Advanced Data Structures and Algorithms

Minimum Spanning Trees

THE GREEDY PARADIGM

Commit to choices one-at-a-time,

never look back,

and hope for the best.

Greedy doesn't always work.

And when it does, it's not always easy to see & prove why it works.

A STRATEGY FOR GREEDY PROOFS

Prove that after each choice, you're not ruling out success.
(i.e. you're not ruling out finding an optimal solution)

- **INDUCTIVE HYPOTHESIS:** After greedy choice t, you haven't ruled out success
- **BASE CASE:** Success is possible before you make any choices
- **INDUCTIVE STEP:** If you haven't ruled out success after choice t, then show that you won't rule out success after choice t+1 (there's an optimal solution that's consistent with the choices we've made so far)
- **CONCLUSION:** If you reach the end of the algorithm and haven't ruled out success then you must have succeeded.

WHAT WE'LL COVER TODAY

- Applications of the greedy algorithm design paradigm to **Minimum Spanning Trees**
  - Prim's algorithm
  - Kruskal's algorithm

MINIMUM SPANNING TREES

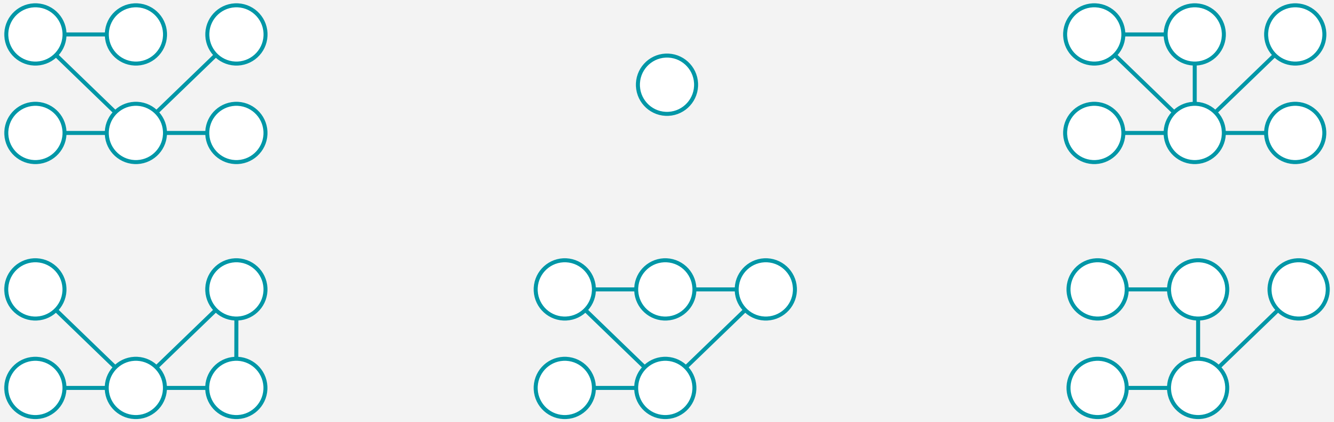
What are minimum spanning trees (MSTs)?

TREES IN GRAPHS

Let's go over some terminology that we'll be using today.

A tree is an undirected, *acyclic*, connected graph.

Which of these graphs are trees?



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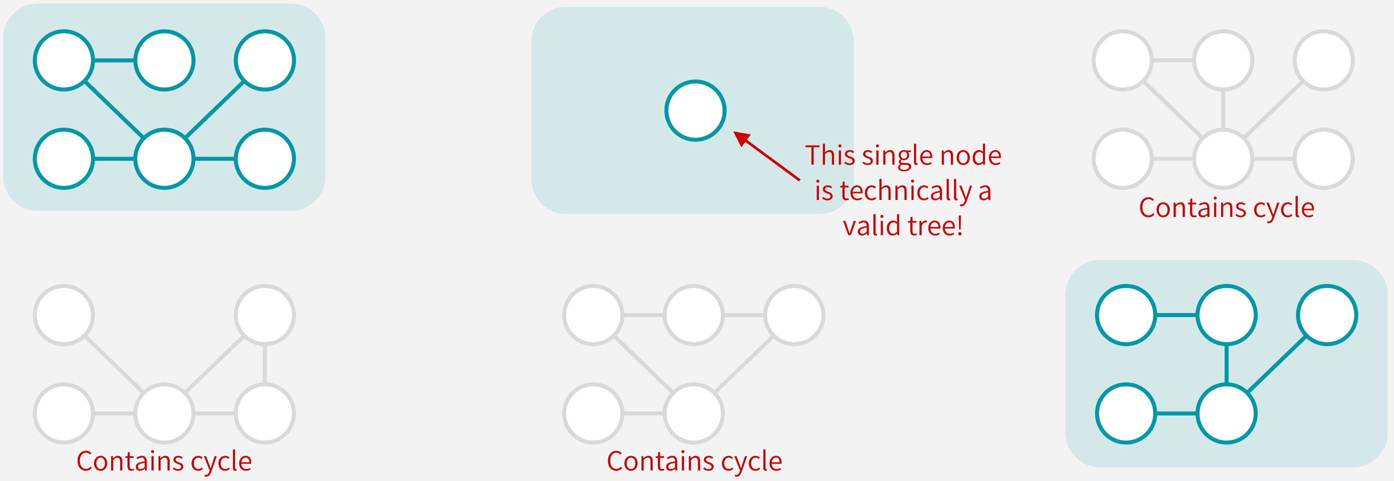
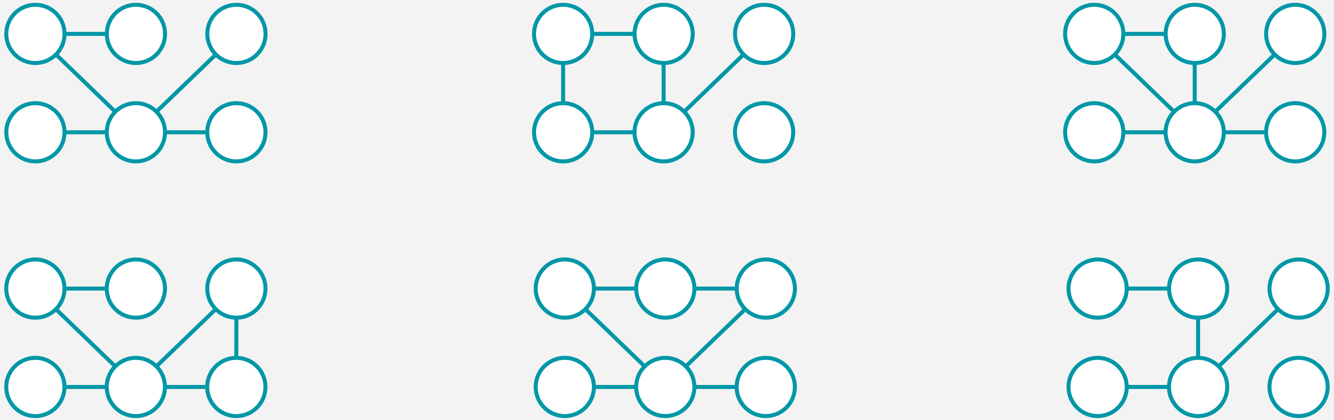


SPANNING TREES

A spanning tree is a tree that connects all of the vertices in the graph

Which of these graphs are spanning trees?



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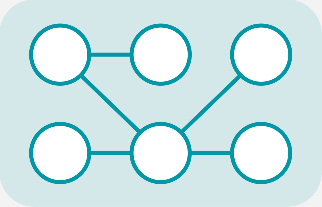

Doesn’t connect all vertices

Not a tree

Not a tree

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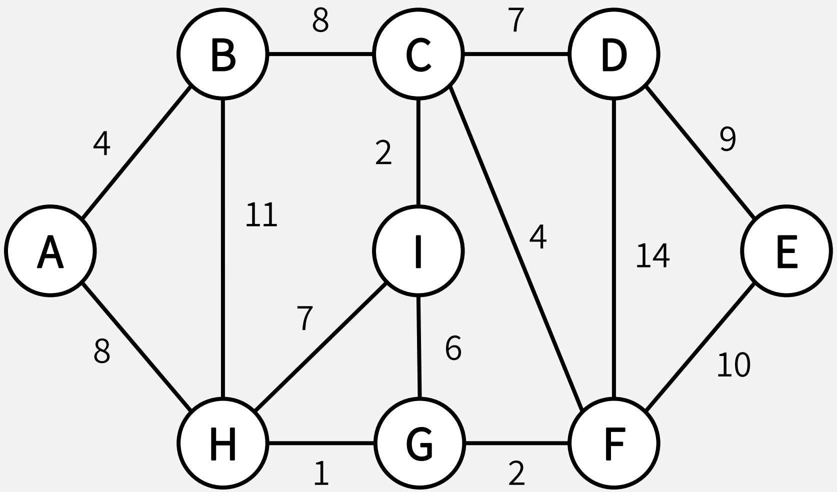
Doesn’t connect all vertices

MINIMUM SPANNING TREES (MSTs)

For the remainder of today, we're going to work with **undirected, weighted, connected graphs.**

The **cost of a spanning tree** is the **sum of the weights on the edges.**

An **MST** of a graph is a spanning tree of the graph with minimum cost.



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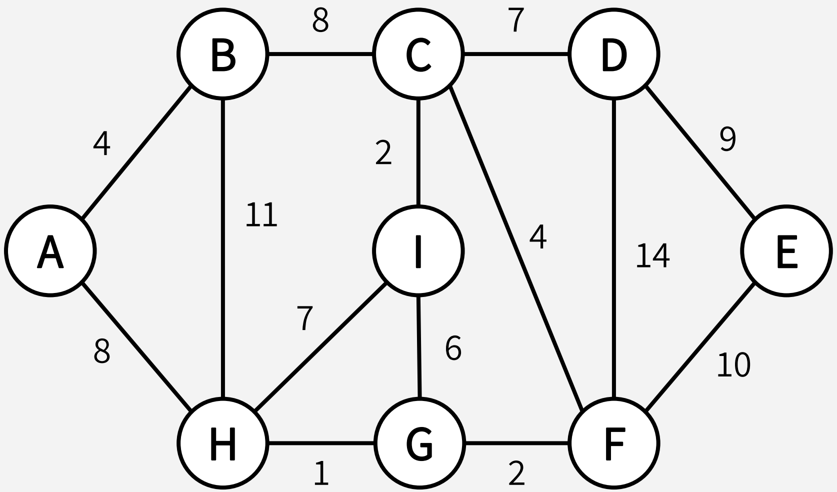

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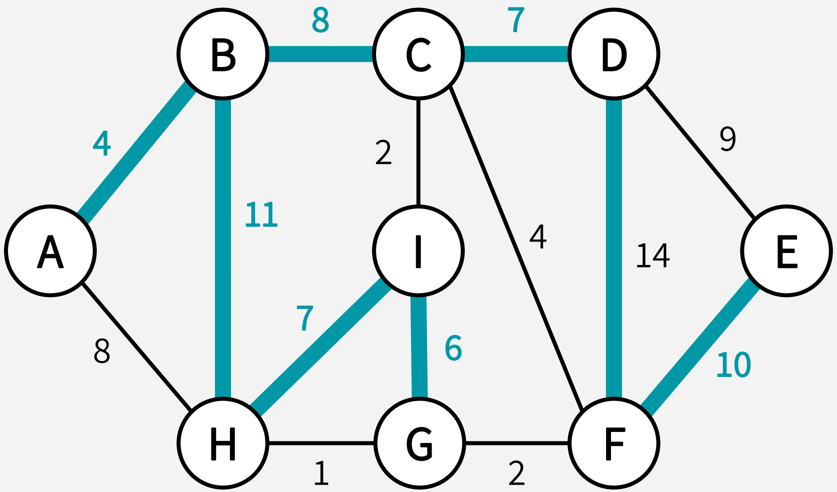

This spanning tree has a cost of **67**.

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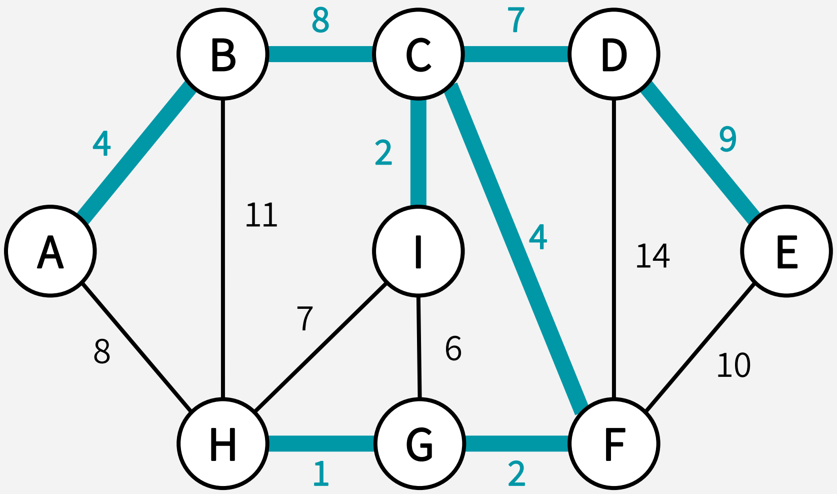
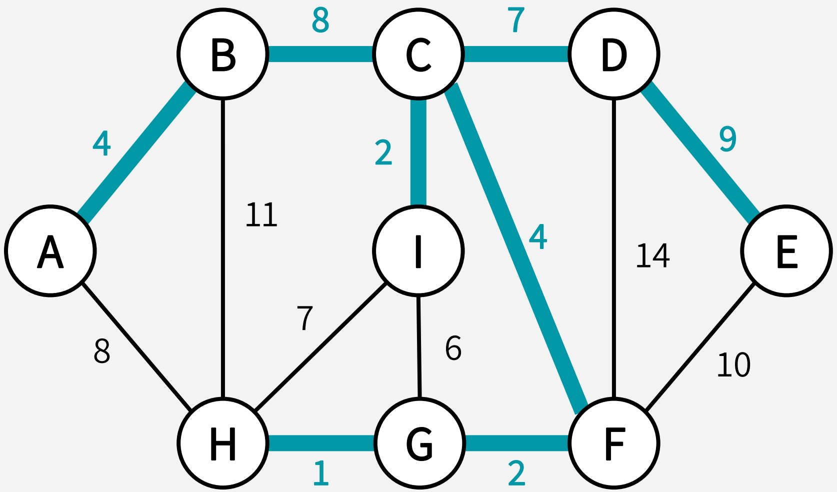


This spanning tree has a cost of **37.**

This is an MST of this graph, since there is no other spanning tree with smaller cost.

MINIMUM SPANNING TREES (MSTs)

The task for today:
Given an undirected, weighted, and connected graph G,
find the minimum spanning tree (as a subset of the G's edges)



We would return this MST.
Sometimes, there may be more than one MST as well, so return any MST of G.

APPLICATIONS OF MSTs

Network design

Find the most cost-effective way to connect cities with roads/water/electricity/phone

Image processing

Image segmentation, which finds connected regions in the image with minimal differences

Cluster analysis

Find clusters in a dataset (one of the algorithms we'll see can be modified slightly to basically do this)

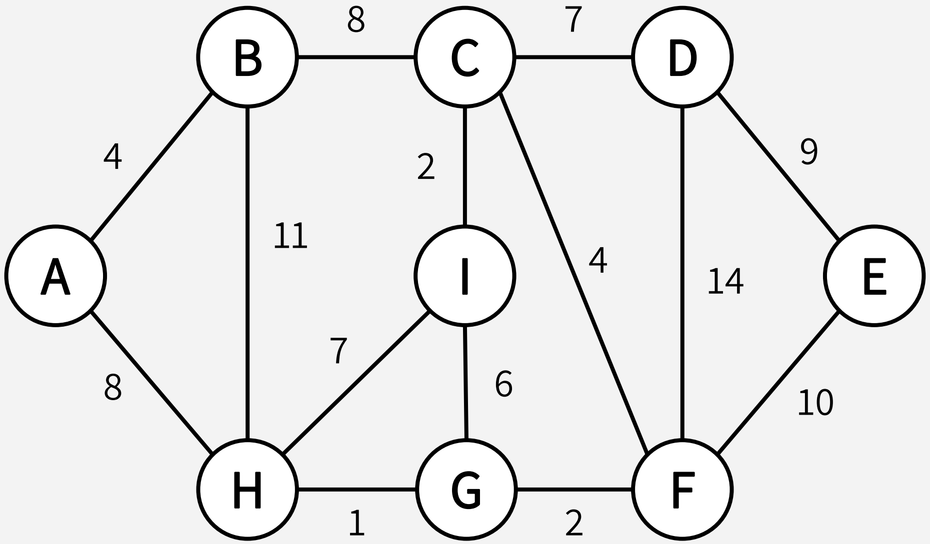
Useful primitive

Finding an MST is often useful as a subroutine or approximation for more advanced graph algorithms

MINIMUM SPANNING TREES (MSTs)

Before we move on with the lecture… why don't you give this a try?

Brainstorm some greedy algorithms to find an MST!

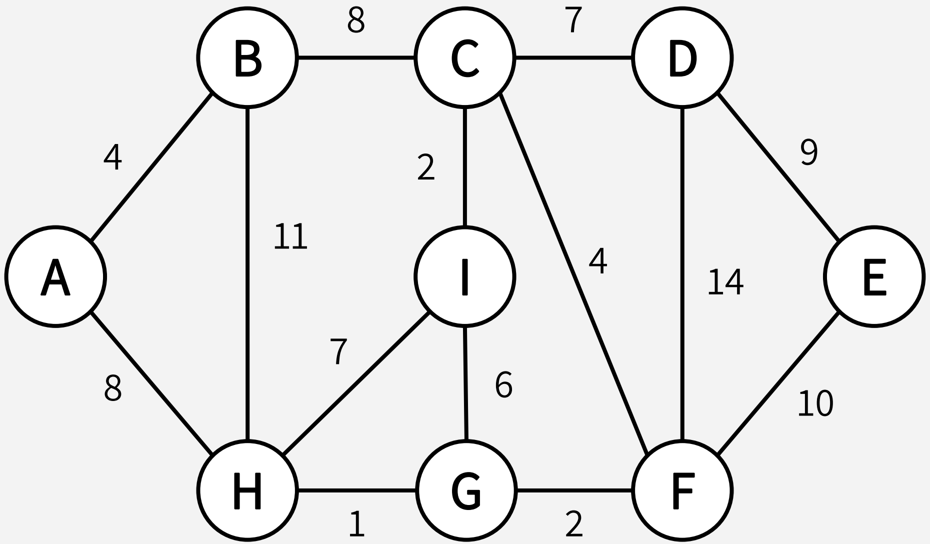


A BRIEF ASIDE: CUTS & "LIGHT" EDGES

What are cuts in graphs? And what can they tell us about MSTs?

CUTS IN GRAPHS

A **cut** is a partition of the vertices into two nonempty parts.



