On Positivity and deriving Kraus operators of a general Quantum dynamical Map

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Tutors:

Abstract

We are given a linear qubit map

$$\rho_{11}(t) = A(t)\rho_{11}(0) + B(t)\rho_{22}(0)$$
$$\rho_{12}(t) = C(t)\rho_{12}(0)$$

Our task is to investigate the conditions for positivity and complete positivity of the map. For a given completely postive map we must derive the kraus operators and determine whether the given map is invertible or not.

Keywords:

Positivity, Complete Positivity, Kraus operators, invertibility.

1. Derivation of full map using properties of a density matrix

From the trace preserving property we know

$$Tr[\rho(t)] = Tr[\rho(0)]$$
$$\implies \rho_{11}(t) + \rho_{22}(t) = \rho_{11}(0) + \rho_{22}(0)$$
$$\implies A(t)\rho_{11}(0) + B(t)\rho_{22}(0) + \rho_{22}(t) = \rho_{11}(0) + \rho_{22}(0)$$
$$\implies \rho_{22}(t) = (1 - A(t))\rho_{11}(0) + (1 - B(t))\rho_{22}(0)$$

From hermiticity property we have,

$$
\begin{gathered}
\rho(t)^{\dagger} = \rho(t) \\
and,\ \rho(0)^{\dagger} = \rho(0) \\
\implies \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix}^{\dagger} = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix} \\
\implies \begin{pmatrix} \rho_{11}(t)^{\dagger} & \rho_{21}(t)^{\dagger} \\ \rho_{12}(t)^{\dagger} & \rho_{22}(t)^{\dagger} \end{pmatrix} = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix} \\
\implies \rho_{11}(t)^{\dagger} = \rho_{11}(t)\ and\ \rho_{22}(t)^{\dagger} = \rho_{22}(t) \\
and\ \rho_{12}(t)^{\dagger} = \rho_{21}(t)\ and\ \rho_{21}(t)^{\dagger} = \rho_{12}(t)
\end{gathered}
$$

and we have,

$$
\begin{gathered}
\rho(0)^{\dagger} = \rho(0) \\
\implies \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix}^{\dagger} = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix} \\
\implies \begin{pmatrix} \rho_{11}(0)^{\dagger} & \rho_{21}(0)^{\dagger} \\ \rho_{12}(0)^{\dagger} & \rho_{22}(0)^{\dagger} \end{pmatrix} = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix} \\
\implies \rho_{11}(0)^{\dagger} = \rho_{11}(0)\ and\ \rho_{22}(0)^{\dagger} = \rho_{22}(0) \\
and\ \rho_{12}(0)^{\dagger} = \rho_{21}(0)\ and\ \rho_{21}(0)^{\dagger} = \rho_{12}(0)
\end{gathered}
$$

From the above two equations we get,

$$
\begin{gathered}
\rho_{12}(0)^{\dagger} = \rho_{21}(0)\ and\ \rho_{21}(0)^{\dagger} = \rho_{12}(0) \\
and\ \rho_{12}(t)^{\dagger} = \rho_{21}(t)\ and\ \rho_{21}(t)^{\dagger} = \rho_{12}(t) \\
\implies \rho_{21}(t) = \rho_{12}(t)^{\dagger} = (C(t)\rho_{12}(0))^{\dagger} \\
\implies \rho_{21}(t) = C^{*}(t)\rho_{12}(0)^{\dagger} = C^{*}\rho 21(0)
\end{gathered}
$$

and $A^{*}(t)$ and $B * (t) = A(t)$ and $B(t)$ respectively
Thus $A(t)$ and $B(t)$ must be real numbers.
Here † is the conjugate transpose operation and * is the complex conjugation operation.

