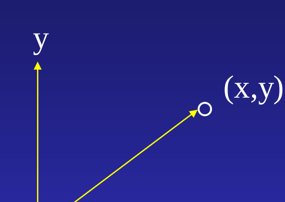
2 - D Transformation

1

Representation of Points



A point (x, y) is represented in 2 – D as a vector : $\begin{bmatrix} x \\ y \end{bmatrix}$

2

Transformation of Points

The initial coordinates x and y are transformed by the matrix **T**, to x^* and y^*.

$$\mathbf{TX} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

3

Identity

When the transformation matrix, **T**, is the identity matrix there is no change in the points, x, y.

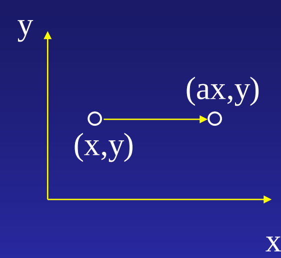
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

4

Scaling

x – scaling

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} ax \\ y \end{bmatrix}$$

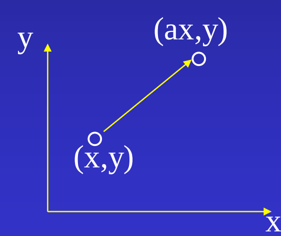


y – scaling

$$\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} x \\ dy \end{bmatrix}$$

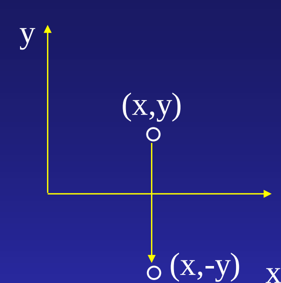
combined x & y scaling

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} ax \\ dy \end{bmatrix}$$



5

Reflection



reflection about x – axis

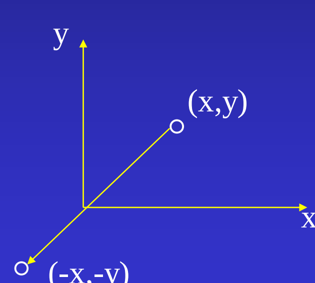
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

reflection about y – axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

reflection through origin

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

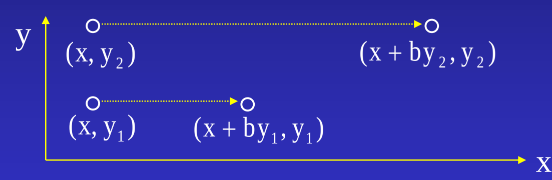


6

Shear

So far all transformations have been on the main diagonal.

Shear, x – coordinate $$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} x + by \\ y \end{bmatrix}$$



Shear, y – coordinate $$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} x \\ cx + y \end{bmatrix}$$

7

Origin

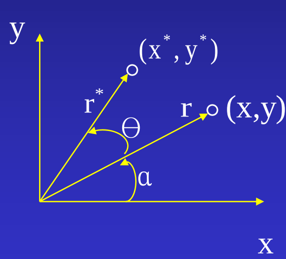
The origin is invariant to these types of trnasformations.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

8

Rotation – 1

Rotation about the origin:



in polar coordinates:

$$x = r\cos\alpha \qquad y = r\sin\alpha$$

$$x^* = r\cos(\alpha + \theta) \qquad y^* = r\sin(\alpha + \theta)$$

where: length of $\vec{r} = \|r\| = r$

Rotation – 2

using some trigonometry

$$\begin{aligned} x^* &= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ &= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta \\ &= x\cos\theta - y\sin\theta \\ &= (\cos\theta)\,x - (\sin\theta)\,y \end{aligned}$$

$$\begin{aligned} y^* &= r\cos\alpha\sin\theta + r\sin\alpha\cos\theta \\ &= (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta \\ &= (\sin\theta)\,x + (\cos\theta)\,y \end{aligned}$$

rotation transformation

$$\begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

Sequential Transformations

Since matrix multiplication is non-commutative, i.e. AB≠BA (in general) $\Rightarrow$ the order of transformation is important

i.e $$\mathbf{T}_b\,\mathbf{T}_a\begin{bmatrix} x \\ y \end{bmatrix} \neq \mathbf{T}_a\,\mathbf{T}_b\begin{bmatrix} x \\ y \end{bmatrix}$$