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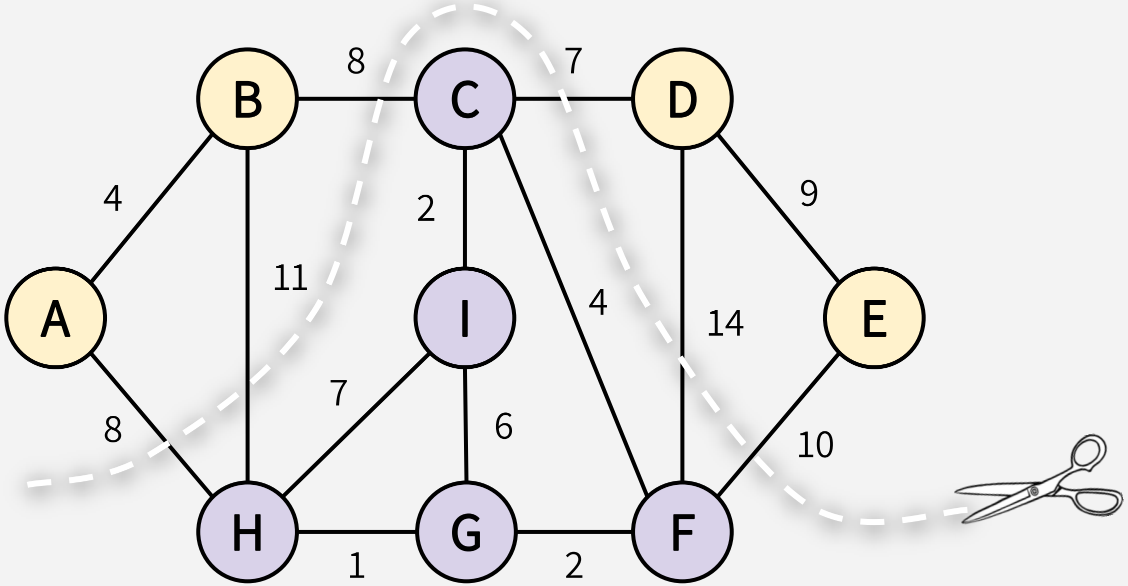

e.g. this is the cut "**{A,B,D,E}** and **{C,I,F,G,H}**"

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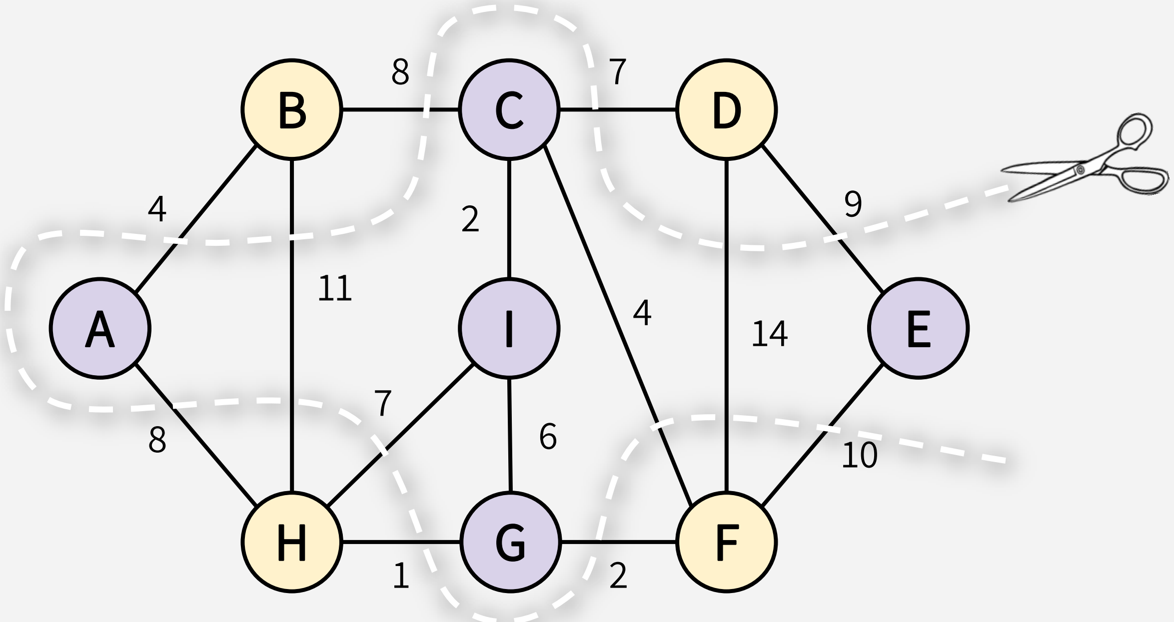

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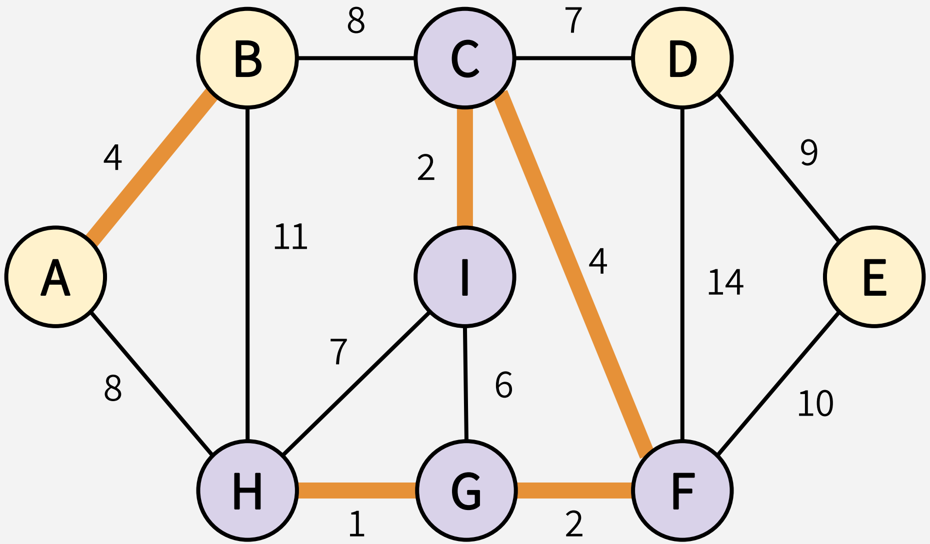
A **cut** is a partition of the vertices into two nonempty parts.

We say a **cut respects a set of edges S** if no edges in S cross the cut



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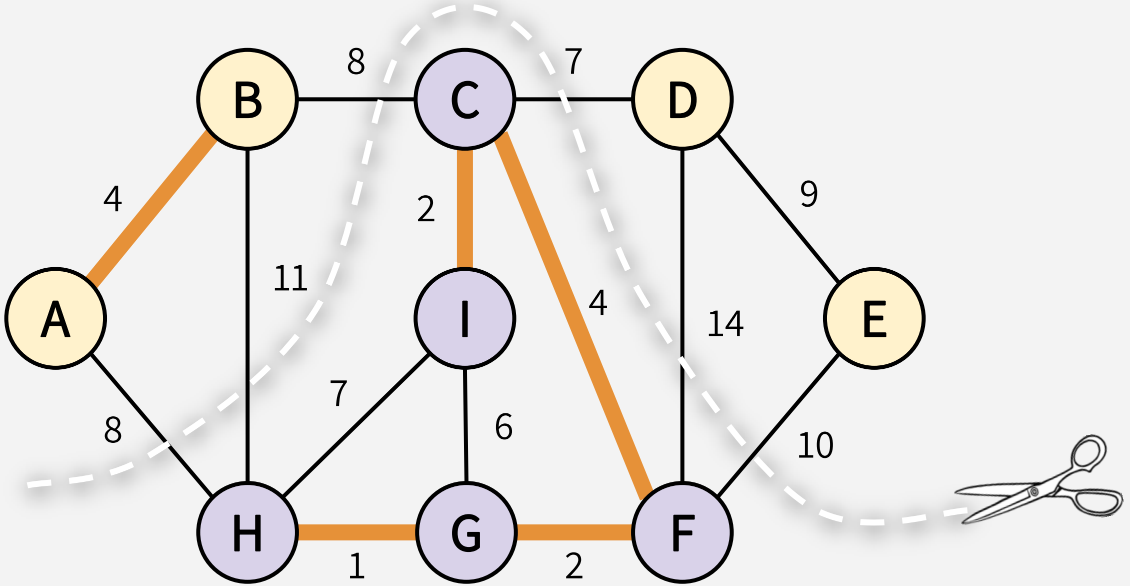
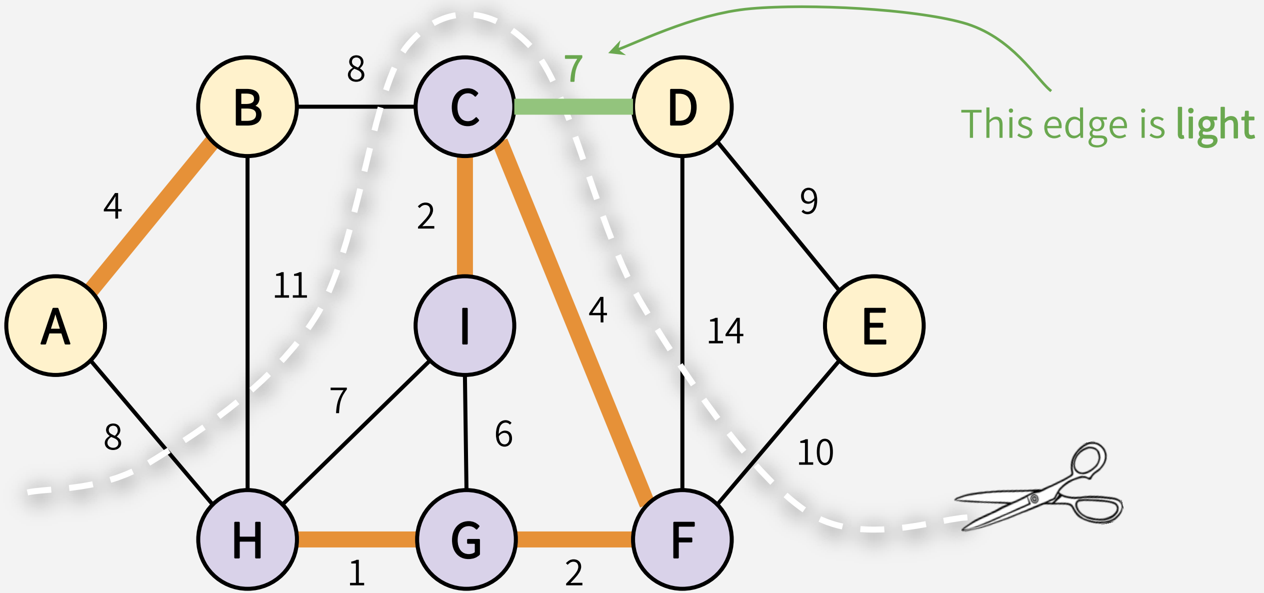
This cut **respects** this orange set of edges!



CUTS IN GRAPHS

A **cut** is a partition of the vertices into two nonempty parts.

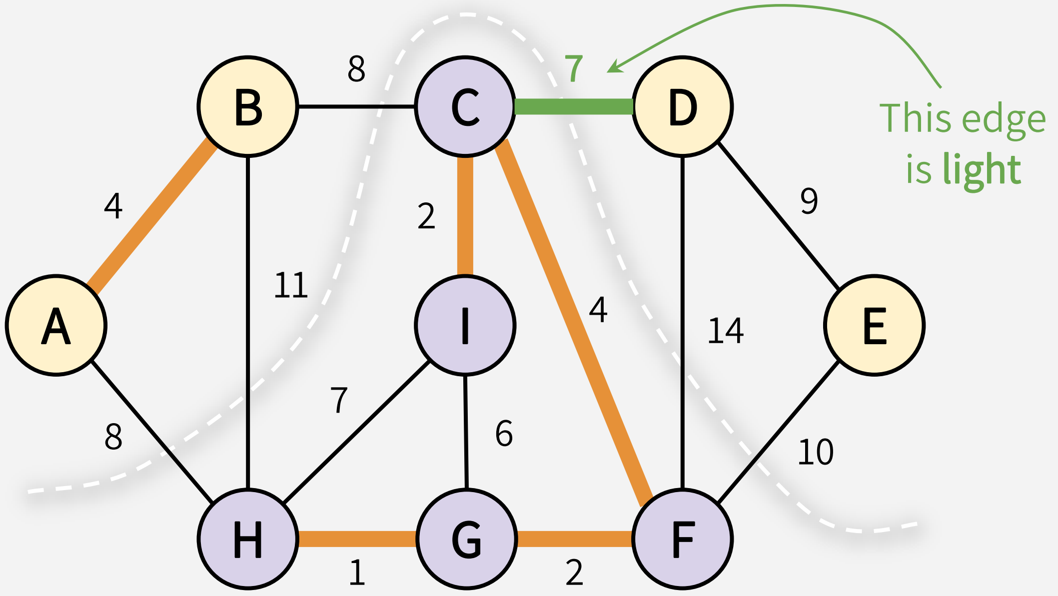
An edge is **light** if it has the smallest weight of any edge crossing the cut



This cut **respects** this orange set of edges!

AN IMPORTANT LEMMA

LEMMA: Consider a cut that respects a set of edges S.
Suppose there exists an MST **T*** containing S. Let (u,v) be a light edge crossing this cut.
Then, there exists an MST containing S **∪** {(u,v)}.



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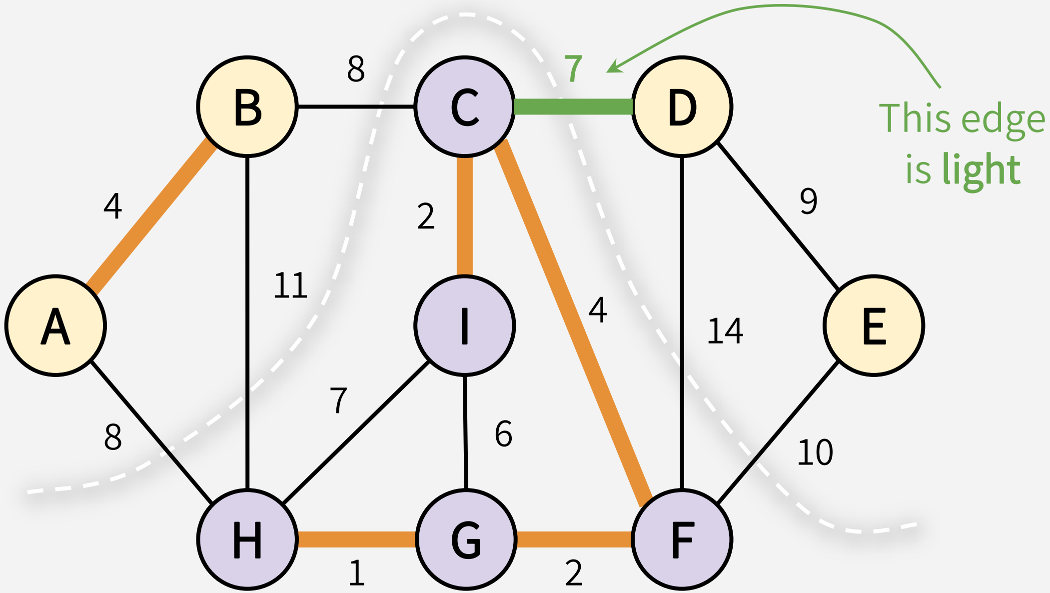
LEMMA: Consider a cut that respects a set of edges S.
Suppose there exists an MST T* containing S. Let (u,v) be a light edge crossing this cut.
Then, there exists an MST containing S ∪ {(u,v)}.

Before we prove this, why is this lemma important?

This is exactly the kind of statement we want for a greedy algorithm: *If we haven't ruled out the possibility of success so far, then adding a light edge still won't rule out success!*

We'll see how this can translate to an algorithm later… let's prove this first!



PROOF OF LEMMA

LEMMA: Consider a cut that respects a set of edges S.
Suppose there exists an MST T* containing S. Let (u,v) be a light edge crossing this cut.
Then, there exists an MST containing S **U** {(u,v)}.

Suppose (u,v) is not in T*.

If it is, then trivially, T* is an MST containing S **U** {(u,v)}.

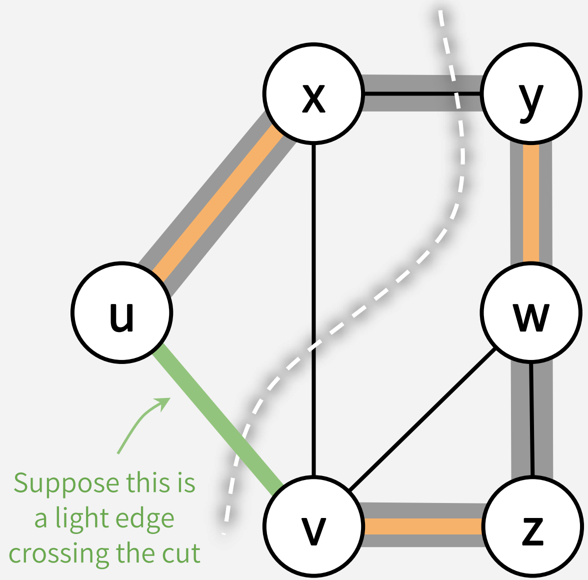
Adding (u,v) to T* will make a cycle.

Since T* is an MST, we know that there must be an edge in T* crossing the cut (in order to connect nodes on opposite sides of the cut).
Call this edge (x,y).

If we replace (x,y) with (u,v) in T*, we end up with an MST T.

Why is T still an MST? Well, since T* was a tree, and we also delete (x,y), then T must also be a tree (no cycles). Since (u,v) is light, then T has at most the cost of T*, so T is also optimal.

Thus, there exists an MST (T) containing S **U** {(u,v)}



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LEMMA: Consider a cut that respects a set of edges S.
Suppose there exists an MST T* containing S. Let (u,v) be a light edge crossing this cut.
Then, there exists an MST containing S ∪ {(u,v)}.

We'll see two famous MST algorithms which each have their own way of greedily claiming the next light edge.

We'll keep this lemma in mind when working out the proofs of correctness for each of the algorithms!

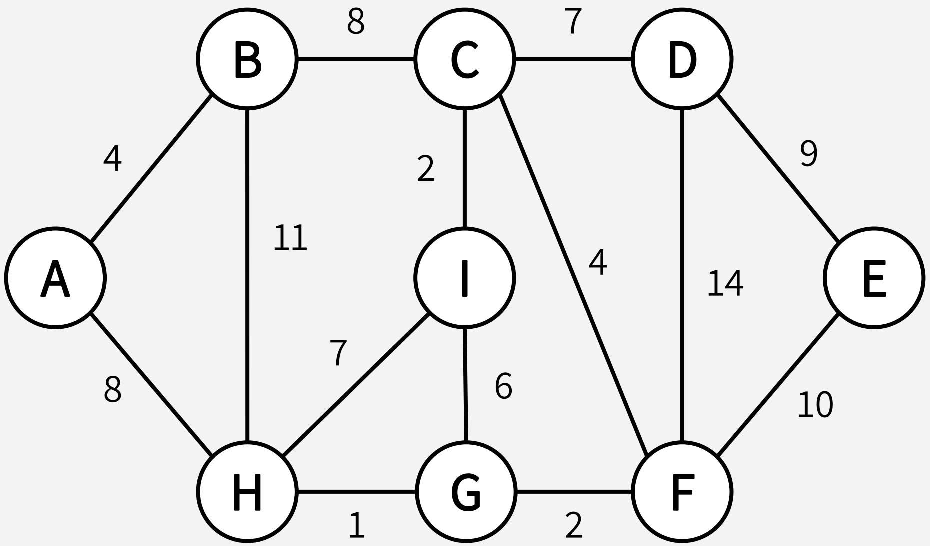
PRIM'S ALGORITHM

Greedily add the closest vertex!

