Chapter 7

Linear Algebra: Matrices, Vectors, Determinants, Linear Systems

7.1 Matrices, Vectors: Addition and Scalar Multiplication

1. Either the dimensions of the matrices do not match, or the elementwise comparison among matrices fails. Thus, they are all different.

2. From the matrix in Example 2,

$$a_{31} = 10 \qquad a_{13} = 81 \tag{7.1.1}$$

$$a_{26} = 96 \qquad a_{33} = 0 \tag{7.1.2}$$

3. The matrices and their sizes are,

$$\text{Example 1} = 3 \times 3 \qquad \text{Example 2} = 3 \times 7 \tag{7.1.3}$$

$$\text{Example 3} = 2 \times 2 \quad \text{and} \quad 2 \times 3 \qquad \text{Example 5} = 3 \times 2 \tag{7.1.4}$$

4. The main diagonals are,

$$\text{Example 1} = \{4, 0, 1\} \qquad \text{Example 2} = \{a_{11}, a_{22}\} \quad \text{and} \quad \{4, -1\} \tag{7.1.5}$$

5. Starting with the given matrix $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 40 & 33 & 81 & 0 & 21 & 47 & 33 \\ 0 & 12 & 78 & 50 & 50 & 96 & 80 \\ 10 & 0 & 0 & 27 & 43 & 78 & 56 \end{bmatrix} \tag{7.1.6}$$

$$\mathbf{B} = \frac{\mathbf{A}}{5} = \begin{bmatrix} 8.0 & 6.6 & 16.2 & 0 & 4.2 & 9.4 & 6.6 \\ 0 & 2.4 & 15.6 & 10.0 & 10.0 & 19.2 & 16.0 \\ 2.0 & 0 & 0 & 5.4 & 8.6 & 15.6 & 11.2 \end{bmatrix} \tag{7.1.7}$$

$$\mathbf{C} = \frac{\mathbf{A}}{10} = \begin{bmatrix} 4.0 & 3.3 & 8.1 & 0 & 2.1 & 4.7 & 3.3 \\ 0 & 1.2 & 7.8 & 5.0 & 5.0 & 9.6 & 8.0 \\ 1.0 & 0 & 0 & 2.7 & 4.3 & 7.8 & 5.6 \end{bmatrix} \tag{7.1.8}$$

6. To convert from kilometers to miles,

$$\mathbf{B} = \frac{\mathbf{A}}{1.6} \tag{7.1.9}$$

7.
- No, since they have different dimensions, regardless of a match in the number of their components.
- Yes
- No, since they are different tyes of mathematical object
- No, since their dimensions do not match even though the number of elements might be the same.

8. Performing the given computations,

$$2\mathbf{A} + 4\mathbf{B} = \begin{bmatrix} 0 & 24 & 16 \\ 32 & 22 & 26 \\ -6 & 16 & -14 \end{bmatrix} \qquad 4\mathbf{B} + 2\mathbf{A} = \begin{bmatrix} 0 & 24 & 16 \\ 32 & 22 & 26 \\ -6 & 16 & -14 \end{bmatrix} \tag{7.1.10}$$

$$0\mathbf{A} + \mathbf{B} = \begin{bmatrix} 0 & 5 & 2 \\ 5 & 3 & 4 \\ -2 & 4 & -2 \end{bmatrix} \qquad 0.4\mathbf{B} - 4.2\mathbf{A} = \begin{bmatrix} 0 & -6.4 & -16 \\ -23.2 & -19.8 & -19.4 \\ -5 & 1.6 & 11.8 \end{bmatrix} \tag{7.1.11}$$

9. Performing the given computations,

$$3\mathbf{A} = \begin{bmatrix} 0 & 6 & 12 \\ 18 & 15 & 15 \\ 3 & 0 & -9 \end{bmatrix} \qquad 0.5\mathbf{B} = \begin{bmatrix} 0 & 2.5 & 1 \\ 2.5 & 1.5 & 2 \\ -1 & 2 & -1 \end{bmatrix} \tag{7.1.12}$$

$$3\mathbf{A} + 0.5\mathbf{B} = \begin{bmatrix} 0 & 8.5 & 13 \\ 20.5 & 16.5 & 17 \\ 2 & 2 & -10 \end{bmatrix} \qquad 3\mathbf{A} + 0.5\mathbf{B} + \mathbf{C} = \text{invalid} \tag{7.1.13}$$

10. Performing the given computations,

$$(4 \cdot 3)\mathbf{A} = \begin{bmatrix} 0 & 24 & 48 \\ 72 & 60 & 60 \\ 12 & 0 & -36 \end{bmatrix} \qquad 4(3\mathbf{A}) = \begin{bmatrix} 0 & 24 & 48 \\ 72 & 60 & 60 \\ 12 & 0 & -36 \end{bmatrix} \tag{7.1.14}$$

$$14\mathbf{B} - 3\mathbf{B} = \begin{bmatrix} 0 & 55 & 22 \\ 55 & 33 & 44 \\ -22 & 44 & -22 \end{bmatrix} \qquad 11\mathbf{B} = \begin{bmatrix} 0 & 55 & 22 \\ 55 & 33 & 44 \\ -22 & 44 & -22 \end{bmatrix} \tag{7.1.15}$$

11. Performing the given computations,

$$8\mathbf{C} + 10\mathbf{D} = \begin{bmatrix} 0 & 26 \\ 34 & 32 \\ 28 & -10 \end{bmatrix} \qquad 2(5\mathbf{D} = 4\mathbf{C}) = \begin{bmatrix} 0 & 26 \\ 34 & 32 \\ 28 & -10 \end{bmatrix} \tag{7.1.16}$$

$$0.6\mathbf{C} - 0.6\mathbf{D} = \begin{bmatrix} 5.4 & 0.6 \\ -4.2 & 2.4 \\ -0.6 & 0.6 \end{bmatrix} \qquad 0.6(\mathbf{C} - \mathbf{D}) = \begin{bmatrix} 5.4 & 0.6 \\ -4.2 & 2.4 \\ -0.6 & 0.6 \end{bmatrix} \tag{7.1.17}$$

12. Performing the given computations,

$$(\mathbf{C} + \mathbf{D}) + \mathbf{E} = \begin{bmatrix} 1 & 5 \\ 6 & 8 \\ 6 & -2 \end{bmatrix} \qquad (\mathbf{D} + \mathbf{E}) + \mathbf{C} = \begin{bmatrix} 1 & 5 \\ 6 & 8 \\ 6 & -2 \end{bmatrix} \tag{7.1.18}$$

$$0(\mathbf{C} - \mathbf{E}) + 4\mathbf{D} = \begin{bmatrix} -16 & 4 \\ 20 & 0 \\ 8 & -4 \end{bmatrix} \qquad \mathbf{A} - 0\mathbf{C} = \begin{bmatrix} 0 & 2 & 4 \\ 6 & 5 & 5 \\ 1 & 0 & -3 \end{bmatrix} \tag{7.1.19}$$

13.

$$(2 \cdot 7)\mathbf{C} = \begin{bmatrix} 70 & 28 \\ -28 & 56 \\ 14 & 0 \end{bmatrix} \qquad 2(7\mathbf{C}) = \begin{bmatrix} 70 & 28 \\ -28 & 56 \\ 14 & 0 \end{bmatrix} \tag{7.1.20}$$

$$-\mathbf{D} + 0\mathbf{E} = \begin{bmatrix} 4 & -1 \\ -5 & 0 \\ -2 & 1 \end{bmatrix} \qquad \mathbf{E} - \mathbf{D} + \mathbf{C} + \mathbf{u} = \text{invalid} \tag{7.1.21}$$

14.

$$(5\mathbf{u} + 5\mathbf{v}) - \frac{1}{2}\mathbf{w} = \begin{bmatrix} 5 \\ 30 \\ -10 \end{bmatrix} \qquad -20(\mathbf{u} + \mathbf{v}) + 2\mathbf{w} = \begin{bmatrix} -20 \\ -120 \\ 40 \end{bmatrix} \tag{7.1.22}$$

$$\mathbf{E} - (\mathbf{u} + \mathbf{v}) = \text{invalid} \qquad 10(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.1.23}$$

15.

$$(\mathbf{u} + \mathbf{v}) - \mathbf{w} = \begin{bmatrix} 5.5 \\ 33 \\ -11 \end{bmatrix} \qquad \mathbf{u} + (\mathbf{v} - \mathbf{w}) = \begin{bmatrix} 5.5 \\ 33 \\ -11 \end{bmatrix} \tag{7.1.24}$$

$$\mathbf{C} + 0\mathbf{w} = \text{dimension mismatch} \qquad 0\mathbf{E} + \mathbf{u} - \mathbf{v} = \text{dimension mismatch} \tag{7.1.25}$$

16.

$$15\mathbf{v} - 3\mathbf{w} - 0\mathbf{u} = \begin{bmatrix} 0 \\ 135 \\ 0 \end{bmatrix} \qquad -3(\mathbf{w} + 15\mathbf{v}) = \begin{bmatrix} 0 \\ 135 \\ 0 \end{bmatrix} \tag{7.1.26}$$

$$\mathbf{D} - \mathbf{u} + 3\mathbf{C} = \text{invalid} \qquad 8.5\mathbf{w} - 11.1\mathbf{u} + 0.4\mathbf{v} = \begin{bmatrix} -59.55 \\ -253.8 \\ 119.1 \end{bmatrix} \tag{7.1.27}$$

17. Resultant force is the sum of all the vectors,

$$\mathbf{F_r} = \mathbf{u} + \mathbf{v} + \mathbf{w} = \begin{bmatrix} -4.5 \\ -27 \\ 9 \end{bmatrix} \tag{7.1.28}$$

18. For the 4 forces to be in equilibrium, their vector sum is $\mathbf{0}$

$$\mathbf{0} = \mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{p} \tag{7.1.29}$$

$$\mathbf{p} = \begin{bmatrix} 4.5 \\ 27 \\ -9 \end{bmatrix} \tag{7.1.30}$$

19. TBC

20. Using matrices to represent network connectivity,

(a) The start and end points of each branch are,

$$1 : A \to X \qquad 2 : B \to A \qquad 3 : C \to A \qquad 4 : B \to X \tag{7.1.31}$$

$$5 : B \to C \qquad 6 : X \to C \tag{7.1.32}$$

This translates to the nodal incidence matrix,

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \tag{7.1.33}$$

(b) The start and end points of each branch are,

$$1 : B \to A \qquad 2 : A \to B \qquad 3 : C \to X \qquad 4 : X \to A \tag{7.1.34}$$

$$5 : C \to A \tag{7.1.35}$$

This translates to the nodal incidence matrix,

$$\begin{bmatrix} -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \tag{7.1.36}$$

The start and end points of each branch are,

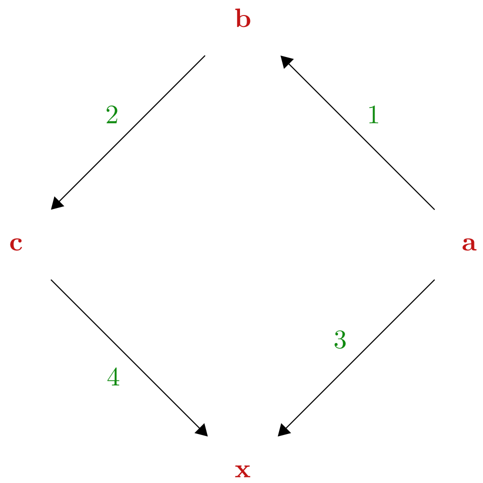
$$1: A \to B \qquad 2: B \to C \qquad 3: C \to X \qquad 4: D \to X \tag{7.1.37}$$

$$5: D \to A \qquad 6: A \to C \qquad 7: C \to A \tag{7.1.38}$$

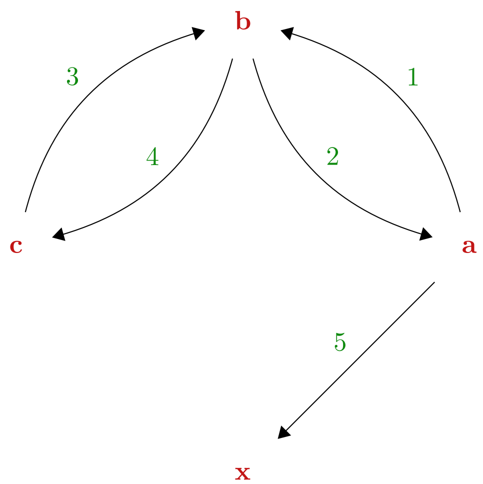
This translates to the nodal incidence matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \tag{7.1.39}$$

(c) Converting the given nodal incidence matrices into graphs,



Converting the given nodal incidence matrices into graphs,



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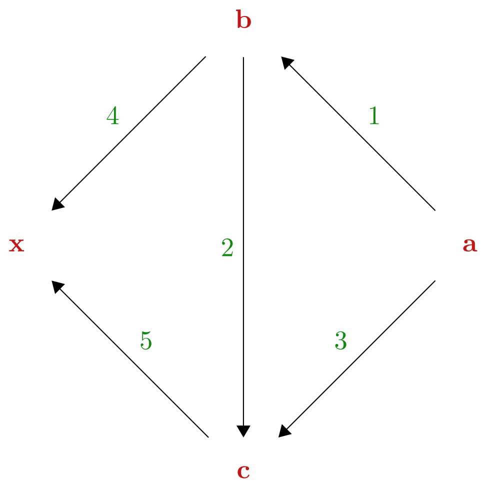

(d) Mesh incidence matrix, with all 4 meshes counter-clockwise.

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{7.1.40}$$

7.2 Matrix Multiplication

1. Matrix multiplication requires the two matrices to have the same inner dimension because the linear transform requires every output variable to be defined as a linear function of every input variable.

$$y_j = f(x_1, x_2, \dots, x_n) \tag{7.2.1}$$

$$\forall\ j \in \{1, \dots, m\} \tag{7.2.2}$$

2. A matrix is skew-symmetric and symmetric. This means that

$$\mathbf{A} = \mathbf{A}^T \qquad \mathbf{A} = -\mathbf{A}^T \tag{7.2.3}$$

$$\implies \mathbf{A} = \mathbf{0} \tag{7.2.4}$$

3. Outer product of a column and row vector, is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} \tag{7.2.5}$$

The ratio of any two columns is fixed as is the ratio of any two rows. This means that not every square matrix can be represented using the above form.

4. A skew symmetric matrix of order n has to have all diagonal entries be zero. All lower traingular entries can be distinct. So the number of different entries is 6 (for order 4) and $0.5(n^2 - n)$ for order n.

5. For symmetric matrices, the diagonal elements are unconstrained. The lower triangular and upper triangular entries match. So the number of distinct entries is $0.5(n^2 + n)$. For order 4, this is 10 entries.

6. Verifying algebra of triangular matrices,

$$\mathbf{U}_1 + \mathbf{U}_2 \quad \text{is upper triangular} \tag{7.2.6}$$

$$\mathbf{U}_1\mathbf{U}_2 \quad \text{is upper triangular} \tag{7.2.7}$$

$$\mathbf{U}_1^2 \quad \text{is upper triangular} \tag{7.2.8}$$

$$\mathbf{U}_1 + \mathbf{L}_1 \quad \text{is not triangular} \tag{7.2.9}$$

$$\mathbf{U}_1\mathbf{L}_1 \quad \text{is not triangular} \tag{7.2.10}$$

$$\mathbf{L}_1\mathbf{L}_2 \quad \text{is lower triangular} \tag{7.2.11}$$

7. Idempotent matrix means,

$$\mathbf{A}^2 = \mathbf{A} \tag{7.2.12}$$

Consider the general 2×2 case,

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{7.2.13}$$

$$b = 0 \qquad \text{or } (a + d) = 1 \tag{7.2.14}$$

$$c = 0 \qquad \text{or } (a + d) = 1 \tag{7.2.15}$$

$$a(1 - a) = bc \tag{7.2.16}$$

$$d(1 - d) = bc \tag{7.2.17}$$

$$\tag{7.2.18}$$

If only one among b or c is zero, then $(a + d) = 1$.

