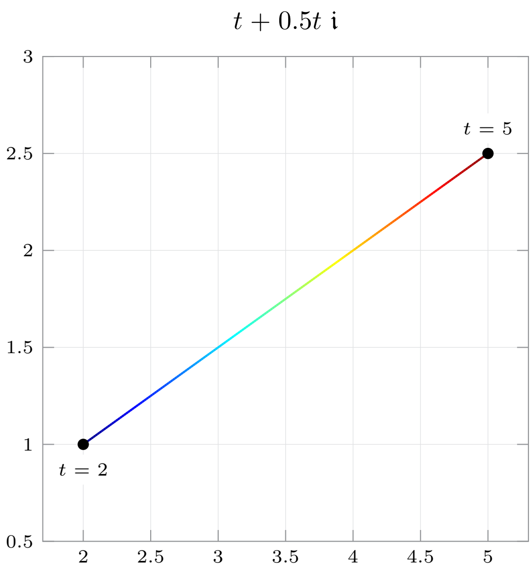
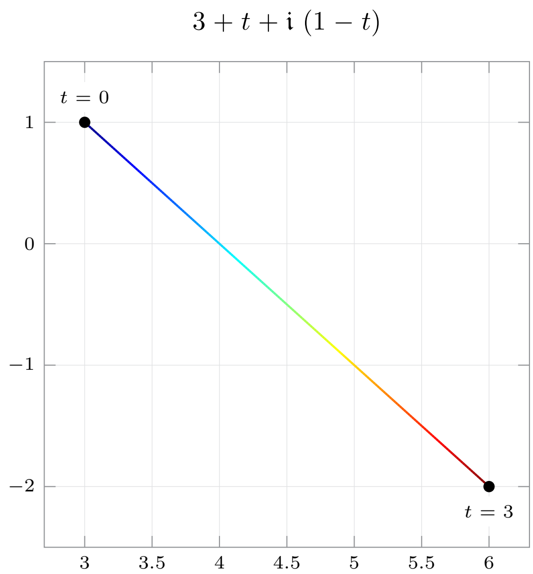
Chapter 14

Complex Integration

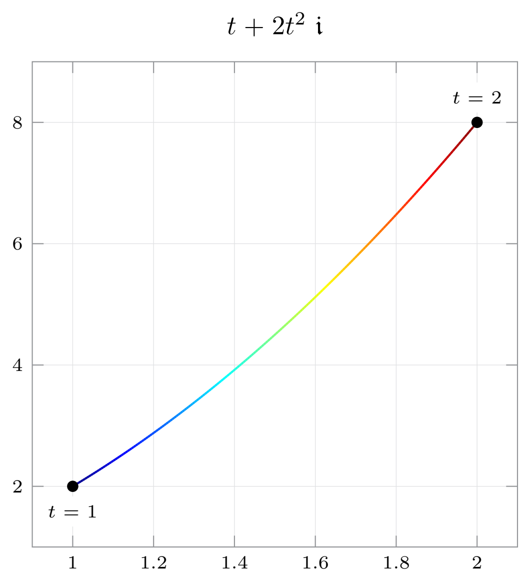
14.1 Line Integral in the Complex Plane

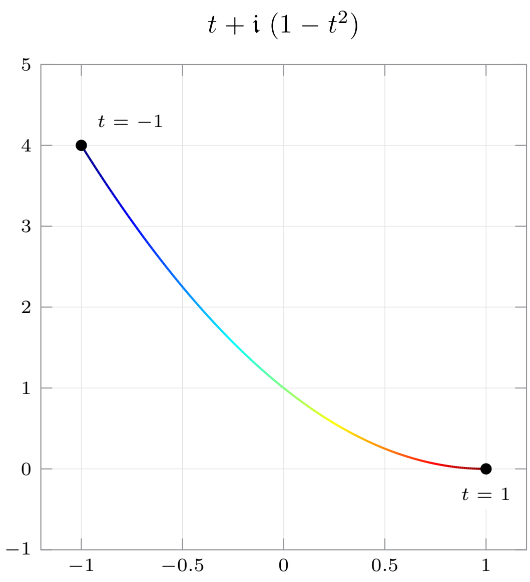
1. Plotting the path
2. Plotting the path





3. Plotting the path
4. Plotting the path



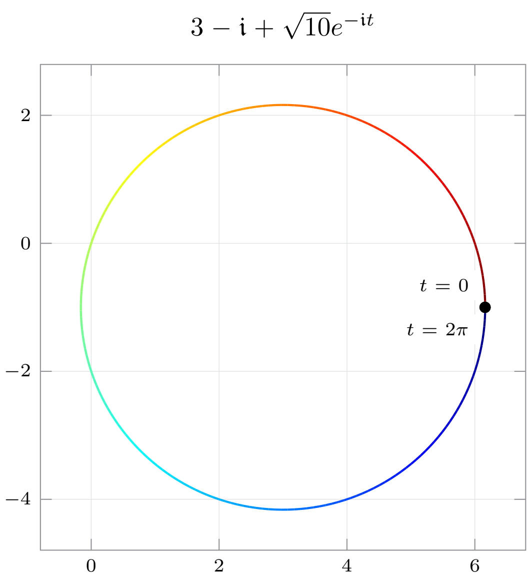


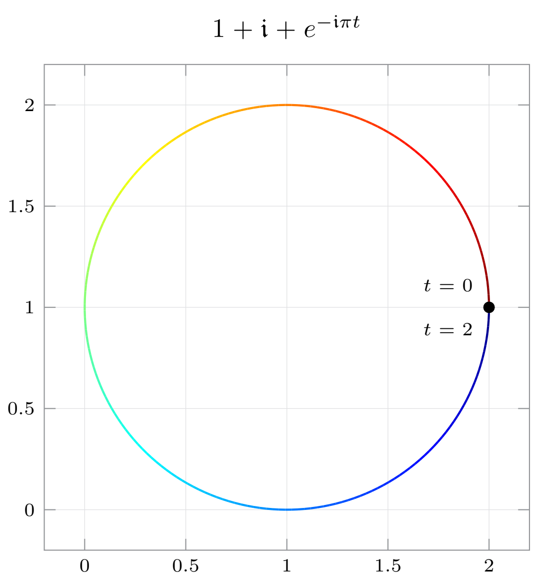
5. Plotting the path

$$z(t) = 3 - \mathrm{i} + \sqrt{10}e^{-\mathrm{i}t} \qquad = 3 + \sqrt{10}\cos t - \mathrm{i}(1 + \sqrt{10}\sin t) \tag{14.1.1}$$

