Advanced Data Structure and Algorithm

2-4 (2-3-4) Trees

(2,4) Trees

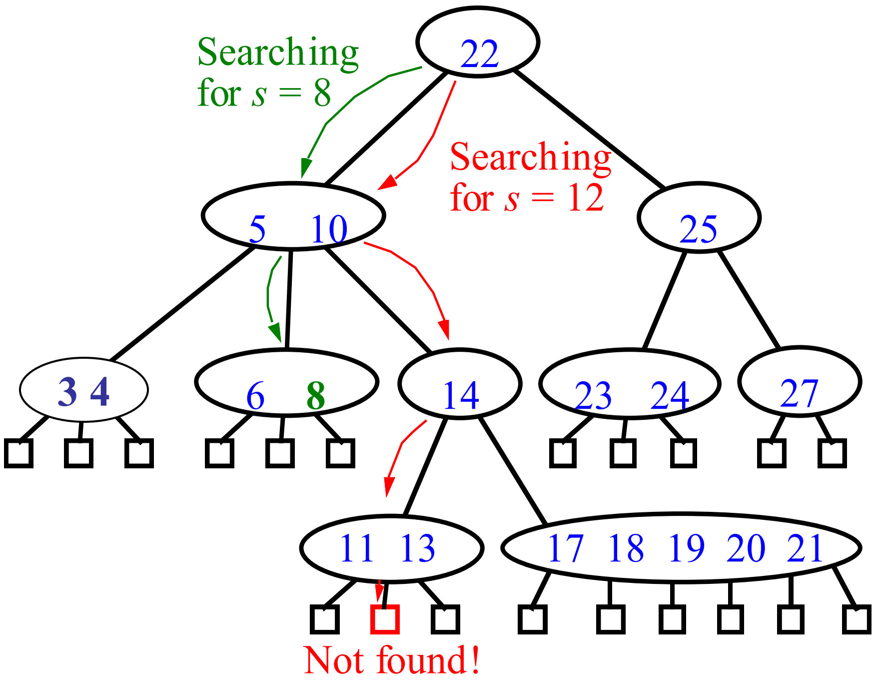
- What are they?
  - They are search Trees (but not binary search trees)
  - They are also known as 2-4, 2-3-4 trees

Multi-way Search Trees

- Each internal node of a multi-way search tree T:
  - has at least two children
  - stores a collection of items of the form (k, x), where k is a key and x is an element
  - contains d - 1 items, where d is the number of children
  - "contains" 2 pseudo-items: $\mathbf{k_0=-\infty}$ **,** $\mathbf{k_d= \infty}$
- Children of each internal node are "between" items
- all keys in the subtree rooted at the child fall between keys of those items
- External nodes are just placeholders

