Chapter 9

Vector Differential Calculus: Grad, Div, Curl

9.1 Vectors in 2-Space and 3-Space

1. Finding the vector using the given points,

$$P = (1,\,1,\,0) \qquad Q = (6,\,2,\,0) \tag{9.1.1}$$

$$(v_x,\,v_y,\,v_z) = (5,\,1,\,0) \qquad |\mathbf{v}| = \sqrt{5^2 + 1^2} = \sqrt{26} \tag{9.1.2}$$

$$\mathbf{v} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{u} = \frac{1}{\sqrt{26}} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \tag{9.1.3}$$

2. Finding the vector using the given points,

$$P = (1,\,1,\,1) \qquad Q = (2,\,2,\,0) \tag{9.1.4}$$

$$(v_x,\,v_y,\,v_z) = (1,\,1,\,-1) \qquad |\mathbf{v}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3} \tag{9.1.5}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{u} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \tag{9.1.6}$$

3. Finding the vector using the given points,

$$P = (-3, 4, -0.5) \qquad Q = (5.5, 0, 1.2) \tag{9.1.7}$$

$$(v_x, v_y, v_z) = (8.5, -4, 1.7) \qquad |\mathbf{v}| = \sqrt{8.5^2 + (-4)^2 + 1.7^2} = \sqrt{91.14} \tag{9.1.8}$$

$$\mathbf{v} = \begin{bmatrix} 8.5 \\ -4 \\ 1.7 \end{bmatrix} \qquad \mathbf{u} = \frac{1}{\sqrt{91.14}} \begin{bmatrix} 8.5 \\ -4 \\ 1.7 \end{bmatrix} \tag{9.1.9}$$

4. Finding the vector using the given points,

$$P = (1, 4, 2) \qquad Q = (-1, -4, -2) \tag{9.1.10}$$

$$(v_x, v_y, v_z) = (-2, -8, -4) \qquad |\mathbf{v}| = \sqrt{(-2)^2 + (-8)^2 + (-4)^2} = \sqrt{84} \tag{9.1.11}$$

$$\mathbf{v} = \begin{bmatrix} -2 \\ -8 \\ -4 \end{bmatrix} \qquad \mathbf{u} = \frac{1}{\sqrt{84}} \begin{bmatrix} -2 \\ -8 \\ -4 \end{bmatrix} \tag{9.1.12}$$

5. Finding the vector using the given points,

$$P = (0, 0, 0) \qquad Q = (2, 1, -2) \tag{9.1.13}$$

$$(v_x, v_y, v_z) = (2, 1, -2) \qquad |\mathbf{v}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3 \tag{9.1.14}$$

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \qquad \mathbf{u} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \tag{9.1.15}$$

6. Finding the terminal point,

$$P = (0, 2, 13) \qquad (v_x, v_y, v_z) = (4, 0, 0) \tag{9.1.16}$$

$$|\mathbf{v}| = \sqrt{4^2} = 4 \qquad Q = (v_x + P_x, v_y + P_y, v_z + P_z) \tag{9.1.17}$$

$$Q = (4, 2, 13) \tag{9.1.18}$$

7. Finding the terminal point,

$$P = \left(\frac{7}{2}, -3, \frac{3}{4}\right) \qquad (v_x, v_y, v_z) = \left(\frac{1}{2}, 3, -\frac{1}{4}\right) \tag{9.1.19}$$

$$|\mathbf{v}| = \sqrt{\frac{1}{4} + 3^2 + \frac{1}{16}} = 4 \qquad Q = (v_x + P_x, v_y + P_y, v_z + P_z) \tag{9.1.20}$$

$$Q = \left(4, 0, \frac{1}{2}\right) \tag{9.1.21}$$

8. Finding the terminal point,

$$P = (0, 0, 0) \qquad (v_x, v_y, v_z) = (13.1, 0.8, -2) \tag{9.1.22}$$

$$|\mathbf{v}| = \sqrt{13.1^2 + 0.8^2 + (-2)^2} = \sqrt{\frac{705}{2}} \qquad Q = (v_x + P_x, v_y + P_y, v_z + P_z) \tag{9.1.23}$$

$$Q = (13.1, 0.8, -2) \tag{9.1.24}$$

9. Finding the terminal point,

$$P = (-6, -1, -4) \qquad (v_x, v_y, v_z) = (6, 1, -4) \tag{9.1.25}$$

$$|\mathbf{v}| = \sqrt{(-8)^2} = 8 \qquad Q = (v_x + P_x, v_y + P_y, v_z + P_z) \tag{9.1.26}$$

$$Q = (0, 0, -8) \tag{9.1.27}$$

10. Finding the terminal point,

$$P = (0, 3, -3) \qquad (v_x, v_y, v_z) = (0, -3, 3) \tag{9.1.28}$$

$$|\mathbf{v}| = \sqrt{(-6)^2 + 6^2} = \sqrt{72} \qquad Q = (v_x + P_x, v_y + P_y, v_z + P_z) \tag{9.1.29}$$

$$Q = (0, -6, 0) \tag{9.1.30}$$

11. Performing the given computations,

$$2\mathbf{a} = (6, 4, 0) \qquad \frac{1}{2}\,\mathbf{a} = \left(\frac{3}{2}, 1, 0\right) \tag{9.1.31}$$

$$-\mathbf{a} = (-3, -2, 0) \tag{9.1.32}$$

12. Performing the given computations,

$$(\mathbf{a}+\mathbf{b}) = (-1, 8, 0) \qquad (\mathbf{a}+\mathbf{b})+\mathbf{c} = (4, 7, 8) \tag{9.1.33}$$

$$(\mathbf{b}+\mathbf{c}) = (1, 5, 8) \qquad \mathbf{a}+(\mathbf{b}+\mathbf{c}) = (4, 7, 8) \tag{9.1.34}$$

$$\tag{9.1.35}$$

13. Performing the given computations,

$$\mathbf{b}+\mathbf{c} = (1, 5, 8) \qquad \mathbf{c}+\mathbf{b} = (1, 5, 8) \tag{9.1.36}$$

14. Performing the given computations,

$$3\mathbf{c} - 6\mathbf{d} = (15, -3, 24) - (0, 0, 24) = (15, -3, 0) \tag{9.1.37}$$

$$3(\mathbf{c} - 2\mathbf{d}) = 3 \cdot (5, -1, 0) = (15, -3, 0) \tag{9.1.38}$$

15. Performing the given computations,

$$7\mathbf{c} - 7\mathbf{b} = (15, -3, 24) - (0, 0, 24) = (15, -3, 0) \tag{9.1.39}$$

$$3(\mathbf{c} - 2\mathbf{d}) = 3 \cdot (5, -1, 0) = (15, -3, 0) \tag{9.1.40}$$

16. Performing the given computations,

$$\frac{9}{2}\mathbf{a} - 3\mathbf{c} = \left(\frac{27}{2}, 9, 0\right) - (15, -3, 24) = \left(-\frac{3}{2}, 12, -24\right) \tag{9.1.41}$$

$$9\ \left(\frac{1}{2}\ \mathbf{a} - \frac{1}{3}\ \mathbf{c}\right) = 9 \cdot \left(\frac{3}{2}, 1, 0\right) - \left(\frac{5}{3}, -\frac{1}{3}, \frac{8}{3}\right) \tag{9.1.42}$$

$$= 9 \cdot \left(-\frac{1}{6}, \frac{4}{3}, -\frac{8}{3}\right) = \left(-\frac{3}{2}, 12, -24\right) \tag{9.1.43}$$

17. Performing the given computations,

$$(7-3)\mathbf{a} = 4 \cdot (3, 2, 0) = (12, 8, 0) \tag{9.1.44}$$

$$7\mathbf{a} - 3\mathbf{a} = (21, 14, 0) - (9, 6, 0) = (12, 8, 0) \tag{9.1.45}$$

18. Performing the given computations,

$$4\mathbf{a} + 3\mathbf{b} = (12, 8, 0) + (-12, 18, 0) = (0, 26, 0) \tag{9.1.46}$$

$$-4\mathbf{a} - 3\mathbf{b} = (-12, -8, 0) + (12, -18, 0) = (0, -26, 0) \tag{9.1.47}$$

19. The laws are,

$$12 - \text{associative} \qquad 13 - \text{commutative} \tag{9.1.48}$$

$$14 - \text{additive} \qquad 15 - \text{additive} \tag{9.1.49}$$

$$16 - \text{additive} \tag{9.1.50}$$

20. Proving equation 4a, using the commutativity of real numbers

$$\mathbf{a}+\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} b_1+a_1 \\ b_2+a_2 \\ \vdots \\ b_n+a_n \end{bmatrix} = \mathbf{b}+\mathbf{a} \tag{9.1.51}$$

Proving equation 4b, using the associativity of real numbers,

$$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{bmatrix} = \begin{bmatrix} u_1+v_1+w_1 \\ u_2+v_2+w_2 \\ \vdots \\ u_3+v_3+w_3 \end{bmatrix} \tag{9.1.52}$$

$$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1+v_1+w_1 \\ u_2+v_2+w_2 \\ \vdots \\ u_3+v_3+w_3 \end{bmatrix} \tag{9.1.53}$$

Proving equation 4c, using 4a and the additive identity for real numbers

$$\mathbf{a}+\mathbf{0} = \begin{bmatrix} a_1+0 \\ a_2+0 \\ \vdots \\ a_n+0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0+a_1 \\ 0+a_2 \\ \vdots \\ 0+a_n \end{bmatrix} = \mathbf{0}+\mathbf{a} \tag{9.1.54}$$

Proving equation 4d, using the additive inverse for real numbers

$$\mathbf{a}+-\mathbf{a} = \begin{bmatrix} a_1-a_1 \\ a_2-a_2 \\ \vdots \\ a_n-a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0} \tag{9.1.55}$$

Proving equation 6a, using distributivity of multiplication in real numbers

$$c\left(\mathbf{a}+\mathbf{b}\right)=c\cdot\begin{bmatrix}a_1+b_1\\a_2+b_2\\\vdots\\a_n+b_n\end{bmatrix}=\begin{bmatrix}c\cdot(a_1+b_1)\\c\cdot(a_2+b_2)\\\vdots\\c\cdot(a_n+b_n)\end{bmatrix}=\begin{bmatrix}ca_1+cb_1\\ca_2+cb_2\\\vdots\\ca_n+cb_n\end{bmatrix}=c\mathbf{a}+c\mathbf{b} \tag{9.1.56}$$

Proving equation 6b, using distributivity of multiplication in real numbers

$$(c+k)\mathbf{a}=c\cdot\begin{bmatrix}(c+k)a_1\\(c+k)a_2\\\vdots\\(c+k)a_n\end{bmatrix}=\begin{bmatrix}ca_1+ka_1\\ca_2+ka_2\\\vdots\\ca_n+ka_n\end{bmatrix}=c\mathbf{a}+k\mathbf{a} \tag{9.1.57}$$

Proving equation 6c, using associativity of multiplication in real numbers

$$c\left(k\mathbf{a}\right)=c\cdot\begin{bmatrix}ka_1\\ka_2\\\vdots\\ka_n\end{bmatrix}=\begin{bmatrix}cka_1\\cka_2\\\vdots\\cka_n\end{bmatrix}=\begin{bmatrix}(ck)\ a_1\\(ck)\ a_2\\\vdots\\(ck)\ a_n\end{bmatrix}=(ck)\ \mathbf{a} \tag{9.1.58}$$

Proving equation 6d, using multiplicative identity in real numbers

$$1\cdot\mathbf{a}=\begin{bmatrix}1\cdot a_1\\1\cdot a_2\\\vdots\\1\cdot a_n\end{bmatrix}=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}=\mathbf{a} \tag{9.1.59}$$

21. Finding the resultant force using vector addition,

$$\mathbf{p}=(2,3,0) \qquad \mathbf{q}=(0,6,1) \tag{9.1.60}$$

$$\mathbf{u}=(2,0,-4) \qquad \mathbf{F}_{\text{res}}=(4,9,-3) \tag{9.1.61}$$

$$|\mathbf{F}_{\text{res}}|=\sqrt{4^2+9^2+(-3)^2} \qquad =\sqrt{106} \tag{9.1.62}$$

22. Finding the resultant force using vector addition,

$$\mathbf{p} = (1, -2, 3) \qquad \mathbf{q} = (3, 21, -16) \tag{9.1.63}$$

$$\mathbf{u} = (-4, -9, 13) \qquad \mathbf{F}_{\text{res}} = (0, 10, 0) \tag{9.1.64}$$

$$|\mathbf{F}_{\text{res}}| = \sqrt{10^2} \qquad = 10 \tag{9.1.65}$$

23. Finding the resultant force using vector addition,

$$\mathbf{p} = (8, -1, 0) \qquad \mathbf{q} = \left(\frac{1}{2}, 0, \frac{4}{3}\right) \tag{9.1.66}$$

$$\mathbf{u} = \left(-\frac{17}{2}, 1, \frac{11}{3}\right) \qquad \mathbf{F}_{\text{res}} = (0, 0, 5) \tag{9.1.67}$$

$$|\mathbf{F}_{\text{res}}| = \sqrt{5^2} \qquad = 5 \tag{9.1.68}$$

24. Finding the resultant force using vector addition,

$$\mathbf{p} = (-1, 2, -3) \qquad \mathbf{q} = (1, 1, 1) \tag{9.1.69}$$

$$\mathbf{u} = (1, -2, 2) \qquad \mathbf{F}_{\text{res}} = (1, 1, 0) \tag{9.1.70}$$

$$|\mathbf{F}_{\text{res}}| = \sqrt{1^2 + 1^2} \qquad = \sqrt{2} \tag{9.1.71}$$

25. Finding the resultant force using vector addition,

$$\mathbf{p} = (3, 1, -6) \qquad \mathbf{q} = (0, 2, 5) \tag{9.1.72}$$

$$\mathbf{u} = (3, -1, -13) \qquad \mathbf{F}_{\text{res}} = (6, 2, -14) \tag{9.1.73}$$

$$|\mathbf{F}_{\text{res}}| = \sqrt{6^2 + 2^2 + (-14)^2} \qquad = \sqrt{236} \tag{9.1.74}$$

26. Equilibrium requires the vector sum to be zero.

$$\mathbf{p} + \mathbf{q} + \mathbf{u} + \mathbf{v} = 0 \qquad \mathbf{v} = -1 \cdot (4, 9, -3) = (-4, -9, 3) \tag{9.1.75}$$

27. Equilibrium requires the vector sum to be zero.

$$\mathbf{p} + \mathbf{q} + \mathbf{u} + \mathbf{v} = 0 \qquad \mathbf{v} = -1 \cdot (0, 0, 5) = (0, 0, -5) \tag{9.1.76}$$

28. Normalizing the vector by making its norm equal to 1,

$$\mathbf{v} = (1, 1, 0) \qquad \hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} \tag{9.1.77}$$

$$\hat{\mathbf{v}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \tag{9.1.78}$$

29. Parallel to the xy-plane requires a vector of the form

$$\mathbf{v} + \mathbf{p} + \mathbf{q} + \mathbf{u} = (F_1, F_2, 0) \qquad (v_1 + 4, v_2 + 9, v_3 - 3) = (F_1, F_2, 0) \tag{9.1.79}$$

$$\mathbf{v} = (v_1, v_2, 3) \tag{9.1.80}$$

30. Parallel to the z-axis requires a vector of the form

$$\mathbf{v} + \mathbf{p} + \mathbf{q} + \mathbf{u} = (0, 0, F_3) \qquad (v_1 + 4, v_2 + 9, v_3 - 3) = (0, 0, F_3) \tag{9.1.81}$$

$$\mathbf{v} = (-4, -9, F_3) \tag{9.1.82}$$

31. Parallel to the xy-plane requires a vector of the form

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = (F_1, F_2, 0) \qquad \begin{bmatrix} 2+1+0 \\ 0+2+3 \\ -7-3+k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix} \tag{9.1.83}$$

$$k = 10 \tag{9.1.84}$$

32. Using the triangle inequality,

$$|\mathbf{p} + \mathbf{q}| \leq |\mathbf{p}| + |\mathbf{q}| \qquad |\mathbf{p} + \mathbf{q}| \leq 10 \tag{9.1.85}$$

33. Using the triangle inequality,

$$|\mathbf{p} + \mathbf{q} + \mathbf{u}| \leq |\mathbf{p}| + |\mathbf{q}| + |\mathbf{u}| \qquad |\mathbf{p} + \mathbf{q} + \mathbf{u}| \leq 18 \tag{9.1.86}$$

34. Finding the relative velocity,

$$\mathbf{v}_A = 550 \text{ mph south-west} = 550 \left(-\frac{\hat{\mathbf{i}}}{\sqrt{2}} - \frac{\hat{\mathbf{j}}}{\sqrt{2}} \right) \tag{9.1.87}$$

$$\mathbf{v}_B = 550 \text{ mph north-west} = 450 \left(-\frac{\hat{\mathbf{i}}}{\sqrt{2}} + \frac{\hat{\mathbf{j}}}{\sqrt{2}} \right) \tag{9.1.88}$$

$$\mathbf{v}_B - \mathbf{v}_A = \left(100 \frac{\hat{\mathbf{i}}}{\sqrt{2}} + 1000 \frac{\hat{\mathbf{j}}}{\sqrt{2}} \right) \tag{9.1.89}$$

35. Finding the relative velocity,

$$\mathbf{v}_A = 22 \text{ knots north-east} = 22 \left(\frac{\hat{\mathbf{i}}}{\sqrt{2}} + \frac{\hat{\mathbf{j}}}{\sqrt{2}} \right) \tag{9.1.90}$$

$$\mathbf{v}_B = -19 \text{ knots west} = 19\hat{\mathbf{i}} \tag{9.1.91}$$

$$\mathbf{v}_B - \mathbf{v}_A = (-19 - 11\sqrt{2})\, \hat{\mathbf{i}} - 11\sqrt{2}\, \hat{\mathbf{j}} \tag{9.1.92}$$

36. The effect of reflecting a vector about a mirror is to reverse the component perpendicular to the mirror's surface and leave unaffected the component parallel to the mirror's surface.

Let the mirrors be aligned along the positive x and y directions.

$$\mathbf{v}_0 = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} \qquad \mathbf{v}_1 = a\hat{\mathbf{i}} - b\hat{\mathbf{j}} \tag{9.1.93}$$

$$\mathbf{v}_2 = -a\hat{\mathbf{i}} - b\hat{\mathbf{j}} \tag{9.1.94}$$

37. Finding the magnitudes of the forces,

$$\mathbf{v} = (l, l) \qquad \mathbf{p} = (0, -1000) \tag{9.1.95}$$

$$\mathbf{u} = (-l, 0) \qquad \mathbf{v} + \mathbf{p} + \mathbf{u} = \mathbf{0} \tag{9.1.96}$$

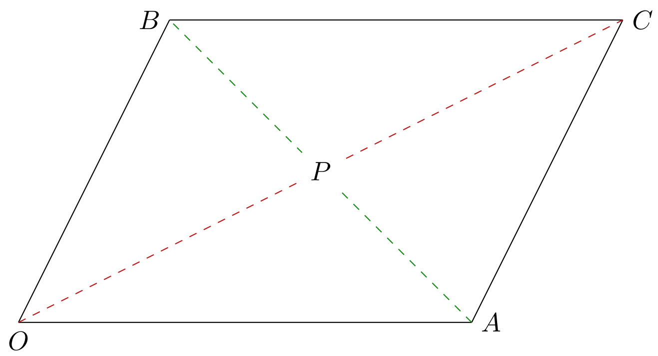
$$l = 1000 \tag{9.1.97}$$

38. Proving geometric identities,

(a) Let P be the midpoint of the main diagonal.
and Q be the midpoint of the other diagonal.

$$P = \frac{\mathbf{a} + \mathbf{b}}{2} \qquad Q = \mathbf{b} + \frac{\mathbf{a} - \mathbf{b}}{2} \tag{9.1.98}$$

P and Q coincide, which means that each diagonal of a parallelogram bisects the other.



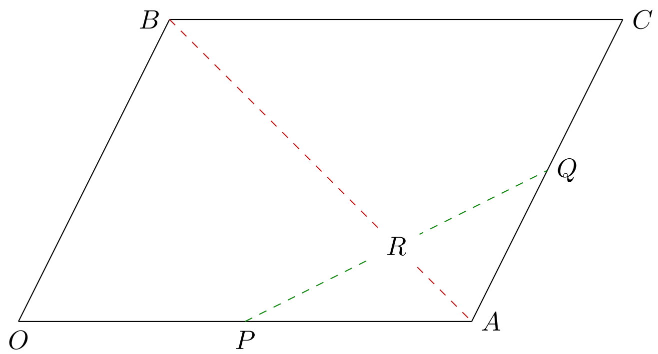
(b) Let P, Q be the midpoints of OA, AC respectively. Further let R be the midpoint of PQ

$$\mathbf{r} = \mathbf{p} + \frac{\mathbf{q} - \mathbf{p}}{2} \qquad = \frac{\mathbf{a}}{2} + \frac{1}{2}\left(\mathbf{a} + \frac{\mathbf{b}}{2} - \frac{\mathbf{a}}{2}\right) \tag{9.1.99}$$

$$= \frac{3\mathbf{a} + \mathbf{b}}{4} \tag{9.1.100}$$

Let this point divide the off-diagonal in the ratio $\lambda : 1 - \lambda$.

$$\mathbf{r} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b}) = \frac{3\mathbf{a} + \mathbf{b}}{4} \qquad \lambda = \frac{3}{4} \tag{9.1.101}$$



(c) Consider the parallellogram with PA and QA as sides. Call the fourth vertex X. Its main diagonal PQ is bisected by and bisects AX

$$\overline{BX} = \frac{1}{2}\,\overline{BA} \qquad \overline{XR} = \frac{1}{2}\,\overline{XA} = \frac{1}{4}\,\overline{BA} \tag{9.1.102}$$

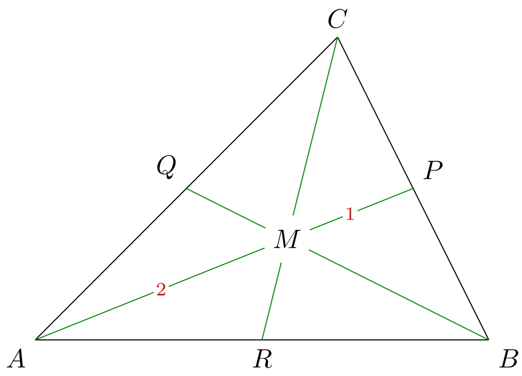
$$\overline{BR} = \frac{3}{4}\,\overline{BA} \tag{9.1.103}$$

(d) Drawing the medians to all 3 sides of a triangle, The point M which divides AP in the ratio $2:1$ is given by

$$\mathbf{m} = \mathbf{a} + \frac{2}{3}(\mathbf{p} - \mathbf{a}) \qquad \mathbf{p} = \frac{\mathbf{b}+\mathbf{c}}{2} \tag{9.1.104}$$

$$\mathbf{m} = \frac{\mathbf{a}}{3} + \left(\frac{\mathbf{b}+\mathbf{c}}{3}\right) \tag{9.1.105}$$

Clearly $\mathbf{m}$ is symmetric in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, which means the other 2 ways of arriving at M will lead to the same position.

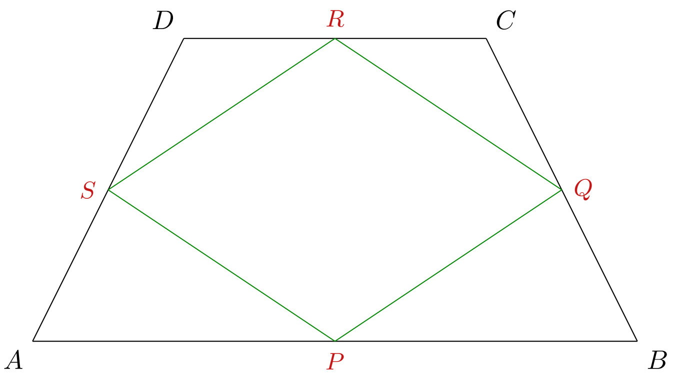


(e) Arbitraty quadrilateral has vertices A, B, C, D with midpoints P, Q, R, S.
Two vectors are parallel if one is a nonzero scalar multiple of the other.

$$\overline{PQ} = \frac{\mathbf{b}+\mathbf{c}}{2} - \frac{\mathbf{b}+\mathbf{a}}{2} = \frac{\mathbf{c}-\mathbf{a}}{2} \qquad \overline{SR} = \frac{\mathbf{c}+\mathbf{d}}{2} - \frac{\mathbf{a}+\mathbf{d}}{2} = \frac{\mathbf{c}-\mathbf{a}}{2} \tag{9.1.106}$$

$$\overline{PS} = \frac{\mathbf{a}+\mathbf{d}}{2} - \frac{\mathbf{a}+\mathbf{b}}{2} = \frac{\mathbf{d}-\mathbf{b}}{2} \qquad \overline{QR} = \frac{\mathbf{d}+\mathbf{c}}{2} - \frac{\mathbf{b}+\mathbf{c}}{2} = \frac{\mathbf{d}-\mathbf{b}}{2} \tag{9.1.107}$$

The quadrilateral $PQRS$ has identical pairs of opposite sides and is therefore a parallellogram by definition.



(f) A parallellopiped is the 3d analog of a parallellogram using three main vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Let the other 4 vertices be

$$\mathbf{a}+\mathbf{b} = \mathbf{p} \qquad \mathbf{b}+\mathbf{c} = \mathbf{q} \tag{9.1.108}$$

$$\mathbf{a}+\mathbf{c} = \mathbf{r} \qquad \mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{s} \tag{9.1.109}$$

The positions of the midpoints of all four body-diagonals are,

$$\mathbf{m}_1 = \frac{\mathbf{p}+\mathbf{c}}{2} \qquad \mathbf{m}_2 = \frac{\mathbf{q}+\mathbf{a}}{2} \tag{9.1.110}$$

$$\mathbf{m}_3 = \frac{\mathbf{r}+\mathbf{b}}{2} \qquad \mathbf{m}_4 = \frac{\mathbf{0}+\mathbf{s}}{2} \tag{9.1.111}$$

Clearly all positions of all four $\mathbf{m}_j$ coincide, which proves the relation.

(g) Consider an n sided regular polygon. Construct it by a set of n vectors whose starting point is the origin and terminal point is P_i.
For a regular polygon, the angle between successive vectors has to be constant and equal to $2\pi/n$.

$$\mathbf{p}_1 = \hat{\mathbf{i}} \qquad \mathbf{p}_{k+1} = \cos\left(\frac{2\pi k}{n}\right)\hat{\mathbf{i}} + \sin\left(\frac{2\pi k}{n}\right)\hat{\mathbf{j}} \tag{9.1.112}$$

$$\tag{9.1.113}$$

Calculating the vector sum,

$$\sum_{k=0}^{n-1} \mathbf{p}_{kx} = \sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) = \sum_{k=0}^{n-1} \text{Re}\left[\exp\left(i\ \frac{2\pi k}{n}\right)\right] \tag{9.1.114}$$

$$= \exp\left(i\ \frac{2\pi k}{n}\right)^{-1} \text{Re}[\exp(i\ 2k\pi) - 1] \tag{9.1.115}$$

Since $z = \exp(i\ 2k\pi) - 1 \equiv 0\ \forall\ k$, this summation is identically zero.
A similar summation for the sine terms involves the imaginary part of this complex number which is also identically zero.

9.2 Inner Product (Dot Product)

1. Performing the given computations,

$$\mathbf{a}\cdot\mathbf{b} = 4 + 0 + 40 = 44 \qquad \mathbf{b}\cdot\mathbf{a} = 4 + 0 + 40 = 44 \tag{9.2.1}$$

$$\mathbf{b}\cdot\mathbf{c} = -8 + 0 + 8 = 0 \tag{9.2.2}$$

2. Performing the given computations,

$$(-3\mathbf{a} + 5\mathbf{c})\cdot\mathbf{b} = -3(\mathbf{a}\cdot\mathbf{b}) + 5(\mathbf{c}\cdot\mathbf{b}) \tag{9.2.3}$$

$$= -3(4 + 0 + 44) + 5(-8 + 0 + 8) \tag{9.2.4}$$

$$= -144 \tag{9.2.5}$$

3. Performing the given computations,

$$|\mathbf{a}| = \sqrt{1^2 + (-3)^2 + 5^2} = \sqrt{35} \qquad |2\mathbf{b}| = 2\sqrt{4^2 + 8^2} = 8\sqrt{5} \tag{9.2.6}$$

$$|-\mathbf{c}| = |-1|\ \sqrt{(-2)^2 + 9^2 + 1^2} = \sqrt{86} \tag{9.2.7}$$

4. Performing the given computations,

$$|\mathbf{a}| = \sqrt{1^2 + (-3)^2 + 5^2} = \sqrt{35} \qquad |\mathbf{b}| = \sqrt{4^2 + 8^2} = 4\sqrt{5} \tag{9.2.8}$$

$$|\mathbf{a} + \mathbf{b}| = \sqrt{(5)^2 + (-3)^2 + 13^2} = \sqrt{203} \tag{9.2.9}$$

5. Performing the given computations,

$$|\mathbf{c}| = \sqrt{(-2)^2 + 9^2 + 1^2} = \sqrt{86} \qquad |\mathbf{b}| = \sqrt{4^2 + 8^2} = 4\sqrt{5} \tag{9.2.10}$$

$$|\mathbf{c} + \mathbf{b}| = \sqrt{2^2 + 9^2 + 9^2} = \sqrt{166} \tag{9.2.11}$$

6. Performing the given computations,

$$|\mathbf{a} + \mathbf{c}|^2 = (-1)^2 + 6^2 + 6^2 = 73 \qquad |\mathbf{a} - \mathbf{c}|^2 = 3^2 + (-12)^2 + 4^2 = 169 \tag{9.2.12}$$

$$|\mathbf{a}|^2 = 1^2 + (-3)^2 + 5^2 = 35 \qquad |\mathbf{c}|^2 = (-2)^2 + 9^2 + 1^2 = 86 \tag{9.2.13}$$

$$|\mathbf{a} + \mathbf{c}|^2 + |\mathbf{a} + \mathbf{c}|^2 = 242 \qquad 2(|\mathbf{a}|^2 + |\mathbf{c}|^2) = 2(35 + 86) = 242 \tag{9.2.14}$$

7. Performing the given computations,

$$|\mathbf{c}| = \sqrt{(-2)^2 + 9^2 + 1^2} = \sqrt{86} \qquad |\mathbf{a}| = \sqrt{1^2 + (-3)^2 + 5^2} = \sqrt{35} \tag{9.2.15}$$

$$|\mathbf{a} \cdot \mathbf{c}| = |-24| = \sqrt{576} \qquad |\mathbf{a}|\ |\mathbf{c}| = \sqrt{3010} \tag{9.2.16}$$

8. Performing the given computations,

$$5\mathbf{a} \cdot 13\mathbf{b} = 5 \cdot 52 - 15 \cdot 0 + 25 \cdot 104 = 2860 \tag{9.2.17}$$

$$65(\mathbf{a} \cdot \mathbf{b}) = 65(4 + 40) = 65(44) = 2860 \tag{9.2.18}$$

9. Performing the given computations,

$$15\mathbf{a} \cdot \mathbf{b} = 15 \cdot 4 - 45 \cdot 0 + 75 \cdot 8 = 660 \tag{9.2.19}$$

$$15\mathbf{a} \cdot \mathbf{c} = -15 \cdot 2 - 45 \cdot 9 + 75 \cdot 1 = -360 \tag{9.2.20}$$

$$15\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 15[1 \cdot 2 - 3 \cdot 9 + 5 \cdot 9] = 15(20) = 300 \tag{9.2.21}$$

10. Performing the given computations,

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 1 \cdot 6 + 3 \cdot 9 + 5 \cdot 7 = 68 \tag{9.2.22}$$

$$(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = 3 \cdot 2 - 3 \cdot 9 - 3 \cdot 1 = -24 \tag{9.2.23}$$

11. The laws being shown are

$$1 : \text{Symmetry} \qquad 4 : \text{Triangle inequality} \tag{9.2.24}$$

$$5 : \text{Triangle inequality} \qquad 6 : \text{Parallellogram equality} \tag{9.2.25}$$

$$7 : \text{Cauchy-Schwarz inequality} \tag{9.2.26}$$

12. If $\mathbf{u} = \mathbf{0}$, nothing is implied since, the definition of inner product makes it an identity.
If $\mathbf{u} \neq \mathbf{0}$,

$$\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{0} \tag{9.2.27}$$

$$\implies \mathbf{v} = \mathbf{w} \tag{9.2.28}$$

13. Using the definition of inner product,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta) \tag{9.2.29}$$

$$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos(\theta)| \tag{9.2.30}$$

$$\tag{9.2.31}$$

Since $|\cos\theta|$ is less than 1 by definition of cosine, the inequality follows.

14. Verifying the triangle inequality

$$|\mathbf{a}| = \sqrt{1^2 + (-3)^2 + 5^2} = \sqrt{35} \qquad |\mathbf{b}| = \sqrt{4^2 + 8^2} = 4\sqrt{5} \tag{9.2.32}$$

$$|\mathbf{a} + \mathbf{b}| = \sqrt{(5)^2 + (-3)^2 + 13^2} = \sqrt{203} \tag{9.2.33}$$

$$\leq |\mathbf{a}| + |\mathbf{b}| \tag{9.2.34}$$

Verifying the Cauchy-Schwarz inequality,

$$|\mathbf{b}| = \sqrt{4^2 + 8^2} = \sqrt{80} \qquad |\mathbf{a}| = \sqrt{1^2 + (-3)^2 + 5^2} = \sqrt{35} \tag{9.2.35}$$

$$|\mathbf{a} \cdot \mathbf{b}| = |44| = \sqrt{1936} \qquad |\mathbf{a}|\,|\mathbf{b}| = \sqrt{2800} \tag{9.2.36}$$

$$\leq |\mathbf{a}|\,|\mathbf{b}| \tag{9.2.37}$$

15. For the parallellogram equality,

$$|\mathbf{a}+\mathbf{b}|^2 = \sum_{k=1}^{n}(a_k+b_k)^2 \qquad |\mathbf{a}-\mathbf{b}|^2 = \sum_{k=1}^{n}(a_k-b_k)^2 \tag{9.2.38}$$

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = \sum_{k=1}^{n}(a_k+b_k)^2 + (a_k+b_k)^2 \tag{9.2.39}$$

$$= \sum_{k=1}^{n} 2a_k^2 + 2b_k^2 \tag{9.2.40}$$

$$= 2\Big(|\mathbf{a}|^2 + |\mathbf{b}|^2 \Big) \tag{9.2.41}$$

This law is the general case of the Pythagoras' law for rectangles. When adjacent sides are no longer perpendicular, a rectangle becomes a parallellogram.

