Chapter 15

Power Series, Taylor Series

15.1 Sequences, Series, Convergence Tests

1. The sequence is bounded but diverges,

$$z_n = \frac{(1+\mathrm{i})^{2n}}{2^n} \qquad z_n = \mathrm{i}^n \tag{15.1.1}$$

2. The sequence is bounded and converges,

$$z_n = \frac{(3+4\mathrm{i})^n}{n!} \qquad \frac{z_{n+1}}{z_n} = \frac{3+4\mathrm{i}}{n+1} \tag{15.1.2}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = 0 \qquad L = 0 + 0\mathrm{i} \tag{15.1.3}$$

3. The sequence is bounded and converges,

$$z_n = \frac{n\pi}{(4+2n\mathrm{i})} = \frac{n\pi}{2}\,\frac{1}{2+n\mathrm{i}} \qquad z_n = \frac{n\pi}{2}\left[\frac{2-n\mathrm{i}}{n^2+4}\right] \tag{15.1.4}$$

$$x_n = \frac{n\pi}{n^2+4} \qquad y_n = -\frac{\pi}{2}\,\frac{n^2}{n^2+4} \tag{15.1.5}$$

$$\lim_{n\to\infty} x_n = 0 \qquad \lim_{n\to\infty} y_n = -\frac{\pi}{2} \tag{15.1.6}$$

$$L = 0 - \frac{\pi}{2}\,\mathrm{i} \tag{15.1.7}$$

4. The sequence is not bounded and diverges,

$$z_n = (1+2\mathrm{i})^n \qquad \frac{z_{n+1}}{z_n} = (1+2\mathrm{i}) \tag{15.1.8}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \sqrt{5} > 1 \tag{15.1.9}$$

5. The sequence is bounded but diverges,

$$z_n = (-1)^n + 10\mathrm{i} \qquad \{z_n\} = \{-1+10\mathrm{i},\, 1+10\mathrm{i}\} \tag{15.1.10}$$

6. The sequence is not bounded and diverges,

$$z_n = \frac{\cos(n\pi\mathrm{i})}{n} \qquad z_n = \frac{\cosh(n\pi)}{n}\,\mathrm{i} \tag{15.1.11}$$

$$\lim_{n\to\infty} \frac{e^{n\pi}}{2n} = \lim_{n\to\infty} \frac{\pi\ e^{n\pi}}{2} \qquad = \infty \tag{15.1.12}$$

$$z_n = \frac{e^{n\pi} + e^{-n\pi}}{2n} \tag{15.1.13}$$

7. The sequence is not bounded and diverges,

$$x_n = n^2 \qquad y_n = \frac{1}{n^2} \tag{15.1.14}$$

$$\lim_{n\to\infty} x_n = \infty \qquad \lim_{n\to\infty} y_n = 0 \tag{15.1.15}$$

8. The sequence is bounded but diverges,

$$z_n = \left[\frac{(1+3\mathrm{i})}{\sqrt{10}}\right]^n \qquad |z_n| = 1^n = 1 \tag{15.1.16}$$

9. The sequence is bounded and converges,

$$z_n = (3+3\mathrm{i})^{-n} = \left[\frac{3-3\mathrm{i}}{18}\right]^n \qquad |z_n| = \frac{1}{(18)^{n/2}} \tag{15.1.17}$$

$$\lim_{n\to\infty} |z_n| = 0 \tag{15.1.18}$$

10. The sequence is bounded but diverges,

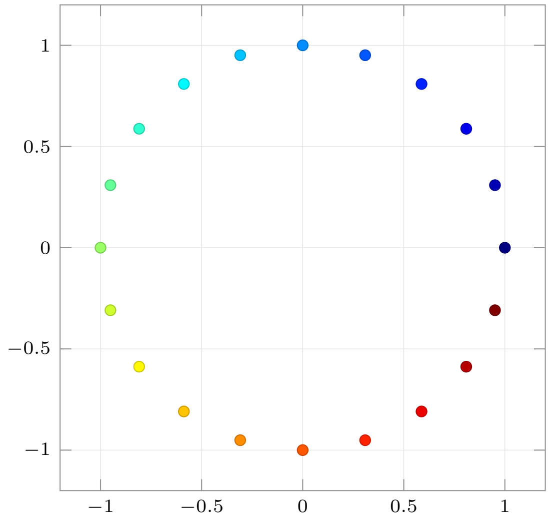
$$x_n = \sin\left(\frac{n\pi}{4}\,\mathfrak{i}\right) \qquad = \left\{0, \pm\frac{1}{\sqrt{2}}, \pm 1\right\} \tag{15.1.19}$$

$$y_n = \mathfrak{i}^n \qquad = \{\pm\mathfrak{i}, \pm 1\} \tag{15.1.20}$$

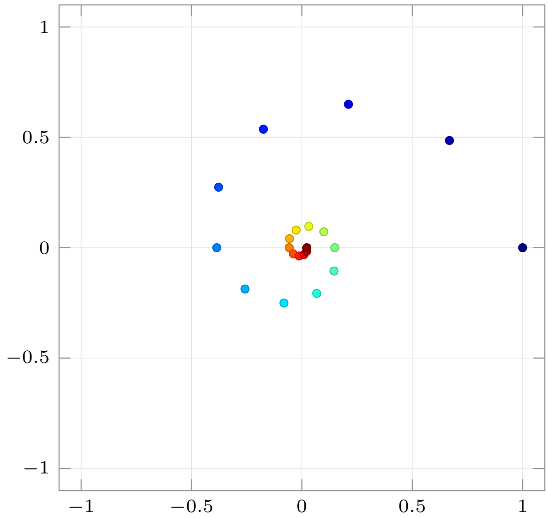
$$\tag{15.1.21}$$

11. Plotting a few types of complex sequences,

Bounded but not convergent



Bounded and convergent



Infinitely many limit points TBC.

12. Using the linearity of the limit operation, the limit of a sum is equal to the sum of the limits.

$$l = a + b\,\mathfrak{i} \qquad l^* = a^* + b^*\,\mathfrak{i} \tag{15.1.22}$$

$$\lim_{n\to\infty} x_n = a_n \qquad \lim_{n\to\infty} y_n = b_n \tag{15.1.23}$$

$$\lim_{n\to\infty} x_n^* = a_n^* \qquad \lim_{n\to\infty} y_n^* = b_n^* \tag{15.1.24}$$

$$\lim_{n\to\infty} z + z^* = l + l^* \tag{15.1.25}$$

13. The forward proof, starting with the fact that $\{x_n\}$, $\{y_n\}$ are bounded.

$$|x_n| \leq a \qquad \forall \quad n > N_1 \tag{15.1.26}$$

$$|y_n| \leq b \qquad \forall \quad n > N_2 \tag{15.1.27}$$

$$N = \max(N_1, N_2) \tag{15.1.28}$$

$$|z_n|^2 \leq a^2 + b^2 \qquad \forall \quad n > N \tag{15.1.29}$$

$$|z_n| \leq |l| \qquad \forall \quad n > N \tag{15.1.30}$$

This proves that z_n is bounded by $l = a + b\mathrm{i}$.

For the reverse proof, start with z_n being bounded.

$$|z_n|^2 \leq M^2 \qquad \forall \quad n > N \tag{15.1.31}$$

$$|x_n|^2 + |y_n|^2 \leq M^2 \qquad \forall \quad n > N \tag{15.1.32}$$

$$|x_n|^2 \leq M^2 \qquad |y_n|^2 \leq M^2 \tag{15.1.33}$$

This means that the real and imaginary parts are bounded sequences. (The other two sides of a right triangle have to be smaller than the hypotenuse).

14. Consider the complex sequence,

$$z_n = [0.9 \exp(\mathrm{i}\pi/4)]^n \qquad = (0.9)^{1/n} \exp\left(\frac{n\pi}{4}\,\mathrm{i}\right) \tag{15.1.34}$$

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} (0.9)^n = l \qquad l = 0 \tag{15.1.35}$$

Comparing the real and imaginary parts of this sequence,

$$x_n = (0.9)^n \cos\left(\frac{n\pi}{4}\right) \qquad y_n = (0.9)^n \sin\left(\frac{n\pi}{4}\right) \tag{15.1.36}$$

$$\lim_{n\to\infty} |x_n| \leq \lim_{n\to\infty} (0.9)^n = a \qquad \lim_{n\to\infty} |y_n| \leq \lim_{n\to\infty} (0.9)^n = b \tag{15.1.37}$$

$$a = 0 \qquad b = 0 \tag{15.1.38}$$

The results match since $l = a + \mathrm{i}\,b$. The reverse verification can also be proved similarly.

15. Consider the complex sequence,

$$z_n = \frac{(1+\mathrm{i})}{2^n} \qquad s = (1+\mathrm{i}) \lim_{n\to\infty} 2^{-n} \tag{15.1.39}$$

$$s = (1+\mathrm{i})\,\frac{1/2}{1-1/2} \qquad s = 1+\mathrm{i} \tag{15.1.40}$$

Comparing the real and imaginary parts of this sequence,

$$x_n = 2^{-n} \qquad y_n = 2^{-n} \tag{15.1.41}$$

$$\lim_{n\to\infty} |u_n| \lim_{n\to\infty} 2^{-n} = u \qquad \lim_{n\to\infty} |v_n| \lim_{n\to\infty} 2^{-n} = v \tag{15.1.42}$$

$$u = 1 \qquad v = 1 \tag{15.1.43}$$

The results match since $s = u + \mathrm{i}\ v$. The reverse verification can also be proved similarly.

16. Using the ratio test, the series sum converges absolutely and is thus convergent.

$$z_n = \frac{(20 + 30\mathrm{i})^n}{n!} \qquad \left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{20 + 30\mathrm{i}}{n+1}\right| \tag{15.1.44}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = 0 < 1 \tag{15.1.45}$$

17. Using the ratio test, is inconclusive.

$$\frac{z_{n+1}}{z_n} = \frac{(-\mathrm{i})\ \ln(n)}{\ln(n+1)} \qquad \left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{\ln(n)}{\ln(n+1)}\right| \tag{15.1.46}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = 1 \tag{15.1.47}$$

Using the fact that $\ln n < n$ for all integers greater than 2,

$$|z_n| \geq \left|\frac{(-\mathrm{i})^n}{n}\right| \qquad |z_n| \geq \frac{1}{n} \tag{15.1.48}$$

Since each term of z_n is larger in absolute value than the sequence $\{1/n\}$ which is known to diverge, it also diverges.