Thus the final map is as follows:

$$\Lambda(\rho) = \begin{pmatrix} A(t)\rho_{11}(0) + B(t)\rho_{22}(0) & C(t)\rho_{12}(0) \\ C^{*}(t)\rho_{21}(0) & (1 - A(t))\rho_{11}(0) + (1 - B(t))\rho_{22}(0) \end{pmatrix}$$

2. Conditions for Positivity Of Linear Quantum Dynamical Map

A positive map is a linear transformation which takes a positive semi-definite matrix to a positive semi-definite matrix.
We can verify if the above map is linear by applying it to a general density matrix and checking whether the resulting density matrix is positive semi-definite.
A hermitian matrix is positive semi-definite if and only if all of it's principal minors are non-negative.A principal minor of a square matrix is one where the indices of the deleted rows are the same as the indices of the deleted columns. For a 2x2 matrix:

1. Deleting nothing leading to the determinant of the matrix.

2. Deleting one row and corresponding column leading to the diagonal elements

Thus for our map, the principal minors are:

$$\Lambda(\rho) = \begin{pmatrix} A(t)\rho_{11}(0) + B(t)\rho_{22}(0) & C(t)\rho_{12}(0) \\ C^{*}(t)\rho_{21}(0) & (1 - A(t))\rho_{11}(0) + (1 - B(t))\rho_{22}(0) \end{pmatrix}$$
$$\implies A(t)\rho_{11}(0) + B(t)\rho_{22}(0) \geq 0,$$
$$(1 - A(t))\rho_{11}(0) + (1 - B(t))\rho_{22}(0) \geq 0,$$
$$(A(t)\rho_{11}(0)+B(t)\rho_{22}(0))*((1-A(t))\rho_{11}(0)+(1-B(t))\rho_{22}(0))-C(t)\rho_{12}(0)*C^{*}(t)\rho_{21}(0) \geq 0$$
$$\implies (A(t)\rho_{11}(0)+B(t)\rho_{22}(0))*((1-A(t))\rho_{11}(0)+(1-B(t))\rho_{22}(0)) \geq C(t)\rho_{12}(0)*C^{*}(t)\rho_{21}(0)$$

The above needs to be true if the original density matrix ρ is positive semi-definite.

Doing the same for ρ we get:

$$
\begin{aligned}
\rho(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix} \\
\implies \rho_{11}(0) \geq 0, \\
\rho_{22}(0) \geq 0, \\
\rho_{11}(0) * \rho_{22}(0) - \rho_{12}(0) * \rho_{21}(0) \geq 0 \\
\rho_{11}(0) * \rho_{22}(0) \geq \rho_{12}(0) * \rho_{21}(0)
\end{aligned}
$$

From the second equation we know that $\rho_{11}(0)$ and $\rho_{22}(0)$ are both positive. Thus,

$$
\begin{aligned}
A(t)\rho_{11}(0) + B(t)\rho_{22}(0) \geq 0, \\
and\ (1 - A(t))\rho_{11}(0) + (1 - B(t))\rho_{22}(0) \geq 0 \\
\implies A(t) \geq -B(t)\frac{\rho_{22}(0)}{\rho_{11}(0)} \\
and\ A(t) \leq \frac{\rho_{22}(0)}{\rho_{11}(0)} + 1 - B(t)\frac{\rho_{22}(0)}{\rho_{11}(0)}
\end{aligned}
$$

Thus $-B(t)\frac{\rho_{22}(0)}{\rho_{11}(0)} \leq A(t) \leq \frac{\rho_{22}(0)}{\rho_{11}(0)} + 1 - B(t)\frac{\rho_{22}(0)}{\rho_{11}(0)}$ is sufficient to make the first two principal minors positive. Now we consider,

$$(A(t)\rho_{11}(0) + B(t)\rho_{22}(0)) * ((1 - A(t))\rho_{11}(0) + (1 - B(t))\rho_{22}(0)) \geq C(t)\rho_{12}(0) * C^*(t)\rho_{21}(0)$$

Here, the LHS of the equation is the product of two positive terms and hence is always positive. Note that $C(t) * C^*(t) \geq 0$ always. Substituting the previously obtained condition for A(t),

$$
\begin{aligned}
(0)*((A(t)\rho_{11}(0)+B(t)\rho_{22}(0))*((1+B(t)\frac{\rho_{22}(0)}{\rho_{11}(0)})\rho_{11}(0)+(1-B(t))\rho_{22}(0))) \geq C(t)*C^*(t)\rho_{12}(0)\rho_{21}(0) \\
\implies |C(t)|^2\rho_{12}(0)\rho_{21}(0) \leq 0 \\
\implies either\ \rho_{12}(0)\rho_{21}(0) \leq 0.
\end{aligned}
$$

These are the conditions for positivity of the map.

