Chapter 17

Conformal Mapping

17.1 Geometry of Analytic Functions: Conformal Mapping

1. The angle doubles which means that the mapping takes

$$Z_1 : \theta \in [0,\, \pi/6] \implies W_1 : \phi \in [0,\, \pi/3] \tag{17.1.1}$$

$$Z_2 : \theta \in [\pi/6,\, \pi/3] \implies W_2 : \phi \in [\pi/3,\, 2\pi/3] \tag{17.1.2}$$

$$Z_3 : \theta \in [\pi/3,\, \pi/2] \implies W_3 : \phi \in [2\pi/3,\, \pi] \tag{17.1.3}$$

The radius is squared which means that the map takes

$$Z_1 = r \in [0,\, 1] \implies W_1 : R \in [0,\, 1] \tag{17.1.4}$$

$$Z_2 = r \in [1,\, \sqrt{2}] \implies W_2 : R \in [1,\, 2] \tag{17.1.5}$$

$$Z_3 = r \in [\sqrt{2},\, 2] \implies W_3 : R \in [2,\, 4] \tag{17.1.6}$$

2. Looking at vertical and horizontal lines under the mapping $f(z) = z^2$,

$$f(z) = (x^2 - y^2) + \mathrm{i}\,(2xy) \qquad x = a \tag{17.1.7}$$

$$u = a^2 - y^2 \qquad v = 2ay \tag{17.1.8}$$

$$\frac{v^2}{4a^2} = a^2 - u \qquad u = a^2 - \left(\frac{v}{2a}\right)^2 \tag{17.1.9}$$

This is a parabola that opens in $-x$ and whose axis is the x axis.

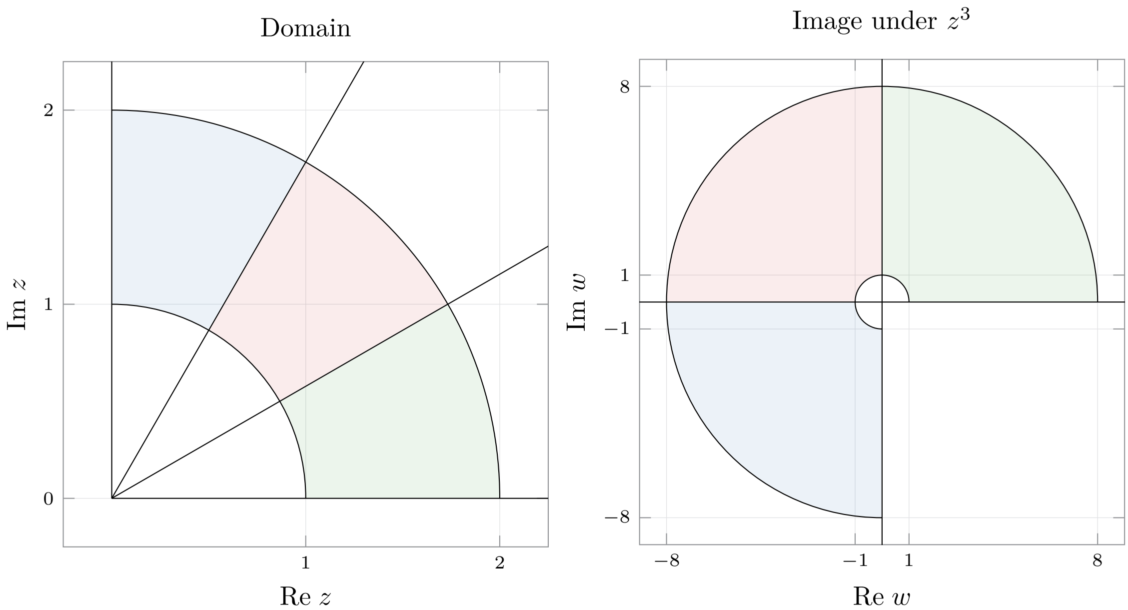
$$y = b \tag{17.1.10}$$

$$u = x^2 - b^2 \qquad v = 2xb \tag{17.1.11}$$

$$\frac{v^2}{4b^2} = u + b^2 \qquad u = \left(\frac{v}{2b}\right)^2 - b^2 \tag{17.1.12}$$

This is a parabola that opens in $+x$ and whose axis is the x axis.

3. Plotting $f(z) = z^3$ using the figure in the text,



4. Since a conformal mapping preserves the magnitude and sense of angles of intersection, except at critical points of $f(z)$, families of orthogonal curves have to remain orthogonal under the mapping.

5. Consider $f(z) = \bar{z}$. Checking it this function is analytic,

$$f(z) = u + \mathrm{i}\, v \qquad = x - \mathrm{i}\, y \tag{17.1.13}$$

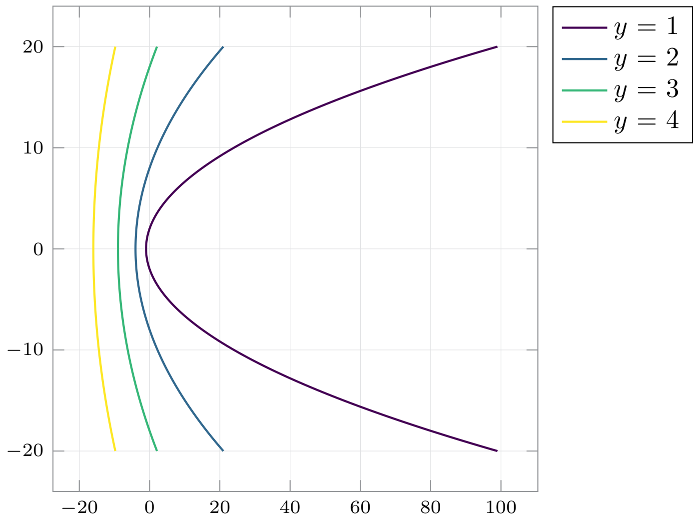
$$u_x \neq v_y \tag{17.1.14}$$

This function is not analytic, which means the map is not conformal. Although it preserves the magnitude of angles it reverses their sense.

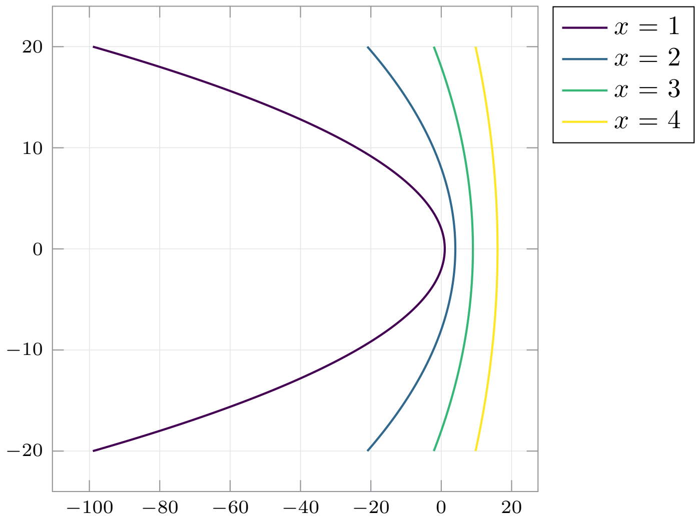
6. The function is,

$$f(z) = z^2 \qquad f(z) = (x^2 - y^2) + \mathrm{i}\,(2xy) \tag{17.1.15}$$

map of horizontal lines



map of vertical lines



7. The function is,

$$f(z) = \mathrm{i}z \qquad f(z) = -y + \mathrm{i}\, x \tag{17.1.16}$$

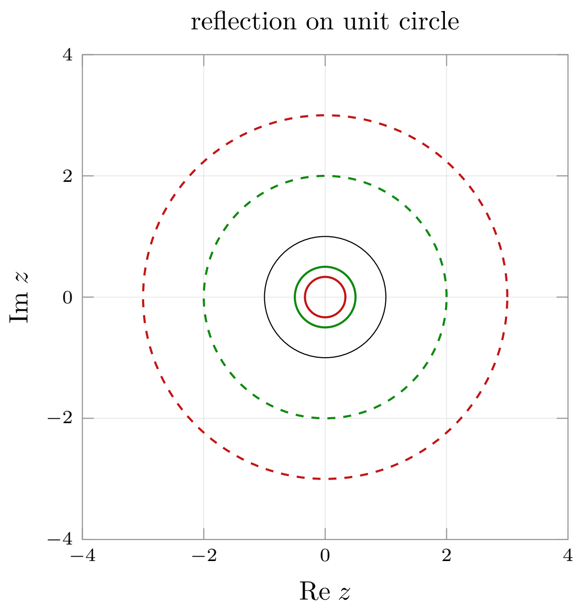
$$z = a + \mathrm{i}\, y \qquad v = a \tag{17.1.17}$$

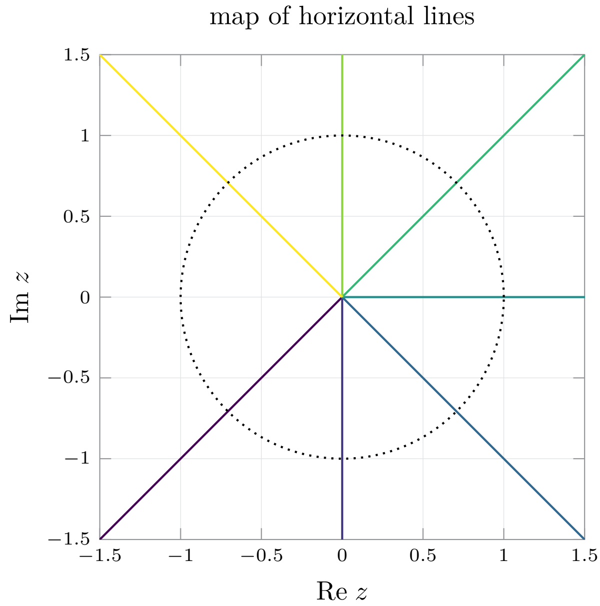
$$z = x + \mathrm{i}\, b \qquad u = -b \tag{17.1.18}$$

The image is a family of lines rotated by 90°.

8. The function is,

$$f(z) = \frac{\bar{z}}{|z|^2} = \frac{1}{z} \qquad f(z) = \frac{1}{r}\ \exp(-\mathrm{i}\theta) \tag{17.1.19}$$

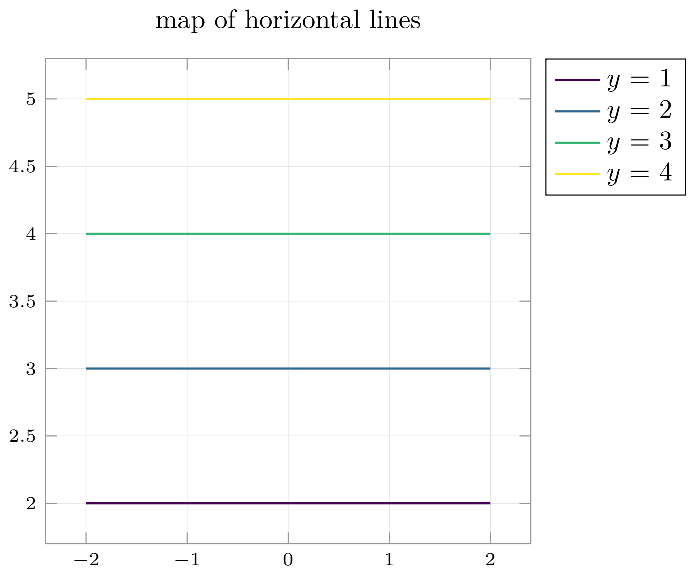


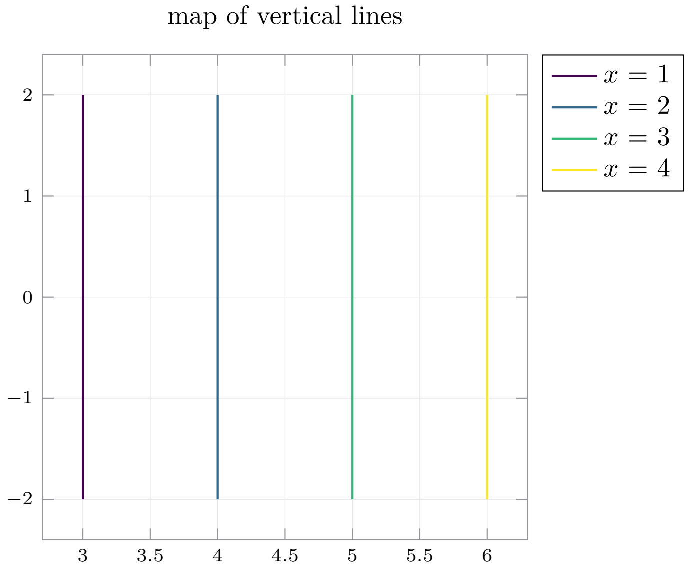


9. The function is a translation,

$$f(z) = z + (2 + \mathrm{i}) \qquad z = x + \mathrm{i}y \tag{17.1.20}$$

$$u = (x + 2) \qquad v = (y + 1) \tag{17.1.21}$$



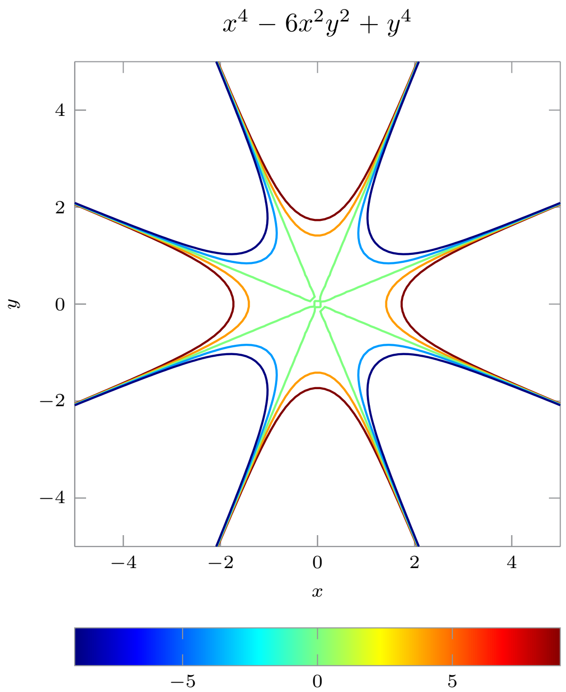


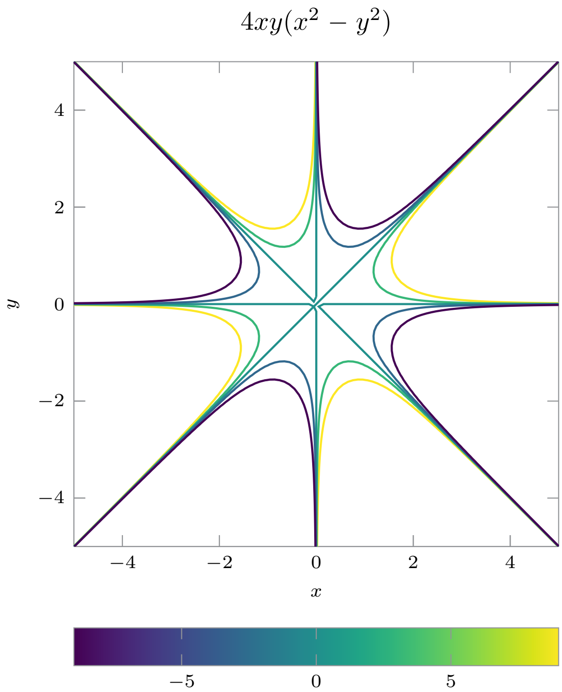
10. Graphing the families of orthogonal curves,

