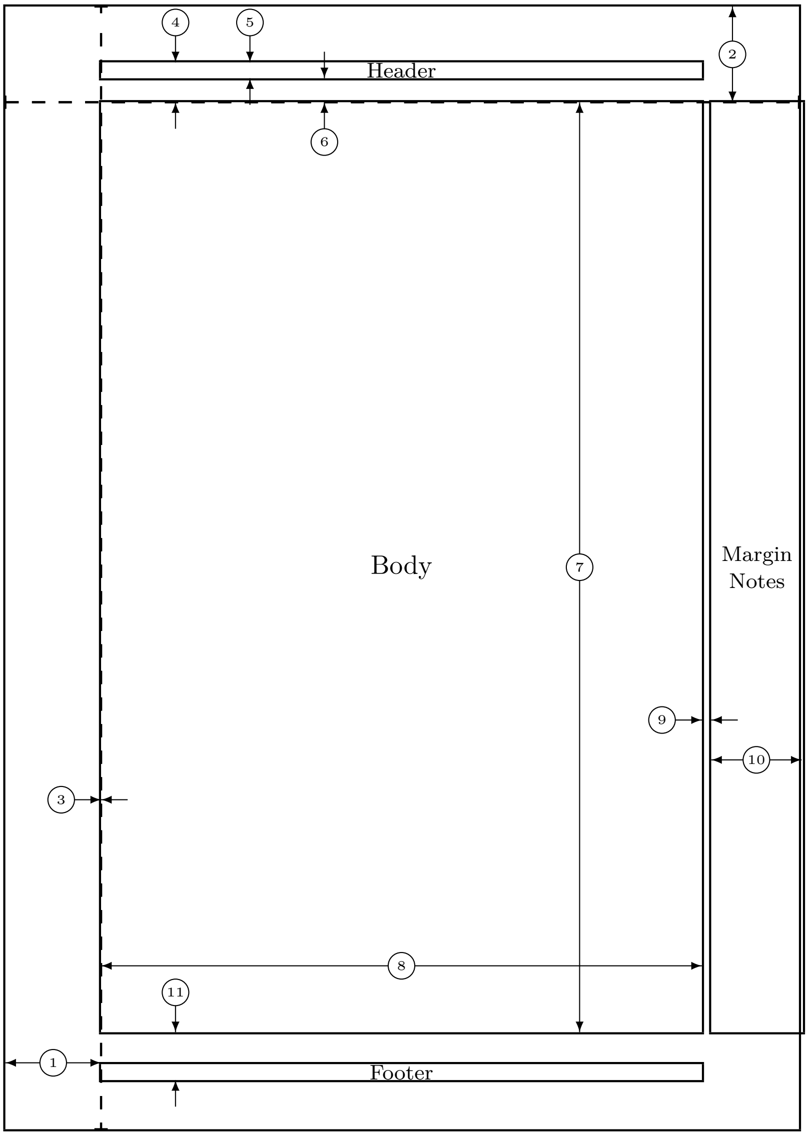

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	\paperwidth = 597pt		\paperheight = 845pt

Anirudh Krishnan

ADVANCED ENGINEERING MATHEMATICS

Notes and Solutions

Erwin Kreyszig

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Chapter 1

First Order ODEs

1.1 Basic Concepts: Modeling

Modeling Converting an engineering problem into a set of mathematical relations

Ordinary Differential Equation An equation that contains one or several derivatives of an unknown function (which only contains a single variable). Shorthand notation for higher order derivatives is:

$$y' := \frac{\mathrm{d}y}{\mathrm{d}x} \qquad y'' := \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \qquad y^{(n)} := \frac{\mathrm{d}^n y}{\mathrm{d}x^n} \tag{1.1.1}$$

Order The highest derivative of the unknown function that is in a given equation

Implicit form ODE represented as $F(x, y, y') = 0$

Explicit form ODE represented as $y' = f(x, y)$

Solution A function $y = h(x)$ on some open interval that satisfies the ODE. This requires $h(x)$ to be defined and differentiable on that interval.

Solution Curve The graph of a solution to an ODE

Open Interval A segment of the real line not including the endpoints. Special cases include the entire real line $(-\infty, \infty)$, and half-infinite intervals of the form $[a, \infty)$ and $(-\infty, b]$

Family of Solutions A set of solutions of an ODE grouped together by the value of the arbitrary constant leftover from integration.

General Solution A solution to an ODE containing an arbitrary constant (denoted by c)

Particular Solution The outcome of fixing the arbitrary constant c in a General solution. This no longer contains any arbitrary constants.

Initial Condition A constraint on c creating a particular solution.

Initial Value Problem An ODE along with an initial condition

Autonomous ODE An ODE not showing the independent variable explicitly $f(x, y) \to f(y)$

$$y' = f(x, y) \qquad y(x_0) = y_0 \tag{1.1.2}$$

The general outline of a mathematical modeling procedure is as follows:

1. Transition from the physical system to its mathematical formulation
2. Using a mathematical method to solve this model
3. Physical interpretation of the results. (Including a sanity check based on the practical nature of the physical system)

1.2 Geometric Meaning of y' = f(x, y)

Slope The slope of the line tangent to a curve at a given point on the curve. Mathematically, this is equal to the fist derivative of the equation of the curve evaluated at that point

Direction field A vector field of the first derivative showing used to reverse engineer the solution to a first order ODE

$$y' = f(x, y) \qquad y'(x_0) = f(x_0, y_0) \tag{1.2.1}$$

Level Curve A curve of the function $f(x, y) = c$ for some constant c. Also called an Isocline.

Euler's method A numeric method for obtaining approximate values of a function at a set of equidistant x values (with separation h).

$$\mathbf{x} = \{x_0, x_1, \ldots, x_n\} \qquad x_i = x_0 + ih \tag{1.2.2}$$

$$y_1 = y_0 + hf(x_0, y_0) \tag{1.2.3}$$

$$\vdots$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{-1}) \tag{1.2.4}$$

1.3 Separable ODEs. Modeling

Separable ODE An ODE which can be expressed in a form where the two variables x and y are on two sides of the equation.

$$g(y)\ y' = f(x) \tag{1.3.1}$$

$$\int g(y)\ \mathrm{d}y = \int f(x)\ \mathrm{d}x + c \tag{1.3.2}$$

Note the introduction of c at the earliest possible step. Being sloppy with introducing the constant of introduction can greatly change the final answer.

Method of separating variables Rewriting an ODE using algebraic manipulation into a separable form.

Reduction to Separable form For equations that are not directly in separable form, consider the variable y/x

$$y' = f\left(\frac{y}{x}\right) \tag{1.3.3}$$

$$u = \frac{y}{x} \qquad \frac{\mathrm{d}u}{f(u) - u} = \frac{\mathrm{d}}{\mathrm{d}x} 1/x \tag{1.3.4}$$

The above form is sometimes called a homogenous ODE.

1.4 Exact ODEs. Integrating Factors

Consider a function $u(x, y)$ with continuous partial derivatives.

Exact Differential Equation An ODE using functions $M(x, y)$ and $N(x, y)$ arranged into the form,

$$M \ \ \mathrm{d}x + N \ \ \mathrm{d}y = 0 \tag{1.4.1}$$

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = du = 0 \tag{1.4.2}$$

$$\frac{\partial M}{\partial y} = \tag{1.4.3}$$

$$diffpux, y = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial N}{\partial x} \qquad \iff \quad u(x, y) = c \tag{1.4.4}$$

Implicit solution A solution of the form $u(x, y) = c$ as opposed to the earlier solutions of the explicit form $y = f(x) + c$. Interconversion may not always be possible.

The general approach to solving an exact ODE is,

1. Integrate M w.r.t x keeping a leftover constant of integration $k(y)$.
2. Differentiate this result w.r.t. y and equate it to N to find dk/dy.
3. Integrate dk/dy w.r.t. y and substitute back to get the general solution.

$$u = \int M \ \ \mathrm{d}x + k(y) \tag{1.4.5}$$

$$\frac{\partial u}{\partial y} = \frac{\mathrm{d}k}{\mathrm{d}y} + \frac{\partial}{\partial y}\left(\int M dx\right) = N \tag{1.4.6}$$

This procedure can also be analogously carried out starting with M.

Integrating Factor For ODEs that are not exact, a pre-factor multiplied to the ODE can reduce it to exact form.

Consider an integrating factor $F(x)$ depending only on x

$$P(x, y)\ \mathrm{d}x + Q(x, y)\ \mathrm{d}y = 0 \tag{1.4.7}$$

$$\frac{\partial FP}{\partial y} = F_y P + F P_y = F_x Q + F Q_x = \frac{\partial FQ}{\partial x} \tag{1.4.8}$$

$$\frac{1}{F}\frac{\mathrm{d}F}{\mathrm{d}x} = R \tag{1.4.9}$$

$$R(x) = \frac{1}{Q}\left[P_y - Q_x\right] \tag{1.4.10}$$

If R depends only on x, then an integrating factor exists of the form,

$$F(x) = \exp\left[\int R(x)\ \mathrm{d}x\right] \tag{1.4.11}$$

An analogous method to find an integrating factor exists for F and therefore R^* depending only on y

$$F^*(y) \quad \text{enables} \quad R^* = \frac{1}{F^*}\frac{\mathrm{d}F^*}{\mathrm{d}y} \tag{1.4.12}$$

$$R^*(y) = \frac{1}{P}\left[Q_x - P_y\right] \tag{1.4.13}$$

$$F^*(y) = \exp\left[\int R^*(y)\ dy\right] \tag{1.4.14}$$

1.5 Linear ODEs, Bernoulli equation, Population Dynamics

Linear ODE An ODE which can be brought to the form,

$$y' + p(x)\ y = r(x) \tag{1.5.1}$$

Here, $r(x)$ is called the input and y is the response to that input and any initial conditions if present. In the standard form, the coefficient of y' is 1.

Homogenous Linear ODE A special case of the linear ODE with input $r(x)$ being 0. This can always be solved using separation of variables.

$$y' + p(x)\ y = 0 \tag{1.5.2}$$

$$y = c\ \exp\left(-\int\ p(x)\ \ \mathrm{d}x\right) \tag{1.5.3}$$

Nonhomogenous Linear ODE An Linear ODE with nonzero input $r(x)$. Its solution is closely related to that of the corresponding homogenous linear ODE.

$$h = \int\ p(x)\ \ \mathrm{d}x \tag{1.5.4}$$

$$y = e^{-h}\left[\int\ e^{h} r(x)\ \ \mathrm{d}x + c\right] \tag{1.5.5}$$

$$= e^{-h}\int\ e^{h} r(x)\ \ \mathrm{d}x \qquad + ce^{-h} \tag{1.5.6}$$

$$= \text{response to input } r(x) \qquad + \text{response to initial condition} \tag{1.5.7}$$

Steady-state solution The part of an ODE's solution which is independent of initial condition, and persists after all transient-state solution is allowed to settle.

Bernoulli's equation A specific form of nonlinear ODE that can be reduced to a linear ODE after change of variables.

$$y' + p(x)\ y = g(x)\ y^{a} \qquad a \neq \{0, 1\} \tag{1.5.8}$$

$$\text{set} \quad u = y^{1-a} \tag{1.5.9}$$

$$u' + (1-a)p(x)\ u = (1-a)\ g(x) \tag{1.5.10}$$

Logistic equation An early population dynamics model which allowed for exponential growth at small initial population levels, with a built-in braking term to prevent infinite growth.

$$y' = Ay - By^{2} \tag{1.5.11}$$

$$\text{stable critical point} \quad y^{*} = \frac{A}{B} \tag{1.5.12}$$

$$\text{unstable critical point} \quad y^{*} = 0 \tag{1.5.13}$$

Autonomous ODE An ODE which has no explicit dependence on the independent variable

$$y' = f(y) \tag{1.5.14}$$

An example is the logistic equation above.

Critical points In an autonomous ODE, zeros of the expression $f(y)$. Also known as equilibrium points (either stable or unstable).

1.6 Orthogonal Trajectories

Orthogonal trajectory A family of curves that intersect another family at right angles

Angle of intersection Angle between the tangents to both curves at the point of intersection

One-parameter family of curves A family of curves $G(x, y, c)$ controlled by a single parameter c

General strategy for finding an orthogonal family:

1. Find an ODE for the starting family of curves which eliminates the parameter
2. Define another ODE $\tilde{y}$, such that

$$y' = f(x, y) \qquad \text{starting family} \tag{1.6.1}$$

$$\tilde{y}\,' = \frac{-1}{f(x, \tilde{y})} \qquad \text{orthogonal family} \tag{1.6.2}$$

3. Solve the new ODE to find a one parameter family of curves orthogonal to the starting family

1.7 Existence and Uniqueness of Solutions for Initial Value Problems

Existence of solution Let $f(x, y)$ be continuous at all points in some rectangle $\mathcal{R}$. Also let it be bounded in $\mathcal{R}$

$$y' = f(x, y) \qquad y(x_0) = y_0 \tag{1.7.1}$$

$$\mathcal{R} : |x - x_0| < a \qquad |y - y_0| < b \tag{1.7.2}$$

$$|f(x, y)| \leq K \qquad \forall (x, y) \in \mathcal{R} \tag{1.7.3}$$

Then, the IVP has at least one solution $y(x)$.

Uniqueness of solution In addition to the above conditions, $\partial F/\partial x$ needs to be continuous and bounded in the rectangle $\mathcal{R}$.

$$|f(x, y)| \leq K \qquad \forall (x, y) \in \mathcal{R} \tag{1.7.4}$$

$$\left|\frac{\partial f(x, y)}{\partial y}\right| \leq M \qquad \forall (x, y) \in \mathcal{R} \tag{1.7.5}$$

Then the IVP has at most one solution. In combination with the existence theorem, The IVP has exactly one solution.

Lipschitz condition For a weaker condition, consider the mean value theorem of differential calculus, which states that some $\tilde{y} \in (y_1, y_2)$ exists for which,

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\left(\frac{\partial f}{\partial y}\right)_{y=\tilde{y}} \tag{1.7.6}$$

The points (x, y_1) and (x, y_2) are in the rectangle $\mathcal{R}$ as defined above. Now, the condition on $\mathrm{d}f/\mathrm{d}y$ can be replaced by the weaker relation,

$$\left|\frac{\partial f(x, y)}{\partial y}\right| \leq M \qquad \forall (x, y) \in \mathcal{R} \tag{1.7.7}$$

$$|f(x, y_2) - f(x, y_1)| \leq M|(y_2 - y_1)| \tag{1.7.8}$$

Continuity of $f(x, y)$ is not enough to guarantee the uniqueness of a solution.

Chapter 2

Second-Order Linear ODEs

2.1 Homogeneous Linear ODEs of Second Order

Linear ODE of second order An ODE which can be written in the form

$$y'' + p(x)y' + q(x)y = r(x) \tag{2.1.1}$$

The form has to be linear in y and all of its derivatives $y^{(n)}$. Standard form requires the coefficient of y'' to be 1.

Homogeneous Linear ODE of second order In the above ODE, if $r(x) \equiv 0$, else the ODE is called non-homogeneous.

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1.2}$$

Superposition principle If y_1, y_2 are any two solutions to a linear homogeneous ODE, then for arbitrary constants c_1, c_2,

$$y = c_1y_1 + c_2y_2 \tag{2.1.3}$$

is also a solution on the same open interval I in which y_1, y_2 were solutions. This is also called the linearity principle.

Initial Value Problem of second order Analogous to the IVP of a first order ODE, a particular solution is found by using,

$$y = c_1y_1 + c_2y_2 \qquad \text{General solution} \tag{2.1.4}$$

$$y(x_0) = K_0 \qquad y'(x_0) = K_1 \tag{2.1.5}$$

Note that both the Initial conditions are to be evaluated at the same x_0.

Linearly Independent solutions Two solutions to an ODE y_1, y_2 are called linearly independent (L.I.), if for constants k_1, k_2,

$$k_1 y_1 + k_2 y_2 = 0 \qquad \text{everywhere on } I \tag{2.1.6}$$

$$\implies k_1 = 0 \quad \text{and} \quad k_2 = 0 \tag{2.1.7}$$

Linearly Dependent solutions y_2 is a scalar multiple of y_1, and the above relation also holds for some k_1, k_2 not both zero.

Basis A system of solutions to an ODE that are L.I., and therefore fully describe a general solution to that ODE. This holds for a linear homogeneous ODE of any higher orders as well.

Since the general solution yields all possible particular solutions using IVPs, there is no singular solution to an ODE of order 2 or higher.

Reduction of order Given one of the solutions to a second order linear homogeneous ODE $y_1$is known, find a basis of solutions by reducing the problem to a first order ODE.

1. y_1 solves the given ODE,

$$y_1'' + p y_1' + q y_1 = 0 \tag{2.1.8}$$

2. Define a new solution $y_2 = u y_1$, and substitute it into the ODE,

$$u'' y_1 + u' \left(2y_1' + p y_1\right) = 0 \tag{2.1.9}$$

3. Define $V = u'$ and $V' = u''$, to get

$$V = \frac{1}{y_1^2} \exp\left(-\int p(x) \ \mathrm{d}x\right) \tag{2.1.10}$$

$$y_2 = u y_1 = y_1 \int V \ \mathrm{d}x \tag{2.1.11}$$

y_2 is automatically L.I. with y_1 unless $V \equiv 0$.

2.2 Homogeneous Linear ODEs with Constant Coefficients

General form of these ODEs for some constants a, b is,

$$y'' + a y' + b = 0 \tag{2.2.1}$$

Characteristic equation The equation derived by substituting $y = e^{\lambda x}$ into the above ODE,

$$\lambda^2 + a\lambda + b = 0 \tag{2.2.2}$$

For $y = e^{\lambda x}$ to be a solution of the ODE, λ has to solve this quadratic equation. The three possible cases are now,

$$a^2 - 4b > 0 \qquad y = c_1 e^{\lambda_1} x + c_2 e^{\lambda_2} x \tag{2.2.3}$$

$$\lambda_1, \lambda_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \tag{2.2.4}$$

$$a^2 = 4b \qquad y = [c_1 + c_2 x] \exp\left(\frac{-ax}{2}\right) \tag{2.2.5}$$

$$a^2 - 4b < 0 \qquad y = [c_1 \cos(\omega x) + c_2 \sin(\omega x)] \exp\left(\frac{-ax}{2}\right) \tag{2.2.6}$$

$$\omega^2 = b - \frac{a^2}{4} \tag{2.2.7}$$

Deriving the above result for imaginary roots λ_1, λ_2 uses,

$$e^{r+it} = e^r(\cos t + i \ \sin t) \tag{2.2.8}$$

$$r = \frac{-ax}{2} \qquad t = \omega x \tag{2.2.9}$$

$$\lambda_1, \lambda_2 = \frac{a}{2} \pm i\omega \tag{2.2.10}$$

2.3 Differential Operators

Operator A transformation that maps a function into another function.

Differential operator An operator D that maps a differentiable function y to its derivative y',

$$Dy \equiv y' \qquad D^2 y \equiv y'' \tag{2.3.1}$$

$$D^n y \equiv y^{(n)} \equiv \frac{\mathrm{d}^n y}{\mathrm{d}x^n} \tag{2.3.2}$$

Identity operator An operator I that maps a function to itself.

$$Iy \equiv y \tag{2.3.3}$$

Second order differential operator Using the operator notation to condense the second order linear ODE, for $P(D)$ being a polynomial in D

$$\mathcal{L} \equiv P(D) = D^2 + aD + bI \tag{2.3.4}$$

$$\mathcal{L}[y] \equiv P(D)y = (D^2 + aD + bI)y = 0 \tag{2.3.5}$$

$$= y'' + ay' + by = 0 \tag{2.3.6}$$

The operator L is a linear operator, which means superposition works,

$$\mathcal{L}[my + nw] = m\ \mathcal{L}[y] + n\ \mathcal{L}[w] \tag{2.3.7}$$

Operator polynomial The characteristic equation $P(\lambda)$ is a regular polynomial, which makes $P(D)$ an operator polynomial that can be manipulated just like any other polynomial.

2.4 Modeling of Free Oscillations of a Mass-Spring System

Hooke's law Law governing the restoring force of a spring, which states that the magnitude of the force is proportional to the displacement from its rest position.

$$F_1 = -ky \tag{2.4.1}$$

Stiffer springs have larger spring constants k.

Newton's second law The net force acting on a system is proportional to the acceleration experienced by it, with the proportionality constant being mass m.

$$F_{\text{net}} = ma \tag{2.4.2}$$

Undamped system A system with no waste of energy by way of damping. Such a system conserves total energy with time.

$$my'' + ky = 0 \tag{2.4.3}$$

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{2.4.4}$$

$$\omega_0 = \sqrt{k/m} \tag{2.4.5}$$

Frequency The number of oscillations per unit time (measured in Hz). An undamped system has a natural frequency f_0,

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{2.4.6}$$

Phase-Amplitude representation A restatement of the $y(t)$ harmonic oscillation showing phase shift δ

and amplitude C,

$$y(t) = C\ cos(\omega_0 t - \delta) \tag{2.4.7}$$

$$C = \sqrt{A^2 + B^2} \qquad\qquad \tan\delta = B/A \tag{2.4.8}$$

Damped system A system with an additional damping force proportional to the velocity, with $c > 0$.

$$F_2 = -cy' \tag{2.4.9}$$

$$my'' + cy' + ky = 0 \tag{2.4.10}$$

Overdamped system Real distinct roots of the characteristic equation, with

$$c^2 - 4mk > 0 \tag{2.4.11}$$

$$y(t) = c_1 \exp[-(\alpha - \beta)t] + c_2 \exp[-(\alpha + \beta)t] \tag{2.4.12}$$

The damping is so strong that the system never gets to oscillate, and settles to rest after a long time.

Critically damped system Characteristic equation has repeated roots. Damping is moderately strong and the system is able to pass through its mean position at most once.

$$c^2 - 4mk = 0 \tag{2.4.13}$$

$$y(t) = (c_1 + c_2 t)\exp(-\alpha t) \tag{2.4.14}$$

Underdamped system Characteristic equation has complex roots. Damping is weak and the system oscillates with amplitude decaying exponentially.

$$c^2 - 4mk < 0 \tag{2.4.15}$$

$$\omega^* = -i\beta = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \tag{2.4.16}$$

$$y(t) = e^{-\alpha t}[A\cos\omega^* t + B\sin\omega^* t] \tag{2.4.17}$$

$$= Ce^{-\alpha t}\ \cos(\omega^* t - \delta) \tag{2.4.18}$$

Free motion Systems with no external driving force, represented by a homogeneous second order linear ODE.

2.5 Euler-Cauchy Equations

Euler Cauchy equation An ODE with the standard form for some constants a, b

$$x^2y'' + axy' + by = 0 \tag{2.5.1}$$

$$y = x^m \tag{2.5.2}$$

Auxiliary equation A quadratic equation in m, and has three possible kinds of solutions, yielding three types of solutions to the ODE.

$$m^2 + (a-1)m + b = 0 \tag{2.5.3}$$

$$(a-1)^2 - 4b > 0 \qquad m_1, m_2 = \frac{(1-a) \pm \sqrt{(1-a)^2 - 4b}}{2} \tag{2.5.4}$$

$$y = c_1x^{m_1} + c_2x^{m_2} \tag{2.5.5}$$

$$(a-1)^2 - 4b = 0 \qquad m_1, m_2 = \frac{(1-a)}{2} \tag{2.5.6}$$

$$y = (c_1 + c_2 \ln x)\, x^{m_1} \tag{2.5.7}$$

$$(a-1)^2 - 4b < 0 \qquad m_1, m_2 = \frac{(1-a) \pm i\sqrt{4b - (1-a)^2}}{2} \tag{2.5.8}$$

$$= \alpha \pm i\beta \tag{2.5.9}$$

$$y = e^{\alpha x}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right] \tag{2.5.10}$$

2.6 Existence and Uniqueness of Solutions. Wronskian

Existence and Uniqueness theorem Consider the general second order linear homogeneous ODE with variable coefficients $p(x)$, $r(x)$, with an IVP given by

$$y'' + py' + qy = 0 \qquad p, q \quad \text{are continuous} \quad \forall x \in \mathcal{I} \tag{2.6.1}$$

$$y(x_0) = K_0, \quad y'(x_0) = K_1 \qquad \text{for some} \quad x_0 \in \mathcal{I} \tag{2.6.2}$$

$$\tag{2.6.3}$$

If $p(x)$, $q(x)$ are continuous in $\mathcal{I}$, then the IVP has a unique solution on $\mathcal{I}$.

Wronskian If y_1, y_2 are two solutions to the above ODE, then

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 \qquad y_1, y_2 \text{ defined on } \mathcal{I} \tag{2.6.4}$$

$$\text{If } W \neq 0 \text{ for some } x \in \mathcal{I} \qquad y_1, y_2 \text{ are linearly independent} \tag{2.6.5}$$

Existence of general solution The above ODE and associated IVP have a general solution on $\mathcal{I}$ of the form,

$$y = c_1 y_1(x) + c_2 y_2(x) \tag{2.6.6}$$

This covers all possible solutions and thus, no singular solution can be found.

2.7 Nonhomogeneous ODEs

Second order linear nonhomegenous ODE General form with $r(x) \not\equiv 0$,

$$y'' + p(x)y' + q(x)y = r(x) \tag{2.7.1}$$

General Solution For a nh-ODE of the form above, let y_p be any solution of the nh-ODE containing no no arbitrary constants.

$$y_{nh}(x) = y_h(x) + y_p(x) \qquad \text{is a solution of the nh-ODE} \tag{2.7.2}$$

$$y_h = c_1 y_1 + c_2 y_2 \qquad \text{is a solution of the h-ODE} \tag{2.7.3}$$

If w and z solve the nh-ODE, then $(w - z)$ solves the h-ODE.

Particular solution Assigning particular values to c_1, c_2 in the general solution above.

A general solution to the nh-ODE includes all possible solutions. There is no singular solution that is unobtainable from the general solution to the nh-ODE.

Method of Undetermined Coefficients An approach to solving nh-ODEs with constant coefficients that uses the functional form of $r(x)$ to make a guess for y_p as given in the following table,

RHS $\mathbf{r(x)}$	Guess $\mathbf{y_p(x)}$
$ke^{\lambda x}$	$Ce^{\lambda x}$
$kx^n \quad n \in \mathcal{N}$	$K_0 + K_1 x + \cdots + K_n x^n$
$k \cos(\omega x)$ $k \sin(\omega x)$	$K \cos(\omega x) + M \sin(\omega x)$
$ke^{\alpha x} \cos(\omega x)$ $ke^{\alpha x} \sin(\omega x)$	$e^{\alpha x} \left[K \cos(\omega x) + M \sin(\omega x)\right]$

$$y'' + ay' + by = r(x) \tag{2.7.4}$$

$$m^2 + am + b = 0 \qquad \text{characteristic equation} \tag{2.7.5}$$

1. *Basic rule:* If $r(x)$ is an individual entry of the table, then y_p is the corresponding entry from the table.
2. *Superposition rule:* Linear superposition of table entries for $r(x)$ implies the same superposition for y_p
3. *Modification rule:* If a term in y_p happens to correspond to a solution given by a single or double root of the characteristic equation, multiply it by x or x^2 repectively.

Stability of solution For solutions corresponding to complex roots of the characteristic equation, the real part has to be negative so that the transient part of the solution decays with time.

$$a^2 - 4b < 0 \qquad \text{complex roots} \tag{2.7.6}$$

$$-a/2 < 0 \qquad \text{stability} \tag{2.7.7}$$

2.8 Modeling: Forced Oscillations, Resonance

Driving force Into the earlier unforced spring-mass system, an external force is introduced by way of the RHS $r(x)$ of the ODE. Also called the input or the forcing function.

$$my'' + cy' + ky = r(x) \tag{2.8.1}$$

Response function The solution $y(x)$ to the nh-ODE where $r(x)$ is the input as defined above. Also called the output function.

Sinusoidal forcing A special practical case of $r(x)$ being sinusoidal.

$$my'' + cy' + ky = F_0 \cos(\omega t) \qquad y_p = a \cos(\omega t) + b \sin(\omega t) \tag{2.8.2}$$

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{[m(\omega_0^2 - \omega^2)]^2 + [\omega c]^2} \qquad b = F_0 \frac{\omega c}{[m(\omega_0^2 - \omega^2)]^2 + [\omega c]^2} \tag{2.8.3}$$

Undamped forced oscillations For the case where damping is negligible, with ω_0 being the natural frequency of the system (from free undamped motion) and ω the frequency of the driving force.

Assuming $\omega \neq \omega_0$ and $c \approxeq 0$,

$$y_p = \frac{F_0}{k[1 - (\omega/\omega_0)^2]} \cos(\omega t) \tag{2.8.4}$$

$$y = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) \tag{2.8.5}$$

Resonance The maximum amplitude of y_p after defining the resonance factor ρ,

$$a_0 = \frac{F_0}{k}\ \rho \qquad\qquad \rho = \frac{1}{1-(\omega/\omega_0)^2} \tag{2.8.6}$$

As $\omega \to \omega_0$, this amplitude goes to infinity. This phenomenon is called resonance.

Amplification c^*/F_0 or equivalently ρ/k is the ratio of the output to input amplitudes.

Resonant oscillations At resonance ($\omega = \omega_0$), the system is governed by the ODE,

$$y'' + \omega_0^2 y = \frac{F_0}{m}\ \cos(\omega_0 t) \tag{2.8.7}$$

$$y_p = \frac{F_0}{2m\omega_0}\ t\ \sin(\omega_0 t) \tag{2.8.8}$$

The linear t term in the output makes the amplitude scale to very large values and can destroy the physical system given enough time.

Beats When ω and ω_0 are close but not equal, the output is a fast sinusoid (summed frequencies) enveloped by a slow sinusoid (subtracted frequencies).

$$y(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}[\cos(\omega_0 t) - \cos(\omega t)] \tag{2.8.9}$$

$$= \frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{\omega_0+\omega}{2}\ t\right)\sin\left(\frac{\omega_0-\omega}{2}\ t\right) \tag{2.8.10}$$

Damped forced oscillations After a long time, the output of a system driven by a sinusoidal force is also a harmonic oscillation with the same frequency.

$$y_p = C^* \cos(\omega t - \eta) \tag{2.8.11}$$

$$\text{amplitude} \qquad C^* = \sqrt{a^2+b^2} = \frac{F_0}{\sqrt{[m(\omega_0^2-\omega^2)]^2 + [\omega c]^2}} \tag{2.8.12}$$

$$\text{phase delay} \qquad \tan\eta = \frac{b}{a} = \frac{\omega c}{m(\omega_0^2-\omega^2)} \tag{2.8.13}$$

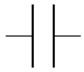
To find the maxima of the aplitude $C^*(\omega)$, differentiation and setting to zero yields,

$$\omega_{\max}^2 = \omega_0^2 - \frac{c^2}{2m} \qquad\qquad \text{if} \quad c^2 < 2mk \tag{2.8.14}$$

$$C^*(\omega_{\max}) = \frac{2mF_0}{c}\ \frac{1}{\sqrt{(2m\omega_o)^2 - c^2}} \qquad\qquad \lim_{c\to 0} C^*(\omega_{\max}) \to \infty \tag{2.8.15}$$

If $c^2 > 2mk$, then $C^*(\omega)$ is a monotonically decreasing function with no peak. $c \to 0$ recovers the undamped forced oscillation result of resonance as seen earlier.

