Chapter 16

Laurent Series, Residue Integration

16.1 Laurent Series

1. Finding the Laurent series,

$$f(z) = \frac{\cos z}{z^4} \qquad f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-4}}{(2n)!} \tag{16.1.1}$$

$$f(z) = \frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \frac{z^4}{8!} - \dots \qquad |z| \in (0, \infty) \tag{16.1.2}$$

2. Finding the Laurent series,

$$w = \frac{1}{z} \qquad f(w) = w^2 e^{-w^2} \tag{16.1.3}$$

$$f(w) = \frac{(-1)^n \ w^{2n+2}}{n!} \qquad f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z^{2n+2}} \tag{16.1.4}$$

$$f(z) = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{2! \ z^6} - \frac{1}{3! \ z^8} + \dots \qquad |z| \in (0, \infty) \tag{16.1.5}$$

3. Finding the Laurent series,

$$f(z) = z^{-3} \ \exp(z^2) \qquad f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n-3} \tag{16.1.6}$$

$$f(z) = \frac{1}{z^3} + \frac{1}{z} + \frac{z}{2!} + \frac{z^3}{3!} + \dots \qquad |z| \in (0, \infty) \tag{16.1.7}$$

4. Finding the Laurent series,

$$f(z) = z^{-2} \ \sin(\pi z) \qquad f(z) = \pi^2 \sum_{n=0}^{\infty} (-1)^n \frac{(\pi z)^{2n-1}}{(2n+1)!} \tag{16.1.8}$$

$$f(z) = \frac{\pi}{z} - \frac{\pi^3 z}{3!} + \frac{\pi^5 z^3}{5!} - \frac{\pi^7 z^5}{7!} + \ldots \qquad |z| \in (0, \infty) \tag{16.1.9}$$

5. Finding the Laurent series,

$$f(z) = \frac{1}{z^2 - z^3} \qquad f(z) = \frac{1}{z^2} \cdot \frac{1}{1-z} \tag{16.1.10}$$

$$f(z) = \sum_{n=0}^{\infty} z^{n-2} \qquad S = \{0, 1\} \tag{16.1.11}$$

$$f(z) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \ldots \qquad |z| \in (0, 1) \tag{16.1.12}$$

6. Finding the Laurent series,

$$f(z) = \frac{\sinh(2z)}{z^2} \qquad f(z) = \sum_{n=0}^{\infty} \frac{2^{2n+1} \ z^{2n-1}}{(2n+1)!} \tag{16.1.13}$$

$$f(z) = \frac{2}{1! \ z} + \frac{2^3 \ z}{3!} + \frac{2^5 \ z^3}{5!} + \frac{2^7 \ z^5}{7!} + \ldots \qquad |z| \in (0, \infty) \tag{16.1.14}$$

7. Finding the Laurent series,

$$w = \frac{1}{z} \qquad f(w) = w^{-3} \cosh(w) \tag{16.1.15}$$

$$f(w) = \sum_{n=0}^{\infty} \frac{w^{2n-3}}{(2n)!} \qquad f(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \ \frac{1}{z^{2n-3}} \tag{16.1.16}$$

$$f(z) = \frac{z^3}{0!} + \frac{z}{2!} + \frac{1}{4! \ z} + \frac{1}{6! \ z^3} + \ldots \qquad |z| \in (0, \infty) \tag{16.1.17}$$

8. Finding the Laurent series,

$$f(z) = \frac{1}{z^2} \cdot \frac{e^z}{1-z} \qquad f(z) = z^{-2} \left[\sum_{n=0}^{\infty} z^n \right] \cdot \left[\sum_{n=0}^{\infty} \frac{z^n}{n!} \right] \tag{16.1.18}$$

$$f(z) = z^{-2} \sum_{n=0}^{\infty} c_n \, z^n \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k} \tag{16.1.19}$$

$$c_n = \sum_{k=0}^{n} \frac{1}{(n-k)!} \qquad f(z) = \frac{1}{z^2} + \frac{2}{z} + \frac{5}{2} + \frac{8z}{3} + \dots \tag{16.1.20}$$

$$S = \{0, 1\} \qquad |z| \in (0, 1) \tag{16.1.21}$$

9. Finding the Laurent series, centered on $z_0 = 1$

$$w = z - 1 \qquad f(w) = e \, \frac{e^w}{w^2} \tag{16.1.22}$$

$$f(w) = e \sum_{n=0}^{\infty} \frac{w^{n-2}}{n!} \qquad f(z) = e \sum_{n=0}^{\infty} \frac{(z-1)^{n-2}}{n!} \tag{16.1.23}$$

Finding the radius of convergence,

$$f(z) = e \left[\frac{1}{0! \, (z-1)^2} + \frac{1}{1! \, (z-1)} + \frac{1}{2} + \frac{(z-1)}{3!} + \frac{(z-1)^2}{4!} + \dots \right] \tag{16.1.24}$$

$$|z - 1| \in (0, \infty) \tag{16.1.25}$$

10. Finding the Laurent series, centered on $z_0 = 3$

$$w = z - 3 \qquad f(w) = \frac{w^2 + 6w + 9 - 3\mathrm{i}}{w^2} \tag{16.1.26}$$

$$f(w) = \frac{9 - 3\mathrm{i}}{w^2} + \frac{6}{w} + 1 \tag{16.1.27}$$

Finding the radius of convergence,

$$f(z) = \frac{9 - 3\mathrm{i}}{(z-3)^2} + \frac{6}{(z-3)} + 1 \qquad |z - 1| \in (0, \infty) \tag{16.1.28}$$

11. Finding the Laurent series, centered on $z_0 = \pi\mathrm{i}$

$$w = z - \pi\mathrm{i} \qquad f(w) = \frac{(w + \pi\mathrm{i})^2}{w^4} \tag{16.1.29}$$

$$f(w) = \frac{w^2 + 2\pi\mathrm{i} \; w - \pi^2}{w^4} \tag{16.1.30}$$

Finding the radius of convergence,

$$f(z) = \frac{1}{(z - \pi\mathrm{i})^2} + \frac{2\pi\mathrm{i}}{(z - \pi\mathrm{i})^3} - \frac{\pi^2}{(z - \pi\mathrm{i})^4} \qquad |z - \pi\mathrm{i}| \in (0, \infty) \tag{16.1.31}$$

12. Finding the Laurent series, centered on $z_0 = \mathrm{i}$

$$w = z - \mathrm{i} \qquad f(w) = \frac{1}{w(w + \mathrm{i})^2} \tag{16.1.32}$$

$$f(w) = \frac{-1}{w} \cdot \sum_{n=0}^{\infty} \binom{-2}{n} \frac{w^n}{\mathrm{i}^n} \qquad \forall \quad |w| < 1 \tag{16.1.33}$$

Finding the radius of convergence,

$$f(z) = \frac{-1}{(z - \mathrm{i})} \left[1 - \frac{2(z - \mathrm{i})}{\mathrm{i}} - \frac{3(z - \mathrm{i})^2}{1} + \frac{4(z - \mathrm{i})^3}{\mathrm{i}} - \dots\right] \tag{16.1.34}$$

$$f(z) = -\frac{1}{(z - \mathrm{i})} + \frac{2}{\mathrm{i}} + 3(z - \mathrm{i}) - \frac{4(z - \mathrm{i})^2}{\mathrm{i}} + \dots \tag{16.1.35}$$

$$|z - \mathrm{i}| \in (0, 1) \tag{16.1.36}$$

13. Finding the Laurent series, centered on $z_0 = \mathrm{i}$

$$w = z - \mathrm{i} \qquad f(w) = \frac{1}{w^2(w + \mathrm{i})^3} \tag{16.1.37}$$

$$f(w) = \frac{-1}{w^2} \cdot \sum_{n=0}^{\infty} \binom{-3}{n} \frac{w^n}{\mathrm{i}^n} \qquad \forall \quad |w| < 1 \tag{16.1.38}$$

Finding the radius of convergence,

$$f(z) = \frac{\mathrm{i}}{(z - \mathrm{i})^2} \left[1 - \frac{3(z - \mathrm{i})}{\mathrm{i}} - 12(z - \mathrm{i})^2 + \frac{30(z - \mathrm{i})^3}{\mathrm{i}} - \dots\right] \tag{16.1.39}$$

$$f(z) = \frac{\mathrm{i}}{(z - \mathrm{i})^2} - \frac{3}{(z - \mathrm{i})} - 6\mathrm{i} + 10(z - \mathrm{i}) + \dots \tag{16.1.40}$$

$$|z - \mathrm{i}| \in (0, 1) \tag{16.1.41}$$

14. Finding the Laurent series, centered on $z_0 = b$

$$w = z - b \qquad\qquad f(w) = e^{ab}\ \frac{e^{aw}}{w} \tag{16.1.42}$$

$$f(w) = e^{ab}\ \cdot \sum_{n=0}^{\infty} \frac{a^n}{n!}\ w^{n-1} \tag{16.1.43}$$

Finding the radius of convergence,

$$f(z) = e^{ab}\left[\frac{1}{w} + a + \frac{a^2\ w}{2!} + \frac{a^3\ w^2}{3!} + \dots\right] \tag{16.1.44}$$

$$|z - b| \in (0,\ \infty) \tag{16.1.45}$$

15. Finding the Laurent series, centered on $z_0 = \pi$

$$w = z - \pi \qquad\qquad f(w) = \frac{\cos(w + \pi)}{w^2} \tag{16.1.46}$$

$$f(w) = -\frac{\cos\,w}{w^2} \qquad\qquad f(w) = \sum_{n=0}^{\infty}(-1)^{n+1}\ \frac{w^{2n-2}}{(2n)!} \tag{16.1.47}$$

Finding the radius of convergence,

$$f(z) = -\frac{1}{0!\ (z-\pi)^2} + \frac{1}{2!} - \frac{(z-\pi)^2}{4!} + \frac{(z-\pi)^4}{6!} + \dots \tag{16.1.48}$$

$$|z - \pi| \in (0,\ \infty) \tag{16.1.49}$$

16. Finding the Laurent series, centered on $z_0 = \pi$

$$w = z - \pi/4 \qquad\qquad f(w) = \frac{\sin(w + \pi/4)}{w^3} \tag{16.1.50}$$

$$f(w) = \frac{\sin\,w - \cos\,w}{\sqrt{2}\ w^3} \qquad\qquad f_1(w) = \sum_{n=0}^{\infty}(-1)^n\ \frac{w^{2n-3}}{\sqrt{2}\ (2n)!} \tag{16.1.51}$$

$$f_2(w) = \sum_{n=0}^{\infty}(-1)^n\ \frac{w^{2n-2}}{\sqrt{2}\ (2n+1)!} \tag{16.1.52}$$

Finding the radius of convergence,

$$f_1(z) = \frac{1}{\sqrt{2}} \left[\frac{1}{(z-\pi/4)^3} - \frac{1}{2\ (z-\pi/4)} + \frac{(z-\pi/4)}{4!} - \frac{(z-\pi/4)^3}{6!} + \dots \right] \tag{16.1.53}$$

$$f_2(z) = \frac{1}{\sqrt{2}} \left[\frac{1}{(z-\pi/4)^2} - \frac{1}{3!} + \frac{(z-\pi/4)^2}{5!} - \frac{(z-\pi/4)^4}{7!} + \dots \right] \tag{16.1.54}$$

$$f(z) = f_2(z) - f_1(z) \tag{16.1.55}$$

$$|z-\pi/4| \in (0, \infty) \tag{16.1.56}$$

17. Program written in `sympy`. Using it to find the Laurent series of a single term,

$$\frac{1}{az+b} = \frac{1/b}{1-(-az/b)} \qquad T(z) = \frac{1}{b} \sum_{n=0}^{\infty} \left(\frac{-az}{b}\right)^n \tag{16.1.57}$$

$$w = \frac{1}{z} \qquad \frac{1}{az+b} = \frac{w}{a+bw} \tag{16.1.58}$$

$$\frac{w/a}{1+(bw/a)} = \frac{w/a}{1-(-bw/a)} \qquad = \frac{w}{a} \sum_{n=0}^{\infty} \left(\frac{-bw}{a}\right)^n \tag{16.1.59}$$

$$L(z) = \sum_{n=0}^{\infty} \frac{-b^n}{a^{n+1}} \frac{1}{z^{n+1}} \tag{16.1.60}$$

Since partial fraction decomposition produces linear factors in the denominators, this procedure takes care of all the factors after decomposition.

