Chapter 10

Vector Integral Calculus, Integral Theorems

10.1 Line Integrals

1. Refer notes. TBC

2. Calculating the work done,

$$\mathbf{F} = \begin{bmatrix} y^2 \\ -x^2 \end{bmatrix} \qquad C : y = 4x^2 \tag{10.1.1}$$

$$\mathbf{r} = \begin{bmatrix} t \\ 4t^2 \end{bmatrix} \qquad P : (0, 0) \qquad Q : (1, 4) \tag{10.1.2}$$

$$\mathbf{r}' = \begin{bmatrix} 1 \\ 8t \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} 16t^4 \\ -t^2 \end{bmatrix} \tag{10.1.3}$$

$$W = \int_0^1 (16t^4 - 8t^3)\ \mathrm{d}t \qquad = \left[\frac{16t^5}{5} - 2t^4\right]_0^1 = 1.2\ \mathrm{J} \tag{10.1.4}$$

3. Calculating the work done,

$$\mathbf{F} = \begin{bmatrix} y^2 \\ -x^2 \end{bmatrix} \qquad C : y = 4x \tag{10.1.5}$$

$$\mathbf{r} = \begin{bmatrix} t \\ 4t \end{bmatrix} \qquad P : (0, 0) \qquad Q : (1, 4) \tag{10.1.6}$$

$$\mathbf{r}' = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} 16t^2 \\ -t^2 \end{bmatrix} \tag{10.1.7}$$

$$W = \int_0^1 12t^2 \, \mathrm{d}t \qquad = \Big[4t^3\Big]_0^1 = 4 \text{ J} \tag{10.1.8}$$

4. Calculating the work done,

$$\mathbf{F} = \begin{bmatrix} xy \\ x^2y^2 \end{bmatrix} \qquad C : y = -x + 2 \tag{10.1.9}$$

$$\mathbf{r} = \begin{bmatrix} t \\ 2 - t \end{bmatrix} \qquad P : (2, 0) \qquad Q : (0, 2) \tag{10.1.10}$$

$$\mathbf{r}' = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} 2t - t^2 \\ 4t^2 + t^4 - 4t^3 \end{bmatrix} \tag{10.1.11}$$

$$W = \int_2^0 (2t - 5t^2 + 4t^3 - t^4) \, \mathrm{d}t \qquad = \left[t^2 - \frac{5t^3}{3} + t^4 - \frac{t^5}{5}\right]_2^0 = \frac{-4}{15} \text{ J} \tag{10.1.12}$$

5. Calculating the work done,

$$\mathbf{F} = \begin{bmatrix} xy \\ x^2y^2 \end{bmatrix} \qquad C : y = \sqrt{4 - x^2} \tag{10.1.13}$$

$$\mathbf{r} = \begin{bmatrix} 2 \cos t \\ 2 \sin t \end{bmatrix} \qquad P : (2, 0) \qquad Q : (0, 2) \tag{10.1.14}$$

$$\mathbf{r}' = \begin{bmatrix} -2 \sin t \\ 2 \cos t \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} 4 \sin t \cos t \\ 16 \sin^2 t \cos^2 t \end{bmatrix} \tag{10.1.15}$$

Evaluating the integral

$$W = \int_0^{\pi/2} (-8\sin^2 t \cos t + 32\sin^2 t \cos^3 t)\, \mathrm{d}t \tag{10.1.16}$$

$$= \left[\frac{-8\sin^3 t}{3} + \frac{32\sin^3 t}{3} - \frac{32\sin^5 t}{5}\right]_0^{\pi/2} = \frac{8}{5}\ \mathrm{J} \tag{10.1.17}$$

6. Finding the integrand,

$$\mathbf{F} = \begin{bmatrix} x - y \\ y - z \\ z - x \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} 2\cos t \\ t \\ 2\sin t \end{bmatrix} \tag{10.1.18}$$

$$P : (2, 0, 0) \qquad Q : (2, 2\pi, 0) \qquad t \in [0, 2\pi] \tag{10.1.19}$$

$$\mathbf{r}' = \begin{bmatrix} -2\sin t \\ 1 \\ 2\cos t \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} 2\cos t - t \\ t - 2\sin t \\ 2\sin t - 2\cos t \end{bmatrix} \tag{10.1.20}$$

Evaluating the integral,

$$W = \int_0^{2\pi} \Big(-2\sin(2t) + 2t\sin t + t - 2\sin t + 2\sin(2t) - 4\cos^2 t \Big)\ \mathrm{d}t \tag{10.1.21}$$

$$= \int_0^{2\pi} \Big(2t\sin t + t - 2\sin t - 2 + 2\cos(2t) \Big)\ \mathrm{d}t \tag{10.1.22}$$

$$= \Big[-2t\cos t + 2\sin t + 0.5t^2 + 2\cos t - 2t + \sin(2t) \Big]_0^{2\pi} = 2\pi^2 - 8\pi \tag{10.1.23}$$

7. Finding the integrand,

$$\mathbf{F} = \begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} \cos t \\ \sin t \\ e^t \end{bmatrix} \tag{10.1.24}$$

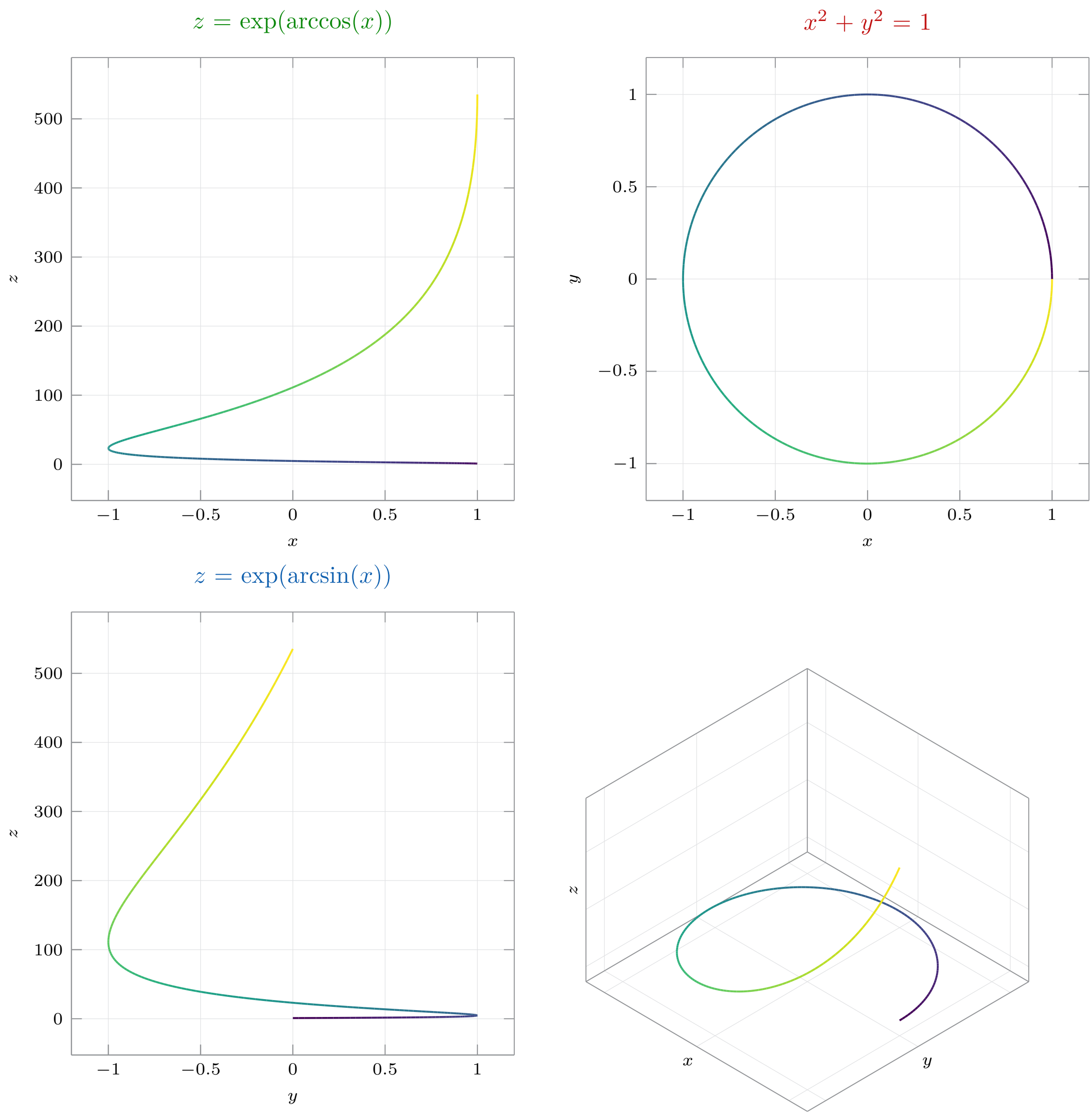
$$P : (1, 0, 1) \qquad Q : (1, 0, e^{2\pi}) \qquad t \in [0, 2\pi] \tag{10.1.25}$$

$$\mathbf{r}' = \begin{bmatrix} -\sin t \\ \cos t \\ e^t \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} \cos^2 t \\ \sin^2 t \\ e^{2t} \end{bmatrix} \tag{10.1.26}$$

Evaluating the integral,

$$W = \int_0^{2\pi} \Big(-\sin t \cos^2 t + \cos t \sin^2 t + e^{3t} \Big) \ \mathrm{d}t \tag{10.1.27}$$

$$= \left[\frac{\cos^3 t + \sin^3 t + e^{3t}}{3} \right]_0^{2\pi} = \frac{-1 + e^{6\pi}}{3} \tag{10.1.28}$$



$z = \exp(\arccos(x))$
$x^2 + y^2 = 1$
$z = \exp(\arcsin(x))$
x
y
z

8. Finding the integrand,

$$\mathbf{F} = \begin{bmatrix} e^x \\ \cosh\, y \\ \sinh\, z \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix} \tag{10.1.29}$$

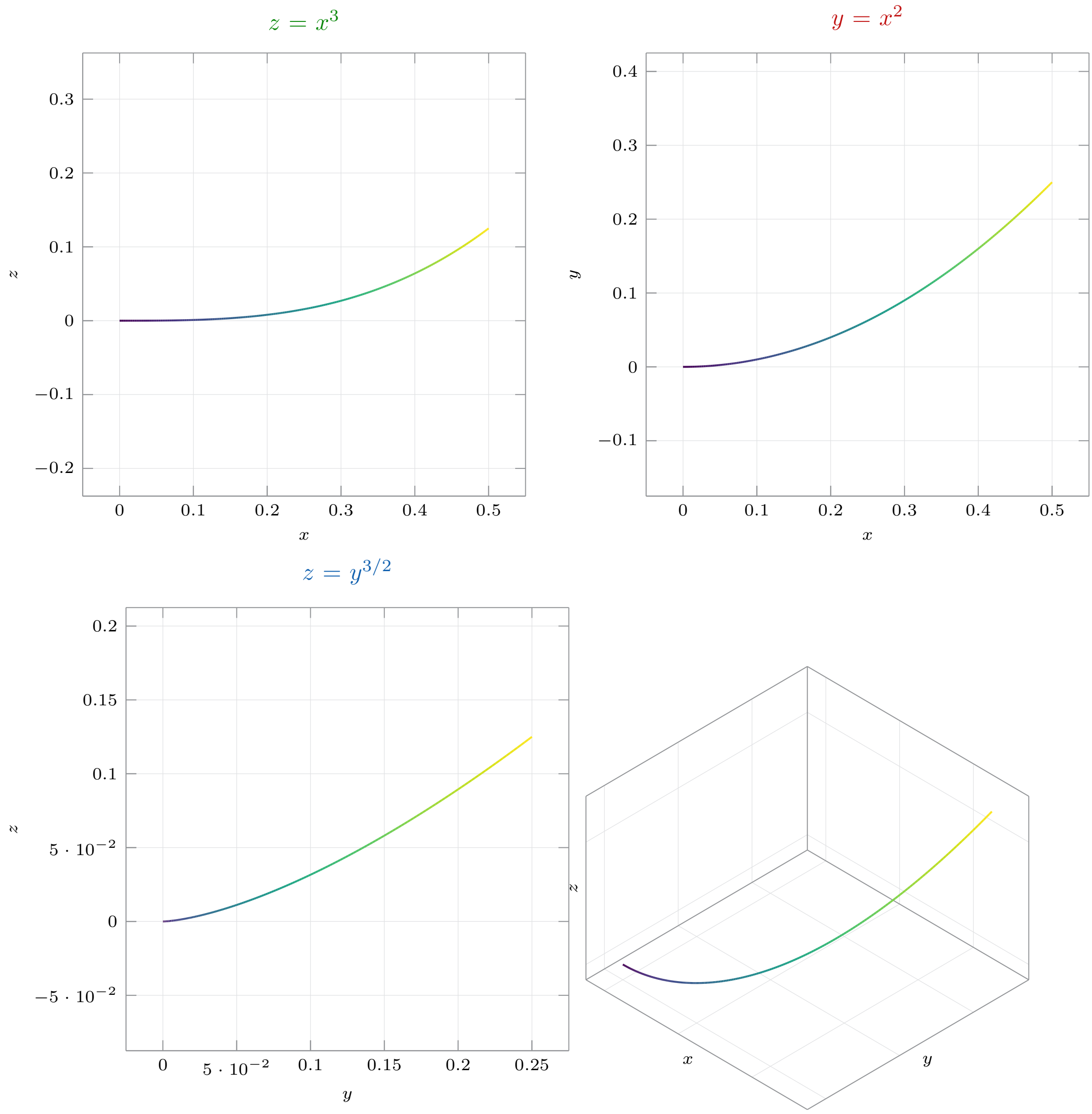
$$P : (0, 0, 0) \qquad Q : \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right) \qquad t \in [0, 1/2] \tag{10.1.30}$$

$$\mathbf{r}' = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} e^t \\ \cosh\, t^2 \\ \sinh\, t^3 \end{bmatrix} \tag{10.1.31}$$

Evaluating the integral,

$$W = \int_0^{1/2} \left(e^t + 2t\cosh(t^2) + 3t^2\sinh(t^3)\right)\, \mathrm{d}t \tag{10.1.32}$$

$$= \Big[e^t + \sinh(t^2) + \cosh(t^3)\Big]_0^{1/2} = e^{1/2} + \sinh(1/4) + \cosh(1/8) - 2 \tag{10.1.33}$$



9. Finding the integrand,

$$\mathbf{F} = \begin{bmatrix} x+y \\ y+z \\ z+x \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} 2t \\ 5t \\ t \end{bmatrix} \tag{10.1.34}$$

$$t \in [a, b] \tag{10.1.35}$$

$$\mathbf{r}' = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} 7t \\ 6t \\ 3t \end{bmatrix} \tag{10.1.36}$$

Evaluating the integral,

$$W = \int_a^b \left(47t\right) \;\mathrm{d}t \qquad\qquad = \left[\frac{47t^2}{2}\right]_a^b \tag{10.1.37}$$

$$t \in [0, 1] \implies W = 23.5 \text{ J} \qquad\qquad t \in [-1, 1] \implies W = 0 \text{ J} \tag{10.1.38}$$

10. Finding the integrand,

$$\mathbf{F} = \begin{bmatrix} x \\ -z \\ 2y \end{bmatrix} \qquad\qquad C_1 : \mathbf{r} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \quad t \in [0, 1] \tag{10.1.39}$$

$$C_2 : \mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix} \quad t \in [0, 1] \qquad\qquad C_3 : \mathbf{r} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} \quad t \in [1, 0] \tag{10.1.40}$$

Evaluating the integral,

$$W = \int_0^1 \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mathrm{d}t + \int_0^1 \begin{bmatrix} 1 \\ -t \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{d}t + \int_1^0 \begin{bmatrix} t \\ -t \\ 2t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathrm{d}t \tag{10.1.41}$$

$$= \left[\frac{t^2}{2} + 2t - t^2\right]_0^1 = \frac{3}{2}\text{J} \tag{10.1.42}$$

11. Finding the integrand,

$$\mathbf{F} = \begin{bmatrix} e^{-x} \\ e^{-y} \\ e^{-z} \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} t \\ t^2 \\ t \end{bmatrix} \tag{10.1.43}$$

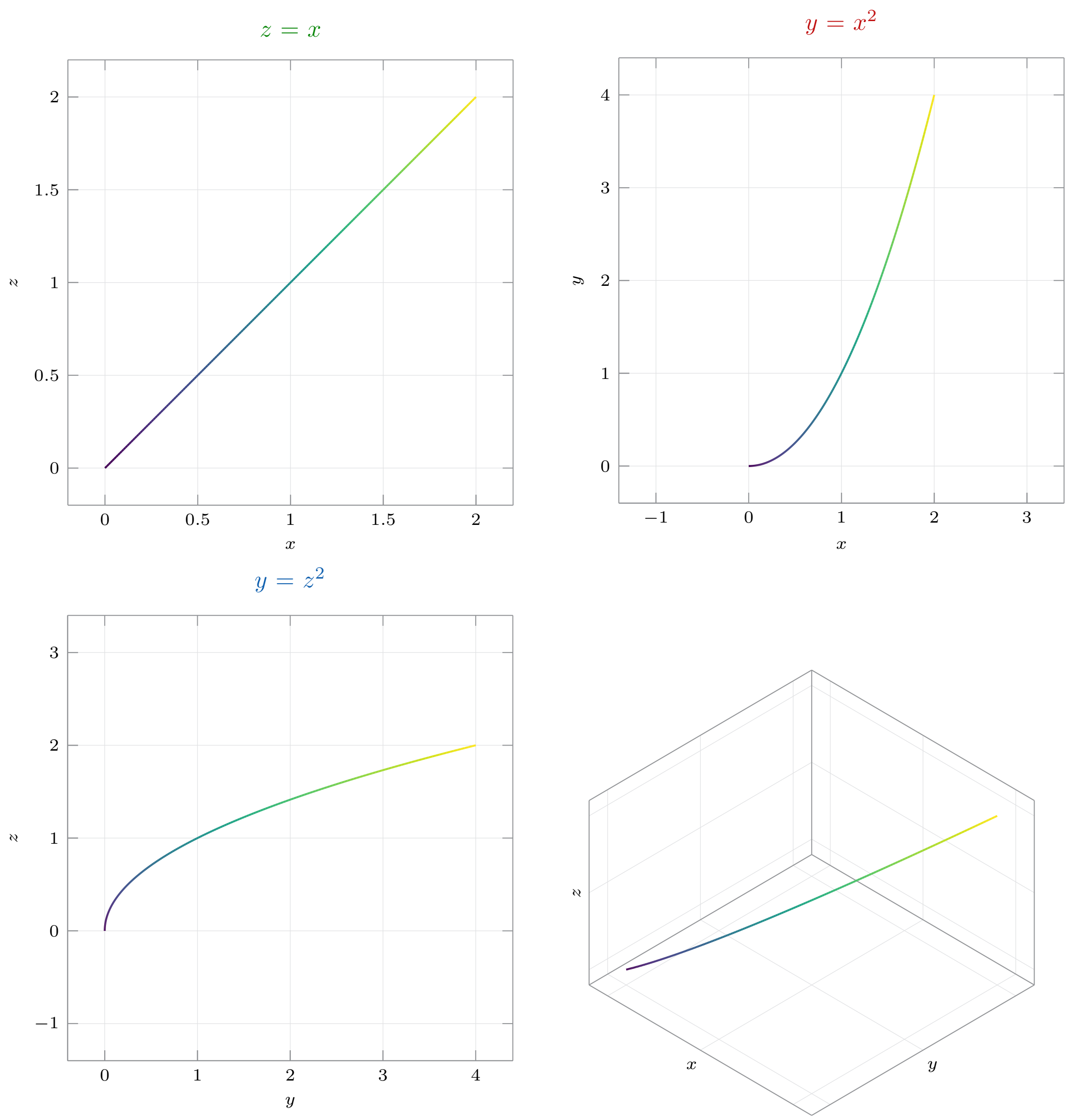
$$P : (0, 0, 0) \qquad Q : (2, 4, 2) \qquad t \in [0, 2] \tag{10.1.44}$$

$$\mathbf{r}' = \begin{bmatrix} 1 \\ 2t \\ 1 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} e^{-t} \\ e^{-t^2} \\ e^{-t} \end{bmatrix} \tag{10.1.45}$$

Evaluating the integral,

$$W = \int_0^2 \left(e^{-t} + 2te^{-t^2} + e^{-t}\right) \, \mathrm{d}t = \left[2e^{-t} + e^{-t^2}\right]_2^0 \tag{10.1.46}$$

$$= 3 - 2e^{-2} - e^{-4} \tag{10.1.47}$$



12. Change of parameter

(a) With the original parametrization

$$\mathbf{F} = \begin{bmatrix} xy \\ -y^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \qquad t \in [0, \pi/2] \tag{10.1.48}$$

$$\mathbf{r}' = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} \cos t \sin t \\ -\sin^2 t \end{bmatrix} \tag{10.1.49}$$

Evaluating the work done,

$$W = \int_0^{\pi/2} \left(-2 \sin^2 t \cos t\right) \, \mathrm{d}t = \left[\frac{-2 \sin^3 t}{3}\right]_0^{\pi/2} = \frac{-2}{3} \tag{10.1.50}$$

Change of parameter $t \to -p$

$$\mathbf{F} = \begin{bmatrix} xy \\ -y^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} \cos p \\ -\sin p \end{bmatrix} \qquad p \in [0, -\pi/2] \tag{10.1.51}$$

$$\mathbf{r}' = \begin{bmatrix} -\sin p \\ -\cos p \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} -\cos p \sin p \\ -\sin^2 p \end{bmatrix} \tag{10.1.52}$$

Evaluating the work done,

$$W = \int_0^{-\pi/2} \left(2\sin^2 p \cos p\right) \, \mathrm{d}p = \left[\frac{2\sin^3 p}{3}\right]_0^{-\pi/2} = \frac{-2}{3} \tag{10.1.53}$$

Change of parameter $t \to p^2$

$$\mathbf{F} = \begin{bmatrix} xy \\ -y^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} \cos p^2 \\ \sin p^2 \end{bmatrix} \qquad p \in [0, \sqrt{\pi/2}] \tag{10.1.54}$$

$$\mathbf{r}' = \begin{bmatrix} -2p\sin p^2 \\ 2p\cos p^2 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} \cos p^2 \sin p^2 \\ -\sin^2(p^2) \end{bmatrix} \tag{10.1.55}$$

Evaluating the work done,

$$W = \int_0^{\sqrt{\pi/2}} \left(-4p\sin^2(p^2)\cos(p^2)\right) \, \mathrm{d}p = -2\left[\frac{\sin^3(p^2)}{3}\right]_0^{\sqrt{\pi/2}} = \frac{-2}{3} \tag{10.1.56}$$

(b) Path with parameter n having the same start and end points.

$$\mathbf{F} = \begin{bmatrix} xy \\ -y^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} t \\ t^n \end{bmatrix} \qquad p \in [0, 1] \tag{10.1.57}$$

$$\mathbf{r}' = \begin{bmatrix} 1 \\ nt^{n-1} \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} t^{n+1} \\ -t^{2n} \end{bmatrix} \tag{10.1.58}$$

Evaluating the work done,

$$W = \int_0^1 \left(t^{n+1} - nt^{3n-1}\right) \, \mathrm{d}t = \left[\frac{t^{n+2}}{n+2} - \frac{t^{3n}}{3}\right]_0^1 = \frac{1}{n+2} - \frac{1}{3} \tag{10.1.59}$$

(c) In the limit of $n \to \infty$, the integration result is $-1/3$.
Direct integration does not yield the same result because the path $y = x^n$ tends to zero identically

over $x \in [0, 1]$, which makes the integrand zero.

13. Consider the infinitesimal work done,

$$\left| \int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} \right| \leq \int_C |\mathbf{F} \cdot \, \mathrm{d}\mathbf{r}| \leq \int_C |\mathbf{F}| \cdot | \, \mathrm{d}\mathbf{r}| \leq ML \tag{10.1.60}$$

The absolute value of an integral is less than the integral of the absolute value.
The absolute dot product of two vectors is less than the product of their magnitudes.
The integration of individual $| \, \mathrm{d}\mathbf{r}|$ over the path is the path length L.