6. Plotting the path

$$z(t) = 1 + \mathrm{i} + e^{-\mathrm{i}\pi t} \qquad = 1 + \cos(\pi t) + \mathrm{i}\,[1 - \sin(\pi t)] \tag{14.1.2}$$



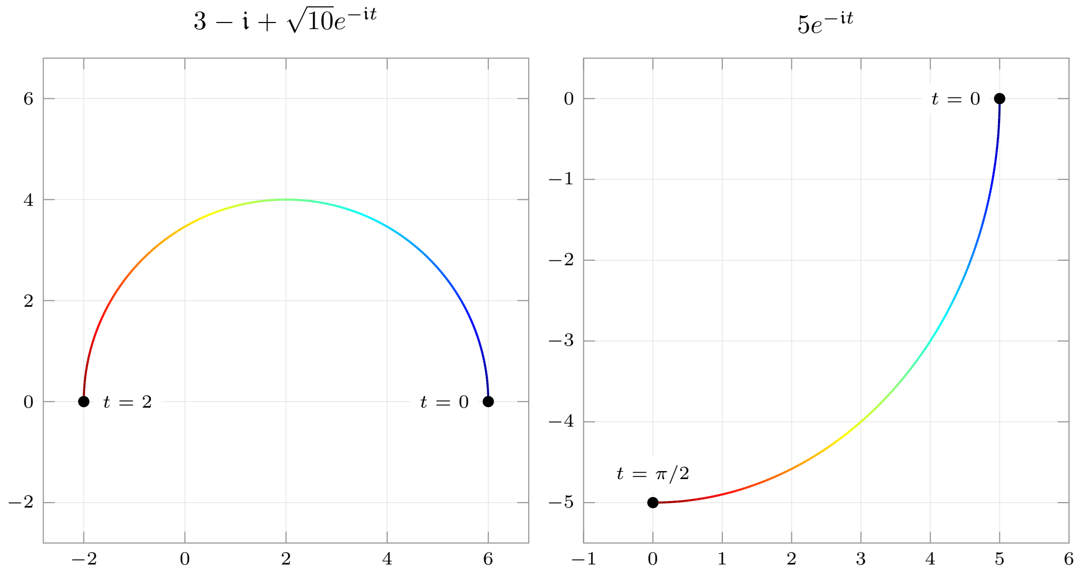


7. Plotting the path

$$z(t) = 2 + 4e^{\mathrm{i}\pi t/2} \qquad = 2 + 4\cos(\pi t/2) + \mathrm{i}\,[4\sin(\pi t/2)] \tag{14.1.3}$$