2.9 Modeling: Electric Circuits

Element	**Resistor**	**Capacitor**	**Inductor**
Notation	R	C	L
Unit	Ω (ohm)	F (Farad)	H (Henry)
Voltage drop	RI	Q/C	$L\,\frac{\mathrm{d}I}{\mathrm{d}t}$
Symbol			

Basic elements of a circuit An RLC circuit has three basic components, shown in the table. The general ODE governing an RLC circuit is,

$$Q = \int I \ \mathrm{d}t \tag{2.9.1}$$

$$LI'' + RI' + \frac{1}{C}\,I = E'(t) \tag{2.9.2}$$

$$LQ'' + RQ' + \frac{1}{C}\,Q = E(t) \qquad \text{corollary} \tag{2.9.3}$$

Sinusoidally driven RLC FOr the specific case of a sinusoidal driving EMF,

$$LI'' + RI' + \frac{1}{C}\,I = E_0\omega\,\cos(\omega t) \tag{2.9.4}$$

$$I_p = a\,\cos(\omega t) + b\,\sin(\omega t) \tag{2.9.5}$$

Reactance A consolidation of capacitance and inductance derived from the complex notation.

$$I_p = a\,\cos(\omega t) + b\,\sin(\omega t)$$

$$S = \omega L - \frac{1}{\omega C} \tag{2.9.6}$$

$$I_p = \frac{E_0}{R^2 + S^2}\left[-\,S\,\cos(\omega t) + R\,\sin(\omega t)\right] \tag{2.9.7}$$

Impedance The complex number analog (Z) of ohmic resistance, used to arrive at the RLC analog of Ohm's law. It is also known as the apparent resistance

$$|Z| = \sqrt{R^2 + S^2} \qquad E_0 = |Z| I_0 \tag{2.9.8}$$

$$I_p = I_0 \sin(\omega t - \theta) \tag{2.9.9}$$

$$I_0 = \sqrt{a^2 + b^2} \qquad \tan(\theta) = -\frac{a}{b} \tag{2.9.10}$$

$$= \frac{E_0}{\sqrt{R^2 + S^2}} \qquad = \frac{S}{R} \tag{2.9.11}$$

Transient current Since a real circuit always has $R > 0$, the transient current always decays exponentially to zero in finite time, with the steady state current being the solution of the nh-ODE.

The equivalences between mechanical and electrical systems are apparent when looking at the extreme similarity between the ODEs used to model both systems.

2.10 Solution by Variation of Parameters

The method of undetermined coefficients is restricted to functions $r(x)$ which are similar to their derivatives $r(x)'$. A more general method is introduced here.

Variation of Parameters If y_1, y_2 form a basis of solutions of the h-ODE, and W is their Wronksian, then the standard form ODE,

$$y'' + py' + qy = r \tag{2.10.1}$$

$$y_h = c_1 y_1 + c_2 y_2 \tag{2.10.2}$$

$$y_p = -y_1 \int \frac{y_2 r}{W} \, dx + y_2 \int \frac{y_1 r}{W} \, dx \tag{2.10.3}$$

Derivation Starting with a generalization of y_p, requires p, q, r to be continuous.

Also, since y_1, y_2 form a basis of solutions to the h-ODE, their W is not zero anywhere on the interval I on which they are defined.

$$y_p = fy_1 + gy_2 \tag{2.10.4}$$

$$y_p' = [f'y_1 + g'y_2] + fy_1' + gy_2' \tag{2.10.5}$$

$$f'y_1 + g'y_2 = 0 \qquad \text{artificial constraint} \tag{2.10.6}$$

$$y_p'' = fy_1'' + gy_2'' + f'y_1' + g'y_2' \tag{2.10.7}$$

$$f'y_1' + g'y_2' = r \qquad \text{substituting into ODE} \tag{2.10.8}$$

$$f' = \frac{-y_2 r}{y_1 y_2' - y_1' y_2} = \frac{-y_2 r}{W} \qquad \text{using } W \neq 0 \tag{2.10.9}$$

$$g' = \frac{y_1 r}{W} \tag{2.10.10}$$

Using the continuity of $r(x)$, the derivatives f', g' can be integrated to obtain f, g and complete the derivation.

Chapter 3

Higher Order Linear ODEs

3.1 Homogeneous Linear ODEs

Linear ODE of nth order The standard form of an ODE on n-th order is,

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \tag{3.1.1}$$

Here, $\{p_i(x)\}$ are any continuous functions of x, and the solution $h(x)$, is defined and n-times differentiable on the interval $\mathcal{I}$ in which the set $\{p_i(x)\}$ are defined and continuous.

Superposition principle If $y_1,\ y_2$ are solutions to a linear h-ODE of order n, then

$$c_1 y_1 + c_2 y_2 = y_3 \tag{3.1.2}$$

is also a solution for some constants c_1, c_2.

Linearly Independent functions A set of functions $y_1(x), \ldots, y_n(x)$ are linearly independent (L.I.) on some interval $\mathcal{I}$ if,

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \Longrightarrow \qquad k_1 = \cdots = k_n = 0 \tag{3.1.3}$$

Conversely, if some solution to the above equation exists for which not all k_i are zero, then the functions are linearly dependent (L.D.)

Basis of solutions A set of L.I. solutions to the linear h-ODE of order n.

General solution Given a basis of solutions $\{y_i\}$ of the h-ODE,

$$y = c_1 y_1 + \cdots + c_n y_n \tag{3.1.4}$$

is a general solution of the ODE. Particular solutions can be obtained by assigning values to the coefficients $\{c_i\}$.

No singular solutions exist which cannot be obtained from the general solution.

Initial Value Problem Given the set of n initial conditions,

$$y(x_0) = K_0 \quad y'(x_0) = K_1 \quad \ldots \quad y^{(n)}(X_0) = K_n \tag{3.1.5}$$

There exists a unique solution for a linear h-ODE with coefficients $\{p_i(x)\}$ continuous on $\mathcal{I}$ given that $x_0 \in \mathcal{I}$.

Wronskian Using the determinant of order n,

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \tag{3.1.6}$$

If $W(x) \neq 0$ for some $x \in \mathcal{I}$, where the coefficients of the h-ODE $\{p_i(x)\}$ are continuous on $\mathcal{I}$, then the solutions $\{y_i(x)\}$ are L.I.

Conversely, if $W(x) = 0$ for some $x = x_0 \in \mathcal{I}$, then $W \equiv 0$ identically for all $x_0 \in \mathcal{I}$ and the functions are L.D.

Existence and Uniqueness If the coefficients $\{p_i(x)\}$ are continuous on some interval $\mathcal{I}$, then the h-ODE has a general solution on $\mathcal{I}$.

Using the fact that the Wronskian is merely the coefficient matrix of the system of linear equations in the unknowns $\{c_i\}$ in the general solution,

The Wronskisan of a solution composed of an L.I. basis of solutions is guaranteed to cover all possible solutions. Thus the general solution is unique.

3.2 Homogeneous Linear ODEs with Constant Coefficients

Standard form In standard form, the h-ODE with constant coefficients of order n, alongside its characteristic equation is,

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 = 0 \tag{3.2.1}$$

$$y_h = e^{\lambda x} \tag{3.2.2}$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots a_1\lambda + a_0 = 0 \tag{3.2.3}$$

Distinct real roots Each distinct root λ_k corresponds to a solution to the ODE $e^{k\lambda}$

$$W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \dots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \dots & \lambda_n e^{\lambda_n x} \\ \lambda_1^2 e^{\lambda_1 x} & \lambda_2^2 e^{\lambda_2 x} & \dots & \lambda_n^2 e^{\lambda_n x} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \dots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix} \tag{3.2.4}$$

Vandermode determinant A simplified version of the above determinant given by,

$$W = \exp\left(x\sum_{k=1}^{n}\lambda_k\right)\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix} \tag{3.2.5}$$

$$= \exp\left(x\sum_{k=1}^{n}\lambda_k\right) \ (-1)^{n(n-1)/2}\cdot\prod_{j=1}^{n}(\lambda_j-\lambda_k) \qquad n \geq j > k \tag{3.2.6}$$

The above expression is simply the difference between all possible combinations of roots. Since the first term is a product of exponentials, it is never zero.

So, the Wronksian provides L.I only if all the roots are distinct, and thus none of the terms $(\lambda_j - \lambda_k)$ are zero.

Repeated roots For repeated roots of multiplicity m, whether real or complex,

$$y_1 = e^{\lambda x} \qquad z_1,\, z_2 = e^{\alpha x}\sin\beta x,\, e^{\alpha x}\cos\beta x \tag{3.2.7}$$

$$y_2 = xe^{\lambda x} \qquad z_3,\, z_4 = xe^{\alpha x}\sin\beta x,\, xe^{\alpha x}\cos\beta x \tag{3.2.8}$$

$$y_3 = x^2e^{\lambda x} \qquad z_5,\, z_6 = x^2e^{\alpha x}\sin\beta x,\, x^2e^{\alpha x}\cos\beta x \tag{3.2.9}$$

$$\vdots \qquad \vdots$$

$$y_m = x^me^{\lambda x} \qquad z_{2m-1},\, z_{2m} = x^me^{\alpha x}\sin\beta x,\, x^me^{\alpha x}\cos\beta x \tag{3.2.10}$$

Other solutions are obtained by multiplying higher powers of x to existing solutions.

To derive the above multiple roots factor, consider the root λ_1 with multiplicity m,

$$\mathcal{L}[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y \tag{3.2.11}$$

$$\mathcal{L}[e^{\lambda x}] = (\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0)\ e^{\lambda x} \tag{3.2.12}$$

$$= (\lambda-\lambda_1)^m\cdot h(\lambda)\cdot e^{\lambda x} \tag{3.2.13}$$

Here $h(\lambda)$ is the leftover polynomial after factoring out all the λ_1.

$$\frac{\partial}{\partial\lambda}\mathcal{L}[e^{\lambda x}] = \mathcal{L}\left[\frac{\partial}{\partial\lambda}e^{\lambda x}\right] = \mathcal{L}[xe^{\lambda x}] \tag{3.2.14}$$

$$= m(\lambda-\lambda_1)^{m-1}\cdot h(\lambda)e^{\lambda x}$$

$$+ (\lambda-\lambda_1)^m\cdot\frac{\partial}{\partial\lambda}[h(\lambda)e^{\lambda x}] \tag{3.2.15}$$

3.2.16

Since $m > 2$, the RHS is zero at $\Lambda = \lambda_1$. This means that $\mathcal{L}[xe^{\lambda x}] = 0$, and thus $xe^{\lambda x}$ is a solution. Further differentiation w.r.t. λ can be used to arrive at further solutions corresponding to λ_1.

3.3 Nonhomogeneous Linear ODEs

Standard form Ensuring the coefficient of $y^{(n)}$ is 1,

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 = r(x) \tag{3.3.1}$$

$$y = y_h(x) + y_p(x) \tag{3.3.2}$$

The Initial value problem is the same as for h-ODEs.

Undetermined coefficients Similar to second order nh-ODEs, higher order ODEs can be solved using the pre-factor x^m to deal with roots of multiplicity m.

Variation of parameters Generalizing to order n,

$$y_p = \sum_{k=1}^{n} y_k \int \frac{W_k}{W}\, r \,\mathrm{d}x \tag{3.3.3}$$

W is the Wronskian of the h-ODE with a basis of solutions $\{y_1, y_2, \ldots, y_n\}$. W_k is the reduced Wronskian produced by replacing the k-th column of W with the column vector $[0\ 0\ \ldots\ 0\ 1]^T$

Chapter 4

Systems of ODEs, Phase Plane, Qualitative Methods

4.1 Systems of ODEs as Models in Engineering Applications

Conversion to system of ODEs Any ODE of order n can be converted to a system of n ODEs of first order. Linear algebra provides easy methods of solving this system of linear equations using eigenvectors and eigenvalues.

$$y^{(n)} = F\left(t,\ y,\ y',\ y'',\ \dots,\ y^{(n-1)}\right) \tag{4.1.1}$$

$$\text{Set} \qquad y_1 = y, \qquad y_2 = y', \qquad \dots \qquad y_n = y^{(n-1)} \tag{4.1.2}$$

$$\begin{aligned}
\text{System becomes} \qquad y_1' &= y_2 \\
y_2' &= y_3 \\
&\vdots \\
y_{n-1}' &= y_n \\
y_n' &= F(t,\ y_1,\ y_2,\ \dots,\ y_n)
\end{aligned} \tag{4.1.3}$$

Eigenvalues When a system of linear equations is expressed in vector form,

$$\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 \\
y_2' &= a_{21}y_1 + a_{22}y_2
\end{aligned} \tag{4.1.4}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4.1.5}$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{4.1.6}$$

For a matrix equation to have a non-trivial solution,

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \tag{4.1.7}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0 \tag{4.1.8}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{vmatrix} \tag{4.1.9}$$

This quadratic equation has solutions λ_1, λ_2 called eigenvalues. The corresponding eigenvectors $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ are found as solutions of the respective systems.

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}^{(1)} = 0 \tag{4.1.10}$$

$$(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{v}^{(2)} = 0 \tag{4.1.11}$$

4.2 Basic Theory of Systems of ODEs, Wronskian

General system of ODEs Using a set of functions $\{f_i\}$,

$$y_1' = f_1(t,\ y_1,\ \ldots,\ y_n) \tag{4.2.1}$$

$$y_2' = f_2(t,\ y_1,\ \ldots,\ y_n) \tag{4.2.2}$$

$$\vdots \tag{4.2.3}$$

$$y_n' = f_n(t,\ y_1,\ \ldots,\ y_n) \tag{4.2.4}$$

converting to vector notation, 4.2.5

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{4.2.6}$$

Introducing the set of solutions as a vector, and another vector for the I.C.

$$\mathbf{y} = \mathbf{h}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{bmatrix} \qquad \mathbf{y}(t_0) = \mathbf{K} = \begin{bmatrix} K(0) \\ K(1) \\ \vdots \\ K_n \end{bmatrix} \tag{4.2.7}$$

Existence and Uniqueness Given the preconditions,

$$\{f_1,\ f_2,\ \ldots,\ f_n\} \qquad \text{are continuous} \tag{4.2.8}$$

$$\left\{\frac{\partial f_i}{\partial y_k}\right\} \qquad \text{are continuous for all } j,\, k \tag{4.2.9}$$

Continuity is guaranteed in some point $(t_0,\ K_0,\ K_1,\ \ldots,\ K_n)$ in this interval of continuity in $(t,\ y_0,\ y_1,\ \ldots,\ y_n)$ space.

Then, a solution to the system of ODEs exists in some interval $t \in (t_0 - \alpha, t_0 + \alpha)$ and is guaranteed to be unique.

Linear System A subset of the above general system of ODEs obeying,

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \tag{4.2.10}$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{4.2.11}$$

The above system is homogeneous if $\mathbf{g} = 0$, with $a_{jk} = \partial f_j / \partial y_k$.

If the elements of $\mathbf{g}$ and $\mathbf{a}$ are continuous functions of t in some interval $t \in (\alpha,\, \beta)$ which contains the point $t = t_0$, then a solution exists and is guaranteed to be unique.

Superposition Principle If $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are solutions of the h-linear system of ODEs on some interval, then

$$\mathbf{y}^{(3)} = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} \tag{4.2.12}$$

is also a solution. (using the linearity of matrix multiplication and of differentiation)

Basis A set of L.I. solutions $\{\mathbf{y}^{(\mathbf{i})}\}$ form a basis in the interval $\mathcal{J}$ if the elements of $\mathbf{a}$ are continuous in $\mathcal{J}$.

A linear combination of this basis is a general solution that contains all possible solutions.

Wronskian The old Wronskian of a set of solutions $\{z_i\}$ becomes a matrix whose columns are members of the basis set $\mathbf{y}^{(\mathbf{i})}$.

The rows are successive derivatives (transformed into subscript variables).

$$\begin{bmatrix} z_1 & z_1 & \ldots & z_n \\ z_1' & z_2' & \ldots & z_n' \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(n-1)} & z_2^{(n-1)} & \ldots & z_n^{(n-1)} \end{bmatrix} \to \begin{bmatrix} y_1^{(1)} & y_1^{(2)} & \ldots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \ldots & y_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{(1)} & y_n^{(2)} & \ldots & y_n^{(n)} \end{bmatrix} \tag{4.2.13}$$

$$W = \det(\mathbf{Y}) = \det \begin{bmatrix} \mathbf{y}^{(\mathbf{1})} & \ldots & \mathbf{y}^{(\mathbf{n})} \end{bmatrix} \tag{4.2.14}$$

$$\tag{4.2.15}$$

The Wronskian is either identically zero everywhere (if L.D.) or nowhere (if L.I.) in the interval $\mathcal{J}$ under consideration.

Fundamental matrix The above matrix $\mathbf{Y}$ is called a fundamental matrix if the solutions $\{\mathbf{y^{(i)}}\}$ form a basis.

$$\mathbf{y} = c_1\mathbf{y^{(1)}} + \cdots + c_2\mathbf{y^{(n)}} \tag{4.2.16}$$

$$\mathbf{y} = \mathbf{Yc} \tag{4.2.17}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \mathbf{y^{(1)}} & \mathbf{y^{(n)}} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \tag{4.2.18}$$

4.3 Constant-Coefficient Systems, Phase Plane Method

Constant coefficient system If the matrix $\mathbf{a}$ has terms independent of t, then the initial guess

$$\mathbf{y}' = \mathbf{Ay} \qquad \mathbf{y} = \mathbf{x}e^{\lambda t} \tag{4.3.1}$$

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{4.3.2}$$

leads to an eigenvalue problem. Here, $\lambda, \mathbf{x}$ are the pairs of eigenvalues and eigenvectors respectively.

L.I. eigenvectors condition For the above matrix $\mathbf{A}$ to have a set of L.I. eigenvectors (as is common in real world applications), conditions are one of

- $\mathbf{A}$ is symmetric. $a_{jk} = a_{kj}$ or $\mathbf{A} = \mathbf{A^T}$.
- $\mathbf{A}$ is skew symmetric. $a_{jk} = -a_{kj}$ or $\mathbf{A} = -\mathbf{A^T}$.
- $\mathbf{A}$ has n distinct eigenvalues.

General solution If $\mathbf{A}$ has a set of L.I. eigenvectors as above, then the corresponding solutions form a basis.

A general solution is of the form,

$$\mathbf{y} = c_1\mathbf{x^{(1)}}e^{\lambda_1 t} + \quad \cdots \quad + c_n\mathbf{x^{(n)}}e^{\lambda_n t} \tag{4.3.3}$$

From this point onwards, the limited case of two member families of ODEs is taken up.

Phase plane The use of a parameter to plot the two solutions y_1 vs. y_2 instead of the usual pair of y vs. t curves.

Critical Points Points in the phase plane where the tangent direction is not defined (for example at $(0, 0)$ in the expression below).

$$\frac{\mathrm{d}y_2}{\mathrm{d}y_1} = \frac{y_2' \,\mathrm{d}t}{y_1' \,\mathrm{d}t} = \frac{y_2'}{y_1'} \tag{4.3.4}$$

$$= \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} \tag{4.3.5}$$

Improper Node A critical point P_0 at which all but two trajectories have the same limiting direction of the tangent L_1. The two exceptional trajectories also have a different (and equal) limit L_2.

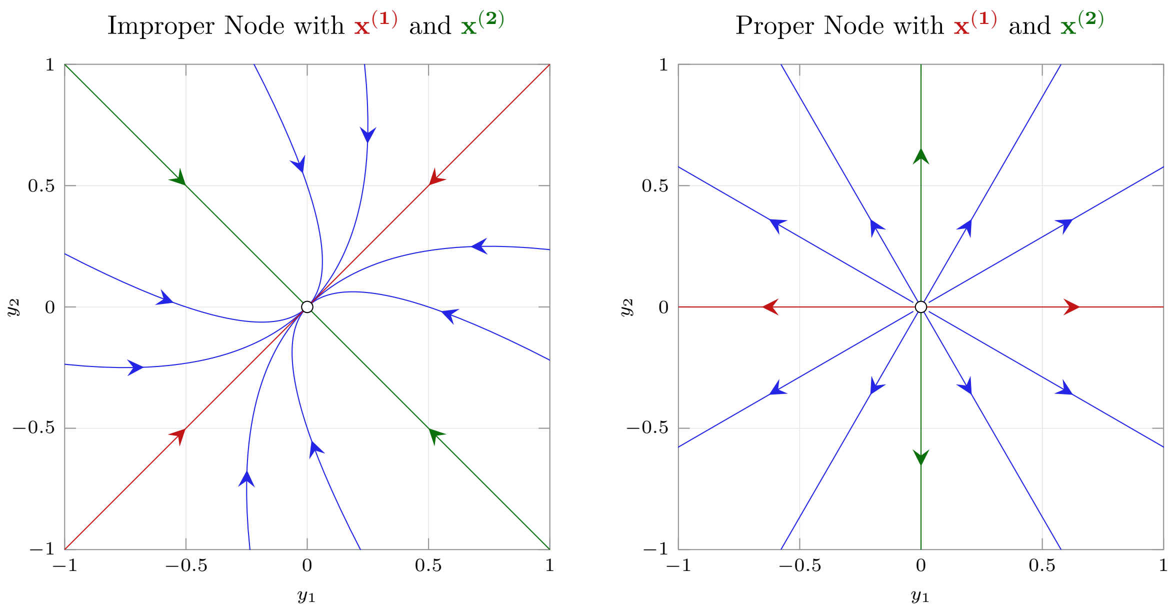
Proper Node A critical point at which every trajectory has a distinct limiting direction, and for any given direction $\mathbf{d}$, there is some trajectory whose limiting direction at P_0 is equal to $\mathbf{d}$.

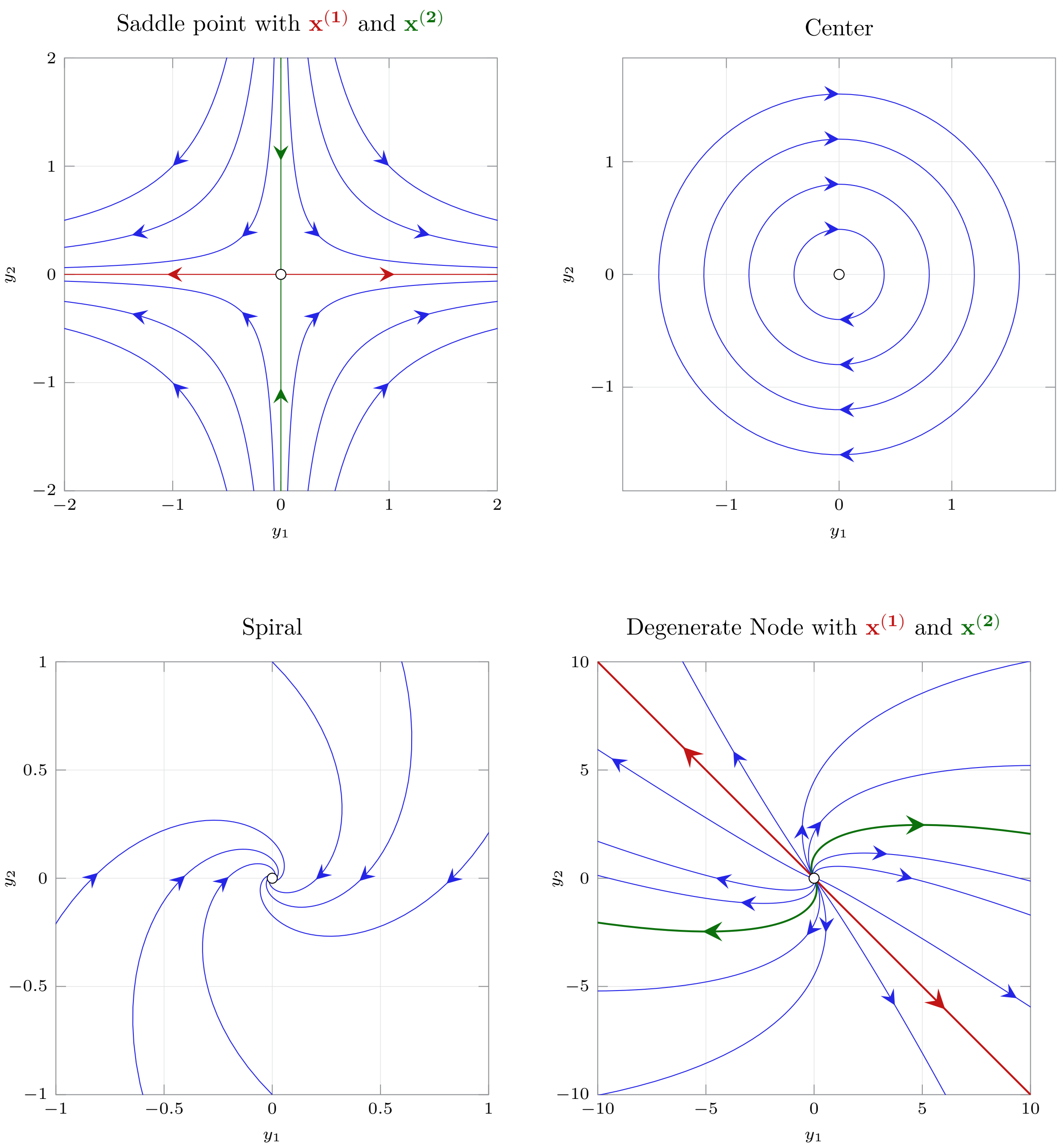
Saddle Point Two incoming and two outgoing trajectories intersect P_0, while all other trajectories bypass P_0.

Center A critical point, enclosed by many closed trajectories, none of which ever pass through P_0.

Spiral A critical point which all trajectories approach as $t \to \infty$ which otherwise resembles a center.

Degenerate Node In the (almost never physical) case when no basis of eigenvectors can be found. The usual condition is that $\mathbf{A}$ is neither symmetric nor skew-symmetric and happens to have degenerate eigenvalues.





4.4 Criteria for Critical Points, Stability

Characteristic equation For the two member system of linear ODEs with constant coefficients,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4.4.1}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - (a_{11} + a_{22})\lambda + \det(\mathbf{A}) = 0 \tag{4.4.2}$$

$$\lambda^2 - p\lambda + q = 0 \tag{4.4.3}$$

Critical point categories Rearranging the characteristic equation into its factors $\lambda_1,\ \lambda_2$,

$$p = \lambda_1 + \lambda_2 \qquad\qquad q = \lambda_1\lambda_2 \tag{4.4.4}$$

$$\text{Discriminant } \Delta = p^2 - 4q \qquad\qquad \Delta = (\lambda_1 - \lambda_2)^2 \tag{4.4.5}$$

Critical Point	**Eigenvalues** λ_1, λ_2
Node	Real, same sign
Saddle point	Real, opposite signs
Center	Purely imaginary
Spiral point	Complex with nonzero real part

Stability From physics, stability is a measure of the effect in the future of a small change in the system at present time.

Stable critical point If for every disk D_ϵ centered on P_0, there is a disk D_δ with $\delta,\ \epsilon > 0$, such that every trajectory which has $P(t = t_1) = P_1 \in D_\delta$ has all its points corresponding to $t \geq t_1$ in D_ϵ.

Attractive critical point Every trajectory in D_δ for a stable critical point, approaches P_0 asymptotically as $t \to \infty$.

Stability	**Condition on p, q**
Stable and attractive	$p < 0$ and $q > 0$
Stable	$p \leq 0$ and $q > 0$
Unstable	$p > 0$ or $q < 0$

4.5 Qualitative Methods for Nonlinear Systems

Assumptions The system of ODEs is autonomous, and the functions $f_1,\ f_2$ are independent of t.

Also, the system of ODEs has finitely many critical points. This means that each critical point is isolated.

For analysis, each critical point is treated as the origin when being analyzed, using the coordinate transformation,

$$P_0 = (a,\ b) \qquad\qquad (a,\ b) \to (0,\ 0) \tag{4.5.1}$$

$$\tilde{y_i} = y_1 - a \qquad\qquad \tilde{y_2} = y_2 - b \tag{4.5.2}$$

Linearization The system is linearlised around its critical point $(0, 0)$ using,

$$\mathbf{y} = \mathbf{f}(\mathbf{y}) \equiv \mathbf{A}\mathbf{y} + \mathbf{h}(\mathbf{y}) \tag{4.5.3}$$

$$y_1' = a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2) \tag{4.5.4}$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2) \tag{4.5.5}$$

Notice $\mathbf{A}$ here is independent of y since $P_0 = (0, 0)$ is a critical point giving.

If f_1, f_2 are continuous and have continuous partial derivatives in a region around P_0, and if $\det(\mathbf{A}) \neq 0$, then the kind and stability of the critical points of the original system are the same as those of the linearized system.

Exceptions occur if the linearized system has equal roots or purely imaginary roots. The general method is as follows,

1. Set up the mathematical model.
2. Identify the critical points as conditions for $\mathbf{y} = 0$
3. Drop all nonlinear terms in order to linearize the system
4. Treat each critical point in turn, transforming coordinates as needed.

Transformation to first order ODE A second order autonomous ODE can be transformed using,

$$F(y, y', y'') = 0 \qquad\qquad y = y_1, \qquad y' = y_2 \tag{4.5.6}$$

$$y'' = \frac{\mathrm{d}y_2}{\mathrm{d}y_1}\, y_2 \tag{4.5.7}$$

Limit Cycle A closed trajectory in phase space into which other trajectories spiral asymptotically. Similar to an attractive node, but instead of a single point in phase space, this is a closed stable trajectory.

In the real world, this requires systems with variable damping (which can also be negative), which push trajectories starting outside and inside the limit cycle inward and outward towards it respectively.

4.6 Nonhomogeneous Linear Systems of ODEs

General solution Assuming $\mathbf{g} \not\equiv \mathbf{0}$ and the entries of $\mathbf{A}$ are continuous on some interval $\mathcal{J}$ of the $t-$axis, a general solution of the nh-system is given by,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{4.6.1}$$

$$\mathbf{y} = \mathbf{y}^{(\mathbf{h})} + \mathbf{y}^{(\mathbf{p})} \tag{4.6.2}$$

Undetermined coefficients The method is analogous to the single ODE procedure, for the same set of candidate functions. The only change is in the modification rule.

Variation of parameters Start with a basis of solutions $\mathbf{Z}$ of the h-ODE system.

$$\mathbf{Z}' = A\mathbf{Z} \tag{4.6.3}$$

$$\mathbf{y}^{(\mathbf{p})} = \mathbf{Z}(t)\mathbf{u}(t) \tag{4.6.4}$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$$

$$\mathbf{Z}\mathbf{u}' = \mathbf{g} \tag{4.6.5}$$

$$\mathbf{u}' = \mathbf{Z}^{-\mathbf{1}}\mathbf{g} \tag{4.6.6}$$

$\mathbf{Z}^{-\mathbf{1}}$ is guaranteed to exist since $\mathbf{Z}$ is a basis and its Wronskian is nonzero.

