

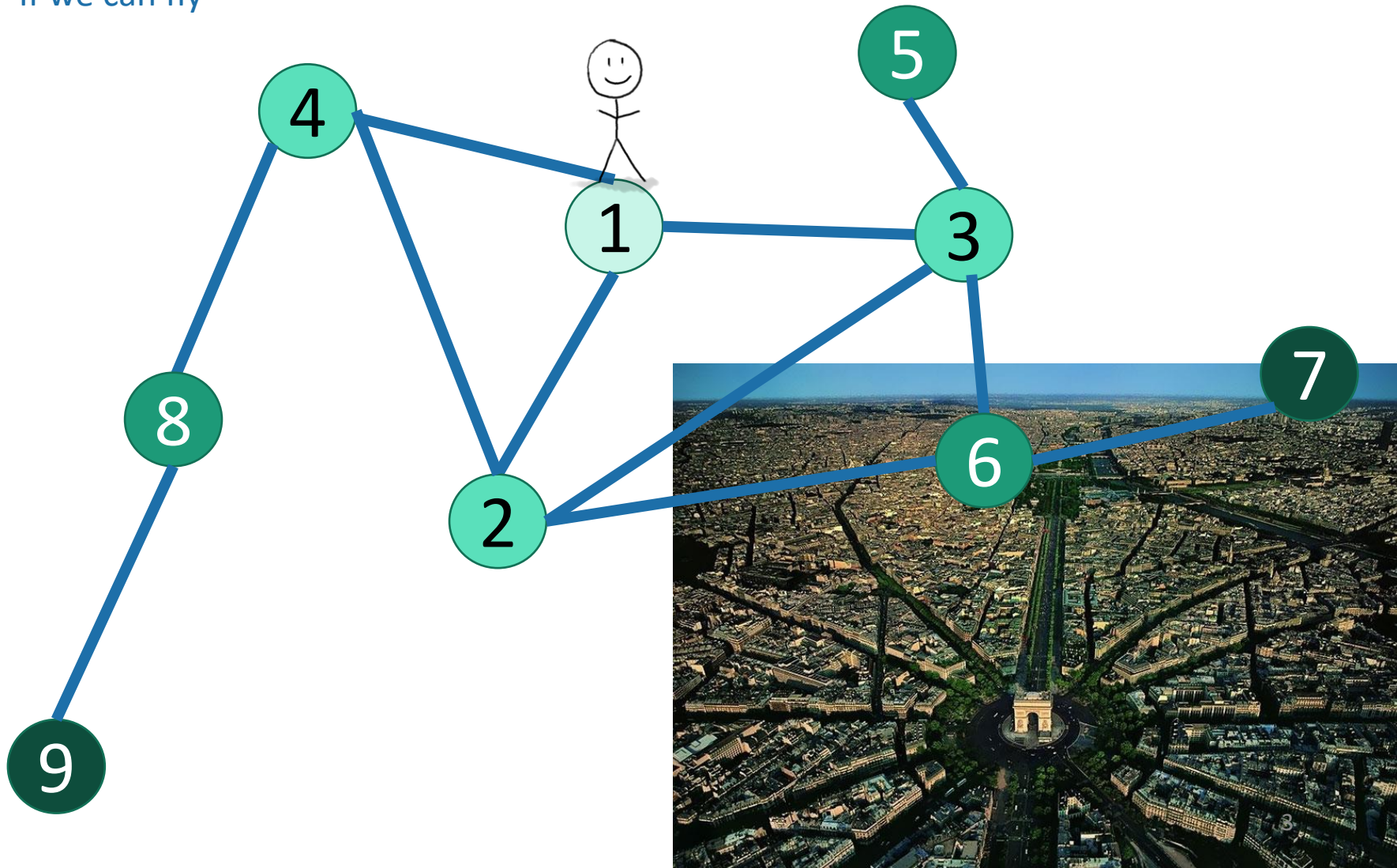
Advanced Data Structures and Algorithms

Breadth First Search (BFS)

Breadth-first search

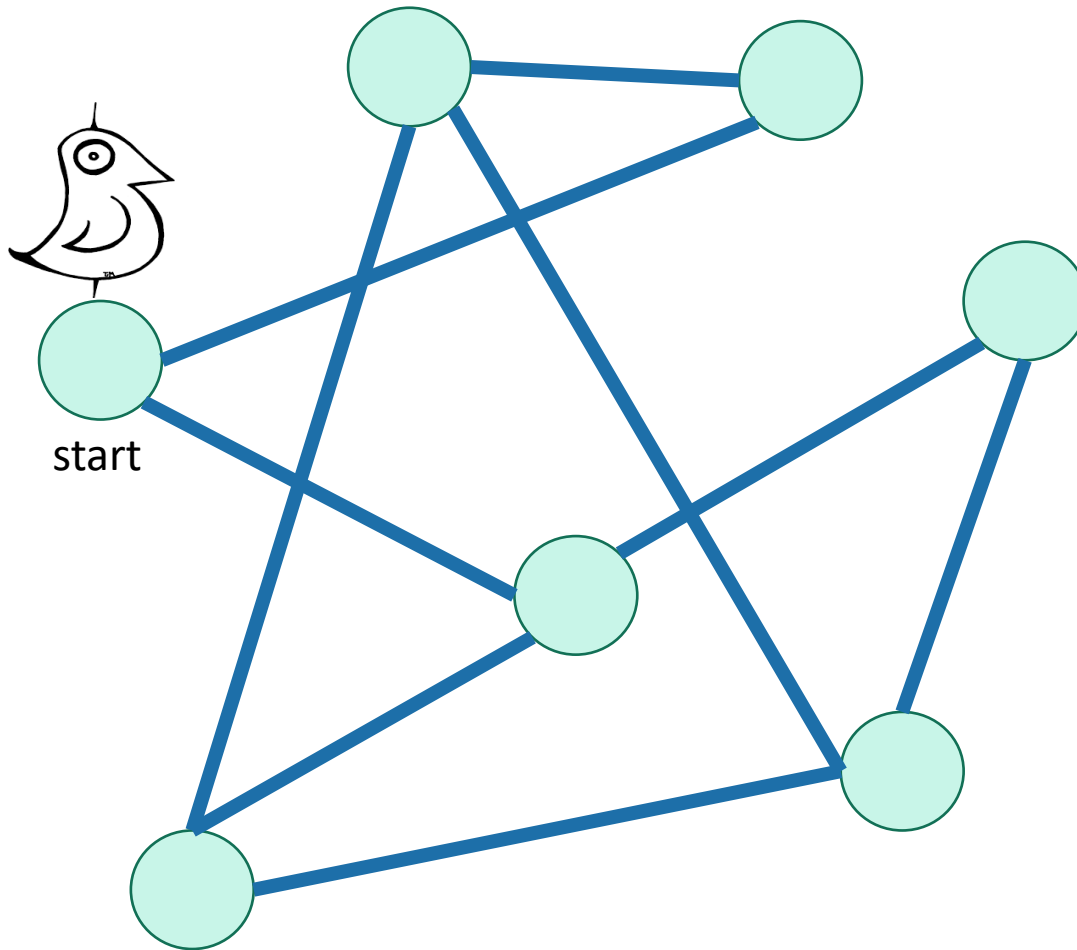
How do we explore a graph?






If we can fly



Breadth-First Search

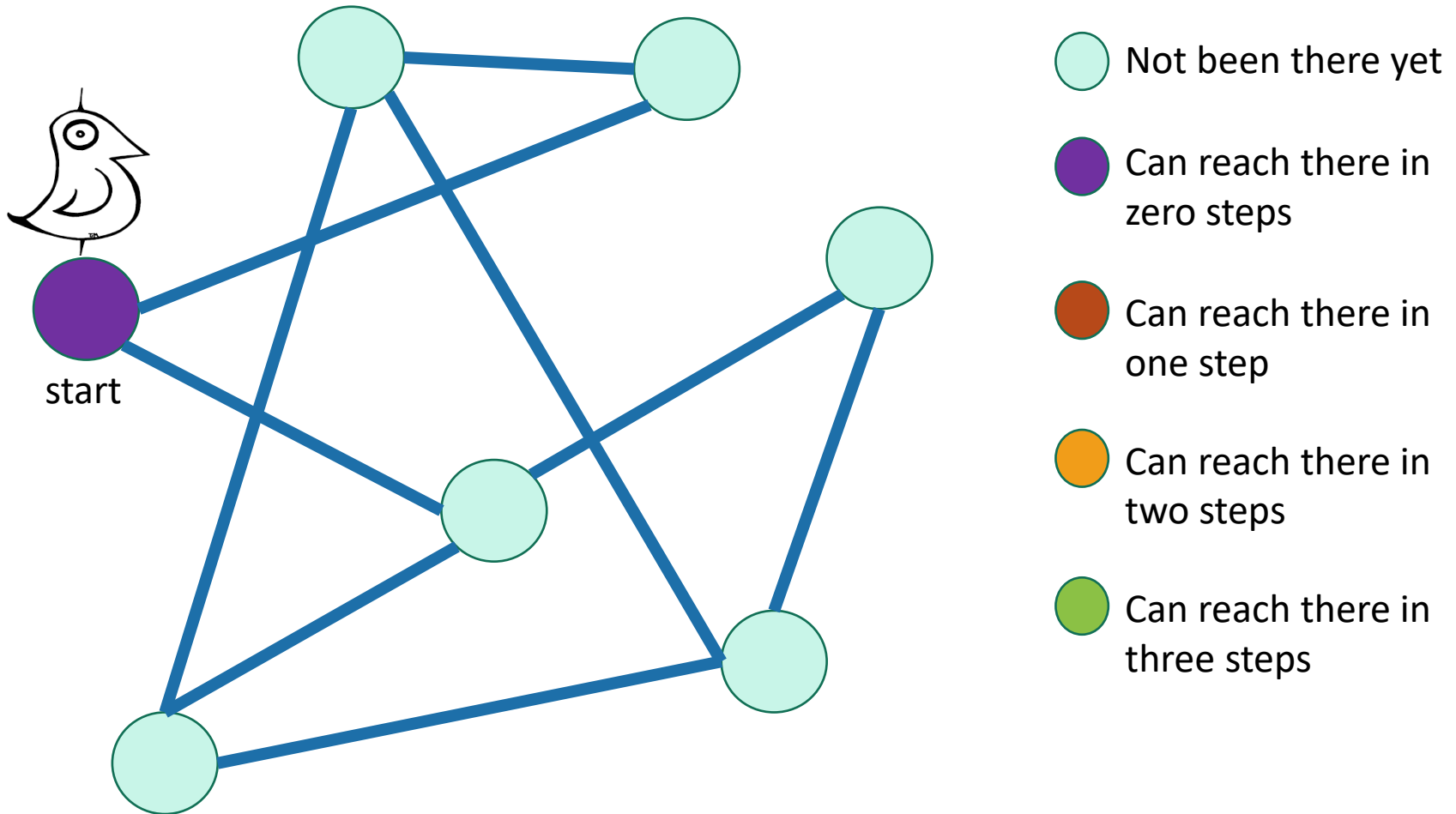
Exploring the world with a bird's-eye view



-  Not been there yet
-  Can reach there in zero steps
-  Can reach there in one step
-  Can reach there in two steps
-  Can reach there in three steps

