Chapter 22

Unconstrained Optimization. Linear Programming

22.1 Unconstrained Optimization: Method of Steepest Descent

1. Looking at Example 1,

$$t^* = \frac{x^2 + 9y^2}{2x^2 + 54y^2} \qquad \mathbf{z}(t) = \begin{bmatrix} (1-2t)\ x \\ (1-6t)\ y \end{bmatrix} \tag{22.1.1}$$

$$\nabla f(\mathbf{z}) = \begin{bmatrix} (2-4t)\ x \\ (6-36t)\ y \end{bmatrix} \qquad \nabla f(\mathbf{x}) = \begin{bmatrix} 2\ x \\ 6\ y \end{bmatrix} \tag{22.1.2}$$

In order to prove that the gradients calculated at $\mathbf{x}$ and $\mathbf{z}$ are perpendicular, and setting $t = t^*$

$$\nabla f(\mathbf{z}) \cdot \nabla f(\mathbf{x}) = (4-8t)\ x^2 + (36-216t)\ y^2 \tag{22.1.3}$$

$$= 4(x^2 + 9y^2) - 8t(x^2 + 27y^2) = 0 \tag{22.1.4}$$

2. Using the method of steepest descent,

$$f(\mathbf{x}) = x_1^2 + x_2^2 \qquad \mathbf{z}(t) = \begin{bmatrix} x_1\ (1-2t) \\ x_2\ (1-2t) \end{bmatrix} \tag{22.1.5}$$

$$g(t) = f\Big[\mathbf{z}(t)\Big] = (1-2t)^2\ (x_1^2 + x_2^2) \qquad t^* = 0.5 \tag{22.1.6}$$

$$\mathbf{z}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22.1.7}$$

The initial guess is that this is a paraboloid with minimum at the origin, which happens to be the very first iteration.

3. Finding the iterative formula,

$$f(\mathbf{x}) = 2(x_1 - 1)^2 + (x_2 + 2)^2 - 6 \tag{22.1.8}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 4x_1 - 4 \\ 2x_2 + 4 \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1(1 - 4t) + 4t \\ x_2(1 - 2t) - 4t \end{bmatrix} \tag{22.1.9}$$

$$g(t) = 2(x_1 - 1)^2(1 - 4t)^2 + (x_2 + 2)^2(1 - 2t)^2 + 6 \tag{22.1.10}$$

$$g'(t) = 0 = 4(x_1 - 1)^2\ (1 - 4t) + (x_2 + 2)^2\ (1 - 2t) \tag{22.1.11}$$

$$t^* = \frac{4(x_1 - 1)^2 + (x_2 + 2)^2}{16(x_1 - 1)^2 + 2(x_2 + 2)^2} \tag{22.1.12}$$

n	x_1	x_2
0	0	0
1	1.333	-1.333
2	0.889	-1.778
3	1.037	-1.926
8	1	-2

4. Finding the iterative formula,

$$f(\mathbf{x}) = (x_1 - 2.5)^2 + 0.5(x_2 - 3)^2 + 14.2 \tag{22.1.13}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 - 5 \\ x_2 - 3 \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1(1 - 2t) + 5t \\ x_2(1 - t) + 3t \end{bmatrix} \tag{22.1.14}$$

$$g(t) = (x_1 - 2.5)^2(1 - 2t)^2 + 0.5(x_2 - 3)^2(1 - t)^2 + 14.2 \tag{22.1.15}$$

$$g'(t) = 0 = 4(x_1 - 2.5)^2\ (1 - 2t) + (x_2 - 3)^2\ (1 - t) \tag{22.1.16}$$

$$t^* = \frac{4(x_1 - 2.5)^2 + (x_2 - 3)^2}{8(x_1 - 2.5)^2 + (x_2 - 3)^2} \tag{22.1.17}$$

n	x_1	x_2
0	3	4
1	2.5	3.5
2	2.5	3.25
3	2.5	3.125
4	2.5	3.0625
5	2.5	3.031 25
10	2.5	3

5. Finding the iterative formula, assuming

$$f(\mathbf{x}) = ax_1 + bx_2 \tag{22.1.18}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1 - at \\ x_2 - bt \end{bmatrix} \tag{22.1.19}$$

$$g(t) = ax_1 + bx_2 - (a^2 + b^2)\, t \tag{22.1.20}$$

$$g'(t) = 0 = -(a^2 + b^2) \tag{22.1.21}$$

No extremum, since the function is linear in x_1 and x_2.

6. This is a saddle point at the origin.

$$f(\mathbf{x}) = x_1^2 - x_2^2 \tag{22.1.22}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1(1-2t) \\ x_2(1+2t) \end{bmatrix} \tag{22.1.23}$$

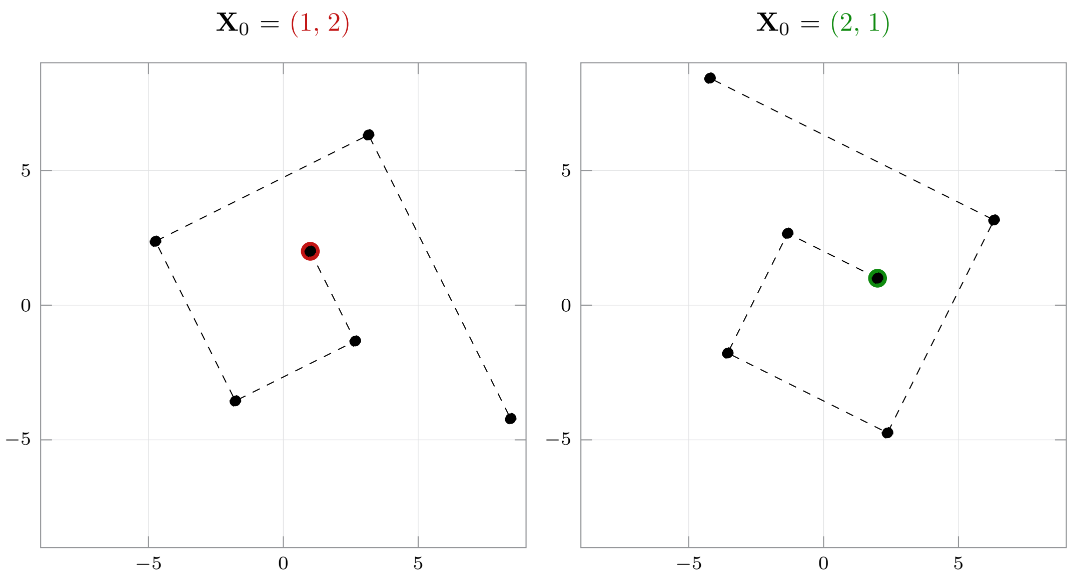
$$g(t) = x_1^2(1-2t)^2 - x_2^2(1+2t)^2 \tag{22.1.24}$$

$$g'(t) = 0 = x_1^2\,(1-2t) + x_2^2\,(1+2t) \tag{22.1.25}$$

$$t^* = \frac{x_1^2 + x_2^2}{2(x_1^2 - x_2^2)} \tag{22.1.26}$$

n	x_1	x_2
0	1	2
1	2.667	−1.333
2	−1.778	−3.556
3	−4.741	2.370
4	3.160	6.321
5	8.428	−4.214

n	x_1	x_2
0	2	1
1	−1.333	2.667
2	−3.556	−1.778
3	2.370	−4.741
4	6.321	3.160
5	−4.214	8.428



7. Using the method of steepest descent

$$f(\mathbf{x}) = x_1^2 + c x_2^2 \tag{22.1.27}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2\ x_1 \\ 2c\ x_2 \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1(1-2t) \\ x_2(1-2ct) \end{bmatrix} \tag{22.1.28}$$

$$g(t) = x_1^2(1-2t)^2 + x_2^2\ c(1-2ct)^2 \tag{22.1.29}$$

$$g'(t) = 0 = x_1^2\ (1-2t) + c^2 x_2^2\ (1-2ct) \tag{22.1.30}$$

$$t^* = \frac{x_1^2 + c^2 x_2^2}{2x_1^2 + 2c^3 x_2^2} \tag{22.1.31}$$

$$\mathbf{z}_1 = \frac{x_1 x_2}{2(x_1^2 + c^3 x_2^2)} \begin{bmatrix} 2c^2(c-1)\ x_2 \\ 2(1-c)\ x_1 \end{bmatrix} = \frac{c-1}{(c+1)} \begin{bmatrix} c \\ -1 \end{bmatrix} \tag{22.1.32}$$

$$\mathbf{z}_2 = \frac{(c-1)^2}{(c+1)^2} \begin{bmatrix} c \\ 1 \end{bmatrix} \tag{22.1.33}$$

Typo in the textbook. TBC.

