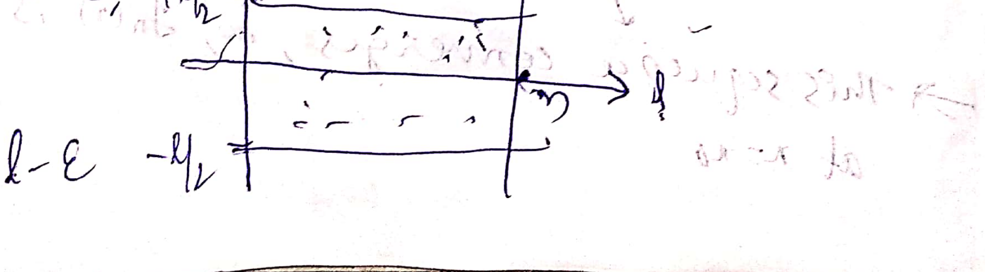
Suppose $\{f_n(x)\}$ is a sequence of f^n's, we say that $f_n(x)$ is point wise, convergent to $f(x)$ if $\lim_{n\to\infty} f_n(x) = f(x)$, for each x in that interval.

$\{f_1(x), f_2(x), \ldots\} \rightarrow f(x).$

Ex: $f_n(x) = \frac{n}{1+nx}$ $\quad f: (0,1) \rightarrow R$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{1+nx} = \lim_{n\to\infty} \frac{n}{n(\frac{1}{n}+x)} = \frac{1}{x} = f(x)$

Let $\{a_n\}$ be a sequence, we can say that sequence is convergent to l, if for each $\epsilon > 0$, $\exists m$ such that $|a_n - l| < \epsilon$, $\forall n \geq m$



Let $\{f_n(x)\}$ be a sequence of functions $\{f_n(x)\}$ is convergent to $f(x)$ or f, if for each $\varepsilon > 0$, $\exists$ m such that $|f_n(x) - f| < \varepsilon$, $\forall n \geq m$ for each x in given interval.

Point wise convergence for a sequence of f^n.

Ex: $f_n(x) = \frac{n}{1+nx}$ $\quad f(x) = \frac{1}{x}$

for $\varepsilon > 0$, $|f_n(x) - f(x)| < \varepsilon$, for each $x \in (0,1]$

$$\left|\frac{n}{1+nx} - \frac{1}{x}\right| < \varepsilon \Rightarrow \left|\frac{nx - 1 - nx}{(1+nx)x}\right| < \varepsilon$$

$$\frac{1}{x(1+nx)} < \varepsilon \Rightarrow (1+nx)x > \varepsilon$$

$$\Rightarrow (1+nx) > \frac{\varepsilon}{x} \qquad nx > \frac{\varepsilon}{x} - 1 \qquad n > \frac{\varepsilon}{x^2} - \frac{1}{x} = m$$

2) Uniform convergence

$\{f_n(x)\}$ is said to uniformly convergent to $f(x)$ if for $\varepsilon > 0$, $\exists$ m such that $|f_n(x) - f(x)| < \varepsilon$, $\forall n \geq m$, & for all x in that interval.

$f_n(x) = \frac{2nx}{1+n^4x^2}$ $\qquad f(x) = 0$

$$\left|\frac{2nx}{1+n^4x^2} - 0\right| < \varepsilon \Rightarrow \qquad 1+n^4x^2 \geq 2n^2x$$

$$\frac{2nx}{1+n^4x^2} \leq \frac{2nx}{2n^2x}$$

$$|f_n(x) - 0| \leq \frac{1}{n}.$$

$\Rightarrow f_n(x)$, if $\exists$ a_n such that $|f_n(x) - f| \geq a_n$

& $\lim_{n \to \infty} a_n = 0$ then $f_n(x)$ is uniformly convergent to f. $\boxed{|f_n(x) - 0| < \frac{1}{n}}$.

$\rightarrow f_n \to f$ pointwise iff $\forall \varepsilon > 0$, $\forall x \in E$, $\exists N \in \mathbb{Z}^+$ s.t.

$\forall n \geq N$ $|f_n(x) - f(x)| < \varepsilon$. [N depends on ε and x]

$\rightarrow f_n \to f$ uniformly iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{Z}^+$, s.t.

$\forall n \geq N$, $\forall x \in E$ $|f_n(x) - f(x)| < \epsilon$

[N depends on ε]

Ex: $f_n(x) = x^n$ on $[0,1]$

if $x = 0$ $f_n(0) = 0^n = 0$

if $x = 1$, $f_n(1) = 1^n = 1 \to 1$

If $0 < x < 1$, $f^n(x) = x^n = 0$

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

point wise convergent

Sequence of functions

By Anirudh

2) $\left| \frac{1}{n+x} - 0 \right| < \varepsilon$ $\forall n \in \mathbb{N}, x \in (0,1)$

$\frac{1}{n+x} < \varepsilon$

$n + x > \frac{1}{\varepsilon}$

both uniform and point wise convergent

$n > \underbrace{\frac{1}{\varepsilon} - x}_{= m}$

max value $\frac{1}{\varepsilon}$
$\rightarrow$ decreases as x increases

$m = \frac{1}{\varepsilon}$.

$\therefore \left\{\frac{1}{n+x}\right\}$ converges to 0.

Ex: $x \in [0,1]$ $\left|\frac{nx}{1+n^2x^2} - 0\right|$; $x = \frac{1}{n}$.

$f_n(x) - f$

$$\left|\frac{nx}{1+n^2x^2}\right| < \varepsilon$$

$$\left|\frac{nx\frac{1}{n}}{1+(n^2x)\frac{1}{n^2}}\right| < \varepsilon$$

$\varepsilon > \frac{1}{2}$

not uniform and point wise conv. but for $\varepsilon < \frac{1}{2}$, it is not true

---

Uniform convergence for series of f^ns.

(Weirstrass M) Theorem:

1) For an series of functions $\{f_n(x)\}$, if the $\exists$ a sequence, a_n of the terms such that $|f_n(x)| \leq a_n$, $\forall n \in N, \forall x \in I$; then, we say $\{f_n(x)\}$ is dominated series & the

2) series Σa_n should be convergent.

2) Every dominating series is Uniformly convergent

Ex: $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$

$\left| \frac{\sin nx}{n^2} \right| \leq \frac{1}{n^2}$, $\sum \frac{1}{n^2}$ is convergent.

My defⁿ:

Mainly, if $\{f_n(x)\}$ is a sequence of fⁿ is said to be pointwise convergence if continuous fⁿ graphs → convergence leads to a non continuous fⁿ.

Said to uniformly convergent, if fⁿ of a continuous graph → convergence leads to a continuous fⁿ.

which means $\{f_n(x)\} \longrightarrow f(x)$ [∵ uniform]
continuous → continuous

Uniform fⁿ preserves continuity, differentiability.

Ordinary Differential Equations — # By Anirudh

- An ordinary differential eq is diff eq in which the dependent variable (y) depends only on one independent variable (x).

∴ $\frac{dy}{dx} = \sin x + \cos x$

- A partial diff eq is the differential eq in which y depends on two or more independent variable x, t...

$\therefore F\left(x, t, y, \frac{\delta y}{\delta x}, \frac{\delta y}{\delta t}, \frac{\delta^2 y}{\delta x^2}, \frac{\delta^2 y}{\delta t^2} \cdots\right) = 0$ $y = x t^2$

Examplo:

$$\frac{dy}{dx} + t\frac{dy}{dt} = 2y$$

- Order of diff eq

order of highest derivative involved in a differential eq is called order of diff

eq: $F\left(x, y, \frac{\delta y}{\delta x}, \frac{\delta^2 y}{\delta x^2} \cdots\right)$

general form of ordinary diff eq.

Ex: $x\frac{dy}{dx} + y = 2$ $\xrightarrow{}$ 1st derivative   order = 1

$\frac{\delta^2 y}{\delta x^2} + 2y = \cos x$ $\xrightarrow{}$ 2nd derivative   order = 2

- Degree ;

Degree of a diff eq is the degree or power of highest order derivative present in eq, after the eq is made free from radicals and fractions in respect of derivatives

Ex: $\frac{\delta^3 y}{\delta x^3} + \left(\frac{\delta y}{\delta x}\right)^2 + xy = 0.$

degree = 1

order = 3 .

Ext

1) $\frac{d^2y}{dx^2} = 0$ Degree = 1, Order = 2

2) $y = x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}}$ Order = 1, Degree = 2

3) $\frac{d^2y}{dx^2} + m^2y = 0$ order = 2, degree = 1

4) $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2\log y$ order = 2, Degree = 1

5) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(a\frac{d^2y}{dx^2}\right)^2$

order = 2

degree = 2

6) $y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\left(\frac{y - x\frac{dy}{dx}}{a}\right)^2 = \left(1+\left(\frac{dy}{dx}\right)^2\right)$$

order = 1

degree = 2

Solution

- Formation of a diff eq under the initial / boundary conditions; and;
- formation of a diff eq from the physical condⁿ called <u>modelling</u>.

1) $y = ax^2 + bx$

$\frac{dy}{dx} = 2ax + b$

$\frac{d^2y}{dx^2} = 2a$ $\boxed{a = \frac{1}{2}\frac{d^2y}{dx^2}}$

$\frac{dy}{dx} - 2 \cdot \frac{1}{2}\frac{d^2y}{dx^2}x = b$

$y = ax^2 + bx$

$y = \frac{1}{2}\frac{d^2y}{dx^2}x^2 + \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)x$

order = 2
degree = 1

** Order of eq formed = No. of constants to be eliminated

---

2) $y = Ae^{2x} + Be^{-2x}$ → 2 constants.
order = 2

$\frac{dy}{dx} = Ae^{2x}(2) + Be^{-2x}(-2)$

$\frac{dy}{dx} = 2(Ae^{2x} - Be^{-2x})$

$\frac{d^2y}{dx^2} = 2(Ae^{2x}(2) - Be^{-2x}(-2))$

$\frac{d^2y}{dx^2} = 4(Ae^{2x} + Be^{-2x})$

$\boxed{\frac{d^2y}{dx^2} = 4y.}$

degree = 1
Order = 2

Solution: Relation between variables which satisfies the given diff eq^n.

General sol^n: No. of arbitrary constants is equal to order of diff eq.

$y = ce^{2x}$ at $x=0$, $y=2$

$\rightarrow \boxed{c=2}$.

Particular sol^n: Obtained from general sol^n by giving particular values to the arbitrary constants. Ex: $y = 2e^{2x}$.

Singular sol^n: Neither a general sol^n nor a particular sol^n. Only some eq^n have singular sol^n [Free of consts].

Ex: $\left(\frac{dy}{dx}\right)^2 - 4y = 0$

$y = (x+a)^2$ is general sol^n.

order of given diff eq is 1.
Also, arbitrary constants in solution is 1 [$\because$ a]

$y = x^2 \rightarrow$ particular sol^n.