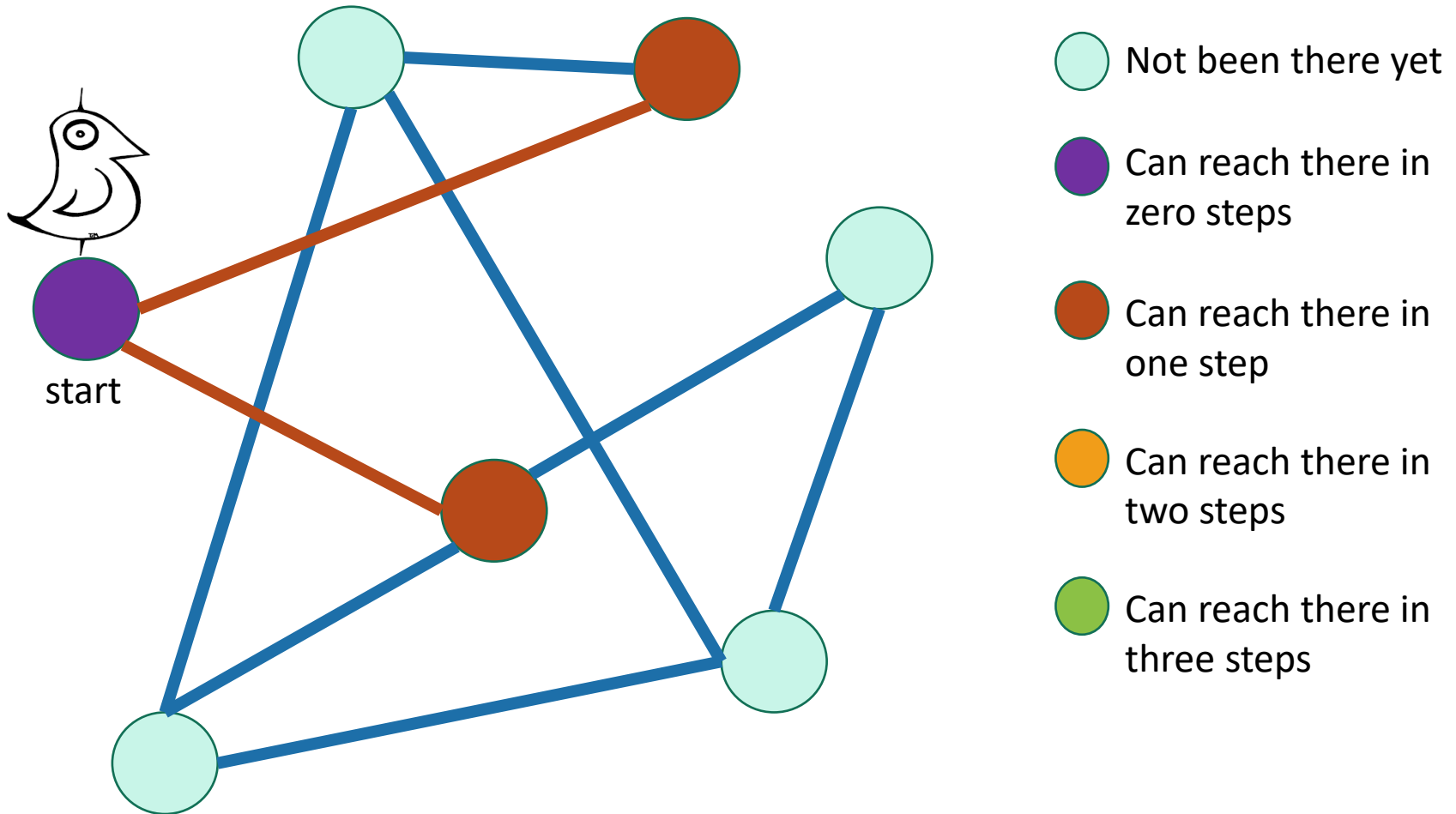
Breadth-First Search

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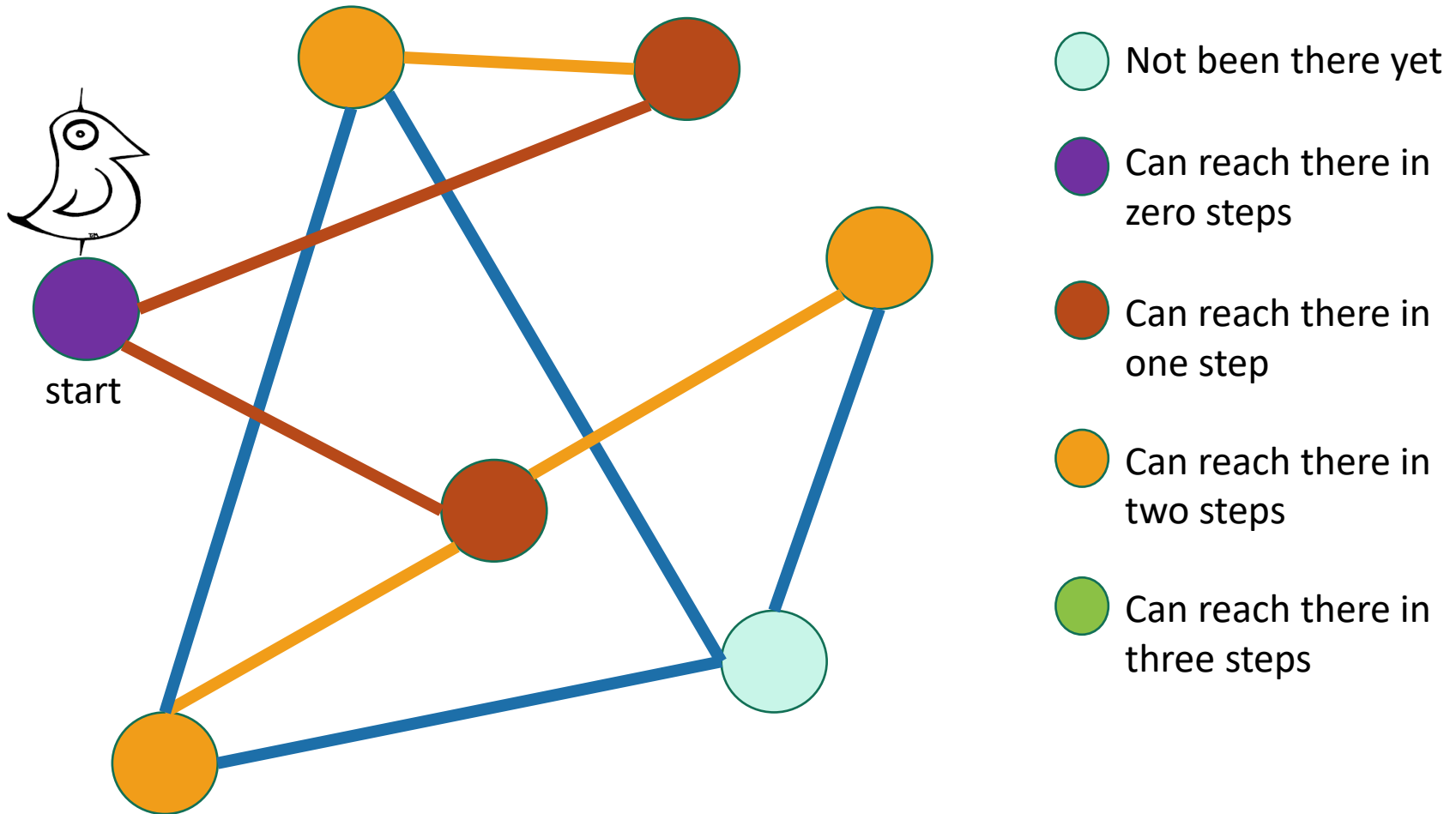
Breadth-First Search

Exploring the world with a bird's-eye view



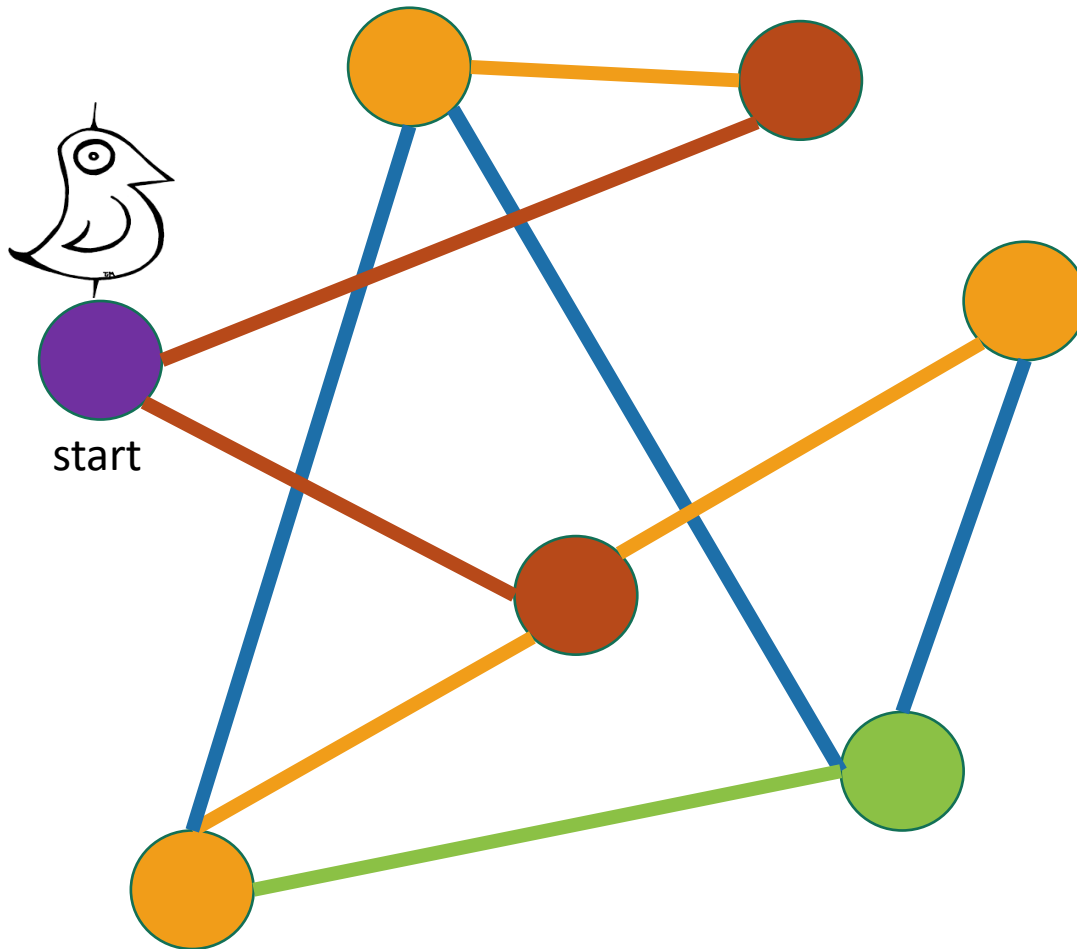
Breadth-First Search

Exploring the world with a bird's-eye view



Breadth-First Search

Exploring the world with a bird's-eye view



Not been there yet

Can reach there in zero steps

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World:
explored!

Same disclaimer as for DFS: you may have seen other ways to implement this,
this will be convenient for us.

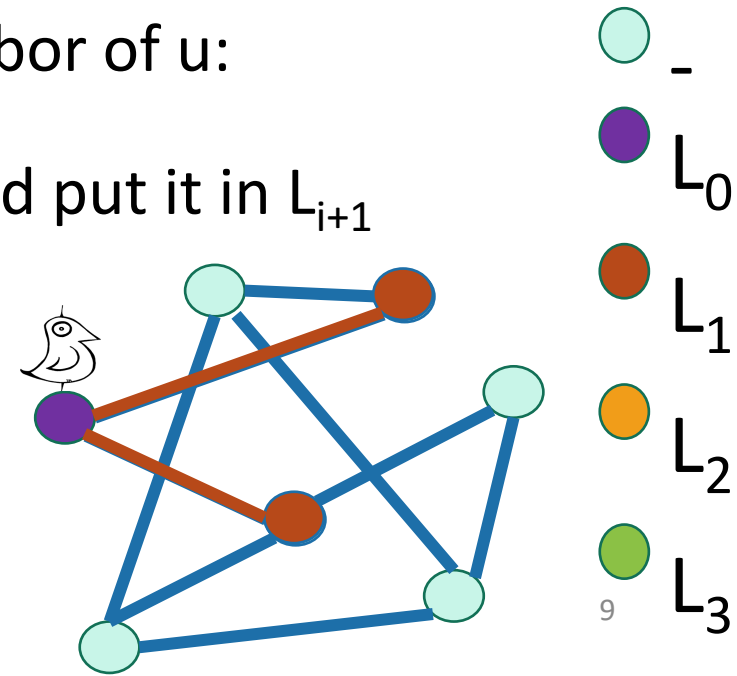
Breadth-First Search

Exploring the world with pseudocode

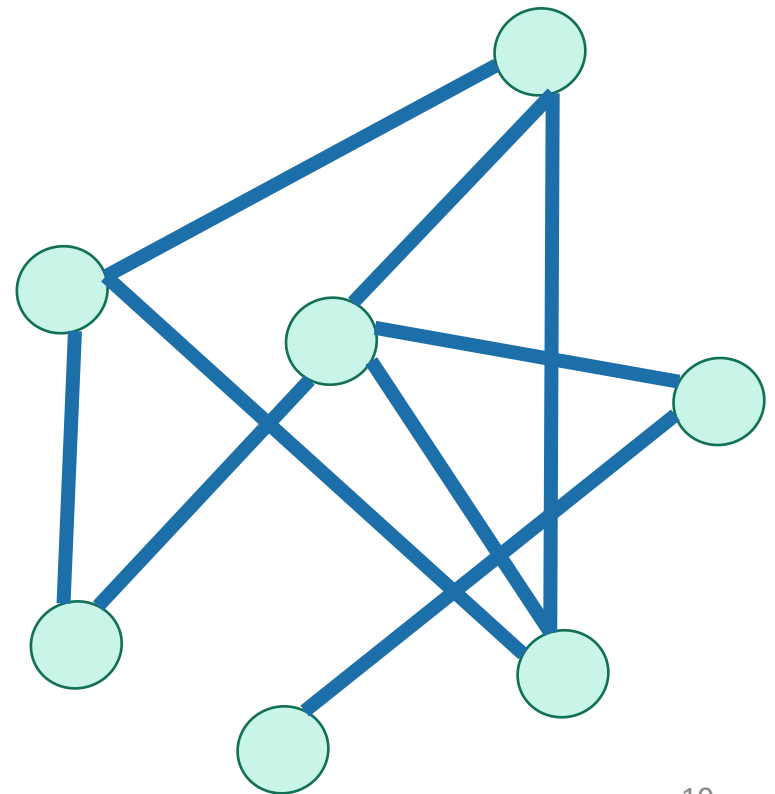
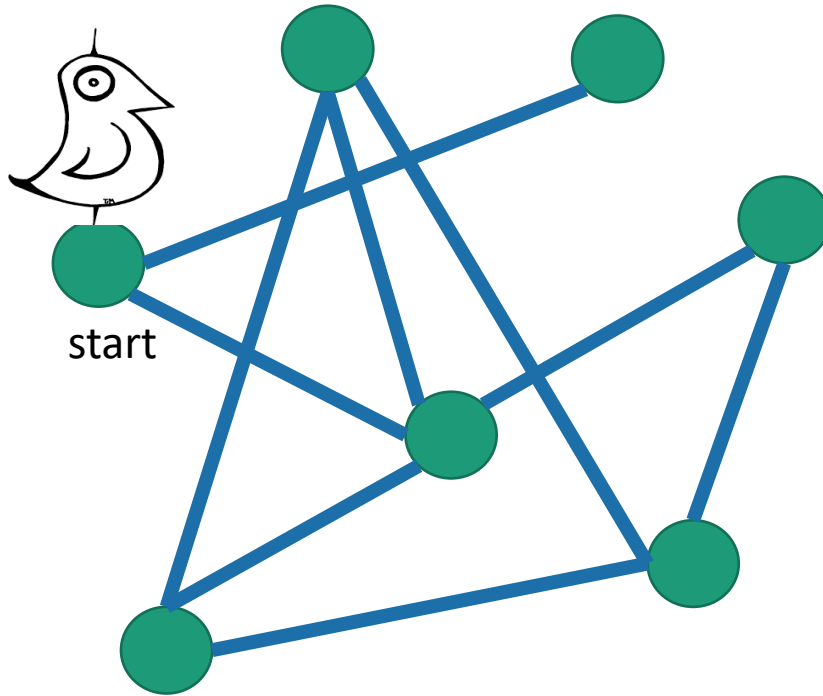
- Set $L_i = []$ for $i=1, \dots, n$
- $L_0 = [w]$, where w is the start node
- Mark w as visited
- **For** $i = 0, \dots, n-1$:
 - **For** u in L_i :
 - **For** each v which is a neighbor of u :
 - **If** v isn't yet visited:
 - mark v as visited, and add v to L_{i+1}