14. Path with parameter n having the same start and end points.

$$\mathbf{F} = \begin{bmatrix} x^2 \\ y \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 3t \\ 4t \end{bmatrix} \qquad p \in [0, 1] \tag{10.1.61}$$

$$\mathbf{r}' = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} 9t^2 \\ 4t \end{bmatrix} \tag{10.1.62}$$

Evaluating the work done,

$$W = \int_0^1 \left(27t^2 + 16t\right) \, \mathrm{d}t = \left[9t^3 + 8t^2\right]_0^1 = 17 \tag{10.1.63}$$

Using the ML inequality, the upper bound is,

$$L = 5 \qquad |\mathbf{F}| = \sqrt{x^4 + y^2} \tag{10.1.64}$$

$$M = \sqrt{97} \qquad ML = 49.24 \tag{10.1.65}$$

15. Integrating the vector function over the path,

$$F = \begin{bmatrix} y^2 \\ z^2 \\ x^2 \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} 3 \cos t \\ 3 \sin t \\ 2t \end{bmatrix} \qquad t \in [0, 4\pi] \tag{10.1.66}$$

$$W_x = \int_0^{4\pi} 9 \sin^2 t \, \, \mathrm{d}t \qquad = \left[\frac{9t}{2} - \frac{9 \sin(2t)}{4}\right]_0^{4\pi} = 18\pi \tag{10.1.67}$$

$$W_y = \int_0^{4\pi} 4t^2 \, \, \mathrm{d}t \qquad = \left[\frac{4t^3}{3}\right]_0^{4\pi} = \frac{256\pi^3}{3} \tag{10.1.68}$$

$$W_z = \int_0^{4\pi} 9 \cos^2 t \, \, \mathrm{d}t \qquad = \left[\frac{9t}{2} + \frac{9 \sin(2t)}{4}\right]_0^{4\pi} = 18\pi \tag{10.1.69}$$

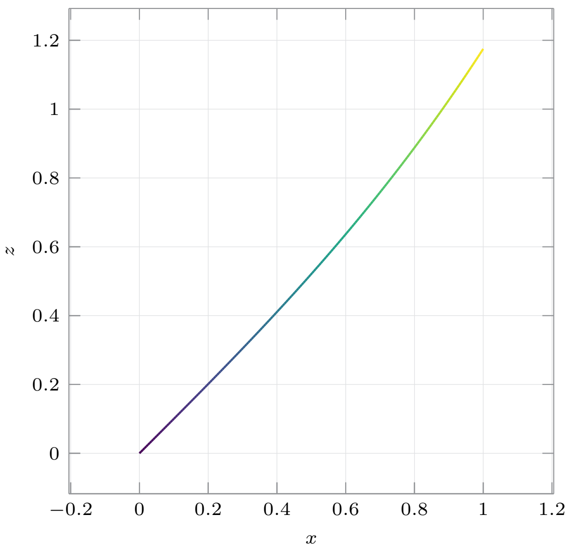
16. Integrating the vector function over the path,

$$F = \begin{bmatrix} 3x + y + 5z \\ 0 \\ 0 \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} t \\ \cosh t \\ \sinh t \end{bmatrix} \quad t \in [0,\, 1] \tag{10.1.70}$$

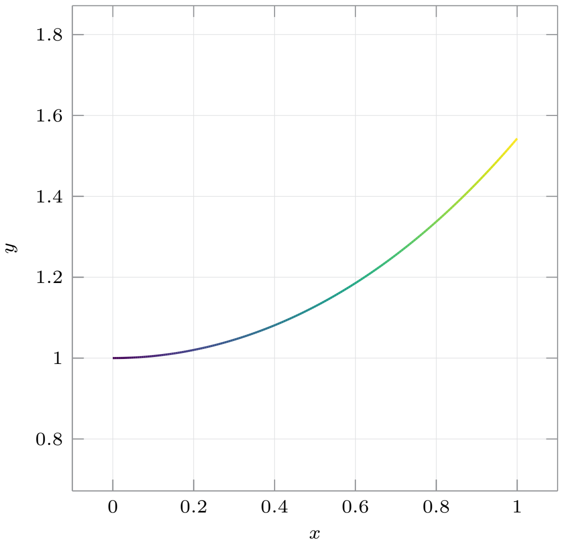
$$W_x = \int_0^1 \Big(3t + \cosh t + 5 \sinh t\Big)\ \mathrm{d}t \qquad = \left[\frac{3t^2}{2} + \sinh t + 5 \cosh t\right]_0^1 \tag{10.1.71}$$

$$= \sinh(1) + 5\cosh(1) - 3.5 \tag{10.1.72}$$

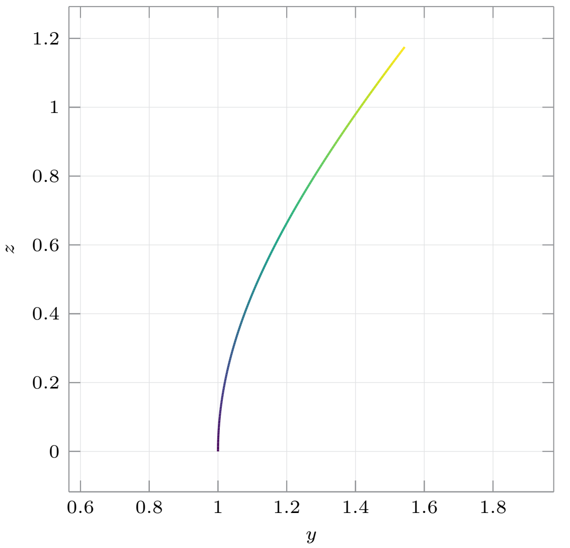
$z = \sinh x$

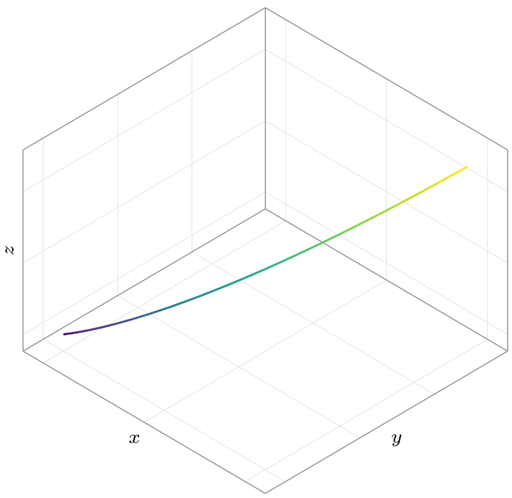


$y = \cosh x$



$y^2 - z^2 = 1$





17. Integrating the vector function over the path,

$$F = \begin{bmatrix} x+y \\ y+z \\ z+x \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} 4\cos t \\ \sin t \\ 0 \end{bmatrix} \qquad t \in [0, \pi] \tag{10.1.73}$$

$$W_x = \int_0^\pi \left(4\cos t + \sin t\right)\, \mathrm{d}t \qquad = \Big[4\sin t - \cos t\Big]_0^\pi = 2 \tag{10.1.74}$$

$$W_y = \int_0^\pi \left(\sin t\right)\, \mathrm{d}t \qquad = \Big[-\cos(t)\Big]_0^\pi = 2 \tag{10.1.75}$$

$$W_z = \int_0^\pi \left(4\cos t\right)\, \mathrm{d}t \qquad = \Big[4\sin t\Big]_0^\pi = 0 \tag{10.1.76}$$

18. Integrating the vector function over the path,

$$F = \begin{bmatrix} y^{1/3} \\ x^{1/3} \\ 0 \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} \cos^3 t \\ \sin^3 t \\ 0 \end{bmatrix} \qquad t \in [0, \pi/4] \tag{10.1.77}$$

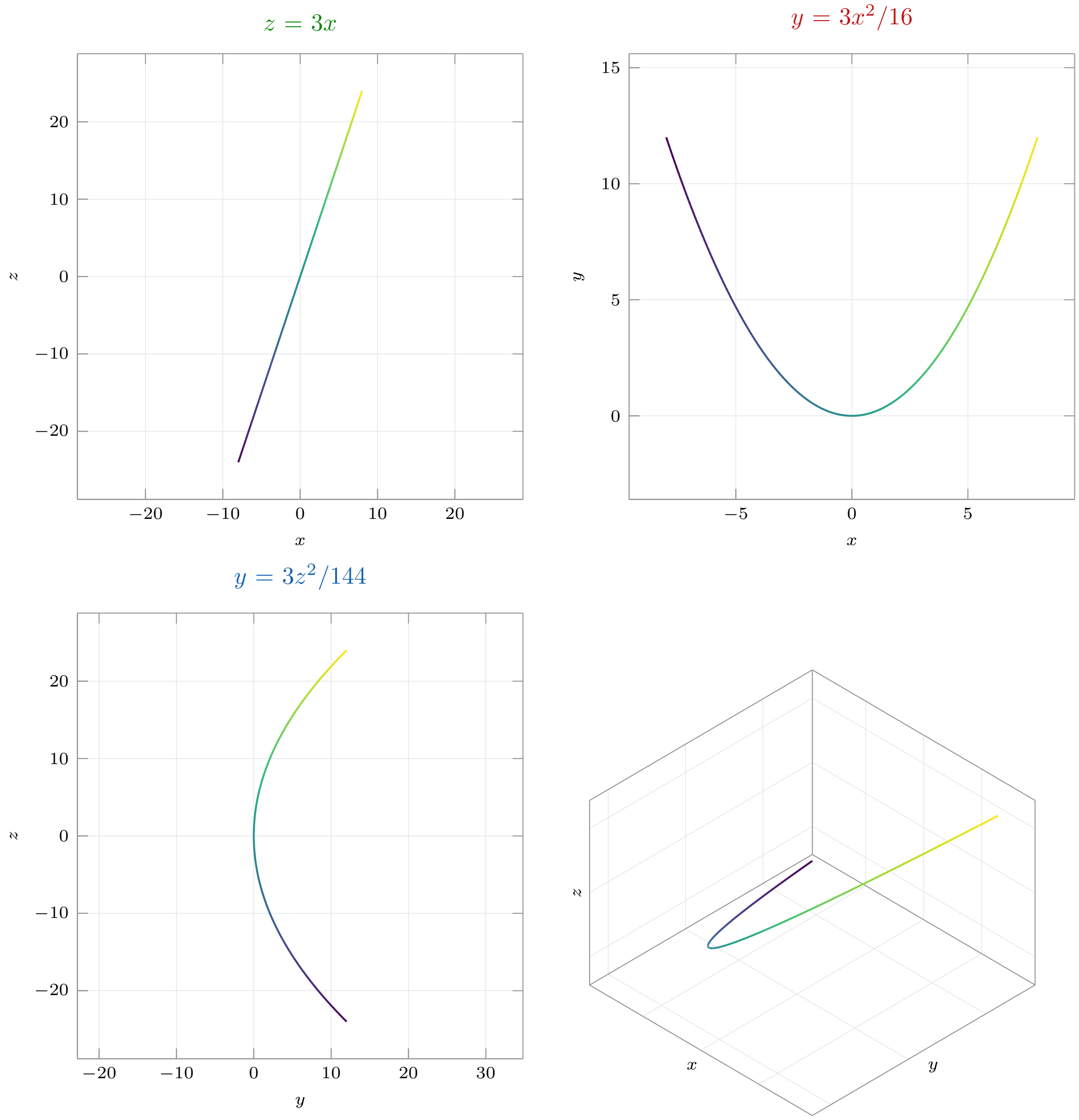
$$W_x = \int_0^{\pi/4} \left(\sin t\right)\, \mathrm{d}t \qquad = \Big[-\cos t\Big]_0^{\pi/4} = 1 - \frac{1}{\sqrt{2}} \tag{10.1.78}$$

$$W_y = \int_0^{\pi/4} \left(\cos t\right)\, \mathrm{d}t \qquad = \Big[\sin t\Big]_0^{\pi/4} = \frac{1}{\sqrt{2}} \tag{10.1.79}$$

19. Integrating the vector function over the path,

$$F = \begin{bmatrix} xyz \\ 0 \\ 0 \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} 4t \\ 3t^2 \\ 12t \end{bmatrix} \qquad t \in [-2, 2] \tag{10.1.80}$$

$$W_x = \int_{-2}^{2} \left(144t^4\right)\, \mathrm{d}t \qquad = \left[\frac{144t^5}{5}\right]_{-2}^{2} = 1843.2 \tag{10.1.81}$$



20. Integrating the vector function over the path,

$$F = \begin{bmatrix} xy \\ yz \\ x^2y^2 \end{bmatrix} \qquad C : \mathbf{r} = \begin{bmatrix} t \\ t \\ e^t \end{bmatrix} \quad t \in [0, 5] \tag{10.1.82}$$

$$W_x = \int_0^5 \left(t^2\right) \mathrm{d}t \qquad = \left[\frac{t^3}{3}\right]_0^5 = \frac{125}{3} \tag{10.1.83}$$

$$W_y = \int_0^5 \left(te^t\right) \mathrm{d}t \qquad = \left[(t-1)e^t\right]_0^5 = 4e^5 + 1 \tag{10.1.84}$$

$$W_z = \int_0^5 \left(t^4\right) \mathrm{d}t \qquad = \left[\frac{t^5}{5}\right]_0^5 = 625 \tag{10.1.85}$$

10.2 Path Independence of Line Integrals

1. Refer notes. TBC.
2. The domain is still not simply connected, as the origin is still excluded from it. The situation does not change.
3. The integrand is exact

$$I = M \,\mathrm{d}x + N \,\mathrm{d}y \qquad I = \partial_x f \;\; \mathrm{d}x + \partial_y f \;\; \mathrm{d}y \tag{10.2.1}$$

$$\int M \,\mathrm{d}x = \sin(0.5x)\cos(2y) + g(y) \qquad \partial_y f = \frac{\mathrm{d}g}{\mathrm{d}y} - 2\sin(0.5x)\sin(2y) \tag{10.2.2}$$

$$\partial_y f = N \qquad \frac{\mathrm{d}g}{\mathrm{d}y} = 0 \tag{10.2.3}$$

$$f = \sin(0.5x)\cos(2y) \qquad A : (\pi/2, \pi) \quad B : (\pi, 0) \tag{10.2.4}$$

$$I = f(B) - f(A) = \Bigg[\sin(0.5x)\cos(2y)\Bigg]_A^B \qquad I = 1 - \frac{1}{\sqrt{2}} \tag{10.2.5}$$

4. The integrand is exact

$$I = M \,\mathrm{d}x + N \,\mathrm{d}y \qquad I = \partial_x f \;\; \mathrm{d}x + \partial_y f \;\; \mathrm{d}y \tag{10.2.6}$$

$$\int M \,\mathrm{d}x = x^2 e^{4y} + g(y) \qquad \partial_y f = \frac{\mathrm{d}g}{\mathrm{d}y} + 4x^2 e^{4y} \tag{10.2.7}$$

$$\partial_y f = N \qquad \frac{\mathrm{d}g}{\mathrm{d}y} = 0 \tag{10.2.8}$$

$$f = x^2 e^{4y} \qquad A : (4, 0) \quad B : (6, 1) \tag{10.2.9}$$

$$I = f(B) - f(A) = \Bigg[x^2 e^{4y}\Bigg]_A^B \qquad I = 36e^4 - 16 \tag{10.2.10}$$

5. The integrand is exact

$$I = M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z = \partial_x f\ \ \mathrm{d}x + \partial_y f\ \ \mathrm{d}y + \partial_z f\ \ \mathrm{d}z \tag{10.2.11}$$

$$\int M\,\mathrm{d}x = f = e^{xy}\sin(z) + g(y,z) \tag{10.2.12}$$

$$\partial_y f = N = \frac{\partial g}{\partial y} + xe^{xy}\sin(z) \quad\implies\quad \frac{\partial g}{\partial y} = 0 \tag{10.2.13}$$

$$g = h(z) \tag{10.2.14}$$

$$\partial_z f = P = \frac{\mathrm{d}h}{\mathrm{d}z} + e^{xy}\cos(z) \quad\implies\quad \frac{\mathrm{d}h}{\mathrm{d}z} = 0 \tag{10.2.15}$$

$$f = e^{xy}\sin(z) \qquad A:(0,0,\pi) \qquad B:(2,1/2,\pi/2) \tag{10.2.16}$$

$$I = f(B) - f(A) = \Big[e^{xy}\sin(z)\Big]_A^B = e \tag{10.2.17}$$

6. The integrand is exact

$$I = M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z = \partial_x f\ \ \mathrm{d}x + \partial_y f\ \ \mathrm{d}y + \partial_z f\ \ \mathrm{d}z \tag{10.2.18}$$

$$\int M\,\mathrm{d}x = f = 0.5\cdot\exp(x^2+y^2+z^2) + g(y,z) \tag{10.2.19}$$

$$\partial_y f = N = \frac{\partial g}{\partial y} + y\exp(x^2+y^2+z^2) \quad\implies\quad \frac{\partial g}{\partial y} = 0 \tag{10.2.20}$$

$$g = h(z) \tag{10.2.21}$$

$$\partial_z f = P = \frac{\mathrm{d}h}{\mathrm{d}z} + z\exp(x^2+y^2+z^2) \quad\implies\quad \frac{\mathrm{d}h}{\mathrm{d}z} = 0 \tag{10.2.22}$$

$$f = 0.5\cdot e^{x^2+y^2+z^2} \qquad A:(0,0,0) \qquad B:(1,1,0) \tag{10.2.23}$$

$$I = f(B) - f(A) = \Big[e^{xy}\sin(z)\Big]_A^B = \frac{e^2-1}{2} \tag{10.2.24}$$

7. The integrand is exact

$$I = M\ \mathrm{d}x + N\ \mathrm{d}y + P\ \mathrm{d}z = \partial_x f \ \ \mathrm{d}x + \partial_y f \ \ \mathrm{d}y + \partial_z f \ \ \mathrm{d}z \quad 10.2.25$$

$$\int M\ \mathrm{d}x = f = y\cosh(xz) + g(y, z) \quad 10.2.26$$

$$\partial_y f = N = \frac{\partial g}{\partial y} + \cosh(xz) \quad \Longrightarrow \quad \frac{\partial g}{\partial y} = 0 \quad 10.2.27$$

$$g = h(z) \quad 10.2.28$$

$$\partial_z f = P = \frac{\mathrm{d}h}{\mathrm{d}z} + xy\sinh(xz) \quad \Longrightarrow \quad \frac{\mathrm{d}h}{\mathrm{d}z} = 0 \quad 10.2.29$$

$$f = y\cosh(xz) \qquad A : (0, 2, 3) \qquad B : (1, 1, 1) \quad 10.2.30$$

$$I = f(B) - f(A) = \Big[y\cosh(xz) \Big]_A^B = \cosh(1) - 2 \quad 10.2.31$$

8. The integrand is exact

$$I = M\ \mathrm{d}x + N\ \mathrm{d}y + P\ \mathrm{d}z = \partial_x f \ \ \mathrm{d}x + \partial_y f \ \ \mathrm{d}y + \partial_z f \ \ \mathrm{d}z \quad 10.2.32$$

$$\int M\ \mathrm{d}x = f = x\cos(yz) + g(y, z) \quad 10.2.33$$

$$\partial_y f = N = \frac{\partial g}{\partial y} - xz\sin(yz) \quad \Longrightarrow \quad \frac{\partial g}{\partial y} = 0 \quad 10.2.34$$

$$g = h(z) \quad 10.2.35$$

$$\partial_z f = P = \frac{\mathrm{d}h}{\mathrm{d}z} - xy\sin(yz) \quad \Longrightarrow \quad \frac{\mathrm{d}h}{\mathrm{d}z} = 0 \quad 10.2.36$$

$$f = x\cos(yz) \qquad A : (5, 3, \pi) \qquad B : (3, \pi, 3) \quad 10.2.37$$

$$I = f(B) - f(A) = \Big[x\cos(yz) \Big]_A^B = 2 \quad 10.2.38$$

9. The integrand is exact

$$I = M \, \mathrm{d}x + N \, \mathrm{d}y + P \, \mathrm{d}z = \partial_x f \ \ \mathrm{d}x + \partial_y f \ \ \mathrm{d}y + \partial_z f \ \ \mathrm{d}z \tag{10.2.39}$$

$$\int M \, \mathrm{d}x = f = e^x \cosh(y) + g(y, z) \tag{10.2.40}$$

$$\partial_y f = N = \frac{\partial g}{\partial y} + e^x \sinh(y) \quad \Longrightarrow \quad \frac{\partial g}{\partial y} = e^z \cosh(y) \tag{10.2.41}$$

$$g = e^z \sinh(y) + h(z) \tag{10.2.42}$$

$$\partial_z f = P = \frac{\mathrm{d}h}{\mathrm{d}z} + e^z \sinh(y) \quad \Longrightarrow \quad \frac{\mathrm{d}h}{\mathrm{d}z} = 0 \tag{10.2.43}$$

$$f = e^x \cosh(y) + e^z \sinh(y) \qquad A : (0, 1, 0) \qquad B : (1, 0, 1) \tag{10.2.44}$$

$$I = f(B) - f(A) = \Big[e^x \cosh(y) + e^z \sinh(y) \Big]_A^B = 0 \tag{10.2.45}$$

10. Path Dependence

(a) Checking path dependence,

$$\mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = x^2 y \ \ \mathrm{d}x + 2xy^2 \ \ \mathrm{d}y \qquad \mathbf{F} = \begin{bmatrix} x^2 y \\ 2xy^2 \\ 0 \end{bmatrix} \tag{10.2.46}$$

$$\nabla \times \mathbf{F} = 2y^2 - x^2 \qquad \nabla \times \mathbf{F} \neq 0 \tag{10.2.47}$$

Since the vector field has nonzero curl, the line integral is path dependent.

(b) Integrating along the first path,

$$\mathbf{F} = \begin{bmatrix} x^2 y \\ 2xy^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} t \\ bt \end{bmatrix} \quad t \in [0, 1] \tag{10.2.48}$$

$$\mathbf{r}' = \begin{bmatrix} 1 \\ b \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} bt^3 \\ 2b^2 t^3 \end{bmatrix} \tag{10.2.49}$$

$$W = \int_0^1 (b + 2b^3) t^3 \ \ \mathrm{d}t \qquad = \left[\frac{b + 2b^3}{4} \ t^4 \right]_0^1 \tag{10.2.50}$$

$$= \frac{b + 2b^3}{4} \tag{10.2.51}$$

Integrating the second path

$$\mathbf{F} = \begin{bmatrix} x^2y \\ 2xy^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 1 \\ t \end{bmatrix} \quad t \in [b, 1] \tag{10.2.52}$$

$$\mathbf{r}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} t \\ 2t^2 \end{bmatrix} \tag{10.2.53}$$

$$W = \int_b^1 2t^2 \ \mathrm{d}t \qquad = \left[\frac{2t^3}{3}\right]_b^1 \tag{10.2.54}$$

$$= \frac{2(1-b^3)}{3} \tag{10.2.55}$$

Optimizing the line integral w.r.t. b,

$$I = \frac{2}{3} + \frac{b}{4} - \frac{b^3}{6} \qquad \frac{\mathrm{d}I}{\mathrm{d}b} = \frac{1}{4} - \frac{b^2}{2} \tag{10.2.56}$$

$$b^* = \frac{1}{\sqrt{2}} \qquad I^* = \frac{2}{3} + \frac{1}{6\sqrt{2}} \tag{10.2.57}$$

(c) Integrating along the third path,

$$\mathbf{F} = \begin{bmatrix} x^2y \\ 2xy^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} ct \\ t \end{bmatrix} \quad t \in [0, 1] \tag{10.2.58}$$

$$\mathbf{r}' = \begin{bmatrix} c \\ 1 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} c^2t^3 \\ 2ct^3 \end{bmatrix} \tag{10.2.59}$$

$$W = \int_0^1 (c^3 + 2c)t^3 \ \mathrm{d}t \qquad = \left[\frac{c^3 + 2c}{4}\ t^4\right]_0^1 \tag{10.2.60}$$

$$= \frac{c^3 + 2c}{4} \tag{10.2.61}$$

Integrating the fourth path

$$\mathbf{F} = \begin{bmatrix} x^2y \\ 2xy^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} t \\ 1 \end{bmatrix} \quad t \in [c, 1] \tag{10.2.62}$$

$$\mathbf{r}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{F}(\mathbf{r}(t)) = \begin{bmatrix} t^2 \\ 2t \end{bmatrix} \tag{10.2.63}$$

$$W = \int_c^1 t^2 \ \mathrm{d}t \qquad = \left[\frac{t^3}{3}\right]_c^1 \tag{10.2.64}$$

$$= \frac{(1-c^3)}{3} \tag{10.2.65}$$

Optimizing the line integral w.r.t. c,

$$I = \frac{4 - c^3 + 6c}{12} \qquad \frac{\mathrm{d}I}{\mathrm{d}c} = \frac{-3c^2 + 6}{12} \tag{10.2.66}$$

$$c^* = \sqrt{2} \quad \text{(out of range [0,1])} \tag{10.2.67}$$

Since the optimal c is out of range, setting $c = 1$ gives $I^* = 3/4$.
Comparing the values at $b = 1$ and $c = 1$

$$b = 1 \implies I_{12} = 3/4 \tag{10.2.68}$$

$$c = 1 \implies I_{34} = 3/4 \tag{10.2.69}$$

11. Checking the differential form in Example 4,

$$F_1 = \frac{-y}{x^2+y^2} \qquad F_2 = \frac{x}{x^2+y^2} \tag{10.2.70}$$

$$\frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad \frac{\partial F_1}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \tag{10.2.71}$$

$$\frac{\partial F_2}{\partial y} = \frac{\partial F_1}{\partial y} \tag{10.2.72}$$

The form is exact. Finding the underlying scalar function,

$$\int F_1 \ \mathrm{d}x = -\arctan(x/y) + C + g(y) \qquad \frac{\partial f}{\partial y} = \frac{\mathrm{d}g}{\mathrm{d}y} + \frac{x}{x^2+y^2} \tag{10.2.73}$$

$$g(y) = 0 \qquad f = \arctan(y/x) \tag{10.2.74}$$

For $x = y = 0$, the function $\arctan(y/x)$ is not defined. So any domain not including this point in $\mathcal{R}$ is acceptable.

12. The centres of the circles lie on the perpendicular bisector of A and

$$l_1 : y = x \qquad l_2 : y = 1 - x \tag{10.2.75}$$

$$C : (\alpha, \beta) = (\alpha, 1 - \alpha) \qquad r^2 = (\alpha - 0.5)^2 + (0.5 - \alpha)^2 + 0.25 \tag{10.2.76}$$

$$= \alpha^2 - 2\alpha + 0.75 \tag{10.2.77}$$

Integrating the vector function $\mathbf{F}$ over the this circle,

$$\mathbf{F} = \begin{bmatrix} x^2 y \\ 2xy^2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} \alpha + r\cos t \\ (1 - \alpha) + r\sin t \end{bmatrix} \quad t \in [0, 2\pi] \tag{10.2.78}$$

$$\mathbf{r}' = \begin{bmatrix} -r\sin t \\ r\cos t \end{bmatrix} \tag{10.2.79}$$

Calculating the line integral using a CAS,

$$W = \frac{\pi r^2}{4}\left(4\alpha^2 - 16\alpha + 8 + r^2\right) \tag{10.2.80}$$

$$W = \frac{\pi}{4}\left(\alpha^2 - 2\alpha + 0.75\right)\left(5\alpha^2 - 18\alpha + 8.75\right) \tag{10.2.81}$$

This is a fourth order polynomial in α. It has one local maximum and no global maximum.

$$W^* = 0.954 \qquad (\alpha^*, \beta^*) = (1.143, -0.143) \tag{10.2.82}$$

13. Checking path independence,

$$\mathbf{F} = \begin{bmatrix} 2xe^{x^2}\cos(2y) \\ -2e^{x^2}\sin(2y) \\ 0 \end{bmatrix} \tag{10.2.83}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 2xe^{x^2}\cos(2y) & -2e^{x^2}\sin(2y) & 0 \end{vmatrix} \tag{10.2.84}$$

$$= \mathbf{0} \tag{10.2.85}$$

This line integral is path independent, and the value of the integral is

$$I = f(b) - f(a) = \left[e^{x^2} \cos(2y) \right]_A^B = e^{a^2} \cos(2b) - 1 \tag{10.2.86}$$

14. Checking path independence,

$$\mathbf{F} = \begin{bmatrix} z \sinh(xy) \\ 0 \\ -x \sinh(xy) \end{bmatrix} \tag{10.2.87}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ z \sinh(xy) & 0 & -x \sinh(xy) \end{vmatrix} = \begin{bmatrix} -x^2 \cosh(xy) \\ 2 \sinh(xy) + xy \cosh(xy) \\ xz \cosh(xy) \end{bmatrix} \tag{10.2.88}$$

This line integral is path dependent.

15. Checking path independence,

$$\mathbf{F} = \begin{bmatrix} x^2 y \\ -4xy^2 \\ 8z^2 x \end{bmatrix} \tag{10.2.89}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ x^2 y & -4xy^2 & 8z^2 x \end{vmatrix} = \begin{bmatrix} 0 \\ -8z^2 \\ -4y^2 + x^2 \end{bmatrix} \tag{10.2.90}$$

This line integral is path dependent.

16. Checking path independence,

$$\mathbf{F} = \begin{bmatrix} e^y \\ xe^y - e^z \\ -ye^z \end{bmatrix} \tag{10.2.91}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ e^y & xe^y - e^z & -ye^z \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10.2.92}$$

This line integral is path independent, and the value of the integral is

$$I = f(b) - f(a) = \Big[xe^y - ye^z \Big]_A^B = ae^b - be^c \tag{10.2.93}$$

17. Checking path independence,

$$\mathbf{F} = \begin{bmatrix} 4y \\ z \\ y - 2z \end{bmatrix} \tag{10.2.94}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 4y & z & y - 2z \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} \tag{10.2.95}$$

This line integral is path dependent.

18. Checking path independence,

$$\mathbf{F} = \begin{bmatrix} yz\,\cos(xy) \\ xz\,\cos(xy) \\ -2\,\sin(xy) \end{bmatrix} \tag{10.2.96}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ yz\,\cos(xy) & xz\,\cos(xy) & -2\,\sin(xy) \end{vmatrix} = \begin{bmatrix} -x\,\cos(xy) \\ 2y\,\cos(xy) \\ 0 \end{bmatrix} \tag{10.2.97}$$

This line integral is path dependent.

