Chapter 11

Fourier Analysis

11.1 Fourier Series

1. The fundamental period of the functions is, by trial and error,

Function	Period	Function	Period
$\cos(x)$	2π	$\sin(x)$	2π
$\cos(2x)$	π	$\sin(2x)$	π
$\cos(\pi x)$	2	$\sin(\pi x)$	2
$\cos(2\pi x)$	1	$\sin(2\pi x)$	1

2. The fundamental period of the functions is, by trial and error,

Function	Period	Function	Period
$\cos(nx)$	$\dfrac{2\pi}{n}$	$\sin(nx)$	$\dfrac{2\pi}{n}$
$\cos\left(\dfrac{2\pi\ x}{k}\right)$	k	$\sin\left(\dfrac{2\pi\ x}{k}\right)$	k
$\cos\left(\dfrac{2\pi n\ x}{k}\right)$	$\dfrac{k}{n}$	$\sin\left(\dfrac{2\pi n\ x}{k}\right)$	$\dfrac{k}{n}$

3. If the functions f, g both have period p,

$$h(x) = af(x) + bg(x) \qquad h(x+p) = af(x+p) + bg(x+p) \tag{11.1.1}$$

$$h(x+p) = af(x) + bg(x) = h(x) \tag{11.1.2}$$

This means that h also has period p

4. $b = 1/a$ proves the second half and is not covered here.

$$f(x+p) = f(p) \qquad a \neq 0 \tag{11.1.3}$$

$$f(ax+q) = f(ax) \qquad \implies f(ax+q) = f(ax+ap) \tag{11.1.4}$$

$$f(ax+ap) = f(ax) \tag{11.1.5}$$

This means that the period scales with the reciprocal of the factor a.

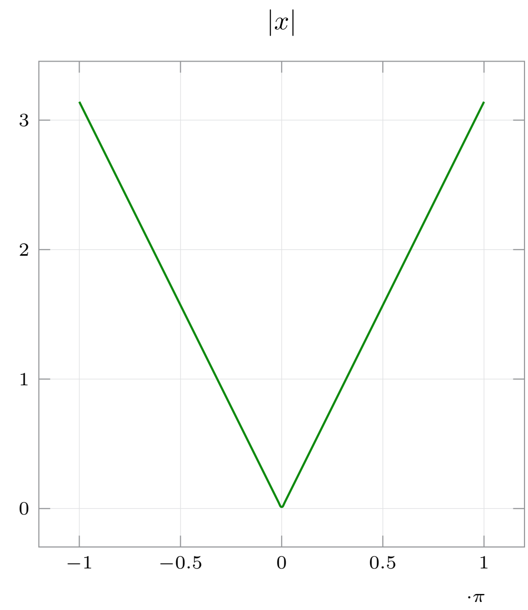
5. Let the function f be a constant function.

$$f(x+p) = f(x) = c \qquad \forall\ c \in \mathcal{R}^+ \tag{11.1.6}$$

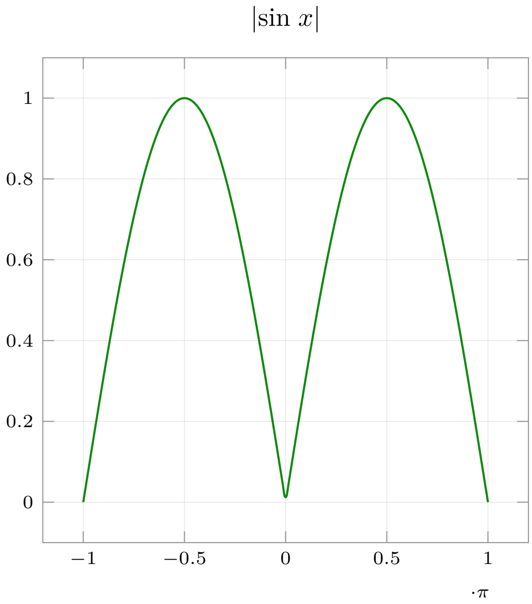
There is no smallest possible choice of p that satisfies this condition.

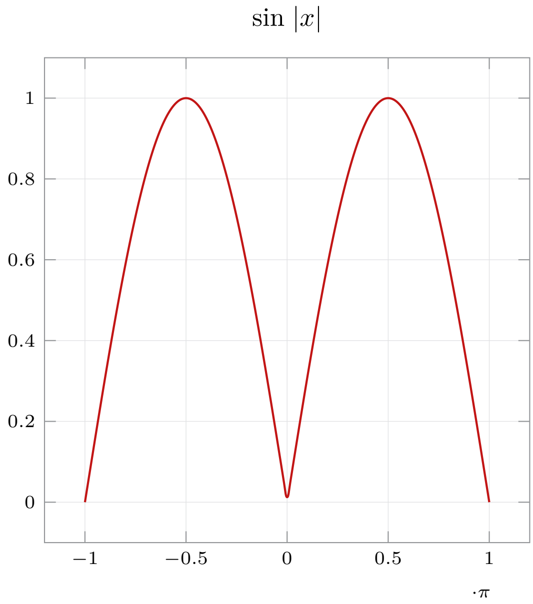
This means that a constant function has any positive real number as a period, but cannot have a fundamental period.

6. Plotting the function in the given domain

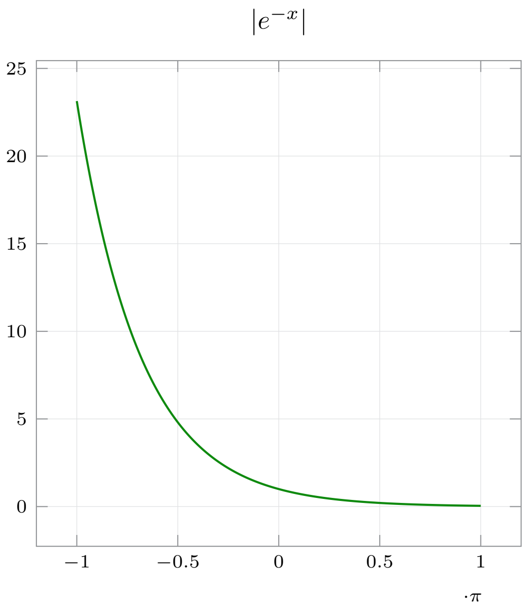


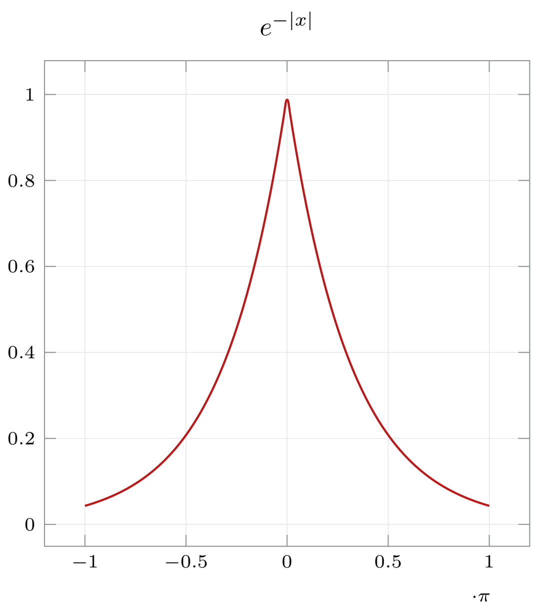
7. Plotting the function in the given domain



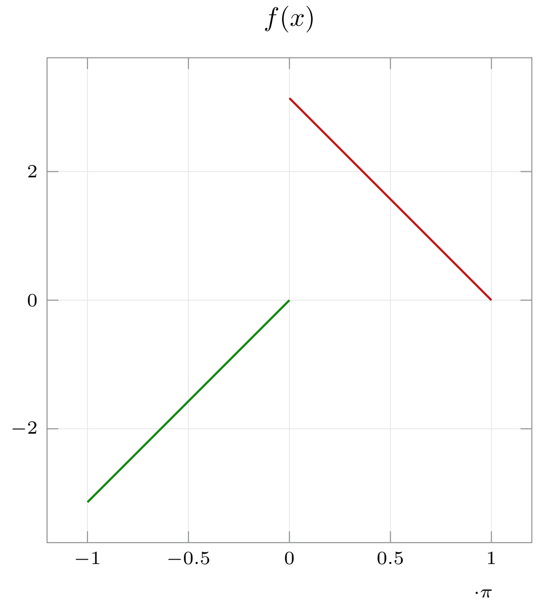


8. Plotting the function in the given domain

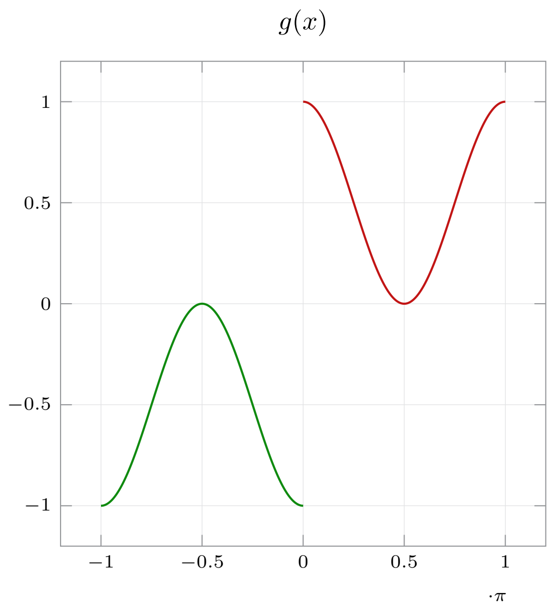




9. Plotting the function in the given domain



10. Plotting the function in the given domain



11. Performing integration by parts,

$$\int_{-\pi}^{\pi} x \;\cos(nx)\;\mathrm{d}x = \left[\frac{x}{n}\;\sin(nx)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n}\;\mathrm{d}x \tag{11.1.7}$$

$$= \left[\frac{\cos(nx)}{n^2}\right]_{-\pi}^{\pi} = 0 \tag{11.1.8}$$

$$\int_{-\pi}^{\pi} x^2 \;\sin(nx)\;\mathrm{d}x = \left[\frac{-x^2}{n}\;\cos(nx)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{2x\;\cos(nx)}{n}\;\mathrm{d}x \tag{11.1.9}$$

$$= \left[\frac{\cos(nx)}{n^2}\right]_{-\pi}^{\pi} = 0 \tag{11.1.10}$$

$$\int_{-\pi}^{\pi} e^{-2x}\;\cos(nx)\;\mathrm{d}x = \left[\frac{e^{-2x}}{n}\;\sin(nx)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{2e^{-2x}\;\sin(nx)}{n}\;\mathrm{d}x \tag{11.1.11}$$

$$= \frac{2}{n}\int_{-\pi}^{\pi} e^{-2x}\;\sin(nx)\;\mathrm{d}x \tag{11.1.12}$$

$$= \left[\frac{-2e^{-2x}}{n^2}\;\cos(nx)\right]_{-\pi}^{\pi} - \frac{4}{n^2}\int_{-\pi}^{\pi} e^{-2x}\;\cos(nx)\;\mathrm{d}x \tag{11.1.13}$$

$$I = \frac{4\;\cos(n\pi)\;\sinh(2\pi)}{n^2} - \frac{4I}{n^2} \tag{11.1.14}$$

$$I = \frac{4\cos(n\pi)\;\sinh(2\pi)}{n^2+4} \tag{11.1.15}$$

12. Finding the Fourier series using Euler's equations

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\;\;\mathrm{d}x = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\;\mathrm{d}x + \int_{0}^{\pi} x\;\mathrm{d}x\right] \tag{11.1.16}$$

$$= \left[\frac{-x^2}{4\pi}\right]_{-\pi}^{0} + \left[\frac{x^2}{4\pi}\right]_{0}^{\pi} = \frac{\pi}{2} \tag{11.1.17}$$

Finding the cosine coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx) \, \mathrm{d}x \tag{11.1.18}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos(nx) \, \mathrm{d}x + \int_{0}^{\pi} x\cos(nx) \, \mathrm{d}x\right] \tag{11.1.19}$$

$$= \frac{1}{\pi}\left[\frac{x \ \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{-\pi} + \frac{1}{\pi}\left[\frac{x \ \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi} \tag{11.1.20}$$

$$= \frac{2(\cos(n\pi)-1)}{\pi n^2} = \begin{cases} 0 & n \quad \text{even} \\ \frac{-4}{\pi n^2} & n \quad \text{odd} \end{cases} \tag{11.1.21}$$

Finding the sine coefficients

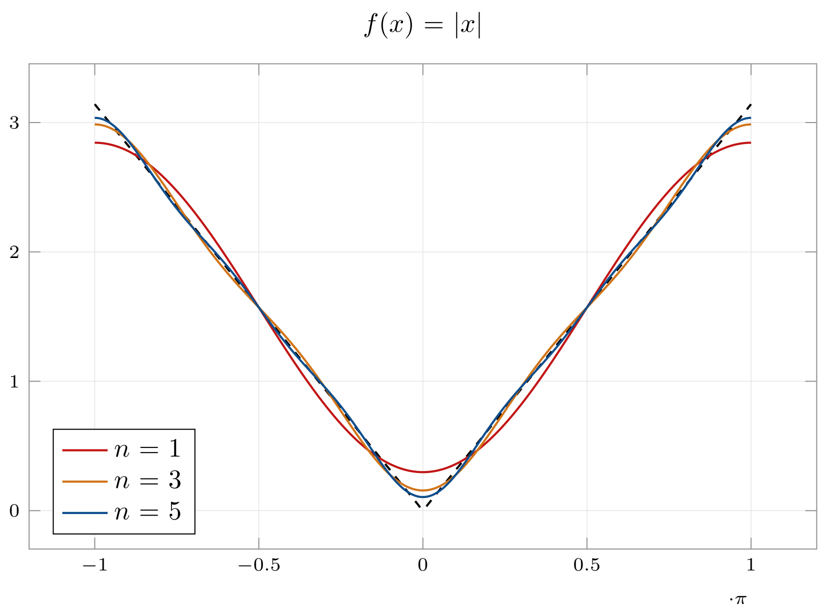
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin(nx) \, \mathrm{d}x \tag{11.1.22}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\sin(nx) \, \mathrm{d}x + \int_{0}^{\pi} x\sin(nx) \, \mathrm{d}x\right] \tag{11.1.23}$$

$$= \frac{1}{\pi}\left[\frac{x \ \cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x \ \cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi}^{0} \tag{11.1.24}$$

$$= \frac{2(\cos(n\pi)-1)}{\pi n^2} = 0 \tag{11.1.25}$$

Graphing the function itself and its partial sums,



13. Finding the Fourier series using Euler's equations

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{2\pi}\left[\int_{-\pi}^{0} x\ \mathrm{d}x + \int_{0}^{\pi} (\pi - x)\ \mathrm{d}x\right] \tag{11.1.26}$$

$$= \left[\frac{x^2}{4\pi}\right]_{-\pi}^{0} - \left[\frac{(\pi - x)^2}{4\pi}\right]_{0}^{\pi} = 0 \tag{11.1.27}$$

Finding the cosine coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\ \mathrm{d}x \tag{11.1.28}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos(nx)\ \mathrm{d}x + \int_{0}^{\pi} (\pi - x)\cos(nx)\ \mathrm{d}x\right] \tag{11.1.29}$$

$$= \frac{1}{\pi}\left[\frac{x\ \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\pi\sin(nx)}{n} - \frac{x\sin(nx)}{n} - \frac{\cos(nx)}{n^2}\right]_{0}^{\pi} \tag{11.1.30}$$

$$= \frac{2(1-\cos(n\pi))}{\pi n^2} = \begin{cases} 0 & n \ \text{even} \\ \frac{4}{\pi n^2} & n \ \text{odd} \end{cases} \tag{11.1.31}$$

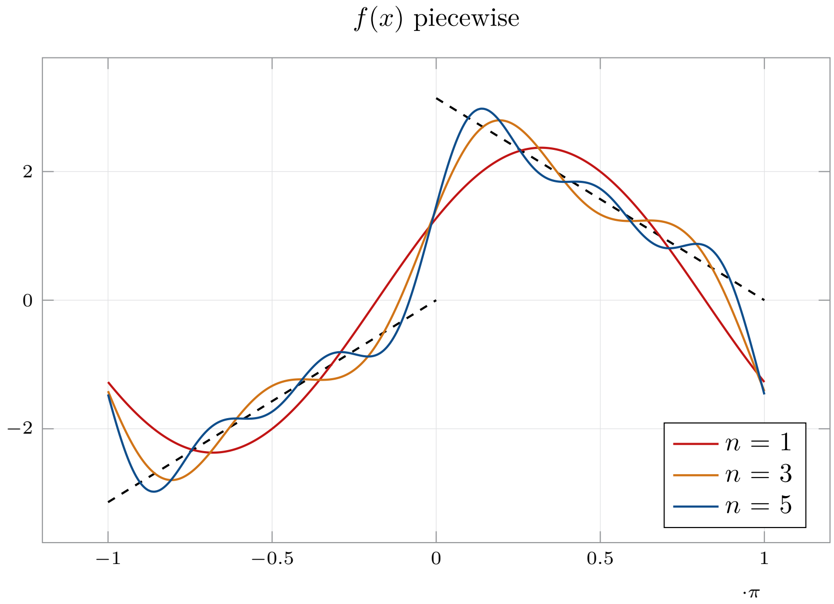
Finding the sine coefficients

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (x)\sin(nx)\ \mathrm{d}x + \int_{0}^{\pi} (\pi - x)\sin(nx)\ \mathrm{d}x\right] \tag{11.1.32}$$

$$= \frac{1}{\pi}\left[-\frac{x\ \cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{\pi\cos(nx)}{n} + \frac{x\ \cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{0}^{\pi} \tag{11.1.33}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n\pi)}{n} + \frac{\pi}{n}\right] = \begin{cases} 0 & n \ \text{even} \\ \frac{2}{n} & n \ \text{odd} \end{cases} \tag{11.1.34}$$

Graphing the function itself and its partial sums,



14. Finding the Fourier series using Euler's equations

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\ \mathrm{d}x \tag{11.1.35}$$

$$= \left[\frac{x^3}{6\pi}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3} \tag{11.1.36}$$

Finding the cosine coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\ \mathrm{d}x \tag{11.1.37}$$

$$= \left[\frac{(n^2x^2-2)\sin(nx)+2nx\cos(nx)}{\pi n^3}\right]_{-\pi}^{\pi} \tag{11.1.38}$$

$$= \frac{4\cos(n\pi)}{n^2} = \begin{cases} \dfrac{4}{n^2} & n \quad \text{even} \\ \dfrac{-4}{n^2} & n \quad \text{odd} \end{cases} \tag{11.1.39}$$

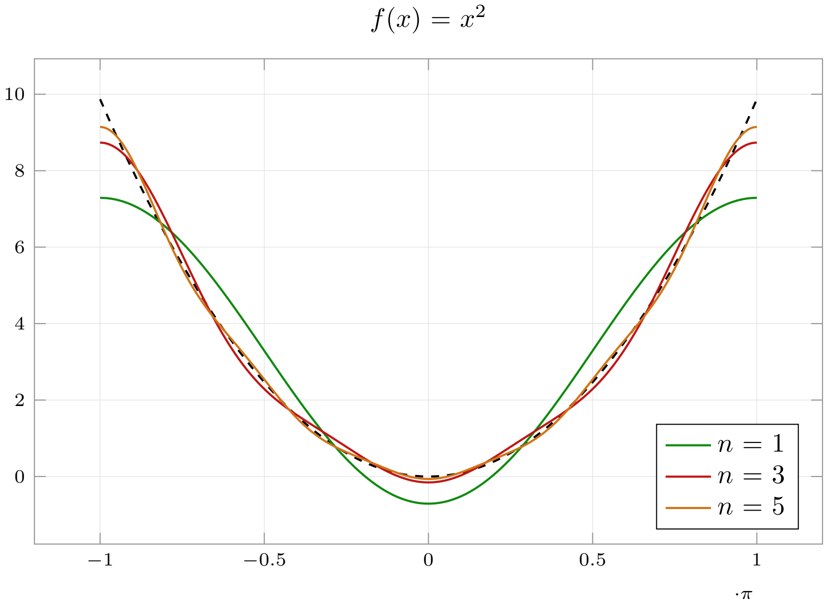
Finding the sine coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\ \mathrm{d}x \tag{11.1.40}$$

$$= \left[\frac{(2-n^2x^2)\cos(nx)+2nx\sin(nx)}{\pi n^3}\right]_{-\pi}^{\pi} \tag{11.1.41}$$

$$= \frac{4\cos(n\pi)}{n^2} = 0 \tag{11.1.42}$$

Graphing the function itself and its partial sums,



15. Finding the Fourier series using Euler's equations

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\ \mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x^2\ \mathrm{d}x \tag{11.1.43}$$

$$= \left[\frac{x^3}{6\pi}\right]_0^{2\pi} = \frac{4\pi^2}{3} \tag{11.1.44}$$

Finding the cosine coefficients

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)\ \mathrm{d}x \tag{11.1.45}$$

$$= \left[\frac{(n^2x^2-2)\sin(nx) + 2nx\cos(nx)}{\pi n^3}\right]_0^{2\pi} = \frac{4}{n^2} \tag{11.1.46}$$

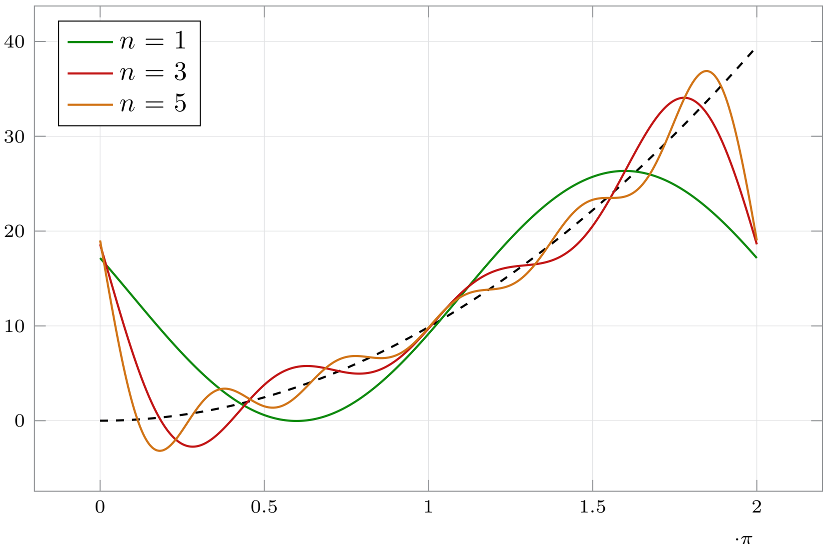
Finding the sine coefficients

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)\ \mathrm{d}x \tag{11.1.47}$$

$$= \left[\frac{(2-n^2x^2)\cos(nx) + 2nx\sin(nx)}{\pi n^3}\right]_0^{2\pi} = \frac{-4\pi}{n} \tag{11.1.48}$$

Graphing the function itself and its partial sums,

$f(x) = x^2$



16. The constant term is zero since the function is odd.

The cosine coefficients are zero since the function is odd.

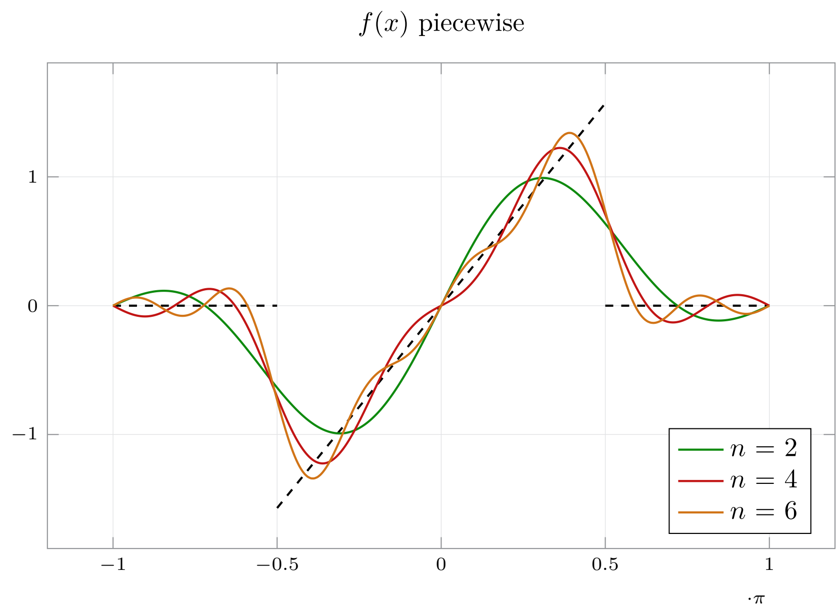
Finding the sine coefficients

$$b_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} x \sin(nx) \ \mathrm{d}x \tag{11.1.49}$$

$$= \frac{1}{\pi} \left[-\frac{x \ \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_{-\pi/2}^{\pi/2} \tag{11.1.50}$$

$$= \begin{cases} \dfrac{-\cos(m\pi)}{n} & n = 2m \\ \dfrac{2}{\pi(2m-1)^2} \ (-1)^{m+1} & n = 2m-1 \end{cases} \tag{11.1.51}$$

Graphing the function itself and its partial sums,



17. Finding the Fourier series using Euler's equations

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(x+\pi)\ \mathrm{d}x + \int_{0}^{\pi}(\pi - x)\ \mathrm{d}x\right] \tag{11.1.52}$$

$$= \left[\frac{(x+\pi)^2}{4\pi}\right]_{-\pi}^{0} - \left[\frac{(\pi - x)^2}{4\pi}\right]_{0}^{\pi} = \frac{\pi}{2} \tag{11.1.53}$$

Finding the cosine coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x \tag{11.1.54}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\cos(nx)\ \mathrm{d}x + \int_{0}^{\pi}(\pi - x)\cos(nx)\ \mathrm{d}x\right] \tag{11.1.55}$$

$$= \frac{1}{\pi}\left[\frac{(x+\pi)\ \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{(\pi - x)\sin(nx)}{n} - \frac{\cos(nx)}{n^2}\right]_{0}^{\pi} \tag{11.1.56}$$

$$= \frac{2(1-\cos(n\pi))}{\pi n^2} = \begin{cases} 0 & n \quad \text{even} \\ \frac{4}{\pi n^2} & n \quad \text{odd} \end{cases} \tag{11.1.57}$$

Finding the sine coefficients

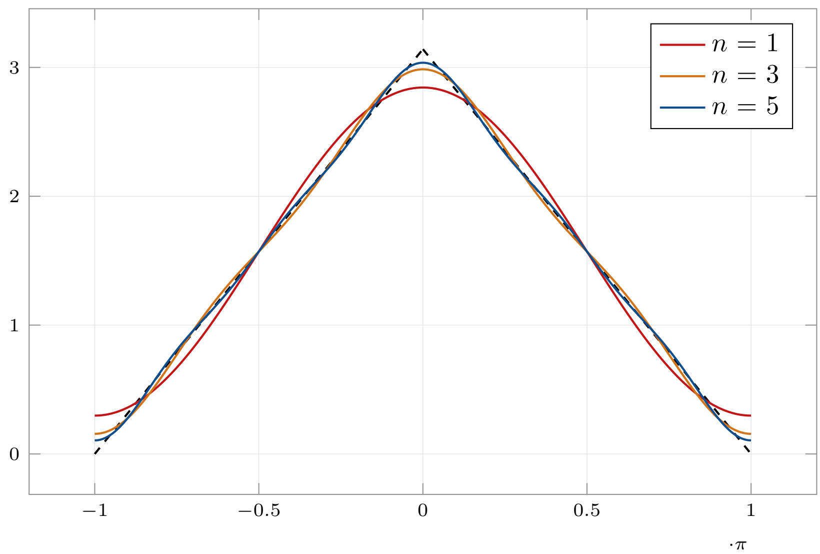
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (x+\pi)\sin(nx)\ \mathrm{d}x + \int_{0}^{\pi} (\pi - x)\sin(nx)\ \mathrm{d}x\right] \quad 11.1.58$$

$$= \frac{1}{\pi}\left[-\frac{(x+\pi)\ \cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{(\pi - x)\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{0}^{\pi} \quad 11.1.59$$

$$= 0 \quad 11.1.60$$

Graphing the function itself and its partial sums,

$f(x)$ piecewise



18. Finding the Fourier series using Euler's equations

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{2\pi}\ \int_{0}^{\pi} (1)\ \mathrm{d}x \quad 11.1.61$$

$$= \left[\frac{x}{2\pi}\right]_{0}^{\pi} = \frac{1}{2} \quad 11.1.62$$

Finding the cosine coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \frac{1}{\pi}\ \int_{0}^{\pi} (1)\cos(nx)\ \mathrm{d}x \quad 11.1.63$$

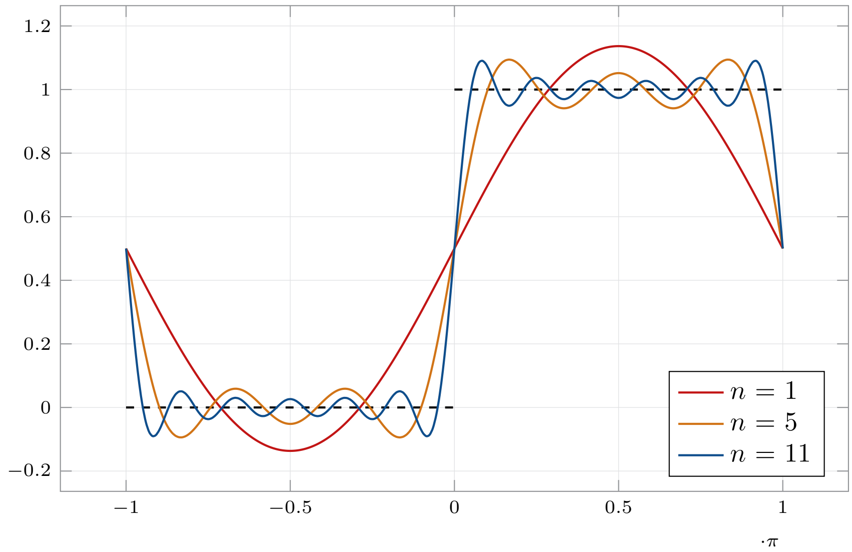
$$= \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_{0}^{\pi} = 0 \quad 11.1.64$$

Finding the sine coefficients

$$b_n = \frac{1}{\pi} \int_0^\pi (1) \sin(nx) \ \mathrm{d}x = \frac{1}{\pi} \left[-\frac{\cos(nx)}{n} \right]_0^\pi = \frac{1-\cos(n\pi)}{n\pi} \tag{11.1.65}$$

Graphing the function itself and its partial sums,

$f(x)$ piecewise



19. Finding the Fourier series using Euler's equations

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x = \frac{1}{2\pi} \int_0^\pi x \ \mathrm{d}x \tag{11.1.66}$$

$$= \left[\frac{x^2}{4\pi} \right]_0^\pi = \frac{\pi}{4} \tag{11.1.67}$$

Finding the cosine coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \ \mathrm{d}x = \frac{1}{\pi} \int_0^\pi x \cos(nx) \ \mathrm{d}x \tag{11.1.68}$$

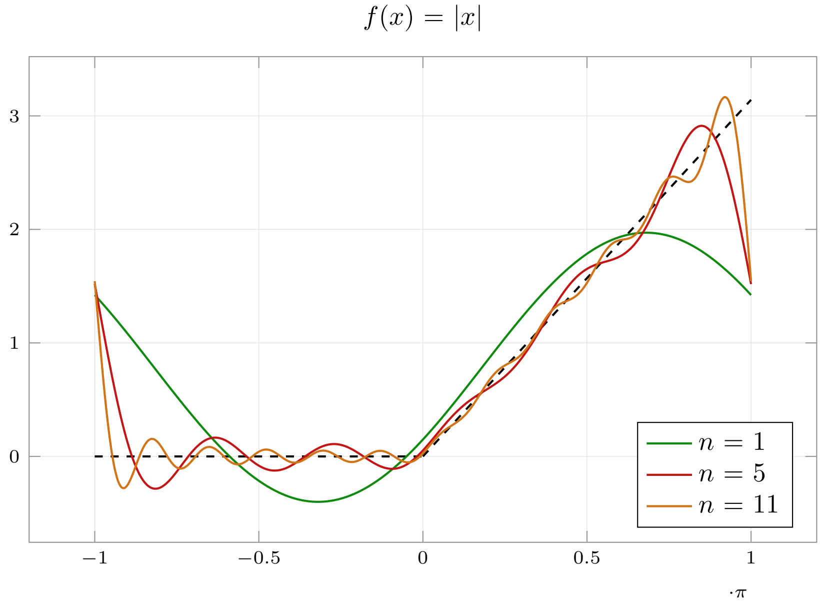
$$= \frac{1}{\pi} \left[\frac{x \ \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^\pi = \frac{(\cos(n\pi) - 1)}{\pi n^2} \tag{11.1.69}$$

Finding the sine coefficients

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \ \mathrm{d}x = \frac{1}{\pi} \int_0^\pi x \sin(nx) \ \mathrm{d}x \tag{11.1.70}$$

$$= \frac{1}{\pi} \left[-\frac{x \ \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_0^\pi = \frac{-\cos(n\pi)}{n} \tag{11.1.71}$$

Graphing the function itself and its partial sums,



20. The constant term is zero since the function is odd.

The cosine coefficients are zero since the function is odd.