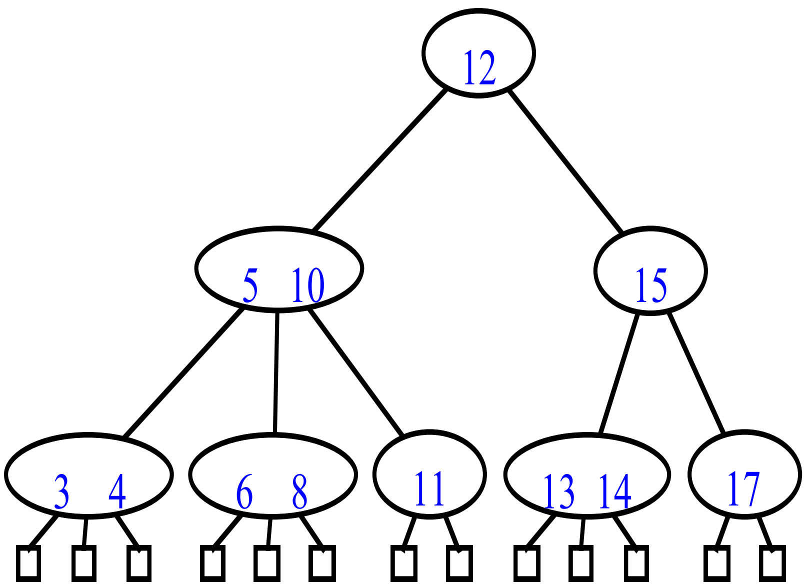
Multi-way Searching

- Similar to binary searching
  - If search key $\mathbf{s<k_1}$ search the leftmost child
  - If $\mathbf{s>k_{d-1}}$, search the rightmost child
- That's it in a binary tree; what about if **d>2**?
  - Find two keys $\mathbf{k_{i-1}}$ and $\mathbf{k_i}$ between which s falls, and search the child $\mathbf{v_i}$.
- What would an in-order traversal look like?



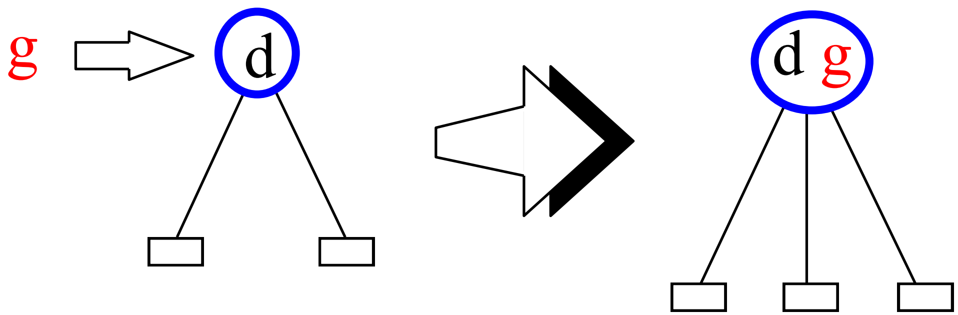
(2,4) Trees

- Properties:
  - At most 4 children
  - All external nodes have same depth
  - Height h of (2,4) tree is $O(\log n)$.
- How is the last fact useful in searching?

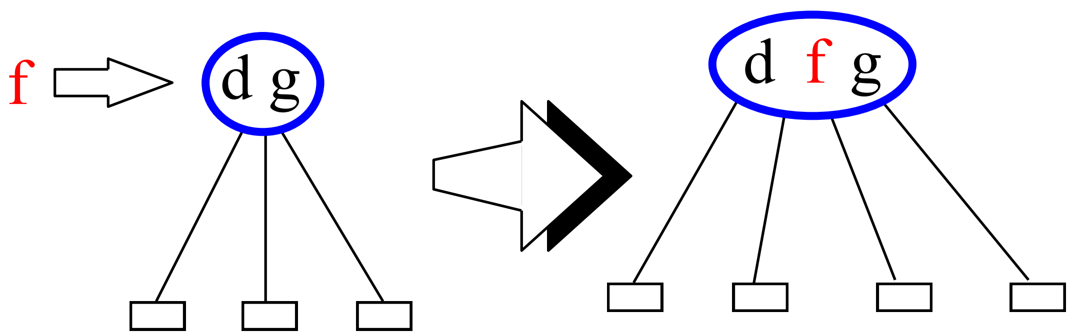


Insertion into (2,4) Trees

- Insert the new key at the lowest internal node reached in the search
- 2-node becomes 3-node