11

Example

transformations – rotation, $\theta = 135^\circ$

reflection thru x – axis

$$\text{let}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

rotate then reflect

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$$

reflect then rotate

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

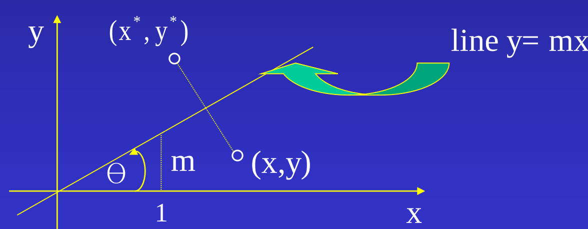
12

Combined Operations – 1

We can use combined operations to determine a more complex transformation.

Reflect about a line thru the origin:

- know how to reflect across axes
- transform line onto x-axis then reflect



13

Combined Operations – 2

1. rotate $(-\Theta)$

$$\mathbf{T}_1 = \frac{1}{\sqrt{m^2+1}}\begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix}$$

$$\sin\Theta = \frac{m}{\sqrt{m^2+1}}$$

$$\cos\Theta = \frac{1}{\sqrt{m^2+1}}$$

2. reflect about $x-$axis

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

3. rotate back (Θ)

$$\mathbf{T}_3 = \frac{1}{\sqrt{m^2+1}}\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}$$

14

Combined Operations – 3

Total transformation: $T = T_3 T_2 T_1$

$$T = \frac{1}{m^2+1}\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix}$$

$$= \frac{1}{m^2+1}\begin{bmatrix} -m^2+1 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

let the line be $x = y \Rightarrow m = 1$:

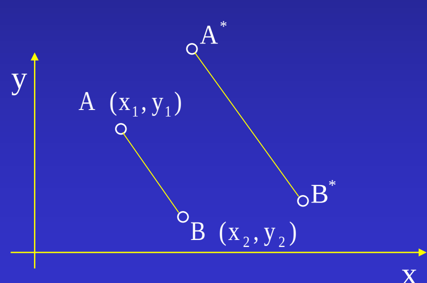
$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

15

Transformation of Lines – 1

Two point define a line,
to transform a line segment → transform end points

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} ax_1+by_1 & ax_2+by_2 \\ cx_1+dy_1 & cx_2+dy_2 \end{bmatrix}$$



Notice every point on *AB* has a corresponding transformed point on *A*B**

16

Transformation of Lines – 2

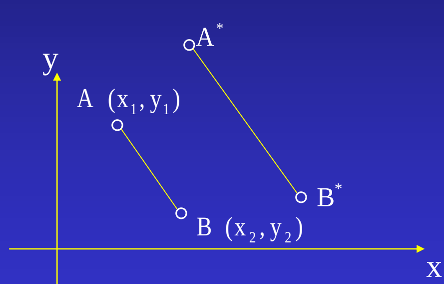
Properties of linear transformations of lines:

1. Straight lines remain straight.
2. Parallel lines remain parallel.
3. Midpoint of line transforms to midpoint.
4. Point of intersection of two lines transforms to point of intersection of transformed lines.

17

Parallel Lines – 1

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix}$$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m^* = \frac{(cx_2 + dy_2) - (cx_1 + dy_1)}{(ax_2 + by_2) - (ax_1 + by_1)}$$

18

Parallel Lines – 2

$$\mathbf{m}^{*}=\frac{(cx_2+dy_2)-(cx_1+dy_1)}{(ax_2+by_2)-(ax_1+by_1)}$$

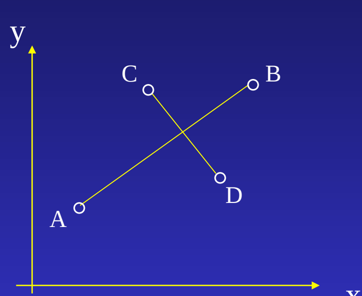
$$=\frac{c(x_2-x_1)+d(y_2-y_1)}{a(x_2-x_1)+b(y_2-y_1)}$$

$$=\frac{c+d\dfrac{(y_2-y_1)}{(x_2-x_1)}}{a+b\dfrac{(y_2-y_1)}{(x_2-x_1)}}=\frac{c+d\mathbf{m}}{a+b\mathbf{m}}$$

Notice the transformed slope, m*, depends only on original slope – not endpoints. So parallel lines transform to parallel lines.