L_i is the set of nodes
we can reach in i
steps from w

Go through all the nodes
in L_i and add their
unvisited neighbors to L_{i+1}



BFS also finds all the nodes reachable from the starting point



It is also a good way to find all the **connected components**.

Running time and extension to directed graphs

- To explore the whole graph, explore the connected components one-by-one.
 - Same argument as DFS: BFS running time is $O(n + m)$
- Like DFS, BFS also works fine on directed graphs.

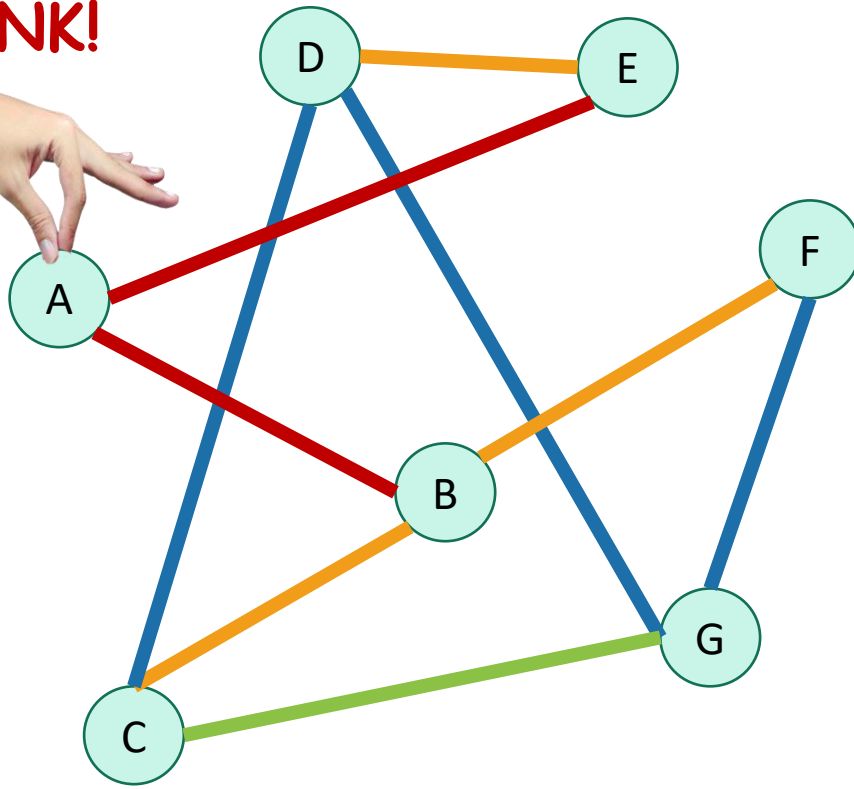
Verify these!



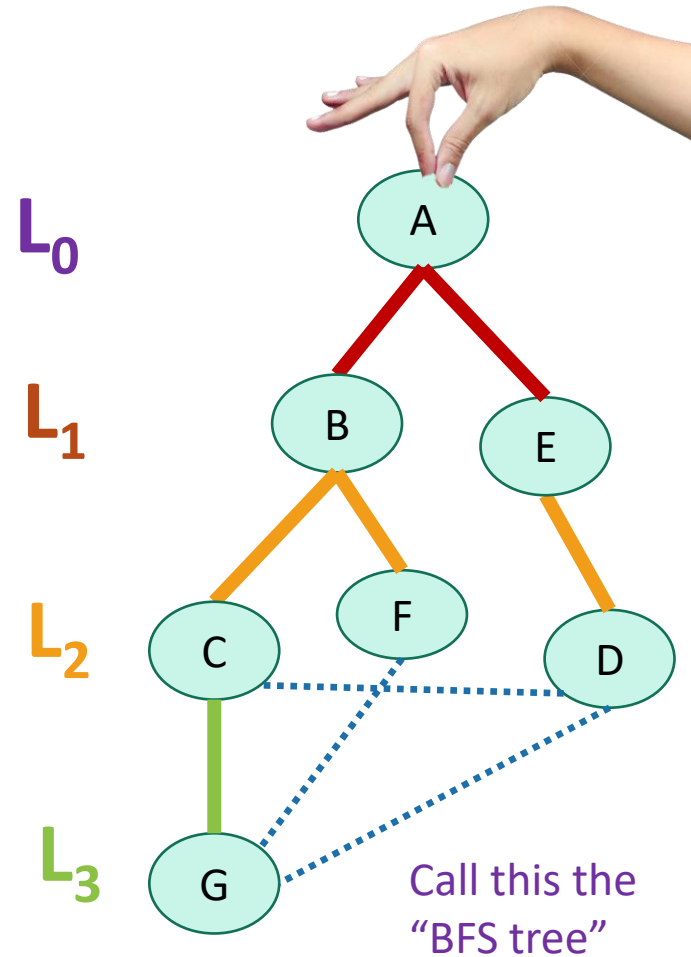
Why is it called breadth-first?

- We are implicitly building a tree:

YOINK!

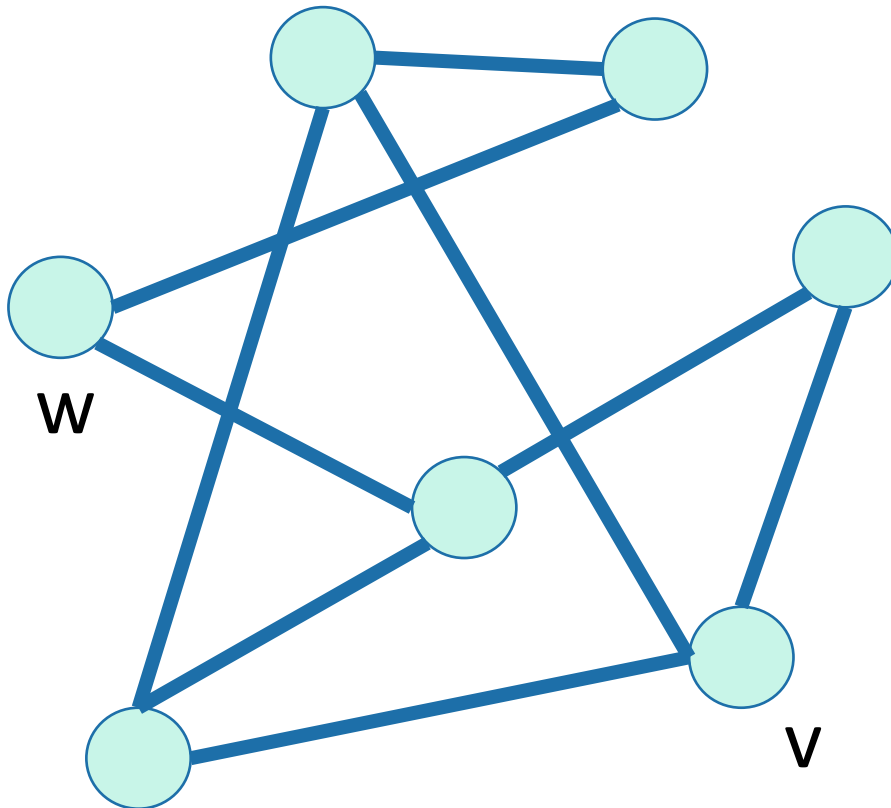


- First we go as broadly as we can.



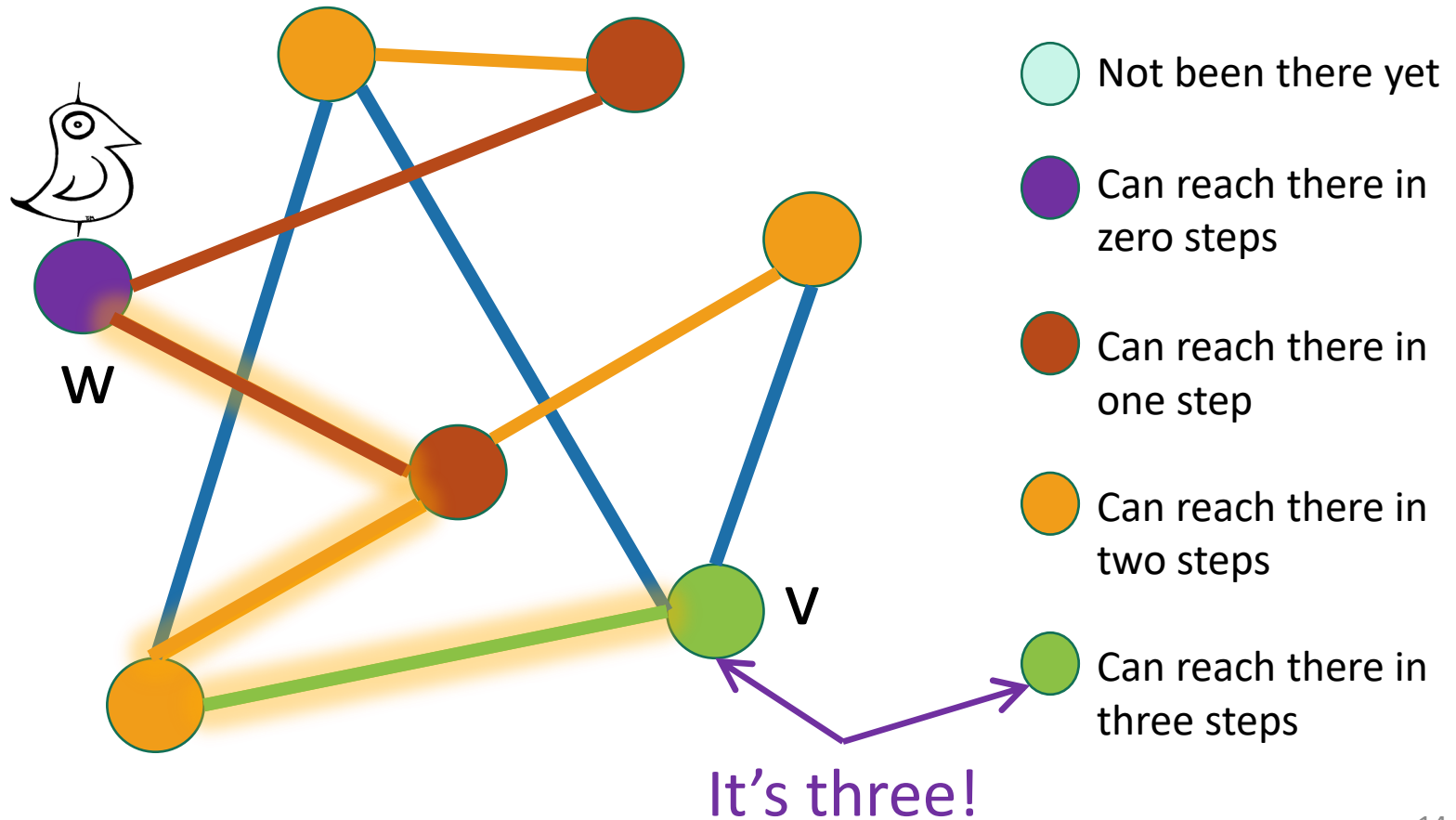
Application of BFS: shortest path

- How long is the shortest path between w and v?



Application of BFS: shortest path

- How long is the shortest path between w and v?



To find the **distance** between w and all other vertices v

The **distance** between two vertices is the number of edges in the shortest path between them.