If both b and c are zero, then the matrix is diagonal with a and d limited to the set $\{0, 1\}$.

8. Nilpotent matrix defined for some positive integer m,

$$\mathbf{B}^m = \mathbf{0} \tag{7.2.19}$$

$$\tag{7.2.20}$$

Examples are any traingular matrix with zero along the diagonal.

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \qquad \begin{bmatrix} c & c \\ -c & -c \end{bmatrix} \tag{7.2.21}$$

9. Transpose is an involution,

$$\mathbf{B} = \mathbf{A}^T \qquad \mathbf{C} = \mathbf{B}^T \tag{7.2.22}$$

$$[b_{jk}] = [a_{kj}] \qquad [c_{jk}] = [b_{kj}] = [a_{jk}] \tag{7.2.23}$$

$$\mathbf{C} = \left(\mathbf{A}^T\right)^T = \mathbf{A} \tag{7.2.24}$$

Transpose is associative under addtion,

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \qquad [c_{jk}] = [a_{jk} + b_{jk}] \tag{7.2.25}$$

$$\mathbf{D} = \mathbf{A}^T + \mathbf{B}^T \qquad [d_{jk}] = [a_{kj} + b_{kj}] \tag{7.2.26}$$

$$\mathbf{C}^T = \mathbf{D} \tag{7.2.27}$$

Transpose is associative under scalar multiplication,

$$(c\mathbf{A})^T = [c\ a_{kj}] = c\ [a_{kj}] = c\ \mathbf{A}^T \tag{7.2.28}$$

10. To show the transpose of the product of two matrices,

$$\mathbf{C} = \mathbf{A}\mathbf{B} \qquad [c_{jk}] = \sum_{l=1}^{n} a_{jl}\ b_{lk} \tag{7.2.29}$$

$$\mathbf{C}^T = \mathbf{D} \qquad [d_{jk}] = [c_{kj}] = \sum_{l=1}^{n} a_{kl}\ b_{lj} \tag{7.2.30}$$

$$\mathbf{B}^T\ \mathbf{A}^T = \mathbf{E} \qquad [e_{jk}] = \sum_{m=1}^{n} b_{mj}\ a_{km} \tag{7.2.31}$$

$$\mathbf{D} = \mathbf{E} \tag{7.2.32}$$

This proves the relation since the two expressions are identical but for the choice of dummy variable l, m inside the summation.

11. Performing the given matrix multiplications,

$$\mathbf{AB} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & -6 \\ -5 & 7 & -12 \\ -5 & -1 & -4 \end{bmatrix} \tag{7.2.33}$$

$$\tag{7.2.34}$$

Not showing the intermediate steps since they are all mental arithmetic.

$$\mathbf{AB}^T = \mathbf{AB} \qquad \mathbf{BA} = \begin{bmatrix} 10 & -5 & -15 \\ -14 & 7 & -3 \\ -2 & -4 & -4 \end{bmatrix} \tag{7.2.35}$$

$$\mathbf{B}^T\ \mathbf{A} = \mathbf{BA} \tag{7.2.36}$$

12. Performing the given matrix multiplication,

$$\mathbf{AA}^T = \begin{bmatrix} 29 & 8 & 6 \\ 8 & 41 & 12 \\ 6 & 12 & 9 \end{bmatrix} \qquad \mathbf{A}^2 = \begin{bmatrix} 23 & -4 & 6 \\ -4 & 17 & 12 \\ 2 & 4 & 19 \end{bmatrix} \tag{7.2.37}$$

$$\mathbf{BB}^T = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \mathbf{B}^2 = \mathbf{BB}^T \tag{7.2.38}$$

13. Performing the given matrix multiplication,

$$\mathbf{CC}^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 13 & -6 \\ 0 & -6 & 4 \end{bmatrix} \qquad \mathbf{BC} = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix} \tag{7.2.39}$$

$$\mathbf{CB} = \text{not defined} \qquad \mathbf{C}^T\ \mathbf{B} = \begin{bmatrix} -9 & 3 & 4 \\ -5 & -1 & 0 \end{bmatrix} \tag{7.2.40}$$

14. Performing the given matrix multiplication,

$$3\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} 10 & 0 & 9 \\ 0 & 1 & 18 \\ 3 & 6 & 10 \end{bmatrix} \tag{7.2.41}$$

$$(3\mathbf{A} - 2\mathbf{B})^T = \begin{bmatrix} 10 & 0 & 3 \\ 0 & 1 & 6 \\ 9 & 18 & 10 \end{bmatrix} \tag{7.2.42}$$

$$3\mathbf{A}^T - 2\mathbf{B}^T = \begin{bmatrix} 12 & -6 & 3 \\ -6 & 3 & 6 \\ 9 & 18 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -6 & 0 \\ -6 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \tag{7.2.43}$$

$$(3\mathbf{A} - 2\mathbf{B})^T \ \mathbf{a}^T = \begin{bmatrix} 10 & 0 & 3 \\ 0 & 1 & 6 \\ 9 & 18 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ -27 \end{bmatrix} \tag{7.2.44}$$

15. Performing the given matrix multiplication,

$$\mathbf{Aa} = \text{not defined} \qquad \mathbf{A} \ \mathbf{a}^T = \begin{bmatrix} 10 \\ -2 \\ -3 \end{bmatrix} \tag{7.2.45}$$

$$(\mathbf{Ab})^T = \begin{bmatrix} 7 & -11 & 3 \end{bmatrix} \qquad \mathbf{b}^T \ \mathbf{A}^T = \mathbf{Ab}^T \tag{7.2.46}$$

16. Performing the given matrix multiplication,

$$\mathbf{BC} = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix} \qquad \mathbf{B} \ \mathbf{C}^T = \text{not defined} \tag{7.2.47}$$

$$(\mathbf{Bb}) = \begin{bmatrix} 0 & -8 & 2 \end{bmatrix} \qquad \mathbf{b}^T \ \mathbf{B} = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix} \tag{7.2.48}$$

17. Performing the given matrix multiplication,

$$\mathbf{ABC} = \mathbf{A} \cdot \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -30 & -18 \\ 45 & 9 \\ 5 & -7 \end{bmatrix} \qquad \mathbf{ABa} = \text{not defined} \tag{7.2.49}$$

$$\mathbf{ABb} = \mathbf{A} \cdot \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 22 \\ 4 \\ -12 \end{bmatrix} \qquad \mathbf{C}\ \mathbf{a}^T = \text{not defined} \tag{7.2.50}$$

18. Performing the given matrix multiplication,

$$\mathbf{ab} = \begin{bmatrix} 1 \end{bmatrix} \qquad \mathbf{ba} = \begin{bmatrix} 3 & -6 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix} \tag{7.2.51}$$

$$\mathbf{aA} = \begin{bmatrix} 8 & -4 & -9 \end{bmatrix} \qquad \mathbf{Bb} = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix} \tag{7.2.52}$$

19. Performing the given matrix multiplication,

$$1.5\mathbf{a} + 3\mathbf{b} = \text{not defined} \qquad 1.5\mathbf{a}^T + 3\mathbf{b} = \begin{bmatrix} 1.5 \\ 0 \\ -3 \end{bmatrix} \tag{7.2.53}$$

$$(\mathbf{A} - \mathbf{B})\mathbf{b} = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 0 & 6 \\ 1 & 2 & 4 \end{bmatrix} \cdot \mathbf{b} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{Ab} - \mathbf{Bb} = (\mathbf{A} - \mathbf{B})\mathbf{b} \tag{7.2.54}$$

20. Performing the given matrix multiplication,

$$\mathbf{b}^T\ \mathbf{A}\mathbf{b} = \mathbf{b}^T \cdot \begin{bmatrix} 7 \\ -11 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix} \tag{7.2.55}$$

$$\mathbf{a}\mathbf{B}\ \mathbf{a}^T = \begin{bmatrix} 7 & -5 & 0 \end{bmatrix} \cdot \mathbf{a}^T = \begin{bmatrix} 17 \end{bmatrix} \tag{7.2.56}$$

$$\mathbf{a}\mathbf{C}\ \mathbf{C}^T = \mathbf{a} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 2 & 13 & -6 \\ 0 & -6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 12 \end{bmatrix} \tag{7.2.57}$$

$$\mathbf{C}^T\ \mathbf{b}\mathbf{a} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \cdot \mathbf{a} = \begin{bmatrix} 5 & -10 & 0 \\ 5 & -10 & 0 \end{bmatrix} \tag{7.2.58}$$

21. Proving the relations for 2×2 matrices,

$$(k\mathbf{A})\ \mathbf{B} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} ka_{11}b_{11} + ka_{12}b_{21} & ka_{11}b_{12} + ka_{12}b_{22} \\ ka_{21}b_{11} + ka_{22}b_{21} & ka_{21}b_{12} + ka_{22}b_{22} \end{bmatrix} \tag{7.2.59}$$

$$= k \cdot \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} = k\ (\mathbf{A}\mathbf{B}) \tag{7.2.60}$$

Proving $2b$,

$$(\mathbf{A}\mathbf{B})\ \mathbf{C} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{7.2.61}$$

$$\mathbf{A}\ (\mathbf{B}\mathbf{C}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{bmatrix} \tag{7.2.62}$$

The elementwise comparison of the two lines yields the equality.
Proving $2c$, using only the first element,

$$\Big[(\mathbf{A} + \mathbf{B})\ \mathbf{C}\Big]_{11} = (a_{11} + b_{11})c_{11} + (a_{12} + b_{12})c_{21} \tag{7.2.63}$$

$$\Big[(\mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C})\Big]_{11} = (a_{11}c_{11} + b_{11}c_{11}) + (a_{12}c_{21} + b_{12}c_{21}) \tag{7.2.64}$$

The equality follows from the distributive law of scalar multiplication.

Proving $2d$, using the first element,

$$\left[\mathbf{C}\ (\mathbf{A}+\mathbf{B})\right]_{11} = c_{11}(a_{11}+b_{11}) + c_{21}(a_{12}+b_{12}) \tag{7.2.65}$$

$$\left[(\mathbf{CA}+\mathbf{CB})\right]_{11} = (c_{11}a_{11}+c_{11}b_{11}) + (c_{12}a_{12}+c_{12}b_{12}) \tag{7.2.66}$$

The equality follows from the distributive law of scalar multiplication.

22. Expressing matrix multiplication in terms of row and column vectors, using superscipt and subscript to indicate a row and column of the matrix.

$$\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \mathbf{Ab}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1\mathbf{b}_1 & \mathbf{a}^2\mathbf{b}_1 & \mathbf{a}^3\mathbf{b}_1 \\ \mathbf{a}^1\mathbf{b}_2 & \mathbf{a}^2\mathbf{b}_2 & \mathbf{a}^3\mathbf{b}_2 \\ \mathbf{a}^1\mathbf{b}_3 & \mathbf{a}^2\mathbf{b}_3 & \mathbf{a}^3\mathbf{b}_3 \end{bmatrix} \tag{7.2.67}$$

This convention is used for this problem and not present in the text.

23. Calculating the product columnwise,

$$\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \mathbf{Ab}_3 \end{bmatrix} = \begin{bmatrix} 10 & -14 & -6 \\ -5 & 7 & -12 \\ -5 & -1 & -4 \end{bmatrix} \tag{7.2.68}$$

24. Given the constraint on $\mathbf{B}$,

$$b_{jk} = j + k \qquad \mathbf{AB} = \mathbf{BA} \tag{7.2.69}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{7.2.70}$$

$$2a_{11} + 3a_{12} = 2a_{11} + 3a_{21} \qquad 3a_{11} + 4a_{12} = 2a_{12} + 3a_{22} \tag{7.2.71}$$

$$2a_{21} + 3a_{22} = 3a_{11} + 4a_{21} \qquad 3a_{21} + 4a_{22} = 3a_{12} + 4a_{22} \tag{7.2.72}$$

This is a system of 4 equations 4 variables. Solving,

$$a_{12} = a_{21} \qquad 3a_{11} + 2a_{12} = 3a_{11} + 2a_{21} \tag{7.2.73}$$

$$a_{22} = a_{11} + \frac{2}{3}\,a_{12} \qquad a_{11} = \text{free} \tag{7.2.74}$$

$$\mathbf{A} = \begin{bmatrix} x & y \\ y & (x + \frac{2y}{3}) \end{bmatrix} \tag{7.2.75}$$

25. Symmetric matrices

(a) Verifying,

$$\mathbf{A} = \mathbf{A}^T \qquad \implies a_{jk} = a_{kj} \tag{7.2.76}$$

$$\mathbf{A} = -\mathbf{A}^T \qquad \implies a_{jk} = -a_{kj} \tag{7.2.77}$$

Examples TBC.

(b) Writing the matrix C in the given form,

$$(\mathbf{C} + \mathbf{C}^T) = \mathbf{D} \qquad \mathbf{D}^T = \mathbf{C}^T + \left(\mathbf{C}^T\right)^T \tag{7.2.78}$$

$$\mathbf{D}^T = \mathbf{D} \tag{7.2.79}$$

$$(\mathbf{C} - \mathbf{C}^T) = \mathbf{E} \qquad \mathbf{E}^T = \mathbf{C}^T - \left(\mathbf{C}^T\right)^T \tag{7.2.80}$$

$$\mathbf{E}^T = -\mathbf{E} \tag{7.2.81}$$

Using the previous result as a starting point,

$$\mathbf{A} = \frac{(\mathbf{A} + \mathbf{A}^T) + (\mathbf{A} - \mathbf{A}^T)}{2} \tag{7.2.82}$$

$$\mathbf{S} = 0.5(\mathbf{A} + \mathbf{A}^T) \qquad \mathbf{T} = 0.5(\mathbf{A} - \mathbf{A}^T) \tag{7.2.83}$$

Representing A and B in this form,

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & 4 \\ 2 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} \tag{7.2.84}$$

$$\mathbf{B} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.2.85}$$

(c) Let the matrices be $\{\mathbf{A}^i\}$ and the scalar coefficients $\{\lambda_i\}$,

$$\mathbf{S} = \sum_{i=1}^{n} \lambda_i \mathbf{A}^i \qquad S_{jk} = \sum_{i=1}^{n} \lambda_i \mathbf{A}^i_{jk} \tag{7.2.86}$$

$$S_{jk} = \sum_{i=1}^{n} \lambda_i \mathbf{A}^i_{kj} = S_{kj} \qquad \mathbf{S} = \mathbf{S}^T \tag{7.2.87}$$

Repeating the procedure for skew symmetric matrices $\{\mathbf{B}^i\}$,

$$\mathbf{P} = \sum_{i=1}^{n} \lambda_i \mathbf{B}^i \qquad P_{jk} = \sum_{i=1}^{n} \lambda_i \mathbf{B}^i_{jk} \tag{7.2.88}$$

$$P_{jk} = -\sum_{i=1}^{n} \lambda_i \mathbf{B}^i_{kj} = -P_{kj} \qquad \mathbf{P} = -\mathbf{P}^T \tag{7.2.89}$$

(d) Given $\mathbf{A}$ and $\mathbf{B}$ are symmetric,

$$(\mathbf{AB})^T = \mathbf{B}^T\ \mathbf{A}^T = \mathbf{BA} \tag{7.2.90}$$

$$(\mathbf{AB})^T = \mathbf{AB} \qquad \implies \mathbf{BA} = \mathbf{AB} \tag{7.2.91}$$

