" M3 Assignment - 01 "

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" REAL ANALYSIS "

1) Determine whether the following sets are bounded (from below, above, or both). If yes, determine their infimum and supremum and check if they are minima or maxima

(i) $S_1 = \{1 + (-1)^n : n \in N\}$;

Sol) Given, $S_1 = \{1 + (-1)^n : n \in N\}$.

clearly, if n is : (a) Even $(-1)^n$ becomes 1 and S_1 becomes 2

(b) odd $(-1)^n$ becomes -1 and S_1 becomes 0.

The set becomes $S_1 = \{0, 2, 0, 2, \ldots\}$

As the values of S_1 are only $\{0, 2\}$, the set is bounded above and bounded below. i.e., set is bounded.

We know that,

If S is bounded above, then supremum is least upper bound. If S is bounded below, infimum is greatest lower bound.

we also know that,

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If $x_0 \in S$ and $x < x_0$ for all $x \in S$, then x_0 is maximum of S.

If $x_0 \in S$ and $x \geq x_0$ for all $x \in S$, then x_0 is minimum of S.

Clearly, in this problem

infimum $= 0$ Minimum $= 0$

Supremum $= 2$ Maximum $= 2$

In this problem,

Infimum $=$ minimum

Supremum $=$ Maximum

Also, the set is bounded.

(ii) $S_2 = \{ \frac{1}{m} + \frac{1}{n} : m, n \in N \}$;

Sol :- If $m=1, n=1$

$S_2 = 1+1 = 2$

If $m=2, n=2$

$S_2 = \frac{1}{2} + \frac{1}{2} = 1$

If $m=2, n=1$

$S_2 = \frac{1}{2} + 1 = \frac{3}{2}$

If $m = \infty, n = \infty$

$S_2 = \frac{1}{\infty} + \frac{1}{\infty} \to 0$

If $m=3, n=1$

$S_2 = \frac{1}{3} + 1 = \frac{4}{3}$

We can clearly see that Maximum of given set is 2. the set tends to 0 if m and n are very large.

the Set is having a range $(0, 2]$

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∴ Set is bounded both below and above i.e., set is bounded.

Infimum is 0; Supremum is 2.

Maximum is 2.

Minimum doesn't exist as 0 is not in the set.

Maximum = Supremum

(iii) $S_3 = \{x \in \mathbb{R} : x^2+x+1 \geq 0\}$.

clearly we know that

$x^2+x+1 \geq 0$ for any value of x

∴ Range of S_3 is $(-\infty, \infty)$

Hence, the set S_3 is "NOT BOUNDED"

As it is not bounded, there are no infimum and supremum and also no max and no minimum.

(iv) $S_4 = \{\cos(n\pi/3) : n \in n\}$.

Sol:- clearly, if n is a multiple of 3, S_4 becomes $\cos k\pi$; the value becomes 1 (or) -1, if n is not multiple of 3, the value is $\frac{1}{2}$ (or) $-\frac{1}{2}$

∴ $S_4 = \{-\frac{1}{2}, -1, 1, \frac{1}{2}\}$ for any n

∴ the set is both bounded above and bounded below (i.e.) set is bounded

Infimum is -1, Supremum is $+1$

Minimum is -1, Maximum is $+1$.

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Infimum = Minimum.
Supremum = Maximum

2) For the following power series find the interval of convergence.

$$\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)} (4x-12)^n$$

Sol: We know that,

By Dalembert's Ratio test

if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$.

series converges if $\lambda < 1$ &
diverges if $\lambda > 1$
test fails if $\lambda = 1$

By applying ratio test,

$$\lambda = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\dfrac{(4x-12)^{n+1}}{(-3)^{2+n+1}((n+1)^2+1)}}{\dfrac{(4x-12)^n}{(-3)^{n+2}(n^2+1)}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{(4x-12)^{n+1}}{(4x-12)^n} \cdot \frac{(-3)^{2+n}(n^2+1)}{(-3)^{3+n}((n+1)^2+1)}$$

$$\Rightarrow \lim_{n\to\infty} (4x-12) \times \frac{(n^2+1)}{-3(n^2+2n+2)}$$

$$= -\frac{1}{3} \lim_{n\to\infty} (4x-12) \frac{(n^2+1)}{(n^2+2n+2)}$$

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$$= -\frac{1}{3}(4x-12) \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n^2})}{\cancel{n^2}(1+\frac{2}{n}+\frac{2}{n^2})}$$

$$= -\frac{1}{3}(4x-12)\frac{(1+0)}{(1+0+0)} \quad [\text{as } n\to\infty \ \tfrac{1}{n}, \tfrac{1}{n^2} \to 0]$$

$$= \frac{12-4x}{3}.$$

Given, that series converges.

