Chapter 21

Numerics for ODEs and PDEs

21.1 Methods for First-Order ODEs

1. Comparing the analytical solution to Euler's method

$$y' = f(x, y) = -0.2y \qquad y(0) = 5 \tag{21.1.1}$$

$$y = 5e^{-0.2x} \tag{21.1.2}$$

n	y_n	Error
1	4.8	0.003947
5	4.07686	0.01679
10	3.32416	0.02344

2. Comparing the analytical solution to Euler's method

$$y' = f(x, y) = \frac{\pi}{2}\sqrt{1-y^2} \qquad y(0) = 0 \tag{21.1.3}$$

$$\arcsin(y) = \frac{\pi x}{2} + c \qquad y = \sin(\pi x/2) \tag{21.1.4}$$

n	y_n	Error
1	0.1571	0.000064
5	0.7264	0.01926
10	1.0117	0.0117

3. Comparing the analytical solution to Euler's method

$$y' = f(x, y) = (y - x)^2 \qquad u = y - x \qquad u' = y' - 1 \tag{21.1.5}$$

$$u' + 1 = u^2 \qquad \frac{\mathrm{d}u}{u^2 - 1} = \mathrm{d}x \tag{21.1.6}$$

$$\tanh^{-1} u = -x + c \qquad y = x - \tanh(x + c) \tag{21.1.7}$$

$$y(0) = 0 \qquad y = x - \tanh(x) \tag{21.1.8}$$

n	y_n	Error
1	0	−0.000332
5	0.02859	−0.009292
10	0.21956	−0.18846

4. Comparing the analytical solution to Euler's method

$$y' = f(x, y) = (y + x)^2 \qquad u = y + x \qquad u' = y' + 1 \tag{21.1.9}$$

$$u' - 1 = u^2 \qquad \frac{\mathrm{d}u}{u^2 + 1} = \mathrm{d}x \tag{21.1.10}$$

$$\tan^{-1} u = x + c \qquad y = -x + \tan(x + c) \tag{21.1.11}$$

$$y(0) = 0 \qquad y = -x + \tan(x) \tag{21.1.12}$$

n	y_n	Error
1	0	−0.000335
5	0.03151	−0.01479
10	0.3964	−0.16101

5. Comparing the analytical solution to improved Euler's method

$$y' = f(x, y) = y \qquad y = ce^x \tag{21.1.13}$$

$$y(0) = 1 \qquad y = e^x \tag{21.1.14}$$

n	y_n	Error
1	1.105	-0.000171
5	1.6474	-0.001274
10	2.7141	-0.004201

6. Comparing the analytical solution to improved Euler's method

$$y' = f(x, y) = 2(1 + y^2) \qquad \tan^{-1} y = 2x + c \tag{21.1.15}$$

$$y(0) = 0 \qquad y = \tan(2x) \tag{21.1.16}$$

n	y_n	Error
1	0.1005	-0.0001653
5	0.54702	-0.0007218
10	1.55379	-0.003618

7. Comparing the analytical solution to improved Euler's method

$$y' = f(x, y) = xy^2 \qquad \frac{-1}{y} = \frac{x^2}{2} + c \tag{21.1.17}$$

$$y(0) = 1 \qquad y = \frac{1}{1 - x^2/2} \tag{21.1.18}$$

n	y_n	Error
1	1.005	-0.000025
5	1.142568	-0.0002889
10	1.98812	-0.01187

8. Comparing the analytical solution to improved Euler's method

$$y' = f(x, y) = y(1 - y) \qquad \frac{\mathrm{d}y}{y(1 - y)} = \mathrm{d}x \tag{21.1.19}$$

$$\ln\left[\frac{y}{y - 1}\right] = x + c \qquad y = \frac{ce^x}{ce^x - 1} \tag{21.1.20}$$

$$y(0) = 0.2 \qquad y = \frac{e^x}{e^x + 4} \tag{21.1.21}$$

n	y_n	Error
1	0.216467	-0.0000135
5	0.291802	-0.0000732
10	0.404462	-0.0001477

9. Comparing the two methods, the simple Euler method has much greater errors for the same number of iterations.

n	Error simple	Error improved
1	-0.005025	-0.000025
5	-0.03547	-0.0002889
10	-0.28715	-0.01187

10. Comparing the two interval sizes, the smaller h gives better errors when reaching the same x_n value.

x_n	$\epsilon_{h/2}$	ϵ_h
0.5	-0.00005221	-0.0002889
1.0	-0.003053	-0.01187

11. Comparing the two methods, the simple Euler method has much greater errors for the same number of iterations.

n	Error simple	Error RK
1	-0.005025	1.0311×10^{-8}
5	-0.03547	3.773×10^{-7}
10	-0.28715	-8.8024×10^{-6}

The error using $y^{(h)}$ and $y^{(2h)}$ is,

$$\epsilon_h = \frac{y^{(h)} - y^{(2h)}}{15} = \frac{204 - 8}{15} \cdot 10^{-6} = 1.3 \times 10^{-5} \tag{21.1.22}$$

12. Comparing the RK method to improved Euler's method

n	Error RK	Error Improved Euler
1	-4.232×10^{-9}	-1.35×10^{-5}
5	-2.096×10^{-8}	-7.32×10^{-5}
10	-3.721×10^{-8}	-1.477×10^{-4}

13. Comparing the RK method to the analytical solution

$$y' = f(x, y) = 1 + y^2 \qquad \frac{\mathrm{d}y}{1 + y^2} = \mathrm{d}x \tag{21.1.23}$$

$$\arctan x = x + c \qquad y = \tan(x + c) \tag{21.1.24}$$

$$y(0) = 0 \qquad y = \tan x \tag{21.1.25}$$

n	RK Classic	Error
1	0.10033	-8.301×10^{-8}
5	0.54631	-1.823×10^{-7}
10	1.5574	-1.282×10^{-6}

14. Comparing the RK method to the analytical solution

$$y' = f(x, y) = (1 - x^{-1})y \qquad \frac{\mathrm{d}y}{y} = (1 - x^{-1})\ \mathrm{d}x \tag{21.1.26}$$

$$\ln y = x - \ln x + c \qquad y = \frac{c\ e^x}{x} \tag{21.1.27}$$

$$y(1) = 1 \qquad y = \frac{e^{x-1}}{x} \tag{21.1.28}$$

n	RK Classic	Error
1	1.0047	-6.845×10^{-8}
5	1.0991	-1.904×10^{-7}
10	1.3591	-2.849×10^{-7}

15. Comparing the RK method to the analytical solution

$$y' = f(x, y) = \sin(2x) - y \tan x \qquad I = \exp\left[\int \tan x\ \ \mathrm{d}x\right] = \sec x \tag{21.1.29}$$

$$\sec x\ y = \int \sin(2x) \sec x\ \ \mathrm{d}x \qquad y = -2\cos^2 x + c\ \ \cos x \tag{21.1.30}$$

$$y(0) = 1 \qquad y = 3\cos x - 2\cos^2 x \tag{21.1.31}$$

n	RK Classic	Error
1	0.00994	-1.812×10^{-8}
5	0.21486	-5.584×10^{-7}
10	0.49674	-6.143×10^{-6}

16. Doubling the step size, approximately boosts the error by a factor of 16.

x_n	ϵ_h	ϵ_{2h}
1.00	6.14×10^{-6}	1.02×10^{-4}
2.00	9.77×10^{-1}	2.75

17. Comparing the RK method to the analytical solution

$$y' = f(x, y) = 4x^3y^2 \qquad \frac{dy}{y^2} = 4x^3 \; dx \tag{21.1.32}$$

$$\frac{-1}{y} = x^4 + c \tag{21.1.33}$$

$$y(0) = 0.5 \qquad y = \frac{1}{2 - x^4} \tag{21.1.34}$$

n	RK Classic	Error
1	0.50002	-3.125×10^{-10}
5	0.51613	-2.846×10^{-8}
10	0.99992	-8.044×10^{-5}

18. Comparing the fourth and third order RK methods,

$$y' = f(x, y) = y + x \qquad y = e^x - x + c \tag{21.1.35}$$

$$y(0) = 0 \qquad y = e^x - x - 1 \tag{21.1.36}$$

x_n	Error RK_3	Error RK_4
0.2	6.942×10^{-5}	2.758×10^{-6}
0.4	1.696×10^{-4}	6.738×10^{-6}
0.6	3.107×10^{-4}	1.234×10^{-5}
0.8	5.06×10^{-4}	2.0103×10^{-5}
1.0	7.724×10^{-4}	3.0692×10^{-5}

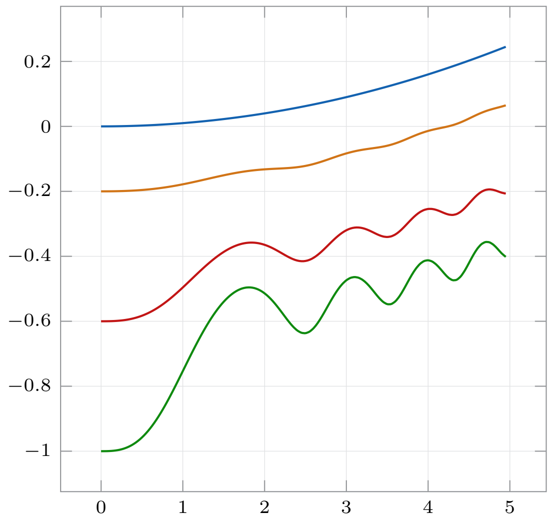
The third order RK method is worse.

19. Comparing methods,

(a) Looking at the errors in the three methods,

Position	Error		
x_n	Euler	Improved Euler	RK
1	2.002×10^{-2}	4.553×10^{-4}	1.12×10^{-6}
3	6.285×10^{-2}	1.208×10^{-2}	1.63×10^{-5}
5	5.074×10^{-2}	9.601×10^{-3}	5.363×10^{-4}

(b) Plotting the function curves for different initial conditions



(c) Comparing the three methods for a monotonically increasing function $y = e^x$, the RK method is far better than the other two methods.

Position	Error		
x_n	Euler	Improved Euler	RK
1	2.299×10^{-1}	1.557×10^{-2}	3.069×10^{-5}
3	4.678	3.432×10^{-1}	6.803×10^{-4}
5	5.302×10^{1}	4.203	8.378×10^{-3}

20. RKF 45 method

(a) Algorithm coded in `numpy`

(b) Tablulating the error in Example 3,

n	RKF_{45}	Error RKF_{45}
1	1.200 334 673	3.0396×10^{-9}
5	2.046 302 512	3.5223×10^{-8}
10	3.557 408 538	8.7516×10^{-7}

(c) TBC

21.2 Multistep Methods

1. Solving analytically,

$$y' = y \qquad y = ce^x \tag{21.2.1}$$

$$y(0) = 1 \qquad y = e^x \tag{21.2.2}$$

n	AM	Error AM
5	1.648 721	5.854×10^{-7}
10	2.718 282	7.615×10^{-7}

2. Solving analytically,

$$y' = 2xy \qquad \ln y = x^2 + c \tag{21.2.3}$$

$$y(0) = 1 \qquad y = \exp(x^2) \tag{21.2.4}$$

n	AM	Error AM
5	1.284 044	1.893×10^{-5}
10	2.718 486	2.046×10^{-4}

3. Solving analytically,

$$y' = 1 + y^2 \qquad \arctan y = x + c \tag{21.2.5}$$

$$y(0) = 0 \qquad y = \tan x \tag{21.2.6}$$

n	AM	Error AM
5	0.546 315	1.2369×10^{-5}
10	1.557 625	2.1773×10^{-4}

4. Comparing the RK and AM methods in Problem 2

x_n	Error RK	Error AM
0.2	1.07×10^{-7}	4.423×10^{-9}
0.4	1.34×10^{-6}	7.559×10^{-6}
0.6	8.00×10^{-6}	3.520×10^{-5}
0.8	3.96×10^{-5}	9.169×10^{-5}
1.0	1.75×10^{-4}	2.045×10^{-4}

5. Comparing the RK and AM methods in Problem 3

x_n	Error RK	Error AM
0.2	2.63×10^{-6}	1.57×10^{-7}
0.4	4.22×10^{-6}	4.86×10^{-6}
0.6	3.41×10^{-6}	2.42×10^{-5}
0.8	1.93×10^{-5}	7.56×10^{-5}
1.0	5.60×10^{-5}	2.18×10^{-4}