Finding the sine coefficients

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} (-\pi/2)\ \sin(nx)\ \mathrm{d}x + \int_{-\pi/2}^{\pi/2} x \sin(nx)\ \mathrm{d}x \right. \tag{11.1.72}$$

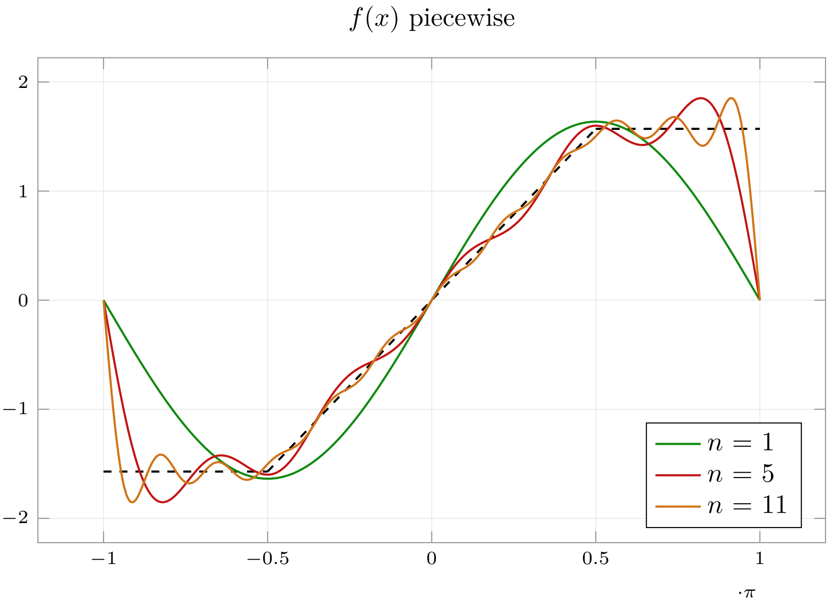
$$\left. + \int_{\pi/2}^{\pi} (\pi/2)\ \sin(nx)\ \mathrm{d}x \right] \tag{11.1.73}$$

$$= \frac{1}{2}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{2}\left[\frac{-\cos(nx)}{n}\right]_{\pi/2}^{\pi} \tag{11.1.74}$$

$$+ \frac{1}{\pi}\left[-\frac{x\ \cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi/2}^{\pi/2} \tag{11.1.75}$$

$$= \frac{-\cos(n\pi)}{n} + \frac{2\sin(n\pi/2)}{\pi n^2} \tag{11.1.76}$$

Graphing the function itself and its partial sums,



21. Finding the Fourier series using Euler's equations

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x = \frac{1}{2\pi} \left[\int_{-\pi}^{0} (-x - \pi) \ \mathrm{d}x + \int_{0}^{\pi} (\pi - x) \ \mathrm{d}x \right] \tag{11.1.77}$$

$$= -\left[\frac{(x + \pi)^2}{4\pi} \right]_{-\pi}^{0} - \left[\frac{(\pi - x)^2}{4\pi} \right]_{0}^{\pi} = 0 \tag{11.1.78}$$

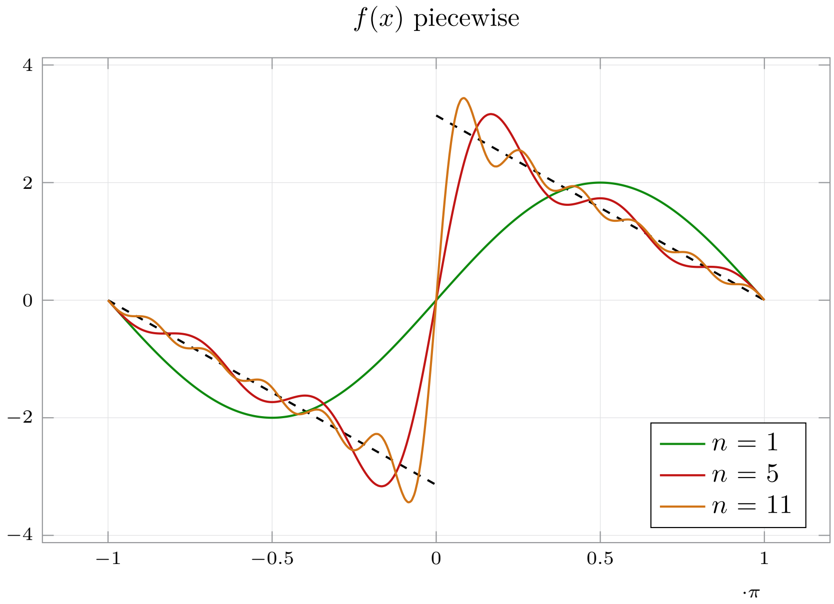
The cosine coefficients for an odd function are zero.
Finding the sine coefficients

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} -(x + \pi) \sin(nx) \ \mathrm{d}x + \int_{0}^{\pi} (\pi - x) \sin(nx) \ \mathrm{d}x \right] \tag{11.1.79}$$

$$= \frac{1}{\pi} \left[\frac{(x + \pi) \ \cos(nx)}{n} - \frac{\sin(nx)}{n^2} \right]_{-\pi}^{0} + \frac{1}{\pi} \left[-\frac{(\pi - x) \cos(nx)}{n} - \frac{\sin(nx)}{n^2} \right]_{0}^{\pi} \tag{11.1.80}$$

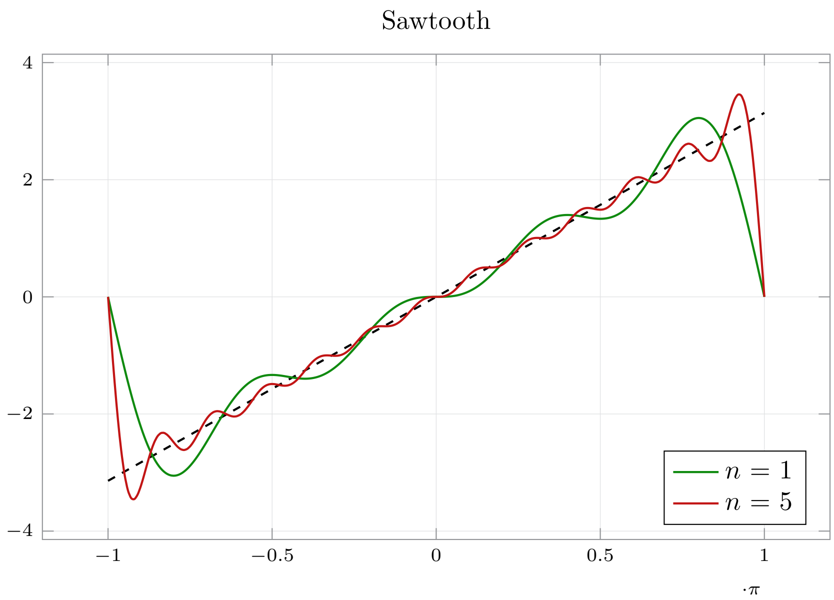
$$= \frac{2}{n} \tag{11.1.81}$$

Graphing the function itself and its partial sums,

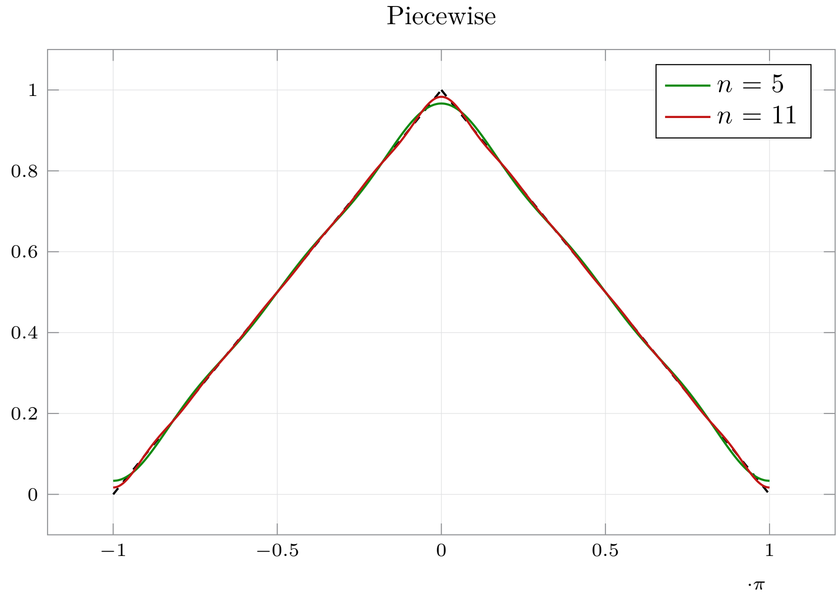


22. Using the Fourier series graphs to identify out the underlying function,

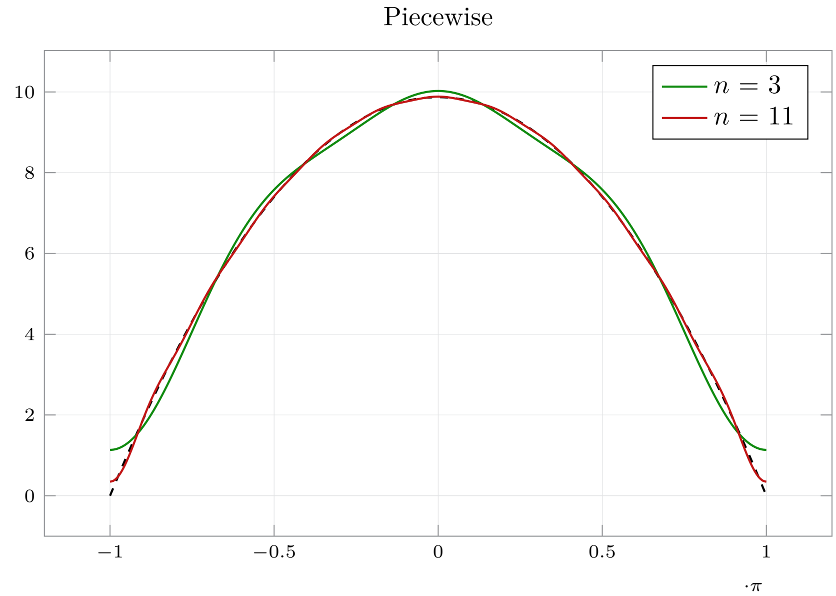
(a) The function is a sawtooth wave $y = x$ with a primary domain of $[-\pi, \pi]$.



(b) The function is a triangle wave with a primary domain of $[-\pi, \pi]$ and range $[0, 1]$.



(c) The function is a downward parabola $y = -x + \pi^2$ with a primary domain of $[-\pi, \pi]$.



23. The average of the left and right handed limits of $f(x)$ at $x = 0$ needs to match the Fourier series expansion at $x = 0$

$$\frac{f(0)^{+} + f(0)^{-}}{2} = \frac{\pi - \pi}{2} = 0 \tag{11.1.82}$$

$$F(x) = 0 + \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx) \tag{11.1.83}$$

$$F(0) = 0 \tag{11.1.84}$$

This verifies the statement.

24. Performing the integration,

$$\int_{-a}^{a} \cos(mx)\ \cos(nx)\ \mathrm{d}x = \frac{1}{2}\int_{-a}^{a} \cos(mx+nx) + \cos(mx-nx)\ \mathrm{d}x \tag{11.1.85}$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha x)}{\alpha} + \frac{\sin(\beta x)}{\beta}\right]_{-a}^{a} \tag{11.1.86}$$

$$I(a) = \frac{\sin(\alpha a)}{\alpha} + \frac{\sin(\beta a)}{\beta} \tag{11.1.87}$$

Here $\alpha = (m+n)$ and $\beta = (m-n)$ with both being nonzero integers.

$$I(a) = 0 \qquad\qquad \implies a = \pi \tag{11.1.88}$$

For $a \to a/k$, the condition becomes α, β are integer multiples of k.
Performing the integration,

$$\int_{-a}^{a} \sin(mx)\ \sin(nx)\ \mathrm{d}x = \frac{1}{2}\int_{-a}^{a} \cos(mx-nx) - \cos(mx+nx)\ \mathrm{d}x \tag{11.1.89}$$

$$= \frac{1}{2}\left[\frac{\sin(\beta x)}{\beta} - \frac{\sin(\alpha x)}{\alpha}\right]_{-a}^{a} \tag{11.1.90}$$

$$I(a) = \frac{\sin(\beta a)}{\beta} - \frac{\sin(\alpha a)}{\alpha} \tag{11.1.91}$$

Performing the integration,

$$\int_{-a}^{a} \sin(mx)\ \cos(nx)\ \mathrm{d}x = \frac{1}{2}\int_{-a}^{a} \cos(mx-nx) - \cos(mx+nx)\ \mathrm{d}x \tag{11.1.92}$$

$$= \frac{-1}{2}\left[\frac{\cos(\beta x)}{\beta} + \frac{\cos(\alpha x)}{\alpha}\right]_{-a}^{a} \tag{11.1.93}$$

$$I(a) = 0 \qquad \text{identically} \tag{11.1.94}$$

25. Order of Fourier coefficients in terms of the discontinuity in f and its higher-order derivatives.
. Start with f being discontinuous at $x = a$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{a} f(x)\ \cos(nx)\ \mathrm{d}x + \frac{1}{\pi}\int_{a}^{\pi} f(x)\ \cos(nx)\ \mathrm{d}x \tag{11.1.95}$$

$$= \frac{[f(a^-) - f(a^+)]\sin(na)}{\pi n} - \frac{1}{n\pi}\left[\int_{-\pi}^{a} f'(x)\sin(nx)\ \mathrm{d}x + \int_{a}^{\pi} f'(x)\sin(nx)\ \mathrm{d}x\right] \tag{11.1.96}$$

Clearly, if f is continuous, the procedure is the exact same acting on f' with an extra $1/n$ factor introduced.

Thus, the order of the Fourier series is n^{-k} depending on the smallest discontinouus derivative of f being $f^{(k-1)}$.

The integral of a sine or consine function over the entire domain is identically zero, which gets rid of the earlier powers of $1/n$ in the Fourier expansion when the current derivative is still continuous.

Integration by parts requires recursive usage of differentiation until some discontinuity is hit.

11.2 Arbitrary Period, Even and Odd Functions, Half-Range Expansions

1. Checking the functions,

$$e^{-x} \neq e^{x} \qquad \text{Neither} \tag{11.2.1}$$

$$e^{-|-x|} = e^{-|x|} \qquad \text{Even} \tag{11.2.2}$$

$$(-x)^3 \ \cos(-nx) = -x^3 \ \cos(nx) \qquad \text{Odd} \tag{11.2.3}$$

$$(-x)^2 \ \tan(-\pi x) = -x^2 \ \tan(\pi x) \qquad \text{Odd} \tag{11.2.4}$$

$$\sinh(-x) - \cosh(-x) = -\sinh(x) - \cosh(x) \qquad \text{Neither} \tag{11.2.5}$$

2. Checking the functions,

$$\sin^2(-x)\, \sin^2(x) \qquad \text{Even} \tag{11.2.6}$$

$$\sin\left((-x)^2\right) = \sin(x^2) \qquad \text{Even} \tag{11.2.7}$$

$$\ln(-x) = \text{not defined} \qquad \text{Neither} \tag{11.2.8}$$

$$\frac{-x}{(-x)^2+1} = -\frac{x}{x^2+1} \qquad \text{Odd} \tag{11.2.9}$$

$$(-x) \ \cot(-x) = x \ \cot(x) \qquad \text{Even} \tag{11.2.10}$$

3. For even functions f, g

$$f(-x) + g(-x) = f(x) + g(x) \qquad \text{Even} \tag{11.2.11}$$

$$f(-x) \cdot g(-x) = f(x) \cdot g(x) \qquad \text{Even} \tag{11.2.12}$$

4. For odd functions f, g

$$f(-x) + g(-x) = -f(x) - g(x) = -[f(x) + g(x)] \qquad \text{Odd} \tag{11.2.13}$$

$$f(-x) \cdot g(-x) = f(x) \cdot g(x) \qquad \text{Even} \tag{11.2.14}$$

5. For an odd function f

$$|f(-x)| = |-f(x)| = |x| \qquad \text{Even} \tag{11.2.15}$$

6. For odd function f and even function g,

$$f(-x) \cdot g(-x) = -f(x) \cdot g(x) \qquad \text{Odd} \tag{11.2.16}$$

7. Functions need to be both even and odd,

$$f(-x) = f(x) \qquad f(-x) = -f(x) \tag{11.2.17}$$

$$f(x) = 0 \tag{11.2.18}$$

8. The function is even with period $p = 2L = 2$.

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(x) \ \mathrm{d}x \tag{11.2.19}$$

$$= \frac{1}{2} \left[\int_{-1}^{0} -x \ \mathrm{d}x + \int_{0}^{1} x \ \mathrm{d}x \right] \tag{11.2.20}$$

$$= \left[\frac{-x^2}{4} \right]_{-1}^{0} + \left[\frac{x^2}{4} \right]_{0}^{1} = \frac{1}{2} \tag{11.2.21}$$

Calculating the Fourier cosine coefficients,

$$a_n = \int_{-1}^{1} f(x) \ \cos(nx) \ \mathrm{d}x \tag{11.2.22}$$

$$= \int_{-1}^{0} -x \cos(n\pi x) \ \mathrm{d}x + \int_{0}^{1} x \cos(n\pi x) \ \mathrm{d}x \tag{11.2.23}$$

$$= -\left[\frac{x \sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2} \right]_{-1}^{0} + \left[\frac{x \sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2} \right]_{0}^{1} \tag{11.2.24}$$

$$= \frac{2}{n^2\pi^2} \left[\cos(n\pi) - 1 \right] \tag{11.2.25}$$

9. The function is odd with period $p = 2L = 4$.

$$a_0 = 0 \qquad a_n = 0 \tag{11.2.26}$$

Calculating the Fourier sine coefficients,

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\ \sin\left(\frac{n\pi x}{2}\right)\ \mathrm{d}x \tag{11.2.27}$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(-1)\sin\left(\frac{n\pi x}{2}\right)\ \mathrm{d}x + \int_{0}^{2}(1)\sin\left(\frac{n\pi x}{2}\right)\ \mathrm{d}x\right] \tag{11.2.28}$$

$$= \left[\frac{1}{n\pi}\ \cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} - \left[\frac{1}{n\pi}\ \cos\left(\frac{n\pi x}{2}\right)\right]_{0}^{2} \tag{11.2.29}$$

$$= \frac{2}{n\pi}\ [1-\cos(n\pi)] \tag{11.2.30}$$

10. The function is odd with period $p = 2L = 8$.

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{11.2.31}$$

Calculating the Fourier sine coefficients,

$$a_n = \frac{1}{4}\int_{-4}^{4} f(x)\ \sin\left(\frac{n\pi x}{4}\right)\ \mathrm{d}x \tag{11.2.32}$$

$$= \frac{1}{4}\left[\int_{-4}^{0}(-x-4)\sin\left(\frac{n\pi x}{4}\right)\ \mathrm{d}x + \int_{0}^{4}(-x+4)\sin\left(\frac{n\pi x}{4}\right)\ \mathrm{d}x\right] \tag{11.2.33}$$

$$= \frac{1}{4}\left[\frac{4(x+4)}{n\pi}\ \cos\left(\frac{n\pi x}{4}\right) - \frac{16}{n^2\pi^2}\ \sin\left(\frac{n\pi x}{4}\right)\right]_{-4}^{0} \tag{11.2.34}$$

$$+ \frac{1}{4}\left[\frac{4(x-4)}{n\pi}\ \cos\left(\frac{n\pi x}{4}\right) - \frac{16}{n^2\pi^2}\ \sin\left(\frac{n\pi x}{4}\right)\right]_{0}^{4} \tag{11.2.35}$$

$$= \frac{8}{n\pi} \tag{11.2.36}$$

11. The function is even with period $p = 2L = 2$.

$$b_n = 0 \tag{11.2.37}$$

Calculating the constant term,

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\ \mathrm{d}x \tag{11.2.38}$$

$$= \frac{1}{2}\left[\int_{-1}^{1} x^2\ \mathrm{d}x\right] = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} \tag{11.2.39}$$

Calculating the Fourier cosine coefficients,

$$a_n = \int_{-1}^{1} x^2 \cos(n\pi x) \, dx \tag{11.2.40}$$

$$= \left[\frac{x^2}{n\pi} \sin(n\pi x) + \frac{2x}{n^2\pi^2} \cos(n\pi x) - \frac{2}{n^3\pi^3} \sin(n\pi x) \right]_{-1}^{1} \tag{11.2.41}$$

$$= \frac{4}{n^2\pi^2} \cos(n\pi) \tag{11.2.42}$$

12. The function is even with period $p = 2L = 4$.

$$b_n = 0 \tag{11.2.43}$$

Calculating the constant term,

$$a_0 = \frac{1}{4} \int_{-1}^{1} f(x) \, dx \tag{11.2.44}$$

$$= \frac{1}{4} \left[\int_{-2}^{2} \left(1 - \frac{x^2}{4}\right) \, dx \right] = \frac{1}{4} \left[x - \frac{x^3}{12} \right]_{-2}^{2} = \frac{2}{3} \tag{11.2.45}$$

Calculating the Fourier cosine coefficients,

$$a_n = \frac{1}{2} \int_{-2}^{2} \left(1 - \frac{x^2}{4}\right) \cos\left(\frac{n\pi x}{2}\right) \, dx \tag{11.2.46}$$

$$= \frac{1}{2} \left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \right]_{-2}^{2} \tag{11.2.47}$$

$$- \frac{1}{8} \left[\frac{2x^2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \frac{16}{n^3\pi^3} \sin\left(\frac{n\pi x}{2}\right) + \frac{8x}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) \right]_{-2}^{2} \tag{11.2.48}$$

$$= \frac{-4}{n^2\pi^2} \cos(n\pi) \tag{11.2.49}$$

13. The function is even with period $p = 2L = 1$.

$$a_0 = \int_{-1/2}^{1/2} f(x) \, dx = \left[\int_{0}^{1/2} x \, dx \right] = \left[\frac{x^2}{2} \right]_{0}^{1/2} = \frac{1}{8} \tag{11.2.50}$$

Calculating the Fourier cosine coefficients,

$$a_n = 2 \int_{-1/2}^{1/2} f(x) \ \cos(nx) \ \mathrm{d}x = 2 \int_0^{1/2} (x) \cos(2n\pi x) \ \mathrm{d}x \tag{11.2.51}$$

$$= 2 \left[\frac{x}{2n\pi} \sin(2n\pi x) + \frac{1}{4n^2\pi^2} \cos(2n\pi x) \right]_0^{1/2} \tag{11.2.52}$$

$$= \frac{1}{2n^2\pi^2} \ [\cos(n\pi) - 1] \tag{11.2.53}$$

Calculating the Fourier sine coefficients,

$$b_n = 2 \int_{-1/2}^{1/2} f(x) \ \sin(nx) \ \mathrm{d}x = 2 \int_0^{1/2} (x) \sin(2n\pi x) \ \mathrm{d}x \tag{11.2.54}$$

$$= 2 \left[- \frac{x}{2n\pi} \cos(2n\pi x) + \frac{1}{4n^2\pi^2} \sin(2n\pi x) \right]_0^{1/2} \tag{11.2.55}$$

$$= \frac{-1}{2n\pi} \ \cos(n\pi) \tag{11.2.56}$$

14. The function is even with period $p = 2L = 1$.

$$a_0 = \int_{-1/2}^{1/2} f(x) \ \mathrm{d}x = \left[\int_{-1/2}^{1/2} \cos(\pi x) \ \mathrm{d}x \right] = \left[\frac{\sin(\pi x)}{\pi} \right]_{-1/2}^{1/2} = \frac{2}{\pi} \tag{11.2.57}$$

Calculating the Fourier cosine coefficients,

$$a_n = 2 \int_{-1/2}^{1/2} f(x) \ \cos(nx) \ \mathrm{d}x = 2 \int_{1/2}^{1/2} \cos(\pi x) \ \cos(2n\pi x) \ \mathrm{d}x \tag{11.2.58}$$

$$= \int_{-1/2}^{1/2} \cos[(2n+1)\pi x] + \cos[(2n-1)\pi x] \ \mathrm{d}x \tag{11.2.59}$$

$$= \left[\frac{\sin[(2n+1)\pi x]}{(2n+1)\pi} + \frac{\sin[(2n-1)\pi x]}{(2n-1)\pi} \right]_{-1/2}^{1/2} \tag{11.2.60}$$

$$= \frac{4 \cdot (-1)^n}{\pi(1+2n)(1-2n)} \tag{11.2.61}$$

15. The function is odd with period $p = 2L = 2\pi$.

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{11.2.62}$$

Calculating the Fourier sine coefficients,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\ \mathrm{d}x \tag{11.2.63}$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin(nx)\ \mathrm{d}x - \int_{\pi/2}^{\pi}(x-\pi)\sin(nx)\ \mathrm{d}x\right] \tag{11.2.64}$$

$$= \frac{2}{\pi}\left[\frac{x\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi/2}^{0} + \frac{2}{\pi}\left[\frac{(x-\pi)\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi/2}^{\pi} \tag{11.2.65}$$

$$= \frac{4}{\pi n^2}\ \sin(n\pi/2) \tag{11.2.66}$$

16. The function is odd with period $p = 2L = 2$.

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{11.2.67}$$

Calculating the Fourier sine coefficients,

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)\ \mathrm{d}x = 2\int_0^1 x^2\sin(n\pi x)\ \mathrm{d}x \tag{11.2.68}$$

$$= 2\left[-\frac{x^2\cos(n\pi x)}{n\pi} + \frac{2x\sin(n\pi x)}{n^2\pi^2} + \frac{2\cos(n\pi x)}{n^3\pi^3}\right]_0^1 \tag{11.2.69}$$

$$= \frac{-2}{n\pi}\cos(n\pi) + \frac{4}{n^3\pi^3}[\cos(n\pi) - 1] \tag{11.2.70}$$

17. The function is even with period $p = 2L = 2$.

$$b_n = 0 \tag{11.2.71}$$

Finding the constant term,

$$a_0 = \int_0^1(-x+1)\,\mathrm{d}x = \left[-\frac{(x-1)^2}{2}\right]_0^1 = \frac{1}{2} \tag{11.2.72}$$

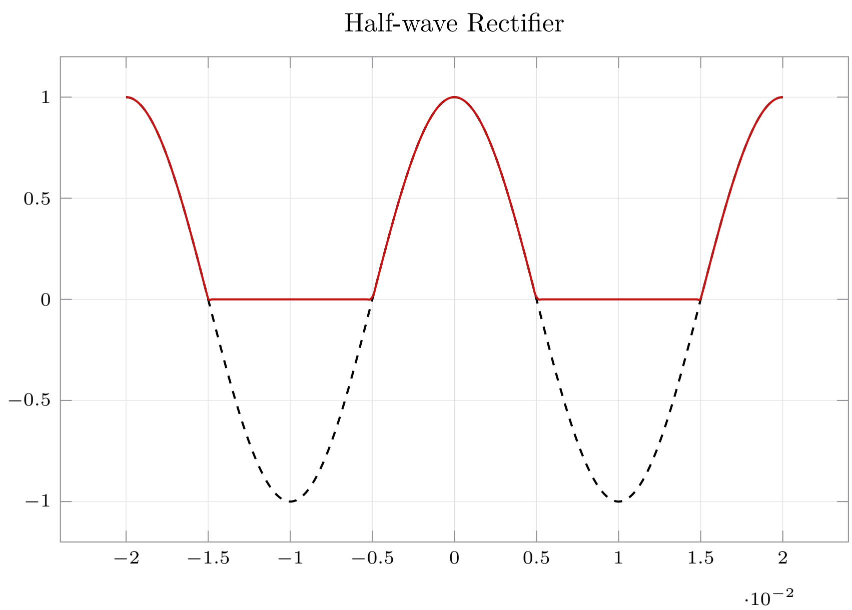
Calculating the Fourier cosine coefficients,

$$a_n = 2 \int_0^1 f(x) \ \cos(nx) \ \mathrm{d}x = 2 \int_0^1 (-x+1) \ \cos(n\pi x) \ \mathrm{d}x \tag{11.2.73}$$

$$= 2 \left[\frac{(1-x)\sin(n\pi x)}{n\pi} - \frac{\cos(n\pi x)}{n^2\pi^2} \right]_0^1 \tag{11.2.74}$$

$$= \frac{2}{n^2\pi^2}[1-\cos(n\pi)] \tag{11.2.75}$$

18. Half-wave rectifier acting on $v(x) = V_0 \cos(100\pi x)$



The function is even with period $p = 2L = 0.02$.

$$b_n = 0 \tag{11.2.76}$$

Finding the constant term,

$$a_0 = \frac{1}{L} \int_0^L f\left(\frac{n\pi x}{L}\right) \ \mathrm{d}x = 100 \int_0^{0.005} V_0 \ \cos\left(100\pi x\right) \ \mathrm{d}x \tag{11.2.77}$$

$$= \left[\frac{V_0 \ \sin(100\pi x)}{\pi} \right]_0^{0.005} = \frac{V_0}{\pi} \tag{11.2.78}$$

Calculating the Fourier cosine coefficients,

$$a_n = 200 \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x \tag{11.2.79}$$

$$= 200 \int_0^{0.005} V_0 \cos(100\pi x) \cos(100n\pi x) \, \mathrm{d}x \tag{11.2.80}$$

$$= 100V_0 \int_0^{0.005} \Big[\cos[(n+1)100\pi x] + \cos[(n-1)100\pi x] \Big] \, \mathrm{d}x \tag{11.2.81}$$

$$= 100V_0 \left[\frac{\sin[(n+1)100\pi\, x]}{(n+1)100\pi} + \frac{\sin[(n-1)100\pi\, x]}{(n-1)100\pi} \right]_0^{1/200} \tag{11.2.82}$$

$$= \frac{V_0}{\pi} \left[\frac{\cos(n\pi/2)}{(n+1)} - \frac{\cos(n\pi/2)}{(n-1)} \right] = \frac{-2V_0}{\pi(n^2-1)} \cos\left(\frac{n\pi}{2}\right) \tag{11.2.83}$$

19. Fourier series expansions of powers of $\cos^3 x$,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^3 x \, \mathrm{d}x \tag{11.2.84}$$

$$= \left[\sin x - \frac{\sin^3 x}{3} \right]_{-\pi}^{\pi} = 0 \tag{11.2.85}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^3 x \cos(nx) \, \mathrm{d}x \tag{11.2.86}$$

$$= \frac{1}{8\pi} \int_{-\pi}^{\pi} \Big[3\cos[(n+1)x] + 3\cos[(n-1)x] \tag{11.2.87}$$

$$+ \cos[(n+3)x] + \cos[(n-3)x] \Big] \, \mathrm{d}x \tag{11.2.88}$$

$$= \frac{1}{8\pi} \Bigg[\frac{3\sin[(n+1)x]}{(n+1)} + \frac{3\sin[(n-1)x]}{(n-1)} + \frac{\sin[(n+3)x]}{(n+3)} \tag{11.2.89}$$

$$+ \frac{\sin[(n-3)x]}{(n-3)} \Bigg]_{-\pi}^{\pi} = 0 \quad \forall \quad n \notin \{1, 3\} \tag{11.2.90}$$

$$a_1 = \frac{3}{4} \qquad a_3 = \frac{1}{4} \tag{11.2.91}$$

A similar Fourier series expansion can be given for $\sin^3(x)$.

