Chapter 5

Series Solutions of ODEs, Special Functions

5.1 Power Series Method

1. TBC. Refer notes.
2. Finding radius of convergence,

$$f(x) = \sum_{m=0}^{\infty} (m+1)m\ x^m \tag{5.1.1}$$

$$\frac{1}{R} = \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| \qquad = \lim_{m\to\infty} \left|\frac{(m+2)(m+1)}{m(m+1)}\right| = 1 \tag{5.1.2}$$

$$R = 1 \tag{5.1.3}$$

3. Finding radius of convergence,

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{k^m}\ x^{2m} \tag{5.1.4}$$

$$\frac{1}{R} = \lim_{m\to\infty} |a_{2m}|^{1/2m} \qquad = \lim_{m\to\infty} \left|\left(\frac{-1}{k}\right)^m\right|^{1/2m} = \frac{1}{\sqrt{k}} \tag{5.1.5}$$

$$R = \sqrt{k} \tag{5.1.6}$$

4. Finding radius of convergence,

$$f(x) = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} \tag{5.1.7}$$

$$\frac{1}{R} = \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right| \qquad = \lim_{m\to\infty} \left| \frac{1}{(2m+2)(2m+3)} \right| = 0 \tag{5.1.8}$$

$$R = \infty \tag{5.1.9}$$

5. Finding radius of convergence,

$$f(x) = \sum_{m=0}^{\infty} \frac{2^m}{3^m}\, x^{2m} \tag{5.1.10}$$

$$\frac{1}{R} = \lim_{m\to\infty} |a_{2m}|^{1/2m} \qquad = \lim_{m\to\infty} \left| \left(\frac{2}{3}\right)^m \right|^{1/2m} = \frac{\sqrt{2}}{\sqrt{3}} \tag{5.1.11}$$

$$R = \sqrt{1.5} \tag{5.1.12}$$

6. Solving by power series method,

$$y = a_0 + \quad a_1 x + \quad a_2 x^2 + \quad a_3 x^3 + \dots \tag{5.1.13}$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \tag{5.1.14}$$

$$xy' = \qquad a_1 x + 2a_2 x^2 + 3a_3 x^3 + \dots \tag{5.1.15}$$

$$\tag{5.1.16}$$

Equating powers of x on both sides,

$$y' + xy' = y \tag{5.1.17}$$

$$a_0 = a_1 \qquad [x^0] \tag{5.1.18}$$

$$a_1 = a_1 + 2a_2 \qquad a_2 = 0 \qquad [x^1] \tag{5.1.19}$$

$$a_2 = 3a_3 + 2a_2 \qquad a_3 = 0 \qquad [x^2] \tag{5.1.20}$$

$$a_3 = 4a_4 + 3a_3 \qquad a_4 = 0 \qquad [x^3] \tag{5.1.21}$$

$$y = a_0(1 + x) \tag{5.1.22}$$

7. Solving by power series method,

$$y = a_0 + \ a_1 x + \ a_2 x^2 + \ a_3 x^3 + \dots \tag{5.1.23}$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \tag{5.1.24}$$

$$-2xy = \quad - 2a_0 x - 2a_1 x^2 - 2a_2 x^3 + \dots \tag{5.1.25}$$

Equating powers of x on both sides,

$$y' = -2xy \tag{5.1.26}$$

$$a_1 = 0 \qquad [x^0] \tag{5.1.27}$$

$$2a_2 = -2a_0 \qquad a_2 = -a_0 \qquad [x^1] \tag{5.1.28}$$

$$3a_3 = -2a_1 \qquad a_3 = 0 \qquad [x^2] \tag{5.1.29}$$

$$4a_4 = -2a_2 \qquad a_4 = \left(\frac{-1^2}{2!}\right) a_0 \qquad [x^3] \tag{5.1.30}$$

$$5a_5 = -2a_3 \qquad a_5 = 0 \qquad [x^4] \tag{5.1.31}$$

$$6a_6 = -2a_4 \qquad a_6 = \frac{-1^3}{3!} a_0 \qquad [x^5] \tag{5.1.32}$$

Assigning the power series to a function,

$$y = a_0 \left[1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \right] \tag{5.1.33}$$

$$= a_0 \sum_{m=0}^{\infty} \frac{(-x^2)^m}{m!} = a_0 \exp(-x^2) \tag{5.1.34}$$

8. Solving by power series method,

$$y = a_0 + \ a_1 x + \ a_2 x^2 + \ a_3 x^3 + \dots \tag{5.1.35}$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \tag{5.1.36}$$

$$xy' = \quad a_1 x + 2a_2 x^2 + 3a_3 x^3 + \dots \tag{5.1.37}$$

Equating powers of x on both sides,

$$xy' - 3y = k \tag{5.1.38}$$

$$-3a_0 = k \qquad a_0 = \frac{-k}{3} \qquad [x^0] \tag{5.1.39}$$

$$a_1 - 3a_1 = 0 \qquad a_1 = 0 \qquad [x^1] \tag{5.1.40}$$

$$2a_2 - 3a_2 = 0 \qquad a_2 = 0 \qquad [x^2] \tag{5.1.41}$$

$$3a_3 - 3a_3 = 0 \qquad a_3 \in \mathcal{R} \qquad [x^3] \tag{5.1.42}$$

$$4a_4 - 3a_4 = 0 \qquad a_4 = 0 \qquad [x^4] \tag{5.1.43}$$

$$ \tag{5.1.44}$$

Assigning the power series to a function,

$$y = \frac{-k}{3} + a_3 x^3 \tag{5.1.45}$$

9. Solving by power series method,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \tag{5.1.46}$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \tag{5.1.47}$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots \tag{5.1.48}$$

Equating powers of x on both sides,

$$y'' + y = 0 \tag{5.1.49}$$

$$a_0 + 2a_2 = 0 \qquad a_2 = \frac{-a_0}{2} \qquad [x^0] \tag{5.1.50}$$

$$6a_3 + a_1 = 0 \qquad a_3 = \frac{-a_1}{6} \qquad [x^1] \tag{5.1.51}$$

$$12a_4 + a_2 = 0 \qquad a_4 = \frac{a_0}{24} \qquad [x^2] \tag{5.1.52}$$

$$20a_5 + a_3 = 0 \qquad a_5 = \frac{a_1}{120} \qquad [x^3] \tag{5.1.53}$$

Assigning the power series to a function,

$$y = a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right] + a_1 \left[1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right] \tag{5.1.54}$$

$$= a_0 \cos(x) + a_1 \sin(x) \tag{5.1.55}$$

10. Solving by power series method,

$$xy = \qquad a_0 x + \quad a_1 x^2 + \quad a_2 x^3 + \dots \tag{5.1.56}$$

$$-y' = -\quad a_1 - 2a_2 x - \quad 3a_3 x^2 - \quad 4a_4 x^3 - \dots \tag{5.1.57}$$

$$y'' = \quad 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots \tag{5.1.58}$$

Equating powers of x on both sides,

$$y'' - y' + xy = 0 \tag{5.1.59}$$

$$2a_2 = a_1 \qquad a_2 = \frac{a_1}{2!} \qquad [x^0] \tag{5.1.60}$$

$$6a_3 + a_0 = 2a_2 \qquad a_3 = \frac{a_1 - a_0}{3!} \qquad [x^1] \tag{5.1.61}$$

$$12a_4 + a_1 = 3a_3 \qquad a_4 = \frac{-a_1 - a_0}{4!} \qquad [x^2] \tag{5.1.62}$$

$$20a_5 + a_2 = 4a_4 \qquad a_5 = \frac{-4a_1 - a_0}{5!} \qquad [x^3] \tag{5.1.63}$$

$$30a_6 + a_3 = 5a_5 \qquad a_6 = \frac{-8a_1 + 3a_0}{6!} \qquad [x^4] \tag{5.1.64}$$

$$42a_7 + a_4 = 6a_6 \qquad a_7 = \frac{-3a_1 + 8a_0}{7!} \qquad [x^5] \tag{5.1.65}$$

Consolidating power series in terms of a_0 and a_1,

$$y = a_0 \left[1 - \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{3x^6}{6!} + \frac{8x^7}{7!} \dots\right] \tag{5.1.66}$$

$$+ a_1 \left[\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{4x^5}{5!} - \frac{8x^6}{6!} - \frac{3x^7}{7!} + \dots\right] \tag{5.1.67}$$

11. Solving by power series method,

$$x^2 y = \qquad a_0 x^2 + \quad a_1 x^3 + \dots \tag{5.1.68}$$

$$-y' = -\quad a_1 - 2a_2 x - \quad 3a_3 x^2 - \quad 4a_4 x^3 - \dots \tag{5.1.69}$$

$$y'' = \quad 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots \tag{5.1.70}$$

Equating powers of x on both sides,

$$y'' - y' + xy = 0 \tag{5.1.71}$$

$$2a_2 = a_1 \qquad a_2 = \frac{a_1}{2!} \qquad [x^0] \tag{5.1.72}$$

$$6a_3 = 2a_2 \qquad a_3 = \frac{a_1}{3!} \qquad [x^1] \tag{5.1.73}$$

$$12a_4 + a_0 = 3a_3 \qquad a_4 = \frac{a_1 - 2a_0}{4!} \qquad [x^2] \tag{5.1.74}$$

$$20a_5 + a_1 = 4a_4 \qquad a_5 = \frac{-5a_1 - 2a_0}{5!} \qquad [x^3] \tag{5.1.75}$$

$$30a_6 + a_2 = 5a_5 \qquad a_6 = \frac{-17a_1 - 2a_0}{6!} \qquad [x^4] \tag{5.1.76}$$

$$42a_7 + a_3 = 6a_6 \qquad a_7 = \frac{-37a_1 - 2a_0}{7!} \qquad [x^5] \tag{5.1.77}$$

Consolidating power series in terms of a_0 and a_1,

$$y = a_0 \left[1 - \frac{2x^4}{4!} - \frac{2x^5}{5!} - \frac{2x^6}{6!} - \frac{2x^7}{7!} \dots\right] \tag{5.1.78}$$

$$+ a_1 \left[\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{5x^5}{5!} - \frac{17x^6}{6!} - \frac{37x^7}{7!} + \dots\right] \tag{5.1.79}$$

12. Trying the general m-th term approach,

$$y'' = \sum_{m=2}^{\infty} a_m \; m(m-1)x^{m-2} \qquad = \sum_{m=0}^{\infty} a_{m+2} \; (m+2)(m+1)x^m \tag{5.1.80}$$

$$x^2y'' = \sum_{m=2}^{\infty} a_m \; (m)(m-1)x^m \tag{5.1.81}$$

$$y' = \sum_{m=1}^{\infty} a_m \; mx^{m-1} \qquad 2xy' = \sum_{m=1}^{\infty} 2a_m \; (m)x^m \tag{5.1.82}$$

Equating powers of x,

$$(1-x^2)y'' + 2y = 2xy' \tag{5.1.83}$$

$$2a_2 + 2a_0 = 0 \qquad a_2 = -a_0 \tag{5.1.84}$$

$$6a_3 = 2a_1 \qquad a_3 = \frac{a_1}{3} \tag{5.1.85}$$

$$(m+1)(m+2)a_{m+2} = (m^2+m-2)a_m \qquad \forall \quad m \geq 2 \tag{5.1.86}$$

$$a_{m+2} = \frac{(m-1)a_m}{(m+1)} \tag{5.1.87}$$

Consolidating terms using a_0 and a_1,

$$y = a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \dots\right] \tag{5.1.88}$$

$$+ a_1\left[x + \frac{x^3}{3} + \frac{x^5}{6} + \frac{x^7}{9} + \frac{x^9}{12} + \dots\right] \tag{5.1.89}$$

13. Trying the general m-th term approach,

$$y'' = \sum_{m=2}^{\infty} a_m\ m(m-1)x^{m-2} \qquad = \sum_{m=0}^{\infty} a_{m+2}\ (m+2)(m+1)x^m \tag{5.1.90}$$

$$x^2 y = \sum_{m=0}^{\infty} a_m\ x^{m+2} \qquad = \sum_{m=2}^{\infty} a_{m-2}\ x^m \tag{5.1.91}$$

Equating powers of x,

$$2a_2 + a_0 = 0 \qquad a_2 = -\frac{a_0}{2!} \tag{5.1.92}$$

$$6a_3 = -a_1 \qquad a_3 = -\frac{a_1}{3!} \tag{5.1.93}$$

$$(m+1)(m+2)a_{m+2} + a_m + a_{m-2} = 0 \qquad \forall \quad m \geq 2 \tag{5.1.94}$$

$$a_{m+2} = \frac{-a_m - a_{m-2}}{(m+2)(m+1)} \tag{5.1.95}$$

Consolidating terms using a_0 and a_1,

$$y = a_0\left[1 - \frac{x^2}{2!} - \frac{x^4}{4!} + 13\frac{x^6}{6!} + \dots\right] \tag{5.1.96}$$

$$+ a_1\left[\frac{x}{1!} - \frac{x^3}{3!} - 5\frac{x^5}{5!} + 25\frac{x^7}{7!} \dots\right] \tag{5.1.97}$$

14. Trying the general m-th term approach,

$$y'' = \sum_{m=2}^{\infty} a_m\ m(m-1)x^{m-2} \qquad = \sum_{m=0}^{\infty} a_{m+2}\ (m+2)(m+1)x^m \tag{5.1.98}$$

$$y' = \sum_{m=1}^{\infty} a_m\ mx^{m-1} \qquad 4xy' = \sum_{m=1}^{\infty} 4a_m\ (m)x^m \tag{5.1.99}$$

$$4x^2y = \sum_{m=0}^{\infty} 4a_m\ x^{m+2} \qquad = \sum_{m=2}^{\infty} 4a_{m-2}\ x^m \tag{5.1.100}$$

Equating powers of x,

$$2a_2 - 2a_0 = 0 \qquad a_2 = a_0 \tag{5.1.101}$$

$$6a_3 - 2a_1 = 4a_1 \qquad a_3 = a_1 \tag{5.1.102}$$

$$(m+1)(m+2)a_{m+2} + 4a_{m-2} = (4m+2)a_m \qquad \forall \quad m \geq 2 \tag{5.1.103}$$

$$a_{m+2} = \frac{(4m+2)a_m - 4a_{m-2}}{(m+2)(m+1)} \tag{5.1.104}$$

Consolidating terms using a_0 and a_1,

$$y = a_0 \left[1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} \dots \right] \tag{5.1.105}$$

$$+ a_1 \left[x + \frac{x^3}{1!} + \frac{x^5}{2!} + \frac{x^7}{3!} + \frac{x^9}{4!} + \dots \right] \tag{5.1.106}$$

$$y = (a_0 + a_1 x)\ e^{x^2} \tag{5.1.107}$$

15. Shifting summation indices,

(a) Using $s \to m+1$

$$f(s) = \sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1}\ x^{s-1} \tag{5.1.108}$$

$$= \frac{6}{5}\ x + \frac{12}{10}\ x^2 + \frac{20}{17}\ x^3 + \frac{30}{26}\ x^4 + \frac{42}{37}\ x^5 + \dots \tag{5.1.109}$$

$$= \sum_{m=1}^{\infty} \frac{(m+1)(m+2)}{(m+1)^2+1}\ x^m \tag{5.1.110}$$

(b) Using $p \to m - 4$

$$g(p) = \sum_{p=1}^{\infty} \frac{p^2}{(p+1)!}\ x^{p+4} \tag{5.1.111}$$

$$= \frac{1}{2!}\ x^5 + \frac{4}{3!}\ x^6 + \frac{9}{4!}\ x^7 + \frac{16}{5!}\ x^8 + \frac{25}{6!}\ x^9 + \dots \tag{5.1.112}$$

$$= \sum_{m=5}^{\infty} \frac{(m-4)^2}{(m-3)!}\ x^m \tag{5.1.113}$$

16. Trying the general m-th term approach,

$$y' = \sum_{m=1}^{\infty} a_m\ m x^{m-1} \qquad y' = \sum_{m=0}^{\infty} a_{m+1}\ (m+1) x^m \tag{5.1.114}$$

$$4y = \sum_{m=0}^{\infty} 4a_m\ x^m \tag{5.1.115}$$

Equating powers of x,

$$y' + 4y = 1 \tag{5.1.116}$$

$$4a_0 + a_1 = 0 \qquad a_1 = -4a_0 \tag{5.1.117}$$

$$(m+1)a_{m+1} + 4a_m = 0 \qquad \forall \quad m \geq 1 \tag{5.1.118}$$

$$a_{m+1} = \frac{-4a_m}{(m+1)} \tag{5.1.119}$$

Consolidating terms using a_0,

$$y = a_0 \left[1 - \frac{4x}{1!} + \frac{4^2 x^2}{2!} - \frac{4^3 x^3}{3!} + \frac{4^4 x^4}{4!} - \dots\right] \tag{5.1.120}$$

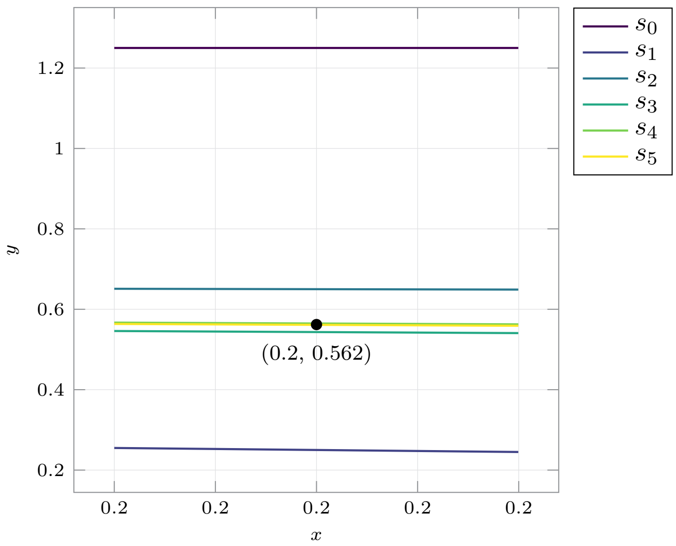
$$y = a_0 e^{-4x} \tag{5.1.121}$$

Applying the I.C. $y(0) = 1.25$, $x_1 = 0.2$,

$$y(0) = a_0 = 1.25 \tag{5.1.122}$$

$$y(x_1) = 1.25 \cdot \exp(-4 \cdot 0.2) \tag{5.1.123}$$

$$= 0.56166 \tag{5.1.124}$$



17. Trying the general m-th term approach,

$$y'' = \sum_{m=2}^{\infty} a_m \; m(m-1)x^{m-2} \qquad = \sum_{m=0}^{\infty} a_{m+2} \; (m+2)(m+1)x^m \tag{5.1.125}$$

$$y' = \sum_{m=1}^{\infty} a_m \; mx^{m-1} \qquad 3xy' = \sum_{m=1}^{\infty} 3a_m \; (m)x^m \tag{5.1.126}$$

Equating powers of x,

$$2a_2 + 2a_0 = 0 \qquad a_2 = -a_0 \tag{5.1.127}$$

$$6a_3 + 5a_1 = 0 \qquad a_3 = \frac{-5}{6}a_1 \tag{5.1.128}$$

$$(m+1)(m+2)a_{m+2} + (3m+2)a_m = 0 \qquad \forall \quad m \geq 2 \tag{5.1.129}$$

$$a_{m+2} = \frac{-(3m+2)a_m}{(m+2)(m+1)} \tag{5.1.130}$$

Consolidating terms using a_0 and a_1,

$$y = a_0 \left[1 - \frac{2x^2}{2!} + \frac{16x^4}{4!} - \frac{224x^6}{6!} + \dots\right] \tag{5.1.131}$$

$$+ a_1 \left[x - \frac{5x^3}{3!} + \frac{55x^5}{5!} - \frac{935x^7}{7!} + \dots\right] \tag{5.1.132}$$