Other functions TBC.

18. Laurent Series

(a) Let there be two Laurent expansions for $f(z)$ with coefficients. Without loss of generality, let the expansions be centered on the origin. $\{a_n,\ b_n\}$.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n\ z^n = \sum_{n=-\infty}^{\infty} b_n\ z^n \tag{16.1.61}$$

$$z^{-m-1}\ f(z) = \sum_{n=-\infty}^{\infty} a_n\ z^{n-m-1} = \sum_{n=-\infty}^{\infty} b_n\ z^{n-m-1} \tag{16.1.62}$$

$$\tag{16.1.63}$$

for some integer m. Consider some closed simple path in the annulus encircling $z_0 = 0$ once ccl. Now, since a Laurent series converges uniformly in its annulus of definition, it can be integrated term-wise.

$$\sum_{n=-\infty}^{\infty} \oint_C a_n \, z^{n-m-1} \, \mathrm{d}z = \sum_{n=-\infty}^{\infty} \oint_C b_n \, z^{n-m-1} \, \mathrm{d}z \tag{16.1.64}$$

$$\oint_C z^\alpha \, \mathrm{d}z = \begin{cases} 2\pi\mathrm{i} & \alpha = -1 \\ 0 & \text{otherwise} \end{cases} \tag{16.1.65}$$

$$a_m = b_m \quad \forall \quad m \in \mathcal{I} \tag{16.1.66}$$

This proves that the two Laurent series are identical and that a Laurent series expansion, if it exists, must be unique.

(b) Looking at the singular points of the function,

$$\cos 1/z) = 0 \qquad \Longrightarrow \frac{1}{z} = n\pi + \frac{\pi}{2} \tag{16.1.67}$$

$$z = \frac{1}{n\pi + \pi/2} \qquad z = \left\{ \frac{2}{\pi}, \frac{2}{3\pi}, \frac{-2}{\pi}, \frac{2}{5\pi}, \frac{-2}{3\pi}, \dots \right\} \tag{16.1.68}$$

There is an infinite number of singular points arbitrarily close to the origin for large enough n. This means that the function can never have a Laurent series that converges for $|z| \in (0, R)$, however small R may be.

(c) Integrating term-wise, given that the Laurent series converges,

$$\frac{e^t - 1}{t} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} \tag{16.1.69}$$

$$\int_0^z f(t) \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)(n+1)!} \tag{16.1.70}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{z^{n-1}}{(n+1)! \, (n+1)} \tag{16.1.71}$$

Integrating term-wise, given that the Laurent series converges,

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!} \tag{16.1.72}$$

$$\int_0^z f(t) \, \mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)(2n+1)!} \tag{16.1.73}$$

$$g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-2}}{(2n+1)! \, (2n+1)} \tag{16.1.74}$$

19. Finding the Taylor series,

$$\frac{1}{1-z^2} = \frac{1/2}{1+z} + \frac{1/2}{1-z} \qquad f(z) = \sum_{n=0}^{\infty} z^{2n} \tag{16.1.75}$$

$$T(z) = 1 + z^2 + z^4 + z^6 + \ldots \qquad |z| < 1 \tag{16.1.76}$$

Finding the Laurent series, with $w = 1/z$,

$$f(w) = \frac{-w^2}{1-w^2} \qquad f(w) = -\sum_{n=0}^{\infty} w^{2n+2} \tag{16.1.77}$$

$$L(z) = -\frac{1}{z^2}\left[1 + \frac{1}{z^2} + \frac{1}{z^4} + \ldots\right] \qquad |z| > 1 \tag{16.1.78}$$

20. Finding the Taylor series, around $z_0 = 1$

$$f(z) = \frac{1}{z} \qquad w = z - 1 \tag{16.1.79}$$

$$f(w) = \frac{1}{1+w} \qquad f(w) = \sum_{n=0}^{\infty}(-1)^n w^n \tag{16.1.80}$$

$$T(z) = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots \qquad |z-1| < 1 \tag{16.1.81}$$

Finding the Laurent series, with $v = 1/w$,

$$f(v) = \frac{v}{1+v} \qquad f(v) = \sum_{n=0}^{\infty}(-1)^n\, v^{n+1} \tag{16.1.82}$$

$$L(z) = \frac{1}{(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \frac{1}{(z-1)^4} + \ldots \qquad |z-1| > 1 \tag{16.1.83}$$

21. Finding the Taylor series, around $z_0 = -\pi/2$

$$f(z) = \frac{\sin z}{z + \pi/2} \qquad w = z + \pi/2 \tag{16.1.84}$$

$$f(w) = \frac{-\cos w}{w} \qquad f(w) = \sum_{n=0}^{\infty}(-1)^{n+1}\,\frac{w^{2n-1}}{(2n)!} \tag{16.1.85}$$

$$T(z) = -\frac{1}{(z+\pi/2)} + \frac{(z+\pi/2)}{2!} - \frac{(z+\pi/2)^3}{4!} + \ldots \qquad |z+\pi/2| > 0 \tag{16.1.86}$$

This also happens to be the Laurent series, since it contains negative powers of $(z - z_0)$.

22. Finding the Taylor series, around $z_0 = \mathfrak{i}$

$$f(z) = \frac{1}{z^2} \qquad w = z - \mathfrak{i} \tag{16.1.87}$$

$$f(w) = \frac{1}{(w+\mathfrak{i})^2} \qquad f(w) = -(1 - w\mathfrak{i})^{-2} \tag{16.1.88}$$

$$T(w) = -\sum_{n=0}^{\infty} \binom{-2}{n} (-\mathfrak{i}w)^n \qquad T(w) = -1 - 2\mathfrak{i}w + 3w^2 + 4w^3 - \dots \tag{16.1.89}$$

$$T(z) = -1 - 2\mathfrak{i}(z - \mathfrak{i}) + 3(z - \mathfrak{i})^2 - \dots \qquad |z - \mathfrak{i}| < 1 \tag{16.1.90}$$

Finding the Laurent series, with $v = 1/w$,

$$f(v) = \frac{v^2}{(1 + \mathfrak{i}v)^2} \qquad f(v) = \sum_{n=0}^{\infty} \binom{-2}{n} (\mathfrak{i}v)^{n+2} \tag{16.1.91}$$

$$f(v) = 1 + 2\mathfrak{i}v - 3v^2 - 4\mathfrak{i}v^3 + \dots \tag{16.1.92}$$

$$L(z) = 1 + \frac{2\mathfrak{i}}{(z - \mathfrak{i})} - \frac{3}{(z - \mathfrak{i})^2} - \frac{4\mathfrak{i}}{(z - \mathfrak{i})^3} + \dots \qquad |z - \mathfrak{i}| > 1 \tag{16.1.93}$$

23. Finding the Taylor series,

$$f(z) = \frac{z^8}{1 - z^4} \qquad f(z) = \sum_{n=0}^{\infty} z^{4n+8} \tag{16.1.94}$$

$$T(z) = z^8 + z^{12} + z^{16} + \dots \qquad |z| < 1 \tag{16.1.95}$$

Finding the Laurent series, with $w = 1/z$,

$$f(w) = \frac{-w^{-4}}{1 - w^4} \qquad f(w) = -\sum_{n=0}^{\infty} w^{4n-4} \tag{16.1.96}$$

$$L(z) = -\frac{1}{z^4}\left[1 + \frac{1}{z^4} + \frac{1}{z^8} + \dots\right] \qquad |z| > 1 \tag{16.1.97}$$

24. Finding the Taylor series, around $z_0 = 1$

$$f(z) = \frac{\sinh z}{(z-1)^4} \qquad w = z - 1 \tag{16.1.98}$$

$$f(w) = \frac{\sinh(w+1)}{w^4} \qquad f(w) = \frac{\sinh(w)\cosh(1) + \cosh(w)\sinh(1)}{w^4} \tag{16.1.99}$$

Treating the two parts separately,

$$T_1(w) = \cosh 1 \left[\frac{1}{w^3} + \frac{1}{3!\ w} + \frac{w}{5!} + \frac{w^3}{7!} + \dots\right] \tag{16.1.100}$$

$$T_1(w) = \cosh 1 \left[\frac{1}{(z-1)^3} + \frac{1}{3!\ (z-1)} + \frac{(z-1)}{5!} + \dots\right] \tag{16.1.101}$$

$$|z-1| > 0 \tag{16.1.102}$$

$$T_2(w) = \sinh 1 \left[\frac{1}{w^4} + \frac{1}{2!\ w^2} + \frac{1}{4!} + \frac{w^2}{6!} + \dots\right] \tag{16.1.103}$$

$$T_1(w) = \sinh 1 \left[\frac{1}{(z-1)^4} + \frac{1}{2!\ (z-1)^2} + \frac{1}{4!} + \frac{(z-1)^2}{6!} + \dots\right] \tag{16.1.104}$$

$$|z-1| > 0 \tag{16.1.105}$$

This also happens to be the Laurent series, since it contains negative powers of $(z - z_0)$.

25. Finding the Taylor series, around $z_0 = \mathrm{i}$

$$f(z) = \frac{z^3 - 2\mathrm{i}z^2}{(z-\mathrm{i})^2} \qquad w = z - \mathrm{i} \tag{16.1.106}$$

$$f(w) = \frac{(w+\mathrm{i})^2(w-\mathrm{i})}{w^2} \qquad f(w) = w + \mathrm{i} + \frac{1}{w} + \frac{\mathrm{i}}{w^2} \tag{16.1.107}$$

$$f(z) = \frac{\mathrm{i}}{(z-\mathrm{i})^2} + \frac{1}{(z-\mathrm{i})} + \mathrm{i} + (z-\mathrm{i}) \tag{16.1.108}$$

This also happens to be the Laurent series, since it contains negative powers of $(z - z_0)$.

16.2 Singularities and Zeros, Infinity

1. Finding the zeros,

$$f(z) = \sin^4(z/2) \qquad z^* = 2n\pi + 0\mathrm{i} \tag{16.2.1}$$

All the zeros are of order 4.

2. Finding the zeros,

$$f(z) = (z^4 - 81)^3 \qquad f(z) = [(z+3)(z-3)(z+3\mathrm{i})(z-3\mathrm{i})]^3 \tag{16.2.2}$$

$$z^* = \{\pm 3, \pm 3\mathrm{i}\} \tag{16.2.3}$$

All the zeros are of order 3.

3. Finding the zeros,

$$f(z) = (z + 81\mathfrak{i})^4 \qquad z^* = -81\ \mathfrak{i} \tag{16.2.4}$$

All the zeros are of order 4.

4. Finding the zeros,

$$f(z) = \tan^2(2z) \qquad z^* = \frac{n\pi}{2} \tag{16.2.5}$$

All the zeros are of order 2.

5. Finding the zeros,

$$f(z) = z^{-2}\ \sin^2(\pi z) \qquad z^* = n \in \mathcal{I} \setminus 0 \tag{16.2.6}$$

All the zeros are of order 2. $n = 0$ is not a zero since the sinc function does not approach zero at the origin.