18. Using the ratio test, the series sum converges absolutely and is thus convergent.

$$z_n = \frac{n^2\ \mathrm{i}^n}{4^n} \qquad \left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(n+1)^2\ \mathrm{i}}{4n^2}\right| \tag{15.1.49}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \frac{1}{4} < 1 \tag{15.1.50}$$

19. Using the ratio test, is inconclusive.

$$z_n = \frac{\mathrm{i}^n}{n^2 - \mathrm{i}} = \frac{\mathrm{i}^n\ (n^2 + \mathrm{i})}{n^4 + 1} \qquad \frac{z_{n+1}}{z_n} = \frac{\mathrm{i}\ [n^2 - \mathrm{i}]}{(n+1)^2 - \mathrm{i}} \tag{15.1.51}$$

$$\left|\frac{z_{n+1}}{z_n}\right|^2 = \frac{n^4 + 1}{(n+1)^4 + 1} \qquad \lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = 1 \tag{15.1.52}$$

Using the allied series, for $n \geq 1$,

$$z_n^* = \frac{1}{n^2} \qquad |z_n^*| = \frac{1}{n^2} \tag{15.1.53}$$

$$|z_n| = \frac{1}{\sqrt{n^4+1}} \qquad |z_n| < |z_n^*| \tag{15.1.54}$$

Since the series z_n^* is convergent, the given series is also convergent.

20. Using the ratio test, is inconclusive.

$$z_n = \frac{n+\mathrm{i}}{3n^2+2\mathrm{i}} \qquad \left|\frac{z_{n+1}}{z_n}\right|^2 = \frac{(n+1)^2+1}{n^2+1}\,\frac{n^4+4/9}{(n+1)^4+4/9} \tag{15.1.55}$$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 1 \tag{15.1.56}$$

Using the allied series, for $n \geq 1$,

$$|z_n| = \sqrt{\frac{n^2+1}{9n^4+4}} \qquad 9n^4+4 \leq 9n^4+9n^2 \tag{15.1.57}$$

$$9n^4+4 \leq 9n^2(n^2+1) \qquad |z_n| \geq \frac{1}{3n} \tag{15.1.58}$$

$$z_n^* = \frac{1}{3n} \tag{15.1.59}$$

Since the series z_n^* is divergent, the given series is also divergent.

21. Using the ratio test, the series sum converges absolutely and is thus convergent.

$$z_n = \frac{(2\pi^2\,\mathrm{i})^n}{(2n+1)!} \qquad \frac{z_{n+1}}{z_n} = \frac{2\pi^2\,\mathrm{i}}{(2n+2)(2n+3)} \tag{15.1.60}$$

$$\left|\frac{z_{n+1}}{z_n}\right| = 2\pi^2\left|\frac{1}{4n^2+10n+6}\right| \qquad \lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 0 \tag{15.1.61}$$

22. Using the comparison test, knowing thet $1/n$ diverges, the given series also diverges.

$$z_n = \frac{1}{\sqrt{n}} \qquad z_n^* = \frac{1}{n} \tag{15.1.62}$$

$$\sqrt{n} \leq n \qquad \forall \quad n \geq 1 \tag{15.1.63}$$

$$z_n \geq z_n^* \tag{15.1.64}$$

23. Using the ratio test, the series sum converges absolutely and is thus convergent.

$$z_n = \frac{(-2\mathrm{i})^n}{(2n)!} \qquad \frac{z_{n+1}}{z_n} = \frac{-2\mathrm{i}}{(n+1)(n+2)} \tag{15.1.65}$$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{2}{n^2+3n+2}\right| \qquad \lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 0 \tag{15.1.66}$$

24. Using the ratio test, the series diverges.

$$z_n = \frac{(3\mathrm{i})^n\ n!}{n^n} \qquad \frac{z_{n+1}}{z_n} = \frac{3\mathrm{i}\ (n+1)\ n^n}{(n+1)^{n+1}} \tag{15.1.67}$$

$$\frac{z_{n+1}}{z_n} = 3\mathrm{i}\left[\frac{n}{n+1}\right]^n \tag{15.1.68}$$

Evaluating the limit of this indeterminate form,

$$\lim_{n\to\infty}\left[\frac{n}{n+1}\right]^n = \exp\left[\lim_{n\to\infty}\frac{\ln(n/n+1)}{1/n}\right] \tag{15.1.69}$$

$$= \exp\left[\lim_{n\to\infty}\frac{-n}{n+1}\right] = \frac{1}{e} \tag{15.1.70}$$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \frac{3}{e} > 1 \tag{15.1.71}$$

25. Using the comparison test, is inconclusive.

$$z_n = \frac{\mathrm{i}^n}{n} \qquad z_n = x_m + \mathrm{i}\ y_m \tag{15.1.72}$$

$$x_m = \frac{(-1)^m}{2m} \qquad y_m = \frac{(-1)^m}{2m+1} \tag{15.1.73}$$

$$x_m^* = \frac{(-1)^m}{m} \qquad y_m^* = \frac{(-1)^m}{m} \tag{15.1.74}$$

$$\lim_{m\to\infty}\frac{y_m}{y_m^*} = \lim_{m\to\infty}\frac{m}{2m+1} = \frac{1}{2} \tag{15.1.75}$$

Since x_m^* is known to converge, x_m also converges because it is term-wise half of x_m^*. By a similar argument, y_m also converges, and thus the given series z_n converges.

26. The difference is that the limit has to be less than 1. For example, $z_n = 1/n$ does fulfil this criterion but diverges, since the limit of the ratio is equal to 1.

27. Consider the series $z_n = 1/n^3$,

$$\frac{z_{n+1}}{z_n} = \frac{n^3}{(n+1)^3} \qquad \lim_{n\to\infty} \frac{z_{n+1}}{z_n} = 1 \tag{15.1.76}$$

$$s_n = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{n^3} \qquad s \leq 1 + \int_1^n \frac{\mathrm{d}x}{x^3} \tag{15.1.77}$$

$$s \leq 1 + \left[\frac{2}{x^2}\right]_n^1 \qquad s \leq 3 - \frac{2}{n^2} \tag{15.1.78}$$

This is another example of a series failing the conditions of Theorem 8, but still satisfying Theorem 7.

28. Code written in `sympy`. Plotting TBC.

29. Given a series converges absolutely. For every given $\epsilon > 0$, however small, there exists an N, such that

$$|z_{N+1}| + |z_{N+2}| + |z_{N+p}| < \epsilon \qquad \forall \qquad n > N \tag{15.1.79}$$

and p some positive integer. Using the generalized triangle inequality,

$$|z_{N+1} + z_{N+2} + \dots + z_{N+p}| \leq |z_{N+1}| + |z_{N+2}| + |z_{N+p}| \tag{15.1.80}$$

This immediately means that replacing the series of absolute values $|z_n|$ with the actual complex numbers z_n does not change the convergent nature of the series.

30. The series converges by the ratio test. Using the result for the infinite series sum of a geometric series,

$$R_n = z_{n+1} + z_{n+2} + \dots \qquad R_n \leq z_{n+1} \left[1 + q + q^2 + \dots\right] \tag{15.1.81}$$

$$|R_n| \leq |z_{n+1}| \quad \frac{1}{1-q} \tag{15.1.82}$$

For the given series,

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{n+1+\mathrm{i}}{n+\mathrm{i}}\right| \cdot \left|\frac{n}{2(n+1)}\right| \tag{15.1.83}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \frac{1}{2} \qquad q = 1/2 \tag{15.1.84}$$

$$0.025 \geq |z_{n+1}| \tag{15.1.85}$$

The smallest n satisfying this inequality is n = 5.

$$s_5 = \frac{31}{32} + \frac{661}{960}\, \mathrm{i} \qquad s = 1 + \frac{\pi}{ln(4)}\, \mathrm{i} \tag{15.1.86}$$

$$|R_5| = 0.03125 < 0.05 \tag{15.1.87}$$

15.2 Power Series

1. Power series require non-negative integer powers of z so that they terminate after finitely many differentiation steps. So, these two series do not qualify.

2. Refer notes. TBC.

3. A power series converges

(a) On the entire complex plane

(b) In an open disc of finite radius centered on z_0

(c) Only at the center of the power series

4. Example 1, using $z_0 = 4 - 3\pi\mathrm{i}$

$$\sum_{m=0}^{\infty} a_m z^m = \frac{1}{1-z} \qquad \frac{1}{1-z} = \frac{1}{(1-z_0)-(z-z_0)} \tag{15.2.1}$$

$$= \left(\frac{1}{1-z_0}\right) \cdot \frac{1}{1-\frac{z-z_0}{1-z_0}} \qquad \frac{1}{1-z_0} = w_0, \qquad \frac{z-z_0}{1-z_0} = w \tag{15.2.2}$$

$$w_0\, [1-w]^{-1} = w_0\, [1+w+w^2+\dots] \tag{15.2.3}$$

Now, the original series can be expressed as a power series around the new center z_0, as

$$S = \frac{1}{-3+3\pi\mathrm{i}}\left[1 + \frac{z-4+3\pi\mathrm{i}}{-3+3\pi\mathrm{i}} + \left(\frac{z-4+3\pi\mathrm{i}}{-3+3\pi\mathrm{i}}\right)^2 + \dots\right] \tag{15.2.4}$$

Exmaple 2,

$$\exp(z) = \exp(z_0)\cdot\exp(z-z_0) \tag{15.2.5}$$

$$1 + z + \frac{z^2}{2!} + \dots = e^{z_0}\left[1 + (z-z_0) + \frac{(z-z_0)^2}{2!} + \dots\right] \tag{15.2.6}$$

Example 3, using the binomial theorem

$$\sum_{n=0}^{\infty} n!\; z^n = \sum_{n=0}^{\infty} n!\; (z - z_0 + z_0)^n \tag{15.2.7}$$

$$= \sum_{n=0}^{\infty} n!\; \sum_{r=0}^{n} \binom{n}{r} \; (z - z_0)^r \; z_0^{1-r} \tag{15.2.8}$$

In Exmaple 1, making the radius of convergence $R = 6$,

$$z \to z/6 \implies 1 + \frac{z}{6} + \left(\frac{z}{6}\right)^2 + \dots \tag{15.2.9}$$

$$R = \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{6^{n+1}}{6^n}\right| = 6 \tag{15.2.10}$$