19

Perpendicular Lines



$$\mathbf{m_{CD}=\frac{-1}{m_{AB}}}\qquad \therefore\ \mathbf{m_{AB}m_{CD}=-1}$$

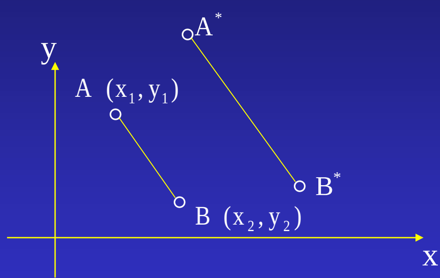
$$\mathbf{m^{*}{}_{AB}=\frac{c+dm_{AB}}{a+bm_{AB}}}$$

$$\mathbf{m^{*}{}_{CD}=\frac{c+dm_{CD}}{a+bm_{CD}}=\frac{c+d\dfrac{(-1)}{m_{AB}}}{a+b\dfrac{(-1)}{m_{AB}}}}$$

In general, $\mathbf{m^{*}_{AB}m^{*}_{CD}\neq -1}$. Therefore perpendicular lines do not remain perpendicular after transformation.

20

Midpoint – 1



$$\text{midpoint of } AB = \begin{bmatrix} \dfrac{x_1 + x_2}{2} \\ \dfrac{y_1 + y_2}{2} \end{bmatrix}$$

$$\text{midpoint of } A^*B^* = \begin{bmatrix} \dfrac{x_1^* + x_2^*}{2} \\ \dfrac{y_1^* + y_2^*}{2} \end{bmatrix}$$

21

Midpoint – 2

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \dfrac{x_1 + x_2}{2} \\ \dfrac{y_1 + y_2}{2} \end{bmatrix} = \begin{bmatrix} \dfrac{a}{2}(x_1 + x_2) + \dfrac{b}{2}(y_1 + y_2) \\ \dfrac{c}{2}(x_1 + x_2) + \dfrac{d}{2}(y_1 + y_2) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{2}(ax_1 + by_1 + ax_2 + by_2) \\ \dfrac{1}{2}(cx_1 + dy_1 + cx_2 + dy_2) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{x_1^* + x_2^*}{2} \\ \dfrac{y_1^* + y_2^*}{2} \end{bmatrix}$$

22

Pure Translation

So far, all transformations have rotated, stretched, reflected, etc, but none have accomplished pure translation.
All transformations have yielded:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Can not get something like:

$$\begin{bmatrix} ax + n \\ dy + m \end{bmatrix}$$

Also, recall that the origin is invariant under this type of transformation.

23

Homogeneous Coordinates – 1

Homogeneous coordinate representation allows transformation of n–dimensional vectors in (n+1)–dimensional space. The transformed n–dimensional results are obtained by projecting the transformed (n+1)–dimensional point back into the n–dimensional space of interest.

ordinary coordinate $\quad$ homogeneous coordinate

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} hx \\ hy \\ h \end{bmatrix} = \begin{bmatrix} X \\ Y \\ H \end{bmatrix}$$

24

Homogeneous Coordinates – 2

Homogeneous coordinates are not unique.

$$\begin{bmatrix}12\\8\\4\end{bmatrix}, \begin{bmatrix}6\\4\\2\end{bmatrix} \text{ and } \begin{bmatrix}3\\2\\1\end{bmatrix}$$

all represent the same cartesian coordinate :

$$\begin{bmatrix}3\\2\end{bmatrix}$$

Homogeneous Coordinates – 3

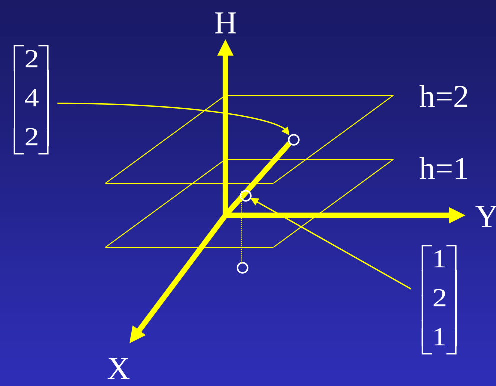
The cartesian coordinate x–y plane can be represented by the h=1 plane in a homogeneous coordinate representation. To obtain the cartesian coordinate which is represented by a homogeneous coordinate divide each entry by h.

homogeneous cartesian

$$\begin{bmatrix}2\\4\\2\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

This can be viewed as a projection onto the h=1 plane.