PRIM’S ALGORITHM: THE IDEA

Greedy choice:

Grow a single tree, & greedily add the shortest edge that could grow our tree



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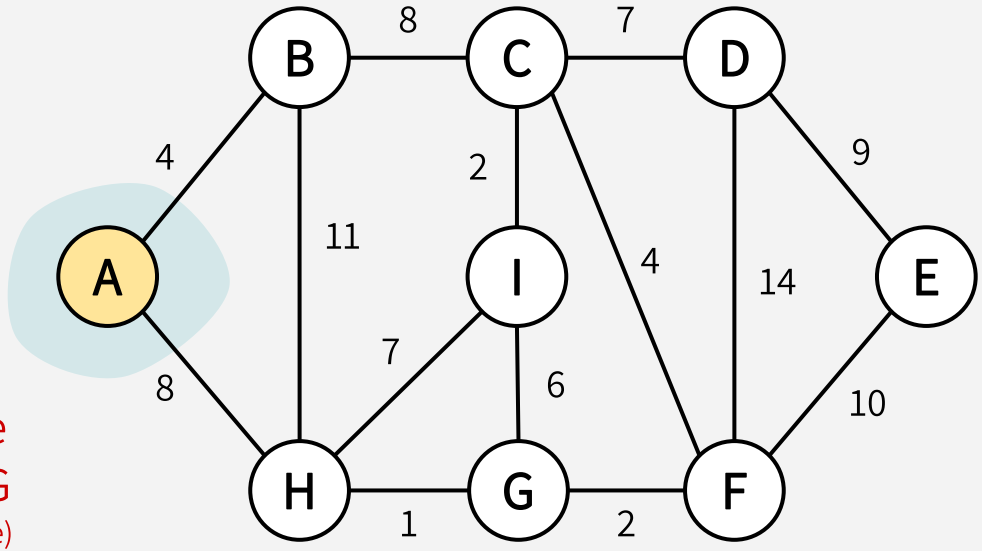

First, we can initialize our tree to contain a single arbitrary node in G
(doesn't matter which node)

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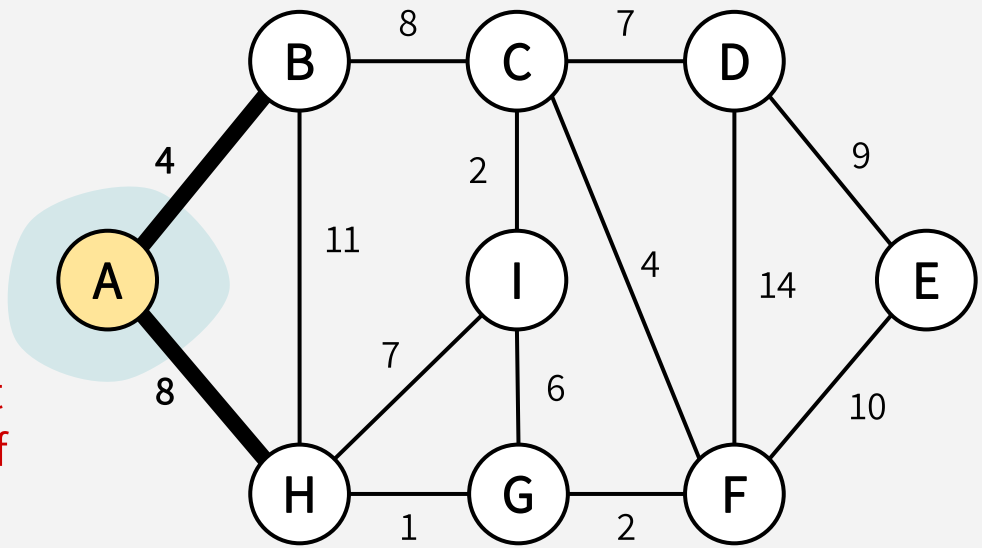

Consider the edges coming out of the "frontier" of our growing tree.

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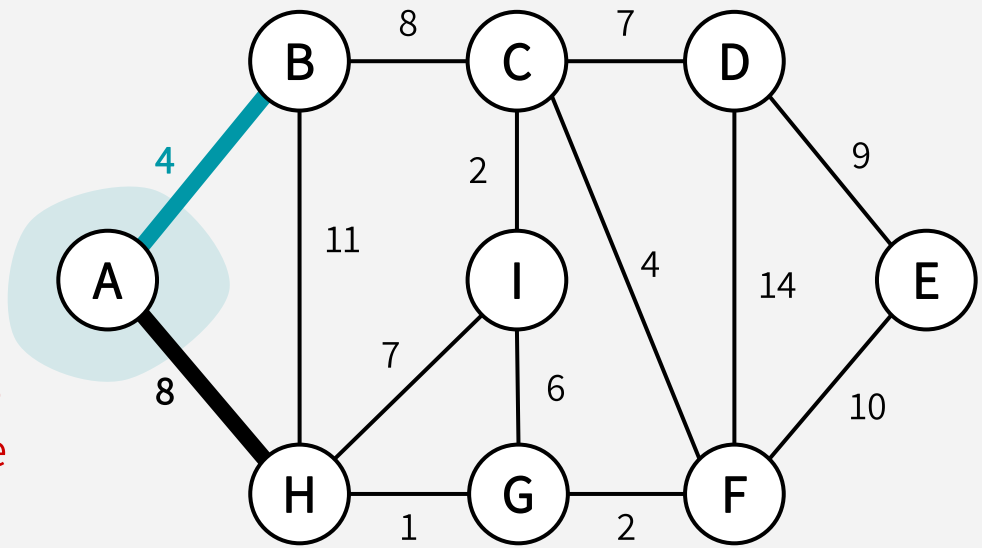

Claim the edge coming out of the "frontier" with the smallest weight

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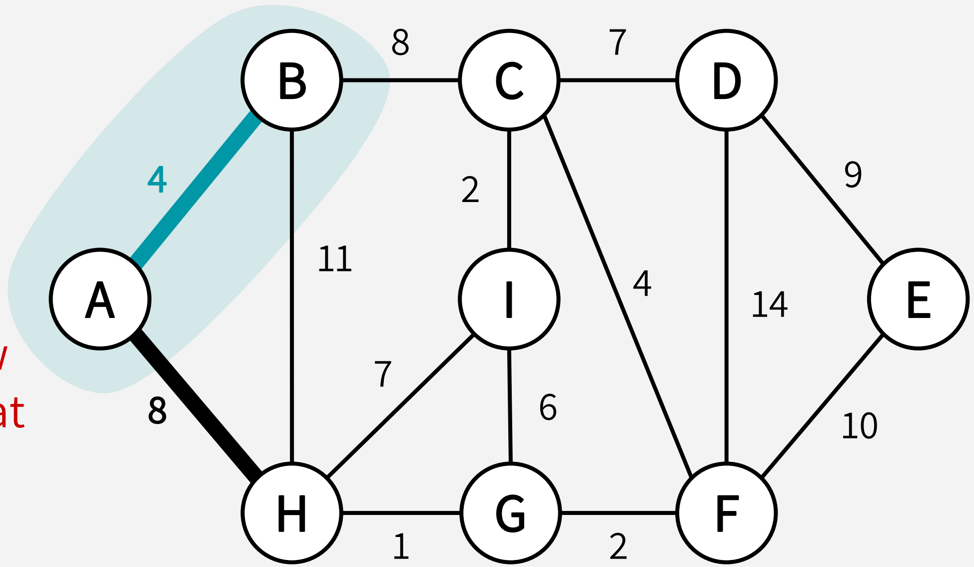

Our tree just grew by one! Now repeat until we reach all the nodes.

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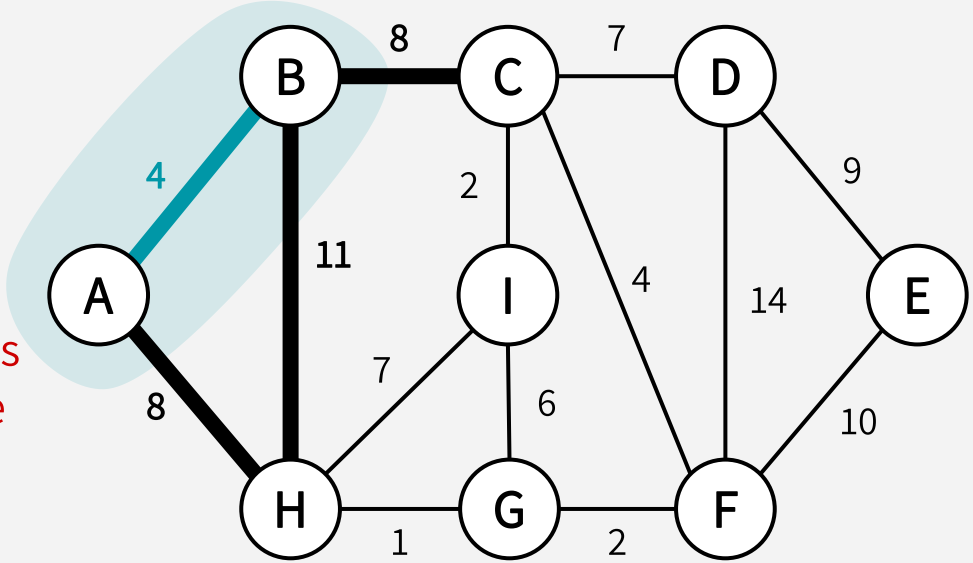

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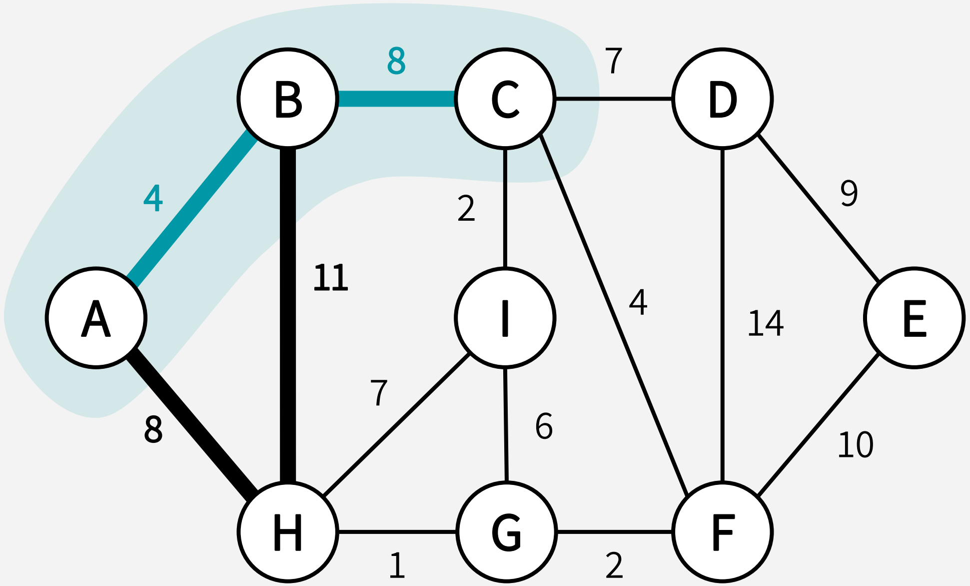
(if there's a tie, choose any)



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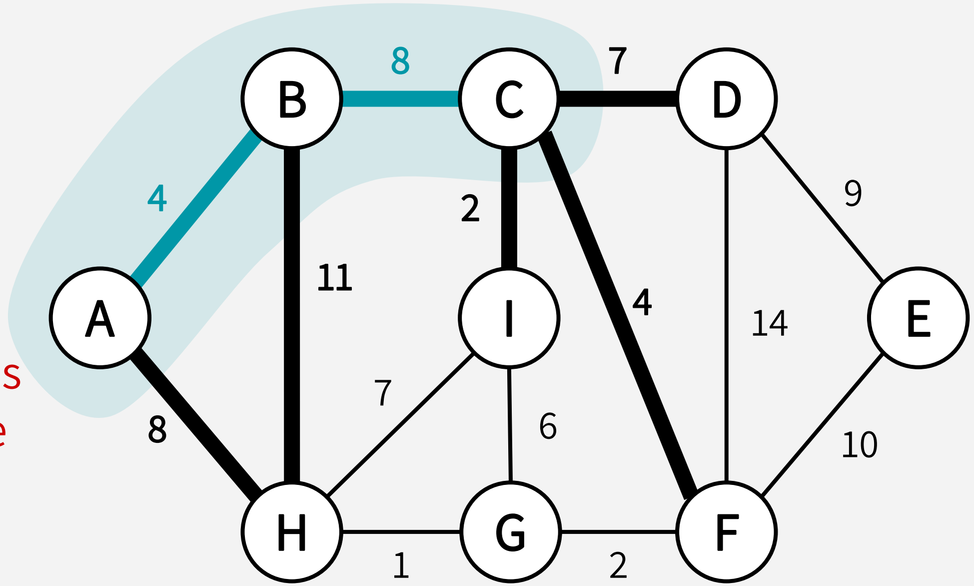

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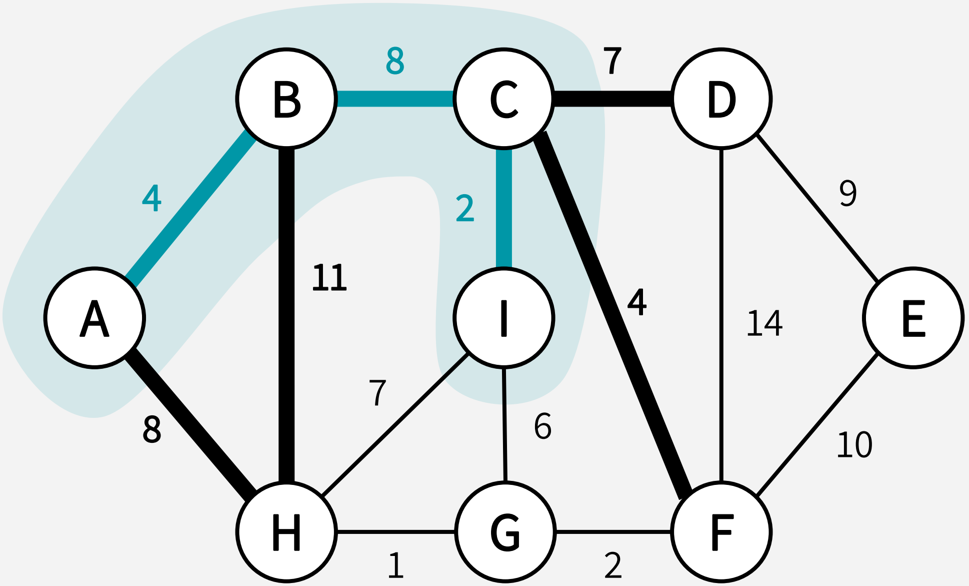

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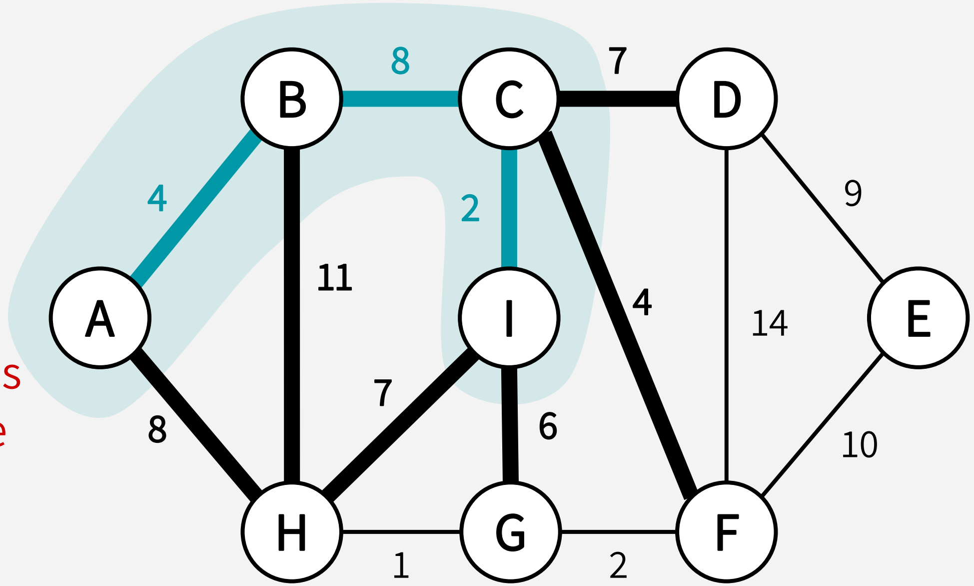

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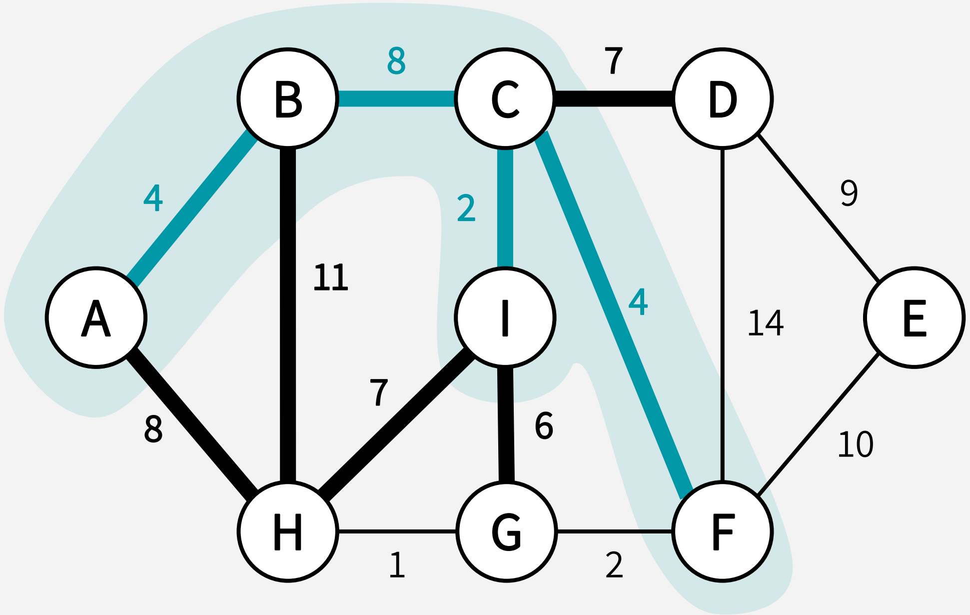