16. For triangle inequality, using Cauchy-Schwarz inequality,

$$|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \tag{9.2.42}$$

$$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) \tag{9.2.43}$$

$$\leq |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\,|\mathbf{a}|\,|\mathbf{b}| \tag{9.2.44}$$

$$\leq (|\mathbf{a}| + |\mathbf{b}|)^2 \tag{9.2.45}$$

Taking the square root of positive quantities proves the inequality.

$$|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \tag{9.2.46}$$

17. To find the work done,

$$\mathbf{F} = (2, 5, 0) \qquad \mathbf{d} = (2, 2, 2) \tag{9.2.47}$$

$$\mathbf{W} = \mathbf{F}\cdot\mathbf{d} \qquad \mathbf{W} = 14\text{ N} \tag{9.2.48}$$

18. To find the work done,

$$\mathbf{F} = (-1, -2, 4) \qquad \mathbf{d} = (6, 7, 5) \tag{9.2.49}$$

$$\mathbf{W} = \mathbf{F}\cdot\mathbf{d} \qquad \mathbf{W} = 0\text{ N} \tag{9.2.50}$$

19. To find the work done,

$$\mathbf{F} = (0, 4, 3) \qquad \mathbf{d} = (-3, -2, 1) \tag{9.2.51}$$

$$\mathbf{W} = \mathbf{F}\cdot\mathbf{d} \qquad \mathbf{W} = -5\text{ N} \tag{9.2.52}$$

20. To find the work done,

$$\mathbf{F} = (6, -3, -3) \qquad \mathbf{d} = (2, -1, -1) \tag{9.2.53}$$

$$\mathbf{W} = \mathbf{F} \cdot \mathbf{d} \qquad \mathbf{W} = 18 \text{ N} \tag{9.2.54}$$

21. Yes, because the associativity of vector addition gives,

$$\mathbf{F}_1 \cdot \mathbf{d} + \mathbf{F}_2 \cdot \mathbf{d} = (\mathbf{F}_1 + \mathbf{F}_2) \cdot \mathbf{d} = \mathbf{F}_{\text{res}} \cdot \mathbf{d} \tag{9.2.55}$$

$$\mathbf{W}_1 + \mathbf{W}_2 = \mathbf{W}_{\text{res}} \tag{9.2.56}$$

22. Finding the angle between the vectors,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{5}{\sqrt{2 \cdot 14}} \tag{9.2.57}$$

$$\theta = \arccos\left(\frac{5}{\sqrt{28}}\right) = 19.11^\circ \tag{9.2.58}$$

23. Finding the angle between the vectors,

$$\cos\theta = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} = \frac{5}{\sqrt{14 \cdot 5}} \tag{9.2.59}$$

$$\theta = \arccos\left(\frac{5}{\sqrt{70}}\right) = 53.3^\circ \tag{9.2.60}$$

24. Finding the angle between the vectors,

$$\cos\theta = \frac{(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{c})}{|(\mathbf{a} + \mathbf{c})|\,|(\mathbf{b} + \mathbf{c})|} = \frac{16}{\sqrt{9 \cdot 29}} \tag{9.2.61}$$

$$\theta = \arccos\left(\frac{16}{\sqrt{261}}\right) = 7.95^\circ \tag{9.2.62}$$

25. Finding the angle between the vectors, with the parameter n,

$$\cos\theta = \frac{(\mathbf{a} + n\mathbf{c}) \cdot (\mathbf{b} + n\mathbf{c})}{|(\mathbf{a} + n\mathbf{c})|\,|(\mathbf{b} + n\mathbf{c})|} = \frac{5n^2 + 6n + 4}{\sqrt{(5n^2 + 2n + 2) \cdot (5n^2 + 10n + 11)}} \tag{9.2.63}$$

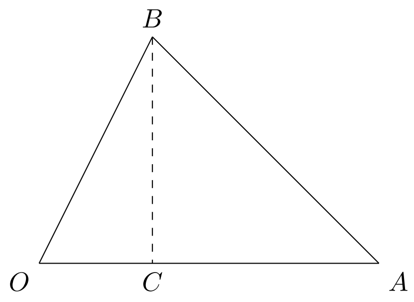
$$\lim_{n \to \infty} \theta = \arccos(1) = 0^\circ \tag{9.2.64}$$

26. Law of cosines, with angle θ between the $\mathbf{a}$, $\mathbf{b}$.

Applying Pythagoras's theorem to the triangle ABC,

$$|\mathbf{a}-\mathbf{b}|^2 = (|\mathbf{a}| - |\mathbf{b}|\cos\theta)^2 + (|\mathbf{b}|\sin\theta)^2 \tag{9.2.65}$$

$$= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta \tag{9.2.66}$$



27. Given $0 \le \alpha \le \beta \le 2\pi$

$$\mathbf{a} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \tag{9.2.67}$$

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos(\beta-\alpha) \qquad \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\beta-\alpha) \tag{9.2.68}$$

28. Using the definition of the inner product,

$$A = (0, 0, 2) \qquad B = (3, 0, 2) \tag{9.2.69}$$

$$C = (1, 1, 1) \tag{9.2.70}$$

$$\cos A = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{|(\mathbf{b}-\mathbf{a})|\,|(\mathbf{c}-\mathbf{a})|} = \frac{3+0+0}{3\cdot\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \angle A = 54.73^\circ \tag{9.2.71}$$

$$\cos B = \frac{(\mathbf{c}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})}{|(\mathbf{c}-\mathbf{b})|\,|(\mathbf{a}-\mathbf{b})|} = \frac{6+0+0}{\sqrt{6}\cdot\sqrt{9}} = \frac{6}{\sqrt{54}} \qquad \angle B = 35.26^\circ \tag{9.2.72}$$

$$\cos C = \frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})}{|(\mathbf{a}-\mathbf{c})|\,|(\mathbf{b}-\mathbf{c})|} = \frac{-2+1+1}{\sqrt{3}\cdot\sqrt{6}} = 0 \qquad \angle C = 90^\circ \tag{9.2.73}$$

29. Finding the angles between adjacent sides.

$$\mathbf{a} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{9.2.74}$$

$$\cos\theta = \frac{12+0}{6\cdot\sqrt{13}} \qquad \theta = 56.31^\circ \tag{9.2.75}$$

$$\phi = 180^\circ - \theta = 123.69^\circ \tag{9.2.76}$$

In a parallellogra, opposite angles are equal and adjacent angles are supplementary. So only one angle needs to be calculated.

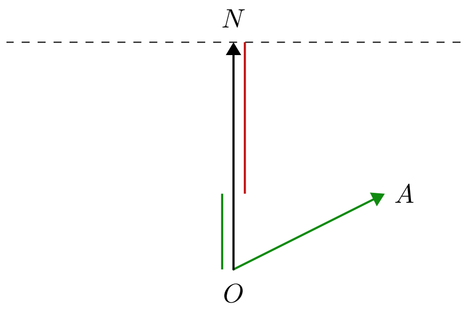
30. Expressing the plane in normal form,

$$\hat{\mathbf{n}} \cdot \mathbf{r} = p \qquad A : (1, 0, 2) \tag{9.2.77}$$

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad p = \frac{9}{\sqrt{11}} \tag{9.2.78}$$

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 27 \\ 9 \\ 9 \end{bmatrix} \tag{9.2.79}$$

$$p = {\color{red} d} + {\color{green} \mathbf{a} \cdot \hat{\mathbf{n}}} \qquad d = \frac{9-5}{\sqrt{11}} = \frac{4}{\sqrt{11}} \tag{9.2.80}$$



Here, the plane is viewed from the side (so it looks like a line). The red and green segments add up to the distance of the plane from the origin.

31. Using the condition for orthogonality,

$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0 \qquad 3a_1 - 8 + 36 = 0 \tag{9.2.81}$$

$$a_1 = \frac{-28}{3} \tag{9.2.82}$$

32. Planes are orthogonal if their normal vectors are orthogonal,

$$\mathbf{n}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{n}_2 = \frac{1}{\sqrt{c^2 + 65}} \begin{bmatrix} 8 \\ -1 \\ c \end{bmatrix} \tag{9.2.83}$$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \qquad \mathbf{c} = -24 \tag{9.2.84}$$

33. Unit vectors are of the form (in $2d$ Cartesian plane)

$$\mathbf{a} \cdot \mathbf{n} = 0 \qquad |\mathbf{a}| = 1 \tag{9.2.85}$$

$$4a_1 + 3a_2 = 0 \qquad a_1^2 + a_2^2 = 1 \tag{9.2.86}$$

$$a_1^2 = \frac{9}{25} \qquad a_2^2 = \frac{16}{25} \tag{9.2.87}$$

$$\hat{\mathbf{b}}_1 = \left(\frac{3}{5}, -\frac{4}{5}\right) \qquad \hat{\mathbf{b}}_2 = \left(-\frac{3}{5}, \frac{4}{5}\right) \tag{9.2.88}$$

34. Each reflection reverses the component orthogonal to that mirror while leaving the other two components unchanged.

After being reflected by all 3 mirrors, all 3 components have been reversed. The angle between $\mathbf{v}_i$ and $\mathbf{v}_f$ is thus 180°

$$\mathbf{v}_i = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \mathbf{v}_f = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix} = -\mathbf{v}_i \tag{9.2.89}$$

$$\mathbf{v}_i \cdot \mathbf{v}_f = |\mathbf{v}_i|^2 \cdot (-1) \qquad \cos\theta = -1 \tag{9.2.90}$$

35. The two diagonals being orthogonal requires,

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0 \qquad |\mathbf{a}|^2 = |\mathbf{b}|^2 \tag{9.2.91}$$

Thus, the lengths of the sides have to be equal, which forces the parallellogram into a square.

36. To find the projection of a vector onto another,

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} = \frac{6}{\sqrt{13}} \tag{9.2.92}$$

37. To find the projection of a vector onto another,

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} = \frac{0}{\sqrt{29}} = 0 \tag{9.2.93}$$

38. To find the projection of a vector onto another,

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} = \frac{-34}{\sqrt{17}} = -2\sqrt{17} \tag{9.2.94}$$

39. To satisfy the condition,

$$\mathbf{a} \cdot \hat{\mathbf{b}} = \mathbf{b} \cdot \hat{\mathbf{a}} \qquad |\mathbf{a}| \cos\theta = |\mathbf{b}| \cos\theta \tag{9.2.95}$$

Either $\cos\theta = 0$ and the vectors are orthogonal or $|\mathbf{a}| = |\mathbf{b}|$ and the vectors are the same length.

40. The component does not change, since the only information about $\mathbf{b}$ needed in the computation is its direction.

9.3 Vector Product (Cross Product)

1. Proving Equation 4, using the commutativity of real numbers

$$l\mathbf{a} \times \mathbf{b} = |l|\,|\mathbf{a}|\,|\mathbf{b}| \sin\theta \qquad\qquad \mathbf{a} \times l\mathbf{b} = |\mathbf{a}|\,|l|\,|\mathbf{b}| \sin\theta \tag{9.3.1}$$

$$l(\mathbf{a} \times \mathbf{b}) = |l|\,(|\mathbf{a}|\,|\mathbf{b}| \sin\theta) \tag{9.3.2}$$

The three expressions are equivalent.

For the specific case of Cartersian $\mathcal{R}^3$, the vectors can be expressed in terms of their 3 components.

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} \tag{9.3.3}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \tag{9.3.4}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{9.3.5}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \tag{9.3.6}$$

The relation follows from the properties of determinant addition

2. The relation with $\mathbf{a} \neq \mathbf{0}$ gives

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \qquad\qquad \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = 0 \tag{9.3.7}$$

$$\implies \mathbf{b} = \mathbf{c} \quad \text{or} \quad \mathbf{a} \parallel (\mathbf{b} - \mathbf{c}) \tag{9.3.8}$$

3. Swapping two rows of a determinant introduces a multiplier of (-1). This proves Equation 6.

Alternatively, enforcing the right-handed triple rule introduces a (-1) factor into the result when swapping $\mathbf{a}$ and $\mathbf{b}$ in the cross product.

To prove Equation 11,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \tag{9.3.9}$$

$$= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \tag{9.3.10}$$

The properties of row swapping in the underlying determinant are used to prove the properties of the box product.

4. Proving the identity,

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2\theta \qquad |\mathbf{a}\cdot\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2\cos^2\theta \tag{9.3.11}$$

$$|\mathbf{a}\times\mathbf{b}|^2 + |\mathbf{a}\cdot\mathbf{b}|^2 = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) \tag{9.3.12}$$

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - |\mathbf{a}\cdot\mathbf{b}|^2} \tag{9.3.13}$$

Verifying the relation for the given vectors,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{vmatrix} = 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}} \qquad |\mathbf{a}\times\mathbf{b}| = \sqrt{96} \tag{9.3.14}$$

$$(\mathbf{a}\cdot\mathbf{b})^2 = 49 \qquad |\mathbf{a}|^2\,|\mathbf{b}|^2 = 5\cdot 29 = 145 \tag{9.3.15}$$

$$\sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - |\mathbf{a}\cdot\mathbf{b}|^2} = \sqrt{96} \tag{9.3.16}$$

5. Replacing $\mathbf{p}$ with $-\mathbf{p}$, reverses the direction of the moment $\mathbf{m}\to-\mathbf{m}$.
   Physically, the body rotates in the opposite direction when the direction of the force is reversed.
6. $\mathbf{r}$ is the position vector of the point whose velocity is to be found. $d\to 2d$ makes $|\mathbf{v}|\to 2\,|\mathbf{v}|$
7. From the details,

$$\boldsymbol{\omega} = 20\ \mathrm{s}^{-1}\,\hat{\mathbf{j}} \qquad \mathbf{r} = (8, 6, 0) \tag{9.3.17}$$

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} \qquad = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 20 & 0 \\ 8 & 6 & 0 \end{vmatrix} \tag{9.3.18}$$

$$\mathbf{v} = -160\ \mathrm{m\,s}^{-1}\,\hat{\mathbf{k}} \qquad |\mathbf{v}| = 160\ \mathrm{m\,s}^{-1} \tag{9.3.19}$$

8. New angular velocity is,

$$\boldsymbol{\omega} = 10\ \mathrm{s}^{-1} \left(\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}\right) \qquad \mathbf{r} = (4,\,2,\,-2) \tag{9.3.20}$$

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \qquad = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{10}{\sqrt{2}} & \frac{10}{\sqrt{2}} & 0 \\ 4 & 2 & -2 \end{vmatrix} \tag{9.3.21}$$

$$= -10\sqrt{2}\,\hat{\mathbf{i}} + 10\sqrt{2}\,\hat{\mathbf{j}} - 10\sqrt{2}\,\hat{\mathbf{k}} \tag{9.3.22}$$

9. Box product is zero,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0 \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \tag{9.3.23}$$

At least 2 of the vectors are L.D. Geometrically, at least 2 of the 3 edge vectors of the parallellopiped are collinear.

10. Refer notes. TBC

11. Performing the given computations,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \tag{9.3.24}$$

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix} \tag{9.3.25}$$

$$\mathbf{a}\cdot\mathbf{b} = -4 \tag{9.3.26}$$

12. Performing the given computations,

$$3\mathbf{c} \times 5\mathbf{d} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 12 & -6 \\ 25 & -5 & 15 \end{vmatrix} = \begin{bmatrix} 150 \\ -195 \\ -315 \end{bmatrix} \tag{9.3.27}$$

$$15\mathbf{d} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 75 & -15 & 45 \\ 1 & 4 & -2 \end{vmatrix} = \begin{bmatrix} -150 \\ 195 \\ 315 \end{bmatrix} \tag{9.3.28}$$

$$15\mathbf{d} \cdot \mathbf{c} = 75 - 60 - 90 = -75 \tag{9.3.29}$$

$$15\mathbf{c} \cdot \mathbf{d} = 75 - 60 - 90 = -75 \tag{9.3.30}$$

$$\tag{9.3.31}$$

13. Performing the given computations,

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & -2 \\ -1 & 3 & 0 \end{vmatrix} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix} \tag{9.3.32}$$

$$\mathbf{a} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 0 \\ 1 & 4 & -2 \end{vmatrix} = \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix} \tag{9.3.33}$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 2 & 0 \\ 1 & 4 & -2 \end{vmatrix} = \begin{bmatrix} -4 \\ -6 \\ -14 \end{bmatrix} \tag{9.3.34}$$

14. Performing the given computations,

$$4\mathbf{b} \times 3\mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -12 & 8 & 0 \\ 3 & 12 & -6 \end{vmatrix} = \begin{bmatrix} -48 \\ -72 \\ -168 \end{bmatrix} \tag{9.3.35}$$

$$12\mathbf{c} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 12 & 48 & -24 \\ -3 & 2 & 0 \end{vmatrix} = \begin{bmatrix} 48 \\ 72 \\ 168 \end{bmatrix} \tag{9.3.36}$$

15. Performing the given computations,

$$(\mathbf{a}+\mathbf{d})\times(\mathbf{d}+\mathbf{a}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 7 & 0 & 3 \\ 7 & 0 & 3 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9.3.37}$$

16. Performing the given computations,

$$(\mathbf{b}\times\mathbf{c})\cdot\mathbf{d} = \begin{vmatrix} 5 & -1 & 3 \\ -3 & 2 & 0 \\ 1 & 4 & -2 \end{vmatrix} = 5(-4)+(6)+3(-14) = -56 \tag{9.3.38}$$

$$\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} -3 & 2 & 0 \\ 1 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} = -3(10)-2(13) = -56 \tag{9.3.39}$$

17. Performing the given computations,

$$(\mathbf{b}\times\mathbf{c})\times\mathbf{d} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 2 & 0 \\ 1 & 4 & -2 \end{vmatrix} \times\mathbf{d} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -4 & -6 & -14 \\ 5 & -1 & 3 \end{vmatrix} = \begin{bmatrix} -32 \\ -58 \\ 34 \end{bmatrix} \tag{9.3.40}$$

$$\mathbf{b}\times(\mathbf{c}\times\mathbf{d}) = \mathbf{b}\times \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 2 & 0 \\ 10 & -13 & -21 \end{vmatrix} = \begin{bmatrix} -42 \\ -63 \\ 19 \end{bmatrix} \tag{9.3.41}$$

18. Performing the given computations,

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix} \times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 7 \\ 2 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -7 \\ 14 \\ 0 \end{bmatrix} \tag{9.3.42}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{a}) = \mathbf{a}\times \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 0 \\ 0 & 0 & -7 \end{vmatrix} = \begin{bmatrix} -7 \\ 14 \\ 0 \end{bmatrix} \tag{9.3.43}$$

19. Performing the given computations,

$$(\mathbf{i}\times\mathbf{j})\cdot\mathbf{k} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \tag{9.3.44}$$

$$(\mathbf{i}\times\mathbf{k})\cdot\mathbf{j} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \tag{9.3.45}$$

9.3.46

20. Performing the given computations,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix} \times \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} \tag{9.3.47}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 7 \\ 10 & -13 & -21 \end{vmatrix} = \begin{bmatrix} 91 \\ 70 \\ 0 \end{bmatrix} \tag{9.3.48}$$

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c})\cdot\mathbf{c} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 2 & 0 \\ 1 & 4 & -2 \end{vmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -14 \\ -56 \\ 28 \end{bmatrix} \tag{9.3.49}$$

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c})\cdot\mathbf{d} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 2 & 0 \\ 1 & 4 & -2 \end{vmatrix} \cdot \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -70 \\ 14 \\ -42 \end{bmatrix} \tag{9.3.50}$$

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c})\cdot\mathbf{c} - (\mathbf{a}\ \mathbf{b}\ \mathbf{c})\cdot\mathbf{d} = \begin{bmatrix} 56 \\ -70 \\ 70 \end{bmatrix} \tag{9.3.51}$$

21. Performing the given computations,

$$4\mathbf{b} \times 3\mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -12 & 8 & 0 \\ 3 & 12 & -6 \end{vmatrix} = \begin{bmatrix} -48 \\ -72 \\ -168 \end{bmatrix} \tag{9.3.52}$$

$$\mathbf{b} \times \mathbf{c} = \cdot \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 2 & 0 \\ 1 & 4 & -2 \end{vmatrix} = \begin{bmatrix} -4 \\ -6 \\ -14 \end{bmatrix} \tag{9.3.53}$$

$$12\,|\mathbf{b} \times \mathbf{c}| = 12\sqrt{16 + 36 + 196} = 24\sqrt{62} \tag{9.3.54}$$

$$\mathbf{c} \times \mathbf{b} = \cdot \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & -2 \\ -3 & 2 & 0 \end{vmatrix} = \begin{bmatrix} 4 \\ 6 \\ 14 \end{bmatrix} \tag{9.3.55}$$

$$12\,|\mathbf{c} \times \mathbf{b}| = 12\sqrt{16 + 36 + 196} = 24\sqrt{62} \tag{9.3.56}$$

22. Performing the given computations,

$$(\mathbf{a}\ \mathbf{c}\ \mathbf{d}) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} = 2(10) - 1(13) = 7 \tag{9.3.57}$$

$$(\mathbf{a} - \mathbf{b} \quad \mathbf{c} - \mathbf{b} \quad \mathbf{d} - \mathbf{b}) = \begin{vmatrix} 5 & -1 & 0 \\ 4 & 2 & -2 \\ 8 & -3 & 3 \end{vmatrix} = 5(0) + 1(28) = 28 \tag{9.3.58}$$

23. Performing the given computations,

$$\mathbf{b} \times \mathbf{b} = 0 \tag{9.3.59}$$

$$(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{b}) = -(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b}) = 0 \tag{9.3.60}$$

$$\mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|^2 = 13 \tag{9.3.61}$$

24. Examples TBC.

(a) Proving the relation for general vectors in $\mathcal{R}^3$,

$$\mathbf{b} \times (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_2 d_3 - c_3 d_2 & c_3 d_1 - c_1 d_3 & c_1 d_2 - c_2 d_1 \end{vmatrix} \tag{9.3.62}$$

$$= c_1(b_2 d_2 + b_3 d_3)\, \hat{\mathbf{i}} + c_2(b_1 d_1 + b_3 d_3)\, \hat{\mathbf{j}} + c_3(b_1 d_1 + b_2 d_2)\, \hat{\mathbf{k}} \tag{9.3.63}$$

$$- [d_1(b_2 c_2 + b_3 c_3)\, \hat{\mathbf{i}} + d_2(b_1 c_1 + b_3 c_3)\, \hat{\mathbf{j}} + d_3(b_1 c_1 + b_2 c_2)\, \hat{\mathbf{k}}] \tag{9.3.64}$$

$$= (b_1 d_1 + b_2 d_2 + b_3 d_3)(c_1\, \hat{\mathbf{i}} + c_2\, \hat{\mathbf{j}} + c_3\, \hat{\mathbf{k}}) \tag{9.3.65}$$

$$- (b_1 c_1 + b_2 c_2 + b_3 c_3)(d_1\, \hat{\mathbf{i}} + d_2\, \hat{\mathbf{j}} + d_3\, \hat{\mathbf{k}}) \tag{9.3.66}$$

$$= (\mathbf{b} \cdot \mathbf{d})\, \mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{d} \tag{9.3.67}$$

(b) Replacing $\mathbf{b}$ with $\mathbf{a} \times \mathbf{b}$ in the result from part a,

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\, \mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\, \mathbf{d} \tag{9.3.68}$$

(c) Let $(\mathbf{c} \times \mathbf{d}) = \mathbf{e}$ for convenience. Taking the dot product of part a with $\mathbf{a}$,

$$\mathbf{b} \times \mathbf{e} \cdot \mathbf{a} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{e} \tag{9.3.69}$$

$$= (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \tag{9.3.70}$$

$$[(\mathbf{b} \cdot \mathbf{d})\, \mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{d}] \cdot \mathbf{a} = (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) \tag{9.3.71}$$

(d) The relations follow from the properties of row swapping in determinants, where a factor of $(-1)^n$ is multiplied for n swaps.

25. Language in question unclear. Force acts on the point A and the moment is to be measured about the point Q

$$\mathbf{r} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \tag{9.3.72}$$

$$\mathbf{m} = \mathbf{r} \times \mathbf{p} \qquad = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} \tag{9.3.73}$$

$$|\mathbf{m}| = 10 \tag{9.3.74}$$

26. Language in question unclear. Force acts on the point A and the moment is to be measured about the point Q

$$\mathbf{r} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \tag{9.3.75}$$

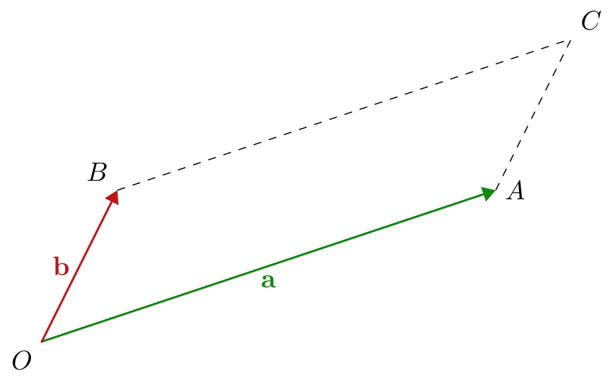
$$\mathbf{m} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 2 \\ 1 & 0 & 3 \end{vmatrix} = \begin{bmatrix} 9 \\ -4 \\ -3 \end{bmatrix} \tag{9.3.76}$$

$$|\mathbf{m}| = \sqrt{81 + 16 + 9} = \sqrt{106} \tag{9.3.77}$$

27. Using cross product of adjacent sides as vectors,

$$\mathbf{a} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{9.3.78}$$

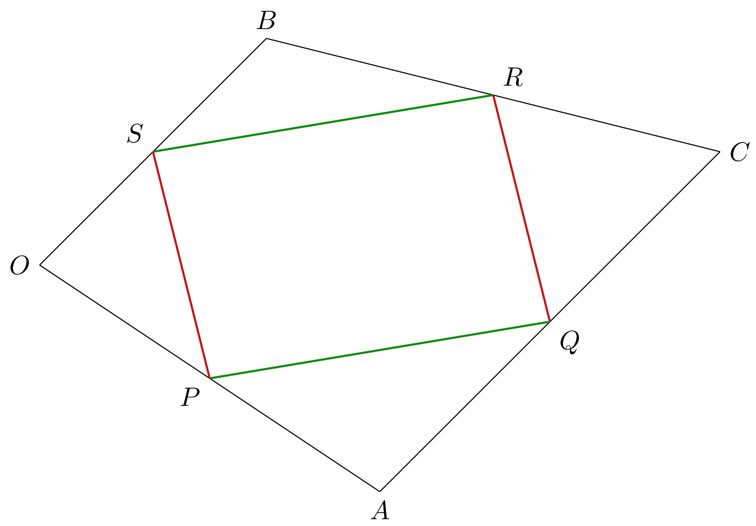
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \qquad \text{Area} = 10 \tag{9.3.79}$$



28. Pairs of opposite sides are identical vectors, which makes it a parallellogram. Using cross product of adjacent sides as vectors,

$$\overline{PQ} = \overline{SR} = \mathbf{a} = \begin{bmatrix} 3 \\ 0.5 \end{bmatrix} \qquad \overline{QR} = \overline{PS} = \mathbf{b} = \begin{bmatrix} -0.5 \\ 2 \end{bmatrix} \tag{9.3.80}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 0.5 & 0 \\ -0.5 & 2 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6.25 \end{bmatrix} \qquad \text{Area} = 10 \tag{9.3.81}$$



29. Area of triangle is half the area of the parallellogram.

$$\mathbf{a} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \tag{9.3.82}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 0 & 4 \\ 2 & 3 & 3 \end{vmatrix} = \begin{bmatrix} -12 \\ 2 \\ 6 \end{bmatrix} \qquad \text{Area} = \frac{1}{2}\left|\mathbf{a} \times \mathbf{b}\right| = \sqrt{46} \tag{9.3.83}$$

30. Using the point A as the reference point.

$$\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -2.25 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -1 \\ 6 \\ 3.75 \end{bmatrix} \tag{9.3.84}$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 0 & -2.25 \\ -1 & 6 & 3.75 \end{vmatrix} = \begin{bmatrix} 13.5 \\ -9 \\ 18 \end{bmatrix} \qquad \hat{\mathbf{n}} = \frac{1}{\sqrt{29}}\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \tag{9.3.85}$$

$$3x - 2y + 4z = 0 \tag{9.3.86}$$

31. Using the point A as the reference point. Answer in appendix is wrong.

$$\mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \tag{9.3.87}$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -5 & 2 \\ 3 & -3 & 3 \end{vmatrix} = \begin{bmatrix} -9 \\ 6 \\ 15 \end{bmatrix} \qquad \hat{\mathbf{n}} = \frac{1}{\sqrt{38}} \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} \tag{9.3.88}$$

$$-3x + 2y + 5z = 23 \tag{9.3.89}$$

32. Volume is equal to box product,

$$\begin{vmatrix} 1 & 1 & 0 \\ -2 & 0 & 2 \\ -2 & 0 & -3 \end{vmatrix} = 6 + 22 = 28 \tag{9.3.90}$$

33. Using the first vertex as the reference point,

$$\begin{vmatrix} 4 & -8 & 2 \\ 6 & 3 & 7 \\ 9 & 6 & 3 \end{vmatrix} = 4(9-42) - 6(-36) + 9(-62) = -474 \tag{9.3.91}$$

$$\text{Area} = \frac{474}{6} = 79 \tag{9.3.92}$$

34. Using the first vertex as the reference point,

$$\begin{vmatrix} 2 & 4 & 6 \\ 7 & 5 & 3 \\ 1 & -1 & 2 \end{vmatrix} = 2(13) - 7(14) + (-18) = -90 \tag{9.3.93}$$

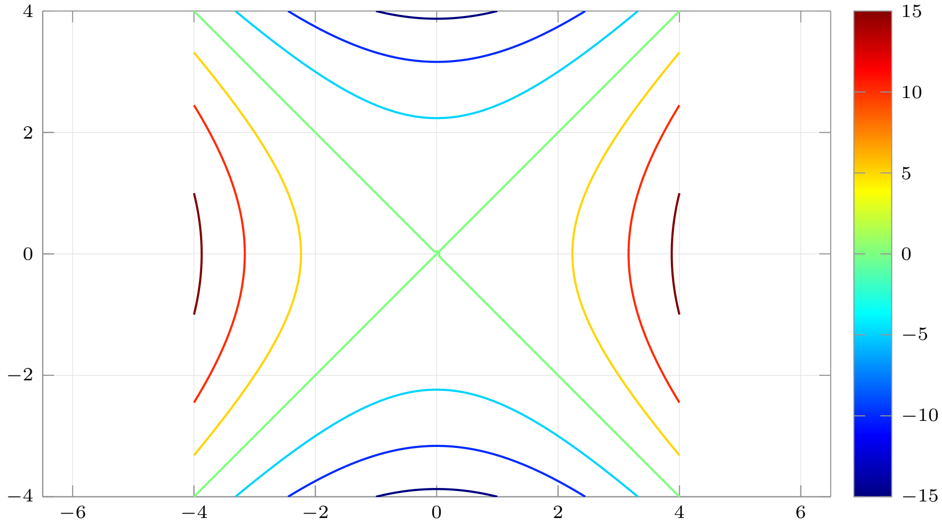
$$\text{Area} = \frac{90}{6} = 15 \tag{9.3.94}$$

35. TBC. Refer notes.

9.4 Vector and Scalar Functions and Their Fields, Vector Calculus: Derivatives

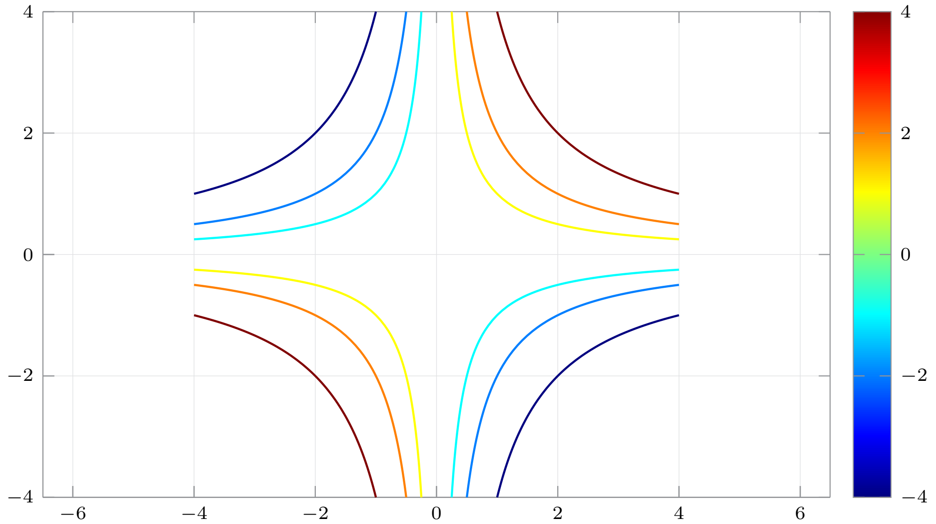
1. Isotherms are hyperbolas.

$c = x^2 - y^2$

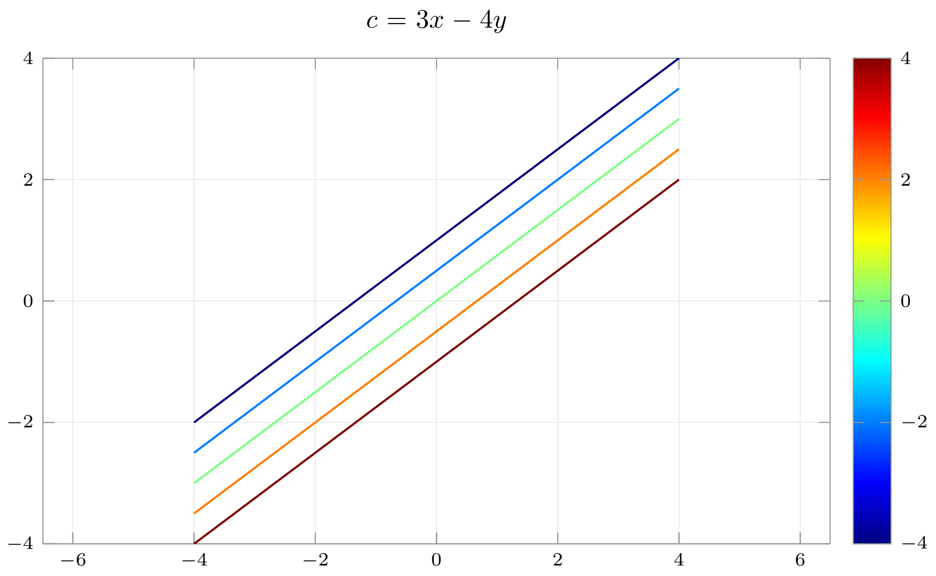


2. Isotherms are hyperbolas.

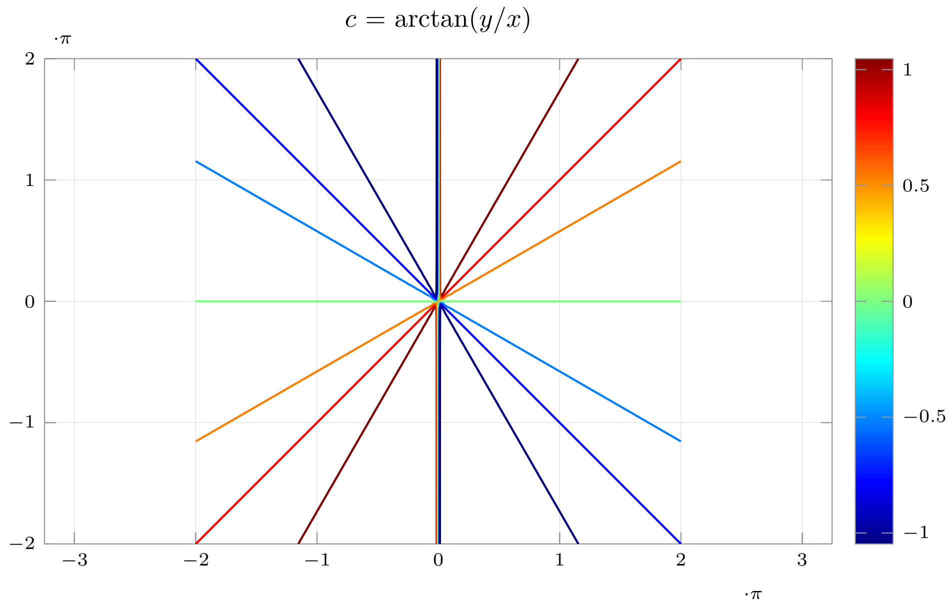
$c = xy$



3. Isotherms are straight lines.

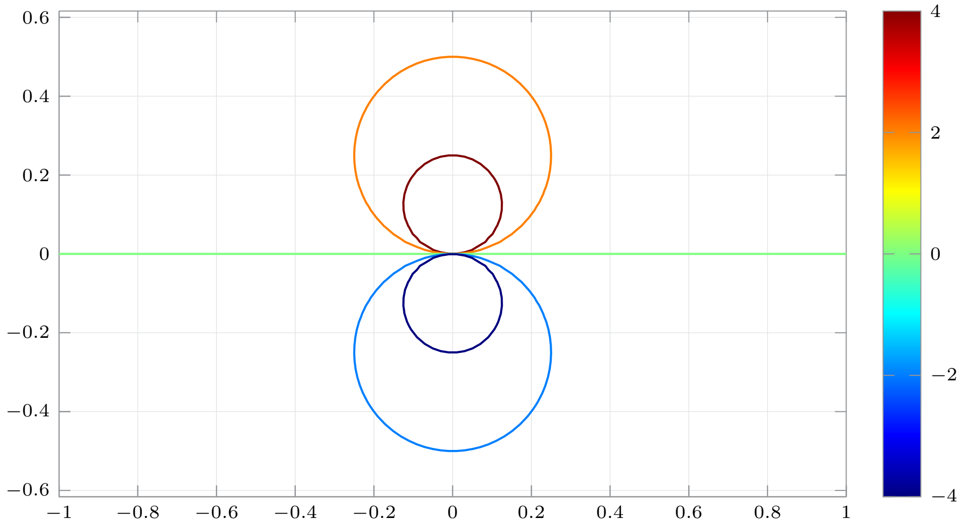


4. Isotherms are straight lines.



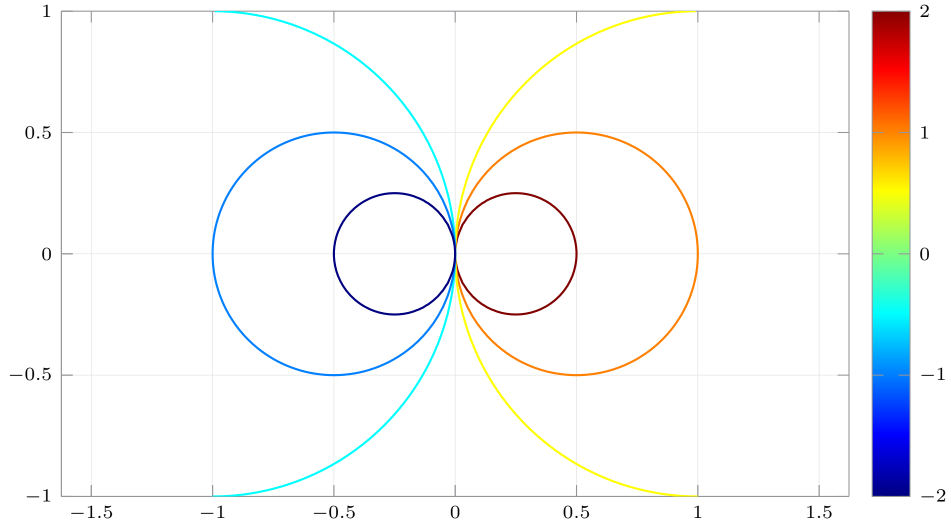
5. Isotherms are circles centered on the y-axis.