- Do a BFS starting at w
- For all v in L_i
 - The shortest path between w and v has length i
 - A shortest path between w and v is given by the path in the BFS tree.
- If we never found v , the distance is infinite.

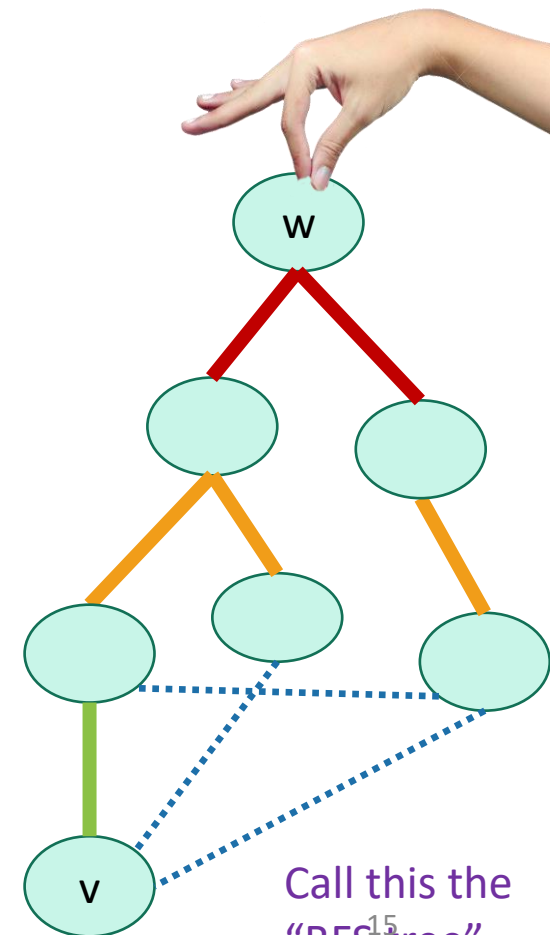
This requires some proof!

L_0

L_1

L_2

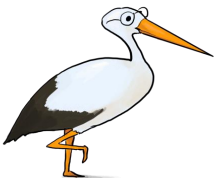
L_3



Call this the
"BFS tree"

15

Modify the BFS pseudocode to return shortest paths!



Proof overview

that the BFS tree behaves like it should

- Proof by induction.
- Inductive hypothesis for j :
 - For all $i < j$ the vertices in L_i have distance i from v .
- Base case:
 - $L_0 = \{v\}$, so we're good.
- Inductive step:
 - Let w be in L_j . Want to show $\text{dist}(v, w) = j$.
 - We know $\text{dist}(v, w) \leq j$, since $\text{dist}(v, w\text{'s parent in } L_{j-1}) = j-1$ by induction, so that gives a path of length j from v to w .
 - On the other hand, $\text{dist}(v, w) \geq j$, since if $\text{dist}(v, w) < j$, w would have shown up in an earlier layer.
 - Thus, $\text{dist}(v, w) = j$.
- Conclusion:
 - For each vertex w in V , if w is in L_j , then $\text{dist}(v, w) = j$.

What have we learned?

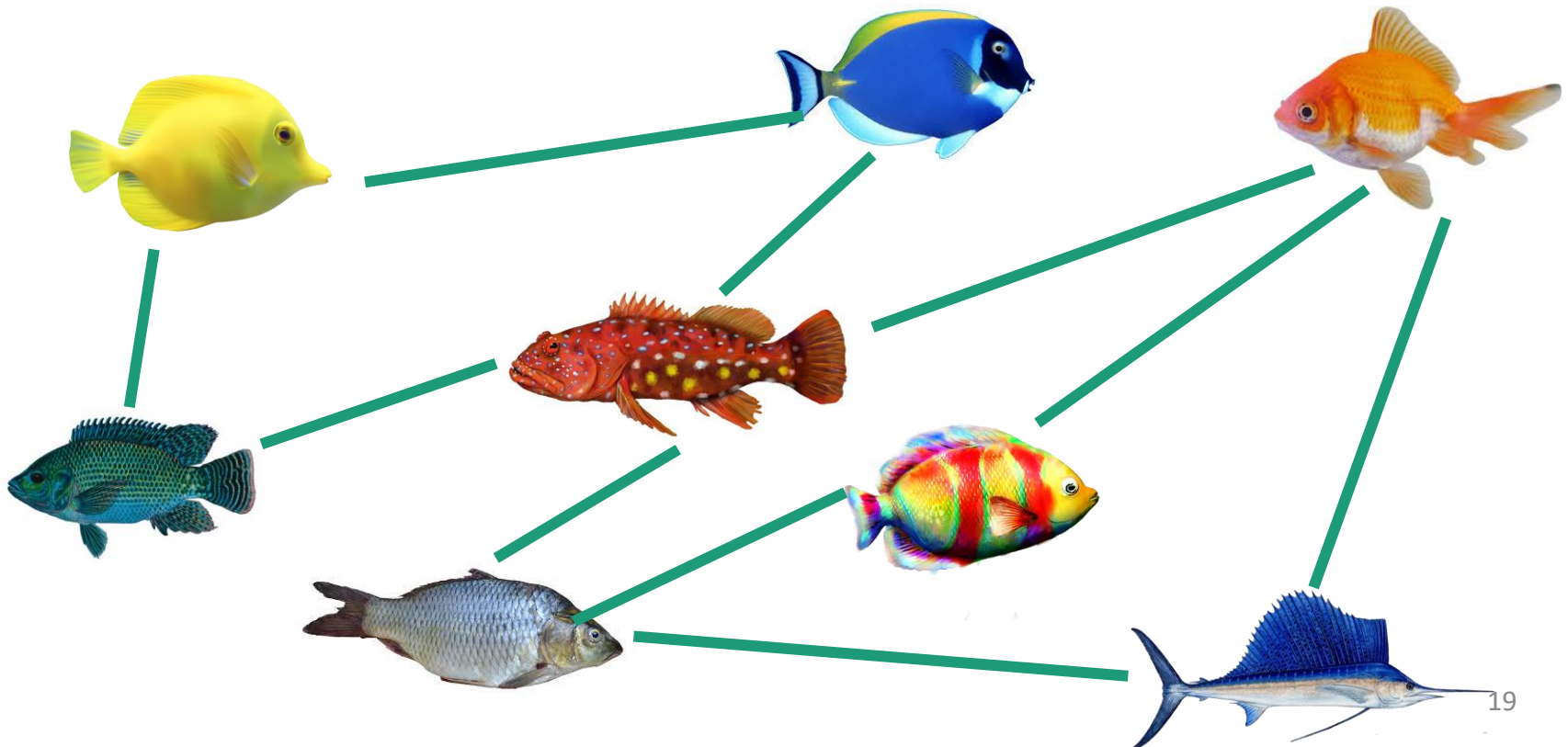
- The BFS tree is useful for computing distances between pairs of vertices.
- We can find the shortest path between u and v in time $O(m)$.

Another application of BFS

- Testing bipartite-ness

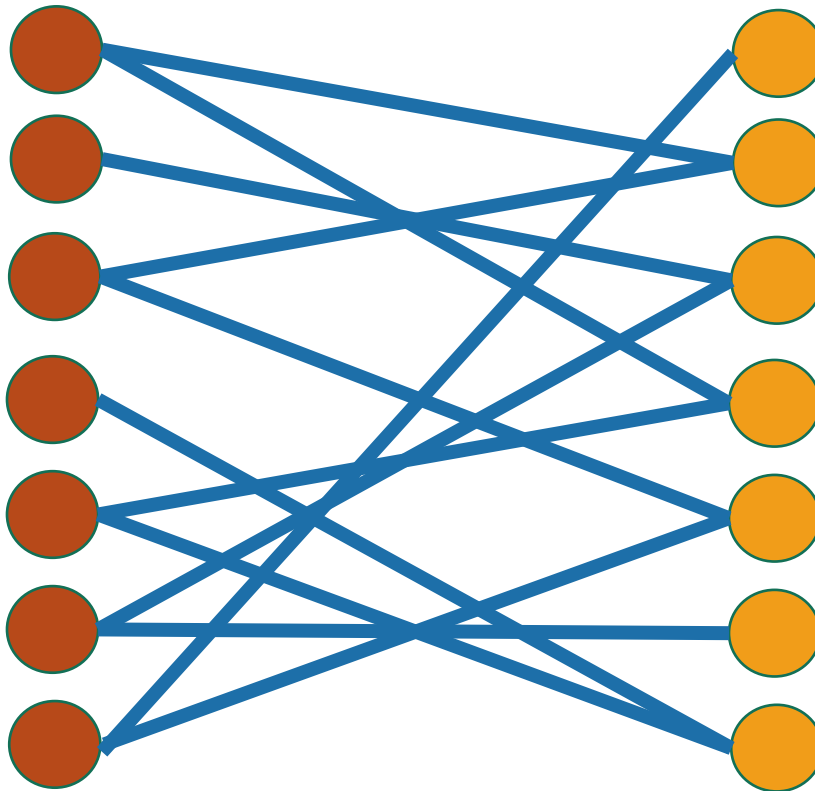
Exercise: fish

- You have a bunch of fish and two fish tanks.
- Some pairs of fish will fight if put in the same tank.
 - Model this as a graph: connected fish will fight.
- Can you put the fish in the two tanks so that there is no fighting?



Bipartite graphs

- A bipartite graph looks like this:



Can color the vertices red and orange so that there are no edges between any same-colored vertices

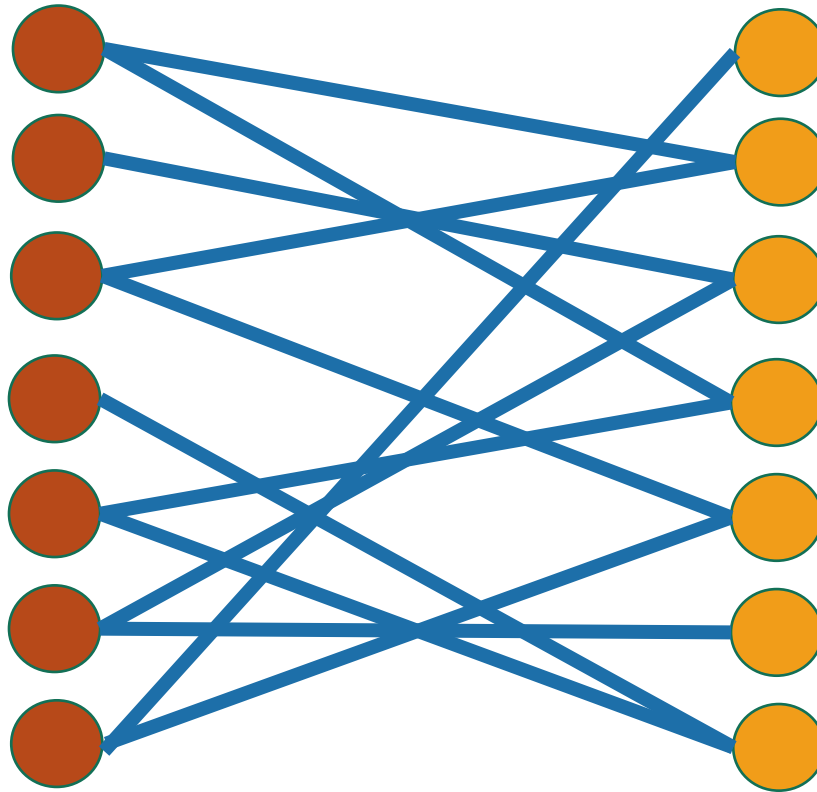
Example:

