Advanced Data Structures and Algorithms

Dynamic Programming and

Floyd-Warshall (All-Pairs Shortest Paths: APSP)

Today

- Bellman-Ford is a special case of ***Dynamic Programming!***
- What is dynamic programming?
  - Warm-up example: Fibonacci numbers
- Another example:
  - Floyd-Warshall Algorithm

Bellman-Ford is an example of…
Dynamic Programming!

Today:

- Example of Dynamic programming:
  - Fibonacci numbers.
  - (And Bellman-Ford)
- What is dynamic programming, exactly?
  - And why is it called "dynamic programming"?
- Another example: Floyd-Warshall algorithm
  - An "all-pairs" shortest path algorithm

Fibonacci Numbers

- Definition:
  - F(n) = F(n-1) + F(n-2), with F(0) = F(1) = 1.
  - The first several are:
    - 1
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    - 2
    - 3
    - 5
    - 8
    - 13, 21, 34, 55, 89, 144,…

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    - 8
    - 13, 21, 34, 55, 89, 144,…
- Question:
  - Given n, what is F(n)?

Candidate algorithm

- **def** Fibonacci(n):
  - **if** n == 0 or n == 1:
    - **return** 1
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- $T(n) = T(n-1) + T(n-2) + O(1)$
- $T(n) \geq T(n-1) + T(n-2)$ for $n \geq 2$

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- T(n) $\geq$ T(n-1) + T(n-2) for n $\geq$ 2
- So T(n) grows *at least* as fast as the Fibonacci numbers themselves…
- Fun fact, that's like ϕ^n where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

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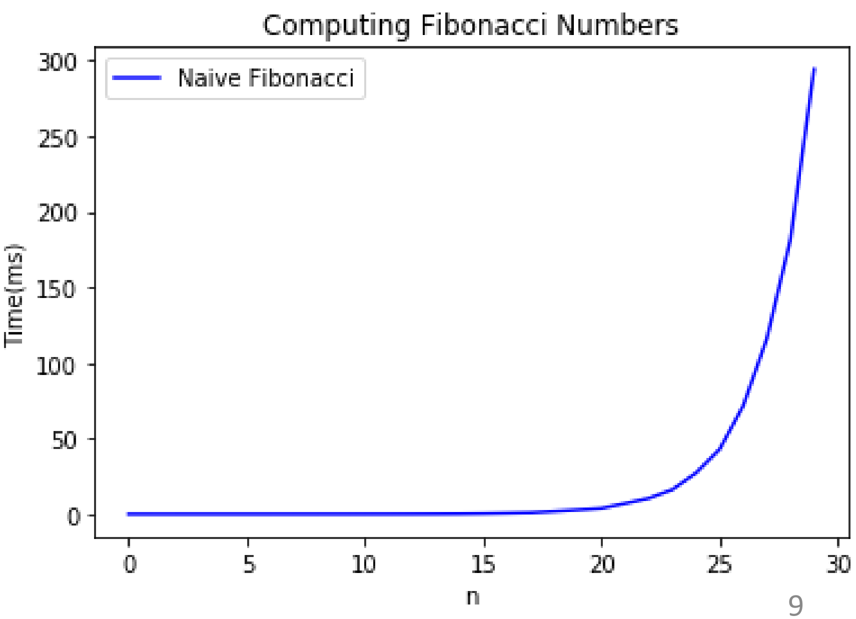
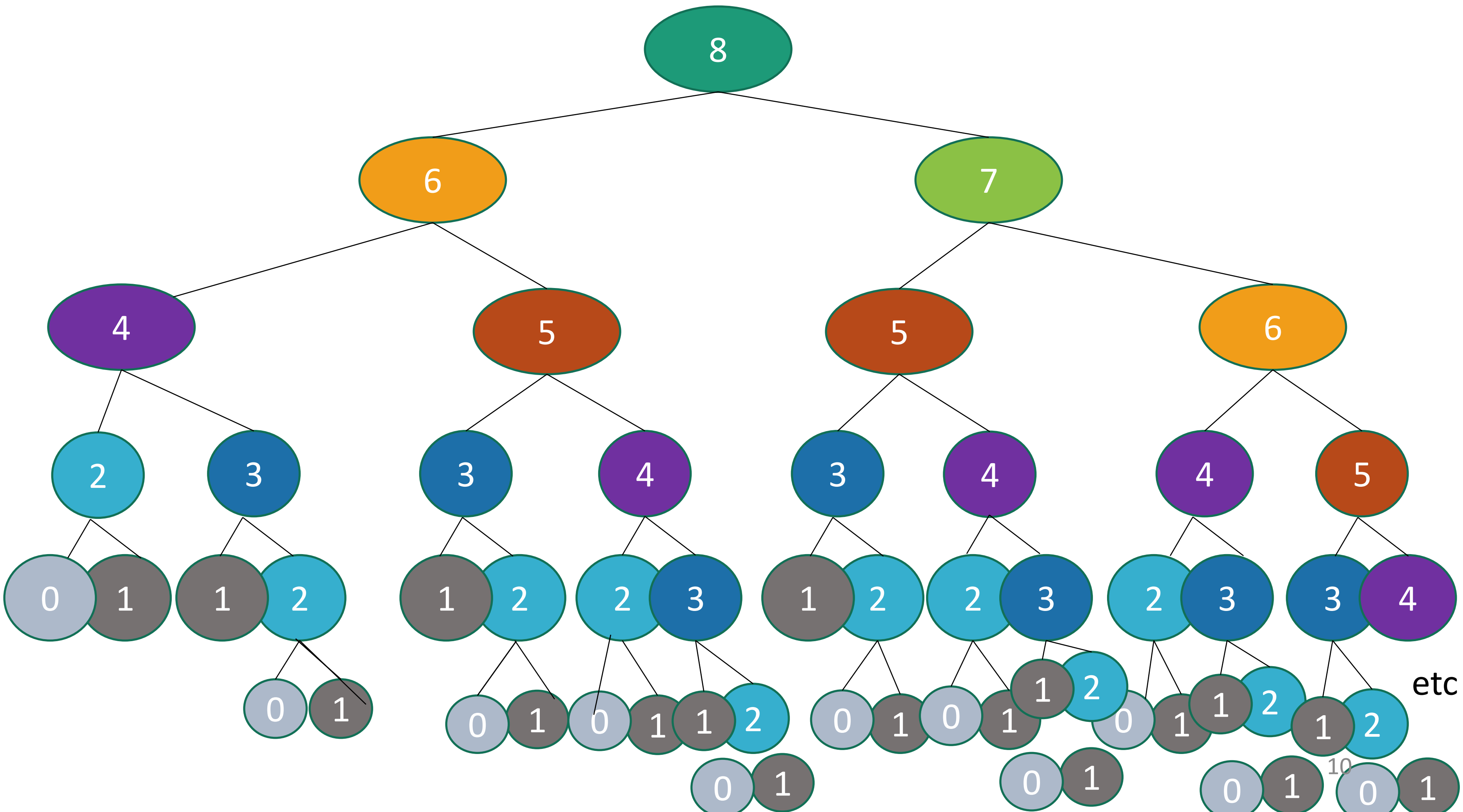
- $T(n) \geq T(n\text{-}1) + T(n\text{-}2)$ for $n \geq 2$
- So $T(n)$ grows *at least* as fast as the Fibonacci numbers themselves…
- Fun fact, that's like ϕ^n where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
- aka, **EXPONENTIALLY QUICKLY ☹**



What’s going on?

Consider Fib(8)



What's going on?

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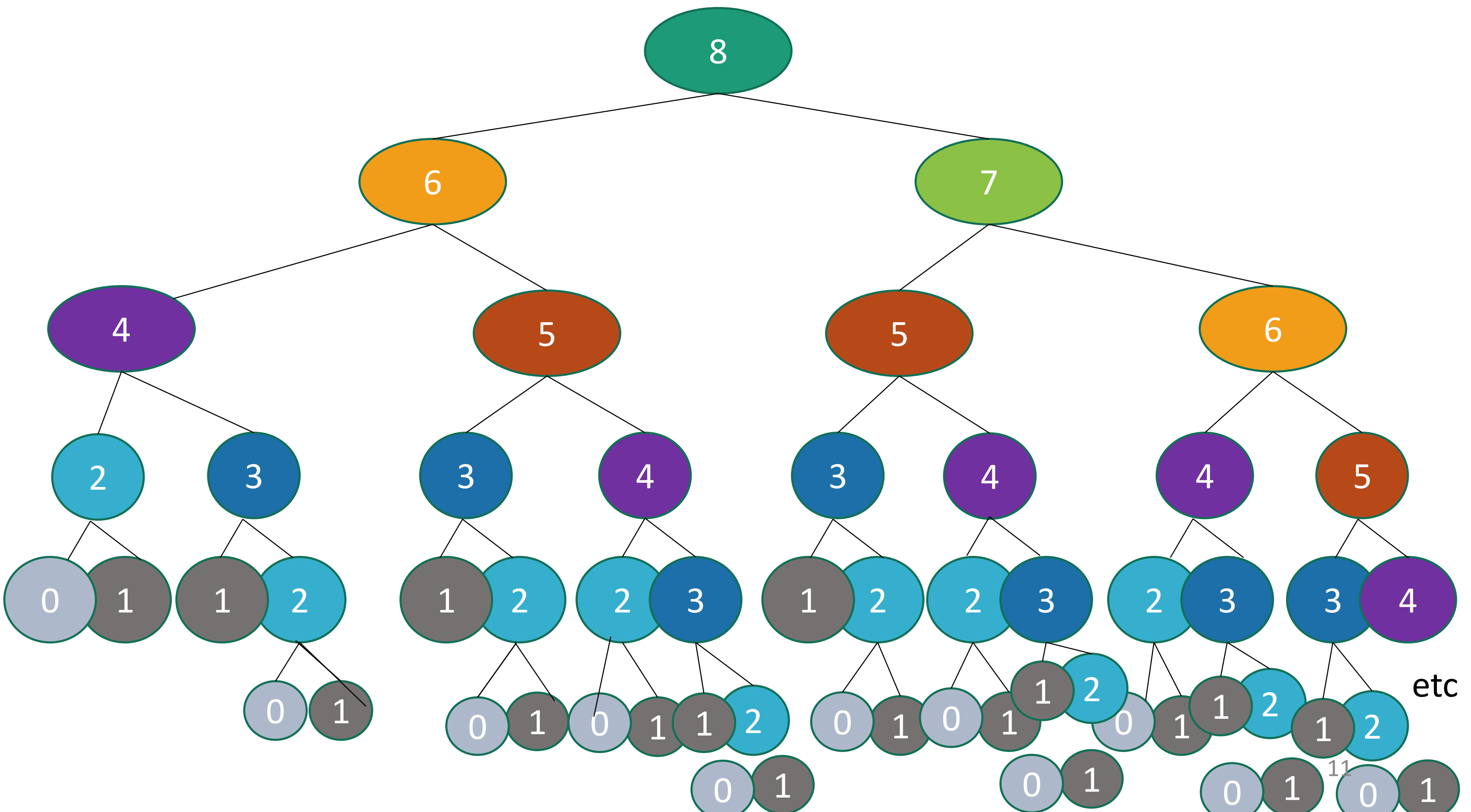
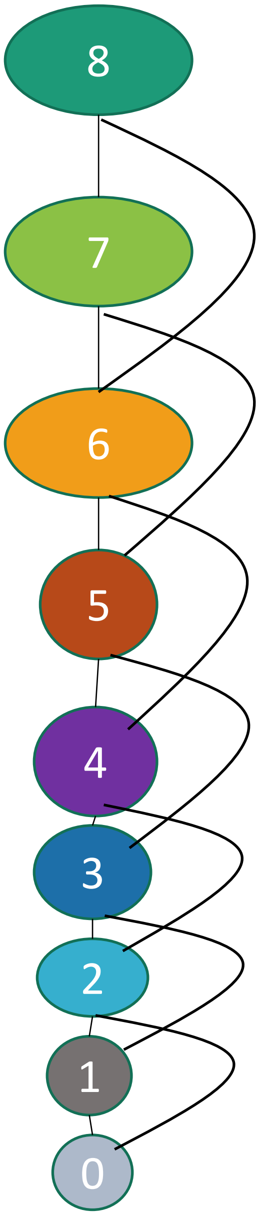
That's a lot of repeated computation!



Maybe this would be better:



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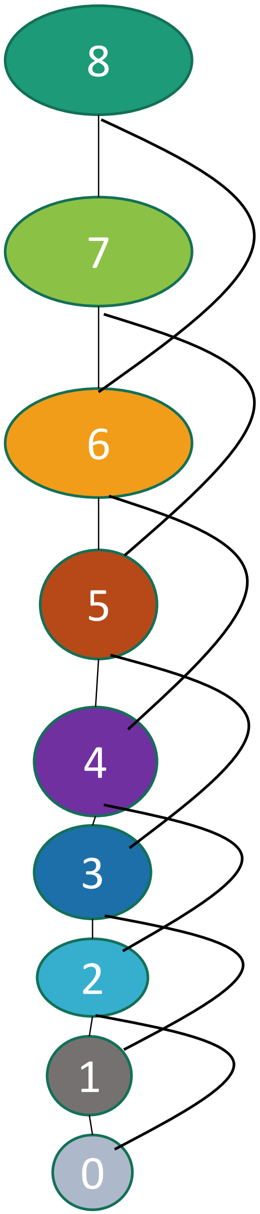

def fasterFibonacci(n):

- F = [1, 1, None, None, …, None]
  - \\ F has length n + 1
- **for** i = 2, …, n:
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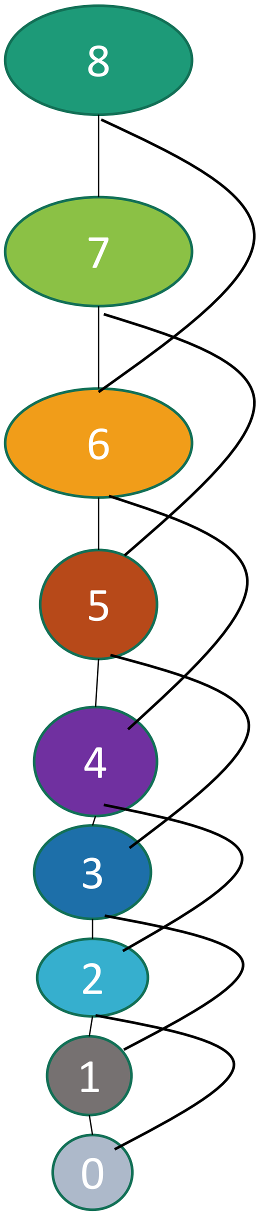

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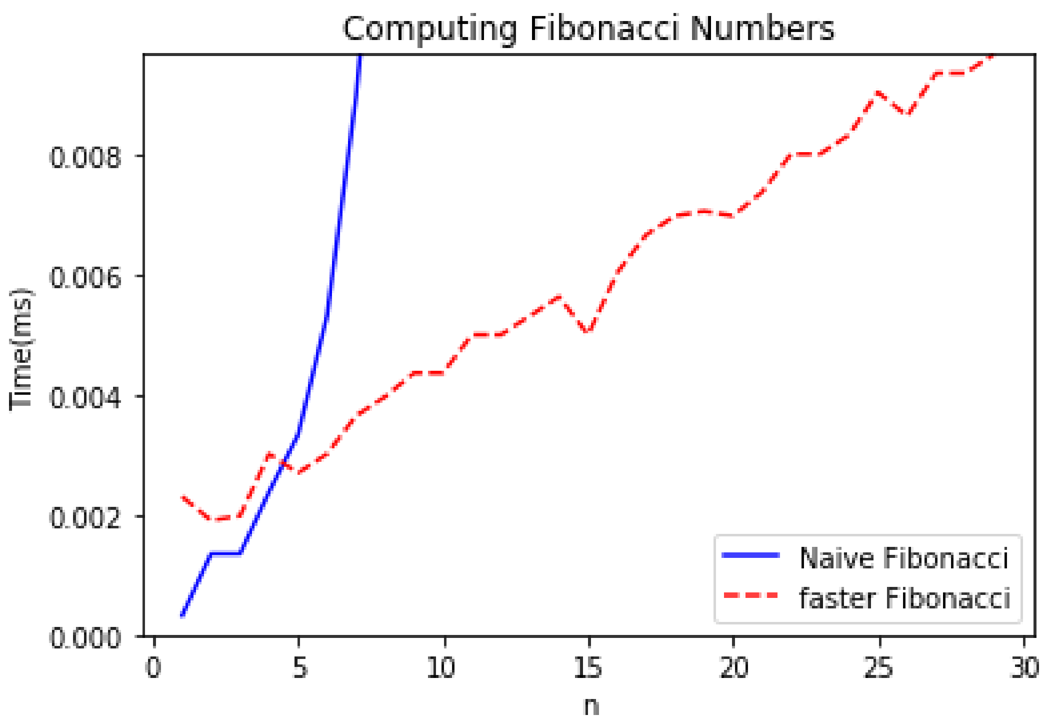
Much better running time!



