Chapter 13

Complex Numbers and Functions, Complex Differentiation

13.1 Complex Numbers and Their Geometric Representation

1. Powers of i,

$$\mathrm{i} = (0, 1) \qquad \mathrm{i}^2 = (0, 1) \cdot (0, 1) = -1 \tag{13.1.1}$$

$$\mathrm{i}^3 = \mathrm{i}^2 \cdot \mathrm{i} = -\mathrm{i} \qquad \mathrm{i}^4 = (\mathrm{i}^2)^2 = (-1)^2 = 1 \tag{13.1.2}$$

This repeats with a period of 4.

$$\mathrm{i} = (0, 1) \qquad \frac{1}{\mathrm{i}} = \frac{-\mathrm{i}}{\mathrm{i}^2} = -\mathrm{i} \tag{13.1.3}$$

$$\frac{1}{\mathrm{i}^2} = \frac{1}{-1} = -1 \qquad \frac{1}{\mathrm{i}^3} = \frac{1}{(-1) \cdot \mathrm{i}} = \mathrm{i} \tag{13.1.4}$$

This also repeats with a period of 4.

2. Plotting the rotated real numbers,

$$z_1 = 1 + \mathrm{i} \qquad \mathrm{i}z_1 = -1 + \mathrm{i} \tag{13.1.5}$$

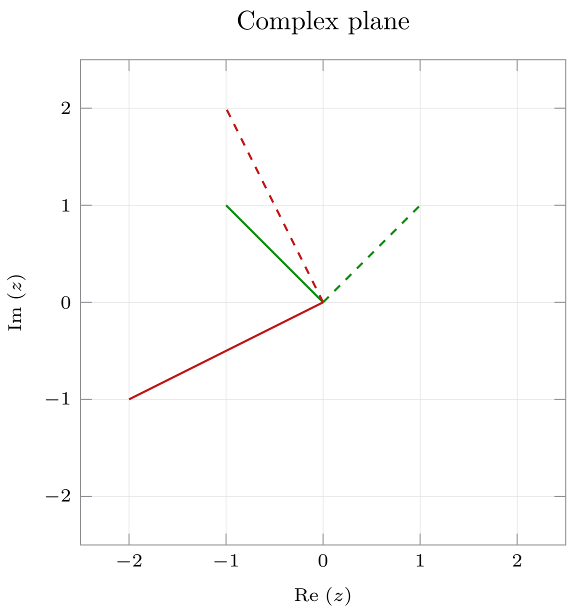
$$z_2 = -1 + 2\mathrm{i} \qquad \mathrm{i}z_2 = -2 - \mathrm{i} \tag{13.1.6}$$

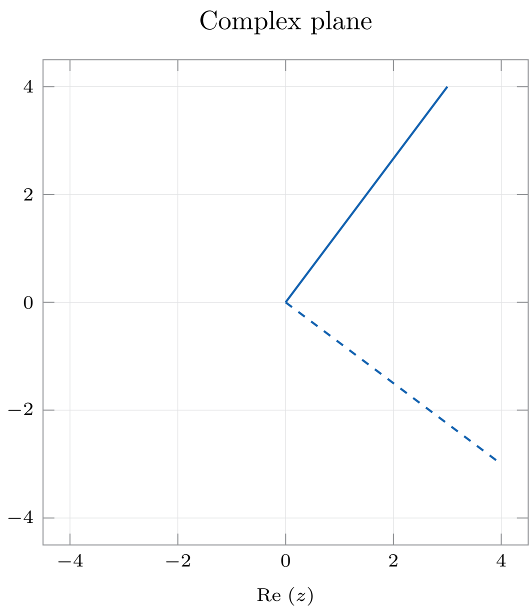
$$z_3 = 4 - 3\mathrm{i} \qquad \mathrm{i}z_3 = 3 + 4\mathrm{i} \tag{13.1.7}$$

Using the geometric verificiation of perpendicular lines having their slopes multiply to -1,

$$m_1 n_1 = 1 \cdot -1 = -1 \qquad m_2 n_2 = -2 \cdot (1/2) = -1 \tag{13.1.8}$$

$$m_3 n_3 = (-3/4) \cdot (4/3) = -1 \tag{13.1.9}$$





3. Verifying the division,

$$z = \frac{x_1 + y_1 \mathrm{i}}{x_2 + y_2 \mathrm{i}} \qquad = \frac{x_1 + y_1 \mathrm{i}}{x_2 + y_2 \mathrm{i}} \cdot \frac{x_2 - y_2 \mathrm{i}}{x_2 - y_2 \mathrm{i}} \tag{13.1.10}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\ \mathrm{i} \tag{13.1.11}$$

Applying the above relation to the given complex number quotient,

$$z = \frac{26 - 18\mathrm{i}}{6 - 2\mathrm{i}} \qquad z = \frac{192 - 56\mathrm{i}}{40} \tag{13.1.12}$$

$$z = 4.8 - 1.4\mathrm{i} \tag{13.1.13}$$

4. Verifying the relations for arithmetic on conjugate complex numbers,

$$\overline{(z_1 + z_2)} = \overline{(-12 + 14\mathrm{i})} = -12 - 14\mathrm{i} \tag{13.1.14}$$

$$\bar{z}_1 + \bar{z}_2 = (-11 - 10\mathrm{i}) + (-1 - 4\mathrm{i}) = -12 - 14\mathrm{i} \tag{13.1.15}$$

$$\overline{(z_1 - z_2)} = \overline{(-10 + 6\mathrm{i})} = -10 - 6\mathrm{i} \tag{13.1.16}$$

$$\bar{z}_1 - \bar{z}_2 = (-11 - 10\mathrm{i}) - (-1 - 4\mathrm{i}) = -10 - 6\mathrm{i} \tag{13.1.17}$$

$$\overline{(z_1 \cdot z_2)} = \overline{(-11 + 10\mathrm{i}) \cdot (-1 + 4\mathrm{i})} = \overline{-29 - 54\mathrm{i}} = -29 + 54\mathrm{i} \tag{13.1.18}$$

$$\bar{z}_1 \cdot \bar{z}_2 = (-11 + 10\mathrm{i}) \cdot (-1 + 4\mathrm{i}) = -29 - 54\mathrm{i} \tag{13.1.19}$$

For division, using the formula,

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{54 + 34\mathrm{i}}}{9} = \frac{54 - 34\mathrm{i}}{9} \tag{13.1.20}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{-11 - 10\mathrm{i}}{-1 - 4\mathrm{i}} = \frac{54 - 34\mathrm{i}}{9} \tag{13.1.21}$$

5. Checking if the real part of z is zero, which makes it purely imaginary,

$$\bar{z} + z = (x + x) + (y - y)\mathrm{i} = 2x + 0\mathrm{i} \tag{13.1.22}$$

$$\bar{z} + z = 0 \iff x = 0 \tag{13.1.23}$$

6. Product of two complex numbers is zero,

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)\ \mathrm{i} = 0 + 0\mathrm{i} \tag{13.1.24}$$

$$x_1x_2 = y_1y_2 \tag{13.1.25}$$

$$x_1y_2 = -x_2y_1 \tag{13.1.26}$$

These two conditions are only satisfied if at least $x_1 = y_1 = 0$. Only one of the two being zero means that the product has either a nonzero real or imaginary part.

7. Commutative law,

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\ \mathrm{i} \tag{13.1.27}$$

$$= (x_2 + x_1) + (y_2 + y_1)\ \mathrm{i} = z_2 + z_1 \tag{13.1.28}$$

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)\ \mathrm{i} \tag{13.1.29}$$

$$= (x_2x_1 - y_2y_1) + (x_2y_1 + x_1y_2)\ \mathrm{i} = z_2 \cdot z_1 \tag{13.1.30}$$

Associative law,

$$(z_1 + z_2) + z_3 = (x_1 + x_2) + x_3 + (y_1 + y_2)\ \mathrm{i} + y_3\ \mathrm{i} \tag{13.1.31}$$

$$= x_1 + (x_2 + x_3) + y_1\ \mathrm{i} + (y_2 + y_3)\ \mathrm{i} = z_1 + (z_2 + z_3) \tag{13.1.32}$$

$$(z_1 \cdot z_2) \cdot z_3 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)\ \mathrm{i} \tag{13.1.33}$$

$$= (x_1x_2x_3 - y_1y_2x_3 - x_1y_2y_3 - x_2y_1y_3) \tag{13.1.34}$$

$$+ (x_1x_3y_2 + x_2x_3y_1 + x_1x_2y_3 - y_1y_2y_3)\ \mathrm{i} \tag{13.1.35}$$

$$= x_1\ (x_2x_3 - y_2y_3) - y_1\ (x_2y_3 + x_3y_2) \tag{13.1.36}$$

$$+ x_1\ (x_2y_3 + x_3y_2)\ \mathrm{i} + y_1\ (x_2x_3 - y_2y_3)\ \mathrm{i} = z_1 \cdot (z_2 \cdot z_3) \tag{13.1.37}$$

Distributive law,

$$z_1(z_2+z_3) = \Big[x_1(x_2+x_3) - y_1(y_2+y_3)\Big] + \Big[x_1(y_2+y_3) + y_1(x_2+x_3)\Big]\ \mathrm{i} \tag{13.1.38}$$

$$= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)\ \mathrm{i} \tag{13.1.39}$$

$$+ (x_1x_3 - y_1y_3) + (x_1y_3 + x_3y_1)\ \mathrm{i} \tag{13.1.40}$$

$$= z_1z_2 + z_1z_3 \tag{13.1.41}$$

Additive null,

$$z + 0 = (x+0) + (y+0)\ \mathrm{i} = (0+x) + (0+y)\ \mathrm{i} = 0 + z = z \tag{13.1.42}$$

Additive inverse,

$$z + 0 = z \qquad \Longrightarrow z + (-z) = 0 \tag{13.1.43}$$

Multiplicative identity,

$$z \cdot 1 = (x \cdot 1 - y_1 \cdot 0) + (x \cdot 0 + 1 \cdot y)\ \mathrm{i} = x + y\mathrm{i} = z \tag{13.1.44}$$

8. Performing the given computations,

$$z_1z_2 = (-2+11\mathrm{i}) \cdot (2-\mathrm{i}) = 7 + 24\mathrm{i} \tag{13.1.45}$$

$$\overline{(z_1z_2)} = (-2-11\mathrm{i}) \cdot (2+\mathrm{i}) = 7 - 24\mathrm{i} \tag{13.1.46}$$

9. Performing the given computations,

$$\mathrm{Re}\ (z_1^2) = \mathrm{Re}\ (\ -117 - 44\mathrm{i}) = -117 \tag{13.1.47}$$

$$\Big[\ \mathrm{Re}\ (z_1)\Big]^2 = (-2)^2 = 4 \tag{13.1.48}$$

10. Performing the given computations,

$$\mathrm{Re}\left(\frac{1}{z_2^2}\right) = \mathrm{Re}\left(\frac{1}{3-4\mathrm{i}}\right) = \mathrm{Re}\left(\frac{3+4\mathrm{i}}{25}\right) = \frac{3}{25} \tag{13.1.49}$$

$$\frac{1}{\mathrm{Re}\ (z_2^2)} = \frac{1}{\mathrm{Re}\ (3-4\mathrm{i})} = \frac{1}{3} \tag{13.1.50}$$

11. Performing the given computations,

$$\frac{(z_1 - z_2)^2}{16} = \frac{(-4+12\mathrm{i})^2}{16} = \frac{-128-96\mathrm{i}}{16} = -8-6\mathrm{i} \tag{13.1.51}$$

$$\left(\frac{z_1}{4} - \frac{z_2}{4}\right)^2 = \left(\frac{-2}{4} + \frac{11}{4}\,\mathrm{i} - \frac{2}{4} + \frac{1}{4}\,\mathrm{i}\right)^2 = (-1+3\mathrm{i})^2 = -8-6\mathrm{i} \tag{13.1.52}$$

12. Performing the given computations,

$$\frac{z_1}{z_2} = \frac{-2+11\mathrm{i}}{2-\mathrm{i}} = \frac{(-2+11\mathrm{i})\cdot(2+\mathrm{i})}{5} = -3+4\mathrm{i} \tag{13.1.53}$$

$$\frac{z_2}{z_1} = \frac{2-\mathrm{i}}{-2+11\mathrm{i}} = \frac{(2-\mathrm{i})\cdot(-2-11\mathrm{i})}{125} = \frac{-3-4\mathrm{i}}{25} \tag{13.1.54}$$

13. Performing the given computations,

$$(z_1+z_2)\cdot(z_1-z_2) = (10\mathrm{i})\cdot(-4+12\mathrm{i}) = -120-40\mathrm{i} \tag{13.1.55}$$

$$z_1^2 - z_2^2 = (-117-44\mathrm{i}) - (3-4\mathrm{i}) = -120-40\mathrm{i} \tag{13.1.56}$$

14. Performing the given computations,

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{-3+4\mathrm{i}} = -3-4\mathrm{i} \tag{13.1.57}$$

$$\frac{\bar{z_1}}{\bar{z_2}} = \frac{-2-11\mathrm{i}}{2+\mathrm{i}} = \frac{(-2-11\mathrm{i})\cdot(2-\mathrm{i})}{5} = -3-4\mathrm{i} \tag{13.1.58}$$

15. Performing the given computations,

$$4\,\frac{z_1+z_2}{z_1-z_2} = 4\,\frac{10\mathrm{i}}{-4+12\mathrm{i}} = \frac{120-40\mathrm{i}}{40} = 3-\mathrm{i} \tag{13.1.59}$$

16. Performing the given computations,

$$\frac{1}{z} = \frac{x-y\mathrm{i}}{x^2+y^2} \qquad \mathrm{Im}\left(\frac{1}{z}\right) = \frac{-y}{x^2+y^2} \tag{13.1.60}$$

$$\frac{1}{z^2} = \frac{1}{x^2-y^2+2xy\mathrm{i}} \qquad \frac{1}{z^2} = \frac{x^2-y^2-2xy\mathrm{i}}{(x^2-y^2)^2+(2xy)^2} \tag{13.1.61}$$

$$\mathrm{Im}\left(\frac{1}{z^2}\right) = \frac{-2xy}{(x^2-y^2)^2+(2xy)^2} \tag{13.1.62}$$

17. Performing the given computations,

$$z^2 = x^2 - y^2 + (2xy)\mathrm{i} \tag{13.1.63}$$

$$z^4 = (x^4 + y^4 - 6x^2y^2) + 4xy(x^2 - y^2)\mathrm{i} \tag{13.1.64}$$

$$\mathrm{Re}\,(z^4) = x^4 + y^4 - 6x^2y^2 \tag{13.1.65}$$

$$\Big[\,\mathrm{Re}\,(z^2)\Big]^2 = (x^2 - y^2)^2 = x^4 + y^4 - 2x^2y^2 \tag{13.1.66}$$

$$\mathrm{Re}\,(z^4) - \Big[\,\mathrm{Re}\,(z^2)\Big]^2 = -4x^2y^2 \tag{13.1.67}$$

18. Performing the given computations,

$$(1+\mathrm{i})^2 = 2\mathrm{i} \qquad (1+\mathrm{i})^{16} = (2\mathrm{i})^8 = 256 \tag{13.1.68}$$

$$\mathrm{Re}\,(256\;z^2) = 256\;(x^2 - y^2) \tag{13.1.69}$$

19. Performing the given computations,

$$\frac{z}{\bar{z}} = \frac{x + y\mathrm{i}}{x - y\mathrm{i}} \qquad \frac{z}{\bar{z}} = \frac{(x^2 - y^2) + 2xy\mathrm{i}}{x^2 + y^2} \tag{13.1.70}$$

$$\mathrm{Re}\left(\frac{z}{\bar{z}}\right) = \frac{(x^2 - y^2)}{x^2 + y^2} \qquad \mathrm{Im}\left(\frac{z}{\bar{z}}\right) = \frac{2xy}{x^2 + y^2} \tag{13.1.71}$$

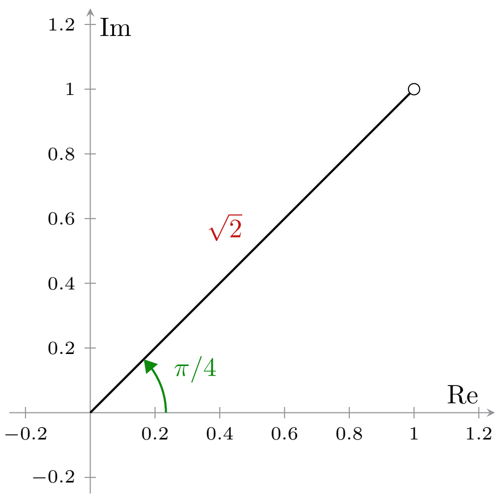
20. Using the result from Problem 16 with $y \to -y$,

$$\mathrm{Im}\left(\frac{1}{\bar{z}^2}\right) = \frac{2xy}{(x^2 - y^2)^2 + (2xy)^2} \tag{13.1.72}$$

13.2 Polar Form of Complex Numbers, Powers and Roots

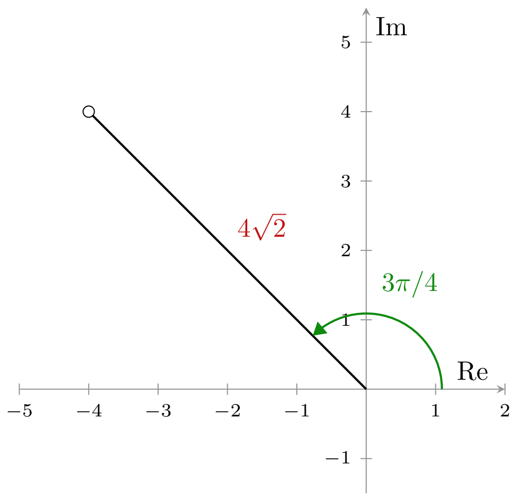
1. Plotting on the complex plane,

$z = 1 + \mathrm{i}$



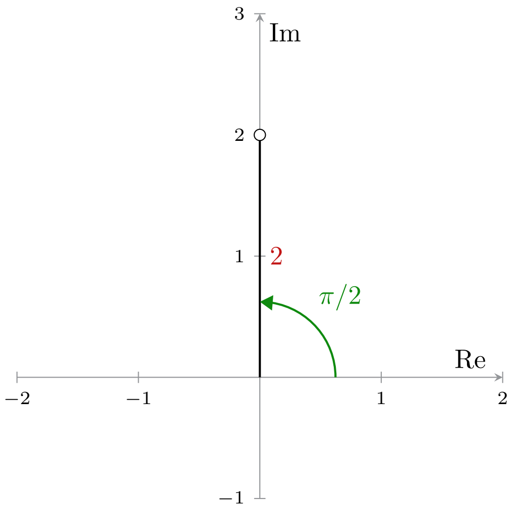
2. Plotting on the complex plane,

$z = -4 + 4\mathrm{i}$

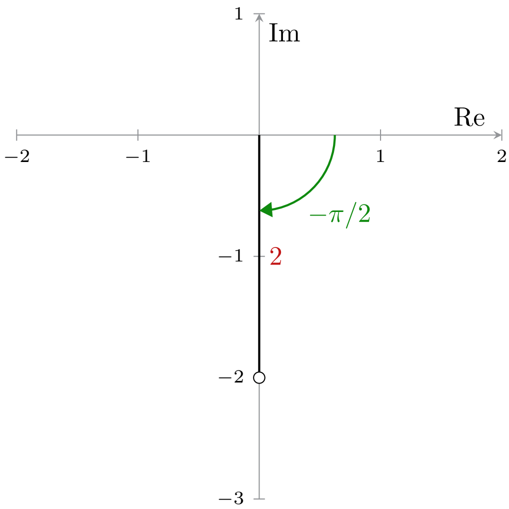


3. Plotting on the complex plane,

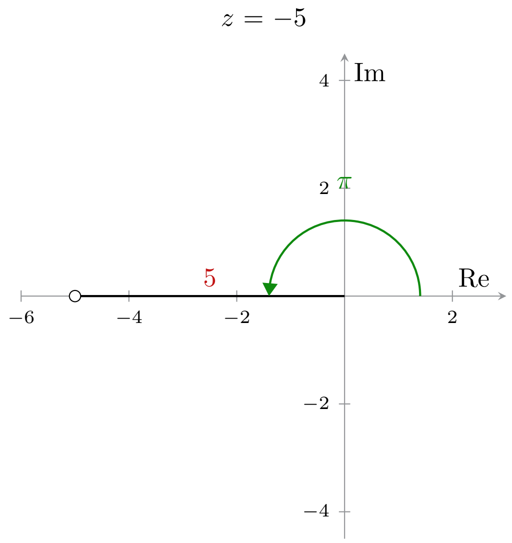
$z = 2\mathrm{i}$



$z = -2\mathrm{i}$

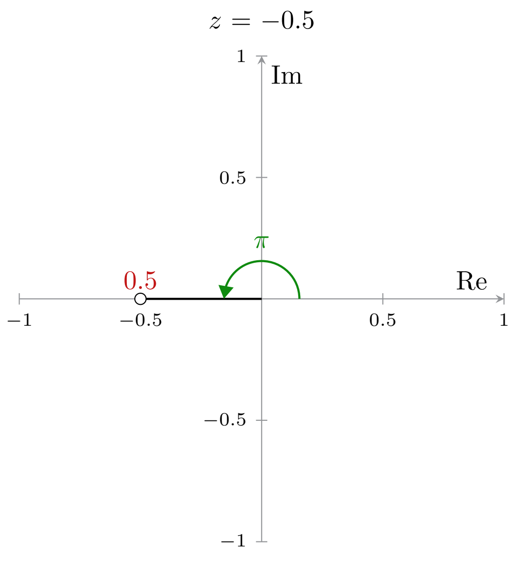


4. Plotting on the complex plane,



5. Simplifying,

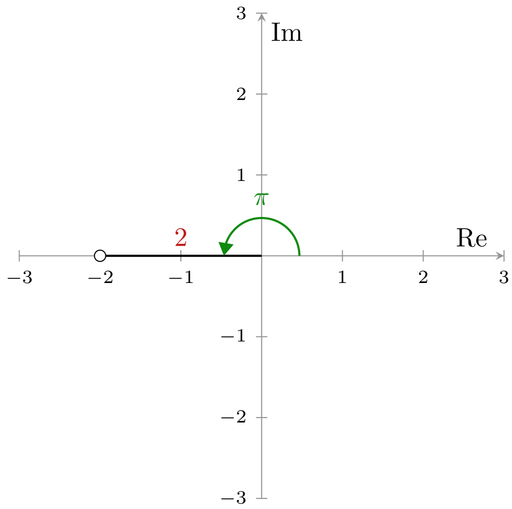
$$z = \frac{\sqrt{2} + \mathrm{i}/3}{-\sqrt{8} - 2\mathrm{i}/3} = \frac{-4 - 2/9}{8 + 4/9} = \frac{-38}{76} = \frac{-1}{2} + 0\ \mathrm{i} \tag{13.2.1}$$



