Chapter 6

Laplace Transforms

6.1 Laplace Transform, Linearity, First Shifting Theorem (s-Shifting)

1. Finding Laplace transform,

$$f(t) = 3t + 12 \tag{6.1.1}$$

$$\mathscr{L}\{f(t)\} = 3\,\mathscr{L}\{t\} + 12\,\mathscr{L}\{1\} \tag{6.1.2}$$

$$= \frac{3 + 12s}{s^2} \tag{6.1.3}$$

2. Finding Laplace transform,

$$f(t) = (a - bt)^2 \tag{6.1.4}$$

$$\mathscr{L}\{f(t)\} = b^2 Lap\{t^2\} - 2ab\,\mathscr{L}\{t\} + a^2\,\mathscr{L}\{1\} \tag{6.1.5}$$

$$= \frac{(a^2)s^2 - (2ab)s + 2b^2}{s^3} \tag{6.1.6}$$

3. Finding Laplace transform,

$$f(t) = \cos(\pi t) \tag{6.1.7}$$

$$\mathscr{L}\{f(t)\} = \frac{s}{s^2 + \pi^2} \tag{6.1.8}$$

4. Finding Laplace transform,

$$f(t) = \cos^2(\omega t) \tag{6.1.9}$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{\frac{1+\cos(2\omega t)}{2}\right\} \tag{6.1.10}$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+4\omega^2)} \tag{6.1.11}$$

$$= \frac{s^2+2\omega^2}{s(s^2+4\omega^2)} \tag{6.1.12}$$

5. Finding Laplace transform,

$$f(t) = e^{2t}\sinh(t) \tag{6.1.13}$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{\frac{e^{3t}-e^{t}}{2}\right\} \tag{6.1.14}$$

$$= \frac{1}{(s-1)(s-3)} \tag{6.1.15}$$

6. Finding Laplace transform,

$$f(t) = e^{-t}\sinh(4t) \tag{6.1.16}$$

$$\mathscr{L}\{f(t)\} = \frac{4}{(s+1)^2-16} \tag{6.1.17}$$

7. Finding Laplace transform,

$$f(t) = \sin(\omega t+\theta) \tag{6.1.18}$$

$$\mathscr{L}\{f(t)\} = \cos\theta\;\; \mathscr{L}\{\sin(\omega t)\} + \sin\theta\;\; \mathscr{L}\{\cos(\omega t)\} \tag{6.1.19}$$

$$= \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2} \tag{6.1.20}$$

8. Finding Laplace transform,

$$f(t) = 1.5\sin(3t-\pi/2) \tag{6.1.21}$$

$$\mathscr{L}\{f(t)\} = \cos(\pi/2)\;\; \mathscr{L}\{\sin(3t)\} + \sin(\pi/2)\;\; \mathscr{L}\{\cos(3t)\} \tag{6.1.22}$$

$$= -\frac{1.5s}{s^2+9} \tag{6.1.23}$$

9. Finding Laplace transform,

$$f(t) = 1 - t \qquad \forall \quad t \in [0, 1] \tag{6.1.24}$$

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}(1-t)\ \ \mathrm{d}t = \left[-\frac{e^{-st}}{s} + \frac{te^{-st}}{s} + \frac{e^{-st}}{s^2}\right]_0^1 \tag{6.1.25}$$

$$= \left.\frac{(st - s + 1)e^{-st}}{s^2}\right|_0^1 = \frac{e^{-s} + s - 1}{s^2} \tag{6.1.26}$$

10. Finding Laplace transform,

$$f(t) = k \qquad \forall \quad t \in [0, c] \tag{6.1.27}$$

$$\mathcal{L}\{f(t)\} = \int_0^c ke^{-st}\ \ \mathrm{d}t = \left[-\frac{ke^{-st}}{s}\right]_0^c \tag{6.1.28}$$

$$= \frac{k}{s} - \frac{ke^{-sc}}{s} = \frac{k(1 - e^{-sc})}{s} \tag{6.1.29}$$

11. Finding Laplace transform,

$$f(t) = t \qquad \forall \quad t \in [0, b] \tag{6.1.30}$$

$$\mathcal{L}\{f(t)\} = \int_0^b e^{-st}(t)\ \ \mathrm{d}t = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^b \tag{6.1.31}$$

$$= \left.\frac{-(st + 1)e^{-st}}{s^2}\right|_0^b = \frac{1 - (1 + bs)e^{-bs}}{s^2} \tag{6.1.32}$$

12. Finding Laplace transform,

$$f(t) = \begin{cases} t & t \in [0, 1) \\ 1 & t \in [1, 2) \end{cases} \tag{6.1.33}$$

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}(t)\ \ \mathrm{d}t + \int_1^2 e^{-st}\ \ \mathrm{d}t \tag{6.1.34}$$

$$= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^2 \tag{6.1.35}$$

$$= \frac{-se^{-s} - e^{-s} + 1}{s^2} + \frac{e^{-s} - e^{-2s}}{s} \tag{6.1.36}$$

$$= \frac{1 - se^{-2s} - e^{-s}}{s^2} \tag{6.1.37}$$

13. Finding Laplace transform,

$$f(t) = \begin{cases} 1 & t \in [0, 1) \\ -1 & t \in [1, 2) \end{cases} \tag{6.1.38}$$

$$\mathscr{L}\{f(t)\} = \int_0^1 e^{-st} \; \mathrm{d}t - \int_1^2 e^{-st} \; \mathrm{d}t \tag{6.1.39}$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^1 + \left[\frac{e^{-st}}{s}\right]_1^2 \tag{6.1.40}$$

$$= \frac{(1 - e^{-s}) + (e^{-2s} - e^{-s})}{s} \tag{6.1.41}$$

$$= \frac{1 + e^{-2s} - 2e^{-s}}{s} \tag{6.1.42}$$

14. Finding Laplace transform,

$$f(t) = \left\{ k \quad t \in [a, b) \right. \tag{6.1.43}$$

$$\mathscr{L}\{f(t)\} = \int_a^b ke^{-st} \; \mathrm{d}t \qquad = \left[\frac{ke^{-st}}{-s}\right]_a^b \tag{6.1.44}$$

$$= \frac{k(e^{-as} - e^{-bs})}{s} \tag{6.1.45}$$

15. Finding Laplace transform,

$$f(t) = 1 - 0.5t \qquad \forall \quad t \in [0, 1] \tag{6.1.46}$$

$$\mathscr{L}\{f(t)\} = \int_0^1 e^{-st}(1 - 0.5t) \; \mathrm{d}t \qquad = \left[-\frac{e^{-st}}{s} + \frac{te^{-st}}{2s} + \frac{e^{-st}}{2s^2}\right]_0^1 \tag{6.1.47}$$

$$= \left.\frac{(st - 2s + 1)e^{-st}}{2s^2}\right|_0^1 \qquad = \frac{(1 - s)e^{-s} + (2s - 1)}{s^2} \tag{6.1.48}$$

16. Finding Laplace transform,

$$f(t) = \begin{cases} 1 & t \in [0, 1) \\ 2 - t & t \in [1, 2) \end{cases} \tag{6.1.49}$$

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st} \ \mathrm{d}t + \int_1^2 (2-t)e^{-st} \ \mathrm{d}t \tag{6.1.50}$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^1 + \left[\frac{2e^{-st}}{-s} + \frac{e^{-st}(1+st)}{s^2}\right]_1^2 \tag{6.1.51}$$

$$= \frac{(s - se^{-s}) + (2se^{-s} - 2se^{-2s}) + (1+2s)e^{-2s} - (1+s)e^{-s}}{s^2} \tag{6.1.52}$$

$$= \frac{-e^{-s} + e^{-2s} + s}{s^2} \tag{6.1.53}$$

17. Converting the table in the text into a table for inverse transforms,

$\mathbf{F(s)}$	$\mathcal{L}^{-1}\{\mathbf{F(s)}\}$	$\mathbf{F(s)}$	$\mathcal{L}^{-1}\{\mathbf{F(s)}\}$
$\frac{1}{s}$	1	$\frac{1}{s^2}$	t
$\frac{1}{s^3}$	$\frac{t^2}{2!}$	$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s^a}\ (a > 0)$	$\frac{t^{a-1}}{\Gamma(a)}$	$\frac{1}{s-a}$	e^{at}
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega t)$
$\frac{s}{s^2-a^2}$	$\cosh(at)$	$\frac{a}{s^2-a^2}$	$\sinh(at)$
$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\sin(\omega t)$

18. From Problem 10,

$$\mathcal{L}\{f\} = \frac{k}{s}\left(1 - e^{-cs}\right) \tag{6.1.54}$$

$$f(t) = \begin{cases} 0 & t \leq 2 \\ 1 & t > 2 \end{cases} \tag{6.1.55}$$

$$\int_c^\infty ke^{-st} \ \mathrm{d}t = \int_0^\infty ke^{-st} \ \mathrm{d}t - \int_0^c ke^{-st} \ \mathrm{d}t \tag{6.1.56}$$

$$\mathcal{L}\{k\} - \mathcal{L}\{f\} = \frac{ke^{-cs}}{s} \tag{6.1.57}$$

$$k = 1 \qquad\qquad c = 2 \tag{6.1.58}$$

$$\mathcal{L}\{f_1\} = \frac{e^{-2s}}{s} \tag{6.1.59}$$

19. Starting from the hyperbolic functions,

$$e^{at} = \frac{\cosh(at) + \sinh(at)}{2} \tag{6.1.60}$$

$$\mathcal{L}\{e^{at}\} = \frac{\mathcal{L}\{\cosh(at)\} + \mathcal{L}\{\sinh(at)\}}{2} \tag{6.1.61}$$

$$= \frac{s+a}{s^2 - a^2} \tag{6.1.62}$$

$$= \frac{1}{s-a} \tag{6.1.63}$$

20. For the function $\exp(t^2)$,

$$f(t) = \exp(t^2) \tag{6.1.64}$$

$$|f(t)| = |\exp(t^2)| \tag{6.1.65}$$

$$\text{Suppose} \quad |\exp(t^2)| \leq M\exp(kt) \tag{6.1.66}$$

for all $t \geq 0$ given k, M are some constants. This requires $t^2 \leq kt$ for all $k \geq 0$. No such k exists by the definition of power law functions. So, this assumption is incorrect.

21. Functions which go to $\pm\infty$ such as $1/x, \ \tan(x)$ do not have Laplace transforms as the integral diverges.

22. For $a = -1/2$,

$$f(t) = t^{-1/2} \tag{6.1.67}$$

$$\mathscr{L}\{f(t)\} = \frac{\Gamma(1/2)}{s^{1/2}} \tag{6.1.68}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} \tag{6.1.69}$$

This function is not defined for $t = 0$ and therefore, does not meet the requirements of the existence theorem in the text. This means that the theorem provides sufficient but not necessary conditions.

23. For some positive scalar consant c, let $u = ct$

$$\mathscr{L}\{f(ct)\} = \int_0^\infty e^{-st} f(ct)\ \ \mathrm{d}t \tag{6.1.70}$$

$$= \frac{1}{c}\int_0^\infty \exp\left(\frac{-su}{c}\right) f(u)\ \ \mathrm{d}u \tag{6.1.71}$$

$$= \frac{1}{c}\ F(s/c) \tag{6.1.72}$$

$$\mathscr{L}\{\cos(\omega t)\} = \frac{1}{\omega}\left[\frac{(s/\omega)}{(s/\omega)^2 + 1}\right] \tag{6.1.73}$$

$$= \frac{s}{s^2 + \omega^2} \tag{6.1.74}$$

24. To prove the inverse Laplace transform is linear,

$$\mathscr{L}^{-1}\{\mathscr{L}\{af(t) + bg(t)\}\} = \mathscr{L}^{-1}\{a\,\mathscr{L}\{f(t)\} + b\,\mathscr{L}\{g(t)\}\} \tag{6.1.75}$$

$$= \mathscr{L}^{-1}\{a\ F(s) + b\ G(s)\} \tag{6.1.76}$$

$$a\ f(t) + b\ g(t) = a\,\mathscr{L}^{-1}\{F(s)\} + b\,\mathscr{L}^{-1}\{G(s)\} \tag{6.1.77}$$

Since the two RHS are equal, $\mathscr{L}^{-1}$ is linear.

25. To find the inverse Laplace transform,

$$F(s) = \frac{0.2s + 1.8}{s^2 + 3.24} \tag{6.1.78}$$

$$= \frac{(0.2)s + (1)1.8}{s^2 + 1.8^2} \tag{6.1.79}$$

$$\mathscr{L}^{-1}\{F(s)\} = 0.2\cos(1.8t) + \sin(1.8t) \tag{6.1.80}$$

26. To find the inverse Laplace transform,

$$F(s) = \frac{5s+1}{s^2-25} \tag{6.1.81}$$

$$= \frac{(5)s+(0.2)5}{s^2-5^2} \tag{6.1.82}$$

$$\mathscr{L}^{-1}\{F(s)\} = 5\cosh(5t) + 0.2\sinh(5t) \tag{6.1.83}$$

27. To find the inverse Laplace transform,

$$F(s) = \frac{s}{(Ls)^2+(n\pi)^2} \tag{6.1.84}$$

$$= \frac{s(L^{-2})}{s^2+(n\pi/L)^2} \tag{6.1.85}$$

$$\mathscr{L}^{-1}\{F(s)\} = \frac{1}{L^2}\ \cos\left(\frac{n\pi}{L}\ t\right) \tag{6.1.86}$$

28. To find the inverse Laplace transform,

$$F(s) = \frac{1}{(s+\sqrt{2})(s-\sqrt{3})} \tag{6.1.87}$$

$$= \frac{p_1}{(s+\sqrt{2})} + \frac{p_2}{(s-\sqrt{3})} \tag{6.1.88}$$

$$p_1 + p_2 = 0 \qquad -\sqrt{3}p_1 + \sqrt{2}p_2 = 1 \tag{6.1.89}$$

$$\mathscr{L}^{-1}\{F(s)\} = \frac{-e^{-\sqrt{2}t} + e^{\sqrt{3}t}}{\sqrt{2}+\sqrt{3}} \tag{6.1.90}$$

29. To find the inverse Laplace transform,

$$F(s) = \frac{12}{s^4} - \frac{228}{s^6} \tag{6.1.91}$$

$$= \frac{(2)3!}{s^4} - \frac{5!}{s^6}\,\frac{228}{120} \tag{6.1.92}$$

$$\mathscr{L}^{-1}\{F(s)\} = 2t^2 - 1.9t^5 \tag{6.1.93}$$

30. To find the inverse Laplace transform,

$$F(s) = \frac{4s+32}{s^2-16} \tag{6.1.94}$$

$$= \frac{(4)s+(8)4}{s^2-4^2} \tag{6.1.95}$$

$$\mathcal{L}^{-1}\{F(s)\} = 4\cosh(4t) + 8\sinh(4t) \tag{6.1.96}$$

31. To find the inverse Laplace transform,

$$F(s) = \frac{s+10}{s^2-s-2} \tag{6.1.97}$$

$$= \frac{s+10}{(s-2)(s+1)} \tag{6.1.98}$$

$$= \frac{p_1}{(s-2)} + \frac{p_2}{(s+1)} \tag{6.1.99}$$

$$p_1 + p_2 = 1 \qquad p_1 - 2p_2 = 10 \tag{6.1.100}$$

$$\mathcal{L}^{-1}\{F(s)\} = 4e^{2t} - 3e^{-t} \tag{6.1.101}$$

32. To find the inverse Laplace transform,

$$F(s) = \frac{1}{(s+a)(s+b)} \tag{6.1.102}$$

$$= \frac{p_1}{(s+a)} + \frac{p_2}{(s+b)} \tag{6.1.103}$$

$$p_1 + p_2 = 0 \qquad bp_1 + ap_2 = 1 \tag{6.1.104}$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{e^{-at} - e^{-bt}}{(b-a)} \tag{6.1.105}$$

33. To find the Laplace transform,

$$f(t) = t^2 e^{-3t} \tag{6.1.106}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} \quad \text{with} \quad s \to s+3 \tag{6.1.107}$$

$$= \frac{2!}{(s+3)^3} \tag{6.1.108}$$

34. To find the Laplace transform,

$$f(t) = ke^{-at}\cos(\omega t) \tag{6.1.109}$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{k\cos(\omega t)\} \qquad \text{with} \quad s \to s + a \tag{6.1.110}$$

$$= \frac{k(s+a)}{(s+a)^2 + \omega^2} \tag{6.1.111}$$

35. To find the Laplace transform,

$$f(t) = 0.5e^{-4.5t}\sin(2\pi t) \tag{6.1.112}$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{0.5\sin(2\pi t)\} \qquad \text{with} \quad s \to s + 4.5 \tag{6.1.113}$$

$$= \frac{0.5 \cdot 2\pi}{(s+4.5)^2 + 4\pi^2} \tag{6.1.114}$$

36. To find the Laplace transform,

$$f(t) = \sinh t \cos t \tag{6.1.115}$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{0.5e^{t}\cos(t)\} - \mathscr{L}\{0.5e^{-t}\cos(t)\} \tag{6.1.116}$$

$$= \frac{0.5(s-1)}{(s-1)^2+1} - \frac{0.5(s+1)}{(s+1)^2+1} \tag{6.1.117}$$

$$= 0.5\left[\frac{(s-1)(s^2+2s+2) - (s+1)(s^2-2s+2)}{(s^2-2s+2)(s^2+2s+2)}\right] \tag{6.1.118}$$

$$= \frac{s^2-2}{s^4+4} \tag{6.1.119}$$

37. To find the inverse Laplace transform,

$$F(s) = \frac{\pi}{(s+\pi)^2} \tag{6.1.120}$$

$$= \pi\,\frac{1}{s^2} \qquad \text{with} \qquad s \to s + \pi \tag{6.1.121}$$

$$\mathscr{L}^{-1}\{F(s)\} = \pi t\, e^{-\pi t} \tag{6.1.122}$$

38. To find the inverse Laplace transform,

$$F(s) = \frac{6}{(s+1)^3} \tag{6.1.123}$$

$$= 3\,\frac{2!}{s^3} \qquad \text{with} \qquad s \to s+1 \tag{6.1.124}$$

$$\mathscr{L}^{-1}\{F(s)\} = 3t^2\ e^{-t} \tag{6.1.125}$$

39. To find the inverse Laplace transform,

$$F(s) = \frac{21}{(s+\sqrt{2})^4} \tag{6.1.126}$$

$$= \frac{21}{6}\,\frac{3!}{s^4} \qquad \text{with} \qquad s \to s+\sqrt{2} \tag{6.1.127}$$

$$\mathscr{L}^{-1}\{F(s)\} = \frac{7t^3}{2}\ e^{-\sqrt{2}t} \tag{6.1.128}$$

40. To find the inverse Laplace transform,

$$F(s) = \frac{4}{s^2-2s-3} \tag{6.1.129}$$

$$= \frac{4}{(s-1)^2-2^2} \tag{6.1.130}$$

$$= \frac{(2)2}{s^2-2^2} \qquad \text{with} \qquad s \to s-1 \tag{6.1.131}$$

$$\mathscr{L}^{-1}\{F(s)\} = 2\sinh(2t)\ e^{t} \tag{6.1.132}$$

$$= e^{3t} - e^{-t} \tag{6.1.133}$$

41. To find the inverse Laplace transform,

$$F(s) = \frac{\pi}{s^2+10\pi s+24\pi^2} \tag{6.1.134}$$

$$= \frac{\pi}{(s+5\pi)^2-\pi^2} \tag{6.1.135}$$

$$= \frac{\pi}{s^2-\pi^2} \qquad \text{with} \qquad s \to s+5\pi \tag{6.1.136}$$

$$\mathscr{L}^{-1}\{F(s)\} = \sinh(\pi t)\ e^{-5\pi t} \tag{6.1.137}$$

42. To find the inverse Laplace transform,

$$F(s) = \frac{a_0}{(s+1)} + \frac{a_1}{(s+1)^2} + \frac{a_2}{(s+1)^3} \tag{6.1.138}$$

$$= \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} \qquad \text{with} \qquad s \to s+1 \tag{6.1.139}$$

$$\mathscr{L}^{-1}\{F(s)\} = \left[a_0 + a_1 t + \frac{a_2 t^2}{2!}\right] e^{-t} \tag{6.1.140}$$

43. To find the inverse Laplace transform,

$$F(s) = \frac{2s-1}{s^2-6s+18} \tag{6.1.141}$$

$$= \frac{2(s-3)}{(s-3)^2+3^2} + \frac{5}{3}\,\frac{3}{(s-3)^2+3^2} \tag{6.1.142}$$

$$= \frac{2s}{s^2+3^2} + \frac{5}{3}\,\frac{3}{s^2+3^2} \qquad \text{with} \qquad s \to s-3 \tag{6.1.143}$$

$$\mathscr{L}^{-1}\{F(s)\} = \left[2\cos(3t) + \frac{5}{3}\sin(3t)\right]\; e^{3t} \tag{6.1.144}$$

44. To find the inverse Laplace transform,

$$F(s) = \frac{a(s+k)+b\pi}{(s+k)^2+\pi^2} \tag{6.1.145}$$

$$= \frac{as+b\pi}{s^2+\pi^2} \qquad \text{with} \qquad s \to s+k \tag{6.1.146}$$

$$\mathscr{L}^{-1}\{F(s)\} = [a\cos(\pi t) + b\sin(\pi t)]\;\; e^{-kt} \tag{6.1.147}$$

45. To find the inverse Laplace transform,

$$F(s) = \frac{k_0(s+a)+k_1}{(s+a)^2} \tag{6.1.148}$$

$$= \frac{k_0}{s} + \frac{k_1}{s^2} \qquad \text{with} \qquad s \to s+a \tag{6.1.149}$$

$$\mathscr{L}^{-1}\{F(s)\} = [k_0 + k_1 t]\; e^{-at} \tag{6.1.150}$$

6.2 Transforms of Derivatives and Integrals, ODEs

1. Solving IVP using Laplace transform,

$$y' + 5.2y = 19.4\sin(2t) \qquad y(0) = 0 \tag{6.2.1}$$

$$R = 19.4\,\mathscr{L}\{\sin(2t)\} = \frac{2(19.4)}{s^2+4} \qquad a = 1 \qquad b = 5.2 \tag{6.2.2}$$

$$a[sY - y(0)] + bY = R \qquad (s+5.2)Y = \frac{38.8}{s^2+4} \tag{6.2.3}$$

Solving the subsidiary equation,

$$Y = \frac{(19.4)\cdot 2}{(s+5.2)(s^2+2^2)} \qquad Y = \frac{p_1}{s+5.2} + \frac{p_2 s + p_3}{s^2+4} \tag{6.2.4}$$

$$p_1 + p_2 = 0 \qquad 4p_1 + 5.2p_3 = 38.8 \tag{6.2.5}$$

$$5.2p_2 + p_3 = 0 \tag{6.2.6}$$

$$Y = \frac{(5/4)}{(s+5.2)} + \frac{(-5s+26)}{4(s^2+4)} \tag{6.2.7}$$

$$y = \frac{5e^{-5.2t} - 5\cos(2t) + 13\sin(2t)}{4} \tag{6.2.8}$$

2. Solving IVP using Laplace transform,

$$y' + 2y = 0 \qquad y(0) = 1.5 \tag{6.2.9}$$

$$R = \mathscr{L}\{0\} = 0 \qquad a = 1 \qquad b = 2 \tag{6.2.10}$$

$$a[sY - y(0)] + bY = R \qquad (s+2)Y - 1.5 = 0 \tag{6.2.11}$$

Solving the subsidiary equation,

$$Y = \frac{1.5}{(s+2)} \qquad y = 1.5e^{-2t} \tag{6.2.12}$$

3. Solving IVP using Laplace transform,

$$y'' - y' - 6y = 0 \qquad y(0) = 11 \qquad y'(0) = 28 \tag{6.2.13}$$

$$R = \mathscr{L}\{0\} = 0 \qquad a = -1 \qquad b = -6 \tag{6.2.14}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.15}$$

$$(s^2 - s - 6)Y = 11(s-1) + 28 \tag{6.2.16}$$

Solving the subsidiary equation,

$$Y = \frac{11(s-1)+28}{s^2-s-6} \qquad Y = \frac{11s+17}{(s-3)(s+2)} \tag{6.2.17}$$

$$Y = \frac{p_1}{s-3} + \frac{p_2}{s+2} \tag{6.2.18}$$

$$p_1 + p_2 = 11 \qquad 2p_1 - 3p_2 = 17 \tag{6.2.19}$$

$$Y = \frac{10}{(s-3)} + \frac{1}{(s+2)} \tag{6.2.20}$$

$$y = 10e^{3t} + e^{-2t} \tag{6.2.21}$$

4. Solving IVP using Laplace transform,

$$y'' + 9y = 10e^{-t} \qquad y(0) = 0 \qquad y'(0) = 0 \tag{6.2.22}$$

$$R = 10\,\mathscr{L}\{e^{-t}\} = \frac{10}{s+1} \qquad a = 0 \qquad b = 9 \tag{6.2.23}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.24}$$

$$(s^2+9)Y = \frac{10}{s+1} \tag{6.2.25}$$

Solving the subsidiary equation,

$$Y = \frac{10}{(s^2+9)(s+1)} \qquad Y = \frac{p_1}{s+1} + \frac{p_2 s + p_3}{s^2+9} \tag{6.2.26}$$

$$p_1 + p_2 = 0 \qquad 9p_1 + p_3 = 10 \tag{6.2.27}$$

$$p_2 + p_3 = 0 \tag{6.2.28}$$

$$Y = \frac{1}{(s+1)} + \frac{-s+1}{(s^2+9)} \tag{6.2.29}$$

$$y = e^{-t} - \cos(3t) + \frac{\sin(3t)}{3} \tag{6.2.30}$$

5. Solving IVP using Laplace transform,

$$y'' - 0.25y = 0 \qquad y(0) = 12 \qquad y'(0) = 0 \tag{6.2.31}$$

$$R = \mathscr{L}\{0\} = 0 \qquad a = 0 \qquad b = -0.25 \tag{6.2.32}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.33}$$

$$(s^2 - 0.25)Y = 12s \tag{6.2.34}$$

Solving the subsidiary equation,

$$Y = \frac{12s}{(s^2 - 0.25)} \qquad y = 12\cosh(0.5t) \tag{6.2.35}$$

6. Solving IVP using Laplace transform,

$$y'' - 6y' + 5y = 29\cos(2t) \qquad y(0) = 3.2 \qquad y'(0) = 6.2 \tag{6.2.36}$$

$$R = 29\,\mathscr{L}\{\cos(2t)\} = \frac{29s}{s^2 + 4} \qquad a = -6 \qquad b = 5 \tag{6.2.37}$$

$$(s^2 + as + b)Y = R + (s + a)y(0) + y'(0) \tag{6.2.38}$$

$$(s^2 - 6s + 5)Y = \frac{29s}{s^2 + 4} + 3.2(s - 6) + 6.2 \tag{6.2.39}$$

Solving the subsidiary equation,

$$Y = \frac{29s}{(s^2 + 4)(s - 5)(s - 1)} + \frac{3.2s - 13}{(s - 5)(s - 1)} \tag{6.2.40}$$

$$= \frac{p_3 s + p_4}{(s^2 + 4)} + \frac{p_1}{(s - 1)} + \frac{p_2}{(s - 5)} \tag{6.2.41}$$

$$Y = \frac{0.2s - 4.8}{(s^2 + 4)} + \frac{1}{(s - 1)} + \frac{2}{(s - 5)} \tag{6.2.42}$$

$$y = e^t + 2e^{5t} + 0.2\cos(2t) - 2.4\sin(2t) \tag{6.2.43}$$

7. Solving IVP using Laplace transform,

$$y'' + 7y' + 12y = 21e^{3t} \qquad y(0) = 3.5 \qquad y'(0) = -10 \tag{6.2.44}$$

$$R = 21\,\mathscr{L}\{e^{3t}\} = \frac{21}{s - 3} \qquad a = 7 \qquad b = 12 \tag{6.2.45}$$

$$(s^2 + as + b)Y = R + (s + a)y(0) + y'(0) \tag{6.2.46}$$

$$(s^2 + 7s + 12)Y = \frac{21}{s - 3} + 3.5(s + 7) - 10 \tag{6.2.47}$$

Solving the subsidiary equation,

$$Y = \frac{3.5s^2 + 4s - 22.5}{(s+4)(s+3)(s-3)} \qquad Y = \frac{p_1}{(s-3)} + \frac{p_2}{(s+3)} + \frac{p_3}{(s+4)} \tag{6.2.48}$$

$$3.5 = p_1 + p_2 + p_3 \qquad 4 = 7p_1 + p_2 \tag{6.2.49}$$

$$-22.5 = 12p_1 - 12p_2 - 9p_3 \tag{6.2.50}$$

$$Y = \frac{0.5}{s-3} + \frac{0.5}{s+3} + \frac{2.5}{s+4} \qquad y = \frac{e^{3t} + e^{-3t} + 5e^{-4t}}{2} \tag{6.2.51}$$

8. Solving IVP using Laplace transform,

$$y'' - 4y' + 4y = 0 \qquad y(0) = 8.1 \qquad y'(0) = 3.9 \tag{6.2.52}$$

$$R = \mathscr{L}\{0\} = 0 \qquad a = -4 \qquad b = 4 \tag{6.2.53}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.54}$$

$$(s^2 - 4s + 4)Y = 8.1(s-4) + 3.9 \tag{6.2.55}$$

Solving the subsidiary equation,

$$Y = \frac{8.1(s-2) - 12.3}{(s-2)^2} \tag{6.2.56}$$

$$y = (8.1 - 12.3t)\, e^{2t} \tag{6.2.57}$$

9. Solving IVP using Laplace transform,

$$y'' - 4y' + 3y = 6t - 8 \qquad y(0) = 0 \qquad y'(0) = 0 \tag{6.2.58}$$

$$R = \mathscr{L}\{6t - 8\} = \frac{6}{s^2} - \frac{8}{s} \qquad a = -4 \qquad b = 3 \tag{6.2.59}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.60}$$

$$(s^2 - 4s + 3)Y = \frac{6 - 8s}{s^2} \tag{6.2.61}$$

Solving the subsidiary equation,

$$Y = \frac{-8s+6}{s^2(s-3)(s-1)} \qquad Y = \frac{p_1}{(s-3)} + \frac{p_2}{(s-1)} + \frac{p_3 s + p_4}{s^2} \tag{6.2.62}$$

$$0 = p_1 + p_2 + p_3 \qquad 0 = -p_1 - 3p_2 - 4p_3 + p_4 \tag{6.2.63}$$

$$-8 = 3p_3 - 4p_4 \qquad 6 = 3p_4 \tag{6.2.64}$$

$$Y = \frac{-1}{s-3} + \frac{1}{s-1} + \frac{2}{s^2} \qquad y = -e^{3t} + e^t + 2t \tag{6.2.65}$$