This was an example of…

Dynamic Programming!

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- It is an algorithm design paradigm
  - like divide-and-conquer is an algorithm design paradigm.

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- It is an algorithm design paradigm
  - like divide-and-conquer is an algorithm design paradigm.
- Usually it is for solving **optimization problems**
  - eg, ***shortest*** path
  - (Fibonacci numbers aren't an optimization problem, but they are a good example…)

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  - Bellman-Ford:

$$d^{(i+1)}[v] \leftarrow \min\{\, d^{(i)}[v],\ \min_u \{d^{(i)}[u] + \mathrm{weight}(u,v)\} \,\}$$

Shortest path with at most i edges from s to v

Shortest path with at most i edges from s to u.

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  - This means that we can save time by solving a sub-problem just once and storing the answer.

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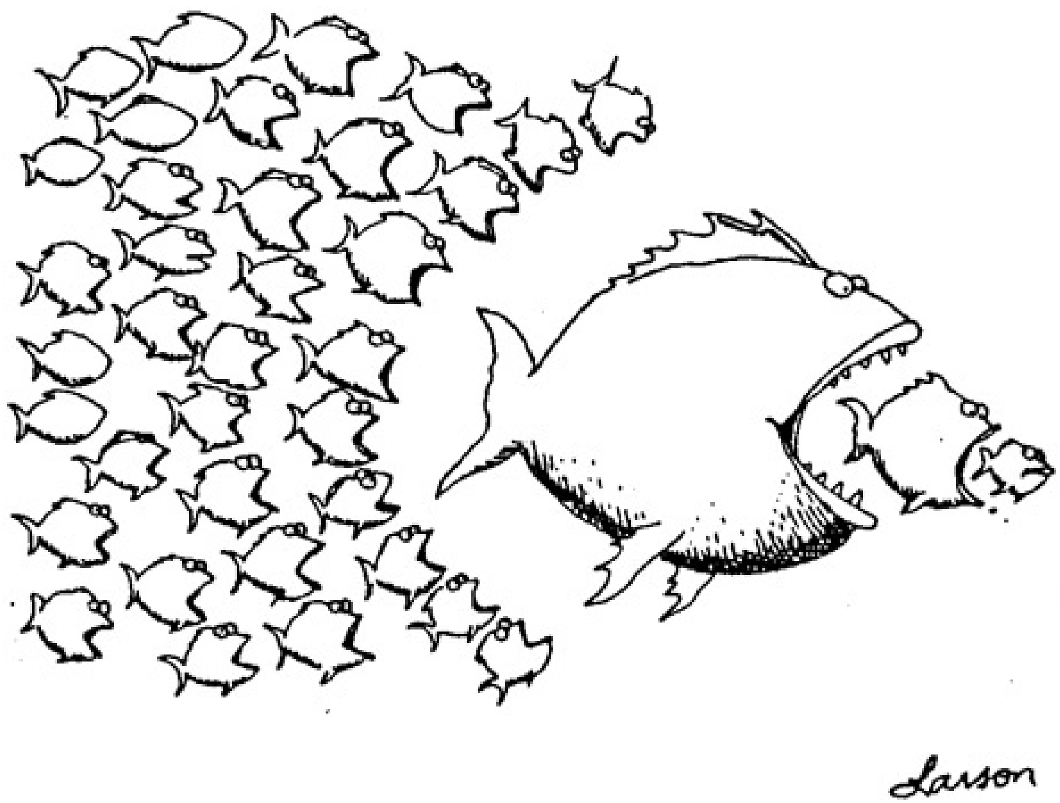
- Using these properties, we can design a ***dynamic programming*** algorithm:
  - Keep a table of solutions to the smaller problems.
  - Use the solutions in the table to solve bigger problems.
  - At the end we can use information we collected along the way to find the solution to the whole thing.

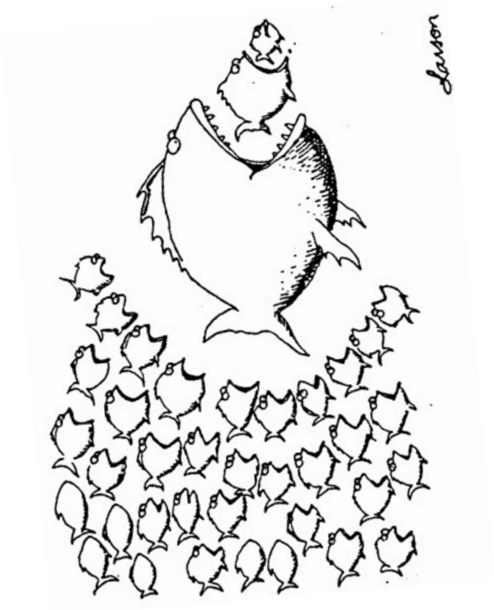
Two ways to think about and/or implement DP algorithms

- Top down
- Bottom up



Bottom up approach

what we just saw.



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- For Fibonacci:
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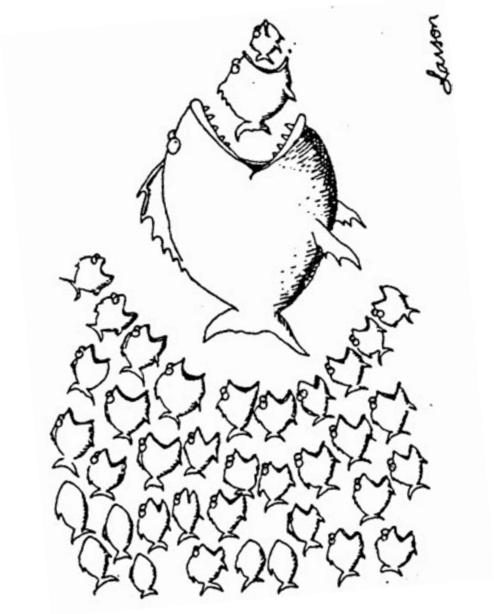

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- For Fibonacci:
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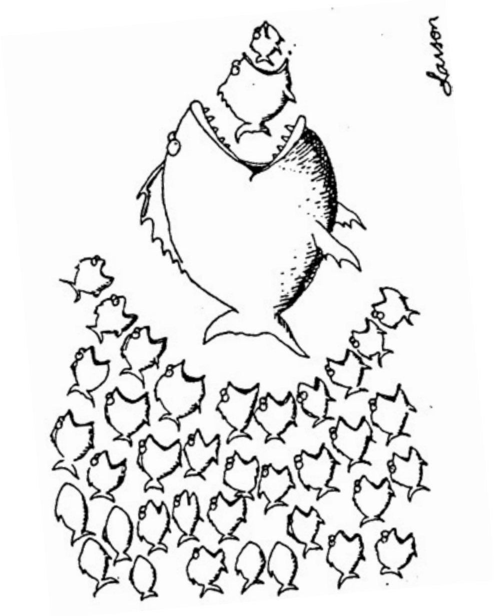

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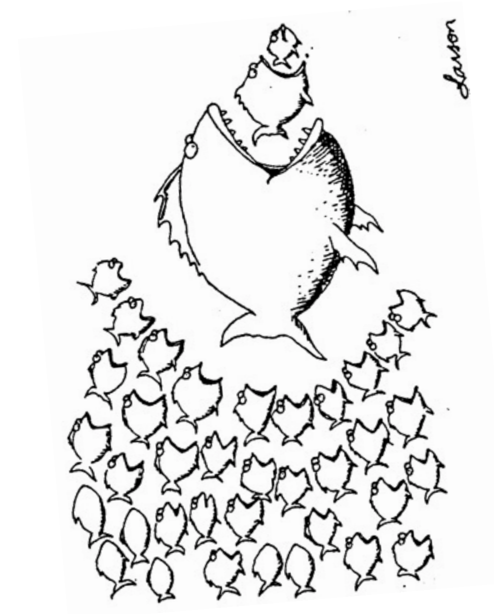

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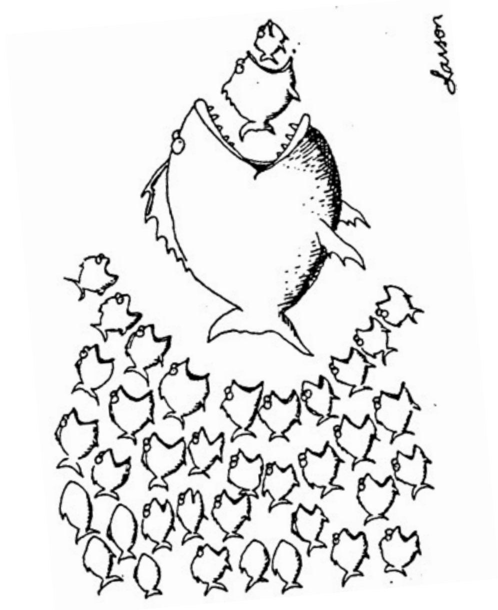
- Then finally solve the real problem.
  - fill in F[n]



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- For Bellman-Ford:
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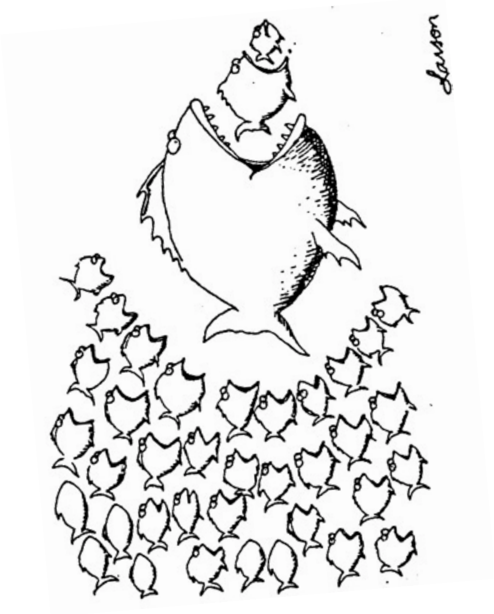

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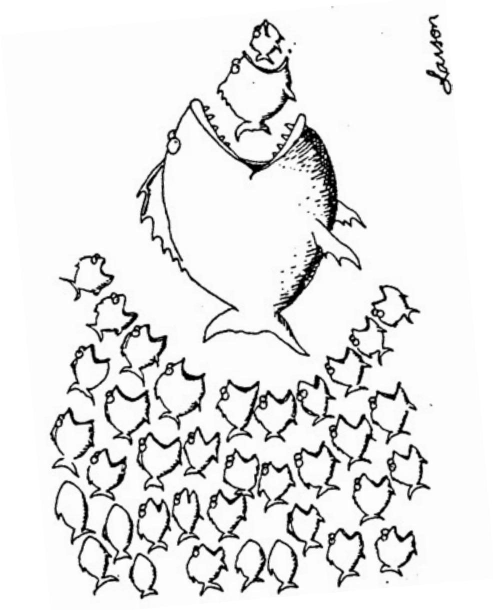

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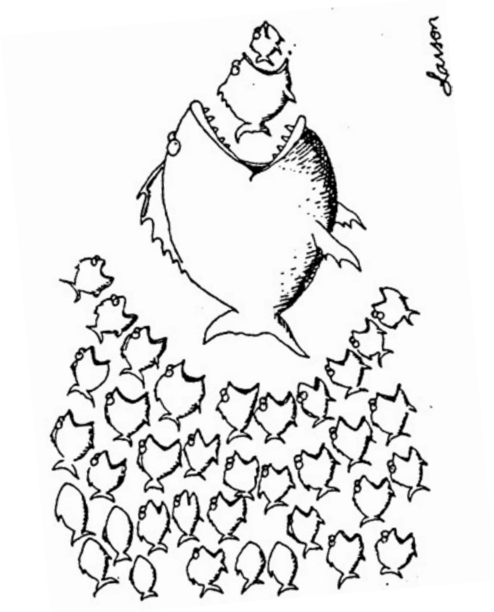

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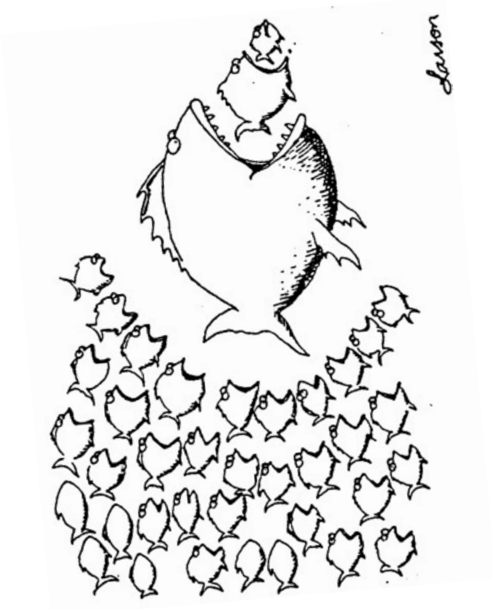

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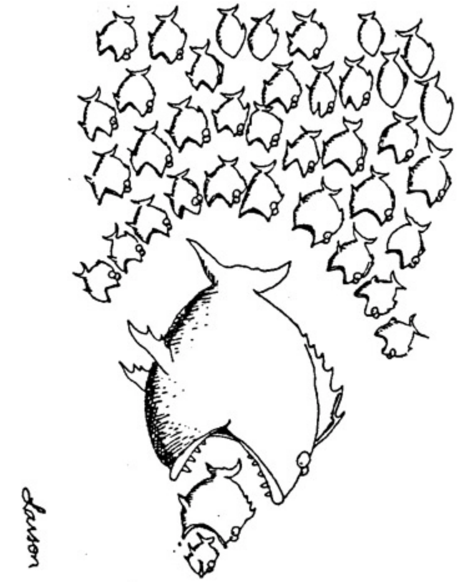
Top down approach



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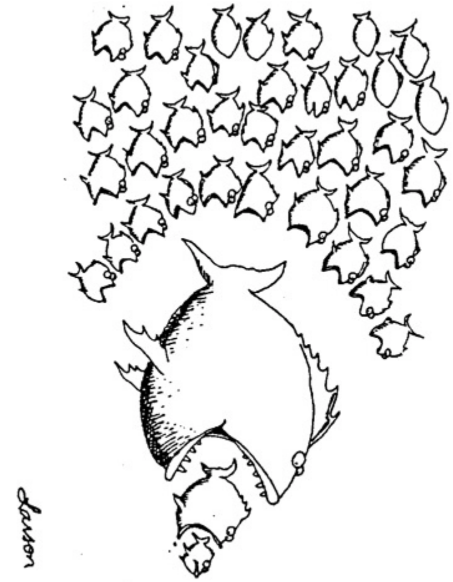

- Think of it like a recursive algorithm.
- To solve the big problem:
  - Recurse to solve smaller problems
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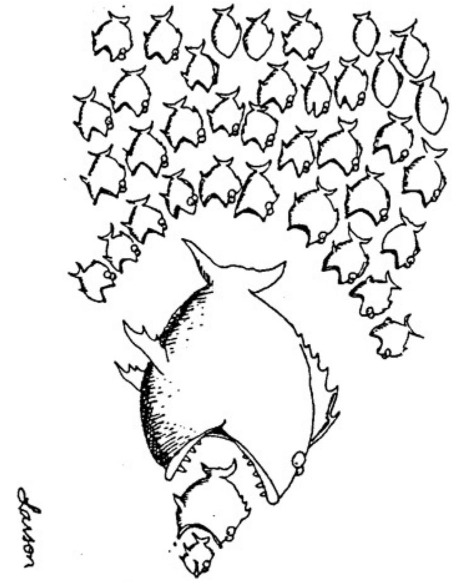
Top down approach



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- The difference from divide and conquer:
  - Keep track of what small problems you've already solved to prevent re-solving the same problem twice.
  - Aka, "**memorization**"