Chapter 5

Series Solutions of ODEs. Special Functions

5.1 Power Series Method

Power Series A function approximated using a polynomial with center x_0 given by,

$$f(x) = \sum_{m=0}^{\infty} a_m \, (x - x_0)^m \tag{5.1.1}$$

Here, the set $\{a_m\}$ are constant coefficients of the series. Also, the set m only includes posittive integers.

Partial sum The n-th partial sum and the n-th remainder of the above power series is,

$$s_n(x) = \sum_{m=0}^{n} a_m \, (x - x_0)^m \tag{5.1.2}$$

$$R_n(x) = f(x) - s_n(x) = \sum_{m=n+1}^{\infty} a_m \, (x - x_0)^m \tag{5.1.3}$$

Convergent series If for some x_1, the sequence of partial sums $\{s_n(x_1)\}$ approaches some finite limit $s(x_1)$, then the series is considered convergent at $x = x_1$

$$\lim_{n\to\infty} s_n(x_1) = s(x_1) \tag{5.1.4}$$

$$= \sum_{m=0}^{\infty} a_m \, (x - x_0)^m \tag{5.1.5}$$

Alternatively, for any positive ϵ there is an N, such that $s_n(x_1)$ for all $n > N$ lies within ϵ distance of the asymptotic limit $s(x_1)$.

$$|R_n(x_1)| = |s(x_1) - s_n(x_1)| < \epsilon \qquad \forall \quad n > N \tag{5.1.6}$$

$$s_n(x_1) \in (s(x_1) - \epsilon, s(x_1) + \epsilon) \qquad \forall \quad n > N \tag{5.1.7}$$

Convergence interval The set of values of x for which the power series converges. Sometimes this set may contain only the element $x = x_0$.

Radius of convergence The half-width of the interval of convergence. Denoted R (from complex notation where the zone of convergence becomes a disk).

$$s(x) \quad \text{converges} \quad \forall \quad |x - x_0| < R \tag{5.1.8}$$

$$\text{and diverges} \quad \forall \quad |x - x_0| > R \tag{5.1.9}$$

$$\frac{1}{R} = \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| \tag{5.1.10}$$

$$\frac{1}{R} = \lim_{m\to\infty} (|a_m|)^{1/m} \tag{5.1.11}$$

The above formulas require the limits to exist and be nonzero. If the limits are ∞, then the series only converges at the center x_0.

Analytic function A function that has a Taylor series representation at $x = x_0$ is analytic at x_0.
If p, q, r are analytic at x_0, then

$$y'' + p(x)y' + q(x)y = r(x)$$

has every solution analytic at x_0 with some radius of convergence $R > 0$.

Operations on Power series The following operations performed on convergent power series result in another power series converging to the result of the same operation on the function itself.

$$y(x) = \sum_{m=0}^{\infty} a_m \, (x - x_0)^m \tag{5.1.12}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{m=1}^{\infty} m a_m \, (x - x_0)^{m-1} \qquad \forall \quad |x - x_0| < R \tag{5.1.13}$$

$$f(x) + g(x) = \sum_{m=0}^{\infty} (a_m + b_m) \, (x - x_0)^m \qquad \forall \quad |x - x_0| < R_f \cap R_g \tag{5.1.14}$$

$$f(x)g(x) = \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m} a_k b_{m-k}\right] \, (x - x_0)^m \tag{5.1.15}$$

$$f(x) \equiv 0 \qquad \implies \{a_m\} \equiv 0 \tag{5.1.16}$$

The last equation follows from polynomials being identically 0 if and only if every single coefficient is identically 0.

5.2 Legendre's Equation, Legendre Polynomials

Legendre ODE In Physics, this ODE is very frequently encountered, which yields a recurrence relation when solved using the power series method,

$$0 = (1 - x^2)y'' - 2xy' + n(n+1)y \tag{5.2.1}$$

$$y = \sum_{m=0}^{\infty} a_m \, x^m \tag{5.2.2}$$

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} \, a_s \tag{5.2.3}$$

$$\tag{5.2.4}$$

The solution in terms of the two free constants a_0 and a_1,

$$y(x) = a_0 y_1 + a_1 y_2 \tag{5.2.5}$$

$$y_1 = 1 - \frac{n(n+1)}{2!} \, x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} \, x^4 - \ldots \tag{5.2.6}$$

$$y_2 = x - \frac{(n-1)(n+2)}{3!} \, x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} \, x^5 - \ldots \tag{5.2.7}$$

$$\tag{5.2.8}$$

Convergence The Legendre polynomials converge only for $|x| < 1$. Since the standard form of the ODE,

$$y'' - \frac{2x}{(1-x^2)} \, y' + \frac{n(n+1)}{(1-x^2)} \, y = 0 \tag{5.2.9}$$

is not analytic at $x = \pm 1$, the convergence interval of the solution is at best $(-1, 1)$.

Legendre's polynomial For n being a non-negative integer, either y_1 or y_2 terminate after finitely many terms depending on n being even or odd.

$$P_n(x) = \sum_{m=0}^{M} (-1^m) \, \frac{(2n-2m)!}{2^n \; m! \; (n-m)! \; (n-2m)!} \, x^{n-2m} \tag{5.2.10}$$

Here, $M = n/2$ or $(n-1)/2$ depending on n even or odd.

The specific choice of coefficients ensures that $Pn(1) = 1$ for all n.

The recurrence relation is the other condition that enables the calculation of the above general formula for the series.

$$P_0(x) = 1 \qquad P_1(x) = x \tag{5.2.11}$$

$$P_2(x) = \frac{1}{2}\,(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}\,(5x^3 - 3x) \tag{5.2.12}$$

$$P_4(x) = \frac{1}{8}\,(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}\,(63x^5 - 70x^3 + 15x) \tag{5.2.13}$$

These polynomials are orthogonal in the interval $[-1, 1]$.

5.3 Extended Power Series Method: Frobenius Method

Frobenius ODE Let $b(x)$, $c(x)$ be any functions analytic at $x = 0$. Then,

$$y'' + \frac{b(x)}{x}\,y' + \frac{c(x)}{x^2}\,y = 0 \tag{5.3.1}$$

has at least one solution with r real or complex and $a_0 \neq 0$,

$$y(x) = x^r \sum_{m=0}^{\infty} a_m\, x^m \tag{5.3.2}$$

The ODE also has a second L.I. solution that looks similar to the first.

Regular point A point x_0 at which the ODE,

$$y'' + p(x)y' + q(x) = 0 \tag{5.3.3}$$

$$p(x),\ \ q(x) \text{ are analytic at } x_0 \tag{5.3.4}$$

Singular point A point x_0 which is not a regular point as defined above.

Indicial equation A quadratic equation in r which indicates the form of the second L.I. solution to the Frobenius ODE.

$$x^2y'' + x\,b(x)y' + c(x)y = 0 \tag{5.3.5}$$

$$\text{Substitute} \qquad b(x) = \sum_{m=0}^{\infty} b_m x^m, \qquad c(x) = \sum_{m=0}^{\infty} c_m x^m \tag{5.3.6}$$

$$\text{Gathering terms with } x^r \qquad r(r-1) + b_0 r + c_0 = 0 \tag{5.3.7}$$

The Euler-Cauchy ODE is the simplest case of the Frobenius ODE and illustrates the three cases arising form the indicial equation described below.

Distinct roots not differing by an integer If r_1, r_2 are the two roots, then a basis of solutions is,

$$y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m \tag{5.3.8}$$

$$y_2(x) = x^{r_2} \sum_{m=0}^{\infty} A_m x^m \tag{5.3.9}$$

where the coefficients $\{a_m\}$, $\{A_m\}$ are found by comparing powers of x^r and higher terms in the ODE.

Repeated root Here, $r_1 = r_2 = (1 - b_0)/2$ and

$$y_1(x) = x^r \sum_{m=0}^{\infty} a_m x^m \tag{5.3.10}$$

$$y_2(x) = y_1 \ln(x) + x^r \sum_{m=0}^{\infty} A_m x^m \qquad (x > 0) \tag{5.3.11}$$

Roots differing by an integer This case also includes complex roots, and the stipulation that $r_1 > r_2$,

$$y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m \tag{5.3.12}$$

$$y_2(x) = k y_1(x) \ln(x) + x^{r_2} \sum_{m=0}^{\infty} A_m x^m \tag{5.3.13}$$

It is possible that $k = 0$ happens because of simplicity in the ODE.

5.4 Bessel's Equation, Bessel Functions Jv(x)

Bessel's ODE The standard form of the Bessel ODE with $\nu \in \mathcal{R}^+ \cup 0$ is,

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \tag{5.4.1}$$

This is often the result of the physical system having cylindrical symmetry.

Power series solution Applying the general power series method to find indicial equation,

$$y = \sum_{m=0}^{\infty} a_m \ x^{m+r} \tag{5.4.2}$$

$$(r+\nu)(r-\nu) = 0 \tag{5.4.3}$$

$$r_1 = \nu(\geq 0) \qquad\qquad r_2 = -\nu \tag{5.4.4}$$

$$a_{2m} = \frac{(-1)^m \ a_0}{2^{2m} \ m! \ (\nu+1)(\nu+2)\dots(\nu+m)} \qquad \forall \ m = \{1, 2, 3, \dots\} \tag{5.4.5}$$

Integer parameter For the special case of $\nu \in \mathcal{I}^+ \cup 0$, assign $\nu \to n$,

$$a_0 = \frac{1}{2^n \ n!} \tag{5.4.6}$$

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} \ m! \ (n+m)!} \qquad \forall \ m = \{1, 2, 3, \dots\} \tag{5.4.7}$$

Function of the first kind By inserting the above recursion relation into the power series solution and noting that odd powers have zero coefficient,

$$J_n(x) = x^n \ \sum_{m=0}^{\infty} \frac{(-1)^m \ x^{2m}}{2^{2m+n} \ m! \ (n+m)!} \tag{5.4.8}$$

This series converges for all x.

Asymptotic behavior All of the $J_\nu(x)$ resemble cosine functions, with gradually decaying amplitudes and zeros not being evenly spaced. This is also evident from the similarity in their series expansions,

$$J_0 = 1 - \frac{x^2}{2^2 \ (1!)^2} + \frac{x^4}{2^4 \ (2!)^2} - \frac{x^6}{2^6 \ (3!)^2} + \dots \tag{5.4.9}$$

$$J_1 = \frac{x}{2} - \frac{x^3}{2^3 \ 1! \ 2!} + \frac{x^5}{2^5 \ 2! \ 3!} - \frac{x^7}{2^7 \ 3! \ 4!} + \dots \tag{5.4.10}$$

$$\lim_{x \to \infty} J_n(x) \approxeq \frac{2}{\pi x} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{5.4.11}$$

Gamma function The generalization of the factorial to non-integer indices.

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} \ t^\nu \ \ \mathrm{d}t \qquad\qquad (\nu > -1) \tag{5.4.12}$$

$$\Gamma(\nu+1) = \nu \ \Gamma(\nu) \tag{5.4.13}$$

$$\Gamma(n+1) = n! \qquad\qquad (n \in 0 \cup \mathcal{I}^+) \tag{5.4.14}$$

Real positive parameter Replacing the factorial with the Gamma function,

$$a_0 = \frac{1}{2^\nu \; \Gamma(\nu + 1)} \tag{5.4.15}$$

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m \; x^{2m}}{2^{2m+\nu} \; m! \; \Gamma(\nu + m + 1)} \tag{5.4.16}$$

ν is called the order of the Bessel function.

Properties of Bessel functions Starting from the power series definition, some properties of $J_\nu(x)$ are,

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^\nu \; J_\nu] = x^\nu \; J_{\nu-1} \tag{5.4.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-\nu} \; J_\nu] = -x^{-\nu} \; J_{\nu+1} \tag{5.4.18}$$

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} \; J_\nu \tag{5.4.19}$$

$$J_{\nu-1} - J_{\nu+1} = 2 \; \frac{\mathrm{d}}{\mathrm{d}x} J_\nu \tag{5.4.20}$$

The second set of relations comes from adding and subtratcing the expansions of the first set.

Half-integer parameter Some basic results,

$$\Gamma(1/2) = \sqrt{\pi} \tag{5.4.21}$$

$$J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \; \sin(x) \qquad\qquad J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \; \cos(x) \tag{5.4.22}$$

5.4.23

General Solution For the special case of $\nu \notin \mathcal{I}$, a straightforward second solution that is L.I. is found by using $-\nu$,

$$y(x) = c_1 \; J_\nu + c_2 \; J_{-\nu} \tag{5.4.24}$$

When ν is an integer, this second function becomes L.D since,

$$J_{-n} = (-1)^n \; J_n \tag{5.4.25}$$

Result follows from the fact that $\Gamma(n + 1)$ is not defined for $n < -1$. In this case, a more involved procedure is necessary to find the second L.I. solution.

5.5 Bessel Functions Yv(x), General Solution

Zero parameter Special case where the ODE reduces to

$$0 = xy'' + y' + xy \qquad r_1 = r_2 = 0 \tag{5.5.1}$$

$$y_2 = J_0 \ln(x) + \sum_{m=1}^{\infty} A_m\, x^m \qquad h_m = \sum_{r=1}^{m} \frac{1}{r} \tag{5.5.2}$$

$$A_{2m} = \frac{(-1)^{m-1}\, h_m}{2^{2m}\, (m!)^2} \tag{5.5.3}$$

$$y_2(x) = J_0(x) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + \frac{11x^6}{13824} + \ldots \tag{5.5.4}$$

Neumann function of order 0 Bessel function of order zero, using the constants

$$a = \frac{2}{\pi} \tag{5.5.5}$$

$$b = \lim_{s\to\infty} \left[1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln(s)\right] = \gamma - \ln(2) \tag{5.5.6}$$

$$Y_0(x) = a(y_2 + bJ_0) \tag{5.5.7}$$

$$= \frac{2}{\pi} \left[J_0\{\ln(x/2) + \gamma\} + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\, h_m}{2^{2m}\, (m!)^2}\, x^{2m}\right] \tag{5.5.8}$$

This function behaves like $\ln x$ for small x

Integer parameter When the parameter is an integer n, another special case is given by

$$Y_n = \lim_{\nu\to n} Y_\nu(x) \tag{5.5.9}$$

$$Y_\nu = \frac{J_\nu \cos(\nu\pi) - J_{-\nu}}{\sin(\nu\pi)} \tag{5.5.10}$$

For non-integer ν, the two functions J_ν and $J_{-\nu}$ are already L.I. and thus, Y_ν is also L.I. of J_ν. By taking the limit above, the expression for Y_n becomes,

$$Y_n = \frac{2}{\pi} J_n\{\ln(x/2) + \gamma\} + \frac{x^n}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}\, (h_m + h_{m+n})}{2^{2m+n}\, m!\, (m+n)!}\, x^{2m} \tag{5.5.11}$$

$$- \frac{1}{nx^n} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n}\, m!}\, x^{2m} \qquad (x > 0) \tag{5.5.12}$$

Some conventions used in the above formula, and a simple result that follows is,

$$h_0 = 0 \qquad \text{(by convention)} \tag{5.5.13}$$

$$Y_{-n} = (-1)^n Y_n \tag{5.5.14}$$

General solution Using the above special cases, a general solution can now be defined using,

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x) \tag{5.5.15}$$

Hankel functions Solutions of Bessel's ODE that are complex for real x, given by linear combinations of J_ν and Y_ν,

$$H_\nu^{(1)} = J_\nu + iY_\nu \tag{5.5.16}$$

$$H_\nu^{(2)} = J_\nu - iY_\nu \tag{5.5.17}$$

These functions are also called Bessel functions of the third kind.

Chapter 6

Laplace Transforms

6.1 Laplace Transform, Linearity, First Shifting Theorem (s-Shifting)

Operational calculus The process of transforming a calculus problem to an algebraic problem. Laplace transforms are one example.

Integral transform The operation of transforming a function in one space (t) to another space (s)by performing an integration.

$$F(s) = \int_0^\infty k(s,\, t)\ f(t)\ \ \mathrm{d}t \tag{6.1.1}$$

Here, the function $k(s,\, t)$ is called the kernel, since it is the bridge function of both variables s and t.

Laplace transform The integral transform with kernel,

$$k(s,\, t) = \exp(-st) \tag{6.1.2}$$

$$F(s) = \mathscr{L}\{f\} \equiv \int_0^\infty e^{-st} f(t)\ \ \mathrm{d}t \tag{6.1.3}$$

Inverse Laplace transform The inverse of the above operation defined as,

$$f(t) \equiv \mathscr{L}^{-1}\{F\} \tag{6.1.4}$$

$$\mathscr{L}^{-1}\{\mathscr{L}\{f\}\} = f \tag{6.1.5}$$

$$\mathscr{L}\{\mathscr{L}^{-1}\{F\}\} = F \tag{6.1.6}$$

Original functions (t domain) are written in small letters and their Laplace transforms (s domain) are written in capital letters.

Linearity of Laplace transform Since integration is a linear opeartion,

$$\mathscr{L}\{af(t) + bg(t)\} = a\,\mathscr{L}\{f(t)\} + b\,\mathscr{L}\{g(t)\} \tag{6.1.7}$$

$$= aF(s) + G(s) \tag{6.1.8}$$

assuming $F(s)$ and $G(s)$ already exist.

$\mathbf{f(t)}$	$\mathbf{F(s)}$	$\mathbf{f(t)}$	$\mathbf{F(s)}$
1	$\frac{1}{s}$	t	$\frac{1}{s^2}$
t^2	$\frac{2!}{s^3}$	t^n	$\frac{n!}{s^{n+1}}$
$t^a\ (a > 0)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$\sinh(at)$	$\frac{a}{s^2-a^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$

First shifting theorem Performing the equivalent of a shift along the s axis in transformed space has the following effect,

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a) \tag{6.1.9}$$

$$e^{at}f(t) = \mathscr{L}^{-1}\{F(s-a)\} \tag{6.1.10}$$

Existence theorem If $f(t)$ is piecewise continuous (only has finite jump discontinuities if any) and satisfies the growth restriction,

$$|f(t)| \leq Me^{kt} \tag{6.1.11}$$

for some $M, k > 0$ for all $tgeq0$. The linear index of e is the fastest rate at which the function is allowed to grow.

A function satisfying these (sufficient) conditions has a Laplace transform.

Uniqueness theorem A Laplace transform, if it exists, is uniquely determined.

Two functions with the same Laplace transform cannot differ over any finite interval. At most they can differ at specific points on the t axis.

If two continuous functions have the same transform, they are identical.

6.2 Transforms of Derivatives and Integrals, ODEs

Transforming derivaties In order to solve ODEs, the effect on $\mathscr{L}\{f(t)\}$ of performing differentiation needs to be used,

$$\mathscr{L}\{f'\} = s\,\mathscr{L}\{f\} - f(0) \tag{6.2.1}$$

$$\mathscr{L}\{f''\} = s^2\,\mathscr{L}\{f\} - sf(0) - f'(0) \tag{6.2.2}$$

These relations require the terms on the RHS to be continuous for all $t \geq 0$ and satisfy the exponential growth restriction.

The terms on the LHS need to be piecewise continuous on every finite interval in $t \geq 0$ The generalised rule for transforming n-th order derivatives is,

$$\mathscr{L}\{f^{(n)}\} = s\,\mathscr{L}\{f^{(n-1)}\} - f^{(n-1)}(0) \tag{6.2.3}$$

$$= s^n\,\mathscr{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \tag{6.2.4}$$

One of the important uses of this relation is to use $\mathscr{L}\{f''\}$ to work backwards and find $\mathscr{L}\{f\}$

Transforming Integrals For a piecewise continuous function $f(t)$ in $t \geq 0$ which obeys the exponential growth restriction for all $t \geq 0$,

$$|f(t)| \leq M\exp(kt) \tag{6.2.5}$$

$$M > 0 \qquad k > 0 \tag{6.2.6}$$

$$\mathscr{L}\left\{\int_0^t f(\tau)\ \mathrm{d}\tau\right\} = \frac{F(s)}{s} \tag{6.2.7}$$

$$\int_0^t f(\tau)\ \mathrm{d}\tau = \mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\} \tag{6.2.8}$$

Subsidiary equation The equation obtained by Laplace transforming the ODE.

$$\mathscr{L}\{y(t)\} = Y(s) \qquad \text{and} \qquad \mathscr{L}\{r(t)\} = R(s) \tag{6.2.9}$$

$$[s^2Y - sy(0) - y'(0)] + a\,[sY - y(0)] + bY = R(s) \tag{6.2.10}$$

$$(s+a)y(0) + y'(0) + R(s) = (s^2 + as + b)Y \tag{6.2.11}$$

Transfer function A function that models the system's output for all possible inputs.

$$Q(s) = \frac{1}{s^2 + as + b} \tag{6.2.12}$$

$$= \frac{Y(s)}{R(s)} \qquad \text{if} \qquad y(0) = y'(0) = 0 \tag{6.2.13}$$

The transfer funtion does not depend on the input $r(t)$ or on the I.C. and only depends on the system itself.

Application to IVPs Consider the standard form of the second order IVP with constant coefficients, whose subsidiary equation is first found as,

$$y'' + ay' + by = r(t) \tag{6.2.14}$$

$$y(0) = K_0 \qquad y'(0) = K_1 \tag{6.2.15}$$

$$Y = [(s+a)y(0) + y'(0)]Q + RQ \tag{6.2.16}$$

The final step is simply the decomposition of $Y(s)$ into partial fractions whose inverse Laplace transforms are standard results.

The advantages of solving ODEs using this method are,

- Avoid having to solve the h-ODE first.
- Avoid having to find a general solution and then apply the I.C to get a particular solution.
- Handle complicated $r(t)$ very easily.
- Initial conditions for $t_0 \neq 0$ are dealt with by a change of variable $u = t - t_0$ so that the new I.C. are at $u = 0$

6.3 Unit Step Function, Second Shifting Theorem

Heaviside function A constant function that is shifted upwards by distance 1, at $t = a$, defined by

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases} \tag{6.3.1}$$

The function is not defined at $t = a$ by convention. Its Laplace transform from the integral definition is,

$$\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s} \tag{6.3.2}$$

Multiplying a function $f(t)$ by the Heaviside function $u(t-a)$ (for some positive a) simply sets the function to 0 $\forall\ x < a$.

Second shifting theorem This theorem deals with a $t-$shifted function with a Heaviside function nullifying it upto the shift distance a.

$$f(t) \to \tilde{f} \equiv f(t-a)u(t-a) \tag{6.3.3}$$

$$\tilde{f} = \begin{cases} 0 & x < a \\ f(t-a) & x > a \end{cases} \tag{6.3.4}$$

$$\mathscr{L}\{\tilde{f}\} = e^{-as}\,\mathscr{L}\{f(t)\} \tag{6.3.5}$$

$$f(t-a)u(t-a) = \mathscr{L}^{-1}\{e^{-as}F(s)\} \tag{6.3.6}$$

A corollary to the above relation is to replace $(t - a) \to t$

$$\mathscr{L}\{f(t)u(t-a)\} = e^{-as}\,\mathscr{L}\{f(t+a)\} \tag{6.3.7}$$

Inputs in engineering problems often have finite duration before they are switched off. This is modeled very easily by the Heaviside function.

Linear sums of Heaviside functions can be used to model functions being nullified for a small time-window before being switched on again, or vice versa.

$$r(t) = \begin{cases} 0 & x < a \\ k & x \in (a, b) \\ 0 & x > b \end{cases} \tag{6.3.8}$$

$$r(t) = k[u(t-a) - u(t-b)] \tag{6.3.9}$$

6.4 Short Impulses, Dirac's Delta Function, Partial Fractions

Finite analog A function that has unit area under the curve and is a finite duration rectangular wave.

$$f_k(t-a) = \begin{cases} 0 & t < a \\ \dfrac{1}{k} & t \in [a, a+k] \\ 0 & t > k \end{cases} \tag{6.4.1}$$

$$\int_0^\infty f_k \,\mathrm{d}t = \int_a^{a+k} \frac{1}{k}\,\mathrm{d}t = 1 \tag{6.4.2}$$

Impulse From, physics, the integral of a force taken over the duration it acts on the system. Consider a time dependent force $F(t)$ acting on the system for a duration δt starting at time t_0.

$$I = \int_{t_0}^{t_0+\delta t} f(t)\,\mathrm{d}t \tag{6.4.3}$$

Dirac's delta function The infiniteismal width limit of the finite rectangular wave defined above. The unit area under the curve is still preserved.

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a) \tag{6.4.4}$$

$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases} \tag{6.4.5}$$

Sifting property The Dirac delta function picks out the value of its coefficient under an integration.

$$\int_0^\infty g(t)\ \delta(t-a) = g(a) \tag{6.4.6}$$

Laplace transform of Dirac's delta Using the limit $k \to 0$ of the finite analog delta fucntion to find the Laplace transform,

$$\mathscr{L}\{f_k(t-a)\} = \frac{e^{-as} - e^{-(a+k)s}}{ks} \tag{6.4.7}$$

$$\mathscr{L}\{\delta(t-a)\} = e^{-as} \tag{6.4.8}$$

Partial fractions In the case of higher powers of polynomial factors in the denominator, the numerator is a generalized polynomial of one less order.

$$\frac{P(nk-1)}{[Q(k)]^n} = \frac{P_1}{Q(k)} + \frac{P_2}{[Q(k)]^2} + \cdots + \frac{P_n}{[Q(k)]^n} \tag{6.4.9}$$

Here, $\{P_1, \ldots, P_n\}$ are each polynomials of order $k-1$

The specific case of $Q(k)$ having only complex roots, requires convolution and is not covered here.

6.5 Convolution, Integral Equations

Multiplication of Laplace transforms Unlike the addition of functions, which leads simply to the addition of their Laplace transforms, multiplication needs to be dealth with using convolution

$$\mathscr{L}\{fg\} \neq \mathscr{L}\{f\}\,\mathscr{L}\{g\} \tag{6.5.1}$$

Convolution The process of filtering a function using the other as a mask, which leads to their respective Laplace transforms being multiplied.

$$h(t) = (f * g)(t) = \int_0^t f(\tau)\ g(t-\tau)\ \ \mathrm{d}\tau \tag{6.5.2}$$

$$H(s) = F(s) \cdot G(s) \tag{6.5.3}$$

This assumes that $f(t)$ and $g(t)$ satisfy the conditions for their Laplace transforms to exist individually.

Properties of convolution Using the integral definition above,

$$f * g = g * f \qquad \text{commutative law} \tag{6.5.4}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{distributive law} \tag{6.5.5}$$

$$(f * g) * \nu = f * (g * \nu) \qquad \text{associative law} \tag{6.5.6}$$

$$f * 0 = 0 \tag{6.5.7}$$

Now for some unusual properties which do not resemble the corresponding properties for the multiplication of real numbers,

$$f * 1 \neq f \qquad \text{in general} \tag{6.5.8}$$

$$f * f \geq 0 \qquad \text{is not guaranteed} \tag{6.5.9}$$

When solving partial fractions, convolution helps deal with the case of repeated complex roots in the denominator.

nh-ODEs using Convolution Consider the specific case of an ODE of the form,

$$y'' + ay' + by = r(t) \qquad y(0) = 0 \quad y'(0) = 0 \tag{6.5.10}$$

$$Y = RQ \qquad y(t) = \int_0^t q(t - \tau)\; r(\tau)\; \mathrm{d}\tau \tag{6.5.11}$$

The limits of integration need to be applied very carefully in case the input $r(t)$ happens to act only for a limited time window.

Integral equations For the specific case where the unknown function $y(t)$ happens to appear in an integral that can be rearranged to resemble a convolution integral, convolution can immediately be used to simplify the Laplace transform.

6.6 Differentiation and Integration of Transforms, ODEs with Variable Coefficients

Differentiation of transforms To find the derivative of the Laplace transform, (w.r.t s), the operation on $f(t)$ is,

$$F'(s) = \frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-st}\; f(t)\; \mathrm{d}t = -\int_0^\infty e^{-st}\; t \cdot f(t)\; \mathrm{d}t \tag{6.6.1}$$

$$F'(s) = -\,\mathscr{L}\{t \cdot f(t)\} \tag{6.6.2}$$

$$\mathscr{L}^{-1}\{F'(s)\} = -t \cdot f(t) \tag{6.6.3}$$

Differentiating the Laplace transform thus is equivalent to multiplying the original function by $(-t)$.

Integration of transforms Assuming the limit of $f(t)/t$ exists for $t \to 0^+$, and the Laplace transform of $f(t)$ exists, then,

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(p)\ \mathrm{d}p \tag{6.6.4}$$

6.6.5

Special Linear ODEs with variable coefficients Using the relation above,

$$\mathscr{L}\{ty'\} = -\frac{\mathrm{d}}{\mathrm{d}s}[sY - y(0)] = -Y - s\frac{\mathrm{d}Y}{\mathrm{d}s} \tag{6.6.6}$$

$$\mathscr{L}\{ty''\} = -\frac{\mathrm{d}}{\mathrm{d}s}[s^2Y - sy(0) - y'(0)] = -2sY - s^2\frac{\mathrm{d}Y}{\mathrm{d}s} + y(0) \tag{6.6.7}$$

Laguerre ODE A special ODE which is amenable to the above Laplace transform manipulation,

$$ty'' + (1-t)y' + ny = 0 \tag{6.6.8}$$

$$(s - s^2)\frac{\mathrm{d}Y}{\mathrm{d}s} + (n+1-s)Y = 0 \qquad \text{(subsidiary eqn.)} \tag{6.6.9}$$

$$Y = \frac{(s-1)^n}{s^{n+1}} \tag{6.6.10}$$

$$l_n = \mathscr{L}^{-1}\{Y\} \tag{6.6.11}$$

By convention, take $l_0 = 1$ and for higher order polynomials, the Rodrigues' formula is

$$l_n = \frac{e^t}{n!}\ \frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^n e^{-t}) \tag{6.6.12}$$

$$l_0 = 1 \tag{6.6.13}$$

$$l_1 = -t + 1 \tag{6.6.14}$$

$$l_2 = \frac{1}{2}\ (t^2 - 4t + 2) \tag{6.6.15}$$

$$l_3 = \frac{1}{6}\ (-t^3 + 9t^2 - 18t + 6) \tag{6.6.16}$$

6.7 Systems of ODEs

First order linear system with constant coefficients The Laplace transform of such a system is,

$$y_1' = a_{11}y_1 + a_{12}y_2 + g_1(t) \tag{6.7.1}$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + g_2(t) \tag{6.7.2}$$

$$(a_{11} - s)Y_1 + (a_{12})Y_2 = -y_1(0) - G_1(s) \tag{6.7.3}$$

$$(a_{21})Y_1 + (a_{22} - s)Y_2 = -y_2(0) - G_2(s) \tag{6.7.4}$$

Written in vector form, (using small and capital letters for functions and Laplace transforms respectively as for scalars),

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{6.7.5}$$

$$(\mathbf{A} - s\mathbf{I})\mathbf{Y} = -\mathbf{y}(0) - \mathbf{G} \tag{6.7.6}$$

This system of equations in $Y_1,\ Y_2$ can be solved and then inverse transformed to solve the given system of ODEs.