So, $\left|\frac{12-4x}{3}\right| < 1$

$$-1 < \frac{12-4x}{3} < 1$$

$$-3 < 12-4x < 3$$

$12-4x < 3$

$9 < 4x$

$\boxed{x > \frac{9}{4}}$

$12-4x > -3$

$4x-12 < 3$

$4x < 15$

$\boxed{x < \frac{15}{4}}$

If $x = 9/4$ $\quad S = \sum_{n=0}^{\infty} \frac{(-3)^n}{(-3)^{2+n}(n^2+1)} = \sum_{n=0}^{\infty} \frac{1}{9(n^2+1)}$

w.k.t $\frac{1}{9(n^2+1)} < \frac{1}{n^2+1} < \frac{1}{n^2} \quad \forall n \in R.$

$\therefore$ By p-test, $\sum \frac{1}{n^2}$ converges.

$\therefore$ By comparision test, $\sum \frac{1}{9(n^2+1)}$ converges at $x = 9/4$

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If $x = 15/4$, $S = \sum_{n=0}^{\infty} \frac{(-3)^n}{(-3)^{2+n}(n^2+1)} = \sum_{n=0}^{\infty} \frac{1}{9(-1)^{2+n}(n^2+1)}$

clearly $|S| = \sum_{n=0}^{\infty} \frac{1}{9(n^2+1)}$ converges as shown above.

So, by absolute convergence, S also converges

$\therefore$ the given series converge in $\left[\frac{9}{4}, \frac{15}{4}\right]$

3) For the following series of functions prove the uniform convergence using Weistrass-M test.

(i) $\sum_{n=0}^{\infty} \frac{x^n}{n}$ on $|x| \leq r < \infty$

Sol: We know that, By weistrass's M test a series $\sum_{n=0}^{\infty} u_n(x)$ will converge on x. if there exists a convergent series $\sum_{n=0}^{\infty} M_n$ such that $|u_n(x)| \leq M_n$ for all n and all $x \in X$.

Given, $|x| \leq r < \infty$

$x^n \leq r^n \quad (\because |x| \leq r)$

$\frac{x^n}{n^n} \leq \frac{r^n}{n^n}$

$\sum M_n = \sum \frac{r^n}{n^n}$

By Ratio test,

$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{r^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{r^n}$

$\Rightarrow \lim_{n\to\infty} \frac{r}{n+1} = 0 < 1$

Clearly, ΣM_n converges — ②.

$\therefore$ By Weistrass M-test, the given series $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$ uniformly converges on $|x| \le r < \infty$.

[By ① and ②]

(ii) $\sum_{0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ on $|x| \le r < 4$.

Sol:- We know that $^nC_r = \frac{n!}{r!(n-r)!}$

$$\sum_{0}^{\infty} \frac{x^n (n!)^2}{(2n)!} = \sum_{0}^{\infty} \frac{x^n}{\frac{(2n)!}{(n!)(n!)}} = \sum_{0}^{\infty} \frac{x^n}{^{2n}C_n}$$

Given $|x| \le r$

$$\frac{x^n}{^{2n}C_n} \le \frac{r^n}{^{2n}C_n}$$

Let $M_n = \frac{r^n}{^{2n}C_n}$

By Ratio test,

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{r^{n+1}}{^{(2n+2)}C_{n+1}} \times \frac{^{2n}C_n}{r^n}$$

$$= \lim_{n\to\infty} \frac{r \times {^{2n}C_n}}{^{2n+2}C_{n+1}} = r \lim_{n\to\infty} \frac{(2n)!}{n! \times n!} \times \frac{[(n+1)!]^2}{(2n+2)!}$$

$$= r \lim_{n\to\infty} \frac{(2n+1)^2}{(2n+2)(2n+1)} = \frac{r}{2} \lim_{n\to\infty} \frac{n+1}{2n+1}$$

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$$= \frac{r}{2} \lim_{n\to\infty} \frac{n}{n}\left[\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right] = \frac{r}{2} \lim_{n\to\infty} \frac{n+1}{2n+1}$$

$$= \frac{r}{2} \lim_{n\to\infty} \left[\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right] = \frac{r}{4} < 1$$

$\because$ $r<4$ It is given in the question

$\therefore$ The series $\sum \frac{r^n}{{}^{2n}C_n}$ is convergent by ratio test $\to$ ②

By ① and ②

$\sum_0^\infty \frac{(n!)^2}{(2n)!} x^n$ is uniformly convergent by Weistrass M-test.