The expansion for $\cos^4(x)$ is

$$\cos^4(x) = \frac{[1+\cos(2x)]^2}{4} = \frac{1+\cos^2(2x)+2\cos(2x)}{4} \tag{11.2.92}$$

$$= \frac{3+4\cos(2x)+\cos(4x)}{8} \tag{11.2.93}$$

This did not require explicit computation of the Fourier coefficients since it is a power of $\cos^2(x)$.

20. Using the Fourier series from Problem 11,

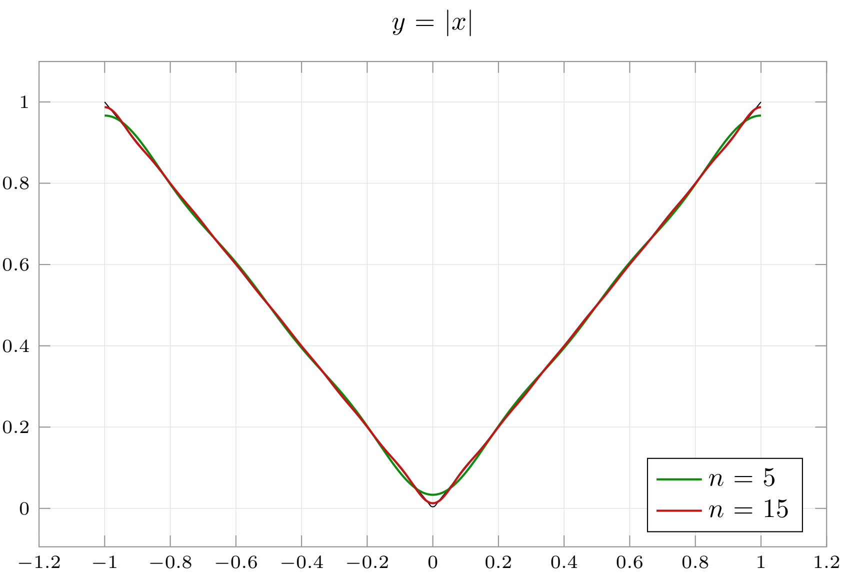
$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4\cos(n\pi)}{n^2\pi^2} \cos(n\pi x) \tag{11.2.94}$$

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \dots\right] \tag{11.2.95}$$

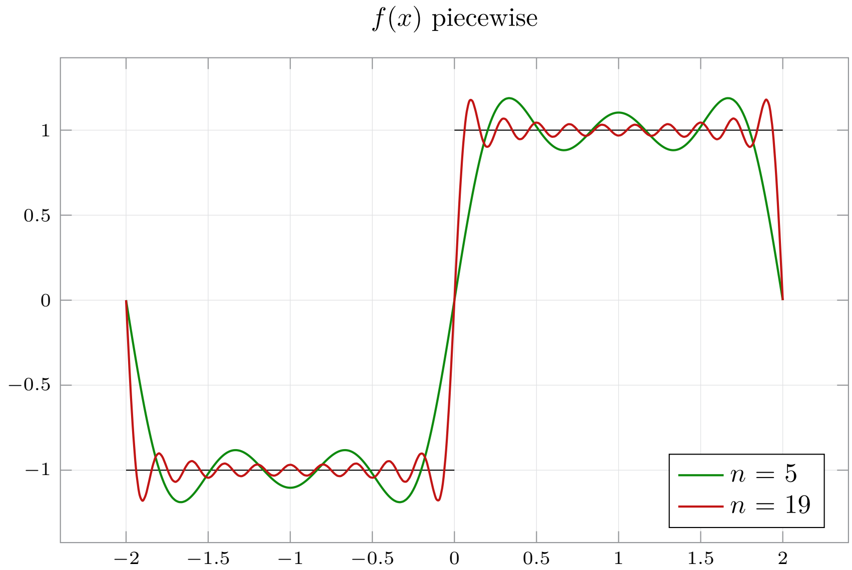
$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{11.2.96}$$

21. Plotting the first few partial sums for

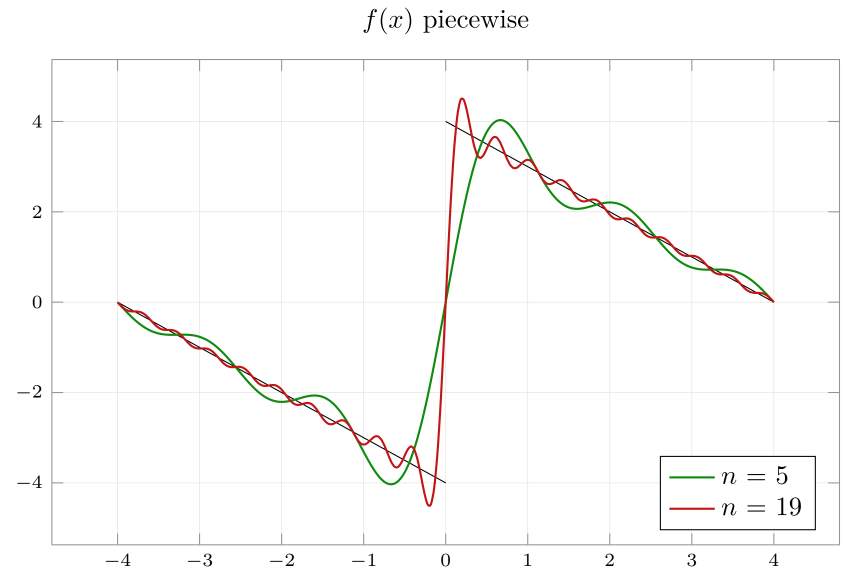
(a) Problem 8



(b) Problem 9

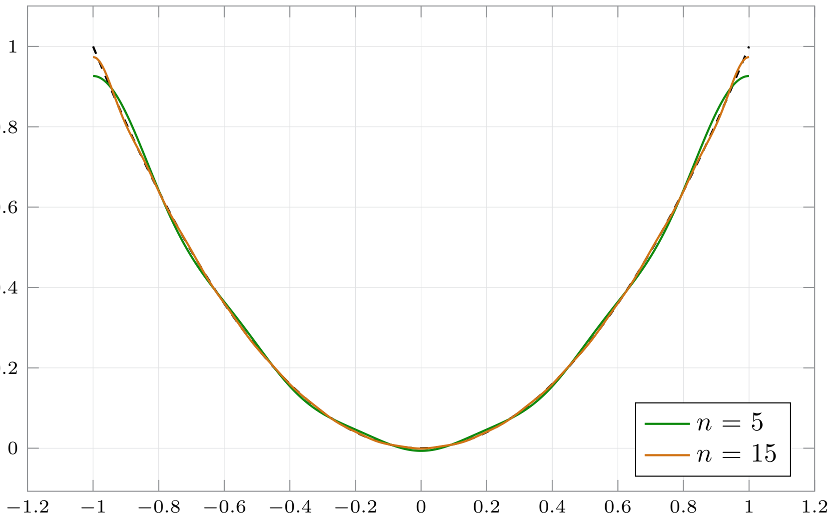


(c) Problem 10



(d) Problem 11

$f(x) = x^2$



22. Using the linearity of Fourier transforms,

$$f(x) = |x| \qquad g(x) = 1 - |x| = 1 - f(x) \tag{11.2.97}$$

$$F(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[\cos(n\pi) - 1\right] \tag{11.2.98}$$

$$G(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[\cos(n\pi) - 1\right] \tag{11.2.99}$$

The inverse mapping $g \to f$ is also as simple.

23. The odd expansion of the given function is, with $p = 2L = 8$

$$f(x) = \begin{cases} -1 & x \in [-4, 0] \\ 1 & x \in [0, 4] \end{cases} \tag{11.2.100}$$

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{11.2.101}$$

Calculating the Fourier sine coefficients,

$$a_n = \frac{1}{4} \int_{-4}^{4} f(x) \ \sin\left(\frac{n\pi x}{4}\right) \ \mathrm{d}x = \frac{1}{2} \int_{0}^{4} (1) \sin\left(\frac{n\pi x}{4}\right) \ \mathrm{d}x \tag{11.2.102}$$

$$= \left[\frac{-2}{n\pi} \ \cos\left(\frac{n\pi x}{4}\right)\right]_0^4 = \frac{2}{n\pi} \left[1 - \cos(n\pi)\right] \tag{11.2.103}$$

The even expansion of the given function is,

$$a_0 = 1 \qquad\qquad a_n = b_n = 0 \tag{11.2.104}$$

24. The odd expansion of the given function is, with $p = 2L = 8$

$$f(x) = \begin{cases} -1 & x \in [-4, -2] \\ 0 & x \in [-2, 2] \\ 1 & x \in [2, 4] \end{cases} \tag{11.2.105}$$

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{11.2.106}$$

Calculating the Fourier sine coefficients,

$$b_n = \frac{1}{4}\int_{-4}^{4} f(x)\ \sin\left(\frac{n\pi x}{4}\right)\ dx = \frac{1}{2}\int_{2}^{4} (1) \sin\left(\frac{n\pi x}{4}\right)\ dx \tag{11.2.107}$$

$$= \left[\frac{-2}{n\pi}\ \cos\left(\frac{n\pi x}{4}\right)\right]_2^4 = \frac{2}{n\pi}\ [\cos(n\pi/2) - \cos(n\pi)] \tag{11.2.108}$$

The even expansion of the given function is, with $p = 2L = 8$

$$f(x) = \begin{cases} 1 & x \in [-4, -2] \\ 0 & x \in [-2, 2] \\ 1 & x \in [2, 4] \end{cases} \tag{11.2.109}$$

$$b_n = 0 \tag{11.2.110}$$

Calculating the Fourier cosine coefficients,

$$a_0 = \frac{1}{8}\int_{-4}^{4} f(x)\ dx = \frac{1}{8}\left[\int_{-4}^{-2} (1)\ dx + \int_{2}^{4} (1)\ dx\right] \tag{11.2.111}$$

$$= \frac{1}{2} \tag{11.2.112}$$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(x)\ \cos\left(\frac{n\pi x}{4}\right)\ dx = \frac{1}{2}\int_{2}^{4} (1) \cos\left(\frac{n\pi x}{4}\right)\ dx \tag{11.2.113}$$

$$= \left[\frac{2}{n\pi}\ \sin\left(\frac{n\pi x}{4}\right)\right]_2^4 = \frac{-2}{n\pi}\ \sin(n\pi/2) \tag{11.2.114}$$

25. The odd expansion of the given function is, with $p = 2L = 2\pi$

$$f(x) = \begin{cases} -x - \pi & x \in [-\pi, 0] \\ -x + \pi & x \in [0, \pi] \end{cases} \tag{11.2.115}$$

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{11.2.116}$$

Calculating the Fourier sine coefficients,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \sin(nx) \ \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin(nx) \ \mathrm{d}x \tag{11.2.117}$$

$$= \frac{2}{\pi} \left[\frac{(x - \pi) \cos(nx)}{n} - \frac{\sin(nx)}{n^2} \right]_0^{\pi} = \frac{-2}{n} \tag{11.2.118}$$

The even expansion of the given function is, with $p = 2L = 2\pi$

$$f(x) = \pi - |x| \tag{11.2.119}$$

$$b_n = 0 \tag{11.2.120}$$

Calculating the Fourier cosine coefficients,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \ \mathrm{d}x \tag{11.2.121}$$

$$= \frac{1}{\pi} \left[\frac{-(x - \pi)^2}{2} \right]_0^{\pi} = \frac{\pi}{2} \tag{11.2.122}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos(nx) \ \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos(nx) \ \mathrm{d}x \tag{11.2.123}$$

$$= \frac{2}{\pi} \left[\frac{(\pi - x) \sin(nx)}{n} - \frac{\cos(nx)}{n^2} \right]_0^{\pi} = \frac{2}{\pi n^2} \left[1 - \cos(n\pi)\right] \tag{11.2.124}$$

26. The odd expansion of the given function is, with $p = 2L = 2\pi$

$$f(x) = \begin{cases} -\pi/2 & x \in [-\pi, -\pi/2] \\ x & x \in [-\pi/2, \pi/2] \\ \pi/2 & x \in [\pi/2, \pi] \end{cases} \tag{11.2.125}$$

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{11.2.126}$$

Calculating the Fourier sine coefficients,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \sin(nx) \ \mathrm{d}x \tag{11.2.127}$$

$$= \frac{2}{\pi} \int_{0}^{\pi/2} (x) \sin(nx) \ \mathrm{d}x + \int_{\pi/2}^{\pi} (\pi/2) \sin(nx) \ \mathrm{d}x \tag{11.2.128}$$

$$= \frac{2}{\pi} \left[-\frac{(x) \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_0^{\pi/2} + \left[-\frac{\cos(nx)}{n} \right]_{\pi/2}^{\pi} \tag{11.2.129}$$

$$= \frac{2 \sin(n\pi/2)}{n\pi^2} - \frac{\cos(n\pi)}{n} \tag{11.2.130}$$

The even expansion of the given function is, with $p = 2L = 2\pi$

$$f(x) = \begin{cases} \pi/2 & x \in [-\pi, -\pi/2] \\ |x| & x \in [-\pi/2, \pi/2] \\ \pi/2 & x \in [\pi/2, \pi] \end{cases} \tag{11.2.131}$$

$$b_n = 0 \tag{11.2.132}$$

Calculating the Fourier cosine coefficients,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x \tag{11.2.133}$$

$$= \frac{1}{\pi} \left[\int_{0}^{\pi/2} (x) \ \mathrm{d}x + \int_{\pi/2}^{\pi} (\pi/2) \ \mathrm{d}x \right] \tag{11.2.134}$$

$$= \left[\frac{x^2}{2\pi} \right]_0^{\pi/2} + \left[\frac{x}{2} \right]_{\pi/2}^{\pi} = \frac{3\pi}{8} \tag{11.2.135}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos(nx) \ \mathrm{d}x \tag{11.2.136}$$

$$= \frac{2}{\pi} \int_{0}^{\pi/2} (x) \cos(nx) \ \mathrm{d}x + \int_{\pi/2}^{\pi} (1) \cos(nx) \ \mathrm{d}x \tag{11.2.137}$$

$$= \frac{2}{\pi} \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^{\pi/2} + \left[\frac{\sin(nx)}{n} \right]_{\pi/2}^{\pi} \tag{11.2.138}$$

$$= \frac{2}{\pi n^2} \left[\cos(n\pi/2) - 1 \right] \tag{11.2.139}$$

27. The odd expansion of the given function is, with $p = 2L = 2\pi$

$$f(x) = \begin{cases} -\pi - x & x \in [-\pi, -\pi/2] \\ -\pi/2 & x \in [-\pi/2, 0] \\ \pi/2 & x \in [0, \pi/2] \\ \pi - x & x \in [\pi/2, \pi] \end{cases} \tag{11.2.140}$$

$$a_0 = 0 \qquad a_n = 0 \tag{11.2.141}$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} (\pi/2)\sin(nx)\,\mathrm{d}x + \int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\,\mathrm{d}x\right] \tag{11.2.142}$$

$$= \left[\frac{-\cos(nx)}{n}\right]_0^{\pi/2} + \frac{2}{\pi}\left[\frac{(x-\pi)\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi/2}^{\pi} \tag{11.2.143}$$

$$= \frac{1}{n} + \frac{2\sin(n\pi/2)}{\pi n^2} \tag{11.2.144}$$

The even expansion of the given function is, with $p = 2L = 2\pi$

$$f(x) = \begin{cases} x + \pi & x \in [-\pi, -\pi/2] \\ \pi/2 & x \in [-\pi/2, \pi/2] \\ \pi - x & x \in [\pi/2, \pi] \end{cases} \tag{11.2.145}$$

$$b_n = 0 \tag{11.2.146}$$

Calculating the Fourier cosine coefficients,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x \tag{11.2.147}$$

$$= \frac{1}{\pi} \left[\int_0^{\pi/2} (\pi/2) \ \mathrm{d}x + \int_{\pi/2}^{\pi} (\pi - x) \ \mathrm{d}x \right] \tag{11.2.148}$$

$$= \left[\frac{-(\pi - x)^2}{2\pi} \right]_{\pi/2}^{\pi} + \left[\frac{x}{2} \right]_0^{\pi/2} = \frac{3\pi}{8} \tag{11.2.149}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos(nx) \ \mathrm{d}x \tag{11.2.150}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} (\pi/2) \cos(nx) \ \mathrm{d}x + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \cos(nx) \ \mathrm{d}x \tag{11.2.151}$$

$$= \frac{2}{\pi} \left[\frac{(\pi - x)\sin(nx)}{n} - \frac{\cos(nx)}{n^2} \right]_{\pi/2}^{\pi} + \left[\frac{\sin(nx)}{n} \right]_0^{\pi/2} \tag{11.2.152}$$

$$= \frac{-2}{\pi n^2} \left[\cos(n\pi) - \cos(n\pi/2) \right] \tag{11.2.153}$$

28. The odd expansion of the given function is, with $p = 2L$

$$f(x) = x \tag{11.2.154}$$

$$a_0 = 0 \qquad a_n = 0 \tag{11.2.155}$$

$$b_n = \frac{2}{L} \int_0^L (x) \sin\left(\frac{n\pi x}{L} \right) \ \mathrm{d}x \tag{11.2.156}$$

$$= \frac{2}{L} \left[-\frac{xL}{n\pi} \ \cos\left(\frac{n\pi x}{L} \right) + \frac{L^2}{n^2\pi^2} \ \sin\left(\frac{n\pi x}{L} \right) \right]_0^L \tag{11.2.157}$$

$$= \frac{-2L}{n\pi} \ \cos(n\pi) \tag{11.2.158}$$

The even expansion of the given function is, with $p = 2L$

$$f(x) = |x| \qquad b_n = 0 \tag{11.2.159}$$

Calculating the Fourier cosine coefficients,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \ \mathrm{d}x = \frac{1}{L} \int_0^L (x) \ \mathrm{d}x \tag{11.2.160}$$

$$= \left[\frac{x^2}{2L}\right]_0^L = \frac{L}{2} \tag{11.2.161}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \ \cos\left(\frac{n\pi x}{L}\right) \ \mathrm{d}x \tag{11.2.162}$$

$$= \frac{2}{L} \int_0^L (x) \cos\left(\frac{n\pi x}{L}\right) \ \mathrm{d}x \tag{11.2.163}$$

$$= \frac{2}{L} \left[\frac{xL}{n\pi} \ \sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2} \ \cos\left(\frac{n\pi x}{L}\right)\right]_0^L \tag{11.2.164}$$

$$= \frac{2L}{\pi^2 n^2} \ [\cos(n\pi) - 1] \tag{11.2.165}$$

29. The odd expansion of the given function is, with $p = 2L = 2\pi$

$$f(x) = \sin(x) \tag{11.2.166}$$

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{11.2.167}$$

$$b_n = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{11.2.168}$$

The even expansion of the given function is, with $p = 2L = 2\pi$

$$f(x) = |\sin(x)| \qquad\qquad b_n = 0 \tag{11.2.169}$$

Calculating the Fourier cosine coefficients,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x = \frac{1}{\pi} \int_0^{\pi} (\sin x) \ \mathrm{d}x \tag{11.2.170}$$

$$= \frac{1}{\pi}\left[- \cos x \right]_0^{\pi} = \frac{2}{\pi} \tag{11.2.171}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos(nx) \ \mathrm{d}x \tag{11.2.172}$$

$$= \frac{2}{\pi} \int_0^{\pi} (\sin x) \cos(nx) \ \mathrm{d}x \tag{11.2.173}$$

$$= \frac{-1}{\pi}\left[\frac{\cos[(1+n)x]}{1+n} + \frac{\cos[(1-n)x]}{1-n}\right]_0^{\pi} \tag{11.2.174}$$

$$= \begin{cases} \frac{-2}{\pi(n^2-1)} \left[1 - \cos[(n+1)\pi]\right] & n \geq 2 \\ 0 & n = 1 \end{cases} \tag{11.2.175}$$

30. The odd expansion of the given function is, with $p = 2L$

$$f(x) = -g(x + \pi) \tag{11.2.176}$$

$$a_0 = 0 \qquad a_n = 0 \tag{11.2.177}$$

$$f(x) = -\sum_{n=1}^{\infty} \left[\frac{1}{n} + \frac{2\sin(n\pi/2)}{\pi n^2}\right] \sin(nx + n\pi) \tag{11.2.178}$$

$$= \sum_{n=1}^{\infty} \left[\frac{-\cos(n\pi)}{n} + \frac{2\sin(n\pi/2)}{\pi n^2}\right] \sin(nx) \tag{11.2.179}$$

The even expansion of the given function is, with $p = 2L$

$$f(x) = g(x + \pi) \qquad b_n = 0 \tag{11.2.180}$$

Calculating the Fourier cosine coefficients,

$$a_0 = \frac{3\pi}{8} \tag{11.2.181}$$

$$a_n = \sum_{n=1}^{\infty} \left[\frac{2}{\pi n^2} \left[\cos(n\pi/2) - \cos(n\pi)\right] \right] \cos(nx + n\pi) \tag{11.2.182}$$

$$= \sum_{n=1}^{\infty} \left[\frac{2}{\pi n^2} \left[\cos(n\pi/2) - 1\right] \right] \cos(nx) \tag{11.2.183}$$

$$\tag{11.2.184}$$

11.3 Forced Oscillations

1. Deriving the terms C_n,

$$A_n = \frac{1}{n\pi D_n} \left[\frac{4(25 - n^2)}{n} \right] \qquad B_n = \frac{1}{n\pi D_n} \left[0.2 \right] \tag{11.3.1}$$

$$C_n = \sqrt{A_n^2 + B_n^2} \qquad = \frac{1}{n\pi D_n} \sqrt{\frac{(25 - n^2)^2 + (0.05n)^2}{n^2/16}} \tag{11.3.2}$$

$$= \frac{4}{n^2 \pi \sqrt{D_n}} \tag{11.3.3}$$

2. The effect of changing k is,

$$C_n \propto \frac{1}{\sqrt{D_n}} \qquad D_n = (k - n^2)^2 + (cn)^2 \tag{11.3.4}$$

$$\tag{11.3.5}$$

The maximum in amplitude shifts from $n = 5$ to $n = 7$, when $k = 7^2$.

The amplitude goes down as k increases, and as c increases.

3. The effect of c is to prevent the output being a pure cosine series by introducing sine terms proportional to the damping.

$$B_n \propto c \qquad c \to 0 \implies B_n \to 0 \tag{11.3.6}$$

$$C_n \to A_n \tag{11.3.7}$$

In the limit of very large c, $B_n \gg A_n$ and the output is completely out of phase with the input.

4. The derivative of the input is,

$$r'(t) = \frac{-4}{n\pi}\ \sin(nt) = \lambda \sin(nt) \qquad C_n = \frac{\lambda}{\sqrt{D_n}} \tag{11.3.8}$$

$$C_{\text{new}} = n\ C_{\text{old}} \tag{11.3.9}$$

Differentiation leads to the amplitude C_n multiplied by a factor of n.

5. The fact that the driving frequency being larger than the resonant frequency makes the output the opposite phase as the input is reflected in those A_n terms being negative.

   No such effect happens as a result of the damping, which means that the B_n terms always remain positive.

6. Solving the ODE,

$$r(t) = \sin(\alpha t) + \sin(\beta t) \qquad \omega^2 \neq \alpha^2, \beta^2 \tag{11.3.10}$$

$$y'' + \omega^2 y = r(t) \tag{11.3.11}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = A_1 \cos(\alpha t) + A_2 \sin(\alpha t) + B_1 \cos(\beta t) + B_2 \sin(\beta t) \tag{11.3.12}$$

$$[\cos(\alpha t)] \qquad 0 = (-\alpha^2 + \omega^2)A_1 \tag{11.3.13}$$

$$[\sin(\alpha t)] \qquad 1 = (-\alpha^2 + \omega^2)A_2 \tag{11.3.14}$$

$$[\cos(\beta t)] \qquad 0 = (-\beta^2 + \omega^2)B_1 \tag{11.3.15}$$

$$[\sin(\beta t)] \qquad 1 = (-\beta^2 + \omega^2)B_2 \tag{11.3.16}$$

$$y_p = \frac{\sin(\alpha t)}{\omega^2 - \alpha^2} + \frac{\sin(\beta t)}{\omega^2 - \beta^2} \tag{11.3.17}$$

Finding the general solution, by solving the h-ODE,

$$y'' + \omega^2 y = 0 \tag{11.3.18}$$

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{11.3.19}$$

$$y = y_h + y_p \tag{11.3.20}$$

7. Solving the ODE,

$$r(t) = \sin(t) \qquad \omega^2 \neq 1 \tag{11.3.21}$$

$$y'' + \omega^2 y = r(t) \tag{11.3.22}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = A_1 \cos(t) + A_2 \sin(t) \tag{11.3.23}$$

$$[\cos(t)] \qquad 0 = (-1 + \omega^2) A_1 \tag{11.3.24}$$

$$[\sin(t)] \qquad 1 = (-1 + \omega^2) A_2 \tag{11.3.25}$$

$$y_p = \frac{\sin(t)}{\omega^2 - 1} \tag{11.3.26}$$

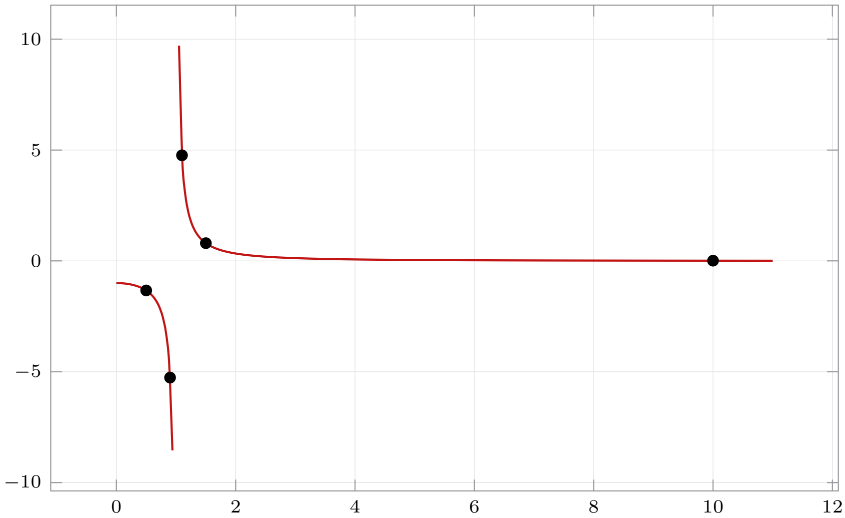
Finding the general solution, by solving the h-ODE,

$$y'' + \omega^2 y = 0 \tag{11.3.27}$$

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{11.3.28}$$

$$y = y_h + y_p \tag{11.3.29}$$

$|y_p|$



8. Finding the Fourier series representation of the input

$$r(t) = \frac{\pi}{4} \ |\cos t| \qquad \forall \quad x \in [-\pi, \pi] \tag{11.3.30}$$

$$p = 2L = 2\pi \tag{11.3.31}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x = \frac{1}{4} \int_0^{\pi} |\cos x| \ \mathrm{d}x \tag{11.3.32}$$

$$= \frac{1}{4} \left[\sin x \right]_0^{\pi/2} + \frac{1}{4} \left[\sin x \right]_{\pi}^{\pi/2} = \frac{1}{2} \tag{11.3.33}$$

Finding the cosine coefficients

$$a_1 = \frac{1}{2}\int_0^{\pi/2} (\cos^2 x)\ \mathrm{d}x + \frac{1}{2}\int_\pi^{\pi/2} (\cos^2 x)\ \mathrm{d}x \tag{11.3.34}$$

$$= \frac{1}{4}\left[x + \frac{\sin(2x)}{2}\right]_0^{\pi/2} + \frac{1}{4}\left[x + \frac{\sin(2x)}{2}\right]_\pi^{\pi/2} = 0 \tag{11.3.35}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos(nx)\ \mathrm{d}x \tag{11.3.36}$$

$$= \frac{1}{2}\int_0^{\pi/2} (\cos x)\cos(nx)\ \mathrm{d}x + \frac{1}{2}\int_\pi^{\pi/2} (\cos x)\cos(nx)\ \mathrm{d}x \tag{11.3.37}$$

$$= \frac{1}{4}\left[\frac{\sin[(1+n)x]}{1+n} + \frac{\sin[(1-n)x]}{1-n}\right]_0^{\pi/2} \tag{11.3.38}$$

$$+ \frac{1}{4}\left[\frac{\sin[(1+n)x]}{1+n} + \frac{\sin[(1-n)x]}{1-n}\right]_\pi^{\pi/2} = \frac{\cos(n\pi/2)}{1-n^2} \tag{11.3.39}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = C + A_n\cos(nt) + B_n\sin(nt) \tag{11.3.40}$$

$$\omega^2 C = \frac{1}{2} \tag{11.3.41}$$

$$\frac{\cos(n\pi/2)}{1-n^2} = (-n^2 + \omega^2)A_n \tag{11.3.42}$$

$$0 = (-n^2 + \omega^2)B_n \tag{11.3.43}$$

$$y_p = \frac{1}{2\omega^2} + \sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{(1-n^2)(\omega^2 - n^2)}\ \cos(nt) \tag{11.3.44}$$

Finding the general solution, by solving the h-ODE,

$$y'' + \omega^2 y = 0 \tag{11.3.45}$$

$$y_h = C_1\cos(\omega t) + C_2\sin(\omega t) \tag{11.3.46}$$

$$y = y_h + y_p \tag{11.3.47}$$

9. In Problem 8, even numbers for n give nonzero terms in the expansion of y_p, which can have zero in the denominator.

This means that no steady state solution exists for even number ω.