6. Solving analytically,

$$y' = (y - x - 1)^2 + 2 \qquad u = y - x - 1 \quad u' = y' - 1 \tag{21.2.7}$$

$$u' = u^2 + 1 \qquad \arctan u = x + c \tag{21.2.8}$$

$$y = 1 + x + \tan(x + c) \qquad y(0) = 1 \tag{21.2.9}$$

$$y = 1 + x + \tan x \tag{21.2.10}$$

n	AM	Error AM
5	2.0463	1.242×10^{-5}
10	3.5576	2.178×10^{-4}

7. Solving analytically,

$$y' = 3y - 12y^2 \qquad \frac{\mathrm{d}y}{y(1-4y)} = 3 \ \mathrm{d}x \tag{21.2.11}$$

$$\ln y - \ln(y - 0.25) = 3x + c \qquad \frac{y}{y - 0.25} = ce^{3x} \tag{21.2.12}$$

$$y(0) = 0.2 \qquad y = \frac{1}{4 + e^{-3x}} \tag{21.2.13}$$

n	AM	Error AM
5	0.236 787	4.43×10^{-6}
10	0.246 926	8.94×10^{-7}

8. Solving analytically,

$$y' = 1 - 4y^2 \qquad \frac{\mathrm{d}y}{1-4y^2} = \mathrm{d}x \tag{21.2.14}$$

$$\frac{\tanh^{-1}(2y)}{2} = x + c \qquad y = \frac{\tanh(2x+c)}{2} \tag{21.2.15}$$

$$y(0) = 0 \qquad y = \frac{\tanh(2x)}{2} \tag{21.2.16}$$

n	AM	Error AM
5	0.380 726	7.1×10^{-5}
10	0.481 949	6.5×10^{-5}

9. Solving analytically,

$$y' = 3x^2(1+y) \qquad \frac{\mathrm{d}y}{1+y} = 3x^2 \ \ \mathrm{d}x \tag{21.2.17}$$

$$\ln(1+y) = x^3 + c \qquad y = c\exp(x^3) - 1 \tag{21.2.18}$$

$$y(0) = 0 \qquad y = \exp(x^3) - 1 \tag{21.2.19}$$

n	AM	Error AM
5	0.015 749	9.607×10^{-7}
10	0.133 156	7.454×10^{-6}

10. Solving analytically,

$$y' = x/y \qquad y \ \ \mathrm{d}y = x \ \ \mathrm{d}x \tag{21.2.20}$$

$$y^2/2 = x^2/2 + c \tag{21.2.21}$$

$$y(1) = 3 \qquad y^2 = x^2 + 8 \tag{21.2.22}$$

n	AM	Error AM
5	3.464	1.274×10^{-6}
10	4.123	1.481×10^{-6}

11. Starting with the Newton backward difference formula,

$$p_3(x) = f_n + r\ \nabla f_n + \frac{r(r+1)}{2}\ \nabla^2 f_n + \frac{r(r+1)(r+2)}{6}\ \nabla^3 f_n \tag{21.2.23}$$

$$\nabla f_n = f_n - f_{n-1} \tag{21.2.24}$$

$$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1} = f_n - 2f_{n-1} + f_{n-2} \tag{21.2.25}$$

$$\nabla^3 f_n = \nabla^2 f_n - \nabla^2 f_{n-1} = f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3} \tag{21.2.26}$$

Deriving the iterative method,

$$\int_0^1 \frac{r(r+1)}{2}\ \mathrm{d}r = \left[\frac{r^3/3 + r^2/2}{2}\right]_0^1 = \frac{5}{12} \tag{21.2.27}$$

$$\int_0^1 \frac{r(r+1)(r+2)}{6}\ \mathrm{d}r = \left[\frac{r^4/4 + r^3 + r^2}{6}\right]_0^1 = \frac{3}{8} \tag{21.2.28}$$

$$\int_{x_n}^{x_{n+1}} p_3(x)\ \mathrm{d}x = h\int_0^1 p_3(r)\ \mathrm{d}r \tag{21.2.29}$$

$$= h\left[f_n + \frac{\nabla f_n}{2} + \frac{5\nabla^2 f_n}{12} + \frac{3\nabla^3 f_n}{8}\right] \tag{21.2.30}$$

Substituting into the expansion

$$y^*_{n+1} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right] \tag{21.2.31}$$

Starting with the Newton backward difference formula,

$$\widetilde{p_3}(x) = f_{n+1} + r\ \nabla f_{n+1} + \frac{r(r+1)}{2}\ \nabla^2 f_{n+1} + \frac{r(r+1)(r+2)}{6}\ \nabla^3 f_{n+1} \tag{21.2.32}$$

$$\nabla f_{n+1} = f_{n+1} - f_n \tag{21.2.33}$$

$$\nabla^2 f_{n+1} = \nabla f_{n+1} - \nabla f_n = f_{n+1} - 2f_n + f_{n-1} \tag{21.2.34}$$

$$\nabla^3 f_{n+1} = \nabla^2 f_{n+1} - \nabla^2 f_n = f_{n+1} - 3f_n + 3f_{n-1} - f_{n-2} \tag{21.2.35}$$

Deriving the iterative method,

$$\int_{-1}^{0} \frac{r(r+1)}{2} \; \mathrm{d}r = \left[\frac{r^3/3 + r^2/2}{2}\right]_{-1}^{0} = \frac{-1}{12} \tag{21.2.36}$$

$$\int_{-1}^{0} \frac{r(r+1)(r+2)}{6} \; \mathrm{d}r = \left[\frac{r^4/4 + r^3 + r^2}{6}\right]_{-1}^{0} = \frac{-1}{24} \tag{21.2.37}$$

$$\int_{x_n}^{x_{n+1}} \widetilde{p_3}(x) \; \mathrm{d}x = h \int_{-1}^{0} \widetilde{p_3}(r) \; \mathrm{d}r \tag{21.2.38}$$

$$= h \left[f_{n+1} - \frac{\nabla f_{n+1}}{2} - \frac{\nabla^2 f_{n+1}}{12} - \frac{\nabla^3 f_{n+1}}{24}\right] \tag{21.2.39}$$

Substituting into the expansion

$$y_{n+1} = y_n + \frac{h}{24} \left[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right] \tag{21.2.40}$$

12. Starting with the Newton backward difference formula,

$$p_2(x) = f_n + r \; \nabla f_n + \frac{r(r+1)}{2} \; \nabla^2 f_n \tag{21.2.41}$$

$$\nabla f_n = f_n - f_{n-1} \tag{21.2.42}$$

$$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1} = f_n - 2 f_{n-1} + f_{n-2} \tag{21.2.43}$$

Deriving the iterative method,

$$\int_{0}^{1} \frac{r(r+1)}{2} \; \mathrm{d}r = \left[\frac{r^3/3 + r^2/2}{2}\right]_{0}^{1} = \frac{5}{12} \tag{21.2.44}$$

$$\int_{x_n}^{x_{n+1}} p_2(x) \; \mathrm{d}x = h \int_{0}^{1} p_3(r) \; \mathrm{d}r \tag{21.2.45}$$

$$= h \left[f_n + \frac{\nabla f_n}{2} + \frac{5 \nabla^2 f_n}{12}\right] \tag{21.2.46}$$

Substituting into the expansion

$$y_{n+1}^{*} = y_n + \frac{h}{12} \left[23 f_n - 16 f_{n-1} + 5 f_{n-2}\right] \tag{21.2.47}$$

Starting with the Newton backward difference formula,

$$\widetilde{p_2}(x) = f_{n+1} + r\ \nabla f_{n+1} + \frac{r(r+1)}{2}\ \nabla^2 f_{n+1} \tag{21.2.48}$$

$$\nabla f_{n+1} = f_{n+1} - f_n \tag{21.2.49}$$

$$\nabla^2 f_{n+1} = \nabla f_{n+1} - \nabla f_n = f_{n+1} - 2f_n + f_{n-1} \tag{21.2.50}$$

Deriving the iterative method,

$$\int_{-1}^{0} \frac{r(r+1)}{2}\ \mathrm{d}r = \left[\frac{r^3/3 + r^2/2}{2}\right]_{-1}^{0} = \frac{-1}{12} \tag{21.2.51}$$

$$\int_{x_n}^{x_{n+1}} \widetilde{p_2}(x)\ \mathrm{d}x = h \int_{-1}^{0} \widetilde{p_2}(r)\ \mathrm{d}r \tag{21.2.52}$$

$$= h\left[f_{n+1} - \frac{\nabla f_{n+1}}{2} - \frac{\nabla^2 f_{n+1}}{12}\right] \tag{21.2.53}$$

Substituting into the expansion

$$y_{n+1} = y_n + \frac{h}{12}\left[5f_{n+1} + 8f_n - f_{n-1}\right] \tag{21.2.54}$$

13. Solving analytically,

$$y' = 2xy \qquad \frac{\mathrm{d}y}{y} = 2x\ \mathrm{d}x \tag{21.2.55}$$

$$\ln y = x^2 + c \qquad y(0) = 1 \tag{21.2.56}$$

$$y = \exp(x^2) \tag{21.2.57}$$

n	AM	Error AM
5	1.284	1.937×10^{-4}
10	2.7198	1.503×10^{-3}

14. Error is proportional to h^3, which means the error should go down by a factor of $1/2^3$ when halving h

n	AM	Error AM
5	3.826	3.826×10^{-5}
10	2.968	2.968×10^{-4}

15. Error is proportional to h^3, which means the error should go down by a factor of $1/2^3$ when halving h

n	AM	Error AM
5	3.826	3.826×10^{-5}
10	2.968	2.968×10^{-4}

16. Adams−Moulton method

(a) Starting with Improved Euler method instead of RK method,

$$y' = f(x, y) = x + y \qquad y(0) = 0 \tag{21.2.58}$$

$$y = e^x - x - 1 \tag{21.2.59}$$

x_n	Error AM (Euler)	Error AM (RK)
1	9.395×10^{-3}	1.2×10^{-5}
2	2.552×10^{-2}	6.0×10^{-6}

(b) Using exact starting values instead of RK values did not provide a significant decrease in error.

n	Error AM (Exact)	Error AM (RK)
5	1.218×10^{-6}	1.274×10^{-6}
10	1.434×10^{-6}	1.481×10^{-6}

(c) TBC. Check stiffness of ODE.

(d) Comparing the RK method with step size $2h$ and the Adams− Moulton method with step size h,

x_n	Error RK with $2h$	Error AM with h
1	2.09×10^{-4}	7.1×10^{-5}
2	1.29×10^{-5}	6.5×10^{-5}

21.3 Methods for Systems and Higher Order ODEs

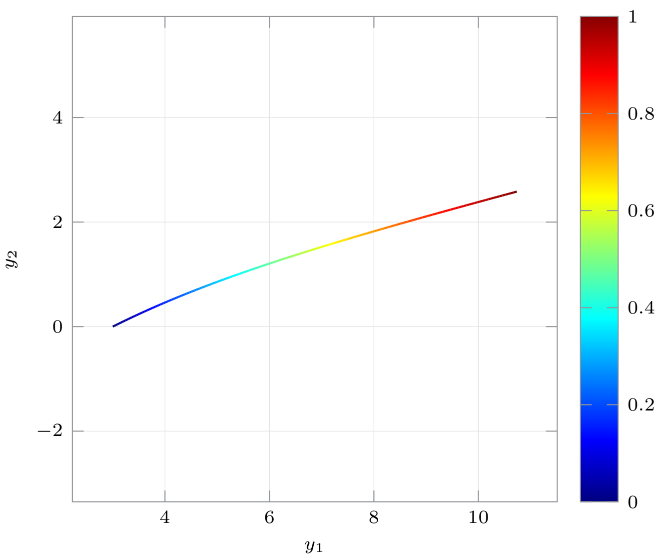
1. Solving the system of ODEs,

$$\mathbf{y}' = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \mathbf{y} \qquad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \tag{21.3.1}$$

$$\lambda_1 = 1, \quad \mathbf{u}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \lambda_1 = -2, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{21.3.2}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{x} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2x} \qquad c_1 = 1,\ c_2 = -1 \tag{21.3.3}$$

n	y_1	ϵ_1	y_2	ϵ_2
1	3.6	0.002	0.3	0.0135
5	6.11	0.112	1.283	0.002
10	10.27	0.47	2.486	0.096