$y = 0$ is a singular solution

First order, First degree diff eqⁿs :

General form :- $F\left(x, y(x), \frac{dy}{dx}\right) = 0.$

Variable Separable Eq's :

General form :- $f(x)\,dx = g(y)\,dy$

Solⁿ is $\int f(x)\,dx = \int g(y)\,dy + c$ is solution

$(1-x)\,dy + (1-y)\,dx = 0$

$(1-x)\,dy = -(1-y)\,dx$

$-\frac{dy}{1-y} = \frac{dx}{1-x}$

$1-y = t \qquad 1-x = m$

$-dy = dt \qquad -dx = dm$

$\int \frac{dt}{t} = -\int \frac{dm}{m} + c$

$\log t = -\log m + c$

$\log(1-y) + \log(1-x) = c$

$\log (1-y)(1-x) = c$

$(1-y)(1-x) = e^{c} = A$

Eq's reducible to variable separable.

Ex: $\frac{dy}{dx} = (4x+y+1)^2$

Taking, $4x+y+1 = t$

$$4\frac{dx}{dt} + \frac{dy}{dt} = \cancel{\ }$$

diff wrt to x

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

$$\frac{dy}{dx} = (4x+y+1)^2$$

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2+4$$

$$\int \frac{dt}{t^2+2^2} = \int dx$$

$$\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x + c$$

General soln $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$

if $y(0) = 1$ $x=0, y=1$

$$\frac{1}{2}\tan^{-1}(1) = 0 + c$$

$$\frac{1}{2}\frac{\pi}{4} = c \quad \boxed{c = \frac{\pi}{8}}$$

Particular soln :- $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + \frac{\pi}{8}$ //

\# written by Anirudh

III Homogeneous eq^n :

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

if $f(tx, ty) = t^n f(x,y)$

where n = degree of eq.

we use $y = vx$

Ex: $(x^2 - y^2)dx = xy\,dy$

$$\frac{dy}{dx} = \frac{(x^2-y^2)}{xy}$$

$y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{x \cdot vx}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x^2}(1-v^2)}{\cancel{x^2}(v)}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-v^2-v^2}{v} = \frac{1-2v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-2v^2}{v}$$

$1 - 2v^2 = m$

$-4v\,dv = dm$

$$\int \frac{dx}{x} = \frac{1}{-4}\int \frac{-4v\,dv}{1-2v^2}$$

$$\log x = \frac{-1}{4}\int \frac{dm}{m}$$

$$1 - \frac{2y^2}{x^2}$$

$$\log x + \frac{1}{4}\log m = c$$

$$\log x + \frac{1}{4}\log\left(1 - 2\left(\frac{y}{x}\right)^2\right) = c$$

$$\log x + \log\left(\frac{x^2-2y^2}{x^2}\right)^{1/4} = c$$

$$\log\left(x\left(\frac{x^2-2y^2}{x^2}\right)^{1/4}\right) = c$$

IV Equations reducible to homogeneous form

form: $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$

Can be reduced to homogeneous form.

Check: whether ratio $\frac{a}{a'}$ and $\frac{b}{b'}$ are equal?

Case I: $\frac{a}{a'} \neq \frac{b}{b'}$. substitute $x = X+h$, $y = Y+k$,

Then, make $(ah+bk+c) = 0$

$(a'h+b'k+c') = 0$

$h = \frac{bc'-b'c}{ab'-a'b}$ $\quad k = \frac{ca'-c'a}{ab'-a'b}$.

Ex: $\frac{dy}{dx} = \frac{x+y+2}{-x+y-4}$

$X = x-h = x+1$

$Y = y-k = y-3$

$x = X+h$ $\quad dx = dX$

$y = Y+k$, $\quad dy = dY$

$\boxed{\frac{a}{a'} \neq \frac{b}{b'} \text{ Case: I}}$

$\frac{dy}{dX} = \frac{X+h+Y+k-2}{-(X+h)+Y+k-4}$

$= \frac{X+Y+h+k-2}{-X+Y-h+k-4}$

① $\to h+k-2=0$

② $\to -h+k-4=0$

$2k-6=0$

$\boxed{k=3}$

Subs., $k=3$ in eq ①

$h+3-2=0$

$\boxed{h=-1}$

$y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x + vx}{-x + vx} = \frac{1+v}{v-1}$$

$$x\frac{dv}{dx} = \frac{v+1}{v-1} - v$$

$$= \frac{v+1 - v^2 + v}{v-1}$$

$$x\frac{dv}{dx} = \frac{1 - v^2 + 2v}{v-1}$$

$1 - v^2 + 2v = m$

$-2v + 2 = dm$

$-2(v-1)dt\ dm$

$$\int \frac{v-1}{1-v^2+2v}\,dv = \frac{dx}{x}$$

$$-\tfrac{1}{2}\int \frac{-2\,dm}{m} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\log(1 - v^2 + 2v) = \log x + c$$

$$\log(1 - v^2 + 2v) + 2\log x = -2c$$

$$(1 - v^2 + 2v)x^2 = e^{-2c} = A$$

$$x^2\left(1 - \frac{y^2}{x^2} + \frac{2y}{x}\right) = A$$

$$\cancel{x^2}\left(\frac{x^2 - y^2 + 2xy}{\cancel{x^2}}\right) = A$$

$$x^2 - y^2 + 2xy = A$$

Gen sol^n $\Rightarrow$ $(x+1)^2 - (y-3)^2 + 2(x+1)(y-3) = A.$

Case III :- If $\frac{a}{a'} = \frac{b}{b'}$

then put $ax + by = t$

Ex :- $\frac{dy}{dx} = \frac{3y + 2x + 4}{4x + 6y + 5}$ → Non-homogeneous

$\frac{2}{4} = \frac{3}{6}$, Case-III :-

Substitution :-

$ax + by = t \Rightarrow \quad 2x + 3y = t.$

$2 + 3\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx} - 2\right)$

$\frac{1}{3}\left[\frac{dt}{dx} - 2\right] = \frac{2x + 3y + 4}{4x + 6y + 5}$

$\frac{1}{3}\left[\frac{dt}{dx} - 2\right] = \frac{t + 4}{2t + 5}$

$\frac{dt}{dx} = \frac{3t + 12}{2t + 5} + 2$

$\frac{dt}{dx} = \frac{7t + 22}{2t + 5}$ → variable separable

$\frac{2}{7} \times \frac{7}{2}\int \frac{2t + 5}{7t + 22}\, dt = \int dx$

$\int \frac{14t + 35}{14t + 44}\, dt = \int dx.$

$= \int \frac{14t + 44 - 9}{14t + 44}\, dt = \int dx$

$$= \int dt - \frac{9\, dt}{14t+44} = \int dx$$

$$= t - \frac{9}{14}\int \frac{dm}{m} = \int x\, dx.$$

$14t+44 = m$

$14\, dt = dm$

$$= \frac{2}{7}\left[t - \frac{9}{14}\log(14t+44)\right] = x + c$$

VI) Linear diff eq^{n}s :-

→ dependent variable y and derivatives should appear only in first order, and should not be multiplied together

Ex:- $\frac{dy}{dx} + y = x$ → linear de

$\frac{dy}{dx} + x + y^2 = 0$ degree of $y = 2$ Non-linear

$y\frac{dy}{dx} + y = 2$ Non linear, dependent v & derivative are multiplied

$\left(\frac{dy}{dx}\right)^2 + y = xy$ Non-linear degree of derivative

→ First order linear differential eq is also called Lebinitiz eq

Solution method for diff eq^ns:-

$$\frac{dy}{dx} + Py = Q.$$

Integrating factor:- is a f^n by which ODE can be multiplied in order to make it integrable.

Let $IF = M(x)$

$$M(x)\left[\frac{dy}{dx} + Py\right] = M \cdot Q(x)$$

LHS. $= M(x)\dfrac{dy}{dx} + M(x) \cdot P(x)\, y$

$$P(x)\, M(x) = M'(x)$$

$$\int P(x) = \int \frac{M'(x)}{M(x)} \qquad \log M(x) = \int P(x)\,dx$$

$$\int P x\,dx = \log M(x) \longrightarrow M(x) = e^{\int P(x)\,dx}$$

Sol:-

$$e^{\int P(x)\,dx}\left[\frac{dy}{dx} + P(x)\, y\right] = Q(x)\, e^{\int P(x)\,dx}$$

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}\, y\right] = Q(x)\, e^{\int P(x)\,dx}$$

Gen sol^n $= e^{\int P(x)\,dx}\, y = \int Q(x)\, e^{\int P(x)\,dx} + C$

$$\boxed{IF.y = \int Q(x)\, IF + C}$$

Ex Solve $x\,dy = (y-3x^2)dx$

$$\frac{dy}{dx} = \frac{y-3x^2}{x}$$

$$\frac{dy}{dx} - \frac{y}{x} = \frac{-3x^2}{x}$$

$$P = -\frac{1}{x} \quad Q = -3x$$

$$IF = e^{\int -\frac{1}{x}dx} = e^{-\ln_e x} = x^{-1}$$

Solt $y.x^{-1} = \int -3x.x^{-1} + c$

$$\frac{y}{x} = -3x + c$$

$$a(x)^{b(x)}$$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$\left(\frac{n}{n(1+\frac{1}{n})}\right)^n$$

$$\lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$$\left(\frac{1}{1+\frac{1}{n}} - 1\right)n$$

$$= e$$

$$e^{\left(\frac{1 - 1 - \frac{1}{n}}{1+\frac{1}{n}}\right)n}$$

$$e^{\frac{-1}{1+\frac{1}{n}}} = \frac{1}{e}$$

$$= e^{\left(\frac{1}{1+\frac{1}{n}} - 1\right)n}$$

$$= e$$

$$\left(\frac{n+1}{n+2}\right)^{(n+1)^2}$$

$$\left(\frac{n}{n+1}\right)^{n^2}$$

$$\frac{(n+1)^{(n+1)^2}}{(n+2)^{(n+1)^2}} \cdot \frac{n^{n^2}}{(n+1)^{n^2}}$$

$$\Rightarrow \boxed{a_n = S_n - S_{n-1}}$$

Non-linear eq^n s,

An eq of form $\frac{dy}{dx} + Py = Qy^{n}$;

P, Q are fn of x alone, n is a constant except 0 and 1 is called a Bernoulli's eq.

$$\frac{1}{y^n}\frac{dy}{dx} + Py^{1-n} = Q.$$

$$y^{1-n} = t$$

d w.r.t. to x

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dt}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} + y^{1-n}P = Q$$

$$\frac{1}{(1-n)}\frac{dt}{dx} + tP = Q$$

$$\frac{dt}{dx} + (1-n)Pt = (1-n)Q \Rightarrow \text{Leibnitz}$$

$$\left\{\begin{array}{l} IF = e^{\int (1-n)P(x)\,dx} \\ \text{Sol}^n \quad t(IF) = \int (1-n)Q \cdot IF \cdot dx. \end{array}\right.$$

Ex 1- $x\frac{dy}{dx} + y = x^3y^6.$

$$\frac{\frac{x}{y^6}\frac{dy}{dx}}{x} + \frac{\frac{1}{y^5}}{x} = \frac{x^3}{x}$$

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$$

$\frac{1}{y^5} = t$

$y^{-5} = t$

$-5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\frac{dt}{dx}$

$-\frac{1}{5}\frac{dt}{dx} + \frac{t}{x} = x^2$

$Q = -5x^2$
$P = \frac{-5}{x}$

$\frac{dt}{dx} - \frac{5t}{x} = -5x^2$

$IF = e^{\int \frac{-5}{x}dx} = e^{-5\log_e x} = e^{\log_e x^{-5}} = \frac{1}{x^5}$

$t \cdot IF = \int Q \cdot IF\, dx$

$t \cdot \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5}\, dx$

$\frac{t}{x^5} = -5\int x^{-3}\, dx$

$= -5\,\frac{x^{-3+1}}{-3+1}$

$= \frac{-5}{-2}x^{-2} + c$

$\frac{t}{x^5} = \frac{5}{2}\frac{1}{x^2} + c$

$= \frac{1}{y^5 x^5} = \frac{5}{2}\frac{1}{x^2} + c.$

Exact diff eq :

Neccessary and sufficient cond^n for diff eq^n $M\,dx + N\,dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

→ Exact means it can be derived from general sol^n directly by differentiating it

Proof of Neccessary cond^n :-

$M\,dx + N\,dy = 0$ is exact prove $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof : Assume $f(x,y) = c$ be general sol^n

$$\partial[f(x,y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$$

Proof of Suffecient cond^n :

$M\,dx + N\,dy = 0$ in D.E and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is true

prove $M\,dx + N\,dy$ is exact eq.