5. Looking at the new power series, with $z^2 = w$

$$S_1 = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \qquad R_1 = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{15.2.11}$$

$$S_2 = a_0 + a_1 w + a_2 w^2 + a_3 w^3 + \dots \tag{15.2.12}$$

Since S_1 converges for $|z| < R$, and S_2 converges for all $|w| < R$

$$|w| = |z|^2 < R \qquad \implies \qquad |z| < \sqrt{R} \tag{15.2.13}$$

6. Radius of convergence,

$$z_0 = -1 + 0\mathrm{i} \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{15.2.14}$$

$$R = \frac{1}{4} \tag{15.2.15}$$

7. Radius of convergence, using the result of Problem 5,

$$s^* = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(z - \frac{\pi}{2}\right)^n \qquad R^* = \lim_{n\to\infty} \left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.2.16}$$

$$R^* = \lim_{n\to\infty} \left|\frac{(2n+2)(2n+1)}{(-1)}\right| \qquad R^* = \infty \tag{15.2.17}$$

$$z_0 = w_0 = \frac{\pi}{2} + 0\mathrm{i} \qquad R = \sqrt{R^*} = \infty \tag{15.2.18}$$

8. Radius of convergence,

$$z_0 = 0 + \pi \mathrm{i} \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.2.19}$$

$$R = \lim_{n\to\infty} \frac{n^n\,(n+1)}{(n+1)^{n+1}} \qquad R = \lim_{n\to\infty} \left[\frac{n}{(n+1)} \right]^n \tag{15.2.20}$$

$$R = \exp\left[\lim_{n\to\infty} \frac{\ln n - \ln(n+1)}{1/n} \right] \qquad R = \exp\left[\lim_{n\to\infty} \frac{-n}{(n+1)} \right] \tag{15.2.21}$$

$$R = \frac{1}{e} \tag{15.2.22}$$

9. Radius of convergence, using the result of Problem 5,

$$s^* = \sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}\,(z - \mathrm{i})^n \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.2.23}$$

$$R^* = \lim_{n\to\infty} \left| \frac{3\,(n-1)}{(n+1)} \right| \qquad R^* = 3 \tag{15.2.24}$$

$$z_0 = w_0 = 0 + \mathrm{i} \qquad R = \sqrt{R^*} = \sqrt{3} \tag{15.2.25}$$

10. Radius of convergence,

$$z_0 = 0 + 2\mathrm{i} \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.2.26}$$

$$R = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \qquad R > \lim_{n\to\infty} \left[\frac{n^{n+1}}{n^n} \right] \tag{15.2.27}$$

$$R = \infty \tag{15.2.28}$$

11. Radius of convergence,

$$z_0 = 0 + 0\mathrm{i} \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.2.29}$$

$$R = \lim_{n\to\infty} \left| \frac{1 + 5\mathrm{i}}{2 - \mathrm{i}} \right| \qquad R = \sqrt{\frac{26}{5}} \tag{15.2.30}$$

12. Radius of convergence,

$$z_0 = 0 + 0\mathrm{i} \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.2.31}$$

$$R = \lim_{n\to\infty} \left| \frac{-8n}{(n+1)} \right| \qquad R = 8 \tag{15.2.32}$$

13. Radius of convergence, using the result from Problem 5, recursively

$$s^* = \sum_{n=0}^{\infty} 16^n \, (z + \mathrm{i})^n \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.2.33}$$

$$R^* = \lim_{n\to\infty} \left| \frac{1}{16} \right| \qquad R^* = \frac{1}{16} \tag{15.2.34}$$

$$z_0 = w_0 = 0 - \mathrm{i} \qquad R = \sqrt[4]{R^*} = \frac{1}{2} \tag{15.2.35}$$

14. Radius of convergence, using the result from Problem 5

$$s^* = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n \, (n!)^2} \, z^n \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.2.36}$$

$$R^* = \lim_{n\to\infty} \left| -4(n+1)^2 \right| \qquad R^* = \infty \tag{15.2.37}$$

$$z_0 = w_0 = 0 + 0\mathrm{i} \qquad R = \sqrt{R^*} = \infty \tag{15.2.38}$$

15. Radius of convergence,

$$z_0 = 0 + 2\mathrm{i} \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.2.39}$$

$$R = \lim_{n\to\infty} \left| \frac{4(n+1)^2}{(2n+1)(2n+2)} \right| \qquad R = \lim_{n\to\infty} \left| \frac{4n^2 + 8n + 4}{4n^2 + 6n + 2} \right| \tag{15.2.40}$$

$$R = 1 \tag{15.2.41}$$

16. Radius of convergence,

$$z_0 = 0 + 0\mathrm{i} \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.2.42}$$

$$R = \lim_{n\to\infty} \left| \frac{2(n+1)^3}{(3n+1)(3n+2)(3n+3)} \right| \qquad R = \lim_{n\to\infty} \left| \frac{2n^3 + O(n^2)}{27n^3 + O(n^2)} \right| \tag{15.2.43}$$

$$R = \frac{2}{27} \tag{15.2.44}$$

17. Radius of convergence, using the result from Problem 5,

$$s^* = \sum_{n=1}^{\infty} \frac{2^n}{n(n+1)}\ z^n \qquad R^* = \lim_{n\to\infty} \left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.2.45}$$

$$R^* = \lim_{n\to\infty} \left|\frac{(n+2)}{2n}\right| \qquad R^* = \frac{1}{2} \tag{15.2.46}$$

$$z_0 = w_0 = 0 + 0\mathrm{i} \qquad R = \sqrt{R^*} = \frac{1}{\sqrt{2}} \tag{15.2.47}$$

18. Radius of convergence, using the result from Problem 5,

$$s^* = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\ n!}\ z^n \qquad R^* = \lim_{n\to\infty} \left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.2.48}$$

$$R^* = \lim_{n\to\infty} \left|\frac{(n+1)(2n+3)}{(-1)(2n+1)}\right| \qquad R^* = \infty \tag{15.2.49}$$

$$z_0 = w_0 = 0 + 0\mathrm{i} \qquad R = \sqrt{R^*} = \infty \tag{15.2.50}$$

19. Code written in `sympy` for the detection of more than one limit point. Rest, TBC.

20. Radius of convergence,

(a) The rate of "decay" of the coefficients should be inversely proportional to the radius of convergence, since this decay is better able to compensate for larger $|z|$ values. Thus,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R} \tag{15.2.51}$$

(b) For the given transformations of coefficients,

$$a_n \to k\ a_n \qquad \Longrightarrow \qquad R \to R \tag{15.2.52}$$

$$a_n \to k^n\ a_n \qquad \Longrightarrow \qquad R \to \frac{R}{|k|} \tag{15.2.53}$$

$$a_n \to \frac{1}{a_n} \qquad \Longrightarrow \qquad R \to \frac{1}{R} \tag{15.2.54}$$

15.2.55

Applications can be the change of a general series into a power series in order to use the results established in this section.

21. Example 6 is a case of two geometric sequences being added term-wise,

$$c_n = a_n + b_n \tag{15.2.56}$$

Using $(-1)^n$ to ensure multiple limit points means that Theorem 6 is not applicable. These limit points lead to different R values that have to be compared to select the strictest one.

22. Comparing the distances from the origin,

$$|z_1| = \sqrt{1000} \qquad |z_2| = \sqrt{997} \tag{15.2.57}$$

$$|z_1| > |z_2| \tag{15.2.58}$$

The second and third statement suggest that a series converging at z_1 has to converge at z_2 since it is closer to the center.

15.3 Functions Given by Power Series

1. Refer notes. TBC.

2. Let the two power series centered around the origin be,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n \tag{15.3.1}$$

$$f(z) + g(z) = h(z) \qquad \sum_{n=0}^{\infty} (a_n + b_n) z^n = \sum_{n=0}^{\infty} c_n z^n \tag{15.3.2}$$

Let the series converge for $|z| < R_1$ and $|z| < R_2$ respectively, with $R_1 < R_2$

$$R_1 = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad R_2 = \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| \tag{15.3.3}$$

$$\tag{15.3.4}$$

Now, the new series is convergent only for $|z| < R_1$ which is the smaller radius of convergence.

3. To prove the limit, which is an indeterminate form,

$$\lim_{n\to\infty} n^{1/n} = \exp\left[\lim_{n\to\infty} \frac{\ln n}{n}\right] = \exp\left[\lim_{n\to\infty} \frac{1}{n}\right] = 1 \tag{15.3.5}$$

4. Using the Cauchy product of two geometric series,

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}\left[\sum_{r=0}^{n} a_r b_{n-r}\right] z^n \qquad a_i = b_i = 1 \quad \forall \quad i \tag{15.3.6}$$

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \tag{15.3.7}$$

By differentiating a single geometric series,

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad f'(z) = \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1} \tag{15.3.8}$$

$$\frac{1}{(1-z)^2} = \sum_{0=n}^{\infty}(n+1)z^n \tag{15.3.9}$$

5. By using the Cauchy-Hadamard formula,

$$a_n = \frac{n(n-1)}{2^n} \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{15.3.10}$$

$$R = \lim_{n\to\infty} \left|\frac{(n-1)2}{(n+1)}\right| = 2 \tag{15.3.11}$$

Using differentiation,

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-2\mathrm{i})^n}{2^n} \qquad f''(z) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} (z-2\mathrm{i})^{n-2} \tag{15.3.12}$$

$$(z-2\mathrm{i})^2 \ f''(z) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} (z-2\mathrm{i})^n \qquad R = \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n}\right| = 2 \tag{15.3.13}$$

6. By using the Cauchy-Hadamard formula,

$$s^* = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\ (4\pi^2)^n} \ z^n \qquad R^* = \lim_{n\to\infty} \left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.3.14}$$

$$R^* = \lim_{n\to\infty} \left|\frac{(-4\pi^2)(2n+3)}{(2n+1)}\right| = 4\pi^2 \qquad R = \sqrt{R^*} = 2\pi \tag{15.3.15}$$

Using differentiation,

$$f'(z) = \sum_{n=0}^{\infty} \left(\frac{-1}{4\pi^2}\right)^n z^{2n} \qquad R^* = \lim_{n\to\infty} \left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.3.16}$$

$$R^* = \lim_{n\to\infty} \left|-4\pi^2\right| = 4\pi^2 \qquad R = \lim_{n\to\infty} \sqrt{R^*} = 2\pi \tag{15.3.17}$$

7. By using the Cauchy-Hadamard formula,

$$s^* = \sum_{n=1}^{\infty} \frac{n}{3^n} \ (z+2\mathrm{i})^n \qquad R^* = \lim_{n\to\infty} \left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.3.18}$$

$$R^* = \lim_{n\to\infty} \left|\frac{3n}{n+1}\right| = 3 \qquad R = \sqrt{R^*} = \sqrt{3} \tag{15.3.19}$$

