Chapter 2

Second-Order Linear ODEs

2.1 Homogeneous Linear ODEs of Second Order

1. y does not occur explicitly.

$$F(x, y', y'') = 0 \tag{2.1.1}$$

$$z = y' \qquad z' = y'' \tag{2.1.2}$$

$$F(x, z, z') = 0 \tag{2.1.3}$$

This is a fist order ODE in z, which can be solved for z. Then,

$$y = \int z \; \mathrm{d}x \tag{2.1.4}$$

2. F is not necessarily linear. x does not occur explicitly.

$$F(y, y', y'') = 0 \tag{2.1.5}$$

$$z = y' \qquad y'' = \frac{\mathrm{d}y'}{\mathrm{d}x} \tag{2.1.6}$$

$$y'' = \frac{\mathrm{d}y'}{\mathrm{d}y}\,\frac{\mathrm{d}y}{\mathrm{d}x} \qquad = z\,\frac{\mathrm{d}z}{\mathrm{d}y} \tag{2.1.7}$$

$$F\left(y, z, z\frac{\mathrm{d}z}{\mathrm{d}y}\right) = 0 \tag{2.1.8}$$

This is a fist order ODE in z, with independent variable y.

$$y = \int z \; \mathrm{d}x \tag{2.1.9}$$

3. Reducing to first order,

$$y'' + y' = 0 \tag{2.1.10}$$

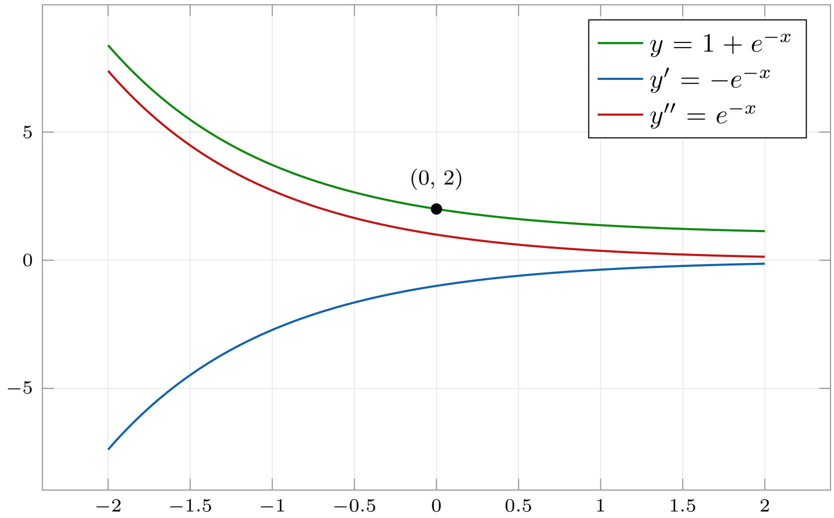
$$z = y' \qquad y'' = z\,\frac{\mathrm{d}z}{\mathrm{d}y} \tag{2.1.11}$$

$$z\,\frac{\mathrm{d}z}{\mathrm{d}y} + z = 0 \qquad \frac{\mathrm{d}z}{\mathrm{d}y} = -1 \tag{2.1.12}$$

$$z = y' = -y + b \qquad \int \frac{1}{y-b}\,\mathrm{d}y = -\int \mathrm{d}x \tag{2.1.13}$$

$$\ln(y-b) = -x + c^* \tag{2.1.14}$$

$$y = b + ce^{-x} \tag{2.1.15}$$



4. Reducing to first order,

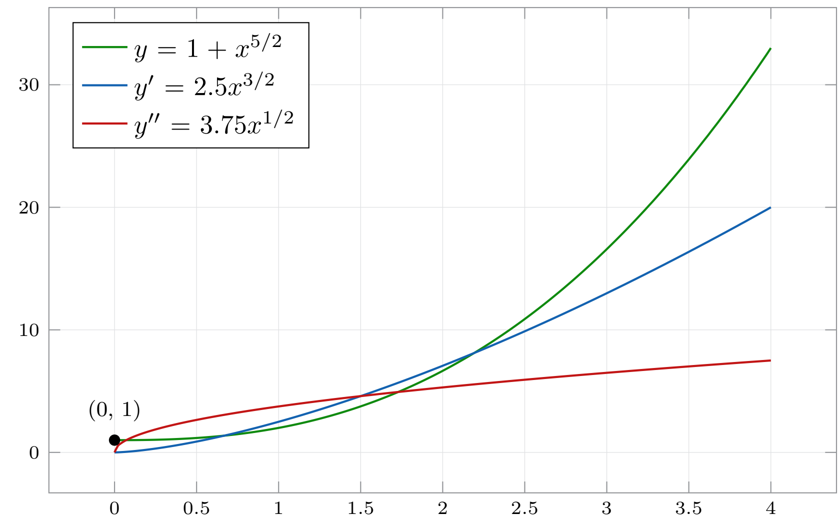
$$2xy'' = 3y' \tag{2.1.16}$$

$$z = y' \qquad y'' = z' \tag{2.1.17}$$

$$2xz' = 3z \tag{2.1.18}$$

$$\ln z = \frac{3}{2}\ln x + c^* \qquad y' = cx^{3/2} \tag{2.1.19}$$

$$y = b + cx^{5/2} \tag{2.1.20}$$



5. Reducing to first order,

$$yy'' = 3y'^2 \tag{2.1.21}$$

$$z = y' \qquad y'' = z\,\frac{\mathrm{d}z}{\mathrm{d}y} \tag{2.1.22}$$

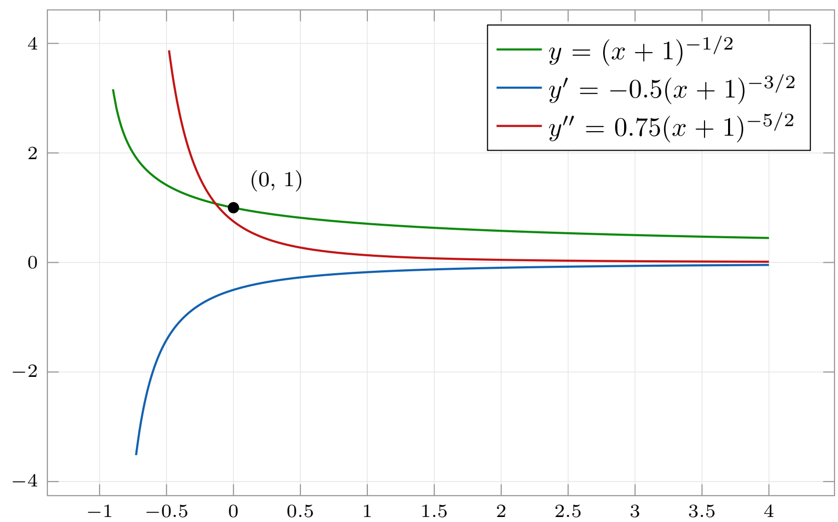
$$yz\,\frac{\mathrm{d}z}{\mathrm{d}y} = 3z^2 \qquad \frac{\mathrm{d}z}{\mathrm{d}y} = \frac{3z}{y} \tag{2.1.23}$$

$$\ln z = y' = 3\ln y + b^* \tag{2.1.24}$$

$$z = by^3 \qquad \int \frac{1}{y^3}\,\mathrm{d}y = b\int \mathrm{d}x \tag{2.1.25}$$

$$\frac{1}{y^2} = bx + c \tag{2.1.26}$$

$$y = (bx + c)^{-1/2} \tag{2.1.27}$$



6. Reducing to first order,

$$xy'' + 2y' + xy = 0 \qquad y_1 = \frac{\cos x}{x} \tag{2.1.28}$$

$$y_2 = uy_1 \qquad p(x) = \frac{2}{x} \tag{2.1.29}$$

$$V = u' = \frac{1}{y_1^2} \exp\left(-\int p(x)\ \mathrm{d}x\right) \tag{2.1.30}$$

$$= \frac{x^2}{\cos^2 x}\, x^{-2} \qquad = \sec^2 x \tag{2.1.31}$$

$$u = \int \sec^2 x\ \mathrm{d}x \qquad = \tan x \tag{2.1.32}$$

$$y = ay_1 + by_2 \tag{2.1.33}$$

$$= \frac{a\cos x}{x} + \frac{b\sin x}{x} \tag{2.1.34}$$

7. Reducing to first order,

$$y'' + y'^3\ \sin y = 0 \tag{2.1.35}$$

$$z = y' \qquad y'' = z\,\frac{\mathrm{d}z}{\mathrm{d}y} \tag{2.1.36}$$

$$z\,\frac{\mathrm{d}z}{\mathrm{d}y} = -z^3 \sin y \qquad \frac{\mathrm{d}z}{\mathrm{d}y} = -z^2 \sin y \tag{2.1.37}$$

$$\frac{-1}{z} = \cos y + b \tag{2.1.38}$$

$$z = \frac{-1}{\cos y + b} \qquad \int (b + \cos y)\ \mathrm{d}y = -\int \mathrm{d}x \tag{2.1.39}$$

$$by + \sin y = -x + c \qquad x = -\sin y + by + c \tag{2.1.40}$$

8. Reducing to first order,

$$y'' = 1 + y'^2 \tag{2.1.41}$$

$$z = y' \qquad y'' = z' \tag{2.1.42}$$

$$z' = 1 + z^2 \tag{2.1.43}$$

$$\arctan(z) = x + b \qquad z = \tan(x + b) \tag{2.1.44}$$

$$y = c + \ln|\sec(x + b)| \tag{2.1.45}$$

9. Reducing to first order,

$$x^2 y'' - 5xy' + 9y = 0 \qquad y_1 = x^3 \tag{2.1.46}$$

$$y_2 = u y_1 \qquad p(x) = \frac{-5}{x} \tag{2.1.47}$$

$$V = u' = \frac{1}{y_1^2} \exp\left(-\int p(x)\ \mathrm{d}x\right) \tag{2.1.48}$$

$$= x^{-6}\ x^5 \qquad = 1/x \tag{2.1.49}$$

$$u = \int 1/x\ \ \mathrm{d}x \qquad = \ln x \tag{2.1.50}$$

$$y = a y_1 + b y_2 \tag{2.1.51}$$

$$= a x^3 + b x^3 \ln x \tag{2.1.52}$$

10. Reducing to first order,

$$y'' + \left(1 + \frac{1}{y}\right) y'^2 = 0 \tag{2.1.53}$$

$$z = y' \qquad y'' = z\frac{\mathrm{d}z}{\mathrm{d}y} \tag{2.1.54}$$

$$z\frac{\mathrm{d}z}{\mathrm{d}y} = -\left(1 + \frac{1}{y}\right) z^2 \qquad \frac{\mathrm{d}z}{\mathrm{d}y} = -\left(1 + \frac{1}{y}\right) z \tag{2.1.55}$$

$$\ln(1/z) = y + \ln y \tag{2.1.56}$$

$$\frac{1}{z} = y e^y \qquad z = y' = \frac{1}{y e^y} \tag{2.1.57}$$

$$by + \sin y = -x + c \qquad x = -\sin y + by + c \tag{2.1.58}$$

11. Curve passes through $y(0) = 0$ and $y'(0) = 1$,

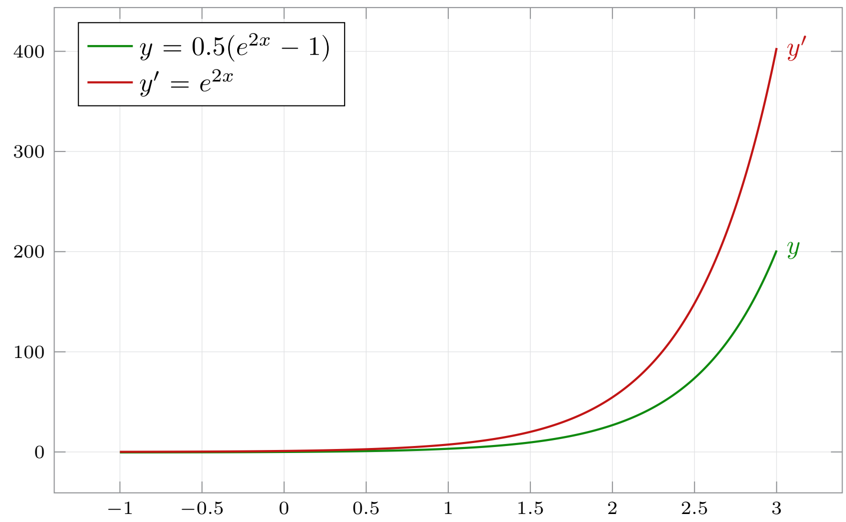
$$y' = z \qquad y'' = z' \tag{2.1.59}$$

$$y'' = 2y' \qquad z' = 2z \tag{2.1.60}$$

$$z = b^* e^{2x} \qquad y = b e^{2x} + c \tag{2.1.61}$$

$$y(0) = b + c = 0 \qquad y'(0) = 2b = 1 \tag{2.1.62}$$

$$b = 0.5 \qquad c = -0.5 \tag{2.1.63}$$



12. Curve passes through $y(-1) = 0$ and $y(1) = 0$,

$$y'' = \sqrt{1 + y'^2} \tag{2.1.64}$$

$$y' = z \qquad y'' = z' \tag{2.1.65}$$

$$z' = \sqrt{1 + z^2} \qquad \int \frac{1}{\sqrt{1+z^2}}\, \mathrm{d}z = x + b \tag{2.1.66}$$

$$\sinh^{-1} z = x + b \qquad z = \sinh(x + b) \tag{2.1.67}$$

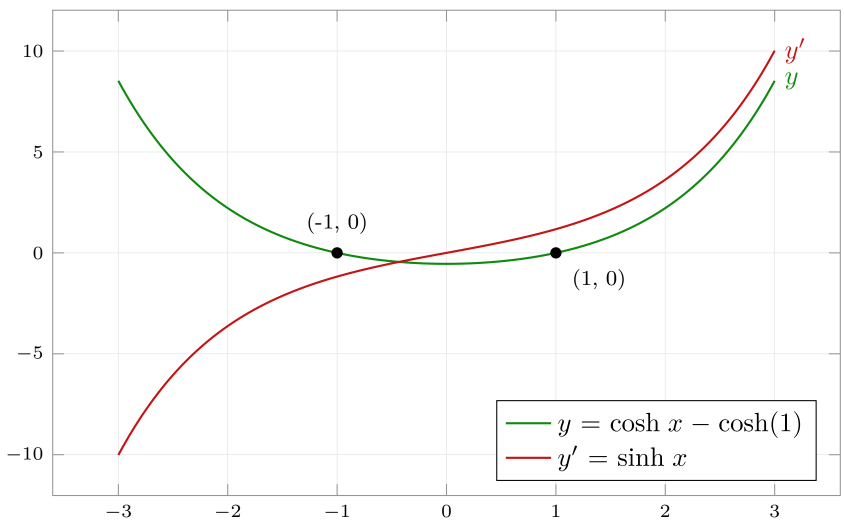
$$y = c + \cosh(x + b) \tag{2.1.68}$$

$$y(1) = c + \cosh(1 + b) = 0 \qquad y(-1) = c + \cosh(-1 + b) = 0 \tag{2.1.69}$$

$$b = -b \quad \text{or} \quad b = 0 \tag{2.1.70}$$

$$b = 0 \qquad c = -\cosh(1) \tag{2.1.71}$$

$$\tag{2.1.72}$$



13. Sum of acceleration and velocity is constant, with initial velocity v_0 and initial position y_0,

$$y'' + y' = \alpha \qquad \alpha > 0 \tag{2.1.73}$$

$$y' = z \qquad y'' = z' \tag{2.1.74}$$

$$z' = \alpha - z \qquad \int \frac{1}{z-\alpha}\, \mathrm{d}z = -t + b^* \tag{2.1.75}$$

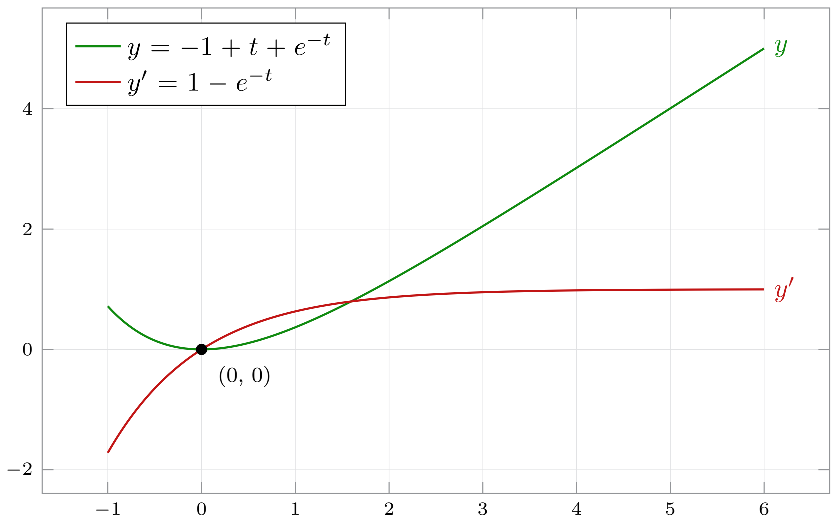
$$z = \alpha + be^{-t} \tag{2.1.76}$$

$$y = \alpha t - be^{-t} + c \tag{2.1.77}$$

$$y(0) = -b + c \qquad y'(0) = \alpha + b \tag{2.1.78}$$

$$y''(0) = -b = \alpha - v_0 \tag{2.1.79}$$

$$y = (y_0 + v_0 - \alpha) + \alpha t + (\alpha - v_0)e^{-t} \tag{2.1.80}$$



14. Velocity is the reciprocal of acceleration, with initial velocity v_0 and initial position y_0,

$$y'' = \frac{1}{y'} \tag{2.1.81}$$

$$y' = z \qquad y'' = z' \tag{2.1.82}$$

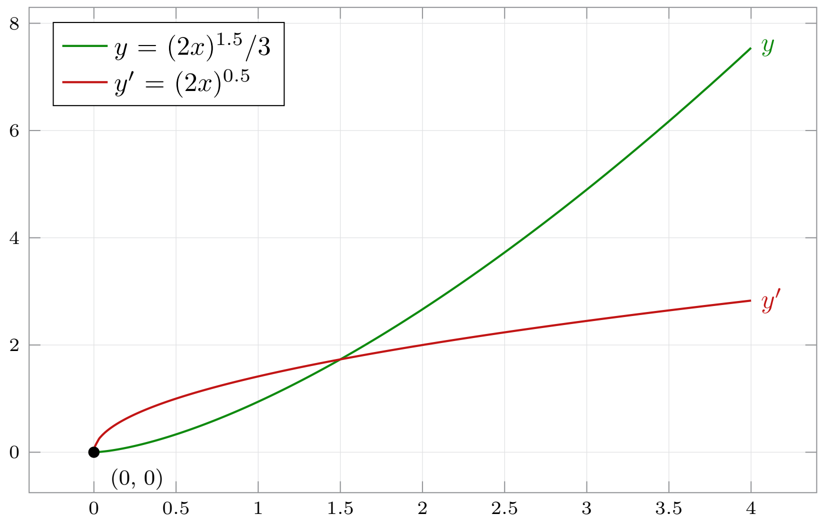
$$z' = \frac{1}{z} \qquad z^2 = 2x + b \tag{2.1.83}$$

$$z = (2x + b)^{1/2} \tag{2.1.84}$$

$$y = \frac{(2x + b)^{3/2}}{3} + c \tag{2.1.85}$$

$$y(0) = \frac{b^{3/2}}{3} + c \qquad y'(0) = b^{1/2} \tag{2.1.86}$$

$$y = \frac{\left(2x + v_0^2\right)^{3/2}}{3} + \left(y_0 - \frac{v_0^3}{3}\right) \tag{2.1.87}$$



15. The given basis is L.I, since for some any non-zero constant λ,

$$y_1 = \cos(2.5x) \qquad y_2 = \sin(2.5x) \tag{2.1.88}$$

$$\frac{y_2}{y_1} = \tan(2.5x) \qquad \not\equiv \lambda \tag{2.1.89}$$

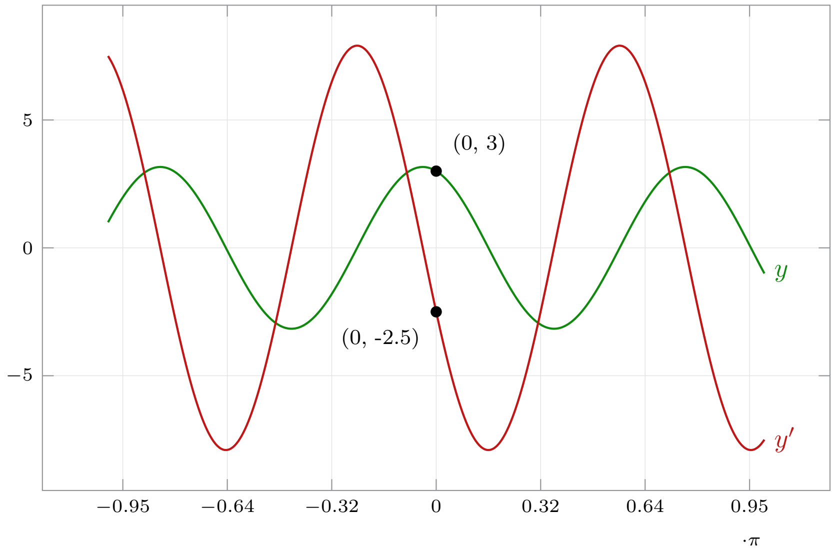
Solving the IVP,

$$4y'' + 25y = 0 \qquad y = a\cos(2.5x) + b\sin(2.5x) \tag{2.1.90}$$

$$y(0) = a = 3 \tag{2.1.91}$$

$$y'(0) = 2.5b = -2.5 \qquad b = -1 \tag{2.1.92}$$

$$y = 3\cos(2.5x) - \sin(2.5x) \tag{2.1.93}$$



16. The given basis is L.I, since for some any non-zero constant λ,

$$y_1 = e^{-0.3x} \qquad y_2 = xe^{-0.3x} \tag{2.1.94}$$

$$\frac{y_2}{y_1} = x \qquad \not\equiv \lambda \tag{2.1.95}$$

Solving the IVP,

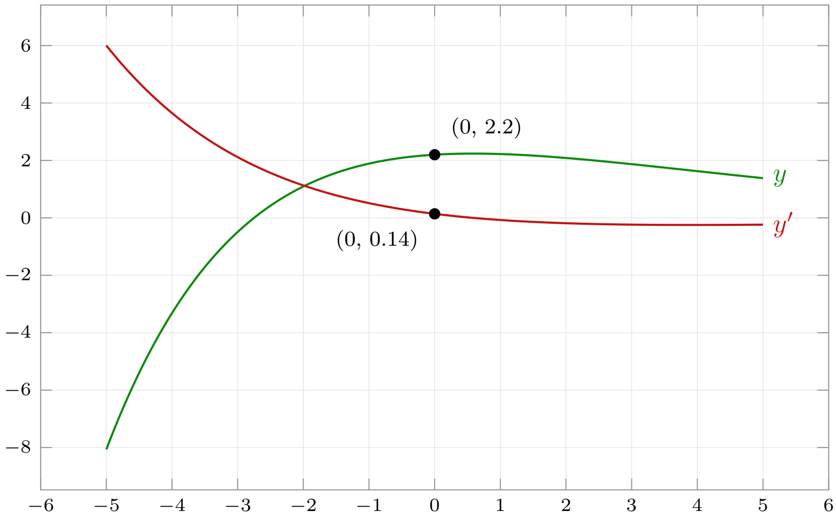
$$y'' + 0.6y' + 0.09y = 0 \qquad y = (a + bx)e^{-0.3x} \tag{2.1.96}$$

$$y(0) = a = 2.2 \tag{2.1.97}$$

$$y' = e^{-0.3x}(b - 0.3a - 0.3bx) \qquad y'(0) = b - 0.66 = 0.14 \tag{2.1.98}$$

$$b = 0.8 \tag{2.1.99}$$

$$y = (2.2 + 0.8x)e^{-0.3x} \tag{2.1.100}$$



17. The given basis is L.I, since for some any non-zero constant λ,

$$y_1 = x^{3/2} \qquad\qquad y_2 = x^{-1/2} \tag{2.1.101}$$

$$\frac{y_1}{y_2} = x^2 \qquad\qquad \not\equiv \lambda \tag{2.1.102}$$

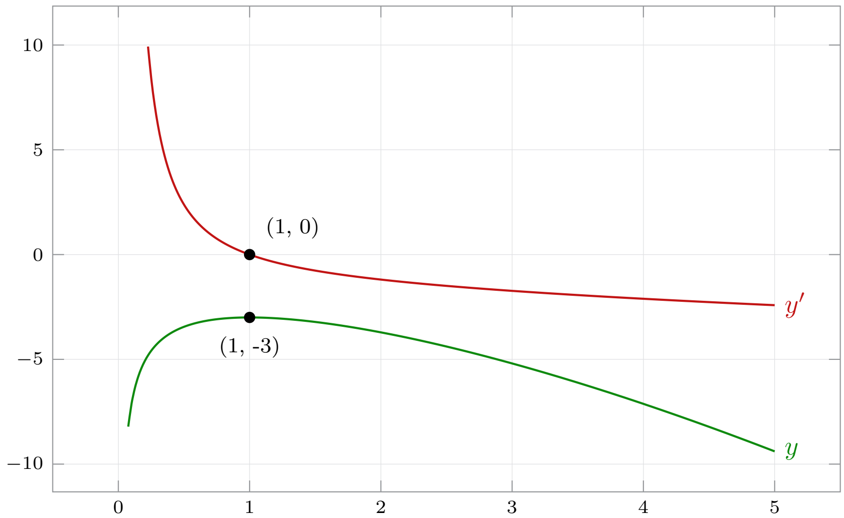
Solving the IVP,

$$4x^2 y'' = 3y \qquad\qquad y = x^{3/2}\left(a + \frac{b}{x^2}\right) \tag{2.1.103}$$

$$y(1) = a + b = -3 \qquad\qquad y'(1) = 1.5a - 0.5b = 0 \tag{2.1.104}$$

$$b = 3a \qquad\qquad a = -3/4 \tag{2.1.105}$$

$$y = -0.75x^{3/2} - 2.25x^{-1/2} \tag{2.1.106}$$



18. The given basis is L.I, since for some any non-zero constant λ,

$$y_1 = x \qquad y_2 = x \ln x \tag{2.1.107}$$

$$\frac{y_2}{y_1} = \ln x \qquad \not\equiv \lambda \tag{2.1.108}$$

Solving the IVP,

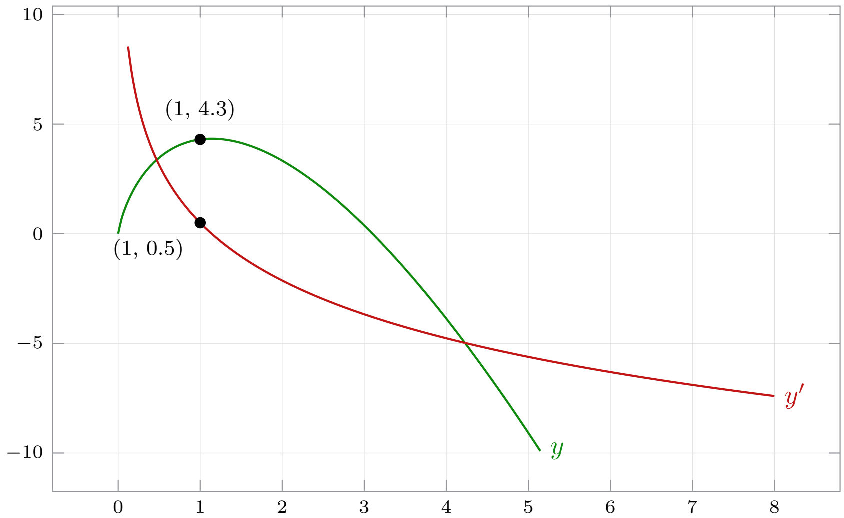
$$x^2 y'' - xy' + y = 0 \qquad y = ax + bx \ln x \tag{2.1.109}$$

$$y(1) = a = 4.3 \tag{2.1.110}$$

$$y' = a + b + b \ln x \qquad y'(1) = a + b = 0.5 \tag{2.1.111}$$

$$b = -3.8 \tag{2.1.112}$$

$$y = 4.3x - 3.8x \ln x \tag{2.1.113}$$



19. The given basis is L.I, since for some any non-zero constant λ,

$$y_1 = e^{-x} \cos x \qquad y_2 = e^{-x} \sin x \tag{2.1.114}$$

$$\frac{y_2}{y_1} = \tan x \qquad \not\equiv \lambda \tag{2.1.115}$$

Solving the IVP,

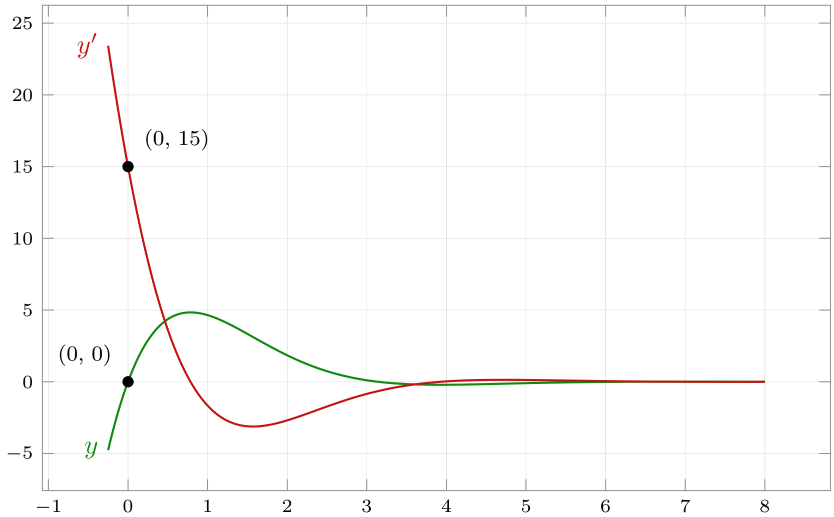
$$y'' - 2y' + 2y = 0 \qquad y = e^{-x}(a \cos x + b \sin x) \tag{2.1.116}$$

$$y(0) = a = 0 \tag{2.1.117}$$

$$y' = e^{-x}[(b-a)\cos x - (a+b)\sin x] \qquad y'(0) = b - a = 15 \tag{2.1.118}$$

$$b = 15 \tag{2.1.119}$$

$$y = 15e^{-x} \sin x \tag{2.1.120}$$



20. Outline of algorithm for program to test if two solutions defined on the interval I are L.I.,

(a) Define the Wronskian for the two functions as,

$$W(y_1, y_2)(x) := y_2 y_1' - y_1 y_2' \tag{2.1.121}$$

(b) If $W \neq 0$ for some $x_0 \in I$, return TRUE.

(c) Else if $W \equiv 0 \quad \forall \quad x \in I$, return FALSE.

2.2 Homogeneous Linear ODEs with Constant Coefficients

1. Solving ODE,

$$4y'' - 25y = 0 \tag{2.2.1}$$

$$a = 0 \qquad b = \frac{-25}{4} \tag{2.2.2}$$

$$a^2 - 4b = 25 > 0 \tag{2.2.3}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm \sqrt{25}}{2} \qquad = \pm\frac{5}{2} \tag{2.2.4}$$

$$y = c_1 e^{5x/2} + c_2 e^{-5x/2} \tag{2.2.5}$$

Checking by substitution,

$$y' = \frac{5c_1}{2} e^{5x/2} - \frac{5c_2}{2} e^{-5x/2} \tag{2.2.6}$$

$$y'' = \frac{25c_1}{4} e^{5x/2} + \frac{25c_2}{4} e^{-5x/2} \tag{2.2.7}$$

$$y'' = \frac{25}{4} y \tag{2.2.8}$$

2. Solving ODE,

$$y'' + 36y = 0 \tag{2.2.9}$$

$$a = 0 \qquad b = 36 \tag{2.2.10}$$

$$a^2 - 4b = -144 < 0 \tag{2.2.11}$$

$$\omega^2 = 36 - \frac{0^2}{4} = 36 \tag{2.2.12}$$

$$y = c_1 \cos(6x) + c_2 \sin(6x) \tag{2.2.13}$$

Checking by substitution,

$$y' = -6c_1 \sin(6x) + 6c_2 \cos(6x) \tag{2.2.14}$$

$$y'' = -36c_2 \cos(6x) - 36c_2 \sin(6x) \tag{2.2.15}$$

$$y'' = -36y \tag{2.2.16}$$

3. Solving ODE,

$$y'' + 6y' + 8.96y = 0 \tag{2.2.17}$$

$$a = 6 \qquad b = 8.96 \tag{2.2.18}$$

$$a^2 - 4b = 36 - 4(8.96) = \frac{4}{25} > 0 \tag{2.2.19}$$

$$\lambda_1, \lambda_2 = \frac{-6 \pm \sqrt{4/25}}{2} = -3.2, -2.8 \tag{2.2.20}$$

$$y = c_1 e^{-3.2x} + c_2 e^{-2.8x} \tag{2.2.21}$$

Checking by substitution,

$$y' = -3.2 c_1 e^{-3.2x} - 2.8 c_2 e^{-2.8x} \tag{2.2.22}$$

$$y'' = 10.24 c_1 e^{-3.2x} + 7.84 c_2 e^{-2.8x} \tag{2.2.23}$$

$$y'' + 6y' + 8.96y = c_1 e^{-3.2x} \, (10.24 - 19.2 + 8.96) \tag{2.2.24}$$

$$+ c_2 e^{-2.8x} \, (7.84 - 16.8 + 8.96) \tag{2.2.25}$$

$$= 0 \tag{2.2.26}$$

4. Solving ODE,

$$y'' + 4y' + (\pi^2 + 4)y = 0 \tag{2.2.27}$$

$$a = 4 \qquad b = (\pi^2 + 4) \tag{2.2.28}$$

$$a^2 - 4b = -4\pi^2 < 0 \tag{2.2.29}$$

$$\omega^2 = (\pi^2 + 4) - \frac{4^2}{4} = \pi^2 \tag{2.2.30}$$

$$y = [c_1 \cos(\pi x) + c_2 \sin(\pi x)] \exp(-2x) \tag{2.2.31}$$

Checking by substitution,

$$y' = e^{-2x} \cos(\pi x)(\pi c_2 - 2c_1) + e^{-2x} \sin(\pi x)(-\pi c_1 - 2c_2) \tag{2.2.32}$$

$$y'' = e^{-2x} \cos(\pi x)(-2\pi c_2 + 4c_1 - \pi^2 c_1 - 2\pi c_2) \tag{2.2.33}$$

$$+ e^{-2x} \sin(\pi x)(2\pi c_1 + 4c_2 - \pi^2 c_2 + 2\pi c_1) \tag{2.2.34}$$

$$y'' + 4y' = c_1 e^{-2x} \cos(\pi x)(-4 - \pi^2) + c_2 e^{-2x} \sin(\pi x)(-4 - \pi^2) \tag{2.2.35}$$

$$= -(4 + \pi^2)y \tag{2.2.36}$$

5. Solving ODE,

$$y'' + 2\pi y' + \pi^2 y = 0 \tag{2.2.37}$$

$$a = 2\pi \qquad b = \pi^2 \tag{2.2.38}$$

$$a^2 - 4b = 0 \tag{2.2.39}$$

$$\lambda_1, \lambda_2 = \frac{-2\pi \pm 0}{2} \qquad = -\pi \tag{2.2.40}$$

$$y = (c_1 + c_2 x)e^{-\pi x} \tag{2.2.41}$$

Checking by substitution,

$$y' = e^{-\pi x}(c_2 - \pi c_1 - \pi c_2 x) \tag{2.2.42}$$

$$y'' = e^{-\pi x}(-\pi c_2 - \pi c_2 + \pi^2 c_1 + \pi^2 c_2 x) \tag{2.2.43}$$

$$y'' + 2\pi y' = e^{-\pi x}(-\pi^2 c_1 - \pi^2 c_2 x) \tag{2.2.44}$$

$$= -\pi^2 y \tag{2.2.45}$$

6. Solving ODE,

$$10y'' - 32y' + 25.6y = 0 \tag{2.2.46}$$

$$a = -3.2 \qquad b = 2.56 \tag{2.2.47}$$

$$a^2 - 4b = 3.2^2 - 4(2.56) = 0 \tag{2.2.48}$$

$$\lambda_1, \lambda_2 = \frac{-3.2 \pm 0}{2} \qquad = -1.6 \tag{2.2.49}$$

$$y = (c_1 + c_2 x)e^{1.6x} \tag{2.2.50}$$

Checking by substitution,

$$y' = e^{1.6x}(c_2 + 1.6c_1 + 1.6c_2 x) \tag{2.2.51}$$

$$y'' = e^{1.6x}(1.6c_2 + 1.6c_2 + 2.56c_1 + 2.56c_2 x) \tag{2.2.52}$$

$$10y'' - 32y' = e^{1.6x}(-25.6c_1 - 25.6c_2 x) \tag{2.2.53}$$

$$= -25.6y \tag{2.2.54}$$

7. Solving ODE,

$$y'' + 4.5y' = 0 \tag{2.2.55}$$

$$a = 4.5 \qquad b = 0 \tag{2.2.56}$$

$$a^2 - 4b = 20.25 > 0 \tag{2.2.57}$$

$$\lambda_1, \lambda_2 = \frac{-4.5 \pm \sqrt{20.25}}{2} \qquad = -4.5, 0 \tag{2.2.58}$$

$$y = c_1 e^{-9x} + c_2 \tag{2.2.59}$$

Checking by substitution,

$$y' = -4.5c_1 e^{-4.5x} \tag{2.2.60}$$

$$y'' = 20.25c_1 e^{-4.5x} \tag{2.2.61}$$

$$y'' + 4.5y' = c_1 e^{-4.5x}(20.25 - 4.5^2) \tag{2.2.62}$$

$$= 0 \tag{2.2.63}$$

8. Solving ODE,

$$y'' + y' + 3.25y = 0 \tag{2.2.64}$$

$$a = 1 \qquad b = 3.25 \tag{2.2.65}$$

$$a^2 - 4b = 1 - 13 = -12 < 0 \tag{2.2.66}$$

$$\omega^2 = 3.25 - \frac{1^2}{4} = 3 \tag{2.2.67}$$

$$y = [c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)] \exp(-x/2) \tag{2.2.68}$$