8. Using the method of steepest descent, leads to a zigzag infinite progression,

$$f(\mathbf{x}) = x_1^2 - x_2 \tag{22.1.34}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 \\ -1 \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1(1-2t) \\ x_2 + t \end{bmatrix} \tag{22.1.35}$$

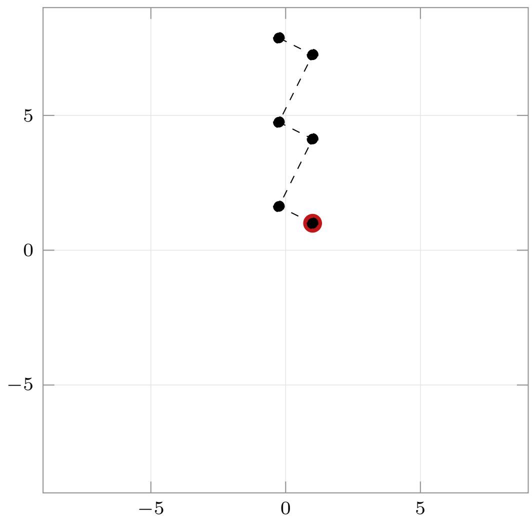
$$g(t) = x_1^2(1-2t)^2 - x_2 - t \tag{22.1.36}$$

$$g'(t) = 0 = -4x_1^2\,(1-2t) - 1 \tag{22.1.37}$$

$$t^* = \frac{4x_1^2 + 1}{8x_1^2} \tag{22.1.38}$$

n	x_1	x_2
0	1	1
1	-0.25	1.625
2	1	4.125
3	-0.25	4.750
4	1	7.250
5	-0.25	7.875

$\mathbf{X}_0 = (1, 1)$



9. Finding the iterative formula,

$$f(\mathbf{x}) = 0.1(x_1 - 0.1)^2 + x_2^2 - 0.001 \tag{22.1.39}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 0.2x_1 - 0.02 \\ 2x_2 \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1(1-0.2t) + 0.02t \\ x_2(1-2t) \end{bmatrix} \tag{22.1.40}$$

$$g(t) = 0.1\,(x_1 - 0.1)^2(1-0.2t)^2 + x_2^2(1-2t)^2 - 0.001 \tag{22.1.41}$$

$$g'(t) = 0 = (x_1 - 0.1)^2\,(1-0.2t) + 100x_2^2\,(1-2t) \tag{22.1.42}$$

$$t^* = \frac{(x_1 - 0.1)^2 + 100x_2^2}{0.2(x_1 - 0.1)^2 + 200x_2^2} \tag{22.1.43}$$

n	x_1	x_2
0	3	3
1	2.708	−0.025
2	0.301	0.207
3	0.280	−0.002
4	0.114	0.014
5	0.112	0.000
6	0.101	0.001

10. Steepest descent

(a) Program written in `numpy`

(b) Using the result from Problem 7,

$$f(\mathbf{x}) = x_1^2 + 4x_2^2 \tag{22.1.44}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2\ x_1 \\ 8\ x_2 \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1(1-2t) \\ x_2(1-8t) \end{bmatrix} \tag{22.1.45}$$

$$g(t) = x_1^2(1-2t)^2 + 4x_2^2\ (1-8t)^2 \tag{22.1.46}$$

$$g'(t) = 0 = x_1^2\ (1-2t) + 16x_2^2\ (1-8t) \tag{22.1.47}$$

$$t^* = \frac{x_1^2 + 16x_2^2}{2x_1^2 + 128x_2^2} \tag{22.1.48}$$

Since the function only has a global minimum at the origin, all initial points converge to this minimum, with convergence faster for points along the major axis.

(c) Finding the iterative method,

$$f(\mathbf{x}) = x_1^2 + x_2^4 \tag{22.1.49}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2\ x_1 \\ 4\ x_2^3 \end{bmatrix}, \qquad \mathbf{z}(t) = \begin{bmatrix} x_1(1-2t) \\ x_2 - 4tx_2^3 \end{bmatrix} \tag{22.1.50}$$

$$g(t) = x_1^2(1-2t)^2 + x_2^4\ (1-4tx_2^2)^4 \tag{22.1.51}$$

$$g'(t) = 0 = x_1^2\ (1-2t) + 16x_2^2\ (1-8t) \tag{22.1.52}$$

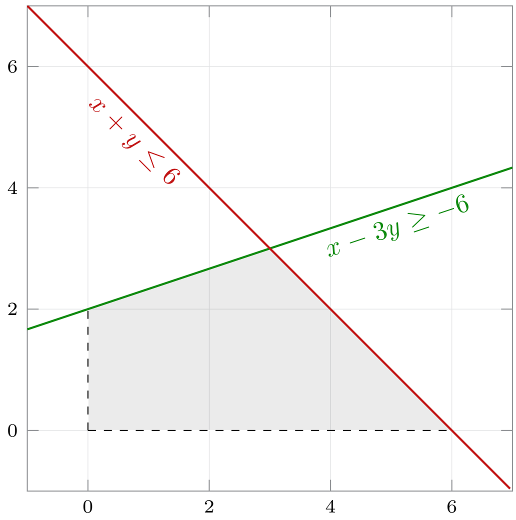
$$t^* = \frac{x_1^2 + 16x_2^2}{2x_1^2 + 128x_2^2} \tag{22.1.53}$$

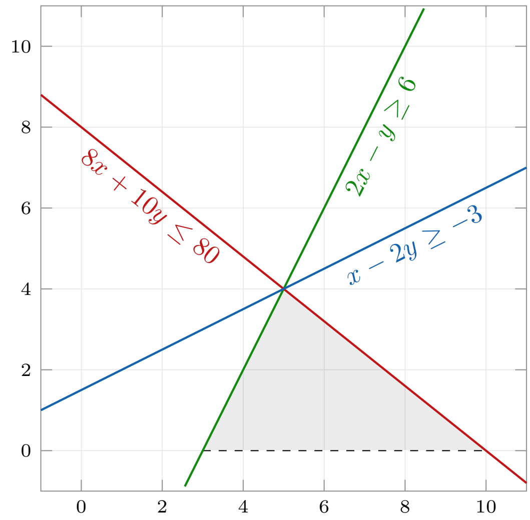
The only minimum is at the origin.