10. Solving IVP using Laplace transform,

$$y'' + 0.04y = 0.02t^2 \qquad y(0) = -25 \qquad y'(0) = 0 \tag{6.2.66}$$

$$R = 0.02\,\mathscr{L}\{t^2\} = \frac{0.04}{s^3} \qquad a = 0 \qquad b = 0.04 \tag{6.2.67}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.68}$$

$$(s^2 + 0.04)Y = \frac{-25s^4 + 0.04}{s^3} \tag{6.2.69}$$

Solving the subsidiary equation,

$$Y = \frac{-25s^4 + 0.04}{s^3(s^2 + 0.04)} \qquad Y = \frac{(0.2 - 5s^2)(0.2 + 5s^2)}{0.2s^3(5s^2 + 0.2)} \tag{6.2.70}$$

$$Y = \frac{1}{s^3} - \frac{25}{s} \qquad y = 0.5t^2 - 25 \tag{6.2.71}$$

11. Solving IVP using Laplace transform,

$$y'' + 3y' + 2.25y = 9t^3 + 64 \qquad y(0) = 1 \qquad y'(0) = 31.5 \tag{6.2.72}$$

$$R = \mathscr{L}\{9t^3 + 64\} = \frac{54}{s^4} + \frac{64}{s} \qquad a = 3 \qquad b = 2.25 \tag{6.2.73}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.74}$$

$$(s^2 + 3s + 2.25)Y = \frac{64s^3 + 54}{s^4} + (s+3) + 31.5 \tag{6.2.75}$$

Solving the subsidiary equation,

$$Y = \frac{s^5 + 34.5s^4 + 64s^3 + 54}{s^4\ (s+1.5)^2} \tag{6.2.76}$$

$$Y = \frac{(s+1.5)+1}{(s+1.5)^2} + \frac{32s^2 - 32s + 24}{s^4} \tag{6.2.77}$$

$$y = e^{-1.5t} + te^{-1.5t} + 32t - 16t^2 + 4t^3 \tag{6.2.78}$$

12. Shifting the IVP using $u = t - 4$,

$$y'' - 2y' - 3y = 0 \qquad u(0) = -3 \qquad u'(0) = -17 \tag{6.2.79}$$

$$R = \mathscr{L}\{0\} = 0 \qquad a = -2 \qquad b = -3 \tag{6.2.80}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.81}$$

$$(s^2 - 2s - 3)Y = 0 - 3(s-2) - 17 \tag{6.2.82}$$

Solving the subsidiary equation,

$$Y = \frac{-3s - 11}{(s-3)(s+1)} \qquad Y = \frac{p_1}{(s-3)} + \frac{p_2}{(s+1)} \tag{6.2.83}$$

$$-3 = p_1 + p_2 \qquad -11 = p_1 - 3p_2 \tag{6.2.84}$$

$$Y = \frac{-5}{(s-3)} + \frac{2}{(s+1)} \qquad y = -5e^{3u} + 2e^{-u} \tag{6.2.85}$$

$$y = -5e^{3(t-4)} + 2e^{-(t-4)} \tag{6.2.86}$$

13. Shifting the IVP using $u = t + 1$,

$$y' - 6y = 0 \qquad u(0) = 4 \tag{6.2.87}$$

$$R = \mathscr{L}\{0\} = 0 \qquad a = 1 \qquad b = -6 \tag{6.2.88}$$

$$a[sY - y(0)] + bY = R \qquad (sY - 4) - 6Y = 0 \tag{6.2.89}$$

Solving the subsidiary equation,

$$Y = \frac{4}{(s-6)} \tag{6.2.90}$$

$$y = 4e^{6u} \qquad y = 4e^{6(t+1)} \tag{6.2.91}$$

14. Shifting the IVP using $u = t - 2$, which changes $r(t) \to r(u)$

$$y'' + 2y' + 5y = 50(u+2) - 100 \qquad u(0) = -4 \qquad u'(0) = 14 \tag{6.2.92}$$

$$R = \mathscr{L}\{50u\} = \frac{50}{s^2} \qquad a = 2 \qquad b = 5 \tag{6.2.93}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.94}$$

$$(s^2 + 2s + 5)Y = \frac{50}{s^2} - 4(s+2) + 14 \tag{6.2.95}$$

Solving the subsidiary equation,

$$Y = \frac{-4s^3 + 6s^2 + 50}{s^2(s^2 + 2s + 5)} \tag{6.2.96}$$

$$Y = \frac{-4s + 10}{s^2} + \frac{4}{(s+1)^2 + 4} \tag{6.2.97}$$

$$y = -4 + 10u + e^{-u}\left[2\sin(2u)\right] \tag{6.2.98}$$

$$y = e^{-(t-2)}\left[2\sin(2t-4)\right] + 10t - 24 \tag{6.2.99}$$

15. Shifting the IVP using $u = t - 1.5$, which changes $r(t) \to r(u)$

$$y'' + 3y' - 4y = 6e^{2u} \qquad u(0) = 4 \qquad u'(0) = 5 \tag{6.2.100}$$

$$R = 6\,\mathscr{L}\{e^{2u}\} = \frac{6}{s-2} \qquad a = 3 \qquad b = -4 \tag{6.2.101}$$

$$(s^2 + as + b)Y = R + (s+a)y(0) + y'(0) \tag{6.2.102}$$

$$(s^2 + 3s - 4)Y = \frac{6}{(s-2)} + 4(s+3) + 5 \tag{6.2.103}$$

Solving the subsidiary equation,

$$Y = \frac{(s+4)(4s-7)}{(s-2)(s+4)(s-1)} \qquad Y = \frac{1}{(s-2)} + \frac{3}{(s-1)} \tag{6.2.104}$$

$$y = e^{2u} + 3e^{u} \qquad y = e^{2t-3} + 3e^{t-1.5} \tag{6.2.105}$$

16. Finding Laplace transform using the derivative,

$$y = t\cos(4t) \qquad y' = -4t\sin(4t) + \cos(4t) \tag{6.2.106}$$

$$y'' = -16t\cos(4t) - 8\sin(4t) \qquad \mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.107}$$

$$-16Y - \frac{32}{s^2+16} = s^2Y - 0 - 1 \qquad Y = \frac{s^2-16}{(s^2+16)^2} \tag{6.2.108}$$

17. Finding Laplace transform using the derivative,

$$y = te^{-at} \qquad y' = e^{-at}[1 - at] \tag{6.2.109}$$

$$\mathscr{L}\{f'\} = s\,\mathscr{L}\{f\} - f(0) \tag{6.2.110}$$

$$\frac{1}{s+a} - aY = sY - 0 \qquad Y = \frac{1}{(s+a)^2} \tag{6.2.111}$$

18. Finding Laplace transform using the derivative,

$$y = \cos^2(2t) \qquad y' = -4\cos(2t)\sin(2t) \tag{6.2.112}$$

$$y'' = -8\cos(4t) \qquad y'' = -8[2\cos^2(2t) - 1] \tag{6.2.113}$$

$$\mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.114}$$

$$-16Y + \frac{8}{s} = s^2Y - s - 0 \qquad Y = \frac{s^2+8}{s(s^2+16)} \tag{6.2.115}$$

19. Finding Laplace transform using the derivative,

$$y = \sin^2(\omega t) \qquad y' = 2\omega\cos(\omega t)\sin(\omega t) \tag{6.2.116}$$

$$y'' = [\omega\sin(2\omega t)]' \qquad y'' = 2\omega^2[1 - 2\sin^2(\omega t)] \tag{6.2.117}$$

$$\mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.118}$$

$$\frac{2\omega^2}{s} - 4\omega^2Y = s^2Y - 0 - 0 \qquad Y = \frac{2\omega^2}{s(s^2+4\omega^2)} \tag{6.2.119}$$

20. Finding Laplace transform using the derivative,

$$y = \sin^4 t \qquad y' = 4\sin^3 t \cos t \tag{6.2.120}$$

$$y'' = 4[-\sin^4 t + 3\sin^2 t \cos^2 t] \qquad y'' = 4[-4\sin^4 t + 3\sin^2 t] \tag{6.2.121}$$

$$\mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.122}$$

$$-16Y + \frac{24}{s(s^2+4)} = s^2Y - 0 - 0 \qquad Y = \frac{24}{s(s^2+4)(s^2+16)} \tag{6.2.123}$$

21. Finding Laplace transform using the derivative,

$$y = \cosh^2 t \qquad y' = 2\cosh t \sinh t \tag{6.2.124}$$

$$y'' = 2[\cosh^2 t + \sinh^2 t] \qquad y'' = 2[2\cosh^2 t - 1] \tag{6.2.125}$$

$$\mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.126}$$

$$4Y - \frac{2}{s} = s^2Y - s - 0 \qquad Y = \frac{s^2-2}{s(s^2-4)} \tag{6.2.127}$$

22. **(a)** Finding Laplace transform using the derivative,

$$y = t\cos(\omega t) \qquad y' = \cos(\omega t) - \omega t\sin(\omega t) \tag{6.2.128}$$

$$y'' = -2\omega\sin(\omega t) - \omega^2 t\cos(\omega t) \tag{6.2.129}$$

$$\mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.130}$$

$$\frac{-2\omega^2}{s^2+\omega^2} - \omega^2 Y = s^2Y - 0 - 1 \qquad Y = \frac{s^2-\omega^2}{(s^2+\omega^2)^2} \tag{6.2.131}$$

(b) Finding Laplace transform using the derivative,

$$y = \frac{\sin(\omega t) - \omega t\cos(\omega t)}{2\omega^3} \tag{6.2.132}$$

$$\mathscr{L}\{f\} = \frac{1}{2\omega^3}\left[\frac{\omega}{s^2+\omega^2} - \frac{\omega(s^2-\omega^2)}{(s^2+\omega^2)^2}\right] \tag{6.2.133}$$

$$Y = \frac{1}{(s^2+\omega^2)^2} \tag{6.2.134}$$

(c) Finding Laplace transform using the derivative,

$$y = \frac{t\sin(\omega t)}{2\omega} \qquad y' = \frac{\omega t\cos(\omega t) + \sin(\omega t)}{2\omega} \tag{6.2.135}$$

$$\mathscr{L}\{f'\} = s\,\mathscr{L}\{f\} - f(0) \tag{6.2.136}$$

$$sY - 0 = \frac{s^2 - \omega^2}{2(s^2+\omega^2)^2} + \frac{1}{2(s^2+\omega^2)} \qquad Y = \frac{s}{(s^2+\omega^2)^2} \tag{6.2.137}$$

(d) Finding Laplace transform using the derivative,

$$y = \frac{\sin(\omega t) + \omega t\cos(\omega t)}{2\omega} \tag{6.2.138}$$

$$\mathscr{L}\{f\} = \frac{1}{2\omega}\left[\frac{\omega}{s^2+\omega^2} + \frac{\omega(s^2-\omega^2)}{(s^2+\omega^2)^2}\right] \tag{6.2.139}$$

$$Y = \frac{s^2}{(s^2+\omega^2)^2} \tag{6.2.140}$$

(e) Finding Laplace transform using the derivative,

$$y = t\cosh(at) \qquad y' = at\sinh(at) + \cosh(at) \tag{6.2.141}$$

$$y'' = 2a\sinh(at) + a^2t\cosh(at) \qquad \mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.142}$$

$$s^2Y - 1 = \frac{2a^2}{s^2-a^2} + a^2Y \qquad Y = \frac{s^2+a^2}{(s^2-a^2)^2} \tag{6.2.143}$$

(f) Finding Laplace transform using the derivative,

$$y = t\sinh(at) \qquad y' = at\cosh(at) + \sinh(at) \tag{6.2.144}$$

$$y'' = 2a\cosh(at) + a^2t\sinh(at) \qquad \mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.145}$$

$$s^2Y = \frac{2as}{s^2-a^2} + a^2Y \qquad Y = \frac{2as}{(s^2-a^2)^2} \tag{6.2.146}$$

23. Using integration to find inverse Laplace transform,

$$G(s) = \frac{F(s)}{s} = \frac{3}{s^2 + 0.25s} \qquad \mathscr{L}\left\{\int_0^t f(\tau)\ \mathrm{d}\tau\right\} = \frac{F(s)}{s} \tag{6.2.147}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{3}{s + 0.25}\right\} \qquad f(\tau) = 3e^{-0.25\tau} \tag{6.2.148}$$

$$g(t) = 3\int_0^t e^{-0.25\tau}\ \mathrm{d}\tau \tag{6.2.149}$$

$$= -12\Big[e^{-0.25\tau}\Big]_0^t \tag{6.2.150}$$

$$= 12(1 - e^{-t/4}) \tag{6.2.151}$$

24. Using integration to find inverse Laplace transform,

$$G(s) = \frac{F(s)}{s} = \frac{20}{s^3 - 2\pi s^2} \qquad G(s) = \mathscr{L}\left\{\int_0^t f(\tau)\ \mathrm{d}\tau\right\} \tag{6.2.152}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{20}{s^2 - 2\pi s}\right\} \qquad f(\tau) = \mathscr{L}^{-1}\left\{\frac{20}{(s-\pi)^2 - \pi^2}\right\} \tag{6.2.153}$$

$$g(t) = \frac{10}{\pi}\int_0^t e^{2\pi\tau} - 1\ \mathrm{d}\tau \tag{6.2.154}$$

$$= \frac{10}{\pi}\left[\frac{e^{2\pi\tau}}{2\pi} - \tau\right]_0^t \tag{6.2.155}$$

$$= \frac{5}{\pi^2}\left[e^{2\pi t} - 1 - 2\pi t\right] \tag{6.2.156}$$

25. Using integration to find inverse Laplace transform,

$$G(s) = \frac{F(s)}{s} = \frac{1}{s(s^2 + \omega^2)} \qquad G(s) = \mathscr{L}\left\{\int_0^t f(\tau)\ \mathrm{d}\tau\right\} \tag{6.2.157}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2 + \omega^2}\right\} \qquad f(\tau) = \frac{\sin(\omega t)}{\omega} \tag{6.2.158}$$

$$g(t) = \frac{1}{\omega}\int_0^t \sin(\omega\tau)\,\mathrm{d}\tau \tag{6.2.159}$$

$$= \frac{-1}{\omega^2}\Big[\cos(\omega\tau)\Big]_0^t \tag{6.2.160}$$

$$= \frac{1 - \cos(\omega t)}{\omega^2} \tag{6.2.161}$$

26. Using integration to find inverse Laplace transform,

$$G(s) = \frac{F(s)}{s} = \frac{1}{s^2(s^2-1)} \qquad G(s) = \mathscr{L}\left\{\int_0^t f(\tau)\ \mathrm{d}\tau\right\} \tag{6.2.162}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s(s+1)(s-1)}\right\} \qquad f(\tau) = \mathscr{L}^{-1}\left\{\frac{-1}{s} + \frac{0.5}{s+1} + \frac{0.5}{s-1}\right\} \tag{6.2.163}$$

$$f(\tau) = -1 + 0.5e^{-\tau} + 0.5e^{\tau} \qquad g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau \tag{6.2.164}$$

$$f(t) = \Bigg[-t - 0.5e^{-\tau} + 0.5e^{\tau}\Bigg]_0^t \tag{6.2.165}$$

$$= -t + \sinh(t) \tag{6.2.166}$$

27. Using integration to find inverse Laplace transform,

$$G(s) = \frac{F(s)}{s} = \frac{s+1}{s^2(s^2+9)} \qquad G(s) = \mathscr{L}\left\{\int_0^t f(\tau)\ \mathrm{d}\tau\right\} \tag{6.2.167}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{s+1}{s(s^2+9)}\right\} \qquad f(\tau) = \frac{1}{9}\left[\frac{1}{s} + \frac{9-s}{s^2+9}\right] \tag{6.2.168}$$

$$f(\tau) = \frac{1}{9}\left[1 + 3\sin(3\tau) - \cos(3\tau)\right] \tag{6.2.169}$$

$$g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau \tag{6.2.170}$$

$$= \frac{1}{9}\left[\tau - \cos(3\tau) - \frac{\sin(3\tau)}{3}\right]_0^t \tag{6.2.171}$$

$$= \frac{t+1-\cos(3t)}{9} - \frac{\sin(3t)}{27} \tag{6.2.172}$$

28. Using integration to find inverse Laplace transform,

$$G(s) = \frac{F(s)}{s} = \frac{3s+4}{s^2(s^2+k^2)} \qquad G(s) = \mathscr{L}\left\{\int_0^t f(\tau)\ \mathrm{d}\tau\right\} \tag{6.2.173}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{3s+4}{s(s^2+k^2)}\right\} \qquad f(\tau) = \frac{1}{k^2}\left[\frac{4}{s} + \frac{3k^2-4s}{s^2+k^2}\right] \tag{6.2.174}$$

$$f(\tau) = \frac{1}{k^2}\left[4 + 3k\ \sin(k\tau) - 4\ \cos(k\tau)\right] \tag{6.2.175}$$

$$g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau \tag{6.2.176}$$

$$= \frac{1}{k^2}\left[4\tau - 3\ \cos(k\tau) - \frac{4\ \sin(k\tau)}{k}\right]_0^t \tag{6.2.177}$$

$$= \frac{4t + 3 - 3\ \cos(kt)}{k^2} - \frac{4\ \sin(kt)}{k^3} \tag{6.2.178}$$

29. Using integration to find inverse Laplace transform,

$$G(s) = \frac{F(s)}{s} = \frac{1}{s^2(s+a)} \qquad G(s) = \mathscr{L}\left\{\int_0^t f(\tau)\ \mathrm{d}\tau\right\} \tag{6.2.179}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s(s+a)}\right\} \qquad F(s) = \frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right] \tag{6.2.180}$$

$$f(\tau) = \frac{1-e^{-a\tau}}{a} \tag{6.2.181}$$

$$g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau \qquad = \frac{1}{a}\left[\tau + \frac{e^{-a\tau}}{a}\right]_0^t \tag{6.2.182}$$

$$= \frac{e^{-at} + at - 1}{a^2} \tag{6.2.183}$$

30. **(a)** Theorems 1 and 2 directly help solve IVPs, whereas Theorem 3 only helps derive complicated $\mathscr{L}^{-1}$ expressions from simple ones.

(b) Let $f(t)$ be continuous except for a finite jump discontinuity at $x = a > 0$,

$$\mathscr{L}\{f'\} = \int_0^{a^-} e^{-st} f'(t)\ \ \mathrm{d}t + \int_{a^+}^{\infty} e^{-st} f'(t)\ \ \mathrm{d}t \tag{6.2.184}$$

$$= \Big[e^{-st} f(t)\Big]_0^{a^-} + \Big[e^{-st} f(t)\Big]_{a^+}^{\infty} \tag{6.2.185}$$

$$+\ s\int_0^{a^-} e^{-st} f(t)\ \ \mathrm{d}t + s\int_{a^+}^{\infty} e^{-st} f(t)\ \ \mathrm{d}t \tag{6.2.186}$$

Solving each integral separately,

$$\mathscr{L}\{f'\} = f(a^-)e^{-as} - f(0) - e^{-as}f(a^+) + sF(s) \tag{6.2.187}$$

$$= s\,\mathscr{L}\{f\} - f(0) - e^{-as}[f(a^+) - f(a^-)] \tag{6.2.188}$$

The existence of $\mathscr{L}\{f\}$ is guaranteed by the fact that it only has a finite jump discontinuity.

(c) Verifying,

$$f(t) = \begin{cases} e^{-t} & t \in (0, 1) \\ 0 & t > 1 \end{cases} \tag{6.2.189}$$

$$\mathscr{L}\{f'\} = s\int_0^1 e^{-st-t}\ \mathrm{d}t - 1 - e^{-s}[0 - e^{-1}] \tag{6.2.190}$$

$$= s\left[\frac{e^{-(s+1)t}}{(s+1)}\right]_1^0 - 1 + e^{-s-1} \tag{6.2.191}$$

$$= \left[1 - e^{-(s+1)}\right]\left[\frac{s}{s+1} - 1\right] \tag{6.2.192}$$

$$= \frac{e^{-(s+1)} - 1}{s+1} \tag{6.2.193}$$

Using the normal method to find $\mathscr{L}\{f'\}$

$$\mathscr{L}\{f'\} = -\int_0^1 e^{-(s+1)t}\ \mathrm{d}t \qquad = \left[\frac{e^{-(s+1)t}}{(s+1)}\right]_0^1 \tag{6.2.194}$$

$$= \frac{e^{-(s+1)} - 1}{(s+1)} \tag{6.2.195}$$

Both methods agree.

(d) Refer notes. TBC.
Laplace transform method deals with finite jump discontinuities well.

6.3 Unit Step Function, Second Shifting Theorem

1. Refer notes. TBC.
First shitfing theorem helps identify the effect on $\mathscr{L}^{-1}$ of translations of the Laplace transfor along the s -axis.

$$F(s) \to F(s-a) \tag{6.3.1}$$

$$\implies \qquad f(t) \to e^{at}f(t) \tag{6.3.2}$$

Second shitfing theorem helps identify the effect on $\mathscr{L}$ of translations of the original function along the t -axis.

$$f(t) \to f(t-a)u(t-a) \tag{6.3.3}$$

$$\implies \qquad F(s) \to e^{-as}F(s) \tag{6.3.4}$$

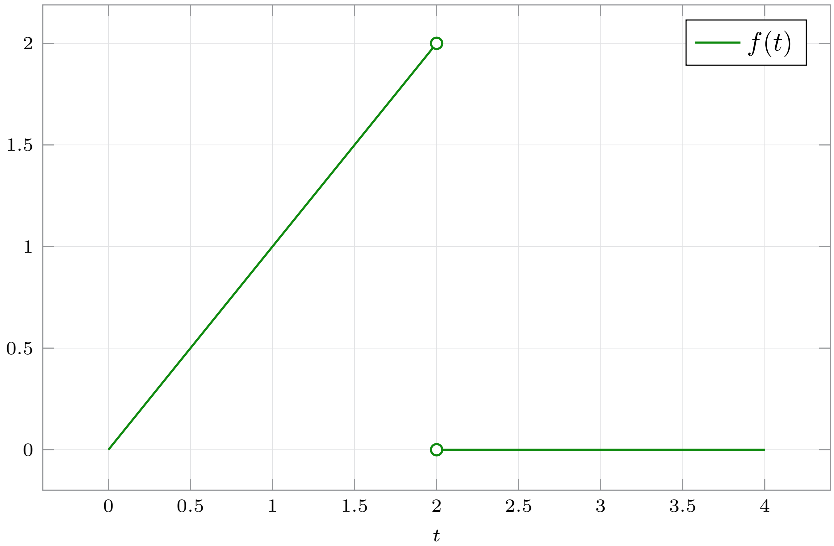
2. Graphing and finding Laplace transform,

$$f(t) = \begin{cases} t & t \in (0, 2) \end{cases} \tag{6.3.5}$$

$$f(t) = t[u(t-0) - u(t-2)] \tag{6.3.6}$$

$$= [(t-0)]u(t-0) - [(t-2)+2]u(t-2) \tag{6.3.7}$$

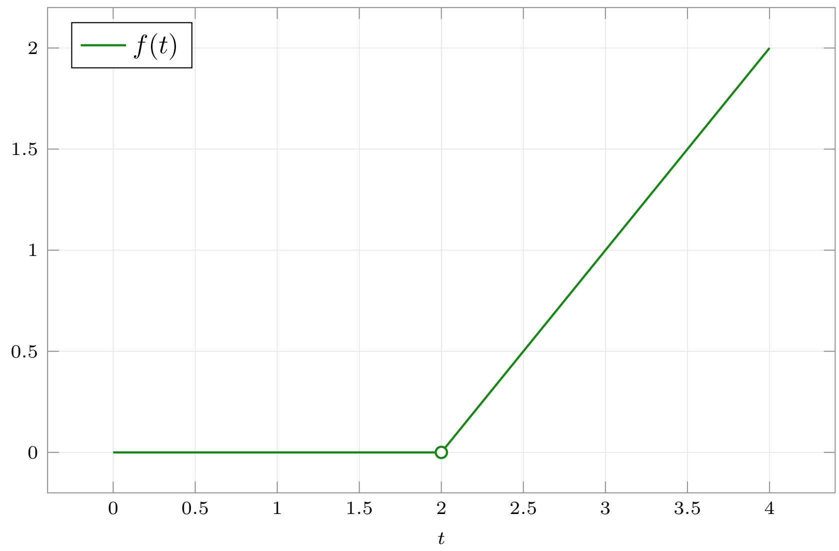
$$F(s) = \frac{1 - e^{-2s}(2s+1)}{s^2} \tag{6.3.8}$$



3. Graphing and finding Laplace transform,

$$f(t) = \begin{cases} t-2 & t > 2 \end{cases} \qquad f(t) = (t-2)[u(t-2)] \tag{6.3.9}$$

$$F(s) = \frac{e^{-2s}}{s^2} \tag{6.3.10}$$



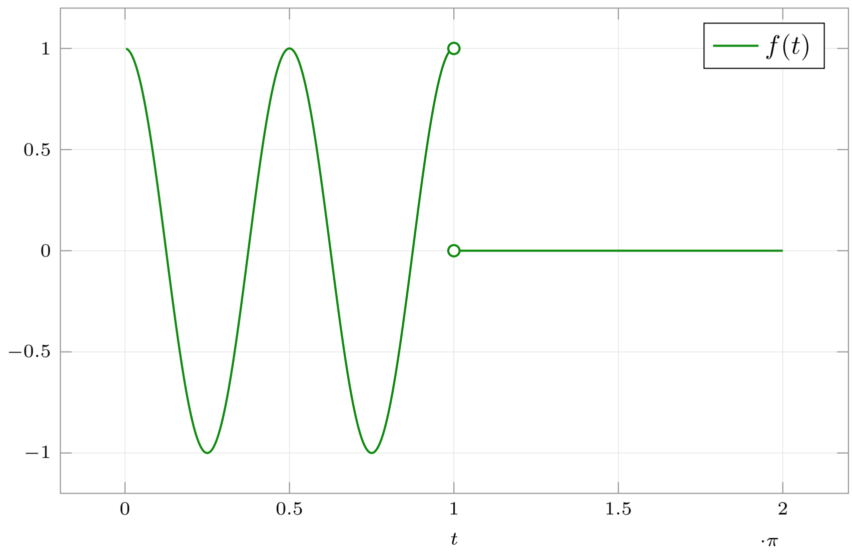
4. Graphing and finding Laplace transform,

$$f(t) = \begin{cases} \cos(4t) & t \in (0, \pi) \end{cases} \tag{6.3.11}$$

$$f(t) = cos(4t)[u(t) - u(t - \pi)] \tag{6.3.12}$$

$$= [\cos(4t)]u(t) - [\cos(4t - 4\pi)]u(t - \pi) \tag{6.3.13}$$

$$F(s) = \frac{s(1 - e^{-\pi s})}{s^2 + 16} \tag{6.3.14}$$



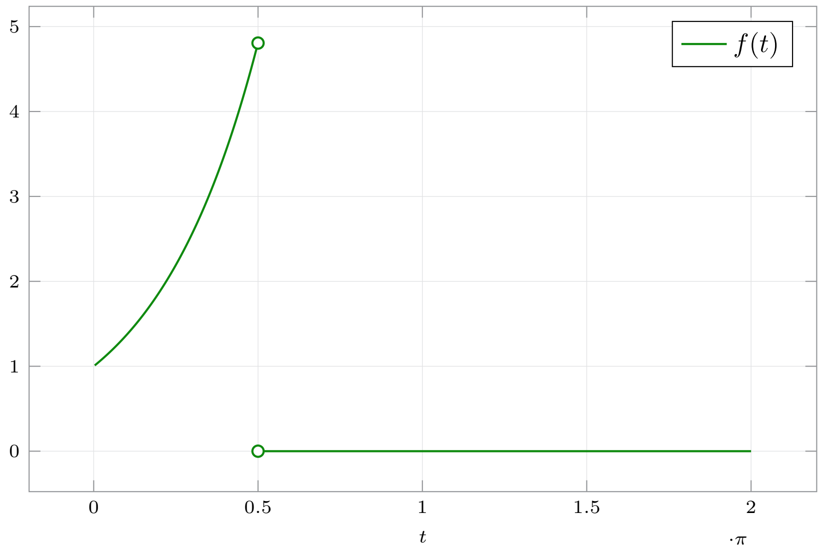
5. Graphing and finding Laplace transform,

$$f(t) = \left\{ e^t \quad t \in (0, \pi/2) \right. \tag{6.3.15}$$

$$f(t) = e^t[u(t) - u(t - \pi/2)] \tag{6.3.16}$$

$$= [e^t]u(t) - [e^{\pi/2}\ e^{t-\pi/2}]u(t - \pi/2) \tag{6.3.17}$$

$$F(s) = \frac{1}{s-1} - \frac{\exp(\pi/2 - \pi s/2)}{s-1} \tag{6.3.18}$$



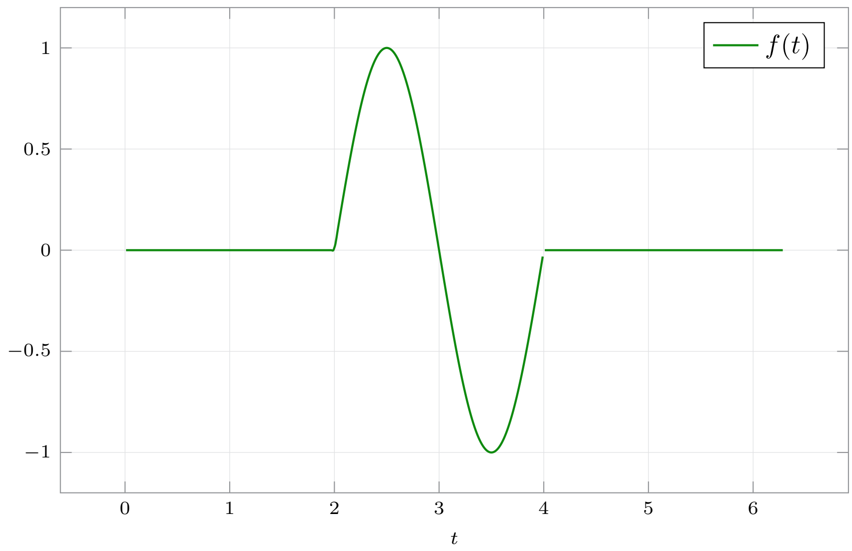
6. Graphing and finding Laplace transform,