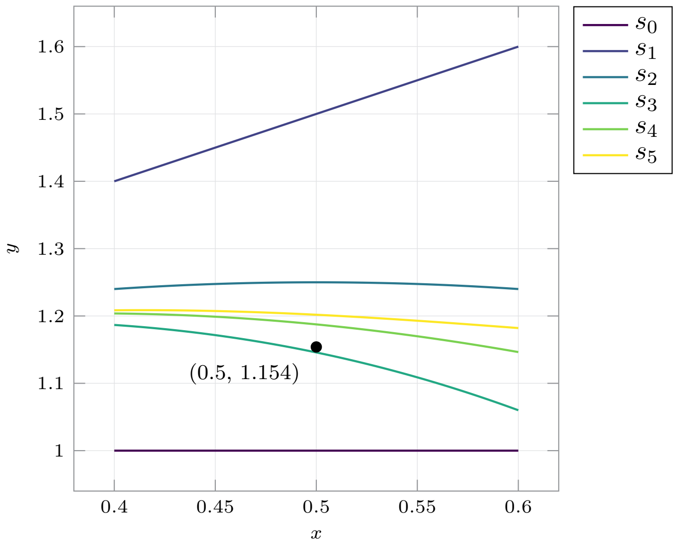
Applying the I.C. $y(0) = 1$, $y'(0) = 1$, $x_1 = 0.5$,

$$y(0) = a_0 = 1 \quad 5.1.133$$

$$y'(0) = a_1 = 1 \quad 5.1.134$$

$$y(x_1) = 1.25 \cdot \exp(-4 \cdot 0.2) \quad 5.1.135$$

$$= 1.15455 \quad 5.1.136$$



18. Trying the general m-th term approach,

$$y'' = \sum_{m=2}^{\infty} a_m \; m(m-1)x^{m-2} \qquad = \sum_{m=0}^{\infty} a_{m+2} \; (m+2)(m+1)x^m \quad 5.1.137$$

$$x^2 y'' = \sum_{m=2}^{\infty} a_m \; (m)(m-1)x^m \quad 5.1.138$$

$$y' = \sum_{m=1}^{\infty} a_m \; m x^{m-1} \qquad 2xy' = \sum_{m=1}^{\infty} 2a_m \; (m)x^m \quad 5.1.139$$

Equating powers of x,

$$(1-x^2)y'' + 30y = 2xy' \tag{5.1.140}$$

$$2a_2 + 30a_0 = 0 \qquad a_2 = -15a_0 \tag{5.1.141}$$

$$6a_3 + 30a_1 = 2a_1 \qquad a_3 = \frac{-14a_1}{3} \tag{5.1.142}$$

$$(m+1)(m+2)a_{m+2} = (m^2+m-30)a_m \qquad \forall \quad m \geq 2 \tag{5.1.143}$$

$$a_{m+2} = \frac{(m+6)(m-5)a_m}{(m+1)(m+2)} \tag{5.1.144}$$

Consolidating terms using a_0 and a_1,

$$y = a_0\left[1 - 15x^2 + 30x^4 - 10x^6 - \frac{15x^8}{7} - \dots\right] \tag{5.1.145}$$

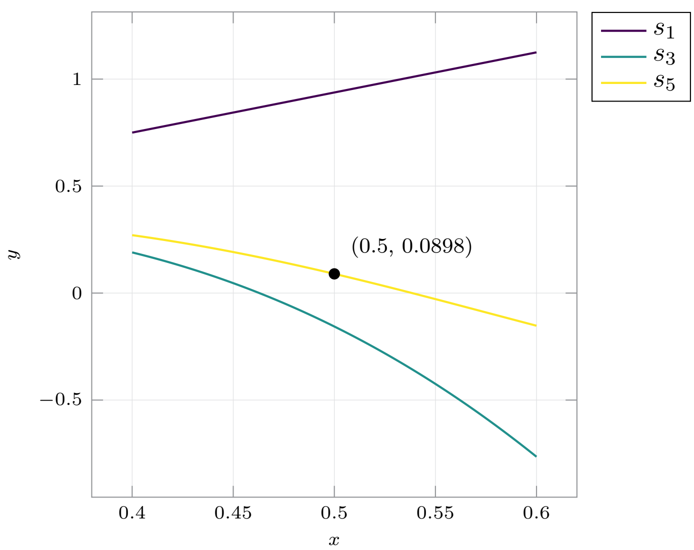
$$+ a_1\left[x - \frac{14x^3}{3} + \frac{21x^5}{5}\right] \tag{5.1.146}$$

Applying the I.C. $y(0) = 0$, $y'(0) = 1.875$, $x_1 = 0.5$,

$$y(0) = a_0 = 0 \tag{5.1.147}$$

$$y'(0) = a_1 = 1.875 \tag{5.1.148}$$

$$y(x_1) = 0.08984 \tag{5.1.149}$$



19. Trying the general m-th term approach,

$$-2y' = \sum_{m=1}^{\infty} -2a_m \; mx^{m-1} \qquad = \sum_{m=0}^{\infty} -2a_{m+1} \; (m+1)x^m \tag{5.1.150}$$

$$xy' = \sum_{m=1}^{\infty} a_m \; (m)x^m \qquad xy = \sum_{m=1}^{\infty} a_{m-1}x^m \tag{5.1.151}$$

Equating powers of x,

$$(x-2)y' = xy \tag{5.1.152}$$

$$-2a_1 = 0 \qquad a_1 = 0 \tag{5.1.153}$$

$$ma_m - 2(m+1)a_{m+1} = a_{m-1} \qquad \forall \quad m \geq 1 \tag{5.1.154}$$

$$a_{m+1} = \frac{ma_m - a_{m-1}}{2(m+1)} \tag{5.1.155}$$

Consolidating terms using a_0 and a_1,

$$y = a_0 \left[1 + 0x - \frac{x^2}{4} - \frac{x^3}{12} + 0x^4 + \frac{x^5}{120} + \dots\right] \tag{5.1.156}$$

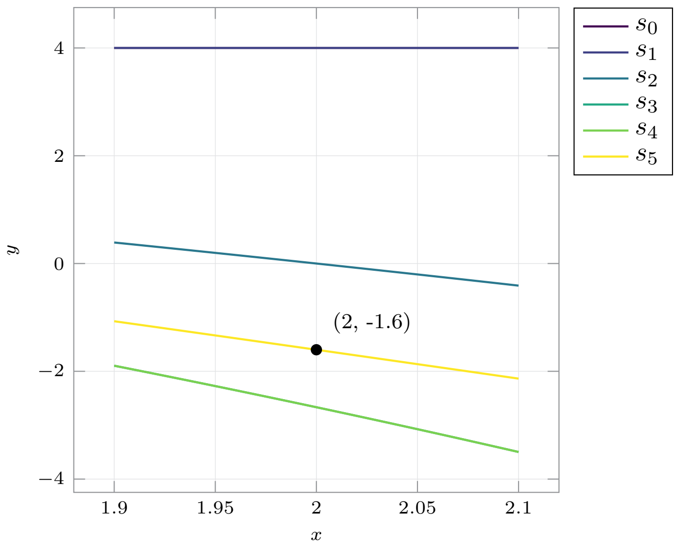
$$\tag{5.1.157}$$

Applying the I.C. $y(0) = 4$, $x_1 = 2$,

$$y(0) = a_0 = 4 \tag{5.1.158}$$

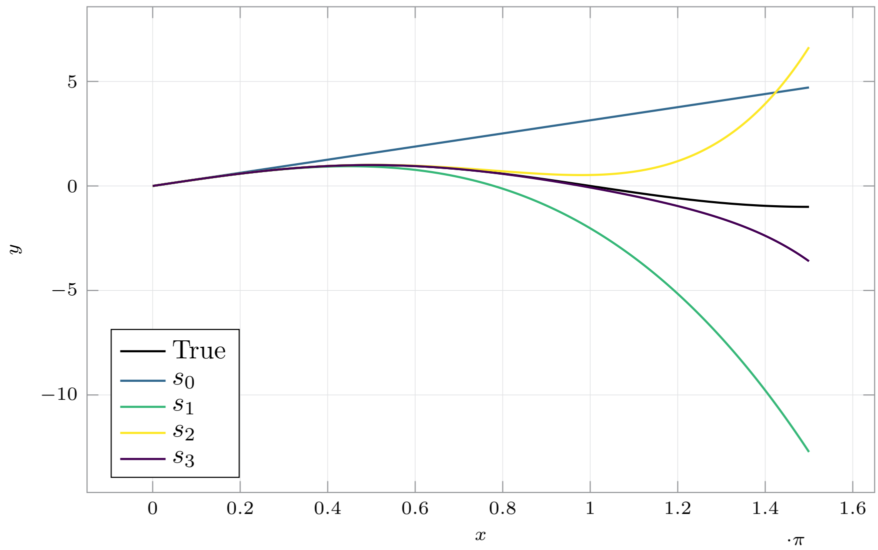
$$y(x_1) = -1.6 \tag{5.1.159}$$

$$z(x_1) = (z-2)^2 e^z \Big|_{z=2} = 0 \tag{5.1.160}$$



20. Graphing the partial sums of the Maclaurin series of

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \tag{5.1.161}$$



From the plot, higher series sums diverge from the true sine curve at larger values of x, which means they are closer approximations.

5.2 Legendre's Equation, Legendre Polynomials

1. Setting $n = 0$,

$$y_1(x) = 1 - \frac{0 \cdot 1}{2!}\, x^2 + \dots \tag{5.2.1}$$

$$= 1 + 0 + 0 + \dots \tag{5.2.2}$$

$$= 1 \tag{5.2.3}$$

$$y_2(x) = x - \frac{-1 \cdot 2}{3!}\, x^3 + \frac{-1 \cdot -3 \cdot 2 \cdot 4}{5!}\, x^5 + \dots \tag{5.2.4}$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \tag{5.2.5}$$

$$= \frac{1}{2}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\right] \tag{5.2.6}$$

$$- \frac{1}{2}\left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots\right] \tag{5.2.7}$$

$$= \frac{1}{2}[\ln(1+x) - \ln(1-x)] \tag{5.2.8}$$

$$= \frac{1}{2}\, \ln\left(\frac{1+x}{1-x}\right) \tag{5.2.9}$$

Solving by separating variables,

$$(1 - x^2)y'' = 2xy' \tag{5.2.10}$$

$$z = y' \qquad z' = y'' \tag{5.2.11}$$

$$(1 - x^2)z' = 2xz \qquad \int \frac{1}{z}\, \mathrm{d}z = \int \frac{2x}{1 - x^2}\, \mathrm{d}x \tag{5.2.12}$$

$$u = 1 - x^2 \qquad \mathrm{d}u = -2x\, \mathrm{d}x \tag{5.2.13}$$

$$\ln(z) = -\ln(1 - x^2) \tag{5.2.14}$$

$$y' = z = \frac{1}{1 - x^2} \qquad y' = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right) \tag{5.2.15}$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + c_3 \tag{5.2.16}$$

2. Setting $n = 1$, and using the result from Problem 1,

$$y_2(x) = x - \frac{0 \cdot 3}{3!}\, x^3 + \ldots \tag{5.2.17}$$

$$= x + 0 + 0 + \ldots \tag{5.2.18}$$

$$= x \tag{5.2.19}$$

$$y_1(x) = 1 - \frac{1 \cdot 2}{2!}\, x^2 + \frac{-1 \cdot 1 \cdot 2 \cdot 4}{4!}\, x^4 - \ldots \tag{5.2.20}$$

$$= 1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \ldots \tag{5.2.21}$$

$$= 1 - x\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right] \tag{5.2.22}$$

$$1 - \frac{x}{2}\, \ln\left(\frac{1+x}{1-x}\right) \tag{5.2.23}$$

$$\tag{5.2.24}$$

3. Deriving the Legendre polynomials from the general term, with $M = \lfloor n/2 \rfloor$

$$P_n(x) = \sum_{m=0}^{M} (-1^m)\, \frac{(2n-2m)!}{2^n\ m!\ (n-m)!\ (n-2m)!}\, x^{n-2m} \tag{5.2.25}$$

$$n = 0 \implies M = 0 \tag{5.2.26}$$

$$P_0(x) = \frac{0!}{2^0\ 0!\ 0!\ 0!}\, x^0 = 1 \tag{5.2.27}$$

$$n = 1 \implies M = 0 \tag{5.2.28}$$

$$P_1(x) = \frac{2!}{2^1\ 0!\ 1!\ 1!}\, x^1 = x \tag{5.2.29}$$

$$n = 2 \implies M = 1 \tag{5.2.30}$$

$$P_2(x) = \frac{4!}{2^2\ 0!\ 2!\ 2!}\, x^2 - \frac{2!}{2^2\ 1!\ 1!\ 0!}\, x^0 = \frac{1}{2}(3x^2 - 1) \tag{5.2.31}$$

$$n = 3 \implies M = 1 \tag{5.2.32}$$

$$P_3(x) = \frac{6!}{2^3\ 0!\ 3!\ 3!}\, x^3 - \frac{4!}{2^3\ 1!\ 2!\ 1!}\, x^1 = \frac{1}{2}(5x^3 - 3x) \tag{5.2.33}$$

For $n > 3$, there are 3 terms in the summation,

$$n = 4 \implies M = 2 \tag{5.2.34}$$

$$P_4(x) = \frac{8!}{2^4\ 0!\ 4!\ 4!}\ x^4 - \frac{6!}{2^4\ 1!\ 3!\ 2!}\ x^2 + \frac{4!}{2^4\ 2!\ 2!\ 0!}\ x^0 \tag{5.2.35}$$

$$= \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{5.2.36}$$

$$n = 5 \implies M = 2 \tag{5.2.37}$$

$$P_5(x) = \frac{10!}{2^5\ 0!\ 5!\ 5!}\ x^5 - \frac{8!}{2^5\ 1!\ 4!\ 3!}\ x^3 + \frac{6!}{2^5\ 2!\ 3!\ 1!}\ x \tag{5.2.38}$$

$$= \frac{1}{8}(63x^5 - 70x^3 + 15x) \tag{5.2.39}$$

4. Verifying that $\{P_i(x)\}$ satisfy the Legendre ODE,

$$P_0(x) = 1 \qquad (1 - x^2)(0) + 0 = 2x(0) \tag{5.2.40}$$

$$P_1(x) = x \qquad (1 - x^2)(0) + 2x = 2x(1) \tag{5.2.41}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad (1 - x^2)(3) + 3(3x^2 - 1) = 2x(3x) \tag{5.2.42}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \qquad (1 - x^2)(30x) + 60x^3 - 36x = 30x^3 - 60 \tag{5.2.43}$$

For P_4, P_5,

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{5.2.44}$$

$$(1 - x^2)(420x^2 - 60) + 20(35x^4 - 30x^2 + 3) = 280x^4 - 120x^2 \tag{5.2.45}$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \tag{5.2.46}$$

$$(1 - x^2)(1260x^3 - 420x) + 1890x^5 - 2100x^3 + 450x = 630x^5 - 420x^3 + 30x \tag{5.2.47}$$

5. For $n = 6, 7$, there are 4 terms in the summation,

$$n = 6 \implies M = 3 \tag{5.2.48}$$

$$P_6(x) = \frac{12!}{2^6\ 0!\ 6!\ 6!}\ x^6 - \frac{10!}{2^6\ 1!\ 5!\ 4!}\ x^4 \tag{5.2.49}$$

$$+ \frac{8!}{2^6\ 2!\ 4!\ 2!}\ x^2 + \frac{6!}{2^6\ 3!\ 3!\ 0!}\ x^0 \tag{5.2.50}$$

$$= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \tag{5.2.51}$$

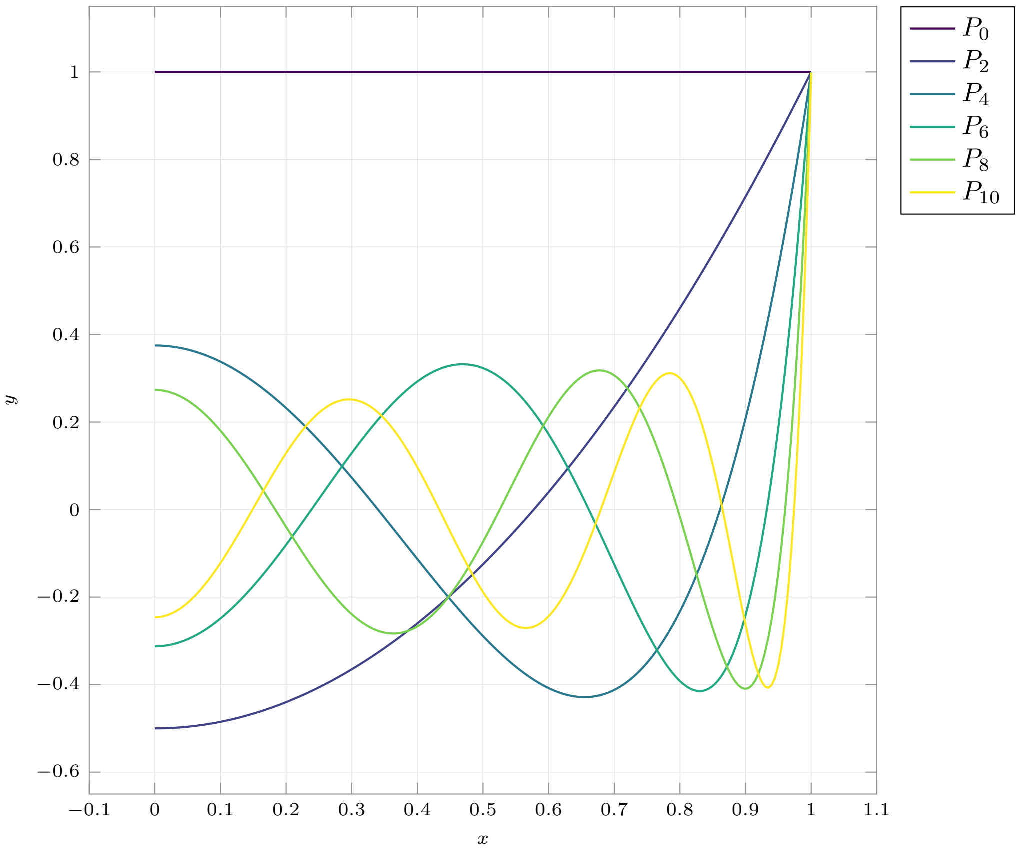
$$n = 7 \implies M = 3 \tag{5.2.52}$$

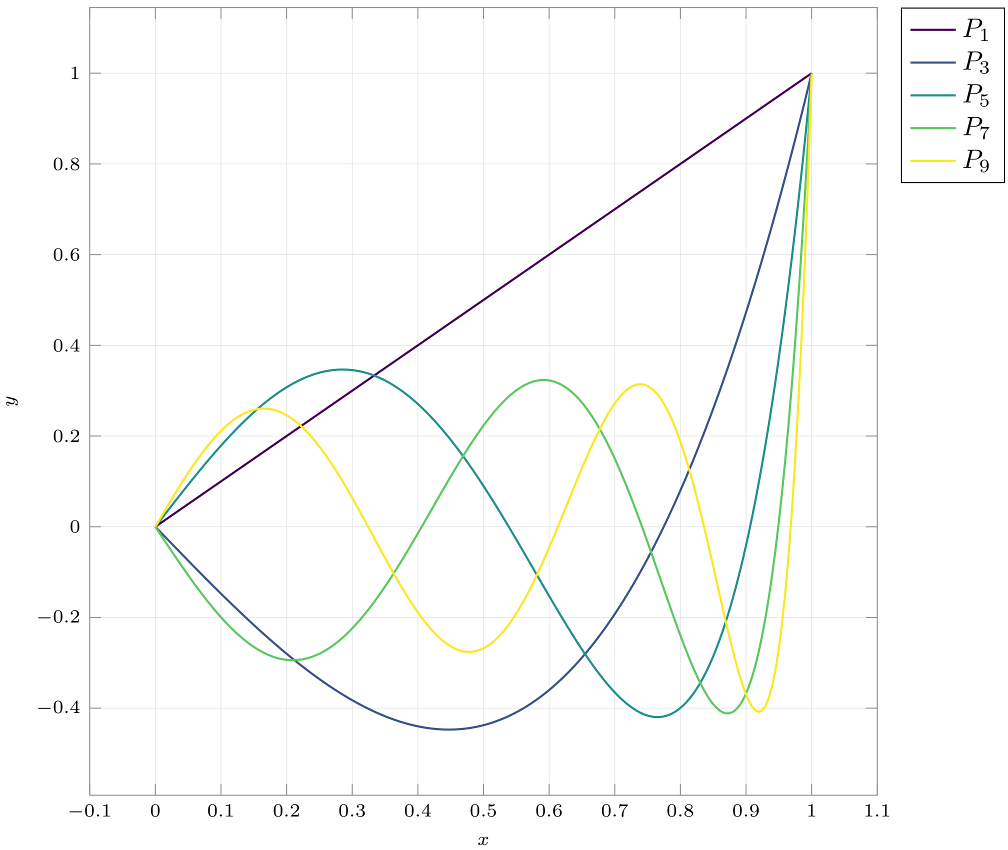
$$P_7(x) = \frac{14!}{2^7\ 0!\ 7!\ 7!}\ x^7 - \frac{12!}{2^7\ 1!\ 6!\ 5!}\ x^5 \tag{5.2.53}$$

