Mathematics Review I
(Basic Terminology)

- Unlike other CS courses, this course is a MATH course…
- We will look at a lot of definitions, theorems and proofs
- This lecture: reviews basic math notation and terminology
  - Set, Sequence, Function, Graph, String…
- Also, common proof techniques
  - By construction, induction, contradiction

Set

- A set is a group of items
- One way to describe a set: list every item in the group inside { }
  - E.g., { 12, 24, 5 } is a set with three items
- When the items in the set has trend: use ...
  - E.g., { 1, 2, 3, 4, ... } means the set of natural numbers
- Or, state the rule
  - E.g., $\{ n \mid n = m^2$ for some positive integer $m \}$ means the set { 1, 4, 9, 16, 25, ... }
- A set with no items is an empty set denoted by { } or $\emptyset$

Set

- The order of describing a set does not matter
  - $\{ 12, 24, 5 \} = \{ 5, 24, 12 \}$
- Repetition of items does not matter too
  - $\{ 5, 5, 5, 1 \} = \{ 1, 5 \}$
- Membership symbol $\in$
  - $5 \in \{ 12, 24, 5 \} \quad 7 \notin \{ 12, 24, 5 \}$

- How many items are in each of the following set?
  - $\{ 3, 4, 5, \ldots, 10 \}$
  - $\{ 2, 3, 3, 4, 4, 2, 1 \}$
  - $\{ 2, \{2\}, \{\{1,2,3,4,5,6\}\} \}$
  - $\emptyset$
  - $\{\emptyset\}$

Set

Given two sets A and B

- we say $A \subseteq B$ (read as A is a **subset** of B) if every item in A also appears in B
  - E.g., A = the set of primes, B = the set of integers
- we say $A \subsetneq B$ (read as A is a **proper subset** of B) if $A \subseteq B$ but $A \neq B$

Warning: Don't be confused with $\in$ and $\subseteq$

- Let $A = \{ 1, 2, 3 \}$. Is $\emptyset \in A$? Is $\emptyset \subseteq A$?

Union, Intersection, Complement

Given two sets A and B

- $A \cup B$ (read as the union of A and B) is the set obtained by combining all elements of A and B in a single set
  - E.g., $A = \{ 1, 2, 4 \}$ $B = \{ 2, 5 \}$
    $A \cup B = \{ 1, 2, 4, 5 \}$
- $A \cap B$ (read as the intersection of A and B) is the set of common items of A and B
  - In the above example, $A \cap B = \{ 2 \}$
- $\overline{A}$ (read as the complement of A) is the set of items under consideration not in A

Set

- The **power set** of A is the set of all subsets of A, denoted by 2^A
  - E.g., $A = \{ 0, 1 \}$

    $2^A = \{ \{\}, \{0\}, \{1\}, \{0,1\} \}$
  - How many items in the above power set of A?
- If A has n items, how many items does its power set contain? Why?

Sequence

- A sequence of items is a list of these items in some order
- One way to describe a sequence: list the items inside ( )
  - (5, 12, 24)
- Order of items inside ( ) matters
  - (5, 12, 24) $\neq$ (12, 5, 24)
- Repetition also matters
  - (5, 12, 24) $\neq$ (5, 12, 12, 24)
- Finite sequences are also called tuples
  - (5, 12, 24) is a 3-tuple
  - (5, 12, 12, 24) is a 4-tuple

Sequence

Given two sets A and B

- The Cartesian product of A and B, denoted by $A \times B$, is the set of all possible 2-tuples with the first item from A and the second item from B
  - E.g., $A = \{1, 2\}$ and $B = \{x, y, z\}$

  $A \times B = \{ (1,x), (1,y), (1,z), (2,x), (2,y), (2,z) \}$

- The Cartesian product of k sets, $A_1, A_2, \ldots, A_k$, denoted by $A_1 \times A_2 \times \cdots \times A_k$, is the set of all possible k-tuples with the i^{th} item from A_i