3. Conditions for Complete Positivity Of Linear Quantum Dynamical Map

For a map Λ to be completely positive,$\Lambda \otimes I_p$ must be positive for all positive integers p.However, a sufficient condition for complete positivity is that corresponding Choi matrix must be positive (Choi, 1975). The Choi matrix for a given linear map Λ is

$$\mathbb{I} \otimes \Lambda(|\psi\rangle\langle\psi|) \geq 0$$

Where $|\psi\rangle$ is the d-dimensional maximally entangled state $(\frac{1}{\sqrt{d}}\Sigma_i^d|ii\rangle)$
Thus for our map $\Lambda(\rho)$, the Choi matrix is

$$C = \begin{pmatrix} \Lambda\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ \Lambda\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{A(t)}{2} & 0 & 0 & \frac{C(t)}{2} \\ 0 & \frac{1-A(t)}{2} & 0 & 0 \\ 0 & 0 & \frac{B(t)}{2} & 0 \\ \frac{C^*(t)}{2} & 0 & 0 & \frac{1-B(t)}{2} \end{pmatrix}$$

For this matrix to be positive semi-definite, it's eigenvalues must be non-negative. We find the eigenvalues as follows: eigen values for C can be obtained through:

$$|C - (\lambda)I| = 0$$

$$\begin{vmatrix} \frac{A(t)}{2} - \lambda & 0 & 0 & \frac{C(t)}{2} \\ 0 & \frac{1-A(t)}{2} - \lambda & 0 & 0 \\ 0 & 0 & \frac{B(t)}{2} - \lambda & 0 \\ \frac{C^*(t)}{2} & 0 & 0 & \frac{1-B(t)}{2} - \lambda \end{vmatrix} = 0$$

The determinant for the above matrix comes out to be

$$((B(t) - 2)(A(t) + 2\lambda - 1)(A(t)B(t) + 2A(t)\lambda - A(t) - 2B(t)\lambda + C(t)C^*(t) - 4\lambda^2 + 2\lambda)$$

Solving this equation we obtain the eigenvalues as

$$\begin{aligned}
\lambda_1 &= \frac{B(t)}{2} \\
\lambda_2 &= \frac{1-A(t)}{2} \\
\lambda_3 &= \frac{a-b}{4} - \frac{\sqrt{(A(t)+B(t))^2 - 2(A(t)+B(t)) + 4C(t)C^*(t)+1}}{4} + \frac{1}{4} \\
\lambda_4 &= \frac{a-b}{4} + \frac{\sqrt{(A(t)+B(t))^2 - 2(A(t)+B(t)) + 4C(t)C^*(t)+1}}{4} + \frac{1}{4}
\end{aligned}$$

As discussed, these 4 eigenvalues must be non-negative for the Choi matrix to be positive semidefinite.

$$\begin{aligned}
\lambda_1 \geq 0 &\implies \frac{B(t)}{2} \geq \implies B(t) \geq 0 \\
\lambda_2 \geq 0 &\implies \frac{1-A(t)}{2} \geq 0 \implies A(t) \leq 1 \\
\lambda_3 \geq 0 &\implies \frac{a-b}{4} + \frac{1}{4} \geq \frac{\sqrt{(A(t)+B(t))^2 - 2(A(t)+B(t)) + 4C(t)C^*(t)+1}}{4} \\
\lambda_4 \geq 0 &\implies \frac{a-b}{4} + \frac{1}{4} \geq -\frac{\sqrt{(A(t)+B(t))^2 - 2(A(t)+B(t)) + 4C(t)C^*(t)+1}}{4}
\end{aligned}$$

