Chapter 4

Systems of ODEs, Phase Plane, Qualitative Methods

4.1 Systems of ODEs as Models in Engineering Applications

1. Looking at the system of ODEs, with tank size V and flow rate f,

$$y_1' = \frac{f}{V}\,y_2 - \frac{f}{V}\,y_1 \tag{4.1.1}$$

$$y_2' = \frac{f}{V}\,y_1 - \frac{f}{V}\,y_2 \tag{4.1.2}$$

Yes, the effect on this system of $f \to 2f$ is the same as the effect of $V \to 0.5V$.

2. With $V_1 = 200$, $V_2 = 100$ the system of ODEs changes to,

$$y_1' = \frac{f}{V_2}\,y_2 - \frac{f}{V_1}\,y_1 \tag{4.1.3}$$

$$y_2' = \frac{f}{V_1}\,y_1 - \frac{f}{V_2}\,y_2 \tag{4.1.4}$$

$$\mathbf{y}' = \begin{bmatrix} -2/200 & 2/100 \\ 2/200 & -2/100 \end{bmatrix}\mathbf{y} \tag{4.1.5}$$

$$\lambda_1 = -0.03 \qquad \mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.1.6}$$

$$\lambda_1 = 0 \qquad \mathbf{v}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{4.1.7}$$

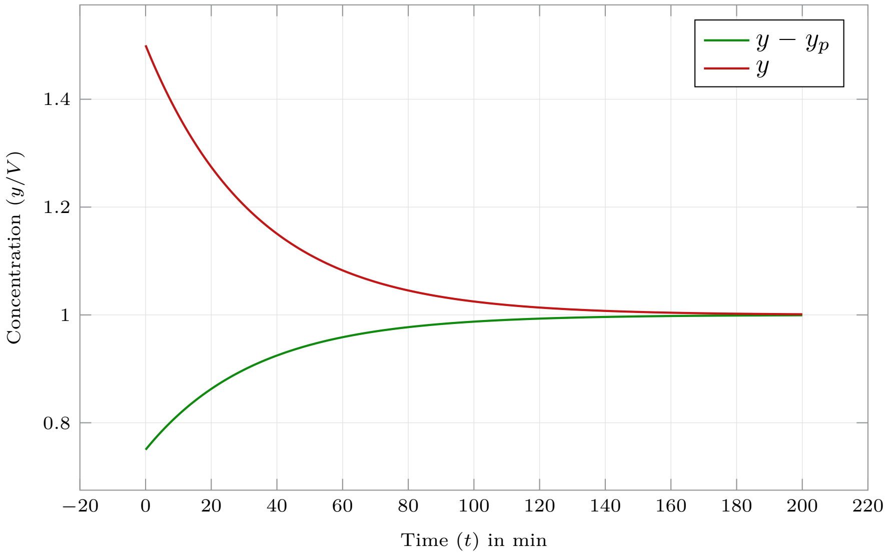
$$\mathbf{y} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-0.03t} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{4.1.8}$$

Using the initial conditions, $y_1(0) = 150$, $y_2(0) = 150$,

$$\mathbf{y}(0) = \begin{bmatrix} -c_1 + 2c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 150 \\ 150 \end{bmatrix} \tag{4.1.9}$$

$$c_1 = 50 \qquad c_2 = 100 \tag{4.1.10}$$

$$\mathbf{y} = \begin{bmatrix} -50 \\ 50 \end{bmatrix} e^{-0.03t} + \begin{bmatrix} 200 \\ 100 \end{bmatrix} \tag{4.1.11}$$



The concentration in both tanks still approaches equality as is physically expected.

3. To derive the eigenvectors

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \tag{4.1.12}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{4.1.13}$$

$$= \begin{vmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{vmatrix} \tag{4.1.14}$$

$$= \lambda^2 + 0.04\lambda \tag{4.1.15}$$

$$\lambda_1, \lambda_2 = 0, -0.04 \tag{4.1.16}$$

$$(-0.02 + 0.04)x_1 + 0.02x_2 = 0 \quad \Longrightarrow \quad \mathbf{v^{(2)}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.1.17}$$

$$(-0.02 + 0)x_1 + 0.02x_2 = 0 \quad \Longrightarrow \quad \mathbf{v^{(1)}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.1.18}$$

4. For a general $a = f/V$, with both tanks having equal volume,

$$\mathbf{A} = \begin{bmatrix} -a & a \\ a & -a \end{bmatrix} \tag{4.1.19}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{4.1.20}$$

$$= \begin{vmatrix} -a - \lambda & a \\ a & -a - \lambda \end{vmatrix} \tag{4.1.21}$$

$$= \lambda^2 + 2a\lambda \tag{4.1.22}$$

$$\lambda_1, \lambda_2 = 0, -2a \tag{4.1.23}$$

$$(-a + 2a)x_1 + ax_2 = 0 \quad \Longrightarrow \quad \mathbf{v^{(2)}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.1.24}$$

$$(-a + 0)x_1 + ax_2 = 0 \quad \Longrightarrow \quad \mathbf{v^{(1)}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.1.25}$$

$$y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2at} \tag{4.1.26}$$

The eigenvectors are independent of a. Only the exponential decay rate depends on a.

5. Upon adding a third tank T_3 connected to T_2,

$$y_1' = \frac{-f}{V}\ y_1 + \frac{f}{V}\ y_2 \tag{4.1.27}$$

$$y_1' = \frac{-2f}{V}\ y_2 + \frac{f}{V}\ y_1 + \frac{f}{V}\ y_3 \tag{4.1.28}$$

$$y_3' = \frac{-f}{V}\ y_3 + \frac{f}{V}\ y_2 \tag{4.1.29}$$

$$ \tag{4.1.30}$$

6. Solving the system above,

$$\mathbf{a} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \qquad \{\lambda_i\} = \{0, -k, -3k\} \tag{4.1.31}$$

$$y = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-3kt} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-kt} \tag{4.1.32}$$

7. $I_1(0) = 0$, $I_2(0) = -3$,

$$\mathbf{J_h} + \mathbf{J_p} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} e^{-0.8t} \tag{4.1.33}$$

$$\mathbf{J}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} \tag{4.1.34}$$

$$c_1 = 1 \qquad c_2 = -5 \tag{4.1.35}$$

8. Remaking the system of ODEs,

$$I_1' = -4I_1 + 4I_2 + 12 \tag{4.1.36}$$

$$I_2' = 0.4I_1' - 0.54I_2 \tag{4.1.37}$$

$$I_2' = -1.6I_1 + 1.06I_2 + 4.8 \tag{4.1.38}$$

$$\mathbf{J}' = \mathbf{Aj} + \mathbf{g} \tag{4.1.39}$$

$$\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -1.6 & 1.06 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 4.8 \end{bmatrix} \tag{4.1.40}$$

$$\tag{4.1.41}$$

Solving the h-ODE,

$$\{\lambda\} = \{-1.5, -1.44\}, \qquad \mathbf{v^{(1)}} = \begin{bmatrix} 1.6 \\ 1 \end{bmatrix}, \qquad \mathbf{v^{(2)}} = \begin{bmatrix} 25/16 \\ 1 \end{bmatrix} \tag{4.1.42}$$

$$\mathbf{I_h} = c_1 \begin{bmatrix} 8 \\ 5 \end{bmatrix} e^{-1.5t} + c_2 \begin{bmatrix} 25 \\ 16 \end{bmatrix} e^{-1.44t} \tag{4.1.43}$$

Solving the nh-ODE,

$$\mathbf{I_p} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{4.1.44}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -1.6 & 1.06 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 4.8 \end{bmatrix} \tag{4.1.45}$$

$$\mathbf{I_p} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \tag{4.1.46}$$

9. $I_1(0) = 28$, $I_2(0) = 14$,

$$\mathbf{J_h} + \mathbf{J_p} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} e^{-0.8t} \tag{4.1.47}$$

$$\mathbf{J_h}(0) = \begin{bmatrix} 28 \\ 14 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} \tag{4.1.48}$$

$$c_1 = 10 \qquad c_2 = 5 \tag{4.1.49}$$

10. By the usual method,

$$y'' + 3y' + 2y = 0 \tag{4.1.50}$$

$$\lambda^2 + 3\lambda + 2 = 0 \tag{4.1.51}$$

$$\{\lambda_i\} = \{-1, -2\} \tag{4.1.52}$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x} \tag{4.1.53}$$

By converting to a system of first order ODEs,

$$y_1 = y \qquad y_2 = y' \tag{4.1.54}$$

$$y_2' = -3y_2 - 2y_1 \tag{4.1.55}$$

$$y_1' = y_2 \tag{4.1.56}$$

$$\mathbf{a} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \{\lambda_i\} = \{-2, -1\} \tag{4.1.57}$$

$$\mathbf{v^{(1)}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \mathbf{v^{(2)}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.1.58}$$

$$y_h = c_1 e^{-2x} + c_2 e^{-x} \tag{4.1.59}$$

Both results match.

11. By the usual method,

$$4y'' - 15y' - 4y = 0 \tag{4.1.60}$$

$$\lambda^2 - 3.75\lambda - 1 = 0 \tag{4.1.61}$$

$$\{\lambda_i\} = \{-0.25, 4\} \tag{4.1.62}$$

$$y_h = c_1 e^{-0.25x} + c_2 e^{4x} \tag{4.1.63}$$

By converting to a system of first order ODEs,

$$y_1 = y \qquad y_2 = y' \tag{4.1.64}$$

$$y_2' = 3.75y_2 + y_1 \tag{4.1.65}$$

$$y_1' = y_2 \tag{4.1.66}$$

$$\mathbf{a} = \begin{bmatrix} 0 & 1 \\ 1 & 3.75 \end{bmatrix} \qquad \{\lambda_i\} = \{-0.25, 4\} \tag{4.1.67}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \tag{4.1.68}$$

$$y_h = c_1 e^{-0.25x} + c_2 e^{4x} \tag{4.1.69}$$

Both results match.

12. By the usual method,

$$y''' + 2y'' - y' - 2y = 0 \tag{4.1.70}$$

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0 \tag{4.1.71}$$

$$\{\lambda_i\} = \{-2, -1, 1\} \tag{4.1.72}$$

$$y_h = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} \tag{4.1.73}$$

By converting to a system of first order ODEs,

$$y_3' = 2y_1 + y_2 - 2y_3 \qquad y_2' = y_3 \tag{4.1.74}$$

$$y_1' = y_2 \tag{4.1.75}$$

$$\mathbf{a} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \qquad \{\lambda_i\} = \{-2, -1, 1\} \tag{4.1.76}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{4.1.77}$$

$$\mathbf{v}^{(\mathbf{3})} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4.1.78}$$

$$y_h = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} \tag{4.1.79}$$

Both results match.

13. By the usual method,

$$y'' + 2y' - 24y = 0 \tag{4.1.80}$$

$$\lambda^2 + 2\lambda - 24 = 0 \tag{4.1.81}$$

$$\{\lambda_i\} = \{-6, 4\} \tag{4.1.82}$$

$$y_h = c_1 e^{-6x} + c_2 e^{4x} \tag{4.1.83}$$

By converting to a system of first order ODEs,

$$y_1 = y \qquad y_2 = y' \tag{4.1.84}$$

$$y_2' = -2y_2 + 24y_1 \tag{4.1.85}$$

$$y_1' = y_2 \tag{4.1.86}$$

$$\mathbf{a} = \begin{bmatrix} 0 & 1 \\ 24 & -2 \end{bmatrix} \qquad \{\lambda_i\} = \{-6, 4\} \tag{4.1.87}$$

$$\mathbf{v^{(1)}} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \qquad \mathbf{v^{(2)}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \tag{4.1.88}$$

$$y_h = c_1 e^{-6x} + c_2 e^{4x} \tag{4.1.89}$$

Both results match.

14. Two spring-mass systems hanging in series.

(a) Setting up the model,

$$m_2 y_2'' = -k_2(y_2 - y_1) \qquad y_2'' = -2y_2 + 2y_1 \tag{4.1.90}$$

$$m_1 y_1'' = k_2(y_2 - y_1) - k_1 y_1 \qquad y_1'' = 2y_2 - 5y_1 \tag{4.1.91}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} + \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{y}'' + \mathbf{A}\mathbf{y} = \mathbf{0} \tag{4.1.92}$$

Using the guess, $\mathbf{y} = \mathbf{x}\cos(\omega t)$,

$$\mathbf{y}'' = -\omega^2 \mathbf{x}\cos(\omega t) \tag{4.1.93}$$

$$\left(-\omega^2 \mathbf{I} + \mathbf{A}\right)\mathbf{x} = 0 \tag{4.1.94}$$

$$\begin{bmatrix} -\omega^2 + 5 & -2 \\ -2 & -\omega^2 + 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{4.1.95}$$

$$\{\omega_i^2\} = \{1, 6\} \tag{4.1.96}$$

$$\mathbf{v^{(1)}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v^{(2)}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{4.1.97}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(t) + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \cos\left(\sqrt{6}t\right) \tag{4.1.98}$$

(b) Initial conditions which guarantee $c_1 = 0$ or $c_2 = 0$ are of interest, since the system oscillates purely according to one mode.

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} c_1 - 2c_2 \\ 2c_1 + c_2 \end{bmatrix} \tag{4.1.99}$$

$$y_2(0) = 2\ y_1(0) \qquad \text{masses displaced in the same direction} \tag{4.1.100}$$

$$y_2(0) = -0.5\ y_1(0) \qquad \text{masses displaced in opposite directions} \tag{4.1.101}$$

The above two kinds of I.C. correspond to only one mode remaining in $\mathbf{y}$.

