Chapter 3

Higher Order Linear ODEs

3.1 Homogeneous Linear ODEs

1. The given functions are solutions, trivially.
Checking if the given solutions are L.I.

$$y^{iv} = 0 \tag{3.1.1}$$

$$y = \{1, x, x^2, x^3\} \tag{3.1.2}$$

$$W = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12 \tag{3.1.3}$$

Since $W \neq 0$, the solutions form a basis.

2. Checking if the given functions are solutions, true

$$y''' - 2y'' - y' + 2y = 0 \tag{3.1.4}$$

$$y = \{e^x, e^{-x}, e^{2x}\} \tag{3.1.5}$$

$$0 = 1 - 2 - 1 + 2 \qquad \cdots\cdots[e^x] \tag{3.1.6}$$

$$0 = -1 - 2 + 1 + 2 \qquad \cdots\cdots[e^{-x}] \tag{3.1.7}$$

$$0 = 8 - 8 - 2 + 2 \qquad \cdots\cdots[e^{2x}] \tag{3.1.8}$$

Checking if the given solutions are L.I.

$$y = \{e^x, e^{-x}, e^{2x}\} \tag{3.1.9}$$

$$W = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = -6e^{2x} \tag{3.1.10}$$

Since $W \neq 0$, the solutions form a basis.

3. Checking if the given functions are solutions, true

$$y^{iv} + 2y'' + y = 0 \tag{3.1.11}$$

$$y = \{\cos x, \sin x, x\cos x, x\sin x\} \tag{3.1.12}$$

$$0 = x\cos x + 4\sin x - 2x\cos x - 4\sin x + x\cos x \qquad \cdots\cdots[x\cos x] \tag{3.1.13}$$

$$0 = x\sin x - 4\cos x - 2x\sin x + 4\cos x + x\sin x \qquad \cdots\cdots[x\sin x] \tag{3.1.14}$$

$$0 = 1 - 2 + 1 \qquad \cdots\cdots[\cos x] \tag{3.1.15}$$

$$0 = 1 - 2 + 1 \qquad \cdots\cdots[\sin x] \tag{3.1.16}$$

Checking if the given solutions are L.I.

$$y = \{\cos x, \sin x, x\cos x, x\sin x\} \tag{3.1.17}$$

$$W = \begin{vmatrix} \cos x & \sin x & x\cos x & x\sin x \\ -\sin x & \cos x & -x\sin x + \cos x & x\cos x + \sin x \\ -\cos x & -\sin x & -x\cos x - 2\sin x & -x\sin x + 2\cos x \\ \sin x & -\cos x & x\sin x - 3\cos x & -x\cos x - 3\sin x \end{vmatrix} = 4 \tag{3.1.18}$$

Since $W \neq 0$, the solutions form a basis.

4. Checking if the given functions are solutions, true

$$0 = y''' + 12y'' + 48y' + 64y \tag{3.1.19}$$

$$y = \{e^{-4x}, xe^{-4x}, x^2e^{-4x}\} \tag{3.1.20}$$

$$0 = -64 + 192 - 192 + 64 \qquad \cdots\cdots[e^{-4x}] \tag{3.1.21}$$

$$0 = e^{-4x}[-192x + 48 + 192x - 96 + 48 - 64x + 64x] \qquad \cdots\cdots[xe^{-4x}] \tag{3.1.22}$$

$$0 = e^{-4x}[(-64 + 192 - 192 + 64)x^2 + (96 - 192 + 96)x - (24 + 24)] \qquad \cdots\cdots[xe^{-4x}] \tag{3.1.23}$$

Checking if the given solutions are L.I.

$$y = \{e^{-4x}, xe^{-4x}, x^2e^{-4x}\} \tag{3.1.24}$$

$$W = \begin{vmatrix} e^{-4x} & xe^{-4x} & x^2e^{-4x} \\ -4e^{-4x} & e^{-4x}(1-4x) & e^{-4x}(2x-4x^2) \\ 16e^{-4x} & e^{-4x}(-8+16x) & e^{-4x}(2-16x+16x^2) \end{vmatrix} = 2e^{-12x} \tag{3.1.25}$$

Since $W \neq 0$, the solutions form a basis.

5. Checking if the given functions are solutions, true

$$0 = y''' + 2y'' + 5y' \tag{3.1.26}$$

$$y = \{1, e^{-x}\cos(2x), e^{-x}\sin(2x)\} \tag{3.1.27}$$

$$0 = 0 + 0 + 0 \qquad \cdots\cdots[1] \tag{3.1.28}$$

$$0 = e^{-x}[\cos(2x)(11-6-5) + \sin(2x)(2+8-10)] \qquad \cdots\cdots[e^{-x}\cos(2x)] \tag{3.1.29}$$

$$0 = e^{-x}[\cos(2x)(-2-8+10) + \sin(2x)(11-6-5)] \qquad \cdots\cdots[e^{-x}\sin(2x)] \tag{3.1.30}$$

Checking if the given solutions are L.I.

$$y = \{1, e^{-x}\cos(2x), e^{-x}\sin(2x)\} \tag{3.1.31}$$

$$W = \begin{vmatrix} 1 & e^{-x}\cos(2x) & e^{-x}\sin(2x) \\ 0 & e^{-x}[-2\sin(2x) - \cos(2x)] & e^{-x}[-\sin(2x) + 2\cos(2x)] \\ 0 & e^{-x}[4\sin(2x) - 3\cos(2x)] & e^{-x}[-3\sin(2x) - 4\cos(2x)] \end{vmatrix} = 10e^{-2x} \tag{3.1.32}$$

Since $W \neq 0$, the solutions form a basis.

6. Checking if the given functions are solutions, true

$$0 = x^3y''' - 3x^2y'' + 3xy' \tag{3.1.33}$$

$$y = \{1, x^2, x^4\} \tag{3.1.34}$$

$$0 = 0 + 0 + 0 \qquad \cdots\cdots[1] \tag{3.1.35}$$

$$0 = 0 - 6x^2 + 6x^2 \qquad \cdots\cdots[x^2] \tag{3.1.36}$$

$$0 = 24x^4 - 36x^4 + 12x^4 \qquad \cdots\cdots[x^4] \tag{3.1.37}$$

Checking if the given solutions are L.I.

$$y = \{1, x^2, x^4\} \tag{3.1.38}$$

$$W = \begin{vmatrix} 1 & x^2 & x^4 \\ 0 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix} = 16x^3 \tag{3.1.39}$$

Since $W \neq 0$, the solutions form a basis.

7. General properties of solutions of n-th order ODEs with constant coefficients.

(a) Relation between solutions of characteristic equation and coefficients,

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{3.1.40}$$

$$\text{Try} \quad y = e^{\lambda x} \tag{3.1.41}$$

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = \lambda^k e^{\lambda x} \tag{3.1.42}$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 = P_n(\lambda) \tag{3.1.43}$$

The relation between the n roots of the polynomial $P_n(\lambda)$, and the coefficients of the characteristic equation is a standard result. Let S_k be the sum of products of roots taken k at a time. Then,

$$a_k = (-1)^k S_k \tag{3.1.44}$$

(b) Reduction of order by substituting

$$z = y^{(k)} \tag{3.1.45}$$

$$z^{(m)} = y^{(k+m)} \tag{3.1.46}$$

extends to higher order ODEs since differentiation of higher order works the same as double differentiation.

(c) Let λ be a root of the characteristic equation with multiplicity m,

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\lambda x} u = \lambda e^{\lambda x} u + e^{\lambda x} \frac{\mathrm{d}u}{\mathrm{d}x} \tag{3.1.47}$$

$$(D - \lambda) e^{\lambda x} u = e^{\lambda x} \frac{\mathrm{d}u}{\mathrm{d}x} \tag{3.1.48}$$

$$(D - \lambda)^m e^{\lambda x} u = e^{\lambda x} D^m u \tag{3.1.49}$$

The LHS is zero by virtue of λ being a repeated root. This means that the RHS is a polynomial of degree $(m-1)$. In the interest of L.I. basis of solutions, each solution corresponding to the repeated root is multiplied by increasing powers of x until x^{m-1}.