Checking by substitution,

$$y' = e^{-x/2} \cos(\sqrt{3}x)(\sqrt{3}c_2 - 0.5c_1) + e^{-x/2} \sin(\sqrt{3}x)(-\sqrt{3}c_1 - 0.5c_2) \tag{2.2.69}$$

$$y'' = e^{-x/2} \cos(\sqrt{3}x)(-0.5\sqrt{3}c_2 + 0.25c_1 - 3c_1 - 0.5\sqrt{3}c_2) \tag{2.2.70}$$

$$+ e^{-x/2} \sin(\sqrt{3}x)(0.5\sqrt{3}c_1 + 0.25c_2 - 3c_2 + 0.5\sqrt{3}c_1) \tag{2.2.71}$$

$$y'' + y' = e^{-x/2} \cos(\sqrt{3}x)(-3.25c_1) + e^{-x/2} \sin(\sqrt{3}x)(-3.25c_2) \tag{2.2.72}$$

$$= -3.25y \tag{2.2.73}$$

9. Solving ODE,

$$y'' + 1.8y' - 2.08y = 0 \tag{2.2.74}$$

$$a = 1.8 \qquad b = -2.08 \tag{2.2.75}$$

$$a^2 - 4b = 1.8^2 - 4(-2.08) = 11.56 > 0 \tag{2.2.76}$$

$$\lambda_1, \lambda_2 = \frac{-1.8 \pm \sqrt{11.56}}{2} = -2.6, 0.8 \tag{2.2.77}$$

$$y = c_1 e^{-2.6x} + c_2 e^{0.8x} \tag{2.2.78}$$

Checking by substitution,

$$y' = -2.6c_1 e^{-2.6x} + 0.8c_2 e^{0.8x} \tag{2.2.79}$$

$$y'' = 5.76c_1 e^{-2.6x} + 0.64c_2 e^{0.8x} \tag{2.2.80}$$

$$y'' + 1.8y' - 2.08y = c_1 e^{-2.6x}\,(6.76 - 4.68 - 2.08) \tag{2.2.81}$$

$$+ c_2 e^{0.8x}\,(0.64 + 1.44 - 2.08) \tag{2.2.82}$$

$$= 0 \tag{2.2.83}$$

10. Solving ODE,

$$0 = 100y'' + 240y' + (196\pi^2 + 144)y \tag{2.2.84}$$

$$a = 2.40 \qquad b = (1.96\pi^2 + 1.44) \tag{2.2.85}$$

$$a^2 - 4b = 2.40^2 - 4(1.96\pi^2 + 1.44) = -\frac{196\pi^2}{25} < 0 \tag{2.2.86}$$

$$\omega^2 = (1.96\pi^2 + 1.44) - \frac{2.4^2}{4} = \frac{49\pi^2}{25} \tag{2.2.87}$$

$$y = [c_1 \cos(1.4\pi x) + c_2 \sin(1.4\pi x)] \exp(-1.2x) \tag{2.2.88}$$

Checking by substitution,

$$y' = e^{-1.2x}\cos(1.4\pi x)(1.4\pi c_2 - 1.2c_1) + e^{-1.2x}\sin(1.4\pi x)(-1.4\pi c_1 - 1.2c_2) \tag{2.2.89}$$

$$y'' = e^{-1.2x}\cos(1.4\pi x)(-1.68\pi c_2 + 1.44c_1 - 1.96\pi^2 c_1 - 1.68\pi c_2) \tag{2.2.90}$$

$$+ e^{-1.2x}\sin(1.4\pi x)(1.68\pi c_1 + 1.44c_2 - 1.96\pi^2 c_2 + 1.68\pi c_1) \tag{2.2.91}$$

$$y'' + 2.4y' = c_1 e^{-1.2x}\cos(1.4\pi x)(-1.44 - 1.96\pi^2) \tag{2.2.92}$$

$$+ c_2 e^{-1.2x}\sin(1.4x)(-1.44 - 1.96\pi^2) \tag{2.2.93}$$

$$= -(1.44 + 1.96\pi^2)y \tag{2.2.94}$$

11. Solving ODE,

$$4y'' - 4y' - 3y = 0 \tag{2.2.95}$$

$$a = -1 \qquad b = -3/4 \tag{2.2.96}$$

$$a^2 - 4b = 1^2 - 4(-3/4) = 4 > 0 \tag{2.2.97}$$

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{4}}{2} \qquad = -0.5, 1.5 \tag{2.2.98}$$

$$y = c_1 e^{1.5x} + c_2 e^{-0.5x} \tag{2.2.99}$$

Checking by substitution,

$$y' = 1.5c_1 e^{1.5x} - 0.5c_2 e^{-0.5x} \tag{2.2.100}$$

$$y'' = 2.25c_1 e^{1.5x} + 0.25c_2 e^{-0.5x} \tag{2.2.101}$$

$$4y'' - 4y' - 3y = c_1 e^{1.5x}\ (9 - 6 - 3) \tag{2.2.102}$$

$$+ c_2 e^{-0.5x}\ (1 + 2 - 3) \tag{2.2.103}$$

$$= 0 \tag{2.2.104}$$

12. Solving ODE,

$$y'' + 9y' + 20y = 0 \tag{2.2.105}$$

$$a = 9 \qquad b = 20 \tag{2.2.106}$$

$$a^2 - 4b = 9^2 - 4(20) = 1 > 0 \tag{2.2.107}$$

$$\lambda_1, \lambda_2 = \frac{-9 \pm \sqrt{1}}{2} \qquad = -5, -4 \tag{2.2.108}$$

$$y = c_1 e^{-1.5x} + c_2 e^{0.5x} \tag{2.2.109}$$

Checking by substitution,

$$y' = -5c_1 e^{-5x} - 4c_2 e^{-4x} \tag{2.2.110}$$

$$y'' = 25c_1 e^{-5x} + 16c_2 e^{-4x} \tag{2.2.111}$$

$$y'' + 9y' + 20y = c_1 e^{-5x} (25 - 45 + 20) \tag{2.2.112}$$

$$+ c_2 e^{-4x} (16 - 36 + 20) \tag{2.2.113}$$

$$= 0 \tag{2.2.114}$$

13. Solving ODE,

$$9y'' - 30y' + 25y = 0 \tag{2.2.115}$$

$$a = -30/9 \qquad b = 25/9 \tag{2.2.116}$$

$$a^2 - 4b = \frac{30}{9}^2 - \frac{100}{9} = 0 \tag{2.2.117}$$

$$\lambda_1, \lambda_2 = \frac{30/9 \pm 0}{2} \qquad = 5/3 \tag{2.2.118}$$

$$y = (c_1 + c_2 x)e^{5x/3} \tag{2.2.119}$$

Checking by substitution,

$$y' = e^{5x/3}\left(c_2 + \frac{5}{3}c_1 + \frac{5}{3}c_2 x\right) \tag{2.2.120}$$

$$y'' = e^{5x/3}\left(\frac{5}{3}c_2 + \frac{5}{3}c_2 + \frac{25}{9}c_1 + \frac{25}{9}c_2 x\right) \tag{2.2.121}$$

$$9y'' - 30y' = e^{5x/3}(-25c_1 - 25c_2 x) \tag{2.2.122}$$

$$= -25y \tag{2.2.123}$$

14. Solving ODE,

$$y'' + 2k^2 y' + k^4 y = 0 \tag{2.2.124}$$

$$a = 2k^2 \qquad b = k^4 \tag{2.2.125}$$

$$a^2 - 4b = 0 \tag{2.2.126}$$

$$\lambda_1, \lambda_2 = \frac{-2k^2 \pm 0}{2} \qquad = k^2 \tag{2.2.127}$$

$$y = (c_1 + c_2 x)e^{-k^2 x} \tag{2.2.128}$$

Checking by substitution,

$$y' = e^{-k^2 x}(c_2 - k^2 c_1 - k^2 c_2 x) \tag{2.2.129}$$

$$y'' = e^{-k^2 x}(-k^2 c_2 - k^2 c_2 + k^4 c_1 + k^4 c_2 x) \tag{2.2.130}$$

$$y'' + 2k^2 y' = e^{-k^2 x}(-k^4 c_1 - k^4 c_2 x) \tag{2.2.131}$$

$$= -k^4 y \tag{2.2.132}$$

15. Solving ODE,

$$0 = y'' + 0.54y' + (\pi + 0.0729)y \tag{2.2.133}$$

$$a = 0.54 \qquad b = (\pi + 0.0729) \tag{2.2.134}$$

$$a^2 - 4b = 0.54^2 - 4(\pi + 0.0729) = -4\pi < 0 \tag{2.2.135}$$

$$\omega^2 = (\pi + 0.0729) - \frac{0.54^2}{4} = \pi \tag{2.2.136}$$

$$y = [c_1 \cos(\sqrt{\pi}x) + c_2 \sin(\sqrt{\pi}x)] \exp(-0.27x) \tag{2.2.137}$$

Checking by substitution,

$$y' = e^{-0.27x} \cos(\sqrt{\pi}x)(\sqrt{\pi}c_2 - 0.27c_1) \quad 2.2.138$$

$$+ e^{-0.27x} \sin(\sqrt{\pi}x)(-\sqrt{\pi}c_1 - 0.27c_2) \quad 2.2.139$$

$$y'' = e^{-0.27x} \cos(\sqrt{\pi}x)(-0.27\sqrt{\pi}c_2 + 0.0729c_1 - \pi c_1 - 0.27\sqrt{\pi}c_2) \quad 2.2.140$$

$$+ e^{-0.27x} \sin(\sqrt{\pi}x)(0.27\sqrt{\pi}c_1 + 0.0729c_2 - \pi c_2 + 0.27\sqrt{\pi}c_1) \quad 2.2.141$$

$$y'' + 0.54y' = c_1 e^{-0.27x} \cos(\sqrt{\pi}x)(-\pi - 0.0729) \quad 2.2.142$$

$$+ c_2 e^{-0.27x} \sin(1.4x)(-\pi - 0.0729) \quad 2.2.143$$

$$= -(\pi + 0.0729)y \quad 2.2.144$$

16. Given the basis, characteristic equation has real distinct roots,

$$\lambda_1 = 2.6 \qquad \lambda_2 = -4.3 \quad 2.2.145$$

$$a = -(\lambda_1 + \lambda_2) \qquad = 1.7 \quad 2.2.146$$

$$b = \lambda_1 \cdot \lambda_2 \qquad = -11.18 \quad 2.2.147$$

$$0 = y'' - 4.3y' - 11.18y \quad 2.2.148$$

17. Given the basis, characteristic equation has real repeated roots,

$$\lambda_1 = \exp(-\sqrt{5}x) \qquad \lambda_2 = \exp(-\sqrt{5}x) \quad 2.2.149$$

$$-a/2 = -\sqrt{5} \qquad a = 2\sqrt{5} \quad 2.2.150$$

$$a^2 - 4b = 0 \qquad b = 5 \quad 2.2.151$$

$$0 = y'' + 2\sqrt{5}y' + 5y \quad 2.2.152$$

18. Given the basis, characteristic equation has imaginary roots,

$$\lambda_1 = \cos(2\pi x) \qquad \lambda_2 = \sin(2\pi x) \quad 2.2.153$$

$$-a/2 = 0 \qquad a = 0 \quad 2.2.154$$

$$b - a^2/4 = \omega^2 \qquad b = 4\pi^2 \quad 2.2.155$$

$$0 = y'' + 4\pi^2 y \quad 2.2.156$$

19. Given the basis, characteristic equation has imaginary roots,

$$\lambda_1 = \exp[(-2+i)x] \qquad \lambda_2 = \exp[(-2-i)x] \tag{2.2.157}$$

$$\lambda_1 + \lambda_2 = e^{-2x}(2\cos x) \qquad a = -2\cos x\ e^{-2x} \tag{2.2.158}$$

$$\lambda_1 \cdot \lambda_2 = e^{-4x} \qquad b = e^{-4x} \tag{2.2.159}$$

To obtain an ODE with constant coefficients, consider the superposition of these solutions,

$$\lambda_3 = \frac{\lambda_1 + \lambda_2}{2} \qquad = e^{-2x}(\cos x) \tag{2.2.160}$$

$$\lambda_4 = \frac{\lambda_1 - \lambda_2}{2i} \qquad = e^{-2x}(\sin x) \tag{2.2.161}$$

$$-a/2 = -2 \qquad a = 4 \tag{2.2.162}$$

$$\omega^2 = b - \frac{a^2}{4} \qquad b = 5 \tag{2.2.163}$$

$$0 = y'' + 4y' + 5y \tag{2.2.164}$$

20. Given the basis, characteristic equation has imaginary roots,

$$\lambda_1 = e^{-3.1x}\cos(2.1x) \qquad \lambda_2 = e^{-3.1x}\sin(2.1x) \tag{2.2.165}$$

$$-a/2 = -3.1 \qquad a = 6.2 \tag{2.2.166}$$

$$b - a^2/4 = \omega^2 \qquad b = \frac{6.2^2}{4} + (2.1)^2 = 14.02 \tag{2.2.167}$$

$$0 = y'' + 6.2y' + 14.02y \tag{2.2.168}$$

21. Solving ODE, with IC $y(0) = 4.6$, $y'(0) = -1.2$,

$$0 = y'' + 25y \tag{2.2.169}$$

$$a = 0 \qquad b = 25 \tag{2.2.170}$$

$$a^2 - 4b = -100 < 0 \tag{2.2.171}$$

$$\omega^2 = 25 - \frac{0^2}{4} = 25 \tag{2.2.172}$$

$$y = c_1\cos(5x) + c_2\sin(5x) \tag{2.2.173}$$

$$y(0) = c_1 = 4.6 \qquad y'(0) = 5c_2 = -1.2 \tag{2.2.174}$$

$$y = 4.6\cos(5x) - 0.24\sin(5x) \tag{2.2.175}$$

Checking by substitution,

$$y' = \cos(5x)(-1.2) + \sin(5x)(-23) \tag{2.2.176}$$

$$y'' = \cos(5x)(-115) + \sin(5x)(6) \tag{2.2.177}$$

$$y'' + 25y' = \cos(5x)(-115 + 115) + \sin(5x)(6 - 6) \tag{2.2.178}$$

$$= 0 \tag{2.2.179}$$

22. Solving ODE, with IC $y(1/2) = 1$, $y'(1/2) = -2$,

$$y'' + 4y' + (\pi^2 + 4)y = 0 \tag{2.2.180}$$

$$a = 4 \qquad b = (\pi^2 + 4) \tag{2.2.181}$$

$$a^2 - 4b = -4\pi^2 < 0 \tag{2.2.182}$$

$$\omega^2 = (\pi^2 + 4) - \frac{4^2}{4} = \pi^2 \tag{2.2.183}$$

$$y = [c_1 \cos(\pi x) + c_2 \sin(\pi x)] \exp(-2x) \tag{2.2.184}$$

$$y(0) = \frac{c_2}{e} = 1 \qquad c_2 = e \tag{2.2.185}$$

$$y'(0) = \frac{1}{e}(-\pi c_1 - 2c_2) = -2 \qquad c_1 = 0 \tag{2.2.186}$$

$$y = e^{-2x}\, (e \sin(\pi x)) \tag{2.2.187}$$

Checking by substitution,

$$y' = e^{-2x} \cos(\pi x)(\pi e) + e^{-2x} \sin(\pi x)(-2e) \tag{2.2.188}$$

$$y'' = e^{-2x} \cos(\pi x)(-2\pi e - 2\pi e) \tag{2.2.189}$$

$$+ e^{-2x} \sin(\pi x)(-\pi^2 e + 4e) \tag{2.2.190}$$

$$y'' + 4y' = e^{-2x+1} \cos(\pi x)(-4\pi + 4\pi) + e^{-2x+1} \sin(\pi x)(-4 - \pi^2) \tag{2.2.191}$$

$$= -(4 + \pi^2)y \tag{2.2.192}$$

23. Solving ODE, with IC $y(0) = 10$, $y'(0) = 0$,

$$y'' + y' - 6y = 0 \tag{2.2.193}$$

$$a = 1 \qquad b = -6 \tag{2.2.194}$$

$$a^2 - 4b = 25 > 0 \tag{2.2.195}$$

$$\lambda_1, \lambda_2 = \frac{-1 \pm \sqrt{25}}{2} \qquad = 2, -3 \tag{2.2.196}$$

$$y = c_1 e^{-3x} + c_2 e^{2x} \tag{2.2.197}$$

$$y(0) = c_1 + c_2 = 10 \qquad y'(0) = -3c_1 + 2c_2 = 0 \tag{2.2.198}$$

$$y = 4e^{-3x} + 6e^{2x} \tag{2.2.199}$$

Checking by substitution,

$$y' = -12e^{-3x} + 12e^{2x} \tag{2.2.200}$$

$$y'' = 36e^{-3x} + 24e^{2x} \tag{2.2.201}$$

$$y'' + y' - 6y = e^{-3x}\,(36 - 12 - 24) \tag{2.2.202}$$

$$+ e^{2x}\,(24 + 12 - 26) \tag{2.2.203}$$

$$= 0 \tag{2.2.204}$$

24. Solving ODE, with IC $y(-2) = e$, $y'(-2) = -e/2$,

$$4y'' - 4y' - 3y = 0 \tag{2.2.205}$$

$$a = -1 \qquad b = -3/4 \tag{2.2.206}$$

$$a^2 - 4b = 1^2 - 4(-3/4) = 4 > 0 \tag{2.2.207}$$

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{4}}{2} \qquad = -0.5, 1.5 \tag{2.2.208}$$

$$y = c_1 e^{1.5x} + c_2 e^{-0.5x} \tag{2.2.209}$$

$$y(-2) = c_1 e^{-3} + c_2 e = e \qquad y'(-2) = 1.5c_1 e^{-3} - 0.5c_2 e = -e/2 \tag{2.2.210}$$

$$y = e^{-0.5x} \tag{2.2.211}$$

Checking by substitution,

$$y' = -0.5e^{-0.5x} \quad 2.2.212$$

$$y'' = +0.25e^{-0.5x} \quad 2.2.213$$

$$4y'' - 4y' - 3y = e^{-0.5x}\,(1 + 2 - 3) \quad 2.2.214$$

$$= 0 \quad 2.2.215$$

25. Solving ODE, with IC $y(0) = 2$, $y'(0) = -2$,

$$y'' - y = 0 \quad 2.2.216$$

$$a = 0 \qquad b = -1 \quad 2.2.217$$

$$a^2 - 4b = 4 > 0 \quad 2.2.218$$

$$\lambda_1, \lambda_2 = \frac{0 \pm \sqrt{4}}{2} \qquad = \pm 1 \quad 2.2.219$$

$$y = c_1 e^{x} + c_2 e^{-x} \quad 2.2.220$$

$$y(0) = c_1 + c_2 = 2 \qquad y'(0) = c_1 - c_2 = -2 \quad 2.2.221$$

$$y = 2e^{-x} \quad 2.2.222$$

Checking by substitution,

$$y' = -2e^{-x} \quad 2.2.223$$

$$y'' = 2e^{-x} \quad 2.2.224$$

$$y'' = y \quad 2.2.225$$

26. Solving ODE, with IC $y(0) = 1$, $y'(0) = 1$,

$$y'' - k^2 y = 0 \tag{2.2.226}$$

$$a = 0 \qquad b = -k^2 \tag{2.2.227}$$

$$a^2 - 4b = 4k^2 > 0 \tag{2.2.228}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm \sqrt{4k^2}}{2} \qquad = \pm k \tag{2.2.229}$$

$$y = c_1 e^{kx} + c_2 e^{-kx} \tag{2.2.230}$$

$$y(0) = c_1 + c_2 = 1/k \qquad y'(0) = c_1 - c_2 = 1/k \tag{2.2.231}$$

$$y = \frac{1}{k} e^{kx} \tag{2.2.232}$$

Checking by substitution,

$$y' = e^{kx} \tag{2.2.233}$$

$$y'' = k e^{kx} \tag{2.2.234}$$

$$y'' = k^2 y \tag{2.2.235}$$

27. Solving ODE, with IC $y(0) = 4.5$, $y'(0) = -4.5\pi - 1$

$$y'' + 2\pi y' + \pi^2 y = 0 \tag{2.2.236}$$

$$a = 2\pi \qquad b = \pi^2 \tag{2.2.237}$$

$$a^2 - 4b = 0 \tag{2.2.238}$$

$$\lambda_1, \lambda_2 = \frac{-2\pi \pm 0}{2} \qquad = -\pi \tag{2.2.239}$$

$$y = (c_1 + c_2 x) e^{-\pi x} \tag{2.2.240}$$

$$y(0) = c_1 = 4.5 \qquad y'(0) = c_2 - \pi c_1 = -4.5\pi - 1 \tag{2.2.241}$$

$$y = (4.5 - x) e^{-\pi x} \tag{2.2.242}$$

Checking by substitution,

$$y' = e^{-\pi x}(-1 - 4.5\pi + \pi x) \tag{2.2.243}$$

$$y'' = e^{-\pi x}(2\pi + 4.5\pi^2 - \pi^2 x) \tag{2.2.244}$$

$$y'' + 2\pi y' = e^{-\pi x}(-4.5\pi^2 + \pi^2 x) \tag{2.2.245}$$

$$= -\pi^2 y \tag{2.2.246}$$

28. Solving ODE, with IC $y(0) = -0.2$, $y'(0) = -0.325$,

$$8y'' - 2y' - y = 0 \tag{2.2.247}$$

$$a = -1/4 \qquad b = -1/8 \tag{2.2.248}$$

$$a^2 - 4b = 9/16 > 0 \tag{2.2.249}$$

$$\lambda_1, \lambda_2 = \frac{1/4 \pm \sqrt{9/16}}{2} \qquad = 0.5, -0.25 \tag{2.2.250}$$

$$y = c_1 e^{0.5x} + c_2 e^{-0.25x} \tag{2.2.251}$$

$$y(0) = c_1 + c_2 = -0.2 \qquad y'(0) = 0.5c_1 - 0.25c_2 = -0.325 \tag{2.2.252}$$

$$y = 1.5e^{0.5x} - 1.7e^{-0.25x} \tag{2.2.253}$$

Checking by substitution,

$$y' = 0.75e^{0.5x} + 0.425e^{-0.25x} \tag{2.2.254}$$

$$y'' = 0.375e^{0.5x} - 0.10625e^{-0.25x} \tag{2.2.255}$$

$$8y'' - 2y' - y = e^{0.5x}\ (3 - 1.5 - 1.5) \tag{2.2.256}$$

$$+ e^{-0.25x}(-0.85 - 0.85 + 1.7) \tag{2.2.257}$$

$$= 0 \tag{2.2.258}$$

29. Solving ODE, with IC $y(0) = 0$, $y'(0) = 1$,

$$0 = y'' + 0.54y' + (\pi + 0.0729)y \tag{2.2.259}$$

$$a = 0.54 \qquad b = (\pi + 0.0729) \tag{2.2.260}$$

$$a^2 - 4b = 0.54^2 - 4(\pi + 0.0729) = -4\pi < 0 \tag{2.2.261}$$

$$\omega^2 = (\pi + 0.0729) - \frac{0.54^2}{4} = \pi \tag{2.2.262}$$

$$y = [c_1 \cos(\sqrt{\pi}x) + c_2 \sin(\sqrt{\pi}x)] \exp(-0.27x) \tag{2.2.263}$$

$$y(0) = c_1 = 0 \tag{2.2.264}$$

$$y'(0) = \sqrt{\pi}c_2 = 1 \tag{2.2.265}$$

$$y = \frac{1}{\sqrt{\pi}} \ \sin(\sqrt{\pi}x) \ \exp(-0.27x) \tag{2.2.266}$$

Checking by substitution,

$$y' = e^{-0.27x} \cos(\sqrt{\pi}x)(1) \tag{2.2.267}$$

$$+ e^{-0.27x} \sin(\sqrt{\pi}x)(-0.27/\sqrt{\pi}) \tag{2.2.268}$$

$$y'' = e^{-0.27x} \cos(\sqrt{\pi}x)(-0.27 - 0.27) \tag{2.2.269}$$

$$+ e^{-0.27x} \sin(\sqrt{\pi}x)(0.27^2/\sqrt{\pi} - \sqrt{\pi}) \tag{2.2.270}$$

$$y'' + 0.54y' = e^{-0.27x} \cos(\sqrt{\pi}x)(-0.54 + 0.54) \tag{2.2.271}$$

$$+ e^{-0.27x} \sin(\sqrt{\pi}x)(-0.27^2/\sqrt{\pi} - \sqrt{\pi}) \tag{2.2.272}$$

$$= -(\pi + 0.0729)y \tag{2.2.273}$$

30. Solving ODE, with IC $y(0) = 3.3$, $y'(0) = 10$,

$$9y'' - 30y' + 25y = 0 \tag{2.2.274}$$

$$a = -30/9 \qquad b = 25/9 \tag{2.2.275}$$

$$a^2 - 4b = \frac{30}{9}^2 - \frac{100}{9} = 0 \tag{2.2.276}$$

$$\lambda_1, \lambda_2 = \frac{30/9 \pm 0}{2} \qquad = 5/3 \tag{2.2.277}$$

$$y = (c_1 + c_2 x)e^{5x/3} \tag{2.2.278}$$

$$y(0) = c_1 = 3.3 \qquad y'(0) = c_2 + \frac{5}{3}c_1 = 10 \tag{2.2.279}$$

$$y = e^{5x/3}(3.3 + 4.5) \tag{2.2.280}$$

Checking by substitution,

$$y' = e^{5x/3}\,(4.5 + 5.5 + 7.5x) \tag{2.2.281}$$

$$y'' = e^{5x/3}\left(15 + \frac{55}{6} + \frac{25}{2}x\right) \tag{2.2.282}$$

$$9y'' - 30y' = 25e^{5x/3}\left(-\frac{165}{50} - \frac{225}{50}c_2 x\right) \tag{2.2.283}$$

$$= -25\,(3.3 + 4.5x)\,e^{5x/3} = -25y \tag{2.2.284}$$

31. Using Wronskian to check if functions are L.I.,

$$y_1 = e^{kx} \qquad y_2 = xe^{kx} \tag{2.2.285}$$

$$W(y_1, y_2) = y_1' y_2 - y_1 y_2' \tag{2.2.286}$$

$$= kxe^{2kx} - (kx + 1)e^{2kx} \qquad = -e^{2kx} \tag{2.2.287}$$

$$W \neq 0 \qquad \text{for some} \quad I \in \mathbb{R} \tag{2.2.288}$$

For $k \neq 0$, the functions are L.I.

32. Using Wronskian to check if functions are L.I.,

$$y_1 = e^{ax} \qquad y_2 = e^{-ax} \tag{2.2.289}$$

$$W(y_1, y_2) = y_1' y_2 - y_1 y_2' \tag{2.2.290}$$

$$= a + a \qquad = 2a \tag{2.2.291}$$

$$W \neq 0 \qquad \text{for some} \quad I \in \mathbb{R}^+ \tag{2.2.292}$$

For $a \neq 0$, the functions are L.I.

33. Using Wronskian to check if functions are L.I.,

$$y_1 = x^2 \qquad y_2 = x^2 \ln x \tag{2.2.293}$$

$$W(y_1, y_2) = y_1' y_2 - y_1 y_2' \tag{2.2.294}$$

$$= 2x^3 \ln x - x^3(1 + 2 \ln x) \qquad = -x^3 \tag{2.2.295}$$

$$W \neq 0 \qquad \text{for some} \quad I \in (1, \infty) \tag{2.2.296}$$

The functions are L.I.

34. Using Wronskian to check if functions are L.I.,

$$y_1 = \ln x \qquad y_2 = \ln x^3 \tag{2.2.297}$$

$$W(y_1, y_2) = y_1' y_2 - y_1 y_2' \tag{2.2.298}$$

$$= \frac{3 \ln x}{x} - \frac{3 \ln x}{x} \qquad = 0 \tag{2.2.299}$$

$$W \equiv 0 \qquad \text{for every} \quad I \in (1, \infty) \tag{2.2.300}$$

The functions are L.D.

35. Using Wronskian to check if functions are L.I.,

$$y_1 = \sin(2x) \qquad y_2 = \cos x \sin x \tag{2.2.301}$$

$$W(y_1, y_2) = y_1' y_2 - y_1 y_2' \tag{2.2.302}$$

$$= 2\cos(2x) \ \ \cos x \sin x - \sin(2x)(\cos^2 - \sin^2 x) \qquad = 0 \tag{2.2.303}$$

$$W \equiv 0 \qquad \text{for every} \quad I \in \mathbb{R}^- \tag{2.2.304}$$

The functions are L.D.

36. Using Wronskian to check if functions are L.I.,

$$y_1 = e^{-x} \cos(x/2) \qquad y_2 = 0 \tag{2.2.305}$$

$$W(y_1, y_2) = y_1' y_2 - y_1 y_2' \tag{2.2.306}$$

$$= 0 \tag{2.2.307}$$

$$W \equiv 0 \qquad \text{for every} \quad I \in [-1, 1] \tag{2.2.308}$$

The functions are L.D.

37. Solving ODE, with IC $y(0) = m$, $y'(0) = -n$,

$$y'' - y = 0 \tag{2.2.309}$$

$$a = 0 \qquad b = -1 \tag{2.2.310}$$

$$a^2 - 4b = 4 > 0 \tag{2.2.311}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm \sqrt{4}}{2} \qquad = \pm 1 \tag{2.2.312}$$

$$y = c_1 e^x + c_2 e^{-x} \tag{2.2.313}$$

$$y(0) = c_1 + c_2 = m \qquad y'(0) = c_1 - c_2 = -n \tag{2.2.314}$$

$$y = \frac{m-n}{2} e^x + \frac{m+n}{2} e^{-x} \tag{2.2.315}$$

Checking by substitution,

$$m = 1 \qquad n = 1 \tag{2.2.316}$$

$$y = e^{-x} \tag{2.2.317}$$

$$m = 1.001 \qquad n = 0.999 \tag{2.2.318}$$

$$y = 0.001 e^x + e^{-x} \tag{2.2.319}$$

The IC has to be balanced $m = n$, for the coefficient of e^x to be 0. Imbalance causes the e^x term to have a nonzero coefficient and the solution to grow exponentially.

38. **(a)** Expressing a, b in terms of λ_1, λ_2,

$$y'' + ay' + by = 0 \qquad \lambda^2 + a\lambda + b = 0 \tag{2.2.320}$$

$$a = -(\lambda_1 + \lambda_2) \qquad b = \lambda_1 \cdot \lambda_2 \tag{2.2.321}$$

Working backwards from the solutions to the three different cases for the roots of the characteristic equation, the above relations can be used to construct the ODE.

(b) By present method,

$$y'' + 4y' = 0 \tag{2.2.322}$$

$$a = 4 \qquad b = 0 \tag{2.2.323}$$

$$a^2 - 4b = 16 > 0 \tag{2.2.324}$$

$$\lambda_1, \lambda_2 = \frac{-4 \pm \sqrt{16}}{2} \qquad = 0, -4 \tag{2.2.325}$$

$$y = c_2 + c_1 e^{-4x} \tag{2.2.326}$$

By reduction to first order,

$$z = y' \qquad z' = y'' \tag{2.2.327}$$

$$z' + 4z = 0 \tag{2.2.328}$$

$$\ln z = -4x + c_1^* \qquad z = c_1 e^{-4x} \tag{2.2.329}$$

$$y = c_1 e^{-4x} + c_2 \tag{2.2.330}$$

Both methods have to match, because the ODE satisfies the conditions for a unique solution. For the general case with some $a \neq 0$ and $b = 0$, By present method,

$$y'' + ay' = 0 \tag{2.2.331}$$

$$b = 0 \tag{2.2.332}$$

$$a^2 - 4b = a^2 > 0 \tag{2.2.333}$$

$$\lambda_1, \lambda_2 = \frac{-a \pm \sqrt{a^2}}{2} \qquad = 0, -a \tag{2.2.334}$$

$$y = c_2 + c_1 e^{-ax} \tag{2.2.335}$$

By reduction to first order,

$$z = y' \qquad z' = y'' \tag{2.2.336}$$

$$z' + az = 0 \tag{2.2.337}$$

$$\ln z = -ax + c_1^* \qquad z = c_1 e^{-ax} \tag{2.2.338}$$

$$y = c_1 e^{-ax} + c_2 \tag{2.2.339}$$

(c) $xe^{\lambda x}$ to be verified against the ODE with repeated roots,

$$y'' + ay' + by = 0 \tag{2.2.340}$$

$$y' = e^{\lambda x}(\lambda x + 1) \tag{2.2.341}$$

$$y'' = e^{\lambda x}(\lambda^2 x + 2\lambda) \tag{2.2.342}$$

$$y'' + ay' + by = e^{\lambda x}\left[(\lambda^2 + a\lambda + b)x + 2\lambda + a\right] \tag{2.2.343}$$

$$b = a^2/4 \qquad \lambda = -a/2 \tag{2.2.344}$$

$$(\lambda^2 + a\lambda + b) = 0 \qquad y'' + ay' + by = 0 \tag{2.2.345}$$

The given ODE is,

$$y'' - y' - 6y = 0 \tag{2.2.346}$$

$$a = -1 \qquad b = -6 \tag{2.2.347}$$

$$a^2 - 4b = 25 > 0 \tag{2.2.348}$$

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{25}}{2} \qquad = 3, -2 \tag{2.2.349}$$

$$y = c_2 e^{3x} + c_1 e^{-2x} \tag{2.2.350}$$

Substituting $y = xe^{\lambda_1 x}$ into the ODE, shows it is not a solution.

$$y' = e^{\lambda_1 x}(1 + \lambda_1 x) \tag{2.2.351}$$

$$y'' = e^{\lambda_1 x}(2\lambda_1 + \lambda_1^2 x) \tag{2.2.352}$$

$$y'' - y' - 6y = e^{\lambda_1 x}\left[2\lambda_1 - 1 + x(\lambda_1^2 - \lambda_1 - 6)\right] \tag{2.2.353}$$

$$= e^{\lambda_1 x}(2\lambda_1 - 1) \neq 0 \tag{2.2.354}$$

Explanation. TBC

(d) Let $\delta \to 0$ with,

$$\lambda_2 = \lambda_1 + \delta \tag{2.2.355}$$

$$e^{\lambda_2 x} = e^{\delta x}\, e^{\lambda_1 x} \tag{2.2.356}$$

$$= (1 + \delta x)\, e^{\lambda_1 x} \tag{2.2.357}$$

$$y_2 = xe^{\lambda_1 x} \tag{2.2.358}$$

Getting rid of the additive component and scalar δ, leaves the second solution $xe^{\lambda_1 x}$. The series approximation only holds for $\delta \to 0$.