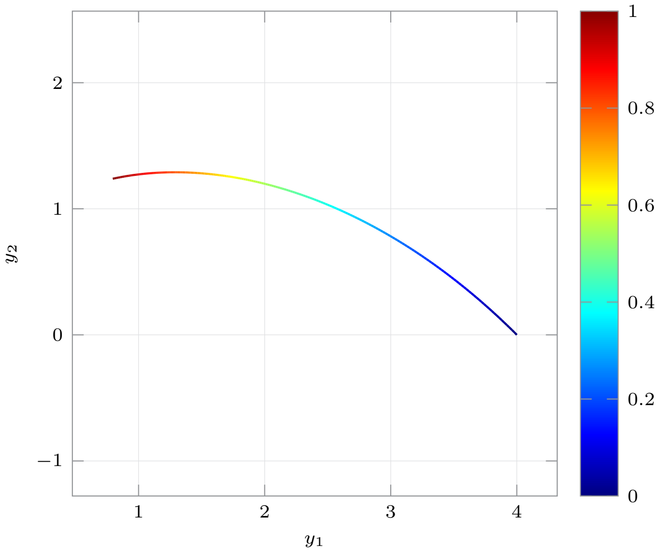
2. Solving the system of ODEs,

$$\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y} \qquad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \tag{21.3.4}$$

$$\lambda_1 = -1 - \mathrm{i}, \quad \mathbf{u}_1 = \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix} \qquad \lambda_1 = -1 + \mathrm{i}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix} \tag{21.3.5}$$

$$\mathbf{y} = \begin{bmatrix} c_1 \sin x + c_2 \cos x \\ c_1 \cos x - c_2 \sin x \end{bmatrix} e^{-x} \qquad c_1 = 4, \ c_2 = 0 \tag{21.3.6}$$

n	y_1	ϵ_1	y_2	ϵ_2
1	0.8	0.149	3.2	0.0096
2	1.28	0.235	2.4	0.0696
5	1.435	0.197	0.517	0.278



3. Solving the system of ODEs,

$$y'' + y/4 = 0 \tag{21.3.7}$$

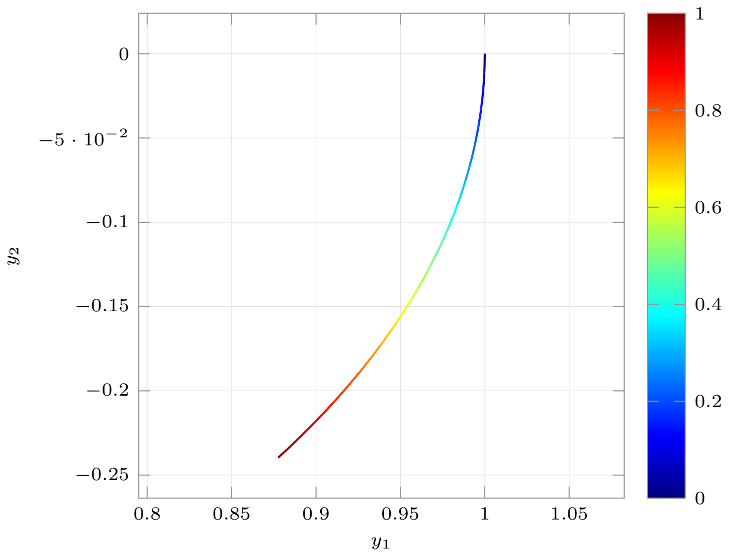
$$y_1' = y_2 \qquad y_2' = -y_1/4 \tag{21.3.8}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1/4 & 0 \end{bmatrix} \mathbf{y} \qquad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{21.3.9}$$

$$\lambda_1 = -\mathrm{i}/2, \quad \mathbf{u}_1 = \begin{bmatrix} 2\mathrm{i} \\ 1 \end{bmatrix} \qquad \lambda_1 = \mathrm{i}/2, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ \mathrm{i} \end{bmatrix} \tag{21.3.10}$$

$$\mathbf{y} = \begin{bmatrix} 2c_1 \sin(x/2) + 2c_2 \cos(x/2) \\ c_1 \cos(x/2) - c_2 \sin(x/2) \end{bmatrix} \qquad c_1 = 0, \; c_2 = 0.5 \tag{21.3.11}$$

n	y_1	ϵ_1	y_2	ϵ_2
1	1	5×10^{-3}	-0.05	8.33×10^{-5}
2	0.99	9.93×10^{-3}	-0.1	1.74×10^{-3}
5	0.9005	2.29×10^{-2}	-0.245	5.3×10^{-3}



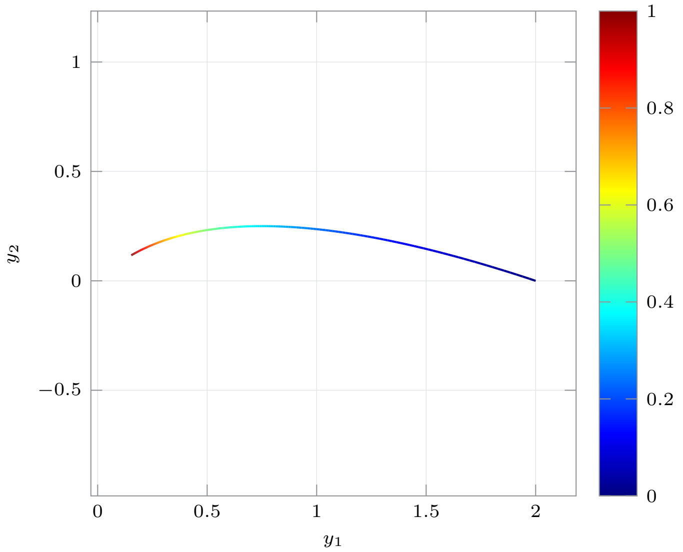
4. Solving the system of ODEs,

$$\mathbf{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} \qquad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{21.3.12}$$

$$\lambda_1 = -2, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_1 = -4, \quad \mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{21.3.13}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2x} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4x} \qquad c_1 = 1, \ c_2 = -1 \tag{21.3.14}$$

n	y_1	ϵ_1	y_2	ϵ_2
1	0.8	0.319	0.4	0.179
2	0.4	0.251	0.32	0.072
5	0.0781	0.075	0.077	0.039



5. Solving the system of ODEs,

$$y'' = x + y \tag{21.3.15}$$

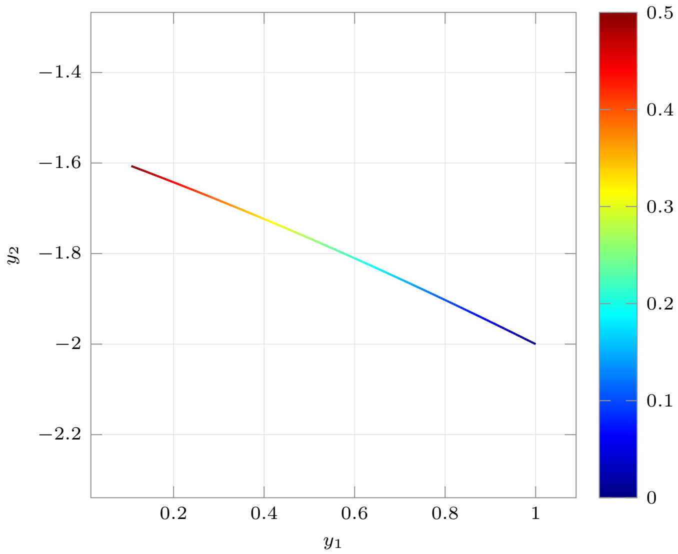
$$y_1' = y_2 \qquad y_2' = x + y_1 \tag{21.3.16}$$

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad y'' + y = x \tag{21.3.17}$$

$$y = Ae^{x} + Be^{-x} + Cx + D \qquad 0 = (C+1)x + D \tag{21.3.18}$$

$$y = e^{-x} - x \tag{21.3.19}$$

n	y_1	ϵ_1	y_2	ϵ_2
1	8	4.8×10^{-3}	-1.9	4.8×10^{-3}
2	0.61	8.7×10^{-3}	-1.81	8.7×10^{-3}
5	0.0905	1.6×10^{-2}	-1.59	1.6×10^{-2}



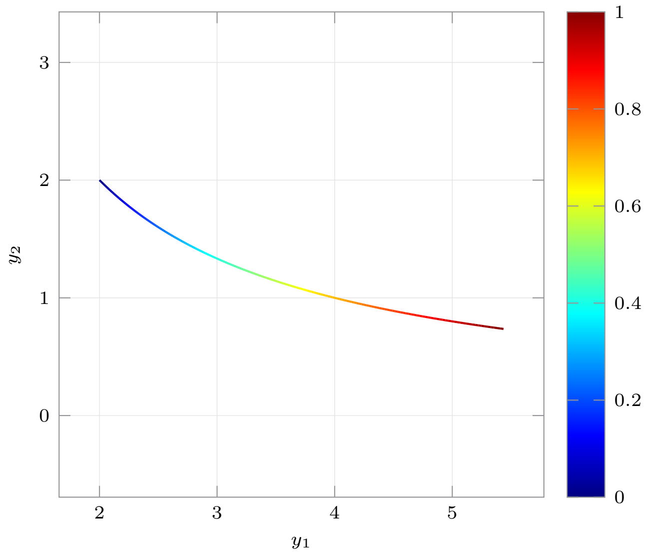
6. Solving the system of ODEs,

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y} \qquad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{21.3.20}$$

$$\lambda_1 = 1, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda_1 = -1, \quad \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{21.3.21}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{x} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-x} \qquad c_1 = 2, \ c_2 = 2 \tag{21.3.22}$$

n	y_1	ϵ_1	y_2	ϵ_2
1	2.2	0.010	1.8	0.0097
2	3.22	0.076	1.181	0.032
5	5.187	0.249	0.697	0.038



7. Comparing the two method for Problem 5,

n	Error			
	Euler y_1	RK y_1	Euler y_2	RK y_2
1	4.8×10^{-3}	8.2×10^{-8}	1.2×10^{-2}	8.2×10^{-8}
2	1.4×10^{-2}	1.5×10^{-7}	1.4×10^{-2}	1.5×10^{-7}
5	1.6×10^{-2}	2.75×10^{-7}	1.6×10^{-2}	2.7×10^{-7}

8. Solving Problem 2 using the RK method instead,

n	y_1	ϵ_1	y_2	ϵ_2
1	0.651	3.99×10^{-5}	3.21	4.26×10^{-5}
2	1.044	5.02×10^{-5}	2.47	8.13×10^{-5}
5	1.24	6.18×10^{-6}	0.795	1.31×10^{-4}

9. Solving Problem 1 using the RK method instead,

n	y_1	ϵ_1	y_2	ϵ_2
1	3.602	2.29×10^{-6}	0.286	2.66×10^{-6}
2	4.125	4.97×10^{-6}	0.551	4.41×10^{-6}
10	10.738	1.26×10^{-5}	2.583	6.35×10^{-6}

10. Solving Problem 4 using the RK method instead,

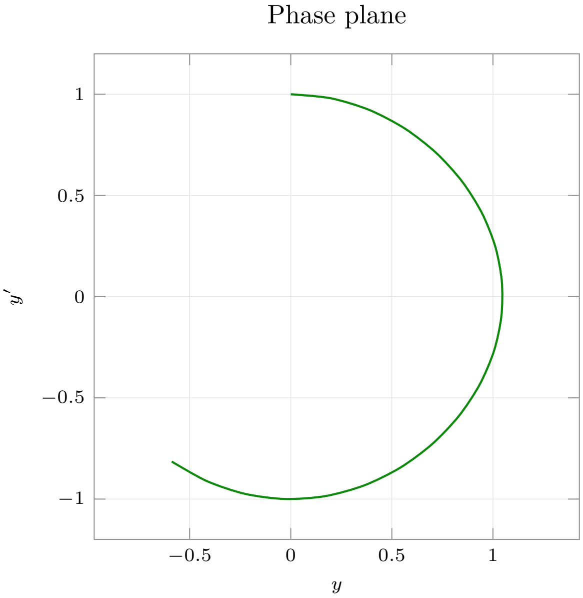
n	y_1	ϵ_1	y_2	ϵ_2
1	1.489	8.25×10^{-5}	0.148	7.74×10^{-5}
2	1.12	1.11×10^{-4}	0.221	1.02×10^{-4}
5	0.503	8.65×10^{-5}	2.232	7.49×10^{-5}