19. Checking path independence, using $w = x^2 + 2y^2 + z^2$

$$\mathbf{F} = \cos(w) \begin{bmatrix} 2x \\ 4y \\ 2z \end{bmatrix} \tag{10.2.98}$$

$$\nabla \times \mathbf{F} = \cos(w) \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 2x & 4y & 2z \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10.2.99}$$

This line integral is path independent, and the value of the integral is

$$I = f(b) - f(a) = \Big[\sin(x^2 + 2y^2 + z^2) \Big]_A^B = \sin(a^2 + 2b^2 + c^2) \tag{10.2.100}$$

20. The three conditions are,

$$\partial_y F_3 = \partial_z F_2 \qquad\qquad \partial_z F_1 = \partial_x F_3 \tag{10.2.101}$$

$$\partial_x F_2 = \partial_y F_1 \qquad\qquad \mathbf{F} = \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \tag{10.2.102}$$

Other simple examples have a different permutation of zero components.

10.3 Calculus Review: Double Integrals

1. TBC.

2. The space between the lines $y = x$ and $y = 2x$ bounded by the vertical lines $x = 0$ and $x = 2$. Trapezium

$$I_1 = \int_x^{2x} x^2 + 2xy + y^2 \ \mathrm{d}y \qquad\qquad I_1 = \left[x^2y + xy^2 + \frac{y^3}{3} \right]_x^{2x} \tag{10.3.1}$$

$$I_1 = x^3 + 3x^3 + \frac{7x^3}{3} = \frac{19x^3}{3} \tag{10.3.2}$$

$$I = \int_0^2 \frac{19x^3}{3} \ \mathrm{d}x \qquad\qquad = \left[\frac{19x^4}{12} \right]_0^2 = \frac{76}{3} \tag{10.3.3}$$

3. The space between the lines $x = -y$ and $x = y$ bounded by the horizontal lines $y = 0$ and $y = 3$. Triangle

$$I_2 = \int_{-y}^{y} x^2 + y^2 \ \mathrm{d}x \qquad\qquad I_2 = \left[\frac{x^3}{3} + xy^2 \right]_{-y}^{y} \tag{10.3.4}$$

$$I_2 = \frac{2y^3}{3} + 2y^3 = \frac{8y^3}{3} \tag{10.3.5}$$

$$I = \int_0^3 \frac{8y^3}{3} \ \mathrm{d}y \qquad\qquad = \left[\frac{2x^4}{3} \right]_0^3 = 54 \tag{10.3.6}$$

4. The space between the lines $x = -y$ and $x = y$ bounded by the horizontal lines $y = 0$ and $y = 3$.

Triangle

$$I = \int_{-3}^{0}\int_{-x}^{3}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x + \int_{0}^{3}\int_{x}^{3}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x \tag{10.3.7}$$

$$= \int_{-3}^{0}\left[x^2y+\frac{y^3}{3}\right]_{-x}^{3}\mathrm{d}x + \int_{0}^{3}\left[x^2y+\frac{y^3}{3}\right]_{x}^{3}\mathrm{d}x \tag{10.3.8}$$

$$= \int_{-3}^{0}\left(3x^2+\frac{4x^3}{3}+9\right)\mathrm{d}x + \int_{0}^{3}\left(3x^2-\frac{4x^3}{3}+9\right)\mathrm{d}x \tag{10.3.9}$$

$$= \left[x^3+\frac{x^4}{3}+9x\right]_{-3}^{0} + \left[x^3-\frac{x^4}{3}+9x\right]_{0}^{3} = 54 \tag{10.3.10}$$

5. The space between the curves $y = x^2$ and $y = x$ bounded by the vertical lines $x = 0$ and $x = 1$.

$$I_1 = \int_{x^2}^{x}(1-2xy)\ \mathrm{d}y \qquad I_1 = \left[y - xy^2\right]_{x^2}^{x} \tag{10.3.11}$$

$$I_1 = x^5 - x^3 - x^2 + x \tag{10.3.12}$$

$$I = \int_{0}^{1}(x^5-x^3-x^2+x)\ \mathrm{d}x \qquad = \left[\frac{x^6}{6}-\frac{x^4}{4}-\frac{x^3}{3}+\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{12} \tag{10.3.13}$$

6. The space between the lines $x = 0$ and $x = y$ bounded by the horizontal lines $y = 0$ and $y = 2$. Triangle

$$I_2 = \int_{0}^{y}\sinh(x+y)\ \mathrm{d}x \qquad I_2 = \left[\cosh(x+y)\right]_{0}^{y} \tag{10.3.14}$$

$$I_2 = \cosh(2y) - \cosh(y) \tag{10.3.15}$$

$$I = \int_{0}^{2}\Big(\cosh(2y)-\cosh(y)\Big)\ \mathrm{d}y \qquad = \left[\frac{\sinh(2y)}{2}-\sinh(y)\right]_{0}^{2} \tag{10.3.16}$$

$$= \frac{\sinh(4)}{2} - \sinh(2) \tag{10.3.17}$$

7. The space between the curves $y = x$ and $y = 2$ bounded by the vertical lines $x = 0$ and $x = 2$.

$$I_1 = \int_x^2 \sinh(x+y)\ \mathrm{d}y \qquad I_1 = \Big[\cosh(x+y)\Big]_x^2 \tag{10.3.18}$$

$$I_1 = \cosh(x+2) - \cosh(2x) \tag{10.3.19}$$

$$I = \int_0^2 \Big(\cosh(x+2) - \cosh(2x)\Big)\ \mathrm{d}x \qquad = \left[\sinh(x+2) - \frac{\sinh(2x)}{2}\right]_0^2 \tag{10.3.20}$$

$$= \frac{\sinh(4)}{2} - \sinh(2) \tag{10.3.21}$$

8. The space between the curves $x = 0$ and $x = \cos(y)$ bounded by the horizontal lines $y = 0$ and $y = \pi/4$. Triangle

$$I_2 = \int_0^{\cos y} (x^2 \sin y)\ \mathrm{d}x \qquad I_2 = \left[\frac{x^3 \sin y}{3}\right]_0^{\cos y} \tag{10.3.22}$$

$$I_2 = \frac{\cos^3 y \sin y}{3} \tag{10.3.23}$$

$$I = \int_0^{\pi/4} \left(\frac{\cos^3 y \sin y}{3}\right)\ \mathrm{d}y \qquad = \left[\frac{-\cos^4 y}{12}\right]_0^{\pi/4} = \frac{1}{16} \tag{10.3.24}$$

9. The double integral is over a rectangle, which simplifies the limits.

$$I = \int_0^3 \left[\int_0^2 (4x^2 + 9y^2)\ \mathrm{d}y\right]\ \mathrm{d}x \qquad = \int_0^3 \Big[4x^2 y + 3y^3\Big]_0^2\ \mathrm{d}x \tag{10.3.25}$$

$$= \int_0^3 (8x^2 + 24)\ \mathrm{d}x \qquad = \left[\frac{8x^3}{3} + 24x\right]_0^3 = 144 \tag{10.3.26}$$

10. Performing the double integral, for the first octant with $x \in [0, 1]$

$$V = \iint_R z(x,y)\ \mathrm{d}x\,\mathrm{d}y \qquad = \int_0^1 \left[\int_0^{1-x^2} (1-x^2)\ \mathrm{d}y\right] \mathrm{d}x \tag{10.3.27}$$

$$= \int_0^1 \Big[y - x^2 y\Big]_0^{1-x^2}\ \mathrm{d}x \qquad = \int_0^1 \Big((1-x^2)^2\Big)\ \mathrm{d}x \tag{10.3.28}$$

$$= \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 = \frac{8}{15} \tag{10.3.29}$$

11. Performing the double integral, using polar coordinates where $r \in [0, 1]$,

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \tag{10.3.30}$$

$$J = \begin{vmatrix} \partial_r x & \partial_\theta x \\ \partial_r y & \partial_\theta y \end{vmatrix} \qquad J = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r \tag{10.3.31}$$

$$I = \iint_R (1 - x^2 - y^2)\, \mathrm{d}x\, \mathrm{d}y \qquad I^* = \iint_{R^*} (1 - r^2)(r)\, \mathrm{d}A \tag{10.3.32}$$

$$I^* = \int_0^1 \left[\int_0^{2\pi} (r - r^3)\, \mathrm{d}\theta \right] \mathrm{d}r \qquad I^* = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2} \tag{10.3.33}$$

12. Using the formula for x-CoM,

$$l_1 : y = \frac{2h}{b}\, x \qquad l_2 : y = -\frac{2h}{b}\, x + 2h \tag{10.3.34}$$

$$M\bar{x} = \int_0^h \left[\int_{by/2h}^{b-by/2h} x\, \mathrm{d}x \right] \mathrm{d}y \qquad \bar{x} = \int_0^h \frac{1}{M} \left[\frac{x^2}{2} \right]_{y^-}^{y^+} \mathrm{d}y \tag{10.3.35}$$

$$= \frac{1}{2M} \int_0^h \left(b^2 - \frac{b^2 y}{h} \right) \mathrm{d}y \qquad \bar{x} = \frac{1}{2M} \left[hb^2 - \frac{hb^2}{2} \right] = \frac{b}{2} \tag{10.3.36}$$

Using the formula for y-CoM,

$$l_1 : y = \frac{2h}{b}\, x \qquad l_2 : y = -\frac{2h}{b}\, x + 2h \tag{10.3.37}$$

$$M\bar{y} = \int_0^h \left[\int_{by/2h}^{b-by/2h} y\, \mathrm{d}x \right] \mathrm{d}y \qquad \bar{y} = \int_0^h \frac{1}{M} \Big[xy \Big]_{y^-}^{y^+} \tag{10.3.38}$$

$$= \frac{1}{M} \int_0^h \left(by - \frac{by^2}{h} \right) \mathrm{d}y \qquad \bar{y} = \frac{1}{M} \left[\frac{bh^2}{6} \right] = \frac{h}{3} \tag{10.3.39}$$

13. Using the formula for x-CoM,

$$l_1 : y = \frac{hx}{b} \qquad l_2 : y = 0 \tag{10.3.40}$$

$$M\bar{x} = \int_0^b \left[\int_0^{hx/b} x\, \mathrm{d}y \right] \mathrm{d}x \qquad \bar{x} = \int_0^b \frac{1}{M} \Big[xy \Big]_0^{hx/b} \mathrm{d}x \tag{10.3.41}$$

$$= \frac{1}{M} \int_0^b \left(\frac{hx^2}{b} \right) \mathrm{d}x \qquad \bar{x} = \frac{1}{M} \left[\frac{hx^3}{3b} \right]_0^b = \frac{2b}{3} \tag{10.3.42}$$

Using the formula for y-CoM,

$$l_1 : y = \frac{hx}{b} \qquad\qquad l_2 : y = 0 \tag{10.3.43}$$

$$M\bar{y} = \int_0^b \left[\int_0^{hx/b} y \, \mathrm{d}y \right] \mathrm{d}x \qquad\qquad \bar{y} = \int_0^b \frac{1}{M} \left[\frac{y^2}{2} \right]_0^{hx/b} \mathrm{d}x \tag{10.3.44}$$

$$= \frac{1}{M} \int_0^b \left(\frac{h^2 x^2}{2b^2} \right) \mathrm{d}x \qquad\qquad \bar{x} = \frac{1}{M} \left[\frac{h^2 x^3}{6b^2} \right]_0^b = \frac{h}{3} \tag{10.3.45}$$

14. Using the polar coordinate transformation, with $J = r$,

$$x = r \cos\theta \qquad\qquad y = r \sin\theta \tag{10.3.46}$$

$$\bar{x} = \frac{1}{M} \int_{R_1}^{R_2} \int_0^{\pi} \left[r^2 \cos\theta \, \mathrm{d}\theta \right] \mathrm{d}r \qquad\qquad \bar{x} = \frac{1}{M} \int_{R_1}^{R_2} r^2 \left[\sin\theta \right]_0^{\pi} \mathrm{d}r = 0 \tag{10.3.47}$$

$$\bar{y} = \frac{1}{M} \int_{R_1}^{R_2} \int_0^{\pi} \left[r^2 \sin\theta \, \mathrm{d}\theta \right] \mathrm{d}r \qquad\qquad = \frac{1}{M} \int_{R_1}^{R_2} r^2 \left[-\cos\theta \right]_0^{\pi} \mathrm{d}r \tag{10.3.48}$$

$$= \frac{2}{M} \int_{R_1}^{R_2} r^2 \, \mathrm{d}r \qquad\qquad \bar{y} = \left[\frac{2r^3}{3M} \right]_{R_1}^{R_2} = \frac{4}{3\pi} \frac{(R_2^3 - R_1^3)}{(R_2^2 - R_1^2)} \tag{10.3.49}$$

15. Using the result from Problem 14, with $R_1 = 0, \ R_2 = r$,

$$\bar{x} = 0 \qquad\qquad \bar{y} = \frac{4r}{3\pi} \tag{10.3.50}$$

16. Using the polar coordinate transformation, with $J = r$,

$$x = r \cos\theta \qquad\qquad y = r \sin\theta \tag{10.3.51}$$

$$\bar{x} = \frac{1}{M} \int_0^R \int_0^{\pi/2} \left[r^2 \cos\theta \, \mathrm{d}\theta \right] \mathrm{d}r \qquad\qquad \bar{x} = \frac{1}{M} \int_0^R r^2 \left[\sin\theta \right]_0^{\pi/2} \mathrm{d}r \tag{10.3.52}$$

$$= \frac{1}{M} \int_0^R r^2 \, \mathrm{d}r \qquad\qquad \bar{x} = \frac{R^3}{3M} = \frac{4R}{3\pi} \tag{10.3.53}$$

$$\bar{y} = \frac{4R}{3\pi} \tag{10.3.54}$$

By the symmetry of the problem, $\bar{y} = \bar{x}$ and the computation can be skipped.

17. Finding I_x

$$I_x = \int_0^b \left[\int_0^{hx/b} y^2 \, \mathrm{d}y \right] \mathrm{d}x \qquad = \int_0^b \left[\frac{y^3}{3} \right]_0^{hx/b} \mathrm{d}x \tag{10.3.55}$$

$$= \int_0^b \frac{h^3}{3b^3} x^3 \, \mathrm{d}x \qquad = \left[\frac{h^3 x^4}{12b^3} \right]_0^b = \frac{h^3 b}{12} \tag{10.3.56}$$

Finding I_y

$$I_y = \int_0^b \left[\int_0^{hx/b} x^2 \, \mathrm{d}y \right] \mathrm{d}x \qquad = \int_0^b \left[x^2 y \right]_0^{hx/b} \mathrm{d}x \tag{10.3.57}$$

$$= \int_0^b \frac{h}{b} x^3 \, \mathrm{d}x \qquad = \left[\frac{hx^4}{4b} \right]_0^b = \frac{hb^3}{4} \tag{10.3.58}$$

Finding I_z for a laminar object in the xy plane

$$I_z = \int_0^b \left[\int_0^{hx/b} (x^2 + y^2) \, \mathrm{d}y \right] \mathrm{d}x \qquad I_z = I_x + I_y = \frac{h^3 b}{12} + \frac{hb^3}{4} \tag{10.3.59}$$

18. Finding I_x

$$I_x = \int_0^h \left[\int_{by/2h}^{b-by/2h} y^2 \, \mathrm{d}x \right] \mathrm{d}y \qquad = \int_0^h \left[xy^2 \right]_{by/2h}^{b-by/2h} \mathrm{d}y \tag{10.3.60}$$

$$= \int_0^h \left(by^2 - \frac{by^3}{h} \right) \mathrm{d}y \qquad = \left[\frac{by^3}{3} - \frac{by^4}{4h} \right]_0^h = \frac{bh^3}{12} \tag{10.3.61}$$

Finding I_y

$$I_y = \int_0^h \left[\int_{by/2h}^{b-by/2h} x^2 \, \mathrm{d}x \right] \mathrm{d}y \qquad = \int_0^h \left[\frac{x^3}{3} \right]_{by/2h}^{b-by/2h} \mathrm{d}y \tag{10.3.62}$$

$$= \frac{b^3}{3} \int_0^h \left(1 - \frac{y}{2h}\right)^3 - \left(\frac{y}{2h}\right)^3 \mathrm{d}y \qquad = \frac{-2hb^3}{12} \left[\left(1 - \frac{y}{2h}\right)^4 + \left(\frac{y}{2h}\right)^4 \right]_0^h \tag{10.3.63}$$

$$= \frac{7b^3 h}{48} \tag{10.3.64}$$

Finding I_z for a laminar object in the xy plane

$$I_z = \int_0^b \left[\int_0^{hx/b} (x^2 + y^2)\ \mathrm{d}y \right] \mathrm{d}x \qquad I_z = I_x + I_y \tag{10.3.65}$$

$$= \frac{bh}{48}\ (7b^2 + 4h^2) \tag{10.3.66}$$

19. Finding the equations of the bounding lines,

$$l_1 : y + \frac{h}{2} = \frac{2h}{a-b}(x + a/2) \qquad l_2 : y + \frac{h}{2} = \frac{2h}{b-a}(x - a/2) \tag{10.3.67}$$

$$x^- = \frac{(2y+h)(a-b)}{4h} - \frac{a}{2} \qquad x^+ = \frac{(2y+h)(b-a)}{4h} + \frac{a}{2} \tag{10.3.68}$$

Finding I_x

$$I_x = \int_{-h/2}^{h/2} \left[\int_{x^-}^{x^+} y^2\ \mathrm{d}x \right] \mathrm{d}y \qquad = \int_{-h/2}^{h/2} \left[xy^2 \right]_{x^-}^{x^+} \mathrm{d}y \tag{10.3.69}$$

$$= \int_{-h/2}^{h/2} \left(2y^2\ x^+ \right) \mathrm{d}y \qquad = \left[\frac{ay^3}{3} + \frac{b-a}{2h}\ \left(\frac{y^4}{4} + \frac{hy^3}{3} \right) \right]_{-h/2}^{h/2} \tag{10.3.70}$$

$$= \frac{ah^3}{12} + \frac{(b-a)}{2h}\ \left(\frac{h^4}{12} \right) \qquad = \frac{(a+b)}{24}\ h^3 \tag{10.3.71}$$

Finding I_y

$$I_y = \int_{-h/2}^{h/2} \left[\int_{x^-}^{x^+} x^2\ \mathrm{d}x \right] \mathrm{d}y \qquad = \int_{-h/2}^{h/2} \left[\frac{x^3}{3} \right]_{x^-}^{x^+} \mathrm{d}y \tag{10.3.72}$$

$$= \int_{-h/2}^{h/2} \left(\frac{2}{3}(x^+)^3 \right) \mathrm{d}y \qquad = \left[\frac{2h}{6(b-a)}(x^+)^4 \right]_{-h/2}^{h/2} \tag{10.3.73}$$

$$= \frac{h}{3(b-a)}\ \left(\frac{b^4 - a^4}{16} \right) \qquad = \frac{h}{48}\ \frac{a^4 - b^4}{a-b} \tag{10.3.74}$$

Finding I_z for a laminar object in the xy plane

$$I_z = \int_0^b \left[\int_0^{hx/b} (x^2 + y^2)\ \mathrm{d}y \right] \mathrm{d}x \qquad I_z = I_x + I_y \tag{10.3.75}$$

$$= \frac{ha^4 - hb^4 + 2h^3(a^2 - b^2)}{48(a-b)} \tag{10.3.76}$$

20. Finding the equations of the bounding lines,

$$l_1 : y = \frac{2h}{a-b}(x + a/2) \qquad l_2 : y = \frac{2h}{b-a}(x - a/2) \tag{10.3.77}$$

$$x^- = \frac{y\,(a-b)}{2h} - \frac{a}{2} \qquad x^+ = \frac{y\,(b-a)}{2h} + \frac{a}{2} \tag{10.3.78}$$

Finding I_x

$$I_x = \int_0^h \left[\int_{x^-}^{x^+} y^2 \, \mathrm{d}x\right] \mathrm{d}y \qquad = \int_0^h \left[xy^2\right]_{x^-}^{x^+} \mathrm{d}y \tag{10.3.79}$$

$$= \int_0^h \left(2y^2\, x^+\right) \mathrm{d}y \qquad = \left[\frac{ay^3}{3} + \left(\frac{b-a}{h}\right)\frac{y^4}{4}\right]_0^h \tag{10.3.80}$$

$$= \frac{ah^3}{3} + \frac{(b-a)h^3}{4} \qquad = \frac{a+3b}{12}\, h^3 \tag{10.3.81}$$

Finding I_y

$$I_y = \int_0^h \left[\int_{x^-}^{x^+} x^2 \, \mathrm{d}x\right] \mathrm{d}y \qquad = \int_0^h \left[\frac{x^3}{3}\right]_{x^-}^{x^+} \mathrm{d}y \tag{10.3.82}$$

$$= \int_0^h \left(\frac{2}{3}(x^+)^3\right) \mathrm{d}y \qquad = \left[\frac{2h}{6(b-a)}(x^+)^4\right]_0^h \tag{10.3.83}$$

$$= \frac{h}{3(b-a)}\left(\frac{b^4 - a^4}{16}\right) \qquad = \frac{h}{48}\,\frac{a^4 - b^4}{a-b} \tag{10.3.84}$$

Finding I_z for a laminar object in the xy plane

$$I_z = \int_0^b \left[\int_0^{hx/b} (x^2 + y^2)\, \mathrm{d}y\right] \mathrm{d}x \qquad I_z = I_x + I_y \tag{10.3.85}$$

$$= \frac{h(a^2 + b^2)(a+b) + 4h^3(a+3b)}{48} \tag{10.3.86}$$

10.4 Green's Theorem in the Plane

1. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} y \\ -x \end{bmatrix} \qquad C : x^2 + y^2 = 1/4 \tag{10.4.1}$$

$$\oint_C \mathbf{F} \cdot \,\mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A \qquad I = -\iint_R 2 \,\mathrm{d}A \tag{10.4.2}$$

$$I = -\int_0^{1/2} \int_0^{2\pi} 2r \,\mathrm{d}r \,\mathrm{d}\theta \qquad = -2\pi \Big[r^2\Big]_0^{1/2} = -\frac{\pi}{2} \tag{10.4.3}$$

2. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} 6y^2 \\ 2x - 2y^4 \end{bmatrix} \qquad C : \text{square with given vertices} \tag{10.4.4}$$

$$\oint_C \mathbf{F} \cdot \,\mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A \qquad I = \iint_R \Big(2 - 12y\Big) \,\mathrm{d}A \tag{10.4.5}$$

$$I = \int_{-2}^{2} \int_{-2}^{2} (2 - 12y) \,\mathrm{d}x \,\mathrm{d}y \qquad = \int_{-2}^{2} \Big[2x - 12xy\Big]_{-2}^{2} \,\mathrm{d}y \tag{10.4.6}$$

$$= \int_{-2}^{2} (8 - 48y) \,\mathrm{d}y \qquad = \Big[8y - 24y^2\Big]_{-2}^{2} = 32 \tag{10.4.7}$$

3. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} x^2 e^y \\ y^2 e^x \end{bmatrix} \qquad C : \text{Rectangle with given vertices} \tag{10.4.8}$$

$$\oint_C \mathbf{F} \cdot \,\mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A \qquad I = \iint_R \Big(y^2 e^x - x^2 e^y\Big) \,\mathrm{d}A \tag{10.4.9}$$

$$I = \int_0^3 \int_0^2 (y^2 e^x - x^2 e^y) \,\mathrm{d}x \,\mathrm{d}y \qquad = \int_0^3 \left[y^2 e^x - \frac{x^3}{3} e^y\right]_0^2 \mathrm{d}y \tag{10.4.10}$$

$$= \int_0^3 \left(y^2(e^2 - 1) - \frac{8e^y}{3}\right) \mathrm{d}y \qquad = \left[\frac{(e^2-1)y^3 - 8e^y}{3}\right]_0^3 \tag{10.4.11}$$

$$= \frac{27e^2 - 19 - 8e^3}{3} \tag{10.4.12}$$

4. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} x\cosh(2y) \\ 2x^2\sinh(2y) \end{bmatrix} \qquad C : y \in [x^2, x] \qquad x \in [0, 1] \tag{10.4.13}$$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A \qquad I = \iint_R \Big(2x\sinh(2y)\Big)\, \mathrm{d}A \tag{10.4.14}$$

$$I = \int_0^1 \int_{x^2}^{x} \Big(2x\sinh(2y)\Big)\, \mathrm{d}y\, \mathrm{d}x \qquad = \int_0^1 \Big[x\cosh(2y)\Big]_{x^2}^{x} \mathrm{d}x \tag{10.4.15}$$

$$= \int_0^1 \Big(x\cosh(2x) - x\cosh(2x^2)\Big)\, \mathrm{d}x \tag{10.4.16}$$

$$= \left[\frac{2x\sinh(2x) - \cosh(2x) - \sinh(2x^2)}{4}\right]_0^1 \qquad = \frac{-e^{-2} + 1}{4} \tag{10.4.17}$$

5. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} x^2 + y^2 \\ x^2 - y^2 \end{bmatrix} \qquad C : y \in [1, 2 - x^2] \qquad x \in [0, 1] \tag{10.4.18}$$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A \qquad I = \iint_R \Big(2x - 2y\Big)\, \mathrm{d}A \tag{10.4.19}$$

$$I = \int_{-1}^1 \int_{1}^{2-x^2} \Big(2x - 2y\Big)\, \mathrm{d}y\, \mathrm{d}x \qquad = \int_{-1}^1 \Big[2xy - y^2\Big]_{1}^{2-x^2} \mathrm{d}x \tag{10.4.20}$$

$$= \int_0^1 \Big(2x - 2x^3 - 3 - x^4 + 4x^2\Big)\, \mathrm{d}x \tag{10.4.21}$$

$$= \left[x^2 - \frac{x^4}{2} - 3x - \frac{x^5}{5} + \frac{4x^3}{3}\right]_{-1}^1 \qquad = \frac{-56}{15} \tag{10.4.22}$$

6. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} \cosh y \\ -\sinh x \end{bmatrix} \qquad C : y \in [x, 3x] \qquad x \in [1, 3] \tag{10.4.23}$$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A \tag{10.4.24}$$

$$I = \iint_R \Big(-\cosh x - \sinh y \Big) \mathrm{d}A \tag{10.4.25}$$

$$I = \int_1^3 \int_x^{3x} \Big(-\cosh x - \sinh y \Big) \mathrm{d}y \, \mathrm{d}x \tag{10.4.26}$$

$$= \int_1^3 \Big[-y \cosh x - \cosh y \Big]_x^{3x} \mathrm{d}x \tag{10.4.27}$$

$$= \int_1^3 \Big(-2x \cosh(x) - \cosh(3x) + \cosh x \Big) \mathrm{d}x \tag{10.4.28}$$

$$= \left[-2x \sinh x + 2 \cosh x - \frac{\sinh(3x)}{3} + \sinh(x) \right]_1^3 = -1379.04 \tag{10.4.29}$$

7. Using Green's theorem,

$$\mathbf{F} = \nabla\big[x^3 \cos^2(xy)\big] = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad C : x \in [0, 2 - x^2] \qquad x \in [0, 1] \tag{10.4.30}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial^2 g}{\partial x \, \partial y} \qquad \frac{\partial F_1}{\partial y} = \frac{\partial^2 g}{\partial y \, \partial x} \tag{10.4.31}$$

Since the vector function is the gradient of an underlying scalar function g, and its second partial derivatives commute, the line integral is identically zero.

8. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} -e^{-x} \cos y \\ -e^{-x} \sin y \end{bmatrix} \qquad C : x \in [0, \sqrt{16 - y^2}] \qquad x \in [-4, 4] \tag{10.4.32}$$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A \qquad = \iint_R \Big(e^{-x} \sin y - e^{-x} \sin y \Big) \mathrm{d}A = 0 \tag{10.4.33}$$

Since the vector function is the gradient of an underlying scalar function $g = e^{-x} \cos(y)$, and its second partial derivatives commute, the line integral is identically zero.

9. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} e^{y/x} \\ e^y \ \ln(x) + 2x \end{bmatrix} \qquad C : y \in [1 + x^4, 2] \qquad x \in [-1, 1] \tag{10.4.34}$$

$$\oint_C \mathbf{F} \cdot \,\mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A \tag{10.4.35}$$

$$I = \iint_R \left(\frac{e^y}{x} + 2 - \frac{e^y}{x} \right) \mathrm{d}A \tag{10.4.36}$$

$$I = \int_{-1}^{1} \int_{1+x^4}^{2} \left(2\right) \mathrm{d}y \,\mathrm{d}x = \int_{-1}^{1} \Big[2y\Big]_{1+x^4}^{2} \mathrm{d}x \tag{10.4.37}$$

$$= \int_{-1}^{1} \left(2 - 2x^4 \right) \mathrm{d}x = \left[2x - \frac{2x^5}{5} \right]_{-1}^{1} = \frac{16}{5} \tag{10.4.38}$$

10. Using Green's theorem,

$$\mathbf{F} = \begin{bmatrix} x^2 y^2 \\ -x/y^2 \end{bmatrix} \qquad C : r \in [1, 2] \qquad \theta \in [\pi/4, \pi/2] \tag{10.4.39}$$

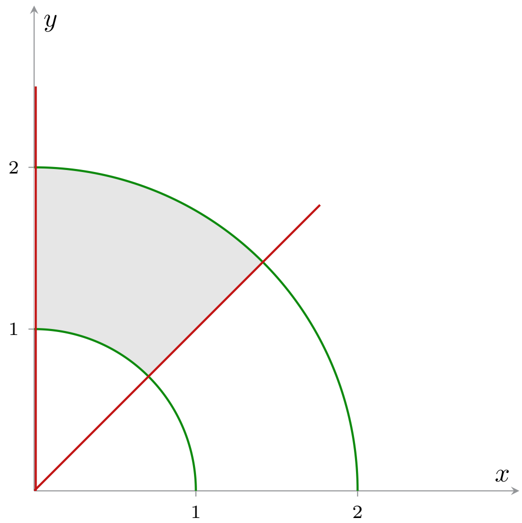
$$\oint_C \mathbf{F} \cdot \,\mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A \tag{10.4.40}$$

$$I = \iint_R \left(\frac{-1}{y^2} - 2x^2 y \right) \mathrm{d}A \tag{10.4.41}$$

$$I = \int_{1}^{2} \int_{\pi/4}^{\pi/2} \left(\frac{-1}{r \sin^2 \theta} - 2r^4 \cos^2 \theta \sin \theta \right) \mathrm{d}\theta \,\mathrm{d}r \tag{10.4.42}$$

$$= \int_{-1}^{1} \left[\frac{\cot(\theta)}{r} + \frac{2r^4}{3} \cos^3 \theta \right]_{\pi/4}^{\pi/2} \mathrm{d}x \tag{10.4.43}$$

$$= \int_{1}^{2} \left(\frac{-1}{r} - \frac{r^4}{3\sqrt{2}} \right) \mathrm{d}x = \left[-\ln(r) - \frac{r^5}{15\sqrt{2}} \right]_{1}^{2} = -\ln(2) - \frac{31\sqrt{2}}{30} \tag{10.4.44}$$



11. Finding the area of a circle using Green's theorem,

$$A = \frac{1}{2} \oint_C (x \, \mathrm{d}y) - (y \, \mathrm{d}x) \qquad x = r \cos\theta \qquad y = r \sin\theta \tag{10.4.45}$$

$$A = \frac{1}{2} \int_0^{2\pi} \left(r^2 \cos^2\theta + r^2 \sin^2\theta \right) \mathrm{d}\theta \qquad = \frac{2\pi r^2}{2} = \pi r^2 \tag{10.4.46}$$

Finding the area of a triangle using Green's theorem, with vertices $(0, 0)$, $(0, b)$ and (b, h)

$$A = \frac{1}{2} \oint_C (x \, \mathrm{d}y) - (y \, \mathrm{d}x) \tag{10.4.47}$$

$$A = \frac{1}{2} \left[\int_0^b (-0) \, \mathrm{d}x + \int_0^h b \, \mathrm{d}y + \int_1^0 (bht) \, \mathrm{d}t - (bht) \, \mathrm{d}t \right] = \frac{bh}{2} \tag{10.4.48}$$

Other examples TBC.

12. Restating Green's theorem,

$$\mathbf{F} = \begin{bmatrix} F_2 \\ -F_1 \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \tag{10.4.49}$$

$$\mathbf{r}' = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \begin{bmatrix} \mathrm{d}_s x \\ \mathrm{d}_s y \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} \mathrm{d}_s y \\ -\mathrm{d}_s x \end{bmatrix} = \hat{\mathbf{n}} \, \mathrm{d}s \tag{10.4.50}$$

$$\mathbf{r}' \cdot \mathbf{n} = 0 \tag{10.4.51}$$

Now, starting from the pre-existing definition of Green's theorem,

$$\iint \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] \mathrm{d}x \, \mathrm{d}y = \iint \left[\nabla \cdot \mathbf{F}\right] \mathrm{d}x \, \mathrm{d}y \tag{10.4.52}$$

$$\oint_C (F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y) = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}s \tag{10.4.53}$$

Starting with the curl of $\mathbf{F}$,

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \tag{10.4.54}$$

$$\mathbf{r}' = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \begin{bmatrix} \mathrm{d}_s x \\ \mathrm{d}_s y \end{bmatrix} \qquad \mathbf{F} \cdot \mathbf{r}' = F_1 \frac{\mathrm{d}x}{\mathrm{d}s} + F_2 \frac{\mathrm{d}y}{\mathrm{d}s} \tag{10.4.55}$$

$$\mathbf{F} \cdot \mathbf{r}' \, \mathrm{d}s = F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y \qquad \iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}} \, \mathrm{d}x \, \mathrm{d}y = \oint_C \mathbf{F} \cdot \mathbf{r}' \, \mathrm{d}s \tag{10.4.56}$$

Verifying the relations for the given example,

$$\mathbf{F} = \begin{bmatrix} 7x \\ -3y \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 2 \cos t \\ 2 \sin t \end{bmatrix} \tag{10.4.57}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = 2 \tag{10.4.58}$$

$$\mathbf{r}(s) = \begin{bmatrix} 2 \cos(s/2) \\ 2 \sin(s/2) \end{bmatrix} \qquad s \in [0, 4\pi] \tag{10.4.59}$$

For the first relation,

$$\iint_R (\nabla \cdot \mathbf{F}) \, \mathrm{d}x \, \mathrm{d}y = \int_0^2 \left[\int_0^{2\pi} 4r \, \mathrm{d}\theta\right] \mathrm{d}r = 16\pi \tag{10.4.60}$$

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \begin{bmatrix} -\sin(s/2) \\ \cos(s/2) \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} \cos(s/2) \\ \sin(s/2) \end{bmatrix} \tag{10.4.61}$$

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} = \oint 14 \cos^2(s/2) - 6 \sin^2(s/2) \, \mathrm{d}s \tag{10.4.62}$$

$$= \Big[7s + 7 \sin(s) - 3s - 3 \sin(s)\Big]_0^{4\pi} = 16\pi \tag{10.4.63}$$

For the second relation,

$$(\nabla \times \mathbf{F}) = \mathbf{0} \qquad \iint_R (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}} \, \mathrm{d}A = 0 \tag{10.4.64}$$

$$\mathbf{F} \cdot \mathbf{r}' = -10 \sin(s) \tag{10.4.65}$$

$$\oint_C \mathbf{F} \cdot \mathbf{r}' \, \mathrm{d}s = -10 \int_0^{4\pi} \sin(s) \, \mathrm{d}s \qquad = \Big[10 \cos(s)\Big]_0^{4\pi} = 0 \tag{10.4.66}$$

Other example TBC.

13. Using the restatement of Green's theorem,

$$\oint_C \frac{\partial w}{\partial n} \, \mathrm{d}s = \iint_R \nabla^2 w \, \mathrm{d}A \qquad C : y \in [0.5x, 2] \qquad x \in [0, 4] \tag{10.4.67}$$

$$w = \cosh(x) \qquad \nabla^2 w = \cosh(x) \tag{10.4.68}$$

$$I = \int_0^4 \left[\int_{0.5x}^2 \cosh(x) \, \mathrm{d}y \right] \mathrm{d}x \qquad = \int_0^4 \Big[y \cosh(x) \Big]_{0.5x}^2 \tag{10.4.69}$$

$$= \int_0^4 \Big(2 \cosh(x) - 0.5x \cosh(x) \Big) \, \mathrm{d}x \tag{10.4.70}$$

$$= \Big[(2 - 0.5x) \sinh(x) + 0.5 \cosh(x) \Big]_0^4 \tag{10.4.71}$$

$$= \frac{\cosh(4) - 1}{2} \tag{10.4.72}$$

14. Using Green's theorem,

$$w = x^2 y + x y^2 \qquad C : r \in [0, 1] \qquad \theta \in [0, \pi/2] \tag{10.4.73}$$

$$\oint_C \frac{\partial w}{\partial n} \, \mathrm{d}s = \iint_R \nabla^2 w \, \mathrm{d}A \tag{10.4.74}$$

$$I = \iint_R \Big(2y + 2x \Big) \, \mathrm{d}A \tag{10.4.75}$$

$$I = \int_0^1 \int_0^{\pi/2} \Big(2r \cos\theta + 2r \sin\theta \Big) \, r \, \mathrm{d}\theta \;\; \mathrm{d}r \tag{10.4.76}$$

$$= \int_0^1 2r^2 \Big[\sin\theta - \cos\theta \Big]_0^{\pi/2} \, \mathrm{d}r \tag{10.4.77}$$

$$= \int_0^1 \Big(4r^2 \Big) \, \mathrm{d}r = \left[\frac{4r^3}{3} \right]_0^1 = \frac{4}{3} \tag{10.4.78}$$

15. Using Green's theorem,

$$w = e^x \cos y + xy^3 \qquad C : y \in [1, 10 - x^2] \qquad x \in [0, 3] \tag{10.4.79}$$

$$\oint_C \frac{\partial w}{\partial n}\,\mathrm{d}s = \iint_R \nabla^2 w\,\mathrm{d}A \tag{10.4.80}$$

$$I = \iint_R \left(e^x \cos y + -e^x \cos y + 6xy \right) \mathrm{d}A \tag{10.4.81}$$

$$I = \int_0^3 \int_1^{10-x^2} \left(6xy\right) \mathrm{d}y\,\mathrm{d}x = \int_0^3 3x \left[y^2 \right]_1^{10-x^2} \mathrm{d}x \tag{10.4.82}$$

$$= \int_0^3 \left(3(99x - 20x^3 + x^5) \right) \mathrm{d}r = 3\left[\frac{99x^2}{2} - 5x^4 + \frac{x^6}{6} \right]_0^3 = 486 \tag{10.4.83}$$

16. Using Green's theorem,

$$w = x^2 + y^2 \qquad C : x^2 + y^2 = 4 \tag{10.4.84}$$

$$\oint_C \frac{\partial w}{\partial n}\,\mathrm{d}s = \iint_R \nabla^2 w\,\mathrm{d}A \tag{10.4.85}$$

$$I = \iint_R \left(2 + 2 \right) \mathrm{d}A \tag{10.4.86}$$

$$I = \int_0^2 \int_0^{2\pi} \left(4\right) r\,\mathrm{d}\theta\ \ \mathrm{d}r \tag{10.4.87}$$

$$= \int_0^2 \left(8\pi r \right) \mathrm{d}r = \left[4\pi r^2 \right]_0^2 = 16\pi \tag{10.4.88}$$

17. Using Green's theorem,

$$w = x^3 - y^3 \qquad C : y \in [0, x^2] \qquad x \in [-2, 2] \tag{10.4.89}$$

$$\oint_C \frac{\partial w}{\partial n}\,\mathrm{d}s = \iint_R \nabla^2 w\,\mathrm{d}A \tag{10.4.90}$$

$$I = \iint_R \left(6x - 6y \right) \mathrm{d}A \tag{10.4.91}$$

$$I = \int_{-2}^2 \left[\int_0^{x^2} \left(6x - 6y\right) \mathrm{d}y \right] \mathrm{d}x = \int_{-2}^2 \left[6xy - 3y^2 \right]_0^{x^2} \mathrm{d}x \tag{10.4.92}$$

$$= \int_{-2}^2 \left(6x^3 - 3x^4 \right) \mathrm{d}x = \left[\frac{3x^4}{2} - \frac{3x^5}{5} \right]_{-2}^2 = -\frac{192}{5} \tag{10.4.93}$$

18. The directional derivative is defined as,

$$\mathbf{F} = \begin{bmatrix} -w \ \partial w/\partial y \\ w \ \partial w/\partial x \end{bmatrix} \qquad \nabla^2 w = 0 \tag{10.4.94}$$

$$\frac{\partial F_2}{\partial x} = w \ \frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 \qquad \frac{\partial F_1}{\partial y} = -w \ \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial w}{\partial y}\right)^2 \tag{10.4.95}$$

$$F_1 \ \mathrm{d}x = F_1 \ \frac{\mathrm{d}x}{\mathrm{d}s} \ \mathrm{d}s \qquad F_2 \ \mathrm{d}y = F_2 \ \frac{\mathrm{d}y}{\mathrm{d}s} \ \mathrm{d}s \tag{10.4.96}$$

$$= -w \ \frac{\partial w}{\partial y} \ \frac{\mathrm{d}x}{\mathrm{d}s} \ \mathrm{d}s \qquad = w \ \frac{\partial w}{\partial x} \ \frac{\mathrm{d}y}{\mathrm{d}s} \ \mathrm{d}s \tag{10.4.97}$$

Rearranging these terms into the Green's function,

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \ \mathrm{d}A = \iint_R \left[w \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \ \mathrm{d}A \tag{10.4.98}$$

$$= \iint_R \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \ \mathrm{d}A \tag{10.4.99}$$

$$\mathbf{n} = \begin{bmatrix} \mathrm{d}y/\mathrm{d}s \\ -\mathrm{d}x/\mathrm{d}s \end{bmatrix} \tag{10.4.100}$$

$$\oint_C F_1 \ \mathrm{d}x + F_2 \ \mathrm{d}y = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}s \tag{10.4.101}$$

$$= \oint_C w \ \frac{\partial w}{\partial n} \ \mathrm{d}s \tag{10.4.102}$$

19. Applying the result from Problem 18,

$$w = e^x \sin y \tag{10.4.103}$$

$$\nabla^2 w = e^x \sin y - e^x \sin y = 0 \tag{10.4.104}$$

$$C : y \in [0, 5] \qquad x \in [0, 2] \tag{10.4.105}$$

$$I = \iint_R \left[w \ \nabla^2 w + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \ \mathrm{d}A \tag{10.4.106}$$

$$= \int_0^2 \left[\int_0^5 \left(e^{2x} \sin^2 y + e^{2x} \cos^2 y\right) \mathrm{d}y\right] \ \mathrm{d}x \tag{10.4.107}$$

$$= \int_0^2 (5e^{2x}) \ \mathrm{d}x = 2.5(e^4 - 1) \tag{10.4.108}$$

20. Applying the result from Problem 18,

$$w = x^2 + y^2 \tag{10.4.109}$$

$$\nabla^2 w = 2 + 2 = 4 \tag{10.4.110}$$

$$C : y \in [0, 1 - x] \qquad x \in [0, 1] \tag{10.4.111}$$

$$I = \iint_R \left[w\ \nabla^2 w + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] \mathrm{d}A \tag{10.4.112}$$

$$= \int_0^1 \left[\int_0^{1-x} \left(4(x^2 + y^2) + 4x^2 + 4y^2 \right) \mathrm{d}y \right] \mathrm{d}x \tag{10.4.113}$$

$$= \int_0^1 \left[\int_0^{1-x} \left(8(x^2 + y^2) \right) \mathrm{d}y \right] \mathrm{d}x \tag{10.4.114}$$

$$= 8 \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} \mathrm{d}x \tag{10.4.115}$$

$$= 8 \int_0^1 \left(x^2 - x^3 + \frac{(1-x)^3}{3} \right) \mathrm{d}x \tag{10.4.116}$$

$$= 8 \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12} \right]_0^1 = \frac{4}{3} \tag{10.4.117}$$

10.5 Surfaces for Surface Integrals

1. Finding the parameter curves in $2d$,

$$\mathbf{r} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{10.5.1}$$

$$u = c \implies x = c \qquad\qquad v = c \implies y = c \tag{10.5.2}$$

The parameter curves are straight lines parallel to one of the axes.

The normal vector is,

$$\mathbf{r}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{10.5.3}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.5.4}$$

2. Finding the parameter curves in $2d$,

$$\mathbf{r} = \begin{bmatrix} u \cos v \\ u \sin v \end{bmatrix} \tag{10.5.5}$$

$$u = c \implies x^2 + y^2 = c^2 \qquad v = c \implies \frac{y}{x} = \tan(c) \tag{10.5.6}$$

The parameter curves are circles centered on the origin and straight lines passing through the origin. The normal vector is,

$$\mathbf{r}_u = \begin{bmatrix} \cos v \\ \sin v \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -u \sin v \\ u \cos v \end{bmatrix} \tag{10.5.7}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} \tag{10.5.8}$$

$$= \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \tag{10.5.9}$$

3. Finding the parameter curves in $3d$,

$$\mathbf{r} = \begin{bmatrix} u \cos v \\ u \sin v \\ \lambda u \end{bmatrix} \tag{10.5.10}$$

$$u = c \implies x^2 + y^2 = c^2 \qquad z = \lambda c \tag{10.5.11}$$

$$v = c \implies \frac{y}{x} = \tan(c) \tag{10.5.12}$$

The parameter curves for constant u are circles in the xy plane centered at the origin at $z = \lambda c$.

For constant v, the parameter curves are straight lines through the origin.

The normal vector is,

$$\mathbf{r}_u = \begin{bmatrix} \cos v \\ \sin v \\ \lambda \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -u \sin v \\ u \cos v \\ 0 \end{bmatrix} \tag{10.5.13}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \qquad = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & \lambda \\ -u \sin v & u \cos v & 0 \end{vmatrix} \tag{10.5.14}$$

$$= \begin{bmatrix} -\lambda u \cos(v) \\ -\lambda u \sin(v) \\ u \end{bmatrix} \tag{10.5.15}$$

4. Finding the parameter curves in $3d$,

$$\mathbf{r} = \begin{bmatrix} a \cos v \\ b \sin v \\ u \end{bmatrix} \tag{10.5.16}$$

$$u = c \implies \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad z = c \tag{10.5.17}$$

$$v = c \implies x = a \cos c \qquad y = b \sin c \qquad z = u \tag{10.5.18}$$

The parameter curves for constant u are ellipses in the xy plane centered at the origin at $z = c$.

For constant v, the parameter curves are straight lines parallel to the z axis intersecting the xy plane at $(a \cos c, b \sin c)$.

The normal vector is,

$$\mathbf{r}_u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -a \sin v \\ b \cos v \\ 0 \end{bmatrix} \tag{10.5.19}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \qquad = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ -a \sin v & b \cos v & 0 \end{vmatrix} \tag{10.5.20}$$

$$= \begin{bmatrix} -b \cos v \\ -a \sin v \\ 0 \end{bmatrix} \tag{10.5.21}$$

5. Finding the parameter curves in $3d$,

$$\mathbf{r} = \begin{bmatrix} u \cos v \\ u \sin v \\ u^2 \end{bmatrix} \tag{10.5.22}$$

$$u = c \implies x^2 + y^2 = c^2 \qquad z = c^2 \tag{10.5.23}$$

$$v = c \implies \frac{y}{x} = \tan(c) \qquad z = u^2 \tag{10.5.24}$$

The parameter curves for constant u are circles in the xy plane centered at the origin at $z = c^2$.
For constant v, the parameter curves are parabolas parallel to the z axis lying in the plane $y = \tan(c)\, x$.
The normal vector is,

$$\mathbf{r}_u = \begin{bmatrix} \cos v \\ \sin v \\ 2u \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -u \sin v \\ u \cos v \\ 0 \end{bmatrix} \tag{10.5.25}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} \tag{10.5.26}$$

$$= \begin{bmatrix} -2u^2 \ \cos v \\ -2u^2 \ \sin v \\ u \end{bmatrix} \tag{10.5.27}$$

6. Finding the parameter curves in $3d$,

$$\mathbf{r} = \begin{bmatrix} u \cos v \\ u \sin v \\ v \end{bmatrix} \tag{10.5.28}$$

$$u = c \implies x = c \cos v \qquad y = c \sin v \qquad z = v \tag{10.5.29}$$

$$v = c \implies \frac{y}{x} = \tan(c) \qquad z = c \tag{10.5.30}$$

The parameter curves for constant u are helices in the xy plane centered at the origin, axis being the z axis and radius c.
For constant v, the parameter curves are straight lines on the xy plane at $z = c$.

The normal vector is,

$$\mathbf{r}_u = \begin{bmatrix} \cos v \\ \sin v \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -u \sin v \\ u \cos v \\ 1 \end{bmatrix} \tag{10.5.31}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} \tag{10.5.32}$$

$$= \begin{bmatrix} \sin v \\ -\cos v \\ u \end{bmatrix} \tag{10.5.33}$$

7. Finding the parameter curves in $3d$,

$$\mathbf{r} = \begin{bmatrix} a \cos v \cos u \\ b \cos v \sin u \\ c \sin v \end{bmatrix} \tag{10.5.34}$$

$$u = \lambda \implies y = x \frac{b \ \tan \lambda}{a} \qquad \frac{z^2}{c^2} + \frac{x^2}{a^2 \cos^2 \lambda} = 1 \tag{10.5.35}$$

$$v = c \implies \frac{y}{x} = \tan(c) \qquad z = c \tag{10.5.36}$$

The parameter curves for constant u are ellipses through the z axis lying in the plane $y = (\tan c)\ bx/a$. For constant v, the parameter curves are ellipses on the xy plane centered at the z axis at $z = c \sin \lambda$. The normal vector is,

$$\mathbf{r}_u = \begin{bmatrix} -a \cos v \sin u \\ b \cos v \cos u \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -a \sin v \cos u \\ -b \sin v \sin u \\ c \cos v \end{bmatrix} \tag{10.5.37}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \cos v \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a \sin u & b \cos u & 0 \\ -a \sin v \cos u & -b \sin v \sin u & c \cos v \end{vmatrix} \tag{10.5.38}$$

$$= \begin{bmatrix} bc \cos u \cos^2 v \\ ac \sin u \cos^2 v \\ ab \sin v \cos v \end{bmatrix} \tag{10.5.39}$$

8. Finding the parameter curves in $3d$,

$$\mathbf{r} = \begin{bmatrix} au \cosh v \\ bu \sinh v \\ u^2 \end{bmatrix} \tag{10.5.40}$$

$$u = \lambda \implies \frac{x^2}{a^2} - \frac{y^2}{b^2} = \lambda^2 \qquad z = \lambda^2 \tag{10.5.41}$$

$$v = c \implies y = x\,\frac{b\ \tanh\lambda}{a} \qquad z = \left(\frac{x}{a\cosh\lambda}\right)^2 \tag{10.5.42}$$

The parameter curves for constant u are hyperbolae on the xy plane at $z = \lambda^2$.
For constant v, the parameter curves are parabolas along the z axis lying on the plane $y = \tanh\lambda(bx/a)$.
The normal vector is,

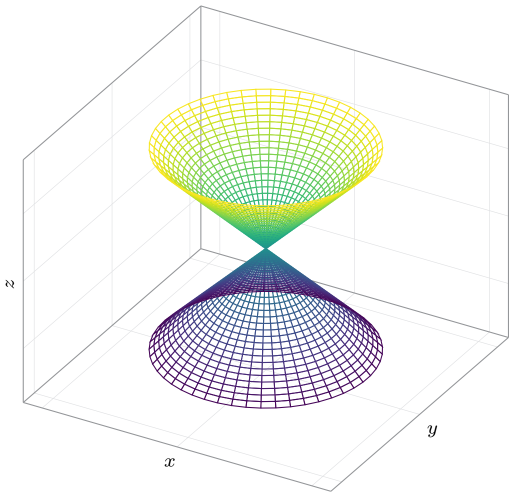
$$\mathbf{r}_u = \begin{bmatrix} a\cosh v \\ b\sinh v \\ 2u \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} au\sinh v \\ bu\cosh v \\ 0 \end{bmatrix} \tag{10.5.43}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a\cosh v & b\sinh v & 2u \\ au\sinh v & bu\cosh v & 0 \end{vmatrix} \tag{10.5.44}$$

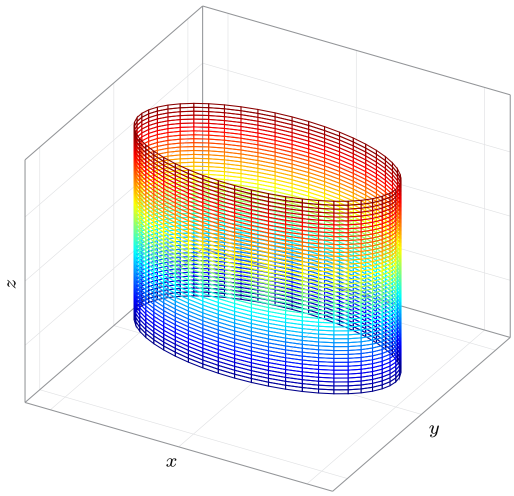
$$= \begin{bmatrix} -2bu^2\cosh v \\ 2au^2\sinh v \\ abu \end{bmatrix} \tag{10.5.45}$$

9. Plotting the various parametric surfaces,

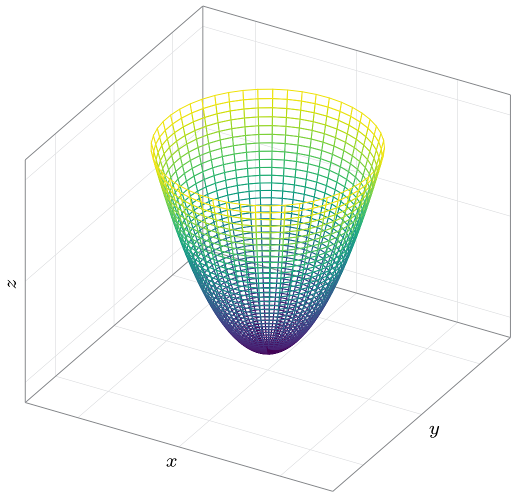
Circular cone



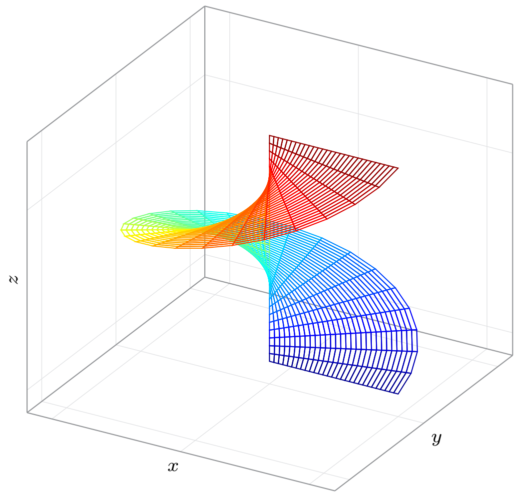
Elliptic cylinder



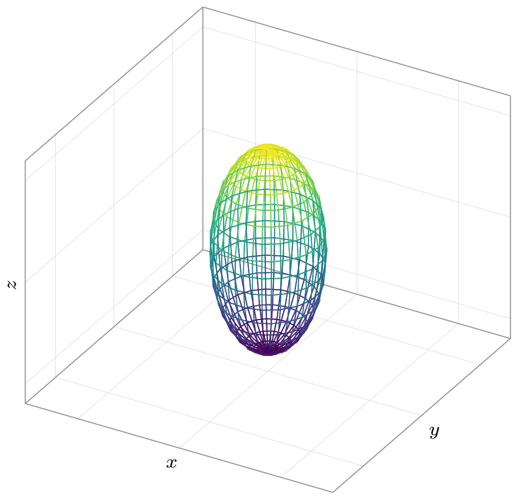
Praboloid of revolution



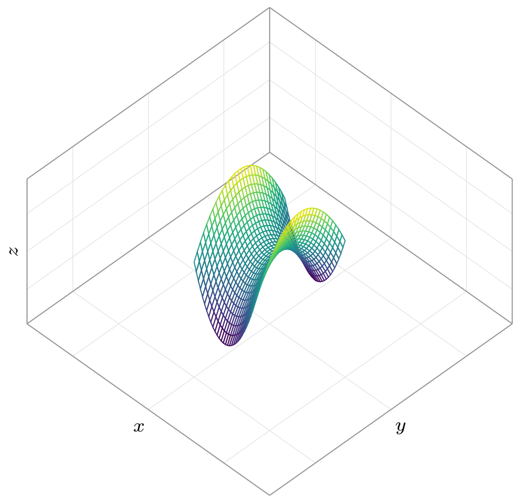
Helicoid



Ellipsoid



Hyperbolic Paraboloid



The effects of varying the parameters a, b, c are not shown here.