(d) For twice repeated root, the derivation is the same as for an ODE of order 2. For higher multiplicity, with $\mu = \lambda + \delta$

$$y_2 = \lim_{\lambda \to \mu} \frac{x^m e^{\mu x} - x^m e^{\lambda x}}{\mu - \lambda} \tag{3.1.50}$$

$$y_2 = \lim_{\delta \to 0} \frac{x^m e^{\lambda x} e^{\delta x} - x^m e^{\lambda x}}{\delta} \tag{3.1.51}$$

$$= \lim_{\delta \to 0} \frac{\mathrm{d}}{\mathrm{d}\delta}(x^m e^{\lambda x} e^{\delta x} - x^m e^{\lambda x}) \tag{3.1.52}$$

$$= x^{m+1} e^{\lambda x} \tag{3.1.53}$$

This result also generalizes from second order to higher order linear h-ODEs with constant coefficients.

8. The given functions are not L.I. on the interval $x \geq 0$, since the zero function is a member of the set and $W \equiv 0$.

9. The given functions are L.I. on the interval $x \geq 0$,

$$W = \begin{vmatrix} \tan x & \cot x & 1 \\ \sec^2 x & -\csc^2 x & 0 \\ 2\sec^2 x \tan x & 2\csc^2 x \cot x & 0 \end{vmatrix} = \frac{2}{\sin^3 x \cos^3 x} \tag{3.1.54}$$

10. The given functions are L.I. on the interval $x \geq 0$,

$$W = \begin{vmatrix} e^{2x} & xe^{2x} & x^2 e^{2x} \\ 2e^{2x} & [1+2x]e^{2x} & [2x+2x^2]e^{2x} \\ 4e^{2x} & [4+4x]e^{2x} & [2+6x+4x^2]e^{2x} \end{vmatrix} = 2(1-x)e^{6x} \tag{3.1.55}$$

11. The given functions are L.I. on the interval $x \geq 0$,

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x & e^x \\ e^x[\cos x - \sin x] & e^x[\cos x + \sin x] & e^x \\ e^x[-2\sin x] & e^x[2\cos x] & e^x \end{vmatrix} = e^{3x} \tag{3.1.56}$$

12. The given functions are not L.I. on the interval $x \geq 0$,
either using the expansion, $\cos(2x) = \cos^2 x - \sin^2 x$ or,

$$W = \begin{vmatrix} \sin^2 x & \cos^2 x & \cos(2x) \\ \sin(2x) & -\sin(2x) & -2\sin(2x) \\ 2\cos(2x) & -2\cos(2x) & -4\cos(2x) \end{vmatrix} = 0 \tag{3.1.57}$$

13. The given functions are L.I. on the interval $x \geq 0$,

$$W = \begin{vmatrix} \sin x & \cos x & \sin(2x) \\ \cos x & -\sin x & 2\cos(2x) \\ -\sin x & -\cos x & -4\sin(2x) \end{vmatrix} = 3\sin(2x) \tag{3.1.58}$$

14. The given functions are not L.I. on the interval $x \geq 0$,
either using the superposition, $2\pi\cos^2 x + 2\pi\sin^2 x = 2\pi$ or,

$$W = \begin{vmatrix} \sin^2 x & \cos^2 x & 2\pi \\ \sin(2x) & -\sin(2x) & 0 \\ 2\cos(2x) & -2\cos(2x) & 0 \end{vmatrix} = 0 \tag{3.1.59}$$

15. The given functions are not L.I. on the interval $x \geq 0$,
either using the superposition, $\cosh 2x + \sinh 2x = e^{2x}$ or,

$$W = \begin{vmatrix} \cosh 2x & \sinh 2x & e^{2x} \\ 2\sinh 2x & 2\cosh 2x & 2e^{2x} \\ 4\cosh 2x & 4\sinh 2x & 4e^{2x} \end{vmatrix} = 0 \tag{3.1.60}$$

16. **(a)** Checking each condition,

(1) If S contains the zero function, can S be linearly independent?
No, if $0 \in S$, then S cannot be an L.I. set

(2) If S is linearly independent on a subinterval $\mathcal{J}$ of $\mathcal{I}$, is it linearly independent on $\mathcal{I}$?
Yes, since $W \equiv 0$ has already been violated when the set S is L.I. on $\mathcal{J}$

(3) If S is linearly dependent on a subinterval $\mathcal{J}$ of $\mathcal{I}$, is it linearly dependent on $\mathcal{I}$?
No, since $W \neq 0$ may be true for some $x \in \mathcal{I} - \mathcal{J}$

(4) If S is linearly independent on $\mathcal{I}$, is it linearly independent on a subinterval $\mathcal{J}$?
No, since the set S might have $W \equiv 0 \ \forall \ \mathcal{J}$ but $W \neq 0$ for some point in $\mathcal{I} - \mathcal{J}$

(5) If S is linearly dependent on $\mathcal{I}$, is it linearly independent on a subinterval $\mathcal{J}$?
No, since $W \equiv 0 \ \forall \ x \in \mathcal{I}$, this has to hold for all subintervals $\mathcal{J}$ as well

(6) If S is linearly dependent on $\mathcal{I}$, and if T contains S, is T linearly dependent on $\mathcal{I}$?
No, since one member of S can be expressed as a linear superposition of other members of S. This means that the superset T is also L.D.

(b) To use the Wronskian on a set of functions S to test their linear independence, they have to be n times differentiable on the interval $\mathcal{I}$ of testing, but need not be continuous.
Other methods might include simple ratios of member functions in S to check for linear independence, or the use of pre-established relations between member functions to come up with linear superpositions which disprove L.I.

3.2 Homogeneous Linear ODEs with Constant Coefficients

1. Solving h-ODE

$$y''' + 25y' = 0 \tag{3.2.1}$$

$$\lambda^3 + 25\lambda = 0 \tag{3.2.2}$$

$$\lambda = \{0, 5i, -5i\} \tag{3.2.3}$$

$$y_h = c_1 + c_2 \cos(5x) + c_3 \sin(5x) \tag{3.2.4}$$

2. Solving h-ODE

$$y^{(iv)} + 2y'' + y = 0 \tag{3.2.5}$$

$$\lambda^4 + 2\lambda^2 + 1 = 0 \tag{3.2.6}$$

$$\lambda = \{i, i, -i, -i\} \tag{3.2.7}$$

$$y_h = [c_1 + c_2 x] \cos x + [c_3 + c_4 x] \sin x \tag{3.2.8}$$

3. Solving h-ODE

$$y^{(iv)} + 4y'' = 0 \tag{3.2.9}$$

$$\lambda^4 + 4\lambda^2 = 0 \tag{3.2.10}$$

$$\lambda = \{0, 0, 2i, -2i\} \tag{3.2.11}$$

$$y_h = [c_1 + c_2 x] + c_3 \cos(2x) + c_4 \sin(2x) \tag{3.2.12}$$

4. Solving h-ODE

$$y''' - y'' - y' + y = 0 \tag{3.2.13}$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0 \tag{3.2.14}$$

$$\lambda = \{-1, 1, 1\} \tag{3.2.15}$$

$$y_h = [c_1 + c_2 x]e^x + c_3 e^{-x} \tag{3.2.16}$$

5. Solving h-ODE

$$y^{iv} + 10y'' + 9y = 0 \tag{3.2.17}$$

$$\lambda^4 + 10\lambda^2 + 9 = 0 \tag{3.2.18}$$

$$\lambda = \{-i, i, -3i, 3i\} \tag{3.2.19}$$

$$y_h = c_1 \cos x + c_2 \sin x + c_3 \cos(3x) + c_4 \sin(3x) \tag{3.2.20}$$

6. Solving h-ODE

$$y^{v} + 8y''' + 16y' = 0 \tag{3.2.21}$$

$$\lambda^5 + 8\lambda^3 + 16\lambda = 0 \tag{3.2.22}$$

$$\lambda = \{0, 2i, 2i, -2i, -2i\} \tag{3.2.23}$$

$$y_h = [c_1 + c_2 x] \cos(2x) + [c_3 + c_4 x] \sin(x) + c_5 \tag{3.2.24}$$

7. Solving the h-ODE,

$$y''' + 3.2y'' + 4.81y' = 0 \tag{3.2.25}$$

$$\lambda^3 + 3.2\lambda^2 + 4.81\lambda = 0 \tag{3.2.26}$$

$$\lambda_1, \lambda_2 = \frac{-3.2 \pm i\sqrt{9}}{2} = -1.6 \pm 1.5i \tag{3.2.27}$$