11. Solving the system of ODEs,

$$y'' = -\sin y \qquad \mathbf{y}(\pi) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{21.3.23}$$

$$y = y_1, \quad y' = y_2 \qquad y_1' = y_2, \quad y_2' = -\sin(y_1) \tag{21.3.24}$$

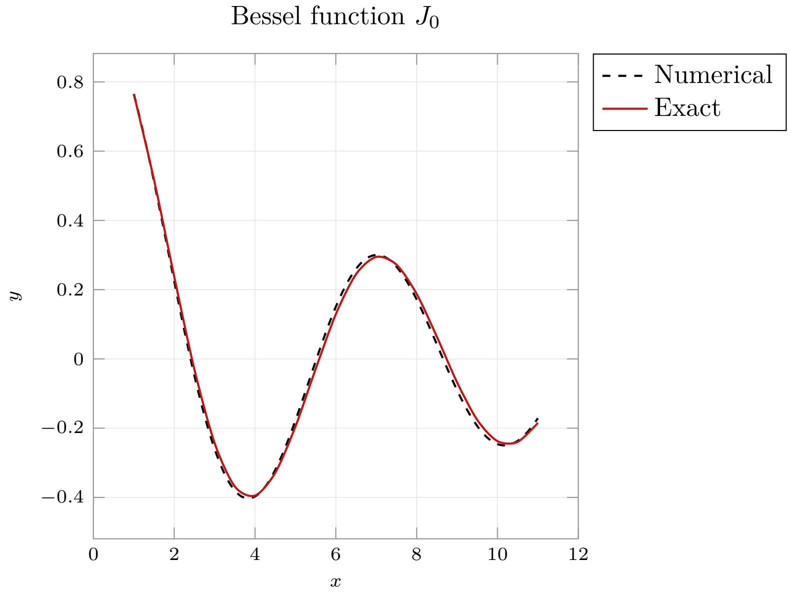
Analytic solution requires elliptic integrals. Plotting the numerical solution of the ODE,



12. Analytic solution requires Bessel function $J_0(x)$. Plotting the numerical solution of the ODE,

$$y'' = -y - \frac{y'}{x} \qquad \mathbf{y}(1) = \begin{bmatrix} 0765198 \\ -0.440051 \end{bmatrix} \tag{21.3.25}$$

$$y = y_1, \quad y' = y_2 \qquad y_1' = y_2, \quad y_2' = -y_1 - \frac{y_2}{x} \tag{21.3.26}$$



13. Verifying the calculations in Example 2,

$$y'' = xy \qquad y = y_1, \quad y' = y_2 \tag{21.3.27}$$

$$y_1' = y_2, \quad y_2' = xy_1 \qquad \mathbf{a} = h\begin{bmatrix} y_{2,n} \\ x_n\, y_{1,n} \end{bmatrix} \tag{21.3.28}$$

$$\mathbf{b} = h\begin{bmatrix} y_{2,n} + a_2/2 \\ (x_n + h/2)(y_{1,n} + a_1/2) \end{bmatrix} \qquad \mathbf{c} = h\begin{bmatrix} y_{2,n} + b_2/2 \\ (x_n + h/2)(y_{1,n} + b_1/2) \end{bmatrix} \tag{21.3.29}$$

$$\mathbf{d} = h\begin{bmatrix} y_{2,n} + c_2 \\ (x_n + h)(y_{1,n} + c_1) \end{bmatrix} \tag{21.3.30}$$

Finally the iterative formula uses these four auxiliary values

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{\mathbf{a} + 2\mathbf{b} + 2\mathbf{c} + \mathbf{d}}{6} \tag{21.3.31}$$

The error values for each timestep do match the values in the text.

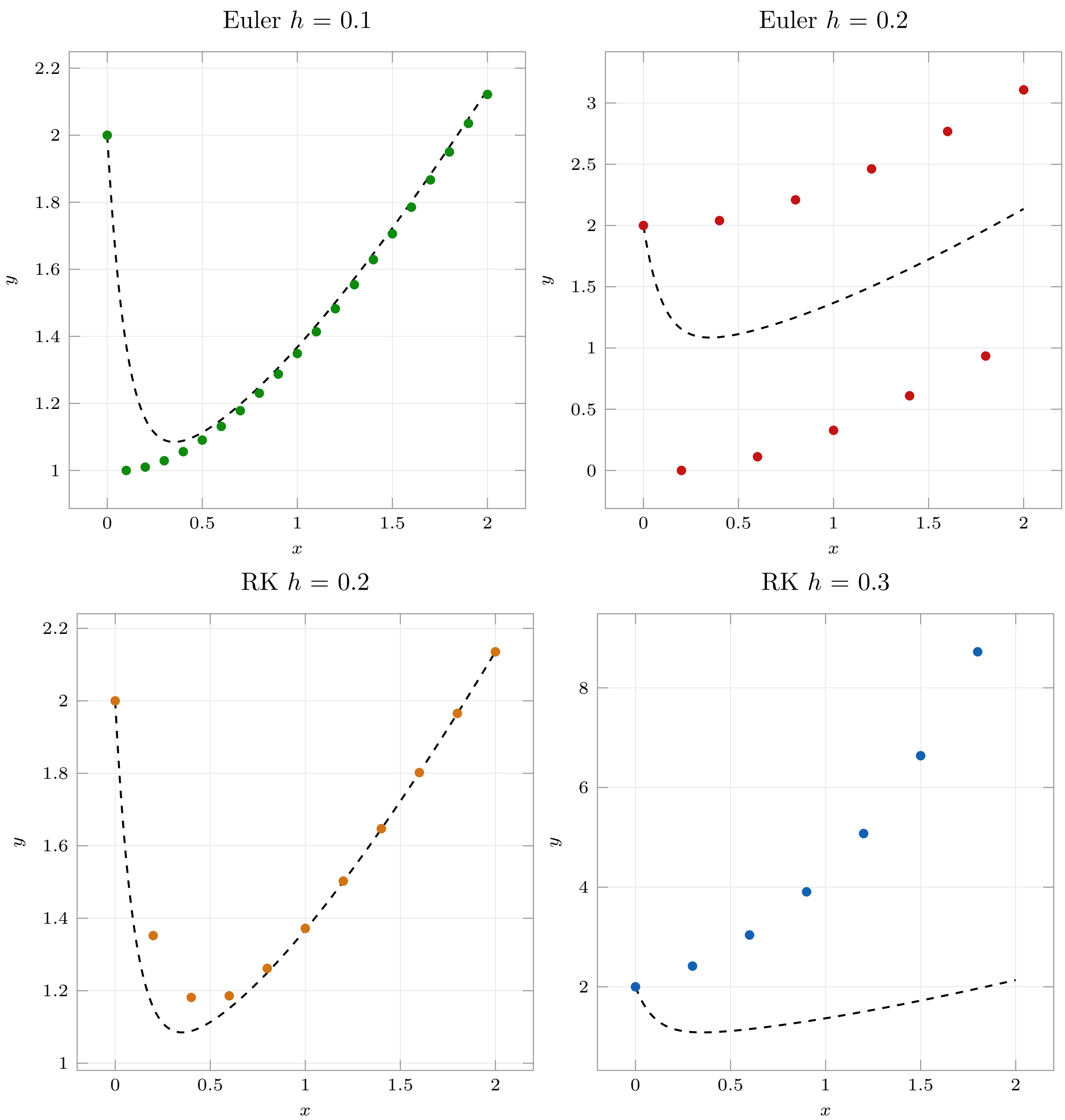
n	y_1	ϵ_1	y_2	ϵ_2
1	0.3037	1.23×10^{-7}	-0.2524	8.35×10^{-7}
2	0.2547	2.47×10^{-7}	-0.2358	1.31×10^{-6}
5	0.1353	3.42×10^{-7}	-0.1591	5.72×10^{-7}

14. Using the RKN method,

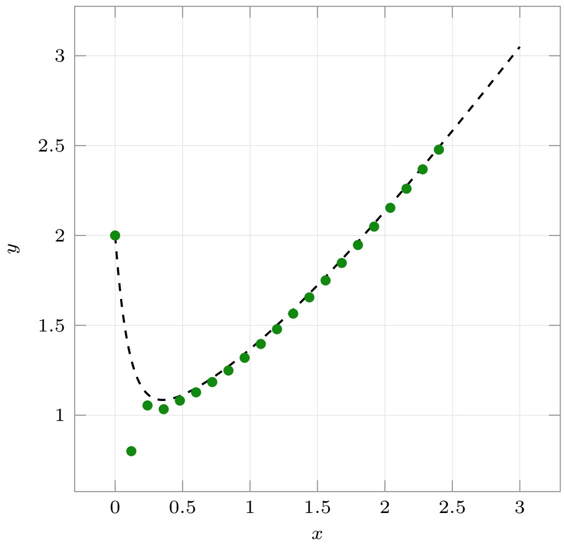
n	y	ϵ
1	0.303 703 031	1.23×10^{-7}
2	0.254 742 107	2.47×10^{-7}
5	0.135 292 178	2.38×10^{-7}

15. The values in the table are verified for the Euler method and RK method for the two values of h.

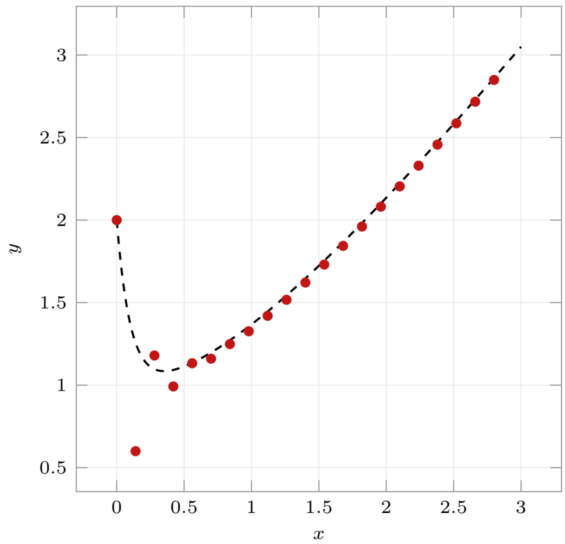
(a) Plotting the four plots from the data in the table,



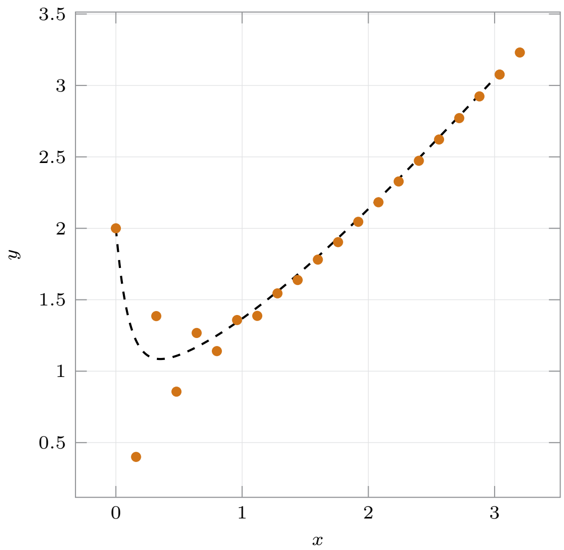
(b) Looking at the stability for various values of h around $h^* = 0.18$, Euler method, $h = 0.12$



