Chapter 8

Linear Algebra: Matrix Eigenvalue Problems

8.1 The Matrix Eigenvalue Problem, Determining Eigenvalues and Eigenvectors

1. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & -0.6 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.1}$$

$$(\lambda - 3)(\lambda + 0.6) = 0 \qquad \{\lambda_i\} = \{-0.6, 3\} \tag{8.1.2}$$

Finding the eigenvectors,

$$\lambda_1 = -0.6 \qquad \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8.1.3}$$

$$\lambda_2 = 3 \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.1.4}$$

2. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.5}$$

$$(\lambda + 0)(\lambda + 0) = 0 \qquad \{\lambda_i\} = \{0, 0\} \tag{8.1.6}$$

Finding the eigenvectors,

$$\lambda_1 = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.1.7}$$

$$\lambda_2 = 0 \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8.1.8}$$

3. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.9}$$

$$(\lambda + 6)(\lambda - 5) + 18 = 0 \qquad \lambda^2 + \lambda - 12 = 0 \tag{8.1.10}$$

$$\{\lambda_i\} = \{-4, 3\} \tag{8.1.11}$$

Finding the eigenvectors,

$$\lambda_1 = -4 \qquad \begin{bmatrix} 9 & -2 \\ 9 & -2 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \end{bmatrix} \tag{8.1.12}$$

$$\lambda_2 = 3 \qquad \begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.1.13}$$

4. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.14}$$

$$(\lambda - 1)(\lambda - 4) - 4 = 0 \qquad \lambda^2 - 5\lambda = 0 \tag{8.1.15}$$

$$\{\lambda_i\} = \{0, 5\} \tag{8.1.16}$$

Finding the eigenvectors,

$$\lambda_1 = 0 \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{8.1.17}$$

$$\lambda_2 = 5 \qquad \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{8.1.18}$$

5. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.19}$$

$$\lambda^2 + 9 = 0 \qquad \{\lambda_i\} = \{-3i, 3i\} \tag{8.1.20}$$

Finding the eigenvectors,

$$\lambda_1 = -3i \qquad \begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{8.1.21}$$

$$\lambda_2 = 3i \qquad \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{8.1.22}$$

6. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.23}$$

$$(\lambda - 1)(\lambda - 3) + 0 = 0 \qquad \{\lambda_i\} = \{1, 3\} \tag{8.1.24}$$

Finding the eigenvectors,

$$\lambda_1 = 1 \qquad \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.1.25}$$

$$\lambda_2 = 3 \qquad \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.1.26}$$

7. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.27}$$

$$\lambda^2 = 0 \qquad \{\lambda_i\} = \{0, 0\} \tag{8.1.28}$$

Finding the eigenvectors,

$$\lambda_1 = 0 \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.1.29}$$

8. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.1.30}$$

$$(\lambda - a)^2 + b^2 = 0 \qquad \lambda^2 - 2a\lambda + (a^2 + b^2) = 0 \tag{8.1.31}$$

$$\{\lambda_i\} = \{a \pm bi\} \tag{8.1.32}$$

Finding the eigenvectors,

$$\lambda_1 = a - bi \qquad \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{8.1.33}$$

$$\lambda_2 = a + bi \qquad \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{8.1.34}$$

9. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.1.35}$$

$$(\lambda - 0.8)^2 + (0.6)^2 = 0 \qquad \lambda^2 - 1.6\lambda + 1 = 0 \tag{8.1.36}$$

$$\{\lambda_i\} = \{0.8 \pm 0.6i\} \tag{8.1.37}$$

Finding the eigenvectors,

$$\lambda_1 = 0.8 - 0.6i \qquad \begin{bmatrix} 0.6i & -0.6 \\ 0.6 & 0.6i \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{8.1.38}$$

$$\lambda_2 = 0.8 + 0.6i \qquad \begin{bmatrix} -0.6i & -0.6 \\ 0.6 & -0.6i \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{8.1.39}$$

10. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.1.40}$$

$$(\lambda - \cos\theta)^2 + (\sin\theta)^2 = 0 \qquad \lambda^2 - 2\lambda\cos\theta + 1 = 0 \tag{8.1.41}$$

$$\{\lambda_i\} = \{\cos\theta \pm i\ \sin\theta\} \tag{8.1.42}$$

Finding the eigenvectors,

$$\lambda_1 = \cos\theta - i\ \sin\theta \qquad \begin{bmatrix} i\ \sin\theta & -\sin\theta \\ \sin\theta & i\ \sin\theta \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$ 8.1.43

$$\lambda_2 = \cos\theta + i\ \sin\theta \qquad \begin{bmatrix} -i\ \sin\theta & -\sin\theta \\ \sin\theta & -i\ \sin\theta \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$ 8.1.44

11. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$ 8.1.45

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0 \qquad \{\lambda_i\} = \{3, 6, 9\}$$ 8.1.46

Finding the eigenvectors,

$$\lambda_1 = 3 \qquad \begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$ 8.1.47

$$\lambda_2 = 6 \qquad \begin{bmatrix} 0 & 2 & -2 \\ 2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$ 8.1.48

$$\lambda_3 = 6 \qquad \begin{bmatrix} -3 & 2 & -2 \\ 2 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$ 8.1.49

12. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$ 8.1.50

$$\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0 \qquad \{\lambda_i\} = \{1, 3, 4\}$$ 8.1.51

Finding the eigenvectors,

$$\lambda_1 = 1 \qquad \begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix} \tag{8.1.52}$$

$$\lambda_2 = 3 \qquad \begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{8.1.53}$$

$$\lambda_3 = 4 \qquad \begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_3 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \tag{8.1.54}$$

13. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.55}$$

$$\lambda^3 - 27\lambda^2 + 243\lambda - 729 = 0 \qquad \{\lambda_i\} = \{9, 9, 9\} \tag{8.1.56}$$

Finding the eigenvectors,

$$\lambda_1 = 1 \qquad \begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \tag{8.1.57}$$

14. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0.5 & 0 \\ 1 & 0 & 4 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.58}$$

$$\lambda^3 - 6.5\lambda^2 + 12\lambda - 4.5 = 0 \qquad \{\lambda_i\} = \{0.5, 3, 3\} \tag{8.1.59}$$

Finding the eigenvectors,

$$\lambda_1 = 0.5 \qquad \begin{bmatrix} 1.5 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 3.5 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{8.1.60}$$

$$\lambda_2 = 3 \qquad \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2.5 & 0 \\ 1 & 0 & 0.5 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{8.1.61}$$

15. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.1.62}$$

$$0 = \lambda^4 + 4\lambda^3 - 10\lambda^2 - 28\lambda - 15 \qquad \{\lambda_i\} = \{-5, -1, -1, 3\} \tag{8.1.63}$$

Finding the eigenvectors,

$$\lambda_1 = -5 \qquad \begin{bmatrix} 4 & 0 & 12 & 0 \\ 0 & 4 & 0 & 12 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & -4 & 4 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} -3 \\ -3 \\ 1 \\ 1 \end{bmatrix} \tag{8.1.64}$$

$$\lambda_2 = -1 \qquad \begin{bmatrix} 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -4 & 0 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{8.1.65}$$

$$\lambda_3 = 3 \qquad \begin{bmatrix} -4 & 0 & 12 & 0 \\ 0 & -4 & 0 & 12 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & -4 & -4 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_4 = \begin{bmatrix} -3 \\ 3 \\ -1 \\ 1 \end{bmatrix} \tag{8.1.66}$$

16. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 4 & 2 \\ 0 & 1 & -2 & 4 \\ 2 & 4 & -1 & -2 \\ 0 & 2 & -2 & 3 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.67}$$

$$0 = \lambda^4 - 22\lambda^2 + 24\lambda + 45 \qquad \{\lambda_i\} = \{-5, -1, 3, 3\} \tag{8.1.68}$$

Finding the eigenvectors,

$$\lambda_1 = -5 \qquad \begin{bmatrix} 2 & 0 & 4 & 2 \\ 0 & 6 & -2 & 4 \\ 2 & 4 & 4 & -2 \\ 0 & 2 & -2 & 8 \end{bmatrix}\mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} -11 \\ 1 \\ 5 \\ 1 \end{bmatrix} \tag{8.1.69}$$

$$\lambda_2 = -1 \qquad \begin{bmatrix} -2 & 0 & 4 & 2 \\ 0 & 2 & -2 & 4 \\ 2 & 4 & 0 & -2 \\ 0 & 2 & -2 & 4 \end{bmatrix}\mathbf{x} = 0 \qquad \mathbf{u}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix} \tag{8.1.70}$$

$$\lambda_3 = 3 \qquad \begin{bmatrix} -6 & 0 & 4 & 2 \\ 0 & -2 & -2 & 4 \\ 2 & 4 & -4 & -2 \\ 0 & 2 & -2 & 0 \end{bmatrix}\mathbf{x} = 0 \qquad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{8.1.71}$$

17. The matrix corresponding to the linear transform is,

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.72}$$

$$\lambda^2 + 1 = 0 \qquad \{\lambda_i\} = \{0 \pm i\} \tag{8.1.73}$$

Finding the eigenvectors, Finding the eigenvectors,

$$\lambda_1 = 0 - i \qquad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{8.1.74}$$

$$\lambda_2 = 0 + i \qquad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \tag{8.1.75}$$

Since the eigenvectors are complex, there exists no vector in $\mathcal{R}^2$ whose direction is preserved under this transform.

18. The matrix corresponding to the linear transform is,

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.76}$$

$$\lambda^2 - 1 = 0 \qquad \{\lambda_i\} = \{-1, 1\} \tag{8.1.77}$$

Finding the eigenvectors, Finding the eigenvectors,

$$\lambda_1 = -1 \qquad \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.1.78}$$

$$\lambda_2 = 1 \qquad \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8.1.79}$$

Any vectors coinciding with the x_1 axis reflect onto themselves.

Any vectors coinciding with the x_2 axis reflect onto their additive inverse. (Since this is Cartesian $\mathcal{R}^2$)

19. The matrix corresponding to the linear transform is,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.80}$$

$$\lambda^2 - \lambda = 0 \qquad \{\lambda_i\} = \{0, 1\} \tag{8.1.81}$$

Finding the eigenvectors, Finding the eigenvectors,

$$\lambda_1 = 0 \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.1.82}$$

$$\lambda_2 = 1 \qquad \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8.1.83}$$

Any vectors coinciding with the x_1 axis project to the zero vector.