$$y_h = e^{-1.6x}[c_1 \cos(1.5x) + c_2 \sin(1.5x)] + c_3 \tag{3.2.28}$$

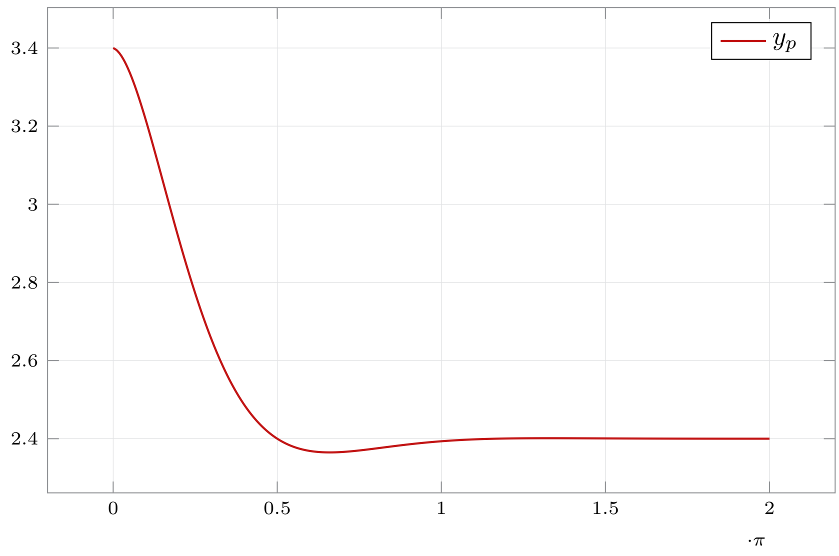
Applying IC $y(0) = 3.4$, $y'(0) = -4.6$, $y''(0) = 9.91$,

$$3.4 = c_1 + c_3 \tag{3.2.29}$$

$$-4.6 = -1.6c_1 + 1.5c_2 \tag{3.2.30}$$

$$9.91 = 0.31c_1 - 4.8c_2 \tag{3.2.31}$$

$$y_h = e^{-1.6x}[\cos(1.5x) - 2\sin(1.5x)] + 2.4 \tag{3.2.32}$$



8. Solving the h-ODE,

$$0 = y''' + 7.5y'' + 14.25y' - 9.125y \quad 3.2.33$$

$$0 = \lambda^3 + 7.5\lambda^2 + 14.25\lambda - 9.125 \quad 3.2.34$$

$$\{\lambda_i\} = \{0.5, -4 \pm 1.5i\} \quad 3.2.35$$

$$y_h = c_1 e^{0.5x} + +e^{-4x}[c_2 \cos(1.5x) + c_3 \sin(1.5x)] \quad 3.2.36$$

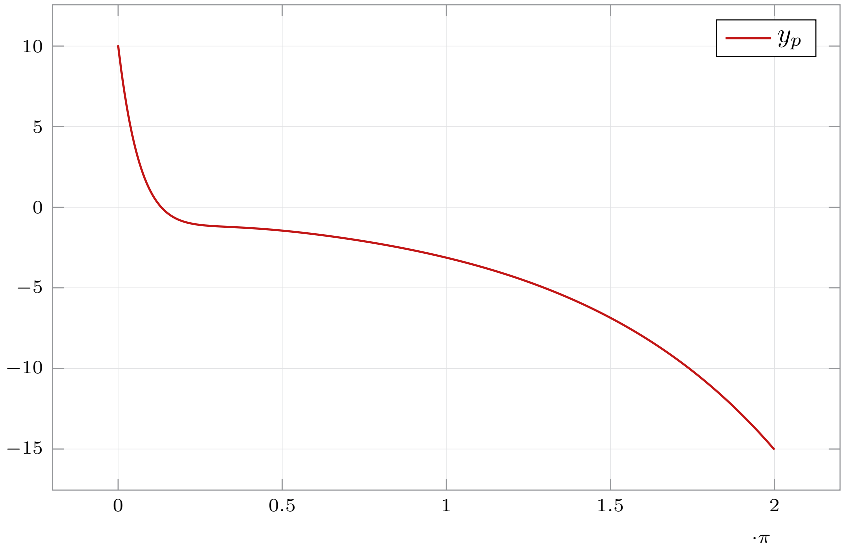
Applying IC $y(0) = 10.05$, $y'(0) = -54.975$, $y''(0) = 257.5125$,

$$10.05 = c_1 + c_2 \quad 3.2.37$$

$$-54.975 = 0.5c_1 + 1.5c_3 - 4c_2 \quad 3.2.38$$

$$257.5125 = 0.25c_1 + 13.75c_2 - 14c_3 \quad 3.2.39$$

$$y_h = -0.65e^{0.5x} + e^{-4x}[10.7 \cos(1.5x) - 7.89 \sin(1.5x)] \quad 3.2.40$$



9. Solving the h-ODE,

$$0 = 4y''' + 8y'' + 41y' + 37y \tag{3.2.41}$$

$$0 = 4\lambda^3 + 8\lambda^2 + 41\lambda + 37 \tag{3.2.42}$$

$$\{\lambda_i\} = \{-1, -0.5 \pm 3i\} \tag{3.2.43}$$

$$y_h = c_1 e^{-x} + e^{-0.5x}[c_2 \cos(3x) + c_3 \sin(3x)] \tag{3.2.44}$$

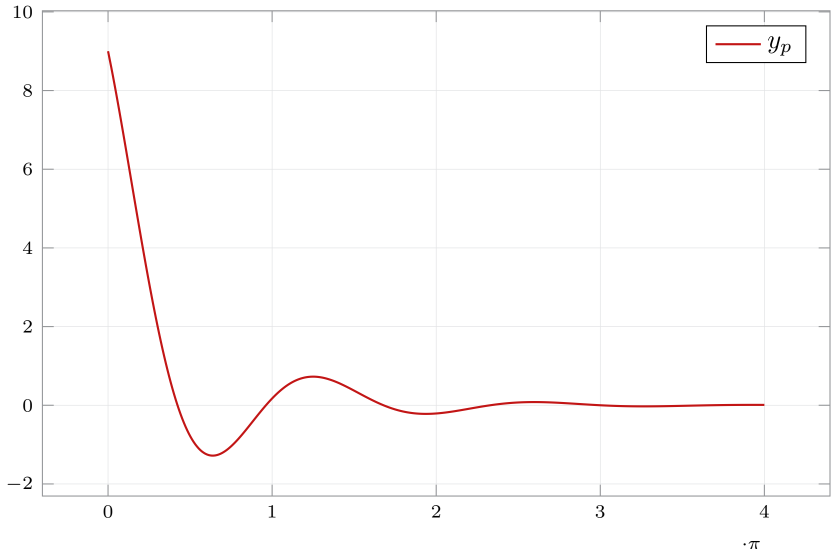
Applying IC $y(0) = 9$, $y'(0) = -6.5$, $y''(0) = -39.75$,

$$9 = c_1 + c_2 \tag{3.2.45}$$

$$-6.5 = -c_1 - 0.5c_2 + 3c_3 \tag{3.2.46}$$

$$-39.75 = c_1 - 8.75c_2 - 3c_3 \tag{3.2.47}$$

$$y_h = 4e^{-x} + e^{-0.5x}[5 \cos(1.5x)] \tag{3.2.48}$$



10. Solving the h-ODE,

$$0 = y^{iv} + 4y \tag{3.2.49}$$

$$0 = \lambda^4 + 4 \tag{3.2.50}$$

$$\{\lambda_i\} = \{-1 \pm i, 1 \pm i\} \tag{3.2.51}$$

$$y_h = e^{x}[c_1 \cos x + c_2 \sin x] + e^{-x}[c_3 \cos x + c_4 \sin x] \tag{3.2.52}$$