10. Finding the Fourier series representation of the input

$$r(t) = \frac{\pi}{4} \ |\sin x| \qquad \forall \quad x \in [-\pi, \pi] \tag{11.3.48}$$

$$p = 2L = 2\pi \tag{11.3.49}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x = \frac{1}{4} \int_0^{\pi} |\sin x| \ \ \mathrm{d}x \tag{11.3.50}$$

$$= \frac{1}{4} \left[-\cos x \right]_0^{\pi} = \frac{1}{2} \tag{11.3.51}$$

Finding the cosine coefficients

$$a_1 = \frac{1}{2} \int_0^{\pi} (\sin x) \cos(x) \ \mathrm{d}x \tag{11.3.52}$$

$$= \left[\frac{-\cos(2x)}{8} \right]_0^{\pi} = 0 \tag{11.3.53}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos(nx) \ \mathrm{d}x = \frac{1}{2} \int_0^{\pi} (\sin x) \cos(nx) \ \mathrm{d}x \tag{11.3.54}$$

$$= -\frac{1}{4} \left[\frac{\cos[(1+n)x]}{1+n} + \frac{\cos[(1-n)x]}{1-n} \right]_0^{\pi} \tag{11.3.55}$$

$$= \frac{1}{2(1-n^2)} \ [\cos(n\pi) + 1] \tag{11.3.56}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = C + A_n \cos(nt) + B_n \sin(nt) \tag{11.3.57}$$

$$\omega^2 C = \frac{1}{2} \tag{11.3.58}$$

$$\frac{1 + \cos(n\pi)}{2(1-n^2)} = (-n^2 + \omega^2) A_n \tag{11.3.59}$$

$$0 = (-n^2 + \omega^2) B_n \tag{11.3.60}$$

$$y_p = \frac{1}{2\omega^2} + \sum_{n=2}^{\infty} \frac{1 + \cos(n\pi)}{2(1-n^2)(\omega^2 - n^2)} \ \cos(nt) \tag{11.3.61}$$

Finding the general solution, by solving the h-ODE,

$$y'' + \omega^2 y = 0 \tag{11.3.62}$$

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{11.3.63}$$

$$y = y_h + y_p \tag{11.3.64}$$

11. Finding the Fourier series representation of the input

$$r(t) = \begin{cases} -1 & x \in [-\pi, 0] \\ 1 & x \in [0, \pi] \end{cases} \tag{11.3.65}$$

$$p = 2L = 2\pi \qquad |\omega| \neq 1, 3, 5, \ldots \tag{11.3.66}$$

$$a_0 = 0 \tag{11.3.67}$$

$$a_n = 0 \tag{11.3.68}$$

Finding the sine coefficients

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \sin(nx) \ \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} (1) \sin(nx) \ \mathrm{d}x \tag{11.3.69}$$

$$= -\frac{2}{\pi} \left[\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2}{n\pi} \left[1 - \cos(n\pi)\right] \tag{11.3.70}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = A_n \cos(nt) + B_n \sin(nt) \tag{11.3.71}$$

$$0 = (-n^2 + \omega^2) A_n \tag{11.3.72}$$

$$\frac{4}{n\pi} = (-n^2 + \omega^2) B_n \tag{11.3.73}$$

$$y_p = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1 - \cos(n\pi)]}{n \ (\omega^2 - n^2)} \ \sin(nt) \tag{11.3.74}$$

Finding the general solution, by solving the h-ODE,

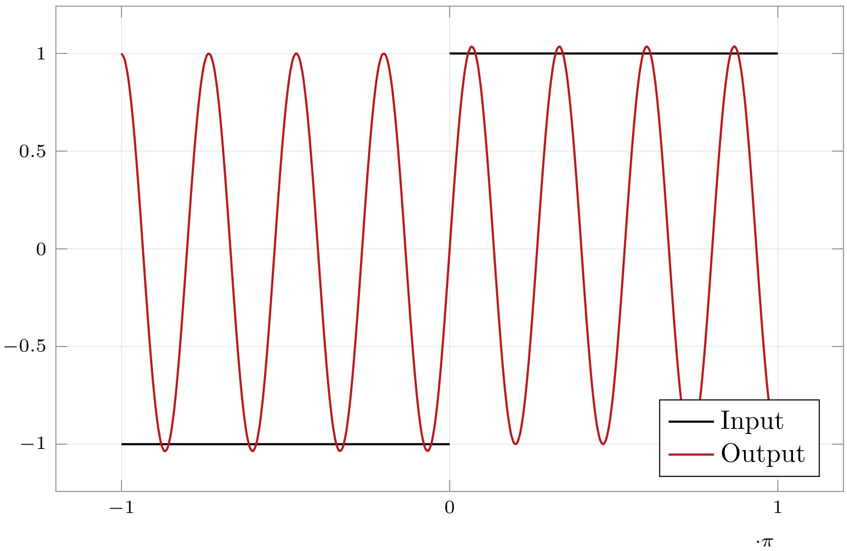
$$y'' + \omega^2 y = 0 \tag{11.3.75}$$

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{11.3.76}$$

$$y = y_h + y_p \tag{11.3.77}$$

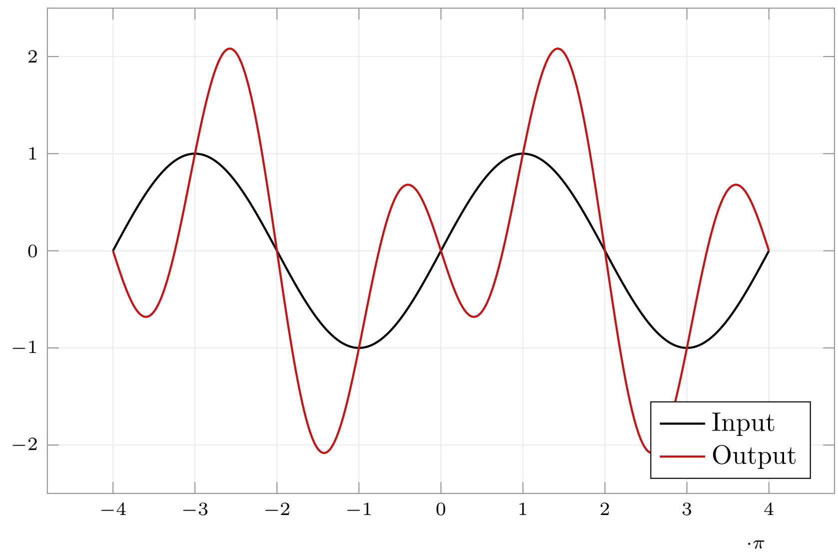
12. Graphing the input and output in Problem 11, with $C_1 = 0$, $C_2 = 1$, and $\omega = 7.5$

Undamped Driven Oscillations



Graphing the input and output in Problem 7, with $C_1 = 0$, $C_2 = 1$, and $\omega = 0.5$

Undamped Driven Oscillations



13. For the damped oscillator, with $k = 1$,

$$D_n = (1 - n^2)^2 + (nc)^2 \tag{11.3.78}$$

$$y_n = P_n \cos(nt) + Q_n \sin(nt) \tag{11.3.79}$$

Consider the two general terms in the input,

$$y'' + cy' + y = a_n \cos(nt) + b_n \sin(nt) \tag{11.3.80}$$

$$a_n = (1 - n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{11.3.81}$$

$$b_n = (1 - n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{11.3.82}$$

$$P_n = \frac{a_n(1 - n^2) - b_n(nc)}{D_n} \tag{11.3.83}$$

$$Q_n = \frac{b_n(1 - n^2) + a_n(nc)}{D_n} \tag{11.3.84}$$

The above system is linear in P_n and Q_n.

14. From Problem 11, the Fourier series representation of the input is,

$$r(t) = \sum_{n=1}^{\infty} \frac{2[1 - \cos(n\pi)]}{n\pi} \sin(nt) \tag{11.3.85}$$

$$y'' + cy' + y = a_n \cos(nt) + b_n \sin(nt) \tag{11.3.86}$$

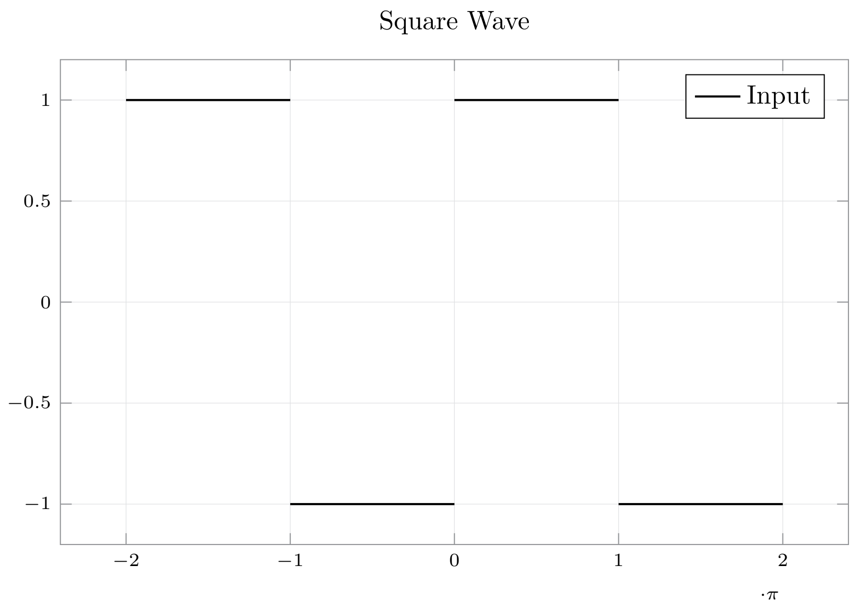
$$0 = (1 - n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{11.3.87}$$

$$b_n = (1 - n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{11.3.88}$$

$$P_n = \frac{-b_n(nc)}{D_n} \tag{11.3.89}$$

$$Q_n = \frac{b_n(1 - n^2)}{D_n} \tag{11.3.90}$$

$$D_n = (1 - n^2)^2 + (nc)^2 \tag{11.3.91}$$



15. Finding the fourier series representation of the input (odd function),

$$a_0 = 0 \qquad a_n = 0 \tag{11.3.92}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 x - x^3)\ \sin(nx)\ \mathrm{d}x \tag{11.3.93}$$

$$= \frac{2}{\pi}\left[\sin(nx)\left(\frac{\pi^2 - 3x^2}{n^2} + \frac{6}{n^4}\right) + \cos(nx)\left(\frac{x(x^2-\pi^2)}{n} - \frac{6x}{n^3}\right)\right]_0^{\pi} \tag{11.3.94}$$

$$= -\frac{12}{n^3}\ \cos(n\pi) \tag{11.3.95}$$

$$r(t) = \sum_{n=1}^{\infty} \frac{-12\cos(n\pi)}{n^3}\ \sin(nt) \tag{11.3.96}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_n\cos(nt) + b_n\sin(nt) \tag{11.3.97}$$

$$0 = (1-n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{11.3.98}$$

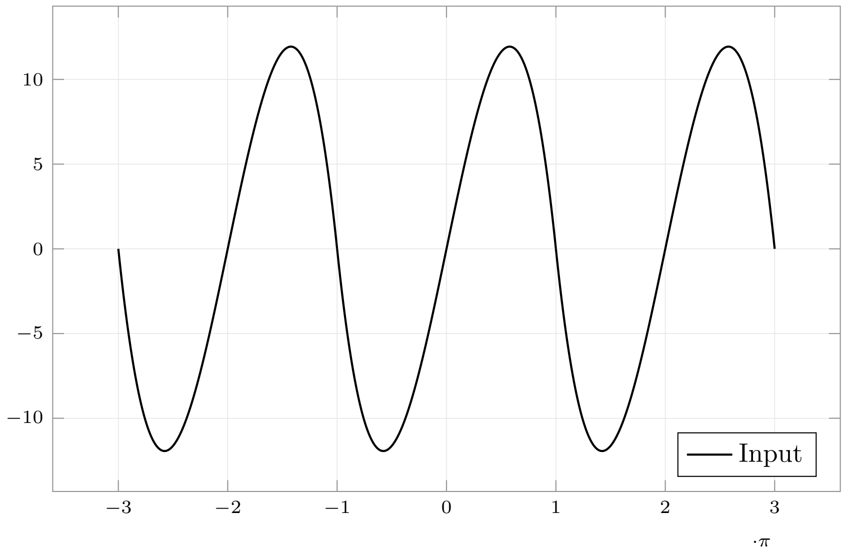
$$b_n = (1-n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{11.3.99}$$

$$P_n = \frac{-b_n(nc)}{D_n} \tag{11.3.100}$$

$$Q_n = \frac{b_n(1-n^2)}{D_n} \tag{11.3.101}$$

$$D_n = (1-n^2)^2 + (nc)^2 \tag{11.3.102}$$

Cubic polynomial wave



16. Finding the fourier series representation of the input(odd function)

$$a_0 = 0 \tag{11.3.103}$$

$$a_n = 0 \tag{11.3.104}$$

Finding the sine coefficients,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \ \mathrm{d}x \tag{11.3.105}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} x \sin(nx) \ \mathrm{d}x + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \sin(nx) \ \mathrm{d}x \tag{11.3.106}$$

$$= \frac{2}{\pi} \left[\frac{\sin(nx)}{n^2} - \frac{x \cos(nx)}{n} \right]_0^{\pi/2} + \frac{2}{\pi} \left[\frac{(x - \pi) \cos(nx)}{n} - \frac{\sin(nx)}{n^2} \right]_{\pi/2}^{\pi} \tag{11.3.107}$$

$$= \frac{4}{\pi n^2} \ \sin(n\pi/2) \tag{11.3.108}$$

$$r(t) = \sum_{n=1}^{\infty} \frac{4 \sin(n\pi/2)}{\pi n^2} \ \sin(nt) \tag{11.3.109}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_n \cos(nt) + b_n \sin(nt) \tag{11.3.110}$$

$$0 = (1 - n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{11.3.111}$$

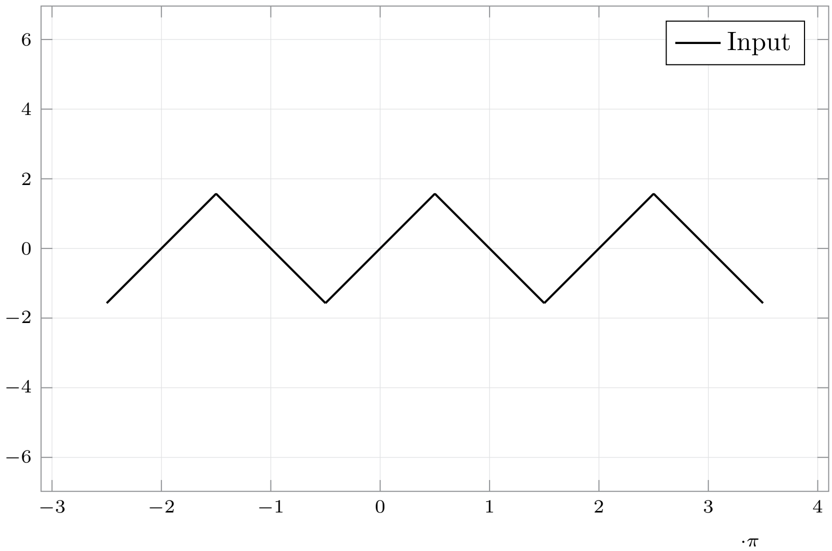
$$b_n = (1 - n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{11.3.112}$$

$$P_n = \frac{-b_n(nc)}{D_n} \tag{11.3.113}$$

$$Q_n = \frac{b_n(1 - n^2)}{D_n} \tag{11.3.114}$$

$$D_n = (1 - n^2)^2 + (nc)^2 \tag{11.3.115}$$

Cubic polynomial wave



17. The second order linear ODE for an RLC circuit with $R = 10$, $L = 1$, $C = 0.1$ is given by,

$$Lj'' + Rj' + \frac{1}{C}\, j = E'(t) \qquad E'(t) = \begin{cases} -100t & t \in [-\pi, 0] \\ 100t & t \in [0, \pi] \end{cases} \tag{11.3.116}$$

Finding the Fourier series representation of the input, (even function),

$$b_n = 0 \tag{11.3.117}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\;\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}(100x)\;\mathrm{d}x \tag{11.3.118}$$

$$= \left[\frac{50x^2}{\pi}\right]_0^{\pi} = 50\pi \tag{11.3.119}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\;\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}(100x)\cos(nx)\;\mathrm{d}x \tag{11.3.120}$$

$$= \frac{200}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi} = \frac{200}{\pi n^2}\left[\cos(n\pi) - 1\right] \tag{11.3.121}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_0 + \sum_{n=0}^{\infty} a_n\cos(nt) + b_n\sin(nt) \tag{11.3.122}$$

$$10P_0 = a_0 = 50\pi \qquad\qquad P_0 = 5\pi \tag{11.3.123}$$

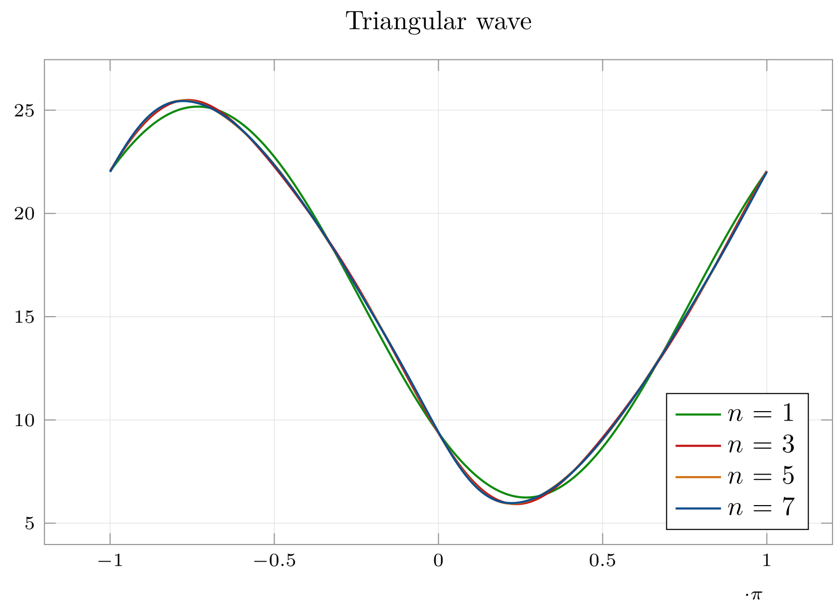
$$y_n = P_n\cos(nt) + Q_n\sin(nt) \tag{11.3.124}$$

$$a_n = (1-n^2)P_n + ncQ_n \qquad\qquad \cdots[\cos(nt)] \tag{11.3.125}$$

$$0 = (1-n^2)Q_n - ncP_n \qquad\qquad \cdots[\sin(nt)] \tag{11.3.126}$$

$$P_n = \frac{a_n(10-n^2)}{D_n} \qquad\qquad Q_n = \frac{a_n(10n)}{D_n} \tag{11.3.127}$$

$$D_n = (10-n^2)^2 + (10n)^2 \tag{11.3.128}$$



18. The second order linear ODE for an RLC circuit with $R = 10$, $L = 1$, $C = 0.1$ is given by,

$$Lj'' + Rj' + \frac{1}{C}\, j = E'(t) \qquad\qquad E'(t) = \begin{cases} 100(1-2t) & t \in [-\pi,\, 0] \\ 100(1+2t) & t \in [0,\, \pi] \end{cases} \tag{11.3.129}$$

Finding the Fourier series representation of the input, (even function),

$$b_n = 0 \tag{11.3.130}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} (100)(1+2x)\ \mathrm{d}x \tag{11.3.131}$$

$$= \frac{100}{\pi}\left[x + x^2\right]_0^{\pi} = 100(1+\pi) \tag{11.3.132}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} (100)(1+2x)\cos(nx)\ \mathrm{d}x \tag{11.3.133}$$

$$= \frac{200}{\pi}\left[\frac{\sin(nx)}{n} + \frac{2x\,\sin(nx)}{n} + \frac{2\,\cos(nx)}{n^2}\right]_0^{\pi} \tag{11.3.134}$$

$$= \frac{400}{\pi n^2}\left[\cos(n\pi) - 1\right] \tag{11.3.135}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_0 + \sum_{n=0}^{\infty} a_n \cos(nt) + b_n \sin(nt) \tag{11.3.136}$$

$$10P_0 = a_0 = 100(1+\pi) \qquad P_0 = 10(1+\pi) \tag{11.3.137}$$

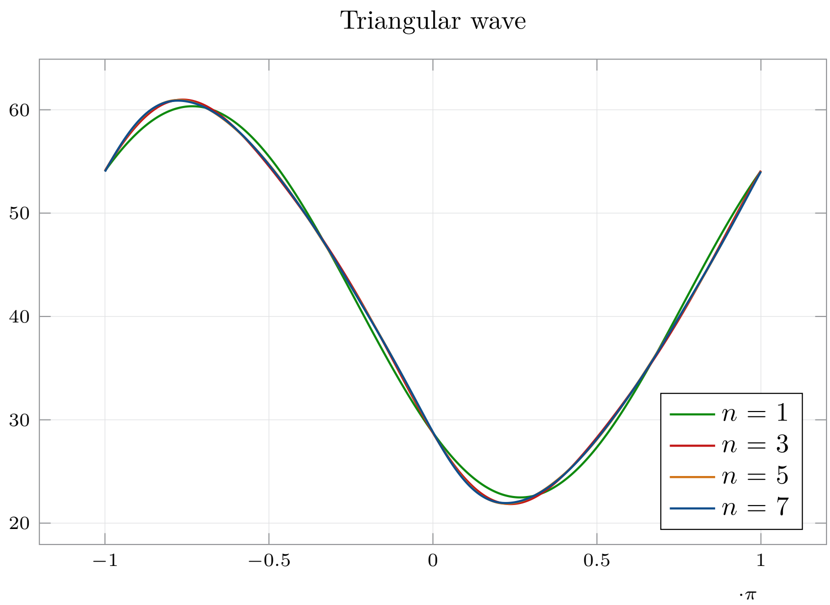
$$y_n = P_n \cos(nt) + Q_n \sin(nt) \tag{11.3.138}$$

$$a_n = (1-n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{11.3.139}$$

$$0 = (1-n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{11.3.140}$$

$$P_n = \frac{a_n(10-n^2)}{D_n} \qquad Q_n = \frac{a_n(10n)}{D_n} \tag{11.3.141}$$

$$D_n = (10-n^2)^2 + (10n)^2 \tag{11.3.142}$$



19. The second order linear ODE for an RLC circuit with $R = 10$, $L = 1$, $C = 0.1$ is given by,

$$Lj'' + Rj' + \frac{1}{C}\, j = E'(t) \qquad E'(t) = 200(\pi^2 - 3t^2) \tag{11.3.143}$$

Finding the Fourier series representation of the input, (even function),

$$b_n = 0 \tag{11.3.144}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} (200)(\pi^2 - 3x^2)\,\mathrm{d}x \tag{11.3.145}$$

$$= \frac{200}{\pi}\left[\pi^2 x - x^3\right]_0^{\pi} = 0 \tag{11.3.146}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} (200)(\pi^2 - 3x^2)\cos(nx)\,\mathrm{d}x \tag{11.3.147}$$

$$= \frac{400}{\pi}\left[\frac{(\pi^2 - 3x^2)\sin(nx)}{n} + \frac{6\sin(nx)}{n^3} - \frac{6x\cos(nx)}{n^2}\right]_0^{\pi} \tag{11.3.148}$$

$$= -\frac{2400\ \cos(n\pi)}{n^2} \tag{11.3.149}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_0 + \sum_{n=0}^{\infty} a_n\cos(nt) + b_n\sin(nt) \tag{11.3.150}$$

$$10P_0 = a_0 = 0 \qquad\qquad P_0 = 0 \tag{11.3.151}$$

$$y_n = P_n\cos(nt) + Q_n\sin(nt) \tag{11.3.152}$$

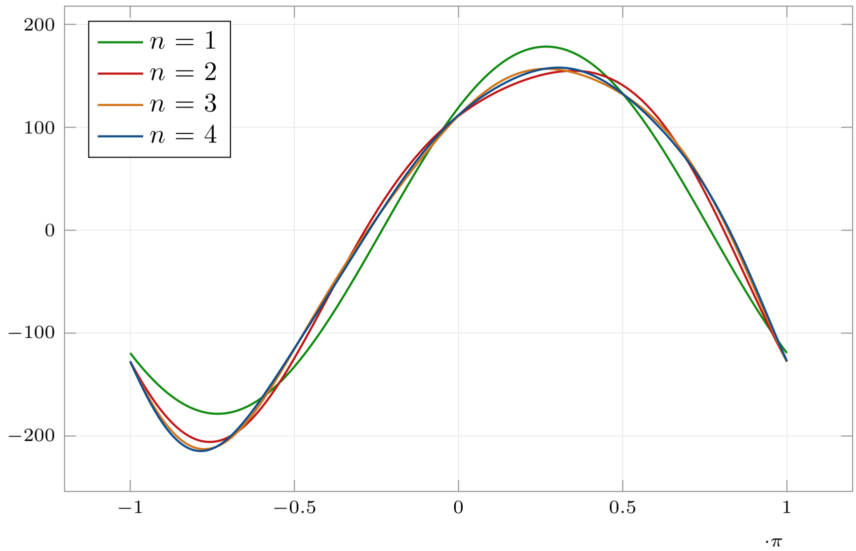
$$a_n = (1 - n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{11.3.153}$$

$$0 = (1 - n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{11.3.154}$$

$$P_n = \frac{a_n(10 - n^2)}{D_n} \qquad\qquad Q_n = \frac{a_n(10n)}{D_n} \tag{11.3.155}$$

$$D_n = (10 - n^2)^2 + (10n)^2 \tag{11.3.156}$$

Cubic polynomial wave

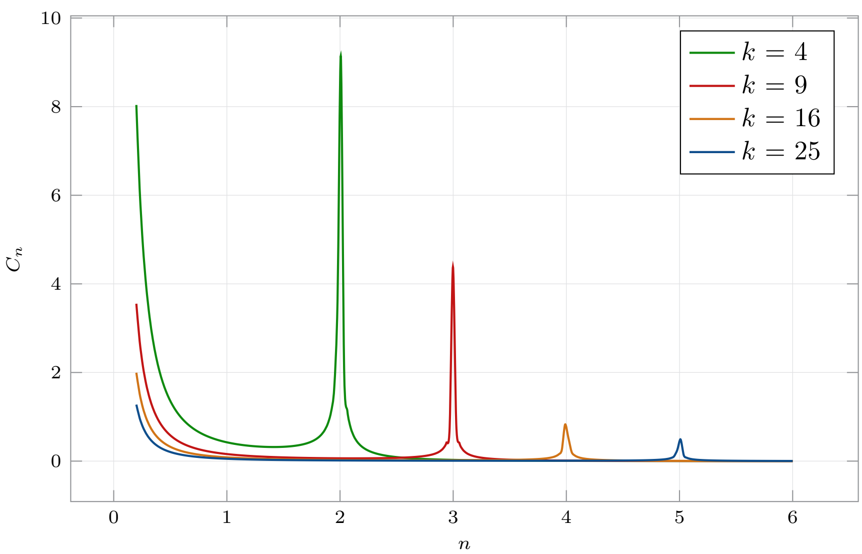


20. Finding the solution to the ODE in Example 1, for general c and k,

$$D_n = (k - n^2)^2 + (cn)^2 \qquad\qquad C_n = \frac{4}{n^2\pi} \cdot \frac{1}{\sqrt{D_n}} \tag{11.3.157}$$

Plotting a graph of C_n vs n for a fixed value of $c = 0.05$, and integer square values of k,

Amplitude of damped driven oscillator



11.4 Approximation by Trigonometric Polynomials

1. From Example 1 in the text,

$$f(x) = x + \pi \qquad x \in [-\pi, \pi] \tag{11.4.1}$$

$$a_0 = \pi \tag{11.4.2}$$

$$a_n = 0 \tag{11.4.3}$$

$$b_n = \frac{-2\cos(n\pi)}{n} \tag{11.4.4}$$

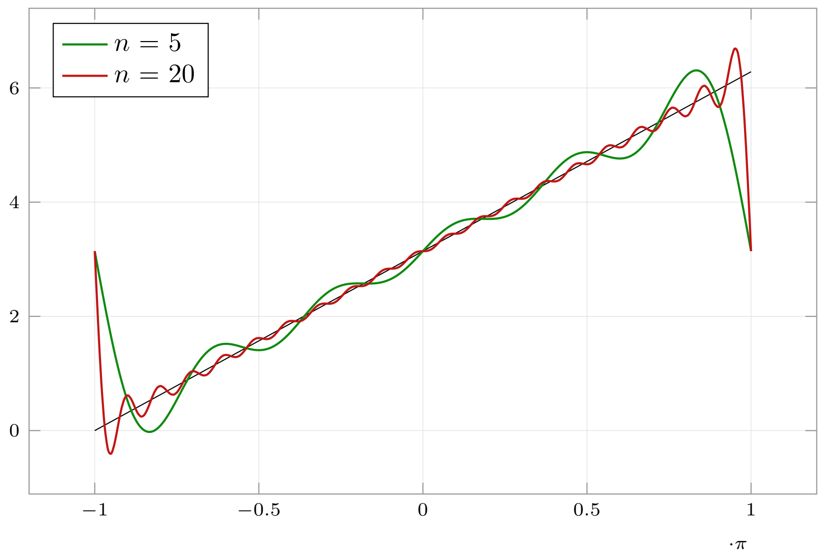
$$\int_{-\pi}^{\pi} f^2 \ dx = \int_{-\pi}^{\pi} (x+\pi)^2 \ dx = \left[\frac{(x+\pi)^3}{3}\right]_{-\pi}^{\pi} = \frac{8\pi^3}{3} \tag{11.4.5}$$

$$E^* = \frac{8\pi^3}{3} - 2\pi^3 - 4\pi \sum_{i=1}^{N} \frac{1}{n^2} \tag{11.4.6}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*	N	E^*
1000	0.01256	6000	0.002094
2000	0.006282	7000	0.001795
3000	0.004188	8000	0.001571
4000	0.003141	9000	0.001396
5000	0.002513	10000	0.001257

Fourier approximation



2. Evaluating the Fourier coefficients,

$$f(x) = x \qquad x \in [-\pi, \pi] \tag{11.4.7}$$

$$a_0 = 0 \tag{11.4.8}$$

$$a_n = 0 \tag{11.4.9}$$

$$b_n = \frac{-2\cos(n\pi)}{n} \tag{11.4.10}$$

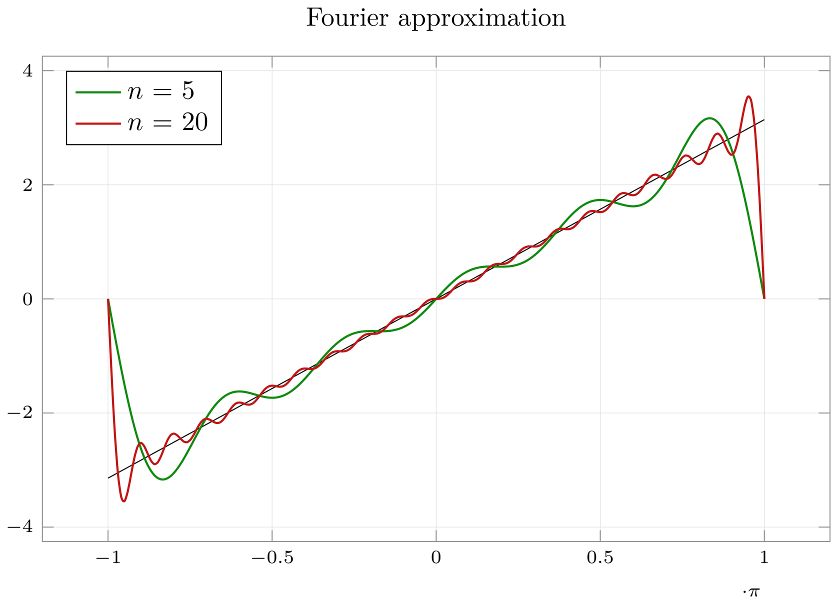
$$\int_{-\pi}^{\pi} f^2 \;\mathrm{d}x = \int_{-\pi}^{\pi} (x)^2 \;\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3} \tag{11.4.11}$$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty} \frac{4}{n^2} \tag{11.4.12}$$

$$E^* = \frac{\pi^3}{3} - 4\pi \sum_{i=1}^{N} \frac{1}{n^2} \tag{11.4.13}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	8.104
2	4.963
3	3.567
4	2.781
5	2.279