(e) Given $\mathbf{A}$ and $\mathbf{B}$ are skew-symmetric,

$$(\mathbf{AB})^T = \mathbf{B}^T\ \mathbf{A}^T = -\mathbf{B} \cdot -\mathbf{A} \tag{7.2.92}$$

$$(\mathbf{AB})^T = -\mathbf{AB} \qquad \implies \mathbf{BA} = -\mathbf{AB} \tag{7.2.93}$$

26. Building the transition matrix, where the column and row indices are the start and end states respectively.

$$\mathbf{T} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \qquad \mathbf{A}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{7.2.94}$$

$$\mathbf{A}_1 = \mathbf{TA}_0 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \qquad \mathbf{A}_2 = \mathbf{TA}_1 = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} \tag{7.2.95}$$

$$\mathbf{A}_3 = \mathbf{TA}_2 = \begin{bmatrix} 0.722 \\ 0.278 \end{bmatrix} \tag{7.2.96}$$

27. Calculating the first few steps of the Markov matrix in Example 13,

$$\mathbf{T} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \tag{7.2.97}$$

$$\mathbf{S}^i = \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix}, \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}, \begin{bmatrix} 17.05 \\ 43.8 \\ 39.15 \end{bmatrix}, \begin{bmatrix} 16.31 \\ 50.66 \\ 33.02 \end{bmatrix}, \begin{bmatrix} 16.48 \\ 55.46 \\ 28.05 \end{bmatrix}, \begin{bmatrix} 17.08 \\ 58.82 \\ 24.08 \end{bmatrix} \tag{7.2.98}$$

$$\begin{bmatrix} 17.84 \\ 61.17 \\ 20.98 \end{bmatrix}, \begin{bmatrix} 18.60 \\ 62.82 \\ 18.56 \end{bmatrix}, \begin{bmatrix} 19.30 \\ 63.97 \\ 16.71 \end{bmatrix}, \begin{bmatrix} 19.91 \\ 64.78 \\ 15.30 \end{bmatrix}, \begin{bmatrix} 20.41 \\ 65.34 \\ 14.23 \end{bmatrix}, \tag{7.2.99}$$

Since other transition matrices and other initial conditions involve the same procedure, they are omitted. TBC

28. The transition matrix and I.C. are,

$$\mathbf{T} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \qquad \mathbf{S}^0 = \begin{bmatrix} 1200 \\ 98800 \end{bmatrix} \tag{7.2.100}$$

$$\mathbf{S}^1 = \begin{bmatrix} 1278 \\ 98722 \end{bmatrix} \qquad \mathbf{S}^2 = \begin{bmatrix} 1347 \\ 98653 \end{bmatrix} \tag{7.2.101}$$

$$\mathbf{S}^3 = \begin{bmatrix} 1409. \\ 98591. \end{bmatrix}, \tag{7.2.102}$$

29. Defining the profit matrix, (solutions are wrong because they have a typo in the first entry of $\mathbf{p}$)

$$\mathbf{A} = \begin{bmatrix} 400 & 60 & 240 \\ 100 & 120 & 500 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 35 \\ 62 \\ 30 \end{bmatrix} \qquad \mathbf{v} = \mathbf{A}\mathbf{p} = \begin{bmatrix} 24920 \\ 25940 \end{bmatrix} \tag{7.2.103}$$

30. Rotation matrices,

(a) Single rotation counterclockwise by θ. Let $R = 1$ for convenience,

$$(x_1, x_2) = \{\cos(\alpha), \sin(\alpha)\} \tag{7.2.104}$$

$$(y_1, y_2) = \{\cos(\alpha + \theta), \sin(\alpha + \theta)\} \tag{7.2.105}$$

$$y_1 = x_1 \cos(\theta) - x_2 \sin(\theta) \tag{7.2.106}$$

$$y_2 = x_2 \cos(\theta) + x_1 \sin(\theta) \tag{7.2.107}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \tag{7.2.108}$$

(b) Repeated rotation by angle θ,

$$\mathbf{A}(n\theta)\ \mathbf{A}(\theta) = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{7.2.109}$$

$$= \begin{bmatrix} \cos(n\theta + \theta) & -\sin(n\theta + \theta) \\ \sin(n\theta + \theta) & \cos(n\theta + \theta) \end{bmatrix} = \mathbf{A}\{(n+1)\theta\} \tag{7.2.110}$$

By induction, this holds for all n since it can be proved by brute force that it holds for $n = 1$.

$$\mathbf{A}(n\theta) = [\mathbf{A}(\theta)]^n \tag{7.2.111}$$

(c) Using the repeated rotation by α and β, and matri multiplication representation,

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \tag{7.2.112}$$

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta \tag{7.2.113}$$

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta \tag{7.2.114}$$

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta \tag{7.2.115}$$

The subtraction formulas depend on the fact that $\sin(-x) = -\sin(x)$ whereas $\cos(-x) = \cos(x)$

(d) A scalar matrix would scale all 3 coordinates equally by the scalar k.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ x_2 \\ 0.5x_3 \end{bmatrix} \tag{7.2.116}$$

$$\mathbf{D}\mathbf{x} = \mathbf{y} \tag{7.2.117}$$

(e) The row index contatining the ones and zeros represents the coordinate that will be unchanged. The other two coordinates undergo rotation by ϕ in the counterclockwise direction.

The three matrices represent rotations in yz, xz and xy planes respectively.

7.3 Linear Systems of Equations, Gauss Elimination

1. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cc|c} 4 & -6 & -11 \\ -3 & 8 & 10 \end{array}\right] \qquad \mathbf{R} = \left[\begin{array}{cc|c} 4 & -6 & -11 \\ 0 & 3.5 & 1.75 \end{array}\right] \tag{7.3.1}$$

$$x_2 = \frac{1.75}{3.5} = 0.5 \qquad x_1 = \frac{-11+3}{4} = -2 \tag{7.3.2}$$

2. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cc|c} 3 & -0.5 & 0.6 \\ 1.5 & 4.5 & 6 \end{array}\right] \qquad \mathbf{R} = \left[\begin{array}{cc|c} 3 & -0.5 & 0.6 \\ 0 & 4.75 & 5.7 \end{array}\right] \tag{7.3.3}$$

$$x_2 = \frac{5.7}{4.75} = 1.2 \qquad x_1 = \frac{0.6+0.6}{3} = 0.4 \tag{7.3.4}$$

3. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{array}\right] \qquad \implies \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{array}\right] \tag{7.3.5}$$

$$\mathbf{R} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & -12.5 & 62.5 \end{array}\right] \tag{7.3.6}$$

$$x_3 = \frac{62.5}{-12.5} = -5 \qquad x_2 = \frac{-6+30}{8} = 3 \tag{7.3.7}$$

$$x_1 = \frac{9-5-3}{1} = 1 \tag{7.3.8}$$

4. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 4 & 1 & 0 & 4 \\ 5 & -3 & 1 & 2 \\ -9 & 2 & -1 & 5 \end{array}\right] \implies \left[\begin{array}{ccc|c} 4 & 1 & 0 & 4 \\ 0 & -4.25 & 1 & -3 \\ 0 & 4.25 & -1 & 14 \end{array}\right] \tag{7.3.9}$$

$$\mathbf{R} = \left[\begin{array}{ccc|c} 4 & 1 & 0 & 4 \\ 0 & -4.25 & 1 & -3 \\ 0 & 0 & 0 & 8 \end{array}\right] \tag{7.3.10}$$

$$\text{no solution} \tag{7.3.11}$$

5. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cc|c} 13 & 12 & -6 \\ -4 & 7 & -73 \\ 11 & -13 & 157 \end{array}\right] = \left[\begin{array}{cc|c} 13 & 12 & -6 \\ 0 & \frac{139}{13} & -\frac{973}{13} \\ 0 & -\frac{301}{13} & \frac{2107}{13} \end{array}\right] \tag{7.3.12}$$

$$\mathbf{R} = \left[\begin{array}{cc|c} 13 & 12 & -6 \\ 0 & \frac{139}{13} & -\frac{973}{13} \\ 0 & 0 & 0 \end{array}\right] \tag{7.3.13}$$

$$x_2 = \frac{-973}{139} = -7 \qquad x_1 = \frac{-6+84}{13} = 6 \tag{7.3.14}$$

6. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 4 & -8 & 3 & 16 \\ -1 & 2 & -5 & -21 \\ 3 & -6 & 1 & 7 \end{array}\right] \implies \left[\begin{array}{ccc|c} 4 & -8 & 3 & 16 \\ 0 & 0 & -\frac{17}{4} & -17 \\ 0 & 0 & -\frac{5}{4} & -5 \end{array}\right] \tag{7.3.15}$$

$$\mathbf{R} = \left[\begin{array}{ccc|c} 4 & -8 & 3 & 16 \\ 0 & 0 & -\frac{17}{4} & -17 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{7.3.16}$$

$$x_3 = \frac{-17}{-17/4} = 4 \qquad x_2 = t_2 \text{ (free)} \tag{7.3.17}$$

$$x_1 = \frac{16-12+8t_2}{4} = 1+2t_2 \tag{7.3.18}$$

7. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ -1 & 1 & -2 & 0 \\ 4 & 0 & 6 & 0 \end{array}\right] \qquad \Longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 0 & 3 & -1.5 & 0 \\ 0 & -8 & 4 & 0 \end{array}\right] \tag{7.3.19}$$

$$\mathbf{R} = \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 0 & 3 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{7.3.20}$$

$$x_3 = \frac{-17}{-17/4} = t_3 \text{ (free)} \qquad x_2 = \frac{1}{2}\, t_3 \tag{7.3.21}$$

$$x_1 = \frac{16 - 12 + 8t_2}{4} = -\frac{3}{2}\, t_3 \tag{7.3.22}$$

8. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right] \qquad \Longrightarrow \left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \end{array}\right] \tag{7.3.23}$$

$$= \left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 4 & 3 & 8 \\ 0 & -\frac{4}{3} & -1 & -\frac{4}{3} \end{array}\right] \qquad \mathbf{R} = \left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & \frac{4}{3} \end{array}\right] \tag{7.3.24}$$

$$\text{no solution} \tag{7.3.25}$$

9. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 0 & -2 & -2 & -8 \\ 3 & 4 & -5 & 13 \end{array}\right] \qquad \mathbf{R} \Longrightarrow \left[\begin{array}{ccc|c} 3 & 4 & -5 & 13 \\ 0 & -2 & -2 & -8 \end{array}\right] \tag{7.3.26}$$

$$x_3 = \frac{-1.5}{-4.5} = t_3 \text{ (free)} \qquad x_2 = 4 - t_3 \tag{7.3.27}$$

$$x_1 = \frac{13 + 5t_3 - 16 + 4t_3}{3} = -1 + 3t_3 \tag{7.3.28}$$

10. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 5 & -7 & 3 & 17 \\ -15 & 21 & -9 & 50 \end{array}\right] \qquad \mathbf{R} \implies \left[\begin{array}{ccc|c} 5 & -7 & 3 & 17 \\ 0 & 0 & 0 & 101 \end{array}\right] \tag{7.3.29}$$

$$\text{no solution} \tag{7.3.30}$$

11. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 0 & 5 & 5 & -10 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{array}\right] \qquad \implies \left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{array}\right] \tag{7.3.31}$$

$$\implies \left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{array}\right] \qquad \mathbf{R} = \left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{7.3.32}$$

$$x_4 = t_4 \text{ (free)} \qquad x_3 = t_3 \text{ (free)} \tag{7.3.33}$$

$$x_2 = 2t_4 - t_3 \qquad x_1 = 1 \tag{7.3.34}$$

12. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 2 & -2 & 4 & 0 & 0 \\ -3 & 3 & -6 & 5 & 15 \\ 1 & -1 & 2 & 0 & 0 \end{array}\right] \qquad \implies \left[\begin{array}{cccc|c} 2 & -2 & 4 & 0 & 0 \\ -3 & 3 & -6 & 5 & 15 \\ 1 & -1 & 2 & 0 & 0 \end{array}\right] \tag{7.3.35}$$

$$\implies \left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{array}\right] \qquad \mathbf{R} = \left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{7.3.36}$$

$$x_4 = t_4 \text{ (free)} \qquad x_3 = t_3 \text{ (free)} \tag{7.3.37}$$

$$x_2 = 2t_4 - t_3 \qquad x_1 = 1 \tag{7.3.38}$$

13. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 0 & 10 & 4 & -2 & -4 \\ -3 & -17 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -34 & 16 & -10 & 4 \end{array}\right] \implies \left[\begin{array}{cccc|c} -3 & -17 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -34 & 16 & -10 & 4 \\ 0 & 10 & 4 & -2 & -4 \end{array}\right] \tag{7.3.39}$$

$$\implies \left[\begin{array}{cccc|c} -3 & -17 & 1 & 2 & 2 \\ 0 & -\frac{14}{3} & \frac{4}{3} & \frac{2}{3} & \frac{20}{3} \\ 0 & -\frac{238}{3} & \frac{56}{3} & -\frac{14}{3} & \frac{28}{3} \\ 0 & 10 & 4 & -2 & -4 \end{array}\right] \implies \left[\begin{array}{cccc|c} -3 & -17 & 1 & 2 & 2 \\ 0 & -\frac{14}{3} & \frac{4}{3} & \frac{2}{3} & \frac{20}{3} \\ 0 & 0 & -4 & -16 & -104 \\ 0 & 0 & \frac{48}{7} & -\frac{4}{7} & \frac{72}{7} \end{array}\right] \tag{7.3.40}$$

$$\mathbf{R} = \left[\begin{array}{cccc|c} -3 & -17 & 1 & 2 & 2 \\ 0 & -\frac{14}{3} & \frac{4}{3} & \frac{2}{3} & \frac{20}{3} \\ 0 & 0 & -4 & -16 & -104 \\ 0 & 0 & 0 & -28 & -168 \end{array}\right] \tag{7.3.41}$$

Back substituion gives,

$$x_4 = \frac{168}{28} = 6 \qquad x_3 = \frac{-104 + 96}{-4} = 2 \tag{7.3.42}$$

$$x_2 = \frac{20 - 12 - 8}{3} = 0 \qquad x_1 = \frac{2 - 12 - 2 + 0}{-3} = 4 \tag{7.3.43}$$

14. The augmented matrix is,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 2 & 3 & 1 & -11 & 1 \\ 5 & -2 & 5 & -4 & 5 \\ 1 & -1 & 3 & -3 & 3 \\ 3 & 4 & -7 & 2 & -7 \end{array}\right] \implies \left[\begin{array}{cccc|c} 2 & 3 & 1 & -11 & 1 \\ 0 & -\frac{19}{2} & \frac{5}{2} & \frac{47}{2} & \frac{5}{2} \\ 0 & -\frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & -\frac{17}{2} & \frac{37}{2} & -\frac{17}{2} \end{array}\right] \tag{7.3.44}$$

$$\implies \left[\begin{array}{cccc|c} 2 & 3 & 1 & -11 & 1 \\ 0 & -\frac{19}{2} & \frac{5}{2} & \frac{47}{2} & \frac{5}{2} \\ 0 & 0 & \frac{35}{19} & -\frac{70}{19} & \frac{35}{19} \\ 0 & 0 & -\frac{164}{19} & \frac{328}{19} & -\frac{164}{19} \end{array}\right] \implies \left[\begin{array}{cccc|c} 2 & 3 & 1 & -11 & 1 \\ 0 & -\frac{19}{2} & \frac{5}{2} & \frac{47}{2} & \frac{5}{2} \\ 0 & 0 & \frac{35}{19} & -\frac{70}{19} & \frac{35}{19} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{7.3.45}$$

$$\mathbf{R} = \left[\begin{array}{cccc|c} 2 & 3 & 1 & -11 & 1 \\ 0 & -19 & 5 & 47 & 5 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{7.3.46}$$

Back substituion gives,

$$x_4 = t_4 \text{ (free)} \qquad x_3 = 1 + 2t_4 \tag{7.3.47}$$

$$x_2 = \frac{5 - 47t_4 - 5 - 10t_4}{-19} = 3t_4 \qquad x_1 = \frac{1 + 11t_4 - 1 - 2t_4 - 9t_4}{2} = 0 \tag{7.3.48}$$

15. Using the facts that,

- Sequence of row operations is a row operation.
- Every row operation has an inverse operation.