$$f(t) = \left\{ \sin(\pi t) \quad t \in (2, 4) \right. \tag{6.3.19}$$

$$f(t) = \sin(\pi t)[u(t - 2) - u(t - 4)] \tag{6.3.20}$$

$$= [\sin(\pi t - 2\pi)]u(t - 2) - [\sin(\pi t - 4\pi)]u(t - 4) \tag{6.3.21}$$

$$F(s) = \frac{\pi[e^{-2s} - e^{-4s}]}{s^2 + \pi^2} \tag{6.3.22}$$



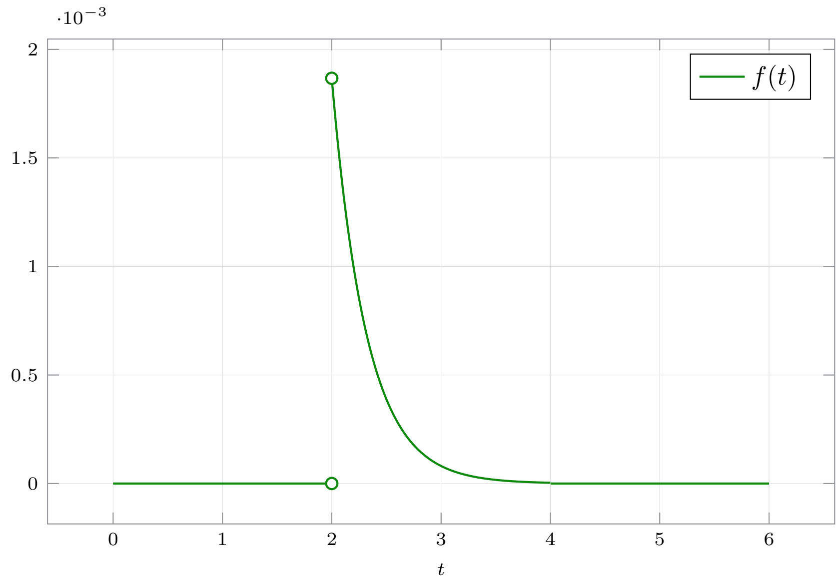
7. Graphing and finding Laplace transform,

$$f(t) = \left\{ e^{-\pi t} \quad t \in (2, 4) \right. \tag{6.3.23}$$

$$f(t) = e^{-\pi t}[u(t-2) - u(t-4)] \tag{6.3.24}$$

$$= e^{-2\pi}[e^{-\pi t + 2\pi}]u(t-2) - e^{-4\pi}[e^{-\pi t + 4\pi}]u(t-4) \tag{6.3.25}$$

$$F(s) = \frac{e^{-2s-2\pi} - e^{-4s-4\pi}}{s+\pi} \tag{6.3.26}$$



8. Graphing and finding Laplace transform,

$$f(t) = \begin{cases} t^2 & t \in (1, 2) \end{cases} \tag{6.3.27}$$

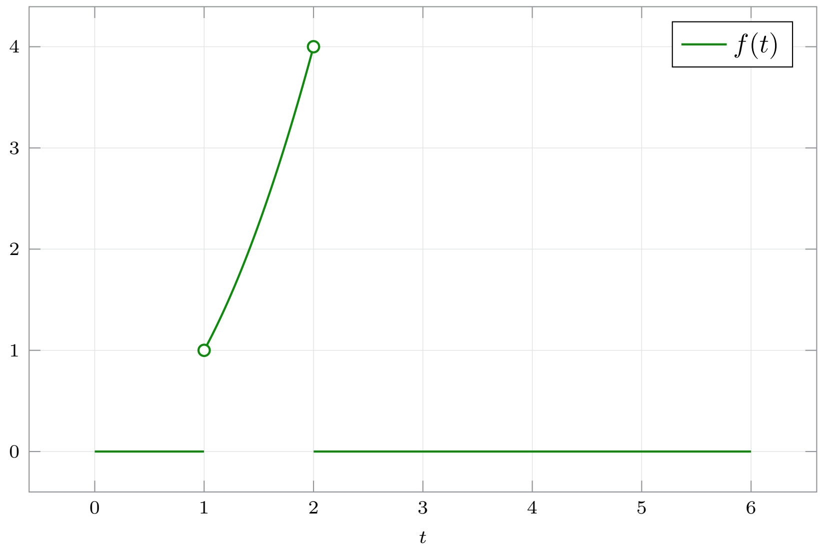
$$f(t) = t^2[u(t-1) - u(t-2)] \tag{6.3.28}$$

$$\mathscr{L}\{g(t)u(t-a)\} = e^{-as}\,\mathscr{L}\{g(t+a)\} \tag{6.3.29}$$

$$F(s) = e^{-s}\,\mathscr{L}\{(t+1)^2\} - e^{-2s}\,\mathscr{L}\{(t+2)^2\} \tag{6.3.30}$$

$$F(s) = e^{-s}\left[\frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right] - e^{-2s}\left[\frac{2!}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] \tag{6.3.31}$$

$$= \frac{e^{-s}(s^2 + 2s + 2) - e^{-2s}(4s^2 + 4s + 2)}{s^3} \tag{6.3.32}$$



9. Graphing and finding Laplace transform,

$$f(t) = \begin{cases} t^2 & t > 1.5 \end{cases} \tag{6.3.33}$$

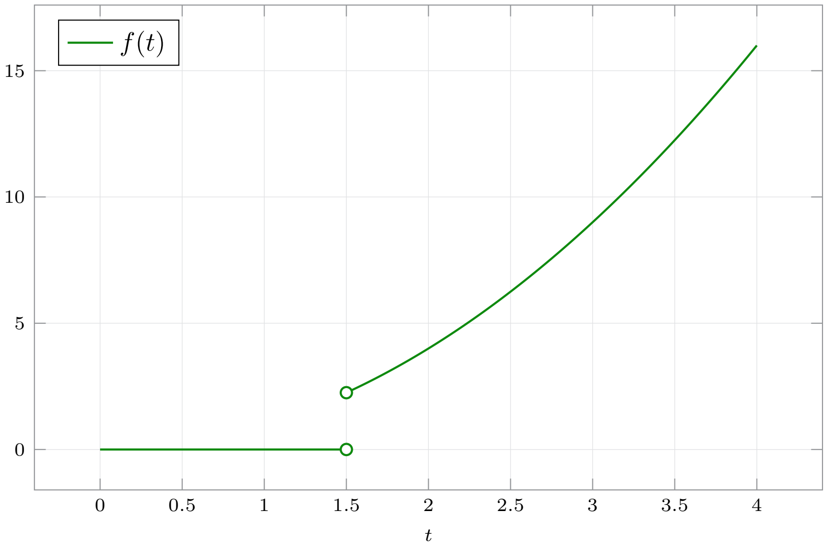
$$f(t) = t^2[u(t-1.5)] \tag{6.3.34}$$

$$\mathscr{L}\{g(t)u(t-a)\} = e^{-as}\,\mathscr{L}\{g(t+a)\} \tag{6.3.35}$$

$$F(s) = e^{-1.5s}\,\mathscr{L}\{(t+1.5)^2\} \tag{6.3.36}$$

$$F(s) = e^{-1.5s}\left[\frac{2!}{s^3} + \frac{3}{s^2} + \frac{2.25}{s}\right] \tag{6.3.37}$$

$$= \frac{e^{-1.5s}(2.25s^2 + 3s + 2)}{s^3} \tag{6.3.38}$$



10. Graphing and finding Laplace transform,

$$f(t) = \begin{cases} \sinh(t) & t \in (0, 2) \end{cases} \tag{6.3.39}$$

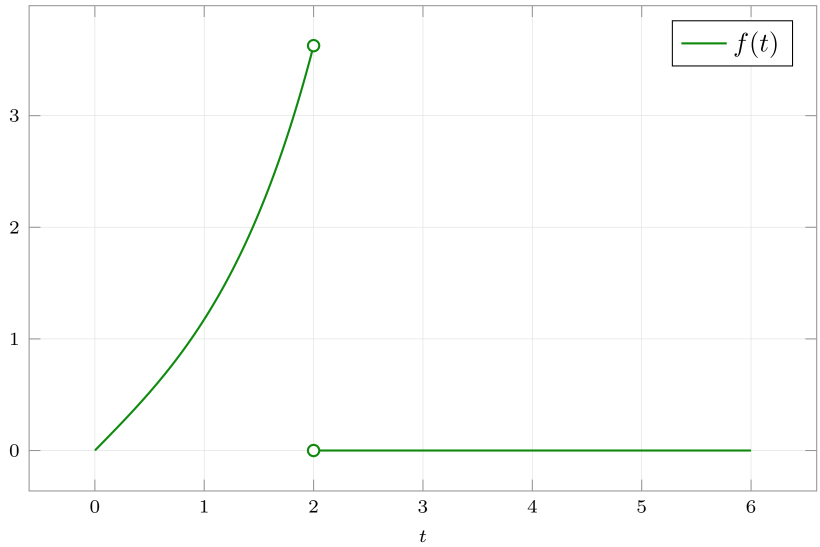
$$f(t) = \sinh(t)[u(t) - u(t-2)] \tag{6.3.40}$$

$$\mathscr{L}\{g(t)u(t-a)\} = e^{-as}\,\mathscr{L}\{g(t+a)\} \tag{6.3.41}$$

$$F(s) = \mathscr{L}\{\sinh(t)\} - e^{-2s}\,\mathscr{L}\{\sinh(t+2)\} \tag{6.3.42}$$

$$F(s) = \frac{1}{s^2-1} - e^{-2s}\left[\frac{e^2}{2(s-1)} - \frac{e^{-2}}{2(s+1)}\right] \tag{6.3.43}$$

$$= \frac{1 - e^{-2s}(s\sinh(2) + \cosh(2))}{s^2-1} \tag{6.3.44}$$



11. Graphing and finding Laplace transform,

$$f(t) = \begin{cases} \sin(t) & t \in (\pi/2, \pi) \end{cases} \tag{6.3.45}$$

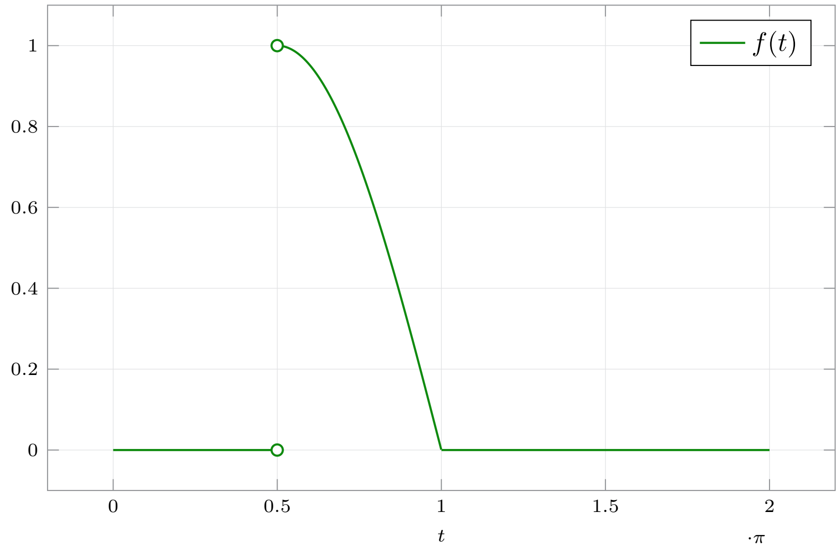
$$f(t) = \sinh(t)[u(t - \pi/2) - u(t - \pi)] \tag{6.3.46}$$

$$\mathscr{L}\{g(t)u(t-a)\} = e^{-as}\,\mathscr{L}\{g(t+a)\} \tag{6.3.47}$$

$$F(s) = e^{-\pi s/2}\,\mathscr{L}\{\sin(t + \pi/2)\} - e^{-\pi s}\,\mathscr{L}\{\sin(t + \pi)\} \tag{6.3.48}$$

$$F(s) = e^{-\pi s/2}\,\mathscr{L}\{\cos(t)\} - e^{-\pi s}\,\mathscr{L}\{-\sin(t)\} \tag{6.3.49}$$

$$F(s) = \frac{e^{-\pi s/2}s + e^{-\pi s}}{s^2 + 1} \tag{6.3.50}$$

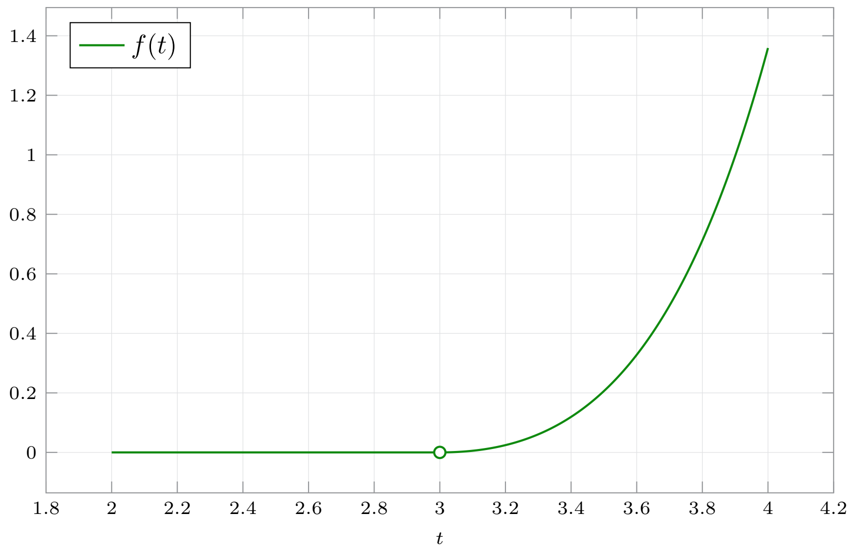


12. Graphing and finding inverse Laplace transform,

$$F(s) = \frac{e^{-3s}}{(s-1)^3} \qquad G(s) = \frac{1}{(s-1)^3} \tag{6.3.51}$$

$$g(t) = \frac{t^2\,e^t}{2} \qquad f(t) = g(t-3)u(t-3) \tag{6.3.52}$$

$$f(t) = \begin{cases} \dfrac{(t-3)^2\,e^{t-3}}{2} & t > 3 \end{cases} \tag{6.3.53}$$

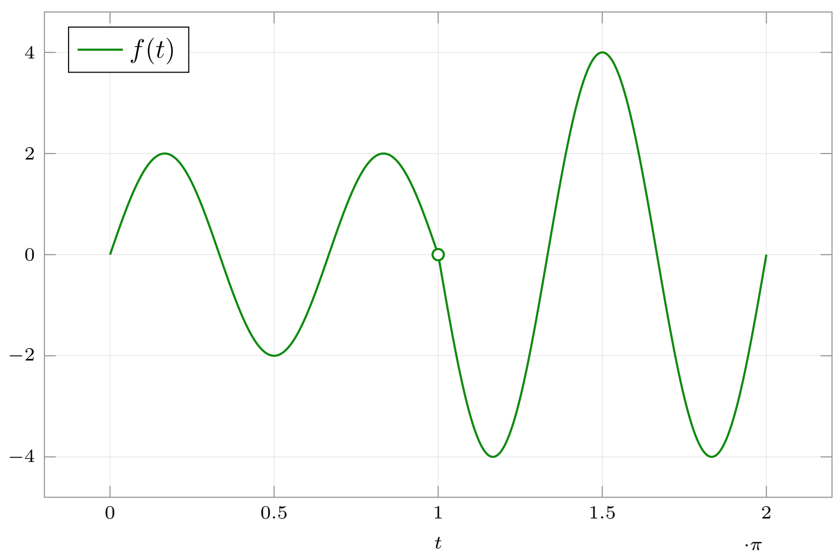


13. Graphing and finding inverse Laplace transform,

$$F(s) = \frac{6(1 - e^{-\pi s})}{s^2 + 9} \qquad G(s) = \frac{3}{s^2 + 9} \tag{6.3.54}$$

$$g(t) = \sin(3t) \qquad f(t) = 2g(t) - 2g(t - \pi)u(t - \pi) \tag{6.3.55}$$

$$f(t) = \begin{cases} 2\sin(3t) & t < \pi \\ 4\sin(3t) & t > \pi \end{cases} \tag{6.3.56}$$

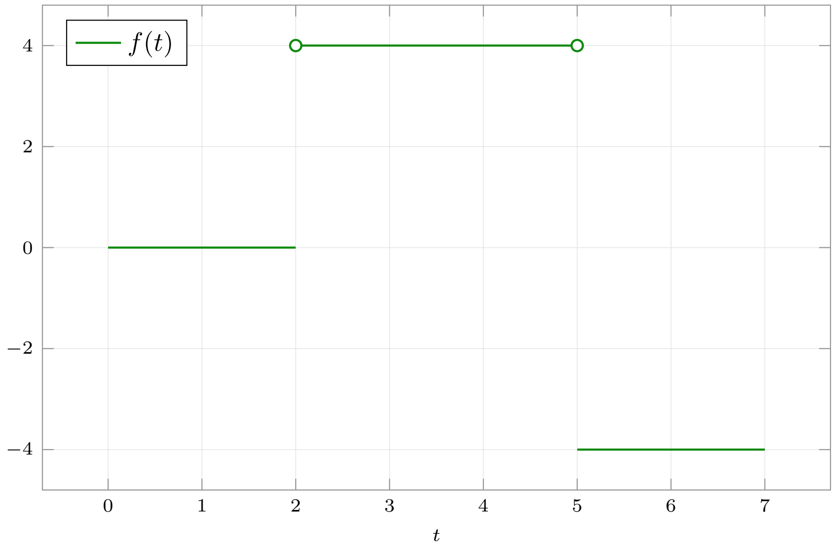


14. Graphing and finding inverse Laplace transform,

$$F(s) = \frac{4e^{-2s} - 8e^{-5s}}{s} \qquad G(s) = \frac{1}{s} \tag{6.3.57}$$

$$g(t) = 1 \qquad f(t) = 4u(t-2) - 8u(t-5) \tag{6.3.58}$$

$$f(t) = \begin{cases} 0 & t < 2 \\ 4 & t \in (2,5) \\ -4 & t > 5 \end{cases} \tag{6.3.59}$$

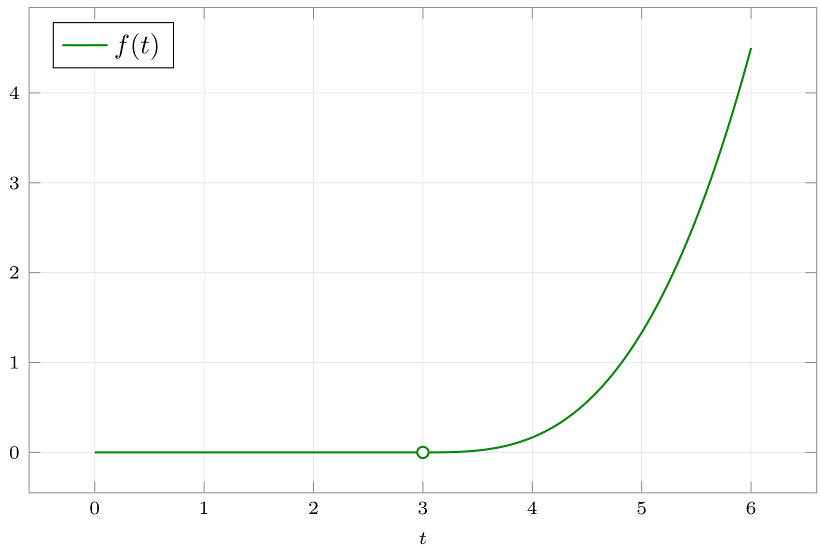


15. Graphing and finding inverse Laplace transform,

$$F(s) = \frac{e^{-3s}}{s^4} \qquad G(s) = \frac{1}{s^4} \tag{6.3.60}$$

$$g(t) = \frac{t^3}{3!} \qquad f(t) = g(t-3)u(t-3) \tag{6.3.61}$$

$$f(t) = \begin{cases} 0 & t < 3 \\ \dfrac{(t-3)^3}{6} & t > 3 \end{cases} \tag{6.3.62}$$



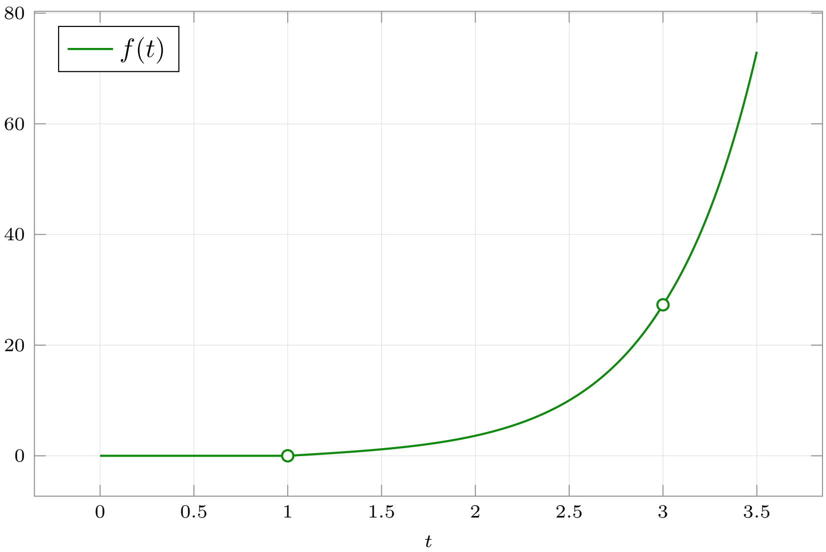
16. Graphing and finding inverse Laplace transform,

$$F(s) = \frac{2(e^{-s} - e^{-3s})}{s^2 - 4} \qquad G(s) = \frac{2}{s^4 - 4} \tag{6.3.63}$$

$$g(t) = \sinh(2t) \tag{6.3.64}$$

$$f(t) = g(t-1)u(t-1) - g(t-3)u(t-3) \tag{6.3.65}$$

$$f(t) = \begin{cases} 0 & t < 1 \\ \sinh(2t-2) & t \in (1,3) \\ \sinh(2t-2) - \sinh(2t-6) & t > 3 \end{cases} \tag{6.3.66}$$



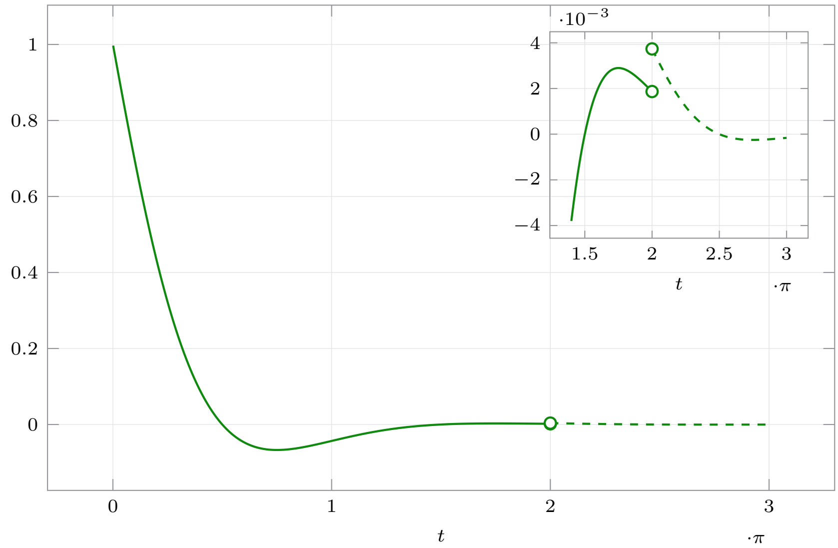
17. Graphing and finding inverse Laplace transform,

$$F(s) = \frac{(s+1)[1+e^{-2\pi(s+1)}]}{(s+1)^2+1} \qquad G(s) = \frac{s(1+e^{-2\pi s})}{s^2+1} \tag{6.3.67}$$

$$g(t) = \cos(t) + \cos(t-2\pi)u(t-2\pi) \tag{6.3.68}$$

$$f(t) = e^{-t}\, g(t) \tag{6.3.69}$$

$$f(t) = \begin{cases} e^{-t}\cos(t) & t < 2\pi \\ 2e^{-t}\cos(t) & t > 2\pi \end{cases} \tag{6.3.70}$$



18. Solving the ODE,

$$0 = 9y'' - 6y' + y \tag{6.3.71}$$

$$y(0) = 3 \qquad y'(0) = 1 \tag{6.3.72}$$

$$0 = 9[s^2Y - sy(0) - y'(0)] - 6[sY - y(0)] + Y \tag{6.3.73}$$

$$0 = Y[9s^2 - 6s + 1] - (9s-6)y(0) - 9y'(0) \tag{6.3.74}$$

$$Y = \frac{9(3s-1)}{(3s-1)^2} = \frac{3}{s-1/3} \tag{6.3.75}$$

$$y = 3e^{t/3} \tag{6.3.76}$$

19. Finding the laplace transform of the input,

$$r(t) = e^{-3t} - e^{-5t} \tag{6.3.77}$$

$$R(s) = \frac{1}{(s+3)} - \frac{1}{(s+5)} = \frac{2}{(s+5)(s+3)} \tag{6.3.78}$$

Solving the ODE,

$$r(t) = y'' + 6y' + 8y \tag{6.3.79}$$

$$y(0) = 0 \qquad y'(0) = 0 \tag{6.3.80}$$

$$R(s) = [s^2Y - sy(0) - y'(0)] + 6[sY - y(0)] + 8Y \tag{6.3.81}$$

$$R(s) = Y[s^2 + 6s + 8] - (s+6)y(0) - y'(0) \tag{6.3.82}$$

$$Y = \frac{2}{(s+5)(s+3)} \frac{1}{(s+2)(s+4)} \tag{6.3.83}$$

$$Y = \frac{1}{3}\left[\frac{1}{(s+2)} - \frac{3}{(s+3)} + \frac{3}{(s+4)} - \frac{1}{(s+5)}\right] \tag{6.3.84}$$

$$y = \frac{e^{-2t} - 3e^{-3t} + 3e^{-4t} - e^{-5t}}{3} \tag{6.3.85}$$

20. Finding the laplace transform of the input,

$$r(t) = 144t^2 \tag{6.3.86}$$

$$R(s) = \frac{288}{s^3} \tag{6.3.87}$$

Solving the ODE,

$$r(t) = y'' + 10y' + 24y \tag{6.3.88}$$

$$y(0) = 19/12 \qquad y'(0) = -5 \tag{6.3.89}$$

$$R(s) = [s^2Y - sy(0) - y'(0)] + 10[sY - y(0)] + 24Y \tag{6.3.90}$$

$$R(s) = Y[s^2 + 10s + 24] - (s + 10)y(0) - y'(0) \tag{6.3.91}$$

$$Y = \frac{1}{(s+4)(s+6)} \left[\frac{288}{s^3} + \frac{19s + 130}{12}\right] \tag{6.3.92}$$

$$Y = \frac{1}{(s+4)(s+6)} \left[\frac{3456 + 19s^4 + 130s^3}{12s^3}\right] \tag{6.3.93}$$

$$Y = \frac{(19/12)s^2 - 5s + 12}{s^3} \tag{6.3.94}$$

$$y = \frac{19}{12} - 5t + 6t^2 \tag{6.3.95}$$

21. Finding the laplace transform of the input,

$$r(t) = \begin{cases} 8\sin(t) & t \in (0, \pi) \\ 0 & t > \pi \end{cases} \qquad r(t) = 8\sin(t)[1 - u(t - \pi)] \tag{6.3.96}$$

$$r(t) = 8\sin(t) + 8\sin(t - \pi)u(t - \pi) \qquad R(s) = \frac{8(1 + e^{-\pi s})}{s^2 + 1} \tag{6.3.97}$$

Solving the ODE,

$$r(t) = y'' + 9y \tag{6.3.98}$$

$$y(0) = 0 \qquad y'(0) = 4 \tag{6.3.99}$$

$$R(s) = [s^2Y - sy(0) - y'(0)] + 9Y \tag{6.3.100}$$

$$R(s) = Y[s^2 + 9] - sy(0) - y'(0) \tag{6.3.101}$$

$$Y = \frac{1}{(s^2+9)} \left[\frac{8e^{-\pi s} + 4s^2 + 12}{s^2 + 1}\right] \tag{6.3.102}$$

$$Y = \frac{1}{s^2+1} + \frac{3}{s^2+9} + e^{-\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] \tag{6.3.103}$$

Restating the function using the piecewise method,

$$y = \sin(t) + \sin(3t) + \sin(t-\pi)u(t-\pi) - \frac{1}{3}\sin(3t-3\pi)u(t-\pi) \tag{6.3.104}$$

$$y = \sin(t)[1-u(t-\pi)] + \sin(3t)[1+(1/3)u(t-\pi)] \tag{6.3.105}$$

$$y = \begin{cases} \sin(t)+\sin(3t) & t \in (0,\pi) \\ \frac{4}{3}\sin(3t) & t > \pi \end{cases} \tag{6.3.106}$$

22. Finding the laplace transform of the input,

$$r(t) = \begin{cases} 4t & t \in (0,1) \\ 8 & t > 1 \end{cases} \qquad r(t) = 4t[1-u(t-1)] + 8[u(t-1)] \tag{6.3.107}$$

$$r(t) = 4t + [4-4(t-1)]u(t-1) \qquad R(s) = \frac{4+4e^{-s}(s-1)}{s^2} \tag{6.3.108}$$

Solving the ODE,

$$r(t) = y'' + 3y' + 2y \tag{6.3.109}$$

$$y(0) = 0 \qquad y'(0) = 0 \tag{6.3.110}$$

$$R(s) = [s^2Y - sy(0) - y'(0)] + 3[sY - y(0)] + 2Y \tag{6.3.111}$$

$$R(s) = Y[s^2+3s+2] - (s+3)y(0) - y'(0) \tag{6.3.112}$$

$$Y = \frac{4+4e^{-s}(s-1)}{s^2(s+1)(s+2)} \tag{6.3.113}$$

$$Y = \frac{-3s+2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2} + e^{-s}\left[\frac{5s-2}{s^2} - \frac{8}{s+1} + \frac{3}{s+2}\right] \tag{6.3.114}$$

Restating the function using the piecewise method,

$$y = -3 + 2t + 4e^{-t} - e^{-2t} + [7 - 2t - 8e^{1-t} + 3e^{2-2t}]u(t-1) \tag{6.3.115}$$

$$y = \begin{cases} -3+2t+4e^{-t}-e^{-2t} & t < 1 \\ 4+(4-8e)e^{-t}+(3e^2-1)e^{-2t} & t > 1 \end{cases} \tag{6.3.116}$$

23. Finding the laplace transform of the input,

$$r(t) = \begin{cases} 3\sin(t) - \cos(t) & t < 2\pi \\ 3\sin(2t) - \cos(2t) & t > 2\pi \end{cases} \quad 6.3.117$$

$$r(t) = [3\sin(t) - \cos(t)][1 - u(t - 2\pi)] + [3\sin(2t) - \cos(2t)]u(t - 2\pi) \quad 6.3.118$$

$$r(t) = 3\sin t - \cos t - [3\sin t - \cos t - 3\sin(2t) + \cos(2t)]u(t - 2\pi) \quad 6.3.119$$

$$R(s) = \frac{3 - s}{s^2 + 1} + e^{-2\pi s}\left[\frac{s - 3}{s^2 + 1} + \frac{6 - s}{s^2 + 4}\right] \quad 6.3.120$$

Solving the ODE,

$$r(t) = y'' + y' - 2y \quad 6.3.121$$

$$y(0) = 1 \qquad y'(0) = 0 \quad 6.3.122$$

$$R(s) = [s^2 Y - sy(0) - y'(0)] + [sY - y(0)] - 2Y \quad 6.3.123$$

$$R(s) = Y[s^2 + s - 2] - (s + 1)y(0) - y'(0) \quad 6.3.124$$

$$Y = \frac{4 + s^3 + s^2}{(s^2 + 1)(s + 2)(s - 1)} + \frac{e^{-2\pi s}}{(s + 2)(s - 1)}\left[\frac{s - 3}{s^2 + 1} + \frac{6 - s}{s^2 + 4}\right] \quad 6.3.125$$

$$Y = \frac{-1}{s^2 + 1} + \frac{1}{s - 1} + e^{-2\pi s}\left[\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4}\right] \quad 6.3.126$$