10. Examples TBC. For the forward case, start with

$$\mathbf{r}_u \cdot \mathbf{r}_v = 0 \tag{10.5.46}$$

$$u = c_1 \implies \mathbf{r}_1'(t) = \mathbf{r}_v \; v' \qquad\qquad v = c_2 \implies \mathbf{r}_2'(t) = \mathbf{r}_u \; u' \tag{10.5.47}$$

If the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are orthogonal, then the tangent vectors to the two curves $\mathbf{r}_1'$ and $\mathbf{r}_2'$ are also orthogonal.

This means that the curves with $u = c_1$ and $v = c_2$ are orthogonal at P on surface S.

The backward proof is exactly these steps in reverse.

11. The normal vector from Problem 5 is

$$\mathbf{N} = \begin{bmatrix} -2u^2 \ \cos v \\ -2u^2 \ \sin v \\ u \end{bmatrix} \qquad\qquad \mathbf{N}(u = 0, v = 0) = \mathbf{0} \tag{10.5.48}$$

To redefine the normal vector and avoid this,

$$\tilde{\mathbf{r}}(u, v) = \begin{bmatrix} u \\ v \\ u^2 + v^2 \end{bmatrix} \qquad \tilde{\mathbf{r}}_u = \begin{bmatrix} 1 \\ 0 \\ 2u \end{bmatrix} \tag{10.5.49}$$

$$\tilde{\mathbf{r}}_v = \begin{bmatrix} 0 \\ 1 \\ 2v \end{bmatrix} \qquad \tilde{\mathbf{N}} = \begin{bmatrix} -2u \\ -2v \\ 1 \end{bmatrix} \tag{10.5.50}$$

$$\tilde{\mathbf{N}}(0, 0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \mathbf{0} \tag{10.5.51}$$

12. Finding the points at which the normal vector is zero,

(a) No such points exist for the given surface.

$$\mathbf{N}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{N} \neq \mathbf{0} \tag{10.5.52}$$

(b) This is caused by the representation.

$$\mathbf{N}_2 = \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \qquad \mathbf{N} = 0 \implies (0, 0, 0) \tag{10.5.53}$$

(c) This is caused by the surface itself.

$$\mathbf{N}_3 = \begin{bmatrix} -\lambda u \cos(v) \\ -\lambda u \sin(v) \\ u \end{bmatrix} \qquad \mathbf{N} = 0 \implies (0, 0, 0) \tag{10.5.54}$$

(d) No such points exist for the given surface.

$$\mathbf{N}_4 = \begin{bmatrix} -b \cos v \\ -a \sin v \\ 0 \end{bmatrix} \qquad \mathbf{N} \neq 0 \tag{10.5.55}$$

(e) This is caused by the representation.

$$\mathbf{N}_5 = \begin{bmatrix} -2u^2 \ \cos v \\ -2u^2 \ \sin v \\ u \end{bmatrix} \qquad \mathbf{N} = \mathbf{0} \implies (0,\, \alpha,\, \beta) \tag{10.5.56}$$

(f) No such points exist for the given surface.

$$\mathbf{N}_6 = \begin{bmatrix} \sin v \\ -\cos v \\ u \end{bmatrix} \qquad \mathbf{N} \neq \mathbf{0} \tag{10.5.57}$$

(g) This is caused by the representation.

$$\mathbf{N}_7 = \begin{bmatrix} bc \cos u \cos^2 v \\ ac \sin u \cos^2 v \\ ab \sin v \cos v \end{bmatrix} \qquad \mathbf{N} = 0 \implies v = n\pi + \frac{\pi}{2} \tag{10.5.58}$$

(h) This is caused by the representation.

$$\mathbf{N}_8 = \begin{bmatrix} -2b \ u^2 \cosh v \\ 2a \ u^2 \sinh v \\ ab \ u \end{bmatrix} \qquad \mathbf{N} = 0 \implies u = 0 \tag{10.5.59}$$

13. Representing the surface as

$$z = f(x,\, y) \qquad \mathbf{r} = \begin{bmatrix} u \\ v \\ f(u,\, v) \end{bmatrix} \tag{10.5.60}$$

$$g \equiv z - f(u,\, v) = 0 \qquad \nabla g = \begin{bmatrix} -\partial_u f \\ -\partial_v f \\ 1 \end{bmatrix} \tag{10.5.61}$$

14. Finding the parametric representation,

$$4x + 3y + 2z = 12 \qquad \mathbf{r} = \begin{bmatrix} u \\ v \\ 6 - 2u - 1.5v \end{bmatrix} \tag{10.5.62}$$

$$\mathbf{r}_u = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 1 \\ -1.5 \end{bmatrix} \tag{10.5.63}$$

The normal vector is,

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -2 \\ 0 & 1 & -1.5 \end{vmatrix} = \begin{bmatrix} 2 \\ 1.5 \\ 1 \end{bmatrix} \tag{10.5.64}$$

The parameter curves are families of parallel straight lines lying on this plane with either a fixed x or y coordinate.

15. Finding the parametric representation,

$$(x-2)^2 + (y+1)^2 = 25 \qquad \mathbf{r} = \begin{bmatrix} 2 + 5\cos u \\ -1 + 5\sin u \\ v \end{bmatrix} \tag{10.5.65}$$

$$\mathbf{r}_u = \begin{bmatrix} -5\sin u \\ 5\cos u \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.5.66}$$

The normal vector is,

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5\sin u & 5\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} 5\cos u \\ 5\sin u \\ 0 \end{bmatrix} \tag{10.5.67}$$

One set of parameter curves are families straight lines parallel to the z axis lying on the perimeter of the cross-section.

The other parameter curves are circles parallel to the xy plane centered on $(2, -1, z)$ and with radius 5.

16. Finding the parametric representation,

$$x^2 + y^2 + \frac{z^2}{9} = 1 \qquad \mathbf{r} = \begin{bmatrix} \cos u \cos v \\ \sin u \cos v \\ 3 \sin v \end{bmatrix} \tag{10.5.68}$$

$$\mathbf{r}_u = \begin{bmatrix} -5 \sin u \\ 5 \cos u \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.5.69}$$

The normal vector is,

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 \sin u & 5 \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} 5 \cos u \\ 5 \sin u \\ 0 \end{bmatrix} \tag{10.5.70}$$

One set of parameter curves are families straight lines parallel to the z axis lying on the perimeter of the cross-section.

The other parameter curves are circles parallel to the xy plane centered on $(2, -1, z)$ and with radius 5.

17. Finding the parametric representation,

$$x^2 + (y + 2.8)^2 + (z - 3.2)^2 = 1.5^2 \qquad \mathbf{r} = \begin{bmatrix} 1.5 \cos u \cos v \\ -2.8 + 1.5 \sin u \cos v \\ 3.2 + 1.5 \sin v \end{bmatrix} \tag{10.5.71}$$

$$\mathbf{r}_u = \begin{bmatrix} -1.5 \sin u \cos v \\ 1.5 \cos u \cos v \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -1.5 \cos u \sin v \\ -1.5 \sin u \sin v \\ 1.5 \cos v \end{bmatrix} \tag{10.5.72}$$

The normal vector is,

$$\mathbf{N} = 2.25 \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin u \cos v & \cos u \cos v & 0 \\ -\cos u \sin v & -\sin u \sin v & \cos v \end{vmatrix} = 2.25 \begin{bmatrix} \cos u \cos^2 v \\ \sin u \cos^2 v \\ \cos v \sin v \end{bmatrix} \tag{10.5.73}$$

One set of parameter curves are circles parallel to the xy plane (latitudes) at elevation $z = 3.2 + 1.5 \sin v$.

The other parameter curves are circles meridians similar to a globe.

18. Finding the parametric representation,

$$\sqrt{x^2+4y^2} = z^2 \qquad \mathbf{r} = \begin{bmatrix} u \ \cos v \\ 0.5u \ \sin v \\ u \end{bmatrix} \tag{10.5.74}$$

$$\mathbf{r}_u = \begin{bmatrix} \cos v \\ 0.5 \sin v \\ 1 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -u \sin v \\ 0.5u \cos v \\ 0 \end{bmatrix} \tag{10.5.75}$$

The normal vector is,

$$\mathbf{N} = 2.25 \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & 0.5 \sin v & 1 \\ -u \sin v & 0.5u \cos v & 0 \end{vmatrix} = \begin{bmatrix} -0.5u \ \cos v \\ u \ \sin v \\ 0.5u \end{bmatrix} \tag{10.5.76}$$

One set of parameter curves are ellipses parallel to the xy plane at elevation $z = u$(latitudes).
The other parameter curves are straight lines passing through the origin lying on the cone's surface.

19. Finding the parametric representation,

$$x^2 - y^2 = 1 \qquad \mathbf{r} = \begin{bmatrix} \cosh v \\ \sinh v \\ u \end{bmatrix} \tag{10.5.77}$$

$$\mathbf{r}_u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} \sinh v \\ \cosh v \\ 0 \end{bmatrix} \tag{10.5.78}$$

The normal vector is,

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ \sinh v & \cosh v & 0 \end{vmatrix} = \begin{bmatrix} -\cosh v \\ \sinh v \\ 0 \end{bmatrix} \tag{10.5.79}$$

One set of parameter curves are hyperbolae parallel to the xy plane at elevation $z = u$.
The other parameter curves are straight lines parallel to the z axis passing through $(\cosh v, \sinh v, 0)$.

20. Tangent planes. Examples TBC.

(a) Using the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$

$$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{N} \qquad \mathbf{r}^* \cdot \mathbf{N} = \mathbf{r}(P) \cdot \mathbf{N}(P) \tag{10.5.80}$$

The tangent plane is defined using the surface normal at P which also happnes to be normal to the plane.

(b) Using the fact that the gradient of a scalar function at a point is the surface normal at that point and the result from part a,

$$g(x,\, y,\, z) = 0 \qquad \mathbf{N} = \nabla g \tag{10.5.81}$$

$$\mathbf{r}^* \cdot (\nabla g) = \mathbf{r}(P) \cdot (\nabla g)(P) \tag{10.5.82}$$

(c) Using the results from parts a and b,

$$S : z = f(x,\, y) \qquad \mathbf{r} = \begin{bmatrix} x \\ y \\ f(x,\, y) \end{bmatrix} \tag{10.5.83}$$

$$\mathbf{r}_x = \begin{bmatrix} 1 \\ 0 \\ \partial_x f \end{bmatrix} \qquad \mathbf{r}_y = \begin{bmatrix} 0 \\ 1 \\ \partial_y f \end{bmatrix} \tag{10.5.84}$$

$$\mathbf{N} = \mathbf{r}_x \times \mathbf{r}_y \qquad \mathbf{N} = \begin{bmatrix} -\partial_x f \\ -\partial_y f \\ 1 \end{bmatrix} \tag{10.5.85}$$

$$\mathbf{r}^* \cdot \mathbf{N} = \mathbf{r}(P) \cdot \mathbf{N}(P) \tag{10.5.86}$$

This can be rearranged into the required form.

10.6 Surface Integrals

1. The surface is a plane, with domain $u \in [0, 1.5], \;\; v \in [-2, 2]$

$$\mathbf{F} = \begin{bmatrix} -x^2 \\ y^2 \\ 0 \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} u \\ v \\ 3u - 2v \end{bmatrix} \tag{10.6.1}$$

$$\mathbf{r}_u = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \tag{10.6.2}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{vmatrix} \qquad \hat{\mathbf{n}} = \frac{1}{\sqrt{14}} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \tag{10.6.3}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \;\, \mathrm{d}A \qquad = \int_{-2}^{2} \left[\int_0^{1.5} \left(3u^2 + 2v^2 \right) \mathrm{d}u \right] \mathrm{d}v \tag{10.6.4}$$

$$= \int_{-2}^{2} \left[u^3 + 2v^2 \; u \right]_0^{1.5} \; \mathrm{d}v \qquad = \int_{-2}^{2} \left(1.5^3 + 3v^2 \right) \; \mathrm{d}v \tag{10.6.5}$$

$$= \left[\frac{27v}{8} + v^3 \right]_{-2}^{2} \qquad = 29.5 \tag{10.6.6}$$

2. The surface is a plane, with domain $u \in [0, 1 - v], \;\; v \in [0, 1]$

$$\mathbf{F} = \begin{bmatrix} e^y \\ e^x \\ 1 \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} u \\ v \\ 1 - u - v \end{bmatrix} \tag{10.6.7}$$

$$\mathbf{r}_u = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{10.6.8}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \qquad \hat{\mathbf{n}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{10.6.9}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}A \qquad = \int_0^1 \left[\int_0^{1-v} \left(e^u + e^v + 1 \right) \mathrm{d}u \right] \mathrm{d}v \tag{10.6.10}$$

$$= \int_0^1 \left[e^u + u\ e^v + u \right]_0^{1-v} \mathrm{d}v \qquad = \int_0^1 \left(e^{1-v} + e^v - ve^v - v \right) \mathrm{d}v \tag{10.6.11}$$

$$= \left[e^v - \frac{v^2}{2} - e^{1-v} + (1-v)e^v \right]_0^1 \qquad = 2e - 3.5 \tag{10.6.12}$$

3. The surface is a sphere in the first octant, $u \in [0, \pi/2], \ \ v \in [0, \pi/2]$

$$\mathbf{F} = \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{bmatrix} \tag{10.6.13}$$

$$\mathbf{r}_u = \begin{bmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -\cos u \sin v \\ \cos u \cos v \\ 0 \end{bmatrix} \tag{10.6.14}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin u \cos v & -\sin u \sin v & \cos u \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} -\cos^2 u \cos v \\ \cos^2 u \sin v \\ -\sin u \cos u \end{bmatrix} \tag{10.6.15}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}A \tag{10.6.16}$$

$$= \int_0^{\pi/2} \left[\int_0^{\pi/2} \left(\cos^3 u \cos v \sin v \right) \mathrm{d}u \right] \mathrm{d}v \tag{10.6.17}$$

$$= \int_0^{\pi/2} \frac{\sin(2v)}{2} \left[\sin u - \frac{\sin^3 u}{3} \right]_0^{\pi/2} \mathrm{d}v \qquad = \int_0^{\pi/2} \left(\frac{\sin(2v)}{3} \right) \mathrm{d}v \tag{10.6.18}$$

$$= \left[\frac{\cos(2v)}{6} \right]_{\pi/2}^0 \qquad = \frac{1}{3} \tag{10.6.19}$$

4. The surface is a circular cylinder oriented along the z axis, with domain $u \in [0, \pi/2], \;\; v \in [0, 2]$

$$\mathbf{F} = \begin{bmatrix} e^y \\ -e^z \\ e^x \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} 5 \cos u \\ 5 \sin u \\ v \end{bmatrix} \tag{10.6.20}$$

$$\mathbf{r}_u = \begin{bmatrix} -5 \sin u \\ 5 \cos u \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.6.21}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 \sin u & 5 \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} 5 \cos u \\ 5 \sin u \\ 0 \end{bmatrix} \tag{10.6.22}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \;\, \mathrm{d}A \tag{10.6.23}$$

$$= \int_0^2 \left[\int_0^{\pi/2} \left(5 \cos u \; e^{5 \sin u} - 5 \sin u \; e^v \right) \mathrm{d}u \right] \mathrm{d}v \tag{10.6.24}$$

$$= \int_0^2 \left[5e^v \cos u + e^{5 \sin u} \right]_0^{\pi/2} \mathrm{d}v \qquad = \int_0^2 \left(e^5 - 5e^v - 1 \right) \; \mathrm{d}v \tag{10.6.25}$$

$$= \left[v \; e^5 - 5e^v - v \right]_0^2 \qquad = 2e^5 - 5e^2 + 3 \tag{10.6.26}$$

5. The surface is a paraboloid of revolution oriented along the z axis, with domain $u \in [0, 4], \;\; v \in [-\pi, \pi]$

$$\mathbf{F} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} u \;\; \cos v \\ u \;\; \sin v \\ u^2 \end{bmatrix} \tag{10.6.27}$$

$$\mathbf{r}_u = \begin{bmatrix} \cos v \\ \sin v \\ 2u \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -u \;\; \sin v \\ u \;\; \cos v \\ 0 \end{bmatrix} \tag{10.6.28}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} -2u^2 \cos v \\ -2u^2 \sin v \\ u \end{bmatrix} \tag{10.6.29}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}A \tag{10.6.30}$$

$$= \int_{-\pi}^{\pi} \left[\int_0^4 \Big(-2\cos^2 v - 2\sin^2 v + 1 \Big) \ u^3 \ \mathrm{d}u \right] \mathrm{d}v \tag{10.6.31}$$

$$= \int_{-\pi}^{\pi} \left[\frac{-u^4}{4} \right]_0^4 \ \mathrm{d}v \qquad = \int_{-\pi}^{\pi} \Big(-64 \Big) \ \mathrm{d}v \tag{10.6.32}$$

$$= -64 \Big[v \Big]_{-\pi}^{\pi} \qquad = -128\pi \tag{10.6.33}$$

6. The surface is a parabolic canal resting on the $z = x$ plane, with domain $v \in [0, u], \ u \in [0, 1]$

$$\mathbf{F} = \begin{bmatrix} \cosh y \\ 0 \\ \sinh x \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} u \\ v \\ u + v^2 \end{bmatrix} \tag{10.6.34}$$

$$\mathbf{r}_u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 1 \\ 2v \end{bmatrix} \tag{10.6.35}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 1 \\ 0 & 1 & 2v \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} -1 \\ -2v \\ 1 \end{bmatrix} \tag{10.6.36}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}A \qquad = \int_0^1 \left[\int_0^u \Big(-\cosh v + \sinh u \Big) \ \mathrm{d}v \right] \mathrm{d}u \tag{10.6.37}$$

$$= \int_0^1 \Big[-\sinh v + v \sinh u \Big]_0^u \ \mathrm{d}u \qquad = \int_0^1 \Big((u-1) \sinh u \Big) \ \mathrm{d}u \tag{10.6.38}$$

$$= \Big[(u-1)\cosh u - \sinh u \Big]_0^1 \qquad = 1 - \sinh(1) \tag{10.6.39}$$

7. The surface is a cylinder $z = x$ plane, with domain $v \in [0, u],\ \ u \in [0, \pi/4]$

$$\mathbf{F} = \begin{bmatrix} 0 \\ \sin y \\ \cos z \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} u^2 \\ u \\ v \end{bmatrix} \tag{10.6.40}$$

$$\mathbf{r}_u = \begin{bmatrix} 2u \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.6.41}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2u & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} 1 \\ -2u \\ 0 \end{bmatrix} \tag{10.6.42}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ \mathrm{d}A \qquad = \int_0^{\pi/4} \left[\int_0^u \Big(-2u \sin u \Big)\, \mathrm{d}v \right] \mathrm{d}u \tag{10.6.43}$$

$$= \int_0^{\pi/4} \Big[(-2u \sin u)\ v \Big]_0^u \ \mathrm{d}u \qquad = \int_0^{\pi/4} \left(-2u^2\ \sin u \right)\ \mathrm{d}u \tag{10.6.44}$$

$$= \Big[(2u^2 - 4) \cos u - 4u \sin u \Big]_0^{\pi/4} \qquad = \frac{\pi^2 - 8\pi - 32}{8\sqrt{2}} + 4 \tag{10.6.45}$$

8. The surface is a cylinder whose axis is the x axis, with domain $v \in [2, 5],\ \ u \in [0, \pi/2]$

$$\mathbf{F} = \begin{bmatrix} \tan(xy) \\ x \\ y \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} v \\ \cos u \\ \sin u \end{bmatrix} \tag{10.6.46}$$

$$\mathbf{r}_u = \begin{bmatrix} 0 \\ -\sin u \\ \cos u \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{10.6.47}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\sin u & \cos u \\ 1 & 0 & 0 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} 0 \\ \cos u \\ \sin u \end{bmatrix} \tag{10.6.48}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \; \mathrm{d}A \tag{10.6.49}$$

$$= \int_0^{\pi/2} \left[\int_2^5 \Big(v \cos u + \cos u \sin u \Big) \, \mathrm{d}v \right] \mathrm{d}u \tag{10.6.50}$$

$$= \int_0^{\pi/2} \left[\frac{v^2 \; \cos u}{2} + \frac{v \; \sin(2u)}{2} \right]_2^5 \; \mathrm{d}u \tag{10.6.51}$$

$$= \int_0^{\pi/2} \Bigg(10.5 \cos u + 1.5 \sin(2u) \Bigg) \; \mathrm{d}u \tag{10.6.52}$$

$$= \Bigg[10.5 \sin(u) - 0.75 \cos(2u) \Bigg]_0^{\pi/2} = 12 \tag{10.6.53}$$

9. The surface is a cylinder along the y axis, with domain $v \in [0, 5], \;\; u \in [\pi/4, \pi/2]$

$$\mathbf{F} = \begin{bmatrix} 0 \\ \sinh z \\ \cosh x \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} 2 \cos u \\ v \\ 2 \sin u \end{bmatrix} \tag{10.6.54}$$

$$\mathbf{r}_u = \begin{bmatrix} -2 \sin u \\ 0 \\ 2 \cos u \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{10.6.55}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 \sin u & 0 & 2 \cos u \\ 0 & 1 & 0 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} -2 \cos u \\ 0 \\ -2 \sin u \end{bmatrix} \tag{10.6.56}$$

Finding the integral, Answer Key is incorrect

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \; \mathrm{d}A \tag{10.6.57}$$

$$= \int_{\pi/4}^{\pi/2} \left[\int_0^5 \Big(-2 \sin u \;\; \cosh(2 \cos u) \Big) \, \mathrm{d}v \right] \mathrm{d}u \tag{10.6.58}$$

$$= 5 \int_{\pi/4}^{\pi/2} \Bigg(-2 \sin u \;\; \cosh(2 \cos u) \Bigg) \; \mathrm{d}u \tag{10.6.59}$$

$$= 5 \Bigg[\sinh(2 \cos u) \Bigg]_{\pi/4}^{\pi/2} = -5 \sinh(\sqrt{2}) \tag{10.6.60}$$

10. The surface is a cone along the z axis, with domain $v \in [0, \pi],\ \ u \in [0, 2]$

$$\mathbf{F} = \begin{bmatrix} y^2 \\ x^2 \\ z^4 \end{bmatrix} \qquad S : \mathbf{r} = \begin{bmatrix} u \cos v \\ u \sin v \\ 4u \end{bmatrix} \tag{10.6.61}$$

$$\mathbf{r}_u = \begin{bmatrix} \cos v \\ \sin v \\ 4 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -u \sin v \\ u \cos v \\ 0 \end{bmatrix} \tag{10.6.62}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 4 \\ -u \sin v & u \cos v & 0 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} -4u \cos v \\ -4u \sin v \\ u \end{bmatrix} \tag{10.6.63}$$

Finding the integral,

$$I = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}A \tag{10.6.64}$$

$$= \int_0^2 \left[\int_0^\pi \Big(-4u^3 \cos v \sin^2 v - 4u^3 \sin v \cos^2 v + 256u^5 \Big) \mathrm{d}v \right] \mathrm{d}u \tag{10.6.65}$$

$$= \int_0^2 \left[\frac{4u^3}{3} \left(-\sin^3 v + \cos^3 v \right) + 256u^5 \ v \right]_0^\pi \ \mathrm{d}u \tag{10.6.66}$$

$$= 5 \int_0^2 \left(\frac{-8u^3}{3} + 256\pi \ u^5 \right) \ \mathrm{d}u \tag{10.6.67}$$

$$= \left[\frac{-2u^4}{3} + \frac{256\pi \ u^6}{6} \right]_0^2 = \frac{-64 + 16384 \ \pi}{6} \tag{10.6.68}$$

11. Sympy program written. TBC.

12. The surface is a plane, with domain $u \in [0, 1-v], \; v \in [0, 1]$

$$G = \cos x + \sin x \qquad S : \mathbf{r} = \begin{bmatrix} u \\ v \\ 1-u-v \end{bmatrix} \tag{10.6.69}$$

$$\mathbf{r}_u = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{10.6.70}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{10.6.71}$$

Finding the integral,

$$I = \iint_S G(\mathbf{r}) \, \mathrm{d}A \tag{10.6.72}$$

$$= \sqrt{3} \int_0^1 \left[\int_0^{1-v} \Big(\cos u + \sin u \Big) \, \mathrm{d}u \right] \mathrm{d}v \tag{10.6.73}$$

$$= \sqrt{3} \int_0^1 \Big[\sin u - \cos u \Big]_0^{1-v} \mathrm{d}v \tag{10.6.74}$$

$$= \sqrt{3} \int_0^1 \Big(\sin(1-v) - \cos(1-v) + 1 \Big) \, \mathrm{d}v \tag{10.6.75}$$

$$= \sqrt{3} \Big[\cos(1-v) + \sin(1-v) + v \Big]_0^1 = \sqrt{3} \, [2 - \cos(1) - \sin(1)] \tag{10.6.76}$$

13. The surface is a plane, with domain $u \in [0, \pi], \; v \in [0, u]$

$$G = x + y + z \qquad C : x + 2y - z = 0 \tag{10.6.77}$$

$$S : \mathbf{r} = \begin{bmatrix} u \\ v \\ u + 2v \end{bmatrix} \qquad \mathbf{N} = \nabla f = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \tag{10.6.78}$$