Chapter 7

Linear Algebra: Matrices, Vectors, Determinants, Linear Systems

7.1 Matrices, Vectors: Addition and Scalar Multiplication

Matrix A rectangular array of numbers in square brackets. A matrix with one row or one column is called a row or column vector respectively.

The matrix is considered square if it has the same number of rows and columns.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \qquad \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{7.1.1}$$

Elements of a matrix are indexed by their row and column address in that order.

Matrix notation A capital boldface letter is used to denote a matrix (or a vector). It can also be represented by putting its general term in square brackets. For an $m \times n$ matrix,

$$\mathbf{A} \equiv [a_{jk}] \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{7.1.2}$$

Main diagonal The diagonal entries of a square matrix from top left to bottom right

$$\{a_{jj}\} \quad \forall \quad j \in \{1, \dots, n\} \tag{7.1.3}$$

Vector A special matrix with one row or one column (called a column vector or row vector respectively).

It is represented by small boldface letters or by the general term in square brackets.

$$\mathbf{v} \equiv [v_j] \equiv \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \mathbf{u} \equiv [u_k] \equiv \begin{bmatrix} u_1 & u_2 \end{bmatrix} \tag{7.1.4}$$

Equality of matrices Two matrices are equal if and only if each element of the two matrices are equal. Matrices of different sizes are automatically not equal.

Addition of matrices Two matrices of equal size are added by element-wise addition of their entries.

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \qquad \Longrightarrow c_{jk} = a_{jk} + b_{jk} \tag{7.1.5}$$

Scalar Multiplication of matrices Multiplying a matrix by a scalar requires elementwise multiplication by that scalar.

$$\mathbf{C} = \lambda \mathbf{A} \qquad \Longrightarrow c_{jk} = \lambda a_{jk} \tag{7.1.6}$$

Properties of Matrices Using the above definitions of addition and scalar multiplication,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \text{commutative} \tag{7.1.7}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \qquad \text{associative} \tag{7.1.8}$$

$$\mathbf{A} + (-\mathbf{A}) = 0 \qquad \text{additive inverse} \tag{7.1.9}$$

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B} \tag{7.1.10}$$

$$1\ \mathbf{A} = \mathbf{A} \tag{7.1.11}$$

7.2 Matrix Multiplication

Product of two matrices Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be multiplied if their inner dimensions match,

$$\mathbf{A} := m \times n \qquad \mathbf{B} := r \times p \tag{7.2.1}$$

$$n = r \qquad \Longrightarrow \mathbf{C} = \mathbf{AB} \tag{7.2.2}$$

$$\mathbf{C} := m \times p \tag{7.2.3}$$

$$c_{jk} = \sum_{l=1}^{n} a_{jl}\ b_{lk} \tag{7.2.4}$$

In the summation above, $l$is a dummy variable traversing over the common inner dimension n. The subscripts j and k have ranges m and p respectively.

Properties Matrix multiplication is not commutative.

$$\mathbf{AB} \neq \mathbf{BA} \qquad \text{in general} \tag{7.2.5}$$

$$\mathbf{AB} = 0 \qquad \not\Rightarrow \ \mathbf{BA} = \mathbf{0} \tag{7.2.6}$$

$$\not\Rightarrow \ \mathbf{A} = \mathbf{0} \tag{7.2.7}$$

$$\not\Rightarrow \ \mathbf{B} = \mathbf{0} \tag{7.2.8}$$

This might be because the product is not defined or because the result happens to be different even when the product is defined.

The second rule follows from the fact that the **0** matrix requires all of its elements to be 0.

Properties similar to multiplication of scalars are,

$$(k\mathbf{A})\ \mathbf{B} = k\ (\mathbf{AB}) = \mathbf{A}\ (k\mathbf{B}) \tag{7.2.9}$$

$$\mathbf{A}\ (\mathbf{BC}) = (\mathbf{AB})\ \mathbf{C} \tag{7.2.10}$$

$$(\mathbf{A} + \mathbf{B})\ \mathbf{C} = \mathbf{AC} + \mathbf{BC} \tag{7.2.11}$$

$$\mathbf{C}\ (\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB} \tag{7.2.12}$$

Parallel computation by computers uses the shortcut,

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix} \tag{7.2.13}$$

Linear Transforms The most direct use of matrix multiplication is in linear transforms of the form,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Ax} \tag{7.2.14}$$

If another step takes the system $\mathbf{x}$ to $\mathbf{w}$, using

$$\mathbf{x} = \mathbf{Bw} \qquad \mathbf{y} = \mathbf{Ax} \tag{7.2.15}$$

$$\mathbf{y} = \mathbf{Cw} \qquad \mathbf{C} = \mathbf{AB} \tag{7.2.16}$$

Thus, the definition of matrix multiplication follows from the act of linear transforms or from systems of linear equations.

Transpose Writing a matrix's rows as columns (or columns as rows). For the special case of a square matrix, the diagonal elements stay in place, as seen here.

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \tag{7.2.17}$$

Some useful propoerties of the transpose are,

$$\left(\mathbf{A}^T\right)^T = A \tag{7.2.18}$$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{7.2.19}$$

$$(c\mathbf{A})^T = c\ \mathbf{A}^T \tag{7.2.20}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\ \mathbf{A}^T \tag{7.2.21}$$

Special names for matrices Some square matrices have special names based on their transpose,

$\mathbf{A}^T = \mathbf{A}$	$a_{jk} = a_{kj}$	symmetric	7.2.22
$\mathbf{A}^T = -\mathbf{A}$	$a_{kk} = 0$	skew-symmetric	7.2.23
$a_{jk} = 0$	$\forall\ j > k$	upper-triangular	7.2.24
$a_{jk} = 0$	$\forall\ j < k$	lower-triangular	7.2.25
$a_{jk} = 0$	$\forall\ j \neq k$	diagonal	7.2.26
$a_{kk} = c$	$\forall\ k \in \{1, \ldots, n\}$	scalar	7.2.27
$a_{kk} = 1$	$\forall\ k \in \{1, \ldots, n\}$	identity ($\mathbf{I}$)	7.2.28

The scalar matrix has the same effect on multiplying by another compatible matrix $\mathbf{A}$ as multiplication by the scalar c.

$$\mathbf{AS} = \mathbf{SA} = c\ \mathbf{A} \tag{7.2.29}$$

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \tag{7.2.30}$$

Stochastic matrix A matrix whose entries are all non-negative and whose columns all sum to 1. They can be used to represent the transition probabilities between states of a system.

Markov process A process in which the current state of the system only depends on the previous state of the system.

No other details of its state history are remembered by the system.

The transition matrix is a stochastic matrix used to convey the probablities of transition from every state of the system to every other state possible.

$$\mathbf{y}_n = \mathbf{A}\ \mathbf{y}_{n-1} \tag{7.2.31}$$

$$\mathbf{y}_n = \mathbf{A}^n\ \mathbf{y}_0 \tag{7.2.32}$$

7.3 Linear Systems of Equations, Gauss Elimination

Linear system A system of m equations in n unknowns is represented as,

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \tag{7.3.1}$$

$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \tag{7.3.2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots = \cdots \tag{7.3.3}$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \tag{7.3.4}$$

7.3.5

Augmented matrix Condensing the matrices $\mathbf{A}$ and $\mathbf{b}$ into one onject, by adding $\mathbf{b}$ after the last column of $\mathbf{A}$,

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ a_{21} & \dots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right] \tag{7.3.6}$$

Elementary row operations Three kindds of manipulations of the rows of a matrix, that leave its determinant unchanged.

- Interchange of two rows
- Adding a constant multiple of one row to another
- Multiplying a row by a nonzero constant

Gauss Elimination Repeated row operations on the augmented matrix in order to convert it into upper triangular form. Then, simple back substitution from the bottom-up can yield one element of the solution at a time.

The general procedure is as follows,

- Eliminate the first variable from all but the first row by performing the appropriate row operations on the augmented matrix.
- Repeat this process for the next variable, keeping in mind that the first $(k-1)$ rows remain unchanged when targeting variable x_k.

- After performing this operation k times, the first k columns of the coefficient matrix will be upper triangular.

Row equivalnce Two linear systems related by a finite number of row operations. They necessarily have the same solution.

Types of linear system A system is called consistent if it has at least one solution.

For a linear system with m equations and n unknowns, (translates to a matrix with m rows and n columns),

$$m > n \implies \text{overdetermined} \tag{7.3.7}$$

$$m = n \implies \text{determined} \tag{7.3.8}$$

$$m < n \implies \text{under-determined} \tag{7.3.9}$$

Infinitely many solutions If reduction to upper triangular form leaves one or more rows completely zero, then the system has infinitely many solutions, and by convention the free variables are denoted by a different alphabet.

No solution Gauss elimination will simply produce a contradictory statement, such as two unequal constants being related by an equality.

Row Echelon form The upper-triangular like form of the coefficient matrix after completion of Gauss elimination. This may have some number or completely zero rows at the bottom.

Starting from the first row, every row will have more zero elements starting from the left edge before the first nonzero element.

$$\begin{bmatrix}\mathbf{A}|\mathbf{b}\end{bmatrix} \iff \begin{bmatrix}\mathbf{R}|\mathbf{f}\end{bmatrix} \tag{7.3.10}$$

$$\begin{bmatrix}\mathbf{R}|\mathbf{f}\end{bmatrix} = \left[\begin{array}{ccccc|c} r_{11} & r_{12} & \dots & \dots & r_{1n} & f_1 \\ 0 & r_{22} & \dots & \dots & r_{2n} & f_2 \\ 0 & \ddots & & \dots & & \vdots \\ 0 & 0 & r_{kk} & \dots & r_{kn} & f_k \\ 0 & 0 & 0 & 0 & 0 & f_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & f_m \end{array}\right] \tag{7.3.11}$$

Here, $k \leq m$

Rank of matrix The number of nonzero rows k in the row echelon form of the matrix $\mathbf{A}$ is called its rank. The rank can be used to classify the type of solutions of the linear system.

- If the rank k is less than the number of rows m, and at least one of the RHS elements $\{f_{k+1}, \dots, f_m\}$ is non-zero, then the system has no solution.
- If the system is consistent ($r = m$ or $r < m$ and all of the RHS elements $\{f_{k+1}, \dots, f_m\}$) are zero, then the system has at least one solution.

7.4 Linear Independence, Rank of a Matrix, Vector Space

Linear Independence of vectors Given a set of vectors with the same number of components $\{\mathbf{a}_i\}$, the equation

$$c_1\mathbf{a}_1 + c_2\mathbf{a_2} + \cdots + c_n\mathbf{a}_n = \mathbf{0} \tag{7.4.1}$$

for some scalars $\{c_i\}$, always has the trivial solution $\{c_i\} = 0$.

If this is the only solution to this equation, then the set of vectors $\{a_i\}$ are considered L.I.

If there is some solution to the above equation which does not require the entire set $\{c_i\}$ to be zero, then the vectors are linearly dependent (L.D.)

Rank of a matrix The maximum number of L.I. row vectors of a matrix.

Rank of a matrix is invariant under elementary row operations.

If the matrix formed by a set of vectors as rows, has rank equal to the number of rows, then the set of vectors are L.I.

Using the result,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) \tag{7.4.2}$$

The rank of a matrix is also equal to the number of L.I. column vectors in it.

Linear dependence of vectors Consider a set of p vectors each having n components. If $n < p$, then the set of vectors are L.D.

Vector Space For a non-empty set of vectors V, whose members all have the same number of components, all linear combinations of elements of V are also members of V.

$$\mathbf{a},\ \mathbf{b} \in V \qquad \implies k_1\mathbf{a} + k_2\mathbf{b} \in V \tag{7.4.3}$$

Here $\{k_1\}$ are real numbers.

Dimension and basis of vector space The maximum number of L.I. vectors in V. (for the special case of finite dimensional spaces)

Such a set of L.I. vectors in V is called a basis of V. Adding another vector to this set would make it L.D.

The number of vectors in a basis of V is equal to $\dim(V)$.

Span The set of all linear combinations of a set of vectors is called the span of those vectors. If this set of vectors is L.I., then they form a basis for that span (which is also a vector space).

Subspace Any non-empty subset of V that obeys the same rules for addition and scalar multiplication as the parent vector space V.

Row space and Column space of a matrix The span of the row vectors or column vectors of a matrix.

The row space and column space of $\mathbf{A}$ have the same dimension, equal to $\text{rank}(\mathbf{A})$.

Null space, nullity For a homogeneous system of equations,

$$\mathbf{Ax} = \mathbf{0} \tag{7.4.4}$$

The solution set of this system is called the null space of $\mathbf{A}$. The dimension of the null space is called the nullity.

7.5 Solutions of Linear Systems: Existence, Uniqueness

Submatrix A matrix obtained by omitting some rows or columns of a larger matrix.

Existence of solutions A linear system of m equations in n variables is consistent (has at least one solution) if and only if the coefficient matrix $\mathbf{A}$ and augmented matrix $\tilde{\mathbf{A}}$ have the same rank.

- The solution is unique if and only if this common rank is equal to n.
- If the common rank $r < n$, then the system has infinitely many solutions.
- If the solutions exist, they can be obtained by Gauss elimination.

Homogeneous Linear System A homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ always has a trivial solution, where A is an $m \times n$ matrix.

Nontrivial solutions exist if and only if $\mathrm{rank}(\mathbf{A}) < n$

Homogeneous linear systems with fewer equations than unknowns $(m < n)$ always has nontrivial solutions.

$$\mathrm{rank}(\mathbf{A}) \le m \qquad\qquad m < n \tag{7.5.1}$$

$$\implies \mathrm{rank}(\mathbf{A}) < n \tag{7.5.2}$$

Solution space Only in the case of homogeneous systems, the trivial and nontrivial solutions together form a vector space called the solution space.

The dimension of the solution space is $n - r$, where the system has n variables, but the matrix $\mathbf{A}$ has rank $r < n$.

This is also called the null space of $\mathbf{A}$, and its dimension is called the nullity. This leads to,

$$\mathrm{rank}(\mathbf{A}) + \mathrm{nullity}(\mathbf{A}) = n \tag{7.5.3}$$

Homogeneous linear systems with fewer equations than unknowns $(m < n)$ always has nontrivial solutions.

Nonhomogeneous Linear Systems Analogous to the theorem for ODEs, the set of all solutions to a nh-linear system is of the form,

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h \tag{7.5.4}$$

where $\mathbf{x}_p$ is a particular solution to the nh-system and $\mathbf{x}_h$ is the set of all solutions to the corresponding h-system.

7.6 Second and Third-order determinants

Second order determinant A square matrix of order 2 has the determinant

$$D = \det(\mathbf{A}) \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{7.6.1}$$

Minor of a matrix element The determinant of the submatrix obtained by removing the row and column in which the element is located. For example, a 3×3 matrix gives,

$$M_{1,1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{2,1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M_{3,1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{7.6.2}$$

$$M_{1,1} = \begin{vmatrix} e & f \\ h & i \end{vmatrix} \qquad M_{2,1} = \begin{vmatrix} b & c \\ h & i \end{vmatrix} \qquad M_{3,1} = \begin{vmatrix} b & c \\ e & f \end{vmatrix} \tag{7.6.3}$$

7.6.4

Cofactor of a matrix In order to calculate higher order determinants,

$$C_{i,j} = (-1)^{i+j} M_{i,j} \tag{7.6.5}$$

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ik} \cdot C_{i,k} \tag{7.6.6}$$

For some column index k. Alternatively the expansion can also be performed along any row.

Third order determinants A square matrix of order 3 has the determinant,

$$D = \det(\mathbf{A}) \equiv \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \tag{7.6.7}$$

$$= a \cdot C_{1,1} + d \cdot C_{2,1} + g \cdot C_{3,1} \tag{7.6.8}$$

$$= a \cdot M_{1,1} - d \cdot M_{2,1} + g \cdot M_{3,1} \tag{7.6.9}$$

7.7 Determinants, Cramer's Rule

Determinant properties A scalar associated to a square matrix, used in linear algebra and solving linera systems.

Properties of determinants Similar to the underlying matrices,

- Determinant is invariant when adding a scalar multiple of one row to another.
- Interchanging rows however, introduces a global multiplier of (-1).
- Multiplying a row by a scalar introduces a global multiplier by the same scalar.

$$\det(c\mathbf{A}) = c^n \ \det(\mathbf{A}) \tag{7.7.1}$$

- Transposing a matrix does not change its determinant.
- A zero row or column makes the determinant zero.
- A row or column being a scalar multiple of another makes the determinant zero.

Relating rank and determinant of a matrix Let $A_{m \times n}$ have non-zero rank r.

rank$(\mathbf{A}) = r \geq 1$ ifand only if $\mathbf{A}$ has at least one $r \times r$ submatrix with nonzero determinant.

The determinant of any submatrix with more than r rows contained in A is zero.

For the specific case of $m = n$ (square matrix)

$$\det(\mathbf{A}) \neq 0 \qquad \Longleftrightarrow \qquad \text{rank}(\mathbf{A}) = n \tag{7.7.2}$$

Cramer's rule A computationally inefficient method of solving linear systems using quotients of determinants.

Consider a system of n equations in n variables

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{7.7.3}$$

$$D_k = D \text{ with k}^{\text{th}} \text{ column replaced by } \mathbf{b} \tag{7.7.4}$$

$$x_k = \frac{D_k}{D} \tag{7.7.5}$$

For the unique solution defined above to exist, the system has to have $\det(\mathbf{A}) \neq 0$.

If the system is homogeneous, $D \neq 0$ guarantees only the trivial solution exists.

7.8 Inverse of a Matrix, Gauss-Jordan Elimination

Inverse of a matrix Analog of the multilicative inverse of a number,

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{7.8.1}$$

This is only defined for square matrices with $\mathbf{I}$ being the identity matrix of order n.

The inverse of a matrix is unique, if it exsits.

Singular matrix A matrix which does not have an inverse.

Existence of inverse The inverse of a matrix exists if and only if it has maximum possible rank n.

or if and only if $\det(\mathbf{A}) \neq 0$.

Gauss-Jordan method Enhancement of Gauss elimination method to find the inverse using the following steps,

- Construct the augmented matrix $\tilde{\mathbf{A}} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right]$
- Apply Gauss elimination to $\mathbf{A}$ until it is triangular

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right] \rightarrow \left[\begin{array}{c|c} \mathbf{U} & \mathbf{H} \end{array}\right] \tag{7.8.2}$$

- Apply further row operations to convert $\mathbf{U}$ into a diagonal matrix with all diagonal entries 1.

$$\left[\begin{array}{c|c} \mathbf{U} & \mathbf{H} \end{array}\right] \to \left[\begin{array}{c|c} \mathbf{I} & \mathbf{K} \end{array}\right] \tag{7.8.3}$$

- The inverse is now directly read from the right half of $\tilde{\mathbf{A}}$.

$$\mathbf{K} = \mathbf{A}^{-1} \tag{7.8.4}$$

Inverse using Cramer's rule Cramer's rule provides a direct formula to calculate the inverse, using the cofactors of det $\mathbf{A}$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \left[C_{jk}\right]^T = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} \tag{7.8.5}$$

For the special case of 2×2 matrices, where this is an easy computation,

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{A}^{-1} \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{7.8.6}$$

Properties of inverse Inverse of a matrix has some properties resembling the transpose,

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1} \tag{7.8.7}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A} \tag{7.8.8}$$

Cancellation laws Laws explaining the unusual properties of matrix multiplication.

- If $\operatorname{rank}(\mathbf{A}) = n$ and $\mathbf{AB} = \mathbf{AC}$ then, $\mathbf{B} = \mathbf{C}$
- If $\operatorname{rank}(\mathbf{A}) = n$ and $\mathbf{AB} = 0$ then, $\mathbf{B} = 0$
- If $\mathbf{AB} = 0$ but $\mathbf{A} \neq 0$ and $\mathbf{B} \neq 0$, then $\operatorname{rank}(\mathbf{B}) < n$ and $\operatorname{rank}(\mathbf{A}) < n$
- If $\mathbf{A}$ is singular, then $\mathbf{AB}$ and $\mathbf{BA}$ are also singular.

Determinant of matrix product The determinant of the product of matrices is given by,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) \equiv \det(\mathbf{A}) \cdot \det(\mathbf{B}) \tag{7.8.9}$$

7.9 Vector Spaces, Inner Product Spaces, Linear Transformations

Real vector space A vector space whose components are ordered n-tuples of real numbers. This is denoted $\mathcal{R}^n$.

Vector addition With any two elements $\mathbf{a}$ and $\mathbf{b}$ of the vector space V, a unique member of V can be associated to the operation $\mathbf{a}+\mathbf{b}$ satisfying,

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} \qquad \text{commutativity} \tag{7.9.1}$$

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}) \qquad \text{associativity} \tag{7.9.2}$$

$$\mathbf{a}+\mathbf{0}=\mathbf{a} \qquad \text{zero vector} \tag{7.9.3}$$

$$\mathbf{a}+(-\mathbf{a})=\mathbf{0} \qquad \text{additive inverse} \tag{7.9.4}$$

$$\tag{7.9.5}$$

Scalar multiplication For every element $\mathbf{a}$ of the vector space V, and a scalar c a unique member of V can be associated to the operation $k\mathbf{a}$ satisfying,

$$k(\mathbf{a}+\mathbf{b})=k\mathbf{a}+k\mathbf{b} \qquad \text{distributivity over vectors} \tag{7.9.6}$$

$$(k+m)\mathbf{a}=k\mathbf{a}+m\mathbf{a} \qquad \text{distributivity over scalars} \tag{7.9.7}$$

$$c\,(k\mathbf{a})=(ck)\,\mathbf{a} \qquad \text{associativity} \tag{7.9.8}$$

$$1\cdot\mathbf{a}=\mathbf{a} \qquad \text{additive identity} \tag{7.9.9}$$

The above axioms are necessary and sufficient to define vector spaces.

Dimension of vector space The maximum size of a set of L.I. vectors in V, such that adding even one more vector to the set would make it L.I.

If a vector space contains a L.I. set of n vectors, no matter how large n is, then V is an infinite dimensional vector space.

Inner Product Given two members $\mathbf{a}$ and $\mathbf{b}$ of a vector space in $\mathcal{R}^n$,

$$\mathbf{a}\cdot\mathbf{b}=(\mathbf{a},\mathbf{b})\equiv\mathbf{a}^T\mathbf{b}=\begin{bmatrix}a_1 & a_2 & \ldots & a_n\end{bmatrix}\cdot\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}=\sum_{l=1}^{n}a_lb_l \tag{7.9.10}$$

Real Inner Product space Consider a real vector space V and two members $\mathbf{a}$ and $\mathbf{b}$.

If there is a real number associated with these vectors denoted by $\mathbf{a} \cdot \mathbf{b}$ (with $p,\ q$ being scalars)

$$(p\mathbf{a} + q\mathbf{b}, \mathbf{c}) = p(\mathbf{a}, \mathbf{c}) + q(\mathbf{b}, \mathbf{c}) \qquad \text{linearity} \qquad 7.9.11$$

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a}) \qquad \text{symmetry} \qquad 7.9.12$$

$$(\mathbf{a}, \mathbf{a}) \geq 0 \qquad 7.9.13$$

$$(\mathbf{a}, \mathbf{a}) = 0 \iff \mathbf{a} = 0 \qquad \text{positive-definite} \qquad 7.9.14$$

Orthogonal Two vectors are orthogonal if their inner product is zero.

Norm The norm (or length) of a vector is the square root of its inner product.

$$\|\mathbf{a}\| \equiv \sqrt{(\mathbf{a}, \mathbf{a})} \qquad 7.9.15$$

$$= \sqrt{a_1^2 + \cdots + a_n^2} \qquad \text{Euclidean space} \qquad 7.9.16$$

Unit vector A vector with $\|\mathbf{a}\| = 1$

Vector inequalities Using the above definition of the norm,

$$|(\mathbf{a}, \mathbf{b})| \leq \|\mathbf{a}\|\ \|\mathbf{b}\| \qquad \text{Cauchy-Shwarz inequality} \qquad 7.9.17$$

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \qquad \text{Triangle inequality} \qquad 7.9.18$$

$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\Big(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\Big) \qquad \text{Parallellogram equality} \qquad 7.9.19$$

Linear Transform A mapping from each vector $\mathbf{x}$ in X to a unique vector $\mathbf{y}$ in Y denoted by

$$F\mathbf{x} = \mathbf{y} \qquad F(x) = y \qquad 7.9.20$$

For all vectors $\mathbf{a}$ and $\mathbf{b}$ in X and for all scalars c,

$$F(\mathbf{a} + \mathbf{b}) = F(\mathbf{a}) + F(\mathbf{b}) \qquad 7.9.21$$

$$F(c\mathbf{a}) = c\ F(\mathbf{a}) \qquad 7.9.22$$

Image The result of a linear transform acting on a source vector. In the above definition, $\mathbf{y}$ is the image of $\mathbf{x}$ under F

Matrices as linear transforms All linear transforms of the form $\mathcal{R}^n \rightarrow \mathcal{R}^m$ can be represented by an $m \times n$ matrix.

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad 7.9.23$$

The matrix $\mathbf{A}$ is said to represent the linear transform F.

Composition of Linear Transforms An ordered application of linear transforms one after the other.

For some vector spaces W, X, Y,

$$F : X \to Y \qquad G : W \to X \tag{7.9.24}$$

$$H \equiv (F \circ G) = FG = F(G) \qquad H : W \to Y \tag{7.9.25}$$

Compositions of linera transforms are also linear. In terms of matrices, this is analogous to multiplication of matrices.

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \mathbf{x} = \mathbf{B}\mathbf{w} \tag{7.9.26}$$

$$\mathbf{y} = \mathbf{C}\mathbf{w} \qquad \mathbf{C} = \mathbf{A}\mathbf{B} \tag{7.9.27}$$

Chapter 8

Linear Algebra: Matrix Eigenvalue Problems

8.1 The Matrix Eigenvalue Problem, Determining Eigenvalues and Eigenvectors

Matrix Eigenvalue problem The process of finding scalars λ and nonzero vectors $\mathbf{v}$ satisfying,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{8.1.1}$$

This equation is always solved by the trivial $\mathbf{v} = \mathbf{0}$ which is not of interest.

Characteristic determinant The determinant which is set to zero in order to obtain non-trivial solutions of the eigenvalue problem,

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = 0 \tag{8.1.2}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \qquad\qquad D(\lambda) = 0 \tag{8.1.3}$$

The eigenvalues are solutions to this polynomial equation in λ. An $n \times n$ matrix has at least one and at most n distinct eigenvalues.

Eigenspace The set of all eigenvectors corresponding to the same eigenvalue, along with the zero vector, form a vector space called the eigenspace of the matrix corresponding to that eigenvalue.

Algebraic multiplicity The order (M_λ) of an eigenvalue λ as a root of the characteristic equation.

Geometric multiplicity The number of L.I. eigenvectors (m_λ) corresponding to a particular eigenvalue.

Defect of eigenvalue The difference,

$$\Delta_\lambda \equiv M_\lambda - m_\lambda \tag{8.1.4}$$

$$\Delta_\lambda \geq 0 \tag{8.1.5}$$

Eigenvalues of transpose Using the properties of determinant, $\mathbf{A}^T$ has the same eigenvalues as $\mathbf{A}$.

8.2 Some Applications of Eigenvalue Problems

Applications Systems which can be reduced to a system of linear equations or a system of linear ODEs can be solved using the eigenvalue problem

Steady state The unity eigenvalue represents the steady state of a system, since the action of the system on the input state causes no change

$$\lambda = 1 \qquad \implies \mathbf{A}\mathbf{x} = x \tag{8.2.1}$$

Interpretation of eigenvectors Usually, the eigenvector can be interpreted as the privileged initial state of the system, which produces a final state in proportion to it.

8.3 Symmetric, Skew-Symmetric, and Orthogonal Matrices

Orthogonal matrix A special square matrix whose transpose is equal to its inverse.

$$\mathbf{A}^T = \mathbf{A}^{-1} \tag{8.3.1}$$

Eigenvalues conditions The eigenvalues of a symmetric matrix are real.

The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

Orthogonal transforms Transforms that are defined as

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \mathbf{A} \text{ is orthogonal} \tag{8.3.2}$$

Any orthogonal transform in Euclidean $\mathcal{R}^2$ is a rotation (that may be combined with a reflection in a straight line).

Any orthogonal transform in Euclidean $\mathcal{R}^3$ is a rotation (that may be combined with a reflection in a plane).

Invariance of Inner product The inner product of two vectors undergoing the same orthogonal transform is preserved. For some orthogonal matrix C,

$$\mathbf{u} = \mathbf{C}\mathbf{a} \qquad \mathbf{v} = \mathbf{C}\mathbf{b} \tag{8.3.3}$$

$$\|\mathbf{a}\| = \|\mathbf{u}\| \qquad \|\mathbf{b}\| = \|\mathbf{v}\| \tag{8.3.4}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{u} \cdot \mathbf{v} \tag{8.3.5}$$

Orthonormality System A system whose elements have nonzero dot product only when the other vector is the same element.

$$\mathbf{a}_j \cdot \mathbf{a}_k = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \tag{8.3.6}$$

A real square matrix is orthogonal if and only if its column vectors form an orthonormal system. (Also applies to its row vectors)

Determinant of orthogonal matrix This is always ± 1.

$$\mathbf{A}\mathbf{A}^T = \mathbf{I} \qquad \det(\mathbf{A}) \cdot \det(\mathbf{A}^T) = 1 \tag{8.3.7}$$

$$\det(\mathbf{A})^2 = 1 \qquad \det(\mathbf{A}) = \pm 1 \tag{8.3.8}$$

Eigenvalues of orthogonal matrix Since the entries are real, the eigenvalues are real or complex conjugate pairs.

Additionally, their absolute value is 1.

8.4 Eigenbases, Diagonalization, Quadratic Forms

Eigenbasis A basis for $\mathcal{R}^n$ formed by the set of eigenvectors of a matrix $\mathbf{A}$. This means,

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \mathbf{x} = \sum_{j=1}^{n} c_j \, \mathbf{b}_j \tag{8.4.1}$$

$$\mathbf{y} = \sum_{j=1}^{n} c_j \left(\lambda_j \mathbf{b}_j\right) \tag{8.4.2}$$

Thus, the multiplication $\mathbf{A}\mathbf{x}$ is replaced by the much simpler linear superposition of eigenvectors.

For the special case of all eigenvalues being distinct, an eigenbasis exists for $\mathcal{R}^n$.

Symmetric matrix eigenbasis A symmetric matrix has an orthonormal basis for eigenvectors for $\mathcal{R}^n$.

Similar matrices Two matrices are similar if

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{8.4.3}$$

and $\mathbf{P}$ is some non-singular matrix.

Such a transformation is called a similarity transformation.

Similarity transformation These transformations preserve eigenvalues and transoform the corresponding eigenvectors by,

$$\mathbf{y}_j = \mathbf{P}^{-1}\mathbf{x}_j \qquad \mu_j = \lambda_j \tag{8.4.4}$$

Diagonalization If a matrix $\mathbf{A}$ of order n, has a basis of eigenvectors, then

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} \tag{8.4.5}$$

$\mathbf{D}$ is diagonal with the eigenvalues being the entries of the main diagonal.