The proof for all three properties of the row equivalence of matrices follows trivially from the above two statements.

16. TBC. Coded in python. Cross verified with Sympy

17. Using KVL and KCL, the system is

$$j_1 + j_2 = j_3 \qquad 2j_1 + j_3 + 2j_1 = 16 \tag{7.3.49}$$

$$4j_2 + j_3 = 32 \tag{7.3.50}$$

$$\mathbf{A} = \left[\begin{array}{cc|c} 5 & 1 & 16 \\ 1 & 5 & 32 \end{array}\right] \implies \left[\begin{array}{cc|c} 5 & 1 & 16 \\ 0 & \frac{24}{5} & \frac{144}{5} \end{array}\right] \tag{7.3.51}$$

Back-substituting gives,

$$j_2 = 6 \qquad j_1 = 2 \tag{7.3.52}$$

$$j_3 = 8 \tag{7.3.53}$$

18. Using KVL and KCL, the system is

$$j_2 + j_3 = j_1 \qquad 4j_1 + 12j_2 = 36 \tag{7.3.54}$$

$$-8j_3 + 12j_2 = 24 \tag{7.3.55}$$

$$\mathbf{A} = \left[\begin{array}{cc|c} 4 & 12 & 36 \\ -8 & 20 & 24 \end{array}\right] \implies \left[\begin{array}{cc|c} 4 & 12 & 36 \\ 0 & 44 & 96 \end{array}\right] \tag{7.3.56}$$

Back-substituting gives,

$$j_2 = \frac{24}{11} \qquad j_1 = \frac{27}{11} \tag{7.3.57}$$

$$j_3 = \frac{3}{11} \tag{7.3.58}$$

19. Using KVL and KCL, the system is

$$-R_1 j_2 = E_0 \qquad R_2 j_3 + R_1 j_2 = 0 \tag{7.3.59}$$

$$j_1 + j_2 = j_3 \tag{7.3.60}$$

$$\mathbf{A} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -R_1 & 0 & E_0 \\ 0 & R_1 & R_2 & 0 \end{array}\right] \qquad \mathbf{R} \implies \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -R_1 & 0 & E_0 \\ 0 & 0 & R_2 & E_0 \end{array}\right] \tag{7.3.61}$$

Back-substituting gives,

$$j_3 = \frac{E_0}{R_2} \qquad j_2 = -\frac{E_0}{R_1} \tag{7.3.62}$$

$$j_1 = \frac{E_0\,(R_1 + R_2)}{R_1 R_2} \tag{7.3.63}$$

20. Wheatstone bridge, with voltage arbitrarily set to 0 and E behind and in front of the voltage source. Now, by KVL,

$$E - R_1 I_1 = R_2 I_2 \qquad E - R_x I_x = R_3 I_3 \tag{7.3.64}$$

For there to be zero current throught the resistor R_0, both its ends have to be at the same voltage.

$$E - R_1 I_1 = E - R_x I_x \qquad R_3 I_3 = R_2 I_2 \tag{7.3.65}$$

$$R_x = R_1 \frac{I_1}{I_x} \qquad R_x = R_1 \frac{I_1\ R_3}{R_2\ I_1} = \frac{R_1 R_3}{R_2} \tag{7.3.66}$$

21. By KVL,

$$x_1 + x_4 = 1000 \qquad x_1 + x_2 = 1600 \tag{7.3.67}$$

$$x_3 + x_4 = 1600 \qquad x_2 + x_3 = 2200 \tag{7.3.68}$$

Solving by Gauss elimination,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 1 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 1 & 0 & 2200 \\ 0 & 0 & 1 & 1 & 1600 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 1 & 1 & 0 & 2200 \\ 0 & 0 & 1 & 1 & 1600 \end{array}\right] \tag{7.3.69}$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 0 & 1 & 1 & 1600 \end{array}\right] \quad \mathbf{R} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{7.3.70}$$

Back substitution gives,

$$x_4 = t_4 \text{ (free)} \qquad x_3 = 1600 - t_4 \tag{7.3.71}$$

$$x_2 = 600 + t_4 \qquad x_1 = 1000 - t_4 \tag{7.3.72}$$

The solution is not unique, as one variable is free.

22. Using Gauss elimination, with supply equaling demand,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 40 \\ 4 & -1 & -1 & 0 & -4 \\ 5 & -2 & 0 & -1 & -16 \\ 0 & 3 & 0 & -1 & 4 \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 40 \\ 0 & -3 & -3 & 0 & -84 \\ 0 & -\frac{9}{2} & -\frac{5}{2} & -1 & -116 \\ 0 & 3 & 0 & -1 & 4 \end{array}\right] \tag{7.3.73}$$

$$= \left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 40 \\ 0 & -3 & -3 & 0 & -84 \\ 0 & 0 & -4 & 2 & -20 \\ 0 & 0 & -3 & -1 & -80 \end{array}\right] \quad \mathbf{R} = \left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 40 \\ 0 & 1 & 1 & 0 & 28 \\ 0 & 0 & -2 & 1 & -10 \\ 0 & 0 & 0 & 1 & 26 \end{array}\right] \tag{7.3.74}$$

Back substitution gives,

$$S_2 = 26 \qquad S_1 = 18 \tag{7.3.75}$$

$$P_2 = 10 \qquad P_1 = 6 \tag{7.3.76}$$

23. Using Gauss elimination, with the number of atoms of each element unchanged during the reaction,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 8 & 0 & 0 & -2 & 0 \end{array}\right] \qquad \mathbf{R} = \left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & \frac{8}{3} & -2 & 0 \end{array}\right] \tag{7.3.77}$$

Back substitution gives,

$$x_4 = t_4 \text{ (free)} \qquad x_3 = \frac{3}{4}\, t_4 \tag{7.3.78}$$

$$x_2 = \frac{5}{4}\, t_4 \qquad x_1 = \frac{1}{4}\, t_4 \tag{7.3.79}$$

The set of smallest positive integers is thus, $\{1, 5, 3, 4\}$

24. Elementary matrices.

(a) Showing the effect of the matrices the $n-$th column of $\mathbf{A}$,

$$\mathbf{E}_1\mathbf{A}_n = \begin{bmatrix} a_{1n} \\ a_{3n} \\ a_{2n} \\ a_{4n} \end{bmatrix} \qquad \mathbf{E}_2\mathbf{A}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ -5a_{1n} + a_{3n} \\ a_{4n} \end{bmatrix} \qquad \mathbf{E}_3\mathbf{A}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ 8a_{4n} \end{bmatrix} \tag{7.3.80}$$

Applying the three elementary operations to the general 4×2 matrix $\mathbf{A}$,

$$\mathbf{B} = \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1(\mathbf{A}) \qquad = \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \tag{7.3.81}$$

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ a_{21} - 5a_{11} & a_{22} - 5a_{12} \\ 8a_{41} & 8a_{42} \end{bmatrix} \tag{7.3.82}$$

Checking if the row operations applied in the reverse order produce the same result,

$$\mathbf{C} = \mathbf{E}_1\mathbf{E}_2\mathbf{E}_3(\mathbf{A}) \qquad = \mathbf{E}_1\mathbf{E}_2\mathbf{E}_3 \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \tag{7.3.83}$$

$$\mathbf{C} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} - 5a_{11} & a_{32} - 5a_{12} \\ a_{21} & a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix} \qquad \mathbf{B} \neq \mathbf{C} \tag{7.3.84}$$

Numerical examples of applying elementary operations TBC.

(b) Self-evident for the specific case of $\mathbf{E}_1$ etc. For the general proof, simply use brute force for each kind of row operation one at a time.

7.4 Linear Independence, Rank of a Matrix, Vector Space

1. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 4 & -2 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \tag{7.4.1}$$

$$\mathbf{R} = \begin{bmatrix} 4 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 4 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.4.2}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 1

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{7.4.3}$$

2. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \tag{7.4.4}$$

$$\mathbf{R} = \begin{bmatrix} a & b \\ 0 & \frac{a^2-b^2}{a} \end{bmatrix} \qquad \mathbf{Q} = \mathbf{R} \tag{7.4.5}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$,

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a^2-b^2 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & a^2-b^2 \end{bmatrix} \tag{7.4.6}$$

- If $a^2 = b^2$ and $a \neq 0$, then rank is 1
- Else if $a^2 \neq b^2$ and $a \neq 0$, rank is 2
- Else if $a = b = 0$, rank is 0
- Else if $a = 0$ and $b \neq 0$, rank is 2

3. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 10 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 10 \end{bmatrix} \tag{7.4.7}$$

$$\mathbf{R} = \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & -\frac{25}{3} & 10 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & -\frac{25}{3} & 10 \end{bmatrix} \tag{7.4.8}$$

$$\mathbf{R} = \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{215}{9} \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{215}{9} \end{bmatrix} \tag{7.4.9}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 3.

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 3 & 0 \\ 0 & 5 & 1 \end{bmatrix} \tag{7.4.10}$$

4. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 0 & 2 \\ 0 & 2 & 6 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 0 & 2 \\ 0 & 2 & 6 \end{bmatrix} \tag{7.4.11}$$

$$\mathbf{R} = \begin{bmatrix} 6 & -4 & 0 \\ 0 & 2 & 6 \\ 0 & -\frac{8}{3} & 2 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 6 & -4 & 0 \\ 0 & 2 & 6 \\ 0 & -\frac{8}{3} & 2 \end{bmatrix} \tag{7.4.12}$$

$$\mathbf{R} = \begin{bmatrix} 6 & -4 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 10 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 6 & -4 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 10 \end{bmatrix} \tag{7.4.13}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 3.

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \tag{7.4.14}$$

5. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 0.2 & -0.1 & 0.4 \\ 0 & 1.1 & -0.3 \\ 0.1 & 0 & -2.1 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 0.2 & 0 & 0.1 \\ -0.1 & 1.1 & 0 \\ 0.4 & -0.3 & -2.1 \end{bmatrix} \tag{7.4.15}$$

$$\mathbf{R} = \begin{bmatrix} 0.2 & -0.1 & 0.4 \\ 0 & 1.1 & -0.3 \\ 0 & 0.05 & -2.3 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0 & 1.1 & 0.05 \\ 0 & -0.3 & -2.3 \end{bmatrix} \tag{7.4.16}$$

$$\mathbf{R} = \begin{bmatrix} 0.2 & -0.1 & 0.4 \\ 0 & 1.1 & -0.3 \\ 0 & 0 & -\frac{503}{220} \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0 & 1.1 & 0.05 \\ 0 & 0 & -\frac{503}{220} \end{bmatrix} \tag{7.4.17}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 3.

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 11 & -3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 22 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{7.4.18}$$

6. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix} \tag{7.4.19}$$

$$\mathbf{R} = \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix} \tag{7.4.20}$$

$$\mathbf{R} = \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.4.21}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 2.

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4 & 0 \end{bmatrix} \tag{7.4.22}$$

7. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} \tag{7.4.23}$$

$$\mathbf{R} = \begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \tag{7.4.24}$$

$$\mathbf{R} = \begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.4.25}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 2.

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{7.4.26}$$

8. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \\ 2 & 16 & 8 & 4 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 2 & 16 & 4 & 2 \\ 4 & 8 & 8 & 16 \\ 8 & 4 & 16 & 8 \\ 16 & 2 & 2 & 4 \end{bmatrix} \tag{7.4.27}$$

$$\mathbf{R} = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & -24 & -60 & -126 \\ 0 & 0 & 0 & -30 \\ 0 & 12 & 0 & -12 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 2 & 16 & 4 & 2 \\ 0 & -24 & 0 & 12 \\ 0 & -60 & 0 & 0 \\ 0 & -126 & -30 & -12 \end{bmatrix} \tag{7.4.28}$$

$$\mathbf{R} = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & -24 & -60 & -126 \\ 0 & 0 & -30 & -75 \\ 0 & 0 & 0 & -30 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 2 & 16 & 4 & 2 \\ 0 & -24 & 0 & 12 \\ 0 & 0 & -30 & -75 \\ 0 & 0 & 0 & -30 \end{bmatrix} \tag{7.4.29}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 2.

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 4 & 10 & 21 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 8 & -2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 5 & 1 \end{bmatrix} \tag{7.4.30}$$

9. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7.4.31}$$

$$\mathbf{R} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & \frac{8}{9} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & \frac{8}{9} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7.4.32}$$

$$\mathbf{R} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & \frac{8}{9} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & \frac{8}{9} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.4.33}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 3.

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 9 & 8 & 9 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 1 & 8 & 1 \\ 0 & 9 & 0 \end{bmatrix} \tag{7.4.34}$$

10. Finding the rank,

$$\mathbf{A} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7.4.35}$$

$$\mathbf{R} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & \frac{8}{9} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & \frac{8}{9} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7.4.36}$$

$$\mathbf{R} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & \frac{8}{9} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & \frac{8}{9} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.4.37}$$

Using the L.I. rows of $\mathbf{R}$ and L.I. columns of $\mathbf{Q}^T$, rank is 3.

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 9 & 8 & 9 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 1 & 8 & 1 \\ 0 & 9 & 0 \end{bmatrix} \tag{7.4.38}$$

11. Proving the general case,

(a) All intermediate points in between A and B can be written as

$$P = \lambda A + (1-\lambda)B \qquad \lambda \in \mathcal{R} \tag{7.4.39}$$

For $a_{jk} = j + k - 1$, for some integer $n > 1$, consider the elements of the first column,

$$a_{1k} = k \qquad a_{nk} = n + k - 1 \tag{7.4.40}$$

$$a_{jk} = (1-\lambda)(k) + (\lambda)(n + k - 1) \qquad a_{jk} = j + k - 1 \tag{7.4.41}$$

$$\lambda = \frac{j-1}{n-1} \tag{7.4.42}$$

The fact that λ is independent of k, means that every intermediate row has a separate λ that can be used to express it as a linear combination of the first and last rows.
All but the first and last row can thus be zeroed out and the rank is 2.

(b) Replacing 1 with c in the above proof only changes the expression for λ to

$$j + k + c = [1 + k + c](1-\lambda) + \lambda[n + k + c] \qquad \lambda = \frac{j-1}{n-1} \tag{7.4.43}$$

Since λ is still independent of k, the result still holds for some general c.