6. Simplifying,

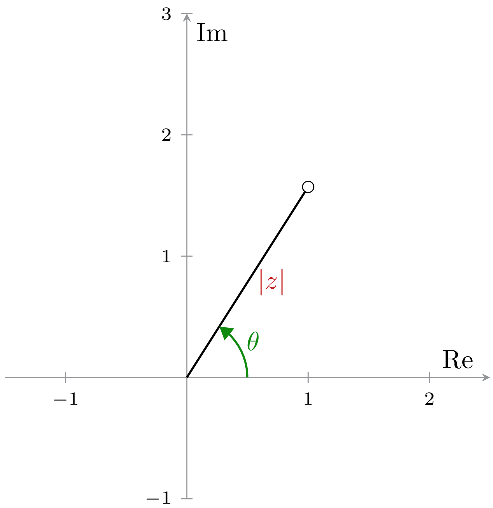
$$z = \frac{\sqrt{3} - 10\mathrm{i}}{-0.5\sqrt{3} + 5\mathrm{i}} = \frac{-51.5}{103/4} = -2 + 0\ \mathrm{i} \tag{13.2.2}$$

$$z = -2$$



7. Plotting on the complex plane, with $\theta = \arctan(\pi/2)$ and $|z| = \sqrt{1 + \pi^2/4}$

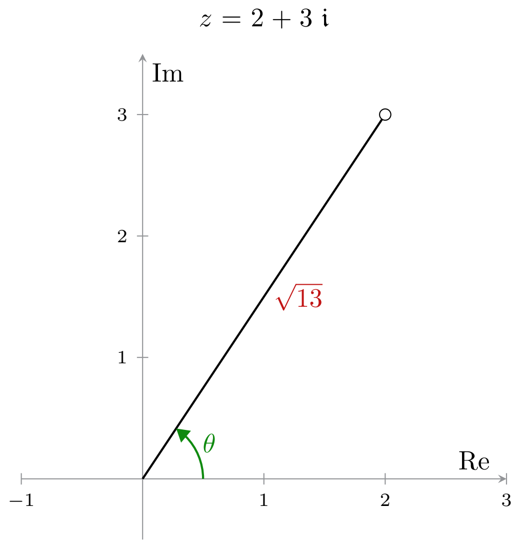
$$z = 1 + \frac{\pi}{2}\,\mathrm{i}$$



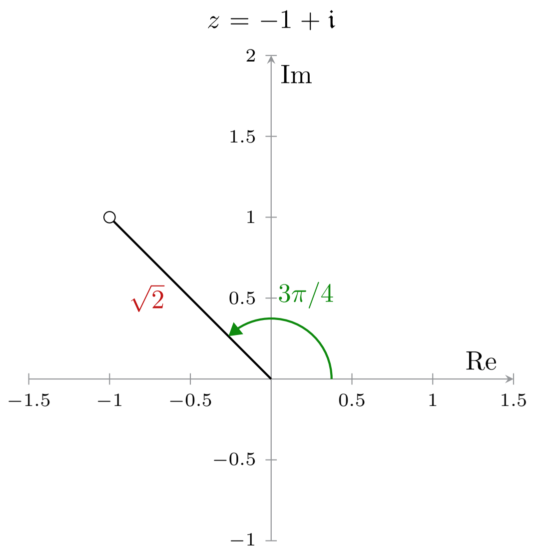
8. Simplifying,

$$z = \frac{-4 + 19\mathrm{i}}{2 + 5\mathrm{i}} = \frac{87 + 58\mathrm{i}}{29} = 3 + 2\mathrm{i} \qquad 13.2.3$$

$$\theta = \arctan(2/3) \qquad 13.2.4$$



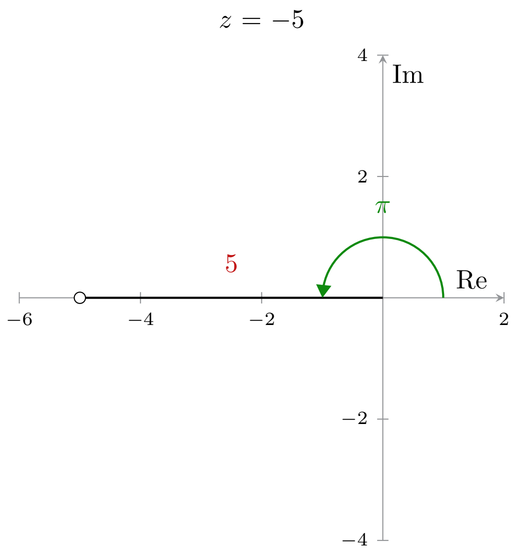
9. Plotting on the complex plane,

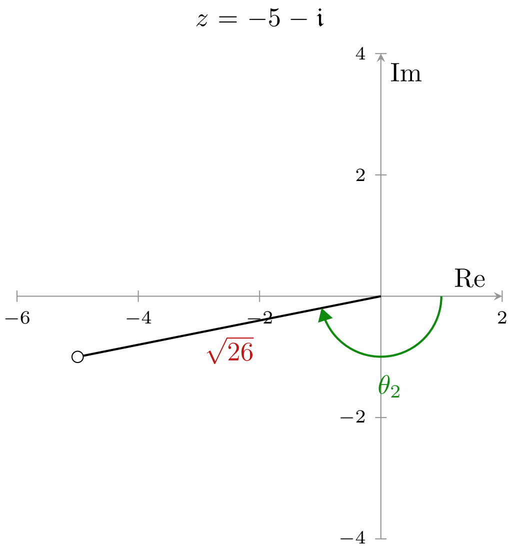


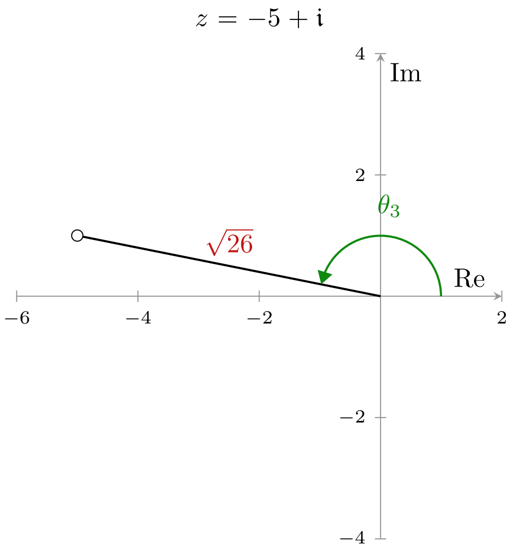
10. Plotting on the complex plane, with

$$\mathrm{Arg}(z_1) = \pi \qquad \mathrm{Arg}(z_2) = -\pi + \arctan(1/5) \tag{13.2.5}$$

$$\mathrm{Arg}(z_3) = \pi + \arctan(-1/5) \tag{13.2.6}$$



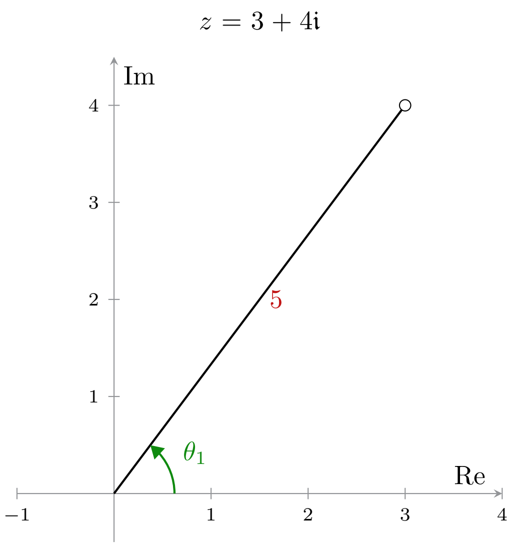


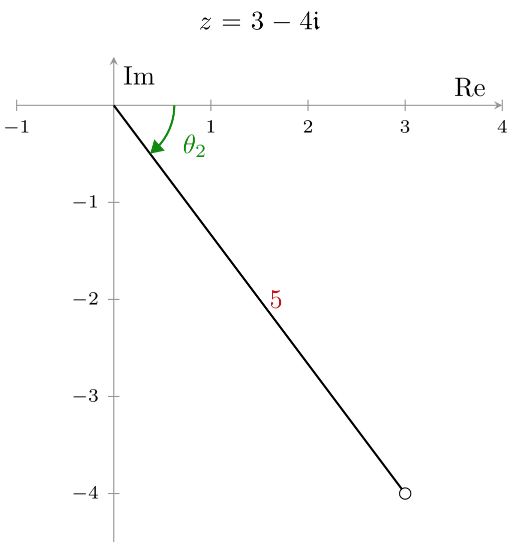


11. Plotting on the complex plane, with

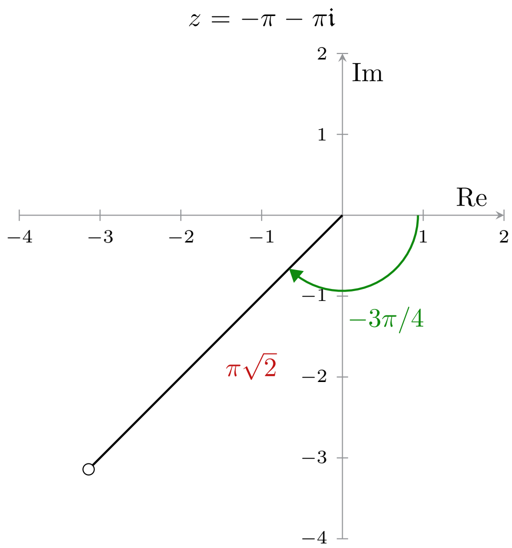
$$\mathrm{Arg}(z_1) = \arctan(4/3) \qquad\qquad \mathrm{Arg}(z_2) = \arctan(-4/3)$$

13.2.7





12. Plotting on the complex plane, with

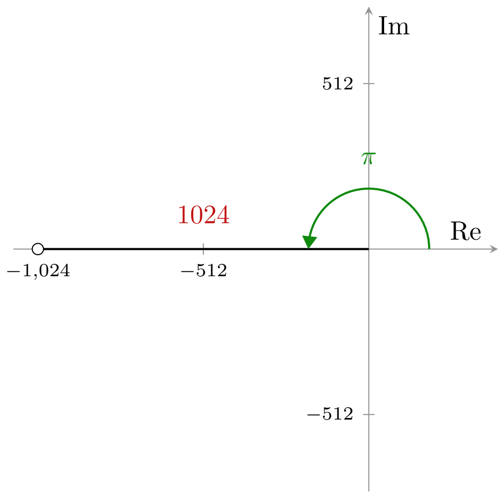


13. Simplifying,

$$w = (1+i) \qquad w^4 = (1+i)^4 = [(1+i)^2]^2 = -4 \tag{13.2.8}$$

$$z = w^{20} \qquad z = (-4)^5 = -1024 + 0\mathrm{i} \tag{13.2.9}$$

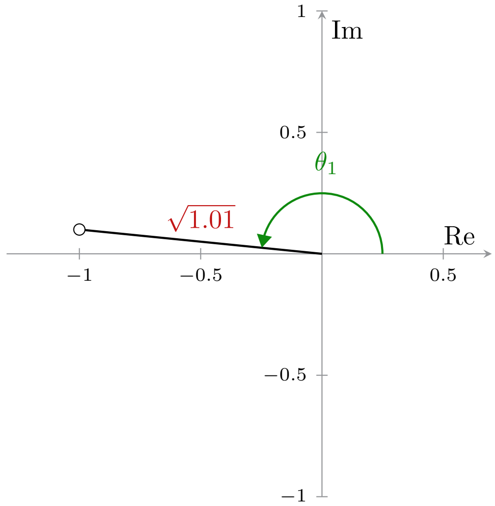
$$z = -1024$$



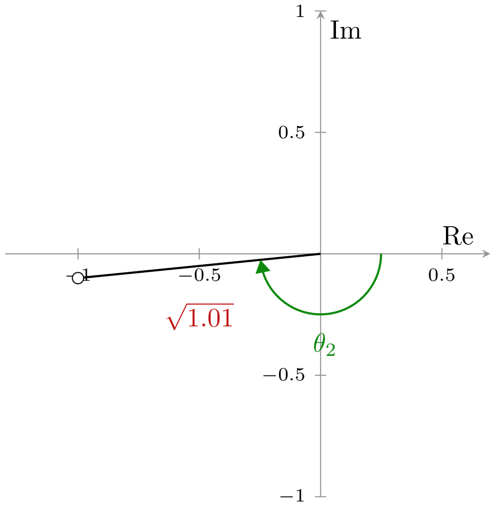
14. Plotting on the complex plane, with

$$\operatorname{Arg}(z_1) = \pi + \arctan(-1/10) \qquad \operatorname{Arg}(z_2) = -\pi + \arctan(1/10)$$ 13.2.10

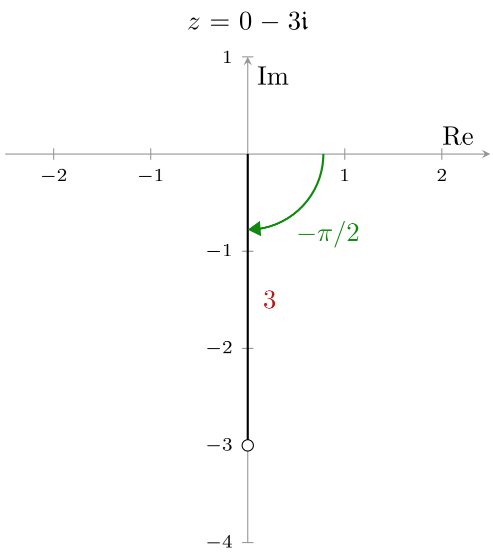
$$z = -1 + 0.1\mathrm{i}$$



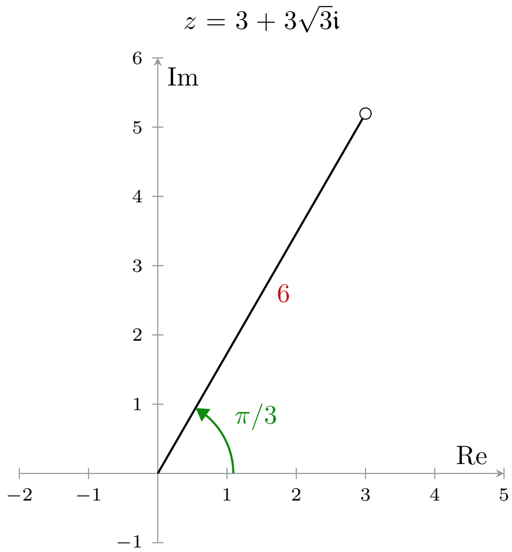
$$z = -1 - 0.1\mathrm{i}$$



15. Using the polar representation, with $r = 3$, $\theta = \pi/2$,

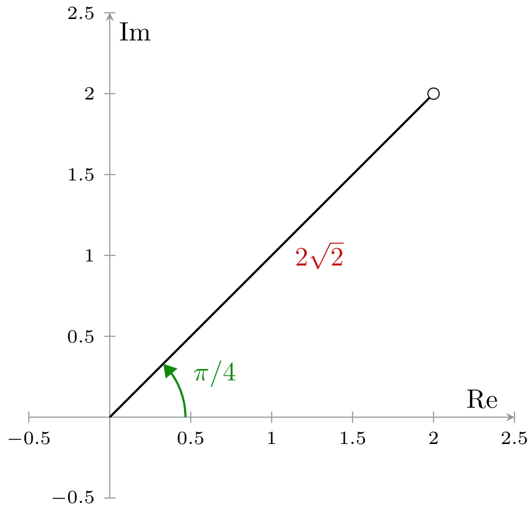


16. Using the polar representation, with $r = 6$, $\theta = \pi/3$,



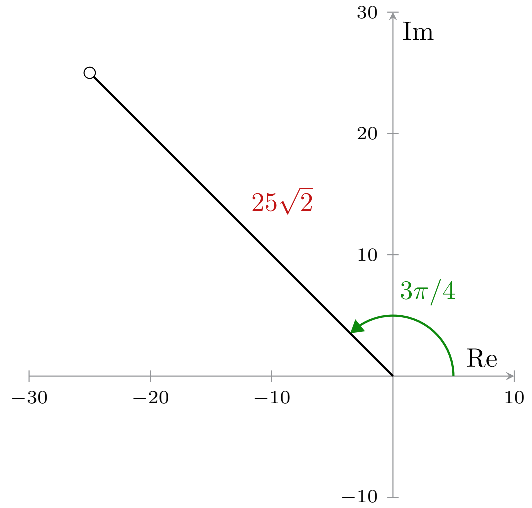
17. Using the polar representation, with $r = \sqrt{8}$, $\theta = \pi/4$,

$$z = 2 + 2\mathrm{i}$$



18. Using the polar representation, with $r = \sqrt{50}$, $\theta = 3\pi/4$,

$$z = -25 + 25\mathrm{i}$$

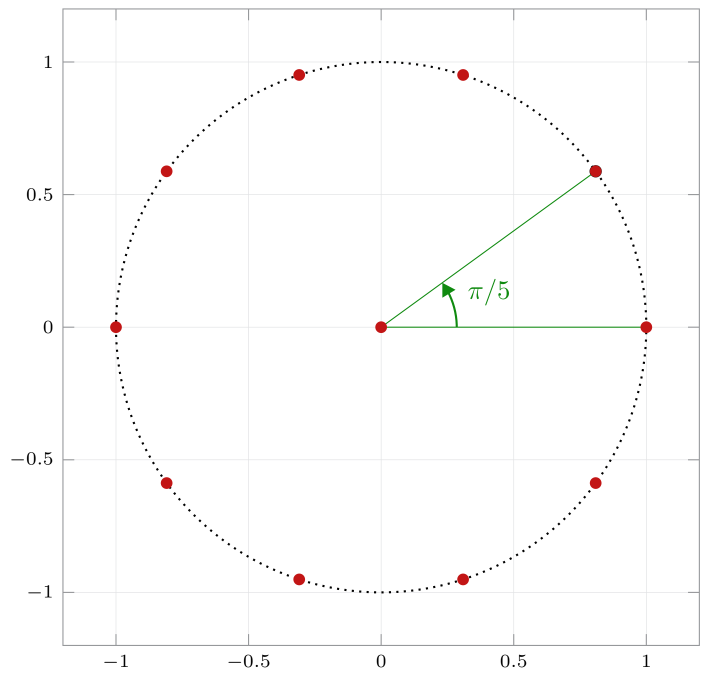


19. Using `numpy`, to find the zeros of -1, for the illustrative case $n = 10$ as illustrative examples.
For a general complex number $z \neq -1$, such that

$$w = z^{1/n} \quad 13.2.11$$

The set of all roots of z is now $\{\omega_k \cdot w\}$. Since the roots have equal magnitude and are evenly oriented around the full 2π angle, this is readily visible in a plot.

$n = 10$



20. Square roots of complex numbers,

(a) To find the n roots of a complex number when $n = 2$,

$$w = \sqrt{z} \qquad\qquad z = r\,[\cos\theta + \mathrm{i}\ \sin\theta] \tag{13.2.12}$$

$$|w_k| = |z|^{1/2} \qquad\qquad = \sqrt{r} \tag{13.2.13}$$

$$\arg\,(w_k) = \frac{\theta + 2k\pi}{2} \qquad\qquad = \frac{\theta}{2} + k\pi \tag{13.2.14}$$

$$w_1 = \sqrt{r}\,[\cos(\theta/2) + \mathrm{i}\ \sin(\theta/2)] \tag{13.2.15}$$

$$w_2 = \sqrt{r}\,[\cos(\pi + \theta/2) + \mathrm{i}\ \sin(\pi + \theta/2)] \tag{13.2.16}$$

Using the properties of trigonometric functions, $w_2 = -w_1$

(b) Using the half angle formulas,

$$\cos^2(\theta/2) = \frac{1 + \cos\theta}{2} \qquad\qquad \sin^2(\theta/2) = \frac{1 - \cos\theta}{2} \tag{13.2.17}$$

$$r\,\cos(\theta/2) = \sqrt{r}\cdot\sqrt{\frac{r + r\cos\theta}{2}} \qquad\qquad r\,\sin(\theta/2) = \sqrt{r}\cdot\sqrt{\frac{r - r\cos\theta}{2}} \tag{13.2.18}$$

Now, using the polar representation of $\sqrt{z}$,

$$w_1 = \sqrt{z} = \sqrt{r}\,\left[\cos(\theta/2) + \mathfrak{i}\ \sin(\theta/2)\right] \tag{13.2.19}$$

$$= \sqrt{\frac{|z| + x}{2}} + \mathfrak{i}\ (\operatorname{sgn} y)\ \sqrt{\frac{|z| - x}{2}} \tag{13.2.20}$$

$$w_2 = -w_1 \tag{13.2.21}$$

The signum comes from the fact that the sign of y determines whether the positive or negative root of $\sin^2\theta$ will be taken in the second step.

(c) Using the new method,

$$z_1 = -14\mathfrak{i} \qquad \sqrt{z_1} = \sqrt{\frac{14+0}{2}} + \mathfrak{i}\ (-1)\cdot\sqrt{\frac{14-0}{2}} \tag{13.2.22}$$

$$\sqrt{z_1} = \sqrt{7} - \sqrt{7}\ \mathfrak{i} \tag{13.2.23}$$

$$z_2 = -9 - 40\mathfrak{i} \qquad \sqrt{z_2} = \sqrt{\frac{41-9}{2}} + \mathfrak{i}\ (-1)\cdot\sqrt{\frac{41+9}{2}} \tag{13.2.24}$$

$$\sqrt{z_2} = 4 - 5\ \mathfrak{i} \tag{13.2.25}$$

$$z_3 = 1 + \sqrt{48}\mathfrak{i} \qquad \sqrt{z_3} = \sqrt{\frac{7+1}{2}} + \mathfrak{i}\ (1)\cdot\sqrt{\frac{7-1}{2}} \tag{13.2.26}$$

$$\sqrt{z_3} = 2 + \sqrt{3}\ \mathfrak{i} \tag{13.2.27}$$

Using the old method,

$$z_1 = -14\mathfrak{i} \qquad \sqrt{z_1} = \sqrt{14}\,\Big[\cos(-\pi/4) + \mathfrak{i}\ \sin(-\pi/4)\Big] \tag{13.2.28}$$

$$\sqrt{z_1} = \sqrt{7} - \sqrt{7}\ \mathfrak{i} \tag{13.2.29}$$

For z_2 whose argument is not readily apparent,

$$z_2 = -9 - 40\mathfrak{i} \qquad \tan\theta = 40/9 \tag{13.2.30}$$

$$\frac{40}{9} = \frac{2\tan(\theta/2)}{1 - \tan^2(\theta/2)} \qquad \tan(\theta/2) = \frac{-5}{4} \tag{13.2.31}$$

$$\sqrt{z_2} = \sqrt{41}\left[\frac{4}{\sqrt{41}} - \mathfrak{i}\ \frac{5}{\sqrt{41}}\right] \qquad \sqrt{z_2} = 4 - 5\ \mathfrak{i} \tag{13.2.32}$$

For z_3 whose argument is not readily apparent,

$$z_3 = 1 + \sqrt{48}\mathrm{i} \qquad \tan\theta = \sqrt{48} \tag{13.2.33}$$

$$\sqrt{48} = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} \qquad \tan(\theta/2) = \frac{\sqrt{3}}{2} \tag{13.2.34}$$

$$\sqrt{z_3} = \sqrt{7}\left[\frac{2}{\sqrt{7}} + \mathrm{i}\ \frac{\sqrt{3}}{\sqrt{7}}\right] \qquad \sqrt{z_3} = 2 + \sqrt{3}\ \mathrm{i} \tag{13.2.35}$$

The newer method is clearly better.