Proof :- $M = \frac{\partial P}{\partial x}$ $N = \frac{\partial P}{\partial y}$

$\frac{\partial M}{\partial y} = \frac{\partial^2 P}{\partial y\, \partial x}$, we know $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 P}{\partial y\, \partial x}$

$\frac{\partial N}{\partial x} = \frac{\partial^2 P}{\partial y\, \partial x}$

Integrating wr rto x

$\Rightarrow \int \frac{\partial N}{\partial x}\, dx = \int \frac{\partial^2 P}{\partial y\, \partial x}\, dx$

$\Rightarrow \int \frac{\partial N}{\partial x}\, dx = \int \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) dx$

$N = \frac{\partial P}{\partial y} + \phi(y)$

$M\,dx + N\,dy = \frac{\partial P}{\partial x}\, dx + \left(\frac{\partial P}{\partial y} + \phi(y)\right) dy$

$= \underline{\frac{\partial P}{\partial x}\, dx + \frac{\partial P}{\partial y}\, dy} + \phi(y)\, dy$

$\boxed{d[F(y)] = \phi(y)\, dy}$

$= dP + \phi(y)\, dy$

$= dP + d(F(y))$

$= d[P + F(y)]$

Solution of Exact diff eq^n

1. $M\,dx + N\,dy = 0$.
2. Integrate M wr to x, considering y as const
3. Integrate N wr to y, having no x terms
4. Sum both $\int M\,dx + \int N\,dy$ to a constant c.

Ex^t $\quad y\sin 2x\,dx = (y^2 + \cos^2 x)\,dy$

$y\sin 2x\,dx - (y^2 + \cos^2 x)\,dy = 0 \Rightarrow M\,dx + N\,dy = 0$

$M = \underset{u\;\;v}{y\sin 2x} \qquad N = -(y^2 + \cos^2 x)$

$$\frac{\partial M}{\partial y} = \cancel{y(\cos 2x(2))} + \sin 2x(1)$$

$$\frac{\partial N}{\partial x} = 0 - 2\cos x(\sin x)$$

$$= \sin 2x$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$ true, so exact

Sol^n $\quad \underset{y\text{ const}}{\int M\,dx} + \underset{\text{no }x\text{ terms}}{\int N\,dy} = c$

$$\int y\sin 2x\,dx + \int -y^2\,dy = c$$

Sol^t $\quad -\frac{y\cos 2x}{2} - \frac{y^3}{3} = c$

2) $\underbrace{(y^2 e^{xy^2} + 4x^3)}_{M}\,dx + \underbrace{(2xy\,e^{xy^2} - 3y^2)}_{N}\,dy = 0$

$\left[\underset{x\text{ const}}{\frac{\partial M}{\partial y}} = e^{xy^2}(2y) + (0) \qquad \underset{y\text{ const}}{\frac{\partial N}{\partial x}} = \right]$ ✗

$\int y^2 e^{xy^2} + 4x^3 \, dx + \int -3y^2 \, dy = c$

y const   No x

$\frac{y^2 e^{xy^2}}{y^2} + \frac{4x^4}{4} - \frac{3y^3}{3} = c$

Sol:- $e^{xy^2} + x^4 - y^3 = c$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[y e^{xy^2} + 4x^3 \right]$

$= y e^{xy^2}(2yx) + e^{xy^2}(1)$

$= 2y e^{xy^2}[xy^2 + 1]$ //

$\frac{\partial N}{\partial x} = y[2x e^{xy^2} y^2 (1) + e^{xy^2}(2)]$

$= 2y e^{xy^2}[xy^2 + 1]$. //

Ans) $e^{xy^2} + x^4 - y^3 = c$

Inexact eq^ns :-

$M\,dx + N\,dy = 0$ but $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Method (I) :-

Regrouping.

Ex:

(a) $y(2xy + e^x)\,dx = e^x\,dy$

$2xy^2 dx + e^x y\,dx = e^x\,dy$

$M = 2xy^2 + e^x y \qquad N = e^x$

$\frac{\delta M}{\delta y} = 2x(2y) + e^x(1) \qquad \frac{\delta N}{\delta x} = e^x$

$\neq$ Not exact

$\Rightarrow \quad 2xy^2 dx + e^x y\,dx - e^x\,dy = 0$

$e^x y\,dx - e^x\,dy = -2xy^2\,dx$

$\frac{ye^x\,dx - e^x\,dy}{y^2} = -2x\,dx$

for this eq. $\frac{\delta N}{\delta x} = \frac{\delta M}{\delta y}$

$d\left(\frac{e^x}{y}\right) = -2x\,dx$

$\int d\left(\frac{e^x}{y}\right) = \int -2x\,dx$

$\frac{e^x}{y} = -\frac{2x^2}{2} + c$

$\Rightarrow \frac{e^x}{y} + x^2 = c$

(b) $x\,dy - y\,dx = a(x^2+y^2)\,dx$

$\frac{x\,dy - y\,dx}{x^2+y^2} = a\,dx$

$\int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \int a\,dx$

$\tan^{-1}\left(\frac{y}{x}\right) = ax + c$

Method : II : Integrating factor of homogeneous

→ In diff eq, $Mdx + Ndy = 0$ is homogeneous eq^n, and $Mx + Ny \neq 0$, the $\frac{1}{Mx+Ny}$ is the integrating factor.

Ex:

$$\underbrace{(x^2y - 2xy^2)}_{M}\,dx - \underbrace{(x^3 - 3x^2y)}_{N}\,dy = 0$$

$$Mx + Ny = (x^2y - 2xy^2)x + (-x^3 + 3x^2y)y.$$

$$\frac{dM}{dy} = x^2 - 4xy$$

$$\frac{dN}{dx} = -3x^2 + 6xy$$

$\neq$

Inexact eq.

Homogeneous,

$$If = \frac{1}{Mx+Ny}$$

$$\frac{1}{Mx+Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

Multiply with $IF = \frac{1}{x^2y^2}$

$$\frac{(x^2y - 2xy^2)}{x^2y^2}\,dx = \frac{(x^3 - 3x^2y)}{x^2y^2}\,dy$$

$$\underbrace{\left(\frac{1}{y} - \frac{2}{x}\right)}_{M}dx - \underbrace{\left(\frac{x}{y^2} - \frac{3}{y}\right)}_{N}dy = 0$$

$$\frac{dM}{dy} = \frac{-1}{y^2} = \frac{dN}{dx} = \frac{-1}{y^2}$$

exact

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx - \int\frac{-3}{y}\,dy = 0.$$

y const   no x

$$\frac{x}{y} - 2\log x + 3\log y = 0 + c$$

Method III: If eq of type $f(xy)\,y\,dx + g(xy)\,x\,dy = 0$

- If eq $M dx + N dy = 0$ be of form $f(xy)\,y\,dx + g(xy)\,x\,dy = 0$, then IF is $\frac{1}{Mx - Ny}$ where $Mx - Ny \neq 0$.

Example: $\underbrace{(1+xy)}_{f(xy)} y\,dx + \underbrace{(1-xy)}_{g(xy)} x\,dy = 0$.

$M = y + xy^2 \qquad N = x - x^2 y$

$$\frac{\partial M}{\partial y} = 1 + x(2y) \neq \frac{\partial N}{\partial x} = 1 - 2xy$$

Inexact eq of Method III.

$$IF = \frac{1}{Mx - Ny} = \frac{1}{yx + x^2y^2 - (xy - x^2y^2)}$$

$$IF = \frac{1}{2x^2y^2}$$

$$\frac{(1+xy)\,y\,dx}{2x^2y^2} + \frac{(1-xy)x}{2x^2y^2}\,dy = 0.$$

$$\frac{(xy^2 + y)\,dx}{2x^2y^2} + \frac{(x - x^2y)\,dy}{2x^2y^2} = 0$$

$$= \left(\frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-1}{2x^2} \cancel{- \frac{1}{2x^3y}} \qquad \frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{2x^2y^2} \qquad \text{exact}$$

$\int M dx + \int N dy = 0$

$\int \frac{1}{2x} + \frac{1}{2x^2 y} dx + \int -\frac{1}{2y} dy = c$

no x terms

y const

$\frac{1}{2}\log x + -\frac{1}{2yx} - \frac{1}{2}\log y = c$

$\frac{1}{2}\log\left(\frac{x}{y}\right) - \frac{1}{2xy} = c$

Inexact eqt

Method 4 and 5 :

Ex 1) $\underbrace{(xy^2 - e^{1/x^3})}_{M} dx - \underbrace{x^2 y}_{N} dy = 0$

$\frac{\partial M}{\partial y} = x(2y)$ $\qquad \frac{\partial N}{\partial x} = -y(2x)$

→ If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a f^n of x alone or const

$= f(x)$ then the IF $= e^{\int f(x) dx}$

→ If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a f^n of y alone or const $= f(y)$

then IF $= e^{\int f(y) dy}$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x2y + (2xy)}{-x^2 y} = \frac{4xy}{-x^2 y} = \frac{-4}{x} = f(x)$

$IF = e^{\int f(x) dx} = e^{-\int \frac{4}{x} dx} = e^{-\ln x^4} = \frac{1}{x^4}$ //

Multiply by $\frac{1}{x^4}$

$$\frac{1}{x^4}\left(xy^2 - e^{1/x^3}\right)dx - \frac{x^2 y}{x^4}\,dy = 0$$

$$\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}\,dy = 0.$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^3}(2y) = \frac{\partial N}{\partial x} = +\frac{2y}{x^3}$$

exact eq

$$\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx + \int 0\,dy = c$$

$$\frac{y^2 x^{-3+1}}{-3+1} - \frac{e^t\,dt}{-3} = c \qquad \frac{1}{x^3} = t$$

$$-\frac{3}{x^4}\,dx = dt$$

$$\frac{y^2 x^{-2}}{-2} + \frac{1}{3}e^t = c \qquad \frac{dx}{x^4} = \frac{dt}{-3}$$

$$-\frac{y^2}{2x^2} + \frac{1}{3}e^{\frac{1}{x^3}} = c$$

Ex ② $\underbrace{(xy^3 + y)}_{M}dx + 2\underbrace{(x^2y^2 + x + y^4)}_{N}dy = 0$

$$\frac{\partial M}{\partial y} = x(3y^2) + 1 \qquad \frac{\partial N}{\partial x} = 2(2xy^2 + 1)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3xy^2 + 1 - 4xy^2 - 2}{M}$$

$$= \frac{-xy^2 - 1}{M} = -\frac{(xy^2 + 1)}{xy^3 + y} = \frac{-1}{y}$$

$= f(y)$

$IF = e^{\int f(y)\,dy} = e^{\int \frac{1}{y} dy} = e^{\log_e y} = y$

$IF = y$

$y(xy^3+y)\,dx + 2y(x^2y^2+x+y^4)\,dy = 0$

$\underbrace{(xy^4+y^2)}_{M}\,dx + \underbrace{(2x^2y^3+2xy+2y^5)}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = x(4)y^3+2y \quad = \quad \frac{\partial N}{\partial x} = 2y^3(2x)+2y$

$\int M\,dx + \int N\,dy = c$

$= \int (xy^4+y^2)\,dx + \int 2y^5\,dy = c$

y const

$y^4\frac{x^2}{2} + y^2x + \cancel{2}\frac{y^6}{\cancel{6}\,3} = c$

$\frac{x^2y^4}{2} + x^2y + \frac{y^6}{3} = c$ //

Method - VI : eq $Mdx+Ndy=0$ is of form

$x^ay^b(my\,dx+nx\,dy)+x^cy^d(py\,dx+qx\,dy)=0$

where a,b,c,d,m,n,p,q are consts.

the x^hy^k is IF. where h,k are const.