(c) Every row is a scalar multiple of the first row, as can be seen by

$$a_{j,k} = 2^{j+k-2} \qquad a_{j+1,k} = 2^{j+k-1} \tag{7.4.44}$$

$$a_{j+1,k} = 2a_{j,k} \tag{7.4.45}$$

Thus, rank is 1.
Other exmaples TBC

12. To prove the relation,

$$\mathbf{C} = (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \qquad \mathbf{C}^T = \mathbf{AB} \tag{7.4.46}$$

$$\text{rank}(\mathbf{C}) = \text{rank}(\mathbf{C}^T) \tag{7.4.47}$$

13. Consider a nilpotent matrix $\mathbf{A}$ and an identity matrix of equal rank $\mathbf{B}$. This counterexample is now,

$$\operatorname{rank}(\mathbf{A}^2) = 0 \qquad \operatorname{rank}(\mathbf{B}^2) = \operatorname{rank}(\mathbf{A}) \tag{7.4.48}$$

14. $\mathbf{A}$ is not a square matrix. Let it have m rows and n columns.

$$m > n \qquad \operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^T) \tag{7.4.49}$$

$$\operatorname{rank}(\mathbf{A}^T) \le n \qquad \operatorname{rank}(\mathbf{A}) < m \tag{7.4.50}$$

Since $\operatorname{rank}(\mathbf{A})$ is less than the number of rows, some of the rows are L.D. and can be row reduced to zero.

A similar proof for the case $m < n$ can be used to prove the column vectors of such a matrix are L.D.

15. $\mathbf{A}$ is a square matrix of order n with L.I. rows.

$$\operatorname{rank}(\mathbf{A}) = n \qquad \operatorname{rank}(\mathbf{A}^T) = \operatorname{rank}(\mathbf{A}) \tag{7.4.51}$$

$$\operatorname{rank}(\mathbf{A}^T) = n \tag{7.4.52}$$

Since $\mathbf{A}^T$ has rank n, $\mathbf{A}$ has L.I. columns.

The exact same procedure can be used for the backwards proof, using the fact that the L.I. rows of $\mathbf{A}^T$ is the same as the L.I. columns of $\mathbf{A}$.

16. Consider the column representation of $\mathbf{AB}$,

$$\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \dots & \mathbf{Ab}_n \end{bmatrix} \tag{7.4.53}$$

Every column vector $\mathbf{Ab}_k$ is a vector in the column space of $\mathbf{A}$. So, the columns of $\mathbf{AB}$ belong to the column space of $\mathbf{A}$.

The dimension of the column space of a vector is equal to its rank.

$$\operatorname{rank}(\mathbf{AB}) \le \operatorname{rank}(\mathbf{A}) \tag{7.4.54}$$

Using the fact that $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^T)$

$$\operatorname{rank}\left((\mathbf{AB})^T\right) = \operatorname{rank}(\mathbf{AB}) \qquad \operatorname{rank}(\mathbf{B}^T\mathbf{A}^T) \le \operatorname{rank}(\mathbf{B}^T) \tag{7.4.55}$$

$$\operatorname{rank}(\mathbf{AB}) \le \operatorname{rank}(\mathbf{B}) \tag{7.4.56}$$

This proves the result.

17. Checking if the set of vectors are L.I, no

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 & 2 \\ 2 & -1 & 3 & 7 \\ 1 & 16 & -12 & -22 \end{bmatrix} \tag{7.4.57}$$

$$= \begin{bmatrix} 1 & 16 & -12 & -22 \\ 0 & -33 & 27 & 51 \\ 0 & -44 & 36 & 68 \end{bmatrix} = \begin{bmatrix} 1 & 16 & -12 & -22 \\ 0 & -33 & 27 & 51 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.4.58}$$

18. Checking if the set of vectors are L.I, yes

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \tag{7.4.59}$$

$$= \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & \frac{1}{12} & \frac{4}{45} & \frac{1}{12} \\ 0 & \frac{3}{40} & \frac{1}{12} & \frac{9}{112} \end{bmatrix} = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & 0 & \frac{1}{180} & \frac{1}{120} \\ 0 & 0 & \frac{1}{120} & \frac{9}{700} \end{bmatrix} \tag{7.4.60}$$

$$= \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & 0 & \frac{1}{180} & \frac{1}{120} \\ 0 & 0 & 0 & \frac{1}{2800} \end{bmatrix} \tag{7.4.61}$$

19. Checking if the set of vectors are L.I, yes

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.4.62}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.4.63}$$

20. Checking if the set of vectors are L.I, no

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \tag{7.4.64}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.4.65}$$

21. Checking if the set of vectors are L.I, no

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 7 \\ 2 & 0 & 0 & 8 \\ 2 & 0 & 0 & 9 \\ 2 & 0 & 1 & 0 \end{bmatrix} \tag{7.4.66}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 7 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.4.67}$$

22. Checking if the set of vectors are L.I, no

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 & 0.2 \\ 0 & 0 & 0 \\ 3 & -0.6 & 1.5 \end{bmatrix} \tag{7.4.68}$$

$$= \begin{bmatrix} 0.4 & -0.2 & 0.2 \\ 0 & \frac{9}{10} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.4.69}$$

23. Checking if the set of vectors are L.I, yes

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 7 & 6 & 5 \\ 9 & 7 & 5 & 3 & 1 \end{bmatrix} \tag{7.4.70}$$

$$= \begin{bmatrix} 9 & 8 & 7 & 6 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \tag{7.4.71}$$

24. Checking if the set of vectors are L.I, no

Since the number of vectors is greater than the number of components.

25. Checking if the set of vectors are L.I, yes

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & -1 & 3 \\ 2 & 2 & 5 & 0 \\ -4 & -4 & -4 & -4 \end{bmatrix} \tag{7.4.72}$$

$$= \begin{bmatrix} 6 & 0 & -1 & 3 \\ 0 & 2 & \frac{16}{3} & -1 \\ 0 & -4 & -\frac{14}{3} & -2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & -1 & 3 \\ 0 & 2 & \frac{16}{3} & -1 \\ 0 & 0 & 6 & -4 \end{bmatrix} \tag{7.4.73}$$

26. Finding the row echelon form,

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \mathbf{v}_5 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 6 & 0 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{bmatrix} \tag{7.4.74}$$

$$= \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 \end{bmatrix} \tag{7.4.75}$$

$$\tag{7.4.76}$$

The linearly independent set of vectors is $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_5$

27. Are the given set of vectors a vector space? yes

$$v_1 - v_2 + 2v_3 = 0 \qquad \in \mathcal{R}^3 \tag{7.4.77}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \Longrightarrow \quad \mathbf{A} + \mathbf{B} \in V \tag{7.4.78}$$

$$\mathbf{A} \in V \qquad \Longrightarrow \quad k\mathbf{A} \in V \tag{7.4.79}$$

The dimension is 2, since the relation between the components has 2 d.o.f.
A basis is any set of 2 L.I. members of V, for example,

$$\mathbf{A} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \tag{7.4.80}$$

28. Are the given set of vectors a vector space? no

$$3v_2 + v_3 = k \qquad \in \mathcal{R}^3 \tag{7.4.81}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \not\Longrightarrow \quad \mathbf{A} + \mathbf{B} \in V \tag{7.4.82}$$

$$\mathbf{A} \in V \qquad \not\Longrightarrow \quad k\mathbf{A} \in V \tag{7.4.83}$$

29. Are the given set of vectors a vector space? no

$$v_1 \geq v_2 \qquad \in \mathcal{R}^2 \tag{7.4.84}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \Longrightarrow \quad \mathbf{A} + \mathbf{B} \in V \tag{7.4.85}$$

$$\mathbf{A} \in V \qquad \not\Longrightarrow \quad k\mathbf{A} \in V \tag{7.4.86}$$

30. Are the given set of vectors a vector space? yes

$$\{v_1, v_2, \ldots, v_{n-2}\} = 0 \qquad \in \mathcal{R}^n \tag{7.4.87}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \Longrightarrow \quad \mathbf{A} + \mathbf{B} \in V \tag{7.4.88}$$

$$\mathbf{A} \in V \qquad \Longrightarrow \quad k\mathbf{A} \in V \tag{7.4.89}$$

The dimension is 2, since the relation between the components has 2 d.o.f.

A basis is any set of 2 L.I. members of V, for example,

$$\mathbf{A} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{7.4.90}$$

31. Are the given set of vectors a vector space? no

$$v_i > 0 \;\forall\; i \qquad \in \mathcal{R}^5 \tag{7.4.91}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \Longrightarrow\ \mathbf{A} + \mathbf{B} \in V \tag{7.4.92}$$

$$\mathbf{A} \in V \qquad \not\Longrightarrow\ k\mathbf{A} \in V \tag{7.4.93}$$

32. Are the given set of vectors a vector space? yes

$$3v_1 - 2v_2 + v_3 = 0 \tag{7.4.94}$$

$$4v_1 + 5v_2 = 0 \qquad \in \mathcal{R}^3 \tag{7.4.95}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \Longrightarrow\ \mathbf{A} + \mathbf{B} \in V \tag{7.4.96}$$

$$\mathbf{A} \in V \qquad \Longrightarrow\ k\mathbf{A} \in V \tag{7.4.97}$$

Since two components can be expressed in terms of the first, there is only 1 d.o.f. and a basis is

$$\mathbf{A} = \begin{bmatrix} 5 \\ -4 \\ -23 \end{bmatrix} \tag{7.4.98}$$

33. Are the given set of vectors a vector space? yes

$$3v_1 - v_3 = 0 \tag{7.4.99}$$

$$2v_1 + 3v_2 - 4v_3 = 0 \qquad \in \mathcal{R}^3 \tag{7.4.100}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \Longrightarrow\ \mathbf{A} + \mathbf{B} \in V \tag{7.4.101}$$

$$\mathbf{A} \in V \qquad \Longrightarrow\ k\mathbf{A} \in V \tag{7.4.102}$$

Since two components can be expressed in terms of the first, there is only 1 d.o.f. and a basis is

$$\mathbf{A} = \begin{bmatrix} 3 \\ 10 \\ 9 \end{bmatrix} \tag{7.4.103}$$

34. Are the given set of vectors a vector space? no

$$|v_j| = 1 \ \forall \ j \in \{1, \dots, n\} \qquad \in \mathcal{R}^n \tag{7.4.104}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \not\Longrightarrow \ \mathbf{A} + \mathbf{B} \in V \tag{7.4.105}$$

$$\mathbf{A} \in V \qquad \not\Longrightarrow \ k\mathbf{A} \in V \tag{7.4.106}$$

35. Are the given set of vectors a vector space? yes

$$v_1 = 2v_2 = 3v_3 = 4v_4 \qquad \in \mathcal{R}^4 \tag{7.4.107}$$

$$\mathbf{A},\ \mathbf{B} \in V \qquad \Longrightarrow \ \mathbf{A} + \mathbf{B} \in V \tag{7.4.108}$$

$$\mathbf{A} \in V \qquad \Longrightarrow \ k\mathbf{A} \in V \tag{7.4.109}$$

Since three components can be expressed in terms of the first, there is only 1 d.o.f. and a basis is

$$\mathbf{A} = \begin{bmatrix} 12 \\ 6 \\ 4 \\ 3 \end{bmatrix} \tag{7.4.110}$$

7.5 Solutions of Linear Systems: Existence, Uniqueness

1. No problem set in this section

7.6 Reference: Second- and Third-Order Determinants

1. No problem set in this section

7.7 Determinants, Cramer's Rule

1. Illustrating Theorem 1 part a using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \tag{7.7.1}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = bc - ad \tag{7.7.2}$$

$$= -\det(\mathbf{M}) \tag{7.7.3}$$

Illustrating Theorem 1 part b using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & b \\ c + \lambda a & d + \lambda b \end{bmatrix} \tag{7.7.4}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = ad - bc + [\lambda ab - \lambda ab] \tag{7.7.5}$$

$$= \det(\mathbf{M}) \tag{7.7.6}$$

Illustrating Theorem 1 part c using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & b \\ \lambda c & \lambda d \end{bmatrix} \tag{7.7.7}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = \lambda ad - \lambda bc \tag{7.7.8}$$

$$= \lambda \ \det(\mathbf{M}) \tag{7.7.9}$$

Illustrating Theorem 2 part a using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \tag{7.7.10}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = bc - ad \tag{7.7.11}$$

$$= -\det(\mathbf{M}) \tag{7.7.12}$$

Illustrating Theorem 2 part b using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & b + \lambda a \\ c & d + \lambda c \end{bmatrix} \tag{7.7.13}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = ad - bc + [\lambda ac - \lambda ac] \tag{7.7.14}$$

$$= \det(\mathbf{M}) \tag{7.7.15}$$

Illustrating Theorem 2 part c using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & \lambda b \\ c & \lambda d \end{bmatrix} \tag{7.7.16}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = \lambda ad - \lambda bc \tag{7.7.17}$$

$$= \lambda \ \det(\mathbf{M}) \tag{7.7.18}$$

Illustrating Theorem 2 part d using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \tag{7.7.19}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = ad - bc \tag{7.7.20}$$

$$= \det(\mathbf{M}) \tag{7.7.21}$$

Illustrating Theorem 2 part e using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \tag{7.7.22}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = 0a - 0c \tag{7.7.23}$$

$$= 0 \tag{7.7.24}$$

Illustrating Theorem 2 part f using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & \mu a \\ c & \mu c \end{bmatrix} \tag{7.7.25}$$

$$\det(\mathbf{M}) = ad - bc \qquad \det(\mathbf{N}) = \mu ac - \mu ac \tag{7.7.26}$$

$$= 0 \tag{7.7.27}$$

2. Expanding a second order determinant in all four ways possible,

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{7.7.28}$$

$$D_{1i} = a(d) - b(c) \qquad D_{2i} = -c(b) + d(a) \tag{7.7.29}$$

$$D_{k1} = a(d) - c(b) \qquad D_{k2} = -b(c) + d(a) \tag{7.7.30}$$

All four expansion match.

3. Illustrating Theorem 1 part a using 3×3 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \tag{7.7.31}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.32}$$

$$\det(\mathbf{N}) = dbi - dch - aei + afh + gec - gbf \tag{7.7.33}$$

$$= -\det(\mathbf{M}) \tag{7.7.34}$$

Illustrating Theorem 1 part b using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & b & c \\ d + \lambda a & e + \lambda b & f + \lambda c \\ g & h & i \end{bmatrix} \tag{7.7.35}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.36}$$

$$\det(\mathbf{N}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.37}$$

$$+ \lambda\Big[abi - ach - abi + ach + gbc - gbc\Big] \tag{7.7.38}$$

$$= \det(\mathbf{M}) \tag{7.7.39}$$

Illustrating Theorem 1 part c using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{7.7.40}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.41}$$

$$\det(\mathbf{N}) = \lambda\Big[aei - afh - dbi + dch + gbf - gec\Big] \tag{7.7.42}$$

$$= \lambda \det(\mathbf{M}) \tag{7.7.43}$$

Illustrating Theorem 2 part a using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix} \tag{7.7.44}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.45}$$

$$\det(\mathbf{N}) = bdi - bfg - eai + ecg + haf - hcd \tag{7.7.46}$$

$$= -\det(\mathbf{M}) \tag{7.7.47}$$

Illustrating Theorem 2 part b using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & b + \lambda a & c \\ d & e + \lambda d & f \\ g & h + \lambda g & i \end{bmatrix} \tag{7.7.48}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.49}$$

$$\det(\mathbf{N}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.50}$$

$$+ \lambda\Big[adi - agf - adi + dgc + gaf - gdc\Big] \tag{7.7.51}$$

$$= \det(\mathbf{M}) \tag{7.7.52}$$

Illustrating Theorem 2 part c using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} \lambda a & b & c \\ \lambda d & e & f \\ \lambda g & h & i \end{bmatrix} \tag{7.7.53}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.54}$$

$$\det(\mathbf{N}) = \lambda\Big[aei - afh - dbi + dch + gbf - gec\Big] \tag{7.7.55}$$

$$= \lambda \det(\mathbf{M}) \tag{7.7.56}$$

Illustrating Theorem 2 part d using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \tag{7.7.57}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.58}$$

$$\det(\mathbf{N}) = aei - afh - bdi + bfg + cdh - ceg \tag{7.7.59}$$

$$= \det(\mathbf{M}) \tag{7.7.60}$$

Illustrating Theorem 2 part e using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & 0 & c \\ d & 0 & f \\ g & 0 & i \end{bmatrix} \tag{7.7.61}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.62}$$

$$\det(\mathbf{N}) = 0 \tag{7.7.63}$$

Illustrating Theorem 2 part f using 2×2 matrices,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} a & \lambda a & c \\ d & \lambda d & f \\ g & \lambda g & i \end{bmatrix} \tag{7.7.64}$$

$$\det(\mathbf{M}) = aei - afh - dbi + dch + gbf - gec \tag{7.7.65}$$

$$\det(\mathbf{N}) = \lambda(adi - agf - dai + dgc + gaf - gdc) \tag{7.7.66}$$

$$= 0 \tag{7.7.67}$$

Evaluating determinant by reuction to triangular form,

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{7.7.68}$$

$$= \begin{bmatrix} a & b & c \\ 0 & e - \frac{bd}{a} & f - \frac{cd}{a} \\ 0 & h - \frac{bg}{a} & i - \frac{cg}{a} \end{bmatrix} \tag{7.7.69}$$

$$= \frac{1}{a} \cdot \begin{bmatrix} a^2 & ab & ac \\ 0 & ae - bd & af - cd \\ 0 & ah - bg & ai - cg \end{bmatrix} \tag{7.7.70}$$

$$= \frac{1}{a} \cdot \begin{bmatrix} a^2 & ab & ac \\ 0 & ae - bd & af - cd \\ 0 & 0 & ai - cg - \frac{(af-cd)(bg-ah)}{ae-bd} \end{bmatrix} \tag{7.7.71}$$

Using the fact that the determinant of a triangular matrix is the product of its diagonal terms,

$$\det(\mathbf{M}) = \frac{(ai - cg)(ae - bd) - (af - cd)(ah - bg)}{a} \tag{7.7.72}$$

$$= aei - cge - bdi + bfg - afh + cdh \tag{7.7.73}$$

$$= a(ei - fh) - d(bi - ch) + g(bf - ce) \tag{7.7.74}$$

This result matches the row major expansion method.

