Chapter 18

Complex Analysis and Potential Theory

18.1 Electrostatic Fields

1. Using the standard result,

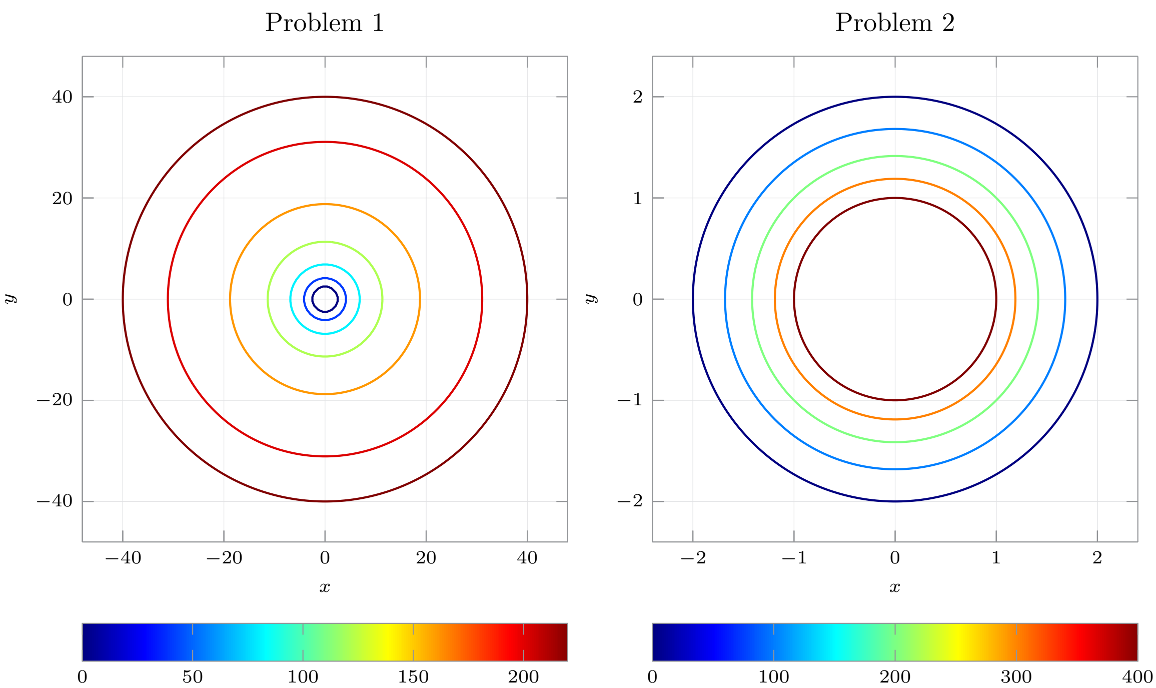
$$\Phi = a \ln r + b \qquad \Phi(2.5) = 0, \qquad \Phi(40) = 220 \tag{18.1.1}$$

$$\Phi = 79.35 * \ln r - 72.7 \tag{18.1.2}$$

2. Using the standard result,

$$\Phi = a \ln r + b \qquad \Phi(1) = 400, \qquad \Phi(2) = 0 \tag{18.1.3}$$

$$\Phi = -577.08 * \ln r + 400 \tag{18.1.4}$$



3. Using the standard result,

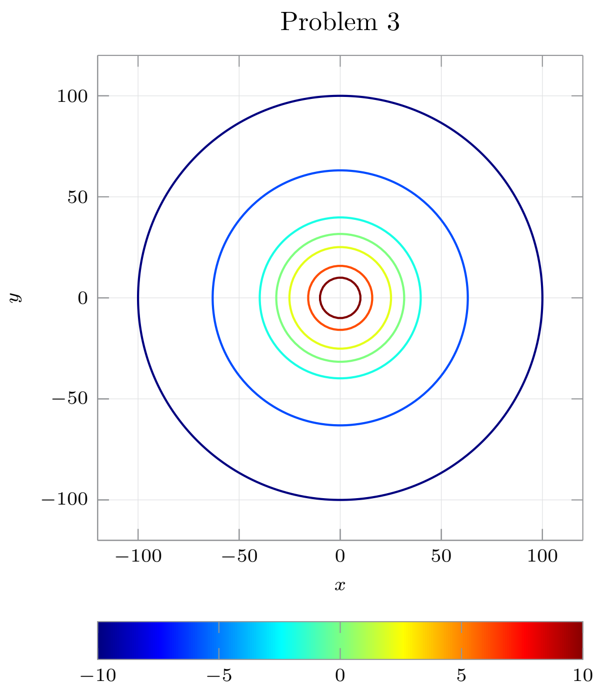
$$\Phi = a \ln r + b \qquad \Phi(10) = 10, \qquad \Phi(100) = -10 \tag{18.1.5}$$

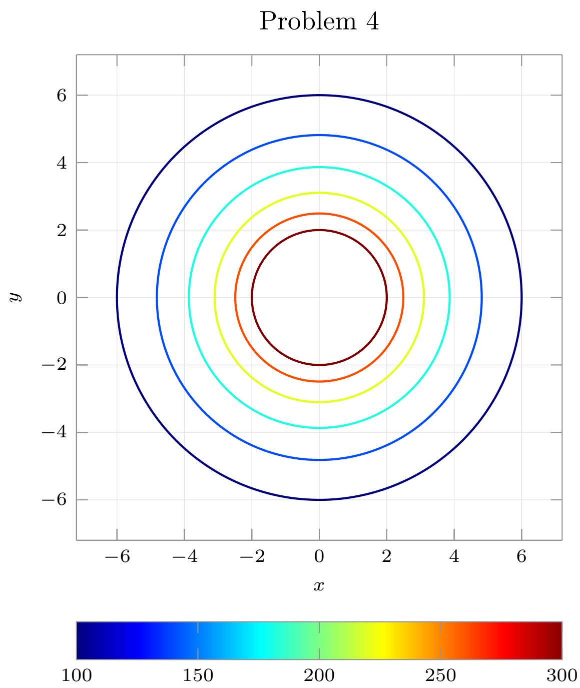
$$\Phi = 79.35 * \ln r - 72.7 \tag{18.1.6}$$

4. Using the standard result,

$$\Phi = a \ln r + b \qquad \Phi(2) = 300, \qquad \Phi(6) = 100 \tag{18.1.7}$$

$$\Phi = -577.08 * \ln r + 400 \qquad \Phi(4) = 173.8 < 200 \tag{18.1.8}$$





Since the logarithm falls faster at the beginning and slower later on, the potential is less than the midway value at the midway radius.

5. Using the standard result,

$$\Phi = ax + b \qquad \Phi(-5) = 250, \qquad \Phi(5) = 500 \tag{18.1.9}$$

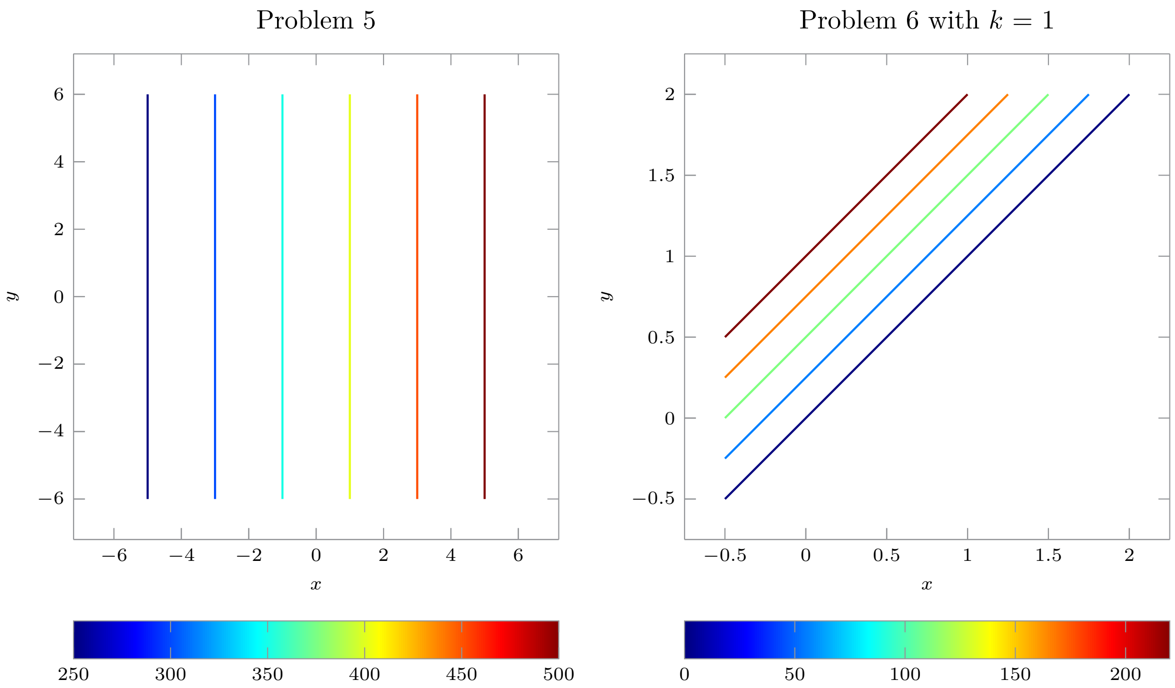
$$\Phi = 25x + 375 \qquad \Psi = 25y \tag{18.1.10}$$

6. Using the standard result,

$$\Phi = ax + b \qquad \Phi(y = x) = 0, \qquad \Phi(y = x + k) = 220 \tag{18.1.11}$$

$$\Phi = a(y - x) + b \qquad \Phi = \frac{220}{k}\,(y - x) \tag{18.1.12}$$

$$\Psi = \frac{-200}{k}\,(x + y) \tag{18.1.13}$$

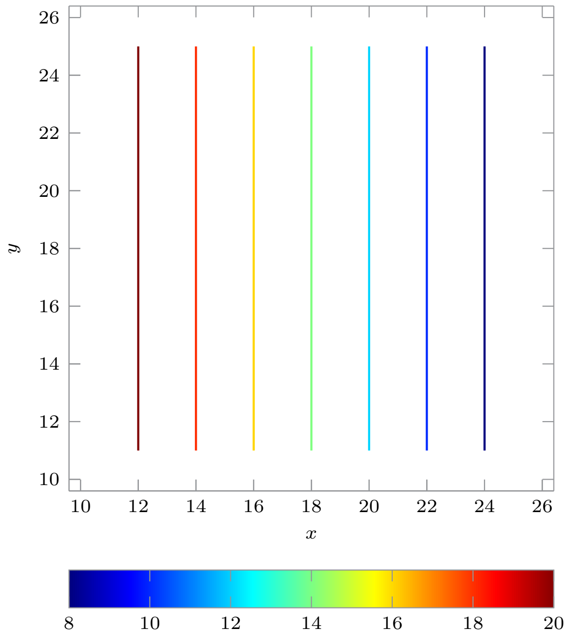


7. Using the standard result,

$$\Phi = ax + b \qquad \Phi(12) = 20, \qquad \Phi(24) = 8 \tag{18.1.14}$$

$$\Phi = -x + 32 \qquad \Psi = -y \tag{18.1.15}$$

Problem 7



8. Plotting the equipotential lines and the lines of force within the same plot,

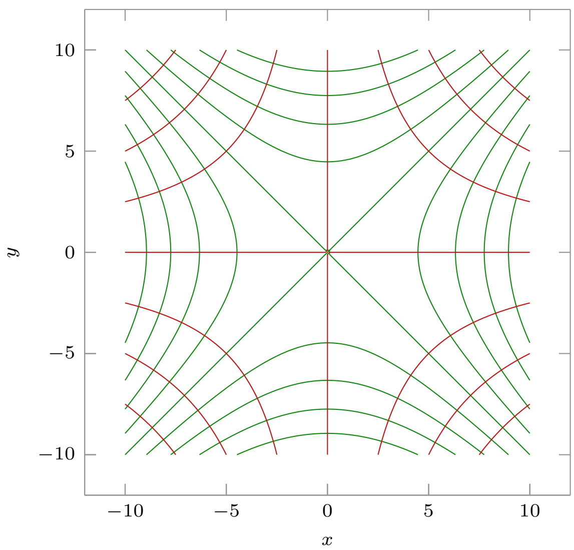
(a) Expanding the function,

$$F(z) = z^2 \qquad\qquad F(z) = (x^2 - y^2) + (2xy)\,\mathfrak{i} \tag{18.1.16}$$

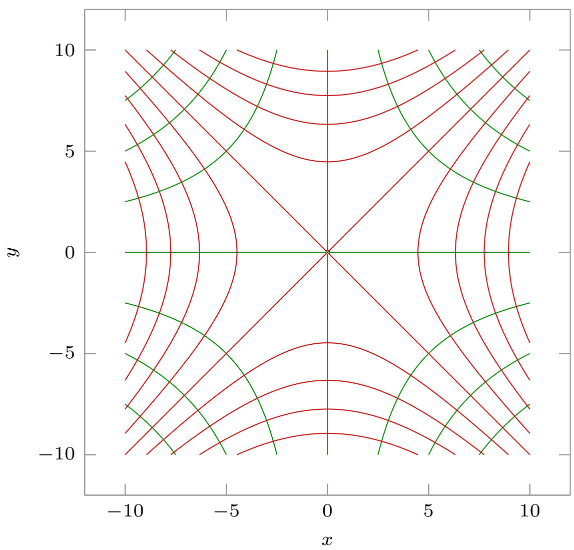
(b) Expanding the function,

$$F(z) = \mathfrak{i}z^2 \qquad\qquad F(z) = -2xy + \mathfrak{i}\,(x^2 - y^2) \tag{18.1.17}$$

$\Phi(x, y)$ and $\Psi(x, y)$



$\Phi(x, y)$ and $\Psi(x, y)$

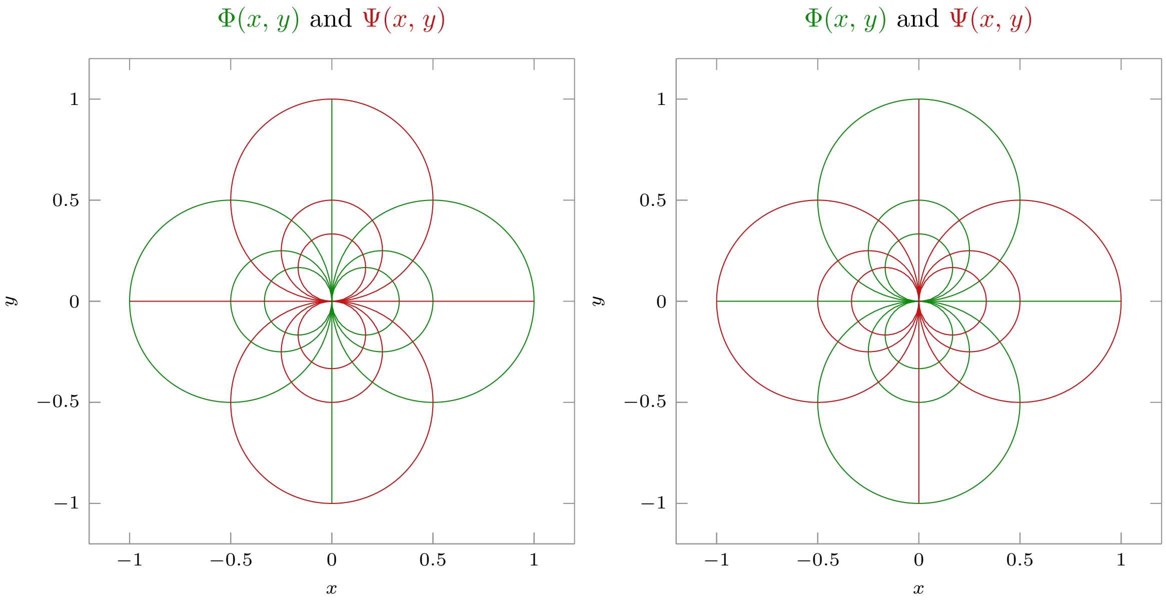


(c) Expanding the function,

$$F(z) = \frac{1}{z} \qquad\qquad F(z) = \frac{x - i\,y}{x^2 + y^2} \tag{18.1.18}$$

(d) Expanding the function,

$$F(z) = \frac{i}{z} \qquad\qquad F(z) = \frac{y + ix}{x^2 + y^2} \tag{18.1.19}$$



9. Checking that the given function is harmonic,

$$\Phi(r, \theta) = \frac{\theta}{\pi} \qquad\qquad r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \tag{18.1.20}$$

The function satisfies Laplace's equation in polar coordinates and is thus harmonic. On the positive and negative real axis, the potential is equal to,

$$\Phi(x > 0, y = 0) = \frac{0}{\pi} = 0 \qquad\qquad \Phi(x < 0, y = 0) = \frac{\pi}{\pi} = 1 \tag{18.1.21}$$

Using the Euler-Cauchy equations in Cartesian coordinates,

$$\Phi_x = \frac{1}{\pi} \cdot \frac{-y}{x^2+y^2} = \Psi_y \qquad -\Phi_y = \frac{1}{\pi} \cdot \frac{-x}{x^2+y^2} = \Psi_x \tag{18.1.22}$$

$$\Psi = \frac{-1}{2\pi} \cdot \ln(x^2+y^2) + f(x) \qquad \Psi = \frac{-1}{2\pi} \cdot \ln(x^2+y^2) + g(y) \tag{18.1.23}$$

$$F(z) = \frac{\theta}{\pi} - \mathrm{i}\,\frac{\ln r}{\pi} \qquad = \frac{-\mathrm{i}}{\pi}\left[\ln r + \mathrm{i}\,\theta\right] \tag{18.1.24}$$

$$F(z) = \frac{-\mathrm{i}\ \mathrm{Ln}\, z}{\pi} \tag{18.1.25}$$

10. The points to be mapped are $(0 \to 1)$, $(\infty \to \mathrm{i})$, $(-1 \to -\mathrm{i})$

$$\frac{w-1}{w+\mathrm{i}} \cdot \left[\frac{2\mathrm{i}}{\mathrm{i}-1}\right] = \frac{z}{z+1} \cdot \left[\frac{\infty+1}{\infty}\right] \qquad \frac{w-1}{w+\mathrm{i}} = \frac{(1+\mathrm{i})z}{2z+2} \tag{18.1.26}$$

$$w(z+2-z\mathrm{i}) = \mathrm{i}z + z + 2 \qquad w = \frac{\mathrm{i}z+1+\mathrm{i}}{z+1+\mathrm{i}} \tag{18.1.27}$$

Using the potential in Problem 9,

$$\Phi(z) = \frac{\theta}{\pi} \qquad |w| = 1 \implies |z+1+\mathrm{i}| = |z+1-\mathrm{i}| \tag{18.1.28}$$

$$\mathrm{Re}\, z > 0 \implies \Phi(z) = 0 \qquad \mathrm{Re}\, z < 0 \implies \Phi(z) = 1 \tag{18.1.29}$$

$$\Phi(0) = 0 \qquad \Phi(\infty) = 0, \qquad \Phi(-1) = \frac{-\pi}{-\pi} = 1 \tag{18.1.30}$$

$$0 = \frac{\mathrm{i}z^* = \mathrm{i}+1}{z+1+\mathrm{i}} \qquad z^* = -1+\mathrm{i} \tag{18.1.31}$$

$$\Phi(-1+\mathrm{i}) = \frac{\mathrm{Arg}(\mathrm{i}+1)}{\pi} = \frac{1}{4} \tag{18.1.32}$$

11. Using brute force,

$$|z-c|^2 = k^2\,|z+c|^2 \qquad (x-c)^2 + y^2 = k^2\Big[(x+c)^2+y^2\Big] \tag{18.1.33}$$

$$x^2+y^2+c^2 = \frac{1+k^2}{1-k^2}\,2cx \qquad \lambda = \frac{1+k^2}{1-k^2} \tag{18.1.34}$$

$$(x-\lambda c)^2 + y^2 = (\lambda^2-1)c^2 \tag{18.1.35}$$

This is the equation of a circle.