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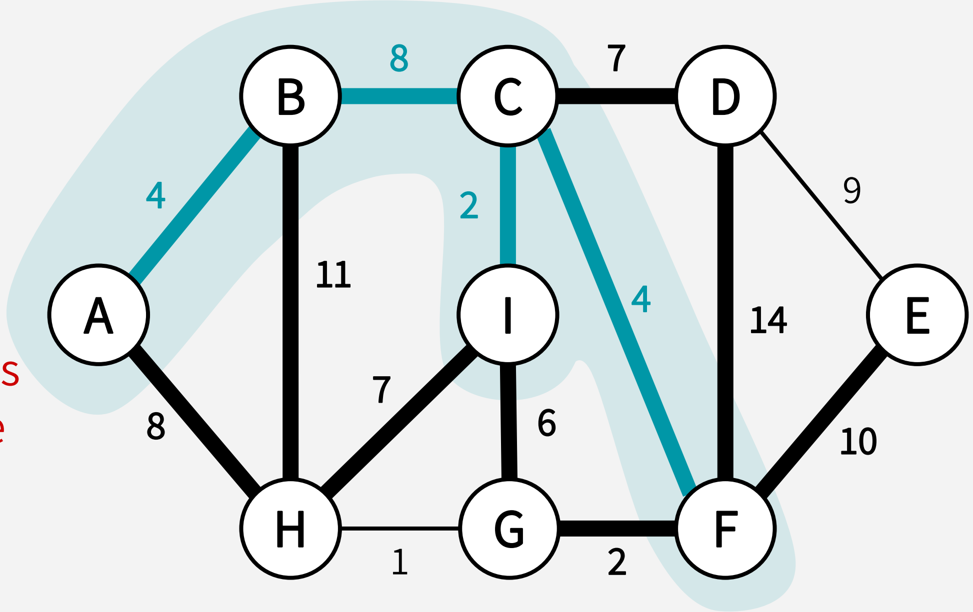

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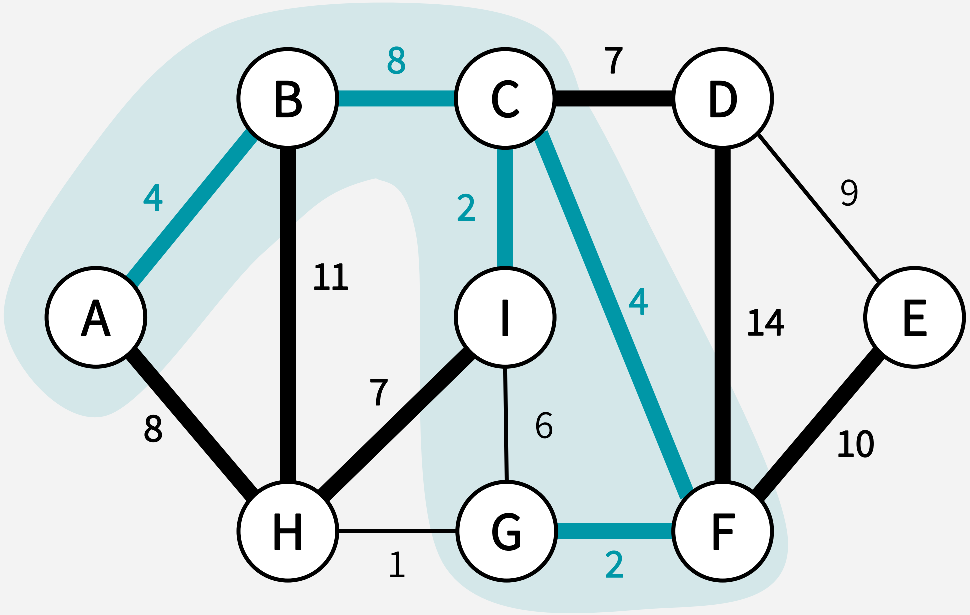

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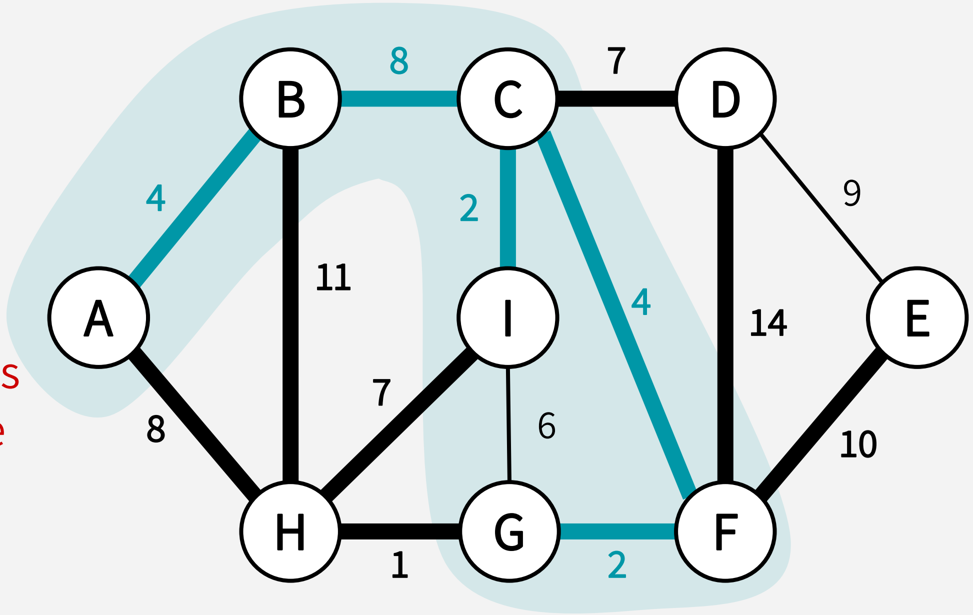

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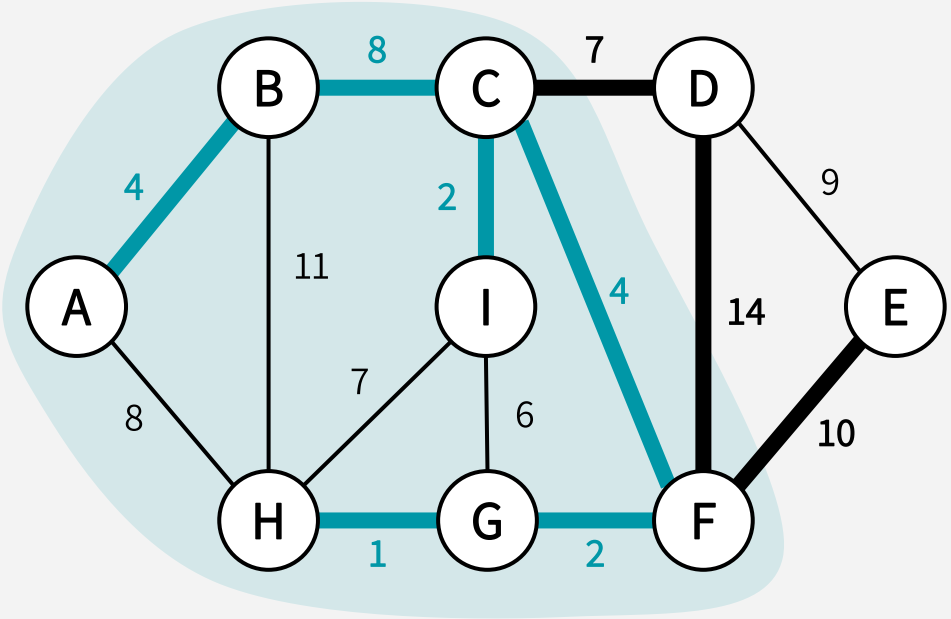

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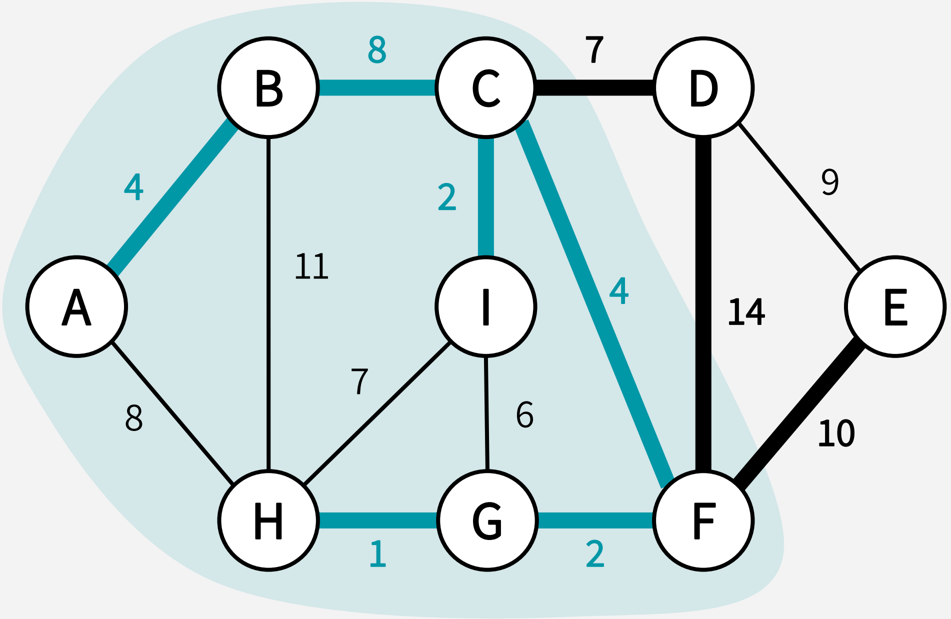

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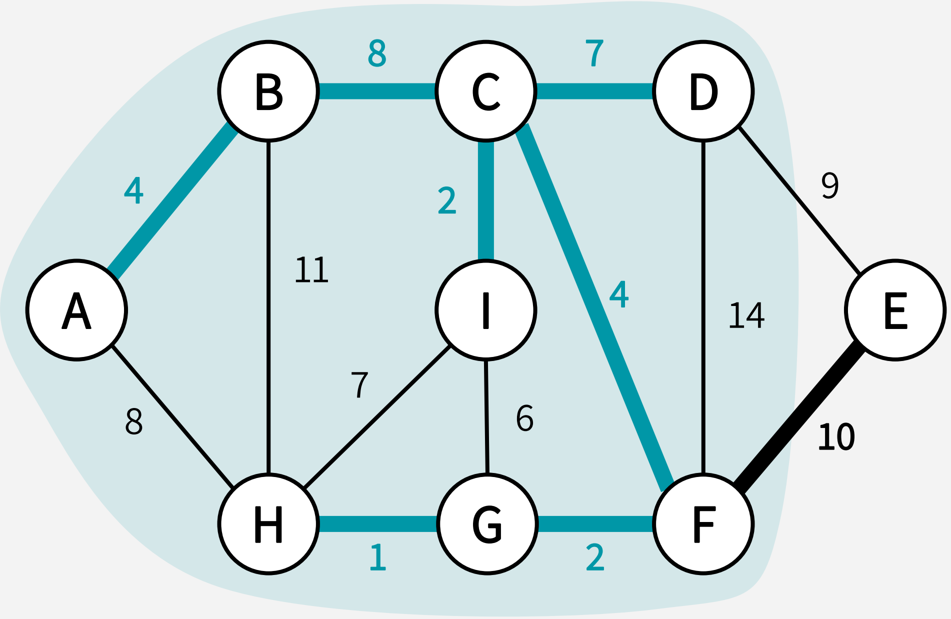

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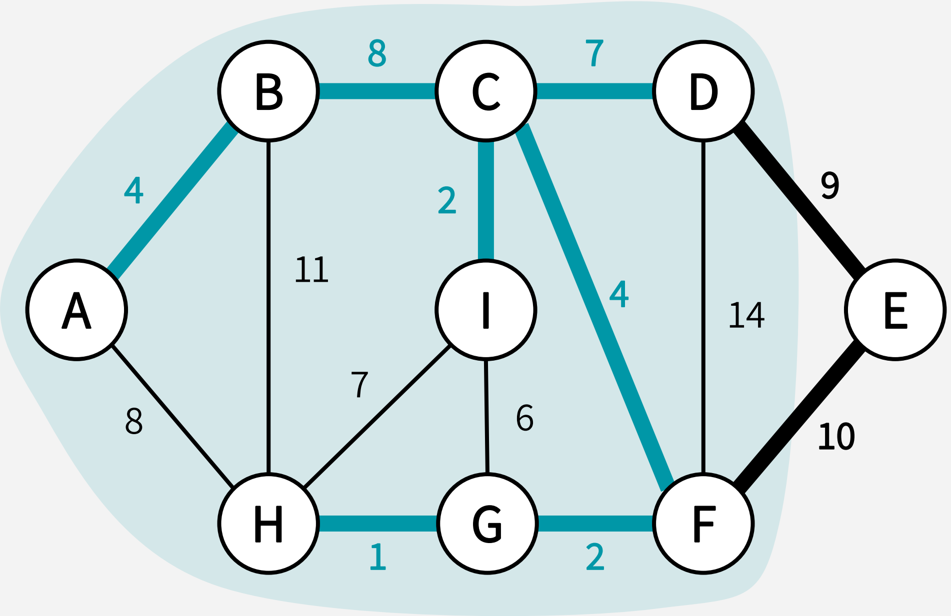

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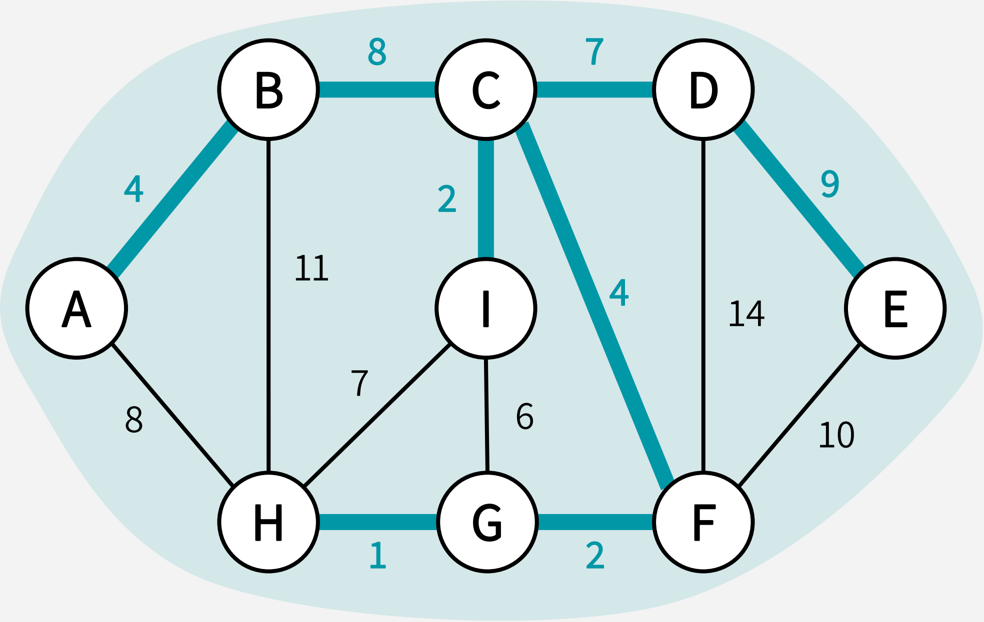

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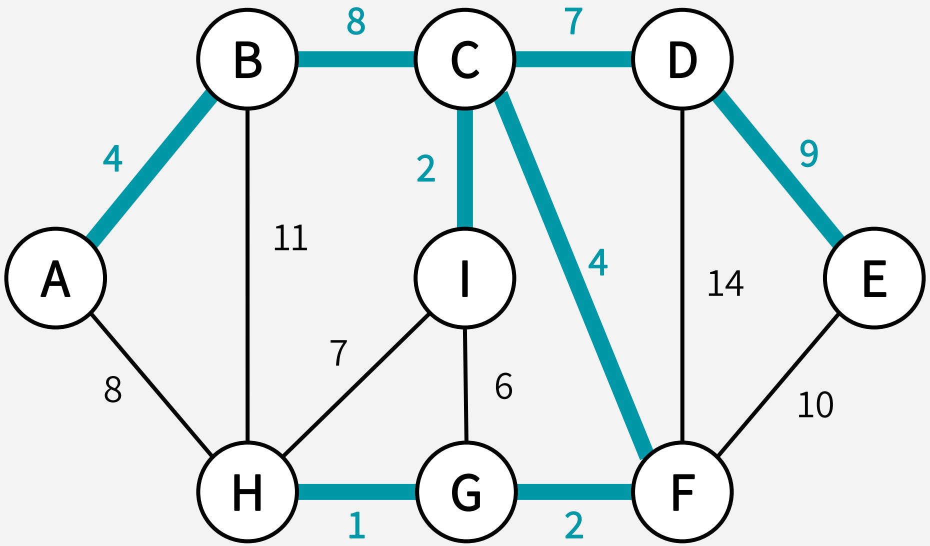

And we're done!

This is our MST.

(with weight 37)

PRIM'S ALGORITHM: SLOW VERSION

```
NAIVE_PRIM(G = (V,E), s):
    MST = {}
    visited = {s}
    while len(visited) < n:
        find the lightest edge (x,v) in E s.t.
            • x in visited
            • v not in visited
        MST.add((x,v))
        visited.add(v)
    return MST
```

If we manually find the lightest edge each iteration, it could be O(m) time per iteration..

(Naive) Runtime: O(nm)

(We'll speed this up by using smart data structures...)

PRIM'S ALGORITHM: SLOW VERSION

```
NAIVE_PRIM(G = (V,E), s):
    MST = {}
```

How should we actually implement this?

Each vertex that's not yet reached by the growing tree keeps track of:

1) the **distance** from itself to the growing spanning tree using *one edge*
2) **how to get to there** (the closest neighbor that's reached by the tree already)

```
    return MST
```

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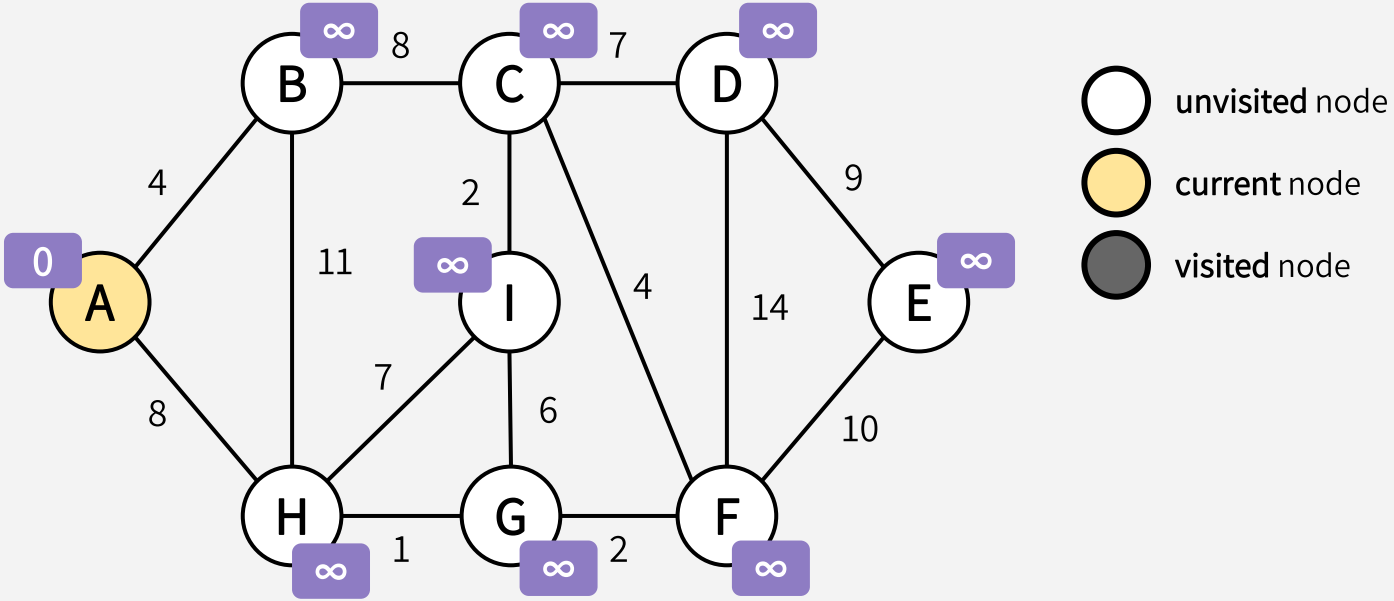

A is part of the growing tree first

HOW DO WE IMPLEMENT THIS?