Finding the integral,

$$I = \iint_S G(\mathbf{r})\ \mathrm{d}A \qquad |\mathbf{N}| = \sqrt{6} \tag{10.6.79}$$

$$= \sqrt{6}\ \int_0^{\pi} \left[\int_0^{u} \Big(2u + 3v\Big)\ \mathrm{d}v \right]\ \mathrm{d}u \qquad = \sqrt{6}\ \int_0^{\pi} \left[2u\ v + \frac{3v^2}{2} \right]_0^{u}\ \mathrm{d}u \tag{10.6.80}$$

$$= \sqrt{6}\ \int_0^{\pi} \left(3.5\ u^2 \right)\ \mathrm{d}u \qquad = \sqrt{6}\ \left[\frac{7u^3}{6} \right]_0^{\pi} = \frac{7\pi^3}{\sqrt{6}} \tag{10.6.81}$$

14. The surface is a sphere in the first and second octants, $u \in [0, \pi/2],\ \ v \in [0, \pi]$

$$g = ax + by + cz \qquad f : x^2 + y^2 + z^2 = 1,\ \ y = 0,\ \ z = 0 \tag{10.6.82}$$

$$\mathbf{N} = \nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} \cos u\ \cos v \\ \cos u\ \sin v \\ \sin u \end{bmatrix} \tag{10.6.83}$$

Finding the integral,

$$I = \iint_S G(\mathbf{r})\ \ \mathrm{d}A \tag{10.6.84}$$

$$|\mathbf{N}| = 2\sqrt{x^2 + y^2 + z^2} = 2 \tag{10.6.85}$$

$$= 2\ \int_0^{\pi} \left[\int_0^{\pi/2} \Big(a\ \cos u\ \cos v + b\ \sin u\ \sin v + c\ \sin u \Big)\ \mathrm{d}u \right]\ \mathrm{d}v \tag{10.6.86}$$

$$= 2\ \int_0^{\pi} \Big[a\ \sin u\ \cos v - b\ \cos u\ \sin v - c\ \cos u \Big]_0^{\pi/2}\ \ \mathrm{d}v \tag{10.6.87}$$

$$= 2\ \int_0^{\pi} \left(a\ \cos v + b\ \sin v + c \right)\ \ \mathrm{d}v \tag{10.6.88}$$

$$= 2\ \Big[a\ \sin v - b\ \cos v + cv \Big]_0^{\pi} = 4b + 2\pi\ c \tag{10.6.89}$$

15. The surface is a sphere in the first and second octants, $u \in [0, 1]$, $v \in [-2, 2]$

$$g = (1 + 9xz)^{3/2} \qquad \mathbf{r} : \begin{bmatrix} u \\ v \\ u^3 \end{bmatrix} \tag{10.6.90}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 3u^2 \\ 0 & 1 & 0 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} -3u^2 \\ 0 \\ 1 \end{bmatrix} \tag{10.6.91}$$

Finding the integral,

$$I = \iint_S G(\mathbf{r}) \ \mathrm{d}A \qquad |\mathbf{N}| = \sqrt{1 + 9u^4} \tag{10.6.92}$$

$$= \int_{-2}^{2} \left[\int_0^1 \left((1 + 9u^4)^2 \right) \mathrm{d}u \right] \mathrm{d}v \tag{10.6.93}$$

$$= \int_{-2}^{2} \left[u + \frac{18u^5}{5} + 9u^9 \right]_0^1 \ \mathrm{d}v \tag{10.6.94}$$

$$= \int_{-2}^{2} \left(13.6 \right) \ \mathrm{d}v = \left[13.6v \right]_{-2}^{2} = 54.4 \tag{10.6.95}$$

16. The surface is a sphere in the first and second octants, $u \in [1, 3]$, $v \in [0, \pi/2]$

$$g = \arctan(y/x) \qquad \mathbf{r} : \begin{bmatrix} u \cos v \\ u \sin v \\ u^2 \end{bmatrix} \tag{10.6.96}$$

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} \qquad \mathbf{N} = \begin{bmatrix} -2u^2 \ \cos v \\ -2u^2 \ \sin v \\ u \end{bmatrix} \tag{10.6.97}$$

Finding the integral,

$$I = \iint_S G(\mathbf{r}) \ \mathrm{d}A \qquad\qquad |\mathbf{N}| = \sqrt{4u^4 + u^2} \tag{10.6.98}$$

$$= \int_0^{\pi/2} \left[\int_1^3 \left(vu \ \sqrt{4u^2+1} \right) \mathrm{d}u \right] \mathrm{d}v \tag{10.6.99}$$

$$= \int_0^{\pi/2} \frac{v}{12} \left[(1+4u^2)^{3/2} \right]_1^3 \ \mathrm{d}v \tag{10.6.100}$$

$$= \int_0^{\pi/2} \left(\frac{37^{3/2} - 5^{3/2}}{12} \ v \right) \ \mathrm{d}v = \left[\frac{\lambda \ v^2}{2} \right]_0^{\pi/2} = 21.98 \tag{10.6.101}$$

17. Youtube videos animating this process.
`https://www.youtube.com/watch?v=X1Q0ipIVFPk`

18. TBC

19. Using the mass per unit area ρ of a surface S,

$$\bar{x} = \frac{1}{M} \ \iint_S x \ \mathrm{d}m = \frac{1}{M} \ \iint_S x\rho \ \ \mathrm{d}A \tag{10.6.102}$$

Similary for the other two coordinates.

20. Using the definition of moment of inertia,

$$I_a = \iint_S d_a^2 \ \ \mathrm{d}m \qquad\qquad I_x = \iint_S (y^2 + z^2) \ \rho \ \ \mathrm{d}A \tag{10.6.103}$$

Similarly for the other coordinates. This uses the fact that the three corrdinates are mutually orthogonal and then Pythogoras' theorem to get $d_x^2 = y^2 + z^2$ and so on.

21. Moment of inertia about the line $y = x, \ \ z = 0$,

$$I_a = \iint_S d_a^2 \ \rho \ \ \mathrm{d}A \qquad\qquad \mathbf{r} = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp} \tag{10.6.104}$$

$$d_a = \frac{|\mathbf{r} \times \mathbf{d}|}{|\mathbf{d}|} \qquad\qquad I_a = \frac{1}{2} \ \iint_S \left[(x-y)^2 + 2z^2 \right] \rho \ \ \mathrm{d}A \tag{10.6.105}$$

This uses the formula for the distance of a point from a given line with direction $\mathbf{d}$ and with $\mathbf{r}$ pointing from any point on the line to the given point P,

$$d_{\perp} = \frac{|\mathbf{r} \times \mathbf{d}|}{|\mathbf{d}|} \tag{10.6.106}$$

22. Moment of inertial about the axis $z = h/2$ lying in the xz plane.

$$S : x^2 + y^2 = 1 \quad z \in [0, h] \tag{10.6.107}$$

$$\mathbf{r} = \begin{bmatrix} \cos v \\ \sin v \\ u \end{bmatrix} \quad u \in [0, h] \quad v \in [0, 2\pi] \tag{10.6.108}$$

$$I_a = \iint_S d_a^2 \, \rho \, \mathrm{d}A = \int_0^h \left[\int_0^{2\pi} \sin^2 v + (u - h/2)^2 \, \mathrm{d}v \right] \mathrm{d}u \tag{10.6.109}$$

$$= \int_0^h \left[\frac{v}{2} - \frac{\sin(2v)}{4} + v \, (u - h/2)^2 \right]_0^{2\pi} \mathrm{d}u \tag{10.6.110}$$

$$= \pi \int_0^h 1 + 2(u - h/2)^2 \, \mathrm{d}u \tag{10.6.111}$$

$$= \pi \left[u + \frac{2}{3} \, (u - h/2)^3 \right]_0^h = \pi \left[h + \frac{h^3}{6} \right] \tag{10.6.112}$$

23. Moment of inertial about the axis z axis. THe limits in the uv plane are $u \in [0, h], \;\; v \in [0, 2\pi]$

$$S : x^2 + y^2 = z^2 \quad z \in [0, h] \tag{10.6.113}$$

$$\mathbf{r} = \begin{bmatrix} u \cos v \\ u \sin v \\ u \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} -u \cos v \\ -u \sin v \\ u \end{bmatrix} \qquad |\mathbf{N}| \qquad = \sqrt{2} u \tag{10.6.114}$$

$$I_a = \iint_S d_a^2 \, \rho \, \mathrm{d}A = \int_0^h \left[\int_0^{2\pi} \sqrt{2} u^3 \left(\cos^2 v + \sin^2 v \right) \, \mathrm{d}v \right] \mathrm{d}u \tag{10.6.115}$$

$$= \sqrt{2} \int_0^h \left[v \, u^3 \right]_0^{2\pi} \mathrm{d}u = 2\sqrt{2}\pi \int_0^h u^3 \, \mathrm{d}u = \frac{\pi \, h^4}{\sqrt{2}} \tag{10.6.116}$$

24. To prove Steiner's theorem, assume that the axis B is the z axis and the other axis is displaced from it in the x direction.

The two axes pass through $(0, 0)$ and $(d, 0)$ respectively in the xy plane.

$$I_B = \iint \left[x^2 + y^2 \right] \rho \;\; \mathrm{d}x \, \mathrm{d}y \qquad I_K = \iint \left[(x - d)^2 + y^2 \right] \rho \;\; \mathrm{d}x \, \mathrm{d}y \tag{10.6.117}$$

$$I_K - I_B = \iint (d^2 - 2dx) \; \rho \, \mathrm{d}x \, \mathrm{d}y \qquad = \iint (d^2 - 2dx) \;\; \mathrm{d}m \tag{10.6.118}$$

The integral $\int x \;\; \mathrm{d}m$ is the x co-ordinate of the center of gravity which is zero by definition. This leaves,

$$I_k = I_B + d^2 \iint \mathrm{d}m = I_B + Md^2 \tag{10.6.119}$$

25. With the axis B being the z axis, and the limits in the uv plane
$u \in [-\pi/2, \pi/2],\ \ v \in [0, 2\pi]$

$$\mathbf{r} = \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} \cos^2 u \cos v \\ \cos^2 u \sin v \\ \cos u \sin u \end{bmatrix} \qquad |\mathbf{N}| = \cos u \tag{10.6.120}$$

$$I_B = \iint_S (y^2 + x^2)\, \mathrm{d}A \qquad = \int_{-\pi/2}^{\pi/2} \left[\int_0^{2\pi} (\cos^3 u)\, \mathrm{d}v \right] \mathrm{d}u \tag{10.6.121}$$

$$= 2\pi \int_{-\pi/2}^{\pi/2} \cos^3 u \ \mathrm{d}u \qquad = 2\pi \left[\sin u - \frac{\sin^3 u}{3} \right]_{-\pi/2}^{\pi/2} \tag{10.6.122}$$

$$= \frac{8\pi}{3} \tag{10.6.123}$$

Using the density $\rho = 1$, the mass $M = A\rho = 4\pi$

$$I_K = I_B + Md^2 = \frac{8\pi}{3} + 4\pi \cdot 1^2 = \frac{20\pi}{3} \tag{10.6.124}$$

26. First fundamental form,

(a) Proving the relation for arc length,

$$\mathrm{d}s^2 = E\ \ \mathrm{d}u^2 + 2F\ \ \mathrm{d}u\, \mathrm{d}v + G\ \ \mathrm{d}v^2 \qquad l = \int_a^b \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\ \ \mathrm{d}t \tag{10.6.125}$$

$$\mathbf{r}' = \mathbf{r}_u\ u' + \mathbf{r}_v\ v' \qquad \mathbf{r}' \cdot \mathbf{r}' = E\ u'^2 + G\ v'^2 + 2F\ u'v' \tag{10.6.126}$$

$$l = \int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\ \ \mathrm{d}t \tag{10.6.127}$$

(b) Angle of intersection between two curves,

$$\mathbf{r}_1 = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \tag{10.6.128}$$

$$\mathbf{a} = \mathbf{r}_u\ g' + \mathbf{r}_v\ h' \qquad \mathbf{b} = \mathbf{r}_u\ p' + \mathbf{r}_v\ q' \tag{10.6.129}$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos \gamma \tag{10.6.130}$$

(c) The length of the normal vector is,

$$|\mathbf{N}|^2 = |\mathbf{r}_u \times \mathbf{r}_v|^2 \qquad = |\mathbf{r}_u|^2\,|\mathbf{r}_v|^2 - |\mathbf{r}_u \cdot \mathbf{r}_v|^2 \tag{10.6.131}$$

$$= EG - F^2 \tag{10.6.132}$$

(d) For polar coordinates,

$$u = r \qquad v = \theta \tag{10.6.133}$$

$$x = u\cos v \qquad y = u\sin v \tag{10.6.134}$$

$$\mathbf{r}_u = \begin{bmatrix} \cos v \\ \sin v \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} -u\sin v \\ u\cos v \end{bmatrix} \tag{10.6.135}$$

$$E = \mathbf{r}_u \cdot \mathbf{r}_u \qquad = 1 \tag{10.6.136}$$

$$G = \mathbf{r}_v \cdot \mathbf{r}_v \qquad = u^2 \tag{10.6.137}$$

$$F = 0 \tag{10.6.138}$$

To find the area of a disk of radius a,

$$A = \iint_S \mathrm{d}A \qquad = \int_0^a \int_0^{2\pi} u \;\mathrm{d}u\,\mathrm{d}v \tag{10.6.139}$$

$$= 2\pi \int_0^a u \;\mathrm{d}u \qquad = 2\pi \left[\frac{a^2}{2}\right] = \pi a^2 \tag{10.6.140}$$

(e) For the torus,

$$\mathbf{r} = \begin{bmatrix} (a + b\cos v)\;\cos u \\ (a + b\cos v)\;\sin u \\ b\sin v \end{bmatrix} \tag{10.6.141}$$

$$\mathbf{r}_u = \begin{bmatrix} -(a + b\cos v)\;\sin u \\ (a + b\cos v)\;\cos u \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} (-b\sin v)\;\cos u \\ (-b\sin v)\;\sin u \\ b\cos v \end{bmatrix} \tag{10.6.142}$$

$$E = a + b\cos v \qquad G = b, \qquad F = 0 \tag{10.6.143}$$

10.6.144

Using the first fundamental form to find the area of the torus,

$$A = \int_0^{2\pi} \left[\int_0^{2\pi} (a + b \cos v) \, \mathrm{d}u \right] b \;\; \mathrm{d}v \tag{10.6.145}$$

$$= 2b\pi \int_0^{2\pi} (a + b \cos v) \, \mathrm{d}v = 2b\pi \left[av + b \sin v \right]_0^{2\pi} \tag{10.6.146}$$

$$= 2b\pi(2\pi a + 0) = 4\pi^2 \; ab \tag{10.6.147}$$

The meridian length is $2\pi b$ and the length of the path traced by the center is $2\pi a$, arrives at the same value of A.

(f) TBC. Sphere, Cone, Cylinder, Torus, Plane, Ellipsoid, Paraboloid, Hyperboloid.

10.7 Triple Integrals, Divergence Theorem of Gauss

1. Finding the mass, given density ρ,

$$\rho = x^2 + y^2 + z^2 \tag{10.7.1}$$

$$T : x \in [-4, 4] \quad y \in [-1, 1] \quad z \in [0, 2] \tag{10.7.2}$$

$$I_1 = \int_{-4}^{4} (x^2 + y^2 + z^2) \;\; \mathrm{d}x = \left[\frac{x^3}{3} + xy^2 + xz^2 \right]_{-4}^{4} = \frac{128}{3} + 8(y^2 + z^2) \tag{10.7.3}$$

$$I_2 = \int_{-1}^{1} I_1 \, \mathrm{d}y = \int_{-1}^{1} \left(\frac{128}{3} + 8y^2 + 8z^2 \right) \mathrm{d}y \tag{10.7.4}$$

$$= \left[\frac{128y + 8y^3}{3} + 8z^2 \; y \right]_{-1}^{1} = \frac{272}{3} + 16z^2 \tag{10.7.5}$$

$$I_3 = \int_0^2 I_2 \, \mathrm{d}z = \int_0^2 \left(\frac{272}{3} + 16z^2 \right) \mathrm{d}z = \left[\frac{272z + 16z^3}{3} \right]_0^2 = 224 \tag{10.7.6}$$

2. Finding the mass, given density ρ,

$$\rho = xyz \quad 10.7.7$$

$$T : x \in [0,\, a] \quad y \in [0,\, b] \quad z \in [0,\, c] \quad 10.7.8$$

$$I_1 = \int_0^a (xyz)\ \mathrm{d}x = \left[yz\ \frac{x^2}{2} \right]_0^a = \frac{a^2}{2}\ yz \quad 10.7.9$$

$$I_2 = \int_0^b I_1\ \mathrm{d}y = \int_0^b \left(\frac{a^2}{2}\ yz \right) \mathrm{d}y \quad 10.7.10$$

$$= \left[\frac{a^2 y^2}{4}\ z \right]_0^b = \frac{a^2 b^2}{4}\ z \quad 10.7.11$$

$$I_3 = \int_0^c I_2\ \mathrm{d}z = \int_0^c \left(a^2 b^2\ \frac{z}{4} \right) \mathrm{d}z = \left[\frac{a^2 b^2\ z^2}{8} \right]_0^c = \frac{(abc)^2}{8} \quad 10.7.12$$

3. Finding the mass, given density ρ,

$$\rho = \exp(-x - y - z) \quad 10.7.13$$

$$T : x \in [0,\, 1 - y] \quad y \in [0,\, 1] \quad z \in [0,\, 2] \quad 10.7.14$$

$$I_1 = \int_0^{1-y} \exp(-x - y - z)\ \mathrm{d}x = \left[-\exp(-y - z)e^{-x} \right]_0^{1-y} \quad 10.7.15$$

$$= \exp(-y - z)[1 - e^{y-1}] \quad 10.7.16$$

$$I_2 = \int_0^b I_1\ \mathrm{d}y = \int_0^1 \left(e^{-y-z} - e^{-z-1} \right) \mathrm{d}y \quad 10.7.17$$

$$= \left[-\ e^{-z} e^{-y} - y\ e^{-z-1} \right]_0^1 = e^{-z}(1 - 2e^{-1}) \quad 10.7.18$$

$$I_3 = \int_0^c I_2\ \mathrm{d}z = \int_0^2 \left(e^{-z}(1 - 2e^{-1}) \right) \mathrm{d}z = \left[(2e^{-1} - 1)e^{-z} \right]_0^2 \quad 10.7.19$$

$$= (2e^{-1} - 1)(e^{-2} - 1) \quad 10.7.20$$

4. Finding the mass, given density ρ,

$$\rho = \exp(-x-y-z) \quad 10.7.21$$

$$T : x \in [0, 3] \quad y \in [0, 3-x] \quad z \in [0, 3-x-y] \quad 10.7.22$$

$$I_1 = \int_0^{3-x-y} \exp(-x-y-z) \;\, \mathrm{d}z = \Bigg[-\exp(-x-y)e^{-z} \Bigg]_0^{3-x-y} \quad 10.7.23$$

$$= \exp(-x-y)[1 - e^{x+y-3}] \quad 10.7.24$$

$$I_2 = \int_0^{3-x} I_1 \; \mathrm{d}y = \int_0^{3-x} \Bigg(e^{-x-y} - e^{-3} \Bigg) \; \mathrm{d}y \quad 10.7.25$$

$$= \Bigg[-e^{-x}e^{-y} - y \; e^{-3} \Bigg]_0^{3-x} = e^{-x} + (x-4)e^{-3} \quad 10.7.26$$

$$I_3 = \int_0^{3} I_2 \; \mathrm{d}z = \int_0^{3} \Bigg(e^{-x} + (x-4)e^{-3} \Bigg) \; \mathrm{d}x \quad 10.7.27$$

$$= \Bigg[-e^{-x} + \frac{(x-4)^2}{2} \; e^{-3} \Bigg]_0^{3} = 1 - 8.5e^{-3} \quad 10.7.28$$

5. Finding the mass, given density ρ,

$$\rho = \sin(2x)\,\cos(2y) \quad 10.7.29$$

$$T : x \in [0, \pi/4] \quad y \in [\pi/4 - x, \pi/4] \quad z \in [0, 6] \quad 10.7.30$$

$$I_1 = \int_{\pi/4-x}^{\pi/4} \sin(2x)\,\cos(2y) \;\, \mathrm{d}y = \Bigg[\frac{\sin(2x)\,\sin(2y)}{2} \Bigg]_{\pi/4-x}^{\pi/4} \quad 10.7.31$$

$$= \frac{\sin(2x)}{2}\,[1 - \cos(2x)] \quad 10.7.32$$

$$I_2 = \int_0^{\pi/4} I_1 \; \mathrm{d}x = \int_0^{\pi/4} \Bigg(\frac{2\,\sin(2x) - \sin(4x)}{4} \Bigg) \; \mathrm{d}x \quad 10.7.33$$

$$= \Bigg[\frac{-\cos(2x)}{4} + \frac{\cos(4x)}{16} \Bigg]_0^{\pi/4} = \frac{1}{8} \quad 10.7.34$$

$$I_3 = \int_0^{6} I_2 \; \mathrm{d}z = \int_0^{6} \Bigg(\frac{1}{8} \Bigg) \; \mathrm{d}z = \Bigg[\frac{z}{8} \Bigg]_0^{6} = \frac{3}{4} \quad 10.7.35$$

6. Finding the mass, given density ρ,

$$\rho = x^2 y^2 z^2 \qquad x = r\cos\theta \qquad z = r\sin\theta \tag{10.7.36}$$

$$T : r \in [0,\, 4] \qquad \theta \in [0,\, 2\pi] \qquad y \in [-4,\, 4] \tag{10.7.37}$$

$$I_1 = \int_0^4 \left(y^2\ r^5 \sin^2\theta \cos^2\theta\right)\ \mathrm{d}r = \left[\frac{y^2\,\sin^2(\theta)\,\cos^2(\theta)\ r^6}{6}\right]_0^4 \tag{10.7.38}$$

$$= \frac{y^2\,\sin^2(\theta)\,\cos^2(\theta)\ 4^6}{6} \tag{10.7.39}$$

$$I_2 = \int_0^{2\pi} I_1\ \mathrm{d}\theta = \int_0^{2\pi} \left(\frac{4^6 y^2}{6}\ \frac{1-\cos(4\theta)}{8}\right)\ \mathrm{d}\theta \tag{10.7.40}$$

$$= \frac{256 y^2}{3}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{2\pi} = \frac{256 y^2}{3}\ [2\pi] \tag{10.7.41}$$

$$I_3 = \int_{-4}^4 I_2\ \mathrm{d}y = \int_{-4}^4 \left(\frac{512\pi}{3}\ y^2\right)\ \mathrm{d}y = \left[\frac{512\pi\ y^3}{9}\right]_{-4}^4 = \frac{4^8\ \pi}{9} \tag{10.7.42}$$

7. Finding the mass, given density ρ,

$$\rho = \arctan(y/x) \qquad x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{10.7.43}$$

$$T : r \in [0,\, a] \qquad \theta \in [0,\, \pi/2] \qquad y \in [0,\, 2\pi] \tag{10.7.44}$$

$$I_1 = \int_0^{2\pi} (\phi)\ \ \mathrm{d}\phi = \left[\frac{\phi^2}{2}\right]_0^{2\pi} = 2\pi^2 \tag{10.7.45}$$

$$I_2 = \int_0^{\pi/2} I_1 \sin\theta\ \ \mathrm{d}\theta = \int_0^{\pi/2} \left(2\pi^2\,\sin(\theta)\right)\ \mathrm{d}\theta \tag{10.7.46}$$

$$= \left[-\,2\pi^2\,\cos(\theta)\right]_0^{\pi/2} = 2\pi^2 \tag{10.7.47}$$

$$I_3 = \int_{-4}^4 I_2\ \mathrm{d}y = \int_0^a \left(2\pi^2\right)\ r^2\ \mathrm{d}r = \left[2\pi^2\frac{r^3}{3}\right]_0^a = \frac{2\pi^2 a^3}{3} \tag{10.7.48}$$

8. Finding the mass, given density ρ,

$$\rho = x^2 + y^2 \qquad x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r \cos\theta \tag{10.7.49}$$

$$T : r \in [0,\, a] \qquad \theta \in [0,\, \pi/2] \qquad y \in [0,\, 2\pi] \tag{10.7.50}$$

$$I_1 = \int_0^{2\pi} (r^2 \sin^2\theta)\ \mathrm{d}\phi = r^2 \sin^2\theta \Big[\phi\Big]_0^{2\pi} = 2\pi r^2 \sin^2\theta \tag{10.7.51}$$

$$I_2 = \int_0^{\pi/2} I_1 \sin\theta\ \mathrm{d}\theta = \int_0^{\pi/2} \left(2\pi r^2 \sin^3(\theta)\right) \mathrm{d}\theta \tag{10.7.52}$$

$$= \left[2\pi r^2 \left(\frac{\cos^3\theta}{3} - \cos\theta\right)\right]_0^{\pi/2} = \frac{4\pi r^2}{3} \tag{10.7.53}$$

$$I_3 = \int_0^a I_2\, r^2\ \mathrm{d}r = \int_0^a \left(\frac{4\pi r^2}{3}\right) r^2\, \mathrm{d}r = \left[4\pi \frac{r^5}{15}\right]_0^a = \frac{4\pi a^5}{15} \tag{10.7.54}$$

9. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} x^2 \\ 0 \\ z^2 \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = 2x + 2z \tag{10.7.55}$$

$$T : x \in [-1,\, 1] \qquad y \in [-3,\, 3] \qquad z \in [0,\, 2] \tag{10.7.56}$$

$$I_1 = \int_{-1}^{1} (2x + 2z)\ \mathrm{d}x = \Big[x^2 + 2zx\Big]_{-1}^{1} = 4z \tag{10.7.57}$$

$$I_2 = \int_{-3}^{3} I_1\, \mathrm{d}y = \int_{-3}^{3} \Big(4z\Big)\, \mathrm{d}y = \Big[4zy\Big]_{-3}^{3} = 24z \tag{10.7.58}$$

$$I_3 = \int_0^2 I_2\, \mathrm{d}z = \int_0^2 \Big(24z\Big)\, \mathrm{d}z = \Big[12z^2\Big]_0^2 = 48 \tag{10.7.59}$$

10. Solving Problem 9 by direct integration over the surface,

$$\mathbf{F} = \begin{bmatrix} x^2 \\ 0 \\ z^2 \end{bmatrix} \tag{10.7.60}$$

$$T : x \in [-1, 1] \qquad y \in [-3, 3] \qquad z \in [0, 2] \tag{10.7.61}$$

$$I_1 = \int_{-1}^{1} \int_{-3}^{3} \left[0^2 \cdot (-1) + 2^2 \cdot 1\right] \; \mathrm{d}x \, \mathrm{d}y = 48 \tag{10.7.62}$$

$$I_2 = \int_{-3}^{3} \int_{0}^{2} \left[(-1) \cdot 1 + 1 \cdot 1\right] \mathrm{d}y \, \mathrm{d}z = 0 \tag{10.7.63}$$

$$I_3 = \int_{-1}^{1} \int_{0}^{2} \left[(-1) \cdot 0 + 1 \cdot 0\right] \mathrm{d}z \, \mathrm{d}x = 0 \tag{10.7.64}$$

The results match. The surfaces integrated are pairs of opposite faces of the cuboid.

11. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} e^x \\ e^y \\ e^z \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = e^x + e^y + e^z \tag{10.7.65}$$

$$T : x \in [-1, 1] \qquad y \in [-1, 1] \qquad z \in [-1, 1] \tag{10.7.66}$$

$$I_1 = \int_{-1}^{1} (e^x + e^y + e^z) \; \mathrm{d}x = \Big[e^x + x(e^y + e^z)\Big]_{-1}^{1} = e - e^{-1} + 2(e^y + e^z) \tag{10.7.67}$$

$$I_2 = \int_{-1}^{1} I_1 \; \mathrm{d}y = \int_{-1}^{1} \left(2(e^y + e^z) + e - e^{-1}\right) \mathrm{d}y \tag{10.7.68}$$

$$= \Big[(2e^z + e - e^{-1}) \; y + 2e^y\Big]_{-1}^{1} = 4e^z + 4e - 4e^{-1} \tag{10.7.69}$$

$$I_3 = \int_{-1}^{1} I_2 \; \mathrm{d}z = \int_{-1}^{1} \left(4(e^z + e - e^{-1})\right) \mathrm{d}z \tag{10.7.70}$$

$$= \Big[4e^z + 4z(e - e^{-1})\Big]_{-1}^{1} = 12(e - e^{-1}) \tag{10.7.71}$$

12. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} x^3 - y^3 \\ y^3 - z^3 \\ z^3 - x^3 \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = 3(x^2 + y^2 + z^2) = 3r^2 \tag{10.7.72}$$

$$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r\cos\theta \tag{10.7.73}$$

$$T : r \in [0, 5] \qquad \theta \in [0, \pi/2] \qquad \phi \in [0, 2\pi] \tag{10.7.74}$$

$$I_1 = \int_0^{2\pi} (3r^2)\ \mathrm{d}\phi = \Big[3r^2\phi\Big]_0^{2\pi} = 6\pi\ r^2 \tag{10.7.75}$$

$$I_2 = \int_0^{\pi/2} I_1 \sin\theta\ \mathrm{d}\theta = \int_0^{\pi/2} \left(6\pi r^2\ \sin\theta\right) \mathrm{d}\theta \tag{10.7.76}$$

$$= \Big[-6\pi r^2\ \cos\theta\Big]_0^{\pi/2} = 6\pi r^2 \tag{10.7.77}$$

$$I_3 = \int_0^5 I_2\ r^2\ \mathrm{d}r = \int_0^5 \left(6\pi r^4\right) \mathrm{d}r = \left[\frac{6\pi}{5}\ r^5\right]_0^5 = 3750\pi \tag{10.7.78}$$

13. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} \sin y \\ \cos x \\ \cos z \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = -\sin z \tag{10.7.79}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{10.7.80}$$

$$T : r \in [0, 2] \qquad \theta \in [0, 2\pi] \qquad z \in [-2, 2] \tag{10.7.81}$$

$$I_1 = \int_0^{2\pi} (-\sin z)\ \mathrm{d}\theta = \Big[-\sin(z)\ \theta\Big]_0^{2\pi} = -2\pi\ \ \sin z \tag{10.7.82}$$

$$I_2 = \int_{-2}^{2} I_1\ \ \mathrm{d}z = \int_{-2}^{2} \left(-2\pi\ \ \sin z\right) \mathrm{d}z \tag{10.7.83}$$

$$= \Big[2\pi\ \ \cos z\Big]_{-2}^{2} = 0 \tag{10.7.84}$$

10.7.85

14. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} \sin\, y \\ \cos\, x \\ \cos\, z \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = -\sin\, z \tag{10.7.86}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{10.7.87}$$

$$T : r \in [0,\, 3] \qquad \theta \in [0,\, 2\pi] \qquad \phi \in [0,\, 2] \tag{10.7.88}$$

$$I_1 = \int_0^{2\pi} (-\sin\, z)\ \ \mathrm{d}\theta = \Big[-\sin(z)\ \theta \Big]_0^{2\pi} = -2\pi\ \ \sin\, z \tag{10.7.89}$$

$$I_2 = \int_0^{2} I_1\ \ \mathrm{d}z = \int_{-2}^{2} \left(-2\pi\ \ \sin\, z \right) \mathrm{d}z \tag{10.7.90}$$

$$= \Big[2\pi\ \ \cos\, z \Big]_0^{2} = 2\pi(\cos 2 - 1) \tag{10.7.91}$$

$$I_3 = \int_0^{3} I_2\ r\ \ \mathrm{d}r = \int_0^{3} \left(2\pi(\cos 2 - 1)\ r \right) \mathrm{d}r \tag{10.7.92}$$

$$= \left[2\pi(\cos 2 - 1)\ \frac{r^2}{2} \right]_0^{3} = 9\pi\ (\cos 2 - 1) \tag{10.7.93}$$

15. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} 2x^2 \\ y^2/2 \\ \sin(\pi z) \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = 4x + y + \pi \ \cos(\pi z) \tag{10.7.94}$$

$$T : x \in [0, 1] \quad y \in [0, 1 - x] \quad z \in [0, 1 - x - y] \tag{10.7.95}$$

$$I_1 = \int_0^{1-x-y} (4x + y + \pi \ \cos(\pi z)) \ \mathrm{d}z = \Big[z(4x + y) + \sin(\pi z) \Big]_0^{1-x-y} \tag{10.7.96}$$

$$= (1 - x - y)(4x + y) + \sin\Big(\pi(1 - x - y)\Big) \tag{10.7.97}$$

$$I_2 = \int_0^{1-x} I_1 \ \mathrm{d}y = \int_0^{1-x} \Big(4x - 4x^2 - 5xy + y - y^2 + \sin(\pi - \pi x - \pi y) \Big) \ \mathrm{d}y \tag{10.7.98}$$

$$= \left[(4x - 4x^2)y + \frac{(1 - 5x) \ y^2}{2} - \frac{y^3}{3} + \frac{\cos(\pi - \pi x - \pi y)}{\pi} \right]_0^{1-x} \tag{10.7.99}$$

$$= \frac{1 + 9x - 21x^2 + 11x^3}{6} + \frac{1 - \cos(\pi - \pi x)}{\pi} \tag{10.7.100}$$

$$I_3 = \int_0^1 I_2 \ \mathrm{d}z = \int_0^1 \left(\frac{1}{6} + \frac{3x}{2} - \frac{7x^2}{2} + \frac{11x^3}{6} + \frac{1 - \cos(\pi - \pi x)}{\pi} \right) \mathrm{d}x \tag{10.7.101}$$

$$= \left[\frac{x}{6} + \frac{3x^2}{4} - \frac{7x^3}{6} + \frac{11x^4}{24} + \frac{x}{\pi} + \frac{\sin(\pi - \pi x)}{\pi^2} \right]_0^1 = \frac{5}{24} + \frac{1}{\pi} \tag{10.7.102}$$

16. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} \cosh x \\ z \\ y \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = \sinh x \tag{10.7.103}$$

$$T : x \in [0,\, 1] \quad y \in [0,\, 1-x] \quad z \in [0,\, 1-x-y] \tag{10.7.104}$$

$$I_1 = \int_0^{1-x-y} (\sinh x) \;\mathrm{d}z = \Big[z \;\sinh x \Big]_0^{1-x-y} = (1-x-y) \;\sinh x \tag{10.7.105}$$

$$I_2 = \int_0^{1-x} I_1 \,\mathrm{d}y = \int_0^{1-x} \Big((1-x-y) \;\sinh x \Big) \,\mathrm{d}y \tag{10.7.106}$$

$$= \left[\sinh x \;\left(y - xy - \frac{y^2}{2} \right) \right]_0^{1-x} = \sinh(x) \;\frac{1 + x^2 - 2x}{2} \tag{10.7.107}$$

$$I_3 = \int_0^1 I_2 \,\mathrm{d}z = \int_0^1 \left((x-1)^2 \;\frac{\sinh(x)}{2} \right) \mathrm{d}x \tag{10.7.108}$$

$$= \left[\sinh(x)\;(x-1) + \cosh(x)\;\frac{3 + x^2 - 2x}{2} \right]_0^1 = \cosh(1) - \frac{3}{2} \tag{10.7.109}$$

17. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = 2x + 2y + 2z \tag{10.7.110}$$

$$x = r \cos \theta \qquad y = r \sin \theta \qquad z = z \tag{10.7.111}$$

$$T : r \in [0,\, z] \qquad \theta \in [0,\, 2\pi] \qquad z \in [0,\, h] \tag{10.7.112}$$

$$I_1 = \int_0^{2\pi} \Big(2r(\cos\theta + \sin\theta) + 2z \Big) \;\mathrm{d}\theta = \Big[2r\;(\sin\theta - \cos\theta) + 2z\;\theta \Big]_0^{2\pi} = 4\pi z \tag{10.7.113}$$

$$I_2 = \int_0^z I_1 \;r \;\;\mathrm{d}r = \int_0^z \Big(4\pi z \; r \Big) \;\mathrm{d}r = \Big[2\pi z \; r^2 \Big]_0^z = 2\pi \; z^3 \tag{10.7.114}$$

$$I_3 = \int_0^h I_2 \;\;\mathrm{d}z = \int_0^h \Big(2\pi \; z^3 \Big) \;\mathrm{d}z = \left[\frac{\pi \; z^4}{2} \right]_0^h = \frac{\pi h^4}{2} \tag{10.7.115}$$

18. Applying the divergence theorem,

$$\mathbf{F} = \begin{bmatrix} xy \\ yz \\ zx \end{bmatrix} \qquad \nabla \cdot \mathbf{F} = y + z + x \tag{10.7.116}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{10.7.117}$$

$$T : r \in [0,\, 2z] \qquad \theta \in [0,\, 2\pi] \qquad z \in [0,\, 2] \tag{10.7.118}$$

$$I_1 = \int_0^{2\pi} \left(r\sin\theta + r\cos\theta + z \right)\ d\theta = \left[-r\cos\theta + r\sin\theta + z\theta \right]_0^{2\pi} = 4\pi z \tag{10.7.119}$$

$$I_2 = \int_0^{2z} I_1\ r\ \ dr = \int_0^{2z} \left(2\pi z\ r \right)\ dr = \left[\pi z\ r^2 \right]_0^{2z} = 4\pi\ z^3 \tag{10.7.120}$$

$$I_3 = \int_0^{2} I_2\ \ dz = \int_0^{2} \left(4\pi\ z^3 \right)\ dz = \left[\pi z^4 \right]_0^{2} = 16\pi \tag{10.7.121}$$

19. Finding the moment of inertia about the x-axis,

$$I_x = \iiint_T (y^2 + z^2)\ \ dx\,dy\,dz \qquad \rho = 1 \tag{10.7.122}$$

$$x \in [-a,\, a] \qquad y \in [-b,\, b] \qquad z \in [-c,\, c] \tag{10.7.123}$$

$$I_1 = \int_{-c}^{c} \left(y^2 + z^2 \right)\ \ dz = \left[zy^2 + \frac{z^3}{3} \right]_{-c}^{c} = 2cy^2 + \frac{2c^3}{3} \tag{10.7.124}$$

$$I_2 = \int_{-b}^{b} I_1\ \ dy = \int_{-b}^{b} \left(2cy^2 + \frac{2c^3}{3} \right)\ dy = \left[\frac{2c\ y^3}{3} + \frac{2c^3\ y}{3} \right]_{-b}^{b} \tag{10.7.125}$$

$$= \frac{4cb(b^2 + c^2)}{3} \tag{10.7.126}$$

$$I_3 = \int_{-a}^{a} I_2\ \ dx = \int_{-a}^{a} \left(\frac{4bc\ (b^2 + c^2)}{3} \right)\ dx = \left[\frac{4bc(b^2 + c^2)}{3}\ x \right]_{-a}^{a} \tag{10.7.127}$$

$$= \frac{8abc}{3}\ (b^2 + c^2) \tag{10.7.128}$$

20. Finding the moment of inertia about the x-axis,

$$I_x = \iiint_T (y^2 + z^2)\ \ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad\qquad \rho = 1 \tag{10.7.129}$$

$$x = r\cos\theta \qquad y = r\sin\theta\cos\phi \qquad z = r\sin\theta\sin\phi \tag{10.7.130}$$

$$T : r \in [0,\, a] \qquad \theta \in [0,\, \pi] \qquad \phi \in [0,\, 2\pi] \tag{10.7.131}$$

$$I_1 = \int_0^{2\pi} (r^2\sin^2\theta)\ \ \mathrm{d}\phi = \Big[\phi\ r^2\sin^2\theta\Big]_0^{2\pi} = 2\pi r^2\sin^2\theta \tag{10.7.132}$$

$$I_2 = \int_0^{\pi} I_1\ (\sin\theta)\ \ \mathrm{d}\theta = \int_0^{\pi}\Big(2\pi r^2(\sin^3\theta)\Big)\ \mathrm{d}\theta \tag{10.7.133}$$

$$= \left[2\pi r^2\ \left(\frac{\cos^3\theta}{3} - \cos\theta\right)\right]_0^{\pi} = \frac{8\pi}{3}\ r^2 \tag{10.7.134}$$

$$I_3 = \int_{-4}^{4} I_2\ r^2\ \ \mathrm{d}r = \int_0^{a}\left(\frac{8\pi}{3}\right)\ r^4\ \mathrm{d}r = \left[2\pi^2\frac{r^3}{3}\right]_0^{a} = \frac{8\pi a^5}{15} \tag{10.7.135}$$

21. Finding the moment of inertia about the x-axis,

$$I_x = \iiint_T (y^2 + z^2)\ \ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad\qquad \rho = 1 \tag{10.7.136}$$

$$x = x \qquad y = r\cos\phi \qquad z = r\sin\phi \tag{10.7.137}$$

$$T : r \in [0,\, a] \qquad x \in [0,\, h] \qquad \phi \in [0,\, 2\pi] \tag{10.7.138}$$

$$I_1 = \int_0^{2\pi} (r^2)\ \ \mathrm{d}\phi = \Big[r^2\ \phi\Big]_0^{2\pi} = 2\pi\ r^2 \tag{10.7.139}$$

$$I_2 = \int_0^{h} I_1\ \ \mathrm{d}x = \int_0^{h}\Big(2\pi r^2\Big)\ \mathrm{d}x = \Big[2\pi r^2\ x\Big]_0^{h} = 2\pi h r^2 \tag{10.7.140}$$

$$I_3 = \int_0^{a} I_2\ r\ \ \mathrm{d}r = \int_0^{a}\Big(2\pi h\ r^3\Big)\ \mathrm{d}r = \left[\frac{\pi h\ r^4}{2}\right]_0^{a} = \frac{\pi h a^4}{2} \tag{10.7.141}$$

22. Finding the moment of inertia about the x-axis,

$$I_x = \iiint_T (y^2 + z^2)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad \rho = 1 \tag{10.7.142}$$

$$x = r^2 \qquad y = r\cos\phi \qquad z = r\sin\phi \tag{10.7.143}$$

$$T : r \in [0, \sqrt{x}] \qquad x \in [0, h] \qquad \phi \in [0, 2\pi] \tag{10.7.144}$$

$$I_1 = \int_0^{2\pi} (r^2)\ \mathrm{d}\phi = \left[r^2\ \phi \right]_0^{2\pi} = 2\pi\ r^2 \tag{10.7.145}$$

$$I_2 = \int_0^{\sqrt{x}} I_1\ r\ \ \mathrm{d}r = \int_0^{\sqrt{x}} \left(2\pi r^3 \right)\ \mathrm{d}r = \left[\pi \frac{r^4}{2} \right]_0^{\sqrt{x}} = \frac{\pi x^2}{2} \tag{10.7.146}$$

$$I_3 = \int_0^h I_2\ \ \mathrm{d}x = \int_0^h \left(\frac{\pi x^2}{2} \right) \mathrm{d}x = \left[\frac{\pi x^3}{6} \right]_0^h = \frac{\pi h^3}{6} \tag{10.7.147}$$

23. Finding the moment of inertia about the x-axis,

$$I_x = \iiint_T (y^2 + z^2)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad \rho = 1 \tag{10.7.148}$$

$$x = r \qquad y = r\cos\phi \qquad z = r\sin\phi \tag{10.7.149}$$

$$T : r \in [0, x] \qquad x \in [0, h] \qquad \phi \in [0, 2\pi] \tag{10.7.150}$$

$$I_1 = \int_0^{2\pi} (r^2)\ \mathrm{d}\phi = \left[r^2\ \phi \right]_0^{2\pi} = 2\pi\ r^2 \tag{10.7.151}$$

$$I_2 = \int_0^{x} I_1\ r\ \ \mathrm{d}r = \int_0^{x} \left(2\pi r^3 \right)\ \mathrm{d}r = \left[\pi \frac{r^4}{2} \right]_0^{x} = \frac{\pi x^4}{2} \tag{10.7.152}$$

$$I_3 = \int_0^h I_2\ \ \mathrm{d}x = \int_0^h \left(\frac{\pi x^4}{2} \right) \mathrm{d}x = \left[\frac{\pi x^5}{10} \right]_0^h = \frac{\pi h^5}{10} \tag{10.7.153}$$

24. The moment of inertia is a measure of how far from the axis the masses are distributed. Using $\sqrt{x} > x$ when $x \in [0, 1]$ and vice versa when $x > 1$.

The envelope in the xz plane is a straight line and parabola respectively. This means that the moment of inertia is smaller in the case of the cone for small h since the masses are closer to the axis.

For large h, $\sqrt{x} < x$ and thus the paraboloid has masses grouped much closer to the origin than the cone, making its I_x smaller.

25. A solid of revolution is symmetric in the azimuthal axis.

$$g = y^2 + z^2 = r^2 \tag{10.7.154}$$

$$I_x = \int_0^h \left[\int_0^r \left(\int_0^{2\pi} g \;\mathrm{d}\phi \right) r \;\mathrm{d}r \right] \;\mathrm{d}x \tag{10.7.155}$$

$$I_x = \frac{2\pi}{4} \int_0^h r(x)^4 \;\mathrm{d}x \tag{10.7.156}$$

Using the above formula to solve

(a) Problem 20,

$$T : x^2 + y^2 + z^2 \leq a^2 \qquad r(x) \in [-\sqrt{a^2 - x^2}, \sqrt{a^2 - x^2}] \tag{10.7.157}$$

$$I_x = \frac{\pi}{2} \int_{-a}^{a} (a^2 - x^2)^2 \;\mathrm{d}x \qquad = \frac{\pi}{2} \left[a^4\, x - \frac{2a^2 x^3}{3} + \frac{x^5}{5} \right]_{-a}^{a} \tag{10.7.158}$$

$$= \frac{\pi}{2} \left[2a^5 - \frac{4a^5}{3} + \frac{2a^5}{5} \right]_{-a}^{a} \qquad = \frac{8\pi\, a^5}{15} \tag{10.7.159}$$

(b) Problem 21,

$$T : y^2 + z^2 \leq a^2 \qquad r(x) \in [-a, a] \tag{10.7.160}$$

$$I_x = \frac{\pi}{2} \int_0^h a^4 \;\mathrm{d}x \qquad = \frac{\pi}{2} \Big[a^4\, x \Big]_0^h = \frac{\pi h\, a^4}{2} \tag{10.7.161}$$

(c) Problem 22,

$$T : y^2 + z^2 \leq a^2 \qquad r(x) \in [-\sqrt{x}, \sqrt{x}] \tag{10.7.162}$$

$$I_x = \frac{\pi}{2} \int_0^h x^2 \;\mathrm{d}x \qquad = \frac{\pi}{2} \left[\frac{x^3}{3} \right]_0^h = \frac{\pi h^3}{6} \tag{10.7.163}$$

(d) Problem 23,

$$T : y^2 + z^2 \leq a^2 \qquad r(x) \in [x, x] \tag{10.7.164}$$

$$I_x = \frac{\pi}{2} \int_0^h x^4 \;\mathrm{d}x \qquad = \frac{\pi}{2} \left[\frac{x^5}{5} \right]_0^h = \frac{\pi h^5}{10} \tag{10.7.165}$$

10.8 Further Applications of the Divergence Theorem

1. Verifying Theorem 1,

$$f = 2z^2 - x^2 - y^2 \qquad \nabla^2 f = -2 - 2 + 4 = 0 \tag{10.8.1}$$

$$\nabla f = \begin{bmatrix} -2x \\ -2y \\ 4z \end{bmatrix} \tag{10.8.2}$$

$$I_1 = \int_0^a \int_0^b (4c \cdot 1 + 0 \cdot -1) \ \mathrm{d}y \, \mathrm{d}x \qquad = 4abc \tag{10.8.3}$$

$$I_2 = \int_0^c \int_0^b (-2a \cdot 1 + 0 \cdot -1) \ \mathrm{d}y \, \mathrm{d}z \qquad = -2abc \tag{10.8.4}$$

$$I_3 = \int_0^c \int_0^a (-2b \cdot 1 + 0 \cdot -1) \ \mathrm{d}x \, \mathrm{d}z \qquad = -2abc \tag{10.8.5}$$

The sum of the three integrals is zero, which verifies the theorem.

2. Verifying Theorem 1,

$$f = x^2 - y^2 \qquad \nabla^2 f = 2 - 2 = 0 \tag{10.8.6}$$

$$\nabla f = \begin{bmatrix} 2x \\ -2y \\ 0 \end{bmatrix} \tag{10.8.7}$$

$$I_1 = \int_0^2 \int_0^{2\pi} \Big(0 + 0\Big) \ r \ \mathrm{d}\phi \, \mathrm{d}r \qquad = 0 \tag{10.8.8}$$

$$I_2 = \int_0^h \int_0^{2\pi} \Big(4\cos(2\phi)\Big) \ (2) \ \mathrm{d}\phi \, \mathrm{d}z \qquad = 0 \tag{10.8.9}$$

The sum of the two integrals is zero, which verifies the theorem.

3. Verifying Green's first form,

$$f = 4y^2 \qquad g = x^2 \tag{10.8.10}$$

$$\mathbf{F} = f \ \nabla g \qquad = \begin{bmatrix} 8xy^2 \\ 0 \\ 0 \end{bmatrix} \tag{10.8.11}$$

$$f \ \nabla^2 g = 8y^2 \qquad \nabla f \cdot \nabla g = 0 \tag{10.8.12}$$

The left-hand side computes to

$$\iiint_T \left(f\ \nabla^2 g + \nabla f \cdot \nabla g\right) = \iiint_T (8y^2)\ \ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \tag{10.8.13}$$

$$= \int_0^1 \int_0^1 \int_0^1 (8y^2)\ \ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = \frac{8}{3} \tag{10.8.14}$$

The right-hand side computes to

$$\iint_S\ f\ (\mathbf{n} \cdot \nabla g)\ \ \mathrm{d}A = I_x + I_y + I_z \tag{10.8.15}$$

$$I_x = \int_0^1 \int_0^1 4y^2\ \left(2 \cdot 1 - 0 \cdot 1\right)\ \mathrm{d}y\ \mathrm{d}z = \frac{8}{3} \tag{10.8.16}$$

$$I_y = \int_0^1 \int_0^1 4y^2\ \left(0\right)\ \mathrm{d}x\ \mathrm{d}z = 0 \tag{10.8.17}$$

$$I_z = \int_0^1 \int_0^1 4y^2\ \left(0\right)\ \mathrm{d}x\ \mathrm{d}y = 0 \tag{10.8.18}$$

Both sides match, verifying the theorem.

4. Verifying Green's first form,

$$f = x \qquad\qquad g = y^2 + z^2 \tag{10.8.19}$$

$$\mathbf{F} = f\ \nabla g \qquad\qquad = \begin{bmatrix} 0 \\ 2xy \\ 2xz \end{bmatrix} \tag{10.8.20}$$

$$f\ \nabla^2 g = 4x \qquad\qquad \nabla f \cdot \nabla g = 0 \tag{10.8.21}$$

The left-hand side computes to

$$\iiint_T \left(f\ \nabla^2 g + \nabla f \cdot \nabla g\right) = \iiint_T (4x)\ \ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \tag{10.8.22}$$

$$= \int_0^3 \int_0^2 \int_0^1 (4x)\ \ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = 12 \tag{10.8.23}$$

The right-hand side computes to

$$\iint_S f\ (\mathbf{n}\cdot\nabla g)\ \mathrm{d}A = I_x + I_y + I_z \tag{10.8.24}$$

$$I_x = \int_0^3 \int_0^2 x\ \Big(0\Big)\ \mathrm{d}y\ \mathrm{d}z = 0 \tag{10.8.25}$$

$$I_y = \int_0^3 \int_0^1 x\ \Big(4\cdot 1 - 0\cdot 1\Big)\ \mathrm{d}x\ \mathrm{d}z = 6 \tag{10.8.26}$$

$$I_z = \int_0^2 \int_0^1 x\ \Big(6\cdot 1 - 0\cdot 1\Big)\ \mathrm{d}x\ \mathrm{d}y = 6 \tag{10.8.27}$$

Both sides match, verifying the theorem.

5. Verifying Green's second form,

$$f = 6y^2 \qquad\qquad g = 2x^2 \tag{10.8.28}$$

$$f\nabla^2 g - g\nabla^2 f = 24(y^2 - x^2) \qquad\qquad \nabla f = \begin{bmatrix} 0 \\ 12y \\ 0 \end{bmatrix} \quad \nabla g = \begin{bmatrix} 4x \\ 0 \\ 0 \end{bmatrix} \tag{10.8.29}$$

The left-hand side computes to

$$\iiint_T \Big(f\ \nabla^2 g - g\ \nabla^2 f\Big) = \iiint_T (24y^2 - 24x^2)\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \tag{10.8.30}$$

$$= \int_0^1 \int_0^1 \int_0^1 24(y^2 - x^2)\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = 0 \tag{10.8.31}$$

The right-hand side computes to

$$\iint_S f\ (\mathbf{n}\cdot\nabla g)\ \mathrm{d}A = I_x + I_y + I_z \tag{10.8.32}$$

$$I_x = \int_0^1 \int_0^1 6y^2\ \Big(4\cdot 1 - 0\cdot 1\Big)\ \mathrm{d}y\ \mathrm{d}z = 8 \tag{10.8.33}$$

$$I_y = \int_0^1 \int_0^1 6y^2\ \Big(0\Big)\ \mathrm{d}x\ \mathrm{d}z = 0 \tag{10.8.34}$$

$$I_z = \int_0^1 \int_0^1 6y^2\ \Big(0\Big)\ \mathrm{d}x\ \mathrm{d}y = 0 \tag{10.8.35}$$

$$\iint_S g\ (\mathbf{n}\cdot\nabla f)\ \ \mathrm{d}A = J_x + J_y + J_z \tag{10.8.36}$$

$$J_x = \int_0^1\int_0^1 2x^2\ \Big(0\Big)\ \mathrm{d}y\ \mathrm{d}z = 0 \tag{10.8.37}$$

$$J_y = \int_0^1\int_0^1 2x^2\ \Big(12\cdot 1 - 0\cdot 1\Big)\ \mathrm{d}x\ \mathrm{d}z = 8 \tag{10.8.38}$$

$$J_z = \int_0^1\int_0^1 2x^2\ \Big(0\Big)\ \mathrm{d}x\ \mathrm{d}y = 0 \tag{10.8.39}$$

Both sides match, verifying the theorem.