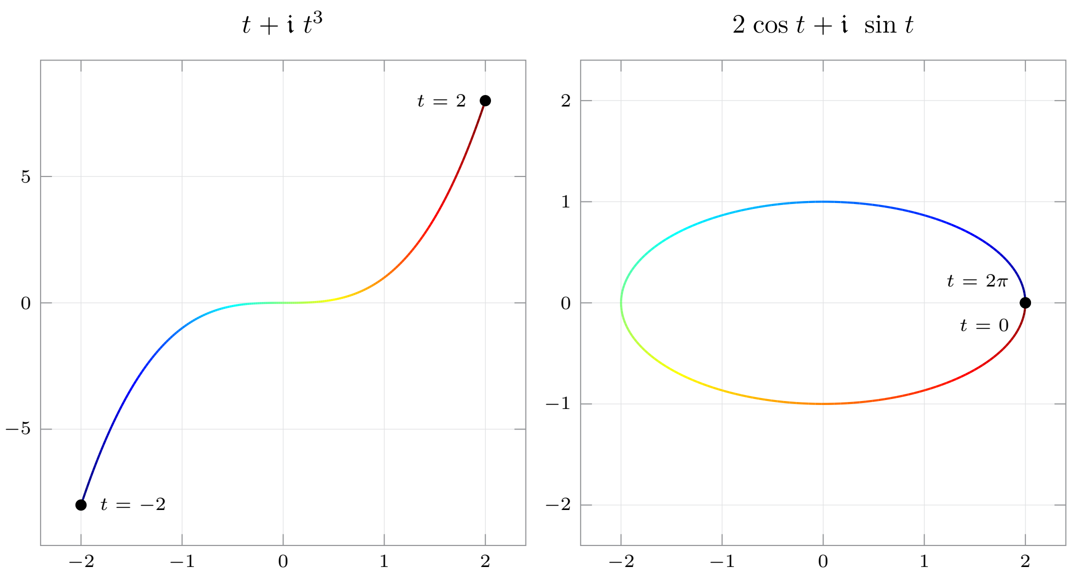
8. Plotting the path

$$z(t) = 5e^{-\mathrm{i}t} \qquad = 5\cos t - \mathrm{i}\,5\sin t \tag{14.1.4}$$



9. Plotting the path

10. Plotting the path



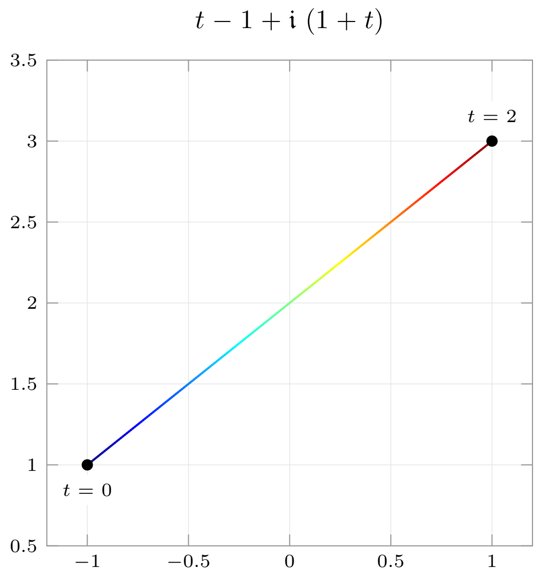
11. Parametrizing the path

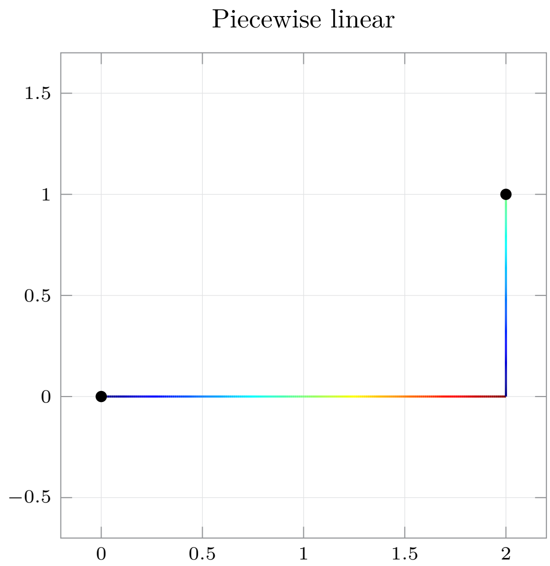
$$z(t) = -1 + t + \mathrm{i}\,[1 + t] \qquad t \in [0, 2] \tag{14.1.5}$$

12. Plotting the path

$$z_1(t) = t + \mathfrak{i}\ 0 \qquad t \in [0,\, 2] \tag{14.1.6}$$

$$z_2(t) = 2 + \mathfrak{i}\ t \qquad t \in [0,\, 1] \tag{14.1.7}$$





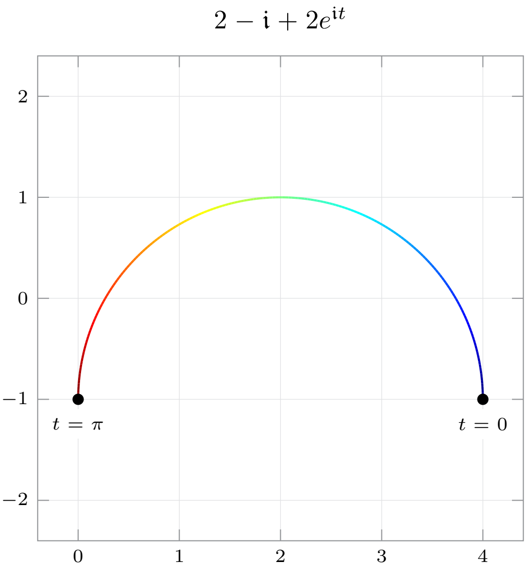
13. Parametrizing the path

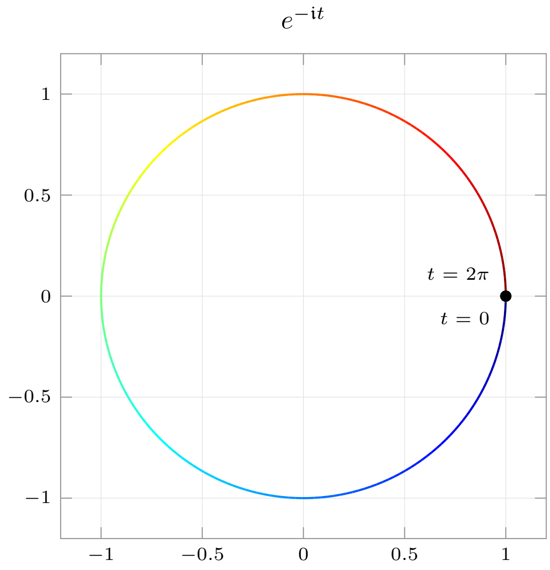
$$z(t) : |z - 2 + \mathfrak{i}| = 2 \qquad z(t) : 2 + 2\cos t + \mathfrak{i}\left[-1 + 2\sin t\right] \tag{14.1.8}$$

$$t \in [0,\, \pi] \tag{14.1.9}$$

14. Plotting the path

$$z(t) = e^{-\mathfrak{i}t} \qquad t \in [0,\, 2\pi] \tag{14.1.10}$$





15. Parametrizing the branch through $2 + 0\ \mathfrak{i}$

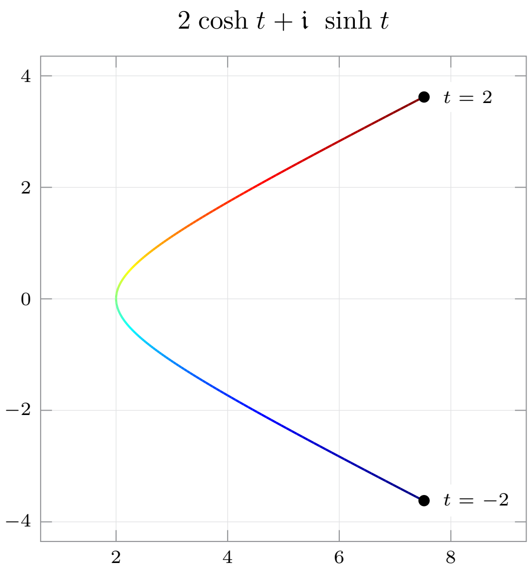
$$z(t) : x^2 - 4y^2 = 4 \qquad\qquad z(t) : 2\cosh t + \mathfrak{i}\ \sinh t \tag{14.1.11}$$