Applying IC $y(0) = 1/2$, $y'(0) = -3/2$, $y''(0) = 5/2$, $y'''(0) = -7/2$,

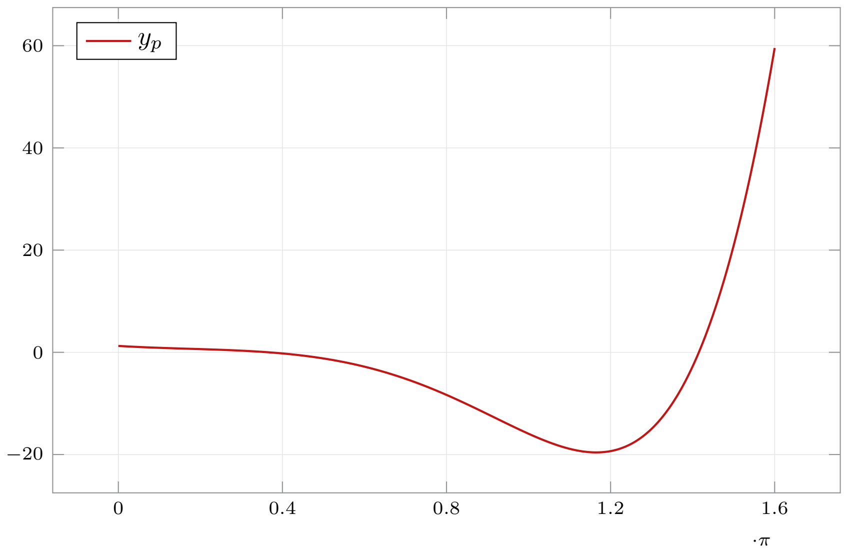
$$1/2 = c_1 + c_2 \tag{3.2.53}$$

$$-3/2 = c_1 + c_2 - c_3 + c_4 \tag{3.2.54}$$

$$5/2 = 2c_2 - 2c_4 \tag{3.2.55}$$

$$-7/2 = -2c_1 + 2c_2 + 2c_3 + 2c_4 \tag{3.2.56}$$

$$y_h = \frac{e^{x}}{16}[11 \cos x - 3 \sin x] + \frac{e^{-x}}{16}[9 \cos x - 23 \sin x] \tag{3.2.57}$$



11. Solving the h-ODE,

$$0 = y^{iv} - 9y'' - 400y \tag{3.2.58}$$

$$0 = \lambda^4 - 9\lambda^2 - 400 \tag{3.2.59}$$

$$\{\lambda_i\} = \{-5, 5, -4i, 4i\} \tag{3.2.60}$$

$$y_h = c_1 \cos(4x) + c_2 \sin(4x) + c_3 e^{-5x} + c_4 e^{5x} \tag{3.2.61}$$

Applying IC $y(0) = 0$, $y'(0) = 0$, $y''(0) = 41$, $y'''(0) = 0$,

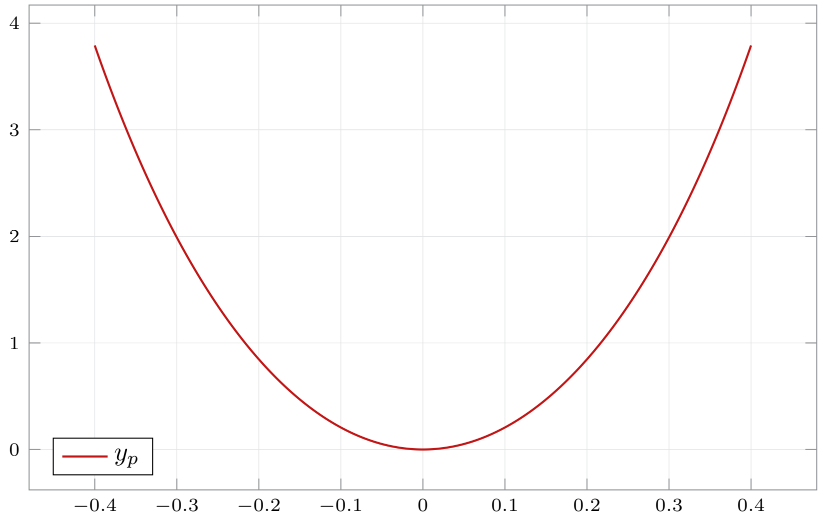
$$0 = c_1 + c_3 + c_4 \tag{3.2.62}$$

$$0 = 4c_2 - 5c_3 + 5c_4 \tag{3.2.63}$$

$$41 = -16c_1 + 25c_3 + 25c_4 \tag{3.2.64}$$

$$0 = -64c_2 - 125c_3 + 125c_4 \tag{3.2.65}$$

$$y_h = -\cos x + 0.5e^{-5x} + 0.5e^{5x} \tag{3.2.66}$$



12. Solving the h-ODE,

$$0 = y^v - 5y''' + 4y' \quad (3.2.67)$$

$$0 = \lambda^5 - 5\lambda^3 + 4\lambda \quad (3.2.68)$$

$$\{\lambda_i\} = \{-2, -1, 0, 1, 2\} \quad (3.2.69)$$

$$y_h = c_1 e^{-2x} + c_2 e^{-x} + c_3 + c_4 e^x + c_5 e^{2x} \quad (3.2.70)$$

Applying IC $y(0) = 0$, $y'(0) = -5$, $y''(0) = 11$, $y'''(0) = -23$, $y^{iv}(0) = 47$,

$$0 = c_1 + c_2 + c_3 + c_4 + c_5 \quad (3.2.71)$$

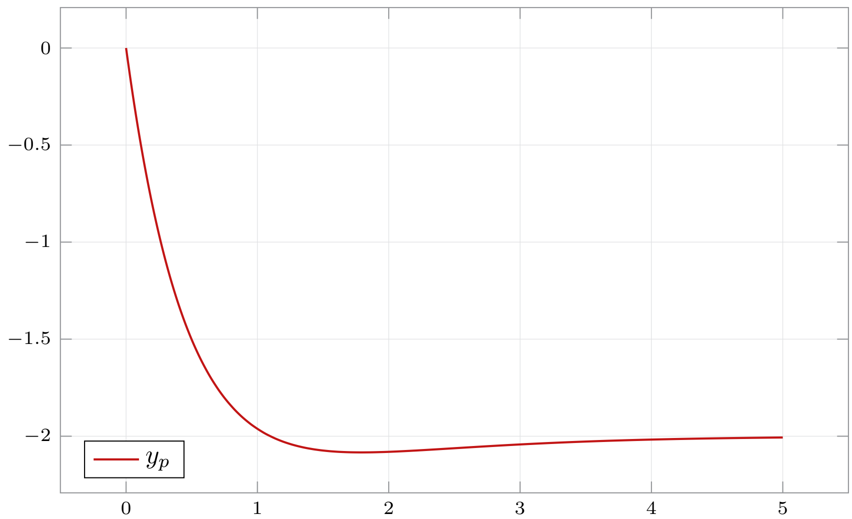
$$-5 = -2c_1 - c_2 + c_4 + 2c_5 \quad (3.2.72)$$

$$11 = 4c_1 + c_2 + c_4 + 4c_5 \quad (3.2.73)$$

$$-23 = -8c_1 - c_2 + c_4 + 8c_5 \quad (3.2.74)$$

$$47 = 16c_1 + c_2 + c_4 + 16c_5 \quad (3.2.75)$$

$$y_h = 3e^{-2x} - e^{-x} - 2 \quad (3.2.76)$$



13. Solving the h-ODE,

$$0 = y^{iv} + 0.45y''' - 0.165y'' + 0.0045y' - 0.00175y \quad (3.2.77)$$

$$0 = \lambda^4 + 0.45\lambda^3 - 0.165\lambda^2 + 0.0045\lambda - 0.00175 \quad (3.2.78)$$

$$\{\lambda_i\} = \{-0.7, 0.25, -0.1i, 0.1i\} \quad (3.2.79)$$

$$y_h = c_1 e^{-0.7x} + c_2 e^{0.25x} + c_3 \cos(0.1x) + c_4 \sin(0.1x) \quad (3.2.80)$$

Applying IC $y(0) = 17.4$, $y'(0) = -2.82$, $y''(0) = 2.0485$, $y'''(0) = -1.458675$,