12. Finding the potential,

$$\Phi(x,y) = -100\,xy + 500 \tag{18.1.36}$$

13. Finding the potential,

$$\arccos(z) = w \qquad z = \cos w \tag{18.1.37}$$

$$z = x + iy \qquad w = u + iv \tag{18.1.38}$$

$$z = \cos u \cosh v - i \ \sin u \sinh v \qquad w = \Phi + i \ \Psi \tag{18.1.39}$$

The equipotential lines are, $\Phi = c$,

$$\frac{x^2}{\cos^2 u} - \frac{y^2}{\sin^2 u} = 1 \qquad \text{Hyperbolae} \tag{18.1.40}$$

The foci of these hyperbolae are $c^2 = \cos^2 u + \sin^2 u = 1$, giving $c = \pm 1$.

14. Since the line $x = 0$ is one of the equipotential lines in the potential plot of Problem 13, This vertical line can be replaced by another equipotential surface without changing the nature of the rest of the lines in the right-half plane.

15. The pair of straight lines is

$$y = \pm \frac{x}{\sqrt{3}} \qquad x^3 - 3xy^2 = 0 \tag{18.1.41}$$

This means that the potential itself is of the form,

$$\Phi(x, y) = a(x^3 - 3xy^2) + b \qquad \Phi(C_1) = 0, \qquad \Phi(C_2) = 220 \tag{18.1.42}$$

$$\Phi = 220x(3y^2 - x^2) \qquad z = x + iy \tag{18.1.43}$$

$$z^3 = x(x^2 - 3y^2) + i \ y(3x^2 - y^2) \qquad \Phi = \text{Re } z^3 \implies \Psi = \text{Im } z^3 \tag{18.1.44}$$

$$F(z) = 220 \ z^3 \tag{18.1.45}$$

18.2 Use of Conformal Mapping

1. Verifying the derivation,

$$f(z) = \frac{z - z_0}{\overline{z_0} z - 1} \qquad z_0 = b + 0i \tag{18.2.1}$$

$$f(0) = r_0 \qquad f(0.8) = -r_0 \tag{18.2.2}$$

$$r_0 = \frac{0 - b}{0 - 1} = b \qquad -r_0 = \frac{0.8 - b}{0.8b - 1} \tag{18.2.3}$$

$$b^2 - 2.5b + 1 = 0 \qquad b^* = \{2, 0.5\} \tag{18.2.4}$$

Since the disk mapping formula requires $|z_0| < 1$, $b^* \neq 2$.

$$b = 0.5 \qquad\qquad f(z) = \frac{2z - 1}{z - 2} \tag{18.2.5}$$

2. The proof in the appendix which does not use the harmonic conjugate of Φ is,

$$\Phi = \Phi^* \Big[u(x, y), \ v(x, y)\Big] \tag{18.2.6}$$

$$\Phi_x = \Phi^*_u \ u_x + \Phi^*_v \ v_x \tag{18.2.7}$$

$$\Phi_{xx} = \Phi^*_u \ u_{xx} + u_x \ (\Phi^*_{uu} \ u_x + \Phi^*_{uv} \ v_x) + \Phi^*_v \ v_{xx} + v_x \ (\Phi^*_{uv} \ u_x + \Phi^*_{vv} \ v_x) \tag{18.2.8}$$

$$\Phi_{xx} + \Phi_{yy} = \Phi^*_u \ \nabla^2 u + \Phi^*_v \ \nabla^2 v + 2\Phi^*_{uv} \ (u_x v_x + u_y v_y) \tag{18.2.9}$$

$$+ \Phi^*_{uu} \ (u_x^2 + u_y^2) + \Phi^*_{vv} \ (v_x^2 + v_y^2) \tag{18.2.10}$$

Since $w = u + \mathfrak{i} \ v$ is analytic, u, v satisfy the Cauchy-Riemann relations. Additionally, u, v are themselves harmonic.

$$\Phi_{xx} + \Phi_{yy} = \Phi^*_{uu} \ (u_x^2 + u_y^2) + \Phi^*_{vv} \ (v_x^2 + v_y^2) \tag{18.2.11}$$

$$= (\Phi^*_{uu} + \Phi^*_{vv})(v_y^2 + v_x^2) \tag{18.2.12}$$

$$\nabla^2 \Phi = \nabla^2 \Phi^* = 0 \tag{18.2.13}$$

3. Trying out a conformal map,

$$f(z) = -\mathfrak{i} z^2 \qquad\qquad f(z) = 2xy + \mathfrak{i} \ (y^2 - x^2) \tag{18.2.14}$$

$$xy \in [0, 1] \qquad \Longrightarrow \qquad u \in [0, 2] \tag{18.2.15}$$

$$\Phi^* = u + \mathfrak{i} \ v \qquad\qquad \Phi^*(0, v) = K_1, \quad \Phi^*(2, v) = K_2 \tag{18.2.16}$$

$$\Phi^*(u, v) = K_1 + \left(\frac{K_2 - K_1}{2}\right) u \tag{18.2.17}$$

Back-converting $\Phi^*(u, v)$ into $\Phi(x, y)$,

$$\Phi(x, y) = K_1 + (K_2 - K_1) \ xy \tag{18.2.18}$$

4. For the given function,,

$$\Phi^*(u,\, v) = 4uv \qquad\qquad w = e^z = u + \mathrm{i}\ v \tag{18.2.19}$$

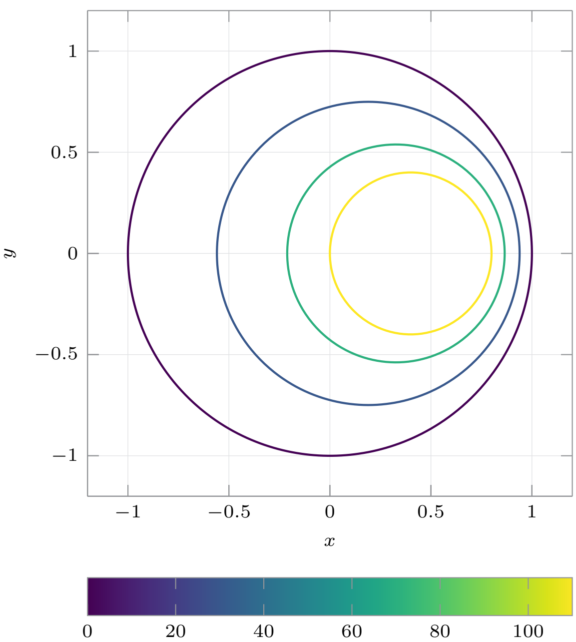
$$w = e^x \cos y + \mathrm{i}\ e^x \sin y \qquad\qquad \Phi(x,\, y) = 2e^{2x}\ \sin(2y) \tag{18.2.20}$$

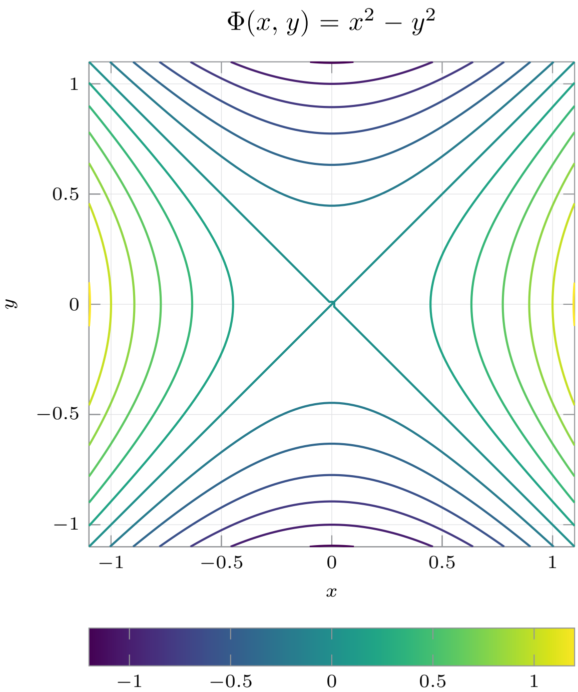
$$\Phi(x,\, 0) = 0, \quad \Phi(x,\, \pi) = 0 \qquad\qquad \Phi(0,\, y) = 2\sin(2y) \tag{18.2.21}$$

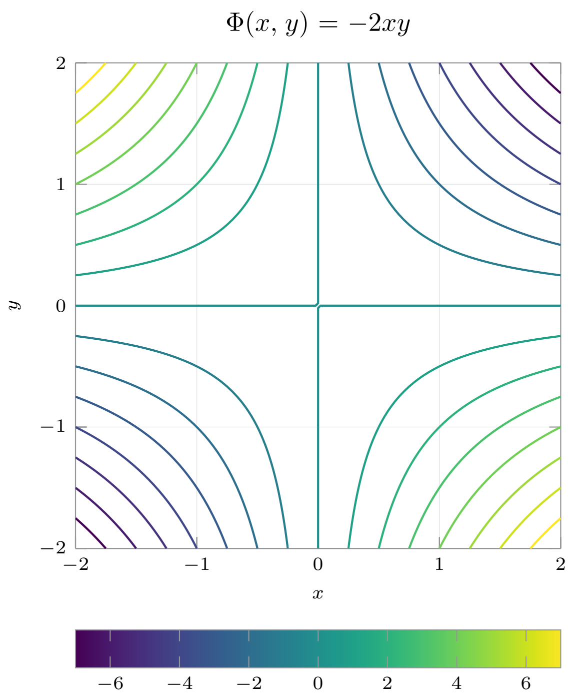
5. Plotting the equipotential lines and the lines of force,

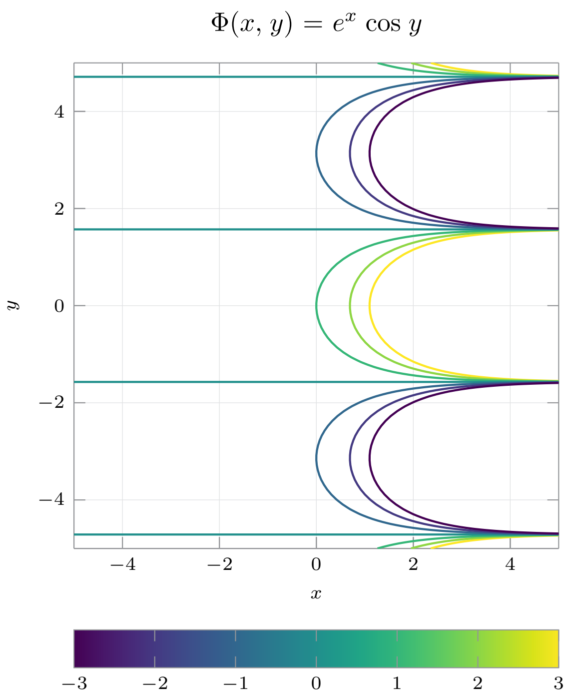
$$F(z) = a\ \mathrm{Ln}\,\frac{2z-1}{z-2} \qquad\qquad \Phi = a\ \ln\left[\frac{(2x-1)^2 + (2y)^2}{(x-2)^2 + y^2}\right] \tag{18.2.22}$$

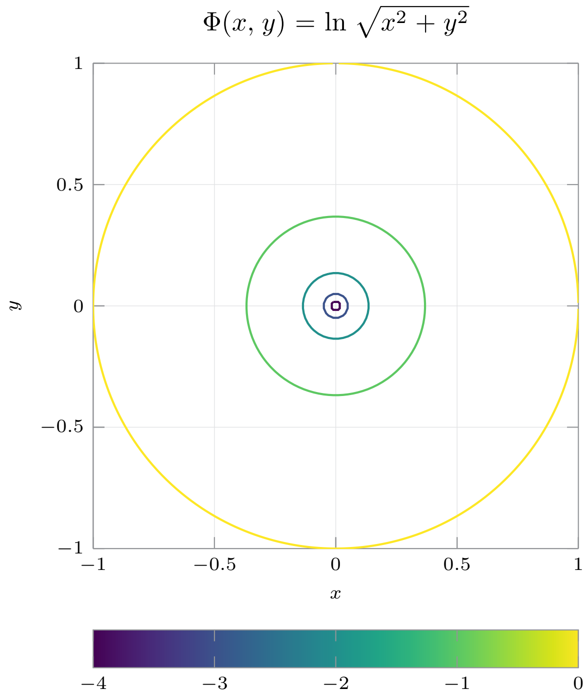
18.2.23











These cylinders are visualized as circular cross-sections in the $z = 0$ plane.

6. Using theorem 1,

$$\Phi^*(u, v) = u^2 - v^2 \qquad w = f(z) = e^z \tag{18.2.24}$$

$$\Phi(x, y) = e^{2x} \cos^2 y - e^{2x} \sin^2 y \qquad \Phi(x, y) = e^{2x} \cos(2y) \tag{18.2.25}$$

$$\Phi_{xx} + \Phi_{yy} = -4e^{2x} \cos(2y) + 4e^{2x} \cos(2y) = 0 \tag{18.2.26}$$

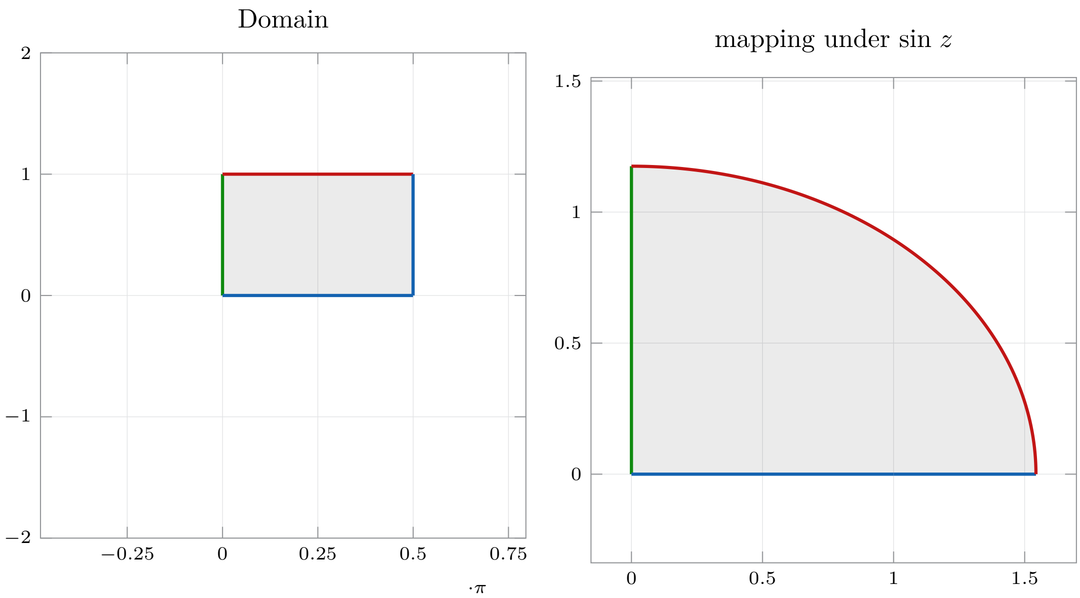
7. Using theorem 1,

$$\Phi^*(u, v) = u^2 - v^2 \qquad w = f(z) = \sin z \tag{18.2.27}$$

$$\Phi(x, y) = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y \tag{18.2.28}$$

$$\Phi(0, y) = -\sinh^2 y \qquad \Phi(\pi/2, y) = \cosh^2 y \tag{18.2.29}$$

$$\Phi(x, 0) = \sin^2 x \qquad \Phi(x, 1) = \sin^2 x \cosh^2 1 - \cos^2 x \sinh^2 1 \tag{18.2.30}$$