(a) The explicit equation for the level curves is,

$$f(z) = z^4 \qquad z = x + \mathfrak{i}\, y \tag{17.1.22}$$

$$u = x^4 - 6x^2y^2 + y^4 \qquad v = 4xy(x^2 - y^2) \tag{17.1.23}$$

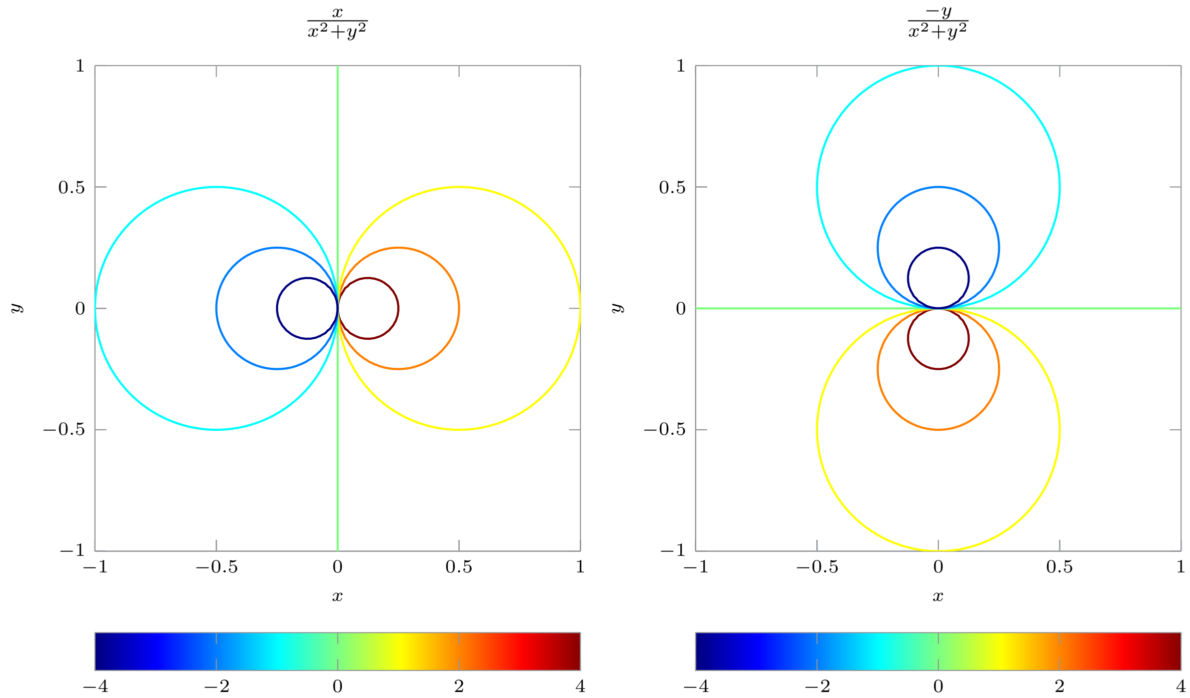




(b) The explicit equation for the level curves is,

$$f(z) = \frac{1}{z} \qquad z = x + \mathrm{i}\, y \tag{17.1.24}$$

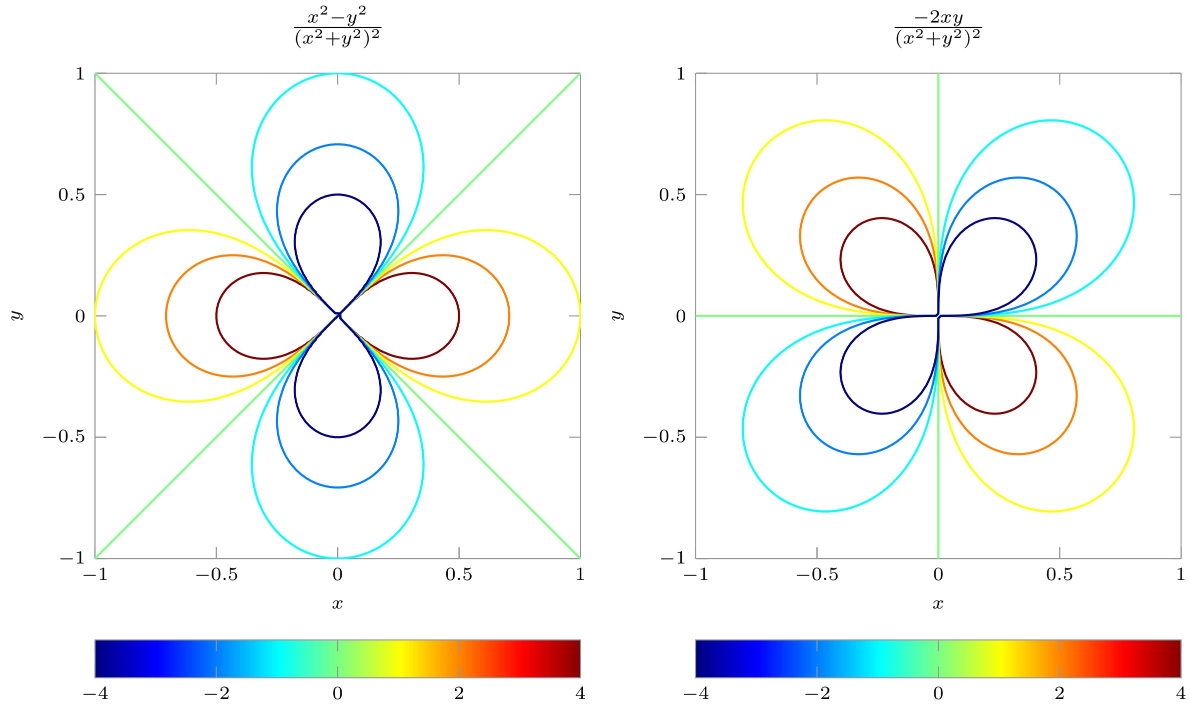
$$u = \frac{x}{x^2 + y^2} \qquad v = -\frac{y}{x^2 + y^2} \tag{17.1.25}$$



(c) The explicit equation for the level curves is,

$$f(z) = \frac{1}{z^2} \qquad z = x + \mathrm{i}\, y \tag{17.1.26}$$

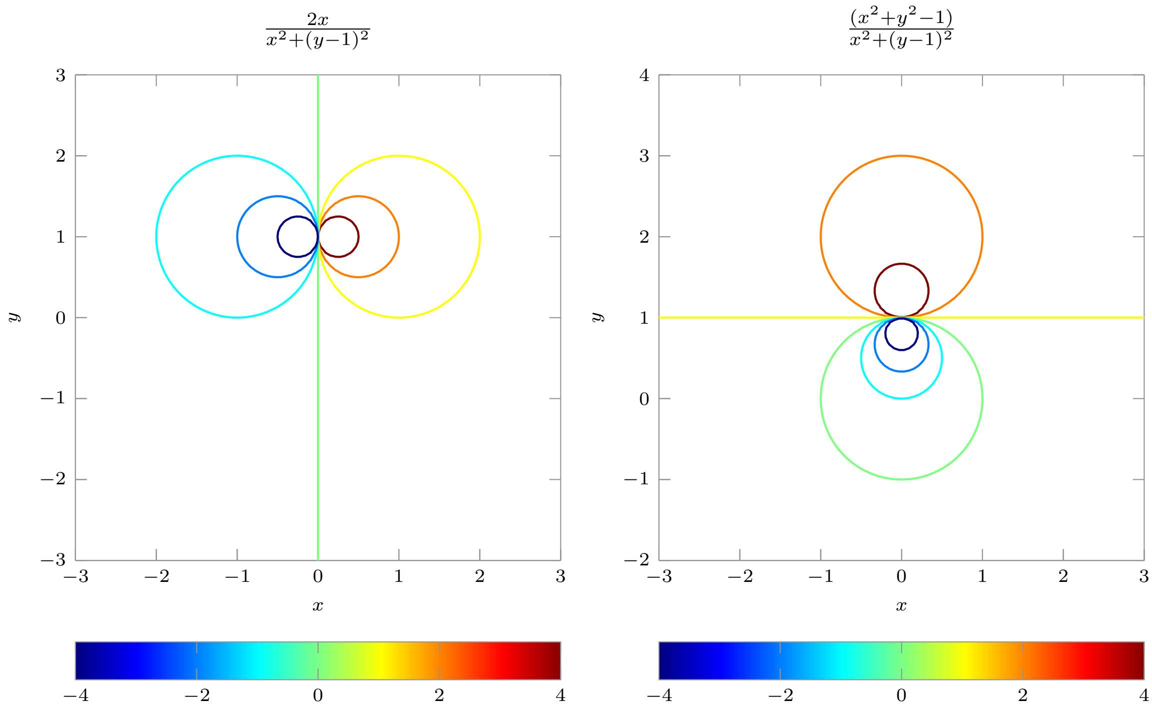
$$u = \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad v = -\frac{2xy}{(x^2 + y^2)^2} \tag{17.1.27}$$



(d) The explicit equation for the level curves is,

$$f(z) = \frac{z + \mathrm{i}}{1 + \mathrm{i}z} \qquad z = x + \mathrm{i}\, y \tag{17.1.28}$$

$$u = \frac{2x}{x^2 + (y-1)^2} \qquad v = \frac{(x^2 + y^2 - 1)}{x^2 + (y-1)^2} \tag{17.1.29}$$



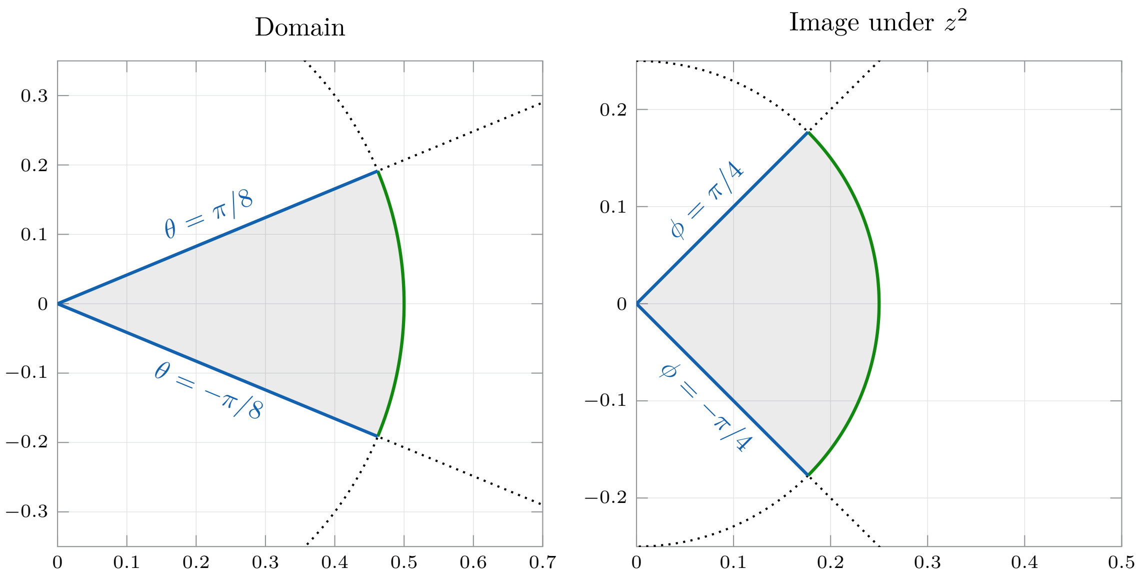
These curves intersect at right angles because the function $f(z)$ is a conformal mapping,

$$z = x + \mathrm{i}\, y \qquad\qquad f(z) = u + \mathrm{i}\, v \tag{17.1.30}$$

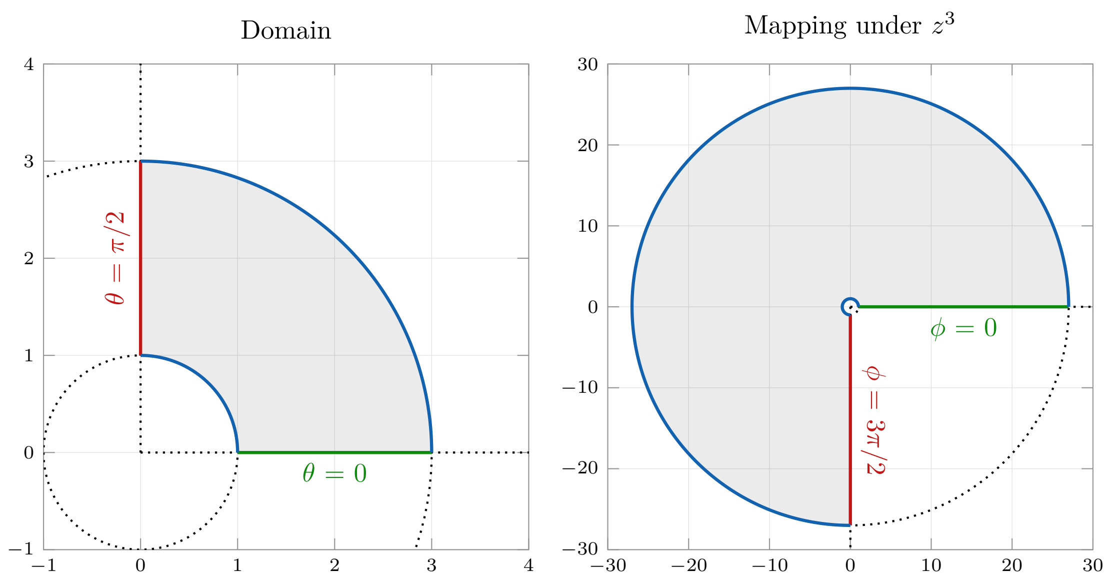
$$(x = a) \perp (y = b) \qquad \Longrightarrow \qquad (u = c_1) \perp (v = c_2) \tag{17.1.31}$$

The real and imaginary parts of z are mutually orthogonal families of curves (horizontal and vertical lines).