Any vectors coinciding with the x_2 axis project onto themselves

20. The projection onto the plane is obtained by subtracting the projection onto the normal vector from the vector itself.

$$\mathbf{v} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{n})\, \mathbf{n} \qquad \mathbf{n} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{8.1.84}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - \begin{bmatrix} 0.5v_1 - 0.5v_2 \\ 0.5v_2 - 0.5v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5v_1 + 0.5v_2 \\ 0.5v_1 + 0.5v_2 \\ v_3 \end{bmatrix} \tag{8.1.85}$$

The matrix corresponding to the linear transform is,

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.1.86}$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0 \qquad \{\lambda_i\} = \{0, 1, 1\} \tag{8.1.87}$$

Finding the eigenvectors,

$$\lambda_1 = 0 \qquad \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{8.1.88}$$

$$\lambda_2 = 1 \qquad \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x} = 0 \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{8.1.89}$$

Any vectors in the plane get mapped onto themselves.

Any vectors along the normal to the plane $\mathbf{n}$ get mapped to the zero vector.

21. Defect is the difference between algebraic and geometric multiplicity.

$$\Delta_\lambda = M_\lambda - m_\lambda \tag{8.1.90}$$

From this problem set, 7, 13, 14, 16 have nonzero defect.

22. From this problem set 1, 3, 4, 11, 12 have multiple eigenvalues, all of which are distinct.

23. Eigenvalues of a real matrix are roots of a polynomial with real coefficients. Since these roots are either real or complex conjugate pairs, the statemenet is true.

24. Starting with the characteristic polynomial,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = P(\lambda) \qquad P(\lambda = 0) = \prod_{i=1}^{n} \lambda_i \tag{8.1.91}$$

$$\prod_{i=1}^{n} \lambda_i = \det(\mathbf{A}) \tag{8.1.92}$$

Thus, the product of eigenvalues is equal to the determinant. If at least one eigenvalue is zero, $\mathbf{A}$ is non-invertible.

Conversely, if $\mathbf{A}$ is non-invertible, then it is singular and at least one eigenvalue is zero.

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \qquad \mathbf{A}^{-1}\ \mathbf{A}\mathbf{v} = \mathbf{A}^{-1}\ \lambda \mathbf{v} = \mathbf{v} \tag{8.1.93}$$

$$\mathbf{A}^{-1}\mathbf{v} = \frac{1}{\lambda}\ \mathbf{v} \tag{8.1.94}$$

If $\mathbf{A}$ has eigenvector λ, then $\mathbf{A}^{-1}$ has corresponding eigenvector λ^{-1}.

25. $\mathbf{A}^T$ has the same characteristic polynomial as $\mathbf{A}$. All further steps are identical, leading to identical eigenvalues. Examples TBC.

8.2 Some Applications of Eigenvalue Problems

1. Finding eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 3 & 1.5 \\ 1.5 & 3 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.2.1}$$

$$0 = (\lambda - 3)^2 - (1.5)^2 \qquad 0 = \lambda^2 - 6\lambda + 6.75 \tag{8.2.2}$$

$$\{\lambda_i\} = \{1.5, 4.5\} \tag{8.2.3}$$

Finding the eigenvectors,

$$\lambda_1 = 1.5 \qquad \begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ -45^\circ \tag{8.2.4}$$

$$\lambda_2 = 4.5 \qquad \begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ 45^\circ \tag{8.2.5}$$

2. Finding eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 2 & 0.4 \\ 0.4 & 2 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.2.6}$$

$$0 = (\lambda - 2)^2 - (0.4)^2 \qquad 0 = \lambda^2 - 4\lambda + 3.84 \tag{8.2.7}$$

$$\{\lambda_i\} = \{1.6, 2.4\} \tag{8.2.8}$$

Finding the eigenvectors,

$$\lambda_1 = 1.6 \qquad \begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ -45^\circ \tag{8.2.9}$$

$$\lambda_2 = 2.4 \qquad \begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ 45^\circ \tag{8.2.10}$$

3. Finding eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.2.11}$$

$$0 = (\lambda - 2)(\lambda - 7) - 6 \qquad 0 = \lambda^2 - 9\lambda + 8 \tag{8.2.12}$$

$$\{\lambda_i\} = \{1, 8\} \tag{8.2.13}$$

Finding the eigenvectors,

$$\lambda_1 = 1 \qquad \begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -\sqrt{6} \end{bmatrix}, \ -67.8^\circ \tag{8.2.14}$$

$$\lambda_2 = 8 \qquad \begin{bmatrix} -1 & \sqrt{6} \\ \sqrt{6} & -6 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_2 = \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}, \ 22.2^\circ \tag{8.2.15}$$

4. Finding eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.2.16}$$

$$0 = (\lambda - 5)(\lambda - 13) - 4 \qquad 0 = \lambda^2 - 18\lambda + 61 \tag{8.2.17}$$

$$\{\lambda_i\} = \{9 \pm 2\sqrt{5}\} \tag{8.2.18}$$

Finding the eigenvectors,

$$\lambda_1 = 9 - 2\sqrt{5} \qquad \begin{bmatrix} -4 + 2\sqrt{5} & 2 \\ 2 & 4 + 2\sqrt{5} \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 - \sqrt{5} \end{bmatrix}, \ -76.72^\circ \tag{8.2.19}$$

$$\lambda_2 = 9 + 2\sqrt{5} \qquad \begin{bmatrix} -4 - 2\sqrt{5} & 2 \\ 2 & 4 - 2\sqrt{5} \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 + \sqrt{5} \end{bmatrix}, \ 13.28^\circ \tag{8.2.20}$$

5. Finding eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.2.21}$$

$$0 = (\lambda - 1)^2 - 0.25 \qquad 0 = \lambda^2 - 2\lambda + 0.75 \tag{8.2.22}$$

$$\{\lambda_i\} = \{0.5, 1.5\} \tag{8.2.23}$$

Finding the eigenvectors,

$$\lambda_1 = 0.5 \qquad \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ -45^\circ \tag{8.2.24}$$

$$\lambda_2 = 1.5 \qquad \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ 45^\circ \tag{8.2.25}$$

6. Finding eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.2.26}$$

$$0 = (\lambda - 1.25)^2 - (0.75)^2 \qquad 0 = \lambda^2 - 2.5\lambda + 1 \tag{8.2.27}$$

$$\{\lambda_i\} = \{0.5, 2\} \tag{8.2.28}$$

Finding the eigenvectors,

$$\lambda_1 = 0.5 \qquad \begin{bmatrix} 0.75 & 0.75 \\ 0.75 & 0.75 \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ -45^\circ \tag{8.2.29}$$

$$\lambda_2 = 2 \qquad \begin{bmatrix} -0.75 & 0.75 \\ 0.75 & -0.75 \end{bmatrix} \mathbf{x} = \mathbf{0} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ 45^\circ \tag{8.2.30}$$

7. Finding eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.2.31}$$

$$0 = (\lambda - 0.2)(\lambda - 0.5) - 0.4 \qquad 0 = \lambda^2 - 0.7\lambda - 0.3 \tag{8.2.32}$$

$$\{\lambda_i\} = \{-0.3, 1\} \tag{8.2.33}$$

Finding the eigenvectors,

$$\lambda_1 = -0.3 \qquad \begin{bmatrix} 0.5 & 0.5 \\ 0.8 & 0.8 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8.2.34}$$

$$\lambda_2 = 1 \qquad \begin{bmatrix} -0.8 & 0.5 \\ 0.8 & -0.5 \end{bmatrix}\mathbf{x} = \mathbf{0} \qquad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \tag{8.2.35}$$

Since a Markov process has steady state when $\lambda = 1$, the steady state is $\mathbf{u}_2$.

8. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.2.36}$$

$$0 = \lambda^3 - 1.6\lambda^2 + 0.65\lambda - 0.05 \qquad \{\lambda_i\} = \{0.1, 0.5, 1\} \tag{8.2.37}$$

Finding the steady state using the unity eigenvalue,

$$\lambda_1 = 1 \qquad \begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.1 & -0.4 \end{bmatrix}\mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{8.2.38}$$

9. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{8.2.39}$$

$$0 = \lambda^3 - 1.1\lambda^2 - 0.02\lambda + 0.12 \qquad \{\lambda_i\} = \{-0.3, 0.4, 1\} \tag{8.2.40}$$

Finding the steady state using the unity eigenvalue,

$$\lambda_1 = 1 \qquad \begin{bmatrix} -0.4 & 0.1 & 0.2 \\ 0.4 & -0.9 & 0.4 \\ 0 & 0.8 & -0.6 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 11 \\ 12 \\ 16 \end{bmatrix} \tag{8.2.41}$$

10. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0 & 9 & 5 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.2.42}$$

$$0 = \lambda^3 - 3.6\lambda - 0.8 \qquad \{\lambda_i\} = \{2, -1 \pm 0.2\sqrt{15}\} \tag{8.2.43}$$

Finding the growth rate using the positive eigenvalue,

$$\lambda_1 = 2 \qquad \begin{bmatrix} -2 & 9 & 5 \\ 0.4 & -2 & 0 \\ 0 & 0.4 & -2 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \frac{1}{31} \cdot \begin{bmatrix} 25 \\ 5 \\ 1 \end{bmatrix} \tag{8.2.44}$$

11. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0 & 3.45 & 0.6 \\ 0.9 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.2.45}$$

$$0 = \lambda^3 - 3.105\lambda - 0.243 \qquad \{\lambda_i\} = \{1.8, -0.9 \pm 0.15\sqrt{30}\} \tag{8.2.46}$$

Finding the growth rate using the positive eigenvalue,

$$\lambda_1 = 1.8 \qquad \begin{bmatrix} 0 & 3.45 & 0.6 \\ 0.9 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 25 \\ 5 \\ 1 \end{bmatrix} \tag{8.2.47}$$

12. Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 2 & 2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.2.48}$$

$$0 = \lambda^3 - 1.5\lambda^2 - 0.5\lambda - 0.05 \qquad \{\lambda_i\} = \{1.8, -0.9 \pm 0.15\sqrt{30}\} \tag{8.2.49}$$

Finding the growth rate using the positive eigenvalue,

$$\lambda_1 = 1.375 \qquad \begin{bmatrix} -1.375 & 3 & 2 & 2 \\ 0.5 & -1.375 & 0 & 0 \\ 0 & 0.5 & -1.375 & 0 \\ 0 & 0 & 0.1 & -1.375 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 103.94 \\ 37.80 \\ 13.75 \\ 1 \end{bmatrix} \tag{8.2.50}$$

13. The total purchase by industry j is,

$$\sum_{k=1}^{n} a_{jk}\, p_j \equiv \mathbf{A}_j \cdot \mathbf{p} \tag{8.2.51}$$

The total revenue of industry j is simply p_j, which gives the equation

$$\mathbf{A}\mathbf{p} = \lambda\, \mathbf{p} \tag{8.2.52}$$

For revenue to be equal to expenditure, $\lambda = 1$.
Finding the eigenvalues,

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{8.2.53}$$

$$0 = \lambda^3 - 0.7\lambda^2 - 0.54\lambda + 0.24 \qquad \{\lambda_i\} = \{-0.66, 0.36, 1\} \tag{8.2.54}$$

Finding the equilibrium price using the unity eigenvalue,

$$\lambda_1 = 1 \qquad \begin{bmatrix} -0.9 & 0.5 & 0 \\ 0.8 & -1 & 0.4 \\ 0.1 & 0.5 & -0.4 \end{bmatrix} \mathbf{x} = 0 \qquad \mathbf{u}_1 = \begin{bmatrix} 10 \\ 18 \\ 25 \end{bmatrix} \tag{8.2.55}$$

14. Since a column is the set of all fractions of a certain quantity, it must sum to 1 by definition. (Similar to a Markov process).