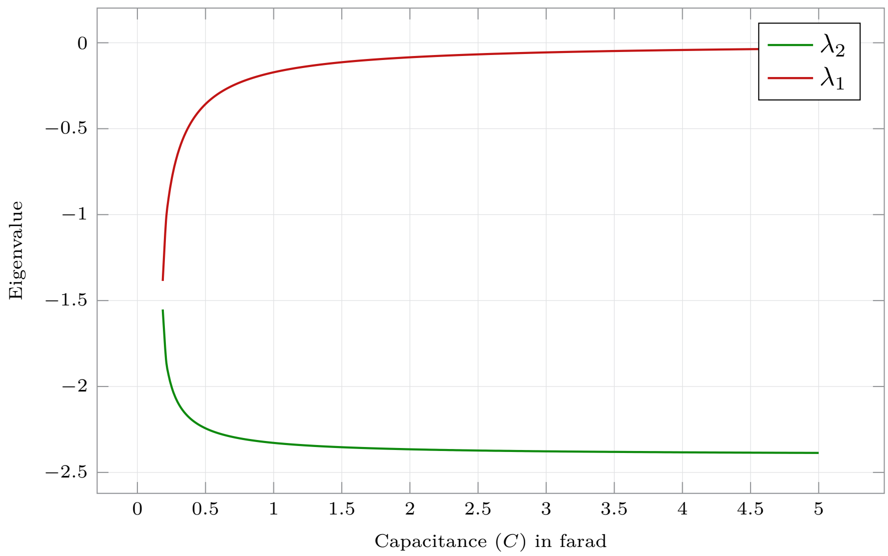
15. From example 2, the system of ODEs is,

$$I_1' = -4I_1 + 4I_2 + 12 \tag{4.1.102}$$

$$I_2' = -1.6I_1 + \left(1.6 - \frac{1}{10C}\right) I_2 + 4.8 \tag{4.1.103}$$

$$\mathbf{A} = \begin{bmatrix} -4 & 4 \\ -1.6 & (1.6 - 0.1/C) \end{bmatrix} \tag{4.1.104}$$

$$\{\lambda_i\} = \frac{-(24C+1) \pm \sqrt{576C^2 - 112C + 1}}{20C} \tag{4.1.105}$$



From the CAS, letting capacitance approach infinity gives

$$\lim_{C \to \infty} \lambda_1 = 0^- \qquad \lim_{C \to \infty} \lambda_2 = -2.4^+ \tag{4.1.106}$$

4.2 Basic Theory of Systems of ODEs, Wronskian

1. No Problem set in section.

4.3 Constant-Coefficient Systems, Phase Plane Method

1. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \tag{4.3.1}$$

$$\lambda^2 - 4 = 0 \qquad \{\lambda_i\} = \{-2, 2\} \tag{4.3.2}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.3.3}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \tag{4.3.4}$$

2. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 6 & 9 \\ 1 & 6 \end{bmatrix} \tag{4.3.5}$$

$$\lambda^2 - 12\lambda + 27 = 0 \qquad \{\lambda_i\} = \{3, 9\} \tag{4.3.6}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{4.3.7}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{9t} \tag{4.3.8}$$

3. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0.5 & 1 \end{bmatrix} \tag{4.3.9}$$

$$\lambda^2 - 2\lambda = 0 \qquad \{\lambda_i\} = \{0, 2\} \tag{4.3.10}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{4.3.11}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} \tag{4.3.12}$$

4. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} -8 & -2 \\ 2 & -4 \end{bmatrix} \tag{4.3.13}$$

$$\lambda^2 + 7\lambda - 9 = 0 \qquad \lambda_1, \lambda_2 = \left\{ \frac{-7 + \sqrt{85}}{2}, \frac{-7 - \sqrt{85}}{2} \right\} \tag{4.3.14}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} 9 - \sqrt{85} \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 9 + \sqrt{85} \\ 1 \end{bmatrix} \tag{4.3.15}$$

$$\mathbf{y} = c_1 \mathbf{v}^{(\mathbf{1})} e^{\lambda_1 t} + c_2 \mathbf{v}^{(\mathbf{2})} e^{\lambda_2 t} \tag{4.3.16}$$

5. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 2 & 5 \\ 5 & 25/2 \end{bmatrix} \tag{4.3.17}$$

$$\lambda^2 - 14.5\lambda = 0 \qquad \lambda_1, \lambda_2 = \{0, 14.5\} \tag{4.3.18}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \tag{4.3.19}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} e^{14.5t} \tag{4.3.20}$$

6. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \tag{4.3.21}$$

$$0 = \lambda^2 - 4\lambda + 8 \qquad \lambda_1, \lambda_2 = \{2 - 2\mathrm{i}, 2 + 2\mathrm{i}\} \tag{4.3.22}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix} \tag{4.3.23}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix} e^{(2-2\mathrm{i})t} + c_2 \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix} e^{(2+2\mathrm{i})t} \tag{4.3.24}$$

$$y_1 = e^{2t}[-c_1 \sin(2t) - c_2 \sin(2t)] \qquad y_2 = e^{2t}[c_1 \cos(2t) + c_2 \cos(2t)] \tag{4.3.25}$$

7. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{4.3.26}$$

$$0 = \lambda^3 + 2\lambda \qquad \lambda_1, \lambda_2 = \left\{0, -\sqrt{2}\,\mathrm{i}, \sqrt{2}\,\mathrm{i}\right\} \tag{4.3.27}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ \sqrt{2}\,\mathrm{i} \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(3)} = \begin{bmatrix} -1 \\ -\sqrt{2}\,\mathrm{i} \\ 1 \end{bmatrix} \tag{4.3.28}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ \sqrt{2}\,\mathrm{i} \\ 1 \end{bmatrix} e^{-\sqrt{2}\,\mathrm{i}t} + c_3 \begin{bmatrix} -1 \\ -\sqrt{2}\,\mathrm{i} \\ 1 \end{bmatrix} e^{\sqrt{2}\,\mathrm{i}t} \tag{4.3.29}$$

$$y_1 = c_1 - c_2 \cos(\sqrt{2}t) - c_3 \cos(\sqrt{2}t) \tag{4.3.30}$$

$$y_2 = c_2\sqrt{2} \sin(\sqrt{2}t) - c_3\sqrt{2} \sin(\sqrt{2}t) \tag{4.3.31}$$

$$y_3 = c_1 + c_2 \cos(\sqrt{2}t) + c_3 \cos(\sqrt{2}t) \tag{4.3.32}$$

Dividing the third eigenvector by $-\mathrm{i}$ produces the textbook result.

8. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 8 & -1 \\ 1 & 10 \end{bmatrix} \tag{4.3.33}$$

$$0 = \lambda^2 - 18\lambda + 81 \qquad \lambda_1, \lambda_2 = \{9, 9\} \tag{4.3.34}$$

$$\mathbf{y}^{(2)} = (\mathbf{x}^{(1)} t + \mathbf{u}) e^{\lambda t} \tag{4.3.35}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad (\mathbf{A} - \lambda \mathbf{I})\mathbf{u} = \mathbf{x}^{(1)} \tag{4.3.36}$$

$$u_1 + u_2 = 1 \qquad \mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.3.37}$$

$$y = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{9t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{9t} \tag{4.3.38}$$

9. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 10 & 10 & -4 \\ -10 & 1 & -14 \\ -4 & -14 & -2 \end{bmatrix} \tag{4.3.39}$$

$$0 = \lambda^3 - 9\lambda - 324\lambda + 2196 \qquad \lambda_1, \lambda_2 = \{-18, 9, 18\} \tag{4.3.40}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}^{(3)} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \tag{4.3.41}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} e^{-18t} + c_2 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} e^{-9t} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{18t} \tag{4.3.42}$$

10. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} \tag{4.3.43}$$

$$0 = \lambda^2 - \lambda - 12 \qquad \lambda_1, \lambda_2 = \{-3, 4\} \tag{4.3.44}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.3.45}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \tag{4.3.46}$$

Using the I.C.,

$$\mathbf{y}(0) = \begin{bmatrix} 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2 \\ 5c_1 + c_2 \end{bmatrix} \tag{4.3.47}$$

$$c_1 = 1 \qquad c_2 = 2 \tag{4.3.48}$$

11. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 2 & 5 \\ -0.5 & -1.5 \end{bmatrix} \tag{4.3.49}$$

$$0 = \lambda^2 - 0.5\lambda - 0.5 \qquad \lambda_1, \lambda_2 = \{-0.5, 1\} \tag{4.3.50}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \tag{4.3.51}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{t} \tag{4.3.52}$$

Using the I.C.,

$$\mathbf{y}(0) = \begin{bmatrix} -12 \\ 0 \end{bmatrix} = \begin{bmatrix} -2c_1 - 5c_2 \\ c_1 + c_2 \end{bmatrix} \tag{4.3.53}$$

$$c_1 = -4 \qquad c_2 = 4 \tag{4.3.54}$$

$$\mathbf{y} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} -20 \\ 4 \end{bmatrix} e^{t} \tag{4.3.55}$$

12. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1/3 & 1 \end{bmatrix} \tag{4.3.56}$$

$$0 = \lambda^2 - 2\lambda \qquad \lambda_1, \lambda_2 = \{0, 2\} \tag{4.3.57}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{4.3.58}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t} \tag{4.3.59}$$

Using the I.C.,

$$\mathbf{y}(0) = \begin{bmatrix} 12 \\ 2 \end{bmatrix} = \begin{bmatrix} -3c_1 + 3c_2 \\ c_1 + c_2 \end{bmatrix} \tag{4.3.60}$$

$$c_1 = -1 \qquad c_2 = 3 \tag{4.3.61}$$

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 9 \\ 3 \end{bmatrix} e^{2t} \tag{4.3.62}$$

13. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.3.63}$$

$$0 = \lambda^2 - 1 \qquad \lambda_1, \lambda_2 = \{-1, 1\} \tag{4.3.64}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.3.65}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} \tag{4.3.66}$$

Using the I.C.,

$$\mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ c_1 + c_2 \end{bmatrix} \tag{4.3.67}$$

$$c_1 = 1 \qquad c_2 = 1 \tag{4.3.68}$$

$$\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} \tag{4.3.69}$$

14. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \tag{4.3.70}$$

$$0 = \lambda^2 + 2\lambda + 2 \qquad \lambda_1, \lambda_2 = \{-1 - \mathrm{i}, -1 + \mathrm{i}\} \tag{4.3.71}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix} \tag{4.3.72}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix} e^{(-1-\mathrm{i})t} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(-1+\mathrm{i})t} \tag{4.3.73}$$

Using the I.C.,

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{i}(-c_1 + c_2) \\ c_1 + c_2 \end{bmatrix} \tag{4.3.74}$$

$$c_1 = 0.5i \qquad c_2 = -0.5i \tag{4.3.75}$$

$$\mathbf{y} = \begin{bmatrix} 0.5 \\ 0.5i \end{bmatrix} e^{(-1-\mathrm{i})t} + \begin{bmatrix} 0.5 \\ -0.5i \end{bmatrix} e^{(-1+i)t} \tag{4.3.76}$$

$$y_1 = e^{-t}\cos(t) \tag{4.3.77}$$

$$y_2 = e^{-t}\sin(t) \tag{4.3.78}$$

15. Solving the homogeneous system of ODEs,

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \tag{4.3.79}$$

$$0 = \lambda^2 - 6\lambda + 5 \qquad \lambda_1, \lambda_2 = \{1, 5\} \tag{4.3.80}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.3.81}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} \tag{4.3.82}$$

Using the I.C.,

$$\mathbf{y}(0) = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ c_1 + c_2 \end{bmatrix} \tag{4.3.83}$$

$$c_1 = -0.5 \qquad c_2 = 0 \tag{4.3.84}$$

$$\mathbf{y} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} e^t \tag{4.3.85}$$

16. Converting system of ODEs into a single ODE,

$$y_1' = 8y_1 - y_2 \qquad y_2' = y_1 + 10y_2 \tag{4.3.86}$$

$$y_2 = 8y_1 - y_1' \qquad 8y_1' - y_1'' = 81y_1 - 10y_1' \tag{4.3.87}$$

$$0 = z'' - 18z' + 81z \qquad \lambda_1, \lambda_2 = 9, 9 \tag{4.3.88}$$

$$z = y_1 = (c_1 + c_2 t)e^{9t} \qquad y_1' = e^{9t}(c_2 + 9c_1 + 9c_2 t) \tag{4.3.89}$$

$$y_2 = e^{9t}(-c_1 - c_2 - c_2 t) \tag{4.3.90}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{9t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{9t} \tag{4.3.91}$$

This matches the other method.

17. Converting system of ODEs into a single ODE,

$$y_1' = -y_1 + y_2 \qquad y_2' = -y_1 - y_2 \tag{4.3.92}$$

$$y_2 = y_1 + y_1' \qquad y_1' + y_1'' = -2y_1 - y_1' \tag{4.3.93}$$

$$0 = z'' + 2z' + 2z \qquad \lambda_1, \lambda_2 = -1 \pm i \tag{4.3.94}$$

$$z = y_1 = [c_1 \cos(t) + c_2 \sin(t)]e^{-t} \tag{4.3.95}$$

$$y_1' = e^{-t}[(-c_1 + c_2)\cos t + (-c_1 - c_2)\sin t] \tag{4.3.96}$$

$$y_2 = e^{-t}[c_2 \cos t - c_1 \sin t] \tag{4.3.97}$$

$$\mathbf{y} = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{-t} \tag{4.3.98}$$

This matches the other method.

18. Solving the homogeneous system of ODEs for the two tanks,

$$y_1' = \frac{4}{200}y_2 - \frac{16}{200}y_1 \qquad y_2' = \frac{16}{200}y_1 - \frac{16}{200}y_2 \tag{4.3.99}$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} = \begin{bmatrix} -0.08 & 0.02 \\ 0.08 & -0.08 \end{bmatrix} \tag{4.3.100}$$

$$0 = \lambda^2 + \frac{4}{25}\lambda + \frac{3}{625} \qquad \lambda_1, \lambda_2 = \left\{\frac{-3}{25}, \frac{-1}{25}\right\} \tag{4.3.101}$$

$$\mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{4.3.102}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-0.12t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.04t} \tag{4.3.103}$$

Using the I.C.,

$$\mathbf{y}(0) = \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ 2c_1 + 2c_2 \end{bmatrix} \tag{4.3.104}$$

$$c_1 = 0 \qquad c_2 = 100 \tag{4.3.105}$$

$$\mathbf{y} = \begin{bmatrix} 100 \\ 200 \end{bmatrix} e^{-0.04t} \tag{4.3.106}$$

19. By KVL on each loop in the circuit,

$$LI_2' + R(I_2 - I_1) = 0 \qquad R(I_1 - I_2) + \frac{1}{C}\int I_1 \ \mathrm{d}t = 0 \tag{4.3.107}$$

$$R(I_1' - I_2') + \frac{1}{C}I_1 = 0 \qquad I_1' = \frac{-1}{RC}I_1 + I_2' \tag{4.3.108}$$

$$I_2' = 0.75(I_1 - I_2) \qquad I_1' = -3.25I_1 - 0.75I_2 \tag{4.3.109}$$

Solving the system of h-ODEs

$$\mathbf{I}' = \mathbf{AI} \qquad \mathbf{A} = \begin{bmatrix} -3.25 & -0.75 \\ 0.75 & -0.75 \end{bmatrix} \tag{4.3.110}$$

$$0 = \lambda^2 + 4\lambda + 3 \qquad \lambda_1, \lambda_2 = \{-3, -1\} \tag{4.3.111}$$

$$\mathbf{v}^{(1)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{4.3.112}$$

$$\mathbf{I} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-t} \tag{4.3.113}$$

20. Graphing the 6 types of critical points,

(a) Improper Node, with $c_1 = c_2 = 1$, passing through $(2, 0)$

(b) Proper Node, with $c_1 = c_2 = 1$, passing through $(1, 1)$

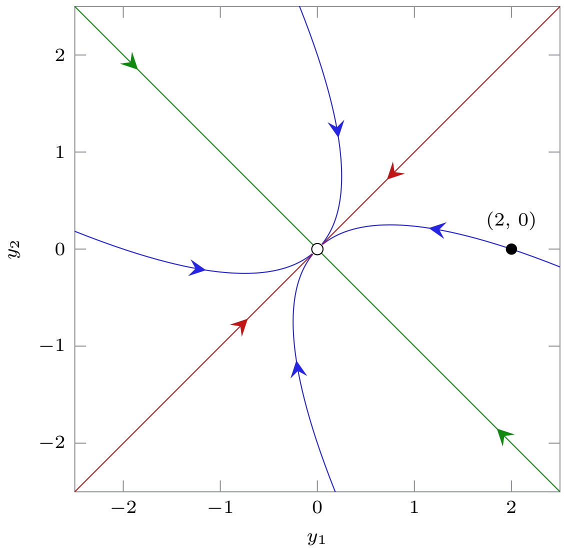
(c) Saddle point, with $c_1 = c_2 = 1$, passing through $(1, 1)$