2.3 Differential Operators

1. Applying the operator to the given functions,

$$L = D^2 + 2D \tag{2.3.1}$$

$$y_1 = \cosh(2x) \qquad Ly_1 = 4\sinh(2x) + 4\sinh(2x) \tag{2.3.2}$$

$$y_2 = e^{-x} + e^{2x} \qquad Ly_2 = -e^{-x} + 8e^{2x} \tag{2.3.3}$$

$$y_3 = \cos x \qquad Ly_3 = -\cos x - 2\sin x \tag{2.3.4}$$

2. Applying the operator to the given functions,

$$L = D - 3I \tag{2.3.5}$$

$$y_1 = 3x^2 + 3x \qquad Ly_1 = -3x + 3 - 9x^2 \tag{2.3.6}$$

$$y_2 = 3e^{3x} \qquad Ly_2 = e^{3x}(9 - 9) = 0 \tag{2.3.7}$$

$$y_3 = \cos(4x) - \sin(4x) \qquad Ly_3 = -7\cos(4x) - \sin(4x) \tag{2.3.8}$$

3. Applying the operator to the given functions,

$$L = (D - 2I)^2 \qquad = D^2 - 4D + 4I \tag{2.3.9}$$

$$y_1 = e^{2x} \qquad Ly_1 = e^{2x}(4 - 8 + 4) = 0 \tag{2.3.10}$$

$$y_2 = xe^{2x} \qquad Ly_2 = e^{2x}(4 + 4x - 4 - 8x + 4x) = 0 \tag{2.3.11}$$

$$y_3 = e^{-2x} \qquad Ly_3 = e^{-2x}(4 + 8 + 4) = 16e^{-2x} \tag{2.3.12}$$

4. Applying the operator to the given functions,

$$L = (D + 6I)^2 \qquad = D^2 + 12D + 36I \tag{2.3.13}$$

$$y_1 = 6x + \sin(6x) \qquad Ly_1 = 72 + 216x + 72\cos(6x) \tag{2.3.14}$$

$$y_2 = xe^{-6x} \qquad Ly_2 = e^{-6x}(36x - 12 + 12 - 72x + 36x) = 0 \tag{2.3.15}$$

5. Applying the operator to the given functions,

$$L = (D - 2I)(D + 3I) \qquad = D^2 + D - 6I \tag{2.3.16}$$

$$y_1 = e^{2x} \qquad Ly_1 = e^{2x}(4 + 2 - 6) = 0 \tag{2.3.17}$$

$$y_2 = xe^{2x} \qquad Ly_2 = e^{2x}(4 + 4x + 1 + 2x - 6x) = 5e^{2x} \tag{2.3.18}$$

$$y_3 = e^{-3x} \qquad Ly_3 = e^{-3x}(9 - 3 - 6) = 0 \tag{2.3.19}$$

6. Factorising $P(D)$ and solving,

$$P(D)y = 0 \qquad (D^2 + 4D + 3.36I)y = 0 \tag{2.3.20}$$

$$0 = \left(D + \frac{6}{5}I\right)\left(D + \frac{14}{5}I\right) y \tag{2.3.21}$$

$$\lambda_1 = -6/5 \qquad y_1 = e^{-1.2x} \tag{2.3.22}$$

$$\lambda_2 = -14/5 \qquad y_2 = e^{-2.8x} \tag{2.3.23}$$

$$y = c_1 e^{-1.2x} + c_2 e^{-2.8x} \tag{2.3.24}$$

7. Factorising $P(D)$ and solving,

$$P(D)y = 0 \qquad (4D^2 - I)y = 0 \tag{2.3.25}$$

$$(2D + I)\,(2D - I)\,y = 0 \tag{2.3.26}$$

$$\lambda_1 = -1/2 \qquad y_1 = e^{-0.5x} \tag{2.3.27}$$

$$\lambda_2 = 1/2 \qquad y_2 = e^{0.5x} \tag{2.3.28}$$

$$y = c_1 e^{-0.5x} + c_2 e^{0.5x} \tag{2.3.29}$$

8. Factorising $P(D)$ and solving,

$$P(D)y = 0 \qquad (D^2 + 3I)y = 0 \tag{2.3.30}$$

$$(D + i\sqrt{3}I)(D - i\sqrt{3}I)y = 0 \tag{2.3.31}$$

$$\lambda_1 = -i\sqrt{3} \qquad y_1 = e^{-i\sqrt{3}x} \tag{2.3.32}$$

$$\lambda_2 = i\sqrt{3} \qquad y_2 = e^{i\sqrt{3}x} \tag{2.3.33}$$

$$y = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) \tag{2.3.34}$$

9. Factorising $P(D)$ and solving,

$$P(D)y = 0 \qquad (D^2 - 4.2D + 4.41I)y = 0 \tag{2.3.35}$$

$$(D - 2.1I)^2 y = 0 \tag{2.3.36}$$

$$\lambda_1 = 2.1 \qquad y_1 = e^{2.1x} \tag{2.3.37}$$

$$\lambda_2 = 2.1 \qquad y_2 = xe^{2.1x} \tag{2.3.38}$$

$$y = (c_1 + c_2 x)e^{2.1x} \tag{2.3.39}$$

10. Factorising $P(D)$ and solving,

$$P(D)y = 0 \qquad (D^2 + 4.8D + 5.76I)y = 0 \tag{2.3.40}$$

$$\left(D + \frac{12}{5}I\right)^2 y = 0 \tag{2.3.41}$$

$$\lambda_1 = -2.4 \qquad y_1 = e^{-2.4x} \tag{2.3.42}$$

$$\lambda_2 = -2.4 \qquad y_2 = xe^{-2.4x} \tag{2.3.43}$$

$$y = (c_1 + c_2 x)e^{-2.4x} \tag{2.3.44}$$

11. Factorising $P(D)$ and solving,

$$P(D)y = 0 \qquad (D^2 - 4D + 3.84I)y = 0 \tag{2.3.45}$$

$$\left(D - \frac{12}{5}I\right)\left(D - \frac{8}{5}I\right) y = 0 \tag{2.3.46}$$

$$\lambda_1 = 2.4 \qquad y_1 = e^{2.4x} \tag{2.3.47}$$

$$\lambda_2 = 1.6 \qquad y_2 = e^{1.6x} \tag{2.3.48}$$

$$y = c_1 e^{1.6x} + c_2 e^{2.4x} \tag{2.3.49}$$

12. Factorising $P(D)$ and solving,

$$P(D)y = 0 \qquad 0 = (D^2 + 3D + 2.5I)y \tag{2.3.50}$$

$$0 = \left(D + \frac{3}{2} - \frac{i}{2}\right)\left(D + \frac{3}{2} + \frac{i}{2}\right) \tag{2.3.51}$$

$$\lambda_1 = -\frac{3}{2} + \frac{i}{2} \qquad y_1 = e^{-1.5x}e^{i\ 0.5x} \tag{2.3.52}$$

$$\lambda_2 = -\frac{3}{2} - \frac{i}{2} \qquad y_2 = e^{-1.5x}e^{-i\ 0.5x} \tag{2.3.53}$$

$$y = [c_1 \cos(x/2) + c_2 \sin(x/2)]e^{-1.5x} \tag{2.3.54}$$

13. To illustrate the linearity of L,

$$c = 4 \qquad k = -6 \tag{2.3.55}$$

$$y = e^{2x} \qquad w = \cos(2x) \tag{2.3.56}$$

$$\mathcal{L}y = P(D)y \qquad = (D^2 + aD + bI)y \tag{2.3.57}$$

$$L(cy + kw) = L[4e^{2x} - 6\cos(2x)] \tag{2.3.58}$$

$$\begin{aligned} &= 16e^{2x} + 24\cos(2x) \\ &+ 8ae^{2x} + 12a\sin(2x) \\ &+ 4be^{2x} - 6b\cos(2x) \end{aligned} \tag{2.3.59}$$

$$= c\ \mathcal{L}y + k\ \mathcal{L}w \tag{2.3.60}$$

For the general case, using the linearity of the D^n and I operators,

$$L(cy + kw) = D^2(cy + kw) + aD(cy + kw) + bI(cy + kw) \tag{2.3.61}$$

$$= (D^2 + aD + bI)\ cy + (D^2 + aD + bI)\ kw \tag{2.3.62}$$

$$= L(cy) + L(kw) \tag{2.3.63}$$

$$= c\ \mathcal{L}y + k\ \mathcal{L}w \tag{2.3.64}$$

14. Testing the given solution, if the ODE has roots μ and λ,

$$0 = D^2 + aD + bI \tag{2.3.65}$$

$$0 = (D - \mu I)(D - \lambda I) \tag{2.3.66}$$

$$y = \frac{e^{\mu x} - e^{\lambda x}}{\mu - \lambda} \tag{2.3.67}$$

$$0 = \frac{\mu^2 e^{\mu x} - \lambda^2 e^{\lambda x}}{\mu - \lambda} + a\left(\frac{\mu e^{\mu x} - \lambda e^{\lambda x}}{\mu - \lambda}\right) + b\left(\frac{e^{\mu x} - e^{\lambda x}}{\mu - \lambda}\right) \tag{2.3.68}$$

$$0 = \frac{e^{\mu x}(\mu^2 + a\mu + b)}{\mu - \lambda} + \frac{e^{\lambda x}(\lambda^2 + a\lambda + b)}{\mu - \lambda} \tag{2.3.69}$$

$$= 0 + 0 \tag{2.3.70}$$

Proving the relation for repeated roots, (here the variable is $a = \mu - \lambda$),

$$\lim_{\mu \to \lambda} \frac{e^{\mu x} - e^{\lambda x}}{\mu - \lambda} = \lim_{a \to 0} \frac{e^{\lambda x}(e^{ax} - 1)}{a} \tag{2.3.71}$$

$$= \lim_{a \to 0} \frac{\mathrm{d}}{\mathrm{d}a}(e^{\lambda x + ax}) \tag{2.3.72}$$

$$= x\ e^{\lambda x + ax} \tag{2.3.73}$$

$$= x e^{\mu x} \tag{2.3.74}$$

Thus, the second root is $xe^{\mu x}$, using L'Hopital rule.

15. To prove linearity of L,

$$L = P(D) = D^2 + aD + bI \tag{2.3.75}$$

$$L(cy + kw) = D^2(cy + kw) + aD(cy + kw) + bI(cy + kw) \tag{2.3.76}$$

$$= D^2 + aD + bI(cy) + v\ (kw) \tag{2.3.77}$$

$$= c\ (D^2 + aD + bI)y + k\ (D^2 + aD + bI)w \tag{2.3.78}$$

$$= c\ \mathcal{L}y + k\ \mathcal{L}w \tag{2.3.79}$$

Since the differential operators D^n are linear operators, their composition L is also linear.

2.4 Modeling of Free Oscillations of a Mass-Spring System

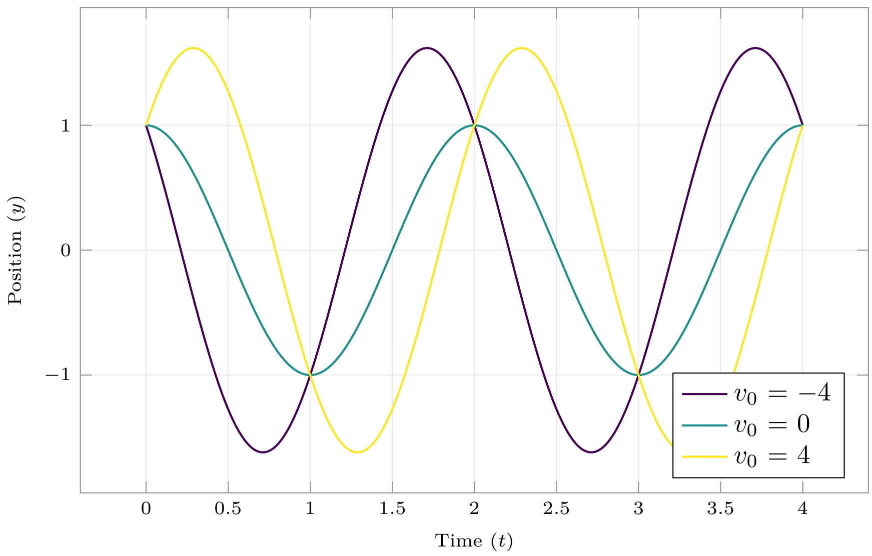
1. Initial velocity v_0 and position y_0, gives

$$y = A \cos \omega_0 t + B \sin \omega_0 t \tag{2.4.1}$$

$$y(0) = A = y_0 \tag{2.4.2}$$

$$y'(0) = B\omega_0 = v_0 \tag{2.4.3}$$

Graphing solutions for $\omega_0 = \pi$, $y_0 = 1$, with the different time series intersecting at $t = n$



2. Given $m = 20$ N and $y_0 = 2$ cm,

$$mg = ky_0 \qquad \text{Hooke's law} \tag{2.4.4}$$

$$k = \frac{20}{0.02} = 1000 \text{ N m}^{-1} \tag{2.4.5}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{500} \tag{2.4.6}$$

$$f_0 = \frac{\omega_0}{2\pi} \qquad = 3.56 \text{ Hz} \tag{2.4.7}$$

$$T = \frac{1}{f_0} \qquad = 0.281 \text{ s} \tag{2.4.8}$$

3. Doubling the mass increases inertia, which decreases natural frequency

$$f_1 = 2\pi\sqrt{\frac{k}{2m}} = \frac{1}{\sqrt{2}}f_0 \tag{2.4.9}$$

Doubling the spring modulus increases natural frequency

$$f_2 = 2\pi\sqrt{\frac{2k}{m}} = \sqrt{2}f_0 \tag{2.4.10}$$

4. No, because the frequency f_0 does not depend on the initial velocity v_0.
The initial velocity only increases the maximum amplitude of the oscillations , not the time period of the motion.

5. Springs in parallel, with $m = 5$, $k_1 = 20$, $k_2 = 45$,

$$mg = k_1 y \qquad \omega_1 = \sqrt{\frac{k_1}{m}} = 2 \text{ N m}^{-1} \tag{2.4.11}$$

$$mg = k_2 y \qquad \omega_2 = \sqrt{\frac{k_2}{m}} = 3 \text{ N m}^{-1} \tag{2.4.12}$$

$$mg = (k_1 + k_2)y = k_3 y \qquad \omega_2 = \sqrt{\frac{k_3}{m}} = 3.61 \text{ N m}^{-1} \tag{2.4.13}$$

Springs in parallel have an effective spring constant given by,

$$k_{\text{par}} = \sum_{i=1}^{n} k_i \tag{2.4.14}$$

6. Springs in series, with $k_1 = 8$, $k_2 = 12$. Consider a point in between the two springs. At static equilibrium,

$$k_1 y_1 = k_2 y_2 \tag{2.4.15}$$

When replacing the springs by an effective spring with constant k_3, and looking at a point between the mass and the bottom-most spring at static equilibrium,

$$k_1 y_1 = k_2 y_2 = k_3 y_3 = mg \tag{2.4.16}$$

$$y_3 = y_1 + y_2 \tag{2.4.17}$$

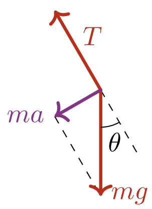
$$\frac{1}{k_3} = \frac{1}{k_1} + \frac{1}{k_2} \tag{2.4.18}$$

The above relation follows from the fact that the total extension of the system of springs in series is equal to the sum of the extension in each individual spring. Springs in parallel have an effective spring constant given by,

$$k_3 = \frac{k_1 k_2}{(k_1 + k_2)} \tag{2.4.19}$$

$$= 4.8 \text{ N m}^{-1} \tag{2.4.20}$$

7. For small angle θ, the relation $\sin\theta \approx \theta$,



For small displacement x,

$$mg\cos\theta = T \qquad \text{along string} \tag{2.4.21}$$

$$mg\sin\theta = ma \qquad \text{along tangent} \tag{2.4.22}$$

$$mg\sin\theta \approxeq mg\theta = ma \qquad a = g\theta = \frac{gx}{l} \tag{2.4.23}$$

$$\omega_0 = \sqrt{\frac{g}{l}} \qquad f_0 = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \tag{2.4.24}$$

8. Let the equilibrium depth of cylinder bottom be y_0, with density of water ρ. Now the entire mass of water is acted upon by the gravitational force on the water above the equal line.

$$(\rho V)g = mg = \rho\,(\pi R^2 y_0)g \tag{2.4.25}$$

When a small downward displacement y happens, the restoring force is,

$$ma = -mg + (\rho\pi g R^2)\,(y_0 + y) \tag{2.4.26}$$

$$= \rho\,(\pi R^2 g y) \tag{2.4.27}$$

$$\omega = \sqrt{\frac{\rho\pi R^2 g}{m}} \qquad T = \frac{2\pi}{\omega} = \sqrt{\frac{4\pi m}{\rho R^2 g}} \tag{2.4.28}$$

$$m = \frac{\rho R^2 g}{4\pi}\,T^2 \qquad = \frac{9.8\cdot 997\cdot 0.09}{\pi} \tag{2.4.29}$$

$$= 279.9\text{ kg} \tag{2.4.30}$$

9. $1L$ of water vibrating in a U-shaped tube, displaced by y upwards,

$$ma = \rho(\pi R^2)(2y)g \qquad \omega^2 = \frac{2\rho g\pi R^2}{m} \tag{2.4.31}$$

$$f = \frac{\omega}{2\pi} = \sqrt{\frac{\rho g R^2}{2\pi(\rho V)}} \qquad = \sqrt{\frac{9.8\cdot(0.01)^2}{2\pi(0.001)}} \tag{2.4.32}$$

$$= 0.395\text{ Hz} \tag{2.4.33}$$

10. **(a)** A pendulum's simple harmonic oscillation is governed by the relation,

$$ma = mg \sin\theta \approxeq mg\theta \quad 2.4.34$$

$$= mg\frac{x}{l} \quad 2.4.35$$

$$\omega_2 = \frac{g}{l} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \quad 2.4.36$$

$$T = 2 \text{ s} \quad 2.4.37$$

In 1 minute, the pendulum has completed 30 ticks.

(b) Given $y_0 = 0.01,\ m = 8,\ v_0 = 0.1,$

$$mg = ky_0 \qquad \text{static equilibrium} \quad 2.4.38$$

$$ma = -mg + k(y + y_0) = ky \quad 2.4.39$$

$$a = \frac{k}{m}y = \frac{gy}{y_0} \qquad \omega = \sqrt{\frac{g}{y_0}} = 31.305 \text{ Hz} \quad 2.4.40$$

$$y = y_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t \quad 2.4.41$$

2.4.42

(c) Torsional spring, with $\theta'(0) = v_0 =$ and $\theta(0) = \theta_0$,

$$\theta'' = \frac{-K}{I_0}\theta \qquad \omega^2 = \frac{K}{I_0} \quad 2.4.43$$

$$\theta = \theta_0 \cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) \quad 2.4.44$$

$$= [0.5325]\cos(3.7t) + [0.0943]\sin(3.7t) \quad 2.4.45$$

11. Consider the standard solution for a damped oscillation,

$$y(t) = c_1 \exp\left[-(\alpha - \beta)t\right] + c_2 \exp\left[-(\alpha + \beta)t\right] \tag{2.4.46}$$

$$y'(0) = -c_1(\alpha - \beta) - c_2(\alpha + \beta) = v_0 \tag{2.4.47}$$

$$y(0) = c_1 + c_2 = y_0 \tag{2.4.48}$$

$$v_0 = -c_1(\alpha - \beta) - (y_0 - c_1)(\alpha + \beta) \tag{2.4.49}$$

$$c_1 = \frac{v_0 + y_0(\alpha + \beta)}{2\beta} \tag{2.4.50}$$

$$c_2 = -\frac{v_0 + y_0(\alpha - \beta)}{2\beta} \tag{2.4.51}$$

12. Checking the zero crossings of overdamped motion, $\alpha > \beta > 0$,

$$y(t) = c_1 \exp\left[-(\alpha - \beta)t\right] + c_2 \exp\left[-(\alpha + \beta)t\right] \tag{2.4.52}$$

$$y(t^*) = 0 \implies \frac{-c_1}{c_2} = \exp(-2\beta t^*) \tag{2.4.53}$$

$$\ln\left(\frac{-c_1}{c_2}\right) = -2\beta t^* \tag{2.4.54}$$

$$t^* = \frac{-1}{2\beta} \ln\left(\frac{-c_1}{c_2}\right) \tag{2.4.55}$$

$$t^* > 0 \implies \frac{-c_1}{c_2} \in (0, 1) \tag{2.4.56}$$

$t^* > 0$, which means that the body crosses mean position at some real time, has at most one solution given by $c_1 c_2 < 0$ and $|c_2| > |c_1|$.

13. For critically damped motion,

$$y(t) = (c_1 + c_2 t)e^{-\alpha t} \qquad y'(t) = (c_2 - \alpha c_1 - \alpha c_2 t)e^{-\alpha t} \tag{2.4.57}$$

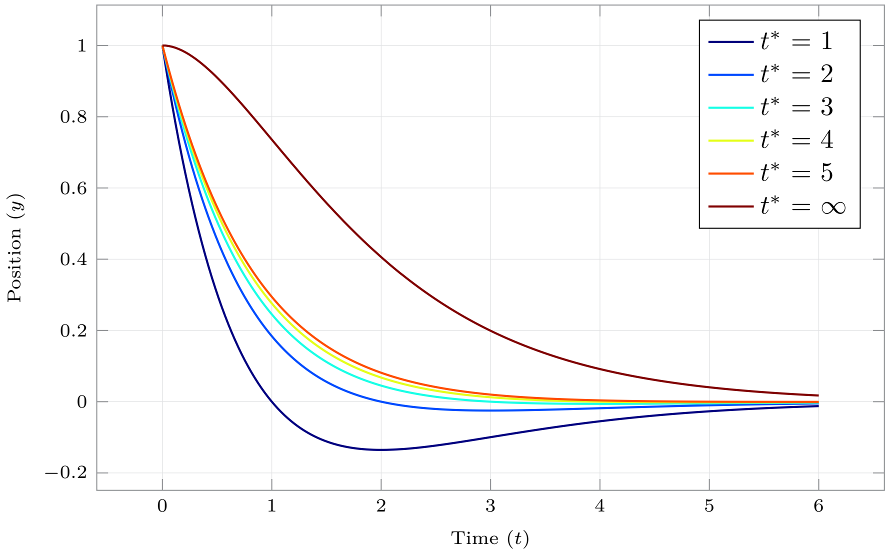
$$y_0 = c_1 \qquad v_0 = c_2 - \alpha c_1 \tag{2.4.58}$$

$$c_2 = v_0 + \alpha y_0 \tag{2.4.59}$$

Plotting graphs for $\alpha = 1,\ y_0 = 1$,

$$y = [1 + (1 + v_0)t]e^{-t} = 0 \tag{2.4.60}$$

$$t = \frac{-1}{1 + v_0} \tag{2.4.61}$$



14. To avoid oscillations,

$$c^2 - 4mk \geq 0 \tag{2.4.62}$$

$$c \geq 2\sqrt{mk} = 2\sqrt{2000 \cdot 4500} \tag{2.4.63}$$

$$c \geq 6000 \text{ kg s}^{-1} \tag{2.4.64}$$

15. Applying binomial approximation,

$$\omega^* = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \omega\sqrt{1 - \frac{c^2}{4mk}} \tag{2.4.65}$$

$$(1-x)^n = \left[1 - nx + \frac{n(n-1)}{2}x^2\right] \tag{2.4.66}$$

$$(1-x)^{1/2} = \left[1 - \frac{x}{2} - \frac{x^2}{8}\right] \tag{2.4.67}$$

$$x = \frac{c^2}{4mk} = \frac{100}{4 \cdot 10 \cdot 90} = 1/36 \tag{2.4.68}$$

$$\omega^* = 0.986 \cdot \omega = 0.986 \cdot \sqrt{\frac{90}{10}} = 2.958\,043\,98 \text{ Hz} \tag{2.4.69}$$

$$\omega_{\text{true}} = 2.958\,039\,89 \text{ Hz} \tag{2.4.70}$$

The binomial approximation using 2 terms provides an answer with an error of 0.00013%.

16. For the maxima of underdamped motion, with $\alpha > 0$ and $\omega > 0$,

$$y(t) = [A \cos(\omega t) + B \sin(\omega t)]e^{-\alpha t} \tag{2.4.71}$$

$$y'(t) = e^{-\alpha t}[(B\omega - A\alpha) \cos(\omega t) - (A\omega + B\alpha) \sin(\omega t)] \tag{2.4.72}$$

$$y'(t) = 0 \implies \tan(\omega t) = \frac{B\omega - A\alpha}{A\omega + B\alpha} \tag{2.4.73}$$

The times at which extrema are reached are equally spaced, as seen by the expression for $y'(t) = 0$ involving only constants. The period of oscillations is,

$$\Delta t = \frac{2\pi}{\omega} \qquad \omega = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \tag{2.4.74}$$

17. To find the maxima of the function,

$$y(t) = e^{-t} \sin t \tag{2.4.75}$$

$$y'(t) = e^{-t}[\cos t - \sin t] \tag{2.4.76}$$

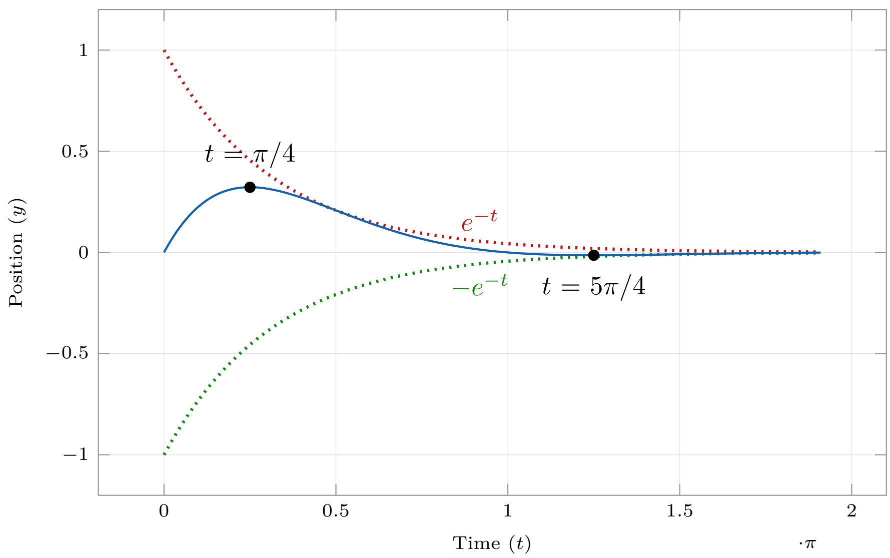
$$y''(t) = e^{-t}[-2 \cos t] \tag{2.4.77}$$

$$y'(t) = 0 \implies \tan t = 1 \tag{2.4.78}$$

$$y''(t) < 0 \implies \cos t > 0 \tag{2.4.79}$$

$$\implies t \in (2n\pi - \pi/2, 2n\pi + \pi/2) \tag{2.4.80}$$

Maxima occurs at $t_{\max} = 2n\pi + \arctan(1)$. The minima occur at $t_{\min} = (2n - 1)\pi + \arctan(1)$ correspondingly.



18. For underdamped oscillations at successive maxima,

$$y_1 = Ce^{-\alpha t_1} \cos(\omega t_1 + \delta) \tag{2.4.81}$$

$$y_2 = Ce^{-\alpha t_2} \cos(\omega t_2 + \delta) \tag{2.4.82}$$

$$\frac{y_2}{y_1} = \exp[-\alpha(t_2 - t_1)] \qquad t_2 - t_1 = \frac{2\pi}{\omega} \tag{2.4.83}$$

$$\ln\left(\frac{y_2}{y_1}\right) = \frac{2\pi\alpha}{\omega} \tag{2.4.84}$$

$$y'' + 2y' + 5y = 0 \tag{2.4.85}$$

$$\lambda_1, \lambda_2 = -1 \pm 2i \qquad y = Ce^{-t}\cos(2t + \delta) \tag{2.4.86}$$

$$\Delta = \frac{2\pi \cdot (1)}{2} \qquad = \pi \tag{2.4.87}$$

19. Time between consecutive maxima is 3 s, mass $m = 0.5$ kg,

$$\frac{y(30)}{y(0)} = \exp\left[-\alpha(t_2 - t_1)\right] \qquad = \exp(-30\alpha) = 0.5 \tag{2.4.88}$$

$$\alpha = \frac{\ln 2}{30} \tag{2.4.89}$$

$$T = \frac{2\pi}{\omega} = 3 \text{ s} \qquad \omega = \frac{2\pi}{3} \tag{2.4.90}$$

$$\alpha = \frac{c}{2m} \qquad c = 0.0231 \text{ kg s}^{-1} \tag{2.4.91}$$

20. **(a)** Solving the general ODE with $y(0) = 1,\ y'(0) = 0$,

$$y'' + cy' + y = 0 \tag{2.4.92}$$

$$\lambda_1, \lambda_2 = \frac{-c}{2} \pm \frac{\sqrt{c^2 - 4}}{2} = -\alpha \pm \beta \tag{2.4.93}$$

$$y = \begin{cases} c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t} & c > 2 \\ (c_1 + c_2 t)e^{-\alpha t} & c = 2 \\ e^{-\alpha t}[c_1 \cos(-i\beta t) + c_2 \sin(-i\beta t)] & c < 2 \end{cases} \tag{2.4.94}$$

Applying the IC to each case individually, overdamped motion gives,

$$c_1 + c_2 = 1 \qquad -c_1(\alpha - \beta) - c_2(\alpha + \beta) = 0 \tag{2.4.95}$$

$$c_1 = \frac{\alpha + \beta}{2\beta} \qquad c_2 = \frac{\beta - \alpha}{2\beta} \tag{2.4.96}$$

critically damped motion gives,

$$c_1 = 1 \qquad -c_1\alpha + c_2 = 0 \tag{2.4.97}$$

$$c_2 = \alpha \tag{2.4.98}$$

underdamped motion gives,

$$c_1 = 1 \qquad -\alpha c_1 - i\beta c_2 = 0 \tag{2.4.99}$$

$$c_2 = \frac{i\alpha}{\beta} \tag{2.4.100}$$

For the alternative IC, $y(0) = 1,\ \ y'(0) = -2$,
Applying the IC to each case individually, overdamped motion gives,

$$c_1 + c_2 = 1 \qquad -c_1(\alpha - \beta) - c_2(\alpha + \beta) = -2 \tag{2.4.101}$$

$$c_1 = \frac{\alpha + \beta - 2}{2\beta} \qquad c_2 = \frac{\beta - \alpha + 2}{2\beta} \tag{2.4.102}$$

critically damped motion gives,

$$c_1 = 1 \qquad -c_1\alpha + c_2 = -2 \tag{2.4.103}$$

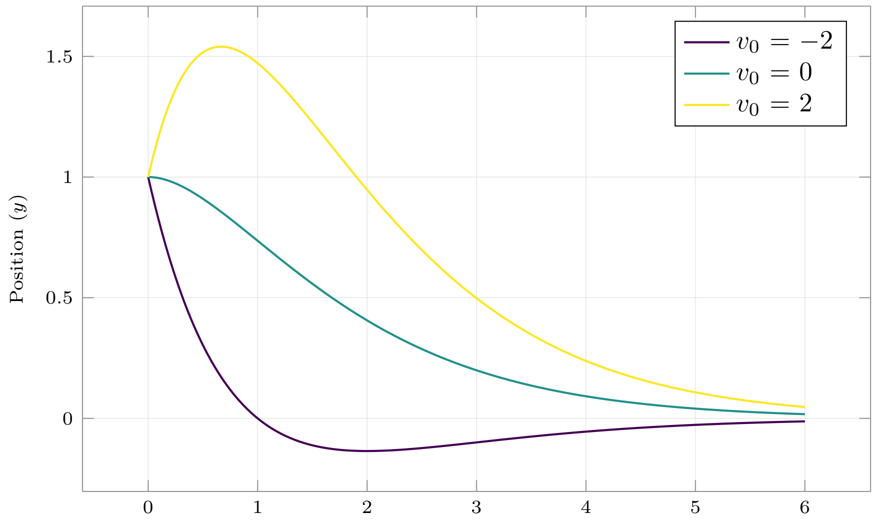
$$c_2 = \alpha - 2 \tag{2.4.104}$$

underdamped motion gives,

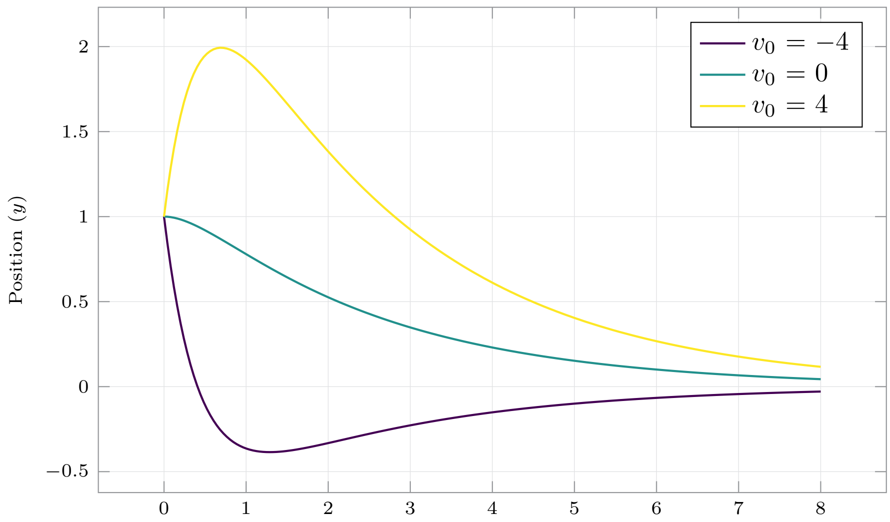
$$c_1 = 1 \qquad -\alpha c_1 - i\beta c_2 = -2 \tag{2.4.105}$$

$$c_2 = \frac{i(\alpha - 2)}{\beta} \tag{2.4.106}$$

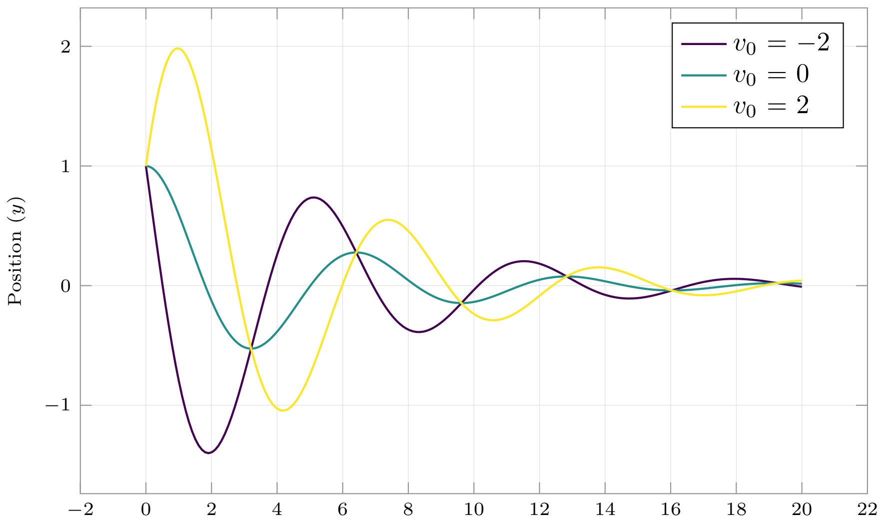
Critically damped



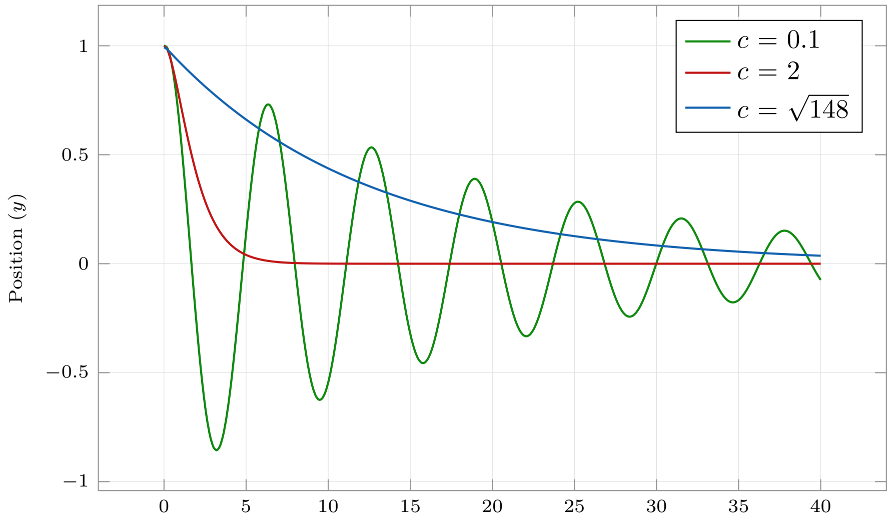
Overdamped



Underdamped



(b) To show the transition of damping levels,



(c) Amplitude decaying to 0.1% of its initial value at t^*,
For critically damped motion,

$$[y_0 + ((c/2)y_0 + v_0)t]e^{-ct/2} = 0.001 \cdot y_0 \tag{2.4.107}$$

This cannot yield an explicit relation, but provides an implicit relation between c and t^*. Similar implicit relations can be established for underdamped and overdamped motion.

(d) Solving analytically, TBC

$$0 = y'' + cy' + y \tag{2.4.108}$$

$$\alpha = c/2 \qquad \beta = \frac{\sqrt{c^2 - 4}}{2} \tag{2.4.109}$$

$$y = \begin{cases} \left(\frac{\beta + \alpha}{2\beta}\right) e^{-(\alpha-\beta)t} + \left(\frac{\beta - \alpha}{2\beta}\right) e^{-(\alpha+\beta)t} & c > 2 \\ (1 + \alpha t)e^{-\alpha t} & c = 2 \\ e^{-\alpha t} \left[\cos(-i\beta t) + \left(\frac{i\alpha}{\beta}\right) \sin(-i\beta t)\right] & c < 2 \end{cases} \tag{2.4.110}$$

(e) The difference is that in B, the system has a zero crossing and in A it does not.

2.5 Euler-Cauchy Equations

1. Auxiliary equation has a double root,

$$(1-a)^2 = 4b \tag{2.5.1}$$

$$y_2 = x^{(1-a)/2} \ln x \tag{2.5.2}$$

$$0 = x^2y'' + axy' + \left[\frac{(1-a)^2}{4}\right] y \tag{2.5.3}$$

$$y_2' = x^{(-1-a)/2} \left[1 + \left(\frac{1-a}{2}\right) \ln x\right] \tag{2.5.4}$$

$$y_2'' = x^{(-3-a)/2} \left[-a + \ln x \left(\frac{a^2-1}{4}\right)\right] \tag{2.5.5}$$

$$x^2y'' + axy' = x^{(1-a)/2} \left[\ln x \left(\frac{-a^2-1+2a}{4}\right)\right] \tag{2.5.6}$$

$$= x^{(1-a)/2} \; \ln x \left[\frac{-(1-a)^2}{4}\right] \tag{2.5.7}$$

$$= \left[\frac{-(1-a)^2}{4}\right] y \tag{2.5.8}$$

Now, if the roots are distinct,

$$(1-a)^2 > 4b \tag{2.5.9}$$

$$y = x^{m_1} \ln x \tag{2.5.10}$$

$$y' = x^{m_1-1}(1 + m_1 \ln x) \tag{2.5.11}$$

$$y'' = x^{m_1-2}(2m_1 - 1 + m_1(m_1-1) \ln x) \tag{2.5.12}$$

$$x^2y'' + axy' + by = x^{m_1}\Big[2m_1 - 1 + a \tag{2.5.13}$$

$$+ \{m_1^2 + (a-1)m_1 + b\} \ln x\Big] \tag{2.5.14}$$

$$= 2x^{m_1} \left[m_1 - \left(\frac{1-a}{2}\right)\right] \tag{2.5.15}$$

Thus, the fact that the roots are distinct makes this not satisfy the ODE. If instead, the roots were repeated, $m = (1-a)/2$ holds and the ODE is satisfied.