$$c = \frac{y}{x^2 + y^2}$$



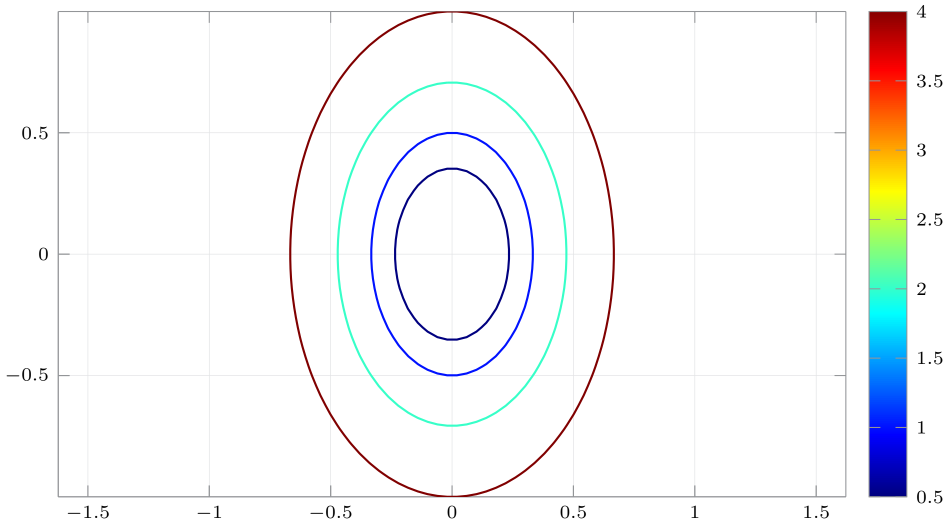
6. Isotherms are circles centered on the x-axis.

$$c = \frac{x}{x^2 + y^2}$$



7. Isotherms are ellipses.

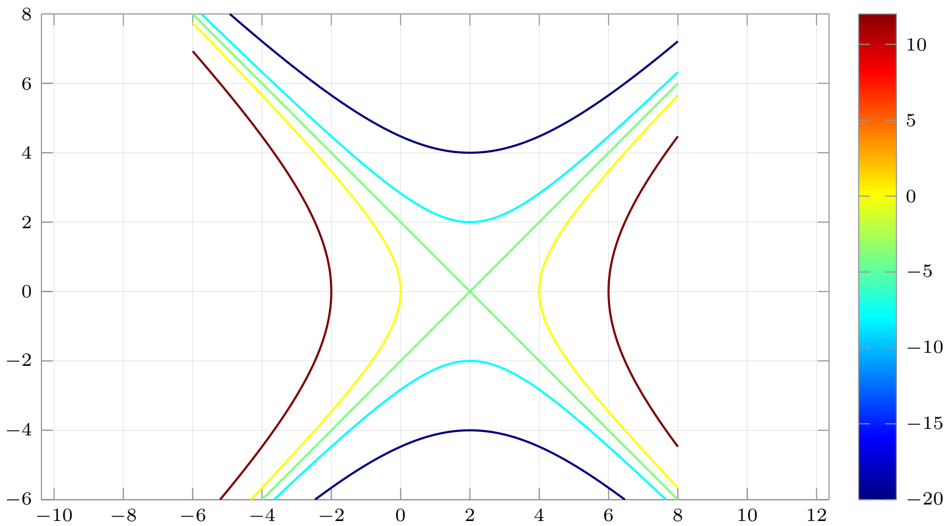
$c = 9x^2 + 4y^2$



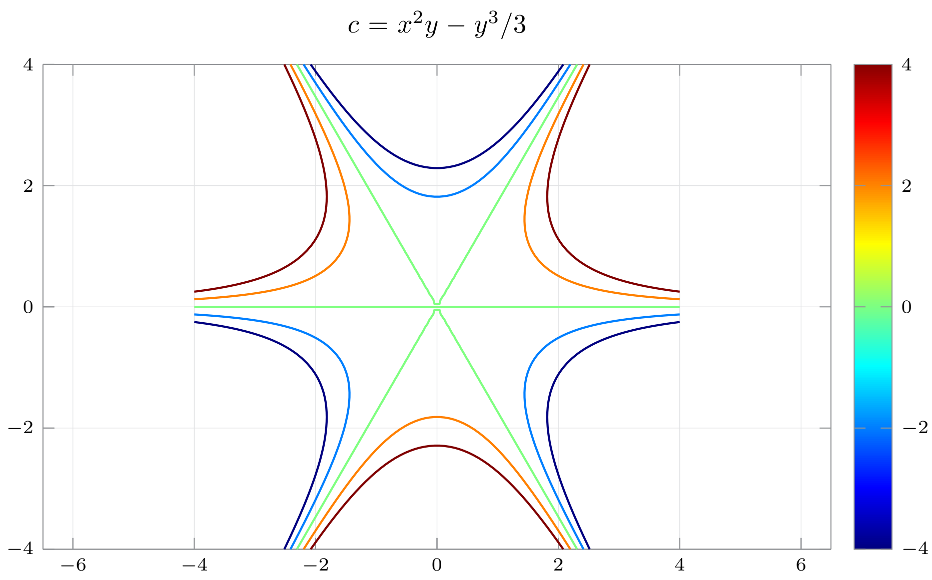
8. Plotting the isotherms using `gnuplot`

(a) The isotherms are hyperbolas.

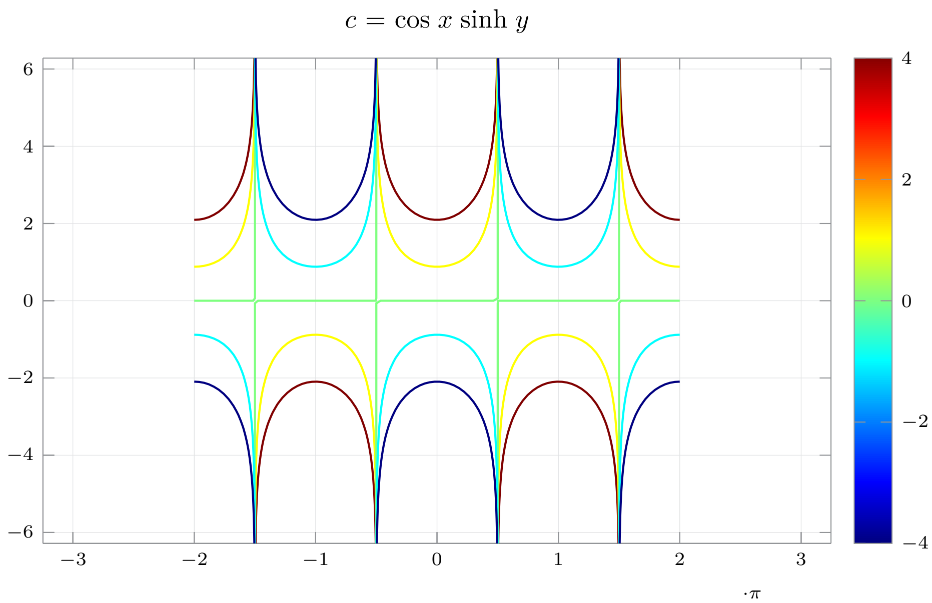
$c = x^2 - 4x - y^2$



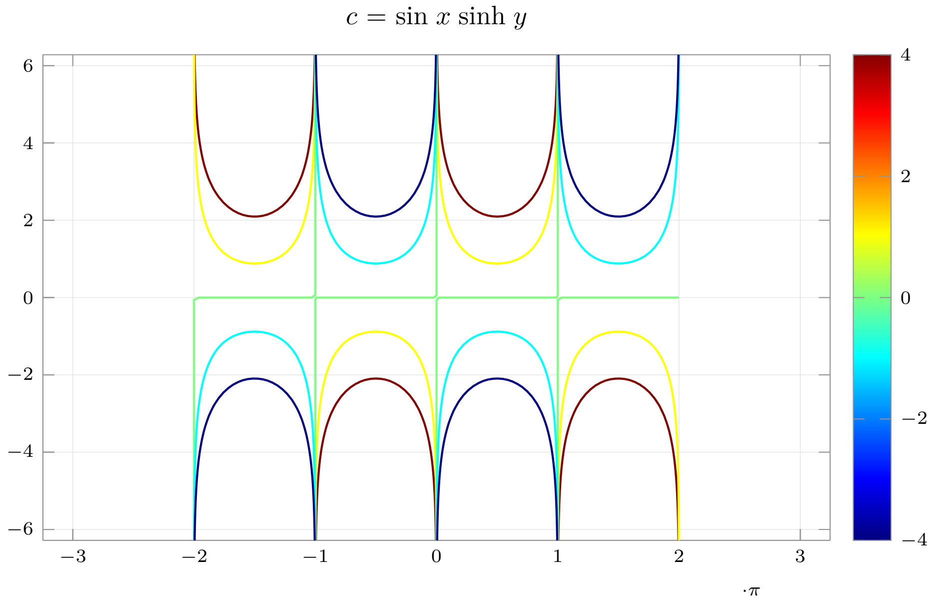
(b) The isotherms are hyperbola analogs with three principal axes.



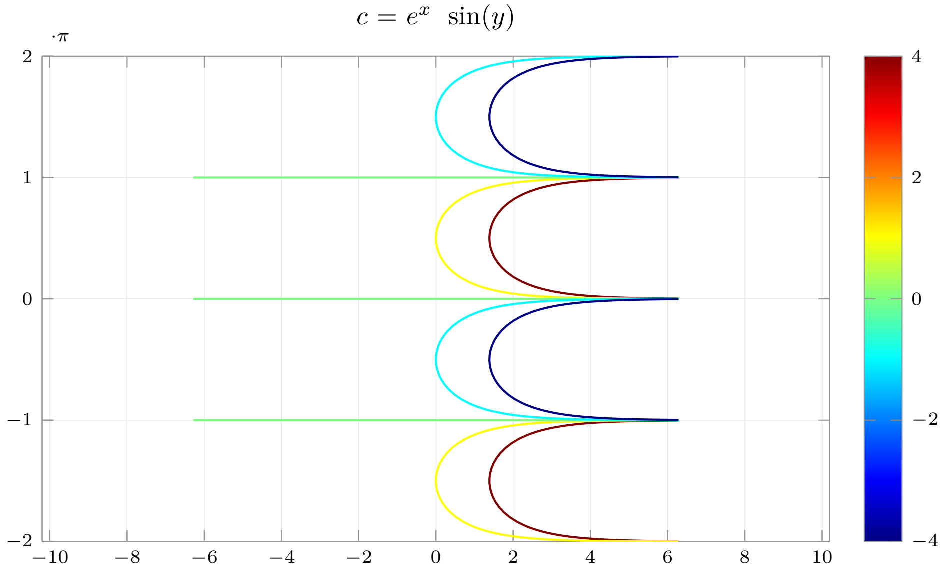
(c) The isotherms are periodic in x.



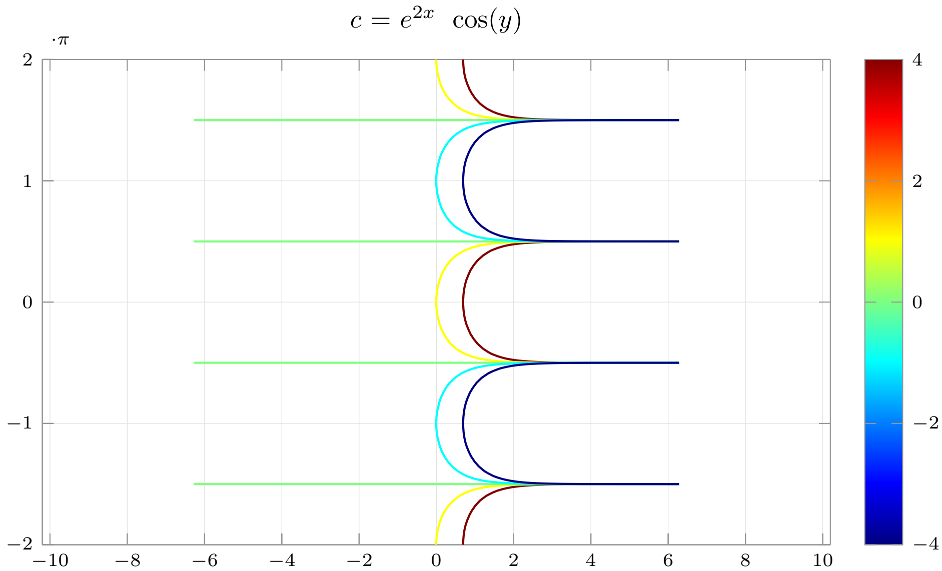
(d) The isotherms are periodic in x.



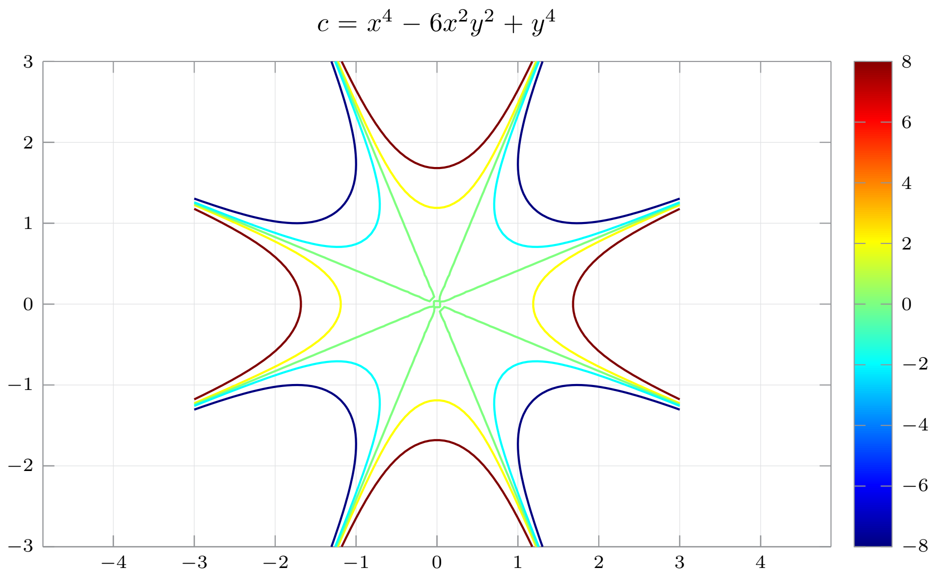
(e) The isotherms are periodic in y.



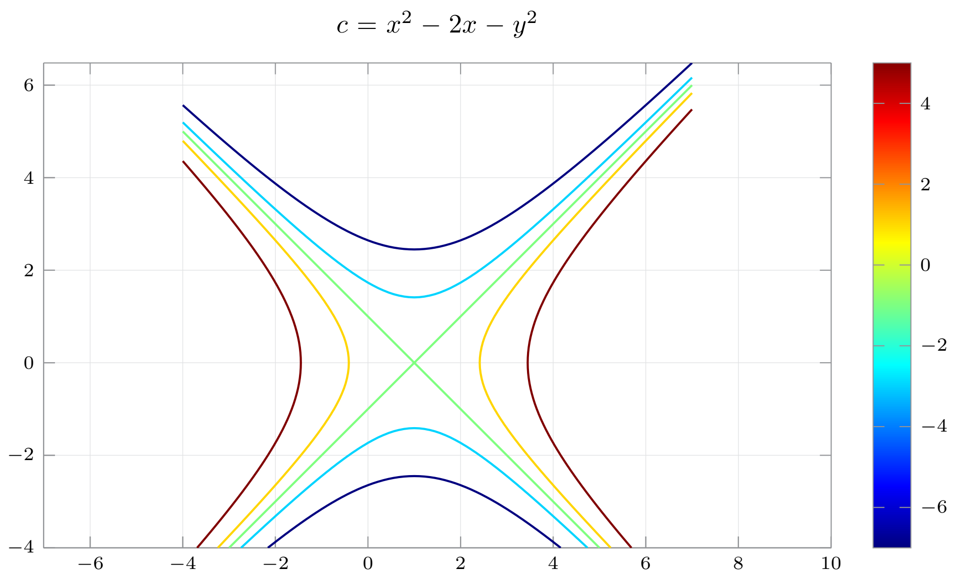
(f) The isotherms are periodic in y.



(g) The isotherms are hyperbola analogs with six principal axes.



(h) The isotherms are hyperbola analogs with six principal axes.



9. The level surfaces are a family of parallel planes with the common normal vector $\mathbf{n} = (4, -3, 2)$

10. For constant z, the cross sections in the xy plane are circles

$$x^2 + y^2 = \frac{c - z_0^2}{9} \tag{9.4.1}$$

For constant x, the cross sections in the yz plane are ellipses

$$y^2 + \frac{z^2}{3^2} = c - x_0^2 \tag{9.4.2}$$

Thus the level surfaces are ellipsoids with circular symmetry about the z axis.

11. Level surfaces are cylinders with an elliptical cross section in the xy plane. The cylindrical axis is the z axis (since it does not appear in the function)

12. The equation can be rewritten as

$$x^2 + y^2 = (z - c)^2 \tag{9.4.3}$$

This is an infinite cone centered at $z = c$, since at constant z, the cross-sections in the xy plane are

$$x^2 + y^2 = (z_0 - c)^2 \geq 0 \tag{9.4.4}$$

13. For constant x, the cross sections in the yz plane are

$$y^2 = z - (x_0^2 + c) \tag{9.4.5}$$

This is a parabola facing upward with vertex at $z = x_0^2 + c$. In 3d, the shape is a paraboloid with vertex at $z = c$

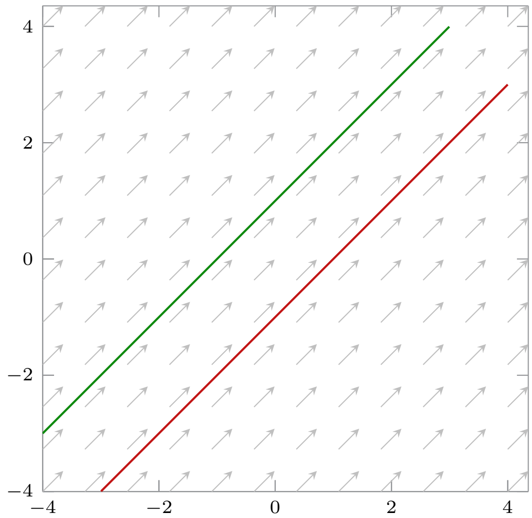
14. At constant z, the cross section in the xy plane are

$$x - y^2 = c \tag{9.4.6}$$

This is a parabola facing the positive x direction with vertex at $x = c$

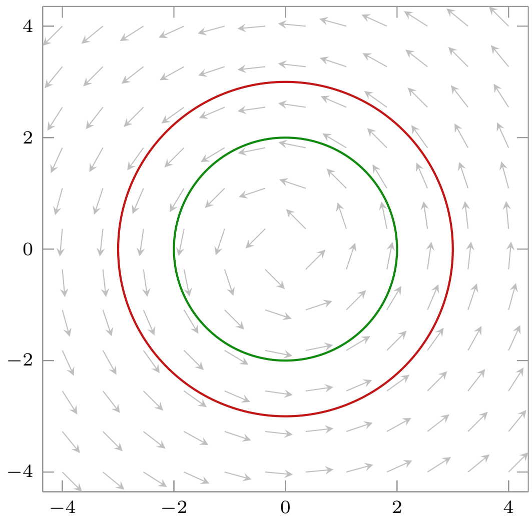
15. Plotting the `quiver` plot, to visualize the $2d$ vector field.

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

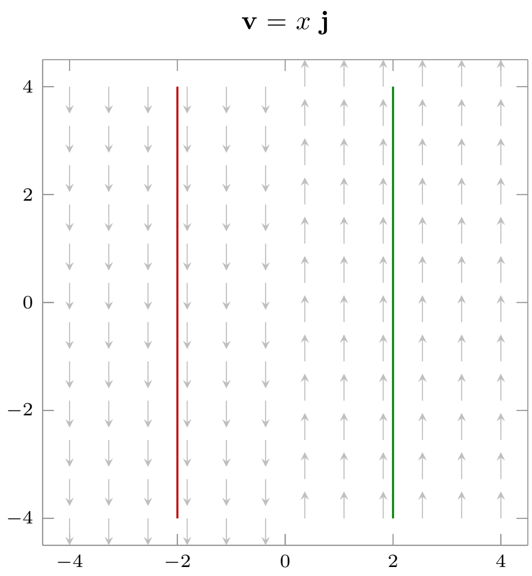


16. Plotting the `quiver` plot, to visualize the $2d$ vector field.

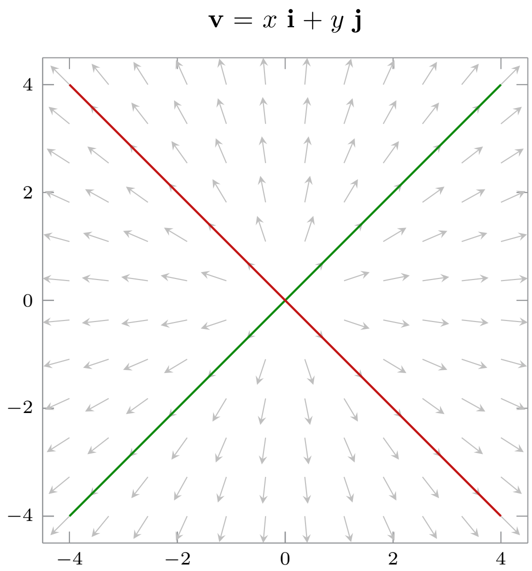
$$\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j}$$



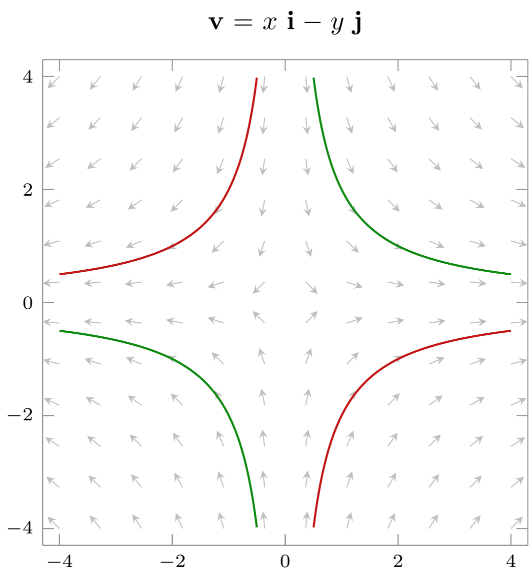
17. Plotting the `quiver` plot, to visualize the $2d$ vector field.



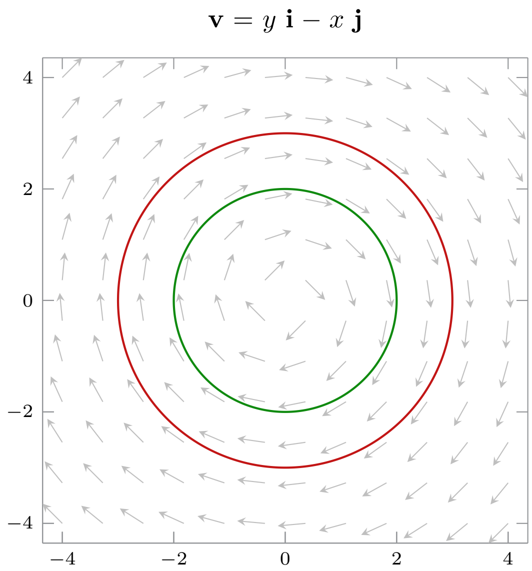
18. Plotting the `quiver` plot, to visualize the $2d$ vector field.



19. Plotting the `quiver` plot, to visualize the $2d$ vector field.

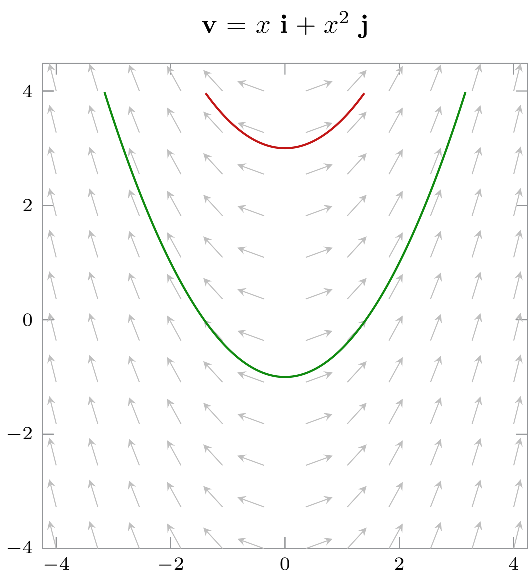


20. Plotting the `quiver` plot, to visualize the $2d$ vector field.

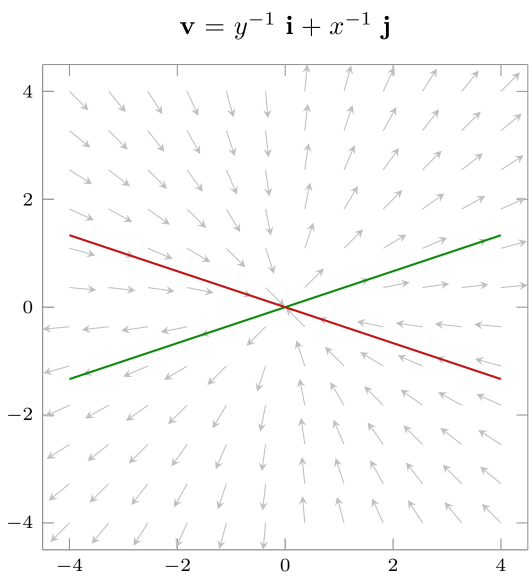


21. Using `quiver` to plot the vector field in $2d$

(a) The level curves are parabolas facing upwards with vertices on the y axis.

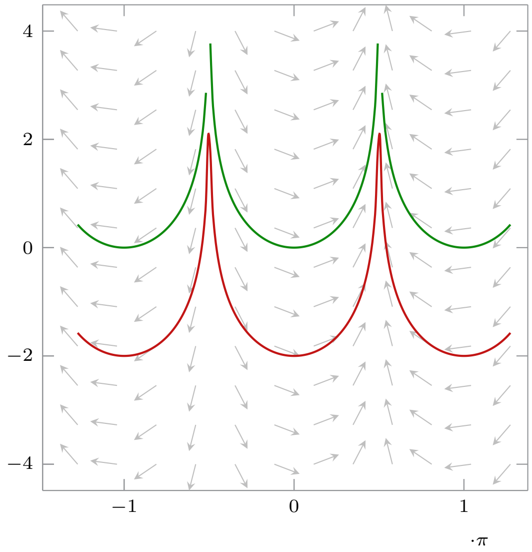


(b) The level curves are straight lines through the origin.



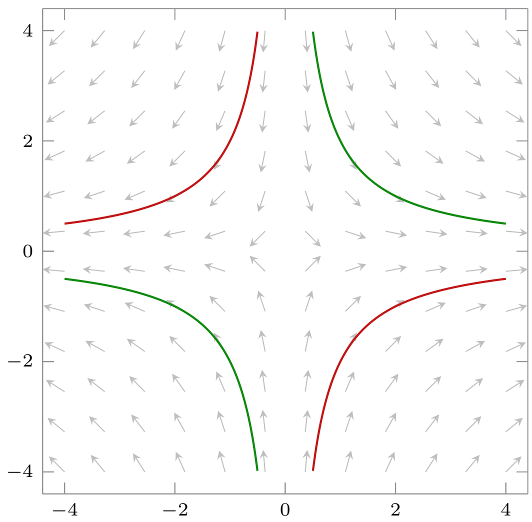
(c) The level curves are of the form $y = \ln|\sec(x)|$.

$\mathbf{v} = \cos(x)\ \mathbf{i} + \sin(x)\ \mathbf{j}$



(d) The level curves are hyperbolas.

$\mathbf{v} = xe^{-(x^2+y^2)}\ \mathbf{i} - ye^{-(x^2+y^2)}\ \mathbf{j}$



22. Differentiating each component,

$$\mathbf{v} = \begin{bmatrix} 3\cos(2t) \\ 3\sin(2t) \\ 4t \end{bmatrix} \qquad \mathbf{v}' = \begin{bmatrix} -6\sin(2t) \\ 6\cos(2t) \\ 4 \end{bmatrix} \tag{9.4.7}$$

$$\mathbf{v}'' = \begin{bmatrix} -12\cos(2t) \\ -12\sin(2t) \\ 0 \end{bmatrix} \tag{9.4.8}$$

23. Differentiating each component,

$$\mathbf{v} = \begin{bmatrix} 3\cos(2t) \\ 3\sin(2t) \\ 4t \end{bmatrix} \qquad \mathbf{v}' = \begin{bmatrix} -6\sin(2t) \\ 6\cos(2t) \\ 4 \end{bmatrix} \tag{9.4.9}$$

$$\mathbf{v}'' = \begin{bmatrix} -12\cos(2t) \\ -12\sin(2t) \\ 0 \end{bmatrix} \tag{9.4.10}$$

24. Using the definition of the derivative as a limit,

$$(\mathbf{v}\cdot\mathbf{u})' = \lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t)\cdot\mathbf{u}(t+\Delta t) - \mathbf{v}(t)\cdot\mathbf{u}(t)}{\Delta t} \tag{9.4.11}$$

$$= \lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t)\cdot\mathbf{u}(t+\Delta t) - \mathbf{v}(t+\Delta t)\cdot\mathbf{u}(t)}{\Delta t} \tag{9.4.12}$$

$$+ \lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t)\cdot\mathbf{u}(t) - \mathbf{v}(t)\cdot\mathbf{u}(t)}{\Delta t} \tag{9.4.13}$$

$$= \lim_{\Delta t\to 0} \mathbf{v}(t+\Delta t)\cdot\mathbf{u}'(t) + \lim_{\Delta t\to 0} \mathbf{v}'(t)\cdot\mathbf{u}(t) \tag{9.4.14}$$

$$= \mathbf{v}(t)\cdot\mathbf{u}'(t) + \mathbf{v}'(t)\cdot\mathbf{u}(t) \tag{9.4.15}$$

Using the definition of the derivative as a limit,

$$(\mathbf{v}\times\mathbf{u})' = \lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t)\times\mathbf{u}(t+\Delta t) - \mathbf{v}(t)\times\mathbf{u}(t)}{\Delta t} \tag{9.4.16}$$

$$= \lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t)\times\mathbf{u}(t+\Delta t) - \mathbf{v}(t+\Delta t)\times\mathbf{u}(t)}{\Delta t} \tag{9.4.17}$$

$$+ \lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t)\times\mathbf{u}(t) - \mathbf{v}(t)\times\mathbf{u}(t)}{\Delta t} \tag{9.4.18}$$

$$= \lim_{\Delta t\to 0} \mathbf{v}(t+\Delta t)\times\mathbf{u}'(t) + \lim_{\Delta t\to 0} \mathbf{v}'(t)\times\mathbf{u}(t) \tag{9.4.19}$$

$$= \mathbf{v}(t)\times\mathbf{u}'(t) + \mathbf{v}'(t)\times\mathbf{u}(t) \tag{9.4.20}$$

Using the above two results,

$$[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]' = \mathbf{a}'\cdot[\mathbf{b}\times\mathbf{c}] + \mathbf{a}\cdot[\mathbf{b}\times\mathbf{c}]' \tag{9.4.21}$$

$$= \mathbf{a}'\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}'\times\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}') \tag{9.4.22}$$

Examples TBC.