$$+ \frac{10!}{2^7\ 2!\ 5!\ 3!}\ x^3 + \frac{8!}{2^7\ 3!\ 4!\ 1!}\ x \tag{5.2.54}$$

$$= \frac{1}{6}(429x^7 - 693x^5 + 315x^3 - 35x) \tag{5.2.55}$$

6. Plotting P_2 to P_10 on common axes,



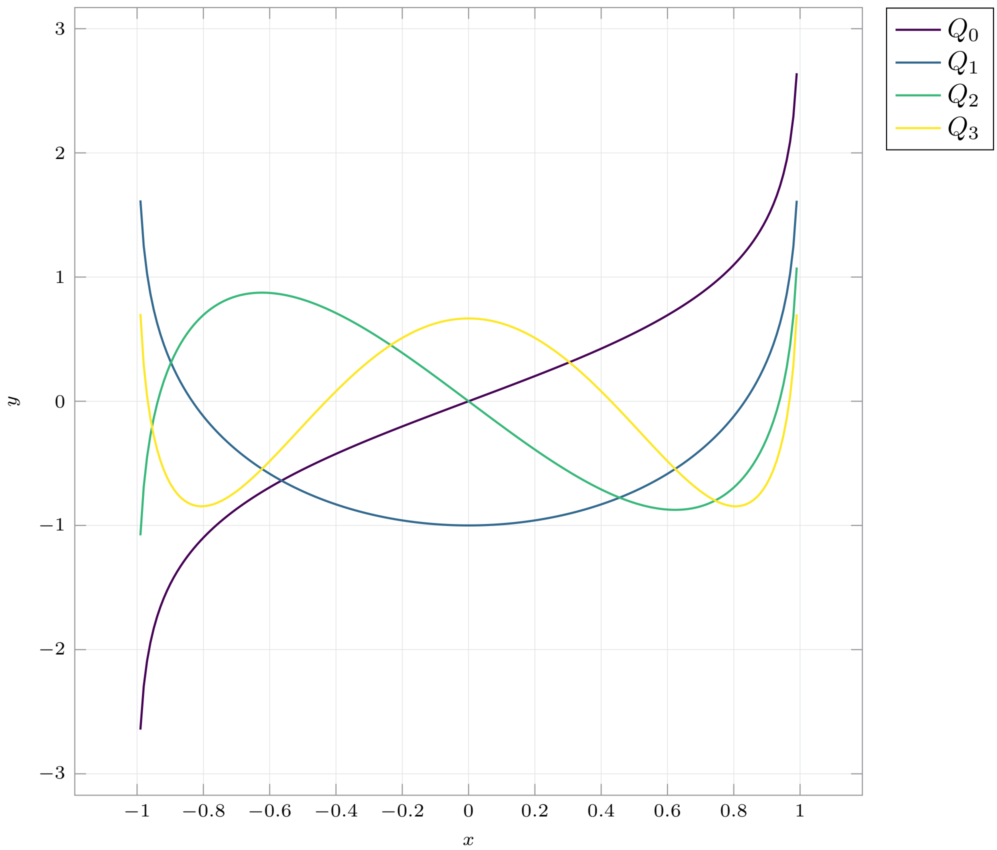


Wthin the interval (0, 1) each polynomial only intersects $y = 0.5$ once, at x_0.

Even	x_0	Odd	x_0
P_2	0.8165	P_1	0.5
P_4	0.9430	P_3	0.9059
P_6	0.9726	P_5	0.9618
P_8	0.984	P_7	0.9794
P_{10}	0.9895	P_9	0.9872

7. TBC. Whenever inbuilt precision limits are reached.

8. Plotting the first few legendre functions of the second kind,



9. Substituting the given expression into the legendre ODE,

$$0 = (1-x^2)y'' - 2xy' + n(n+1)y = \tag{5.2.56}$$

$$y = a_s x^s + a_{s+1}x^{s+1} + a_{s+2}x^{s+2} \tag{5.2.57}$$

$$0 = x^{s+2}[-(s+2)(s+1)a_{s+2} - 2(s+2)a_{s+2} + n(n+1)a_{s+2}] \tag{5.2.58}$$

$$+ x^{s+1}[-(s+1)sa_{s+1}] - 2(s+1)a_{s+1} + n(n+1)a_{s+1} \tag{5.2.59}$$

$$+ x^s[(s+2)(s+1)a_{s+2} - s(s-1)a_s - 2(s)a_s + n(n+1)a_s] \tag{5.2.60}$$

$$+ x^{s-1}[(s+1)sa_{s+1}] \tag{5.2.61}$$

$$+ x^{s-2}[s(s-1)a_s] \tag{5.2.62}$$

Since only x^s has two different coefficients, setting the expression to zero,

$$a_{s+2} = a_s \frac{s^2+s-n^2-n}{(s+1)(s+2)} \tag{5.2.63}$$

$$= -a_s \frac{n^2 - ns + ns - s^2 + n - s}{(s+1)(s+2)} \tag{5.2.64}$$

$$= -a_s \frac{(n-2)(1+s+n)}{(s+1)(s+2)} \tag{5.2.65}$$

10. Generating function given by,

$$G(u, x) = \sum_{n=0}^{\infty} f_n(x)\ u^n \tag{5.2.66}$$

(a) For Legendre polynomials,

$$G(u, x) = (1 - 2xu + u^2)^{-1/2} \tag{5.2.67}$$

$$= (1 - v)^{-1/2} \tag{5.2.68}$$

$$= 1 + \frac{v}{2} + \frac{3v^2}{4 \cdot 2!} + \frac{15v^3}{8 \cdot 3!} + \dots \tag{5.2.69}$$

$$= 1 + \frac{2xu - u^2}{2} + \frac{3(2xu - u^2)^2}{8} + \frac{15(2xu - u^2)^3}{48} + \dots \tag{5.2.70}$$

$$= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}\ k!k!}\ v^k = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}\ k!k!}\ u^k\ (2x - u)^k \tag{5.2.71}$$

Gathering the coefficients of u^m, using the binomial expansion of $(2x - u)^k$. Here, r is used as the summation index when finding $f_n(x)$, with $r \in \{0, \dots, \lfloor n/2 \rfloor\}$

$$0 \leq m \leq k \tag{5.2.72}$$

$$g(u, x) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^k\ k!}\ u^k\ \sum_{m=0}^{k} \frac{1}{(k - m)!m!}\ (-u)^m (x)^{n-m} \tag{5.2.73}$$

$$f_n(x) = \frac{(2n)!}{2^n\ n!\ (n)!\ 0!}\ x^n - \frac{(2n - 2)!}{2^{n-1}\ (n - 1)!\ (n - 2)!1!}\ x^{n-2} \tag{5.2.74}$$

$$+ \frac{(2n - 4)!}{2^{n-2}\ (n - 2)!\ (n - 4)!2!}\ x^{n-4} - \dots \tag{5.2.75}$$

$$= \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r\ (2n - 2r)!}{2^{n-r}\ (n - r)!\ (n - 2r)!\ r!}\ x^{n-2r} \tag{5.2.76}$$

The above expression is exactly the n-th Legendre polynomial, and simutaneously the coefficient $f_n(x)$ of u^n in the binomial expansion of the generating function.

(b) By the cartesian distane rule,

$$r^2 = (A_{1x} - A_{2x})^2 + (A_{1y} - A_{2y})^2 \tag{5.2.77}$$

$$r^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta \tag{5.2.78}$$

$$\frac{1}{r} = \left(r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta\right)^{-1/2} \tag{5.2.79}$$

$$= \frac{1}{r_2}\left(1 + \frac{r_1^2}{r_2^2} - \frac{2r_1}{r_2}\cos\theta\right)^{-1/2} \tag{5.2.80}$$

$$\frac{r_1}{r_2} \to u \qquad \cos\theta \to x \tag{5.2.81}$$

$$= \frac{1}{r_2}(1 + u^2 - 2ux)^{-1/2} \tag{5.2.82}$$

$$= \frac{1}{r_2}\sum_{m=0}^{\infty} P_m(\cos\theta)\left(\frac{r_1}{r_2}\right)^m \tag{5.2.83}$$

This result follows from the previous part.

(c) Using the generating function at $x = 1$

$$G(u, 1) = (1 - 2u + u^2)^{-1/2} = (1 - u)^{-1} \tag{5.2.84}$$

$$= \sum_{k=0}^{\infty} u^k = \sum_{n=0}^{\infty} P_n(x) u^n \tag{5.2.85}$$

$$P_n(1) = 1 \qquad \forall n \tag{5.2.86}$$

Using the generating function at $x = -1$

$$G(u, -1) = (1 + 2u + u^2)^{-1/2} = (1 + u)^{-1} \tag{5.2.87}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)(-2)\ldots(-k)}{k!} u^k = \sum_{n=0}^{\infty} (-1)^n u^n \tag{5.2.88}$$

$$P_n(-1) = (-1)^n \quad \forall \quad n \tag{5.2.89}$$

Using the generating function at $x = 0$

$$G(u, 0) = (1 + u^2)^{-1/2} \tag{5.2.90}$$

$$= \sum_{k=0}^{\infty} \frac{(-1/2)(-3/2)\dots(1/2-k)}{k!} u^{2k} \tag{5.2.91}$$

$$P_n(0) = 0 \quad \forall \quad n = 2k+1 \tag{5.2.92}$$

$$P_n(0) = \frac{(-1)^k \ [1 \cdot 3 \cdots (2k-1)]}{2^k \ k!} \quad \forall \quad n = 2k \tag{5.2.93}$$

11. Reducing to the Legendre ODE,

$$(a^2 - x^2)y'' - 2xy' + n(n+1)y = 0 \tag{5.2.94}$$

$$x = au \qquad \mathrm{d}x = a \ \ \mathrm{d}u \tag{5.2.95}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{a}\frac{\mathrm{d}y}{\mathrm{d}u} \tag{5.2.96}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{a}\frac{\mathrm{d}^2y}{\mathrm{d}u^2}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{a^2}\frac{\mathrm{d}^2y}{\mathrm{d}u^2} \tag{5.2.97}$$

$$\left[\frac{a^2 - (au)^2}{a^2}\right] \frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{2au}{a}\frac{\mathrm{d}y}{\mathrm{d}u} + (n)(n+1)y = 0 \tag{5.2.98}$$

$$(1-u^2)\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - 2u\frac{\mathrm{d}y}{\mathrm{d}u} + n(n+1)y = 0 \tag{5.2.99}$$

$$y = c_1 P_n(u) + c_2 Q_n(u) \tag{5.2.100}$$

$$= c_1 P_n(x/a) + c_2 Q_n(x/a) \tag{5.2.101}$$

12. Rodriguez formula,

$$(x^2-1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{2n-2k}(-1)^k \tag{5.2.102}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n = \sum_{k=0}^{M}(-1)^k \frac{n! \ (2n-2k)!}{(n-k)! \ k! \ (n-2k)!} x^{n-2k} \tag{5.2.103}$$

$$\left[\frac{1}{2^n \ n!}\right] \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n = \sum_{k=0}^{M}(-1)^k \frac{(2n-2k)!}{2^n \ k! \ (n-k)! \ (n-2k)!} x^{n-2k} \tag{5.2.104}$$

$$= P_n(x) \tag{5.2.105}$$

Here, $M = \lfloor n/2 \rfloor$ and the expression matches the Legendre polynomial formula given.

13. Applying Rodriguez formula for $n = 0$ to $n = 5$,

$$P_0(x) = \frac{1}{2^0 \ 0!} \ \left[(x^2-1)^0\right] = 1 \tag{5.2.106}$$

$$P_1(x) = \frac{1}{2^1 \ 1!} \ \frac{\mathrm{d}}{\mathrm{d}x} \left[(x^2-1)^1\right] = \frac{1}{2}(2x) = x \tag{5.2.107}$$

$$P_2(x) = \frac{1}{2^2 \ 2!} \ \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left[(x^2-1)^2\right] = \frac{1}{8}(12x^2-4) = \frac{1}{2}(3x^2-1) \tag{5.2.108}$$

$$P_3(x) = \frac{1}{2^3 \ 3!} \ \frac{\mathrm{d}^3}{\mathrm{d}x^3} \left[(x^2-1)^3\right] = \frac{1}{48}(120x^3-72x) = \frac{1}{2}(5x^3-3x) \tag{5.2.109}$$

$$P_4(x) = \frac{1}{2^4 \ 4!} \ \frac{\mathrm{d}^4}{\mathrm{d}x^4} \left[(x^2-1)^4\right] = \frac{1}{8}(35x^4-30x^2+3) \tag{5.2.110}$$

$$P_5(x) = \frac{1}{2^5 \ 5!} \ \frac{\mathrm{d}^5}{\mathrm{d}x^5} \left[(x^2-1)^5\right] = \frac{1}{8}(63x^5-70x^3+15x) \tag{5.2.111}$$

14. Bonnet's recursion, differentiating w.r.t. u,

$$\frac{x-u}{(1-2ux+u^2)^{3/2}} = \sum_{n=1}^{\infty} nP_n(x) \ u^{n-1} \tag{5.2.112}$$

$$(x-u)\sum_{n=0}^{\infty} P_n(x) \ u^n = (1-2ux+u^2)\sum_{n=1}^{\infty} nP_n(x) \ u^{n-1} \tag{5.2.113}$$

$$= (1-2ux+u^2)\sum_{n=0}^{\infty}(n+1)P_{n+1}(x) \ u^n \tag{5.2.114}$$

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1} \tag{5.2.115}$$

$$(n+1) \ P_{n+1} = (2n+1)x \ P_n - n \ P_{n-1} \tag{5.2.116}$$

Manual calculations TBC.

15. Finding and cross-checking associated Legendre functions,

(a) $n = k = 1$,

$$P_1^1(x) = (1-x^2)^{1/2} \ \frac{\mathrm{d}}{\mathrm{d}x} \ [p_1(x)] \tag{5.2.117}$$

$$= \sqrt{1-x^2} = \mu \tag{5.2.118}$$

$$2xy' = \frac{-2x^2}{\mu} \tag{5.2.119}$$

$$(1-x^2)y'' + \left(2 - \frac{1}{1-x^2}\right) y = \frac{-1}{\mu} + \frac{(1-2x^2)}{\mu} = \frac{-2x^2}{\mu} \tag{5.2.120}$$

(b) $n = 2, k = 1,$

$$P_2^1(x) = \sqrt{1-x^2}\,\frac{\mathrm{d}}{\mathrm{d}x}\,[p_2(x)] \tag{5.2.121}$$

$$= 3x\sqrt{1-x^2} = 3x\mu \tag{5.2.122}$$

$$2xy' = \frac{-2x^2}{\mu} = \frac{6x-12x^3}{\mu} \tag{5.2.123}$$

$$(1-x^2)y'' + \left(6 - \frac{1}{1-x^2}\right)y = \frac{6x^3-9x}{\mu} + \frac{(15x-18x^3)}{\mu} \tag{5.2.124}$$

$$= \frac{(6x-12x^3)}{\mu} \tag{5.2.125}$$

(c) $n = 2, k = 2,$

$$P_2^2(x) = (1-x^2)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,[p_2(x)] \tag{5.2.126}$$

$$= 3(1-x^2) \tag{5.2.127}$$

$$2xy' = -12x^2 \tag{5.2.128}$$

$$(1-x^2)y'' + \left(6 - \frac{4}{1-x^2}\right)y = -6 + 6x^2 + (6-18x^2) \tag{5.2.129}$$

$$= -12x^2 \tag{5.2.130}$$

(d) $n = 4, k = 2,$

$$P_4^2(x) = (1-x^2)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,[p_4(x)] \tag{5.2.131}$$

$$= \frac{15(7x^2-1)(1-x^2)}{2} \tag{5.2.132}$$

$$2xy' = -60x^2(7x^2-4) \tag{5.2.133}$$

$$(1-x^2)y'' + \left(20 - \frac{4}{1-x^2}\right)y = (1-x^2)(120-630x^2) + 15(8-10x^2)(7x^2-1) \tag{5.2.134}$$

$$= -420x^4 + 240x^2 \tag{5.2.135}$$

5.3 Extended Power Series Method: Frobenius Method

1. TBC. Refer notes and chapter end exercises from C2 and C5.

2. Finding indicial equation, using the coefficients of the lowest power, after $x + 2 \to x$,

$$y'' + \frac{y'}{(x+2)} - \frac{y}{(x+2)^2} = 0 \tag{5.3.1}$$

$$x^2y'' + xy' - y = 0 \tag{5.3.2}$$

$$r(r-1) + r - 1 = 0 \qquad r_1 = -1,\ r_2 = 1 \tag{5.3.3}$$

$$\tag{5.3.4}$$

Finding the first solution,

$$y_1 = \sum_{m=1}^{\infty} a_{m-1}\ x^m \qquad xy_1' = \sum_{m=1}^{\infty} a_{m-1}\ mx^m \tag{5.3.5}$$

$$x^2y_1'' = \sum_{m=2}^{\infty} a_{m-1}\ m(m-1)x^m \tag{5.3.6}$$

$$a_{m-1}(m^2 - 1) = 0 \tag{5.3.7}$$

$$y_1 = x \tag{5.3.8}$$

Finding the second solution using reduction of order,

$$y_2 = gy_1 \qquad y_2' = g + xg' \tag{5.3.9}$$

$$y_2'' = 2g' + xg'' \tag{5.3.10}$$

$$0 = x^2(2g' + xg'') + x(g + xg') - gx \tag{5.3.11}$$

$$0 = g''(x^3) + g'(3x^2) \qquad h = g' \tag{5.3.12}$$

$$h' = -h\frac{3}{x} \qquad \ln(h) = -3\ln(x) \tag{5.3.13}$$

$$g' = x^{-3} \qquad g = \frac{1}{2x^2} \tag{5.3.14}$$

$$y_2 = gy_1 = \frac{1}{2x} \tag{5.3.15}$$

After reversing the change of variables, the general solution is

$$y = c_1y_1 + c_2y_2 \tag{5.3.16}$$

$$= c_1(x+2) + \frac{c_2}{(x+2)} \tag{5.3.17}$$

3. Finding indicial equation, using the coefficients of the lowest power,

$$x^2y'' + (2)xy' + (x^2)y = 0 \tag{5.3.18}$$

$$r(r-1) + 2r + 0 = 0 \qquad r_1 = 0,\ r_2 = -1 \tag{5.3.19}$$

Finding the first solution,

$$y_1 = \sum_{m=0}^{\infty} a_m\ x^m \qquad xy_1' = \sum_{m=1}^{\infty} a_m\ mx^m \tag{5.3.20}$$

$$x^2y_1'' = \sum_{m=2}^{\infty} a_m\ m(m-1)x^m \qquad a_m = -a_{m-2}\ \frac{1}{m(m+1)} \tag{5.3.21}$$

$$y_1 = a_0\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right] \qquad y_1 = \frac{\sin(x)}{x} \tag{5.3.22}$$

Finding the second solution using reduction of order,

$$y_2 = gy_1 \tag{5.3.23}$$

$$y_2' = g\ \frac{x\ \cos(x) - \sin(x)}{x^2} + g'\ \frac{\sin(x)}{x} \tag{5.3.24}$$

$$y_2'' = g''\ \frac{\sin(x)}{x} + 2g'\ \frac{x\ \cos(x) - \sin(x)}{x^2} + g\ \frac{(2-x^2)\ \sin(x) - 2x\ \cos(x)}{x^3} \tag{5.3.25}$$

$$0 = g'\ [2x\ \cos(x) - 2\ \sin(x) + 2\ \sin(x)] + g''\ [x\ \sin(x)] \tag{5.3.26}$$

$$0 = g''[x\ \sin(x)] + g'[2x\ \cos(x)] \tag{5.3.27}$$

Solving the reduced order equation,

$$h = g' \qquad h' = -h\ [2\ \cot(x)] \tag{5.3.28}$$

$$\ln(h) = -2\ \ln(\sin\ x) \qquad g' = \frac{1}{\sin^2\ x} \tag{5.3.29}$$

$$g = -\cot(x) \qquad y_2 = gy_1 = \frac{\cos(x)}{x} \tag{5.3.30}$$

4. Finding indicial equation, using the coefficients of the lowest power,

$$x^2y'' + (x)y = 0 \tag{5.3.31}$$

$$r(r-1) + 0r + 0 = 0 \qquad r_1 = 1,\ r_2 = 0 \tag{5.3.32}$$

Finding the first solution,

$$y_1 = \sum_{m=1}^{\infty} a_{m-1}\, x^m \qquad\qquad xy_1'' = \sum_{m=1}^{\infty} a_m\, m(m+1)x^m \tag{5.3.33}$$

$$a_m = \frac{-a_{m-1}}{(m)(m+1)} \tag{5.3.34}$$

$$y_1 = x\left[1 - \frac{x}{2!} + \frac{x^2}{2\cdot 3!} - \frac{x^3}{6\cdot 4!} + \dots\right] \qquad\qquad y_1 = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(m-1)!\; m!}\, x^m \tag{5.3.35}$$