6. Finding the zeros,

$$f(z) = \cosh^4(z) \qquad z^* = \left[n\pi + \frac{\pi}{2}\right]\ \mathfrak{i} \tag{16.2.7}$$

All the zeros are of order 4.

7. Finding the zeros,

$$f(z) = z^4 + (1 - 8\mathfrak{i})z^2 - 8\mathfrak{i} \qquad f(z) = (z^2 + 1)(z^2 - 8\mathfrak{i}) \tag{16.2.8}$$

$$z^* = \{\pm\mathfrak{i},\ 2 + 2\mathfrak{i},\ -2 - 2\mathfrak{i}\} \tag{16.2.9}$$

All the zeros are of order 1.

8. Finding the zeros,

$$f(z) = (\sin z - 1)^3 \qquad z^* = 2n\pi + \frac{\pi}{2} \tag{16.2.10}$$

All the zeros are of order 3.

9. Finding the zeros,

$$f(z) = \sin(2z)\cos(2z) \qquad f(z) = 0.5\ \sin(4z) \tag{16.2.11}$$

$$z^* = \frac{n\pi}{4} \tag{16.2.12}$$

All the zeros are of order 1.

10. Finding the zeros,

$$f(z) = (z^2 - 8)^3 \; [\exp(z^2) - 1] \tag{16.2.13}$$

$$e^x[\cos y + \mathfrak{i} \sin y] = 1 \qquad\qquad z^* = \sqrt{n\pi}\; (1 + \mathfrak{i}) \tag{16.2.14}$$

$$(z^2 - 8) = 0 \qquad\qquad z^* = \pm 2\sqrt{2}, \;\text{ order } 3 \tag{16.2.15}$$

All the zeros are of order 1.

11. Given that $f(z)$ has a zero of order n,

$$f(z) = (z - z_0)^n \; g(z) \qquad\qquad g(z_0) \neq 0 \tag{16.2.16}$$

$$f^2(z) = (z - z_0)^{2n} \; g^2(z) \qquad\qquad g^2(z_0) \neq 0 \tag{16.2.17}$$

Thus, $f^2(z)$ has a zero of order $2n$ at z_0

12. Zeros,

(a) Let $g(z) \equiv f'(z)$,

$$g(z_0) = g'(z_0) = \cdots = g^{(n-2)}(z_0) = 0 \qquad\qquad g^{(n-1)}(z_0) \neq 0 \tag{16.2.18}$$

Thus, $g(z)$ has a zero of order $(n-1)$ at $z = z_0$.

(b) Since $f(z)$ is analytic at $z = z_0$ and has a zero of order n at z_0,

$$f(z) = (z - z_0)^n \; g(z) \qquad\qquad g(z_0) \neq 0 \tag{16.2.19}$$

$$\frac{1}{f(z)} = \frac{1}{(z - z_0)^n} \cdot \frac{1}{g(z)} \qquad\qquad \frac{1}{g(z_0)} \neq 0 \tag{16.2.20}$$

Thus, the reciprocal of $f(z)$ has a pole of order n at z_0.

(c) A nonconstant analytic function $f(z)$, by Liouville's theorem, has to be unbounded.

$$g(z) = f(z) - k \tag{16.2.21}$$

$g(z)$ is also a nonconstant analytic function. By Theorem 3, the zeros of $g(z)$ are isolated.

(d) Consider the difference function,

$$g(z) = f_1(z) - f_2(z) \qquad\qquad g(z_n) = 0 \quad \forall \quad \{z_n\} \tag{16.2.22}$$

This sequence is convergent. Suppose it converges to w.

$$|z_n - w| < \epsilon \qquad\qquad \forall \quad n > N(\epsilon) \tag{16.2.23}$$

This means $z = w$ is not an isolated zero of $g(z)$. This means that $g(z)$, if analytic, has to be a constant function.

$$g(z) \equiv 0 \qquad \Longrightarrow \quad f_1(z) \equiv f_2(z) \tag{16.2.24}$$

13. Finding the singularities, using the fact that the denominator is zero at every singularity.

$$z_1 = -2\mathrm{i} \qquad O_1 = 2 \tag{16.2.25}$$

$$z_2 = \mathrm{i} \qquad O_2 = 2 \tag{16.2.26}$$

14. Finding the singularities, using the fact that the denominator is zero at every singularity.

$$z_1 = \mathrm{i} \qquad O_1 = 3 \tag{16.2.27}$$

This is the only singularity since e^z is entire.

15. Finding the singularities,

$$w = \frac{1}{z} \qquad g(w) = \frac{1}{w} \exp\left[\frac{w^2}{(1 - wz_0)^2}\right] \tag{16.2.28}$$

$$w_1 = 0 \qquad \text{simple pole} \tag{16.2.29}$$

$$z_1 = \infty \qquad O_1 = 1 \quad \text{(simple pole)} \tag{16.2.30}$$

Finding essential singularities,

$$\exp\left[\frac{1}{(z - E_1)^2}\right] = \sum_{n=0}^{\infty} \frac{(z - E_1)^{-2n}}{n!} \qquad E_1 = 1 + \mathrm{i} \tag{16.2.31}$$

Since this is an infinite Laurent series, this is an essential singularity,

16. Finding the singularities, using the fact that the denominator is zero at every singularity.

$$\cos(\pi z) = 0 \qquad \Longrightarrow \quad z = n + 0.5 \quad \forall \quad n \in \mathcal{I} \tag{16.2.32}$$

$$z_1 = n + 0.5 \qquad O_1 = 1 \tag{16.2.33}$$

Finding singularities at infinity,

$$\cos(\pi/w) = 0 \qquad \Longrightarrow \quad w^* = \frac{1}{n + 0.5} \tag{16.2.34}$$

$$E_1 = \infty \tag{16.2.35}$$

Since $g(w)$ has an essential zero at the origin, $f(z)$ has an essential singularity at infinity.

17. Finding the singularities, using the fact that the denominator is zero at every singularity.

$$\sin^4 z = 0 \qquad \Longrightarrow \quad z = n\pi \quad \forall \quad n \in \mathcal{I} \tag{16.2.36}$$

$$z_1 = n\pi \qquad O_1 = 4 \tag{16.2.37}$$

Finding singularities at infinity,

$$\sin^4(1/w) = 0 \qquad \Longrightarrow \quad w^* = \frac{1}{n\pi} \tag{16.2.38}$$

$$E_1 = \infty \tag{16.2.39}$$

Since $g(w)$ has an essential zero at the origin, $f(z)$ has an essential singularity at infinity.

18. Finding the singularities,

$$w = \frac{1}{z} \qquad g(w) = \frac{1}{w^3} \exp\left[\frac{w}{w-1}\right] \tag{16.2.40}$$

$$w_1 = 0 \qquad \text{triple pole} \tag{16.2.41}$$

$$z_1 = \infty \qquad O_1 = 3 \quad \text{(triple pole)} \tag{16.2.42}$$

Finding essential singularities,

$$\exp\left[\frac{1}{(z-E_1)^2}\right] = \sum_{n=0}^{\infty} \frac{(z-E_1)^{-2n}}{n!} \qquad E_1 = 1 \tag{16.2.43}$$

Since this is an infinite Laurent series, this is an essential singularity,

19. Finding the singularities, using the fact that the denominator is zero at every singularity.

$$f(z) = e^{-z} + \frac{1}{1-e^z} \qquad \Longrightarrow \quad z = 2n\pi\ \mathrm{i} \quad \forall \quad n \in \mathcal{I} \tag{16.2.44}$$

$$z_1 = 2n\pi\ \mathrm{i} \qquad O_1 = 1 \tag{16.2.45}$$

Finding singularities at infinity,

$$g(w) = e^{-1/w} + \frac{1}{1-e^{1/w}} \qquad \Longrightarrow \quad \lim_{w\to 0} g(w) = 0 \tag{16.2.46}$$

$$E_1 = \infty \tag{16.2.47}$$

Since the denominator of $g(w)$ has an essential zero at the origin, $f(z)$ has an essential singularity at infinity.

20. Finding the singularities, using the fact that the denominator is zero at every singularity.

$$f(z) = \frac{\cos z + \sin z}{\cos(2z)} \qquad \implies z = \frac{n\pi}{2} + \frac{\pi}{4} \quad \forall \quad n \in \mathcal{I} \tag{16.2.48}$$

$$z_1 = (2n+1)\,\frac{\pi}{4} \qquad O_1 = 1 \tag{16.2.49}$$

Finding singularities at infinity,

$$g(w) = \frac{\cos(1/w) + \sin(1/w)}{\cos(2/w)} = 0 \qquad \implies w^* = \phi \tag{16.2.50}$$

Since $g(w)$ is analytic at the origin, $f(z)$ is analytic at infinity.

21. Finding the singularities, using the fact that the denominator is zero at every singularity.

$$f(z) = e^{1/(z-1)} \cdot \frac{1}{e^z - 1} \qquad \implies z = 2n\mathrm{i}\ \mathrm{i} \quad \forall \quad n \in \mathcal{I} \tag{16.2.51}$$

$$z_1 = 2n\pi\ \mathrm{i} \qquad O_1 = 1 \tag{16.2.52}$$

The Laurent series around $z = 1$, is

$$L(z=1) = \sum_{m=0}^{\infty} \frac{1}{m!\,(z-1)^m} \tag{16.2.53}$$

This means that there is an essential singularity at $z = 1$.
Finding singularities at infinity,

$$g(w) = e^{w/(1-w)} \cdot \frac{1}{e^{1/w} - 1} \qquad \implies w^* = \frac{1}{2n\pi\ \mathrm{i}} \tag{16.2.54}$$

Since $g(w)$ has an essential singularity at the origin, $f(z)$ has an essential singularity at infinity.

22. Finding the singularities, using the fact that the denominator is zero at every singularity.

$$f(z) = \frac{\sin z}{z - \pi} \qquad = -\frac{\sin(z-\pi)}{z-\pi} \tag{16.2.55}$$

The Laurent series around $z = \pi$, is

$$L(z=\pi) = \sum_{m=0}^{\infty} (-1)^{m+1} \frac{(z-\pi)^{2m}}{(2m+1)!} \tag{16.2.56}$$

This means that there is an essential singularity at $z = \pi$.

Finding singularities at infinity,

$$g(w) = e^{w/(1-w)} \cdot \frac{1}{e^{1/w} - 1} \qquad \Longrightarrow \; w^* = \frac{1}{2n\pi \; \mathrm{i}} \tag{16.2.57}$$

Since $g(w)$ has an essential singularity at the origin, $f(z)$ has an essential singularity at infinity.

23. Using the same steps as in Example 3,

$$f(z) = \exp\left(\frac{1}{z^2}\right) \qquad f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{2n} \; n!} \tag{16.2.58}$$

Since the principal part has infinitely many negative powers of z, the function has an essential singularity at $z = 0$

$$z = r\,\exp(\mathrm{i}\theta) \qquad f = \exp(r^{-2} e^{-2\mathrm{i}\theta}) \tag{16.2.59}$$

$$f(z) = c_0\,\exp(\mathrm{i}\alpha) \qquad \frac{\cos(2\theta) - \mathrm{i}\;\sin(2\theta)}{r^2} = \ln\,c_0 + \mathrm{i}\alpha \tag{16.2.60}$$

$$\cos(2\theta) = r^2\;\ln\,c_0 \qquad -\sin(2\theta) = r^2\alpha \tag{16.2.61}$$

$$r^4 = \frac{1}{\ln^2\,c_0 + \alpha^2} \qquad \tan(2\theta) = -\frac{\alpha}{\ln\,c_0} \tag{16.2.62}$$

Increasing α by multiples of 2π, does not change c_0 while making r arbitrarily small. This means there is no limit when approaching $z = 0$.