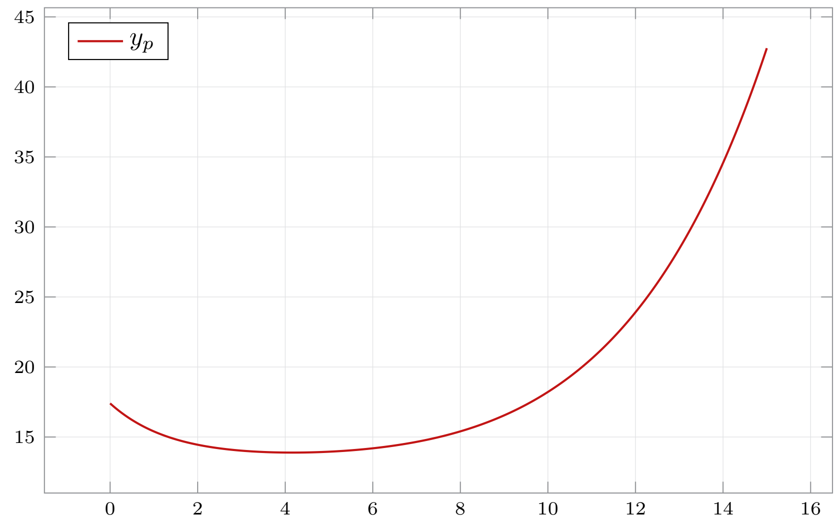
$$17.4 = c_1 + c_2 + c_3 \quad (3.2.81)$$

$$-2.82 = -0.7c_1 + 0.25c_2 + 0.1c_4 \quad (3.2.82)$$

$$2.0485 = 0.49c_1 + 0.0625c_2 - 0.01c_3 \quad (3.2.83)$$

$$-1.458675 = -0.343c_1 + 0.015625c_2 - 0.001c_4 \quad (3.2.84)$$

$$y_h = 4.3e^{-0.7x} + e^{0.25x} + 12.1\cos(0.1x) - 0.6\sin(0.1x) \quad (3.2.85)$$



14. (a) For a constant coefficient ODE, if one solution λ_1 is known,

$$(\lambda - \lambda_1) \cdot h(\lambda) = P(\lambda) \tag{3.2.86}$$

The remainder after factoring out $(\lambda - \lambda_1)$ is also a polynomial in λ corresponding to the ODE obtained after reduction of order.

(b) y_1 is a known solution of

$$0 = y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y \tag{3.2.87}$$

$$\text{Let} \quad y_2 = uy_1 \tag{3.2.88}$$

$$y_2' = u'y_1 + uy_1' \tag{3.2.89}$$

$$y_2'' = u''y_1 + uy_1'' + 2u'y_1' \tag{3.2.90}$$

$$y_2''' = u'''y_1 + 3u''y_1' + uy_1''' + 3u'y_1'' \tag{3.2.91}$$

$$\begin{aligned} 0 &= u'''[y_1] + u''[3y_1' + p_2y_1] + u'[3y_1'' + 2p_2y_1' + p_1y_1] \\ &\quad + u[y_1''' + p_2y_1'' + p_1y_1' + p_0y_1] \end{aligned} \tag{3.2.92}$$

Setting $z = u'$ and thus, $u = \int z(x)\,\mathrm{d}x$, the ODE of 1 lesser order is given by,

$$y_1z'' + (3y_1' + p_2y_1)z' + (3y_1'' + 2p_2y_1' + p_1y_1)z = 0 \tag{3.2.93}$$

(c) Applying the above method with $y_1 = x$,

$$x^3y''' - 3x^2y'' + (6 - x^2)xy' - (6 - x^2)y = 0 \tag{3.2.94}$$

$$y_2 = ux \qquad y_2' = u + xu' \tag{3.2.95}$$

$$y_2'' = 2u' + xu'' \qquad y_2''' = 3u'' + xu''' \tag{3.2.96}$$

$$u'''[x^4] + u'[-x^4] = 0 \tag{3.2.97}$$

$$u''' - u' = 0 \tag{3.2.98}$$

$$\text{Let} \quad u' = z \qquad z'' - z = 0 \tag{3.2.99}$$

$$z = c_1e^{-x} + c_2e^x \tag{3.2.100}$$

$$u = c_1e^{-x} + c_2e^x \tag{3.2.101}$$

$$y_2 = c_1\ xe^x + c_2\ xe^{-x} \tag{3.2.102}$$

(d) TBC. Trying Euler Cauchy third order ODEs with a basis given by,

$$\text{Basis } S = \{1,\, x,\, x^2\} \tag{3.2.103}$$

$$x^2y''' - 2xy'' + 2y' = 0 \tag{3.2.104}$$

By inspection, one of the solutions is x^0. Reduction of order is also trivial by assigning $z = y'$.

$$x^2z'' - 2xz' + 2z = 0 \tag{3.2.105}$$

This can be solved using the usual methods for Euler-Cauchy ODEs to obtain x, x^2 as the other two solutions to the oringinal ODE.

3.3 Nonhomogeneous Linear ODEs

1. Solving the h-ODE,

$$0 = y''' + 3y'' + 3y' + y \qquad 0 = \lambda^3 + 3\lambda^2 + 3\lambda + 1 \tag{3.3.1}$$

$$\{\lambda_i\} = \{-1, -1, -1\} \qquad y_h = [c_1 + c_2x + c_3x^2]e^{-x} \tag{3.3.2}$$

Solving the nh-ODE by undetermined coefficients,

$$y''' + 3y'' + 3y' + y = e^x - x - 1 \tag{3.3.3}$$

$$y_p = Ke^x + M + Nx \tag{3.3.4}$$

$$1 = K + 3K + 3K + K \qquad \cdots\cdots[e^x] \tag{3.3.5}$$

$$-1 = 3N + M \qquad \cdots\cdots[1] \tag{3.3.6}$$

$$-1 = N \qquad \cdots\cdots[x] \tag{3.3.7}$$

$$y_p = \frac{e^x}{8} + 2 - x \tag{3.3.8}$$

$$y = y_h + y_p \tag{3.3.9}$$

2. Solving the h-ODE,

$$0 = y''' + 2y'' - y' - 2y \qquad 0 = \lambda^3 + 2\lambda^2 - 2\lambda - 2 \tag{3.3.10}$$

$$\{\lambda_i\} = \{-2, -1, 1\} \qquad y_h = c_1e^{-2x} + c_2e^{-x} + c_3e^x \tag{3.3.11}$$