22.2 Linear Programming

1. Plotting the linear inequalities,

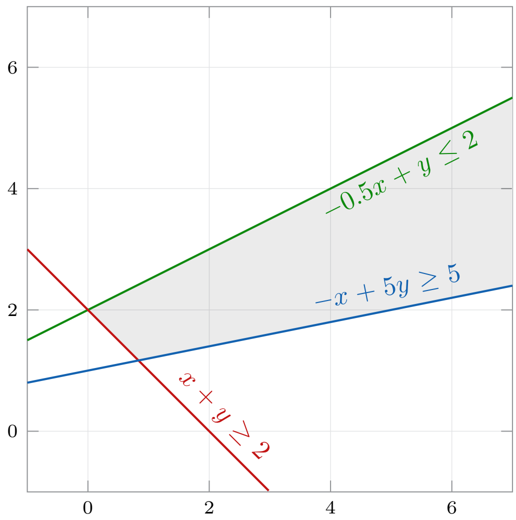
2. Plotting the linear inequalities,

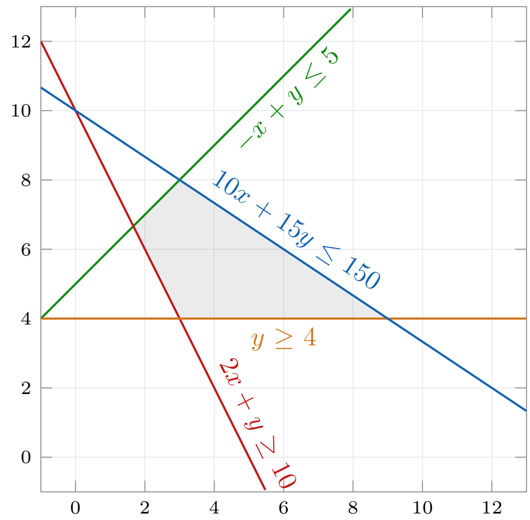




3. Plotting the linear inequalities,

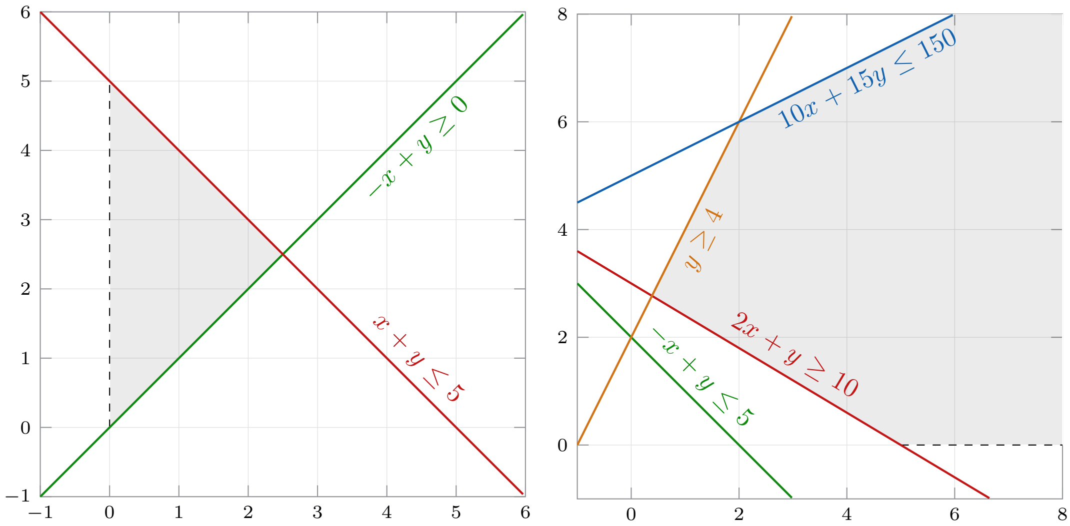
4. Plotting the linear inequalities,





5. Plotting the linear inequalities,

6. Plotting the linear inequalities,



7. Since the function $a_1x_1 + a_2x_2$ has straight line contours that vary monotonically, the optimal value must lie along an edge or at a vertex of the quadrangle.

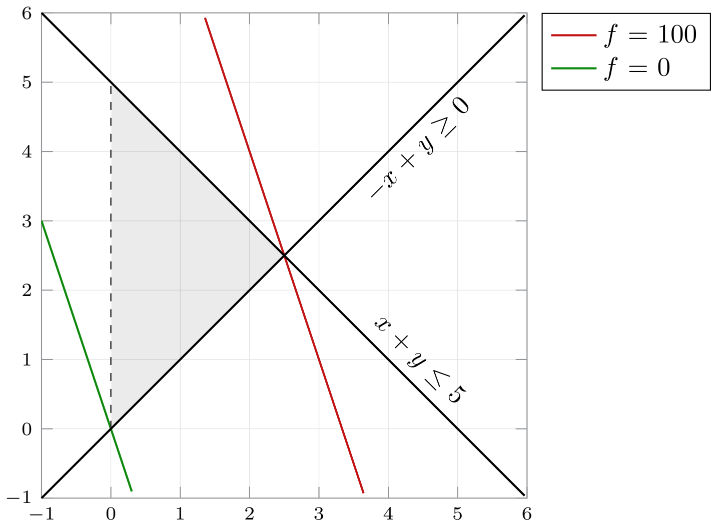
8. Every inequality (constraint) needs a slack variable. By convention, it is useful to have them all be the same sign, and nonnegative is merely convention.

9. The two slack variables are the two linear inequalities in Problem 1, which refer to the time taken to produce the units on machines M_1 and M_2 respectively.