- are in tank A
- are in tank B
- — ● if the fish fight

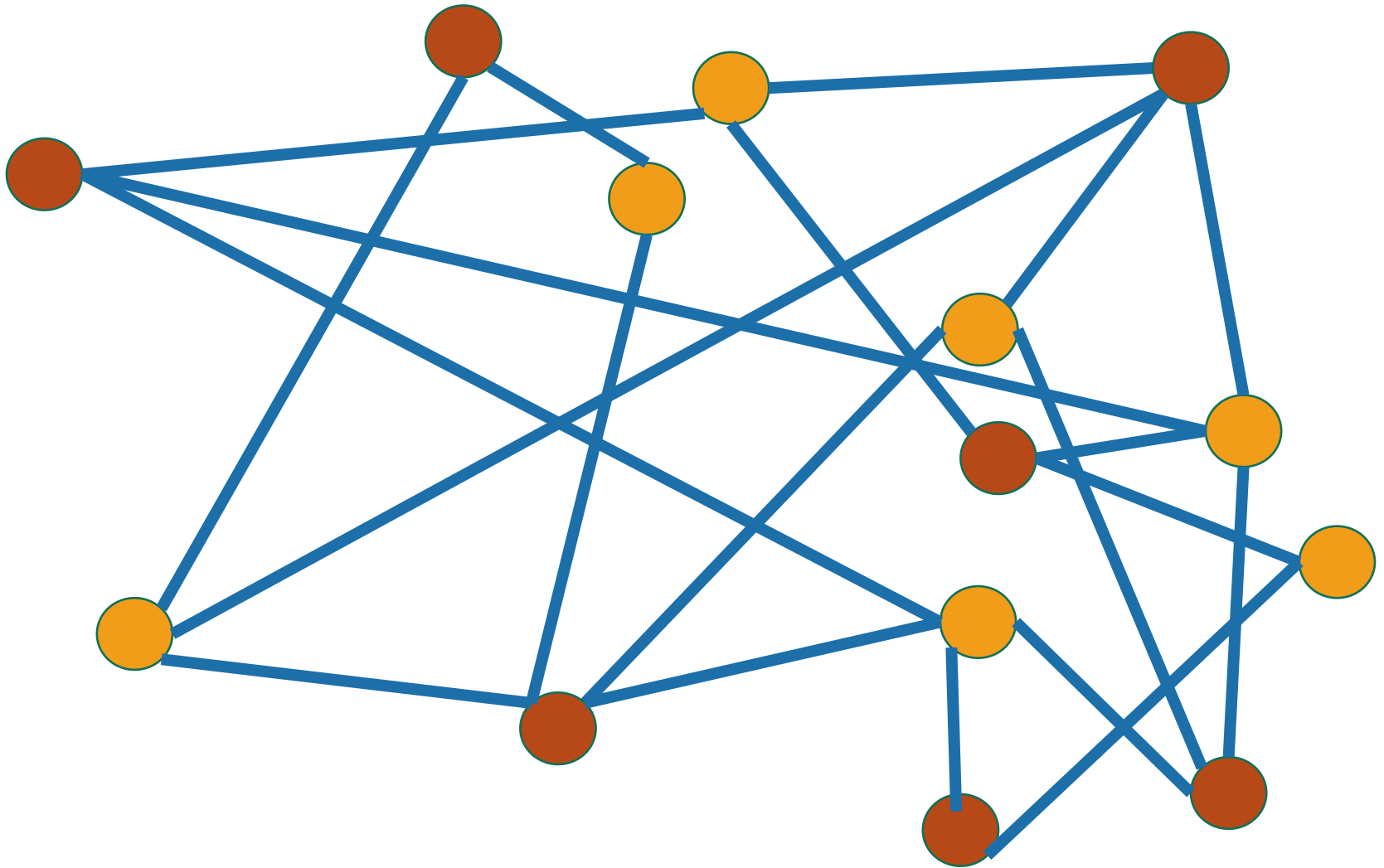
Example:

- are students
- are classes
- — ● if the student is enrolled in the class

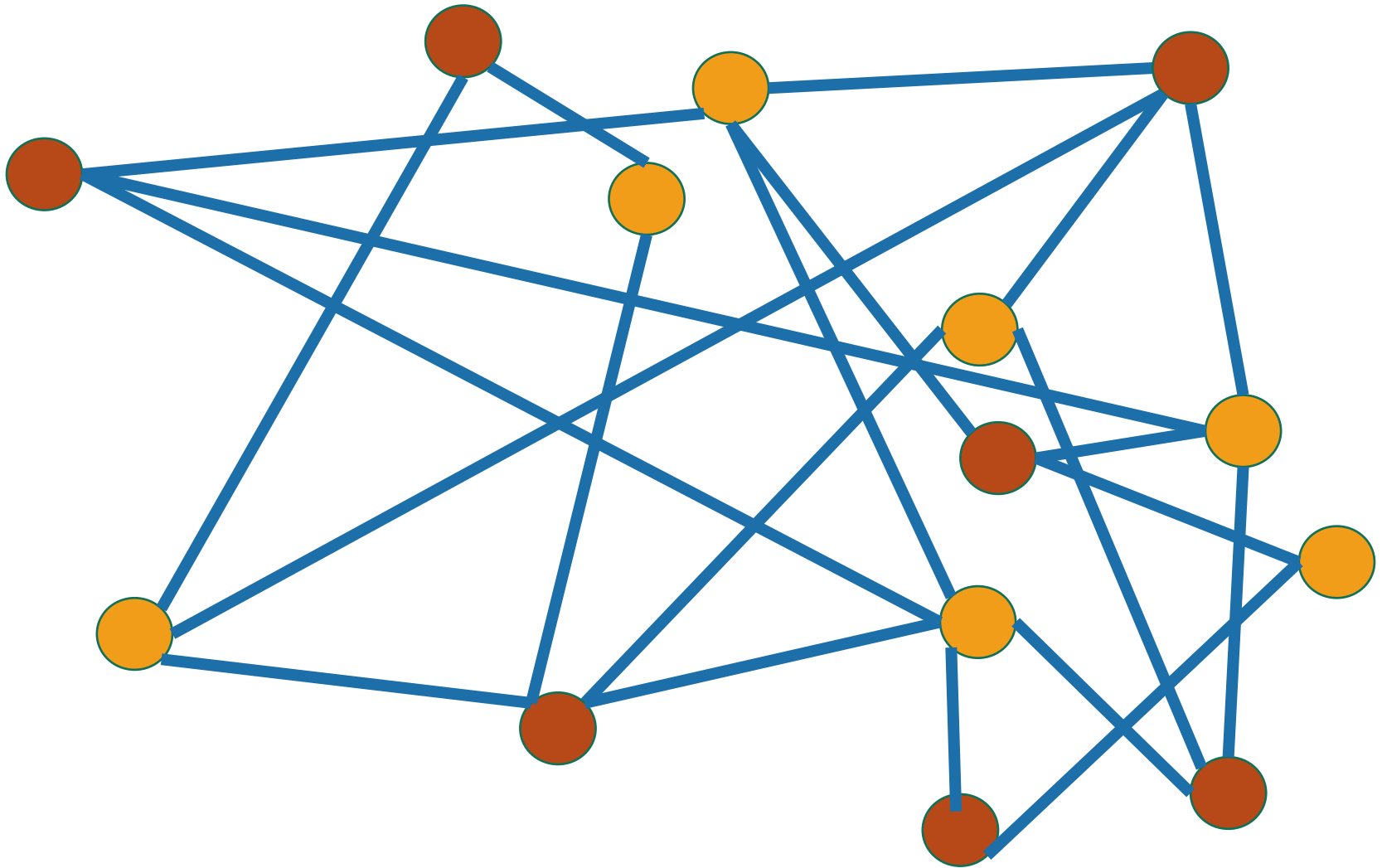
Is this graph bipartite?



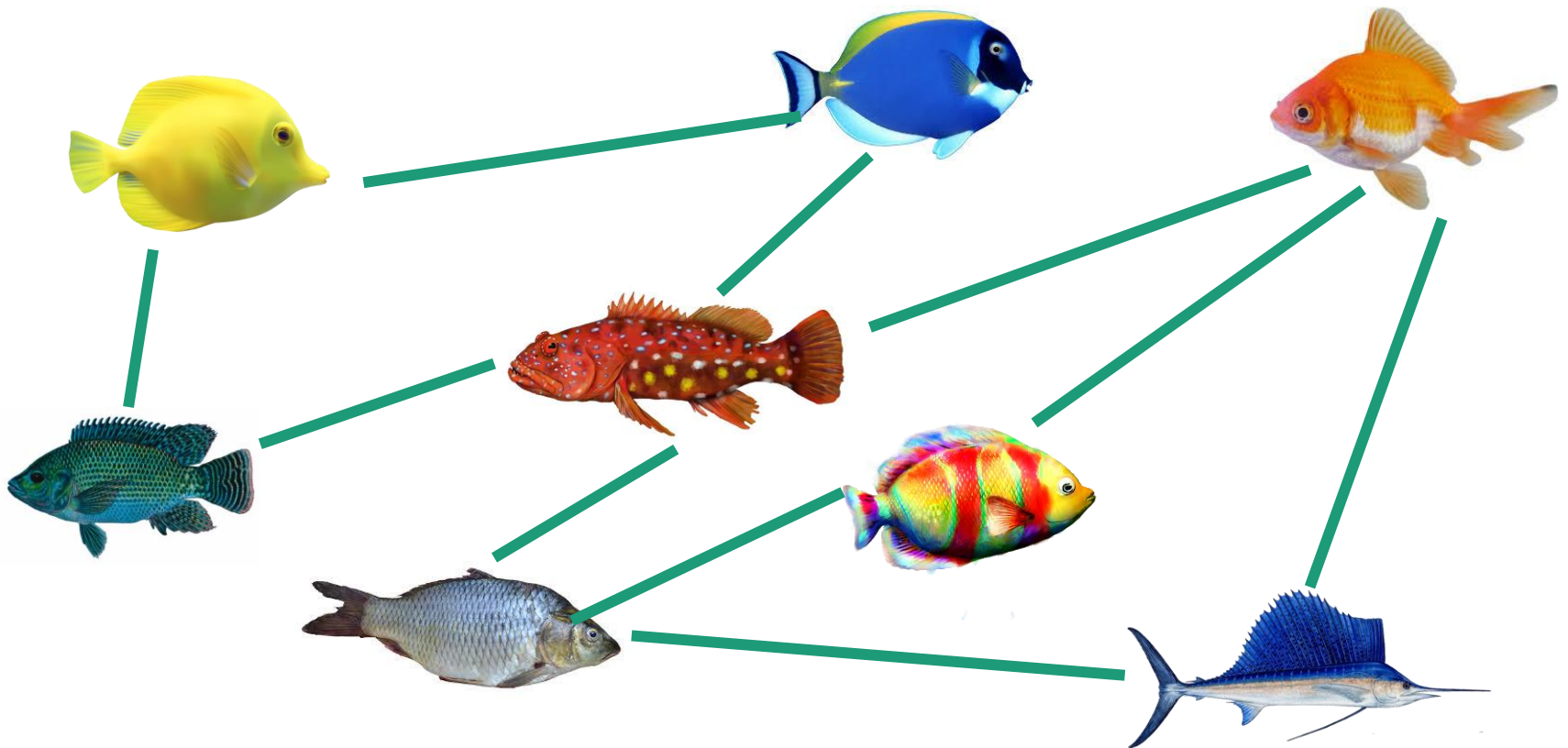
How about this one?



How about this one?

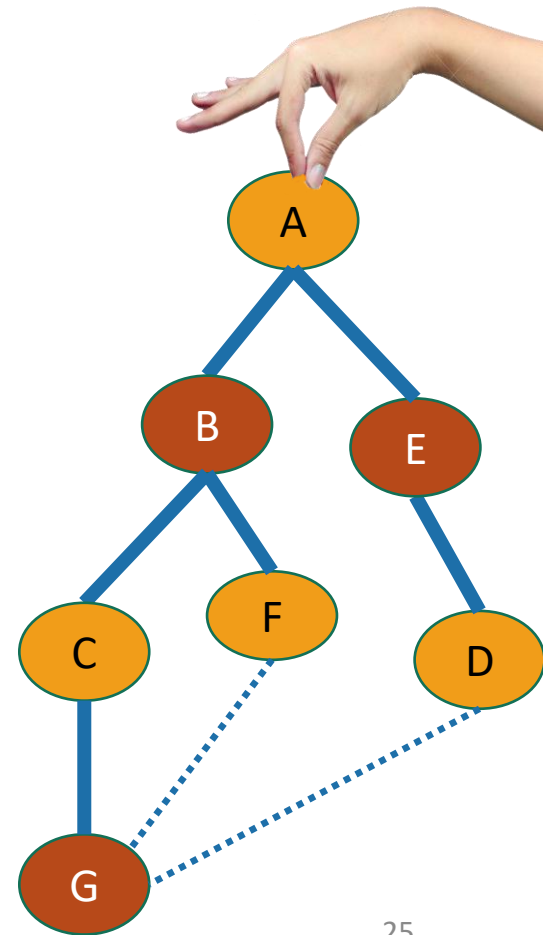


This one?



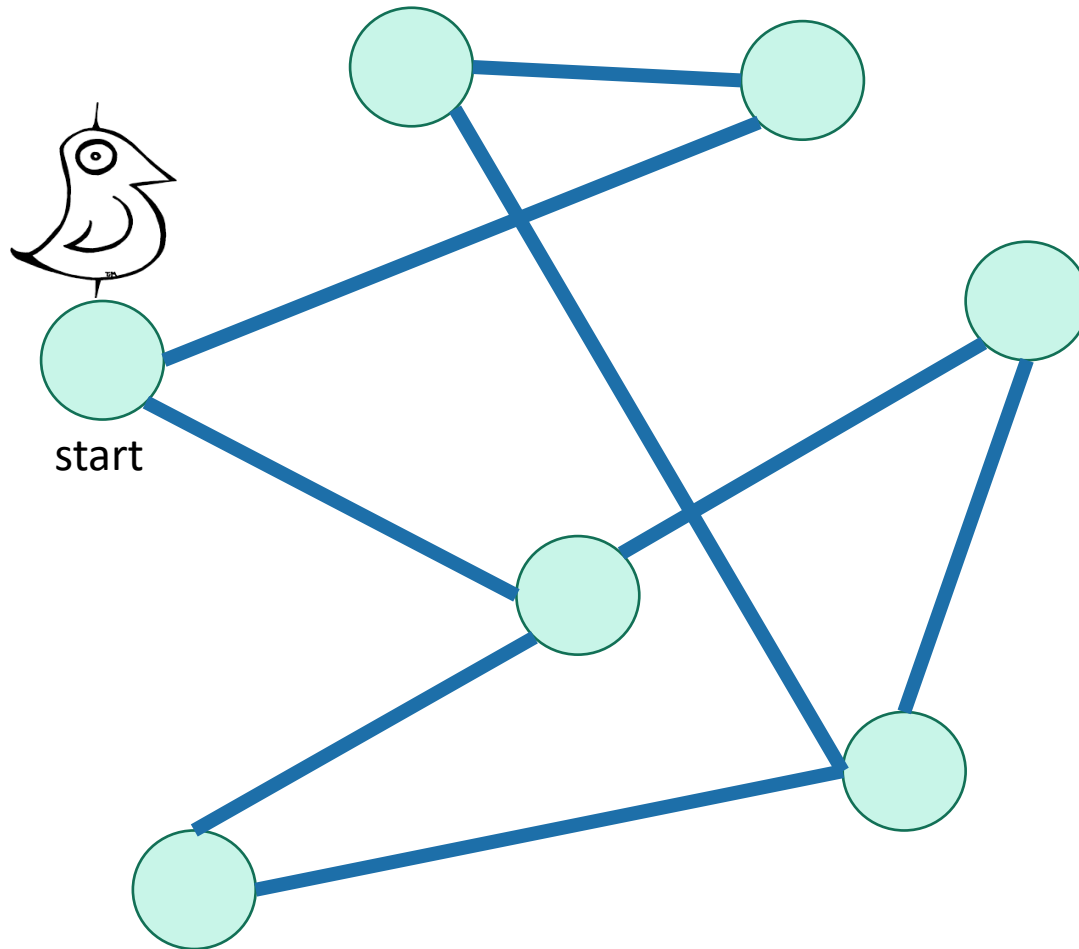
Application of BFS: Testing Bipartiteness






- Color the levels of the BFS tree in alternating colors.
- If you never color two connected nodes the same color, then it is bipartite.
- Otherwise, it's not.