Consider adding to the first row of $\mathbf{A}-\mathbf{I}$, every other row. This leaves the determinant unchanged.

$$\mathbf{A}-\mathbf{I} = \begin{bmatrix} v_1-1 & v_3 & v_5 \\ v_2 & v_4-1 & v_6 \\ 1-v_1-v_2 & 1-v_3-v_4 & -v_5-v_6 \end{bmatrix} \tag{8.2.56}$$

$$\det(\mathbf{A}-\mathbf{I}) = \begin{bmatrix} 0 & 0 & 0 \\ v_2 & v_4-1 & v_6 \\ 1-v_1-v_2 & 1-v_3-v_4 & -v_5-v_6 \end{bmatrix} = 0 \tag{8.2.57}$$

This means that $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$ will always have the solution $\lambda = 1$. (Illustrated for $n = 3$, but this result holds for general n).

15. Given the equation,

$$\mathbf{x}-\mathbf{A}\mathbf{x} = y \qquad (\mathbf{I}-\mathbf{A})\mathbf{x} = \mathbf{y} \tag{8.2.58}$$

$$\mathbf{B}\mathbf{x} = \begin{bmatrix} 0.9 & -0.4 & -0.2 \\ -0.5 & 1 & -0.9 \\ -0.1 & -0.4 & 0.6 \end{bmatrix}\mathbf{x} = \mathbf{y} \qquad \mathbf{x} = \mathbf{B}^{-1}\mathbf{y} \tag{8.2.59}$$

$$\mathbf{y} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \frac{11}{15} \\ \frac{71}{90} \\ \frac{11}{9} \end{bmatrix} \tag{8.2.60}$$

16. Sum of the main diagonal entries is,

$$\det(\mathbf{A}-\lambda\mathbf{I}) = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n) \tag{8.2.61}$$

$$= (-1)^n\Big[\lambda^n - \mathrm{tr}(\mathbf{A})\ \lambda^{n-1} + \ldots\Big] \tag{8.2.62}$$

Comparing coefficients of λ^{n-1} proves the relation

$$\mathrm{tr}(\mathbf{A}) = \sum \lambda_i \tag{8.2.63}$$

17. Starting with $\mathbf{B} = \mathbf{A} - k\mathbf{I}$, which has eigenvalues $\{\mu_i\}$ and eigenvectors $\{\mathbf{b}_i\}$

$$\det(\mathbf{B}-\mu\mathbf{I}) = 0 \qquad \det\Big[\mathbf{A}-(k+\mu)\mathbf{I}\Big] = 0 \tag{8.2.64}$$

8.2.65

So, if the eigenvales of $\mathbf{A}$ form the set $\{\lambda_i\}$, then the corresponding set of eigenvalues of $\mathbf{B}$ is $\{\lambda_i - k\}$.

Solving for the eigenvector μ_1,

$$\mathbf{Bx} = \mu_1 \mathbf{x} \qquad\qquad \mathbf{Ax} = \lambda_1 \mathbf{x} \tag{8.2.66}$$

$$\mathbf{Ax} - k\mathbf{x} = \mu_1 \mathbf{x} \tag{8.2.67}$$

These are the same equation, which means they have the same solution. Thus, spectral shifts do not change the eigenvector.

18. Scalar multiples of $\mathbf{A}$,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \qquad\qquad \det(k\mathbf{A} - \mu \mathbf{I}) = 0 \tag{8.2.68}$$

$$\det \Big[k(\mathbf{A} - \lambda \mathbf{I}) \Big] = k^n \ 0 = 0 \qquad\qquad \mu = k\lambda \tag{8.2.69}$$

The eigenvectors are found using,

$$(\mathbf{A} - \lambda_1 \mathbf{I}) \ \mathbf{x} = 0 \qquad\qquad (k\mathbf{A} - k\lambda_1 \mathbf{I}) \ \mathbf{x} = 0 \tag{8.2.70}$$

Since the second equation is the same as the first, the solutions are the same.

Looking at positive integer powers of $\mathbf{A}$, by induction

$$\text{If} \quad \mathbf{A}^m \mathbf{x} = \lambda^m \mathbf{x} \tag{8.2.71}$$

$$\text{then} \quad \mathbf{A}^{m+1} \mathbf{x} = \mathbf{A} \ (\mathbf{A}^m \mathbf{x}) = \mathbf{A} \ \lambda^m \mathbf{x} = \lambda^{m+1} \mathbf{x} \tag{8.2.72}$$

Since the above relation is true for $m = 1$, induction makes it true for all positive integers.

The eigenvectors are found using the equation,

$$\mathbf{A} \mathbf{x}_1 = \lambda_1 \ \mathbf{x}_1 \tag{8.2.73}$$

$$\mathbf{A}^m \mathbf{x}_1 = \lambda_1 \ \mathbf{A}^m \ \mathbf{x}_1 = \lambda_1^m \mathbf{x}_1 \tag{8.2.74}$$

Thus the eigenvectors remain unchanged.

19. Looking at the sum of matrices,

$$\mathbf{Ax} = \lambda \ \mathbf{x} \qquad\qquad \mathbf{Bx} = \mu \ \mathbf{x} \tag{8.2.75}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{x} = (\lambda + \mu) \ \mathbf{x} \tag{8.2.76}$$

In combination with the results of Prob. 17, 18, the result is proved trivially.

20. Some terms of a Leslie matrix are zero by defintion, since an organism can only go from the current age range to the next higher age range.

Now, with $b, c, d, e > 0$,

$$\mathbf{L} = \begin{bmatrix} a & b & c \\ d & 0 & 0 \\ 0 & e & 0 \end{bmatrix} \tag{8.2.77}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (a - \lambda)(\lambda^2) - d(-b\lambda - ce) = 0 \tag{8.2.78}$$

$$\lambda^3 - a\lambda^2 - bd\lambda - cde = 0 \tag{8.2.79}$$

$$cde > 0 \implies \lambda_1 \lambda_2 \lambda_3 > 0 \tag{8.2.80}$$

This means none of the roots are zero, and either no roots are negative or two roots are negative.

Thus, at least one root is guaranteed positive.

For the case of the other two roots being complex, their product is always positive.

8.3 Symmetric, Skew-Symmetric, and Orthogonal Matrices

1. The matrix is orthogonal

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} = \mathbf{A}^T \tag{8.3.1}$$

$$0 = (\lambda - 0.8)^2 + 0.6^2 \qquad \{\lambda_i\} = \{0.8 \pm 0.6i\} \tag{8.3.2}$$

$$|\lambda_i| = 1 \quad \forall \quad i \tag{8.3.3}$$

The eigenvalues are real or complex conjugate pairs, with absolute value 1.

2. The matrix is not special

$$\mathbf{A} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{8.3.4}$$

$$0 = (\lambda - a)^2 + b^2 \qquad \{\lambda_i\} = \{a \pm bi\} \tag{8.3.5}$$

$$|\lambda_i| = \sqrt{a^2 + b^2} \quad \forall \quad i \tag{8.3.6}$$

Theorem 5 holds if $a^2 + b^2 = 1$, and the matrix becomes orthogonal.

3. The matrix is not special

$$\mathbf{A} = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{\sqrt{68}} \begin{bmatrix} 2 & -8 \\ 8 & 2 \end{bmatrix} \tag{8.3.7}$$

$$0 = (\lambda - 2)^2 + 8^2 \qquad \{\lambda_i\} = \{2 \pm 8i\} \tag{8.3.8}$$

$$|\lambda_i| = \sqrt{68} \quad \forall \quad i \tag{8.3.9}$$

4. The matrix is orthogonal

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{8.3.10}$$

$$0 = (\lambda - \cos\theta)^2 + \sin^2\theta \qquad \{\lambda_i\} = \{\cos\theta \pm \sin\theta\} \tag{8.3.11}$$

$$|\lambda_i| = 1 \quad \forall \quad i \tag{8.3.12}$$

The eigenvalues are real or complex conjugate pairs, with absolute value 1.

5. The matrix is symmetric

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix} \qquad \mathbf{A}^{T} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix} \tag{8.3.13}$$

$$\mathbf{A}^{-1} = \frac{1}{36} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 30 & 12 \\ 0 & 12 & 12 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{5}{6} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{8.3.14}$$

$$0 = \lambda^3 - 13\lambda^2 + 48\lambda - 36 \qquad \{\lambda_i\} = \{1, 6, 6\} \tag{8.3.15}$$

The eigenvalues are real.

6. The matrix is symmetric

$$\mathbf{A} = \begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix} \tag{8.3.16}$$

$$\mathbf{A}^T = \begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix} \tag{8.3.17}$$

$$\mathbf{A}^{-1} = \frac{1}{a^2 + ak - 2k^2} \begin{bmatrix} (a+k) & -k & -k \\ -k & (a+k) & -k \\ -k & -k & (a+k) \end{bmatrix} \tag{8.3.18}$$

$$0 = \lambda^3 - 3a\lambda^2 + 3(a^2 - k^2)\lambda - (a^3 + 3ak^2 - 2k^3) \tag{8.3.19}$$

$$\{\lambda_i\} = \{a + 2k, a - k, a - k\} \tag{8.3.20}$$

The eigenvalues are real.