(d) Center, with $c_1 = c_2 = 1$, passing through $(1, 1)$

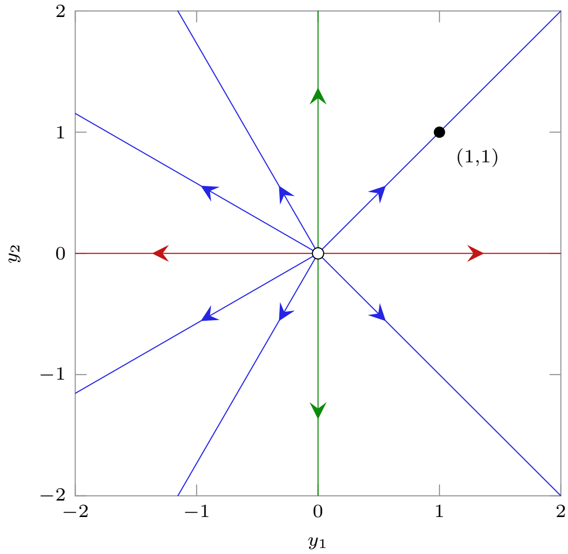
(e) Spiral, with $c_1 = c_2 = 1$, passing through $(0, 1)$

(f) Improper Node, with $c_1 = -1, \ c_2 = 1$, passing through $(-1, 2)$

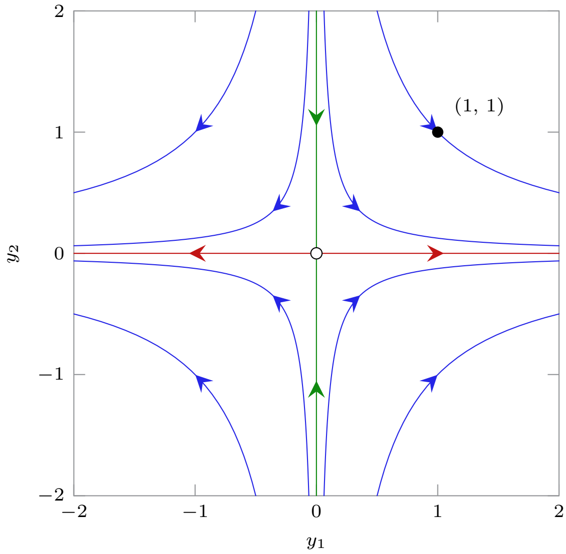
Improper Node with $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$



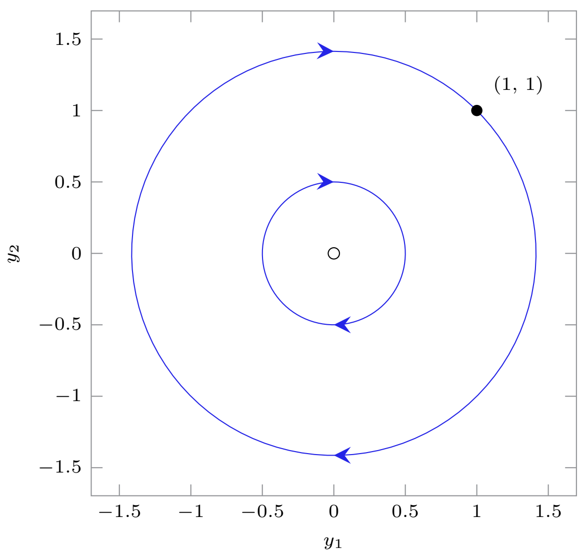
Proper Node with $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$

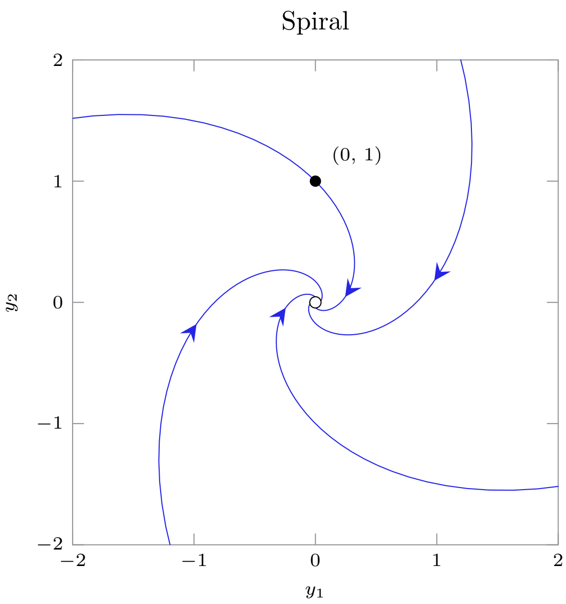


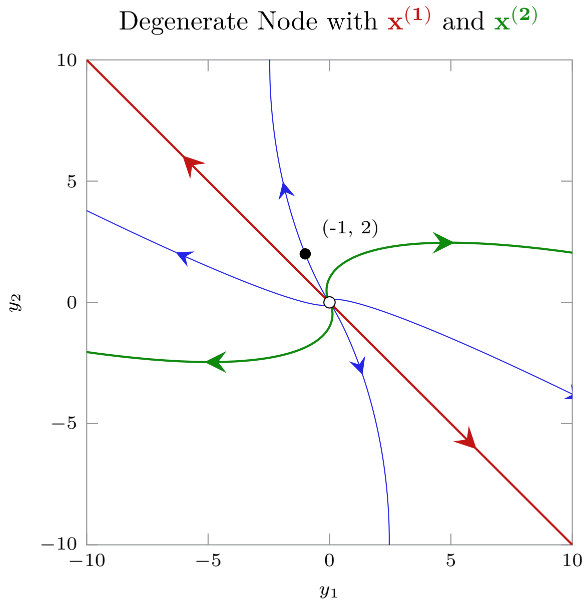
Saddle point with $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$



Center







4.4 Criteria for Critical Points, Stability

1. Finding the type and stability of critical point,

$$y_1' = y_1 \qquad\qquad y_2' = 2y_2 \tag{4.4.1}$$

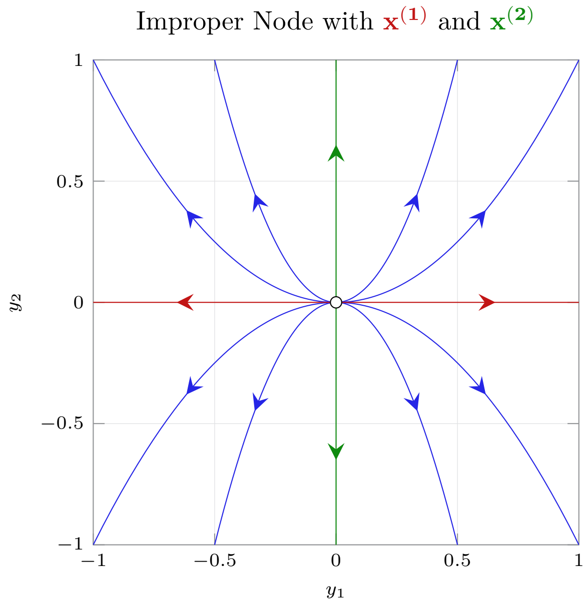
$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad\qquad \lambda^2 - 3\lambda + 2 = 0 \tag{4.4.2}$$

Using $p = 3,\ \ q = 2,\ \ \Delta = 1$ gives an unstable improper node at $(0, 0)$.

$$\lambda_1 = 1 \qquad\qquad \mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.4.3}$$

$$\lambda_2 = 2 \qquad\qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.4.4}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} \tag{4.4.5}$$



2. Finding the type and stability of critical point,

$$y_1' = -4y_1 \qquad y_2' = -3y_2 \tag{4.4.6}$$

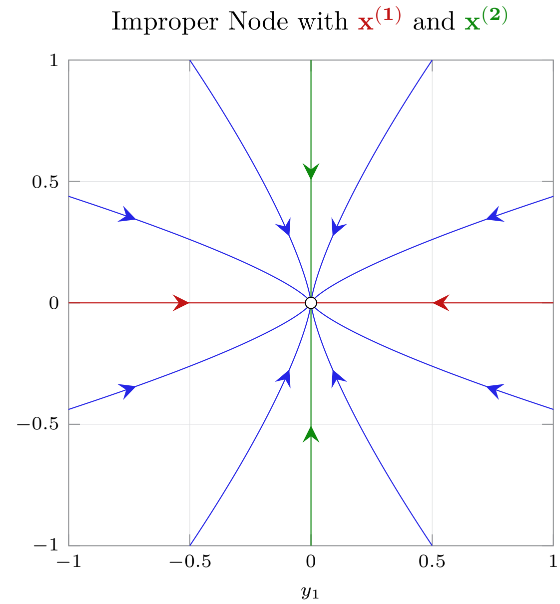
$$\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \qquad \lambda^2 + 7\lambda + 12 = 0 \tag{4.4.7}$$

Using $p = -7,\ q = 12,\ \Delta = 1$ gives an improper node at $(0, 0)$.

$$\lambda_1 = -4 \qquad \mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.4.8}$$

$$\lambda_2 = -3 \qquad \mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.4.9}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t} \tag{4.4.10}$$



3. Finding the type and stability of critical point,

$$y_1' = y_2 \qquad y_2' = -9y_1 \tag{4.4.11}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \qquad \lambda^2 + 9 = 0 \tag{4.4.12}$$

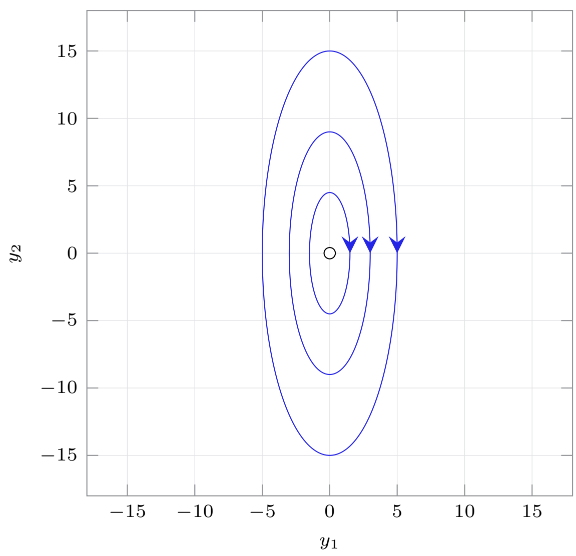
Using $p = 0,\ q = 9,\ \Delta = -36$ gives a center at $(0, 0)$.

$$\lambda_1 = -3\,\mathrm{i} \qquad \mathbf{v}^{(1)} = \begin{bmatrix} \mathrm{i} \\ 3 \end{bmatrix} \tag{4.4.13}$$

$$\lambda_2 = 3\,\mathrm{i} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -\mathrm{i} \\ 3 \end{bmatrix} \tag{4.4.14}$$

$$\mathbf{y} = c_1 \begin{bmatrix} \sin(3t) \\ 3\cos(3t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(3t) \\ 3\cos(3t) \end{bmatrix} \tag{4.4.15}$$

Center



4. Finding the type and stability of critical point,

$$y_1' = 2y_1 + y_2 \qquad y_2' = 5y_1 - 2y_2 \tag{4.4.16}$$

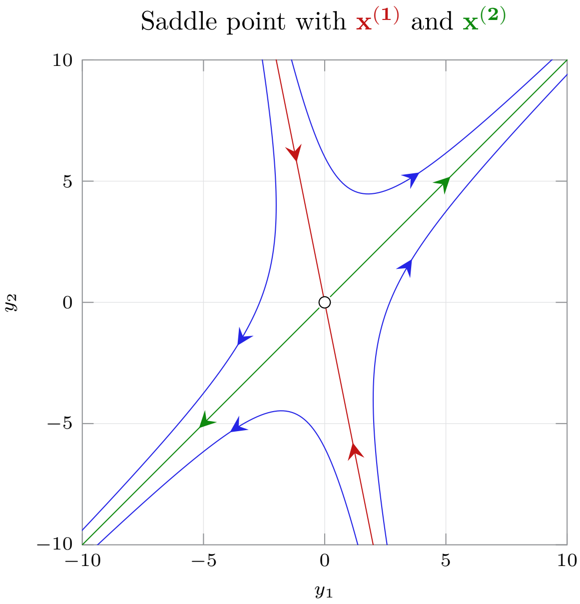
$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix} \qquad \lambda^2 - 9 = 0 \tag{4.4.17}$$

Using $p = 0,\ q = -9,\ \Delta = 36$ gives a saddle point at $(0, 0)$.

$$\lambda_1 = -3 \qquad \mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \tag{4.4.18}$$

$$\lambda_2 = 3 \qquad \mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.4.19}$$

$$\mathbf{y} = c_1 e^{-3t} \begin{bmatrix} -1 \\ 5 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.4.20}$$



5. Finding the type and stability of critical point,

$$y_1' = -2y_1 + 2y_2 \qquad\qquad y_2' = -2y_1 - 2y_2 \tag{4.4.21}$$

$$\mathbf{A} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \qquad\qquad \lambda^2 + 4\lambda + 8 = 0 \tag{4.4.22}$$

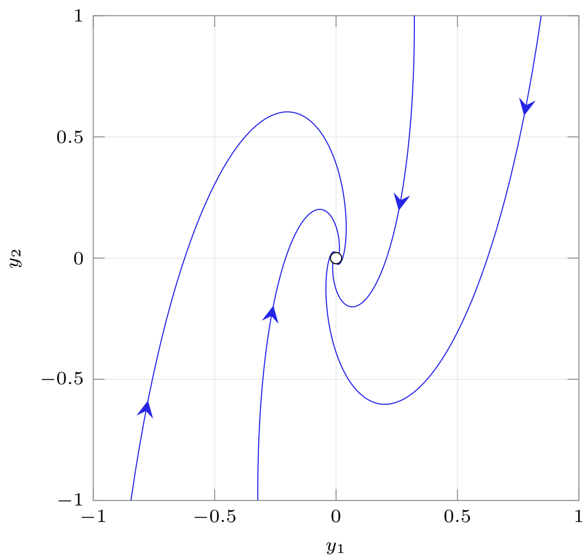
Using $p = 0,\ \ q = 9,\ \ \Delta = -36$ gives a center at $(0, 0)$.

$$\lambda_1 = -2 - 2i \qquad\qquad \mathbf{v}^{(1)} = \begin{bmatrix} i \\ 1 \end{bmatrix} \tag{4.4.23}$$

$$\lambda_2 = -2 + 2i \qquad\qquad \mathbf{v}^{(2)} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \tag{4.4.24}$$

$$\mathbf{y} = c_1 e^{-2t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} \tag{4.4.25}$$

