5 Let G be a group with a finite number of elements. Show that there must be an odd number of elements $x \in G$ with the property that $x^3 = e$.

Proof: Since $e^3 = e$ we know that e has this property. Suppose that some $x \neq e$ has this property, so $x^3 = e$. Then $(x^{-1})^3 = (x^3)^{-1} = e^{-1} = e$ so x^{-1} has this property too. Moreover $x \neq x^{-1}$ since if this were so we would have $x^2 = e$ and this, coupled with $x^3 = e$, gives $x = e$.

Thus e has this property and the other elements with this property are in pairs, meaning there are an odd number of elements with this property.

6 Prove that a group G is Abelian iff $\forall a, b \in G$ we have $(ab)^2 = a^2b^2$.

Proof: Assume G is Abelian, we will prove that $\forall a, b \in G$ we have $(ab)^2 = a^2b^2$. Let $a, b \in G$. Then $(ab)^2 = abab = aabb = a^2b^2$.

For the converse assume that $\forall a, b \in G$ we have $(ab)^2 = a^2b^2$, we will show G is Abelian. Let $a, b \in G$. Then

$$\begin{aligned}
(ab)^2 &= a^2b^2 \\
abab &= aabb \\
a^{-1}ababb^{-1} &= a^{-1}aabbb^{-1} \\
ba &= ab
\end{aligned}$$