Using integration,

$$f(z) = \sum_{n=0}^{\infty} \frac{(z+2\mathrm{i})^n}{3^n} \qquad f'(z) = \sum_{n=1}^{\infty} \frac{n}{3^n}\ (z+2\mathrm{i})^{n-1} \tag{15.3.20}$$

$$(z+2\mathrm{i})\ f'(z) = \sum_{n=1}^{\infty} \frac{n}{3^n}\ (z+2\mathrm{i})^n \qquad R^* = \lim_{n\to\infty} \left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.3.21}$$

$$R^* = \lim_{n\to\infty} \left|\frac{3^{n+1}}{3^n}\right| = 3 \qquad R = \lim_{n\to\infty} \sqrt{R^*} = \sqrt{3} \tag{15.3.22}$$

8. By using the Cauchy-Hadamard formula,

$$s = \sum_{n=1}^{\infty} \frac{5^n}{n(n+1)}\ z^n \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{15.3.23}$$

$$R = \lim_{n\to\infty} \left|\frac{(n+2)}{5n}\right| = \frac{1}{5} \tag{15.3.24}$$

Using integration twice,

$$f(z) = \sum_{n=0}^{\infty} 5^n\ z^n \tag{15.3.25}$$

$$g(z) = \int \left[\int f\ \mathrm{d}z\right] \mathrm{d}z \qquad g(z) = \sum_{n=0}^{\infty} \frac{5^n}{n(n+1)}\ z^{n+2} \tag{15.3.26}$$

$$z^{-2}\ g(z) = \sum_{n=1}^{\infty} \frac{5^n}{n(n+1)}\ z^n \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{15.3.27}$$

$$R = \lim_{n\to\infty} \left|\frac{n+2}{5n}\right| = \frac{1}{5} \tag{15.3.28}$$

9. By using the Cauchy-Hadamard formula,

$$s^* = \sum_{n=1}^{\infty} \frac{(-2)^n}{n(n+1)(n+2)}\ z^n \qquad R^* = \lim_{n\to\infty} \left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.3.29}$$

$$R^* = \lim_{n\to\infty} \left|\frac{(n+3)}{-2n}\right| = \frac{1}{2} \qquad R = \sqrt{R^*} = \frac{1}{\sqrt{2}} \tag{15.3.30}$$

Using integration thrice,

$$f(z) = \sum_{n=0}^{\infty} (-2)^n \; z^n \qquad g(z) = \sum_{n=0}^{\infty} \frac{(-2)^n}{n(n+1)(n+2)} \; z^{n+3} \tag{15.3.31}$$

$$z^{-3} \; g(z) = \sum_{n=1}^{\infty} \frac{(-2)^n}{n(n+1)(n+2)} \; z^n \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.3.32}$$

$$R^* = \lim_{n\to\infty} \left| \frac{1}{-2} \right| \qquad R = \sqrt{R^*} = \frac{1}{\sqrt{2}} \tag{15.3.33}$$

10. By using the Cauchy-Hadamard formula,

$$s = \sum_{n=k}^{\infty} \frac{n!}{(n-k)! \; k! \; 2^n} \; z^n \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.3.34}$$

$$R = \lim_{n\to\infty} \left| \frac{2(n+1-k)}{(n+1)} \right| = 2 \tag{15.3.35}$$

Using differentiation k times,

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{2} \right)^n \qquad g(z) = f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{2^n \; (n-k)!} \; z^{n-k} \tag{15.3.36}$$

$$\frac{z^k}{k!} \; g(z) = \sum_{n=1}^{\infty} \binom{n}{k} \left(\frac{z}{2} \right)^n \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.3.37}$$

$$R = 2 \tag{15.3.38}$$

11. By using the Cauchy-Hadamard formula,

$$s^* = \sum_{n=0}^{\infty} \frac{3^n \; n(n+1)}{7^n} \; (z+2)^n \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.3.39}$$

$$R^* = \lim_{n\to\infty} \left| \frac{7n}{3(n+2)} \right| = \frac{7}{3} \qquad R = \sqrt{R^*} = \sqrt{\frac{7}{3}} \tag{15.3.40}$$

Using differentiation twice,

$$f(z) = \sum_{n=0}^{\infty} (3/7)^n \ (z+2)^n \qquad g = f'' = \sum_{n=2}^{\infty} (3/7)^n \ n(n+1) \ (z+2)^{n-2} \tag{15.3.41}$$

$$z^2 \ g(z) = \sum_{n=2}^{\infty} (3/7)^n \ n(n+1) \ (z+2)^n \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.3.42}$$

$$R^* = \lim_{n\to\infty} \left| \frac{7}{3} \right| \qquad R = \sqrt{R^*} = \sqrt{\frac{7}{3}} \tag{15.3.43}$$

12. By using the Cauchy-Hadamard formula, setting $z^2 = w$

$$s^* = \sum_{n=0}^{\infty} \frac{2n(2n-1)}{n^n} \ z^{2n} \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.3.44}$$

$$R^* = \lim_{n\to\infty} \left| \frac{n(2n-1) \ (n+1)^n}{(2n+1) \ n^n} \right| \qquad R^* = \left| \frac{n(2n-1)}{(2n+1)} \ \left(1 + \frac{1}{n}\right)^n \right| \tag{15.3.45}$$

$$R^* = \infty \qquad R = \sqrt{R^*} = \infty \tag{15.3.46}$$

Using differentiation twice,

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{n^n} \qquad g = f'' = \sum_{n=2}^{\infty} \frac{(2n)(2n-1)}{n^n} \ z^{2n-2} \tag{15.3.47}$$

$$f^*(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^n} \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.3.48}$$

$$R^* = \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^n \right| \qquad R^* = \lim_{n\to\infty} \left| \left(1 + \frac{1}{n}\right)^n \right| \tag{15.3.49}$$

$$R^* = \infty \qquad R = \sqrt{R^*} = \infty \tag{15.3.50}$$

13. By using the Cauchy-Hadamard formula

$$s = \sum_{n=0}^{\infty} \binom{n+k}{k}^{-1} \ z^{n+k} \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{15.3.51}$$

$$R = \lim_{n\to\infty} \left| \frac{(n+1+k)}{(n+1)} \right| \qquad R = 1 \tag{15.3.52}$$

Using differentiation k times,

$$f(z) = \sum_{n=0}^{\infty} \binom{n+k}{k}^{-1} z^{n+k} \qquad g = f^{(k)} = \sum_{n=0}^{\infty} k!\ z^n \tag{15.3.53}$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad R = 1 \tag{15.3.54}$$

14. By using the Cauchy-Hadamard formula

$$s = \sum_{n=0}^{\infty} \binom{n+m}{m} z^n \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{15.3.55}$$

$$R = \lim_{n\to\infty} \left|\frac{(n+1)}{(n+m+1)}\right| \qquad R = 1 \tag{15.3.56}$$

Using differentiation m times,

$$f(z) = \sum_{n=0}^{\infty} z^{n+m} \qquad g = f^{(m)} = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!}\ z^n \tag{15.3.57}$$

$$\frac{1}{m!}\ g = \sum_{n=0}^{\infty} \binom{n+m}{m} z^n \tag{15.3.58}$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad R = 1 \tag{15.3.59}$$

15. By using the Cauchy-Hadamard formula

$$s = \sum_{n=2}^{\infty} \frac{4^n\ n(n-1)}{3^n}\ (z-\mathrm{i})^n \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{15.3.60}$$

$$R = \lim_{n\to\infty} \left|\frac{3(n-1)}{4(n+1)}\right| \qquad R = \frac{3}{4} \tag{15.3.61}$$

Using differentiation m times,

$$f(z) = \sum_{n=0}^{\infty} (4/3)^n\ (z-\mathrm{i})^n \qquad g = f'' = \sum_{n=2}^{\infty} (4/3)^n\ n(n-1)\ (z-\mathrm{i})^{n-2} \tag{15.3.62}$$

$$(z-\mathrm{i})^2\ g = \sum_{n=2}^{\infty} (4/3)^n\ n(n-1)\ (z-\mathrm{i})^n \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{3}{4} \tag{15.3.63}$$

16. For an even function, given that its power series is unique,

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \tag{15.3.64}$$

$$f(-z) = a_0 - a_1 z + a_2 z^2 - a_3 z^3 + \dots \tag{15.3.65}$$

$$f(-z) = f(z) \tag{15.3.66}$$

$$a^{2m+1} = -a^{2m+1} \qquad \implies a^{2m+1} = 0 \tag{15.3.67}$$

Thus, the coefficients of odd powers are zero. Example is cosine function

17. For an even function, given that its power series is unique,

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \tag{15.3.68}$$

$$f(-z) = a_0 - a_1 z + a_2 z^2 - a_3 z^3 + \dots \tag{15.3.69}$$

$$f(-z) = -f(z) \tag{15.3.70}$$

$$a^{2m} = -a^{2m} \qquad \implies a^{2m} = 0 \tag{15.3.71}$$

Thus, the coefficients of even powers are zero. Example is the sine function

18. Using the Cauchy product,

$$(1+z)^p \cdot (1+z)^q = \sum_{r=0}^{p+q} \left[\sum_{n=0}^{r} a_n \cdot b_{r-n} \right] x^r \tag{15.3.72}$$

$$(1+z)^p = \sum_{n=0}^{p} \binom{p}{n} z^n \qquad \implies \qquad a_n = \binom{p}{n} \tag{15.3.73}$$

$$(1+z)^q = \sum_{n=0}^{q} \binom{q}{n} z^n \qquad \implies \qquad b_n = \binom{q}{n} \tag{15.3.74}$$

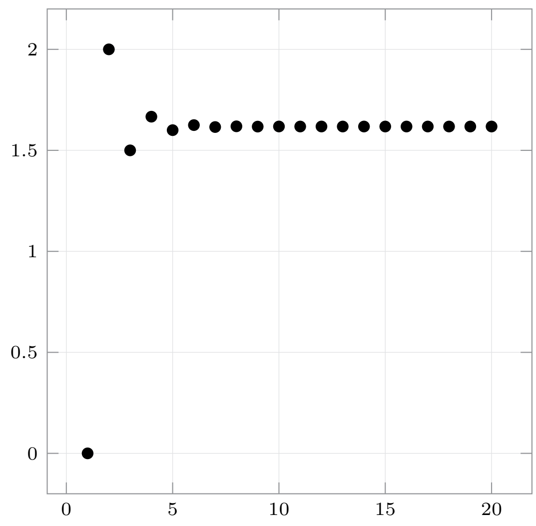
$$\binom{p}{n} \cdot \binom{q}{r-n} = \binom{p+q}{r} \tag{15.3.75}$$