Finding the second solution using standard forumla,

$$y_2 = ky_1 \ln(x) + \sum_{m=0}^{\infty} A_m x^m \tag{5.3.36}$$

$$y_2'' = k\left[y_1'' \ln(x) - \frac{y_1}{x^2} + \frac{2y_1'}{x}\right] + \sum_{m=2}^{\infty} A_m\; m(m-1)x^{m-2} \tag{5.3.37}$$

$$0 = kx \ln(x)y_1'' - \frac{ky_1}{x} + 2ky_1' + k \ln(x)y_1 \tag{5.3.38}$$

$$+ 1 + \sum_{m=1}^{\infty} [A_m + A_{m+1}\; m(m+1)]x^m \tag{5.3.39}$$

$$-(k+1) = k\left[\sum_{m=1}^{\infty} \frac{(-1)^m(2m+1)}{m!\;(m+1)!}\; x^m\right] + \sum_{m=1}^{\infty} [A_m + A_{m+1}\; m(m+1)]x^m \tag{5.3.40}$$

$$k = -1 \tag{5.3.41}$$

$$A_{m+1} = \left[\frac{(-1)^m(2m+1)}{m!\;(m+1)!} - A_m\right]\; \frac{1}{m(m+1)} \tag{5.3.42}$$

$$y_2(x) = -y_1(x) \ln(x) + 1 + \sum_{m=1}^{\infty} A_m x^m \tag{5.3.43}$$

5. Finding indicial equation, using the coefficients of the lowest power,

$$xy'' + (2x+1)y' + (x+1)y = 0 \tag{5.3.44}$$

$$x^2y'' + (2x+1)\; xy' + x(x+1)\; y = 0 \tag{5.3.45}$$

$$r(r-1) + r + 0 = 0 \qquad\qquad r_1 = 0,\;\; r_2 = 0 \tag{5.3.46}$$

Finding the first solution, by matching coefficients

$$y_1 = x^0 \sum_{m=0}^{\infty} a_m x^m \qquad x^2 y_1'' = \sum_{m=2}^{\infty} a_m \, m(m-1) x^m \tag{5.3.47}$$

$$x y_1' = \sum_{m=1}^{\infty} a_m \, m x^m \qquad x^2 y_1' = \sum_{m=2}^{\infty} a_{m-1} \, (m-1) x^m \tag{5.3.48}$$

$$a_0 = 1 \qquad a_1 = -a_0 = -1 \tag{5.3.49}$$

$$0 = m^2 \, a_m + (2m-1) \, a_{m-1} + a_{m-2} \tag{5.3.50}$$

$$a_m = -\frac{(2m-1) \, a_{m-1} + a_{m-2}}{m^2} \tag{5.3.51}$$

$$y_1 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \dots \tag{5.3.52}$$

$$= \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} = \exp(-x) \tag{5.3.53}$$

Finding second root by reduction of order,

$$y_2 = g y_1 \qquad y_2' = g' y_1 + g y_1' \tag{5.3.54}$$

$$y_2'' = g'' y_1 + 2 g' y_1' + g y_1'' \tag{5.3.55}$$

$$0 = x(g'' y_1 + 2 g' y_1' + g y_1'') \tag{5.3.56}$$

$$+ (2x+1)(g' y_1 + g y_1') + (x+1) g y_1 \tag{5.3.57}$$

$$g''[x y_1] = -g'[2 x y_1' + (2x+1) y_1] \qquad g'' x = -g' \tag{5.3.58}$$

$$\ln(g') = -\ln(x) \qquad g = \ln(x) \tag{5.3.59}$$

$$y_2 = \exp(-x) \;\; \ln(x) \tag{5.3.60}$$

6. Finding indicial equation, using the coefficients of the lowest power,

$$x y'' + 2x^3 y' + (x^2 - 2) y = 0 \tag{5.3.61}$$

$$x^2 y'' + (2x^3) \, x y' + (x^3 - 2x) \, y = 0 \tag{5.3.62}$$

$$r(r-1) + 0r + 0 = 0 \qquad r_1 = 1, \;\; r_2 = 0 \tag{5.3.63}$$

Finding the first solution, by matching coefficients

$$y_1 = \sum_{m=1}^{\infty} a_{m-1} x^m \qquad xy_1'' = \sum_{m=1}^{\infty} a_m \ m(m+1) x^m \tag{5.3.64}$$

$$x^3 y_1' = \sum_{m=3}^{\infty} a_{m-3} \ (m-2) x^m \qquad x^2 y_1 = \sum_{m=3}^{\infty} a_{m-3} \ x^m \tag{5.3.65}$$

$$2a_1 = 2a_0 \qquad a_1 = a_0 = 1 \tag{5.3.66}$$

$$6a_2 = 2a_1 \qquad a_2 = a_1/3 = 1/3 \tag{5.3.67}$$

$$0 = m(m+1) \ a_m + (2m-3) \ a_{m-3} - 2a_{m-1} \tag{5.3.68}$$

$$a_m = \frac{2a_{m-1} - (2m-3)a_{m-3}}{m(m+1)} \tag{5.3.69}$$

$$y_1 = x + x^2 + \frac{x^3}{3} - \frac{7x^4}{36} - \frac{97x^5}{360} - \dots \tag{5.3.70}$$

Finding the second solution using standard forumla,

$$y_2 = k y_1 \ln(x) + \sum_{m=0}^{\infty} A_m x^m \tag{5.3.71}$$

$$y_2' = \frac{k y_1}{x} + k \ln(x) y_1' + \sum_{m=0}^{\infty} A_{m+1} \ (m+1) x^m \tag{5.3.72}$$

$$y_2'' = k \left[y_1'' \ln(x) - \frac{y_1}{x^2} + \frac{2y_1'}{x} \right] + \sum_{m=2}^{\infty} A_m \ (m)(m-1) x^{m-2} \tag{5.3.73}$$

$$0 = \frac{-k y_1}{x} + 2k y_1' + 2k x^2 y_1 \tag{5.3.74}$$

$$+ \sum_{m=1}^{\infty} A_{m+1} \ m(m+1) x^m + \sum_{m=3}^{\infty} 2A_{m-2} \ (m-2) x^m \tag{5.3.75}$$

$$- \sum_{m=0}^{\infty} 2A_m \ x^m + \sum_{m=2}^{\infty} A_{m-2} x^m \tag{5.3.76}$$

Equating coefficients of x^0 and x^1,

$$0 = (k - 2A_0) + (2A_2 - 2A_1 + 3k)x + (-k/3 + 2k + 6A_3 - 2A_2 + A_0)x^2 \tag{5.3.77}$$

$$k = 2 \qquad A_0 = 1 \qquad A_2 = A_1 - 3 \qquad A_3 = \frac{A_2 - 13}{3} \tag{5.3.78}$$

$$+ \sum_{m=1}^{\infty} \frac{(-1)^m (2m+1)}{m!\,(m+1)!}\, x^m + \sum_{m=1}^{\infty} [A_m + A_{m+1}\, m(m+1)] x^m \tag{5.3.79}$$

$$k = -1 \tag{5.3.80}$$

$$A_{m+1} = \left[\frac{(-1)^m (2m+1)}{m!\,(m+1)!} - A_m\right] \frac{1}{m(m+1)} \tag{5.3.81}$$

$$y_2(x) = 2y_1(x)\ln(x) + 1 + A_1\, x + (A_1 - 3)\, x^2 + \frac{A_1 - 16}{3}\, x^3 + \dots \tag{5.3.82}$$

No elegant closed form for the higher powers. Need to manually calculate the rest of the $\{A_m\}$ for $m \geq 2$.

7. Finding indicial equation, using the coefficients of the lowest power,

$$y'' + (x-1)y = 0 \tag{5.3.83}$$

$$x^2 y'' + (x^3 - x^2)\, y = 0 \tag{5.3.84}$$

$$r(r-1) + 0r + 0 = 0 \qquad\qquad r_1 = 1,\ r_2 = 0 \tag{5.3.85}$$

Finding the first solution, by matching coefficients

$$y_1 = \sum_{m=1}^{\infty} a_{m-1} x^m \qquad\qquad y_1'' = \sum_{m=0}^{\infty} a_{m+1}\,(m+2)(m+1) x^m \tag{5.3.86}$$

$$x y_1 = \sum_{m=2}^{\infty} a_{m-2}\, x^m \tag{5.3.87}$$

$$2a_1 = 0 \qquad\qquad a_1 = 0 \tag{5.3.88}$$

$$6a_2 = a_0 \qquad\qquad a_2 = 1/6 \tag{5.3.89}$$

$$a_{m-1} = a_{m-2} + (m+2)(m+1)\, a_{m+1} \qquad\qquad a_m = \frac{a_{m-2} - a_{m-3}}{(m+1)m} \tag{5.3.90}$$

$$y_1 = x + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{120} - \frac{x^6}{120} + \dots \tag{5.3.91}$$

Finding the second solution using standard forumla,

$$y_2 = ky_1 \ln(x) + \sum_{m=0}^{\infty} A_m x^m \tag{5.3.92}$$

$$y_2' = \frac{ky_1}{x} + k\ln(x)y_1' + \sum_{m=0}^{\infty} A_{m+1}\ (m+1)x^m \tag{5.3.93}$$

$$y_2'' = k\left[y_1''\ln(x) - \frac{y_1}{x^2} + \frac{2y_1'}{x}\right] + \sum_{m=2}^{\infty} A_m\ (m)(m-1)x^{m-2} \tag{5.3.94}$$

$$0 = \frac{-ky_1}{x^2} + \frac{2ky_1'}{x} \tag{5.3.95}$$

$$+ \sum_{m=0}^{\infty} A_{m+2}\ (m+1)(m+2)x^m + \sum_{m=1}^{\infty} 2A_{m-1}\ x^m - \sum_{m=0}^{\infty} A_m\ x^m \tag{5.3.96}$$

Finding coefficients of x^0 and x^1 terms,

$$0 = -\frac{k}{x} - \frac{kx}{6} + \frac{kx^2}{12} + \frac{2k}{x} + kx - \frac{2kx^2}{3} \tag{5.3.97}$$

$$+ (2A_2 - A_0) + (6A_3 + 2A_0 - A_1)x + (12A_4 + 2A_1 - A_2)x^2 + \dots \tag{5.3.98}$$

$$k = 0 \qquad A_0 = 1 \qquad A_1 = 0 \text{ (arbitrary)} \tag{5.3.99}$$

$$y_2 = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{24} + \dots \tag{5.3.100}$$

There is no elegant closed form for the rest of the coefficients $\{A_i\}$ and successive comparison of higher powers of x needs to be used to find them.

8. Finding indicial equation, using the coefficients of the lowest power,

$$xy'' + y' - xy = 0 \tag{5.3.101}$$

$$x^2y'' + xy' - (x^2)y = 0 \tag{5.3.102}$$

$$r(r-1) + r + 0 = 0 \qquad\qquad r_1 = 0,\ r_2 = 0 \tag{5.3.103}$$

Finding the first solution, by matching coefficients

$$y_1 = \sum_{m=0}^{\infty} a_m x^m \qquad xy_1'' = \sum_{m=1}^{\infty} a_{m+1}\,(m+1)(m)x^m \tag{5.3.104}$$

$$xy_1 = \sum_{m=1}^{\infty} a_{m-1}\,x^m \qquad y' = \sum_{m=0}^{\infty} a_{m+1}\,(m+1)x^m \tag{5.3.105}$$

$$a_1 = 0 \tag{5.3.106}$$

$$2a_2 + 2a_2 = a_0 \qquad a_2 = 1/4 \tag{5.3.107}$$

$$a_{m-1} = (m+1)^2\,a_{m+1} \qquad a_m = \frac{a_{m-2}}{m^2} \tag{5.3.108}$$

$$y_1 = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\,4^2} + \frac{x^6}{2^2\,4^2\,6^2} + \dots \tag{5.3.109}$$

$$y_1 = \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m}\,(m!)^2} \tag{5.3.110}$$

Finding second solution by using standard result,

$$y_2 = y_1\,\ln(x) + \sum_{m=1}^{\infty} A_m\,x^m \tag{5.3.111}$$

$$y_2' = y_1'\,\ln(x) + \frac{y_1}{x} + \sum_{m=0}^{\infty} A_{m+1}\,(m+1)x^m \tag{5.3.112}$$

$$y_2'' = y_1''\,\ln(x) + \frac{2y_1'}{x} - \frac{y_1}{x^2} + \sum_{m=0}^{\infty} A_{m+2}\,(m+1)(m+2)x^m \tag{5.3.113}$$

$$0 = 2y_1' + \sum_{m=1}^{\infty} A_{m+1}\,m(m+1)x^m + \sum_{m=0}^{\infty} A_{m+1}\,(m+1)x^m - \sum_{m=2}^{\infty} A_{m-1}\,x^m \tag{5.3.114}$$

Equating coefficients of x^0, x^1,

$$0 = 0 + x/2 + 2A_2x + A_1 + 2A_2x \qquad A_2 = -1/8 \quad A_1 = 0 \tag{5.3.115}$$

$$A_2 = 16A_4 + 1/8 \qquad A_4 = -1/64 \tag{5.3.116}$$

$$A_4 = 36A_6 + 1/192 \qquad A_6 = -1/1728 \tag{5.3.117}$$

$$A_1 = 9A_3 \qquad A_3 = 0 \tag{5.3.118}$$

$$A_3 = 25A_5 \qquad A_5 = 0 \tag{5.3.119}$$

$$y_2 = y_1\,\ln(x) - \left[\frac{x^2}{8} + \frac{x^4}{64} + \frac{x^6}{1728} + \dots\right] \tag{5.3.120}$$

9. Finding indicial equation, using the limits as $x \to 0$ of $b(x)$ and $c(x)$ after rewriting the ODE in standard form,

$$2x(x-1)y'' - (x+1)y' + y = 0 \tag{5.3.121}$$

$$x^2y'' - \frac{(x+1)}{2(x-1)}\,xy' + \frac{x}{2(x-1)}\,y = 0 \tag{5.3.122}$$

$$r(r-1) + \frac{r}{2} + 0 = 0 \tag{5.3.123}$$

$$r_1 = 1/2 \qquad\qquad r_2 = 0 \tag{5.3.124}$$

Finding the first solution using $r_1 = 1/2$, and noting that the series never truncates upon differentiation of fractional powers,

$$y_1 = \sum_{m=0}^{\infty} a_m x^{m+0.5} \tag{5.3.125}$$

$$y_1' = \sum_{m=-1}^{\infty} a_{m+1}\,(m+1.5)x^{m+0.5} \qquad xy_1' = \sum_{m=0}^{\infty} a_m\,(m+0.5)x^{m+0.5} \tag{5.3.126}$$

$$y_1'' = \sum_{m=0}^{\infty} a_m\,(m+0.5)(m-0.5)x^{m-1.5} \qquad x^2y_1'' = \sum_{m=0}^{\infty} a_m\,(m+0.5)(m-0.5)x^{m+0.5} \tag{5.3.127}$$

$$xy_1'' = \sum_{m=-1}^{\infty} a_{m+1}\,(m+1.5)(m+0.5)x^{m+0.5} \tag{5.3.128}$$

Finding the recursive relation for higher coefficients,

$$0 = 2a_m(m+0.5)(m-0.5) - 2a_{m+1}(m+1.5)(m+0.5) \tag{5.3.129}$$

$$- a_m(m+0.5) - a_{m+1}(m+1.5) + a_m \tag{5.3.130}$$

$$a_{m+1} = a_m\,\frac{m(2m-1)}{(m+1.5)(2m+1.5)} \tag{5.3.131}$$

$$y_1 = x^{1/2} \tag{5.3.132}$$

Finding the second solution using $r_2 = 0$,

$$y_1 = \sum_{m=0}^{\infty} a_m x^m \tag{5.3.133}$$

$$y_1' = \sum_{m=0}^{\infty} a_{m+1}\ (m+1)x^m \qquad xy_1' = \sum_{m=1}^{\infty} a_m\ (m)x^m \tag{5.3.134}$$

$$y_1'' = \sum_{m=0}^{\infty} a_{m+2}\ (m+2)(m+1)x^m \qquad x^2y_1'' = \sum_{m=2}^{\infty} a_m\ (m)(m-1)x^m \tag{5.3.135}$$

$$xy_1'' = \sum_{m=1}^{\infty} a_{m+1}\ (m+1)(m)x^m \tag{5.3.136}$$

Finding the recursive relation for higher coefficients,

$$0 = 2m(m-1)\ a_m - 2m(m+1)\ a_{m+1} - m\ a_m - (m+1)\ a_{m+1} + a_m \tag{5.3.137}$$

$$a_{m+1} = \frac{(2m-1)(m-1)}{(2m+1)(m+1)}\ a_m \tag{5.3.138}$$

$$a_1 = a_0 = 1 \qquad a_2 = a_3 = \cdots = 0 \tag{5.3.139}$$

$$y_2 = 1 + x \tag{5.3.140}$$

10. Finding indicial equation, using the limits as $x \to 0$ of $b(x)$ and $c(x)$ after rewriting the ODE in standard form,

$$xy'' + 2y' + 4xy = 0 \tag{5.3.141}$$

$$x^2y'' + (2)\ xy' + (4x^2)\ y = 0 \tag{5.3.142}$$

$$r(r-1) + 2r + 0 = 0 \tag{5.3.143}$$

$$r_1 = 0 \qquad r_2 = -1 \tag{5.3.144}$$

Finding the first solution using $r_1 = 0$,

$$x^2y'' = \sum_{m=2}^{\infty} a_m\ m(m-1)x^m \qquad xy' = \sum_{m=1}^{\infty} a_m\ mx^m \tag{5.3.145}$$

$$x^2y = \sum_{m=2}^{\infty} a_{m-2}\ x^m \tag{5.3.146}$$

$$a_1 = 0 = a_3 = a_5 = \dots \tag{5.3.147}$$

Finding the recursive relation for higher coefficients,

$$0 = m(m-1)\, a_m + 2m\, a_m + 4a_{m-2} \tag{5.3.148}$$

$$a_m = \frac{-4}{m(m+1)} a_{m-2} \tag{5.3.149}$$

$$y_1 = 1 - \frac{2^2x^2}{3!} + \frac{2^4x^4}{5!} - \frac{2^6x^6}{7!} + \dots \tag{5.3.150}$$

$$y_1 = \frac{1}{2x}\left[2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots\right] \tag{5.3.151}$$

$$y_1 = \frac{\sin(2x)}{2x} \tag{5.3.152}$$

Finding second root by reduction of order,

$$y_2 = g y_1 \qquad y_2' = g'y_1 + g y_1' \tag{5.3.153}$$

$$y_2'' = g''y_1 + 2g'y_1' + g y_1'' \tag{5.3.154}$$

$$0 = x(g''y_1 + 2g'y_1' + g y_1'') \tag{5.3.155}$$

$$+ 2(g'y_1 + g y_1') + (4x) g y_1 \tag{5.3.156}$$

$$g''[xy_1] = -g'[2xy_1' + 2y_1] \tag{5.3.157}$$

$$g'' = -g'[4\cot(2x)] \tag{5.3.158}$$

$$\ln(g') = -2\ln(|\sin(2x)|) \qquad g' = \csc^2(2x) \tag{5.3.159}$$

$$g = -\frac{\cot(2x)}{2} \qquad y_2 = \frac{\cos(2x)}{2x} \tag{5.3.160}$$

11. Finding indicial equation, using the limits as $x \to 0$ of $b(x)$ and $c(x)$ after rewriting the ODE in standard form,

$$xy'' + (2-2x)y' + (x-2)y = 0 \tag{5.3.161}$$

$$x^2y'' + (2-2x)\, xy' + (x^2 - 2x)\, y = 0 \tag{5.3.162}$$

$$r(r-1) + 2r + 0 = 0 \tag{5.3.163}$$

$$r_1 = 0 \qquad r_2 = -1 \tag{5.3.164}$$

Finding the first solution using $r_1 = 0$,

$$xy'' = \sum_{m=1}^{\infty} a_{m+1}\ (m+1)(m)x^m \tag{5.3.165}$$

$$2y' = 2\sum_{m=0}^{\infty} a_{m+1}\ (m+1)x^m \qquad -2xy' = -2\sum_{m=1}^{\infty} a_m\ mx^m \tag{5.3.166}$$