(d) TBC. Both methods are coded in `sympy`.

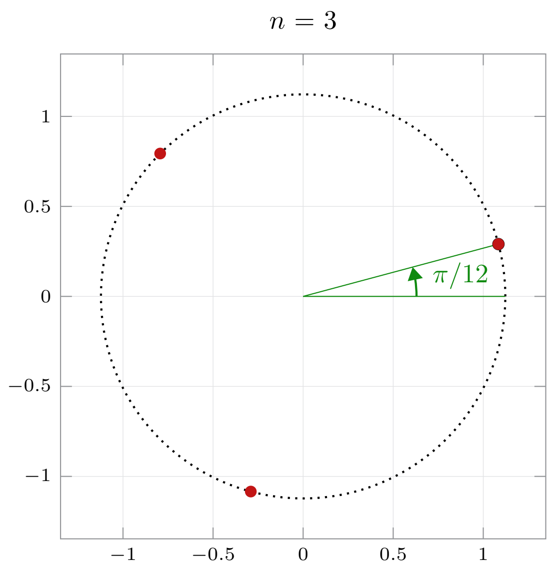
21. Finding the roots of the complex number,

$$w_1 = (1+i)^{1/3} \tag{13.2.36}$$

$$w_1 = 2^{1/6}\ [\cos(\pi/12) + \mathrm{i}\ \sin(\pi/12)] \tag{13.2.37}$$

$$w_2 = \omega \cdot w_1 = 2^{1/6}\ [\cos(5\pi/12) + \mathrm{i}\ \sin(5\pi/12)] \tag{13.2.38}$$

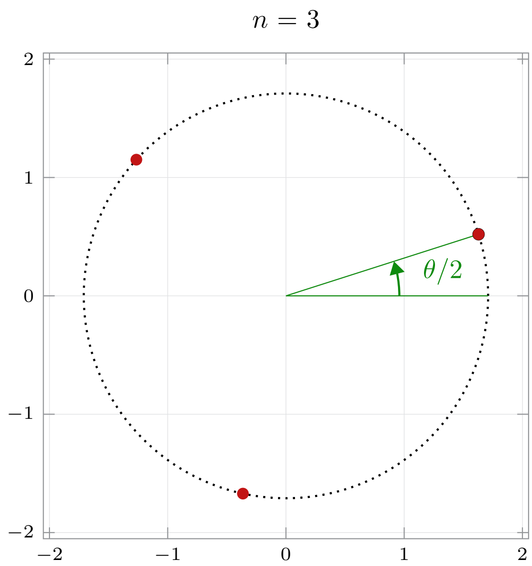
$$w_3 = \omega^2 \cdot w_1 = 2^{1/6}\ [\cos(9\pi/12) + \mathrm{i}\ \sin(9\pi/12)] \tag{13.2.39}$$



22. Finding the roots of the complex number,

$$w_1 = (3+4i)^{1/3} \qquad \tan(\theta) = \frac{4}{3} \tag{13.2.40}$$

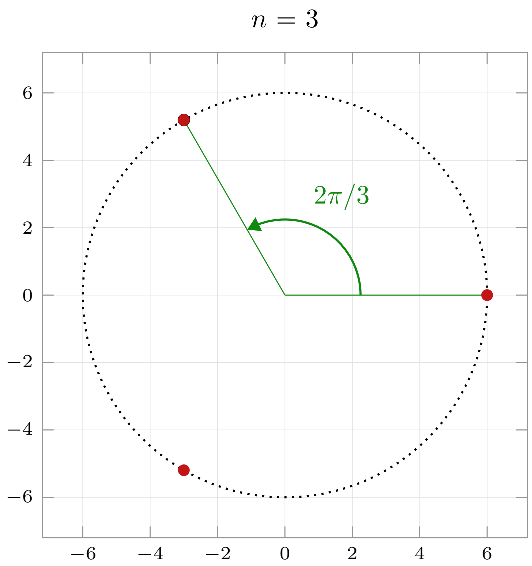
$$w_1 = 5^{1/6}\left[\frac{2}{\sqrt{5}} + \mathrm{i}\ \frac{1}{\sqrt{5}}\right] \qquad \tan(\theta/2) = \frac{1}{2} \tag{13.2.41}$$



23. Finding the roots of the complex number,

$$w_1 = (216)^{1/3} \qquad \tan(\theta) = 0 \tag{13.2.42}$$

$$w_1 = 6 + 0\,\mathrm{i} \qquad \tan(\theta/2) = 0 \tag{13.2.43}$$

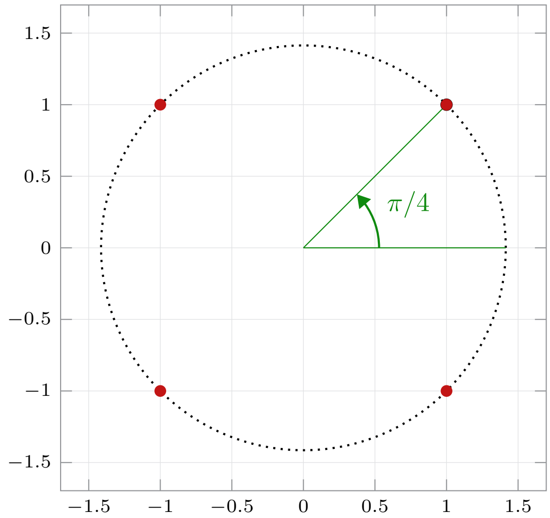


24. Finding the roots of the complex number,

$$w_1 = (-4)^{1/4} \qquad \tan(\theta) = 0 \tag{13.2.44}$$

$$w_1 = \sqrt{2}\,[\cos(\pi/4) + \mathrm{i}\ \sin(\pi/4)] \qquad w_1 = 1 + \mathrm{i} \tag{13.2.45}$$

$n = 4$

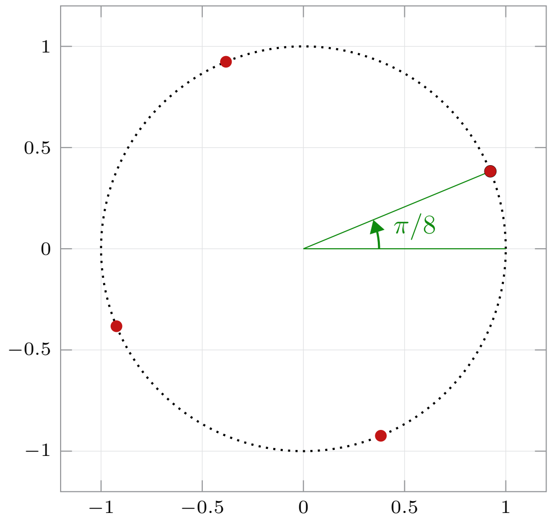


25. Finding the roots of the complex number,

$$w_1 = (\mathrm{i})^{1/4} \qquad \tan(\theta) = \frac{\pi}{2} \tag{13.2.46}$$

$$w_1 = \cos(\pi/8) + \mathrm{i}\ \sin(\pi/8) \qquad w_1 = 1 + \mathrm{i} \tag{13.2.47}$$

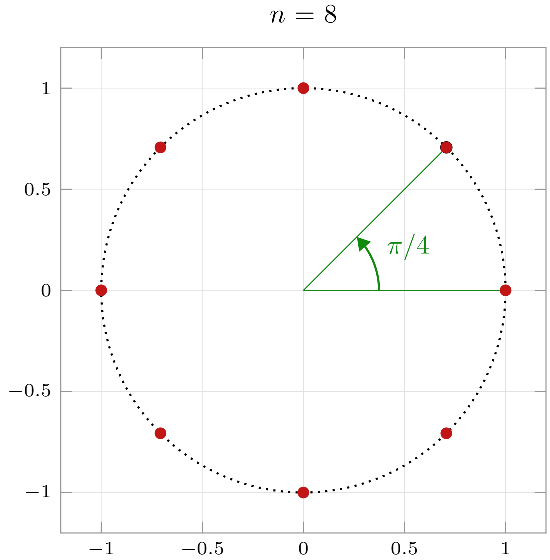
$n = 4$



26. Finding the roots of the complex number,

$$w_1 = (1)^{1/8} \qquad \tan(\theta) = 0 \tag{13.2.48}$$

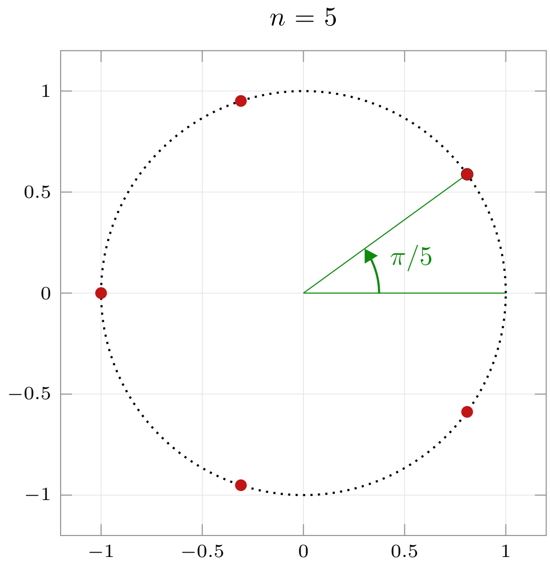
$$w_1 = \cos(\pi/4) + \mathrm{i}\ \sin(\pi/4) \qquad w_1 = \frac{1}{\sqrt{2}}\,[1 + \mathrm{i}] \tag{13.2.49}$$



27. Finding the roots of the complex number,

$$w_1 = (-1)^{1/5} \qquad \theta = \pi \tag{13.2.50}$$

$$w_1 = \cos(\pi/5) + \mathrm{i}\ \sin(\pi/5) \tag{13.2.51}$$



28. Solving the equation,

$$0 = z^2 - (6 - 2\mathrm{i})\ z + (17 - 6\mathrm{i}) \qquad D = 32 - 24\mathrm{i} - 68 + 24\mathrm{i} = -36 \tag{13.2.52}$$

$$\sqrt{D} = \pm 6\mathrm{i} \qquad z = \frac{6 - 2\mathrm{i} \pm 6\mathrm{i}}{2} \tag{13.2.53}$$

$$z = 3 - 4\mathrm{i},\ \ 3 + 2\mathrm{i} \tag{13.2.54}$$

29. Solving the equation,

$$0 = z^2 + z + (1 - \mathrm{i}) \qquad D = 1 - 4 + 4\mathrm{i} = -3 + 4\mathrm{i} \tag{13.2.55}$$

$$\sqrt{D} = 1 + 2\ \mathrm{i},\ \ -1 - 2\ \mathrm{i} \qquad z = \mathrm{i},\ \ -1 - \mathrm{i} \tag{13.2.56}$$

30. Solving the equation,

$$0 = z^4 + 324 \qquad D = -1296 \tag{13.2.57}$$

$$\sqrt{D} = \pm 36\ \mathrm{i} \qquad z^2 = \pm 18\ \mathrm{i} \tag{13.2.58}$$

$$(36\mathrm{i})^{1/2} = 3\ [1 + \mathrm{i}],\ \ 3\ [-1 - \mathrm{i}] \qquad (-36\mathrm{i})^{1/2} = 3\ [1 - \mathrm{i}],\ \ 3\ [-1 + \mathrm{i}] \tag{13.2.59}$$

$$z^4 + 324 = (z^2 - 6 + 18) \cdot (z^2 + 6 + 18) \tag{13.2.60}$$

The last step uses pairs of roots which are complex conjugates, which is a necessary condition for quadratic polynomials with real coefficients.

31. Solving the equation, with $w = z^2$

$$0 = w^2 - 6\mathrm{i}\ w + 16 \qquad D = -36 - 64 = -100 \tag{13.2.61}$$

$$\sqrt{D} = \pm 10\ \mathrm{i} \qquad w = 3\mathrm{i} \pm 5\mathrm{i} = -2\mathrm{i},\ \ 8\mathrm{i} \tag{13.2.62}$$

$$\sqrt{w_1} = \sqrt{2}\ [1 - \mathrm{i}],\ \ \sqrt{2}\ [-1 + \mathrm{i}] \qquad \sqrt{w_2} = 2\ [1 + \mathrm{i}],\ \ 2\ [-1 - \mathrm{i}] \tag{13.2.63}$$

32. Verifying the triangle inequality,

$$|z_1| = |3 + \mathrm{i}| = \sqrt{10} \qquad |z_2| = |-2 + 4\mathrm{i}| = \sqrt{20} \tag{13.2.64}$$

$$z_1 + z_2 = 1 + 5\mathrm{i} \qquad |z_1 + z_2| = \sqrt{26} \tag{13.2.65}$$

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{13.2.66}$$

33. Proving the triangle inequality,

$$|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z_1} + \bar{z_2}) = z_1\bar{z_1} + z_2\bar{z_2} + z_1\bar{z_2} + \bar{z_1}z_2 \tag{13.2.67}$$

$$= |z_1|^2 + |z_2|^2 + 2 \cdot \mathrm{Re}\ (z_1\bar{z_2}) \tag{13.2.68}$$

$$\le |z_1|^2 + |z_2|^2 + 2\ |z_1\ \ \bar{z_2}| \tag{13.2.69}$$

$$\le |z_1|^2 + |z_2|^2 + 2\ |z_1|\ \ |z_2| \tag{13.2.70}$$

Taking the positive square root of both sides proves the inequality. The real part of a complex number is not larger than its modulus. The modulus of a complex number is equal to its conjugate.

34. Starting with the properties of sine and cosine functions,

$$\text{Re } z = x = |z| \ \cos\theta \qquad \text{Im } z = y = |z| \ \sin\theta \tag{13.2.71}$$

$$\cos\theta \leq 1 \implies x \leq |z| \tag{13.2.72}$$

$$\sin\theta \leq 1 \implies y \leq |z| \tag{13.2.73}$$

35. The name comes from the fact that the left side is each of the diagonals of a parallellogram and the right side is each of its adjacent sides.
Using the fact that $|w|^2 = w\bar{w}$,

$$|z_1 + z_2|^2 = z_1\bar{z_1} + z_2\bar{z_2} + z_1\bar{z_2} + \bar{z_1}z_2 \tag{13.2.74}$$

$$= |z_1|^2 + |z_2|^2 + 2 \text{ Re } (z_1\bar{z_2}) \tag{13.2.75}$$

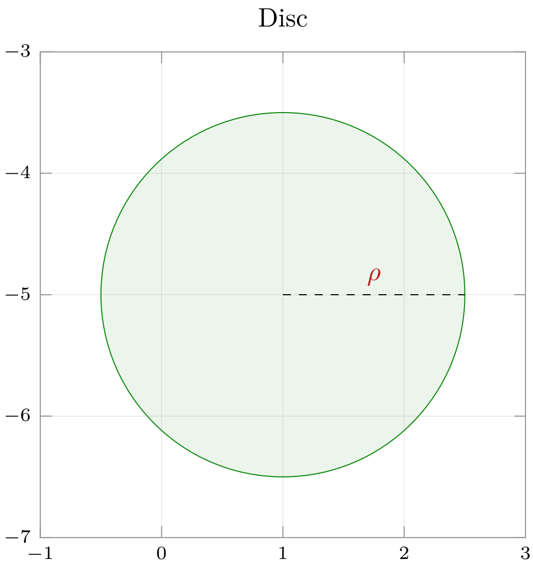
$$|z_1 - z_2|^2 = z_1\bar{z_1} + z_2\bar{z_2} - z_1\bar{z_2} - \bar{z_1}z_2 \tag{13.2.76}$$

$$= |z_1|^2 + |z_2|^2 - 2 \text{ Re } (z_1\bar{z_2}) \tag{13.2.77}$$

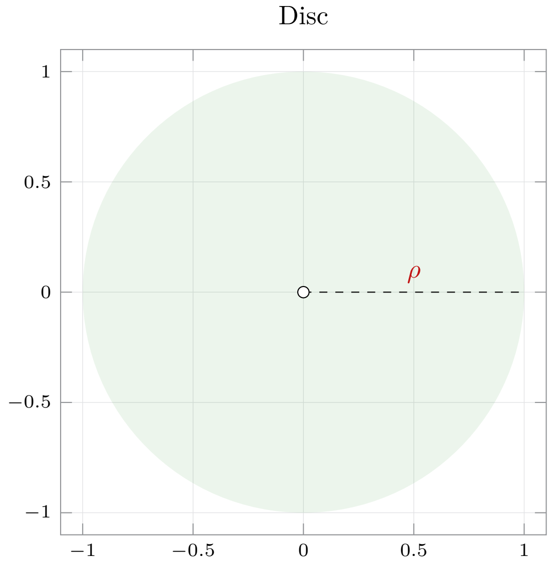
$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2 \, |z_1|^2 + 2 \, |z_2|^2 \tag{13.2.78}$$

13.3 Derivative, Analytic Function

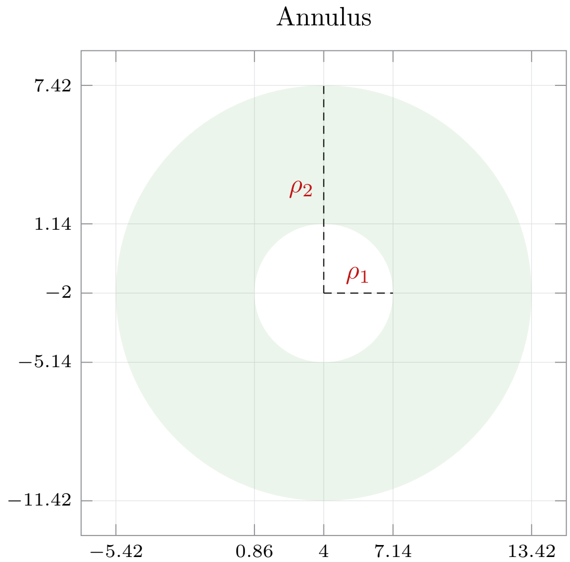
1. Plotting the region in the complex plane, with center of circle $z_0 = 1 - 5i$ and radius of circle $\rho = 1.5$



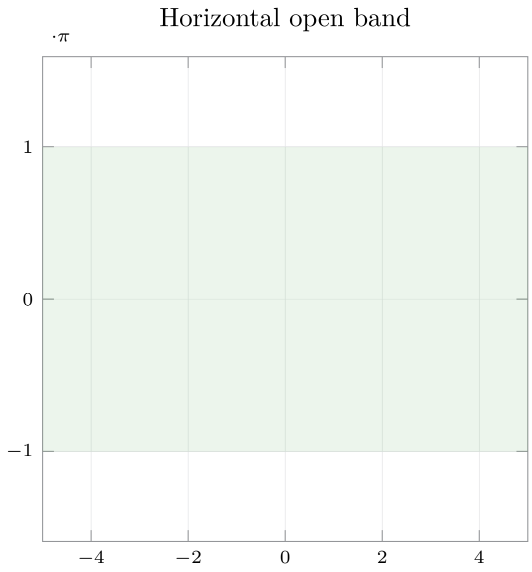
2. Plotting the region in the complex plane, with center of circle $z_0 = 0$ and radius of circle 1



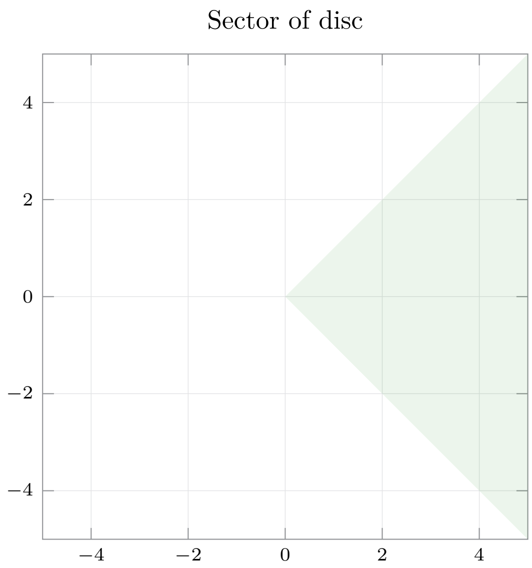
3. Plotting the region in the complex plane, with center of circle $z_0 = 4 - 2\mathrm{i}$ and radius of circle in the range $(\pi, 3\pi)$



4. Plotting the region in the complex plane,



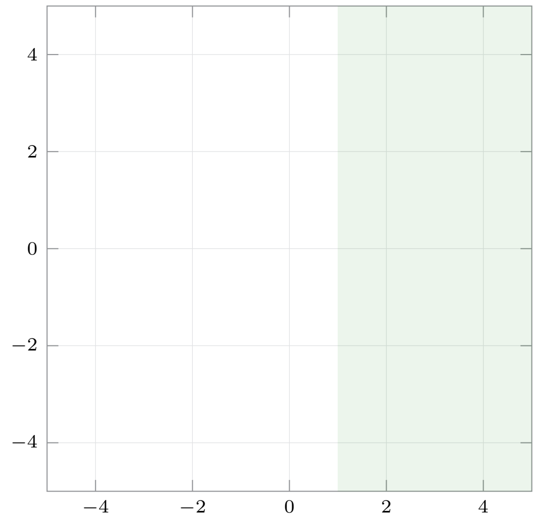
5. Plotting the region in the complex plane,



6. Plotting the region in the complex plane,

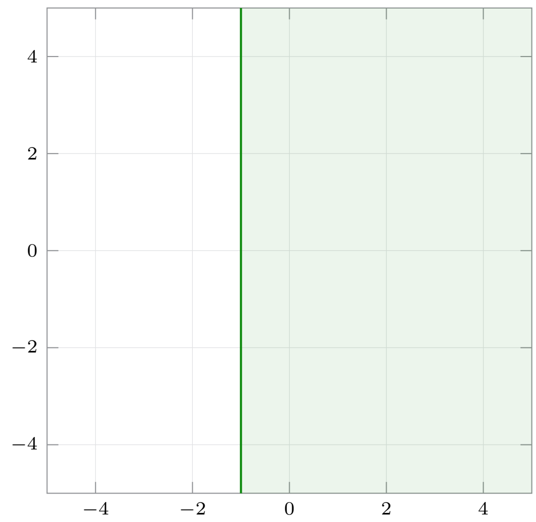
$$\operatorname{Re}\left(\frac{1}{z}\right) < 1 \qquad\qquad \operatorname{Re}\left(z\right) > 1$$ 13.3.1

Vertical band of exclusion



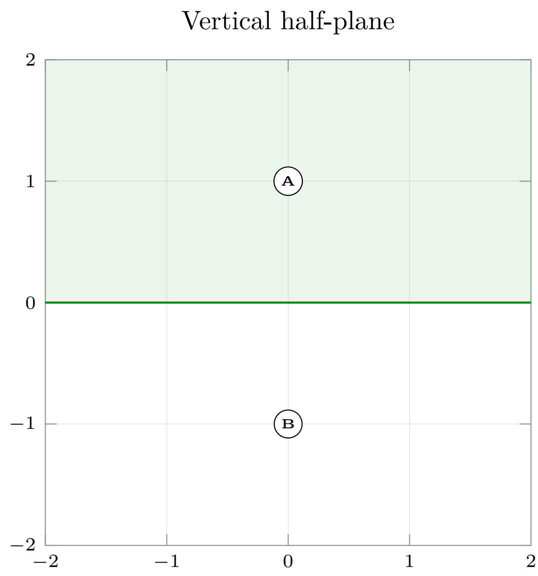
7. Plotting the region in the complex plane,

Off center horizontal half-plane



8. The distance of z from $A = -\mathfrak{i}$ is not less than its distance from $B = \mathfrak{i}$. The perpendicular bisector of the line joining these two points happens to be the real axis.