Breadth-First Search

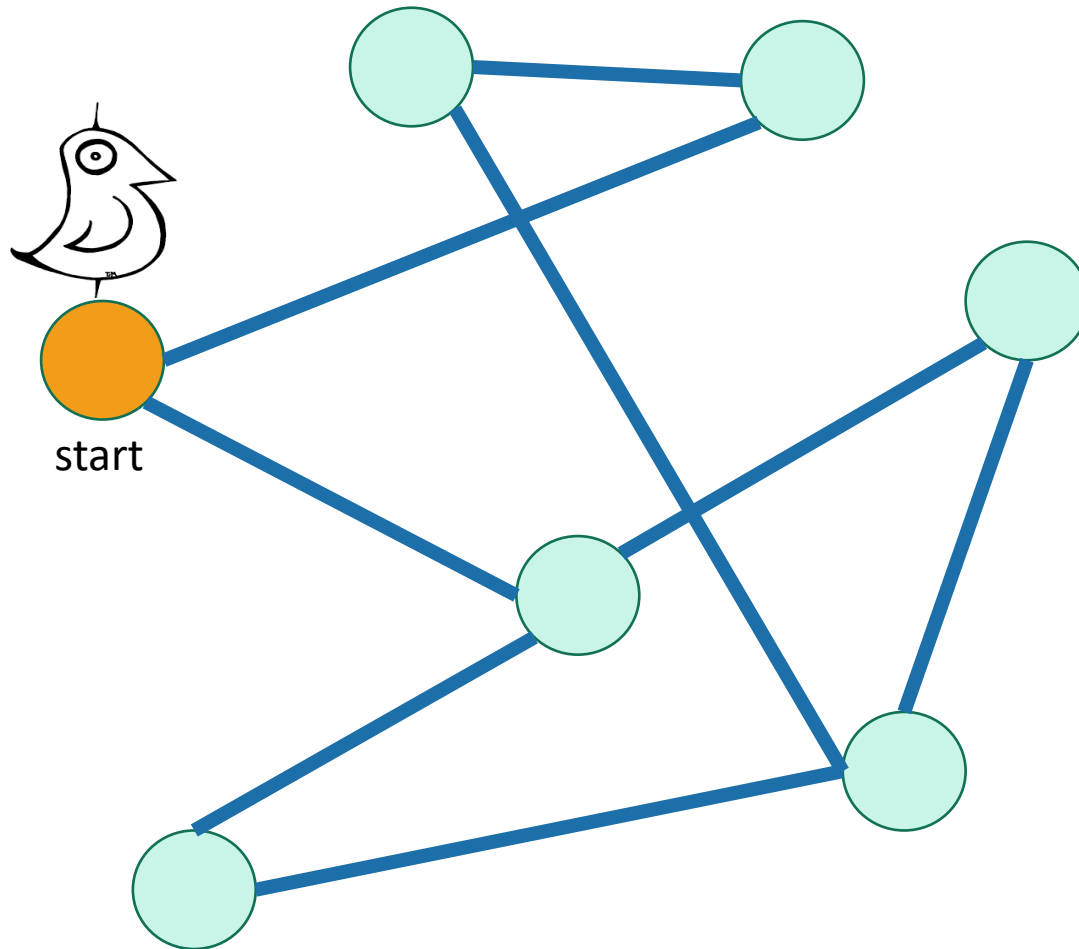
For testing bipartite-ness



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Breadth-First Search

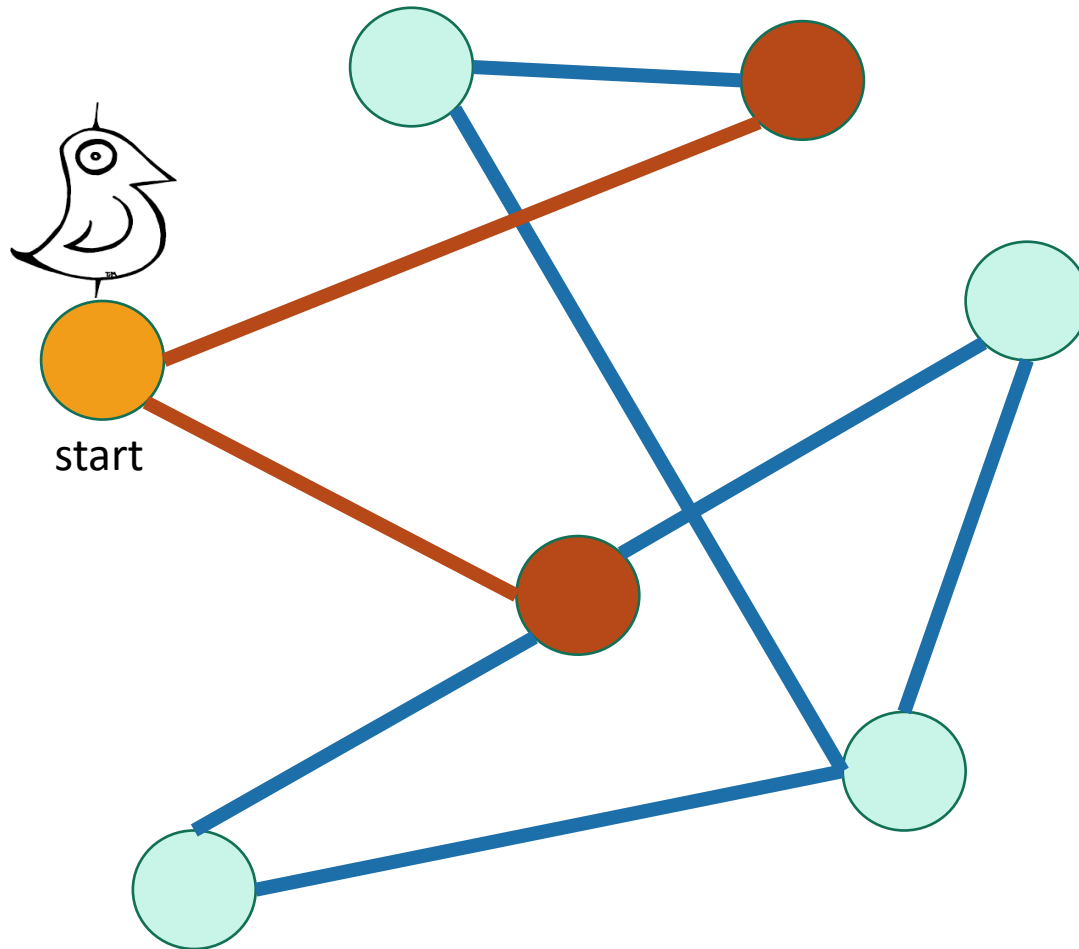
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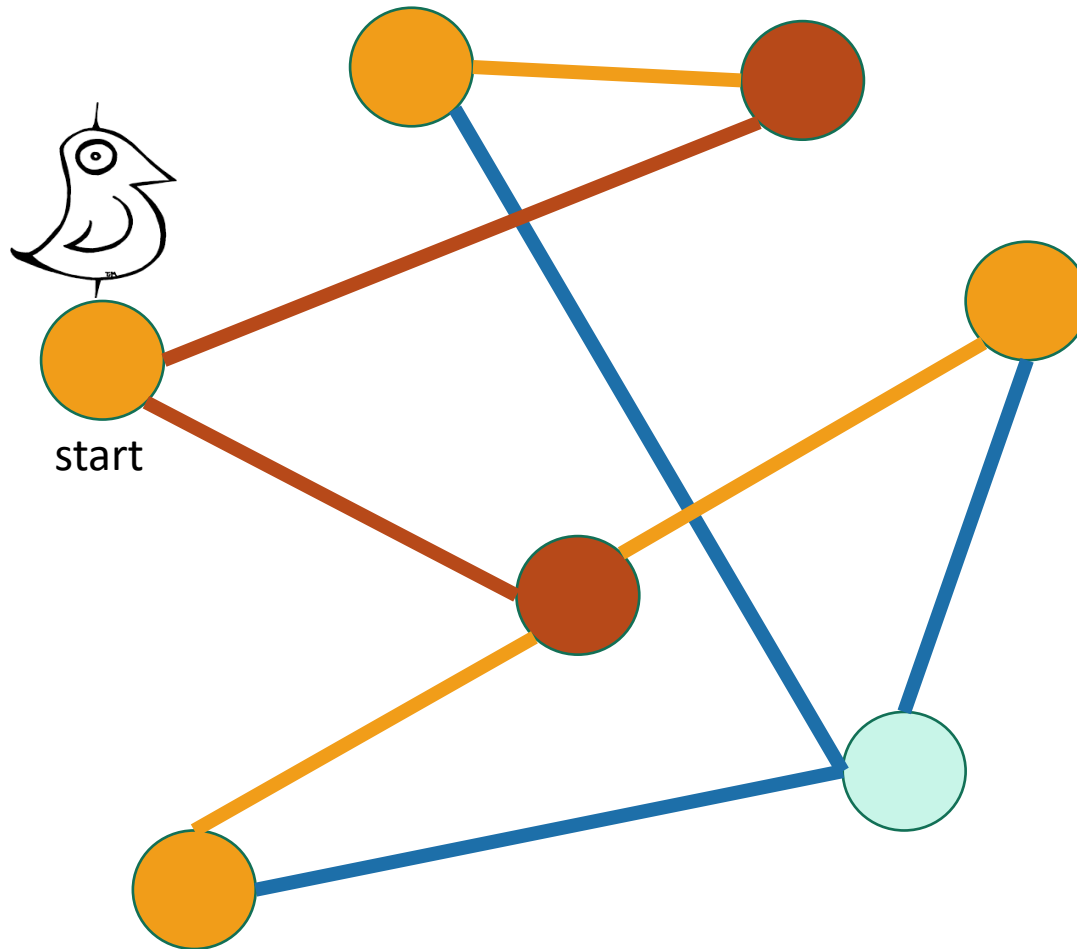
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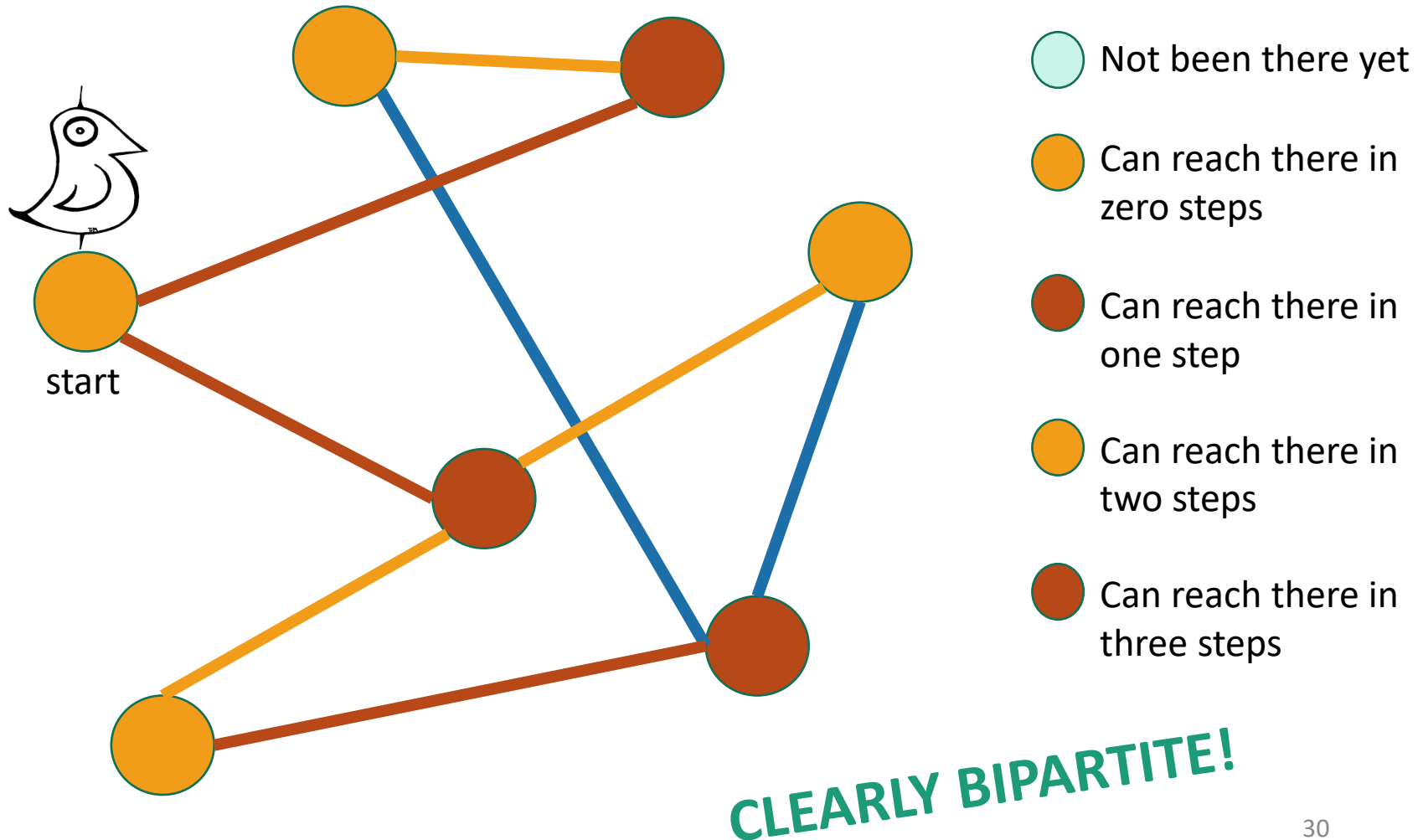
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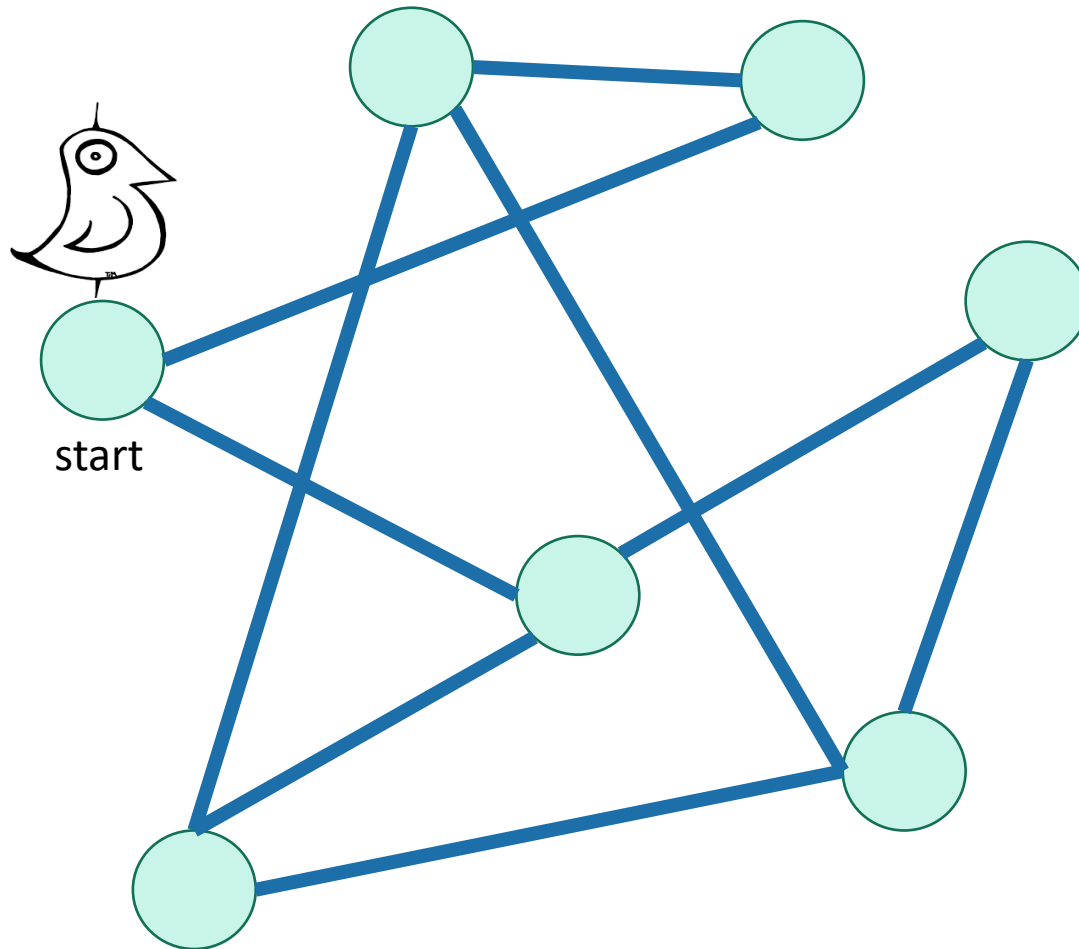
Breadth-First Search






