③ fourier Transform is a linear operator

$$\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha \mathcal{F}(f(x)) + \beta \mathcal{F}(g(x))$$

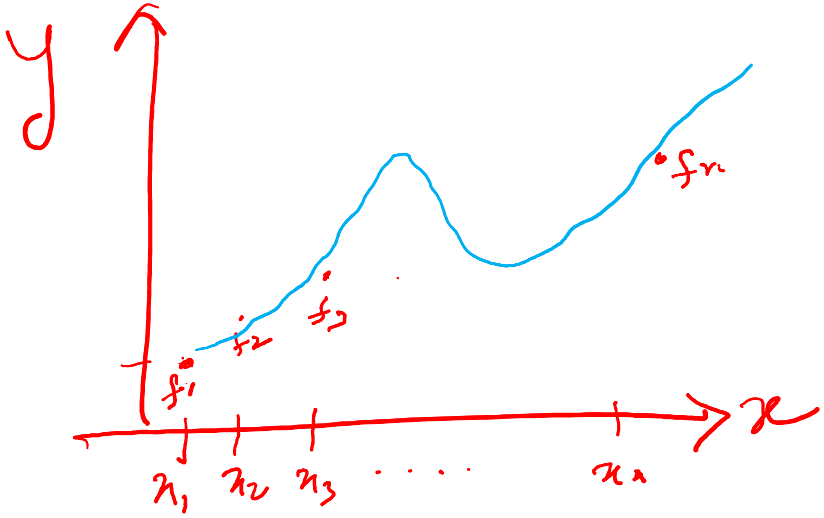
$$\mathcal{F}^{-1}(\alpha \hat{f}(\omega) + \beta \hat{g}(\omega)) = \alpha \mathcal{F}^{-1}(\hat{f}(\omega)) + \beta \mathcal{F}^{-1}(\hat{g}(\omega))$$

④ Parseval's theorem

$$\underbrace{\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega}_{} = \underbrace{2\pi \int_{-\infty}^{\infty} |f(x)|^2 \, dx}_{}$$

Discrete Fourier Transform

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_n \end{bmatrix}$$



$$\hat{f}_k = \sum_{j=0}^{n-1} f_j \, e^{-i2\pi jk/n}$$

$$f_k = \left[\sum_{j=0}^{n-1} \hat{f}_j \, e^{i2\pi jk/n} \right] \cdot \frac{1}{n}$$

$\{f_0, f_1, f_2 \ldots f_n\} \overset{DFT}{\Longrightarrow} \{\hat{f}_0, \hat{f}_1, \ldots \hat{f}_n\}$

$\hat{f}$:- how much of each freq. is required to reconstruct the data in f.

$$f_j \, \underbrace{e^{-i 2\pi j k / n}} \Rightarrow e^{-2\pi i / n} \Rightarrow \text{"}\omega_n\text{"} \to \text{fundamental freq.}$$

[approximation in form of sine & cosines with n discrete values]

$$\hat{f}_k = \sum_{j=0}^{n-1} f_j \, e^{-i 2\pi j k / n}$$

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_n \end{bmatrix} = \begin{array}{c|c} & j \\ \hline k & \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & & & & \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix} \end{array} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$

DFT $\longrightarrow$ Mathematical Transform

FFT $\longrightarrow$ computationally efficient way of computing DFT

FAST FOURIER TRANSFORM

$$DFT = [\;]_n[\;]_n \rightarrow \underline{O(n^2)}$$

$$FFT \longrightarrow O(n\underbrace{\log n}) \;\{$$

$$[\quad\quad]_{n^2} \longrightarrow n\log n$$

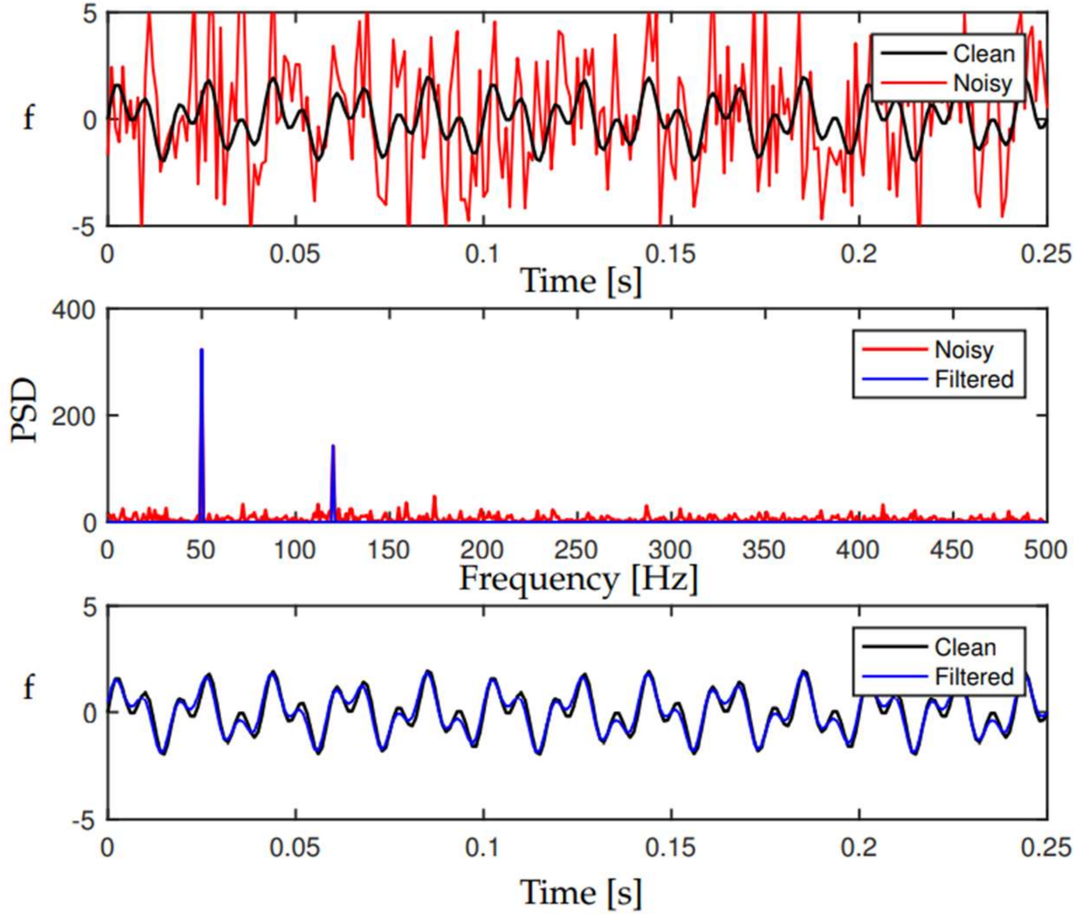
Ex $\hat{f} = F_{1024}\, f \qquad \{ n = 1024 \text{ or } 2^{10} \}$