24. Verifying Theorem 1,

$$f(z) = \frac{1 - z^2}{z^3} \qquad z_0 = \{0\} \tag{16.2.63}$$

$$\lim_{z \to 0} f(z) = \frac{\to 1}{\to 0} = \infty \tag{16.2.64}$$

Since the function has poles at $z = z_0$, its Laurent series is of the form

$$f(z) = \sum_{m=0}^{\infty} a_m\,(z - z_0)^m + \sum_{n=1}^{N} \frac{b_n}{(z - z_0)^n} \tag{16.2.65}$$

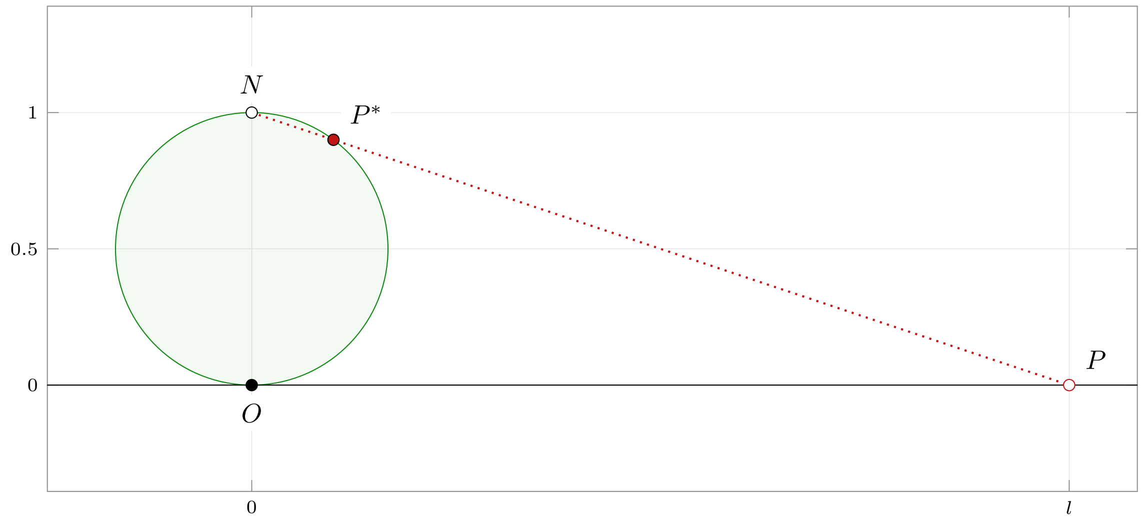
for some finite N. The first series is approaches zero as $z \to z_0$.

The second series approaches infinity.

25. Riemann sphere,

(a) The region $|z| > 100$

Riemann sphere



The region of the sphere corresponding to $|z| \geq l$ is,

$$\angle ONP \geq \arctan(l) \quad 16.2.66$$

(b) The lower half plane maps onto the western hemisphere.

(c) Using the result from part a,

$$|z| = \tan(\alpha) \qquad \tan \alpha \in [0.5, 2] \quad 16.2.67$$

$$\alpha \in [\arctan(0.5), \arctan(2)] \quad 16.2.68$$

16.3 Residue Integration Method

1. Finding the residues,

$$f(z) = \frac{9z + \mathrm{i}}{z(z+\mathrm{i})(z-\mathrm{i})} \quad 16.3.1$$

$$z_1 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = -5\mathrm{i} \quad 16.3.2$$

$$z_2 = -\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = 4\mathrm{i} \quad 16.3.3$$

$$z_3 = 0 \qquad \operatorname*{Res}_{z=z_3} f(z) = \mathrm{i} \quad 16.3.4$$

2. Finding the residues,

$$f(z) = \frac{50z}{z^3 + 2z^2 - 7z + 4} \qquad f(z) = \frac{50z}{(z-1)^2(z+4)} \tag{16.3.5}$$

$$z_1 = -4 \qquad \operatorname*{Res}_{z=z_1} f(z) = -8 \tag{16.3.6}$$

$$z_2 = 1 \qquad \operatorname*{Res}_{z=z_2} f(z) = \lim_{z\to z_2} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{50z}{z+4}\right] \tag{16.3.7}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \lim_{z\to z_2} \frac{200}{(z+4)^2} \qquad \operatorname*{Res}_{z=z_2} f(z) = 8 \tag{16.3.8}$$

3. Finding the residues,

$$f(z) = \frac{\sin(2z)}{z^6} \tag{16.3.9}$$

$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{1}{5!}\frac{\mathrm{d}^5}{\mathrm{d}z^5}\left[\sin(2z)\right] \tag{16.3.10}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{2^5\cos(2z)}{5!} \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{4}{15} \tag{16.3.11}$$

4. Finding the residues,

$$f(z) = \frac{\cos(z)}{z^4} \tag{16.3.12}$$

$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\left[\cos(z)\right] \tag{16.3.13}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{\sin z}{3!} \qquad \operatorname*{Res}_{z=z_1} f(z) = 0 \tag{16.3.14}$$

5. Finding the residues,

$$f(z) = \frac{8}{1+z^2} \qquad f(z) = \frac{8}{(z+\mathrm{i})(z-\mathrm{i})} \tag{16.3.15}$$

$$z_1 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = -4\mathrm{i} \tag{16.3.16}$$

$$z_2 = -\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = 4\mathrm{i} \tag{16.3.17}$$

6. Finding the residues, using the Laurent series,

$$f(z) = \tan z \qquad f(z) = \frac{p(z)}{q(z)} = \frac{\sin z}{\cos z} \tag{16.3.18}$$

$$z_n = n\pi + \frac{\pi}{2} \qquad \operatorname*{Res}_{z=z_n} f(z) = \lim_{z \to z_n} \frac{p(z_n)}{q'(z_n)} \tag{16.3.19}$$

$$\operatorname*{Res}_{z=z_n} f(z) = -1 \tag{16.3.20}$$

7. Finding the residues, using the Laurent series,

$$f(z) = \cot(\pi z) \qquad f(z) = \frac{p(z)}{q(z)} = \frac{\cos(\pi z)}{\sin(\pi z)} \tag{16.3.21}$$

$$z_n = n \quad \in \mathcal{I} \qquad \operatorname*{Res}_{z=z_n} f(z) = \lim_{z \to z_n} \frac{p(z_n)}{q'(z_n)} \tag{16.3.22}$$

$$\operatorname*{Res}_{z=z_n} f(z) = \frac{1}{\pi} \tag{16.3.23}$$

8. Finding the residues,

$$f(z) = \frac{\pi}{(z^2-1)^2} \qquad f(z) = \frac{\pi}{(z-1)^2(z+1)^2} \tag{16.3.24}$$

$$z_1 = 1 \qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z \to z_1} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\pi}{(z+1)^2}\right] \tag{16.3.25}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z \to z_1} \frac{-2\pi}{(z+1)^3} \qquad \operatorname*{Res}_{z=z_1} f(z) = -\frac{\pi}{4} \tag{16.3.26}$$

$$z_2 = -1 \qquad \operatorname*{Res}_{z=z_2} f(z) = \lim_{z \to z_2} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\pi}{(z-1)^2}\right] \tag{16.3.27}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \lim_{z \to z_2} \frac{-2\pi}{(z-1)^3} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{\pi}{4} \tag{16.3.28}$$

9. Finding the residues,

$$f(z) = \frac{1}{1-e^z} \tag{16.3.29}$$

$$z_n = 2n\pi\,\mathrm{i} \qquad \operatorname*{Res}_{z=z_n} f(z) = \lim_{z \to z_n} \frac{p(z_n)}{q'(z_n)} \tag{16.3.30}$$

$$\operatorname*{Res}_{z=z_n} f(z) = -1 \tag{16.3.31}$$

10. Finding the residues,

$$f(z) = \frac{z^4}{z^2 - \mathrm{i}z + 2} \qquad f(z) = \frac{z^4}{(z-2\mathrm{i})(z+\mathrm{i})} \tag{16.3.32}$$

$$z_1 = 2\mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = -\frac{16\mathrm{i}}{3} \tag{16.3.33}$$

$$z_2 = -\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{\mathrm{i}}{3} \tag{16.3.34}$$

11. Finding the residues,

$$f(z) = \frac{e^z}{(z-\pi\mathrm{i})^3} \tag{16.3.35}$$

$$z_1 = \pi\mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[e^z\right] \tag{16.3.36}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{e^z}{2!} \qquad \operatorname*{Res}_{z=z_1} f(z) = -\frac{1}{2} \tag{16.3.37}$$

12. Finding the residues, using the Laurent expansion,

$$f(z) = e^{1/(1-z)} \qquad z_n = 1 \tag{16.3.38}$$

$$w = z - 1 \qquad f(w) = e^{-1/w} \tag{16.3.39}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{w^n\ n!} \qquad b_1 = -1 \tag{16.3.40}$$

$$\operatorname*{Res}_{z=z_n} f(z) = b_1 \qquad \operatorname*{Res}_{z=z_n} f(z) = -1 \tag{16.3.41}$$

13. Program written in `sympy` anc results verified. Program only works for poles of finite order.

14. Finding the residues,

$$f(z) = \frac{z-23}{z^2-4z-5} \qquad f(z) = \frac{z-23}{(z-5)(z+1)} \tag{16.3.42}$$

$$z_1 = 5 \qquad \operatorname*{Res}_{z=z_1} f(z) = -3 \tag{16.3.43}$$

$$z_2 = -1 \qquad \operatorname*{Res}_{z=z_2} f(z) = 4 \tag{16.3.44}$$

Finding the integral,

$$C : |z - 2 - \mathrm{i}| = 3.2 \qquad I = 2\pi\mathrm{i}\ (4-3) = 2\pi\mathrm{i} \tag{16.3.45}$$

15. Finding the residues,

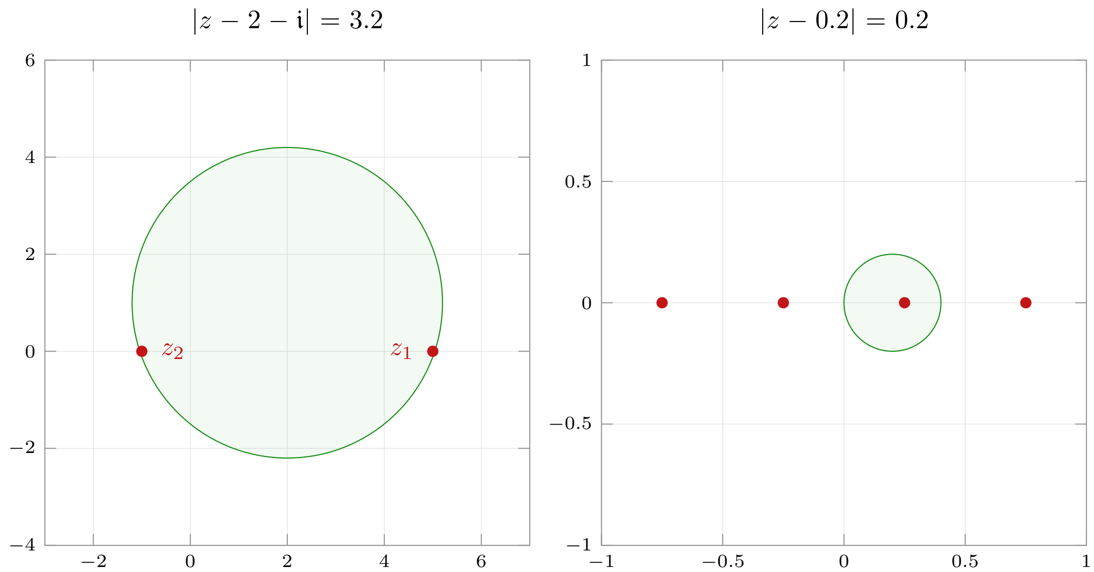
$$f(z) = \tan(2\pi z) \qquad f(z) = \frac{\sin(2\pi z)}{\cos(2\pi z)} \tag{16.3.46}$$

$$z_n = \frac{n}{2} + \frac{1}{4} \qquad \operatorname*{Res}_{z=z_n} f(z) = \frac{p(z_n)}{q'(z_n)} \tag{16.3.47}$$

$$= -\frac{1}{2\pi} \tag{16.3.48}$$

Finding the integral,

$$C : |z - 0.2| = 0.2 \qquad I = 2\pi\mathrm{i}\,(-1) = -\mathrm{i} \tag{16.3.49}$$