Spiral



6. Finding the type and stability of critical point,

$$y_1' = -6y_1 - y_2 \qquad y_2' = -9y_1 - 6y_2 \tag{4.4.26}$$

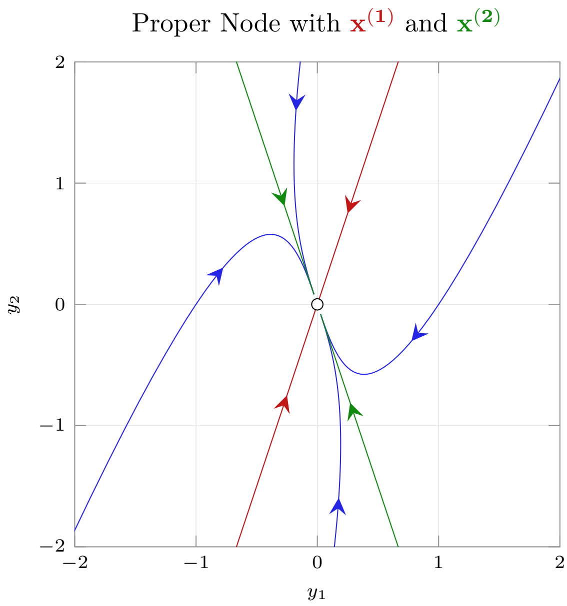
$$\mathbf{A} = \begin{bmatrix} -6 & -1 \\ -9 & -6 \end{bmatrix} \qquad \lambda^2 + 12\lambda + 27 = 0 \tag{4.4.27}$$

Using $p = -12,\ q = 27,\ \Delta = 36$ gives a center at $(0, 0)$.

$$\lambda_1 = -9 \qquad \mathbf{v^{(1)}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{4.4.28}$$

$$\lambda_2 = -3 \qquad \mathbf{v^{(2)}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{4.4.29}$$

$$\mathbf{y} = c_1 e^{-9t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{4.4.30}$$



7. Finding the type and stability of critical point,

$$y_1' = y_1 + 2y_2 \qquad y_2' = 2y_1 + y_2 \tag{4.4.31}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \lambda^2 - 2\lambda - 3 = 0 \tag{4.4.32}$$

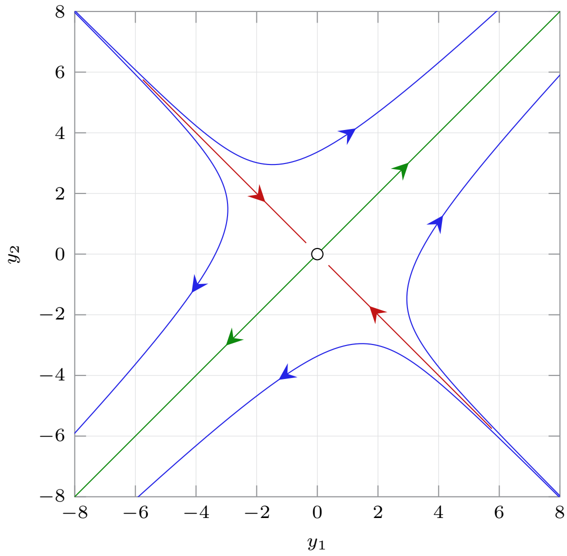
Using $p = 2,\ q = -3,\ \Delta = 16$ gives a saddle point at $(0, 0)$.

$$\lambda_1 = -1 \qquad \mathbf{v^{(1)}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.4.33}$$

$$\lambda_2 = 3 \qquad \mathbf{v^{(2)}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.4.34}$$

$$\mathbf{y} = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.4.35}$$

Saddle point with $\mathbf{x^{(1)}}$ and $\mathbf{x^{(2)}}$



8. Finding the type and stability of critical point,

$$y_1' = -y_1 + 4y_2 \qquad y_2' = 3y_1 - 2y_2 \tag{4.4.36}$$

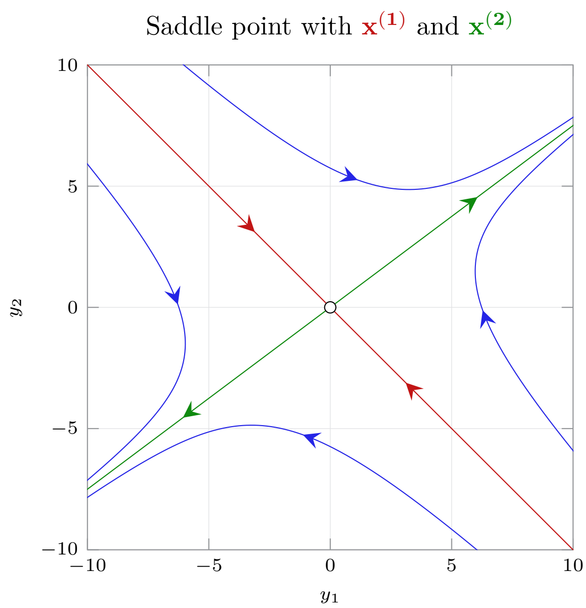
$$\mathbf{A} = \begin{bmatrix} -1 & 4 \\ 3 & -2 \end{bmatrix} \qquad \lambda^2 + 3\lambda - 10 = 0 \tag{4.4.37}$$

Using $p = -3,\ q = -10,\ \Delta = 49$ gives a saddle point at $(0, 0)$.

$$\lambda_1 = -5 \qquad \mathbf{v^{(1)}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.4.38}$$

$$\lambda_2 = 2 \qquad \mathbf{v^{(2)}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \tag{4.4.39}$$

$$\mathbf{y} = c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \tag{4.4.40}$$



9. Finding the type and stability of critical point,

$$y_1' = 4y_1 + y_2 \qquad\qquad y_2' = 4y_1 + 4y_2 \tag{4.4.41}$$

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 4 & 4 \end{bmatrix} \qquad\qquad \lambda^2 - 8\lambda + 12 = 0 \tag{4.4.42}$$

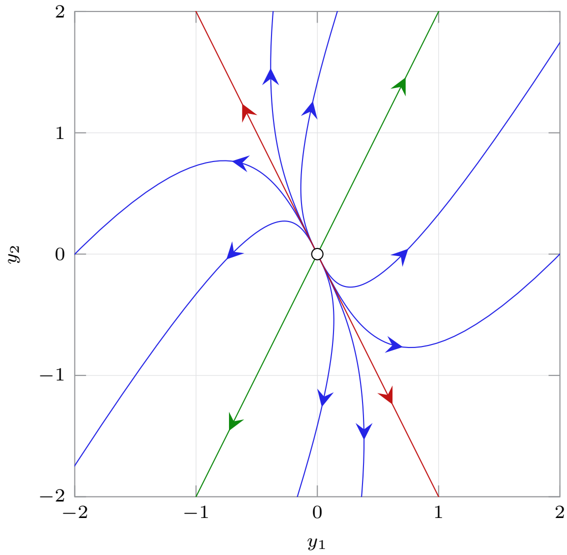
Using $p = 8,\ \ q = 12,\ \ \Delta = 16$ gives an improper node at $(0, 0)$.

$$\lambda_1 = 2 \qquad\qquad \mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \tag{4.4.43}$$

$$\lambda_2 = 6 \qquad\qquad \mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{4.4.44}$$

$$\mathbf{y} = c_1 e^{2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{4.4.45}$$

Improper Node with $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$



10. Finding the type and stability of critical point,

$$y_1' = y_2 \qquad y_2' = -5y_1 - 2y_2 \tag{4.4.46}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \qquad \lambda^2 - 8\lambda + 12 = 0 \tag{4.4.47}$$

Using $p = 8,\ q = 12,\ \Delta = 16$ gives a spiral at $(0, 0)$.

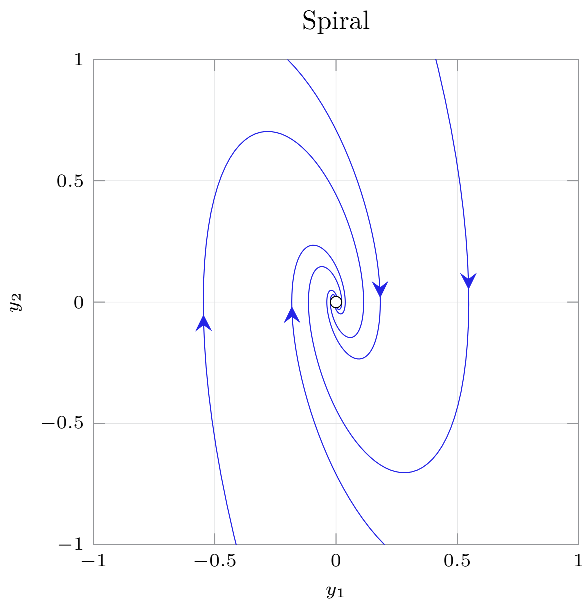
$$\lambda_1 = -1 - 2i \qquad \mathbf{v}^{(1)} = \begin{bmatrix} -0.2 + 0.4i \\ 1 \end{bmatrix} \tag{4.4.48}$$

$$\lambda_2 = -1 + 2i \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -0.2 - 0.4i \\ 1 \end{bmatrix} \tag{4.4.49}$$

$$\mathbf{y} = c_1 e^{(-1-2i)t} \begin{bmatrix} -0.2 + 0.4i \\ 1 \end{bmatrix} + c_2 e^{(-1+2i)t} \begin{bmatrix} -0.2 - 0.4i \\ 1 \end{bmatrix} \tag{4.4.50}$$

$$\mathbf{y} = e^{-t} c_1 \begin{bmatrix} -0.2\cos(2t) + 0.4\sin(2t) \\ \cos(2t) \end{bmatrix}$$

$$+ e^{-t} c_2 \begin{bmatrix} -0.2\cos(2t) + 0.4\sin(2t) \\ \cos(2t) \end{bmatrix} \tag{4.4.51}$$



11. Damped oscillations,

$$y'' + 2y' + 2y = 0 \qquad y_1 = y, \quad y_2 = y' \tag{4.4.52}$$

$$y_2' = -2y_1 - 2y_2 \qquad y_1' = y_2 \tag{4.4.53}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \qquad \lambda^2 + 2\lambda + 2 = 0 \tag{4.4.54}$$

Using $p = -2, \ q = 2, \ \Delta = -4$ gives a spiral at $(0, 0)$.

$$\lambda_1 = -1 - \mathrm{i} \qquad \mathbf{v}^{(1)} = \begin{bmatrix} -1 + \mathrm{i} \\ 2 \end{bmatrix} \tag{4.4.55}$$

$$\lambda_2 = -1 + \mathrm{i} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -1 - \mathrm{i} \\ 2 \end{bmatrix} \tag{4.4.56}$$

$$\mathbf{y} = c_1 e^{(-1-\mathrm{i})t} \begin{bmatrix} -1 + \mathrm{i} \\ 2 \end{bmatrix} + c_2 e^{(-1+\mathrm{i})t} \begin{bmatrix} -1 - \mathrm{i} \\ 2 \end{bmatrix} \tag{4.4.57}$$

$$\mathbf{y} = e^{-t} c_1 \begin{bmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \end{bmatrix} + e^{-t} c_2 \begin{bmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \end{bmatrix} \tag{4.4.58}$$

12. Harmonic oscillations,

$$y'' + \frac{1}{9}y = 0 \qquad y_1 = y, \qquad y_2 = y' \tag{4.4.59}$$

$$y_2' = -\frac{1}{9}y_1 \qquad y_1' = y_2 \tag{4.4.60}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/9 & 0 \end{bmatrix} \qquad \lambda^2 + \frac{1}{9} = 0 \tag{4.4.61}$$

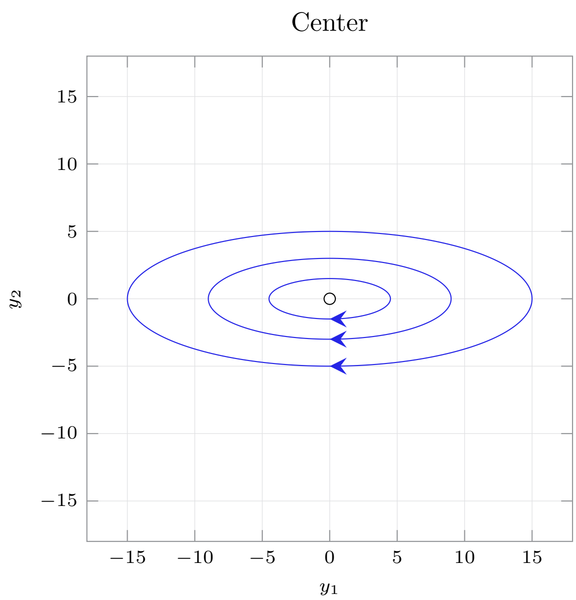
Using $p = 0,\ q = 1/9,\ \Delta = -4/9$ gives a center at $(0, 0)$.

$$\lambda_1 = \frac{-\mathrm{i}}{3} \qquad \mathbf{v}^{(1)} = \begin{bmatrix} 3i \\ 1 \end{bmatrix} \tag{4.4.62}$$

$$\lambda_2 = \frac{\mathrm{i}}{3} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -3i \\ 1 \end{bmatrix} \tag{4.4.63}$$

$$\mathbf{y} = c_1 e^{-t\ \mathrm{i}/3} \begin{bmatrix} 3\ \mathrm{i} \\ 1 \end{bmatrix} + c_2 e^{t\ \mathrm{i}/3} \begin{bmatrix} -3\ \mathrm{i} \\ 1 \end{bmatrix} \tag{4.4.64}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 3\ \sin(t/3) \\ \cos(t/3) \end{bmatrix} + c_2 \begin{bmatrix} 3\ \sin(t/3) \\ \cos(t/3) \end{bmatrix} \tag{4.4.65}$$

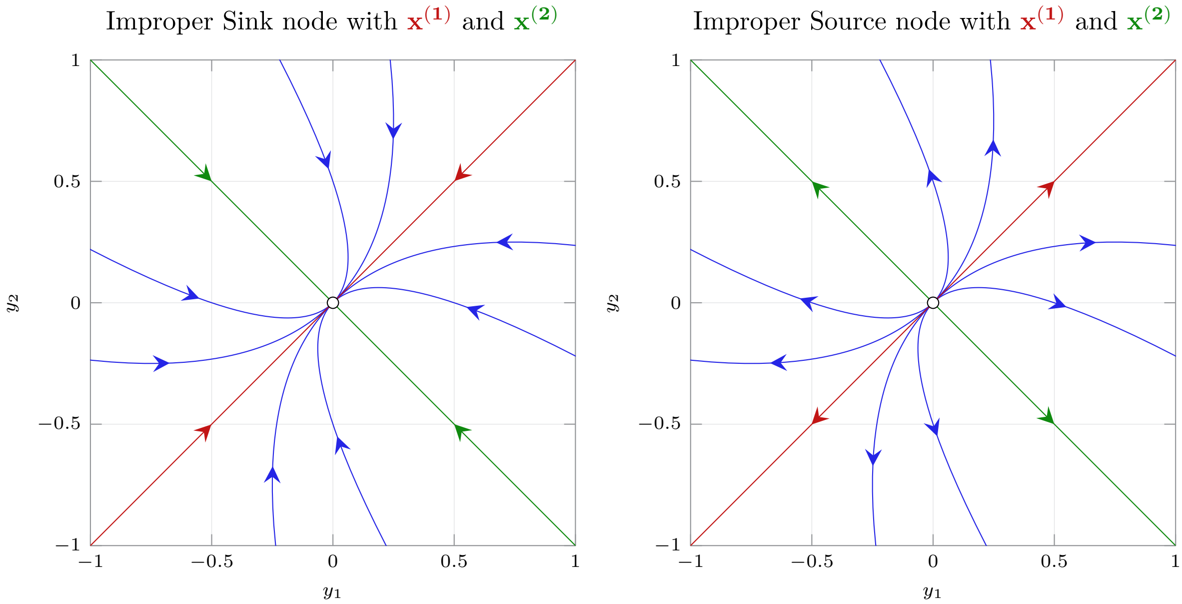


13. Refer to notes for sections 4.3 and 4.4

14. From example 1, the general solution after making the change $-t \to \tau$ is, the nature of the critical point remains the same, except that the sink becomes a source.