Substituting for boundary conditions, i.e, A(t) = 1 and B(t) = 0 in inequality for λ_3 we get:

$$\begin{aligned}
\frac{1-0}{4} + \frac{1}{4} &\geq \frac{\sqrt{(1+0)^2 - 2(1+0) + 4C(t)C^*(t)+1}}{4} \\
&\implies 2 \geq \sqrt{1 - 2 + 4C(t)C^*(t) + 1} \\
&\implies 2 \geq 2\sqrt{|C(t)|^2} \\
&|C(t)| \leq 1
\end{aligned}$$

Since $\lambda_3 \geq \implies \lambda_4 \geq 0$ as the square root is taken to be a positive value, the condition for C(t) is sufficient for both λ_3 and λ_4.

Squaring equations for λ_3 we get

$$
\begin{aligned}
(A(t)-B(t)+1)^2 &\geq (A(t)+B(t))^2-2(A(t)+B(t))+4C(t)C^*(t)+1\\
\implies A(t)^2+B(t)^2+1+2A(t)-2B(t)-2A(t)B(t) &\geq A(t)^2+B(t)^2+2A(t)B(t)-2A(t)-2B(t)+4C(t)C^*(t)+1\\
\implies 4A(t) &\geq 4A(t)B(t)+4C(t)C^*(t)\\
\implies A(t)(1-B(t)) &\geq |C(t)|^2
\end{aligned}
$$

Since $|C(t)|^2 \geq 0$ always, $A(t)(1-B(t)) \geq 0$ should hold true. Thus either, $A(t) \geq 0$ *and* $B(t) \leq 1$ or $A(t) \leq 0$ *and* $B(t) \geq 1$
We note that the principal minors must be non-negative for a positive semi-definite matrix. Thus:

$$
\begin{gathered}
0 \leq A(t) \leq 1,\\
0 \leq B(t) \leq 1,\\
\mathit{and}\ |C(t)| \leq 1
\end{gathered}
$$

Result obtained from squaring both sides of the λ_4 equation will be the same. This concludes the conditions for complete-positivity of our linear dynamic map.

4. Additional notes on Positivity and Complete Positivity

Positive maps may not be completely positive.
Consider the transposition operator, which is a positive linear map. However, when applied to the density matrix of a bell state we see that the resulting matrix is negative

$$
M = \begin{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}^T & \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}^T \\ \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}^T & \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^T \end{pmatrix}
$$

$$
= \begin{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{pmatrix}
$$

The principal minor corresponding to deleting the third row and third column yields the 3x3 matrix whose determinant is $\frac{1}{2}(0-\frac{1}{2}*\frac{1}{2}) = \frac{1}{2}*(-\frac{1}{4}) = -\frac{1}{8}$, and thus this matrix is not positive semi-definite.

An interesting application of positive maps is detecting entanglement which is difficult to do in higher dimension.
Horodeckis (Horodecki et al., 1996) gave a characterization of separability as: a quantum state is separable if and only if for every positive map, the resulting matrix is positive semi-definite.
Hence, to show that a quantum state ρ_{AB} is entangled, it suffices to find a single positive map ϕ who's partial action on the quantum state renders it non-positive semi-definite; such a map is called an entanglement witness, for instance the partial transposition operation.

5. Krauss Operators

Let the map given be Λ, then the Choi state for this map is:

$$C = \begin{pmatrix} \Lambda\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ \Lambda\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{A(t)}{2} & 0 & 0 & \frac{C(t)}{2} \\ 0 & \frac{1-A(t)}{2} & 0 & 0 \\ 0 & 0 & \frac{B(t)}{2} & 0 \\ \frac{C^*(t)}{2} & 0 & 0 & \frac{1-B(t)}{2} \end{pmatrix}$$

eigen values for C can be obtained through:

$$|C - (\lambda)I| = 0$$

$$\begin{vmatrix} \frac{A(t)}{2} - \lambda & 0 & 0 & \frac{C(t)}{2} \\ 0 & \frac{1-A(t)}{2} - \lambda & 0 & 0 \\ 0 & 0 & \frac{B(t)}{2} - \lambda & 0 \\ \frac{C^*(t)}{2} & 0 & 0 & \frac{1-B(t)}{2} - \lambda \end{vmatrix} = 0$$