16. Finding the residues,

$$f(z) = \exp(1/z) \qquad f(z) = \sum_{n=0}^{\infty} \frac{1}{z^n\, n!} \tag{16.3.50}$$

$$b_1 = 1 \qquad C : |z| = 1 \tag{16.3.51}$$

$$I = 2\pi\mathrm{i}\, b_1 = 2\pi\mathrm{i} \tag{16.3.52}$$

17. Finding the residues,

$$f(z) = \frac{e^z}{\cos z} \qquad f(z) = \frac{p(z)}{q(z)} \tag{16.3.53}$$

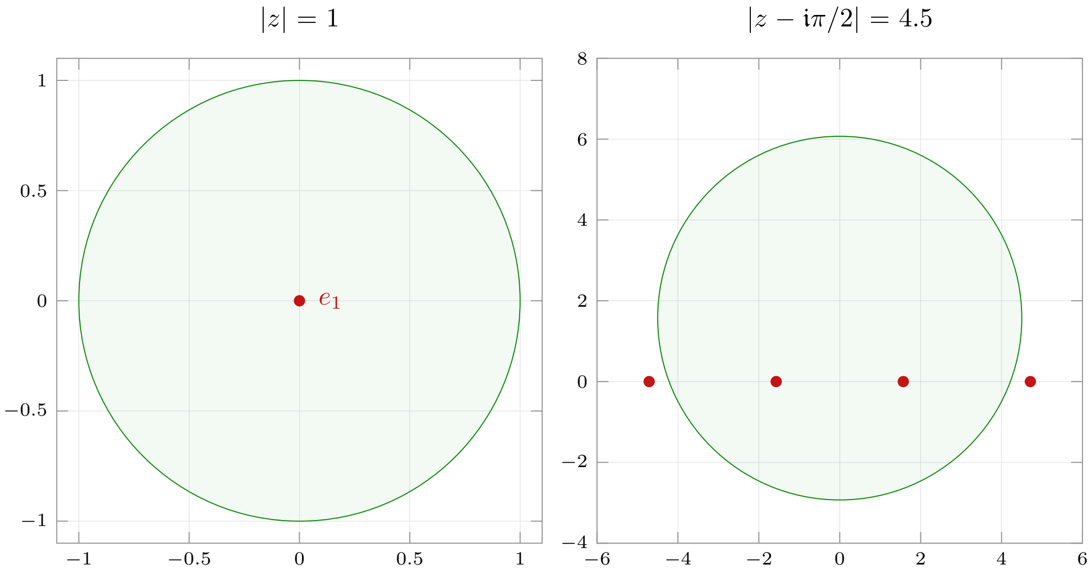
$$z_n = n\pi + \frac{\pi}{2} \qquad \operatorname*{Res}_{z=z_n} f(z) = \frac{p(z_n)}{q'(z_n)} \tag{16.3.54}$$

$$\operatorname*{Res}_{z=z_n} f(z) = \frac{\exp(n\pi + \pi/2)}{(-1)^{n+1}} \tag{16.3.55}$$

Finding the integral,

$$C : |z - 0.2| = 0.2 \qquad I = 2\pi\mathfrak{i}\ (e^{-\pi/2} - e^{\pi/2}) \tag{16.3.56}$$

$$I = -4\pi\mathfrak{i}\ \sinh(\pi/2) \tag{16.3.57}$$



18. Finding the residues,

$$f(z) = \frac{z+1}{z^4 - 2z^3} \qquad f(z) = \frac{z+1}{z^3(z-2)} \tag{16.3.58}$$

$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{1}{2!}\ \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{z+1}{z-2}\right] \tag{16.3.59}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{1}{2}\ \frac{6}{(z-2)^3} \qquad \operatorname*{Res}_{z=z_1} f(z) = -\frac{3}{8} \tag{16.3.60}$$

$$z_2 = 2 \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{3}{8} \tag{16.3.61}$$

Finding the integral,

$$C : |z - 1| = 2 \qquad I = 2\pi\mathrm{i}\,(-3/8 + 3/8) \tag{16.3.62}$$

$$I = 0 \tag{16.3.63}$$

19. Finding the residues,

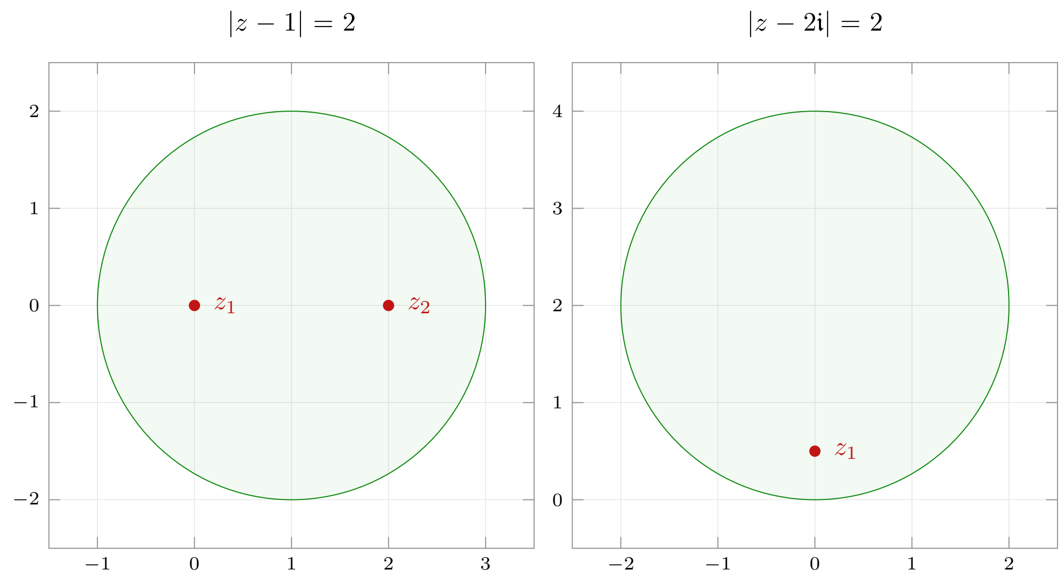
$$f(z) = \frac{\sinh z}{2z - \mathrm{i}} \tag{16.3.64}$$

$$z_1 = 0.5\mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \sin(0.5)\ \frac{\mathrm{i}}{2} \tag{16.3.65}$$

Finding the integral,

$$C : |z - 2\mathrm{i}| = 2 \qquad I = 2\pi\mathrm{i}\left[\frac{\mathrm{i}\ \sin(0.5)}{2}\right] \tag{16.3.66}$$

$$I = -\pi\ \sin(0.5) \tag{16.3.67}$$



20. Finding the residues,

$$f(z) = \frac{1}{(z^2+1)^3} \qquad f(z) = \frac{1}{(z+\mathrm{i})^3(z-\mathrm{i})^3} \tag{16.3.68}$$

$$z_1 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{1}{(z+\mathrm{i})^3}\right] \tag{16.3.69}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{1}{2}\frac{12}{(z+\mathrm{i})^5} \qquad \operatorname*{Res}_{z=z_1} f(z) = -\frac{3\mathrm{i}}{16} \tag{16.3.70}$$

$$z_2 = -\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{1}{(z-\mathrm{i})^3}\right] \tag{16.3.71}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \lim_{z\to z_2} \frac{1}{2}\frac{12}{(z-\mathrm{i})^5} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{3\mathrm{i}}{16} \tag{16.3.72}$$

Finding the integral,

$$C : |z-\mathrm{i}| = 3 \qquad I = 2\pi\mathrm{i}\ (3\mathrm{i}/16 - 3\mathrm{i}/16) \tag{16.3.73}$$

$$I = 0 \tag{16.3.74}$$

21. Finding the residues,

$$f(z) = \frac{\cos(\pi z)}{z^5} \tag{16.3.75}$$

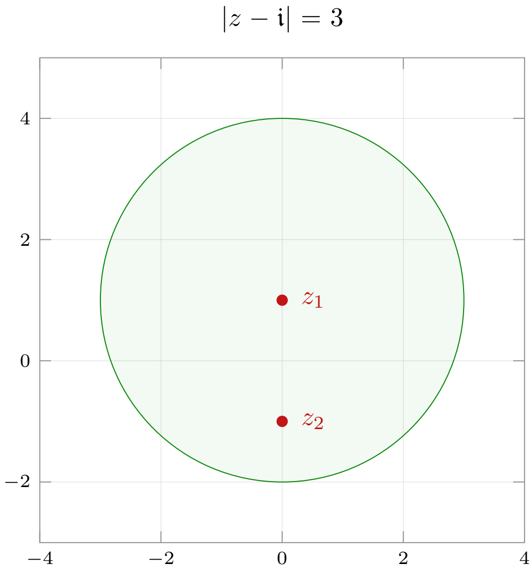
$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{1}{4!}\frac{\mathrm{d}^4}{\mathrm{d}z^4}\left[\cos(\pi z)\right] \tag{16.3.76}$$

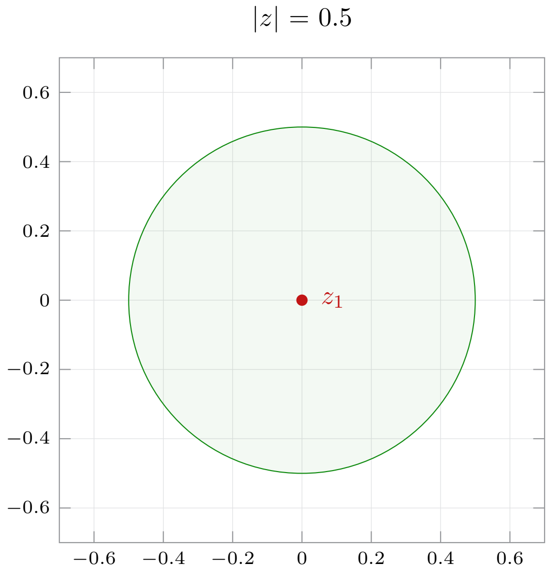
$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{1}{4!}\pi^4 \ \cos(\pi z) \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{\pi^4}{24} \tag{16.3.77}$$

Finding the integral,

$$C : |z| = 0.5 \qquad I = 2\pi\mathrm{i}\left[\frac{\pi^4}{24}\right] \tag{16.3.78}$$

$$I = \frac{\pi^5}{12}\ \mathrm{i} \tag{16.3.79}$$





22. Finding the residues,

$$f(z) = \frac{z^2 \sin z}{4z^2 - 1} \qquad f(z) = \frac{0.25\ z^2 \sin z}{(z + 0.5)(z - 0.5)} \tag{16.3.80}$$

$$z_1 = 0.5 \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{\sin(0.5)}{16} \tag{16.3.81}$$

$$z_2 = -0.5 \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{\sin(0.5)}{16} \tag{16.3.82}$$