19. Refer to the Chapter 5 on the power series method for solving ODEs.

20. Fibonacci sequence,

    (a) Plotting the ratios of successive Fibonacci numbers, the limit is 1.618

$n = 20$



(b) The Fibonacci sequence is computed and $a_{12} = 233$ is verified. The total number off rabbits is equal to the number of rabbits born to pairs ready to breed and the number of pairs too young to breed.

$$a_n = 2a_{n-2} + (a_{n-1} - a_{n-2}) \quad \text{15.3.76}$$

$$= a_{n-2} + a_{n-1} \quad \text{15.3.77}$$

(c) Using the Cauchy product,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{15.3.78}$$

$$z\ f(z) = \sum_{n=0}^{\infty} a_n z^{n+1} \qquad z\ f(z) = \sum_{n=1}^{\infty} a_{n-1} z^n \quad \text{15.3.79}$$

$$z^2\ f(z) = \sum_{n=0}^{\infty} a_n z^{n+2} \qquad z^2\ f(z) = \sum_{n=2}^{\infty} a_{n-2} z^n \quad \text{15.3.80}$$

$$g(z) = f(z) - zf(z) - z^2 f(z) \quad \text{15.3.81}$$

$$g(z) = a_0 + (a_1 - a_0)\ z + \sum_{n=2}^{\infty} \Big[a_n - a_{n-1} - a_{n-2}\Big]\ z^n \quad \text{15.3.82}$$

For $a_0 = 1$ and $a_1 = a_0$ and $a_n = a_{n-1} + a_{n-2}$, the function

$$g(z) = 1 \qquad f(z) = \frac{1}{1 - z - z^2} \quad \text{15.3.83}$$

This is the generating function of the Fibonacci sequence.

15.4 Taylor and Maclaurin Series

1. Refer notes. TBC.

   The changes come from the fact that the complex plane is a 2d plane as opposed to the 1D real line, for real calculus.

2. Finding the Maclaurin series for Example 5,

$$f(z) = \frac{1}{1+z^2} \qquad = \frac{1}{1-(-z^2)} \tag{15.4.1}$$

$$= \sum_{n=0}^{\infty} (-z^2)^n \qquad = 1 - z^2 + z^4 - z^6 + \dots \tag{15.4.2}$$

$$\left|-z^2\right| < 1 \qquad \implies \qquad |z| < 1 \tag{15.4.3}$$

   Finding the Maclaurin series for Example 6,

$$f(z) = \arctan(z) \qquad f'(z) = g(z) = \frac{1}{1+z^2} \tag{15.4.4}$$

$$g(z) = \sum_{n=0}^{\infty} (-1)^n \ z^{2n} \qquad \int g(z) \ \mathrm{d}z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} \tag{15.4.5}$$

$$f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots \tag{15.4.6}$$

   The resulting branch of arctan is the principal branch such that its real part satisfies $|u| < \pi/2$.

3. Finding the Maclaurin series,

$$f(z) = \sin(2z^2) \qquad = \sum_{n=0}^{\infty} (-1)^n \ \frac{(2z^2)^{2n+1}}{2n+1} \tag{15.4.7}$$

$$\sin(2z^2) = 2z^2 - \frac{(2z^2)^3}{3!} + \frac{(2z^2)^5}{5!} - \dots \qquad = 2z^2 - \frac{4z^6}{3} + \frac{4z^{10}}{15} - \dots \tag{15.4.8}$$

$$R = \infty \tag{15.4.9}$$

   Since $\sin(z)$ itself is entire.

4. Finding the Maclaurin series,

$$f(z) = \frac{z+2}{1-z^2} \qquad = \frac{0.5}{1+z} + \frac{1.5}{1-z} \tag{15.4.10}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad = 1 + z + z^2 + z^3 + \dots \tag{15.4.11}$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n \qquad = 1 - z + z^2 - z^3 + \dots \tag{15.4.12}$$

$$f(z) = 2 + z + 2z^2 + z^3 + \dots \qquad R = 1 \tag{15.4.13}$$

Since the geometric series converges for $|z| < 1$

5. Finding the Maclaurin series

$$f(z) = \frac{1}{2+z^4} \qquad = \frac{0.5}{1+0.5z^4} \tag{15.4.14}$$

$$w = \frac{z^2}{\sqrt{2}} \qquad f(w) = \frac{0.5}{1+w^2} \tag{15.4.15}$$

$$f(w) = 0.5\left[1 - w^2 + w^4 - w^6 + \dots\right] \qquad \forall \ = |w| < 1 \tag{15.4.16}$$

$$f(z) = \frac{1}{2} - \frac{z^4}{4} + \frac{z^8}{8} - \frac{z^{12}}{16} + \dots \qquad \forall \ = |z| < 2^{1/4} \tag{15.4.17}$$

6. Finding the Maclaurin series

$$f(z) = \frac{1}{1+3\mathrm{i}z} \tag{15.4.18}$$

$$w = 3\mathrm{i}z \qquad f(w) = \frac{1}{1+w} \tag{15.4.19}$$

$$f(w) = 1 - w + w^2 - w^3 + \dots \qquad \forall \ = |w| < 1 \tag{15.4.20}$$

$$f(z) = 1 - 3\mathrm{i}\ z - 9z^2 + 27\mathrm{i}\ z^3 + \dots \qquad \forall \ = |z| < \frac{1}{3} \tag{15.4.21}$$

7. Finding the Maclaurin series

$$f(z) = \cos^2(z/2) \qquad f(z) = \frac{1+\cos(z)}{2} \tag{15.4.22}$$

$$f(z) = 0.5 + 0.5\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right] \qquad = 1 - \frac{z^2}{4} + \frac{z^4}{48} - \frac{z^6}{1440} + \dots \tag{15.4.23}$$

$$R = \infty \tag{15.4.24}$$

Since the cosine function itself is entire.

8. Finding the Maclaurin series

$$f(z) = \sin^2 z \qquad f(z) = \frac{1 - \cos(2z)}{2} \tag{15.4.25}$$

$$f(z) = 0.5 - 0.5\left[1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \dots\right] \qquad R = \infty \tag{15.4.26}$$

$$= z^2 - \frac{z^4}{3} + \frac{2z^6}{45} \tag{15.4.27}$$

Since the cosine function itself is entire.

9. Integrating term-wise,

$$f(t) = \exp(-t^2/2) \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\ n!}\ t^{2n} \tag{15.4.28}$$

$$\int_0^z f\ \ \mathrm{d}t = \left[\left(\frac{-1}{2}\right)^n \frac{t^{2n+1}}{(2n+1)\ n!}\right]_0^z \qquad = \left(\frac{-1}{2}\right)^n \frac{z^{2n+1}}{(2n+1)\ n!} \tag{15.4.29}$$

$$g(z) = \int_0^z f\ \ \mathrm{d}t = \frac{z}{1\cdot 1\cdot 1} - \frac{z^3}{2\cdot 3\cdot 1} + \frac{z^5}{4\cdot 5\cdot 2} + \dots \qquad g(z) = z - \frac{z^3}{6} + \frac{z^5}{40} - \dots \tag{15.4.30}$$

$$R = \infty \tag{15.4.31}$$

Since the exponential function itself is entire.

10. Using the result from Problem 10,

$$g(z) = \int_0^z \exp(-t^2)\ \ \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n\ z^{2n+1}}{(2n+1)\ n!} \tag{15.4.32}$$

$$f(z) = \exp(z^2) = \sum_{m=0}^{\infty} \frac{z^{2m}}{m!} \tag{15.4.33}$$

$$h(z) = f(z)\cdot g(z) = \sum_{r=0}^{\infty} a_r\ x^r \tag{15.4.34}$$

Matching powers of x on both sides,

$$r = 2m + 2n + 1 \qquad M(n) = \frac{r-1}{2} - m \tag{15.4.35}$$

$$a_r = \sum_{n=0}^{M} \frac{(-1)^n}{(2n+1)\ n!\ m!} \tag{15.4.36}$$

Since the exponential function itself is entire.

11. Integrating term-wise,

$$f(t) = \sin(t^2) \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n \ t^{4n+2}}{(2n+1)!} \tag{15.4.37}$$

$$g(z) = \int_0^z f \ \ \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n \ z^{4n+3}}{(2n+1)! \ (4n+3)} \qquad = \frac{z^3}{1! \ 3} - \frac{z^7}{3! \ 7} + \frac{z^{11}}{5! \ 11} - \ldots \tag{15.4.38}$$

$$R = \infty \tag{15.4.39}$$

12. Integrating term-wise,

$$f(t) = \cos(t^2) \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n \ t^{4n}}{(2n)!} \tag{15.4.40}$$

$$g(z) = \int_0^z f \ \ \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n \ z^{4n+1}}{(2n)! \ (4n+1)} \qquad = \frac{z}{0! \ 1} - \frac{z^5}{2! \ 5} + \frac{z^9}{4! \ 9} - \ldots \tag{15.4.41}$$

$$R = \infty \tag{15.4.42}$$

13. Integrating term-wise,

$$f(t) = \exp(-t^2) \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n \ t^{2n}}{n!} \tag{15.4.43}$$

$$g(z) = \int_0^z f \ \ \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n \ z^{2n+1}}{n! \ (2n+1)} \qquad \mathrm{erf} \ z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \ \ \mathrm{d}t \tag{15.4.44}$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{z}{0! \ 1} - \frac{z^3}{1! \ 3} + \frac{z^5}{2! \ 5} - \ldots \right] \qquad R = \infty \tag{15.4.45}$$

14. Integrating term-wise,

$$f(t) = \frac{\sin(t)}{t} \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n \ t^{2n}}{(2n+1)!} \tag{15.4.46}$$

$$g(z) = \int_0^z f \ \ \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n \ z^{2n+1}}{(2n+1)! \ (2n+1)} \qquad = \frac{z}{1! \ 1} - \frac{z^3}{3! \ 3} + \frac{z^5}{5! \ 5} - \ldots \tag{15.4.47}$$

$$R = \infty \tag{15.4.48}$$

15. Finding the Maclaurin series,

(a) Using the coefficient formula, and `sympy` to calculate the derivatives,

n	E_n	n	E_n
0	1	12	2702765
2	−1	14	−199360981
4	5	16	19391512145
6	−61	18	−2404879675441
8	1385	20	370371188237525
10	−50521		