(iii) $\sum_0^\infty \frac{1}{x^2+2^n}$ on R

Sol: We know that

$$\frac{1}{x^2+2^n} < \frac{1}{2^n} = T_n \to ①$$

By Ratio test,

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} < 1$$

$\therefore$ $\sum \frac{1}{2^n}$ is convergent by Ratio test. $\to$ ②

By ① and ②

the given series $\sum_0^\infty \frac{1}{x^2+2^n}$ is uniformly convergent by Weistrass M-test.

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4) Show that the sequence defined by

$a_1 = 2 \quad a_{n+1} = \frac{1}{3-a_n}$

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

Sol:- (i) To prove $0 < a_n \leq 2$

Base case:- If $n=1$, $a_1 = 2$ $0 < a_1 \leq 2$

It is proved.

Inductive step:- Let us assume that it is true $[0 < a_k \leq 2]$, we need to prove for $0 < a_{k+1} \leq 2$ is true.

We know that, $a_{k+1} = \frac{1}{3-a_k}$

$0 < a_k \leq 2$

$-2 \leq -a_k < 0$

$3-2 \leq 3-a_k < 3$

$\frac{1}{3} < \frac{1}{3-a_k} \leq 1$

$0 < \frac{1}{3} < a_{k+1} \leq 1 < 2$

$\therefore 0 < a_k \leq 2$

Hence, we proved that $0 < a_n \leq 2$ by induction $\forall n \in \mathbb{N}$

(ii) We need to prove a_n is decreasing (i.e.)

$a_{n+1} < a_n$

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<u>Base case</u>: If $n=1$

$a_n = 2$; $n=2$, $a_n = a_2 = \frac{1}{3-2} = 1$

$a_2 < a_1$

It is proved for $n=1$.

<u>Inductive Hypothesis</u>: Let us assume that

$a_{k+1} < a_k$ [$\because$ $n=k$ is true]

We need to prove that $n=k+1$ is true (i.e)

$a_{k+2} < a_{k+1}$ [To be proved]

$a_{k+1} < a_k$

$-a_{k+1} > -a_k$

$3 - a_{k+1} > 3 - a_k$

$\frac{1}{3-a_{k+1}} < \frac{1}{3-a_k}$

$\therefore$ $a_{k+2} < a_{k+1}$

Hence, it is true for $n=k+1$.

$\therefore$ $a_{n+1} < a_n$ is true by induction $\forall n \in N$.

Hence, sequence is monotonically decreasing.

Clearly, it is convergent as it is monotonic seq and bounded.

$\lim_{n\to\infty} a_{n+1} = k$. (assume)

i.e.) $\lim_{n\to\infty} a_n = k$

$k = \lim_{n\to\infty} \frac{1}{3-a_n} = \frac{1}{3-k}$

$k(3-k) = 1$

$k^2 - 3k + 1 = 0$

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$k^2 - 3k + 1 = 0$

$$k = \frac{3 \pm \sqrt{3^2 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

if, $k = \frac{3 + \sqrt{5}}{2} = 2.6 > 2$, it does n't satisfy

the condition $0 < a_n \leq 2$

So, $\because$ $k = \frac{3 - \sqrt{5}}{2}$.

$\therefore \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n = \frac{3 - \sqrt{5}}{2}$, it is finite.

$\therefore$ Sequence is convergent with limit $\frac{3 - \sqrt{5}}{2}$.

5) Discuss convergence of the following sequence using sandwich theorem

$$x_n = \frac{[\alpha] + [2\alpha] + \ldots [n\alpha]}{n^2}$$

where $[x]$ denotes the greatest integer less than or equal to the real number x, and α is an arbitrary real number.

Sol:- Given,

$$x_n = \frac{[\alpha] + [2\alpha] + \ldots [n\alpha]}{n^2}$$

We know that, $x - 1 < [x] \leq x \quad \forall x \in R$

Sol:- $\alpha - 1 < [\alpha] \leq \alpha$

$2\alpha - 1 < [2\alpha] \leq 2\alpha$

$\vdots$

$n\alpha - 1 < [n\alpha] \leq n\alpha$

$$\frac{\frac{n(n+1)}{2}\alpha - n}{n^2} < \frac{[\alpha]+[2\alpha]+\dots+[n\alpha]}{n^2} < \frac{\frac{n(n+1)}{2}\alpha}{n^2}$$

$$\frac{\cancel{n}(n+1)\alpha}{2n^{\cancel{2}}} - \frac{\cancel{n}}{n^{\cancel{2}}} < x_n < \left[\left[\frac{n+1}{2n}\right]\alpha\right]$$

$$\lim_{n\to\infty}\left[\left(\frac{n+1}{2n}\right)\alpha - \frac{1}{n}\right] = \lim_{n\to\infty}\left[\frac{\cancel{n}\left[1+\frac{1}{n}\right]}{2\cancel{n}}\alpha - \frac{1}{n}\right]$$

$$= \frac{\alpha}{2} = LHS$$

$$\lim_{n\to\infty}\left[\frac{n+1}{2n}\right]\alpha = \lim_{n\to\infty}\left[\frac{\cancel{n}\left[1+\frac{1}{n}\right]\alpha}{2\cancel{n}}\right] = \frac{\alpha}{2}$$