25. Finding the component-wise partial derivatives,

$$\mathbf{v} = \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix} \qquad \frac{\partial \mathbf{v}}{\partial x} = \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix} \tag{9.4.23}$$

$$\frac{\partial \mathbf{v}}{\partial y} = \begin{bmatrix} -e^x \sin y \\ e^x \cos y \end{bmatrix} \tag{9.4.24}$$

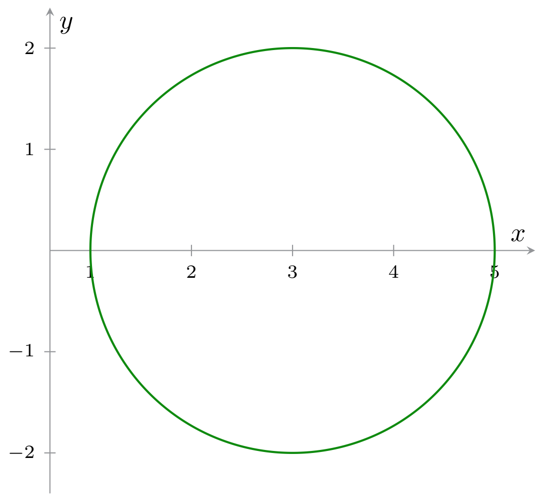
$$\mathbf{u} = \begin{bmatrix} \cos x \cosh y \\ -\sin x \sinh y \end{bmatrix} \qquad \frac{\partial \mathbf{u}}{\partial x} = \begin{bmatrix} -\sin x \cosh y \\ -\cos x \sinh y \end{bmatrix} \tag{9.4.25}$$

$$\frac{\partial \mathbf{u}}{\partial y} = \begin{bmatrix} \cos x \sinh y \\ -\sin x \cosh y \end{bmatrix} \tag{9.4.26}$$

26. TBC. Refer notes.

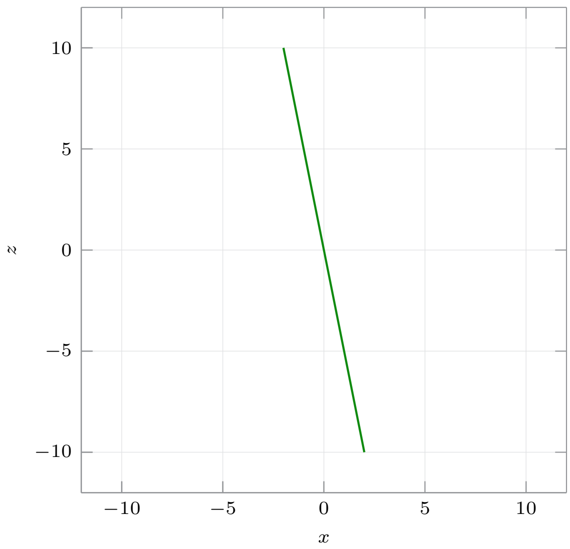
9.5 Curves, Arc Length, Curvature, Torsion

1. The parametric curve is a circle in the xy plane at $z = 0$

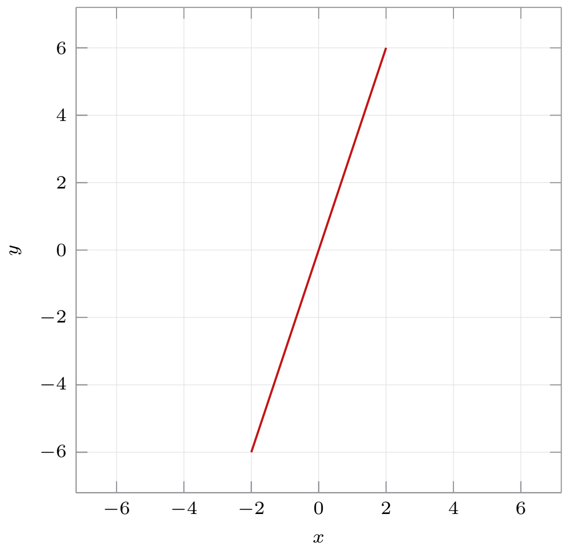


2. The parametric curve is a straight line in $3d$ space.

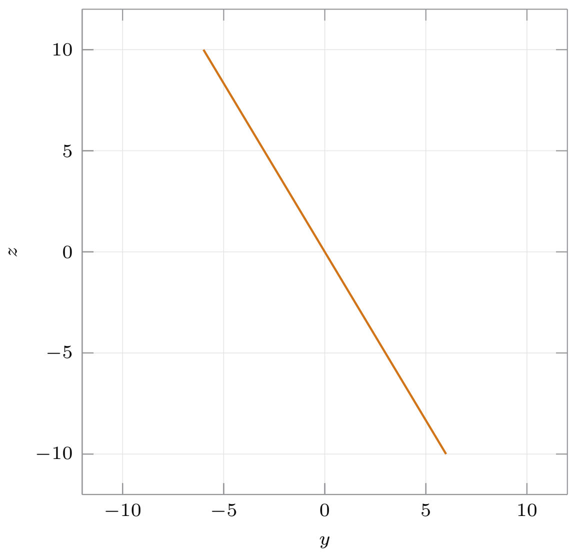
$z = -5x$



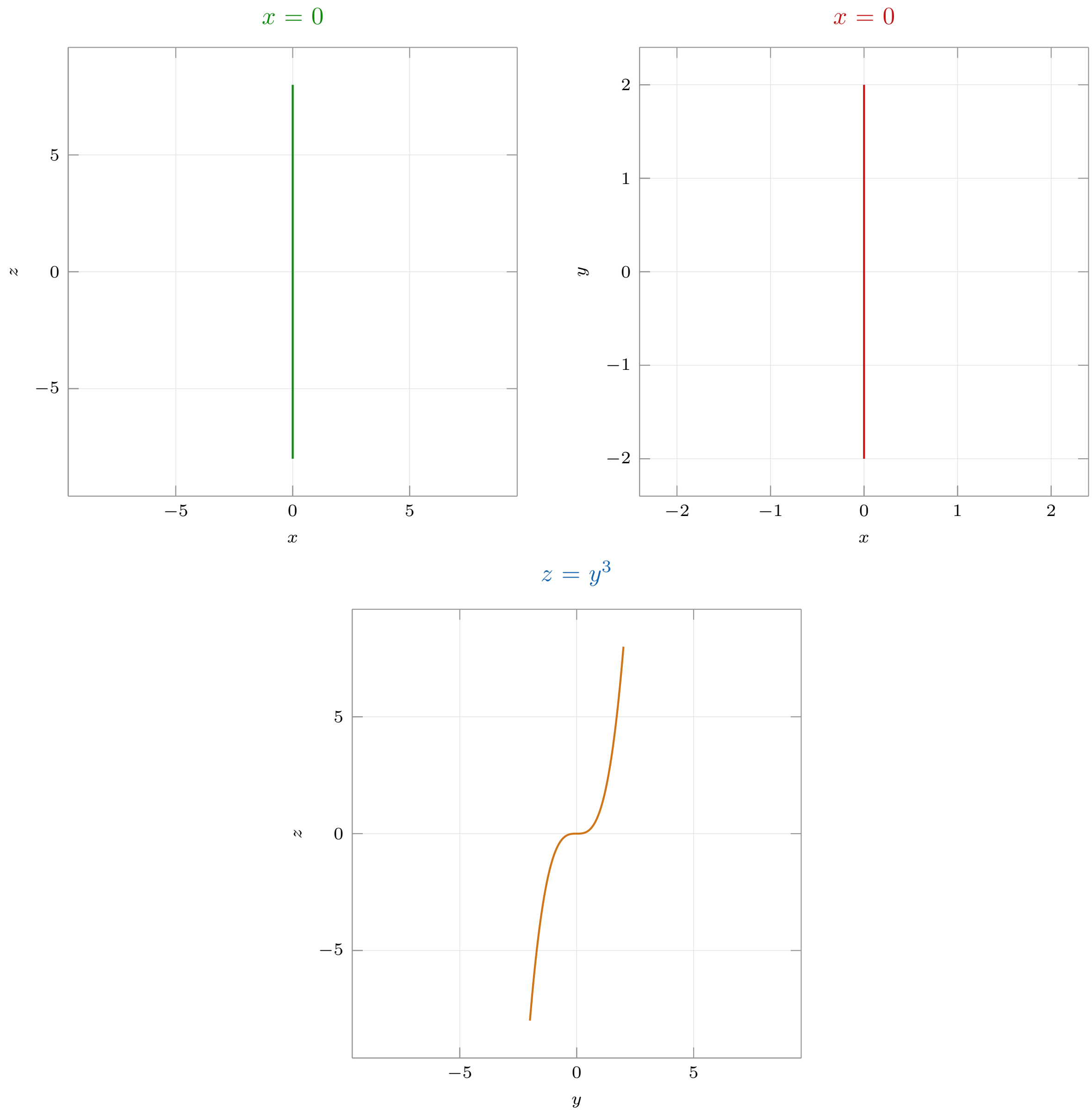
$y = 3x$



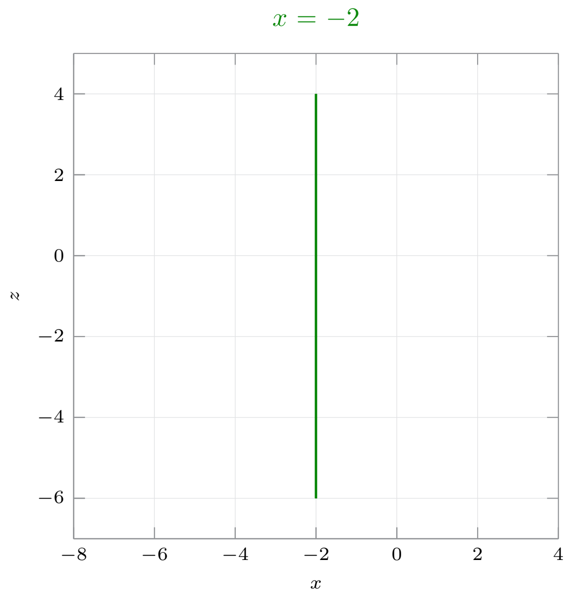
$z = \frac{-5y}{3}$

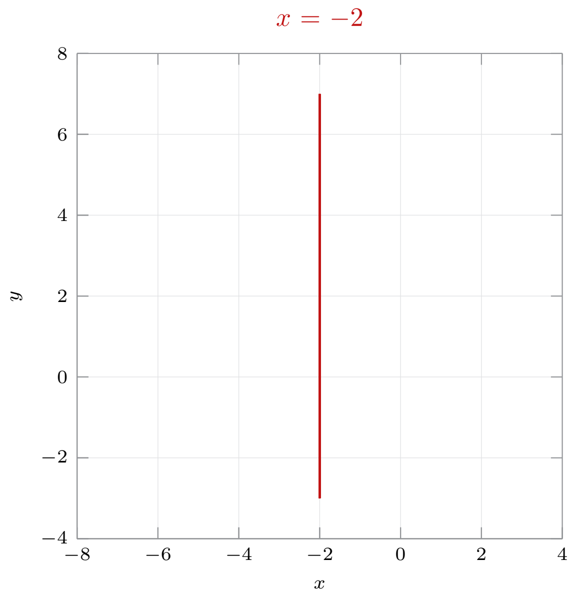


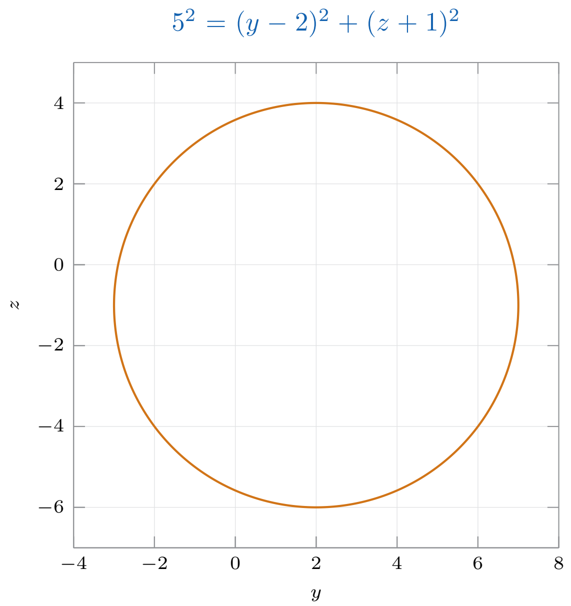
3. The parametric curve is a polynomial in the yz space.



4. The parametric curve is a circle in the yz space.

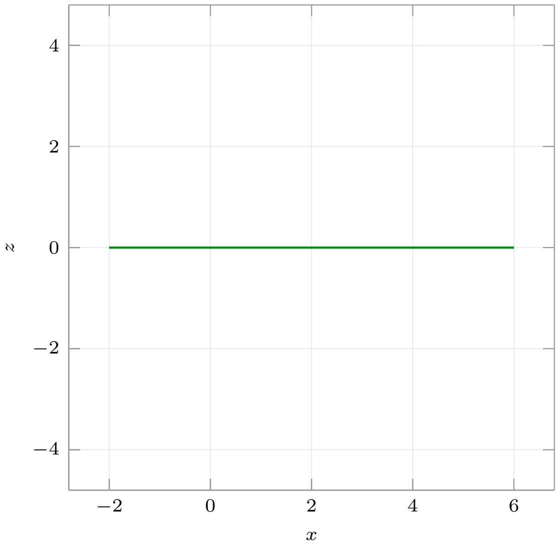




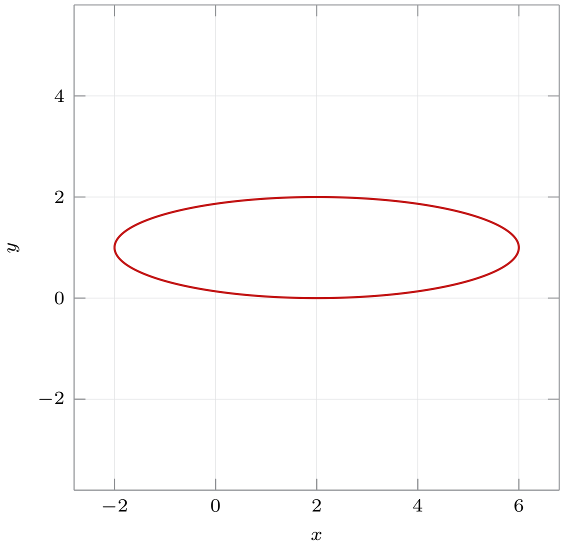


5. The parametric curve is an ellipse in the xy space.

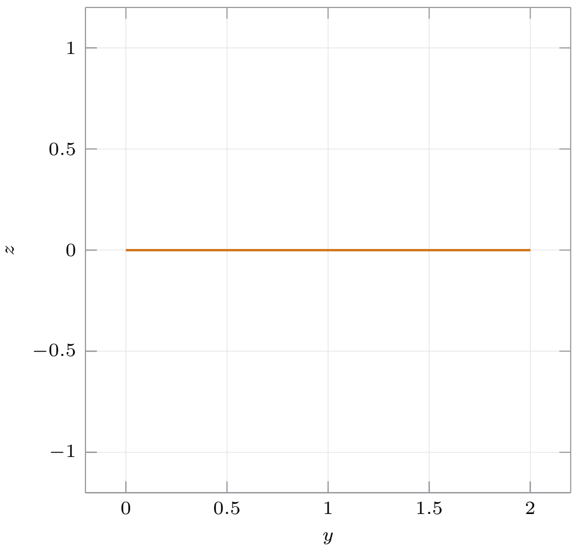
$z = 0$



$\frac{(x-2)^2}{16} + \frac{(y-1)^2}{1} = 1$

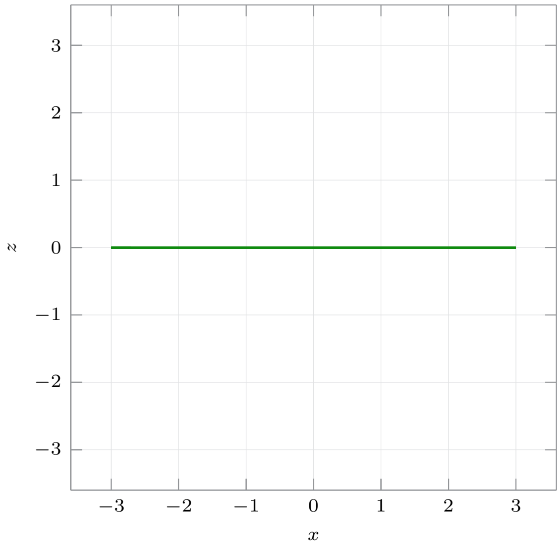


$z = 0$

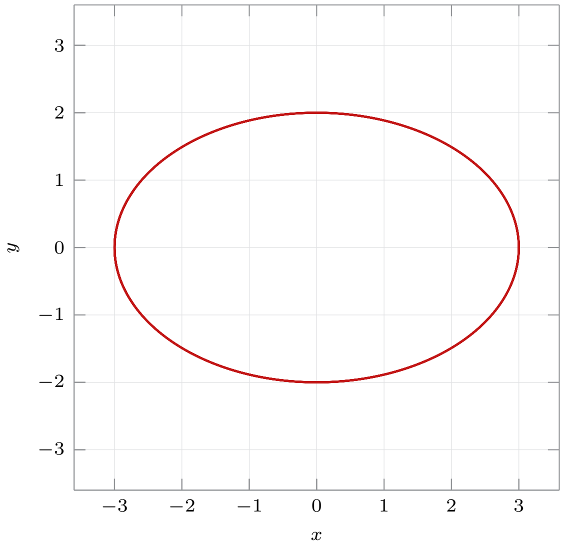


6. The parametric curve is an ellipse in the xy space.

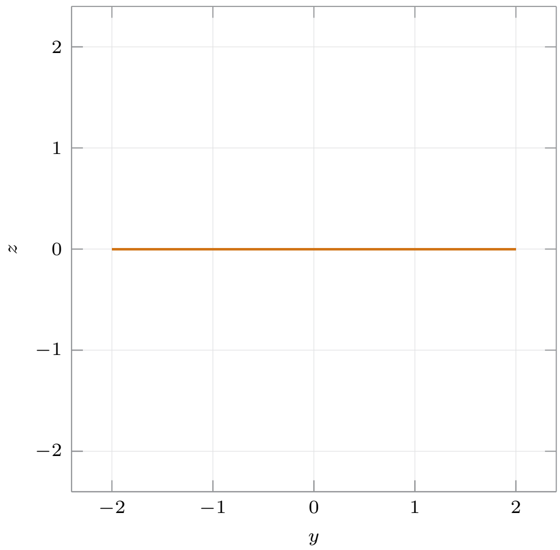
$z = 0$



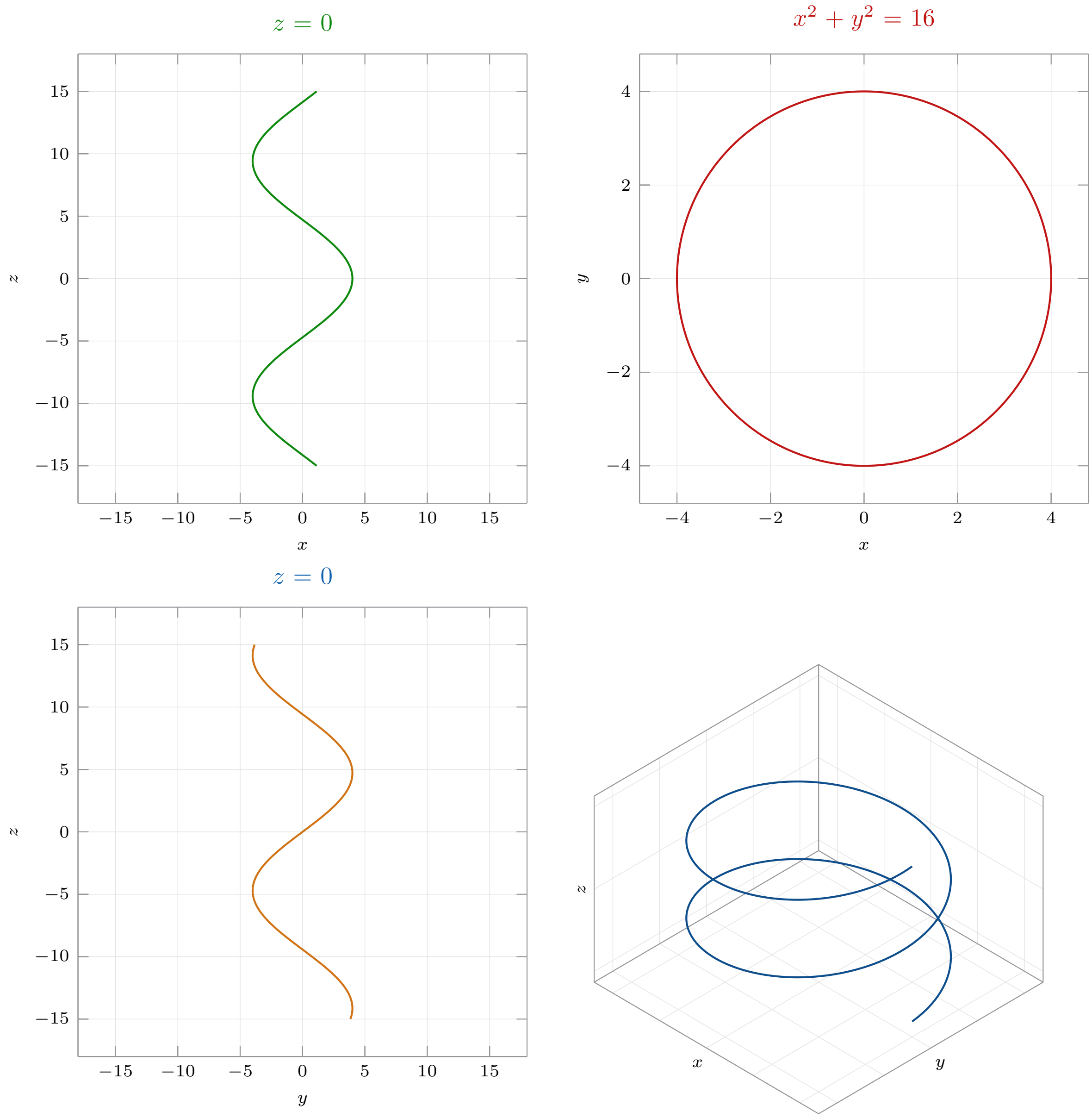
$\frac{x^2}{9} + \frac{y^2}{4} = 1$



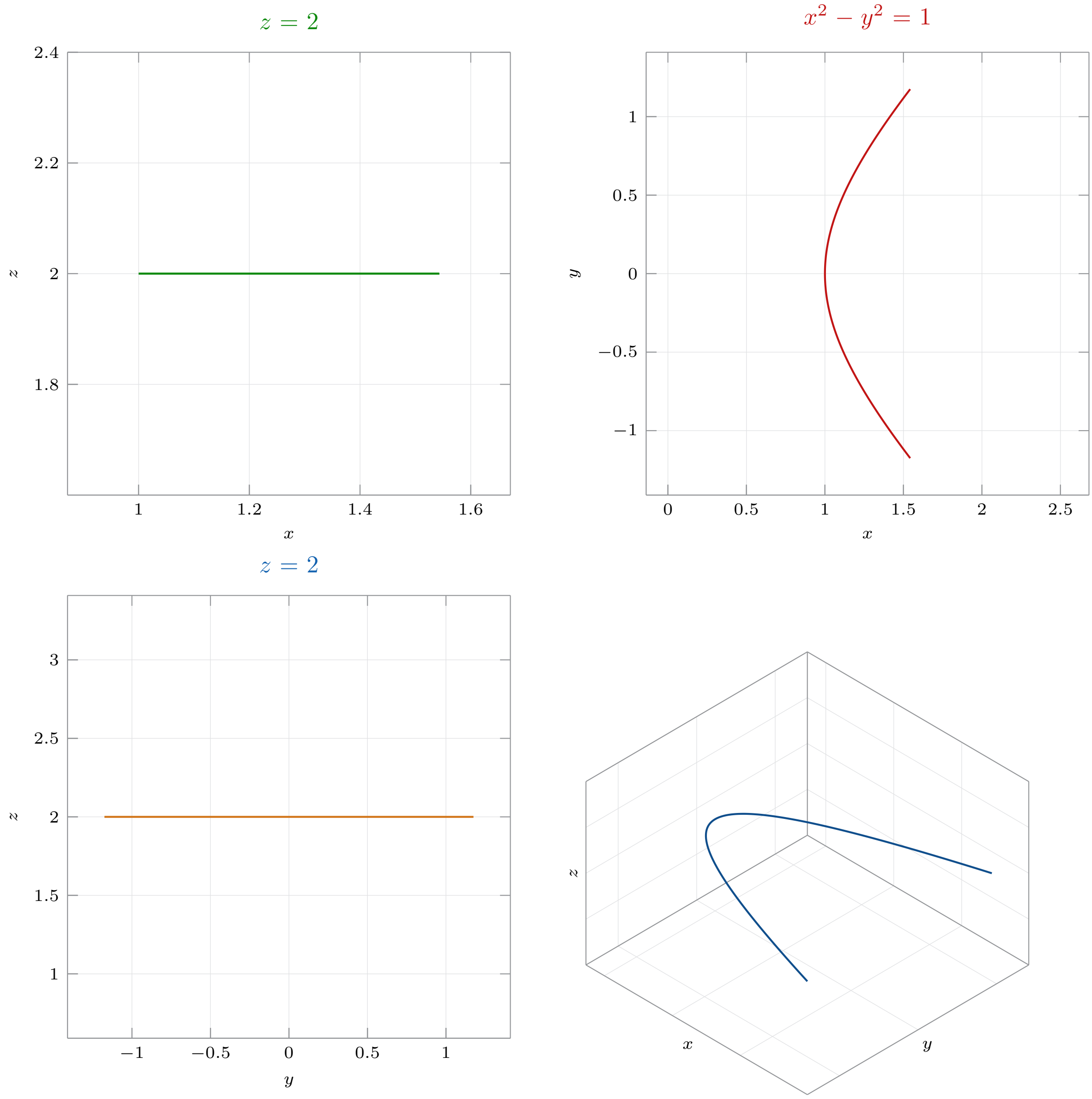
$z = 0$



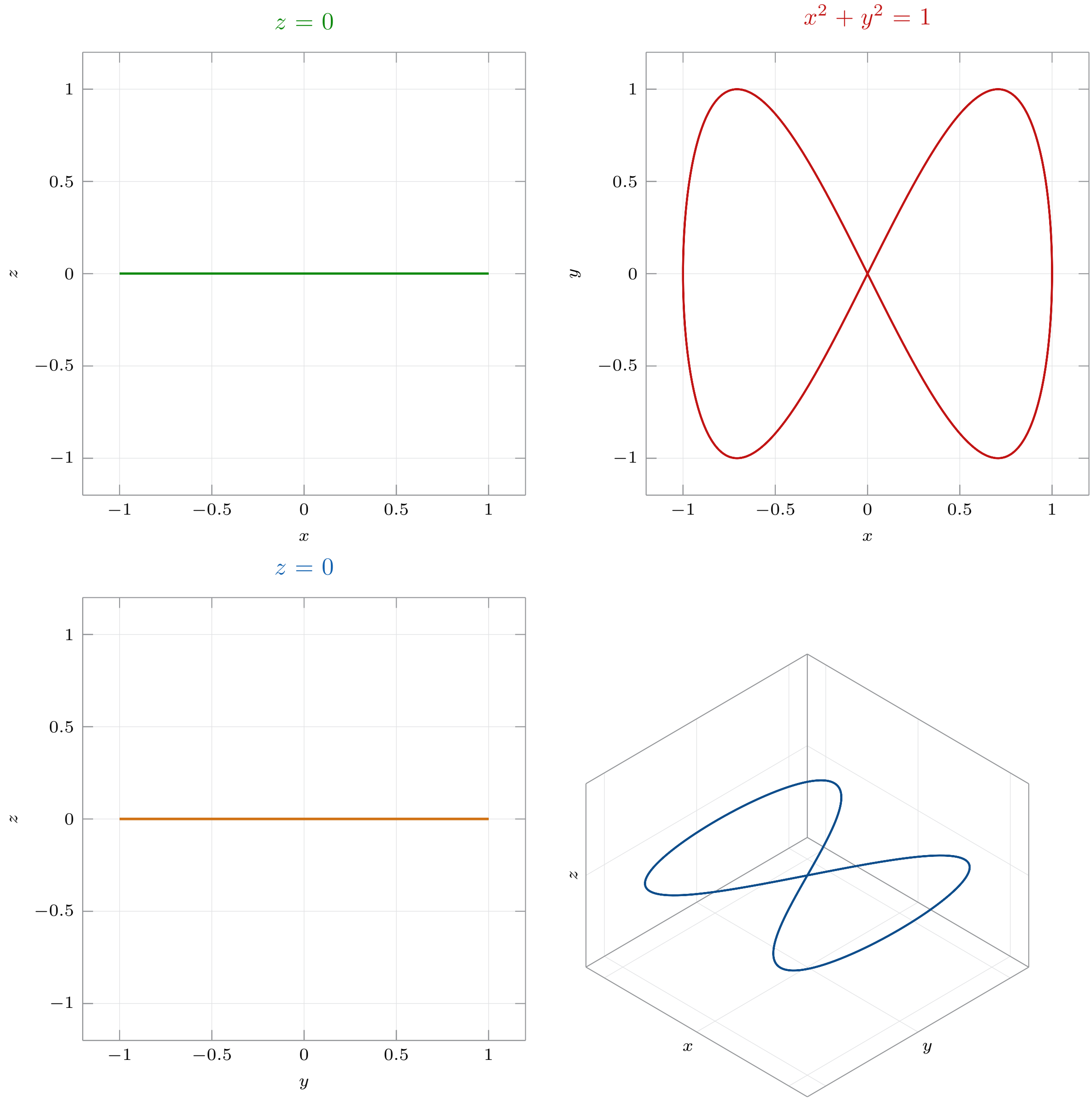
7. The parametric curve is a circular spiral upwards in the z direction.



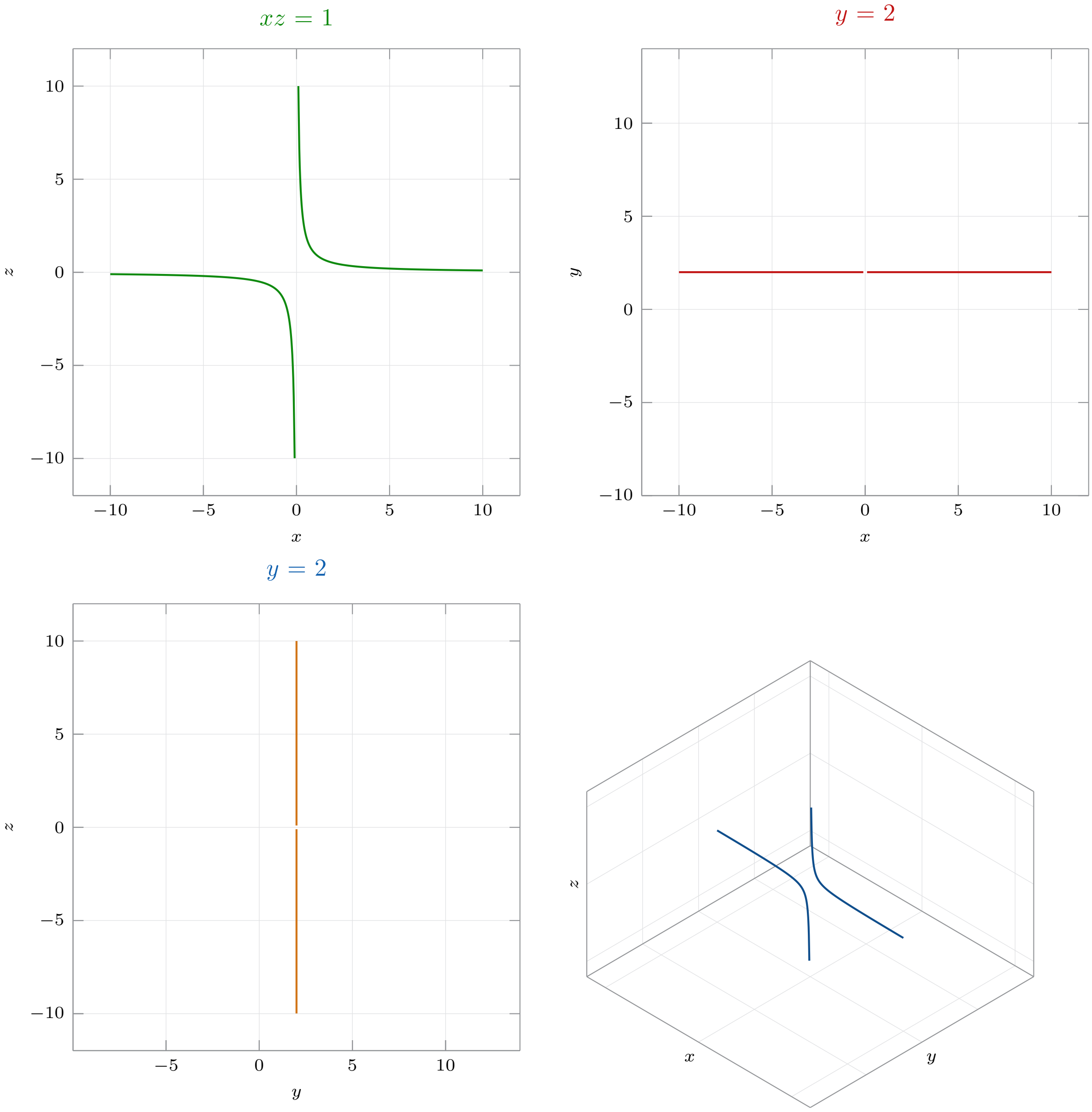
8. The parametric curve is the positive part of a hyperbola in the xy plane.



9. The parametric curve is a closed periodic curve in the xy plane.



10. The parametric curve is a hyperbola in the xz space.



11. Finding the parametric representation,

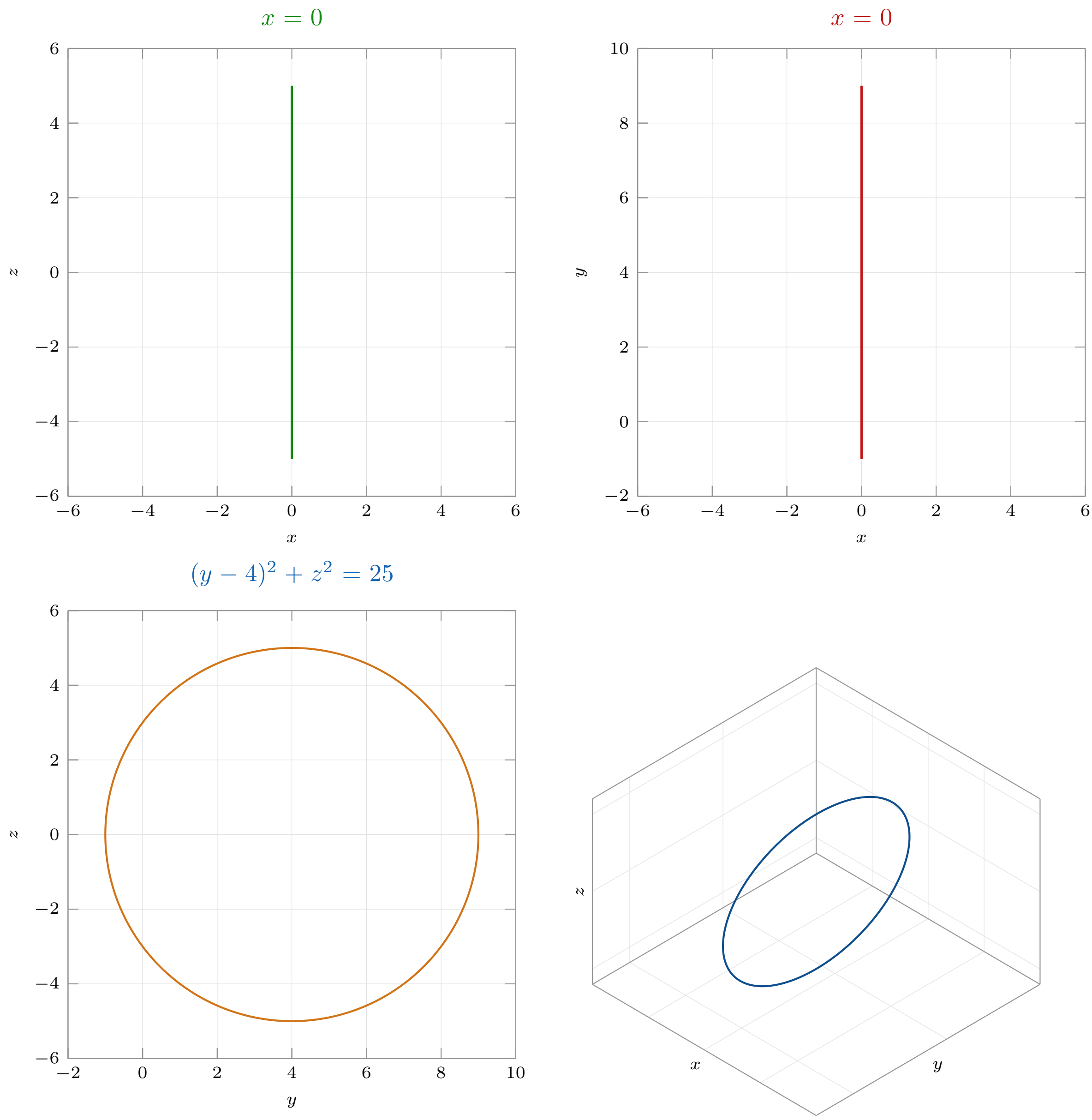
$$z = 1 \qquad (x-3)^2 + (y-2)^2 = 13 \tag{9.5.1}$$

$$\mathbf{v}(t) = \begin{bmatrix} 3 + \sqrt{13}\cos(t) \\ 2 + \sqrt{13}\sin(t) \\ 1 \end{bmatrix} \tag{9.5.2}$$

12. Finding the parametric representation,

$$x = 0 \qquad (y-4)^2 + z^2 = 5^2 \tag{9.5.3}$$

$$\mathbf{v}(t) = \begin{bmatrix} 0 \\ 4 + 5\cos(t) \\ 5\sin(t) \end{bmatrix} \tag{9.5.4}$$



13. Line through a point in a particular direction

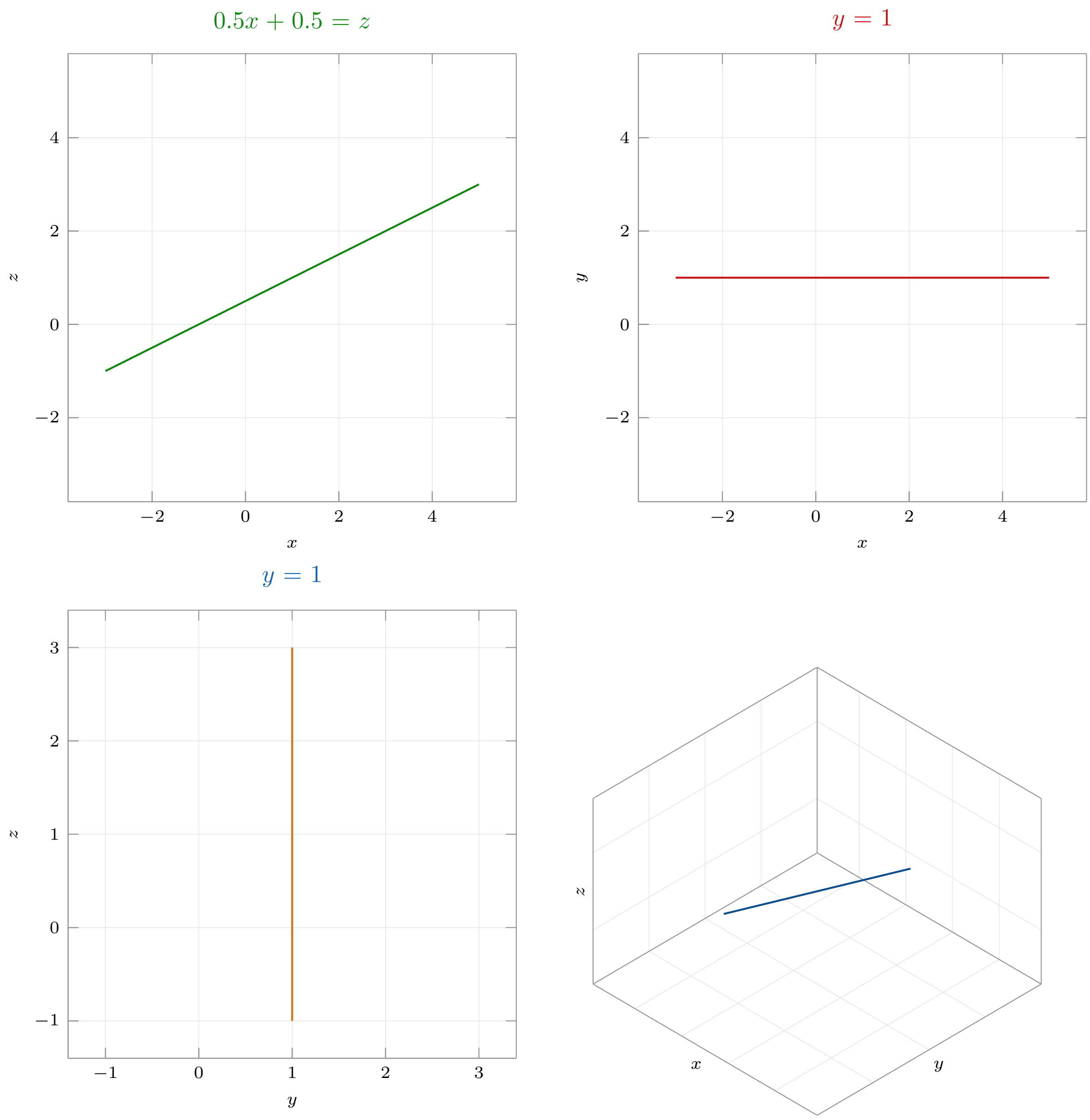
$$A : (2, 1, 3) \qquad \hat{\mathbf{n}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \tag{9.5.5}$$

$$\mathbf{v}(t) = \begin{bmatrix} 2 + \frac{1}{\sqrt{5}}\ t \\ 1 + \frac{2}{\sqrt{5}}\ t \\ 3 \end{bmatrix} \tag{9.5.6}$$