(b) Using the Cauchy product of the two series,

$$f(z) = e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \tag{15.4.49}$$

$$g(z) = 1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \dots \tag{15.4.50}$$

$$f(z)\cdot g(z) = z + \left[\frac{1}{2!} + B_1\right]z^2 + \left[\frac{1}{3!} + \frac{B_1}{2!} + \frac{B_2}{2!}\right]z^3 + \dots \tag{15.4.51}$$

$$B_1 = \frac{-1}{2} \qquad B_2 = \frac{1}{6} \tag{15.4.52}$$

$$0 = \frac{B_3}{3!\ 1!} + \frac{B_2}{2!\ 2!} + \frac{B_1}{1!\ 3!} + \frac{1}{4!\ 0!} \qquad B_3 = 0 \tag{15.4.53}$$

$$0 = \frac{B_4}{4!\ 1!} + \frac{B_3}{3!\ 2!} + \frac{B_2}{2!\ 3!} + \frac{B_1}{1!\ 4!} + \frac{1}{5!\ 0!} \qquad B_4 = -\frac{1}{30} \tag{15.4.54}$$

$$0 = \frac{B_5}{5!\ 1!} + \frac{B_4}{4!\ 2!} + \frac{B_3}{3!\ 3!} + \frac{B_2}{2!\ 4!} + \frac{B_1}{1!\ 5!} + \frac{1}{0!\ 6!} \qquad B_5 = 0 \tag{15.4.55}$$

$$0 = \frac{B_6}{6!\ 1!} + \frac{B_4}{4!\ 3!} + \frac{B_2}{2!\ 5!} + \frac{B_1}{1!\ 6!} + \frac{1}{0!\ 6!} \qquad B_6 = \frac{1}{42} \tag{15.4.56}$$

The rest of the Bernoulli numbers can be found by recursively solving forwards starting with the lower powers in z.

(c) Using the definition of complex sine and cosine,

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{\mathrm{i}}\frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{e^{\mathrm{i}z} + e^{-\mathrm{i}z}} \qquad = \frac{1}{\mathrm{i}}\frac{(e^{\mathrm{i}z} - e^{-\mathrm{i}z})^2}{e^{2\mathrm{i}z} - e^{-2\mathrm{i}z}} \tag{15.4.57}$$

$$w = e^{2\mathrm{i}z} \qquad \tan(z) = -\mathrm{i}\ \frac{w + (1/w) - 2}{w - (1/w)} \tag{15.4.58}$$

$$= -\mathrm{i}\ \frac{w^2 - 2w + 1}{w^2 - 1} \tag{15.4.59}$$

$$w^2 - 2w + 1 = a + b(w+1) + c(w^2 - 1) \qquad c = 1 \quad b = -2 \quad a = 4 \tag{15.4.60}$$

Splitting the numerator using these partial fractions,

$$\tan z = -\mathrm{i}\left[1 - \frac{2}{w-1} + \frac{4}{w^2-1}\right] \tag{15.4.61}$$

$$= \frac{2\mathrm{i}}{e^{2\mathrm{i}z}-1} - \frac{4\mathrm{i}}{e^{4\mathrm{i}z}-1} - \mathrm{i} \tag{15.4.62}$$

$$= \frac{1}{z}\left[\sum_{n=0}^{\infty} \frac{B_n}{n!}\,(2\mathrm{i}z)^n - \sum_{n=0}^{\infty} \frac{B_n}{n!}\,(4\mathrm{i}z)^n - \mathrm{i}z\right] \tag{15.4.63}$$

Using $B_0 = 1$, $B_1 = -0.5$, the coefficients of z^0, z^1 cancel. This leaves z^2 terms onwards.
Since $\tan z$ is an odd function, the Bernoulli numbers, (which are the only nonzero part of the coefficients) have to vanish.
Setting $n = 2m$

$$\tan z = \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!}\,(-1)^m\,\left[2^{2m} - 4^{2m}\right]\,z^{2m-1} \tag{15.4.64}$$

16. Starting with,

$$f(z) = (1-z^2)^{-1/2} \tag{15.4.65}$$

$$= 1 - \frac{z^2}{2} + \frac{1\cdot 3}{2^2\ 2!}\,z^4 - \frac{1\cdot 3\cdot 5}{2^3\ 3!}\,z^6 + \dots \tag{15.4.66}$$

$$= 1 - \frac{z^2}{2} + \frac{1\cdot 3}{2\cdot 4}\,z^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,z^6 + \dots \tag{15.4.67}$$

$$\int f\ \mathrm{d}z = z - \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \dots \tag{15.4.68}$$

Since the left hand side integrates to $\tan z$, this is the Taylor series required. The series converges only for $|z| < 1$ since this is the radius of convergence of the binomial theorem.

17. Using Maclaurin series,

(a) Proving the derivative formulas using term-wise differentiation of the Maclaurin series.

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad\qquad f'(z) = \sum_{n=1}^{\infty} \frac{n\ z^{n-1}}{n!} \tag{15.4.69}$$

$$= \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} \qquad\qquad = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z \tag{15.4.70}$$

$$f(z) = \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad f'(z) = \sum_{n=1}^{\infty} (-1)^n \frac{2n\ z^{2n-1}}{(2n)!} \tag{15.4.71}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n-1)!} \qquad = \sin z \tag{15.4.72}$$

$$f(z) = \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad f'(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\ z^{2n}}{(2n+1)!} \tag{15.4.73}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad = \cos z \tag{15.4.74}$$

$$f(z) = \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad f'(z) = \sum_{n=1}^{\infty} \frac{2n\ z^{2n-1}}{(2n)!} \tag{15.4.75}$$

$$= \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!} \qquad = \sinh z \tag{15.4.76}$$

$$f(z) = \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad f'(z) = \sum_{n=0}^{\infty} \frac{(2n+1)\ z^{2n}}{(2n+1)!} \tag{15.4.77}$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad = \cosh z \tag{15.4.78}$$

$$f(z) = \mathrm{Ln}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} \qquad f'(z) = \sum_{n=1}^{\infty} (-1)^{n+1}\ z^{n-1} \tag{15.4.79}$$

$$= \sum_{n=0}^{\infty} (-1)^n\ z^n \qquad = \frac{1}{1+z} \tag{15.4.80}$$

(b) Starting from the LHS, and noting that odd powers of z cancel,

$$e^{\mathrm{i}z} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}z)^n}{n!} \qquad e^{-\mathrm{i}z} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}z)^n}{n!} \tag{15.4.81}$$

$$\frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2} = \sum_{m=0}^{\infty} \frac{(\mathrm{i}z)^{2m}}{(2m)!} \qquad = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!} \tag{15.4.82}$$

$$= \cos z \tag{15.4.83}$$

(c) Using the Taylor series of the sine function,

$$\sin z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad z = \mathfrak{i}y \neq 0 \tag{15.4.84}$$

$$\sin z = \sum_{n=0}^{\infty} \mathfrak{i}\,(-1)^n\, \frac{y^{2n+1}}{(2n+1)!} \tag{15.4.85}$$

Given $y \neq 0$, all terms in the expansion have the same sign and are nonzero. This means that $\sin z \neq 0$

18. Expanding as a Taylor series around z_0,

$$f(z) = \frac{1}{z} \qquad z_0 = \mathfrak{i} \tag{15.4.86}$$

$$f(z) = \frac{1}{z_0 + z - z_0} \qquad = \frac{1/z_0}{1 + \frac{z-z_0}{z_0}} \tag{15.4.87}$$

$$f(z) = \frac{1}{z_0} \sum_{n=0}^{\infty} \left(\frac{-1}{z_0}\right)^n (z - z_0)^n \qquad f(z) = -\sum_{n=0}^{\infty} \mathfrak{i}^{n+1}\,(z - \mathfrak{i})^n \tag{15.4.88}$$

$$|z - \mathfrak{i}| < |\mathfrak{i}| \qquad R = 1 \tag{15.4.89}$$

19. Expanding as a Taylor series around z_0,

$$f(z) = \frac{1}{1-z} \qquad z_0 = \mathfrak{i} \tag{15.4.90}$$

$$f(z) = \frac{1}{1 - z_0 - (z - z_0)} \qquad = \frac{1}{1 - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{1-z_0}} \tag{15.4.91}$$

$$f(z) = \frac{1}{1 - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{1 - z_0}\right)^n \qquad f(z) = \frac{1}{1 - \mathfrak{i}} \sum_{n=0}^{\infty} \left(\frac{z - \mathfrak{i}}{1 - \mathfrak{i}}\right)^n \tag{15.4.92}$$

$$= \sum_{n=0}^{\infty} \left[\frac{1+i}{2}\right]^{n+1} (z - \mathfrak{i})^n \qquad R = |z - \mathfrak{i}| < \sqrt{2} \tag{15.4.93}$$

20. Expanding as a Taylor series around z_0,

$$f(z) = \cos^2 z \qquad z_0 = \pi/2 \tag{15.4.94}$$

$$f(z) = \frac{1 + \cos(2z)}{2} \qquad = \frac{1 + \cos[2(z - \pi/2) + \pi]}{2} \tag{15.4.95}$$

$$f(z) = \frac{1 - \cos[2(z - \pi/2)]}{2} \qquad f(z) = \sum_{n=1}^{\infty} (-1)^{n+1}\, \frac{2^{2n-1}(z - \pi/2)^{2n}}{(2n)!} \tag{15.4.96}$$

$$R = \infty \tag{15.4.97}$$

21. Expanding as a Taylor series around z_0,

$$f(z) = \sin z \qquad z_0 = \pi/2 \tag{15.4.98}$$

$$f(z) = \sin(z - \pi/2 + \pi/2) \qquad = \cos(z - \pi/2) \tag{15.4.99}$$

$$f(z) = \sum_{n=1}^{\infty} (-1)^n \frac{(z - \pi/2)^{2n}}{(2n)!} \qquad R = \infty \tag{15.4.100}$$

22. The Maclaurin series of $\cosh z$ is directly the answer

$$f(z) = \sum_{n=0}^{\infty} \frac{(z - \pi \mathrm{i})^{2n}}{(2n)!} \qquad R = \infty \tag{15.4.101}$$