$= RHS$

$\therefore$ By Sandwich theorem,

$$\lim_{n\to\infty}\left[\left[\frac{n+1}{2n}\right]\alpha - \frac{1}{n}\right] = \lim_{n\to\infty}\left[\frac{n+1}{2n}\right]\alpha = \frac{\alpha}{2}.$$

$\therefore \lim_{n\to\infty} x_n = \frac{\alpha}{2}$ which is finite

Hence, x_n is convergent.

6) Prove the following theorem :- The neccessary condition for a monotonic sequence to be convergent is it is bounded.

Sol:- We know that,
Every bounded sequence have LUB and highest lower bound

Let a bounded sequence $\{s_n\}$ be montonic increasing. Let S denote its range, which is bounded. By completeness axiom, S has supremum (LUB) as mentioned above.

Let Sup (S_n) be supremum of M.

Let ϵ be any pre-assigned positive number.

Since, $M-\epsilon$ is number less than supremum M, there exists atleast an member S_m such that

$$S_m > M-\epsilon.$$

As $\{S_n\}$ is monotonic increasing.

$$S_n \geq S_m > M-\epsilon \quad \forall\, n \geq m \rightarrow (1)$$

Since, M is supremum

$$S_n \leq M < M+\epsilon \quad \forall\, n \rightarrow (2)$$

By (1) and (2), we can say that

$$M-\epsilon < S_n < M+\epsilon \; ; \; \forall\, n \geq m$$

$$|S_n - M| < \epsilon \; ; \; \forall\, n \geq m$$

∴ It is proved that S_n converges and $\lim_{n\to\infty} S_n = M$ if it is Monotonic Increasing and bounded.

Similarly, we can prove for Monotonic decreasing.

Hence, the neccessary and sufficient cond^n for monotonic seq to be convergent is it is bounded.

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7) Evaluate the following.

$$\lim_{x \to 0} \frac{x^3+2x^2+x}{x^2+2x}$$

Sol :- Clearly it is of the form $0/0$.

By applying L-Hospital's Rule,

$$\underset{x\to 0}{Lt} \frac{3x^3+2x^2+x}{x^2+2x} = \underset{x\to 0}{Lt} \frac{\frac{d}{dx}(x^3+2x^2+x)}{\frac{d}{dx}(x^2+2x)}$$

$$\underset{x\to 0}{Lt} \frac{3x^2+4x+1}{2x+2} = \frac{3(0)+4(0)+1}{2(0)+2} = \boxed{\frac{1}{2}}$$

(b) Find the relationship b/w a and b so that f is defined by

$$f(x) = \begin{cases} ax+1 & ; \text{ if } x \le 3 \\ bx+3 & ; \text{ if } x > 3 \end{cases}$$

is continuous at $x=3$

Sol :- For $F(x)$ be continuous at $x=3$

$$\underset{x\to 3^-}{Lt} f(x) = \underset{x\to 3}{Lt} f(x) = \underset{x\to 3^+}{Lt} f(x)$$

$$\underset{x\to 3^-}{Lt} (ax+1) = 3a+1$$

$$\underset{x\to 3^+}{Lt} (bx+3) = 3b+3$$

$$\Rightarrow 3a+1 = 3b+3$$

$$3a-3b = 2$$

$$\boxed{a-b=\frac{2}{3}}$$

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(c) Prove that greatest integer fn defined by $f(x) = [x]$; $0 < x < 3$ is not differentiable at $x = 1$ and $x = 2$:

Sol: Given, $f(x) = [x]$, $0 < x < 3$.

$$f(x) = \begin{cases} 0 ; & 0 < x < 1 \\ 1 ; & 1 \leq x < 2 \\ 2 ; & 2 \leq x < 3 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = 0 \qquad \lim_{x \to 1^+} f(x) = 1$$

$$\lim_{x \to 1^-} [f(x)] \neq \lim_{x \to 1^+} [f(x)].$$

$$\lim_{x \to 2^-} f(x) = 1, \quad \lim_{x \to 2^+} f(x) = 2$$

$$\lim_{x \to 2^-} [f(x)] \neq \lim_{x \to 2^+} [f(x)]$$

$\therefore$ $f(x)$ is not differentiable at $x = 1$ & $x = 2$ since, it is discontinuous at those points