This follows from the indices of the exponential becoming both positive.

$$y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2\tau} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4\tau} \tag{4.4.66}$$



15. The new system is, for some small perturbation $\mu > 0$,

$$\mathbf{y} = \begin{bmatrix} \mu & 1 \\ -4 & \mu \end{bmatrix} \qquad 0 = \lambda^2 - (2\mu)\lambda + (4 + \mu^2) \tag{4.4.67}$$

$$p = 2\mu \qquad q = (4 + \mu^2) \qquad \Delta = -16 \tag{4.4.68}$$

$$\lambda_1 = \mu - 2\,\mathrm{i} \qquad \mathbf{v}^{(1)} = \begin{bmatrix} \mathrm{i} \\ 2 \end{bmatrix} \tag{4.4.69}$$

$$\lambda_2 = \mu + 2\,\mathrm{i} \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -\mathrm{i} \\ 2 \end{bmatrix} \tag{4.4.70}$$

$$\mathbf{y} = e^{\mu t}(c_1 + c_2) \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix} \tag{4.4.71}$$

This spiral is symmetric in y_1 and y_2 and for the special case $\mu = 0$, reduces to a center. In this problem, $\mu = 0.1$.

16. From the previous problem, $\mu \neq 0$, causes the center to become a spiral, which is a source ($\mu > 0$) or sink ($\mu < 0$).

17. Types of perturbation $a_{jk} \to a_{jk} + b$ required to produce different kinds of critical points.

(a) To get a saddle point,

$$\mathbf{A} = \begin{bmatrix} b & 1+b \\ -4+b & b \end{bmatrix} \tag{4.4.72}$$

$$0 = \lambda^2 + 2b\lambda + (3b+4) \tag{4.4.73}$$

$$q = (3b+4) < 0 \qquad b \in \left(-\infty, \frac{-4}{3}\right) \tag{4.4.74}$$

$$\Delta = b^2 - 3b - 4 > 0 \qquad b \in (-\infty, -1) \cup (4, \infty) \tag{4.4.75}$$

$$\text{Saddle point} \implies b \in \left(-\infty, \frac{-4}{3}\right) \tag{4.4.76}$$

(b) To get a stable and attractive node.

$$\mathbf{A} = \begin{bmatrix} b & 1+b \\ -4+b & b \end{bmatrix} \tag{4.4.77}$$

$$0 = \lambda^2 + 2b\lambda + (3b+4) \tag{4.4.78}$$

$$q = (3b+4) > 0 \qquad b \in \left(\frac{-4}{3}, \infty\right) \tag{4.4.79}$$

$$p = -2b < 0 \qquad b \in (0, \infty) \tag{4.4.80}$$

$$\Delta = b^2 - 3b - 4 \geq 0 \qquad b \in (-\infty, -1] \cup [4, \infty) \tag{4.4.81}$$

$$\text{Stable and attractive node} \implies b \in [4, \infty) \tag{4.4.82}$$

(c) To get a stable and attractive spiral.

$$\mathbf{A} = \begin{bmatrix} b & 1+b \\ -4+b & b \end{bmatrix} \tag{4.4.83}$$

$$0 = \lambda^2 + 2b\lambda + (3b+4) \tag{4.4.84}$$

$$q = (3b+4) > 0 \qquad b \in \left(\frac{-4}{3}, \infty\right) \tag{4.4.85}$$

$$\Delta = b^2 - 3b - 4 < 0 \qquad b \in (-1, 4) \tag{4.4.86}$$

$$p = -2b < 0 \qquad b \in (0, \infty) \tag{4.4.87}$$

$$\text{Stable and attractive spiral} \implies b \in (0, 4) \tag{4.4.88}$$

(d) To get a unstable spiral,

$$\mathbf{A} = \begin{bmatrix} b & 1+b \\ -4+b & b \end{bmatrix} \tag{4.4.89}$$

$$0 = \lambda^2 + 2b\lambda + (3b+4) \tag{4.4.90}$$

$$q = (3b+4) > 0 \qquad b \in \left(\frac{-4}{3}, \infty\right) \tag{4.4.91}$$

$$\Delta = b^2 - 3b - 4 < 0 \qquad b \in (-1, 4) \tag{4.4.92}$$

$$p = -2b > 0 \qquad b \in (-\infty, 0) \tag{4.4.93}$$

$$\text{Unstable spiral} \implies b \in (-1, 0) \tag{4.4.94}$$

(e) To get a unstable node,

$$\mathbf{A} = \begin{bmatrix} b & 1+b \\ -4+b & b \end{bmatrix} \tag{4.4.95}$$

$$0 = \lambda^2 + 2b\lambda + (3b+4) \tag{4.4.96}$$

$$q = (3b+4) > 0 \qquad b \in \left(\frac{-4}{3}, \infty\right) \tag{4.4.97}$$

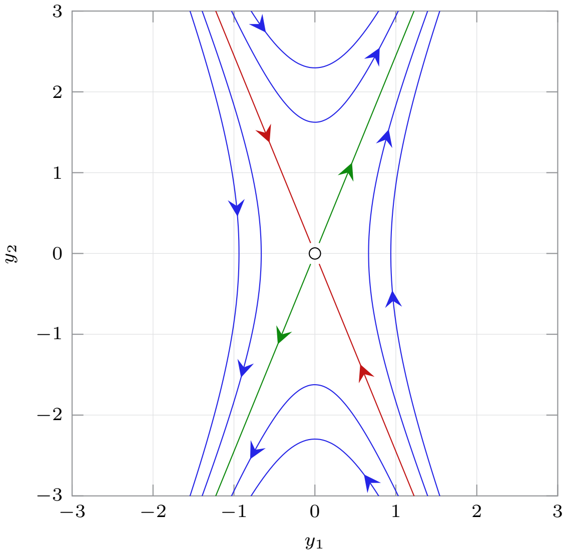
$$\Delta = b^2 - 3b - 4 \geq 0 \qquad b \in (-\infty, -1] \cup [4, \infty) \tag{4.4.98}$$

$$p = -2b > 0 \qquad b \in (-\infty, 0) \tag{4.4.99}$$

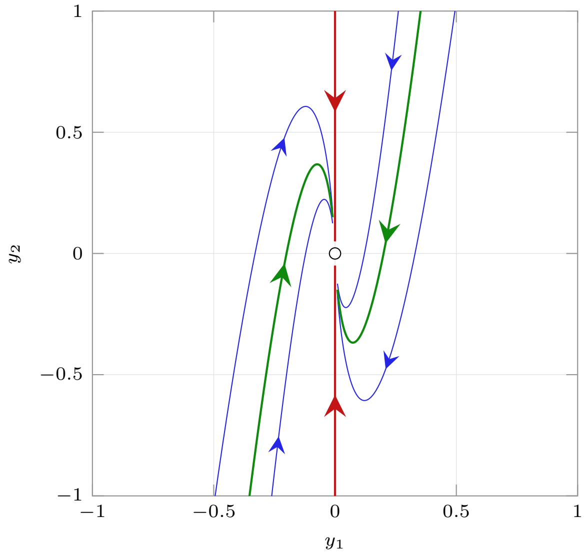
$$\text{Unstable spiral} \implies b \in \left(\frac{-4}{3}, -1\right] \tag{4.4.100}$$

18. Plotting some phase portraits. Cant show the effect of continuous change in b on paper.

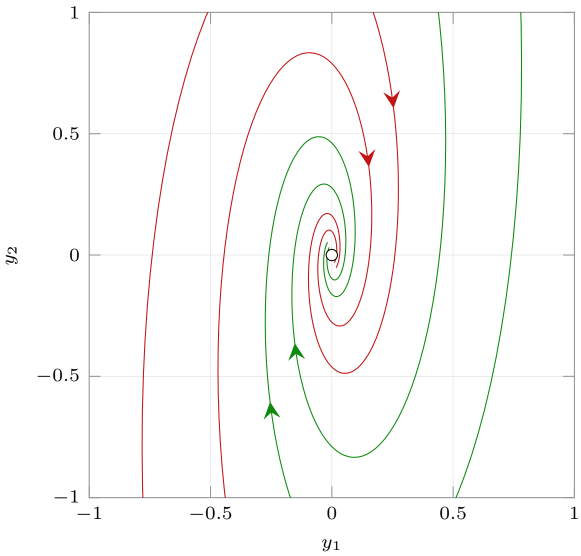
Saddle point with $b = -2$
$\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$



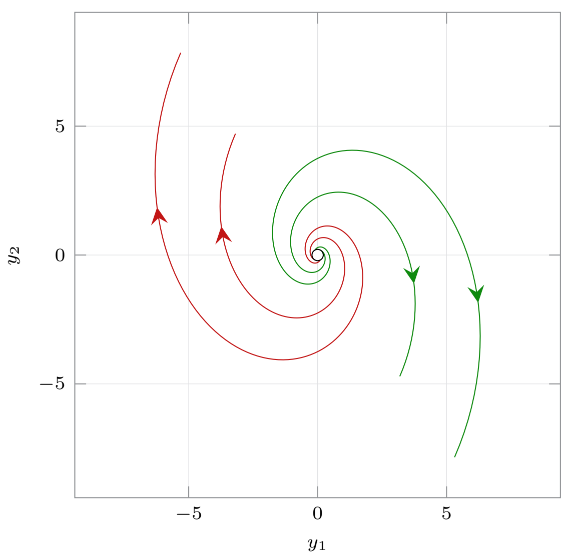
Degenerate Node with $b = -1$
$\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$

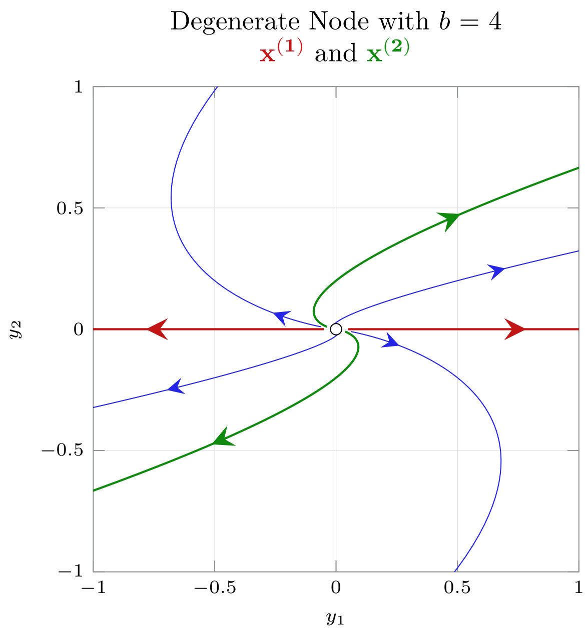


Stable Spiral with $b = -0.5$



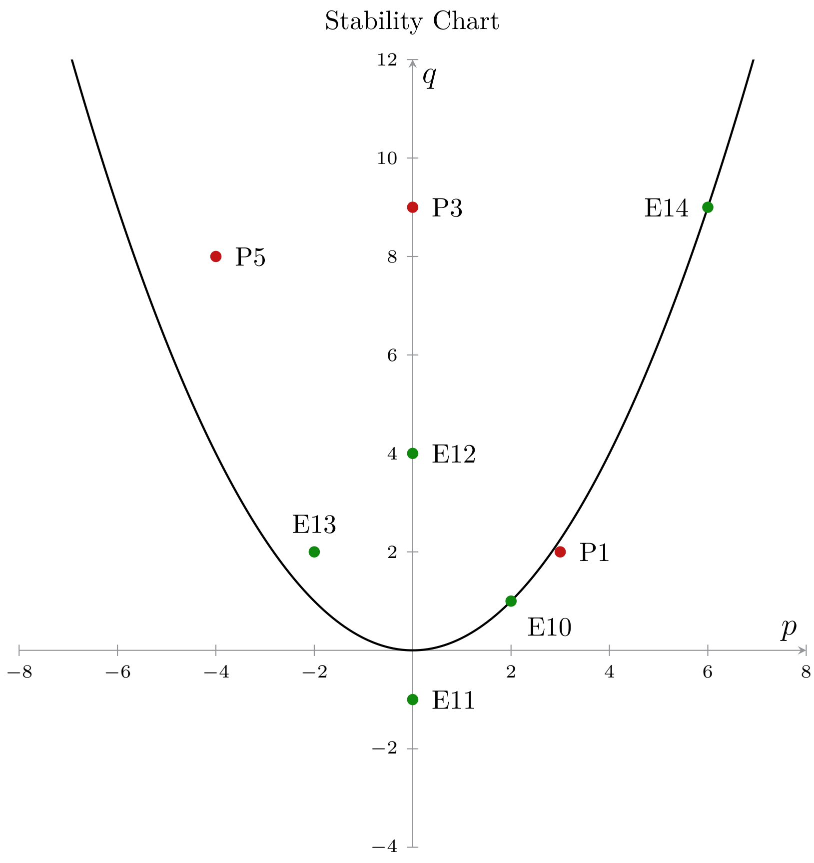
Unstable Spiral with $b = 1$





19. TBC.

20. Plotting the points on the $p - q$ plane,



4.5 Qualitative Methods for Nonlinear Systems

1. Consider the closed trajectory intersecting with the two axes. This is an ellipse with major axis $2a$ and minor axis $2b$.

Intersection	Position y	Velocity y'	Trajectory
$+x$ axis	a	0	Right edge
$-x$ axis	$-a$	0	Left edge
$+y$ axis	0	b	Mean position traveling left
$-y$ axis	0	$-b$	Mean position traveling right

For the open trajectory, the penduum never reverses angular direction. Maxima and minima in the open trajectory represent the bottom and top of the vertical plane circular trajectory.