Example of top-down Fibonacci

- define a global list F = [1,1,None, None, …, None]
- **def** Fibonacci(n):
  - **if** F[n] != None:
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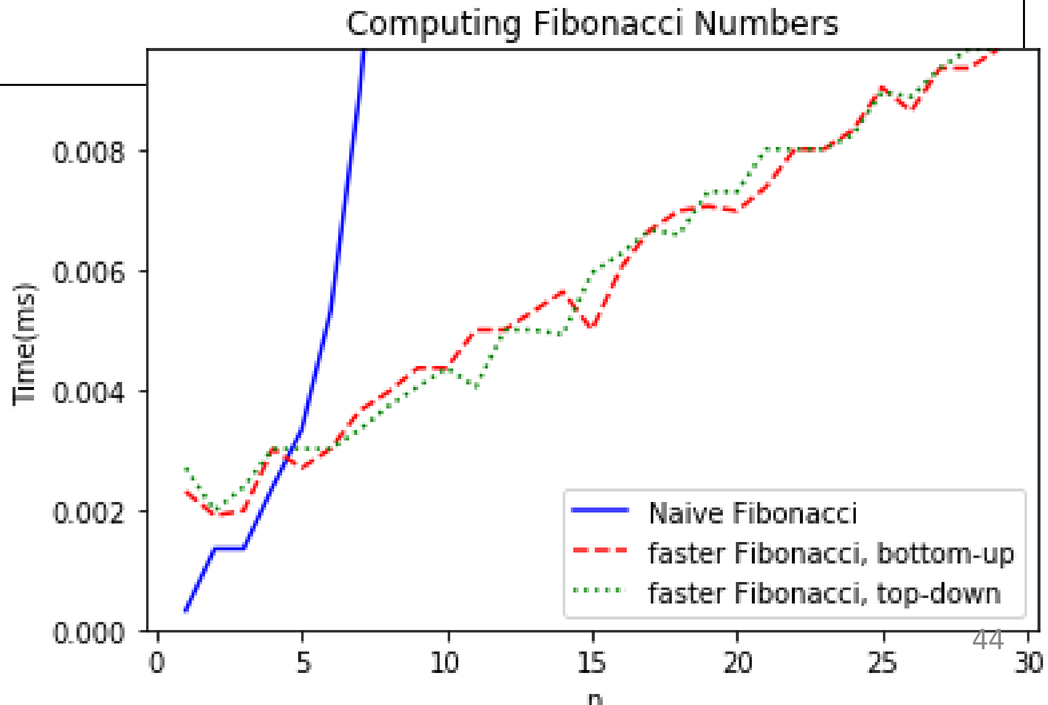
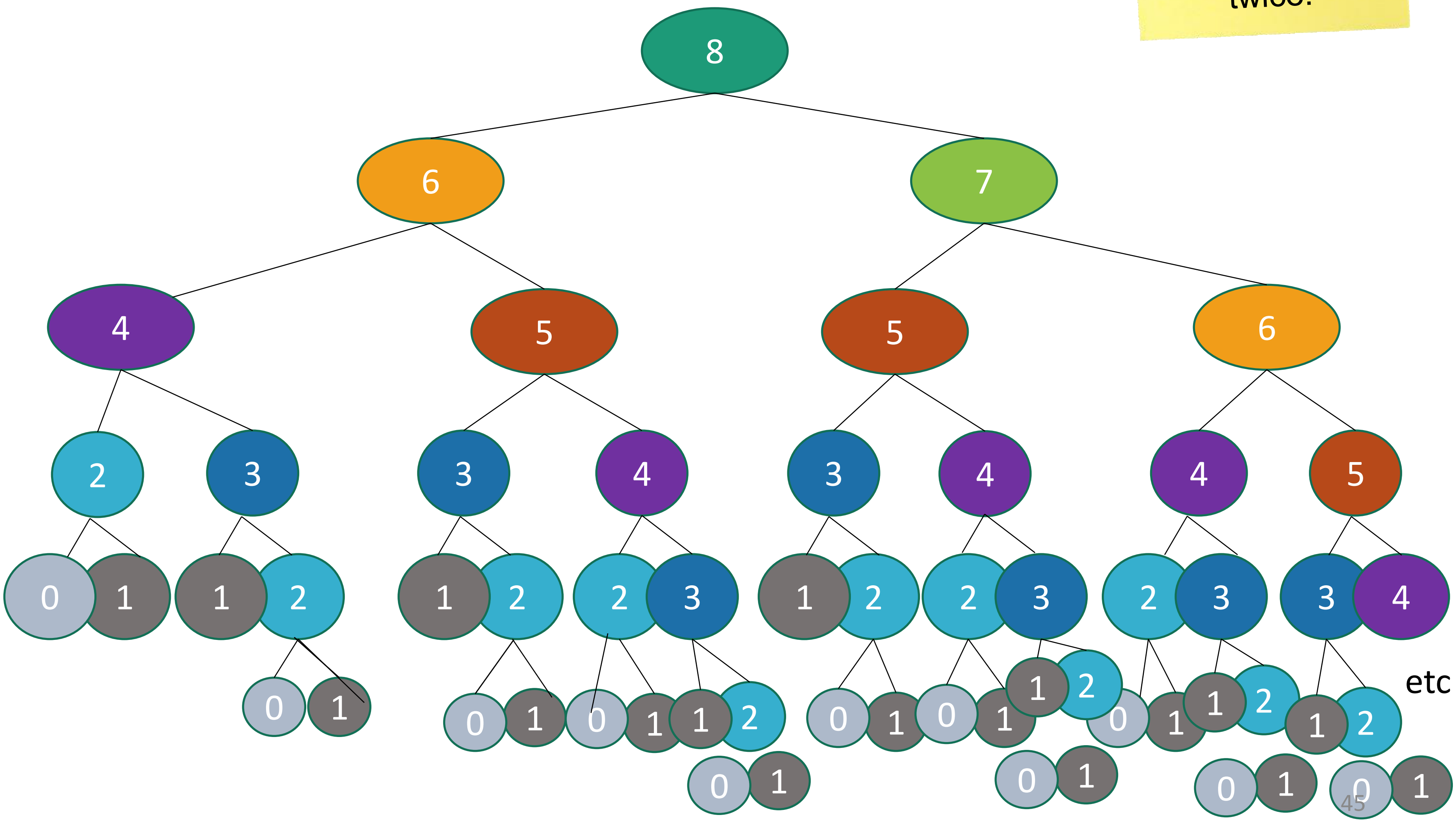


Memorization visualization

Collapse repeated nodes and don't do the same work twice!



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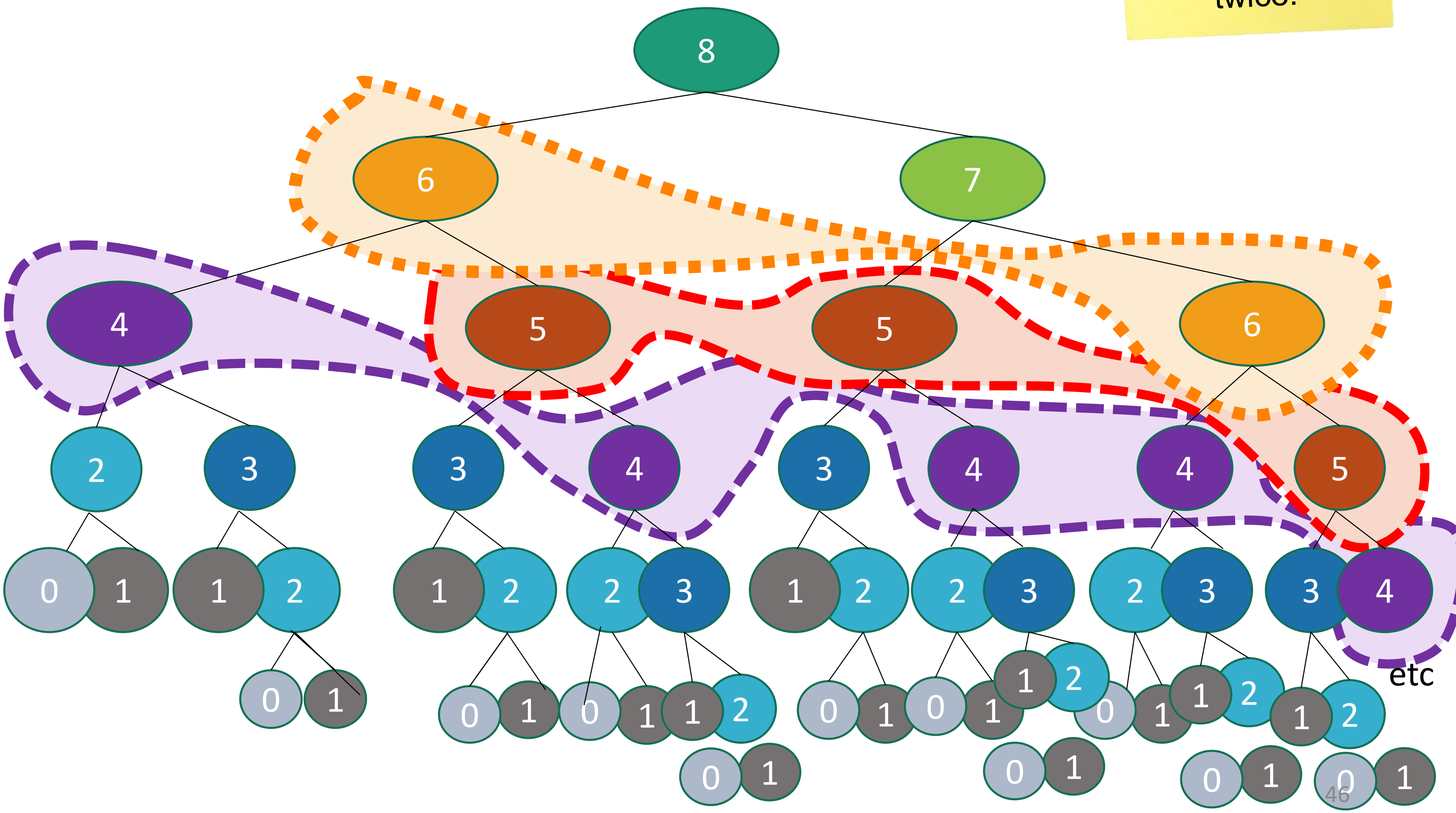

Memorization Visualization

ctd

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But otherwise treat it like the same old recursive algorithm.

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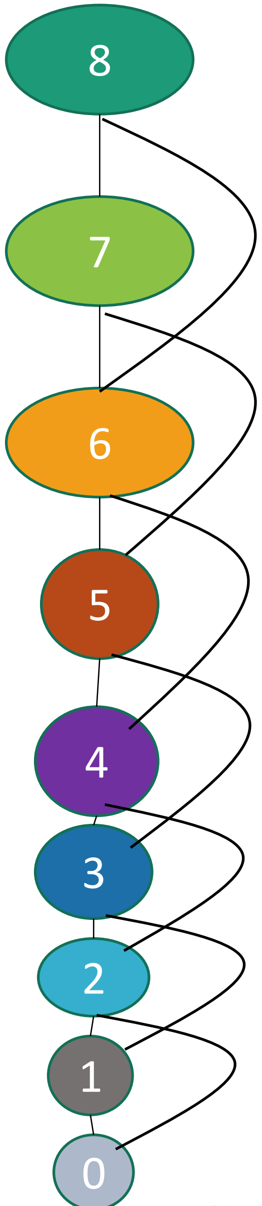
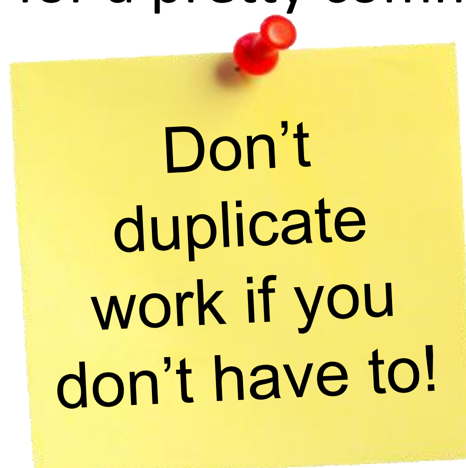


What have we learned?

- ***Dynamic programming:***
  - Paradigm in algorithm design.
  - Uses **optimal substructure**
  - Uses **overlapping subproblems**
  - Can be implemented **bottom-up** or **top-down.**
  - It's a fancy name for a pretty common-sense idea:



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Manipulating computer code in an action movie?

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- Programming refers to finding the optimal "program."
  - as in, a shortest route is a *plan* aka a *program*.

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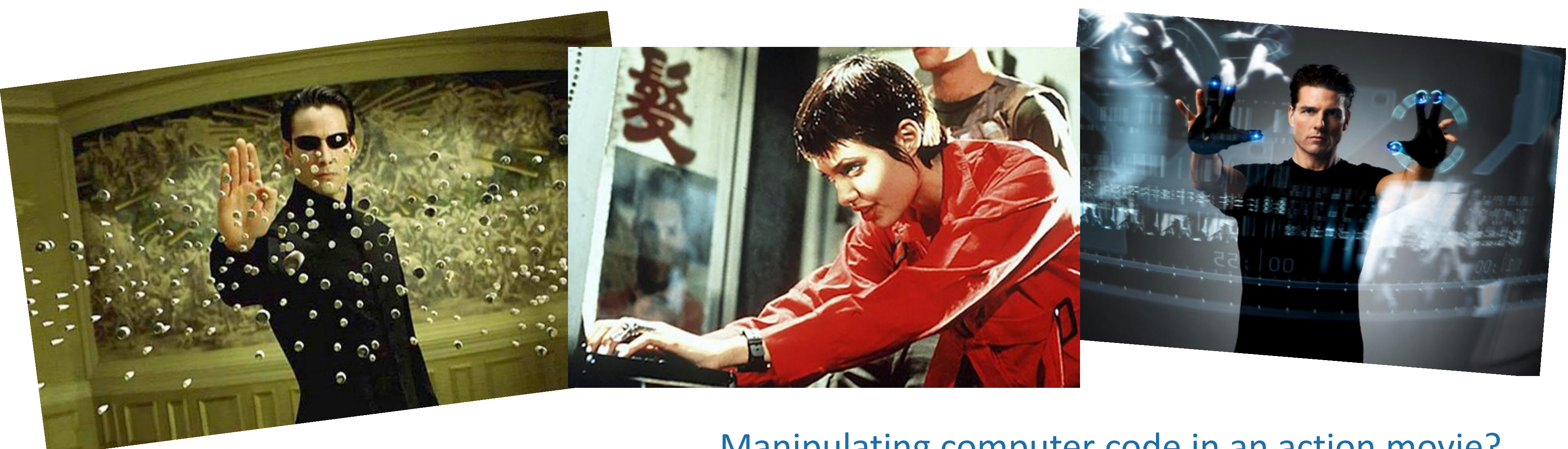
- But also it's just a fancy-sounding name.



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Why “*dynamic programming*” ?

- Richard Bellman invented the name in the 1950’s.
- At the time, he was working for the RAND Corporation, and projects needed flashy names to get funded.

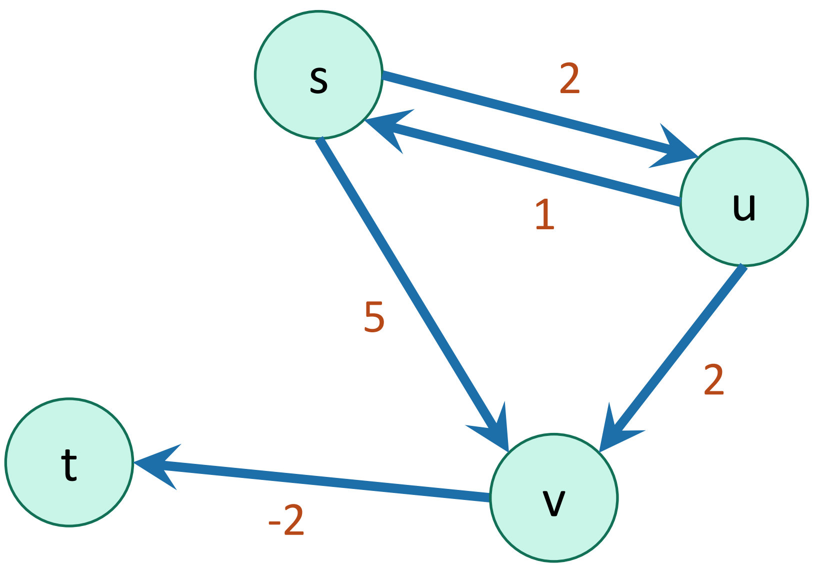
Floyd-Warshall Algorithm

Another example of Dynamic Programming

Floyd-Warshall Algorithm

Another example of Dynamic Programming

- This is an algorithm for **All-Pairs Shortest Paths** (APSP)
  - That is, I want to know the shortest path from u to v for **ALL pairs** u,v of vertices in the graph.
  - Not just from a special single source s.