10. No, if one of the edges of the quadrangle is parallel to the objective function.

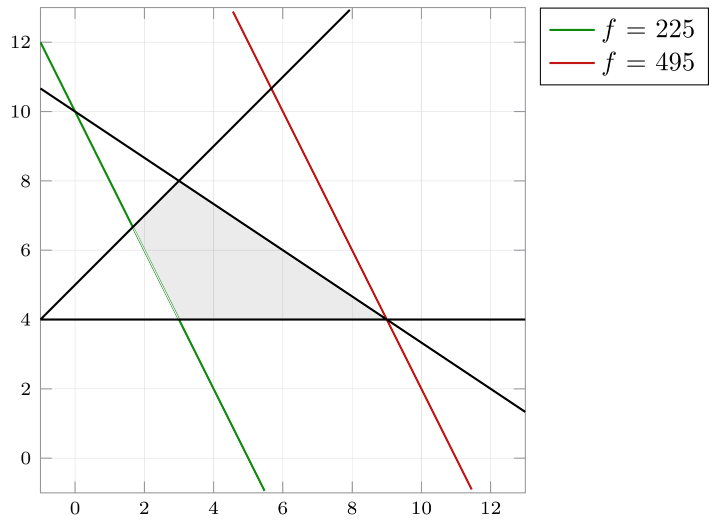
11. Maximizing the objective function,

$$f(\mathbf{x}) = 30x_1 + 10x_2 \tag{22.2.1}$$



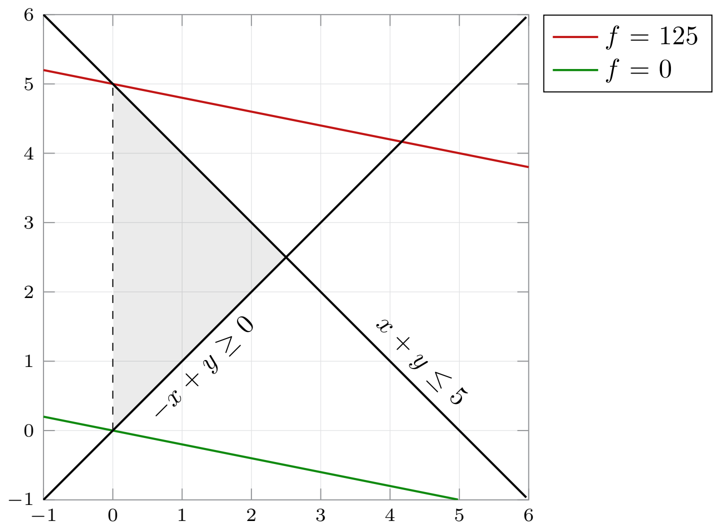
12. Minimizing the objective function,

$$f(\mathbf{x}) = 45x_1 + 22.5x_2 \tag{22.2.2}$$



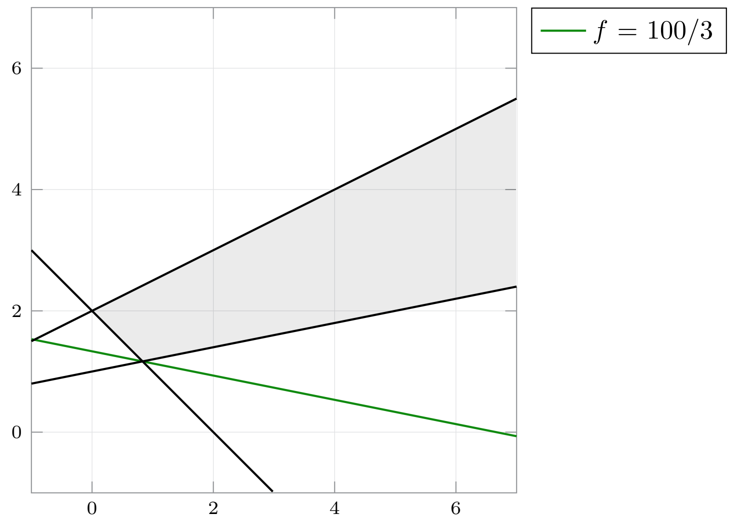
13. Maximizing the objective function,

$$f(\mathbf{x}) = 5x_1 + 25x_2$$ 22.2.3



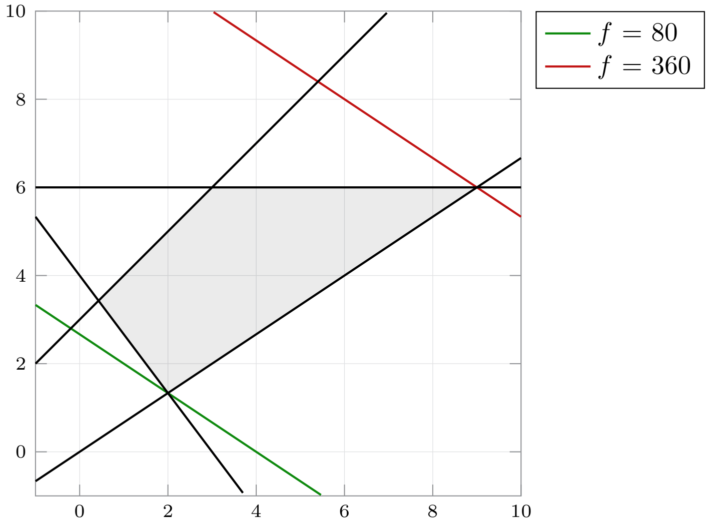
14. Minimizing the objective function,

$$f(\mathbf{x}) = 5x_1 + 25x_2$$ 22.2.4



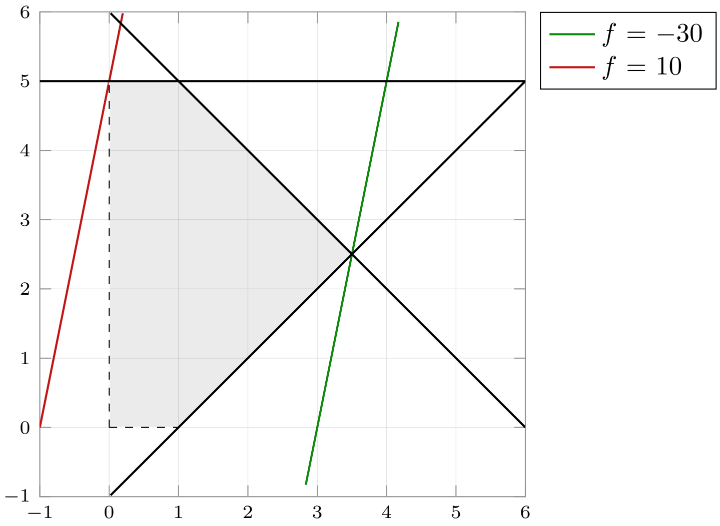
15. Maximizing the objective function,

$$f(\mathbf{x}) = 20x_1 + 30x_2$$ 22.2.5



16. Maximizing the objective function,

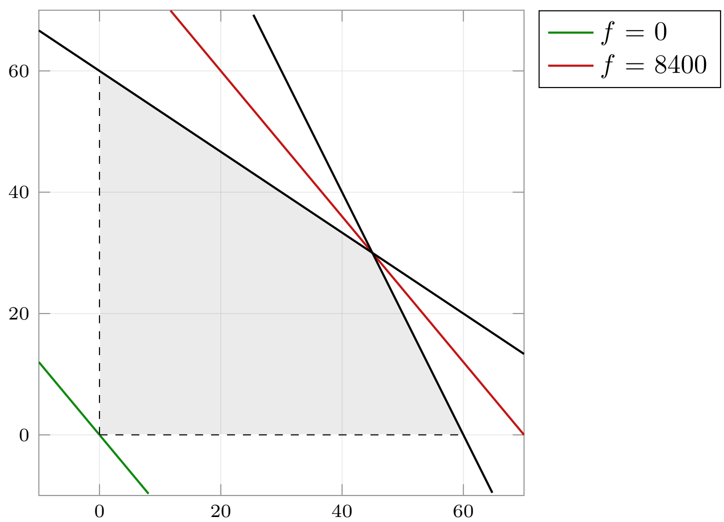
$$f(\mathbf{x}) = -10x_1 + 2x_2$$ 22.2.6



17. The problem is,

$$f(\mathbf{x}) = 120x_1 + 100x_2 \qquad 0.5x_1 + 0.75x_2 \le 45 \tag{22.2.7}$$

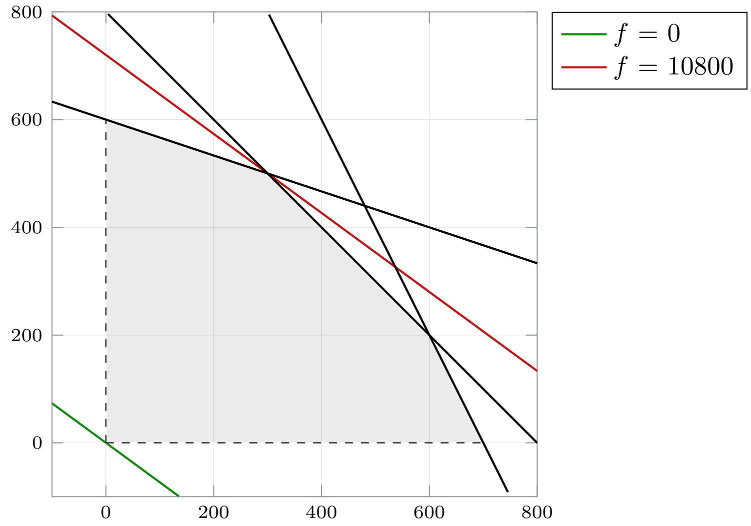
$$0.5x_1 + 0.25x_2 \le 30 \qquad x_1, x_2 \ge 0 \tag{22.2.8}$$



18. The problem is, with nonnegative x_i

$$f(\mathbf{x}) = 11x_1 + 15x_2 \qquad 2x_1 + x_2 \le 1400 \tag{22.2.9}$$

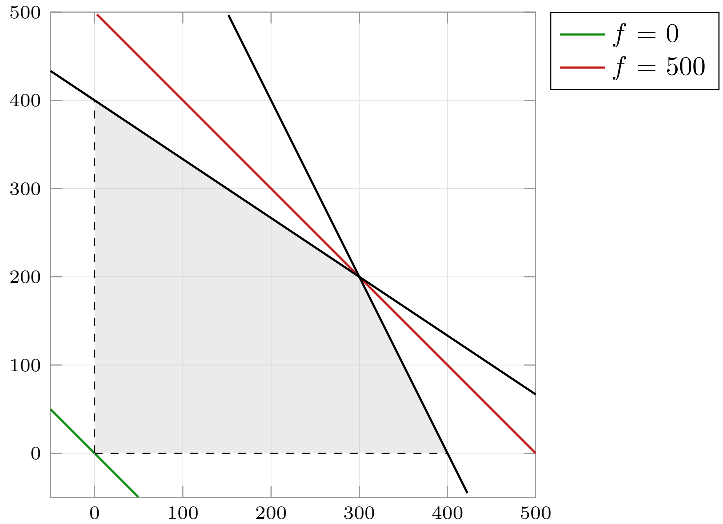
$$x_1 + x_2 \le 800 \qquad x_1 + 3x_2 \le 1800 \tag{22.2.10}$$



19. The problem is, with nonnegative x_i

$$f(\mathbf{x}) = x_1 + x_2 \qquad 2x_1 + 3x_2 \leq 1200 \tag{22.2.11}$$

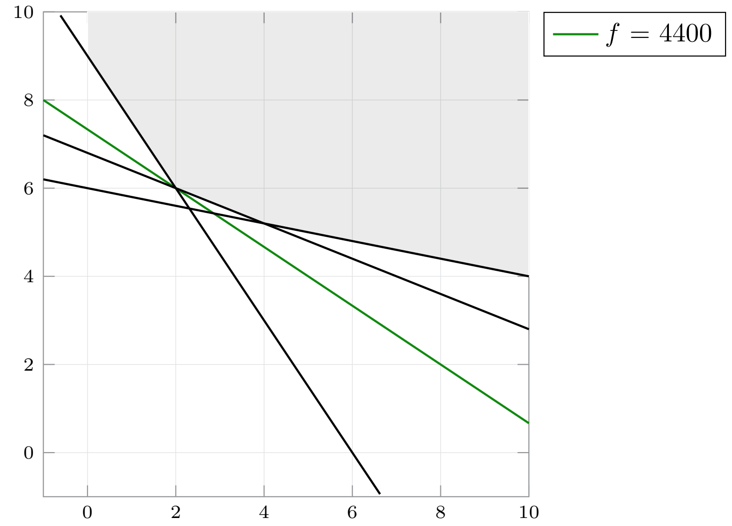
$$4x_1 + 2x_2 \leq 1600 \tag{22.2.12}$$



20. The problem is, with nonnegative x_i

$$f(\mathbf{x}) = 400x_1 + 600x_2 \qquad 3000x_1 + 2000x_2 \geq 18000 \tag{22.2.13}$$

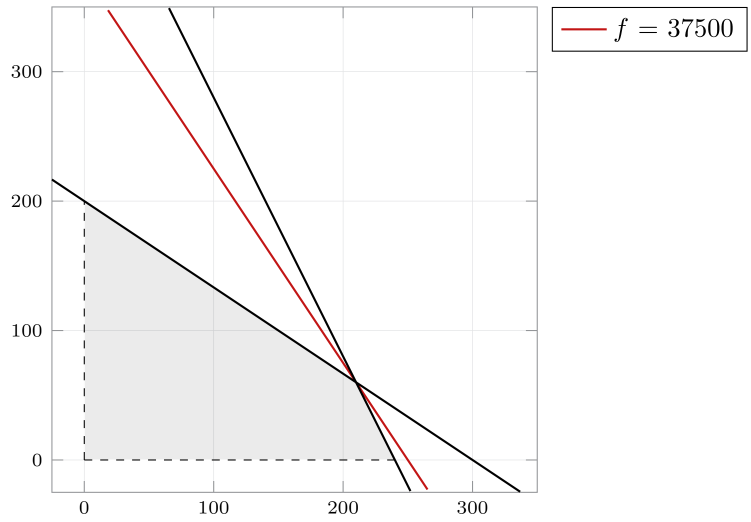
$$2000x_1 + 5000x_2 \geq 34000 \qquad 300x_1 + 1500x_2 \geq 9000 \tag{22.2.14}$$