2. Finding the general solution

$$x^2y'' - 20y = 0 \qquad a = 0, \quad b = -20 \tag{2.5.16}$$

$$(a-1)^2 - 4b = 81 > 0 \qquad m_1, m_2 = -4, 5 \tag{2.5.17}$$

$$y = c_1x^{-4} + c_2x^5 \tag{2.5.18}$$

3. Finding the general solution

$$5x^2y'' + 23xy' + 16.2y = 0 \qquad a = 4.6, \quad b = 3.24 \tag{2.5.19}$$

$$(a-1)^2 - 4b = 0 \qquad m_1, m_2 = -1.8 \tag{2.5.20}$$

$$y = (c_1 + c_2 \ln x)x^{-1.8} \tag{2.5.21}$$

4. Finding the general solution

$$x^2y'' + 2xy' = 0 \qquad a = 2, \quad b = 0 \tag{2.5.22}$$

$$(a-1)^2 - 4b = 1 > 0 \qquad m_1, m_2 = -1, 0 \tag{2.5.23}$$

$$y = c_1 + c_2x^{-1} \tag{2.5.24}$$

5. Finding the general solution

$$4x^2y'' + 5y = 0 \qquad a = 0, \quad b = 5/4 \tag{2.5.25}$$

$$(a-1)^2 - 4b = -4 < 0 \qquad m_1, m_2 = 0.5 \pm i \tag{2.5.26}$$

$$y = e^{0.5x}[A\cos(\ln x) + B\sin(\ln x)] \tag{2.5.27}$$

6. Finding the general solution

$$x^2y'' + 0.7xy' - 0.1y = 0 \qquad a = 0.7, \quad b = -0.1 \tag{2.5.28}$$

$$(a-1)^2 - 4b = 0.49 > 0 \qquad m_1, m_2 = 0.5, -0.2 \tag{2.5.29}$$

$$y = c_1x^{0.5} + c_2x^{-0.2} \tag{2.5.30}$$

7. Finding the general solution

$$(x^2D^2 - 4xD + 6I)y = 0 \qquad a = -4, \quad b = 6 \tag{2.5.31}$$

$$(a-1)^2 - 4b = 1 > 0 \qquad m_1, m_2 = 3, 2 \tag{2.5.32}$$

$$y = c_1x^3 + c_2x^2 \tag{2.5.33}$$

8. Finding the general solution

$$(x^2D^2 - 3xD + 4I)y = 0 \qquad a = -3, \quad b = 4 \tag{2.5.34}$$

$$(a-1)^2 - 4b = 0 \qquad m_1, m_2 = 2 \tag{2.5.35}$$

$$y = [c_1 + c_2 \ln x]x^2 \tag{2.5.36}$$

9. Finding the general solution

$$(x^2D^2 - 0.2xD + 0.36I)y = 0 \qquad a = -0.2, \quad b = 0.36 \tag{2.5.37}$$

$$(a-1)^2 - 4b = 0 \qquad m_1, m_2 = 0.6 \tag{2.5.38}$$

$$y = [c_1 + c_2 \ln x]x^{0.6} \tag{2.5.39}$$

10. Finding the general solution

$$(x^2D^2 - xD + 5I)y = 0 \qquad a = -1, \quad b = 5 \tag{2.5.40}$$

$$(a-1)^2 - 4b = -16 < 0 \qquad m_1, m_2 = 1 \pm 2i \tag{2.5.41}$$

$$y = e^x[A\cos(2\ln x) + B\sin(2\ln x)] \tag{2.5.42}$$

11. Finding the general solution

$$(x^2D^2 - 3xD + 10I)y = 0 \qquad a = -3, \quad b = 10 \tag{2.5.43}$$

$$(a-1)^2 - 4b = -24 < 0 \qquad m_1, m_2 = 2 \pm \sqrt{6}i \tag{2.5.44}$$

$$y = x^2[A\cos(\sqrt{6}\ln x) + B\sin(\sqrt{6}\ln x)] \tag{2.5.45}$$

12. Solving the IVP $y(1) = 0.4,\ y'(1) = 0$ and graphing,

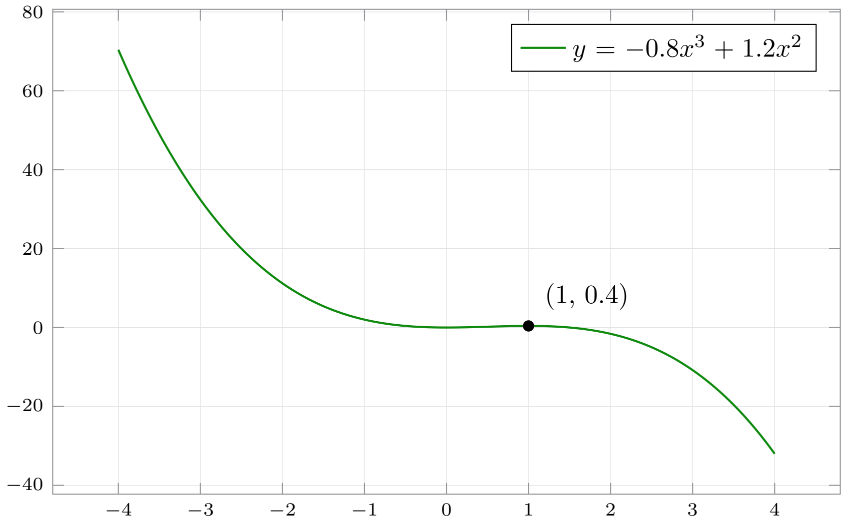
$$x^2y'' - 4xy' + 6y = 0 \qquad a = -4, \quad b = 6 \tag{2.5.46}$$

$$(a-1)^2 - 4b = 1 > 0 \qquad m_1, m_2 = 3, 2 \tag{2.5.47}$$

$$y = c_1x^3 + c_2x^2 \tag{2.5.48}$$

$$y(1) = c_1 + c_2 = 0.4 \qquad y'(1) = 3c_1 + 2c_2 = 0 \tag{2.5.49}$$

$$c_1 = -0.8 \qquad c_2 = 1.2 \tag{2.5.50}$$



13. Solving the IVP $y(1) = 1,\ y'(1) = -1.5$ and graphing,

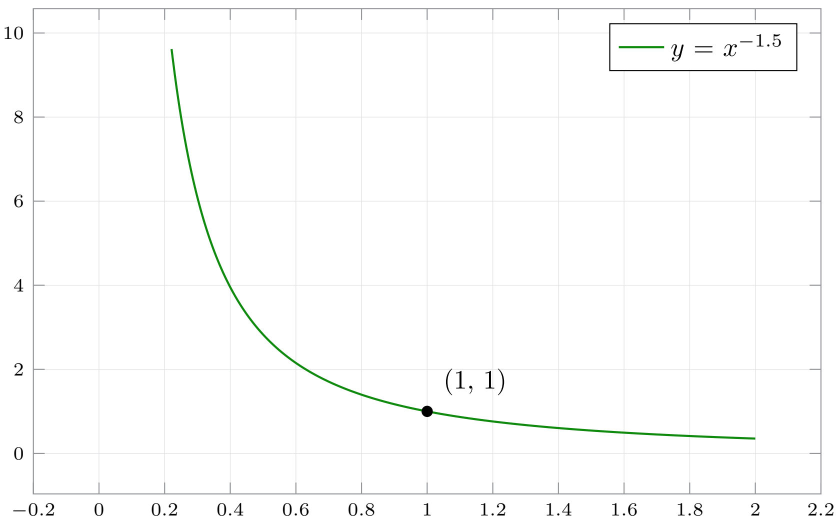
$$x^2y'' + 3xy' + 0.75y = 0 \qquad a = 3, \quad b = 0.75 \tag{2.5.51}$$

$$(a-1)^2 - 4b = 1 > 0 \qquad m_1, m_2 = -1.5, -0.5 \tag{2.5.52}$$

$$y = c_1x^{-1.5} + c_2x^{-0.5} \tag{2.5.53}$$

$$y(1) = c_1 + c_2 = 1 \qquad y'(1) = -1.5c_1 - 0.5c_2 = -1.5 \tag{2.5.54}$$

$$c_1 = 1 \qquad c_2 = 0 \tag{2.5.55}$$



14. Solving the IVP $y(1) = 0,\ \ y'(1) = 2.5$ and graphing,

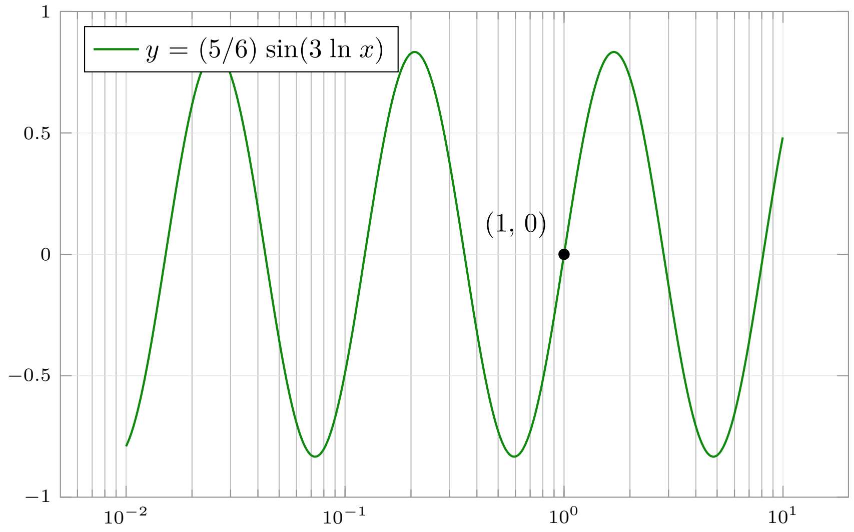
$$x^2y'' + xy' + 9y = 0 \qquad a = 1, \quad b = 9 \tag{2.5.56}$$

$$(a-1)^2 - 4b = -36 < 0 \qquad m_1, m_2 = 0 \pm 3i \tag{2.5.57}$$

$$y = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x) \tag{2.5.58}$$

$$y(1) = c_1 = 0 \tag{2.5.59}$$

$$y'(1) = 3c_2 = 2.5 \qquad c_2 = \frac{5}{6} \tag{2.5.60}$$



15. Solving the IVP $y(1) = 3.6,\ \ y'(1) = 0.4$ and graphing,

$$x^2y'' + 3xy' + y = 0 \tag{2.5.61}$$

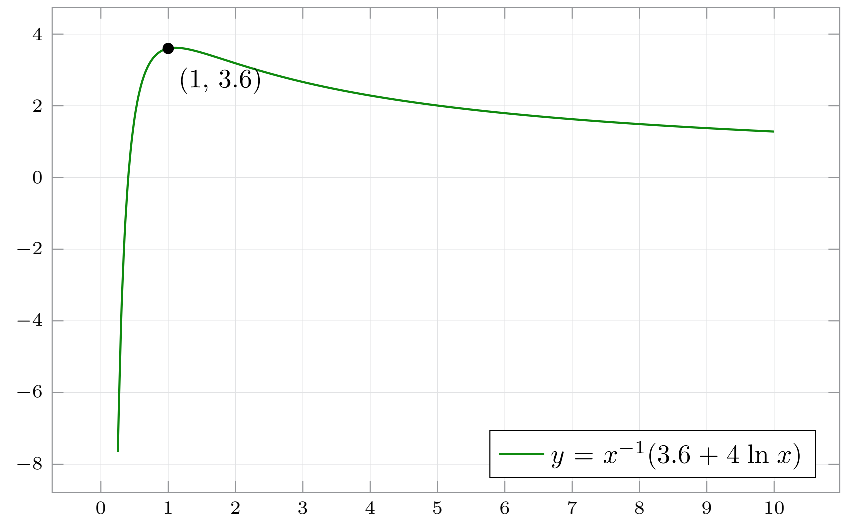
$$a = 3, \quad b = 1 \tag{2.5.62}$$

$$(a-1)^2 - 4b = 0 \qquad m_1, m_2 = -1 \tag{2.5.63}$$

$$y = (c_1 + c_2 \ln x)x^{-1} \tag{2.5.64}$$

$$y(1) = c_1 = 3.6 \qquad y'(1) = c_2 - c_1 = 0.4 \tag{2.5.65}$$

$$c_2 = 4.0 \tag{2.5.66}$$



16. Solving the IVP $y(1) = -\pi,\ y'(1) = 2\pi$ and graphing,

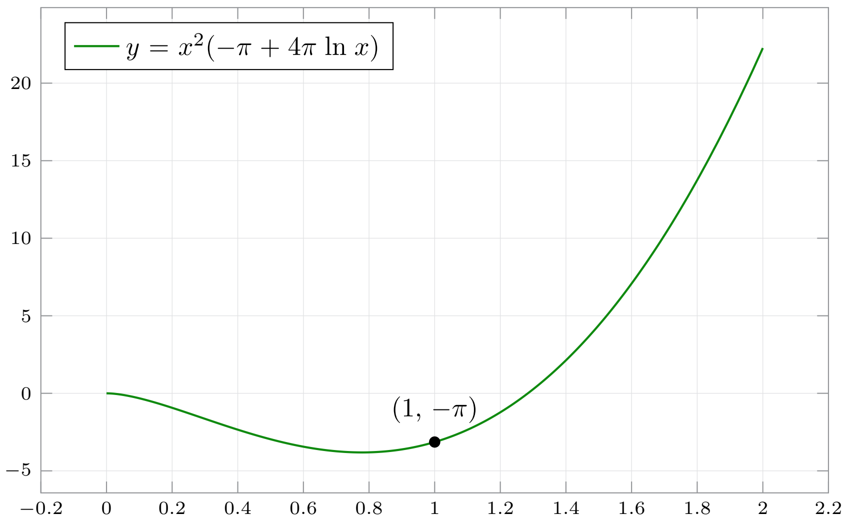
$$(x^2D^2 - 3xD + 4I)y = 0 \qquad a = -3, \quad b = 4 \tag{2.5.67}$$

$$(a-1)^2 - 4b = 0 \qquad m_1, m_2 = 2 \tag{2.5.68}$$

$$y = [c_1 + c_2 \ln x]x^2 \tag{2.5.69}$$

$$y(1) = c_1 = -\pi \qquad y'(1) = c_2 + 2c_1 = 2\pi \tag{2.5.70}$$

$$c_2 = 4\pi \tag{2.5.71}$$



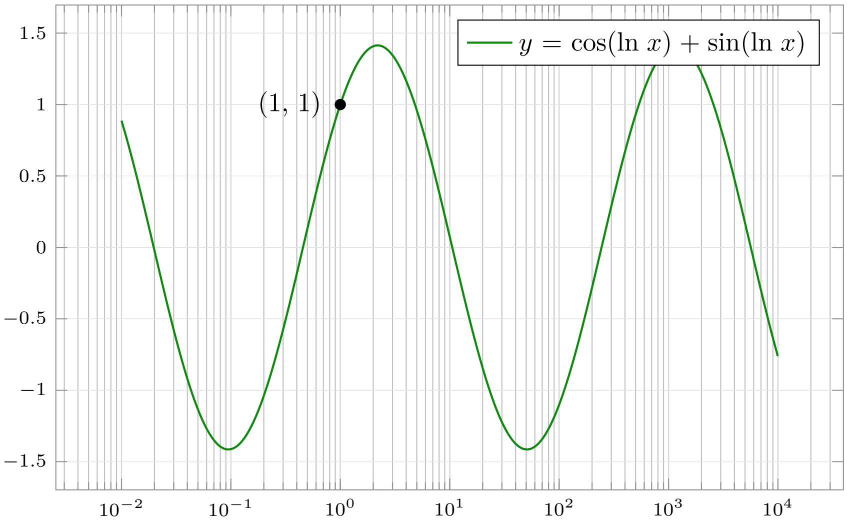
17. Solving the IVP $y(1) = 1,\ y'(1) = 1$ and graphing,

$$(x^2D^2 + xD + I)y = 0 \qquad a = 1, \quad b = 1 \tag{2.5.72}$$

$$(a-1)^2 - 4b = -4 < 0 \qquad m_1, m_2 = 0 \pm i \tag{2.5.73}$$

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) \tag{2.5.74}$$

$$y(1) = c_1 = 1 \qquad y'(1) = c_2 = 1 \tag{2.5.75}$$



18. Solving the IVP $y(1) = 1,\ y'(1) = 0$ and graphing,

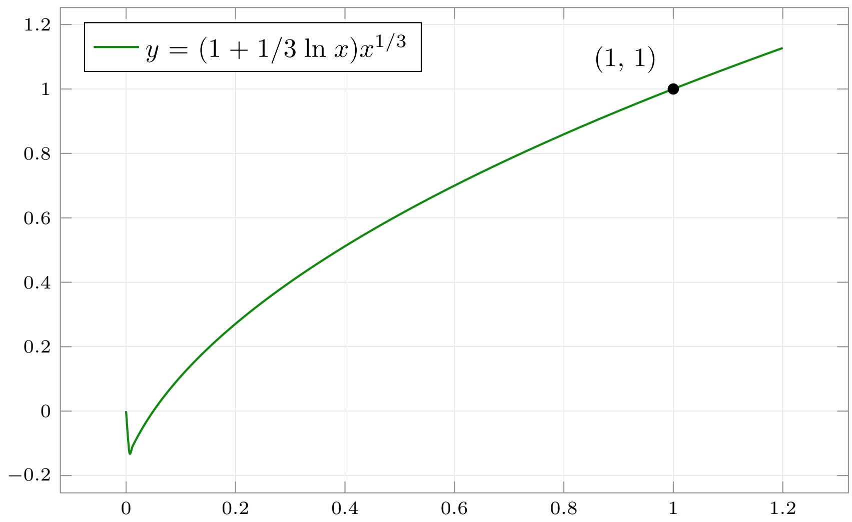
$$(9x^2D^2 + 3xD + I)y = 0 \qquad a = 1/3, \quad b = 1/9 \tag{2.5.76}$$

$$(a-1)^2 - 4b = 0 \qquad m_1, m_2 = 1/3 \tag{2.5.77}$$

$$y = x^{1/3}(c_1 + c_2 \ln x) \tag{2.5.78}$$

$$y(1) = c_1 = 1 \qquad y'(1) = \frac{c_1}{3} + c_2 = 0 \tag{2.5.79}$$

$$c_2 = -1/3 \tag{2.5.80}$$



19. Solving the IVP $y(1) = 0.1,\ \ y'(1) = -4.5$ and graphing,

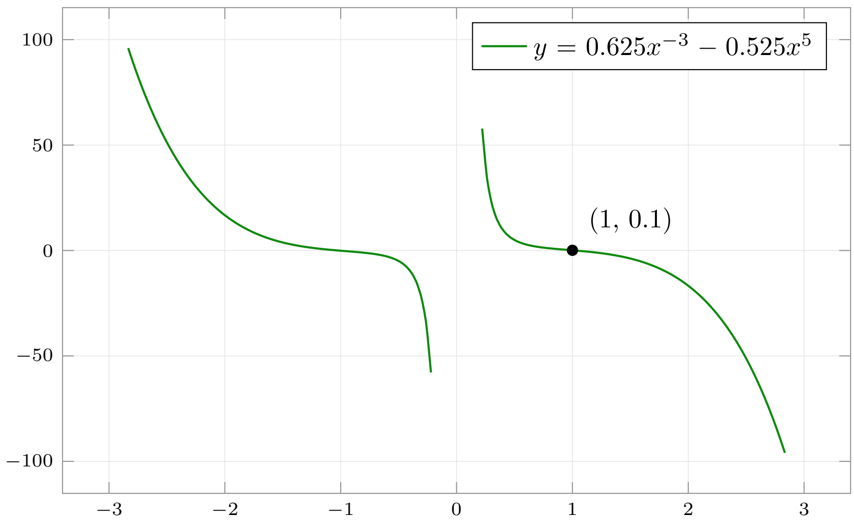
$$(x^2D^2 - xD - 15I)y = 0 \qquad a = -1, \quad b = -15 \tag{2.5.81}$$

$$(a-1)^2 - 4b = 64 > 0 \qquad m_1, m_2 = -3, 5 \tag{2.5.82}$$

$$y = c_1x^{-3} + c_2x^5 \tag{2.5.83}$$

$$y(1) = c_1 + c_2 = 0.1 \qquad y'(1) = -3c_1 + 5c_2 = -4.5 \tag{2.5.84}$$

$$c_2 = -0.525 \qquad c_1 = 0.625 \tag{2.5.85}$$



20. (a) By reduction of order with one solution $y_1 = x^{(1-a)/2}$, and the auxiliary equation having repeated roots.

$$0 = m^2 + (a-1)m + by \tag{2.5.86}$$

$$b = \frac{(1-a)^2}{4} \tag{2.5.87}$$

$$0 = x^2y'' + axy' + \frac{(1-a)^2}{4}y \tag{2.5.88}$$

$$y_2 = uy_1 = ux^{(1-a)/2} \tag{2.5.89}$$

$$y_2' = u'x^{(1-a)/2} + \frac{u(1-a)}{2}x^{(-1-a)/2} \tag{2.5.90}$$

$$y_2'' = u''x^{(1-a)/2} + u'(1-a)x^{(-1-a)/2} + \frac{u(a^2-1)}{4}x^{(-3-a)/2} \tag{2.5.91}$$

$$0 = u''x^{(5-a)/2} + u'(1-a)x^{(3-a)/2} + \frac{u(a^2-1)}{4}x^{(1-a)/2} \tag{2.5.92}$$

$$+ au'x^{(3-a)/2} + \frac{au(1-a)}{2}x^{(1-a)/2} + \frac{u(1-a)^2}{4}x^{(1-a)/2} \tag{2.5.93}$$

$$0 = u''x^{(5-a)/2} + u'x^{(3-a)/2} = u''x + u' \tag{2.5.94}$$

$$z = u' \qquad z' = u'' \tag{2.5.95}$$

$$z' = \frac{-z}{x} \qquad \Longrightarrow z = \frac{1}{x} \qquad \Longrightarrow u = \ln x \tag{2.5.96}$$

$$y_2 = y_1 \ln x \tag{2.5.97}$$

(b) Let $y_1 = x^m$ and $y_2 = x^{m+s}$,

$$0 = y'' + (1-2m-s)y' + (m^2+ms)y \tag{2.5.98}$$

$$y_3 = \frac{x^{m+s} - x^m}{(m+s)-s} = \frac{x^{m+s} - x^m}{s} \tag{2.5.99}$$

$$\lim_{s\to 0} y_3 = \lim_{s\to 0} \frac{\mathrm{d}}{\mathrm{d}s}x^{m+s} \cdot \frac{1}{\mathrm{d}s/\mathrm{d}s} \tag{2.5.100}$$

$$= \lim_{s\to 0} x^m \frac{\mathrm{d}}{\mathrm{d}s} \exp(s \ \ln x) \tag{2.5.101}$$

$$= \lim_{s\to 0} x^m \ (x^s \ln x) = x^m \ln x \tag{2.5.102}$$

(c) In the critical case, with repeated roots $b = (1-a)^2/4$,

$$0 = x^2 y'' + axy' + by \tag{2.5.103}$$

$$y_2 = x^{(1-a)/2} \ln x \tag{2.5.104}$$

$$y_2' = x^{(-1-a)/2} \left[1 + \left(\frac{1-a}{2}\right) \ln x\right] \tag{2.5.105}$$

$$y_2'' = x^{(-3-a)/2} \left[-a + \left(\frac{a^2-1}{4}\right) \ln x\right] \tag{2.5.106}$$

$$x^2 y'' + axy' + by = x^{(1-a)/2} \left[\frac{a^2 - 1 + 2a - 2a^2 + (1-a)^2}{4}\right] \ln x \tag{2.5.107}$$

$$= 0 \tag{2.5.108}$$

Thus, $x^{(1-a)/2} \ln x$ is indeed a solution to the ODE when the auxiliary equation has repeated roots.

(d) Substitute $x = e^t \qquad (x > 0)$,

$$x^2 y'' + axy' + by = 0 \tag{2.5.109}$$

$$y' = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = e^{-t} \frac{\mathrm{d}y}{\mathrm{d}t} \tag{2.5.110}$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{1}{\mathrm{d}x/\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) \tag{2.5.111}$$

$$= e^{-2t}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right) \tag{2.5.112}$$

$$e^{2t} y'' + ae^t y' + by = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (a-1)\frac{\mathrm{d}y}{\mathrm{d}t} + by \tag{2.5.113}$$

(e) Let $y_1, y_2 = e^{\lambda_1 t}, e^{\lambda_2 t}$ be solutions to the above ODE in y, t, with $\lambda_2 = \lambda_1 + \delta$

$$\lambda^2 + (a-1)\lambda + b = 0 \tag{2.5.114}$$

$$b = (a-1)^2/4 \tag{2.5.115}$$

$$\text{By superposition} \quad y_3 = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \tag{2.5.116}$$

$$\lim_{\delta \to 0} y_3 = \lim_{\delta \to 0} \frac{\mathrm{d}y_3}{\mathrm{d}\delta} \tag{2.5.117}$$

$$= \lim_{\delta \to 0} \frac{\mathrm{d}}{\mathrm{d}\delta} \frac{e^{\lambda_1 t}(e^{\delta t} - 1)}{\delta} \tag{2.5.118}$$

$$= \lim_{\delta \to 0} t e^{\lambda_1 t} e^{\delta t} = t e^{\lambda_1 t} \tag{2.5.119}$$

The two solutions are $e^{\lambda t}$ and $te^{\lambda t}$, which when converted back to functions of x, become x^{λ} and $x^{\lambda}\ln x$.

2.6 Existence and Uniqueness of Solutions, Wronskian

1.

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 \qquad W = -(y_1' y_2 - y_1 y_2') \tag{2.6.1}$$

$$= \frac{y_1 y_2' - y_1' y_2}{y_1^2}\, y_1^2 \qquad = -\frac{(y_1' y_2 - y_1 y_2')}{y_2^2}\, y_2^2 \tag{2.6.2}$$

$$= \left(\frac{y_2}{y_1}\right)'\, y_1^2 \qquad = -\left(\frac{y_1}{y_2}\right)'\, y_2^2 \tag{2.6.3}$$

$$\text{if } y_1 \neq 0 \qquad \text{if } y_2 \neq 0 \tag{2.6.4}$$

2. the functions are L.I. by the quotient method,

$$y_1 = e^{4x} \qquad y_2 = e^{-1.5x} \tag{2.6.5}$$

$$Q(x) = \frac{y_1}{y_2} = e^{5.5x} \qquad Q \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.6}$$

the functions are L.I. using the Wronskian,

$$W(y_1, y_2) = -1.5e^{2.5x} - 4e^{2.5x} \tag{2.6.7}$$

$$= -5.5e^{2.5x} \tag{2.6.8}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.9}$$

3. the functions are L.I. by the quotient method,

$$y_1 = e^{-0.4x} \qquad y_2 = e^{-2.6x} \tag{2.6.10}$$

$$Q(x) = \frac{y_1}{y_2} = e^{2.2x} \qquad Q \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.11}$$

the functions are L.I. using the Wronskian,

$$W(y_1, y_2) = -2.6e^{-3x} + 0.4e^{-3x} \tag{2.6.12}$$

$$= -2.2e^{-3x} \tag{2.6.13}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.14}$$

4. the functions are L.I. by the quotient method,

$$y_1 = x \qquad\qquad y_2 = 1/x \tag{2.6.15}$$

$$Q(x) = \frac{y_1}{y_2} = x^2 \qquad\qquad Q \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.16}$$

the functions are L.I. using the Wronskian,

$$W(y_1, y_2) = \frac{-1}{x} - \frac{1}{x} \tag{2.6.17}$$

$$= \frac{-2}{x} \tag{2.6.18}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.19}$$

5. the functions are L.I. by the quotient method,

$$y_1 = x^3 \qquad\qquad y_2 = x^2 \tag{2.6.20}$$

$$Q(x) = \frac{y_1}{y_2} = x \qquad\qquad Q \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.21}$$

the functions are L.I. using the Wronskian,

$$W(y_1, y_2) = 2x^4 - 3x^4 \tag{2.6.22}$$

$$= -x^4 \tag{2.6.23}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.24}$$

6. the functions are L.I. by the quotient method,

$$y_1 = e^{-x}\cos(\omega x) \qquad\qquad y_2 = e^{-x}\sin(\omega x) \tag{2.6.25}$$

$$Q(x) = \frac{y_1}{y_2} = \cot(\omega x) \qquad\qquad Q \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.26}$$

the functions are L.I. using the Wronskian,

$$W(y_1, y_2) = e^{-2x}\cos(\omega x)[\omega\cos(\omega x) - \sin(\omega x)] \tag{2.6.27}$$

$$- e^{-2x}\sin(\omega x)[-\omega\sin(\omega x) - \cos(\omega x)] \tag{2.6.28}$$

$$= e^{-2x}[\omega] \tag{2.6.29}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.30}$$

7. the functions are L.I. by the quotient method,

$$y_1 = \cosh ax \qquad y_2 = \sinh ax \tag{2.6.31}$$

$$Q(x) = \frac{y_2}{y_1} = \tanh ax \qquad Q \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.32}$$

the functions are L.I. using the Wronskian,

$$W(y_1, y_2) = a \cosh^2 ax - a \sinh^2 ax \tag{2.6.33}$$

$$= a \tag{2.6.34}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.35}$$

8. the functions are L.I. by the quotient method,

$$y_1 = x^k \cos(\ln x) \qquad y_2 = x^k \sin(\ln x) \tag{2.6.36}$$

$$Q(x) = \frac{y_1}{y_2} = \cot(\ln x) \qquad Q \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.37}$$

the functions are L.I. using the Wronskian,

$$W(y_1, y_2) = x^{2k-2} \cos(\ln x)[k \sin(\ln x) + \cos(\ln x)] \tag{2.6.38}$$

$$- x^{2k-2} \sin(\ln x)[- \sin(\ln x) + k \cos(\ln x)] \tag{2.6.39}$$

$$= x^{2k-2} \tag{2.6.40}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.41}$$

9. the functions are L.I. using the Wronskian,

$$y_1 = \cos(5x) \qquad y_2 = \sin(5x) \tag{2.6.42}$$

$$W(y_1, y_2) = 5 \cos^2(5x) + 5 \sin^2(5x) \tag{2.6.43}$$

$$= 5 \tag{2.6.44}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.45}$$

to construct the ODE given these solutions,

$$\alpha = 0 \qquad \beta = 5 \tag{2.6.46}$$

$$\lambda_1, \lambda_2 = 0 \pm 5i \qquad a = 0, \quad b = 25 \tag{2.6.47}$$

$$y'' + 25y = 0 \tag{2.6.48}$$

solving the IVP $y(0) = 3,\ y'(0) = -5$,

$$c_1 = 3 \qquad 5c_2 = -5 \tag{2.6.49}$$

$$y = 3\cos(5x) - \sin(5x) \tag{2.6.50}$$

10. the functions are L.I. using the Wronskian,

$$y_1 = x^{m_1} \qquad y_2 = x^{m_2} \tag{2.6.51}$$

$$W(y_1, y_2) = x^{m_1+m_2-1}[m_2 - m_1] \tag{2.6.52}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.53}$$

to construct the Euler-Cauchy ODE given these solutions,

$$\lambda_1, \lambda_2 = m_1, m_2 \tag{2.6.54}$$

$$(a-1) = -(m_1 + m_2) \qquad b = m_1 m_2 \tag{2.6.55}$$

$$x^2y'' + (1 - m_1 - m_2)xy' + m_1m_2y = 0 \tag{2.6.56}$$

solving the IVP $y(1) = -2,\ y'(1) = 2m_1 - 4m_2$,

$$c_1 + c_2 = -2 \qquad m_1c_1 + m_2c_2 = 2m_1 - 4m_2 \tag{2.6.57}$$

$$y = 2x^{m_1} - 4x^{m_2} \tag{2.6.58}$$

11. the functions are L.I. using the Wronskian,

$$y_1 = e^{-2.5x}\cos(0.3x) \qquad y_2 = e^{-2.5x}\sin(0.3x) \tag{2.6.59}$$

$$W(y_1, y_2) = e^{-5x}\cos(0.3x)[0.3\cos(0.3x) - 2.5\sin(0.3x)] \tag{2.6.60}$$

$$- e^{-5x}\sin(0.3x)[-0.3\sin(0.3x) - 2.5\cos(0.3x)] \tag{2.6.61}$$

$$= e^{-5x}[0.3] \tag{2.6.62}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.63}$$

to construct the ODE given these solutions,

$$\lambda_1, \lambda_2 = -2.5 \pm 0.3i \tag{2.6.64}$$

$$a = 5 \qquad b = 6.34 \tag{2.6.65}$$

$$y'' + 5y' + 6.34y = 0 \tag{2.6.66}$$

solving the IVP $y(0) = 3,\ y'(0) = -7.5$,

$$c_1 = 3 \qquad -2.5c_1 + 0.3c_2 = -7.5 \tag{2.6.67}$$

$$c_2 = 0 \tag{2.6.68}$$

$$y = 3e^{-2.5x}\cos(0.3x) \tag{2.6.69}$$

12. the functions are L.I. using the Wronskian,

$$y_1 = x^2 \qquad y_2 = x^2 \ln x \tag{2.6.70}$$

$$W(y_1, y_2) = x^3[1 + 2\ln x - 2\ln x] \tag{2.6.71}$$

$$= x^3 \tag{2.6.72}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.73}$$

to construct the Euler-Cauchy ODE given these solutions,

$$\lambda_1, \lambda_2 = 2 \tag{2.6.74}$$

$$(a-1) = -4 \qquad b = 4 \tag{2.6.75}$$

$$x^2y'' - 3xy' + 4y = 0 \tag{2.6.76}$$

solving the IVP $y(1) = 4,\ y'(1) = 6$,

$$c_1 = 4 \qquad 2c_1 + c_2 = 6 \tag{2.6.77}$$

$$c_2 = -2 \tag{2.6.78}$$

$$y = 4x^2 - 2x^2\ln x \tag{2.6.79}$$

13. the functions are L.I. using the Wronskian,

$$y_1 = 1 \qquad y_2 = e^{-2x} \tag{2.6.80}$$

$$W(y_1, y_2) = -2e^{-2x} \tag{2.6.81}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.82}$$

to construct the ODE given these solutions,

$$\lambda_1, \lambda_2 = 0, -2 \tag{2.6.83}$$

$$a = 2 \qquad\qquad b = 0 \tag{2.6.84}$$

$$y'' + 2y' = 0 \tag{2.6.85}$$

solving the IVP $y(0) = 1,\ y'(0) = -1$,

$$c_1 + c_2 = 1 \qquad\qquad -2c_2 = -1 \tag{2.6.86}$$

$$c_2 = 0.5 \qquad\qquad c_1 = 0.5 \tag{2.6.87}$$

$$y = 0.5(1 + e^{-2x}) \tag{2.6.88}$$

14. the functions are L.I. using the Wronskian,

$$y_1 = e^{-kx}\cos(\pi x) \qquad\qquad y_2 = e^{-kx}\sin(\pi x) \tag{2.6.89}$$

$$W(y_1, y_2) = e^{-2kx}\cos(\pi x)[\pi\cos(\pi x) - k\sin(\pi x)] \tag{2.6.90}$$

$$- e^{-2kx}\sin(\pi x)[-\pi\sin(\pi x) - k\cos(\pi x)] \tag{2.6.91}$$

$$= \pi e^{-2kx} \tag{2.6.92}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.93}$$

to construct the ODE given these solutions,

$$\alpha = -k \qquad\qquad \beta = \pi \tag{2.6.94}$$

$$\lambda_1, \lambda_2 = -k \pm i\pi \qquad\qquad a = 2k, \quad b = k^2 + \pi^2 \tag{2.6.95}$$

$$y'' + 2ky' + (k^2 + \pi^2)y = 0 \tag{2.6.96}$$

solving the IVP $y(0) = 1,\ y'(0) = -k - \pi$,

$$c_1 = 1 \qquad\qquad -kc_1 + \pi c_2 = -k - \pi \tag{2.6.97}$$

$$c_2 = -1 \tag{2.6.98}$$

$$y = e^{-kx}\left[\cos(kx) - \sin(kx)\right] \tag{2.6.99}$$

15. the functions are L.I. using the Wronskian,

$$y_1 = \cosh(1.8x) \qquad y_2 = \sinh(1.8x) \tag{2.6.100}$$

$$W(y_1, y_2) = 1.8\cosh^2 x - 1.8\sinh^2 x \tag{2.6.101}$$

$$= 1.8 \tag{2.6.102}$$

$$W \neq 0 \text{ for some } x \in \mathcal{R} \tag{2.6.103}$$

to construct the ODE given these solutions,

$$\lambda_1, \lambda_2 = \pm 1.8 \qquad a = 0, \quad b = 3.24 \tag{2.6.104}$$

$$y'' - 3.24y = 0 \tag{2.6.105}$$

solving the IVP $y(0) = 14.20,\ y'(0) = 16.38$,

$$c_1 = 14.20 \qquad 1.8c_2 = 16.38 \tag{2.6.106}$$

$$c_2 = 9.1 \tag{2.6.107}$$

$$y = 14.20\cosh x + 9.1\sinh x \tag{2.6.108}$$

16. **(a)** Solving the ODE using both kinds of basis functions,

$$y'' - y = 0 \tag{2.6.109}$$

$$y = c_1 e^x + c_2 e^{-x} \qquad \text{exponential functions} \tag{2.6.110}$$

$$y = c_3\cosh(x) + c_4\sinh(x) \qquad \text{hyperbolic functions} \tag{2.6.111}$$

$$c_3 = 0.5(c_1 + c_2) \qquad c_4 = 0.5(c_1 - c_2) \tag{2.6.112}$$

(b) Claim : Solutions of a basis set can be 0 at the same point (say X).

$$y_1(X) = y_2(X) = 0 \tag{2.6.113}$$

$$c_1 y_1(X) + c_2 y_2(X) = 0 \qquad \text{without needing } c_1, c_2 = 0 \tag{2.6.114}$$

$$y_2(X) = \frac{-c_1}{c_2} y_1 \tag{2.6.115}$$

$$W(X) = \Bigg[y_1 y_2' - y_2' y_1 \Bigg]_{x=X} \tag{2.6.116}$$

$$= \frac{-c_2}{c_1} y_1(X) y_1'(X) + \frac{c_2}{c_1} y_1'(X) y_1(X) \tag{2.6.117}$$

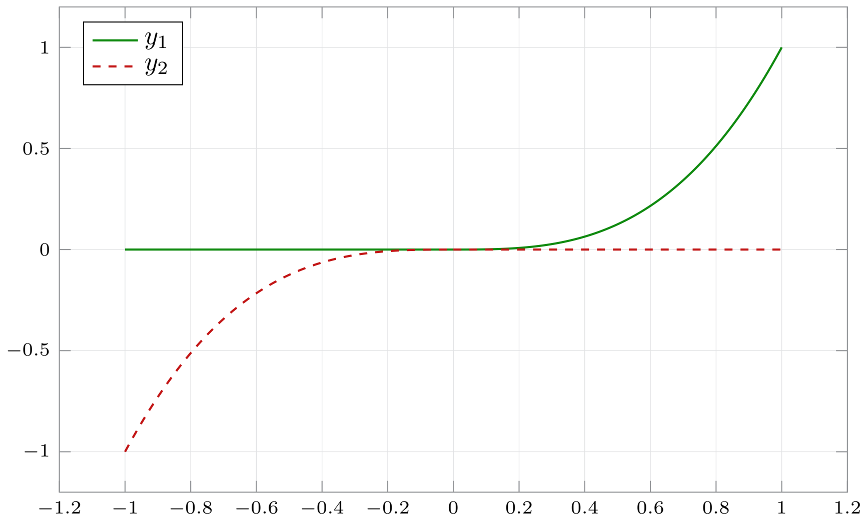
$$= 0 \tag{2.6.118}$$

The fact that $W(X) = 0$ means that the two functions are linearly dependent in $\mathcal{I}$. But this contradicts the assumption that they are members of a basis set. This contradiction falsifies the original claim.