7. The matrix is skew-symmetric

$$\mathbf{A} = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix} = -\mathbf{A} \tag{8.3.21}$$

$$\det(\mathbf{A}) = 0 \qquad \mathbf{A}^{-1} \text{ does not exist} \tag{8.3.22}$$

$$0 = \lambda^3 + 625\lambda \qquad \{\lambda_i\} = \{0, 0 \pm 25i\} \tag{8.3.23}$$

The eigenvalues are either zero or purely imaginary.

8. The matrix is orthogonal

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{8.3.24}$$

$$\det(\mathbf{A}) = 1 \qquad \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \mathbf{A}^T \tag{8.3.25}$$

$$0 = (1 - \lambda)(\lambda^2 - 2\lambda\cos\theta + 1) \qquad \{\lambda_i\} = \{1, \cos\theta \pm i\sin\theta\} \tag{8.3.26}$$

The eigenvalues are either real or complex conjuagate pairs with $|\lambda_i| = 1$.

9. The matrix is orthogonal

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{8.3.27}$$

$$\det(\mathbf{A}) = 1 \qquad \mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \mathbf{A}^T \tag{8.3.28}$$

$$0 = \lambda^3 - \lambda^2 + \lambda - 1 \qquad \{\lambda_i\} = \{1, 0 \pm i\} \tag{8.3.29}$$

The eigenvalues are either real or complex conjuagate pairs with $|\lambda_i| = 1$.

10. The matrix is orthogonal

$$\mathbf{A} = \frac{1}{9}\begin{bmatrix} 4 & 8 & 1 \\ -7 & 4 & -4 \\ -4 & 1 & 8 \end{bmatrix} \qquad \mathbf{A}^T = \frac{1}{9}\begin{bmatrix} 4 & -7 & -4 \\ 8 & 4 & 1 \\ 1 & -4 & 8 \end{bmatrix} \tag{8.3.30}$$

$$\det(\mathbf{A}) = 1 \qquad \mathbf{A}^{-1} = \frac{1}{9}\begin{bmatrix} 4 & -7 & -4 \\ 8 & 4 & 1 \\ 1 & -4 & 8 \end{bmatrix} = \mathbf{A}^T \tag{8.3.31}$$

$$0 = \lambda^3 - \frac{16}{9}\,\lambda^2 + \frac{16}{9}\,\lambda - 1 \qquad \{\lambda_i\} = \left\{1, \frac{7}{18} \pm \frac{\sqrt{275}}{18} i\right\} \tag{8.3.32}$$

The eigenvalues are either real or complex conjuagate pairs with $|\lambda_i| = 1$.

11. Refer notes. Exmaples TBC. (Take from exercises)

12. Properties of orthogonal matrices.

    (a) Let $\mathbf{A}$, $\mathbf{B}$ be orthogonal,

$$\mathbf{C} = \mathbf{A}\mathbf{B} \qquad \mathbf{C}^T = \mathbf{B}^T\mathbf{A}^T \tag{8.3.33}$$

$$\mathbf{C}^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \mathbf{C}^{-1} = \mathbf{C}^T \tag{8.3.34}$$

    Their product is also orthogonal.

(b) Checking if the matrix is orthogonal,

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{8.3.35}$$

$$0 = (\lambda - \cos\theta)^2 + \sin^2\theta \qquad \{\lambda_i\} = \{\cos\theta \pm \sin\theta\} \tag{8.3.36}$$

$$|\lambda_i| = 1 \quad \forall \quad i \tag{8.3.37}$$

Checking the orthonormality of rows,

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = 0 \qquad \mathbf{a}_1 \cdot \mathbf{a}_1 = 1 \tag{8.3.38}$$

$$\mathbf{a}_2 \cdot \mathbf{a}_2 = 1 \tag{8.3.39}$$

The rows and columns do form an orthonormal system. The inverse transformation is rotation by angle $-\theta$

(c) The eigenvalues of $\mathbf{A}^m$ are $\{\lambda_i^m\}$ as shown in Problem 18, section 8.2

$$\mathbf{A} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{8.3.40}$$

$$0 = (\lambda - a)^2 + b^2 \qquad \{\lambda_i\} = \{a \pm bi\} \tag{8.3.41}$$

$$|\lambda_i| = \sqrt{a^2+b^2} \quad \forall \quad i \tag{8.3.42}$$

Assuming the matrix is orthogonal, defining

$$\theta = \arctan\left(\frac{b}{a}\right) \qquad R = \sqrt{a^2+b^2} \tag{8.3.43}$$

$$\lambda_1 = R\exp(i\ \theta) \qquad \lambda_1^m = R^m \exp(i\ m\theta) \tag{8.3.44}$$

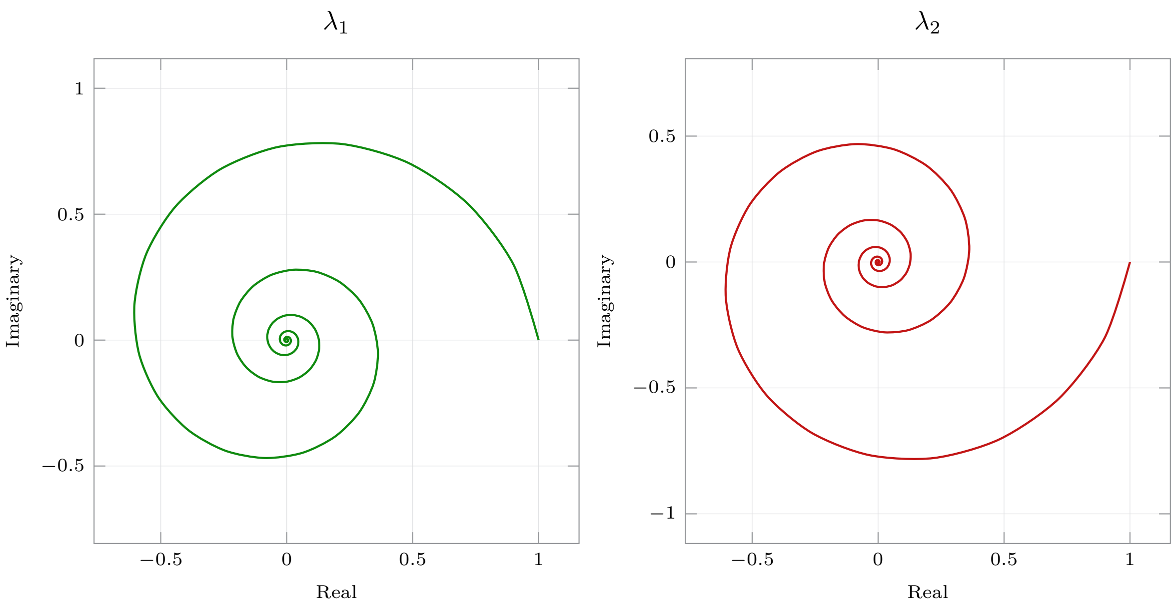
$$\lambda_2 = R\exp(-i\ \theta) \qquad \lambda_2^m = R^m \exp(-i\ m\theta) \tag{8.3.45}$$

The matrix $\mathbf{A}$ corresponds to rotation by 36.87° in the clockwise direction.
There is no asymptotic eigenvalue since $R = 1$.

(d) Since $R < 1$, the eigenvalues decay to zero with increasing m.

$$\lambda_1^m = R^m\cos(m\theta) + i\ R^m\sin(m\theta) \tag{8.3.46}$$

$$\lambda_2^m = R^m\cos(m\theta) - i\ R^m\sin(m\theta) \tag{8.3.47}$$



The eigenvalues approach the limit $0 \pm 0i$ along a spiral.

(e) Counter-clockwise rotation through 30° in the 2-d plane,

$$\mathbf{A} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \tag{8.3.48}$$

13. Verifying,

$$\mathbf{A} = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} = \mathbf{A} \tag{8.3.49}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad \mathbf{B}^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix} = -\mathbf{B} \tag{8.3.50}$$

$$\mathbf{C} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix} \qquad \mathbf{C}^T = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 1 & -2 \end{bmatrix} \tag{8.3.51}$$

$$\det(\mathbf{C}) = -1 \qquad \mathbf{C}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 1 & -2 \end{bmatrix} = \mathbf{A}^T \tag{8.3.52}$$

14. Refer Problem 7 for skew-symmetric matrix.

Checking the eigenvalues of the matrix,

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \qquad (\lambda - 3)^2 - 4 = 0 \tag{8.3.53}$$

$$\{\lambda_i\} = \{1, 5\} \tag{8.3.54}$$

This does not violate the theorem, because it only has the forward implication.
Checking the given matrix,

$$\mathbf{M} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \det(\mathbf{M}) = \frac{-12 + 2(-6) + (-3)}{27} = -1 \tag{8.3.55}$$

Refer to Problem 4.

15. Checking,

$$\mathbf{A}\mathbf{x}_j = \lambda_j \mathbf{x}_j \qquad \mathbf{B}\mathbf{x}_k = \mu_k \mathbf{x}_k \tag{8.3.56}$$

Only eigenvectors common to the two sets satisfy the required relation. So the relation is not true in general.

16. Different eigenvalues of a symmetric matrix,

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \qquad \mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2 \tag{8.3.57}$$

$$\mathbf{x}_1^T \mathbf{A}^T = \lambda_1 \mathbf{x}_1^T \qquad \mathbf{x}_1^T \mathbf{A}^T \mathbf{x}_2 = \lambda_1 \ \mathbf{x}_1^T \mathbf{x}_2 \tag{8.3.58}$$

$$\mathbf{A}^T = \mathbf{A} \qquad \lambda_2 \ \mathbf{x}_1^T \mathbf{x}_2 = \lambda_1 \ \mathbf{x}_1^T \mathbf{x}_2 \tag{8.3.59}$$

Since the eigenvectors correspond to distinct eigenvalues, $\lambda_2 \neq \lambda_1$.

$$\lambda_1 \neq \lambda_2 \implies \mathbf{x}_1^T \mathbf{x}_2 = 0 \tag{8.3.60}$$

17. Let $\mathbf{B}^T = -\mathbf{B}$,

$$\mathbf{B}^{-1} \ \mathbf{B} = \mathbf{I} \qquad \mathbf{B}^T \left(\mathbf{B}^{-1}\right)^T = \mathbf{I} \tag{8.3.61}$$

$$\mathbf{B} \left(\mathbf{B}^{-1}\right)^T = -\mathbf{I} \qquad \mathbf{B}^{-1}\mathbf{B} \left(\mathbf{B}^{-1}\right)^T = -\mathbf{B}^{-1}\mathbf{I} \tag{8.3.62}$$

$$\left(\mathbf{B}^{-1}\right)^T = -\mathbf{B}^{-1} \tag{8.3.63}$$

Thus, the inverse of a skew-symmetric matrix is also skew-symmetric.