21. The problem is, with nonnegative x_i

$$f(\mathbf{x}) = 150x_1 + 100x_2 \tag{22.2.15}$$

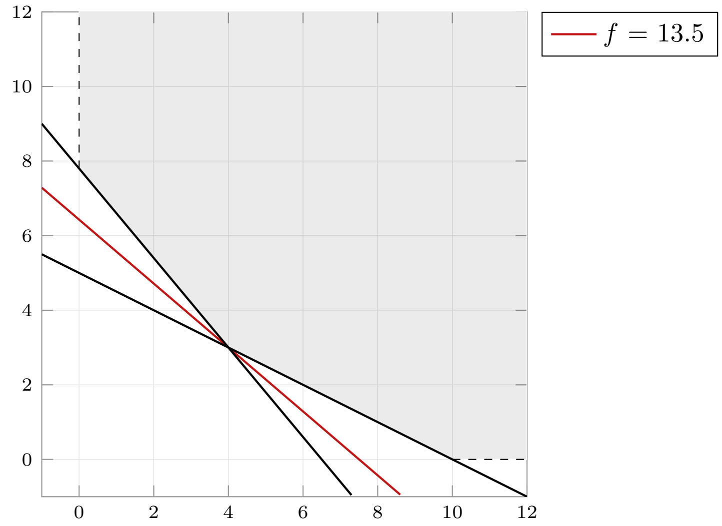
$$\frac{x_1}{3} + \frac{x_2}{2} \leq 100 \qquad \frac{x_1}{3} + \frac{x_2}{6} \leq 80 \tag{22.2.16}$$



22. The problem is, with nonnegative x_i

$$f(\mathbf{x}) = 1.8x_1 + 2.1x_2 \tag{22.2.17}$$

$$600x_1 + 500x_2 \geq 3900 \qquad 15x_1 + 30x_2 \geq 150 \tag{22.2.18}$$



22.3 Simplex Method

1. Using the simplex method, for nonnegative x_i

$$z = 40x_1 + 88x_2 \tag{22.3.1}$$

$$2x_1 + 8x_2 \leq 60 \qquad 2x_1 + 8x_2 + x_3 = 60 \tag{22.3.2}$$

$$5x_1 + 2x_2 \leq 60 \qquad 5x_1 + 2x_2 + x_4 = 60 \tag{22.3.3}$$

z	x_1	x_2	x_3	x_4	b
1	-40	-88	0	0	0
0	2	8	1	0	60
0	5	2	0	1	60

The pivot element in the first row is -40, the smallest quotient is 12 which makes $R2, C1$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	-72	0	8	480
0	0	7.2	1	-0.4	36
0	5	2	0	1	60

The pivot element in the first row is -72, the smallest quotient is 5 which makes $R1, C2$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	0	10	4	840
0	0	7.2	1	-0.4	36
0	5	0	$-\frac{5}{18}$	$\frac{10}{9}$	50

Stopping criterion is reached, and the optimal solution is

$$z\left(\frac{50}{5}, \frac{36}{7.2}\right) = z(10, 5) = 840 \tag{22.3.4}$$

2. Using the simplex method, for nonnegative x_i

$$z = 40x_1 + 88x_2 \tag{22.3.5}$$

$$8x_1 + 2x_2 \leq 60 \qquad 8x_1 + 2x_2 + x_3 = 60 \tag{22.3.6}$$

$$2x_1 + 5x_2 \leq 60 \qquad 2x_1 + 5x_2 + x_4 = 60 \tag{22.3.7}$$

z	x_1	x_2	x_3	x_4	b
1	-40	-88	0	0	0
0	8	2	1	0	60
0	2	5	0	1	60

The pivot element in the first row is -40, the smallest quotient is 7.5 which makes $R1$, $C1$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	-78	5	0	300
0	8	2	1	0	60
0	0	4.5	-0.25	1	45

The pivot element in the first row is -78, the smallest quotient is 10 which makes $R2$, $C2$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	0	$\frac{2}{3}$	$\frac{52}{3}$	1080
0	8	0	$\frac{10}{9}$	$-\frac{4}{9}$	40
0	0	4.5	-0.25	1	45

Stopping criterion is reached, and the optimal solution is

$$z\left(\frac{40}{8}, \frac{45}{4.5}\right) = z(5, 10) = 1080 \tag{22.3.8}$$

3. Using the simplex method, for nonnegative x_i

$$z = 3x_1 + 2x_2 \tag{22.3.9}$$

$$3x_1 + 4x_2 \leq 60 \qquad 3x_1 + 4x_2 + x_3 = 60 \tag{22.3.10}$$

$$4x_1 + 3x_2 \leq 60 \qquad 4x_1 + 3x_2 + x_4 = 60 \tag{22.3.11}$$

$$10x_1 + 2x_2 \leq 120 \qquad 10x_1 + 2x_2 + x_5 = 120 \tag{22.3.12}$$

z	x_1	x_2	x_3	x_4	x_5	b
1	-3	-2	0	0	0	0
0	3	4	1	0	0	60
0	4	3	0	1	0	60
0	10	2	0	0	1	120

The pivot element in the first row is -3, the smallest quotient is 12 which makes $R1, C3$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	-1.4	0	0	0.3	36
0	0	3.4	1	0	-0.3	24
0	0	2.2	0	1	-0.4	12
0	10	2	0	0	1	120

The pivot element in the first row is -1.4, the smallest quotient is $\frac{60}{11}$ which makes $R2, C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	0	0	$\frac{7}{11}$	$\frac{1}{22}$	$\frac{480}{11}$
0	0	0	1	$-\frac{17}{11}$	$\frac{7}{22}$	$\frac{60}{11}$
0	0	2.2	0	1	-0.4	12
0	10	0	0	$-\frac{10}{11}$	$\frac{15}{11}$	$\frac{1200}{11}$

Stopping criterion is reached, and the optimal solution is

$$z\left(\frac{120}{11}, \frac{60}{11}\right) = \frac{480}{11} \tag{22.3.13}$$

4. Using the simplex method, for nonnegative x_i

$$z = x_1 + x_2 \tag{22.3.14}$$

$$3x_1 + 4x_2 \le 550 \qquad 3x_1 + 4x_2 + x_3 = 550 \tag{22.3.15}$$

$$5x_1 + 4x_2 \le 650 \qquad 5x_1 + 4x_2 + x_4 = 650 \tag{22.3.16}$$

z	x_1	x_2	x_3	x_4	b
1	-1	-1	0	0	0
0	3	4	1	0	550
0	5	4	0	1	650

The pivot element in the first row is -1, the smallest quotient is 130 which makes $R2, C1$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	-0.2	0	0.2	130
0	0	1.6	1	-0.6	160
0	5	4	0	1	650

The pivot element in the first row is -0.2, the smallest quotient is 100 which makes $R1, C2$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	0	$\frac{1}{8}$	$\frac{1}{8}$	150
0	0	1.6	1	-0.6	160
0	5	0	-2.5	2.5	250

Stopping criterion is reached, and the optimal solution is

$$z(50, 100) = 150 \tag{22.3.17}$$

5. Using the simplex method, for nonnegative x_i

$$z = 5x_1 - 20x_2 \tag{22.3.18}$$

$$-2x_1 + 10x_2 \leq 5 \qquad -2x_1 + 10x_2 + x_3 = 5 \tag{22.3.19}$$

$$2x_1 + 5x_2 \leq 10 \qquad 2x_1 + 5x_2 + x_4 = 10 \tag{22.3.20}$$

z	x_1	x_2	x_3	x_4	b
1	-5	20	0	0	0
0	-2	10	1	0	5
0	2	5	0	1	10

The pivot element in the first row is 20, the smallest quotient is 0.5 which makes $R1$, $C2$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	-1	0	-2	0	-10
0	-2	10	1	0	5
0	3	0	-0.5	1	7.5