(c) Refer previous part. Instead of $y_1(X) = y_2(X) = 0$, the new claim is that $y_1'(X) = y_2'(X) = 0$. Procedure is the same from the Wronskian step onward.

(d) These formulas exist if one of the solutions is not the trivial solution, which is true for most physical systems.

(e) Sketching,



Wronskian is zero in $(-1, 0)$ and $(0, 1)$ separately because one of the two functions is zero. It is also zero at $x = 0$.
However, over the entire interval $(-1, 1)$, they are L.I, because,

$$c_1 y_1 + c_2 y_2 = 0 \tag{2.6.119}$$

$$x < 0 \qquad \implies c_2 \neq 0 \tag{2.6.120}$$

$$x \geq 0 \qquad \implies c_1 \neq 0 \tag{2.6.121}$$

Looking at the functions and their derivatives, both are continuous in $(-1, 1)$, which means that these functions can be solutions to an ODE.
Consider an Euler-Cauchy equation with $m_1, m_2 = 0, 3$

$$(a - 1) = -3 \qquad b = 0 \tag{2.6.122}$$

$$m^2 - 3m = 0 \qquad x^2 y'' - 2xy' = 0 \tag{2.6.123}$$

$$y'' - \frac{2}{x} y' = 0 \tag{2.6.124}$$

If the interval under consideration is $(-1, 1)$, then $p(x) = 2/x$ is discontinuous at $x = 0$, which means that Theorem 2 does not apply to this ODE.

(f) Eliminating $q(x)$,

$$y_1'' + p(x)y_1' + q(x)y_1 = 0 \tag{2.6.125}$$

$$y_2'' + p(x)y_2' + q(x)y_2 = 0 \tag{2.6.126}$$

$$y_2'' + py_2' - y_2\left(\frac{y_1'' + py_1'}{y_1}\right) = 0 \tag{2.6.127}$$

$$y_1y_2'' - y_1''y_2 + p(y_1y_2' - y_1'y_2) = 0 \tag{2.6.128}$$

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(y_1y_2' - y_1'y_2) \tag{2.6.129}$$

$$\frac{\mathrm{d}W}{\mathrm{d}x} = y_1y_2'' - y_1''y_2 + y_1'y_2' - y_1'y_2' = y_1y_2'' - y_1''y_2 \tag{2.6.130}$$

$$W' + pW = 0 \tag{2.6.131}$$

$$\int_{x_0}^{x} \frac{1}{W}\ \mathrm{d}W = -\int_{x_0}^{t} p(t)\ \ \mathrm{d}t \tag{2.6.132}$$

$$\frac{W(x)}{W(x_0)} = \exp\left(-\int_{x_0}^{x} p(t)\ \ \mathrm{d}t\right) \tag{2.6.133}$$

Applying Abel's formula to problem 6, TBC

2.7 Nonhomogeneous ODEs

1. Solving the h-ODE,

$$y'' + 5y' + 4y = 0 \qquad a = 5, \quad b = 4 \tag{2.7.1}$$

$$\lambda_1, \lambda_2 = \frac{-5 \pm \sqrt{9}}{2} \qquad = -4, -1 \tag{2.7.2}$$

$$y_h = c_1e^{-4x} + c_2e^{-x} \tag{2.7.3}$$

Solving the nh-ODE,

$$y'' + 5y' + 4y = 10e^{-3x} \tag{2.7.4}$$

$$y_p = Ke^{-3x} \qquad \text{basic rule} \tag{2.7.5}$$

$$e^{-3x}(9K - 15K + 4K) = 10e^{-3x} \qquad K = -5 \tag{2.7.6}$$

$$y_p = -5e^{-3x} \tag{2.7.7}$$

$$y = y_p + y_h \tag{2.7.8}$$

2. Solving the h-ODE,

$$0 = 10y'' + 50y' + 57.6y \qquad a = 5, \quad b = 5.76 \tag{2.7.9}$$

$$\lambda_1, \lambda_2 = \frac{-5 \pm \sqrt{1.96}}{2} \qquad = -3.2, -1.8 \tag{2.7.10}$$

$$y_h = c_1 e^{-3.2x} + c_2 e^{-1.8x} \tag{2.7.11}$$

Solving the nh-ODE,

$$10y'' + 50y' + 57.6y = \cos x \tag{2.7.12}$$

$$y_p = K \cos x + M \sin x \qquad \text{basic rule} \tag{2.7.13}$$

$$47.6K + 50M = 1 \qquad 47.6M - 50K = 0 \tag{2.7.14}$$

$$K = 0.00998 \qquad M = 0.0105 \tag{2.7.15}$$

$$y_p = 0.00998 \cos x + 0.0105 \sin x \tag{2.7.16}$$

$$y = y_p + y_h \tag{2.7.17}$$

3. Solving the h-ODE,

$$0 = y'' + 3y' + 2y \qquad a = 3, \quad b = 2 \tag{2.7.18}$$

$$\lambda_1, \lambda_2 = \frac{-3 \pm \sqrt{1}}{2} \qquad = -1, -2 \tag{2.7.19}$$

$$y_h = c_1 e^{-1x} + c_2 e^{-2x} \tag{2.7.20}$$

Solving the nh-ODE,

$$y'' + 3y' + 2y = 12x^2 \tag{2.7.21}$$

$$y_p = K_0 + K_1 x + K_2 x^2 \qquad \text{basic rule} \tag{2.7.22}$$

$$2K_2 x^2 = 12x^2 \qquad 2K_2 + 3K_1 + 2K_0 = 0 \tag{2.7.23}$$

$$6K_2 + 2K_1 = 0 \qquad K_2 = 6 \tag{2.7.24}$$

$$K_1 = -18 \qquad K_0 = 21 \tag{2.7.25}$$

$$y_p = 21 - 18x + 6x^2 \tag{2.7.26}$$

$$y = y_p + y_h \tag{2.7.27}$$

4. Solving the h-ODE,

$$0 = y'' - 9y \qquad a = 0, \quad b = -9 \tag{2.7.28}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm \sqrt{36}}{2} \qquad = -3, 3 \tag{2.7.29}$$

$$y_h = c_1 e^{-3x} + c_2 e^{3x} \tag{2.7.30}$$

Solving the nh-ODE,

$$y'' - 9y = 18\cos(\pi x) \tag{2.7.31}$$

$$y_p = K\cos(\pi x) + M\sin(\pi x) \qquad \text{basic rule} \tag{2.7.32}$$

$$-9K - \pi^2 K = 18 \qquad -\pi^2 M - 9M = 0 \tag{2.7.33}$$

$$K = \frac{-18}{\pi^2 + 9} \qquad M = 0 \tag{2.7.34}$$

$$y_p = \frac{-18\cos(\pi x)}{\pi^2 + 9} \tag{2.7.35}$$

$$y = y_p + y_h \tag{2.7.36}$$

5. Solving the h-ODE,

$$0 = y'' + 4y' + 4y \qquad a = 4, \quad b = 4 \tag{2.7.37}$$

$$\lambda_1, \lambda_2 = \frac{-4 \pm \sqrt{0}}{2} \qquad = -2, -2 \tag{2.7.38}$$

$$y_h = (c_1 + c_2 x)e^{-2x} \tag{2.7.39}$$

Solving the nh-ODE,

$$y'' + 4y' + 4y = e^{-x}\cos x \tag{2.7.40}$$

$$y_p = e^{-x}(K\cos x + M\sin x) \qquad \text{basic rule} \tag{2.7.41}$$

$$y_p' = e^{-x}[(M - K)\cos x - (K + M)\sin x] \tag{2.7.42}$$

$$y_p'' = e^{-x}[(-2M)\cos x + (2K)\sin x] \tag{2.7.43}$$

$$2M = 1 \qquad -2K = 0 \tag{2.7.44}$$

$$M = 0.5 \qquad K = 0 \tag{2.7.45}$$

$$y_p = 0.5e^{-x}\sin(x) \tag{2.7.46}$$

$$y = y_p + y_h \tag{2.7.47}$$

6. Solving the h-ODE,

$$0 = y'' + y' + (\pi^2 + 0.25)y \qquad a = 1, \quad b = \pi^2 + 0.25 \tag{2.7.48}$$

$$\lambda_1, \lambda_2 = \frac{-1 \pm i\sqrt{4\pi^2}}{2} \qquad = -0.5 \pm i\pi \tag{2.7.49}$$

$$y_h = e^{-0.5x}[c_1 \cos(\pi x) + c_2 \sin(\pi x)] \tag{2.7.50}$$

Solving the nh-ODE,

$$e^{-0.5x} \sin(\pi x) = y'' + y' + (\pi^2 + 0.25)y \tag{2.7.51}$$

$$y_p = xe^{-0.5x}[K \cos(\pi x) + M \sin(\pi x)] \qquad \text{modification} \tag{2.7.52}$$

$$[x \sin] \qquad 0 = -(\pi^2 - 0.25)M + \pi K - 0.5M - \pi K + 0.25M + \pi^2 M \tag{2.7.53}$$

$$[x \cos] \qquad 0 = -\pi M + (0.25 - \pi^2)K + \pi M - 0.5K + \pi^2 K + 0.25K \tag{2.7.54}$$

$$[\sin] \qquad 1 = -M - 2\pi K + M \tag{2.7.55}$$

$$[\cos] \qquad 0 = 2\pi M - K + K \tag{2.7.56}$$

$$M = 0 \qquad K = \frac{-1}{2\pi} \tag{2.7.57}$$

$$y_p = \frac{-x}{2\pi} e^{-0.5x} \cos(\pi x) \tag{2.7.58}$$

$$y = y_p + y_h \tag{2.7.59}$$

7. Solving the h-ODE,

$$0 = y'' + 2y' + 0.75y \qquad a = 2, \quad b = 0.75 \tag{2.7.60}$$

$$\lambda_1, \lambda_2 = \frac{-2 \pm \sqrt{1}}{2} \qquad = -1.5, -0.5 \tag{2.7.61}$$

$$y_h = c_1 e^{-1.5x} + c_2 e^{-0.5x} \tag{2.7.62}$$

Solving the nh-ODE,

$$y'' + 2y' + 0.75y = 3e^x + 4.5x \tag{2.7.63}$$

$$y_p = K_0 + K_1 x + Me^x \qquad \text{superposition} \tag{2.7.64}$$

$$y_p' = K_1 + Me^x \tag{2.7.65}$$

$$y_p'' = Me^x \tag{2.7.66}$$

$$M + 2M + 0.75M = 3 \qquad 2K_1 + 0.75K_0 = 0 \tag{2.7.67}$$

$$0.75K_1 = 4.5 \qquad K_1 = 6 \tag{2.7.68}$$

$$K_0 = -16 \qquad M = 0.8 \tag{2.7.69}$$

$$y_p = 0.8e^x - 16 + 6x \tag{2.7.70}$$

$$y = y_p + y_h \tag{2.7.71}$$

8. Solving the h-ODE,

$$0 = 3y'' + 27y \qquad a = 0, \quad b = 9 \tag{2.7.72}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{36}}{2} \qquad = 0 \pm 3i \tag{2.7.73}$$

$$y_h = c_1 \cos(3x) + c_2 \sin(3x) \tag{2.7.74}$$

Solving the nh-ODE,

$$3y'' + 27y = 3\cos x + \cos(3x) \tag{2.7.75}$$

$$y_p = [K_1 \cos x + K_2 \sin x] + x[M_1 \cos(3x) + M_2 \sin(3x)] \qquad \text{modification} \tag{2.7.76}$$

$$y_p'' = -K_1 \cos x - K_2 \sin x \tag{2.7.77}$$

$$+ (6M_2 - 9M_1 x)\cos(3x) - (9M_2 x + 6M_1)\sin(3x) \tag{2.7.78}$$

$$[\cos(x)] \qquad 3 = -3K_1 + 27K_1 \tag{2.7.79}$$

$$[\sin(x)] \qquad 0 = -3K_2 + 27K_2 \tag{2.7.80}$$

$$[\cos(3x)] \qquad 1 = 18M_2 - 27M_1 x + 27M_1 x \tag{2.7.81}$$

$$[\sin(3x)] \qquad 0 = -27M_2 x - 18M_1 + 27M_2 x \tag{2.7.82}$$

$$y_p = \frac{1}{8}\cos x + \frac{x}{18}\sin(3x) \tag{2.7.83}$$

$$y = y_p + y_h \tag{2.7.84}$$

9. Solving the h-ODE,

$$0 = y'' - 16y \qquad a = 0, \quad b = -16 \tag{2.7.85}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm \sqrt{64}}{2} \qquad = -4, 4 \tag{2.7.86}$$

$$y_h = c_1 e^{-4x} + c_2 e^{4x} \tag{2.7.87}$$

Solving the nh-ODE,

$$y'' - 16y = 9.6e^{4x} + 30e^{x} \tag{2.7.88}$$

$$y_p = Me^{x} + Kxe^{4x} \qquad \text{modification} \tag{2.7.89}$$

$$y_p'' = Me^{x} + Ke^{4x}(8 + 16x) \tag{2.7.90}$$

$$[e^{4x}] \qquad 9.6 = 8K + 16Kx - 16Kx \tag{2.7.91}$$

$$[e^{x}] \qquad 30 = M - 16M \tag{2.7.92}$$

$$y_p = -2e^{x} + 1.2xe^{4x} \tag{2.7.93}$$

$$y = y_p + y_h \tag{2.7.94}$$

10. Solving the h-ODE,

$$0 = y'' + 2y' + y \qquad a = 2, \quad b = 1 \tag{2.7.95}$$

$$\lambda_1, \lambda_2 = \frac{-2 \pm \sqrt{0}}{2} \qquad = -1, -1 \tag{2.7.96}$$

$$y_h = (c_1 + c_2 x)e^{-x} \tag{2.7.97}$$

Solving the nh-ODE,

$$y'' + 2y' + y = 2x \sin x \tag{2.7.98}$$

$$y_p = [K_0 + K_1 x] \cos x + [M_0 + M_1 x] \sin x \qquad \text{modification} \tag{2.7.99}$$

$$y_p' = \cos x[K_1 + M_0 + M_1 x] + \sin x[M_1 - K_0 - K_1 x] \tag{2.7.100}$$

$$y_p'' = \cos x[2M_1 - K_0 - K_1 x] + \sin x[-2K_1 - M_0 - M_1 x] \tag{2.7.101}$$

$$[\cos(x)] \qquad 0 = 2M_1 - K_0 + 2K_1 + 2M_0 + K_0 \tag{2.7.102}$$

$$[\sin(x)] \qquad 0 = -2K_1 - M_0 + 2M_1 - 2K_0 + M_0 \tag{2.7.103}$$

$$[x \cos(x)] \qquad 0 = -K_1 + 2M_1 + K_1 \tag{2.7.104}$$

$$[x \sin(x)] \qquad 2 = -M_1 - 2K_1 + M_1 \tag{2.7.105}$$

$$y_p = (1 - x) \cos x + \sin x \tag{2.7.106}$$

$$y = y_p + y_h \tag{2.7.107}$$

11. Solving the h-ODE,

$$0 = y'' + 3y \qquad a = 0, \quad b = 3 \tag{2.7.108}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{12}}{2} \qquad = 0 \pm \sqrt{3}i \tag{2.7.109}$$

$$y_h = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) \tag{2.7.110}$$

Solving the nh-ODE,

$$y'' + 3y = 18x^2 \tag{2.7.111}$$

$$y_p = K_0 + K_1 x + K_2 x^2 \qquad \text{basic} \tag{2.7.112}$$

$$[x^2] \qquad 18 = 3K_2 \tag{2.7.113}$$

$$[x] \qquad 0 = 3K_1 \tag{2.7.114}$$

$$[1] \qquad 0 = 2K_2 + 3K_0 \tag{2.7.115}$$

$$y_p = -4 + 6x^2 \tag{2.7.116}$$

$$y = y_p + y_h \tag{2.7.117}$$

Solving the IVP, $y(0) = -3,\ \ y'(0) = 0$,

$$y(0) = -3 = -4 + c_1 \tag{2.7.118}$$

$$y'(0) = 0 = \sqrt{3}c_2 \tag{2.7.119}$$

$$y = -4 + 6x^2 + \cos(\sqrt{3}x) \tag{2.7.120}$$

12. Solving the h-ODE,

$$0 = y'' + 4y \qquad\qquad a = 0, \quad b = 4 \tag{2.7.121}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{16}}{2} \qquad\qquad = 0 \pm 2i \tag{2.7.122}$$

$$y_h = c_1 \cos(2x) + c_2 \sin(2x) \tag{2.7.123}$$

Solving the nh-ODE,

$$y'' + 4y = -12 \sin(2x) \tag{2.7.124}$$

$$y_p = Kx \cos(2x) + Mx \sin(2x) \tag{2.7.125}$$

$$y_p' = \cos(2x)[K + 2Mx] + \sin(2x)[M - 2Kx] \tag{2.7.126}$$

$$y_p'' = \cos(2x)[4M - 4Kx] + \sin(2x)[-4K - 4Mx] \tag{2.7.127}$$

$$[\sin(2x)] \qquad -12 = -4K \tag{2.7.128}$$

$$[\cos(2x)] \qquad 0 = 4M \tag{2.7.129}$$

$$y_p = 3x \cos(2x) \tag{2.7.130}$$

$$y = y_p + y_h \tag{2.7.131}$$

Solving the IVP, $y(0) = 1.8,\ \ y'(0) = 5$,

$$y(0) = 1.8 = c_1 \tag{2.7.132}$$

$$y'(0) = 5 = 2c_2 \tag{2.7.133}$$

$$y = (1.8 + 3x) \cos(2x) + 2.5 \sin(2x) \tag{2.7.134}$$

13. Solving the h-ODE,

$$0 = 8y'' - 6y' + y \qquad a = -3/4, \quad b = 1/8 \tag{2.7.135}$$

$$\lambda_1, \lambda_2 = \frac{3/4 \pm \sqrt{1/16}}{2} \qquad = 0.5, 0.25 \tag{2.7.136}$$

$$y_h = c_1 e^{0.25x} + c_2 e^{0.5x} \tag{2.7.137}$$

Solving the nh-ODE,

$$8y'' - 6y' + y = 3e^x + 3x^{-x} \tag{2.7.138}$$

$$y_p = Ke^x + Me^{-x} \qquad \text{basic} \tag{2.7.139}$$

$$[e^x] \qquad 3 = 8K - 6K + K \tag{2.7.140}$$

$$[e^{-x}] \qquad 3 = 8M + 6M + M \tag{2.7.141}$$

$$y_p = e^x + 0.2e^{-x} \tag{2.7.142}$$

$$y = y_p + y_h \tag{2.7.143}$$

Solving the IVP, $y(0) = 0.2, \; y'(0) = 0.05$,

$$y(0) = 0.2 = 1.2 + c_1 + c_2 \tag{2.7.144}$$

$$y'(0) = 0.05 = 0.8 + 0.25c_1 + 0.5c_2 \tag{2.7.145}$$

$$y = e^{0.25x} - 2e^{0.5x} + e^x + 0.2e^{-x} \tag{2.7.146}$$

14. Solving the h-ODE,

$$0 = y'' + 4y' + 4y \qquad a = 4, \quad b = 4 \tag{2.7.147}$$

$$\lambda_1, \lambda_2 = \frac{-4 \pm \sqrt{0}}{2} \qquad = -2, -2 \tag{2.7.148}$$

$$y_h = (c_1 + c_2 x)e^{-2x} \tag{2.7.149}$$

Solving the nh-ODE,

$$y'' + 4y' + 4y = e^{-2x}\sin(2x) \tag{2.7.150}$$

$$y_p = e^{-2x}(K\cos(2x) + M\sin(2x)) \qquad \text{basic} \tag{2.7.151}$$

$$y_p' = e^{-2x}[\cos(2x)(2M - 2K) - \sin(2x)(2M + 2K)] \tag{2.7.152}$$

$$y_p'' = e^{-2x}[\cos(2x)(-8M) + \sin(2x)(8K)] \tag{2.7.153}$$

$$[e^{-2x}\cos(2x)] \qquad 0 = -8M + 8M - 8K + 4K \tag{2.7.154}$$

$$[e^{-2x}\sin(2x)] \qquad 1 = 8K - 8M - 8K + 4M \tag{2.7.155}$$

$$y_p = e^{-2x}[-0.25\sin(2x)] \tag{2.7.156}$$

$$y = y_p + y_h \tag{2.7.157}$$

Solving the IVP, $y(0) = 1,\ y'(0) = -1.5$,

$$y(0) = 1 = c_1 \tag{2.7.158}$$

$$y'(0) = -1.5 = -0.5 + c_2 - 2c_1 \tag{2.7.159}$$

$$y = -0.25e^{-2x}\sin(2x) + (1 + x)e^{-2x} \tag{2.7.160}$$

15. Solving the h-ODE,

$$0 = x^2y'' - 3xy' + 3y \qquad a = -3, \quad b = 3 \tag{2.7.161}$$

$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{4}}{2} \qquad = 1, 3 \tag{2.7.162}$$

$$y_h = c_1x + c_2x^3 \tag{2.7.163}$$

Solving the nh-ODE,

$$x^2y'' - 3xy' + 3y = 3\ln x - 4 \tag{2.7.164}$$

$$y_p = \ln x \tag{2.7.165}$$

$$y = y_p + y_h \tag{2.7.166}$$

Solving the IVP, $y(1) = 0,\ \ y'(1) = 1$,

$$y(1) = 0 = c_1 + c_2 \tag{2.7.167}$$

$$y'(1) = 1 = c_1 + 3c_2 + 1 \tag{2.7.168}$$

$$y = \ln x \tag{2.7.169}$$

16. Solving the h-ODE,

$$0 = y'' - 2y' \qquad a = -2, \quad b = 0 \tag{2.7.170}$$

$$\lambda_1, \lambda_2 = \frac{2 \pm \sqrt{4}}{2} \qquad = 0, 2 \tag{2.7.171}$$

$$y_h = c_1 + c_2 e^{2x} \tag{2.7.172}$$

Solving the nh-ODE,

$$y'' - 2y' = 6e^{2x} - 4e^{-2x} \qquad y_p = Kxe^{2x} + Me^{-2x} \quad \text{modif} \tag{2.7.173}$$

$$y_p' = Ke^{2x}[1 + 2x] - 2Me^{-2x} \qquad y_p'' = Ke^{2x}[4 + 4x] + 4Me^{-2x} \tag{2.7.174}$$

$$[e^{2x}] \quad 6 = 4K - 2K \qquad [xe^{2x}] \quad 0 = 4K - 4K \tag{2.7.175}$$

$$[e^{-2x}] \quad -4 = 4M + 4M \tag{2.7.176}$$

$$y_p = 3xe^{2x} - 0.5e^{-2x} \qquad y = y_p + y_h \tag{2.7.177}$$

Solving the IVP, $y(0) = -1,\ \ y'(0) = 6$,

$$y(0) = -1 = c_1 + c_2 - 0.5 \tag{2.7.178}$$

$$y'(0) = 6 = 2c_2 + 1 + 3 \tag{2.7.179}$$

$$y = (3x + 1)e^{2x} - 0.5e^{-2x} - 1.5 \tag{2.7.180}$$

17. Solving the h-ODE,

$$0 = y'' + 0.2y' + 0.26y \qquad a = 0.2, \quad b = 0.26 \tag{2.7.181}$$

$$\lambda_1, \lambda_2 = \frac{-0.2 \pm i\sqrt{1}}{2} \qquad = -0.1 \pm 0.5i \tag{2.7.182}$$

$$y_h = e^{-0.1x}[c_1 \cos(0.5x) + c_2 \sin(0.5x)] \tag{2.7.183}$$

Solving the nh-ODE,

$$y'' + 0.2y' + 0.26y = 1.22e^{0.5x} \qquad y_p = Ke^{0.5x} \qquad \text{basic} \tag{2.7.184}$$

$$[e^{0.5x}] \qquad 1.22 = 0.25K + 0.1K + 0.26K \qquad K = 2 \tag{2.7.185}$$

$$y_p = {\color{red}2e^{0.5x}} \qquad y = {\color{red}y_p} + {\color{green}y_h} \tag{2.7.186}$$

Solving the IVP, $y(0) = 3.5,\ \ y'(0) = 0.35$,

$$y(0) = 3.5 = c_1 + 2 \tag{2.7.187}$$

$$y'(0) = 0.35 = -0.1c_2 + 0.5c_2 + 1 \tag{2.7.188}$$

$$y = e^{-0.1x}[1.5\cos(0.5x) - \sin(0.5x)] + 2e^{0.5x} \tag{2.7.189}$$

18. Solving the h-ODE,

$$0 = y'' + 2y' + 10y \qquad a = 2, \quad b = 10 \tag{2.7.190}$$

$$\lambda_1, \lambda_2 = \frac{-2 \pm i\sqrt{36}}{2} \qquad = -1 \pm 3i \tag{2.7.191}$$

$$y_h = {\color{green}e^{-x}[c_1\cos(3x) + c_2\sin(3x)]} \tag{2.7.192}$$

Solving the nh-ODE,

$$y'' + 2y' + 10y = 17\sin x + 37\sin(3x) \tag{2.7.193}$$

$$y_p = K_1\cos x + K_2\sin x + M_1\cos(3x) + M_2\sin(3x) \qquad \text{basic} \tag{2.7.194}$$

$$[\cos x] \qquad 0 = -K_1 + 2K_2 + 10K_1 \tag{2.7.195}$$

$$[\sin x] \qquad 17 = -K_2 - 2K_1 + 10K_2 \tag{2.7.196}$$

$$[\cos(3x)] \qquad 0 = -9M_1 + 6M_2 + 10M_1 \tag{2.7.197}$$

$$[\sin(3x)] \qquad 37 = -9M_2 - 6M_1 + 10M_2 \tag{2.7.198}$$

$$y_p = {\color{red}-0.4\cos x + 1.8\sin x - 6\cos(3x) + \sin(3x)} \tag{2.7.199}$$

$$y = {\color{red}y_p} + {\color{green}y_h} \tag{2.7.200}$$

Solving the IVP, $y(0) = 6.6,\ \ y'(0) = -2.2,$

$$y(0) = 6.6 = c_1 - 0.4 - 6 \tag{2.7.201}$$

$$y'(0) = -2.2 = -c_1 + 3c_2 + 1.8 + 3 \tag{2.7.202}$$

$$y = e^{-x}[13\cos(3x) + 2\sin(3x)] \tag{2.7.203}$$

$$- 0.4\cos x + 1.8\sin x - 6\cos(3x) + \sin(3x) \tag{2.7.204}$$

19. (a) To look at the effect of changing initial conditions, consider the IVP,

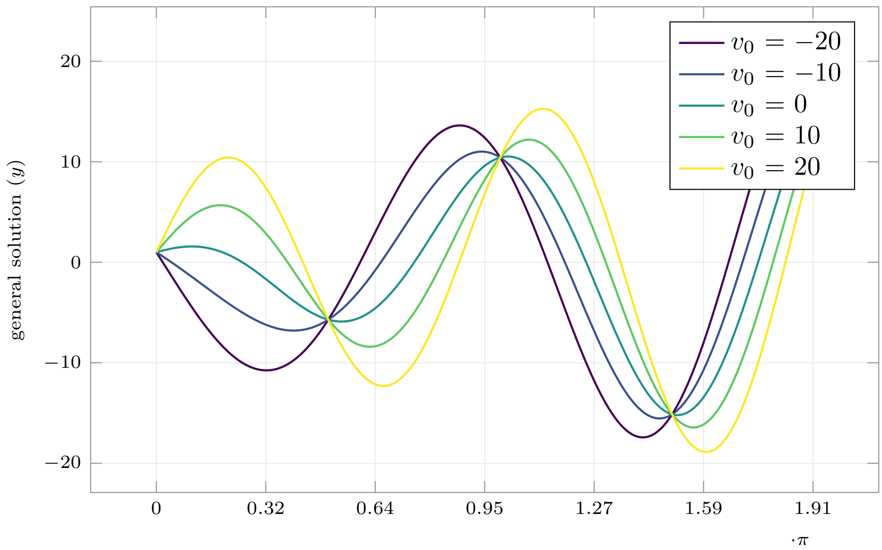
$$y'' + 4y = -12\sin(2x) \tag{2.7.205}$$

$$y_h = y_0\cos(2x) + \frac{v_0}{2}\sin(2x) \tag{2.7.206}$$

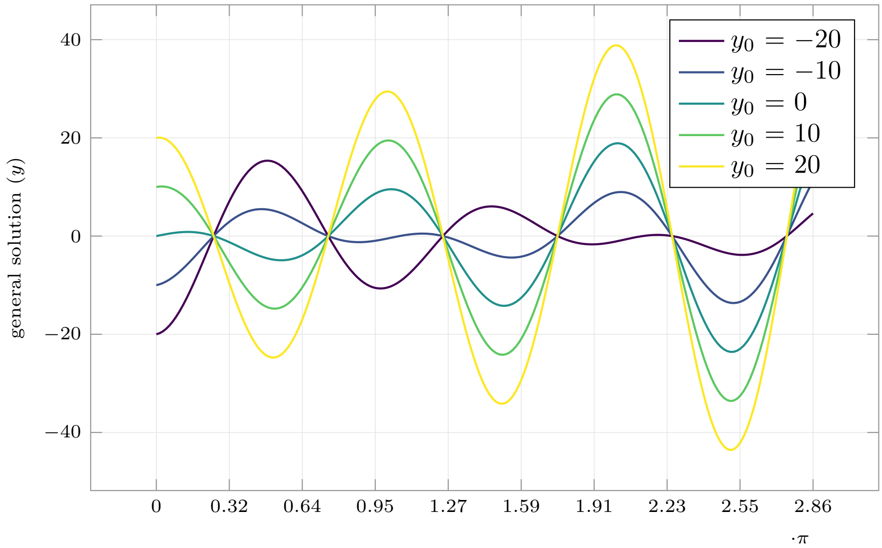
$$y_p = 3x\cos(2x) \tag{2.7.207}$$

The nh-ODE solution y_p does not depend on IC and is not plotted here. Looking at the effect of varying $y_0,\ \ v_0$ on y and then on just y_h,