Finding the integral,

$$C : |z| = 1 \qquad I = 2\pi\mathrm{i}\ (R_1 + R_2) \tag{16.3.83}$$

$$I = \frac{\pi \sin(0.5)}{4}\ \mathrm{i} \tag{16.3.84}$$

23. Finding the residues,

$$f(z) = \frac{30z^2 - 23z + 5}{(2z - 1)^2(3z - 1)} \qquad f(z) = \frac{1}{12} \cdot \frac{30z^2 - 23z + 5}{(z - 1/2)^2(z - 1/3)} \tag{16.3.85}$$

$$z_1 = 0.5 \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{1}{12}\ \frac{\mathrm{d}}{\mathrm{d}z} \left[\frac{30z^2 - 23z + 5}{(z - 1/3)}\right] \tag{16.3.86}$$

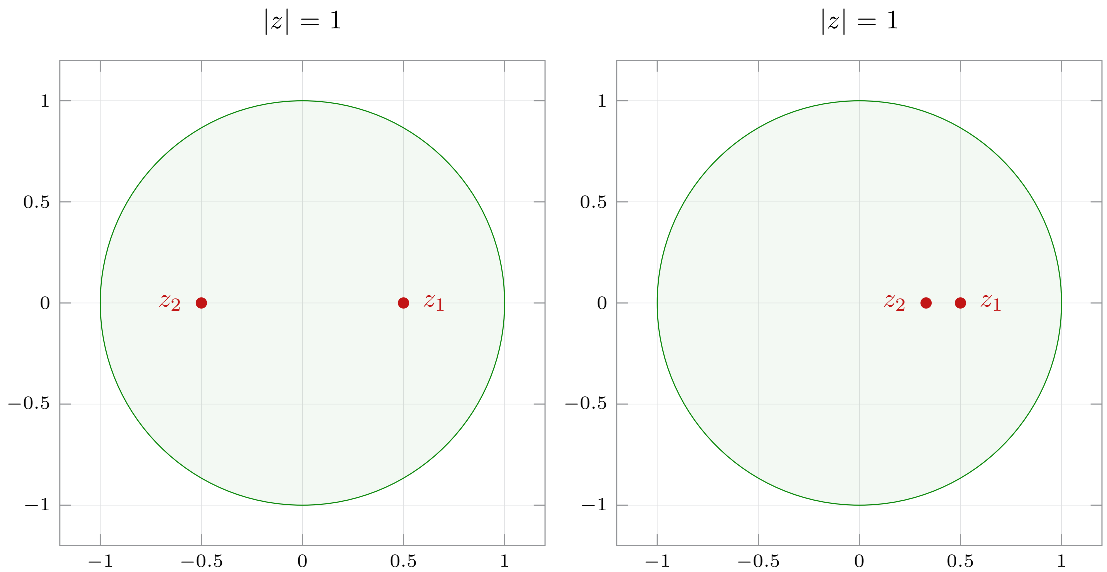
$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z \to z_1} \frac{1}{2}\ \frac{45z^2 - 30z + 4}{(3z - 1)^2} \qquad \operatorname*{Res}_{z=z_1} f(z) = 0.5 \tag{16.3.87}$$

$$z_2 = 1/3 \qquad \operatorname*{Res}_{z=z_2} f(z) = 2 \tag{16.3.88}$$

Finding the integral,

$$C : |z| = 1 \qquad\qquad I = 2\pi\mathfrak{i}\ (0.5 + 2) \tag{16.3.89}$$

$$I = 5\pi\mathfrak{i} \tag{16.3.90}$$



24. Finding the residues,

$$f(z) = \frac{e^{-z^2}}{\sin(4z)} \tag{16.3.91}$$

$$z_n = \frac{n\pi}{4} \qquad\qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_n} \frac{e^{-z^2}}{4\,\cos(4z)} \tag{16.3.92}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \exp\left[-\left(\frac{n\pi}{4}\right)^2\right] \cdot \frac{(-1)^n}{4} \tag{16.3.93}$$

Finding the integral,

$$C : |z| = 1.5 \qquad\qquad I = 2\pi\mathfrak{i}\ (R_{-1} + R_0 + R_1) \tag{16.3.94}$$

$$I = 2\pi\mathfrak{i}\left[\frac{1}{4} - \frac{e^{-\pi^2/16}}{2}\right] \tag{16.3.95}$$

25. Finding the residues,

$$f(z) = \frac{z\cosh(\pi z)}{z^4 + 13z^2 + 36} \qquad f(z) = \frac{z\cosh(\pi z)}{(z^2+9)(z^2+4)} \tag{16.3.96}$$

$$z_1 = 3\mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{1}{10} \tag{16.3.97}$$

$$z_2 = -3\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{1}{10} \tag{16.3.98}$$

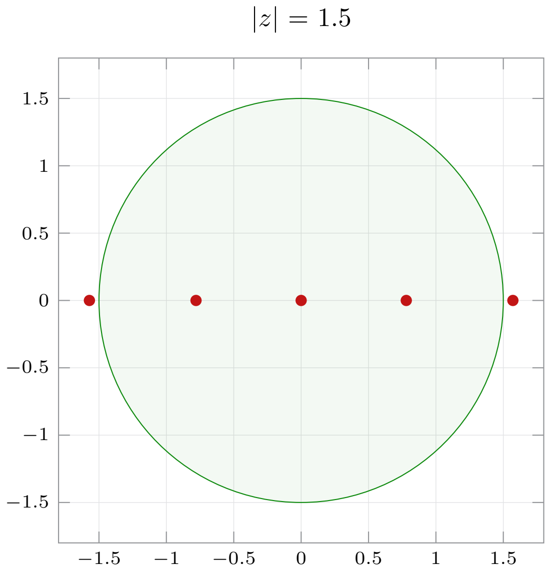
$$z_3 = 2\mathrm{i} \qquad \operatorname*{Res}_{z=z_3} f(z) = \frac{1}{10} \tag{16.3.99}$$

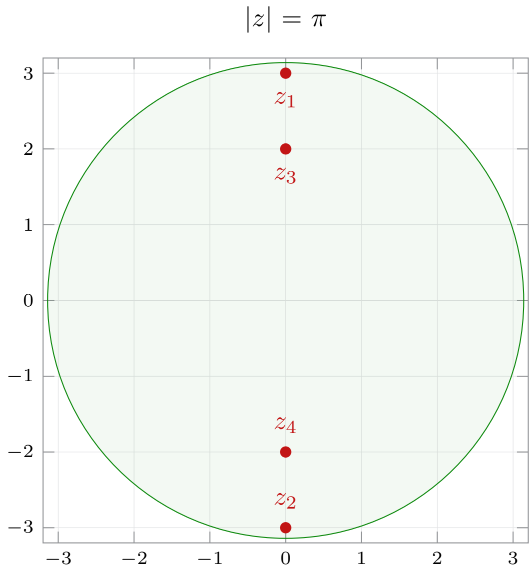
$$z_4 = -2\mathrm{i} \qquad \operatorname*{Res}_{z=z_4} f(z) = \frac{1}{10} \tag{16.3.100}$$

Finding the integral,

$$C : |z| = \pi \qquad I = 2\pi\mathrm{i} \cdot \frac{4}{10} \tag{16.3.101}$$

$$I = 0.8\pi\ \mathrm{i} \tag{16.3.102}$$





16.4 Residue Integration of Real Integrals

1. Calculating the integral, with $|k| > 1$

$$I = \frac{1}{2} \int_{-\pi}^{\pi} \frac{2}{k - \cos\theta} \, \mathrm{d}\theta \qquad = \oint_C \frac{2}{2k - z - (1/z)} \frac{\mathrm{d}z}{\mathrm{i}z} \tag{16.4.1}$$

$$= \oint_C \frac{2\mathrm{i}}{(z-\alpha)(z-\beta)} \, \mathrm{d}z \qquad \alpha, \beta = k \pm \sqrt{k^2 - 1} \tag{16.4.2}$$

Finding the residues of the poles that lie inside the unit circle for $k > 1$,

$$\beta = k - \sqrt{k^2 - 1} \qquad \underset{z=\beta}{\mathrm{Res}}\, g(z) = \frac{-\mathrm{i}}{\sqrt{k^2 - 1}} \tag{16.4.3}$$

$$I = \frac{2\pi}{\sqrt{k^2 - 1}} \tag{16.4.4}$$

Finding the residues of the poles that lie inside the unit circle for $k < -1$,

$$\alpha = k + \sqrt{k^2 - 1} \qquad \underset{z=\alpha}{\mathrm{Res}}\, g(z) = \frac{\mathrm{i}}{\sqrt{k^2 - 1}} \tag{16.4.5}$$

$$I = \frac{-2\pi}{\sqrt{k^2 - 1}} \tag{16.4.6}$$

2. Calculating the integral,

$$I = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1}{\pi + 3\cos\theta} \, \mathrm{d}\theta \qquad = \oint_C \frac{1}{2\pi + 3z + 3/z} \frac{\mathrm{d}z}{\mathrm{i}z} \tag{16.4.7}$$

$$= \oint_C \frac{-\mathrm{i}/3}{z^2 + (2\pi/3)z + 1} \, \mathrm{d}z \qquad = \oint_C \frac{-\mathrm{i}/3}{(z-\alpha)(z-\beta)} \, \mathrm{d}z \tag{16.4.8}$$

$$\alpha, \beta = \frac{-\pi}{3} \pm \sqrt{\frac{\pi^2}{9} - 1} \tag{16.4.9}$$

Finding the residues of the poles that lie inside the unit circle,

$$\alpha = \frac{-\pi}{3} + \sqrt{\frac{\pi^2}{9} - 1} \qquad \underset{z=\alpha}{\mathrm{Res}}\, g(z) = \frac{-0.5\mathrm{i}}{\sqrt{\pi^2 - 9}} \tag{16.4.10}$$

$$I = \frac{\pi}{\sqrt{\pi^2 - 9}} \tag{16.4.11}$$

3. Calculating the integral,

$$I = \int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\ \mathrm{d}\theta \qquad = \oint_C \frac{2\mathrm{i} + z - 1/z}{\mathrm{i}\,[6 + z + (1/z)]}\ \frac{\mathrm{d}z}{\mathrm{i}z} \tag{16.4.12}$$

$$= -\oint_C \frac{z^2 + 2\mathrm{i}z - 1}{(z^2+6z+1)z}\ \mathrm{d}z \qquad = \oint_C \frac{-(z^2+2\mathrm{i}z-1)}{z(z-\alpha)(z-\beta)}\ \mathrm{d}z \tag{16.4.13}$$

$$\alpha,\, \beta = -3 \pm 2\sqrt{2} \tag{16.4.14}$$

Finding the residues of the poles that lie inside the unit circle,

$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} g(z) = 1 \tag{16.4.15}$$

$$\alpha = -3 + \sqrt{8} \qquad \operatorname*{Res}_{z=\alpha} g(z) = -1 - \frac{\mathrm{i}}{2\sqrt{2}} \tag{16.4.16}$$

$$I = 2\pi\mathrm{i}\cdot\frac{-\mathrm{i}}{2\sqrt{2}} \qquad I = \frac{\pi}{\sqrt{2}} \tag{16.4.17}$$