18. Let $\mathbf{A}$ be a skew-symmetric matrix of odd order,

$$\mathbf{A} = -\mathbf{A}^T \qquad \det(\mathbf{A}^T) = \det(\mathbf{A}) \tag{8.3.64}$$

$$\det(-\mathbf{A}) = \det(\mathbf{A}) \qquad (-1)^n \det(\mathbf{A}) = \det(\mathbf{A}) \tag{8.3.65}$$

$$\text{n odd} \implies \det(\mathbf{A}) = 0 \tag{8.3.66}$$

So, all such matrices are singular. The claim is false.

19. Let the matrix be orthogonal

$$\mathbf{A}^T = \mathbf{A}^{-1} \qquad \det(\mathbf{A}) \neq 0 \tag{8.3.67}$$

From the result in Problem 18, skew-symmetric matrices of odd order cannot have an inverse, and thus cannot be orthogonal. So, the claim is false.

20. The matrix is of odd order, symmetric, and orthogonal.

$$\mathbf{A}^T = \mathbf{A}^{-1} \qquad \mathbf{A}^T = \mathbf{A} \tag{8.3.68}$$

$$\implies \mathbf{A} = \mathbf{A}^{-1} \tag{8.3.69}$$

All such matrices are diagonal. This makes the claim false.

8.4 Eigenbases, Diagonalization, Quadratic Forms

1. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad 0 = \lambda^2 - 9 - 16 \tag{8.4.1}$$

$$\{\lambda_i\} = \{-5, 5\} \tag{8.4.2}$$

$$\lambda_1 = -5 \qquad \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\mathbf{x} = \mathbf{0}, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{8.4.3}$$

$$\lambda_2 = 5 \qquad \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\mathbf{x} = \mathbf{0}, \ \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{8.4.4}$$

Finding the eigenvalues and eigenvectors of $\hat{\mathbf{A}}$

$$\mathbf{P} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \qquad \hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix} \tag{8.4.5}$$

$$0 = \lambda^2 - 625 + 600 \qquad \{\lambda_i\} = \{-5, 5\} \tag{8.4.6}$$

$$\lambda_1 = -5 \qquad \begin{bmatrix} -20 & 12 \\ -50 & 30 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{y}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \tag{8.4.7}$$

$$\lambda_2 = 5 \qquad \begin{bmatrix} -30 & 12 \\ -50 & 20 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{y}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \tag{8.4.8}$$

The eigenvalues match. Verifying the eigenvector relationship,

$$\mathbf{P}\mathbf{y}_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \propto \mathbf{x}_1 \qquad \mathbf{P}\mathbf{y}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \propto \mathbf{x}_2 \tag{8.4.9}$$

2. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \qquad 0 = \lambda^2 - 1 \tag{8.4.10}$$

$$\{\lambda_i\} = \{-1, 1\} \tag{8.4.11}$$

$$\lambda_1 = -1 \qquad \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8.4.12}$$

$$\lambda_2 = 1 \qquad \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.4.13}$$

Finding the eigenvalues and eigenvectors of $\hat{\mathbf{A}}$

$$\mathbf{P} = \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix} \qquad \hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -29 & 20 \\ -42 & 29 \end{bmatrix} \tag{8.4.14}$$

$$0 = \lambda^2 - 841 + 840 \qquad \{\lambda_i\} = \{-1, 1\} \tag{8.4.15}$$

$$\lambda_1 = -1 \qquad \begin{bmatrix} -28 & 20 \\ -42 & 30 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{y}_1 = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \tag{8.4.16}$$

$$\lambda_2 = 1 \qquad \begin{bmatrix} -30 & 20 \\ -42 & 28 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{y}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{8.4.17}$$

The eigenvalues match. Verifying the eigenvector relationship,

$$\mathbf{P}\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \propto \mathbf{x}_1 \qquad \mathbf{P}\mathbf{y}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \propto \mathbf{x}_2 \tag{8.4.18}$$

3. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} \qquad 0 = \lambda^2 - 10\lambda + 24 \tag{8.4.19}$$

$$\{\lambda_i\} = \{4, 6\} \tag{8.4.20}$$

$$\lambda_1 = 4 \qquad \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.4.21}$$

$$\lambda_2 = 6 \qquad \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{8.4.22}$$

Finding the eigenvalues and eigenvectors of $\hat{\mathbf{A}}$

$$\mathbf{P} = \begin{bmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{bmatrix} \qquad \hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 3.008 & -0.544 \\ 5.456 & 6.992 \end{bmatrix} \tag{8.4.23}$$

$$0 = \lambda^2 - 10\lambda + 24 \qquad \{\lambda_i\} = \{4, 6\} \tag{8.4.24}$$

$$\lambda_1 = 4 \qquad \begin{bmatrix} -0.992 & -0.544 \\ 5.456 & 2.992 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{y}_1 = \begin{bmatrix} -17 \\ 31 \end{bmatrix} \tag{8.4.25}$$

$$\lambda_2 = 6 \qquad \begin{bmatrix} -2.992 & -0.544 \\ 5.456 & 0.992 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{y}_2 = \begin{bmatrix} -2 \\ 11 \end{bmatrix} \tag{8.4.26}$$

The eigenvalues match. Verifying the eigenvector relationship,

$$\mathbf{P}\mathbf{y}_1 = \begin{bmatrix} 25 \\ 25 \end{bmatrix} \propto \mathbf{x}_1 \qquad \mathbf{P}\mathbf{y}_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \propto \mathbf{x}_2 \tag{8.4.27}$$

4. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \qquad 0 = \lambda^3 - 4\lambda^2 + \lambda + 6 \qquad \{\lambda_i\} = \{-1, 2, 3\} \tag{8.4.28}$$

$$\lambda_1 = -1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3 \tag{8.4.29}$$

$$\mathbf{x}_1 = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{8.4.30}$$

Finding the eigenvalues and eigenvectors of $\hat{\mathbf{A}}$

$$\mathbf{P} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix} \qquad \hat{\mathbf{A}} = \begin{bmatrix} 15 & 0 & 26 \\ 6 & 3 & 10 \\ -8 & 0 & -14 \end{bmatrix} \qquad 0 = \lambda^3 - 4\lambda^2 + \lambda + 6 \tag{8.4.31}$$

$$\lambda_1 = -1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3 \tag{8.4.32}$$

$$\mathbf{x}_1 = \begin{bmatrix} -26 \\ -1 \\ 16 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{8.4.33}$$

The eigenvalues match. Verifying the eigenvector relationship,

$$\mathbf{P}\mathbf{y}_1 = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \propto \mathbf{x}_1 \qquad \mathbf{P}\mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \propto \mathbf{x}_2 \qquad \mathbf{P}\mathbf{y}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \propto \mathbf{x}_3 \tag{8.4.34}$$

5. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{bmatrix} \qquad 0 = \lambda^3 - 14\lambda^2 + 40\lambda \qquad \{\lambda_i\} = \{0, 4, 10\} \tag{8.4.35}$$

$$\lambda_1 = 0 \qquad \lambda_2 = 4 \qquad \lambda_3 = 10 \tag{8.4.36}$$

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{8.4.37}$$

Finding the eigenvalues and eigenvectors of $\hat{\mathbf{A}}$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \hat{\mathbf{A}} = \begin{bmatrix} 4 & 3 & -9 \\ 0 & -5 & 15 \\ 0 & -5 & 15 \end{bmatrix} \qquad 0 = \lambda^3 - 14\lambda^2 + 40\lambda \tag{8.4.38}$$

$$\lambda_1 = 0 \qquad \lambda_2 = 4 \qquad \lambda_3 = 10 \tag{8.4.39}$$

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{8.4.40}$$

The eigenvalues match. Verifying the eigenvector relationship,

$$\mathbf{P}\mathbf{y}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \propto \mathbf{x}_1 \qquad \mathbf{P}\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \propto \mathbf{x}_2 \qquad \mathbf{P}\mathbf{y}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \propto \mathbf{x}_3 \tag{8.4.41}$$

6. Similar matrices

(a) Diagonalizing the matrix does not change its eigenvalues.

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} \qquad \mathrm{tr}(\mathbf{D}) = \sum_{k=1}^{n} \lambda_k \tag{8.4.42}$$

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}) \qquad \mathrm{tr}(\mathbf{X}^{-1}\mathbf{AX}) = \mathrm{tr}(\mathbf{AXX}^{-1}) \tag{8.4.43}$$

$$\mathrm{tr}(\mathbf{D}) = \mathrm{tr}(\mathbf{A}) \qquad \implies \mathrm{tr}(\mathbf{A}) = \sum_{k=1}^{n} \lambda_k \tag{8.4.44}$$

Using the eigenvalues calculated earlier,

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad \{\lambda_k\} = \{-5, 5\} \qquad \mathrm{tr}(\mathbf{A}) = 0 = \sum \lambda_k \tag{8.4.45}$$

$$\mathbf{B} = \begin{bmatrix} 8 & -4 \\ 4 & 2 \end{bmatrix} \qquad \{\lambda_k\} = \{4, 6\} \qquad \mathrm{tr}(\mathbf{B}) = 10 = \sum \lambda_k \tag{8.4.46}$$

$$\mathbf{C} = \begin{bmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{bmatrix} \qquad \{\lambda_k\} = \{0, 4, 10\} \qquad \mathrm{tr}(\mathbf{C}) = 14 = \sum \lambda_k \tag{8.4.47}$$

(b) Proving the result,

$$\mathrm{tr}(\mathbf{AB}) = \sum_{i=1}^{n} (\mathbf{AB})_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} \tag{8.4.48}$$

$$\mathrm{tr}(\mathbf{BA}) = \sum_{i=1}^{n} (\mathbf{BA})_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}a_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ik}a_{ki} = \sum_{k=1}^{n} (\mathbf{AB})_{kk} \tag{8.4.49}$$

Refer to part a for the rest of the proof.