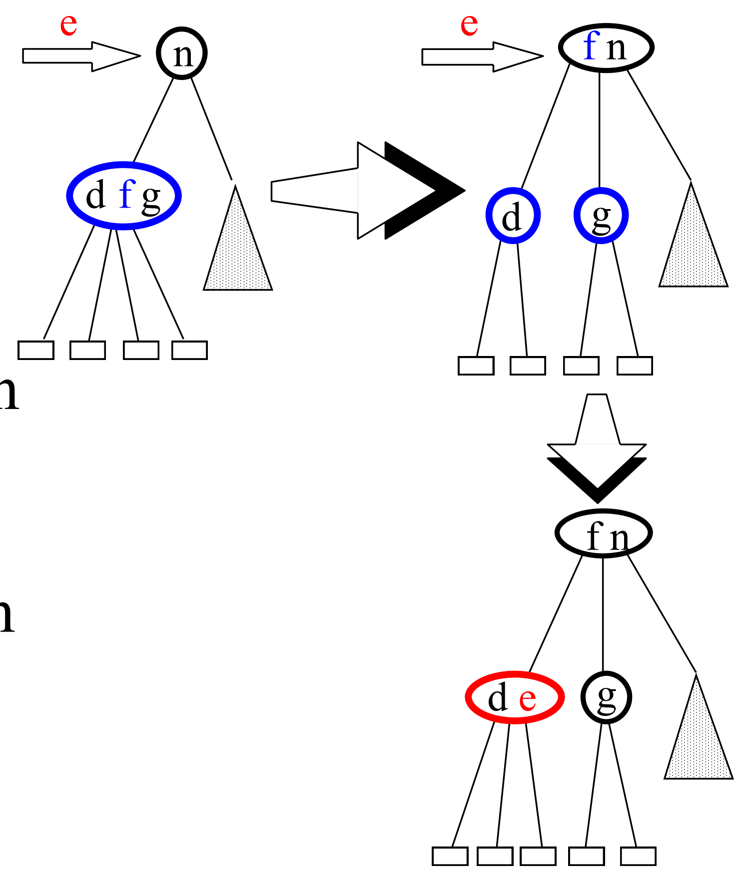
- 3-node becomes 4-node



- What about a 4-node?
- We can't insert another key!

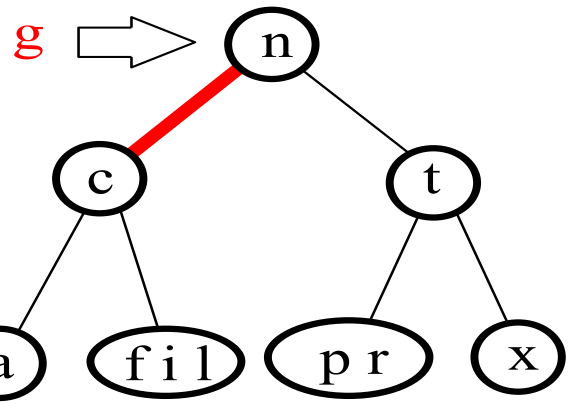
Top Down Insertion

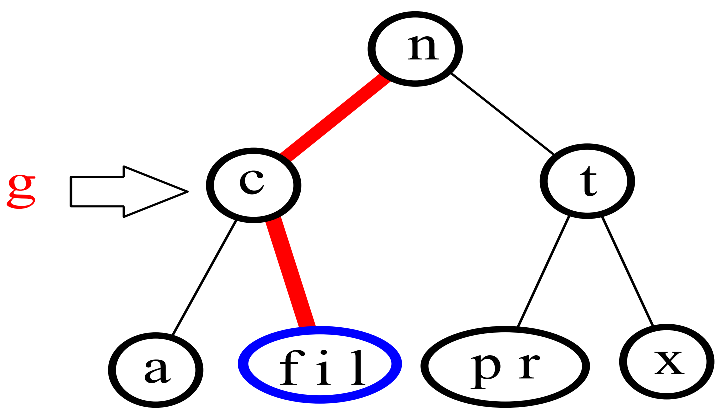
- In our way down the tree, whenever we reach a 4-node, we break it up into two 2-nodes, and move the middle element up into the parent node

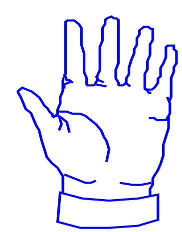


- Now we can perform the insertion using one of the previous two cases
- Since we follow this method from the root down to the leaf, it is called ***top down insertion***