For testing bipartite-ness



Breadth-First Search

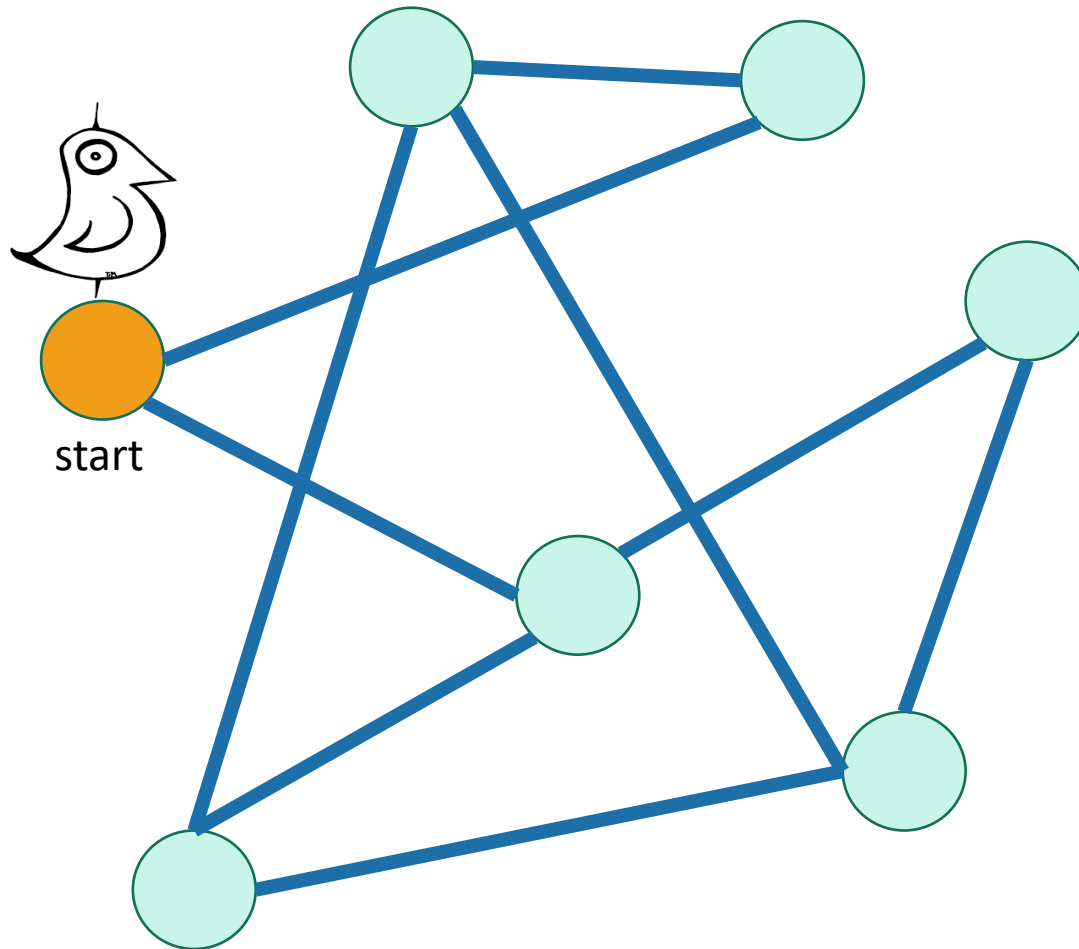
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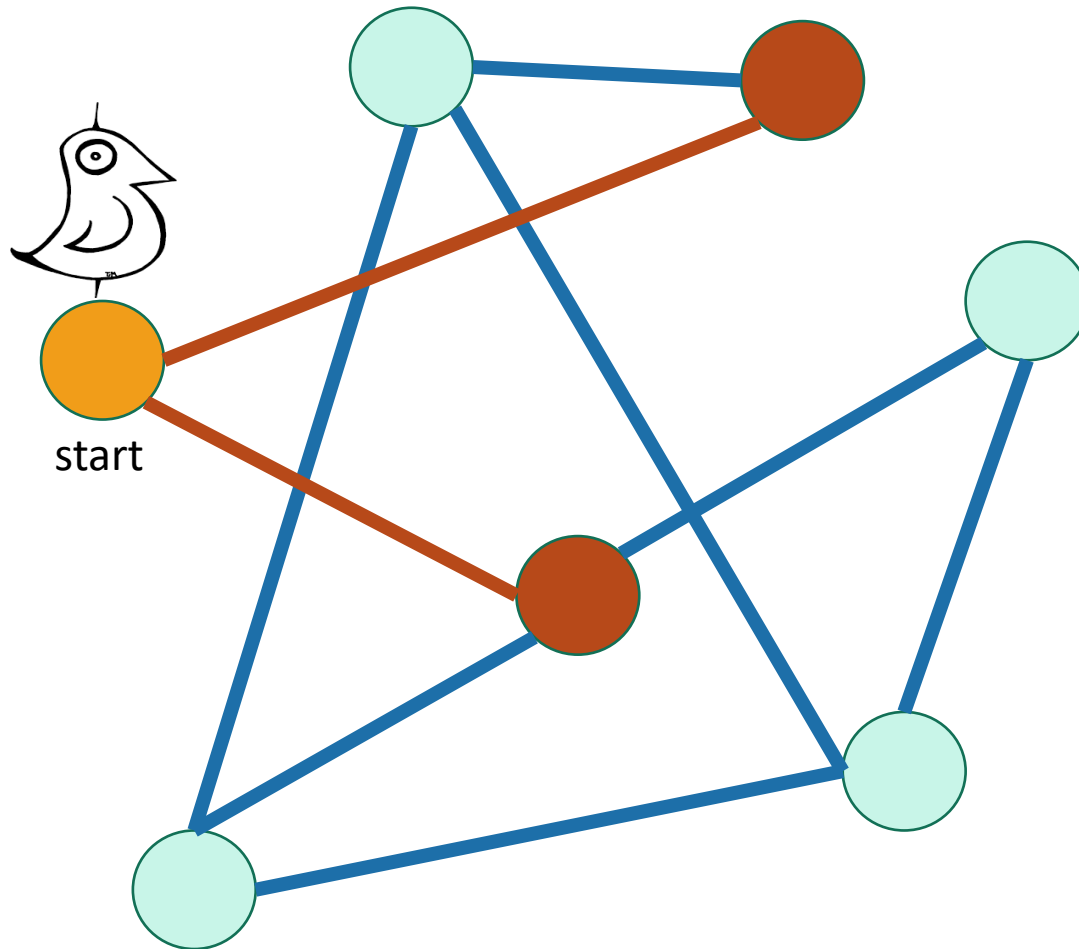
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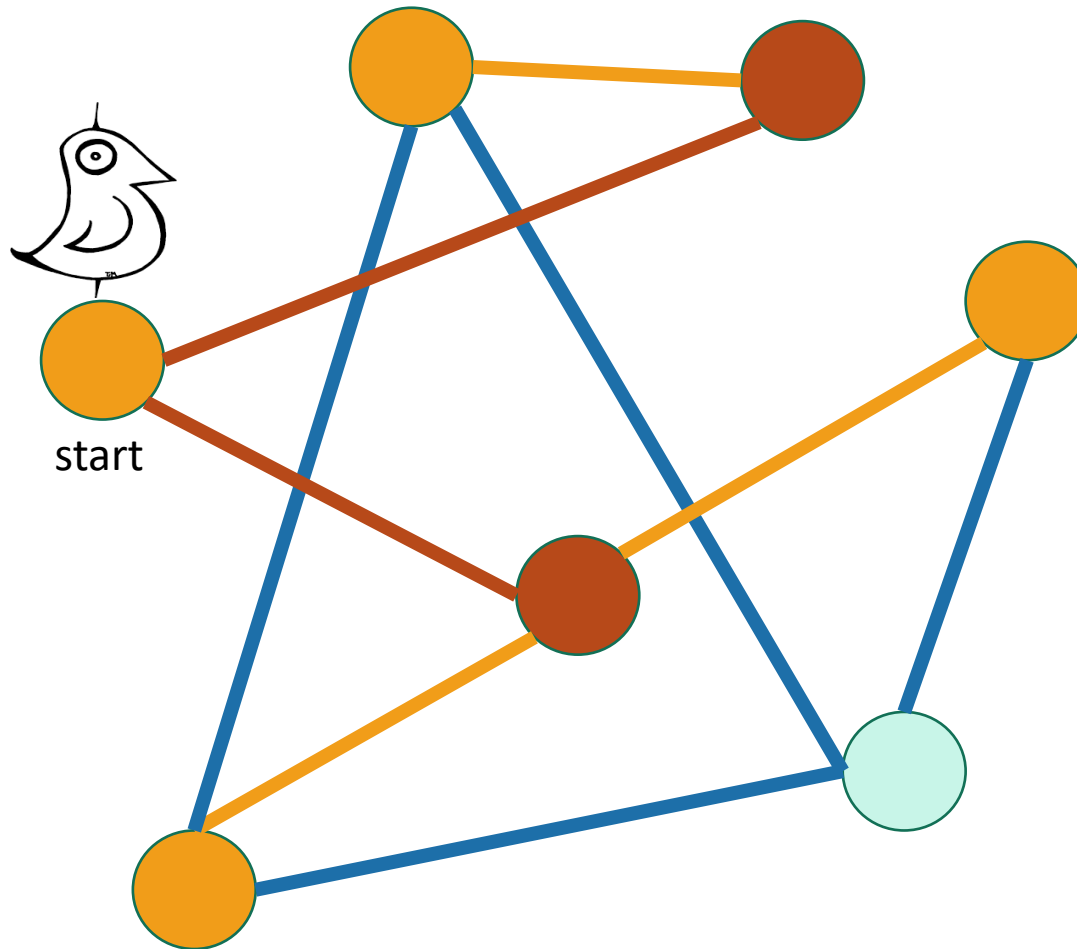
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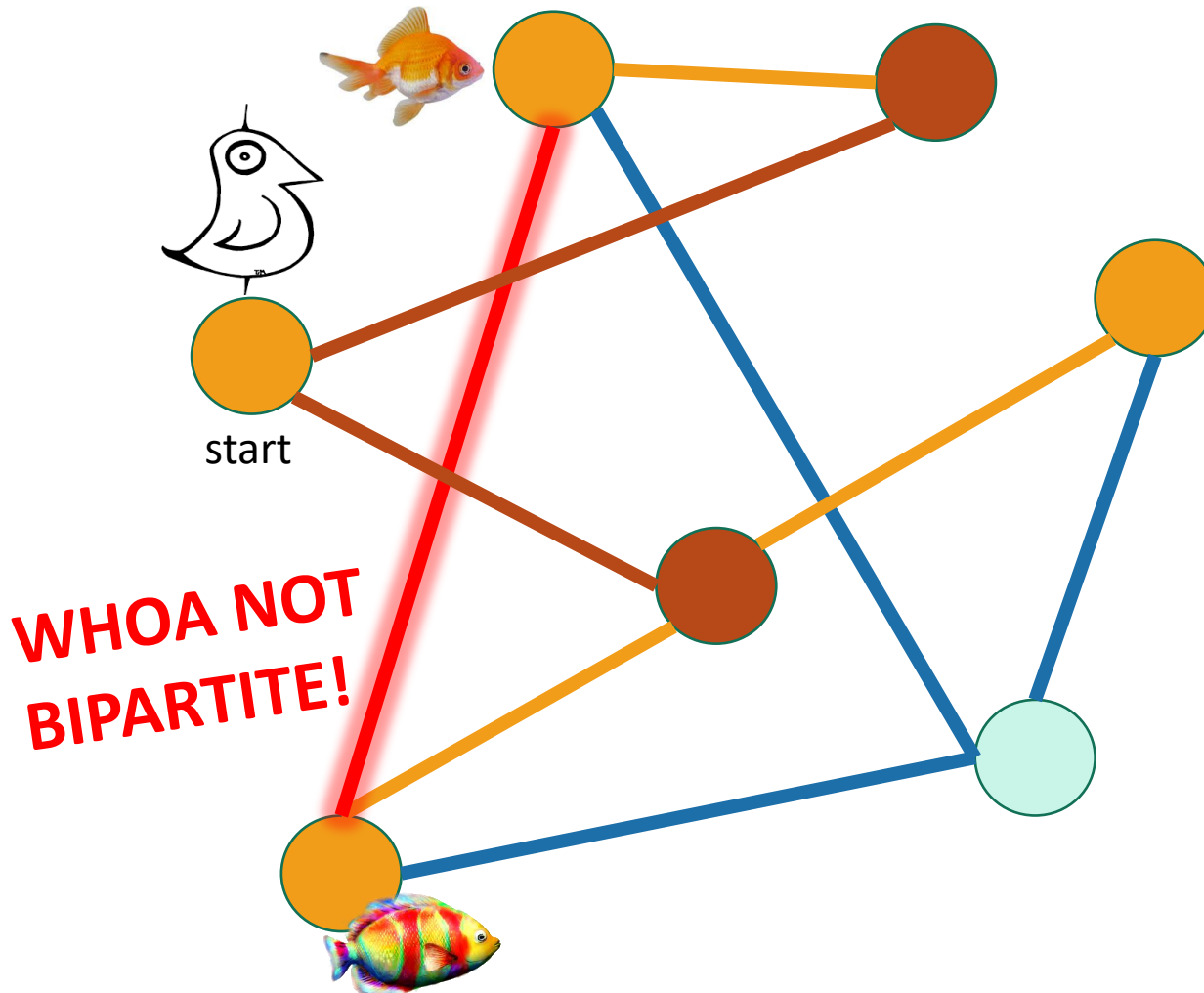
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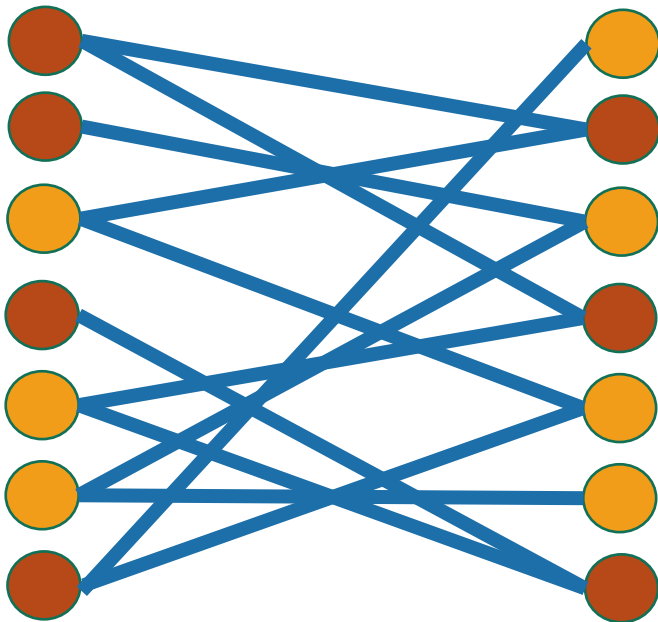
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Hang on now.