$$-2y = -2\sum_{m=0}^{\infty} a_m\ x^m \qquad xy = \sum_{m=1}^{\infty} a_{m-1}\ x^m \tag{5.3.167}$$

$$2a_1 - 2a_0 = 0 \qquad a_1 = a_0 = 1 \tag{5.3.168}$$

$$6a_2 = 4a_1 - a_0 \qquad a_2 = 1/2 \tag{5.3.169}$$

Finding the recursive relation for higher coefficients,

$$0 = (m^2 + 3m + 2)\ a_{m+1} - (2m+2)\ a_m + a_{m-1} \tag{5.3.170}$$

$$a_{m+1} = \frac{2(m+1)\ a_m - a_{m-1}}{(m+1)(m+2)} \tag{5.3.171}$$

$$y_1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots \tag{5.3.172}$$

$$y_1 = \exp(x) \tag{5.3.173}$$

Finding second root by reduction of order,

$$y_2 = gy_1 \qquad y_2' = g'y_1 + gy_1' \tag{5.3.174}$$

$$y_2'' = g''y_1 + 2g'y_1' + gy_1'' \tag{5.3.175}$$

$$0 = x(g''y_1 + 2g'y_1' + gy_1'') \tag{5.3.176}$$

$$+ (2-2x)(g'y_1 + gy_1') + (x-2)gy_1 \tag{5.3.177}$$

$$g''[xy_1] = -g'[2xy_1' + 2y_1 - 2xy_1] \tag{5.3.178}$$

$$g''[x] = -g'[2] \tag{5.3.179}$$

$$\ln(g') = -2\ln(x) \qquad g' = x^{-2} \tag{5.3.180}$$

$$g = \frac{-1}{x} \qquad y_2 = \frac{\exp(x)}{x} \tag{5.3.181}$$

12. Finding indicial equation, using the limits as $x \to 0$ of $b(x)$ and $c(x)$ after rewriting the ODE in standard form,

$$x^2y'' + (6)\ xy' + (4x^2 + 6)\ y = 0 \tag{5.3.182}$$

$$r^2 + 5r + 6 = 0 \tag{5.3.183}$$

$$r_1 = -3 \qquad\qquad r_2 = -2 \tag{5.3.184}$$

Finding the first solution using $r_1 = -3$,

$$y = \sum_{m=-3}^{\infty} a_{m+3}\ x^m \qquad\qquad x^2y = \sum_{m=-1}^{\infty} a_{m+1}\ x^m \tag{5.3.185}$$

$$y' = \sum_{m=-4}^{\infty} a_{m+4}\ (m+1)x^m \qquad\qquad xy' = \sum_{m=-3}^{\infty} a_{m+3}\ (m)x^m \tag{5.3.186}$$

$$x^2y'' = \sum_{m=-3}^{\infty} a_{m+3}\ (m)(m-1)x^m \tag{5.3.187}$$

$$12a_0 - 18a_0 + 6a_0 = 0 \qquad\qquad a_0 = 0 \tag{5.3.188}$$

$$6a_1 - 12a_1 + 6a_1 = 0 \qquad\qquad a_1 = 1 \text{ (free)} \tag{5.3.189}$$

Finding the recursive relation for higher coefficients,

$$0 = (m^2 + 5m + 6)\ a_{m+3} + 4a_{m+1} \tag{5.3.190}$$

$$a_{m+3} = \frac{-4a_{m+1}}{(m+2)(m+3)} \tag{5.3.191}$$

$$y_1 = \frac{1}{x^3}\left[x - \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} - \frac{2^6x^7}{7!} + \ldots\right] \tag{5.3.192}$$

$$y_1 = \frac{\sin(2x)}{2x^3} \tag{5.3.193}$$

Finding second root by reduction of order,

$$y_2 = g y_1 \qquad y_2' = g' y_1 + g y_1' \tag{5.3.194}$$

$$y_2'' = g'' y_1 + 2g' y_1' + g y_1'' \tag{5.3.195}$$

$$0 = x^2 (g'' y_1 + 2g' y_1' + g y_1'') \tag{5.3.196}$$

$$+ (6x)(g' y_1 + g y_1') + (4x^2 + 6) g y_1 \tag{5.3.197}$$

$$g''[y_1] = -g' \left[2y_1' + 6\frac{y_1}{x}\right] \tag{5.3.198}$$

$$g'' = -g'[4\cot(2x)] \tag{5.3.199}$$

$$\ln(g') = -2\ln(|\sin(2x)|) \qquad g' = \csc^2(2x) \tag{5.3.200}$$

$$g = -\frac{\cot(2x)}{2} \qquad y_2 = \frac{\cos(2x)}{4x^3} \tag{5.3.201}$$

13. Finding indicial equation, using the coefficients of the lowest power,

$$xy'' + (-2x + 1)y' + (x - 1)y = 0 \tag{5.3.202}$$

$$x^2 y'' + (-2x + 1)\ xy' + x(x - 1)\ y = 0 \tag{5.3.203}$$

$$r(r - 1) + r + 0 = 0 \qquad r_1 = 0,\ \ r_2 = 0 \tag{5.3.204}$$

Finding the first solution, by matching coefficients

$$y_1 = \sum_{m=0}^{\infty} a_m x^m \qquad x y_1 = \sum_{m=1}^{\infty} a_{m-1} x^m \tag{5.3.205}$$

$$x y_1'' = \sum_{m=1}^{\infty} a_{m+1}\ (m + 1)(m) x^m \tag{5.3.206}$$

$$y_1' = \sum_{m=0}^{\infty} a_{m+1}\ (m + 1) x^m \qquad x y_1' = \sum_{m=1}^{\infty} a_m\ (m) x^m \tag{5.3.207}$$

$$a_1 - a_0 = 0 \qquad a_1 = a_0 = 1 \tag{5.3.208}$$

$$(2m + 1)\ a_m = (m^2 + 2m + 1)\ a_{m+1} + a_{m-1} \tag{5.3.209}$$

$$a_{m+1} = \frac{(2m + 1)\ a_m - a_{m-1}}{(m + 1)^2} \tag{5.3.210}$$

$$y_1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \tag{5.3.211}$$

$$= \exp(x) \tag{5.3.212}$$

Finding second root by reduction of order,

$$y_2 = g y_1 \qquad y_2' = g' y_1 + g y_1' \tag{5.3.213}$$

$$y_2'' = g'' y_1 + 2 g' y_1' + g y_1'' \tag{5.3.214}$$

$$0 = x(g'' y_1 + 2g' y_1' + g y_1'') \tag{5.3.215}$$

$$+ (1-2x)(g' y_1 + g y_1') + (x-1) g y_1 \tag{5.3.216}$$

$$g''[x y_1] = -g'[2x y_1' + (1-2x) y_1] \qquad g'' x = -g' \tag{5.3.217}$$

$$\ln(g') = -\ln(x) \qquad g = \ln(x) \tag{5.3.218}$$

$$y_2 = \exp(x)\ \ \ln(x) \tag{5.3.219}$$

14. Hypergeometric ODE,

(a) Indicial equation,

$$x(1-x)\ y'' + [c - (a+b+1)x]\ y' - ab\ y = 0 \tag{5.3.220}$$

$$(1-x)\ x^2 y'' + [c - (a+b+1)x]\ xy' - (abx)\ y = 0 \tag{5.3.221}$$

$$r(r-1) + cr + 0 = 0 \tag{5.3.222}$$

$$r_1 = 0 \qquad r_2 = 1 - c \tag{5.3.223}$$

Applying Frobenius method with $r_1 = 0$,

$$y = \sum_{m=0}^{\infty} j_m\ x^m \tag{5.3.224}$$

$$y' = \sum_{m=0}^{\infty} j_{m+1}\ (m+1) x^m \qquad xy' = \sum_{m=1}^{\infty} j_m\ m x^m \tag{5.3.225}$$

$$y'' = \sum_{m=2}^{\infty} j_m\ m(m-1) x^{m-2} \qquad x^2 y'' = \sum_{m=2}^{\infty} j_m\ m(m-1) x^m \tag{5.3.226}$$

$$xy'' = \sum_{m=1}^{\infty} j_{m+1}\ m(m+1) x^m \tag{5.3.227}$$

$$c\ j_1 = ab\ j_0 \qquad j_1 = \frac{ab}{c}\ j_0 = \frac{ab}{c} \tag{5.3.228}$$

Finding recursive relation for higher coefficients, using $j_0 = 1$

$$j_{m+1} = \frac{(m+a)(m+b)}{(m+c)(m+1)}\, j_m \tag{5.3.229}$$

$$y = 1 + \frac{ab}{1!\ c}\, x + \frac{a(a+1)\ b(b+1)}{2!\ c(c+1)}\, x^2 \tag{5.3.230}$$

$$+ \frac{a(a+1)(a+2)\ b(b+1)(b+2)}{3!\ c(c+1)(c+2)}\, x^3 + \dots \tag{5.3.231}$$

To arrive at the geometric series sum,

$$F(1, 1, 1\ ;\ x) = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \tag{5.3.232}$$

This is also true for $a = 1$, $b = c$ or $b = 1$, $a = c$ since those two expresisons will cancel.

(b) For the infinite series to reduce to a polynomial, it must truncate. This requires b or a to be non-positive integers. Performing the ratio test,

$$\frac{t_{m+1}}{t_m} = \frac{(a+m-1)(b+m-1)}{m\ (c+m-1)}\, x \tag{5.3.233}$$

$$\lim_{m\to\infty} \frac{t_{m+1}}{t_m} = x \tag{5.3.234}$$

For the ratio test to guarantee convergence, $|x| < 1$.

(c) Showing elementray functions to be special cases of the solution to the hypergeometric ODE,

$$F(-n, b, b\ ;\ -x) = 1 + nx + \frac{n(n-1)}{2!}\, x^2 + \frac{n(n-1)(n-2)}{3!}\, x^3 + \dots \tag{5.3.235}$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\ r!}\, x^r = (1+x)^n \tag{5.3.236}$$

Starting with $F(1-n, 1, 2\ ;\ x)$,

$$F(1-n, 1, 2\ ;\ x) = 1 - \frac{(n-1)\ 1!}{1!\ 2!}\, x + \frac{(n-1)(n-2)\ 2!}{2!\ 3!}\, x^2 + \dots \tag{5.3.237}$$

$$+ \frac{(n-1)(n-2)\dots(1)\ (n-1)!}{(n-1)!\ n!}\, (-1)^n x^{n-1} \tag{5.3.238}$$

$$= \frac{-1}{nx}\left[-nx + \frac{n(n-1)}{2!}\, x^2 - \frac{n(n-1)(n-2)}{3!}\, x^3 + \dots\right] \tag{5.3.239}$$

$$= \frac{-1}{nx}[(1-x)^n - 1] \tag{5.3.240}$$

$$(1-x)^n = 1 - nx\ F(1-n, 1, 2\ ;\ x) \tag{5.3.241}$$

Starting with $a = 1/2$, $b = 1$, $c = 3/2$,

$$F(1/2, 1, 3/2\ ; -x^2) = 1 - \frac{1}{1!\ 3}\ x^2 + \frac{(3/4)\ 2!}{2!\ (15/4)}\ x^4 - \frac{(15/8)\ 3!}{3!\ (105/8)}\ x^6 + \ldots \quad \text{5.3.242}$$

$$= \frac{1}{x}\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right] \quad \text{5.3.243}$$

$$x\ F(1/2, 1, 3/2\ ; -x^2) = \arctan(x) \quad \text{5.3.244}$$

Starting with $a = 1/2$, $b = 1/2$, $c = 3/2$,

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\ ; x^2\right) = 1 + \frac{(1/4)}{1!\ (6/4)}\ x^2 + \frac{(9/16)}{2!\ (15/4)}\ x^4 + \frac{(15/8)(15/8)}{3!\ (105/8)}\ x^6 + \ldots \quad \text{5.3.245}$$

$$= \frac{1}{x}\left[x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{6!\ x^7}{4^3\ 3!\ 3!\ 7} + \ldots\right] \quad \text{5.3.246}$$

$$x\ F(1/2, 1, 3/2\ ; -x^2) = \arcsin(x) \quad \text{5.3.247}$$

Starting with $a = 1 = b$, $c = 2$,

$$F(1, 1, 2\ ; -x) = 1 - \frac{1!\ 1!}{1!\ 2!}\ x + \frac{2!\ 2!}{2!\ 3!}\ x^2 - \frac{3!\ 3!}{3!\ 4!}\ x^3 + \ldots \quad \text{5.3.248}$$

$$= \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right] = \frac{\ln(x)}{x} \quad \text{5.3.249}$$

$$x\ F(1, 1, 2\ ; -x) = \frac{\ln(x)}{x} \quad \text{5.3.250}$$

Starting with $a = 1/2$, $b = 1$, $c = 3/2$,

$$F(1/2, 1, 3/2\ ; x^2) = 1 + \frac{1}{1!\ 3}\ x^2 + \frac{(3/4)\ 2!}{2!\ (15/4)}\ x^4 + \frac{(15/8)\ 3!}{3!\ (105/8)}\ x^6 + \ldots \quad \text{5.3.251}$$

$$= \frac{1}{x}\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right] \quad \text{5.3.252}$$

$$= \frac{1}{2x}[\ln(1 + x) - \ln(1 - x)] \quad \text{5.3.253}$$

$$2x\ F(1/2, 1, 3/2\ ; -x^2) = \frac{\ln(1 + x)}{\ln(1 - x)} \quad \text{5.3.254}$$

More relations TBC.

(d) Frobenius method with $r_2 = 1 - c$,

$$y = \sum_{m=0}^{\infty} j_m \; x^{m-c+1} \tag{5.3.255}$$

$$y' = \sum_{m=-1}^{\infty} j_{m+1} \; (m-c+2)x^{m-c+1} \tag{5.3.256}$$

$$xy' = \sum_{m=0}^{\infty} j_m \; (m-c+1)x^{m-c+1} \tag{5.3.257}$$

$$y'' = \sum_{m=-2}^{\infty} j_{m+2} \; (m-c+3)(m-c+2)x^{m-c+1} \tag{5.3.258}$$

$$x^2y'' = \sum_{m=0}^{\infty} j_m \; (m-c+1)(m-c)x^{m-c+1} \tag{5.3.259}$$

$$xy'' = \sum_{m=-1}^{\infty} j_{m+1} \; (m-c+2)(m-c+1)x^{m-c+1} \tag{5.3.260}$$

$$0 = j_{m+1} \; (m-c+2)(m-c+1) - j_m \; (m-c+1)(m-c) \tag{5.3.261}$$

$$+ j_{m+1} \; c(m-c+2) - j_m \; (a+b+1)(m-c+1) - j_m \; (ab) \tag{5.3.262}$$

$$j_1 = j_0 \; \frac{(a-c+1)(b-c+1)}{1 \cdot (2-c)} \tag{5.3.263}$$

$$j_2 = j_1 \; \frac{(a-c+2)(b-c+2)}{2 \cdot (3-c)} \tag{5.3.264}$$

Finding coefficients of x^{1-c} and x^{2-c},

$$y_2 = x^{1-c}\Bigg[1 + \frac{(a-c+1)(b-c+1)}{1! \; (2-c)} \; x \tag{5.3.265}$$

$$+ \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2! \; (2-c)(3-c)} \; x^2 + \ldots \Bigg] \tag{5.3.266}$$

$$= x^{(1-c)} \; F(a-c+1, b-c+1, 2-c \; ; x) \tag{5.3.267}$$

The correspondence to the hypergeometric fucntion is readily seen from the form of the solution $y_2(x)$.

(e) General hypergeometric function,

$$x = \frac{t - t_1}{t_2 - t_1} \qquad x(1-x) = \frac{(t_2 - t)(t - t_1)}{(t_2 - t_1)^2} \tag{5.3.268}$$

$$(t_2 - t_1) = A^2 - 4B \qquad t_1 t_2 = B \tag{5.3.269}$$

$$x(1-x) = \frac{t^2 + At + B}{t_2 - t_1} \tag{5.3.270}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{1}{(t_2 - t_1)} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\,\frac{1}{(t_2 - t_1)^2} \tag{5.3.271}$$

$$(Ct + D) = C(t_2 - t_1)\,x + [Ct_1 + D] \tag{5.3.272}$$

Consolidating all terms,

$$(t^2 + At + B)\,\ddot{y} = \frac{x(x-1)}{(t_2 - t_1)}\,y'' \tag{5.3.273}$$

$$(Ct + D)\,\dot{y} = [(a + b + 1)x - c]y' \tag{5.3.274}$$

$$K = ab \tag{5.3.275}$$

$$Ct_1 + D = -c(t_2 - t_1) \tag{5.3.276}$$

$$C = a + b + 1 \tag{5.3.277}$$

This reduces the general hypergeometric equation to the special form involving a, b, c.

15. Solving using hypergeometric ODE,

$$0 = x(1-x)y'' - (0.5 + 3x)y' - y \tag{5.3.278}$$

$$c = -0.5 \qquad (a + b) = 2 \qquad ab = 1 \tag{5.3.279}$$

$$c = \frac{-1}{2} \qquad a = 1 \qquad b = 1 \tag{5.3.280}$$

$$y_1 = F\left(1, 1, \frac{-1}{2}\;; x\right) \qquad y_2 = x^{3/2} F\left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}\;; x\right) \tag{5.3.281}$$

16. Solving using hypergeometric ODE,

$$0 = x(1-x)y'' + (0.5+2x)y' - 2y \tag{5.3.282}$$

$$c = 0.5 \qquad (a+b) = -3 \qquad ab = 2 \tag{5.3.283}$$

$$c = \frac{1}{2} \qquad a = -2 \qquad b = -1 \tag{5.3.284}$$

$$y_1 = F\left(-2, -1, \frac{1}{2}\ ;x\right) \qquad y_2 = x^{-1/2}F\left(\frac{-3}{2}, \frac{-1}{2}, \frac{3}{2}\ ;x\right) \tag{5.3.285}$$

17. Solving using hypergeometric ODE,

$$0 = x(1-x)y'' + (0.25)y' + 2y \tag{5.3.286}$$

$$c = 0.25 \qquad (a+b) = -1 \qquad ab = -2 \tag{5.3.287}$$

$$c = \frac{1}{4} \qquad a = -2 \qquad b = 1 \tag{5.3.288}$$

$$y_1 = F\left(-2, 1, \frac{1}{4}\ ;x\right) \qquad y_2 = x^{3/4}F\left(\frac{-5}{4}, \frac{7}{4}, \frac{7}{4}\ ;x\right) \tag{5.3.289}$$

$$y_1 = 1 - 8x + \frac{32x^2}{5} \tag{5.3.290}$$

18. Solving using general hypergeometric form,

$$(t^2 - 3t + 2)\ddot{y} - (0.5)\dot{y} + (0.25)y = 0 \tag{5.3.291}$$

$$A = -2 \qquad B = 2 \qquad C = 0 \qquad D = -0.5 \qquad K = 0.25 \tag{5.3.292}$$

$$t_1 = 1 \qquad t_2 = 2 \qquad a+b = -1 \qquad c = 0.5 \qquad ab = 0.25 \tag{5.3.293}$$

$$a = \frac{-1}{2} \qquad b = \frac{-1}{2} \qquad c = \frac{1}{4} \qquad x = t-1 \tag{5.3.294}$$

$$y_1 = F\left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{4}\ ;(t-1)\right) \tag{5.3.295}$$

$$y_1 = (t-1)^{3/4}F\left(\frac{1}{4}, \frac{1}{4}, \frac{7}{4}\ ;(t-1)\right) \tag{5.3.296}$$

19. Solving using general hypergeometric form,

$$(t^2 - 5t + 6)\ddot{y} + (t - 1.5)\dot{y} - (4)y = 0 \tag{5.3.297}$$

$$A = -5 \qquad B = 6 \qquad C = 1 \qquad D = -1.5 \qquad K = -4 \tag{5.3.298}$$

$$t_1 = 2 \qquad t_2 = 3 \qquad a + b = 0 \qquad c = -0.5 \qquad ab = -4 \tag{5.3.299}$$

$$a = 2 \qquad b = -2 \qquad c = \frac{-1}{2} \qquad x = t - 2 \tag{5.3.300}$$

$$y_1 = F\left(2, -2, \frac{-1}{2}\ ; t - 2\right) \tag{5.3.301}$$

$$y_1 = (t - 2)^{1/2} F\left(\frac{7}{2}, \frac{-1}{2}, \frac{5}{2}\ ; t - 2\right) \tag{5.3.302}$$

20. Solving using general hypergeometric form,

$$(t^2 + t)\ddot{y} + (t/3)\dot{y} - (1/3)y = 0 \tag{5.3.303}$$

$$A = 1 \qquad B = 0 \qquad C = 1/3 \qquad D = 0 \qquad K = -1/3 \tag{5.3.304}$$

$$t_1 = -1 \qquad t_2 = 0 \qquad a + b = -2/3 \qquad c = 1/3 \qquad ab = -1/3 \tag{5.3.305}$$

$$a = -1 \qquad b = \frac{1}{3} \qquad c = \frac{1}{3} \qquad x = t + 1 \tag{5.3.306}$$

$$y_1 = F\left(-1, \frac{1}{3}, \frac{1}{3}\ ; t + 1\right) \tag{5.3.307}$$

$$y_1 = (t + 1)^{2/3} F\left(\frac{-1}{3}, 1, \frac{5}{3}\ ; t + 1\right) \tag{5.3.308}$$

5.4 Bessel's Equation, Bessel Functions Jv(x)

1. For Bessel functions with integer parameter,

$$J_n = x^n \sum_{m=0}^{\infty} \frac{(-1)^m\ x^{2m}}{2^{2m+n}\ m!\ (n+m)!} \tag{5.4.1}$$

$$\text{Ratio test} \quad \frac{a_{m+1}}{a_m} = \frac{-x^2}{2^2(m+1)(n+m+1)} \tag{5.4.2}$$

$$\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = 0 \tag{5.4.3}$$

This guarantees convergence for all x. In case $n < 0$, the ratio test can only be applied on terms with $m + n > 0$.