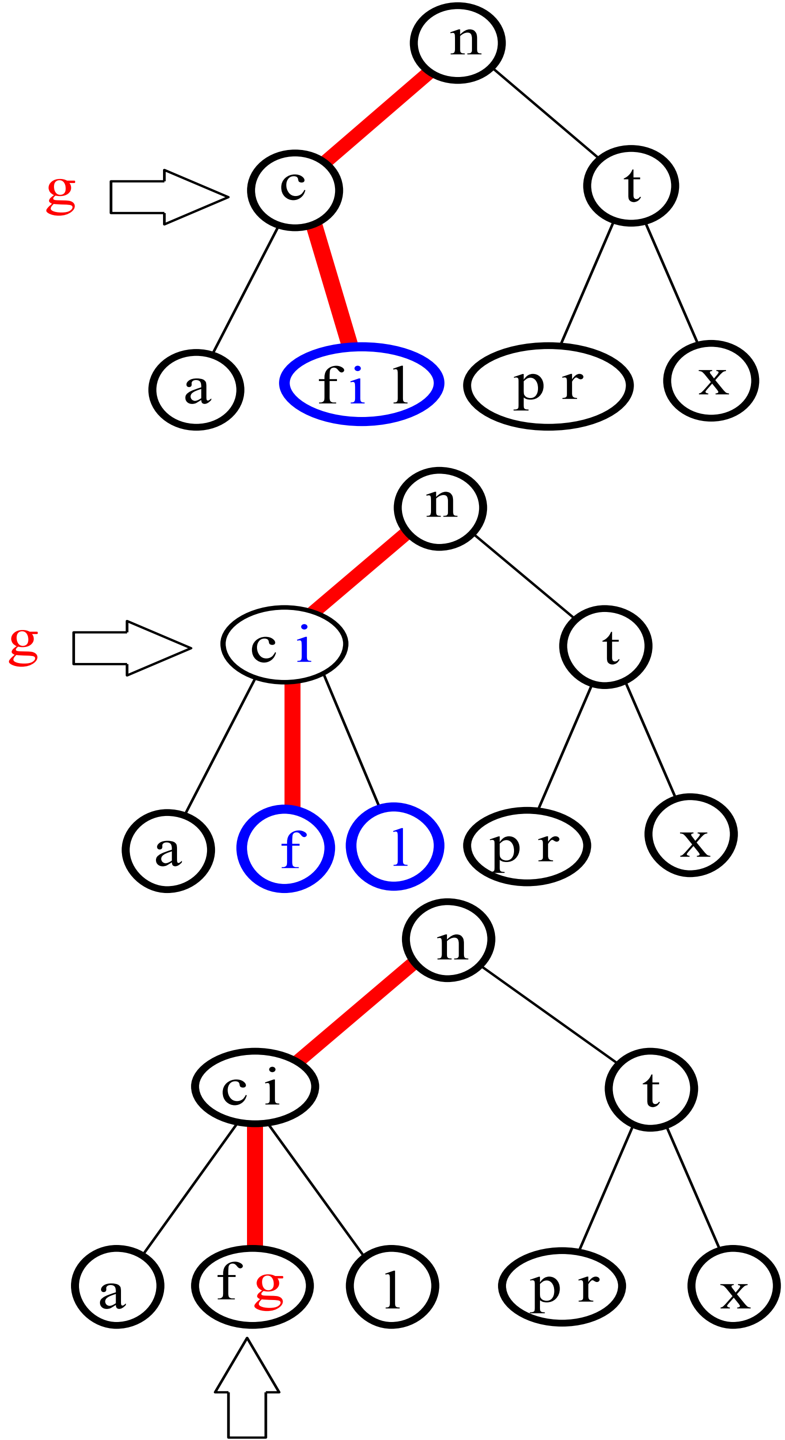
An Example

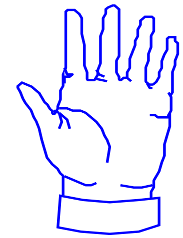






Whoa, cowboy





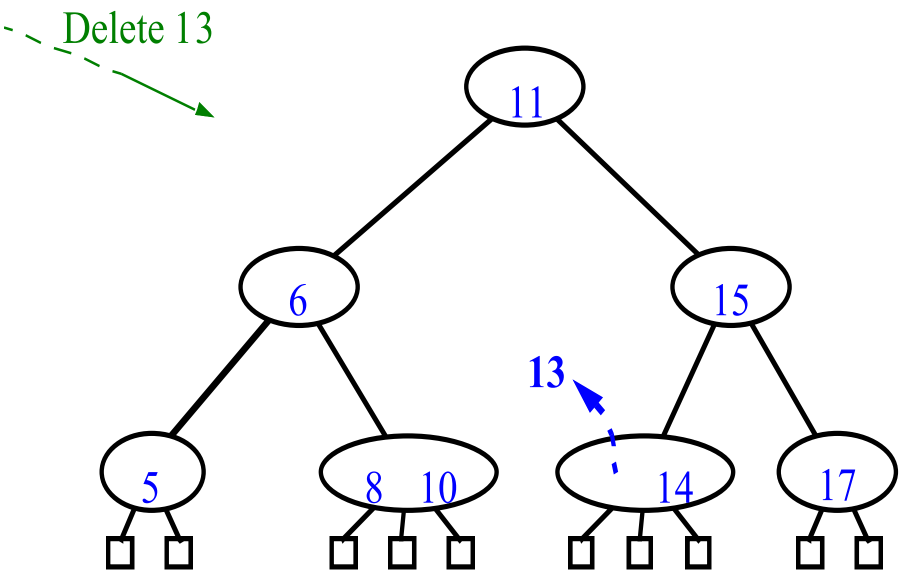
Whoa, cowboy

Time Complexity of Insertion in (2,4) Trees

- Time complexity:
- A search visits O(log N) nodes
- An insertion requires O(log N) node splits
- Each node split takes constant time
- Hence, operations ***Search*** and ***Insert*** each take time O(log N)
- **Notes**:
  - Instead of doing splits top-down, we can perform them bottom-up starting at the insertion node, and only when needed. This is called ***bottom-up*** insertion.
  - A deletion can be performed by fusing nodes (inverse of splitting)
- Let's take a look!

(2,4) Deletion

- A little trickier
- First of all, find the key with a simple multi-way search
- If the item to delete has non-external children, reduce to the case where deletable item is at the bottom of the tree:
  - Find item which precedes it in in-order traversal
  - Swap them
  - Remove the item
- Easy, right?
- ...but what about removing from 2-nodes?