After multiplication by x^hy^k, the given eq becomes exact.

Ex:- $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$.

$IF = x^h y^k$. $\qquad \int M\,dx + N\,dy = 0$

$x^h y^k [(2y + 6xy^2)\,dx + (3x + 8x^2 y)\,dy = 0]$

$\Rightarrow (2x^h y^{k+1} + 6x^{h+1} y^{k+2})\,dx + (3x^{h+1} y^k + 8x^{h+2} y^{k+1})\,dy$

$\frac{\delta M}{\delta y} = 2x^h (k+1) y^k + 6(k+2) x^{h+1} y^{k+1}$

$\frac{\delta N}{\delta x} = 3(h+1) x^h y^k + 8(h+2) x^{h+1} y^{k+1}$

$3(h+1) = 2(k+1) \rightarrow (1)$.

$3\,\cancel{6}(k+2) = \overset{4}{\cancel{8}}(h+2) \rightarrow (2)$

$3h + 3 = 2k + 2 \rightarrow (1) \qquad 3k + 6 = 4h + 8 \ (2)$

$3h - 2k + 1 = 0 \rightarrow (1) \qquad 4h - 3k + 2 = 0 \ (2)$.

Solving $h = 1, k = 2$.

$IF = xy^2$.

$xy^2 ((2y + 6xy^2)\,dx + (3x + 8x^2 y)\,dy] = 0$

$(2xy^3 + 6x^2 y^4)\,dx + (3x^2 y^2 + 8x^3 y^3)\,dy = 0$

$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x} \rightarrow$ Satisfies.

$\int M dx + \int N dy = c$

$\Rightarrow \int (2xy^3 + 6x^2 y^4)\,dx + \int 0\,dy = c$

$= \cancel{2} y^3 \frac{x^2}{\cancel{2}} + \overset{2}{\cancel{6}} y^4 \frac{x^3}{\cancel{3}} = c$

$\Rightarrow x^2 y^3 + 2x^3 y^4 = c$

First order higher degree eq^ns :

- General eq of a diff eq of the first order but not first degree (say nth degree) is

$$\left(\frac{dy}{dx}\right)^n + A_1\left(\frac{dy}{dx}\right)^{n-1} + A_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots + A_n = 0$$

Let $p = \frac{dy}{dx}$

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots + A_{n-1}p + A_n = 0$$

↳ first order nth degree

$$f(x, y, p) = 0.$$

Case I: Eq^ns solvable for p.

where you can resolve eq^ns into factors.

Suppose

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots A_{n-1}p + A_n = 0.$$

$$\underbrace{(p - f_1(x,y))}_{=0} \underbrace{(p - f_2(x,y))}_{=0} \underbrace{(p - f_3(x,y))}_{=0} \cdots \underbrace{(p - f_n(x,y))}_{=0} = 0.$$

Solutions: $\phi_1(x, y, c_1)$ $\phi_2(x, y, c_2)$ $\cdots$ $\phi_n(x, y, c_n)$.

without loss of generality,

$$c_1 = c_2 = c_3 = \cdots = c_n \} = c$$

$\therefore$ $\phi_1(x, y, c)$, $\phi_2(x, y, c)$, $\cdots$ $\phi_n(x, y, c)$

Combined sol^n

Ex 1: $\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0.$

$$p^2 - 5p + 6 = 0$$

$$(p-3)(p-2) = 0$$

$$p = 3 \qquad p = 2$$

$$\frac{dy}{dx} = 3 \qquad \frac{dy}{dx} = 2$$

$$y = 3x + c \quad , \quad y = 2x + c$$

(2). $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}.$

$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}.$$

Case - II: Cannot be factorized

M-1: Eq^ns solvable for x.

M-2: Eq^ns solvable for y.

ii) If DE $f(x, y, p) = 0$ is solvable for y,
then $y = f(x, p)$

diff w.r to x

$\frac{dy}{dx} = p = F(x, p, \frac{dp}{dx})$ → eq in two variable x, p

↓ Solving

$$f(x, p, c) = 0$$

eliminate p: $f(x, y, c)$; eliminate p by $\begin{cases} x = f_1(p, c) \\ y = f_2(p, c) \end{cases}$

Ext $y + px = p^2 x^4$

Case (ii) r - Cannot be resolved to factors

(i) Eqⁿ's solvable for y
(ii) Eqⁿs solvable for x
(iii) clairaut's eq.

(i) Eq solvable for y ↓

first order nth degree

diff w.r.to x | $y = f(x, p)$ → eq solvable for y,

↳ $\frac{dy}{dx} = p = F(x, p, \frac{dp}{dx})$

↓ solving

$f(x, p, c) = 0$

$f(x, y, c)$
eliminate p

Not eliminate
$x = f_1(p, c)$ → sol 1
$y = f_2(p, c)$ → sol 2
constitute to general solⁿ.

Ext $y + px = p^2 x^4$

$y = p^2 x^4 - px$ → eqⁿ solvable for y.

$\frac{dy}{dx} = p^2(4x^3) + x^4(2p)\frac{dp}{dx} - p(1) - x\frac{dp}{dx}$

↓

$p = 4x^3 p^2 + x^4 2p \frac{dp}{dx} - x\frac{dp}{dx} - p$

$$2p - 4x^3p^2 - 2x^4p\frac{dp}{dx} + x\frac{dp}{dx} = 0$$

$$2p(1-2x^3p) + x\frac{dp}{dx}(1-2x^3p) = 0$$

$$\left(2p + x\frac{dp}{dx}\right)(1-2x^3p) = 0$$

$$2p + x\frac{dp}{dx} = 0 \qquad 1-2x^3p = 0$$

1st term

$$2p = -x\frac{dp}{dx}$$

$$\frac{2dx}{x} = \frac{-dp}{p}$$

$$2\ln x = -\ln p + c.$$

$$\ln x^2 + \ln p = c_1$$

$$\ln x^2p = c_1$$

$$x^2p = e^{c_1} = c$$

$$\boxed{p = \frac{c}{x^2}}$$

subst in original eq.

$$y = p^2x^4 - px$$

$$y = \frac{c^2}{\cancel{x^4}}\cancel{x^4} - \frac{c}{x^{\cancel{2}}}\cancel{x}$$

$$y = c^2 - \frac{c}{x} \rightarrow \text{General sol}^n$$

2nd term

$$1 = 2x^3p$$

$$p = \frac{1}{2x^3}$$

$$\int\frac{dy}{dx} = \int\frac{1}{2}x^{-3}$$

$$y = \frac{1}{2}\frac{x^{-2}}{-2} + c_1$$

$$-4y = \frac{1}{x^2} + c_1$$

$$y = -\frac{1}{4x^2} + c$$

Check using given eq

$$p = \frac{1}{2x^3}$$

$$y = -\frac{1}{4x^2} + c$$

original eq is

$$y = p^2x^4 - px$$

$$-\frac{1}{4x^2} + c = \left(\frac{1}{2x^3}\right)^2 x^4 - \left(\frac{1}{2x^3}\right)x$$

$$-\frac{1}{4x^2} + c = \frac{1}{4x^2} - \frac{1}{2x^2}$$

$$-\frac{1}{4x^2} + c = -\frac{1}{4x^2}$$

$$\boxed{c = 0}$$

$$y = -\frac{1}{4x^2} + c$$

$$\boxed{y = \frac{-1}{4x^2} + 0} \to \text{free of consts}$$

& singular sol^n.

① GS = $y + \frac{c}{x} = c^2$

② SS = $y = \frac{-1}{4x^2}$

(2) $y = 2px - p^2$

$y = 2px - p^2$ (eq solvable for y)

$$\frac{dy}{dx} = 2\left(p(1) + x\frac{dp}{dx}\right) - 2p\frac{dp}{dx}$$

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$p = 2p + (2x - 2p)\frac{dp}{dx}$$

$$-p = (2x - 2p)\frac{dp}{dx}$$

$$-p = 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$-p\frac{dx}{dp} = 2x - 2p$$

$$\frac{dx}{dp} = -\frac{2x}{p} + \frac{2p}{p}$$

$$\frac{dx}{dp} + \frac{2}{p}(x) = 2$$ → Leibnitz form

$$IF = e^{\int \frac{2}{p}dp} = e^{2\ln p} = p^2 \quad \boxed{IF = p^2}$$

$$x(IF) = \int 2(IF)\,dp + c$$

$$xp^2 = 2\int p^2\,dp + c$$

$$xp^2 = \frac{2p^3}{3} + c$$

$$x = \frac{2p^3}{3p^2} + \frac{c}{p^2}$$

$$\boxed{x = \frac{2}{3}p + \frac{c}{p^2}}$$ → sol.

substitute in original eq

$$y = 2px - p^2$$

$$y = 2p\left(\frac{2p}{3} + \frac{c}{p^2}\right) - p^2$$

$$y = \frac{4p^2}{3} + \frac{2c}{p} - p^2$$

$$\boxed{y = \frac{p^2}{3} + \frac{2c}{p}}$$ Not eliminating p

Sol 1 and sol 2 constitute to general solution

Eq solvable for x & $f(x, p, y) = 0$.