Functions

- A function takes an input and produces an output
- If f is a function, which gives an output b when input is a, we write

$$f(a) = b$$

- For a particular function f, the set of all possible input is called f's domain
- The outputs of a function come from a set called f's range

Functions

- To describe the property of a function that it has domain D and range R, we write

$$f : D \rightarrow R$$

- E.g., The function add (to add two numbers) will have an input of two integers, and output of an integer
  - We write: add: $Z \times Z \rightarrow Z$

Deterministic Finite Automaton and Non-deterministic Finite Automaton

DFA and NFA

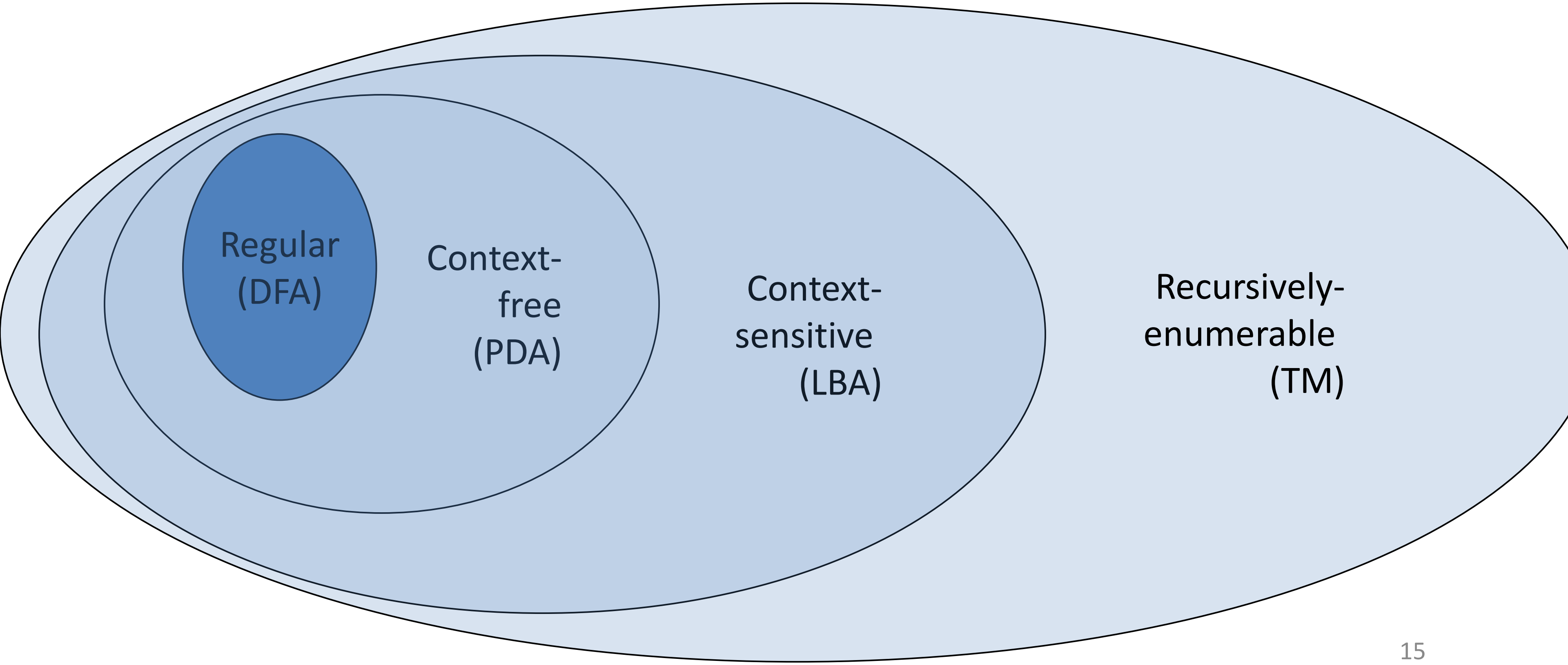
(Finite State Machines)

What is Automata Theory?