4. Computaion of a determinant of order n requires the computation of n determinants of order $(n-1)$. Starting with $n = 1$,

$$n = 1 \implies \text{comp} = 1 \tag{7.7.75}$$

$$n = 2 \implies \text{comp} = 2 \tag{7.7.76}$$

$$n = 3 \implies \text{comp} = 3 \cdot 2 \tag{7.7.77}$$

$$n = 4 \implies \text{comp} = 4 \cdot 3 \cdot 2 \tag{7.7.78}$$

From this pattern it is easy to extrapolate the result that $n!$ computations are required for a determinant of order n.

5. Multiplying a matrix by a scalar imvolves scaling every element by the scalar. However, dividing a determinant by a scalar only involves dividing one column by that scalar.

   Since n such divisions have to be performed to convert $\det(\mathbf{kA})$ into $\det(\mathbf{A})$, the pre-factor is k^n

6. Making a list of all 9 majors possible,

$$M_{1,1} = \begin{vmatrix} e & f \\ h & i \end{vmatrix} \qquad M_{2,1} = \begin{vmatrix} b & c \\ h & i \end{vmatrix} \qquad M_{3,1} = \begin{vmatrix} b & c \\ e & f \end{vmatrix} \tag{7.7.79}$$

$$M_{1,2} = \begin{vmatrix} d & f \\ g & i \end{vmatrix} \qquad M_{2,2} = \begin{vmatrix} a & c \\ g & i \end{vmatrix} \qquad M_{3,2} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} \tag{7.7.80}$$

$$M_{1,3} = \begin{vmatrix} d & e \\ g & h \end{vmatrix} \qquad M_{2,3} = \begin{vmatrix} a & b \\ g & h \end{vmatrix} \qquad M_{3,3} = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \tag{7.7.81}$$

The cofactors are obtained by the formula

$$C_{j,k} = (-1)^{j+k}\, M_{j,k} \tag{7.7.82}$$

7. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix} \qquad = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{7.7.83}$$

$$= \cos(\alpha+\beta) \tag{7.7.84}$$

8. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} 0.4 & 4.9 \\ 1.5 & -1.3 \end{vmatrix} \qquad = 0.4(-1.3) - 1.5(4.9) \tag{7.7.85}$$

$$= -7.87 \tag{7.7.86}$$

9. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{vmatrix} \qquad = \cos(n\theta)\cos(n\theta) + \sin(n\theta)\sin(n\theta) \tag{7.7.87}$$

$$= \cos(0) = 1 \tag{7.7.88}$$

10. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{vmatrix} \qquad = \cosh^2(t) - \sinh^2(t) \tag{7.7.89}$$

$$= 1 \tag{7.7.90}$$

11. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} 4 & -1 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{vmatrix} = 4(10) - 0 + 0 \tag{7.7.91}$$

$$= 40 \tag{7.7.92}$$

12. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a(a^2 - bc) - c(ab - c^2) + b(b^2 - ac) \tag{7.7.93}$$

$$= a^3 + b^3 + c^3 - 3abc \tag{7.7.94}$$

13. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} 0 & 4 & -1 & 5 \\ -4 & 0 & 3 & -2 \\ 1 & -3 & 0 & 1 \\ -5 & 2 & -1 & 0 \end{vmatrix} \tag{7.7.95}$$

$$= 4 \cdot \begin{vmatrix} 4 & -1 & 5 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & -1 & 5 \\ 0 & 3 & -2 \\ 2 & -1 & 0 \end{vmatrix} + 5 \cdot \begin{vmatrix} 4 & -1 & 5 \\ 0 & 3 & -2 \\ -3 & 0 & 1 \end{vmatrix} \tag{7.7.96}$$

$$= 4\left[4(1) + 3(5) + 2(-1)\right] + 1 \cdot \left[4(-2) + 2(-13)\right] + 5 \cdot \left[4(3) - 3(-13)\right] \tag{7.7.97}$$

$$= 4(17) + 1(-34) + 5(51) = 289 \tag{7.7.98}$$

14. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} 4 & 7 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 7 & 0 & 0 \\ 0 & \frac{9}{2} & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -2 & 2 \end{vmatrix} \quad 7.7.99$$

$$= \begin{vmatrix} 4 & 7 & 0 & 0 \\ 0 & \frac{9}{2} & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 12 \end{vmatrix} = \frac{4 \cdot 9 \cdot 1 \cdot 12}{2} \quad 7.7.100$$

$$= 216 \quad 7.7.101$$

15. Finding the determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{vmatrix} \quad 7.7.102$$

$$= \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \\ 0 & 0 & 0 & -16 \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot (-16) \quad 7.7.103$$

$$= -64 \quad 7.7.104$$

16. Starting with a matrix of order 2,

$$\det(M^{(2)}) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \quad 7.7.105$$

$$\det(M^{(3)}) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \quad 7.7.106$$

$$\det(M^{(4)}) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \quad 7.7.107$$

Thus, the general matrix of order n can be recast into upper triangular form with the last diagonal term being $(n-1)$ and all other diagonal terms -1. Thus,

$$\det(\mathbf{M}^{(n)}) = (-1)^{n-1}\,(n-1) \tag{7.7.108}$$

Interpretation in terms of incidence matrix and simplex TBC.

17. Row reduction gives,

$$\mathbf{M} = \begin{bmatrix} 4 & 9 \\ -8 & -6 \\ 16 & 12 \end{bmatrix} \qquad = \begin{bmatrix} 4 & 9 \\ 0 & 12 \\ 0 & -24 \end{bmatrix} \tag{7.7.109}$$

$$= \begin{bmatrix} 4 & 9 \\ 0 & 12 \\ 0 & 0 \end{bmatrix} \qquad \operatorname{rank}(\mathbf{M}) = 2 \tag{7.7.110}$$

Starting with some 2×2 matrix with nonzero determinant,

$$\det(\mathbf{N}) = \begin{vmatrix} 4 & 9 \\ -8 & -6 \end{vmatrix} \qquad = -24 + 72 \neq 0 \tag{7.7.111}$$

$$\operatorname{rank}(\mathbf{M}) = \operatorname{rank}(\mathbf{N}) = 2 \tag{7.7.112}$$

18. Row reduction gives,

$$\mathbf{M} = \begin{bmatrix} 0 & 4 & -6 \\ 4 & 0 & 10 \\ -6 & 10 & 0 \end{bmatrix} \qquad = \begin{bmatrix} -6 & 10 & 0 \\ 0 & 4 & -6 \\ 4 & 0 & 10 \end{bmatrix} \tag{7.7.113}$$

$$= \begin{bmatrix} -6 & 10 & 0 \\ 0 & 4 & -6 \\ 0 & \frac{20}{3} & 10 \end{bmatrix} \qquad = \begin{bmatrix} -6 & 10 & 0 \\ 0 & 4 & -6 \\ 0 & 0 & 20 \end{bmatrix} \tag{7.7.114}$$

$$\operatorname{rank}(\mathbf{M}) = 3 \tag{7.7.115}$$

Starting with $\mathbf{M}$ itself,

$$\det(\mathbf{M}) = \begin{vmatrix} 0 & 4 & -6 \\ 4 & 0 & 10 \\ -6 & 10 & 0 \end{vmatrix} = -4(60) - 6(40) \neq 0 \tag{7.7.116}$$

$$\text{rank}(\mathbf{M}) = 3 \tag{7.7.117}$$

The results match.

19. Row reduction gives,

$$\mathbf{M} = \begin{bmatrix} 1 & 5 & 2 & 2 \\ 1 & 3 & 2 & 6 \\ 4 & 0 & 8 & 48 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 2 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & -20 & 0 & 40 \end{bmatrix} \tag{7.7.118}$$

$$= \begin{bmatrix} 1 & 5 & 2 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.7.119}$$

$$\text{rank}(\mathbf{M}) = 2 \tag{7.7.120}$$

Starting with some 3×3 submatrix, all of them have determinant zero.
Looking for some 2×2 submatrix with nonzero determinant,

$$\det(\mathbf{M}) = \begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix} = -2 \neq 0 \tag{7.7.121}$$

$$\text{rank}(\mathbf{M}) = \text{rank}(\mathbf{N}) = 2 \tag{7.7.122}$$

$$\tag{7.7.123}$$

The results match.

20. Using the determinant to find the locus,

(a) A straight line passing through two points,

$$\det(\mathbf{M}) = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} x - x_1 & y - y_1 & 0 \\ x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \end{vmatrix} \tag{7.7.124}$$

$$= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x - x_1 & y - y_1 \end{vmatrix} \tag{7.7.125}$$

This can be rearranged into the required form, if the denominators are nonzero.

(b) A plane passing through 3 given points,

$$\det(\mathbf{N}) = \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 & 0 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 & 0 \end{vmatrix} \quad 7.7.126$$

$$= \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} \quad 7.7.127$$

The expansion of this determinant is not as elegant, but the correspondence to the 2-D case is clear.
Applying the formula to the given points,

$$\det(\mathbf{M}) = 0 = \begin{vmatrix} (x-1) & (y-1) & (z-1) \\ 2 & 1 & 5 \\ 4 & -1 & 4 \end{vmatrix} \quad 7.7.128$$

$$0 = 9(x-1) + 12(y-1) - 6(z-1) \quad 7.7.129$$

$$15 = 9x + 12y - 6z \quad 7.7.130$$

(c) Circle in two dimensions through three given points.

$$0 = a(x^2+y^2) + bx + cy + d \quad 7.7.131$$

$$\det(\mathbf{M}) = \begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} \quad 7.7.132$$

$$= \begin{vmatrix} x^2+y^2-(x_1^2+y_1^2) & x-x_1 & y-y_1 & 0 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2-(x_1^2+y_1^2) & x_2-x_1 & y_2-y_1 & 0 \\ x_3^2+y_3^2-(x_1^2+y_1^2) & x_3-x_1 & y_3-y_1 & 0 \end{vmatrix} \quad 7.7.133$$

$$= \begin{vmatrix} x^2+y^2-(x_1^2+y_1^2) & x-x_1 & y-y_1 \\ x_2^2+y_2^2-(x_1^2+y_1^2) & x_2-x_1 & y_2-y_1 \\ x_3^2+y_3^2-(x_1^2+y_1^2) & x_3-x_1 & y_3-y_1 \end{vmatrix} \quad 7.7.134$$

Using the determinant to find the circle passing through the three given points (2, 6), (6, 4), (7, 1),

$$\det(\mathbf{M}) = \begin{vmatrix} x^2+y^2-(40) & x-2 & y-6 \\ 52-(40) & 4 & -2 \\ 50-(40) & 5 & -5 \end{vmatrix} \tag{7.7.135}$$

$$= \begin{vmatrix} x^2+y^2-(40) & x-2 & y-6 \\ 12 & 4 & -2 \\ 10 & 5 & -5 \end{vmatrix} \tag{7.7.136}$$

$$0 = (x^2+y^2-40)(-10)+(x-2)(40)+(y-6)(20) \tag{7.7.137}$$

$$0 = x^2+y^2-40-4x+8-2y+12 \tag{7.7.138}$$

$$5^2 = (x-2)^2+(y-1)^2 \tag{7.7.139}$$

(d) Sphere in three dimensions through four given points.

$$0 = a(x^2+y^2+z^2)+bx+cy+dz+e \tag{7.7.140}$$

$$\det(\mathbf{M}) = \begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} \tag{7.7.141}$$

$$= \begin{vmatrix} x^2+y^2+z^2-(x_1^2+y_1^2+z_1^2) & x-x_1 & y-y_1 & z-z_1 \\ x_2^2+y_2^2+z_2^2-(x_1^2+y_1^2+z_1^2) & x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3^2+y_3^2+z_3^2-(x_1^2+y_1^2+z_1^2) & x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4^2+y_4^2+z_4^2-(x_1^2+y_1^2+z_1^2) & x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} \tag{7.7.142}$$

Using the determinant to find the circle passing through the four given points

$(0, 0, 5),\ \ (4, 0, 1),\ \ (0, 4, 1),\ \ (0, 0, -3),$

$$\det(\mathbf{M}) = \begin{vmatrix} x^2+y^2+z^2-(25) & x & y & z-5 \\ 17-(25) & 4 & 0 & -4 \\ 17-(25) & 0 & 4 & -4 \\ 9-(25) & 0 & 0 & -8 \end{vmatrix} \tag{7.7.143}$$

$$= \begin{vmatrix} x^2+y^2+z^2-(25) & x & y & z-5 \\ -8 & 4 & 0 & -4 \\ -8 & 0 & 4 & -4 \\ -16 & 0 & 0 & -8 \end{vmatrix} \tag{7.7.144}$$

$$0 = (x^2+y^2+z^2-25)\big[4(-32)\big] - x\big[4(0)\big] \tag{7.7.145}$$

$$+\, y\big[(-4)(0)\big] - (z-5)\big[(-4)(64)\big] \tag{7.7.146}$$

$$0 = x^2+y^2+z^2-25-2z+10 \tag{7.7.147}$$

$$4^2 = x^2+y^2+(z-1)^2 \tag{7.7.148}$$

(e) General conic section has the equation,

$$ax^2+bxy+cy^2+dx+ey+f=0 \tag{7.7.149}$$

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = \det(\mathbf{M}) = 0 \tag{7.7.150}$$