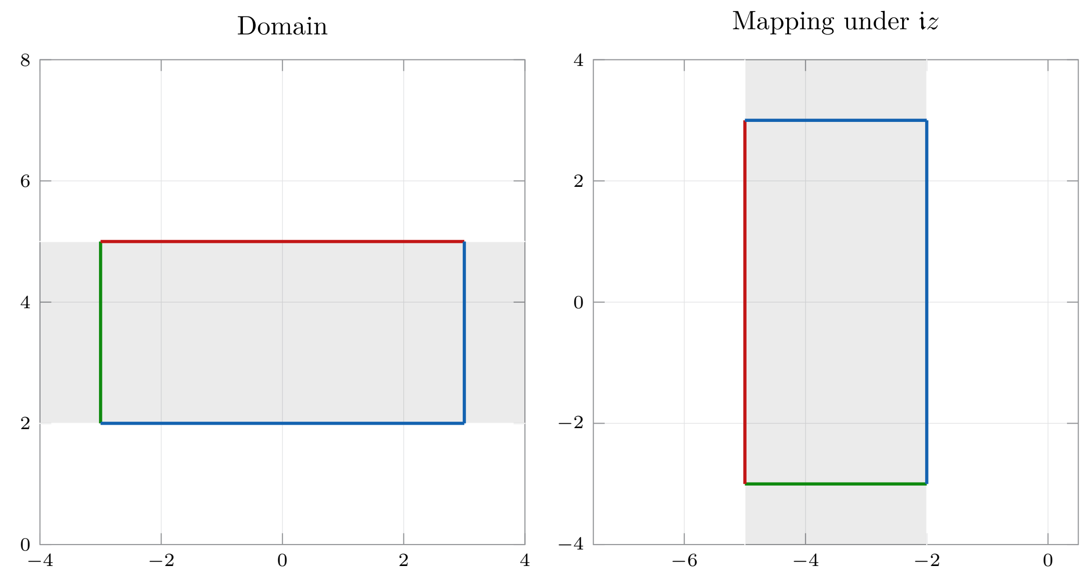
11. Plotting the result of the mapping,



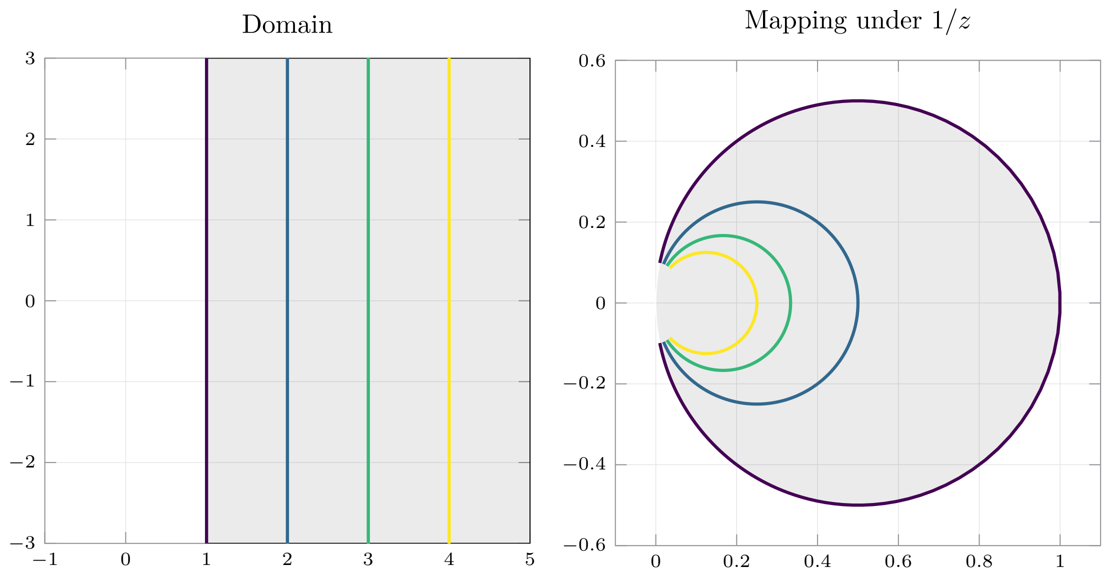
12. Plotting the result of the mapping,



13. Plotting the result of the mapping,

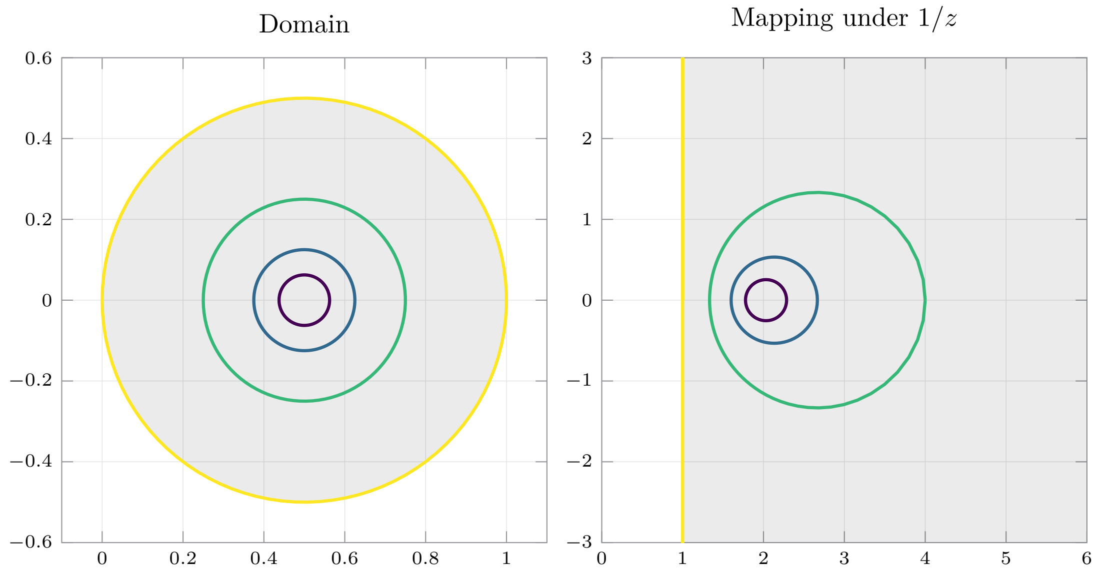


14. Plotting the result of the mapping,

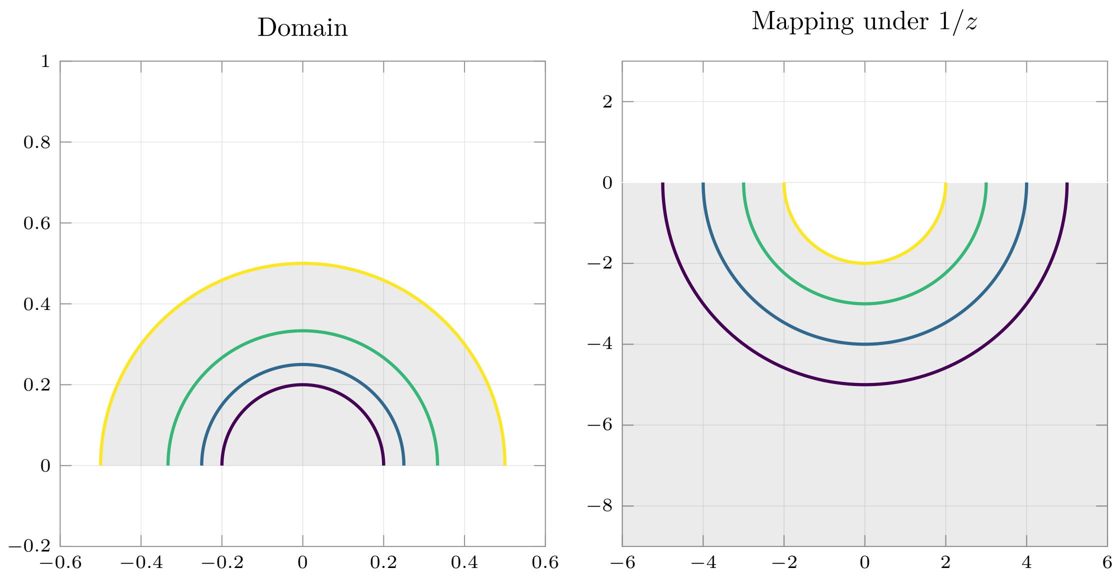


15. Plotting the result of the mapping, (inverse of Problem 14)

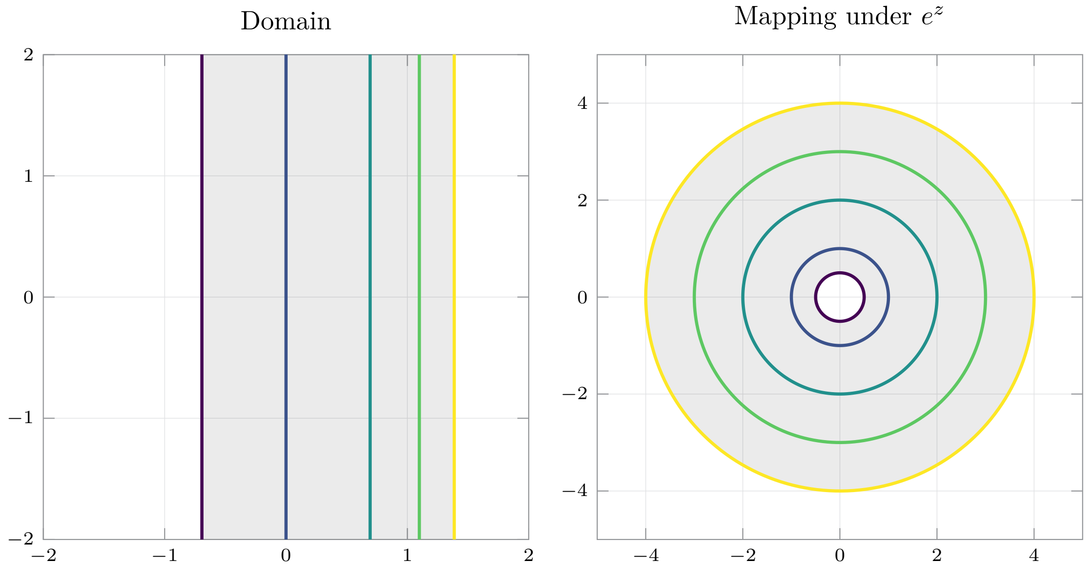
$$z = 0.5 + r\cos\theta + \mathrm{i}\, r\sin\theta \qquad\qquad f(z) = \frac{1}{z} = \frac{(0.5 + r\cos\theta) - \mathrm{i}\,(r\sin\theta)}{1/4 + r^2 + r\cos\theta} \tag{17.1.32}$$



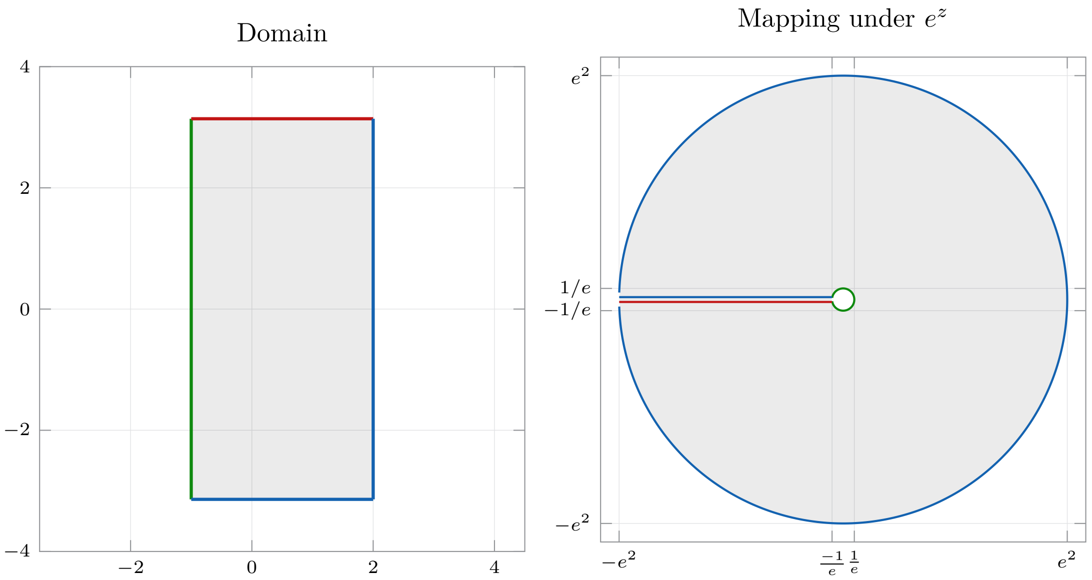
16. Plotting the result of the mapping,



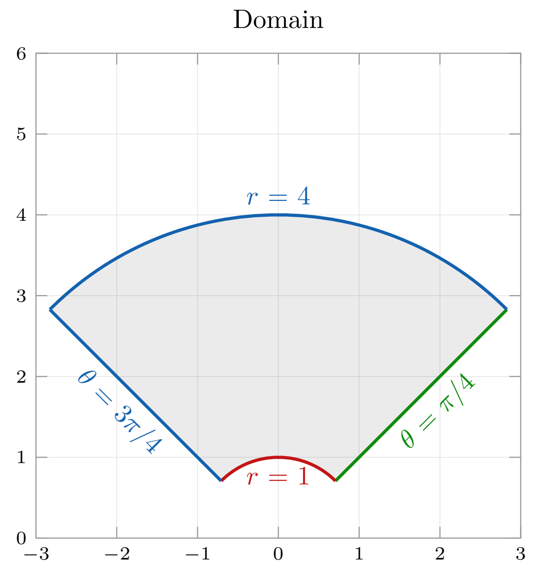
17. Plotting the result of the mapping,

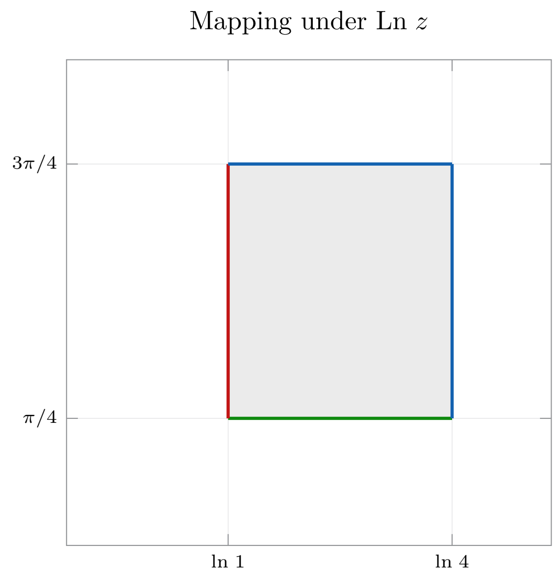


18. Plotting the result of the mapping, noting that the negative real line is not part of the image.

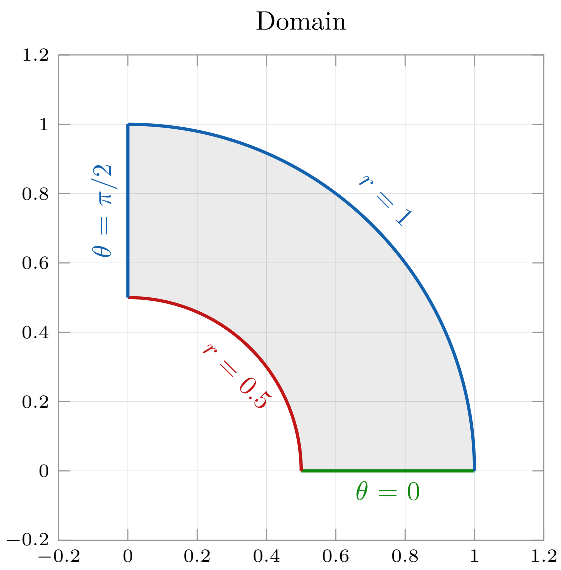


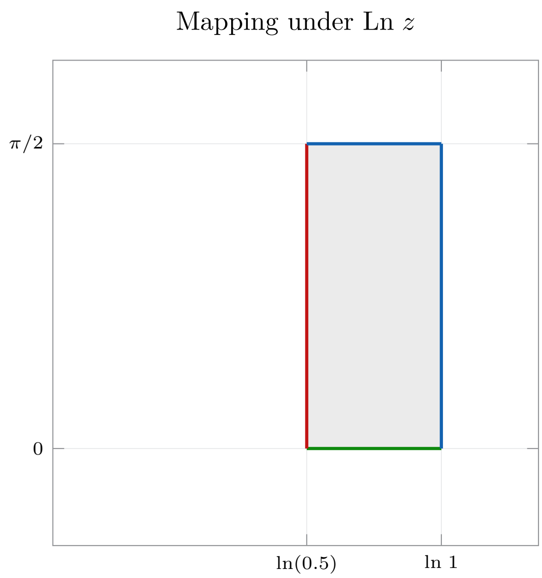
19. Plotting the result of the mapping,





20. Plotting the result of the mapping,





21. The critical points are,

$$f(z) = az^3 + bz^2 + cz + d \qquad f'(z) = 3az^2 + 2bz + c \tag{17.1.33}$$

$$z^* = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a} \tag{17.1.34}$$

22. The critical points are,

$$f(z) = z^2 + z^{-2} \qquad f'(z) = 2z - 2z^{-3} \tag{17.1.35}$$

$$z^* = \pm 1, \pm \mathrm{i} \tag{17.1.36}$$

23. The critical points are,

$$f(z) = \frac{z + 0.5}{4z^2 + 2} \qquad f'(z) = \frac{4z^2 + 2 - (z + 0.5)(8z)}{(4z^2 + 2)^2} \tag{17.1.37}$$

$$f'(z) = \frac{-4z^2 - 4z + 2}{(4z^2 + 2)^2} \qquad f'(z) = 0 \tag{17.1.38}$$

$$\implies \quad z^2 + z - 0.5 = 0 \qquad z^* = \frac{-1 \pm \sqrt{3}}{2} \tag{17.1.39}$$

24. The critical points are,

$$f(z) = \exp(z^5 - 80z) \qquad f'(z) = 5(z^4 - 16)\exp(z^5 - 80z) \tag{17.1.40}$$

$$z^* = \pm 2, \pm 2\mathfrak{i} \tag{17.1.41}$$

25. The critical points are,

$$f(z) = \cosh z \qquad f'(z) = \sinh z \tag{17.1.42}$$

$$z^* = 0 + \mathfrak{i}\, n\pi \tag{17.1.43}$$

26. The critical points are,

$$f(z) = \sin(\pi z) \qquad f'(z) = \pi \cos(\pi z) \tag{17.1.44}$$

$$z^* = n + 0.5 \tag{17.1.45}$$

27. Let the first $(k - 1)$ derivatives of $f(z)$ be zero at $z = z_0$ but not the next higher derivative. Also, $f(z)$ is analytic at z_0

$$f(z) = f(z_0) + (z - z_0)^k\, g(z) \qquad g(z_0) \neq 0 \tag{17.1.46}$$

$$f(z) - f(z_0) = (z - z_0)^k\, g(z) \tag{17.1.47}$$

Let the angle θ between two curves C_1, C_2 become the angle ϕ between the curves K_1, K_2 under the mapping. Let z_1, z_2 respectively approach z_0 along the curves K_1, K_2

$$\text{Arg}\left[f(z) - f(z_0)\right] = k \text{ Arg}\left(z - z_0\right) + \text{Arg } g(z) \tag{17.1.48}$$

$$\phi = \lim_{z_1 \to z_0} \lim_{z_2 \to z_0} \text{Arg}\left[\frac{f(z_1) - f(z_0)}{f(z_2) - f(z_0)}\right] \tag{17.1.49}$$

$$= \lim_{z_1 \to z_0} \lim_{z_2 \to z_0} k \text{ Arg}\left[\frac{(z_1 - z_0)}{(z_2 - z_0)}\right] + \text{Arg}\left[\frac{g(z_1)}{g(z_2)}\right] \tag{17.1.50}$$

$$= k\theta + 0 \tag{17.1.51}$$

Thus, the angle gets magnified k times. Examples are $f(z) = z^2$, z^3, z^4 as seen in the problem set.