14. Line through a point in a particular direction

$$A : (1, 1, 1) \qquad \hat{\mathbf{n}} = \frac{1}{\sqrt{20}} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \tag{9.5.7}$$

$$\mathbf{v}(t) = \begin{bmatrix} 1 + \frac{4}{\sqrt{20}}\ t \\ 1 \\ 1 + \frac{2}{\sqrt{20}}\ t \end{bmatrix} \tag{9.5.8}$$



15. Eliminating x in favor of a unified parameter t,

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 4t - 1 \\ 5t \end{bmatrix} \tag{9.5.9}$$

16. Points lie on the intersection of the two curves

$$C_1 : x^2 + y^2 = 1 \qquad C_2 : z = y \tag{9.5.10}$$

$$\mathbf{v} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ \sin(t) \end{bmatrix} \tag{9.5.11}$$

17. Points lie on the intersection of the two curves

$$C_1 : x^2 + 2y^2 = 2 \qquad\qquad C_2 : z = y \tag{9.5.12}$$

$$\mathbf{v} = \begin{bmatrix} \sqrt{2}\cos(t) \\ \sin(t) \\ \sin(t) \end{bmatrix} \tag{9.5.13}$$

18. Points lie on the intersection of the two curves

$$C_1 : x^2 + 2y^2 = 25 \qquad\qquad C_2 : z = 2\arctan(y/x) \tag{9.5.14}$$

$$\mathbf{v} = \begin{bmatrix} 5\cos(t) \\ 5\sin(t) \\ 2t \end{bmatrix} \tag{9.5.15}$$

19. Points lie on the intersection of the two curves

$$C_1 : 4x^2 - 3y^2 = 4 \qquad\qquad C_2 : z = 2 \tag{9.5.16}$$

$$\mathbf{v} = \begin{bmatrix} \cosh(t) \\ \frac{2}{\sqrt{3}}\sinh(t) \\ 2 \end{bmatrix} \tag{9.5.17}$$

20. Points lie on the intersection of the two curves

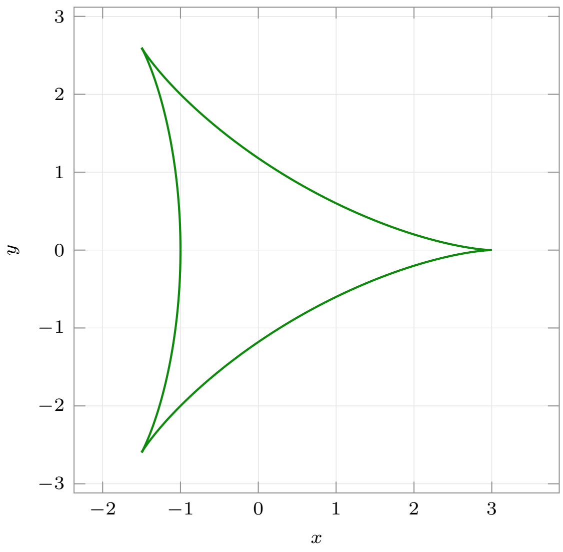
$$C_1 : 2x - y + 3z = 2 \qquad\qquad C_2 : x + 2y - z = 3 \tag{9.5.18}$$

$$\mathbf{v} = \begin{bmatrix} t \\ -t + 2.2 \\ 1.4 - t \end{bmatrix} \tag{9.5.19}$$

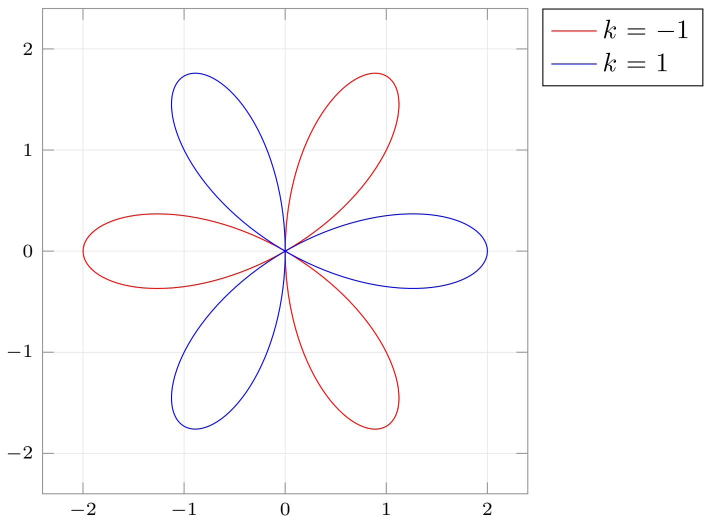
21. Orientation reverses because $t \to (-t)$ makes the change in parameter $\Delta t \to -\Delta t$. This means that traversing over the curve happens in the opposite sense.

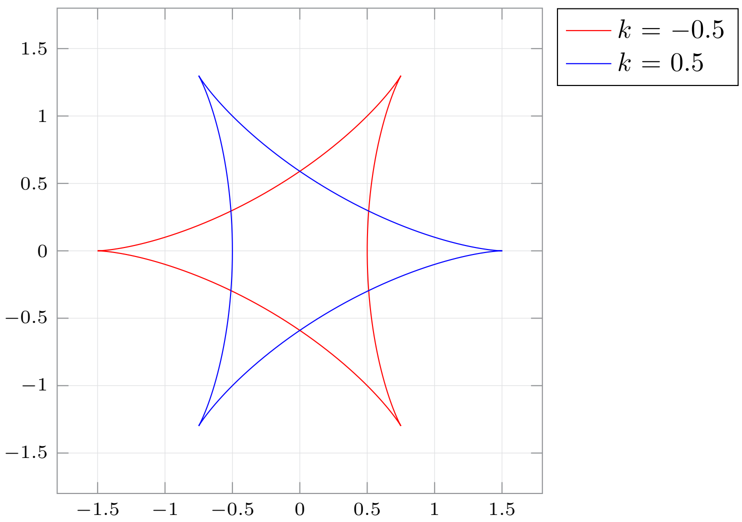
22. Graphing the curves using `pgfplots`

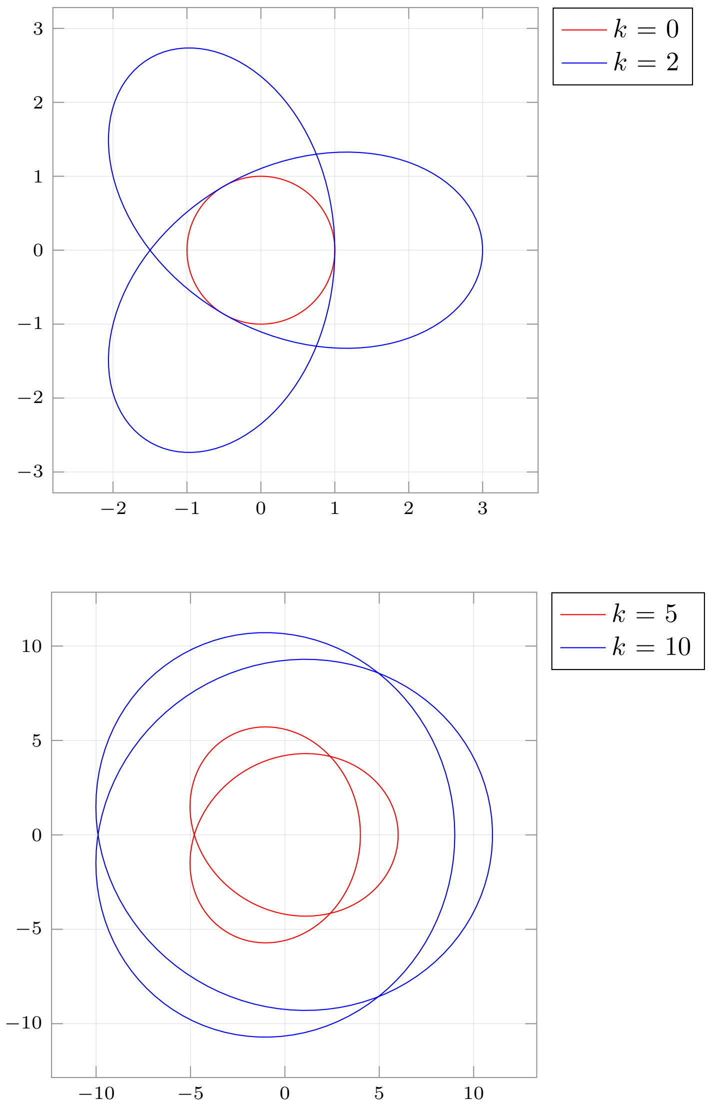
(a) Steiner's hypocycloid



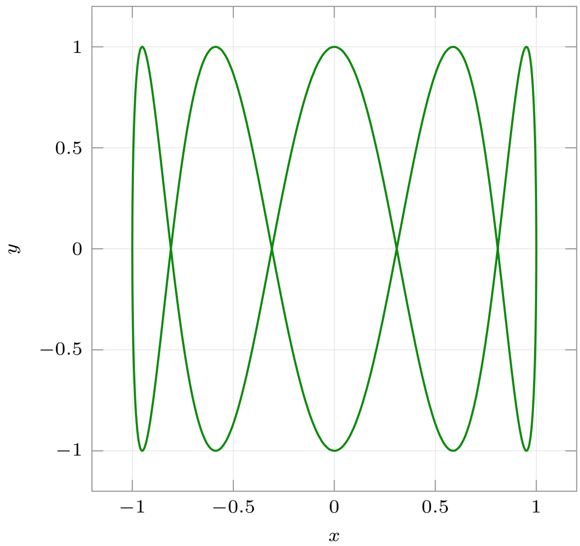
(b) Plotting the family of curves



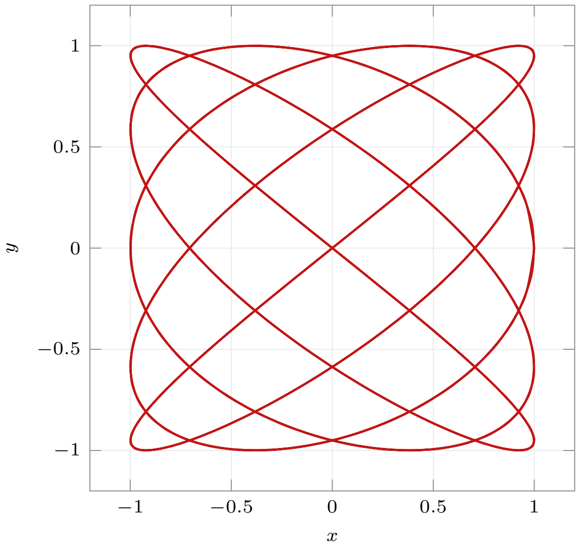




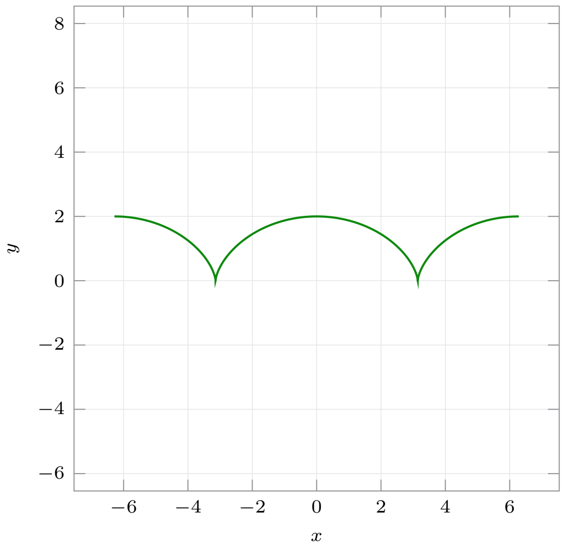
(c) Lissajous' curve, closed for rational k



(d) Lissajous' curve, closed for rational k

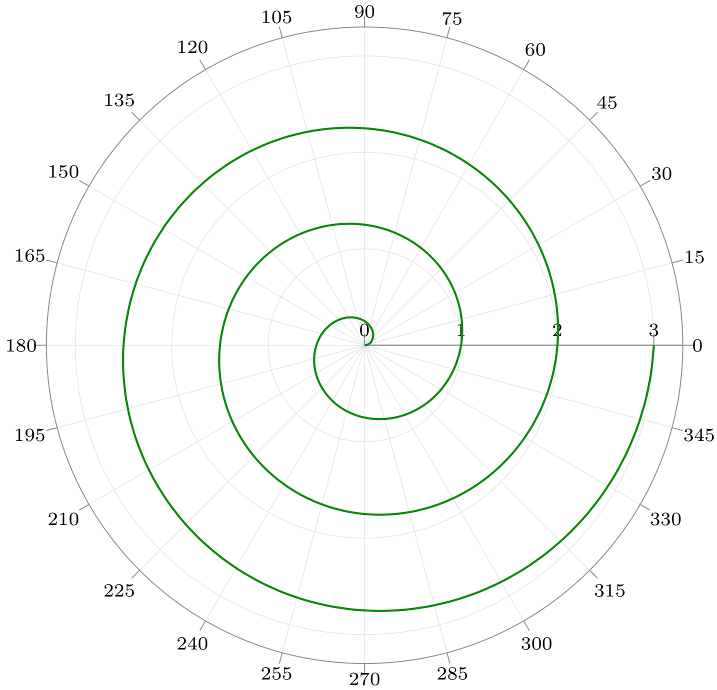


(e) Cycloid using $R = \omega = 1$ for simplicity,

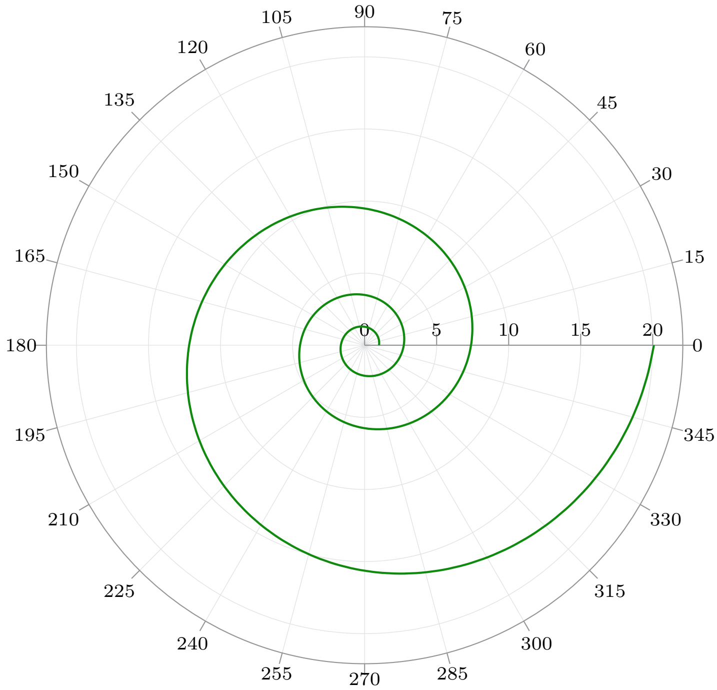


23. Graphing the polar plots using `pgfplots`

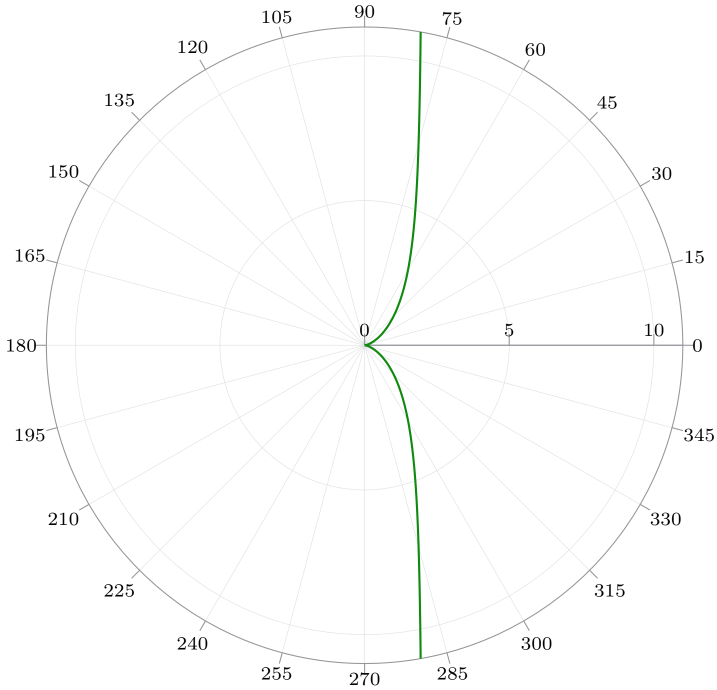
(a) Spiral of Archimedes



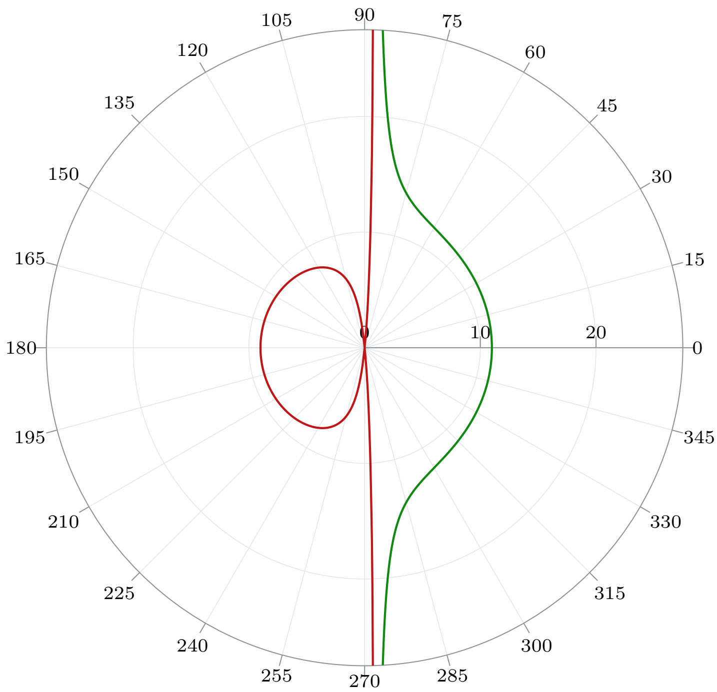
(b) Logarithmic spiral



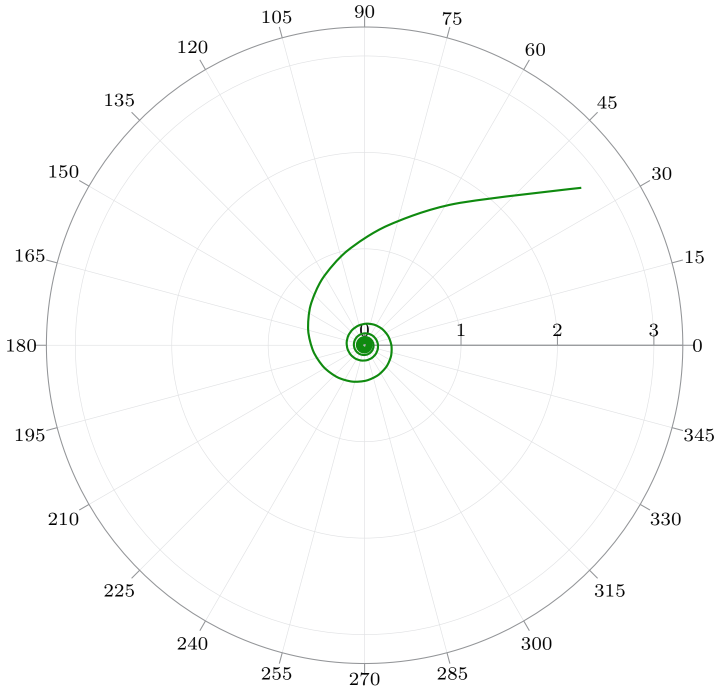
(c) Cissoid of Diocles



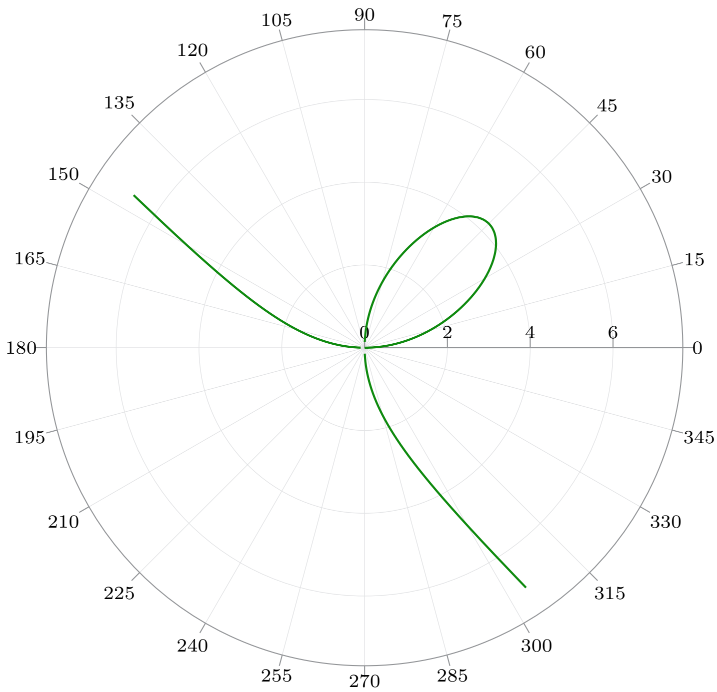
(d) Conchoid of Nichomedes



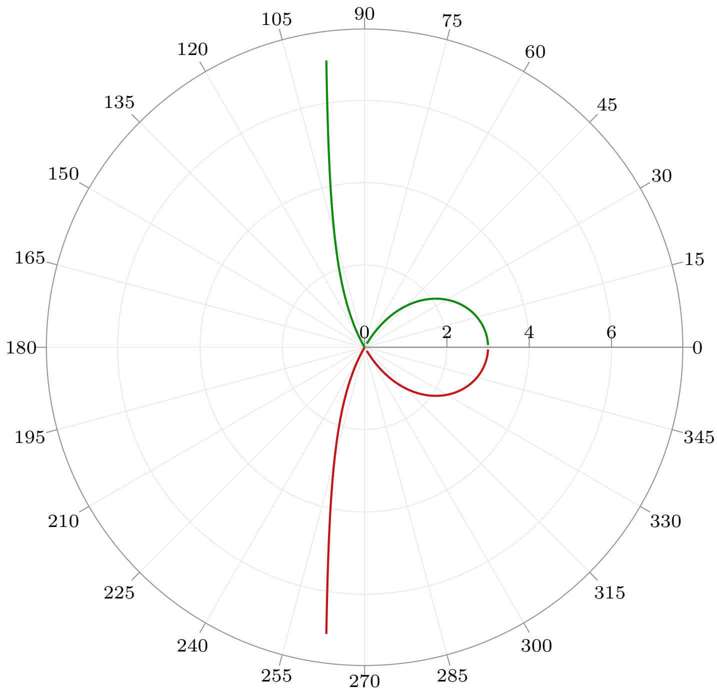
(e) Hyperbolic spiral



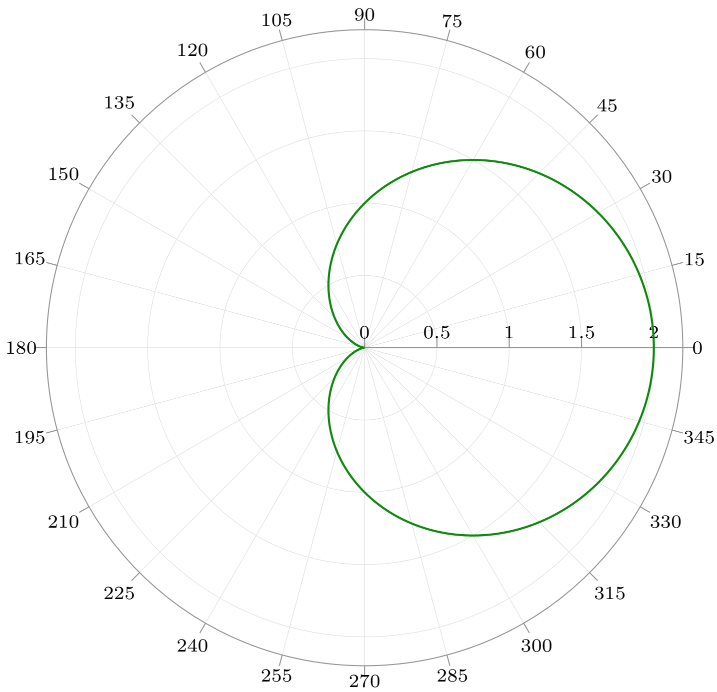
(f) Folium of Descartes



(g) Maclaurin's trisectrix



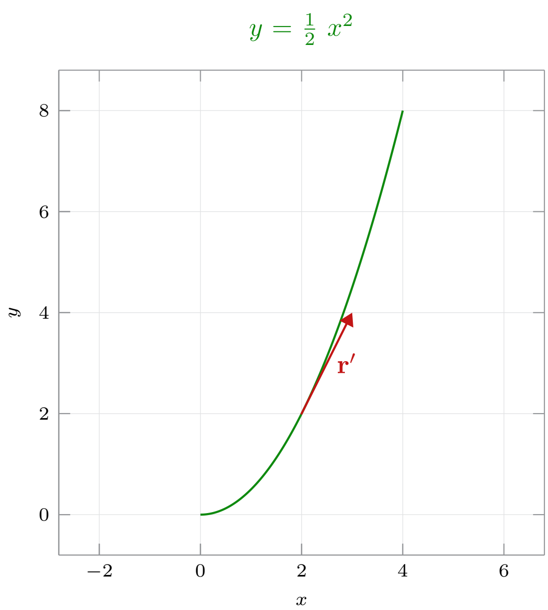
(h) Pascal's snail



24. Finding the tangent vector,

$$\mathbf{r}(t) = \begin{bmatrix} t \\ 0.5t^2 \\ 1 \end{bmatrix} \qquad P = (2, 2, 1) \tag{9.5.20}$$

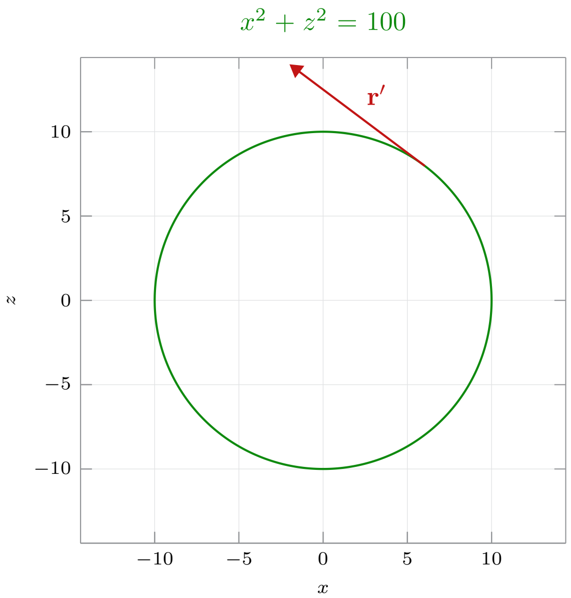
$$\mathbf{r}'(t) = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix} \qquad \mathbf{u}' = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \tag{9.5.21}$$



25. Finding the tangent vector,

$$\mathbf{r}(t) = \begin{bmatrix} 10\,\cos(t) \\ 1 \\ 10\,\sin(t) \end{bmatrix} \qquad P = (6,\,1,\,8) \tag{9.5.22}$$

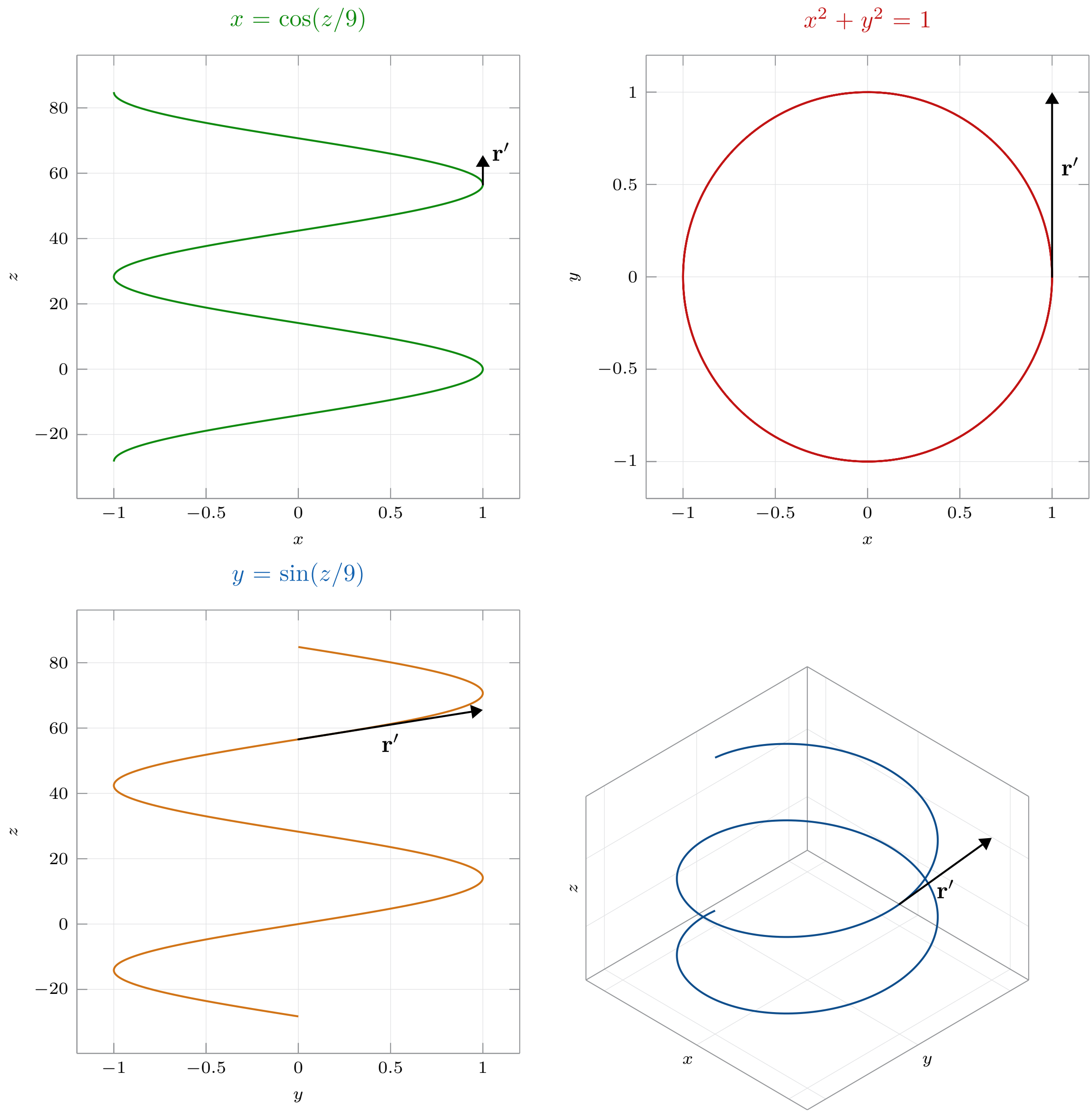
$$\mathbf{r}'(t) = \begin{bmatrix} -10\,\sin(t) \\ 0 \\ 10\,\cos(t) \end{bmatrix} \qquad \mathbf{u}' = \begin{bmatrix} -0.8 \\ 0 \\ 0.6 \end{bmatrix} \tag{9.5.23}$$



26. Finding the tangent vector,

$$\mathbf{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 9t \end{bmatrix} \qquad P = (1, 0, 18\pi) \tag{9.5.24}$$

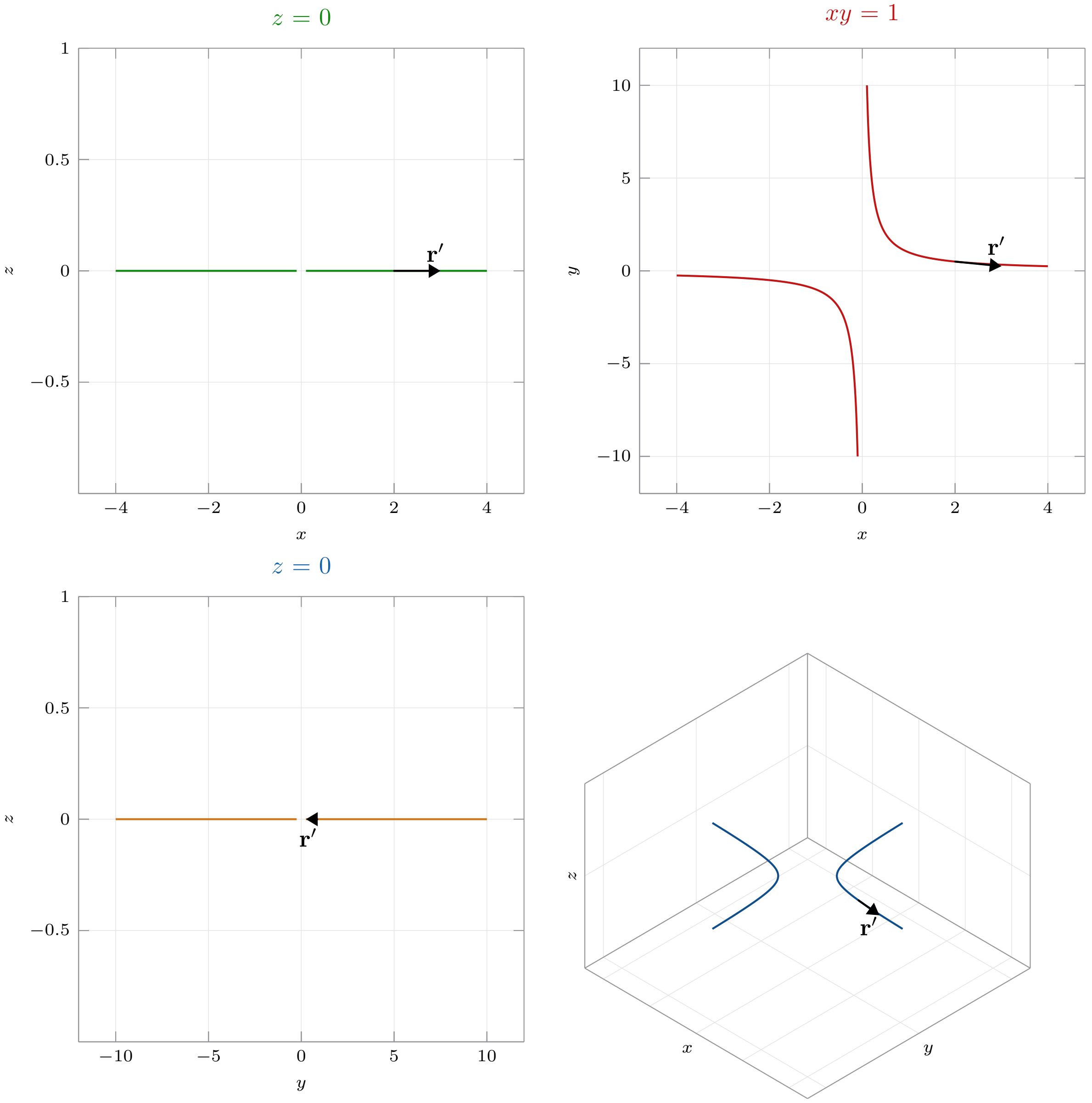
$$\mathbf{r}'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 9 \end{bmatrix} \qquad \mathbf{u}' = \frac{1}{\sqrt{82}} \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix} \tag{9.5.25}$$



27. Finding the tangent vector,

$$\mathbf{r}(t) = \begin{bmatrix} t \\ 1/t \\ 0 \end{bmatrix} \qquad P = (2,\, 1/2,\, 0) \tag{9.5.26}$$