(c) Finding the relation,

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP} \qquad \mathbf{C} = \mathbf{PAP}^{-1} \tag{8.4.50}$$

$$\mathbf{A} = \mathbf{PBP}^{-1} \qquad \mathbf{C} = (\mathbf{P})^2\ \mathbf{B}\ \left(\mathbf{P}^{-1}\right)^2 \tag{8.4.51}$$

(d) Swapping the columns of $\mathbf{X}$ swaps the position of the corresponding eigenvalues in the diagonal of $\mathbf{D}$.

7. Matrices with all eigenvalues equal can have no eigenbasis

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \{\lambda_k\} = \{1, 1\} \tag{8.4.52}$$

$$\lambda = 1 \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.4.53}$$

Clearly, this is not a basis for $\mathcal{R}^2$.
Similarly, for order 3,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{\lambda_k\} = \{1, 1, 1\} \tag{8.4.54}$$

$$\lambda = 1 \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{8.4.55}$$

This fails to be a basis for $\mathcal{R}^3$

8. TBC

9. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad 0 = \lambda^2 - 5\lambda \tag{8.4.56}$$

$$\{\lambda_i\} = \{0, 5\} \tag{8.4.57}$$

$$\lambda_1 = 0 \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{8.4.58}$$

$$\lambda_2 = 5 \qquad \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{8.4.59}$$

Diagonalizing,

$$\mathbf{X} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{X}^{-1} = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \tag{8.4.60}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \mathbf{D} = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \tag{8.4.61}$$

10. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \qquad 0 = \lambda^2 - 1 \tag{8.4.62}$$

$$\{\lambda_i\} = \{-1, 1\} \tag{8.4.63}$$

$$\lambda_1 = -1 \qquad \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8.4.64}$$

$$\lambda_2 = 1 \qquad \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.4.65}$$

Diagonalizing,

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{X}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \tag{8.4.66}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \mathbf{D} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8.4.67}$$

11. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix} \qquad 0 = \lambda^2 + 3\lambda - 10 \tag{8.4.68}$$

$$\{\lambda_i\} = \{-5, 2\} \tag{8.4.69}$$

$$\lambda_1 = -5 \qquad \begin{bmatrix} -14 & 7 \\ -42 & 21 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{8.4.70}$$

$$\lambda_2 = 2 \qquad \begin{bmatrix} -21 & 7 \\ -42 & 14 \end{bmatrix}\mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{8.4.71}$$

Diagonalizing,

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{X}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \tag{8.4.72}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \mathbf{D} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ -10 & 6 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} \tag{8.4.73}$$

12. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} -4.3 & 7.7 \\ 1.3 & 9.3 \end{bmatrix} \qquad 0 = \lambda^2 - 5\lambda - 50 \tag{8.4.74}$$

$$\{\lambda_i\} = \{-5, 10\} \tag{8.4.75}$$

$$\lambda_1 = -5 \qquad \begin{bmatrix} 0.7 & 7.7 \\ 1.3 & 14.3 \end{bmatrix}\mathbf{x} = \mathbf{0}, \ \ \mathbf{x}_1 = \begin{bmatrix} -11 \\ 1 \end{bmatrix} \tag{8.4.76}$$

$$\lambda_2 = 10 \qquad \begin{bmatrix} -14.3 & 7.7 \\ 1.3 & -0.7 \end{bmatrix}\mathbf{x} = \mathbf{0}, \ \ \mathbf{x}_2 = \begin{bmatrix} 7 \\ 13 \end{bmatrix} \tag{8.4.77}$$

Diagonalizing,

$$\mathbf{X} = \begin{bmatrix} -11 & 7 \\ 1 & 13 \end{bmatrix} \qquad \mathbf{X}^{-1} = \frac{1}{-150}\begin{bmatrix} 13 & -7 \\ -1 & -11 \end{bmatrix} \tag{8.4.78}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \mathbf{D} = \frac{1}{-150}\begin{bmatrix} 13 & -7 \\ -1 & -11 \end{bmatrix}\begin{bmatrix} 55 & 70 \\ -5 & 130 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} -5 & 0 \\ 0 & 10 \end{bmatrix} \tag{8.4.79}$$

13. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{bmatrix} \qquad 0 = \lambda^3 - 3\lambda^2 - 6\lambda + 8 \qquad \{\lambda_k\} = \{-2, 1, 4\} \tag{8.4.80}$$

$$\lambda_1 = -2 \qquad \lambda_2 = 1 \qquad \lambda_3 = 4 \tag{8.4.81}$$

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{8.4.82}$$

Diagonalizing,

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix} \qquad \mathbf{X}^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{8.4.83}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{8.4.84}$$

14. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} -5 & -6 & 6 \\ -9 & -8 & 12 \\ -12 & -12 & 16 \end{bmatrix} \qquad 0 = \lambda^3 - 3\lambda^2 - 6\lambda + 8 \qquad \{\lambda_k\} = \{-2, 1, 4\} \tag{8.4.85}$$

$$\lambda_1 = -2 \qquad \lambda_2 = 1 \qquad \lambda_3 = 4 \tag{8.4.86}$$

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{8.4.87}$$

Diagonalizing,

$$\mathbf{X} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \qquad \mathbf{X}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} \tag{8.4.88}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{8.4.89}$$

15. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{bmatrix} \qquad 0 = \lambda^3 - 16\lambda^2 + 65\lambda - 50 \qquad \{\lambda_k\} = \{1, 5, 10\} \tag{8.4.90}$$

$$\lambda_1 = 1 \qquad \lambda_2 = 5 \qquad \lambda_3 = 10 \tag{8.4.91}$$

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{8.4.92}$$

Diagonalizing,

$$\mathbf{X} = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{X}^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \tag{8.4.93}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix} \tag{8.4.94}$$

16. Finding the eigenvalues and eigenvectors of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad 0 = \lambda^3 + 2\lambda^2 - 8\lambda \qquad \{\lambda_k\} = \{-4, 0, 2\} \tag{8.4.95}$$

$$\lambda_1 = -4 \qquad \lambda_2 = 0 \qquad \lambda_3 = 2 \tag{8.4.96}$$

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{8.4.97}$$

Diagonalizing,

$$\mathbf{X} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{X}^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{8.4.98}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{8.4.99}$$

17. The quadratic form is,

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix} \tag{8.4.100}$$

$$0 = (\lambda - 7)^2 - 3^2 \qquad \{\lambda_i\} = \{4, 10\} \tag{8.4.101}$$

$$\lambda_1 = 4 \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8.4.102}$$

$$\lambda_2 = 10 \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.4.103}$$

Finding the principal axes in terms of the old axes,

$$\mathbf{x} = \mathbf{X}\mathbf{y} \qquad \mathbf{X} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{8.4.104}$$

$$Q = \sum_{i=1}^{n} \lambda_i \ y_i^2 \qquad 200 = Q = 4y_1^2 + 10y_2^2 \tag{8.4.105}$$

$$1 = \frac{y_1^2}{50} + \frac{y_2^2}{20} \tag{8.4.106}$$

This is an ellipse with the axes along $-45°$ and $45°$

18. The quadratic form is,

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \tag{8.4.107}$$

$$0 = \lambda^2 - 9 - 16 \qquad \{\lambda_i\} = \{-5, 5\} \tag{8.4.108}$$

$$\lambda_1 = -5 \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{8.4.109}$$

$$\lambda_2 = 5 \qquad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{8.4.110}$$

Finding the principal axes in terms of the old axes,

$$\mathbf{x} = \mathbf{X} \mathbf{y} \qquad \mathbf{X} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \tag{8.4.111}$$

$$Q = \sum_{i=1}^{n} \lambda_i \ y_i^2 \qquad 10 = Q = -5y_1^2 + 5y_2^2 \tag{8.4.112}$$

$$1 = -\frac{y_1^2}{2} + \frac{y_2^2}{2} \tag{8.4.113}$$

This is an hyperbola with the axes along $-63.4°$ and $26.56°$

19. The quadratic form is,

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 3 & 11 \\ 11 & 3 \end{bmatrix} \tag{8.4.114}$$

$$0 = (\lambda - 3)^2 - 11^2 \qquad \{\lambda_i\} = \{-8, 14\} \tag{8.4.115}$$

$$\lambda_1 = -8 \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8.4.116}$$

$$\lambda_2 = 14 \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.4.117}$$

Finding the principal axes in terms of the old axes,

$$\mathbf{x} = \mathbf{X}\mathbf{y} \qquad \mathbf{X} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{8.4.118}$$

$$Q = \sum_{i=1}^{n} \lambda_i \; y_i^2 \qquad 0 = Q = -8y_1^2 + 14y_2^2 \tag{8.4.119}$$

$$y_2 = \pm \left(\frac{2}{\sqrt{7}} \right) \; y_1 \tag{8.4.120}$$

This is a pair of straight lines. with the axes rotated by 45°

20. The quadratic form is,

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \tag{8.4.121}$$

$$0 = \lambda^2 - 10\lambda \qquad \{\lambda_i\} = \{0, 10\} \tag{8.4.122}$$

$$\lambda_1 = 0 \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \tag{8.4.123}$$

$$\lambda_2 = 10 \qquad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{8.4.124}$$

Finding the principal axes in terms of the old axes,

$$\mathbf{x} = \mathbf{X}\mathbf{y} \qquad \mathbf{X} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \tag{8.4.125}$$

$$Q = \sum_{i=1}^{n} \lambda_i \; y_i^2 \qquad 10 = Q = 10y_2^2 \tag{8.4.126}$$

$$y_2 = \pm 1 \tag{8.4.127}$$

This is a pair of straight lines. with the axes rotated by $-71.56°$

21. The quadratic form is,

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix} \tag{8.4.128}$$

$$0 = (\lambda - 1)^2 - 6^2 \qquad \{\lambda_i\} = \{-5, 7\} \tag{8.4.129}$$

$$\lambda_1 = -5 \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.4.130}$$

$$\lambda_2 = 7 \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8.4.131}$$

Finding the principal axes in terms of the old axes,

$$\mathbf{x} = \mathbf{X} \mathbf{y} \qquad \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{8.4.132}$$

$$Q = \sum_{i=1}^{n} \lambda_i \; y_i^2 \qquad 70 = Q = -5y_1^2 + 7y_2^2 \tag{8.4.133}$$

$$1 = -\frac{y_1^2}{14} + \frac{y_2^2}{10} \tag{8.4.134}$$

This is a hyperbola, with the principal axes at 45° and −45°

22. The quadratic form is,

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} \tag{8.4.135}$$

$$0 = \lambda^2 - 17\lambda + 16 \qquad \{\lambda_i\} = \{1, 16\} \tag{8.4.136}$$

$$\lambda_1 = 1 \qquad \mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{8.4.137}$$

$$\lambda_2 = 16 \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{8.4.138}$$