2. Reducing to Bessel ODE form,

$$0 = x^2 y'' + xy' + \left(x^2 - \frac{2^2}{7^2}\right) y \qquad \nu = \frac{2}{7} \tag{5.4.4}$$

$$y_1 = J_{2/7} \qquad y_2 = J_{-2/7} \tag{5.4.5}$$

3. Reducing to Bessel ODE form,

$$0 = xy'' + y' + \frac{1}{4}\, y \qquad \sqrt{x} = z \tag{5.4.6}$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} = \frac{1}{2z} \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{2z}\left[\frac{\ddot{y}}{2z} - \frac{\dot{y}}{2z^2}\right] \tag{5.4.7}$$

$$0 = \ddot{y}\,\frac{z^2}{4z^2} + \dot{y}\,\left[\frac{1}{2z} - \frac{z^2}{4z^3}\right] + y\,\frac{1}{4} \qquad 0 = \ddot{y} + \frac{\dot{y}}{z} + y \tag{5.4.8}$$

$$0 = z^2\ddot{y} + z\dot{y} + (z^2 - 0^2)y \tag{5.4.9}$$

$$y_1 = J_0(\sqrt{x}) \tag{5.4.10}$$

Second L.I. solution requires $Y_\nu(\sqrt{x})$.

4. Reducing to Bessel form ODE,

$$0 = y'' + \left(e^{-2x} - \frac{1}{9}\right) y \qquad z = e^{-x} \tag{5.4.11}$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -z \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = z(\dot{y} + z\ddot{y}) \tag{5.4.12}$$

$$0 = z^2\ddot{y} + z\dot{y} + \left(z^2 - \frac{1}{3^2}\right) y \qquad \nu = 1/3 \tag{5.4.13}$$

$$y_1 = J_{1/3}(e^{-x}) \qquad y_2 = J_{-1/3}(e^{-x}) \tag{5.4.14}$$

5. Reducing to Bessel form ODE,

$$0 = x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2)y \qquad z = \lambda x \tag{5.4.15}$$

$$y' = \lambda\dot{y} \qquad y'' = \lambda^2\ddot{y} \tag{5.4.16}$$

$$0 = z^2\ddot{y} + z\dot{y} + (z^2 - \nu^2)y \tag{5.4.17}$$

$$y_1 = J_\nu(\lambda x) \qquad y_2 = J_{-\nu}(\lambda x) \tag{5.4.18}$$

provided $\nu \notin \mathcal{I}$.

6. Transforming both dependent and independent variable,

$$0 = x^2y'' + \left(x + \frac{3}{4}\right)\frac{y}{4} \tag{5.4.19}$$

$$y = u\sqrt{x} \qquad \sqrt{x} = z \tag{5.4.20}$$

$$y' = \frac{1}{2z}\left(z\dot{u} + u\right) \qquad y'' = \frac{1}{2z}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\dot{u}}{2} + \frac{u}{2z}\right) \tag{5.4.21}$$

$$y'' = \frac{1}{2z}\left(\frac{\ddot{u}}{2} + \frac{\dot{u}}{2z} - \frac{u}{2z^2}\right) \tag{5.4.22}$$

$$0 = \frac{z^3\ddot{u}}{4} + \frac{z^2\dot{u}}{4} - \frac{zu}{4} + \frac{uz^3}{4} + \frac{3uz}{16} \qquad 0 = z^2\ddot{u} + z\dot{u} + u\left(z^2 - \frac{1}{4}\right) \tag{5.4.23}$$

$$y_1 = \sqrt{x}\ J_{1/2}(\sqrt{x}) \qquad y_2 = \sqrt{x}\ J_{-1/2}(\sqrt{x}) \tag{5.4.24}$$

7. Transforming both dependent and independent variable,

$$0 = x^2y'' + xy' + (x^2 - 1)\ \frac{y}{4} \qquad x = 2z \tag{5.4.25}$$

$$y' = \frac{\dot{y}}{2} \qquad y'' = \frac{\ddot{y}}{4} \tag{5.4.26}$$

$$0 = \ddot{y} + \dot{y} + \left(z^2 - \frac{1}{4}\right)y \tag{5.4.27}$$

$$y_1 = J_{1/2}\left(\frac{x}{2}\right) = \frac{\sin(x/2)}{\sqrt{x}} \qquad y_2 = J_{-1/2}\left(\frac{x}{2}\right) = \frac{\cos(x/2)}{\sqrt{x}} \tag{5.4.28}$$

8. Transforming both dependent and independent variable,

$$0 = (2x + 1)^2y'' + 2(2x + 1)y' + 16x(x + 1)y \qquad 2x + 1 = z \tag{5.4.29}$$

$$y' = 2\dot{y} \qquad y'' = 4\ddot{y} \tag{5.4.30}$$

$$0 = z^2\ddot{y} + z\dot{y} + (z^2 - 1)y \tag{5.4.31}$$

$$y_1 = J_1(2x + 1) \tag{5.4.32}$$

Second L.I. solution requires $Y_1(x)$.

9. Transforming both dependent and independent variable,

$$0 = xy'' + (2\nu + 1)y' + xy \qquad y = x^{-\nu} u \tag{5.4.33}$$

$$y' = x^{-\nu} u' - \nu x^{-\nu-1} u \tag{5.4.34}$$

$$y'' = x^{-\nu} u'' - 2\nu x^{-\nu-1} u' + (\nu)(\nu+1) x^{-\nu-2} u \tag{5.4.35}$$

$$0 = u''[x^{-\nu+1}] + u'[x^{-\nu}] + u[(x^2 - \nu^2) x^{-\nu-1}] \tag{5.4.36}$$

$$0 = x^2 u'' + xu' + u(x^2 - \nu^2) \tag{5.4.37}$$

$$y_1 = x^{-\nu}\ J_\nu(x) \qquad y_1 = x^{-\nu}\ J_{-\nu}(x) \tag{5.4.38}$$

provided $\nu \notin \mathcal{I}$.

10. Transforming both dependent and independent variable,

$$0 = x^2 y'' + (1 - 2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y \tag{5.4.39}$$

$$y = x^\nu u \qquad x^\nu = z \tag{5.4.40}$$

$$y' = \nu x^{\nu-1}[z\dot{u} + u] \qquad y' = \nu z^{1-1/\nu}[z\dot{u} + u] \tag{5.4.41}$$

$$y'' = \nu^2 z^{1-2/\nu} \left[z^2 \ddot{u} + z\dot{u}(3 - 1/\nu) + u(1 - 1/\nu)\right] \tag{5.4.42}$$

$$0 = \ddot{u}(z^2) + \dot{u}[z] + u[z^2 - \nu^2] \tag{5.4.43}$$

$$u_1 = J_\nu(z) \qquad u_2 = J_{-\nu}(z) \tag{5.4.44}$$

$$y_1 = x^\nu\ J_\nu(x^\nu) \qquad y_2 = x^\nu\ J_{-\nu}(x^\nu) \tag{5.4.45}$$

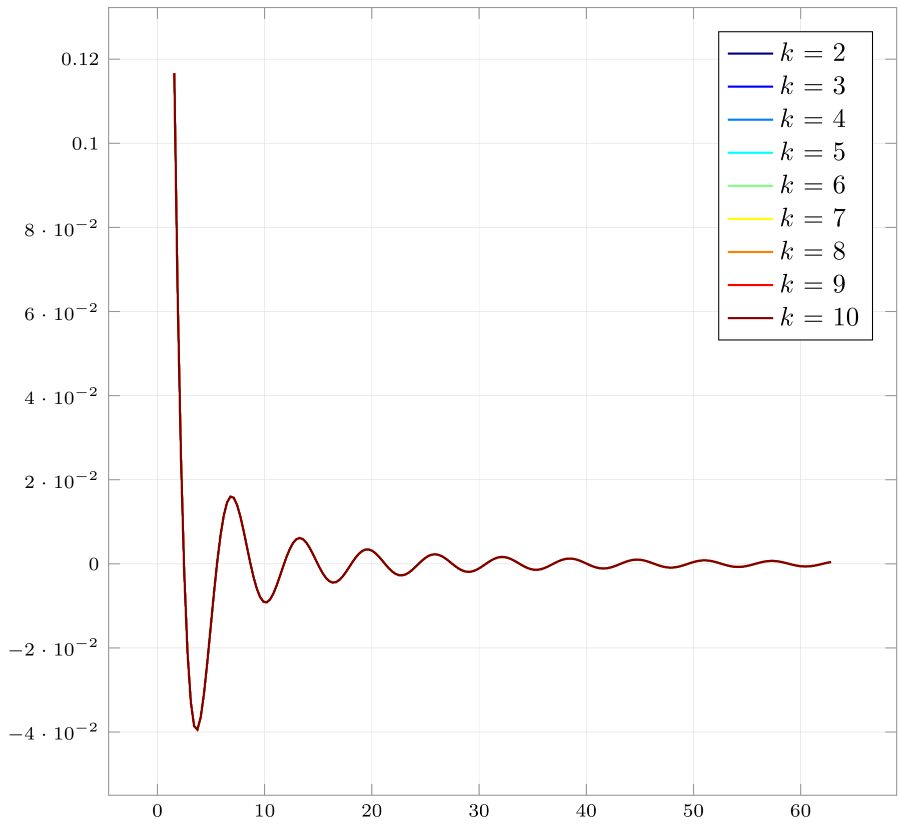
provided $\nu \notin \mathcal{I}$.

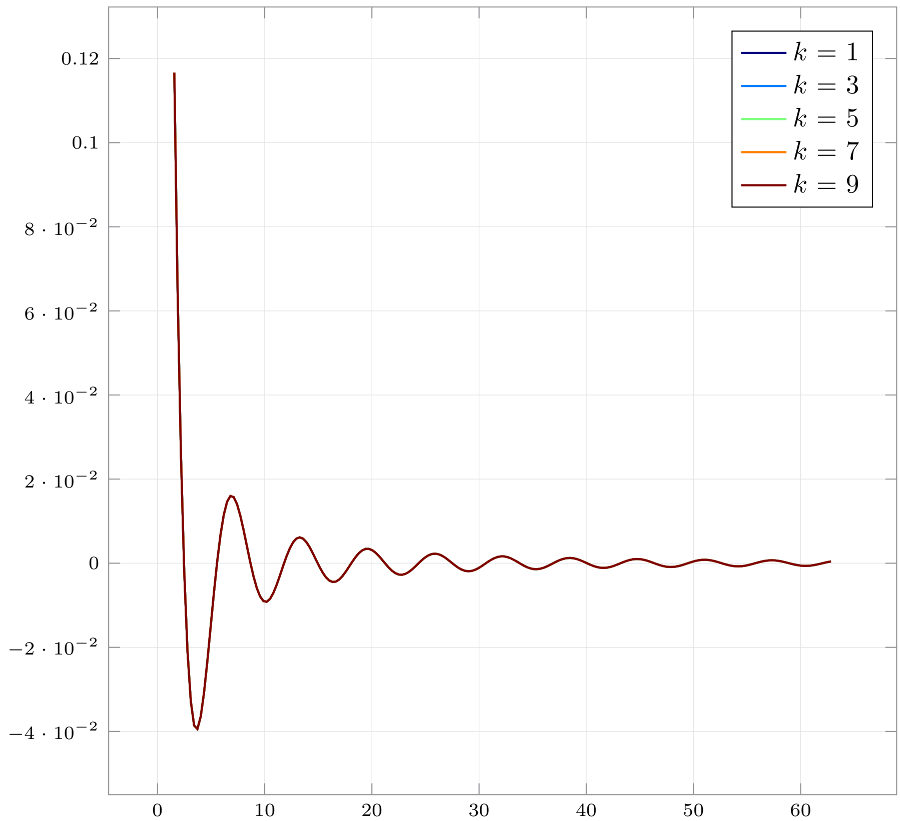
11. Graphing solutions of given ODE with varying k, where $k \in \mathcal{I}$ gives elementary functions as solutions.

$$y'' + \frac{k}{x}\, y' + y = 0 \tag{5.4.46}$$

$$y(0) = 1 \qquad y'(0) = 0 \tag{5.4.47}$$

$$y_1 = x^{(1-k)/2}\ J_{(k-1)/2}(x) \tag{5.4.48}$$


0.12
0.1
$8 \cdot 10^{-2}$
$6 \cdot 10^{-2}$
$4 \cdot 10^{-2}$
$2 \cdot 10^{-2}$
0
$-2 \cdot 10^{-2}$
$-4 \cdot 10^{-2}$
0
10
20
30
40
50
60
$k = 2$
$k = 3$
$k = 4$
$k = 5$
$k = 6$
$k = 7$
$k = 8$
$k = 9$
$k = 10$


0.12
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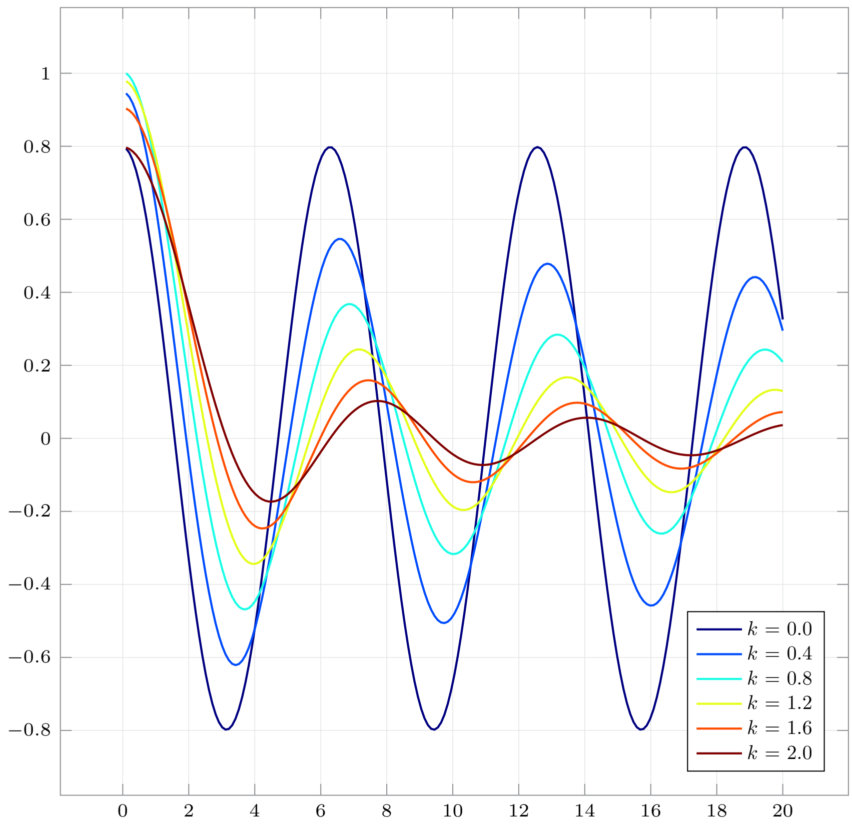
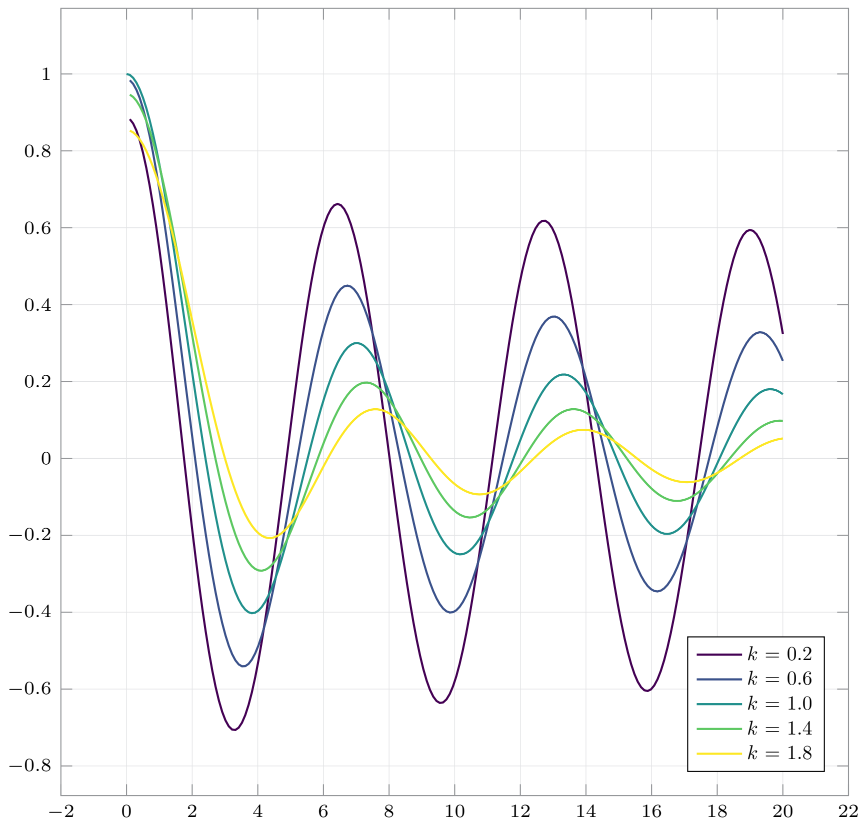
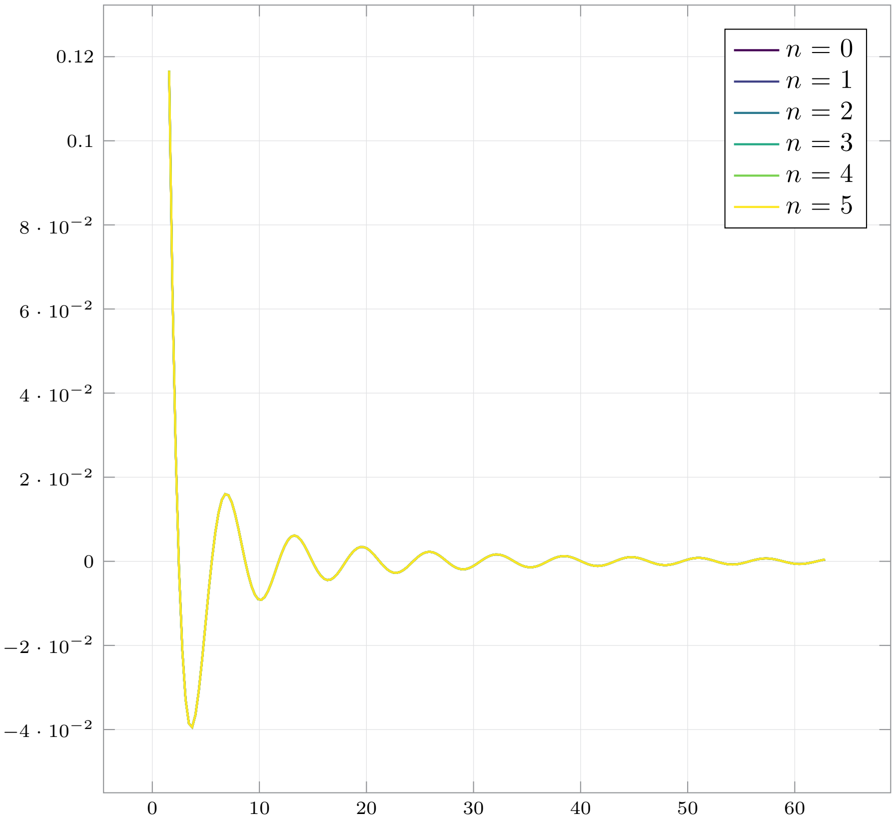
$k = 1$
$k = 3$
$k = 5$
$k = 7$
$k = 9$





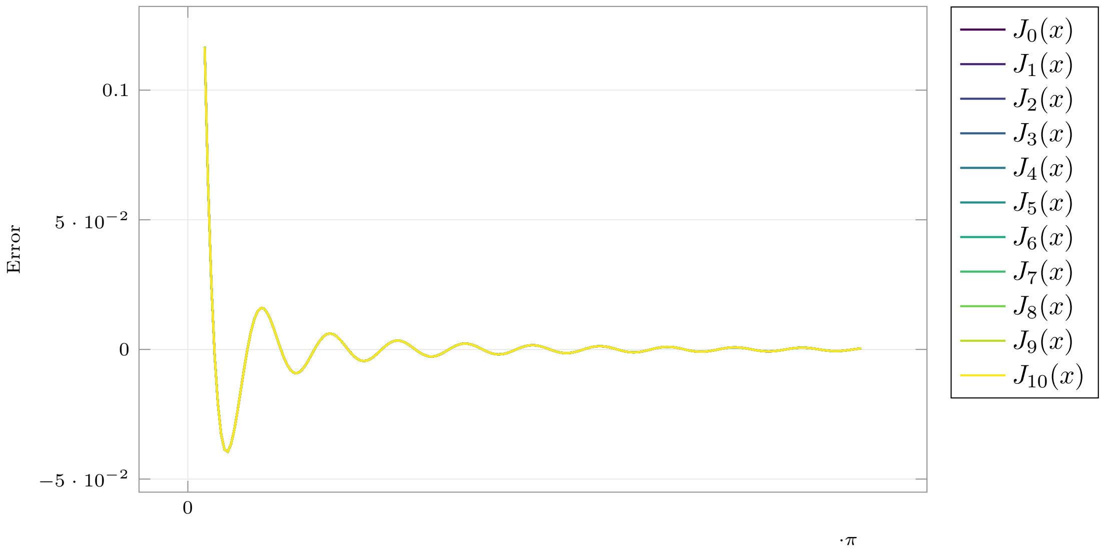
The locations of the zeros and extrema shift forward in x with increasing k. This looks like an increasingly damped sinusoidal oscillation with the envelope not being exponential.