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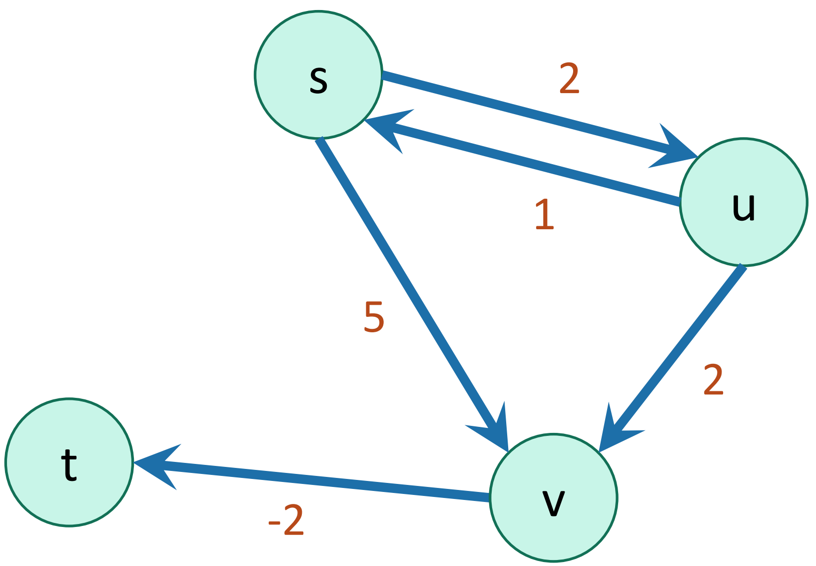
Destination

Source	s	u	v	t
s	0	2	4	2
u	1	0	2	0
v	∞	∞	0	-2
t	∞	∞	∞	0



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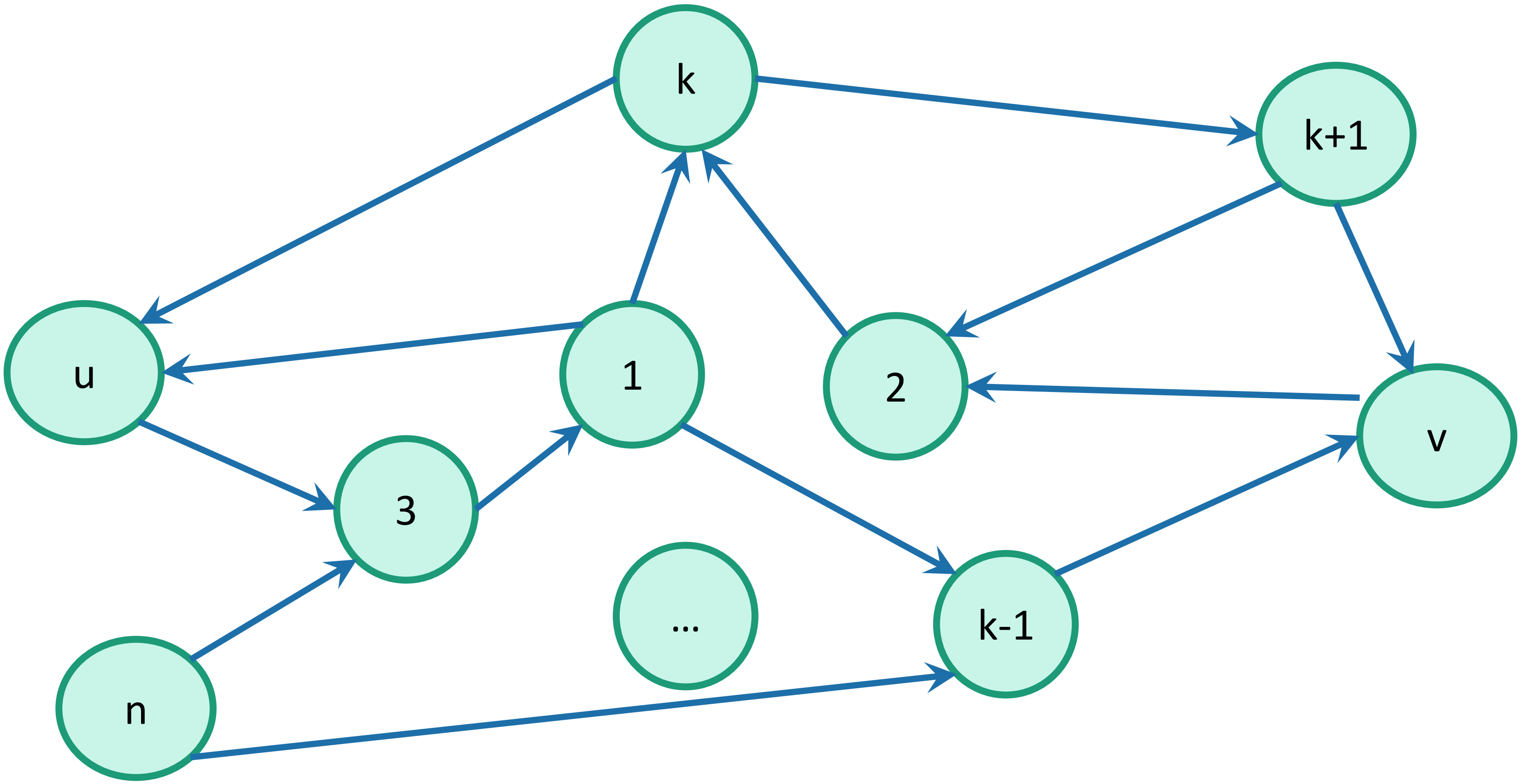
Can we do better?

Label the vertices 1,2,…,n

Optimal substructure



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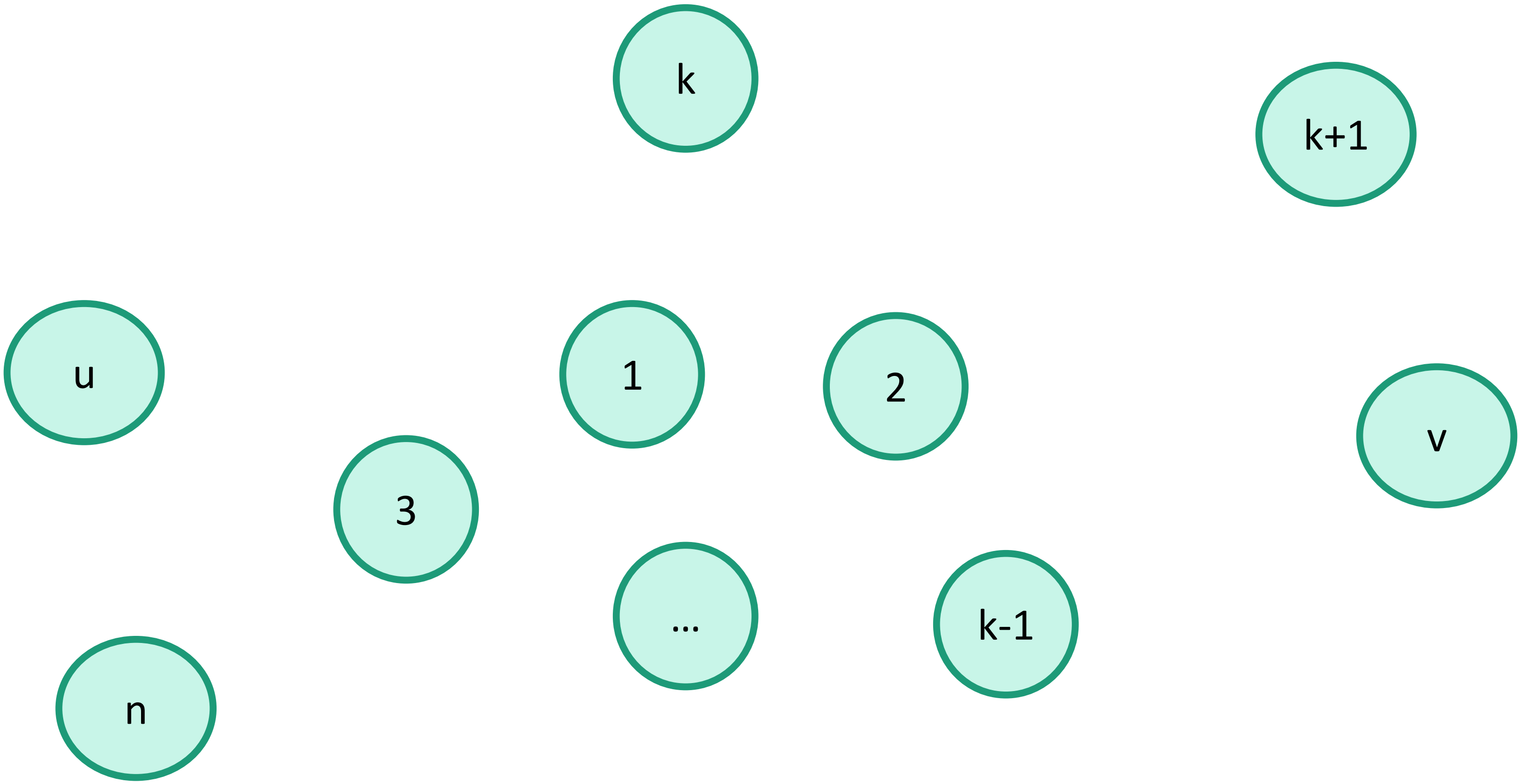
(We omit some edges in the picture below – meant to be a cartoon, not an example).



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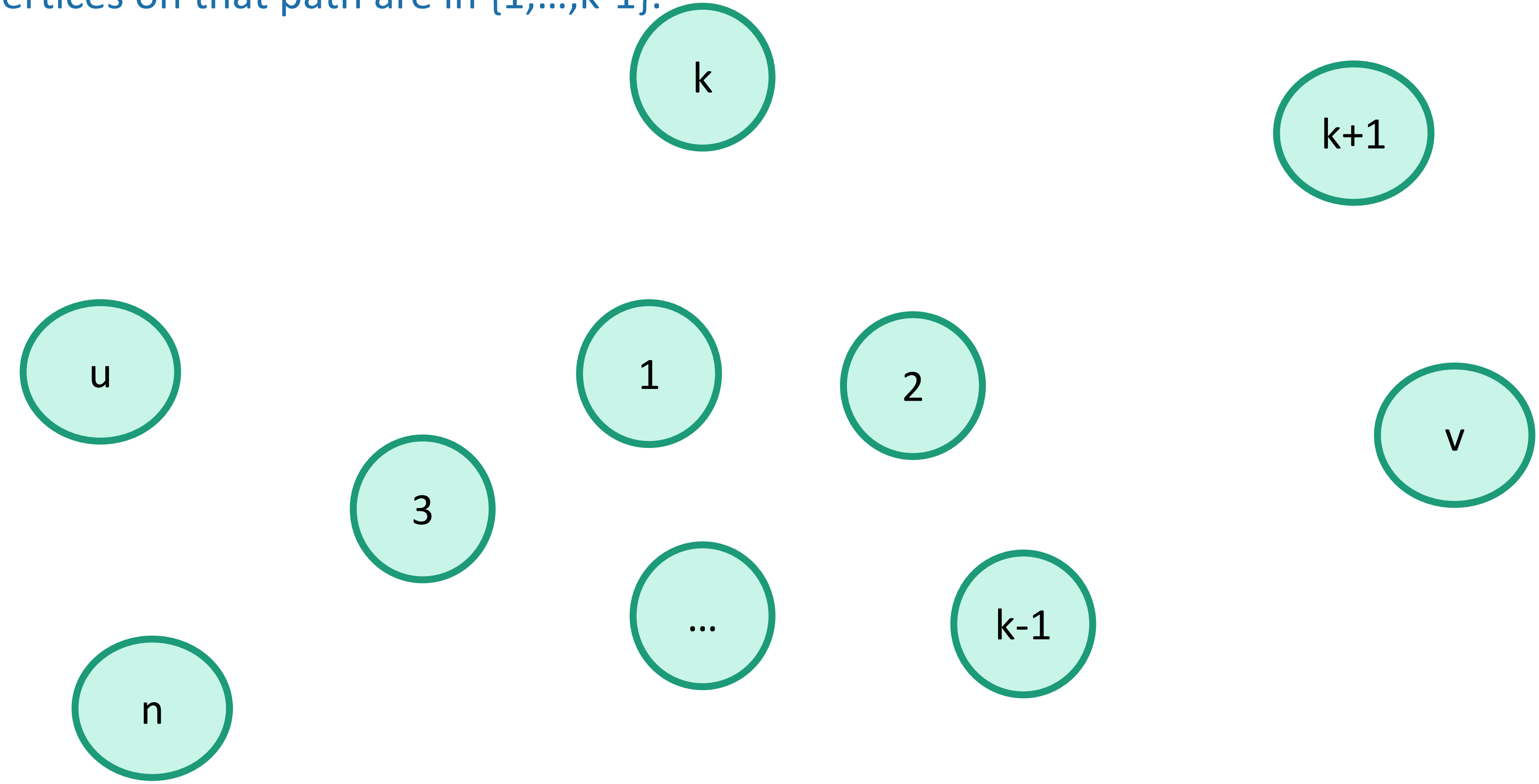
Sub-problem(k-1):

For all pairs, u,v, find the cost of the shortest path from u to v, so that all the internal vertices on that path are in {1,…,k-1}.



Optimal substructure

Label the vertices 1,2,…,n

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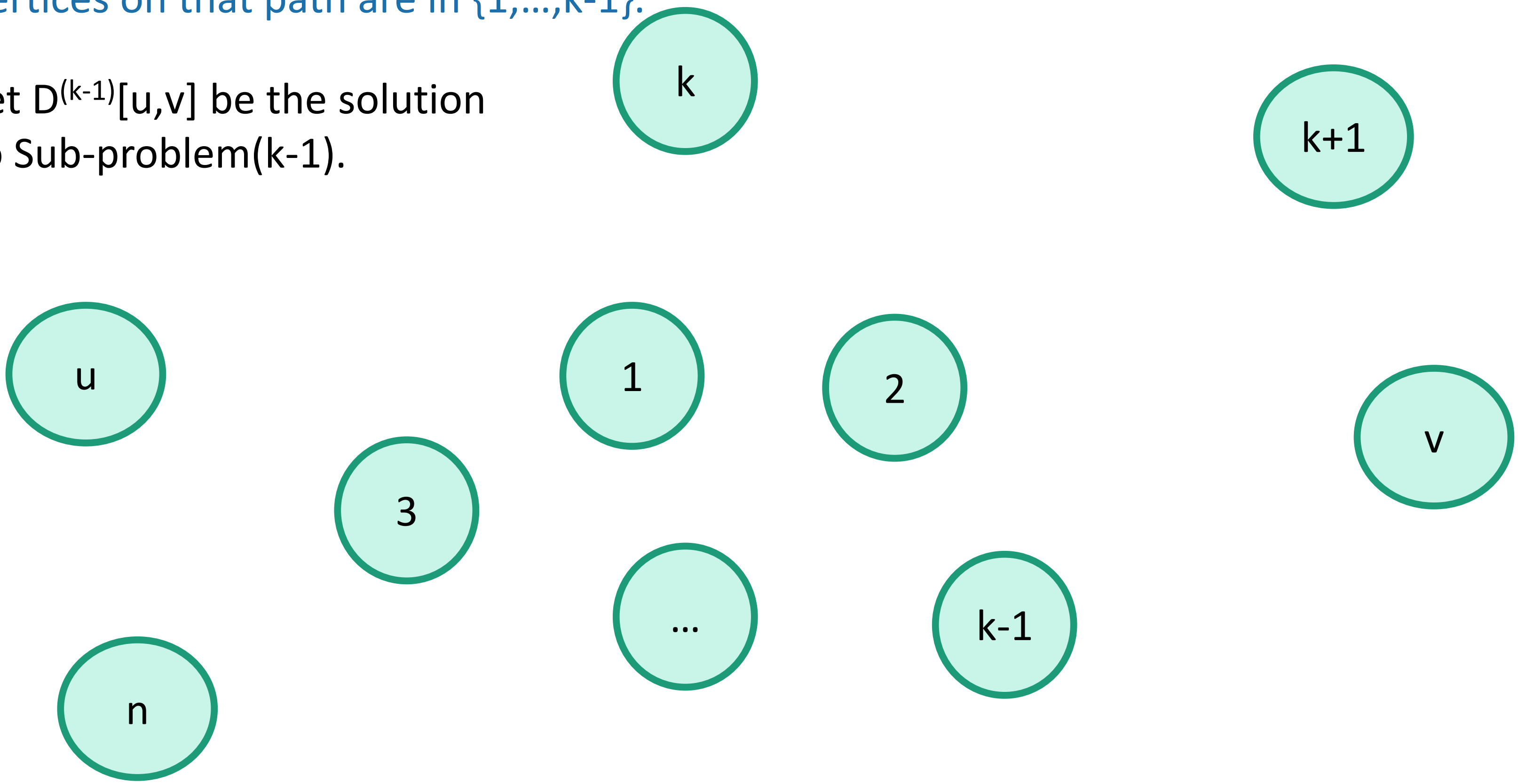