3. Evaluating the Fourier coefficients,

$$f(x) = |x| \qquad x \in [-\pi, \pi] \tag{11.4.14}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \;\mathrm{d}x = \frac{1}{\pi} \int_{0}^{\pi} x \;\mathrm{d}x = \left[\frac{x^2}{2\pi}\right]_0^{\pi} = \frac{\pi}{2} \tag{11.4.15}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \;\mathrm{d}x = \frac{2}{\pi} \int_{0}^{\pi} x \cos(nx) \;\mathrm{d}x \tag{11.4.16}$$

$$= \frac{2}{\pi} \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2} [\cos(n\pi - 1)] \tag{11.4.17}$$

$$b_n = 0 \tag{11.4.18}$$

Calculating the error function,

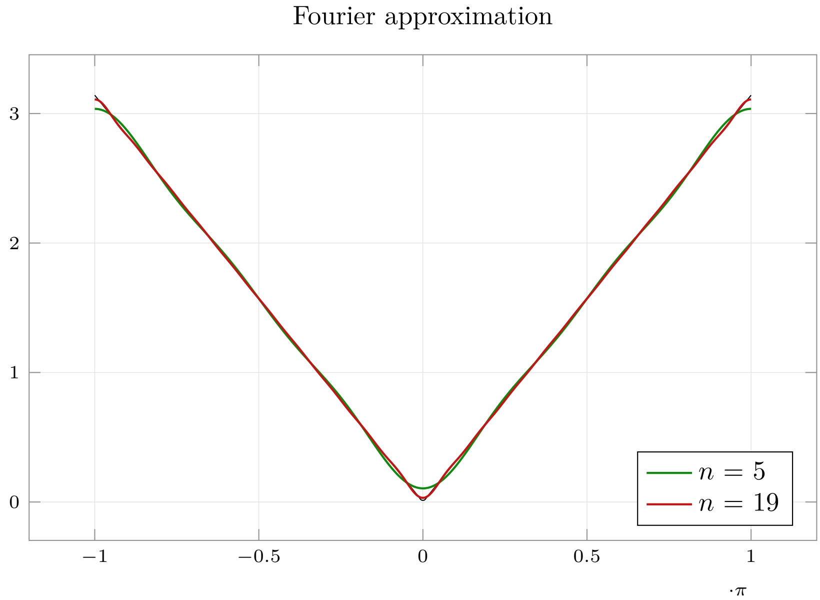
$$\int_{-\pi}^{\pi} f^2 \;\mathrm{d}x = \int_{-\pi}^{\pi} (x)^2 \;\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3} \tag{11.4.19}$$

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{\pi^2}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{[\cos(n\pi) - 1]^2}{n^4} \tag{11.4.20}$$

$$E^* = \frac{\pi^3}{6} - \frac{4}{\pi} \sum_{i=1}^{N} \frac{[\cos(n\pi) - 1]^2}{n^4} \tag{11.4.21}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	0.0747
3	0.0118
5	0.0037
7	0.0016
9	0.00083



4. Evaluating the Fourier coefficients,

$$f(x) = x^2 \qquad x \in [-\pi, \pi] \tag{11.4.22}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\;\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x^2\;\mathrm{d}x = \left[\frac{x^3}{3\pi}\right]_0^{\pi} = \frac{\pi^2}{3} \tag{11.4.23}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\;\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\;\mathrm{d}x \tag{11.4.24}$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin(nx)}{n} - \frac{2\sin(nx)}{n^3} + \frac{2x\cos(nx)}{n^2}\right]_0^{\pi} = \frac{4\cos(n\pi)}{n^2} \tag{11.4.25}$$

$$b_n = 0 \tag{11.4.26}$$

Calculating the error function,

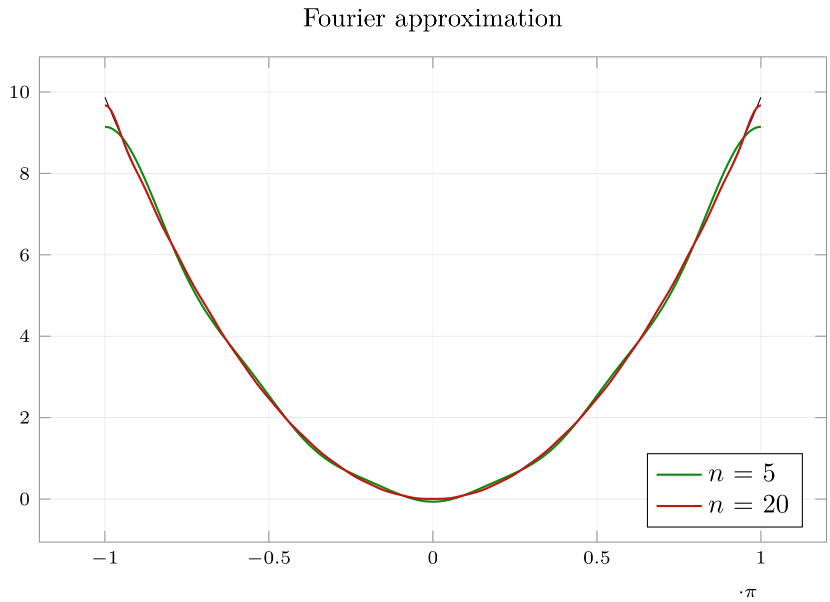
$$\int_{-\pi}^{\pi} f^2\;\mathrm{d}x = \int_{-\pi}^{\pi}(x)^4\;\mathrm{d}x = \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{5} \tag{11.4.27}$$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} \tag{11.4.28}$$

$$E^* = \frac{8\pi^5}{45} - 16\pi\sum_{i=1}^{N}\frac{1}{n^4} \tag{11.4.29}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	4.138
2	0.9964
3	0.3758
4	0.1795
5	0.0991



5. Evaluating the Fourier coefficients,

$$f(x) = \begin{cases} -1 & x \in [-\pi, 0] \\ 1 & x \in [0, \pi] \end{cases} \tag{11.4.30}$$

$$a_0 = 0 \tag{11.4.31}$$

$$a_n = 0 \tag{11.4.32}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \ \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} \sin(nx) \ \mathrm{d}x \tag{11.4.33}$$

$$= \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2}{n\pi} \left[1 - \cos(n\pi)\right] \tag{11.4.34}$$

Calculating the error function,

$$\int_{-\pi}^{\pi} f^2 \ \mathrm{d}x = \int_{-\pi}^{\pi} (1) \ \mathrm{d}x = \Big[x\Big]_{-\pi}^{\pi} = 2\pi \tag{11.4.35}$$

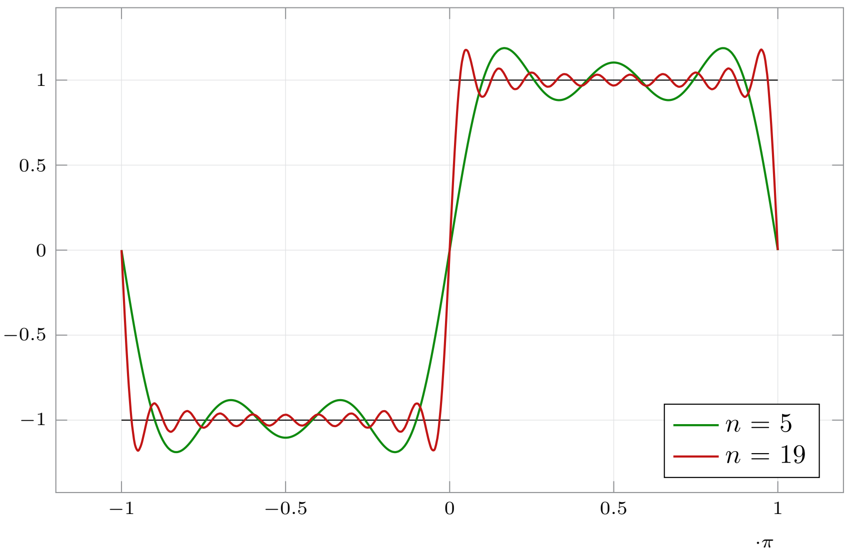
$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{[1 - \cos(n\pi)]^2}{n^2} \tag{11.4.36}$$

$$E^* = 2\pi - \frac{4}{\pi} \sum_{i=1}^{N} \frac{[1 - \cos(n\pi)]^2}{n^2} \tag{11.4.37}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	1.1902
2	0.6243
3	0.4206
4	0.3167
5	0.2538

Fourier approximation



6. The discontinuity at $x = 0$ in Problem 5 makes the Fourier series a very bad approximation to the function around $x = 0$. This makes the errors much larger.

7. Evaluating the Fourier coefficients,

$$f(x) = x^3 \qquad x \in [-\pi, \pi] \tag{11.4.38}$$

$$a_0 = 0 \tag{11.4.39}$$

$$a_n = 0 \tag{11.4.40}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \ \mathrm{d}x = \frac{2}{\pi} \int_{0}^{\pi} x^3 \ \sin(nx) \ \mathrm{d}x \tag{11.4.41}$$

$$= \frac{2}{\pi} \left[\sin(nx) \left(\frac{3x^2}{n^2} - \frac{6}{n^4} \right) + \cos(nx) \left(\frac{-x^3}{n} + \frac{6x}{n^3} \right) \right]_0^{\pi} \tag{11.4.42}$$

$$= \cos(n\pi) \left[\frac{12}{n^3} - \frac{2\pi^2}{n} \right] \tag{11.4.43}$$

Calculating the error function,

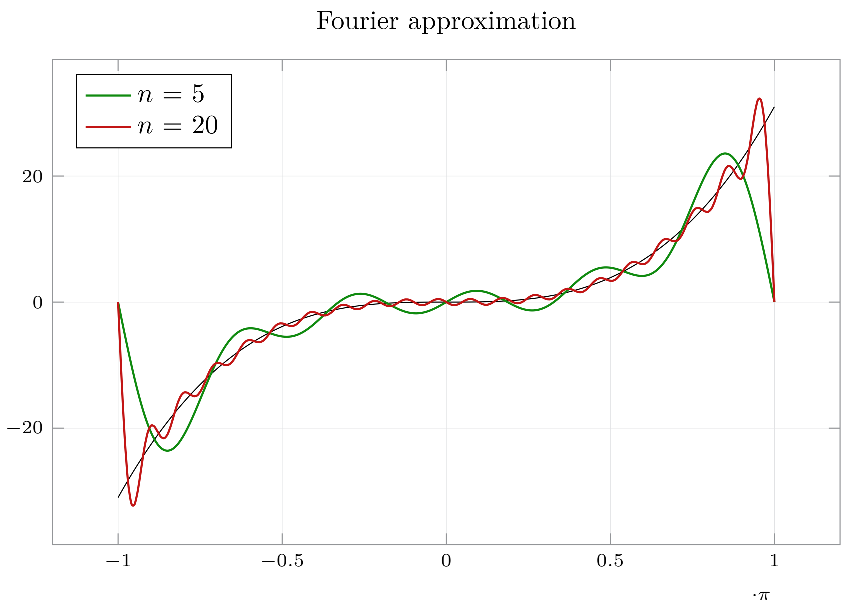
$$\int_{-\pi}^{\pi} f^2 \ \mathrm{d}x = \int_{-\pi}^{\pi} x^6 \ \mathrm{d}x = \left[\frac{x^7}{7} \right]_{-\pi}^{\pi} = \frac{2\pi^7}{7} \tag{11.4.44}$$

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \sum_{n=1}^{\infty} \left[\frac{12}{n^3} - \frac{2\pi^2}{n} \right]^2 \tag{11.4.45}$$

$$E^* = \frac{2\pi^7}{7} - \pi \sum_{i=1}^{N} \left[\frac{12}{n^3} - \frac{2\pi^2}{n} \right]^2 \tag{11.4.46}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	674.774
10	116.065
100	12.1793
500	2.4457
1000	1.2235



8. Evaluating the Fourier coefficients,

$$f(x) = |\sin(x)| \qquad x \in [-\pi, \pi] \tag{11.4.47}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \sin(x)\ \mathrm{d}x = \frac{1}{\pi}\Big[-\cos(x)\Big]_0^{\pi} = \frac{2}{\pi} \tag{11.4.48}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\ \mathrm{d}x \tag{11.4.49}$$

$$= \frac{2}{\pi}\left[\frac{n\sin(x)\sin(nx) + \cos(x)\cos(nx)}{n^2 - 1}\right]_0^{\pi} \tag{11.4.50}$$

$$= \frac{-2}{\pi(n^2-1)}\ [1 + \cos(n\pi)] \tag{11.4.51}$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\ \mathrm{d}x = \left[\frac{-\cos(2x)}{2\pi}\right]_0^{\pi} = 0 \tag{11.4.52}$$

$$b_n = 0 \tag{11.4.53}$$

Calculating the error function,

$$\int_{-\pi}^{\pi} f^2\ \mathrm{d}x = \int_{-\pi}^{\pi} \sin^2(x)\ \mathrm{d}x = \left[\frac{x}{2} + \frac{\sin(2x)}{4}\right]_{-\pi}^{\pi} = \pi \tag{11.4.54}$$

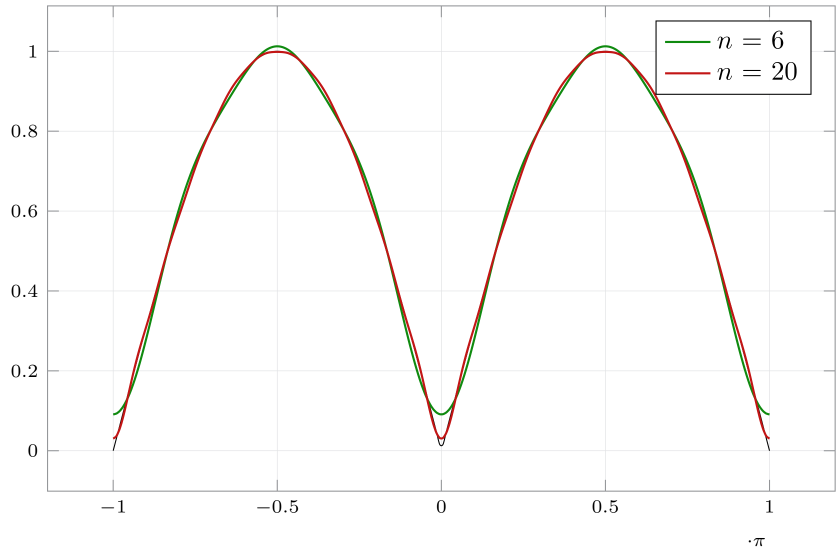
$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{8}{\pi^2} + \frac{4}{\pi^2}\sum_{n=2}^{\infty}\frac{[1+\cos(n\pi)]^2}{(n^2-1)^2} \tag{11.4.55}$$

$$E^* = \pi - \frac{8}{\pi} - \frac{4}{\pi}\sum_{n=2}^{N}\frac{[1+\cos(n\pi)]^2}{(n^2-1)^2} \tag{11.4.56}$$

Using sympy to evaluate the minimum error for various values of N,

N	E^*
2	0.0292
4	0.00659
6	0.002436
8	0.001153
10	0.000634

Fourier approximation



9. The minimized square error is a series of squares of Fourier coefficients, which are all nonnegative. The negative scalar factor makes the function monotonically decreasing in N.

10. The more trigonometric the actual function is, the faster E^* decreases with increasing N. Compare Problems $2 - 8$ using sympy to program $E^*(N)$.

11. From Example 1 in Section 11.1, the Fourier series expansion is

$$f(x) = \begin{cases} -1 & x \in [-\pi, 0] \\ 1 & x \in [0, \pi] \end{cases} \tag{11.4.57}$$

$$a_0 = a_n = 0 \tag{11.4.58}$$

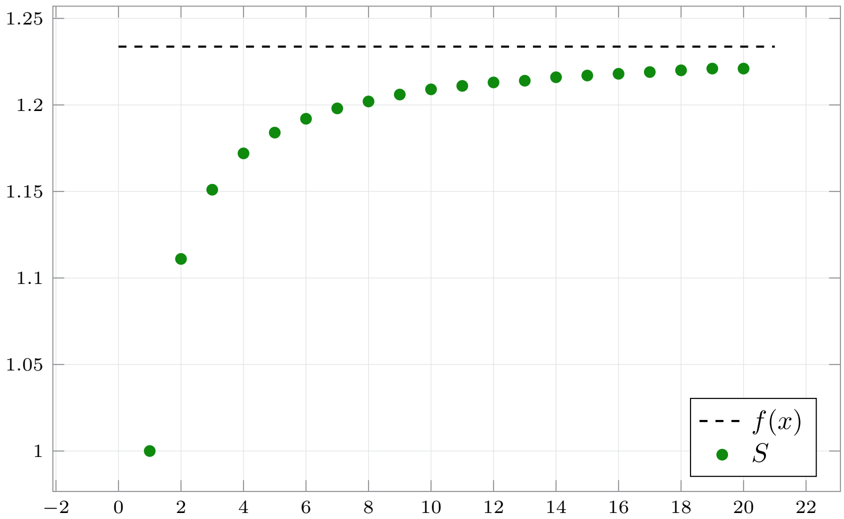
$$b_n = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right] \tag{11.4.59}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\ \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{11.4.60}$$

$$2 = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-\cos(n\pi)}{n}\right]^2 \tag{11.4.61}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \tag{11.4.62}$$

Fourier series partial sums



12. From Problem 14 in Section 11.1, the Fourier series expansion is

$$f(x) = x^2 \tag{11.4.63}$$

$$a_0 = \frac{\pi^2}{3} \tag{11.4.64}$$

$$a_n = \frac{4\,\cos(n\pi)}{n^2} \tag{11.4.65}$$

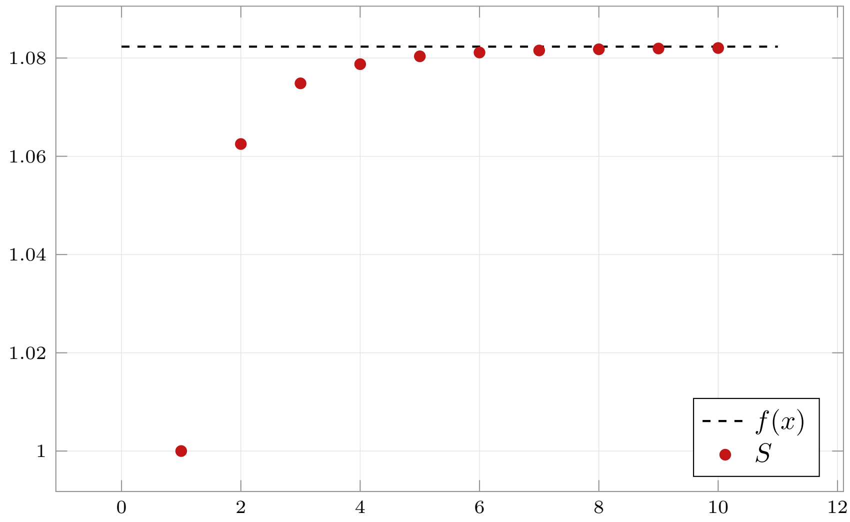
$$b_n = 0 \tag{11.4.66}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\ \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{11.4.67}$$

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right]^2 \tag{11.4.68}$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \tag{11.4.69}$$

Fourier series partial sums



13. From Problem 17 in Section 11.1, the Fourier series expansion is

$$f(x) = \begin{cases} x + \pi & x \in [-\pi, 0] \\ -x + \pi & x \in [0, \pi] \end{cases} \tag{11.4.70}$$

$$a_0 = \frac{\pi}{2} \tag{11.4.71}$$

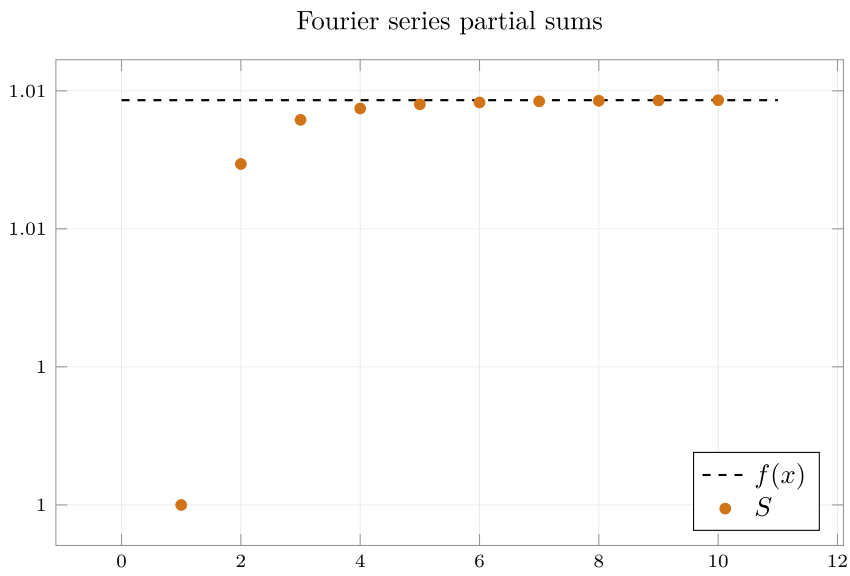
$$a_n = \frac{2}{\pi n^2} \left[1 - \cos(n\pi)\right] \tag{11.4.72}$$

$$b_n = 0 \tag{11.4.73}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \ \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{11.4.74}$$

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1 - \cos(n\pi)}{n^2}\right]^2 \tag{11.4.75}$$

$$\frac{\pi^4}{99} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \tag{11.4.76}$$



14. Using Parseval's identity,

$$f(x) = \cos^2(x) = \frac{1 + \cos(2x)}{2} \tag{11.4.77}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{11.4.78}$$

$$\int_{-\pi}^{\pi} \cos^4(x) \, \mathrm{d}x = \pi \left[\frac{2}{2^2} + \frac{1}{4} \right] = \frac{3\pi}{4} \tag{11.4.79}$$

15. Using Parseval's identity,

$$f(x) = \cos^3(x) = \frac{3\cos(x) + \cos(3x)}{4} \tag{11.4.80}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{11.4.81}$$

$$\int_{-\pi}^{\pi} \cos^6(x) \, \mathrm{d}x = \pi \left[0 + \frac{9}{16} + \frac{1}{16} \right] = \frac{5\pi}{8} \tag{11.4.82}$$

11.5 Sturm-Liouville Problems, Orthogonal Functions

1. (a) For Case III, where $p(a) = 0,\ \ p(b) \neq 0,$

$$(\lambda_m - \lambda_n) \int_a^b (r\ y_m\ y_n)\ \ \mathrm{d}x = p(b)\ z(b) - p(a)\ z(a) \tag{11.5.1}$$

$$z(x) = y_n'(x)\ y_m(x) - y_m'(x)\ y_n(x) \tag{11.5.2}$$

From the boundary conditions,

$$k_1\ y_n(b) + k_2\ y_n'(b) = 0 \tag{11.5.3}$$

$$k_1\ y_m(b) + k_2\ y_m'(b) = 0 \tag{11.5.4}$$

$$k_2\ z(b) = 0 \qquad \text{Eliminating } k_1 \tag{11.5.5}$$

Here, assume $k_2 \neq 0$, since at least one of $k_1,\ k_2$ has to be nonzero. The argument for the opposite case is identical

$$k_2 \neq 0 \implies z(b) = 0 \tag{11.5.6}$$

$$p(a) = 0,\ \ z(b) = 0 \implies y_m,\ \ y_n \text{ are orthogonal} \tag{11.5.7}$$

(b) For Case IV, where $p(a) \neq 0,\ \ p(b) \neq 0,$

$$(\lambda_m - \lambda_n) \int_a^b (r\ y_m\ y_n)\ \ \mathrm{d}x = p(b)\ z(b) - p(a)\ z(a) \tag{11.5.8}$$

$$z(x) = y_n'(x)\ y_m(x) - y_m'(x)\ y_n(x) \tag{11.5.9}$$

From the boundary conditions,

$$k_1\ y_n(b) + k_2\ y_n'(b) = 0 \tag{11.5.10}$$

$$k_1\ y_m(b) + k_2\ y_m'(b) = 0 \tag{11.5.11}$$

$$k_2\ z(b) = 0 \qquad \text{Eliminating } k_1 \tag{11.5.12}$$

Here, assume $k_2 \neq 0$, since at least one of $k_1,\ k_2$ has to be nonzero. The argument for the opposite case is identical.
Additionally, the same process leads to

$$k_2\ z(a) = 0 \tag{11.5.13}$$

$$k_2 \neq 0 \implies z(b) = z(a) = 0 \tag{11.5.14}$$

$$z(a) = 0,\ \ z(b) = 0 \implies y_m,\ \ y_n \text{ are orthogonal} \tag{11.5.15}$$

2. Proving that a scalar multiple of an eigenfunctino is also an eigenfunction,

$$z_m = c\ y_m \qquad\qquad c \neq 0 \tag{11.5.16}$$

$$\Big[py_m'\Big]' + \Big[q + \lambda r\Big] y = 0 \tag{11.5.17}$$

$$\Big[pz_m'\Big]' + \Big[q + \lambda r\Big] z = p' z_m' + p\ z_m'' + \Big[q + \lambda r\Big] z_m \tag{11.5.18}$$

$$= c\ (p' y_m') + c\ (p\ y_m'') + c\ \Big[q + \lambda r\Big] y_m \tag{11.5.19}$$

$$= 0 \tag{11.5.20}$$

By the linearity of differentiation, it is trivial to see that z_m also satisfies the boundary conditions of the Sturm-Liouville problem.

3. Given that $\{y_m\}$ is an orthogonal set under the weight function $r(x) = 1$ in the interval $x \in [a, b]$,

$$\int_a^b r(x)\ y_m(x)\ y_n(x)\ \ \mathrm{d}x = 0 \qquad\qquad \forall\ m \neq n \tag{11.5.21}$$

Making the substitution $x = ct + k$ for some $c > 0$ and constants c, k,

$$x = a \implies t_a = \frac{a - k}{c} \tag{11.5.22}$$

$$x = b \implies t_b = \frac{b - k}{c} \tag{11.5.23}$$

$$\mathrm{d}x = c\ \ \mathrm{d}t \tag{11.5.24}$$

$$\int_{t_a}^{t_b} y_m(ct + k)\ y_n(ct + k)\ c\ \ \mathrm{d}t = 0 \qquad\qquad \forall\ m \neq n \tag{11.5.25}$$

4. Using Problem 3, and setting $c = \pi,\ \ k = 0$,

$$\int_{-1}^{1} (1)\ \ \cos(m\pi t)\ \ \cos(n\pi t)\ \pi\ \ \mathrm{d}t = 0 \tag{11.5.26}$$

$$\int_{-1}^{1} (1)\ \ \cos(m\pi t)\ \ \sin(n\pi t)\ \pi\ \ \mathrm{d}t = 0 \tag{11.5.27}$$

$$\int_{-1}^{1} (1)\ \ \sin(m\pi t)\ \ \sin(n\pi t)\ \pi\ \ \mathrm{d}t = 0 \tag{11.5.28}$$

For all $m \neq n$, which proves their orthogonality in the domain $t \in [-1, 1]$

5. Legendre polynomials in $\cos\theta$, using the substitution $\cos\theta = x$,

$$r(\theta) = \sin\theta \qquad \theta \in [0, \pi] \tag{11.5.29}$$

$$\cos\theta = x \qquad \mathrm{d}x = -\sin\theta \ \ \mathrm{d}\theta \tag{11.5.30}$$

$$\int_0^{\pi} P_n(\cos\theta)\ P_m(\cos\theta)\ \sin(\theta)\ \mathrm{d}\theta = \int_{-1}^{1} P_n(x)\ P_m(x)\ \mathrm{d}x \tag{11.5.31}$$

Looking at the Legendre ODE which yields Legendre polynomials as eigenfunctions,

$$(1-x^2)\ y'' - 2x\ y' + n(n+1)\ y = 0 \qquad \left[(1-x^2)\ y'\right]' + \lambda\ y = 0 \tag{11.5.32}$$

$$\lambda = n(n+1) \qquad p(x) = 1 - x^2 \tag{11.5.33}$$

$$q(x) = 0 \qquad r(x) = 1 \tag{11.5.34}$$

Since the Legendre polynomials for integer n are solutions to the ODE, they are eigenfunctions of the Sturm-Lioville equation and the orthogonality relation holds.

6. Transforming variables,

$$0 = y'' + fy' + (g + \lambda h)y \qquad q = gp \qquad r = hp \tag{11.5.35}$$

$$p = \exp\left(\int f\ \ \mathrm{d}x\right) \qquad p' = f \cdot \exp\left(\int f\ \ \mathrm{d}x\right) = fp \tag{11.5.36}$$

$$0 = py'' + (fp)y' + (gp + \lambda\ hp)y \qquad 0 = \left[py'\right]' + (q + \lambda\ r)y \tag{11.5.37}$$

The advantage of reframing the original ODE as a Sturm-Liouville ODE is that a set of orthogonal solutions are guaranteed to exist.