The region is simply $\text{Im}\,(z) \geq 0$



9. Refer notes. Use Problems 1 − 8 as examples.

10. Finding the real and imaginary parts of the function.

$$f(z) = u(x, y) + \mathrm{i}\, v(x, y) \qquad f(z) = 5z^2 - 12z + (3 + 2\mathrm{i}) \tag{13.3.2}$$

$$u = 5(x^2 - y^2) - 12x + 3 \qquad v = 10xy - 12y + 2 \tag{13.3.3}$$

$$P = 4 - 3\,\mathrm{i} \qquad f(P) = 10 - 82\,\mathrm{i} \tag{13.3.4}$$

11. Finding the real and imaginary parts of the function.

$$f(z) = u(x, y) + \mathrm{i}\, v(x, y) \qquad f(z) = \frac{1}{1 - z} \tag{13.3.5}$$

$$f(z) = \frac{1}{(1 - x) - y\,\mathrm{i}} \qquad f(z) = \frac{(1 - x) + y\,\mathrm{i}}{(1 - x)^2 + y^2} \tag{13.3.6}$$

$$u = \frac{1 - x}{(1 - x)^2 + y^2} \qquad v = \frac{y}{(1 - x)^2 + y^2} \tag{13.3.7}$$

$$P = 1 - \mathrm{i} \qquad f(P) = 0 - 1\,\mathrm{i} \tag{13.3.8}$$

12. Finding the real and imaginary parts of the function.

$$f(z) = u(x, y) + v(x, y)\,\mathrm{i} \qquad f(z) = \frac{z - 2}{z + 2} \tag{13.3.9}$$

$$f(z) = \frac{(x - 2) + y\,\mathrm{i}}{(x + 2) + y\,\mathrm{i}} \qquad f(z) = \frac{(x^2 + y^2 - 4) + (4y)\,\mathrm{i}}{(x + 2)^2 + y^2} \tag{13.3.10}$$

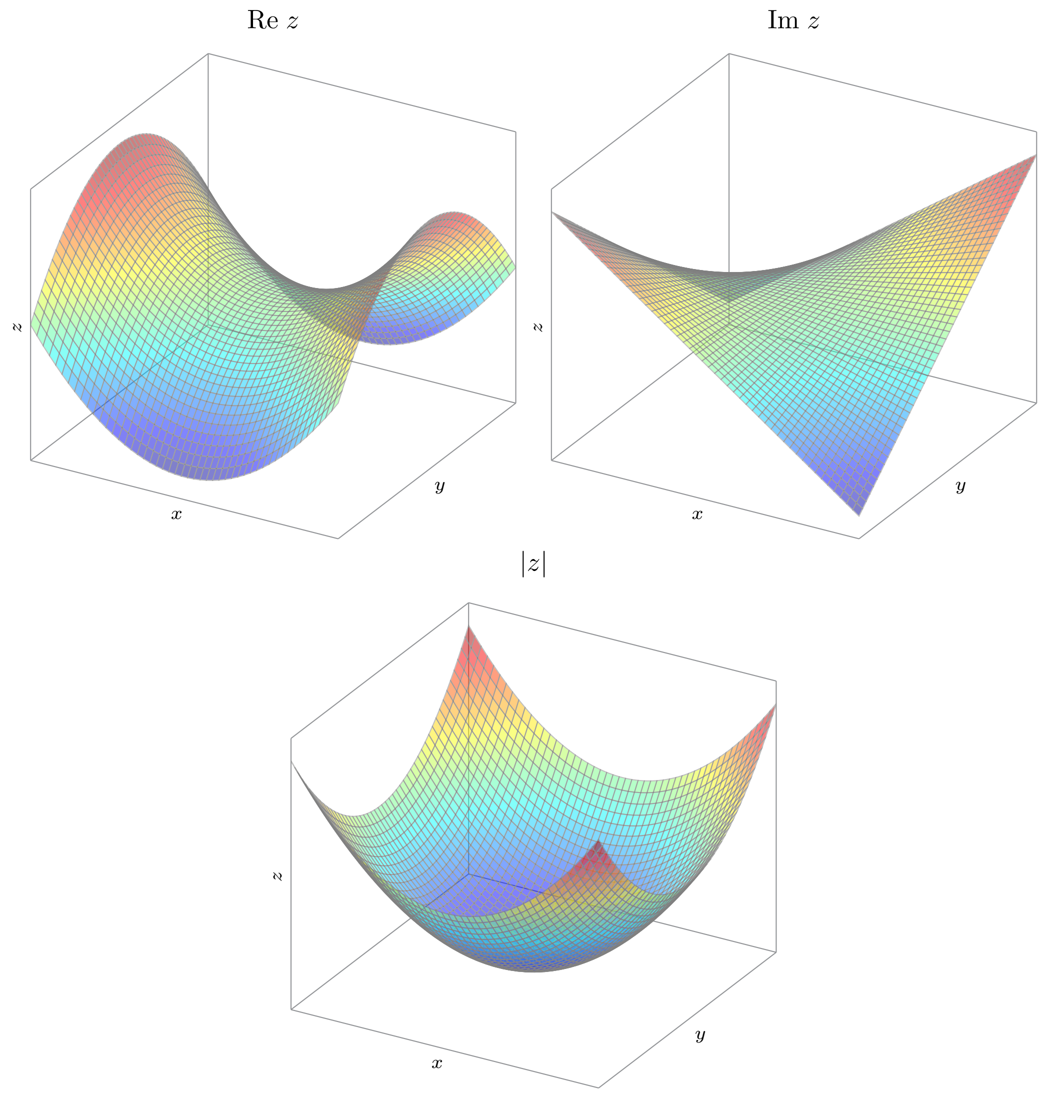
$$u = \frac{x^2 + y^2 - 4}{(x + 2)^2 + y^2} \qquad v = \frac{4y}{(x + 2)^2 + y^2} \tag{13.3.11}$$

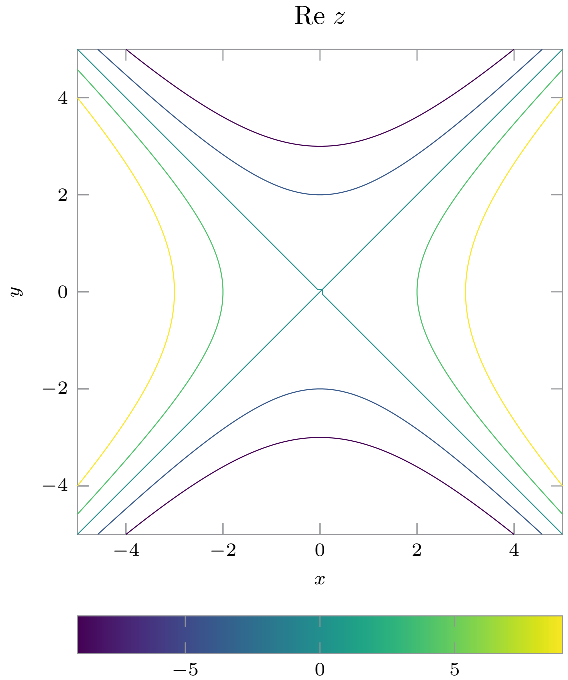
$$P = 8\mathrm{i} \qquad f(P) = 15/17 + 8/17\,\mathrm{i} \tag{13.3.12}$$

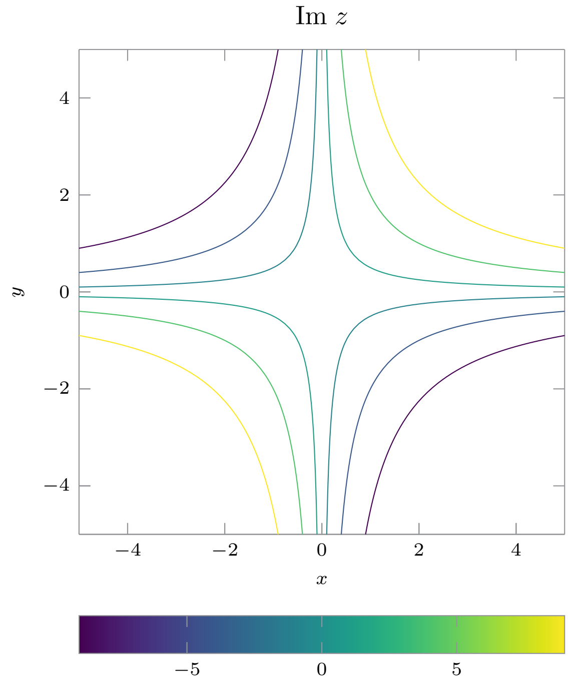
13. Plotting the graphs separately to avoid clutter.

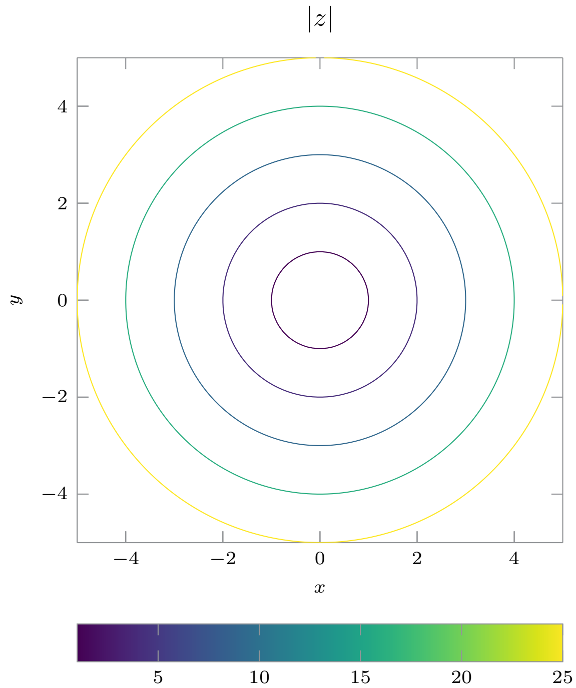
(a) Finding the real and imaginary parts of the function.

$$f(z) = u(x, y) + v(x, y)\ \mathrm{i} \qquad\qquad f(z) = z^2 = (x^2 - y^2) + (2xy)\ \mathrm{i} \tag{13.3.13}$$





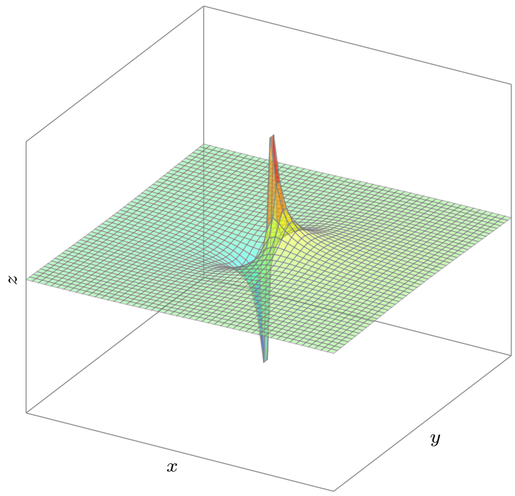




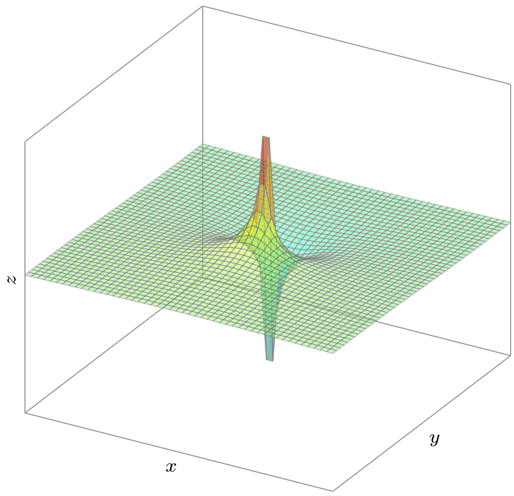
(b) Finding the real and imaginary parts of the function.

$$f(z) = u(x, y) + v(x, y)\,\mathfrak{i} \qquad\qquad f(z) = \frac{1}{z} = \frac{x - \mathfrak{i}\,y}{x^2 + y^2} \tag{13.3.14}$$

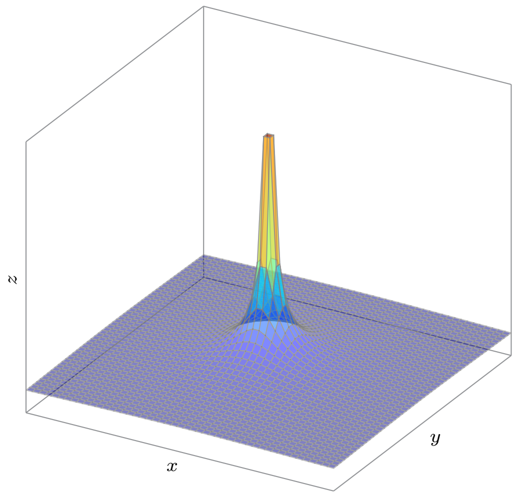
$\mathrm{Re}\, z$

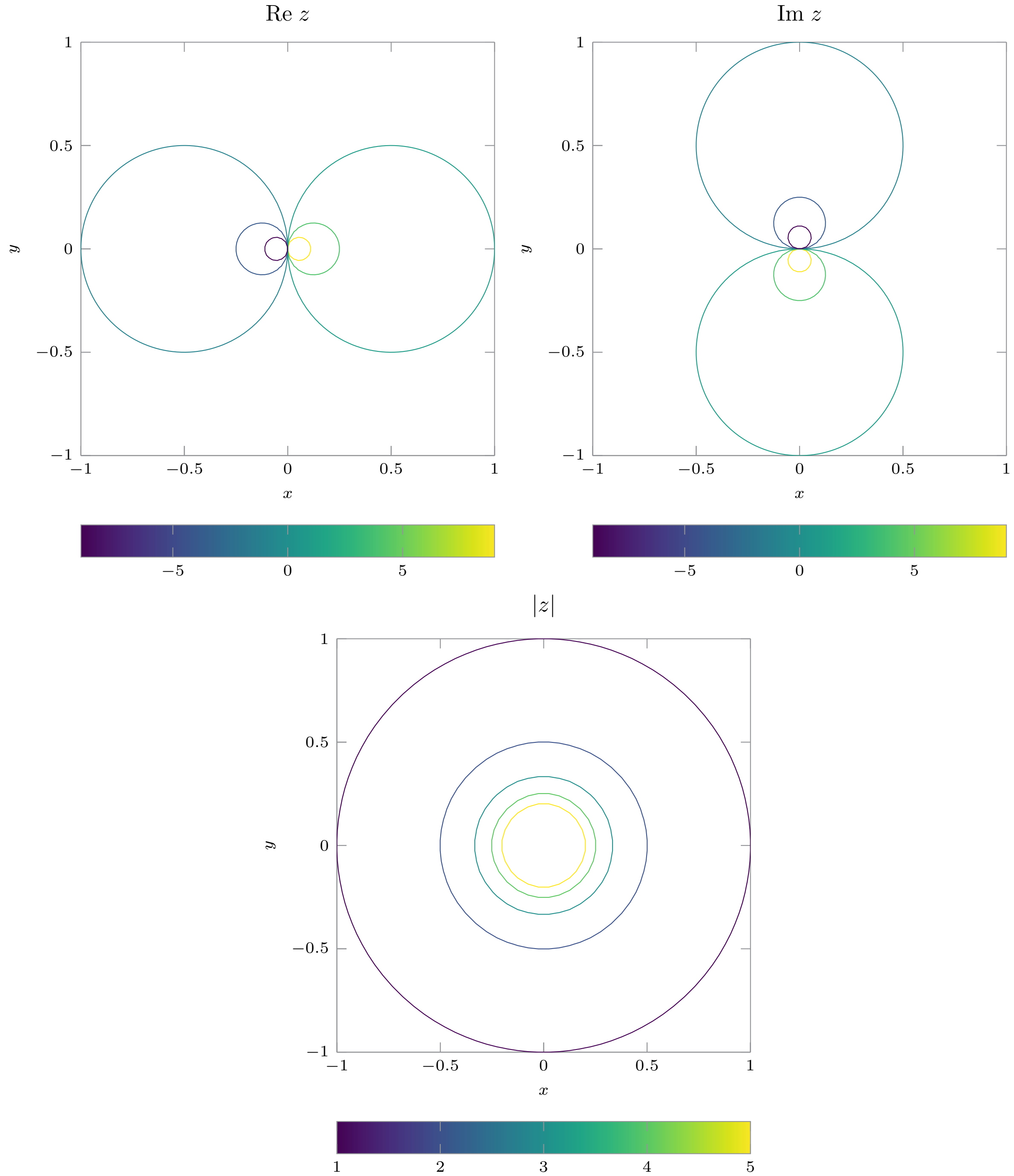


$\mathrm{Im}\, z$



$|z|$

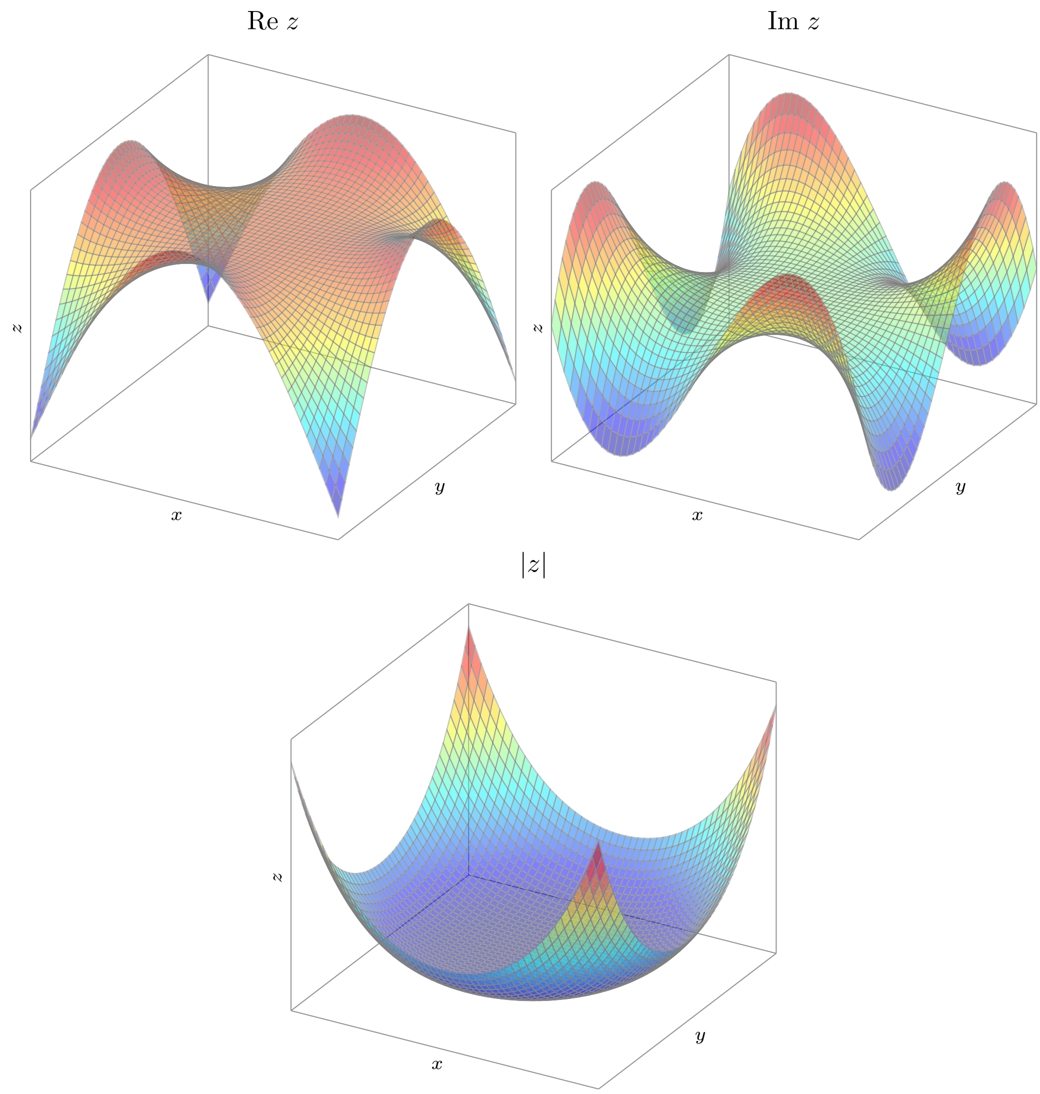


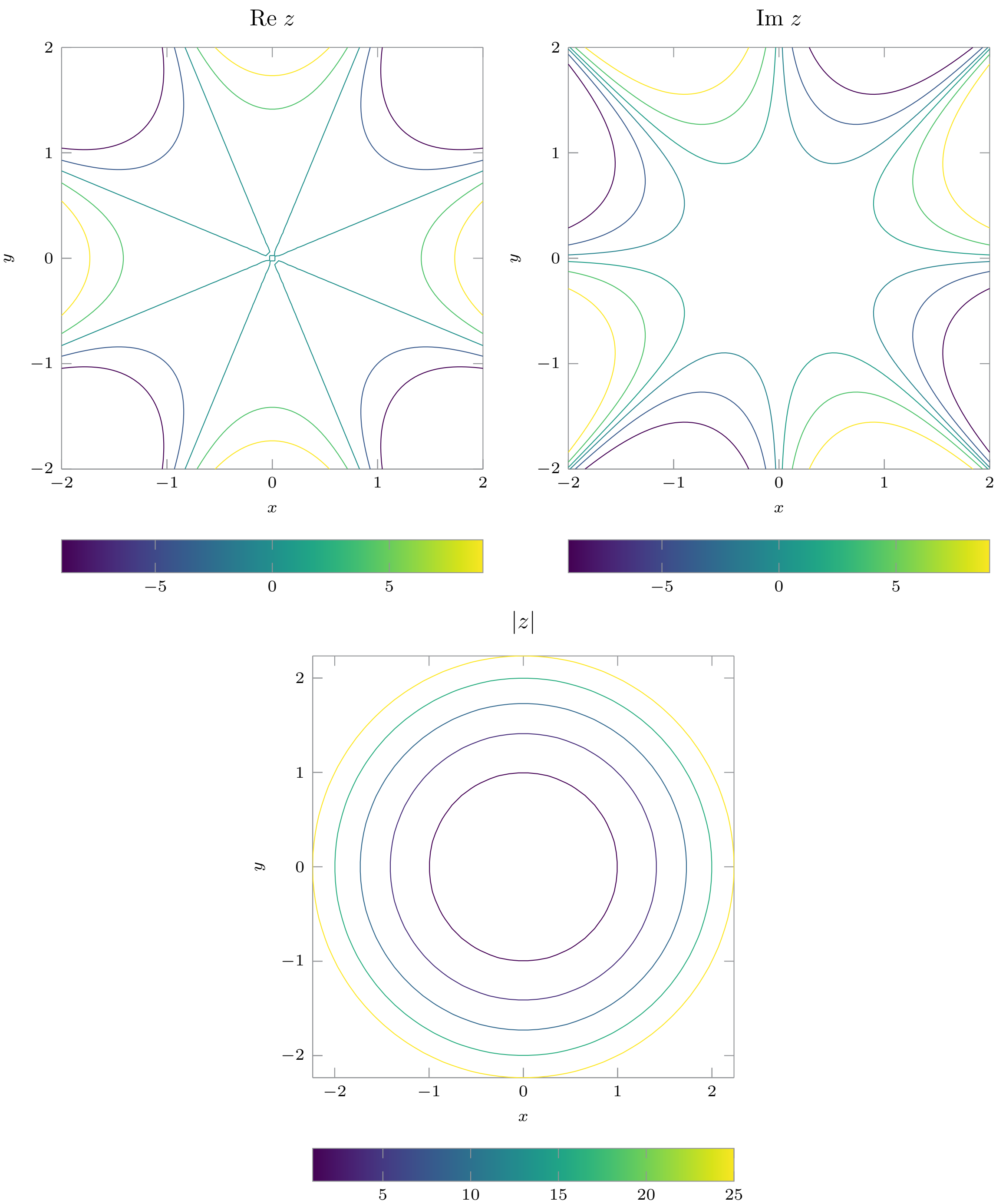


(c) Finding the real and imaginary parts of the function.

$$f(z) = u(x, y) + v(x, y)\,\mathfrak{i} \tag{13.3.15}$$

$$f(z) = z^4 = (x^4 + y^4 - 6x^2y^2) + (4xy)(x^2 - y^2)\,\mathfrak{i} \tag{13.3.16}$$


Re z
Im z
$|z|$
x
y
z



14. Testing the continuity at the origin,

$$f(z) = \frac{\text{Re}\,(z^2)}{|z|} = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \qquad y = mx \tag{13.3.17}$$

$$f(z) = \frac{1 - m^2}{\sqrt{1 + m^2}}\, x \qquad \lim_{x \to 0} f(z) = 0 \quad \forall \quad m \in \mathcal{R} \tag{13.3.18}$$

Thus, the function is continuous.