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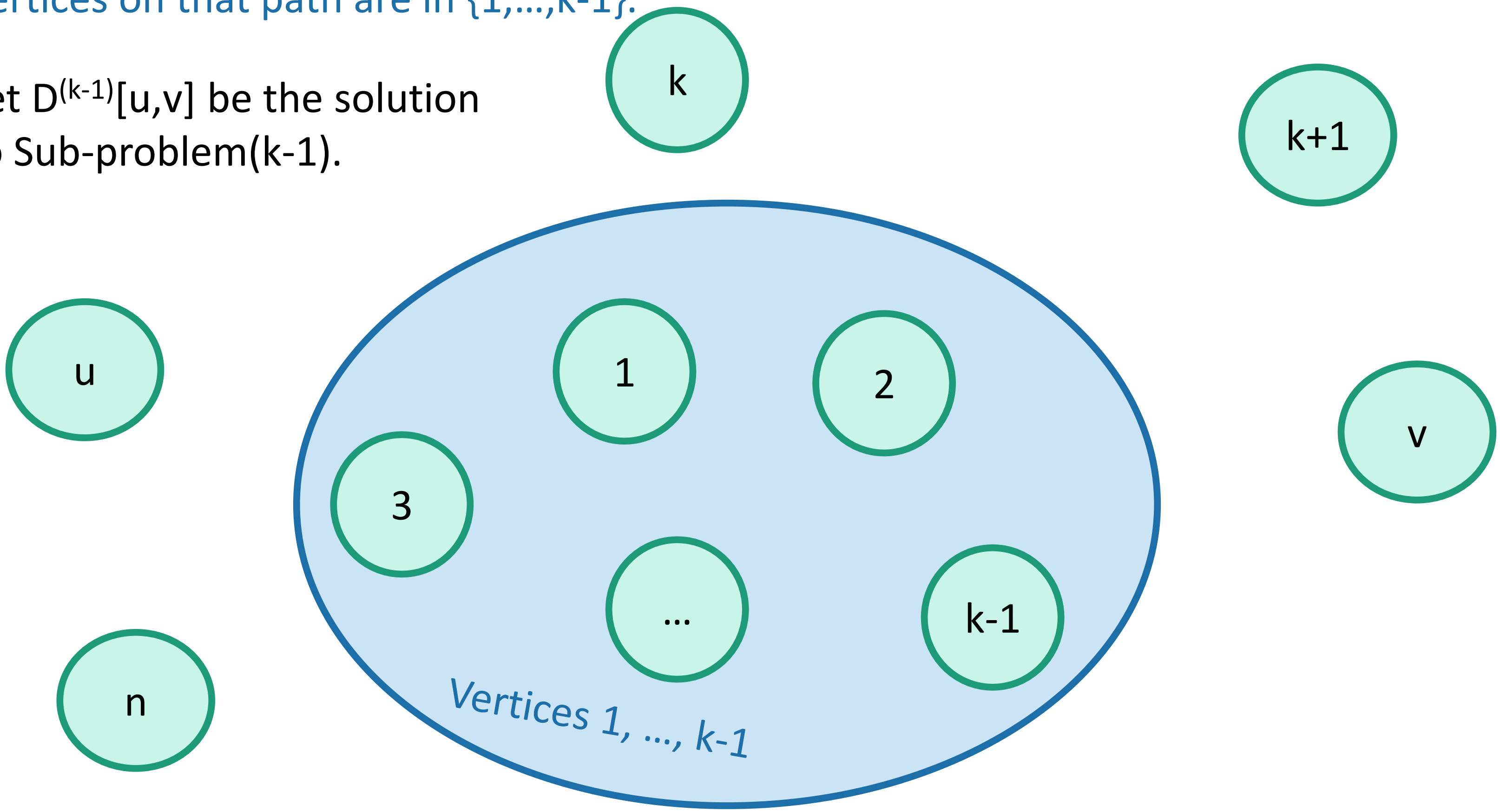

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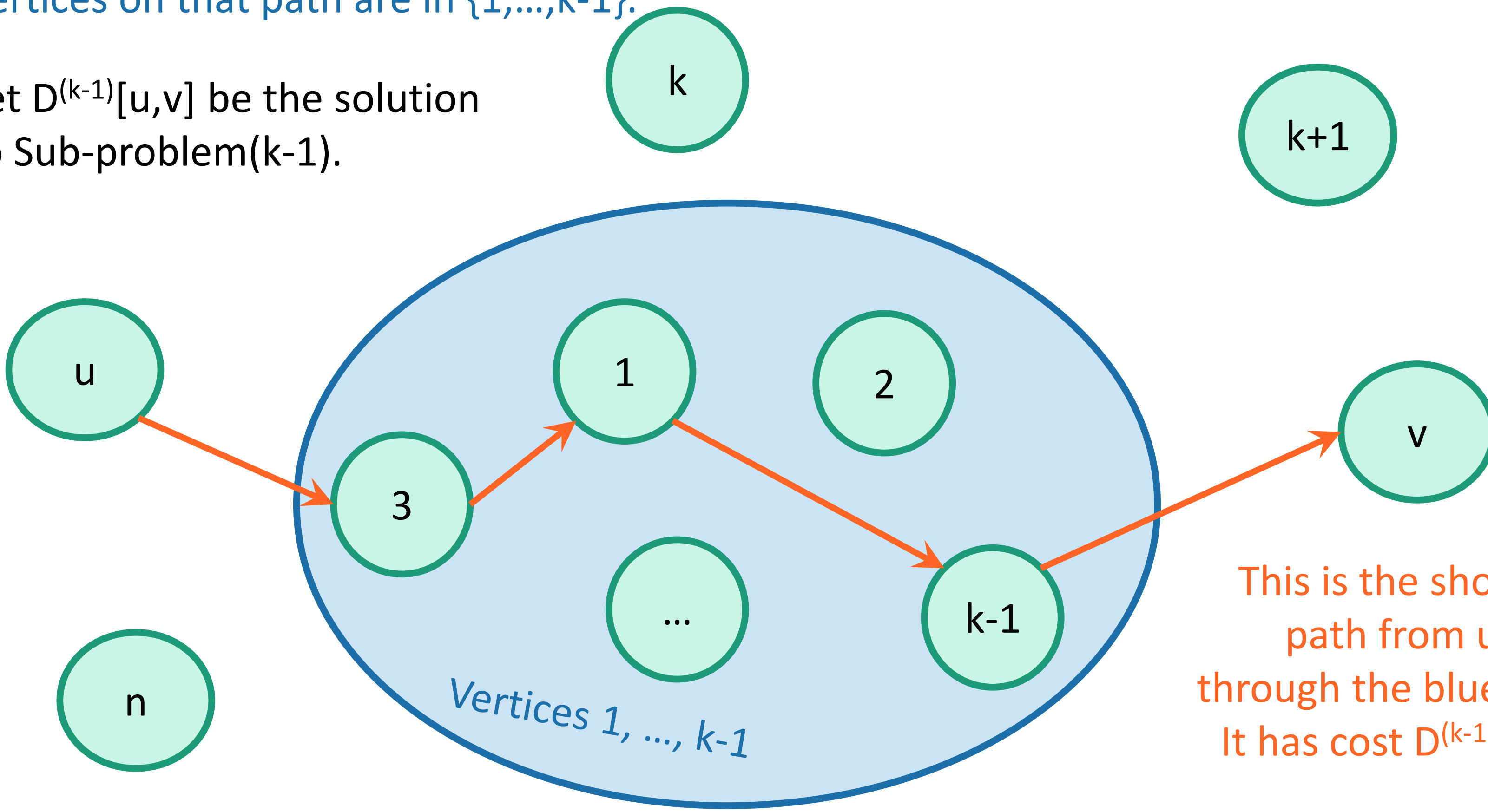

This is the shortest path from u to v through the blue set. It has cost $D^{(k-1)}[u,v]$

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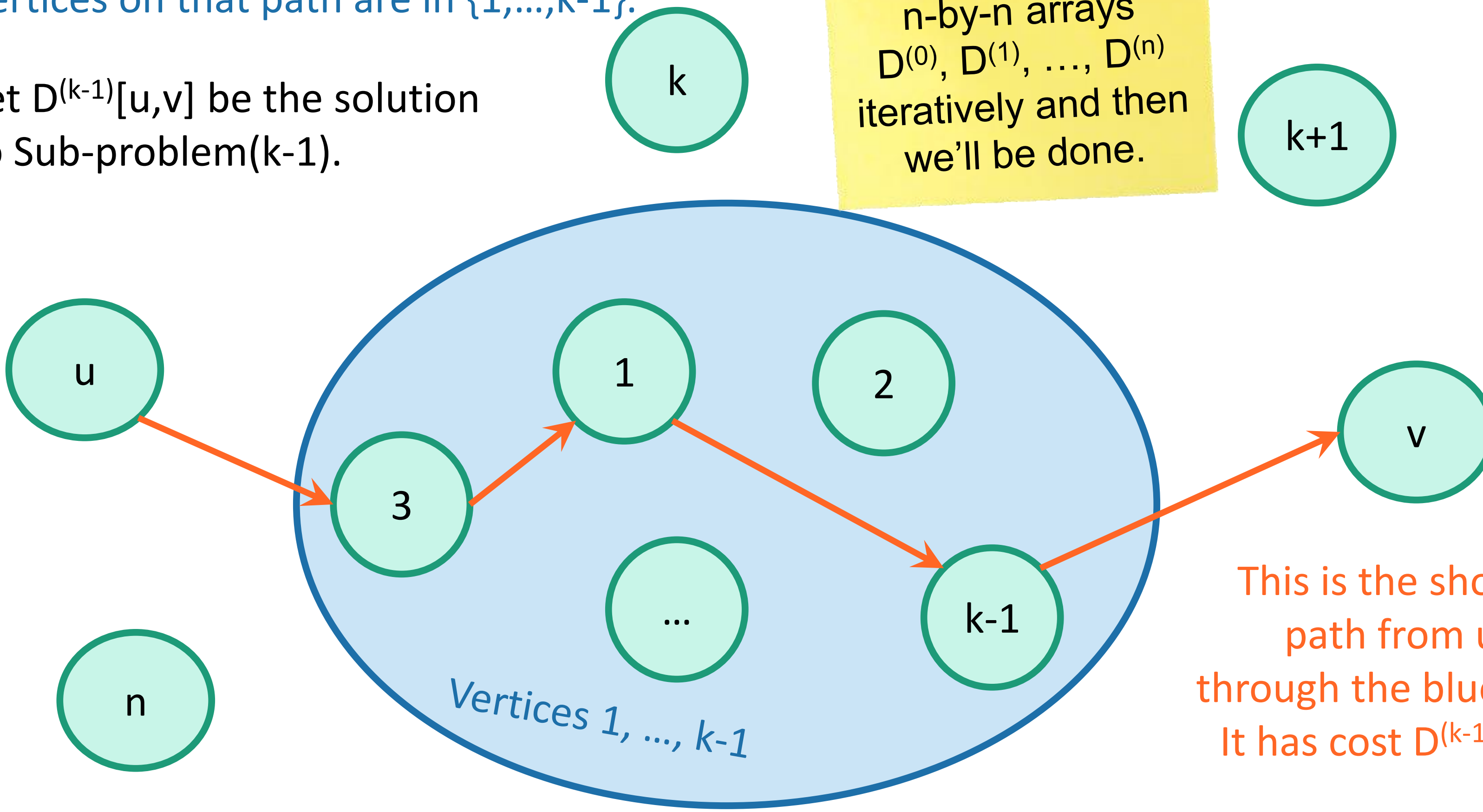

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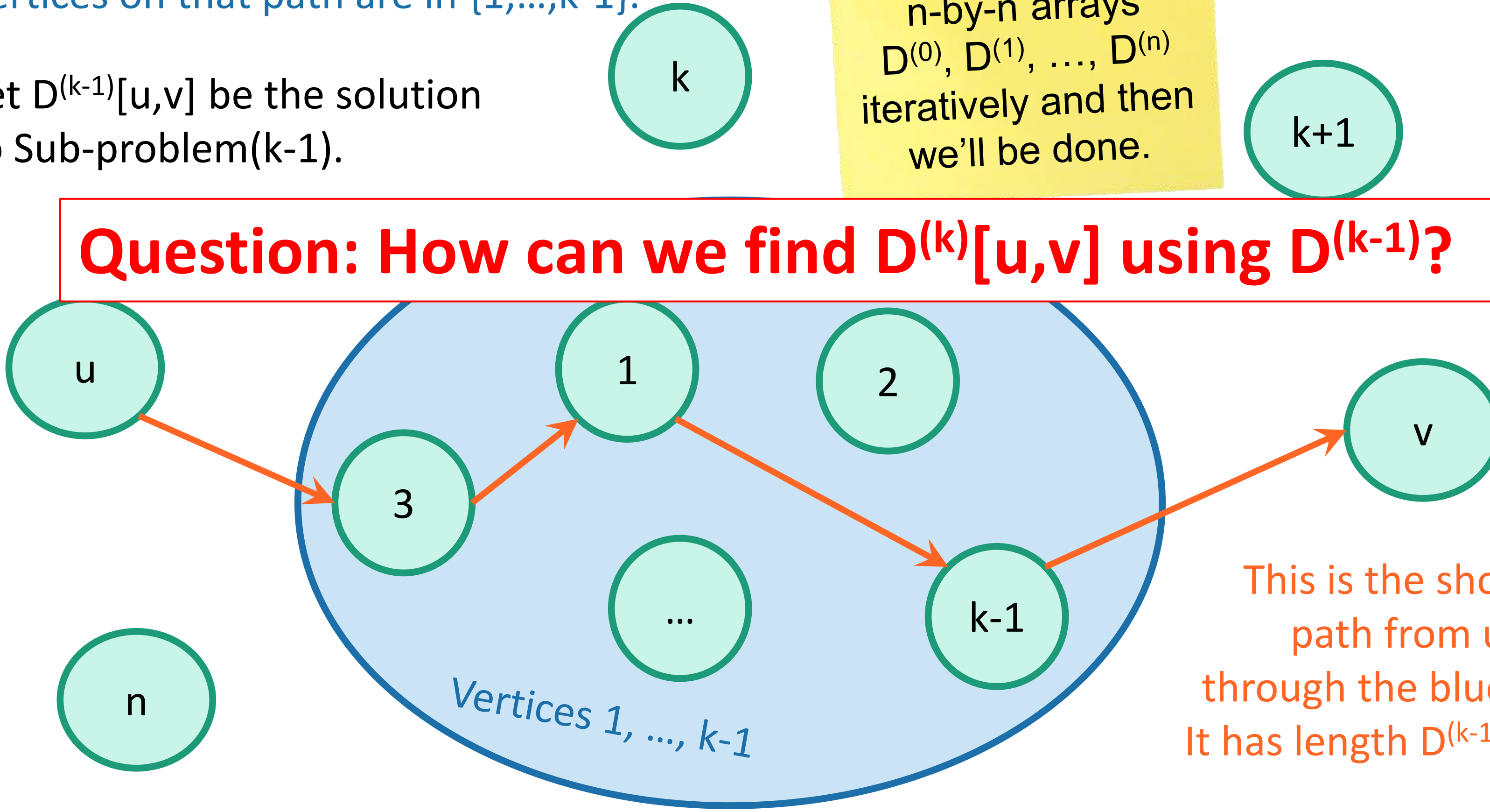