28. Refer to Problem 27

29. Finding the magnification ratio,

$$f(z) = z^2/2 \qquad f'(z) = z \tag{17.1.52}$$

$$M = |f'(z)| = |z| \qquad J = |f'(z)|^2 = |z|^2 \tag{17.1.53}$$

$M = 1$ at the unit circle

30. Finding the magnification ratio,

$$f(z) = z^3 \qquad f'(z) = 3z^2 \tag{17.1.54}$$

$$M = |f'(z)| = 3\,|z|^2 \qquad J = |f'(z)|^2 = 9\,|z|^4 \tag{17.1.55}$$

$$M = 1 \qquad \Longrightarrow \qquad |z| = \frac{1}{\sqrt{3}} \tag{17.1.56}$$

31. Finding the magnification ratio,

$$f(z) = z^{-1} \qquad f'(z) = -z^{-2} \tag{17.1.57}$$

$$M = |f'(z)| = |z|^{-2} \qquad J = |f'(z)|^2 = |z|^{-4} \tag{17.1.58}$$

$$M = 1 \qquad \Longrightarrow \qquad |z| = 1 \tag{17.1.59}$$

32. Finding the magnification ratio,

$$f(z) = z^{-2} \qquad f'(z) = -2z^{-3} \tag{17.1.60}$$

$$M = |f'(z)| = 2\,|z|^{-3} \qquad J = |f'(z)|^2 = 4\,|z|^{-6} \tag{17.1.61}$$

$$M = 1 \qquad \Longrightarrow \qquad |z| = 2^{1/3} \tag{17.1.62}$$

33. Finding the magnification ratio,

$$f(z) = e^z \qquad f'(z) = e^z \tag{17.1.63}$$

$$M = |f'(z)| = e^x \qquad J = |f'(z)|^2 = e^{2x} \tag{17.1.64}$$

$$M = 1 \qquad \implies \qquad x = 0 \tag{17.1.65}$$

34. Finding the magnification ratio,

$$f(z) = \frac{z+1}{2z-2} \qquad f'(z) = \frac{-1}{(z-1)^2} \tag{17.1.66}$$

$$M = |f'(z)| = \frac{1}{|z-1|^2} \qquad J = |f'(z)|^2 = \frac{1}{|z-1|^4} \tag{17.1.67}$$

$$M = 1 \qquad \implies \qquad |z-1| = 1 \tag{17.1.68}$$

35. Finding the magnification ratio,

$$f(z) = \operatorname{Ln} z \qquad f'(z) = \frac{1}{z} \tag{17.1.69}$$

$$M = |f'(z)| = \frac{1}{|z|} \qquad J = |f'(z)|^2 = \frac{1}{|z|^2} \tag{17.1.70}$$

$$M = 1 \qquad \implies \qquad |z| = 1 \tag{17.1.71}$$

17.2 Linear Fractional Transformations (Mobius Transformations)

1. Proving the theorem,

$$w = \frac{az+b}{cz+d} \qquad w = \frac{bc-ad}{c\,(cz+d)} + \frac{c}{d} \tag{17.2.1}$$

$$w_1 = cz \qquad w_2 = w_1 + d \tag{17.2.2}$$

$$w_3 = \frac{1}{w_2} \qquad w_4 = Kw_3 \tag{17.2.3}$$

Since $ad \neq bc$ is a precondition for Mobius transformations, $K \neq 0$. A composition of the four basic kinds of linear fractional maps yields the general mapping.

For the special case of $c = 0$,

$$w = \frac{az+b}{d} \qquad\qquad w_1 = az \tag{17.2.4}$$

$$w_2 = w_1 + b \qquad\qquad w_3 = \frac{w_2}{d} \tag{17.2.5}$$

2. Substituting w into $v(w)$ gives,

$$v = \frac{a^*w + b^*}{c^*w + d^*} \tag{17.2.6}$$

$$w = \frac{az+b}{cz+d} \tag{17.2.7}$$

$$v = \frac{a^*az + a^*b + b^*cz + b^*d}{cz+d} \cdot \frac{cz+d}{c^*az + c^*b + d^*cz + d^*d} \tag{17.2.8}$$

$$v = \frac{(a^*a + b^*c)z + a^*b + b^*d}{(c^*a + d^*c)z + c^*b + d^*d} \tag{17.2.9}$$

To check if this satisfies the condition on the determinant,

$$(ad - bc) \neq 0 \qquad\qquad (a^*d^* - b^*c^*) \neq 0 \tag{17.2.10}$$

$$M = (a^*a + b^*c)(c^*b + d^*d) - (a^*b + b^*d)(c^*a + d^*c) \tag{17.2.11}$$

$$= (ad - bc)(a^*d^* - b^*c^*) \neq 0 \tag{17.2.12}$$

Thus, $g(w)$ is also an L.F.T.

3. Finding the inverse of the coefficient matrix M,

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad\qquad \mathbf{M}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{17.2.13}$$

This matches the coefficient matrix of the inverse mapping.

$$w = f(v) \qquad\qquad v = g(w) = g[f(v)] \tag{17.2.14}$$

$$M_2M_1 = \begin{bmatrix} aa^* + b^*c & a^*b + b^*d \\ c^*a + d^*c & c^*b + d^*d \end{bmatrix} \tag{17.2.15}$$

This result matches the composition of two LFTs from Problem 2.

4. Starting with a vertical line,

$$z = a + y\mathrm{i} \qquad f(z) = \frac{1}{z} \tag{17.2.16}$$

$$f(z) = \frac{a - y\mathrm{i}}{a^2 + y^2} \tag{17.2.17}$$

$$z = \frac{b}{\cos\theta}\ \exp(\mathrm{i}\theta) \qquad \theta \in (-\pi/2, \pi/2) \tag{17.2.18}$$

$$w = \frac{\cos\theta}{b}\ \exp(-\mathrm{i}\theta) \qquad w = \frac{1+\cos(2\theta)}{2b} - \mathrm{i}\,\frac{\sin(2\theta)}{2b} \tag{17.2.19}$$

This is the equation of a circle with center $1/2b$ and radius $1/2b$, which means it passes through the origin.

Refer to Problem set 17.1, problem 14.

5. Deriving the inverse transform,

$$g(w) = z \qquad \frac{ew + f}{gw + h} = z \tag{17.2.20}$$

$$w = \frac{az + b}{cz + d} \qquad z = \frac{eaz + eb + cfz + df}{agz + bg + chz + dh} \tag{17.2.21}$$

$$ea + cf = 1 \qquad eb + df = 0 \tag{17.2.22}$$

$$ag + ch = 0 \qquad bg + dh = 1 \tag{17.2.23}$$

$$f = \frac{-b}{ad - bc} \qquad e = \frac{d}{ad - bc} \tag{17.2.24}$$

$$h = \frac{a}{ad - bc} \qquad g = \frac{-c}{ad - bc} \tag{17.2.25}$$

The reverse proof follows the exact same process, simply exchaning the start and end points.

6. Finding the fixed points of the special mappings,

$$f_1(z) = z \qquad z_1 \in \mathcal{C} \tag{17.2.26}$$

$$f_2(z) = \bar{z} \qquad z_2 = \mathcal{R} \tag{17.2.27}$$

$$f_3 = \frac{1}{z} \qquad z_3 = \pm 1 + 0\,\mathrm{i} \tag{17.2.28}$$

$$f_4 = az, \qquad |a| = 1 \qquad z_4 = 0 \tag{17.2.29}$$

$$f_5 = z + b, \qquad b \neq 0 \qquad z_5 = \phi \tag{17.2.30}$$

7. Finding the inverse transform, using matrix inversion,

$$\mathbf{M} = \begin{bmatrix} 0 & \mathrm{i} \\ 2 & -1 \end{bmatrix} \qquad \mathbf{M}^{-1} = \frac{1}{(-2\mathrm{i})} \begin{bmatrix} -1 & -\mathrm{i} \\ -2 & 0 \end{bmatrix} \tag{17.2.31}$$

$$z = \frac{w + \mathrm{i}}{2w} \qquad w = \frac{\mathrm{i}}{2z - 1} \tag{17.2.32}$$

8. Finding the inverse transform, using matrix inversion,

$$\mathbf{M} = \begin{bmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{bmatrix} \qquad \mathbf{M}^{-1} = \frac{1}{(2\mathrm{i})} \begin{bmatrix} \mathrm{i} & \mathrm{i} \\ -1 & 1 \end{bmatrix} \tag{17.2.33}$$

$$z = \frac{\mathrm{i}(w + 1)}{-w + 1} \qquad w = \frac{z - \mathrm{i}}{z + \mathrm{i}} \tag{17.2.34}$$

9. Finding the inverse transform, using matrix inversion,

$$\mathbf{M} = \begin{bmatrix} 1 & -\mathrm{i} \\ 3\mathrm{i} & 4 \end{bmatrix} \qquad \mathbf{M}^{-1} = \begin{bmatrix} 4 & \mathrm{i} \\ -3\mathrm{i} & 1 \end{bmatrix} \tag{17.2.35}$$

$$z = \frac{4w + \mathrm{i}}{-3\mathrm{i}\ w + 1} \qquad w = \frac{z - \mathrm{i}}{3\mathrm{i}\ z + 4} \tag{17.2.36}$$

10. Finding the inverse transform, using matrix inversion,

$$\mathbf{M} = \begin{bmatrix} 1 & -0.5\mathrm{i} \\ -0.5\mathrm{i} & -1 \end{bmatrix} \qquad \mathbf{M}^{-1} = -0.75 \begin{bmatrix} -1 & 0.5\mathrm{i} \\ 0.5\mathrm{i} & 1 \end{bmatrix} \tag{17.2.37}$$

$$z = \frac{-w + 0.5\mathrm{i}}{0.5\mathrm{i}\ w + 1} \qquad w = \frac{-z + 0.5\mathrm{i}}{0.5\mathrm{i}\ z + 1} \tag{17.2.38}$$

11. Finding the fixed points,

$$w = (a + \mathrm{i}b)\ z^2 \qquad z^* = (a + \mathrm{i}b)\ (z^*)^2 \tag{17.2.39}$$

$$z_1 = 0 \qquad z_2 = \frac{1}{a + \mathrm{i}b} \tag{17.2.40}$$

12. Finding the fixed points,

$$w = z - 3\mathrm{i} \qquad z^* = z^* - 3\mathrm{i} \tag{17.2.41}$$

No fixed points exist.

13. Finding the fixed points, (not an LFT)

$$w = 16\,z^5 \qquad z^* = 16\,(z^*)^5 \tag{17.2.42}$$

$$z_1 = 0 \qquad z_2 = \pm\frac{1}{2},\ \pm\frac{\mathrm{i}}{2} \tag{17.2.43}$$

14. Finding the fixed points,

$$w = az + b \qquad z^* = az^* + b \tag{17.2.44}$$

$$z_1 = \frac{b}{1-a} \tag{17.2.45}$$

15. Finding the fixed points,

$$w = \frac{\mathrm{i}z + 4}{2z - 5\mathrm{i}} \qquad 2(z^*)^2 - 6\mathrm{i}\,z^* - 4 = 0 \tag{17.2.46}$$

$$z_1 = \mathrm{i} \qquad z_2 = 2\mathrm{i} \tag{17.2.47}$$

16. Finding the fixed points, given $a \neq 1$

$$w = \frac{a\mathrm{i}\,z - 1}{z + a\mathrm{i}} \qquad (z^*)^2 + 1 = 0 \tag{17.2.48}$$

$$z_1 = \mathrm{i} \qquad z_2 = -\mathrm{i} \tag{17.2.49}$$

17. Rotation about the origin

18. Reflection about the unit sphere $w = 1/z$

19. Brute forcing,

$$\frac{a\mathrm{i} + b}{c\mathrm{i} + d} = \mathrm{i} \qquad \frac{-a\mathrm{i} + b}{-c\mathrm{i} + d} = -\mathrm{i} \tag{17.2.50}$$

$$a\mathrm{i} + b = -c + d\mathrm{i} \qquad -a\mathrm{i} + b = c - d\mathrm{i} \tag{17.2.51}$$

$$f(w) = \frac{az + b}{-bz + a} \tag{17.2.52}$$

20. Any translation by a nonzero complex number.

17.3 Special Linear Fractional Transformations

1. For Examples $1 - 5$ from the text

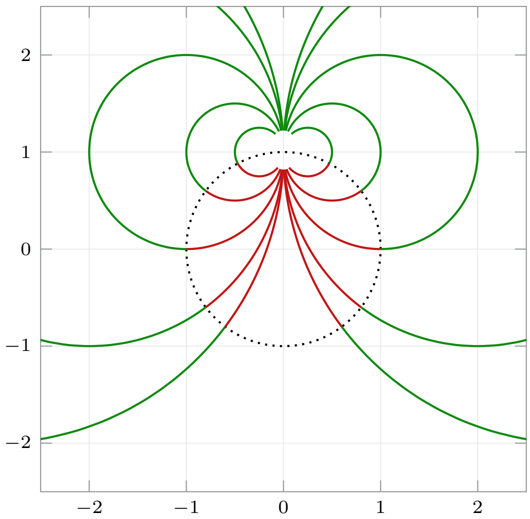
(a) Plotting the mapping of lines and concentric circles,

$$w = \frac{z - \mathfrak{i}}{-\mathfrak{i}z + 1} = \frac{x + (y-1)\,\mathfrak{i}}{(y+1) - x\,\mathfrak{i}} \tag{17.3.1}$$

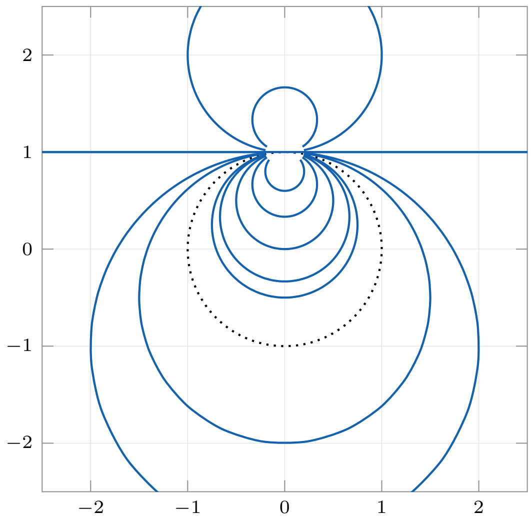
$$w = \frac{2x + \mathfrak{i}\,(x^2 + y^2 - 1)}{x^2 + (y+1)^2} \tag{17.3.2}$$

In the first plot, the red and green parts represent the upper and lower half plane segments of vertical lines respectively.
In the second plot, the blue and yellow curves represent horizontal lines in the lower and upper half planes respectively.