Each vertex that's not yet reached by the growing tree keeps track of:

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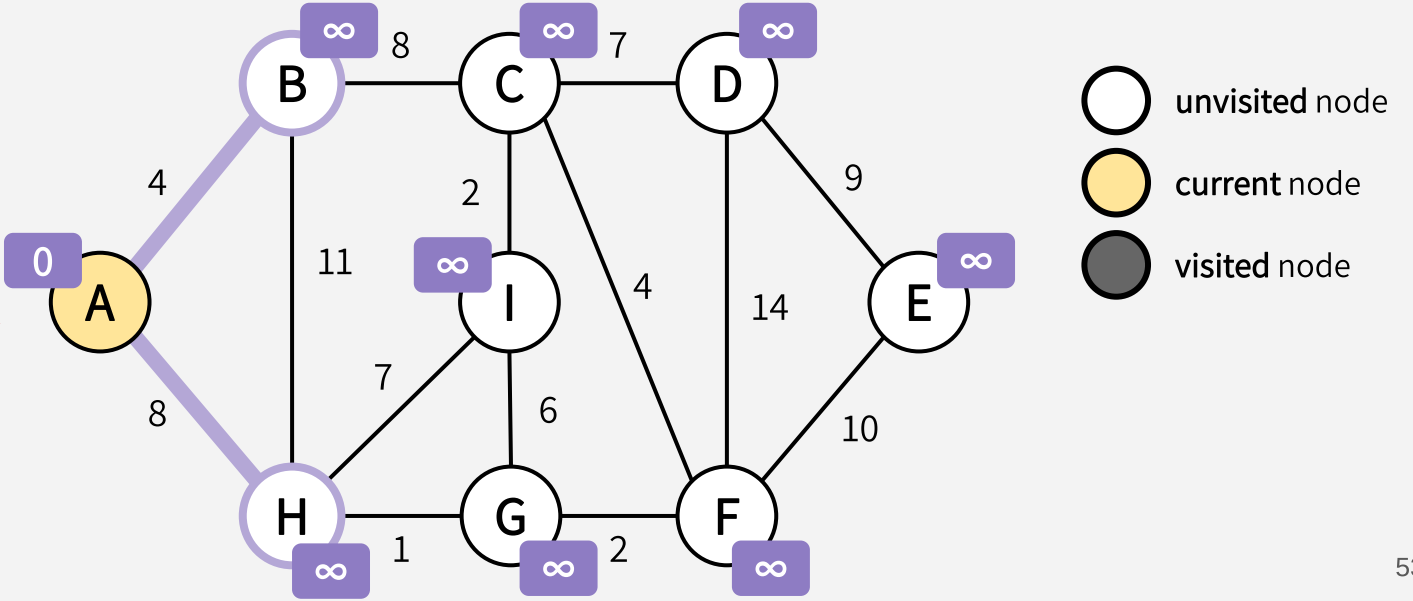
2) **how to get to there** (the closest neighbor that's reached by the tree already)

Now that A got added, see if any of its neighbors are closer to the tree because of it!



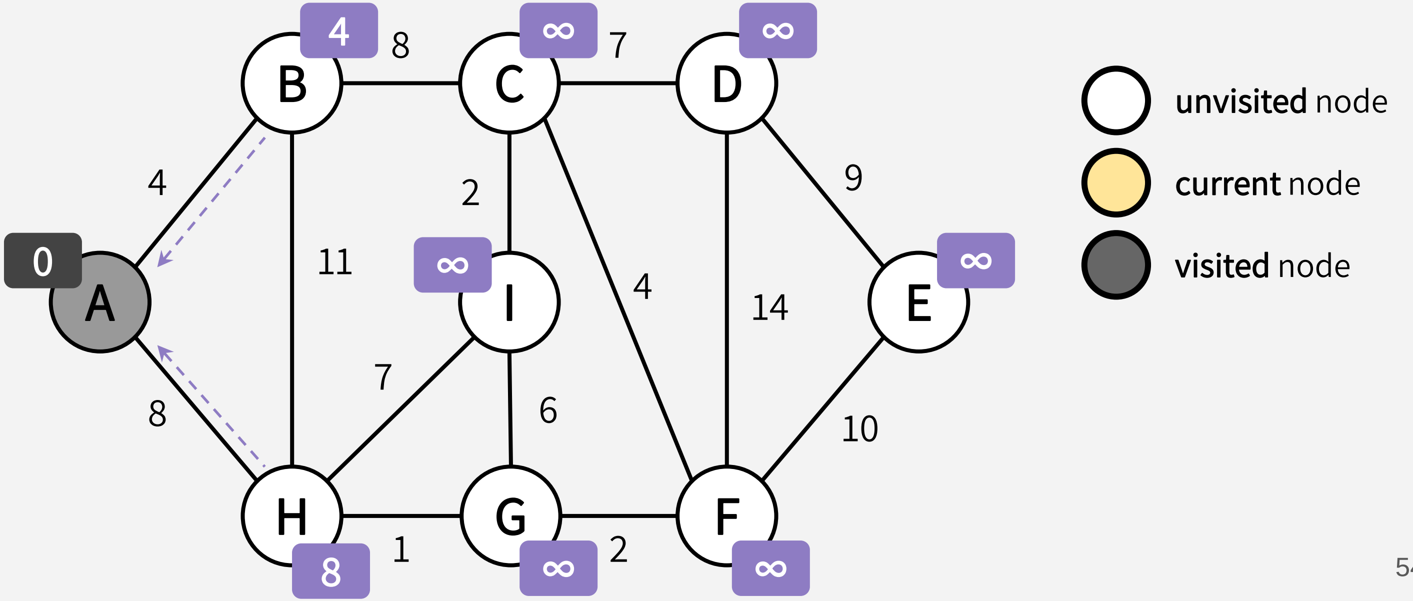
HOW DO WE IMPLEMENT THIS?

Each vertex that's not yet reached by the growing tree keeps track of:

1) the **distance** from itself to the growing spanning tree using *one edge*
2) **how to get to there** (the closest neighbor that's reached by the tree already)

Update their estimates, and now A is officially done.

Time to choose the lightest edge on the frontier (i.e. the edge whose endpoint has the lowest distance stored)



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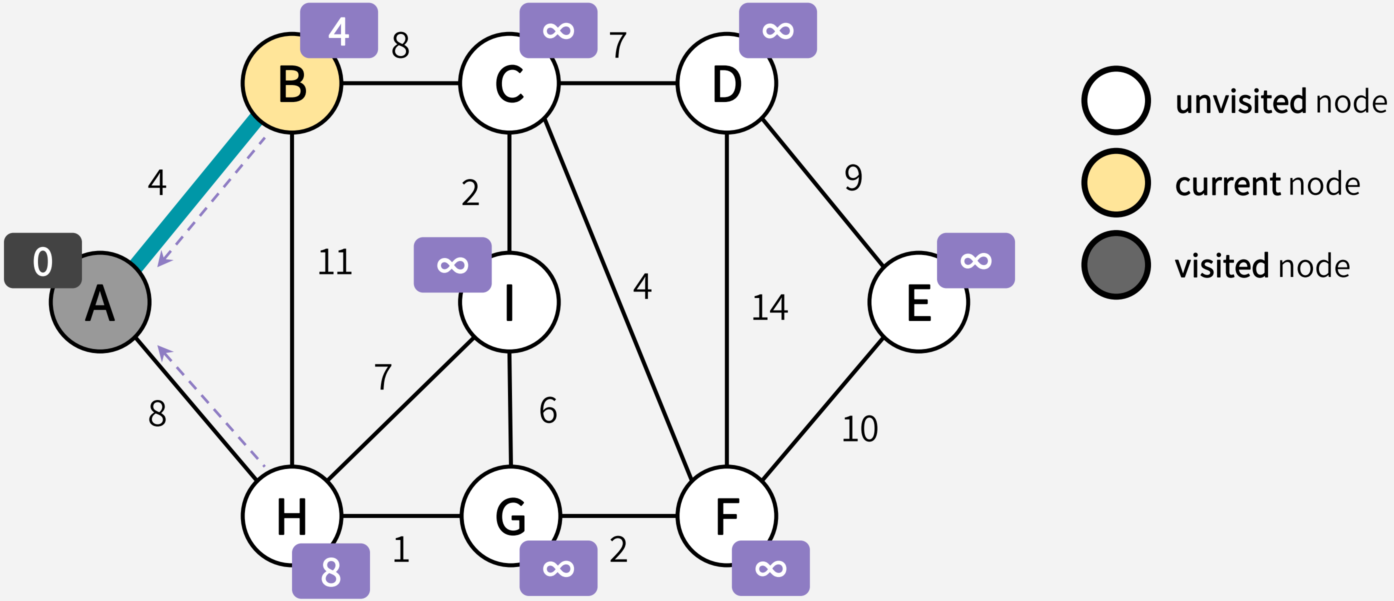
B is the closest node to the growing tree.

Since we recorded how to get to the tree from B, we know which edge to add.



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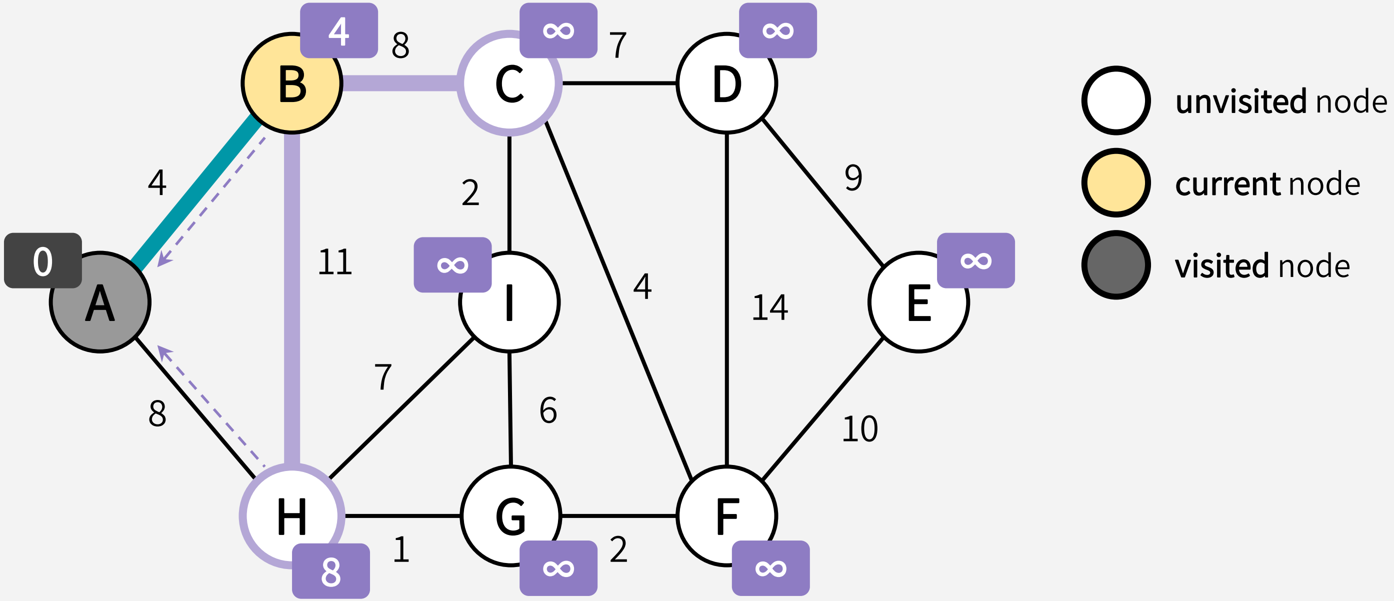
Each vertex that's not yet reached by the growing tree keeps track of:

1) the **distance** from itself to the growing spanning tree using *one edge*
2) **how to get to there** (the closest neighbor that's reached by the tree already)

Now that B is reached by the tree, see if any of its neighbors are closer to the tree because of it!



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Each vertex that's not yet reached by the growing tree keeps track of:

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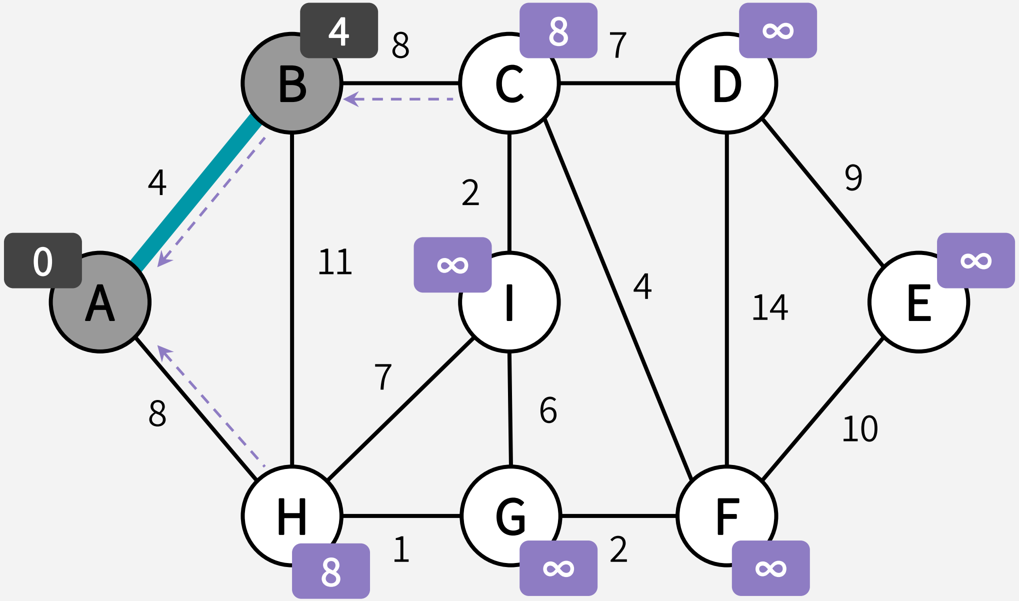
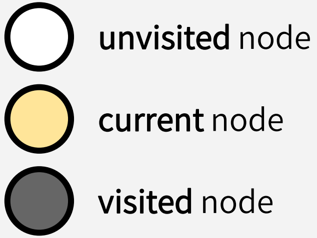
2) **how to get to there** (the closest neighbor that's reached by the tree already)

Update their estimates, and now B is officially done.

Time to choose the lightest edge on the frontier (i.e. the edge whose endpoint has the lowest distance stored)





HOW DO WE IMPLEMENT THIS?

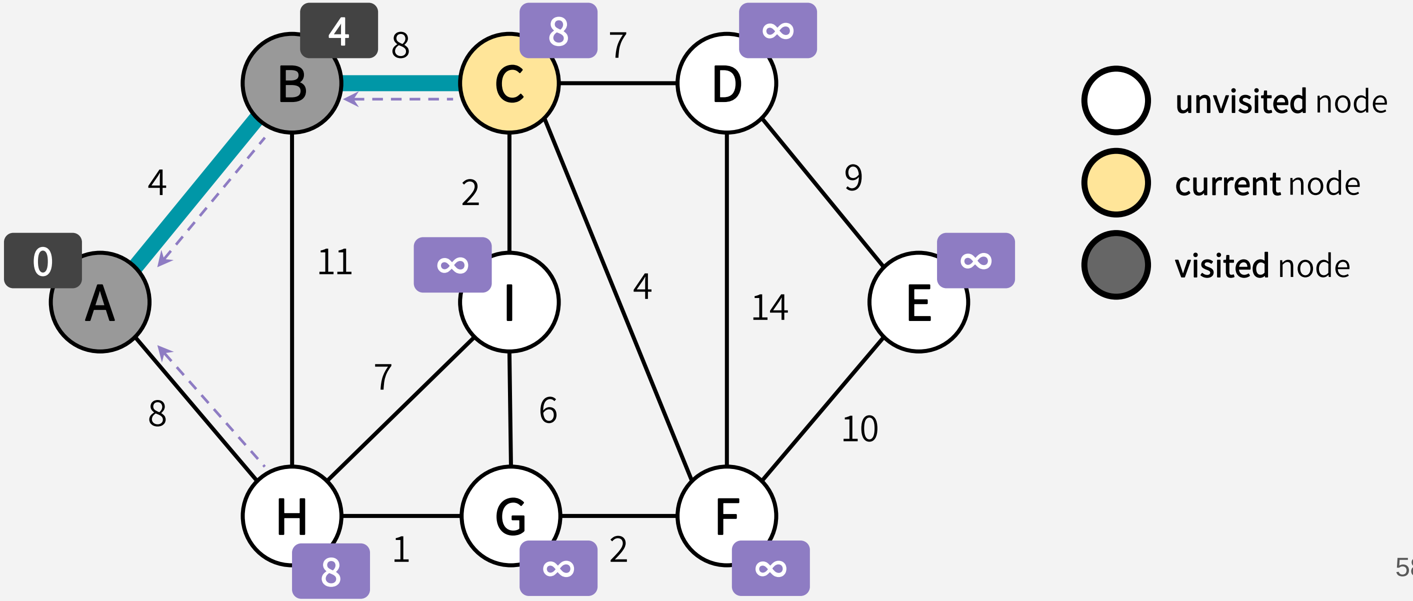
Each vertex that's not yet reached by the growing tree keeps track of:

1) the **distance** from itself to the growing spanning tree using *one edge*
2) **how to get to there** (the closest neighbor that's reached by the tree already)

C is the closest node to the growing tree.

(technically a tie, but let's choose C)

Since we recorded how to get to the tree from C, we know which edge to add.

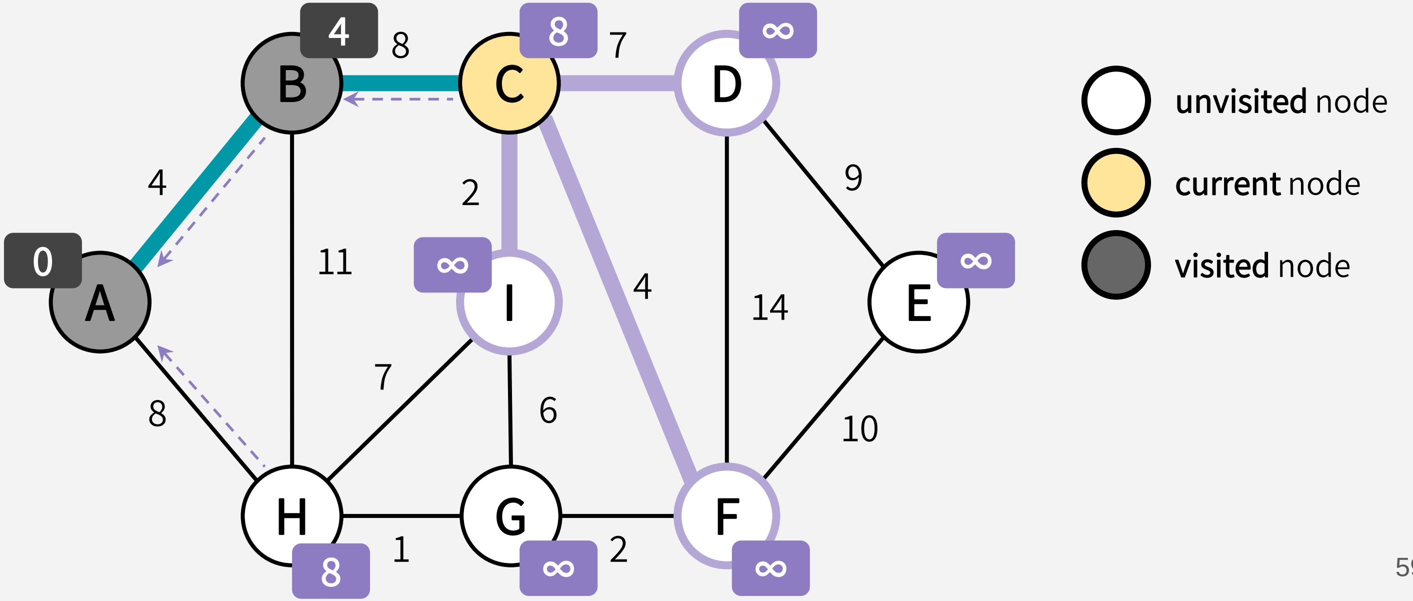


HOW DO WE IMPLEMENT THIS?