Stopping criterion is reached, and the optimal solution is

$$x_2 = 0.5 \qquad x_4 = 7.5 \tag{22.3.21}$$

$$x_1 = 0 \qquad z(0, 1/2) = -10 \tag{22.3.22}$$

6. Using the simplex method, for nonnegative x_i

$$z = x_1 + x_2 \tag{22.3.23}$$

$$2x_1 + 3x_2 \leq 1200 \qquad 2x_1 + 3x_2 + x_3 = 1200 \tag{22.3.24}$$

$$4x_1 + 2x_2 \leq 1600 \qquad 4x_1 + 2x_2 + x_4 = 1600 \tag{22.3.25}$$

z	x_1	x_2	x_3	x_4	b
1	-1	-1	0	0	0
0	2	3	1	0	1200
0	4	2	0	1	1600

The pivot element in the first row is -1, the smallest quotient is 400 which makes $R2, C1$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	-0.5	0	0.5	800
0	0	2	1	-0.5	400
0	4	2	0	1	1600

The pivot element in the first row is -0.5, the smallest quotient is 200 which makes $R1, C2$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	0	0.25	$\frac{7}{16}$	900
0	0	2	1	-0.5	400
0	4	0	-1	1.5	1200

Stopping criterion is reached, and the optimal solution is

$$z(300, 200) = 500 \tag{22.3.26}$$

7. Using the simplex method, for nonnegative x_i

$$0.1x_1 + 0.1x_2 + 0.2x_3 + 0.2x_4 = z \tag{22.3.27}$$

$$12x_1 + 8x_2 + 6x_3 + 4x_4 + x_5 = 120 \tag{22.3.28}$$

$$3x_1 + 6x_2 + 12x_3 + 24x_4 + x_6 = 180 \tag{22.3.29}$$

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	-0.1	-0.1	-0.2	-0.2	0	0	0
0	12	8	6	4	1	0	120
0	3	6	12	24	0	1	180

The pivot element in the first row is -0.1, the smallest quotient is 10 which makes $R1, C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	$-\frac{1}{30}$	-0.15	$-\frac{1}{6}$	$\frac{1}{120}$	0	1
0	12	8	6	4	1	0	120
0	0	4	10.5	23	-0.25	1	150

The pivot element in the first row is $-1/30$, the smallest quotient is 15 which makes $R1, C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	$\frac{1}{20}$	0	$-\frac{1}{8}$	$-\frac{3}{20}$	$\frac{1}{80}$	0	1.5
0	12	8	6	4	1	0	120
0	-6	0	7.5	21	-0.75	1	90

The pivot element in the first row is $-1/8$, the smallest quotient is 12 which makes $R2, C3$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	$-\frac{1}{20}$	0	0	$\frac{1}{5}$	0	$\frac{1}{60}$	3
0	$\frac{84}{5}$	8	0	$-\frac{64}{5}$	$\frac{8}{5}$	-0.8	48
0	-6	0	7.5	21	-0.75	1	90

The pivot element in the first row is $-1/20$, the smallest quotient is $\frac{20}{7}$ which makes $R1, C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	$\frac{1}{42}$	0	$\frac{17}{105}$	$\frac{1}{210}$	$\frac{1}{70}$	$\frac{22}{7}$
0	$\frac{84}{5}$	8	0	$-\frac{64}{5}$	$\frac{8}{5}$	-0.8	48
0	0	$\frac{20}{7}$	7.5	$\frac{115}{7}$	$-\frac{5}{28}$	$\frac{5}{7}$	$\frac{750}{7}$

Stopping criterion is reached, and the optimal solution is

$$z\left(\frac{20}{7}, 0, \frac{100}{7}, 0\right) = \frac{22}{7} \tag{22.3.30}$$

8. Using the simplex method, for nonnegative x_i

$$z = 90x_1 + 50x_2 \tag{22.3.31}$$

$$x_1 + 3x_2 \leq 18 \qquad x_1 + 3x_2 + x_3 = 18 \tag{22.3.32}$$

$$x_1 + x_2 \leq 10 \qquad x_1 + x_2 + x_4 = 10 \tag{22.3.33}$$

$$3x_1 + x_2 \leq 24 \qquad 3x_1 + x_2 + x_5 = 24 \tag{22.3.34}$$

z	x_1	x_2	x_3	x_4	x_5	b
1	-90	-50	0	0	0	0
0	1	3	1	0	0	18
0	1	1	0	1	0	10
0	3	1	0	0	1	24

The pivot element in the first row is -90, the smallest quotient is 8 which makes $R3, C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	-20	0	0	30	720
0	0	$\frac{8}{3}$	1	0	$-\frac{1}{3}$	10
0	0	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	2
0	3	1	0	0	1	24

The pivot element in the first row is -20, the smallest quotient is 3 which makes $R2, C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	0	0	30	20	780
0	0	0	1	-4	1	2
0	0	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	2
0	3	0	0	-1.5	0.5	21

Stopping criterion is reached, and the optimal solution is

$$z(7, 3) = 780$$ 22.3.35

9. Using the simplex method, for nonnegative x_i

$$z = 2x_1 + x_2 + 3x_3 \qquad 4x_1 + 3x_2 + 6x_3 = 12$$ 22.3.36

z	x_1	x_2	x_3	b
1	-2	-1	-3	0
0	4	3	6	12

The pivot element in the first row is -2, the smallest quotient is 3 which makes $R1, C1$ the pivot.

z	x_1	x_2	x_3	b
1	0	0.5	0	6
0	4	3	6	12

Stopping criterion is reached, and the optimal solution is on the line segment from $(3, 0, 0)$ to $(0, 0, 2)$.

$$x_2 = 0 \qquad\qquad 2x_1 + 3x_3 = 6 \tag{22.3.37}$$

10. Using the simplex method, for nonnegative x_i

$$z = 4x_1 - 10x_2 - 20x_3 \tag{22.3.38}$$

$$3x_1 + 4x_2 + 5x_3 \leq 60 \qquad\qquad 3x_1 + 4x_2 + 5x_3 + x_4 = 60 \tag{22.3.39}$$

$$2x_1 + x_2 \leq 20 \qquad\qquad 2x_1 + x_2 + x_5 = 20 \tag{22.3.40}$$

$$2x_1 + 3x_3 \leq 30 \qquad\qquad 2x_1 + 3x_3 + x_6 = 30 \tag{22.3.41}$$

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	-4	10	20	0	0	0	0
0	3	4	5	1	0	0	60
0	2	1	0	0	1	0	20
0	2	0	3	0	0	1	30

The pivot element in the first row is 10, the smallest quotient is 15 which makes $R1, C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	-11.5	0	7.5	-2.5	0	0	-150
0	3	4	5	1	0	0	60
0	1.25	0	-1.25	-0.25	1	0	5
0	2	0	3	0	0	1	30

The pivot element in the first row is 7.5, the smallest quotient is 10 which makes $R3, C3$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	-16.5	0	0	-2.5	0	-2.5	-225
0	$-\frac{1}{3}$	4	0	1	0	$-\frac{5}{3}$	10
0	1.25	0	-1.25	-0.25	1	0	5
0	2	0	3	0	0	1	30

Stopping criterion is reached, and the optimal solution is

$$x_1 = 0 \qquad z(0, 2.5, 10) = -225 \tag{22.3.42}$$

11. Using the simplex method, for nonnegative x_i

$$z = 1.8x_1 + 2.1x_2 \tag{22.3.43}$$

$$600x_1 + 500x_2 \geq 3900 \qquad 600x_1 + 500x_2 = 3900 + x_3 \tag{22.3.44}$$

$$15x_1 + 30x_2 \geq 150 \qquad 15x_1 + 30x_2 = 150 + x_4 \tag{22.3.45}$$

z	x_1	x_2	x_3	x_4	b
1	-1.8	-2.1	0	0	0
0	600	500	-1	0	3900
0	15	30	0	-1	150

The pivot element in the first row is -1.8, the smallest quotient is 6.5 which makes $R1, C1$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	$-\frac{3}{5}$	-0.003	0	$\frac{117}{10}$
0	600	500	-1	0	3900
0	0	$\frac{35}{2}$	$\frac{1}{40}$	-1	$\frac{105}{2}$

The pivot element in the first row is $-3/5$, the smallest quotient is 3 which makes $R2, C2$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	0	$-\frac{3}{1400}$	$-\frac{6}{175}$	$\frac{27}{2}$
0	600	0	$-\frac{12}{7}$	$\frac{200}{7}$	2400
0	0	$\frac{35}{2}$	$\frac{1}{40}$	-1	$\frac{105}{2}$