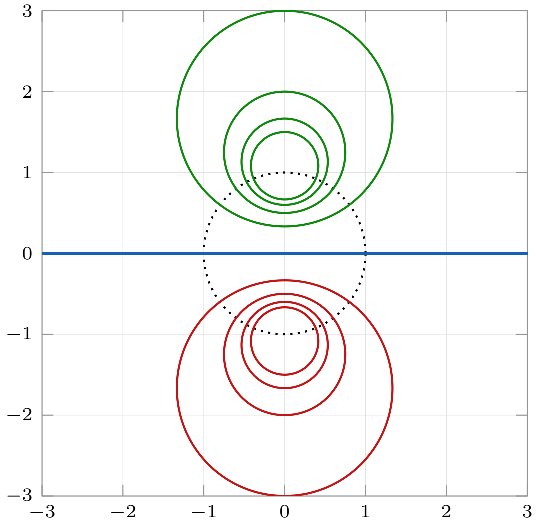
map of vertical lines



map of horizontal lines



map of concentric circles



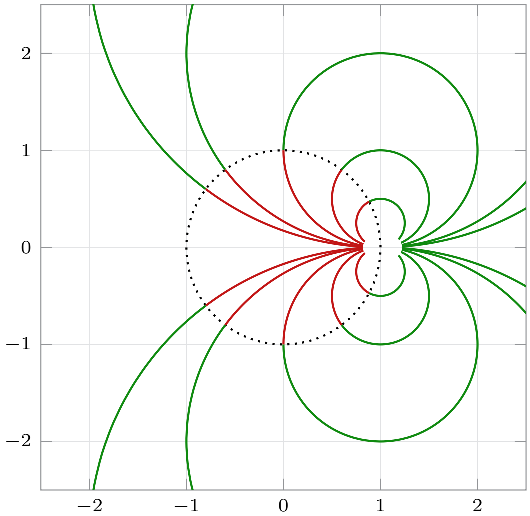
(b) Plotting the mapping of lines and concentric circles,

$$w = \frac{z - \mathrm{i}}{z + \mathrm{i}} = \frac{x + (y-1)\,\mathrm{i}}{x + (y+1)\,\mathrm{i}} \tag{17.3.3}$$

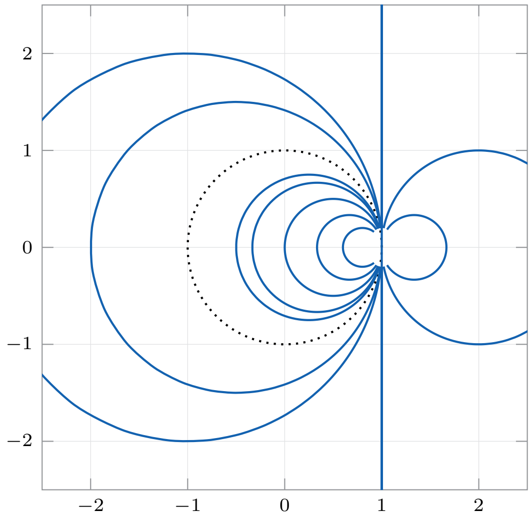
$$w = \frac{(x^2 + y^2 - 1) + \mathrm{i}\,(-2x)}{x^2 + (y+1)^2} \tag{17.3.4}$$

Same as part a except rotated 90°

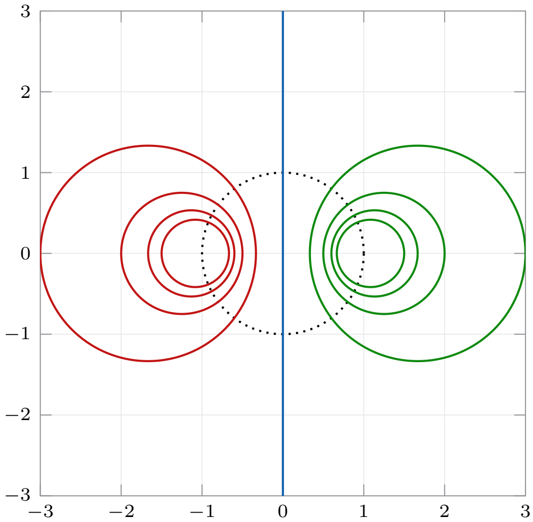
map of vertical lines



map of horizontal lines



map of concentric circles



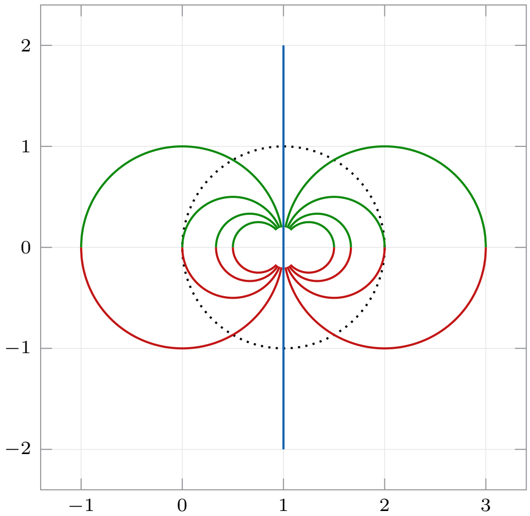
(c) Plotting the mapping of lines and concentric circles,

$$w = \frac{z + 1}{z - 1} = \frac{(x+1) + y\,\mathrm{i}}{(x-1) + y\,\mathrm{i}} \tag{17.3.5}$$

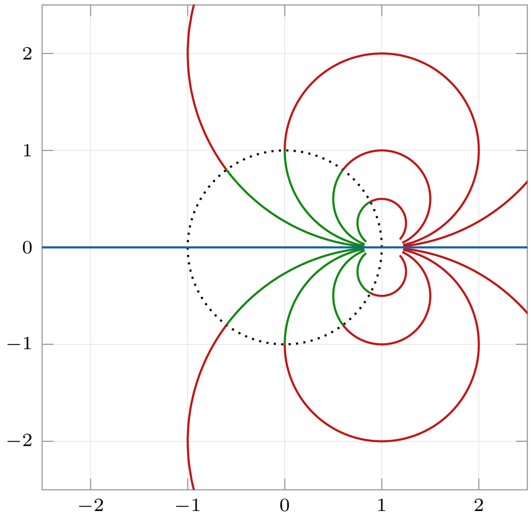
$$w = \frac{(x^2 + y^2 - 1) + \mathrm{i}\,(-2y)}{(x-1)^2 + y^2} \tag{17.3.6}$$

Vertical lines are mapped depending on their distance from $x = 1$.

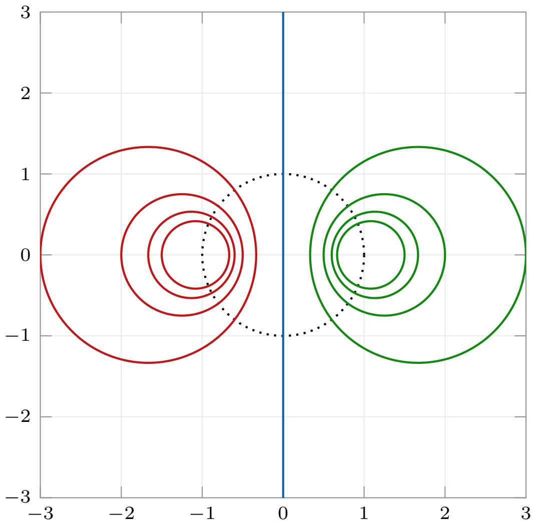
map of vertical lines



map of horizontal lines



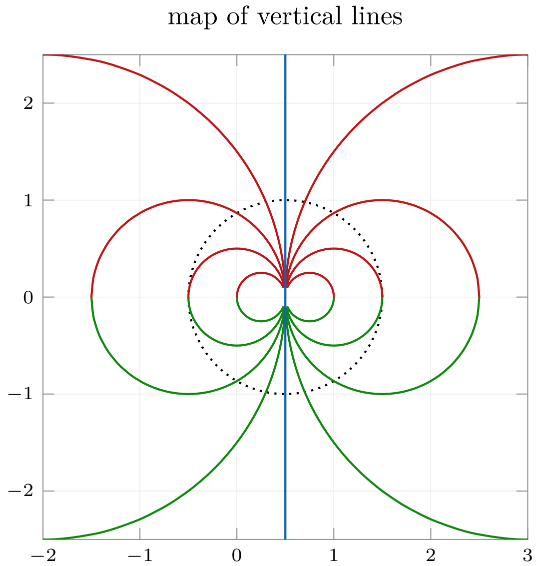
map of concentric circles

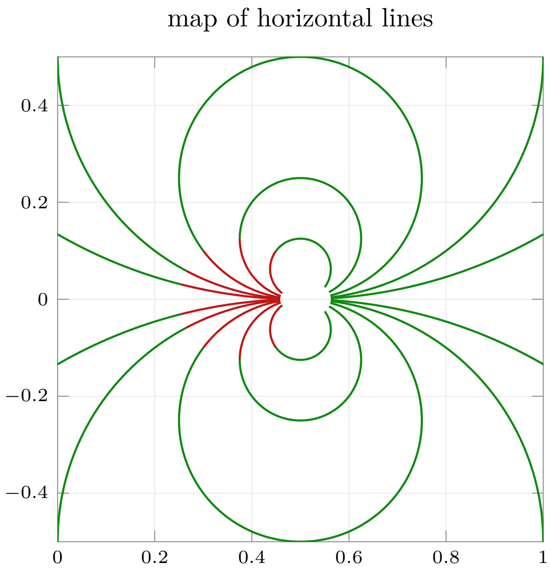


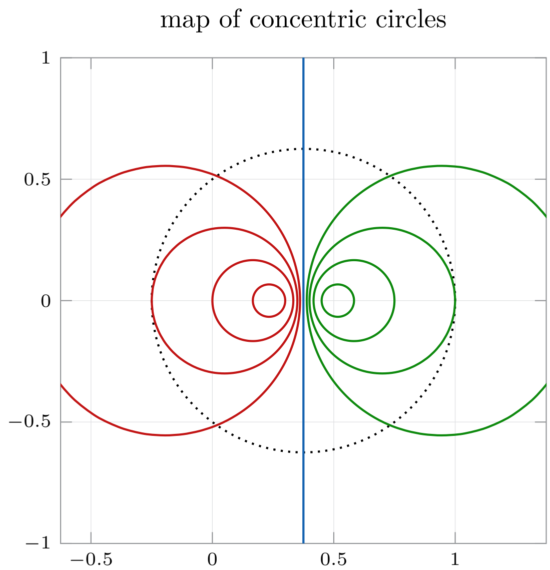
(d) Plotting the mapping of lines and concentric circles,

$$w = \frac{z+1}{2z+4} = \frac{(x+1) + y\,\mathrm{i}}{(2x+4) + 2y\,\mathrm{i}} \tag{17.3.7}$$

$$w = \frac{(2x^2 + 2y^2 + 6x + 4) + \mathrm{i}\,(2y)}{(2x+4)^2 + (2y)^2} \tag{17.3.8}$$



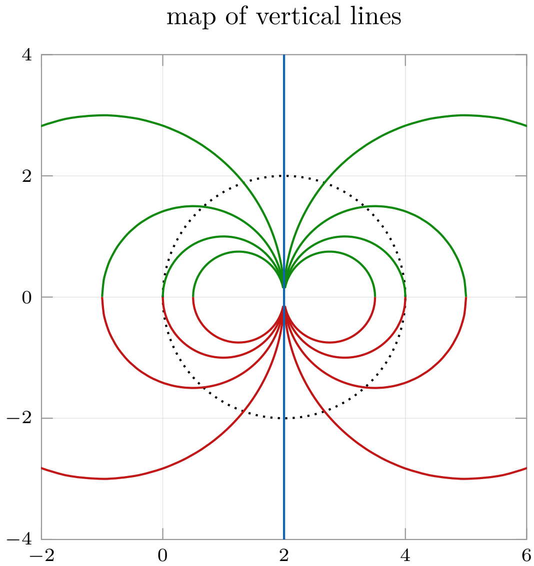


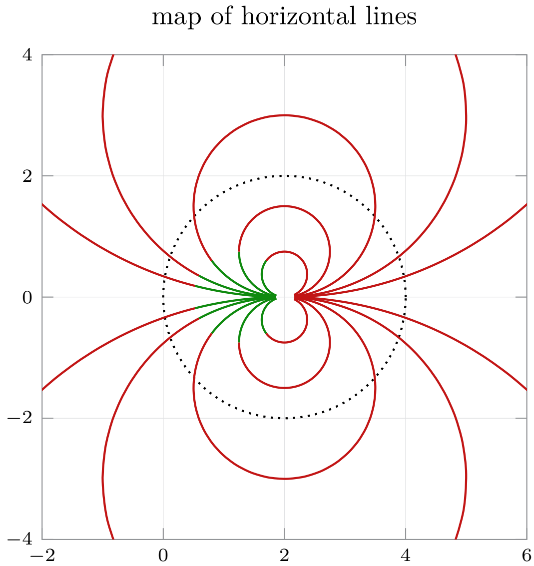


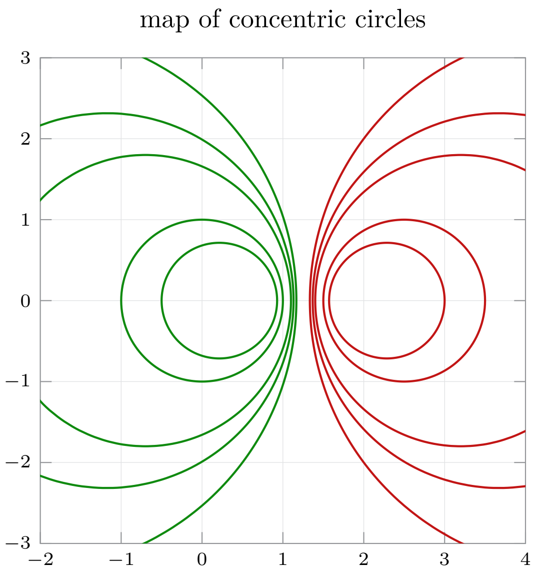
(e) Plotting the mapping of lines and concentric circles,

$$w = \frac{2z-1}{z-2} = \frac{(2x-1)+2y\,\mathrm{i}}{(x-2)+y\,\mathrm{i}} \tag{17.3.9}$$

$$w = \frac{(2x^2+2y^2-5x+2)-\mathrm{i}\,(3y)}{(x-2)^2+y^2} \tag{17.3.10}$$







Cannot do animations on paper. Performed in `sympy`.

2. Finding the inverse,

$$w = \frac{z - \mathrm{i}}{-\mathrm{i}z + 1} \qquad z = \frac{w + \mathrm{i}}{\mathrm{i}w + 1} \tag{17.3.11}$$

$$w = u + \mathrm{i}v \qquad z = \frac{u + (v+1)\,\mathrm{i}}{(1-v) + \mathrm{i}\,u} \tag{17.3.12}$$

$$z = \frac{2u + \mathrm{i}\,(1 - u^2 - v^2)}{(v-1)^2 + u^2} \tag{17.3.13}$$

$$z = c + y\mathrm{i} \qquad \Longrightarrow \qquad 0 = u^2 - \frac{2u}{c} + (v-1)^2 \tag{17.3.14}$$

$$\frac{1}{c^2} = \left[u - \frac{1}{c}\right]^2 + (v-1)^2 \tag{17.3.15}$$

3. f has an inverse, call it g.

$$w = f(z^*) = z^* \qquad\qquad g(z^*) = g[f(z^*)] = z^* \tag{17.3.16}$$

This means that z^* is also a fixed point of the inverse map g.

4. Given the fixed points,

$$\frac{a(1)+b}{c(1)+d} = 1 \qquad\qquad \frac{a(-1)+b}{c(-1)+d} = -1 \tag{17.3.17}$$

$$a+b = c+d \qquad\qquad b-a = c-d \tag{17.3.18}$$

$$b = c \qquad\qquad a = d \tag{17.3.19}$$

$$f(z) = \frac{az+b}{bz+a} \tag{17.3.20}$$

A third mapping is from $0 \to -\mathrm{i}$,

$$b = -\mathrm{i}a \qquad\qquad f(z) = \frac{z-\mathrm{i}}{-\mathrm{i}z+1} \tag{17.3.21}$$

The third mapped point is thus necessary to uniquely specify the L.F.T.