Solving the nh-ODE by undetermined coefficients,

$$y''' + 2y'' - y' - 2y = 1 - 4x^3 \tag{3.3.12}$$

$$y_p = K_0 + K_1x + K_2x^2 + K_3x^3 \tag{3.3.13}$$

$$1 = 6K_3 + 4K_2 - K_1 - 2K_0 \qquad \cdots\cdots[1] \tag{3.3.14}$$

$$0 = 12K_3 - 2K_2 - 2K_1 \qquad \cdots\cdots[x] \tag{3.3.15}$$

$$0 = -3K_3 - 2K_2 \qquad \cdots\cdots[x^2] \tag{3.3.16}$$

$$-4 = -2K_3 \qquad \cdots\cdots[x^3] \tag{3.3.17}$$

$$y_p = -8 + 15x - 3x^2 + 2x^3 \tag{3.3.18}$$

$$y = y_h + y_p \tag{3.3.19}$$

3. Solving the h-ODE,

$$0 = y^{iv} + 10y'' + 9y \tag{3.3.20}$$

$$0 = \lambda^4 + 10\lambda^2 + 9 \tag{3.3.21}$$

$$\{\lambda_i\} = \{-i, i, -3i, 3i\} \tag{3.3.22}$$

$$y_h = c_1 \cos x + c_2 \sin x + c_3 \cos(3x) + c_4 \sin(3x) \tag{3.3.23}$$

Solving the nh-ODE by undetermined coefficients,

$$y^{iv} + 10y'' + 9y = 6.5 \sinh(2x) \tag{3.3.24}$$

$$y_p = K \cosh(2x) + M \sinh(2x) \tag{3.3.25}$$

$$0 = 16K + 40K + 9K \qquad \cdots\cdots[\cosh(2x)] \tag{3.3.26}$$

$$6.5 = 16M + 40M + 9M \qquad \cdots\cdots[\sinh(2x)] \tag{3.3.27}$$

$$y_p = 0.1 \sinh(2x) \tag{3.3.28}$$

$$y = y_h + y_p \tag{3.3.29}$$

4. Solving the h-ODE,

$$0 = y''' + 3y'' - 5y' - 39y \tag{3.3.30}$$

$$0 = \lambda^3 + 3\lambda^2 - 5\lambda - 39 \tag{3.3.31}$$

$$\{\lambda_i\} = \{3, -3 \pm 2i\} \tag{3.3.32}$$

$$y_h = e^{-3x}[c_1 \cos(2x) + c_2 \sin(2x)] + c_3 e^{3x} \tag{3.3.33}$$

Solving the nh-ODE by undetermined coefficients,

$$y''' + 3y'' - 5y' - 39y = -300 \cos(x) \tag{3.3.34}$$

$$y_p = K \cos(x) + M \sin(x) \tag{3.3.35}$$

$$-300 = -39K - 5M - 3K - M \qquad \cdots\cdots[\cos(x)] \tag{3.3.36}$$

$$0 = -39M + 5K - 3M + K \qquad \cdots\cdots[\sin(x)] \tag{3.3.37}$$

$$y_p = 7 \cos x + \sin x \tag{3.3.38}$$

$$y = y_h + y_p \tag{3.3.39}$$

5. Solving the h-ODE,

$$0 = x^3 y''' + x^2 y'' - 2xy' + 2y \tag{3.3.40}$$

$$0 = m(m-1)(m-2) + m(m-1) - 2m + 2 \tag{3.3.41}$$

$$\{\lambda_i\} = \{-1, 1, 2\} \tag{3.3.42}$$

$$y_h = c_1 x^{-1} + c_2 x + c_3 x^2 \tag{3.3.43}$$

Solving the nh-ODE by variation of parameters,

$$x^3y''' + x^2y'' - 2xy' + 2y = x^{-2} \tag{3.3.44}$$

$$W = \begin{vmatrix} x^{-1} & x & x^2 \\ -x^{-2} & 1 & 2x \\ 2x^{-3} & 0 & 2 \end{vmatrix} = \frac{6}{x} \qquad r(x) = x^{-5} \tag{3.3.45}$$

$$W_1 = \begin{vmatrix} 0 & x & x^2 \\ 0 & 1 & 2x \\ 1 & 0 & 2 \end{vmatrix} = x^2 \qquad W_2 = \begin{vmatrix} x^{-1} & 0 & x^2 \\ -x^{-2} & 0 & 2x \\ 2x^{-3} & 1 & 2 \end{vmatrix} = -3 \tag{3.3.46}$$

$$W_3 = \begin{vmatrix} x^{-1} & x & 0 \\ -x^{-2} & 1 & 0 \\ 2x^{-3} & 0 & 1 \end{vmatrix} = \frac{2}{x} \tag{3.3.47}$$

$$T_1 = x^{-1} \int \frac{x^{-2}}{6} \, \mathrm{d}x = -\frac{x^{-2}}{6} \tag{3.3.48}$$

$$T_2 = x \int \frac{x^{-4}}{-2} \, \mathrm{d}x = \frac{x^{-2}}{6} \tag{3.3.49}$$

$$T_3 = x^2 \int \frac{x^{-5}}{3} \, \mathrm{d}x = \frac{-x^{-2}}{12} \tag{3.3.50}$$

$$y_p = \frac{-1}{12x^2} \tag{3.3.51}$$

$$y = y_h + y_p \tag{3.3.52}$$

6. Solving the h-ODE,

$$0 = y''' + 4y' \qquad 0 = \lambda^3 + 4\lambda \tag{3.3.53}$$

$$\{\lambda_i\} = \{0, -2i, 2i\} \qquad y_h = c_1 \cos(2x) + c_2 \sin(2x) + c_3 \tag{3.3.54}$$

Solving the nh-ODE by undetermined coefficients,

$$y''' + 4y' = \sin x \tag{3.3.55}$$

$$y_p = K\cos(x) + M\sin(x) \tag{3.3.56}$$

$$0 = -M + 4M \qquad \cdots\cdots[\cos(x)] \tag{3.3.57}$$

$$1 = K - 4K \qquad \cdots\cdots[\sin(x)] \tag{3.3.58}$$

$$y_p = \frac{-1}{3}\cos x \tag{3.3.59}$$

$$y = y_h + y_p \tag{3.3.60}$$

7. Solving the h-ODE,

$$0 = y''' - 9y'' + 27y' - 27y \qquad 0 = \lambda^3 - 9\lambda^2 + 27\lambda - 27 \tag{3.3.61}$$

$$\{\lambda_i\} = \{3, 3, 3\} \qquad y_h = [c_1 + c_2 x + c_3 x^2]e^{3x} \tag{3.3.62}$$

Solving the nh-ODE by undetermined coefficients,

$$y''' - 9y'' + 27y' - 27y = 27\sin(3x) \tag{3.3.63}$$

$$y_p = K\cos(3x) + M\sin(3x) \tag{3.3.64}$$

$$0 = -27K + 81M + 81K - 27M \qquad \cdots\cdots[\cos(3x)] \tag{3.3.65}$$

$$27 = -27M - 81K + 81M + 27K \qquad \cdots\cdots[\sin(3x)] \tag{3.3.66}$$

$$y_p = \frac{-1}{4}\cos(3x) + \frac{1}{4}\sin(3x) \tag{3.3.67}$$

$$y = y_h + y_p \tag{3.3.68}$$

8. Solving the h-ODE,

$$0 = y^{iv} - 5y'' + 4y \qquad 0 = \lambda^4 - 5\lambda^2 + 4 \tag{3.3.69}$$

$$\{\lambda_i\} = \{-2, -1, 1, 2\} \qquad y_h = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x} \tag{3.3.70}$$

Solving the nh-ODE by undetermined coefficients,

$$y^{iv} - 5y'' + 4y = 10e^{-3x} \tag{3.3.71}$$

$$y_p = Ke^{3x} \tag{3.3.72}$$

$$10 = 81K - 45K + 4K \qquad \cdots\cdots [e^{-3x}] \tag{3.3.73}$$

$$y_p = 0.25e^{-3x} \tag{3.3.74}$$

$$y = y_h + y_p \tag{3.3.75}$$

Applying the IC $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$

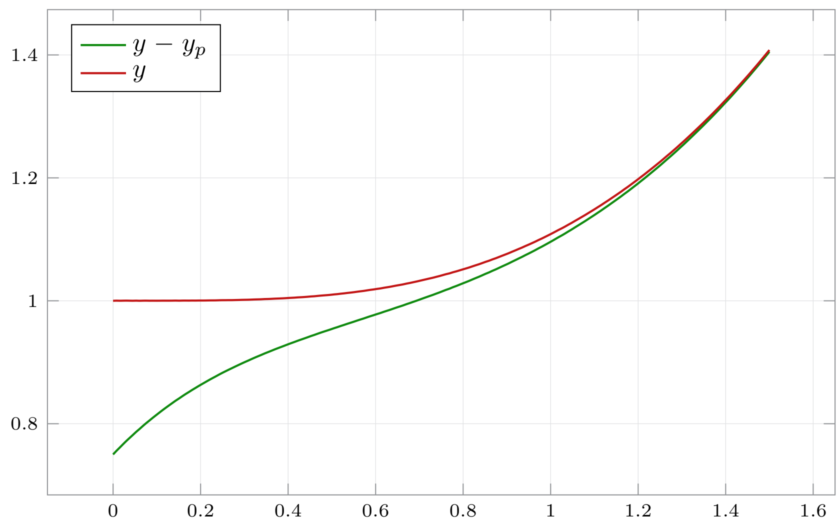
$$1 = c_1 + c_2 + c_3 + c_4 + 0.25 \tag{3.3.76}$$