Additionally, $\mathbf{X}$ is simply a matrix composed of these eigenvectors as columns.

$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X} \tag{8.4.6}$$

for some positive integer m.

Quadratic Form A quadratic form Q in a vector $\mathbf{x}$ is a sum of the n^2 terms. It is a scalar defined as,

$$Q \equiv \mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_{j=1}^{n} \sum_{k=1}^{n} x_j \ a_{jk} \ j_k \tag{8.4.7}$$

$$\equiv \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{12} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{8.4.8}$$

Additionally, $\mathbf{A}$ is called the coefficient matrix of this form.

Symmetric coefficient matrix Any coefficient matrix can be replaced in the quadratic form by its another symmetric matrix using,

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{A}^T}{2} \qquad Q = \mathbf{x}^T \mathbf{C} \mathbf{x} = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{8.4.9}$$

Canonical form Also called principal axis form. Starting with,

$$\mathbf{A} = \mathbf{A}^T \qquad \mathbf{A} = \mathbf{X} \mathbf{D} \mathbf{X}^{-1} \tag{8.4.10}$$

$$\mathbf{X}^{-1} = \mathbf{X}^T \qquad Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{8.4.11}$$

Since an orthonormal basis of eigenvectors is guaranteed.

Principal axis theorem Any quadratic form can be transformed into the simplified form, (with symmetric matrix $\mathbf{A}$)

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \mathbf{y} = \mathbf{X}^{-1} \mathbf{x} \tag{8.4.12}$$

$$Q = \mathbf{y}^T \mathbf{D} \mathbf{y} = \sum_{j=1}^{n} \lambda_j \ y_j^2 \tag{8.4.13}$$

Here, the eigenvalues need not be distinct.

8.5 Complex Matrices and Forms

Conjugate transpose The process of applying a complex conjugate operation before taking the transpose of a matrix. Useful in quantum physics etc.

$$\mathbf{A}^\dagger \equiv \left(\bar{\mathbf{A}}\right)^T \tag{8.5.1}$$

Special complex matrices Some complex matrices are special analogous to the symmetric and skew-symmetric matrices defined earlier

$$\mathbf{A}^\dagger = \mathbf{A} \qquad \text{Hermitian} \tag{8.5.2}$$

$$\mathbf{A}^\dagger = -\mathbf{A} \qquad \text{Skew-Hermitian} \tag{8.5.3}$$

$$\mathbf{A}^\dagger = \mathbf{A}^{-1} \qquad \text{Unitary} \tag{8.5.4}$$

Eigenvalues of special matrices The eigenvalues of

- Hermitian matrices are real
- skew-Hermitian matrices are either zero or purely imaginary
- unitary matrix have absolute value 1

Inner Product For complex numbers, the inner product generalizes to

$$\mathbf{a} \cdot \mathbf{b} \equiv \mathbf{a}^\dagger \mathbf{b} \tag{8.5.5}$$

Norm of complex vector Even when the components of a vector are complex numbers ($\mathbf{v} \in \mathcal{C}^n$),

$$\|\mathbf{a}\| \equiv \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^\dagger \mathbf{a}} = \sum_{i=1}^{n} |a_i|^2 \tag{8.5.6}$$

Unitary transformation A transformation of the form $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is unitary, preserves the inner product and the norm.

Unitary system A set of vectors obeying

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^\dagger \, \mathbf{a}_k = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \tag{8.5.7}$$

Unitary matrix The determinant of a unitary matrix has absolute value 1.

A complex square matrix is unitary if and only if its column vectors form a unitary system. (also its row vectors)

Eigenbasis A Hermitian, skew-Hermitian, or unitary matrix has a basis of eigenvectors for $\mathcal{C}^n$ that form a unitary system.

Complex quadratic forms Similar to quadratic real forms, complex forms are defined as,

$$\mathbf{x}^\dagger \mathbf{A} \mathbf{x} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \, \bar{x}_j x_k \tag{8.5.8}$$

This is a summation of n^2 terms that can now be complex. For the special cases of

- Hermitian matrices, the form is real.
- skew-Hermitian matrices, the form is zero or purely imaginary.

Chapter 9

Vector Differential Calculus: Grad, Div, Curl

9.1 Vectors in 2-Space and 3-Space

Scalar A quantity determined solely by its magnitude.

Vector A quantity determined by magnitude and direction. It is represented by a directed line segment (an arrow).

Norm The length of a vector represented by $|\mathbf{v}|$.

Translation Displacement without rotation (as a linear transform or operation , usually acting on vectors)

Unit vector A vector with length 1

Equality of vectors Two vectors are equal if and only if their magnitudes and directions are both equal.

$$\mathbf{a} = \mathbf{b} \iff |\mathbf{a}| = |\mathbf{b}| \quad \text{and} \quad \hat{\mathbf{a}} = \hat{\mathbf{b}} \tag{9.1.1}$$

Components of a vector If a vector has initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$, then

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \equiv \begin{bmatrix} (x_2 - x_1) \\ (y_2 - y_1) \\ (z_2 - z_1) \end{bmatrix} \qquad |\mathbf{a}| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{9.1.2}$$

This definition of a vector's components is specific to a Cartesian coordinate system in 3-d.

Position vector A vector whose initial point is the origin and terminal point is a given point in 3-d space.

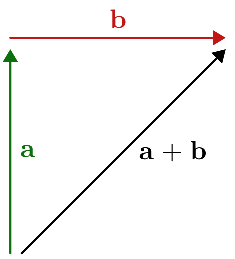
Algebraic vector Given a fixed Cartesian coordinate system, every ordered triple of real numbers (a_1, a_2, a_3) is associated to a unique vector $\mathbf{a}$ (and vice versa). Only the zero vector has no direction.

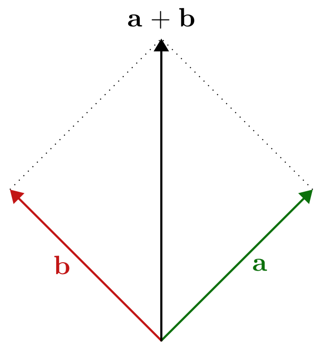
A vector equation is the same as a scalar equation for each of its components.

Vector addition Adding two vectors is performed by adding their corresponding components.

$$\mathbf{a} + \mathbf{b} \equiv \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} \tag{9.1.3}$$

Geometrically, the initial point of $\mathbf{b}$ is placed at the terminal point of $\mathbf{a}$ and the vector sum is drawn from the initial point of $\mathbf{a}$ to the terminal point of $\mathbf{b}$.





The paralleogram rule is used in physics to calculate the resultant (vector sum) of many forces acting on the same point.

Properties of vector addition Similar to scalar addition,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \qquad \text{commutative} \tag{9.1.4}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \qquad \text{associative} \tag{9.1.5}$$

$$\mathbf{a} + \mathbf{0} = \mathbf{a} \qquad \text{additive null} \tag{9.1.6}$$

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0} \qquad \text{additive inverse} \tag{9.1.7}$$

Scalar multiplication Multiplying a vector by a scalar multiplies it by every component of the vector.

$$c\ \mathbf{v} \equiv \begin{bmatrix} c\ v_1 \\ c\ v_2 \\ c\ v_3 \end{bmatrix} \qquad |c\ \mathbf{v}| = |c|\ |\mathbf{v}| \tag{9.1.8}$$

Geometrically, if $c < 0$, then $c\mathbf{v}$ faces the opposite direction.

Properties of scalar multiplication Similar to scalars,

$$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b} \qquad (c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a} \tag{9.1.9}$$

$$c\ (k\mathbf{a}) = (ck)\ \mathbf{a} \qquad 1 \cdot \mathbf{a} = \mathbf{a} \tag{9.1.10}$$

$$(-1) \cdot \mathbf{a} = -\mathbf{a} \qquad 0 \cdot \mathbf{a} = \mathbf{0} \tag{9.1.11}$$

Standard basis $\mathcal{R}^3$ represented geometrically by 3-d Cartesian space, has dimension 3, and needs a set of 3 basis vectors defined by

$$\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{9.1.12}$$

$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} \tag{9.1.13}$$

9.2 Inner Product (Dot Product)

Inner product The product of lengths of two vectors and the cosine of the angle between them. This is a scalar.

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} |\mathbf{a}|\ |\mathbf{b}| \cos(\theta) & |\mathbf{a}| \neq 0 \text{ and } \ |\mathbf{b}| \neq 0 \\ 0 & |\mathbf{a}| = 0 \text{ or } \ |\mathbf{b}| = 0 \end{cases} \tag{9.2.1}$$

Here θ is the smaller angle between the vectors measured after making their initial points coincide.

In terms of the components,

$$\mathbf{a} \cdot \mathbf{b} \equiv \sum_{k=1}^{n} a_k b_k \tag{9.2.2}$$

Orthogonal vectors Two vectors are perpendicular if and only if their inner product is zero.

Length of a vector The square root of the inner product of a vector with itself. This is geomatrically equal to its magnitude.

Additionally, the angle between two nonzero vectors can be found using,

$$|\mathbf{v}| \equiv \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{9.2.3}$$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|} \tag{9.2.4}$$

Properties of inner product Some properties similar to multiplication,

$$(p\mathbf{a} + q\mathbf{b}) \cdot \mathbf{c} = p\mathbf{a} \cdot \mathbf{c} + q\mathbf{b} \cdot \mathbf{c} \qquad \text{linear} \tag{9.2.5}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \qquad \text{symmetry} \tag{9.2.6}$$

$$\mathbf{a} \cdot \mathbf{a} \geq 0 \tag{9.2.7}$$

$$\mathbf{a} \cdot \mathbf{a} = 0 \iff \mathbf{a} = \mathbf{0} \qquad \text{positive-definite} \tag{9.2.8}$$

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \qquad \text{distributive} \tag{9.2.9}$$

Inequality relations The fact that $|\cos\theta| \leq 1$ gives,

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \qquad \text{Triangle inequality} \tag{9.2.10}$$

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}| \qquad \text{Cauchy-Schwraz inequality} \tag{9.2.11}$$

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \qquad \text{parallellogram equality} \tag{9.2.12}$$

Orthogonal projection The orthogonal projection (or component) of a vector along a line whose unit vector is $\hat{\mathbf{b}}$ is defined as

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} = |\mathbf{a}| \cos\theta \tag{9.2.13}$$

Here θ is the angle between the vector $\mathbf{a}$ and the line it is projected upon.

A plane can be defined in Cartesian 3d by requiring the projection of all points on it upon the normal vector to be constant.

$$\hat{\mathbf{n}} \cdot \mathbf{r} = p \tag{9.2.14}$$

Here, p is the distance of $\hat{\mathbf{n}}$ from the origin. All planes parallel to this plane simply vary in their p value.

9.3 Vector Product (Cross Product)

Cross product Geometrically, a vector orthogonal to the two given vectors whose magnitude is equal to the area of the parallellogram formed by them.

$$\mathbf{v} \equiv \mathbf{a} \times \mathbf{b} \tag{9.3.1}$$

- If $\mathbf{a} = 0$ or $\mathbf{b} = 0$, then by definition, $\mathbf{v} = \mathbf{0}$.
- Else if $\mathbf{a}$ and $\mathbf{b}$ are parallel, such that the angle between then is 0 or π, then again $\mathbf{v} = \mathbf{0}$

- Outside of these special cases, the magnitude of the cross product is

$$|\mathbf{v}| \equiv |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta \tag{9.3.2}$$

$$\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{9.3.3}$$

  Its direction is such that $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$ form a right-handed triple.

Right handed triple If the fingers of the right hand are on the plane formed by $\mathbf{a}$ and $\mathbf{b}$, then the right thumb points in the direction of $\mathbf{c}$.

This convention picks one of the two possible choices for the direction of $\mathbf{c}$.

Properties of cross product Similar to scalar multiplication,

$$(p\mathbf{a}) \times \mathbf{b} = p(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (p\mathbf{b}) \tag{9.3.4}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \tag{9.3.5}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \qquad \text{distributive} \tag{9.3.6}$$

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) \qquad \text{anti-commutative} \tag{9.3.7}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \qquad \text{not associative} \tag{9.3.8}$$

Scalar Triple product Also called the mixed product or box product. This is a scalar resulting from three input vectors.

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \tag{9.3.9}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{9.3.10}$$

Properties of box product Some important properties,

- Geometrically, it is the volume of the parallellopiped with the three vectors as the edge vectors.
- The dot and cross can be interchanged

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \tag{9.3.11}$$

- Three vectors in Cartesian $\mathcal{R}^3$ are L.I. if and only if their box product is nonzero.

9.4 Vector and Scalar Functions and Their Fields, Vector Calculus: Derivatives

Vector function Functions of scalars that produce vectors as outputs. Such a function defines a vector field in the domain of definiton.

$$\mathbf{v}(P) = \begin{bmatrix} v_1(P) \\ v_2(P) \\ v_3(P) \end{bmatrix} \qquad \mathbf{v}(x, y, z) = \begin{bmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{bmatrix} \tag{9.4.1}$$

A vector function may additionally depend on some other variable such as time.

Scalar function Functions of scalars that produce scalars as outputs. Such a function defines a scalar field in the domain of definiton.

$$f \equiv f(P)f \qquad \equiv f(x, y, z) \tag{9.4.2}$$

A scalar function may additionally depend on some other variable such as time.

The value of the scalar function is independent of the choice of coordinate system.

Vector calculus Many properties outlined here carry over from regular calculus.

Convergence A sequence of vectors $\{\mathbf{a}_k\}$ is said to converge if there exists some vector $\mathbf{a}$ such that

$$\lim_{n\to\infty} |\mathbf{a}_n - \mathbf{a}| = 0 \tag{9.4.3}$$

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a} \qquad \text{(Limit vector)} \tag{9.4.4}$$

The corresponding definition of the convergence of vector functions of a scalar are identical to the standard definition of the convergence of scalar functions $f(x)$ and omitted here.

Continuity If a vector function $\mathbf{v}(t)$ is defined in some neighbourhood of $t = t_0$ (including t_0 itself) and

$$\lim_{t\to t_0} \mathbf{v}(t) = \mathbf{v}(t_0) \tag{9.4.5}$$

then, the vector function is continuous at $t = t_0$.

Derivative of vector function The derivative of $\mathbf{v}(t)$ at $t = t_0$ exists, is the limit

$$\lim_{\Delta t\to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} \equiv \mathbf{v}'(t) \tag{9.4.6}$$

assuming this limit exists.

When a coordinate system is defined, each component of the vector function is differentiated separately to yield the derivative of the vector.

Properties of vector derivatives Similar to the properties of scalar derivatives,

$$(c\mathbf{v})' = c\,\mathbf{v}' \tag{9.4.7}$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}' \tag{9.4.8}$$

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' \tag{9.4.9}$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}' \tag{9.4.10}$$

$$(\mathbf{u}\ \mathbf{v}\ \mathbf{w})' = (\mathbf{u}'\ \mathbf{v}\ \mathbf{w}) + (\mathbf{u}\ \mathbf{v}'\ \mathbf{w}) + (\mathbf{u}\ \mathbf{v}\ \mathbf{w}') \tag{9.4.11}$$

Partial derivatives of vector function The operation to be applied to the vector function is simply applied to each component individually.

$$\mathbf{v}(\{t_k\}) \equiv \mathbf{v}(t_1, t_2, \dots, t_n) \tag{9.4.12}$$

$$\frac{\partial \mathbf{v}}{\partial t_1} = \frac{\partial v_1}{\partial t_1}\,\hat{\mathbf{i}} + \frac{\partial v_2}{\partial t_1}\,\hat{\mathbf{j}} + \frac{\partial v_3}{\partial t_1}\,\hat{\mathbf{k}} \tag{9.4.13}$$

$$\frac{\partial^2 \mathbf{v}}{\partial t_1\,\partial t_2} = \frac{\partial^2 v_1}{\partial t_1\,\partial t_2}\,\hat{\mathbf{i}} + \frac{\partial^2 v_2}{\partial t_1\,\partial t_2}\,\hat{\mathbf{j}} + \frac{\partial^2 v_3}{\partial t_1\,\partial t_2}\,\hat{\mathbf{k}} \tag{9.4.14}$$

9.5 Curves, Arc Length, Curvature, Torsion

Parametric vector functions Replacing the independent variable in each component of a vector by a common parameter (t) yields,

$$\mathbf{v}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}} \tag{9.5.1}$$

This makes the curve oriented in the direction of increasing t. (called the positive sense on the curce C)

Twisted curve A curve that does not lie in a plane in $3d$ space. Else, it is called a plane curve.

Simple curve A curve which does not touch or intersect itself. There is a unique value of the parameter t for every point of the curve,

Tangent vector The limiting position of a straight line through P on curve C and another close point Q

$$P : \mathbf{r}(t) \qquad\qquad Q : \mathbf{r}(t + \Delta t) \tag{9.5.2}$$

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \tag{9.5.3}$$

If $\mathbf{r} \neq \mathbf{0}$, it is called the tangent vector to curve C at point P.

Parametric straight line The equation of a line passing through $\mathbf{a}$ in the direction of $\mathbf{b}$ is given by

$$\mathbf{v} = \mathbf{a} + t\mathbf{b} \tag{9.5.4}$$

$$\mathbf{q}(w) = \mathbf{r} + w\mathbf{r}' \qquad \text{Tangent line} \tag{9.5.5}$$

Here, both $\mathbf{r}$ and $\mathbf{r}'$ are functions of the original parameter t.

Length of curve If a curve C is specified using a paramteric vector functionm $\mathbf{r}(t)$, then the length of a curve corresponding to $t \in [a, b]$ is given by

$$l = \int_a^b \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\ \mathrm{d}t \tag{9.5.6}$$

assuming $\mathbf{r}$ is differentiable.

Arc length of a curve The length of a curve from $t = a$ to a variable end point $t = w$. It is defined as

$$s(w) = \int_a^w \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\ \mathrm{d}t \tag{9.5.7}$$

Linear element From the definition of Pythagoras theorem, the linear element in $3d$ Cartesian space is

$$\mathrm{d}s^2 \equiv \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{9.5.8}$$

Trajectories In mechanics, the parameter is usually time (t) and a curve represents the path taken by an object through $3d$ space.

The first and second derivaties of the position $\mathbf{r}(t)$ represent the velocity vector and acceleration vector respectively.

$$\mathbf{v}(t) \equiv \mathbf{r}'(t) \qquad \mathbf{a}(t) \equiv \mathbf{r}''(t) \tag{9.5.9}$$

The acceleration can be split into tangential and normal components.

Defining the unit tangent vector in terms of the arc length s,

$$\mathbf{u}(s) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{r}'(s) \tag{9.5.10}$$

$$\mathbf{v}(t) = \mathbf{u}(s)\ \frac{\mathrm{d}s}{\mathrm{d}t} \tag{9.5.11}$$

$$\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{u}(\mathbf{s})\ \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + \mathbf{u}(s)\ \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \tag{9.5.12}$$

$$\tag{9.5.13}$$

Since the tangent vector $\mathbf{u}(s)$ is always a unit vector, the first acceleration term has to be perpendicular to the velocity.

Normal acceleration The acceleration vector can be split into two components, normal and tangential using the projection rule,

$$\mathbf{a}(t) = \mathbf{a}_{\text{norm}} + \mathbf{a}_{\text{tangent}} \qquad \mathbf{a}_{\text{tangent}} = \mathbf{a} \cdot \hat{\mathbf{v}} \tag{9.5.14}$$

Curvature The rate of change of the unit tangent vector at a point P on the curve C, give by

$$\kappa(s) \equiv \left|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{u}(s)\right| = \left|\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathbf{r}(s)\right| \tag{9.5.15}$$

Here, the arc length (s) is used as the parameter of the curve instead of the usual t.

Normal plane The plane through P on curve C whose normal vector is the tangent vector $\mathbf{u}$.

Osculating plane The plane spanned by the tangent vector and its derivative (which happens to be the normal vector at that point).

This plane contains the osculating (kissing) circle to the curve C at the point P. Its normal vector is $\mathbf{b} = \mathbf{u} \times \mathbf{n}$.

Rectifying plane The third plane which is spanned by $\mathbf{u}$ and $\mathbf{b}$. Its normal vector is the unit normal vector $\mathbf{u}' = \mathbf{p}$.

Torsion The rate of change of the osculating plane defined as,

$$\mathbf{p} \equiv \frac{1}{\kappa}\,\mathbf{u}' \qquad \text{unit principal normal vector} \tag{9.5.16}$$

$$\mathbf{b} \equiv \mathbf{u} \times \mathbf{p} \qquad \text{unit binormal vector} \tag{9.5.17}$$

$$|\tau(s)| = |\mathbf{b}'(s)| \tag{9.5.18}$$

$$\tau(s) = -\mathbf{p}(s) \cdot \mathbf{b}'(s) \tag{9.5.19}$$

Since the unit tangent vector $\mathbf{u}$ has constant magnitude, its derivate is orthogonal to itself by definition. The right handed triple of vectors $\mathbf{u}$, $\mathbf{p}$, $\mathbf{b}$ are defined at each point on the curve.

9.6 Calculus Review: Functions of Several Variables

Chain rule Consider a mapping from the domain B in the uv plane onto the domain D in xyz space using the mapping,

$$x \equiv x(u, v) \qquad y \equiv y(u, v) \qquad z \equiv z(u, v) \tag{9.6.1}$$

These functions are continuous and have continuous first partial derivatives in B.

Further, let $f \equiv f(x, y, z)$ be continouous and have continouous first partial derivatices in D. Then,

$$w = f\Big(x(u, v),\ \ y(u, v),\ \ z(u, v)\Big) \tag{9.6.2}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \tag{9.6.3}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \tag{9.6.4}$$

Mean Value theorem With $f(x, y, z)$ and D as defined above, and any two points P_0 and P connected by a straight line that lies entirely in D,

$$P_0 : (x_0, y_0, z_0) \qquad P : (x_0 + h, y_0 + k, z_0 + l) \tag{9.6.5}$$

$$f(x_0 + h, y_0 + k, z_0 + l) - f(x, y, z) = h\,\frac{\partial f}{\partial x} + k\,\frac{\partial f}{\partial y} + l\,\frac{\partial f}{\partial z} \tag{9.6.6}$$

This is a generalization of the much more familiar special case,

$$f \equiv f(x) \qquad\qquad f(x_0 + h) - f(x_0) = h\,\frac{\partial f}{\partial x} \tag{9.6.7}$$

9.7 Gradient of a Scalar Field, Directional Derivative

Gradient A vector function derived from a scalar function f in the $3d$ Cartesian space, defined by,

$$\nabla f \equiv \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}} \tag{9.7.1}$$

The differential operator for $3d$ Cartsian coordinates is defined as

$$\nabla \equiv \frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}\,\hat{\mathbf{k}} \tag{9.7.2}$$

Directional derivative The directional derivative of a function f at a point of P in the direction of $\mathbf{b}$,

$$D_{\mathbf{b}} f \equiv \frac{\mathrm{d}f}{\mathrm{d}s} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s} \tag{9.7.3}$$

Here, Q is a point on the line passing through P in the direction of $\mathbf{b}$, and s is the distance between these two points.

Defining the line in terms of a unit vector $\mathbf{b}$ and arc length s from an intitial point $\mathbf{p_0}$,

$$\mathbf{r} = \mathbf{p_0} + s\,\mathbf{b} \qquad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{b} \tag{9.7.4}$$

$$D_{\mathbf{b}} f = \frac{\partial f}{\partial s} = \hat{\mathbf{b}} \cdot \nabla f \tag{9.7.5}$$

Gradient is a vector The magnitude and direction of the gradient are independent of the choice of Cartesian coordinates.

Also, the gradient (if it is a nonzero vector) points in the direction of maximum increase of the function at that point.

Normal vector to surface For a function f whose level surface is defined as $S : f(x, y, z) = c$,

$$0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \qquad 0 = \nabla f \cdot \mathbf{r}' \tag{9.7.6}$$

Since the gradient is perpendicluar all possible tangent vectors at the point P on surface S, it is the surface normal vector at P.

Potentials A scalar field whose gradient happens to be a vector field.

$$\mathbf{v}(P) = \nabla f(P) \tag{9.7.7}$$

The corresponding vector field is called conservative.

Gravitational field THe force of gravitation between two particles is the best known exmaple of a conservative force.

For two particles at $P_0 : (x_0, y_0, z_0)$ and $P : (x, y, z)$ separated by distance r.

$$\mathbf{p} = \frac{-c}{r^3}\,\mathbf{r} = \frac{-c}{r^3}\begin{bmatrix} (x - x_0) \\ (y - y_0) \\ (z - z_0) \end{bmatrix} \qquad \phi(x, y, z) = \frac{c}{r} \tag{9.7.8}$$

$$\mathbf{p} = \nabla \phi \tag{9.7.9}$$

The vector field of force produced by any mass distribution in a space is the gradient of a scalar field (potential) ϕ which satisfies Laplace's equation in any region free of matter.

Laplace's equation A partial differential equation defined as

$$\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{9.7.10}$$

Laplacian The operator which condenses the Laplacian equation, defined as

$$\nabla \cdot \nabla f \equiv \nabla^2 f \tag{9.7.11}$$

$$\nabla^2 \equiv \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{9.7.12}$$

9.8 Divergence of a Vector Field

Divergence This obtains a scalar field from a vector field. Consider a differentiable vector function $\mathbf{v}$ with components in $3d$ Cartesian space, given by

$$\mathbf{v} = \begin{bmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{bmatrix} \tag{9.8.1}$$

$$\nabla \cdot \mathbf{v} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \tag{9.8.2}$$

Note that the dot product notation is not the usual multiplication. It represents the partial derivative acting on the components of $\mathbf{v}$.

Invariance of divergence Since the divergence is a scalar field, it is independent of the choice of coordinate system.

It only depends on the point P in space and the form of $\mathbf{v}$ for a particular choice of coordinate system.

Fluid flow An example of divergence used in physical systems is the flow of a fluid that has density ρ in a region R with no sources and sinks.

Let $\mathbf{v}$ be the velocity vector of the fluid.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{9.8.3}$$

This is the continuity equation of a compressible fluid flow

Solenoidal vector field A special case of the above system is steady flow, which means that ρ is independent of time, and constant fluid density. This gives the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0 \tag{9.8.4}$$

Such a vector field is called solenoidal.

9.9 Curl of a Vector Field

Curl Let $\mathbf{v}$ be a differentiable vector function in $3d$ defined in Cartesian coordinates,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{9.9.1}$$

This assumes the coordinate system is right handed.

The curl is a vector independent of the choice of coordinate system.

Curl of rotating body The curl is in the direction of the axis of rotation and its magnitude is equal to twice the angular speed.

Relating curl, grad and div A vector function that is the gradient of a continouous differentiable scalar function is irrotational.

$$\nabla \times (\nabla f) = 0 \qquad \text{Irrotational} \tag{9.9.2}$$

For a vector function $\mathbf{v}$ that is twice continuously differentiable,

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0 \tag{9.9.3}$$

These relations are proved using the fact that second order partial derivaties commute.

Chapter 10

Vector Integral Calculus, Integral Theorems

10.1 Line Integrals

Curve integral A generalization of the usual definite integral, which now integrates over a general one-dimensional curve in space, instead of the x axis.

$$I = \int_a^b f(x) \ \mathrm{d}x \tag{10.1.1}$$

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad t \in [a, b] \tag{10.1.2}$$

The path of integration is parametrized using a single parameter (t).

If the initial and terminal point of the curve coincide, it is called a closed path.

Smooth curve A curve which has a unique tangent defined at each point that varies continuously when traversing C.

$\mathbf{r}(t)$ is differentiable and $\mathbf{r}'(t)$ is continuous and not the zero vector at any point along C.

Line Integral Curve integrals are commonly called line integrals, even if the path itself is not a straight line. They are defined for some vector function $\mathbf{F}(\mathbf{r})$ over a curve C by,

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \ \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \ \mathrm{d}t \tag{10.1.3}$$

$$\int_C (F_1 \ \mathrm{d}x + F_2 \ \mathrm{d}y + F_3 \ \mathrm{d}z) = \int_a^b (F_1 x' + F_2 y' + F_3 z') \ \mathrm{d}t \tag{10.1.4}$$

The above definition requires right-handed $3d$ Cartesian coordinates.

The differentiation is w.r.t. the parameter t.

The line integral can be thought of as the sum of many infinitesimal vectors each of which is the tangential component of **F** at **r**. Thus,

$$\mathrm{d}I = \mathbf{F} \cdot \frac{\mathbf{r}'}{|\mathbf{r}'|} \tag{10.1.5}$$

The integrand is piecewise continuous if **F** is continuous and the curve C is at least piecewise smooth.

Properties of line integral Similar to the definite scalar integral,

$$\int_C k\mathbf{F} \cdot \mathrm{d}\mathbf{r} = k \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{10.1.6}$$

$$\int_C (\mathbf{F} + \mathbf{G}) \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} \tag{10.1.7}$$

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{10.1.8}$$

Splitting a path into pieces that have the same orientation does not change the line integral.

Invariance of line integral Any representation of the path C that preserve the orientation of the curve, do not change the line integral.

Work done by a force When the path is no longer a straight line, the total work done by a force is the sum of many small displacements along the path C

$$\mathrm{d}W = \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \qquad W = \int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \tag{10.1.9}$$

Vector line integral Without taking the dot product, the line integral can be evaluated separately for each component to give,

$$\int_C \mathbf{F}(\mathbf{r}) \ \mathrm{d}t = \int_a^b \mathbf{F}(\mathbf{r}(t)) \ \mathrm{d}t = \begin{bmatrix} \int_a^b F_1(\mathbf{r}(t)) \ \mathrm{d}t \\ \int_a^b F_2(\mathbf{r}(t)) \ \mathrm{d}t \\ \int_a^b F_3(\mathbf{r}(t)) \ \mathrm{d}t \end{bmatrix} \tag{10.1.10}$$

Path Dependence Generally, the path taken affects the outcome of the line integral even if the function $\mathbf{F}(\mathbf{r})$ and the endpoints remain the same.

10.2 Path Independence of Line Integrals

Path Independene A line integral is path independent in domain D if for all pairs of points (A, B) in this domain, the path integral has the same value regardless of the path taken from A to B.

Gradient of scalar function A line integral (with continouous vector components in the integrand) is path independent if and only if the vector happens to be the gradient of some scalar function in the same domain.

$$\mathbf{F} = \nabla f \qquad \Longleftrightarrow \qquad \int_A^B \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = f(B) - f(A) \tag{10.2.1}$$

This scalar function f is called the potential of the vector field $\mathbf{F}$.

Closed line integral A line integral is path independent if and only if its value around any possible closed path is zero.

Conservative field In physics, if the work done by a force $\mathbf{F}$ is path independent, then the vector field is called conservative. Else, it is called dissipative.