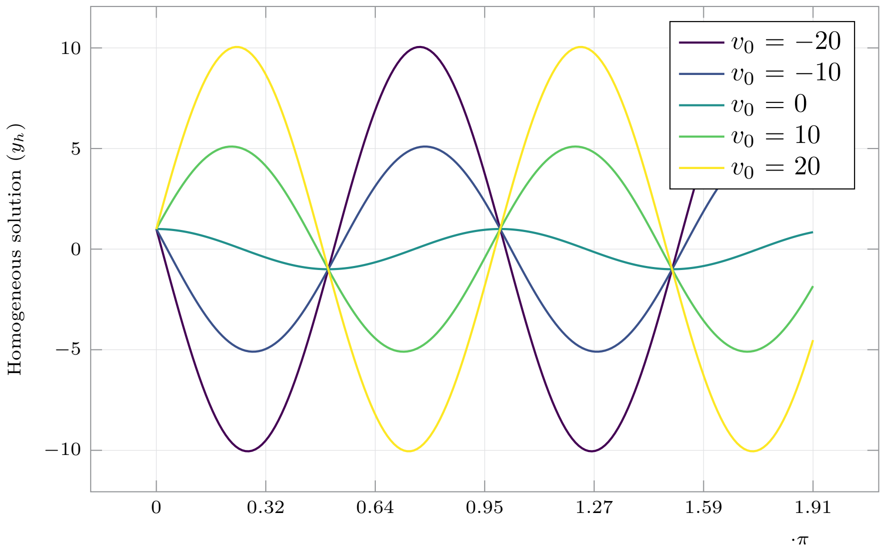
$y_0 = 1,\ \ v_0 = [-20 : 10 : 20]$



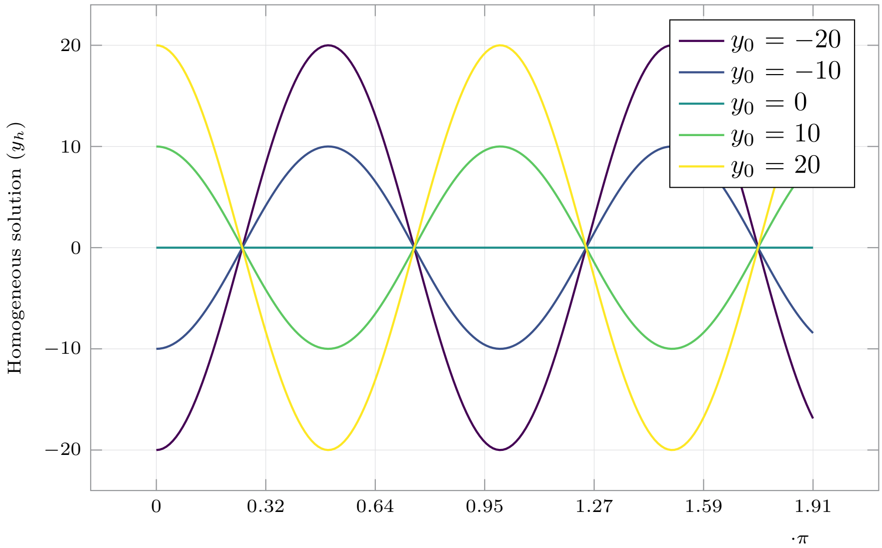
$v_0 = 0, \ y_0 = [-20 : 10 : 20]$



$y_0 = 1, \ v_0 = [-20 : 10 : 20]$



$v_0 = 0, \;\; y_0 = [-20 : 10 : 20]$



(b) Repeating the above procedure but for y_h decaying to zero,

$$y'' + 3y' + 2y = 10 \tag{2.7.208}$$

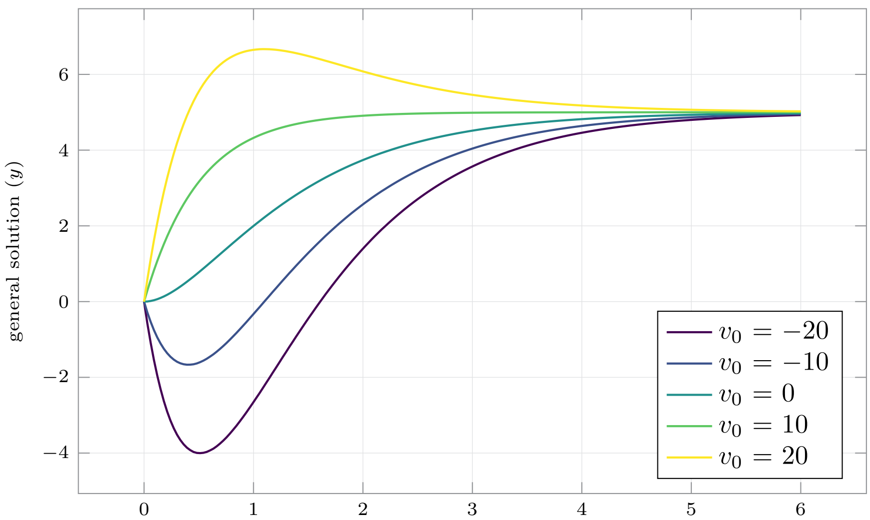
$$y_h = c_1 e^{-x} + c_2 e^{-2x} \tag{2.7.209}$$

$$y_p = 5 \tag{2.7.210}$$

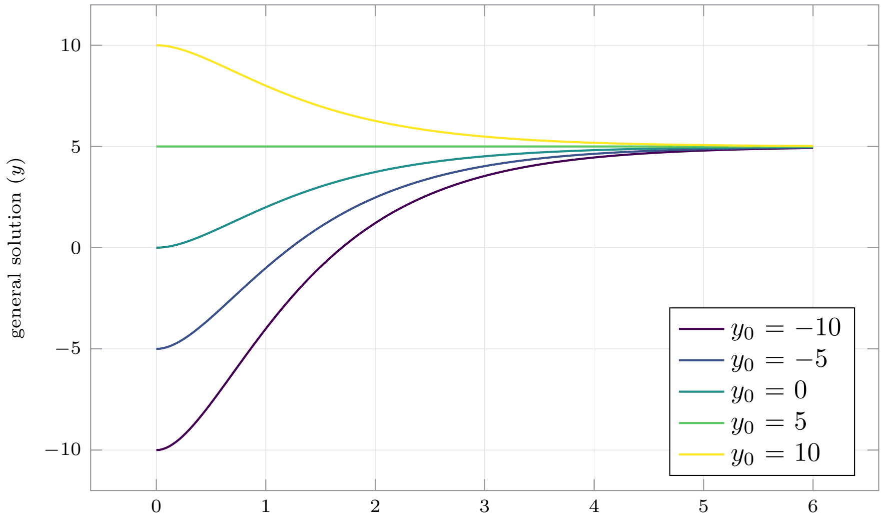
$$y = -(y_0 + v_0 - 5)e^{-2x} + (2y_0 + v_0 - 10)e^{-x} \tag{2.7.211}$$

$$+ 5 \tag{2.7.212}$$

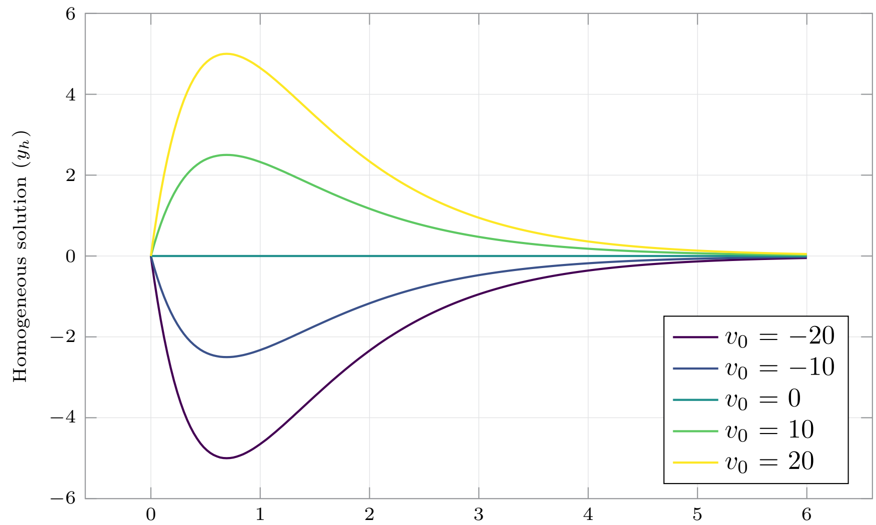
$y_0 = 0, \;\; v_0 = [-20 : 10 : 20]$



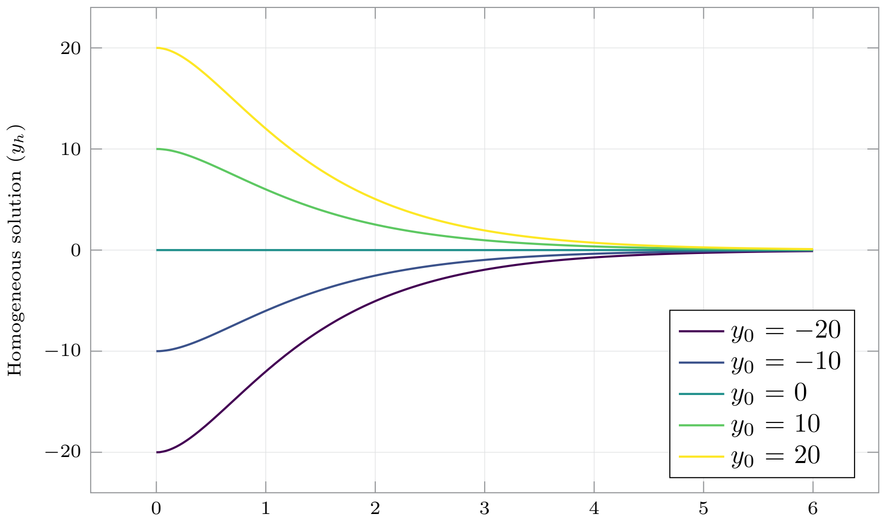
$v_0 = 0, \ y_0 = [-10 : -5 : 10]$



$y_0 = 0, \ v_0 = [-20 : 10 : 20]$



$v_0 = 0, \ y_0 = [-20 : 10 : 20]$



(c) Repeating the above procedure but for y_h growing with time,

$$y'' - y = \sin x \tag{2.7.213}$$

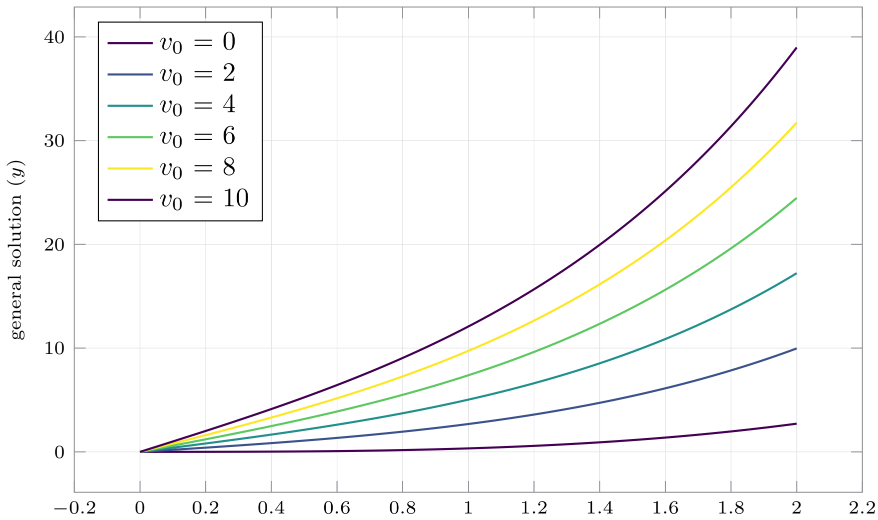
$$y_h = c_1 e^{x} + c_2 e^{-x} \tag{2.7.214}$$

$$y_p = -\sin x \tag{2.7.215}$$

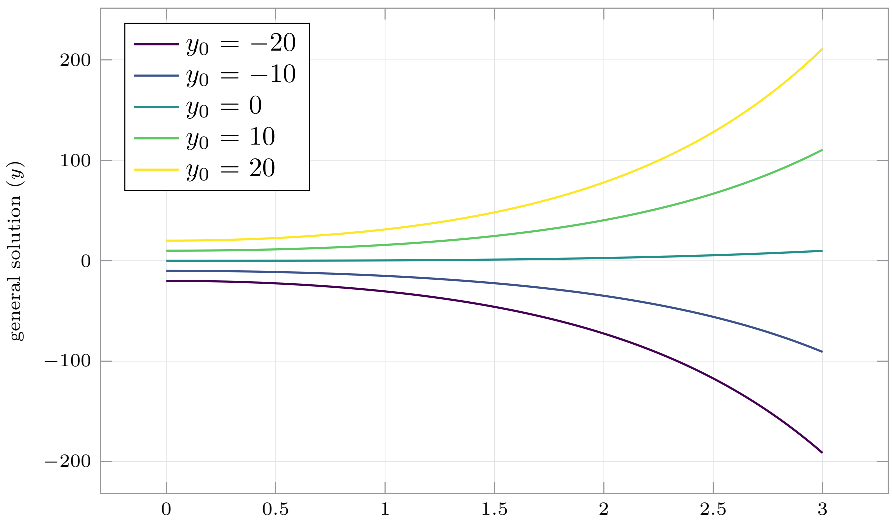
$$y = 0.5(y_0 - v_0 - 1)e^{-x} \tag{2.7.216}$$

$$+\, 0.5(y_0 + v_0 + 1)e^{x} - \sin x \tag{2.7.217}$$

$y_0 = 0, \;\; v_0 = [0 : 2 : 10]$



$v_0 = 0, \;\; y_0 = [-20 : 10 : 20]$



The other 2 plots are not shown because of the negligible impact of y_p on the general solution.

(d) For the y_h part of the solution to not be present in y, back-calculate the IC so that the coefficients of y_h turn out to be zero.

$$y'' + 3y' + 2y = 10 \tag{2.7.218}$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x} \tag{2.7.219}$$

$$y_p = 5 \tag{2.7.220}$$

$$y = -(y_0 + v_0 - 5)e^{-2x} + (2y_0 + v_0 - 10)e^{-x} + 5 \tag{2.7.221}$$

$$y_0 + v_0 = 5, \qquad 2y_0 + v_0 = 10 \tag{2.7.222}$$

$$y_0 = 5, \qquad v_0 = 0 \tag{2.7.223}$$

$$y = y_p = 5 \tag{2.7.224}$$

This graph has been plotted above.

(e) Modification rule being used for simple and double root. TBC

20. Extending the method of undetermined coefficients to products of functions in the table as $r(x)$, the guess also becomes a product of the corresponding guess functions from the table. The rule of modifying guesses to account for overlap with solutions of the h-ODE still applies.

For Euler-Cauchy equations, the transformation $x = e^t, \ x > 0$, which leads to an ODE with constant coefficients. The modifier for a single or double root matching the guess function is $\ln x$ and $\ln^2 x$ respectively.

In the table, P_n is a polynomial of degree n. The reverse transformation to construct this table is

$$e^{\lambda x} \to x^{\lambda} \tag{2.7.225}$$

$$\omega x \to \beta \ln x \tag{2.7.226}$$

$$x \to \ln x \tag{2.7.227}$$

RHS $\mathbf{r(x)}$	Guess $\mathbf{y_p(x)}$
kx^{λ}	Cx^{λ}
$kx^{\lambda} \cdot P_n(\ln x) \quad n \in \mathcal{N}$	$x^{\lambda} \cdot Q_n(\ln x)$
$k \cos(\beta \ln x)$ $k \sin(\beta \ln x)$	$K \cos(\beta \ln x) + M \sin(\beta \ln x)$
$kx^{\lambda} \cos(\beta \ln x)$ $kx^{\lambda} \sin(\beta \ln x)$	$x^{\lambda} [\cos(\beta \ln x) + M \sin(\beta \ln x)]$

2.8 Modeling: Forced Oscillations, Resonance

1. Refer to chapter notes.

2. Spring mass systems with harmonic oscillations as steady state solutions. y_p has to be sinusoidal and y_h has to decay to zero.

From Problem set 2.7, Problems - 2, 4, 10, 14, 18.

This includes problems where the steady state solution does not have constant amplitude.

3. Solving the h-ODE,

$$0 = y'' + 6y' + 8y \qquad a = 6, \quad b = 8 \tag{2.8.1}$$

$$\lambda_1, \lambda_2 = \frac{-6 \pm \sqrt{4}}{2} \qquad = -2, -4 \tag{2.8.2}$$

$$y_h = c_1 e^{-2x} + c_2 e^{-4x} \tag{2.8.3}$$

Solving the nh-ODE,

$$y'' + 6y + 8y = 42.5\cos(2x) \tag{2.8.4}$$

$$y_p = K\cos(2x) + M\sin(2x) \qquad \text{basic rule} \tag{2.8.5}$$

$$42.5 = -4K + 12M + 8K \qquad \cdots\cdots[\cos(2x)] \tag{2.8.6}$$

$$0 = -4M - 12K + 8M \qquad \cdots\cdots[\sin(2x)] \tag{2.8.7}$$

$$K = \frac{17}{16} \qquad M = \frac{51}{16} \tag{2.8.8}$$

$$y_p = \frac{17}{16}[\cos(2x) + 3\sin(2x)] \tag{2.8.9}$$

$$y = y_p + y_h \tag{2.8.10}$$

4. Solving the h-ODE,

$$0 = y'' + 2.5y' + 10y \qquad a = 2.5, \quad b = 10 \tag{2.8.11}$$

$$\lambda_1, \lambda_2 = \frac{-2.5 \pm i\,5.809}{2} \qquad = -1.25 \pm i\,2.9045 \tag{2.8.12}$$

$$y_h = e^{-1.25x}[c_1\cos(2.9045x) + c_2\sin(2.9045x)] \tag{2.8.13}$$

Solving the nh-ODE,

$$y'' + 2.5y + 10y = -13.6\sin(4x) \tag{2.8.14}$$

$$y_p = K\cos(4x) + M\sin(4x) \qquad \text{basic rule} \tag{2.8.15}$$

$$0 = -16K + 10M + 10K \qquad \cdots\cdots[\cos(2x)] \tag{2.8.16}$$

$$-13.6 = -16M - 10K + 10M \qquad \cdots\cdots[\sin(2x)] \tag{2.8.17}$$

$$K = 1 \qquad M = 0.6 \tag{2.8.18}$$

$$y_p = \cos(4x) + 0.6\sin(4x) \tag{2.8.19}$$

$$y = y_p + y_h \tag{2.8.20}$$

5. Solving the h-ODE,

$$0 = y'' + y' + 4.25y \qquad a = 1, \quad b = 4.25 \tag{2.8.21}$$

$$\lambda_1, \lambda_2 = \frac{-1 \pm 4i}{2} \qquad = -0.5 \pm 2i \tag{2.8.22}$$

$$y_h = e^{-0.5x}[c_1\cos(2x) + c_2\sin(2x)] \tag{2.8.23}$$

Solving the nh-ODE,

$$y'' + y' + 4.25y = 22.1\cos(4.5x) \tag{2.8.24}$$

$$y_p = K\cos(4.5x) + M\sin(4.5x) \qquad \text{basic rule} \tag{2.8.25}$$

$$22.1 = -20.25K + 4.5M + 4.25K \qquad \cdots\cdots[\cos(4.5x)] \tag{2.8.26}$$

$$0 = -20.25M - 4.5K + 4.25M \qquad \cdots\cdots[\sin(4.5x)] \tag{2.8.27}$$

$$K = -1.28 \qquad M = 0.36 \tag{2.8.28}$$

$$y_p = -1.28\cos(4.5x) + 0.36\sin(4.5x) \tag{2.8.29}$$

$$y = y_p + y_h \tag{2.8.30}$$

6. Solving the h-ODE,

$$0 = y'' + 4y' + 3y \qquad a = 4, \quad b = 3 \tag{2.8.31}$$

$$\lambda_1, \lambda_2 = \frac{-4 \pm \sqrt{4}}{2} \qquad = -3, -1 \tag{2.8.32}$$

$$y_h = c_1 e^{-3x} + c_2 e^{-x} \tag{2.8.33}$$

Solving the nh-ODE,

$$y'' + 4y' + 3y = \cos x + \frac{1}{3}\cos(3x) \tag{2.8.34}$$

$$y_p = K_1 \cos(x) + K_2 \sin(x) + M_1 \cos(3x) + M_2 \sin(3x) \tag{2.8.35}$$

$$1 = -K_1 + 4K_2 + 3K_1 \qquad \cdots\cdots[\cos(x)] \tag{2.8.36}$$

$$0 = -K_2 - 4K_1 + 3K_2 \qquad \cdots\cdots[\sin(x)] \tag{2.8.37}$$

$$1/3 = -9M_1 + 12M_2 + 3M_1 \qquad \cdots\cdots[\cos(3x)] \tag{2.8.38}$$

$$0 = -9M_2 - 12M_1 + 3M_2 \qquad \cdots\cdots[\sin(x)] \tag{2.8.39}$$

$$K_1 = 0.1, \qquad K_2 = 0.2, \qquad M_1 = -1/90, \qquad M_2 = 1/45 \tag{2.8.40}$$

$$y_p = \frac{1}{10}\cos(x) + \frac{1}{5}\sin(x) + \frac{-1}{90}\cos(3x) + \frac{1}{45}\sin(3x) \tag{2.8.41}$$

$$y = y_p + y_h \tag{2.8.42}$$

7. Solving the h-ODE,

$$0 = 4y'' + 12y' + 9y \qquad a = 3, \quad b = 9/4 \tag{2.8.43}$$

$$\lambda_1, \lambda_2 = \frac{-3 \pm \sqrt{0}}{2} \qquad = -1.5, -1.5 \tag{2.8.44}$$

$$y_h = (c_1 + c_2 x)e^{-1.5x} \tag{2.8.45}$$

Solving the nh-ODE,

$$4y'' + 12y' + 9y = 225 - 75\sin(3x) \tag{2.8.46}$$

$$y_p = K_1 \cos(3x) + K_2 \sin(3x) + M_0 \tag{2.8.47}$$

$$0 = -36K_1 + 36K_2 + 9K_1 \qquad \cdots\cdots[\cos(3x)] \tag{2.8.48}$$

$$-75 = -36K_2 - 36K_1 + 9K_2 \qquad \cdots\cdots[\sin(3x)] \tag{2.8.49}$$

$$[x^0] \qquad 225 = 9M_0 \tag{2.8.50}$$

$$M_0 = 25, \qquad K_1 = 4/3, \qquad K_2 = 1 \tag{2.8.51}$$

$$y_p = \frac{4}{3}\cos(3x) + \sin(3x) + 25 \tag{2.8.52}$$

$$y = y_p + y_h \tag{2.8.53}$$

8. Solving the h-ODE,

$$0 = 2y'' + 4y' + 6.5y \qquad a = 2, \quad b = 13/4 \tag{2.8.54}$$

$$\lambda_1, \lambda_2 = \frac{-2 \pm i\sqrt{9}}{2} \qquad = -1 \pm 1.5i \tag{2.8.55}$$

$$y_h = [c_1 \sin(1.5x) + c_2 \cos(1.5x)]e^{-x} \tag{2.8.56}$$

Solving the nh-ODE,

$$2y'' + 4y' + 6.5y = 4 \sin(1.5x) \tag{2.8.57}$$

$$y_p = K \cos(1.5x) + M \sin(1.5x) \tag{2.8.58}$$

$$0 = -4.5K + 6M + 6.5K \qquad \cdots\cdots[\cos(1.5x)] \tag{2.8.59}$$

$$4 = -4.5M - 6K + 6.5M \qquad \cdots\cdots[\sin(1.5x)] \tag{2.8.60}$$

$$K = 0.2, \qquad M = -0.6 \tag{2.8.61}$$

$$y_p = \frac{1}{5} \cos(1.5x) + \frac{-3}{5} \sin(1.5x) \tag{2.8.62}$$

$$y = y_p + y_h \tag{2.8.63}$$

9. Solving the h-ODE,

$$0 = y'' + 3y' + 3.25y \qquad a = 3, \quad b = 3.25 \tag{2.8.64}$$

$$\lambda_1, \lambda_2 = \frac{-3 \pm i\sqrt{4}}{2} \qquad = -1.5 \pm i \tag{2.8.65}$$

$$y_h = [c_1 \sin(x) + c_2 \cos(x)]e^{-1.5x} \tag{2.8.66}$$

Solving the nh-ODE,

$$y'' + 3y' + 3.25y = 3 \cos x - 1.5 \sin x \tag{2.8.67}$$

$$y_p = K \cos x + M \sin x \tag{2.8.68}$$

$$3 = -K + 3M + 3.25K \qquad \cdots\cdots[\cos(x)] \tag{2.8.69}$$

$$-1.5 = -M - 3K + 3.25M \qquad \cdots\cdots[\sin(x)] \tag{2.8.70}$$

$$K = 0.8, \qquad M = 0.4 \tag{2.8.71}$$

$$y_p = \frac{4}{5} \cos x + \frac{2}{5} \sin x \tag{2.8.72}$$

$$y = y_p + y_h \tag{2.8.73}$$

10. Solving the h-ODE,

$$0 = y'' + 16y \qquad a = 0, \quad b = 16 \tag{2.8.74}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{64}}{2} \qquad = 0 \pm 4i \tag{2.8.75}$$

$$y_h = c_1 \sin(4x) + c_2 \cos(4x) \tag{2.8.76}$$

Solving the nh-ODE,

$$y'' + 16y = 56\cos(4x) \tag{2.8.77}$$

$$y_p = Kx\cos(4x) + Mx\sin(4x) \tag{2.8.78}$$

$$y_p' = [K + 4Mx]\cos(4x) + [M - 4Kx]\sin(4x) \tag{2.8.79}$$

$$y_p'' = [8M - 16Kx]\cos(4x) + [-8K - 16Mx]\sin(4x) \tag{2.8.80}$$

$$56 = 8M \qquad \cdots\cdots[\cos(4x)] \tag{2.8.81}$$

$$0 = -8K \qquad \cdots\cdots[\sin(4x)] \tag{2.8.82}$$

$$K = 0, \qquad M = 7 \tag{2.8.83}$$

$$y_p = 7x\sin(4x) \tag{2.8.84}$$

$$y = y_p + y_h \tag{2.8.85}$$

11. Solving the h-ODE,

$$0 = y'' + 2y \qquad a = 0, \quad b = 2 \tag{2.8.86}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{8}}{2} \qquad = 0 \pm \sqrt{2}i \tag{2.8.87}$$

$$y_h = c_1 \sin(\sqrt{2}x) + c_2 \cos(\sqrt{2}x) \tag{2.8.88}$$

Solving the nh-ODE,

$$y'' + 2y = \cos(\sqrt{2}x) + \sin(\sqrt{2}x) \quad 2.8.89$$

$$y_p = Kx\cos(\sqrt{2}x) + Mx\sin(\sqrt{2}x) \quad 2.8.90$$

$$y_p' = [K + \sqrt{2}Mx]\cos(\sqrt{2}x) + [M - \sqrt{2}Kx]\sin(\sqrt{2}x) \quad 2.8.91$$

$$y_p'' = [2\sqrt{2}M - 2Kx]\cos(\sqrt{2}x) + [-2\sqrt{2}K - 2Mx]\sin(\sqrt{2}x) \quad 2.8.92$$

$$1 = 2\sqrt{2}M \qquad \cdots\cdots[\cos(\sqrt{2}x)] \quad 2.8.93$$

$$1 = -2\sqrt{2}K \qquad \cdots\cdots[\cos(\sqrt{2}x)] \quad 2.8.94$$

$$K = \frac{-1}{2\sqrt{2}}, \qquad M = \frac{1}{2\sqrt{2}} \quad 2.8.95$$

$$y_p = \frac{-x\cos(\sqrt{2}x) + x\sin(\sqrt{2}x)}{2\sqrt{2}} \quad 2.8.96$$

$$y = y_p + y_h \quad 2.8.97$$

12. Solving the h-ODE,

$$0 = y'' + 2y' + 5y \qquad a = 2, \quad b = 5 \quad 2.8.98$$

$$\lambda_1, \lambda_2 = \frac{-2 \pm i\sqrt{16}}{2} \qquad = -1 \pm 2i \quad 2.8.99$$

$$y_h = [c_1\sin(2x) + c_2\cos(2x)]e^{-x} \quad 2.8.100$$

Solving the nh-ODE,

$$y'' + 2y' + 5y = 4\cos x + 8\sin x \quad 2.8.101$$

$$y_p = K\cos x + M\sin x \quad 2.8.102$$

$$4 = -K + 2M + 5K \qquad \cdots\cdots[\cos x] \quad 2.8.103$$

$$8 = -M - 2K + 5M \qquad \cdots\cdots[\sin x] \quad 2.8.104$$

$$K = 0, \qquad M = 2 \quad 2.8.105$$

$$y_p = 2\sin x \quad 2.8.106$$

$$y = y_p + y_h \quad 2.8.107$$

13. Solving the h-ODE,

$$0 = y'' + y \qquad a = 0, \quad b = 1 \tag{2.8.108}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{4}}{2} \qquad = 0 \pm i \tag{2.8.109}$$

$$y_h = c_1 \sin(x) + c_2 \cos(x) \tag{2.8.110}$$

Solving the nh-ODE,

$$y'' + y = \cos(\omega x) \tag{2.8.111}$$

$$y_p = K \cos(\omega x) + M \sin(\omega x) \tag{2.8.112}$$

$$1 = -\omega^2 K + K \qquad \cdots\cdots[\cos(\omega x)] \tag{2.8.113}$$

$$0 = -\omega^2 M + M \qquad \cdots\cdots[\sin(\omega x)] \tag{2.8.114}$$

$$K = \frac{1}{1-\omega^2}, \qquad M = 0 \quad (\text{since } \omega^2 \neq 1) \tag{2.8.115}$$

$$y_p = \frac{1}{1-\omega^2} \cos(\omega x) \tag{2.8.116}$$

$$y = y_p + y_h \tag{2.8.117}$$

14. Solving the h-ODE,

$$0 = y'' + y \qquad a = 0, \quad b = 1 \tag{2.8.118}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{4}}{2} \qquad = 0 \pm i \tag{2.8.119}$$

$$y_h = c_1 \sin(x) + c_2 \cos(x) \tag{2.8.120}$$

Solving the nh-ODE,

$$y'' + y = 5e^{-x}\cos x \tag{2.8.121}$$

$$y_p = e^{-x}[K\cos x + M\sin x] \tag{2.8.122}$$

$$y_p' = e^{-x}[(-K+M)\cos x + (-M-K)\sin x] \tag{2.8.123}$$

$$y_p'' = e^{-x}[(-2M)\cos x + (2K)\sin x] \tag{2.8.124}$$

$$5 = -2M + K \qquad \cdots\cdots[e^{-x}\cos x] \tag{2.8.125}$$

$$0 = 2K + M \qquad \cdots\cdots[e^{-x}\sin x] \tag{2.8.126}$$

$$K = 1, \qquad M = -2 \tag{2.8.127}$$

$$y_p = e^{-x}[\cos x - 2\sin x] \tag{2.8.128}$$

$$y = y_p + y_h \tag{2.8.129}$$

15. Solving the h-ODE,

$$0 = y'' + 4y' + 8y \qquad a = 4, \quad b = 8 \tag{2.8.130}$$

$$\lambda_1, \lambda_2 = \frac{-4 \pm i\sqrt{16}}{2} \qquad = -2 \pm 2i \tag{2.8.131}$$

$$y_h = [c_1\sin(2x) + c_2\cos(2x)]e^{-2x} \tag{2.8.132}$$

Solving the nh-ODE,

$$y'' + 4y' + 8y = 2\cos(2x) + \sin(2x) \tag{2.8.133}$$

$$y_p = K\cos(2x) + M\sin(2x) \tag{2.8.134}$$

$$2 = -4K + 8M + 8K \qquad \cdots\cdots[\cos(2x)] \tag{2.8.135}$$

$$1 = -4M - 8K + 8M \qquad \cdots\cdots[\sin(2x)] \tag{2.8.136}$$

$$K = 0, \qquad M = 1/4 \tag{2.8.137}$$

$$y_p = 0.25\sin(2x) \tag{2.8.138}$$

$$y = y_p + y_h \tag{2.8.139}$$

16. Solving the h-ODE,

$$0 = y'' + 25y \qquad a = 0, \quad b = 25 \tag{2.8.140}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{100}}{2} \qquad = 0 \pm 5i \tag{2.8.141}$$

$$y_h = c_1 \cos(5x) + c_2 \sin(5x) \tag{2.8.142}$$

Solving the nh-ODE, with IC $y(0) = 1$, $y'(0) = 1$,

$$y'' + 25y = 24 \sin x \tag{2.8.143}$$

$$y_p = K \cos(x) + M \sin(x) \tag{2.8.144}$$

$$0 = -K + 25K \qquad \cdots\cdots [\cos(x)] \tag{2.8.145}$$

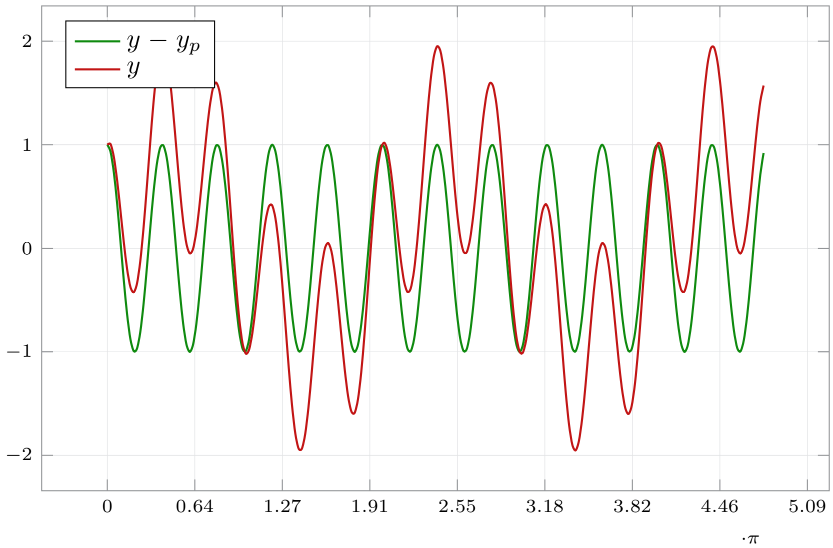
$$24 = -M + 25M \qquad \cdots\cdots [\sin(x)] \tag{2.8.146}$$

$$K = 0, \qquad M = 1 \tag{2.8.147}$$

$$y_p = \sin(x) \tag{2.8.148}$$

$$y(0) = 1 = c_1 \qquad y'(0) = 1 = 5c_2 + 1 \tag{2.8.149}$$

$$y = \sin x + \cos(5x) \tag{2.8.150}$$



17. Solving the h-ODE,

$$0 = y'' + 4y \qquad a = 0, \quad b = 4 \tag{2.8.151}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{16}}{2} \qquad = 0 \pm 2i \tag{2.8.152}$$

$$y_h = c_1 \cos(2x) + c_2 \sin(2x) \tag{2.8.153}$$

Solving the nh-ODE, with IC $y(0) = 0$, $y'(0) = \frac{3}{35}$,

$$y'' + 4y = \sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) \tag{2.8.154}$$

$$y_p = M_1 \sin x + M_2 \sin(3x) + M_3 \sin(5x) \tag{2.8.155}$$

$$1 = -M_1 + 4M_1 \qquad \cdots\cdots[\sin(x)] \tag{2.8.156}$$

$$1/3 = -9M_2 + 4M_2 \qquad \cdots\cdots[\sin(3x)] \tag{2.8.157}$$

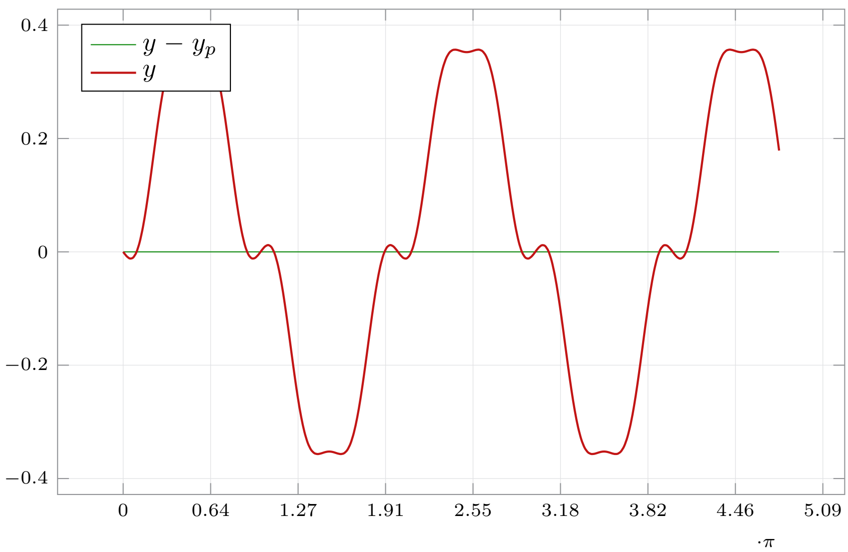
$$1/5 = -25M_3 + 4M_3 \qquad \cdots\cdots[\sin(5x)] \tag{2.8.158}$$

$$M_1 = 1/3 \qquad M_2 = -1/15 \qquad M_3 = -1/105 \tag{2.8.159}$$

$$y_p = \frac{1}{3}\sin(x) - \frac{1}{15}\sin(3x) - \frac{1}{105}\sin(5x) \tag{2.8.160}$$

$$y(0) = 0 = c_1 \qquad y'(0) = \frac{3}{35} = 2c_2 + \frac{3}{35} \tag{2.8.161}$$

$$y = \frac{1}{3}\sin(x) - \frac{1}{15}\sin(3x) - \frac{1}{105}\sin(5x) + 0 \tag{2.8.162}$$



18. Solving the h-ODE,

$$0 = y'' + 8y' + 17y \qquad a = 8, \quad b = 17 \tag{2.8.163}$$

$$\lambda_1, \lambda_2 = \frac{-8 \pm i\sqrt{4}}{2} \qquad = -4 \pm i \tag{2.8.164}$$

$$y_h = [c_1 \cos(x) + c_2 \sin(x)]e^{-4x} \tag{2.8.165}$$

Solving the nh-ODE, with IC $y(0) = -5.4$, $y'(0) = 9.4$,

$$y'' + 8y' + 17y = 474.5 \sin(0.5x) \tag{2.8.166}$$

$$y_p = K \cos(0.5x) + M \sin(0.5x) \tag{2.8.167}$$

$$0 = -0.25K + 4M + 17K \qquad \cdots\cdots[\cos(0.5x)] \tag{2.8.168}$$

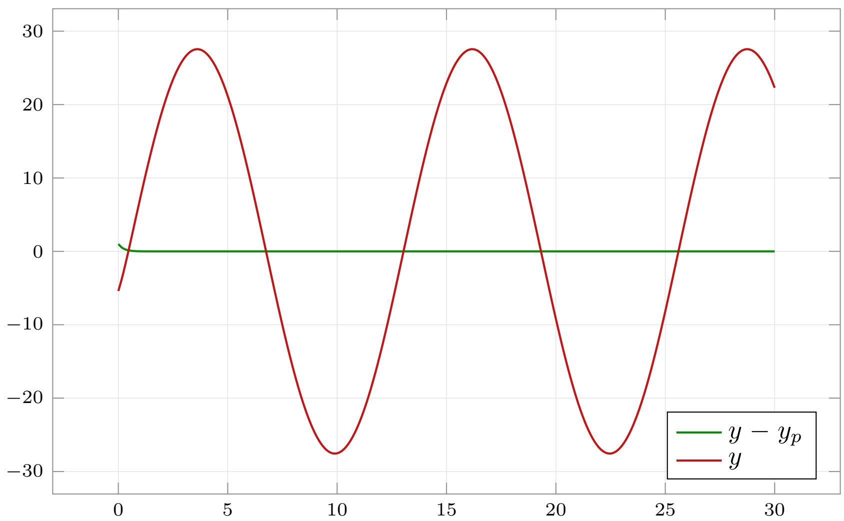
$$474.5 = -0.25M - 4K + 17M \qquad \cdots\cdots[\sin(0.5x)] \tag{2.8.169}$$

$$K = \frac{-32}{5} \qquad M = \frac{134}{5} \tag{2.8.170}$$

$$y_p = \frac{-32 \cos(0.5x) + 134 \sin(0.5x)}{5} \tag{2.8.171}$$

$$y(0) = -5.4 = c_1 - 6.4 \qquad y'(0) = -4 = (-4c_1 + c_2) \tag{2.8.172}$$

$$y = \frac{-32 \cos(0.5x) + 134 \sin(0.5x)}{5} + e^{-4x} \cos x \tag{2.8.173}$$



19. Solving the h-ODE,

$$0 = y'' + 2y' + 2y \qquad a = 2, \quad b = 2 \tag{2.8.174}$$

$$\lambda_1, \lambda_2 = \frac{-2 \pm i\sqrt{4}}{2} \qquad = -1 \pm i \tag{2.8.175}$$

$$y_h = [c_1 \cos(x) + c_2 \sin(x)]e^{-x} \tag{2.8.176}$$

Solving the nh-ODE,

$$y'' + 2y' + 2y = e^{-0.5x} \sin(0.5x) \tag{2.8.177}$$

$$y_p = e^{-0.5x}[K \cos(0.5x) + M \sin(0.5x)] \tag{2.8.178}$$

$$y_p' = e^{-0.5x}[(-0.5K + 0.5M) \cos(0.5x) \tag{2.8.179}$$

$$+ (-0.5M - 0.5K) \sin(0.5x)] \tag{2.8.180}$$

$$y_p'' = e^{-0.5x}[(-0.5M) \cos(0.5x) + (0.5K) \sin(0.5x)] \tag{2.8.181}$$

$$0 = -0.5M - K + M + 2K \qquad \cdots\cdots[e^{-0.5x} \cos(0.5x)] \tag{2.8.182}$$

$$1 = 0.5K - M - K + 2M \qquad \cdots\cdots[e^{-0.5x} \sin(0.5x)] \tag{2.8.183}$$

$$K = \frac{-2}{5} \qquad M = \frac{4}{5} \tag{2.8.184}$$

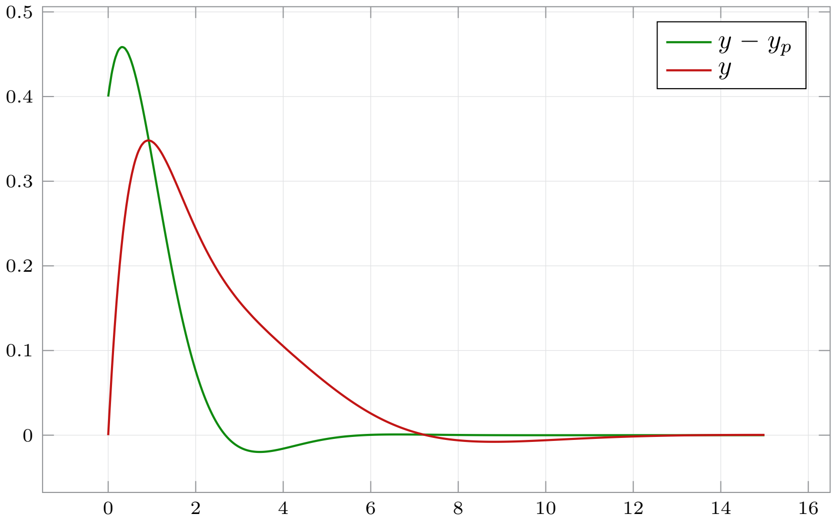
$$y_p = \left[\frac{-2 \cos(0.5x) + 4 \sin(0.5x)}{5}\right] e^{-0.5x} \tag{2.8.185}$$