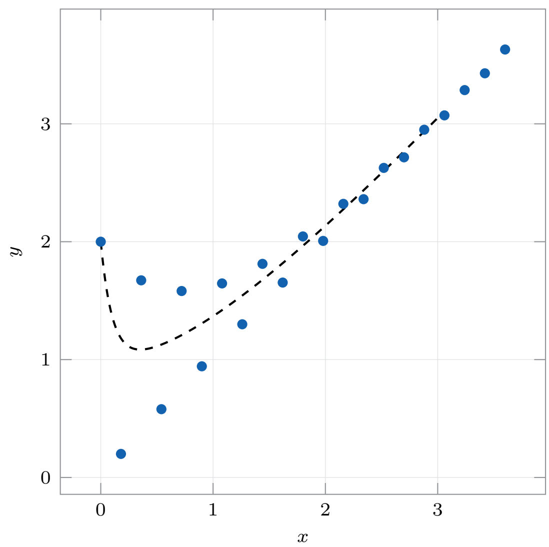
Euler method, $h = 0.14$



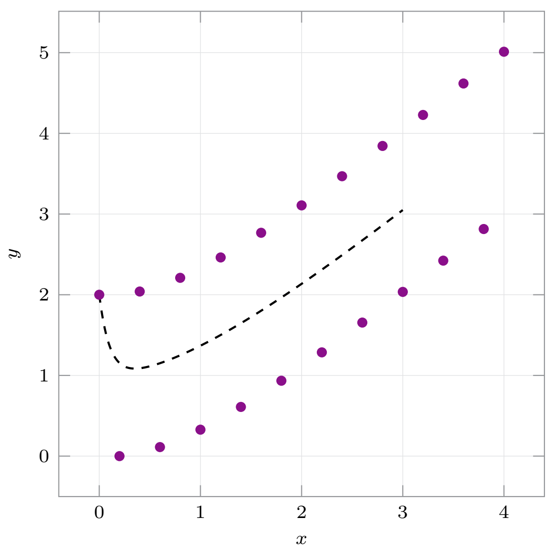
Euler method, $h = 0.16$



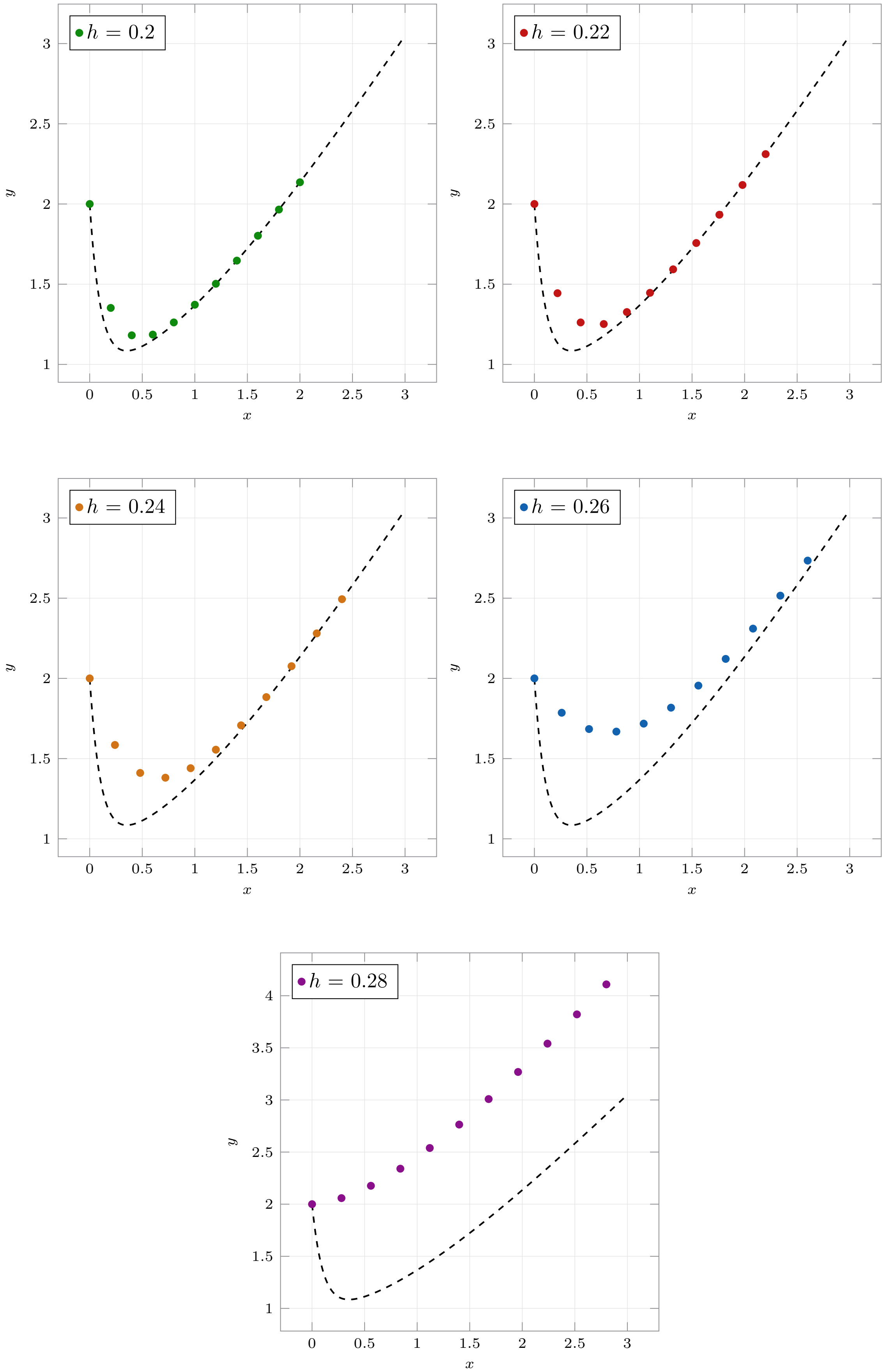
Euler method, $h = 0.18$



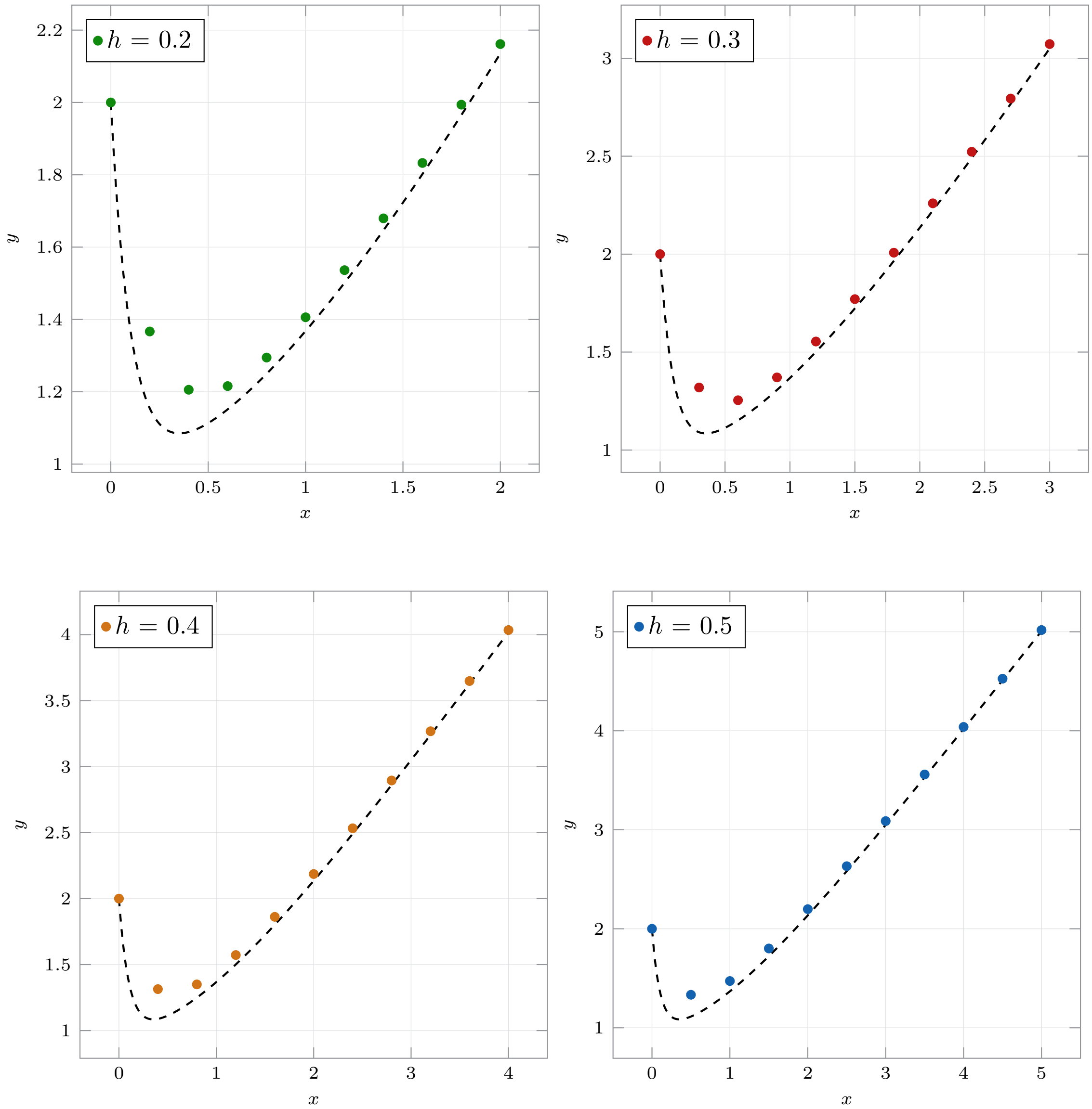
Euler method, $h = 0.2$



(c) RK method for values of h between 0.2 and 0.3, the numerical approximations deviates at $h \approxeq 0.25$.



(d) The backward Euler method provides the implicit relation,



There is no instability even for large values of h.

21.4 Methods for Elliptic PDEs

1. Deriving the relations, by using the first 2 terms of the Taylor series,

$$u(x, y + k) = u + k\, u_y + \frac{k^2}{2!}\, u_{yy} = \ldots \tag{21.4.1}$$

$$u(x, y - k) = u - k\, u_y + \frac{k^2}{2!}\, u_{yy} = \ldots \tag{21.4.2}$$

$$u_y \approxeq \frac{u(x, y + k) - u(x, y - k)}{2k} \tag{21.4.3}$$

Next, adding these two Taylor series,

$$u(x, y + k) + u(x, y - k) = 2u + k^2\, u_{yy} \tag{21.4.4}$$

$$u_{yy} \approxeq \frac{u(x, y + k) - 2u + u(x, y - k)}{k^2} \tag{21.4.5}$$

To get the mixed derivative,

$$\frac{\partial u_y}{\partial x} = \frac{u_x(x, y + k) - u_x(x, y - k)}{2k} \tag{21.4.6}$$

$$= \frac{u(x + h, y + k) - u(x - h, y + k)}{4hk} - \frac{u(x + h, y - k) - u(x - h, y - k)}{4hk} \tag{21.4.7}$$

$$u_{xy} \approxeq \frac{u(x + h, y + k) - u(x - h, y + k) - u(x + h, y - k) + u(x - h, y - k)}{4hk} \tag{21.4.8}$$

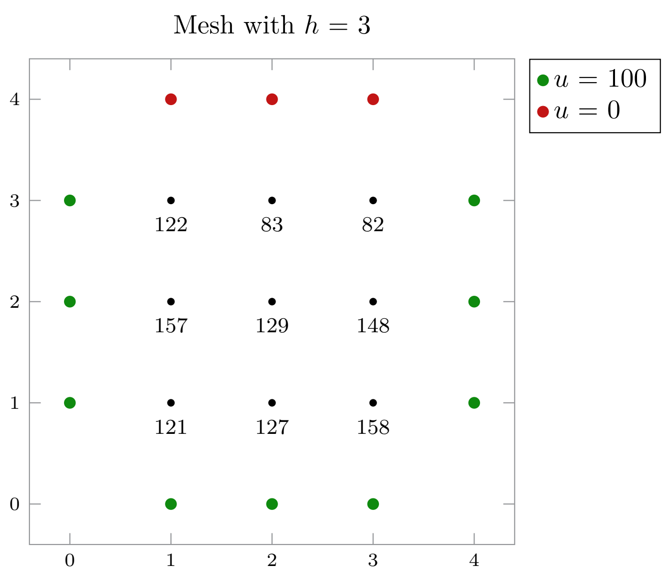
2. Gauss-Seidel method written in `numpy`. It takes 7 iterations for $3S$ values to be achieved.
3. Since the boundary conditions are symmetric horizontally, mirrorring the columns should not change the results.

$$u_{11} = u_{21} \qquad u_{12} = u_{22} \tag{21.4.9}$$

$$-3u_{11} + u_{12} = -200 \qquad u_{11} - 3u_{12} = -100 \tag{21.4.10}$$

$$u_{12} = 62.5 \qquad u_{11} = 87.5 \tag{21.4.11}$$

4. Replacing $h = 4$, with the smaller $h = 3$ gives $M = N = 4$ and a set of 9 mesh points.



The exact solution using Cramer's rule is computed using `numpy`

5. Using the given boundary conditions, and 5 Gauss-Siedel iterations,

$$\mathbf{P} = \begin{bmatrix} 105 & 115 \\ 105 & 155 \end{bmatrix} \qquad \widetilde{\mathbf{P}} = \begin{bmatrix} 104.98 & 114.97 \\ 104.94 & 154.96 \end{bmatrix}$$ 21.4.12