Exact differential form If the integrand $\mathbf{F} \cdot \, \mathrm{d}\mathbf{r}$ is equal to an infinitesimal change in some scalar function f,

$$\mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y + F_3 \, \mathrm{d}z \tag{10.2.2}$$

$$= \frac{\partial f}{\partial x} \, \mathrm{d}x + \frac{\partial f}{\partial y} \, \mathrm{d}y + \frac{\partial f}{\partial z} \, \mathrm{d}z = \, \mathrm{d}f \tag{10.2.3}$$

Here, $f(x, y, z)$ is some differentiable function in the domain D.

This reduces to the condition that the above equation is exact if and only if the vector function $\mathbf{F}$ is the gradient of the differentiable scalar function f everywhere in the domain D.

Simply connected domain Any closed curve in the domain D can be continuously shrunk to any point in D without leaving D.

Criterion for exactness Let the components of a vector function be continuous and have continuous first partial derivatives in a domain D. Then,

- If the differential form is exact in D then the vector field is irrotational.

$$\mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = \, \mathrm{d}f \qquad \Longrightarrow \qquad \nabla \times F = 0 \tag{10.2.4}$$

- Conversely, if the vector field $\mathbf{F}$ is irrotational in a simply connected domain D, then the line integral is path independent.

10.3 Calculus Review: Double Integrals

Area Double integration happens over any bounded region in $\mathcal{R}^2$ whose boundary curve has a unique tanget at all points.

The only exception is a finite number of cusps. (Discontinuity in the tangent vector over the boundary).

This is analogous to a regular integral having at most a finite number of jump discontinuities.

Definition of double integral Assuming some function $f(x, y)$ is continuous in $\mathcal{R}$ and this region is bounded by finitely many smooth curves, the integral over the area is

$$I \equiv \iint_R f(x, y) \, \mathrm{d}A = \iint_R f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{10.3.1}$$

Properties of double integral Similar to the definite single integral,

$$\iint_R kf \ \mathrm{d}A = k \iint_R f \ \mathrm{d}A \tag{10.3.2}$$

$$\iint_R (f+g) \ \mathrm{d}A = \iint_R f \ \mathrm{d}A + \iint_R g \ \mathrm{d}A \tag{10.3.3}$$

$$\iint_R (f+g) \ \mathrm{d}A = \iint_{R_1} f \ \mathrm{d}A + \iint_{R_2} f \ \mathrm{d}A \tag{10.3.4}$$

The area to be integrated over can be split into parts just like the line segment for the case of the single integral.

Mean value theorem If the area of integration R as defined above is also simply connected, then

$$\iint_R f(x, y) \ \mathrm{d}A = f(x_0, y_0) \ A \tag{10.3.5}$$

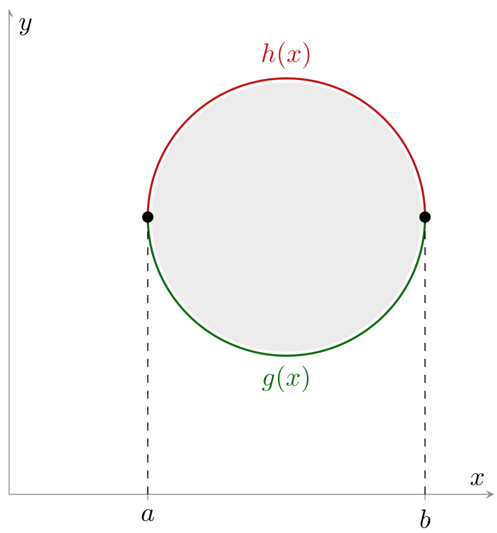
for some point (x_0, y_0) in R and A being the total area of R.

Evaluation Sucessively integrating over x then y is the most common means of performing double integration.

$$\iint_R f \ \mathrm{d}x \ \mathrm{d}y = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y) \ \mathrm{d}y \right] \mathrm{d}x \tag{10.3.6}$$

Here, x is considered a constant when performing the integration over y.

The order of integration can also be reversed if the computation happens to be simpler.



If the region R cannot be represented in the above form, it must at least be divisible into a finite number of areas that can.

Applications The double integral of the unit function is simply the area enclosed by the boundary curves.

$$\iint_R \mathrm{d}A = A_{\text{total}} \tag{10.3.7}$$

The volume beneath a surface $z = f(x, y) > 0$ and the xy plane is equal to its double integral over the area projected onto the xy plane.

$$\iint_R f(x, y) \ \mathrm{d}x \ \ \mathrm{d}y = V \tag{10.3.8}$$

This is analogous to the area between the x axis and a one dimensional curve being its definite integral.

Change of variables Starting with the single integral analog

$$\int_a^b f(x) \, \mathrm{d}x = \int_\alpha^\beta f\Big(x(u)\Big) \frac{\mathrm{d}x}{\mathrm{d}u} \, \mathrm{d}u \tag{10.3.9}$$

$$\iint_R f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \iint_{R^*} f\Big(x(u, v), \ \ y(u, v)\Big) \ \ |J| \ \ \mathrm{d}u \, \mathrm{d}v \tag{10.3.10}$$

Jacobian A determinant that supplies the partial derivatives of each of the old variables w.r.t. each of the new variables.

$$J = \begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \tag{10.3.11}$$

Assuming the functions $x(u, v)$ and $y(u, v)$ are continuous and have continuous first partial derivatives in some region R^* in the uv plane.

There has to be a bijective mapping from (u, v) in R^* to (x, y) in R. The Jacobian has the same sign throughout R^*

10.4 Green's Theorem in the Plane

Utility Converting a double integral over a region R into a line integral over its boundary in order to simplify computations.

Green' theorem For a closed region R in 2 dimensions bounded by a finite number of smooth curves. Call this boundary C.

Let $F_1(x, y)$ and $F_2(x, y)$ be continuous functions in R. Let the derivatives $\partial F_1/\partial y$ and $\partial F_2/\partial x$ also be continuous in a domain that is a superset of R.

$$\iint_R \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right) \ \mathrm{d}A = \oint_C F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y \tag{10.4.1}$$

$$\iint (\nabla \times \mathbf{F}) \cdot \mathbf{k} \ \ \mathrm{d}A = \oint_C \mathbf{F} \cdot \ \mathrm{d}\mathbf{r} \tag{10.4.2}$$

The region R is to the left when moving along its boundary in order to preserve orientation. (in accordance with the direction of cross product)

Proof General proof is complex and not covered here. For a specific kind of region R which can be represented by boundary curves in x or y,

$$\iint_R \frac{\partial F_1}{\partial y} \; \mathrm{d}A = \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} \; \mathrm{d}y \right] \; \mathrm{d}x \tag{10.4.3}$$

$$= \int_a^b \left[F_1\big(x,\, v(x)\big) - F_1\big(x,\, u(x)\big) \right] \mathrm{d}x \tag{10.4.4}$$

$$= -\int_a^b F_1\big(x,\, u(x)\big) - \int_b^a F_1\big(x,\, v(x)\big) = -\oint_C F_1\big(x,\, y\big) \tag{10.4.5}$$

Since $u(x)$ and $v(x)$ both travel from $x = a$ to $x = b$, the above relation is a round trip starting and ending at $x = a$, which reduces it to the closed line integral needed.

A similar procedure for the closed region traversed from $y = c$ to $y = d$, via the two curves $p(y)$ and $q(y)$ gives the other half of the proof.

Area of region Using Green's theorem to bypass the double integration,

$$\iint_R \mathrm{d}A = \frac{1}{2} \oint_C (x \; \mathrm{d}y) - (y \; \mathrm{d}x) \qquad \text{cartesian} \tag{10.4.6}$$

$$= \frac{1}{2} \oint_C r^2 \; \mathrm{d}\theta \qquad \text{polar} \tag{10.4.7}$$

Normal derivative Consider a scalar functionm $w(x,\, y)$ that is continuous and has continuous first partial derivatives in the region R.

$$\mathbf{r}' = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \qquad \mathbf{r}' \cdot \mathbf{n} = 0 \tag{10.4.8}$$

Here, the parameter s is the arc length of the boundary curve. This ensures that the derivative $\mathbf{r}'$ is the unit tangent vector.

The outward facing vector perpendicular to $\mathbf{r}'$ is the unit normal vector $\mathbf{n}$.

The component of the gradient along the unit normal vector is called the normal derivative.

$$\nabla w \cdot \mathbf{n} = \frac{\partial w}{\partial n} \tag{10.4.9}$$

Relating Laplacian and normal derivative Using the substitutions,

$$F_1 = -\frac{\partial w}{\partial y} \qquad F_2 = \frac{\partial w}{\partial x} \tag{10.4.10}$$

$$\iint_R \nabla^2 w \;\; \mathrm{d}A = \oint_C \frac{\partial w}{\partial n} \;\; \mathrm{d}s \tag{10.4.11}$$

10.5 Surfaces for Surface Integrals

Parametric representation Similar to curves, surfaces can be parametrized using two parameters (u, v) to give

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv \begin{bmatrix} x(u, v) \\ y(y, v) \\ z(u, v) \end{bmatrix} \tag{10.5.1}$$

This maps every point (u, v) in a region R of the uv plane onto the surface S in $\mathcal{R}^3$.

Tangent Plane Consider one possible curve C on surface S that passes through a given point P.

Since the curve can be parametrized using a single parameter (t), the chain rule gives,

$$u, \;\; v \equiv u(t), \;\; v(t) \qquad\qquad \tilde{\mathbf{r}}(t) \equiv \mathbf{r}\Big(u(t), \;\; v(t)\Big) \tag{10.5.2}$$

$$\frac{\mathrm{d}\tilde{\mathbf{r}}}{\mathrm{d}t} = \tilde{\mathbf{r}}' = \frac{\partial \mathbf{r}}{\partial u}\, u' + \frac{\partial \mathbf{r}}{\partial v}\, v' \qquad\qquad \tilde{\mathbf{r}}' = \mathbf{r}_u\, u' + \mathbf{r}_v\, v' \tag{10.5.3}$$

Here the functions $u(t)$ and $v(t)$ are continuous and have continuous derivatives w.r.t. t.

Surface Normal vector The partial derivatives are assumed L.I. and thus span the tangent plane at P. Their cross product becomes the normal to the surface at P

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0} \tag{10.5.4}$$

Using the fact that the gradient of a level curve at a given point is the surface normal vector,

$$g(x, y, z) = 0 \quad \implies \quad \mathbf{N} = \nabla g \tag{10.5.5}$$

Smooth surface A surface whose normal vector depends continuously on the points of the surface.

At best, the surfaces encountered in practical applications can have a finite number of smooth portions.

10.6 Surface Integrals

Parametrized surface integral If the surface S is parametrized using the parameters u, vwhich lie in region R of the uv plane, then,

$$S: \;\; \mathbf{r}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix} \tag{10.6.1}$$

This surface is piecewise smooth with a normal vector defined at every point.

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \qquad \hat{\mathbf{n}} = \frac{\mathbf{N}}{|\mathbf{N}|} \tag{10.6.2}$$

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}A = \iint_R \mathbf{F}(\mathbf{r}) \cdot \mathbf{N} \ \mathrm{d}u \, \mathrm{d}v \tag{10.6.3}$$

The area element of the actual surface $\hat{\mathbf{n}} \ \mathrm{d}A$ maps on to the area element $|\mathbf{N}| \ \mathrm{d}u \, \mathrm{d}v$ of the parameter plane R.

Flux In physics, the flux through a surface is the amount of some physical quantity (mass, heat) passing through the surface.

It can mathematically be represented as the component of a vector field normal to the surface.

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \ \mathrm{d}A = \iint_S F_1 \ \mathrm{d}y \, \mathrm{d}z + F_2 \ \mathrm{d}x \, \mathrm{d}z + F_3 \ \mathrm{d}x \, \mathrm{d}y \tag{10.6.4}$$

Orientation of Surfaces Changing the orientation of the surface (by choosing $-\mathbf{n}$ instead of $\mathbf{n}$), the value of the surface integral gets multiplied by -1.

Non-Orientable surface Usually surfaces which have a positive direction of the normal vector at a given point P also have the same positive direction of the normal vector at all other points that can be reached by smoothly translating P along the surface.

Surfaces like the Mobius strip are non-oreintable, because they do not satisfy this condition.

Surface Integral disregarding orientation Consider a scalar function of position $G(\mathbf{r})$ to be integrated over a surface,

$$\iint_S G(\mathbf{r}) \ \mathrm{d}A = \iint_R G(\mathbf{r}) \ |\mathbf{N}| \ \ \mathrm{d}u \, \mathrm{d}v \tag{10.6.5}$$

Here, the normal vector and position vector are both parametrized using (u, v).

One application is to find the mass of a surface, in which case the scalar function $G(\mathbf{r})$ is its area density.

The total area of a surface A can be found using

$$A = \iint_S \mathrm{d}A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \ \ \mathrm{d}u \, \mathrm{d}v \tag{10.6.6}$$

Parametrization using coordinates If the parameters chosen happen to be the x and y coordinates themselves, then,

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix} \tag{10.6.7}$$

$$\iint_S G(\mathbf{r}) \ \mathrm{d}A = \iint_{R^*} G(x, y, f) \left[\sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\right] \ \mathrm{d}x \, \mathrm{d}y \tag{10.6.8}$$

Here the parameter plane R^* is the xy plane and geometrically is the projection of the surface S onto the xy plane.

By convention the normal vector points upwards away from the xy plane.

10.7 Triple Integrals, Divergence Theorem of Gauss

Triple Integral An integral over a closed, bounded $3d$ region in space. This volume is bounded by finitely many smooth surfaces.

$$I = \iiint_T f(x, y, z) \ \mathrm{d}V \tag{10.7.1}$$

Gauss' Divergence theorem Triple integrals over a volume can be transformed into double integrals over the boundary surface using the divergence of the position vector,

$$\mathbf{r} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \qquad \nabla \cdot \mathbf{r} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{10.7.2}$$

For some continuous vector function $\mathbf{F}$ in the region T which is continuous and has continuous first partial derivatives in some domain containing T,

$$\iiint_T \nabla \cdot F \ \mathrm{d}V = \iint_S \mathbf{F} \cdot \mathbf{n} \ \mathrm{d}S \tag{10.7.3}$$

$$\iiint_T \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}z = \iint_S F_1 \ \mathrm{d}y \ \mathrm{d}z + F_2 \ \mathrm{d}x \ \mathrm{d}z + F_3 \ \mathrm{d}x \ \mathrm{d}y \tag{10.7.4}$$

10.7.5

Co-ordinate Invariance of divergence Since the divergence is a scalar function of the position P in space, it is invariant under change of co-ordinate system.

$$\nabla \cdot \mathbf{F}(P) = \lim_{d(T) \to 0} \frac{1}{V(T)} \iint_{S(T)} \mathbf{F} \cdot \mathbf{n} \ \mathrm{d}A \tag{10.7.6}$$

Here, $V(T)$ is the volume of the region in space T.

$S(T)$ is the boundary surface of the region T. $d(T)$ is the distance of the points in T from a specific point P chosen in T to satisfy the mean value theorem for triple integrals.

10.8 Further Applications of the Divergence Theorem

Fluid flow Consider an incompressible fluid with density $\rho = 1$, with steady flow that does not varyin time.

If the outward normal vector is $\mathbf{n}$ and the fluid flow is characterized by the velocity field $\mathbf{v}$, then

$$\iint_S \mathbf{v} \cdot \mathbf{n} \ \mathrm{d}A \tag{10.8.1}$$

represents the total mass of fluid flowing out of the region T bounded by the surface S.

There are no sources or sinks in a region T if and only if $\nabla \cdot \mathbf{v} = 0$ everywhere in T.

Heat equation Since heat flows in the direction of decreasing temperature at a rate proportional to the gradient,

$$\mathbf{v} = -K \ \nabla U \qquad\qquad \nabla \cdot \mathbf{v} = -K \ \nabla^2 U \tag{10.8.2}$$

where U is the temperature, t is the time and K is the thermal conductivity of the body.

Equating the rate of decrease of heat of the region T to the total heat flowing out of the bounding surface S,

$$\frac{\partial U}{\partial t} = c^2 \ \nabla^2 U \tag{10.8.3}$$

where c is the thermal diffusivity of the material. This equation is also called the diffusion equation.

Potential theory Looking at solutions of Laplace's equation,

$$\nabla^2 f = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{10.8.4}$$

Any solution of Laplace's equation f with continuous second-order partial derivatives is called a harmonic function.

Solutions of Laplace's equation Using the definition of directional derivative to define the normal derivative in the direction of the outward normal,

$$\frac{\partial f}{\partial n} \equiv \nabla f \cdot \mathbf{n} \tag{10.8.5}$$

If the underlying scalar function f is such that $\mathbf{F} = \nabla f$,

$$\iiint_T \nabla^2 f \ \mathrm{d}V = \iint_S \frac{\partial f}{\partial n} \ \mathrm{d}A \tag{10.8.6}$$

If f is a harmonic function then the integral of the normal derivative over the bounding surface S of some region in space T is zero.

Green's first formula A special case of Green's theorem when the vector function $\mathbf{F}$ is

$$\mathbf{F} = f \ \nabla g \qquad\qquad \nabla \cdot \mathbf{F} = f \ \nabla^2 g + (\nabla f) \cdot (\nabla g) \tag{10.8.7}$$

Substituting into Green's theorem gives,

$$\iiint_T \left[f\ \nabla^2 g + (\nabla f)\cdot(\nabla g)\right]\ \mathrm{d}V = \iint_S\ f\ \frac{\partial g}{\partial n}\ \mathrm{d}A \qquad 10.8.8$$

Green's second formula An even more special case of Green's theorem, using the symmetry in Green's first formula upon interchange of f and g.

$$\iiint_T \left[f\ \nabla^2 g - g\ \nabla^2 f\right]\ \mathrm{d}V = \iint_S \left(f\ \frac{\partial g}{\partial n} - g\ \frac{\partial f}{\partial n}\right)\ \mathrm{d}A \qquad 10.8.9$$

Uniqueness of Harmonic functions Let f be harmonic in some domain D and equal to zero at every point of the bounding surface S of a region in space T as defined above.

Then, f is identically zero in T

This harmonic function f is uniquely determined in T by its values on the bounding surface S

10.9 Stokes' Theorem

Stokes' theorem A generalization of Green's theorem in the plane to the full $3d$ space, using the curl.

Consider a surface S whose bounding curve is parametrized using the arc length s, with the outward normal unit vector being $\mathbf{n}$

$$\iint_R (\nabla\times\mathbf{F})\cdot\mathbf{n}\ \ \mathrm{d}A = \oint_C\ \mathbf{F}\cdot\mathbf{r}'(s)\ \ \mathrm{d}s \qquad 10.9.1$$

The unit tangent vector $\mathbf{r}'$ is differentiated w.r.t. the arc length s.

The orientation of the curve is the same convention as for the cross product.

$$\mathbf{n}\ \ \mathrm{d}A = \mathbf{N}\ \ \mathrm{d}u\ \mathrm{d}v \qquad 10.9.2$$

$$\mathbf{r}'\ \ \mathrm{d}s = \ \mathrm{d}x\hat{\mathbf{i}} + \ \mathrm{d}y\hat{\mathbf{j}} + \ \mathrm{d}z\hat{\mathbf{k}} \qquad 10.9.3$$

In terms of the xyz coordinate system,

$$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) N_3\right]\ \mathrm{d}u\ \mathrm{d}v \qquad 10.9.4$$

$$= \oint_C\ F_1\ \ \mathrm{d}x + F_2\ \ \mathrm{d}y + F_3\ \ \mathrm{d}z \qquad 10.9.5$$

Relation to Green's theorem Green's theorem in the plane is a special case of Stokes' theorem,

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix} \qquad \nabla \times \mathbf{F} = \begin{bmatrix} -\partial_z F_2 \\ \partial_z F_1 \\ \partial_x F_2 - \partial_y F_1 \end{bmatrix} \tag{10.9.6}$$

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{r}' = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ 0 \end{bmatrix} \tag{10.9.7}$$

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \oint_C F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y \tag{10.9.8}$$

Physical meaning of curl Consider a disk of radius r_0 whose boundary is C_0 and the enlosed area is S_0,

$$\oint_{C_0} \mathbf{v} \cdot \mathbf{r}' \, \mathrm{d}s = \iint_{S_0} (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, \mathrm{d}A \tag{10.9.9}$$

$$= \Big[(\nabla \times \mathbf{v}) \cdot \mathbf{n} \Big]_Q A_0 \tag{10.9.10}$$

This is the $2d$ analog of the mean value theorem with the area of the disk A_0 and Q being some point in the area for which the equality holds.

$$\Big[(\nabla \times \mathbf{v}) \cdot \mathbf{n} \Big]_Q = \lim_{r_0 \to 0} \frac{1}{A_0} \oint_{C_0} \Big(\mathbf{v} \cdot \mathbf{r}' \Big) \, \mathrm{d}s \tag{10.9.11}$$

If $\mathbf{v}$ is the velocity vector, then the component of the curl in the outward normal direction can be visualized as a measure of the circulation of the fluid flow around the point Q.

Path Independence The proof for path independence of a line integral is straightforward using Stokes' theorem.

Starting with the fact that the curl is zero, which means that the vector function is the divergence of a scalar field, Stokes' theorem makes the line integral over any closed path identically zero.

Chapter 11

Fourier Analysis

11.1 Fourier Series

Periodic function A function defined on the real line, except possibly at a finite number of points such that,

$$f(x+p) = f(x) \qquad \forall\ x \in \mathcal{R} \tag{11.1.1}$$

$$f(x+np) = f(x) \qquad \forall\ n \in \mathcal{I} \tag{11.1.2}$$

for some positive real p, which is called the period.

The smallest positive period is called the fundamental period.

Trigonometric system A family of periodic functions all having period 2π which are the simplest basis used to represent all periodic functions.

$$T = \{1,\ \cos x,\ \sin x,\ \cos(2x),\ \sin(2x), \dots,\ \cos(nx),\ \sin(nx), \dots\} \tag{11.1.3}$$

Fourier Series A periodic function si represented as a linear combination of the above basis with each member assigned a coefficient (called the Fourier coefficient).

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{11.1.4}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ \mathrm{d}x \tag{11.1.5}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \cos(nx)\ \mathrm{d}x \tag{11.1.6}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ \sin(nx)\ \mathrm{d}x \tag{11.1.7}$$

Orthogonality of a trigonometric system If integral over one period (taken to be $-\pi$ to π by convention) of the product of any two members of the basis is zero, then the basis is orthogonal.

$$\int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\ \mathrm{d}x = 0 \qquad n \neq m \tag{11.1.8}$$

$$\int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\ \mathrm{d}x = 0 \qquad n \neq m \tag{11.1.9}$$

$$\int_{-\pi}^{\pi} \cos(nx)\ \sin(mx)\ \mathrm{d}x = 0 \qquad n, m \in \mathcal{I} \tag{11.1.10}$$

The Euler formulas for the Fourier coefficients are derived from the application of the orthogonality condition to the Fouier series definition.

Convergence of Fourier series Let $f(x)$ be periodic with period 2π and be piecewise continuous in $[-\pi, \pi]$. Also, let it have both left-handed and right-handed derivatives defined everywhere in this interval.

Then, its Fourier series converges and is equal to $f(x)$ at all points except at the finitely many points of discontinuity of $f(x)$.

At such points, the series converges to the average of the left and right-handed limits of $f(x)$.

11.2 Arbitrary Period, Even and Odd Functions, Half-Range Expansions

Functions with arbitrary period Instead of the standard period $p = 2\pi$, generalizing to $p = 2L$ simply involves a change of variable

$$2\pi \to 2L \implies x \to \frac{\pi x}{L} \tag{11.2.1}$$

$$f(x) \to a_0 + \sum_{n=0}^{\infty} \left[a_n\ \cos\left(\frac{n\pi x}{L}\right) + b_n\ \sin\left(\frac{n\pi x}{L}\right)\right] \tag{11.2.2}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\ \mathrm{d}x \tag{11.2.3}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x)\ \cos\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \tag{11.2.4}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x)\ \sin\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \tag{11.2.5}$$

Even and Odd functions Even functions and odd functions can be represented just by a Fourier cosine and sine series respectively.

$$f(-x) = f(x) \implies b_n = 0 \qquad \text{even} \tag{11.2.6}$$

$$f(-x) = -f(x) \implies a_0 = a_n = 0 \qquad \text{odd} \tag{11.2.7}$$

Lineraity The Fourier series is linear under addition and scalar multiplication.

$$f(x) + g(x) \to F(x) + G(x) \tag{11.2.8}$$

$$c\ f(x) \to c\ F(x) \tag{11.2.9}$$

Where the uppercase is the fourier series expansion of the lowercase function.

Half-range expansions Extending the function specified in the domain $[0, L]$ into the domain $[-L, L]$ as either an even or odd function in order to simplify the computation of Fourier coefficients.

11.3 Forced Oscillations

Standard form ODE A second order linear ODE is very common in physical systems undergoing forced damped oscillations.

$$my'' + cy' + ky = r(t) \tag{11.3.1}$$

Here, the output $y(t)$ is the solution to the ODE, corresponding to the input $r(t)$. The constant coefficients m, c, k characterize the system.

Solution to ODE Since the input can be represented as a Fourier series, the output can also be decomposed into a sum of outputs corresponding to each input term.

The amplitude of each of the output terms happens to be a function of the input frequency and typically, one or two frequencies dominate the output in most real world systems.

11.4 Approximation by Trigonometric Polynomials

Trigonometric polynomial An approximation of a function using a series of trigonometric functions. The fourier series expansion happens to be an example.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \Big[A_n \cos(nx) + B_n \sin(nx)\Big] \tag{11.4.1}$$

Here, N is called the order.

Square Error The error in this approximation is

$$E = \int_{-\pi}^{\pi} (f - F)^2 \ \mathrm{d}x \tag{11.4.2}$$

This error is a measure of the agreement between the approximation f and the actual function F over the entire period of the trigonometric function.

Minimum square error The squared error of a trigonometric polynomial F with fixed order N is smallest in the domainm $[-\pi, \pi]$ if the coefficients are the Fourier coefficients $A_n = a_n$, $B_n = b_n$

$$E^* = \int_{-\pi}^{\pi} f^2 \ \mathrm{d}x - \pi \left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right] \tag{11.4.3}$$

The above squared error can only decrease with increasing N.

Bessel's Inequality Given a function f and fourier coefficients $a_0, \{a_n\}, \{b_n\}$, the condition on minimized squared error gives,

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \ \mathrm{d}x \tag{11.4.4}$$

Parseval's identity The integral of the square of a function over its standardized period $[-\pi, \pi]$ is equal to the sum of the squares of its Fourier coefficients. (rough analog of the Pythagoras theorem)

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \ \mathrm{d}x \tag{11.4.5}$$

11.5 Sturm-Liouville Problems, Orthogonal Functions

Sturm-Liouville problem A second order ODE, along with boundary conditions of the form,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x) \frac{\mathrm{d}y}{\mathrm{d}x}\right] + \left[q(x) + \lambda r(x)\right] y = 0 \tag{11.5.1}$$

$$k_1 y + k_2 y' = 0 \qquad \text{at } x = a \tag{11.5.2}$$

$$l_1 y + l_2 y' = 0 \qquad \text{at } x = b \tag{11.5.3}$$

for some interval $x \in [a, b]$. λ is a paramter and the two k, l are real constants.

If p, q, r, p' are real valued and continuous in the interval $[a, b]$ and r is the same sign throughout, then all eigenvalues λ of the Sturm-Liouville equation are real.

Orthogonal functions Using a weight function $r(x) > 0$, two functions are orthogonal in the interval $x \in [a, b]$ if,

$$(y_m, y_n) = \int_a^b r(x) \ y_m(x) \ y_n(x) \ \mathrm{d}x = 0 \qquad \forall \quad m \neq n \tag{11.5.4}$$

Norm of function The square integral of the function $f(x)$ with respect to the weight function $r(x)$.

$$\|y_n\| = \sqrt{(y_n, y_n)} = \sqrt{\int_a^b r(x) \ y_n^2(x) \ \mathrm{d}x} \tag{11.5.5}$$

Orthonormal functions A set of functions that are orthogonal in some interval $x \in [a, b]$ and additionally all have unit norm. Using the Kronecker Delta functions,

$$(y_m, y_n) = \int_a^b r(x)\ y_m(x)\ y_n(x)\ \ \mathrm{d}x = \delta_{mn} \tag{11.5.6}$$

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{11.5.7}$$

The weight function is not mentioned when it is the identically equal to 1.

Eigenfunctions of Sturm-Lioville problems Let $y_m(x)$ and $y_n(x)$ be eigenfunctions that correspond to different eigenvalues λ_m and λ_n of the Sturm-Liouville problem.

Then, y_m, y_n are orthogonal on the interval $[a, b]$ with respect to their weight function $r(x)$

If $p(a) = p(b)$, then the boundary conditions also become periodic,

$$y(a) = y(b) \qquad\qquad y'(a) = y'(b) \tag{11.5.8}$$

Orthogonal system Many real world systems can be cast into Sturm-Liouville form leading to a set of orthonormal basis functions. Examples include the Bessel functions, Legendre polynomials and Fourier series expansions.

Chapter 12

Fourier Analysis

12.1 Forced Oscillations

1. Deriving the terms C_n,

$$A_n = \frac{1}{n\pi D_n}\left[\frac{4(25-n^2)}{n}\right] \qquad B_n = \frac{1}{n\pi D_n}\left[0.2\right] \tag{12.1.1}$$

$$C_n = \sqrt{A_n^2 + B_n^2} \qquad = \frac{1}{n\pi D_n}\sqrt{\frac{(25-n^2)^2 + (0.05n)^2}{n^2/16}} \tag{12.1.2}$$

$$= \frac{4}{n^2\pi\sqrt{D_n}} \tag{12.1.3}$$

2. The effect of changing k is,

$$C_n \propto \frac{1}{\sqrt{D_n}} \qquad D_n = (k-n^2)^2 + (cn)^2 \tag{12.1.4}$$

$$\tag{12.1.5}$$

The maximum in amplitude shifts from $n = 5$ to $n = 7$, when $k = 7^2$.

The amplitude goes down as k increases, and as c increases.

3. The effect of c is to prevent the output being a pure cosine series by introducing sine terms proportional to the damping.

$$B_n \propto c \qquad c \to 0 \implies B_n \to 0 \tag{12.1.6}$$

$$C_n \to A_n \tag{12.1.7}$$

In the limit of very large c, $B_n \gg A_n$ and the output is completely out of phase with the input.

4. The derivative of the input is,

$$r'(t) = \frac{-4}{n\pi}\ \sin(nt) = \lambda \sin(nt) \qquad\qquad C_n = \frac{\lambda}{\sqrt{D_n}} \tag{12.1.8}$$

$$C_{\text{new}} = n\ C_{\text{old}} \tag{12.1.9}$$

Differentiation leads to the amplitude C_n multiplied by a factor of n.

5. The fact that the driving frequency being larger than the resonant frequency makes the output the opposite phase as the input is reflected in those A_n terms being negative.

No such effect happens as a result of the damping, which means that the B_n terms always remain positive.