Finding the principal axes in terms of the old axes,

$$\mathbf{x} = \mathbf{X}\mathbf{y} \qquad \mathbf{X} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \tag{8.4.139}$$

$$Q = \sum_{i=1}^{n} \lambda_i \ y_i^2 \qquad 16 = Q = y_1^2 + 16y_2^2 \tag{8.4.140}$$

$$1 = \frac{y_1^2}{16} + y_2^2 \tag{8.4.141}$$

This is an ellipse, with the principal axes at $-26.56°$ and $63.43°$

23. The quadratic form is,

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix} \tag{8.4.142}$$

$$0 = \lambda^2 - 13\lambda - 2028 \qquad \{\lambda_i\} = \{-39, 52\} \tag{8.4.143}$$

$$\lambda_1 = 39 \qquad \mathbf{x}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \tag{8.4.144}$$

$$\lambda_2 = -52 \qquad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{8.4.145}$$

Finding the principal axes in terms of the old axes,

$$\mathbf{x} = \mathbf{X}\mathbf{y} \qquad \mathbf{X} = \begin{bmatrix} -3 & 2 \\ 2 & 3 \end{bmatrix} \tag{8.4.146}$$

$$Q = \sum_{i=1}^{n} \lambda_i \ y_i^2 \qquad 156 = Q = -39y_1^2 + 52y_2^2 \tag{8.4.147}$$

$$1 = -\frac{y_1^2}{4} + \frac{y_2^2}{3} \tag{8.4.148}$$

This is an hyperbola, with the principal axes at $-33.69°$ and $56.31°$

24. Principal axis theorem,

(a) Positive definite, assume all eigenvalues are positive

$$Q = \sum_{k=1}^{n} \lambda_k \ y_k^2 \qquad \mathbf{x} \neq 0 \implies \mathbf{y} \neq 0 \tag{8.4.149}$$

$$\implies Q > 0 \tag{8.4.150}$$

The sum of squares not all of which are zero and all of which have a positive coefficient is guaranteed positive.
Assume Q is positive definite,

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} \qquad \mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{8.4.151}$$

$$\text{If} \quad \lambda < 0 \qquad Q = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda\mathbf{x}^T\mathbf{x} \tag{8.4.152}$$

$$\mathbf{x}^T\mathbf{x} > 0 \qquad \forall\ \mathbf{x} \neq \mathbf{0} \tag{8.4.153}$$

$$Q < 0 \tag{8.4.154}$$

Thus, a contradiction occurs if $\mathbf{A}$ has any negative eigenvalues, which makes it necessary to have all eigenvalues positive.

(b) Positive definite, assume all eigenvalues are negative

$$Q = \sum_{k=1}^{n} \lambda_k\ y_k^2 \qquad \mathbf{x} \neq 0 \implies \mathbf{y} \neq 0 \tag{8.4.155}$$

$$\implies Q < 0 \tag{8.4.156}$$

The sum of squares not all of which are zero and all of which have a negative coefficient is guaranteed negative.
Assume Q is negative definite,

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} \qquad \mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{8.4.157}$$

$$\text{If} \quad \lambda > 0 \qquad Q = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda\mathbf{x}^T\mathbf{x} \tag{8.4.158}$$

$$\mathbf{x}^T\mathbf{x} > 0 \qquad \forall\ \mathbf{x} \neq \mathbf{0} \tag{8.4.159}$$

$$Q > 0 \tag{8.4.160}$$

Thus, a contradiction occurs if $\mathbf{A}$ has any positive eigenvalues, which makes it necessary to have all eigenvalues negative.

(c) Indefinite. Eigenvalues are both positive and negative. Neither of the above two special cases. Contradictions occur like the special cases above, which prove both the forward and backward statements.

(d) From Prob. 22,

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} \tag{8.4.161}$$

$$M_1 = 4 > 0 \qquad M_2 = \det(\mathbf{A}) = 16 > 0 \tag{8.4.162}$$

$$\tag{8.4.163}$$

All principal minors are positive. So, the form is positive definite.

From Prob. 23,

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix} \tag{8.4.164}$$

$$M_1 = -11 < 0 \tag{8.4.165}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \implies Q = -11 \tag{8.4.166}$$

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \implies Q = 24 \tag{8.4.167}$$

There exist some non-zero vectors $\mathbf{x}$ for which Q takes positive and negative values. Thus, it is indefinite.

8.5 Complex Matrices and Forms

1. $\mathbf{A}$ is Hermitian

$$\mathbf{A} = \begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix} \qquad 0 = (\lambda - 6)^2 - 1 \tag{8.5.1}$$

$$\{\lambda_i\} = \{5, 7\} \tag{8.5.2}$$

$$\lambda_1 = 5 \qquad \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_1 = \begin{bmatrix} i \\ -1 \end{bmatrix} \tag{8.5.3}$$

$$\lambda_2 = 7 \qquad \begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}, \;\; \mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} \tag{8.5.4}$$

2. $\mathbf{A}$ is skew-Hermitian

$$\mathbf{A} = \begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix} \qquad 0 = \lambda^2 - i\lambda + 2 \tag{8.5.5}$$

$$\{\lambda_i\} = \{-i, 2i\} \tag{8.5.6}$$

$$\lambda_1 = -i \qquad \begin{bmatrix} 2i & 1+i \\ -1+i & i \end{bmatrix}\mathbf{x} = \mathbf{0},\ \mathbf{x}_1 = \begin{bmatrix} 1+i \\ -2i \end{bmatrix} \tag{8.5.7}$$

$$\lambda_2 = 2i \qquad \begin{bmatrix} -i & 1+i \\ -1+i & -2i \end{bmatrix}\mathbf{x} = \mathbf{0},\ \mathbf{x}_2 = \begin{bmatrix} 1+i \\ i \end{bmatrix} \tag{8.5.8}$$

3. $\mathbf{A}$ is Unitary

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{i\sqrt{3}}{2} \\ \frac{i\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad 0 = \lambda^2 - \lambda + 1 \tag{8.5.9}$$

$$\{\lambda_i\} = \left\{\frac{1 \pm \sqrt{3}i}{2}\right\} \qquad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-i\sqrt{3}}{2} \\ \frac{-i\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \tag{8.5.10}$$

$$\lambda_1 = \frac{1 - \sqrt{3}i}{2} \qquad \begin{bmatrix} \frac{i\sqrt{3}}{2} & \frac{i\sqrt{3}}{2} \\ \frac{i\sqrt{3}}{2} & \frac{i\sqrt{3}}{2} \end{bmatrix}\mathbf{x} = \mathbf{0},\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8.5.11}$$

$$\lambda_2 = \frac{1 + \sqrt{3}i}{2} \qquad \begin{bmatrix} -\frac{i\sqrt{3}}{2} & \frac{i\sqrt{3}}{2} \\ \frac{i\sqrt{3}}{2} & -\frac{i\sqrt{3}}{2} \end{bmatrix}\mathbf{x} = \mathbf{0},\ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.5.12}$$

4. $\mathbf{A}$ is Unitary

$$\mathbf{A} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \qquad 0 = \lambda^2 + 1 \tag{8.5.13}$$

$$\{\lambda_i\} = \{0 \pm i\} \qquad \mathbf{A}^{-1} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \tag{8.5.14}$$

$$\lambda_1 = -i \qquad \begin{bmatrix} i & i \\ i & i \end{bmatrix}\mathbf{x} = \mathbf{0},\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8.5.15}$$

$$\lambda_2 = i \qquad \begin{bmatrix} -i & i \\ i & -i \end{bmatrix}\mathbf{x} = \mathbf{0},\ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.5.16}$$

5. $\mathbf{A}$ is skew-Hermitian and Unitary

$$\mathbf{A} = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \qquad 0 = \lambda^3 - i\lambda^2 + \lambda - i \qquad \{\lambda_i\} = \{-i, i, i\} \tag{8.5.17}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \tag{8.5.18}$$

$$\lambda_1 = -i \qquad \lambda_2 = i \tag{8.5.19}$$

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{8.5.20}$$

6. $\mathbf{A}$ is Hermitian

$$\mathbf{A} = \begin{bmatrix} 0 & 2+2i & 0 \\ 2-2i & 0 & 2+2i \\ 0 & 2-2i & 0 \end{bmatrix} \qquad 0 = \lambda^3 - 16\lambda \qquad \{\lambda_i\} = \{-4, 0, 4\} \tag{8.5.21}$$

$$\mathbf{A}^{-1} = \text{not defined} \tag{8.5.22}$$

$$\lambda_1 = -4 \qquad \lambda_2 = 0 \qquad \lambda_2 = 4 \tag{8.5.23}$$

$$\mathbf{x}_1 = 4\begin{bmatrix} i \\ -1-i \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} i \\ 1+i \\ 1 \end{bmatrix} \tag{8.5.24}$$

7. Pauli spin matrices

$$\mathbf{S}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{S}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \mathbf{S}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8.5.25}$$

$$\mathbf{S}_x^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{S}_y^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{S}_z^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8.5.26}$$

$$0 = \lambda^2 - 1 \qquad 0 = \lambda^2 - 1 \qquad 0 = \lambda^2 - 1 \tag{8.5.27}$$

$$\{\lambda_i\} = \{-1, 1\} \qquad \{\lambda_j\} = \{-1, 1\} \qquad \{\lambda_k\} = \{-1, 1\} \tag{8.5.28}$$

Finding the eigenvectors

$$\lambda_{x,1} = -1 \qquad \lambda_{y,1} = -1 \qquad \lambda_{z,1} = -1 \tag{8.5.29}$$

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \mathbf{y}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \mathbf{z}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8.5.30}$$

$$\lambda_{x,2} = 1 \qquad \lambda_{y,2} = 1 \qquad \lambda_{z,2} = 1 \tag{8.5.31}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{y}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.5.32}$$

8. $\mathbf{A}$ is Hermitian

$$\mathbf{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} \qquad 0 = \lambda^2 - 11\lambda + 18 \qquad \{\lambda_i\} = \{2, 9\} \tag{8.5.33}$$

$$\lambda_1 = 2 \qquad \mathbf{0} = \begin{bmatrix} 2 & 1-3i \\ 1+3i & 5 \end{bmatrix}\mathbf{x} \qquad \mathbf{x}_1 = \begin{bmatrix} -1+3i \\ 2 \end{bmatrix} \tag{8.5.34}$$

$$\lambda_2 = 9 \qquad \mathbf{0} = \begin{bmatrix} -5 & 1-3i \\ 1+3i & -2 \end{bmatrix}\mathbf{x} \qquad \mathbf{x}_2 = \begin{bmatrix} 1-3i \\ 5 \end{bmatrix} \tag{8.5.35}$$