When the trajectory intersects the y_2 axis, the pendulum is at the mean position moving at maximum angular speed (because its potential energy is lowest at this point in the cyclic path).

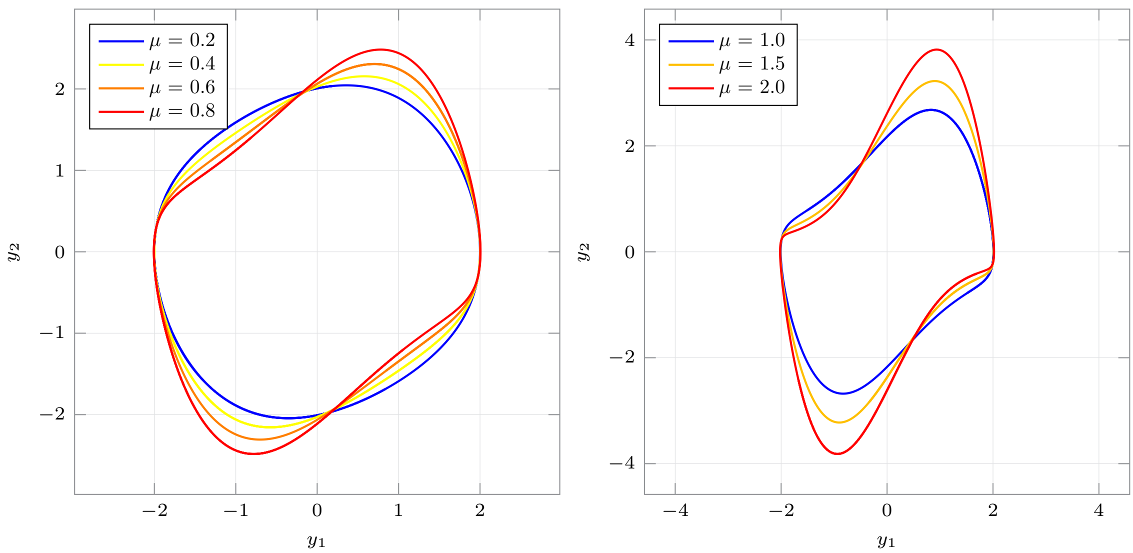
2. A limit cycle is the limiting set of a different trajectory that may not have had its initial conditions close to the limit cycle. The limit cycle is the steady state behaviour after all transient effects have worn off.

A closed trajectory surrounding a center is constrained to start and always stay on this trajectory.

3. System of ODEs for Van der Pol oscillator is,

$$y_1' = y_2 \qquad\qquad y_2' = \mu(1 - y_1^2)y_2 - y_1 \tag{4.5.1}$$

After simulating until steady state, the attractor is plotted here for different values of μ,



All the simulations have initial condition $(1, 0)$. The spiraling of the individual trajectories into the limit cycle is not plotted here, for the sake of clarity.

The limit cycle starts very closely resembling the circle for $\mu \approxeq 0$, then deforms into the shape in the graph smoothly as μ increases.

4. Finding critical points,

$$y_1' = 4y_1 - y_1^2 \qquad y_2' = y_2 \tag{4.5.2}$$

$$y_1(4 - y_1) = 0 \qquad y_2 = 0 \tag{4.5.3}$$

$$P_1 = (0, 0) \qquad P_2 = (4, 0) \tag{4.5.4}$$

Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 - 5\lambda + 4 \tag{4.5.5}$$

$$p < 0 \qquad q > 0 \qquad \Delta > 0 \qquad \text{Improper node} \tag{4.5.6}$$

Linearizing the system for P_2

$$z_1 = y_1 - 4 \qquad z_2 = y_2 \tag{4.5.7}$$

$$z_1' = 4(z_1 + 4) - (z_1 + 4)^2 \qquad z_2' = z_2 \tag{4.5.8}$$

$$\mathbf{y}' = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 3\lambda - 4 \tag{4.5.9}$$

$$p < 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.10}$$

5. Finding critical points,

$$y_1' = y_2 \qquad y_2' = -y_1 + 0.5y_1^2 \tag{4.5.11}$$

$$y_1(0.5y_1 - 1) = 0 \qquad y_2 = 0 \tag{4.5.12}$$

$$P_1 = (0, 0) \qquad P_2 = (2, 0) \tag{4.5.13}$$

Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 1 \tag{4.5.14}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.15}$$

Linearizing the system for P_2

$$z_1 = y_1 - 2 \qquad z_2 = y_2 \tag{4.5.16}$$

$$z_1' = z_2 \qquad z_2' = 0.5(z_1 + 2)z_1 \tag{4.5.17}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 - 1 \tag{4.5.18}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.19}$$

6. Finding critical points,

$$y_1' = y_2 \qquad y_2' = -y_1(1 + y_1) \tag{4.5.20}$$

$$y_1(y_1 + 1) = 0 \qquad y_2 = 0 \tag{4.5.21}$$

$$P_1 = (0, 0) \qquad P_2 = (-1, 0) \tag{4.5.22}$$

Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 1 \tag{4.5.23}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.24}$$

Linearizing the system for P_2

$$z_1 = y_1 + 1 \qquad z_2 = y_2 \tag{4.5.25}$$

$$z_1' = z_2 \qquad z_2' = -(z_1 - 1)z_1 \tag{4.5.26}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 - 1 \tag{4.5.27}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.28}$$

7. Finding critical points,

$$y_1' = -y_1 + y_2(1 - y_2) \qquad y_2' = -y_1 - y_2 \tag{4.5.29}$$

$$-y_1 + y_2(1 - y_2) = 0 \qquad y_1 + y_2 = 0 \tag{4.5.30}$$

$$P_1 = (0, 0) \qquad P_2 = (-2, 2) \tag{4.5.31}$$

Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 2\lambda + 2 \tag{4.5.32}$$

$$p < 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Attractive Spiral} \tag{4.5.33}$$

Linearizing the system for P_2

$$z_1 = y_1 + 2 \qquad z_2 = y_2 - 2 \tag{4.5.34}$$

$$z_1' = 2 - z_1 - (z_2 + 2)(z_2 + 1) \qquad z_2' = 2 - z_1 - z_2 - 2 \tag{4.5.35}$$

$$\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ -1 & -1 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 2\lambda - 2 \tag{4.5.36}$$

$$p < 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.37}$$

8. Finding critical points,

$$y_1' = y_2(1 - y_2) \qquad y_2' = y_1(1 - y_1) \tag{4.5.38}$$

$$y_2(1 - y_2) = 0 \qquad y_1(1 - y_1) = 0 \tag{4.5.39}$$

$$P_1 = (0, 0) \qquad P_2 = (0, 1) \tag{4.5.40}$$

$$P_3 = (1, 0) \qquad P_4 = (1, 1) \tag{4.5.41}$$

Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 - 1 \tag{4.5.42}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle Point} \tag{4.5.43}$$

Linearizing the system for P_2

$$z_1 = y_1 \qquad z_2 = y_2 - 1 \tag{4.5.44}$$

$$z_1' = -z_2(z_2 + 1) \qquad z_2' = z_1(1 - z_1) \tag{4.5.45}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 1 \tag{4.5.46}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.47}$$

Linearizing the system for P_3

$$z_1 = y_1 - 1 \qquad z_2 = y_2 \tag{4.5.48}$$

$$z_1' = z_2(1 - z_2) \qquad z_2' = -z_1(1 + z_1) \tag{4.5.49}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 + 1 \tag{4.5.50}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.51}$$

Linearizing the system for P_4

$$z_1 = y_1 - 1 \qquad z_2 = y_2 - 1 \tag{4.5.52}$$

$$z_1' = -z_2(1 + z_2) \qquad z_2' = -z_1(1 + z_1) \tag{4.5.53}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 - 1 \tag{4.5.54}$$

$$p = 0 \qquad q > 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.55}$$

9. Finding critical points,

$$y'' - 9y + y^3 = 0 \tag{4.5.56}$$

$$y_1' = y_2 \qquad y_2' = y_1(9 - y_1^2) \tag{4.5.57}$$

$$y_2 = 0 \qquad y_1(y_1 - 3)(y_1 + 3) = 0 \tag{4.5.58}$$

$$P_1 = (0, 0) \qquad P_2 = (-3, 0) \tag{4.5.59}$$

$$P_3 = (3, 0) \tag{4.5.60}$$

Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 - 2 \tag{4.5.61}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.62}$$

Linearizing the system for P_2

$$z_1 = y_1 + 3 \qquad z_2 = y_2 \tag{4.5.63}$$

$$z_1' = z_2 \qquad z_2' = (z_1 - 3)(z_1)(6 - z_1) \tag{4.5.64}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -18 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 18 \tag{4.5.65}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.66}$$

Linearizing the system for P_3

$$z_1 = y_1 - 3 \qquad z_2 = y_2 \tag{4.5.67}$$

$$z_1' = z_2 \qquad z_2' = -(z_1 + 3)(z_1)(6 + z_1) \tag{4.5.68}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -18 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 18 \tag{4.5.69}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.70}$$

10. Finding critical points,

$$y'' + y - y^3 = 0 \tag{4.5.71}$$

$$y_1' = y_2 \qquad y_2' = y_1(y_1^2 - 1) \tag{4.5.72}$$

$$y_2 = 0 \qquad y_1(y_1 + 1)(y_1 - 1) = 0 \tag{4.5.73}$$

$$P_1 = (0, 0) \qquad P_2 = (-1, 0) \tag{4.5.74}$$

$$P_3 = (1, 0) \tag{4.5.75}$$

Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + 1 \tag{4.5.76}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.77}$$

Linearizing the system for P_2

$$z_1 = y_1 + 1 \qquad z_2 = y_2 \tag{4.5.78}$$

$$z_1' = z_2 \qquad z_2' = (z_1 - 1)(z_1 - 2)(z_1) \tag{4.5.79}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 - 2 \tag{4.5.80}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.81}$$

Linearizing the system for P_3

$$z_1 = y_1 - 1 \qquad z_2 = y_2 \tag{4.5.82}$$

$$z_1' = z_2 \qquad z_2' = (z_1 + 1)(z_1)(2 + z_1) \tag{4.5.83}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 - 2 \tag{4.5.84}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.85}$$

11. Finding critical points,

$$y'' + \cos y = 0 \tag{4.5.86}$$

$$y_1' = y_2 \qquad y_2' = -\cos(y_1) \tag{4.5.87}$$

$$y_2 = 0 \qquad \cos(y_1) = 0 \tag{4.5.88}$$

$$P_1 = (2n\pi - \pi/2, 0) \qquad P_2 = (2n\pi + \pi/2, 0) \tag{4.5.89}$$

$$\tag{4.5.90}$$

Linearizing the system for P_1

$$z_1 = y_1 - 2n\pi + \pi/2 \qquad z_2 = y_2 \tag{4.5.91}$$

$$z_1' = z_2 \qquad z_2' = -\cos(z_1 + 2n\pi - \pi/2) \tag{4.5.92}$$

$$= -\sin(z_1 + 2n\pi) \tag{4.5.93}$$

$$= -z_1 + \frac{1}{6}z_1^3 \tag{4.5.94}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 + 1 \tag{4.5.95}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.96}$$

Linearizing the system for P_2

$$z_1 = y_1 - 2n\pi - \pi/2 \qquad z_2 = y_2 \tag{4.5.97}$$

$$z_1' = z_2 \qquad z_2' = -\cos(z_1 + 2n\pi + \pi/2) \tag{4.5.98}$$

$$= \sin(z_1 + 2n\pi) \tag{4.5.99}$$

$$= z_1 - \frac{1}{6}z_1^3 \tag{4.5.100}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 - 1 \tag{4.5.101}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.102}$$

12. Finding critical points,

$$y'' + 9y + y^2 = 0 \tag{4.5.103}$$

$$y_1' = y_2 \qquad y_2' = -y_1(y_1 + 9) \tag{4.5.104}$$

$$y_2 = 0 \qquad y_1(y_1 + 9) = 0 \tag{4.5.105}$$

$$P_1 = (0, 0) \qquad P_2 = (-9, 0) \tag{4.5.106}$$

$$\tag{4.5.107}$$

Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 + 9 \tag{4.5.108}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.109}$$

Linearizing the system for P_2

$$z_1 = y_1 + 9 \qquad z_2 = y_2 \tag{4.5.110}$$

$$z_1' = z_2 \qquad z_2' = (9 - z_1)(z_1) \tag{4.5.111}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 - 9 \tag{4.5.112}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.113}$$

13. Finding critical points,

$$y'' + \sin y = 0 \tag{4.5.114}$$

$$y_1' = y_2 \qquad y_2' = -\sin(y_1) \tag{4.5.115}$$

$$y_2 = 0 \qquad \cos(y_1) = 0 \tag{4.5.116}$$

$$P_1 = (2n\pi, 0) \qquad P_2 = (2n\pi + \pi, 0) \tag{4.5.117}$$

$$\tag{4.5.118}$$

Linearizing the system for P_1

$$z_1 = y_1 - 2n\pi \qquad z_2 = y_2 \tag{4.5.119}$$

$$z_1' = z_2 \qquad z_2' = -\sin(z_1 + 2n\pi) \tag{4.5.120}$$

$$= -z_1 + \frac{1}{6}z_1^3 \tag{4.5.121}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 + 1 \tag{4.5.122}$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \tag{4.5.123}$$

Linearizing the system for P_2

$$z_1 = y_1 - 2n\pi - \pi \qquad z_2 = y_2 \tag{4.5.124}$$

$$z_1' = z_2 \qquad z_2' = -\sin(z_1 + 2n\pi + \pi) \tag{4.5.125}$$

$$= z_1 - \frac{1}{6}z_1^3 \tag{4.5.126}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{y} \qquad 0 = \lambda^2 - 1 \tag{4.5.127}$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \tag{4.5.128}$$

14. Ploting the graphs,

(a) Van der Pol equation,

$$y_1' = y_2 \qquad y_2' = \mu y_2 - y_1 - \mu y_1^2 y_2 \tag{4.5.129}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \qquad 0 = \lambda^2 - \mu\lambda + 1 \tag{4.5.130}$$

$$p = \mu \qquad q = 1 \qquad \Delta = \mu^2 - 4 \tag{4.5.131}$$

For $\mu = 0$, $\mu > 0$, $\mu < 0$, the critical point at $(0, 0)$ is a center, spiral source and spiral sink respectively.

(b) Rayleigh equation,

$$z'' - \mu\left(1 - \frac{z'^2}{3}\right)z' + z = 0 \tag{4.5.132}$$

$$z''' - \mu\left(1 - \frac{z'^2}{3}\right)z'' + \mu z'\left(\frac{2z'}{3}\right)z'' + z' = 0 \tag{4.5.133}$$

$$z''' - (1 - z'^2)\mu z'' + z' = 0 \tag{4.5.134}$$

$$\tag{4.5.135}$$

Using the substitution $z' \to y$, the above equation reduces to the Van der Pol ODE.