15. Testing the continuity at the origin,

$$f(z) = |z|^2 \cdot \text{Im}\,(1/z) \qquad f(z) = -y \tag{13.3.19}$$

$$\lim_{(x,y)\to(0,0)} f(z) = 0 \quad \forall \quad m \ \in \mathcal{R} \tag{13.3.20}$$

Thus, the function is continuous.

16. Testing the continuity at the origin,

$$f(z) = \frac{\text{Im}\,(z^2)}{|z|^2} \qquad f(z) = \frac{2xy}{x^2 + y^2} \tag{13.3.21}$$

$$y = mx \qquad \implies \qquad f(z) = \frac{2m}{(1 + m^2)} \tag{13.3.22}$$

$$\lim_{(x,y)\to(0,0)} f(z) = \frac{2m}{1 + m^2} \tag{13.3.23}$$

Thus, the limit at the origin depends on the direction of approach, making $f(z)$ discontinuous.

17. Testing the continuity at the origin,

$$f(z) = \frac{\text{Re}\,(z)}{1 - |z|} \qquad f(z) = \frac{x}{1 - \sqrt{x^2 + y^2}} \tag{13.3.24}$$

$$\lim_{(x,y)\to(0,0)} f(z) = \frac{\to 0}{\to 1} = 0 \tag{13.3.25}$$

Thus, the function is continuous.

18. To find the derivative of the complex function,

$$f(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}} \qquad f'(z) = \frac{(z + \mathrm{i}) - (z - \mathrm{i})}{(z + \mathrm{i})^2} \tag{13.3.26}$$

$$f'(z) = \frac{2\mathrm{i}}{(z + \mathrm{i})^2} \qquad f'(\mathrm{i}) = \frac{-\mathrm{i}}{2} \tag{13.3.27}$$

19. To find the derivative of the complex function,

$$f(z) = (z - 4\mathrm{i})^8 \qquad f'(z) = 8\ (z - 4\mathrm{i})^7 \tag{13.3.28}$$

$$f'(3 + 4\mathrm{i}) = 17496 \tag{13.3.29}$$

20. To find the derivative of the complex function,

$$f(z) = \frac{1.5z + 2\mathrm{i}}{3\mathrm{i}\ z - 4} \qquad f'(z) = \frac{4.5\mathrm{i}\ z - 6 - 4.5\mathrm{i}\ z + 6}{(3\mathrm{i}\ z - 4)^2} = 0 \tag{13.3.30}$$

This is because $f(z)$ simplifies to $(1/2\mathfrak{i})$ and is a constant.

21. To find the derivative of the complex function, Typo in question

$$f(z) = \mathfrak{i}\ (1-z)^{-n} \qquad f'(z) = n\mathfrak{i}\ (1-z)^{-n-1} \tag{13.3.31}$$

$$P = 0 \qquad f'(P) = n\mathfrak{i} \tag{13.3.32}$$

22. To find the derivative of the complex function,

$$f(z) = (\mathfrak{i}z^3 + 3z^2)^3 \qquad f'(z) = 3\ (\mathfrak{i}z^3 + 3z^2)^2 \cdot (3\mathfrak{i}\ z)(z - 2\mathfrak{i}) \tag{13.3.33}$$

$$P = 2\mathfrak{i} \qquad f'(P) = 0 \tag{13.3.34}$$

23. To find the derivative of the complex function,

$$f(z) = \frac{z^3}{(z+\mathfrak{i})^3} \qquad f'(z) = \frac{3z^2\ \mathfrak{i}}{(z+\mathfrak{i})^4} \tag{13.3.35}$$

$$P = \mathfrak{i} \qquad f'(P) = \frac{-3}{16}\ \mathfrak{i} \tag{13.3.36}$$

24. Team Project

(a) Using the linearity of the limit operator,

$$f(z) = \text{Re}\ f(z) + \mathfrak{i}\ \ \text{Im}\ f(z) \qquad \lim_{z\to z_0} f(z) = l = \text{Re}\ (l) + \mathfrak{i}\ \ \text{Im}\ (l) \tag{13.3.37}$$

$$\lim_{z\to z_0} \text{Re}\ f(z) = \text{Re}\ (l) \qquad \lim_{z\to z_0} \text{Im}\ f(z) = \text{Im}\ (l) \tag{13.3.38}$$

$$\tag{13.3.39}$$

(b) Suppose the limit is not unique. Let the two possible limits be A, B at $z = z_0$

$$|z - z_0| < \delta_1 \qquad \implies |f(z) - A| < \epsilon/2 \tag{13.3.40}$$

$$|z - z_0| < \delta_2 \qquad \implies |f(z) - B| < \epsilon/2 \tag{13.3.41}$$

for some fixed value of ϵ.
Let δ be the smaller among $\delta_1 > 0,\ \ \delta_2 > 0$.

$$|A - B| = |A - f(z) + f(z) - B| \tag{13.3.42}$$

$$\leq |A - f(z)| + |f(z) - B| \tag{13.3.43}$$

$$\leq \epsilon \tag{13.3.44}$$

the distance between the two limits $|A - B|$ is less than ϵ for any choice of positive real epsilon, however small.

This means that $|A - B| = 0$ and the limit is unique, if it exists.

(c) Given a infinite series of complex numbers $\{z_i\}$ such that

$$\lim_{n \to \infty} z_n = a \qquad \lim_{z \to a} f(z) = f(a) \tag{13.3.45}$$

Consider some sufficiently large n such that,

$$|z_n - a| < \delta \qquad \forall \quad n > N \tag{13.3.46}$$

$$\implies \quad |f(z_n) - f(a)| < \epsilon \qquad \implies \lim_{n \to \infty} f(z_n) = f(a) \tag{13.3.47}$$

(d) If $f(z)$ is differentiable at $z = z_0$ then this limit exists.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{13.3.48}$$

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} (z - z_0) \cdot f'(z_0) = 0 \tag{13.3.49}$$

$$\implies \quad \lim_{z \to z_0} f(z) = f(z_0) \tag{13.3.50}$$

This uses the linearity of the limit operator. The last expression implies that $f(z)$ is continuous at z_0.

(e) Checking the differentiability,

$$\lim_{z \to z_0} \frac{x - x_0}{z - z_0} = L \tag{13.3.51}$$

Consider two approaches, one vertical $z_1 = x_0 + \mathrm{i}\ y$, and the other horizontal $z_2 = x + \mathrm{i}\ y_0$

$$L_1 = \lim_{z_1 \to z_0} \frac{x - x_0}{z - z_0} = \lim_{y \to y_0} \frac{x_0 - x_0}{y - y_0} = 0 \tag{13.3.52}$$

$$L_2 = \lim_{z_2 \to z_0} \frac{x - x_0}{z - z_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1 \tag{13.3.53}$$

Since the two limits at the same point z_0 are different, the function is not differentiable at z_0, regardless of where in the complex plane z_0 is located.

Another such function is $g(z) = \text{Im}\ (z)$

(f) Checking the differentiability,

$$\lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x^2 + \Delta y^2 + 2x_0 \Delta x + 2y_0 \Delta y}{\Delta x + \mathrm{i}\ \Delta y} \tag{13.3.54}$$

Consider two approaches, one vertical $z_1 = 0 + \mathrm{i}\ \Delta y$, and the other horizontal $z_2 = \Delta x + 0\ \mathrm{i}$

$$L_1 = \lim_{\Delta y \to 0} \frac{\Delta y(\Delta y + 2y_0)}{\mathrm{i}\ \Delta y} = \frac{2y_0}{\mathrm{i}} \tag{13.3.55}$$

$$L_2 = \lim_{\Delta x \to 0} \frac{\Delta x(\Delta x + 2x_0)}{\Delta x} = 2x_0 \tag{13.3.56}$$

Since the two limits at the same point z_0 are different, the function is not differentiable at z_0, regardless of where in the complex plane z_0 is located except the origin.
Even when $x_0 = y_0 = 0$, the function is not differentiable everywhere in the neighbourhood of z_0. Hence, the function is nowhere analytic.

25. Refer notes. TBC.

13.4 Cauchy-Riemann Equations, Laplace's Equation

1. Deriving the Cauchy-Riemann equations in polar form,

$$f(z) = u(r, \theta) + \mathrm{i}\ v(r, \theta) \qquad (x, y) = (r \cos\theta, r \sin\theta) \tag{13.4.1}$$

$$u_x = v_y \qquad u_y = -v_x \tag{13.4.2}$$

Using partial differentiation,

$$u_r = u_x\ x_r + u_y\ y_r \qquad u_\theta = u_x\ x_\theta + u_y\ y_\theta \tag{13.4.3}$$

$$u_r = u_x\ \cos\theta + u_y\ \sin\theta \qquad u_\theta = -u_x\ r \sin\theta + u_y\ r\cos\theta \tag{13.4.4}$$

$$v_r = v_x\ \cos\theta + v_y\ \sin\theta \qquad v_\theta = -v_x\ r\sin\theta + v_y\ r\cos\theta \tag{13.4.5}$$

$$\tag{13.4.6}$$

Equating after dividing by r,

$$u_r = u_x \cos\theta + u_y \sin\theta = \frac{v_y\ r\cos\theta - v_x\ r\sin\theta}{r} = \frac{v_\theta}{r} \tag{13.4.7}$$

$$v_r = v_x \cos\theta + v_y \sin\theta = \frac{-u_y\ r\cos\theta + u_x\ r\sin\theta}{r} = -\frac{u_\theta}{r} \tag{13.4.8}$$

2. The function is not analytic.

$$f(z) = \mathrm{i}\ z\bar{z} = 0 + \mathrm{i}\ (x^2 + y^2) \tag{13.4.9}$$

$$u_x = 0 \qquad u_y = 0 \tag{13.4.10}$$

$$v_x = 2x \qquad v_y = 2y \tag{13.4.11}$$

3. The function is analytic.

$$f(z) = e^{-2x}\,[\cos(2y) - \mathrm{i}\ \sin(2y)] \tag{13.4.12}$$

$$u_x = -2e^{-2x}\cos(2y) \qquad u_y = -2e^{-2x}\sin(2y) \tag{13.4.13}$$

$$v_x = 2e^{-2x}\sin(2y) \qquad v_y = -2e^{-2x}\cos(2y) \tag{13.4.14}$$

4. The function is not analytic.

$$f(z) = e^{x}\,[\cos(y) - \mathrm{i}\ \sin(y)] \tag{13.4.15}$$

$$u_x = e^{x}\cos(y) \qquad u_y = -e^{x}\sin(y) \tag{13.4.16}$$

$$v_x = -e^{x}\sin(y) \qquad v_y = -e^{x}\cos(y) \tag{13.4.17}$$

5. The function is not analytic.

$$f(z) = \mathrm{Re}\,(z^2) - \mathrm{i}\ \mathrm{Im}\,(z^2) \qquad f(z) = x^2 - y^2 - (2xy)\,\mathrm{i} \tag{13.4.18}$$

$$u_x = 2x \qquad u_y = -2y \tag{13.4.19}$$

$$v_x = -2y \qquad v_y = -2x \tag{13.4.20}$$

6. The function is analytic.

$$f(z) = \frac{1}{z - z^5} = \frac{1}{z(z+1)(z-1)(z+\mathrm{i})(z-\mathrm{i})} \tag{13.4.21}$$

$$= -\frac{1}{z} + \frac{0.25}{z+1} + \frac{0.25}{z-1} + \frac{0.25}{z+\mathrm{i}} + \frac{0.25}{z-\mathrm{i}} \tag{13.4.22}$$

$$= -\frac{x - \mathrm{i}\,y}{x^2+y^2} + \frac{1}{4}\left[\frac{x+1-\mathrm{i}\,y}{(x+1)^2+y^2} + \frac{x-1-\mathrm{i}\,y}{(x-1)^2+y^2}\right. \tag{13.4.23}$$

$$\left. + \frac{x - \mathrm{i}\,(y+1)}{x^2+(y+1)^2} + \frac{x - \mathrm{i}\,(y-1)}{x^2+(y-1)^2}\right] \tag{13.4.24}$$

Finding the first partial derivatives,

$$u_x = \frac{x^2 - y^2}{[x^2 + y^2]^2} + \frac{1}{4}\left[\frac{y^2 - (x+1)^2}{[(x+1)^2 + y^2]^2} + \frac{y^2 - (x-1)^2}{[(x-1)^2 + y^2]^2}\right. \tag{13.4.25}$$

$$\left. + \frac{(y+1)^2 - x^2}{[x^2 + (y+1)^2]^2} + \frac{(y-1)^2 - x^2}{[x^2 + (y-1)^2]^2}\right] \tag{13.4.26}$$

$$v_x = \frac{-2xy}{[x^2 + y^2]^2} + \frac{1}{4}\left[\frac{2y(x+1)}{[(x+1)^2 + y^2]^2} + \frac{2y(x-1)}{[(x-1)^2 + y^2]^2}\right. \tag{13.4.27}$$

$$\left. + \frac{2x(y+1)}{[x^2 + (y+1)^2]^2} + \frac{2x(y-1)}{[x^2 + (y-1)^2]^2}\right] \tag{13.4.28}$$

Finding the first partial derivatives,

$$u_y = \frac{2xy}{[x^2 + y^2]^2} - \frac{1}{4}\left[\frac{2y(x+1)}{[(x+1)^2 + y^2]^2} + \frac{2y(x-1)}{[(x-1)^2 + y^2]^2}\right. \tag{13.4.29}$$

$$\left. + \frac{2x(y+1)}{[x^2 + (y+1)^2]^2} + \frac{2x(y-1)}{[x^2 + (y-1)^2]^2}\right] \tag{13.4.30}$$

$$v_y = \frac{x^2 - y^2}{[x^2 + y^2]^2} + \frac{1}{4}\left[\frac{y^2 - (x+1)^2}{[(x+1)^2 + y^2]^2} + \frac{y^2 - (x-1)^2}{[(x-1)^2 + y^2]^2}\right. \tag{13.4.31}$$

$$\left. + \frac{(y+1)^2 - x^2}{[x^2 + (y+1)^2]^2} + \frac{(y-1)^2 - x^2}{[x^2 + (y-1)^2]^2}\right] \tag{13.4.32}$$

7. The function is analytic.

$$f(z) = \frac{\mathrm{i}}{z^8} \qquad f(z) = e^{\mathrm{i}\;\pi/2} \cdot r^{-8}\; e^{-\mathrm{i}\;8\theta} \tag{13.4.33}$$

$$u = r^{-8}\,\sin(8\theta) \qquad v = r^{-8}\,\cos(8\theta) \tag{13.4.34}$$

$$u_r = -8r^{-9}\,\sin(8\theta) \qquad u_\theta = 8r^{-8}\,\cos(8\theta) \tag{13.4.35}$$

$$v_r = -8r^{-9}\,\cos(8\theta) \qquad v_\theta = -8r^{-8}\,\sin(8\theta) \tag{13.4.36}$$

$$u_r = \frac{1}{r}\,v_\theta \qquad v_r = \frac{-1}{r}\,u_\theta \tag{13.4.37}$$

8. The function is not analytic.

$$f(z) = \text{Arg}\,(2\pi z) \qquad f(z) = \text{Arg}\left(2\pi r\ e^{\mathrm{i}\ \theta}\right) = \theta \tag{13.4.38}$$

$$u_r = 0 \qquad u_\theta = 1 \tag{13.4.39}$$

$$v_r = 0 \qquad v_\theta = 0 \tag{13.4.40}$$

$$u_r = \frac{1}{r}\ v_\theta \qquad v_r = \frac{-1}{r}\ u_\theta \tag{13.4.41}$$

9. The function is analytic, barring the points where the denominator is zero.

$$f(z) = \frac{3\pi^2}{z^3 + 4\pi^2 z} = \frac{3\pi^2}{r^3\ e^{\mathrm{i}\ 3\theta} + 4\pi^2 r\ e^{\mathrm{i}\ \theta}} \tag{13.4.42}$$

$$\overline{f(z)} = \frac{3\pi^2}{r^3\ e^{-\mathrm{i}\ 3\theta} + 4\pi^2 r\ e^{-\mathrm{i}\ \theta}} \tag{13.4.43}$$

$$f_r = -3\pi^2 \left[\frac{3r^2 e^{\mathrm{i}\ 3\theta} + 4\pi^2 e^{\mathrm{i}\ \theta}}{(r^3\ e^{\mathrm{i}\ 3\theta} + 4\pi^2 r\ e^{\mathrm{i}\ \theta})^2}\right] \tag{13.4.44}$$

$$\bar{f}_r = -3\pi^2 \left[\frac{3r^2 e^{-\mathrm{i}\ 3\theta} + 4\pi^2 e^{-\mathrm{i}\ \theta}}{(r^3\ e^{-\mathrm{i}\ 3\theta} + 4\pi^2 r\ e^{-\mathrm{i}\ \theta})^2}\right] \tag{13.4.45}$$

$$f_\theta = -3\pi^2\ \mathrm{i} \left[\frac{3r^3 e^{\mathrm{i}\ 3\theta} + 4\pi^2 r e^{\mathrm{i}\ \theta}}{(r^3\ e^{\mathrm{i}\ 3\theta} + 4\pi^2 r\ e^{\mathrm{i}\ \theta})^2}\right] \tag{13.4.46}$$

$$\bar{f}_\theta = 3\pi^2\ \mathrm{i} \left[\frac{3r^3 e^{-\mathrm{i}\ 3\theta} + 4\pi^2 r e^{-\mathrm{i}\ \theta}}{(r^3\ e^{-\mathrm{i}\ 3\theta} + 4\pi^2 r\ e^{-\mathrm{i}\ \theta})^2}\right] \tag{13.4.47}$$

$$u_r = \frac{f_r + \bar{f}_r}{2} = \frac{1}{2}\left[\frac{f_\theta - \bar{f}_\theta}{r\mathrm{i}}\right] = \frac{v_\theta}{r} \tag{13.4.48}$$

$$v_r = \frac{f_r - \bar{f}_r}{2\mathrm{i}} = \frac{1}{2\mathrm{i}}\left[\frac{f_\theta + \bar{f}_\theta}{r\mathrm{i}}\right] = -\frac{u_\theta}{r} \tag{13.4.49}$$

This is a more elegant method of solving Problem 6.