- *Study of abstract computing devices, or “machines”*
- Automaton = an abstract computing device
    - Note: A “device” need not even be a physical hardware!
- A fundamental question in computer science:
    - Find out what different models of machines can do and cannot do
    - The *theory of computation*
- Computability vs. Complexity

The Chomsky Hierachy

- A containment hierarchy of classes of formal languages



Notation and Definitions

- Alphabet
- String
- Language
- Operations on languages

Strings

- An alphabet is any finite set of distinct symbols
  - $\{0, 1\}$, $\{0,1,2,\ldots,9\}$, $\{a,b,c\}$
  - We denote a generic alphabet by Σ
- A string is any finite-length sequence of elements of Σ.
- e.g., if $\Sigma = \{a, b\}$ then *a*, *aba*, *aaaa*, ..., *abababbaab* are some strings over the alphabet Σ

Strings

- The length of a string ω is the number of symbols in ω. We denote it by $|\omega|$. $|aba| = 3$.
- The symbol ϵ denotes a special string called the empty string
  - ϵ has length 0
- String concatenation
  - If $\omega = a_1, \ldots, a_n$ and $\nu = b_1, \ldots, b_m$ then $\omega \cdot \nu$ (or $\omega\nu$) $= a_1, \ldots, a_n b_1, \ldots, b_m$
  - Concatenation is associative with ϵ as the identity element.
- If $a \in \Sigma$, we use a^n to denote a string of n a's concatenated
  - $\Sigma = \{0, 1\}, 0^5 = 00000$
  - $a^0 = \epsilon$
  - $a^{n+1} = a^n a$

Strings

- The reverse of a string ω is denoted by ω^R.
  - $\omega^R = a_n, \ldots, a_1$
- A substring y of a string ω is a string such that $\omega = xyz$ with $|x|, |y|, |z| \geq 0$ and $|x| + |y| + |z| = |\omega|$
- If $\omega = xy$ with $|x|, |y| \geq 0$ and $|x| + |y| = |\omega|$, then x is prefix of ω and y is a suffix of ω.
  - For $\omega = abaab$,
    - ϵ, a, aba, and $abaab$ are some prefixes
    - ϵ, $abaab$, aab, and $baab$ are some suffixes.

Strings

- The set of all possible strings over Σ is denoted by Σ^*.
- We define $\Sigma^0 = \{\epsilon\}$ and $\Sigma^n = \Sigma^{n-1} \cdot \Sigma$
  - with some abuse of the concatenation notation applying to sets of strings now
- So $\Sigma^n = \{\omega | \omega = xy \text{ and } x \in \Sigma^{n-1} \text{ and } y \in \Sigma\}$
- $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots \Sigma^n \cup \cdots = \bigcup_0^\infty \Sigma^i$
  - Alternatively, $\Sigma^* = \{x_1 x_2 \ldots x_n | n \geq 0 \text{ and } x_i \in \Sigma \text{ for all } i\}$
- Φ denotes the empty set of strings $\Phi = \{\}$,
  - but $\Phi^* = \{\epsilon\}$

- Σ = { a, b, c}
- Σ^1 = { a, b, c}
- Σ^2 = { aa, ab, ac, ba, bb, bc, ca, cb, cc}