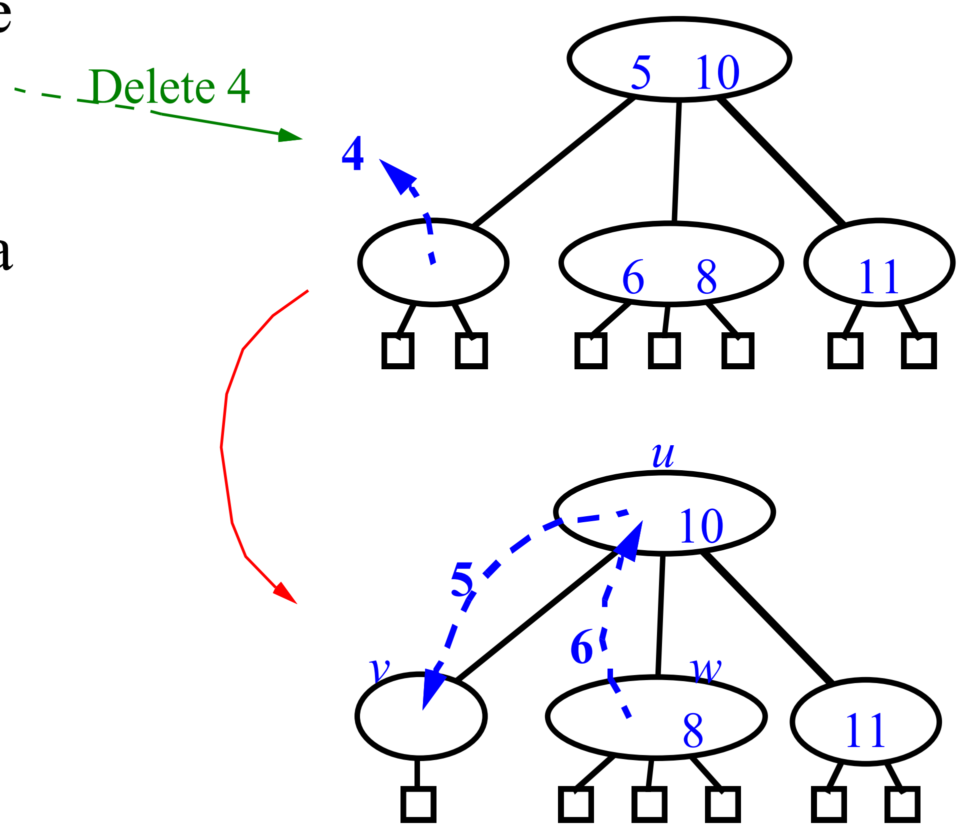


(2,4) Deletion (cont.)

- Not enough items in the node

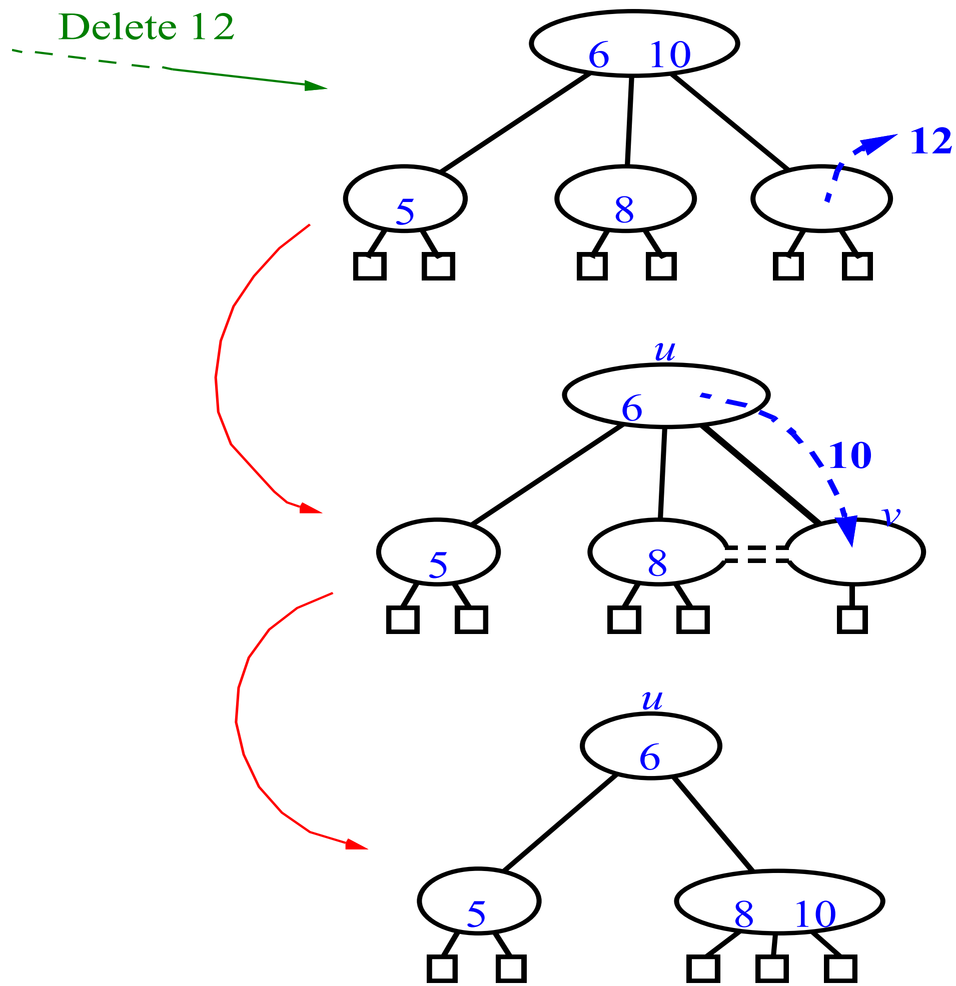
  - *underflow*
- Pull an item from the parent, replace it with an item from a sibling

  - *called transfer*
- Still not good enough! What happens if siblings are 2-nodes?
- Could we just pull one item from the parent?
- No. Too many children
- But maybe...