8. The equipotential lines and electromagnetic lines of force are interchanged

$$\Phi^*(u, v) = 2uv \qquad w = \sin z \tag{18.2.31}$$

$$\Phi(x, y) = \frac{\sin(2x) \sinh(2y)}{2} \tag{18.2.32}$$

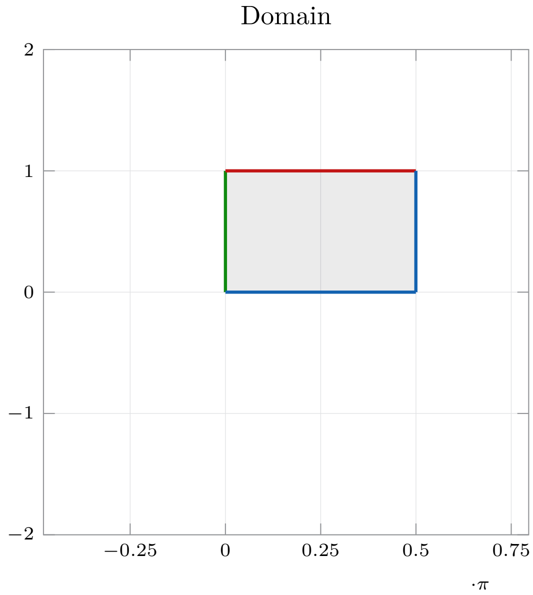
9. The potential and the mapped region change,

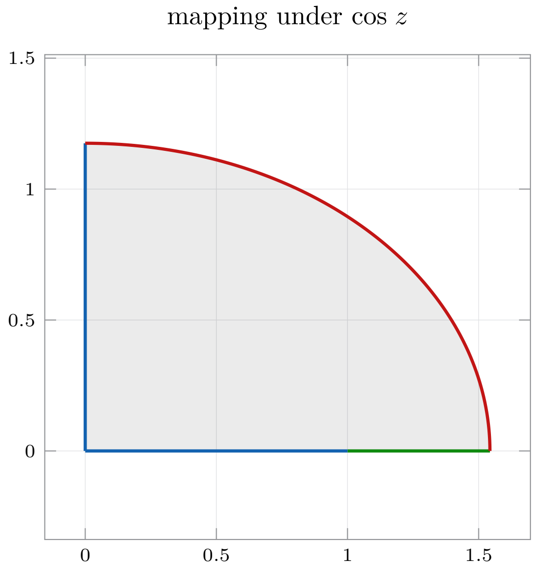
$$\Phi^*(u, v) = u^2 - v^2 \qquad w = f(z) = \cos z \tag{18.2.33}$$

$$\Phi(x, y) = \cos^2 x \cosh^2 y - \sin^2 x \sinh^2 y \tag{18.2.34}$$

$$\Phi(0, y) = \cosh^2 y \qquad \Phi(\pi/2, y) = -\sinh^2 y \tag{18.2.35}$$

$$\Phi(x, 0) = \cos^2 x \qquad \Phi(x, 1) = \cos^2 x \cosh^2 1 - \sin^2 x \sinh^2 1 \tag{18.2.36}$$





10. Using the procedure in Example 1,

$$w = \frac{z-b}{bz-1} \qquad w(0) = b = r_0 \tag{18.2.37}$$

$$w(2c) = -r_0 = \frac{2c - r_0}{r_0(2c) - 1} \tag{18.2.38}$$

This is a quadratic equation in r_0 whose only solution with $|r_0| < 1/2$ is

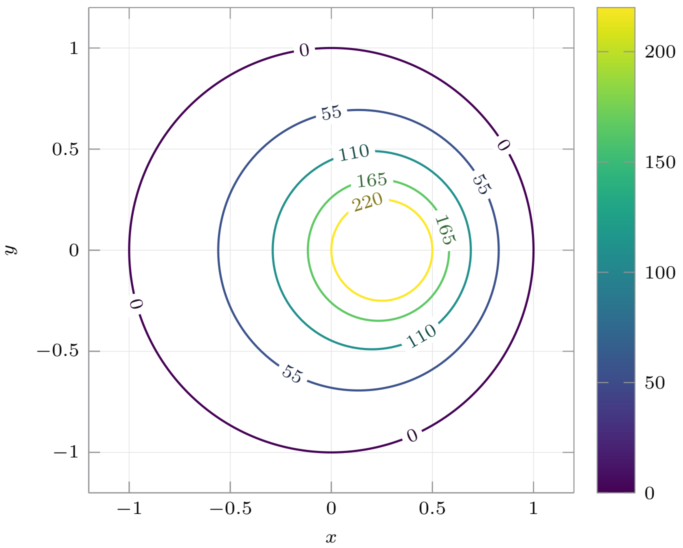
$$r_0 = \frac{1 - \sqrt{1-4c^2}}{2c} \qquad c = 0.25 \implies r_0 = 2 - \sqrt{3} \tag{18.2.39}$$

$$\Phi^*(u, v) = \text{Re } F^*(w) = a \ln |w| + k \qquad a = \frac{220}{\ln(2-\sqrt{3})} = -167.05 \tag{18.2.40}$$

$$b = 0 \tag{18.2.41}$$

Back-transforming the potential into the z plane,

$$\Phi(x, y) = a \ln \left| \frac{z - r_0}{r_0 z - 1} \right| \tag{18.2.42}$$



On increasing c, the equipotential lines bunch up to the right and get sparse to the left.

11. The mapping is,

$$f(z) = \frac{1+z}{1-z} \qquad f(z) = \frac{(1 - x^2 - y^2) + \mathrm{i}\,(2y)}{(1-x)^2 + y^2} \tag{18.2.43}$$

$$|z| = 1 \implies u = 0 \qquad y > 0 \implies v > 0 \quad \text{and vice versa} \tag{18.2.44}$$

The upper and lower half unit circle are mapped onto the first and second quadrants in the w plane. Let the potential of the top and bottom semicircles be $\pm k$.

$$\Phi^*(u, v) = \frac{2k}{\pi} \arctan(v/u) \qquad F^*(w) = \frac{-2k\mathrm{i}}{\pi} \operatorname{Ln} w \tag{18.2.45}$$

$$F(z) = \frac{-2k\mathrm{i}}{\pi} \operatorname{Ln} \frac{1+z}{1-z} \qquad \Phi(x, y) = \frac{2k}{\pi} \operatorname{Arg} \frac{1+z}{1-z} \tag{18.2.46}$$

12. Mapping the y axis onto the w plane,

$$x = 0 \implies f(z) = \frac{1 - y^2 + \mathrm{i}\,2y}{1+y^2} \qquad |f(z)| = \left| \frac{1 + y^4 + 2y^2}{(1+y^2)^2} \right| = 1 \tag{18.2.47}$$

This means that the map is the unit circle.

13. The mapping is conformal and the rays shown in the w plane make equally spaced (angular). Since angles are preserved under the reverse mapping, the equipotential lines shown in the z plane also have to be equally (angular) spaced at $z = -1$.

14. Using the fact that the rate of change of potential is proportional to

$$x = 0 \qquad \implies \Phi(0, y) \propto \arctan\left(\frac{2y}{1-y^2}\right) \tag{18.2.48}$$

$$\frac{\partial \Phi}{\partial y} \propto \frac{2}{1+y^2} \tag{18.2.49}$$

Clearly the rate of change of potential along the y axis is larger, the closer the point is to the origin.

15. Using the fact that squaring doubles the argument, with $k = 3000$

$$w = z^2 \qquad F^*(w) = \frac{-2k\mathrm{i}}{\pi} \ \mathrm{Ln}\ w \tag{18.2.50}$$

$$F(w) = \frac{-4k\mathrm{i}}{\pi} \ \mathrm{Ln}\ z \qquad \Phi = \frac{4k}{\pi} \ \mathrm{Arg}(z) \tag{18.2.51}$$

16. Using Problem 15, with $k = 3000$

$$w = z^4 \qquad F^*(w) = \frac{-2k\mathrm{i}}{\pi} \ \mathrm{Ln}\ w \tag{18.2.52}$$

$$F(w) = \frac{-8k\mathrm{i}}{\pi} \ \mathrm{Ln}\ z \qquad \Phi = \frac{8k}{\pi} \ \mathrm{Arg}(z) \tag{18.2.53}$$

17. Using the three mapped points, ($\mathrm{i}/2 \to 0$), ($0.6 + 0.8\mathrm{i} \to -1$) and ($-0.6 + 0.8\mathrm{i} \to 1$),

$$\frac{w-0}{w-1}\left[\frac{-2}{-1}\right] = \frac{z - 0.5\mathrm{i}}{z + 0.6 - 0.8\mathrm{i}}\left[\frac{1.2}{0.6 + 0.3\mathrm{i}}\right] \tag{18.2.54}$$

$$\frac{2w}{w-1} = \frac{1.2z - 0.6\mathrm{i}}{(0.6 + 0.3\mathrm{i})z + (0.6 - 0.3\mathrm{i})} \tag{18.2.55}$$

$$w = \frac{-2z + \mathrm{i}}{\mathrm{i}z + 2} \tag{18.2.56}$$

18. By explicit calculation,

$$w = \frac{-2x - (2y+1)\ \mathrm{i}}{(-y+2) + \mathrm{i}\ x} \qquad \mathrm{Re}\ w = \frac{-3x}{(y-2)^2 + x^2} \tag{18.2.57}$$

Equipotential lines require $\mathrm{Re}\ w = 1/c$, reduce to circles

$$(y-2)^2 + x^2 + 3cx = 0 \qquad (y-2)^2 + (x + 1.5c)^2 = 2.25c^2 \tag{18.2.58}$$

19. Using the result from Example 3, and the transformation,

$$w = (z - 2) \qquad \mathrm{Arg}\ w = 0 \implies \Phi^* = 0 \tag{18.2.59}$$

$$\mathrm{Arg}\ w = \pi \implies \Phi^* = 5 \qquad \Phi = \frac{5}{\pi}\ \mathrm{Arg}\ z - 2 \tag{18.2.60}$$

$$F(z) = \frac{-5\mathrm{i}}{\pi} \ \mathrm{Ln}(z-2) \tag{18.2.61}$$

20. Using the three mapped points, $(-a \to 0)$, $(0 \to 1)$ and $(a \to \infty)$,

$$\frac{w}{w-\infty}\left[\frac{1-\infty}{1}\right] = \frac{z+a}{z-a}\left[\frac{-a}{a}\right] \qquad w = \frac{z+a}{a-z} \tag{18.2.62}$$

In the w plane, the positive real axis is at $\Phi^* = V_0$ and the negative real axis is at $\Phi^* = 0$.

$$F^*(w) = \frac{\mathrm{i}V_0}{\pi}\ \mathrm{Ln}\ w \qquad \Phi = \frac{-V_0}{\pi}\ \mathrm{Arg}\left[\frac{z+a}{a-z}\right] \tag{18.2.63}$$

18.3 Heat Problems

1. Proceeding directly,

$$T(x,0) = 20 \qquad T(x,d) = 100 \tag{18.3.1}$$

$$T(x,y) = 20 + \frac{80y}{d} \tag{18.3.2}$$

Using the result from Example 1,

$$F^*(w) = 20 + \frac{80w}{d}w \qquad w = -\mathrm{i}z \tag{18.3.3}$$

$$F(z) = 20 + \frac{80}{d}(-\mathrm{i}x+y) \qquad T(x,y) = 20 + \frac{80y}{d} \tag{18.3.4}$$

The results match. The conformal mapping was a rotation by $-\pi/2$.

2. Using the conformal mapping,

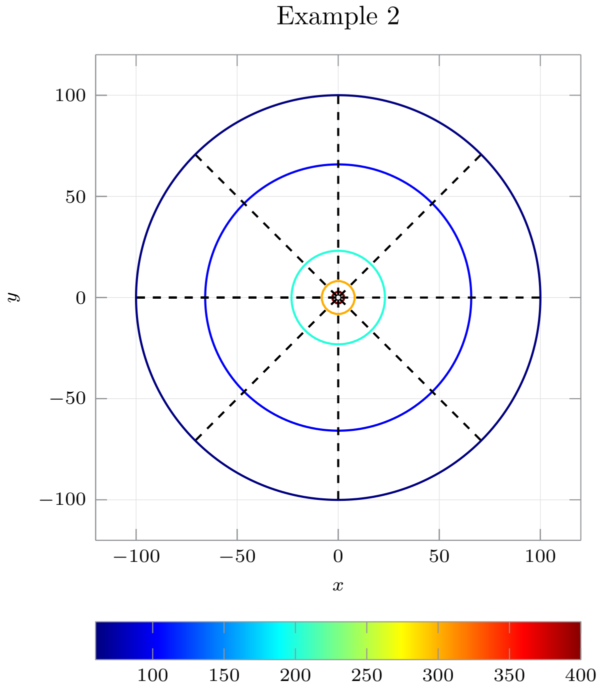
$$w = e^{\mathrm{i}\pi/4}\ z \qquad u + \mathrm{i}\ v = \frac{(x-y) + \mathrm{i}\ (x+y)}{\sqrt{2}} \tag{18.3.5}$$

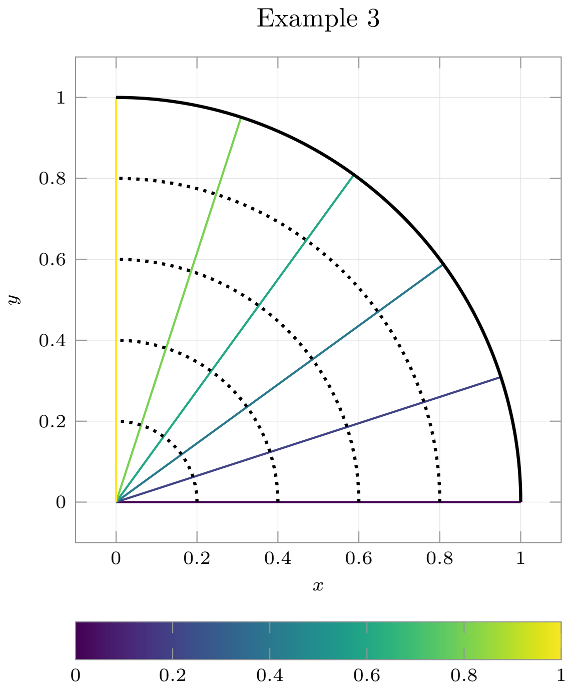
$$T^*(-\sqrt{8},v) = 40, \quad T^*(\sqrt{8},v) = -20 \qquad T^*(u,v) = au + b \tag{18.3.6}$$

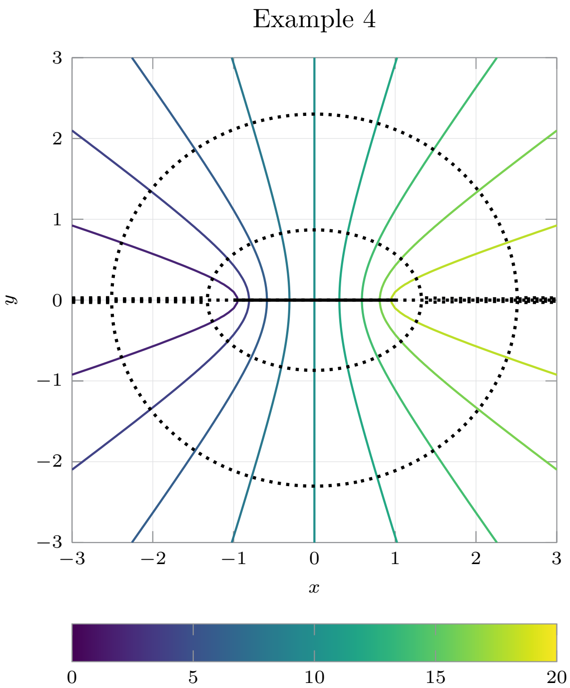
Using the boundary conditions and then back-transforming to the z plane,

$$T^*(u,v) = 10 - \frac{15u}{\sqrt{2}} \qquad T(x,y) = 10 + 7.5(y-x) \tag{18.3.7}$$

3. Plotting the isotherms and heat flow lines,







4. Using the conformal map,

$$f(z) = -\mathrm{i}z^2 = u + \mathrm{i}\ v \qquad f(z) = (2xy) + \mathrm{i}\ (y^2 - x^2) \tag{18.3.8}$$

$$T^*(0, v) = 200, \quad T^*(20, v) = 0 \qquad T^*(u, v) = 200 - 10u \tag{18.3.9}$$

$$T(x, y) = 200 - 20xy \qquad F(z) = 200 + 10\mathrm{i}\ z^2 \tag{18.3.10}$$

5. Using the conformal map,

$$f(z) = -\mathrm{i}\,\mathrm{Ln}\, z = u + \mathrm{i}\, v \qquad f(z) = \mathrm{Arg}\, z - \mathrm{i}\ln|z| \tag{18.3.11}$$

$$T^*(-\pi/4, v) = -20, \quad T^*(\pi/4, v) = 20 \qquad T^*(u, v) = \frac{80u}{\pi} \tag{18.3.12}$$

$$T(x, y) = \frac{80}{\pi}\ \mathrm{Arg}\, z \qquad F(z) = \frac{-80\mathrm{i}}{\pi}\ \mathrm{Ln}\, z \tag{18.3.13}$$

6. Using the conformal map,

$$f(z) = -\mathrm{i}\,\mathrm{Ln}\, z = u + \mathrm{i}\, v \qquad f(z) = \mathrm{Arg}\, z - \mathrm{i}\ln|z| \tag{18.3.14}$$

$$T^*(0, v) = 0, \quad T^*(\pi/3, v) = 50 \qquad T^*(u, v) = \frac{150u}{\pi} \tag{18.3.15}$$

$$T(x, y) = \frac{150}{\pi}\ \mathrm{Arg}\, z \qquad F(z) = \frac{-150\mathrm{i}}{\pi}\ \mathrm{Ln}\, z \tag{18.3.16}$$

7. Using the conformal map,

$$f(z) = -\mathrm{i}\,\mathrm{Ln}\, z = u + \mathrm{i}\, v \qquad f(z) = \mathrm{Arg}\, z - \mathrm{i}\ln|z| \tag{18.3.17}$$

$$T^*(0, v) = T_1, \quad T^*(\pi/2, v) = T_2 \qquad T^*(u, v) = T_1 + \frac{2(T_2 - T_1)}{\pi}\ u \tag{18.3.18}$$

$$T(x, y) = T_1 + \frac{2(T_2 - T_1)}{\pi}\ \mathrm{Arg}\, z \qquad F(z) = T_1 - \frac{2(T_2 - T_1)\mathrm{i}}{\pi}\ \mathrm{Ln}\, z \tag{18.3.19}$$