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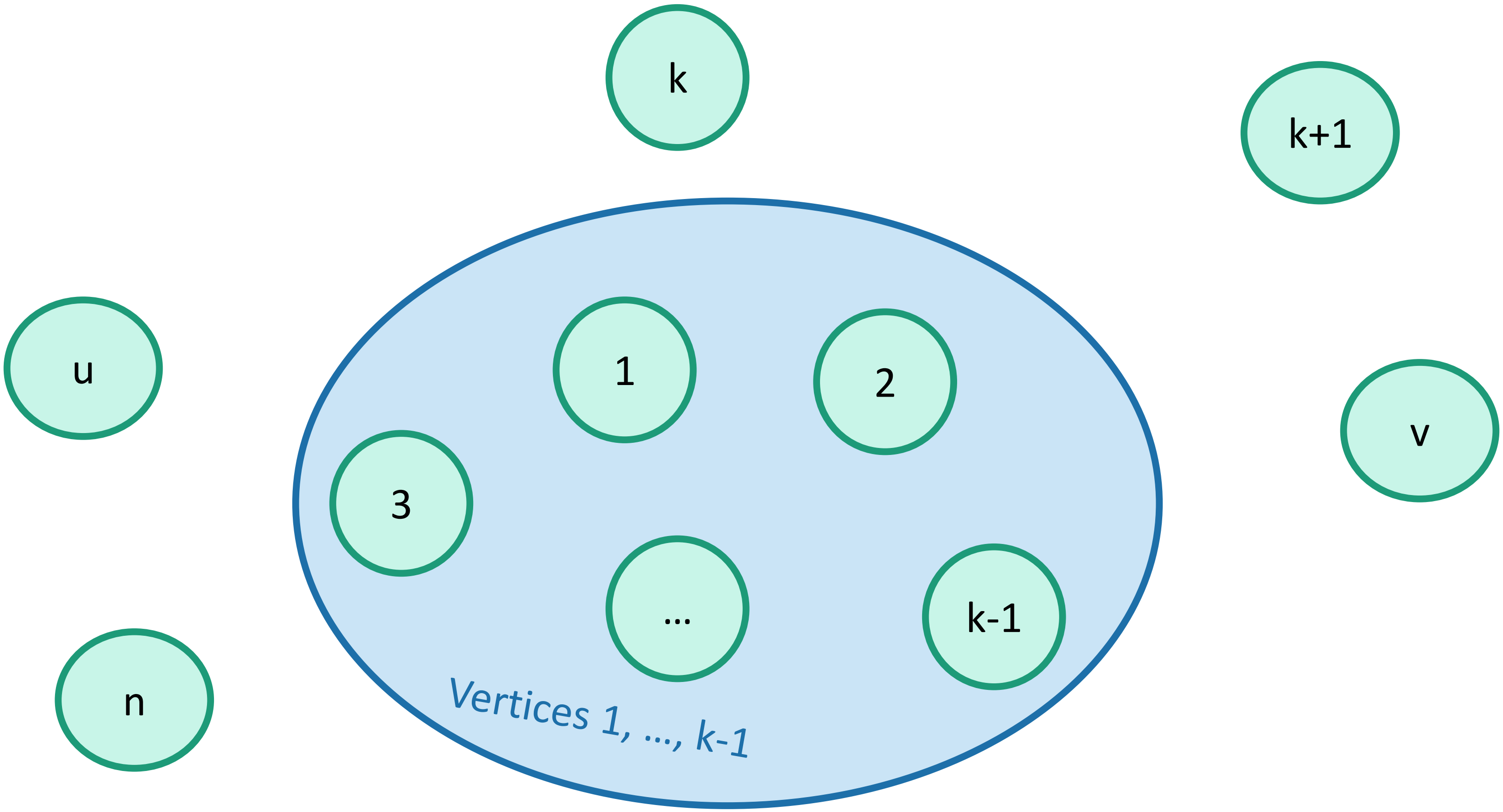

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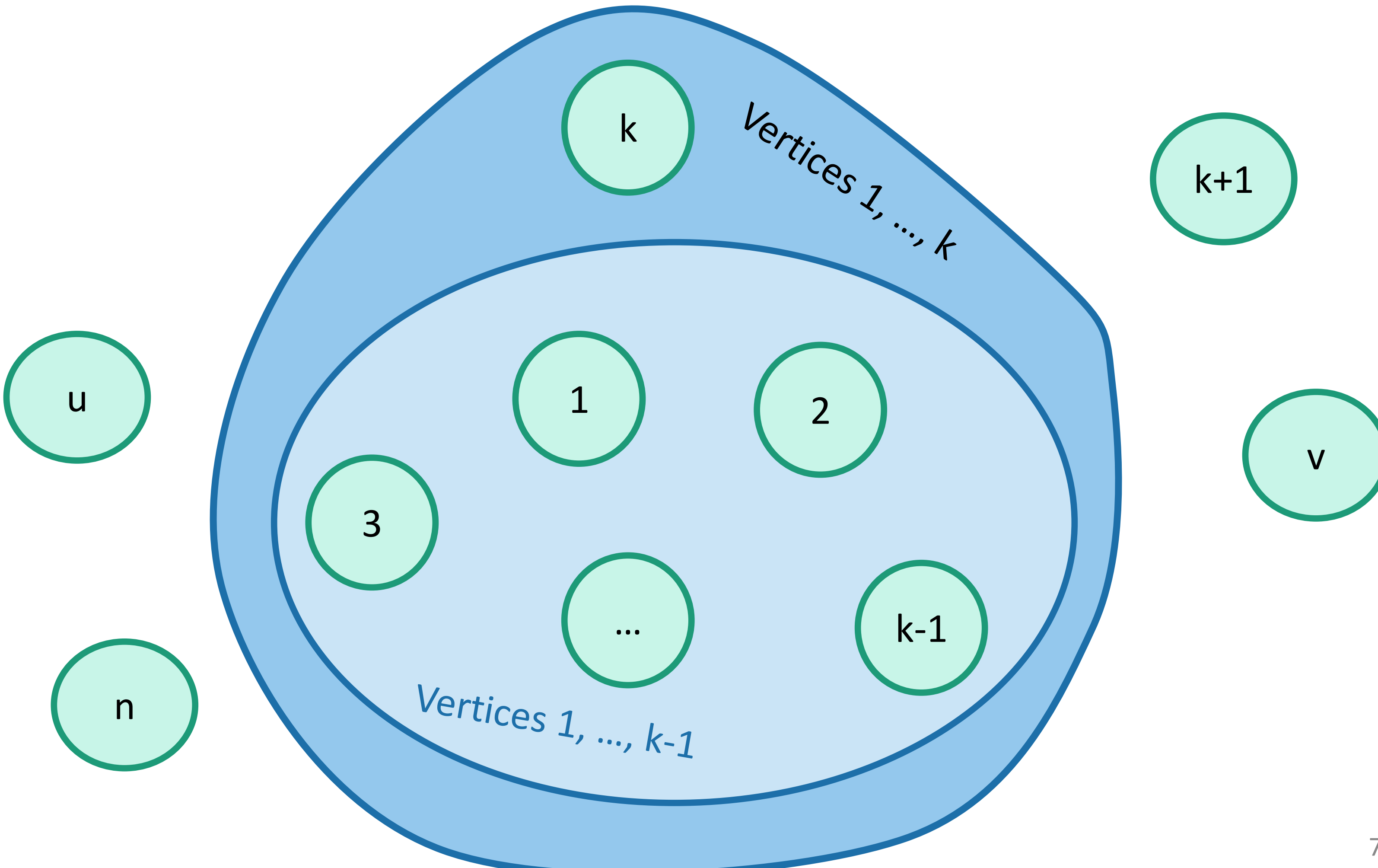

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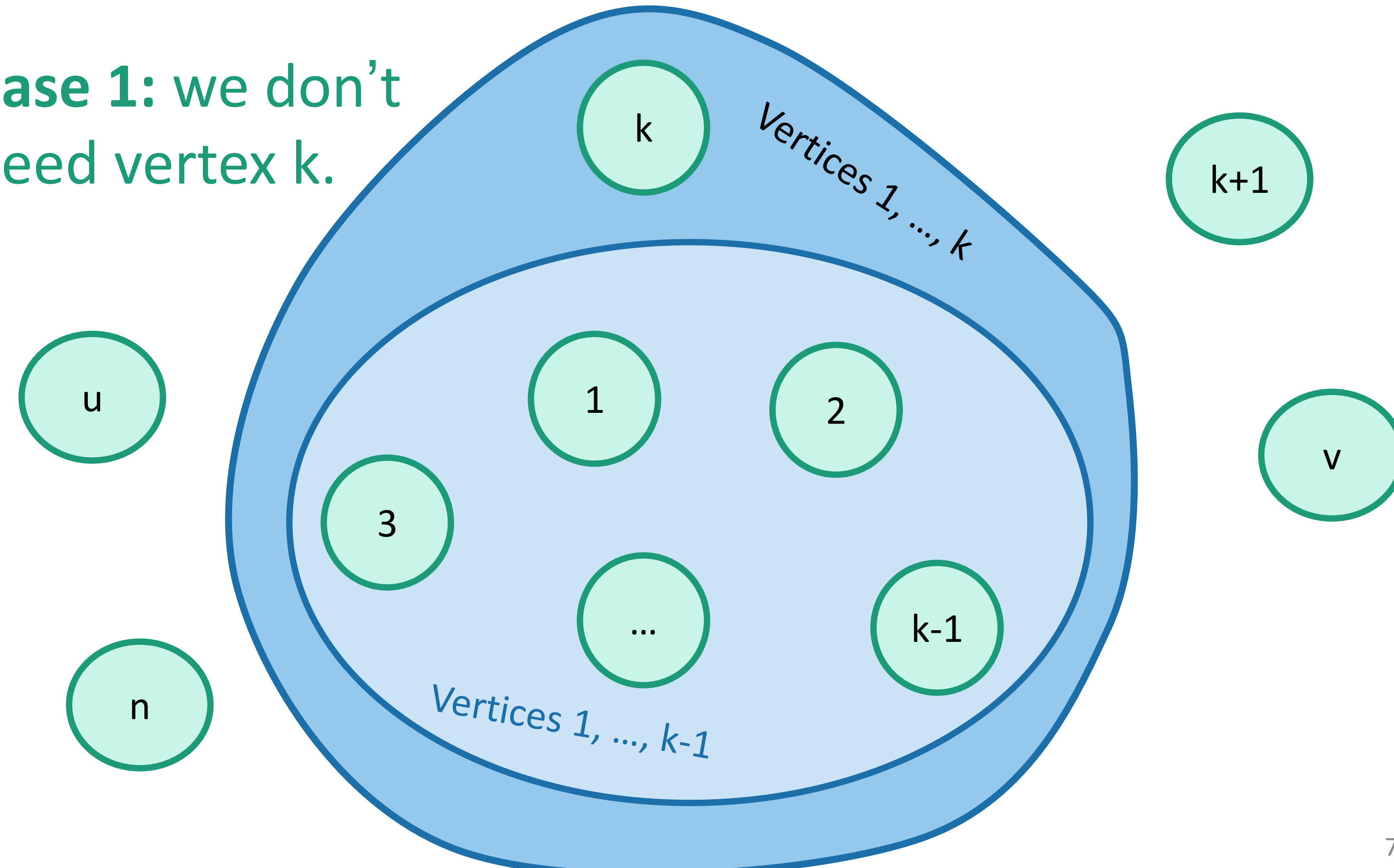

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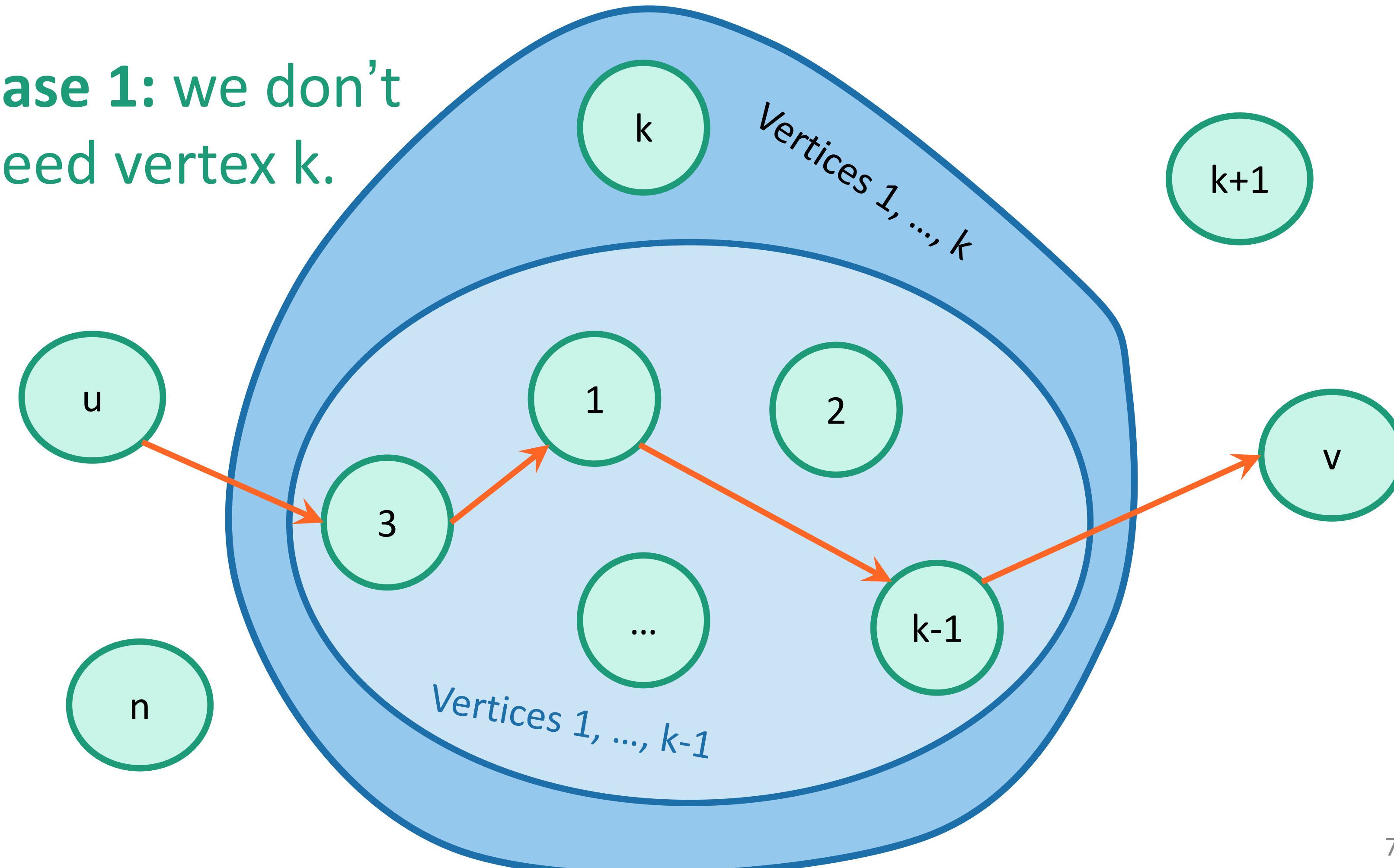

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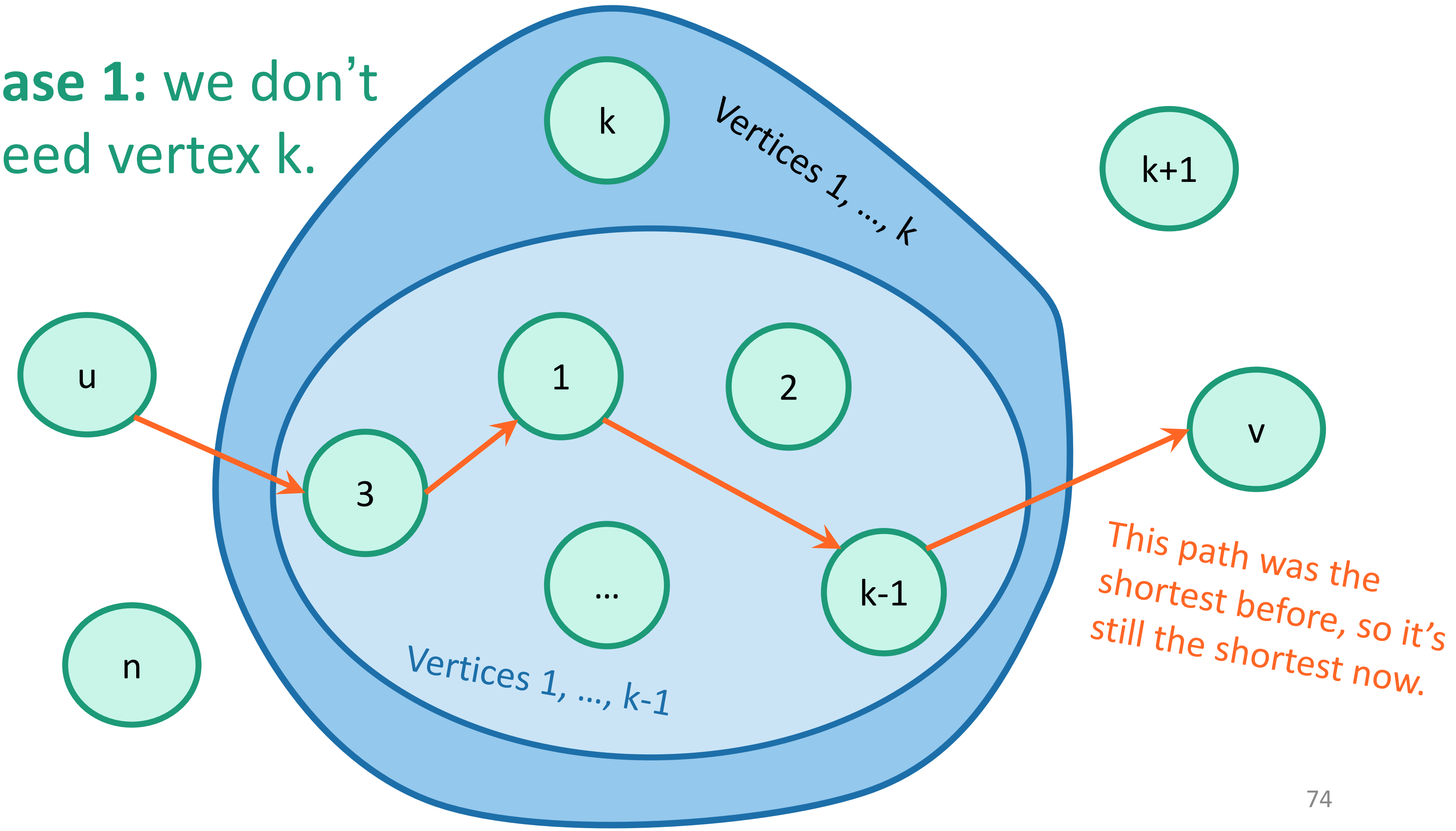