Stopping criterion is reached, and the optimal solution is

$$z(4, 3) = 13.5 \tag{22.3.46}$$

12. Using the simplex method, for nonnegative x_i

$$z = 2x_1 + 3x_2 + x_3 \tag{22.3.47}$$

$$x_1 + x_2 + x_3 \leq 4.8 \qquad x_1 + x_2 + x_3 + x_4 = 4.8 \tag{22.3.48}$$

$$10x_1 + x_3 \leq 9.9 \qquad 10x_1 + x_3 + x_5 = 9.9 \tag{22.3.49}$$

$$x_2 - x_3 \leq 0.2 \qquad x_2 - x_3 + x_6 = 0.2 \tag{22.3.50}$$

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	-2	-3	-1	0	0	0	0
0	1	1	1	1	0	0	4.8
0	10	0	1	0	1	0	9.9
0	0	1	-1	0	0	1	0.2

The pivot element in the first row is -2, the smallest quotient is 0.99 which makes $R2, C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	-3	-0.8	0	0.2	0	1.98
0	0	1	0.9	1	-0.1	0	3.81
0	10	0	1	0	1	0	9.9
0	0	1	-1	0	0	1	0.2

The pivot element in the first row is -3, the smallest quotient is 0.2 which makes $R3$, $C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	0	-3.8	0	0.2	3	2.58
0	0	0	1.9	1	-0.1	-1	3.61
0	10	0	1	0	1	0	9.9
0	0	1	-1	0	0	1	0.2

The pivot element in the first row is -3.8, the smallest quotient is 1.9 which makes $R1$, $C3$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	0	0	2	0	1	9.8
0	0	0	1.9	1	-0.1	-1	3.61
0	10	0	0	$-\frac{10}{19}$	$\frac{20}{19}$	$\frac{10}{19}$	8
0	0	1	0	$\frac{10}{19}$	$-\frac{1}{19}$	$\frac{9}{19}$	2.1

Stopping criterion is reached, and the optimal solution is

$$x_1 = 0 \qquad z(0.8, 2.1, 1.9) = 9.8 \tag{22.3.51}$$

13. Using the simplex method, for nonnegative x_i

$$z = 34x_1 + 29x_2 + 32x_3 \tag{22.3.52}$$

$$8x_1 + 2x_2 + x_3 \leq 54 \qquad 8x_1 + 2x_2 + x_3 + x_4 = 54 \tag{22.3.53}$$

$$3x_1 + 8x_2 + 2x_3 \leq 59 \qquad 3x_1 + 8x_2 + 2x_3 + x_5 = 59 \tag{22.3.54}$$

$$x_1 + x_2 + 5x_3 \leq 39 \qquad x_1 + x_2 + 5x_3 + x_6 = 39 \tag{22.3.55}$$

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	-34	-29	-32	0	0	0	0
0	8	2	1	1	0	0	54
0	3	8	2	0	1	0	59
0	1	1	5	0	0	1	39

The pivot element in the first row is -34, the smallest quotient is 6.75 which makes $R1$, $C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	$-\frac{193}{9}$	$-\frac{254}{9}$	$\frac{34}{9}$	0	0	204
0	8	2	1	1	0	0	54
0	0	$\frac{29}{4}$	$\frac{13}{8}$	$-\frac{3}{8}$	1	0	$\frac{155}{4}$
0	0	0.75	$\frac{39}{8}$	$-\frac{1}{8}$	0	1	$\frac{129}{4}$

The pivot element in the first row is $-193/9$, the smallest quotient is $155/29$ which makes $R2$, $C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	0	$-\frac{12223}{522}$	$\frac{1393}{522}$	$\frac{772}{261}$	0	$\frac{83159}{261}$
0	8	0	$\frac{16}{29}$	$\frac{32}{29}$	$-\frac{8}{29}$	0	$\frac{1256}{29}$
0	0	$\frac{29}{4}$	$\frac{13}{8}$	$-\frac{3}{8}$	1	0	$\frac{155}{4}$
0	0	0	$\frac{273}{58}$	$-\frac{5}{58}$	$-\frac{3}{29}$	1	$\frac{819}{29}$

The pivot element in the first row is $-12223/522$, the smallest quotient is 6 which makes $R3$, $C3$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	0	0	$\frac{5503}{2457}$	$\frac{667}{273}$	$\frac{12223}{2457}$	$\frac{4132}{9}$
0	8	0	0	$\frac{304}{273}$	$-\frac{24}{91}$	$-\frac{32}{273}$	40
0	0	$\frac{29}{4}$	0	$-\frac{29}{84}$	$\frac{29}{28}$	$-\frac{29}{84}$	29
0	0	0	$\frac{273}{58}$	$-\frac{5}{58}$	$-\frac{3}{29}$	1	$\frac{819}{29}$

Stopping criterion is reached, and the optimal solution is

$$z(5, 4, 6) = 478 \tag{22.3.56}$$

14. Using the simplex method, for nonnegative x_i

$$z = 2x_1 + 3x_2 \tag{22.3.57}$$

$$5x_1 + 3x_2 \geq 105 \qquad 5x_1 + 3x_2 + x_3 = 105 \tag{22.3.58}$$

$$3x_1 + 6x_2 \geq 126 \qquad 3x_1 + 6x_2 + x_4 = 126 \tag{22.3.59}$$

z	x_1	x_2	x_3	x_4	b
1	-2	-3	0	0	0
0	5	3	1	0	105
0	3	6	0	1	126

The pivot element in the first row is -2, the smallest quotient is 21 which makes $R1, C1$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	$-\frac{9}{5}$	0.4	0	42
0	5	3	1	0	105
0	0	$\frac{21}{5}$	$-\frac{3}{5}$	1	63

The pivot element in the first row is -1.8, the smallest quotient is 15 which makes $R2, C2$ the pivot.

z	x_1	x_2	x_3	x_4	b
1	0	0	$\frac{1}{7}$	$\frac{3}{7}$	69
0	5	0	$\frac{10}{7}$	$-\frac{5}{7}$	60
0	0	$\frac{21}{5}$	$-\frac{3}{5}$	1	63

Stopping criterion is reached, and the optimal solution is

$$z(12, 15) = 69 \quad \text{22.3.60}$$

15. Programs written in `numpy`. They were used to produce the tables above when for all the problems in this set.

22.4 Simplex Method: Difficulties

1. Using the simplex method, for nonnegative x_i

$$z = 7x_1 + 14x_2 \quad \text{22.4.1}$$

$$x_1 \le 6 \qquad x_1 + x_3 = 6 \quad \text{22.4.2}$$

$$x_2 \le 3 \qquad x_2 + x_4 = 3 \quad \text{22.4.3}$$

$$7x_1 + 14x_2 \le 84 \qquad 7x_2 + 14x_2 + x_5 = 84 \quad \text{22.4.4}$$

z	x_1	x_2	x_3	x_4	x_5	b
1	−7	−14	0	0	0	0
0	1	0	1	0	0	6
0	0	1	0	1	0	3
0	7	14	0	0	1	84

The pivot element in the first row is 1, the smallest quotient is 6 which makes $R1, C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	-14	7	0	0	42
0	1	0	1	0	0	6
0	0	1	0	1	0	3
0	0	14	-7	0	1	42

The pivot element in the first row is -14, the smallest quotient is 3 which makes $R2$, $C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	0	7	14	0	84
0	1	0	1	0	0	6
0	0	1	0	1	0	3
0	0	0	-7	-14	1	0

Stopping criterion is reached, and the optimal solution is

$$z(6, 3) = 84 \tag{22.4.5}$$

2. Using the simplex method, for nonnegative x_i

$$z = 7x_1 + 14x_2 \tag{22.4.6}$$

$$x_1 \leq 6 \qquad x_1 + x_3 = 6 \tag{22.4.7}$$

$$x_2 \leq 84 \qquad x_2 + x_4 = 84 \tag{22.4.8}$$

$$7x_1 + 14x_2 \leq 3 \qquad 7x_2 + 14x_2 + x_5 = 3 \tag{22.4.9}$$

z	x_1	x_2	x_3	x_4	x_5	b
1	-7	-14	0	0	0	0
0	1	0	1	0	0	6
0	0	1	0	1	0	84
0	7	14	0	0	1	3

The pivot element in the first row is 1, the smallest quotient is 6 which makes $R1, C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	0	0	0	1	3
0	0	-2	1	0	$-\frac{1}{7}$	5.57
0	0	1	0	1	0	84
0	7	14	0	0	1	3

Stopping criterion is reached, and the optimal solution is

$$z\left(\frac{3}{7}, 0\right) = 3 \tag{22.4.10}$$

3. Using the simplex method, for nonnegative x_i

$$z = x_1 + x_2 \tag{22.4.11}$$

$$3x_1 + 2x_2 \leq 180 \qquad 3x_1 + 2x_2 + x_3 = 180 \tag{22.4.12}$$

$$4x_1 + 6x_2 \leq 200 \qquad 4x_1 + 6x_2 + x_4 = 200 \tag{22.4.13}$$

$$5x_1 + 3x_2 \leq 160 \qquad 5x_2 + 3x_2 + x_5 = 160 \tag{22.4.14}$$

z	x_1	x_2	x_3	x_4	x_5	b
1	-1	-1	0	0	0	0
0	3	2	1	0	0	180
0	4	6	0	1	0	200
0	5	3	0	0	1	160