(c) Duffing equation,

$$y'' + \omega_0^2 y + \beta y^3 = 0 \qquad 4.5.136$$

$$y_1' = y_2 \qquad y_2' = -y_1(\omega_0^2 + \beta y_1^2) \qquad 4.5.137$$

$$y_2 = 0 \qquad -y_1(\omega_0^2 + \beta y_1^2) = 0 \qquad 4.5.138$$

$$P_1 = (0, 0) \qquad P_2 = (-\alpha, 0) \qquad 4.5.139$$

$$P_2 = (\alpha, 0) \qquad \alpha = \frac{\omega_0}{\sqrt{-\beta}} \qquad 4.5.140$$

P_2 and P_3 require $\beta < 0$ in order to exist.
Linearizing the system for P_1

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 + \omega_0^2 \qquad 4.5.141$$

$$p = 0 \qquad q > 0 \qquad \Delta < 0 \qquad \text{Center} \qquad 4.5.142$$

Linearizing the system for P_2

$$z_1 = y_1 + \alpha \qquad z_2 = y_2 \qquad 4.5.143$$

$$z_1' = z_2 \qquad z_2' = (\alpha - z_1)(\beta z_1)(z_1 - 2\alpha) \qquad 4.5.144$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 - \omega_0^2 \qquad 4.5.145$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \qquad 4.5.146$$

Linearizing the system for P_2

$$z_1 = y_1 - \alpha \qquad z_2 = y_2 \qquad 4.5.147$$

$$z_1' = z_2 \qquad z_2' = -(\alpha + z_1)(\beta z_1)(z_1 + 2\alpha) \qquad 4.5.148$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{bmatrix} \mathbf{y} \qquad 0 = \lambda^2 - \omega_0^2 \qquad 4.5.149$$

$$p = 0 \qquad q < 0 \qquad \Delta > 0 \qquad \text{Saddle point} \qquad 4.5.150$$

15. Reframing ODE as a phase plane parametric function,

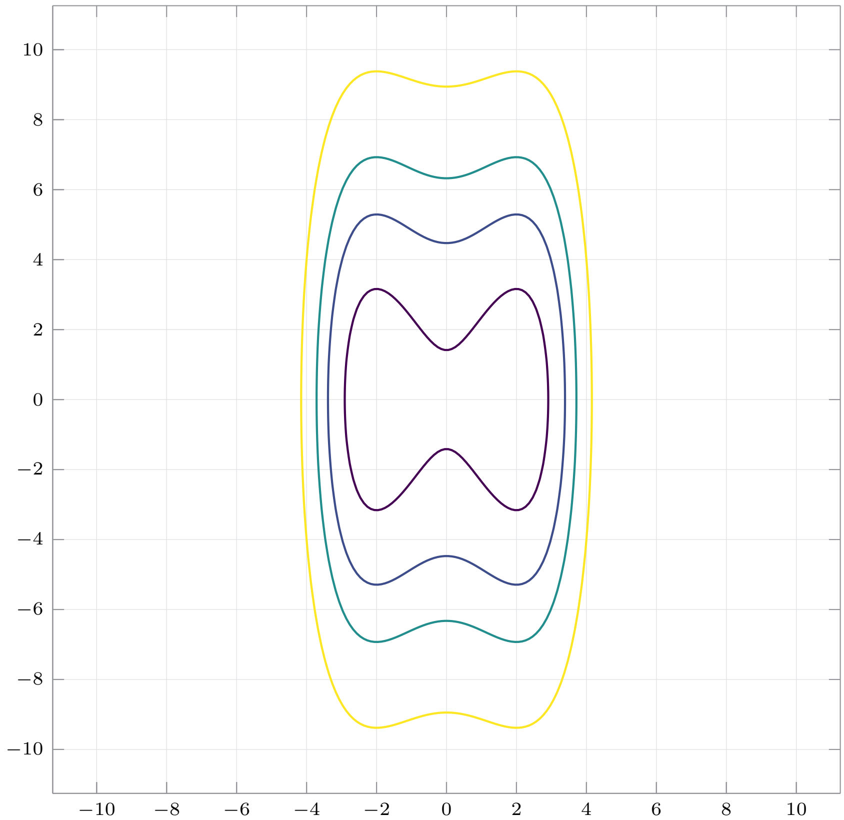
$$y'' - 4y + y^3 = 0 \tag{4.5.151}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}y_1}\, y_2 = 4y_1 - y_1^3 \tag{4.5.152}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}y_1} = \frac{y_1(4 - y_1^2)}{y_2} \tag{4.5.153}$$

$$0.5y_2^2 = 2y_1^2 - 0.25y_1^4 + C \tag{4.5.154}$$

Drawing contour plots for this equation in $y_1,\ y_2$.



4.6 Nonhomogeneous Linear Systems of ODEs

1. Following the proof for second order linear nh-ODEs, Start with the fact that the solution $\mathbf{y}^{(\mathbf{h})}$ of the h-ODE includes all possible solutions.

Let $\mathbf{y}^*$ be a solution of the nh-ODE on the interval $\mathcal{I}$ and some point $t_0 \in \mathcal{I}$.

$$\mathbf{y} = \mathbf{y}^{(\mathbf{p})} + \mathbf{y}^{(\mathbf{h})} \text{ be a solution of the nh-ODE} \tag{4.6.1}$$

$$\mathbf{Y} \equiv \mathbf{y}^* - \mathbf{y}^{(\mathbf{p})} \tag{4.6.2}$$

$$\mathbf{Y} \text{ solves the h-ODE} \tag{4.6.3}$$

$$\mathbf{Y}(t_0) = \mathbf{y}^*(t_0) - \mathbf{y}^{(\mathbf{p})}(t_0) \tag{4.6.4}$$

$$\mathbf{Y}'(t_0) = \mathbf{y}^{*'}(t_0) - \mathbf{y}^{(\mathbf{p})'}(t_0) \tag{4.6.5}$$

$$\tag{4.6.6}$$

For any I.C. in $\mathcal{I}$, there exists a unique c_1, c_2 which goes into $\mathbf{y}^{(\mathbf{h})}$. Since $\mathbf{y}^{(\mathbf{h})}$ and $\mathbf{y}^*$ no longer have any arbitrary constants, $\mathbf{Y}$ has also been uniquely determined.

This proves that every possible IVP has a solution contained in the general solution of the nh-ODE and that no singular solution exists.

2. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 10\cos(t) \\ 10\sin(t) \end{bmatrix} \qquad \lambda^2 - 4 = 0 \tag{4.6.7}$$

$$\lambda_1 = -2 \qquad \mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{4.6.8}$$

$$\lambda_2 = 2 \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.6.9}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \tag{4.6.10}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1\cos(t) + B_1\sin(t) \\ A_2\cos(t) + B_2\sin(t) \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} 10\cos(t) \\ 10\sin(t) \end{bmatrix} \tag{4.6.11}$$

$$\mathbf{y}^{(\mathbf{p})'} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.12}$$

$$-A_1\sin(t) + B_1\cos(t) = [A_1 + A_2 + 10]\cos(t) \qquad + [B_1 + B_2]\sin(t) \tag{4.6.13}$$

$$-A_2\sin(t) + B_2\cos(t) = [3A_1 - A_2]\cos(t) \qquad + [3B_1 - B_2 + 10]\sin(t) \tag{4.6.14}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} -2 \\ -8 \end{bmatrix} \cos(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin(t) \tag{4.6.15}$$

3. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^{3t} \\ -3e^{3t} \end{bmatrix} \qquad \lambda^2 - 1 = 0 \tag{4.6.16}$$

$$\lambda_1 = -1 \qquad \mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.6.17}$$

$$\lambda_2 = 1 \qquad \mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.6.18}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} \tag{4.6.19}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1 e^{3t} \\ A_2 e^{3t} \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} e^{3t} \\ -3e^{3t} \end{bmatrix} \tag{4.6.20}$$

$$\mathbf{y}^{(\mathbf{p})\prime} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.21}$$

$$3A_1 = A_2 + 1 \qquad 3A_2 = A_1 - 3 \tag{4.6.22}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{3t} \tag{4.6.23}$$

4. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 4 & -8 \\ 2 & -6 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2\cosh(t) \\ \cosh(t) + 2\sinh(t) \end{bmatrix} \qquad \lambda^2 + 2\lambda - 8 = 0 \tag{4.6.24}$$

$$\lambda_1 = -4 \qquad \mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.6.25}$$

$$\lambda_2 = 2 \qquad \mathbf{v}^{(2)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{4.6.26}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{2t} \tag{4.6.27}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1 \cosh(t) + B_1 \sinh(t) \\ A_2 \cosh(t) + B_2 \sinh(t) \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} 2 \cosh(t) \\ \cosh(t) + 2 \sinh(t) \end{bmatrix} \tag{4.6.28}$$

$$\mathbf{y}^{(\mathbf{p})'} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.29}$$

$$B_1 = 4A_1 - 8A_2 + 2 \qquad A_1 = 4B_1 - 8B_2 \tag{4.6.30}$$

$$B_2 = 2A_1 - 6A_2 + 1 \qquad A_2 = 2B_1 - 6B_2 + 2 \tag{4.6.31}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sinh(t) \tag{4.6.32}$$

5. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0.6t \\ -2.5t \end{bmatrix} \qquad \lambda^2 - 7\lambda + 10 = 0 \tag{4.6.33}$$

$$\lambda_1 = 2 \qquad \mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \tag{4.6.34}$$

$$\lambda_2 = 5 \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.6.35}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} \tag{4.6.36}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1 t + B_1 \\ A_2 t + B_2 \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} 0.6t \\ -2.5t \end{bmatrix} \tag{4.6.37}$$

$$\mathbf{y}^{(\mathbf{p})'} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.38}$$

$$A_1 = [4A_1 + A_2 + 0.6]t + [4B_1 + B_2] \tag{4.6.39}$$

$$A_2 = [2A_1 + 3A_2 - 2.5]t + [2B_1 + 3B_2] \tag{4.6.40}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} -0.43 \\ 1.12 \end{bmatrix} t + \begin{bmatrix} -0.241 \\ 0.534 \end{bmatrix} \tag{4.6.41}$$

6. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ -16t^2 + 2 \end{bmatrix} \qquad \lambda^2 - 16 = 0 \tag{4.6.42}$$

$$\lambda_1 = -4 \qquad \mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.6.43}$$

$$\lambda_2 = 4 \qquad \mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.6.44}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \tag{4.6.45}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1 t^2 + B_1 t + D_1 \\ A_2 t^2 + B_2 t + D_2 \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} 0 \\ -16t^2 + 2 \end{bmatrix} \tag{4.6.46}$$

$$\mathbf{y}^{(\mathbf{p})'} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.47}$$

$$[2A_1]t + [B_1] = 4A_2 t^2 + 4B_2 t + 4D_2 \tag{4.6.48}$$

$$[2A_2]t + [B_2] = [4A_1 - 16]t^2 + 4B_1 t + [4D_1 + 2] \tag{4.6.49}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 2 \end{bmatrix} t \tag{4.6.50}$$

7. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 11t + 15 \\ 3e^{-t} - 15t - 20 \end{bmatrix} \qquad \lambda^2 - 3\lambda + 2 = 0 \tag{4.6.51}$$

$$\lambda_1 = 1 \qquad \mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.6.52}$$

$$\lambda_2 = 2 \qquad \mathbf{v}^{(2)} = \begin{bmatrix} -4 \\ 5 \end{bmatrix} \tag{4.6.53}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} e^{2t} \tag{4.6.54}$$

Solving the nh-ODE,

$$\mathbf{y^{(p)}} = \begin{bmatrix} A_1e^{-t} + B_1t + D_1 \\ A_2e^{-t} + B_2t + D_2 \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} 11t + 15 \\ 3e^{-t} - 15t - 20 \end{bmatrix} \tag{4.6.55}$$

$$\mathbf{y^{(p)}}' = \mathbf{A}\mathbf{y^{(p)}} + \mathbf{g} \tag{4.6.56}$$

$$[2A_1]t + [B_1] = 4A_2t^2 + 4B_2t + 4D_2 \tag{4.6.57}$$

$$[2A_2]t + [B_2] = [4A_1 - 16]t^2 + 4B_1t + [4D_1 + 2] \tag{4.6.58}$$

$$\mathbf{y^{(p)}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 2 \end{bmatrix} t \tag{4.6.59}$$

8. As in the case of a second order linear ODE, the method of undetermined coefficients requires a guess to be the superposition of the candidate functions that make up $\mathbf{g}$.

If the modification rule comes into play, only that part of the guess needs to be modified which appears in $\mathbf{g}$.

9. When using the modification rule, and if $\mathbf{u}$ is the eigenvector corresponding to λ

$$\mathbf{y^{(p)}} = [\mathbf{u}t + \mathbf{v}]\exp(\lambda t) \tag{4.6.60}$$

The linear system of equation in $a,\ v_1,\ v_2$ yields a straight line results and not a single point. This means that there is one DOF in choosing $\mathbf{v}$ becuase the system is missing one equation.

This resembles the single DOF in choosing eigenvectors corresponding to each eigenvalue.

10. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 5 \\ -6 \end{bmatrix} e^t \qquad \lambda^2 - 3\lambda + 2 = 0 \tag{4.6.61}$$

$$\lambda_1 = 1 \qquad \mathbf{v^{(1)}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.6.62}$$

$$\lambda_2 = 2 \qquad \mathbf{v^{(2)}} = \begin{bmatrix} -4 \\ 5 \end{bmatrix} \tag{4.6.63}$$

$$\mathbf{y^{(h)}} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} e^{2t} \tag{4.6.64}$$

Solving the nh-ODE, using the modification rule

$$\mathbf{y^{(p)}} = \left(\begin{bmatrix} -a \\ a \end{bmatrix} t + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right) e^t \qquad \mathbf{g} = \begin{bmatrix} 5 \\ -6 \end{bmatrix} e^t \tag{4.6.65}$$

$$\mathbf{y^{(p)}}' = \mathbf{A}\mathbf{y^{(p)}} + \mathbf{g} \tag{4.6.66}$$

$$\begin{bmatrix} -a + B_1 - at \\ a + B_2 + at \end{bmatrix} = \begin{bmatrix} -at - 3B_1 - 4B_2 + 5 \\ at + 5B_1 + 6B_2 - 6 \end{bmatrix} \tag{4.6.67}$$

$$\begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} a + 5 \\ a + 6 \end{bmatrix} \qquad a = 1 \tag{4.6.68}$$

$$\mathbf{y^{(p)}} = \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}\right) e^t \tag{4.6.69}$$

Applying the I.C.,

$$\begin{bmatrix} 19 \\ -23 \end{bmatrix} = \begin{bmatrix} -c_1 - 4c_2 \\ c_1 + 5c_2 + 1.5 \end{bmatrix} \tag{4.6.70}$$

$$c_1 = 3 \qquad c_2 = -6.5 \tag{4.6.71}$$

$$\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 22 \\ -32.5 \end{bmatrix} e^{2t} + \begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix} e^{2t} \tag{4.6.72}$$

11. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t} \qquad \lambda^2 - 1 = 0 \tag{4.6.73}$$

$$\lambda_1 = -1 \qquad \mathbf{v^{(1)}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.6.74}$$

$$\lambda_2 = 1 \qquad \mathbf{v^{(2)}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.6.75}$$

$$\mathbf{y^{(h)}} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t \tag{4.6.76}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{2t} \qquad \mathbf{g} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t} \tag{4.6.77}$$

$$\mathbf{y}^{(\mathbf{p})'} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.78}$$

$$2A_1 = A_2 + 6 \qquad 2A_2 = A_1 - 1 \tag{4.6.79}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix} e^{2t} \tag{4.6.80}$$

Applying the I.C.,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 + 11/3 \\ c_1 + c_2 + 4/3 \end{bmatrix} \tag{4.6.81}$$

$$c_1 = 3 \qquad c_2 = -6.5 \tag{4.6.82}$$

$$\mathbf{y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} e^{t} + \begin{bmatrix} -2/3 \\ 2/3 \end{bmatrix} e^{-t} + \begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix} e^{2t} \tag{4.6.83}$$

12. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -t^2 + 6t \\ -t^2 + t - 1 \end{bmatrix} e^{2t} \qquad \lambda^2 - 2\lambda - 3 = 0 \tag{4.6.84}$$

$$\lambda_1 = -1 \qquad \mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{4.6.85}$$

$$\lambda_2 = 3 \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{4.6.86}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} \tag{4.6.87}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1t^2 + B_1t + D_1 \\ A_2t^2 + B_2t + D_2 \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} -t^2 + 6t \\ -t^2 + t - 1 \end{bmatrix} \tag{4.6.88}$$

$$\mathbf{y}^{(\mathbf{p})'} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.89}$$

$$2A_1 = B_1 + 4B_2 + 6 \qquad 2A_2 = B_1 + B_2 + 1 \tag{4.6.90}$$

$$B_1 = D_1 + 4D_2 \qquad B_2 = D_1 + D_2 - 1 \tag{4.6.91}$$

$$0 = A_1 + 4A_2 - 1 \qquad 0 = A_1 + A_2 - 1 \tag{4.6.92}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} t^2 + t \\ -t \end{bmatrix} \tag{4.6.93}$$

Applying the I.C.,

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2c_1 + 2c_2 \\ c_1 + c_2 \end{bmatrix} \tag{4.6.94}$$

$$c_1 = -1 \qquad c_2 = 0 \tag{4.6.95}$$

$$\mathbf{y} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} t^2 + t \\ -t \end{bmatrix} \tag{4.6.96}$$

13. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -5\sin(t) \\ 17\cos(2t) \end{bmatrix} \qquad \lambda^2 + 4 = 0 \tag{4.6.97}$$

$$\lambda_1 = -2i \qquad \mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} i \\ 2 \end{bmatrix} \tag{4.6.98}$$

$$\lambda_2 = 2i \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} -i \\ 2 \end{bmatrix} \tag{4.6.99}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} i \\ 2 \end{bmatrix} e^{-2it} + c_2 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} \tag{4.6.100}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} \tag{4.6.101}$$

Solving the nh-ODE,

$$\mathbf{y^{(p)}} = \begin{bmatrix} A_1 \cos(t) + B_1 \sin(t) \\ A_2 \cos(t) + B_2 \sin(t) \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} -5 \sin(t) \\ 17 \cos(t) \end{bmatrix} \tag{4.6.102}$$

$$\mathbf{y^{(p)}}' = \mathbf{A}\mathbf{y^{(p)}} + \mathbf{g} \tag{4.6.103}$$

$$B_1 = A_2 \qquad -A_1 = B_2 - 5 \tag{4.6.104}$$

$$B_2 = -4A_1 + 17 \qquad -A_2 = -4B_1 \tag{4.6.105}$$

$$\mathbf{y^{(p)}} = \begin{bmatrix} 4 \cos(t) \\ \sin(t) \end{bmatrix} \tag{4.6.106}$$

Applying the I.C.,

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} c_2 + 4 \\ 2c_1 \end{bmatrix} \tag{4.6.107}$$

$$c_1 = 1 \qquad c_2 = 1 \tag{4.6.108}$$

$$\mathbf{y} = \begin{bmatrix} \cos(2t) + \sin(2t) \\ 2 \cos(2t) - 2 \sin(2t) \end{bmatrix} + \begin{bmatrix} 4 \cos(t) \\ \sin(t) \end{bmatrix} \tag{4.6.109}$$

14. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 5e^{t} \\ -20e^{-t} \end{bmatrix} \qquad \lambda^2 + 4 = 0 \tag{4.6.110}$$

$$\lambda_1 = -2\ \mathrm{i} \qquad \mathbf{v^{(1)}} = \begin{bmatrix} 2\ \mathrm{i} \\ 1 \end{bmatrix} \tag{4.6.111}$$

$$\lambda_2 = 2\ \mathrm{i} \qquad \mathbf{v^{(2)}} = \begin{bmatrix} -2\ \mathrm{i} \\ 1 \end{bmatrix} \tag{4.6.112}$$

$$\mathbf{y^{(h)}} = c_1 \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{-2it} + c_2 \begin{bmatrix} 2 \\ i \end{bmatrix} e^{2it} \tag{4.6.113}$$

$$\mathbf{y^{(h)}} = c_1 \begin{bmatrix} 2 \sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 \begin{bmatrix} 2 \cos(2t) \\ -\sin(2t) \end{bmatrix} \tag{4.6.114}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1 e^t + B_1 e^{-t} \\ A_2 e^t + B_2 e^{-t} \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} 5e^t \\ -20e^{-t} \end{bmatrix} \tag{4.6.115}$$

$$\mathbf{y}^{(\mathbf{p})'} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.116}$$

$$A_1 = 4A_2 + 5 \qquad -B_1 = 4B_2 \tag{4.6.117}$$

$$A_2 = -A_1 \qquad -B_2 = -B_1 - 20 \tag{4.6.118}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} e^t - 16e^{-t} \\ -e^t + 4e^{-t} \end{bmatrix} \tag{4.6.119}$$

Applying the I.C.,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_2 - 15 \\ c_1 + 3 \end{bmatrix} \tag{4.6.120}$$

$$c_1 = -3 \qquad c_2 = 8 \tag{4.6.121}$$

$$\mathbf{y} = \begin{bmatrix} 16\cos(2t) - 6\sin(2t) \\ -3\cos(2t) - 8\sin(2t) \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + \begin{bmatrix} -16 \\ 4 \end{bmatrix} e^{-t} \tag{4.6.122}$$

15. Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{2t} - 2t \\ t + 1 \end{bmatrix} \qquad \lambda^2 - 1 = 0 \tag{4.6.123}$$

$$\lambda_1 = -1 \qquad \mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.6.124}$$

$$\lambda_2 = 1 \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.6.125}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \tag{4.6.126}$$

Solving the nh-ODE,

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} A_1e^{2t} + B_1t + D_1 \\ A_2e^{2t} + B_2t + D_2 \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} e^{2t} - 2t \\ t + 1 \end{bmatrix} \tag{4.6.127}$$

$$\mathbf{y}^{(\mathbf{p})'} = \mathbf{A}\mathbf{y}^{(\mathbf{p})} + \mathbf{g} \tag{4.6.128}$$

$$2A_1 = A_1 + 2A_2 + 1 \qquad 2A_2 = -A_2 \tag{4.6.129}$$

$$B_1 = [B_1 + 2B_2 - 2]t + [D_1 + 2D_2] \qquad B_2 = [-B_2 + 1]t + [-D_2 + 1] \tag{4.6.130}$$

$$\mathbf{y}^{(\mathbf{p})} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.6.131}$$

Applying the I.C.,

$$\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 + 1 \\ c_1 \end{bmatrix} \tag{4.6.132}$$

$$c_1 = -4 \qquad c_2 = -4 \tag{4.6.133}$$

$$\mathbf{y} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} e^{-t} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} e^{t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t \tag{4.6.134}$$

16. Refer to notes

17. Setting up the model,

$$LI_1' + R_1(I_1 - I_2) - E = 0 \tag{4.6.135}$$

$$\frac{1}{C}\int I_2 \ \mathrm{d}t + R_2I_2 + R_1(I_2 - I_1) = 0 \tag{4.6.136}$$

$$\frac{1}{C}I_2 + I_2'(R_2 + R_1) - I_1'R_1 = 0 \tag{4.6.137}$$

$$-\frac{1}{C}I_2 + R_1\left[\frac{R_1(I_2 - I_1) + E}{L}\right] = I_2'(R_1 + R_2) \tag{4.6.138}$$

$$\frac{R_1(I_2 - I_1) + E}{L} = I_1' \tag{4.6.139}$$

Applying the given values to the above equation,

$$I_1' = -2I_1 + 2I_2 + 200 \tag{4.6.140}$$

$$I_2' = -0.4I_I + 0.2I_2 + 40 \tag{4.6.141}$$

Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} -2 & 2 \\ -0.4 & 0.2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 200 \\ 40 \end{bmatrix} \qquad 0 = \lambda^2 + 1.8\lambda + 0.4 \tag{4.6.142}$$

$$\lambda_1 = -0.9 - \sqrt{0.41} \qquad \mathbf{v^{(1)}} = \begin{bmatrix} 2 \\ 1.1 - \sqrt{0.41} \end{bmatrix} \tag{4.6.143}$$

$$\lambda_2 = -0.9 + \sqrt{0.41} \qquad \mathbf{v^{(2)}} = \begin{bmatrix} 2 \\ 1.1 + \sqrt{0.41} \end{bmatrix} \tag{4.6.144}$$

$$\mathbf{y^{(h)}} = c_1\mathbf{v^{(1)}}e^{\lambda_1 t} + c_2\mathbf{v^{(2)}}e^{\lambda_2 t} \tag{4.6.145}$$

Solving the nh-ODE,

$$\mathbf{y^{(p)}} = \begin{bmatrix} B_1t + D_1 \\ B_2t + D_2 \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} 200 \\ 40 \end{bmatrix} \tag{4.6.146}$$

$$\mathbf{y^{(p)}}' = \mathbf{A}\mathbf{y^{(p)}} + \mathbf{g} \tag{4.6.147}$$

$$B_1 = [-2B_1 + 2B_2]t + [-2D_1 + 2D_2 + 200] \tag{4.6.148}$$

$$B_2 = [-0.4B_1 + 0.2B_2]t + [-0.4D_1 + 0.2D_2 + 40] \tag{4.6.149}$$

$$\mathbf{y^{(p)}} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \tag{4.6.150}$$

18. Now $E = 440\sin(t)$, Solving the nh-ODE,

$$\mathbf{y^{(p)}} = \begin{bmatrix} A_1\cos(t) + B_1\sin(t) \\ A_2\cos(t) + B_2\sin(t) \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} -440\sin(t) \\ -88\sin(t) \end{bmatrix} \tag{4.6.151}$$

$$\mathbf{y^{(p)}}' = \mathbf{A}\mathbf{y^{(p)}} + \mathbf{g} \tag{4.6.152}$$

$$B_1 = -2A_1 + 2A_2 \qquad B_2 = -0.4A_1 + 0.2A_2 \tag{4.6.153}$$

$$-A_1 = -2B_1 + 2B_2 - 440 \qquad -A_2 = -0.4B_1 + 0.2B_2 - 88 \tag{4.6.154}$$

$$\mathbf{y^{(p)}} = \begin{bmatrix} 352/3 \\ 44/3 \end{bmatrix}\cos(t) + \begin{bmatrix} -616/3 \\ -44 \end{bmatrix}\sin(t) \tag{4.6.155}$$

19. Applying the I.C, $I_1(0) = I_2(0) = Q(0) = 0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_1 + 2c_2 + 100 \\ (1.1 - \sqrt{0.41})c_1 + (1.1 + \sqrt{0.41})4c_2 \end{bmatrix} \tag{4.6.156}$$

$$c_1 = -4 \qquad c_2 = -4 \tag{4.6.157}$$

$$\mathbf{y} = -67.9\mathbf{v}^{(\mathbf{1})}e^{\lambda_1 t} + 17.9\mathbf{v}^{(\mathbf{2})}e^{\lambda_2 t} + \begin{bmatrix} 100 \\ 0 \end{bmatrix} \tag{4.6.158}$$

20. Setting up the model,

$$L_1 I_1' + R_1(I_1 - I_2) + R_2 I_1 - E = 0 \tag{4.6.159}$$

$$L_2 I_2' + R_2(I_2 - I_1) = 0 \tag{4.6.160}$$

$$-3I_1 + 1.25I_2 + 125 = I_1' \tag{4.6.161}$$

$$1.4I_1 - 1.4I_2 = I_2' \tag{4.6.162}$$

Solving the h-ODE,

$$\mathbf{y}' = \begin{bmatrix} -3 & 1.25 \\ 1.4 & -1.4 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 125 \\ 0 \end{bmatrix} \qquad 0 = \lambda^2 + (22/5)\lambda + (49/20) \tag{4.6.163}$$

$$\lambda_1 = -2.2 - \sqrt{2.39} \qquad \mathbf{v}^{(\mathbf{1})} = \begin{bmatrix} -8 - \sqrt{239} \\ 14 \end{bmatrix} \tag{4.6.164}$$

$$\lambda_2 = -2.2 + \sqrt{2.39} \qquad \mathbf{v}^{(\mathbf{2})} = \begin{bmatrix} -8 + \sqrt{239} \\ 14 \end{bmatrix} \tag{4.6.165}$$

$$\mathbf{y}^{(\mathbf{h})} = c_1\mathbf{v}^{(\mathbf{1})}e^{\lambda_1 t} + c_2\mathbf{v}^{(\mathbf{2})}e^{\lambda_2 t} \tag{4.6.166}$$

Solving the nh-ODE,

$$\mathbf{y^{(p)}} = \begin{bmatrix} B_1 t + D_1 \\ B_2 t + D_2 \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} 125 \\ 0 \end{bmatrix} \tag{4.6.167}$$

$$\mathbf{y^{(p)}}' = \mathbf{A}\mathbf{y^{(p)}} + \mathbf{g} \tag{4.6.168}$$

$$B_1 = [-3B_1 + 1.25B_2]t + [-3D_1 + 1.25D_2 + 125] \tag{4.6.169}$$

$$B_2 = [1.4B_1 - 1.4B_2]t + [1.4D_1 - 1.4D_2] \tag{4.6.170}$$

$$\mathbf{y^{(p)}} = \begin{bmatrix} 500/7 \\ 500/7 \end{bmatrix} \tag{4.6.171}$$

Applying the I.C, $I_1(0) = I_2(0)$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (-8 - \sqrt{239})c_1 + (-8 + \sqrt{239})c_2 + 500/7 \\ 14c_1 + 14c_2 + 500/7 \end{bmatrix} \tag{4.6.172}$$

$$c_1 = -4 \qquad c_2 = -4 \tag{4.6.173}$$

$$\mathbf{y} = 1.079\mathbf{v^{(1)}}e^{\lambda_1 t} - 6.181\mathbf{v^{(2)}}e^{\lambda_2 t} + \begin{bmatrix} 500/7 \\ 500/7 \end{bmatrix} \tag{4.6.174}$$