21. Solving by Cramer's rule,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cc|c} 3 & -5 & 15.5 \\ 6 & 16 & 5 \end{array}\right] \tag{7.7.151}$$

$$D = \begin{vmatrix} 3 & -5 \\ 6 & 16 \end{vmatrix} = 78 \qquad D_1 = \begin{vmatrix} 15.5 & -5 \\ 5 & 16 \end{vmatrix} = 273 \tag{7.7.152}$$

$$D_2 = \begin{vmatrix} 3 & 15.5 \\ 6 & 5 \end{vmatrix} = -78 \qquad \mathbf{x} = \begin{bmatrix} 3.5 \\ -1 \end{bmatrix} \tag{7.7.153}$$

Solving by Gauss elimination,

22. Solving by Cramer's rule,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cc|c} 2 & -4 & -24 \\ 5 & 2 & 0 \end{array}\right] \tag{7.7.154}$$

$$D = \begin{vmatrix} 2 & -4 \\ 5 & 2 \end{vmatrix} = 24 \qquad D_1 = \begin{vmatrix} -24 & -4 \\ 0 & 2 \end{vmatrix} = -48 \tag{7.7.155}$$

$$D_2 = \begin{vmatrix} 2 & -24 \\ 5 & 0 \end{vmatrix} = 120 \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \tag{7.7.156}$$

Solving by Gauss elimination,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cc|c} 2 & -4 & -24 \\ 5 & 2 & 0 \end{array}\right] \qquad = \left[\begin{array}{cc|c} 2 & -4 & -24 \\ 0 & 12 & 60 \end{array}\right] \tag{7.7.157}$$

$$\mathbf{u} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \tag{7.7.158}$$

23. Solving by Cramer's rule,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 0 & 3 & -4 & 16 \\ 2 & -5 & 7 & -27 \\ -1 & 0 & -9 & 9 \end{array}\right] \qquad D = \begin{vmatrix} 0 & 3 & -4 \\ 2 & -5 & 7 \\ -1 & 0 & -9 \end{vmatrix} = 53 \tag{7.7.159}$$

$$D_1 = \begin{vmatrix} 16 & 3 & -4 \\ -27 & -5 & 7 \\ 9 & 0 & -9 \end{vmatrix} = 0 \qquad D_2 = \begin{vmatrix} 0 & 16 & -4 \\ 2 & -27 & 7 \\ -1 & 9 & -9 \end{vmatrix} = 212 \tag{7.7.160}$$

$$D_3 = \begin{vmatrix} 0 & 3 & 16 \\ 2 & -5 & -27 \\ -1 & 0 & 9 \end{vmatrix} = -53 \qquad \mathbf{x} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \tag{7.7.161}$$

Solving by Gauss elimination,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 0 & 3 & -4 & 16 \\ 2 & -5 & 7 & -27 \\ -1 & 0 & -9 & 9 \end{array}\right] \qquad = \left[\begin{array}{ccc|c} 2 & -5 & 7 & -27 \\ 0 & 3 & -4 & 16 \\ 0 & -\frac{5}{2} & -\frac{11}{2} & -\frac{9}{2} \end{array}\right] \tag{7.7.162}$$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 7 & -27 \\ 0 & 3 & -4 & 16 \\ 0 & 0 & -\frac{53}{6} & \frac{53}{6} \end{array}\right] \qquad \mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \tag{7.7.163}$$

24. Solving by Cramer's rule,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 3 & -2 & 1 & 13 \\ -2 & 1 & 4 & 11 \\ 1 & 4 & -5 & -31 \end{array}\right] \qquad D = \begin{vmatrix} 3 & -2 & 1 \\ -2 & 1 & 4 \\ 1 & 4 & -5 \end{vmatrix} = -60 \tag{7.7.164}$$

$$D_1 = \begin{vmatrix} 13 & -2 & 1 \\ 11 & 1 & 4 \\ -31 & 4 & -5 \end{vmatrix} = -60 \qquad D_2 = \begin{vmatrix} 3 & 13 & 1 \\ -2 & 11 & 4 \\ 1 & -31 & -5 \end{vmatrix} = 180 \tag{7.7.165}$$

$$D_3 = \begin{vmatrix} 3 & -2 & 13 \\ -2 & 1 & 11 \\ 1 & 4 & -31 \end{vmatrix} = -240 \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \tag{7.7.166}$$

Solving by Gauss elimination,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 1 & 4 & -5 & -31 \\ 3 & -2 & 1 & 13 \\ -2 & 1 & 4 & 11 \end{array}\right] \qquad = \left[\begin{array}{ccc|c} 1 & 4 & -5 & -31 \\ 0 & -14 & 16 & 106 \\ 0 & 9 & -6 & -51 \end{array}\right] \tag{7.7.167}$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & -5 & -31 \\ 0 & -14 & 16 & 106 \\ 0 & 0 & \frac{30}{7} & \frac{120}{7} \end{array}\right] \qquad \mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \tag{7.7.168}$$

25. Solving by Cramer's rule,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} -4 & 1 & 1 & 0 & -10 \\ 0 & 1 & 1 & -4 & 10 \\ 1 & 0 & -4 & 1 & -7 \\ 1 & -4 & 0 & 1 & 1 \end{array}\right] \qquad D = \begin{vmatrix} -4 & 0 & 0 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -8 & 16 \\ 0 & 0 & 0 & -6 \end{vmatrix} = -192 \tag{7.7.169}$$

$$D_1 = \begin{vmatrix} -10 & 1 & 1 & 0 \\ 10 & 1 & 1 & -4 \\ -7 & 0 & -4 & 1 \\ 1 & -4 & 0 & 1 \end{vmatrix} = -576 \qquad D_2 = \begin{vmatrix} -4 & -10 & 1 & 0 \\ 0 & 10 & 1 & -4 \\ 1 & -7 & -4 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = 0 \tag{7.7.170}$$

$$D_3 = \begin{vmatrix} -4 & 1 & -10 & 0 \\ 0 & 1 & 10 & -4 \\ 1 & 0 & -7 & 1 \\ 1 & -4 & 1 & 1 \end{vmatrix} = -384 \qquad D_4 = \begin{vmatrix} -4 & 1 & 1 & -10 \\ 0 & 1 & 1 & 10 \\ 1 & 0 & -4 & -7 \\ 1 & -4 & 0 & 1 \end{vmatrix} = 384 \tag{7.7.171}$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -2 \end{bmatrix} \tag{7.7.172}$$

Solving by Gauss elimination,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} -4 & 1 & 1 & 0 & -10 \\ 0 & 1 & 1 & -4 & 10 \\ 1 & 0 & -4 & 1 & -7 \\ 1 & -4 & 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & 1 \\ 0 & 4 & -4 & 0 & -8 \\ 0 & -15 & 1 & 4 & -6 \\ 0 & 1 & 1 & -4 & 10 \end{array}\right] \tag{7.7.173}$$

$$= \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & 1 \\ 0 & 4 & -4 & 0 & -8 \\ 0 & 0 & -14 & 4 & -36 \\ 0 & 0 & 2 & -4 & 12 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & 1 \\ 0 & 4 & -4 & 0 & -8 \\ 0 & 0 & -14 & 4 & -36 \\ 0 & 0 & 0 & -\frac{24}{7} & \frac{48}{7} \end{array}\right] \tag{7.7.174}$$

$$\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -2 \end{bmatrix} \tag{7.7.175}$$

7.8 Inverse of a Matrix, Gauss-Jordan Elimination

1. Finding inverse using direct formula

$$\mathbf{A} = \begin{bmatrix} 1.8 & -2.32 \\ -0.25 & 0.6 \end{bmatrix} \qquad \det(\mathbf{A}) = 1.8(0.6) - 0.25(2.32) = \frac{1}{2} \tag{7.8.1}$$

$$\mathbf{A}^{-1} = 2 \cdot \begin{bmatrix} 0.6 & 2.32 \\ 0.25 & 1.8 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} 1.2 & 4.64 \\ 0.5 & 3.6 \end{bmatrix} \tag{7.8.2}$$

Verifying,

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1.8(1.2) - 2.32(0.5) & 1.8(4.64) - 2.32(3.6) \\ -0.25(1.2) + 0.6(0.5) & -0.25(4.64) + 0.6(3.6) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.8.3}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1.2(1.8) - 4.64(0.25) & 0.5(1.8) - 3.6(0.25) \\ 1.2(-2.32) + 4.64(0.6) & 0.5(-2.32) + 3.6(0.6) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.8.4}$$

2. Finding inverse using direct formula

$$\mathbf{A} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix} \qquad \det(\mathbf{A}) = 1 \tag{7.8.5}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix} \tag{7.8.6}$$

Verified using a CAS

3. Finding inverse using Gauss-Jordan elimination

$$\begin{bmatrix}\mathbf{A}|\mathbf{I}\end{bmatrix} = \left[\begin{array}{ccc|ccc} 0.3 & -0.1 & 0.5 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 1 & 0 \\ 5 & 0 & 9 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 0.3 & -0.1 & 0.5 & 1 & 0 & 0 \\ 0 & \frac{20}{3} & \frac{2}{3} & -\frac{20}{3} & 1 & 0 \\ 0 & \frac{5}{3} & \frac{2}{3} & -\frac{50}{3} & 0 & 1 \end{array}\right] \tag{7.8.7}$$

$$= \left[\begin{array}{ccc|ccc} 0.3 & -0.1 & 0.5 & 1 & 0 & 0 \\ 0 & \frac{20}{3} & \frac{2}{3} & -\frac{20}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{45}{3} & -\frac{1}{4} & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 0.3 & 0 & \frac{51}{100} & \frac{9}{10} & \frac{3}{200} & 0 \\ 0 & \frac{20}{3} & \frac{2}{3} & -\frac{20}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{45}{3} & -\frac{1}{4} & 1 \end{array}\right] \tag{7.8.8}$$

$$= \left[\begin{array}{ccc|ccc} 0.3 & 0 & 0 & \frac{81}{5} & \frac{27}{100} & -1.02 \\ 0 & \frac{20}{3} & 0 & \frac{40}{3} & \frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & \frac{1}{2} & -\frac{45}{3} & -\frac{1}{4} & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 54 & 0.9 & -3.4 \\ 0 & 1 & 0 & 2 & 0.2 & -0.2 \\ 0 & 0 & 1 & -30 & -0.5 & 2 \end{array}\right] \tag{7.8.9}$$

Verified using a CAS

4. Finding inverse using Gauss-Jordan elimination

$$\left[\mathbf{A}|\mathbf{I}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 0.1 & 1 & 0 & 0 \\ 0 & -0.4 & 0 & 0 & 1 & 0 \\ 2.5 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2.5 & 0 & 0 & 0 & 0 & 1 \\ 0 & -0.4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.1 & 1 & 0 & 0 \end{array}\right] \tag{7.8.10}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0.4 \\ 0 & 1 & 0 & 0 & -2.5 & 0 \\ 0 & 0 & 1 & 10 & 0 & 0 \end{array}\right] \tag{7.8.11}$$

Verified using a CAS

5. Finding inverse using Gauss-Jordan elimination

$$\left[\mathbf{A}|\mathbf{I}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 5 & 4 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & 1 & -5 & 0 & 1 \end{array}\right] \tag{7.8.12}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -4 & 1 \end{array}\right] \tag{7.8.13}$$

Verified using a CAS

6. Finding inverse using Gauss-Jordan elimination

$$\left[\mathbf{A}|\mathbf{I}\right] = \left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 13 & 0 & 1 & 0 \\ 0 & 3 & 5 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 13 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & -\frac{3}{8} & 1 \end{array}\right] \tag{7.8.14}$$

$$\left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 0 & 0 & 40 & -104 \\ 0 & 0 & \frac{1}{8} & 0 & -\frac{3}{8} & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.25 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & -13 \\ 0 & 0 & 1 & 0 & -3 & 8 \end{array}\right] \tag{7.8.15}$$

7. Finding inverse using Gauss-Jordan elimination

$$\left[\mathbf{A}|\mathbf{I}\right] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{7.8.16}$$

Verified using a CAS

8. Finding inverse using Gauss-Jordan elimination

$$\left[\mathbf{A}|\mathbf{I}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \tag{7.8.17}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right] = \text{singular, non-invertible} \tag{7.8.18}$$

9. Finding inverse using Gauss-Jordan elimination

$$\left[\mathbf{A}|\mathbf{I}\right] = \left[\begin{array}{ccc|ccc} 0 & 8 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \end{array}\right] \tag{7.8.19}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \end{array}\right] \tag{7.8.20}$$

Verified using a CAS

10. Finding inverse using Gauss-Jordan elimination

$$\left[\mathbf{A}|\mathbf{I}\right] = \frac{1}{3} \cdot \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 3 & 0 & 0 \\ -2 & 2 & 1 & 0 & 3 & 0 \\ 1 & 2 & -2 & 0 & 0 & 3 \end{array}\right] = \frac{1}{3} \cdot \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 3 & 0 & 0 \\ 0 & 3 & 3 & 3 & 3 & 0 \\ 0 & 1.5 & -3 & -1.5 & 0 & 3 \end{array}\right] \tag{7.8.21}$$

$$= \frac{1}{3} \cdot \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 3 & 0 & 0 \\ 0 & 3 & 3 & 3 & 3 & 0 \\ 0 & 0 & -4.5 & -3 & -1.5 & 3 \end{array}\right] = \frac{1}{3} \cdot \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 2 & -1 & 0 \\ 0 & 3 & 3 & 3 & 3 & 0 \\ 0 & 0 & -4.5 & -3 & -1.5 & 3 \end{array}\right] \tag{7.8.22}$$

$$= \frac{1}{3} \cdot \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{4}{3} & -\frac{4}{3} & \frac{2}{3} \\ 0 & 3 & 0 & 1 & 2 & 2 \\ 0 & 0 & -4.5 & -3 & -1.5 & 3 \end{array}\right] = \frac{1}{3} \cdot \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 2 & -2 & 1 \\ 0 & 3 & 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 2 & 1 & -2 \end{array}\right] \tag{7.8.23}$$

Verified using a CAS

11. Verifying the relation,

$$\mathbf{A} = \begin{bmatrix} 1.8 & -2.32 \\ -0.25 & 0.6 \end{bmatrix} \qquad \mathbf{A}^2 = \begin{bmatrix} 3.82 & -5.568 \\ -0.6 & 0.94 \end{bmatrix} \tag{7.8.24}$$

$$\left(\mathbf{A}^2\right)^{-1} = \begin{bmatrix} 3.76 & 22.272 \\ 2.4 & 15.28 \end{bmatrix} \tag{7.8.25}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1.2 & 4.64 \\ 0.5 & 3.6 \end{bmatrix} \qquad \left(\mathbf{A}^{-1}\right)^2 = \begin{bmatrix} 3.76 & 22.272 \\ 2.4 & 15.28 \end{bmatrix} \tag{7.8.26}$$

12. Proving the relation,

$$\left(\mathbf{A}^2\right)^{-1} = \left(\mathbf{A}\mathbf{A}\right)^{-1} = \mathbf{A}^{-1}\mathbf{A}^{-1} = \left(\mathbf{A}^{-1}\right)^2 \tag{7.8.27}$$

13. Verifying the relation,

$$\mathbf{A} = \begin{bmatrix} 1.8 & -2.32 \\ -0.25 & 0.6 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1.8 & -0.25 \\ -2.32 & 0.6 \end{bmatrix} \tag{7.8.28}$$

$$\left(\mathbf{A}^T\right)^{-1} = \begin{bmatrix} 1.2 & 0.5 \\ 4.64 & 3.6 \end{bmatrix} \tag{7.8.29}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1.2 & 4.64 \\ 0.5 & 3.6 \end{bmatrix} \qquad \left(\mathbf{A}^{-1}\right)^T = \begin{bmatrix} 1.2 & 0.5 \\ 4.64 & 3.6 \end{bmatrix} \tag{7.8.30}$$

14. Proving the relation,

$$\left(\mathbf{A}\mathbf{A}^{-1}\right)^T = \left(\mathbf{A}^{-1}\right)^T\mathbf{A}^T = \mathbf{I} \tag{7.8.31}$$

$$\left(\mathbf{A}^{-1}\right)^T\mathbf{A}^T\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^T\right)^{-1} \tag{7.8.32}$$

$$\left(\mathbf{A}^{-1}\right)^T = \left(\mathbf{A}^T\right)^{-1} \tag{7.8.33}$$

15. Proving the relation,

$$\left(\mathbf{A}\mathbf{A}^{-1}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{-1}\mathbf{A}^{-1} = \mathbf{I} \tag{7.8.34}$$

$$\left(\mathbf{A}^{-1}\right)^{-1}\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}\mathbf{A} \tag{7.8.35}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A} \tag{7.8.36}$$

16. The matrix and its inverse are rotation by 2θ and -2θ respectively. This shows that multiplying by the inverse of a matrix corresponds to performing the inverse linear transoform.