7. Reframing as a Sturm-Lioville problem,

$$y'' + \lambda y = 0 \qquad y(0) = 0 \qquad y(10) = 0 \tag{11.5.38}$$

$$f = 0 \qquad p = \exp\left(\int_0^{10} f\ \ \mathrm{d}x\right) = 1 \tag{11.5.39}$$

$$g = 0 \qquad q = gp = 0 \tag{11.5.40}$$

$$h = 1 \qquad r = hp = 1 \tag{11.5.41}$$

Solving the ODE for negative eigenvalues,

$$0 = \left[py'\right]' + \left[q + \lambda\, r\right]y \qquad \lambda = -\nu^2 \tag{11.5.42}$$

$$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} \qquad y(0) = 0 = c_1 + c_2 \tag{11.5.43}$$

$$y(10) = 0 = c_1 e^{10\nu} + c_2 e^{-10\nu} \qquad c_1 = c_2 = 0 \tag{11.5.44}$$

Solving the ODE for positive eigenvalues,

$$0 = \left[py'\right]' + \left[q + \lambda\, r\right]y \qquad \lambda = \nu^2 \tag{11.5.45}$$

$$y(x) = c_1 \cos(\nu x) + c_2 \sin(\nu x) \qquad y(0) = 0 = c_1 \tag{11.5.46}$$

$$y(10) = 0 = c_1 \cos(10\nu) + c_2 \sin(10\nu) \qquad 10\nu = n\pi \tag{11.5.47}$$

For $\lambda = 0$, only the trivial solution exists. The eigenfunctions and corresponding eigenvalues are,

$$y_n(x) = \sin\left(\frac{n\pi}{10}\, x\right) \qquad \lambda_n = \left(\frac{n\pi}{10}\right)^2 \tag{11.5.48}$$

8. Solving the ODE for negative eigenvalues, using Problem 7,

$$0 = \left[py'\right]' + \left[q + \lambda\, r\right]y \qquad \lambda = -\nu^2 \tag{11.5.49}$$

$$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} \qquad y(0) = 0 = c_1 + c_2 \tag{11.5.50}$$

$$y(L) = 0 = c_1 e^{L\nu} + c_2 e^{-L\nu} \qquad c_1 = c_2 = 0 \tag{11.5.51}$$

Solving the ODE for positive eigenvalues,

$$0 = \left[py'\right]' + \left[q + \lambda\, r\right]y \qquad \lambda = \nu^2 \tag{11.5.52}$$

$$y(x) = c_1 \cos(\nu x) + c_2 \sin(\nu x) \qquad y(0) = 0 = c_1 \tag{11.5.53}$$

$$y(L) = 0 = c_1 \cos(L\nu) + c_2 \sin(L\nu) \qquad L\nu = n\pi \tag{11.5.54}$$

For $\lambda = 0$, only the trivial solution exists. The eigenfunctions and corresponding eigenvalues are,

$$y_n(x) = \sin\left(\frac{n\pi}{L}\, x\right) \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{11.5.55}$$

9. Solving the ODE for negative eigenvalues, using Problem 7,

$$0 = \left[py'\right]' + \left[q + \lambda\, r\right]y \qquad \lambda = -\nu^2 \tag{11.5.56}$$

$$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} \qquad y(0) = 0 = c_1 + c_2 \tag{11.5.57}$$

$$y'(L) = 0 = \nu c_1 e^{\nu L} - \nu c_2 e^{-\nu L} \qquad c_1 = c_2 = 0 \tag{11.5.58}$$

Solving the ODE for positive eigenvalues,

$$0 = \left[py'\right]' + \left[q + \lambda\, r\right]y \qquad \lambda = \nu^2 \tag{11.5.59}$$

$$y(x) = c_1 \cos(\nu x) + c_2 \sin(\nu x) \qquad y(0) = 0 = c_1 \tag{11.5.60}$$

$$y'(L) = 0 = \nu c_2 \cos(L\nu) - \nu c_1 \sin(L\nu) \qquad L\nu = \frac{(2n-1)\pi}{2} \tag{11.5.61}$$

For $\lambda = 0$, only the trivial solution exists. The eigenfunctions and corresponding eigenvalues are,

$$y_n(x) = \sin\left(\frac{(2n-1)\pi}{2L}\, x\right) \qquad \lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \tag{11.5.62}$$

for integers $n = \{1, 2, 3, \dots\}$

10. Solving the ODE for negative eigenvalues, using Problem 7,

$$0 = \left[py'\right]' + \left[q + \lambda\, r\right]y \qquad \lambda = -\nu^2 \tag{11.5.63}$$

$$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} \tag{11.5.64}$$

$$y(0) = y(1) \qquad c_1 + c_2 = c_1 e^{\nu} + c_2 e^{-\nu} \tag{11.5.65}$$

$$y'(0) = y'(1) \qquad \nu(c_1 - c_2) = \nu(c_1 e^{\nu} - c_2 e^{-\nu}) \tag{11.5.66}$$

$$c_1 = 0 \qquad c_2 = 0 \tag{11.5.67}$$

Solving the ODE for positive eigenvalues,

$$0 = \Big[py'\Big]' + \Big[q + \lambda\ r\Big]y \qquad \lambda = \nu^2 \tag{11.5.68}$$

$$y(x) = c_1\cos(\nu x) + c_2\sin(\nu x) \tag{11.5.69}$$

$$y(0) = y(1) \qquad c_1 = c_1\cos(\nu) + c_2\sin(\nu) \tag{11.5.70}$$

$$y'(0) = y'(1) \qquad \nu c_2 = -\nu c_1\sin(\nu) + \nu c_2\cos(\nu) \tag{11.5.71}$$

$$c_2 = c_1\frac{1-\cos(\nu)}{\sin(\nu)} \qquad 0 = c_1\left(1 + \left(\frac{1-\cos\nu}{\sin\nu}\right)^2\right) \tag{11.5.72}$$

$$c_1 = 0 \qquad c_2 = 0 \tag{11.5.73}$$

The above case requires $\sin(\nu) \neq 0$. Looking at this special case,

$$\nu = n\pi \qquad c_1 = c_1\cos(n\pi) \tag{11.5.74}$$

$$c_2 = c_2\cos(n\pi) \qquad \cos(n\pi) = 1 \implies n = 2k \tag{11.5.75}$$

A nontrivial solution is now

$$y_k(x) = c_1\cos(2k\pi\ x) + c_2\sin(2k\pi\ x) \qquad \lambda_k = (2k\pi)^2 \tag{11.5.76}$$

To prove orthogonality use the result from Problem 4

$$\int_{-1}^{1}(1)\ y_m\ y_n\ \ \mathrm{d}x = 0 \tag{11.5.77}$$

11. Reframing as a Sturm-Lioville problem,

$$0 = \left(\frac{y'}{x}\right)' + \frac{\lambda+1}{x^3}\ y \qquad x = e^t \tag{11.5.78}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^t = x \qquad y' = \dot{y}\ \frac{\mathrm{d}t}{\mathrm{d}x} = \dot{y}\ e^{-t} \tag{11.5.79}$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}t}[\dot{y}\ e^{-t}]\ \frac{\mathrm{d}t}{\mathrm{d}x} \qquad y'' = \ddot{y}e^{-2t} - \dot{y}e^{-2t} \tag{11.5.80}$$

$$0 = \frac{y''}{x} - \frac{y'}{x^2} + \frac{\lambda+1}{x^3}\ y \qquad 0 = \ddot{y} - 2\dot{y} + (\lambda+1)y \tag{11.5.81}$$

This is an second order linear ODE with constant coefficients.

$$\mu = \frac{2 \pm \sqrt{4 - 4(\lambda+1)}}{2} \qquad \mu_1,\ \mu_2 = 1 \pm \sqrt{-\lambda} \tag{11.5.82}$$

For the case where $\lambda = -\nu^2$,

$$y = c_1 e^{(1+\nu)t} + c_2 e^{(1-\nu)t} \qquad y(t=0) = y(t=\pi) = 0 \tag{11.5.83}$$

$$0 = c_1 + c_2 \qquad 0 = c_1 e^{(1+\nu)\pi} + c_2 e^{(1-\nu)\pi} \tag{11.5.84}$$

$$c_1 = 0 \qquad c_2 = 0 \tag{11.5.85}$$

This leads to the trivial solution.
For the case where $\lambda = \nu^2$,

$$y = e^t \left[c_1 \cos(\nu t) + c_2 \sin(\nu t)\right] \qquad y(t=0) = y(t=\pi) = 0 \tag{11.5.86}$$

$$0 = c_1 \qquad 0 = c_2 \ \sin(\nu\pi) \tag{11.5.87}$$

$$c_1 = 0 \qquad \nu = n \tag{11.5.88}$$

$$y_n(x) = e^t \ \sin(nt) \tag{11.5.89}$$

For $\lambda = 0$,

$$y = (c_1 + c_2 t)\ e^t \qquad y(t=0) = y(t=\pi) = 0 \tag{11.5.90}$$

$$0 = c_1 \qquad 0 = c_1 + c_2\pi \tag{11.5.91}$$

This also leads to the trivial solution.
Reverting to the original variable x,

$$y_n(x) = x \ \sin(n \ \ln x) \qquad \lambda_n = n^2 \tag{11.5.92}$$

Checking for orthogonality,

$$I = \int_1^{e^\pi} x^2 \ \sin(n \ln x) \sin(m \ln x) \ (x^{-3}) \ \mathrm{d}x \tag{11.5.93}$$

$$\ln(x) = u \qquad \frac{1}{x} \ \mathrm{d}x = \ \mathrm{d}u \tag{11.5.94}$$

$$I = \int_0^{\pi} \sin(nu) \ \sin(mu) \ \mathrm{d}u \tag{11.5.95}$$

This is proven orthogonal already.

12. Using the result from Problem 11,

$$0 = y'' - 2y' + (\lambda + 1)y \tag{11.5.96}$$

$$\mu = \frac{2 \pm \sqrt{4 - 4(\lambda+1)}}{2} \qquad \mu_1, \ \mu_2 = 1 \pm \sqrt{-\lambda} \tag{11.5.97}$$

For the case where $\lambda = -\nu^2$,

$$y = c_1 e^{(1+\nu)x} + c_2 e^{(1-\nu)x} \qquad y(0) = y(1) = 0 \tag{11.5.98}$$

$$0 = c_1 + c_2 \qquad 0 = c_1 e^{(1+\nu)} + c_2 e^{(1-\nu)} \tag{11.5.99}$$

$$c_1 = 0 \qquad c_2 = 0 \tag{11.5.100}$$

This leads to the trivial solution.
For the case where $\lambda = \nu^2$,

$$y = e^x \left[c_1 \cos(\nu x) + c_2 \sin(\nu x)\right] \qquad y(0) = y(1) = 0 \tag{11.5.101}$$

$$0 = c_1 \qquad 0 = c_2 \sin(\nu) \tag{11.5.102}$$

$$c_1 = 0 \qquad \nu = n\pi \tag{11.5.103}$$

$$y_n(x) = e^x \sin(n\pi x) \qquad \lambda_n = (n\pi)^2 \tag{11.5.104}$$

For $\lambda = 0$,

$$y = (c_1 + c_2 x)\, e^x \qquad y(0) = y(1) = 0 \tag{11.5.105}$$

$$0 = c_1 \qquad 0 = c_1 + c_2 \tag{11.5.106}$$

This also leads to the trivial solution.

13. Using the result from Problem 11,

$$0 = y'' + 8y' + (\lambda + 16)y \tag{11.5.107}$$

$$\mu = \frac{-8 \pm \sqrt{64 - 4(\lambda + 16)}}{2} \qquad \mu_1,\ \mu_2 = -4 \pm \sqrt{-\lambda} \tag{11.5.108}$$

For the case where $\lambda = -\nu^2$,

$$y = c_1 e^{(-4+\nu)x} + c_2 e^{(-4-\nu)x} \qquad y(0) = y(\pi) = 0 \tag{11.5.109}$$

$$0 = c_1 + c_2 \qquad 0 = c_1 e^{(-4+\nu)\pi} + c_2 e^{(-4-\nu)\pi} \tag{11.5.110}$$

$$c_1 = 0 \qquad c_2 = 0 \tag{11.5.111}$$

This leads to the trivial solution.

For the case where $\lambda = \nu^2$,

$$y = e^{-4x}\left[c_1 \cos(\nu x) + c_2 \sin(\nu x)\right] \qquad y(0) = y(\pi) = 0 \tag{11.5.112}$$

$$0 = c_1 \qquad 0 = c_2 \ \sin(\nu\pi) \tag{11.5.113}$$

$$c_1 = 0 \qquad \nu = n \tag{11.5.114}$$

$$y_n(x) = e^{-4x} \ \sin(nx) \qquad \lambda_n = n^2 \tag{11.5.115}$$

For $\lambda = 0$,

$$y = (c_1 + c_2 x)\ e^{-4x} \qquad y(0) = y(\pi) = 0 \tag{11.5.116}$$

$$0 = c_1 \qquad 0 = c_1 + c_2 \tag{11.5.117}$$

This also leads to the trivial solution.

14. Special families of orthogonal polynomials,

(a) Chebyshev polynomials of the first kind, with $\arccos(x) = \theta$

$$T_n(x) = \cos(n \ \arccos(x)) = \cos(n\theta) \tag{11.5.118}$$

$$T_0(x) = \cos(0) = 1 \tag{11.5.119}$$

$$T_1(x) = \cos(\arccos(x)) = x \tag{11.5.120}$$

$$T_2(x) = \cos(2 \ \arccos(x)) = 2\cos^2(\theta) - 1 = 2x^2 - 1 \tag{11.5.121}$$

$$T_3(x) = \cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta) = 4x^3 - 3x \tag{11.5.122}$$

Chebyshev polynomials of the second kind, with $\arccos(x) = \theta$

$$U_n(x) = \frac{\sin[(n+1)\arccos(x)]}{\sqrt{1-x^2}} = \frac{\sin[(n+1)\theta]}{\sin(\theta)} \tag{11.5.123}$$

$$U_0(x) = \cos(0) = 1 \tag{11.5.124}$$

$$U_1(x) = \frac{\sin(2\theta)}{\sin\theta} = 2\cos\theta = 2x \tag{11.5.125}$$

$$U_2(x) = \frac{\sin(3\theta)}{\sin\theta} = 3\cos^2(\theta) - \sin^2(\theta) = 4x^2 - 1 \tag{11.5.126}$$

$$U_3(x) = \frac{\sin(4\theta)}{\sin(\theta)} = 4\cos^3(\theta) - 4\cos(\theta)\sin^2(\theta) = 8x^3 - 4x \tag{11.5.127}$$

Checking the orthogonality of the polynomials $T_n(x)$,

$$\arccos(x) = \theta \qquad \frac{-1}{\sqrt{1-x^2}} \, \mathrm{d}x = \mathrm{d}\theta \tag{11.5.128}$$

$$r(x) = \frac{1}{\sqrt{1-x^2}} \qquad I = \int_{-1}^{1} T_n(x) \; T_m(x) \; \frac{\mathrm{d}x}{\sqrt{1-x^2}} \tag{11.5.129}$$

$$I = \int_0^{\pi} \cos(n\theta) \;\; \cos(m\theta) \;\; \mathrm{d}\theta \tag{11.5.130}$$

This is known to be orthogonal which proves the relation.
Verifying the set $\{T_n\}$ satisfy the Chebyshev ODE,

$$(1-x^2)y'' - xy' + n^2 y = 0 \tag{11.5.131}$$

$$n = 0 \implies (1-x^2)(0) - x(0) + 0(1) = 0 \tag{11.5.132}$$

$$n = 1 \implies (1-x^2)(0) - x(1) + 1(x) = 0 \tag{11.5.133}$$

$$n = 2 \implies (1-x^2)(4) - x(4x) + 4(2x^2-1) = 0 \tag{11.5.134}$$

$$n = 3 \implies (1-x^2)(24x) - x(12x^2-3) + 9(4x^3-3x) = 0 \tag{11.5.135}$$

(b) LaGuerre polynomials,

$$L_n(x) = \frac{e^x}{n!} \, \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x}) \tag{11.5.136}$$

$$L_1(x) = \frac{e^x}{1!} \, \frac{\mathrm{d}}{\mathrm{d}x}[xe^{-x}] = 1 - x \tag{11.5.137}$$

$$L_2(x) = \frac{e^x}{2!} \, \frac{\mathrm{d}^2}{\mathrm{d}x^2}[x^2e^{-x}] = \frac{2-4x+x^2}{2} = 1 - 2x + \frac{x^2}{2} \tag{11.5.138}$$

$$L_3(x) = \frac{e^x}{3!} \, \frac{\mathrm{d}^3}{\mathrm{d}x^3}[x^3e^{-x}] = \frac{6-18x+9x^2-x^3}{6} = 1 - 3x + \frac{3x^2}{2} - \frac{x^3}{6} \tag{11.5.139}$$

To prove orthogonality, consider $L_n, \;\; L_k$ with $k < n$, without loss of generality. Now, since integration is linear, the polynomial L_k is a linear combination of powers of x.

$$I = \int_0^{\infty} e^{-x} \; x^k \; L_n(x) \;\; \mathrm{d}x \tag{11.5.140}$$

$$= \int_0^{\infty} e^{-x} \; x^k \; \frac{e^x}{n!} \, \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x}) \;\; \mathrm{d}x \tag{11.5.141}$$

$$I = \left[\frac{x^k}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^n e^{-x})\right]_0^{\infty} - \int_0^{\infty} \frac{kx^{k-1}}{n!} \;\; \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^n e^{-x}) \;\; \mathrm{d}x \tag{11.5.142}$$

The first term above is always zero for all positive $(n-k)$, since the polynomial $e^{-\infty} = 0$ and

$0^k = 0$

$$\frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}(x^n e^{-x}) = e^{-x} \cdot Q(x) \tag{11.5.143}$$

After k such integrations by part,

$$I = \left[(-1)^k \ \frac{k!}{n!} \ \frac{\mathrm{d}^{n-k-1}}{\mathrm{d}x^{n-k-1}}(x^n e^{-x})\right]_0^\infty = 0 \tag{11.5.144}$$

11.6 Orthogonal Series, Generalized Fourier Series

1. Expanding into a Fourier-Legendre series, neglecting the integrals of odd functions in $[-1, 1]$

$$f(x) = 63x^5 - 90x^3 + 35x \tag{11.6.1}$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.2}$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} (63x^5 - 90x^3 + 35x)(x)\ \ \mathrm{d}x \qquad = 8 \tag{11.6.3}$$

$$a_3 = \frac{3}{2} \int_{-1}^{1} (63x^5 - 90x^3 + 35x)(2.5x^3 - 1.5x)\ \ \mathrm{d}x \qquad = -8 \tag{11.6.4}$$

$$a_5 = \frac{11}{16} \int_{-1}^{1} (63x^5 - 90x^3 + 35x)(63x^5 - 70x^3 + 15x)\ \ \mathrm{d}x \qquad = 8 \tag{11.6.5}$$

$$f(x) = 8P_1 - 8P_3 + 8P_5 \tag{11.6.6}$$

2. Expanding into a Fourier-Legendre series, neglecting the integrals of odd functions in $[-1, 1]$

$$f(x) = (x+1)^2 \tag{11.6.7}$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.8}$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} (x+1)^2(1)\ \mathrm{d}x \qquad = \frac{4}{3} \tag{11.6.9}$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} (x^3 + 2x^2 + x)\ \mathrm{d}x \qquad = 2 \tag{11.6.10}$$

$$a_5 = \frac{5}{4} \int_{-1}^{1} (x+1)^2(3x^2 - 1)\ \mathrm{d}x \qquad = \frac{2}{3} \tag{11.6.11}$$

$$f(x) = \frac{4P_0 + 6P_1 + 2P_2}{3} \tag{11.6.12}$$

3. Expanding into a Fourier-Legendre series, neglecting the integrals of odd functions in $[-1, 1]$

$$f(x) = 1 - x^4 \tag{11.6.13}$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.14}$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} (1 - x^4)\ \mathrm{d}x \qquad = \frac{4}{5} \tag{11.6.15}$$

$$a_2 = \frac{5}{4} \int_{-1}^{1} (1 - x^4)(3x^2 - 1)\ \mathrm{d}x \qquad = \frac{-4}{7} \tag{11.6.16}$$

$$a_4 = \frac{9}{16} \int_{-1}^{1} (1 - x^4)(35x^4 - 30x^2 + 3)\ \mathrm{d}x \qquad = \frac{-8}{35} \tag{11.6.17}$$

$$f(x) = \frac{28P_0 - 20P_2 - 8P_4}{35} \tag{11.6.18}$$

4. By observation,

$$1 = P_0 \qquad x = P_1 \tag{11.6.19}$$

$$x^2 = \frac{2P_2 + P_0}{3} \qquad x^3 = \frac{2P_3 + 3P_1}{5} \tag{11.6.20}$$

$$x^4 = \frac{8P_4 + 20P_2 + 7P_0}{35} \tag{11.6.21}$$

5. Assume $f(x)$ is odd. Then $f(x)P_n(x)$ is also odd for even n. Further, the integral of an odd function in a region symmetric about the origin is zero. This means that the coefficients of odd Legendre polynomials is zero.

This means that the Fourier-Legendre expansion will only contain odd n terms.

The proof for odd functions $g(x)$ is the exact same. Examples are problems 1, 2, 3, 4 above.

6. Suppose f is not a constant function and its MacLaurin series only contains terms of the form x^{4m}. Its Fourier-Legendre polynomials cannot contain odd terms by observation.

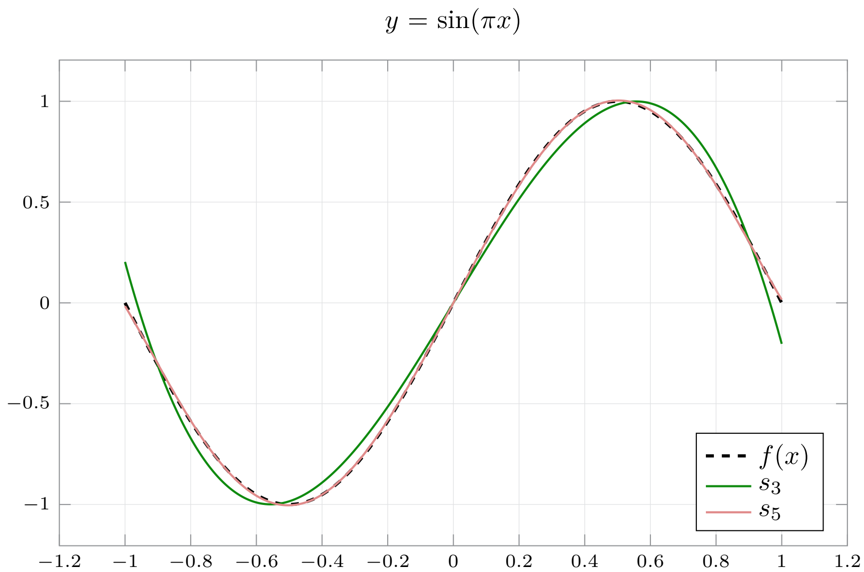
Further, even Legendre polynomials contain terms of the form x^{4m+2}, which can be made to cancel out when expanding $f(x)$ in terms of the even Legendre polynomials.

7. Changing the coefficient of x^m inside $f(x)$, changes the coefficients of all the Legendre polynomials P_m, P_{m-2}, P_{m-4} and so on.

8. Finding the Fourier-Legendre expansion,

$$f(x) = \sin(\pi x) \tag{11.6.22}$$

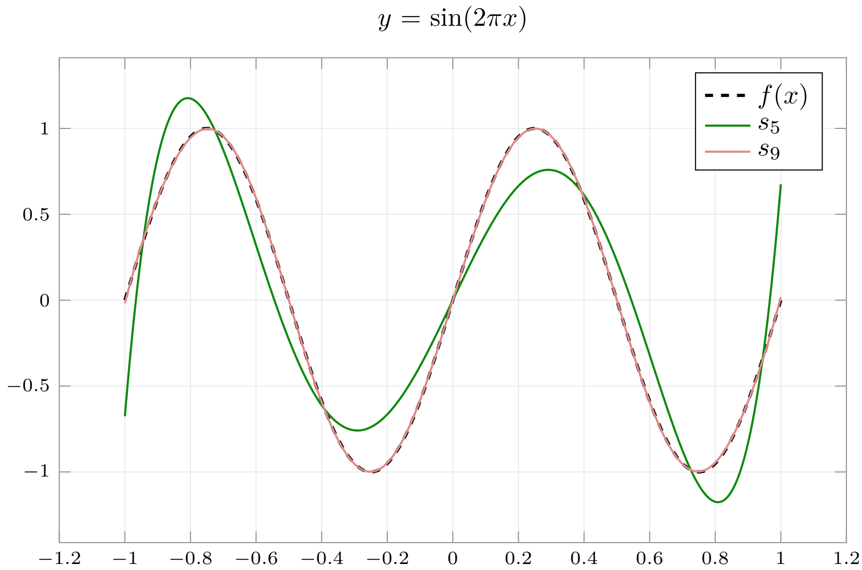
$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.23}$$



9. Finding the Fourier-Legendre expansion,

$$f(x) = \sin(2\pi x) \tag{11.6.24}$$

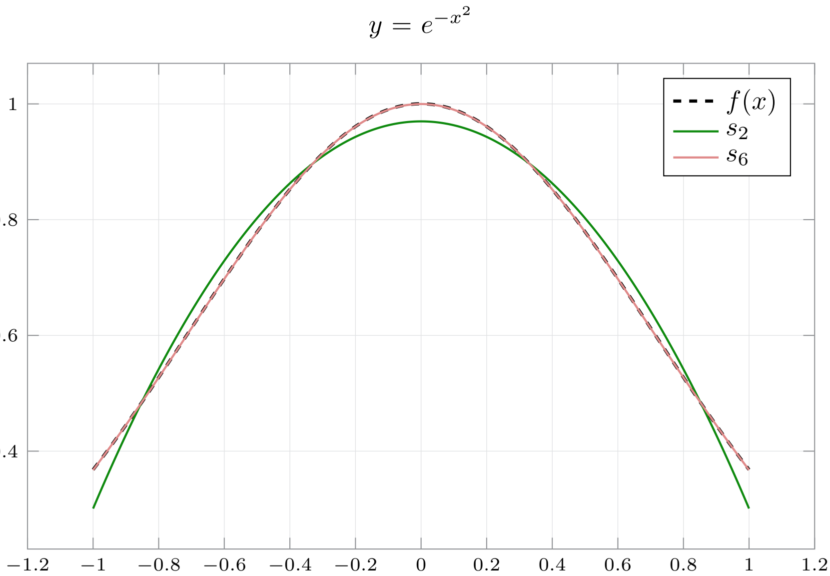
$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.25}$$



10. Finding the Fourier-Legendre expansion,

$$f(x) = \exp(-x^2) \tag{11.6.26}$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.27}$$

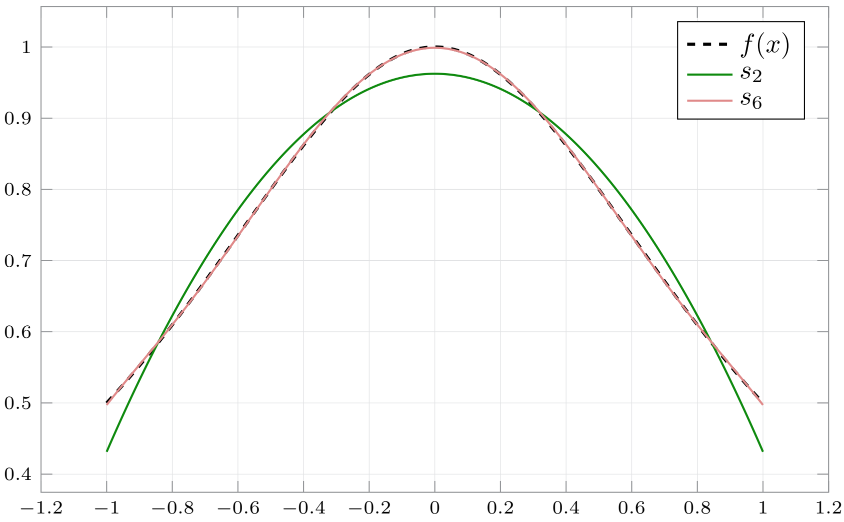


11. Finding the Fourier-Legendre expansion,

$$f(x) = \frac{1}{1+x^2} \tag{11.6.28}$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.29}$$

$y = \frac{1}{1+x^2}$

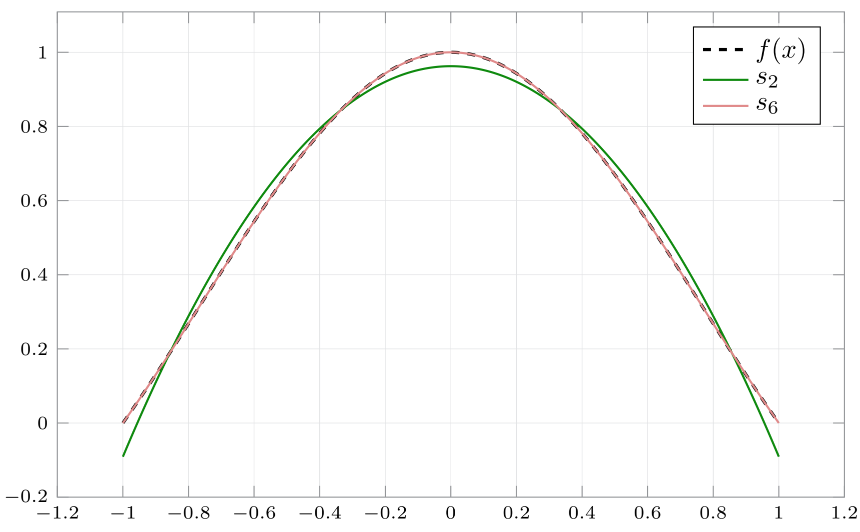


12. Finding the Fourier-Legendre expansion, with $\alpha = \alpha_{0,1}$

$$f(x) = \frac{1}{1+x^2} \tag{11.6.30}$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.31}$$

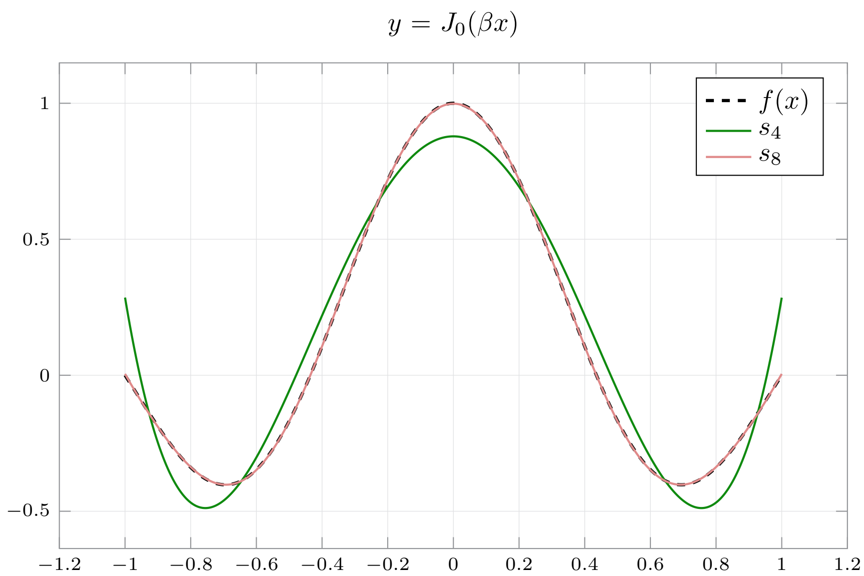
$y = J_0(\alpha x)$



13. Finding the Fourier-Legendre expansion, with $\beta = \alpha_{0,2}$

$$f(x) = \frac{1}{1+x^2} \tag{11.6.32}$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\ P_m(x)\ \ \mathrm{d}x \qquad \|P_m\| = \sqrt{\frac{2}{2m+1}} \tag{11.6.33}$$



14. Hermite's polynomials

(a) For small values of n,

$$H_n(x) = (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2/2}) \tag{11.6.34}$$

$$H_1(x) = -e^{x^2/2} \frac{\mathrm{d}}{\mathrm{d}x}(e^{-x^2/2}) = x \tag{11.6.35}$$

$$H_2(x) = e^{x^2/2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{-x^2/2}) = -1 + x^2 \tag{11.6.36}$$

$$H_3(x) = -e^{x^2/2} \frac{\mathrm{d}^3}{\mathrm{d}x^3}(e^{-x^2/2}) = x^3 - 3x \tag{11.6.37}$$

$$H_4(x) = e^{x^2/2} \frac{\mathrm{d}^4}{\mathrm{d}x^4}(e^{-x^2/2}) = x^4 - 6x^2 + 3 \tag{11.6.38}$$

(b) The Maclaurin series is given by,

$$f(t=0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t=0)}{n!}\; t^n \tag{11.6.39}$$

$$f(t) = \exp\left(tx - \frac{t^2}{2}\right) = \exp\left[\frac{x^2}{2} - \frac{(x-t)^2}{2}\right] \tag{11.6.40}$$

$$\frac{\mathrm{d}^n f}{\mathrm{d}t^n} = e^{x^2/2}\; \frac{\mathrm{d}^n}{\mathrm{d}t^n} \exp\left[\frac{-(x-t)^2}{2}\right] \tag{11.6.41}$$

$$z = (x-t) \qquad\qquad t = 0 \to z = x \tag{11.6.42}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} = (-1)^n\; \frac{\mathrm{d}^n}{\mathrm{d}z^n} \tag{11.6.43}$$

$$\frac{\mathrm{d}^n f}{\mathrm{d}t^n} = (-1)^n\; e^{x^2/2}\; \frac{\mathrm{d}^n}{\mathrm{d}z^n}(e^{-z^2/2}) \tag{11.6.44}$$

$$\left.\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right|_{t=0} = \left[(-1)^n\; e^{x^2/2}\; \frac{\mathrm{d}^n}{\mathrm{d}z^n}(e^{-z^2/2})\right]_{z=x} \tag{11.6.45}$$

$$= (-1)^n\; e^{x^2/2}\; \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2/2}) = H_n(x) \tag{11.6.46}$$

Thus, the given function f is a generating function of the Hermite polynomials.