8. Using the conformal map,

$$f(z) = -\mathrm{i}\,\mathrm{Ln}\,(z - a) = u + \mathrm{i}\, v \qquad f(z) = \mathrm{Arg}\,(z - a) - \mathrm{i}\ln|z - a| \tag{18.3.20}$$

$$T^*(0, v) = T_1, \quad T^*(\pi, v) = T_2 \qquad T^*(u, v) = T_1 + \frac{(T_2 - T_1)u}{\pi} \tag{18.3.21}$$

$$T(x, y) = T_1 + \frac{(T_2 - T_1)}{\pi}\ \mathrm{Arg}\,(z - a) \qquad F(z) = T_1 - \frac{(T_2 - T_1)\mathrm{i}}{\pi}\ \mathrm{Ln}\,(z - a) \tag{18.3.22}$$

9. Consider the conformal map acting on the upper half z plane,

$$f(z) = -\mathrm{i}\Big[\,\mathrm{Ln}(z - b) - \mathrm{Ln}(z - a)\Big] = u + \mathrm{i}\, v \tag{18.3.23}$$

$$x > b \implies x + 0\mathrm{i} \to u = 0 - 0 \tag{18.3.24}$$

$$x < a \implies x + 0\mathrm{i} \to u = \pi - \pi \tag{18.3.25}$$

$$x \in (a, b) \implies x + 0\mathrm{i} \to u = \pi - 0 \tag{18.3.26}$$

The boundary conditions in the w plane are,

$$T^*(0,\, v) = 0 \qquad\qquad T^*(\pi,\, v) = T_1 \tag{18.3.27}$$

$$T^*(u,\, v) = \frac{T_1}{\pi}\; u \qquad\qquad T(x,\, y) = \frac{-\mathrm{i}T_1}{\pi}\; \text{Ln}\left[\frac{z-b}{z-a}\right] \tag{18.3.28}$$

10. Consider the conformal map acting on the upper half z plane,

$$T(x,\, y) = T_1 + \frac{(T_2 - T_1)}{\pi}\, \text{Arg}\,(z-b) + \frac{(T_3 - T_2)}{\pi}\, \text{Arg}\,(z-a) \tag{18.3.29}$$

$$x > b \implies T = T_1 + 0 + 0 \tag{18.3.30}$$

$$x \in (a,\, b) \implies T = T_1 + (T_2 - T_1) + 0 \tag{18.3.31}$$

$$x < a \implies T = T_1 + (T_2 - T_1) + (T_3 - T_2) \tag{18.3.32}$$

The boundary conditions in the w plane are,

$$\Phi(x,\, y) = T_1 - \frac{\mathrm{i}}{\pi}\Big[(T_2 - T_1)\, \text{Ln}(z-b) + (T_3 - T_2)\, \text{Ln}(z-a)\Big] \tag{18.3.33}$$

This is a generalization of the result from Problem 9.

11. Consider the conformal map acting on the upper half z plane,

$$f(z) = -\mathrm{i}\Big[\,\text{Ln}(z-1) - \text{Ln}(z+1)\Big] = u + \mathrm{i}\; v \tag{18.3.34}$$

$$x > 1 \implies x + 0\mathrm{i} \to u = 0 - 0 \tag{18.3.35}$$

$$x < -1 \implies x + 0\mathrm{i} \to u = \pi - \pi \tag{18.3.36}$$

$$x \in (-1,\, 1) \implies x + 0\mathrm{i} \to u = \pi - 0 \tag{18.3.37}$$

The boundary conditions in the w plane are,

$$T^*(0,\, v) = 0 \qquad\qquad T^*(\pi,\, v) = 100 \tag{18.3.38}$$

$$T^*(u,\, v) = \frac{100}{\pi}\; u \qquad\qquad T(x,\, y) = \frac{-100\mathrm{i}}{\pi}\; \text{Ln}\left[\frac{z-1}{z+1}\right] \tag{18.3.39}$$

12. Consider the conformal map,

$$w = \cosh z \qquad u + \mathrm{i}v = \cosh x \cos y + \mathrm{i}\ \sinh x \sin y \tag{18.3.40}$$

$$x > 0, \quad y = 0 \qquad \Longrightarrow \qquad v = 0, \quad u > 1 \tag{18.3.41}$$

$$x > 0, \quad y = \pi \qquad \Longrightarrow \qquad v = 0, \quad u < -1 \tag{18.3.42}$$

$$x = 0, \quad y \in [0, \pi] \qquad \Longrightarrow \qquad v = 0, \quad u \in [-1, 1] \tag{18.3.43}$$

This is exactly the boundary conditions in Problem 12,

$$T(w) = \frac{-100\mathrm{i}}{\pi}\ \mathrm{Ln}\left[\frac{w-1}{w+1}\right] \qquad T(z) = \frac{-100\mathrm{i}}{\pi}\ \mathrm{Ln}\left[\frac{\cosh(z)-1}{\cosh(z)+1}\right] \tag{18.3.44}$$

13. Once again, this problem transforms to Problem 11, using

$$w = z^2 \tag{18.3.45}$$

$$x > 1, \quad y = 0 \implies u > 1, \quad v = 0 \tag{18.3.46}$$

$$x < 1, \quad y = 0 \implies u \in [0, 1], \quad v = 0 \tag{18.3.47}$$

$$y > 1, \quad x = 0 \implies u < -1, \quad v = 0 \tag{18.3.48}$$

$$y < 1, \quad x = 0 \implies u < [-1, 0], \quad v = 0 \tag{18.3.49}$$

Using the result from Problem 11,

$$T(w) = \frac{-100\mathrm{i}}{\pi}\ \mathrm{Ln}\left[\frac{w-1}{w+1}\right] \qquad T(z) = \frac{-100\mathrm{i}}{\pi}\ \mathrm{Ln}\left[\frac{z^2-1}{z^2+1}\right] \tag{18.3.50}$$

14. From Example 3 in the text,

$$\left[\frac{\partial T}{\partial r}\right]_{r=R_1} = 0 \qquad \left[\frac{\partial T}{\partial r}\right]_{r=R_2} = 0 \tag{18.3.51}$$

$$T(R_1 < r < R_2, 0) = 0 \qquad T(R_1 < r < R_2, \pi/2) = 200 \tag{18.3.52}$$

$$T(r, \theta) = \frac{400}{\pi}\ \theta \qquad F(z) = \frac{-400\mathrm{i}}{\pi}\ \mathrm{Ln}\ z \tag{18.3.53}$$

15. From Example 3 in the text,

$$T(r, \theta) = -20 + \frac{80}{\pi/4}\ \theta \qquad F(z) = -20 - \frac{320\mathrm{i}}{\pi}\ \mathrm{Ln}\ z \tag{18.3.54}$$

16. From Example 3 in the text,

$$T(r,\theta) = 20 + \frac{480}{\pi/3}\,\theta \qquad\qquad F(z) = 20 - \frac{1440\mathrm{i}}{\pi}\ \mathrm{Ln}\ z \tag{18.3.55}$$

17. Since in Example 4 from the text, the y axis is at the mean potential, this problem can be extrapolated to $T_2 = 200$, $T_1 = 0$

$$T(w) = \frac{0+200}{2} + \frac{100}{\pi/2}\ \mathrm{Re}\ (\ \arcsin\ z) \tag{18.3.56}$$

$$F(z) = 100 + \frac{200}{\pi}\ \arcsin(z) \tag{18.3.57}$$

18. Using Example 3 from the text,

$$T(r,\theta) = 100 - \frac{130}{\pi/2}\,\theta \qquad\qquad T(r,\theta) = 100 + \frac{260\mathrm{i}}{\pi}\ \mathrm{Ln}\ z \tag{18.3.58}$$

19. Three plates at $x < -1$, $x \in (-1, 1)$ and $x > 1$, each at potential 0 V, 100 V and 0 V respectively.

20. The temperature on the minor arc and major arc between Z_1 and Z_2 is 10 °C and 200 °C respectively. Find the temperature everywhere inside the unit disk.

18.4 Fluid Flow

1. The velocity must be continuous and have continuous first partial derivatives. Then, it can satisfy the incompressible and irrotational conditions.

2. The speed in Example 1 is given by

$$F(z) = z^2 \qquad\qquad V = \overline{F'(z)} = 2(x - \mathrm{i}y) \tag{18.4.1}$$

$$|V| = 2\sqrt{x^2+y^2} \qquad\qquad C : x^2 + y^2 = k \tag{18.4.2}$$

The contours of constant speed are circles in the first quadrant centered on the origin.

3. The speed is,

$$F'(z) = 1 - \frac{1}{z^2} \qquad\qquad V = \overline{F'(z)} = 1 + \frac{1}{y^2} \tag{18.4.3}$$

The speed is greatest at $y = \pm 1$ and therefore at $z = \pm\mathrm{i}$

4. The speed along the cylinder wall is,

$$F'(z) = 1 - \frac{1}{z^2} \qquad z = re^{\mathrm{i}\theta} \tag{18.4.4}$$

$$F'(z) = 1 - \frac{e^{-2\mathrm{i}\theta}}{r^2} \qquad V = \overline{F'(z)} = 1 - \frac{e^{2\mathrm{i}\theta}}{r^2} \tag{18.4.5}$$

$$|r| = 1 \implies V(\theta) = \sqrt{2 - 2\cos(2\theta)} = 2\,|\sin\,\theta| \tag{18.4.6}$$

The speed is greatest at $\theta = \pm\pi/2$ and therefore at $z = \pm\mathrm{i}$

5. Starting from the complex potential,

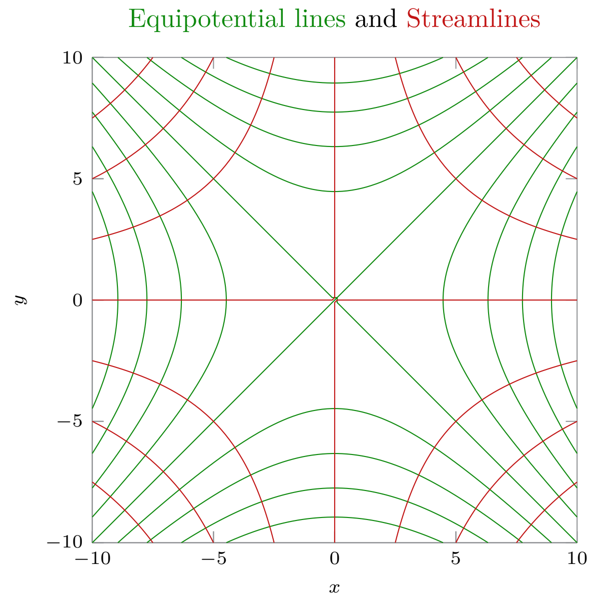
$$F'(z) = 1 - \frac{1}{z^2} \qquad F'(z) = 1 - \frac{x^2 - y^2 - (2xy)\,\mathrm{i}}{(x^2 + y^2)^2} \tag{18.4.7}$$

$$V_1 = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad V_2 = \frac{-2xy}{(x^2 + y^2)^2} \tag{18.4.8}$$

$$\frac{\partial V_1}{\partial y} = \frac{2y\,(3x^2 - y^2)}{(x^2 + y^2)^3} \qquad \frac{\partial V_2}{\partial x} = \frac{(3x^2 - y^2)\,2y}{(x^2 + y^2)^3} \tag{18.4.9}$$

Since these expressions are equal, the curl in two dimensions is zero, making the flow irrotational.

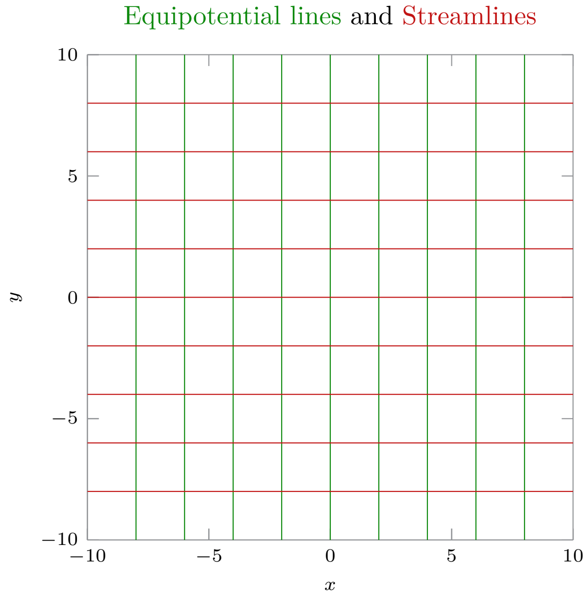
6. Plotting the entire upper half plane,



7. Plotting the flow,

$$F(z) = z \qquad V = \overline{F'(z)} = 1 \tag{18.4.10}$$

$$\Phi + \mathrm{i}\,\Psi = x + \mathrm{i}\,y \tag{18.4.11}$$

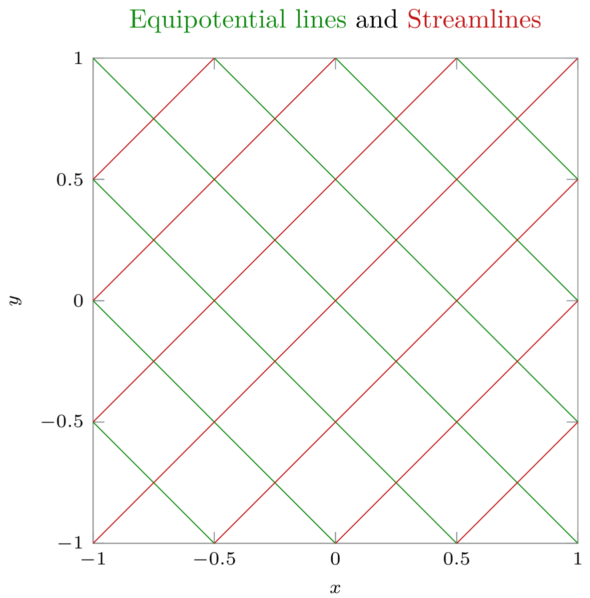


8. Given the velocity vector field,

$$|V| = K > 0 \qquad \text{Arg } V = \pi/4 \tag{18.4.12}$$

$$V = K \cdot \frac{1 + \mathfrak{i}}{\sqrt{2}} \qquad F'(z) = K \cdot \frac{1 - \mathfrak{i}}{\sqrt{2}} \tag{18.4.13}$$

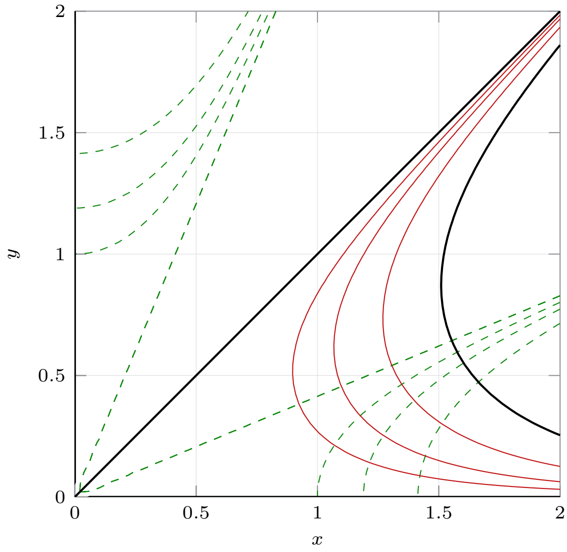
$$F(z) = K \cdot \frac{(x + y) + \mathfrak{i}\,(y - x)}{\sqrt{2}} \tag{18.4.14}$$



9. An initial guess could be $F(z) = z^4$ instead of z^2,

$$F(z) = z^4 = (x^4 + y^4 - 6x^2y^2) + \mathfrak{i}\; 4xy(x^2 - y^2) \tag{18.4.15}$$