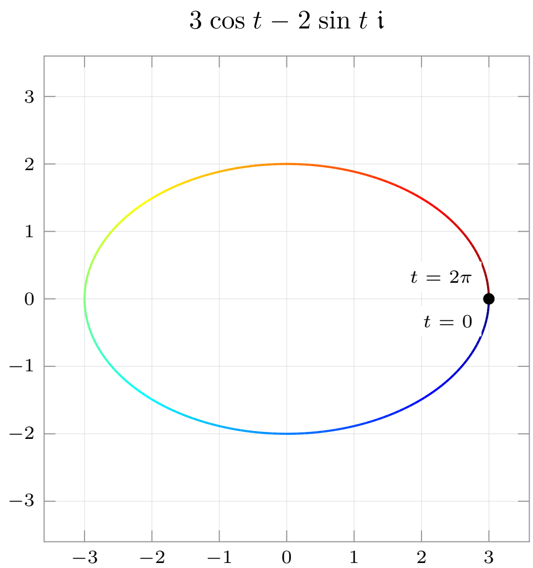
$$t \in [0, \pi] \tag{14.1.12}$$

16. Parametrizing the path

$$z(t) : 4x^2 + 9y^2 = 36 \qquad\qquad z(t) : 3\cos t - 2\sin t\ \mathfrak{i} \tag{14.1.13}$$

$$t \in [0, 2\pi] \tag{14.1.14}$$



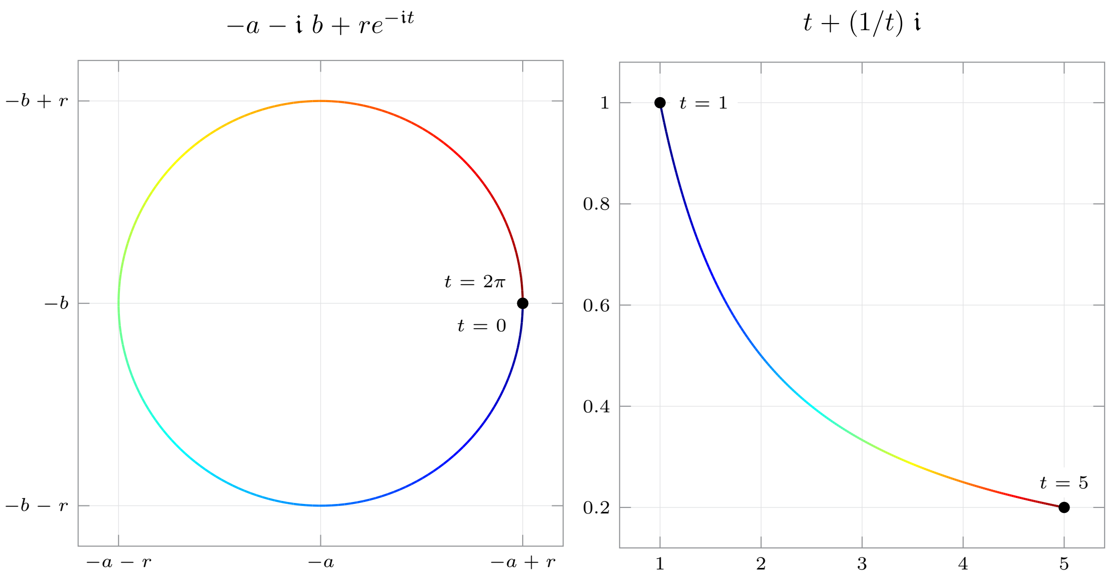


17. Plotting the path

$$z(t) = -a - \mathfrak{i}\, b + re^{-\mathfrak{i}t} \qquad t \in [0,\, 2\pi] \tag{14.1.15}$$

18. Plotting the path

$$z(t) = t + \frac{1}{t}\, \mathfrak{i} \qquad t \in [1,\, 5] \tag{14.1.16}$$



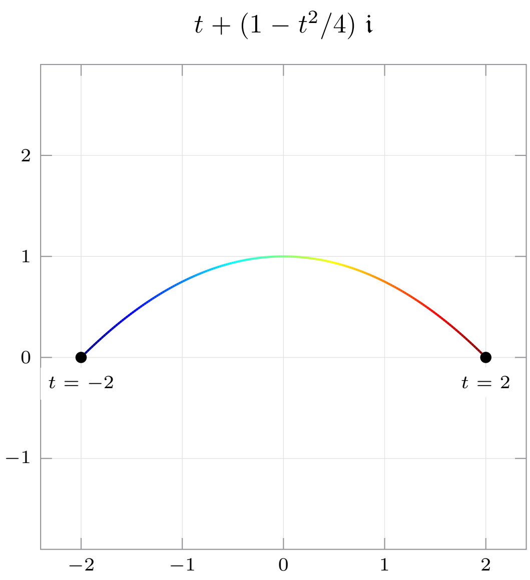
19. Plotting the path

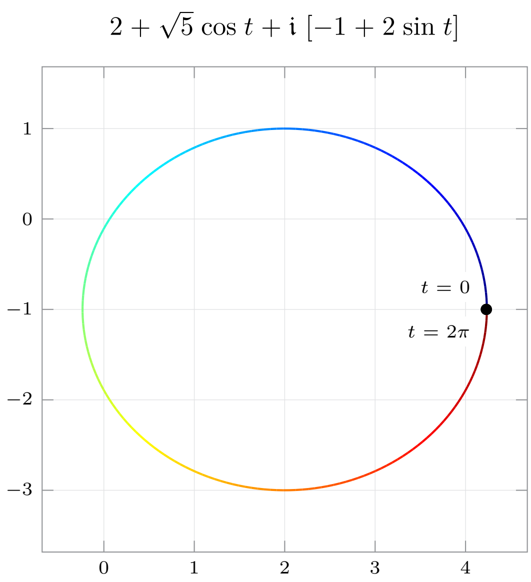
$$z(t) = t + (1 - t^2/4)\, \mathfrak{i} \qquad t \in [-2,\, 2] \tag{14.1.17}$$