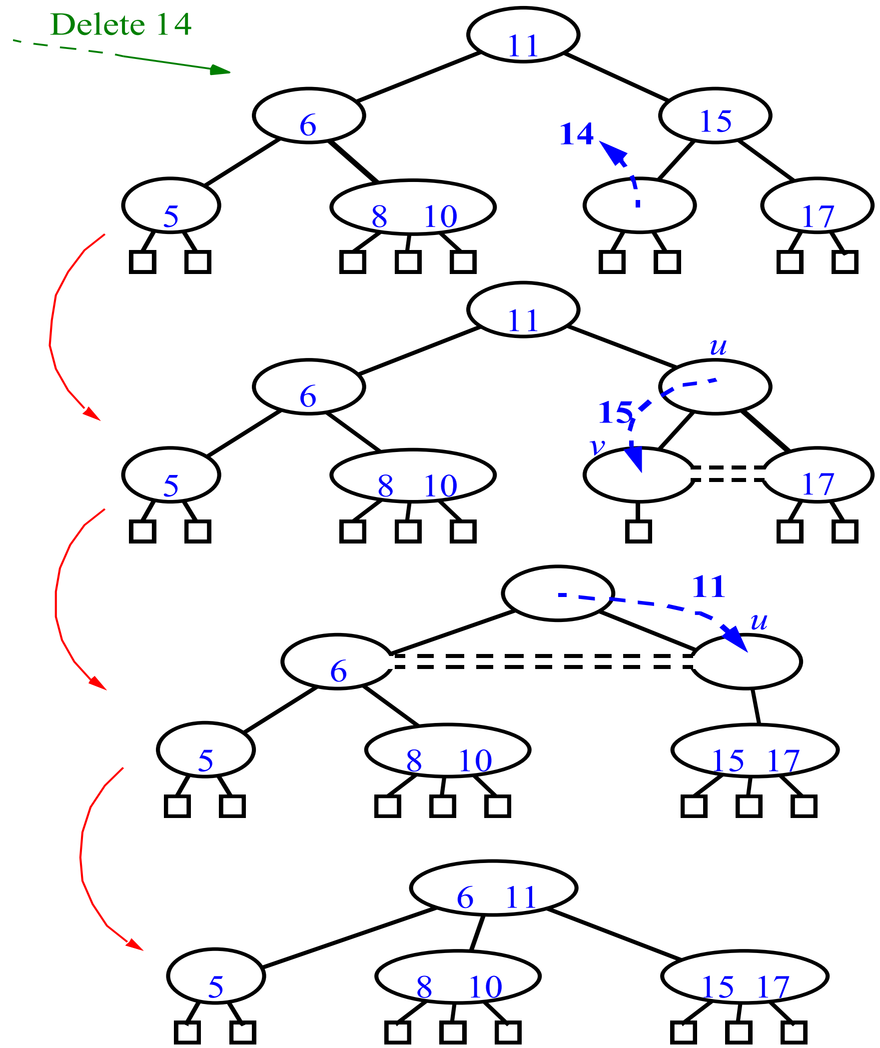
(2,4) Deletion (cont.)

- We know that the node's sibling is just a 2-node
- So we *fuse* them into one (after stealing an item from the parent, of course)
- Last special case: what if the parent was a 2-node?



(2,4) Deletion (cont.)

- Underflow can cascade up the tree, too.



(2,4) Conclusion

- The height of a (2,4) tree is $O(\log n)$.
- Split, transfer, and fusion each take $O(1)$.
- Search, insertion and deletion each take $O(\log n)$.
- Why are we doing this?
  - -(2,4) trees are fun! Why else would we do it?
  - -Well, there's another reason, too.
  - -They're pretty fundamental to the idea of Red-Black trees as well.