6. Using the given boundary conditions, and 5 Gauss-Siedel iterations,

$$\mathbf{P} = \begin{bmatrix} -2 & 2 \\ -11 & -16 \end{bmatrix} \qquad \widetilde{\mathbf{P}} = \begin{bmatrix} -1.67 & 2.16 \\ -10.83 & -15.91 \end{bmatrix}$$ 21.4.13

7. Using the given boundary conditions, and 5 Gauss-Siedel iterations,

$$\mathbf{P} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \widetilde{\mathbf{P}} = \begin{bmatrix} 0.293 & 0.146 \\ 0.146 & 0.073 \end{bmatrix}$$ 21.4.14

8. Using the given boundary conditions, and 5 Gauss-Siedel iterations,

$$\mathbf{P} = \begin{bmatrix} 165 & 165 \\ 165 & 165 \end{bmatrix} \qquad \widetilde{\mathbf{P}} = \begin{bmatrix} 164.81 & 164.9 \\ 164.9 & 164.95 \end{bmatrix}$$ 21.4.15

9. Using the given boundary conditions, and 10 Gauss-Siedel iterations,

$$\mathbf{P} = \begin{bmatrix} 0.108 & 0.108 \\ 0.325 & 0.325 \end{bmatrix} \qquad \widetilde{\mathbf{P}} = \begin{bmatrix} 0.108 & 0.108 \\ 0.325 & 0.325 \end{bmatrix}$$ 21.4.16

10. Using the given boundary conditions, and 5 Gauss-Siedel iterations,

$$\mathbf{P} = \begin{bmatrix} 2 & -5 \\ -5 & -62 \end{bmatrix} \qquad \widetilde{\mathbf{P}} = \begin{bmatrix} -1.58 & -4.79 \\ -4.79 & -61.89 \end{bmatrix}$$ 21.4.17

11. Using the coarse grid, and 10 Gauss-Siedel iterations,

$$\mathbf{P} = \begin{bmatrix} -66 \\ 66 \end{bmatrix} \qquad \widetilde{\mathbf{P}} = \begin{bmatrix} -65.99987 \\ 66.00003 \end{bmatrix}$$ 21.4.18

Using the finer grid, and 10 Gauss-Siedel iterations,

$$\mathbf{P} = \begin{bmatrix} 92.9 & 87.4 & 92.9 \\ 64.2 & 54.0 & 64.2 \\ 0 & 0 & 0 \\ -64.2 & -54.0 & -64.2 \\ -92.9 & -87.4 & -92.9 \end{bmatrix} \qquad \widetilde{\mathbf{P}} = \begin{bmatrix} 92.18 & 86.63 & 92.18 \\ 63.22 & 52.86 & 63.22 \\ 0.90 & 1.005 & 0.56 \\ -63.22 & -52.86 & -63.22 \\ -92.18 & -86.63 & -92.18 \end{bmatrix} \tag{21.4.19}$$

12. Using the given boundary conditions, and 10 Gauss-Siedel iterations, to compare the two different initial guesses.

$$\widetilde{\mathbf{P}}_{100} = \begin{bmatrix} 0.10853845 & 0.10839581 \\ 0.32490216 & 0.32483085 \end{bmatrix} \qquad \widetilde{\mathbf{P}}_{0} = \begin{bmatrix} 0.10825235 & 0.10825276 \\ 0.32475911 & 0.32475932 \end{bmatrix} \tag{21.4.20}$$

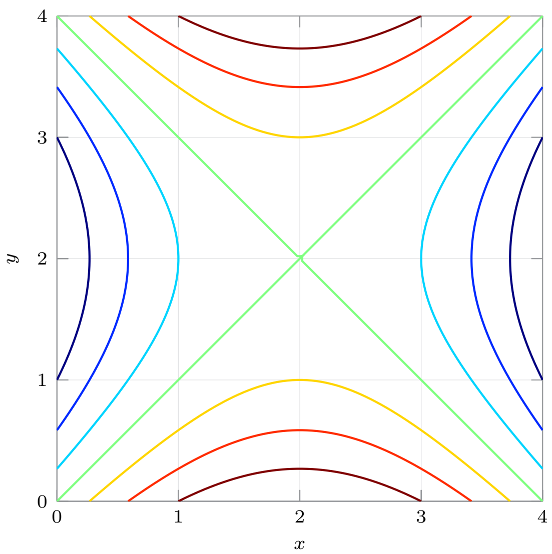
The newer guess provides a better result.

13. Using the finer grid, and 10 Gauss-Siedel iterations, setting all mesh points to initial guess 25 °C

$$\mathbf{P} = \begin{bmatrix} 25 & 18.75 & 25 \\ 31.25 & 25 & 31.25 \\ 25 & 18.75 & 25 \end{bmatrix} \tag{21.4.21}$$

$$\widetilde{\mathbf{P}} = \begin{bmatrix} 25 & 18.74999 & 24.999999 \\ 31.25000002 & 25 & 31.25 \\ 25.0000001 & 18.75 & 25 \end{bmatrix} \tag{21.4.22}$$

14. Plotting rough isotherms using the results from Problem 13,



15. Using the finer grid, and 10 Gauss-Siedel iterations, setting all mesh points to initial guess 25 °C

$$\mathbf{P} = \begin{bmatrix} 0 & 0.25 & 0 \\ -0.25 & 0 & -0.25 \\ 0 & 0.25 & 0 \end{bmatrix} \tag{21.4.23}$$

$$\widetilde{\mathbf{P}} = \begin{bmatrix} 0 & 0.25 & 4.65 \times 10^{-10} \\ -0.25 & 0 & -0.25 \\ -4.65 \times 10^{-10} & 0.25 & 0 \end{bmatrix} \tag{21.4.24}$$

16. Using 5 steps of the ADI method,

$$\mathbf{P}_{\mathrm{ADI}} = \begin{bmatrix} 1.092 \times 10^{-1} & 1.092 \times 10^{-1} \\ 3.2476 \times 10^{-1} & 3.2476 \times 10^{-1} \end{bmatrix} \tag{21.4.25}$$

$$\mathbf{P}_{\mathrm{Lieb}} = \begin{bmatrix} 0.108 & 0.108 \\ 0.325 & 0.325 \end{bmatrix} \tag{21.4.26}$$

17. Using the improved ADI method, the optimal value of p is

$$p^* = 2\sin(\pi/3) = \sqrt{3} \qquad \mathbf{P}^* = \begin{bmatrix} 0.077 & 0.0987 \\ 0.308 & 0.318 \end{bmatrix} \tag{21.4.27}$$

$$p = 2 \qquad \mathbf{P} = \begin{bmatrix} 0.0849 & 0.109 \\ 0.3170 & 0.323 \end{bmatrix} \tag{21.4.28}$$

After one iteration, the improved ADI method is closer to the exact values.

18. Laplace equation

(a) Code written in `numpy`

(b) The exact solution using Gaussian elimination is,

$$\mathbf{P}_{\mathrm{G}} = \begin{bmatrix} 159.545454 & 170.151515 & 156.515151 & 110.454545 \\ 138.030303 & 144.545454 & 125.454545 & 75.303030 \\ 138.030303 & 144.545454 & 125.454545 & 75.303030 \\ 159.545454 & 170.151515 & 156.515151 & 110.454545 \end{bmatrix} \tag{21.4.29}$$

$$\mathbf{P}_{\mathrm{num}} = \begin{bmatrix} 159.51469 & 170.11124 & 156.482572 & 110.438255 \\ 138.990032 & 144.492739 & 125.411898 & 75.281707 \\ 138.997723 & 144.502807 & 125.420043 & 75.285779 \\ 159.52916 & 170.13019 & 156.497900 & 110.445919 \end{bmatrix} \tag{21.4.30}$$

$$\epsilon = 5.27 \times 10^{-2} \tag{21.4.31}$$

21.5 Neumann and Mixed Problems. Irregular Boundary

1. Solving equation 3 by Gaussian elimination,

$$\widetilde{\mathbf{x}} = \begin{bmatrix} 0.07686 & 0.19099 & 0.86646 & 1.81211 \end{bmatrix} \tag{21.5.1}$$

This matches the result shown in the text.

2. Mixed BVP with normal derivative at right edge.

$$\nabla^2 u = f(x, y) = 2(x^2 + y^2) \qquad h = 1 \tag{21.5.2}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -0 - 0 + 4 \\ -0 + 10 \\ -0 - u_{41} + 20 \\ -0 - 9 + 10 \\ -36 + 16 \\ -u_{42} + 81 \end{bmatrix} \tag{21.5.3}$$

$$2 \cdot 6 = u_{41} - u_{21} \qquad 2 \cdot 24 = u_{42} - u_{22} \tag{21.5.4}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 10 \\ 8 \\ 1 \\ -20 \\ 33 \end{bmatrix} \tag{21.5.5}$$

The exact solution using Gaussian elimination is,

$$\mathbf{u} = \begin{bmatrix} -2.16 & -4.24 & -6.38 & -0.396 & 1.57 & -9.06 \end{bmatrix} \tag{21.5.6}$$

3. TBC

4. Mixed BVP with normal derivative at right edge.

$$\nabla^2 u = f(x, y) = 2(x^2 + y^2) \qquad h = 0.5 \tag{21.5.7}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{01} \\ u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -u_{-1,1} - 0 + 1 \\ -0.25 - 0.75 \\ -1 - 0 \\ -u_{41} - 2.25 + 1.25 \end{bmatrix} \tag{21.5.8}$$

$$1 \cdot 0 = u_{-1,1} - u_{11} \qquad 1 \cdot 3 = u_{41} - u_{21} \tag{21.5.9}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 2 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} u_{01} \\ u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -7 \end{bmatrix} \tag{21.5.10}$$

The exact solution using Gaussian elimination is,

$$\mathbf{u} = \begin{bmatrix} -0.0111 & 0.4778 & 0.9222 & 2.2111 \end{bmatrix} \tag{21.5.11}$$

5. Mixed BVP with normal derivative at right edge.

$$\nabla^2 u = f(x, y) = 0 \qquad\qquad h = 0.5 \tag{21.5.12}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -0 - 0 \\ -0 - 0.375 \\ -u_{13} - 0 \\ -u_{23} - 3 \end{bmatrix} \tag{21.5.13}$$

$$1 \cdot 3 = u_{13} - u_{11} \qquad\qquad 1 \cdot 6 = u_{23} - u_{21} \tag{21.5.14}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ -0.375 \\ -3 \\ -9 \end{bmatrix} \tag{21.5.15}$$

The exact solution using Gaussian elimination is,

$$\mathbf{u} = \begin{bmatrix} 0.766 & 1.109 & 1.956 & 3.293 \end{bmatrix} \tag{21.5.16}$$

6. Mixed BVP with normal derivative at top edge.

$$\nabla^2 u = f(x, y) = -\pi^2 y \ \sin(\pi x/3) \qquad h = 1 \tag{21.5.17}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \\ u_{13} \\ u_{23} \end{bmatrix} \tag{21.5.18}$$

$$\mathbf{b} = \begin{bmatrix} -0 - 0 - \pi^2\sqrt{3}/2 \\ -0 - 0 - \pi^2\sqrt{3}/2 \\ -0 - \pi^2\sqrt{3} \\ -0 - \pi^2\sqrt{3} \\ -0 - u_{14} - \pi^2 \ (3\sqrt{3}/2) \\ -0 - u_{24} - \pi^2 \ (3\sqrt{3}/2) \end{bmatrix} \tag{21.5.19}$$

$$2 \cdot \frac{9\sqrt{3}}{2} = u_{14} - u_{12} \qquad 2 \cdot \frac{9\sqrt{3}}{2} = u_{24} - u_{22} \tag{21.5.20}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \\ u_{13} \\ u_{23} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -\pi^2\sqrt{3}/2 \\ -\pi^2\sqrt{3}/2 \\ -\pi^2\sqrt{3} \\ -\pi^2\sqrt{3} \\ (-9 - 1.5\pi^2)\sqrt{3} \\ (-9 - 1.5\pi^2)\sqrt{3} \end{bmatrix} \tag{21.5.21}$$

$$\tag{21.5.22}$$

The exact solution using Gaussian elimination is,

$$\mathbf{u} = \begin{bmatrix} 8.463 & 8.463 & 16.844 & 16.844 & 24.972 & 24.972 \end{bmatrix} \tag{21.5.23}$$