10. The function is analytic.

$$f(z) = \ln|z| + \mathrm{i}\ \ \text{Arg}\,(z) \qquad f(z) = \ln(r) + \mathrm{i}\ \theta \tag{13.4.50}$$

$$u_r = \frac{1}{r} \qquad u_\theta = 0 \tag{13.4.51}$$

$$v_r = 0 \qquad v_\theta = 1 \tag{13.4.52}$$

$$u_x = v_y \qquad u_y = -v_x \tag{13.4.53}$$

11. The function is analytic.

$$f(z) = \cos x \cosh y - \mathrm{i} \;\sin x \sinh y \tag{13.4.54}$$

$$u_x = -\sin x \cosh y \qquad u_y = \cos x \sinh y \tag{13.4.55}$$

$$v_x = -\cos x \sinh y \qquad v_y = -\sin x \cosh y \tag{13.4.56}$$

$$u_x = v_y \qquad u_y = -v_x \tag{13.4.57}$$

12. The function is not harmonic,

$$u(x, y) = x^2 + y^2 \tag{13.4.58}$$

$$u_{xx} = 2 \qquad u_{yy} = 2 \tag{13.4.59}$$

$$u_{xx} + u_{yy} \neq 0 \tag{13.4.60}$$

13. The function is harmonic,

$$u(x, y) = xy \tag{13.4.61}$$

$$u_{xx} = 0 \qquad u_{yy} = 0 \tag{13.4.62}$$

$$u_{xx} + u_{yy} = 0 \tag{13.4.63}$$

Using the Cauchy Riemann equations,

$$v_y = u_x = y \qquad v_x = -u_y = -x \tag{13.4.64}$$

$$v = \frac{y^2}{2} + f(x) + c_1 \qquad v = -\frac{x^2}{2} + g(y) + c_2 \tag{13.4.65}$$

$$v(x, y) = \frac{y^2 - x^2}{2} \tag{13.4.66}$$

The analytic function is, (with c real)

$$f(z) = \frac{1}{2}\left[2xy + \mathrm{i}\;(y^2 - x^2) + c\right] \tag{13.4.67}$$

$$= \frac{-\mathrm{i}\;(z^2 + c)}{2} \tag{13.4.68}$$

14. The function is harmonic,

$$v(x, y) = xy \tag{13.4.69}$$

$$v_{xx} = 0 \qquad v_{yy} = 0 \tag{13.4.70}$$

$$v_{xx} + v_{yy} = 0 \tag{13.4.71}$$

Using the Cauchy Riemann equations,

$$u_y = v_x = y \qquad u_x = -v_y = -x \tag{13.4.72}$$

$$u = \frac{y^2}{2} + f(x) + c_1 \qquad u = -\frac{x^2}{2} + g(y) + c_2 \tag{13.4.73}$$

$$u(x, y) = \frac{y^2 - x^2}{2} \tag{13.4.74}$$

The analytic function is, (with c real)

$$f(z) = \frac{1}{2}\left[y^2 - x^2 + \mathrm{i}\,(2xy) + c\right] \tag{13.4.75}$$

$$= \frac{(z^2 + c)}{2} \tag{13.4.76}$$

15. The function is harmonic,

$$u(x, y) = \frac{\cos\theta}{r} \tag{13.4.77}$$

$$u_r = -\frac{\cos\theta}{r^2} \qquad u_\theta = -\frac{\sin\theta}{r} \tag{13.4.78}$$

$$u_{rr} = \frac{2\cos\theta}{r^3} \qquad u_{\theta\theta} = -\frac{\cos\theta}{r} \tag{13.4.79}$$

$$u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\theta\theta} = \frac{\cos\theta}{r^3}\,(2 - 1 - 1) \qquad = 0 \tag{13.4.80}$$

Using the Cauchy Riemann equations,

$$v_r = \frac{-1}{r}\,u_\theta = \frac{\sin\theta}{r^2} \qquad v_\theta = r\,u_r = -\frac{\cos\theta}{r} \tag{13.4.81}$$

$$v = -\frac{\sin\theta}{r} + f(\theta) \qquad v = -\frac{\sin\theta}{r} + g(r) \tag{13.4.82}$$

$$v(x, y) = -\frac{\sin\theta}{r} \tag{13.4.83}$$

The analytic function is, (with c real)

$$f(z) = \frac{\bar{z}}{|z|^2} + c = \frac{1}{z} + c \tag{13.4.84}$$

16. The function is harmonic,

$$u(x, y) = \sin x \cosh y \tag{13.4.85}$$

$$u_{xx} = -\sin x \cosh y \qquad u_{yy} = \sin x \cosh y \tag{13.4.86}$$

$$u_{xx} + u_{yy} = 0 \tag{13.4.87}$$

Using the Cauchy Riemann equations,

$$v_x = -u_y = -\sin x \sinh y \qquad v_y = u_x = \cos x \cosh y \tag{13.4.88}$$

$$v = \cos x \sinh y + f(x) + c_1 \qquad v = \cos x \sinh y + g(y) + c_2 \tag{13.4.89}$$

$$v(x, y) = \cos x \sinh y \tag{13.4.90}$$

The analytic function is, (with c real)

$$f(z) = (\sin x \cosh y) + \mathfrak{i}\,(\cos x \sinh y + c) \tag{13.4.91}$$

$$\sin(z) + \mathfrak{i}\, c \tag{13.4.92}$$

This uses the Euler's formula to define the sine function on the complex plane.

17. The function is harmonic,

$$v(x, y) = (2x + 1)y \tag{13.4.93}$$

$$v_{xx} = 0 \qquad v_{yy} = 0 \tag{13.4.94}$$

$$v_{xx} + v_{yy} = 0 \tag{13.4.95}$$

Using the Cauchy Riemann equations,

$$u_x = v_y = (2x + 1) \qquad u_y = -v_x = -2y \tag{13.4.96}$$

$$u = x^2 + x + g(y) + c_1 \qquad u = -y^2 + f(x) + c_2 \tag{13.4.97}$$

$$u(x, y) = x^2 + x - y^2 + c \tag{13.4.98}$$

The analytic function is, (with c real)

$$f(z) = (x^2 + x - y^2) + \mathrm{i}\,(2xy + y) + c \tag{13.4.99}$$

$$= z^2 + z + c \tag{13.4.100}$$

18. The function is harmonic,

$$u(x,\,y) = x^3 - 3xy^2 \tag{13.4.101}$$

$$u_{xx} = 6x \qquad\qquad u_{yy} = -6x \tag{13.4.102}$$

$$u_{xx} + u_{yy} = 0 \tag{13.4.103}$$

Using the Cauchy Riemann equations,

$$v_x = -u_y = 6xy \qquad\qquad v_y = u_x = 3(x^2 - y^2) \tag{13.4.104}$$

$$v = 3x^2y + g(y) + c_1 \qquad\qquad v = 3x^2y - y^3 + f(x) + c_2 \tag{13.4.105}$$

$$v(x,\,y) = 3x^2y - y^3 + c \tag{13.4.106}$$

The analytic function is, (with c real)

$$f(z) = (x^3 - 3xy^2) + \mathrm{i}\,(3x^2y - y^3) + c \tag{13.4.107}$$

$$= z^3 + c \tag{13.4.108}$$

19. The function is not harmonic,

$$v(x,\,y) = e^x\,\sin(2y) \tag{13.4.109}$$

$$v_{xx} = e^x\,\sin(2y) \qquad\qquad v_{yy} = -4e^x\,\sin(2y) \tag{13.4.110}$$

$$v_{xx} + v_{yy} \neq 0 \tag{13.4.111}$$

20. Starting with the Cauchy Euler equations,

$$u_x = v_y \qquad\qquad u_y = -v_x \tag{13.4.112}$$

$$u_{xy} = v_{yy} \qquad\qquad u_{yx} = -v_{xx} \tag{13.4.113}$$

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} \qquad\qquad = 0 \tag{13.4.114}$$

$$\implies \qquad \nabla^2 v = 0 \tag{13.4.115}$$

21. For the given function to be harmonic,

$$u(x,\,y) = e^{\pi x}\cos(ay) \qquad u_{xx} + u_{yy} = 0 \tag{13.4.116}$$

$$u_{xx} = \pi^2 e^{\pi x}\cos(ay) \qquad u_{yy} = -a^2 e^{\pi x}\cos(ay) \tag{13.4.117}$$

$$\nabla^2 u = 0 \qquad \implies \quad a = \pi \tag{13.4.118}$$

To find the harmonic conjugate,

$$v_y = u_x = \pi e^{\pi x}\cos(\pi y) \qquad v_x = -u_y = \pi e^{\pi x}\sin(\pi y) \tag{13.4.119}$$

$$v = e^{\pi x}\sin(\pi y) + f(x) + c_1 \qquad v = e^{\pi x}\sin(\pi y) + g(y) + c_2 \tag{13.4.120}$$

$$v(x,\,y) = e^{\pi x}\sin(\pi y) \tag{13.4.121}$$

22. For the given function to be harmonic,

$$u(x,\,y) = \cos(ax)\cosh(2y) \qquad u_{xx} + u_{yy} = 0 \tag{13.4.122}$$

$$u_{xx} = -a^2\cos(ax)\cosh(2y) \qquad u_{yy} = 4\cos(ax)\cosh(2y) \tag{13.4.123}$$

$$\nabla^2 u = 0 \qquad \implies \quad a = 2 \tag{13.4.124}$$

To find the harmonic conjugate,

$$v_y = u_x = -2\sin(2x)\cosh(2y) \qquad v_x = -u_y = -2\cos(2x)\sinh(2y) \tag{13.4.125}$$

$$v(x,\,y) = -\sin(2x)\sinh(2y) \tag{13.4.126}$$

23. For the given function to be harmonic,

$$u(x,\,y) = ax^3 + bxy \qquad u_{xx} + u_{yy} = 0 \tag{13.4.127}$$

$$u_{xx} = 6ax \qquad u_{yy} = 0 \tag{13.4.128}$$

$$b = \text{free} \qquad \implies \quad a = 0 \tag{13.4.129}$$

To find the harmonic conjugate,

$$v_y = u_x = by \qquad v_x = -u_y = -bx \tag{13.4.130}$$

$$v = \frac{by^2}{2} + f(x) \qquad v = -\frac{bx^2}{2} + f(y) \tag{13.4.131}$$

$$v(x,\,y) = \frac{b}{2}\,(y^2 - x^2) \tag{13.4.132}$$

24. For the given function to be harmonic,

$$u(x,\, y) = \cosh(ax)\,\cos(y) \qquad u_{xx} + u_{yy} = 0 \tag{13.4.133}$$

$$u_{xx} = a^2\,\cosh(ax)\,\cos(y) \qquad u_{yy} = -\,\cosh(ax)\,\cos(y) \tag{13.4.134}$$

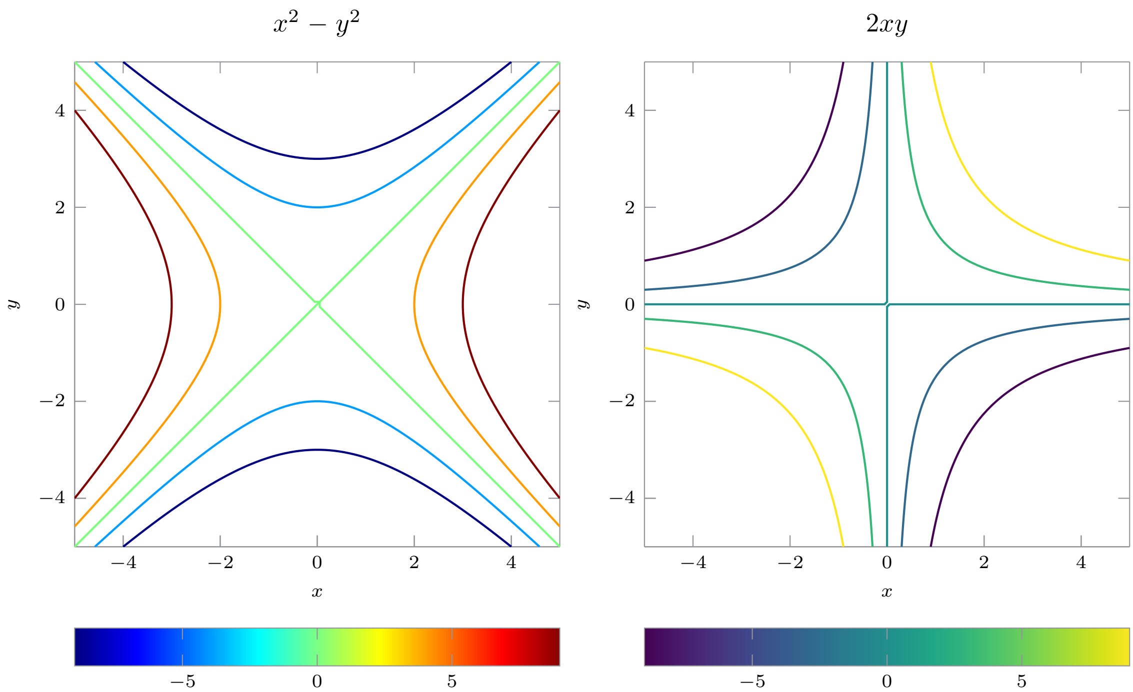
$$\nabla^2 u = 0 \qquad \implies \qquad a = 1 \tag{13.4.135}$$

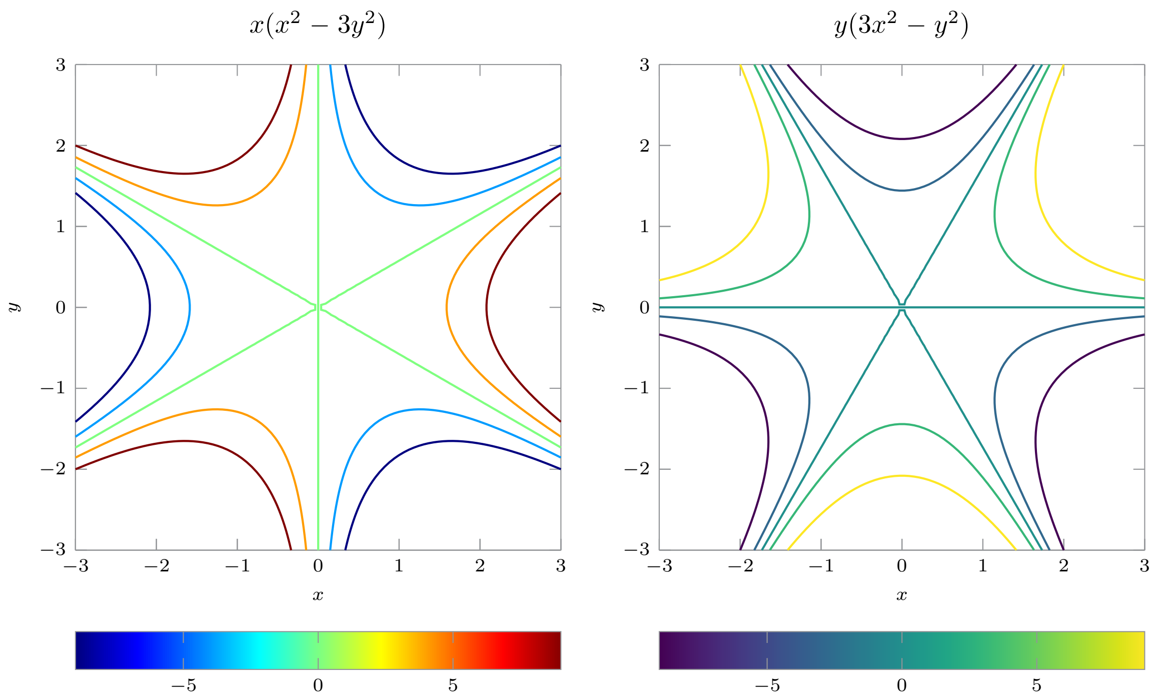
To find the harmonic conjugate,

$$v_y = u_x = \sinh(x)\,\cos(y) \qquad v_x = -u_y = \cosh(x)\,\sin(y) \tag{13.4.136}$$

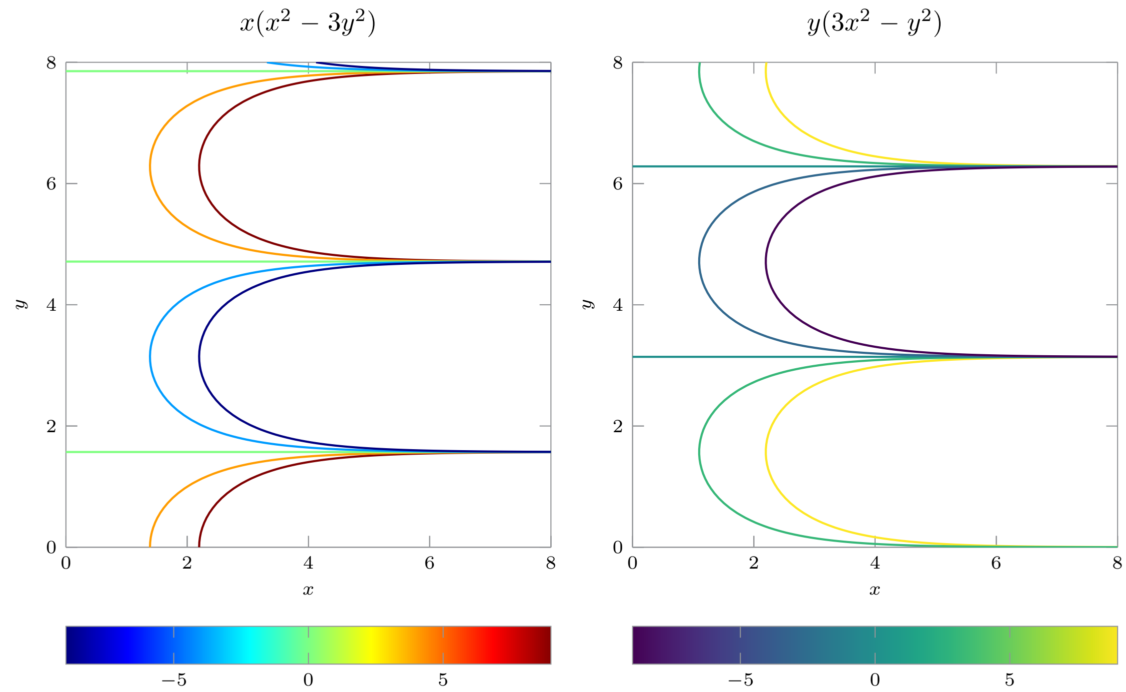
$$v(x,\, y) = \sinh(x)\,\sin(y) \tag{13.4.137}$$

25. The equipotential lines of harmonic conjugates happen to be orthogonal families of curves.





26. Applying the program to the given function,



27. Given v is the harmonic conjugate of u,

$$u_x = v_y \qquad u_y = -v_x \tag{13.4.138}$$

$$v_x = (-u)_y \qquad v_y = -(-u)_x \tag{13.4.139}$$

Thus $(-u)$ is the harmonic conjugate of v.

28. Use Problem 14.

29. Starting with equations 4, 5 and substituting the Cauchy-Riemann relations,

$$f'(z) = u_x + \mathrm{i}\ v_x \qquad u_y = -v_x \tag{13.4.140}$$

$$f'(z) = u_x - \mathrm{i}\ u_y \tag{13.4.141}$$

$$f'(z) = -\mathrm{i}u_y + v_y \qquad f'(z) = v_y + \mathrm{i}\ v_x \tag{13.4.142}$$

30. Conditions for a constant complex function

(a) Checking the given function,

$$f(z) = c + \mathrm{i}\ v(x, y) \tag{13.4.143}$$

$$u_x = v_y \implies 0 = v_y \tag{13.4.144}$$

$$u_y = -v_x \implies 0 = v_x \tag{13.4.145}$$

$$v = c_1 + f(x) \qquad v_x = \frac{\mathrm{d}f}{\mathrm{d}x} = 0 \tag{13.4.146}$$

$$v(x, y) = c \tag{13.4.147}$$

Thus, $f(z)$ is a constant function.

(b) Checking the given function,

$$f(z) = u(x, y) + c\ \mathrm{i} \tag{13.4.148}$$

$$u_x = v_y \implies 0 = u_x \tag{13.4.149}$$

$$u_y = -v_x \implies 0 = u_y \tag{13.4.150}$$

$$u = c_1 + f(x) \qquad u_x = \frac{\mathrm{d}f}{\mathrm{d}x} = 0 \tag{13.4.151}$$

$$u(x, y) = c \tag{13.4.152}$$

Thus, $f(z)$ is a constant function.

(c) Checking the given function,

$$f'(z) = 0 \tag{13.4.153}$$

$$u_x = \mathrm{i}\ u_y \implies u_x = -\mathrm{i}\ v_x \tag{13.4.154}$$

$$v_y = -\mathrm{i}\ v_x \implies v_y = \mathrm{i}\ u_y \tag{13.4.155}$$

$$u_x = u_y = 0 \qquad v_x = v_y = 0 \tag{13.4.156}$$

$$f(x, y) = c \tag{13.4.157}$$

Since a real and complex number are being equated, the only solution is for both sides to be identically zero.

(d) Checking the given function,

$$|f(z)| = c \qquad |f(z)|^2 = u^2 + v^2 = c^2 \tag{13.4.158}$$

$$uu_x + vv_x = 0 \qquad uu_y + vv_y = 0 \tag{13.4.159}$$

$$-uv_x + vu_x = 0 \qquad \left(u^2 + v^2\right) u_x = 0 \tag{13.4.160}$$

Similarly, all four first order partial derivatives are constrained to be zero. This means $f(z) = c$

13.5 Exponential Function

1. To prove that e^z is entire,

$$f(z) = e^z = e^x \ (\cos y + \mathfrak{i} \ \sin y) \qquad f(z) = u(x, y) + \mathfrak{i} \ v(x, y) \tag{13.5.1}$$

$$u_x = e^x \cos y \qquad u_y = -e^x \sin y \tag{13.5.2}$$

$$v_x = e^x \sin y \qquad v_y = e^x \cos y \tag{13.5.3}$$

$$u_x = v_y \qquad u_y = -v_x \tag{13.5.4}$$

Since e^a, $\sin(a)$, $\cos(a)$ are themselves defined for all real a, the Euler-Cauchy relations are satisfied everywhere in the complex plane.

This makes the function entire.