5. The three pairs of mapped points are $(0 \to -1)$, $(1 \to -\mathrm{i})$, $(\infty \to 1)$

$$\frac{w+1}{w-1} \cdot \left[\frac{-1-\mathrm{i}}{1-\mathrm{i}}\right] = \frac{z}{z-\infty} \cdot \left[\frac{1-\infty}{1}\right] \qquad\qquad \frac{w+1}{w-1} = \mathrm{i}z \tag{17.3.22}$$

$$w+1 = w\,\mathrm{i}z - \mathrm{i}z \qquad\qquad w = \frac{z-\mathrm{i}}{z+\mathrm{i}} \tag{17.3.23}$$

6. The three pairs of mapped points are $(-2 \to \infty)$, $(0 \to 1/4)$, $(2 \to 3/8)$

$$\frac{w-\infty}{w-3/8} \cdot \left[\frac{1/4-3/8}{1/4-\infty}\right] = \frac{z+2}{z-2} \cdot \left[\frac{-2}{2}\right] \qquad\qquad \frac{1}{8w-3} = \frac{z+2}{z-2} \tag{17.3.24}$$

$$4z+4 = 8wz+16w \qquad\qquad w = \frac{z+1}{2z+4} \tag{17.3.25}$$

The inverse and fixed points are,

$$z = \frac{4w-1}{-2w+1} \qquad\qquad z^* = \frac{z^*+1}{2z^*+4} \tag{17.3.26}$$

$$2(z^*)^2 + 3z^* - 1 = 0 \qquad\qquad z^* = \frac{-3 \pm \sqrt{17}}{4} \tag{17.3.27}$$

7. Deriving the disk mapping function,

$$f(z) = \frac{az+b}{cz+d} \qquad f(z_0) = 0 \tag{17.3.28}$$

$$|z| = 1 \implies |w| = 1 \tag{17.3.29}$$

$$b = -az_0 \qquad f(z) = \frac{a(z-z_0)}{cz+d} \tag{17.3.30}$$

$$f(z) = \frac{z-z_0}{c^*z+d^*} \tag{17.3.31}$$

$$|z| = 1 \implies |z-z_0| = |\bar{z}|\,|z-z_0| \tag{17.3.32}$$

$$= |z|\,|\bar{z}-\bar{z}_0| \qquad |c^*z+d^*| = |1-z\cdot\bar{z}_0| \tag{17.3.33}$$

By convention, $d^* = 1$, $c^* = \bar{z}_0$ and the formula is proved.

8. The three pairs of mapped points are $(0 \to 1)$, $(1 \to 1/2)$, $(2 \to 1/3)$

$$\frac{w-1}{w-1/3}\cdot\left[\frac{1/2-1/3}{1/2-1}\right] = \frac{z}{z-2}\cdot\left[\frac{-1}{1}\right] \qquad \frac{w-1}{3w-1} = \frac{z}{z-2} \tag{17.3.34}$$

$$-2w+2 = 2wz \qquad w = \frac{1}{z+1} \tag{17.3.35}$$

9. The three pairs of mapped points are $(1 \to \mathrm{i})$, $(\mathrm{i} \to -1)$, $(-1 \to -\mathrm{i})$

$$\frac{w-\mathrm{i}}{w+\mathrm{i}}\cdot\left[\frac{-1+\mathrm{i}}{-1-\mathrm{i}}\right] = \frac{z-1}{z+1}\cdot\left[\frac{1+\mathrm{i}}{-1+\mathrm{i}}\right] \qquad \frac{w-\mathrm{i}}{w+\mathrm{i}} = \frac{z-1}{z+1} \tag{17.3.36}$$

$$2w = 2\mathrm{i}z \qquad w = \mathrm{i}z \tag{17.3.37}$$

10. The three pairs of mapped points are $(0 \to -1)$, $(-\mathrm{i} \to 0)$, $(\mathrm{i} \to \infty)$

$$\frac{w+1}{w-\infty}\cdot\left[\frac{-\infty}{1}\right] = \frac{z}{z-\mathrm{i}}\cdot\left[\frac{-2\mathrm{i}}{-\mathrm{i}}\right] \qquad \frac{w+1}{1} = \frac{2z}{z-\mathrm{i}} \tag{17.3.38}$$

$$wz - \mathrm{i}w - \mathrm{i} = z \qquad w = \frac{z+\mathrm{i}}{z-\mathrm{i}} \tag{17.3.39}$$

11. The three pairs of mapped points are $(-1 \to -\mathrm{i})$, $(0 \to -1)$, $(1 \to \mathrm{i})$

$$\frac{w+\mathrm{i}}{w-\mathrm{i}}\cdot\left[\frac{-1-\mathrm{i}}{-1+\mathrm{i}}\right] = \frac{z+1}{z-1}\cdot\left[\frac{-1}{1}\right] \qquad \frac{-\mathrm{i}w+1}{w-\mathrm{i}} = \frac{z+1}{z-1} \tag{17.3.40}$$

$$w(-\mathrm{i}z+\mathrm{i}-z-1) = -\mathrm{i}z-\mathrm{i}-z+1 \qquad w = \frac{z+\mathrm{i}}{z-\mathrm{i}} \tag{17.3.41}$$

12. The three pairs of mapped points are $(0 \to -1)$, $(2\mathrm{i} \to 0)$, $(-2\mathrm{i} \to \infty)$

$$\frac{w+1}{w-\infty} \cdot \left[\frac{0-\infty}{1}\right] = \frac{z}{z+2\mathrm{i}} \cdot \left[\frac{4\mathrm{i}}{2\mathrm{i}}\right] \qquad \frac{w+1}{1} = \frac{2z}{z+2\mathrm{i}} \tag{17.3.42}$$

$$w(z+2\mathrm{i}) = z - 2\mathrm{i} \qquad w = \frac{z-2\mathrm{i}}{z+2\mathrm{i}} \tag{17.3.43}$$

13. The three pairs of mapped points are $(0 \to \infty)$, $(1 \to 1)$, $(\infty \to 0)$

$$w = \frac{1}{z} \tag{17.3.44}$$

14. The three pairs of mapped points are $(-1 \to 1)$, $(0 \to 1+\mathrm{i})$, $(1 \to 1+2\mathrm{i})$

$$\frac{w-1}{w-1-2\mathrm{i}} \cdot \left[\frac{-\mathrm{i}}{\mathrm{i}}\right] = \frac{z+1}{z-1} \cdot \left[\frac{-1}{1}\right] \qquad \frac{w-1}{w-1-2\mathrm{i}} = \frac{z+1}{z-1} \tag{17.3.45}$$

$$w = 1 + \mathrm{i} + \mathrm{i}z \qquad w = \mathrm{i}\, z + (1+\mathrm{i}) \tag{17.3.46}$$

15. The three pairs of mapped points are $(1 \to 0)$, $(\mathrm{i} \to -1-\mathrm{i})$, $(2 \to -1/2)$

$$\frac{w}{w+1/2} \cdot \left[\frac{1/2+\mathrm{i}}{1+\mathrm{i}}\right] = \frac{z-1}{z-2} \cdot \left[\frac{2-\mathrm{i}}{1-\mathrm{i}}\right] \qquad \frac{w}{2w+1} = \frac{z-1}{z-2} \tag{17.3.47}$$

$$w(z-2-2z+2) = z-1 \qquad w = \frac{1}{z} - 1 \tag{17.3.48}$$

16. The three pairs of mapped points are $(-3/2 \to 0)$, $(0 \to 3/2)$, $(1 \to 1)$

$$\frac{w}{w-1} \cdot \left[\frac{1/2}{3/2}\right] = \frac{z+3/2}{z-1} \cdot \left[\frac{-1}{3/2}\right] \qquad \frac{w}{3w-3} = \frac{2z+3}{-3z+3} \tag{17.3.49}$$

$$w(-3z+3-6z-9) = -6z-9 \qquad w = \frac{2z+3}{3z+2} \tag{17.3.50}$$

17. The L.F.T. is, (using the disk mapping formula),

$$w = \frac{az+b}{cz+d} \qquad 0 = \frac{a\mathrm{i}}{2} + b \tag{17.3.51}$$

$$w = \frac{(z-0.5\mathrm{i})}{cz+d} \tag{17.3.52}$$

$$|z_0| < 1 \qquad \Longrightarrow \qquad w = \frac{2z-\mathrm{i}}{-\mathrm{i}z-2} \tag{17.3.53}$$

18. The L.F.T. is,

$$w = \frac{az+b}{cz+d} \qquad\qquad u = \frac{ax+b}{cx+d} \tag{17.3.54}$$

$$\text{Im } (ax+b)(\bar{c}x+\bar{d}) = 0 \qquad\qquad \text{Im } a\bar{c}\, x^2 + (b\bar{c}+a\bar{d})\, x + b\bar{d} = 0 \tag{17.3.55}$$

This is a quadratic equation in x that is identically zero.

$$x^2\ (a_2c_1 - a_1c_2) + x\ (b_2c_1 - b_1c_2 + a_2d_1 - a_1d_2) + (b_2d_1 - b_1d_2) = 0 \tag{17.3.56}$$

$$F(0) = 0 \qquad \Longrightarrow \qquad \frac{d_1}{d_2} = \frac{b_1}{b_2} \tag{17.3.57}$$

$$F(1) = 0 = F(-1) \qquad \Longrightarrow \qquad \frac{a_1}{a_2} = \frac{c_1}{c_2} \tag{17.3.58}$$

$$(a_2b_1 - a_1b_2)\left(-\frac{c_2}{a_2} + \frac{d_1}{b_1} + \frac{d_2}{b_2} - \frac{c_1}{a_1}\right) = 0 \tag{17.3.59}$$

$$(a_2b_1 - a_1b_2)\left(-\frac{c_2}{a_2} + \frac{d_1}{b_1}\right) = 0 \tag{17.3.60}$$

The second term being zero, violates the condition $ad \neq bc$, which forces the first term to be zero. Thus,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2} \tag{17.3.61}$$

19. The mapping $f(z) = z^4$ takes the given region into the upper half plane. Next, in order to map the upper half plane onto the unit disk,

$$w = \frac{z^4 - \mathrm{i}}{-\mathrm{i}z^4 + 1} \tag{17.3.62}$$

This is a composition of two L.F.T. operations.

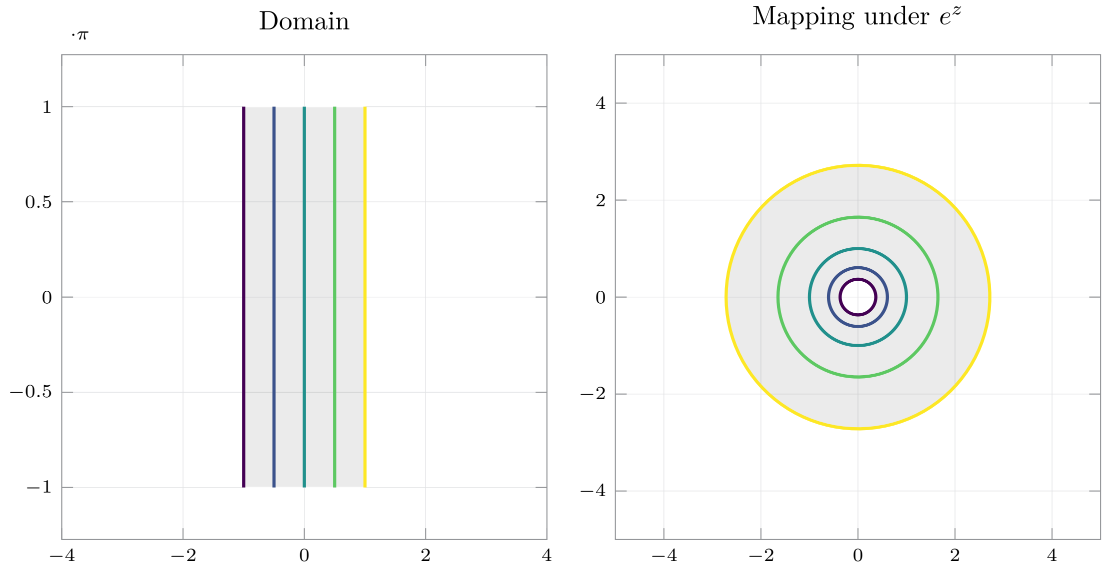
20. The mapping $f(z) = z^2$ takes the first quadrant into the upper half plane. Next, in order to map the upper half plane onto the unit disk,

$$w = \frac{z^2 - \mathrm{i}}{-\mathrm{i}z^2 + 1} \tag{17.3.63}$$

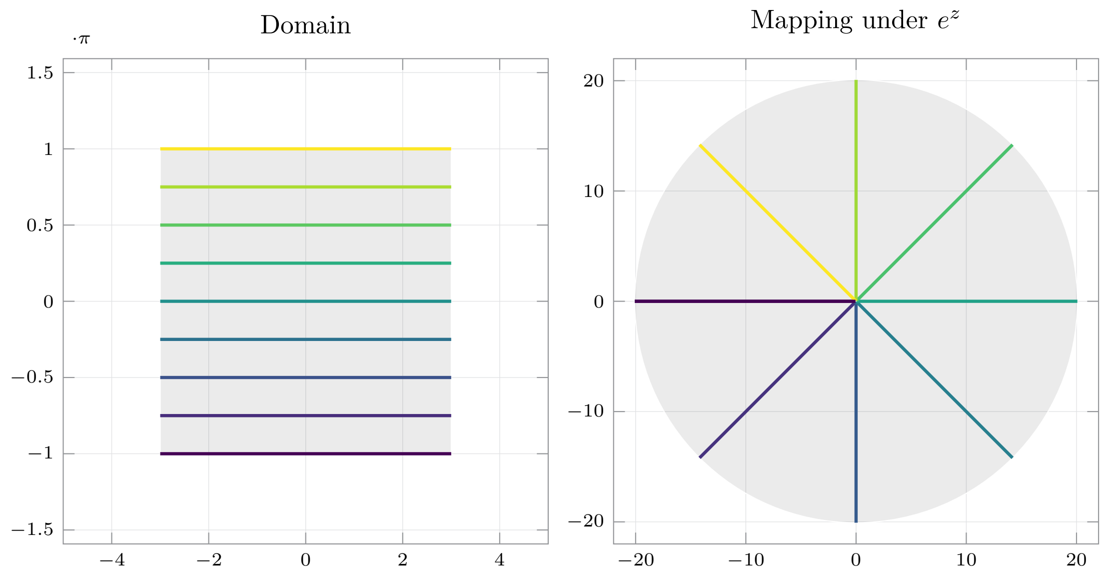
This is a composition of two L.F.T. operations.