20. Plotting the path

$$C : 4(x-2)^2 + 5(y+1)^2 = 20 \qquad z(t) = 2 + \sqrt{5}\cos t + \mathfrak{i}\left[-1 + 2\sin t\right] \tag{14.1.18}$$

$$t \in [0,\, 2\pi] \tag{14.1.19}$$





21. The function is not analytic, so the first method cannot be used.

$$z(t) = t + \mathrm{i}\ t \qquad t \in [1, 3] \tag{14.1.20}$$

$$f(z) = \mathrm{Re}\ (z) = t \qquad \dot{z}(t) = 1 + \mathrm{i} \tag{14.1.21}$$

$$I = \int_1^3 (t) \cdot (1 + \mathrm{i})\ \ \mathrm{d}t \qquad = (1 + \mathrm{i}) \cdot \left[\frac{t^2}{2}\right]_1^3 \tag{14.1.22}$$

$$I = 4 + 4\ \mathrm{i} \tag{14.1.23}$$

22. The function is not analytic, so the first method cannot be used.

$$z(t) = t + \mathrm{i}\ \left[1 + \frac{(t-1)^2}{2}\right] \qquad t \in [1, 3] \tag{14.1.24}$$

$$f(z) = \mathrm{Re}\ (z) = t \qquad \dot{z}(t) = 1 + \mathrm{i}\ (t - 1) \tag{14.1.25}$$

$$I = \int_1^3 (t) \cdot \left[1 + \mathrm{i}\ (t-1)\right]\ \ \mathrm{d}t \qquad = \left[\frac{t^2}{2}\right]_1^3 + \mathrm{i}\ \left[\frac{t^3}{3} - \frac{t^2}{2}\right]_1^3 \tag{14.1.26}$$

$$I = 4 + \frac{14}{3}\ \mathrm{i} \tag{14.1.27}$$

23. The function is analytic, so the first method can be used.

$$I = \int_{z_a}^{z_b} e^z\ \ \mathrm{d}z \qquad = \left[e^z\right]_{z_a}^{z_b} \tag{14.1.28}$$

$$I = 1 - (-1) = 2 \tag{14.1.29}$$

24. The function is analytic, so the first method can be used.

$$I = \int_{z_a}^{z_b} \cos(2z) \ \mathrm{d}z = \left[\frac{\sin(2z)}{2}\right]_{z_a}^{z_b} \tag{14.1.30}$$

$$I = \frac{\sin(2\pi\mathrm{i}) - \sin(-2\pi\mathrm{i})}{2} = \sin(2\pi\mathrm{i}) = \mathrm{i} \ \sinh(2\pi) \tag{14.1.31}$$

25. The function is analytic, so the first method can be used.

$$I = \int_{z_a}^{z_b} z \ e^{z^2} \ \mathrm{d}z = \left[\frac{e^{z^2}}{2}\right]_{z_a}^{z_b} \tag{14.1.32}$$

$$I = \frac{e^{-1} - e}{2} = -\sinh(1) \tag{14.1.33}$$

26. The function is not analytic at the origin, so the first method cannot be used.

$$z(t) = \cos t + \mathrm{i} \ \sin t \qquad t \in [0, 2\pi] \tag{14.1.34}$$

$$f(z) = z + \frac{1}{z} \qquad f[z(t)] = 2 \cos t \tag{14.1.35}$$

$$\dot{z}(t) = -\sin t + \mathrm{i} \ \cos t \tag{14.1.36}$$

$$I = \int_0^{2\pi} (2 \cos t) \cdot \Big[-\sin t + \mathrm{i} \ \cos t \Big] \ \mathrm{d}t = \mathrm{i} \left[t + \frac{\sin(2t)}{2}\right]_0^{2\pi} \tag{14.1.37}$$

$$I = 2\pi \ \mathrm{i} \tag{14.1.38}$$

27. The function is analytic, so the first method can be used.

$$I = \int_{z_a}^{z_b} \sec^2 z \ \mathrm{d}z = \Big[\tan z\Big]_{z_a}^{z_b} \tag{14.1.39}$$

$$I = \tan(\mathrm{i}\pi/4) - 1 = -1 + \mathrm{i} \ \tanh(\pi/4) \tag{14.1.40}$$

28. The function is not analytic in the domain, so the first method cannot be used.
Using the result from the text on $(z - z_0)^m$,

$$I = 10\pi \ \mathrm{i} \tag{14.1.41}$$

29. The function is not analytic, so the first method cannot be used.

$$I_1 = 0 \tag{14.1.42}$$

$$z_2(t) = (1-t) + t\,\mathrm{i} \qquad t \in [0, 1] \tag{14.1.43}$$

$$I_2 = \int_0^1 (-1+\mathrm{i}) \cdot \left(2t - 2t^2\right)\, \mathrm{d}t = \frac{1}{3}\,(-1+\mathrm{i}) \qquad (1, 0) \to (0, 1) \tag{14.1.44}$$

$$I_3 = 0 \tag{14.1.45}$$

$$I = \frac{1}{3}\,(-1+\mathrm{i}) \tag{14.1.46}$$

30. The function is not analytic, so the first method cannot be used.

$$I_1 = \int_0^1 (\mathrm{i}) \cdot \left(-t^2\right)\, \mathrm{d}t = \frac{-1}{3}\,\mathrm{i} \qquad (0, 0) \to (0, 1) \tag{14.1.47}$$

$$I_2 = \int_0^1 (1) \cdot \left(t^2 - 1\right)\, \mathrm{d}t = \frac{-2}{3} \qquad (0, 1) \to (1, 1) \tag{14.1.48}$$

$$I_3 = \int_1^0 (\mathrm{i}) \cdot \left(1 - t^2\right)\, \mathrm{d}t = \frac{-2}{3}\,\mathrm{i} \qquad (1, 1) \to (1, 0) \tag{14.1.49}$$

$$I_4 = \int_1^0 (1) \cdot \left(t^2\right)\, \mathrm{d}t = \frac{-1}{3} \qquad (1, 0) \to (0, 0) \tag{14.1.50}$$