<u>Homogeneous Coordinates</u> – 4



- Thus there are infinite number of homogeneous coordinates which represent a given point in cartesian coordinates.

27

<u>Homogeneous Coordinates</u> – 5

Consider the homogeneous coordinate:

$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ by definition the ordinary cartesian coordinate is:

$$\begin{bmatrix} \frac{x}{0} \\ \frac{y}{0} \end{bmatrix} \Rightarrow \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$ the point at infinity!

This can be thought of as the point on the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ at ∞

Recall – computers don't like infinity so that gives a method to handle it.

28

General 2 – D transformations

$$\begin{bmatrix} a & b & m \\ c & d & n \\ p & q & s \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by+m \\ cx+dy+n \\ px+qy+s \end{bmatrix}$$

Thus the ordinary phsical coordinates :

$$x^* = \frac{ax+by+m}{px+qy+s} \qquad y^* = \frac{cx+dy+n}{px+qy+s}$$

Four of these element a, b, c and d have all ready been discussed. The remaining elements will be considered next.

Pure Translations

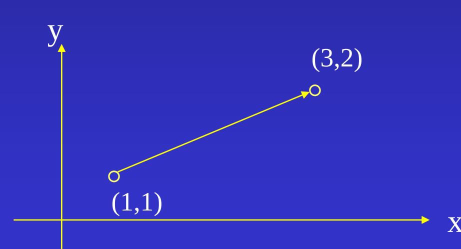
This 3 dimensional transformation allows us to perform translation in 2 – space.

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{m} \\ \mathbf{0} & \mathbf{1} & \mathbf{n} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{x+m} \\ \mathbf{y+n} \\ \mathbf{1} \end{bmatrix}$$

Example: want to move the point (1,1) to (3,2)

i.e. 2 units in x and y unit in y

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$



Overall Scaling

The s term controls overall uniform scaling :

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{s} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{s} \end{bmatrix} \Rightarrow \quad \mathbf{x}^* = \frac{\mathbf{x}}{\mathbf{s}} \quad \mathbf{y}^* = \frac{\mathbf{y}}{\mathbf{s}}$$

if $s < 1$ ⇨ enlarge

$s > 1$ ⇨ reduce

distinct from scaling with a & d, these are called "local" scaling

All points end up on the **h**=s plane must projected back onto **h**=1 plane

Projection

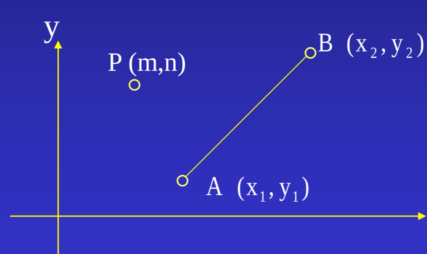
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & q & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ px + qy + 1 \end{bmatrix} \Rightarrow \quad x^* = \frac{x}{px + qy + 1} \quad y^* = \frac{y}{px + qy + 1}$$

So, a transformed point lands on the plane **h**=p**x**+q**y**+1 and must be projected back onto **h**=1.

i.e. each (**x,y**) ends up on a different h plane

Combined Operations – 1

Consider rotation about a point other than the origin.



Steps :

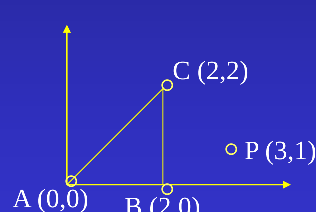
1. translate **P** to origin
2. rotate **AB** about **P**
3. translate back

Combined Operations – 2

$$T = \begin{bmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -m \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Theta & -\sin\Theta & -m(\cos\Theta - 1) + n\sin\Theta \\ \sin\Theta & \cos\Theta & -m\sin\Theta - n(\cos\Theta - 1) \\ 0 & 0 & 1 \end{bmatrix}$$