Restating the function using the piecewise method,

$$y = -\sin(t) + e^t + [\sin(t) - 0.5\sin(2t)]u(t - 2\pi) \quad 6.3.127$$

$$y = \begin{cases} -\sin(t) + e^t & t < 2\pi \\ -0.5\sin(2t) + e^t & t > 2\pi \end{cases} \quad 6.3.128$$

24. Finding the laplace transform of the input,

$$r(t) = \begin{cases} 1 & t < 1 \\ 0 & t > 1 \end{cases} \qquad r(t) = [1 - u(t - 1)] \quad 6.3.129$$

$$R(s) = \frac{1 - e^{-s}}{s} \quad 6.3.130$$

Solving the ODE,

$$r(t) = y'' + 3y' + 2y \tag{6.3.131}$$

$$y(0) = 0 \qquad y'(0) = 0 \tag{6.3.132}$$

$$R(s) = [s^2Y - sy(0) - y'(0)] + 3[sY - y(0)] + 2Y \tag{6.3.133}$$

$$R(s) = Y[s^2 + 3s + 2] - (s + 2)y(0) - y'(0) \tag{6.3.134}$$

$$Y = \frac{1 - e^{-s}}{s(s + 2)(s + 1)} \tag{6.3.135}$$

$$Y = [1 - e^{-s}]\left[\frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2}\right] \tag{6.3.136}$$

Restating the function using the piecewise method,

$$y = 0.5 - e^{-t} + 0.5e^{-2t} - \left[0.5 - e^{1-t} + 0.5e^{2-2t}\right]u(t - 1) \tag{6.3.137}$$

$$y = \begin{cases} 0.5 - e^{-t} + 0.5e^{-2t} & t < 1 \\ e^{-t}[e - 1] + 0.5e^{-2t}[1 - e^2] & t > 1 \end{cases} \tag{6.3.138}$$

25. Finding the laplace transform of the input,

$$r(t) = \begin{cases} t & t < 1 \\ 0 & t > 1 \end{cases} \qquad r(t) = t[1 - u(t - 1)] \tag{6.3.139}$$

$$r(t) = t - [(t - 1) + 1]u(t - 1) \qquad R(s) = \frac{1}{s^2} - e^{-s}\,\frac{(1 + s)}{s^2} \tag{6.3.140}$$

Solving the ODE,

$$r(t) = y'' + y \qquad y(0) = 0 \qquad y'(0) = 0 \tag{6.3.141}$$

$$R(s) = [s^2Y - sy(0) - y'(0)] + Y \qquad R(s) = Y[s^2 + 1] - (s)y(0) - y'(0) \tag{6.3.142}$$

$$Y = \frac{1 - e^{-s}(1 + s)}{s^2(s^2 + 1)} \tag{6.3.143}$$

$$Y = \frac{1}{s^2} - \frac{1}{s^2 + 1} - e^{-s}\left[\frac{s + 1}{s^2} - \frac{(s + 1)}{s^2 + 1}\right] \tag{6.3.144}$$

Restating the function using the piecewise method,

$$y = t - \sin(t) - \Big[t - \cos(t-1) - \sin(t-1)\Big]u(t-1) \tag{6.3.145}$$

$$y = \begin{cases} t - \sin(t) & t < 1 \\ -\sin(t) + \cos(t-1) + \sin(t-1) & t > 1 \end{cases} \tag{6.3.146}$$

26. Finding the laplace transform of the input, with $x = t - \pi$

$$r(t) = \begin{cases} 10\sin(x+\pi) & x \in (-\pi, \pi) \\ 0 & x > \pi \end{cases} \qquad r(t) = [-10\sin(x)][1 - u(x-\pi)] \tag{6.3.147}$$

$$r(t) = -10\sin(x) - [10\sin(x-\pi)]u(x-\pi) \qquad R(s) = -\frac{10\,(1 + e^{-\pi s})}{s^2+1} \tag{6.3.148}$$

Solving the ODE,

$$r(t) = y'' + 2y' + 5y \tag{6.3.149}$$

$$y(0) = 1 \qquad y'(0) = 2e^{-\pi} - 2 \tag{6.3.150}$$

$$R(s) = [s^2Y - sy(0) - y'(0)] + 2[sY - y(0)] + 5Y \tag{6.3.151}$$

$$R(s) = Y[s^2 + 2s + 5] - (s+2)y(0) - y'(0) \tag{6.3.152}$$

$$Y = \frac{-10 + s^3 + s + 2e^{-\pi}(s^2+1)}{(s^2+1)(s^2+2s+5)} - \frac{10e^{-\pi s}}{(s^2+1)(s^2+2s+5)} \tag{6.3.153}$$

$$Y = \frac{s-2}{s^2+1} + \frac{2e^{-\pi}}{s^2+2s+5} + e^{-\pi s}\left[\frac{s-2}{s^2+1} - \frac{(s+1)-1}{(s+1)^2+4}\right] \tag{6.3.154}$$

Restating the function using the piecewise method,

$$y = \cos x - 2\sin x + e^{-\pi - x}\sin(2x) \tag{6.3.155}$$

$$+ \Big[-\cos(x) + 2\sin(x) - e^{-x+\pi}\{\cos(2x) - 0.5\sin(2x)\}\Big]u(x-\pi) \tag{6.3.156}$$

$$y = -\cos(t) + 2\sin(t) + e^{-t}\sin(2t) \tag{6.3.157}$$

$$+ \Big[\cos(t) - 2\sin(t) - e^{-t+2\pi}\{\cos(2t) - 0.5\sin(2t)\}\Big]u(t-2\pi) \tag{6.3.158}$$

$$y = \begin{cases} 2\sin(t) - \cos(t) + e^{-t}\sin(2t) & t < \pi \\ e^{-t}\sin(2t)[1 + 0.5e^{2\pi}] - e^{-t+2\pi}\cos(2t) & t > \pi \end{cases} \tag{6.3.159}$$

27. Finding the laplace transform of the input, with $x = t - 1$

$$r(t) = \begin{cases} 8(x+1)^2 & x \in (-1, 4) \\ 0 & x > 4 \end{cases} \tag{6.3.160}$$

$$r(t) = 8[x^2 + 2x + 1][1 - u(x-4)] \tag{6.3.161}$$

$$r(t) = 8[x^2 + 2x + 1] - 8u(x-4)[(x-4)^2 + 5^2 + 10(x-4)] \tag{6.3.162}$$

$$R(s) = \frac{8(2 + 2s + s^2)}{s^3} - 8e^{-4s}\left[\frac{2 + 10s + 25s^2}{s^3}\right] \tag{6.3.163}$$

Solving the ODE,

$$r(t) = y'' + 4y \tag{6.3.164}$$

$$y(0) = 1 + \cos(2) \qquad y'(0) = 4 - 2\sin(2) \tag{6.3.165}$$

$$R(s) = [s^2Y - sy(0) - y'(0)] + 4Y \tag{6.3.166}$$

$$R(s) = Y[s^2 + 4] - (s)y(0) - y'(0) \tag{6.3.167}$$

$$Y = \frac{-10 + s^3 + s + 2e^{-\pi}(s^2+1)}{(s^2+1)(s^2+2s+5)} - \frac{10e^{-\pi s}}{(s^2+1)(s^2+2s+5)} \tag{6.3.168}$$

$$Y = \frac{s-2}{s^2+1} + \frac{2e^{-\pi}}{s^2+2s+5} + e^{-\pi s}\left[\frac{s-2}{s^2+1} - \frac{(s+1)-1}{(s+1)^2+4}\right] \tag{6.3.169}$$

Restating the function using the piecewise method,

$$y = \cos x - 2\sin x + e^{-\pi - x}\sin(2x) \tag{6.3.170}$$

$$+ \Big[-\cos(x) + 2\sin(x) - e^{-x+\pi}\{\cos(2x) - 0.5\sin(2x)\}\Big]u(x-\pi) \tag{6.3.171}$$

$$y = -\cos(t) + 2\sin(t) + e^{-t}\sin(2t) \tag{6.3.172}$$

$$+ \Big[\cos(t) - 2\sin(t) - e^{-t+2\pi}\{\cos(2t) - 0.5\sin(2t)\}\Big]u(t-2\pi) \tag{6.3.173}$$

$$y = \begin{cases} 2\sin(t) - \cos(t) + e^{-t}\sin(2t) & t < \pi \\ e^{-t}\sin(2t)[1 + 0.5e^{2\pi}] - e^{-t+2\pi}\cos(2t) & t > \pi \end{cases} \tag{6.3.174}$$

28. Solving the given RL circuit,

$$v(t) = \begin{cases} 0 & t \in (0, \pi) \\ 40\sin(t) & t > \pi \end{cases} \qquad v(t) = [40\sin(t)]u(t-\pi) \tag{6.3.175}$$

$$v(t) = [-40\sin(t-\pi)]u(t-\pi) \qquad V(s) = \frac{-40e^{-\pi s}}{s^2+1} \tag{6.3.176}$$

Solving the ODE, with $R = 1000$, $L = 1$

$$v(t) = j' + 1000j \tag{6.3.177}$$

$$y(0) = 0 \tag{6.3.178}$$

$$V(s) = sJ - j(0) + 1000J \tag{6.3.179}$$

$$V(s) = J[s + 1000] - 0 \tag{6.3.180}$$

$$J = \frac{-40e^{-\pi s}}{(s^2+1)(s+1000)} \qquad \mu = \frac{40}{10^6+1} \tag{6.3.181}$$

$$J = \mu e^{-\pi s}\left[\frac{(s-1000)}{s^2+1} - \frac{1}{s+1000}\right] \tag{6.3.182}$$

Restating the function using the piecewise method,

$$j = \mu[-\cos t + 1000\sin t - e^{-1000(t-\pi)}]U(t-\pi) \tag{6.3.183}$$

$$y = \begin{cases} 0 & t < \pi \\ \mu[-\cos(t) + 1000\sin(t) - e^{-1000(t-\pi)}] & t > \pi \end{cases} \tag{6.3.184}$$

29. Solving the given RL circuit,

$$v(t) = \begin{cases} 490e^{-5t} & t < 1 \\ 0 & t > 1 \end{cases} \qquad v(t) = 490e^{-5t}[1 - u(t-1)] \tag{6.3.185}$$

$$v(t) = 490e^{-5t} - [490e^{-5}\ \exp(-5t+5)]u(t-1) \tag{6.3.186}$$

$$V(s) = \frac{490}{s+5} - \frac{490e^{-(s+5)}}{s+5} \tag{6.3.187}$$

Solving the ODE, with $R = 25$, $L = 0.1$

$$v(t) = 0.1j' + 25j \tag{6.3.188}$$

$$y(0) = 0 \tag{6.3.189}$$

$$V(s) = 0.1sJ - 0.1j(0) + 25J \tag{6.3.190}$$

$$V(s) = J[0.1s + 25] - 0 \tag{6.3.191}$$

$$J = \frac{4900}{(s+5)(s+250)} - \frac{4900e^{-(s+5)}}{(s+5)(s+250)} \tag{6.3.192}$$

$$J = [1 - e^{-(s+5)}]\left[\frac{20}{s+5} - \frac{20}{s+250}\right] \tag{6.3.193}$$

Restating the function using the piecewise method,

$$j = 20e^{-5t} - 20e^{-250t} - e^{-5}\Big[20e^{-5t+5} - 20e^{-250t+250}\Big]u[t-1] \tag{6.3.194}$$

$$y = \begin{cases} 20e^{-5t} - 20e^{-250t} & t < 1 \\ 20e^{-250t}[e^{245} - 1] & t > 1 \end{cases} \tag{6.3.195}$$

30. Solving the given RL circuit,

$$v(t) = \begin{cases} 200t & t < 2 \\ 0 & t > 2 \end{cases} \qquad v(t) = 200t[1 - u(t-2)] \tag{6.3.196}$$

$$v(t) = 200t - 200\Big[(t-2) + 2\Big]u(t-2) \tag{6.3.197}$$

$$V(s) = \frac{200}{s^2} - 200e^{-2s}\left[\frac{1+2s}{s^2}\right] \tag{6.3.198}$$

Solving the ODE, with $R = 10$, $L = 0.5$

$$v(t) = 0.5j' + 10j \tag{6.3.199}$$

$$y(0) = 0 \tag{6.3.200}$$

$$V(s) = 0.5sJ - 0.5j(0) + 10J \tag{6.3.201}$$

$$V(s) = J[0.5s + 10] - 0 \tag{6.3.202}$$

$$J = \frac{400}{s^2(s+20)} - \frac{400e^{-2s}(1+2s)}{s^2(s+20)} \tag{6.3.203}$$

$$J = \left[\frac{-s+20}{s^2} + \frac{1}{s+20}\right] - e^{-2s}\left[\frac{39s+20}{s^2} - \frac{39}{s+20}\right] \tag{6.3.204}$$

Restating the function using the piecewise method,

$$j = -1 + 20t + e^{-20t} + \Big[1 - 20t + 39e^{-20t+40}\Big]u(t-2) \tag{6.3.205}$$

$$y = \begin{cases} -1 + 20t + e^{-20t} & t < 2 \\ [39e^{40} + 1]e^{-20t} & t > 2 \end{cases} \tag{6.3.206}$$

31. Initial charge $q(0) = CV_0$, switch is closed at $t = 0$,

$$Rq' + \frac{q}{C} = 0 \qquad\qquad RCq' = -q \tag{6.3.207}$$

$$(RC)[sQ - q(0)] + Q = 0 \qquad\qquad Q = \frac{(RC)\cdot CV_0}{RCs + 1} \tag{6.3.208}$$

$$Q = \frac{CV_0}{s + (1/RC)} \qquad\qquad q(t) = CV_0 \ \exp\left(\frac{-t}{RC}\right) \tag{6.3.209}$$

32. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 0 & t < 4 \\ 14 \times 10^6 e^{-3t} & t > 4 \end{cases} \tag{6.3.210}$$

$$v(t) = [14 \times 10^6 e^{-3t}]u(t-4) \qquad\qquad v(t) = [14 \times 10^6 e^{-12} e^{-3t+12}]u(t-4) \tag{6.3.211}$$

$$V(s) = \frac{[14 \times 10^6 e^{-12}]e^{-4s}}{s+3} \tag{6.3.212}$$

Initial charge $q(0) = 0,\ \ R = 10,\ \ C = 0.01$,

$$Rq' + \frac{q}{C} = v(t) \tag{6.3.213}$$

$$10[sQ - q(0)] + 100Q = V(s) \tag{6.3.214}$$

$$Q = \frac{14 \times 10^5 \ e^{-12-4s}}{(s+3)(s+10)} \tag{6.3.215}$$

$$Q = e^{-4s-12}(2 \times 10^5) \ \left[\frac{1}{(s+3)} - \frac{1}{(s+10)}\right] \tag{6.3.216}$$

$$q(t) = 2 \times 10^5 e^{-12}\Big[e^{-3t+12} - e^{-10t+40}\Big]u(t-4) \tag{6.3.217}$$

Restating the charge and current in piecewise form,

$$q(t) = 2 \times 10^5 \Big[e^{-3t} - e^{-10t+28}\Big] u(t-4) \tag{6.3.218}$$

$$q'(t) = j(t) = \begin{cases} 0 & t < 4 \\ -6 \times 10^5 e^{-3t} + 2 \times 10^6 e^{28} e^{-10t} & t > 4 \end{cases} \tag{6.3.219}$$

33. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 0 & t < 2 \\ 100(t-2) & t > 2 \end{cases} \tag{6.3.220}$$

$$v(t) = [100(t-2)]u(t-2) \qquad V(s) = \frac{100e^{-2s}}{s^2} \tag{6.3.221}$$

Initial charge $q(0) = 0$, $R = 10$, $C = 0.01$,

$$v(t) = Rq' + \frac{q}{C} \tag{6.3.222}$$

$$V(s) = 10[sQ - q(0)] + 100Q \tag{6.3.223}$$

$$Q = \frac{10e^{-2s}}{s^2(s+10)} \tag{6.3.224}$$

$$Q = \frac{e^{-2s}}{10} \left[\frac{-s+10}{s^2} + \frac{1}{s+10}\right] \tag{6.3.225}$$

$$q(t) = 0.1\Big[-21 + 10t + e^{-10t+20}\Big]u(t-2) \tag{6.3.226}$$

Restating the charge and current in piecewise form,

$$q(t) = 0.1\Big[-21 + 10t + e^{-10t+20}\Big]u(t-2) \tag{6.3.227}$$

$$q'(t) = j(t) = \begin{cases} 0 & t < 2 \\ 1 - e^{-10t+20} & t > 2 \end{cases} \tag{6.3.228}$$

34. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 0 & t < 0.5 \\ 100 & t \in (0.5, 0.6) \\ 0 & t > 0.6 \end{cases} \tag{6.3.229}$$

$$v(t) = 100[u(t-0.5) - u(t-0.6)] \qquad V(s) = \frac{100[e^{-0.5s} - e^{-0.6s}]}{s} \tag{6.3.230}$$

Initial charge $q(0) = 0,\ R = 10,\ C = 0.01,$

$$v(t) = Rq' + \frac{q}{C} \tag{6.3.231}$$

$$V(s) = 10[sQ - q(0)] + 100Q \tag{6.3.232}$$

$$Q = \frac{10[e^{-0.5s} - e^{-0.6s}]}{s(s+10)} \tag{6.3.233}$$

$$Q = [e^{-0.5s} - e^{-0.6s}]\left[\frac{1}{s} - \frac{1}{s+10}\right] \tag{6.3.234}$$

$$q(t) = \left[1 - e^{-10t+5}\right]u(t-0.5) - \left[1 - e^{-10t+6}\right]u(t-0.6) \tag{6.3.235}$$

Restating the charge and current in piecewise form,

$$q(t) = \begin{cases} 0 & t < 0.5 \\ 1 - e^{5}(e^{-10t}) & t \in (0.5, 0.6) \\ e^{-10t}[e^{6} - e^{5}] & t > 0.6 \end{cases} \tag{6.3.236}$$

$$q'(t) = j(t) = \begin{cases} 0 & t < 0.5 \\ 10e^{5}(e^{-10t}) & t \in (0.5, 0.6) \\ -10[e^{6} - e^{5}]e^{-10t} & t > 0.6 \end{cases} \tag{6.3.237}$$

Looking at the jumps in current when the voltage source is switched on and off,

$$j(0.5^{+}) - j(0.5^{-}) = 10 \tag{6.3.238}$$

$$j(0.6^{+}) - j(0.6^{-}) = -10 + 10e^{-1} - 10e^{-1} = -10 \tag{6.3.239}$$

That part of the current flowing through the circuit because of the resistor, instantly gets turned on and off respectively along with the voltage source.

35. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 0 & t < \pi \\ -9900\cos(t) & t \in (\pi, 3\pi) \\ 0 & t > 3\pi \end{cases} \tag{6.3.240}$$

$$v'(t) = 9900\sin(t)[u(t-\pi) - u(t-3\pi)] \tag{6.3.241}$$

$$v'(t) = -9900\sin(t-\pi)[u(t-\pi)] + 9900\sin(t-3\pi)[u(t-3\pi)] \tag{6.3.242}$$

$$V'(s) = \frac{-9900[e^{-\pi s} - e^{-3\pi s}]}{s^2+1} \tag{6.3.243}$$

Initial charge $q(0) = 0,\ L = 1,\ C = 0.01,$
This gives, $j(0) = j'(0) = 0$

$$v'(t) = Lj'' + \frac{j}{C} \tag{6.3.244}$$

$$V'(s) = [s^2 J - sj(0) - j'(0)] + 100J \tag{6.3.245}$$

$$J = \frac{-9900[e^{-\pi s} - e^{-3\pi s}]}{(s^2+1)(s^2+100)} \tag{6.3.246}$$

$$J = [e^{-\pi s} - e^{-3\pi s}]\left[\frac{-100}{s^2+1} + \frac{100}{s^2+100}\right] \tag{6.3.247}$$

$$j(t) = \Big[100\sin(t) + 10\sin(10t)\Big][u(t-\pi) - u(t-3\pi)] \tag{6.3.248}$$

Restating the charge and current in piecewise form,

$$j(t) = \begin{cases} 0 & t < \pi \\ 100\sin(t) + 10\sin(10t) & t \in (\pi, 3\pi) \\ 0 & t > 3\pi \end{cases} \tag{6.3.249}$$

36. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 200t - \dfrac{200t^3}{3} & t < 1 \\ 0 & t > 1 \end{cases} \tag{6.3.250}$$

$$v'(t) = 200(1 - t^2)\,[1 - u(t-1)] \tag{6.3.251}$$

$$v'(t) = 200(1 - t^2) + 200\Big[(t-1)^2 + 2(t-1)\Big][u(t-1)] \tag{6.3.252}$$

$$V'(s) = \frac{200(s^2 - 2)}{s^3} + \frac{400(1+s)e^{-s}}{s^3} \tag{6.3.253}$$

Initial charge $q(0) = 0$, $L = 1$, $C = 0.25$,
This gives, $j(0) = j'(0) = 0$

$$v'(t) = Lj'' + \frac{j}{C} \tag{6.3.254}$$

$$V'(s) = [s^2J - sj(0) - j'(0)] + 4J \tag{6.3.255}$$

$$J = \frac{200(s^2-2)}{s^3(s^2+4)} + \frac{400(s+1)e^{-s}}{s^3(s^2+4)} \tag{6.3.256}$$

$$J = \frac{75s^2 - 100}{s^3} - \frac{75s}{s^2+4} + \left[\frac{-25s^2 + 100s + 100}{s^3} + \frac{25s - 100}{s^2+4}\right] e^{-s} \tag{6.3.257}$$

$$j(t) = 75 - 50t^2 - 75\cos(2t) \tag{6.3.258}$$

$$+ \Big[-75 + 50t^2 + 25\cos(2t-2) - 50\sin(2t-2)\Big]u(t-1) \tag{6.3.259}$$

Restating the charge and current in piecewise form,

$$j(t) = \begin{cases} 75 - 50t^2 - 75\cos(2t) & t < 1 \\ -75\cos(2t) + 25\cos(2t-2) - 50\sin(2t-2) & t > 1 \end{cases} \tag{6.3.260}$$

37. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 78\sin(t) & t < \pi \\ 0 & t > \pi \end{cases} \tag{6.3.261}$$

$$v'(t) = 78\cos(t)[1 - u(t-\pi)] \tag{6.3.262}$$

$$v'(t) = 78\cos(t) + [78\cos(t-\pi)]u(t-\pi) \tag{6.3.263}$$

$$V'(s) = \frac{78s}{s^2+1} + e^{-\pi s}\,\frac{78s}{s^2+1} \tag{6.3.264}$$

Initial charge $q(0) = 0,\ \ L = 0.5,\ \ C = 0.05,$
This gives, $j(0) = j'(0) = 0$

$$v'(t) = Lj'' + \frac{j}{C} \tag{6.3.265}$$

$$V'(s) = 0.5[s^2 J - sj(0) - j'(0)] + 20J \tag{6.3.266}$$

$$J = \frac{156s(1 + e^{-\pi s})}{(s^2 + 1)(s^2 + 40)} \tag{6.3.267}$$

$$J = (1 + e^{-\pi s}) \left[\frac{4s}{s^2 + 1} - \frac{4s}{s^2 + 40}\right] \tag{6.3.268}$$

$$j(t) = 4\ \cos(t) - 4\ \cos(\sqrt{40}t) - [4\ \cos(t) + 4\ \cos(\sqrt{40}t - \sqrt{40}\pi)]u(t - \pi) \tag{6.3.269}$$

Restating the charge and current in piecewise form,

$$j(t) = \begin{cases} 4\ \cos(t) - 4\ \cos(\sqrt{40}t) & t < \pi \\ \\ -4\ \cos(\sqrt{40}t) - 4\ \cos(\sqrt{40}t - \sqrt{40}\pi) & t > \pi \end{cases} \tag{6.3.270}$$

38. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 34e^{-t} & t < 4 \\ \\ 0 & t > 4 \end{cases} \tag{6.3.271}$$

$$v(t) = 34e^{-t}[1 - u(t - 4)] \tag{6.3.272}$$

$$v(t) = 34e^{-t} - [34e^{-4}\ e^{-(t-4)}]u(t - 4) \tag{6.3.273}$$

$$V(s) = \frac{34}{s + 1} - \frac{34e^{-4}e^{-4s}}{s + 1} \tag{6.3.274}$$

Initial charge $q(0) = 0,\ \ R = 4,\ \ L = 1,\ \ C = 0.05,$

This gives, $j(0) = j'(0) = 0$

$$v(t) = Lj' + Rj + \frac{1}{C}\int_0^t j \ \mathrm{d}t \tag{6.3.275}$$

$$V(s) = sJ - j(0) + 4J + \frac{20J}{s} \tag{6.3.276}$$

$$V(s) = \frac{J(s^2 + 4s + 20)}{s} \tag{6.3.277}$$

$$J = \frac{34s}{(s+1)(s^2 + 4s + 20)} - \frac{34e^{-4}e^{-4s} \ s}{(s+1)(s^2 + 4s + 20)} \tag{6.3.278}$$

$$J = [1 - e^{-4-4s}]\left[\frac{-2}{s+1} + \frac{2(s+2) + 36}{(s+2)^2 + 16}\right] \tag{6.3.279}$$

$$j(t) = -2e^{-t} + e^{-2t}[2\cos(4t) + 9\sin(4t)] \tag{6.3.280}$$

$$+ \left[2e^{-t} - e^{-2t+4}\left\{2\cos(4t-16) + 9\sin(4t-16)\right\}\right]u(t-4) \tag{6.3.281}$$

Restating the charge and current in piecewise form,

$$j(t) = \begin{cases} -2e^{-t} + e^{-2t}[2\cos(4t) + 9\sin(4t)] & t < 4 \\ e^{-2t}\Big[2\cos(4t) + 9\sin(4t)\Big] - e^{-2t+4}\Big\{2\cos(4t-16) + 9\sin(4t-16)\Big\} & t > 4 \end{cases} \tag{6.3.282}$$

39. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 1000 & t < 2 \\ 0 & t > 2 \end{cases} \tag{6.3.283}$$

$$v(t) = 1000[1 - u(t-2)] \tag{6.3.284}$$

$$V(s) = \frac{1000(1 - e^{-2s})}{s} \tag{6.3.285}$$

Initial charge $q(0) = 0, \ R = 2, \ L = 1, \ C = 0.5,$

This gives, $j(0) = j'(0) = 0$

$$v(t) = Lj' + Rj + \frac{1}{C}\int_0^t j \ \mathrm{d}t \tag{6.3.286}$$

$$V(s) = sJ - j(0) + 2J + \frac{2J}{s} \tag{6.3.287}$$

$$V(s) = \frac{J(s^2 + 2s + 2)}{s} \tag{6.3.288}$$

$$J = \frac{1000(1 - e^{-2s})}{(s^2 + 2s + 2)} \tag{6.3.289}$$

$$J = [1 - e^{-2s}]\left[\frac{1000}{(s+1)^2 + 1}\right] \tag{6.3.290}$$

$$j(t) = 1000e^{-t}\sin(t) - \left[1000e^{-t+2}\sin(t-2)\right]u(t-2) \tag{6.3.291}$$

Restating the charge and current in piecewise form,

$$j(t) = \begin{cases} 1000e^{-t}\sin(t) & t < 2 \\ 1000e^{-t}[\sin(t) - e^2\sin(t-2)] & t > 2 \end{cases} \tag{6.3.292}$$

40. Finding the Laplace transform of the input,

$$v(t) = \begin{cases} 255\sin(t) & t < 2\pi \\ 0 & t > 2\pi \end{cases} \tag{6.3.293}$$

$$v(t) = 255\sin(t)[1 - u(t - 2\pi)] \tag{6.3.294}$$

$$V(s) = \frac{255}{s^2+1}[1 - e^{-2\pi s}] \tag{6.3.295}$$

Initial charge $q(0) = 0, \ R = 2, \ L = 1, \ C = 0.1,$

This gives, $j(0) = j'(0) = 0$

$$v(t) = Lj' + Rj + \frac{1}{C}\int_0^t j \ \, \mathrm{d}t \tag{6.3.296}$$

$$V(s) = sJ - j(0) + 2J + \frac{10J}{s} \tag{6.3.297}$$

$$V(s) = \frac{J(s^2 + 2s + 10)}{s} \tag{6.3.298}$$

$$J = \frac{255s(1 - e^{-2\pi s})}{(s^2 + 2s + 10)(s^2 + 1)} \tag{6.3.299}$$

$$J = [1 - e^{-2\pi s}]\left[\frac{27s + 6}{s^2 + 1} - \frac{27(s + 1) + 33}{(s + 1)^2 + 9}\right] \tag{6.3.300}$$

$$j(t) = 27\cos(t) + 6\sin(t) - e^{-t}\Big[27\cos(3t) + 11\sin(3t)\Big] \tag{6.3.301}$$

$$- \Big[27\cos(t) + 6\sin(t) - e^{-t+2\pi}\Big\{27\cos(3t) + 11\sin(3t)\Big\}\Big]u(t - 2\pi) \tag{6.3.302}$$

Restating the charge and current in piecewise form,

$$j(t) = \begin{cases} 27\cos(t) + 6\sin(t) - e^{-t}\Big[27\cos(3t) + 11\sin(3t)\Big] & t < 2\pi \\ e^{-t}\Big\{27\cos(3t) + 11\sin(3t)\Big\}[e^{2\pi} - 1] & t > 2\pi \end{cases} \tag{6.3.303}$$