$$I = -1 - \mathrm{i} \tag{14.1.51}$$

31. Program written in `sympy`

32. Performing both integrations,

$$\int_a^b z^2\, \mathrm{d}z = \left[\frac{z^3}{3}\right]_a^b = \frac{(1+\mathrm{i})^3 - (-1-\mathrm{i})^3}{3} \tag{14.1.52}$$

$$= \frac{2}{3}\,(4 + 2\mathrm{i}) \tag{14.1.53}$$

$$\int_b^a z^2\, \mathrm{d}z = \left[\frac{z^3}{3}\right]_b^a = \frac{(-1-\mathrm{i})^3 - (1+\mathrm{i})^3}{3} \tag{14.1.54}$$

$$= \frac{-2}{3}\,(4 + 2\mathrm{i}) \tag{14.1.55}$$

33. Performing both integrations,

$$f[z(t)] = \frac{\bar{z}}{|z|^2} = \cos t - \mathfrak{i}\ \sin t \tag{14.1.56}$$

$$C_1 : \cos t + \mathfrak{i}\ \sin t \qquad \theta \in [0, \pi] \tag{14.1.57}$$

$$I_1 = \int_0^\pi (\cos t - \mathfrak{i}\ \sin t)(-\sin t + \mathfrak{i}\ \cos t)\ \mathrm{d}t = \pi\ \mathfrak{i} \tag{14.1.58}$$

$$C_2 : \cos t + \mathfrak{i}\ \sin t \qquad \theta \in [\pi, 2\pi] \tag{14.1.59}$$

$$I_2 = \int_\pi^\pi (\cos t - \mathfrak{i}\ \sin t)(-\sin t + \mathfrak{i}\ \cos t)\ \mathrm{d}t = \pi\ \mathfrak{i} \tag{14.1.60}$$

Thus, the standard result $I_1 + I_2 = 2\pi\ \mathfrak{i}$ matches the integration by partition of paths.

34. Integration methods,

(a) Refer notes. TBC.

(b) From the first method,

$$I_1 = \int_a^b z^4\ \mathrm{d}z \qquad = \left[\frac{z^5}{5}\right]_{-2\mathfrak{i}}^{2\mathfrak{i}} = \frac{64}{5}\ \mathfrak{i} \tag{14.1.61}$$

From the second method,

$$z(t) : 2\cos t + \mathfrak{i}\ 2\sin t \qquad t \in [-\pi/2, \pi/2] \tag{14.1.62}$$

$$f[z(t)] = 16\cos(4t) + \mathfrak{i}\ 16\sin(4t) \qquad \dot{z}(t) = -2\sin t + \mathfrak{i}\ 2\cos t \tag{14.1.63}$$

$$f[z(t)] \cdot \dot{z}(t) = -32\ [\sin(5t) - \mathfrak{i}\cos(5t)] \qquad I_2 = \int_{-\pi/2}^{\pi/2} f[z(t)]\ \dot{z}(t)\ \ \mathrm{d}t \tag{14.1.64}$$

$$= \frac{32}{5}\left[\cos(5t) + \mathfrak{i}\ \sin(5t)\right]_{-\pi/2}^{\pi/2} \qquad = \frac{64}{5}\ \mathfrak{i} \tag{14.1.65}$$

Both methods match, but the first method is much easier.

$$I_1 = \int_a^b e^{2z}\ \mathrm{d}z \qquad = \left[\frac{e^{2z}}{2}\right]_0^{1+2\mathfrak{i}} = \frac{e^{2+4\mathfrak{i}} - e^0}{2} \tag{14.1.66}$$

From the second method,

$$z(t) : t + \mathfrak{i}\ (2t) \qquad t \in [0, 1] \tag{14.1.67}$$

$$f[z(t)] = e^{(2+4\mathfrak{i})t} \qquad \dot{z}(t) = 1 + 2\ \mathfrak{i} \tag{14.1.68}$$

$$I_2 = \int_0^1 f[z(t)]\ \dot{z}(t)\ \ \mathrm{d}t \qquad = \frac{e^{2+4\mathfrak{i}} - e^0}{2} \tag{14.1.69}$$

Both methods match, but the first method is much easier.

(c) To show the path dependence of a non-analytic function,

$$z(t) : t + \mathfrak{i}\ (a \sin t) \qquad t \in [0, \pi] \tag{14.1.70}$$

$$f[z(t)] = \text{Re}\ (z) = t \qquad \dot{z}(t) = 1 + \mathfrak{i}\ (a \cos t) \tag{14.1.71}$$

$$I_2 = \int_0^\pi f[z(t)]\ \dot{z}(t)\ \ \mathrm{d}t \qquad = \int_0^\pi \Big[t + \mathfrak{i}\ (at \cos t)\Big]\ \ \mathrm{d}t \tag{14.1.72}$$

$$= \left[\frac{t^2}{2} + at \sin t + a \cos t\right]_0^\pi \qquad = \frac{\pi^2}{2} - 2a \tag{14.1.73}$$

Using another non-analytic function,

$$z(t) : t + \mathfrak{i}\ (a \sin t) \qquad t \in [0, \pi] \tag{14.1.74}$$

$$f[z(t)] = \text{Re}\ (z^2) \qquad = t^2 - \frac{a^2[1 - \cos(2t)]}{2} \tag{14.1.75}$$

$$\dot{z}(t) = 1 + \mathfrak{i}\ (a \cos t) \tag{14.1.76}$$

$$I_2 = \int_0^\pi f[z(t)] \tag{14.1.77}$$

Performing the integrations on the real and imaginary part,

$$I_{2r} = \int_0^\pi \left[t^2 - \frac{a^2}{2} + \frac{a^2 \cos(2t)}{2}\right]\ \mathrm{d}t = \frac{\pi^3}{3} - \frac{a^2\pi}{2} \tag{14.1.78}$$

$$I_{2m} = \int_0^\pi \left[at^2 \cos t - \frac{a^3}{2}\ \ \cos t + \frac{a^3[\cos(3t) + \cos t]}{4}\right]\ \mathrm{d}t \tag{14.1.79}$$

$$= a\left[(t^2 - 2) \sin t + 2t \cos t\right]_0^\pi + 0 + 0 = -2at \tag{14.1.80}$$