6. Solving the ODE,

$$r(t) = \sin(\alpha t) + \sin(\beta t) \qquad\qquad \omega^2 \neq \alpha^2, \beta^2 \tag{12.1.10}$$

$$y'' + \omega^2 y = r(t) \tag{12.1.11}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = A_1 \cos(\alpha t) + A_2 \sin(\alpha t) + B_1 \cos(\beta t) + B_2 \sin(\beta t) \tag{12.1.12}$$

$$[\cos(\alpha t)] \qquad 0 = (-\alpha^2 + \omega^2)A_1 \tag{12.1.13}$$

$$[\sin(\alpha t)] \qquad 1 = (-\alpha^2 + \omega^2)A_2 \tag{12.1.14}$$

$$[\cos(\beta t)] \qquad 0 = (-\beta^2 + \omega^2)B_1 \tag{12.1.15}$$

$$[\sin(\beta t)] \qquad 1 = (-\beta^2 + \omega^2)B_2 \tag{12.1.16}$$

$$y_p = \frac{\sin(\alpha t)}{\omega^2 - \alpha^2} + \frac{\sin(\beta t)}{\omega^2 - \beta^2} \tag{12.1.17}$$

Finding the general solution, by solving the h-ODE,

$$y'' + \omega^2 y = 0 \tag{12.1.18}$$

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{12.1.19}$$

$$y = y_h + y_p \tag{12.1.20}$$

7. Solving the ODE,

$$r(t) = \sin(t) \qquad\qquad \omega^2 \neq 1 \tag{12.1.21}$$

$$y'' + \omega^2 y = r(t) \tag{12.1.22}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = A_1 \cos(t) + A_2 \sin(t) \tag{12.1.23}$$

$$[\cos(t)] \qquad 0 = (-1 + \omega^2) A_1 \tag{12.1.24}$$

$$[\sin(t)] \qquad 1 = (-1 + \omega^2) A_2 \tag{12.1.25}$$

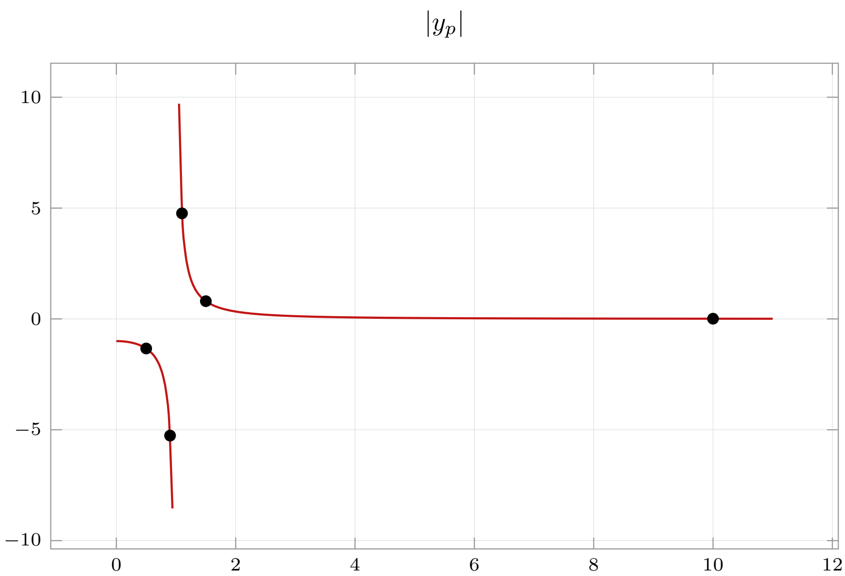
$$y_p = \frac{\sin(t)}{\omega^2 - 1} \tag{12.1.26}$$

Finding the general solution, by solving the h-ODE,

$$y'' + \omega^2 y = 0 \tag{12.1.27}$$

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{12.1.28}$$

$$y = y_h + y_p \tag{12.1.29}$$



8. Finding the Fourier series representation of the input

$$r(t) = \frac{\pi}{4} \quad |\cos t| \qquad\qquad \forall \quad x \in [-\pi, \pi] \tag{12.1.30}$$

$$p = 2L = 2\pi \tag{12.1.31}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \;\; \mathrm{d}x = \frac{1}{4} \int_0^{\pi} |\cos x| \quad \mathrm{d}x \tag{12.1.32}$$

$$= \frac{1}{4} \left[\sin x \right]_0^{\pi/2} + \frac{1}{4} \left[\sin x \right]_{\pi}^{\pi/2} = \frac{1}{2} \tag{12.1.33}$$

Finding the cosine coefficients

$$a_1 = \frac{1}{2}\int_0^{\pi/2} (\cos^2 x)\ \mathrm{d}x + \frac{1}{2}\int_{\pi}^{\pi/2} (\cos^2 x)\ \mathrm{d}x \tag{12.1.34}$$

$$= \frac{1}{4}\left[x + \frac{\sin(2x)}{2}\right]_0^{\pi/2} + \frac{1}{4}\left[x + \frac{\sin(2x)}{2}\right]_{\pi}^{\pi/2} = 0 \tag{12.1.35}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos(nx)\ \mathrm{d}x \tag{12.1.36}$$

$$= \frac{1}{2}\int_0^{\pi/2} (\cos x)\cos(nx)\ \mathrm{d}x + \frac{1}{2}\int_{\pi}^{\pi/2} (\cos x)\cos(nx)\ \mathrm{d}x \tag{12.1.37}$$

$$= \frac{1}{4}\left[\frac{\sin[(1+n)x]}{1+n} + \frac{\sin[(1-n)x]}{1-n}\right]_0^{\pi/2} \tag{12.1.38}$$

$$+ \frac{1}{4}\left[\frac{\sin[(1+n)x]}{1+n} + \frac{\sin[(1-n)x]}{1-n}\right]_{\pi}^{\pi/2} = \frac{\cos(n\pi/2)}{1-n^2} \tag{12.1.39}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = C + A_n\cos(nt) + B_n\sin(nt) \tag{12.1.40}$$

$$\omega^2 C = \frac{1}{2} \tag{12.1.41}$$

$$\frac{\cos(n\pi/2)}{1-n^2} = (-n^2 + \omega^2)A_n \tag{12.1.42}$$

$$0 = (-n^2 + \omega^2)B_n \tag{12.1.43}$$

$$y_p = \frac{1}{2\omega^2} + \sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{(1-n^2)(\omega^2 - n^2)}\ \cos(nt) \tag{12.1.44}$$

Finding the general solution, by solving the h-ODE,

$$y'' + \omega^2 y = 0 \tag{12.1.45}$$

$$y_h = C_1\cos(\omega t) + C_2\sin(\omega t) \tag{12.1.46}$$

$$y = y_h + y_p \tag{12.1.47}$$

9. In Problem 8, even numbers for n give nonzero terms in the expansion of y_p, which can have zero in the denominator.

This means that no steady state solution exists for even number ω.

10. Finding the Fourier series representation of the input

$$r(t) = \frac{\pi}{4} \; |\sin x| \qquad \forall \quad x \in [-\pi, \pi] \tag{12.1.48}$$

$$p = 2L = 2\pi \tag{12.1.49}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \; \mathrm{d}x = \frac{1}{4} \int_0^{\pi} |\sin x| \; \mathrm{d}x \tag{12.1.50}$$

$$= \frac{1}{4} \Big[- \cos x \Big]_0^{\pi} = \frac{1}{2} \tag{12.1.51}$$

Finding the cosine coefficients

$$a_1 = \frac{1}{2} \int_0^{\pi} (\sin x) \cos(x) \; \mathrm{d}x \tag{12.1.52}$$

$$= \left[\frac{- \cos(2x)}{8} \right]_0^{\pi} = 0 \tag{12.1.53}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \; \cos(nx) \; \mathrm{d}x = \frac{1}{2} \int_0^{\pi} (\sin x) \cos(nx) \; \mathrm{d}x \tag{12.1.54}$$

$$= -\frac{1}{4} \left[\frac{\cos[(1+n)x]}{1+n} + \frac{\cos[(1-n)x]}{1-n} \right]_0^{\pi} \tag{12.1.55}$$

$$= \frac{1}{2(1-n^2)} \; [\cos(n\pi) + 1] \tag{12.1.56}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = C + A_n \cos(nt) + B_n \sin(nt) \tag{12.1.57}$$

$$\omega^2 C = \frac{1}{2} \tag{12.1.58}$$

$$\frac{1 + \cos(n\pi)}{2(1-n^2)} = (-n^2 + \omega^2) A_n \tag{12.1.59}$$

$$0 = (-n^2 + \omega^2) B_n \tag{12.1.60}$$

$$y_p = \frac{1}{2\omega^2} + \sum_{n=2}^{\infty} \frac{1 + \cos(n\pi)}{2(1-n^2)(\omega^2 - n^2)} \; \cos(nt) \tag{12.1.61}$$

Finding the general solution, by solving the h-ODE,

$$y'' + \omega^2 y = 0 \tag{12.1.62}$$

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{12.1.63}$$

$$y = y_h + y_p \tag{12.1.64}$$

11. Finding the Fourier series representation of the input

$$r(t) = \begin{cases} -1 & x \in [-\pi, 0] \\ 1 & x \in [0, \pi] \end{cases} \tag{12.1.65}$$

$$p = 2L = 2\pi \qquad |\omega| \neq 1, 3, 5, \ldots \tag{12.1.66}$$

$$a_0 = 0 \tag{12.1.67}$$

$$a_n = 0 \tag{12.1.68}$$

Finding the sine coefficients

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \sin(nx) \ \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} (1) \sin(nx) \ \mathrm{d}x \tag{12.1.69}$$

$$= -\frac{2}{\pi} \left[\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2}{n\pi} \left[1 - \cos(n\pi)\right] \tag{12.1.70}$$

Using a guess for the solution, and solving the nh-ODE,

$$y_p = A_n \cos(nt) + B_n \sin(nt) \tag{12.1.71}$$

$$0 = (-n^2 + \omega^2) A_n \tag{12.1.72}$$

$$\frac{4}{n\pi} = (-n^2 + \omega^2) B_n \tag{12.1.73}$$

$$y_p = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1 - \cos(n\pi)]}{n \ (\omega^2 - n^2)} \ \sin(nt) \tag{12.1.74}$$

Finding the general solution, by solving the h-ODE,

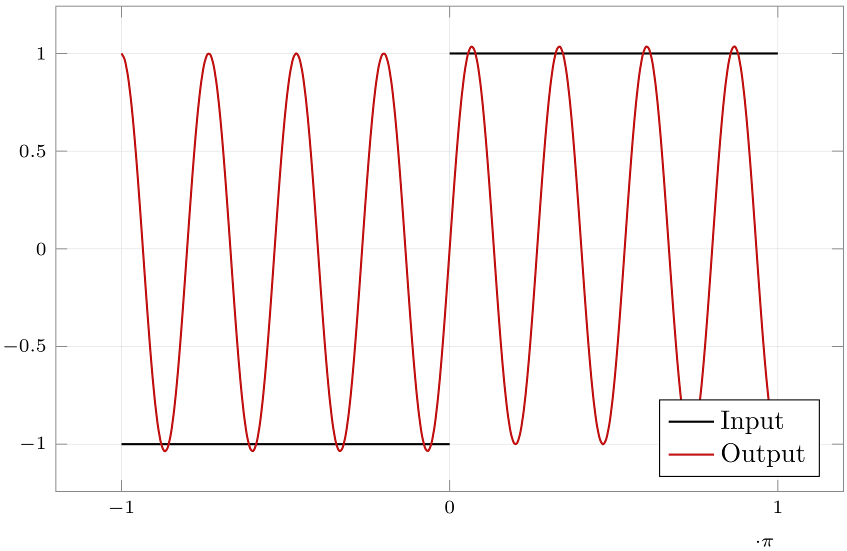
$$y'' + \omega^2 y = 0 \tag{12.1.75}$$

$$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{12.1.76}$$

$$y = y_h + y_p \tag{12.1.77}$$

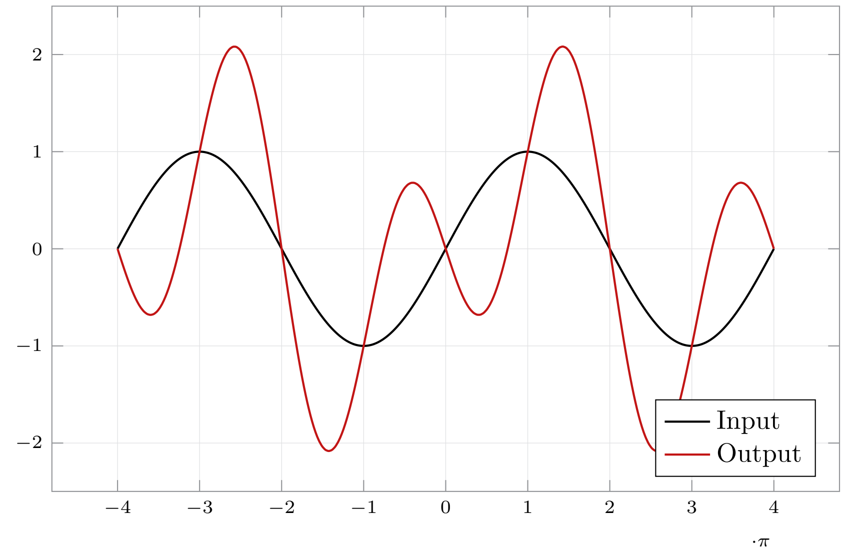
12. Graphing the input and output in Problem 11, with $C_1 = 0$, $C_2 = 1$, and $\omega = 7.5$

Undamped Driven Oscillations



Graphing the input and output in Problem 7, with $C_1 = 0$, $C_2 = 1$, and $\omega = 0.5$

Undamped Driven Oscillations



13. For the damped oscillator, with $k = 1$,

$$D_n = (1 - n^2)^2 + (nc)^2 \quad \text{12.1.78}$$

$$y_n = P_n \cos(nt) + Q_n \sin(nt) \quad \text{12.1.79}$$

Consider the two general terms in the input,

$$y'' + cy' + y = a_n \cos(nt) + b_n \sin(nt) \tag{12.1.80}$$

$$a_n = (1 - n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{12.1.81}$$

$$b_n = (1 - n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{12.1.82}$$

$$P_n = \frac{a_n(1 - n^2) - b_n(nc)}{D_n} \tag{12.1.83}$$

$$Q_n = \frac{b_n(1 - n^2) + a_n(nc)}{D_n} \tag{12.1.84}$$

The above system is linear in P_n and Q_n.

14. From Problem 11, the Fourier series representation of the input is,

$$r(t) = \sum_{n=1}^{\infty} \frac{2[1 - \cos(n\pi)]}{n\pi} \sin(nt) \tag{12.1.85}$$

$$y'' + cy' + y = a_n \cos(nt) + b_n \sin(nt) \tag{12.1.86}$$

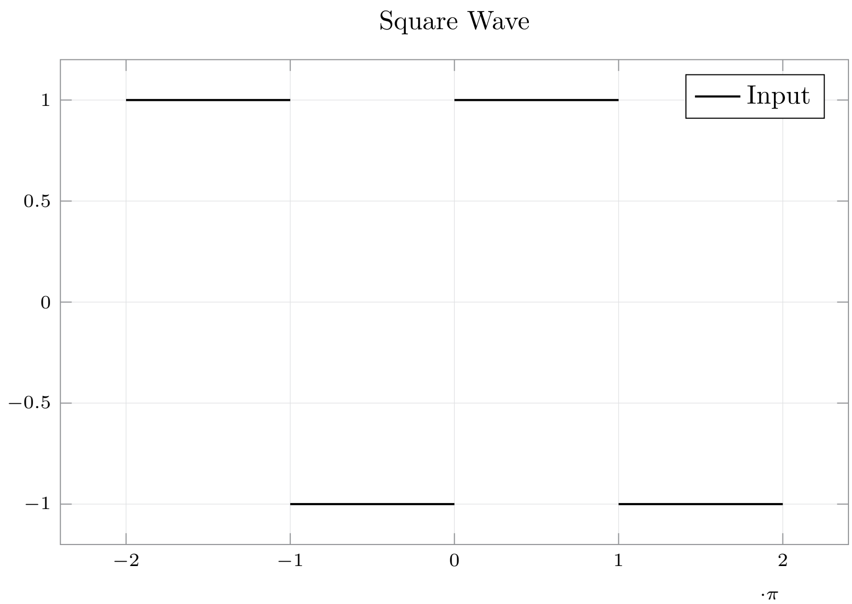
$$0 = (1 - n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{12.1.87}$$

$$b_n = (1 - n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{12.1.88}$$

$$P_n = \frac{-b_n(nc)}{D_n} \tag{12.1.89}$$

$$Q_n = \frac{b_n(1 - n^2)}{D_n} \tag{12.1.90}$$

$$D_n = (1 - n^2)^2 + (nc)^2 \tag{12.1.91}$$



15. Finding the fourier series representation of the input (odd function),

$$a_0 = 0 \qquad\qquad a_n = 0 \tag{12.1.92}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 x - x^3)\ \sin(nx)\ \mathrm{d}x \tag{12.1.93}$$

$$= \frac{2}{\pi}\left[\sin(nx)\left(\frac{\pi^2 - 3x^2}{n^2} + \frac{6}{n^4}\right) + \cos(nx)\left(\frac{x(x^2-\pi^2)}{n} - \frac{6x}{n^3}\right)\right]_0^{\pi} \tag{12.1.94}$$

$$= -\frac{12}{n^3}\ \cos(n\pi) \tag{12.1.95}$$

$$r(t) = \sum_{n=1}^{\infty} \frac{-12\cos(n\pi)}{n^3}\ \sin(nt) \tag{12.1.96}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_n\cos(nt) + b_n\sin(nt) \tag{12.1.97}$$

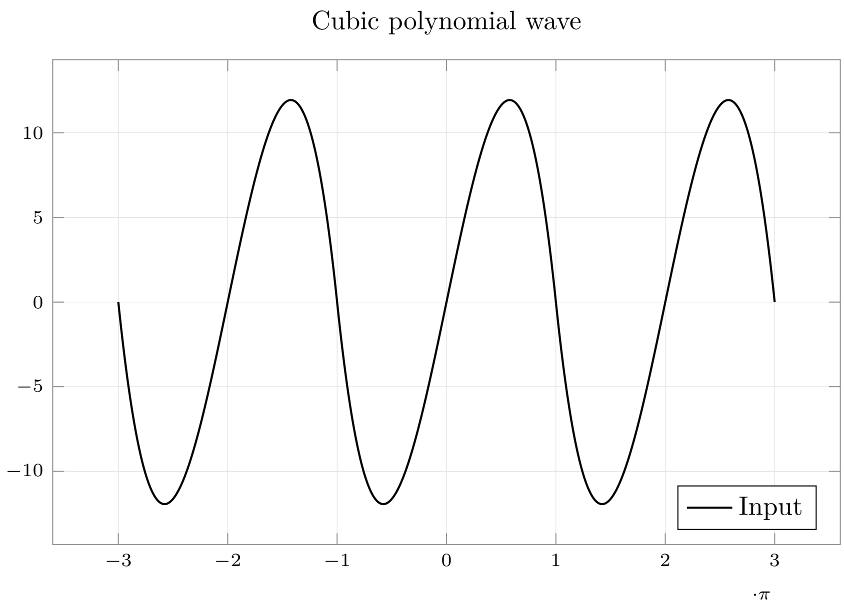
$$0 = (1-n^2)P_n + ncQ_n \qquad \cdots[\cos(nt)] \tag{12.1.98}$$

$$b_n = (1-n^2)Q_n - ncP_n \qquad \cdots[\sin(nt)] \tag{12.1.99}$$

$$P_n = \frac{-b_n(nc)}{D_n} \tag{12.1.100}$$

$$Q_n = \frac{b_n(1-n^2)}{D_n} \tag{12.1.101}$$

$$D_n = (1-n^2)^2 + (nc)^2 \tag{12.1.102}$$



16. Finding the fourier series representation of the input(odd function)

$$a_0 = 0 \quad (12.1.103)$$

$$a_n = 0 \quad (12.1.104)$$

Finding the sine coefficients,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\ \mathrm{d}x \quad (12.1.105)$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)\ \mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\ \mathrm{d}x \quad (12.1.106)$$

$$= \frac{2}{\pi}\left[\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right]_0^{\pi/2} + \frac{2}{\pi}\left[\frac{(x-\pi)\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi/2}^{\pi} \quad (12.1.107)$$

$$= \frac{4}{\pi n^2}\ \sin(n\pi/2) \quad (12.1.108)$$

$$r(t) = \sum_{n=1}^{\infty}\frac{4\sin(n\pi/2)}{\pi n^2}\ \sin(nt) \quad (12.1.109)$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_n \cos(nt) + b_n \sin(nt) \tag{12.1.110}$$

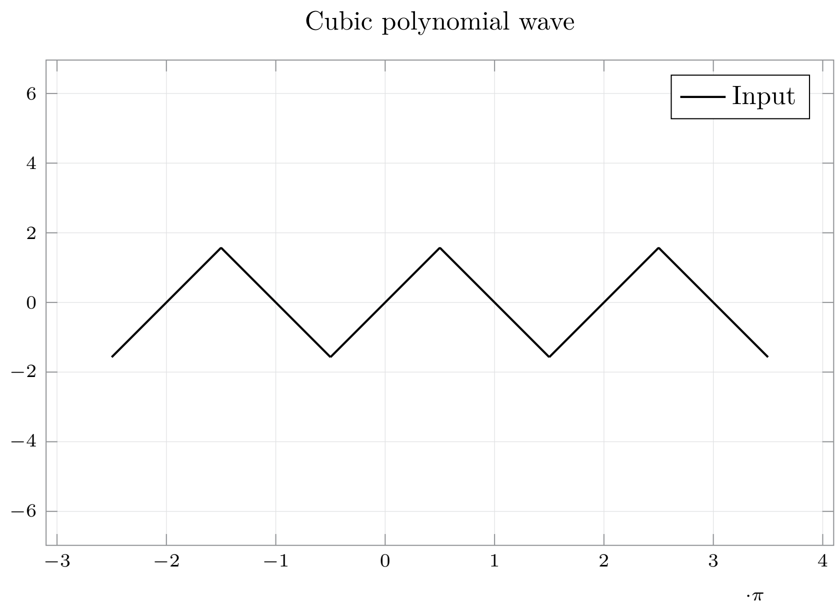
$$0 = (1 - n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{12.1.111}$$

$$b_n = (1 - n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{12.1.112}$$

$$P_n = \frac{-b_n(nc)}{D_n} \tag{12.1.113}$$

$$Q_n = \frac{b_n(1 - n^2)}{D_n} \tag{12.1.114}$$

$$D_n = (1 - n^2)^2 + (nc)^2 \tag{12.1.115}$$



17. The second order linear ODE for an RLC circuit with $R = 10$, $L = 1$, $C = 0.1$ is given by,

$$Lj'' + Rj' + \frac{1}{C}\,j = E'(t) \qquad E'(t) = \begin{cases} -100t & t \in [-\pi, 0] \\ 100t & t \in [0, \pi] \end{cases} \tag{12.1.116}$$

Finding the Fourier series representation of the input, (even function),

$$b_n = 0 \tag{12.1.117}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ \mathrm{d}x = \frac{1}{\pi} \int_0^{\pi} (100x) \, \mathrm{d}x \tag{12.1.118}$$

$$= \left[\frac{50x^2}{\pi} \right]_0^{\pi} = 50\pi \tag{12.1.119}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \ \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} (100x) \cos(nx) \, \mathrm{d}x \tag{12.1.120}$$

$$= \frac{200}{\pi} \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^{\pi} = \frac{200}{\pi n^2} \left[\cos(n\pi) - 1 \right] \tag{12.1.121}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_0 + \sum_{n=0}^{\infty} a_n \cos(nt) + b_n \sin(nt) \tag{12.1.122}$$

$$10P_0 = a_0 = 50\pi \qquad P_0 = 5\pi \tag{12.1.123}$$

$$y_n = P_n \cos(nt) + Q_n \sin(nt) \tag{12.1.124}$$

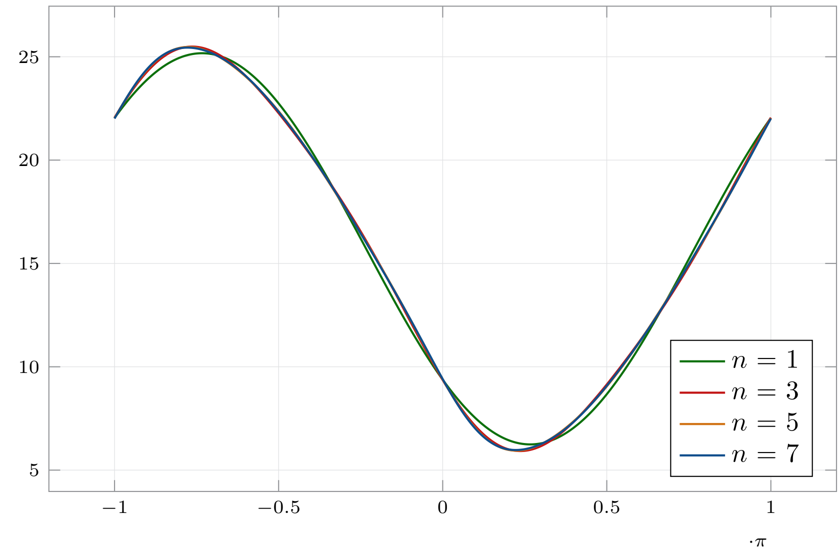
$$a_n = (1 - n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{12.1.125}$$

$$0 = (1 - n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{12.1.126}$$

$$P_n = \frac{a_n(10 - n^2)}{D_n} \qquad Q_n = \frac{a_n(10n)}{D_n} \tag{12.1.127}$$

$$D_n = (10 - n^2)^2 + (10n)^2 \tag{12.1.128}$$

Triangular wave



18. The second order linear ODE for an RLC circuit with $R = 10$, $L = 1$, $C = 0.1$ is given by,

$$Lj'' + Rj' + \frac{1}{C}\,j = E'(t) \qquad E'(t) = \begin{cases} 100(1-2t) & t \in [-\pi, 0] \\ 100(1+2t) & t \in [0, \pi] \end{cases} \tag{12.1.129}$$

Finding the Fourier series representation of the input, (even function),

$$b_n = 0 \tag{12.1.130}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} (100)(1+2x)\,\mathrm{d}x \tag{12.1.131}$$

$$= \frac{100}{\pi}\Big[x + x^2\Big]_0^{\pi} = 100(1+\pi) \tag{12.1.132}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} (100)(1+2x)\cos(nx)\,\mathrm{d}x \tag{12.1.133}$$

$$= \frac{200}{\pi}\left[\frac{\sin(nx)}{n} + \frac{2x\sin(nx)}{n} + \frac{2\cos(nx)}{n^2}\right]_0^{\pi} \tag{12.1.134}$$

$$= \frac{400}{\pi n^2}\,[\cos(n\pi) - 1] \tag{12.1.135}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_0 + \sum_{n=0}^{\infty} a_n \cos(nt) + b_n \sin(nt) \tag{12.1.136}$$

$$10P_0 = a_0 = 100(1+\pi) \qquad P_0 = 10(1+\pi) \tag{12.1.137}$$

$$y_n = P_n \cos(nt) + Q_n \sin(nt) \tag{12.1.138}$$

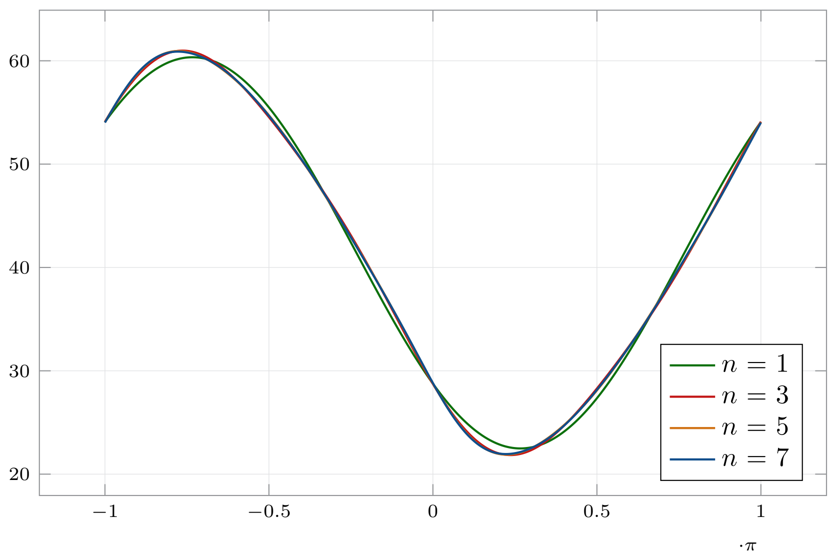
$$a_n = (1-n^2)P_n + ncQ_n \qquad \cdots [\cos(nt)] \tag{12.1.139}$$

$$0 = (1-n^2)Q_n - ncP_n \qquad \cdots [\sin(nt)] \tag{12.1.140}$$

$$P_n = \frac{a_n(10-n^2)}{D_n} \qquad Q_n = \frac{a_n(10n)}{D_n} \tag{12.1.141}$$

$$D_n = (10-n^2)^2 + (10n)^2 \tag{12.1.142}$$

Triangular wave



19. The second order linear ODE for an RLC circuit with $R = 10$, $L = 1$, $C = 0.1$ is given by,

$$Lj'' + Rj' + \frac{1}{C}\,j = E'(t) \qquad E'(t) = 200(\pi^2 - 3t^2) \tag{12.1.143}$$

Finding the Fourier series representation of the input, (even function),

$$b_n = 0 \tag{12.1.144}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}(200)(\pi^2 - 3x^2)\,\mathrm{d}x \tag{12.1.145}$$

$$= \frac{200}{\pi}\left[\pi^2 x - x^3\right]_0^{\pi} = 0 \tag{12.1.146}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}(200)(\pi^2 - 3x^2)\cos(nx)\,\mathrm{d}x \tag{12.1.147}$$

$$= \frac{400}{\pi}\left[\frac{(\pi^2 - 3x^2)\sin(nx)}{n} + \frac{6\sin(nx)}{n^3} - \frac{6x\cos(nx)}{n^2}\right]_0^{\pi} \tag{12.1.148}$$

$$= -\frac{2400\ \cos(n\pi)}{n^2} \tag{12.1.149}$$

Using the standard result for a sinusoidal input to a damped oscillator,

$$y'' + cy' + y = a_0 + \sum_{n=0}^{\infty} a_n\cos(nt) + b_n\sin(nt) \tag{12.1.150}$$

$$10P_0 = a_0 = 0 \qquad\qquad P_0 = 0 \tag{12.1.151}$$

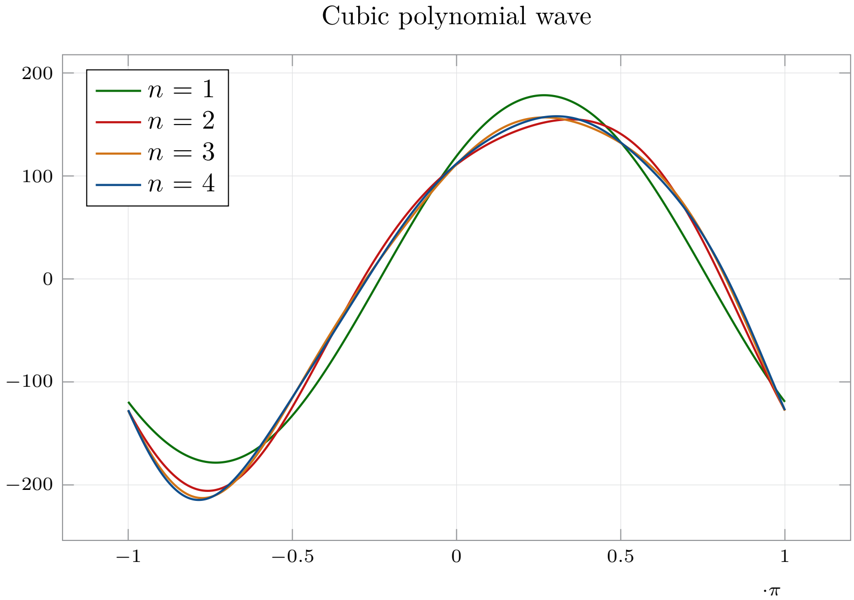
$$y_n = P_n\cos(nt) + Q_n\sin(nt) \tag{12.1.152}$$

$$a_n = (1 - n^2)P_n + ncQ_n \qquad \cdots[\cos(nt)] \tag{12.1.153}$$

$$0 = (1 - n^2)Q_n - ncP_n \qquad \cdots[\sin(nt)] \tag{12.1.154}$$

$$P_n = \frac{a_n(10 - n^2)}{D_n} \qquad\qquad Q_n = \frac{a_n(10n)}{D_n} \tag{12.1.155}$$