6.4 Short Impulses, Dirac's Delta Function, Partial Fractions

1. Modeling a simple damped harmonic oscillation,

(a) Keeping k constant and decreasing c to zero, and aplying an impulse at $t = \pi$,

$$y'' + cy' + ky = \delta(t - \pi) \tag{6.4.1}$$

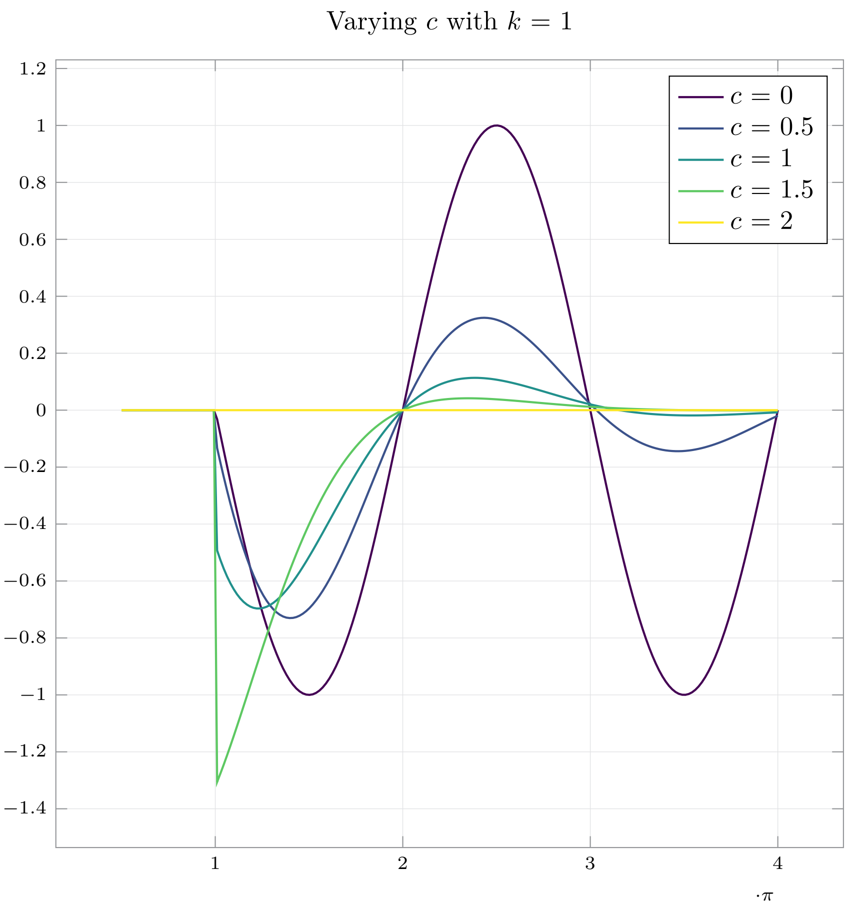
$$s^2Y - sy(0) - y'(0) + c[sY - y(0)] + kY = e^{-\pi s} \tag{6.4.2}$$

$$Y[s^2 + cs + k] - y(0)[s + c] - y'(0) = e^{-\pi s} \tag{6.4.3}$$

For simplicity, let $y(0) = y'(0) = 0, \ \ k = 1$

$$Y = \frac{e^{-\pi s}}{(s + c/2)^2 + (1 - c^2/4)} \qquad \lambda = \sqrt{1 - c^2/4} \tag{6.4.4}$$

$$y = \left[\frac{e^{-c(t-\pi)/2} \ \sin(\lambda t - \lambda\pi)}{\lambda}\right] u(t - \pi) \tag{6.4.5}$$



(b) Varying k with $c = 0$, and setting $y(0) = y'(0) = 0$,

$$y'' + ky = \delta(t - \pi) \tag{6.4.6}$$

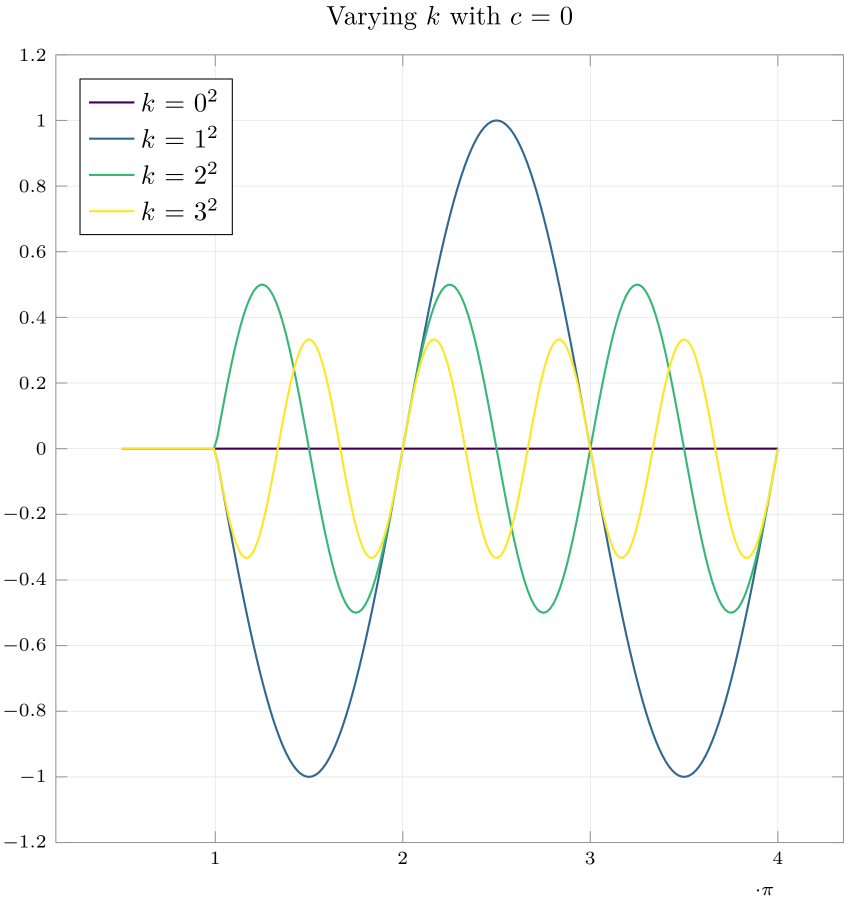
$$s^2Y - sy(0) - y'(0) + kY = e^{-\pi s} \tag{6.4.7}$$

$$Y[s^2 + k] - y(0)[s] - y'(0) = e^{-\pi s} \tag{6.4.8}$$

For simplicity, let $y(0) = y'(0) = 0, \;\; k = 1$

$$Y = \frac{e^{-\pi s}}{s^2 + k} \qquad\qquad y = \left[\frac{\sin(\sqrt{k}t - \sqrt{k}\pi)}{\sqrt{k}}\right] u(t - \pi) \tag{6.4.9}$$

The frequency of the sinusoidal oscillations varies as $2\pi\sqrt{k}$ while the amplitude varies as $1/\sqrt{k}$. Representative plots are plotted for $k = 0, 1, 4, 9$



(c) Keeping k constant and decreasing c to zero, and aplying an impulse at $t = \pi$ and another negative impulse at $t = 3\pi$,

$$y'' + cy' + ky = \delta(t - \pi) - \delta(t - 3\pi) \tag{6.4.10}$$

$$s^2Y - sy(0) - y'(0) + c[sY - y(0)] + kY = e^{-\pi s} - e^{-3\pi s} \tag{6.4.11}$$

$$Y[s^2 + cs + k] - y(0)[s + c] - y'(0) = e^{-\pi s} - e^{-3\pi s} \tag{6.4.12}$$

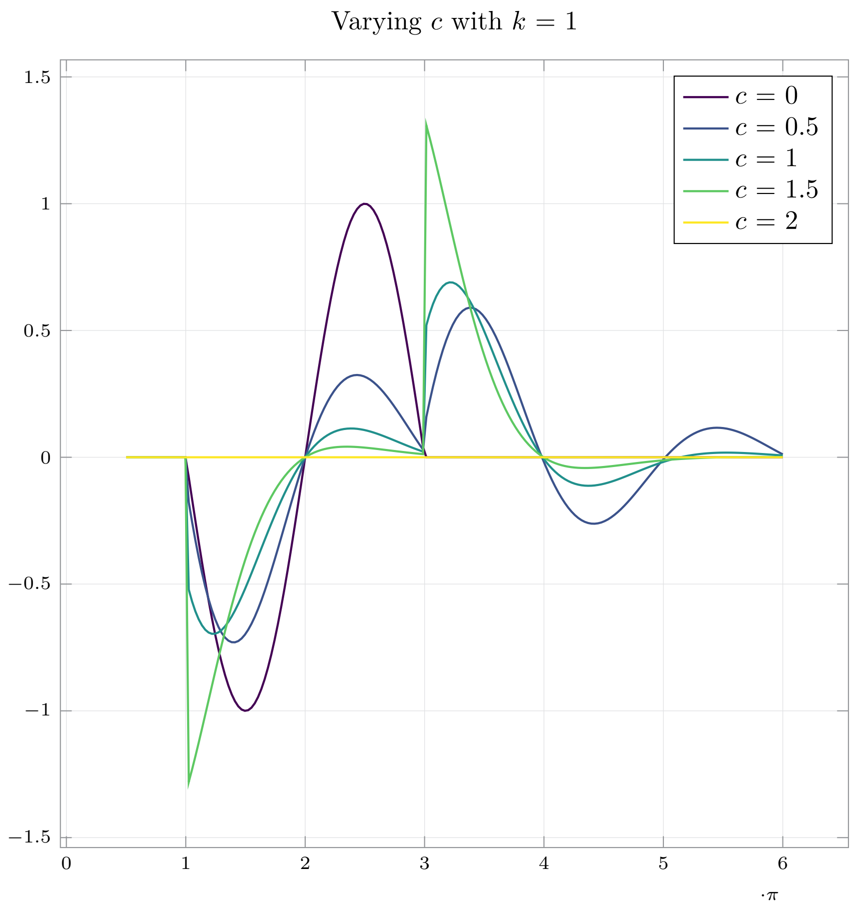
For simplicity, let $y(0) = y'(0) = 0,\ \ k = 1$

$$Y = \frac{e^{-\pi s} - e^{-3\pi s}}{(s + c/2)^2 + (1 - c^2/4)} \tag{6.4.13}$$

$$\lambda = \sqrt{1 - c^2/4} \tag{6.4.14}$$

$$y = \left[\frac{e^{-c(t-\pi)/2}\ \sin(\lambda t - \lambda\pi)}{\lambda}\right] u(t - \pi) \tag{6.4.15}$$

$$- \left[\frac{e^{-c(t-3\pi)/2}\ \sin(\lambda t - 3\lambda\pi)}{\lambda}\right] u(t - 3\pi) \tag{6.4.16}$$



(d) Varying k with $c = 0$, and setting $y(0) = y'(0) = 0$,

$$y'' + ky = \delta(t - \pi) - \delta(t - 3\pi) \tag{6.4.17}$$

$$s^2Y - sy(0) - y'(0) + kY = e^{-\pi s} - e^{-3\pi s} \tag{6.4.18}$$

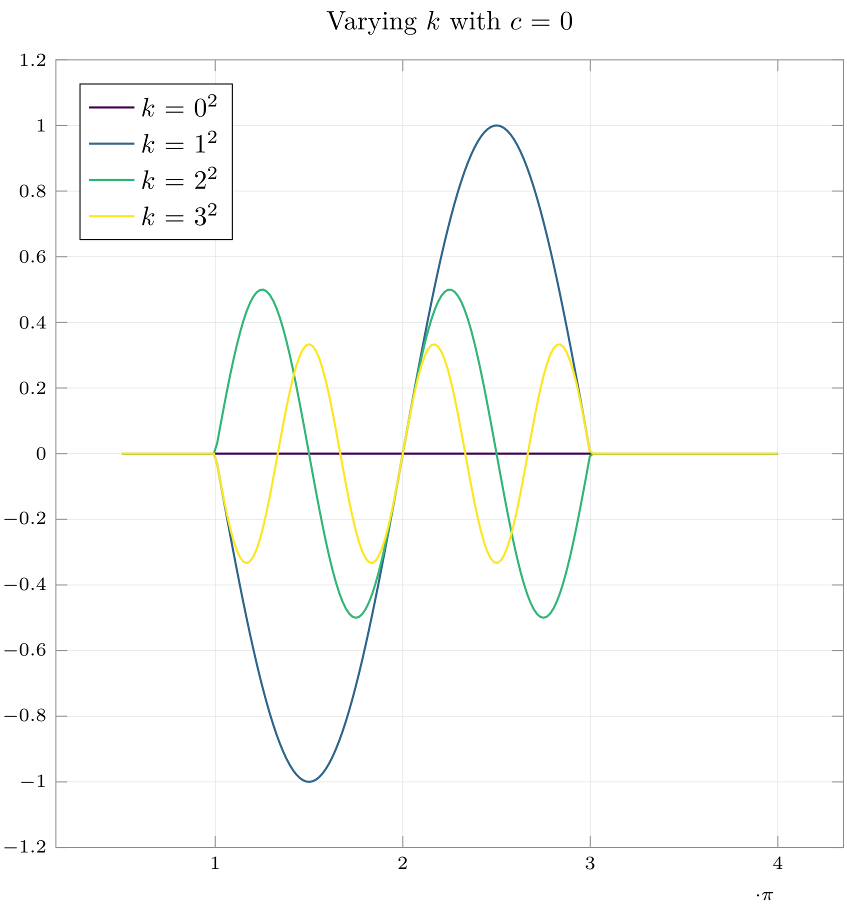
$$Y[s^2 + k] - y(0)[s] - y'(0) = e^{-\pi s} - e^{-3\pi s} \tag{6.4.19}$$

For simplicity, let $y(0) = y'(0) = 0,\ \ k = 1$

$$Y = \frac{e^{-\pi s} - e^{-3\pi s}}{s^2 + k} \tag{6.4.20}$$

$$y = \left[\frac{\sin(\sqrt{k}t - \sqrt{k}\pi)}{\sqrt{k}}\right] u(t - \pi) - \left[\frac{\sin(\sqrt{k}t - 3\sqrt{k}\pi)}{\sqrt{k}}\right] u(t - 3\pi) \tag{6.4.21}$$

The frequency of the sinusoidal oscillations varies as $2\pi\sqrt{k}$ while the amplitude varies as $1/\sqrt{k}$. Representative plots are plotted for $k = 0, 1, 4, 9$



The very elegant result of the system going back to rest when given a negative impulse at $t = 3\pi$ is easily observed in the plots. This requires the system to have zero daming and therefore not lose any energy with time.

2. From Example 1 in the text,

(a) Finding the solution for a general k, with impulse magnitude $1/k$,

$$y'' + 3y' + 2y = [u(t-1) - t(t-1-k)]\frac{1}{k} \tag{6.4.22}$$

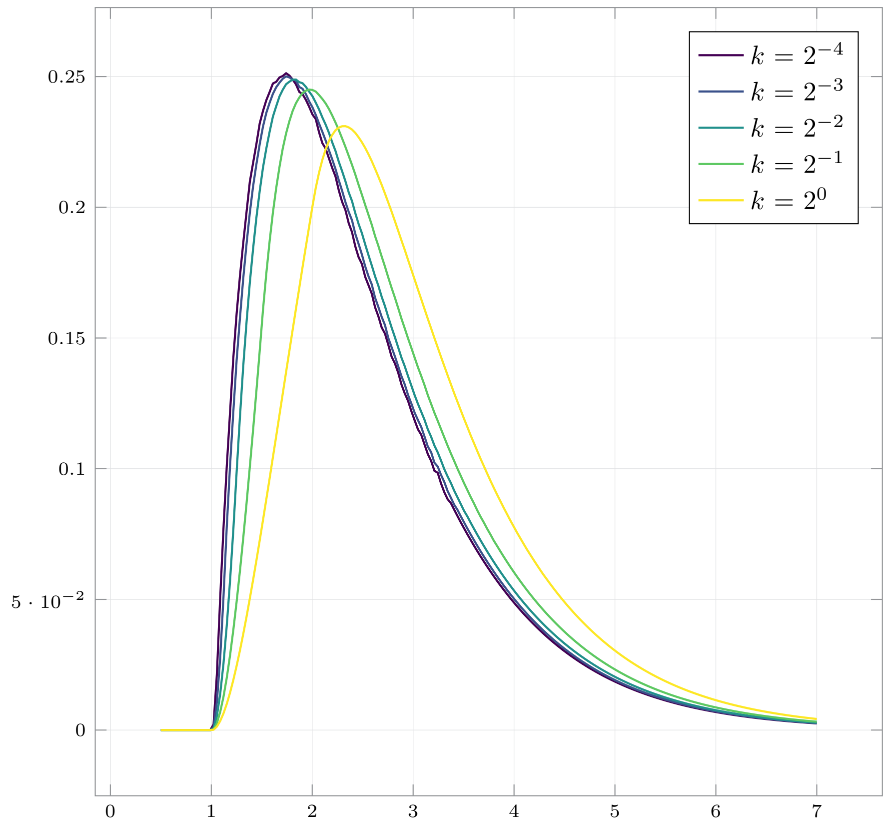
$$y(0) = 0 \qquad y'(0) = 0 \tag{6.4.23}$$

$$Y[s^2 + 3s + 2] = \frac{e^{-s} - e^{-s-ks}}{ks} \tag{6.4.24}$$

$$Y = \frac{e^{-s} - e^{-s-ks}}{ks(s+1)(s+2)} \tag{6.4.25}$$

$$g(t) = \frac{1 - 2e^{-t} + e^{-2t}}{2k} \tag{6.4.26}$$

$$y = f(t-1)u(t-1) - f(t-1-k)u(t-1-k) \tag{6.4.27}$$



CAS issues with providing a smoothed plot at small values of k.

(b) Let the impulse be applied at $t = a$,

$$y'' + 3y' + 2y = \delta(t-a) \tag{6.4.28}$$

$$y(0) = 1 \qquad y'(0) = 0 \tag{6.4.29}$$

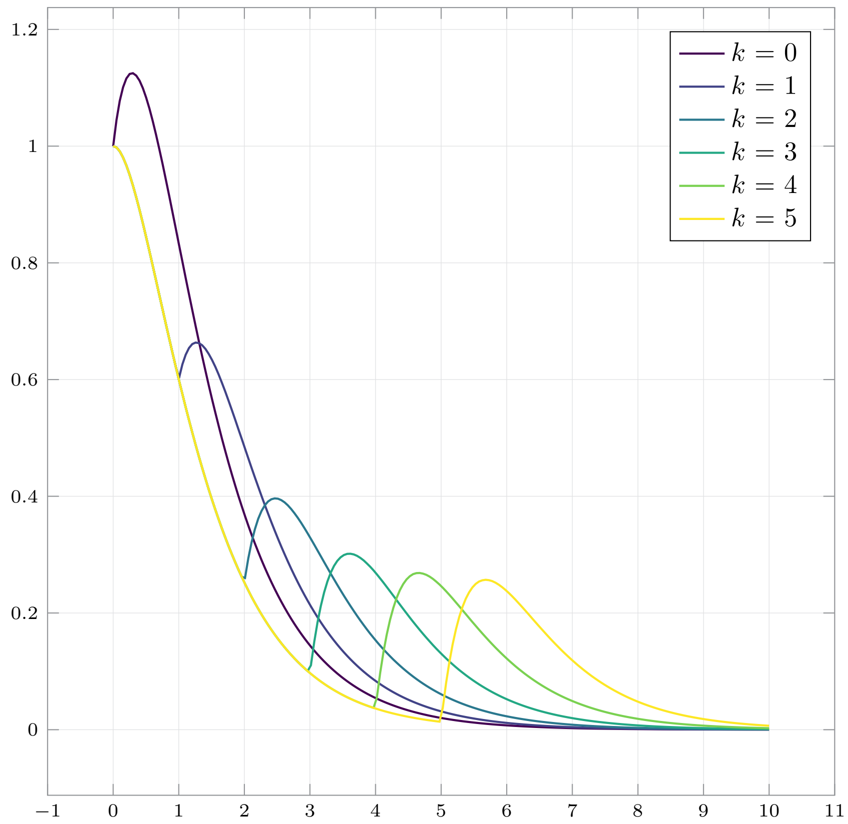
$$Y[s^2 + 3s + 2] - (s+3) = e^{-as} \tag{6.4.30}$$

$$Y = \frac{e^{-as} + (s+3)}{(s+1)(s+2)} \tag{6.4.31}$$

$$Y = \frac{2}{s+1} - \frac{1}{s+2} + e^{-as}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \tag{6.4.32}$$

$$y(t) = 2e^{-t} - e^{-2t} + [e^{-t+a} - e^{-2t+2a}]u(t-a) \tag{6.4.33}$$

$$y = \begin{cases} 2e^{-t} - e^{-2t} & t < a \\ (2+e^{a})e^{-t} - (1+e^{2a})e^{-2t} & t > a \end{cases} \tag{6.4.34}$$



For the effect of two equal and opposite impulses one after the other on a system with no damping, see Problem 1 part c, where the system does indeed go back to its initial state after both these impulses have been applied.

The response does depend on a as can be seen from the coefficients of the decay terms.

A linear scaling of the impulse b would simply scale the output after the impulse by the same factor.

3. Solving the IVP,

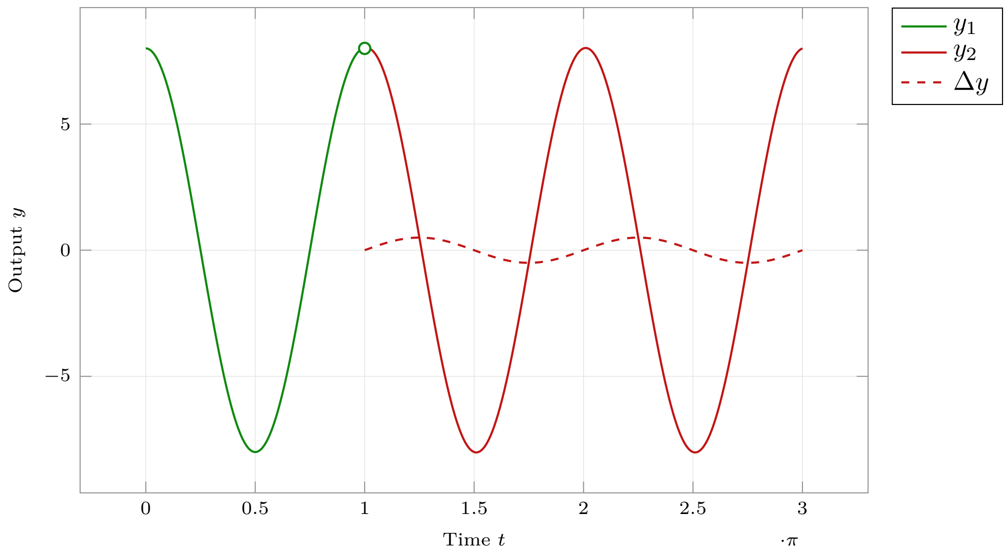
$$y'' + 4y = \delta(t - \pi) \qquad y(0) = 8 \qquad y'(0) = 0 \tag{6.4.35}$$

$$s^2Y - sy(0) - y'(0) + 4Y = e^{-\pi s} \qquad Y = \frac{e^{-\pi s} + 8s}{s^2 + 4} \tag{6.4.36}$$

Restating the output as in piecewise form,

$$y = 8\cos(2t) + [0.5\sin(2t)]u(t - \pi) \tag{6.4.37}$$

$$y = \begin{cases} 8\cos(2t) & y < \pi \\ 8\cos(2t) + 0.5\sin(2t) & t > \pi \end{cases} \tag{6.4.38}$$



4. Solving the IVP,

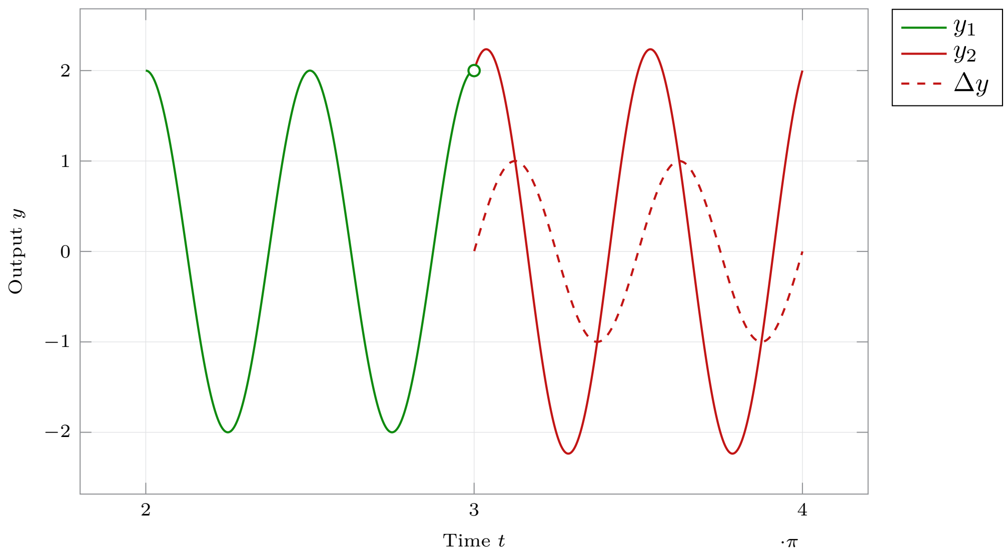
$$y'' + 16y = 4\delta(t - 3\pi) \qquad y(0) = 2 \qquad y'(0) = 0 \tag{6.4.39}$$

$$s^2 Y - sy(0) - y'(0) + 16Y = 4e^{-3\pi s} \qquad Y = \frac{4e^{-3\pi s} + 2s}{s^2 + 16} \tag{6.4.40}$$

Restating the output as in piecewise form,

$$y = 2\cos(4t) + [\sin(4t)]u(t - 3\pi) \tag{6.4.41}$$

$$y = \begin{cases} 2\cos(4t) & y < 3\pi \\ 2\cos(4t) + \sin(4t) & t > 3\pi \end{cases} \tag{6.4.42}$$



5. Solving the IVP,

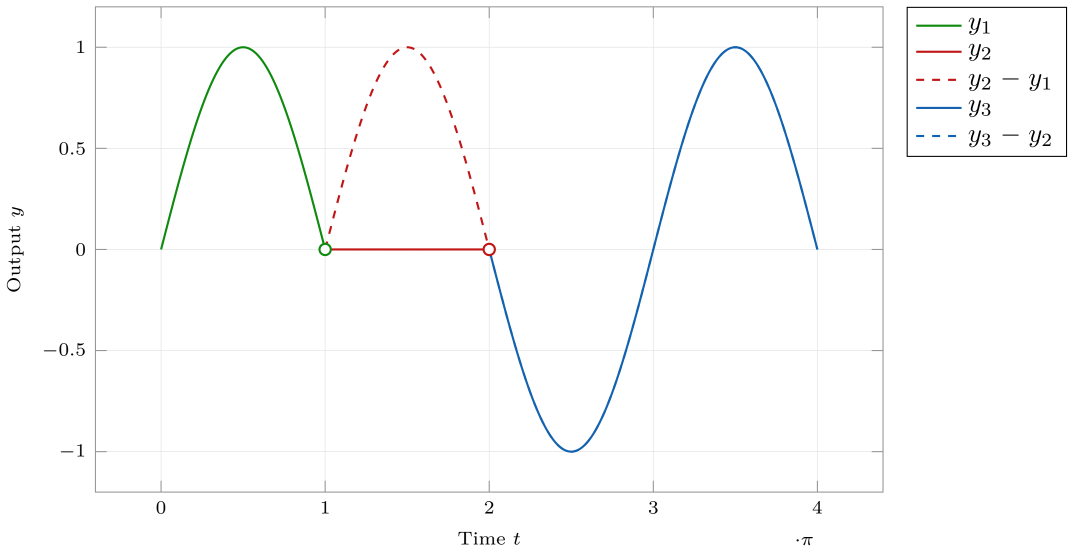
$$y'' + y = \delta(t-\pi) - \delta(t-2\pi) \qquad y(0) = 0 \qquad y'(0) = 1 \tag{6.4.43}$$

$$s^2Y - sy(0) - y'(0) + Y = e^{-\pi s} - e^{-2\pi s} \qquad Y = \frac{e^{-\pi s} - e^{-2\pi s} + 1}{s^2+1} \tag{6.4.44}$$

Restating the output as in piecewise form,

$$y = \sin(t) - [\sin(t)]u(t-\pi) - [\sin(t)]u(t-2\pi) \tag{6.4.45}$$

$$y = \begin{cases} \sin(t) & y < \pi \\ 0 & y \in (\pi, 2\pi) \\ -\sin(t) & t > 2\pi \end{cases} \tag{6.4.46}$$



6. Solving the IVP,

$$y'' + 4y' + 5y = \delta(t-1) \tag{6.4.47}$$

$$y(0) = 0 \qquad y'(0) = 3 \tag{6.4.48}$$

$$e^{-s} = s^2Y - sy(0) - y'(0) + 4(sY - y(0)) + 5Y \tag{6.4.49}$$

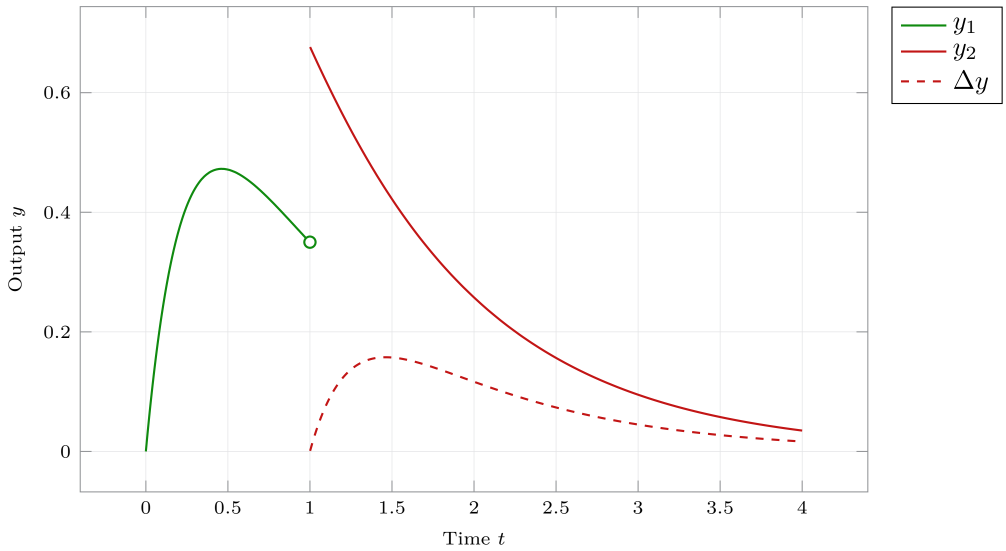
$$Y = \frac{e^{-s} + 3}{(s+4)(s+1)} \tag{6.4.50}$$

$$= \frac{1}{s+1} - \frac{1}{s+4} + e^{-s}\left[\frac{(1/3)}{s+1} - \frac{(1/3)}{s+4}\right] \tag{6.4.51}$$

Restating the output as in piecewise form,

$$y = e^{-t} - e^{-4t} + \left[\frac{e^{-t+1} - e^{-4t+4}}{3}\right] u(t-1) \tag{6.4.52}$$

$$y = \begin{cases} e^{-t} - e^{-4t} & t < 1 \\ [1 + e/3]e^{-t} - [1 + e^4/3]e^{-4t} & t > 1 \end{cases} \tag{6.4.53}$$



7. Solving the IVP,

$$4y'' + 24y' + 37y = 17e^{-t} + \delta(t - 0.5) \tag{6.4.54}$$

$$y(0) = 1 \qquad y'(0) = 1 \tag{6.4.55}$$

$$\frac{17}{s+1} + e^{-0.5s} = 4[s^2Y - sy(0) - y'(0)] + 24[sY - y(0)] + 37Y \tag{6.4.56}$$