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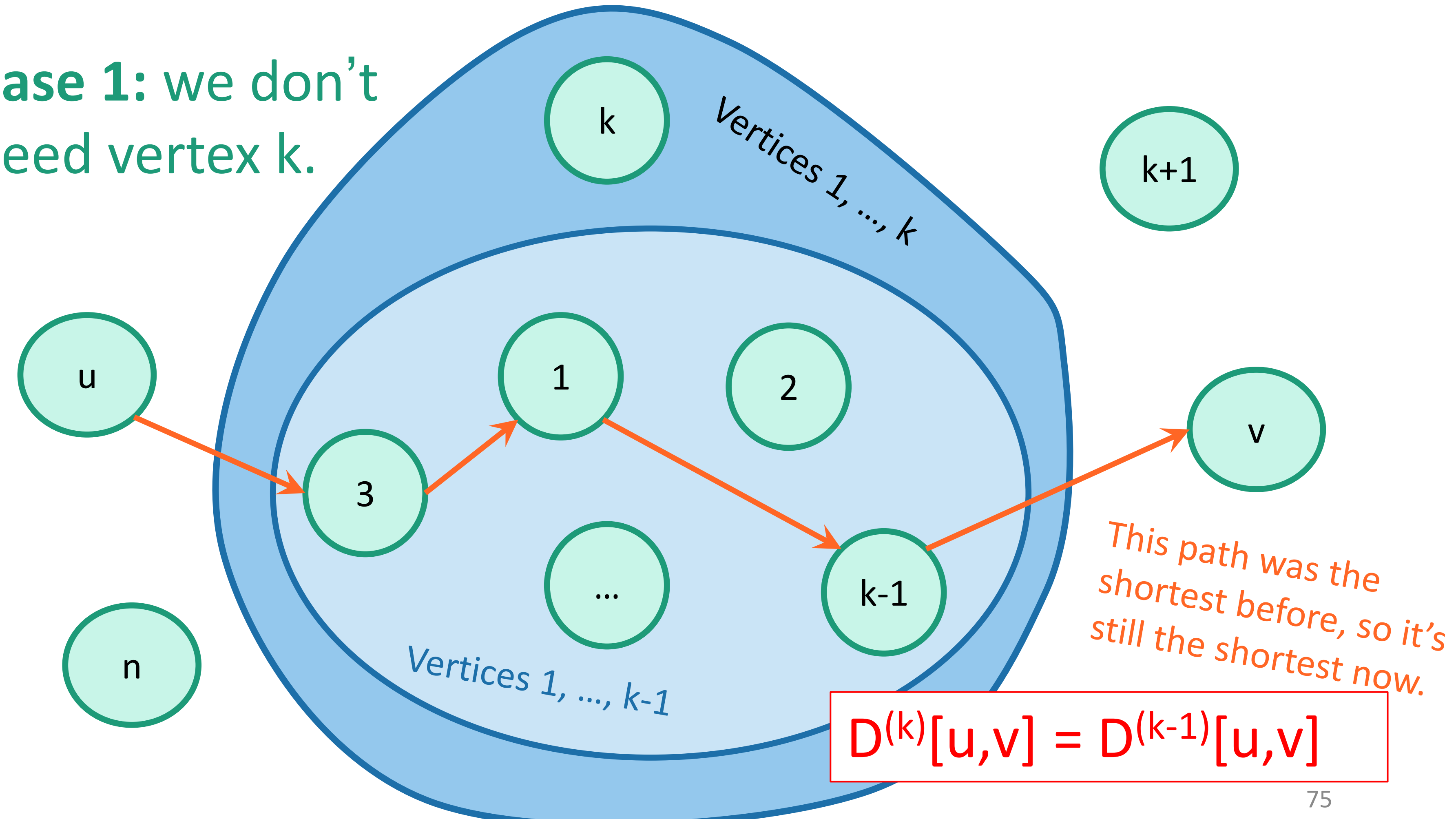

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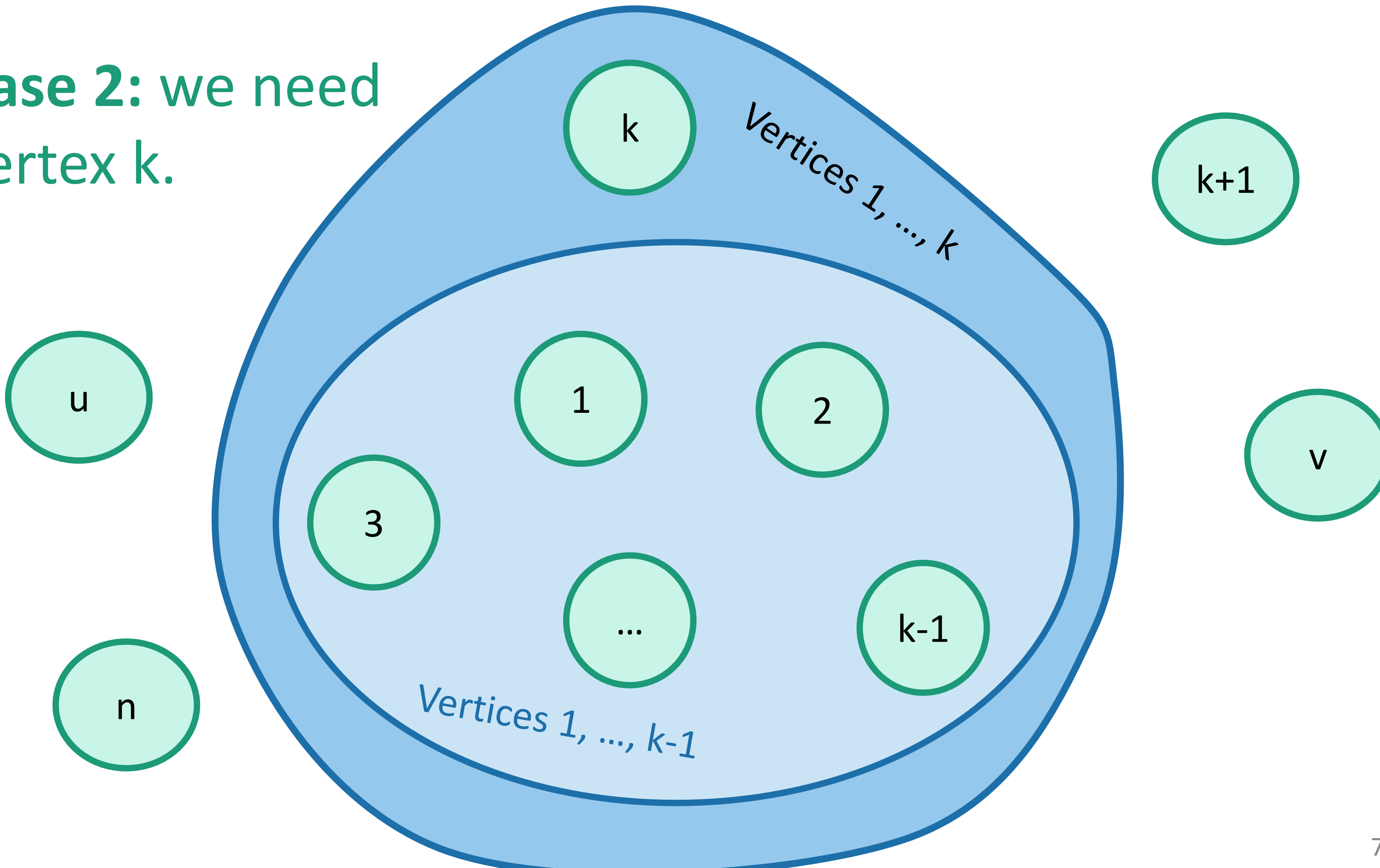

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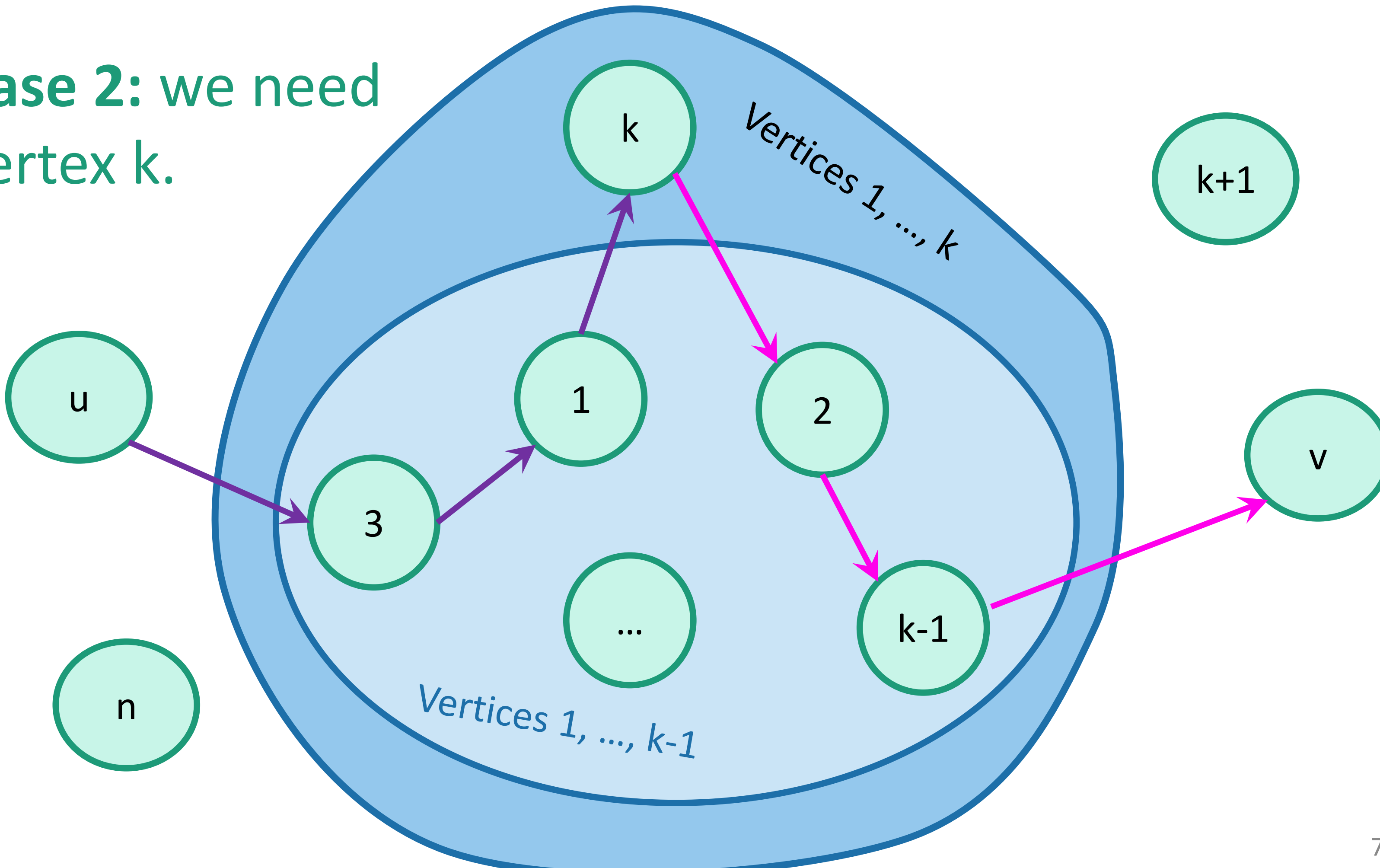

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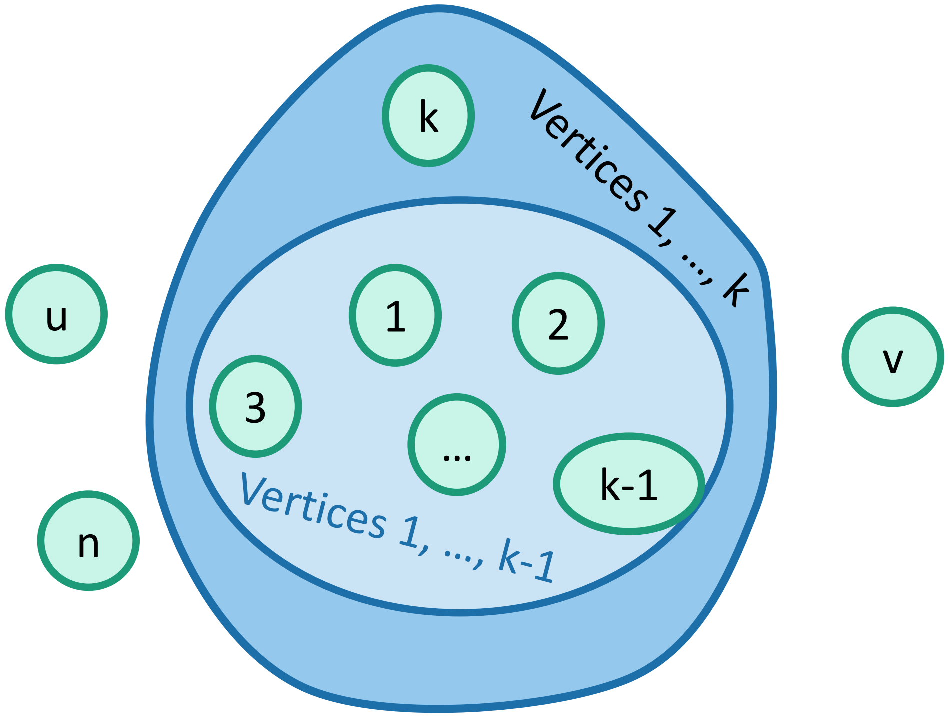

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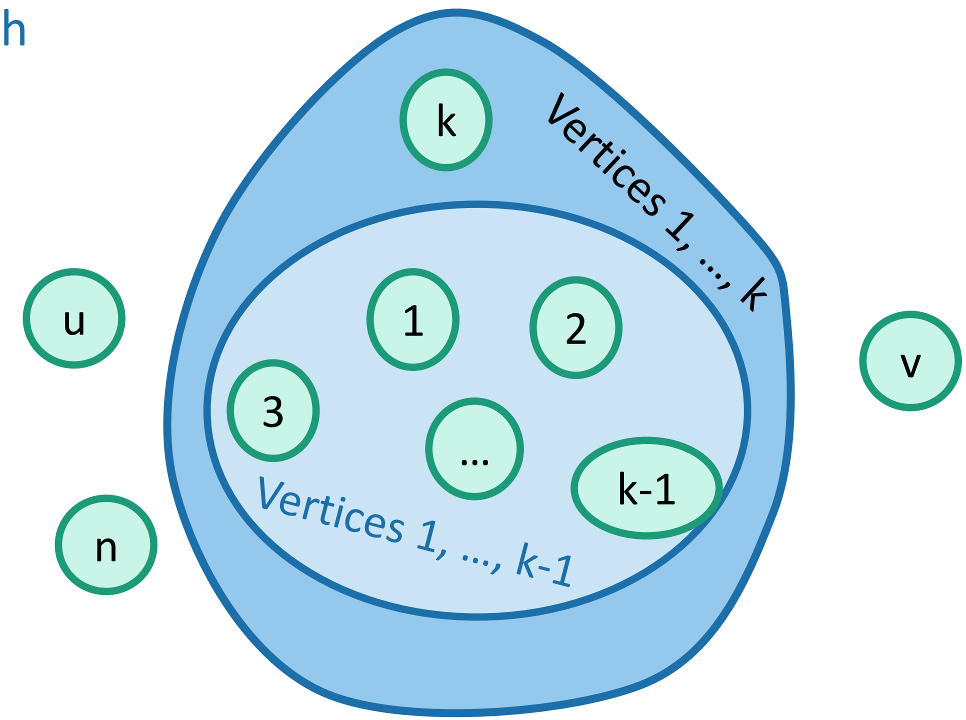