(c) Differentiating with respect to x gives,

$$\exp\left(tx - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(x)\; \frac{t^n}{n!} \tag{11.6.47}$$

$$t\;\; \exp\left(tx - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n'(x)\; \frac{t^n}{n!} \tag{11.6.48}$$

$$\sum_{n=0}^{\infty} H_n'(x)\; \frac{t^n}{n!} = \sum_{n=0}^{\infty}(n+1)H_n(x)\; \frac{t^{n+1}}{(n+1)!} \tag{11.6.49}$$

Equating coefficients of t^n, gives,

$$H_n' = n \cdot H_{n-1} \tag{11.6.50}$$

(d) Checking orthogonality on the real line, assuming $n < m$

$$r(x) = e^{-x^2/2} \tag{11.6.51}$$

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\ H_n(x)\ H_m(x)\ \mathrm{d}x \tag{11.6.52}$$

$$= \int_{-\infty}^{\infty} (-1)^m\ H_n\ \frac{\mathrm{d}^m}{\mathrm{d}x^m}(e^{-x^2/2})\ \mathrm{d}x \tag{11.6.53}$$

$$= (-1)^m \left[H_n\ \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}(e^{-x^2/2})\right]_{-\infty}^{\infty} \tag{11.6.54}$$

$$- (-1)^m \int_{-\infty}^{\infty} (nH_{n-1})\ \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}(e^{-x^2/2})\ \mathrm{d}x \tag{11.6.55}$$

Since $\exp(-x^2/2)$ is always dominant over any polynomial in in x, the first term in the integration by parts is always zero. Repetitive integration by parts yields,

$$I = (-1)^{m+n}\ (n!)\ \int_{-\infty}^{\infty} (H_0)\ \frac{\mathrm{d}^{m-n}}{\mathrm{d}x^{m-n}}(e^{-x^2/2})\ \mathrm{d}x \tag{11.6.56}$$

$$= (-1)^{m+n}\ n!\ H_0\ \left[\frac{\mathrm{d}^{m-n-1}}{\mathrm{d}x^{m-n-1}}(e^{-x^2/2})\right]_{-\infty}^{\infty} = 0 \tag{11.6.57}$$

(e) Differentiating with respect to t,

$$H_n(x) = (-1)^n\ e^{x^2/2}\ \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2/2}) \tag{11.6.58}$$

$$H_n'(x) = (-1)^n\ xe^{x^2/2}\ \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2/2}) + (-1)^n\ e^{x^2/2}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(e^{-x^2/2}) \tag{11.6.59}$$

$$H_n' = x \cdot H_n - H_{n+1} \tag{11.6.60}$$

Rewriting with $(n-1)$ instead of n,

$$H_n' = x \cdot H_n - H_{n+1} \tag{11.6.61}$$

$$H_n'' = H_n + xH_n' - H_{n+1}' \tag{11.6.62}$$

$$= H_n + xH_n' - (n+1)H_n \tag{11.6.63}$$

$$= xH_n' - n\ H_n \tag{11.6.64}$$

$$y'' = xy' - ny \tag{11.6.65}$$

Checking if $w = e^{-x^2/4}\, y$ solves Weber's equation,

$$w' = e^{-x^2/4}\, y' - \frac{x}{2}\, e^{-x^2/4}\, y \tag{11.6.66}$$

$$w'' = e^{-x^2/4}\, y'' - x\, e^{-x^2/4}\, y' + \left(\frac{x^2}{4} - \frac{1}{2}\right) e^{-x^2/4}\, y \tag{11.6.67}$$

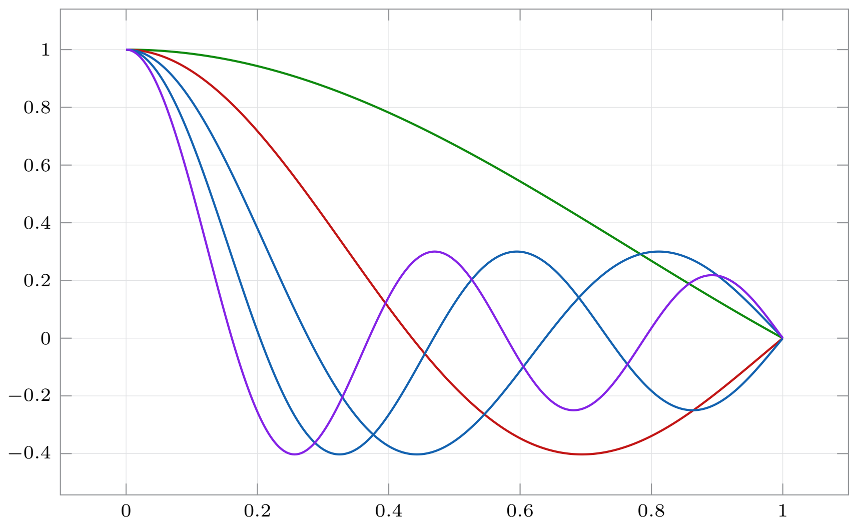
$$w'' = e^{-x^2/4}\, (-ny) + \left(\frac{x^2}{4} - \frac{1}{2}\right) e^{-x^2/4}\, y \tag{11.6.68}$$

$$w'' = -w \left[n + \frac{1}{2} - \frac{x^2}{4}\right] \tag{11.6.69}$$

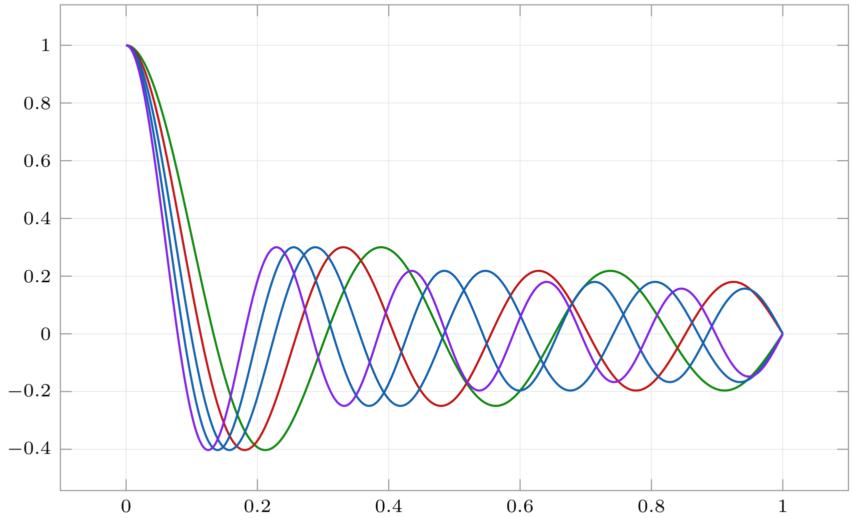
15. Using a CAS to plot Fourier-Bessel expansions,

(a) Plotting the first 10 functions in the family $\{J_0(\alpha_{0,k}\ x)\}$

$$y = J_0(\alpha x),\ \ \alpha \in [1, 5]$$



$y = J_0(\alpha x),\ \ \alpha \in [6, 10]$



(b) Since $J_0(x)$ is an even function, the program can only handle even functions. Program written in `sympy`. Trial runs TBC.

(c) Let $f(x) = 1$. This is an even function and thus can be expanded in terms of $J_0(\alpha\ x)$.

$$[x^\nu\ J_\nu(x)]' = x^\nu\ J_{\nu-1}(x) \tag{11.6.70}$$

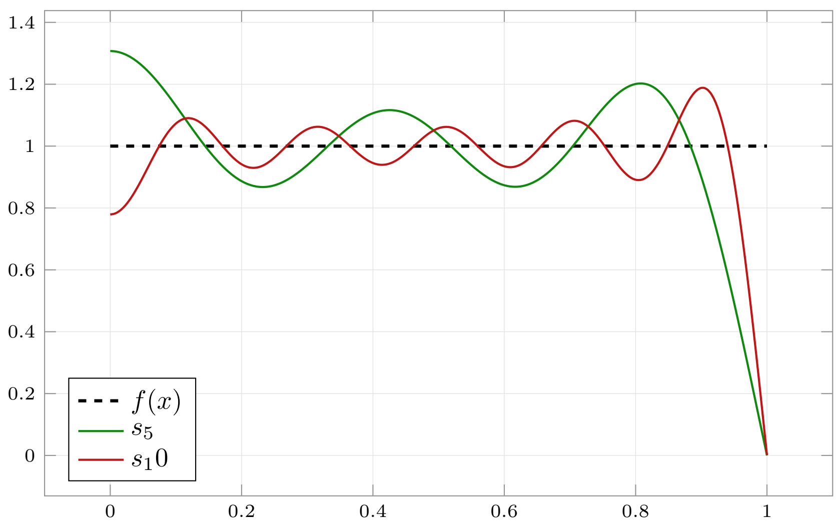
$$a_m = \frac{2}{J_1^2(\lambda)} \int_0^1 x\ J_0(\lambda x)\ \ \mathrm{d}x \tag{11.6.71}$$

$$= \left[\frac{2}{J_1^2(\lambda)} \cdot \frac{x J_1(\lambda x)}{\lambda}\right]_0^1 \tag{11.6.72}$$

$$= \frac{2}{\lambda \cdot J_1(\lambda)} \tag{11.6.73}$$

Here, λ is shorthand for $\alpha_{0,m}$, the m^{th} root of J_0

$y(x) = 1$



The convergence of the series is very slow because it is very dissimilar to a sinusoidal function.

11.7 Fourier Integral

1. Calculating the Fourier cosine integral of $f(x)$,

$$f(x) = \pi e^{-x} \qquad \forall\ x > 0 \tag{11.7.1}$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\ \cos(wu)\ \mathrm{d}u = \int_{0}^{\infty} e^{-u}\ \cos(wu)\ \mathrm{d}u \tag{11.7.2}$$

$$= \left[e^{-u}\ \frac{\sin(wu)}{w}\right]_0^{\infty} + \int_0^{\infty} e^{-u}\ \frac{\sin(wu)}{w}\ \mathrm{d}u \tag{11.7.3}$$

$$= 0 - \left[e^{-u}\ \frac{\cos(wu)}{w^2}\right]_0^{\infty} - \int_0^{\infty} e^{-u}\ \frac{\cos(wu)}{w^2}\ \mathrm{d}u = \frac{1 - A(w)}{w^2} \tag{11.7.4}$$

$$A(w) = \frac{1}{1 + w^2} \tag{11.7.5}$$

Calculating the Fourier sine integral of $f(x)$,

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\ \sin(wu)\ \mathrm{d}u = \int_{0}^{\infty} e^{-u}\ \sin(wu)\ \mathrm{d}u \tag{11.7.6}$$

$$= \left[e^{-u}\ \frac{-\cos(wu)}{w}\right]_0^{\infty} - \int_0^{\infty} e^{-u}\ \frac{\cos(wu)}{w}\ \mathrm{d}u \tag{11.7.7}$$

$$= \frac{1}{w} - \left[e^{-u}\ \frac{\sin(wu)}{w^2}\right]_0^{\infty} - \int_0^{\infty} e^{-u}\ \frac{\sin(wu)}{w^2}\ \mathrm{d}u = \frac{w - B(w)}{w^2} \tag{11.7.8}$$

$$B(w) = \frac{w}{1 + w^2} \tag{11.7.9}$$

Writing out the fourier integral of $f(x)$,

$$f(x) = \int_0^{\infty} \left[\frac{\cos(xw)}{1 + w^2}\right] + \left[\frac{w\ \sin(xw)}{1 + w^2}\right]\ \mathrm{d}w \tag{11.7.10}$$

The value of $f(x)$ at the jump discontinuity $x = 0$ is equal to the average of the left-handed limit (0) and right-handed limit π.

2. Calculating the Fourier sine integral of $f(x)$, since a cosine term is absent from the expression,

$$f(x) = \begin{cases} \frac{\pi}{2} \ \sin(x) & x \in [0, \pi] \\ 0 & x > \pi \end{cases} \tag{11.7.11}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u = \int_0^\pi \sin(u) \ \sin(wu) \ \mathrm{d}u \tag{11.7.12}$$

$$= \int_0^\pi \frac{\cos[(1-w)u] - \cos[(1+w)u]}{4} \ \mathrm{d}u \tag{11.7.13}$$

$$= \frac{1}{2}\left[\frac{\sin[(1-w)u]}{1-w} - \frac{\sin[(1+w)u]}{1+w}\right]_0^\pi = \frac{\sin(w\pi)}{(1-w^2)} \tag{11.7.14}$$

Writing out the fourier integral of $f(x)$,

$$f(x) = \int_0^\infty \left[\frac{\sin(\pi w)}{1-w^2} \ \sin(xw)\right] \ \mathrm{d}w \tag{11.7.15}$$

3. Calculating the Fourier sine integral of $f(x)$, since a cosine term is absent from the expression,

$$f(x) = \begin{cases} \frac{\pi}{2} & x \in (0, \pi) \\ 0 & x > \pi \end{cases} \tag{11.7.16}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u = \int_0^\pi \sin(wu) \ \mathrm{d}u \tag{11.7.17}$$

$$= \int_0^\pi \frac{\cos[(1-w)u] - \cos[(1+w)u]}{4} \ \mathrm{d}u \tag{11.7.18}$$

$$= \left[\frac{-\cos(wu)}{w}\right]_0^\pi = \frac{1-\cos(w\pi)}{w} \tag{11.7.19}$$

Writing out the fourier integral of $f(x)$,

$$f(x) = \int_0^\infty \left[\frac{1-\cos(\pi w)}{w} \ \sin(xw)\right] \ \mathrm{d}w \tag{11.7.20}$$

4. Calculating the Fourier cosine integral of $f(x)$, since a sine term is absent from the expression,

$$f(x) = \begin{cases} \dfrac{\pi}{2} \ \cos(x) & |x| \in (0, \pi/2) \\ \\ 0 & |x| \geq \pi/2 \end{cases} \tag{11.7.21}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \cos(wu) \ \mathrm{d}u = \int_0^{\pi/2} \cos(u) \ \cos(wu) \ \mathrm{d}u \tag{11.7.22}$$

$$= \int_0^{\pi/2} \frac{\cos[(1-w)u] + \cos[(1+w)u]}{2} \ \mathrm{d}u \tag{11.7.23}$$

$$= \frac{1}{2}\left[\frac{\sin[(1-w)u]}{(1-w)} + \frac{\sin[(1+w)u]}{(1+w)}\right]_0^{\pi/2} = \frac{\cos(w\pi/2)}{1-w^2} \tag{11.7.24}$$

Writing out the fourier integral of $f(x)$,

$$f(x) = \int_0^\infty \left[\frac{\cos(w\pi/2)}{1-w^2} \ \cos(xw)\right] \ \mathrm{d}w \tag{11.7.25}$$

5. Calculating the Fourier sine integral of $f(x)$, since a cosine term is absent from the expression,

$$f(x) = \begin{cases} \dfrac{\pi x}{2} & x \in (0, 1) \\ \\ \dfrac{\pi}{4} & x = 1 \\ \\ 0 & x > 1 \end{cases} \tag{11.7.26}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u = \int_0^1 (u) \ \sin(wu) \ \mathrm{d}u \tag{11.7.27}$$

$$= \left[\frac{\sin(wu)}{w^2} - \frac{u \ \cos(wu)}{w}\right]_0^1 = \frac{\sin(w) - w \ \cos(w)}{w^2} \tag{11.7.28}$$

Writing out the fourier integral of $f(x)$,

$$f(x) = \int_0^\infty \left[\frac{\sin(w) - w \ \cos(w)}{w^2} \ \sin(xw)\right] \ \mathrm{d}w \tag{11.7.29}$$

The value of $f(x)$ at the jump discontinuity $x = 1$ is equal to the average of the left-handed limit $(\pi/2)$ and right-handed limit (0).

6. Calculating the Fourier sine integral of $f(x)$, since a cosine term is absent from the expression, using the result from Problem 1,

$$f(x) = \frac{\pi e^{-x}}{2} \ \cos(x) \qquad \forall\ x > 0 \tag{11.7.30}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u)\ \sin(wu)\ \mathrm{d}u = \int_0^\infty (e^{-u})\ \cos(u)\ \sin(wu)\ \mathrm{d}u \tag{11.7.31}$$

$$= \frac{1}{2} \int_0^\infty e^{-u} \left[\sin[(1+w)u] - \sin[(1-w)u] \right] \ \mathrm{d}u \tag{11.7.32}$$

$$= \frac{1}{2} \left[\frac{1+w}{1+(1+w)^2} - \frac{1-w}{1+(1-w)^2} \right] = \frac{w^3}{w^4+4} \tag{11.7.33}$$

7. Calculating the Fourier cosine integral,

$$f(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & x > 1 \end{cases} \tag{11.7.34}$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(u)\ \cos(wu)\ \mathrm{d}u = \frac{2}{\pi} \int_0^1 (1)\ \cos(wu)\ \mathrm{d}u \tag{11.7.35}$$

$$= \frac{2}{\pi} \left[\frac{\sin(wu)}{w} \right]_0^1 = \frac{2}{\pi} \cdot \frac{\sin(w)}{w} \tag{11.7.36}$$

8. Calculating the Fourier cosine integral,

$$f(x) = \begin{cases} x^2 & x \in (0,1) \\ 0 & x > 1 \end{cases} \tag{11.7.37}$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(u)\ \cos(wu)\ \mathrm{d}u = \frac{2}{\pi} \int_0^1 (u^2)\ \cos(wu)\ \mathrm{d}u \tag{11.7.38}$$

$$= \frac{2}{\pi} \left[\frac{w^2u^2 - 2}{w^3}\ \sin(wu) + \frac{2u}{w^2}\ \cos(wu) \right]_0^1 \tag{11.7.39}$$

$$= \frac{2}{\pi} \left[\frac{w^2 - 2}{w^3}\ \sin(w) + \frac{2}{w^2}\ \cos(w) \right] \tag{11.7.40}$$

9. Calculating the Fourier cosine integral, using the Laplace integral ($k = 1$),

$$f(x) = \frac{1}{1+x^2} \qquad \forall\ x > 0 \tag{11.7.41}$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(u)\ \cos(wu)\ \mathrm{d}u = \frac{2}{\pi} \int_0^\infty \frac{1}{1+u^2}\ \cos(wu)\ \mathrm{d}u \tag{11.7.42}$$

$$= e^{-w} \qquad (w > 0) \tag{11.7.43}$$

10. Calculating the Fourier cosine integral,

$$f(x) = \begin{cases} a^2 - x^2 & x \in (0, a) \\ 0 & x > a \end{cases} \tag{11.7.44}$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(u)\ \cos(wu)\ \mathrm{d}u = \frac{2}{\pi} \int_0^a (a^2 - u^2)\ \cos(wu)\ \mathrm{d}u \tag{11.7.45}$$

$$= \frac{2}{\pi} \left[\frac{2 + a^2w^2 - w^2u^2}{w^3}\ \sin(wu) - \frac{2u}{w^2}\ \cos(wu) \right]_0^a \tag{11.7.46}$$

$$= \frac{2}{\pi} \left[\frac{2}{w^3}\ \sin(wa) - \frac{2a}{w^2}\ \cos(wa) \right] \tag{11.7.47}$$

11. Calculating the Fourier cosine integral,

$$f(x) = \begin{cases} \sin(x) & x \in (0, \pi) \\ 0 & x > \pi \end{cases} \tag{11.7.48}$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(u)\ \cos(wu)\ \mathrm{d}u = \frac{2}{\pi} \int_0^\pi \sin(u)\ \cos(wu)\ \mathrm{d}u \tag{11.7.49}$$

$$= \frac{-1}{\pi} \left[\frac{\cos[(1+w)u]}{1+w} + \frac{\cos[(1-w)u]}{1-w} \right]_0^\pi \tag{11.7.50}$$

$$= \frac{2}{\pi} \left[\frac{1 + \cos(\pi w)}{1 - w^2} \right] \tag{11.7.51}$$

12. Calculating the Fourier cosine integral, using the recursive nature of integration by parts,

$$f(x) = \begin{cases} e^{-x} & x \in (0,\, a) \\ 0 & x > a \end{cases} \tag{11.7.52}$$

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(u)\ \cos(wu)\ \mathrm{d}u = \frac{2}{\pi}\int_0^{a} (e^{-u})\ \cos(wu)\ \mathrm{d}u \tag{11.7.53}$$

$$= \frac{2}{\pi}\left[e^{-u}\ \frac{\sin(wu)}{w}\right]_0^a + \frac{2}{\pi}\int_0^{a} e^{-u}\ \frac{\sin(wu)}{w}\ \mathrm{d}u \tag{11.7.54}$$

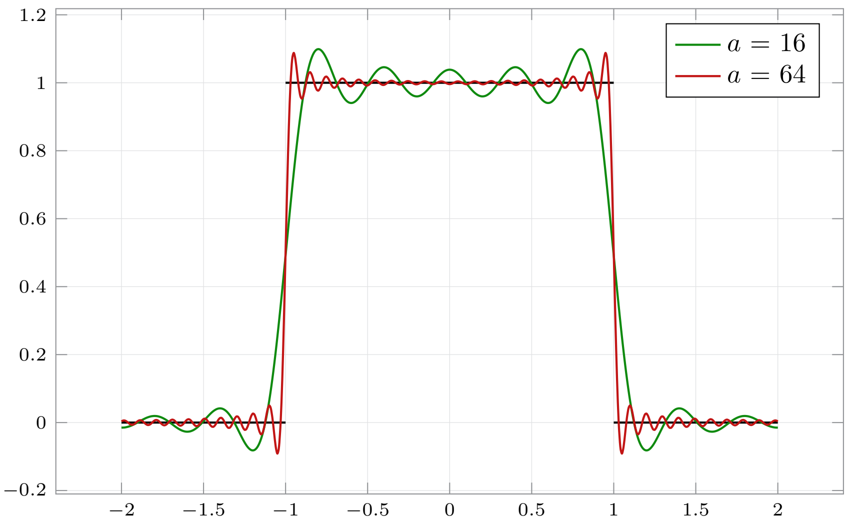
$$= \frac{2e^{-a}}{\pi w}\ \sin(wa) + \left[\frac{-2e^{-u}}{\pi w^2}\ \cos(wu)\right]_0^a - \frac{2}{\pi}\int_0^{a} e^{-u}\ \frac{\cos(wu)}{w^2}\ \mathrm{d}u \tag{11.7.55}$$

$$= \frac{2}{\pi}\left[\frac{we^{-a}\ \sin(wa) - e^{-a}\ \cos(wa) + 1}{w^2}\right] - \frac{B(w)}{w^2} \tag{11.7.56}$$

$$A(w) = \frac{2}{\pi}\left[\frac{we^{-a}\ \sin(wa) - e^{-a}\ \cos(wa) + 1}{1 + w^2}\right] \tag{11.7.57}$$

13. Graphing the integral function in Problem 7 using a CAS,

$$f(x) = \int_0^{\infty} \frac{2}{\pi} \cdot \frac{\sin(w)}{w}\ \cos(xw)\ \mathrm{d}w \tag{11.7.58}$$



Graphing the integral function in Problem 9 using a CAS,

$$f(x) = \int_0^{\infty} e^{-w}\ \cos(xw)\ \mathrm{d}w \tag{11.7.59}$$

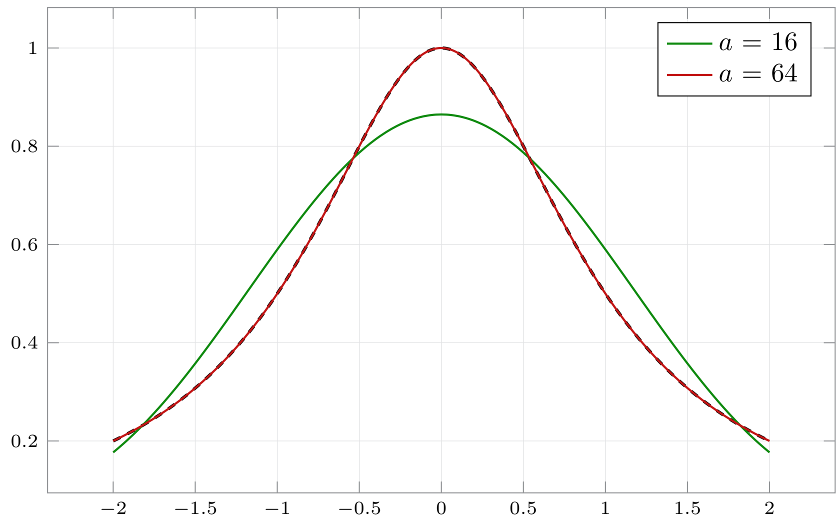


Fig 11 TBC. `Sympy` getting stuck on function definition.

14. Properties of Fourier cosine and sine integrals

(a) Using the fourier cosine integral,

$$f(ax) = \int_0^\infty A(u) \;\; \cos(u \; ax) \;\; du \qquad\qquad a > 0 \tag{11.7.60}$$

$$w = au \qquad\qquad dw = a \;\; du \tag{11.7.61}$$

$$f(ax) = \frac{1}{a} \int_0^\infty A\left(\frac{w}{a}\right) \;\; \cos(wx) \;\; dw \tag{11.7.62}$$

Using the fact that an odd function times an even function is odd,

$$A(w) = \frac{2}{\pi} \int_0^\infty f(u) \;\; \cos(wu) \;\; du \tag{11.7.63}$$

$$-\frac{dA}{dw} = \frac{2}{\pi} \int_0^\infty u \; f(u) \;\; \sin(wu) \;\; du \tag{11.7.64}$$

$$g(x) = x \cdot f(x) = \int_0^\infty \left[\frac{2}{\pi} \int_0^\infty g(u) \;\; \sin(wu) \;\; du\right] \; \sin(wx) \;\; dw \tag{11.7.65}$$

$$= \int_0^\infty \left[-\frac{dA}{dw}\right] \;\; \sin(wx) \;\; dw \tag{11.7.66}$$

Performing the differentiation twice,

$$A(w) = \frac{2}{\pi} \int_0^\infty f(u)\ \cos(wu)\ \mathrm{d}u \tag{11.7.67}$$

$$-\frac{\mathrm{d}A}{\mathrm{d}w} = \frac{2}{\pi} \int_0^\infty u\ f(u)\ \sin(wu)\ \mathrm{d}u \tag{11.7.68}$$

$$-\frac{\mathrm{d}^2A}{\mathrm{d}w^2} = \frac{2}{\pi} \int_0^\infty u^2\ f(u)\ \cos(wu)\ \mathrm{d}u \tag{11.7.69}$$

$$g(x) = x^2 \cdot f(x) = \int_0^\infty \left[\frac{2}{\pi} \int_0^\infty g(u)\ \cos(wu)\ \mathrm{d}u\right] \cos(wx)\ \mathrm{d}w \tag{11.7.70}$$

$$= \int_0^\infty \left[-\frac{\mathrm{d}^2A}{\mathrm{d}w^2}\right] \cos(wx)\ \mathrm{d}w \tag{11.7.71}$$

(b) Using the above results to solve Problem 8,

$$A(w) = \frac{2}{\pi} \cdot \frac{\sin(w)}{w} \tag{11.7.72}$$

$$-\frac{\mathrm{d}^2A}{\mathrm{d}w^2} = \frac{2}{\pi} \cdot \left[\frac{w^2-2}{w^3}\ \sin(w) + \frac{2}{w^2}\ \cos(w)\right] \tag{11.7.73}$$

which agrees with the earlier solution.

(c) Verifying the relation,

$$f(x) = \begin{cases} 1 & x \in (0,\, a) \\ 0 & x > a \end{cases} \qquad A(w) = \frac{2a}{\pi} \cdot \frac{\sin(w)}{w} \tag{11.7.74}$$

$$\frac{\mathrm{d}A}{\mathrm{d}w} = \frac{2a}{\pi} \cdot \left[\frac{\cos(w)}{w} - \frac{\sin(w)}{w^2}\right] \tag{11.7.75}$$

$$g(x) = \begin{cases} x & x \in (0,\, a) \\ 0 & x > a \end{cases} \qquad B(w) = \frac{2}{\pi} \int_0^1 u\ \sin(wu)\ \mathrm{d}u \tag{11.7.76}$$

$$= \frac{2}{\pi}\left[\frac{\sin(wu)}{w^2} - \frac{u\cos(wu)}{w}\right]_0^1 \qquad B(w) = -\frac{\mathrm{d}A}{\mathrm{d}w} \tag{11.7.77}$$

(d) Finding similar formulas for Fourier sine integrals,

$$f(ax) = \int_0^\infty B(u) \ \sin(u \ ax) \ \mathrm{d}u \qquad a > 0 \tag{11.7.78}$$

$$w = au \qquad \mathrm{d}w = a \ \mathrm{d}u \tag{11.7.79}$$

$$f(ax) = \frac{1}{a} \int_0^\infty B\left(\frac{w}{a}\right) \ \sin(wx) \ \mathrm{d}w \tag{11.7.80}$$

Using the fact that an odd function times an odd function is even,

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u \tag{11.7.81}$$

$$\frac{\mathrm{d}B}{\mathrm{d}w} = \frac{2}{\pi} \int_0^\infty u \ f(u) \ \cos(wu) \ \mathrm{d}u \tag{11.7.82}$$

$$g(x) = x \cdot f(x) = \int_0^\infty \left[\frac{2}{\pi} \int_0^\infty g(u) \ \cos(wu) \ \mathrm{d}u\right] \cos(wx) \ \mathrm{d}w \tag{11.7.83}$$

$$= \int_0^\infty \left[\frac{\mathrm{d}B}{\mathrm{d}w}\right] \cos(wx) \ \mathrm{d}w \tag{11.7.84}$$

Performing the differentiation twice,

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u \tag{11.7.85}$$

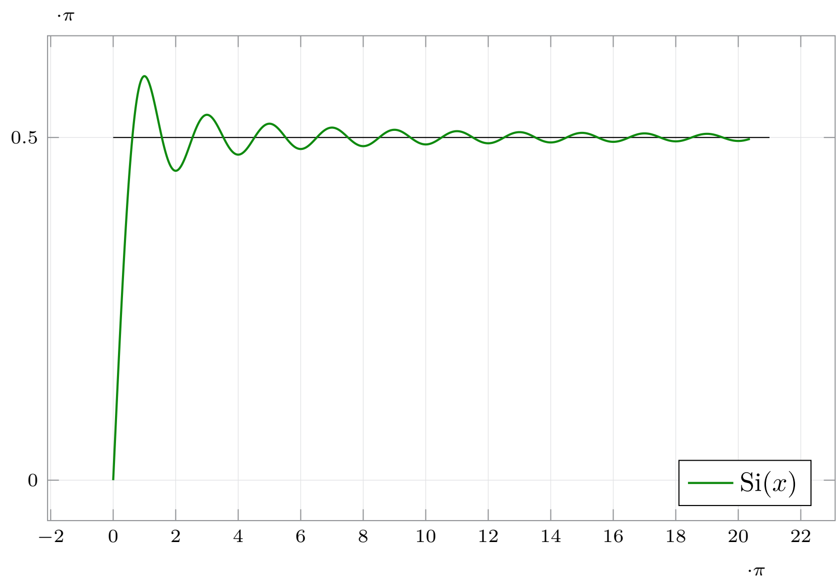
$$\frac{\mathrm{d}B}{\mathrm{d}w} = \frac{2}{\pi} \int_0^\infty u \ f(u) \ \cos(wu) \ \mathrm{d}u \tag{11.7.86}$$

$$-\frac{\mathrm{d}^2B}{\mathrm{d}w^2} = \frac{2}{\pi} \int_0^\infty u^2 \ f(u) \ \sin(wu) \ \mathrm{d}u \tag{11.7.87}$$

$$g(x) = x^2 \cdot f(x) = \int_0^\infty \left[\frac{2}{\pi} \int_0^\infty g(u) \ \sin(wu) \ \mathrm{d}u\right] \sin(wx) \ \mathrm{d}w \tag{11.7.88}$$

$$= \int_0^\infty \left[-\frac{\mathrm{d}^2B}{\mathrm{d}w^2}\right] \cos(wx) \ \mathrm{d}w \tag{11.7.89}$$

15. Plotting the sine integral and seeing the convergence of the extrema to $y = \pi/2$,



The Gibbs phenomenon at $x = 0$ moves closer and closer to the $y - axis$ as the approximation improves.