17.4 Conformal Mapping by Other Functions

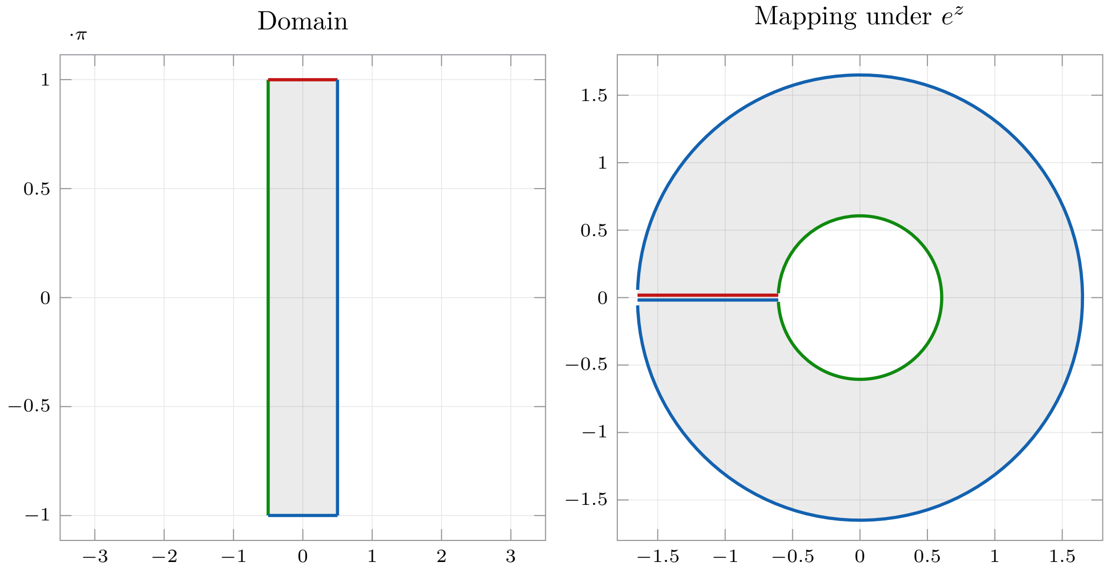
1. Mapping under $f(z) = e^z$



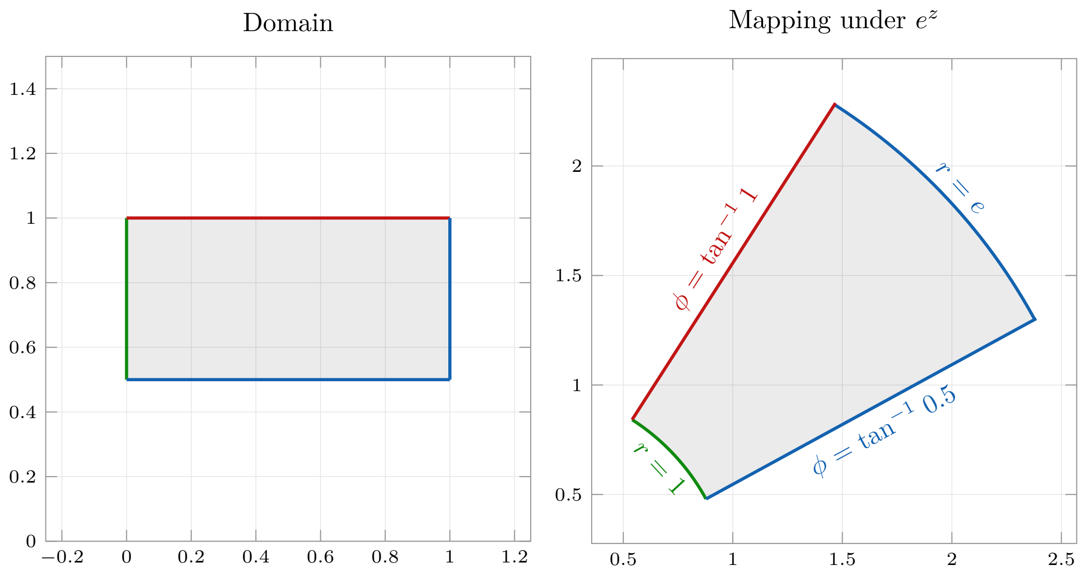
2. Mapping under $f(z) = e^z$



3. Mapping under $f(z) = e^z$

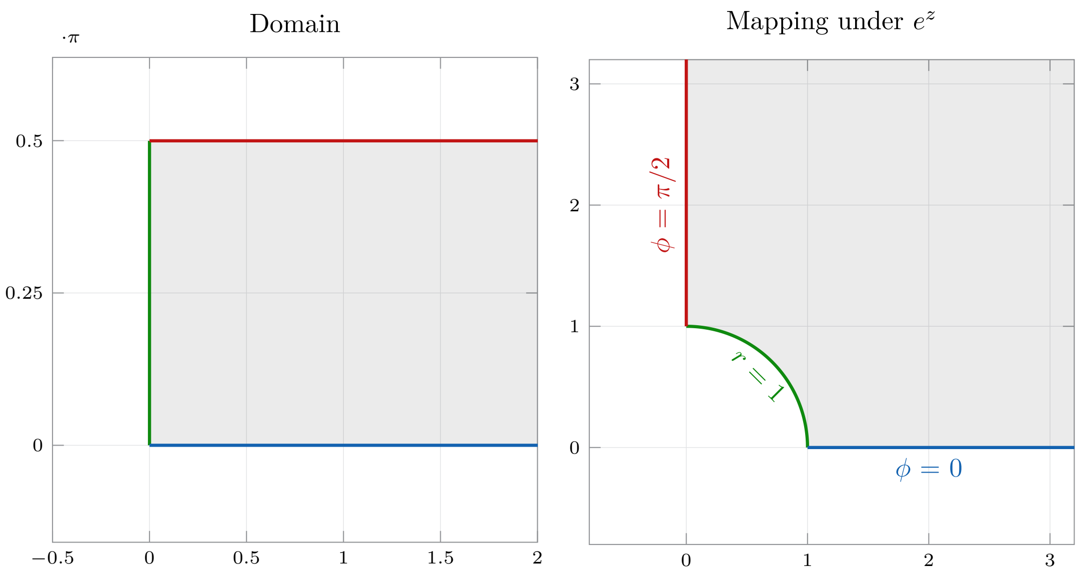


4. Mapping under $f(z) = e^z$

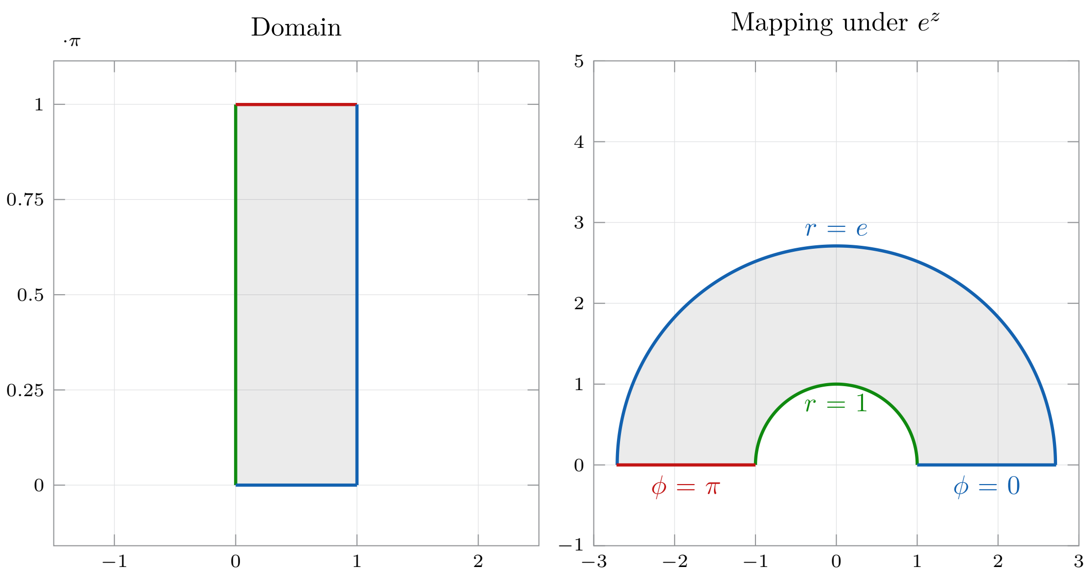


5. This maps onto the entire w-plane except the origin.

6. Mapping under $f(z) = e^z$



7. Mapping under $f(z) = e^z$



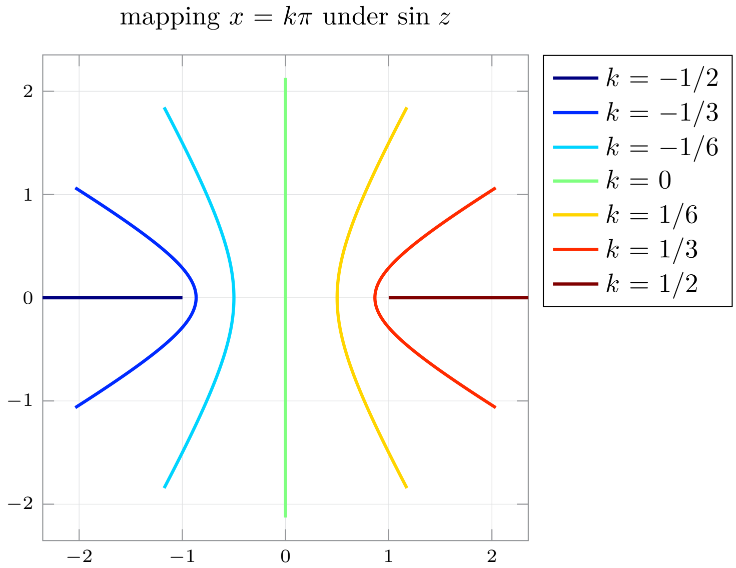
8. Since e^z is periodic in y with $\exp(z + 2\pi \mathrm{i}) = e^z$, the full annulus is repeatedly mapped over when increasing y beyond 2π.

9. To find the points at which the function is not conformal,

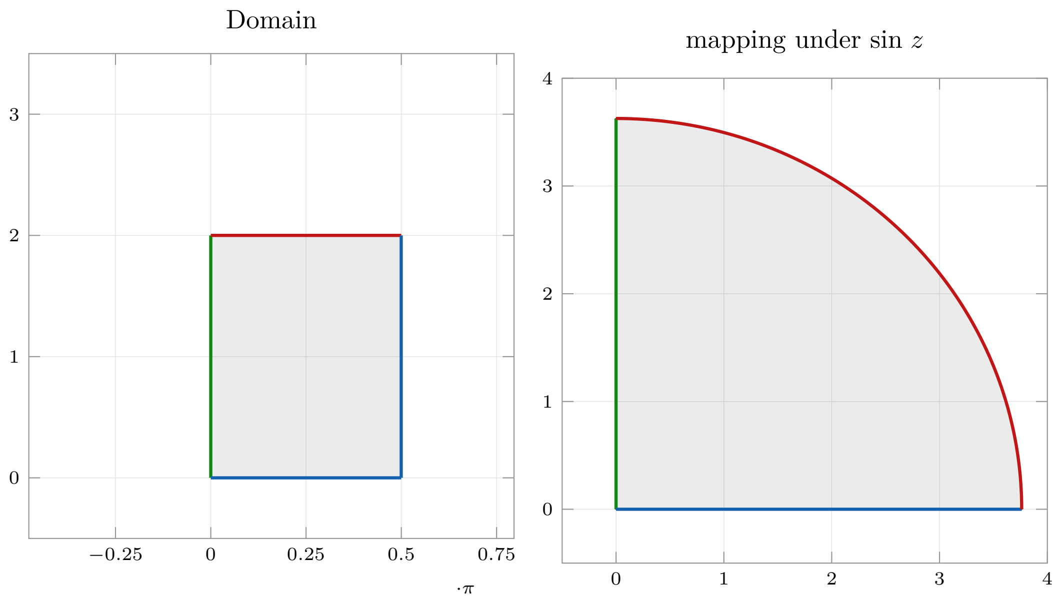
$$f(z) = \sin z \qquad f'(z) = \cos z \tag{17.4.1}$$

$$z^* = n\pi + \frac{\pi}{2} \tag{17.4.2}$$

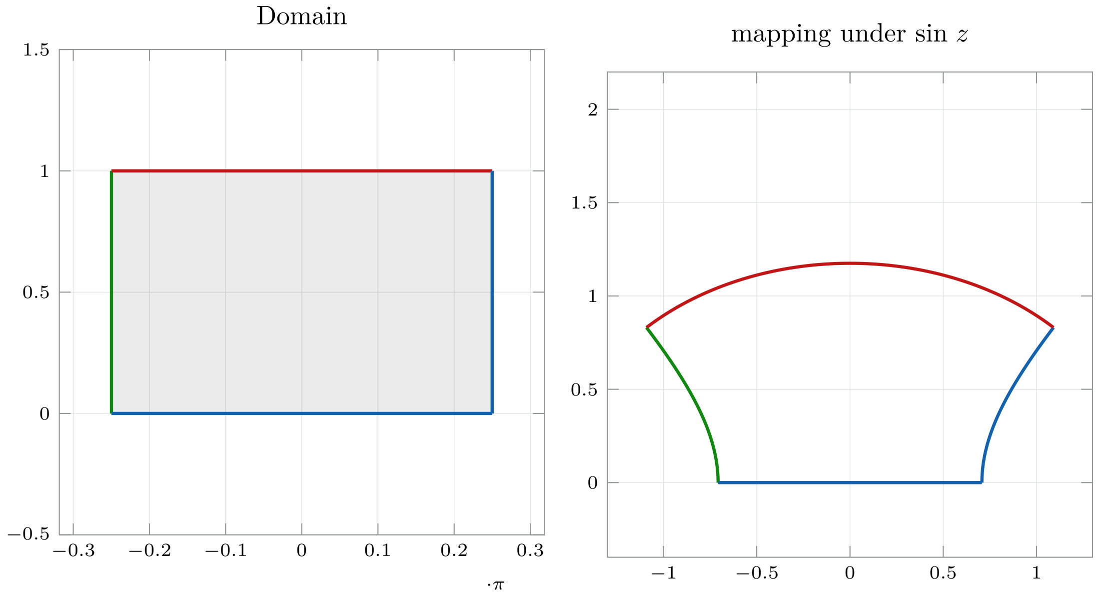
10. Plotting the mapping using $f(z) = \sin z$



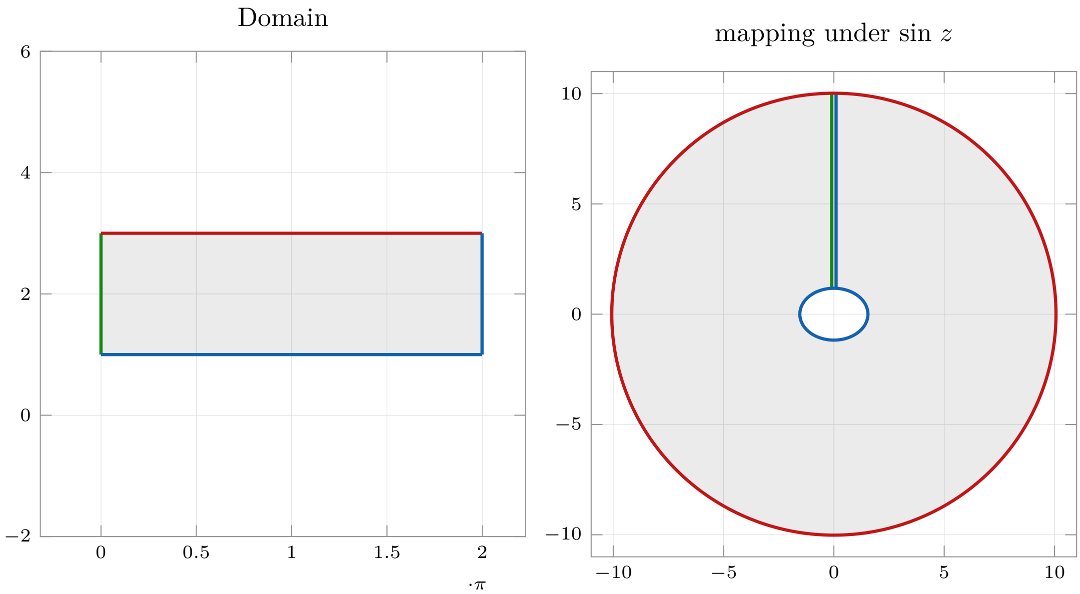
11. Plotting the mapping using $f(z) = \sin z$



12. Plotting the mapping using $f(z) = \sin z$

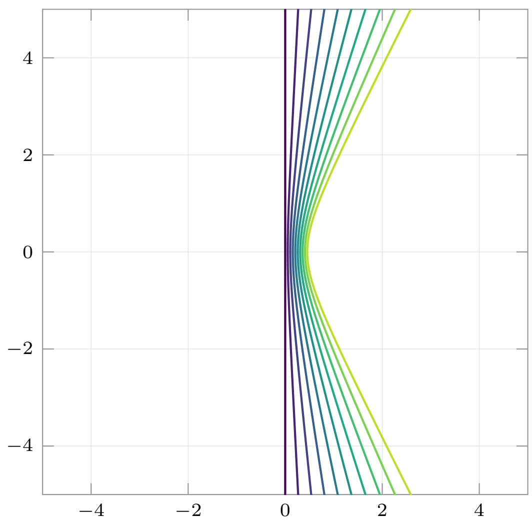


13. Plotting the mapping using $f(z) = \sin z$



14. Plotting the mapping using $f(z) = \sin z$

mapping $x \in (0, \pi/6)$ under sin z



15. The transformation is,

$$\cosh(z) = \cos(\mathfrak{i}z) = \sin(\mathfrak{i}z + \pi/2) \quad 17.4.3$$

The initial point z is rotated ccl. by $\pi/2$ and then translated by $\pi/2$ before the sine operation.