6. Verifying Green's second form,

$$f = x^2 \qquad\qquad g = y^4 \tag{10.8.40}$$

$$f\nabla^2 g - g\nabla^2 f = 12x^2y^2 - 2y^4 \qquad \nabla f = \begin{bmatrix} 2x \\ 0 \\ 0 \end{bmatrix} \quad \nabla g = \begin{bmatrix} 0 \\ 4y^3 \\ 0 \end{bmatrix} \tag{10.8.41}$$

The left-hand side computes to

$$\iiint_T \Big(f\ \nabla^2 g - g\ \nabla^2 f\Big) = \iiint_T (12x^2y^2 - 2y^4)\ \ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \tag{10.8.42}$$

$$= \int_0^1\int_0^1\int_0^1 (12x^2y^2 - 2y^4)\ \ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = \frac{4}{3} - \frac{2}{5} \tag{10.8.43}$$

The right-hand side computes to

$$\iint_S f\ (\mathbf{n}\cdot\nabla g)\ \ \mathrm{d}A = I_x + I_y + I_z \tag{10.8.44}$$

$$I_x = \int_0^1\int_0^1 x^2\ \Big(0\Big)\ \mathrm{d}y\ \mathrm{d}z = 0 \tag{10.8.45}$$

$$I_y = \int_0^1\int_0^1 x^2\ \Big(4\cdot 1 - 0\cdot 1\Big)\ \mathrm{d}x\ \mathrm{d}z = \frac{4}{3} \tag{10.8.46}$$

$$I_z = \int_0^1\int_0^1 x^2\ \Big(0\Big)\ \mathrm{d}x\ \mathrm{d}y = 0 \tag{10.8.47}$$

$$\iint_S g\,(\mathbf{n}\cdot\nabla f)\ \mathrm{d}A = J_x + J_y + J_z \tag{10.8.48}$$

$$J_x = \int_0^1\int_0^1 y^4\left(2\cdot 1 - 0\cdot 1\right)\,\mathrm{d}y\,\mathrm{d}z = \frac{2}{5} \tag{10.8.49}$$

$$J_y = \int_0^1\int_0^1 y^4\left(0\right)\,\mathrm{d}x\,\mathrm{d}z = 8 \tag{10.8.50}$$

$$J_z = \int_0^1\int_0^1 y^4\left(0\right)\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{10.8.51}$$

Both sides match, verifying the theorem.

7. Using Green's theorem to find the volume,

$$\iiint_T (\nabla\cdot\mathbf{F})\ \mathrm{d}V = \iint_S (F_1)\ \mathrm{d}y\,\mathrm{d}z + (F_2)\ \mathrm{d}x\,\mathrm{d}z + (F_3)\,\mathrm{d}x\,\mathrm{d}y \tag{10.8.52}$$

$$\mathbf{F}_a = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{F}_b = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \quad \mathbf{F}_c = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \quad \mathbf{F}_d = \frac{1}{3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{10.8.53}$$

Each of the four equalities are obtained by applying Green's theorem to these vector functions one by one.

8. Circular cone of height h and radius of base a, with the cone facing upwards for convenience.

$$3V = \iint_S (x)\ \mathrm{d}y\,\mathrm{d}z + (y)\ \mathrm{d}x\,\mathrm{d}z + (z)\,\mathrm{d}x\,\mathrm{d}y \tag{10.8.54}$$

$$I_1 = \int_0^a\int_0^{2\pi}(h)\ r\ \mathrm{d}r\,\mathrm{d}\phi = \int_0^a 2\pi(h)\ r\ \ \mathrm{d}r = \pi h a^2 \tag{10.8.55}$$

$$V = \frac{\pi a^2}{3}\ h \tag{10.8.56}$$

9. Circular cone of height h and radius of base a, with the cone facing upwards for convenience.

$$3V = \iint_S (x)\ \mathrm{d}y\,\mathrm{d}z + (y)\ \mathrm{d}x\,\mathrm{d}z + (z)\,\mathrm{d}x\,\mathrm{d}y \tag{10.8.57}$$

$$I_1 = \int_0^a\int_0^{2\pi}(0)\ r\ \mathrm{d}r\,\mathrm{d}\phi = 0 \tag{10.8.58}$$

$$I_2 = \int_0^{2\pi}\int_0^{\pi/2}(a)\ a^2\sin\theta\ \mathrm{d}\theta\,\mathrm{d}\phi = 2\pi a^3\int_0^{\pi/2}\sin\theta\ \mathrm{d}\theta = 2\pi a^3 \tag{10.8.59}$$

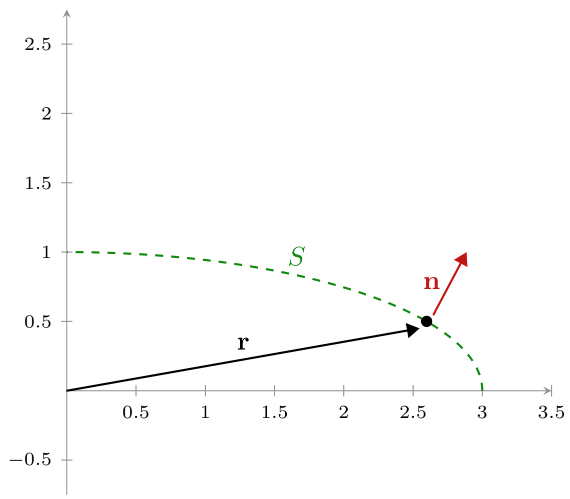
$$V = \frac{0 + 2\pi a^3}{3} = \frac{2\pi}{3}\ a^3 \tag{10.8.60}$$

10. A variable point P has the position vector

$$P : \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{r} \cdot \mathbf{n} = |\mathbf{r}| \cdot 1 \cdot \cos\phi \tag{10.8.61}$$

$$\iiint_T \nabla \cdot \mathbf{r} \ \mathrm{d}A = \iint_S \mathbf{r} \cdot \mathbf{n} \ \mathrm{d}A \qquad \iiint_T \mathrm{d}V = \frac{1}{3} \iint_S |\mathbf{r}| \cos\phi \, \mathrm{d}A \tag{10.8.62}$$

$$V = \frac{1}{3} \iint_S r \cos\phi \ \mathrm{d}A \tag{10.8.63}$$



11. For the special case of a sphere, with $\phi = 0$,

$$V = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} (r) \ (r) \ \mathrm{d}\theta \ (r \sin\theta) \ \mathrm{d}\phi \tag{10.8.64}$$

$$= \frac{a^3}{3} \int_0^{\pi} 2\pi \ \sin\theta \ \mathrm{d}\theta = \left[\frac{2\pi a^3}{3} \ \cos(\theta) \right]_{\pi}^{0} = \frac{4\pi}{3} \ a^3 \tag{10.8.65}$$

12. Potential Theory. Examples TBC.

(a) Using the first form of Green's theorem, with $f = g$

$$\iiint_T \left(f \nabla^2 g + \nabla f \cdot \nabla g \right) \mathrm{d}V = \iint_S \left(f \ \frac{\partial g}{\partial n} \right) \mathrm{d}A \tag{10.8.66}$$

$$\iiint_T \ |\nabla g|^2 \ \ \mathrm{d}V = \iint_S \left(g \ \frac{\partial g}{\partial n} \right) \mathrm{d}A \tag{10.8.67}$$

(b) Using the result from part a, starting with the fact that everywhere in T

$$\frac{\partial g}{\partial n} = 0 \qquad \implies |\nabla g| = 0 \tag{10.8.68}$$

$$\implies \nabla g = \mathbf{0} \qquad \implies g = \text{constant} \tag{10.8.69}$$

(c) Using the second form of Green's theorem with f, g both being harmonic, proves the result.

(d) Let $h = f - g$, which is also harmonic, and using the result from part b,

$$\frac{\partial h}{\partial n} = 0 \qquad \implies |\nabla h| = 0 \tag{10.8.70}$$

$$\implies \nabla h = \mathbf{0} \qquad \implies h = \text{constant} \tag{10.8.71}$$

(e) Replace $\mathbf{F}$ with ∇f in the definition of the coordinate independent divergence,

$$\nabla \cdot \mathbf{F}(P) = \lim_{d(T)\to 0} \frac{1}{V(T)} \iint_{S(T)} \mathbf{F} \cdot \mathbf{n} \; \mathrm{d}A \tag{10.8.72}$$

$$\nabla^2 f = \lim_{d(T)\to 0} \frac{1}{V(T)} \iint_{S(T)} \left(\frac{\partial f}{\partial n}\right) \; \mathrm{d}A \tag{10.8.73}$$

10.9 Stokes' Theorem

1. Calculating the surface integral directly,

$$\mathbf{F} = \begin{bmatrix} z^2 \\ -x^2 \\ 0 \end{bmatrix} \qquad \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ z^2 & -x^2 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 2z \\ -2x \end{bmatrix} \tag{10.9.1}$$

$$\mathbf{n} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} u \\ v \\ v \end{bmatrix} \quad u \in [0, 1] \quad v \in [0, 4] \tag{10.9.2}$$

Performing the integration,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \;\; \mathrm{d}u \, \mathrm{d}v = \int_0^1 \left[\int_0^4 (-2v - 2u) \;\; \mathrm{d}v \right] \mathrm{d}u \tag{10.9.3}$$

$$= \int_0^1 \left[(-v^2 - 2uv) \right]_0^4 \; \mathrm{d}u \tag{10.9.4}$$

$$= \int_0^1 (-16 - 8u) \; \mathrm{d}u = \left[-16u - 4u^2 \right]_0^1 = -20 \tag{10.9.5}$$

2. Calculating the surface integral directly,

$$\mathbf{F} = \begin{bmatrix} -13 \sin y \\ 3 \sinh z \\ x \end{bmatrix} \qquad \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ -13 \sin y & 3 \sinh z & x \end{vmatrix} = \begin{bmatrix} -3 \cosh z \\ -1 \\ 13 \cos y \end{bmatrix} \tag{10.9.6}$$

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} u \\ v \\ 2 \end{bmatrix} \quad u \in [0, 4] \quad v \in [0, \pi/2] \tag{10.9.7}$$

Performing the integration,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \;\; \mathrm{d}u \, \mathrm{d}v = \int_0^4 \left[\int_0^{\pi/2} (13 \cos v) \;\; \mathrm{d}v \right] \mathrm{d}u \tag{10.9.8}$$

$$= \int_0^4 \left[13 \sin v \right]_0^{\pi/2} \; \mathrm{d}u \tag{10.9.9}$$

$$= \int_0^4 (13) \; \mathrm{d}u = \left[13u \right]_0^4 = 52 \tag{10.9.10}$$

3. Calculating the surface integral directly,

$$\mathbf{F} = \begin{bmatrix} e^{-z} \\ e^{-z}\cos y \\ e^{-z}\sin y \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} u \\ v \\ v^2/2 \end{bmatrix} \quad u \in [-1, 1] \quad v \in [0, 1] \tag{10.9.11}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ e^{-z} & e^{-z}\cos y & e^{-z}\sin y \end{vmatrix} = \begin{bmatrix} 2e^{-z}\cos y \\ -e^{-z} \\ 0 \end{bmatrix} \tag{10.9.12}$$

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & v \end{vmatrix} = \begin{bmatrix} 0 \\ -v \\ 1 \end{bmatrix} \tag{10.9.13}$$

Performing the integration,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \ \mathrm{d}u \, \mathrm{d}v = \int_{-1}^{1} \left[\int_0^1 (ve^{-v^2/2}) \ \mathrm{d}v \right] \mathrm{d}u \tag{10.9.14}$$

$$= \int_{-1}^{1} \left[-\, e^{-v^2/2} \right]_0^1 \mathrm{d}u \tag{10.9.15}$$

$$= \int_{-1}^{1} (1 - e^{-0.5}) \ \mathrm{d}u = \Big[\lambda u \Big]_{-1}^{1} = 2(1 - e^{-0.5}) \tag{10.9.16}$$

4. Calculating the surface integral directly,

$$\mathbf{F} = \begin{bmatrix} z^2 \\ -x^2 \\ 0 \end{bmatrix} \qquad \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ z^2 & -x^2 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 2z \\ -2x \end{bmatrix} \tag{10.9.17}$$

$$\mathbf{r} = \begin{bmatrix} u \\ v \\ uv \end{bmatrix} \quad u \in [0, 1] \quad v \in [0, 4] \qquad \mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = \begin{bmatrix} -v \\ -u \\ 1 \end{bmatrix} \tag{10.9.18}$$

Performing the integration,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \;\; du\, dv = \int_0^1 \left[\int_0^4 (-2u^2 v - 2u) \;\; dv \right] du \tag{10.9.19}$$

$$= \int_0^1 \left[(-u^2 v^2 - 2uv) \right]_0^4 du \tag{10.9.20}$$

$$= \int_0^1 (-16u^2 - 8u)\, du = \left[\frac{-16u^3}{3} - 4u^2 \right]_0^1 = \frac{-28}{3} \tag{10.9.21}$$

5. Calculating the surface integral directly,

$$\mathbf{F} = \begin{bmatrix} z^2 \\ 1.5x \\ 0 \end{bmatrix} \qquad \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ z^2 & 1.5x & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 2z \\ 1.5 \end{bmatrix} \tag{10.9.22}$$

$$\mathbf{r} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad u \in [0, a] \quad v \in [0, a] \qquad \mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.9.23}$$

Performing the integration,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \;\; du\, dv = \int_0^a \left[\int_0^a (1.5) \;\; dv \right] du \tag{10.9.24}$$

$$= \int_0^a \left[(1.5v) \right]_0^a du \tag{10.9.25}$$

$$= \int_0^a (1.5a)\, du = \left[1.5au \right]_0^a = 1.5a^2 \tag{10.9.26}$$

6. Calculating the surface integral directly,

$$\mathbf{F} = \begin{bmatrix} y^3 \\ -x^3 \\ 0 \end{bmatrix} \qquad \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ y^3 & -x^3 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3(x^2 + y^2) \end{bmatrix} \tag{10.9.27}$$

$$g = z - (x^2 + y^2) \qquad \mathbf{n} = \nabla g = \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix} \tag{10.9.28}$$

Performing the integration,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \;\; \mathrm{d}u \; \mathrm{d}v = \int_0^1 \left[\int_0^{2\pi} (-3r^2) \; r \; \mathrm{d}\phi \right] \mathrm{d}r \tag{10.9.29}$$

$$= \int_0^1 \left[-\, 3r^3 \; \phi \right]_0^{2\pi} \; \mathrm{d}\phi \tag{10.9.30}$$

$$= \int_0^1 (-6\pi \; r^3) \; \mathrm{d}r = \left[\frac{-6\pi}{4} \; r^4 \right]_0^1 = \frac{-3\pi}{2} \tag{10.9.31}$$

7. Calculating the surface integral directly,

$$\mathbf{F} = \begin{bmatrix} e^y \\ e^z \\ e^x \end{bmatrix} \qquad \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ e^y & e^z & e^x \end{vmatrix} = \begin{bmatrix} -e^z \\ -e^x \\ -e^y \end{bmatrix} \tag{10.9.32}$$

$$\mathbf{r} = \begin{bmatrix} u \\ v \\ u^2 \end{bmatrix} \quad u \in [0, 2] \quad v \in [0, 1] \qquad \mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 2u \\ 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -2u \\ 0 \\ 1 \end{bmatrix} \tag{10.9.33}$$

Performing the integration,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \;\; \mathrm{d}u \; \mathrm{d}v = \int_0^2 \left[\int_0^1 (2ue^{u^2} - e^v) \;\; \mathrm{d}v \right] \mathrm{d}u \tag{10.9.34}$$

$$= \int_0^2 \left[2ue^{u^2} \; v - e^v \right]_0^1 \; \mathrm{d}u \tag{10.9.35}$$

$$= \int_0^2 2u \; e^{u^2} + (1 - e) \; \mathrm{d}u = \left[e^{u^2} + u(1 - e) \right]_0^2 \tag{10.9.36}$$

$$= e^4 + 1 - 2e \tag{10.9.37}$$

8. Calculating the surface integral directly,

$$\mathbf{F} = \begin{bmatrix} z^2 \\ x^2 \\ y^2 \end{bmatrix} \tag{10.9.38}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ z^2 & x^2 & y^2 \end{vmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2x \end{bmatrix} = \begin{bmatrix} 2r \sin \phi \\ 2r \\ 2r \cos \phi \end{bmatrix} \tag{10.9.39}$$

$$\mathbf{r} = \begin{bmatrix} r \cos \phi \\ r \sin \phi \\ r \end{bmatrix} \quad r \in [0,\, h] \quad \phi \in [0,\, \pi] \tag{10.9.40}$$

$$\mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos \phi & \sin \phi & 1 \\ -r \sin \phi & r \cos \phi & 0 \end{vmatrix} = \begin{bmatrix} -r \cos \phi \\ -r \sin \phi \\ r \end{bmatrix} \tag{10.9.41}$$

Performing the integration,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \;\; \mathrm{d}u \; \mathrm{d}v = \int_0^h \left[\int_0^\pi (-r^2 \sin(2\phi) - 2r^2 \sin \phi + 2r^2 \cos \phi)\; r \; \mathrm{d}\phi \right] \mathrm{d}r \tag{10.9.42}$$

$$= \int_0^h r^3 \left[2 \sin \phi + 2 \cos \phi + \frac{\cos(2\phi)}{2} \right]_0^\pi \mathrm{d}\phi \tag{10.9.43}$$

$$= \int_0^h (-4\, r^3) \; \mathrm{d}r = \Big[-r^4 \Big]_0^h = -h^4 \tag{10.9.44}$$

9. Verifying Stokes' theorem in Problem 5,

$$\mathbf{F} = \begin{bmatrix} z^2 \\ 1.5x \\ 0 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad u \in [0,\, a] \quad v \in [0,\, a] \tag{10.9.45}$$

$$\oint_C \mathbf{F} \cdot \mathbf{r}'(s) \;\; \mathrm{d}s = I_1 + I_2 + I_3 + I_4 \tag{10.9.46}$$

$$I_1 = \int_0^a 1^2 \;\; \mathrm{d}x = a \qquad I_2 = \int_0^a (1.5a) \;\; \mathrm{d}y = 1.5a^2 \tag{10.9.47}$$

$$I_3 = \int_a^0 1^2 \;\; \mathrm{d}x = -a \qquad I_4 = \int_a^0 (1.5 \cdot 0) \;\; \mathrm{d}y = 0 \tag{10.9.48}$$

The results match.

10. Verifying Stokes' theorem in Problem 5,

$$\mathbf{F} = \begin{bmatrix} y^3 \\ -x^3 \\ 0 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} \cos\,\phi \\ \sin\,\phi \\ 0 \end{bmatrix} \quad \phi \in [0,\, 2\pi] \tag{10.9.49}$$

$$\oint_C \mathbf{F}\cdot\mathbf{r}'(s)\;\, \mathrm{d}s = \int_0^{2\pi} (-\sin^4\,\phi - \cos^4\,\phi)\;\, \mathrm{d}\phi \qquad = \left[\frac{-\sin(4\phi) - 12\phi}{16}\right]_0^{2\pi} \tag{10.9.50}$$

$$= \frac{-3\pi}{2} \tag{10.9.51}$$

The results match.

11. Calculating,

$$\mathbf{F} = \frac{1}{x^2+y^2}\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \qquad \nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ -y/r^2 & x/r^2 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10.9.52}$$

The left-hand side of Stokes' theorem gives,

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\;\, \mathrm{d}A = 0 \tag{10.9.53}$$

The right-hand side of Stokes' theorem gives,

$$\mathbf{F} = \frac{1}{r^2}\begin{bmatrix} -r\,\sin\,\phi \\ r\,\cos\,\phi \\ 0 \end{bmatrix} \qquad \mathbf{r}' = \begin{bmatrix} -\sin\,\phi \\ \cos\,\phi \\ 0 \end{bmatrix} \tag{10.9.54}$$

$$\oint_C \mathbf{F}\cdot\mathbf{r}'(s)\;\, \mathrm{d}s = \int_0^{2\pi}\;\, \mathrm{d}\phi = 2\pi \tag{10.9.55}$$

The mismatch is because **F** is not continuous everywhere in S. An extra factor of -1 needs to be added to the result of the RHS to make the orientation clockwise.

12. Refer notes. TBC

13. Applying reverse Stokes' theorem,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ -5y & 4x & z \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} \tag{10.9.56}$$

$$\mathbf{r} = \begin{bmatrix} u \cos v \\ u \sin v \\ 4 \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.9.57}$$

Calculating the surface integral,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ \mathrm{d}A = \int_0^4 \left[\int_0^{2\pi} 9\ (r)\ \ \mathrm{d}\phi \right] \mathrm{d}r \tag{10.9.58}$$

$$= \int_0^4 18\pi r\ \mathrm{d}r = \left[9r^2 \right]_0^4 = 144\pi \tag{10.9.59}$$

14. Applying reverse Stokes' theorem,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ z^3 & x^3 & y^3 \end{vmatrix} = \begin{bmatrix} 3y^2 \\ 3z^2 \\ 3x^2 \end{bmatrix} \tag{10.9.60}$$

$$\mathbf{r} = \begin{bmatrix} 2 \\ r \cos \phi \\ r \sin \phi \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{10.9.61}$$

Calculating the surface integral,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ \mathrm{d}A = \int_0^3 \left[\int_0^{2\pi} 3r^2\ \ \cos^2 \phi\ (r)\ \ \mathrm{d}\phi \right] \mathrm{d}r \tag{10.9.62}$$

$$= \int_0^3 \left[3r^3 \ \left(\frac{\phi}{2} + \frac{\sin(2\phi)}{4} \right) \right]_0^{2\pi} \mathrm{d}r \tag{10.9.63}$$

$$= \int_0^3 \left(3\pi r^3\right) \mathrm{d}r = \left[\frac{3\pi}{4}\ r^4 \right]_0^3 = \frac{243\pi}{4} \tag{10.9.64}$$

15. Applying reverse Stokes' theorem,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ y^2 & x^2 & z + x \end{vmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2x - 2y \end{bmatrix} \quad (10.9.65)$$

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (10.9.66)$$

Calculating the surface integral,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \;\, \mathrm{d}A = \int_0^1 \int_0^x (2x - 2y) \;\, \mathrm{d}y \, \mathrm{d}x \quad (10.9.67)$$

$$= \int_0^1 \left[2xy - y^2 \right]_0^x \mathrm{d}x = \int_0^1 \left(x^2 \right) \;\, \mathrm{d}x \quad (10.9.68)$$

$$= \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} \quad (10.9.69)$$

16. Applying reverse Stokes' theorem,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ e^y & 0 & e^x \end{vmatrix} = \begin{bmatrix} 0 \\ -e^x \\ -e^y \end{bmatrix} \quad (10.9.70)$$

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (10.9.71)$$

Calculating the surface integral,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \;\, \mathrm{d}A = \int_0^1 \int_0^x (-e^y) \;\, \mathrm{d}y \, \mathrm{d}x \quad (10.9.72)$$

$$= \int_0^1 \left[-e^y \right]_0^x \mathrm{d}x = \int_0^1 \left(1 - e^x \right) \;\, \mathrm{d}x \quad (10.9.73)$$

$$= \left[x - e^x \right]_0^1 = 2 - e \quad (10.9.74)$$

17. Applying reverse Stokes' theorem,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 0 & z^3 & 0 \end{vmatrix} = \begin{bmatrix} -3z^2 \\ 0 \\ 0 \end{bmatrix} \tag{10.9.75}$$

$$\mathbf{r} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ z \end{bmatrix} \tag{10.9.76}$$

$$g = x^2 + y^2 - 1 \qquad \nabla g = \begin{bmatrix} 2x \\ 2y \\ 0 \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \tag{10.9.77}$$

Calculating the surface integral,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \ \mathrm{d}A = \int_0^1 \int_0^{\pi/2} (-3z^2 \ \cos\phi)\ (1)\ \mathrm{d}\phi\ \mathrm{d}z \tag{10.9.78}$$

$$= \int_0^1 \Big[-3z^2 \sin\phi \Big]_0^{\pi/2} \ \mathrm{d}z = \int_0^1 \left(-3z^2\right) \ \mathrm{d}z \tag{10.9.79}$$

$$= \Big[-z^3 \Big]_0^1 = -1 \tag{10.9.80}$$

18. Applying reverse Stokes' theorem,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ -y & 2z & 0 \end{vmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \tag{10.9.81}$$

$$\mathbf{r} = \begin{bmatrix} x \\ \cos\phi \\ \sin\phi \end{bmatrix} \tag{10.9.82}$$

$$g = y^2 + z^2 - 4 \qquad \nabla g = \begin{bmatrix} 0 \\ 2y \\ 2z \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \end{bmatrix} \tag{10.9.83}$$

Calculating the surface integral,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \; \mathrm{d}A = \int_0^h \int_0^\pi (2 \sin \phi) \; \mathrm{d}\phi \, \mathrm{d}x \tag{10.9.84}$$

$$= \int_0^h \Big[-2 \cos \phi \Big]_0^\pi \mathrm{d}x = \int_0^h (4) \; \mathrm{d}x \tag{10.9.85}$$

$$= \Big[4z \Big]_0^h = 4h \tag{10.9.86}$$

19. Applying reverse Stokes' theorem,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ z & e^z & 0 \end{vmatrix} = \begin{bmatrix} -e^z \\ 1 \\ 0 \end{bmatrix} \tag{10.9.87}$$

$$\mathbf{r} = \begin{bmatrix} r \cos \phi \\ r \sin \phi \\ r \end{bmatrix} \tag{10.9.88}$$

$$g = x^2 + y^2 - z^2 \qquad \nabla g = \begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} r \cos \phi \\ r \sin \phi \\ -r \end{bmatrix} \tag{10.9.89}$$

Calculating the surface integral,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \; \mathrm{d}A = \int_0^1 \int_0^{\pi/2} (-e^r \; r \cos \phi + r \sin \phi) \; \mathrm{d}\phi \, \mathrm{d}r \tag{10.9.90}$$

$$= \int_0^1 \Big[-re^r \sin \phi - r \cos \phi \Big]_0^{\pi/2} \mathrm{d}r \tag{10.9.91}$$

$$= \int_0^1 (-re^r + r) \; \mathrm{d}r = \left[e^r(1-r) + \frac{r^2}{2} \right]_0^1 = -\frac{1}{2} \tag{10.9.92}$$

20. Applying reverse Stokes' theorem,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 0 & \cos x & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\sin x \end{bmatrix} \tag{10.9.93}$$

$$\mathbf{r} = \begin{bmatrix} x \\ \cos\phi \\ \sin\phi \end{bmatrix} \tag{10.9.94}$$

$$g = y^2 + z^2 - 4 \qquad \nabla g = \begin{bmatrix} 0 \\ 2y \\ 2z \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} 0 \\ 2\cos\phi \\ 2\sin\phi \end{bmatrix} \tag{10.9.95}$$

Calculating the surface integral,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \ \mathrm{d}A = \int_0^{\pi} \int_0^{\pi/2} (-2\sin x \sin\phi) \ \mathrm{d}\phi \, \mathrm{d}x \tag{10.9.96}$$

$$= \int_0^{\pi} \Big[2\sin x \cos\phi\Big]_0^{\pi/2} \ \mathrm{d}x = \int_0^{\pi} (-2\sin x) \ \mathrm{d}x \tag{10.9.97}$$

$$= \Big[2\cos x\Big]_0^{\pi} = -4 \tag{10.9.98}$$