2. Finding the functional form and modulus,

$$z = 3 + 4\mathfrak{i} \qquad e^z = e^3 \ (\cos 4 + \sin 4 \ \mathfrak{i}) \tag{13.5.5}$$

$$|e^z| = e^x = e^3 \tag{13.5.6}$$

3. Finding the functional form and modulus,

$$z = 2\pi\mathfrak{i}(1 + \mathfrak{i}) \qquad e^z = e^{-2\pi} \ [\cos(2\pi) + \sin(2\pi) \ \mathfrak{i}] \tag{13.5.7}$$

$$e^z = e^{-2\pi} \qquad |e^z| = e^x = e^{-2\pi} \tag{13.5.8}$$

4. Finding the functional form and modulus,

$$z = 0.6 - 1.8\mathfrak{i} \qquad e^z = e^{0.6} \ [\cos(1.8) - \sin(1.8) \ \mathfrak{i}] \tag{13.5.9}$$

$$|e^z| = e^x = e^{0.6} \tag{13.5.10}$$

5. Finding the functional form and modulus,

$$z = 2 + 3\pi \mathrm{i} \qquad e^z = e^2 \left[\cos(3\pi) + \sin(3\pi)\ \mathrm{i}\right] \tag{13.5.11}$$

$$e^z = -e^2 \qquad |e^z| = e^x = e^2 \tag{13.5.12}$$

6. Finding the functional form and modulus,

$$z = \frac{11\pi}{2}\ \mathrm{i} \qquad e^z = e^0 \left[\cos(5.5\pi) + \sin(5.5\pi)\ \mathrm{i}\right] \tag{13.5.13}$$

$$e^z = -\mathrm{i} \qquad |e^z| = e^x = e^0 = 1 \tag{13.5.14}$$

7. Finding the functional form and modulus,

$$z = \sqrt{2} + \frac{\pi}{2}\ \mathrm{i} \qquad e^z = e^{\sqrt{2}} \left[\cos(\pi/2) + \sin(\pi/2)\ \mathrm{i}\right] \tag{13.5.15}$$

$$e^z = e^{\sqrt{2}}\ \mathrm{i} \qquad |e^z| = e^x = e^{\sqrt{2}} \tag{13.5.16}$$

8. Finding the polar form,

$$z = e\ \exp(\mathrm{i}\theta) \qquad z^{1/n} = r^{1/n} \exp\left(\frac{\mathrm{i}\theta}{n}\right) \tag{13.5.17}$$

9. Finding the polar form,

$$z = 4 + 3\mathrm{i} \qquad r = \sqrt{3^2 + 4^2} = 5 \tag{13.5.18}$$

$$\theta = \arctan(3/4) \qquad z = 5\ \exp(\mathrm{i}\ \theta) \tag{13.5.19}$$

10. Finding the polar form,

$$z_1 = \sqrt{i} \qquad r_1 = \sqrt{1} = 1 \tag{13.5.20}$$

$$\theta_1 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \qquad z_1 = 1 \cdot \exp\left(\mathrm{i}\ \frac{\pi}{4}\right) \tag{13.5.21}$$

$$z_2 = \sqrt{-i} \qquad r_2 = \sqrt{1} = 1 \tag{13.5.22}$$

$$\theta_2 = \frac{1}{2} \cdot \frac{-\pi}{2} = \frac{-\pi}{4} \qquad z_2 = 1 \cdot \exp\left(-\ \mathrm{i}\ \frac{\pi}{4}\right) \tag{13.5.23}$$

11. Finding the polar form,

$$z = -6.3 + 0\ \mathrm{i} \qquad z = 6.3\ \exp(\mathrm{i}\ \pi) \tag{13.5.24}$$

12. Finding the polar form,

$$z = re^{\mathrm{i}\,\theta} \qquad w = \frac{1}{1-z} = \frac{1-\bar{z}}{|1-z|^2} \tag{13.5.25}$$

$$w = \frac{1 - r\cos\theta + r\sin\theta\ \mathrm{i}}{(1-r\cos\theta)^2 + (r\sin\theta)^2} \qquad w = \frac{(1-r\cos\theta) + \mathrm{i}\,(r\sin\theta)}{1 + r^2 - 2r\cos\theta} \tag{13.5.26}$$

$$|w| = \frac{1}{1 + r^2 - 2r\cos\theta} \qquad \mathrm{Arg}\,(w) = \frac{r\sin\theta}{1 - r\cos\theta} \tag{13.5.27}$$

13. Finding the polar form,

$$z = 1 + \mathrm{i} \qquad z = \sqrt{2}\,\exp(\mathrm{i}\,\pi/4) \tag{13.5.28}$$

14. Finding the real and imaginary parts of the function,

$$f(z) = e^{-\pi z} \qquad \mathrm{Re}\,f = e^{-\pi x}\cos(\pi y) \tag{13.5.29}$$

$$\mathrm{Im}\,f = -e^{-\pi x}\sin(\pi y) \tag{13.5.30}$$

15. Finding the real and imaginary parts of the function,

$$f(z) = e^{z^2} \qquad f(z) = \exp(x^2 - y^2 + 2xy\ \mathrm{i}) \tag{13.5.31}$$

$$\mathrm{Re}\,f = e^{x^2-y^2}\cos(2xy) \qquad \mathrm{Im}\,f = e^{x^2-y^2}\sin(2xy) \tag{13.5.32}$$

16. Finding the real and imaginary parts of the function,

$$f(z) = e^{1/z} \qquad f(z) = \exp\left(\frac{x - \mathrm{i}\,y}{x^2+y^2}\right) \tag{13.5.33}$$

$$\mathrm{Re}\,f = e^{x/(x^2+y^2)}\cos\left(\frac{y}{x^2+y^2}\right) \qquad \mathrm{Im}\,f = -e^{x/(x^2+y^2)}\sin\left(\frac{y}{x^2+y^2}\right) \tag{13.5.34}$$

17. Finding the real and imaginary parts of the function,

$$f(z) = e^{z^3} \qquad f(z) = \exp(x^3 - 3xy^2 - y^3\mathrm{i} + 3x^2y\mathrm{i}) \tag{13.5.35}$$

$$\mathrm{Re}\,f = e^{x^3-3xy^2}\cos(3x^2y - y^3) \qquad \mathrm{Im}\,f = e^{x^3-3xy^2}\sin(3x^2y - y^3) \tag{13.5.36}$$

18. Properties of the exponential function

(a) Since this power is not defined at $z = 0 + 0\ \mathrm{i}$, the function $e^{1/z}$ is not entire.

Checking $e^{\bar{z}}$,

$$e^{\bar{z}} = e^{x}\ (\cos y - \mathrm{i}\ \sin y) \tag{13.5.37}$$

$$u_x = e^{x}\cos(y) \qquad v_y = -e^{x}\cos(y) \tag{13.5.38}$$

$$u_y = -e^{x}\sin(y) \qquad v_x = -e^{x}\sin(y) \tag{13.5.39}$$

$$\tag{13.5.40}$$

This function does not satisfy Cauchy-Riemann relations and is not entire.
Checking the third function,

$$f(z) = e^{x}\ [\cos(ky) + \mathrm{i}\ \sin(ky)] \tag{13.5.41}$$

$$u_x = e^{x}\cos(ky) \qquad v_y = ke^{x}\cos(ky) \tag{13.5.42}$$

$$u_y = -ke^{x}\sin(ky) \qquad v_x = e^{x}\sin(ky) \tag{13.5.43}$$

This function is entire if $k = 1$.

(b) Condition for e^z to be real,

$$\text{Im}\ (e^{z}) = e^{x}\ \sin(y) = 0 \qquad \implies \qquad y = n\pi \tag{13.5.44}$$

Condition for $|e^{-z}| < 1$,

$$e^{-z} = e^{-x}\ [\cos(y) - \mathrm{i}\ \sin(y)] \qquad \left|e^{-z}\right| = e^{-x} \tag{13.5.45}$$

$$\left|e^{-z}\right| < 1 \qquad \implies \qquad e^{x} > 1 \tag{13.5.46}$$

$$x > 0 \tag{13.5.47}$$

Condition for $e^{\bar{z}} = \overline{e^{z}}$,

$$e^{\bar{z}} = e^{x}\ [\cos(y) - \mathrm{i}\ \sin(y)] \qquad \overline{e^{z}} = e^{x}\ [\cos(y) - \mathrm{i}\ \sin(y)] \tag{13.5.48}$$

This relation is identically true for all z.

(c) The given function is harmonic, using

$$\frac{x^2 - y^2}{2} = \lambda \tag{13.5.49}$$

$$u = e^{xy} \ \cos\lambda \tag{13.5.50}$$

$$u_x = y e^{xy} \cos\lambda - x e^{xy} \sin\lambda \tag{13.5.51}$$

$$u_{xx} = e^{xy} \cos\lambda \, [y^2 - x^2] - e^{xy} \ \sin\lambda \, [1 + 2xy] \tag{13.5.52}$$

$$u_y = x e^{xy} \cos\lambda + y e^{xy} \sin\lambda \tag{13.5.53}$$

$$u_{yy} = e^{xy} \cos\lambda \, [x^2 - y^2] + e^{xy} \ \sin\lambda \, [1 + 2xy] \tag{13.5.54}$$

$$u_{xx} + u_{yy} = 0 \tag{13.5.55}$$

Attempting to find a complex function whose real or imaginary part is $u(x, y)$,

$$f(z) = \exp(-\mathrm{i} \ z^2/2) = \exp\left[xy + \frac{y^2 - x^2}{2} \ \mathrm{i}\right] \tag{13.5.56}$$

$$\mathrm{Re} \ f = e^{xy} \cos\left(\frac{y^2 - x^2}{2}\right) \tag{13.5.57}$$

$$\mathrm{Im} \ f = e^{xy} \sin\left(\frac{y^2 - x^2}{2}\right) \tag{13.5.58}$$

Since $f(z)$ is entire, its imaginary part is the harmonic conjugate of u.

(d) Using the two constraints,

$$f'(z) = f(z) \qquad f(z) = u(x, y) + \mathrm{i} \ v(x, y) \tag{13.5.59}$$

$$u_x + \mathrm{i} \ v_x = u + \mathrm{i} \ v \qquad -\mathrm{i} \ u_y + v_y = u + \mathrm{i} \ v \tag{13.5.60}$$

$$u = A(x) \cdot B(y) \qquad u = c_1 e^x \cdot B(y) \tag{13.5.61}$$

$$u_{xx} + u_{yy} = 0 \qquad B(y) = c_2 \cos(y) + c_3 \sin(y) \tag{13.5.62}$$

Applying the "boundary condition",

$$f(x + 0 \ \mathrm{i}) = e^x \qquad \implies \qquad u(x, y) = e^x \ [\cos y + c_3 \sin(y)] \tag{13.5.63}$$

$$v(x, y) = e^x [c_5 \sin y] \qquad u_x = v_y \tag{13.5.64}$$

$$\implies \quad c_5 = 1 \qquad c_3 = 0 \tag{13.5.65}$$

$$f(z) = e^x \cos y + \mathrm{i} \ e^x \sin y \tag{13.5.66}$$

The parallel computation leading to $v(x, y)$ is omitted.

19. Finding the solutions,

$$e^z = 1 \tag{13.5.67}$$

$$x = 0 \qquad y = 2n\pi \tag{13.5.68}$$

20. Finding the solutions,

$$e^z = 4 + 3\,\mathrm{i} \tag{13.5.69}$$

$$e^x = \sqrt{4^2 + 3^2} = 5 \qquad x = \ln 5 \tag{13.5.70}$$

$$y = \arctan 3/4 + 2n\pi \tag{13.5.71}$$

21. No solution exists, since the modulus of e^z is e^x which is always a positive real number.

22. Finding the solutions,

$$e^z = -2 + 0\,\mathrm{i} \tag{13.5.72}$$

$$e^x = \sqrt{-2^2 + 0^2} = 2 \qquad x = \ln 2 \tag{13.5.73}$$

$$\cos y = -1 \qquad \sin y = 0 \qquad y = \pi + 2n\pi \tag{13.5.74}$$

13.6 Trigonometric and Hyperbolic Functions, Euler's Formula

1. To prove these identities,

$$\cosh z = \cosh(x + \mathrm{i}y) = \cosh x \cosh(\mathrm{i}y) + \sinh x \sinh(\mathrm{i}y) \tag{13.6.1}$$

$$= \cosh x \cos y + \mathrm{i}\ \sinh x \sin y \tag{13.6.2}$$

$$\sinh z = \cosh(x + \mathrm{i}y) = \sinh x \cosh(\mathrm{i}y) + \cosh x \sinh(\mathrm{i}y) \tag{13.6.3}$$

$$= \sinh x \cos y + \mathrm{i}\ \cosh x \sin y \tag{13.6.4}$$

2. To prove these identities,

$$\cosh(z_1 + z_2) = \cosh(x_1 + x_2)\cosh[\mathrm{i}(y_1 + y_2)] + \sinh(x_1 + x_2)\sinh[\mathrm{i}(y_2 + y_2)] \tag{13.6.5}$$

$$= \cosh(x_1 + x_2)\cos(y_1 + y_2) + \mathrm{i}\ \sinh(x_1 + x_2)\sin(y_2 + y_2) \tag{13.6.6}$$

$$= (\cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2)\,(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) \tag{13.6.7}$$

$$+\,\mathrm{i}\ (\sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2)\,(\sin y_1 \cos y_2 + \cos y_1 \sin y_2) \tag{13.6.8}$$

Collecting terms and simplifying,

$$\cosh(z_1 + z_2) = \Big[\cosh x_1 \cos y_1 + \mathrm{i}\ \sinh x_1 \sin y_1\Big] \Big[\cosh x_2 \cos y_2 + \mathrm{i}\ \sinh x_2 \sin y_2\Big] \tag{13.6.9}$$

$$+ \Big[\sinh x_1 \cos y_1 + \mathrm{i}\ \cosh x_1 \sin y_1\Big] \Big[\sinh x_2 \cos y_2 + \mathrm{i}\ \cosh x_2 \sin y_2\Big] \tag{13.6.10}$$

$$= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \tag{13.6.11}$$

Starting with the right-hand side instead,

$$\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 = -\mathrm{i}\ \sin(\mathrm{i}z_1)\cos(\mathrm{i}z_2) - \mathrm{i}\ \cos(\mathrm{i}z_1)\sin(\mathrm{i}z_2) \tag{13.6.12}$$

$$= -\mathrm{i}\Big[\sin(\mathrm{i}z_1 + \mathrm{i}z_2)\Big] \tag{13.6.13}$$

$$= -\mathrm{i}^2 \sinh(z_1 + z_2) = \sinh(z_1 + z_2) \tag{13.6.14}$$

This is a much simpler approach.

3. Using the relations between trigonometric and hyperbolic functions,

$$\cosh^2(z) - \sinh^2(z) = \cos^2(\mathrm{i}z) - \mathrm{i}^2\ \sin^2(\mathrm{i}z) \tag{13.6.15}$$

$$= \cos^2(\mathrm{i}z) + \sin^2(\mathrm{i}z) = 1 \tag{13.6.16}$$

$$\cosh^2(z) + \sinh^2(z) = \cos^2(\mathrm{i}z) + \mathrm{i}^2\ \sin^2(\mathrm{i}z) \tag{13.6.17}$$

$$= \cos^2(\mathrm{i}z) - \sin^2(\mathrm{i}z) = \cos(2\mathrm{i}z) \tag{13.6.18}$$

$$= \cosh(2z) \tag{13.6.19}$$

4. Since hyperbolic functions are linear combinations of exponential functions, they are also entire. Using the relations,

$$\cos z = \cosh(\mathrm{i}z) \qquad\qquad \sin z = -\mathrm{i}\ \sinh(\mathrm{i}z) \tag{13.6.20}$$

The trigonometric functions are also entire.

5. The first function is harmonic,

$$\cos z = \cosh(\mathrm{i}z) = \cosh(-y + \mathrm{i}x) \tag{13.6.21}$$

$$= \cosh(y)\cos(x) - \mathrm{i}\ \sinh(y)\sin(x) \tag{13.6.22}$$

$$\mathrm{Im}\ (\cos z) = f(z) = -\sinh y \sin x \tag{13.6.23}$$

$$f_{xx} + f_{yy} = \sinh y \sin x - \sinh y \sin x = 0 \tag{13.6.24}$$

The second function is harmonic,

$$\sin z = -\mathrm{i}\ \sinh(\mathrm{i}z) = -\mathrm{i}\ \sinh(-y + \mathrm{i}x) \tag{13.6.25}$$

$$= -\mathrm{i}\left[-\sinh(y)\cos(x) + \mathrm{i}\ \cosh(y)\sin(x)\right] \tag{13.6.26}$$

$$\text{Re}\,(\sin z) = g(z) = \cosh y \sin x \tag{13.6.27}$$

$$g_{xx} + g_{yy} = -\cosh y \sin x + \cosh y \sin x = 0 \tag{13.6.28}$$

6. Calculating,

$$\sin(2\pi\ \mathrm{i}) = \mathrm{i}\ \sinh(2\pi) \tag{13.6.29}$$

$$\tag{13.6.30}$$

7. Calculating,

$$\cos(\mathrm{i}) = \cosh(1) \tag{13.6.31}$$

$$\sin(\mathrm{i}) = \mathrm{i}\ \sinh(1) \tag{13.6.32}$$

8. Calculating,

$$\cos(\pi\ \mathrm{i}) = \cosh(\pi) \tag{13.6.33}$$

$$\cosh(\pi\ \mathrm{i}) = \cos(\pi) = -1 \tag{13.6.34}$$

9. Calculating,

$$\cosh(-1 + 2\mathrm{i}) = \cosh(1)\cos(2) - \mathrm{i}\ \sinh(1)\sin(2) \tag{13.6.35}$$

$$\cos(-2 - \mathrm{i}) = \cosh(1)\cos(2) - \mathrm{i}\ \sinh(1)\sin(2) \tag{13.6.36}$$

10. Calculating,

$$\sinh(3 + 4\mathrm{i}) = \sinh(3)\cos(4) + \mathrm{i}\ \cosh(3)\sin(4) \tag{13.6.37}$$

$$\cosh(3 + 4\mathrm{i}) = \cosh(3)\cos(4) + \mathrm{i}\ \sinh(3)\sin(4) \tag{13.6.38}$$

11. Calculating,

$$\sin(\pi\ \mathrm{i}) = \mathrm{i}\ \sinh(\pi) \tag{13.6.39}$$

$$\cos\left(\frac{\pi}{2} - \pi\ \mathrm{i}\right) = \sin(\pi\ \mathrm{i}) = \mathrm{i}\ \sinh(\pi) \tag{13.6.40}$$

12. Calculating,

$$\cos(\mathrm{i}\ \pi/2) = \cosh(\pi/2) \tag{13.6.41}$$

$$\cos\left(\frac{\pi}{2} + \mathrm{i}\ \frac{\pi}{2}\right) = -\sin(\mathrm{i}\ \pi/2) = -\mathrm{i}\ \sinh(\pi/2) \tag{13.6.42}$$

13. Proving $\cos(z)$ is even,

$$\cos(-z) = \cos(-x - \mathrm{i}\ y) = \cos(-x)\cosh(-y) - \mathrm{i}\ \sin(-x)\sinh(-y) \tag{13.6.43}$$

$$= \cos(x)\cosh(y) - \mathrm{i}\ \sin(x)\sinh(y) = \cos(z) \tag{13.6.44}$$

Proving $\sin(z)$ is odd,

$$\sin(-z) = \sin(-x - \mathrm{i}\ y) = \sin(-x)\cosh(-y) + \mathrm{i}\ \cos(-x)\sinh(-y) \tag{13.6.45}$$

$$= -\sin(x)\cosh(y) - \mathrm{i}\ \cos(x)\sinh(y) = -\sin(z) \tag{13.6.46}$$

14. Proving the first inequality,

$$|\cos z|^2 = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y \qquad = \cos^2 x + \sinh^2 y \tag{13.6.47}$$

$$|\cos z| \geq |\sinh y| \tag{13.6.48}$$

$$|\cos z|^2 = \cosh^2 y - \sin^2 x \qquad |\cos z| \leq \cosh y \tag{13.6.49}$$

Proving the second inequality,

$$|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y \qquad = \sin^2 x + \sinh^2 y \tag{13.6.50}$$

$$|\sin z| \geq |\sinh y| \tag{13.6.51}$$

$$|\sin z|^2 = \cosh^2 y - \cos^2 x \qquad |\sin z| \leq \cosh y \tag{13.6.52}$$

Since the hyperbolic functions in real numbers are not bounded, the complex sine and cosine functions are also not bounded.

15. Calculating directly,

$$\sin z_1 \cos z_2 = \left[\sin x_1 \cosh y_1 + \mathrm{i}\ \cos x_1 \sinh y_1\right]\left[\cos x_2 \cosh y_2 - \mathrm{i}\ \sin x_2 \sinh y_2\right] \tag{13.6.53}$$

$$= \left[\sin x_1 \cos(\mathrm{i}\ y_1) + \cos x_1 \sin(\mathrm{i}\ y_1)\right]\left[\cos x_2 \cos(\mathrm{i}\ y_2) - \sin x_2 \sin(\mathrm{i}\ y_2)\right] \tag{13.6.54}$$

$$\sin(z_1 + z_2) = \sin(x_1 + x_2)\cos[\mathrm{i}(y_1 + y_2)] + \cos(x_1 + x_2)\sin[\mathrm{i}(y_1 + y_2)] \tag{13.6.55}$$

$$\sin(z_1 - z_2) = \sin(x_1 - x_2)\cos[\mathrm{i}(y_1 - y_2)] + \cos(x_1 - x_2)\sin[\mathrm{i}(y_1 - y_2)] \tag{13.6.56}$$

Using the properties of sines and cosine addition,

$$\frac{\sin(z_1 + z_2) + \sin(z_1 - z_2)}{2} = \sin x_1 \cos x_2 \cos(\mathrm{i}y_1) \cos(\mathrm{i}y_2) \tag{13.6.57}$$

$$- \cos x_2 \sin x_1 \sin(\mathrm{i}y_1) \sin(\mathrm{i}y_2)$$

$$+ \cos x_1 \cos x_2 \sin(\mathrm{i}y_1) \cos(\mathrm{i}y_2)$$

$$- \sin x_1 \sin x_2 \cos(\mathrm{i}y_1) \sin(\mathrm{i}y_2) \tag{13.6.58}$$

The two expressions match.

16. Finding all solutions,

$$\sin z = 100 \tag{13.6.59}$$

$$\sin x \cosh y = 100 \qquad \cos x \sinh y = 0 \tag{13.6.60}$$

$$x = 2n\pi + \frac{\pi}{2} \qquad y = \cosh^{-1}(100) \tag{13.6.61}$$

17. Finding all solutions,

$$\cosh z = 0 \tag{13.6.62}$$

$$\cos y \cosh x = 0 \qquad \sinh x \sin y = 0 \tag{13.6.63}$$

$$x = 0 \qquad y = n\pi + \frac{\pi}{2} \tag{13.6.64}$$

18. Finding all solutions,

$$\cosh z = -1 \tag{13.6.65}$$

$$\cos y \cosh x = -1 \qquad \sinh x \sin y = 0 \tag{13.6.66}$$

$$x_1 = 0 \qquad \Longrightarrow \qquad y_1 = (2n + 1)\pi \tag{13.6.67}$$

$$y_2 = (2m + 1)\pi \qquad \Longrightarrow \qquad x_2 = 0 \tag{13.6.68}$$

$$\tag{13.6.69}$$

Since (x_1, y_1) and (x_2, y_2) are the same solution, there is only one distinct family of solutions.

19. Finding all solutions,

$$\sinh z = 0 \tag{13.6.70}$$

$$\sinh x \cos y = 0 \qquad \cosh x \sin y = 0 \tag{13.6.71}$$

$$y = n\pi \qquad x = 0 \tag{13.6.72}$$

20. Finding the expression for $\tan(z)$,

$$\tan z = \frac{\sin z}{\cos z} = \frac{\sin x \cos x + \mathfrak{i}\ \sinh y \cosh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \tag{13.6.73}$$

$$\text{Re} \tan(z) = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y} \tag{13.6.74}$$

$$\text{Im} \tan(z) = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y} \tag{13.6.75}$$

13.7 Logarithm, General Power, Principal Value

1. Performing the computations in Example 1,

$$\ln(1 + 0\mathfrak{i}) = \ln(1) + \mathfrak{i}\ (0 + 2n\pi) \qquad = \mathfrak{i}\ 2n\pi \tag{13.7.1}$$

$$\ln(4 + 0\mathfrak{i}) = \ln(4) + \mathfrak{i}\ (0 + 2n\pi) \qquad = 2\ln(2) + \mathfrak{i}\ 2n\pi \tag{13.7.2}$$

$$\ln(-1 + 0\mathfrak{i}) = \ln(1) + \mathfrak{i}\ (-\pi + 2n\pi) \qquad = \mathfrak{i}\ (2n - 1)\pi \tag{13.7.3}$$

$$\ln(-4 + 0\mathfrak{i}) = \ln(4) + \mathfrak{i}\ (\pi + 2n\pi) \qquad = 2\ln(2) + \mathfrak{i}\ (2n - 1)\pi \tag{13.7.4}$$

$$\ln(\mathfrak{i}) = \ln(1) + \mathfrak{i}\ (\pi/2 + 2n\pi) \qquad = \mathfrak{i}\left[2n\pi + \frac{\pi}{2}\right] \tag{13.7.5}$$

$$\ln(4\mathfrak{i}) = \ln(4) + \mathfrak{i}\ (\pi/2 + 2n\pi) \qquad = 2\ln(2) + \mathfrak{i}\left[2n\pi + \frac{\pi}{2}\right] \tag{13.7.6}$$

$$\ln(-4\mathfrak{i}) = \ln(4) + \mathfrak{i}\ (-\pi/2 + 2n\pi) \qquad = 2\ln(2) + \mathfrak{i}\left[2n\pi - \frac{\pi}{2}\right] \tag{13.7.7}$$

$$\ln(3 - 4\mathfrak{i}) = \ln(5) - \mathfrak{i}\ [\arctan(4/3) + 2n\pi] \tag{13.7.8}$$

2. Verifying the relations,

$$z_1 = -\mathrm{i} \qquad z_2 = -1 \tag{13.7.9}$$

$$\ln(z_1 \cdot z_2) = \ln(\mathrm{i}) \qquad = 0 + \mathrm{i}\ \frac{\pi}{2} \tag{13.7.10}$$

$$\ln\ z_1 = \ln(-\mathrm{i}) \qquad = 0 - \mathrm{i}\ \frac{\pi}{2} \tag{13.7.11}$$

$$\ln\ z_2 = \ln(-1) \qquad = 0 + \mathrm{i}\ \pi \tag{13.7.12}$$

$$\ln(z_1 \cdot z_2) = \ln\ z_1 + \ln\ z_2 \tag{13.7.13}$$

$$\ln(z_1/z_2) = \ln(\mathrm{i}) \qquad = 0 + \mathrm{i}\ \frac{\pi}{2} \tag{13.7.14}$$

$$\ln(z_1) - \ln(z_2) = -\mathrm{i}\ \frac{3\pi}{2} \qquad = \mathrm{i}\ \frac{\pi}{2} \tag{13.7.15}$$

3. Using the Cauchy Riemann relations in polar form,

$$\ln\ z = \ln\ r + \mathrm{i}\ (\theta + c) \tag{13.7.16}$$

$$u_r = \frac{1}{r} \qquad u_\theta = 0 \tag{13.7.17}$$

$$v_r = 0 \qquad v_\theta = 1 \tag{13.7.18}$$

$$u_r = \frac{1}{r}\ v_\theta \qquad v_r = \frac{-1}{r}\ u_\theta \tag{13.7.19}$$

Since the Cauchy-Riemann relations hold, the function is analytic, except at the origin and the negative real axis (which is a branch cut).