16. Finding the points at which the function is not conformal,

$$f(z) = \cosh(2\pi z) \qquad f'(z) = 2\pi \sinh(2\pi z) \quad 17.4.4$$

$$z^* = 0 \quad 17.4.5$$

17. The starting point is a region bounded by,

$$x > 0 \quad y > 0 \qquad xy < \pi \quad 17.4.6$$

$$z = x + \mathfrak{i}\, y \qquad 0.5z^2 = 0.5(x^2 - y^2) + \mathfrak{i}\,(xy) \quad 17.4.7$$

$$\text{Im}\,(0.5z^2) \in (0, \pi) \qquad w = 0.5z^2 \quad 17.4.8$$

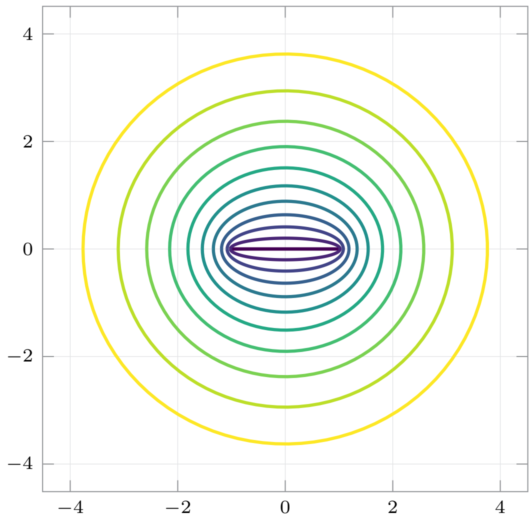
$$e^w = \exp(\text{Re}\, w)\,[\cos \text{Im}\, w + \mathfrak{i}\, \sin \text{Im}\, w] \quad 17.4.9$$

The function $\exp(0.5z^2)$ clearly has the entire upper half plane as its image, as required.

18. Using the formula,

$$\cos z = \cos x \cosh y - \mathfrak{i}\, \sin x \sinh y \quad 17.4.10$$

Since all of the values of cos z are surveyed for $x \in [0, \pi]$, mapping $y = k$ under $\cos z$



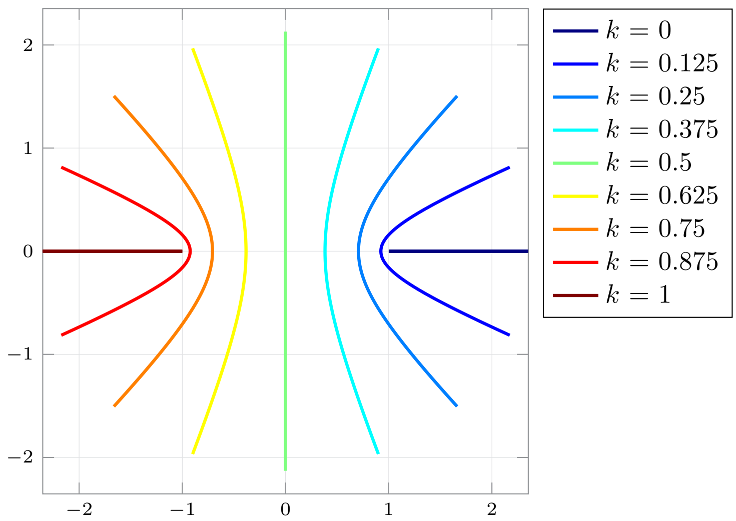
The ellipses approach circles, as $\cosh(x) \approxeq \sinh(x)$ for large x. Also, since the cosine function is even, negative values of k are not considered here.

19. Using the formula,

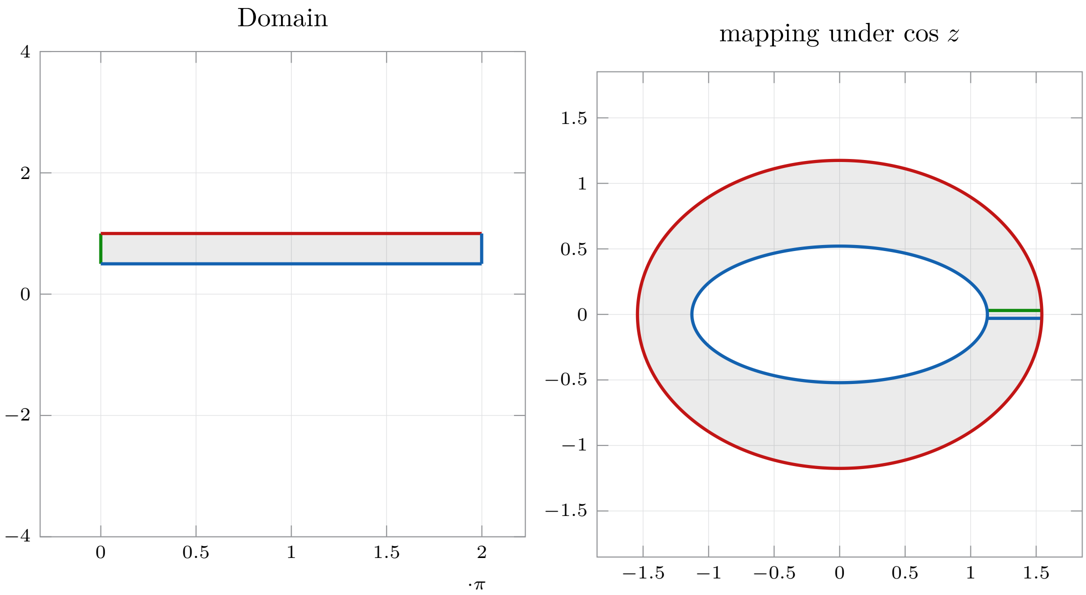
$$\cos z = \cos x \cosh y - \mathfrak{i} \ \sin x \sinh y \qquad \text{17.4.11}$$

Since all of the values of $\cos z$ are surveyed for $x \in [0, \pi]$,

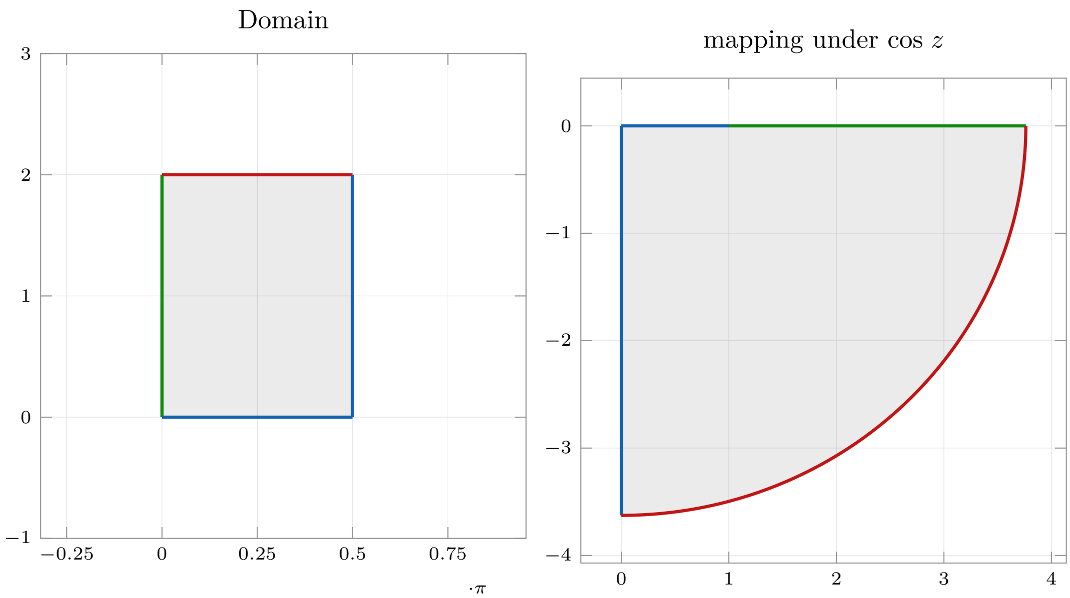
mapping $x = k\pi$ under $\cos z$



20. Plotting the mapping using $f(z) = \cos z$

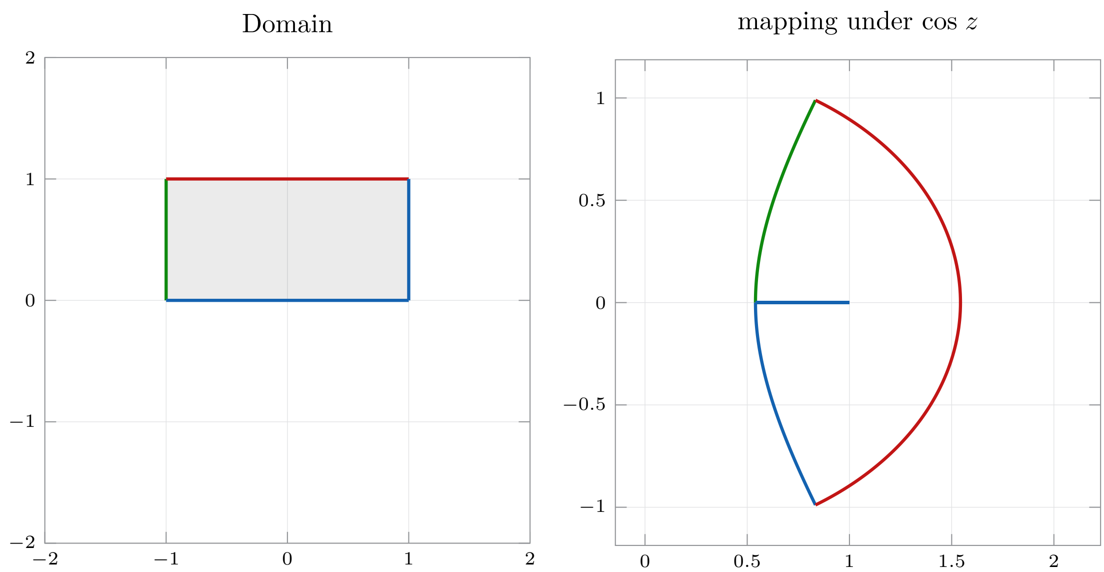


21. Plotting the mapping using $f(z) = \cos z$

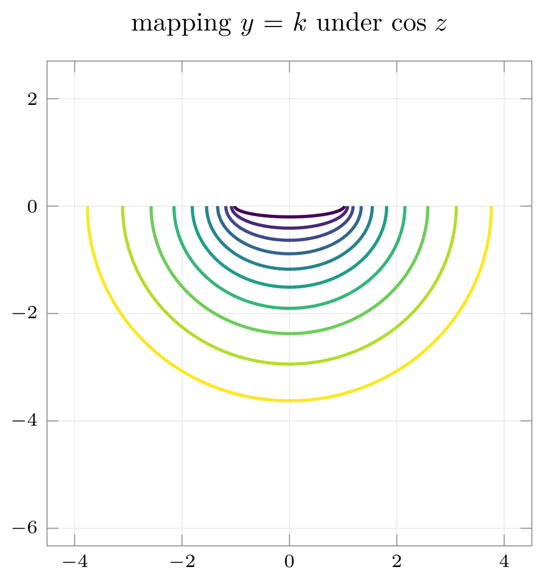


From Problem 11, the answer is simply $\sin(z + \pi/2)$, which happens to be the fourth quadrant of the ellipse instead of the first.

22. Plotting the mapping using $f(z) = \cos z$



23. Plotting the lower half plane for $x \in (\pi, 2\pi)$,



These are lower halves of ellipses in the uv plane

24. The function is,

$$\operatorname{Ln} z = \ln(r) + \mathfrak{i}\,\theta \tag{17.4.12}$$

The image is simply a rectangle with $u \in [\ln 2, \ln 3]$ and $v \in [\pi/4, \pi/2]$

25. Looking at the function,

$$w = \frac{z-1}{z+1} \qquad\qquad w = \frac{(x^2+y^2-1)+y\mathfrak{i}}{(x+1)^2+y^2} \tag{17.4.13}$$

$$y \geq 0 \quad \Longrightarrow \quad \operatorname{Im} w \geq 0 \tag{17.4.14}$$

Since w is restricted to the upper half plane, the complex logarithm maps it to the upper half plane,

$$\text{Ln } w = \ln |w| + \mathrm{i} \ \text{Arg } w \tag{17.4.15}$$

$$\text{Im } w \geq 0 \qquad \implies \qquad \text{Arg } w \in [0, \pi] \tag{17.4.16}$$

17.5 Riemann Surfaces

1. The function is double valued, with branch point $z = 0$,

$$|w| = \frac{1}{2} \qquad \text{Arg } w = \frac{\text{Arg } z}{2} \tag{17.5.1}$$

$$\theta \in [0, 4\pi] \qquad \implies \qquad \phi \in [0, 2\pi] \tag{17.5.2}$$

This means that w moves once around the circle $|w| = 0.5$.

2. Using the given substitution,

$$z - 1 = r_1 \exp(\mathrm{i}\alpha) \qquad z - 2 = r_2 \exp(\mathrm{i}\beta) \tag{17.5.3}$$

$$f(z) = \sqrt{r_1 r_2} \ \exp[0.5\mathrm{i}(\alpha + \beta)] \qquad \alpha \to \alpha + 2\pi \tag{17.5.4}$$

$$\beta \to \beta + 2\pi \qquad \implies \qquad \frac{\alpha + \beta}{2} \to \frac{\alpha + \beta}{2} + 2\pi \tag{17.5.5}$$

Branch points are at $z = 1$, $z = 2$ since $w = 0$ is the image of only one point $z = 0$.

3. Four sheets (since the function is 4-valued originally) with a branch point at $z = -1$.

4. Finding the branch points, using the fact that $z = 0$ only maps to one image.

$$f(z) = \sqrt{\mathrm{i}z - 2 + \mathrm{i}} \qquad z^* = \frac{2 - \mathrm{i}}{\mathrm{i}} = -1 - 2\mathrm{i} \tag{17.5.6}$$

$$s = 2 \tag{17.5.7}$$

5. Finding the branch points, using the fact that $w^{1/3}$ has three images except for $w = 0$,

$$f(z) = z^2 + (4z + \mathrm{i})^{1/3} \qquad z^* = -\frac{\mathrm{i}}{4} \tag{17.5.8}$$

$$s = 3 \tag{17.5.9}$$

6. Finding the branch points, using the fact that $\ln(w)$ has infinitely many images except for $w = 1$,

$$f(z) = \ln(6z - 2\mathrm{i}) \qquad z^* = \frac{2 + 2\mathrm{i}}{6} \tag{17.5.10}$$

$$s = \infty \tag{17.5.11}$$

7. Finding the branch points, using the fact that $w^{1/n}$ has n images except for $w = 0$,

$$f(z) = (z - z_0)^{1/n} \qquad z^* = z_0 \tag{17.5.12}$$

$$s = n \tag{17.5.13}$$

8. Finding the branch points, using the fact that $\sqrt{w}$ has 2 images except for $w = 0$,

$$f(z) = e^{\sqrt{z}} \qquad z^* = 0 \tag{17.5.14}$$

$$s = 2 \tag{17.5.15}$$

Using the fact that e^w is never zero, the second function has no branch points.

9. Finding the branch points, using the fact that $\sqrt{w}$ has 2 images except for $w = 0$,

$$f(z) = \sqrt{z^3 + z} \qquad z^* = 0, \pm\mathrm{i} \tag{17.5.16}$$

$$s = 2 \tag{17.5.17}$$

10. Finding the branch points, using the fact that $\sqrt{w}$ has 2 images except for $w = 0$,

$$f(z) = \sqrt{(4 - z^2)(1 - z^2)} \qquad z^* = \pm 1, \pm 2 \tag{17.5.18}$$

$$s = 2 \tag{17.5.19}$$