4. Calculating the integral,

$$I = \int_0^{2\pi} \frac{1+4\cos\theta}{17-8\cos\theta}\ \mathrm{d}\theta \qquad = \oint_C \frac{2+4z+4/z}{34-8z-(8/z)}\ \frac{\mathrm{d}z}{\mathrm{i}z} \tag{16.4.18}$$

$$= \frac{\mathrm{i}}{2}\oint_C \frac{z^2+0.5z+1}{(z^2-4.25z+1)z}\ \mathrm{d}z \qquad = \frac{\mathrm{i}}{2}\oint_C \frac{z^2+0.5z+1}{(z-\alpha)(z-\beta)z}\ \mathrm{d}z \tag{16.4.19}$$

$$\alpha,\, \beta = 4,\, 0.25 \tag{16.4.20}$$

Finding the residues of the poles that lie inside the unit circle,

$$\beta = 0.25 \qquad \operatorname*{Res}_{z=\beta} g(z) = -\frac{19}{30}\,\mathrm{i} \tag{16.4.21}$$

$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} g(z) = \frac{1}{2}\,\mathrm{i} \tag{16.4.22}$$

$$I = 2\pi\mathrm{i}\cdot\frac{-2\mathrm{i}}{15} \qquad I = \frac{4\pi}{15} \tag{16.4.23}$$

5. Calculating the integral,

$$I = \int_0^{2\pi} \frac{\cos^2\theta}{5-4\cos\theta}\ \mathrm{d}\theta \qquad = \oint_C \frac{(z^2+1)^2}{(10z-4z^2-4)}\ \frac{\mathrm{d}z}{2\mathrm{i}\ z^2} \tag{16.4.24}$$

$$= \frac{\mathrm{i}}{8}\oint_C \frac{(z^2+1)^2}{(z^2-2.5z+1)\ z^2}\ \mathrm{d}z \qquad = \frac{\mathrm{i}}{8}\oint_C \frac{(z^2+1)^2}{(z-\alpha)(z-\beta)\ z^2}\ \mathrm{d}z \tag{16.4.25}$$

$$\alpha,\, \beta = 0.5,\, 2 \tag{16.4.26}$$

Finding the residues of the poles that lie inside the unit circle,

$$\alpha = 0.5 \qquad \operatorname*{Res}_{z=\alpha} g(z) = -\frac{25}{48}\,\mathrm{i} \tag{16.4.27}$$

$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} g(z) = \lim_{z\to z_1} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z^2+1)^2}{z^2-2.5z+1}\right] \tag{16.4.28}$$

$$\operatorname*{Res}_{z=z_1} g(z) = \frac{5}{16}\,\mathrm{i} \tag{16.4.29}$$

$$I = 2\pi\mathrm{i}\cdot\frac{-5\mathrm{i}}{24} \qquad I = \frac{5\pi}{12} \tag{16.4.30}$$

6. Calculating the integral,

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\,\mathrm{d}\theta \qquad = \oint_C \frac{(z^2-1)^2/(-4z^2)}{(10z-4z^2-4)/(2z)}\,\frac{\mathrm{d}z}{\mathrm{i}z} \tag{16.4.31}$$

$$= \frac{-\mathrm{i}}{8}\oint_C \frac{(z^2-1)^2}{(z^2-2.5z+1)\,(z^2)}\,\mathrm{d}z \qquad = \frac{-\mathrm{i}}{8}\oint_C \frac{(z^2-1)^2}{(z-\alpha)(z-\beta)\,z^2}\,\mathrm{d}z \tag{16.4.32}$$

$$\alpha,\beta = 0.5,\,2 \tag{16.4.33}$$

Finding the residues of the poles that lie inside the unit circle,

$$\alpha = 0.5 \qquad \operatorname*{Res}_{z=\alpha} g(z) = \frac{3}{16}\,\mathrm{i} \tag{16.4.34}$$

$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} g(z) = \lim_{z\to z_1} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z^2-1)^2}{z^2-2.5z-1}\right] \tag{16.4.35}$$

$$\operatorname*{Res}_{z=z_1} g(z) = -\frac{5}{16}\,\mathrm{i} \tag{16.4.36}$$

$$I = 2\pi\mathrm{i}\cdot\frac{-\mathrm{i}}{8} \qquad I = \frac{\pi}{4} \tag{16.4.37}$$

7. Calculating the integral, assuming $|a| > 1$

$$I = \int_0^{2\pi} \frac{a}{a-\sin\theta}\,\mathrm{d}\theta \qquad = \oint_C \frac{2a\mathrm{i}}{2a\mathrm{i}-z+(1/z)}\,\frac{\mathrm{d}z}{\mathrm{i}z} \tag{16.4.38}$$

$$= \mathrm{i}\oint_C \frac{2a\,\mathrm{i}}{z^2-2a\mathrm{i}\,z-1}\,\mathrm{d}z \qquad = \oint_C \frac{-2a}{(z-\alpha)(z-\beta)}\,\mathrm{d}z \tag{16.4.39}$$

$$\alpha,\beta = (a \pm \sqrt{a^2-1})\,\mathrm{i} \tag{16.4.40}$$

Finding the residues of the poles that lie inside the unit circle for $k > 1$,

$$\beta = (a - \sqrt{a^2 - 1})\ \mathrm{i} \qquad \operatorname*{Res}_{z=\beta} g(z) = \frac{-a\mathrm{i}}{\sqrt{a^2 - 1}} \tag{16.4.41}$$

$$I = \frac{2\pi a}{\sqrt{a^2 - 1}} \tag{16.4.42}$$

8. Calculating the integral,

$$I = \int_0^{2\pi} \frac{1}{8 - 2\sin\theta}\ \mathrm{d}\theta \qquad = \oint_C \frac{2\mathrm{i}}{16\mathrm{i} - 2z + (2/z)}\ \frac{\mathrm{d}z}{\mathrm{i}z} \tag{16.4.43}$$

$$= \oint_C \frac{-1}{z^2 - 8\mathrm{i}\ z - 1}\ \mathrm{d}z \qquad = \oint_C \frac{-1}{(z-\alpha)(z-\beta)}\ \mathrm{d}z \tag{16.4.44}$$

$$\alpha, \beta = (4 \pm \sqrt{15})\ \mathrm{i} \tag{16.4.45}$$

Finding the residues of the poles that lie inside the unit circle for $k > 1$,

$$\beta = (4 - \sqrt{15})\ \mathrm{i} \qquad \operatorname*{Res}_{z=\beta} g(z) = \frac{-\mathrm{i}}{2\sqrt{15}} \tag{16.4.46}$$

$$I = \frac{\pi}{\sqrt{15}} \tag{16.4.47}$$

9. Calculating the integral,

$$I = \int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos(2\theta)}\ \mathrm{d}\theta \qquad I = \int_0^{2\pi} \frac{\cos\theta}{25 - 24\cos^2\theta}\ \mathrm{d}\theta \tag{16.4.48}$$

$$I = \oint_C \frac{(z^2+1)(2)}{100z^2 - 24(z^2+1)^2}\ \frac{\mathrm{d}z}{\mathrm{i}} \qquad = 0.5\mathrm{i} \oint_C \frac{z^2+1}{6z^4 - 13z^2 + 6}\ \mathrm{d}z \tag{16.4.49}$$

$$= \frac{\mathrm{i}}{12} \oint_C \frac{z^2+1}{(z^2 - 3/2)(z^2 - 2/3)}\ \mathrm{d}z \tag{16.4.50}$$

Finding the residues of the poles that lie inside the unit circle

$$\alpha = \sqrt{2/3} \qquad \operatorname*{Res}_{z=\alpha} g(z) = -\frac{\sqrt{6}}{24}\ \mathrm{i} \tag{16.4.51}$$

$$\beta = -\sqrt{2/3} \qquad \operatorname*{Res}_{z=\beta} g(z) = \frac{\sqrt{6}}{24}\ \mathrm{i} \tag{16.4.52}$$

$$I = 0 \tag{16.4.53}$$

The integrand is odd for $\theta \in [0, \pi]$ around $\theta = \pi/2$, and is also odd for $\theta \in [\pi, 2\pi]$ around $\theta = 1.5\pi$, This makes the integral zero.

10. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^3} \qquad f(z) = \frac{1}{(1+z^2)^3} \tag{16.4.54}$$

$$z_1 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{1}{(z+\mathrm{i})^3}\right] \tag{16.4.55}$$

$$= \lim_{z\to z_1} \frac{6}{(z+\mathrm{i})^5} \qquad = \frac{-3}{16}\,\mathrm{i} \tag{16.4.56}$$

Evalauting the integral,

$$I = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I = \frac{3\pi}{8} \tag{16.4.57}$$

11. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^2} \qquad f(z) = \frac{1}{(1+z^2)^2} \tag{16.4.58}$$

$$z_1 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{(z+\mathrm{i})^2}\right] \tag{16.4.59}$$

$$= \lim_{z\to z_1} \frac{-2}{(z+\mathrm{i})^3} \qquad = \frac{-1}{4}\,\mathrm{i} \tag{16.4.60}$$

Evalauting the integral,

$$I = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I = \frac{\pi}{2} \tag{16.4.61}$$

12. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2-2x+5)^2} \qquad f(z) = \frac{1}{(z^2-2z+5)^2} \tag{16.4.62}$$

$$\alpha, \beta = 1 \pm 2\mathrm{i} \tag{16.4.63}$$

$$z_1 = 1+2\mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{(z-1+2\mathrm{i})^2}\right] \tag{16.4.64}$$

$$= \lim_{z\to z_1} \frac{-2}{(z-1+2\mathrm{i})^3} \qquad = \frac{-1}{32}\,\mathrm{i} \tag{16.4.65}$$

Evalauting the integral,

$$I = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I = \frac{\pi}{16} \tag{16.4.66}$$

13. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+4)} \qquad f(z) = \frac{z}{(z^2+1)(z^2+4)} \tag{16.4.67}$$

$$z_1 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{1}{6} \tag{16.4.68}$$

$$z_2 = 2\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{-1}{6} \tag{16.4.69}$$

Evalauting the integral,

$$I = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I = 0 \tag{16.4.70}$$

14. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{(x^2+1)}{(x^4+1)} \qquad f(z) = \frac{(z^2+1)}{(z^2+\mathrm{i})(z^2-\mathrm{i})} \tag{16.4.71}$$

$$z_1 = \frac{-1+\mathrm{i}}{\sqrt{2}} \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{-\mathrm{i}}{\sqrt{8}} \tag{16.4.72}$$

$$z_2 = \frac{1+\mathrm{i}}{\sqrt{2}} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{-\mathrm{i}}{\sqrt{8}} \tag{16.4.73}$$

Evalauting the integral,

$$I = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I = \sqrt{2}\pi \tag{16.4.74}$$

15. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^6+1)} \qquad f(z) = \frac{z^2}{(z^3+\mathrm{i})(z^3-\mathrm{i})} \tag{16.4.75}$$

$$z_1 = \frac{-\sqrt{3}+\mathrm{i}}{2} \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{-\mathrm{i}}{6} \tag{16.4.76}$$

$$z_2 = \frac{\sqrt{3}+\mathrm{i}}{2} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{-\mathrm{i}}{6} \tag{16.4.77}$$

$$z_3 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_3} f(z) = \frac{\mathrm{i}}{6} \tag{16.4.78}$$

Evalauting the integral,

$$I = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad\qquad I = \frac{\pi}{3} \tag{16.4.79}$$

16. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{(x^2+1)^2}\ \mathrm{d}x \qquad\qquad g(z) = \frac{\exp(2z\mathrm{i})}{(z^2+1)^2} \tag{16.4.80}$$

$$z_1 = \mathrm{i} \qquad\qquad \operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\exp(2z\mathrm{i})}{(z+\mathrm{i})^2}\right] \tag{16.4.81}$$

$$= \lim_{z\to z_1} \frac{(2\mathrm{i}z - 4)\ \exp(2z\mathrm{i})}{(z+\mathrm{i})^3} \qquad\qquad = \frac{-3e^{-2}}{4}\ \mathrm{i} \tag{16.4.82}$$

$$\tag{16.4.83}$$

Evalauting the integral,

$$I = -2\pi \sum_k \operatorname{Im}\left[\operatorname*{Res}_{z=z_k} g(z)\right] \qquad\qquad I = \frac{3\pi}{2e^2} \tag{16.4.84}$$

17. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\sin(3x)}{(x^4+1)}\ \mathrm{d}x \qquad\qquad g(z) = \frac{\exp(3z\mathrm{i})}{(z^2+\mathrm{i})(z^2-\mathrm{i})} \tag{16.4.85}$$

$$z_1 = \frac{1+\mathrm{i}}{\sqrt{2}} \qquad\qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{\exp[(3/\sqrt{2})\ (-1+\mathrm{i})]}{\sqrt{8}\ (-1+\mathrm{i})} \tag{16.4.86}$$

$$z_2 = \frac{-1+\mathrm{i}}{\sqrt{2}} \qquad\qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{\exp[(-3/\sqrt{2})\ (1+\mathrm{i})]}{\sqrt{8}\ (1+\mathrm{i})} \tag{16.4.87}$$

Evalauting the integral,

$$I = 2\pi \sum_k \operatorname{Re}\left[\operatorname*{Res}_{z=z_k} g(z)\right] \tag{16.4.88}$$

$$I = 2\pi\left[\frac{e^{-k}}{4\sqrt{2}}\ (\cos k - \sin k - \cos k + \sin k)\right] \tag{16.4.89}$$

$$I = 0 \tag{16.4.90}$$

Since the integrand is odd, the integral must vanish.

18. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\cos(4x)}{(x^4+5x^2+4)}\ \mathrm{d}x \qquad g(z) = \frac{\exp(4z\mathrm{i})}{(z^2+1)(z^2+4)} \tag{16.4.91}$$

$$z_1 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{e^{-4}}{6\mathrm{i}} \tag{16.4.92}$$

$$z_2 = 2\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{e^{-8}}{-3(4\mathrm{i})} \tag{16.4.93}$$

Evalauting the integral,

$$I = -2\pi \sum_k \operatorname{Im}\left[\operatorname*{Res}_{z=z_k} g(z)\right] \qquad I = -2\pi\left[\frac{-e^4}{6} + \frac{e^{-8}}{12}\right] \tag{16.4.94}$$

$$I = \frac{\pi}{6}\left[-e^{-8} + 2e^{-4}\right] \tag{16.4.95}$$

19. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^4-1)} \qquad f(z) = \frac{1}{(z^2+1)(z^2-1)} \tag{16.4.96}$$

$$z_1 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{\mathrm{i}}{4} \tag{16.4.97}$$

$$z_2 = 1 \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{1}{4} \tag{16.4.98}$$

$$z_2 = -1 \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{-1}{4} \tag{16.4.99}$$

Evalauting the integral,

$$I_1 = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I_1 = 2\pi\mathrm{i}\ (\mathrm{i}/4) \tag{16.4.100}$$

$$I_2 = \pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_j} f(z) \qquad I_2 = \pi\mathrm{i}\ (0) \tag{16.4.101}$$

$$I = I_1 + I_2 = \frac{-\pi}{2} \tag{16.4.102}$$

20. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{x \ \mathrm{d}x}{(8 - x^3)} \qquad f(z) = \frac{-z}{(z-2)(z^2+2z+4)} \tag{16.4.103}$$

$$z_1 = 2 \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{-1}{6} \tag{16.4.104}$$

$$z_2 = -1 + \sqrt{3}\ \mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{-\sqrt{3} + \mathrm{i}}{12\ (\mathrm{i})} \tag{16.4.105}$$

Evalauting the integral,

$$I_1 = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I_1 = \frac{\pi}{6}\ (-\sqrt{3} + \mathrm{i}) \tag{16.4.106}$$

$$I_2 = \pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_j} f(z) \qquad I_2 = -\frac{\pi\mathrm{i}}{6} \tag{16.4.107}$$

$$I = I_1 + I_2 = -\frac{\pi}{2\sqrt{3}} \tag{16.4.108}$$

21. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{(x-1)(x^2+4)} \ \mathrm{d}x \qquad g(z) = \frac{\exp(z\mathrm{i})}{(z-1)(z+2\mathrm{i})(z-2\mathrm{i})} \tag{16.4.109}$$

$$z_1 = 1 \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{[\cos 1 + \mathrm{i}\ \ \sin 1]}{5} \tag{16.4.110}$$

$$z_2 = 2\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{e^{-2}(\mathrm{i} - 2)}{20} \tag{16.4.111}$$

Evalauting the integral,

$$I_1 = 2\pi \sum_k \mathrm{Re}\left[\operatorname*{Res}_{z=z_k} g(z)\right] \qquad I_1 = -\frac{\pi e^{-2}}{5} \tag{16.4.112}$$

$$I_2 = \pi \sum_k \mathrm{Re}\left[\operatorname*{Res}_{z=z_j} g(z)\right] \qquad I_2 = \frac{\pi \cos 1}{5} \tag{16.4.113}$$

$$I = I_1 + I_2 = \frac{\pi}{5}\ [\cos 1 - e^{-2}] \tag{16.4.114}$$

22. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2 - \mathrm{i}x)} \qquad f(z) = \frac{1}{z(z-\mathrm{i})} \tag{16.4.115}$$

$$z_1 = 0 \qquad \operatorname*{Res}_{z=z_1} f(z) = \mathrm{i} \tag{16.4.116}$$

$$z_2 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = -\mathrm{i} \tag{16.4.117}$$

Evalauting the integral,

$$I_1 = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I_1 = 2\pi \tag{16.4.118}$$

$$I_2 = \pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_j} f(z) \qquad I_2 = -\pi \tag{16.4.119}$$

$$I = I_1 + I_2 = \pi \tag{16.4.120}$$

23. Same as Problem 19

24. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^4 + 3x^2 - 4)} \qquad f(z) = \frac{1}{(x^2+4)(x^2-1)} \tag{16.4.121}$$

$$z_1 = -1 \qquad \operatorname*{Res}_{z=z_1} f(z) = \frac{-1}{10} \tag{16.4.122}$$

$$z_2 = 1 \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{1}{10} \tag{16.4.123}$$

$$z_2 = 2\mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{\mathrm{i}}{20} \tag{16.4.124}$$

Evalauting the integral,

$$I_1 = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I_1 = -\frac{\pi}{10} \tag{16.4.125}$$

$$I_2 = \pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_j} f(z) \qquad I_2 = 0 \tag{16.4.126}$$

$$I = I_1 + I_2 = \frac{-\pi}{10} \tag{16.4.127}$$

25. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{x+5}{(x(x^2-1))}\,\mathrm{d}x \qquad f(z) = \frac{x+5}{x(x+1)(x-1)} \tag{16.4.128}$$

$$z_1 = -1 \qquad \operatorname*{Res}_{z=z_1} f(z) = 2 \tag{16.4.129}$$

$$z_2 = 1 \qquad \operatorname*{Res}_{z=z_2} f(z) = 3 \tag{16.4.130}$$

$$z_2 = 0 \qquad \operatorname*{Res}_{z=z_2} f(z) = -5 \tag{16.4.131}$$

Evalauting the integral,

$$I_1 = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I_1 = 0 \tag{16.4.132}$$

$$I_2 = \pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_j} f(z) \qquad I_2 = \pi\mathrm{i}\ (0) \tag{16.4.133}$$

$$I = I_1 + I_2 = 0 \tag{16.4.134}$$

26. Evaluating the improper integral,

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4-1}\,\mathrm{d}x \qquad f(z) = \frac{x^2}{(x^2+1)(x^2-1)} \tag{16.4.135}$$

$$z_1 = -1 \qquad \operatorname*{Res}_{z=z_1} f(z) = -1/4 \tag{16.4.136}$$

$$z_2 = 1 \qquad \operatorname*{Res}_{z=z_2} f(z) = 1/4 \tag{16.4.137}$$

$$z_2 = \mathrm{i} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{-\mathrm{i}}{4} \tag{16.4.138}$$

Evalauting the integral,

$$I_1 = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_k} f(z) \qquad I_1 = \frac{\pi}{2} \tag{16.4.139}$$

$$I_2 = \pi\mathrm{i} \sum_k \operatorname*{Res}_{z=z_j} f(z) \qquad I_2 = \pi\mathrm{i}\ (0) \tag{16.4.140}$$

$$I = I_1 + I_2 = \frac{\pi}{2} \tag{16.4.141}$$

27. The function has $N > 1$ simple poles on the real axis, by way of N distinct linear factors in its denominator.

Looking at the sum of the residues for the special case of $N = 3$,

$$\sum_{z=z_j} \operatorname*{Res}_{z=z_j} f(z) = \frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} \tag{16.4.142}$$

$$= \frac{c-b+a-c+b-a}{(a-b)(b-c)(c-a)} = 0 \tag{16.4.143}$$

For the general case, TBC

28. Real integrals,

(a) Starting with equation 9,

$$\int_{-\infty}^{\infty} f(x)\ \exp(\mathrm{i}sx)\ \mathrm{d}x = 2\pi\mathrm{i} \sum_j \operatorname*{Res}_{z=z_j} \left[f(z)e^{\mathrm{i}sz}\right] \tag{16.4.144}$$

$$\int_{-\infty}^{\infty} f(x)\ \cos(sx)\ \mathrm{d}x + \int_{-\infty}^{\infty} f(x)\ \mathrm{i}\ \sin(sx)\ \mathrm{d}x = 2\pi\mathrm{i} \sum_j \text{Re } w_j + \mathrm{i}\ \text{Im } w_j \tag{16.4.145}$$

Equating the real and imaginary parts,

$$\int_{-\infty}^{\infty} f(x)\ \cos(sx)\ \mathrm{d}x = -2\pi \sum_j \text{Im } w_j \tag{16.4.146}$$

$$\int_{-\infty}^{\infty} f(x)\ \sin(sx)\ \mathrm{d}x = 2\pi \sum_j \text{Re } w_j \tag{16.4.147}$$

$$w_j = \operatorname*{Res}_{z=z_j} \left[f(z)\ \exp(\mathrm{i}sz)\right] \tag{16.4.148}$$

(b) Performing the integration, one side at a time,

$$\int_0^a e^{-x^2}\ \cos(2bx)\ \mathrm{d}x = \frac{1}{2} \int_{-a}^{a} e^{-x^2}\ \cos(2bx)\ \mathrm{d}x \tag{16.4.149}$$

$$= \frac{1}{2} \int_{-a}^{a} e^{-x^2}\ \exp(2bx\ \mathrm{i})\ \mathrm{d}x \tag{16.4.150}$$

$$= \frac{e^{-b^2}}{2} \int_{-a}^{a} \exp[-(x-b\mathrm{i})^2]\ \mathrm{d}x \tag{16.4.151}$$

Using the substitution, $w = x - b\mathrm{i}$, and noting that the contour integral over the two vertical sides of the rectangle vanishes at $R \to \infty$, Cauchy's integral theorem for an entire function gives,

$$I = \frac{e^{-b^2}}{2} \int_{-a-b\mathrm{i}}^{a-b\mathrm{i}} e^{-w^2}\ \mathrm{d}w = \int_{-a}^{a} e^{-w^2}\ \mathrm{d}w \tag{16.4.152}$$

$$= e^{-b^2} \int_0^a e^{-x^2}\ \mathrm{d}x = \frac{e^{-b^2}\sqrt{\pi}}{2} \tag{16.4.153}$$

(c) These problems have odd integrands with symmetric limits. Their value is thus zero.