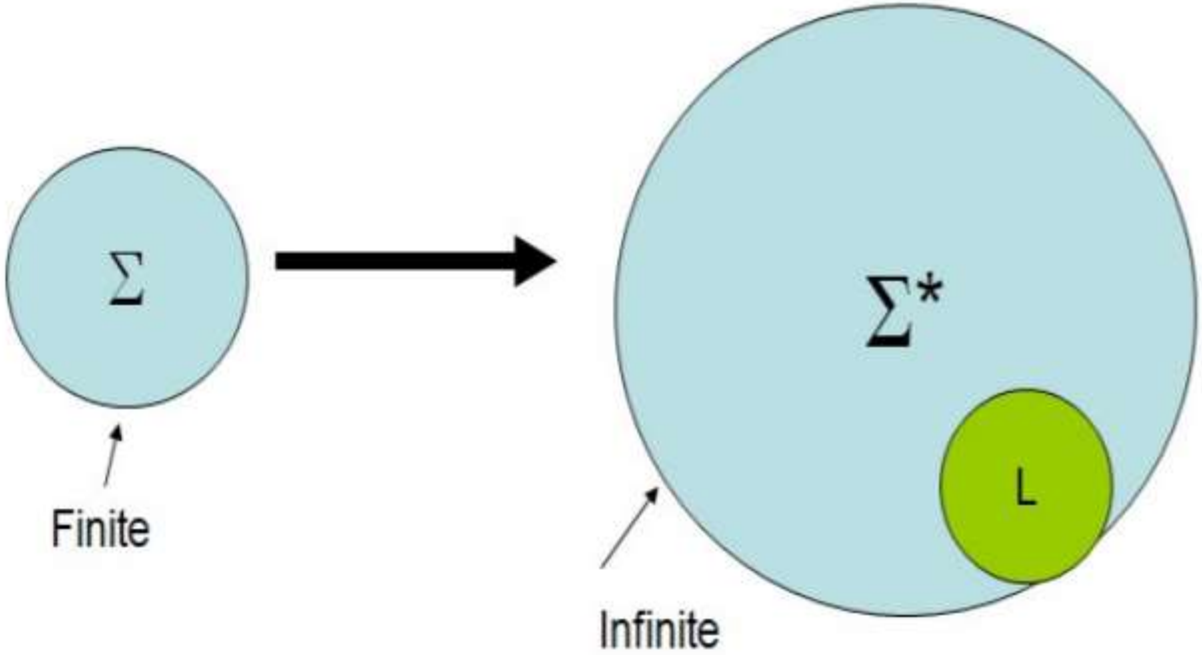
Σ^0 = ε for any Σ

Strings

- Σ^* is a countably infinite set of finite length strings
- If x is a string, we write x^n for the string obtained by concatenating n copies of x.
  - $(aab)^3 = aabaabaab$
  - $(aab)^0 = \epsilon$

Languages

- A language L over Σ is any subset of Σ^*



- L can be finite or (countably) infinite

Some Languages

- $L = \Sigma^*$ – The mother of all languages!
- $L = \{a, ab, aab\}$ – A fine finite language.
  - Description by enumeration
- $L = \{a^n b^n : n \geq 0\} = \{\epsilon, \text{ ab}, \text{ aabb}, \text{ aaabbb}, \ldots\}$
- $L = \{\omega | n_a(\omega) \text{ is even}\}$
  - $n_x(\omega)$ denotes the number of occurrences of x in ω
  - all strings with even number of a's.
- $L = \{\omega | \omega = \omega^R\}$
  - All strings which are the same as their reverses – palindromes.
- $L = \{\omega | \omega = xx\}$
  - All strings formed by duplicating some string once.
- $L = \{\omega | \omega$ is a syntactically correct Java program $\}$

Languages

- Since languages are sets, all usual set operations such as intersection and union, etc. are defined.
- Complementation is defined with respect to the universe Σ^* : $\overline{L} = \Sigma^* - L$

Languages

- If L, L_1 and L_2 are languages:
  - $L_1 \cdot L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
  - $L^0 = \{\epsilon\}$ and $L^n = L^{n-1} \cdot L$
  - $L^* = \bigcup_0^\infty L^i$
  - $L^+ = \bigcup_1^\infty L^i$

Sets of Languages

- The power set of Σ^*, the set of all its subsets, is denoted as 2^{Σ^*}