7. Mixed BVP with normal derivative at upper edge.

$$\nabla^2 u = f(x, y) = 2(x^2 + y^2) \qquad h = 0.5 \tag{21.5.24}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -110 - 110 \\ -110 - 110 \\ -110 - u_{13} \\ -110 - u_{23} \end{bmatrix} \tag{21.5.25}$$

$$1 \cdot 110 = u_{13} - u_{11} \qquad 1 \cdot 110 = u_{23} - u_{21} \tag{21.5.26}$$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -220 \\ -220 \\ -220 \\ -220 \end{bmatrix} \tag{21.5.27}$$

The exact solution using Gaussian elimination is,

$$\mathbf{u} = \begin{bmatrix} 125.71 & 125.71 & 157.14 & 157.14 \end{bmatrix} \tag{21.5.28}$$

8. For $a = b = 1/2$,

$$\nabla^2 u_O = \frac{2}{h^2} \left[\frac{4u_A}{3} + \frac{4u_B}{3} + \frac{2u_P}{3} + \frac{2u_Q}{3} - 4u_O \right] \tag{21.5.29}$$

$$\mathbf{S} = \begin{bmatrix} \cdot & 4/3 & \cdot \\ 2/3 & -4 & 4/3 \\ \cdot & 2/3 & \cdot \end{bmatrix} \tag{21.5.30}$$

9. Using the Taylor expansion and keeping terms upto second order,

$$u_A = u_O + (ah)\,\frac{\partial u_o}{\partial x} + \frac{(ah)^2}{2!}\,\frac{\partial^2 u_O}{\partial x^2} + \dots \tag{21.5.31}$$

$$u_P = u_O - (h)\,\frac{\partial u_o}{\partial x} + \frac{(h)^2}{2!}\,\frac{\partial^2 u_O}{\partial x^2} + \dots \tag{21.5.32}$$

$$\frac{\partial^2 u_O}{\partial x^2} \approxeq \frac{2}{h^2}\left[\frac{u_A}{a(1+a)} + \frac{u_P}{(1+a)} - \frac{u_O}{a}\right] \tag{21.5.33}$$

$$\frac{\partial^2 u_O}{\partial y^2} \approxeq \frac{2}{h^2}\left[\frac{u_B}{b(1+b)} + \frac{u_Q}{(1+b)} - \frac{u_O}{b}\right] \tag{21.5.34}$$

$$\nabla^2 u_O \approxeq \frac{2}{h^2}\left[\frac{u_B}{b(1+b)} + \frac{u_Q}{(1+b)} + \frac{u_A}{a(1+a)} + \frac{u_P}{(1+a)} - \frac{(a+b)u_O}{ab}\right] \tag{21.5.35}$$

10. For the general case where the distances to the points A, B, P, Q are ah, bh, ph, qh respectively.

$$u_A = u_O + (ah)\,\frac{\partial u_o}{\partial x} + \frac{(ah)^2}{2!}\,\frac{\partial^2 u_O}{\partial x^2} + \dots \tag{21.5.36}$$

$$u_P = u_O - (ph)\,\frac{\partial u_o}{\partial x} + \frac{(ph)^2}{2!}\,\frac{\partial^2 u_O}{\partial x^2} + \dots \tag{21.5.37}$$

$$\frac{\partial^2 u_O}{\partial x^2} \approxeq \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_P}{p(a+p)} - \frac{u_O}{ap}\right] \tag{21.5.38}$$

$$\frac{\partial^2 u_O}{\partial y^2} \approxeq \frac{2}{h^2}\left[\frac{u_B}{b(b+q)} + \frac{u_Q}{q(b+q)} - \frac{u_O}{bq}\right] \tag{21.5.39}$$

$$\nabla^2 u_O \approxeq \frac{2}{h^2}\left[\frac{u_B}{b(b+q)} + \frac{u_Q}{q(b+q)} + \frac{u_A}{a(a+p)} + \frac{u_P}{p(a+p)} - \frac{(ap+bq)u_O}{apbq}\right] \tag{21.5.40}$$

11. For Example 2 in the text,

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1.2 & -5 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1.2 & 1.2 & -6 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -27-0 \\ -216-296(9/5) \\ -0+702 \\ 352(9/5)+936(9/5) \end{bmatrix} \tag{21.5.41}$$

$$\mathbf{b} = \begin{bmatrix} -27 \\ -748.8 \\ 702 \\ 2318.4 \end{bmatrix} \tag{21.5.42}$$

The second and fourth rows are doubled as compared to the text, but the linear system itself is the same.

12. Using Gaussian elimination,

$$\mathbf{u} = \begin{bmatrix} -55.57 & 49.17 & -298.46 & -436.26 \end{bmatrix} \tag{21.5.43}$$

13. Solving the irregular boundary $y = 4.5 - x$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & \frac{4}{3} & \frac{4}{3} & -8 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3-0 \\ -6-6 \\ -0-0 \\ -(8/3)2.5-(8/3)(1) \end{bmatrix} \tag{21.5.44}$$

$$\mathbf{b} = \begin{bmatrix} -3 \\ -12 \\ 0 \\ -\frac{28}{3} \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 2 \end{bmatrix} \tag{21.5.45}$$

14. Solving with the unknown potential V on the boundary

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & \frac{4}{3} & \frac{4}{3} & -8 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -0-0 \\ -0-V \\ -0-V \\ -(8/3)V-(8/3)V \end{bmatrix} \tag{21.5.46}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ -V \\ -V \\ -\frac{16}{3}V \end{bmatrix} \qquad \mathbf{u} = \frac{V}{19} \begin{bmatrix} 5 \\ 10 \\ 10 \\ 16 \end{bmatrix} \tag{21.5.47}$$

$$\frac{5V}{19} = 220 \qquad V = 836 \tag{21.5.48}$$

15. Solving the system with $u = 0$ on the outer portion and $u = 100$ on the axes,

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & \frac{4}{3} & \frac{4}{3} & -8 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -100 - 100 \\ -100 - 0 \\ -100 - 0 \\ -(8/3)0 - (8/3)0 \end{bmatrix} \tag{21.5.49}$$

$$\mathbf{b} = \begin{bmatrix} -200 \\ -100 \\ -100 \\ 0 \end{bmatrix} \qquad \mathbf{u} = \frac{100}{19} \begin{bmatrix} 14 \\ 9 \\ 9 \\ 3 \end{bmatrix} \tag{21.5.50}$$

16. Solving the Poisson equation with $f(x, y) = 2$ and $h = 1$

$$\mathbf{Au} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 0 \\ 4/3 & 0 & -6 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 - 0 + 2 \\ 2 - 0 + 0.5 \\ 2 + 2 - 0 - (8/3)(0) \end{bmatrix} \tag{21.5.51}$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 2.5 \\ 4 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} -1.5 \\ -1 \\ -1 \end{bmatrix} \tag{21.5.52}$$

21.6 Methods for Parabolic PDEs

1. Deriving the non-dimensional version,

$$u_t = c^2\, u_{xx} \tag{21.6.1}$$

$$v = \frac{x}{L} \qquad u_x = u_v \cdot v_x = \frac{u_v}{L} \tag{21.6.2}$$

$$u_{xx} = \frac{\partial}{\partial x}\, \frac{u_v}{L} = \frac{\partial}{\partial x}\, \frac{u_{vv}}{L}\, v_x \qquad u_{xx} = \frac{u_{vv}}{L^2} \tag{21.6.3}$$

$$q = \frac{c^2\, t}{L^2} \qquad \frac{\partial u}{\partial t} = u_q\, q_t = u_q\, \frac{c^2}{L^2} \tag{21.6.4}$$

$$u_q = u_{vv} \tag{21.6.5}$$

Now, q and v are the dimensionless version of time and position.

2. The difference approximation is,

$$u_{xx} = \frac{1}{h}\left[\frac{u_{i+1} - u_i}{h} - \frac{u_i - u_{i-1}}{h}\right] \qquad = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{21.6.6}$$

Using a forward difference for the time derivative, since information is not available for negative time,

$$u_t = \frac{u_{j+1} - u_j}{k} \tag{21.6.7}$$

The subscripts corresponding to the other coordinate are omitted for clarity.

3. Deriving the relation,

$$u_{i,j+1} = u_{ij} + \frac{k}{h^2}\left[u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right] \qquad r = \frac{k}{h^2} \tag{21.6.8}$$

$$u_{i,j+1} = (1 - 2r)\, u_{ij} + r\, (u_{i+1,j} + u_{i-1,j}) \tag{21.6.9}$$

4. Comparison of methods

   (a) Code written in `numpy`

   (b) Comparing the explicit method and the CN method,

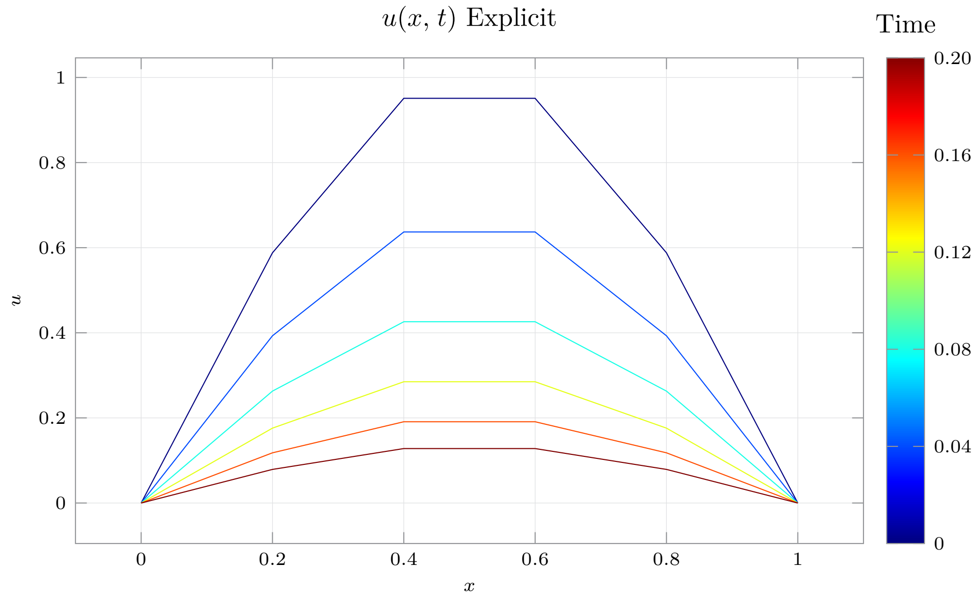
t	**Explicit**		**Crank-Nicholson**	
	$x = 0.2$	$x = 0.4$	$x = 0.2$	$x = 0.4$
0.04	0.393 432	0.636 586	0.399 274	0.646 039
0.08	0.263 342	0.426 096	0.271 221	0.438 844
0.12	0.176 267	0.285 206	0.184 236	0.298 100
0.16	0.117 983	0.190 901	0.125 149	0.202 495
0.20	0.078 972	0.127 779	0.085 012	0.137 552

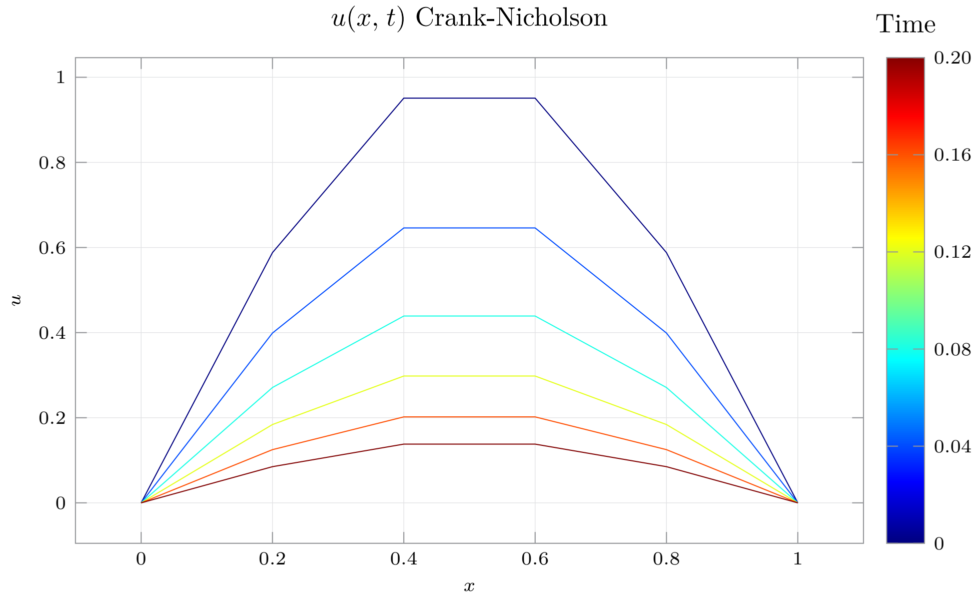
The maximum error at each time step for both methods is,

t	$\epsilon_{\text{explicit}}$	ϵ_{CN}
0.04	4.26×10^{-3}	5.19×10^{-3}
0.08	5.72×10^{-3}	7.02×10^{-3}
0.12	5.76×10^{-3}	7.13×10^{-3}
0.16	5.16×10^{-3}	6.43×10^{-3}
0.20	4.33×10^{-3}	5.43×10^{-3}

The accuracies are similar, in spite of the explicit method requiring 4 times as many time steps.