23. Expanding as a Taylor series around z_0,

$$f(z) = (z + \mathrm{i})^{-2} \qquad z_0 = \mathrm{i} \tag{15.4.102}$$

$$f(z) = (2\mathrm{i} + z - \mathrm{i})^{-2} \qquad = -\frac{1}{4} \left(1 + \frac{z - \mathrm{i}}{2\mathrm{i}}\right)^{-2} \tag{15.4.103}$$

Using the binomial theorem, given $|z - \mathrm{i}| < 2$,

$$f(z) = -\frac{1}{4} \left[1 - 2\left(\frac{z - \mathrm{i}}{2\mathrm{i}}\right) + \frac{2 \cdot 3}{2!} \left(\frac{z - \mathrm{i}}{2\mathrm{i}}\right)^2 - \dots\right] \tag{15.4.104}$$

$$= \sum_{n=0}^{\infty} \binom{-2}{n} \left[\frac{1}{2\mathrm{i}}\right]^{n+2} (z - \mathrm{i})^n \tag{15.4.105}$$

24. Expanding as a Taylor series around z_0,

$$f(z) = \exp[z(z - 2)] \qquad z_0 = 1 \tag{15.4.106}$$

$$f(z) = \exp[(z - 1 + 1)(z - 1 - 1)] \qquad = \exp[(z - 1)^2 - 1] = \frac{\exp[(z - 1)^2]}{e} \tag{15.4.107}$$

$$f(z) = \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z - 1)^{2n}}{n!} \qquad R = \infty \tag{15.4.108}$$

25. Expanding as a Taylor series around z_0,

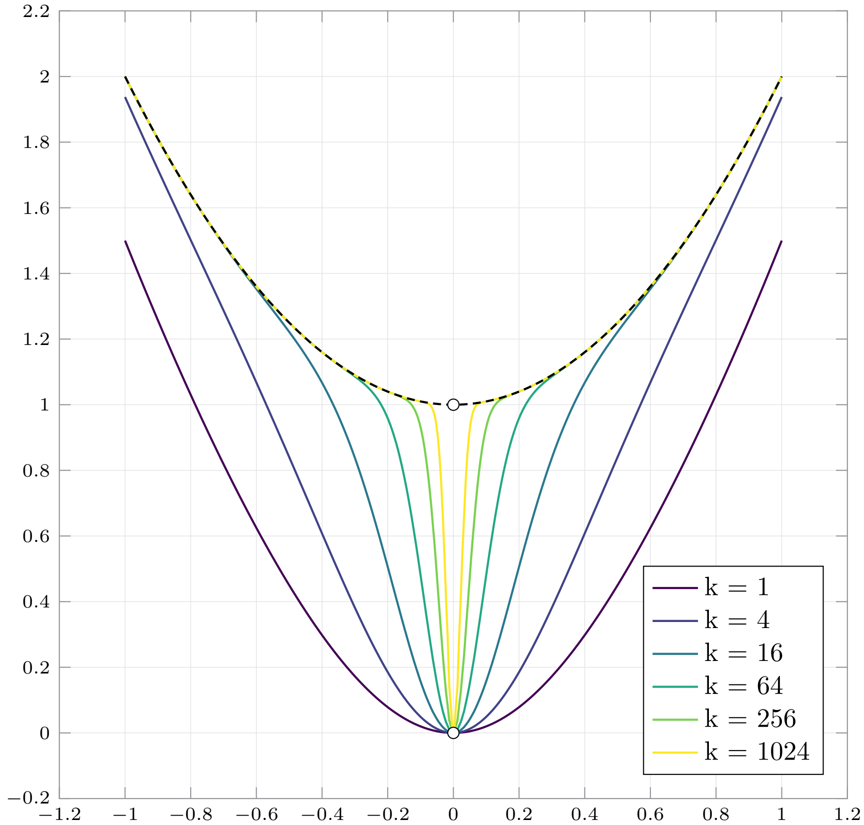
$$f(z) = \sinh(2z - \pi) \qquad z_0 = \mathfrak{i}/2 \tag{15.4.109}$$

$$f(z) = \sinh[2(z - \mathfrak{i}/2)] \qquad f(z) = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}\,(z - \mathfrak{i}/2)^{2n+1} \tag{15.4.110}$$

$$R = \infty \tag{15.4.111}$$

15.5 Uniform Convergence

1. Plotting the graph,



2. Finding the radius of convergence,

$$a_n = \left(\frac{n+2}{7n-3}\right)^n \qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{15.5.1}$$

$$R = \lim_{n\to\infty} \frac{7n+4}{n+3}\left[\frac{(n+2)(7n+4)}{(n+3)(7n-3)}\right]^n \qquad R = 7 \tag{15.5.2}$$

$$|z| \le r \qquad r < 7 \tag{15.5.3}$$

3. Finding the radius of convergence,

$$s^* = \frac{1}{3^n}\,(z+\mathrm{i})^n \qquad R^* = \lim_{n\to\infty}\left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.5.4}$$

$$R^* = \lim_{n\to\infty} 3 \qquad R = \sqrt{R^*} = \sqrt{3} \tag{15.5.5}$$

$$|z+\mathrm{i}| \le r \qquad r < \sqrt{3} \tag{15.5.6}$$

4. Finding the radius of convergence,

$$t_n = \frac{3^n\,(1-\mathrm{i})^n}{n!}\,(z-\mathrm{i})^n \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \tag{15.5.7}$$

$$R = \lim_{n\to\infty}\left|\frac{(n+1)}{1\,(1-\mathrm{i})}\right| \qquad R = \infty \tag{15.5.8}$$

$$|z-\mathrm{i}| \le r \qquad r < \infty \tag{15.5.9}$$

5. Finding the radius of convergence,

$$t_n = \frac{4^n\,n(n-1)}{2}\,(z+0.5\mathrm{i})^n \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \tag{15.5.10}$$

$$R = \lim_{n\to\infty}\left|\frac{(n-1)}{4\,(n+1)}\right| \qquad R = \frac{1}{4} \tag{15.5.11}$$

$$|z+0.5\mathrm{i}| \le r \qquad r < \frac{1}{4} \tag{15.5.12}$$

6. Finding the radius of convergence,

$$s^* = 2^n\,\tanh(n^2)\;z^n \qquad R^* = \lim_{n\to\infty}\left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.5.13}$$

$$R^* = \lim_{n\to\infty}\left|\frac{\tanh(n^2)}{2\;\tanh(n^2+1)}\right| = \frac{1}{2} \qquad R = \sqrt{R^*} = \frac{1}{\sqrt{2}} \tag{15.5.14}$$

$$|z| \le r \qquad r < \frac{1}{\sqrt{2}} \tag{15.5.15}$$

7. Finding the radius of convergence,

$$t_n = \frac{n!}{n^2}\,(z+0.5\mathrm{i})^n \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \tag{15.5.16}$$

$$R = \lim_{n\to\infty}\left|\frac{n+1}{n^2}\right| \qquad R = 0 \tag{15.5.17}$$

Series has zero radius of convergence and thus, is uniformly convergent nowhere.

8. Finding the radius of convergence,

$$s^* = \frac{3^n}{n(n+1)}\ (z-1)^n \qquad R^* = \lim_{n\to\infty}\left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.5.18}$$

$$R^* = \lim_{n\to\infty}\left|\frac{(n+2)}{3n}\right| = \frac{1}{3} \qquad R = \sqrt{R^*} = \frac{1}{\sqrt{3}} \tag{15.5.19}$$

$$|z| \le r \qquad r < \frac{1}{\sqrt{3}} \tag{15.5.20}$$

9. Finding the radius of convergence,

$$t_n = \frac{(-1)^n}{2^n\ n^2}\ (z-2\mathrm{i})^n \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \tag{15.5.21}$$

$$R = \lim_{n\to\infty}\left|\frac{(-2)(n+1)^2}{n^2}\right| \qquad R = 2 \tag{15.5.22}$$

$$|z-2\mathrm{i}| \le r \qquad r < 2 \tag{15.5.23}$$

10. Finding the radius of convergence,

$$t_n^* = \frac{z^n}{(2n)!} \qquad R^* = \lim_{n\to\infty}\left|\frac{a_n^*}{a_{n+1}^*}\right| \tag{15.5.24}$$

$$R^* = \lim_{n\to\infty}\left|\frac{(2n+2)!}{(2n)!}\right| \qquad R^* = \infty \tag{15.5.25}$$

$$R = \sqrt{R^*} = \infty \tag{15.5.26}$$

The series is uniformly convergent in any finite disk.

11. Using the Weierstrass M-test,

$$t_n = \frac{z^n}{n^2} \qquad G : |z| \le 1 \tag{15.5.27}$$

$$|t_n| = \left|\frac{z^n}{n^2}\right| \qquad |t_n| = \frac{|z|^n}{n^2} \tag{15.5.28}$$

$$|t_n| \le \frac{1}{n^2} \tag{15.5.29}$$

The series is uniformly convergent in the closed disk $|z| \le 1$.

12. Using the Weierstrass M-test,

$$t_n = \frac{z^n}{n^3 \ \cosh(n\,|z|)} \qquad G : |z| \le 1 \tag{15.5.30}$$

$$|t_n| = \left|\frac{z^n}{n^3 \ \cosh(n\,|z|)}\right| \qquad n^3 \cosh(n\,|z|) \ge n^3 \tag{15.5.31}$$

$$|t_n| \le \frac{1}{n^3} \tag{15.5.32}$$

The series is uniformly convergent in the closed disk $|z| \le 1$.
This used the fact that for real $x \ge 0$, $\cosh x \ge 1$,

13. Using the Weierstrass M-test,

$$t_n = \frac{\sin^n(|z|)}{n^2} \qquad G : \text{all of } \mathcal{C} \tag{15.5.33}$$

$$\sin^n x \le 1 \qquad \forall \qquad x \in \mathcal{R} \qquad |t_n| \le \frac{1}{n^2} \tag{15.5.34}$$

The series is uniformly convergent in all of $\mathcal{C}$.
This uses the fact that n^{-2} is a convergent series.

14. Using the Weierstrass M-test,

$$t_n = \frac{z^n}{|z|^{2n} + 1} \qquad G : 2 \le |z| \le 10 \tag{15.5.35}$$

$$|t_n| = \frac{|z|^n}{|z|^{2n} + 1} \qquad |t_n| \le \frac{1}{|z|^n} \le \frac{1}{2^n} \tag{15.5.36}$$

The series is uniformly convergent in the closed annulus $2 \le |z| \le 10$.
This used the fact that for the geometric series converges for $q = 0.5$.