4. Proving formula $4a$,

$$\exp(\ln\ z) = \exp[\ln\ r + \mathrm{i}\ \theta] \qquad = e^{\ln\ r}\ [\cos\ \theta + \mathrm{i}\ \ \sin\ \theta] \tag{13.7.20}$$

$$= r\ [\cos\ \theta + \mathrm{i}\ \ \sin\ \theta] \qquad = z \tag{13.7.21}$$

Proving formula $4b$,

$$\ln(e^z) = \ln(e^x\ \cos\ y + \mathrm{i}\ e^x\ \sin\ y) \qquad = \ln(e^x) + \mathrm{i}\ (y + 2n\pi) \tag{13.7.22}$$

$$= x + \mathrm{i}y + \mathrm{i}\ (2n\pi) \qquad = z + \mathrm{i}\ 2n\pi \tag{13.7.23}$$

5. Finding the principal logarithm,

$$z = -11 + 0\mathrm{i} \qquad \ln\ z = \ln(11) - \mathrm{i}\ \pi \tag{13.7.24}$$

$$\mathrm{Ln}\ z = \ln(11) + \mathrm{i}\ \pi \tag{13.7.25}$$

6. Finding the principal logarithm,

$$z = 4 + 4\mathrm{i} \qquad \ln z = \ln(\sqrt{32}) + \mathrm{i}\ \frac{\pi}{4} \tag{13.7.26}$$

$$\mathrm{Ln}\ z = \frac{5}{2}\ \ln 2 + \mathrm{i}\ \frac{\pi}{4} \tag{13.7.27}$$

7. Finding the principal logarithm,

$$z = 4 - 4\mathrm{i} \qquad \ln z = \ln(\sqrt{32}) - \mathrm{i}\ \frac{\pi}{4} \tag{13.7.28}$$

$$\mathrm{Ln}\ z = \frac{5}{2}\ \ln 2 - \mathrm{i}\ \frac{\pi}{4} \tag{13.7.29}$$

8. Finding the principal logarithm,

$$z_1 = 1 + \mathrm{i} \qquad \mathrm{Ln}\ z_1 = \frac{1}{2}\ \ln(2) + \mathrm{i}\ \frac{\pi}{4} \tag{13.7.30}$$

$$z_2 = 1 - \mathrm{i} \qquad \mathrm{Ln}\ z_2 = \frac{1}{2}\ \ln(2) - \mathrm{i}\ \frac{\pi}{4} \tag{13.7.31}$$

$$\tag{13.7.32}$$

9. Finding the principal logarithm,

$$z = 0.6 + 0.8\mathrm{i} \qquad \ln z = \ln(1) + \mathrm{i}\ \arctan(4/3) \tag{13.7.33}$$

$$\mathrm{Ln}\ z = \mathrm{i}\ \arctan(4/3) \tag{13.7.34}$$

10. Finding the principal logarithm,

$$z_1 = -15 + 0.1\ \mathrm{i} \tag{13.7.35}$$

$$\ln z_1 = \frac{1}{2}\ \ln(225.01) + \mathrm{i}\ [\pi - \arctan(1/150)] \tag{13.7.36}$$

$$z_2 = -15 - 0.1\ \mathrm{i} \tag{13.7.37}$$

$$\ln z_2 = \frac{1}{2}\ \ln(225.01) + \mathrm{i}\ \arctan(1/150) \tag{13.7.38}$$

$$\mathrm{Ln}\ z_2 = \frac{1}{2}\ \ln(225.01) + \mathrm{i}\ [\arctan(1/150) - \pi] \tag{13.7.39}$$

11. Finding the principal logarithm,

$$z = 0 + e\ \mathrm{i} \qquad \ln z = \ln(e) + \mathrm{i}\ \frac{\pi}{2} \tag{13.7.40}$$

$$\mathrm{Ln}\ z = 1 + \mathrm{i}\ \frac{\pi}{2} \tag{13.7.41}$$

12. Graphing some solutions in the complex plane,

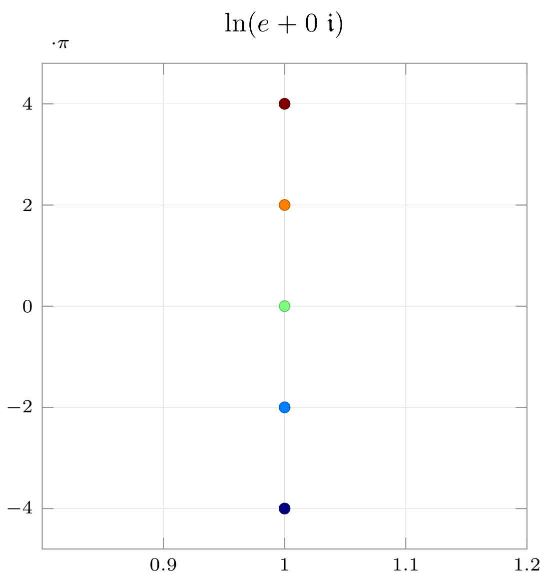
$$z = e + 0\ \mathrm{i} \qquad \ln z = \ln(e) + 2n\pi\ \mathrm{i} \tag{13.7.42}$$

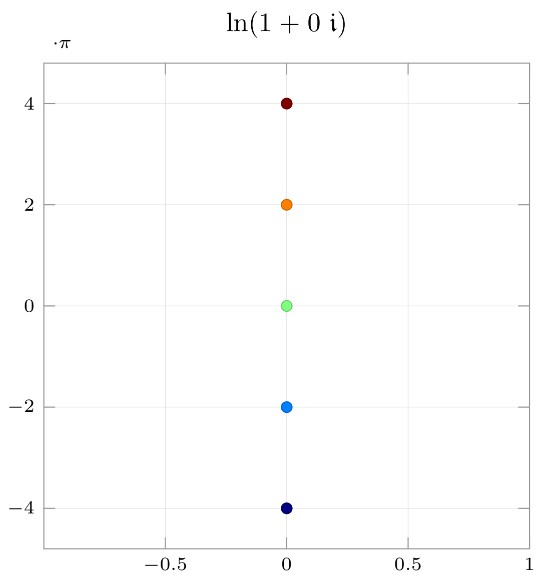
$$\ln z = 1 + 2n\pi\ \mathrm{i} \tag{13.7.43}$$

13. Graphing some solutions in the complex plane,

$$z = 1 + 0\ \mathrm{i} \qquad \ln z = \ln(1) + 2n\pi\ \mathrm{i} \tag{13.7.44}$$

$$\ln z = 0 + 2n\pi\ \mathrm{i} \tag{13.7.45}$$



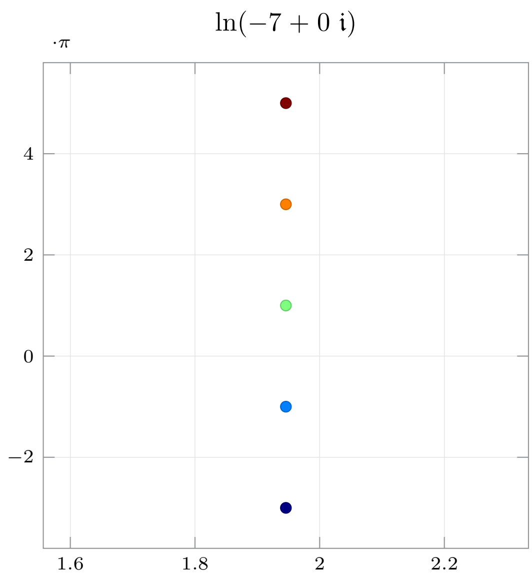


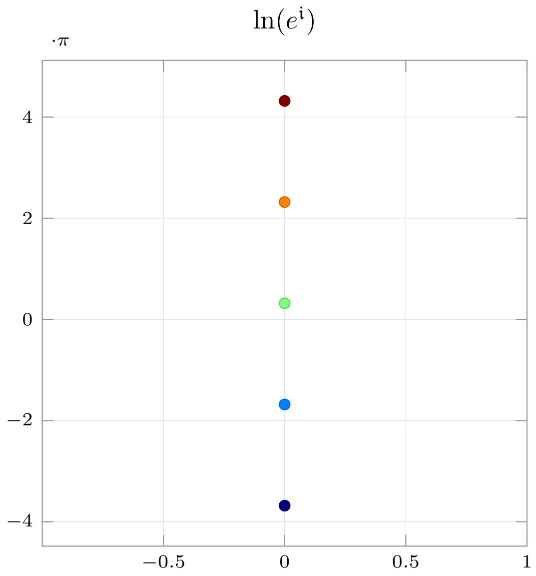
14. Graphing some solutions in the complex plane,

$$z = -7 + 0\ \mathrm{i} \qquad \ln z = \ln(7) + (2n+1)\pi\ \mathrm{i} \tag{13.7.46}$$

15. Graphing some solutions in the complex plane,

$$z = \exp(0 + \mathrm{i}) \qquad \ln z = \mathrm{i} + 2n\pi\ \mathrm{i} \tag{13.7.47}$$





16. Graphing some solutions in the complex plane,

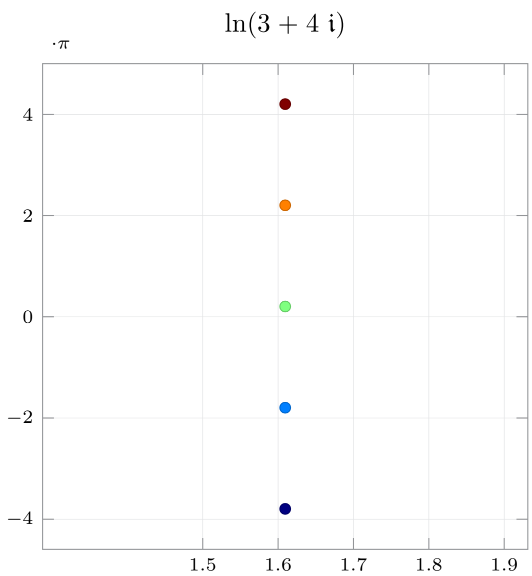
$$z = 4 + 3\,\mathrm{i} \qquad \ln z = \ln(5) + (\arctan(3/4) + 2n\pi)\,\mathrm{i} \tag{13.7.48}$$

17. Graphing some solutions in the complex plane,

$$z_1 = \mathrm{i}^2 = -1 \qquad \ln z_1 = \ln(-1 + 0\mathrm{i}) = (2n+1)\pi\,\mathrm{i} \tag{13.7.49}$$

$$z_2 = \mathrm{i} \qquad 2\ln z_2 = 2\ln(0 + \mathrm{i}) = (4n+1)\pi\,\mathrm{i} \tag{13.7.50}$$

13.7.51



18. Solving,

$$\ln z = -\frac{\pi}{2}\ \mathfrak{i} \qquad e^{\ln z} = \exp(-\pi/2\ \mathfrak{i}) = -\mathfrak{i} \tag{13.7.52}$$

$$z + 2n\pi\ \mathfrak{i} = -\mathfrak{i} \qquad z = -\mathfrak{i} + 2n\pi\ \mathfrak{i} \tag{13.7.53}$$

19. Solving,

$$\ln z = 4 - 3\mathfrak{i} \qquad e^{\ln z} = \exp(4 - 3\mathfrak{i}) \tag{13.7.54}$$

$$z + 2n\pi\ \mathfrak{i} = e^4\ (\cos 3 - \mathfrak{i}\ \sin 3) \tag{13.7.55}$$

20. Solving,

$$\ln z = e - \pi\mathfrak{i} \qquad e^{\ln z} = \exp(e - \pi\mathfrak{i}) \tag{13.7.56}$$

$$z + 2n\pi\ \mathfrak{i} = e^e\ (\cos \pi - \mathfrak{i}\ \sin \pi) \qquad z = -e^e \tag{13.7.57}$$

21. Solving,

$$\ln z = 0.6 + 0.4\mathfrak{i} \qquad e^{\ln z} = \exp(0.6 + 0.4\mathfrak{i}) \tag{13.7.58}$$

$$z + 2n\pi\ \mathfrak{i} = e^{0.6}\ [\cos(0.4) + \mathfrak{i}\ \sin(0.4)] \tag{13.7.59}$$

22. Finding the principal value,

$$(2\mathfrak{i})^{2\mathfrak{i}} = \exp[2\mathfrak{i}\ \ln(2\mathfrak{i})] \qquad \ln(2\mathfrak{i}) = \ln(2) + \mathfrak{i}\ \frac{\pi}{2} \tag{13.7.60}$$

$$z = \exp[\ln(4)\ \mathfrak{i} - \pi] \qquad z = e^{-\pi}\ [\cos(\ln 4) + \mathfrak{i}\ \sin(\ln 4)] \tag{13.7.61}$$

23. Finding the principal value,

$$(1+\mathfrak{i})^{1-\mathfrak{i}} = \exp[(1-\mathfrak{i})\ \ln(1+\mathfrak{i})] \tag{13.7.62}$$

$$\ln(1+\mathfrak{i}) = \ln(\sqrt{2}) + \mathfrak{i}\ \frac{\pi}{4} \tag{13.7.63}$$

$$z = \exp[\ln(\sqrt{2}) + \pi/4 + \mathfrak{i}\ (\pi/4 - \ln\sqrt{2})] \tag{13.7.64}$$

$$z = \sqrt{2}e^{\pi/4}\ [\cos(\pi/4 - \ln\sqrt{2}) + \mathfrak{i}\ \sin(\pi/4 - \ln\sqrt{2})] \tag{13.7.65}$$

24. Finding the principal value,

$$(1-\mathrm{i})^{1+\mathrm{i}} = \exp[(1+\mathrm{i})\ \ln(1-\mathrm{i})] \tag{13.7.66}$$

$$\ln(1-\mathrm{i}) = \ln(\sqrt{2}) - \mathrm{i}\ \frac{\pi}{4} \tag{13.7.67}$$

$$z = \exp[\ln(\sqrt{2}) + \pi/4 + \mathrm{i}\ (\ln\sqrt{2} - \pi/4)] \tag{13.7.68}$$

$$z = \sqrt{2}e^{\pi/4}\ [\cos(\ln\sqrt{2} - \pi/4) + \mathrm{i}\ \ \sin(\ln\sqrt{2} - \pi/4)] \tag{13.7.69}$$

25. Finding the principal value,

$$(-3)^{3-\mathrm{i}} = \exp[(3-\mathrm{i})\ \ln(-3)] \tag{13.7.70}$$

$$\ln(-3) = \ln 3 + \mathrm{i}\ \pi \tag{13.7.71}$$

$$z = \exp[3\ln 3 + \pi + \mathrm{i}\ (3\pi - \ln 3)] \tag{13.7.72}$$

$$z = 27e^{\pi}\ [-\cos(\ln 3) + \mathrm{i}\ \ \sin(\ln 3)] \tag{13.7.73}$$

26. Finding the principal value,

$$\mathrm{i}^{\mathrm{i}} = \exp[\mathrm{i}\ \ln(\mathrm{i})] \qquad \ln(\mathrm{i}) = \ln(1) + \mathrm{i}\ \frac{\pi}{2} \tag{13.7.74}$$

$$z = \exp(-\pi/2) \tag{13.7.75}$$

27. Finding the principal value,

$$(-1)^{2-\mathrm{i}} = \exp[(2-\mathrm{i})\ \ln(-1)] \tag{13.7.76}$$

$$\ln(-1) = \ln 1 + \mathrm{i}\ \pi \tag{13.7.77}$$

$$z = \exp[\pi + \mathrm{i}\ 2\pi] \tag{13.7.78}$$

$$z = e^{\pi}\ [\cos(2\pi) + \mathrm{i}\ \ \sin(2\pi)] = e^{\pi} \tag{13.7.79}$$

28. Finding the principal value,

$$(3+4\mathrm{i})^{1/3} = \exp[(1/3)\ \ln(3+4\mathrm{i})] \tag{13.7.80}$$

$$\ln(3+4\mathrm{i}) = \ln 5 + \mathrm{i}\ \ \arctan(4/3) \tag{13.7.81}$$

$$z = \exp\left[\frac{\ln 5}{3} + \mathrm{i}\ \frac{\arctan(4/3)}{3}\right] \tag{13.7.82}$$

$$z = 5^{1/3}\left[\cos\left(\frac{\arctan(4/3)}{3}\right) + \mathrm{i}\ \ \sin\left(\frac{\arctan(4/3)}{3}\right)\right] \tag{13.7.83}$$

29. Starting with problem 23,

$$z_1 = (x + iy)^{a+ib} = \exp\left[(a + ib)\ (\ln x \cos y + i \ln x \sin y)\right] \qquad 13.7.84$$

$$= \exp(\alpha + i\ \beta) \qquad 13.7.85$$

$$z_2 = (x - iy)^{a-ib} = \exp\left[(a - ib)\ (\ln x \cos y - i \ln x \sin y)\right] \qquad 13.7.86$$

$$= \exp(\alpha - i\ \beta) \qquad 13.7.87$$

Clearly, $z_2 = \bar{z}_1$, which provides the answer to problem 24.

30. Inverse trigonometric functions,

(a) Finding the expression,

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2} = \frac{e^{2iw} + 1}{2e^{iw}} \qquad t = e^{iw} \qquad 13.7.88$$

$$t^2 - 2zt + 1 = 0 \qquad t = z \pm \sqrt{z^2 - 1} \qquad 13.7.89$$

$$\arccos(z) = w = -i\ \ln\left(z + \sqrt{z^2 - 1}\right) \qquad 13.7.90$$

(b) Finding the expression,

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{2iw} - 1}{2i\ e^{iw}} \qquad t = e^{iw} \qquad 13.7.91$$

$$t^2 - 2i\ zt - 1 = 0 \qquad t = iz \pm \sqrt{1 - z^2} \qquad 13.7.92$$

$$\arcsin(z) = w = -i\ \ln\left(iz + \sqrt{1 - z^2}\right) \qquad 13.7.93$$

(c) Finding the expression,

$$z = \cosh w = \frac{e^{w} + e^{-w}}{2} = \frac{e^{2w} + 1}{2e^{w}} \qquad t = e^{w} \qquad 13.7.94$$

$$t^2 - 2zt + 1 = 0 \qquad t = z \pm \sqrt{z^2 - 1} \qquad 13.7.95$$

$$\cosh^{-1} z = w = \ln\left(z + \sqrt{z^2 - 1}\right) \qquad 13.7.96$$

(d) Finding the expression,

$$z = \sinh w = \frac{e^w - e^{-w}}{2} = \frac{e^{2w} - 1}{2e^w} \qquad t = e^w \tag{13.7.97}$$

$$t^2 - 2zt - 1 = 0 \qquad t = z \pm \sqrt{1 + z^2} \tag{13.7.98}$$

$$\sinh^{-1}(z) = w = \ln\left(\mathrm{i}z + \sqrt{1 + z^2}\right) \tag{13.7.99}$$

(e) Finding the expression,

$$z = \tan w = \frac{1}{\mathrm{i}} \frac{e^{\mathrm{i}w} - e^{-\mathrm{i}w}}{e^{\mathrm{i}w} + e^{-\mathrm{i}w}} = \frac{e^{2\mathrm{i}w} - 1}{\mathrm{i}\, e^{2\mathrm{i}w} + \mathrm{i}} \qquad t = e^{\mathrm{i}w} \tag{13.7.100}$$

$$t^2 - 1 = \mathrm{i}zt^2 + \mathrm{i}z \qquad t^2 = \frac{\mathrm{i}z + 1}{1 - \mathrm{i}z} \tag{13.7.101}$$

$$t = \sqrt{\frac{\mathrm{i} - z}{\mathrm{i} + z}} = \left(\frac{\mathrm{i} + z}{\mathrm{i} - z}\right)^{-1/2} \tag{13.7.102}$$

$$\arctan(z) = w = \frac{\mathrm{i}}{2} \ln\left[\frac{\mathrm{i} + z}{\mathrm{i} - z}\right] \tag{13.7.103}$$

(f) Finding the expression,

$$z = \tanh w = \frac{e^w - e^{-w}}{e^w + e^{-w}} = \frac{e^{2w} - 1}{e^{2w} + 1} \qquad t = e^w \tag{13.7.104}$$

$$t^2 - 1 = zt^2 + z \qquad t^2 = \frac{z + 1}{1 - z} \tag{13.7.105}$$

$$t = \sqrt{\frac{1 + z}{1 - z}} \tag{13.7.106}$$

$$\tanh^{-1}(z) = w = \frac{1}{2} \ln\left[\frac{1 + z}{1 - z}\right] \tag{13.7.107}$$

(g) To show the many-valued nature of $\sin^{-1} z$, look at the fact that sin and cos are periodic with period 2π. This means that $z \to z + 2n\pi$ makes no change to the output of $\sin(z)$.
The converse of this statement is that $\arcsin(z)$ is many valued with separation $\Delta w = 2n\pi$.
Since $\sqrt{1 - z^2}$ can have two possible results which differ by a factor of (-1), the two possible values of w differ by π
This provides the second branch of solutions $w_1 + (2n + 1)\pi$.