(i) $x = f(p, y) \rightarrow$ eq solvable for x

diff w.r.to y $\frac{dx}{dy} = \frac{1}{p}$

$\frac{dx}{dy} = \frac{1}{p} = f(y, p, \frac{dp}{dy})$

$\downarrow$

$f(y, p, c)$

eliminate p

$f(x, y, c)$

Cannot eliminate p

$y = f_1(p, c) \rightarrow$ sol 1

$x = f_2(p, c) \rightarrow$ sol 2

Ex: $x = y + a \log p$

Eq solvable for x

so, $\frac{dx}{dy} = 1 + \frac{a}{p}\frac{dp}{dy}$

$\frac{1}{p} = 1 + \frac{a}{p}\frac{dp}{dy}$

$\frac{1-p}{\cancel{p}} = \frac{a}{\cancel{p}}\frac{dp}{dy}$

$1-p = m$

$-dp = dm$

$\int \frac{dy}{a} = \int \frac{dp}{1-p}$

$\int \frac{dy}{a} = \int \frac{-dm}{m}$

$\frac{y}{a} = -\log(1-p) + c$

$\boxed{y = -a\log(1-p) + c} \rightarrow$ sol

$x = y + a\log p$

$x = -a\log(1-p) + c + a\log p$

$$\boxed{x = a\log\frac{p}{1-p} + c} \rightarrow (2)$$

Eq (1) and eq (2) constitute in general eq.

Method 3 : Clairaut's eqⁿ

$\boxed{y = px + f(p)} \rightarrow$ function of p $y = f(x, p)$

$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$(x + f'(p))\frac{dp}{dx} = 0$

$\frac{dp}{dx} = 0$

$\boxed{p = c}$

$y = px + f(p)$

$\boxed{y = cx + f(c)} \rightarrow$

General sol

$x + f'(p) = 0$

$\Downarrow$

Singular s

$y = cx + f(c)$ General solⁿ

Singular solⁿ

(1) diff w.r.to c

(2) eliminate c

Ex : $p = \sin(y - xp)$

$\sin^{-1}p = y - xp \Rightarrow y = xp + \sin^{-1}p$

$y = xp + f(p)$

Clairaut's eq

General sol $p=c$

$y = cx + \sin^{-1}c \Rightarrow$ GS

~~diff~~

Singular solⁿ

$y = cx + \sin^{-1}c$

diff w.r.t c

$0 = x + \frac{1}{\sqrt{1-c^2}}$

$\sqrt{1-c^2} = \frac{-1}{x}$

$1-c^2 = \frac{1}{x^2} \Rightarrow 1-\frac{1}{x^2} = c^2$

$c^2 = \frac{x^2-1}{x^2}$

$c = \pm\sqrt{\frac{x^2-1}{x^2}}$

$y = \pm\frac{\sqrt{x^2-1}}{\cancel{x}}(\cancel{x}) + \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$

19/6/2020

LDE:

Dependent variable, derivatives appear only int its first degree and are not multipled together.

→ First order LDE

$\frac{dy}{dx} + Py = Q$

P, Q are f^ns of x.

LDE with nth order

- Linear differential eq with nth variable co-effe eq.
- LDE with nth order constant co-effecient eq.

LDE with nth variable coeffecient eq:

General form of LDE with variable coeffecient

$$\frac{d^n y}{dx^n} + A_1(x)\frac{d^{n-1}y}{dx^{n-1}} + A_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots A_n(x)y = Q(x)$$

$A_1, A_2, A_3 \ldots\ldots A_n$, $Q(x)$ are f^n of x only

LDE with nth order const co-effecient eq:

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots k_n y = Q(x)$$

$k_1, k_2, \ldots k_n$ are consts.

$Q(x)$ is f^n of x.

Homogeneous and Non-Homogeneous linear Diff eq

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots k_n y = Q(x)$$

If $Q(x) = 0$ Homogeneous LDE

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots k_n y = 0$$

If $Q(x) \neq 0$.

Non homogeneous LDE.

- Two solutions $y_1(x)$ and $y_2(x)$ are so be said to be linearly independent on the defined interval I if $c_1 y_1(x) + c_2 y_2(x) = 0$ only if $c_1 = c_2 = 0$.
- It is said to be linearly dependent if $c_1 y_1(x) + c_2 y_2(x) = 0$ holds for some non zero constant

Wronskian of determinant

check the linear independent of solution

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \quad \text{linearly independent}$$

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0 \quad \text{linearly independent}$$

Fundamental theorem of homogeneous linear DE's

linear homogeneous Linear

→ If y_1, y_2 are the sol^n of linear diff eq^n

$c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is also a solution of linear D.E [homogeneous]

★ Does not hold for non-linear and non-homogeneous.

Proof :

If y_1, y_2 are sol^n of the eq

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = 0$$

LDE Homogeneous

y_1 is solution

$$\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \dots + k_n y_1 = 0 \rightarrow ①$$

y_2 is solution.

$$\frac{d^n y_2}{dx^n} + \frac{d^{n-1} y_2}{dx^n} + \dots + k_n y_2 = 0$$

$u = c_1 y_1 + c_2 y_2$

$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1} y}{dx^n} + \dots + k_n u = 0$$

$$\Rightarrow \frac{d^n(c_1y_1+c_2y_2)}{dx^n} + k_1\frac{d^{n-1}(c_1y_1+c_2y_2)}{dx^{n-1}}$$

$$+ \cdots k_n(c_1y_1+c_2y_2) = 0$$

$$\Rightarrow c_1\left(\frac{d^ny_1}{dx^n} + k_1\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots k_ny_1\right)$$

①

$$+ c_2\left(\frac{d^ny_2}{dx^n} + k_1\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots k_ny_2\right) = 0$$

②

$$c_1(0) + c_2(0) = 0$$

$c_1y_1 + c_2y_2$ is solution of n^{th} order homogeneous LDE.

$\therefore$ $y_1, y_2, \ldots y_n$ are solutions of homogeneous LDE then $c_1y_1 + c_2y_2 + \cdots c_ny_n$ is also a solution

---

→ **Solution of n^{th} order linear DE with constant coeffecients**

① Consider first order first degree LDE

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \rightarrow \text{lebinitz eq.}$$

$$IF = e^{\int P(x)\,dx}$$

Sol $y \cdot (e^{\int P dx}) = \int Q(x)\, e^{\int P(x) dx} dx + C$

$y = \underbrace{e^{-\int P(x) dx} \int Q(x)\, e^{\int P(x) dx}}_{\text{II part}} + \underbrace{C e^{-\int P(x) dx}}_{\text{I part}}$

II part
Particular integral

I part
&
Complementary function

$\frac{dy}{dx} + P(x) y = Q(x)$ → Leibnitz

→ If $Q(x) = 0$ ⇒ Complementary solⁿ which satisfies homogeneous version of LDE

$\frac{dy}{dx} + Py = 0$

$\frac{dy}{y} = -P dx$ → Variable separable

$\ln y = -\int P dx + C_1$

$y = e^{-\int P(x) dx + C_1}$

$y = e^{C_1} e^{-\int P(x) dx}$

$\boxed{y = C e^{-\int P(x) dx}}$

General sol = complementary solⁿ + particular integral

→ Solⁿ which satisfies the homogeneous version of LDE

→ No of constants = order of D.E.

→ $Q(x) = 0$

Solⁿ that satisfies the non-homog version of LDE

→ No constants

Sol^n of n^{th} order LDE with constant coeff.

$y_{complete}$ = y complementary f^n + y particular integral

Method of complementary f^n:-

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_{n-1} \frac{dy}{dx}$$
$$+ k_n y = Q(x).$$

D operator form : $\frac{d}{dx} = D$ or $\frac{dy}{dx} = D(y)$

$\frac{d^n y}{dx^n} = D^n y$

$\therefore (D^n y + k_1 D^{n-1} y + k_2 D^{n-2} y + \dots k_n y)$

$= Q(x) \rightarrow$ D operator form

$(D - m_1)(D - m_2) \dots\dots (D - m_n) y = 0.$

& $m_1, m_2, \dots\dots m_n$ are roots of eq^n.

To find complementary f^n:- $Q(x) = 0.$

$(D - m_1)(D - m_2) \dots (D - m_n) y = 0$

Auxillary eq
Characteristic eq.

$Dy - m_1 y = 0$

$\frac{dy}{dx} = m_1 y$

$\int \frac{dy}{y} = \int m_1 dx$

$\ln y = m_1 x + C$

$y_1 = e^{m_1 x + c_1} = c e^{m_1 x}$

$y_2 = c e^{m_2 x}$

$y_3 = c e^{m_3 x}$

$\vdots$

$y_n = c e^{m_n x}$

$\rightarrow$ $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} \dots c_n e^{m_n x}$

is also a Sol^n of Homoglnous LDE

Case (i) :- Roots are real & distinct

$\boxed{y_{comp} f^n = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots c_n e^{m_n x}}$

Case (ii) :- If roots are repeating

$(D^n + k_1 D^{n-1} + \dots k_n) y = 0,$

$m_1 = m_2 \dots = m_n = m$ is repeating n times

$y_{cf} = c_1 e^{mx} + c_2 e^{mx} + \dots c_n e^{mx} = (c_1 + \dots c_n) e^{mx}$

Ex :- If roots are repeating $(D^n + k_1 D^{n-1} + k_2 D^{n-2}$

$+ \dots k_n) y = 0$

$m_1 = m_2 = \dots = m_n = m$ is repeating n times

$\boxed{y_{cf} = c e^{mx} x}$

$y_{cf} = c_1 e^{mx} + c_2 e^{mx} + \dots c_n e^{mx} = (c_1 + \dots c_n) e^{mx}$

$= c_1 e^{mx} \cancel{x}$

Sol $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

$(D^2 + 4D + 4)y = 0 \rightarrow$ Auxillary eq

Roots $\rightarrow -2, -2$

$y_{cf} = c_1 e^{-2x} + c_2 e^{-2x} = c e^{-2x}$ ✗

Assume $= y_{cf} = v(x) e^{-2x}$

$y' = -2e^{-2x} v(x) + v'(x) e^{-2x}$

$= (v' - 2v) e^{-2x}$

$y'' = (v'' - 2v') e^{-2x} + (-2e^{-2x})(v' - 2v)$

$(D^2 + 4D + 4)y = 0$

$[(v'' - 2v')e^{-2x} - 2e^{-2x}(v' - 2v)] + 4(\cancel{-2e^{-2x}}(v' - 2v)e^{-2x})$

$= 0$

$+ 4)y = 0.$

$e^{-2x}[v'' - 2v' - 2v' + 4v + 4v' - 8v$
$+ 4v] = 0$

$v'' = 0$

$e^{-2x} v'' = 0$

$\frac{d^2 v}{dx^2} = 0$

$\frac{dv}{dx} = c_1$

$\int dv = \int c_1 dx$

$v = c_1 x + c_2$

$v(x) = c_1 x + c_2$

$y_{cf} = (c_1 x + c_2) e^{-2x}$

Assume 3 repeating roots

$y = v(x)\, e^{-mx}$

$v''' = 0$

$v = (c_1 + c_2 x + c_3 x^2)$

Generalise,

m_1, m_2 ... n times

$v(x) = c_1 + c_2 x + c_3 x^2 + \dots c_n x^{n-1}$

$y_{cf} = (c_1 + c_2 x + c_3 x^2 + \dots c_n x^{n-1})\, e^{mx}$

Case iii) Roots are imaginary.