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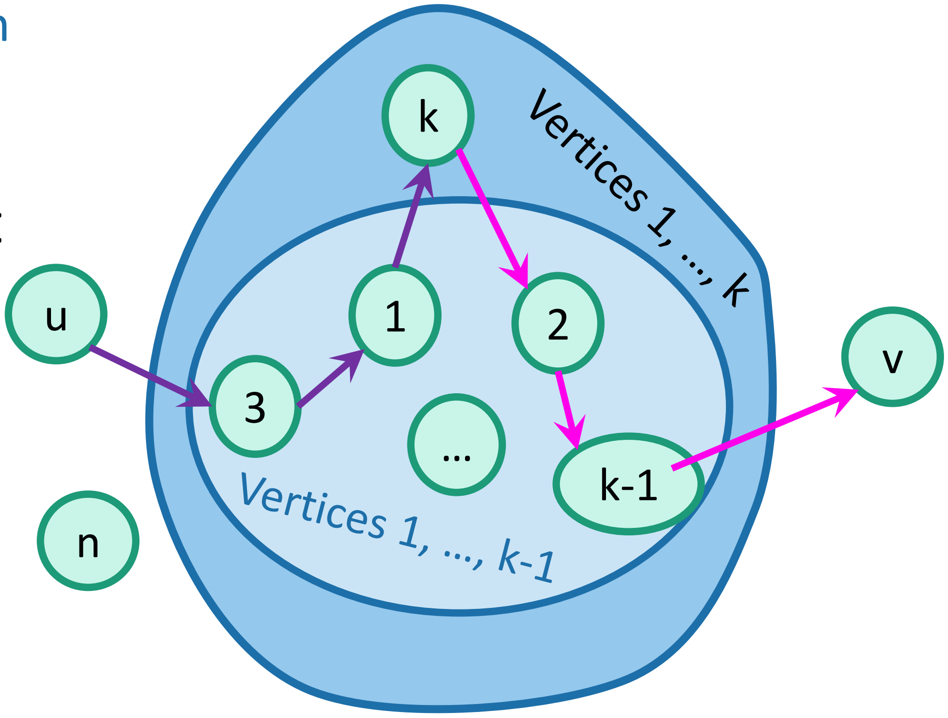

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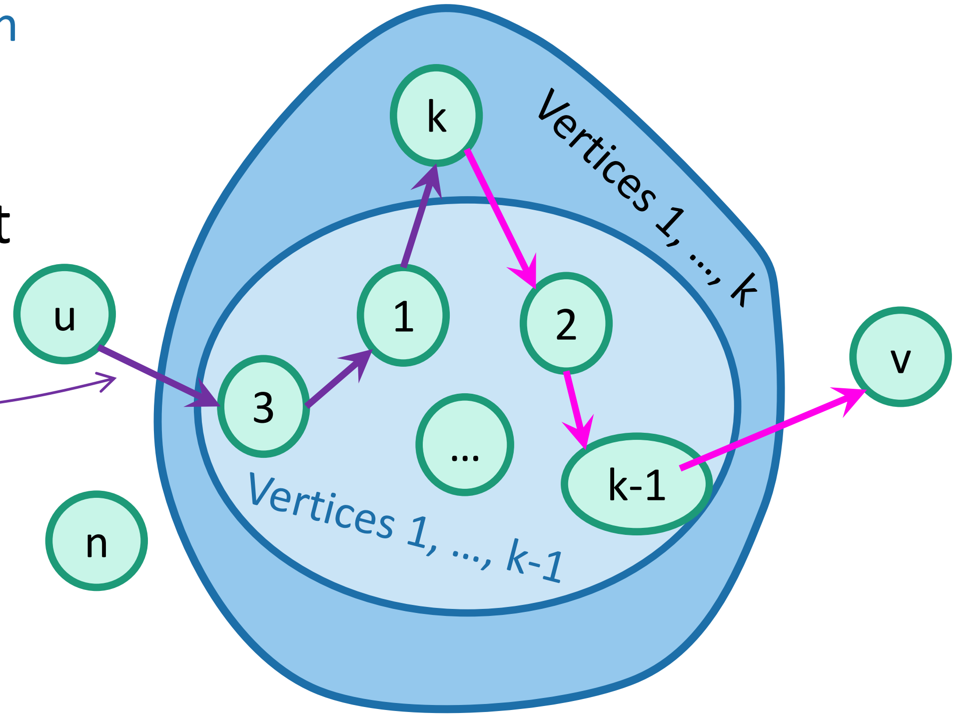

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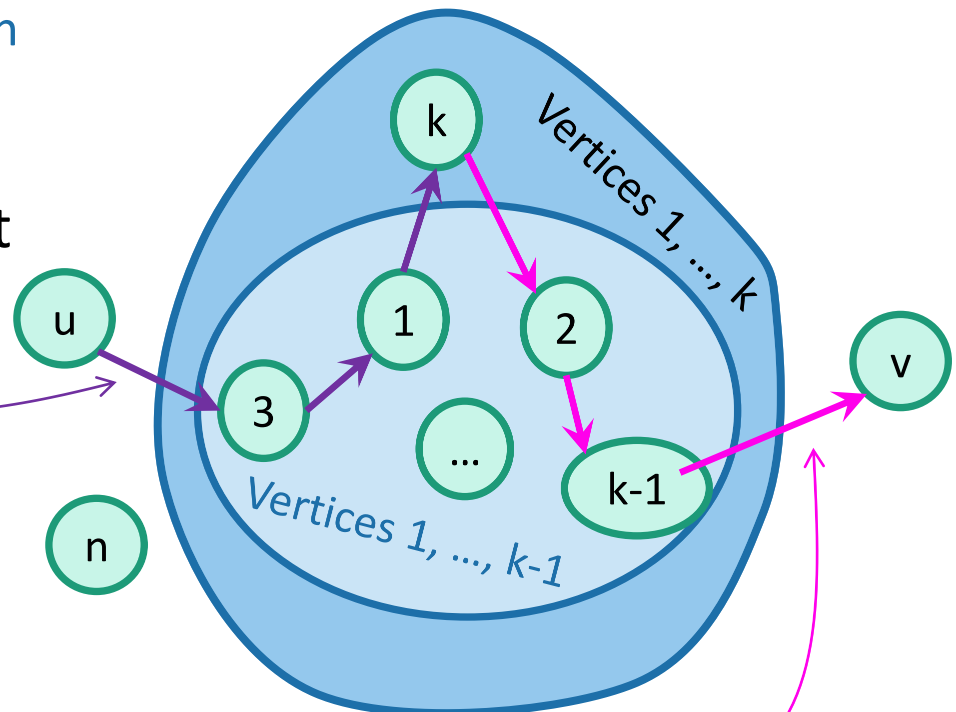

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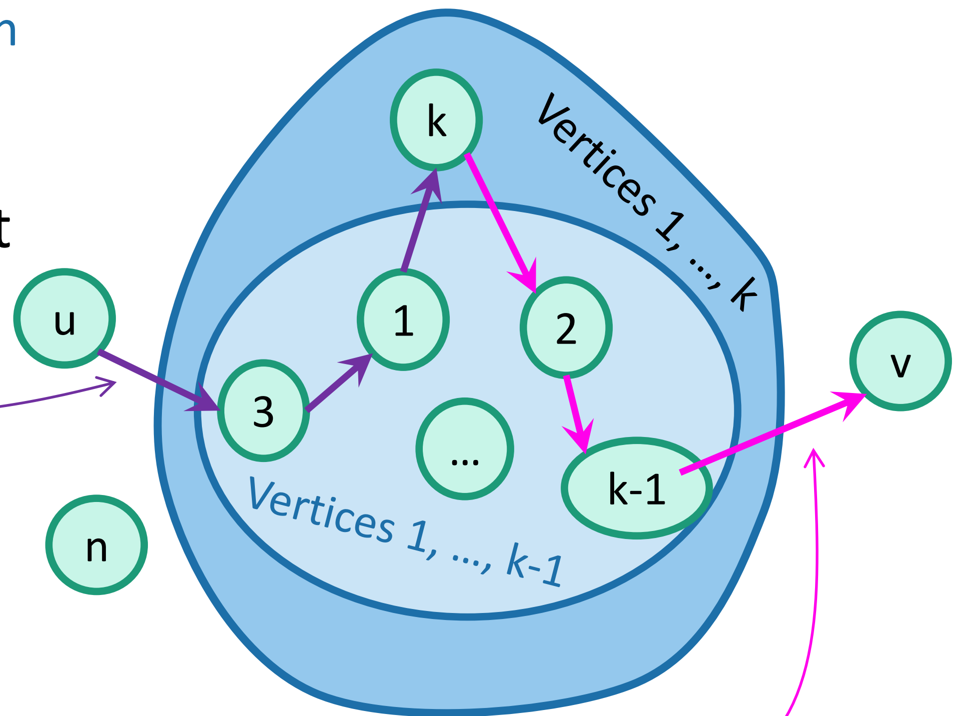

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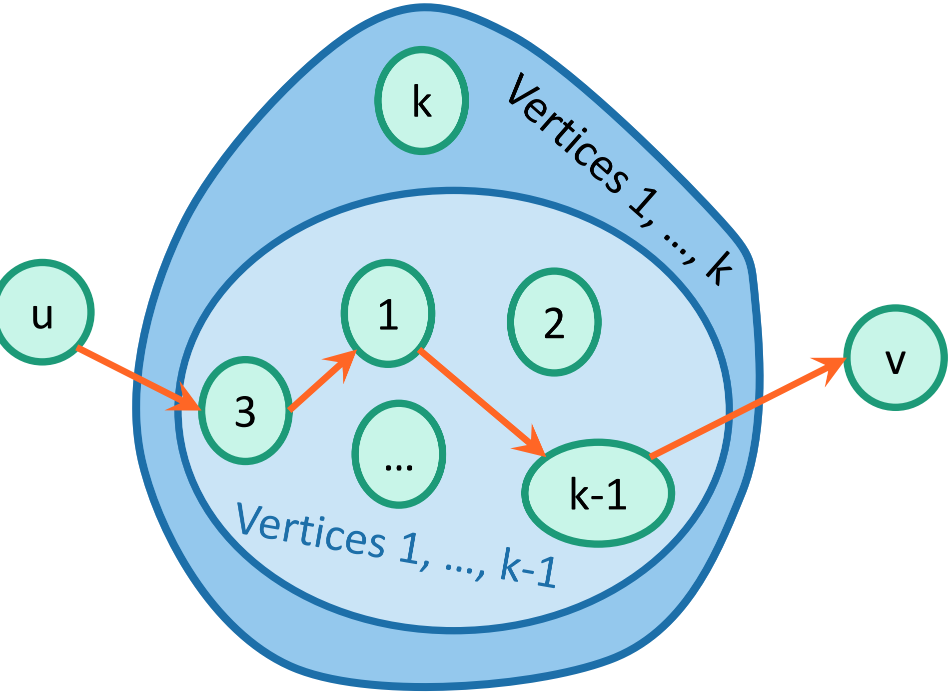

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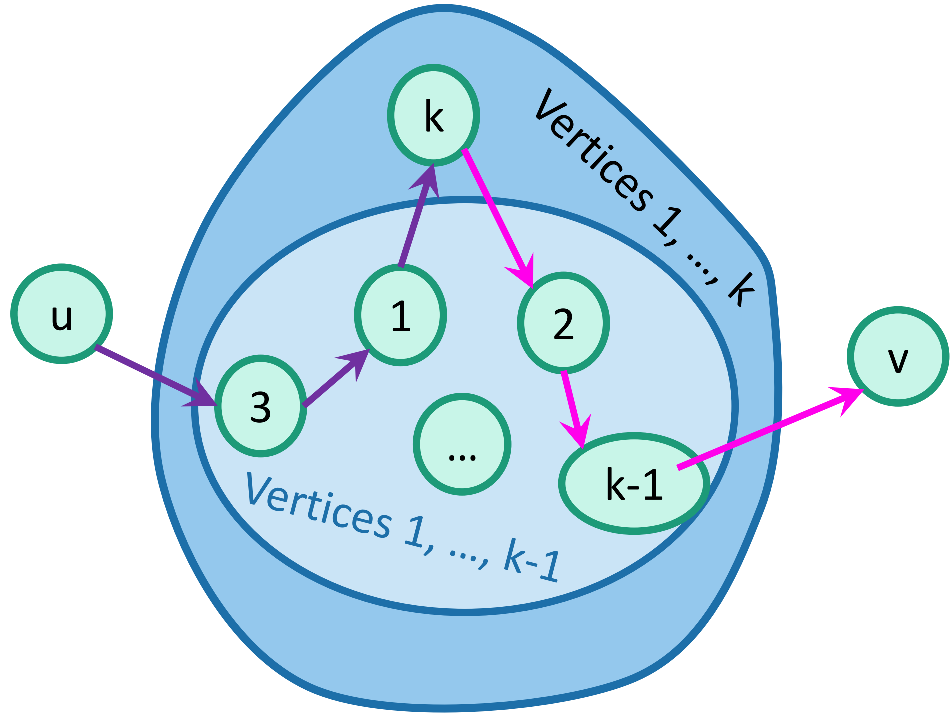

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- Optimal substructure:
  - We can solve the big problem using solutions to smaller problems.
- Overlapping sub-problems:
  - $D^{(k-1)}[k,v]$ can be used to help compute $D^{(k)}[u,v]$ for lots of different u's.

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- Using our ***Dynamic programming*** paradigm, this immediately gives us an algorithm!

Floyd-Warshall algorithm

- Initialize n-by-n arrays $D^{(k)}$ for k = 0,…,n
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This is a bottom-up ***Dynamic programming*** algorithm.

We’ve basically just shown

- Theorem:

  If there are no negative cycles in a weighted directed graph G, then the Floyd-Warshall algorithm, running on G, returns a matrix $D^{(n)}$ so that:

  $$D^{(n)}[u,v] = \text{distance between u and v in G.}$$

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As with Bellman-Ford, we don't really need to store all n of the $D^{(k)}$.

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- Just like Bellman-Ford, Floyd-Warshall can detect negative cycles:
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- Algorithm:
  - Run Floyd-Warshall as before.
  - If there is some v so that $D^{(n)}[v,v] < 0$:
    - **return** negative cycle.

What have we learned?

- The Floyd-Warshall algorithm is another example of *dynamic programming*.
- It computes All Pairs Shortest Paths in a directed weighted graph in time $O(n^3)$.

Recap

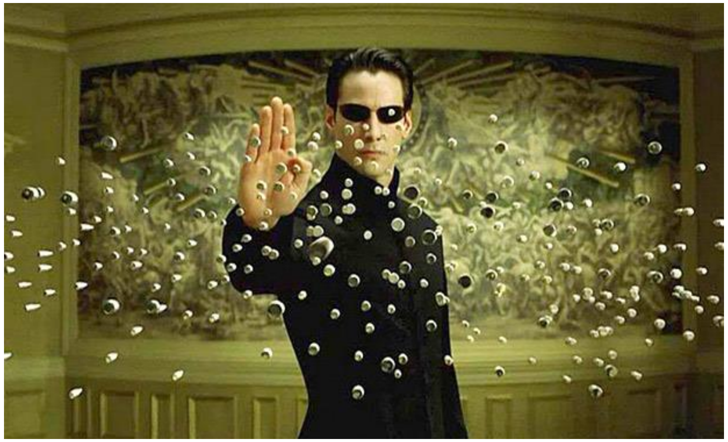
- ***Two shortest-path algorithms:***
  - Bellman-Ford for single-source shortest path
  - Floyd-Warshall for all-pairs shortest path

- ***Dynamic programming!***
  - This is a fancy name for:
    - Break up an optimization problem into smaller problems
      - The optimal solutions to the sub-problems should be sub-solutions to the original problem.
    - Build the optimal solution iteratively by filling in a table of sub-solutions.
      - Take advantage of overlapping sub-problems!

Next time

- More examples of ***dynamic programming***!

We will stop bullets with our action-packed coding skills, and also maybe find longest common subsequences.



Acknowledgement

- Stanford University