Solving for eigen values we get:

$$\lambda_1 = \frac{B(t)}{2}$$

$$\lambda_2 = \frac{1 - A(t)}{2}$$

$$\lambda_3 = \frac{a-b}{4} - \frac{\sqrt{(A(t)+B(t))^2 - 2(A(t)+B(t)) + 4C(t)C^*(t) + 1}}{4} + \frac{1}{4}$$

$$\lambda_4 = \frac{a-b}{4} + \frac{\sqrt{(A(t)+B(t))^2 - 2(A(t)+B(t)) + 4C(t)C^*(t) + 1}}{4} + \frac{1}{4}$$

Corresponding eigen vectors:

$$\lambda_1 : \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 : \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 : \begin{pmatrix} \frac{-2C(t)}{A(t)+B(t)-1+\sqrt{(1-A(t)-B(t))^2+4|C(t)|^2}} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_4 : \begin{pmatrix} \frac{-2C(t)}{A(t)+B(t)-1-\sqrt{(1-A(t)-B(t))^2+4|C(t)|^2}} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

For each eigen value

$$\lambda_i$$

with corresponding eigen vector

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Krauss operator,

$$K_i = \sqrt{\lambda_i} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Hence we get,

$$K_1 = \sqrt{\frac{B(t)}{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$K_2 = \sqrt{\frac{1-A(t)}{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$K_3 = \sqrt{\lambda_3} \begin{pmatrix} \frac{-2C(t)}{A(t)+B(t)-1+\sqrt{(1-A(t)-B(t))^2+4|C(t)|^2}} & 0 \\ 0 & 1 \end{pmatrix}$$
$$K_4 = \sqrt{\lambda_4} \begin{pmatrix} \frac{-2C(t)}{A(t)+B(t)-1-\sqrt{(1-A(t)-B(t))^2+4|C(t)|^2}} & 0 \\ 0 & 1 \end{pmatrix}$$

Krauss operators obey the completeness condition,

$$\sum_{i=1}^{4} K_i^\dagger K_i = I$$

and it is straight forward to verify this. Note that A(t) and B(t) are real, and A(t)* = A(t) and B(t)* = B(t),

6. Conditions for Invertibility Of Linear Quantum Dynamical Map

Consider a convenient orthonormal set

$$\{G_a\}$$

,with properties,

$$G_a^\dagger = G_a$$

and

$$Tr(G_i G_j) = \delta_{ij}$$

.

Our Map can be represented as,

$$\Phi(\rho(0)) = [F(t)r(0)]G^T$$

$$F_{ij} = Tr(G_i \Lambda(G_j)) i = 0, 1, 2, 3$$

where

$$G_i = \{I/2, \sigma_x/2, \sigma_y/2, \sigma_z/2\}$$

$$F_{00} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})) == \frac{1}{2}Tr(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}A(t)+B(t) & 0\\ 0 & 2-A(t)-B(t)\end{pmatrix})$$

$$= \frac{1}{2}Tr(\begin{pmatrix}A(t)+B(t) & 0\\ 0 & 2-A(t)-B(t)\end{pmatrix}) = 1$$

$$F_{01} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \\ C(t)^* & 0\end{pmatrix}) = \frac{1}{2}Tr(\begin{pmatrix}0 & C(t)\\ C(t)^* & 0\end{pmatrix}) = 0$$

$$F_{02} \;=\; Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix})) \;=\; \frac{1}{2}Tr(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -iC(t)\\ iC(t)^* & 0\end{pmatrix}) \;=\; 0$$

$$F_{03} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}A(t)-B(t) & 0\\ 0 & B(t)-A(t)\end{pmatrix}) = 0$$

$$F_{10} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}A(t)+B(t) & 0\\ 0 & 2-A(t)-B(t)\end{pmatrix})$$

$$= Tr\begin{pmatrix}0 & 2-A(t)-B(t)\\ A(t)+B(t) & 0\end{pmatrix}) = 0$$

$$F_{11} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & C(t)\\ C^*(t) & 0\end{pmatrix})$$