   (c) Plotting the two approximations,





(d) Keeping h constant, and looking at the error as a function of r,

r	$\epsilon_{\text{explicit}}$	ϵ_{CN}	r	$\epsilon_{\text{explicit}}$	ϵ_{CN}
0.01	2.21×10^{-3}	5.98×10^{-4}	1.5	1.2×10^{-1}	1.59×10^{-3}
0.1	4.63×10^{-3}	5.00×10^{-3}	2	2.07×10^{-1}	6.49×10^{-3}
0.5	1.05×10^{-2}	2.49×10^{-2}	2.5	25.79	1.81×10^{-2}
1	7.13×10^{-3}	6.85×10^{-2}	3	2432	3.27×10^{-2}

The explicit method is unstable for $r > 1/2$

5. Using the explicit method with

$$h = 1, \quad k = 0.5 \qquad\qquad r = \frac{k}{h^2} = 0.5$$ 21.6.10

r	$u(2, t)$	r	$u(2, t)$
0	1.6	3	1.125
0.5	1.5	3.5	1.070 312
1	1.4	4	1.015 625
1.5	1.325	4.5	0.966 797
2	1.25	5	0.917 969
2.5	1.1875		

6. Using the explicit method with

$$h = 0.2, \quad k = 0.01 \qquad\qquad r = \frac{k}{h^2} = 0.25 \tag{21.6.11}$$

t	x = 0.2		x = 0.4	
	Explicit	Exact	Explicit	Exact
0.08	0.105	0.108	0.170	0.175

7. Using the explicit method with

$$h = 0.2, \quad k = 0.01 \qquad\qquad r = \frac{k}{h^2} = 0.25 \tag{21.6.12}$$

t	$x = 0.2$		$x = 0.4$	
	$r = 0.25$	$r = 0.5$	$r = 0.25$	$r = 0.5$
0.04	0.156	0.15	0.254	0.25
0.08	0.105	0.100	0.170	0.162

The larger r gives much worse approximations compared to Problem 6.

8. Using the explicit method with

$$h = 0.2, \quad k = 0.01 \qquad\qquad r = \frac{k}{h^2} = 0.25 \tag{21.6.13}$$

t	$x = 0.2$	$x = 0.4$
0.01	0.2	0.35
0.02	0.1875	0.3125
0.03	0.171 875	0.281 25
0.04	0.156 25	0.253 906
0.05	0.141 602	0.229 492

9. At the end of 5 time steps,

$$\begin{bmatrix} 0 & 0.062793 & 0.093359 & 0.083643 & 0.04707 & 0 \end{bmatrix} \tag{21.6.14}$$

10. TBC

11. At the end of 2 time steps, with $h = 0.2$ and $r = 1$

$$\begin{bmatrix} 0 & 0.045333 & 0.067218 & 0.06708 & 0.039378 & 0 \end{bmatrix} \tag{21.6.15}$$

12. Using the CN method for $x = 0.2, 0.4$ due to symmetry at $t = 0.2$

$$h = 0.2, \quad k = 0.04 \qquad\qquad r = \frac{k}{h^2} = 1 \tag{21.6.16}$$

$$\tilde{\mathbf{u}} = \begin{bmatrix} 0.033869 & 0.054798 \end{bmatrix} \qquad\qquad \mathbf{u}_{\text{Ex}} = \begin{bmatrix} 0.033091 & 0.053543 \end{bmatrix} \tag{21.6.17}$$

13. At the end of 5 time steps, with $h = 0.2$ and $r = 1$

$$\begin{bmatrix} 0 & 0.065375 & 0.106034 & 0.105646 & 0.06543 & 0 \end{bmatrix} \tag{21.6.18}$$

14. At the end of 20 time steps, with $h = 0.1$ and $r = 1$ giving $t = 0.2$.
For comparison, only the points $x = 0.2, 0.4, 0.6, 0.8$ are noted as in Problem 15.

$$\begin{bmatrix} 0 & 0.021376 & 0.034587 & 0.0.034587 & 0.021376 & 0 \end{bmatrix} \tag{21.6.19}$$

15. At the end of 5 time steps, with $h = 0.2$ and $r = 1$ giving $t = 0.2$

$$\begin{bmatrix} 0 & 0.021919 & 0.035467 & 0.035467 & 0.021919 & 0 \end{bmatrix} \tag{21.6.20}$$

The differences are larger at points farther from the boundary.

21.7 Method for Hyperbolic PDEs

1. Showing only the result for the last timestep,

$$h = 0.2, \qquad k = 0.2, \qquad r = \frac{k^2}{h^2} = 1 \tag{21.7.1}$$

$$\mathbf{u} = \begin{bmatrix} 0 & -0.05 & -0.1 & -0.15 & -0.2 & 0 \end{bmatrix} \tag{21.7.2}$$

2. Showing only the result for the last timestep,

$$h = 0.2, \qquad k = 0.2, \qquad r = \frac{k^2}{h^2} = 1 \tag{21.7.3}$$

$$\mathbf{u} = \begin{bmatrix} 0 & 0.032 & 0.096 & 0.144 & -0.128 & 0 \end{bmatrix} \tag{21.7.4}$$

3. Showing only the result for the last timestep,

$$h = 0.2, \qquad k = 0.2, \qquad r = \frac{k^2}{h^2} = 1 \tag{21.7.5}$$

$$\mathbf{u} = \begin{bmatrix} 0 & 0.032 & 0.048 & 0.48 & 0.032 & 0 \end{bmatrix} \tag{21.7.6}$$

4. Using D'Alembert's solution to the wave equation, with the initial position and displacement given by $f(x)$ and $g(x)$ respectively.

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \ \mathrm{d}s \tag{21.7.7}$$

$$c = 1, \qquad \Delta t = k \tag{21.7.8}$$

$$u(x_i, 1) = \frac{u_{i-1,0} + u_{i+1,0}}{2} + \frac{1}{2} \int_{x-k}^{x+k} g(s) \ \mathrm{d}s \tag{21.7.9}$$

If $g(s)$ is a constant function, then the integral is $2k\ g(x_i)$, and this equation simplifies to the one in this section in the text.

5. Showing only the result for the last timestep, using the symmetry of the problem to tabulate only the left half of the bar.

$$h = 0.1, \qquad k = 0.1, \qquad r = \frac{k^2}{h^2} = 1 \tag{21.7.10}$$

$$\begin{bmatrix} u(t = 0.1) \\ u(t = 0.2) \end{bmatrix} = \begin{bmatrix} 0 & 0.354492 & 0.766 & 1.271 & 1.678508 & 1.834017 & \dots \\ 0 & 0.575017 & 0.934508 & 1.135492 & 1.296 & 1.357017 & \dots \end{bmatrix} \tag{21.7.11}$$

6. Applying the B.C. at the left edge,

$$u_{01} = \frac{u_{-1,0} + u_{1,0}}{2} \qquad\qquad 2h \cdot 0.2j = u_{-1,j} - u_{1,j} \tag{21.7.12}$$

$$u_{0,j+1} = u_{-1,j} + u_{1,j} - u_{0,j-1} \qquad\qquad = 0.4jh + 2u_{1,j} - u_{0,j-1} \tag{21.7.13}$$

At the end of 5 time steps,

$$\mathbf{u} = \begin{bmatrix} 0.852 & 1.728 & 2.168 & 2.762 & 3.24 & 4 \end{bmatrix} \tag{21.7.14}$$

7. Using the given values, at $t = 0.4$

$$h = 0.2, \qquad k = 0.2, \qquad r = \frac{k^2}{h^2} = 1 \tag{21.7.15}$$

$$\mathbf{u} = \begin{bmatrix} 0 & 0.190211 & 0.307768 & 0.307768 & 0.190211 & 0 \end{bmatrix} \tag{21.7.16}$$

Using D'Alembert's solution to the wave equation,

$$u(x,\, t) = \frac{f(x+ct) - f(x-ct)}{2} + \frac{1}{2}\int_{x-ct}^{x+ct} g(s)\ \mathrm{d}s \tag{21.7.17}$$

$$u(x,\, t) = 0 + 0 + \frac{1}{2}\int x - tx + t\sin(\pi s)\ \mathrm{d}s \tag{21.7.18}$$

$$u(x,\, t) = \frac{1}{2\pi}\Bigg[\ \cos(\pi s)\Bigg]_{x+t}^{x-t} \tag{21.7.19}$$

$$u(x,\, t) = \frac{\sin(\pi x)\ \ \sin(\pi t)}{\pi} \tag{21.7.20}$$

$$\mathbf{u}_{\mathrm{Ex}} = \begin{bmatrix} 0 & 0.1779 & 0.2879 & 0.2879 & 0.1779 & 0 \end{bmatrix} \tag{21.7.21}$$

8. Comparing the accuracy upon using a finer grid, for the matching positions as in Problem 7,

$$\mathbf{u}_{0.2} = \begin{bmatrix} 0 & 0.190211 & 0.307768 & 0.307768 & 0.190211 & 0 \end{bmatrix} \tag{21.7.22}$$

$$\mathbf{u}_{0.1} = \begin{bmatrix} 0 & 0.180902 & 0.292705 & 0.292705 & 0.180902 & 0 \end{bmatrix} \tag{21.7.23}$$

$$\mathbf{u}_{\mathrm{Ex}} = \begin{bmatrix} 0 & 0.1779 & 0.2879 & 0.2879 & 0.1779 & 0 \end{bmatrix} \tag{21.7.24}$$

The smaller value of h gives a more accurate result.

9. Using the formula 8 in the text, the wave is

$$\begin{bmatrix} 0 & 0.354492 & 0.766 & 1.271 & 1.678508 & 1.834017 & \dots \end{bmatrix} \tag{21.7.25}$$

10. From D'Alembert's solution, with initial velocity being zero,

$$u(x,\, t) = \frac{f(x-t) + f(x+t)}{2} \qquad\qquad r^* = 1, \qquad k = h \tag{21.7.26}$$

$$u_{i1} = \frac{f(ih-h) + f(ih+h)}{2} \qquad\qquad u_{i1} = f(ih,\, h) = u(ih,\, k) \tag{21.7.27}$$

Thus, u_{i1} are the exact values $u(ih, k)$. Next, for $j = 2$,

$$u_{i2} = u_{i-1,1} + u_{i+1,1} - u_{i,0} \qquad = u(ih - h, k) + u(ih + h, k) - f(ih) \tag{21.7.28}$$

$$= \frac{f(ih - 2h) + f(ih + 2h)}{2} \qquad = u(ih, 2h) = u(ih, 2k) \tag{21.7.29}$$

Thus, u_{i2} are the exact values. Now, assuming the relation holds for u_{ij} and $u_{i,j-1}$,

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \tag{21.7.30}$$

$$= \frac{f(ih - h - jh) + f(ih + h + jh)}{2} = u(ih, jh + h) \tag{21.7.31}$$

Thus, $u_{i,j+1}$ are also the exact values. Since $j = 1$ and $j = 2$ are already shown to be exact, induction proves the relation for all higher values of j.