12. **(a)** Graphing on common axes, and using the asymptotic approximation for $J_n(x)$,



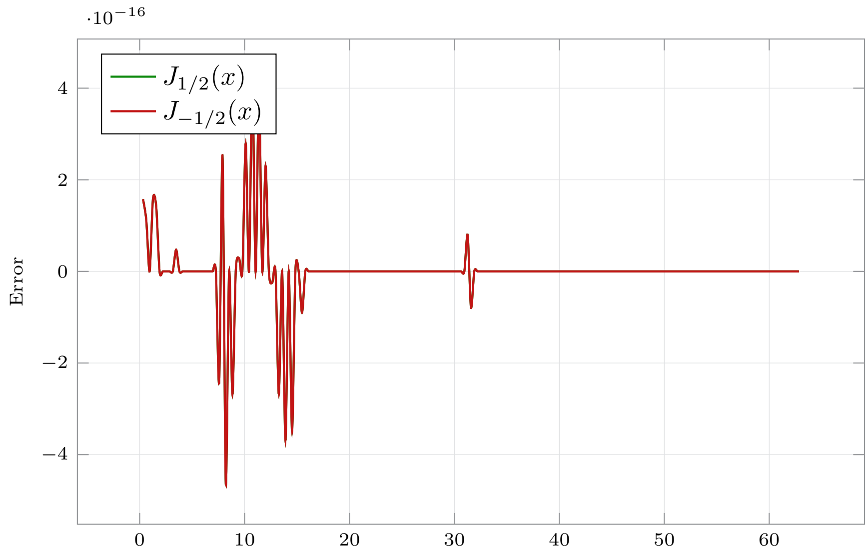
After the transients have decayed, the $J_n(x)$ practically resembles $k\cos(x)$ for even n and $k\sin(x)$ for odd n.

(b) Checking the difference function between the J_n and its approximation, the difference goes to zero for around $x_n = 200\pi$.

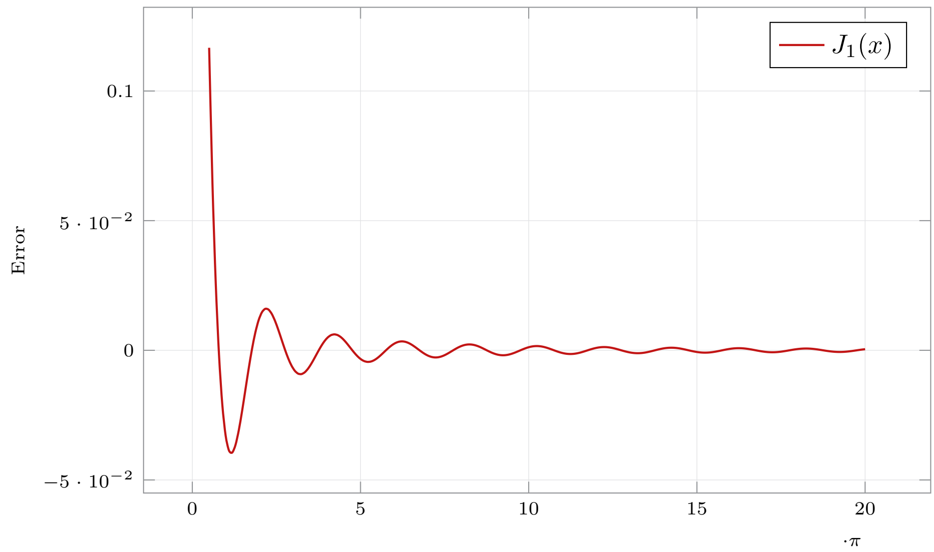


x_n decreases with increasing n, since the decay is faster.

(c) Using $\nu = \pm 1/2$, the error is analytically zero. To machine precision, this is also seen in the plot.



(d) Looking at the error in the approximation formula for fixed n,

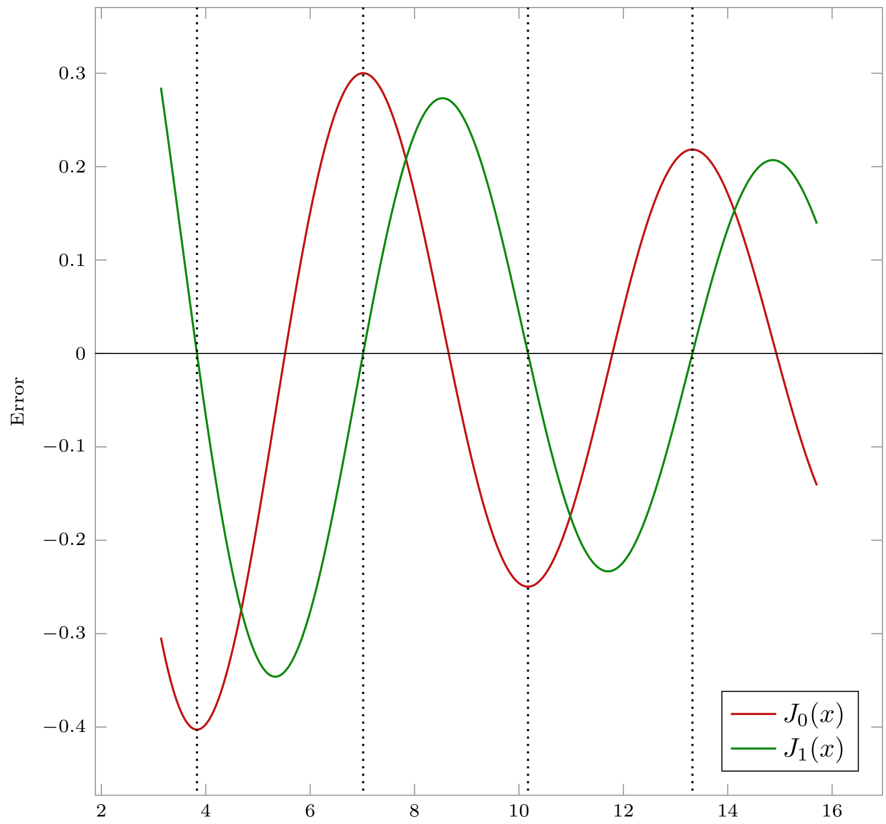


(e) Looking at the functions J_0 and J_1, the extrema of J_0 seem to occur at the zeros of J_1.

$$2J_0' = J_{-1} - J_1 = -2J_1 \tag{5.4.49}$$

$$J_{-n} = -1^n J_n \qquad \forall\ n \in \mathcal{I}^+ \tag{5.4.50}$$

From the above relation, this is proved to be true.



13. Using Rolle's theorem, and the fact that $J_n(x)$ are continuous and differentiable in $\mathcal{R}^+$,

$$J_n(a) = J_n(b) = 0 \qquad a^{-n} J_n(a) = b^{-n} J_n(b) = 0 \tag{5.4.51}$$

$$[x^{-n} J_n(x)]' = 0 \qquad \text{for some } c \in (a, b) \tag{5.4.52}$$

$$c^{-n} J_{n+1}(c) = 0 \qquad J_{n+1}(c) = 0 \tag{5.4.53}$$

If a, b are consecutive zeros, then c is guaranteed to be the only extremum point within (a, b). This makes it the only zero of J_{n+1} within (a, b).

14. Using the approximation formula, it gets better as x increases. This is seen as the reduction in error for successively higher zeros

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{5.4.54}$$

J_0

Approx.	Accurate	Error
2.3562	2.4048	0.0486
5.4978	5.5201	0.0223
8.6394	8.6537	0.0143
11.7810	11.7915	0.0106
14.9226	14.9309	0.0084
18.0642	18.0711	0.0069
21.2058	21.2116	0.0059
24.3473	24.3525	0.0051

J_1

Approx.	Accurate	Error
3.9270	3.8317	-0.0953
7.0686	7.0156	-0.0530
10.2102	10.1735	-0.0367
13.3518	13.3237	-0.0281
16.4934	16.4706	-0.0227
19.6350	19.6159	-0.0191
22.7765	22.7601	-0.0165
25.9181	25.9037	-0.0145

15. Special case of Problem 13 with $n = 0$

16. Using the definition of v,

$$y = uv \qquad v = \exp\left(-\frac{1}{2}\int p \ \mathrm{d}x\right) \tag{5.4.55}$$

$$y' = u'v + uv' \qquad y' = u''v + 2u'v' + uv'' \tag{5.4.56}$$

$$v' = \frac{-pv}{2} \qquad v'' = -\frac{pv'}{2} - \frac{p'v}{2} \tag{5.4.57}$$

$$0 = y'' + py' + qy \tag{5.4.58}$$

$$0 = u''[v] + u'[-pv + pv] + u\left[\frac{-2p'v - p^2v + 4qv}{4}\right] \tag{5.4.59}$$

$$0 = u'' + u\left[q - \frac{p^2}{4} - \frac{p'}{2}\right] \tag{5.4.60}$$

5.4.61

17. According to Problem 16, the substitution is

$$y = u\,\exp\left(-\frac{1}{2}\int p \ \mathrm{d}x\right) \qquad 0 = y'' + \frac{1}{x}\,y' + \left(1 - \frac{\nu^2}{x^2}\right)\,y \tag{5.4.62}$$

$$p(x) = \frac{1}{x} \qquad y = u\,\exp[\ln(x^{-1/2})] \tag{5.4.63}$$

$$0 = u'' + u\left[1 - \frac{\nu^2}{x^2} - \frac{1}{4x^2} + \frac{1}{2x^2}\right] \tag{5.4.64}$$

$$0 = x^2u'' + u[x^2 - \nu^2 + 1/4] \tag{5.4.65}$$

18. Let $\nu = \pm 0.5$, then,

$$0 = x^2 u'' + u\left[x^2 - 1/4 + 1/4\right] \qquad 0 = u'' + u \tag{5.4.66}$$

$$u = c_1 \cos(x) + c_2 \sin(x) \qquad y = \frac{c_1}{\sqrt{x}}\cos(x) + \frac{c_2}{\sqrt{x}}\sin(x) \tag{5.4.67}$$

Since J_ν and $J_{-\nu}$ are L.I. solutions, from the series expansions of $\sin(x)$ and $\cos(x)$, it can be deduced that,

$$J_{1/2} = \frac{c_1}{\sqrt{x}}\sin(x) \qquad J_{-1/2} = \frac{c_2}{\sqrt{x}}\cos(x) \tag{5.4.68}$$

$$c_1 = c_2 = \sqrt{\frac{2}{\pi}} \tag{5.4.69}$$

The normalization arises form the choice of a_0 in the power series definition of J_ν.

19. Using the derivative recursion relations,

$$2J_0' = J_{-1} - J_1 \qquad J_{-1} = -J_1 \tag{5.4.70}$$

$$J_0' = -J_1 \tag{5.4.71}$$

$$[x^1 J_1]' = x^1 J_0 \qquad xJ_1' + J_1 = xJ_0 \tag{5.4.72}$$

$$J_1' = J_0 - \frac{J_1}{x} \tag{5.4.73}$$

$$2J_2' = J_1 - J_3 \qquad J_2' = \frac{J_1 - J_3}{2} \tag{5.4.74}$$

20. Using the recursion relation to perform double derivative,

$$[x^\nu J_\nu]'' = [x^\nu J_{\nu-1}]' \tag{5.4.75}$$

$$= [x^{2\nu-1}(x^{1-\nu} J_{\nu-1})]' \tag{5.4.76}$$

$$\text{RHS} = x^{2\nu-1}[x^{1-\nu} J_{\nu-1}]' + (2\nu-1)x^{2\nu-2}[x^{1-\nu} J_{\nu-1}] \tag{5.4.77}$$

$$= -x^\nu J_\nu + \frac{(2\nu-1)}{x}\,[x^\nu J_\nu]' \tag{5.4.78}$$

$$= -x^\nu J_\nu + (2\nu-1)x^{\nu-1}J_\nu' + (2\nu-1)\nu x^{\nu-2}J_\nu \tag{5.4.79}$$

$$\text{LHS} = [x^\nu J_\nu' + \nu x^{\nu-1} J_\nu]' \tag{5.4.80}$$

$$= x^\nu J_\nu'' + 2\nu x^{\nu-1} J_\nu' + \nu(\nu-1)x^{\nu-2}J_\nu \tag{5.4.81}$$

$$0 = J_\nu''\,[x^\nu] + J_\nu'\,[2\nu - 2\nu + 1]x^{\nu-1} - J_\nu\,[x^2 - \nu^2]x^{\nu-2} \tag{5.4.82}$$

$$0 = x^2 J_\nu'' + xJ_\nu' + (x^2 - \nu^2)J_\nu \tag{5.4.83}$$

This proves that J_ν is a solution to the ODE,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{5.4.84}$$

21. Integrating,

$$\int x^\nu J_{\nu-1}\ \mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}[x^\nu J_\nu]\ \mathrm{d}x \tag{5.4.85}$$

$$= x^\nu J_\nu + c \tag{5.4.86}$$

22. Integrating,

$$\int x^{-\nu} J_{\nu+1}\ \mathrm{d}x = -\int \frac{\mathrm{d}}{\mathrm{d}x}[x^{-\nu} J_\nu]\ \mathrm{d}x \tag{5.4.87}$$

$$= -x^{-\nu} J_\nu + c \tag{5.4.88}$$

Using recurrence relation,

$$2J_\nu' = J_{\nu-1} - J_{\nu+1} \tag{5.4.89}$$

$$\int J_{\nu+1}\ \mathrm{d}x = \int J_{\nu-1}\ \mathrm{d}x - 2\int \frac{\mathrm{d}}{\mathrm{d}x}J_\nu\ \mathrm{d}x \tag{5.4.90}$$

$$= \int J_{\nu-1}\ \mathrm{d}x - 2J_\nu \tag{5.4.91}$$

$$\tag{5.4.92}$$

23. Using recurrence relation,

$$\int xJ_0\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}[xJ_1]\,\mathrm{d}x = xJ_1 \tag{5.4.93}$$

$$-\int J_1\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}[J_0]\,\mathrm{d}x = J_0 \tag{5.4.94}$$

$$[xJ_1]' = xJ_1' + J_1 = xJ_0 \tag{5.4.95}$$

$$\tag{5.4.96}$$

Applying these relations,

$$\int x^2 J_0 \, \mathrm{d}x = x \int x J_0 \, \mathrm{d}x - \int \left[\int x J_0 \, \mathrm{d}x \right] \, \mathrm{d}x \tag{5.4.97}$$

$$= x^2 J_1 - \int x J_1 \, \mathrm{d}x \tag{5.4.98}$$

$$= x^2 J_1 - x \int J_1 \, \mathrm{d}x + \int J_0 \, \mathrm{d}x \tag{5.4.99}$$

$$= x^2 J_1 + x J_0 - \int J_0 \, \mathrm{d}x \tag{5.4.100}$$

$$\tag{5.4.101}$$

24. Using recurrence relations,

$$\int x^{-1} J_4 \, \mathrm{d}x = \int (-x^2)(-x^{-3} J_4) \, \mathrm{d}x \tag{5.4.102}$$

$$= -x^2 (x^{-3} J_3) + \int 2x (x^{-3} J_3) \, \mathrm{d}x \tag{5.4.103}$$

$$= -x^{-1} J_3 - \int (2)(-x^{-2} J_3) \, \mathrm{d}x \tag{5.4.104}$$

$$= -\frac{J_3}{x} - \frac{2J_2}{x^2} + c \tag{5.4.105}$$

25. Using recurrence relation, such that the result does not have any powers of x, and only linear combinations of loewr order J_n,

$$\int J_5 \, \mathrm{d}x = \int J_3 \, \mathrm{d}x - 2 \int J_4' \, \mathrm{d}x \tag{5.4.106}$$

$$= -2J_4 + \int J_3 \, \mathrm{d}x \tag{5.4.107}$$

$$= -2J_4 + \int J_1 \, \mathrm{d}x - 2 \int J_2' \, \mathrm{d}x \tag{5.4.108}$$

$$= -2J_4 - 2J_2 - J_0 + c \tag{5.4.109}$$

5.5 Bessel Functions Yv(x), General Solution

1. Since $\nu \in \mathcal{I}$,

$$x^2y'' + xy' + (x^2 - 4^2)y = 0 \tag{5.5.1}$$

$$\nu = 4 \tag{5.5.2}$$

$$y_1 = J_4(x) \tag{5.5.3}$$

$$y_2 = Y_4(x) \tag{5.5.4}$$

2. Substituting $u = yx^2$,

$$0 = xy'' + 5y' + xy \qquad y = \frac{u}{x^2} \tag{5.5.5}$$

$$y' = \frac{u'}{x^2} - \frac{2u}{x^3} \qquad y'' = \frac{u''}{x^2} - \frac{4u'}{x^3} + \frac{6u}{x^4} \tag{5.5.6}$$

$$0 = u''\left[\frac{1}{x}\right] + u'\left[\frac{-4+5}{x^2}\right] + u\left[\frac{6 - 10 + x^2}{x^3}\right] \tag{5.5.7}$$

$$0 = x^2u'' + xu' + u\left[x^2 - 2^2\right] \tag{5.5.8}$$

$$u_1 = J_2(x) \qquad u_2 = Y_2(x) \tag{5.5.9}$$

$$y_1 = x^{-2}\ J_2(x) \qquad y_2 = x^{-2}\ Y_2(x) \tag{5.5.10}$$

Since $\nu \in \mathcal{I}$, the Neumann function is necessary.