$$0 = -2c_1 - c_2 + c_3 + 2c_4 + -0.75 \tag{3.3.77}$$

$$0 = 4c_1 + c_2 + c_3 + 4c_4 + 2.25 \tag{3.3.78}$$

$$0 = -8c_1 - c_2 + c_3 + 8c_4 + -6.75 \tag{3.3.79}$$

$$y = -e^{-2x} + 1.5e^{-x} + 0.25e^{x} + 0.25e^{-3x} \tag{3.3.80}$$



9. Solving the h-ODE,

$$0 = y^{iv} + 5y'' + 4y \tag{3.3.81}$$

$$0 = \lambda^4 + 5\lambda^2 + 4 \tag{3.3.82}$$

$$\{\lambda_i\} = \{-2i, -i, i, 2i\} \tag{3.3.83}$$

$$y_h = c_1 \cos x + c_2 \sin x + c_3 \cos(2x) + c_4 \sin(2x) \tag{3.3.84}$$

Solving the nh-ODE by undetermined coefficients,

$$y^{iv} + 5y'' + 4y = 90\,\sin(4x) \tag{3.3.85}$$

$$y_p = K\,\sin(4x) \tag{3.3.86}$$

$$90 = 256K - 80K + 4K \qquad \cdots\cdots[\sin(4x)] \tag{3.3.87}$$

$$y_p = \color{red}{0.5\,\sin(4x)} \tag{3.3.88}$$

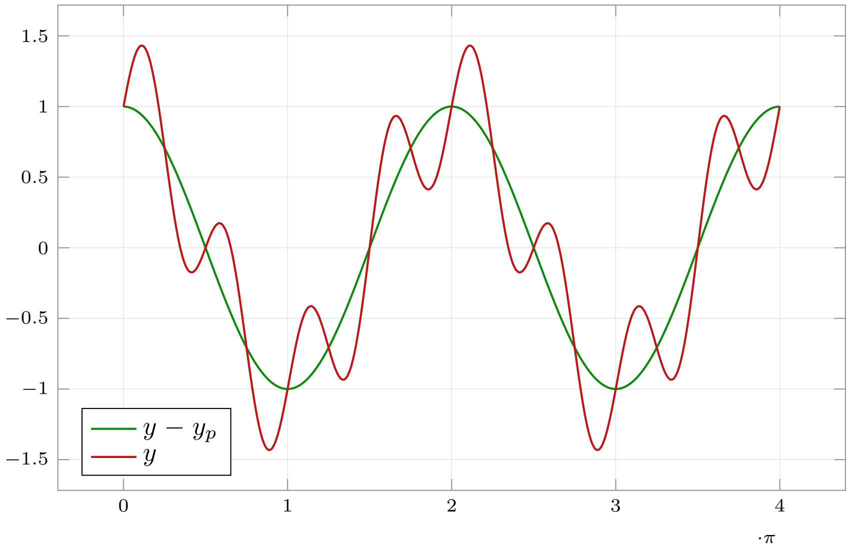
$$y = \color{green}{y_h} + \color{red}{y_p} \tag{3.3.89}$$

Applying the IC $y(0) = 1$, $y'(0) = 2$, $y''(0) = -1$, $y'''(0) = -32$

$$1 = c_1 + c_3 \qquad 2 = c_2 + 2c_4 + 2 \tag{3.3.90}$$

$$-1 = -c_1 - 4c_3 \qquad -32 = -c_2 - 8c_4 - 32 \tag{3.3.91}$$

$$y = \color{green}{\cos x} + \color{red}{0.5\,\sin(4x)} \tag{3.3.92}$$



10. Solving the h-ODE,

$$0 = x^3y''' + xy' - y \qquad 0 = m(m-1)(m-2) + m - 1 \tag{3.3.93}$$

$$\{\lambda_i\} = \{1, 1, 1\} \qquad y_h = \color{green}{[c_1 + c_2\ln x + c_3\ln^2 x]x} \tag{3.3.94}$$

Solving the nh-ODE by undetermined coefficients,

$$x^3y''' + xy' - y = x^2 \tag{3.3.95}$$

$$y_p = K_0 + K_1x + K_2x^2 \tag{3.3.96}$$

$$0 = K_1 - K_1 \qquad \cdots\cdots[x] \tag{3.3.97}$$

$$1 = 2K_2 - K_2 \qquad \cdots\cdots[x^2] \tag{3.3.98}$$

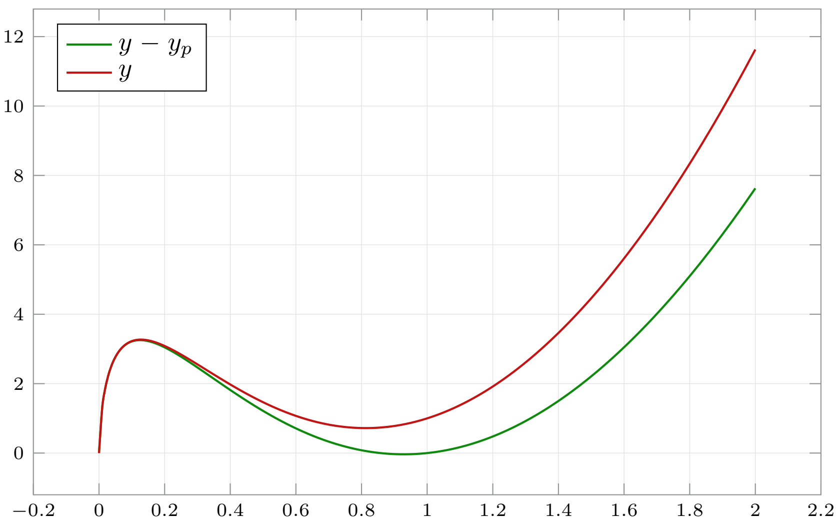
$$y_p = x^2 \tag{3.3.99}$$

$$y = y_h + y_p \tag{3.3.100}$$

Applying the IC $y(1) = 1$, $y'(1) = 3$, $y''(1) = 14$

$$1 = c_1 + 1 \qquad 3 = c_1 + c_2 + 2 \tag{3.3.101}$$

$$14 = c_2 + 2c_3 + 2 \qquad y = x[\ln x + 6.5\ln^2 x] + x^2 \tag{3.3.102}$$



11. Solving the h-ODE,

$$0 = y''' - 2y'' - 3y' \qquad 0 = \lambda^3 - 2\lambda^2 - 3\lambda \tag{3.3.103}$$

$$\{\lambda_i\} = \{-1, 0, 3\} \qquad y_h = c_1 + c_2e^{-x} + c_3e^{3x} \tag{3.3.104}$$

Solving the nh-ODE by undetermined coefficients,

$$y''' - 2y'' - 3y' = 74e^{-3x} \sin x \tag{3.3.105}$$

$$y_p = e^{-3x}[K \cos x + M \sin x] \tag{3.3.106}$$

$$0 = (-18 - 16 + 9)K + (26 + 12 - 3)M \qquad \cdots\cdots[e^{-3x} \cos x] \tag{3.3.107}$$

$$74 = (-26 - 12 + 3)K + (-18 - 16 + 9)M \qquad \cdots\cdots[e^{-3x} \sin x] \tag{3.3.108}$$

$$y_p = -e^{-3x}[1.4 \cos x + \sin x] \tag{3.3.109}$$

$$y = y_h + y_p \tag{3.3.110}$$

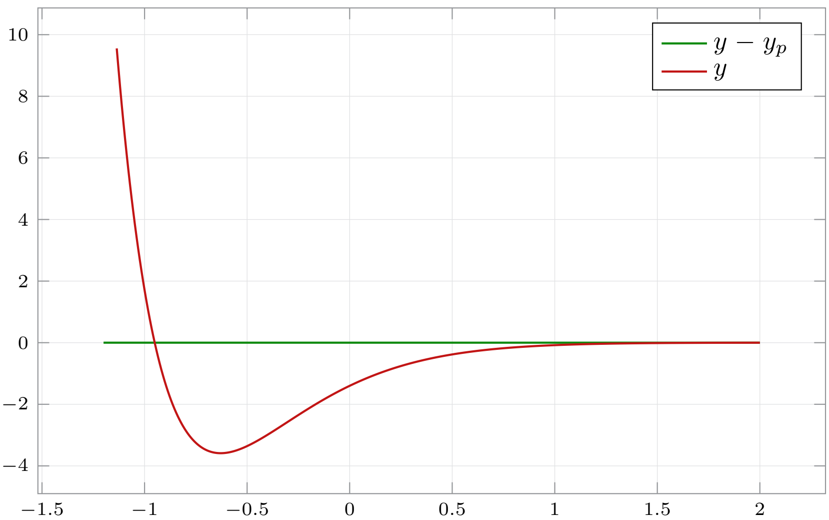
Applying the IC $y(0) = -1.4$, $y'(0) = 3.2$, $y''(0) = -5.2$