Solving the IC, $y(0) = 0$, $y'(0) = 1$,

$$y(0) = 0 = c_1 - 0.4 \tag{2.8.186}$$

$$y'(0) = 1 = (-c_1 + c_2) + 0.2 + 0.4 \tag{2.8.187}$$

$$y = e^{-0.5x}[-0.4 \cos(0.5x) + 0.8 \sin(0.5x)] + e^{-x}[0.4 \cos x + 0.8 \sin x] \tag{2.8.188}$$



20. Solving the h-ODE,

$$0 = y'' + 5y \qquad a = 0, \quad b = 5 \tag{2.8.189}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{20}}{2} \qquad = 0 \pm i\sqrt{5} \tag{2.8.190}$$

$$y_h = c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x) \tag{2.8.191}$$

Solving the nh-ODE,

$$y'' + 5y = \cos(\pi x) - \sin(\pi x) \tag{2.8.192}$$

$$y_p = K \cos(\pi x) + M \sin(\pi x) \tag{2.8.193}$$

$$1 = -\pi^2 K + 5K \qquad \cdots\cdots[\cos(\pi x)] \tag{2.8.194}$$

$$-1 = -\pi^2 M + 5M \qquad \cdots\cdots[\sin(\pi x)] \tag{2.8.195}$$

$$K = \frac{-1}{\pi^2 - 5} \qquad M = \frac{1}{\pi^2 - 5} \tag{2.8.196}$$

$$y_p = \frac{-\cos(\pi x) + \sin(\pi x)}{\pi^2 - 5} \tag{2.8.197}$$

Solving the IC, $y(0) = 0$, $y'(0) = 0$,

$$y(0) = 0 = c_1 - \frac{1}{\pi^2 - 5} \tag{2.8.198}$$

$$y'(0) = 0 = \sqrt{5}c_2 + \frac{\pi}{\pi^2 - 5} \tag{2.8.199}$$

$$y = \frac{-\cos(\pi x) + \sin(\pi x)}{\pi^2 - 5} + \frac{\cos(\sqrt{5}x) - (\pi/\sqrt{5})\sin(\sqrt{5}x)}{\pi^2 - 5} \tag{2.8.200}$$

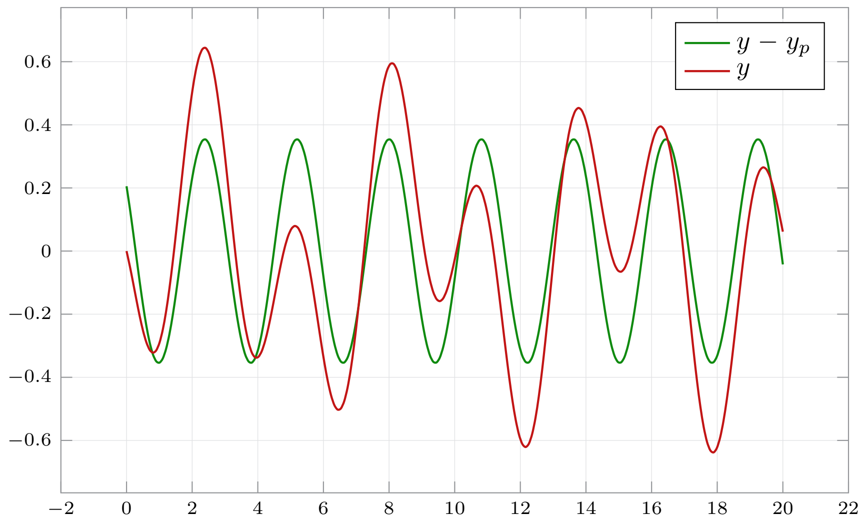


Figure does not agree with $y'(0) = 0$, TBC investigate

21. Beats formula,

$$y = Y_0[\cos(\omega t) - \cos(\omega_0 t)] \tag{2.8.201}$$

$$= Y_0\left[\cos\left(\frac{\omega t + \omega_0 t}{2} + \frac{\omega t - \omega_0 t}{2}\right) - \cos\left(\frac{\omega t + \omega_0 t}{2} - \frac{\omega t - \omega_0 t}{2}\right)\right] \tag{2.8.202}$$

$$= Y_0[\cos(a + b) - \cos(a - b)] \tag{2.8.203}$$

$$= Y_0[-2\sin a \sin b] = -2Y_0\left[\sin\left(\frac{\omega + \omega_0}{2}\,t\right)\sin\left(\frac{\omega - \omega_0}{2}\,t\right)\right] \tag{2.8.204}$$

$$= 2Y_0\left[\sin\left(\frac{\omega_0 + \omega}{2}\,t\right)\sin\left(\frac{\omega_0 - \omega}{2}\,t\right)\right] \tag{2.8.205}$$

In a damped forced harmonic oscillator, $a > 0$, and the solutions have an exponential decay multiplier $\exp(-ax/2)$. Thus, beats will quickly die out along with the rest of the transient response y_h leaving only the steady state response y_p remaining.

No damping, implies $a = 0$ and the beats persist to the point of being noticeable.

22. Solving the h-ODE,

$$0 = y'' + 25y \qquad a = 0, \quad b = 25 \tag{2.8.206}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{100}}{2} \qquad = 0 \pm 5i \tag{2.8.207}$$

$$y_h = c_1\cos(5x) + c_2\sin(5x) \tag{2.8.208}$$

Solving the nh-ODE,

$$y'' + 25y = 99\ \cos(4.9x) \tag{2.8.209}$$

$$y_p = K\ \cos(4.9x) \tag{2.8.210}$$

$$99 = -4.9^2 K + 25K \qquad \cdots\cdots[\cos(4.9x)] \tag{2.8.211}$$

$$K = 100.0 \qquad M = 0 \tag{2.8.212}$$

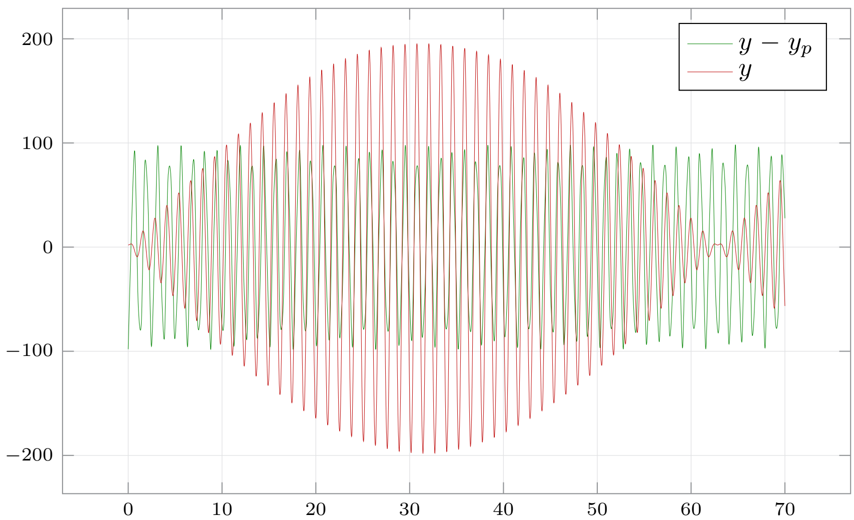
$$y_p = 100\ \cos(4.9x) \tag{2.8.213}$$

Solving the IC, $y(0) = 2$, $y'(0) = 0$,

$$y(0) = 2 = c_1 + 100 \tag{2.8.214}$$

$$y'(0) = 0 = 5c_2 \tag{2.8.215}$$

$$y = 100\ \cos(4.9x) - 98\ \cos(5x) \tag{2.8.216}$$

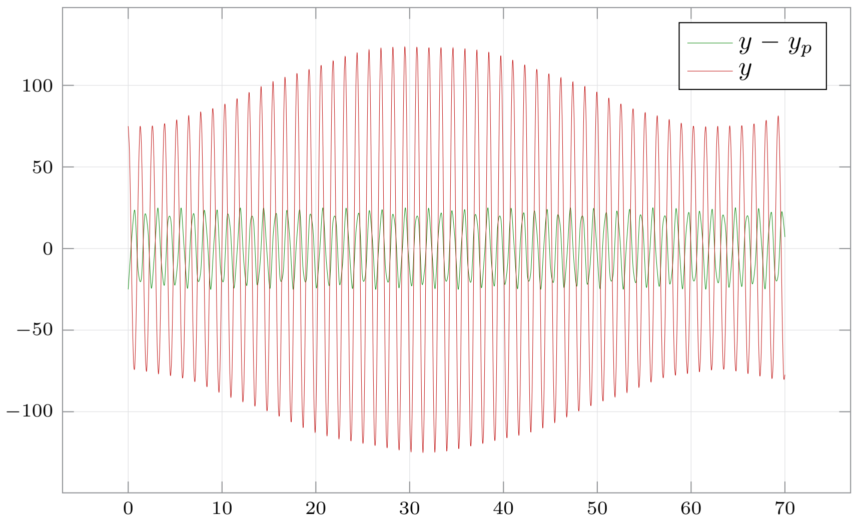


Solving the changed IC, $y(0) = 75$, $y'(0) = 0$,

$$y(0) = 75 = c_1 + 100 \tag{2.8.217}$$

$$y'(0) = 0 = 5c_2 \tag{2.8.218}$$

$$y = 100\ \cos(4.9x) - 25\ \cos(5x) \tag{2.8.219}$$



Changing the frequency of the input to move it closer to the natural frequency, makes the envelope frequency $(\omega_0 - \omega)/2$ smaller, and the envelope time period larger.

23. (a) Deriving the maximum amplitude as a function of input frequency,

$$C^* = \frac{F_0}{\sqrt{[m(\omega_0^2 - \omega^2)]^2 + [\omega c]^2}} = \frac{F_0}{\sqrt{R(\omega)}} \quad 2.8.220$$

$$\frac{\mathrm{d}C^*}{\mathrm{d}\omega} = \frac{-F_0 R^{-3/2}}{2} \frac{\mathrm{d}R}{\mathrm{d}\omega} \quad 2.8.221$$

$$= \frac{-F_0 R^{-3/2}}{2}[2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2] \quad 2.8.222$$

$$\frac{\mathrm{d}C^*}{\mathrm{d}\omega} = 0 \implies c^2 = 2m^2(\omega_0^2 - \omega^2) \quad 2.8.223$$

$$\omega^2 = \omega_0^2 - \frac{c^2}{2m^2} = \frac{2mk - c^2}{2m^2} \quad 2.8.224$$

$$2.8.225$$

For $c^2 > 2mk$, no real solution exists for $\mathrm{d}C^*/\mathrm{d}\omega = 0$, and there is no extremum of $C(\omega)$, and thus no peak in the function.
For $c^2 < 2mk$,

$$\omega_{\text{max}}^2 = \omega_0^2 - \frac{c^2}{2m^2} \quad 2.8.226$$

$$C^*(\omega_{\text{max}}) = F_0 \left[c^2\omega_0^2 - \frac{c^4}{4m^2}\right]^{-1/2} \quad 2.8.227$$

$$= \frac{2mF_0}{c}\,[4m^2\omega_0^2 - c^2]^{-1/2} \quad 2.8.228$$

(b) Looking at the variation of C^* with c,

$$S(c) = Pc^2 + Q, \qquad P = \omega^2, \qquad Q = m^2(\omega_0^2 - \omega^2)^2 \tag{2.8.229}$$

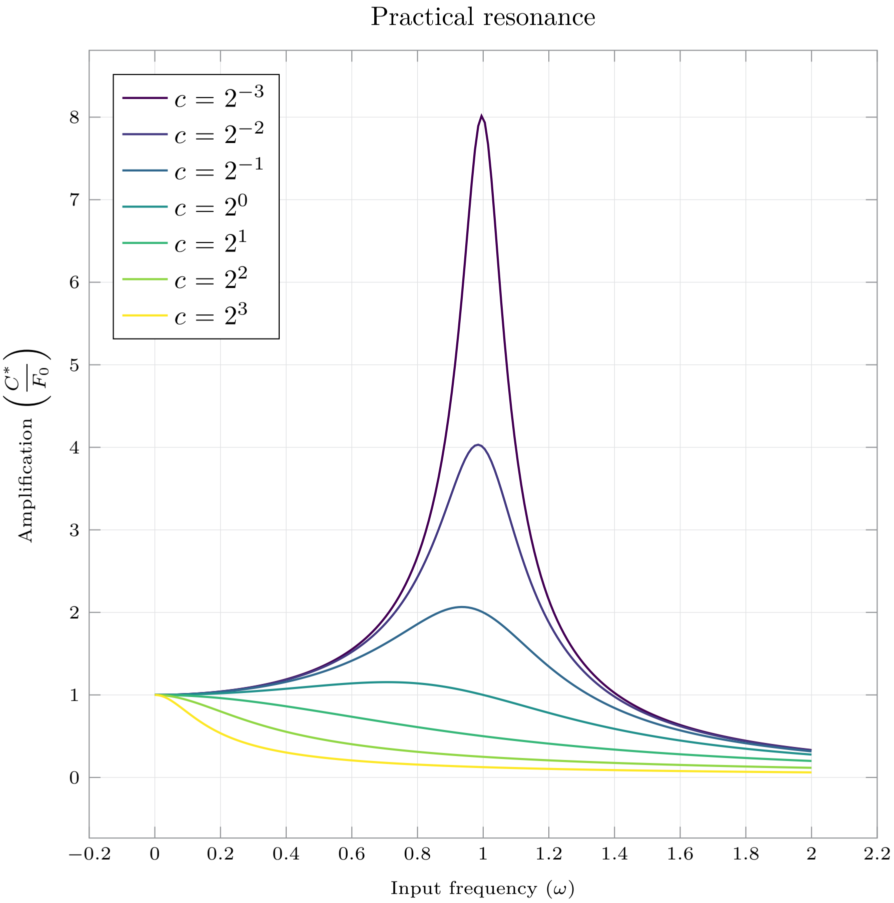
$$\frac{\mathrm{d}C^*}{\mathrm{d}c} = \frac{-F_0}{2}[S(c)]^{-3/2}\,\frac{\mathrm{d}S}{\mathrm{d}c} \tag{2.8.230}$$

$$= \frac{-F_0}{2}[Pc^2 + Q]^{-3/2}\;2Pc \tag{2.8.231}$$

$$\tag{2.8.232}$$

For $P, Q > 0$, $S(c)$ is always positive, which means that $\mathrm{d}C^*/\mathrm{d}c$ is negative and C^* increases for decreasing c.

(c) Graphing the variation of C^* vs ω for differing values of c,
Dummy values are $m = 1$, $k = 1$, $\omega_0 = 1$



(d) Solving the h-ODE, with $m = k = 1$, $c = 0.25$,

$$\omega_{\max} = \sqrt{\left[1 - \frac{c^2}{2}\right]} = 0.9843 \text{ Hz} \tag{2.8.233}$$

$$0 = y'' + 0.25y' + y \qquad a = 0.25, \quad b = 1 \tag{2.8.234}$$

$$\lambda_1, \lambda_2 = \frac{-.25 \pm i\sqrt{3.9375}}{2} = -0.125 \pm 0.9922i \tag{2.8.235}$$

$$y_h = [c_1 \cos(0.9922x) + c_2 \sin(0.9922x)]e^{-0.125x} \tag{2.8.236}$$

Solving the nh-ODE,

$$y'' + 0.25y' + y = \cos(3x) + \cos(0.984x) \tag{2.8.237}$$

$$y_p = 3.9985 \sin(0.984x) + 0.516 \cos(0.984x) \tag{2.8.238}$$

$$+ 0.0116 \sin(3x) - 0.124 \cos(3x) \tag{2.8.239}$$

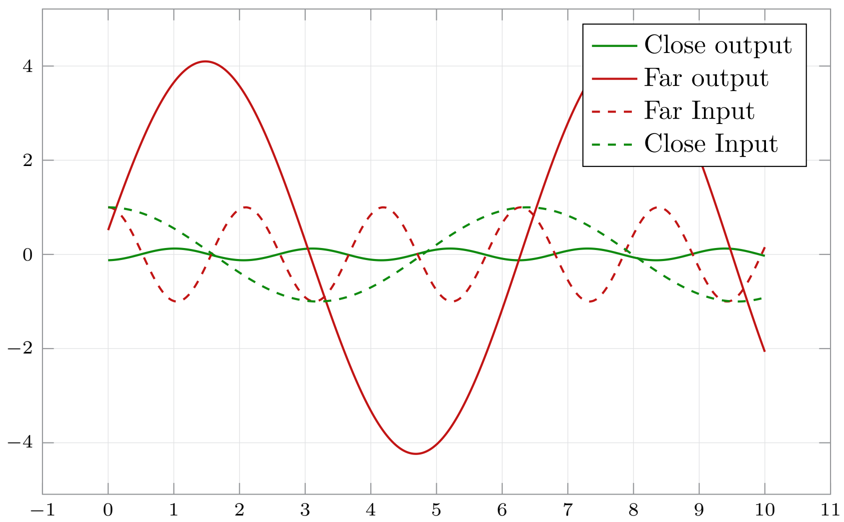
Solving the IC, $y(0) = 2$, $y'(0) = 0$,

$$y(0) = 2 = c_1 + 100 \tag{2.8.240}$$

$$y'(0) = 0 = 5c_2 \tag{2.8.241}$$

$$y = y_p + y_h \tag{2.8.242}$$

The response amplification is much larger for the input frequency close to the practical resonance frequency than the other input farther away, as seen in the coefficients of the output function.



From the plot, the amplification of the inputs is clearly different.

(e) TBC.

24. Piecewise continuous forcing function, Solving the h-ODE,

$$0 = y'' + y \qquad a = 0, \quad b = 1 \tag{2.8.243}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{4}}{2} \qquad = 0 \pm i \tag{2.8.244}$$

$$y_h = c_1 \cos(x) + c_2 \sin(x) \tag{2.8.245}$$

Solving the nh-ODE,

$$y'' + y = 1 - \frac{x^2}{\pi^2} \tag{2.8.246}$$

$$y_p = K_0 + K_1 x + K_2 x^2 \tag{2.8.247}$$

$$1 = 2K_2 + K_0 \qquad \cdots\cdots[x^0] \tag{2.8.248}$$

$$\frac{-1}{\pi^2} = K_2 \qquad \cdots\cdots[x^2] \tag{2.8.249}$$

$$K_0 = 1 + \frac{2}{\pi^2}, \qquad K_1 = 0, \qquad K_2 = \frac{-1}{\pi^2} \tag{2.8.250}$$

$$y_p = \frac{2 + \pi^2 - x^2}{\pi^2} \tag{2.8.251}$$

Solving the IC $y(0) = 0,\ y'(0) = 0$,

$$y(0) = 0 = c_1 + \frac{2 + \pi^2}{\pi^2} \tag{2.8.252}$$

$$y'(0) = 0 = c_2 \tag{2.8.253}$$

$$y_1 = \frac{2 + \pi^2 - x^2}{\pi^2} - \frac{2 + \pi^2}{\pi^2} \cos x \tag{2.8.254}$$

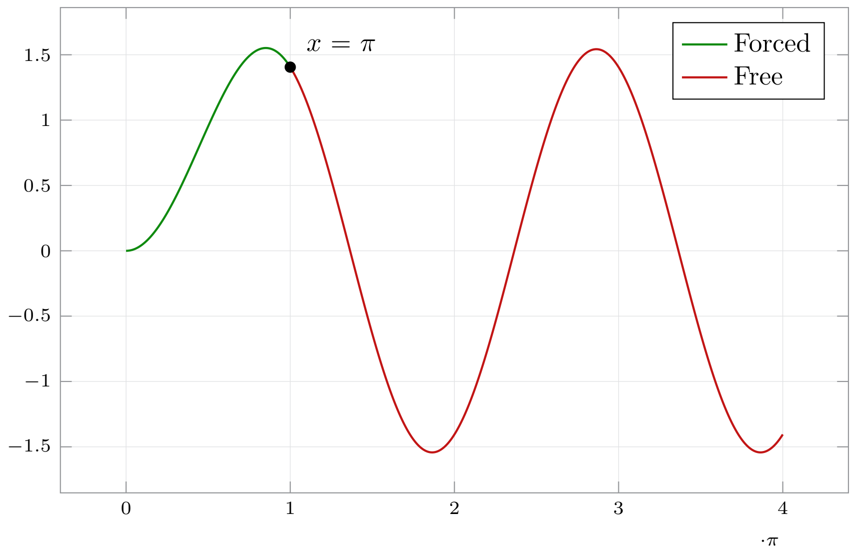
To guarantee continuity of the output and its derivative at $x = \pi$,

$$y_2 = M \cos x + N \sin x \qquad x > \pi \tag{2.8.255}$$

$$\lim_{x \to \pi^-} y_1 = \frac{4 + \pi^2}{\pi^2} \qquad \lim_{x \to \pi^+} y_2 = -M \tag{2.8.256}$$

$$\lim_{x \to \pi^-} y_1' = \frac{-2}{\pi} \qquad \lim_{x \to \pi^+} y_2' = -N \tag{2.8.257}$$

$$y_2 = -\frac{4 + \pi^2}{\pi^2} \cos x + \frac{2}{\pi} \sin x \tag{2.8.258}$$



25. Solving the h-ODE,

$$0 = y'' + y \qquad a = 0, \quad b = 1 \tag{2.8.259}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{4}}{2} \qquad = 0 \pm i \tag{2.8.260}$$

$$y_h = c_1 \sin(x) + c_2 \cos(x) \tag{2.8.261}$$

Solving the nh-ODE,

$$y'' + y = \cos(\omega x) \tag{2.8.262}$$

$$y_p = K \cos(\omega x) + M \sin(\omega x) \tag{2.8.263}$$

$$1 = -\omega^2 K + K \qquad \cdots\cdots[\cos(\omega x)] \tag{2.8.264}$$

$$0 = -\omega^2 M + M \qquad \cdots\cdots[\sin(\omega x)] \tag{2.8.265}$$

$$K = \frac{1}{1 - \omega^2}, \qquad M = 0 \quad (\text{since } \omega^2 \neq 1) \tag{2.8.266}$$

$$y_p = \frac{1}{1 - \omega^2} \cos(\omega x) \tag{2.8.267}$$

$$y = y_p + y_h \tag{2.8.268}$$

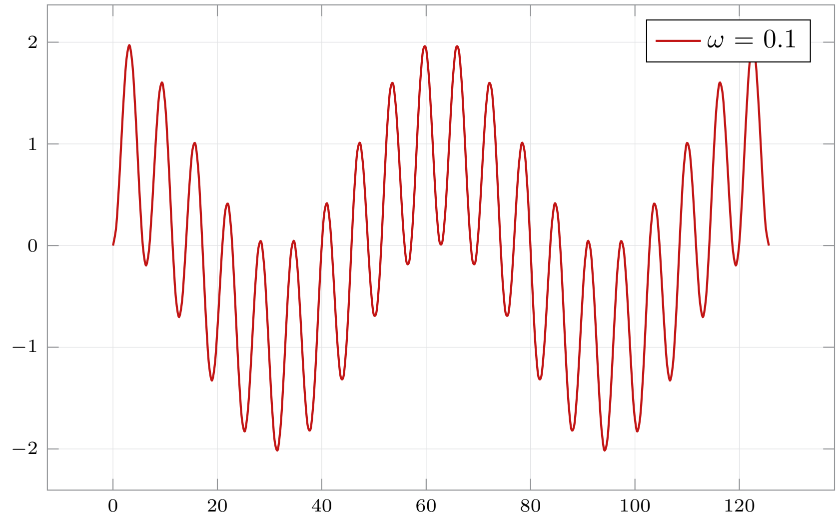
Solving the IVP $y(0) = 0,\ y'(0) = 0,$

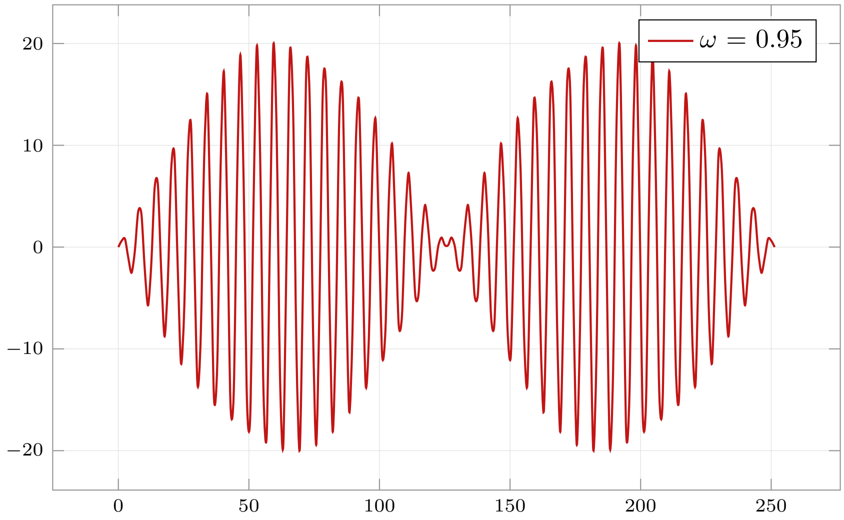
$$y(0) = 0 = c_2 + \frac{1}{1-\omega^2} \tag{2.8.269}$$

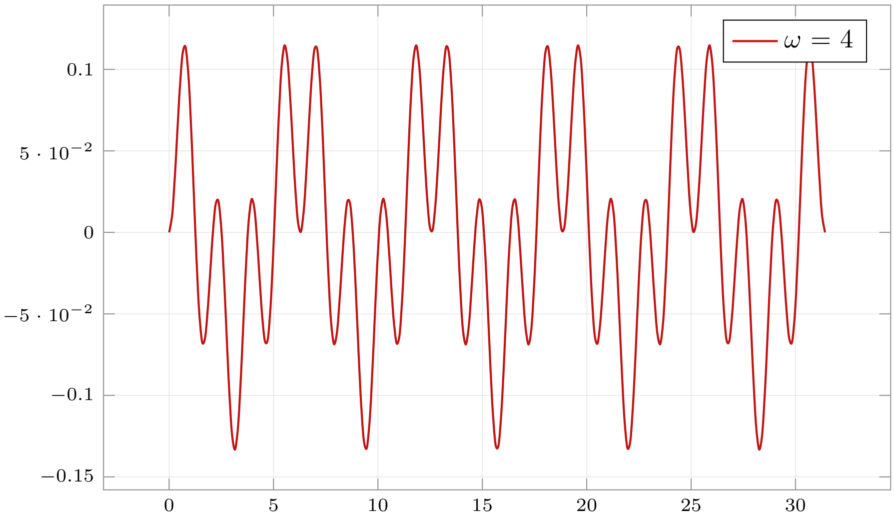
$$y'(0) = 0 = c_1 \tag{2.8.270}$$

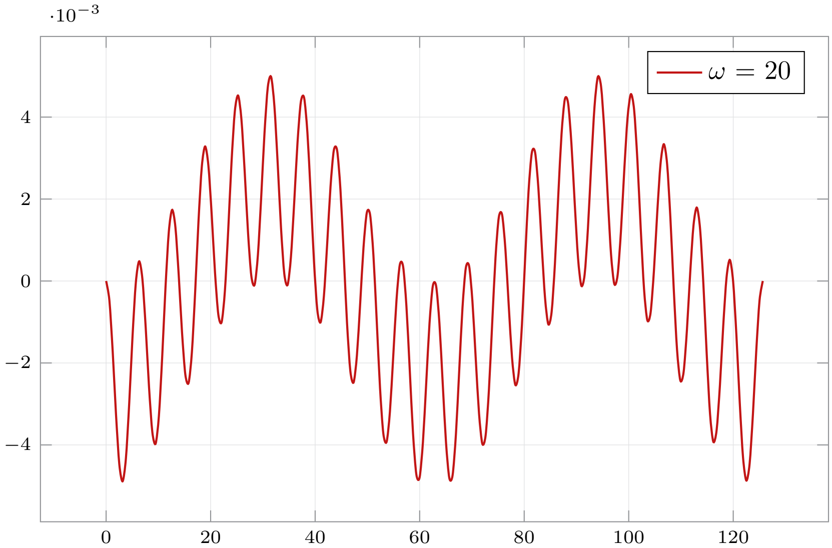
$$y = \frac{1}{1-\omega^2}[\cos(\omega x) - \cos(x)] \tag{2.8.271}$$

$$= \frac{2}{1-\omega^2} \sin\left(\frac{x+\omega x}{2}\right) \sin\left(\frac{x-\omega x}{2}\right) \tag{2.8.272}$$









2.9 Modeling: Electric Circuits

1. Solving the h-ODE,

$$0 = RI' + \frac{1}{C}\, I \tag{2.9.1}$$

$$\int \frac{1}{I}\, \mathrm{d}I = \frac{-1}{RC} \int \mathrm{d}t \tag{2.9.2}$$

$$I_h = p_1 \exp\left(\frac{-t}{RC}\right) \tag{2.9.3}$$

$$I_p = 0 \tag{2.9.4}$$

$$I = I_p + I_h \tag{2.9.5}$$

After the transient current exponentially decays to zero, the steady state current is zero, because the capacitor has charged up fully. Note that this result is independent of the constant emf E_0.

2. RL circuit with sinusoidally varying E,

$$\omega E_0 \cos(\omega t) = RI' + \frac{1}{C}\, I \tag{2.9.6}$$

$$S = \omega L - \frac{1}{\omega C} \tag{2.9.7}$$

$$I_p = \frac{E_0}{R^2 + S^2}\left[- S\cos(\omega t) + R\sin(\omega t)\right] \tag{2.9.8}$$

$$I_p = \frac{E_0 \cdot (\omega C)^2}{(\omega RC)^2 + 1}\left[R\sin(\omega t) + \frac{1}{\omega C}\cos(\omega t)\right] \tag{2.9.9}$$

$$I_h = c_1 \exp\left(\frac{-t}{RC}\right) \tag{2.9.10}$$

$$I = I_p + I_h \tag{2.9.11}$$

3. For an RL circuit with a constant emf,

$$E = LI' + RI \tag{2.9.12}$$

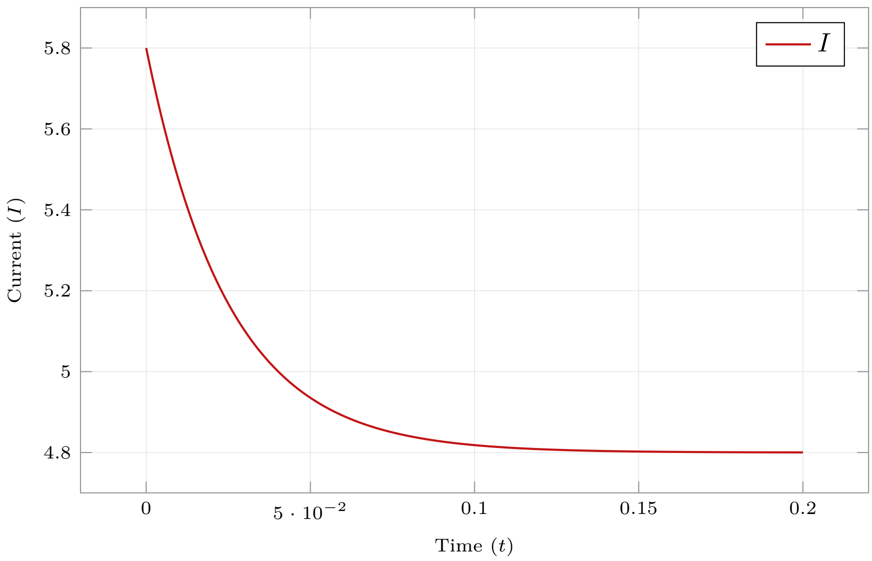
$$\int \frac{1}{(RI - E)}\, \mathrm{d}I = \frac{-1}{L}\int \mathrm{d}t \tag{2.9.13}$$

$$\frac{1}{R}\ln(RI - E) = \frac{-t}{L} + c_1 \tag{2.9.14}$$

$$I = \frac{E}{R} + c_1 \exp\left(\frac{-Rt}{L}\right) \tag{2.9.15}$$

$$I = I_p + I_h \tag{2.9.16}$$

Graphing the current for $L = 0.25,\ \ R = 10,\ \ E = 48,$



4. For an RL circuit with a constant emf, solving the h-ODE

$$E_0 \sin(\omega t) = LI' + RI \tag{2.9.17}$$

$$E_0\omega \cos(\omega t) = LI'' + RI' \qquad a = R/L \qquad b = 0 \tag{2.9.18}$$

$$\lambda_1, \lambda_2 = \frac{-R/L \pm R/L}{2} = \frac{-R}{L}, 0 \tag{2.9.19}$$

$$I_h = c_1 + c_2 \exp\left(\frac{-Rt}{L}\right) \tag{2.9.20}$$

Solving the nh-ODE,

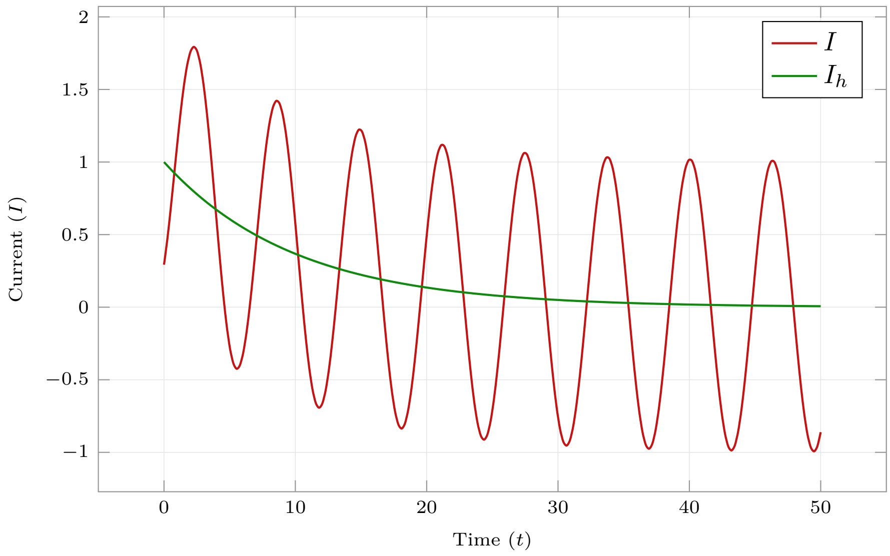
$$S = \omega L - \frac{1}{\omega C} \tag{2.9.21}$$

$$I_p = \frac{E_0}{R^2 + S^2} \Big[- S \cos(\omega t) + R \sin(\omega t) \Big] \tag{2.9.22}$$

$$I_p = \frac{E_0}{R^2 + (\omega L)^2} \Big[R \sin(\omega t) - \omega L \cos(\omega t) \Big] \tag{2.9.23}$$

$$I = I_p + I_h \tag{2.9.24}$$

Graphing the current for $L = 0.25, \;\; R = 1, \;\; E = 10, \;\; \omega = 4,$



5. For an LC circuit with a negligible resistance and sinusoidal emf, solving the h-ODE, given $L = 0.5,\ \ C = 0.005,\ \ \omega = 1,\ \ E_0 = 1,$

$$\cos t = LI'' + \frac{1}{C}\ I \qquad\qquad a = 0 \qquad b = \frac{-4}{LC} \tag{2.9.25}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{4/LC}}{2} \qquad\qquad = 0 \pm \frac{i}{\sqrt{LC}} \tag{2.9.26}$$

$$I_h = c_1\ \cos(20t) + c_2\ \sin(20t) \tag{2.9.27}$$

Solving the nh-ODE,

$$S = \omega L - \frac{1}{\omega C} = \frac{-399}{2} \tag{2.9.28}$$

$$I_p = \frac{E_0}{R^2 + S^2}\ \Big[- S\ \cos(t) + R\ \sin(t) \Big] \tag{2.9.29}$$

$$I_p = \frac{-E_0}{S}\ [\cos(t)] \tag{2.9.30}$$

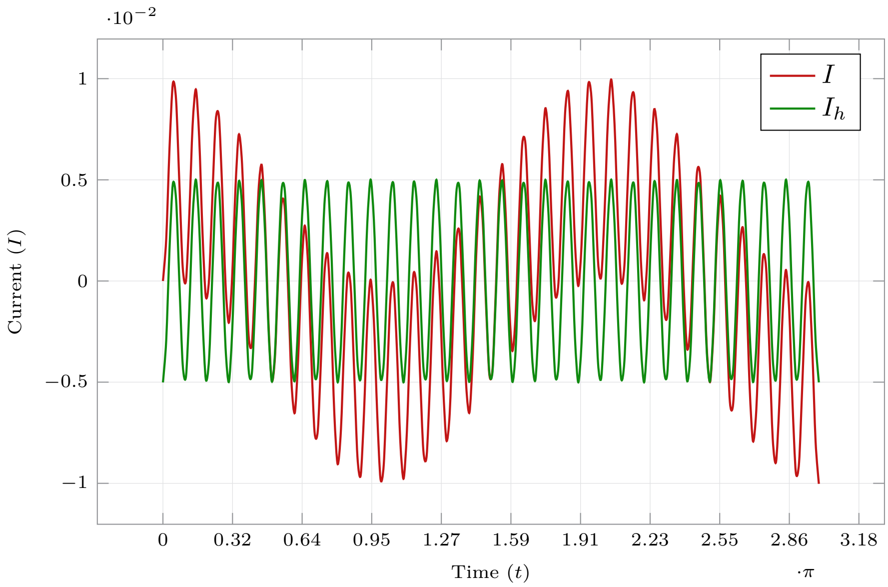
$$I = I_p + I_h \tag{2.9.31}$$

Applying the IC $Q(0) = 0,\ \ I(0) = 0,$

$$I(0) = 0 = c_1 - \frac{E_0}{S} \tag{2.9.32}$$

$$Q(0) = 0 \implies I'(0) = 0 = 20c_2 \tag{2.9.33}$$

$$I = \frac{E_0}{S}\ \cos(20t) - \frac{E_0}{S}\ \ \cos(t) \tag{2.9.34}$$



6. For an LC circuit with a negligible resistance and sinusoidal emf, solving the h-ODE, given $L = 0.5,\ \ C = 0.005,\ \ \omega = 1,\ \ E_t = 2t^2$,

$$4t = LI'' + \frac{1}{C}\ I \qquad a = 0 \qquad b = \frac{-4}{LC} \tag{2.9.35}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{4/LC}}{2} \qquad = 0 \pm \frac{i}{\sqrt{LC}} \tag{2.9.36}$$

$$I_h = c_1 \cos(20t) + c_2 \sin(20t) \tag{2.9.37}$$

Solving the nh-ODE,

$$I_p = K_0 + K_1 t \tag{2.9.38}$$

$$K_0 = 0, \qquad K_1 = 4C \tag{2.9.39}$$

$$I_p = 4Ct \tag{2.9.40}$$

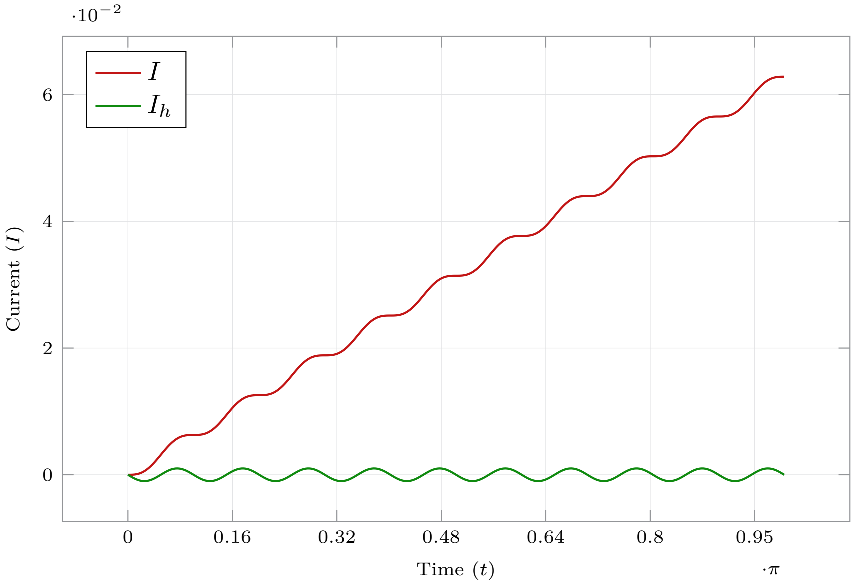
$$I = I_p + I_h \tag{2.9.41}$$

Applying the IC $Q(0) = 0,\ \ I(0) = 0$,

$$I(0) = 0 = c_1 \tag{2.9.42}$$

$$Q(0) = 0 \implies I'(0) = 0 = 20c_2 + 4C \tag{2.9.43}$$

$$I = \frac{-C}{5}\sin(20t) + 4Ct \tag{2.9.44}$$



7. Maximize I_0, given R, L are constant,

$$I_0 = E_0 \left[R^2 + (\omega L)^2 + \frac{1}{(\omega C)^2} - \frac{2L}{C} \right]^{-1/2} \tag{2.9.45}$$

$$\frac{\mathrm{d}I_0}{\mathrm{d}C} = \frac{-E_0}{2} \cdot \left(\frac{-2}{\omega^2 C^3} + \frac{2L}{C^2} \right) \cdot [R^2 + S^2]^{-3/2} \tag{2.9.46}$$

$$\frac{\mathrm{d}I_0}{\mathrm{d}C} = 0 \implies \frac{1}{\omega^2} = LC \implies S = 0 \tag{2.9.47}$$

At zero complex impedance, the resistance of the circuit is minimized thus maximizing the current amplitude.