Equipotential lines and Streamlines

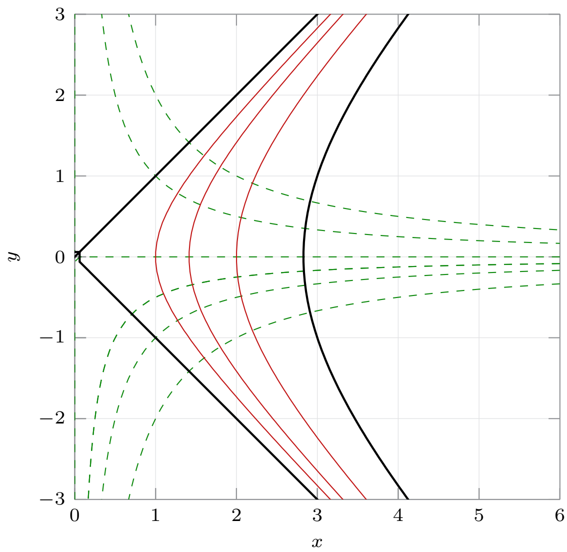


10. An initial guess could be $F(z) = \mathrm{i}z^2$ instead of z^2,

$$F(z) = \mathrm{i}z^2 = -2xy + \mathrm{i}\,(x^2 - y^2) \qquad F'(z) = 2z\,\mathrm{i} = -2y + 2x\,\mathrm{i} \tag{18.4.16}$$

$$V = -2y - 2x\,\mathrm{i} \tag{18.4.17}$$

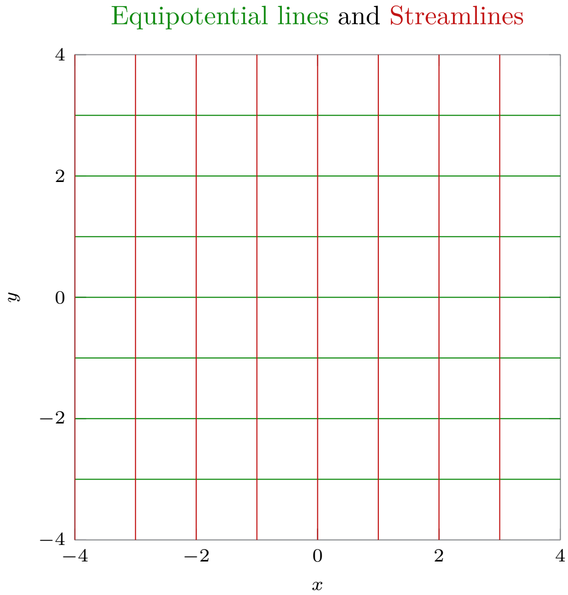
Equipotential lines and Streamlines



11. For some positive real K

$$F(z) = -\mathrm{i}\,Kz \qquad F(z) = Ky - Kx\,\mathrm{i} \tag{18.4.18}$$

$$V = \overline{F'(z)} = K\mathrm{i} \tag{18.4.19}$$

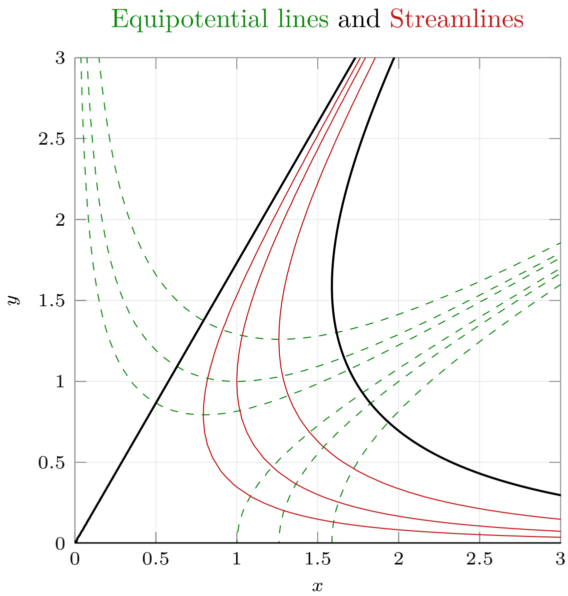


12. Starting from Problem 11,

$$F(z) = -Kz\,\mathrm{i} = w \qquad\qquad G(w) = \frac{w^2}{-K^2} = z^2 \tag{18.4.20}$$

13. An initial guess could be $F(z) = z^3$ instead of z^2,

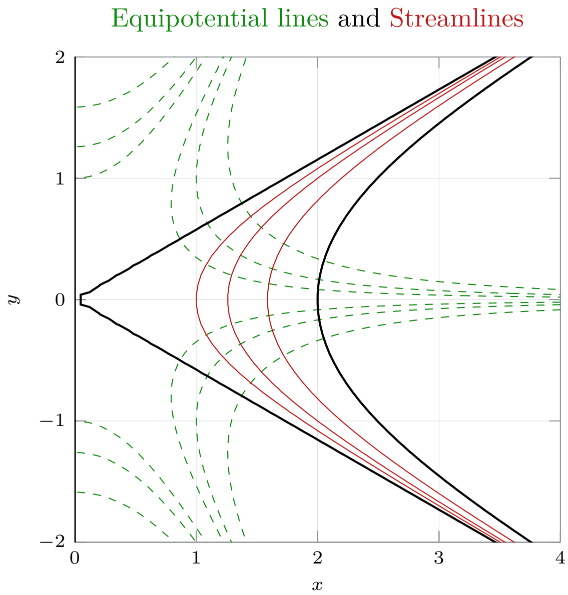
$$F(z) = z^4 = x(x^2 - 3y^2) + \mathrm{i}\ y(3x^2 - y^2) \tag{18.4.21}$$



14. Starting with $F(z) = \mathrm{i}z^3$,

$$F(z) = \mathrm{i}z^3 = y(y^2 - 3x^2) + \mathrm{i}\; x(x^2 - 3y^2) \qquad F'(z) = 3\mathrm{i}\; z^2 = -6xy + 3\mathrm{i}\;(x^2 - y^2) \tag{18.4.22}$$

$$\tag{18.4.23}$$



15. Scaling Example 2, to a cylinder of radius r_0,

$$F(z) = \frac{z}{r_0} + \frac{r_0}{z} \qquad F'(z) = \frac{1}{r_0} - \frac{r_0}{z^2} \tag{18.4.24}$$

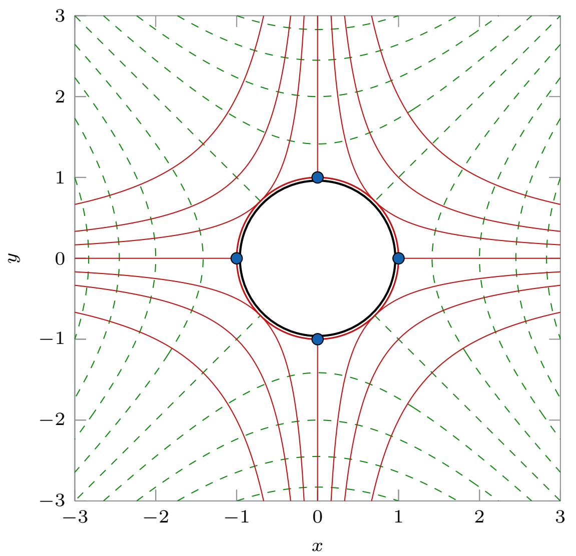
The stagnation points are now at $z = \pm r_0$, and this model reduces to Example 2, when $r_0 \to 1$. Alternatively, scaling the x and y axes by a factor r_0 leads to the cylinder having radius r_0 in the new coordinate system.

16. Replacing z with z^2 in Example 2,

$$F(z) = z^2 + z^{-2} \qquad F'(z) = 2z - \frac{2}{z^3} \tag{18.4.25}$$

$$F'(z) = 0 \qquad \Longrightarrow \qquad z^4 = 1 \tag{18.4.26}$$

The stagnation points are $z = \pm 1, \pm\mathrm{i}$



Since the fluid cannot cross the cylinder border (the unit disk), it is forced to follow the border of the cylinder for that duration.

17. Using the velocity potential

$$F(z) = \arccos z = \arccos \beta - \mathrm{i}\ \operatorname{sgn}(y)\ \ln\left[\alpha + \sqrt{\alpha^2 - 1}\right] \tag{18.4.27}$$

$$\Psi(x, y) = c \quad \Longrightarrow \quad \alpha + \sqrt{\alpha^2 - 1} = c \tag{18.4.28}$$

$$c^2 - 2c\alpha = -1 \quad \Longrightarrow \quad \alpha = \frac{c^2 + 1}{2c} \tag{18.4.29}$$

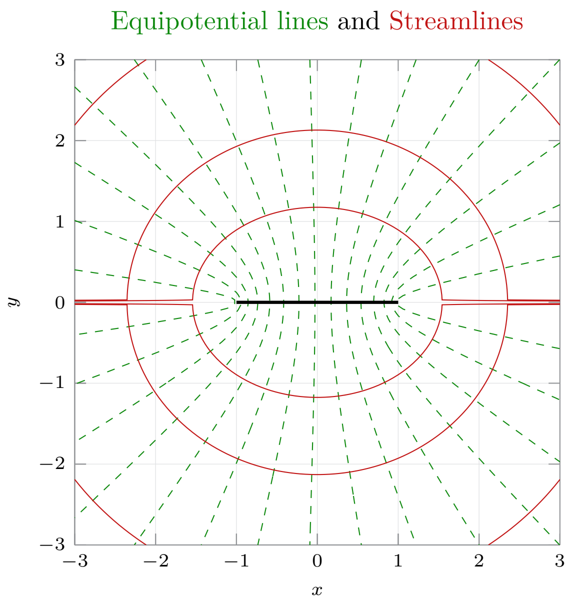
$$\alpha = \frac{\sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2}}{2} = b \tag{18.4.30}$$

$$2b^2 = y^2 + x^2 + 1 + \sqrt{y^4 + y^2(2x^2 + 2) + (x^2 - 1)^2} \tag{18.4.31}$$

$$x^2(b^2 - 1) + b^2 y^2 = b^4 - b^2 \tag{18.4.32}$$

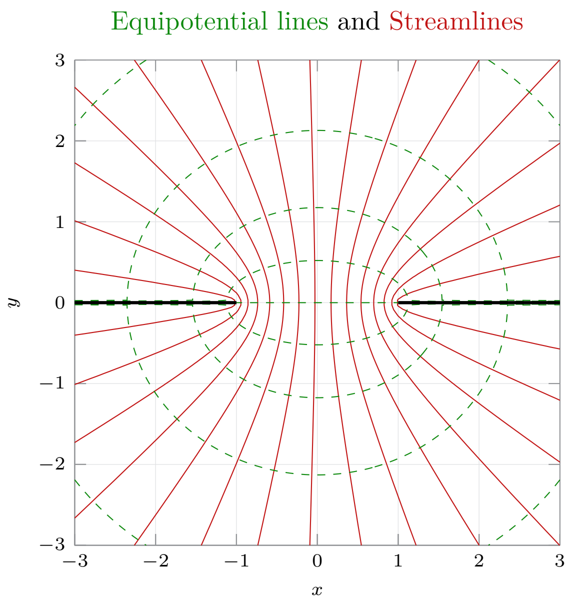
$$\frac{x^2}{b^2} + \frac{y^2}{b^2 - 1} = 1 \tag{18.4.33}$$

This is a family of confocal ellipses with major axis being the x axis and foci ± 1.



18. This is a rotation by $-\pi/2$ of Problem 17, as shown by the relation,

$$F(z) = \cosh^{-1} z = -\mathfrak{i} \cos^{-1} z \tag{18.4.34}$$



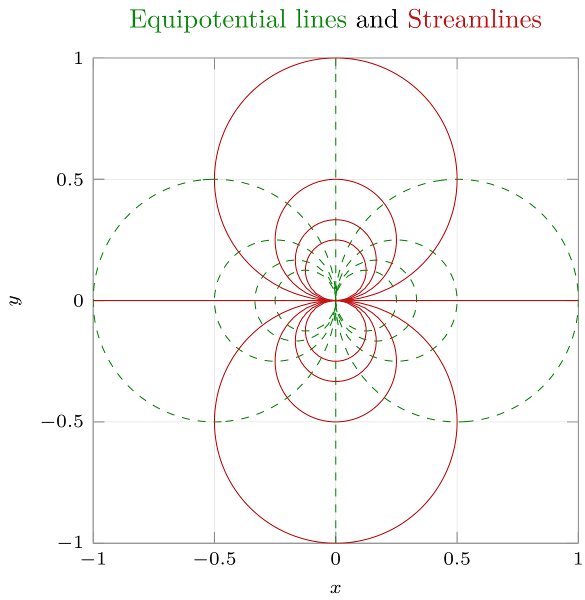
19. Starting from the velocity potential,

$$F(z) = \frac{1}{z} = \Phi + \mathfrak{i}\,\Psi \qquad\qquad F(z) = \frac{\bar{z}}{|z|^2} \tag{18.4.35}$$

$$\Psi(x,\,y) = \frac{-y}{x^2+y^2} = \frac{1}{c} \qquad\qquad \Longrightarrow\; 0 = x^2 + y^2 + cy \tag{18.4.36}$$

$$x^2 + (y + c/2)^2 = (c/2)^2 \tag{18.4.37}$$

This is a family of circles with center on the y axis that pass through the origin.



20. The complex logarithm,

(a) Looking at the function,

$$F(z) = \frac{c}{2\pi} \ \mathrm{Ln}\, z \qquad = \frac{c}{2\pi} \left(\ln r + \mathrm{i}\ \ \mathrm{Arg}\ z\right) \tag{18.4.38}$$

$$c > 0 \implies \Psi = \frac{c}{2\pi} \ \arctan(y/x) = \theta \qquad y = \tan\left[\frac{2\pi\theta}{c}\right] x \tag{18.4.39}$$

This is a straight line. Looking at the velocity,

$$F'(z) = \frac{c}{2\pi} \cdot \frac{1}{z} \qquad V = \frac{c}{2\pi} \cdot \frac{x + \mathrm{i}\ y}{x^2 + y^2} \tag{18.4.40}$$

If $c > 0$, then the velocity is pointed away from the origin and vice versa.

(b) The function is,

$$F(z) = \frac{-\mathrm{i}K}{2\pi} \ \ln z \qquad \Phi + \mathrm{i}\ \Psi = \frac{K}{2\pi} \left[\theta - \mathrm{i} \ln r\right] \tag{18.4.41}$$

$$V = \frac{K}{2\pi} \cdot \frac{y - \mathrm{i}x}{x^2 + y^2} \tag{18.4.42}$$

The streamlines correspond to constant modulus r, circles centered on the origin. From the complex velocity, it is ccl as can be checked at $\pm r_0$, $\pm r_0 \mathrm{i}$.

(c) The complex conjugate and derivative operation are distributive under addition.

$$F_3 = F_1 + F_2 \qquad \overline{F_3'} = \overline{F_1'} + \overline{F_2'} \tag{18.4.43}$$

$$\implies V_3 \quad = V_1 + V_2 \tag{18.4.44}$$

(d) Using the superposition principle on the velocity potential,

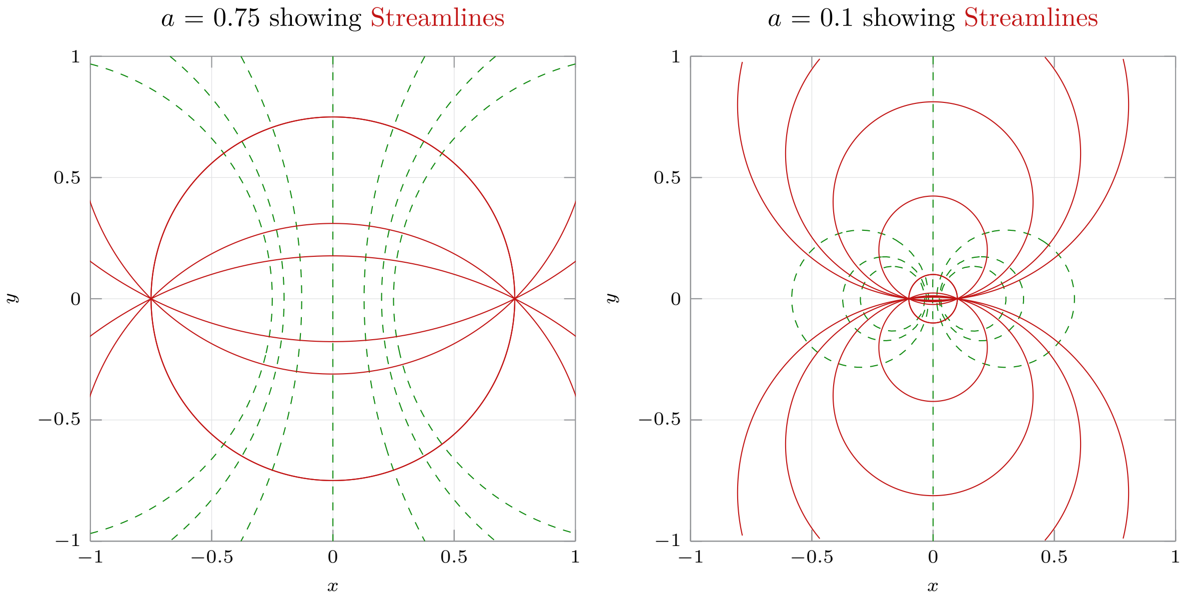
$$F(z) = \frac{1}{2\pi} \ln\left(\frac{z+a}{z-a}\right) \qquad \Phi(x, y) = \frac{1}{4\pi} \ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2} \tag{18.4.45}$$

$$\Psi(x, y) = \mathrm{Arg}(z+a) - \mathrm{Arg}(z-a) \qquad 0 = x^2 + y^2 - a^2 + 2ayc \tag{18.4.46}$$

$$\tag{18.4.47}$$

The family of circles is,

$$x^2 + (y - ac)^2 = a^2(1 + c^2) \tag{18.4.48}$$



(e) Adding the two potentials,

$$F(z) = \frac{-K\mathrm{i}}{2\pi} \ln z + z + \frac{1}{z} \qquad V = 1 - \frac{1}{\bar{z}^2} + \frac{K\mathrm{i}}{2\pi\ \bar{z}} \tag{18.4.49}$$

$$V = 0 \qquad \implies 0 = \bar{z}^2 - 1 + \frac{K\mathrm{i}}{2\pi}\ \bar{z} \tag{18.4.50}$$

This is a quadratic in $\bar{z}$, with stagnation points,

$$\bar{z}^* = -\frac{K\mathrm{i}}{4\pi} \pm \sqrt{\frac{-K^2}{16\pi^2} + 1} \qquad z^* = \frac{K\mathrm{i}}{4\pi} \pm \sqrt{\frac{-K^2}{16\pi^2} + 1} \tag{18.4.51}$$

$$K = 0 \qquad \implies z^* = \pm 1 \tag{18.4.52}$$

$$K = 4\pi \qquad \implies z^* = \mathrm{i}, \mathrm{i} \tag{18.4.53}$$

$$K = 4\pi c, \qquad c > 1 \qquad z^* = \mathrm{i}\left[c \pm \sqrt{c^2 - 1}\right] \tag{18.4.54}$$

For all $c > 1$, $z_a^* < \mathrm{i}$ while $z_b^* > \mathrm{i}$. Only the stagnation point outside the cylinder is physically meaningful.