$$= \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & 0 \\ 0 & F_{512} \end{bmatrix} \begin{bmatrix} f_{even} \\ f_{odds} \end{bmatrix} \qquad \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix}$$

$$D_{512} = \begin{bmatrix} 1 & 0 & & 0 \\ & \omega & & \\ 0 & & \omega^2 & 0 \\ & & \ddots & \\ 0 & & 0 & \omega^{512} \end{bmatrix}$$

$$\left\{ \underrightarrow{F_{1024} \rightarrow F_{512} \rightarrow F_{256} \rightarrow F_{128}} \rightarrow \cdots \rightarrow F_2 \right\}$$

$$F_2 = \begin{bmatrix} - & - \\ - & - \end{bmatrix}_{2\times 2}$$



Clean
Noisy
f
Time [s]
Noisy
Filtered
PSD
Frequency [Hz]
Clean
Filtered
f
Time [s]

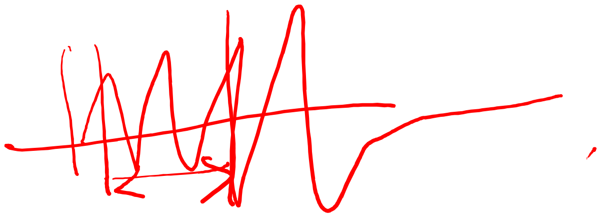
Wavelet Transform

FFT → basis f^n → cosine & sine

Signals which are non periodic, finite, discontinuous

EEG → non-stationary

① → Perform fourier analysis over short time windows known as Short time fourier transform (STFTs)



STFT

Wavelet transform

finite Basis function → wavelets → scaled over a

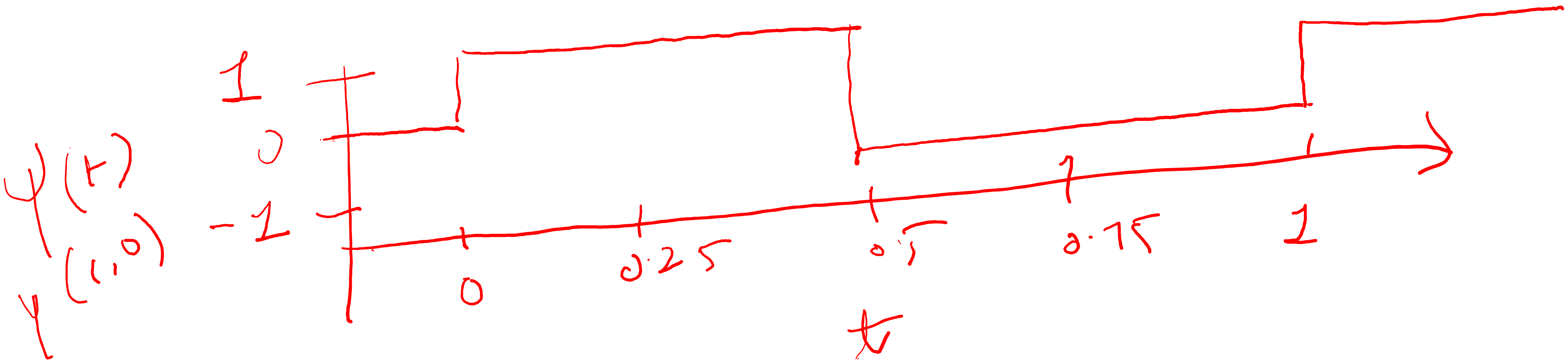
single finite length waveform → Mother Wavelets

$\psi(t) \rightarrow$ mother wavelet

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right)$$

Haar Wavelet

$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & -\frac{1}{2} \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Continuous WT

$$W_\psi(f)(a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(t)\, \underline{\psi_{a,b}}(t)\, dt$$

Inverse CWT

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_\psi(f)(a,b)\, \underline{\psi_{a,b}}(t)\, \frac{1}{a^2}\, da\, db$$

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\psi(\omega)|^2}{|\omega|}\, d\omega$$

DWT

$$D_{\psi}(f)(j,k) = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(t)\, \psi_{j,k}(t)\, dt$$

$$\psi_{j,k}(t) = \frac{1}{a_0^{j}} \psi\left(\frac{\theta - kb}{a_0^{j}}\right)$$