8. Finding the steady state current,

$$0.5I'' + 4I' + \frac{1}{0.1} I = \frac{\mathrm{d}}{\mathrm{d}t} 500 \sin(2t) = 1000 \cos(2t) \tag{2.9.48}$$

$$S = \omega L - \frac{1}{\omega C} = -4 \tag{2.9.49}$$

$$I_p = \frac{E_0}{R^2 + S^2} \Big[-S \cos(\omega t) + R \sin(\omega t) \Big] \tag{2.9.50}$$

$$I_p = \frac{125}{2} [\cos(2t) + \sin(2t)] \tag{2.9.51}$$

$$\tag{2.9.52}$$

9. Finding the steady state current,

$$0.1I'' + 4I' + \frac{1}{0.05}\ I = \frac{\mathrm{d}}{\mathrm{d}t}110 = 0 \tag{2.9.53}$$

$$I_p = Kt + M \tag{2.9.54}$$

$$0 = 4K + 20M \qquad \cdots\cdots[t^0] \tag{2.9.55}$$

$$0 = 20K \qquad \cdots\cdots[t] \tag{2.9.56}$$

$$I_p = 0 \tag{2.9.57}$$

2.9.58

10. Finding the steady state current,

$$I'' + 2I' + 20I = \frac{\mathrm{d}}{\mathrm{d}t}157\sin(3t) = 471\cos(3t) \tag{2.9.59}$$

$$S = \omega L - \frac{1}{\omega C} = -\frac{11}{3} \tag{2.9.60}$$

$$I_p = \frac{E_0}{R^2 + S^2}\Big[-S\cos(\omega t) + R\sin(\omega t)\Big] \tag{2.9.61}$$

$$I_p = 33\cos(2t) + 18\sin(2t) \tag{2.9.62}$$

2.9.63

11. Finding the steady state current,

$$0.4I'' + 12I' + 80I = \frac{\mathrm{d}}{\mathrm{d}t}220\sin(10t) = 2200\cos(10t) \tag{2.9.64}$$

$$S = \omega L - \frac{1}{\omega C} = -4 \tag{2.9.65}$$

$$I_p = \frac{E_0}{R^2 + S^2}\Big[-S\cos(\omega t) + R\sin(\omega t)\Big] \tag{2.9.66}$$

$$I_p = 5.5\,[\cos(10t) + 3\sin(10t)] \tag{2.9.67}$$

2.9.68

12. Finding the steady state current,

$$0.1I'' + 0.2I' + \frac{1}{2}\ I = \frac{\mathrm{d}}{\mathrm{d}t}220\sin(314t) = 69080\cos(314t) \tag{2.9.69}$$

$$S = \omega L - \frac{1}{\omega C} = 31.4 \tag{2.9.70}$$

$$I_p = \frac{E_0}{R^2 + S^2}\Big[-S\cos(\omega t) + R\sin(\omega t)\Big] \tag{2.9.71}$$

$$I_p = -7.00534\cos(10t) + 0.04462\sin(10t) \tag{2.9.72}$$

$$\tag{2.9.73}$$

13. Finding the steady state current,

$$1.2I'' + 12I' + \frac{3000}{20}\ I = \frac{\mathrm{d}}{\mathrm{d}t}12000\sin(25t) = 300000\cos(25t) \tag{2.9.74}$$

$$S = \omega L - \frac{1}{\omega C} = 24 \tag{2.9.75}$$

$$I_p = \frac{E_0}{R^2 + S^2}\Big[-S\cos(\omega t) + R\sin(\omega t)\Big] \tag{2.9.76}$$

$$I_p = -400\cos(25t) + 200\sin(25t) \tag{2.9.77}$$

$$\tag{2.9.78}$$

14. If $R > 0$, then prove I_h decays to zero with time, given $R, L, C > 0$,

$$E_0\omega\cos(\omega t) = LI'' + RI' + \frac{1}{C}\ I \tag{2.9.79}$$

$$a = \frac{R}{L} \qquad\qquad b = \frac{1}{LC} \tag{2.9.80}$$

$$\alpha = \frac{R}{2L} \qquad\qquad \beta = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}} \tag{2.9.81}$$

$$I_h = e^{-\alpha t}[c_1\cos(\beta t) + c_2\sin(\beta t)] \qquad\qquad \beta < 0 \tag{2.9.82}$$

$$I_h = c_1 e^{-(\alpha+\beta)t} + c_2 e^{-(\alpha-\beta)t} \qquad\qquad \beta > 0 \tag{2.9.83}$$

Since $0 < \beta < \alpha$ is guaranteed for the case $\beta > 0$, the two terms both have a negative exponent. In both cases, exponential decay is unavoidable for $R > 0$.

15. From the above problem,

$$\beta = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}} \tag{2.9.84}$$

$$\beta = 0 \implies R_{\text{crit}} = 2\sqrt{\frac{L}{C}} \tag{2.9.85}$$

$$R \begin{cases} = R_{\text{crit}} & \text{critically damped} \\ < R_{\text{crit}} & \text{underdamped} \\ > R_{\text{crit}} & \text{overdamped} \end{cases} \tag{2.9.86}$$

16. Solving the h-ODE,

$$0 = 0.2I'' + 8I' + \frac{1000}{12.5}\, I \qquad\qquad a = 40 \qquad b = 400 \tag{2.9.87}$$

$$\lambda_1, \lambda_2 = \frac{-40 \pm \sqrt{0}}{2} \qquad\qquad = -20, -20 \tag{2.9.88}$$

$$I_h = (c_1 + c_2 t)e^{-20t} \tag{2.9.89}$$

Solving the nh-ODE,

$$0.2I'' + 8I' + \frac{1000}{12.5}\, I = \frac{\mathrm{d}}{\mathrm{d}t} 100\sin(10t) = 1000\cos(10t) \tag{2.9.90}$$

$$S = \omega L - \frac{1}{\omega C} = -6 \tag{2.9.91}$$

$$I_p = \frac{E_0}{R^2 + S^2}\Big[-S\cos(\omega t) + R\sin(\omega t)\Big] \tag{2.9.92}$$

$$I_p = 6\cos(25t) + 12\sin(25t) \tag{2.9.93}$$

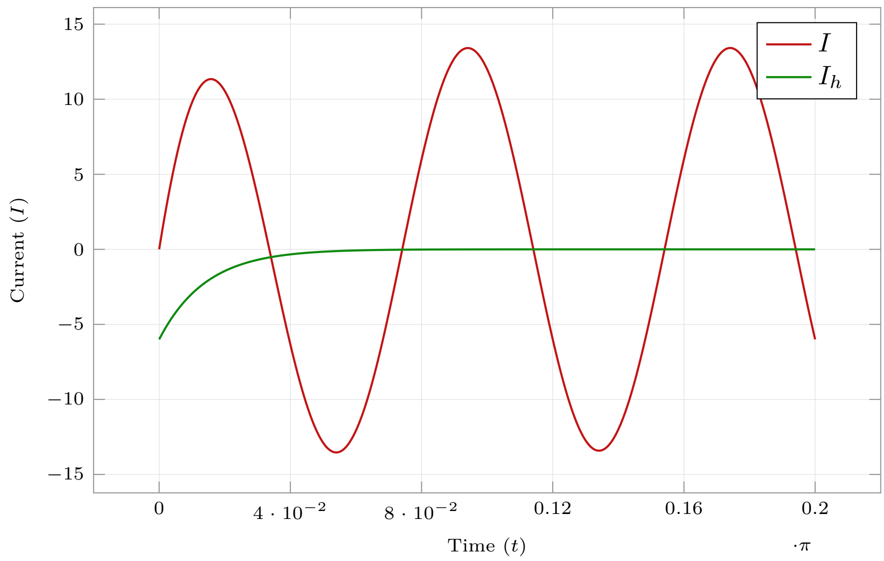
$$\tag{2.9.94}$$

Applying the IC $Q(0) = 0,\ I(0) = 0$,

$$I(0) = 0 = c_1 + 6 \tag{2.9.95}$$

$$Q(0) = 0 \implies I'(0) = 0 = 300 + c_2 - 20c_1 \tag{2.9.96}$$

$$I = [-6 + 14.7t]e^{-20t} + 6[\cos(25t) + 2\sin(25t)] \tag{2.9.97}$$



17. Solving the h-ODE,

$$0 = I'' + 6I' + 25I \qquad a = 6 \qquad b = 25 \tag{2.9.98}$$

$$\lambda_1, \lambda_2 = \frac{-6 \pm i\sqrt{64}}{2} = -3 \pm 4i \tag{2.9.99}$$

$$I_h = [c_1 \cos(4t) + c_2 \sin(4t)]e^{-3t} \tag{2.9.100}$$

Solving the nh-ODE,

$$I'' + 6I' + 25I = -600 \sin t + 2400 \cos t \tag{2.9.101}$$

$$I_p = K \cos t + M \sin t \tag{2.9.102}$$

$$2400 = -K + 6M + 25K \qquad \cdots\cdots[\cos t] \tag{2.9.103}$$

$$-600 = -M - 6K + 25M \qquad \cdots\cdots[\sin t] \tag{2.9.104}$$

$$I_p = 100 \cos(t) \tag{2.9.105}$$

$$\tag{2.9.106}$$

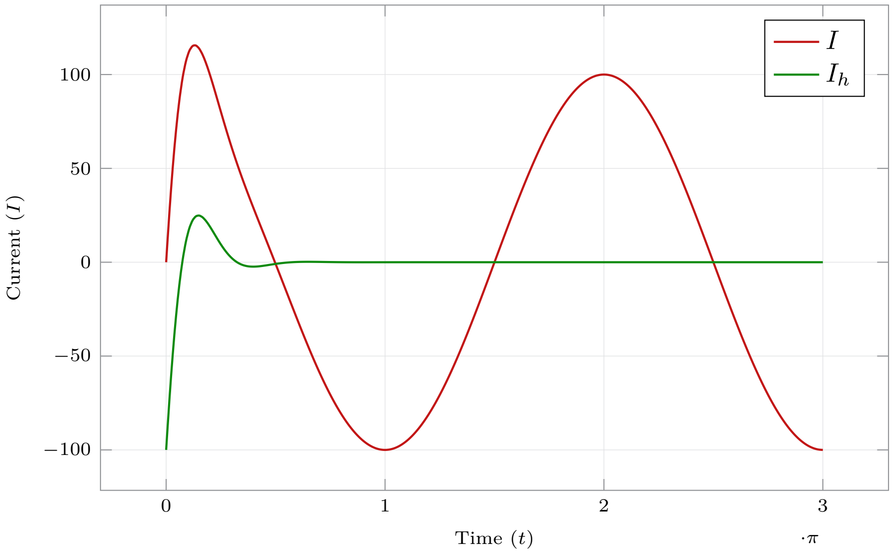
Applying the IC $Q(0) = 0,\ I(0) = 0$,

$$I'(0) + 6I(0) + 25Q(0) = E(0) = 600 \tag{2.9.107}$$

$$I(0) = 0 = 100 + c_1 \tag{2.9.108}$$

$$Q(0) = 0 \implies I'(0) = 600 = 4c_2 - 3c_1 \tag{2.9.109}$$

$$I = [-100 \cos(4t) + 75 \sin(4t)]e^{-3t} + 100 \cos(t) \tag{2.9.110}$$



18. Solving the h-ODE,

$$0 = I'' + 18I' + 80I \qquad a = 18 \qquad b = 80 \tag{2.9.111}$$

$$\lambda_1, \lambda_2 = \frac{-18 \pm \sqrt{4}}{2} = -8, -10 \tag{2.9.112}$$

$$I_h = c_1 e^{-10t} + c_2 e^{-8t} \tag{2.9.113}$$

Solving the nh-ODE,

$$I'' + 18I' + 80I = -8200 \sin(10t) \tag{2.9.114}$$

$$I_p = K \cos(10t) + M \sin(10t) \tag{2.9.115}$$

$$0 = -100K + 180M + 80K \qquad \cdots\cdots[\cos t] \tag{2.9.116}$$

$$-8200 = -100M - 180K + 80M \qquad \cdots\cdots[\sin t] \tag{2.9.117}$$

$$I_p = 45 \cos(10t) + 5 \sin(10t) \tag{2.9.118}$$

$$\tag{2.9.119}$$

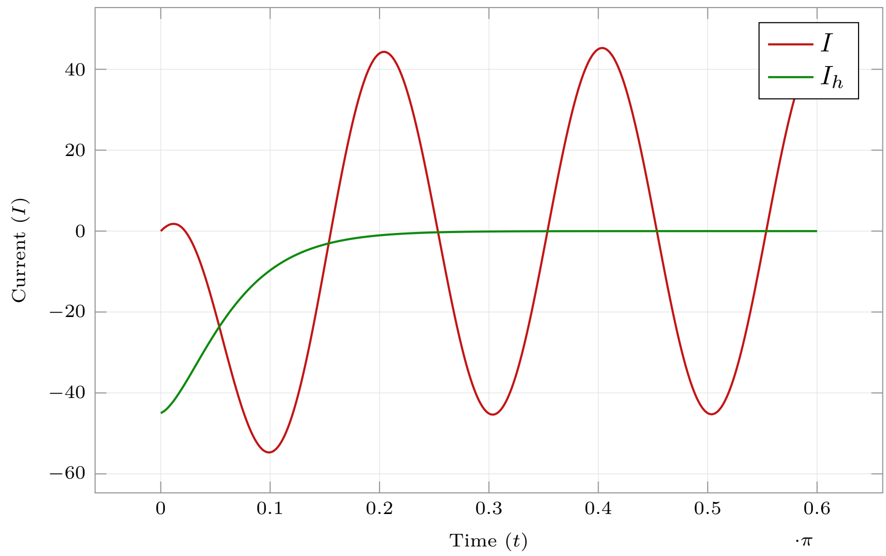
Applying the IC $Q(0) = 0,\ I(0) = 0$,

$$E(0) = 820 = I'(0) + 18I(0) + 80Q(0) \tag{2.9.120}$$

$$I(0) = 0 = 45 + c_1 + c_2 \tag{2.9.121}$$

$$Q(0) = 0 \implies I'(0) = 820 = -8c_2 - 10c_1 + 50 \tag{2.9.122}$$

$$I = 160e^{-10t} - 205e^{-8t} + 45 \cos(10t) + 5 \sin(10t) \tag{2.9.123}$$



19. To understand the underlying similarities in the modeling of spring-mass systems and electrical circuits,

Spring Mass		**Electrical Circuit**	
Mass	m	L	Inductance
Damping constant	c	R	Resistance
Spring modulus	k	$1/C$	reciprocal of Capacitance
Driving Force	$F_0 \cos(\omega t)$	$E_0\omega \cos(\omega t)$	derivative of EMF
Displacement	$y(t)$	$I(t)$	Current

$$5I'' + 10I' + 60I = 220 \cos(10t) \tag{2.9.124}$$

$$I'' + 2I' + 12I = 44 \cos(10t) \tag{2.9.125}$$

From the above table, $L = 1$ H, $R = 2\,\Omega$, $C = 1/12$F,
the derivative of the emf becomes, $E(t)' = 44 \cos(10t)$V, which means, $E(t) = 4.4 \sin(10t)$V

20. Deriving the solution to the electrical system ODE using complex numbers,

$$L\tilde{I}'' + R\tilde{I}' + \frac{1}{C}\,\tilde{I} = E_0\omega\,\exp(i\omega t) \tag{2.9.126}$$

$$I_p = K\,\exp(i\omega t) \tag{2.9.127}$$

$$I_p' = i\omega K\,\exp(i\omega t) \tag{2.9.128}$$

$$I_p'' = -\omega^2 K\,\exp(i\omega t) \tag{2.9.129}$$

$$K\,\exp(i\omega t)[-\omega^2 L + iR\omega + 1/C] = E_0 i\omega\,\exp(i\omega t) \tag{2.9.130}$$

$$K = \frac{E_0 i}{-S + iR} = \frac{E_0}{R + iS} \tag{2.9.131}$$

$$|I_p| = \sqrt{K\bar{K}} = \frac{E_0}{\sqrt{S^2 + R^2}} \tag{2.9.132}$$

$$\text{given } R > 0, \qquad \arg Z = \frac{S}{R} = \tan(-\theta) \tag{2.9.133}$$

These results agree with the text.

2.10 Solution by Variation of Parameters

1. Solving the h-ODE,

$$y'' + 9y = 0 \qquad\qquad a = 0 \qquad b = 9 \tag{2.10.1}$$

$$\lambda_1,\,\lambda_2 = \frac{0 \pm i\sqrt{36}}{2} \qquad\qquad = 0 \pm 3i \tag{2.10.2}$$

$$y_h = c_1\,\cos(3x) + c_2\,\sin(3x) \tag{2.10.3}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = \sec(3x) \tag{2.10.4}$$

$$W(y_1,\,y_2) = 3\,\cos^2(3x) + 3\,\sin^2(3x) = 3 \tag{2.10.5}$$

$$f' = \frac{-y_2 r}{W} = \frac{-\tan(3x)}{3} \qquad\qquad g' = \frac{y_1 r}{W} = \frac{1}{3} \tag{2.10.6}$$

$$f = \frac{1}{9}\ln|\cos(3x)| \qquad\qquad g = \frac{x}{3} \tag{2.10.7}$$

$$y_p = \frac{\ln|\cos(3x)|}{9}\,\cos(3x) + \frac{x}{3}\,\sin(3x) \tag{2.10.8}$$

$$y = y_h + y_p \tag{2.10.9}$$

2. Solving the h-ODE,

$$y'' + 9y = 0 \qquad a = 0 \qquad b = 9 \tag{2.10.10}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{36}}{2} \qquad = 0 \pm 3i \tag{2.10.11}$$

$$y_h = c_1 \cos(3x) + c_2 \sin(3x) \tag{2.10.12}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = \csc(3x) \tag{2.10.13}$$

$$W(y_1, y_2) = 3\cos^2(3x) + 3\sin^2(3x) = 3 \tag{2.10.14}$$

$$f' = \frac{-y_2 r}{W} = \frac{-1}{3} \qquad g' = \frac{y_1 r}{W} = \frac{\cot(3x)}{3} \tag{2.10.15}$$

$$f = \frac{-x}{3} \qquad g = \frac{\ln|\sin(3x)|}{9} \tag{2.10.16}$$

$$y_p = \frac{-x}{3}\cos(3x) + \frac{\ln|\sin(3x)|}{9}\sin(3x) \tag{2.10.17}$$

$$y = y_h + y_p \tag{2.10.18}$$

3. Solving the h-ODE,

$$x^2 y'' - 2xy' + 2y = 0 \qquad a - 1 = -3 \qquad b = 2 \tag{2.10.19}$$

$$\lambda_1, \lambda_2 = \frac{3 \pm \sqrt{1}}{2} \qquad = 2, 1 \tag{2.10.20}$$

$$y_h = c_1 x + c_2 x^2 \tag{2.10.21}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = x\sin(x) \tag{2.10.22}$$

$$W(y_1, y_2) = 2x^2 - x^2 = x^2 \tag{2.10.23}$$

$$f' = \frac{-y_2 r}{W} = -x\sin(x) \qquad g' = \frac{y_1 r}{W} = \sin(x) \tag{2.10.24}$$

$$f = x\cos(x) - \sin(x) \qquad g = -\cos(x) \tag{2.10.25}$$

$$y_p = -x\sin(x) \tag{2.10.26}$$

$$y = y_h + y_p \tag{2.10.27}$$

4. Solving the h-ODE,

$$y'' - 4y' + 5y = 0 \qquad a = -4 \qquad b = 5 \tag{2.10.28}$$

$$\lambda_1, \lambda_2 = \frac{4 \pm i\sqrt{4}}{2} \qquad = 2 \pm i \tag{2.10.29}$$

$$y_h = [c_1 \cos(x) + c_2 \sin(x)]e^{2x} \tag{2.10.30}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = e^{2x} \csc x \tag{2.10.31}$$

$$W(y_1, y_2) = e^{4x} \cos(x)[\cos x + 2 \sin x] \tag{2.10.32}$$

$$- e^{4x} \sin x[- \sin x + 2 \cos x] \tag{2.10.33}$$

$$= e^{4x} \tag{2.10.34}$$

$$f' = \frac{-y_2 r}{W} = -1 \qquad g' = \frac{y_1 r}{W} = \cot x \tag{2.10.35}$$

$$f = -x \qquad g = \ln|\sin(x)| \tag{2.10.36}$$

$$y_p = e^{2x}[-x \cos x + \ln(|\sin x|) \sin x] \tag{2.10.37}$$

$$y = y_h + y_p \tag{2.10.38}$$

5. Solving the h-ODE,

$$y'' + y = 0 \qquad a = 0 \qquad b = 1 \tag{2.10.39}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{4}}{2} \qquad = 0 \pm i \tag{2.10.40}$$

$$y_h = c_1 \cos(x) + c_2 \sin(x) \tag{2.10.41}$$

Solving the nh-ODE by undetermined coefficients,

$$y'' + y = \cos x - \sin x \tag{2.10.42}$$

$$y_p = Kx \cos x + Mx \sin x \tag{2.10.43}$$

$$y_p' = \cos x[K + Mx] + \sin x[M - Kx] \tag{2.10.44}$$

$$y_p'' = \cos x[2M - Kx] + \sin x[-2K - Mx] \tag{2.10.45}$$

$$1 = 2M \qquad \cdots\cdots[\cos x] \tag{2.10.46}$$

$$-1 = -2K \qquad \cdots\cdots[\sin x] \tag{2.10.47}$$

$$y_p = \frac{x}{2}[\cos x + \sin x] \tag{2.10.48}$$

$$y = y_h + y_p \tag{2.10.49}$$

6. Solving the h-ODE,

$$y'' + 6y' + 9y = 0 \qquad a = 6 \qquad b = 9 \tag{2.10.50}$$

$$\lambda_1, \lambda_2 = \frac{-6 \pm i\sqrt{0}}{2} \qquad = -3, -3 \tag{2.10.51}$$

$$y_h = (c_1 + c_2 x)e^{-3x} \tag{2.10.52}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = \frac{16}{x^2 + 1}\, e^{-3x} \tag{2.10.53}$$

$$W(y_1, y_2) = e^{-6x} \tag{2.10.54}$$

$$f' = \frac{-y_2 r}{W} = \frac{-16x}{1 + x^2} \qquad g' = \frac{y_1 r}{W} = \frac{16}{1 + x^2} \tag{2.10.55}$$

$$f = -8 \ln(1 + x^2) \qquad g = 16 \arctan x \tag{2.10.56}$$

$$y_p = e^{-3x}[-8 \ln(1 + x^2) + 16x \arctan x] \tag{2.10.57}$$

$$y = y_h + y_p \tag{2.10.58}$$

7. Solving the h-ODE,

$$y'' - 4y' + 4y = 0 \qquad a = -4 \qquad b = 4 \tag{2.10.59}$$

$$\lambda_1, \lambda_2 = \frac{4 \pm i\sqrt{0}}{2} \qquad = 2, 2 \tag{2.10.60}$$

$$y_h = (c_1 + c_2 x)e^{2x} \tag{2.10.61}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = 6e^{2x}x^{-4} \tag{2.10.62}$$

$$W(y_1, y_2) = e^{4x} \tag{2.10.63}$$

$$f' = \frac{-y_2 r}{W} = -6x^{-3} \qquad g' = \frac{y_1 r}{W} = 6x^{-4} \tag{2.10.64}$$

$$f = 3x^{-2} \qquad g = -2x^{-3} \tag{2.10.65}$$

$$y_p = e^{2x}[x^{-2}] \tag{2.10.66}$$

$$y = y_h + y_p \tag{2.10.67}$$

8. Solving the h-ODE,

$$y'' + 4y = 0 \qquad a = 0 \qquad b = 4 \tag{2.10.68}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm i\sqrt{16}}{2} \qquad = 0 \pm 2i \tag{2.10.69}$$

$$y_h = c_1 \cos(2x) + c_2 \sin(2x) \tag{2.10.70}$$

Solving the nh-ODE by undetermined coefficients,

$$y'' + 4y = \cosh(2x) = 0.5e^{2x} + 0.5e^{-2x} \tag{2.10.71}$$

$$y_p = K \cosh(2x) \tag{2.10.72}$$

$$1 = 4K + 4K \qquad \cdots\cdots[\cosh(2x)] \tag{2.10.73}$$

$$y_p = \frac{1}{8}\cosh(2x) \tag{2.10.74}$$

$$y = y_h + y_p \tag{2.10.75}$$

9. Solving the h-ODE,

$$y'' - 2y' + 1y = 0 \qquad a = -2 \qquad b = 1 \tag{2.10.76}$$

$$\lambda_1, \lambda_2 = \frac{2 \pm i\sqrt{0}}{2} = 1, 1 \tag{2.10.77}$$

$$y_h = (c_1 + c_2 x)e^x \tag{2.10.78}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = 35e^x x^{1.5} \tag{2.10.79}$$

$$W(y_1, y_2) = e^{2x} \tag{2.10.80}$$

$$f' = \frac{-y_2 r}{W} = -35x^{2.5} \qquad g' = \frac{y_1 r}{W} = 35x^{1.5} \tag{2.10.81}$$

$$f = -10x^{3.5} \qquad g = 14x^{2.5} \tag{2.10.82}$$

$$y_p = 4e^x x^{3.5} \tag{2.10.83}$$

$$y = y_h + y_p \tag{2.10.84}$$

10. Solving the h-ODE,

$$y'' + 2y' + 2y = 0 \qquad a = 2 \qquad b = 2 \tag{2.10.85}$$

$$\lambda_1, \lambda_2 = \frac{-2 \pm i\sqrt{4}}{2} = -1 \pm i \tag{2.10.86}$$

$$y_h = [c_1 \cos x + c_2 \sin x]e^{-x} \tag{2.10.87}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = 4e^{-x} \sec^3 x \tag{2.10.88}$$

$$W(y_1, y_2) = e^{-2x} \tag{2.10.89}$$

$$f' = \frac{-y_2 r}{W} = -4 \tan x \sec^2 x \qquad g' = \frac{y_1 r}{W} = 4 \sec^2 x \tag{2.10.90}$$

$$f = -2 \tan^2 x \qquad g = 4 \tan x \tag{2.10.91}$$

$$y_p = e^{-x} \sin x \; [2 \tan x] \tag{2.10.92}$$

$$y = y_h + y_p \tag{2.10.93}$$

11. Solving the h-ODE,

$$x^2y'' - 4xy' + 6y = 0 \qquad a - 1 = -5 \qquad b = 6 \tag{2.10.94}$$

$$\lambda_1, \lambda_2 = \frac{5 \pm \sqrt{1}}{2} = 2, 3 \tag{2.10.95}$$

$$y_h = c_1x^2 + c_2x^3 \tag{2.10.96}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = 21x^{-6} \tag{2.10.97}$$

$$W(y_1, y_2) = x^4 \tag{2.10.98}$$

$$f' = \frac{-y_2r}{W} = -21x^{-7} \qquad g' = \frac{y_1r}{W} = 21x^{-8} \tag{2.10.99}$$

$$f = \frac{21}{6}x^{-6} \qquad g = -3x^{-7} \tag{2.10.100}$$

$$y_p = 0.5x^{-4} \tag{2.10.101}$$

$$y = y_h + y_p \tag{2.10.102}$$

12. Solving the h-ODE,

$$y'' - y = 0 \qquad a = 0 \qquad b = -1 \tag{2.10.103}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm \sqrt{4}}{2} = -1, 1 \tag{2.10.104}$$

$$y_h = c_1e^{-x} + c_2e^{x} \tag{2.10.105}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = \frac{1}{\cosh x} \tag{2.10.106}$$

$$W(y_1, y_2) = 2 \tag{2.10.107}$$

$$f' = \frac{-y_2r}{W} = \frac{-e^{2x}}{1 + e^{2x}} \qquad g' = \frac{y_1r}{W} = \frac{e^{-2x}}{e^{-2x} + 1} \tag{2.10.108}$$

$$f = -0.5\,\ln(1 + e^{2x}) \qquad g = -0.5\,\ln(1 + e^{-2x}) \tag{2.10.109}$$

$$y_p = -0.5[e^{-x}\,\ln(1 + e^{2x}) + e^{x}\,\ln(1 + e^{-2x})] \tag{2.10.110}$$

$$y = y_h + y_p \tag{2.10.111}$$

13. Solving the h-ODE,

$$x^2y'' + xy' - 9y = 0 \qquad a - 1 = 0 \qquad b = -9 \tag{2.10.112}$$

$$\lambda_1, \lambda_2 = \frac{0 \pm \sqrt{36}}{2} = -3, 3 \tag{2.10.113}$$

$$y_h = c_1x^{-3} + c_2x^3 \tag{2.10.114}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = 48x^3 \tag{2.10.115}$$

$$W(y_1, y_2) = 6x^{-1} \tag{2.10.116}$$

$$f' = \frac{-y_2r}{W} = -8x^7 \qquad g' = \frac{y_1r}{W} = 8x \tag{2.10.117}$$

$$f = -x^8 \qquad g = 4x^2 \tag{2.10.118}$$

$$y_p = 3x^5 \tag{2.10.119}$$

$$y = y_h + y_p \tag{2.10.120}$$

14. Applying both methods to compare

(a) Solving the h-ODE,

$$y'' + 4y' + 3y = 0 \qquad a = 4 \qquad b = 3 \tag{2.10.121}$$

$$\lambda_1, \lambda_2 = \frac{-4 \pm \sqrt{4}}{2} = -3, -1 \tag{2.10.122}$$

$$y_h = c_1e^{-3x} + c_2e^{-x} \tag{2.10.123}$$

Solving the nh-ODE by variation of parameters,

$$r(x) = 65\cos(2x) \tag{2.10.124}$$

$$W(y_1, y_2) = 2e^{-4x} \tag{2.10.125}$$

$$f' = \frac{-y_2r}{W} = \frac{-65\cos(2x)e^{3x}}{2} \qquad g' = \frac{y_1r}{W} = \frac{65\cos(2x)e^{x}}{2} \tag{2.10.126}$$

$$f = \frac{-5e^{3x}}{2}[3\cos(2x) + 2\sin(2x)] \qquad g = \frac{13e^{x}}{2}[\cos(2x) + 2\sin(2x)] \tag{2.10.127}$$

$$y_p = -\cos(2x) + 8\sin(2x) \tag{2.10.128}$$

$$y = y_h + y_p \tag{2.10.129}$$

Solving the nh-ODE by undetermined coefficients,

$$y'' + 4y' + 3y = 65\cos(2x) \tag{2.10.130}$$

$$y_p = K\cos(2x) + M\sin(2x) \tag{2.10.131}$$

$$65 = -4K + 8M + 3K \qquad \cdots\cdots[\cos(2x)] \tag{2.10.132}$$

$$0 = -4M - 8K + 3M \qquad \cdots\cdots[\sin(2x)] \tag{2.10.133}$$

$$y_p = -\cos(2x) + 8\sin(2x) \tag{2.10.134}$$

$$y = y_h + y_p \tag{2.10.135}$$

Both methods match, but method 2 is simpler.

(b) Solving the h-ODE,

$$y'' - 2y' + y = 0 \qquad a = -2 \qquad b = 1 \tag{2.10.136}$$

$$\lambda_1, \lambda_2 = \frac{2 \pm \sqrt{0}}{2} \qquad = 1, 1 \tag{2.10.137}$$

$$y_h = [c_1 + c_2 x]e^x \tag{2.10.138}$$

Solving the nh-ODE by variation of parameters,

$$r_1(x) = 35e^x x^{1.5} \tag{2.10.139}$$

$$W(y_1, y_2) = e^{2x} \tag{2.10.140}$$

$$f' = \frac{-y_2 r}{W} = -35x^{2.5} \qquad g' = \frac{y_1 r}{W} = 35x^{1.5} \tag{2.10.141}$$

$$f = -10x^{3.5} \qquad g = 14x^{2.5} \tag{2.10.142}$$

$$y_{p1} = 4e^x x^{3.5} \tag{2.10.143}$$

Solving the nh-ODE by undetermined coefficients,

$$y'' - 2y' + y = x^2 \tag{2.10.144}$$

$$y_{p2} = K_2x^2 + K_1x + K_0 \tag{2.10.145}$$

$$1 = K_2 \qquad \cdots\cdots[x^2] \tag{2.10.146}$$

$$0 = -4K_2 + K_1 \qquad \cdots\cdots[x] \tag{2.10.147}$$

$$0 = 2K_2 - 2K_1 + K_0 \qquad \cdots\cdots[1] \tag{2.10.148}$$

$$y_{p2} = x^2 + 4x + 6 \tag{2.10.149}$$

$$y = [c_1 + c_2x]e^x + 4e^x x^{3.5} + x^2 + 4x + 6 \tag{2.10.150}$$

Each of the terms in $r(x)$ are more amenable to one of the methods.

(c) Refer Problem set 2.7, Question 20 for the method of undetermined coefficients applied to Euler-Cauchy ODEs.