$$= Tr\begin{pmatrix}C^*(t) & 0\\ 0 & C\end{pmatrix}) = \frac{C^*(t)+C(t)}{2}$$

$$F_{12} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -iC(t)\\ iC^*(t) & 0\end{pmatrix})$$

$$= Tr\begin{pmatrix}iC^*(t)t & 0\\ 0 & -iC(t)\end{pmatrix}) = \frac{iC^*(t)-iC(t)}{2}$$

$$F_{13} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}A(t)-B(t) & 0\\ 0 & B(t)-A(t)\end{pmatrix})$$

$$= Tr\begin{pmatrix}0 & -\\ - & 0\end{pmatrix}) = 0$$

$$F_{20} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}A(t)+B(t) & 0\\ 0 & 2-B(t)-A(t)\end{pmatrix})$$

$$= Tr\begin{pmatrix}0 & -\\ - & 0\end{pmatrix}) = 0$$

$$F_{21} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & C(t)\\ C^*(t) & 0\end{pmatrix})$$
$$= Tr\begin{pmatrix}-iC^* & 0\\ 0 & iC(t)\end{pmatrix}) = \frac{-iC^*(t) + iC(t)}{2}$$

$$F_{22} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -iC(t)\\ iC^*(t) & 0\end{pmatrix})$$
$$= Tr\begin{pmatrix}C^* & 0\\ 0 & C(t)\end{pmatrix}) = \frac{C^*(t) + C(t)}{2}$$

$$F_{23} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}A(t) - B(t) & 0\\ 0 & -A(t) + B(t)\end{pmatrix})$$
$$= 0$$

$$F_{30} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}A(t) + B(t) & 0\\ 0 & 2 - A(t) - B(t)\end{pmatrix})$$
$$= Tr\begin{pmatrix}A(t) + B(t) & 0\\ 0 & A(t) + B(t) - 2\end{pmatrix} = A(t) + B(t) - 1$$

$$F_{31} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & C(t)\\ C^*(t) & 0\end{pmatrix}) = 0$$

$$F_{32} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -iC(t)\\ iC^*(t) & 0\end{pmatrix}) = 0$$

$$F_{33} = Tr(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Lambda(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix})) = \frac{1}{2}Tr(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}A(t) - B(t) & 0\\ 0 & B(t) - A(t)\end{pmatrix})$$
$$= Tr\begin{pmatrix}A(t) - B(t) & 0\\ 0 & A(t) - B(t)\end{pmatrix} = A(t) - B(t)$$

$$F(t) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{C^*(t)+C(t)}{2} & \frac{iC^*(t)-iC(t)}{2} & 0\\ 0 & \frac{-iC^*(t)+iC(t)}{2} & \frac{C^*(t)+C(t)}{2} & 0\\ A(t) + B(t) - 1 & 0 & 0 & A(t) - B(t)\end{pmatrix}$$

F(t) must be invertible for the operation to be invertible.Hence F(t) shouldn't be singular for the operation to be invertible.

$$Det(F(t)) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C^*(t)+C(t)}{2} & \frac{iC^*(t)-iC(t)}{2} & 0 \\ 0 & \frac{-iC^*(t)+iC(t)}{2} & \frac{C^*(t)+C(t)}{2} & 0 \\ A(t)+B(t)-1 & 0 & 0 & A(t)-B(t) \end{vmatrix}$$

$$Det(F(t)) = |C|^2(A(t) - B(t))$$

For the operation to invertible, A(t) != B(t) and C(t) != 0.

7. Conclusion

In this report, we determined the conditions for positivity and complete positivity of the linear quantum dynamical map. Additionally, we derived it's terms from trace preserving property and hermiticity of density matrices.

$$\rho_{11}(t) = A(t)\rho_{11}(0) + B(t)\rho_{22}(0)$$
$$\rho_{12}(t) = C(t)\rho_{12}(0)$$

We derived it's kraus operators and found the conditions for this map to be invertible.

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