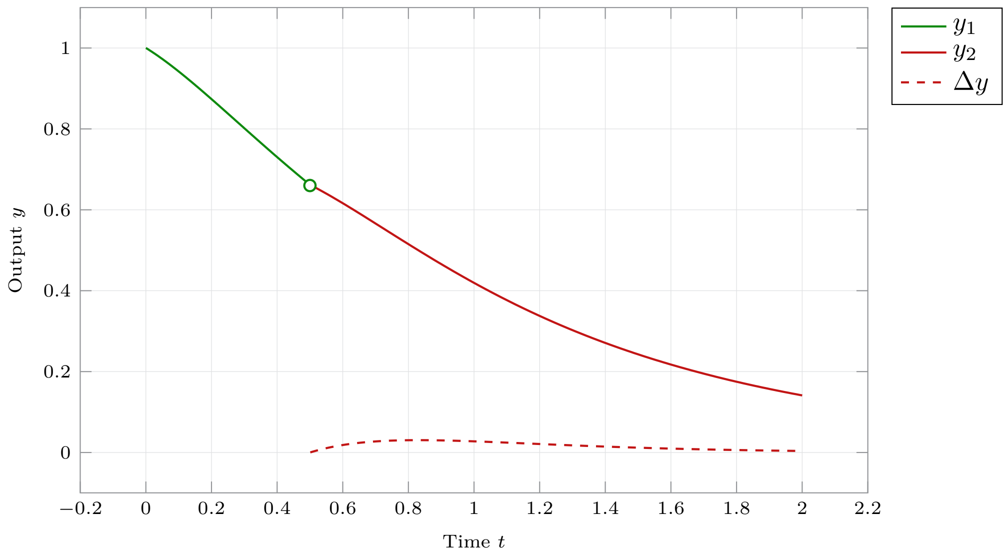
$$Y(4s^2 + 24s + 37) = \frac{17}{s+1} + e^{-0.5s} + 4s + 28 \tag{6.4.57}$$

$$Y = \frac{1}{s+1} + \frac{2}{(s+3)^2 + 0.25} + e^{-0.5s}\left[\frac{0.25}{(s+3)^2 + 0.25}\right] \tag{6.4.58}$$

Restating the output as in piecewise form,

$$y = e^{-t} + e^{-3t}4\sin(0.5t) + \left[0.5e^{-3t+1.5}\sin(0.5t - 0.25)\right] u(t - 0.5) \tag{6.4.59}$$

$$y = \begin{cases} e^{-t} + e^{-3t}4\sin(0.5t) & t < 0.5 \\ e^{-t} + e^{-3t}\left[4\sin(0.5t) + 0.5e^{1.5}\sin(0.5t - 0.25)\right] & t > 0.5 \end{cases} \tag{6.4.60}$$



8. Solving the IVP,

$$y'' + 3y' + 2y = 10\sin(t) + 10\delta(t-1) \tag{6.4.61}$$

$$y(0) = 1 \qquad y'(0) = -1 \tag{6.4.62}$$

$$\frac{10}{s^2+1} + 10e^{-s} = [s^2Y - sy(0) - y'(0)] + 3[sY - y(0)] + 2Y \tag{6.4.63}$$

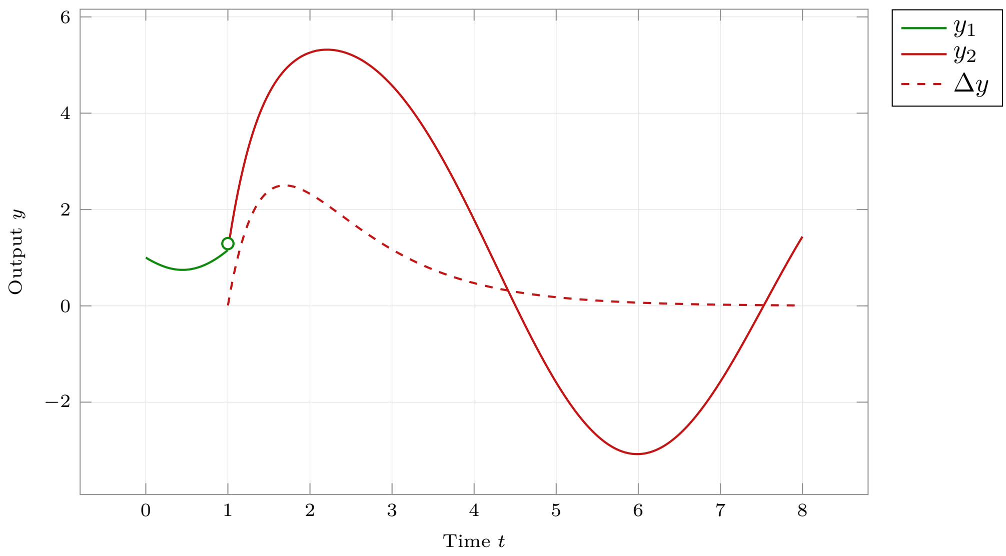
$$Y(s^2+3s+2) = \frac{10}{s^2+1} + 10e^{-s} + (s+2) \tag{6.4.64}$$

$$Y = \frac{6}{s+1} - \frac{2}{s+2} + \frac{1-3s}{s^2+1} + 10e^{-s}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \tag{6.4.65}$$

Restating the output as in piecewise form,

$$y = 6e^{-t} - 2e^{-2t} + \sin(t) - 3\cos(t) + 10u(t-1)\left[e^{-t+1} - e^{-2t+2}\right] \tag{6.4.66}$$

$$y = \begin{cases} 6e^{-t} - 2e^{-2t} + \sin(t) - 3\cos(t) & t < 1 \\ [6+10e]e^{-t} - [2+10e^2]e^{-2t} + \sin(t) - 3\cos(t) & t > 1 \end{cases} \tag{6.4.67}$$



9. Solving the IVP,

$$y'' + 4y' + 5y = [1 - u(t-10)]e^t - e^{10}\delta(t-10) \tag{6.4.68}$$

$$y(0) = 0 \qquad y'(0) = 1 \tag{6.4.69}$$

$$\frac{1}{s-1} - e^{10-10s}\left[\frac{s}{s-1}\right] = [s^2Y - sy(0) - y'(0)] + 4[sY - y(0)] + 5Y \tag{6.4.70}$$

$$Y(s^2 + 4s + 5) = \frac{s}{s-1}\,[1 - e^{10-10s}] \tag{6.4.71}$$

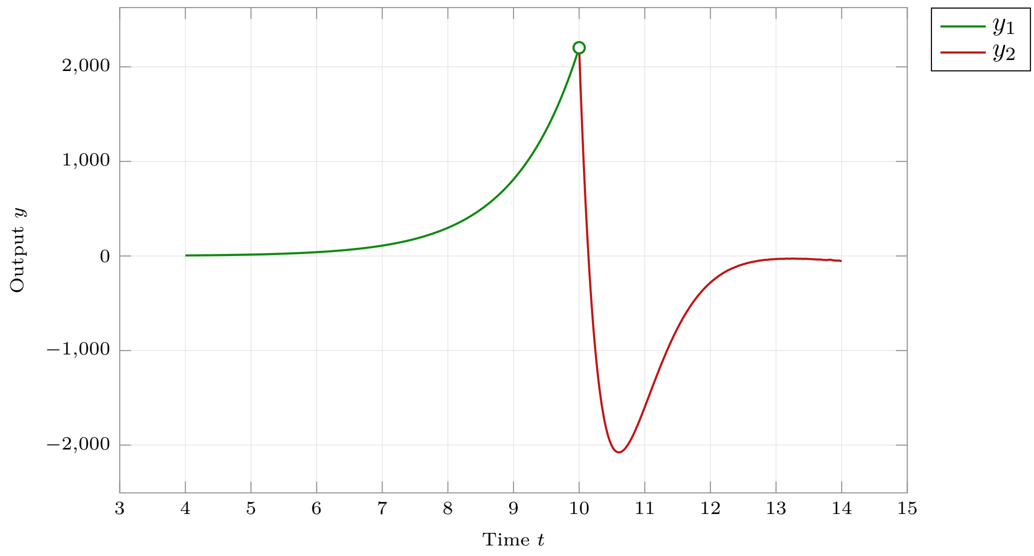
$$Y = \left[\frac{0.1}{(s-1)} + \frac{0.7 - 0.1(s+2)}{(s+2)^2 + 1}\right]\ [1 - e^{10-10s}] \tag{6.4.72}$$

Restating the output as in piecewise form,

$$y = 0.1e^t + 0.1e^{-2t}\Big[7\sin(t) - \cos(t)\Big] \tag{6.4.73}$$

$$-\ 0.1\ u(t-10)\Big[e^t + e^{-2t+30}\Big\{7\sin(t-10) - \cos(t-10)\Big\}\Big] \tag{6.4.74}$$

$$y = \begin{cases} 0.1e^t + 0.1e^{-2t}\Big[7\sin(t) - \cos(t)\Big] & t < 10 \\ 0.1\Big[7e^{-2t}\sin(t) - e^{-2t}\cos(t) - 7e^{-2t+30}\sin(t-10) & \\ +e^{-2t+30}\cos(t-10)\Big] & t > 10 \end{cases} \tag{6.4.75}$$



10. Solving the IVP,

$$y'' + 5y' + 6y = \delta(t - \pi/2) + \cos(t)\ u(t - \pi) \tag{6.4.76}$$

$$y(0) = 0 \qquad y'(0) = 0 \tag{6.4.77}$$

$$\frac{-s}{s^2+1} + e^{-\pi s/2} = [s^2Y - sy(0) - y'(0)] + 5[sY - y(0)] + 6Y \tag{6.4.78}$$

$$Y(s^2 + 5s + 6) = \frac{-s\ e^{-\pi s}}{s^2+1} + e^{-\pi s/2} \tag{6.4.79}$$

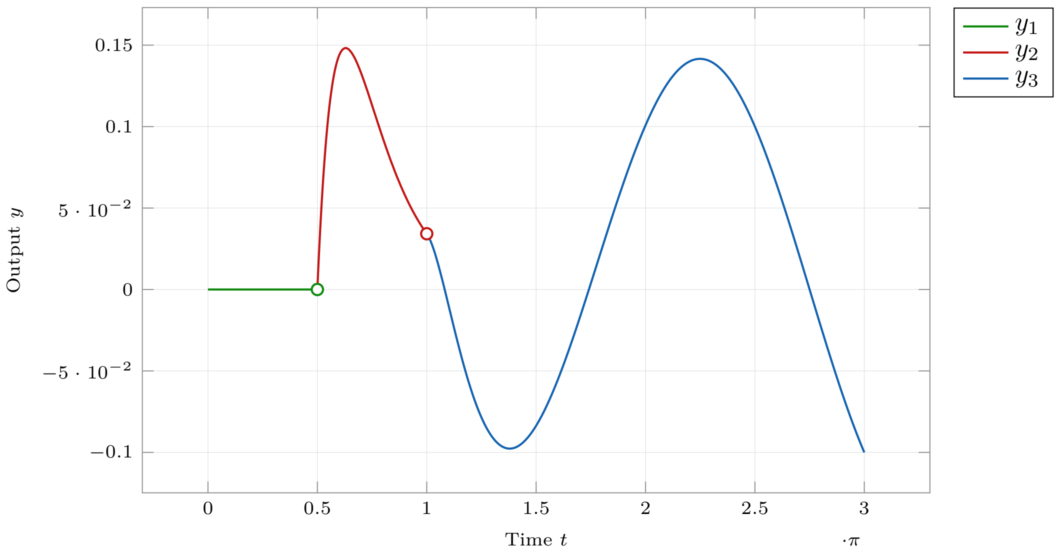
$$Y = e^{-\pi s}\left[\frac{-0.1s - 0.1}{s^2+1} + \frac{0.4}{s+2} - \frac{0.3}{s+3}\right] + e^{-\pi s/2}\left[\frac{1}{s+2} - \frac{1}{s+3}\right] \tag{6.4.80}$$

Restating the output as in piecewise form,

$$y = u(t-\pi)\Big[0.1\cos(t) + 0.1\sin(t) + 0.4e^{-2t+2\pi} - 0.3e^{-3t+3\pi}\Big] \tag{6.4.81}$$

$$+\, u(t - \pi/2)\Big[e^{-2t+\pi} - e^{-3t+1.5\pi}\Big] \tag{6.4.82}$$

$$y = \begin{cases} 0 & t < \pi/2 \\ e^{\pi-2t} - e^{1.5\pi-3t} & t \in (\pi/2,\, \pi) \\ 0.1\cos(t) + 0.1\sin(t) + e^{\pi-2t}[1 + 0.4e^{\pi}] - e^{1.5\pi-3t}[1 + 0.3e^{1.5\pi}] & t > \pi \end{cases} \tag{6.4.83}$$



11. Solving the IVP,

$$y'' + 5y' + 6y = u(t-1) + \delta(t-2) \tag{6.4.84}$$

$$y(0) = 0 \qquad y'(0) = 1 \tag{6.4.85}$$

$$\frac{e^{-s}}{s} + e^{-2s} = [s^2 Y - sy(0) - y'(0)] + 5[sY - y(0)] + 6Y \tag{6.4.86}$$

$$Y(s^2 + 5s + 6) = \frac{e^{-s}}{s} + e^{-2s} + 1 \tag{6.4.87}$$

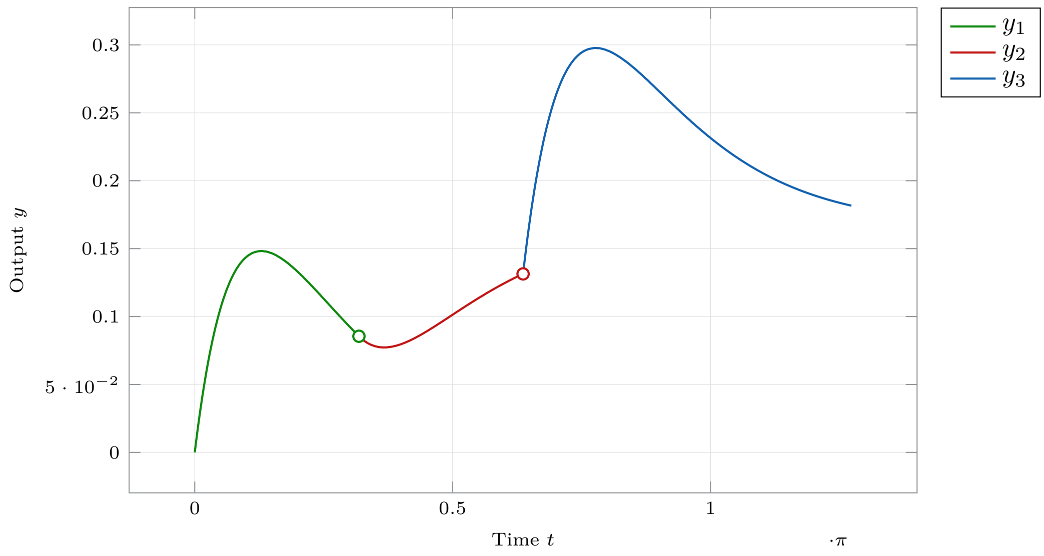
$$Y = e^{-s}\left[\frac{(1/6)}{s} - \frac{(1/2)}{s+2} + \frac{(1/3)}{s+3}\right] + [1 + e^{-2s}]\left[\frac{1}{s+2} - \frac{1}{s+3}\right] \tag{6.4.88}$$

Restating the output as in piecewise form,

$$y = e^{-2t} - e^{-3t} + u(t-1)\left[\frac{1 - 3e^{-2(t-1)} + 2e^{-3(t-1)}}{6}\right] \tag{6.4.89}$$

$$+ u(t-2)\left[e^{-2(t-2)} - e^{-3(t-2)}\right] \tag{6.4.90}$$

$$y = \begin{cases} e^{-2t} - e^{-3t} & t < 1 \\ (1/6) + e^{-2t}[1 - 0.5e^2] - e^{-3t}[1 - e^3/3] & t \in (1, 2) \\ (1/6) + e^{-2t}[1 - 0.5e^2 + e^4] - e^{-3t}[1 - e^3/3 + e^6] & t > 2 \end{cases} \tag{6.4.91}$$



12. Solving the IVP,

$$y'' + 2y' + 5y = 25t - 100\delta(t - \pi) \quad \text{6.4.92}$$

$$y(0) = -2 \qquad y'(0) = 5 \quad \text{6.4.93}$$

$$\frac{25}{s^2} - 100e^{-\pi s} = [s^2Y - sy(0) - y'(0)] + 2[sY - y(0)] + 5Y \quad \text{6.4.94}$$

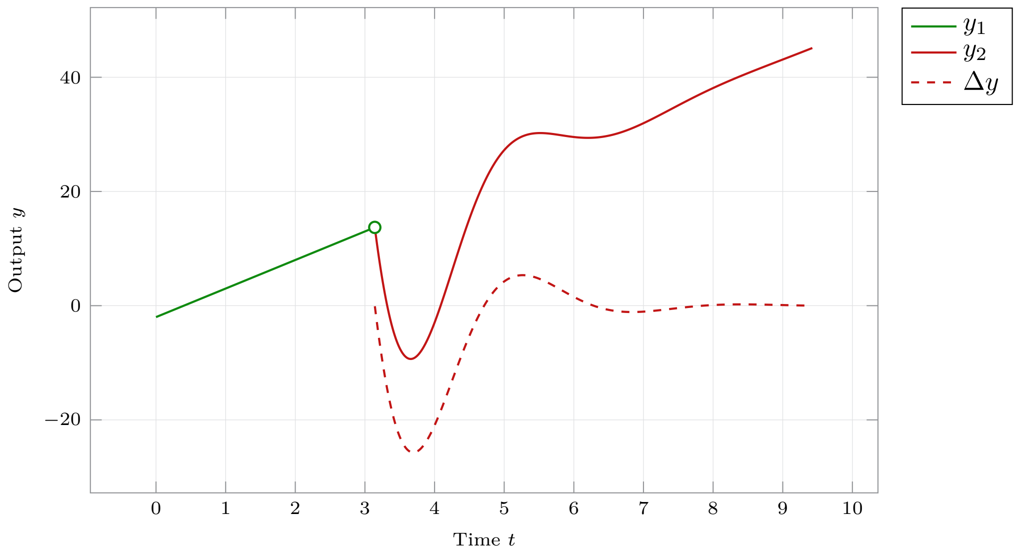
$$Y(s^2 + 2s + 5) = \frac{25}{s^2} - 2s + 1 - 100e^{-\pi s} \quad \text{6.4.95}$$

$$Y = \frac{-2s + 5}{s^2} - e^{-\pi s}\left[\frac{100}{(s + 1)^2 + 4}\right] \quad \text{6.4.96}$$

Restating the output as in piecewise form,

$$y = -2 + 5t - u(t - \pi)\Big[50e^{-t+\pi}\sin(2t)\Big] \quad \text{6.4.97}$$

$$y = \begin{cases} -2 + 5t & t < \pi \\ -2 + 5t - 50e^{\pi - t}\sin(2t) & t > \pi \end{cases} \quad \text{6.4.98}$$



13. Heaviside formulas,

(a) a is a simple root. After decomposition into fractions, the other terms are guaranteed not to have $(s-a)$ in their denominator.

$$\frac{F(s)}{G(s)} = \frac{A}{(s-a)} + \text{other terms} \tag{6.4.99}$$

$$\frac{(s-a)F(s)}{G(s)} = A + (s-a)[\text{other terms}] \tag{6.4.100}$$

$$\lim_{s\to a}\frac{(s-a)F(s)}{G(s)} = A + (0)[\text{other terms}] \tag{6.4.101}$$

$$= A \tag{6.4.102}$$

(b) a is a root of order m,

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{(s-a)} + \text{other terms} \tag{6.4.103}$$

$$\frac{(s-a)^m F(s)}{G(s)} = A_m + (s-a)A_{m-1} + \cdots + (s-a)^{m-1}A_1 \tag{6.4.104}$$

$$+ (s-a)^m[\text{other terms}] \tag{6.4.105}$$

$$\lim_{s\to a}\frac{(s-a)^m F(s)}{G(s)} = A_m + 0\ [\text{all other terms}] = A_m \tag{6.4.106}$$

For the other terms, consider the $(m-k)$ th derivative,

$$\frac{\mathrm{d}^{m-k}}{\mathrm{d}s^{m-k}}\frac{(s-a)^m F(s)}{G(s)} = (m-k)!\ A_k + (s-a)[\text{some terms}] \tag{6.4.107}$$

$$+ (s-a)^m\ [\text{other terms}] \tag{6.4.108}$$

Taking the limit $s \to a$ makes every term except $(m-k)!\ A_k$ go to zero, which proves the relation.

14. **(a)** For a piecewise continuous function with period p, and some positive integer k,

$$\mathscr{L}\{f(t)\} = \int_0^p e^{-st} f(t)\ \mathrm{d}t + \int_p^{2p} e^{-st} f(t)\ \mathrm{d}t + \dots \tag{6.4.109}$$

$$\text{Substitute}\quad t - kp \to r \tag{6.4.110}$$

$$\int_{kp}^{(k+1)p} e^{-st} f(t)\ \mathrm{d}t = \int_0^p e^{-s(r+kp)} f(r+kp)\ \mathrm{d}r \tag{6.4.111}$$

$$= \int_0^p e^{-kps}\ e^{-sr} f(r)\ \mathrm{d}r \tag{6.4.112}$$

$$\mathscr{L}\{f(t)\} = \sum_{k=0}^{\infty} e^{-kps} \int_0^p e^{-sr} f(r)\ \mathrm{d}r \tag{6.4.113}$$

$$= \frac{1}{1 - e^{-ps}} \int_0^p e^{-sr} f(r)\ \mathrm{d}r \tag{6.4.114}$$

(b) Using the formula from part a, with $p = 2\pi/\omega$,

$$f_1(t) = \sin(\omega t)[1 - u(t - \pi/\omega)] \tag{6.4.115}$$

$$\mathscr{L}\{f_1(t)\} = \frac{\omega}{s^2 + \omega^2}[1 + e^{-\pi s/\omega}] \tag{6.4.116}$$

$$\mathscr{L}\{f(t)\} = \left(\frac{1}{1 - e^{-2\pi s/\omega}}\right) \frac{\omega[1 + e^{-\pi s/\omega}]}{s^2 + \omega^2} \tag{6.4.117}$$

(c) Full wave rectifier simply changes p to π/ω,

$$\mathscr{L}\{f(t)\} = \left[\frac{1 + e^{-\pi s/\omega}}{1 - e^{-\pi s/\omega}}\right] \frac{\omega}{s^2 + \omega^2} \tag{6.4.118}$$

$$= \frac{\omega}{s^2 + \omega^2} \left[\frac{1 + e^{-2\mu}}{1 - e^{-2\mu}}\right] \qquad \left(\frac{\pi s}{2\omega} = \mu\right) \tag{6.4.119}$$

$$= \frac{\omega}{s^2 + \omega^2} \coth(\mu) \tag{6.4.120}$$

(d) For the saw-tooth wave, whose slope is k/p and period is p,

$$f(t) = (k/p)t\,[1 - u(t-p)] \tag{6.4.121}$$

$$\mathscr{L}\{f_1(t)\} = \frac{(k/p)}{s^2} - e^{-ps}\left[\frac{(k/p)(1+ps)}{s^2}\right] \tag{6.4.122}$$

$$\mathscr{L}\{f(t)\} = \frac{(k/p)}{s^2} - \frac{ke^{-ps}}{s(1-e^{-ps})} \tag{6.4.123}$$

15. Given that the staircase function $g(t)$ is the difference of two simpler functions,

$$g(t) = (k/p)t - \Big[(k/p)t - (k/p)t\;u(t-p)\Big] \tag{6.4.124}$$

$$G(s) = \frac{ke^{-ps}}{s(1-e^{-ps})} \tag{6.4.125}$$

6.5 Convolution, Integral Equations

1. Finding convolution using the integral definition,

$$1 * 1 = \int_0^t f(r)\;g(t-r)\;\;\mathrm{d}r \qquad = \int_0^t (1)(1)\;\;\mathrm{d}r \tag{6.5.1}$$

$$= \Big[r\Big]_0^t \qquad = t \tag{6.5.2}$$

2. Finding convolution using the integral definition,

$$1 * \sin(\omega t) = \int_0^t f(r)\;g(t-r)\;\;\mathrm{d}r \qquad = \int_0^t \sin(\omega r)(1)\;\;\mathrm{d}r \tag{6.5.3}$$

$$= \left[\frac{\cos(\omega r)}{\omega}\right]_t^0 \qquad = \frac{1-\cos(\omega t)}{\omega} \tag{6.5.4}$$

3. Finding convolution using the integral definition,

$$e^t * e^{-t} = \int_0^t f(r)\;g(t-r)\;\;\mathrm{d}r \qquad = \int_0^t e^r\;e^{r-t}\;\;\mathrm{d}r \tag{6.5.5}$$

$$= e^{-t}\left[\frac{e^{2r}}{2}\right]_0^t \qquad = \sinh(t) \tag{6.5.6}$$

4. Finding convolution using the integral definition,

$$\cos(\omega t) * \cos(\omega t) = \int_0^t f(r)\ g(t-r)\ \ \mathrm{d}r \tag{6.5.7}$$

$$= \int_0^t \cos(\omega r)\ \ \cos(\omega t - \omega r)\ \ \mathrm{d}r \tag{6.5.8}$$

$$= \frac{1}{2}\int_0^t [\cos(\omega t) + \cos(2\omega r - \omega t)]\ \ \mathrm{d}r \tag{6.5.9}$$

$$= \left[\frac{r\ \cos(\omega t)}{2} + \frac{\sin(2\omega r - \omega t)}{4\omega}\right]_0^t \tag{6.5.10}$$

$$= \frac{\omega t\ \cos(\omega t) + \sin(\omega t)}{2\omega} \tag{6.5.11}$$

5. Finding convolution using the integral definition,

$$\sin(\omega t) * \cos(\omega t) = \int_0^t f(r)\ g(t-r)\ \ \mathrm{d}r \tag{6.5.12}$$

$$= \int_0^t \sin(\omega r)\ \ \cos(\omega t - \omega r)\ \ \mathrm{d}r \tag{6.5.13}$$

$$= \frac{1}{2}\int_0^t [\sin(\omega t) + \sin(2\omega r - \omega t)]\ \ \mathrm{d}r \tag{6.5.14}$$

$$= \left[\frac{r\ \sin(\omega t)}{2} - \frac{\cos(2\omega r - \omega t)}{4\omega}\right]_0^t \tag{6.5.15}$$

$$= \frac{t\ \sin(\omega t)}{2} \tag{6.5.16}$$

6. Finding convolution using the integral definition,

$$e^{at} * e^{bt} = \int_0^t f(r)\ g(t-r)\ \ \mathrm{d}r \qquad = \int_0^t e^{ar}\ e^{bt-br}\ \ \mathrm{d}r \tag{6.5.17}$$

$$= e^{bt}\int_0^t e^{(a-b)r}\ \ \mathrm{d}r \qquad = \left[\frac{e^{bt}}{(a-b)}\ e^{(a-b)r}\right]_0^t \tag{6.5.18}$$

$$= \frac{e^{at} - e^{bt}}{(a-b)} \tag{6.5.19}$$

7. Finding convolution using the integral definition,

$$t * e^t = \int_0^t f(r)\ g(t-r)\ \ \mathrm{d}r \qquad = \int_0^t r\ e^{t-r}\ \ \mathrm{d}r \tag{6.5.20}$$

$$= e^t \int_0^t r\ e^{-r}\ \ \mathrm{d}r \qquad = e^t \Big[-e^{-r}(1+r) \Big]_0^t \tag{6.5.21}$$

$$= -(1+t) + e^t \tag{6.5.22}$$

8. Using convolution to solve the integral equation,

$$2t = y(t) + 4\int_0^t y(r)\ (t-r)\ \ \mathrm{d}r \tag{6.5.23}$$

$$f(t) = y(t) \qquad g(t) = t \tag{6.5.24}$$

$$F(s) = Y \qquad G(s) = \frac{1}{s^2} \tag{6.5.25}$$

$$\frac{2}{s^2} = Y + \frac{4Y}{s^2} \qquad Y = \frac{2}{s^2+4} \tag{6.5.26}$$

$$y(t) = \sin(2t) \tag{6.5.27}$$

9. Using convolution to solve the integral equation,

$$1 = y(t) - \int_0^t y(r)\ \ \mathrm{d}r \tag{6.5.28}$$

$$f(t) = y(t) \qquad g(t) = 1 \tag{6.5.29}$$

$$F(s) = Y \qquad G(s) = \frac{1}{s} \tag{6.5.30}$$

$$\frac{1}{s} = Y - \frac{Y}{s} \qquad Y = \frac{1}{s-1} \tag{6.5.31}$$

$$y(t) = e^t \tag{6.5.32}$$

10. Using convolution to solve the integral equation,

$$\sin(2t) = y(t) - \int_0^t y(r)\ \sin(2t - 2r)\ \mathrm{d}r \tag{6.5.33}$$

$$f(t) = y(t) \qquad\qquad g(t) = \sin(2t) \tag{6.5.34}$$

$$F(s) = Y \qquad\qquad G(s) = \frac{2}{s^2 + 4} \tag{6.5.35}$$

$$\frac{2}{s^2 + 4} = Y - \frac{2Y}{s^2 + 4} \qquad\qquad Y = \frac{2}{s^2 + 2} \tag{6.5.36}$$

$$y(t) = \sqrt{2}\sin(\sqrt{2}t) \tag{6.5.37}$$

11. Using convolution to solve the integral equation,

$$1 = y(t) + \int_0^t y(r)\ (t - r)\ \mathrm{d}r \tag{6.5.38}$$

$$f(t) = y(t) \qquad\qquad g(t) = t \tag{6.5.39}$$

$$F(s) = Y \qquad\qquad G(s) = \frac{1}{s^2} \tag{6.5.40}$$

$$\frac{1}{s} = Y + \frac{Y}{s^2} \qquad\qquad Y = \frac{s}{s^2 + 1} \tag{6.5.41}$$

$$y(t) = \cos(t) \tag{6.5.42}$$

12. Using convolution to solve the integral equation,

$$t + e^t = y(t) + \int_0^t y(r)\ \cosh(t - r)\ \mathrm{d}r \tag{6.5.43}$$

$$f(t) = y(t) \qquad\qquad g(t) = \cosh(t) \tag{6.5.44}$$

$$F(s) = Y \qquad\qquad G(s) = \frac{s}{s^2 - 1} \tag{6.5.45}$$

$$\frac{1}{s^2} + \frac{1}{(s-1)} = Y + \frac{sY}{s^2 - 1} \qquad\qquad Y = \frac{s + 1}{s^2} \tag{6.5.46}$$

$$y(t) = 1 + t \tag{6.5.47}$$

13. Using convolution to solve the integral equation,

$$te^t = y(t) + 2e^t \int_0^t y(r)\, e^{-r} \;\mathrm{d}r \tag{6.5.48}$$

$$f(t) = y(t) \qquad g(t) = e^t \tag{6.5.49}$$

$$F(s) = Y \qquad G(s) = \frac{1}{s-1} \tag{6.5.50}$$

$$\frac{1}{(s-1)^2} = Y + \frac{2Y}{s-1} \qquad Y = \frac{1}{(s-1)(s+1)} \tag{6.5.51}$$

$$y(t) = \sinh(t) \tag{6.5.52}$$

14. Using convolution to solve the integral equation,

$$2 - \frac{t^2}{2} = y(t) - \int_0^t y(r)\;(t-r) \;\mathrm{d}r \tag{6.5.53}$$

$$f(t) = y(t) \qquad g(t) = t \tag{6.5.54}$$

$$F(s) = Y \qquad G(s) = \frac{1}{s^2} \tag{6.5.55}$$

$$\frac{2}{s} - \frac{1}{s^3} = Y - \frac{Y}{s^2} \qquad Y = \frac{(2s^2-1)}{s(s^2-1)} \tag{6.5.56}$$

$$Y = \frac{1}{s} + \frac{0.5}{s+1} + \frac{0.5}{s-1} \qquad y(t) = 1 + \cosh(t) \tag{6.5.57}$$

15. Variation of parameter k,

(a) Solving the general integral equation

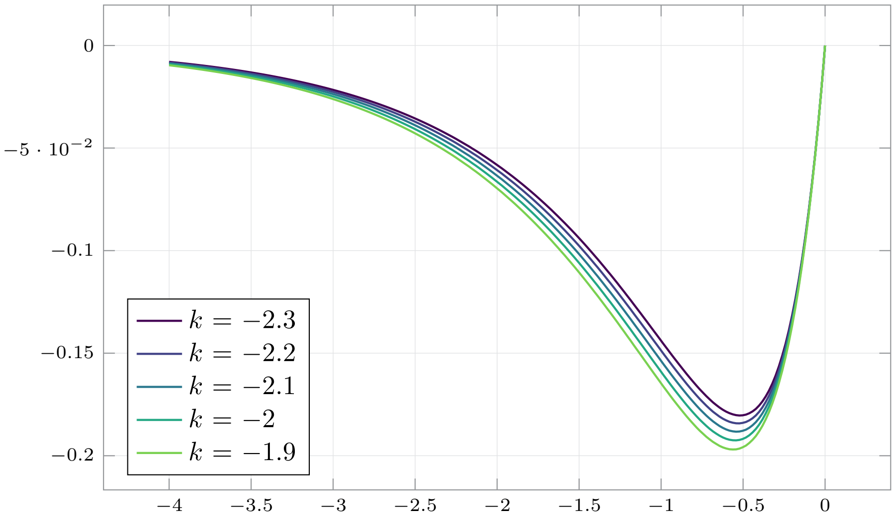
$$te^t = y(t) + ke^t \int_0^t y(r)\, e^{-r} \;\mathrm{d}r \tag{6.5.58}$$

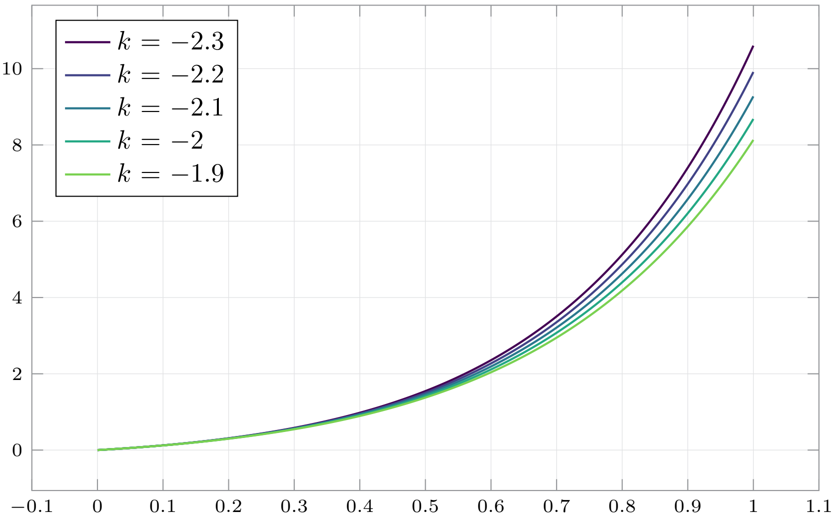
$$f(t) = y(t) \qquad g(t) = e^t \tag{6.5.59}$$

$$F(s) = Y \qquad G(s) = \frac{1}{s-1} \tag{6.5.60}$$

$$\frac{1}{(s-1)^2} = Y + \frac{kY}{s-1} \qquad Y = \frac{1}{(s-1)(s-1+k)} \tag{6.5.61}$$

$$Y = \frac{(1/k)}{s-1} - \frac{(1/k)}{s-1+k} \qquad y = \frac{e^t - e^{(1-k)t}}{k} \tag{6.5.62}$$





(b) TBC. I suspect the changes are gradual and nothing drastic happens in the other integral equations as well.