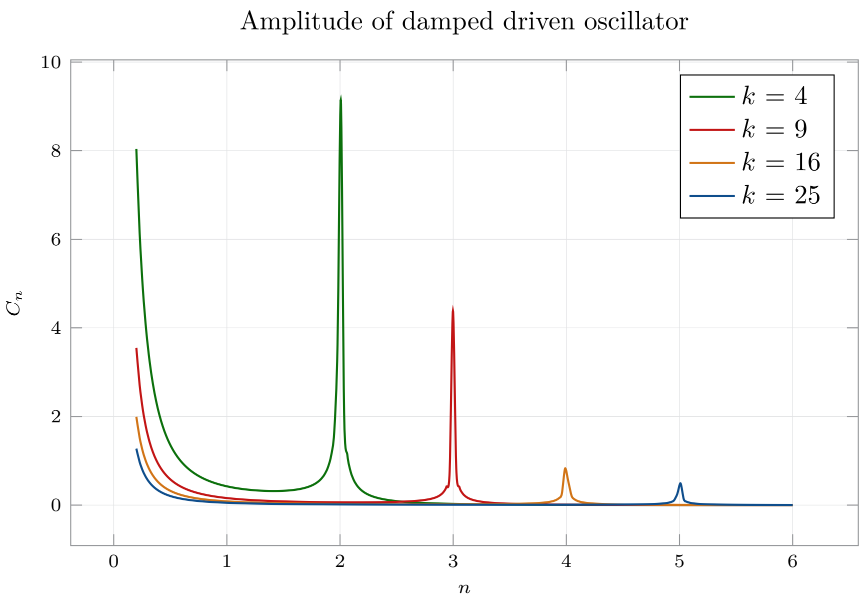
$$D_n = (10 - n^2)^2 + (10n)^2 \tag{12.1.156}$$



20. Finding the solution to the ODE in Example 1, for general c and k,

$$D_n = (k - n^2)^2 + (cn)^2 \qquad\qquad C_n = \frac{4}{n^2\pi} \cdot \frac{1}{\sqrt{D_n}} \tag{12.1.157}$$

Plotting a graph of C_n vs n for a fixed value of $c = 0.05$, and integer square values of k,



12.2 Approximation by Trigonometric Polynomials

1. From Example 1 in the text,

$$f(x) = x + \pi \qquad x \in [-\pi, \pi] \tag{12.2.1}$$

$$a_0 = \pi \tag{12.2.2}$$

$$a_n = 0 \tag{12.2.3}$$

$$b_n = \frac{-2\cos(n\pi)}{n} \tag{12.2.4}$$

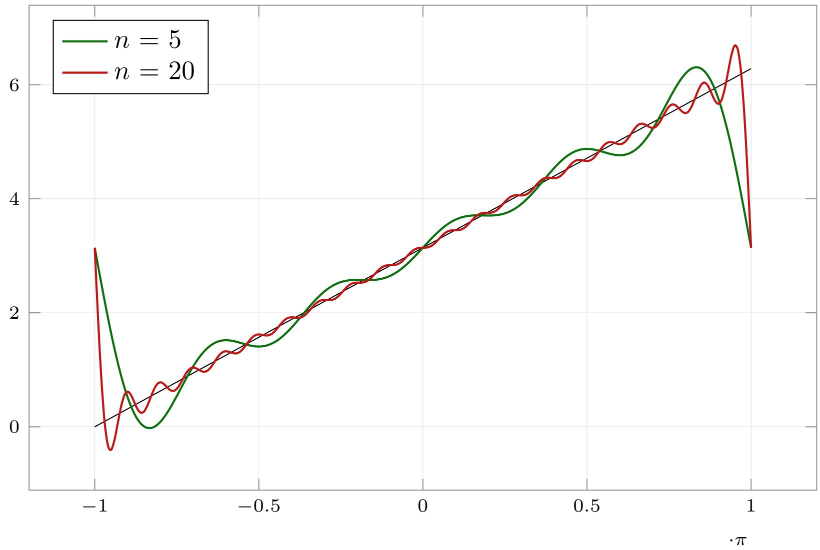
$$\int_{-\pi}^{\pi} f^2 \ \mathrm{d}x = \int_{-\pi}^{\pi} (x+\pi)^2 \ \mathrm{d}x = \left[\frac{(x+\pi)^3}{3}\right]_{-\pi}^{\pi} = \frac{8\pi^3}{3} \tag{12.2.5}$$

$$E^* = \frac{8\pi^3}{3} - 2\pi^3 - 4\pi \sum_{i=1}^{N} \frac{1}{n^2} \tag{12.2.6}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*	N	E^*
1000	0.01256	6000	0.002094
2000	0.006282	7000	0.001795
3000	0.004188	8000	0.001571
4000	0.003141	9000	0.001396
5000	0.002513	10000	0.001257

Fourier approximation



2. Evaluating the Fourier coefficients,

$$f(x) = x \qquad x \in [-\pi, \pi] \tag{12.2.7}$$

$$a_0 = 0 \tag{12.2.8}$$

$$a_n = 0 \tag{12.2.9}$$

$$b_n = \frac{-2\cos(n\pi)}{n} \tag{12.2.10}$$

$$\int_{-\pi}^{\pi} f^2 \ \mathrm{d}x = \int_{-\pi}^{\pi} (x)^2 \ \mathrm{d}x = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3} \tag{12.2.11}$$

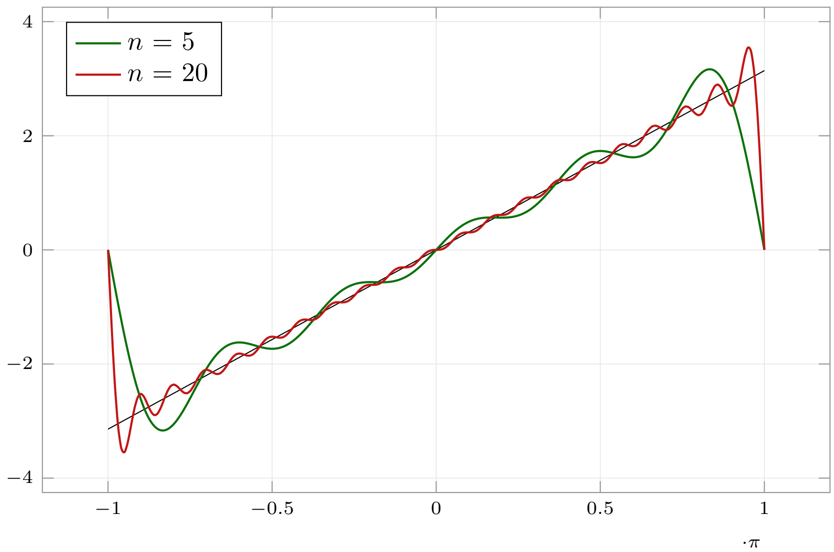
$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty} \frac{4}{n^2} \tag{12.2.12}$$

$$E^* = \frac{\pi^3}{3} - 4\pi \sum_{i=1}^{N} \frac{1}{n^2} \tag{12.2.13}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	8.104
2	4.963
3	3.567
4	2.781
5	2.279

Fourier approximation



3. Evaluating the Fourier coefficients,

$$f(x) = |x| \qquad x \in [-\pi, \pi] \tag{12.2.14}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \;\mathrm{d}x = \frac{1}{\pi} \int_{0}^{\pi} x \;\mathrm{d}x = \left[\frac{x^2}{2\pi}\right]_0^{\pi} = \frac{\pi}{2} \tag{12.2.15}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \;\mathrm{d}x = \frac{2}{\pi} \int_{0}^{\pi} x \cos(nx) \;\mathrm{d}x \tag{12.2.16}$$

$$= \frac{2}{\pi} \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2} [\cos(n\pi - 1)] \tag{12.2.17}$$

$$b_n = 0 \tag{12.2.18}$$

Calculating the error function,

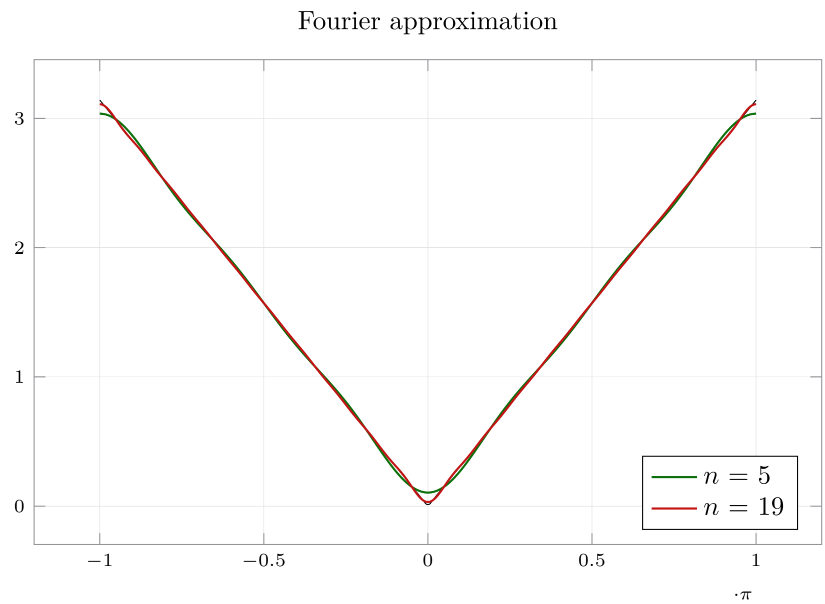
$$\int_{-\pi}^{\pi} f^2 \;\mathrm{d}x = \int_{-\pi}^{\pi} (x)^2 \;\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3} \tag{12.2.19}$$

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{\pi^2}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{[\cos(n\pi) - 1]^2}{n^4} \tag{12.2.20}$$

$$E^* = \frac{\pi^3}{6} - \frac{4}{\pi} \sum_{i=1}^{N} \frac{[\cos(n\pi) - 1]^2}{n^4} \tag{12.2.21}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	0.0747
3	0.0118
5	0.0037
7	0.0016
9	0.00083



4. Evaluating the Fourier coefficients,

$$f(x) = x^2 \qquad x \in [-\pi, \pi] \tag{12.2.22}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \ \mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x^2 \ \mathrm{d}x = \left[\frac{x^3}{3\pi}\right]_0^{\pi} = \frac{\pi^2}{3} \tag{12.2.23}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) \ \mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx) \ \mathrm{d}x \tag{12.2.24}$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin(nx)}{n} - \frac{2\sin(nx)}{n^3} + \frac{2x\cos(nx)}{n^2}\right]_0^{\pi} = \frac{4\cos(n\pi)}{n^2} \tag{12.2.25}$$

$$b_n = 0 \tag{12.2.26}$$

Calculating the error function,

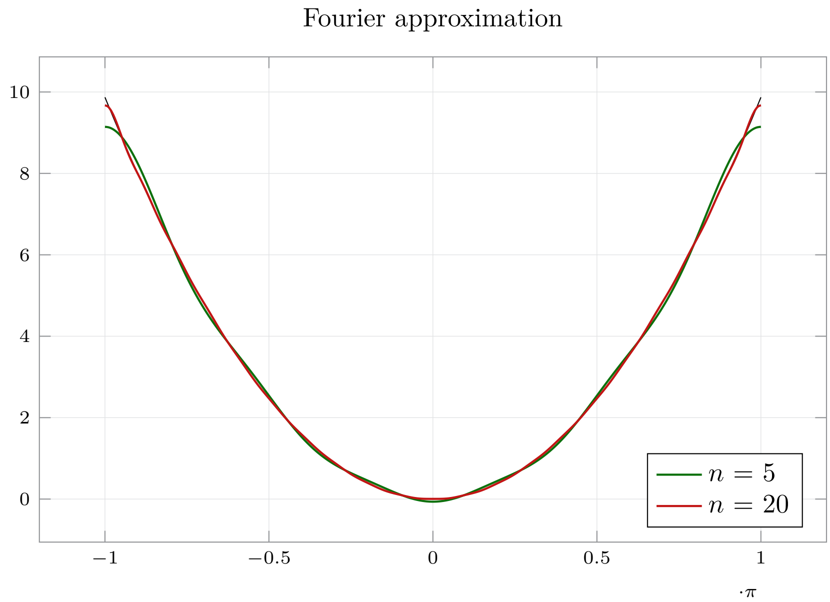
$$\int_{-\pi}^{\pi} f^2 \ \mathrm{d}x = \int_{-\pi}^{\pi} (x)^4 \ \mathrm{d}x = \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{5} \tag{12.2.27}$$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} \tag{12.2.28}$$

$$E^* = \frac{8\pi^5}{45} - 16\pi\sum_{i=1}^{N}\frac{1}{n^4} \tag{12.2.29}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	4.138
2	0.9964
3	0.3758
4	0.1795
5	0.0991



5. Evaluating the Fourier coefficients,

$$f(x) = \begin{cases} -1 & x \in [-\pi, 0] \\ 1 & x \in [0, \pi] \end{cases} \tag{12.2.30}$$

$$a_0 = 0 \tag{12.2.31}$$

$$a_n = 0 \tag{12.2.32}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ \mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\ \mathrm{d}x \tag{12.2.33}$$

$$= \left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right] \tag{12.2.34}$$

Calculating the error function,

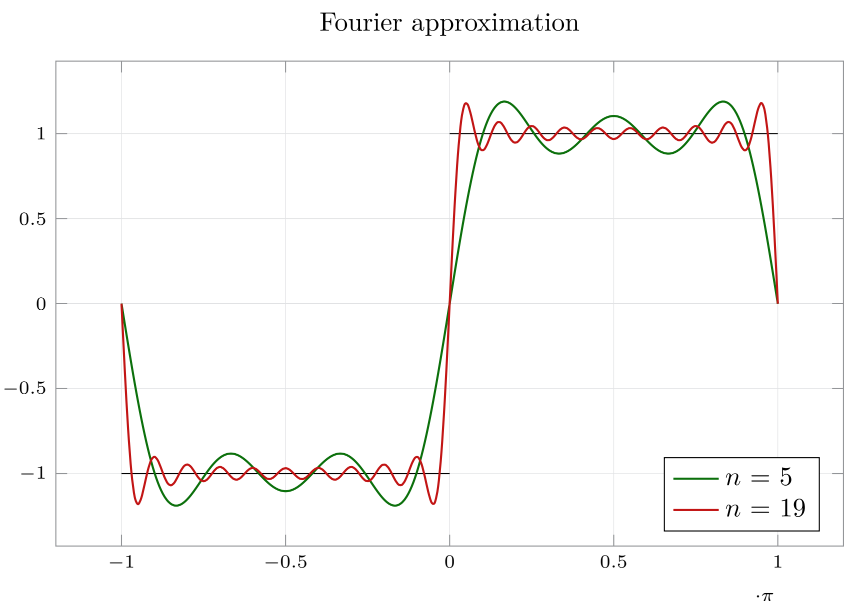
$$\int_{-\pi}^{\pi} f^2 \, \mathrm{d}x = \int_{-\pi}^{\pi} (1) \, \mathrm{d}x = \Big[x \Big]_{-\pi}^{\pi} = 2\pi \tag{12.2.35}$$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{[1-\cos(n\pi)]^2}{n^2} \tag{12.2.36}$$

$$E^* = 2\pi - \frac{4}{\pi} \sum_{i=1}^{N} \frac{[1-\cos(n\pi)]^2}{n^2} \tag{12.2.37}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	1.1902
2	0.6243
3	0.4206
4	0.3167
5	0.2538



6. The discontinuity at $x = 0$ in Problem 5 makes the Fourier series a very bad approximation to the function around $x = 0$. This makes the errors much larger.

7. Evaluating the Fourier coefficients,

$$f(x) = x^3 \qquad\qquad x \in [-\pi, \pi] \tag{12.2.38}$$

$$a_0 = 0 \tag{12.2.39}$$

$$a_n = 0 \tag{12.2.40}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\ \mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x^3\ \sin(nx)\ \mathrm{d}x \tag{12.2.41}$$

$$= \frac{2}{\pi}\left[\sin(nx)\left(\frac{3x^2}{n^2} - \frac{6}{n^4}\right) + \cos(nx)\left(\frac{-x^3}{n} + \frac{6x}{n^3}\right)\right]_0^{\pi} \tag{12.2.42}$$

$$= \cos(n\pi)\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right] \tag{12.2.43}$$

Calculating the error function,

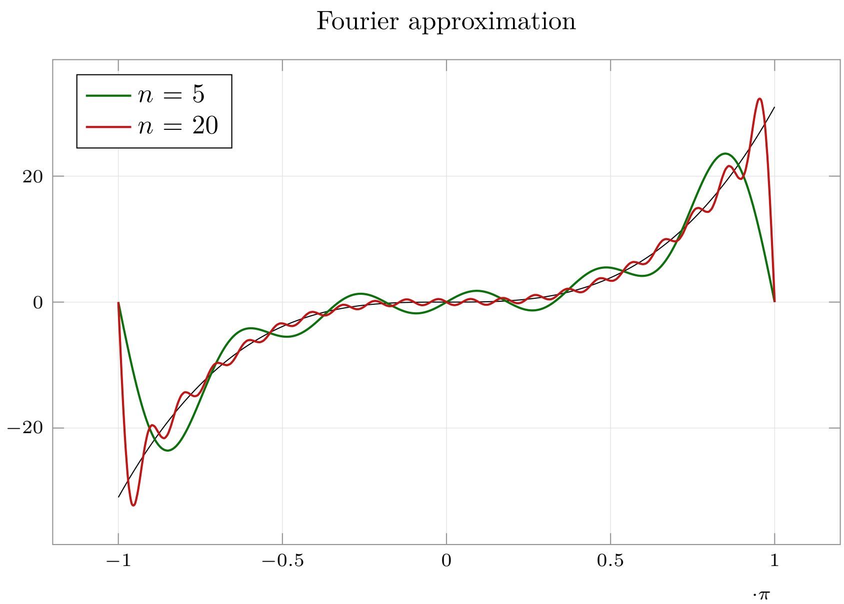
$$\int_{-\pi}^{\pi} f^2\ \mathrm{d}x = \int_{-\pi}^{\pi} x^6\ \mathrm{d}x = \left[\frac{x^7}{7}\right]_{-\pi}^{\pi} = \frac{2\pi^7}{7} \tag{12.2.44}$$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty}\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]^2 \tag{12.2.45}$$

$$E^* = \frac{2\pi^7}{7} - \pi\sum_{i=1}^{N}\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]^2 \tag{12.2.46}$$

Using `sympy` to evaluate the minimum error for various values of N,

N	E^*
1	674.774
10	116.065
100	12.1793
500	2.4457
1000	1.2235



8. Evaluating the Fourier coefficients,

$$f(x) = |\sin(x)| \qquad x \in [-\pi, \pi] \tag{12.2.47}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\;\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin(x)\;\mathrm{d}x = \frac{1}{\pi}\Big[-\cos(x)\Big]_0^{\pi} = \frac{2}{\pi} \tag{12.2.48}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\;\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(nx)\;\mathrm{d}x \tag{12.2.49}$$

$$= \frac{2}{\pi}\left[\frac{n\sin(x)\sin(nx) + \cos(x)\cos(nx)}{n^2 - 1}\right]_0^{\pi} \tag{12.2.50}$$

$$= \frac{-2}{\pi(n^2-1)}\;[1 + \cos(n\pi)] \tag{12.2.51}$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin(2x)\;\mathrm{d}x = \left[\frac{-\cos(2x)}{2\pi}\right]_0^{\pi} = 0 \tag{12.2.52}$$

$$b_n = 0 \tag{12.2.53}$$

Calculating the error function,

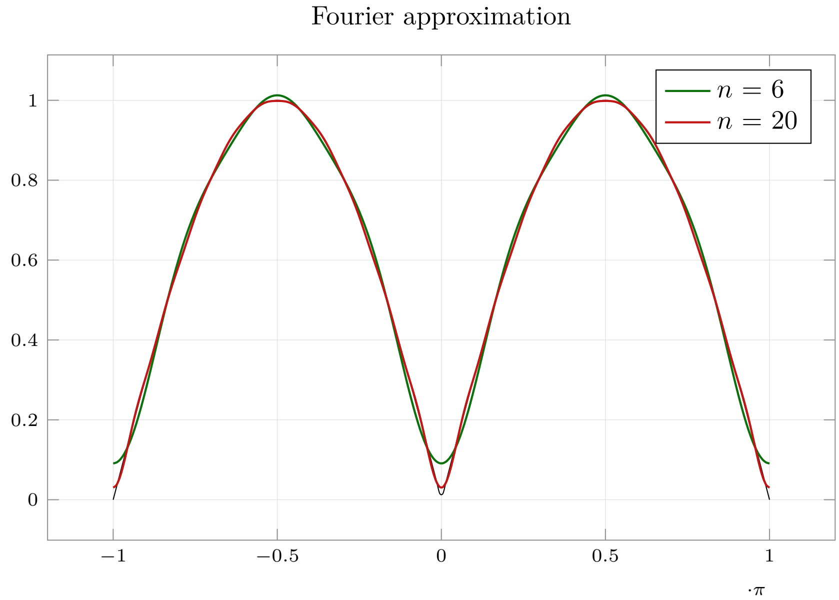
$$\int_{-\pi}^{\pi} f^2\;\mathrm{d}x = \int_{-\pi}^{\pi}\sin^2(x)\;\mathrm{d}x = \left[\frac{x}{2} + \frac{\sin(2x)}{4}\right]_{-\pi}^{\pi} = \pi \tag{12.2.54}$$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{8}{\pi^2} + \frac{4}{\pi^2}\sum_{n=2}^{\infty}\frac{[1+\cos(n\pi)]^2}{(n^2-1)^2} \tag{12.2.55}$$

$$E^* = \pi - \frac{8}{\pi} - \frac{4}{\pi}\sum_{n=2}^{N}\frac{[1+\cos(n\pi)]^2}{(n^2-1)^2} \tag{12.2.56}$$

Using sympy to evaluate the minimum error for various values of N,

N	E^*
2	0.0292
4	0.00659
6	0.002436
8	0.001153
10	0.000634



9. The minimized square error is a series of squares of Fourier coefficients, which are all nonnegative. The negative scalar factor makes the function monotonically decreasing in N.

10. The more trigonometric the actual function is, the faster E^* decreases with increasing N. Compare Problems 2 − 8 using sympy to program $E^*(N)$.

11. From Example 1 in Section 11.1, the Fourier series expansion is

$$f(x) = \begin{cases} -1 & x \in [-\pi, 0] \\ 1 & x \in [0, \pi] \end{cases} \tag{12.2.57}$$

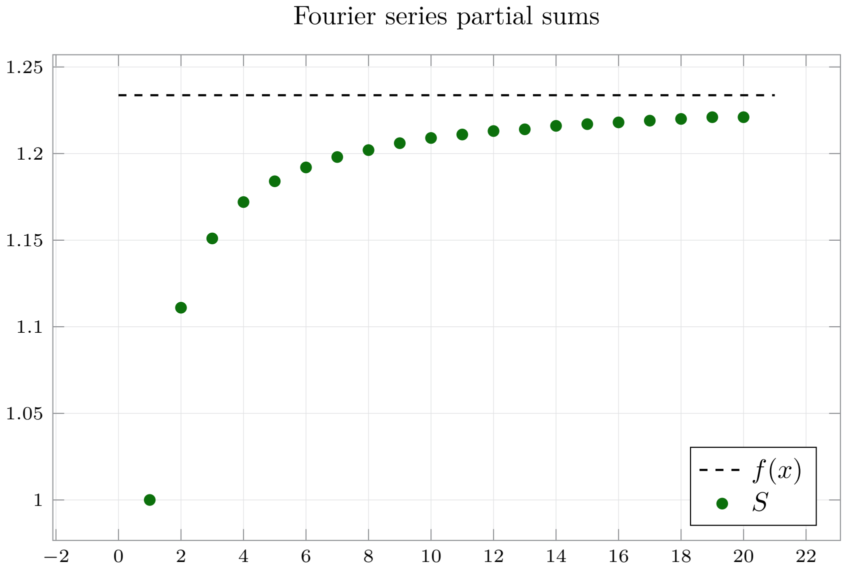
$$a_0 = a_n = 0 \tag{12.2.58}$$

$$b_n = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right] \tag{12.2.59}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\ \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{12.2.60}$$

$$2 = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-\cos(n\pi)}{n}\right]^2 \tag{12.2.61}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \tag{12.2.62}$$



12. From Problem 14 in Section 11.1, the Fourier series expansion is

$$f(x) = x^2 \tag{12.2.63}$$

$$a_0 = \frac{\pi^2}{3} \tag{12.2.64}$$

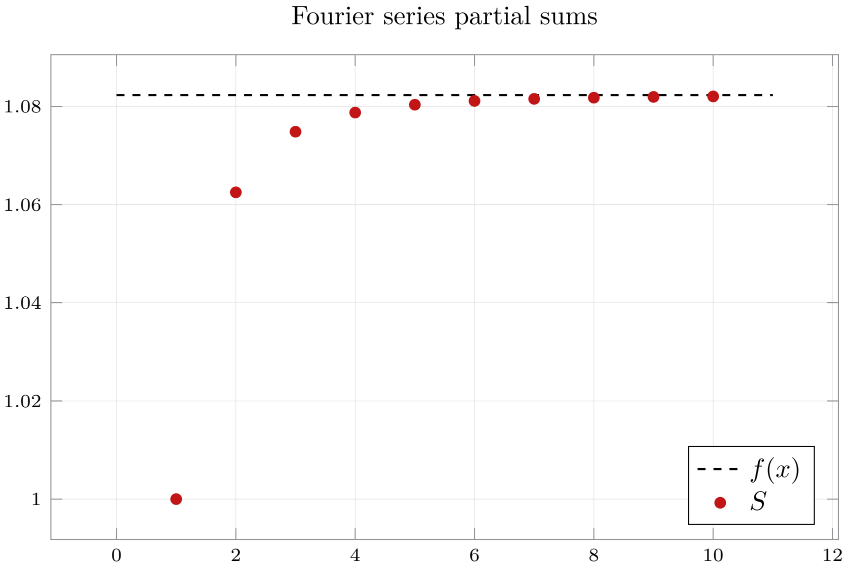
$$a_n = \frac{4\cos(n\pi)}{n^2} \tag{12.2.65}$$

$$b_n = 0 \tag{12.2.66}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\ \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{12.2.67}$$

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right]^2 \tag{12.2.68}$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \tag{12.2.69}$$



13. From Problem 17 in Section 11.1, the Fourier series expansion is

$$f(x) = \begin{cases} x + \pi & x \in [-\pi, 0] \\ -x + \pi & x \in [0, \pi] \end{cases} \tag{12.2.70}$$

$$a_0 = \frac{\pi}{2} \tag{12.2.71}$$

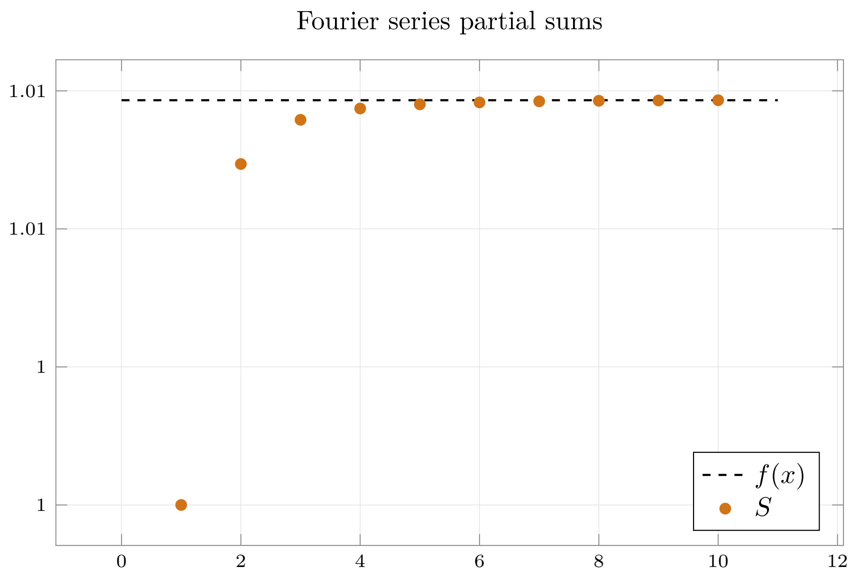
$$a_n = \frac{2}{\pi n^2} [1 - \cos(n\pi)] \tag{12.2.72}$$

$$b_n = 0 \tag{12.2.73}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \ \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{12.2.74}$$

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1 - \cos(n\pi)}{n^2} \right]^2 \tag{12.2.75}$$

$$\frac{\pi^4}{99} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \tag{12.2.76}$$



14. Using Parseval's identity,

$$f(x) = \cos^2(x) = \frac{1+\cos(2x)}{2} \tag{12.2.77}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\ \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{12.2.78}$$

$$\int_{-\pi}^{\pi} \cos^4(x)\ \mathrm{d}x = \pi\left[\frac{2}{2^2} + \frac{1}{4}\right] = \frac{3\pi}{4} \tag{12.2.79}$$

15. Using Parseval's identity,

$$f(x) = \cos^3(x) = \frac{3\cos(x) + \cos(3x)}{4} \tag{12.2.80}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\ \mathrm{d}x = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{12.2.81}$$

$$\int_{-\pi}^{\pi} \cos^6(x)\ \mathrm{d}x = \pi\left[0 + \frac{9}{16} + \frac{1}{16}\right] = \frac{5\pi}{8} \tag{12.2.82}$$

12.3 Sturm-Liouville Problems, Orthogonal Functions

1.