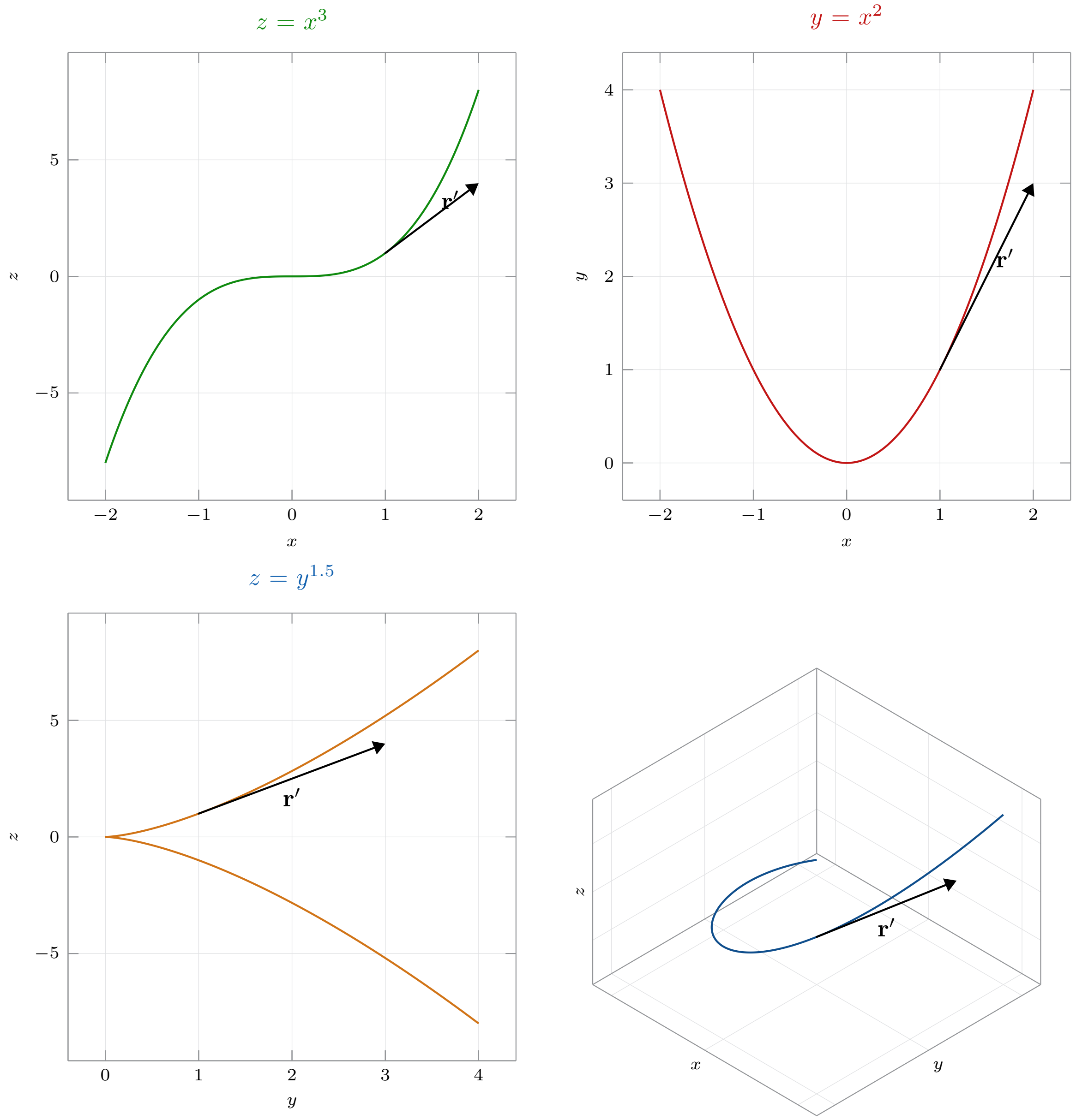
$$\mathbf{r}'(t) = \begin{bmatrix} 1 \\ -(t)^{-2} \\ 0 \end{bmatrix} \qquad \mathbf{u}' = \frac{2}{\sqrt{5}} \begin{bmatrix} 1 \\ -0.25 \\ 0 \end{bmatrix} \tag{9.5.27}$$



28. Finding the tangent vector,

$$\mathbf{r}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix} \qquad P = (1, 1, 1) \tag{9.5.28}$$

$$\mathbf{r}'(t) = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} \qquad \mathbf{u}' = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{9.5.29}$$



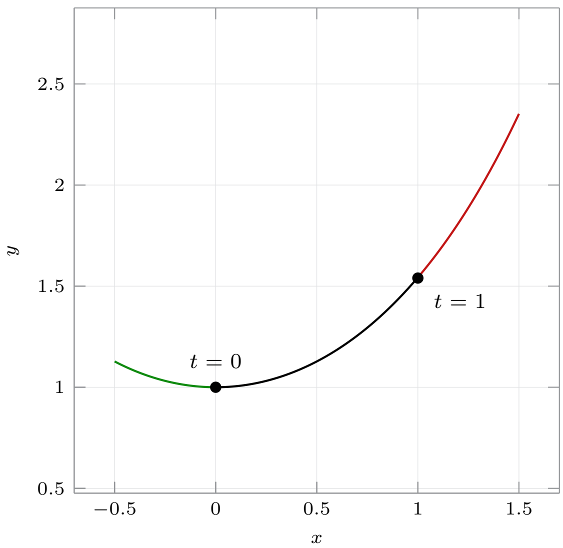
29. Finding the arc length,

$$\mathbf{r}(t) = \begin{bmatrix} t \\ \cosh(t) \end{bmatrix} \qquad t \in [0,\, 1] \tag{9.5.30}$$

$$s = \int_0^1 \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\ \mathrm{d}t \qquad \mathbf{r}' = \begin{bmatrix} 1 \\ \sinh(t) \end{bmatrix} \tag{9.5.31}$$

$$s = \int_0^1 \cosh(t)\ \mathrm{d}t \qquad = \Big[\sinh(t)\Big]_0^1 \tag{9.5.32}$$

$$s = \sinh(1) \tag{9.5.33}$$

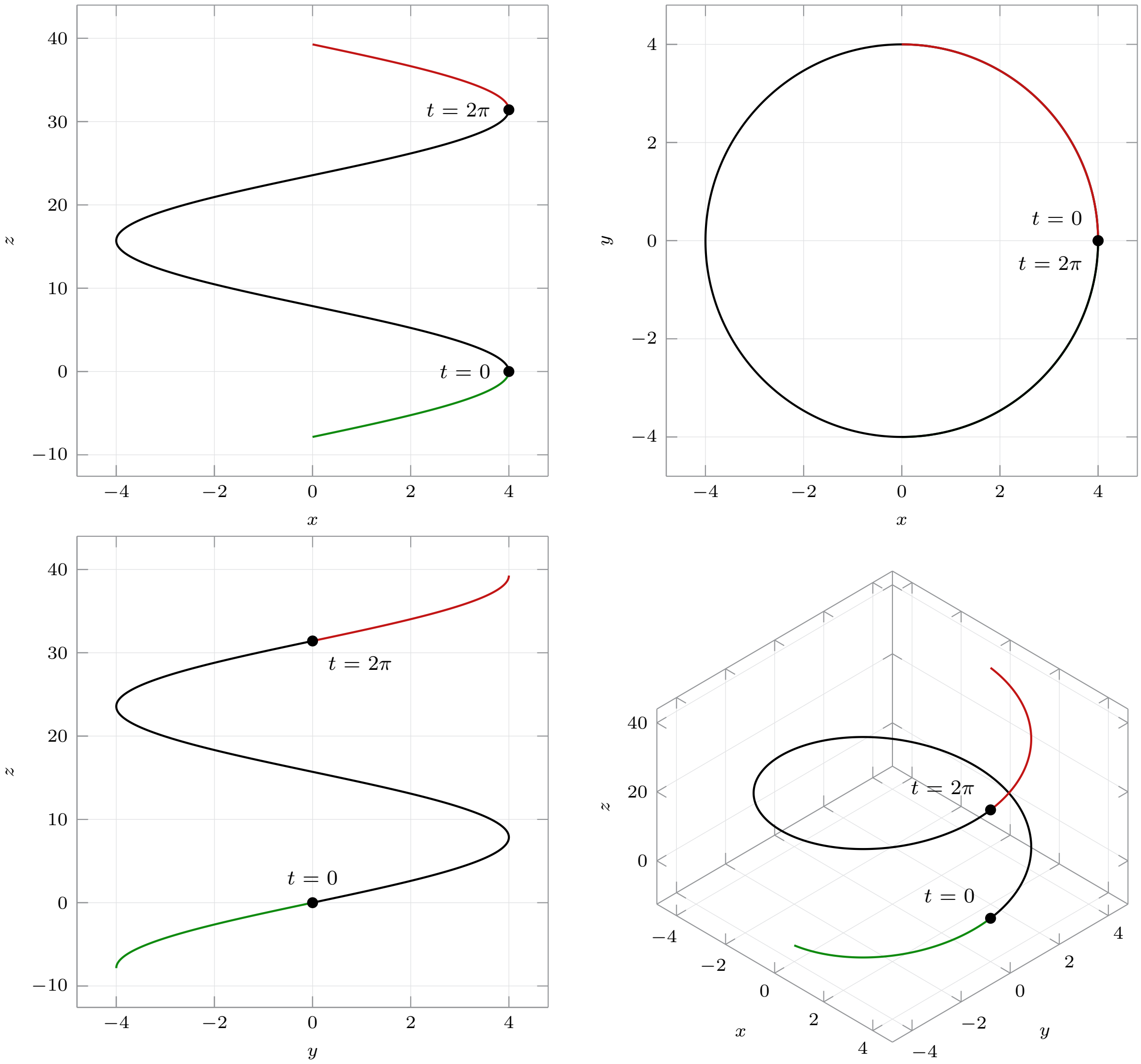


30. Finding the arc length,

$$\mathbf{r}(t) = \begin{bmatrix} 4\cos(t) \\ 4\sin(t) \\ 5t \end{bmatrix} \qquad t \in [0,\, 2\pi] \qquad \mathbf{r}' = \begin{bmatrix} -4\sin(t) \\ 4\cos(t) \\ 5 \end{bmatrix} \tag{9.5.34}$$

$$s = \int_0^{2\pi} \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\; \mathrm{d}t \qquad = \int_0^{2\pi} \sqrt{41}\; \mathrm{d}t \qquad = \left[\sqrt{41}t\right]_0^{2\pi} \tag{9.5.35}$$

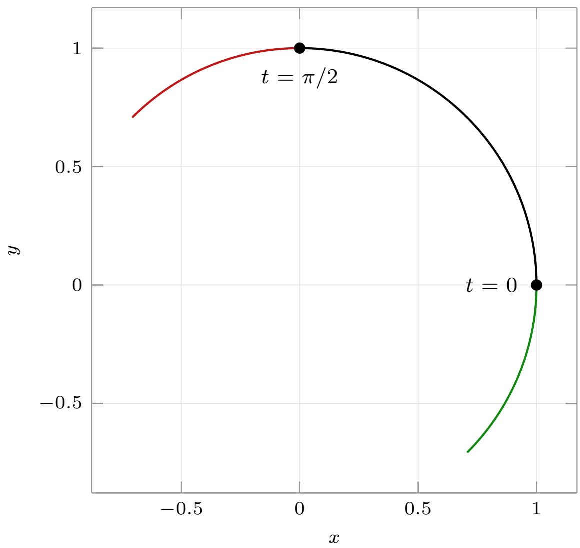
$$= 2\pi\sqrt{41} \tag{9.5.36}$$



31. Finding the arc length,

$$\mathbf{r}(t) = \begin{bmatrix} a\,\cos(t) \\ a\,\sin(t) \end{bmatrix} \qquad t \in [0,\, \pi/2] \qquad \mathbf{r}' = \begin{bmatrix} -a\,\sin(t) \\ a\,\cos(t) \end{bmatrix} \tag{9.5.37}$$

$$s = \int_0^{\pi/2} \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\ \mathrm{d}t \qquad s = \int_0^{\pi/2} a\ \mathrm{d}t \qquad = \Big[at\Big]_0^{\pi/2} = \frac{\pi a}{2} \tag{9.5.38}$$

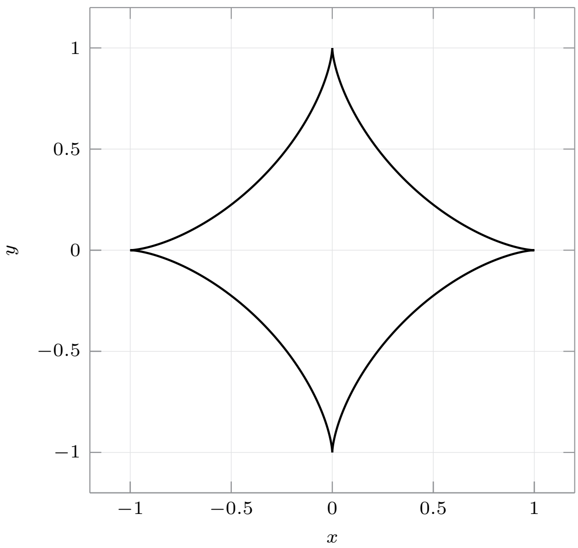


32. Finding the arc length,

$$\mathbf{r}(t) = \begin{bmatrix} a\,\cos^3(t) \\ a\,\sin^3(t) \end{bmatrix} \qquad t \in [0,\, 2\pi] \qquad \mathbf{r}' = \begin{bmatrix} -1.5a\,\cos(t)\,\sin(2t) \\ 1.5a\,\sin(t)\,\sin(2t) \end{bmatrix} \tag{9.5.39}$$

$$s = \int_0^{2\pi} \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\ \mathrm{d}t \qquad = 6\,|a| \int_0^{\pi/2} \sin(2t)\ \mathrm{d}t \qquad = \Big[-3a\,\cos(2t)\Big]_0^{\pi/2} \tag{9.5.40}$$

$$= 6\,|a| \tag{9.5.41}$$



33. Starting with the arc length formula,

$$l = \int_a^b \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, \mathrm{d}t \qquad\qquad C : y = f(x), \quad z = 0 \tag{9.5.42}$$

$$l = \int_a^b \sqrt{1 + \frac{\mathrm{d}y}{\mathrm{d}x}^2}\, \mathrm{d}x \tag{9.5.43}$$

There is no parameter t involved here, since every point on the curve is of the form $(x, f(x))$

34. Polar coordinates, with the vector function having parameter θ. The radial distance ρ itself is also a function of θ.

$$\rho = \sqrt{x^2 + y^2} \qquad\qquad \theta = \arctan(y/x) \tag{9.5.44}$$

$$\mathbf{r} = \begin{bmatrix} \rho\cos(\theta) \\ \rho\sin(\theta) \end{bmatrix} \qquad\qquad \mathbf{r}' = \begin{bmatrix} \rho'\cos(\theta) - \rho\sin(\theta) \\ \rho'\sin(\theta) + \rho\cos(\theta) \end{bmatrix} \tag{9.5.45}$$

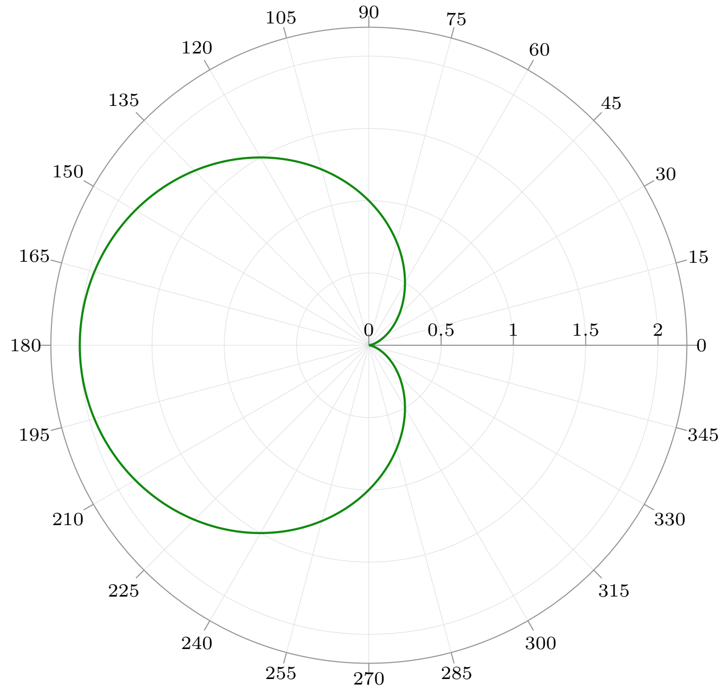
$$l = \int_\alpha^\beta \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, \mathrm{d}\theta \qquad\qquad = \int_\alpha^\beta \sqrt{\rho^2 + \rho'^2}\, \mathrm{d}\theta \tag{9.5.46}$$

The cardioid is given by,

$$C : \rho = a(1 - \cos\theta) \tag{9.5.47}$$

$$l = \int_0^{2\pi} a\sqrt{1 + \cos^2\theta - 2\cos\theta + \sin^2\theta}\, \mathrm{d}\theta \tag{9.5.48}$$

$$= \int_0^{2\pi} 2a\sin(\theta/2)\, \mathrm{d}\theta = -4a\Big[\cos(\theta/2)\Big]_0^{2\pi} = 8a \tag{9.5.49}$$

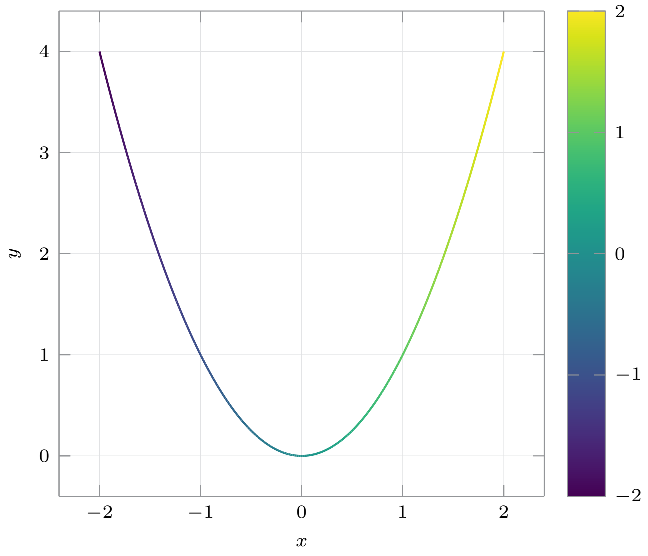


35. Finding the velocity and acceleration vectors,

$$\mathbf{r}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix} \tag{9.5.50}$$

$$\mathbf{a}(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \mathbf{a}_{\text{tan}} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} \tag{9.5.51}$$

$$\mathbf{a}_{\text{tan}} = \frac{4t}{1+4t^2}\begin{bmatrix} 1 \\ 2t \end{bmatrix} \qquad \mathbf{a}_{\text{norm}} = \frac{1}{1+4t^2}\begin{bmatrix} -4t \\ 2 \end{bmatrix} \tag{9.5.52}$$

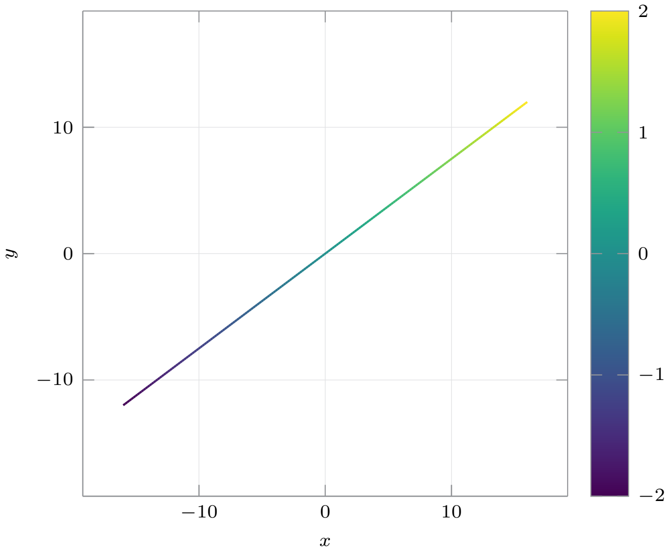


36. Finding the velocity and acceleration vectors,

$$\mathbf{r}(t) = \begin{bmatrix} 8t \\ 6t \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \tag{9.5.53}$$

$$\mathbf{a}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{a}_{\tan} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \tag{9.5.54}$$

$$\mathbf{a}_{\tan} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{a}_{\text{norm}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9.5.55}$$

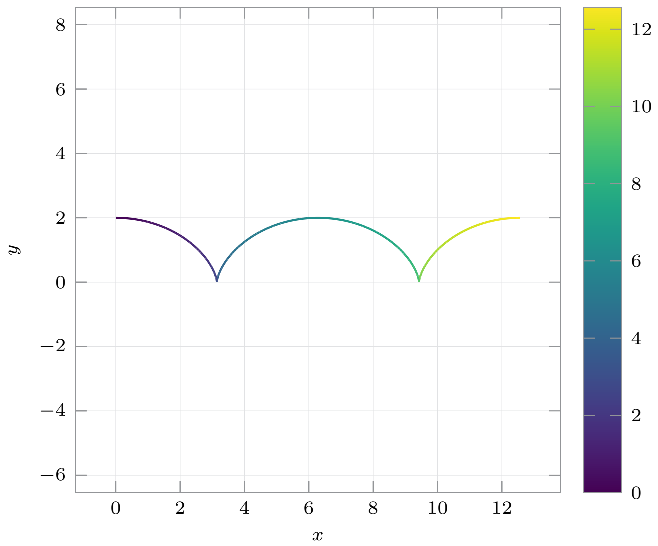


37. Finding the velocity and acceleration vectors when the $y = y_{\max}$,

$$\mathbf{r}(t) = \begin{bmatrix} R\sin(\omega t) + Rt \\ R\cos(\omega t) + R \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} R\omega\cos(\omega t) + R \\ -R\omega\sin(\omega t) \end{bmatrix} \tag{9.5.56}$$

$$\mathbf{a}(t) = \begin{bmatrix} -R\omega^2\sin(\omega t) \\ -R\omega^2\cos(\omega t) \end{bmatrix} \tag{9.5.57}$$

$$\mathbf{v}(0) = \begin{bmatrix} R(1+\omega) \\ 0 \end{bmatrix} \qquad \mathbf{a}(0) = \begin{bmatrix} 0 \\ -R\omega^2 \end{bmatrix} \tag{9.5.58}$$

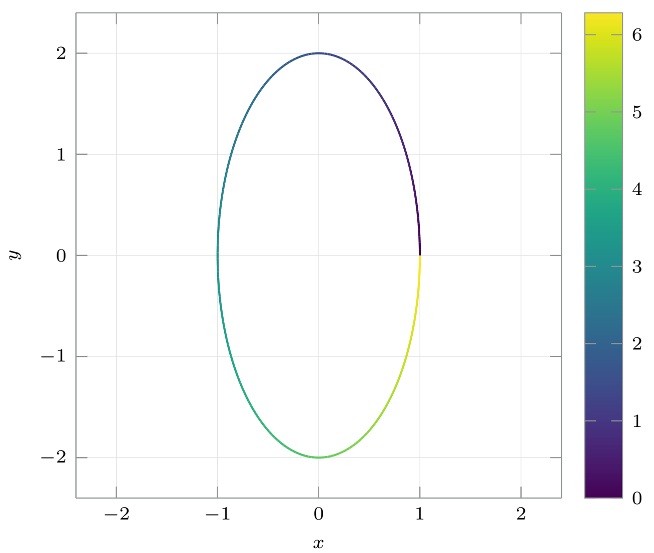


38. Finding the velocity and acceleration vectors,

$$\mathbf{r}(t) = \begin{bmatrix} \cos t \\ 2 \sin t \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} -\sin t \\ 2 \cos t \end{bmatrix} \tag{9.5.59}$$

$$\mathbf{a}(t) = \begin{bmatrix} -\cos t \\ -2 \sin t \end{bmatrix} \qquad \mathbf{a}_{\tan} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \tag{9.5.60}$$

$$\mathbf{a}_{\tan} = \frac{-3 \sin t \cos t}{4 \cos^2 t + \sin^2 t} \begin{bmatrix} -\sin t \\ 2 \cos t \end{bmatrix} \qquad \mathbf{a}_{\text{norm}} = \frac{1}{4 \cos^2 t + \sin^2 t} \begin{bmatrix} -4 \cos t \\ -2 \sin t \end{bmatrix} \tag{9.5.61}$$

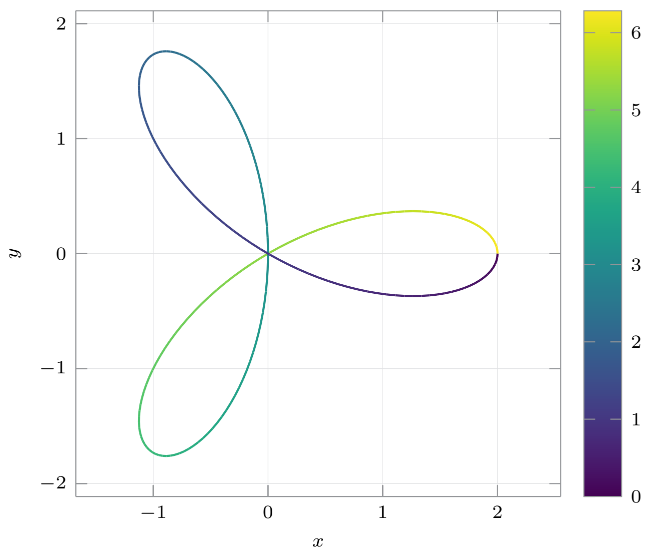


39. Finding the velocity and acceleration vectors,

$$\mathbf{r}(t) = \begin{bmatrix} \cos t + \cos(2t) \\ \sin t - \sin(2t) \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} -\sin(t) - 2\sin(2t) \\ \cos(t) - 2\cos(2t) \end{bmatrix} \tag{9.5.62}$$

$$\mathbf{a}(t) = \begin{bmatrix} -\cos(t) - 4\cos(2t) \\ -\sin(t) + 4\sin(2t) \end{bmatrix} \qquad \mathbf{a}_{\text{tan}} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} \tag{9.5.63}$$

$$\mathbf{a}_{\text{tan}} = \frac{6\sin(3t)}{5 - 4\cos(3t)}\,\mathbf{v}(t) \qquad \mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}} \tag{9.5.64}$$

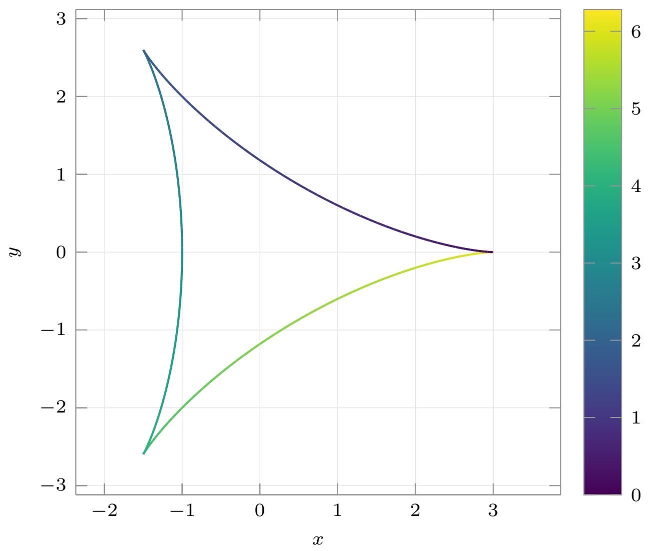


40. Finding the velocity and acceleration vectors,

$$\mathbf{r}(t) = \begin{bmatrix} 2\cos t + \cos(2t) \\ 2\sin t - \sin(2t) \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} -2\sin(t) - 2\sin(2t) \\ 2\cos(t) - 2\cos(2t) \end{bmatrix} \tag{9.5.65}$$

$$\mathbf{a}(t) = \begin{bmatrix} -2\cos(t) - 4\cos(2t) \\ -2\sin(t) + 4\sin(2t) \end{bmatrix} \qquad \mathbf{a}_{\text{tan}} = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} \tag{9.5.66}$$

$$\mathbf{a}_{\text{tan}} = \frac{1.5\sin(3t)}{1-\cos(3t)}\,\mathbf{v}(t) \qquad \mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}} \tag{9.5.67}$$

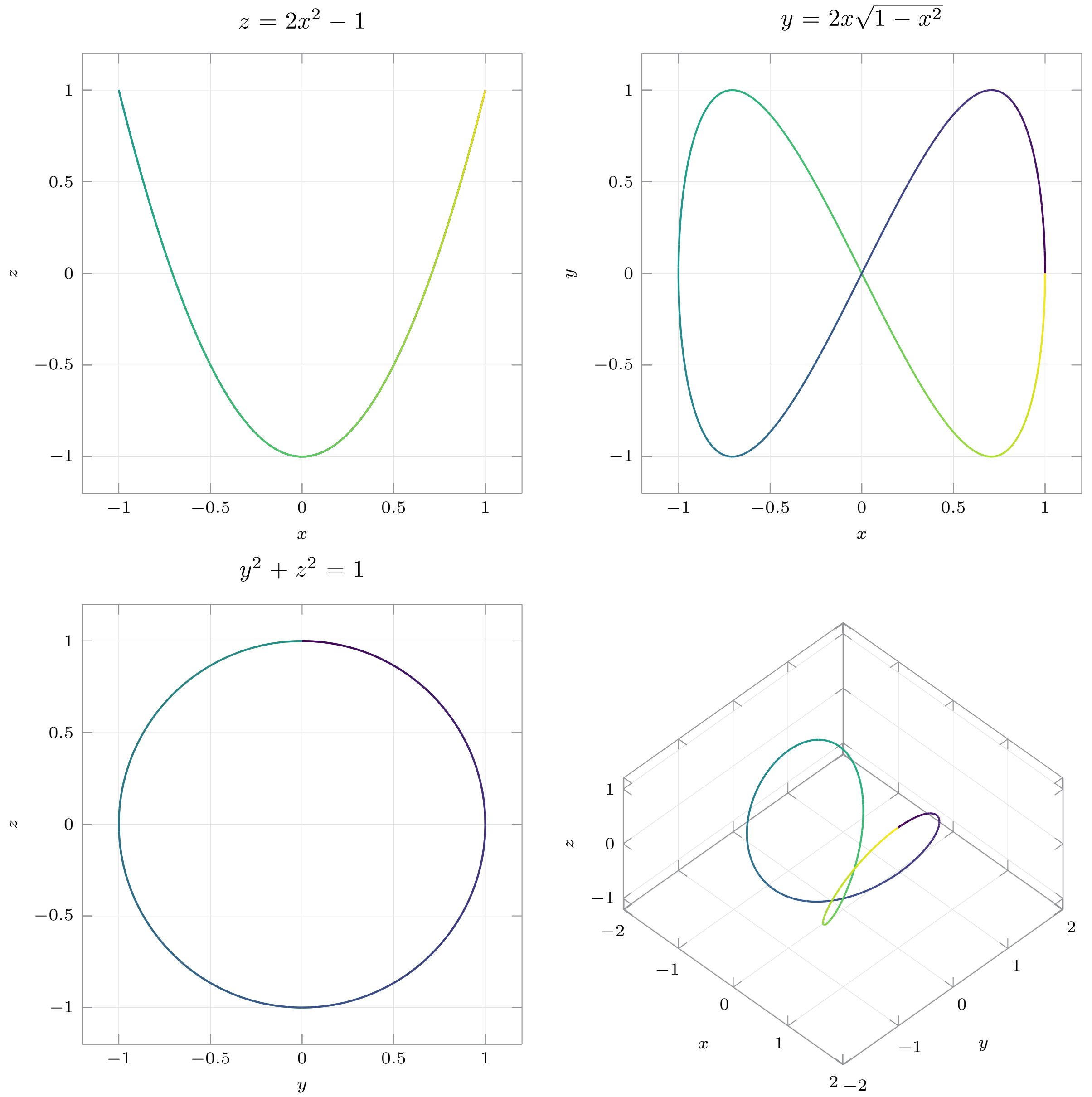


41. Finding the velocity and acceleration vectors,

$$\mathbf{r}(t) = \begin{bmatrix} \cos t \\ \sin(2t) \\ \cos(2t) \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} -\sin t \\ 2\cos(2t) \\ -2\sin(2t) \end{bmatrix} \tag{9.5.68}$$

$$\mathbf{a}(t) = \begin{bmatrix} -\cos t \\ -4\sin(2t) \\ -4\cos(2t) \end{bmatrix} \qquad \mathbf{a}_{\text{tan}} = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} \tag{9.5.69}$$

$$\mathbf{a}_{\text{tan}} = \frac{\sin t\cos t}{4+\sin^2 t}\,\mathbf{v}(t) \qquad \mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}} \tag{9.5.70}$$

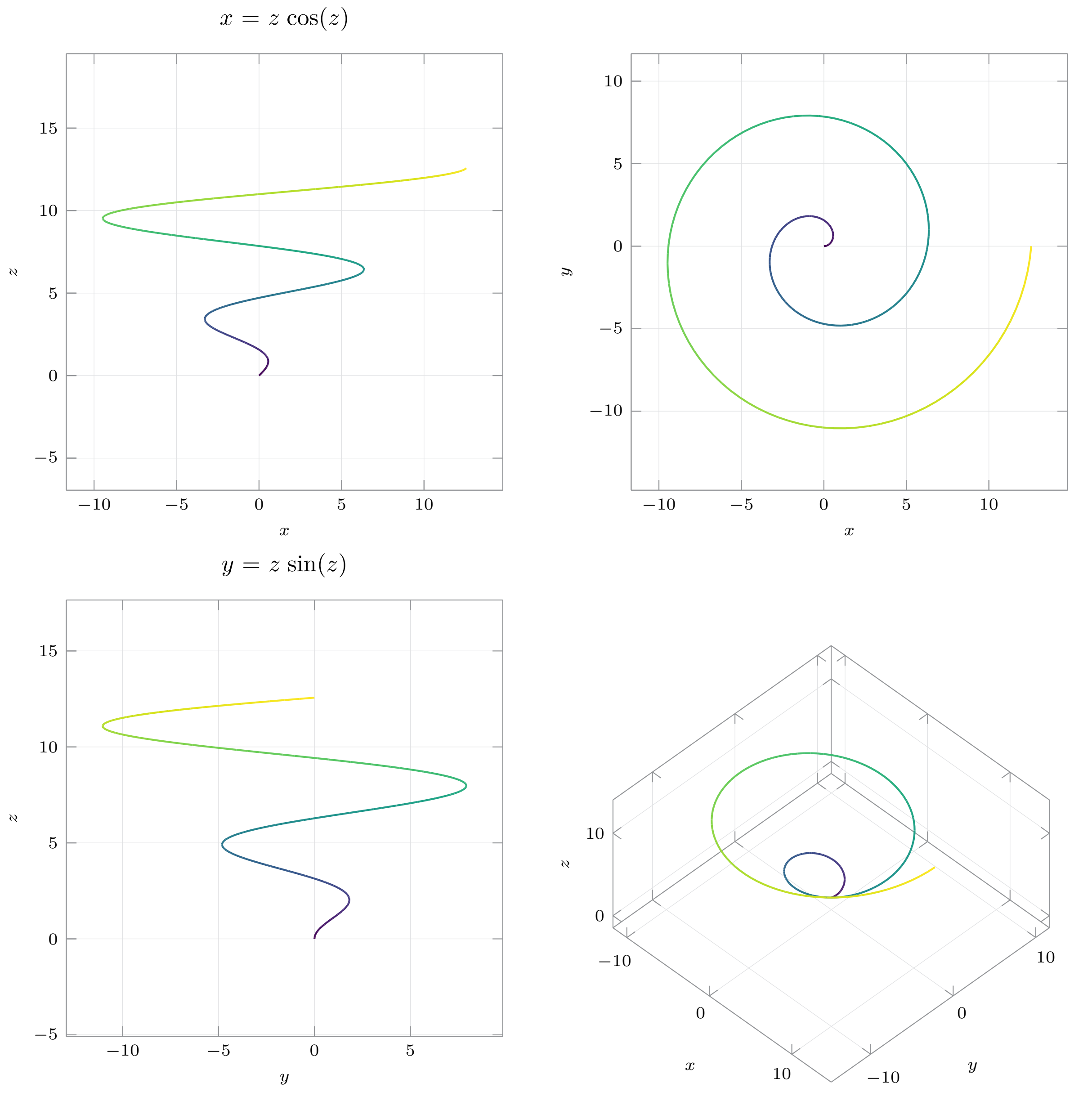


42. Finding the velocity and acceleration vectors,

$$\mathbf{r}(t) = \begin{bmatrix} ct\cos t \\ ct\sin t \\ ct \end{bmatrix} \qquad \mathbf{v}(t) = \begin{bmatrix} -ct\sin t + c\cos t \\ ct\cos t + c\sin t \\ c \end{bmatrix} \tag{9.5.71}$$

$$\mathbf{a}(t) = \begin{bmatrix} -ct\cos t - 2c\sin t \\ -ct\sin t + 2c\cos t \\ 0 \end{bmatrix} \qquad \mathbf{a}_{\text{tan}} = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} \tag{9.5.72}$$

$$\mathbf{a}_{\text{tan}} = \frac{t}{1+t^2}\,\mathbf{v}(t) \qquad \mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}} \tag{9.5.73}$$