16. Calculating the Fourier sine integral

$$f(x) = \begin{cases} x & x \in (0, a) \\ 0 & x > a \end{cases} \tag{11.7.90}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u = \frac{2}{\pi} \int_0^a (u) \ \sin(wu) \ \mathrm{d}u \tag{11.7.91}$$

$$= \frac{2}{\pi} \left[\frac{\sin(wu) - wu \ \cos(wu)}{w^2} \right]_0^a = \frac{2}{\pi} \left[\frac{\sin(wa) - wa \ \cos(wa)}{w^2} \right] \tag{11.7.92}$$

17. Calculating the Fourier sine integral

$$f(x) = \begin{cases} 1 & x \in (0, 1) \\ 0 & x > 1 \end{cases} \tag{11.7.93}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u = \frac{2}{\pi} \int_0^1 (1) \ \sin(wu) \ \mathrm{d}u \tag{11.7.94}$$

$$= \frac{2}{\pi} \left[\frac{- \cos(wu)}{w} \right]_0^1 = \frac{2}{\pi} \left[\frac{1 - \cos(w)}{w} \right] \tag{11.7.95}$$

18. Calculating the Fourier sine integral,

$$f(x) = \begin{cases} \cos(x) & x \in (0, \pi) \\ 0 & x > \pi \end{cases} \tag{11.7.96}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u = \frac{2}{\pi} \int_0^\pi \cos(u) \ \sin(wu) \ \mathrm{d}u \tag{11.7.97}$$

$$= \frac{1}{\pi} \left[-\frac{\cos[(1+w)u]}{1+w} + \frac{\cos[(1-w)u]}{1-w} \right]_0^\pi \tag{11.7.98}$$

$$= \frac{-\cos(wu) - 1}{1-w} + \frac{1 + \cos(wu)}{1+w} = \frac{2}{\pi} \left[\frac{w}{w^2 - 1} \ [1 + \cos(\pi w)] \right] \tag{11.7.99}$$

19. Calculating the Fourier sine integral, using the recursive nature of integration by parts,

$$f(x) = \begin{cases} e^x & x \in (0, 1) \\ 0 & x > 1 \end{cases} \tag{11.7.100}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(u) \ \sin(wu) \ \mathrm{d}u = \frac{2}{\pi} \int_0^1 (e^u) \ \sin(wu) \ \mathrm{d}u \tag{11.7.101}$$

$$= \frac{-2}{\pi} \left[e^u \ \frac{\cos(wu)}{w} \right]_0^1 + \frac{2}{\pi} \int_0^1 e^u \ \frac{\cos(wu)}{w} \ \mathrm{d}u \tag{11.7.102}$$

$$= \frac{2}{\pi w} \ [1 - e \ \cos(w)] + \left[\frac{2e^u}{\pi w^2} \ \sin(wu) \right]_0^1 - \frac{2}{\pi} \int_0^1 e^u \ \frac{\sin(wu)}{w^2} \ \mathrm{d}u \tag{11.7.103}$$

$$= \frac{2}{\pi} \left[\frac{w - we \ \cos(w) + e \ \sin(w)}{w^2} \right] - \frac{B(w)}{w^2} \tag{11.7.104}$$

$$B(w) = \frac{2}{\pi} \left[\frac{w - we \ \cos(w) + e \ \sin(w)}{1 + w^2} \right] \tag{11.7.105}$$

20. Calculating the Fourier sine integral, using the recursive nature of integration by parts,

$$f(x) = \begin{cases} e^{-x} & x \in (0,\,1) \\ 0 & x > 1 \end{cases} \tag{11.7.106}$$

$$B(w) = \frac{2}{\pi}\int_0^\infty f(u)\ \sin(wu)\ \mathrm{d}u = \frac{2}{\pi}\int_0^1 (e^{-u})\ \sin(wu)\ \mathrm{d}u \tag{11.7.107}$$

$$= \frac{-2}{\pi}\left[e^{-u}\ \frac{\cos(wu)}{w}\right]_0^1 - \frac{2}{\pi}\int_0^1 e^{-u}\ \frac{\cos(wu)}{w}\ \mathrm{d}u \tag{11.7.108}$$

$$= \frac{2}{\pi w}\ [1 - e^{-1}\ \cos(w)] - \left[\frac{2e^{-u}}{\pi w^2}\ \sin(wu)\right]_0^1 - \frac{2}{\pi}\int_0^1 e^{-u}\ \frac{\sin(wu)}{w^2}\ \mathrm{d}u \tag{11.7.109}$$

$$= \frac{2}{\pi}\left[\frac{w - we^{-1}\ \cos(w) - e^{-1}\ \sin(w)}{w^2}\right] - \frac{B(w)}{w^2} \tag{11.7.110}$$

$$B(w) = \frac{2}{\pi}\left[\frac{w - we^{-1}\ \cos(w) - e^{-1}\ \sin(w)}{1 + w^2}\right] \tag{11.7.111}$$

11.8 Fourier Cosine and Sine Transforms

1. Finding the Fourier cosine transform,

$$f(x) = \begin{cases} 1 & x \in (0,\,1) \\ -1 & x \in (1,\,2) \\ 0 & x > 2 \end{cases} \tag{11.8.1}$$

$$\widehat{f_c}(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ \cos(wx)\ \mathrm{d}x \tag{11.8.2}$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 (1)\ \cos(wx)\ \mathrm{d}x + \int_1^2 (-1)\ \cos(wx)\ \mathrm{d}x\right] \tag{11.8.3}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin(wx)}{w}\right]_0^1 - \sqrt{\frac{2}{\pi}}\left[\frac{\sin(wx)}{w}\right]_1^2 = \sqrt{\frac{2}{\pi}}\left[\frac{2\ \sin(w) - \sin(2w)}{w}\right] \tag{11.8.4}$$

2. Finding the Fourier cosine transform,

$$\widehat{f}_c(w) = \sqrt{\frac{2}{\pi}} \left[\frac{2\,\sin(w) - \sin(2w)}{w} \right] \tag{11.8.5}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f}_c(w)\ \cos(wx)\ \mathrm{d}w \tag{11.8.6}$$

$$= \frac{2}{\pi} \int_0^\infty \left[\frac{2\,\sin(w) - \sin(2w)}{w} \right] \cos(wx)\ \mathrm{d}w \tag{11.8.7}$$

$$I_1 = \frac{2}{\pi} \int_0^\infty \left[\frac{\sin[(1+x)w]}{w} + \frac{\sin[(1-x)w]}{w} \right] \mathrm{d}w \tag{11.8.8}$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2}\ \mathrm{sgn}(1+x) + \frac{\pi}{2}\ \mathrm{sgn}(1-x) \right] = \mathrm{sgn}(1+x) + \mathrm{sgn}(1-x) \tag{11.8.9}$$

$$I_2 = \frac{-1}{\pi} \int_0^\infty \left[\frac{\sin[(2+x)w]}{w} + \frac{\sin[(2-x)w]}{w} \right] \mathrm{d}w \tag{11.8.10}$$

$$= \frac{-1}{\pi} \left[\frac{\pi}{2}\ \mathrm{sgn}(2+x) + \frac{\pi}{2}\ \mathrm{sgn}(2-x) \right] = \frac{-1}{2} \left[\mathrm{sgn}(2+x) + \mathrm{sgn}(2-x) \right] \tag{11.8.11}$$

$$f(x) = \begin{cases} 1 & x \in (0,1) \\ -1 & x \in (1,2) \\ 0 & x > 2 \end{cases} \tag{11.8.12}$$

3. Finding the Fourier cosine transform,

$$f(x) = \begin{cases} x & x \in (0,2) \\ 0 & x > 2 \end{cases} \tag{11.8.13}$$

$$\widehat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \cos(wx)\ \mathrm{d}x = \sqrt{\frac{2}{\pi}} \left[\int_0^2 (x)\ \cos(wx)\ \mathrm{d}x \right] \tag{11.8.14}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{x\ \sin(wx)}{w} + \frac{\cos(wx)}{w^2} \right]_0^2 = \sqrt{\frac{2}{\pi}} \left[\frac{\cos(2w) - 1 + 2w\,\sin(2w)}{w^2} \right] \tag{11.8.15}$$

4. Finding the Fourier cosine transform,

$$f(x) = e^{-ax} \qquad (a > 0) \tag{11.8.16}$$

$$f''(x) = a^2 \; f(x) \tag{11.8.17}$$

$$\mathscr{F}_c\{f''\} = -w^2 \;\; \mathscr{F}_c\{f\} - \sqrt{\frac{2}{\pi}} \; f'(0) \tag{11.8.18}$$

$$a^2 \;\; \mathscr{F}_c\{f\} = -w^2 \;\; \mathscr{F}_c\{f\} - \sqrt{\frac{2}{\pi}}(-a) \tag{11.8.19}$$

$$\mathscr{F}_c\{f\} = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + w^2}\right] \tag{11.8.20}$$

5. Finding the Fourier cosine transform,

$$f(x) = \begin{cases} x^2 & x \in (0, 1) \\ 0 & x > 1 \end{cases} \tag{11.8.21}$$

$$\widehat{f_c}(w) = \sqrt{\frac{2}{\pi}} \;\; \int_0^\infty \; f(x) \;\; \cos(wx) \;\; \mathrm{d}x = \sqrt{\frac{2}{\pi}} \left[\int_0^1 (x^2) \;\; \cos(wx) \;\; \mathrm{d}x \right] \tag{11.8.22}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{x^2 \sin(wx)}{w} - \frac{2 \sin(wx)}{w^3} + \frac{2x \cos(wx)}{w^2}\right]_0^1 \tag{11.8.23}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{w^2 - 2}{w^3} \;\; \sin(w) + \frac{2}{w^2} \;\; \cos(w)\right] \tag{11.8.24}$$

6. Finding the Fourier cosine transform directly,

$$g(x) = \begin{cases} 2 & x \in (0, 1) \\ 0 & x > 1 \end{cases} \tag{11.8.25}$$

$$\widehat{f_c}(w) = \sqrt{\frac{2}{\pi}} \;\; \int_0^\infty \; f(x) \;\; \cos(wx) \;\; \mathrm{d}x = \sqrt{\frac{2}{\pi}} \left[\int_0^1 (2) \;\; \cos(wx) \;\; \mathrm{d}x \right] \tag{11.8.26}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin(wx)}{w}\right]_0^1 = \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin(w)}{w}\right] \tag{11.8.27}$$

Trying to find it using the double derivative relation,

$$f''(x) = g(x) \tag{11.8.28}$$

$$\mathscr{F}_c\{f''\} = -w^2 \ \mathscr{F}_c\{f\} - \sqrt{\frac{2}{\pi}} \ f'(0) \tag{11.8.29}$$

$$\mathscr{F}_c\{g\} = \sqrt{\frac{2}{\pi}}\left[\left(-w + \frac{2}{w}\right) \ \sin(w) - 2\cos(w)\right] \tag{11.8.30}$$

The results do not match since $f(x)$ is not continuous on the real line. This makes the second method invalid.

7. Outside of the limit $x \to 0^+$, both functions are continuous and eligible. Looking at this limit,

$$\lim_{x\to 0^+} \frac{\sin(x)}{x} = \lim_{x\to 0^+} \cos(x) = 1 \tag{11.8.31}$$

The limit is an indeterminate form that does exist using L'Hospital rule. Yes

$$\lim_{x\to 0^+} \frac{\cos(x)}{x} = \text{LDNE} \tag{11.8.32}$$

The limit does not exist. No

8. A function is absolutely integrable if the following integral exists and is finite.

$$\int_{-\infty}^{\infty} |f(x)| \ \ \mathrm{d}x = \int_{-\infty}^{\infty} |k| \ \ \mathrm{d}x = \infty \tag{11.8.33}$$

So, this function does not have Fourier cosine and sine transforms.

9. Finding the Fourier sine transform directly, using the standard result,

$$f(x) = e^{-ax} \qquad (a > 0) \tag{11.8.34}$$

$$\widehat{f_s}(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \ \sin(wx) \ \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (e^{-ax}) \ \sin(wx) \ \mathrm{d}x \tag{11.8.35}$$

$$= \sqrt{\frac{2}{\pi}}\left[- e^{-ax} \quad \frac{a\sin(wx) + w\cos(wx)}{a^2 + w^2}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\left[\frac{w}{a^2 + w^2}\right] \tag{11.8.36}$$

10. Finding it using the double derivative relation,

$$f''(x) = a^2\ f(x) \tag{11.8.37}$$

$$\mathscr{F}_s\{f''\} = -w^2\ \mathscr{F}_s\{f\} + \sqrt{\frac{2}{\pi}}\ wf(0) \tag{11.8.38}$$

$$\mathscr{F}_s\{g\} = \sqrt{\frac{2}{\pi}}\left[\frac{w}{a^2 + w^2}\right] \tag{11.8.39}$$

The results do not match since $f(x)$ is not continuous on the real line. This makes the second method invalid.

11. Finding the Fourier sine transform,

$$f(x) = \begin{cases} x^2 & x \in (0, 1) \\ 0 & x > 1 \end{cases} \tag{11.8.40}$$

$$\widehat{f_s}(w) = \sqrt{\frac{2}{\pi}}\ \int_0^\infty f(x)\ \sin(wx)\ \mathrm{d}x = \sqrt{\frac{2}{\pi}}\left[\int_0^1 (x^2)\ \sin(wx)\ \mathrm{d}x\right] \tag{11.8.41}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2x\sin(wx)}{w^2} + \frac{2\cos(wx)}{w^3} - \frac{x^2\cos(wx)}{w}\right]_0^1 \tag{11.8.42}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2}{w^2}\ \sin(w) + \frac{(2 - w^2)}{w^3}\ \cos(w) - \frac{2}{w^3}\right] \tag{11.8.43}$$

12. Finding the Fourier sine transform using the derivative relation,

$$g(x) = e^{-x^2/2} \qquad f(x) = xe^{-x^2/2} = -g'(x) \tag{11.8.44}$$

$$\mathscr{F}_s\{g'(x)\} = -w\ \ \mathscr{F}_c\{g(x)\} \qquad \mathscr{F}_s\{f\} = w\ \ \mathscr{F}_c\{e^{-x^2/2}\} \tag{11.8.45}$$

$$\mathscr{F}_s\{f\} = we^{-w^2/2} \tag{11.8.46}$$

13. Finding the Fourier sine transform using the derivative relation,

$$f(x) = e^{-x} \qquad f'(x) = -f(x) \tag{11.8.47}$$

$$\mathscr{F}_c\{f'(x)\} = w\ \ \mathscr{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\ f(0) \qquad -\mathscr{F}_c\{e^{-x}\} = w\ \ \mathscr{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} \tag{11.8.48}$$

$$\mathscr{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi w}}\left[\frac{-1}{1 + w^2} + 1\right] = \sqrt{\frac{2}{\pi}}\left[\frac{w}{1 + w^2}\right] \tag{11.8.49}$$

14. Using the formulas in the Table,

$$\mathscr{F}_s\left\{\frac{1}{\sqrt{x}}\right\} = \frac{1}{\sqrt{w}} \qquad \mathscr{F}_s\{x^{a-1}\} = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(a)}{w^a}\,\sin\left(\frac{a\pi}{2}\right) \tag{11.8.50}$$

$$a = 1/2 \qquad \frac{1}{\sqrt{w}} = \frac{\Gamma(1/2)}{\sqrt{\pi w}} \tag{11.8.51}$$

$$\Gamma(1/2) = \sqrt{\pi} \tag{11.8.52}$$

15. TBC. Refer notes.

- Direct computation
- Derivative relation
- Second derivative relation

11.9 Fourier Transform, Discrete and Fast Fourier Transforms

1. Proving the relations using the definition of i,

$$i \cdot i = -1 \qquad \frac{1}{i} = \frac{i}{-1} = -i \tag{11.9.1}$$

$$e^{-ix} = \cos(-x) + i\ \sin(-x) \qquad = \cos(x) - i\ \sin(x) \tag{11.9.2}$$

$$e^{ix} + e^{-ix} = \cos(x) + i\ \sin(x) + \cos(x) - i\ \sin(x) \qquad = 2\cos(x) \tag{11.9.3}$$

$$e^{ix} - e^{-ix} = \cos(x) + i\ \sin(x) - \cos(x) + i\ \sin(x) \qquad = 2i\ \sin(x) \tag{11.9.4}$$

Using the Taylor series expansions of $\cos(kx)$ and $\sin(kx)$,

$$\sin(kx) = kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \dots \tag{11.9.5}$$

$$\cos(kx) = 1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \dots \tag{11.9.6}$$

$$\cos(kx) + i\ \sin(kx) = 1 + (ikx) + \frac{(ikx)^2}{2!} + \frac{(ikx)^3}{3!} + \frac{(ikx)^4}{4!} + \dots \tag{11.9.7}$$

$$= \exp(ikx) \tag{11.9.8}$$

2. Finding the Fourier transform by integration,

$$f(x) = \begin{cases} e^{2ix} & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.9}$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx} \;\, \mathrm{d}x \tag{11.9.10}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ix(2-w)} \;\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ix(2-w)}}{i(2-w)}\right]_{-1}^{1} \tag{11.9.11}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2\sin(2-w)}{(2-w)} \tag{11.9.12}$$

3. Finding the Fourier transform by integration, assuming $a < b$

$$f(x) = \begin{cases} 1 & x \in (a, b) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.13}$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx} \;\, \mathrm{d}x \tag{11.9.14}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{-iwx} \;\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{(-iw)}\right]_{a}^{b} \tag{11.9.15}$$

$$= \frac{i}{\sqrt{2\pi}} \frac{e^{-iwb} - e^{-iwa}}{w} \tag{11.9.16}$$

4. Finding the Fourier transform by integration, assuming $k > 0$

$$f(x) = \begin{cases} e^{kx} & x < 0 \\ 0 & x > 0 \end{cases} \tag{11.9.17}$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx} \;\, \mathrm{d}x \tag{11.9.18}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(k-iw)x} \;\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(k-iw)x}}{(k-iw)}\right]_{-\infty}^{0} \tag{11.9.19}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{k-iw} = \frac{1}{\sqrt{2\pi}} \frac{k+iw}{k^2+w^2} \tag{11.9.20}$$

5. Finding the Fourier transform by integration, assuming $a > 0$

$$f(x) = \begin{cases} e^{x} & x \in (-a,\, a) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.21}$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\; e^{-iwx} \;\; \mathrm{d}x \tag{11.9.22}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{(1-iw)x} \;\; \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1-iw)x}}{(1-iw)} \right]_{-a}^{a} \tag{11.9.23}$$

$$= \frac{1}{\sqrt{2\pi}} \; \frac{e^{(1-iw)a} - e^{-(1-iw)a}}{1 - iw} \tag{11.9.24}$$

6. Finding the Fourier transform by integration, assuming $a > 0$

$$f(x) = \begin{cases} e^{x} & x < 0 \\ e^{-x} & x > 0 \end{cases} \tag{11.9.25}$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\; e^{-iwx} \;\; \mathrm{d}x \tag{11.9.26}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(1-iw)x} \;\; \mathrm{d}x + \int_{0}^{\infty} e^{(-1-iw)x} \;\; \mathrm{d}x \tag{11.9.27}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1-iw)x}}{(1-iw)} \right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(-1-iw)x}}{(-1-iw)} \right]_{0}^{\infty} \tag{11.9.28}$$

$$= \frac{1}{\sqrt{2\pi}} \; \frac{2}{1 + w^2} \tag{11.9.29}$$

7. Finding the Fourier transform by integration, assuming $a > 0$

$$f(x) = \begin{cases} x & x \in (0, a) \\ 0 & \text{otherwise} \end{cases} \qquad 11.9.30$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx} \;\mathrm{d}x = \int_0^a x\, e^{-iwx} \;\mathrm{d}x \qquad 11.9.31$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{xe^{-iwx}}{-iw}\right]_0^a - \frac{1}{\sqrt{2\pi}} \int_0^a \frac{e^{-iwx}}{-iw} \;\mathrm{d}x \qquad 11.9.32$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{ae^{-iwa}}{-iw}\right] + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{w^2}\right]_0^a \qquad 11.9.33$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{(iwa+1)e^{-iwa} - 1}{w^2}\right] \qquad 11.9.34$$

8. Finding the Fourier transform by integration,

$$f(x) = \begin{cases} xe^{-x} & x \in (-1, 0) \\ 0 & \text{otherwise} \end{cases} \qquad 11.9.35$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx} \;\mathrm{d}x = \int_{-1}^{0} x\, e^{-(1+iw)x} \;\mathrm{d}x \qquad 11.9.36$$

$$= \frac{1}{\sqrt{2\pi}} \left[- \frac{1 + (1+iw)x}{(1+iw)^2}\, e^{-(1+iw)x}\right]_{-1}^{0} \qquad 11.9.37$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-iw\, e^{(1+iw)} - 1}{(1+iw)^2}\right] \qquad 11.9.38$$

9. Finding the Fourier transform by integration,

$$f(x) = \begin{cases} |x| & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.39}$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{-iwx}\ \mathrm{d}x \tag{11.9.40}$$

$$= \int_{-1}^{0} (-x)\ e^{-iwx}\ \mathrm{d}x + \int_{0}^{1} (x)\ e^{-iwx}\ \mathrm{d}x \tag{11.9.41}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1+iwx}{-w^2}\ e^{-iwx}\right]_{-1}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{1+iwx}{-w^2}\ e^{-iwx}\right]_{1}^{0} \tag{11.9.42}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{\cos(w) + w\ \sin(w) - 1}{w^2}\right] \tag{11.9.43}$$

10. Finding the Fourier transform by integration,

$$f(x) = \begin{cases} x & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.44}$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{-iwx}\ \mathrm{d}x \tag{11.9.45}$$

$$= \sqrt{\frac{1}{2\pi}} \int_{-1}^{1} (x)\ e^{-iwx}\ \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[\frac{1+iwx}{w^2}\ e^{-iwx}\right]_{-1}^{1} \tag{11.9.46}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{(1+iw)e^{-iw} - (1-iw)e^{iw}}{w^2}\right] \tag{11.9.47}$$

$$= \frac{2i}{\sqrt{2\pi}} \left[\frac{w\ \ \cos(w) + \sin(w)}{w^2}\right] \tag{11.9.48}$$

11. Finding the Fourier transform by integration,

$$f(x) = \begin{cases} -1 & x \in (-1, 0) \\ 1 & x \in (0, 1) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.49}$$

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{-iwx}\ \ \mathrm{d}x \tag{11.9.50}$$

$$= \sqrt{\frac{1}{2\pi}} \int_{-1}^{0} (-1)\ e^{-iwx}\ \ \mathrm{d}x + \frac{1}{\sqrt{2\pi}} \int_{0}^{1} (1)\ e^{-iwx}\ \ \mathrm{d}x \tag{11.9.51}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{iw}\right]_{-1}^{0} - \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{iw}\right]_{0}^{1} \tag{11.9.52}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - e^{iw} - e^{-iw} + 1}{iw}\right] = \frac{2}{\sqrt{2\pi}} \left[\frac{1 - \cos(w)}{iw}\right] \tag{11.9.53}$$

12. Using the table,

$$f(x) = \begin{cases} xe^{-x} & x > 0 \\ 0 & x < 0 \end{cases} \qquad g(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases} \tag{11.9.54}$$

$$\mathscr{F}\{g\} = \frac{1}{\sqrt{2\pi}} \frac{1}{1 + iw} \qquad f'(x) = (1 - x)e^{-x} = g(x) - f(x) \tag{11.9.55}$$

$$\mathscr{F}\{f'\} = iw\ \ \mathscr{F}\{f\} = \mathscr{F}\{g - f\} \qquad \mathscr{F}\{f\} = \frac{1}{\sqrt{2\pi}} \frac{1}{(1 + iw)^2} \tag{11.9.56}$$

13. Using the table,

$$f(x) = e^{-x^2/2} \qquad \mathscr{F}\{e^{-ax^2}\} = \frac{1}{\sqrt{2a}}\ e^{-w^2/4a} \qquad (a > 0) \tag{11.9.57}$$

$$\mathscr{F}\{f(x)\} = e^{-w^2/2} \tag{11.9.58}$$

14. Obtaining formula 7 from formula 8,

$$f(x) = \begin{cases} e^{iax} & x \in (b,\, c) \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} e^{iax} & x \in (-b,\, b) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.59}$$

$$\mathscr{F}\{f\} = \frac{i}{\sqrt{2\pi}} \frac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w} \qquad b \to -c \tag{11.9.60}$$

$$\mathscr{F}\{g\} = \frac{i}{\sqrt{2\pi}} \frac{e^{-ic(a-w)} - e^{ic(a-w)}}{a - w} \qquad \mathscr{F}\{g\} = \frac{2}{\sqrt{2\pi}} \frac{\sin[c(w-a)]}{w - a} \tag{11.9.61}$$

Which mathces the formula in the table with $b \leftrightarrow c$

15. Obtaining formula 1 from formula 2,

$$f(x) = \begin{cases} 1 & x \in (b,\, c) \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} 1 & x \in (-b,\, b) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.62}$$

$$\mathscr{F}\{f\} = \frac{1}{\sqrt{2\pi}} \frac{e^{-ibw} - e^{-icw}}{iw} \qquad b \to -c \tag{11.9.63}$$

$$\mathscr{F}\{g\} = \frac{1}{\sqrt{2\pi}} \frac{e^{icw} - e^{-icw}}{iw} \qquad \mathscr{F}\{g\} = \frac{2}{\sqrt{2\pi}} \frac{\sin(cw)}{w} \tag{11.9.64}$$

Which matches the formula in the table with $b \leftrightarrow c$

16. Shifting,

(a) Shifting in x,

$$\mathscr{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)\; e^{-iwx} \;\, dx \tag{11.9.65}$$

$$y = (x - a) \qquad dy = dx \tag{11.9.66}$$

$$\mathscr{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(y)\; e^{-iwy}\; e^{-iwa} \;\, dy \tag{11.9.67}$$

$$= e^{-iwa} \;\; \mathscr{F}\{f(x)\} \tag{11.9.68}$$

(b) Obtaining formula 1 from formula 2,

$$\frac{b+c}{2} = \alpha \qquad \frac{c-b}{2} = \beta \tag{11.9.69}$$

$$f(x) = \begin{cases} 1 & x \in (\alpha - \beta,\, \alpha + \beta) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.70}$$

Using the fourier transform from the table and x shifting,

$$\mathscr{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \frac{e^{-iw(\alpha-\beta)} - e^{-iw(\alpha+\beta)}}{iw} \tag{11.9.71}$$

$$= \frac{e^{-iw\alpha}}{\sqrt{2\pi}} \frac{e^{iw\beta} - e^{-iw\beta}}{iw} = e^{-iw\alpha}\left[\sqrt{\frac{2}{\pi}} \frac{\sin(\beta w)}{w}\right] \tag{11.9.72}$$

$$g(x) = \begin{cases} 1 & x \in (-\beta, \beta) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.73}$$

$$f(x) = g(x - \alpha) \tag{11.9.74}$$

This proves the relation.

(c) Shifting in w,

$$\mathscr{F}^{-1}\{\widehat{f}(w-a)\} = \int_{-\infty}^{\infty} \widehat{f}(w-a)\; e^{iwx} \;\; \mathrm{d}w \tag{11.9.75}$$

$$y = (w - a) \qquad\qquad \mathrm{d}y = \mathrm{d}w \tag{11.9.76}$$

$$\mathscr{F}\{f(w-a)\} = \int_{-\infty}^{\infty} \widehat{f}(y)\; e^{ixy}\; e^{ixa} \;\; \mathrm{d}y \tag{11.9.77}$$

$$= e^{iax} \;\; \mathscr{F}^{-1}\{\widehat{f}(w)\} = e^{iax} \cdot f(x) \tag{11.9.78}$$

(d) Obtaining formula 7 from formula 1,

$$f(x) = \begin{cases} 1 & x \in (-b, b) \\ 0 & \text{otherwise} \end{cases} \qquad\qquad g(x) = \begin{cases} e^{iax} & x \in (-b, b) \\ 0 & \text{otherwise} \end{cases} \tag{11.9.79}$$

$$\mathscr{F}\{f\} = \sqrt{\frac{2}{\pi}} \frac{\sin(bw)}{w} \qquad\qquad \mathscr{F}\{g\} = \mathscr{F}\{e^{iax} \cdot f\} \tag{11.9.80}$$

$$\mathscr{F}\{g\} = \sqrt{\frac{2}{\pi}} \frac{\sin(bw - ba)}{w - a} \tag{11.9.81}$$

Formula 8 is similarly derived from formula 2 by simple substitution.

17. The derivative relation cannot be used becuase its requirements are not satisfied by Problem 9

18. Here, $n = 4$

$$w = \exp\left(\frac{-2\pi\ i}{N}\right) = -i \tag{11.9.82}$$

$$\mathbf{F}_4 = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{11.9.83}$$

$$\mathbf{f} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad \widehat{\mathbf{f}} = \mathbf{F}_4\ \mathbf{f} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix} \tag{11.9.84}$$

19. Here, $n = 4$, and the general signal has 4 samples.

$$w = \exp\left(\frac{-2\pi\ i}{N}\right) = -i \tag{11.9.85}$$

$$\mathbf{F}_4 = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{11.9.86}$$

$$\mathbf{f} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad \widehat{\mathbf{f}} = \mathbf{F}_4\ \mathbf{f} = \begin{bmatrix} (a+b+c+d) \\ (a-c)-i\ (b-d) \\ (a-b+c-d) \\ (a-c)+i\ (b-d) \end{bmatrix} \tag{11.9.87}$$

20. Finding the inverse matrix of $\mathbf{F}_4$ in Example 4,

$$\mathbf{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \qquad \mathbf{F}_4^{-1} = \frac{1}{4}\ \mathbf{F}_4^{\dagger} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \tag{11.9.88}$$

$$\mathbf{f} = \mathbf{F}_4^{-1}\ \widehat{\mathbf{f}} = \mathbf{F}_4^{-1} \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix} \qquad \mathbf{f} = \frac{1}{4} \begin{bmatrix} 0 \\ 4 \\ 16 \\ 39 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \tag{11.9.89}$$

21. Here, $n = 2$, and the general signal has 4 samples.

$$w = \exp\left(\frac{-2\pi\ i}{N}\right) = -1 \qquad \mathbf{F}_2 = \begin{bmatrix} w^0 & w^0 \\ w^0 & w^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{11.9.90}$$

$$\mathbf{f} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \widehat{\mathbf{f}} = \mathbf{F}_2\ \mathbf{f} = \begin{bmatrix} a+b \\ a-b \end{bmatrix} \tag{11.9.91}$$

22. Finding the inverse matrix of $\mathbf{F}_2$,

$$\mathbf{F}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{F}_2^{-1} = \frac{1}{2}\ \mathbf{F}_2^{\dagger} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{11.9.92}$$

$$\mathbf{f} = \mathbf{F}_2^{-1}\ \widehat{\mathbf{f}} = \mathbf{F}_2^{-1}\begin{bmatrix} a+b \\ a-b \end{bmatrix} \qquad \mathbf{f} = \frac{1}{2}\begin{bmatrix} 2a \\ 2b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{11.9.93}$$

23. For $N = 8$,

$$z = \exp\left(\frac{-2\pi\ i}{8}\right) \qquad = \cos(\pi/4) - i\ \sin(\pi/4) = \frac{1-i}{\sqrt{2}} \tag{11.9.94}$$

$$z^2 = \frac{(1-i)^2}{2} = \frac{1+(-1)-2i}{2} = -i \qquad z^2 = w_4 \implies z = w_8 \tag{11.9.95}$$

24. For $w = 8$, the DFT matrix is,

$$\mathbf{F}_8 = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\ w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^{18} & w^{21} \\ w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} & w^{24} & w^{28} \\ w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} & w^{30} & w^{35} \\ w^0 & w^6 & w^{12} & w^{18} & w^{24} & w^{30} & w^{36} & w^{42} \\ w^0 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} & w^{49} \end{bmatrix} \tag{11.9.96}$$

$$z^0 = 1 \qquad z = \frac{1-i}{\sqrt{2}} \qquad z^2 = -i \qquad z^3 = \frac{-1-i}{\sqrt{2}} \tag{11.9.97}$$

$$z^4 = -1 \qquad z^5 = \frac{-1+i}{\sqrt{2}} \qquad z^6 = i \qquad z^7 = \frac{1+i}{\sqrt{2}} \tag{11.9.98}$$

$$z^{r+8} = z^r \tag{11.9.99}$$

25. TBC. Performed using CAS. Coded in `sympy`