Each vertex that's not yet reached by the growing tree keeps track of:

1) the **distance** from itself to the growing spanning tree using *one edge*
2) **how to get to there** (the closest neighbor that's reached by the tree already)

Now that C is reached by the tree, see if any of its neighbors are closer to the tree because of it!



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Each vertex that's not yet reached by the growing tree keeps track of:

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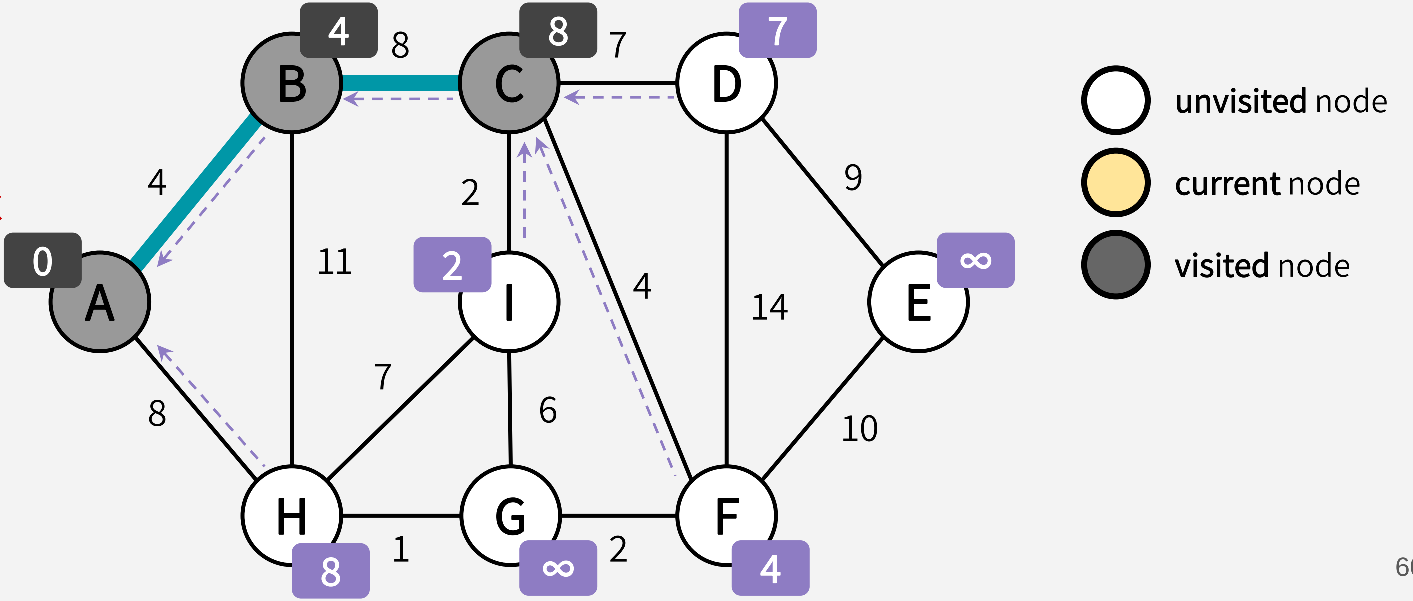
Update their estimates, and now C is officially done.

Time to choose the lightest edge on the frontier (i.e. the edge whose endpoint has the lowest distance stored)



HOW DO WE IMPLEMENT THIS?

Each vertex that's not yet reached by the growing tree keeps track of:

1) the **distance** from itself to the growing spanning tree using *one edge*
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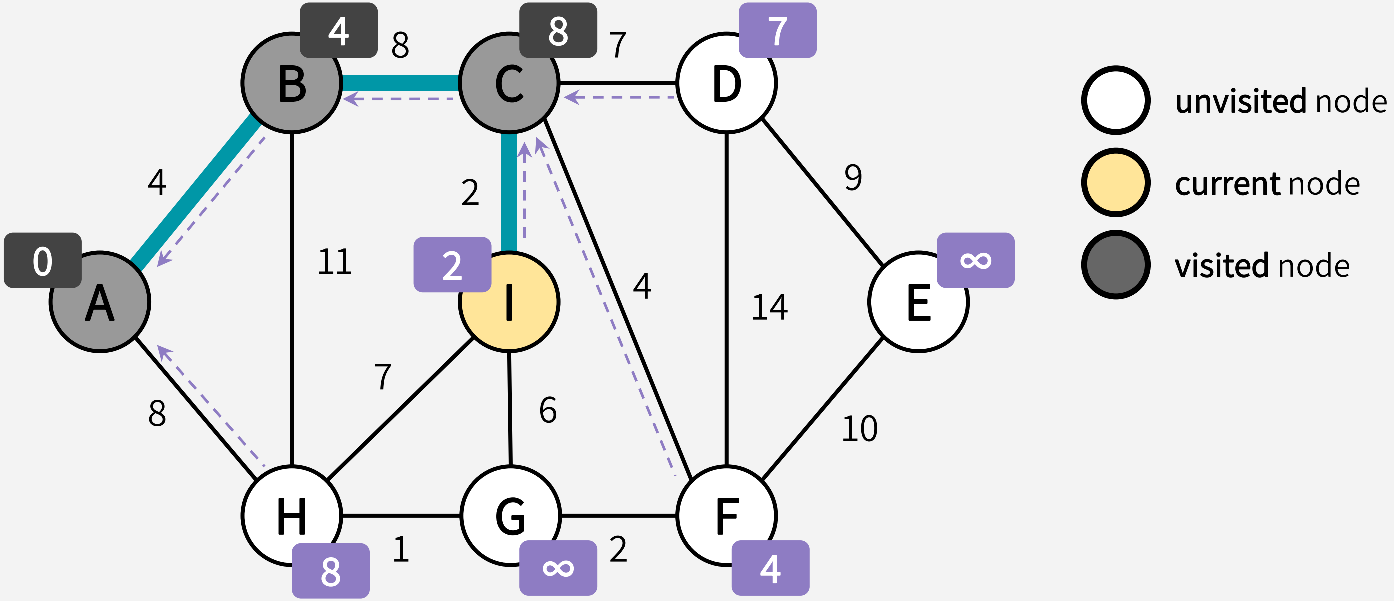
I is the closest node to the growing tree.

Since we recorded how to get to the tree from I, we know which edge to add.



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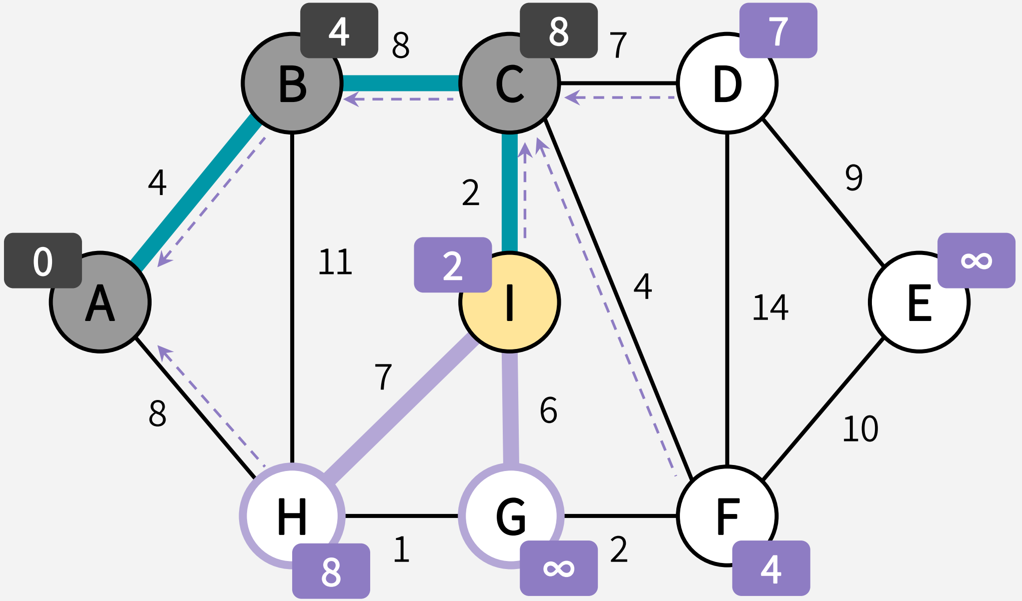
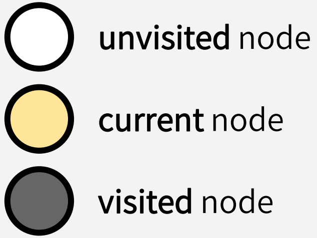
Now that I is reached by the tree, see if any of its neighbors are closer to the tree because of it!





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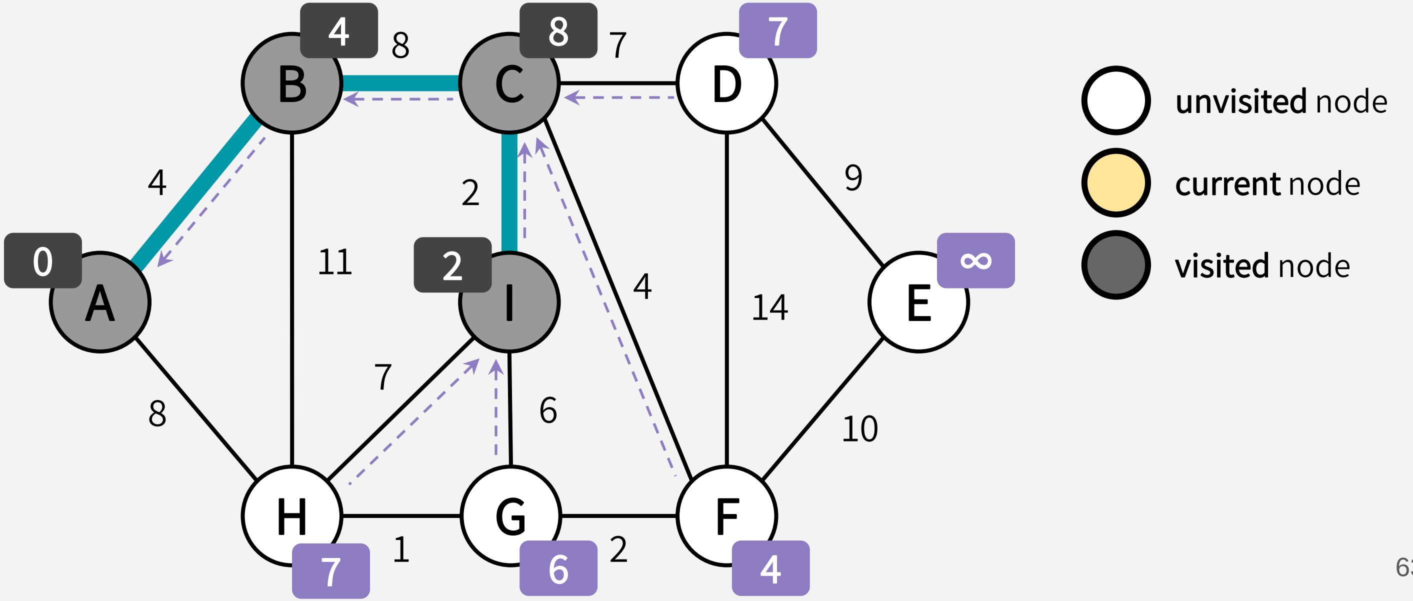
Update their estimates, and now I is officially done.

Time to choose the lightest edge on the frontier (i.e. the edge whose endpoint has the lowest distance stored)



PRIM'S ALGORITHM: PSEUDOCODE

```
PRIM(G = (V,E), s):
   MST = {}
   visited = {s}
   for all v besides s: d[v] = ∞ and k[v] = NULL
   for each neighbor v of s: d[v] = w(s,v) and k[v] = s
   while len(visited) < n:
      x = unvisited vertex v with smallest d[v] value
      MST.add((K[x], x))
      for each unreached neighbor v of x:
         d[v] = min(w(x,v), d[v])
         if d[v] was updated: k[v] = x
      visited.add(x)
   return MST
```

k[v] stores the the node in the growing tree that is closest to v (using one edge)

Runtime (using RB-Tree): O(m log n)

(Exact same structure as Dijkstra! We will see, Dijkstra's runtime depends on the data structure used for a priority queue.)