43. Earth revolving around the sun,

$$\mathbf{a} = \begin{bmatrix} -R\omega^2 \cos(\omega t) \\ -R\omega^2 \sin(\omega t) \end{bmatrix} \qquad |\mathbf{v}| = 30 \text{ km s}^{-1} \tag{9.5.74}$$

$$\omega = \frac{2\pi}{365 \cdot 86400} \qquad \mathbf{a}_c = v\omega = 5.98 \times 10^{-6} \text{ km s}^{-2} \tag{9.5.75}$$

44. Moon revolving around the Earth,

$$R = 3.85 \times 10^8 \text{ m} \qquad T = 2.36 \times 10^6 \text{ s} \tag{9.5.76}$$

$$a_c = \frac{R(4\pi^2)}{T^2} \qquad = 2.73 \times 10^{-3} \text{ m s}^{-2} \tag{9.5.77}$$

45. Satellite revolving around the Earth,

$$R = 6\,501\,750 \text{ m} \qquad g = 9.4488 \text{ m s}^{-2} \tag{9.5.78}$$

$$mg = F_{\text{centr}} = \frac{mv^2}{R} \qquad v = 7.84 \text{ km s}^{-1} \tag{9.5.79}$$

46. Satellite revolving around the Earth,

$$R = 7097.207 \text{ km} \qquad T = 6000 \text{ s} \tag{9.5.80}$$

$$mg = F_{\text{centr}} = mR\omega^2 \qquad g = \frac{R4\pi^2}{T^2} = 7.783 \text{ m s}^{-2} \tag{9.5.81}$$

47. Circle of radius a

$$\mathbf{r}(t) = \begin{bmatrix} a\cos(t) \\ a\sin(t) \end{bmatrix} \qquad \mathbf{r}'(t) = \begin{bmatrix} -a\sin(t) \\ a\cos(t) \end{bmatrix} \tag{9.5.82}$$

$$s(t) = \int_0^t \sqrt{\mathbf{r}'(w) \cdot \mathbf{r}'(w)}\ \mathrm{d}w \qquad = \Big[aw\Big]_0^t = at \tag{9.5.83}$$

Rewriting the vector function using the arc length as parameter,

$$\mathbf{r}(s) = \begin{bmatrix} a\cos(s/a) \\ a\sin(s/a) \end{bmatrix} \qquad \mathbf{r}'(s) = \begin{bmatrix} -\sin(s/a) \\ \cos(s/a) \end{bmatrix} \tag{9.5.84}$$

$$\mathbf{r}''(s) = \frac{1}{a}\begin{bmatrix} -\cos(s/a) \\ -\sin(s/a) \end{bmatrix} \qquad \kappa = |\mathbf{r}''(s)| = \frac{1}{a} \tag{9.5.85}$$

48. By the definition of curvature,

$$s(t) = \int_0^t \sqrt{\mathbf{r}'(w) \cdot \mathbf{r}'(w)}\ \mathrm{d}w \tag{9.5.86}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{r}'(t)| \tag{9.5.87}$$

$$\frac{\mathrm{d}\mathbf{r}(s)}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} \cdot \frac{1}{|\mathbf{r}'(t)|} \tag{9.5.88}$$

$$\frac{\mathrm{d}^2\mathbf{r}(s)}{\mathrm{d}s^2} = \frac{\mathrm{d}\mathbf{r}'(t)}{\mathrm{d}s}\frac{1}{|\mathbf{r}'(t)|} + \mathbf{r}'(t)\ \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{|\mathbf{r}'(t)|} \tag{9.5.89}$$

$$= \frac{\mathbf{r}''(t)}{|\mathbf{r}'(t)|^2} + \mathbf{r}'(t)\ \frac{\mathrm{d}t}{\mathrm{d}s}\ \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)]^{-1/2} \tag{9.5.90}$$

$$= \frac{\mathbf{r}''(t)}{|\mathbf{r}'(t)|^2} - \mathbf{r}'(t)\ \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|^4} \tag{9.5.91}$$

$$= \frac{\mathbf{r}''(t)\left[\mathbf{r}'(t) \cdot \mathbf{r}'(t)\right] - \mathbf{r}'(t)\left[\mathbf{r}'(t) \cdot \mathbf{r}''(t)\right]}{|\mathbf{r}'(t)|^4} \tag{9.5.92}$$

$$= \frac{\mathbf{r}'(t) \times \left[\mathbf{r}''(t) \times \mathbf{r}'(t)\right]}{|\mathbf{r}'(t)|^4} \tag{9.5.93}$$

Using the formula of curvature parametrized by arc length,

$$\kappa(s) = \left|\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathbf{r}(s)\right| \tag{9.5.94}$$

$$= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \tag{9.5.95}$$

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\ \ |\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}|^2 \tag{9.5.96}$$

$$= \frac{\sqrt{|\mathbf{r}'(t)|^2\ \ |\mathbf{r}''(t)|^2 - \left[\mathbf{r}'(t) \cdot \mathbf{r}''\right]^2}}{|\mathbf{r}'(t)|^3} \tag{9.5.97}$$

$$= \frac{\sqrt{|\mathbf{r}'(t)|^2\ \ |\mathbf{r}''(t)|^2 - \left[\mathbf{r}'(t) \cdot \mathbf{r}''\right]^2}}{|\mathbf{r}'(t)|^3} \tag{9.5.98}$$

49. Using the result from Prob. 48,

$$\mathbf{r} = \begin{bmatrix} x \\ y(x) \end{bmatrix} \qquad \mathbf{r}' = \begin{bmatrix} 1 \\ y'(x) \end{bmatrix} \tag{9.5.99}$$

$$\mathbf{r}'' = \begin{bmatrix} 0 \\ y''(x) \end{bmatrix} \tag{9.5.100}$$

$$\kappa(x) = \frac{\sqrt{(1+y'^2)(y''^2) - y'^2 \cdot y''^2}}{(1+y'^2)^{3/2}} \qquad \kappa(x) = \frac{|y''|}{(1+y'^2)^{3/2}} \tag{9.5.101}$$

50. Using the two given relations,

$$\mathbf{b} = \mathbf{u} \times \mathbf{p} \qquad \tau(s) = -\mathbf{p}(s) \cdot \mathbf{b}'(s) \tag{9.5.102}$$

$$\tau(s) = -\mathbf{p}(s) \cdot \left[\mathbf{u}' \times \mathbf{p} + \mathbf{u} \times \mathbf{p}'\right] \qquad \tau(s) = -\mathbf{p} \cdot \mathbf{u} \times \mathbf{p}' \tag{9.5.103}$$

$$\tau(s) = \mathbf{u} \cdot (\mathbf{p} \times \mathbf{p}') \tag{9.5.104}$$

$$\mathbf{p} = \frac{1}{\kappa}\,\mathbf{u}' \qquad \mathbf{u} = \mathbf{r}' \tag{9.5.105}$$

$$\tau(s) = \frac{1}{\kappa^2}\,\mathbf{r}' \cdot (\mathbf{r}'' \times \mathbf{r}''') \tag{9.5.106}$$

51. Changing the derivative from s to t,

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}s} = \mathbf{r}'\,\frac{\mathrm{d}t}{\mathrm{d}s} \tag{9.5.107}$$

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}s}\,\frac{\mathrm{d}t}{\mathrm{d}s} + \mathbf{r}'\,\frac{\mathrm{d}^2t}{\mathrm{d}s^2} \tag{9.5.108}$$

$$= \mathbf{r}''\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \mathbf{r}'\,\frac{\mathrm{d}^2t}{\mathrm{d}s^2} \tag{9.5.109}$$

$$\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}s^3} = \mathbf{r}'''\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^3 + 3\mathbf{r}''\,\frac{\mathrm{d}t}{\mathrm{d}s}\,\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \mathbf{r}'\,\frac{\mathrm{d}^3t}{\mathrm{d}s^3} \tag{9.5.110}$$

Substituting these into the result from Prob. 50, and using the results from Prob. 48,

$$\tau(t) = \frac{1}{\kappa^2}\left[\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^3 (\mathbf{r}' \times \mathbf{r}'') \cdot \frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}s^3}\right] \tag{9.5.111}$$

$$\tau(t) = \frac{1}{\kappa^2}\left[\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^6 (\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''\right] \tag{9.5.112}$$

$$= \frac{(\mathbf{r}' \cdot \mathbf{r}')^{-3}}{\kappa^2}\left[(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''\right] \tag{9.5.113}$$

$$= \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{(\mathbf{r}' \cdot \mathbf{r}')(\mathbf{r}'' \cdot \mathbf{r}'') - (\mathbf{r}' \cdot \mathbf{r}'')^2} \tag{9.5.114}$$

52. Parametrizing the helix by arc length,

$$s = \int_0^t \sqrt{\mathbf{r}'(w) \cdot \mathbf{r}'(w)}\ \mathrm{d}w \qquad = \int_0^t \sqrt{a^2 + c^2}\ \mathrm{d}w \tag{9.5.115}$$

$$= \int_0^t K\ \mathrm{d}w \qquad = Kt \tag{9.5.116}$$

$$\mathbf{r}(s) = \begin{bmatrix} a\cos(s/K) \\ a\sin(s/K) \\ c(s/K) \end{bmatrix} \tag{9.5.117}$$

Finding curvature,

$$\kappa(s) = |\mathbf{r}''(s)| \tag{9.5.118}$$

$$\mathbf{r}''(\mathbf{s}) = \begin{bmatrix} (-a/K^2)\cos(s/K) \\ (-a/K^2)\sin(s/K) \\ 0 \end{bmatrix} \tag{9.5.119}$$

$$\kappa(s) = \frac{a}{K^2} \tag{9.5.120}$$

$$\mathbf{b} = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\kappa} \tag{9.5.121}$$

$$= \frac{K^2}{a}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(a/K)\sin(s/K) & (a/K)\cos(s/K) & (c/K) \\ -(a/K^2)\cos(s/K) & -(a/K^2)\sin(s/K) & 0 \end{vmatrix} \tag{9.5.122}$$

$$= \begin{bmatrix} (-c/K)\sin(s/K) \\ (c/K)\cos(s/K) \\ a/K \end{bmatrix} \tag{9.5.123}$$

Finding torsion,

$$\mathbf{b}'(s) = \begin{bmatrix} (-c/K^2)\cos(s/K) \\ -(c/K^2)\sin(s/K) \\ 0 \end{bmatrix} \qquad \mathbf{p}(s) = \begin{bmatrix} -\cos(s/K) \\ -\sin(s/K) \\ 0 \end{bmatrix} \tag{9.5.124}$$

$$\tau(s) = -\mathbf{p} \cdot \mathbf{b} \qquad |\tau(s)| = \frac{c}{K^2} \tag{9.5.125}$$

53. Using the result from Prob. 51,

$$\mathbf{r}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix} \qquad \mathbf{r}'(t) = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} \tag{9.5.126}$$

$$\mathbf{r}''(t) = \begin{bmatrix} 0 \\ 2 \\ 6t \end{bmatrix} \qquad \mathbf{r}'''(t) = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \tag{9.5.127}$$

$$(\mathbf{r}' \; \mathbf{r}'' \; \mathbf{r}''') = 12 \qquad \tau(t) = \frac{3}{1 + 9t^2 + 9t^4} \tag{9.5.128}$$

54. From the definition of torsion,

$$\kappa(s) = |\mathbf{u}'(s)| \qquad |\mathbf{p}| = 1 \tag{9.5.129}$$

$$\mathbf{p} = \frac{\mathbf{u}'}{\kappa} \qquad \mathbf{u}' = \kappa\, \mathbf{p} \tag{9.5.130}$$

Using the right-handed triple of **u**, **p**, **b**,

$$\mathbf{p}' \cdot \mathbf{p} = 0 \qquad \implies \mathbf{p}' = \lambda\mathbf{u} + \mu\mathbf{b} \tag{9.5.131}$$

$$\mathbf{b}' = \mathbf{u}' \times \mathbf{p} + \mathbf{u} \times \mathbf{p}' \qquad \mathbf{b}' = \mathbf{u} \times \mathbf{p}' \tag{9.5.132}$$

$$\mathbf{b}' = -\mu\, \mathbf{p} \tag{9.5.133}$$

From the definition of torsion,

$$|\tau| = |\mathbf{b}'| \qquad \implies \mathbf{b}' = -\tau\, \mathbf{p} \tag{9.5.134}$$

The third Frenet relation,

$$\mathbf{p}' = \lambda\mathbf{u} + \tau\mathbf{b} \qquad \mathbf{p}' = \mathbf{b}' \times \mathbf{u} + \mathbf{b} \times \mathbf{u}' \tag{9.5.135}$$

$$= -\tau(\mathbf{p} \times \mathbf{u}) + \kappa(\mathbf{b} \times \mathbf{p}) \qquad = \tau\, \mathbf{b} - \kappa\, \mathbf{u} \tag{9.5.136}$$

Most of the calculations are simply the relation between three mutually orthogonal unit vectors and their cross products.

55. Refer to problem 52. Already solved using the specified method.

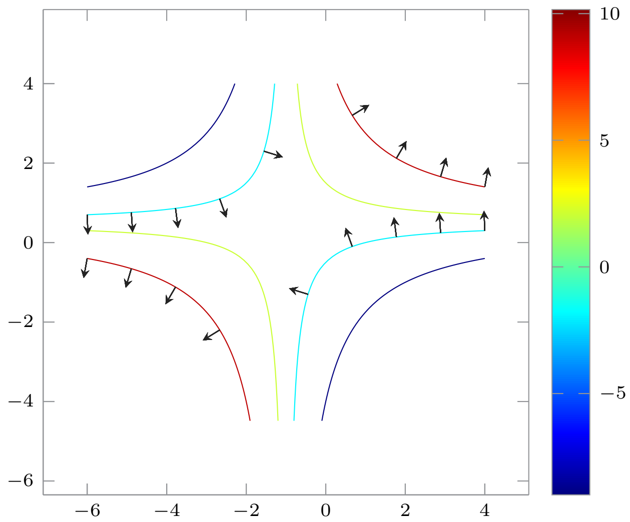
9.6 Review: Functions of Several Variables

1. No problem set in this section

9.7 Gradient of a Scalar Field, Directional Derivative

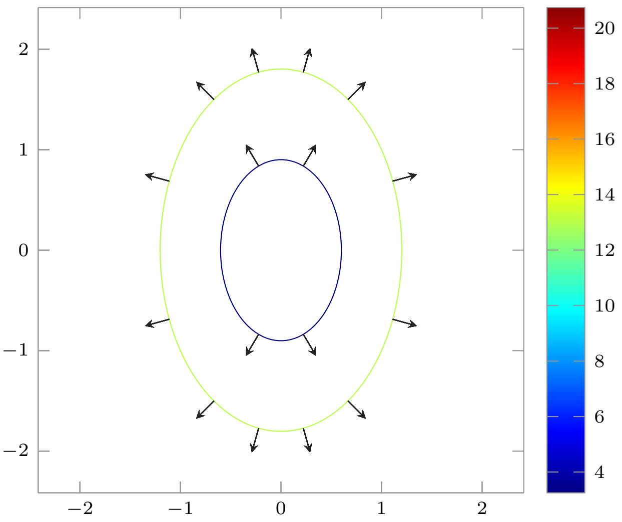
1. Finding the gradient,

$$f = (x+1)(2y-1) \qquad \nabla f = \begin{bmatrix} 2y-1 \\ 2x+2 \end{bmatrix} \tag{9.7.1}$$



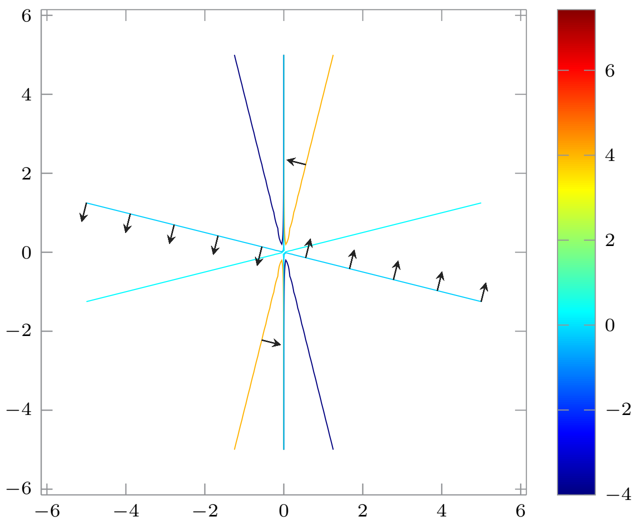
2. Finding the gradient,

$$f = 9x^2 + 4y^2 \qquad \nabla f = \begin{bmatrix} 18x \\ 8y \end{bmatrix} \tag{9.7.2}$$



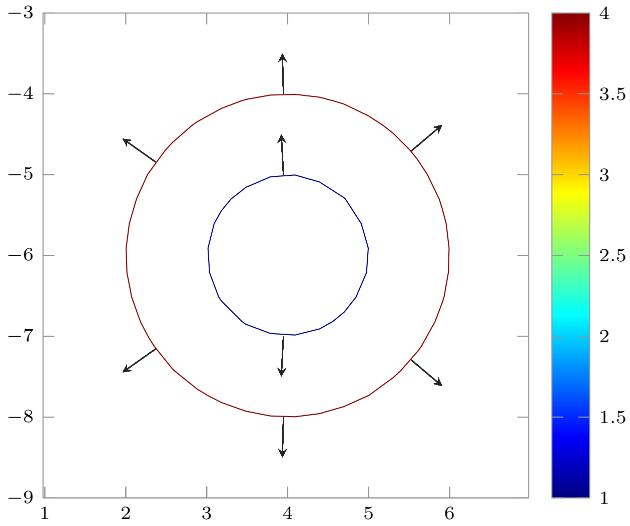
3. Finding the gradient,

$$f = \frac{y}{x} \qquad \nabla f = \begin{bmatrix} -y/x^2 \\ 1/x \end{bmatrix} \tag{9.7.3}$$



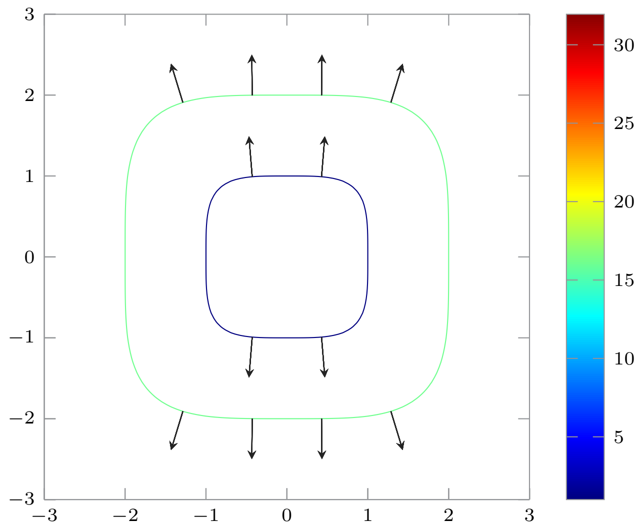
4. Finding the gradient,

$$f = (y+6)^2 + (x-4)^2 \qquad \nabla f = \begin{bmatrix} 2(x-4) \\ 2(y+6) \end{bmatrix} \tag{9.7.4}$$



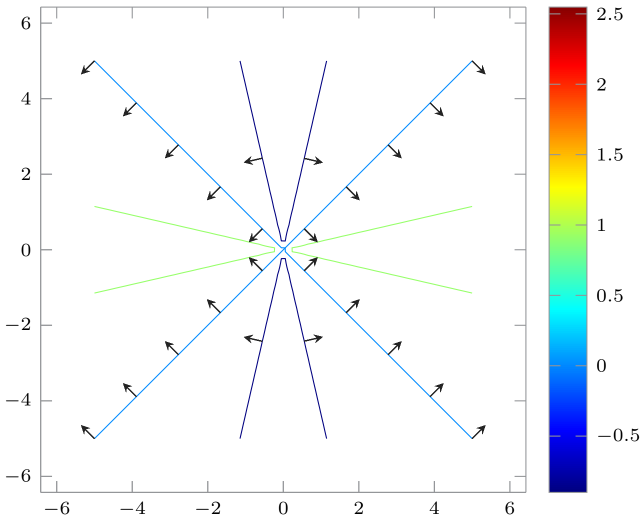
5. Finding the gradient,

$$f = x^4 + y^4 \qquad \nabla f = \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix} \tag{9.7.5}$$



6. Finding the gradient,

$$f = \frac{x^2 - y^2}{x^2 + y^2} \qquad \nabla f = \frac{1}{(x^2 + y^2)^2} \begin{bmatrix} 4xy^2 \\ -4x^2 y \end{bmatrix} \tag{9.7.6}$$



7. Proving the relation

$$\nabla(f^n) = \frac{\partial f^n}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f^n}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f^n}{\partial z}\,\hat{\mathbf{k}} \tag{9.7.7}$$

$$\nabla(f^n) = nf^{n-1}\left[\frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}\right] = nf^{n-1}\,\nabla f \tag{9.7.8}$$

8. Proving the relation

$$\nabla(fg) = \frac{\partial (fg)}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial (fg)}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial (fg)}{\partial z}\,\hat{\mathbf{k}} \tag{9.7.9}$$

$$\nabla(f^n) = g\left[\frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}\right] + f\left[\frac{\partial g}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\,\hat{\mathbf{k}}\right] \tag{9.7.10}$$

$$= g\,\nabla f + f\,\nabla g \tag{9.7.11}$$

9. Proving the relation

$$\nabla(f/g) = \frac{\partial (f/g)}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial (f/g)}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial (f/g)}{\partial z}\,\hat{\mathbf{k}} \tag{9.7.12}$$

$$\nabla(f^n) = \frac{1}{g}\left[\frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}\right] - \frac{f}{g^2}\left[\frac{\partial g}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\,\hat{\mathbf{k}}\right] \tag{9.7.13}$$

$$= \frac{1}{g^2}(g\,\nabla f - f\,\nabla g) \tag{9.7.14}$$

10. Proving the relation

$$\nabla^2(fg) = \nabla \cdot \left[g \, \nabla f + f \, \nabla g\right] \tag{9.7.15}$$

$$\nabla \cdot (g\nabla f) = \left[\frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}\,\hat{\mathbf{k}}\right] \cdot \left[g\frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + g\frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + g\frac{\partial f}{\partial z}\,\hat{\mathbf{k}}\right] \tag{9.7.16}$$

$$= g \, \nabla^2 f + \nabla f \cdot \nabla g \tag{9.7.17}$$

$$\nabla^2(fg) = g \, \nabla^2 f + 2\nabla g \cdot \nabla f + f \, \nabla^2 g \tag{9.7.18}$$

11. Calculating the gradient,

$$f = xy \qquad P : (-4, 5) \tag{9.7.19}$$

$$\nabla f = \begin{bmatrix} y \\ x \end{bmatrix} \qquad \left[\nabla f\right]_P = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \tag{9.7.20}$$

12. Calculating the gradient,

$$f = \frac{x}{x^2 + y^2} \qquad P : (1, 1) \tag{9.7.21}$$

$$\nabla f = \frac{1}{(x^2 + y^2)^2} \begin{bmatrix} y^2 - x^2 \\ -2xy \end{bmatrix} \qquad \left[\nabla f\right]_P = \frac{1}{4} \begin{bmatrix} 0 \\ -2 \end{bmatrix} \tag{9.7.22}$$

13. Calculating the gradient,

$$f = \ln(x^2 + y^2) \qquad P : (8, 6) \tag{9.7.23}$$

$$\nabla f = \frac{1}{(x^2 + y^2)} \begin{bmatrix} 2x \\ 2y \end{bmatrix} \qquad \left[\nabla f\right]_P = \frac{1}{100} \begin{bmatrix} 16 \\ 12 \end{bmatrix} \tag{9.7.24}$$

14. Calculating the gradient,

$$f = (x^2 + y^2 + z^2)^{-1/2} \qquad P : (12, 0, 16) \tag{9.7.25}$$

$$\nabla f = \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \left[\nabla f\right]_P = \frac{-1}{8000} \begin{bmatrix} 12 \\ 0 \\ 16 \end{bmatrix} \tag{9.7.26}$$

15. Calculating the gradient,

$$f = 4x^2 + 9y^2 + z^2 \qquad P : (5, -1, -11) \tag{9.7.27}$$

$$\nabla f = \begin{bmatrix} 8x \\ 18y \\ 2z \end{bmatrix} \qquad \left[\nabla f\right]_P = \begin{bmatrix} 40 \\ -18 \\ -22 \end{bmatrix} \tag{9.7.28}$$

16. Calculating the gradient,

$$f = 25x^2 + 9y^2 + 16z^2 \qquad P : (a, b, c) \tag{9.7.29}$$

$$\nabla f = \begin{bmatrix} 50x \\ 18y \\ 32z \end{bmatrix} \qquad \left[\nabla f\right]_P = \begin{bmatrix} 50a \\ 18b \\ 32c \end{bmatrix} \tag{9.7.30}$$

Since this gradient is in the direction of the origin,

$$\begin{bmatrix} 50a \\ 18b \\ 32c \end{bmatrix} \propto \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{9.7.31}$$

$$\tag{9.7.32}$$

Points on one of the 3 axes will have two out of 3 components zero, which satisfies the relation. THe origin is also a valid point.

17. Calculating the gradient,

$$f = 25x^2 + 4y^2 \qquad P : (a, b) \tag{9.7.33}$$

$$\nabla f = \begin{bmatrix} 50x \\ 8y \end{bmatrix} \qquad \left[\nabla f\right]_P = \begin{bmatrix} 50a \\ 8b \end{bmatrix} \tag{9.7.34}$$

Since this gradient is in the direction of the origin,

$$\begin{bmatrix} 50a \\ 8b \end{bmatrix} \propto \begin{bmatrix} a \\ b \end{bmatrix} \tag{9.7.35}$$

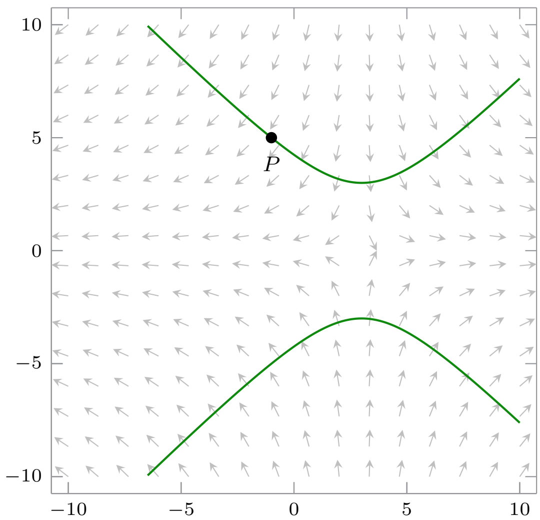
$$\tag{9.7.36}$$

Points on one of the 2 axes will have one out of 2 components zero, which satisfies the relation. THe origin is also a valid point.

18. Finding the gradient,

$$f = x^2 - 6x - y^2 \qquad P : (-1, 5) \tag{9.7.37}$$

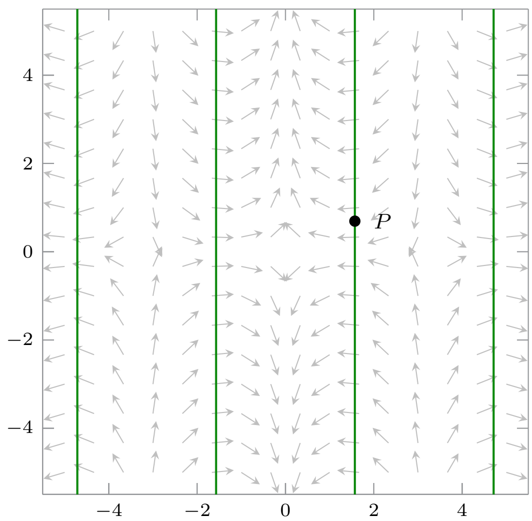
$$\nabla f = \begin{bmatrix} 2x - 6 \\ -2y \end{bmatrix} \qquad \Big[\nabla f\Big]_P = \begin{bmatrix} -8 \\ -10 \end{bmatrix} \tag{9.7.38}$$



19. Finding the gradient,

$$f = \cos(x)\cosh(y) \qquad P : (\pi/2, \ln 2) \tag{9.7.39}$$

$$\nabla f = \begin{bmatrix} -\sin(x)\cosh(y) \\ \cos(x)\sinh(y) \end{bmatrix} \qquad \Big[\nabla f\Big]_P = \begin{bmatrix} -1.25 \\ 0 \end{bmatrix} \tag{9.7.40}$$



20. Finding the gradient,

$$f = x + \frac{x}{x^2 + y^2} \qquad\qquad P : (1, 1) \tag{9.7.41}$$

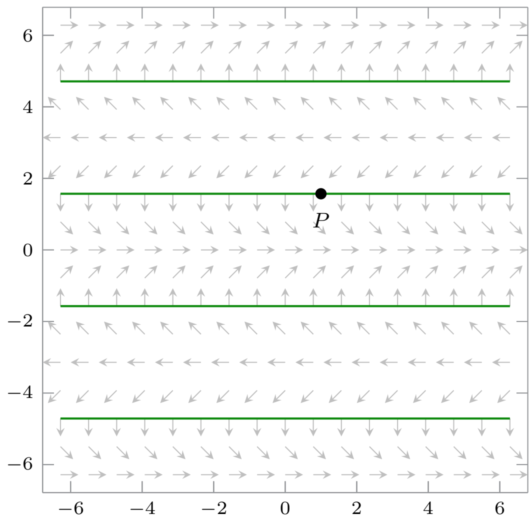
$$\nabla f = \begin{bmatrix} 1 + \frac{y^2 - x^2}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} \end{bmatrix} \qquad\qquad \Big[\nabla f\Big]_P = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \tag{9.7.42}$$

Plot TBC

21. Finding the gradient,

$$f = e^x \cos(y) \qquad\qquad P : (1, \pi/2) \tag{9.7.43}$$

$$\nabla f = \begin{bmatrix} e^x \cos(y) \\ -e^x \sin(y) \end{bmatrix} \qquad\qquad \Big[\nabla f\Big]_P = \begin{bmatrix} 0 \\ -e \end{bmatrix} \tag{9.7.44}$$



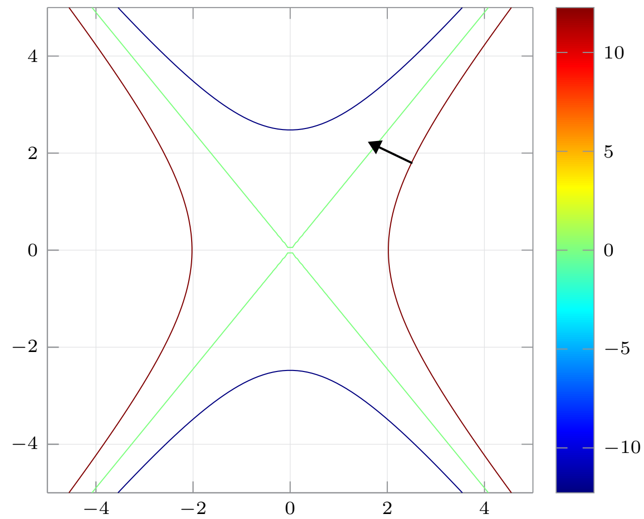
22. For the gradient to be vertically upward, $y = \frac{(2n+1)\pi}{2}$

23. For the gradient to be horizontal, $y = n\pi$.

24. Finding the gradient, and reversing it

$$f = 3x^2 - 2y^2 \qquad\qquad P : (2.5, 1.8) \tag{9.7.45}$$

$$\nabla f = \begin{bmatrix} 6x \\ -4y \end{bmatrix} \qquad\qquad \Big[\nabla f\Big]_P = -\begin{bmatrix} 15 \\ -7.2 \end{bmatrix} \tag{9.7.46}$$



25. Finding the gradient, and reversing it

$$f = \frac{z}{x^2 + y^2} \qquad P : (0, 1, 2) \tag{9.7.47}$$

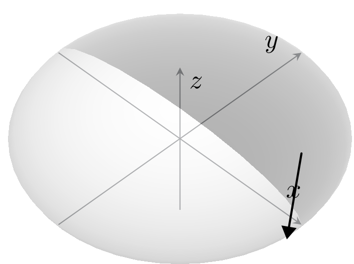
$$\nabla f = \frac{1}{(x^2 + y^2)^2} \begin{bmatrix} -2xz \\ -2yz \\ x^2 + y^2 \end{bmatrix} \qquad \left[\nabla f\right]_P = -\begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix} \tag{9.7.48}$$

Figure TBC

26. Finding the gradient, and reversing it

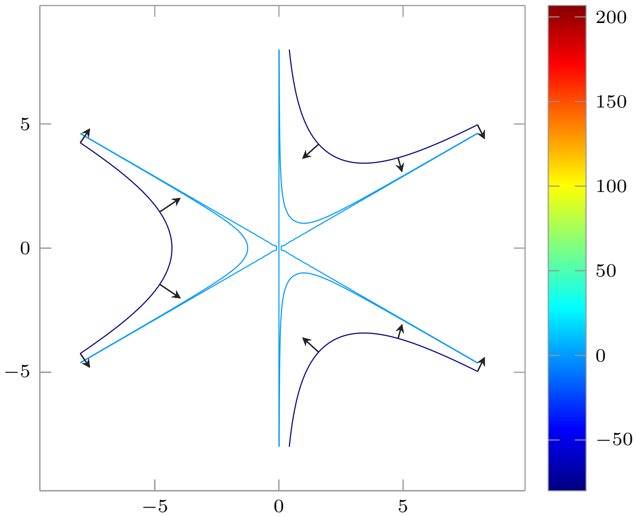
$$f = x^2 + y^2 + 4z^2 \qquad P : (2, -1, 2) \tag{9.7.49}$$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 8z \end{bmatrix} \qquad \left[\nabla f\right]_P = -\begin{bmatrix} 4 \\ -2 \\ 16 \end{bmatrix} \tag{9.7.50}$$



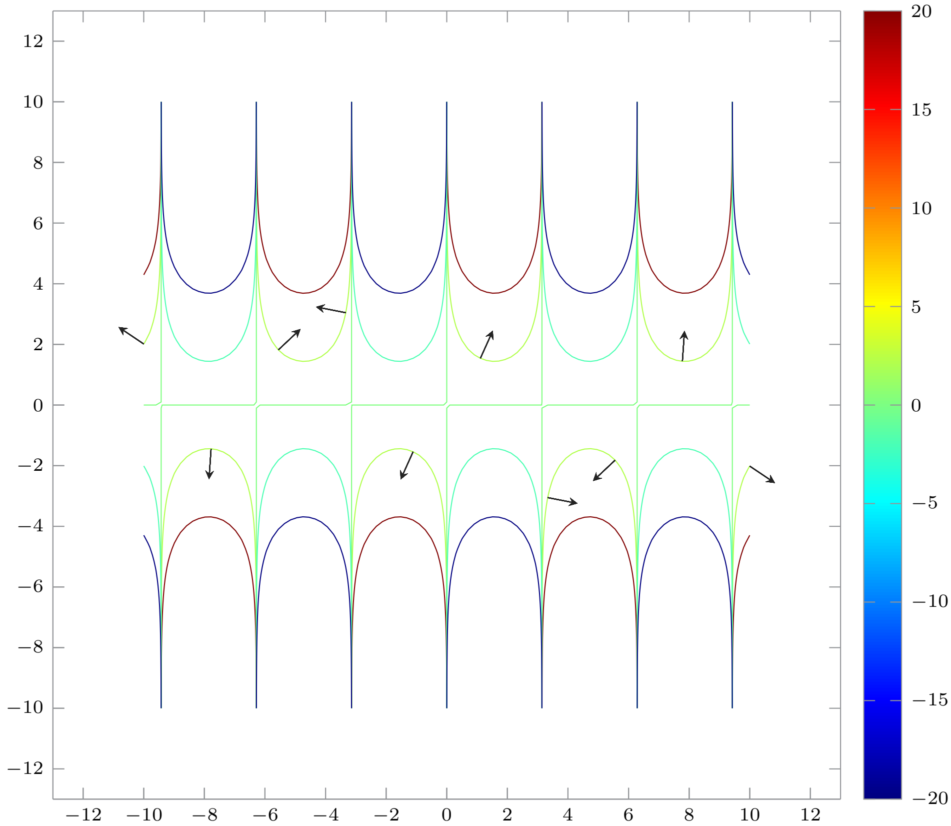
27. **(a)** Finding the gradient,

$$f = x^3 - 3xy^2 \qquad \nabla f = \begin{bmatrix} 3x^2 - 3y^2 \\ -6xy \end{bmatrix} \tag{9.7.51}$$



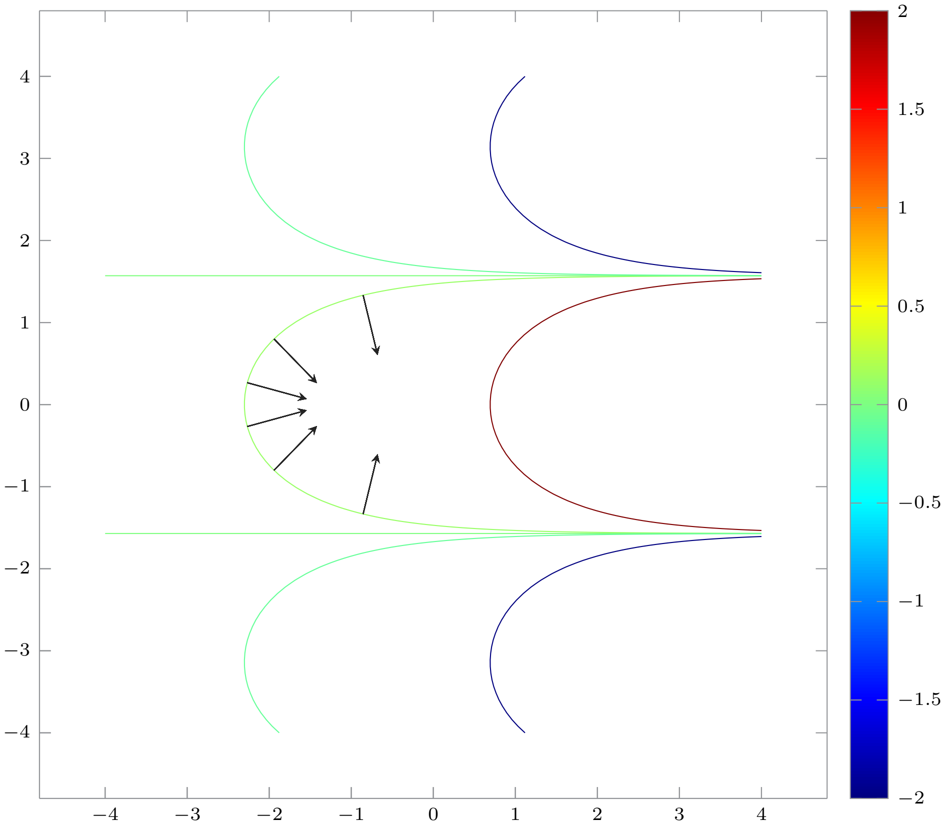
(b) Finding the gradient,

$$f = \sin(x)\sinh(y) \qquad \nabla f = \begin{bmatrix} \cos(x)\sinh(y) \\ \sin(x)\cosh(y) \end{bmatrix} \tag{9.7.52}$$



(c) Finding the gradient,

$$f = e^x \cos(y) \qquad \nabla f = \begin{bmatrix} e^x \cos(y) \\ -e^x \sin(y) \end{bmatrix} \tag{9.7.53}$$



28. Finding the gradient,

$$f = 3000 - x^2 - 9y^2 \qquad P : (4, 1) \tag{9.7.54}$$

$$\nabla f = \begin{bmatrix} -2x \\ -18y \end{bmatrix} \qquad \Big[\nabla f\Big]_P = \begin{bmatrix} -8 \\ -18 \end{bmatrix} \tag{9.7.55}$$

29. Absolute value of gradient is larger P than at Q for two points in the same scalar field. The magnitude of the gradient is the local rate of change (steepness). So, the scalar field has a larger rate of change around P than Q.