$\alpha \pm i\beta \Rightarrow \alpha + i\beta, \ \alpha - i\beta$

$(D - (\alpha \pm i\beta))\, y = (D - (\alpha - i\beta))\,(D - (\alpha + i\beta))\, y = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad m_1 \qquad\qquad m_2$

$\Rightarrow y_{cf} = c_1 e^{(\alpha - i\beta)x} + c_2 e^{(\alpha + i\beta)x}$

$y_{cf} = e^{\alpha x}\left[c_1 e^{-i\beta x} + c_2 e^{i\beta x}\right]$

$= e^{\alpha x}\left[c_1(\cos\beta x - i\sin\beta x) + c_2(\cos\beta x + i\sin\beta x)\right]$

$= e^{\alpha x}\left[\underbrace{(c_1 + c_2)}_{A_1}\cos\beta x + i\underbrace{(c_2 - c_1)}_{A_2}\sin\beta x)\right]$

$= e^{\alpha x}\left[A_1 \cos\beta x + iA_2 \sin\beta x\right]$

$$\boxed{y_{cf} = e^{\alpha x}\left[A_1 \cos\beta x + iA_2 \sin\beta x\right]}$$

where $A_1 = c_1 + c_2$, $A_2 = c_2 - c_1$

To find complimentary f^n.

(1) Roots are real and distinct. ($m_1, \ldots m_n$ roots)

$$y_{cf} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}$$

(2) Roots are repeating ($m_1, m_2, \ldots n$ times roots)

$$y_{cf} = (c_1 + c_2 x + c_3 x^2 + \ldots c_n x^{n-1}) e^{mx}$$

(3) Roots are imaginary roots ($\alpha \pm i\beta$)

$$y_{cf} = e^{\alpha x} [c_1 \cos \beta x + i c_2 \sin \beta x]$$

(4) Roots are $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2$.

$$y_{cf} = e^{\alpha_1 x} [c_1 \cos \beta_1 x + i c_2 \sin \beta_1 x] + e^{\alpha_2 x} [c_3 \cos \beta_2 x + i c_4 \sin \beta_2 x]$$

(5) $\alpha \pm i\beta$, $\alpha \pm i\beta$ 2 times

$$y_{cf} = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + i (c_3 + c_4 x) \sin \beta x]$$

Ex:

(1) $\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 0$

$D^2 y + 5Dy + 6y = 0$.

$(D^2 + 5D + 6) y = 0$

$(D^2 + 3D + 2D + 6) y = 0$

$(D(D+3) + 2(D+3)) y = 0$

$(D+3)(D+2) y = 0$

Roots are $-3, -2$, real & distinct.

$y_{cf} = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y_{cf} = c_1 e^{-3x} + c_2 e^{-2x}$

(2) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

$(D^2 + 6D + 9)y = 0$ A.E.

$(D+3)^2 y = 0.$

roots $= -3, -3$

$y_{cf} = (c_1 + c_2 x)e^{-3x}$

(3) $(D^3 + D^2 + 4D + 4)y = 0.$

$-1, \pm 2i.$

$[D^2(D+1) + 4(D+1)]y = 0$

$((D+1)(D^2+4))y = 0$

$\rightarrow y_{cf_1} = c_1 e^{-x}$

$\rightarrow y_{cf_2} = [c_2 \cos 2x + i c_3 \sin 2x]$

$y_{cf} = y_{cf_1} + y_{cf_2}$

(4) $(D^4 - 4D^2 + 4)y = 0.$

$(D \pm \sqrt{2})(D \pm \sqrt{2}) = 0$

$\pm\sqrt{2}, \pm\sqrt{2}$

$\alpha \pm i\beta$
$\alpha = \sqrt{2}$ $\beta = 0$

$y_{cf} = \cancel{e^0 (c_1 \cos} c_2 x$

$$y_{cf} = e^{\sqrt{2}x}\{(c_1+c_2x)\} + (c_1+c_2x)e^{-\sqrt{2}x}$$

5) $(D^2+1)^3 y = 0$

$D^2+1=0$

$D = \pm i, \pm i, \pm i$

$$y_{cf} = (c_1+c_2x+c_3x^2)\cos x + (c_4+c_5x+c_6x^2)\sin \beta x$$

(6) $(D^4+4)y=0$

$D^4 = -4$

$(D^4+4D+4D^2) - 4D^2$

$= (D^2+2)^2 - (2D)^2$

$= (D^2+2-2D)(D^2+2D+2) = 0$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$D^2-2D+2=0 \quad \overset{\alpha}{1} \pm \overset{\beta}{i}$

$D^2+2D+2=0 \quad -1 \pm i$

$$y_1 cf = e^{x}[c_1\cos\beta x + c_2\sin\beta x] + e^{-x}[c_3\cos\beta x + c_4\sin\beta x]$$

Particular Integrals :- 27/10/2020

① Polynomial operators :-

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots k_n y = Q(x)$$

$$(D^n y + k_1 D^{n-1} y + \ldots k_n y) = Q(x)$$

$$\boxed{P(D) y = Q(x)} \qquad \boxed{y_{PI} = \frac{1}{P(D)} Q(x)}$$

where $P(D) = D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots k_n$

② Inverse operator :-

If $Q(x)$ is differentiable, D^{-1} or $\frac{1}{D}$ is inverse operator such that $\frac{1}{D}(D Q) = Q(x)$ $\qquad D = \frac{d}{dx}$

$$\frac{1}{D}\left(\frac{1}{dx} Q\right) = Q(x)$$

$$\frac{Q}{D} = \int Q\,dx$$

If $Q(x)$ is contⁿ fⁿ, $\frac{1}{P(D)} Q(x)$ is a fⁿ of x with no arbitrary consts such that

$$\boxed{P(D)\left[\frac{1}{P(D)} Q\right] = Q(x)}$$

y_{cp} = no. of arbitrary consts

$P(D)\, y = Q(x)$ $\rightarrow$ n^{th} order LDE with const co-effecients

$y_{complete} = y_{cf} + y_{ps}$

$P(D)\, y = 0$ $\rightarrow$ find y_{cf}

For PI: $y_{PI} = \frac{1}{P(D)} Q(x)$

Case (i) Suppose if $P(D)$ has single root $(D-\alpha)$

$$y_{PI} = \frac{1}{(D-\alpha)} Q(x)$$

Multiply $(D-\alpha)$ on both sides of eq

$$(D-\alpha)\, y = (D-\alpha) \underbrace{\left[\frac{1}{(D-\alpha)} Q(x)\right]}_{y}$$ [Inverse operator]

$$(D-\alpha)\, y = Q(x)$$

$$\frac{dy}{dx} - \alpha y = Q(x) \rightarrow \text{Leibnitz eq}$$

$$IF = e^{-\int \alpha\, dx} = e^{-\alpha x}$$

G.S: $y\, e^{-\alpha x} = \int Q(x)\, e^{-\alpha x} dx$

$$\boxed{y_{P1} = e^{\alpha x} \int Q(x)\, e^{-\alpha x} dx}$$

$$\boxed{\therefore\ y = \frac{1}{(D-\alpha)} Q(x) = e^{\alpha x} \int Q(x)\, e^{-\alpha x} dx}$$

Case (ii) :- $P(D)$ has 2 roots $\Rightarrow (D-\alpha)(D-\beta)$

$$y_{P1} = \frac{1 \cdot Q(x)}{(D-\alpha)(D-\beta)}$$ → repeated integral, → partial Fract

$$= \frac{1}{(D-\beta)}\left[\frac{1}{(D-\alpha)} Q(x)\right]$$

$$= \frac{1}{(D-\beta)}\left[\underbrace{e^{\alpha x}\int Q(x)\, e^{-\alpha x}\,dx}_{V(x)}\right]$$

$$= \frac{V}{}$$

$$= e^{\beta x}\left[\int \left(e^{\alpha x}\int Q(x)\, e^{-\alpha x}\,dx\right) e^{-\beta x}\,dx\right]$$

Using partia fractions:

$$\rightarrow \frac{1}{(D-\alpha)(D-\beta)} Q(x) = \frac{1}{\alpha-\beta}\left[\frac{1}{D-\alpha} - \frac{1}{D-\beta}\right] Q(x)$$

$$= \frac{1}{(\alpha-\beta)}\left[e^{\alpha x}\int Q(x)\, e^{-\alpha x}\,dx - e^{\beta x}\int Q(x)\, e^{-\beta x}\,dx\right]$$

Ex:- $(D-2)\, y = e^{2x}$

$Q(x) = e^{2x}$

$\alpha = 2$

A) $y = \frac{1}{(D-2)} e^{2x}$

$= e^{2x}\int e^{-2x}\, e^{2x}\, dx$

$y_{P1} = e^{2x}\, x$

Ans) $y_{comp} = y_{cf} + y_{PI}$

To find y_{cf}

$(D-2)y = 0.$

Root $= 2$, real & distinct

$y = c_1 e^{m_1 x}$

$y_{cf} = c_1 e^{2x} \rightarrow$ comp f.

$y_{PI} = e^{2x} x$

y complete $= y_{cf} + y_{PI}$

$$\boxed{y_{com} = c_1 e^{2x} + x e^{2x}}$$

2) $\frac{1}{(D-1)(D-2)} e^{2x} = ?$

$(D-1)(D-2) = 0$

Roots are 1, 2.

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y_{cf} = c_1 e^{x} + c_2 e^{2x}$.

$y_{PI} = \frac{1}{(D-1)} \left[\frac{1}{(D-2)} e^{2x} \right] = \frac{1}{(D-1)} \left[x e^{2x} \right]$

$= e^{x} \int (x e^{2x}) e^{-x} dx$

$= e^{x} \int x e^{x} dx$

$y_{PI} = e^{x} (x e^{x} - e^{x}),$

$$y_{comp}' = \underbrace{c_1 e^x + c_2 e^{2x}}_{cf} + \underbrace{e^x (e^x x - e^x)}_{PI}$$

<u>Variation of parameters method</u> :-

If eq is in the form of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = g(x)$ (<u>2nd order eq</u>.)

P, q, $g(x)$ are f^n of x. The particular Integer is given by

$$y_{PI} = -y_1 \int \frac{y_2 g(x)\,dx}{W} + y_2 \int \frac{y_1 g(x)\,dx}{W}$$

y_1 y_2 are solutions of eq

here w = wronskian $= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$

If wronskian is $\neq 0$, $y_1\, y_2$ are linearly independent.

<u>Proof</u> :- $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = g(x) \rightarrow (1)$

If $y_1\, y_2$ are sol^n of eq (1)

$$y_{cf} = c_1 y_1 + c_2 y_2$$

Assume $y_{PI} = u(x)\, y_1 + v(x)\, y_2$ satisfies the solution

$$y' = u' y_1 + u y_1' + v' y_2 + v y_2'$$

$$= u y_1' + v y_2' + \underbrace{(u' y_1 + v' y_2)}_{0}$$

$$y'' = u y_1'' + u' y_1' + v' y_2' + v y_2''$$

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = Q(x)$$

Subst

$$\Rightarrow (u y_1'' + u' y_1' + v' y_2' + v y_2'') + p(u y_1' + v y_2')$$

$$+ q(u y_1 + v y_2) = Q(x)$$

$$u(y_1'' + p y_1' + q y_1) + v(y_2'' + p y_2' + q y_2)$$

$$\Rightarrow \quad = 0 \qquad\qquad = 0$$

$$+ u' y_1' + v' y_2' = Q(x)$$

as $D^2y + PDy + qy = 0$

sol are $y_1, y_2 = 0$

$c_1 y_1 + c_2 y_2 = 0$

$$u' y_1' + v' y_2' = Q(x)$$

$$u' y_1 + v' y_2 = 0$$

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ Q(x) & y_2' \end{vmatrix}}{W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \qquad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & Q(x) \end{vmatrix}}{W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$u' = \frac{0 - y_2 Q(x)}{W} = \frac{-y_2 Q(x)}{W}$$

$$v' = \frac{y_1 Q(x)}{W}$$

$$y = u(x) y_1 + v(x) y_2$$

$$y_p = -y_1 \int \frac{y_2 Q(x)}{W} dx + y_2 \int \frac{y_1 Q(x)}{W} dx$$

Hence proved.