3. Substituting $z = x^2$,

$$0 = 9x^2y'' + 9xy' + (36x^4 - 16)y \qquad z = x^2 \tag{5.5.11}$$

$$y' = 2x\dot{y} = 2\sqrt{z}\dot{y} \qquad y'' = 4z\ddot{y} + 2\dot{y} \tag{5.5.12}$$

$$0 = \ddot{y}\left[36z^2\right] + \dot{y}\left[18z + 18z\right] + y\left[36z^2 - 16\right] \tag{5.5.13}$$

$$0 = \ddot{y}\left[z^2\right] + \dot{y}\left[z\right] + y\left[z^2 - \frac{2^2}{3^2}\right] \tag{5.5.14}$$

$$y_1 = J_{2/3}(z) \qquad y_2 = J_{-2/3}(z) \tag{5.5.15}$$

$$y_1 = J_{2/3}(x^2) \qquad y_2 = J_{-2/3}(x^2) \tag{5.5.16}$$

4. Substituting $y = u\sqrt{x}$, $z = (2/3)x^{3/2}$,

$$0 = y'' + xy \tag{5.5.17}$$

$$y = u\sqrt{x} \qquad z = (2/3)x^{3/2} \tag{5.5.18}$$

$$y' = (1.5z)^{1/3}\left[(1.5z)^{1/3}\dot{u} + \frac{(1.5z)^{-2/3}}{2}u\right] \tag{5.5.19}$$

$$= (1.5z)^{2/3}\ \dot{u} + \frac{(1.5z)^{-1/3}}{2}\ u \tag{5.5.20}$$

$$y'' = (1.5z)^{1/3}\left[(1.5z)^{2/3}\ \ddot{u} + (1.5)(1.5z)^{-1/3}\ \dot{u} - \frac{(1.5z)^{-4/3}}{4}\ u\right] \tag{5.5.21}$$

$$= 1.5z\ \ddot{u} + 1.5\ \dot{u} - (0.25)(1.5z)^{-1}\ u \tag{5.5.22}$$

$$0 = z\ddot{u} + \dot{u} + u\left[z - \frac{1}{9z}\right] \tag{5.5.23}$$

$$= z^2\ddot{u} + z\dot{u} + u\left[z^2 - \frac{1}{3^2}\right] \tag{5.5.24}$$

$$u_1 = J_{1/3}(z) \qquad u_2 = J_{-1/3}(z) \tag{5.5.25}$$

$$y_1 = \sqrt{x}\ J_{1/3}\left(\frac{2x^{3/2}}{3}\right) \qquad y_2 = \sqrt{x}\ J_{-1/3}\left(\frac{2x^{3/2}}{3}\right) \tag{5.5.26}$$

5. Substituting $\sqrt{x} = z$,

$$0 = 4xy'' + 4y' + y \qquad z = \sqrt{x} \tag{5.5.27}$$

$$y' = \frac{\dot{y}}{2\sqrt{x}} = \frac{\dot{y}}{2z} \qquad y'' = \frac{1}{2z}\left[\frac{\ddot{y}}{2z} - \frac{\dot{y}}{2z^2}\right] \tag{5.5.28}$$

$$0 = \ddot{y} + \dot{y}\left[\frac{2}{z} - \frac{1}{z}\right] + y \qquad 0 = z^2\ddot{y} + z\dot{y} + z^2 y \tag{5.5.29}$$

$$y_1 = J_0(z) \qquad y_2 = Y_0(z) \tag{5.5.30}$$

$$y_1 = J_0(\sqrt{x}) \qquad y_2 = Y_0(\sqrt{x}) \tag{5.5.31}$$

Since $\nu \in \mathcal{I}$, the Neumann function is necessary.

6. Substituting $12\sqrt{x} = z$,

$$0 = xy'' + y' + 36y \qquad\qquad z = 12\sqrt{x} \tag{5.5.32}$$

$$y' = \frac{6\dot{y}}{\sqrt{x}} = \frac{72\dot{y}}{z} \qquad\qquad y'' = \frac{72^2}{z}\left[\frac{\ddot{y}}{z} - \frac{\dot{y}}{z^2}\right] \tag{5.5.33}$$

$$0 = 36\ddot{y} + \frac{36\dot{y}}{z} + 36y \qquad\qquad 0 = z^2\ddot{y} + z\dot{y} + (z^2 - 0)y \tag{5.5.34}$$

$$y_1 = J_0(z) \qquad\qquad y_2 = Y_0(z) \tag{5.5.35}$$

$$y_1 = J_0(12\sqrt{x}) \qquad\qquad y_2 = Y_0(12\sqrt{x}) \tag{5.5.36}$$

Since $\nu \in \mathcal{I}$, the Neumann function is necessary.

7. Substituting $y = u\sqrt{x},\;\; z = kx^2/2$,

$$0 = y'' + k^2x^2y \tag{5.5.37}$$

$$y = u\sqrt{x} \qquad\qquad z = \frac{kx^2}{2} \tag{5.5.38}$$

$$y' = \sqrt{2kz}\left[(2z/k)^{1/4}\;\dot{u} + \frac{(2z/k)^{-3/4}}{2k}\;u\right] \tag{5.5.39}$$

$$= (8kz^3)^{1/4}\;\dot{u} + (32z/k)^{-1/4}\;u \tag{5.5.40}$$

$$y'' = \sqrt{2kz}\left[(8kz^3)^{1/4}\;\ddot{u} + \frac{3(8k)^{1/4}}{4z^{1/4}}\;\dot{u} + \frac{(8k)^{1/4}}{4z^{1/4}}\;\dot{u} - \frac{k^{1/4}}{4z^{5/4}\;32^{1/4}}\;u\right] \tag{5.5.41}$$

$$= (32k^3z^5)^{1/4}\;\ddot{u} + \dot{u}[(32k^3z)^{1/4}] - 0.25u\;(k/2z)^{3/4} \tag{5.5.42}$$

$$0 = \ddot{u}\;[(32k^3z^5)^{1/4}] + \dot{u}\;[(32k^3z)^{1/4}] + u\left[(2kz)(2z/k)^{1/4} - 0.25(k/2z)^{3/4}\right] \tag{5.5.43}$$

$$= z^{5/4}\ddot{u} + z^{1/4}\dot{u} + u\left[z^{5/4} - \frac{z^{-3/4}}{16}\right] \tag{5.5.44}$$

$$= z^2\ddot{u} + z\dot{u} + u\left[z^2 - \frac{1}{16}\right] \tag{5.5.45}$$

$$u_1 = J_{1/4}(z) \qquad\qquad u_2 = J_{-1/4}(z) \tag{5.5.46}$$

$$y_1 = \sqrt{x}\;J_{1/4}\left(\frac{kx^2}{2}\right) \qquad\qquad y_2 = \sqrt{x}\;J_{-1/4}\left(\frac{kx^2}{2}\right) \tag{5.5.47}$$

8. Substituting $y = u\sqrt{x}$, $z = kx^3/3$,

$$0 = y'' + k^2x^4 y \tag{5.5.48}$$

$$y = u\sqrt{x} \qquad z = \frac{kx^3}{3} \tag{5.5.49}$$

$$y' = u'x^{1/2} + \frac{ux^{-1/2}}{2} \tag{5.5.50}$$

$$y'' = u''x^{1/2} + u'x^{-1/2} - \frac{ux^{-3/2}}{4} \tag{5.5.51}$$

$$0 = u''x^{1/2} + u'x^{-1/2} + u\left[k^2x^{9/2} - 0.25x^{-3/2}\right] \tag{5.5.52}$$

$$= u''x + u' + \frac{u}{x}[k^2x^6 - 0.25] \tag{5.5.53}$$

Replacing x with z,

$$u' = \dot{u}\; kx^2 = (9kz^2)^{1/3}\;\dot{u} \tag{5.5.54}$$

$$u'' = (9kz^2)^{1/3}\left[\ddot{u}\;(9kz^2)^{1/3} + \dot{u}\;(9k)^{1/3}\frac{2}{3z^{1/3}}\right] \tag{5.5.55}$$

$$= (9kz^2)^{2/3}\;\ddot{u} + (2/3)\dot{u}\;(9k)^{2/3}z^{1/3} \tag{5.5.56}$$

$$0 = \ddot{u}\;\left[(9z^2)\right] + \dot{u}\;[6z + 3z] + u\;[9z^2 - 1/4] \tag{5.5.57}$$

$$0 = z^2\ddot{u} + z\dot{u} + u\left[z^2 - \frac{1}{6^2}\right] \tag{5.5.58}$$

$$u_1 = J_{1/6}(z) \qquad u_2 = J_{-1/6}(z) \tag{5.5.59}$$

$$y_1 = \sqrt{x}\;J_{1/6}\left(\frac{kx^3}{3}\right) \qquad y_2 = \sqrt{x}\;J_{-1/6}\left(\frac{kx^3}{3}\right) \tag{5.5.60}$$

9. Substituting $y = ux^3$,

$$0 = xy'' - 5y' + xy \qquad y = ux^3 \tag{5.5.61}$$

$$y' = x^3\;u' + 3x^2\;u \qquad y'' = x^3\;u'' + 6x^2\;u' + 6x\;u \tag{5.5.62}$$

$$0 = u''\;x^4 + u'\;x^3 + u\;[6x^2 - 15x^2 + x^4] \tag{5.5.63}$$

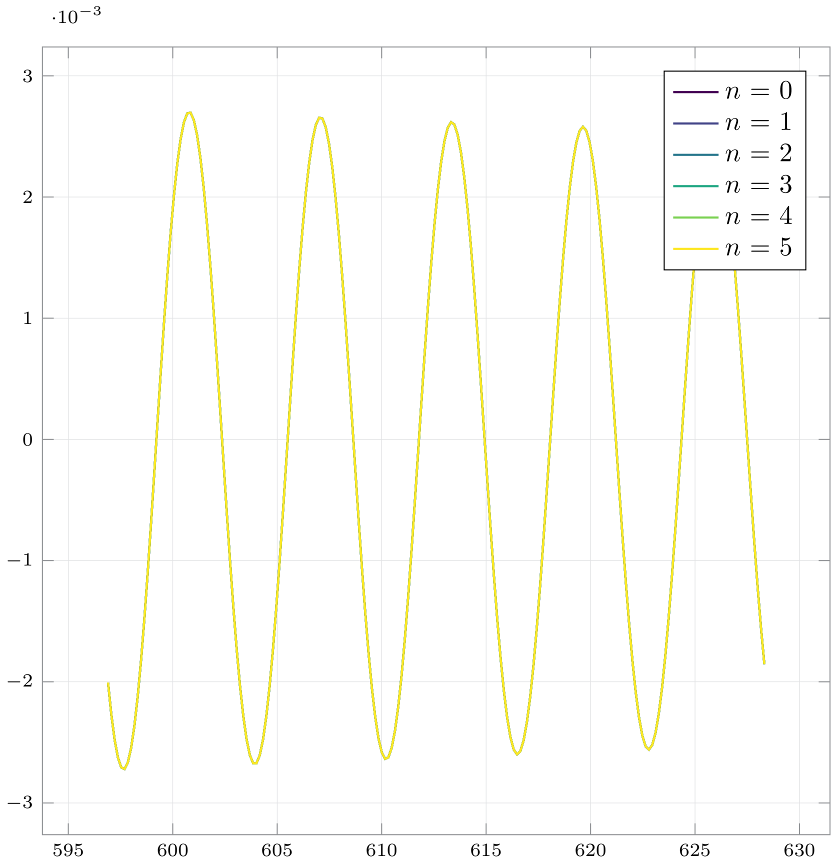
$$0 = x^2u'' + xu' + u\;[x^2 - 3^2] \tag{5.5.64}$$

$$u_1 = J_3(x) \qquad u_2 = Y_3(x) \tag{5.5.65}$$

$$y_1 = x^3\;J_3(x) \qquad y_2 = x^3\;Y_3(x) \tag{5.5.66}$$

Since $\nu \in \mathcal{I}$, the Neumann function is necessary.

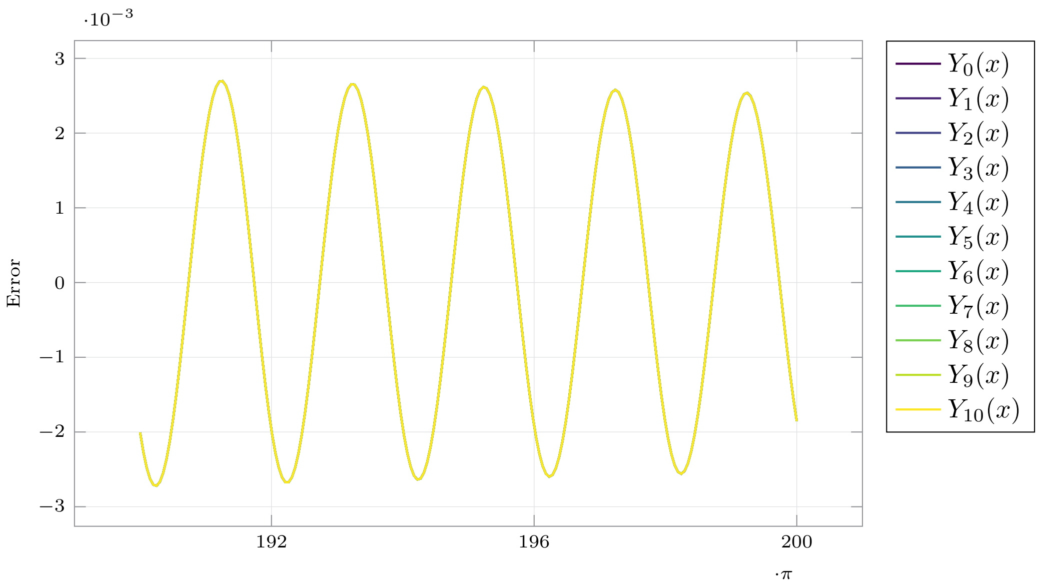
10. (a) Graphing on common axes, and using the asymptotic approximation for $J_n(x)$,

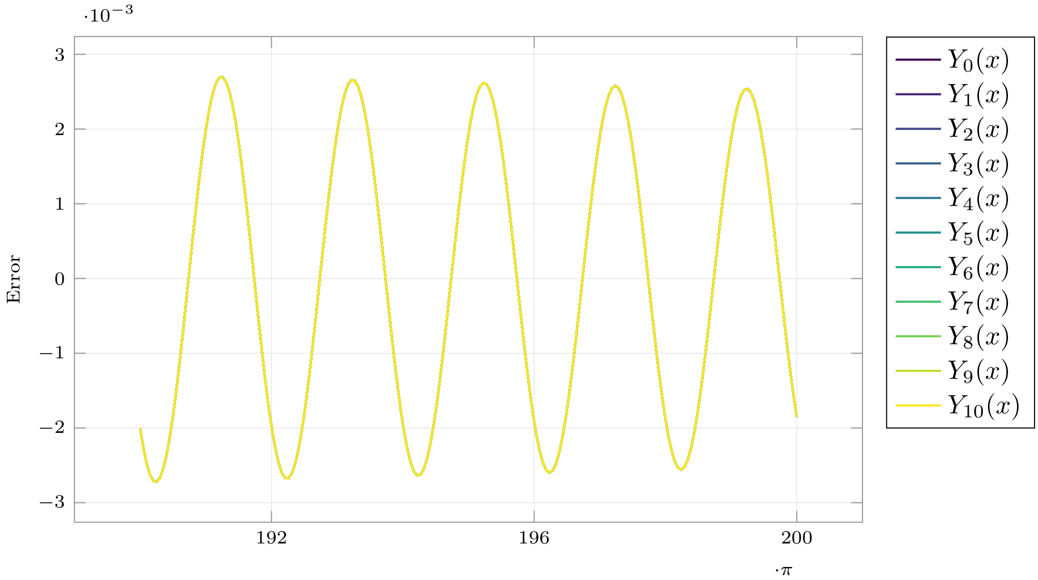


After the transients have decayed, the $Y_n(x)$ practically resembles $k\sin(x)$ for even n and $k\cos(x)$ for odd n.

Similar to the relation for J_n, extrema of Y_1 correspond to zero crossings of Y_0.

(b) Checking the difference function between the Y_n and its approximation, the difference goes to zero for around $x_n = 200\pi$. A second plot is shown at large x to see the small magnitude of error in the approximation.





x_n increases with increasing n, since the decay is slower.

(c) Using the approximation formula, it gets better as x increases. This is seen as the reduction in error for successively higher zeros

$$Y_n(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{5.5.67}$$

Y_0

Approx.	Accurate	Error
0.7854	0.8936	0.1082
3.9270	3.9577	0.0307
7.0686	7.0861	0.0175
10.2102	10.2223	0.0122
13.3518	13.3611	0.0093
16.4934	16.5009	0.0076
19.6350	19.6413	0.0064
22.7765	22.7820	0.0055
25.9181	25.9230	0.0048
29.0597	29.0640	0.0043

(d) Repeating the above procedure for Y_1 and Y_2, the approximation still works better for later zeros, but as the order increases, the approximation clearly gets worse for the same zero crossings.

Y_1

Approx.	Accurate	Error
2.3562	2.1971	-0.1591
5.4978	5.4297	-0.0681
8.6394	8.5960	-0.0434
11.7810	11.7492	-0.0318
14.9226	14.8974	-0.0251
18.0642	18.0434	-0.0208
21.2058	21.1881	-0.0177
24.3473	24.3319	-0.0154
27.4889	27.4753	-0.0136
30.6305	30.6183	-0.0122

J_1

Approx.	Accurate	Error
3.9270	3.3842	-0.5427
7.0686	6.7938	-0.2748
10.2102	10.0235	-0.1867
13.3518	13.2100	-0.1418
16.4934	16.3790	-0.1144
19.6350	19.5390	-0.0959
22.7765	22.6940	-0.0826
25.9181	25.8456	-0.0725
29.0597	28.9951	-0.0647
32.2013	32.1430	-0.0583

11. Suppose the two Hankel functions are L.D., then there exists some k (constant) for which,

$$H_\nu^{(1)} = kH_\nu^{(2)} \tag{5.5.68}$$

$$J_\nu = kJ_\nu \tag{5.5.69}$$

$$kY_\nu = -kY_\nu \tag{5.5.70}$$

No such k exists which means that the two solutions are L.I.

Since J_ν and Y_ν are themselves solutions of the Bessel ODE, Hankel functions are also solutions to the Bessel ODE by being linear superpositions of Bessel and Neumann functions.

12. Modified Bessel function,

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \tag{5.5.71}$$

$$0 = x^2 y'' + xy' - (x^2 + \nu^2)y \tag{5.5.72}$$

$$0 = x^2[-i^{-\nu} J_\nu''(ix)] + x[i^{1-\nu} J_\nu'(ix)] - i^{-\nu}(x^2 + \nu^2)J_\nu(ix) \tag{5.5.73}$$

$$z = ix \tag{5.5.74}$$

$$0 = z^2 J_\nu''(z) + zJ_\nu'(z) + (z^2 - \nu^2)J_\nu(z) \tag{5.5.75}$$

The fact that the last equality holds by the definition of J_ν, means $I_\nu(x)$ satisfies the given ODE.

13. Using the power series definition fo J_ν,

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \quad (5.5.76)$$

$$= i^{-\nu}(ix)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m\ (ix)^{2m}}{2^{2m+\nu}\ m!\ \Gamma(\nu+m+1)} \quad (5.5.77)$$

$$= x^\nu \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m+\nu}\ m!\ \Gamma(\nu+m+1)} \quad (5.5.78)$$

14. From the power series definition of I_ν in Problem 14, all terms are nonzero for $x \neq 0$, which means the function is nonzero and monotonically increasing in x, for $x \in \mathcal{R}$ and is real.

Real ν TBC.

To prove the relation, for some integer n,

$$I_{-n}(x) = i^n J_{-n}(ix) \quad (5.5.79)$$

$$= i^n (-1)^n J_n(ix) \quad (5.5.80)$$

$$= (-i)^n \frac{I_n(x)}{i^{-n}} \quad (5.5.81)$$

$$= (-i^2)^n I_n(x) = I_n(x) \quad (5.5.82)$$

15. Modified Bessel functions of the third kind,

$$K_\nu(x) = \frac{\pi}{2\ \sin(\nu\pi)}[I_{-\nu}(x) - I_\nu(x)] \quad (5.5.83)$$

$$I_{-\nu}(x) = i^\nu J_{-\nu}(ix) \quad (5.5.84)$$

5.5.85

Since I_ν already satisfies the ODE from Problem 12, checking $I_{-\nu}$,

$$0 = -x^2[i^\nu J''_{-\nu}(ix)] + ix[i^\nu J'_{-\nu}(ix)] - (x^2+\nu^2) i^\nu J_{-\nu}(ix) \quad (5.5.86)$$

$$z = ix \quad (5.5.87)$$

$$0 = z^2 J''_{-\nu}(z) + z J'_\nu(z) + (z^2 - \nu^2) J_{-\nu}(z) \quad (5.5.88)$$

5.5.89

Since $J_{-\nu}$ satisfies the original Bessel ODE, $I_{-\nu}(x)$ satisfies the given ODE. Since the given expression is a linear combination of I_ν and $I_{-\nu}$, it also solves the ODE