Using $f(z) = z$ which is analytic,

$$z(t) : t + \mathrm{i}\ (a \sin t) \qquad t \in [0, \pi] \tag{14.1.81}$$

$$f[z(t)] = t + \mathrm{i}\ (a \sin t) \qquad \dot{z}(t) = 1 + \mathrm{i}\ (a \cos t) \tag{14.1.82}$$

$$I_2 = \int_0^\pi f[z(t)]\ \dot{z}(t)\ \ \mathrm{d}t \tag{14.1.83}$$

$$= \int_0^\pi \left[t - \frac{a^2 \sin(2t)}{2} + a\mathrm{i}\ (t \cos t + \sin t)\right]\ \ \mathrm{d}t \tag{14.1.84}$$

$$= \left[\frac{t^2}{2} + \frac{a^2 \cos(2t)}{4} + a\mathrm{i}\ (t \sin t)\right]_0^\pi = \frac{\pi^2}{4} \tag{14.1.85}$$

The integrals dependent on the path cancel, which makes the result path independent.

(d) Performing the integration on $f(z) = \text{Re}\ z$,

$$z(t) : a \cos t + \mathrm{i}\ \ \sin t \qquad t \in [-\pi/2, \pi/2] \tag{14.1.86}$$

$$f[z(t)] = \text{Re}\ (z) = a \cos t \qquad \dot{z}(t) = -a \sin t + \mathrm{i}\ \ \cos t \tag{14.1.87}$$

$$I_2 = \int_{-\pi/2}^{\pi/2} f[z(t)]\ \dot{z}(t)\ \ \mathrm{d}t \tag{14.1.88}$$

$$= \int_{-\pi/2}^{\pi/2} \left[- \frac{a^2 \sin(2t)}{2} + \mathrm{i}\ a \cos^2 t\right]\ \ \mathrm{d}t \tag{14.1.89}$$

$$= \left[\frac{a^2 \cos(2t)}{4} + a\mathrm{i}\ \frac{2t + \sin(2t)}{4}\right]_{-\pi/2}^{\pi/2} \qquad = \frac{a\pi}{2}\ \mathrm{i} \tag{14.1.90}$$

To find the bound,

$$M = 3 \qquad L = 2\sqrt{2} \tag{14.1.91}$$

From the fact that $\text{Re}\ z \leq 3$ and the length of the path L. This happens to be a very liberal upper bound.

14.2 Cauchy's Integral Theorem

1. Verifying, using path deformation to make a unit circle,

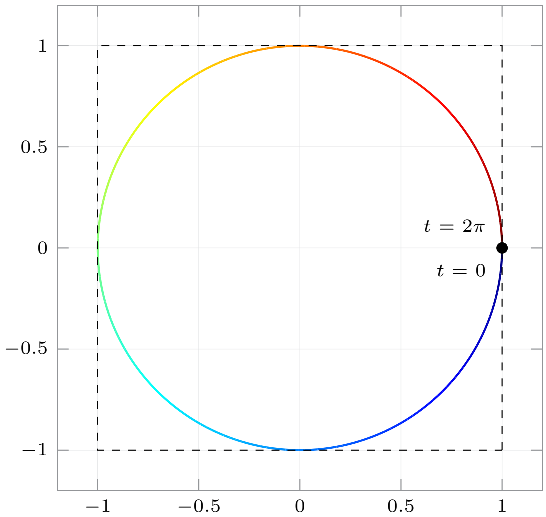
$$z(t) = e^{\mathrm{i}t} \qquad t \in [0, 2\pi] \tag{14.2.1}$$

$$f[z(t)] = e^{2\mathrm{i}t} \qquad \dot{z}(t) = \mathrm{i}\, e^{\mathrm{i}t} \tag{14.2.2}$$

$$I = \int_0^{2\pi} e^{2\mathrm{i}t}\ (\mathrm{i}\ e^{\mathrm{i}t})\ \mathrm{d}t = \left[\frac{e^{3\mathrm{i}t}}{3}\right]_0^{2\pi} \tag{14.2.3}$$

$$I = 0 \tag{14.2.4}$$

Path is deformed to unit circle



2. Checking the points where the functions are not analytic,

(a) $f(z) = 1/z$ is not analytic at $z = 0$. So any domain that does not contain the origin is usable.

(b) The function is,

$$g(z) = \frac{\exp(1/z^2)}{z^2 + 16} \tag{14.2.5}$$

This function is not analytic at $z = 0, 4\mathrm{i}, -4\mathrm{i}$, so any domain excluding these points is usable.

3. Yes. From example 6, deformation of path does not change the value of the integral, since the square and the unit circle enclose points where the function is analytic.
The only point where $f(z) = z^{-2}$ is not analytic is at the origin.

4. No, since this violates the theorem of path deformation. The values of both integrals being different makes it so that the function is not analytic everywhere in between the two paths.

5. For the given function,

$$f(z) = \frac{\cos(z^2)}{z^4 + 1} \qquad S = \{\pm 1, \pm \mathrm{i}\} \tag{14.2.6}$$

The points in set S are points where $f(z)$ is not analytic. So, the domain of analyticity is 5 connected.

6. Integrating over the first path,

$$z(t) = (1 + \mathrm{i})\ t \qquad t \in [0, 1] \tag{14.2.7}$$

$$\dot{z}(t) = (1 + \mathrm{i}) \qquad f[z(t)] = \exp[(1 + \mathrm{i})t] \tag{14.2.8}$$

$$I_1 = \int_0^1 (1 + \mathrm{i})\ e^{(1+\mathrm{i})t}\ \ \mathrm{d}t \qquad = e^{1+\mathrm{i}} - e^0 \tag{14.2.9}$$

Integrating over the second path,

$$I_2 = \int_0^1 (1)\ e^{t}\ \ \mathrm{d}t \qquad = e^{1} - e^0 \tag{14.2.10}$$

$$I_3 = \int_0^1 (\mathrm{i})\ e^{1+\mathrm{i}t}\ \ \mathrm{d}t \qquad = e^{1+\mathrm{i}} - e^1 \tag{14.2.11}$$

$$\tag{14.2.12}$$

Since $I_2 + I_3 = I_1$, path independence is verified.