Exp $\frac{d^2y}{dx^2} + 4y = \tan 2x$

To find y_{cf}.

$$\frac{d^2y}{dx^2} + 4y = 0$$

$$(D^2+4)y = 0$$

$$D^2 = -4$$

$$D = \pm 2i$$

$\alpha = 0 \quad \beta = 2$

$$y_{cf} = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$$

$$y_{cf} = e^0 [c_1 \cos 2x + c_2 \sin 2x]$$

y_{PI}, variation of parameters method

$y_1 = \cos 2x \qquad y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos 2x \cos 2x + 2\sin 2x \sin 2x$$

$\boxed{W=2}$

$$= 2(\cos^2 2x + \sin^2 2x) = 2(1) = 2$$

$$y_{PI} = -y_1 \int \frac{y_2 g(x)}{W}\,dx + y_2 \int \frac{y_1 g(x)\,dx}{W}$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2}\,dx + \frac{\sin 2x}{2} \int \cos 2x \tan 2x\, dx$$

$$= -\int \cos 2x \cdot \frac{\sin 2x \cdot \sin 2x}{2 \cos 2x}\, dx + \frac{1}{2}\sin 2x \int \cos 2x \frac{\sin 2x}{\cos 2x}\, dx$$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

$$= -\frac{1}{2}\int \sin^2 2x + \frac{1}{2}\int \sin^2 2x\, dx. \quad (X)$$

$$= -\frac{\cos 2x}{2}\int \frac{\sin^2 2x}{\cos 2x}\, dx + \frac{\sin 2x}{2}\int \cos 2x \frac{\sin 2x}{\cos 2x}\, dx$$

$$= -\frac{\cos 2x}{2}\int \left(\frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\right) dx + \frac{\sin 2x}{2}\int \sin 2x\, dx$$

$$= -\frac{\cos 2x}{2}\int (\sec 2x - \cos 2x)\, dx + \frac{\sin 2x}{2}\int \sin 2x\, dx$$

$$= -\frac{\cos 2x}{2}\left[\log(\sec 2x + \tan 2x) - \frac{\sin 2x}{2}\right]$$

$$+ \frac{\sin 2x}{2}\left[-\frac{\cos 2x}{2}\right].$$

Method 3 :- To find particular Integral

$$P(D)\, y = Q(x)$$

$$y_{P1} = \frac{1}{P(D)} Q(x)$$

Case (i) If $Q(x) = e^{ax}$

$$P(D)y = e^{ax}$$

$$y_{P1} = \frac{1}{P(D)} e^{ax} \qquad P(D) = D^n + K_1 D^{n-1} \dots$$

$$De^{ax} = \frac{d}{dx} e^{ax} = a e^{ax}, \quad D^2 e^{ax} = a^2 e^{ax}$$

$$D^n e^{ax} = a^n e^{ax}$$

$$(D^n + D^{n-1} + \dots) e^{ax} = (a^n + a^{n-1} + \dots a) e^{ax}$$

$$P(D) e^{ax} = P(a) e^{ax}$$

$$\frac{1}{P(D)} [P(D) e^{ax}] = \frac{1}{P(D)} [P(a) e^{ax}]$$

using inverse operator def.

$$e^{ax} = P(a) \frac{1}{P(D)} e^{ax}$$

$$\boxed{y_{P1} = \frac{1}{P(D)} e^{ax} = \frac{e^{ax}}{P(a)}} \quad \text{Provided } P(a) \neq 0$$

If $P(a) = 0$

$$P(D) = (D-a)\phi(D) \qquad P(D) = D^n + D^{n-1} + \dots = (D-m_1)\underbrace{(D-m_2)\dots(}_{\phi(D)}$$

provided $\phi(a) \neq 0$.

$$\frac{1}{P(D)} e^{ax} = \frac{1}{(D-a)\phi(D)} e^{ax}$$

$$= \frac{1}{(D-a)} \left[\frac{e^{ax}}{\phi(D)}\right]$$

$$y_{P1} = \frac{1}{(D-a)}\left[\frac{e^{ax}}{\phi(a)}\right] \qquad \left[\frac{1}{(D-\alpha)}g(x) = e^{\alpha x}\int g(x)e^{-\alpha x}dx\right]$$

$$= e^{ax}\int \frac{e^{ax}}{\phi(a)} e^{-ax} dx = e^{ax}\int \frac{1}{\phi(a)}dx$$

$$y_{P1} = \frac{e^{ax} x}{\phi(a)}$$

If $P(a) = 0$

$$y_{P1} = \frac{1}{D-a}\frac{e^{ax}}{\phi(a)} = \frac{x e^{ax}}{\phi(a)} = \frac{x e^{ax}}{P'(a)}$$

$$P(D) = (D-a)\phi(D)$$

$$P'(D) = (D-a)\phi'(D) + \phi(D)$$

If $D = a$ $\quad P'(a) = \phi(a)$

$$y_{P1} = \frac{1}{P(D)}e^{ax} = \frac{e^{ax}}{P(a)} \quad \text{if } P(a) \neq 0$$

If $P(a) = 0$.

$$= \frac{x e^{ax}}{P'(a)} \quad \text{if } P'(a) \neq 0$$

If $P'(a) \neq 0$.

$$= \frac{x^2 e^{ax}}{P''(a)} \quad \text{if } P''(a) \neq 0$$

Summary:

$\boxed{\text{If } g(x) = e^{ax}}$

$P(D)\, y = e^{ax}$

$y_{PI} = \frac{1}{P(D)} e^{ax} \quad \frac{e^{ax}}{P(a)}$, provided $P(a) \neq 0$

If $P(a) = 0 \quad = \frac{x\, e^{ax}}{P'(a)}$ provided $P'(a) \neq 0$

If $P'(a) = 0 \quad = \frac{x^2 e^{ax}}{P''(a)}$ Provided $P''(a) \neq 0$

example: $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

To find y_{cf}

roots $(D+2)(D-1)^2 y = 0$

$-2, 1, 1.$

$y_1 = c_1 e^{-2x}$

$y_2 = (c_2 + c_3 x)\, e^{x}$

Total $y_{cf} = (c_1 + c_2) e^{x} + c_3 e^{-2x}$

To find y_{PI}

$y_{PI} = \frac{1}{(D+2)(D-1)^2} (e^{-2x} + 2\sinh x)$

$= \frac{e^{-2x}}{(D+2)(D-1)^2} + \frac{1}{(D+2)(D-1)^2} 2\sinh x.$

$$= \frac{e^{-2x}}{(D+2)(D-1)^2} + \frac{2\cdot\frac{e^{x}-e^{-x}}{2}}{(D+2)(D-1)^2}$$

$$= \frac{e^{-2x}}{P(D)} + \frac{e^{x}}{P(D)} - \frac{e^{-x}}{P(D)}$$

$d_{3} \quad d_{2} \quad d_{1}$

① $\frac{1}{P(D)} e^{-x} = \frac{e^{-x}}{(D+2)(D-1)^2}$ $\quad a=-1$

$$\boxed{\frac{1}{P(D)} e^{ax} = \frac{e^{ax}}{P(a)}}$$

$$= \frac{e^{-x}}{(-1+2)(-1-1)^2} = \frac{e^{-x}}{4}$$

③ $\frac{e^{-2x}}{P(D)} = \frac{e^{-2x}}{(D+2)(D-1)^2} = \frac{1}{(D+2)}\left[\frac{e^{-2x}}{(D-1)^2}\right]$

$1, 1, -2$

$a=-2$

$$= \frac{1}{(D+2)}\left[\frac{e^{-2x}}{(-2-1)^2}\right] = \frac{1}{(D+2)}\left[\frac{e^{-2x}}{9}\right]$$

since $P(a) = 0$

Now, $\frac{x e^{ax}}{P'(a)}$ So, $\frac{x e^{-2x}}{9(1)} = \frac{x e^{-2x}}{9}$.

② $\frac{e^{x}}{P(D)} = \frac{e^{x}}{(D+2)(D-1)^2} = \frac{1}{(D-1)^2}\left[\frac{e^{x}}{D+2}\right]$

$1+2$

$-2, 1, 1$

$a=1$

$$= \frac{1}{(D-1)^2}\left[\frac{e^{x}}{3}\right]$$

$$= \frac{1}{(D-1)^2}\left[\frac{e^x}{3}\right] \qquad a=1$$

$P(D) = (D-1)^2$

$P(a) = 0$

$$= \frac{x^2 e^x}{3(D-1)^2} = \frac{x^2 e^x}{3P''(D)} \qquad P'(a) = 0$$

$P'(D) = 2(D-1)$

$P''(D) = 2$

$P''(a) \neq 0$

$$= \frac{x^2 e^x}{3P''(1)} = \frac{x^2 e^x}{3(2)}$$

$P''(1) = 2$

$$= \frac{x^2 e^x}{6}$$

$y_{complete} = y_{cf} + y_{P.I}$

$$y_{complete} = (c_1 + c_2 x)e^x + c_3 e^{-2x} + \frac{x^2 e^x}{6} + \frac{x e^{-2x}}{9} + \frac{e^{-x}}{4}.$$

Case II :

$$P(D)y = \sin(ax+b) \Rightarrow y_{P.I} = \frac{1}{P(D)}\sin(ax+b)$$

$$D(\sin(ax+b)) = \frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$$

$$\boxed{D^2 \sin(ax+b) = -a^2 \sin(ax+b)}$$

$$D^3 \sin(ax+b) = -a^3 \cos(ax+b).$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

$$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$$

$$p(D) = D^2 \quad P(D^2) \sin(ax+b) = P(-a^2) \sin(ax+b)$$

$$\Rightarrow \frac{1}{P(D^2)} \left[P(D^2) \sin(ax+b) \right] = \frac{1}{P(D^2)} \left[P(-a^2) \sin(ax+b) \right]$$

$$\sin(ax+b) = P(-a^2) \frac{1}{P(D^2)} \sin(ax+b)$$

$$\Rightarrow$$

$$y_{P_1} = \frac{1}{P(D^2)} \sin(ax+b) = \frac{1}{P(-a^2)} \sin(ax+b)$$

$$P(-a^2) \neq 0.$$

$$D^2 = -a^2$$

$$D^4 = (-a^2)^2 = a^4$$

If $P(-a^2) = 0$

$$\sin(ax+b) = \text{Img part of } e^{i(ax+b)}$$

$$= \text{Img } \frac{1}{P(D^2)} e^{i(ax+b)}$$

$$D = ai$$

$$\text{Img} = \frac{e^{i(ax+b)}}{P(-a^2)} \longrightarrow P(-a^2) \neq 0$$

If $P(-a^2) = 0$ $\quad y_{P_1} = \dfrac{x\, e^{i(ax+b)}}{P(-a^2)}$

Summary :- If $g(x) = \sin(ax+b)$ or $\cos(ax+b)$

$P(D^2)y = \sin(ax+b)$ or $\cos(ax+b)$

$y_{PI} = \frac{1}{P(D^2)} \sin(ax+b) = \frac{1}{P(-a^2)} \sin(ax+b)$

provided $P(-a^2) \neq 0$

If $P(-a^2) = 0$ $\quad = \frac{x \sin(ax+b)}{P'(-a^2)}$ provided $P'(-a^2) \neq 0$