16. Proving properties of convolution,

(a) Commutativity, with $p = t - r$

$$g * f = \int_0^t g(r)\ f(t-r)\ \mathrm{d}r \tag{6.5.63}$$

$$= \int_t^0 g(t-p)\ f(p)\ \mathrm{d}(t-p) \tag{6.5.64}$$

$$= \int_0^t f(p)\ g(t-p)\ \mathrm{d}p \tag{6.5.65}$$

$$= f * g \tag{6.5.66}$$

(b) Associativity, with $p = t - r$

$$(f * g) * \nu = \int_{p=0}^{t} \left[\int_{r=0}^{p} f(r)\ g(p-r)\ \mathrm{d}r \right] \nu(t-p)\ \mathrm{d}p \tag{6.5.67}$$

$$= \iint_{0 \le r \le p \le t} f(r)\ g(p-r)\ \nu(t-p)\ \ \mathrm{d}r\ \mathrm{d}p \tag{6.5.68}$$

$$= \int_{r=0}^{t} f(r) \int_{p=r}^{t} g(p-r)\ \nu(t-p)\ \ \mathrm{d}p\ \mathrm{d}r \tag{6.5.69}$$

$$= \int_{r=0}^{t} f(r) \int_{p=0}^{t-r} g(p)\ \nu(t-r-p)\ \ \mathrm{d}p\ \mathrm{d}r \tag{6.5.70}$$

$$= \int_{r=0}^{t} [f(r)]\ [(g * \nu)(t-r)]\ \mathrm{d}r \tag{6.5.71}$$

$$= f * (g * \nu) \tag{6.5.72}$$

(c) Distributivity, using the fact that integration is a linear operation,

$$f * (g_1 + g_2) = \int_0^t f(r)\ [(g_1 + g_2)(t-r)]\ \ \mathrm{d}r \tag{6.5.73}$$

$$= \int_0^t f(r)\ g_1(t-r)\ \ \mathrm{d}r + \int_0^t f(r)\ g_2(t-r)\ \ \mathrm{d}r \tag{6.5.74}$$

$$= f * g_1 + f * g_2 \tag{6.5.75}$$

(d) Dirac's delta function,

$$f_k(t) = \begin{cases} 1/k & t \in [0, k] \\ 0 & \text{otherwise} \end{cases} \tag{6.5.76}$$

$$\int_0^{\infty} g(t) f_k(t)\ \ \mathrm{d}t = \frac{1}{k} \int_0^k g(t)\ \ \mathrm{d}t \tag{6.5.77}$$

$$= g(\kappa) \tag{6.5.78}$$

For some κ in the interval $[0, k]$. In the limit $k \to 0$, this means that $\kappa \to 0$, and the result of the integration is $g(0)$.

(e) Unspecified driving force $r(t)$ with Laplace transform $R(s)$,

$$y'' + \omega^2 y = r(t) \tag{6.5.79}$$

$$y(0) = K_1 \qquad y'(0) = K_2 \tag{6.5.80}$$

$$s^2 Y - sK_1 - K_2 + \omega^2 Y = R \tag{6.5.81}$$

$$\frac{R + K_2 + sK_1}{s^2 + \omega^2} = Y \tag{6.5.82}$$

$$\frac{1}{\omega}\left[r(t) * \sin(\omega t)\right] + K_1 \cos(\omega t) + \frac{K_2}{\omega}\sin(\omega t) = y(t) \tag{6.5.83}$$

The rule $H = FG \implies h(t) = (f * g)(t)$ is used here.

17. Finding invese Laplace transform by convolution,

$$Y = \frac{5.5}{(s + 1.5)(s - 4)} \tag{6.5.84}$$

$$F(s) = \frac{1}{(s + 1.5)} \qquad G(s) = \frac{1}{(s - 4)} \tag{6.5.85}$$

$$f(t) = e^{-1.5t} \qquad g(t) = e^{4t} \tag{6.5.86}$$

$$f * g = \int_0^t f(r)\, g(t - r)\, \mathrm{d}r \qquad = \int_0^t e^{-1.5r} e^{4t - 4r}\, \mathrm{d}r \tag{6.5.87}$$

$$= e^{4t} \int_0^t e^{-5.5r}\, \mathrm{d}r \qquad = e^{4t}\left[\frac{e^{-5.5r}}{-5.5}\right]_0^t \tag{6.5.88}$$

$$= \frac{e^{4t} - e^{-1.5t}}{5.5} \qquad y = e^{4t} - e^{-1.5t} \tag{6.5.89}$$

18. Finding invese Laplace transform by convolution,

$$Y = \frac{1}{(s - a)^2} \tag{6.5.90}$$

$$F(s) = \frac{1}{(s - a)} \qquad G(s) = \frac{1}{(s - a)} \tag{6.5.91}$$

$$f(t) = e^{at} \qquad g(t) = e^{at} \tag{6.5.92}$$

$$f * g = \int_0^t f(r)\, g(t - r)\, \mathrm{d}r \qquad = \int_0^t e^{ar} e^{at - ar}\, \mathrm{d}r \tag{6.5.93}$$

$$= e^{at} \int_0^t (1)\, \mathrm{d}r \qquad = t\, e^{at} \tag{6.5.94}$$

19. Finding invese Laplace transform by convolution,

$$Y = \frac{2\pi s}{(s^2+\pi^2)^2} \tag{6.5.95}$$

$$F(s) = \frac{2\pi}{(s^2+\pi^2)} \qquad G(s) = \frac{s}{(s^2+\pi^2)} \tag{6.5.96}$$

$$f(t) = 2\,\sin(\pi t) \qquad g(t) = \cos(\pi t) \tag{6.5.97}$$

$$f * g = \int_0^t f(r)\,g(t-r)\,\mathrm{d}r \qquad = 2\int_0^t \sin(\pi r)\,\cos(\pi t - \pi r)\,\mathrm{d}r \tag{6.5.98}$$

$$= \int_0^t [\sin(2\pi r - \pi t) + \sin(\pi t)]\,\mathrm{d}r \qquad = \Bigg[r\,\sin(\pi t) + \cos(\pi t - 2\pi r) \Bigg]_0^t \tag{6.5.99}$$

$$y = t\,\sin(\pi t) \tag{6.5.100}$$

20. Finding invese Laplace transform by convolution,

$$Y = \frac{9}{s(s+3)} \tag{6.5.101}$$

$$F(s) = \frac{1}{(s+3)} \qquad G(s) = \frac{1}{s} \tag{6.5.102}$$

$$f(t) = e^{-3t} \qquad g(t) = 1 \tag{6.5.103}$$

$$f * g = \int_0^t f(r)\,g(t-r)\,\mathrm{d}r \qquad = \int_0^t e^{-3r}\,\mathrm{d}r \tag{6.5.104}$$

$$= \Bigg[\frac{e^{-3r}}{3} \Bigg]_t^0 \qquad = \frac{1-e^{-3t}}{3} \tag{6.5.105}$$

$$y = 3\,[1 - e^{-3t}] \tag{6.5.106}$$

21. Finding invese Laplace transform by convolution,

$$Y = \frac{\omega}{s^2(s^2+\omega^2)} \tag{6.5.107}$$

$$F(s) = \frac{1}{s^2} \qquad G(s) = \frac{\omega}{s^2+\omega^2} \tag{6.5.108}$$

$$f(t) = t \qquad g(t) = \sin(\omega t) \tag{6.5.109}$$

$$f * g = \int_0^t f(r)\ g(t-r)\ \mathrm{d}r \qquad = \int_0^t (t-r)\sin(\omega r)\ \mathrm{d}r \tag{6.5.110}$$

$$= \left[\frac{-t\cos(\omega r)}{\omega} + \frac{\omega r\cos(\omega r) - \sin(\omega r)}{\omega^2}\right]_0^t \tag{6.5.111}$$

$$y = \frac{-\sin(\omega t) + \omega t}{\omega^2} \tag{6.5.112}$$

22. Finding invese Laplace transform by convolution,

$$Y = \frac{e^{-as}}{s(s-2)} \tag{6.5.113}$$

$$F(s) = \frac{e^{-as}}{s} \qquad G(s) = \frac{1}{s-2} \tag{6.5.114}$$

$$f(t) = u(t-a) \qquad g(t) = e^{2t} \tag{6.5.115}$$

$$f * g = \int_0^t f(r)\ g(t-r)\ \mathrm{d}r \qquad = \int_0^t u(r-a)e^{2t-2r}\ \mathrm{d}r \tag{6.5.116}$$

$$= e^{2t}\int_a^t e^{-2r}\ \mathrm{d}r \qquad = e^{2t}\left[\frac{e^{-2r}}{2}\right]_t^a \tag{6.5.117}$$

$$y = \begin{cases} 0 & t < a \\ \dfrac{e^{2(t-a)} - 1}{2} & t > a \end{cases} \tag{6.5.118}$$

23. Finding invese Laplace transform by convolution,

$$Y = \frac{40.5}{s(s^2-9)} \tag{6.5.119}$$

$$F(s) = \frac{1}{s} \qquad G(s) = \frac{3}{s^2-9} \tag{6.5.120}$$

$$f(t) = 1 \qquad g(t) = \sinh(3t) \tag{6.5.121}$$

$$f * g = \int_0^t f(r)\ g(t-r)\ \mathrm{d}r \qquad = \int_0^t \sinh(3r)\ \mathrm{d}r \tag{6.5.122}$$

$$= \left[\frac{\cosh(3r)}{3}\right]_0^t \qquad y = \frac{27[\cosh(3t)-1]}{6} \tag{6.5.123}$$

24. Finding invese Laplace transform by convolution,

$$Y = \frac{240}{(s^2+1)(s^2+25)} \tag{6.5.124}$$

$$F(s) = \frac{1}{s^2+1} \qquad G(s) = \frac{5}{s^2+25} \tag{6.5.125}$$

$$f(t) = \sin(t) \qquad g(t) = \sin(5t) \tag{6.5.126}$$

$$f * g = \int_0^t f(r)\ g(t-r)\ \mathrm{d}r \qquad = \int_0^t \sin(r)\ \ \sin(5t-5r)\ \mathrm{d}r \tag{6.5.127}$$

$$= 0.5\int_0^t [\cos(6r-5t) - \cos(4r-5t)]\ \mathrm{d}r \qquad = \left[\frac{\sin(6r-5t)}{6} - \frac{\sin(4r-5t)}{4}\right]_0^t \tag{6.5.128}$$

$$f * g = \frac{\sin(t)+\sin(5t)}{12} + \frac{\sin(t)-\sin(5t)}{8} \qquad y = 5\ \sin(t) - \sin(5t) \tag{6.5.129}$$

25. Finding invese Laplace transform by convolution,

$$Y = \frac{18s}{(s^2+36)^2} \tag{6.5.130}$$

$$F(s) = \frac{6}{s^2+36} \qquad G(s) = \frac{s}{s^2+36} \tag{6.5.131}$$

$$f(t) = \sin(6t) \qquad g(t) = \cos(6t) \tag{6.5.132}$$

$$f * g = \int_0^t f(r)\ g(t-r)\ \mathrm{d}r \qquad = \int_0^t \sin(6r)\ \ \cos(6t-6r)\ \mathrm{d}r \tag{6.5.133}$$

$$= 0.5\int_0^t [\sin(6t) + \sin(12r-6t)]\ \mathrm{d}r \qquad = \left[\frac{r\sin(6t)}{2} - \frac{\cos(12r-6t)}{24}\right]_0^t \tag{6.5.134}$$

$$f * g = \frac{t\sin(6t)}{2} \qquad y = \frac{3t\sin(6t)}{2} \tag{6.5.135}$$

26. Solving instead by partial fractions,

$$F(s) = \frac{5.5}{(s+1.5)(s-4)} \qquad = \frac{1}{s-4} - \frac{1}{s+1.5} \tag{6.5.136}$$

$$y(t) = e^{4t} - e^{-1.5t} \tag{6.5.137}$$

$$F(s) = \frac{\omega}{s^2(s^2+\omega^2)} \qquad = \frac{1}{\omega}\left[\frac{1}{s^2} - \frac{1}{s^2+\omega^2}\right] \tag{6.5.138}$$

$$y(t) = \frac{\omega t - \sin(\omega t)}{\omega^2} \tag{6.5.139}$$

$$F(s) = \frac{40.5}{s(s^2-9)} \qquad = \frac{-4.5}{s} + \frac{(9/4)}{s+3} + \frac{(9/4)}{s-3} \tag{6.5.140}$$

$$y(t) = 4.5[-1+\cosh(3t)] \tag{6.5.141}$$

6.6 Differentiation and Integration of Transforms, ODEs with Variable Coefficients

1. Refer notes. TBC.

2. Using differentiation to find the Laplace transform,

$$g(t) = 3t\,\sinh(4t) \qquad F(s) = 3\,\mathscr{L}\{\sinh(4t)\} \tag{6.6.1}$$

$$F = \frac{12}{s^2 - 16} \qquad G(s) = -F'(s) \tag{6.6.2}$$

$$G = \frac{24s}{(s^2 - 16)^2} \tag{6.6.3}$$

3. Using differentiation to find the Laplace transform,

$$g(t) = 0.5t\; e^{-3t} \qquad F(s) = 0.5\,\mathscr{L}\{e^{-3t}\} \tag{6.6.4}$$

$$F = \frac{0.5}{s + 3} \qquad G(s) = -F'(s) \tag{6.6.5}$$

$$G = \frac{0.5}{(s + 3)^2} \tag{6.6.6}$$

4. Using differentiation to find the Laplace transform,

$$g(t) = t\; e^{-t}\,\cos(t) \qquad F(s) = \mathscr{L}\{e^{-t}\,\cos(t)\} \tag{6.6.7}$$

$$F = \frac{(s + 1)}{(s + 1)^2 + 1} \qquad G(s) = -F'(s) \tag{6.6.8}$$

$$G = \frac{(s + 1)(2s + 2) - (s^2 + 2s + 2)}{[(s + 1)^2 + 1]^2} \qquad G = \frac{s^2 + 2s}{(s^2 + 2s + 2)^2} \tag{6.6.9}$$

5. Using differentiation to find the Laplace transform,

$$g(t) = t\;\;\cos(\omega t) \qquad F(s) = \mathscr{L}\{\cos(\omega t)\} \tag{6.6.10}$$

$$F = \frac{s}{s^2 + \omega^2} \qquad G(s) = -F'(s) \tag{6.6.11}$$

$$G = \frac{s(2s) - s^2 - \omega^2}{[s^2 + \omega^2]^2} \qquad G = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2} \tag{6.6.12}$$

6. Using differentiation to find the Laplace transform,

$$g(t) = t^2 \ \sin(3t) \qquad F(s) = \mathscr{L}\{\sin(3t)\} \tag{6.6.13}$$

$$F = \frac{3}{s^2+9} \qquad G(s) = F''(s) \tag{6.6.14}$$

$$G = \left[\frac{-6s}{(s^2+9)^2}\right]' \qquad G = \frac{-6(s^2+9)^2 + (24s^2)(s^2+9)}{(s^2+9)^2} \tag{6.6.15}$$

$$G = \frac{18(s^2-3)}{(s^2+9)^3} \tag{6.6.16}$$

7. Using differentiation to find the Laplace transform,

$$g(t) = t^2 \ \cosh(2t) \qquad F(s) = \mathscr{L}\{\cosh(2t)\} \tag{6.6.17}$$

$$F = \frac{s}{s^2-4} \qquad G(s) = (-1)^2 F''(s) \tag{6.6.18}$$

$$G = \left[\frac{-(s^2+4)}{(s^2-4)^2}\right]' \qquad G = \frac{-(2s)(s^2-4)^2 + (4s)(s^2-4)(s^2+4)}{(s^2-4)^4} \tag{6.6.19}$$

$$G = \frac{2s^3+24s}{(s^2-4)^3} \tag{6.6.20}$$

8. Using differentiation to find the Laplace transform,

$$g(t) = t\ e^{-kt}\sin(t) \qquad F(s) = \mathscr{L}\{e^{-kt}\sin(t)\} \tag{6.6.21}$$

$$F = \frac{1}{(s+k)^2+1} \qquad G(s) = -F'(s) \tag{6.6.22}$$

$$G = \frac{2(s+k)}{[(s+k)^2+1]^2} \tag{6.6.23}$$

9. Using differentiation to find the Laplace transform,

$$g(t) = 0.5t^2 \ \sin(\pi t) \qquad F(s) = \mathscr{L}\{0.5\sin(\pi t)\} \tag{6.6.24}$$

$$F = \frac{0.5\pi}{s^2+\pi^2} \qquad G(s) = (-1)^2 F''(s) \tag{6.6.25}$$

$$G = \left[\frac{-\pi s}{(s^2+\pi^2)^2}\right]' \qquad G = \frac{-\pi(s^2+\pi^2)^2 + \pi(4s^2)(s^2+\pi^2)}{(s^2+\pi^2)^4} \tag{6.6.26}$$

$$G = \frac{\pi(3s^2-\pi^2)}{(s^2+\pi)^3} \tag{6.6.27}$$

10. Using differentiation to find the Laplace transform,

$$g(t) = t^n\ e^{kt} \qquad F(s) = \mathscr{L}\{e^{kt}\} \tag{6.6.28}$$

$$F = \frac{1}{s-k} \qquad G(s) = (-1)^n F^{(n)}(s) \tag{6.6.29}$$

$$G = \left[\frac{-\pi s}{(s^2+\pi^2)^2}\right]' \qquad G = (-1)^n\ \frac{(-1)^n\ n!}{(s-k)^{n+1}} \tag{6.6.30}$$

$$G = \frac{n!}{(s-k)^{n+1}} \tag{6.6.31}$$

11. Using differentiation to find the Laplace transform,

$$g(t) = 4t\ \ \cos(0.5\pi t) \qquad F(s) = 4\,\mathscr{L}\{\cos(0.5\pi t)\} \tag{6.6.32}$$

$$F = \frac{4s}{s^2+0.25\pi^2} \qquad G(s) = -F'(s) \tag{6.6.33}$$

$$G = \frac{-4s^2-\pi^2+4s(2s)}{(s^2+0.25\pi^2)^2} \qquad G = \frac{4s^2-\pi^2}{(s^2+0.25\pi^2)^2} \tag{6.6.34}$$

12. Laguerre's polynomials using explicit algorithm,

(a) Using a CAS,

$$l_n = \frac{e^t}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^n e^{-t}) \tag{6.6.35}$$

$$l_0 = 1 \tag{6.6.36}$$

$$l_1 = 1 - x \tag{6.6.37}$$

$$l_2 = \frac{x^2}{2} - 2x + 1 \tag{6.6.38}$$

$$l_3 = -\frac{x^3}{6} + \frac{3x^2}{2} - 3x + 1 \tag{6.6.39}$$

$$l_4 = \frac{x^4}{24} - \frac{2x^3}{3} + 3x^2 - 4x + 1 \tag{6.6.40}$$

$$l_5 = -\frac{x^5}{120} + \frac{5x^4}{24} - \frac{5x^3}{3} + 5x^2 - 5x + 1 \tag{6.6.41}$$

$$l_6 = \frac{x^6}{720} - \frac{x^5}{20} + \frac{5x^4}{8} - \frac{10x^3}{3} + \frac{15x^2}{2} - 6x + 1 \tag{6.6.42}$$

$$l_7 = -\frac{x^7}{5040} + \frac{7x^6}{720} - \frac{7x^5}{40} + \frac{35x^4}{24} - \frac{35x^3}{6} + \frac{21x^2}{2} - 7x + 1 \tag{6.6.43}$$

$$l_8 = \frac{x^8}{40320} - \frac{x^7}{630} + \frac{7x^6}{180} - \frac{7x^5}{15} + \frac{35x^4}{12} - \frac{28x^3}{3} + 14x^2 - 8x + 1 \tag{6.6.44}$$

$$l_9 = -\frac{x^9}{362880} + \frac{x^8}{4480} - \frac{x^7}{140} + \frac{7x^6}{60} - \frac{21x^5}{20} + \frac{21x^4}{4} - 14x^3 + 18x^2 - 9x + 1 \tag{6.6.45}$$

$$l_{10} = \frac{x^{10}}{3628800} - \frac{x^9}{36288} + \frac{x^8}{896} - \frac{x^7}{42} + \frac{7x^6}{24} - \frac{21x^5}{10} + \frac{35x^4}{4} - 20x^3 + \frac{45x^2}{2} \tag{6.6.46}$$

$$- 10x + 1 \tag{6.6.47}$$

The fact that these polynomials satisfy the Laguerre ODE is verified using the CAS (not shown here).

$$ty'' + (1-t)y' + ny = 0 \tag{6.6.48}$$

(b) Using the Laplace transform of the Laguerre polynomials,

$$Y = \frac{(s-1)^n}{s^{n+1}} \qquad Y = \sum_{m=0}^{n} \binom{n}{m} \frac{s^{n-m}\,(-1)^m}{s^{n+1}} \tag{6.6.49}$$

$$Y = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{s^{m+1}} \qquad y = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{m!}\,t^m \tag{6.6.50}$$

(c) TBC. Everything on CAS behind the scenes

(d) Given the generating function,

$$\frac{1}{(1-x)^{p+1}} = \sum_{m=0}^{\infty} \frac{(m+p)!}{m!\ p!}\ x^m \tag{6.6.51}$$

$$\sum_{n=0}^{\infty} l_n(t)\ x^n = \frac{1}{(1-x)} \exp\left(\frac{tx}{x-1}\right) \tag{6.6.52}$$

$$= \left[\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{t^p\ x^p}{(1-x)^{p+1}}\right] \tag{6.6.53}$$

$$= \sum_{p=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^p\ (m+p)!}{m!\ (p!)^2}\ t^p\ x^{p+m} \tag{6.6.54}$$

Let $n = m + p$ in order to collect the coefficients of x^n. THen the coefficient of t^p is

$$\frac{(-1)^p\ n!}{(p!)^2\ (n-p)!} = \frac{(-1)^p}{p!} \binom{n}{p} \tag{6.6.55}$$

which aligns with the general term of the Laguerre polynomials in part c above.

13. TBC. Draw graphs.

14. Finding inverse Laplace transform,

$$Y(s) = \frac{s}{(s^2+16)^2} \tag{6.6.56}$$

$$F(s) = \frac{s}{s^2+16} \qquad G(s) = \frac{4}{s^2+16} \tag{6.6.57}$$

$$f(t) = \cos(4t) \qquad g(t) = \sin(4t) \tag{6.6.58}$$

$$f * g = \int_0^\infty \cos(4r)\ \sin(4t-4r)\ \text{d}r \qquad = 0.5 \int_0^\infty \Big[\sin(4t) + \sin(8r-4t)\Big]\ \text{d}r \tag{6.6.59}$$

$$= \left[\frac{8r\ \sin(4t) - \cos(8r-4t)}{16}\right]_0^t \qquad = \frac{t\ \sin(4t)}{2} \tag{6.6.60}$$

$$y(t) = \frac{t\ \sin(4t)}{8} \tag{6.6.61}$$

15. Finding inverse Laplace transform,

$$Y(s) = \frac{s}{(s^2-9)^2} \tag{6.6.62}$$

$$F(s) = \frac{s}{s^2-9} \qquad G(s) = \frac{3}{s^2-9} \tag{6.6.63}$$

$$f(t) = \cosh(3t) \qquad g(t) = \sinh(3t) \tag{6.6.64}$$

$$f*g = \int_0^\infty \cosh(3r)\ \sinh(3t-3r)\,\mathrm{d}r \qquad = 0.5\int_0^\infty \Big[\sinh(3t) - \sinh(6r-3t)\Big]\,\mathrm{d}r \tag{6.6.65}$$

$$= \left[\frac{6r\ \sinh(3t) - \cosh(6r-3t)}{12}\right]_0^t \qquad = \frac{t\ \sinh(3t)}{2} \tag{6.6.66}$$

$$y(t) = \frac{t\ \sinh(3t)}{6} \tag{6.6.67}$$

16. Finding inverse Laplace transform,

$$Y(s) = \frac{2(s+3)}{[(s+3)^2+1]^2} \qquad G(s) = \int_s^\infty F(p)\,\mathrm{d}p \tag{6.6.68}$$

$$G(s) = \left[\frac{-1}{(p+3)^2+1}\right]_s^\infty \qquad G(s) = \frac{1}{(s+3)^2+1} \tag{6.6.69}$$

$$g(t) = e^{-3t}\sin(t) \qquad f(t) = te^{-3t}\ \sin(t) \tag{6.6.70}$$

17. Finding inverse Laplace transform,

$$Y(s) = \ln\left(\frac{s}{s-1}\right) \qquad H(s) = Y'(s) = \frac{-1}{s(s-1)} \tag{6.6.71}$$

$$F(s) = \frac{1}{s} \qquad G(s) = \frac{1}{(s-1)} \tag{6.6.72}$$

$$f(t) = 1 \qquad g(t) = e^t \tag{6.6.73}$$

$$f*g = \int_0^t e^r\ \ \mathrm{d}r \qquad f*g = e^t - 1 \tag{6.6.74}$$

$$h(t) = 1 - e^t \qquad y(t) = \frac{e^t-1}{t} \tag{6.6.75}$$

18. Finding inverse Laplace transform,

$$Y(s) = \text{arccot}\left(\frac{s}{\pi}\right) \qquad H(s) = Y'(s) = \frac{1}{\pi}\frac{-\pi^2}{\pi^2 + s^2} \tag{6.6.76}$$

$$h(t) = -\sin(\pi t) \qquad y(t) = \frac{\sin(\pi t)}{t} \tag{6.6.77}$$

19. Finding inverse Laplace transform,

$$Y(s) = \ln\left[\frac{s^2+1}{(s-1)^2}\right] \qquad H(s) = Y'(s) = \frac{2s}{s^2+1} - \frac{2}{s-1} \tag{6.6.78}$$

$$h(t) = 2\cos(t) - 2e^t \qquad y(t) = \frac{2e^t - 2\cos(t)}{t} \tag{6.6.79}$$

20. Finding inverse Laplace transform,

$$Y(s) = \ln\left[\frac{s+a}{s+b}\right] \qquad H(s) = Y'(s) = \frac{1}{s+a} - \frac{1}{s+b} \tag{6.6.80}$$

$$h(t) = e^{-at} - e^{-bt} \qquad y(t) = \frac{e^{-bt} - e^{-at}}{t} \tag{6.6.81}$$