The pivot element in the first row is -1, the smallest quotient is $3/5$ which makes $R3, C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	-0.4	0	0	0.2	32
0	0	0.2	1	0	-0.6	84
0	0	3.6	0	1	-0.8	72
0	5	3	0	0	1	160

The pivot element in the first row is -0.4, the smallest quotient is 20 which makes $R2, C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	0	0	1.111	0.111	40
0	0	0	1	-0.0556	-0.556	80
0	0	3.6	0	1	-0.8	72
0	5	0	0	-0.833	1.667	100

Stopping criterion is reached, and the optimal solution is

$$z(20, 20) = 40 \tag{22.4.15}$$

4. Using the simplex method, for nonnegative x_i

$$z = 300x_1 + 500x_2 \tag{22.4.16}$$

$$2x_1 + 8x_2 \le 60 \qquad 2x_1 + 8x_2 + x_3 = 60 \tag{22.4.17}$$

$$2x_1 + x_2 \le 30 \qquad 2x_1 + x_2 + x_4 = 30 \tag{22.4.18}$$

$$4x_1 + 4x_2 \le 60 \qquad 4x_2 + 4x_2 + x_5 = 60 \tag{22.4.19}$$

z	x_1	x_2	x_3	x_4	x_5	b
1	-300	-500	0	0	0	0
0	2	8	1	0	0	60
0	2	1	0	1	0	30
0	4	4	0	0	1	60

The pivot element in the first row is -300, the smallest quotient is 15 which makes $R2$, $C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	-350	0	150	0	4500
0	0	7	1	-1	0	30
0	2	1	0	1	0	30
0	0	2	0	-2	1	0

The pivot element in the first row is -350, the smallest quotient is $30/7$ which makes $R1$, $C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	0	0	-200	175	4500
0	0	0	1	6	-3.5	30
0	2	0	0	2	-0.5	30
0	0	2	0	-2	1	0

The pivot element in the first row is -200, the smallest quotient is 0 which makes $R3$, $C3$ the pivot, but this does not lead to an increase in z. Using the second-smallest quotient instead,

z	x_1	x_2	x_3	x_4	x_5	b
1	0	0	0	0	$\frac{175}{3}$	5500
0	0	0	1	6	-3.5	30
0	2	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	20
0	0	2	0	0	$-\frac{1}{6}$	10

Stopping criterion is reached, and the optimal solution is

$$z(10, 5) = 5500$$ 22.4.20

5. Using the simplex method, for nonnegative x_i

$$z = 300x_1 + 500x_2 \qquad \text{22.4.21}$$

$$2x_1 + 8x_2 \leq 60 \qquad 2x_1 + 8x_2 + x_3 = 60 \qquad \text{22.4.22}$$

$$2x_1 + x_2 \leq 60 \qquad 2x_1 + x_2 + x_4 = 60 \qquad \text{22.4.23}$$

$$4x_1 + 4x_2 \leq 30 \qquad 4x_2 + 4x_2 + x_5 = 30 \qquad \text{22.4.24}$$

z	x_1	x_2	x_3	x_4	x_5	b
1	-300	-500	0	0	0	0
0	2	8	1	0	0	60
0	2	1	0	1	0	60
0	4	4	0	0	1	30

The pivot element in the first row is -300, the smallest quotient is 7.5 which makes $R3, C1$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	-200	0	0	75	2250
0	0	6	1	0	-0.5	45
0	0	-1	0	1	-0.5	45
0	4	4	0	0	1	30

The pivot element in the first row is -200, the smallest quotient is 4.5 which makes $R1, C2$ the pivot.

z	x_1	x_2	x_3	x_4	x_5	b
1	0	0	$\frac{100}{3}$	0	$\frac{175}{3}$	3750
0	0	6	1	0	-0.5	45
0	0	0	$\frac{1}{6}$	1	$-\frac{29}{150}$	52.5
0	4	0	$-\frac{2}{3}$	0	$\frac{4}{3}$	0

Stopping criterion is reached, and the optimal solution is

$$z(0, 7.5) = 3750 \tag{22.4.25}$$

6. Using the simplex method, for nonnegative x_i

$$z = x_1 + x_2 + x_3 \tag{22.4.26}$$

$$5x_1 + 6x_2 + 7x_3 \leq 12 \qquad 5x_1 + 6x_2 + 7x_3 + x_4 = 12 \tag{22.4.27}$$

$$7x_1 + 4x_2 + x_3 \leq 12 \qquad 7x_1 + 4x_2 + x_3 + x_5 = 12 \tag{22.4.28}$$

Using the `numpy` algorithm, Stopping criterion is reached, and the optimal solution is

$$z\left(\frac{12}{11}, \frac{12}{11}\right) = \frac{24}{11} \tag{22.4.29}$$

7. Using the simplex method, for nonnegative x_i

$$z = 5x_1 + 8x_2 + 4x_3 \tag{22.4.30}$$

$$x_1 + x_3 + x_5 = 1 \qquad x_2 + x_3 + x_4 = 1 \tag{22.4.31}$$

Using the `numpy` algorithm, Stopping criterion is reached, and the optimal solution is

$$z(1, 1, 0) = 13 \tag{22.4.32}$$

8. Using the simplex method, for nonnegative x_i

$$z = -4x_1 + x_2 \tag{22.4.33}$$

$$x_1 + x_2 - x_3 = 2 \qquad -2x_1 + 3x_2 + x_4 = 1 \tag{22.4.34}$$

$$5x_1 + 4x_2 + x_5 = 50 \tag{22.4.35}$$

Introducing an artificial variable x_6,

$$x_3 = -2 + x_1 + x_2 + x_6 \qquad x_6 \geq 0 \tag{22.4.36}$$

$$\hat{z} = -4x_1 + x_2 - Mx_6 \qquad \hat{z} = (-4 + M)x_1 + (M + 1)x_2 - Mx_3 - 2M \tag{22.4.37}$$

Starting the simplex procedure,

$\hat{z}$	x_1	x_2	x_3	x_4	x_5	x_6	b
1	$-M+4$	$-M-1$	M	0	0	0	$-2M$
0	1	1	-1	0	0	0	2
0	-2	3	0	1	0	0	1
0	5	4	0	0	1	0	50
0	1	1	-1	0	0	1	2

The pivot element in the first row is $-M+4$, the smallest quotient is 2 which makes $R1$, $C1$ the pivot.

$\hat{z}$	x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	-5	4	0	0	0	-8
0	1	1	-1	0	0	0	2
0	0	5	-2	1	0	0	5
0	0	-1	5	0	1	0	40
0	0	0	0	0	0	1	0

The artificial variable can now be removed from the system,

The pivot element in the first row is $(-4-M)$, the smallest quotient is 2 which makes $R1$, $C1$ the pivot.

$\hat{z}$	x_1	x_2	x_3	x_4	x_5	b
1	0	-5	4	0	0	-8
0	1	1	-1	0	0	2
0	0	5	-2	1	0	5
0	0	-1	5	0	1	40

The pivot element in the first row is -5, the smallest quotient is 1 which makes $R2$, $C2$ the pivot.

$\hat{z}$	x_1	x_2	x_3	x_4	x_5	b
1	0	0	2	1	0	−3
0	1	0	−0.6	−0.2	0	1
0	0	5	−2	1	0	5
0	0	0	4.6	0.2	1	41

Stopping criterion is reached, and the optimal solution is

$$\hat{z}(1, 1) = -3 \qquad \hat{z} = 3 \tag{22.4.38}$$

9. Using the simplex method, for nonnegative x_i

$$z = 2x_1 + 3x_2 + 2x_3 \tag{22.4.39}$$

$$x_1 + 2x_2 - 4x_3 + x_4 = 2 \qquad x_1 + 2x_2 + 2x_3 + x_5 = 5 \tag{22.4.40}$$

Introducing an artificial variable x_6,

$$x_3 = -2 + x_1 + x_2 + x_6 \qquad x_6 \geq 0 \tag{22.4.41}$$

$$\hat{z} = -4x_1 + x_2 - Mx_6 \qquad \hat{z} = (-4 + M)x_1 + (M + 1)x_2 - Mx_3 - 2M \tag{22.4.42}$$

Using the `numpy` algorithm,
Stopping criterion is reached, and the optimal solution is

$$\hat{z}(4, 0, 1/2) = 9 \tag{22.4.43}$$