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          if d[v] was updated: k[v] = x
      visited.add(x)
   return MST
```

k[v] stores the the node in the growing tree that is closest to v (using one edge)

Runtime (using Fibonacci Heap): O(m + n log n)

(Exact same structure as Dijkstra! We will see, Dijkstra's runtime depends on the data structure used for a priority queue.)

PRIM'S ALGORITHM: CORRECTNESS

Let's follow our framework from before:

Prove that after each choice, you're not ruling out success.
(i.e. you're not ruling out finding an optimal solution)

- **INDUCTIVE HYPOTHESIS:** After greedy choice t, you haven't ruled out success
- **BASE CASE:** Success is possible before you make any choices
- **INDUCTIVE STEP:** If you haven't ruled out success after choice t, then show that you won't rule out success after choice t+1 (let's elaborate on this!)
- **CONCLUSION:** If you reach the end of the algorithm and haven't ruled out success then you must have succeeded

PRIM'S ALGORITHM: CORRECTNESS

Let's follow our framework from before:

Prove that after each choice, you're not ruling out success.
(i.e. you're not ruling out finding an optimal solution)

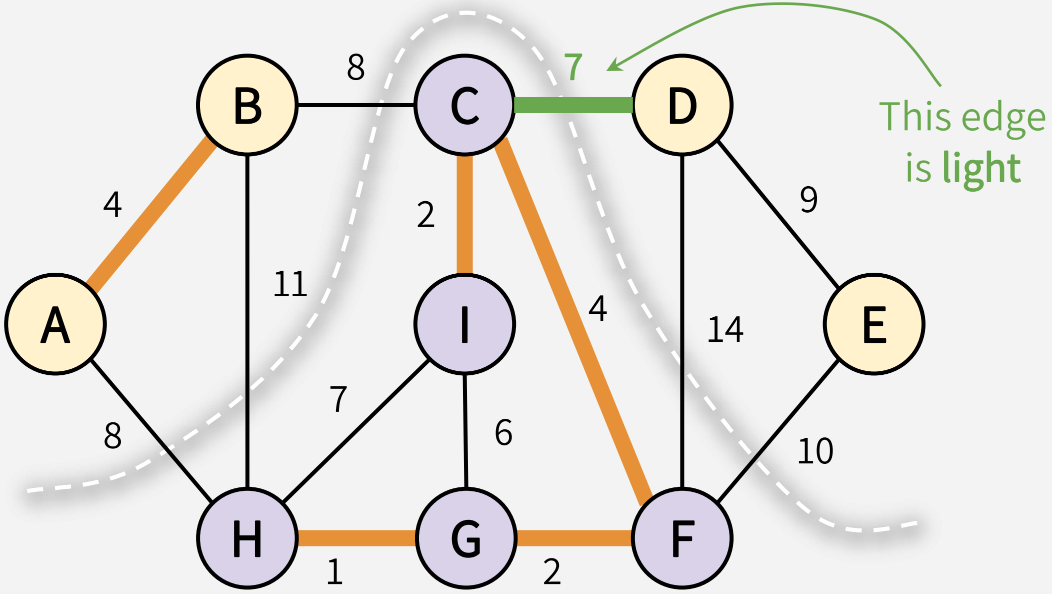
Our greedy choice in Prim's: **choosing the lightest edge on our frontier**

"Not ruling out success": **there's still an MST that extends our current set of edges**

- **CONCLUSION:** If you reach the end of the algorithm and haven't ruled out success then you must have succeeded

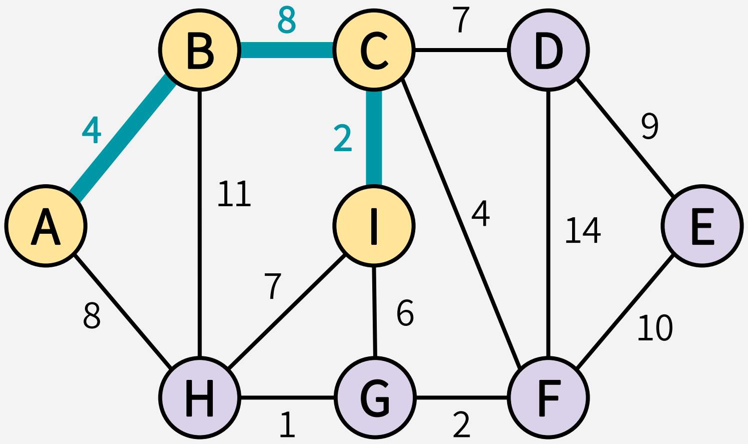
REMEMBER OUR LEMMA

LEMMA: Consider a cut that respects a set of edges S.
Suppose there exists an MST T* containing S. Let (u,v) be a light edge crossing this cut.
Then, there exists an MST containing S **U** {(u,v)}.



PRIM'S ALGORITHM: CORRECTNESS

Inductive Step (sketch): Suppose we've already chosen a set **S** of k edges, and there's an MST T* consistent with those choices. Then, Prim's chooses the *lightest edge on the frontier*, so we need to show there's an MST consistent with this new set of edges.



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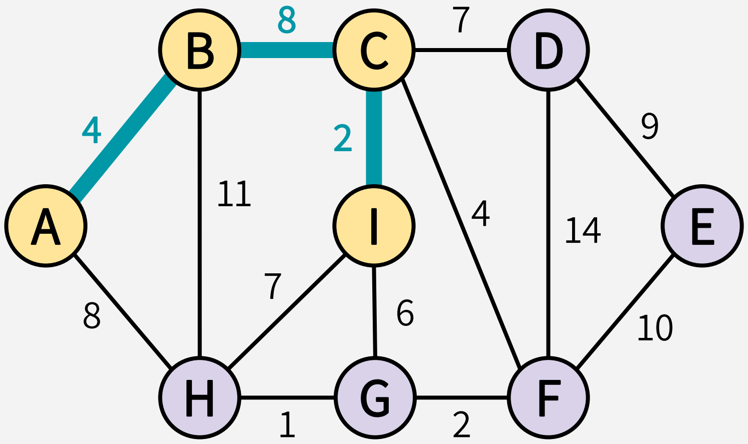
Inductive Step (sketch): Suppose we've already chosen a set **S** of k edges, and there's an MST T* consistent with those choices. Then, Prim's chooses the *lightest edge on the frontier*, so we need to show there's an MST consistent with this new set of edges.

Suppose our choices S so far don't rule out success.
This means there is an MST T* that contains S.



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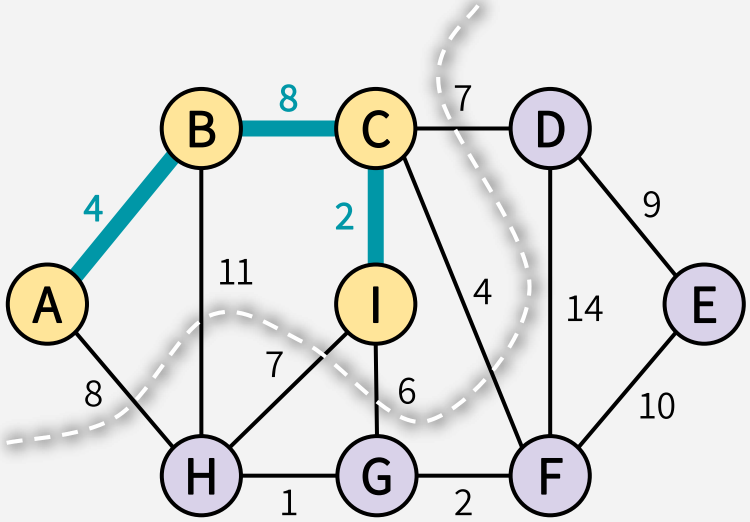
Consider the cut {visited, unvisited}.

This cut respects the set of edges S.



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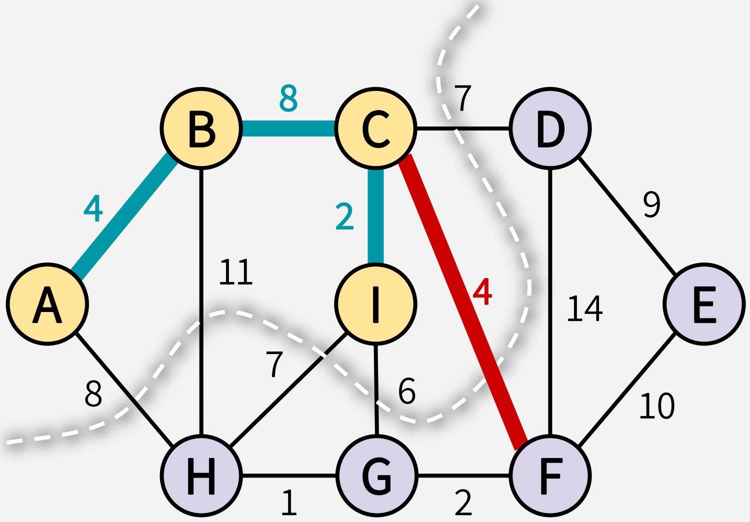
The next edge we add is a **light edge** on this cut.

This is the smallest weight edge that crosses the cut, i.e. the *frontier* of our growing tree.



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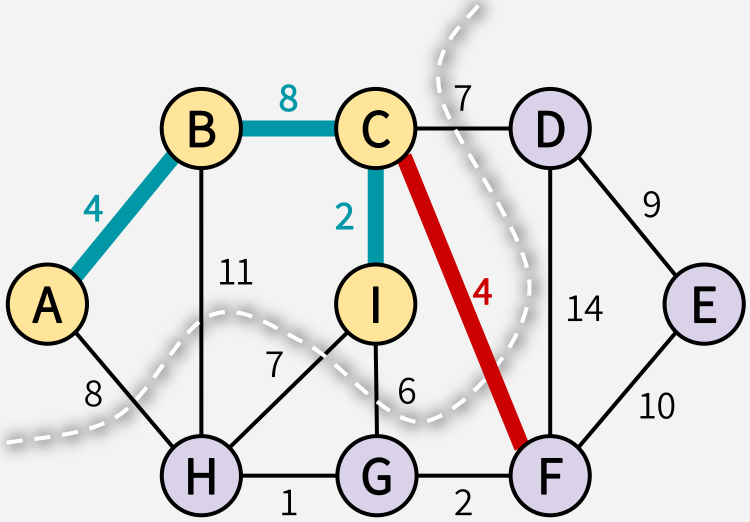
Consider the cut {visited, unvisited}.

This cut respects the set of edges S.

The next edge we add is a light edge on this cut.

This is the smallest weight edge that crosses the cut, i.e. the *frontier* of our growing tree.

By our Lemma, once we add this **light edge**, there is still an MST that is consistent with our new set of edges. Thus, we haven't ruled out success!



PRIM'S ALGORITHM: CORRECTNESS

INDUCTIVE HYPOTHESIS

After adding the t^{th} edge, there is an MST that contains the edges added so far.

BASE CASE

After adding the 0^{th} edge, there exists an MST with the edges added so far.

INDUCTIVE STEP *(weak induction)*

If the inductive hypothesis holds for t (i.e. the edge choices so far are safe), then it holds for t+1, as there is still an MST that contains these t+1 edges. **We proved this by considering the cut between visited & unvisited nodes (i.e. the "frontier) and invoking our Lemma from earlier in class.**

CONCLUSION

After adding the $(n-1)^{st}$ edge, there exists an MST containing the edges added so far. A tree containing n–1 edges is already a spanning tree, so the tree we have must be a minimum spanning tree.

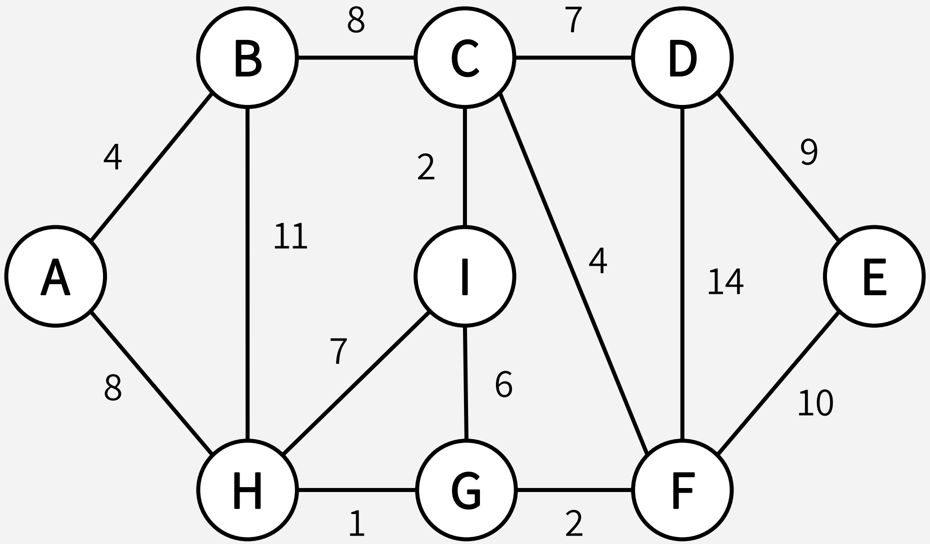
KRUSKAL'S ALGORITHM

Greedily add the cheapest edge!

KRUSKAL'S ALGORITHM: THE IDEA

Greedy choice:

Maintain a forest of trees, & greedily add the cheapest edge to combine trees



KRUSKAL'S ALGORITHM: THE IDEA

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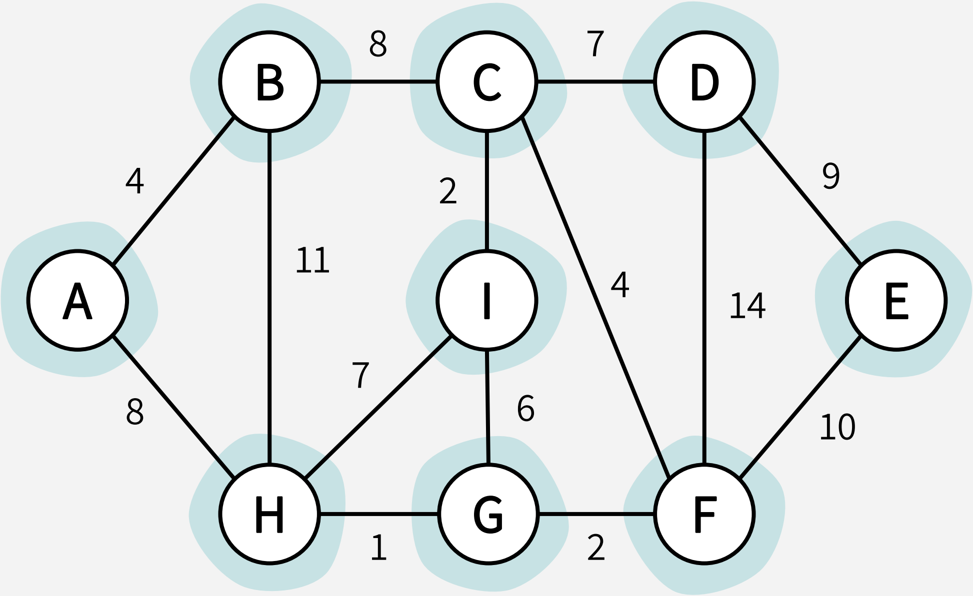
Maintain a forest of trees, & greedily add the cheapest edge to combine trees

Every node on its own starts as an individual tree in this forest



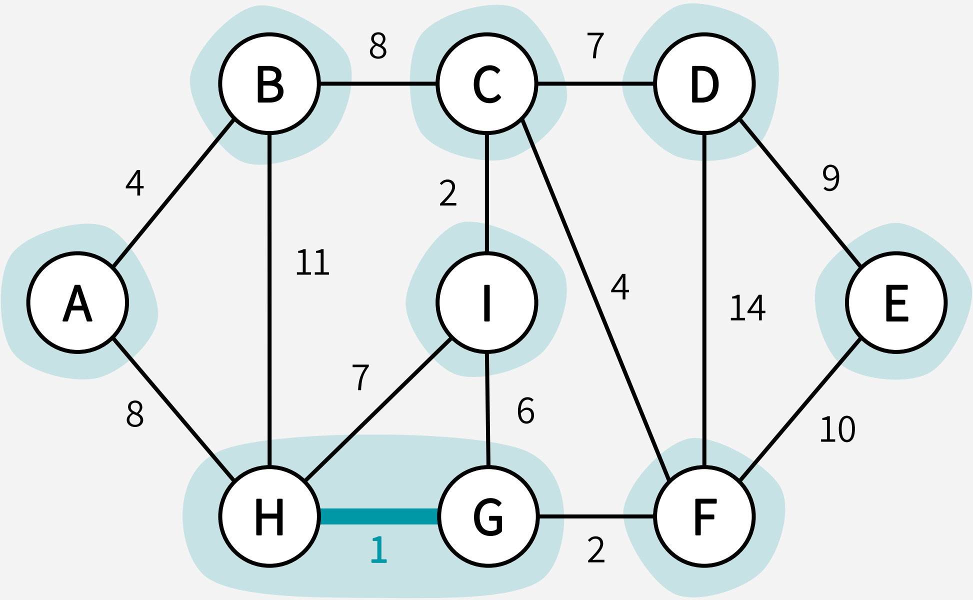
KRUSKAL'S ALGORITHM: THE IDEA

Greedy choice:

Maintain a forest of trees, & greedily add the cheapest edge to combine trees

Choose the cheapest edge that would combine two trees

(i.e. that won't cause a cycle)



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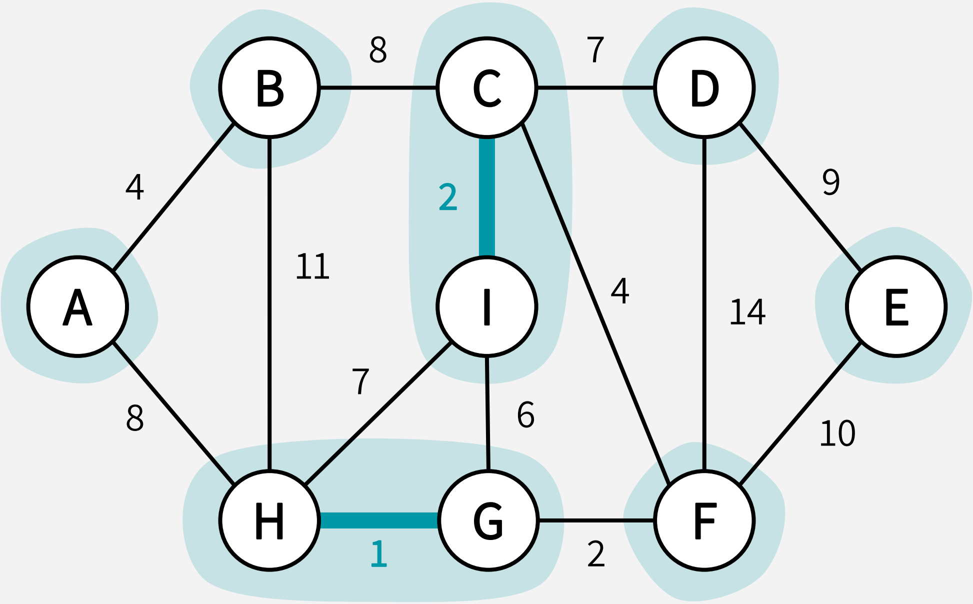

If there's a tie, choose one of the edges

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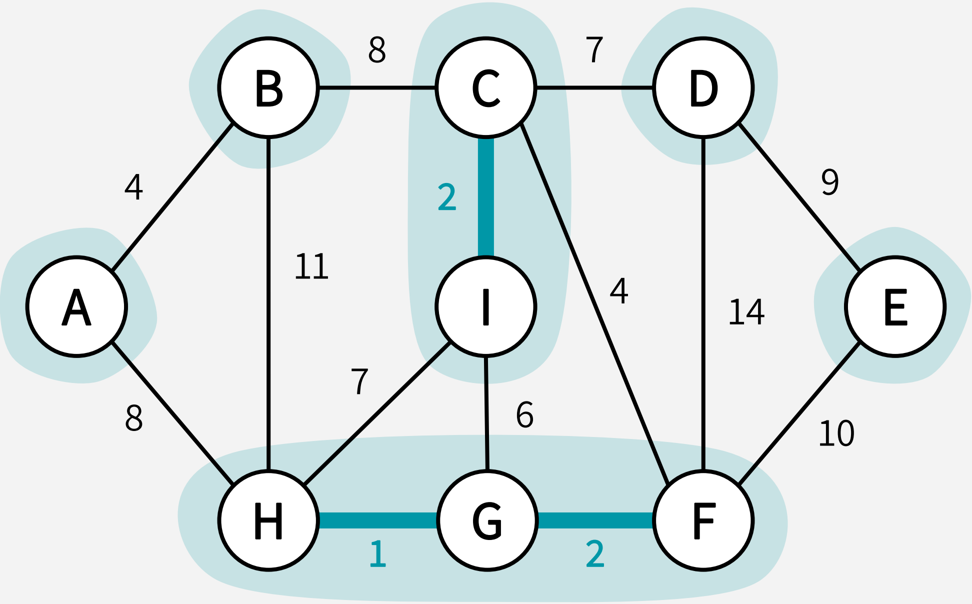

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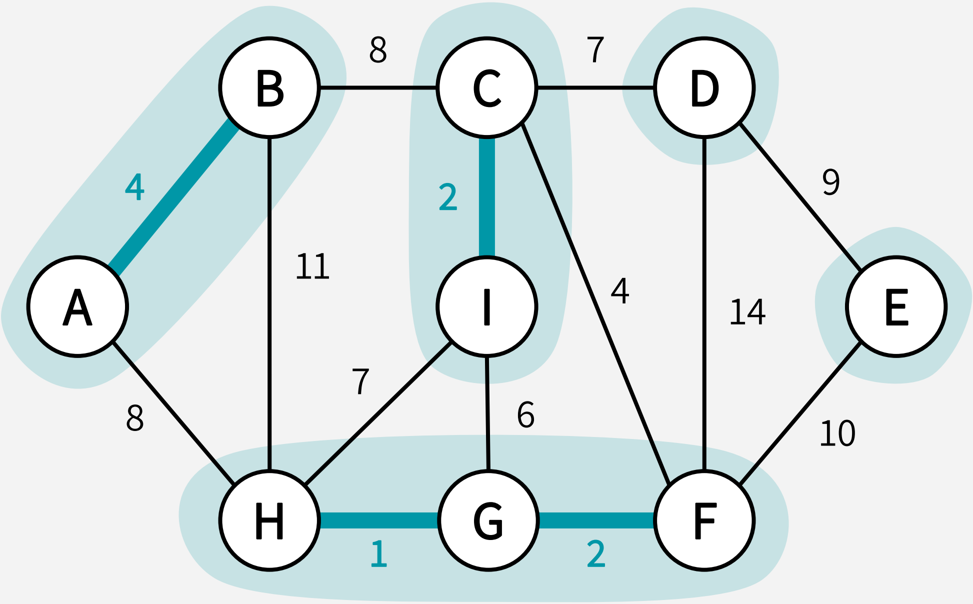

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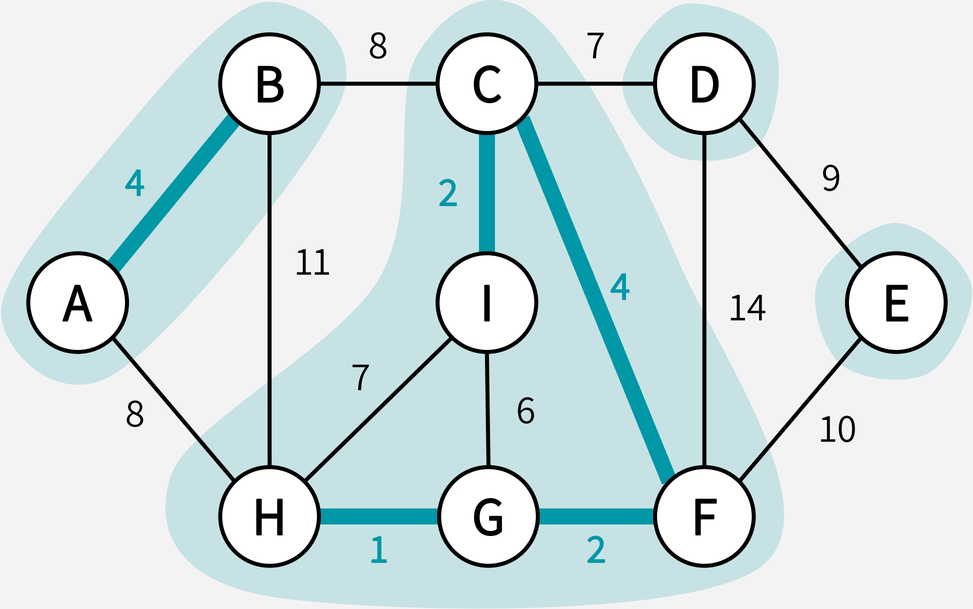

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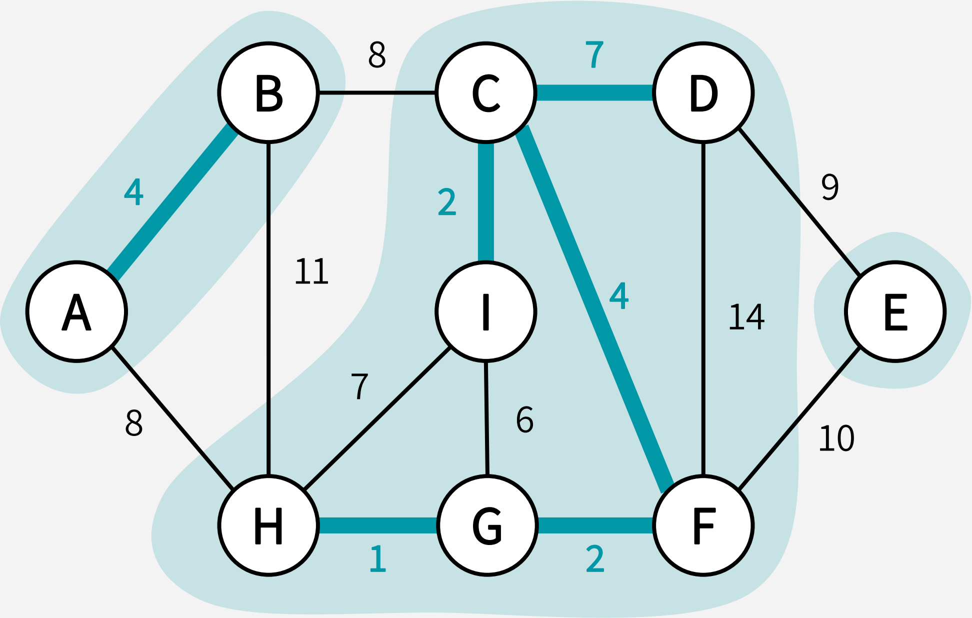

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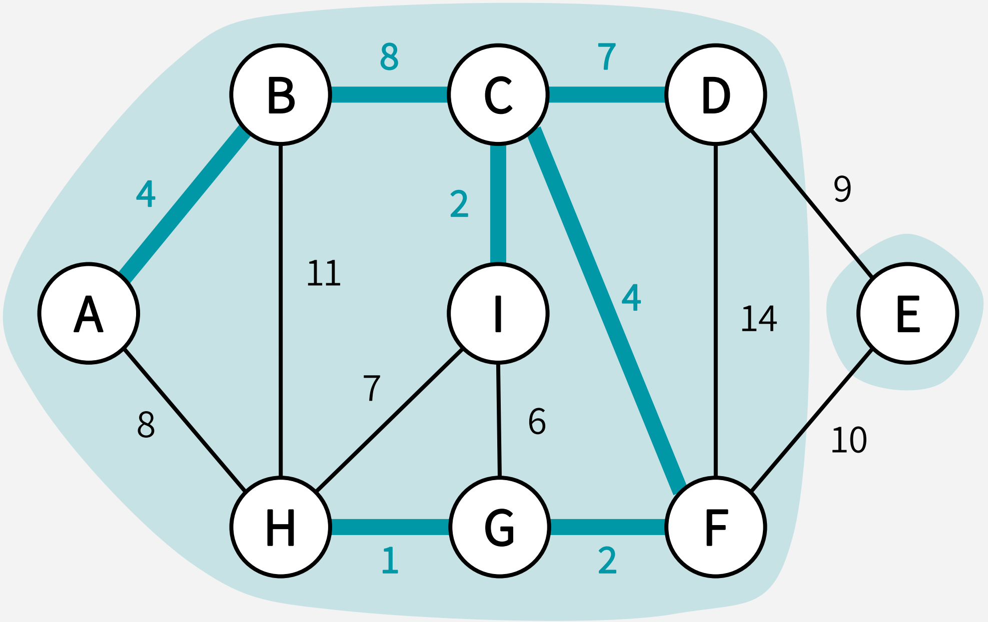

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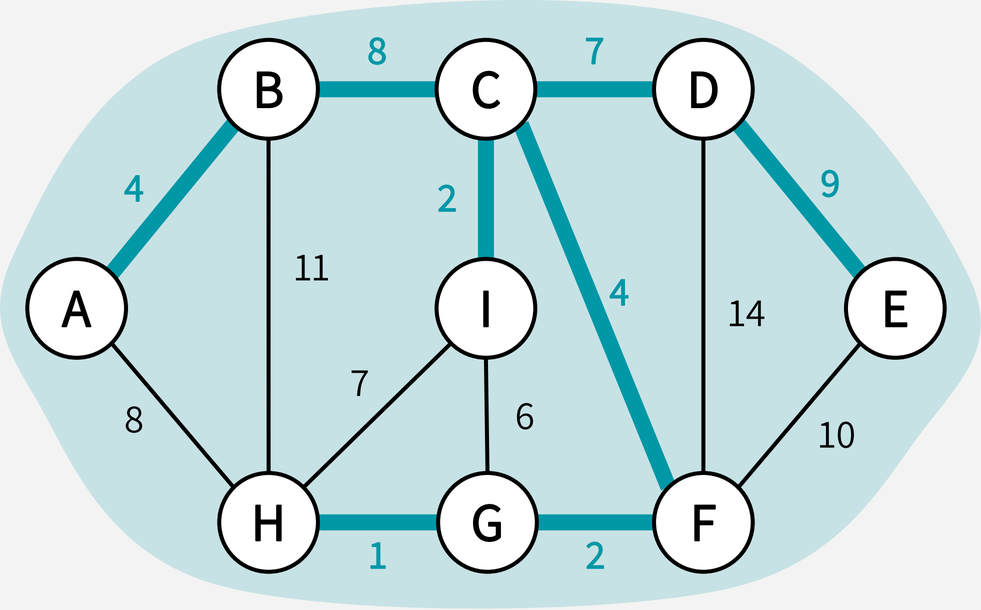

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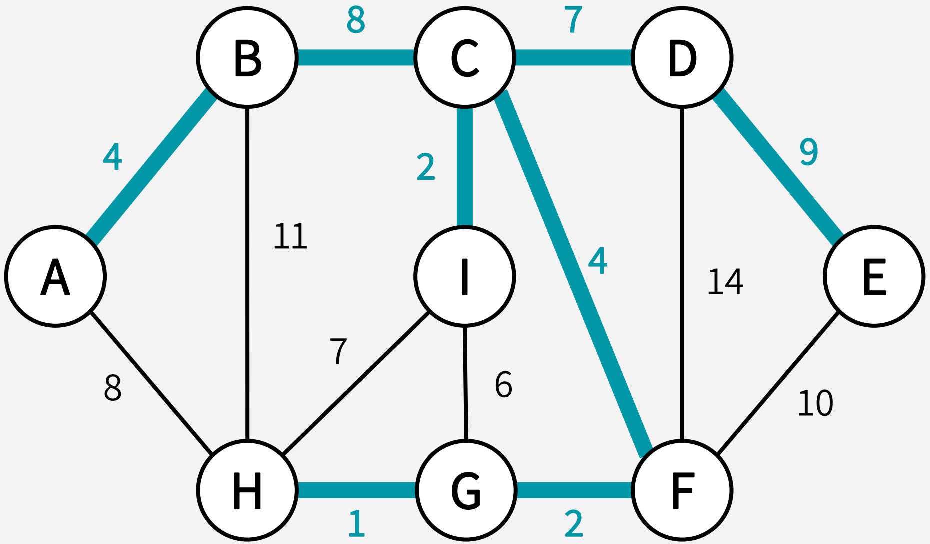

We're done!
This is the MST.

KRUSKAL'S ALGORITHM: PSEUDOCODE