9.8 Divergence of a Vector Field

1. Finding the divergence,

$$\mathbf{v} = \begin{bmatrix} x^2 \\ 4y^2 \\ 9z^2 \end{bmatrix} \qquad P : (-1, 0, 0.5) \tag{9.8.1}$$

$$\nabla \cdot \mathbf{v} = 2x + 8y + 18z \qquad \Big[\nabla \cdot \mathbf{v}\Big]_P = 7 \tag{9.8.2}$$

2. Finding the divergence,

$$\mathbf{v} = \begin{bmatrix} 0 \\ \cos(xyz) \\ \sin(xyz) \end{bmatrix} \qquad P : (2, \pi/2, 0) \tag{9.8.3}$$

$$\nabla \cdot \mathbf{v} = 0 - xz\sin(xyz) + xy\cos(xyz) \qquad \Big[\nabla \cdot \mathbf{v}\Big]_P = \pi \tag{9.8.4}$$

3. Finding the divergence,

$$\mathbf{v} = \frac{1}{x^2 + y^2}\begin{bmatrix} x \\ y \end{bmatrix} \tag{9.8.5}$$

$$\nabla \cdot \mathbf{v} = \frac{x^2 - y^2}{(x^2 + y^2)^2}\,(-1 + 1) \qquad = 0 \tag{9.8.6}$$

Field is solenoidal

4. Finding the divergence,

$$\mathbf{v} = \begin{bmatrix} v_1(y, z) \\ v_2(z, x) \\ v_3(x, y) \end{bmatrix} \qquad P : (3, 1, -1) \tag{9.8.7}$$

$$\nabla \cdot \mathbf{v} = 0 + 0 + 0 = 0 \qquad \Big[\nabla \cdot \mathbf{v}\Big]_P = 0 \tag{9.8.8}$$

Field is solenoidal

5. Finding the divergence,

$$\mathbf{v} = x^2y^2z^2\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad P : (3, -1, 4) \tag{9.8.9}$$

$$\nabla \cdot \mathbf{v} = y^2z^2(3x^2) + x^2z^2(3y^2) + x^2y^2(3z^2) \qquad = (3xyz)^2 \tag{9.8.10}$$

$$\Big[\nabla \cdot \mathbf{v}\Big]_P = 1296 \tag{9.8.11}$$

6. Finding the divergence,

$$\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9.8.12}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{(x^2 + y^2 + z^2)^{5/2}}\left[(y^2 + z^2 - 2x^2) + (x^2 + z^2 - 2y^2) + (x^2 + y^2 - 2z^2)\right] \tag{9.8.13}$$

$$= 0 \tag{9.8.14}$$

Field is solenoidal

7. Finding v_3, given the field is solenoidal

$$\mathbf{v} = \begin{bmatrix} e^x\cos(y) \\ e^x\sin(y) \\ v_3 \end{bmatrix} \tag{9.8.15}$$

$$\nabla \cdot \mathbf{v} = e^x\cos(y) + e^x\cos(y) + \frac{\partial v_3}{\partial z} = 0 \tag{9.8.16}$$

$$\frac{\partial v_3}{\partial z} = -2e^x\cos(y) \tag{9.8.17}$$

$$z = (-2e^x\cos(y))\, z + f(x, y) \tag{9.8.18}$$

8. Finding the condition on v_3,

   (a) divergence is always positive

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ v_3 \end{bmatrix} \qquad \nabla \cdot \mathbf{v} = 1 + 1 + \frac{\partial v_3}{\partial z} > 0 \tag{9.8.19}$$

$$v_3 > -2z + c + f(x, y) \tag{9.8.20}$$

(b) divergence depends on $|z|$

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ v_3 \end{bmatrix} \qquad \nabla \cdot \mathbf{v} = 1 + 1 + \frac{\partial v_3}{\partial z} \tag{9.8.21}$$

$$v_3 = -z\left(1 + \frac{|z|}{2}\right) + c + f(x, y) \qquad \frac{\partial v_3}{\partial z} = -\,|z| - 1 \tag{9.8.22}$$

$$\nabla \cdot \mathbf{v} = 1 - |z| = \begin{cases} > 0 & |z| < 1 \\ < 0 & |z| > 1 \end{cases} \tag{9.8.23}$$

9. Proving the relations,

(a) Scalar multiplication,

$$\nabla \cdot (k\mathbf{v}) = \frac{\partial(kv_1)}{\partial x} + \frac{\partial(kv_2)}{\partial y} + \frac{\partial(kv_3)}{\partial z} \qquad = k\left[\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right] \tag{9.8.24}$$

$$= k\ \nabla \cdot \mathbf{v} \tag{9.8.25}$$

(b) Scalar function f

$$\nabla \cdot (f\mathbf{v}) = \frac{\partial(fv_1)}{\partial x} + \frac{\partial(fv_2)}{\partial y} + \frac{\partial(fv_3)}{\partial z} \tag{9.8.26}$$

$$= f\left[\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right] + v_1\frac{\partial f}{\partial x} + v_2\frac{\partial f}{\partial y} + v_3\frac{\partial f}{\partial z} \tag{9.8.27}$$

$$= f\ \nabla \cdot \mathbf{v} + \mathbf{v} \cdot (\nabla f) \tag{9.8.28}$$

Verifying this result,

$$f = \exp(xyz) \tag{9.8.29}$$

$$\mathbf{v} = \begin{bmatrix} ax \\ by \\ cz \end{bmatrix} \tag{9.8.30}$$

$$\nabla \cdot (f\mathbf{v}) = e^{xyz}[a + axyz + b + byzx + c + czxy] \tag{9.8.31}$$

$$= e^{xyz}[a + b + c] + ax(yz\ e^{xyz}) + by(xz\ e^{xyz}) + cz(xy\ e^{xyz}) \tag{9.8.32}$$

$$= f\ (\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot (\nabla f) \tag{9.8.33}$$

Solving problem 6 using this result,

$$f = (x^2 + y^2 + z^2)^{-3/2} \qquad \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9.8.34}$$

$$f\ (\nabla \cdot v) = 3f \qquad \mathbf{v} \cdot (\nabla f) = \frac{-3f\ (x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)} \tag{9.8.35}$$

$$\nabla \cdot (f\mathbf{v}) = 0 \tag{9.8.36}$$

(c) Scalar functions f, g

$$\nabla \cdot (f\nabla g) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right) \tag{9.8.37}$$

$$= f\left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right] + v_1\ \frac{\partial f}{\partial x} + v_2\ \frac{\partial f}{\partial y} + v_3\ \frac{\partial f}{\partial z} \tag{9.8.38}$$

$$= f\ (\nabla^2 g) + \nabla f \cdot \nabla g \tag{9.8.39}$$

Verifying the result for the given functions,

$$f = x^2 - y^2 \tag{9.8.40}$$

$$g = \exp(x + y) \tag{9.8.41}$$

$$f\ \nabla^2 g = (x^2 - y^2)\ [2\exp(x + y)] \tag{9.8.42}$$

$$\nabla f \cdot \nabla g = \exp(x + y)(2x - 2y) \tag{9.8.43}$$

$$f\ (\nabla g) = x^2 - y^2 \begin{bmatrix} \exp(x + y) \\ \exp(x + y) \end{bmatrix} \tag{9.8.44}$$

$$\nabla \cdot (f\ \nabla g) = \exp(x + y)[x^2 - y^2 + 2x + x^2 - y^2 - 2y] \tag{9.8.45}$$

$$= f\ (\nabla^2 g) + \nabla f \cdot \nabla g \tag{9.8.46}$$

(d) Using the result from part c,

$$\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\ (\nabla^2 g) + \nabla f \cdot \nabla g \tag{9.8.47}$$

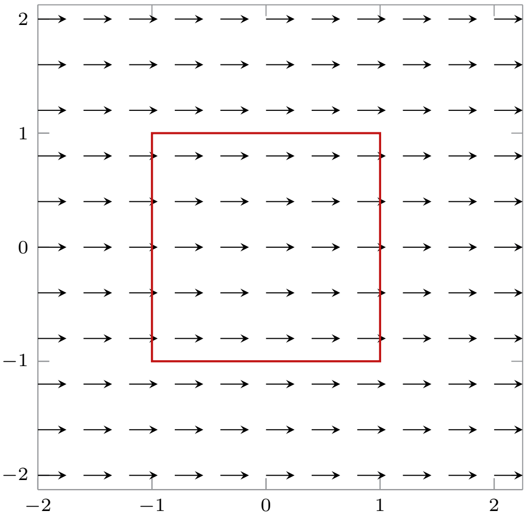
$$- g\ (\nabla^2 f) - \nabla g \cdot \nabla f \tag{9.8.48}$$

$$= f\ (\nabla^2 g) - g\ (\nabla^2 f) \tag{9.8.49}$$

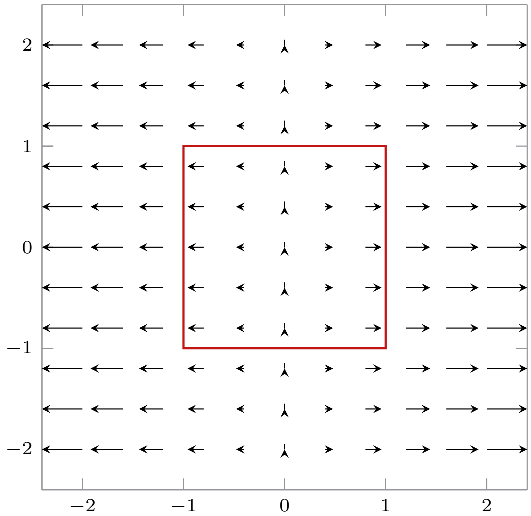
(e) Other examples TBC.

10. Graphing the velocity fields,

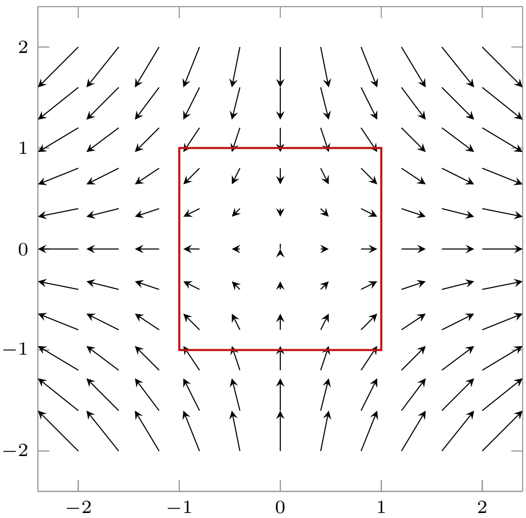
$\mathbf{v} = \hat{\mathbf{i}}$



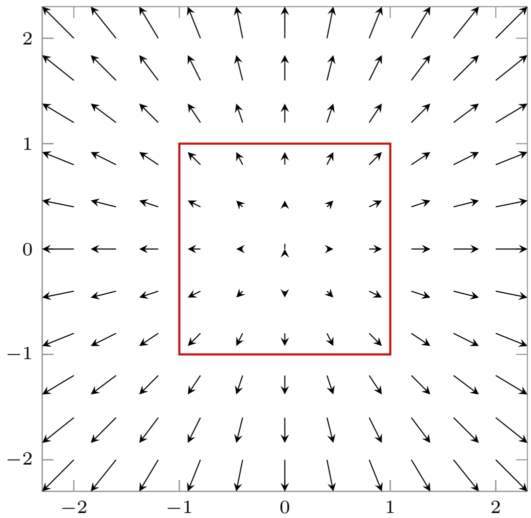
$\mathbf{v} = x\,\hat{\mathbf{i}}$

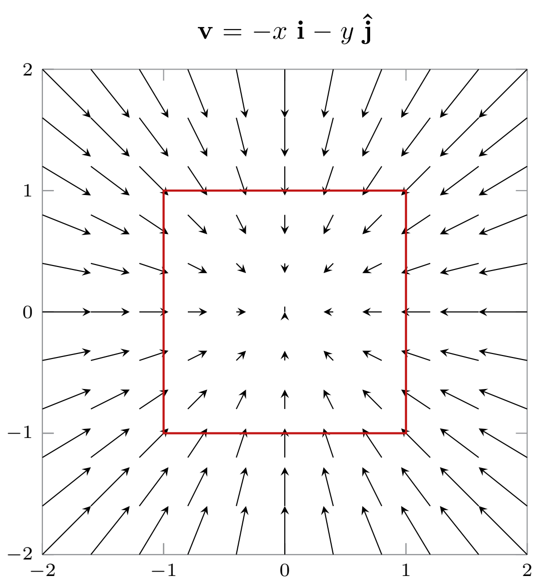


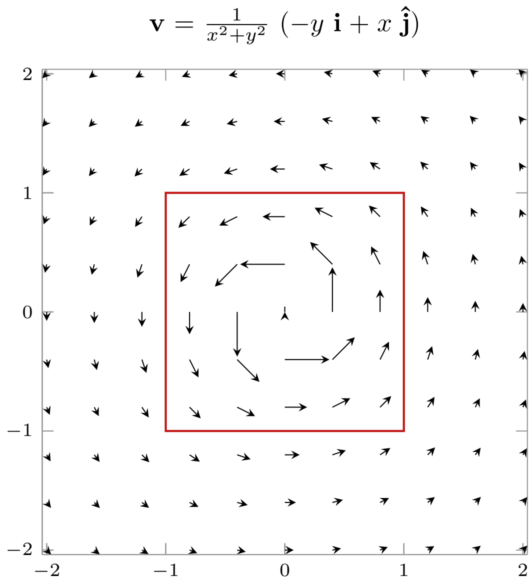
$\mathbf{v} = x\,\mathbf{i} - y\,\hat{\mathbf{j}}$



$\mathbf{v} = x\,\mathbf{i} + y\,\hat{\mathbf{j}}$







Flux from looking at the vector fields can be estimated to be either positive, negative or zero in the figures.

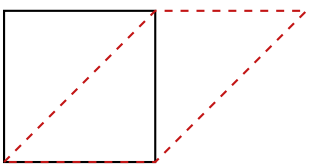
11. For incompressible flow,

$$\mathbf{v} = \mathbf{r}' = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} \qquad \nabla \cdot \mathbf{v} = 0 \tag{9.8.50}$$

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} x_0 + y_0 t \\ y_0 \\ z_0 \end{bmatrix} \tag{9.8.51}$$

Consider the cube as a stack of xz planes. The planes move in the x direction at different speeds depending on their initial y coordinate.

The points have no motion in the x or z directions. This means that each plane simply moves in the x direction by a different amount.



The area of a parallelogram made from deforming a square is the same as that square. This means that the volume of the fluid at $t = 1$ remains the same as the initial cube.

12. For compressible flow,

$$\mathbf{v} = \mathbf{r}' = \begin{bmatrix} x(t) \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 \exp(t) \\ c_2 \\ c_3 \end{bmatrix} \tag{9.8.52}$$

Consider the cube as a stack of yz planes. The planes move in the x direction at different speeds depending on their x coordinate.

The points have no motion in the y or z directions. This means that each plane simply moves in the x direction by a different amount.

For the slice initially at $x = 1$, $c_1 = 1$. This slice has moved to $x = e$ at time $t = 1$. Since the slice at $x = 0$ never moves, the volume is a cuboid of dimensions $e \times 1 \times 1$.

13. For rotational flow in a cylinder about the z axis, with constant angular velocity $\boldsymbol{\omega}$

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \begin{bmatrix} -\omega y \\ \omega x \\ 0 \end{bmatrix} \tag{9.8.53}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{9.8.54}$$

This flow is solenoidal. This is an incompressible flow in a circle centered around the origin, which has zero flux through its perimeter.

14. Considering only the first component,

$$\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \dots \qquad \nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x} + \dots \tag{9.8.55}$$

$$u_1 = v_1 + f(y, z) + c \tag{9.8.56}$$

There is very little constraint on $\mathbf{u}$ to satisfy this condition.

15. Using the definition of the Laplacian,

$$f = \cos^2(x) + \sin^2(y) \qquad \nabla f = \begin{bmatrix} -\sin(2x) \\ \sin(2y) \end{bmatrix} \tag{9.8.57}$$

$$\nabla \cdot (\nabla f) = -2\cos(2x) + 2\cos(2y) \tag{9.8.58}$$

$$\nabla^2 f = \frac{\partial}{\partial x}[-\sin(2x)] + \frac{\partial}{\partial y}[\sin(2y)] \qquad = -2\cos(2x) + 2\cos(2y) \tag{9.8.59}$$

16. Using the definition of the Laplacian,

$$f = \exp(xyz) \qquad \nabla f = e^{xyz}\begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} \tag{9.8.60}$$

$$\nabla \cdot (\nabla f) = e^{xyz}\Big[(yz)^2 + (xz)^2 + (xy)^2\Big] \tag{9.8.61}$$

$$\nabla^2 f = \frac{\partial}{\partial x}\Big[yz\ e^{xyz}\Big] + \frac{\partial}{\partial y}\Big[xz\ e^{xyz}\Big] + \frac{\partial}{\partial z}\Big[xy\ e^{xyz}\Big] \tag{9.8.62}$$

$$= e^{xyz}\Big[(yz)^2 + (xz)^2 + (xy)^2\Big] \tag{9.8.63}$$

17. Using the definition of the Laplacian,

$$f = \ln(x^2 + y^2) \tag{9.8.64}$$

$$\nabla f = \frac{1}{x^2 + y^2}\begin{bmatrix} 2x \\ 2y \end{bmatrix} \tag{9.8.65}$$

$$\nabla \cdot (\nabla f) = \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2} + \frac{2x^2 + 2y^2 - 4y^2}{(x^2 + y^2)^2} = 0 \tag{9.8.66}$$

$$\nabla^2 f = \frac{\partial}{\partial x}\left[\frac{2x}{x^2 + y^2}\right] + \frac{\partial}{\partial y}\left[\frac{2y}{x^2 + y^2}\right] \tag{9.8.67}$$

$$= \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2} + \frac{2x^2 + 2y^2 - 4y^2}{(x^2 + y^2)^2} = 0 \tag{9.8.68}$$

18. Using the definition of the Laplacian,

$$f = z - \sqrt{x^2 + y^2} = z - r \tag{9.8.69}$$

$$\nabla f = \begin{bmatrix} -x\ (x^2 + y^2)^{-1/2} \\ -y\ (x^2 + y^2)^{-1/2} \\ 1 \end{bmatrix} \tag{9.8.70}$$

$$\nabla \cdot (\nabla f) = \frac{-r^2 + x^2 - r^2 + y^2}{r^3} = \frac{-1}{\sqrt{x^2 + y^2}} \tag{9.8.71}$$

$$\nabla^2 f = \frac{\partial}{\partial x}\left[\frac{-x}{r}\right] + \frac{\partial}{\partial y}\left[\frac{-y}{r}\right] \tag{9.8.72}$$

$$= \frac{-r^2 + x^2}{r^3} + \frac{-r^2 + y^2}{r^3} = \frac{-1}{\sqrt{x^2 + y^2}} \tag{9.8.73}$$

19. Using the definition of the Laplacian,

$$f = \frac{1}{x^2 + y^2 + z^2} = \frac{1}{r^2} \tag{9.8.74}$$

$$\nabla f = \frac{-2}{r^4} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9.8.75}$$

$$\nabla \cdot (\nabla f) = \frac{-2r^2 + 8x^2}{r^6} + \frac{-2r^2 + 8y^2}{r^6} + \frac{-2r^2 + 8z^2}{r^6} = \frac{2}{(x^2 + y^2 + z^2)^2} \tag{9.8.76}$$

$$\nabla^2 f = \frac{\partial}{\partial x}\left[\frac{-2x}{r^4}\right] + \frac{\partial}{\partial y}\left[\frac{-2y}{r^4}\right] + \frac{\partial}{\partial z}\left[\frac{-2z}{r^4}\right] \tag{9.8.77}$$

$$= \frac{-2r^2 + 8x^2}{r^6} + \frac{-2r^2 + 8y^2}{r^6} + \frac{-2r^2 + 8z^2}{r^6} = \frac{2}{(x^2 + y^2 + z^2)^2} \tag{9.8.78}$$

20. Using the definition of the Laplacian,

$$f = e^{2x} \cosh(2y) \tag{9.8.79}$$

$$\nabla f = \begin{bmatrix} 2e^{2x} \cosh(2y) \\ 2e^{2x} \sinh(2y) \end{bmatrix} \tag{9.8.80}$$

$$\nabla \cdot (\nabla f) = 4e^{2x} \cosh(2y) + 4e^{2x} \cosh(2y) = 8e^{2x} \cosh(2y) \tag{9.8.81}$$

$$\nabla^2 f = 8e^{2x} \cosh(2y) \tag{9.8.82}$$

9.9 Curl of a Vector Field

1. Refer notes. TBC.

2. Direction of the curl

(a) Vector is parallel to the yz plane. This means its x component is zero.

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix} \partial v_3/\partial y - \partial v_2/\partial z \\ -\partial v_3/\partial x \\ \partial v_2/\partial x \end{bmatrix} \tag{9.9.1}$$

(b) If $\mathbf{v}$ is also independent of x, then its curl is perpendicular to the yz plane.

$$\nabla \times \mathbf{v} = \begin{bmatrix} \partial v_3/\partial y - \partial v_2/\partial z \\ 0 \\ 0 \end{bmatrix} \tag{9.9.2}$$

3. Proving Theorem 2,

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix} \tag{9.9.3}$$

$$= \left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right] \hat{\mathbf{i}} + \dots \tag{9.9.4}$$

$$= 0 \tag{9.9.5}$$

The other two components are also zero.

$$\nabla \cdot (\nabla \times f) = \frac{\partial}{\partial x}\left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right] + \dots \tag{9.9.6}$$

$$= 0 \tag{9.9.7}$$

There are 6 total terms which cancel out using the commutativity of partial differentiation. Examples TBC.

4. Finding the curl,

$$\mathbf{v} = \begin{bmatrix} 2y^2 \\ 5x \\ 0 \end{bmatrix} \qquad \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 & 5x & 0 \end{vmatrix} \tag{9.9.8}$$

$$\nabla \times \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 5 - 4y \end{bmatrix} \tag{9.9.9}$$

5. Finding the curl,

$$\mathbf{v} = xyz\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2(yz) & y^2(xz) & z^2(xy) \end{vmatrix} \tag{9.9.10}$$

$$\nabla\times\mathbf{v} = \begin{bmatrix} x(z^2-y^2) \\ y(x^2-z^2) \\ z(y^2-x^2) \end{bmatrix} \tag{9.9.11}$$

6. Finding the curl,

$$\mathbf{v} = \frac{1}{(x^2+y^2+z^2)^{3/2}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/r^3 & y/r^3 & z/r^3 \end{vmatrix} \tag{9.9.12}$$

$$\nabla\times\mathbf{v} = \frac{1}{r^5}\begin{bmatrix} -3zy+3yz \\ 3zx-3xz \\ -3xy+3yx \end{bmatrix} \qquad = \mathbf{0} \tag{9.9.13}$$

This vector field is irrotational.

7. Finding the curl,

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ e^{-x}\,\sin(y) \end{bmatrix} \qquad \nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & e^{-x}\,\sin(y) \end{vmatrix} \tag{9.9.14}$$

$$\nabla\times\mathbf{v} = \begin{bmatrix} e^{-x}\,\cos(y) \\ e^{-x}\,\sin(y) \\ 0 \end{bmatrix} \tag{9.9.15}$$

8. Finding the curl,

$$\mathbf{v} = \begin{bmatrix} e^{-z^2} \\ e^{-x^2} \\ e^{-y^2} \end{bmatrix} \qquad \nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{-z^2} & e^{-x^2} & e^{-y^2} \end{vmatrix} \tag{9.9.16}$$

$$\nabla\times\mathbf{v} = \begin{bmatrix} -2y\;e^{-y^2} \\ -2z\;e^{-z^2} \\ -2x\;e^{-x^2} \end{bmatrix} \tag{9.9.17}$$

9. Checking the given fluid flow,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 3z^2 & 0 \end{vmatrix} = \begin{bmatrix} -6z \\ 0 \\ 0 \end{bmatrix} \tag{9.9.18}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{9.9.19}$$

The field is incompressible.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \begin{bmatrix} 0 \\ 3z^2 \\ 0 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} c_1 \\ 3z_0^2\ t + c_2 \\ c_3 \end{bmatrix} \tag{9.9.20}$$

10. Checking the given fluid flow,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sec x & \csc x & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\cot x \csc x \end{bmatrix} \tag{9.9.21}$$

$$\nabla \cdot \mathbf{v} = \tan x \sec x \tag{9.9.22}$$

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \begin{bmatrix} \sec x \\ \csc x \\ 0 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} \arcsin(t + c_1) \\ \ln(t + c_1) + c_2 \\ c_3 \end{bmatrix} \tag{9.9.23}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sec x \qquad x = \arcsin(t + c_1) \tag{9.9.24}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \csc x \qquad \mathrm{d}y = \frac{\mathrm{d}t}{t + c_1} \tag{9.9.25}$$

$$y = \ln(t + c_1) + c_2 \qquad z = c_3 \tag{9.9.26}$$

11. Checking the given fluid flow,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -2x & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \tag{9.9.27}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{9.9.28}$$

The field is incompressible

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \begin{bmatrix} y \\ -2x \\ 0 \end{bmatrix} \tag{9.9.29}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -2x \tag{9.9.30}$$

$$x = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) \qquad y = \sqrt{2}c_1 \cos(\sqrt{2}t) - \sqrt{2}c_2 \sin(\sqrt{2}t) \tag{9.9.31}$$

$$z = c_3 \tag{9.9.32}$$

12. Checking the given fluid flow,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & \pi \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \tag{9.9.33}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{9.9.34}$$

The field is incompressible

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \begin{bmatrix} -y \\ x \\ \pi \end{bmatrix} \tag{9.9.35}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -2x \tag{9.9.36}$$

$$x = c_1 \cos(t) + c_2 \sin(t) \qquad y = -c_2 \cos(t) + c_1 \sin(t) \tag{9.9.37}$$

$$z = \pi t + z_0 \tag{9.9.38}$$

13. Checking the given fluid flow,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & -z \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9.9.39}$$

$$\nabla \cdot \mathbf{v} = 1 \tag{9.9.40}$$

The field is irrotational

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \begin{bmatrix} x \\ y \\ -z \end{bmatrix} \tag{9.9.41}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \qquad x = c_1 e^t \tag{9.9.42}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y \qquad y = c_2 e^t \tag{9.9.43}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -z \qquad z = c_3 e^{-t} \tag{9.9.44}$$

14. Proving the relations,

(a) Using the linearity of differentiation, or the properties of deterinant addition

$$\nabla \times (\mathbf{u} + \mathbf{v}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 + u_1 & v_2 + u_2 & v_3 + u_3 \end{vmatrix} \tag{9.9.45}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_1 & u_2 & u_3 \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{9.9.46}$$

$$= \nabla \times \mathbf{u} + \nabla \times \mathbf{v} \tag{9.9.47}$$

(b) Calculating,

$$\nabla \cdot (\nabla \times \mathbf{v}) = \frac{\partial}{\partial x}\left[\partial_y v_3 - \partial_z v_2\right] + \frac{\partial}{\partial y}\left[\partial_z v_1 - \partial_x v_3\right] + \frac{\partial}{\partial z}\left[\partial_x v_2 - \partial_y v_1\right] \tag{9.9.48}$$

$$= \left[\partial_x \partial_y v_3 - \partial_x \partial_z v_2\right] + \left[\partial_y \partial_z v_1 - \partial_y \partial_x v_3\right] + \left[\partial_z \partial_x v_2 - \partial_z \partial_y v_1\right] \tag{9.9.49}$$

$$= 0 \tag{9.9.50}$$

(c) Calculating,

$$\nabla \times (f\mathbf{v}) = \left[\frac{\partial(fv_3)}{\partial y} - \frac{\partial(fv_2)}{\partial z}\right]\hat{\mathbf{i}} + \left[\frac{\partial(fv_1)}{\partial z} - \frac{\partial(fv_3)}{\partial x}\right]\hat{\mathbf{j}} \tag{9.9.51}$$

$$+ \left[\frac{\partial(fv_2)}{\partial x} - \frac{\partial(fv_1)}{\partial y}\right]\hat{\mathbf{k}} \tag{9.9.52}$$

$$= f(\nabla \times \mathbf{v}) + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x f & \partial_y f & \partial_z f \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{9.9.53}$$

$$= f(\nabla \times \mathbf{v}) + (\nabla f \times \mathbf{v}) \tag{9.9.54}$$

(d) Calculating,

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} \tag{9.9.55}$$

$$= \left[\partial_y \partial_z f - \partial_z \partial_y f\right]\hat{\mathbf{i}} + \cdots = \mathbf{0} \tag{9.9.56}$$

The other components also cancel resulting in the zero vector.

(e) Calculating,

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \frac{\partial}{\partial x}\Big[a_2 b_3 - a_3 b_2\Big] + \frac{\partial}{\partial y}\Big[a_3 b_1 - a_1 b_3\Big] + \frac{\partial}{\partial z}\Big[a_1 b_2 - a_2 b_1\Big] \tag{9.9.57}$$

$$= -a_1\left[\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right] - a_2\left[\frac{\partial b_1}{\partial z} - \frac{\partial b_3}{\partial x}\right] - a_3\left[\frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y}\right] \tag{9.9.58}$$

$$+ b_1\left[\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right] + b_2\left[\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right] + b_3\left[\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right] \tag{9.9.59}$$

$$= \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{9.9.60}$$

15. Calculating,

$$\nabla \times (\mathbf{u}+\mathbf{v}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ y+yz & z+zx & x+xy \end{vmatrix} = \begin{bmatrix} x-1-x \\ -1-y+y \\ z-1-z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \tag{9.9.61}$$

$$\nabla \times \mathbf{u} + \nabla \times \mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} x-x \\ y-y \\ z-z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \tag{9.9.62}$$

As seen above, $\nabla \times \mathbf{v} = \mathbf{0}$

16. Calculating,

$$\nabla \times (g\mathbf{v}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ (x+y+z)(yz) & (x+y+z)(zx) & (x+y+z)(xy) \end{vmatrix} \tag{9.9.63}$$

$$= \begin{bmatrix} (x+2y+z-x-y-2z)x \\ (x+y+2z-2x-y-z)y \\ (2x+y+z-x-2y-z)z \end{bmatrix} = \begin{bmatrix} (y-z)x \\ (z-x)y \\ (x-y)z \end{bmatrix} \tag{9.9.64}$$

$$\nabla g \times \mathbf{v} + g(\nabla \times \mathbf{v}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ yz & zx & xy \end{vmatrix} + g \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ yz & zx & xy \end{vmatrix} \tag{9.9.65}$$

$$= \begin{bmatrix} (y-z)x \\ (z-x)y \\ (x-y)z \end{bmatrix} + \begin{bmatrix} x-x \\ -y+y \\ z-z \end{bmatrix} = \begin{bmatrix} (y-z)x \\ (z-x)y \\ (x-y)z \end{bmatrix} \tag{9.9.66}$$

17. Calculating,

$$\mathbf{v}\cdot(\nabla\times\mathbf{u}) = \mathbf{v}\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ y & z & x\end{vmatrix} = -yz - zx - xy \tag{9.9.67}$$

$$\mathbf{u}\cdot(\nabla\times\mathbf{v}) = \mathbf{u}\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ yz & zx & xy\end{vmatrix} = y(0) + z(0) + x(0) = 0 \tag{9.9.68}$$

$$\mathbf{u}\cdot(\nabla\times\mathbf{u}) = \mathbf{u}\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ y & z & x\end{vmatrix} = -(x+y+z) \tag{9.9.69}$$

18. Calculating,

$$\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \nabla\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ y & z & x\\ yz & zx & xy\end{vmatrix} = -(xy+yz+zx) \tag{9.9.70}$$

$$\tag{9.9.71}$$

From Problem 17,

$$\mathbf{v}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{v}) = -(xy+yz+zx) \tag{9.9.72}$$

19. Calculating,

$$\nabla\times(g\mathbf{u}+\mathbf{v}) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ (x+y+2z)y & (2x+y+z)z & (x+2y+z)x\end{vmatrix} \tag{9.9.73}$$

$$= \begin{bmatrix}2x-2x-y-2z\\ -2x-2y-z+2y\\ 2z-x-2z-2y\end{bmatrix} = \begin{bmatrix}-y-2z\\ -z-2x\\ -x-2y\end{bmatrix} \tag{9.9.74}$$

Using the formulae,

$$\nabla g \times \mathbf{u} + g(\nabla \times \mathbf{u}) + \nabla \times v = \begin{bmatrix} x - z \\ y - x \\ z - y \end{bmatrix} + (x + y + z) \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9.9.75}$$

$$= \begin{bmatrix} -y - 2z \\ -z - 2x \\ -x - 2y \end{bmatrix} \tag{9.9.76}$$

The calculation of $\nabla \times (g\mathbf{u})$ is part of the above process.

20. Calculating,

$$\nabla(fg) = \begin{bmatrix} (2x + y + z)(yz) \\ (x + 2y + z)(zx) \\ (x + y + 2z)(xy) \end{bmatrix} \tag{9.9.77}$$

$$\nabla \cdot [\nabla(fg)] = \begin{bmatrix} 2yz \\ 2zx \\ 2xy \end{bmatrix} \tag{9.9.78}$$