18.5 Poisson's Integral Formula for Potentials

1. Starting from the Poisson integral formula,

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} F(z^*) \left[\frac{|z^*|^2 - |z|^2}{|(z^* - z)|^2} \right] \, \mathrm{d}\alpha \tag{18.5.1}$$

The quotient in the integrand is real and can be written as the real part of

$$\frac{z^* + z}{z^* - z} = \frac{|z^*|^2 - |z|^2 + 2\mathrm{i} \;\; \mathrm{Im}\,(z\overline{z^*})}{|z^* - z|^2} \tag{18.5.2}$$

Using a Fourier series expansion on the LHS,

$$\frac{1 + z/z^*}{1 - z/z^*} = (1 + z/z^*) \sum_{n=0}^{\infty} \left(\frac{z}{z^*}\right)^n = 1 + 2 \sum_{n=0}^{\infty} \left(\frac{z}{z^*}\right)^n \tag{18.5.3}$$

The real part of the RHS is K,

$$K = 1 + 2 \sum_{n=1}^{\infty} \mathrm{Re} \left[\left(\frac{z}{z^*}\right)^n \right] \tag{18.5.4}$$

$$z^* = R \exp(\mathrm{i}\alpha), \qquad z = r \exp(\mathrm{i}\theta) \tag{18.5.5}$$

$$K = 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos(n\theta - n\alpha) \tag{18.5.6}$$

Substituting into the Poisson integral formula,

$$\Phi(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(R, \alpha) \left[1 + 2 \sum_{n=1}^{\infty} \cos(n\theta) \cos(n\alpha) + \sin(n\theta) \sin(n\alpha) \right] \, \mathrm{d}\alpha \tag{18.5.7}$$

$$\Phi(r, \theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta) \tag{18.5.8}$$

This reduces to the Fourier expansion of $\Phi(R, \theta)$ on the boundary $r = R$.

2. The numerator is,

$$-\left(z^* \bar{z} - |z^*|^2 - \bar{z} z^* + |z|^2 \right) = |z^*|^2 - |z|^2 \tag{18.5.9}$$

3. Checking if the elements in the infinite series are harmonic,

$$\nabla^2\Phi = \frac{1}{r}\Phi_r + \Phi_{rr} + \frac{1}{r^2}\Phi_{\theta\theta} = 0 \tag{18.5.10}$$

$$\Phi_n(r,\theta) = \frac{r^n}{R^n}\Big[a_n\cos(n\theta) + b_n\sin(n\theta)\Big] \tag{18.5.11}$$

$$0 = \frac{a_n r^{n-2}}{R^n}\ \cos(n\theta)\ \Bigg[n + (n^2 - n) - n^2\Bigg] + \dots \tag{18.5.12}$$

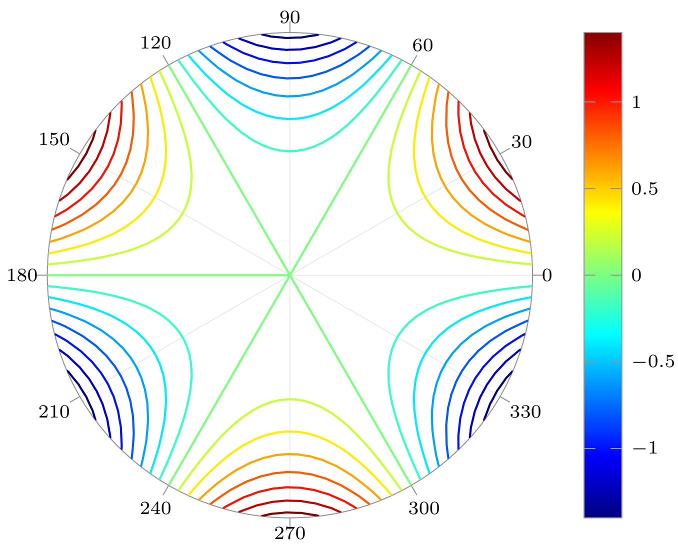
4. The Fourier expansion of an even function has to contain even terms, and thus cannot contain any sine terms.

5. Since the function is odd,

$$\Phi(1,\theta) = 1.5\sin(3\theta) \qquad\qquad a_0 = a_n = 0 \tag{18.5.13}$$

$$b_n = \frac{1.5}{\pi}\int_0^{2\pi} \sin(3\alpha)\ \sin(n\alpha)\ d\alpha \qquad\qquad b_n = \begin{cases} 1.5 & n = 3 \\ 0 & \text{otherwise} \end{cases} \tag{18.5.14}$$

$$\Phi(r,\theta) = 1.5r^3\sin(3\theta) \tag{18.5.15}$$

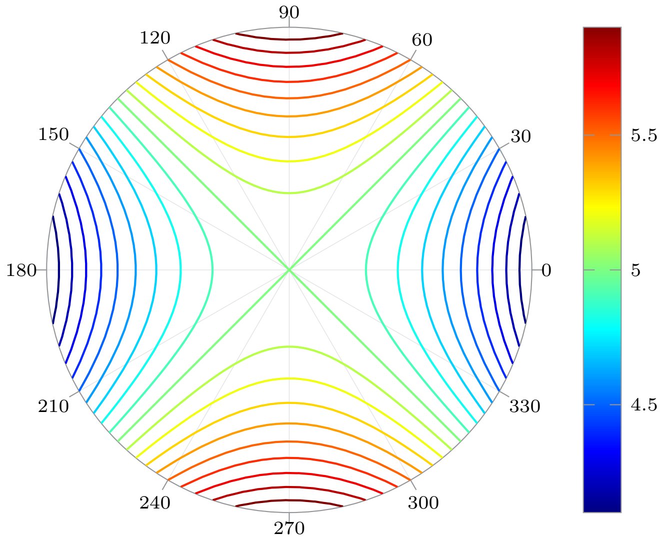


6. Since the function is even, and is already a Fourier series,

$$\Phi(1,\theta) = 5 - \cos(2\theta) \qquad\qquad b_n = 0 \tag{18.5.16}$$

$$a_0 = 5 \qquad\qquad a_n = \begin{cases} -1 & n = 2 \\ 0 & \text{otherwise} \end{cases} \tag{18.5.17}$$

$$\Phi(r,\theta) = 5 - r^2\cos(2\theta) \tag{18.5.18}$$



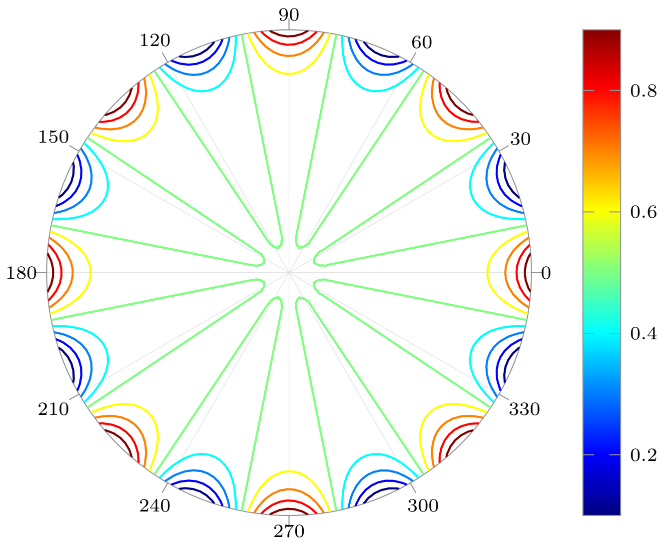
7. Since the function is even, and is already a Fourier series,

$$\Phi(1, \theta) = a \cos^2(4\theta) \qquad = \frac{a}{2} + \frac{a \cos(8\theta)}{2} \tag{18.5.19}$$

$$b_n = 0 \tag{18.5.20}$$

$$a_0 = a/2 \qquad a_n = \begin{cases} a/2 & n = 8 \\ 0 & \text{otherwise} \end{cases} \tag{18.5.21}$$

$$\Phi(r, \theta) = \frac{a}{2} + \frac{ar^8}{2} \cos(8\theta) \tag{18.5.22}$$



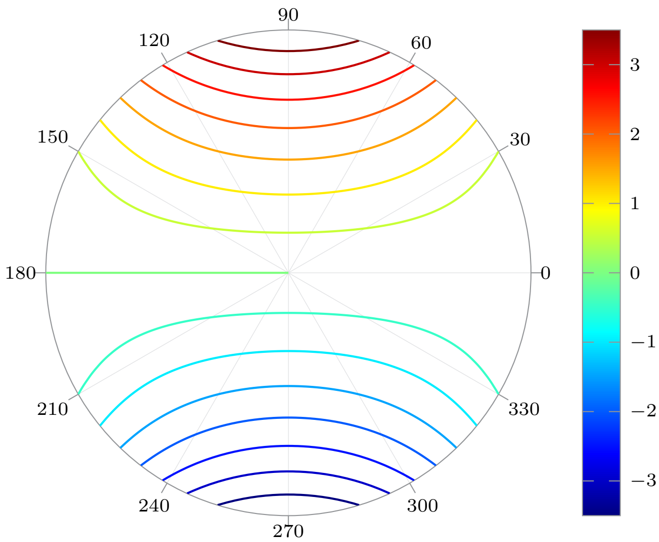
8. Since the function is even, and is already a Fourier series,

$$\Phi(1,\theta) = 4\sin^3\theta \qquad = 3\sin\theta - \sin(3\theta) \tag{18.5.23}$$

$$b_n = \begin{cases} 3 & n = 1 \\ -1 & n = 3 \\ 0 & \text{otherwise} \end{cases} \tag{18.5.24}$$

$$a_0 = 0 \qquad a_n = 0 \tag{18.5.25}$$

$$\Phi(r,\theta) = 3r\sin(\theta) - r^3\sin(3\theta) \tag{18.5.26}$$



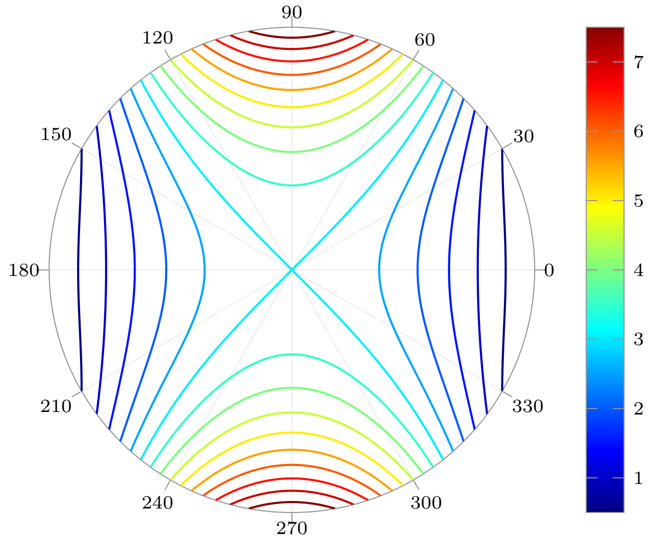
9. Since the function is even, and is already a Fourier series,

$$\Phi(1,\theta) = 8\sin^4\theta \qquad = 3 - 4\cos(2\theta) + \cos(4\theta) \tag{18.5.27}$$

$$b_n = 0 \tag{18.5.28}$$

$$a_0 = 3 \qquad a_n = \begin{cases} -4 & n = 2 \\ 1 & n = 4 \\ 0 & \text{otherwise} \end{cases} \tag{18.5.29}$$

$$\Phi(r,\theta) = 3 - 4r^2\cos(2\theta) + r^4\cos(4\theta) \tag{18.5.30}$$



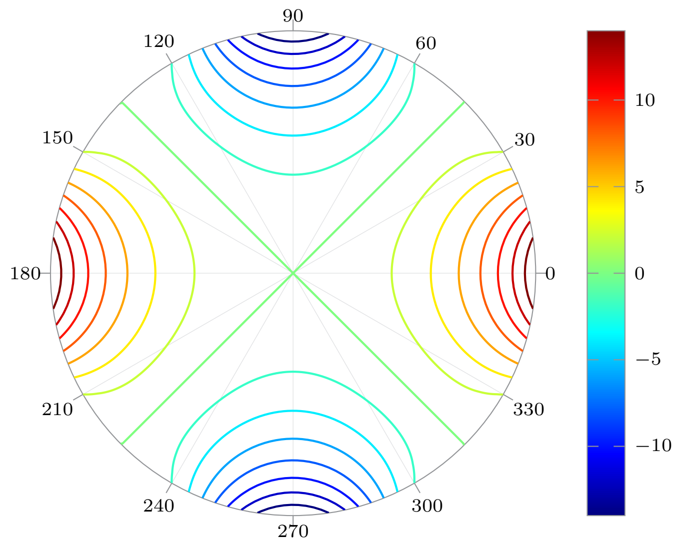
10. Since the function is even, and is already a Fourier series,

$$\Phi(1, \theta) = 16\cos^3(2\theta) \qquad = 12\cos(2\theta) + 4\cos(6\theta) \tag{18.5.31}$$

$$b_n = 0 \tag{18.5.32}$$

$$a_0 = 0 \qquad a_n = \begin{cases} 12 & n = 2 \\ 4 & n = 6 \\ 0 & \text{otherwise} \end{cases} \tag{18.5.33}$$

$$\Phi(r, \theta) = 12r^2\cos(2\theta) + 4r^6\cos(6\theta) \tag{18.5.34}$$



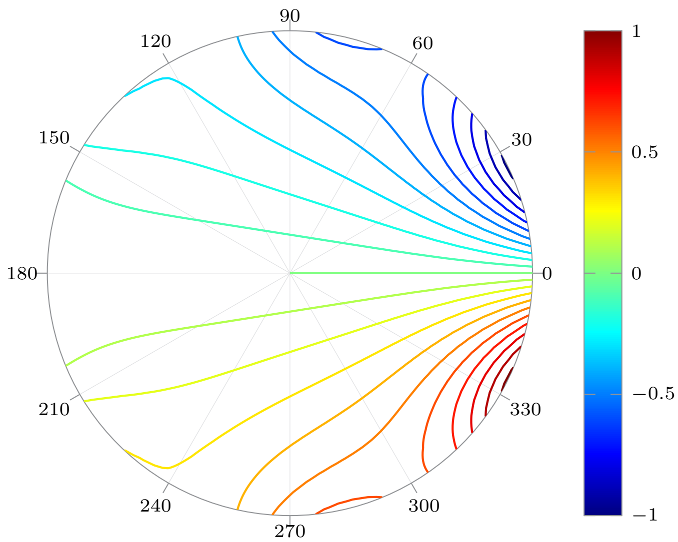
11. Since the function is odd,

$$\Phi(1,\,\theta) = \frac{\theta}{\pi} \qquad \theta \in (-\pi,\,\pi) \tag{18.5.35}$$

$$b_n = \frac{2}{\pi}\int_0^\pi \frac{\alpha}{\pi}\ \sin(n\alpha)\ \ d\alpha \qquad b_n = 2\left[\frac{\sin(n\alpha) - n\alpha\cos(n\alpha)}{\pi^2 n^2}\right]_0^\pi \tag{18.5.36}$$

$$b_n = \frac{-2\cos(n\pi)}{n\pi} \tag{18.5.37}$$

$$a_0 = 0 \qquad a_n = 0 \tag{18.5.38}$$



12. Since the function is odd,

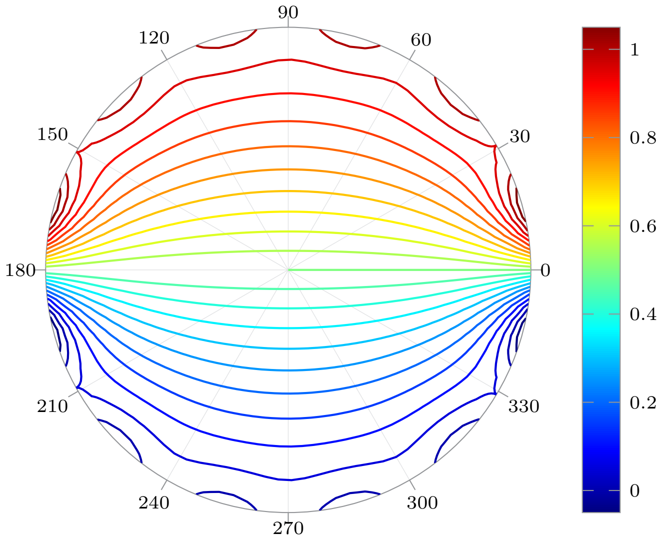
$$\Phi(1,\,\theta) = k \qquad \theta \in (0,\,\pi) \tag{18.5.39}$$

$$b_n = \frac{1}{\pi}\int_0^\pi k\ \sin(n\alpha)\ \ d\alpha \qquad b_n = \left[\frac{-k\cos(n\alpha)}{\pi n}\right]_0^\pi \tag{18.5.40}$$

$$b_n = \frac{k}{n\pi}\,[1-\cos(n\pi)] \tag{18.5.41}$$

$$a_0 = \frac{1}{2\pi}\int_0^\pi k\ \ d\alpha \qquad a_0 = \frac{k}{2} \tag{18.5.42}$$

$$a_n = \frac{1}{\pi}\int_0^\pi k\ \cos(n\alpha)\ \ d\alpha \qquad a_n = 0 \tag{18.5.43}$$