Example:



rotate 90°

about **P**

Combined Operations – 3

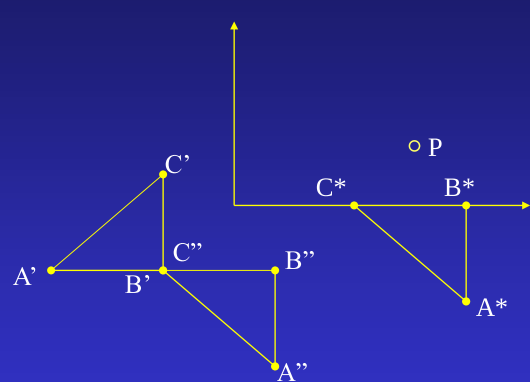
$$1.\ \begin{bmatrix}1&0&-3\\0&1&-1\\0&0&1\end{bmatrix}\begin{array}{c}\begin{matrix}A&B&C\end{matrix}\\\begin{bmatrix}0&2&2\\0&1&2\\1&1&1\end{bmatrix}\end{array}=\begin{array}{c}\begin{matrix}A'&B'&C'\end{matrix}\\\begin{bmatrix}-3&-1&-1\\-1&-1&1\\1&1&1\end{bmatrix}\end{array}$$

$$2.\ \begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}-3&-1&-1\\-1&-1&1\\1&1&1\end{bmatrix}=\begin{array}{c}\begin{matrix}A''&B''&C''\end{matrix}\\\begin{bmatrix}1&1&-1\\-3&-1&-1\\1&1&1\end{bmatrix}\end{array}$$

$$3.\ \begin{bmatrix}1&0&3\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&-1\\-3&-1&-1\\1&1&1\end{bmatrix}=\begin{array}{c}\begin{matrix}A^*&B^*&C^*\end{matrix}\\\begin{bmatrix}4&4&2\\-2&0&0\\1&1&1\end{bmatrix}\end{array}$$

35

Combined Operations – 4



36

Conclusion

$$T = \left[\begin{array}{cc|c} a & b & m \\ c & d & n \\ \hline p & q & s \end{array}\right]$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ local scaling, shear, rotation about origin (stretching)

$\begin{bmatrix} m \\ n \end{bmatrix} \Rightarrow$ translation

$[s] \Rightarrow$ overall scaling

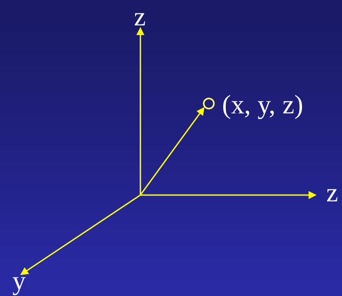
$[p \quad q] \Rightarrow$ projection

37

3 - D Transformation

38

4 – D Homogeneous Coordinates



cartesian $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

homogeneous $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

$$\begin{bmatrix} a & b & c & l \\ d & e & f & m \\ g & h & i & n \\ p & q & r & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ H \end{bmatrix}$$

project back onto $h = 1$ $\Rightarrow$

$$\begin{bmatrix} x^* = \dfrac{X}{H} \\ y^* = \dfrac{Y}{H} \\ z^* = \dfrac{Z}{H} \\ 1 \end{bmatrix}$$

39

General 3 – D Transformation

$$T = \left[\begin{array}{ccc:c} a & b & c & l \\ d & e & f & m \\ g & h & i & n \\ \hdashline p & q & r & s \end{array}\right]$$

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow$ **local scaling, shear, rotate and reflect**

$\begin{bmatrix} l \\ m \\ n \end{bmatrix} \Rightarrow$ **translate**

$[p \quad q \quad r] \Rightarrow$ **perspective transforma tion**

$[s] \Rightarrow$ **overall scaling**

40

Local Scaling

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ ey \\ iz \\ 1 \end{bmatrix}$$

41

Overall Scaling

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ s \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \frac{x}{s} \\ \frac{y}{s} \\ \frac{z}{s} \\ 1 \end{bmatrix}$$

$$s < 1 \Rightarrow \text{enlarge}$$
$$s > 1 \Rightarrow \text{reduce}$$

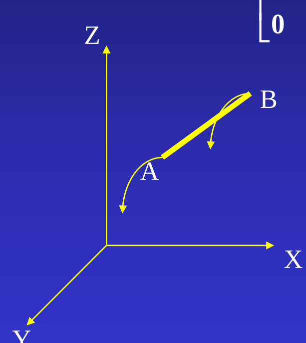
42

Shear

$$\begin{bmatrix} 1 & b & c & 0 \\ d & 1 & f & 0 \\ g & h & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + by + cz \\ y + dx + fz \\ z + gx + hy \\ 1 \end{bmatrix}$$