$$-1.4 = c_1 + c_2 + c_3 - 1.4 \tag{3.3.111}$$

$$3.2 = -c_2 + 3c_3 + 3.2 \tag{3.3.112}$$

$$-5.2 = c_2 + 9c_3 - 5.2 \tag{3.3.113}$$

$$y = 0 + e^{-3x}[-1.4 \cos x - \sin x] \tag{3.3.114}$$



12. Solving the h-ODE,

$$0 = y''' - 2y'' - 9y' + 18y \tag{3.3.115}$$

$$0 = \lambda^3 - 2\lambda^2 - 9\lambda + 18 \tag{3.3.116}$$

$$\{\lambda_i\} = \{-3, 2, 3\} \tag{3.3.117}$$

$$y_h = c_1 e^{-3x} + c_2 e^{2x} + c_3 e^{3x} \tag{3.3.118}$$

Solving the nh-ODE by undetermined coefficients,

$$y''' - 2y'' - 9y' + 18y = e^{2x} \tag{3.3.119}$$

$$y_p = e^{2x}[Mx + N] \tag{3.3.120}$$

$$1 = -5M \qquad \cdots\cdots[e^{2x}] \tag{3.3.121}$$

$$0 = 0 \qquad \cdots\cdots[xe^{2x}] \tag{3.3.122}$$

$$y_p = \frac{-1}{5}xe^{2x} \tag{3.3.123}$$

$$y = y_h + y_p \tag{3.3.124}$$

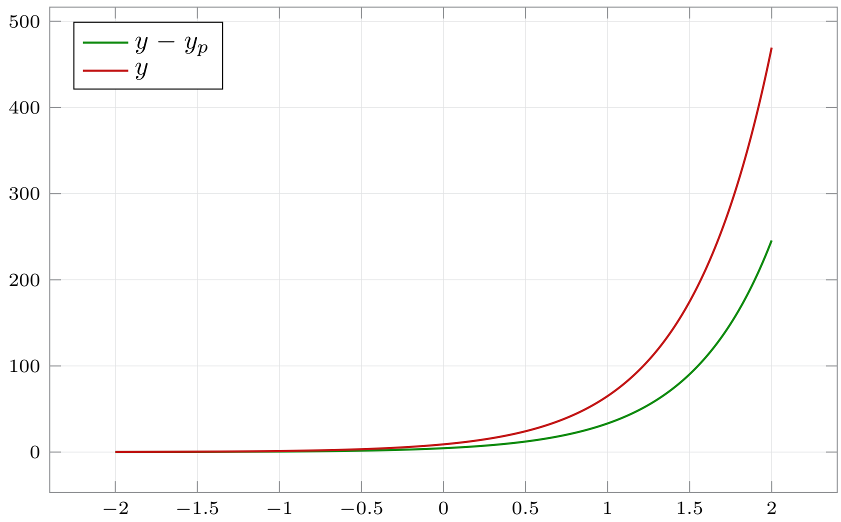
Applying the IC $y(0) = 4.5$, $y'(0) = 8.8$, $y''(0) = 17.2$

$$4.5 = c_1 + c_2 + c_3 \tag{3.3.125}$$

$$8.8 = -3c_1 + 2c_2 + 3c_3 - 0.2 \tag{3.3.126}$$

$$17.2 = 9c_1 + 4c_2 + 9c_3 - 0.8 \tag{3.3.127}$$

$$y = 4.5e^{2x} + [-0.2x]e^{2x} \tag{3.3.128}$$



13. Solving the h-ODE,

$$0 = y''' - 4y' \tag{3.3.129}$$

$$0 = \lambda^3 - 4\lambda \tag{3.3.130}$$

$$\{\lambda_i\} = \{-2, 0, 2\} \tag{3.3.131}$$

$$y_h = c_1e^{-2x} + c_2 + c_3e^{2x} \tag{3.3.132}$$

Solving the nh-ODE by undetermined coefficients,

$$y''' - 4y' = 10\cos x + 5\sin x \tag{3.3.133}$$

$$y_p = M\cos x + N\sin x \tag{3.3.134}$$

$$10 = -4N - N \qquad \cdots\cdots[\cos x] \tag{3.3.135}$$

$$5 = 4M + M \qquad \cdots\cdots[\sin x] \tag{3.3.136}$$

$$y_p = \cos x - 2\sin x \tag{3.3.137}$$

$$y = y_h + y_p \tag{3.3.138}$$

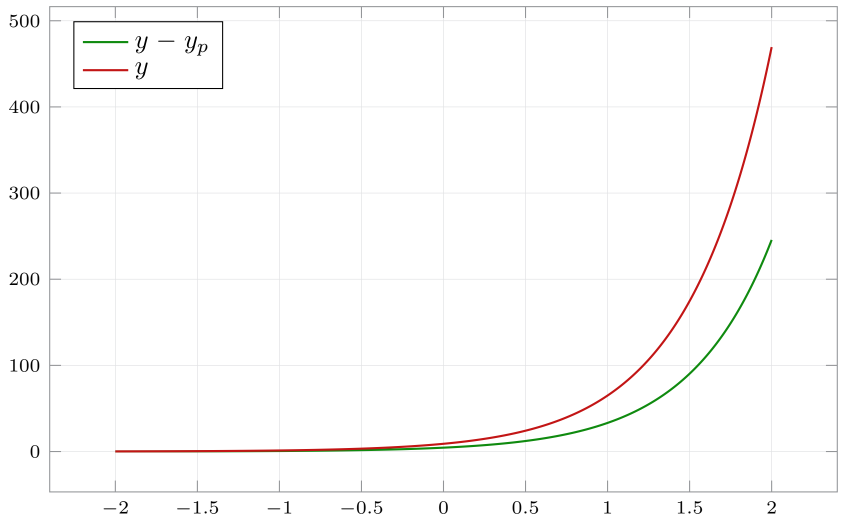
Applying the IC $y(0) = 3$, $y'(0) = -2$, $y''(0) = -1$

$$3 = c_1 + c_2 + c_3 + 1 \tag{3.3.139}$$

$$-2 = -2c_1 + 2c_3 - 2 \tag{3.3.140}$$

$$-1 = 4c_1 + 4c_3 - 1 \tag{3.3.141}$$

$$y = 2 + \cos x - 2\sin x \tag{3.3.142}$$



14. TBC

15. Refer chapter notes from chapter 2 and 3. Taking $a \neq b \neq c$

$$(D-a)(D-b)(D-c)y = e^{ax} \tag{3.3.143}$$

$$W = \begin{vmatrix} e^{ax} & e^{bx} & e^{cx} \\ ae^{ax} & be^{bx} & ce^{cx} \\ a^2e^{ax} & b^2e^{bx} & c^2e^{cx} \end{vmatrix} = (-a^2b + a^2c + ab^2 - ac^2 - b^2c + bc^2)e^{x(a+b+c)} \tag{3.3.144}$$

$$r(x) = e^{ax} \tag{3.3.145}$$

$$W_1 = \begin{vmatrix} 0 & e^{bx} & e^{cx} \\ 0 & be^{bx} & ce^{cx} \\ 1 & b^2e^{bx} & c^2e^{cx} \end{vmatrix} = (-b+c)e^{x(b+c)} \qquad \frac{W_1}{W} = k_1e^{-ax} \tag{3.3.146}$$

$$W_2 = \begin{vmatrix} e^{ax} & 0 & e^{cx} \\ ae^{ax} & 0 & ce^{cx} \\ a^2e^{ax} & 1 & c^2e^{cx} \end{vmatrix} = (a-c)e^{x(a+c)} \qquad \frac{W_2}{W} = k_2e^{-bx} \tag{3.3.147}$$

$$W_3 = \begin{vmatrix} e^{a} & e^{b} & 0 \\ ae^{a} & be^{b} & 0 \\ a^2e^{a} & b^2e^{b} & 1 \end{vmatrix} = (-a+b)e^{x(a+b)} \qquad \frac{W_3}{W} = k_3e^{-cx} \tag{3.3.148}$$

$$T_1 = y_1 \int \frac{W_1}{W}\, r \ \ \mathrm{d}x = e^{ax} \cdot m_1 x = xe^{ax} \tag{3.3.149}$$

$$T_2 = y_2 \int \frac{W_2}{W}\, r \ \ \mathrm{d}x = e^{bx} \cdot m_2 e^{(a-b)x} = e^{bx} \tag{3.3.150}$$

$$T_3 = y_3 \int \frac{W_3}{W}\, r \ \ \mathrm{d}x = e^{cx} \cdot m_3 e^{(a-c)x} = e^{ax} \tag{3.3.151}$$

From variation of parameters, the only new solution is xe^{ax} which matches the result from the undetermined co-efficients method. The set of k, m are some constants.