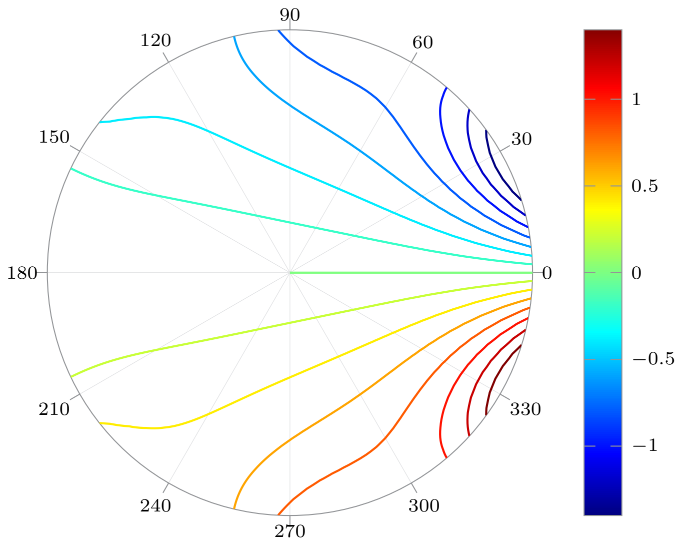
13. Since the function is odd,

$$\Phi(1,\,\theta) = \theta \qquad \theta \in (-\pi/2,\,\pi/2) \tag{18.5.44}$$

$$a_n = 0 \qquad a_0 = 0 \tag{18.5.45}$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \alpha\ \sin(n\alpha)\ \ \mathrm{d}\alpha \qquad b_n = \left[\frac{\sin(n\alpha) - n\alpha\,\cos(n\alpha)}{\pi n^2}\right]_{-\pi/2}^{\pi/2} \tag{18.5.46}$$

$$b_n = \frac{2\,\sin(n\pi/2)}{\pi n^2} - \frac{\cos(n\pi/2)}{n} \tag{18.5.47}$$



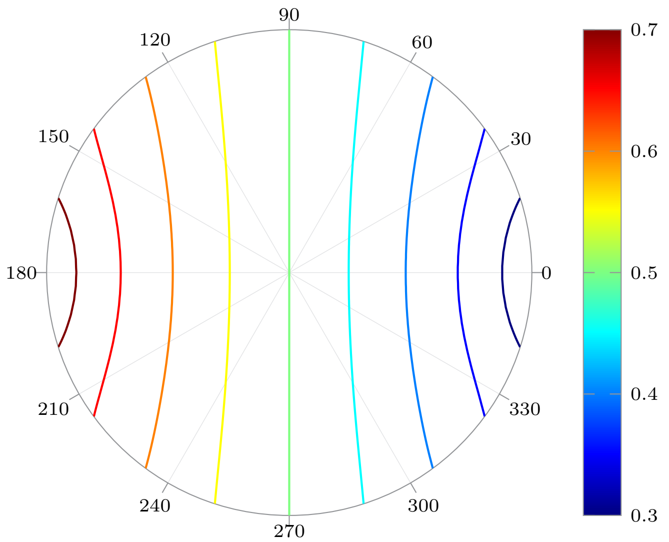
14. Since the function is even,

$$\Phi(1, \theta) = \frac{|\theta|}{\pi} \qquad \theta \in (-\pi, \pi) \tag{18.5.48}$$

$$a_0 = \frac{2}{\pi} \int_0^\pi \frac{\alpha}{\pi} \, \mathrm{d}\alpha \qquad a_0 = \frac{1}{2} \tag{18.5.49}$$

$$a_n = \frac{1}{\pi} \int_0^\pi \frac{\alpha}{\pi} \cos(n\alpha) \, \mathrm{d}\alpha \qquad a_n = \left[\frac{n\alpha \sin(n\alpha) + \cos(n\alpha)}{\pi^2 n^2}\right]_0^\pi \tag{18.5.50}$$

$$a_n = \frac{\cos(n\pi) - 1}{n^2\pi^2} \qquad b_n = 0 \tag{18.5.51}$$



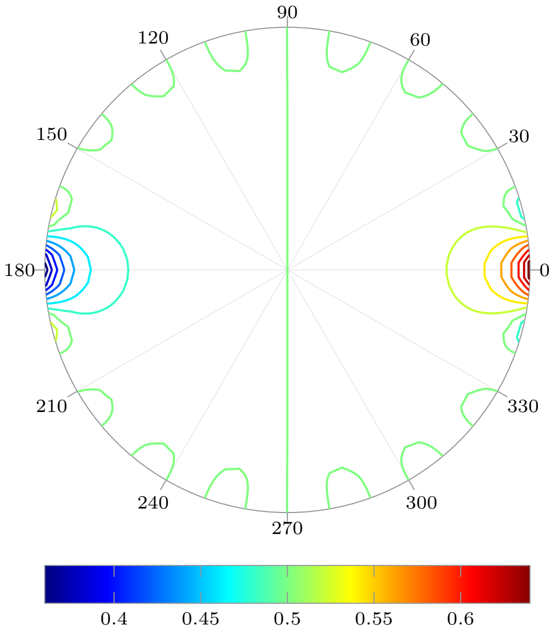
15. Since the function is even,

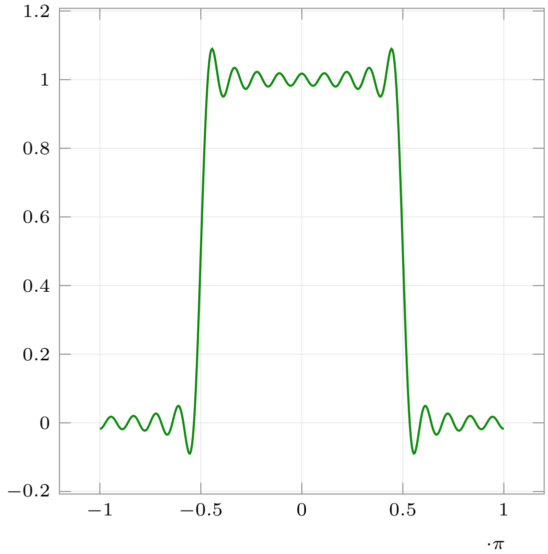
$$\Phi(1, \theta) = 1 \qquad \theta \in (-\pi/2, \pi/2) \tag{18.5.52}$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi/2} (1) \, \mathrm{d}\alpha \qquad a_0 = \frac{1}{2} \tag{18.5.53}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi/2} \cos(n\alpha) \, \mathrm{d}\alpha \qquad a_n = 2\left[\frac{\sin(n\alpha)}{n\pi}\right]_0^{\pi/2} \tag{18.5.54}$$

$$a_n = \frac{2\sin(n\pi/2)}{n\pi} \qquad b_n = 0 \tag{18.5.55}$$





16. Since the function is odd,

$$\Phi(1,\theta) = \begin{cases} \theta + \pi & \theta \in (-\pi, 0) \\ \theta - \pi & \theta \in (0, \pi) \end{cases} \tag{18.5.56}$$

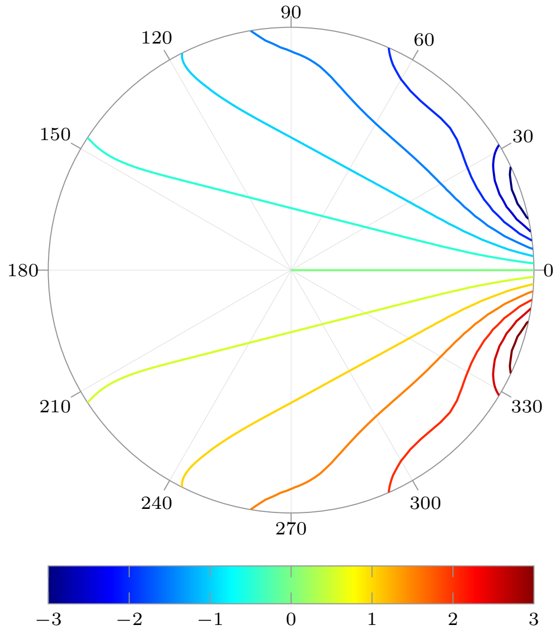
$$a_0 = 0 \tag{18.5.57}$$

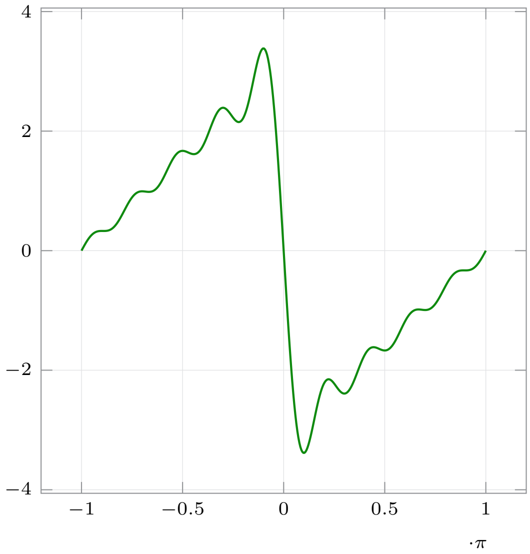
$$a_n = 0 \tag{18.5.58}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (\alpha + \pi)\ \sin(n\alpha)\ \mathrm{d}\alpha + \frac{1}{\pi}\int_{0}^{\pi} (\alpha - \pi)\ \sin(n\alpha)\ \mathrm{d}\alpha \tag{18.5.59}$$

$$= \left[\frac{\sin(n\alpha) - n(\alpha+\pi)\cos(n\alpha)}{n^2\pi}\right]_{-\pi}^{0} + \left[\frac{\sin(n\alpha) + n(\pi-\alpha)\cos(n\alpha)}{n^2\pi}\right]_{0}^{\pi} \tag{18.5.60}$$

$$b_n = \frac{-2}{n} \tag{18.5.61}$$



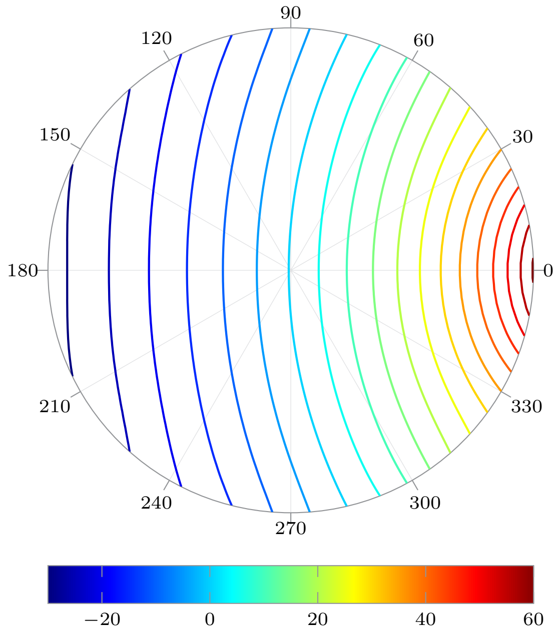


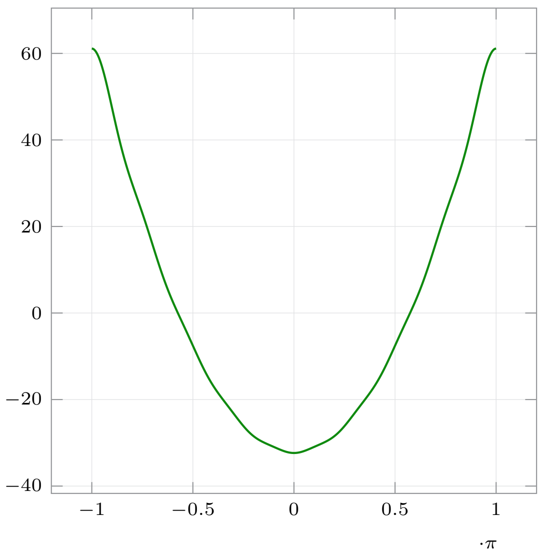
17. Since the function is even,

$$\Phi(1,\,\theta) = \frac{\theta^2}{\pi^2} \qquad \theta \in (-\pi,\,\pi) \qquad \text{18.5.62}$$

$$b_n = 0 \qquad a_0 = \frac{1}{\pi}\int_0^\pi \frac{\alpha^2}{\pi^2}\ \mathrm{d}\alpha = \frac{1}{3} \qquad \text{18.5.63}$$

$$a_n = \frac{2}{\pi}\int_0^\pi \frac{\alpha^2}{\pi^2}\ \cos(n\alpha)\ \ \mathrm{d}\alpha \qquad a_n = \frac{4\cos(n\pi)}{n^2\pi^2} \qquad \text{18.5.64}$$





18. Since the function is even,

$$\Phi(1,\,\theta) = \begin{cases} 0 & \theta \in (-\pi,\,0) \\ \theta & \theta \in (0,\,\pi) \end{cases} \tag{18.5.65}$$

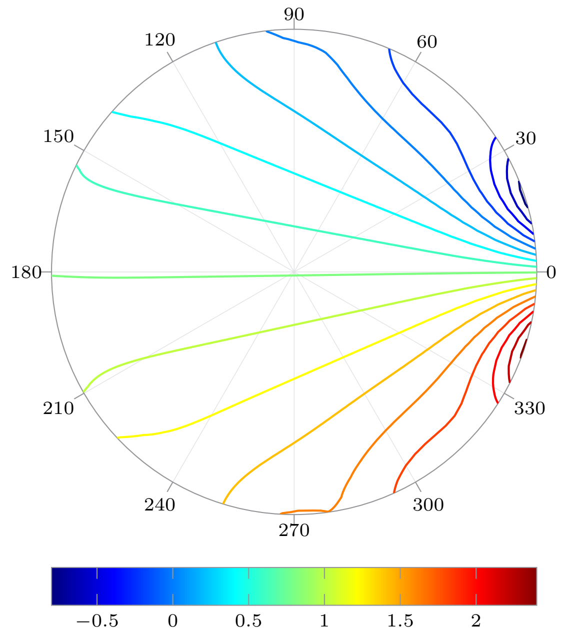
$$a_0 = \frac{1}{2\pi}\int_0^{\pi} \alpha \;\; \mathrm{d}\alpha \qquad\qquad a_0 = \frac{\pi}{4} \tag{18.5.66}$$

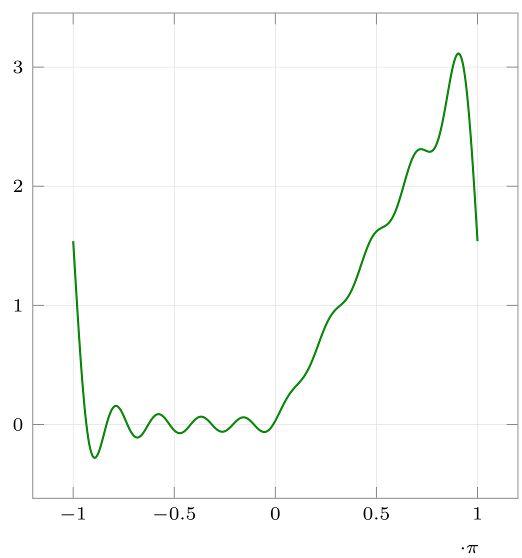
$$a_n = \frac{1}{\pi}\int_0^{\pi} \alpha \;\; \cos(n\alpha) \;\; \mathrm{d}\alpha \qquad\qquad a_n = \left[\frac{n\alpha\,\sin(n\alpha) + \cos(n\alpha)}{\pi n^2}\right]_0^{\pi} \tag{18.5.67}$$

$$a_n = \frac{\cos(n\pi) - 1}{\pi n^2} \tag{18.5.68}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \alpha \;\; \sin(n\alpha) \;\; \mathrm{d}\alpha \qquad\qquad b_n = \left[\frac{\sin(n\alpha) - n\alpha\,\cos(n\alpha)}{\pi n^2}\right]_0^{\pi} \tag{18.5.69}$$

$$b_n = \frac{-\cos(n\pi)}{n} \tag{18.5.70}$$





19. Coded in `sympy`. TBC.

20. Potential in a disk

(a) Using the Fourier series expansion of $\Phi(r, \theta)$,

$$\Phi(0, \theta) = a_0 = I_0 + \sum_{n=1}^{\infty} I_{n,c} + I_{n,s} \tag{18.5.71}$$

$$I_0 = \frac{1}{\pi R^2} \int_0^R \int_{-\pi}^{\pi} a_0 \ \mathrm{d}r \ (r) \ \mathrm{d}\alpha = a_0 \tag{18.5.72}$$

$$I_{n,c} = \frac{1}{\pi R^2} \int_0^R \int_{-\pi}^{\pi} a_n \ r \cos(n\alpha) \ \mathrm{d}r \ \ \mathrm{d}\alpha \tag{18.5.73}$$

$$\int_{-\pi}^{\pi} \cos(n\alpha) \ \ \mathrm{d}\alpha = \left[\frac{\sin(n\alpha)}{n}\right]_{-\pi}^{\pi} = 0 \tag{18.5.74}$$

$$I_{n,s} = \frac{1}{\pi R^2} \int_0^R \int_{-\pi}^{\pi} a_n \ r \sin(n\alpha) \ \mathrm{d}r \ \ \mathrm{d}\alpha \tag{18.5.75}$$

$$\int_{-\pi}^{\pi} \sin(n\alpha) \ \ \mathrm{d}\alpha = \left[\frac{\cos(n\alpha)}{n}\right]_{\pi}^{-\pi} = 0 \tag{18.5.76}$$

Since all the integrals vanish except for I_0, the relation is true.