$\mathbf{B}$ is skew-Hermitian

$$\mathbf{B} = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} \qquad 0 = \lambda^2 - 2i\lambda + 8 \qquad \{\lambda_i\} = \{-2i, 4i\} \tag{8.5.36}$$

$$\lambda_1 = -2i \qquad \mathbf{0} = \begin{bmatrix} 5i & 2+i \\ -2+i & i \end{bmatrix}\mathbf{x} \qquad \mathbf{y}_1 = \begin{bmatrix} -1+2i \\ 5 \end{bmatrix} \tag{8.5.37}$$

$$\lambda_2 = 4i \qquad \mathbf{0} = \begin{bmatrix} -i & 2+i \\ -2+i & -5i \end{bmatrix}\mathbf{x} \qquad \mathbf{y}_2 = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix} \tag{8.5.38}$$

C is Unitary

$$\mathbf{A} = \begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix} \qquad 0 = \lambda^2 - i\lambda - 1 \qquad \{\lambda_i\} = \left\{ \frac{i \pm \sqrt{3}}{2} \right\} \tag{8.5.39}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{-i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-i}{2} \end{bmatrix} \tag{8.5.40}$$

$$\lambda_1 = \frac{i - \sqrt{3}}{2} \qquad \mathbf{0} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \mathbf{x} \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8.5.41}$$

$$\lambda_2 = \frac{i + \sqrt{3}}{2} \qquad \mathbf{0} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \mathbf{x} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8.5.42}$$

9. **A** is Hermitian,

$$\mathbf{A} = \begin{bmatrix} 4 & 3 - 2i \\ 3 + 2i & -4 \end{bmatrix} \qquad \mathbf{A}^{\dagger} = \begin{bmatrix} 4 & 3 - 2i \\ 3 + 2i & -4 \end{bmatrix} \tag{8.5.43}$$

$$\mathbf{x} = \begin{bmatrix} -4i \\ 2 + 2i \end{bmatrix} \qquad \mathbf{A}\mathbf{x} = \begin{bmatrix} 10 - 14i \\ -20i \end{bmatrix} \tag{8.5.44}$$

$$\mathbf{x}^{\dagger} = \begin{bmatrix} 4i & (2 - 2i) \end{bmatrix} \qquad \mathbf{x}^{\dagger}\mathbf{A}\mathbf{x} = 16 \tag{8.5.45}$$

10. **A** is skew-Hermitian,

$$\mathbf{A} = \begin{bmatrix} i & -2 + 3i \\ 2 + 3i & 0 \end{bmatrix} \qquad \mathbf{A}^{\dagger} = \begin{bmatrix} -i & 2 - 3i \\ 2 - 3i & 0 \end{bmatrix} \tag{8.5.46}$$

$$\mathbf{x} = \begin{bmatrix} 2i \\ 8 \end{bmatrix} \qquad \mathbf{A}\mathbf{x} = \begin{bmatrix} -18 + 24i \\ 4i - 6 \end{bmatrix} \tag{8.5.47}$$

$$\mathbf{x}^{\dagger} = \begin{bmatrix} -2i & 8 \end{bmatrix} \qquad \mathbf{x}^{\dagger}\mathbf{A}\mathbf{x} = 68i \tag{8.5.48}$$

11. $\mathbf{A}$ is skew-Hermitian,

$$\mathbf{A} = \begin{bmatrix} i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix} \qquad \mathbf{A}^\dagger = \begin{bmatrix} -i & -1 & -2-i \\ 1 & 0 & -3i \\ 2-i & -3i & -i \end{bmatrix} \tag{8.5.49}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix} \qquad \mathbf{Ax} = \begin{bmatrix} 1 \\ 2 \\ -4+i \end{bmatrix} \tag{8.5.50}$$

$$\mathbf{x}^\dagger = \begin{bmatrix} 1 & -i & i \end{bmatrix} \qquad \mathbf{x}^\dagger\mathbf{Ax} = -6i \tag{8.5.51}$$

12. $\mathbf{A}$ is Hermitian,

$$\mathbf{A} = \begin{bmatrix} 1 & i & 4 \\ -i & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix} \qquad \mathbf{A}^\dagger = \begin{bmatrix} 1 & i & 4 \\ -i & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix} \tag{8.5.52}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix} \qquad \mathbf{Ax} = \begin{bmatrix} -4i \\ 2i \\ 4-2i \end{bmatrix} \tag{8.5.53}$$

$$\mathbf{x}^\dagger = \begin{bmatrix} 1 & -i & i \end{bmatrix} \qquad \mathbf{x}^\dagger\mathbf{Ax} = 4 \tag{8.5.54}$$

13. Given $\mathbf{A}$ is Hermitian, $\mathbf{B}$ is skew-Hermitian and $\mathbf{C}$ is Unitary,

$$(\mathbf{ABC})^\dagger = \mathbf{C}^\dagger\ \mathbf{B}^\dagger\ \mathbf{A}^\dagger \qquad = -\mathbf{C}^{-1}\ \mathbf{BA} \tag{8.5.55}$$

14. Product of matrices, where $\mathbf{A}$ is Hermitian, $\mathbf{B}$ is skew-Hermitian

$$(\mathbf{BA})^\dagger = \mathbf{A}^\dagger\mathbf{B}^\dagger = -\mathbf{AB} \tag{8.5.56}$$

Showing this for the matrices in example 2,

$$\mathbf{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} \tag{8.5.57}$$

$$\mathbf{A}^\dagger = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} \qquad \mathbf{B}^\dagger = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix} \tag{8.5.58}$$

$$\mathbf{AB} = \begin{bmatrix} 1+19i & 5+3i \\ -23+10i & -1 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} -1+19i & 23+10i \\ -5+3i & 1 \end{bmatrix} \tag{8.5.59}$$

$$(\mathbf{BA})^\dagger = \begin{bmatrix} -1-19i & -5-3i \\ 23-10i & 1 \end{bmatrix} \tag{8.5.60}$$

The relation holds.

15. Consider any square matrix $\mathbf{A}$

$$\frac{\mathbf{A}+\mathbf{A}^\dagger}{2} \text{ is Hermitian} \qquad \frac{\mathbf{A}-\mathbf{A}^\dagger}{2} \text{ is skew-Hermitian} \tag{8.5.61}$$

$$\mathbf{A} = \mathbf{H} + \mathbf{S} \qquad \mathbf{A} = \left(\frac{\mathbf{A}+\mathbf{A}^\dagger}{2}\right) + \left(\frac{\mathbf{A}-\mathbf{A}^\dagger}{2}\right) \tag{8.5.62}$$

Examples TBC.

16. Product of unitary matrices is unitary

$$(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\ \mathbf{A}^\dagger \qquad = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{8.5.63}$$

$$= (\mathbf{AB})^{-1} \tag{8.5.64}$$

Inverse of unitary matrix, is also unitary

$$\mathbf{A}^\dagger = \mathbf{A}^{-1} \qquad \left(\mathbf{A}^{-1}\right)^\dagger = \mathbf{A} = \left(\mathbf{A}^{-1}\right)^{-1} \tag{8.5.65}$$

Examples TBC

17. $\mathbf{C}$ is Unitary

$$\mathbf{A} = \begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix} \qquad 0 = \lambda^2 - i\lambda - 1 \qquad \{\lambda_i\} = \left\{\frac{i \pm \sqrt{3}}{2}\right\} \tag{8.5.66}$$

Using the fact that raising a matrix to a positive integral power raises its eigenvalues to the same power,

$$\{\lambda_i\} = \left\{\exp\left(\frac{i\pi}{6}\right), \exp\left(\frac{i5\pi}{6}\right)\right\} \qquad \{\mu_i\} = \{1, 1\} \tag{8.5.67}$$

This requires both eigenvalues to me made into the form $\exp(2n\pi)$. Clearly, the smallest such power is 12.

18. Normal matrices.

$$\text{Let} \quad \mathbf{A} = \mathbf{A}^\dagger \qquad \mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^2 = \mathbf{A}^\dagger\mathbf{A} \tag{8.5.68}$$

$$\text{Let} \quad \mathbf{B} = -\mathbf{B}^\dagger \qquad \mathbf{B}\mathbf{B}^\dagger = -\mathbf{B}^2 = \mathbf{B}^\dagger\mathbf{B} \tag{8.5.69}$$

$$\text{Let} \quad \mathbf{C}^{-1} = \mathbf{C}^\dagger \qquad \mathbf{C}\mathbf{C}^\dagger = \mathbf{I} = \mathbf{C}^{-1}\mathbf{C} = \mathbf{C}^\dagger\mathbf{C} \tag{8.5.70}$$

Examples TBC.

19. From Problem 15,

$$\mathbf{A} = \mathbf{H} + \mathbf{S} \qquad = \left(\frac{\mathbf{A} + \mathbf{A}^\dagger}{2}\right) + \left(\frac{\mathbf{A} - \mathbf{A}^\dagger}{2}\right) \tag{8.5.71}$$

$$\mathbf{A}\mathbf{A}^\dagger = (\mathbf{H} + \mathbf{S})(\mathbf{H} + \mathbf{S})^\dagger \qquad = \mathbf{HH} + \mathbf{SH} - \mathbf{HS} - \mathbf{SS} \tag{8.5.72}$$

$$\mathbf{A}^\dagger\mathbf{A} = (\mathbf{H} + \mathbf{S})^\dagger(\mathbf{H} + \mathbf{S}) \qquad = \mathbf{HH} + \mathbf{HS} - \mathbf{SH} - \mathbf{SS} \tag{8.5.73}$$

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A} \iff \mathbf{HS} = \mathbf{SH} \tag{8.5.74}$$

20. Simple matrix that is not normal

$$\mathbf{A} = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \qquad \mathbf{A}^\dagger = \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix} \tag{8.5.75}$$

$$\mathbf{A}\mathbf{A}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{A}^\dagger\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{8.5.76}$$

Normal matrix that is not Hermitian, skew-Hermitian or Unitary,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^\dagger = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{8.5.77}$$

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A} \tag{8.5.78}$$