```
KRUSKAL_NOT_VERY_DETAILED(G = (V,E)):
    E_SORTED = E sorted by weight in non-decreasing order
    MST = {}
    for v in V:
        put v in its own tree
    for (u,v) in E_SORTED:
        if u's tree and v's tree are not the same:
            MST.add((u,v))
            merge u's tree with v's tree
    return MST
```

KRUSKAL'S ALGORITHM: PSEUDOCODE

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To implement these lines, we'll use a *Union-Find data structure,* which supports 3 operations: **MAKE_SET(x)**, **FIND(x)**, and **UNION(x,y)**

KRUSKAL'S ALGORITHM: PSEUDOCODE

`MAKE_SET(x)`: creates a set {x} in O(1)
`FIND(x)`: returns the set containing x in O(1)
`UNION(x,y)`: merges the sets containing x and y in O(1)

To implement these lines, we'll use a *Union-Find data structure,* which supports 3 operations: **`MAKE_SET(x)`**, **`FIND(x)`**, and **`UNION(x,y)`**

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        if FIND(u) != FIND(v):
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            UNION(u,v)
    return MST
```

Basically, the time to sort the edge weights dominates the runtime.
$O(m \log m) = O(m \log n)$, since $m \leq n^2$

(With union-find data structure) **Runtime = $O(m \log n)$**

KRUSKAL'S ALGORITHM: PSEUDOCODE

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            UNION(u,v)
    return MST
```

If the edge weights are of appropriate values and RadixSort can be applied instead

(With union-find data structure & RadixSort) **Runtime = O(m)**

KRUSKAL'S CORRECTNESS

Let's follow our framework from before:

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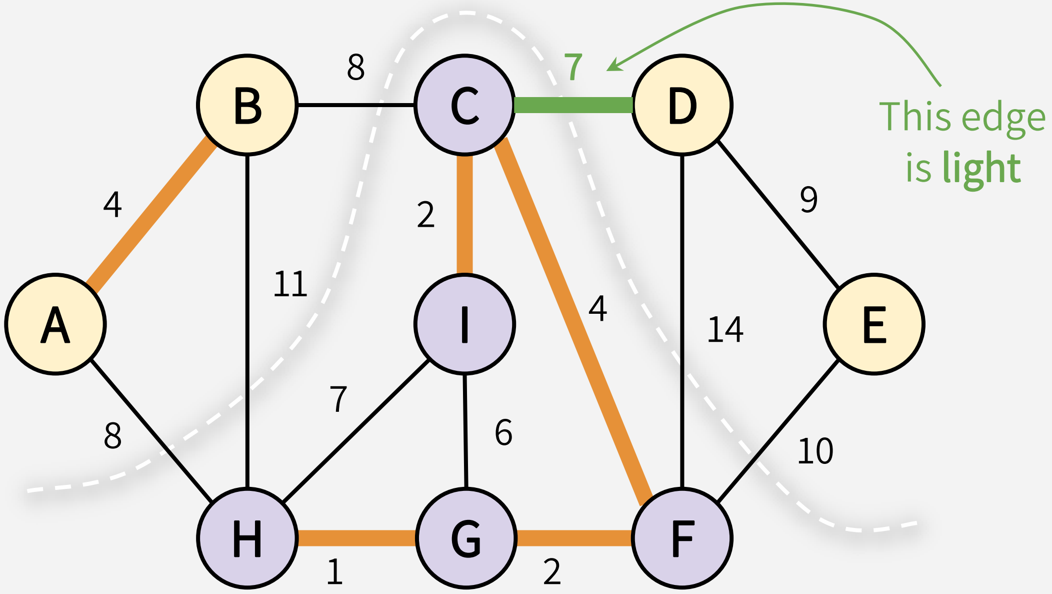
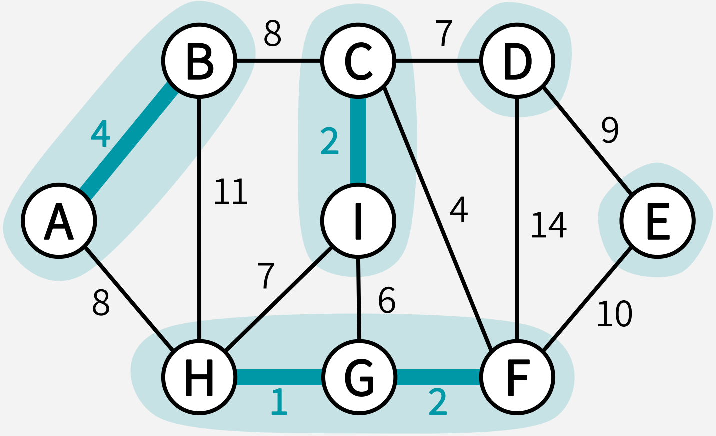


KRUSKAL'S CORRECTNESS

Inductive Step (sketch): Suppose we've already chosen a set **S** k edges, and there's an MST T* consistent with those choices. Then, Kruskal's adds the *cheapest edge that would merge 2 trees*, so we show there's still an MST consistent with this new set of edges.



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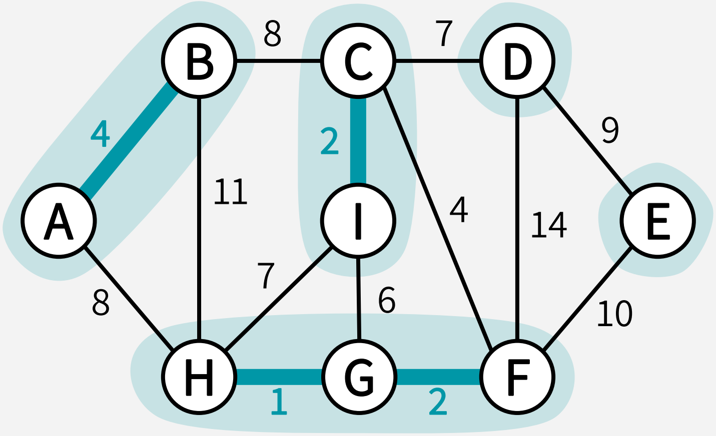
Suppose our choices S so far don't rule out success.
This means there is an MST T* that contains S.



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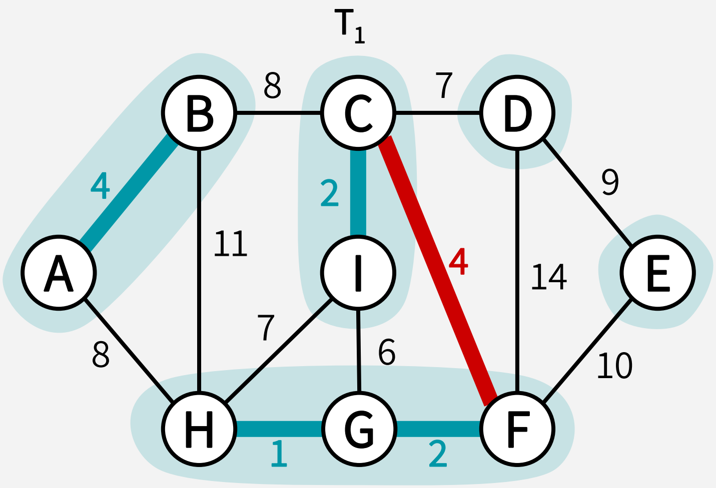

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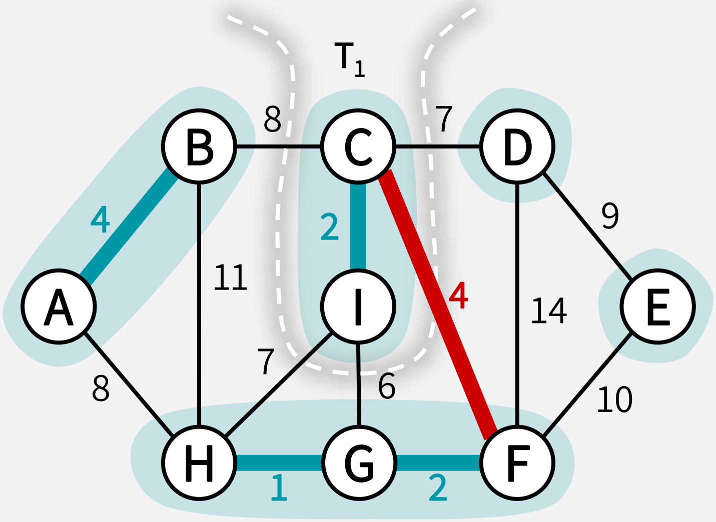
This cut respects S. Our new edge is *light* for the cut (it's the cheapest edge after all).



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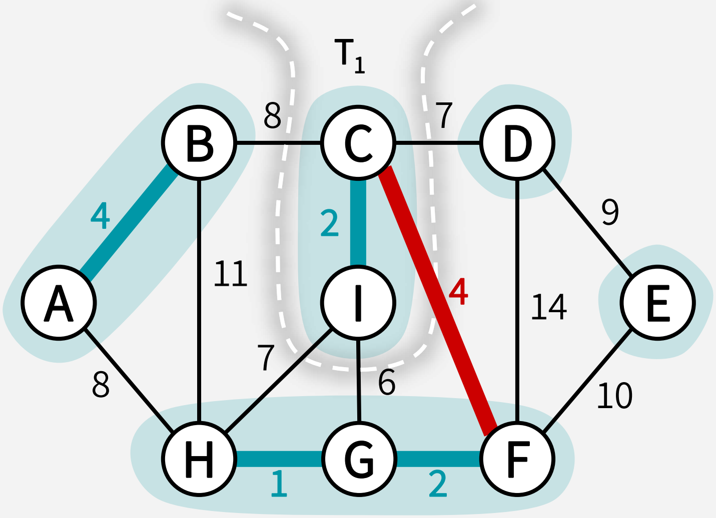
By our Lemma, once we add this **light edge**, there is still an MST that is consistent with our new set of edges. Thus, we haven't ruled out success!



KRUSKAL'S CORRECTNESS

INDUCTIVE HYPOTHESIS

After adding the t^{th} edge, there is an MST that contains the edges added so far.

BASE CASE

After adding the 0^{th} edge, there exists an MST with the edges added so far.

INDUCTIVE STEP *(weak induction)*

If the inductive hypothesis holds for t (i.e. the edge choices so far are safe), then it holds for t+1, as there is still an MST that contains these t+1 edges. **We proved this by considering the cut between the tree living at one endpoint of our chosen edge & all remaining vertices, & invoking our favorite Lemma.**

CONCLUSION

After adding the $(n–1)^{st}$ edge, there exists an MST containing the edges added so far. A tree containing n–1 edges is already a spanning tree, so the tree we have must be a minimum spanning tree.

PRIM'S vs. KRUSKAL'S

Prim's Algorithm

Grows a single tree by greedily adding the cheapest edge on the "frontier" of the growing tree.

Runtime (RB-tree): **$O(m \log n)$**
Runtime (Fibonacci Heap): **$O(m + n \log n)$**

Prim's may be better on dense graphs (where m is ~n^2) if you can't RadixSort edge weights

Kruskal's Algorithm

Maintains a forest and greedily chooses the cheapest edge that would be able to merge two trees

Runtime (union-find data struct.): **$O(m \log n)$**
Runtime (union-find + radixSort) : **$O(m)$**

Kruskal's may be better on sparse graphs if you *can* RadixSort edge weights

Both are greedy algorithms, with similar reasoning (that piggyback off of our lemma).

Optimal substructure: subgraphs generated by cuts — the way to make safe choices is to choose light edges crossing the cut.

CAN WE DO BETTER?

The algorithms are all comparison-based!

Karger-Klein Tarjan (1995)

O(m) expected time *randomized* algorithm

Chazelle (2000)

$O(m \cdot \alpha(n))$ time *deterministic* algorithm

Pettie-Ramachandran (2002)

O(optimal # of comparisons… whatever that is (i.e. if there exists an algo which uses X comparisons, this algo will run in time O(X)) time deterministic algorithm

This bound is unknown!
For now, we know it's $\Omega(n)$ and $O(m \cdot \alpha(n))$.

Acknowledgement

- Stanford University