6.7 Systems of ODEs

1. Solving Examples 1 and 2 in Sec. 4.1 using the Laplace transform method,

(a) The system of ODEs in Example 1 is,

$$y_1' = 0.02y_2 - 0.02y_1 \qquad y_2' = 0.02y_1 - 0.02y_2 \tag{6.7.1}$$

$$y_1(0) = 0 \qquad y_2(0) = 150 \tag{6.7.2}$$

$$sY_1 - y_1(0) = 0.02Y_2 - 0.02Y_1 \qquad sY_2 - y_2(0) = 0.02Y_1 - 0.02Y_2 \tag{6.7.3}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{75}{s(25s+1)} \qquad Y_2 = \frac{3750s + 75}{s(25s+1)} \tag{6.7.4}$$

$$Y_1 = \frac{75}{s} - \frac{75}{s+0.04} \qquad Y_2 = \frac{75}{s} + \frac{75}{s+0.04} \tag{6.7.5}$$

$$y_1 = 75(1 - e^{-0.04t}) \qquad y_2 = 75(1 + e^{-0.04t}) \tag{6.7.6}$$

The system of ODEs in Example 2 is,

$$j_1' = -4j_1 + 4j_2 + 12 \qquad j_2' = -1.6j_1 + 1.2j_2 + 4.8 \tag{6.7.7}$$

$$j_1(0) = 0 \qquad j_2(0) = 0 \tag{6.7.8}$$

$$sJ_1 = 4J_2 - 4J_1 + \frac{12}{s} \qquad sJ_2 = -1.6J_1 + 1.2J_2 + \frac{4.8}{s} \tag{6.7.9}$$

Solving the system for J_1, J_2,

$$J_1 = \frac{60s + 24}{s(5s^2 + 14s + 8)} \qquad J_2 = \frac{24}{5s^2 + 14s + 8} \tag{6.7.10}$$

$$J_1 = \frac{3}{s} - \frac{8}{s+2} + \frac{5}{s+0.8} \qquad J_2 = \frac{-4}{s+2} + \frac{4}{s+0.8} \tag{6.7.11}$$

$$j_1 = 3 - 8e^{-2t} + 5e^{-0.8t} \qquad j_2 = -4e^{-2t} + 4e^{-0.8t} \tag{6.7.12}$$

(b) Solving the systems 8 in Section 4.3,

$$y_1' = -3y_1 + y_2 \qquad y_2' = y_1 - 3y_2 \tag{6.7.13}$$

$$y_1(0) = a \qquad y_2(0) = b \tag{6.7.14}$$

$$sY_1 - y_1(0) = -3Y_1 + Y_2 \qquad sY_2 - y_2(0) = Y_1 - 3Y_2 \tag{6.7.15}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{a(s+3) + b}{(s+2)(s+4)} \qquad Y_2 = \frac{a + b(s+3)}{(s+2)(s+4)} \tag{6.7.16}$$

$$Y_1 = \frac{a+b}{2(s+2)} + \frac{a-b}{2(s+4)} \qquad Y_2 = \frac{a+b}{2(s+2)} - \frac{a-b}{2(s+4)} \tag{6.7.17}$$

$$y_1 = c_1e^{-2t} + c_2e^{-4t} \qquad y_2 = c_1e^{-2t} - c_2e^{-4t} \tag{6.7.18}$$

Solving the system 11 in Section 4.3,

$$y_1' = y_1 \qquad y_2' = -y_2 \tag{6.7.19}$$

$$y_1(0) = a \qquad y_2(0) = b \tag{6.7.20}$$

$$sY_1 - y_1(0) = Y_1 \qquad sY_2 - y_2(0) = -Y_2 \tag{6.7.21}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{a}{(s-1)} \qquad Y_2 = \frac{b}{(s+1)} \tag{6.7.22}$$

$$y_1 = c_1 e^{t} \qquad y_2 = c_2 e^{-t} \tag{6.7.23}$$

Solving the system 12 in Section 4.3,

$$y_1' = y_2 \qquad y_2' = -4y_1 \tag{6.7.24}$$

$$y_1(0) = a \qquad y_2(0) = b \tag{6.7.25}$$

$$sY_1 - y_1(0) = Y_2 \qquad sY_2 - y_2(0) = -4Y_1 \tag{6.7.26}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{as+b}{(s^2+4)} \qquad Y_2 = \frac{-4a+bs}{(s^2+4)} \tag{6.7.27}$$

$$y_1 = a\cos(2t) + 0.5b\sin(2t) \qquad y_2 = b\cos(2t) - 2a\sin(2t) \tag{6.7.28}$$

Solving the system 13 in Section 4.3,

$$y_1' = -y_1 + y_2 \qquad y_2' = -y_1 - y_2 \tag{6.7.29}$$

$$y_1(0) = a \qquad y_2(0) = b \tag{6.7.30}$$

$$sY_1 - y_1(0) = -Y_1 + Y_2 \qquad sY_2 - y_2(0) = -Y_1 - Y_2 \tag{6.7.31}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{a(s+1)+b}{(s+1)^2+1} \qquad Y_2 = \frac{b(s+1)-a}{(s+1)^2+1} \tag{6.7.32}$$

$$y_1 = e^{-t}\Big[a\cos(t) + b\sin(t)\Big] \qquad y_2 = e^{-t}\Big[b\cos(t) - a\sin(t)\Big] \tag{6.7.33}$$

(c) Solving the system 3 in Section 4.6,

$$y_1' = -3y_1 + y_2 - 6e^{-2t} \qquad y_2' = y_1 - 3y_2 + 2e^{-2t} \tag{6.7.34}$$

$$y_1(0) = a \qquad y_2(0) = b \tag{6.7.35}$$

$$sY_1 - y_1(0) = -3Y_1 + Y_2 - \frac{6}{s+2} \qquad sY_2 - y_2(0) = Y_1 - 3Y_2 + \frac{2}{s+2} \tag{6.7.36}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{(s+2)(as+3a+b)+2(s^2+s-5)}{(s+2)^2(s+4)} \tag{6.7.37}$$

$$Y_1 = \frac{a-b+7}{2(s+4)} + \frac{a+b-3}{2(s+2)} - \frac{3}{(s+2)^2} \tag{6.7.38}$$

$$Y_2 = \frac{(s+2)(a+3b+3s)+2(s^3+7s^2+16s+9)}{(s+2)^2(s+4)} \tag{6.7.39}$$

$$Y_2 = \frac{-a+b-7}{2(s+4)} + \frac{a+b+3}{2(s+2)} - \frac{3}{(s+2)^2} + 2 \tag{6.7.40}$$

$$y_1 = c_1 e^{-4t} + c_2 e^{-2t} + 1.5[-1-2t]\ e^{-2t} \tag{6.7.41}$$

$$y_2 = -c_1 e^{-4t} + c_2 e^{-2t} + 1.5[1-2t]\ e^{-2t} + 2\delta(t) \tag{6.7.42}$$

TBC. What is the dirac delta function doing here?

2. Solving using Laplace transforms,

$$y_1' = -y_2 \qquad y_2' = -y_1 + 2\cos(t) \tag{6.7.43}$$

$$y_1(0) = 1 \qquad y_2(0) = 0 \tag{6.7.44}$$

$$sY_1 - y_1(0) = -Y_2 \qquad sY_2 - y_2(0) = -Y_1 + \frac{2s}{s^2+1} \tag{6.7.45}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{s}{s^2+1} \qquad Y_2 = \frac{1}{s^2+1} \tag{6.7.46}$$

$$y_1 = \cos(t) \qquad y_2 = \sin(t) \tag{6.7.47}$$

3. Solving using Laplace transforms,

$$y_1' = -y_1 + 4y_2 \qquad y_2' = 3y_1 - 2y_2 \tag{6.7.48}$$

$$y_1(0) = 3 \qquad y_2(0) = 4 \tag{6.7.49}$$

$$sY_1 - y_1(0) = -Y_1 + 4Y_2 \qquad sY_2 - y_2(0) = 3Y_1 - 2Y_2 \tag{6.7.50}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{3s+22}{(s+5)(s-2)} \qquad Y_2 = \frac{4s+13}{(s+5)(s-2)} \tag{6.7.51}$$

$$Y_1 = \frac{-1}{(s+5)} + \frac{4}{(s-2)} \qquad Y_2 = \frac{1}{(s+5)} + \frac{3}{(s-2)} \tag{6.7.52}$$

$$y_1 = -e^{-5t} + 4e^{2t} \qquad y_2 = e^{-5t} + 3e^{2t} \tag{6.7.53}$$

4. Solving using Laplace transforms,

$$y_1' = 4y_2 - 8\cos(4t) \qquad y_2' = -3y_1 - 9\sin(4t) \tag{6.7.54}$$

$$y_1(0) = 0 \qquad y_2(0) = 3 \tag{6.7.55}$$

$$sY_1 - y_1(0) = 4Y_2 - \frac{8s}{s^2+16} \qquad sY_2 - y_2(0) = -3Y_1 - \frac{36}{s^2+16} \tag{6.7.56}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{4}{s^2+16} \qquad Y_2 = \frac{3s}{(s^2+16)} \tag{6.7.57}$$

$$y_1 = \sin(4t) \qquad y_2 = 3\cos(4t) \tag{6.7.58}$$

5. Solving using Laplace transforms,

$$y_1' = y_2 + 1 - u(t-1) \qquad y_2' = -y_1 + 1 - u(t-1) \tag{6.7.59}$$

$$y_1(0) = 0 \qquad y_2(0) = 0 \tag{6.7.60}$$

$$sY_1 - y_1(0) = Y_2 + \frac{1-e^{-s}}{s} \qquad sY_2 - y_2(0) = -Y_1 + \frac{1-e^{-s}}{s} \tag{6.7.61}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{(1-e^{-s})(s+1)}{s(s^2+1)} \qquad Y_2 = \frac{(1-e^{-s})(s-1)}{s(s^2+1)} \tag{6.7.62}$$

$$Y_1 = [1-e^{-s}]\left[\frac{1}{s} + \frac{1-s}{s^2+1}\right] \qquad Y_2 = [1-e^{-s}]\left[\frac{-1}{s} + \frac{s+1}{s^2+1}\right] \tag{6.7.63}$$

$$\tag{6.7.64}$$

Writing the result in piecewise form,

$$y_1 = 1 + \sin(t) - \cos(t) - \Big[1 + \sin(t-1) - \cos(t-1)\Big]u(t-1) \tag{6.7.65}$$

$$y_2 = -1 + \sin(t) + \cos(t) + \Big[1 - \sin(t-1) - \cos(t-1)\Big]u(t-1) \tag{6.7.66}$$

$$y_1 = \begin{cases} 1 + \sin(t) - \cos(t) & t < 1 \\ \sin(t) - \sin(t-1) + \cos(t) + \cos(t-1) & t > 1 \end{cases} \tag{6.7.67}$$

$$y_2 = \begin{cases} -1 + \sin(t) + \cos(t) & t < 1 \\ \sin(t) - \sin(t-1) + \cos(t) - \cos(t-1) & t > 1 \end{cases} \tag{6.7.68}$$

6. Solving using Laplace transforms,

$$y_1' = 5y_1 + y_2 \qquad y_2' = y_1 + 5y_2 \tag{6.7.69}$$

$$y_1(0) = 1 \qquad y_2(0) = -3 \tag{6.7.70}$$

$$sY_1 - y_1(0) = 5Y_2 + Y_2 \qquad sY_2 - y_2(0) = Y_1 + 5Y_2 \tag{6.7.71}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{s-8}{s^2 - 10s + 24} \qquad Y_2 = \frac{16-3s}{s^2 - 10s + 24} \tag{6.7.72}$$

$$Y_1 = \frac{2}{s-4} - \frac{1}{s-6} \qquad Y_2 = \frac{-2}{s-4} - \frac{1}{s-6} \tag{6.7.73}$$

$$y_1 = 2e^{4t} - e^{6t} \qquad y_2 = -2e^{4t} - e^{6t} \tag{6.7.74}$$

7. Solving using Laplace transforms,

$$y_1' = 2y_1 - 4y_2 + e^t\, u(t-1) \qquad y_2' = y_1 - 3y_2 + e^t\, u(t-1) \tag{6.7.75}$$

$$y_1(0) = 3 \qquad y_2(0) = 0 \tag{6.7.76}$$

$$sY_1 - y_1(0) = 2Y_1 - 4Y_2 + \frac{e^{-s+1}}{s-1} \qquad sY_2 - y_2(0) = Y_1 - 3Y_2 + \frac{e^{-s+1}}{s-1} \tag{6.7.77}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{3s+9+e^{-s+1}}{s^2+s-2} \qquad Y_2 = \frac{3+e^{-s+1}}{s^2+s-2} \tag{6.7.78}$$

$$Y_1 = \frac{4}{s-1} - \frac{1}{s+2} + e^{-s+1}\left[\frac{(1/3)}{s-1} - \frac{(1/3)}{s+2}\right] \tag{6.7.79}$$

$$Y_2 = \frac{1}{s-1} - \frac{1}{s+2} + e^{-s+1}\left[\frac{(1/3)}{s-1} - \frac{(1/3)}{s+2}\right] \tag{6.7.80}$$

$$\tag{6.7.81}$$

Writing the result in piecewise form,

$$y_1 = 4e^t - e^{-2t} + \left[\frac{e^t - e^{-2t+3}}{3}\right]u(t-1) \tag{6.7.82}$$

$$y_2 = e^t - e^{-2t} + \left[\frac{e^t - e^{-2t+3}}{3}\right]u(t-1) \tag{6.7.83}$$

$$y_1 = \begin{cases} 4e^t - e^{-2t} & t < 1 \\ \frac{13}{3}\,e^t - \left[1 + \frac{e^3}{3}\right]\,e^{-2t} & t > 1 \end{cases} \tag{6.7.84}$$

$$y_2 = \begin{cases} e^t - e^{-2t} & t < 1 \\ \frac{4}{3}\,e^t - \left[1 + \frac{e^3}{3}\right]\,e^{-2t} & t > 1 \end{cases} \tag{6.7.85}$$

8. Solving using Laplace transforms,

$$y_1' = -2y_1 + 3y_2 \qquad y_2' = 4y_1 - y_2 \tag{6.7.86}$$

$$y_1(0) = 4 \qquad y_2(0) = 3 \tag{6.7.87}$$

$$sY_1 - y_1(0) = -2Y_1 + 3Y_2 \qquad sY_2 - y_2(0) = 4Y_1 - Y_2 \tag{6.7.88}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{4s+13}{s^2+3s-10} \qquad Y_2 = \frac{3s+22}{s^2+3s-10} \tag{6.7.89}$$

$$Y_1 = \frac{3}{s-2} + \frac{1}{s+5} \qquad Y_2 = \frac{4}{s-2} - \frac{1}{s+5} \tag{6.7.90}$$

$$y_1 = 3e^{2t} + e^{-5t} \qquad y_2 = 4e^{2t} - e^{-5t} \tag{6.7.91}$$

$$\tag{6.7.92}$$

9. Solving using Laplace transforms,

$$y_1' = 4y_1 + y_2 \qquad y_2' = -y_1 + 2y_2 \tag{6.7.93}$$

$$y_1(0) = 3 \qquad y_2(0) = 1 \tag{6.7.94}$$

$$sY_1 - y_1(0) = 4Y_1 + Y_2 \qquad sY_2 - y_2(0) = -Y_1 + 2Y_2 \tag{6.7.95}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{3(s-3)+4}{(s-3)^2} \qquad Y_2 = \frac{(s-3)-4}{(s-3)^2} \tag{6.7.96}$$

$$y_1 = [3+4t]\, e^{3t} \qquad y_2 = [1-4t]e^{3t} \tag{6.7.97}$$

$$\tag{6.7.98}$$

10. Solving using Laplace transforms,

$$y_1' = -y_2 \qquad y_2' = -y_1 + 2\cos(t)\Big[1 - u(t-2\pi)\Big] \tag{6.7.99}$$

$$y_1(0) = 1 \qquad y_2(0) = 0 \tag{6.7.100}$$

$$sY_1 - y_1(0) = -Y_2 \qquad sY_2 - y_2(0) = -Y_1 + \frac{2s(1-e^{-2\pi s})}{s^2+1} \tag{6.7.101}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{s^3 - s + 2se^{-2\pi s}}{(s^4-1)} \tag{6.7.102}$$

$$Y_2 = \frac{s^2 - 1 - 2s^2e^{-2\pi s}}{s^4-1} \tag{6.7.103}$$

$$Y_1 = \frac{s}{s^2+1} + e^{-2\pi s}\left[\frac{s}{s^2-1} - \frac{s}{s^2+1}\right] \tag{6.7.104}$$

$$Y_2 = \frac{1}{s^2+1} + e^{-2\pi s}\left[\frac{-1}{s^2-1} - \frac{1}{s^2+1}\right] \tag{6.7.105}$$

$$\tag{6.7.106}$$

Writing the result in piecewise form,

$$y_1 = \cos(t) + \Big[\sinh(t-2\pi) - \cos(t)\Big]u(t-2\pi) \tag{6.7.107}$$

$$y_2 = \sin(t) + \Big[-\cosh(t-2\pi) - \sin(t)\Big]u(t-2\pi) \tag{6.7.108}$$

$$y_1 = \begin{cases} \cos(t) & t < 2\pi \\ \sinh(t-2\pi) & t > 2\pi \end{cases} \tag{6.7.109}$$

$$y_2 = \begin{cases} \sin(t) & t < 2\pi \\ -\cosh(t-2\pi) & t > 2\pi \end{cases} \tag{6.7.110}$$

11. Solving using Laplace transforms,

$$y_1'' = y_1 + 3y_2 \qquad y_2'' = 4y_1 - 4e^t \tag{6.7.111}$$

$$y_1(0) = 2 \qquad y_2(0) = 1 \tag{6.7.112}$$

$$y_1'(0) = 3 \qquad y_2'(0) = 2 \tag{6.7.113}$$

$$s^2Y_1 - sy_1(0) - y_1'(0) = Y_1 + 3Y_2 \tag{6.7.114}$$

$$s^2Y_2 - sy_2(0) - y_2'(0) = 4Y_1 - \frac{4}{s-1} \tag{6.7.115}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{2s-3}{s^2-3s+2} \qquad Y_2 = \frac{1}{s-2} \tag{6.7.116}$$

$$Y_1 = \frac{1}{s-1} + \frac{1}{s-2} \qquad Y_2 = \frac{1}{s-2} \tag{6.7.117}$$

$$y_1 = e^t + e^{2t} \qquad y_2 = e^{2t} \tag{6.7.118}$$

12. Solving using Laplace transforms,

$$y_1'' = -2y_1 + 2y_2 \qquad y_2'' = 2y_1 - 5y_2 \tag{6.7.119}$$

$$y_1(0) = 1 \qquad y_2(0) = 3 \tag{6.7.120}$$

$$y_1'(0) = 0 \qquad y_2'(0) = 0 \tag{6.7.121}$$

$$s^2Y_1 - sy_1(0) - y_1'(0) = -2Y_1 + 2Y_2 \tag{6.7.122}$$

$$s^2Y_2 - sy_2(0) - y_2'(0) = 2Y_1 - 5Y_2 \tag{6.7.123}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{s(s^2+11)}{(s^4+7s^2+6)} \qquad Y_2 = \frac{s(3s^2+8)}{(s^4+7s^2+6)} \tag{6.7.124}$$

$$Y_1 = \frac{2s}{s^2+1} - \frac{s}{s^2+6} \qquad Y_2 = \frac{s}{s^2+1} + \frac{2s}{s^2+6} \tag{6.7.125}$$

$$y_1 = 2\cos(t) - \cos(\sqrt{6}t) \qquad y_2 = \cos(t) + 2\cos(\sqrt{6}t) \tag{6.7.126}$$

13. Solving using Laplace transforms,

$$y_1'' = -y_2 - 101\sin(10t) \qquad y_2'' = -y_1 + 101\sin(10t) \tag{6.7.127}$$

$$y_1(0) = 0 \qquad y_2(0) = 8 \tag{6.7.128}$$

$$y_1'(0) = 6 \qquad y_2'(0) = -6 \tag{6.7.129}$$

$$s^2Y_1 - sy_1(0) - y_1'(0) = -Y_2 - \frac{1010}{s^2+100} \tag{6.7.130}$$

$$s^2Y_2 - sy_2(0) - y_2'(0) = -Y_1 + \frac{1010}{s^2+100} \tag{6.7.131}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{-6s^3+14s^2+390s+410}{-s^5+s^4-101s^3+101s^2-100s+100} \tag{6.7.132}$$

$$Y_2 = \frac{-2(4s^4-7s^3+407s^2-205s+205)}{-s^5+s^4-101s^3+101s^2-100s+100} \tag{6.7.133}$$

$$Y_1 = \frac{-4}{s-1} + \frac{4s}{s^2+1} + \frac{10}{s^2+100} \qquad Y_2 = \frac{4}{s-1} + \frac{4s}{s^2+1} - \frac{10}{s^2+100} \tag{6.7.134}$$

$$y_1 = -4e^t + \sin(10t) + 4\cos(t) \qquad y_2 = 4e^t - \sin(10t) + 4\cos(t) \tag{6.7.135}$$

14. Solving using Laplace transforms,

$$4y_1' + y_2' - 2y_3' = 0 \qquad -2y_1 + y_3' = 1 \tag{6.7.136}$$

$$2y_2' - 4y_3' = -16t \tag{6.7.137}$$

$$y_1(0) = 2 \qquad y_2(0) = 0 \tag{6.7.138}$$

$$y_3(0) = 0 \tag{6.7.139}$$

$$4[sY_1 - 2] + sY_2 - 2[sY_3] = 0 \qquad -2[sY_1 + 2] + sY_3 = \frac{1}{s} \tag{6.7.140}$$

$$2[sY_2] - 4[sY_3] = \frac{-16}{s^2} \tag{6.7.141}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{2}{s} + \frac{2}{s^3} \qquad Y_2 = \frac{2}{s^2} \qquad Y_3 = \frac{1}{s^2} + \frac{4}{s^3} \tag{6.7.142}$$

$$y_1 = 2 + t^2 \qquad y_2 = 2t \qquad y_3 = t + 2t^2 \tag{6.7.143}$$

15. Solving using Laplace transforms,

$$y_1' + y_2' = 2\sinh(t) \qquad y_2' + y_3' = e^t \tag{6.7.144}$$

$$y_1' + y_3' = 2e^t + e^{-t} \tag{6.7.145}$$

$$y_1(0) = 1 \qquad y_2(0) = 1 \tag{6.7.146}$$

$$y_3(0) = 0 \tag{6.7.147}$$

$$[sY_1 - 1] + [sY_2 - 1] = \frac{2}{s^2 - 1} \qquad [sY_2 - 1] + sY_3 = \frac{1}{s - 1} \tag{6.7.148}$$

$$[sY_1 - 1] + sY_3 = \frac{2}{s - 1} + \frac{1}{s + 1} \tag{6.7.149}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{1}{s - 1} \qquad Y_2 = \frac{1}{s + 1} \qquad Y_3 = \frac{2}{s^2 - 1} \tag{6.7.150}$$

$$y_1 = e^t \qquad y_2 = e^{-t} \qquad y_3 = 2\sinh(t) \tag{6.7.151}$$

16. Model in Example 3 with $k = 4$,

$$y_1(0) = 1 \qquad y_2(0) = 1 \tag{6.7.152}$$

$$y_1'(0) = 1 \qquad y_2'(0) = -1 \tag{6.7.153}$$

$$y_1'' = -4y_1 + 4(y_2 - y_1) + 11\sin(t) \tag{6.7.154}$$

$$y_2'' = -4(y_2 - y_1) - 4y_2 - 11\sin(t) \tag{6.7.155}$$

$$s^2Y_1 - s - 1 = -8Y_1 + 4Y_2 + \frac{11}{s^2 + 1} \tag{6.7.156}$$

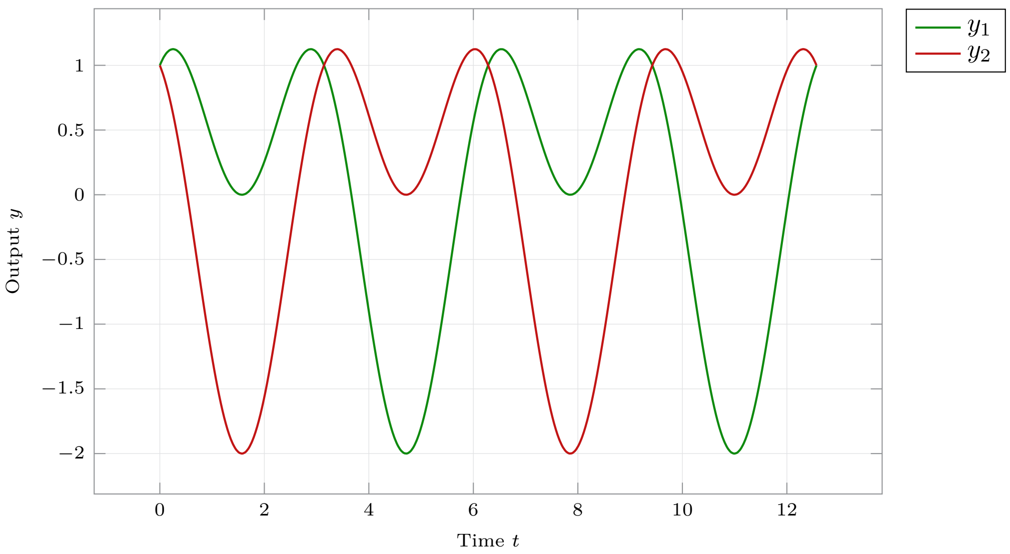
$$s^2Y_2 - s + 1 = 4Y_1 - 8Y_2 - \frac{11}{s^2 + 1} \tag{6.7.157}$$

Solving the system of equations in Y_1, Y_2,

$$Y_1 = \frac{s^3 + s^2 + s + 4}{s^4 + 5s^2 + 4} \qquad Y_2 = \frac{s^3 - s^2 + s - 4}{s^4 + 5s^2 + 4} \tag{6.7.158}$$

$$Y_1 = \frac{1}{(s^2 + 1)} + \frac{s}{s^2 + 4} \qquad Y_2 = \frac{-1}{(s^2 + 1)} + \frac{s}{s^2 + 4} \tag{6.7.159}$$

$$y_1 = \sin(t) + \cos(2t) \qquad y_1 = -\sin(t) + \cos(2t) \tag{6.7.160}$$



The difference between the motion of the two masses is $2\sin(t)$ which means that the masses periodically coincide and then move apart.

17. Varying one of four I.C. at a time, TBC.

Plot using scipy ode solver and store results to table.

18. In Example 1 all flows are doubled,

$$y_1' = -0.16y_1 + 0.04y_2 + 12 \qquad y_2' = 0.16y_1 - 0.16y_2 \tag{6.7.161}$$

$$y_1(0) = 0 \qquad y_2(0) = 150 \tag{6.7.162}$$

$$sY_1 = -0.16Y_1 + 0.04Y_2 + \frac{12}{s} \qquad sY_2 - 150 = 0.16Y_1 - 0.16Y_2 \tag{6.7.163}$$

Solving the system for Y_1, Y_2,

$$Y_1 = \frac{150(75s + 8)}{s(625s^2 + 200s + 12)} \qquad Y_2 = \frac{150(625s^2 + 100s + 8)}{s(625s^2 + 200s + 12)} \tag{6.7.164}$$

$$Y_1 = \frac{100}{s} - \frac{(75/2)}{s + 0.08} - \frac{(125/2)}{s + 0.24} \qquad Y_1 = \frac{100}{s} - \frac{75}{s + 0.08} + \frac{125}{s + 0.24} \tag{6.7.165}$$

$$y_1 = 100 - (75/2)e^{-0.08t} - (125/2)e^{-0.24t} \qquad y_1 = 100 - 75e^{-0.08t} + 125e^{-0.24t} \tag{6.7.166}$$

By intuition, doubling all the flow rates, simply fast forwards time by a factor of two, which means that setting $2t \to \tau$ recovers the old solution

19. Using KVL, the model is,

$$4j_1 + 8(j_1 - j_2) + 2j_1' = 390\cos(t) \qquad 8j_2 + 8(j_2 - j_1) + 4j_2' = 0 \tag{6.7.167}$$

$$j_1(0) = 0 \qquad j_2(0) = 0 \tag{6.7.168}$$

$$2sJ_1 + 12J_1 - 8J_2 = \frac{390s}{s^2+1} \qquad 4sJ_2 - 8J_1 + 16J_2 = 0 \tag{6.7.169}$$

Solving this system for $J_1,\ J_2$,

$$J_1 = \frac{195s(s+4)}{s^4 + 10s^3 + 17s^2 + 10s + 16} \qquad J_2 = \frac{390s}{s^4 + 10s^3 + 17s^2 + 10s + 16} \tag{6.7.170}$$

$$J_1 = \frac{42s+15}{s^2+1} - \frac{26}{s+2} - \frac{16}{s+8} \qquad J_2 = \frac{18s+12}{s^2+1} - \frac{26}{s+2} + \frac{8}{s+8} \tag{6.7.171}$$

$$j_1 = 42\cos(t) + 15\sin(t) - 26e^{-2t} - 16e^{-8t} \tag{6.7.172}$$

$$j_2 = 18\cos(t) + 12\sin(t) - 26e^{-2t} + 8e^{-8t} \tag{6.7.173}$$

20. With the voltage only acting for $t < 2\pi$, the model is,

$$4j_1 + 8(j_1 - j_2) + 2j_1' = 390\cos(t)\Big[1 - u(t - 2\pi)\Big] \tag{6.7.174}$$

$$8j_2 + 8(j_2 - j_1) + 4j_2' = 0 \tag{6.7.175}$$

$$j_1(0) = 0 \tag{6.7.176}$$

$$j_2(0) = 0 \tag{6.7.177}$$

$$2sJ_1 + 12J_1 - 8J_2 = \frac{390s(1 - e^{-2\pi s})}{s^2+1} \tag{6.7.178}$$

$$4sJ_2 - 8J_1 + 16J_2 = 0 \tag{6.7.179}$$

Solving this system for $J_1,\ J_2$,

$$J_1 = \frac{195 - e^{-2\pi s}[195(s)(s+4)]}{s^4 + 10s^3 + 17s^2 + 10s + 16} \tag{6.7.180}$$

$$J_2 = \frac{390 - e^{-2\pi s}[390s]}{s^4 + 10s^3 + 17s^2 + 10s + 16} \tag{6.7.181}$$

$$J_1 = \left[\frac{42s + 15}{s^2 + 1} - \frac{26}{s+2} - \frac{16}{s+8}\right][1 - e^{-2\pi s}] \tag{6.7.182}$$

$$J_2 = \left[\frac{18s + 12}{s^2 + 1} - \frac{26}{s+2} + \frac{8}{s+8}\right][1 - e^{-2\pi s}] \tag{6.7.183}$$

$$j_1 = 42\cos(t) + 15\sin(t) - 26e^{-2t} - 16e^{-8t} \tag{6.7.184}$$

$$- u(t - 2\pi)\Big[42\cos(t) + 15\sin(t) - 26e^{-2t+4\pi} - 16e^{-8t+16\pi}\Big] \tag{6.7.185}$$

$$j_2 = 18\cos(t) + 12\sin(t) - 26e^{-2t} + 8e^{-8t} \tag{6.7.186}$$

$$- u(t - 2\pi)\Big[18\cos(t) + 12\sin(t) - 26e^{-2t+4\pi} + 8e^{-8t+16\pi}\Big] \tag{6.7.187}$$

When the external EMF turns off, the sinusoidal component of the response also turns off, and the only response left is the exponential decay to zero.

When the EMF is still on, the response is exactly what it was in Problem 19.