15. Finding the radius of convergence,

$$t_n = \frac{(n!)^2}{(2n)!}\ z^n \qquad G : |z| \le 3 \tag{15.5.37}$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \qquad R = \lim_{n\to\infty} \left|\frac{(2n+1)(2n+2)}{(n+1)^2}\right| \tag{15.5.38}$$

$$R = 4 \tag{15.5.39}$$

The series is uniformly convergent in G since G lies completely inside the zone of convergence.

16. Using the Weierstrass M-test,

$$t_n = \frac{\tanh^n(|z|)}{n(n+1)} \qquad G : \text{all of } \mathcal{C} \tag{15.5.40}$$

$$\tanh(x) \in (-1, 1) \quad \forall \quad x \in \mathcal{R} \qquad \tanh^n(x) \in (-1, 1) \tag{15.5.41}$$

$$|t_n| \le \frac{1}{n(n+1)} \qquad |t_n| \le \frac{1}{n^2} \tag{15.5.42}$$

The series is uniformly convergent in all of $\mathcal{C}$.
This used the fact that $1/n^2$ converges.

17. Finding the radius of convergence,

$$t_n^* = \frac{\pi^n}{n^4}\, z^n \qquad R^* = \lim_{n\to\infty} \left| \frac{a_n^*}{a_{n+1}^*} \right| \tag{15.5.43}$$

$$R^* = \lim_{n\to\infty} \left| \frac{(n+1)^4}{\pi\ n^4} \right| \qquad R^* = \frac{1}{\pi} \tag{15.5.44}$$

$$R = \sqrt{R^*} = \frac{1}{\sqrt{\pi}} = 0.56419 \tag{15.5.45}$$

The series is uniformly convergent in the given closed disk $|z| \le 0.56$.

18. Weierstrass M-test,

(a) Proving the theorem,

$$|f_n(z)| \le M_n \qquad \forall \quad z \in G \tag{15.5.46}$$

$$s = \sum_{n=0}^{\infty} M_n \tag{15.5.47}$$

$$s_k = \sum_{n=0}^{k} M_n \qquad s = \sum_{n=0}^{\infty} M_n \tag{15.5.48}$$

$$R_k = s - s_k \qquad |R_k| = \sum_{n=k+1}^{\infty} f_n \tag{15.5.49}$$

$$|R_k| \le \sum_{n=k+1}^{\infty} |f_n| \qquad |R_k| \le \sum_{n=k+1}^{\infty} M_n \tag{15.5.50}$$

This proves that the sequence f_k is uniformly convergent.

(b) Let $\{g_m'\}$ be a sequence of continuous terms in G and let this sequence be U.C. in G,

$$G(z) = \sum_{m=0}^{\infty} g_m(z) \tag{15.5.51}$$

Further, the series $\{g'_m(z)\}$ is U.C. and has continuous terms.
From the term-wise integration theorem,

$$H(z) = \sum_{m=0}^{\infty} g'_m(z) \qquad \int_C H(z)\ \mathrm{d}z = \sum_{m=0}^{\infty} \int_C g'_m(z)\ \mathrm{d}z \tag{15.5.52}$$

$$\int_C H(z)\ \mathrm{d}z = \sum_{m=0}^{\infty} g_m(z) \qquad \int_C H(z)\ \mathrm{d}z = G(z) \tag{15.5.53}$$

$$H(z) = G'(z) \tag{15.5.54}$$

Thus, the sum of the series of derivatives is equal to the derivative of the original series sum.

(c) In the conditions for uniform convergence, the absolute value of R_n does not depend on z anywhere within region G.
This means that a series being U.C. in a region G makes it U.C. in all subregions that are infinite sets.
The converse is false. Trivial.

(d) Given that $\left|1 + z^2\right| > 1$, the geometric series formula is applicable,

$$s_n(z) = (1 + z^2) - \frac{1}{(1 + z^2)^n} \tag{15.5.55}$$

Thus, the region of convergence of the series is $\left|1 + z^2\right| > 1$.

(e) Using the geometric series formula with $x \neq 0$

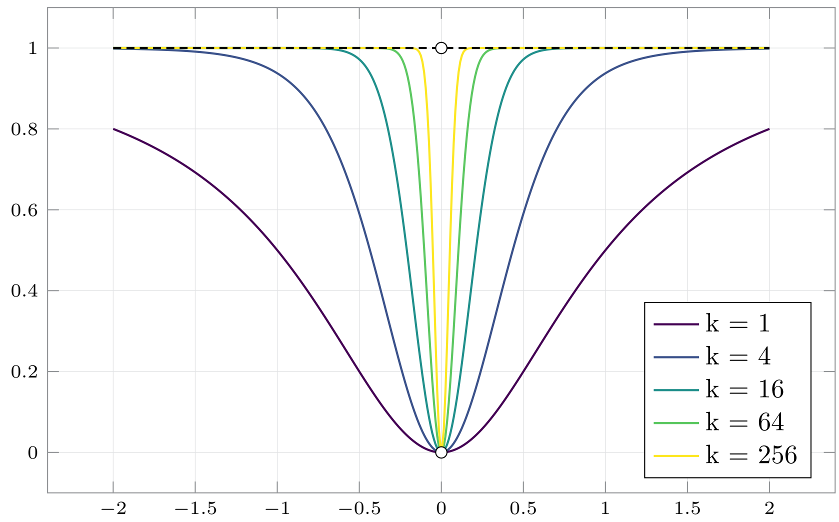
$$q = \frac{1}{1 + x^2} < 1 \qquad s = \frac{1}{1 - q} \tag{15.5.56}$$

$$\sum_{m=1}^{\infty} (1 + x^2)^{-m} = -1 + \frac{1 + x^2}{x^2} \qquad S = -x^2 + 1 + x^2 = 1 \tag{15.5.57}$$

For the special case with $x = 0,\ \ S = 0$
In order to plot the graphs, using the finite series sum of the geometric series,

$$s_n = \frac{1 - q^{n+1}}{1 - q} \qquad S_n = -x^2 + (1 + x^2)\left[1 - \frac{1}{(1 + x^2)^{n+1}}\right] \tag{15.5.58}$$

$$S_n = 1 - \frac{1}{(1 + x^2)^n} \tag{15.5.59}$$



19. From section 12.6,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t} \qquad \lambda_n = \frac{cn\pi}{L} \tag{15.5.60}$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x \qquad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{15.5.61}$$

Looking at an upper bound for $|B_n|$,

$$|B_n| = \left|\frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x\right| \tag{15.5.62}$$

$$|B_n| \leq \frac{2}{L}\int_0^L \left|f(x) \sin\left(\frac{n\pi x}{L}\right)\right| \, \mathrm{d}x \tag{15.5.63}$$

$$|B_n| \leq \frac{2}{L}\int_0^L |f(x)| \, \mathrm{d}x \tag{15.5.64}$$

Assuming $f(x)$ is finite in this interval, and this integral computes to K, it is independent of n.

$$|B_n| \leq K \qquad \forall \quad n \geq 1 \tag{15.5.65}$$

$$|u_n| \leq K \exp(-\lambda_n^2 t) \tag{15.5.66}$$

$$t \geq t_0 > 0 \qquad \Longrightarrow \qquad e^{-\lambda_n^2 t} \leq e^{-\lambda_n^2 t_0} \tag{15.5.67}$$

$$t \geq t_0 > 0 \qquad \Longrightarrow \qquad |u_n| \leq \tag{15.5.68}$$

$$|u_n| \leq K \exp(-\lambda_n^2 t_0) \tag{15.5.69}$$

This series is now term-wise bounded in absolute value by a series of constant functions, (independent of x, t). Using the Weierstrass M-test, it is uniformly convergent.

In combination with U.C., and the fact that each term of the series is continuous $\forall \quad t \geq t_0 > 0$, the

series sum $u(x,\, t)$ is also continuous $\forall \quad t \geq t_0 > 0$, and $\forall \quad x \in [0,\, L]$.
The boundary conditions are,

$$u(0,\, t) = 0 \quad \forall \quad t \geq 0 \qquad\qquad u(L,\, t) = 0 \quad \forall \quad t \geq 0 \tag{15.5.70}$$

$$u_n(0,\, t) = B_n \sin(0) \;\; \exp(-\lambda_n^2 t) = 0 \qquad u_n(L,\, t) = B_n \sin(n\pi) \;\; \exp(c^2 n^2 t) = 0 \tag{15.5.71}$$

$$\tag{15.5.72}$$

Using Theorem 2, the series sum also has to satisfy the boundary conditions since every term of the series does so already $\forall \quad t \geq t_0$.

20. Looking at the upper bound for the time derivative of u_n,

$$\frac{\partial u_n}{\partial t} = B_n \;\sin\left(\frac{n\pi x}{L}\right) (-\lambda_n^2)\; e^{-\lambda_n^2 t} \tag{15.5.73}$$

$$\left|\frac{\partial u_n}{\partial t}\right| \leq |B_n| \;\; (\lambda_n^2) \;\; \left|e^{-\lambda_n^2 t}\right| \tag{15.5.74}$$

$$\leq \lambda_n^2 \; K \; e^{-\lambda_n^2 t_0} \qquad\qquad \forall \quad t \geq t_0 > 0 \tag{15.5.75}$$

Using the ratio test on the expanded series after taking the time derivative,

$$\left|\frac{a'_{n+1}}{a'_n}\right| \leq \frac{\lambda_{n+1}^2}{\lambda_n^2} \cdot \frac{\exp(-\lambda_{n+1}^2 t_0)}{\exp(-\lambda_n^2 t_0)} \tag{15.5.76}$$

$$\left|\frac{a'_{n+1}}{a'_n}\right| \leq \frac{(n+1)^2}{n^2} \cdot \exp\left[-\frac{c\pi(2n+1)}{L}\; t_0\right] \tag{15.5.77}$$

$$\lim_{n\to\infty} \left|\frac{a'_{n+1}}{a'_n}\right| = 0 \tag{15.5.78}$$

Since $\partial_t u_n$ is bounded in absolute value by a series of constant functions independent of x, t and the ratio test is satisfied, term-wise differentiation is permissible.

$$\sum_{n=0}^{\infty} \frac{\partial u_n}{\partial t} = \frac{\partial u}{\partial t} \tag{15.5.79}$$

By the exact same logic, the series can be differentiated term-wise w.r.t. x twice, to yield

$$\sum_{n=0}^{\infty} \frac{\partial^2 u_n}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \tag{15.5.80}$$

Since each term of the series satisfies the heat equation and the series sum can be differentiated term-wise, the series sum itself also satisfies the heat equation.