Rotation about the x–axis

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



x coordinate

stays the same

Rotation about the y – axis

$$T = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

45

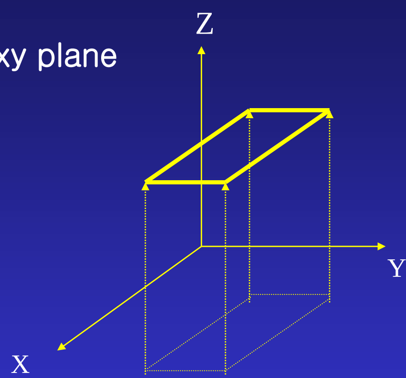
Rotation about the z – axis

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

46

Reflection thru Coordinate Plane

xy plane



$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: other reflections are similar

47

Translation

$$\begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + l \\ y + m \\ z + n \\ 1 \end{bmatrix}$$

48

<u>Combined Operation</u> – 1

Reflect thru z = 2 plane.

1. translate z = 2 to z = 0 (xy plane)
2. reflect thru xy plane
3. translate back

$$T = \overset{T_3}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{T_2}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{T_1}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

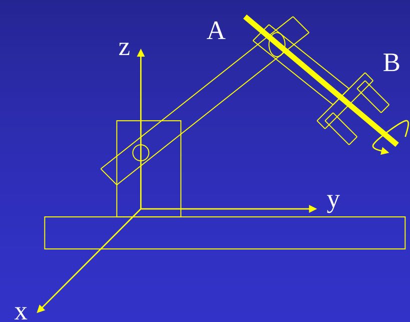
<u>Combined Operation</u> – 2

Rotate about a line parallel to the z – axis thru the point (l, m, n)

$$T = \overset{T_3}{\begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{T_2}{\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{T_1}{\begin{bmatrix} 1 & 0 & 0 & -l \\ 0 & 1 & 0 & -m \\ 0 & 0 & 1 & -n \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Rotation about Arbitrary Axes – 1

Rotation about an arbitrary axes in space are frequently found in engineer application.



51

Rotation about Arbitrary Axes – 2

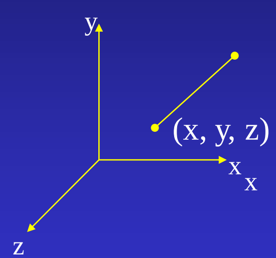
Sequence of steps:

1. Translate the arbitrary axis so that one of its end and points coincides with the origin.
2. Rotate about the x and y axes to align the arbitrary axis with the positive z – axis.
3. Rotate about the z – axis by the desired angle, θ
4. Apply reverse rotations about y and x axes to bring the arbitrary axis back to its original position with respect to the origin.
5. Apply reverse translations to place the arbitrary axis back in its initial position.

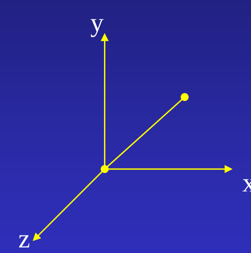
52

Step 1

Translate the arbitrary axis so that one of its end points coincides with the origin.



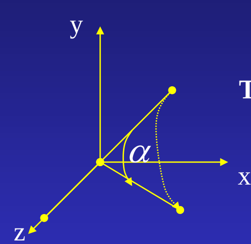
$$T_{tr} = \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



53

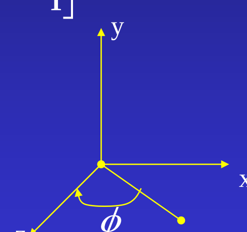
Step 2

Rotate about the x and y axes to align the arbitrary axis with the positive z – axis.



$$\mathbf{T_{rx}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\mathbf{T_{ry}(\varphi) = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$



54

Step 3

Rotate about the z – axis by the desired angle, $\boldsymbol{\theta}$

$$T_{rz}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Concatenation

Steps 4 and 5 use the similar transformations as steps 2 and 1 respectively.

$$T_{rarb} =$$

$$T_{tr}(x, y, z)\, T_{rx}(-\alpha)\, T_{ry}(-\varphi) T_{rz}(\theta) T_{ry}(\varphi)\, T_{rx}(\alpha)\, T_{tr}(-x, -y, -z)$$