- Just because **this** coloring doesn't work, why does that mean that there is **no** coloring that works?



I can come up
with plenty of bad
colorings on this
legitimately
bipartite graph...

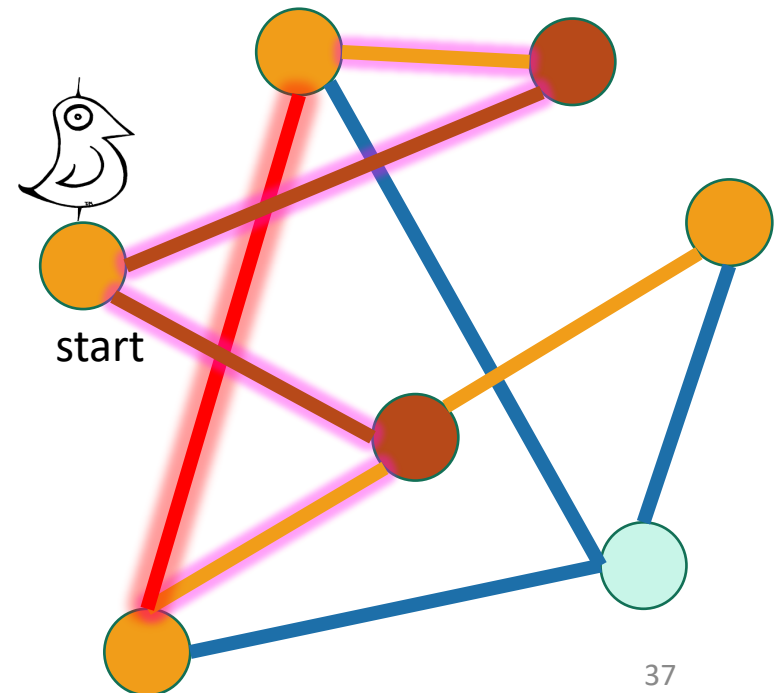
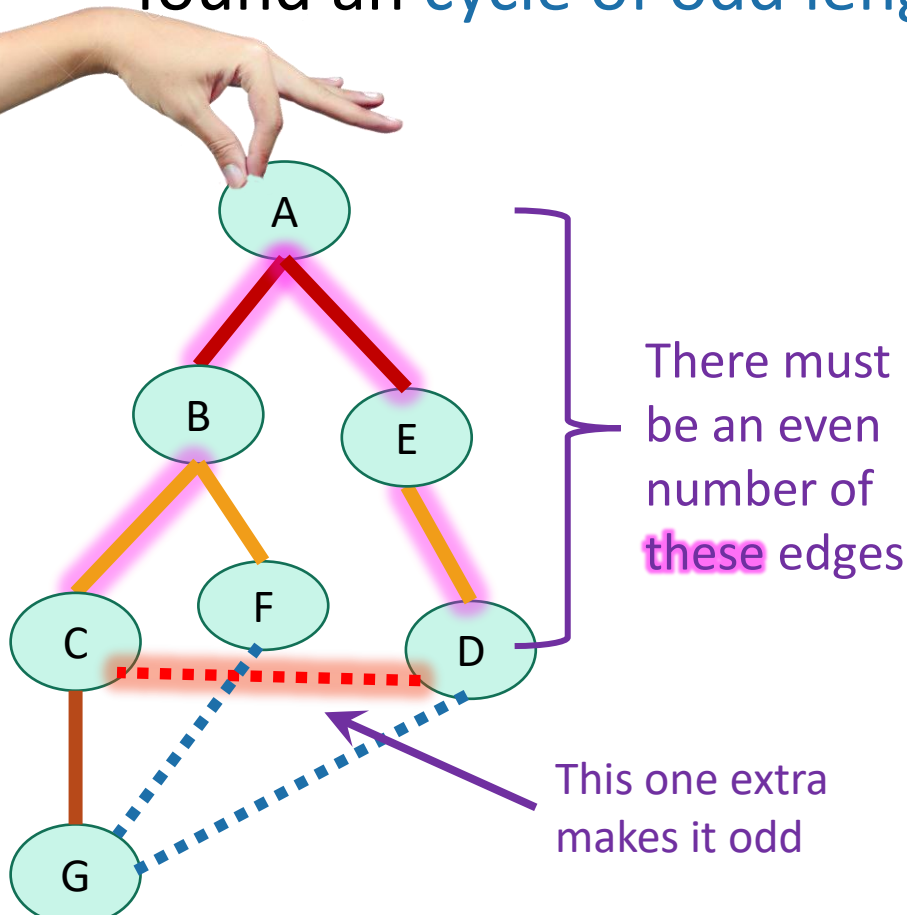


Make this proof
sketch formal!



Some proof required

- If BFS colors two neighbors the same color, then it's found an **cycle of odd length** in the graph.

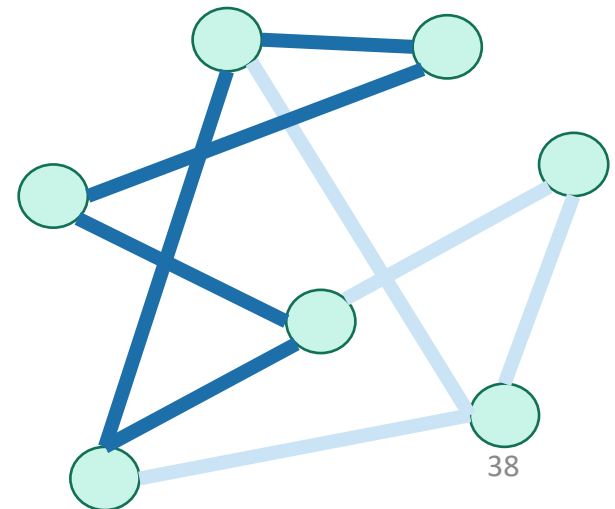


Make this proof
sketch formal!



Some proof required

- If BFS colors two neighbors the same color, then it's found an **cycle of odd length** in the graph.
- But you can **never** color an odd cycle with two colors so that no two neighbors have the same color.
 - [Fun exercise!]
- So you can't legitimately color the whole graph either.
- **Thus it's not bipartite.**



What have we learned?

BFS can be used to detect bipartite-ness in time $O(n + m)$.



Acknowledgement

- Stanford University