(b) Laplace's equation in polar coordinates is,

$$\frac{1}{r}\Phi_r + \Phi_{rr} + \frac{1}{r^2}\Phi_{\theta\theta} = 0 \qquad\qquad \Phi(r, \theta) = F(r) \cdot G(\theta) \tag{18.5.77}$$

$$G \cdot (rF') + G \cdot (r^2 F'') = -F \cdot \ddot{G} \tag{18.5.78}$$

Separating the variables,

$$-k^2 = \frac{\ddot{G}}{G} \qquad\qquad G_k = p_1 \cos(k\theta) + p_2 \sin(k\theta) \tag{18.5.79}$$

$$0 = r^2 F'' + rF' - k^2 F \qquad\qquad F_k = q_1 r^k + q_2 r^{-k} \tag{18.5.80}$$

Since $G_k(\theta + 2\pi) = G_k(\theta)$,

$$G_k(\theta + 2\pi) = p_1 \cos(2k\pi + k\theta) + p_2 \sin(2k\pi + k\theta) \tag{18.5.81}$$

$$k \in \mathcal{I} \tag{18.5.82}$$

Since the function F_k has to be valid at $r = 0$, $q_2 = 0$. Consolidating the p, q parameters,

$$\Phi_n(R, \theta) = R^n \left[p_n \cos(n\theta) + q_n \sin(n\theta)\right] \tag{18.5.83}$$

Since the potential reduces to the Fourier series of the function at its boundary,

$$p_n = \frac{a_n}{R^n} \qquad\qquad q_n = \frac{b_n}{R^n} \tag{18.5.84}$$

$$p_0 = a_0 \tag{18.5.85}$$

Finally, the potential is,

$$\Phi(r,\,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \Big[a_n\,\cos(n\theta) + b_n\,\sin(n\theta)\Big] \tag{18.5.86}$$

(c) Using the Cauchy Riemann equations for polar coordinates,

$$\Phi_r = \frac{\Psi_\theta}{r} \qquad\qquad \Psi_r = -\frac{\Phi_\theta}{r} \tag{18.5.87}$$

$$\Psi_\theta = \sum_{n=1}^{\infty} n \left(\frac{r}{R}\right)^n \Big[a_n\,\cos(n\theta) + b_n\,\sin(n\theta)\Big] \tag{18.5.88}$$

$$\Psi = \sum_{n=1}^{\infty} \frac{r^n}{R^n} \Big[a_n\,\sin(n\theta) - b_n\,\cos(n\theta)\Big] + F(r) + C \tag{18.5.89}$$

$$\Psi_r = \sum_{n=1}^{\infty} \frac{nr^{n-1}}{R^n} \Big[a_n\,\sin(n\theta) - b_n\,\cos(n\theta)\Big] \tag{18.5.90}$$

$$\Psi = \sum_{n=1}^{\infty} \frac{r^n}{R^n} \Big[a_n\,\sin(n\theta) - b_n\,\cos(n\theta)\Big] + G(\theta) + C \tag{18.5.91}$$

Comparing the two expressions for $\Psi(r,\,\theta)$,

$$\Psi = C + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \Big[a_n\,\sin(n\theta) - b_n\,\cos(n\theta)\Big] \tag{18.5.92}$$

Since C is the mean value of Ψ over the boundary,

$$\int_0^{2\pi} \Psi(R,\,\alpha)\ \ \mathrm{d}\alpha = 2\pi C \tag{18.5.93}$$

$$C = \frac{1}{2\pi} \int_0^{2\pi} \Psi(R,\,\alpha)\ \ \mathrm{d}\alpha \tag{18.5.94}$$

(d) A series for the complex potential $\Phi + \mathrm{i}\ \Psi$

$$F(z) = a_0 + \mathrm{i}\ C + \frac{r^n\ a_n}{R^n}\ \Big[\cos(n\theta) + \mathrm{i}\ \ \sin(n\theta)\Big] + \frac{r^n\ b_n}{R^n}\ \Big[\sin(n\theta) - \mathrm{i}\ \ \cos(n\theta)\Big] \tag{18.5.95}$$

$$= a_0 + \mathrm{i}\ C + \Big[a_n - \mathrm{i}\ b_n\Big]\ \left[\frac{r\ \exp(\mathrm{i}\theta)}{R}\right]^n \tag{18.5.96}$$

$$F(r,\,\theta) = \frac{1}{2\pi}\int_0^{2\pi} F(R,\,\alpha)\ \ \mathrm{d}\alpha + \sum_{n=1}^{\infty}\Big[a_n - \mathrm{i}\ b_n\Big]\cdot\left(\frac{z}{R}\right)^n \tag{18.5.97}$$

18.6 General Properties of Harmonic Functions

1. Verifying Theorem 1,

$$F(z) = (z+1)^3 \qquad\qquad z_0 = 2.5 \tag{18.6.1}$$

$$F(z_0) = (3.5)^3 \qquad\qquad I = \oint_C F(z)\ \ \mathrm{d}z \tag{18.6.2}$$

$$I = \frac{1}{2\pi}\ \int_0^{2\pi}(3.5 + re^{\mathrm{i}\theta})^3\ \ \mathrm{d}\theta \tag{18.6.3}$$

$$I = \frac{1}{2\pi}\left[3.5^3\theta + 3r(3.5)^2\ \frac{e^{\mathrm{i}\theta}}{\mathrm{i}} + 10.5r^2\ \frac{e^{2\mathrm{i}\theta}}{2\mathrm{i}} + r^3\ \frac{e^{3\mathrm{i}\theta}}{3\mathrm{i}}\right]_0^{2\pi} \tag{18.6.4}$$

$$I = (3.5)^3 = F(z_0) \tag{18.6.5}$$

2. Verifying Theorem 1,

$$F(z) = 2z^4 \qquad\qquad z_0 = -2 \tag{18.6.6}$$

$$F(z_0) = 2(-2)^4 = 32 \qquad\qquad I = \oint_C F(z)\ \ \mathrm{d}z \tag{18.6.7}$$

$$I = \frac{1}{2\pi}\ \int_0^{2\pi} 2(-2 + re^{\mathrm{i}\theta})^4\ \ \mathrm{d}\theta \qquad\qquad I = \frac{1}{\pi}\ \Big[(-2)^4\ \theta + f(e^{\mathrm{i}\theta})\Big]_0^{2\pi} \tag{18.6.8}$$

$$I = 32 \tag{18.6.9}$$

3. Verifying Theorem 1,

$$F(z) = (3z - 2)^2 \qquad z_0 = 4 \tag{18.6.10}$$

$$F(z_0) = 100 \qquad I = \oint_C F(z) \; \mathrm{d}z \tag{18.6.11}$$

$$I = \frac{1}{2\pi} \int_0^{2\pi} (10 + 3re^{\mathrm{i}\theta})^2 \; \mathrm{d}\theta \qquad I = \frac{1}{2\pi} \left[(10)^2 \; \theta + f(e^{\mathrm{i}\theta}) \right]_0^{2\pi} \tag{18.6.12}$$

$$I = 100 \tag{18.6.13}$$

4. Verifying Theorem 1, using $G(z) = 1/F(z)$

$$G(z) = (z - 1)^2 \qquad z_0 = -1 \tag{18.6.14}$$

$$G(z_0) = 4 = \frac{1}{F(z_0)} \qquad I_g = \oint_C F(z) \; \mathrm{d}z \tag{18.6.15}$$

$$I_g = \frac{1}{2\pi} \int_0^{2\pi} (-2 - re^{\mathrm{i}\theta})^2 \; \mathrm{d}\theta \qquad I = \frac{1}{2\pi} \left[(-2)^2 \; \theta + f(e^{\mathrm{i}\theta}) \right]_0^{2\pi} \tag{18.6.16}$$

$$I_g = 4 = \frac{1}{I_f} \tag{18.6.17}$$

Since $G(z)$ is never zero in the unit disk centered on z_0, $F(z)$ is analytic in this disk and the theorem holds for F as well.

5. The theorem does not hold, but this is not a violation of the theorem.

$$F(z) = \sqrt{x^2 + y^2} + 0 \; \mathrm{i} \qquad F(z) = u(x, y) + \mathrm{i} \; v(x, y) \tag{18.6.18}$$

$$u_x \neq v_y \qquad u_y \neq -v_x \tag{18.6.19}$$

Since the Cauchy-Riemann relations do not hold, $|z|$ is not analytic.

6. Using Poisson's integral formula, with the point z_0 being the origin using the transform, $w = z - z_0$

$$\Phi(0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(R, \alpha) \; \frac{R^2}{R^2} \; \mathrm{d}\alpha = \frac{1}{2\pi R} \int_0^{2\pi} \Phi(R, \alpha) \; R \, \mathrm{d}\alpha \tag{18.6.20}$$

The last expression is the mean value of Φ on the boundary C, for C being any circle centered on z_0.

7. Verifying the second statement of Theorem 2,

$$\Phi(x, y) = (x - 1)(y - 1) \qquad (x_0, y_0) = (2, -2) \tag{18.6.21}$$

$$\Phi(x_0, y_0) = -3 \tag{18.6.22}$$

Integrating over a disk of radius 1 centered at z_0,

$$I = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} (1 + \cos\alpha)(-3 + \sin\alpha)\ (r)\ \mathrm{d}r\ \ \mathrm{d}\alpha \tag{18.6.23}$$

$$I_\alpha = \left[-3\alpha - 3\sin\alpha - \cos\alpha - \frac{\cos(2\alpha)}{2} \right]_0^{2\pi} = -6\pi \tag{18.6.24}$$

$$I = \int_0^1 (-6r)\ \ \mathrm{d}r = -3 \tag{18.6.25}$$

8. Verifying the second statement of Theorem 2,

$$\Phi(x, y) = x^2 - y^2 \qquad\qquad (x_0, y_0) = (3, 8) \tag{18.6.26}$$

$$\Phi(x_0, y_0) = -55 \tag{18.6.27}$$

Integrating over a disk of radius 1 centered at z_0,

$$I = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \Big[(3 + \cos\alpha)^2 - (8 + \sin\alpha)^2\Big]\ (r)\ \mathrm{d}r\ \ \mathrm{d}\alpha \tag{18.6.28}$$

$$I_\alpha = \left[-55\alpha + \frac{\sin(2\alpha)}{2} + 6\sin\alpha + 16\cos\alpha \right]_0^{2\pi} = -110\pi \tag{18.6.29}$$

$$I = \int_0^1 (-110r)\ \ \mathrm{d}r = -55 \tag{18.6.30}$$

9. Verifying the second statement of Theorem 2,

$$\Phi(x, y) = x + y + xy \qquad\qquad (x_0, y_0) = (1, 1) \tag{18.6.31}$$

$$\Phi(x_0, y_0) = 3 \tag{18.6.32}$$

Integrating over a disk of radius 1 centered at z_0,

$$I = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \Big[3 + 2\cos\alpha + 2\sin\alpha + 0.5\sin(2\alpha)\Big]\ (r)\ \mathrm{d}r\ \ \mathrm{d}\alpha \tag{18.6.33}$$

$$I_\alpha = \left[3\alpha - \frac{\cos(2\alpha)}{2} - 2\cos\alpha + 2\sin\alpha \right]_0^{2\pi} = 6\pi \tag{18.6.34}$$

$$I = \int_0^1 (6r)\ \ \mathrm{d}r = 3 \tag{18.6.35}$$

10. Let the maximum value of $|F(z)|$ be M, which the function attains at an interior point z_0 of the region R

$$M = |F(z_0)| \le \frac{1}{2\pi}\left|\oint_{C_1+C_2} \frac{F(z)}{z-z_0}\ \mathrm{d}z\right| \tag{18.6.36}$$

$$\le \frac{1}{2\pi}\left|\oint_{C_1} \frac{F(z)}{z-z_0}\ \mathrm{d}z\right| + \frac{1}{2\pi}\left|\oint_{C_2} \frac{F(z)}{z-z_0}\ \mathrm{d}z\right| \tag{18.6.37}$$

$$|F(z)| \le M - k \qquad \forall \qquad z \text{ on } C_1 \tag{18.6.38}$$

Here $K > 0$. The continuity of $F(z)$ on C, ensures that there exists some arc of the circular boundary of length L_1 on which $|F(z)| \le M - k$

$$M \le \frac{1}{2\pi}\ \frac{M-k}{r}\ L_1 + \frac{1}{2\pi}\ \frac{M}{r}\ (2\pi r - L_1) \tag{18.6.39}$$

$$\le M - \frac{k}{2\pi r}\ L_1 \tag{18.6.40}$$

This is a contradiction since $M < M$, which means the initial assumption is wrong.

11. Plotting the surfaces $F(x, y)$, and checking the positions of the extrema, (interactive plots not shown here)

. Looking at problems 7, 8, 9

(a) Saddle point at $(1, 1)$, with no other extrema.

(b) Saddle point at the origin, with no other extrema.

(c) Saddle point at $(-1, -1)$ with no other extrema.

12. Maximum modulus theorem,

(a) For the function $F(z) = z^2$,

$$F(z) = z^2 \qquad\qquad R : x \in [1, 5] \quad y \in [2, 4] \tag{18.6.41}$$

$$|F(z)| = x^2 + y^2 \qquad\qquad P^* = (0, 0) \qquad \text{(minimum)} \tag{18.6.42}$$

$$\max_R |F(z)| = \max_{x\in R} x^2 + \max_{y \in R} y^2 \tag{18.6.43}$$

Clearly this value is reached at the point $(5, 4)$, which lies on the boundary.
For the function $F(z) = \sin z$,

$$F(z) = \sin z \qquad\qquad R : r \in [0, 1] \quad \theta \in [0, 2\pi] \tag{18.6.44}$$

$$|F(z)| = \sqrt{\sin^2[r \cos\theta] + \sinh^2[r \sin\theta]} \tag{18.6.45}$$

Clearly this value is reached at the boundary since both $\sin^2(r \cos\theta)$ and $\sinh^2(r \sin\theta)$ are both increasing functions of r for the interval $r \in [0, 1]$.

For the function $F(z) = e^z$,

$$F(z) = e^z \qquad R : \text{any bounded domain} \tag{18.6.46}$$

$$|F(z)| = e^x \qquad \max_R |F(z)| = \max_{x\in R} e^x \tag{18.6.47}$$

Clearly this value is reached at the boundary since e^x is an increasing function of x in any bounded interval $[x_a, x_b]$

(b) Since $|z|$ is not analytic at the origin, and the region $|z| \le 2$ contains this point, Theorem 3 does not apply. (Use Cauchy-Riemann equations to check).

(c) Consider the vertical line $L : \pi/2 + y\mathrm{i}$

$$\sin(z) = \cosh y \qquad \max_R\, [F(z)] = \max_{y\in R}\, [\cosh y] \tag{18.6.48}$$

$$\tag{18.6.49}$$

Any region containing $\pi/2 + 0\mathrm{i}$ must contain points not on the real axis, where the value of $\sin z$ is greater than the value at this point.

(d) Let $f(z) \ne 0$ everywhere in the region R. By Theorem 3, $|F(z)|$ must take its maximum and minimum values at the boundary.

This means that $F(z)$ has the same maximum and minimum value making it a constant function.

This is a contradiction which means that the assumption is incorrect.

13. Finding the maximum in the unit disk,

$$F(z) = \cos z \qquad |F(z)| = \sqrt{\sinh^2 y + \cos^2 x} \tag{18.6.50}$$

$$\max_{(1,\theta)} |F(z)| = \max_\theta \sqrt{\sinh^2(\sin\theta) + \cos^2(\cos\theta)} \qquad \theta^* = \frac{\pi}{2}, \frac{3\pi}{2} \tag{18.6.51}$$

$$\max |F(z)| = \sqrt{1 + \sinh^2(1)} \tag{18.6.52}$$

14. Finding the maximum in the unit disk,

$$F(z) = \exp(z^2) \qquad |F(z)| = e^{x^2 - y^2} \tag{18.6.53}$$

$$\max_{(1,\theta)} |F(z)| = \max_\theta\, [\exp(\cos 2\theta)] \qquad \theta^* = 0, \pi \tag{18.6.54}$$

$$\max |F(z)| = e \tag{18.6.55}$$

15. Finding the maximum in the unit disk,

$$F(z) = \sinh(2z) \qquad |F(z)| = \sqrt{\sinh^2(2x) + \sin^2(2y)} \tag{18.6.56}$$

$$\max_{(1,\theta)} |F(z)| = \max_{\theta} \sqrt{\sinh^2(2\cos\theta) + \sin^2(2\sin\theta)} \qquad \theta^* = 0, \pi \tag{18.6.57}$$

$$\max |F(z)| = \sinh(2) \tag{18.6.58}$$

16. Finding the maximum in the unit disk,

$$F(z) = az + b, \qquad a \neq 0 \qquad |F(z)| = \left|Ae^{\mathrm{i}(\alpha+\theta)} + Be^{\mathrm{i}\beta}\right| \tag{18.6.59}$$

$$\theta^* = \beta - \alpha \tag{18.6.60}$$

$$\max |F(z)| = |a| + |b| \tag{18.6.61}$$

The maximum is obtained by rotating a so that it faces the same direction as b. The transform $az = ae^{\mathrm{i}\theta}$ is the act of rotating a by some angle without changing its modulus.

By the triangle inequality,

$$|a| + |b| \geq |a + b| \geq ae^{\mathrm{i}\theta} + b \tag{18.6.62}$$

17. Finding the maximum in the unit disk,

$$F(z) = 2z^2 - 2 \qquad |F(z)| = \sqrt{(2x^2 - 2y^2 - 2)^2 + (4xy)^2} \tag{18.6.63}$$

$$\max_{(1,\theta)} |F(z)| = \max_{\theta} \left[2\sqrt{2 - 2\cos(2\theta)}\right] \qquad \theta^* = \frac{\pi}{2}, \frac{3\pi}{2} \tag{18.6.64}$$

$$\max |F(z)| = 4 \tag{18.6.65}$$

18. Finding the maximum in the given rectangle,

$$F(z) = e^x \ \sin y \qquad R : x \in [a, b], \quad y \in [0, 2\pi] \tag{18.6.66}$$

$$x^* = b \qquad y^* = \frac{\pi}{2}, \frac{3\pi}{2} \tag{18.6.67}$$

$$\max_{R} |F(z)| = e^b \tag{18.6.68}$$

Clearly, this value is reached on the boundary of the rectangle. Since e^x is a monotonically increasing function of x, the maxima is easy to find.

19. Looking at Φ and its harmonic conjugate Ψ,

$$F(z) = \Phi(x, y) + \mathrm{i}\ \Psi(x, y) \qquad\qquad F(z) = x + \mathrm{i}y \tag{18.6.69}$$

$$\max_R \Phi = \max x \in Rx \qquad\qquad \max_R \Psi = \max y \in Ry \tag{18.6.70}$$

Clearly, this counterexample proves that Φ and Ψ need not reach their maximum at the same point in the region R.

20. Finding the maximum in the upper half unit disc in R^*,

$$F^*(w) = e^w \qquad\qquad \Phi^* + \mathrm{i}\ \Psi^* = (e^u \cos v) + \mathrm{i}\ (e^u \sin v) \tag{18.6.71}$$

$$R : |w| \leq 1, \quad v \geq 0 \qquad\qquad (u_1, v_1) = (1, 0) \tag{18.6.72}$$

The inverse mapping yields the region R as

$$R : x \geq 0, \quad y \geq 0, \quad |z| \leq 1 \qquad\qquad u + \mathrm{i}\ v = (x^2 - y^2) + \mathrm{i}\ (2xy) \tag{18.6.73}$$

$$\Phi(x, y) = \exp(x^2 - y^2)\ \cos(2xy) \qquad\qquad (x_1, y_1) = (1, 0) \tag{18.6.74}$$

This happens to be the pre-image of (u_1, y_1), under the mapping $w = z^2$. The mapping is conformal and bijective, which means that this is not a chance occurrence.