If $P'(-a^2) = 0$ $\quad = \frac{x^2 \sin(ax+b)}{P''(-a^2)}$ provided $P''(-a^2) \neq 0$

Ex :- $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

$(D^3 + 4D)y = \sin 2x$

$y_{PI} = \frac{1}{(D^3+4D)} \sin 2x \rightarrow a = 2$

$\rightarrow D^2 = -a^2 = -4$

$= \frac{1}{D(D^2+4)} \sin 2x$

$P(D) = D^3 + 4D$

$P'(D) = 3D^2 + 4$

$= \frac{x \sin 2x}{P'(-4)}$

$P'(-4) = 3(-4) + 4$
$= -8$

$$\boxed{y_{PI} = \frac{x \sin 2x}{-8}}$$

$D^2 = -a^2$

Ex: $\frac{1}{(D-1)} \sin 2x \rightarrow e^{-ax} \int g(x) e^{-ax} dx$

$D^2 = -a^2 = -4$

rationalize

$$= \frac{(D+1)}{(D^2-1)} \sin 2x = \frac{(D+1)\sin 2x}{(-4-1)}$$

$$= \frac{(D+1)\sin 2x}{(-5)} = \frac{2\cos 2x + \sin 2x}{-5} = y_{P1}$$

Case (iii): If $g(x) = x^m$

$$P(D)\, y = x^m$$

$$y_{P1} = \frac{x^m}{P(D)} = [P(D)]^{-1} x^m$$

expand using binomial expansion

expand until D^m.

Ex: $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

$(D^2 + D)y = x^2 + 2x + 4$

$$y_{P1} = \frac{(x+2)^2}{D(D+1)}$$

$$y_{P1} = \frac{(1+D)^{-1}}{D}(x+2)^2$$

$$y_{P1} = \frac{(1 - D + D^2 - D^3 - -)}{D}(x+2)^2$$

$$= \frac{1}{D}\left[(x+2)^2 - D(x+2)^2 + D^2(x+2)^2\right]$$

$$= \frac{1}{D}\left[x^2+\cancel{2x}+4 - (2x+\cancel{2}) + (2)\right]$$

$$\frac{1}{D}\left[x^2+4\right] = \int (x^2+4)\,dx = \frac{x^3}{3}+4x$$

$\frac{1}{D}$ is integra

Case iv :- If $g(x) = e^{ax}V(x)$

$$P(D)\,y = e^{ax}V(x) \Rightarrow y_{P_1} = \frac{1}{P(D)}e^{ax}V(x)$$

let us assume $u(x)$

$$D(e^{ax}u) = (De^{ax})u + e^{ax}Du$$

$$= ae^{ax}u + e^{ax}Du = e^{ax}(D+a)u$$

$$D^2e^{ax}u = D\left[e^{ax}(D+a)u\right] = ae^{ax}(D+a)u + e^{ax}(D^2u + Dau)$$

$$= e^{ax}\left[D^2u + aDu + aDu + a^2u\right]$$

$$= e^{ax}(D+a)^2u$$

Take $P(D) = D$

$$D^3e^{ax}u = e^{ax}(D+a)^3u.$$

$$P(D)\,e^{ax}u = e^{ax}P(D+a)u.$$

exponentional shifting of polynomial operators

$$\frac{1}{P(D)}\left[P(D)\, e^{ax} u\right] = \frac{1}{P(D)}\left[e^{ax} P(D+a) u\right].$$

$$e^{ax} u = \frac{1}{P(D)}\left[e^{ax} P(D+a) \cdot u\right]$$

Let $P(D+a)u = V(x)$

$$u = \frac{V(x)}{P(D+a)}$$

$$\therefore e^{ax} u = \frac{1}{P(D)}\left[e^{ax} P(D+a) u\right]$$

$$e^{ax} \frac{1}{P(D+a)} V(x) = \frac{1}{P(D)}\left[e^{ax} V(x)\right]$$

$$\Rightarrow \boxed{y_{P1} = \frac{1}{P(D)} e^{ax} V(x) = e^{ax} \frac{1}{P(D+a)} V(x)}$$

Ex: y $(D^2-2D+4)y = e^x \cos x$

$$y_{P1} = \frac{e^x \cos x}{(D^2-2D+4)} \Rightarrow Q = \text{product } e^{ax} V(x)$$

$a = 1$

$D \to D+1$

$$= \frac{e^x \cos x}{(D+1)^2 - 2(D+1) + 4} = \frac{e^x \cdot 1}{D^2+2D+1-2D-2+4}$$

$$= e^x \left[\frac{\cos x}{D^2+3}\right], a = 1 \quad D^2 = -a^2 \quad D^2 = -1$$

$$= \frac{e^x \cos x}{-1+3} = \frac{e^x \cos x}{2}.$$

Case (V) :- If $Q(x) = x\,V(x)$

$P(D)\, y = x\, V(x)$

$$y_{PI} = \frac{1}{P(D)} x\, V(x) = \left[x - \frac{1}{P(D)} P'(D) \right] \frac{V(x)}{P(D)}$$

Proof :- Leibnitz rule of successive differentiation

$$D^n(uv) = (D^n u)v + {}^nC_1 (D^{n-1}u)(Dv) + {}^nC_2 (D^{n-2}u)(D^2 v) + \ldots + {}^nC_n (u)(D^n v)$$

$Q(x) = xu$

$$D^n(ux) = (D^n u)(x) + {}^nC_1 (D^{n-1}u)(Dx)$$

$$D^n(ux) = (D^n u)x + {}^nC_1 (D^{n-1}u)$$

If $D^n = P(D) \Rightarrow P'(D) = n D^{n-1}$

$$P(D)(ux) = P(D)\, u(x) + \cancel{P'(D)}\, u$$
$$n D^{n-1} u$$

$$= P(D)\, u(x) + P'(D)\, u$$

$$P(D)(u(x)) = x P(D) u + P'(D) u \qquad \boxed{y_{PI} = \frac{1}{P(D)} x V(x)}$$

- Assume, $P(D) u = V(x)$

$$u = \frac{1}{P(D)} V(x)$$

$$P(D) \left[\frac{x}{P(D)} V(x) \right] = x V(x) + P'(D) \frac{u}{P(D)} V(x)$$

$$\cancel{x\,V(x)} = x V(x) + P'(D)\frac{1}{P(D)} V(x)$$

$$P(D)\left[x \frac{1}{P(D)} V(x)\right]$$

$$\Rightarrow \frac{1}{P(D)}\left[P(D)\left[x \cdot \frac{1}{P(D)} V(x)\right]\right] = \frac{1}{P(D)} \cdot x V(x)$$

$$+ \frac{1}{P(D)}\left[P'(D)\frac{1}{P(D)} V(x)\right]$$

$$\Rightarrow x \frac{1}{P(D)} V(x) - \frac{1}{P(D)}\left[P'(D)\frac{1}{P(D)} V(x)\right) = \frac{1}{P(D)} x V(x)$$

$$\boxed{y_{P1} = \frac{1}{P(D)} x V(x) = \left(x - \frac{P'(D)}{P(D)}\right]\frac{1 V(x)}{P(D)}}$$

Ex: $(D+1)^2 y = x\cos x$

$$y_{P1} = \frac{1}{(D+1)^2} x\cos x$$

$$= \left[x - \frac{1 \cdot 2(D+1)}{(D+1)^2}\right] \times \frac{\cos x}{(D+1)^2}$$

$$= \left[x - \frac{2}{(D+1)}\right] \times \frac{\cos x}{D^2+2D+1}$$

$a = 1$

$D^2 = -a^2$

$D^2 = -1$

$$= \left(x - \frac{2}{(D+1)}\right)\frac{\cos x}{\cancel{1}+2D\cancel{+1}}$$

$$= \left(x - \frac{2}{(D+1)}\right)\frac{1}{2D}\cos x$$

$$= \left(x - \frac{2}{D+1}\right) \frac{1}{2} \int \cos x \, dx$$

$$= \left(x - \frac{2}{D+1}\right) \frac{1}{2} \sin x$$

$$= \frac{x \sin x}{2} - \frac{1}{(D+1)} \sin x \rightarrow \text{case (ii)}$$

$$= \frac{x \sin x}{2} - \frac{(D-1)}{(D^2-1)} \sin x$$

$a = 1$

$D^2 = -a^2$

$D^2 = -1$

$$= \frac{x \sin x}{2} - \frac{(D-1)}{(-1-1)} \sin x$$

$$= \frac{x \sin x}{2} + \frac{(D-1)}{2} \sin x$$

$$= \frac{x \sin x}{2} + \frac{D \sin x}{2} - \frac{\sin x}{2}$$

$$= \frac{x \sin x}{2} + \frac{\cos x}{2} - \frac{\sin x}{2}$$

$$= \frac{x \sin x + \cos x - \sin x}{2}$$

$$y_{P_1} = \frac{\sin x (x-1) + \cos x}{2}$$

- nth order linear differ eq with variable coeff

$$\frac{d^n y}{dx^n} + A_1(x) \frac{dy^{n-1}}{dx^{n-1}} + A_2(x) \frac{dy^{n-2}}{dx^{n-2}} \cdots + A_n(x) y = \phi$$

$\downarrow$

Reduce variable co-eff to const co-eff

Cauchy's Euler diff eq

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots k_{n-1} x \frac{dy}{dx}$$

$$+ k_n y = Q(x)$$

Reduced to const co-effecrcant.

Substitute $e^t = x$ $\quad t = \log x$

$$\frac{dt}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dt} = \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$x \frac{dy}{dx} = \frac{dy}{dt} = Py \qquad \frac{d}{dt} = P$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} \quad \frac{1}{x}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2} \qquad D = \frac{d}{dt}$$

$$= \frac{1}{x^2}(D^2y - Py) = \frac{1}{x^2} D(D-1)y$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} D(D-1)y$$

$$\boxed{x^2\frac{d^2y}{dx^2} = D(D-1)y}$$

Substitution: $e^t = x$; $x\frac{dy}{dx} = Dy$,

$$x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \ldots x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

Exp 2 $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$ ..2

Cauchy's Euler eq

$x = e^t$

$x\frac{dy}{dx} = Dy$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$D(D-1)y - Dy + y = t$

$(D^2 - D - D + 1)y = t$

$(D^2 - 2D + 1)y = t$

$y = \frac{t}{(D-1)^2}$ Reduced to Const.

$2 \xrightarrow{2} 3$

$1 \xrightarrow{3} 4$