17. Consider the Gauss jordan elimination process

$$\begin{bmatrix}\mathbf{U} | \mathbf{I}\end{bmatrix} \implies \begin{bmatrix}\mathbf{I} | \mathbf{H}\end{bmatrix} \tag{7.8.37}$$

The row operations that convert $\mathbf{U}$ to $\mathbf{I}$, automatically leave H triangular.

18. Prob 7 shows a matrix that interchanges Row 1 and Row 2. Its inverse is a matrix that interchanges Row 2 and Row 1.

This happens to be the same matrix since the linear transform is also its own inverse.

19. Finding the inverse by the direct cofactor formula, with the cofactor matrix being $\mathbf{B}$,

$$\mathbf{A} = \begin{bmatrix} 0.3 & -0.1 & 0.5 \\ 2 & 6 & 4 \\ 5 & 0 & 9 \end{bmatrix} \qquad \det(\mathbf{A}) = 0.3(54) - 2(-0.9) + 5(-3.4) = 1 \tag{7.8.38}$$

$$\mathbf{B} = \begin{bmatrix} 54 & 2 & -30 \\ 0.9 & 0.2 & -0.5 \\ -3.4 & -0.2 & 2 \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{B}^T \tag{7.8.39}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 54 & 0.9 & -3.4 \\ 2 & 0.2 & -0.2 \\ -30 & -0.5 & 2 \end{bmatrix} \tag{7.8.40}$$

which matches the Gauss-Jordan procedure from Problem 3

20. Finding the inverse by the direct cofactor formula, with the cofactor matrix being $\mathbf{B}$,

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 13 \\ 0 & 3 & 5 \end{bmatrix} \qquad \det(\mathbf{A}) = -4(1) = -4 \tag{7.8.41}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -20 & 12 \\ 0 & 52 & -32 \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{B}^T \tag{7.8.42}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & 5 & -13 \\ 0 & -3 & 8 \end{bmatrix} \tag{7.8.43}$$

which matches the Gauss-Jordan procedure from Problem 6

7.9 Vector Spaces, Inner Product Spaces, Linear Transformations

1. By observation, three possible bases for $\mathcal{R}^2$ are

$$\begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{7.9.1}$$

$$\begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{7.9.2}$$

2. Consider two possible representations of $\mathbf{v}$,

$$\mathbf{v} = \sum_{i=1}^{n} c_i \ \mathbf{a}_{(i)} \qquad = \sum_{i=1}^{n} b_i \ \mathbf{a}_{(i)} \tag{7.9.3}$$

$$\mathbf{0} = \sum_{i=1}^{n} (c_i - b_i) \ a_{(i)} \tag{7.9.4}$$

Since the set $\{a_{(i)}\}$ form a basis for the vector space V, the only solution to the last equation has to be $\{c_i - b_i\} = 0$.

Thus, the representation of $\mathbf{v}$ in a given basis is unique.

3. Checking if the given set is a vector space, yes

$$-v_1 + 2v_2 + 3v_3 = 0 \qquad -4v_1 + v_2 + v_3 = 0 \tag{7.9.5}$$

$$\mathbf{v} \in \mathcal{R}^3 \tag{7.9.6}$$

$$\mathbf{a}, \ \mathbf{b} \in V \qquad \implies \mathbf{a} + \mathbf{b} \in V \tag{7.9.7}$$

$$\mathbf{a} \in V \qquad \implies k\mathbf{a} \in V \tag{7.9.8}$$

Finding the dimension and basis,

$$v_2 = -\frac{11}{7} \ v_3 \qquad v_1 = -\frac{1}{7} \ v_3 \tag{7.9.9}$$

$$\dim(V) = 1 \qquad \begin{bmatrix} \mathbf{u}_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \\ 7 \end{bmatrix} \tag{7.9.10}$$

4. Checking if the given set is a vector space, yes

$$\mathbf{M} = \begin{bmatrix} 0 & v_1 & v_2 \\ -v_1 & 0 & v_3 \\ -v_2 & -v_3 & 0 \end{bmatrix} \tag{7.9.11}$$

$$\mathbf{M},\ \mathbf{N} \in V \implies \mathbf{M} + \mathbf{N} \in V \tag{7.9.12}$$

$$\mathbf{M} \in V \implies k\mathbf{M} \in V \tag{7.9.13}$$

Finding the dimension and basis,

$$\dim(V) = 3 \qquad \mathbf{U}_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.9.14}$$

$$\mathbf{U}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{U}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{7.9.15}$$

5. Checking if the given set is a vector space, no

$$P(x) = v_1 + v_2 x + v_3 x^2 + v_4 x^3 + v_5 x^4 \qquad \{v_i\} \geq 0 \tag{7.9.16}$$

$$P(x),\ Q(x) \in V \implies P(x) + Q(x) \in V \tag{7.9.17}$$

$$P(x) \in V \implies kP(x) \notin V \tag{7.9.18}$$

Finding the dimension and basis,

$$\dim(V) = 3 \qquad \mathbf{U}_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.9.19}$$

$$\mathbf{U}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{U}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{7.9.20}$$

6. Checking if the given set is a vector space, yes

$$y(x) = v_1 \cos(2x) + v_2 \sin(2x) \qquad \{v_i\} \in \mathcal{R} \tag{7.9.21}$$

$$y(x),\ \ z(x) \in V \qquad \Longrightarrow\ y(x) + z(x) \in V \tag{7.9.22}$$

$$y(x) \in V \qquad \Longrightarrow\ ky(x) \notin V \tag{7.9.23}$$

Finding the dimension and basis,

$$\dim(V) = 2 \tag{7.9.24}$$

$$u_1(x) = \cos(2x) \qquad u_2(x) = \sin(2x) \tag{7.9.25}$$

7. Checking if the given set is a vector space, yes

$$y(x) = (v_1 x + v_2)\ e^{-x} \qquad \{v_i\} \in \mathcal{R} \tag{7.9.26}$$

$$y(x),\ \ z(x) \in V \qquad \Longrightarrow\ y(x) + z(x) \in V \tag{7.9.27}$$

$$y(x) \in V \qquad \Longrightarrow\ ky(x) \notin V \tag{7.9.28}$$

Finding the dimension and basis,

$$\dim(V) = 2 \tag{7.9.29}$$

$$u_1(x) = e^{-x} \qquad u_2(x) = xe^{-x} \tag{7.9.30}$$

8. Checking if the given set is a vector space, yes

$$\mathbf{A} \to n \times n \qquad \det(\mathbf{A}) = 0 \tag{7.9.31}$$

$$\mathbf{M},\ \ \mathbf{N} \in V \qquad \Longrightarrow\ \mathbf{M} + \mathbf{N} \notin V \tag{7.9.32}$$

$$\tag{7.9.33}$$

An example is the addition of two matrices, each of which are the zero matrix and some columns of the identity matrix.

$$\mathbf{A} + \mathbf{B} = \mathbf{I} \tag{7.9.34}$$

$$\det(\mathbf{A}) = \det(\mathbf{B}) = 0 \qquad \det(\mathbf{A} + \mathbf{B}) = 1 \tag{7.9.35}$$

9. Checking if the given set is a vector space, yes

$$\mathbf{A} = \begin{bmatrix} v_1 & v_2 \\ v_3 & -v_1 \end{bmatrix} \qquad \{v_i\} \in \mathcal{R} \tag{7.9.36}$$

$$\mathbf{M},\ \mathbf{N} \in V \qquad \implies \mathbf{M} + \mathbf{N} \in V \tag{7.9.37}$$

$$\mathbf{M} \in V \qquad \implies k\mathbf{M} \notin V \tag{7.9.38}$$

Finding the dimension and basis,

$$\dim(V) = 3 \qquad \mathbf{U}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7.9.39}$$

$$\mathbf{U}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{U}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{7.9.40}$$

10. Checking if the given set is a vector space, yes

$$\mathbf{A} = \begin{bmatrix} 3v_1 & v_2 \\ 0 & v_3 \\ -5v_1 & v_4 \end{bmatrix} \qquad \{v_i\} \in \mathcal{R} \tag{7.9.41}$$

$$\mathbf{M},\ \mathbf{N} \in V \qquad \implies \mathbf{M} + \mathbf{N} \in V \tag{7.9.42}$$

$$\mathbf{M} \in V \qquad \implies k\mathbf{M} \notin V \tag{7.9.43}$$

Finding the dimension and basis,

$$\dim(V) = 4 \tag{7.9.44}$$

$$\mathbf{U}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \\ -5 & 0 \end{bmatrix} \qquad \mathbf{U}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.9.45}$$

$$\mathbf{U}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{U}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.9.46}$$

11. Finding the inverse of the matrix $\mathbf{A}$ that represents the transform,

$$\mathbf{y} = \begin{bmatrix} 0.5 & -0.5 \\ 1.5 & -2.5 \end{bmatrix} \mathbf{x} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{7.9.47}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \cdot \begin{bmatrix} -2.5 & 0.5 \\ -1.5 & 0.5 \end{bmatrix} \qquad = \begin{bmatrix} 5 & -1 \\ 3 & -1 \end{bmatrix} \tag{7.9.48}$$

12. Finding the inverse of the matrix $\mathbf{A}$ that represents the transform,

$$\mathbf{y} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \mathbf{x} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{7.9.49}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \cdot \begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix} \qquad = \begin{bmatrix} -0.2 & 0.4 \\ 0.8 & -0.6 \end{bmatrix} \tag{7.9.50}$$

13. Finding the inverse of the matrix $\mathbf{A}$ that represents the transform,

$$\mathbf{y} = \begin{bmatrix} 5 & 3 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \mathbf{x} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{7.9.51}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \cdot \begin{bmatrix} 2 & -10 & -7 \\ -3 & 16 & 11 \\ 0 & 1 & 1 \end{bmatrix}^T \qquad = \begin{bmatrix} 2 & -3 & 0 \\ -10 & 16 & 1 \\ -7 & 11 & 1 \end{bmatrix} \tag{7.9.52}$$

14. Finding the inverse of the matrix $\mathbf{A}$ that represents the transform,

$$\mathbf{y} = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0 & -0.2 & 0.1 \\ 0.1 & 0 & 0.1 \end{bmatrix} \mathbf{x} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{7.9.53}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \cdot \begin{bmatrix} -0.02 & 0.01 & 0.02 \\ 0.01 & 0.02 & -0.01 \\ -0.01 & -0.02 & -0.04 \end{bmatrix}^T \qquad = -2 \cdot \begin{bmatrix} -2 & 1 & 2 \\ 1 & 2 & -1 \\ -1 & -2 & -4 \end{bmatrix}^T \tag{7.9.54}$$

$$= \begin{bmatrix} 4 & -2 & 2 \\ -2 & -4 & 4 \\ -4 & 2 & 8 \end{bmatrix} \tag{7.9.55}$$

15. The Euclidean norm is,

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} \qquad \|\mathbf{u}\| = \sqrt{3^2 + 1^2 + (-4)^2} = \sqrt{26} \tag{7.9.56}$$

16. The Euclidean norm is,

$$\mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ -\frac{1}{2} \\ -\frac{1}{3} \end{bmatrix} \qquad \|\mathbf{u}\| = \sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{4} + \frac{1}{9}} = \frac{\sqrt{26}}{6} \tag{7.9.57}$$

17. The Euclidean norm is,

$$\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}^T \tag{7.9.58}$$

$$\|\mathbf{u}\| = \sqrt{5} \tag{7.9.59}$$

18. The Euclidean norm is,

$$\mathbf{u} = \begin{bmatrix} -4 \\ 8 \\ -1 \end{bmatrix} \qquad \|\mathbf{u}\| = \sqrt{(-4)^2 + 8^2 + (-1)^2} = 9 \tag{7.9.60}$$

19. The Euclidean norm is,

$$\mathbf{u} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} \qquad \|\mathbf{u}\| = \frac{\sqrt{4+4+1}}{3} = 1 \tag{7.9.61}$$

20. The Euclidean norm is,

$$\mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \|\mathbf{u}\| = \frac{\sqrt{4}}{2} = 1 \tag{7.9.62}$$

21. To ensure orthogonality,

$$\mathbf{u}\cdot\mathbf{v} = 0 \qquad 10 + \frac{k}{2} + 0 + 0 = 0 \tag{7.9.63}$$

$$k = -20 \tag{7.9.64}$$

22. To ensure orthogonality,

$$\mathbf{u}\cdot\mathbf{v} = 0 \qquad 2v_1 + v_3 = 0 \tag{7.9.65}$$

$$\mathbf{M},\ \mathbf{N} \in V \qquad \implies \mathbf{M}+\mathbf{N} \in V \tag{7.9.66}$$

$$\mathbf{M} \in V \qquad \implies k\mathbf{M} \in V \tag{7.9.67}$$

So, V does form a vector space. In 3d Euclidean space, this is a plane orthogonal to the given normal vector.

23. Verifying Triangle inequality,

$$\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} \qquad \|\mathbf{a}\| = \sqrt{26} \tag{7.9.68}$$

$$\mathbf{b} = \begin{bmatrix} -4 \\ 8 \\ -1 \end{bmatrix} \qquad \|\mathbf{b}\| = 9 \tag{7.9.69}$$

$$\mathbf{a}+\mathbf{b} = \begin{bmatrix} -1 \\ 9 \\ -5 \end{bmatrix} \qquad \|\mathbf{a}+\mathbf{b}\| = \sqrt{(-1)^2 + 9^2 + (-5)^2} = \sqrt{107} \tag{7.9.70}$$

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \tag{7.9.71}$$

24. Verifying Cauchy-Schwarz inequality,

$$\mathbf{a} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ -\frac{1}{2} \\ -\frac{1}{3} \end{bmatrix} \qquad \|\mathbf{a}\| = \sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{4} + \frac{1}{9}} = \frac{\sqrt{26}}{6} \tag{7.9.72}$$

$$\mathbf{b} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} \qquad \|\mathbf{b}\| = \frac{\sqrt{4+4+1}}{3} = 1 \tag{7.9.73}$$

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{3} + \frac{2}{9} - \frac{1}{6} = \frac{7}{18} \tag{7.9.74}$$

$$\|\mathbf{a} \cdot \mathbf{b}\| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\| \tag{7.9.75}$$

25. Verifying the Parallellogram equality,

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} \qquad \|\mathbf{a} + \mathbf{b}\|^2 = 90 \tag{7.9.76}$$

$$\mathbf{a} - \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad \|\mathbf{a} - \mathbf{b}\|^2 = 14 \tag{7.9.77}$$

$$\|\mathbf{a}\|^2 = 38 \qquad \|\mathbf{b}\|^2 = 14 \tag{7.9.78}$$

$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right) \tag{7.9.79}$$