7. The distance between $z_1 = 0 + 2\mathrm{i}$ and $z_2 = 2 + 0\mathrm{i}$ is equal to $2\sqrt{2}$. This number is larger than 2 but less than 3.

So Yes and No.

8. Cauchy's integral theorem,

(a) Refer notes for facts. Examples TBC.

(b) Resolving into partial fractions,

$$f(z) = \frac{8z + 12\ \mathrm{i}}{4z^2 + 1} \qquad = \frac{a + b\ i}{2z + \mathrm{i}} + \frac{c + d\ \mathrm{i}}{2z - \mathrm{i}} \tag{14.2.13}$$

$$4 + 0\mathrm{i} = (a + c) + (b + d)\ \mathrm{i} \qquad 12\ \mathrm{i} = (b - d) + \mathrm{i}\ (c - a) \tag{14.2.14}$$

$$f(z) = \frac{-2}{z + 0.5\mathrm{i}} + \frac{4}{z - 0.5\mathrm{i}} \tag{14.2.15}$$

Since the unit circle encloses both the points, $(0, -0.5)$ and $(0, 0.5)$, the result is,

$$I = I_1 + I_2 = -2(2\pi\ \mathrm{i}) + 4(2\pi\ \mathrm{i}) = 4\pi\ \mathrm{i} \tag{14.2.16}$$

Resolving into partial fractions,

$$g(z) = \frac{z + 1}{z^2 + 2z} \qquad = \frac{a + b\ i}{z + 2} + \frac{c + d\ \mathrm{i}}{z} \tag{14.2.17}$$

$$1 + 0\mathrm{i} = (a + c) + (b + d)\ \mathrm{i} \qquad 1 + 0\mathrm{i} = (2c) + \mathrm{i}\ (2d) \tag{14.2.18}$$

$$g(z) = \frac{0.5}{z + 2} + \frac{0.5}{z} \tag{14.2.19}$$

Since the unit circle encloses both the points, (0, 0) but not (−2, 0), the result is,

$$I = 0 + I_2 = 0 + 0.5(2\pi\ \mathrm{i}) = \pi\ \mathrm{i} \tag{14.2.20}$$

(c) From the earlier team project, the deformation of paths led to no change in the integral only when the function was analytic over the region of path deformation, in agreement with the theorem.
For the new path, an analytic function yields,

$$z(t) = t + \mathrm{i}\ at(1-t) \qquad \dot{z}(t) = 1 + \mathrm{i}\ a(1-2t) \tag{14.2.21}$$

$$f(z) = z \tag{14.2.22}$$

$$I_1 = \int_0^1 t - a^2(t - 3t^2 + 2t^3)\ \mathrm{d}t \qquad = \left[\frac{t^2}{2} - \frac{a^2(6t^2 - 12t^3 + 6t^4)}{12}\right]_0^1 \tag{14.2.23}$$

$$= \frac{1}{2} \tag{14.2.24}$$

$$I_2 = \int_0^1 a(2t - 3t^2)\ \mathrm{d}t \qquad = \left[a(t^2 - t^3)\right]_0^1 = 0 \tag{14.2.25}$$

Whereas a non-analytic function yields,

$$z(t) = t + \mathrm{i}\ at(1-t) \qquad \dot{z}(t) = 1 + \mathrm{i}\ a(1-2t) \tag{14.2.26}$$

$$f(z) = \mathrm{Re}\ z = t \tag{14.2.27}$$

$$I_1 = \int_0^1 t + a\mathrm{i}\ (t - 2t^2)\ \mathrm{d}t \qquad = \left[\frac{t^2}{2} + a\mathrm{i}\ \frac{(3t^2 - 4t^3)}{6}\right]_0^1 \tag{14.2.28}$$

$$= \frac{1}{2} - \frac{a}{6}\ \mathrm{i} \tag{14.2.29}$$

Notice the dependence on a for the non-analytic function.

9. Cauchy's theorem is applicable, since the function is entire.

$$I = \oint_C \exp(-z^2)\ \mathrm{d}z = 0 \tag{14.2.30}$$

10. Cauchy's theorem is applicable, since the function only has poles outside the unit circle.

$$\tan(z/4) = 0 \qquad \Longrightarrow \qquad z = 4(n\pi + \pi/2) \tag{14.2.31}$$

$$S = \{2\pi + 4n\pi\} \qquad = \{-2\pi, 2\pi, 6\pi, \dots\} \tag{14.2.32}$$

$$I = \oint_C \exp(-z^2)\ \mathrm{d}z = 0 \tag{14.2.33}$$

11. Cauchy's theorem is not applicable.

$$2z - 1 = 0 \qquad \Longrightarrow \qquad z = 0.5 + 0\mathrm{i} \tag{14.2.34}$$

$$I = \oint_C \frac{0.5}{z - 0.5} \, \mathrm{d}z \qquad = \mathrm{i}\, \pi \tag{14.2.35}$$

12. Cauchy's theorem is not applicable. Using the standard result for the integral of z^n around the unit circle,

$$\bar{z}^3 = 0 \qquad \Longrightarrow \qquad z = 0 + 0\mathrm{i} \tag{14.2.36}$$

$$I = \oint_C \frac{|z|^6}{z^3} \, \mathrm{d}z \qquad = 0 \tag{14.2.37}$$

13. Cauchy's theorem is applicable, since $(1.1)^{1/4}$ has a modulus greater than one.

$$\bar{z}^4 = 1.1 \qquad \Longrightarrow \qquad z = (1.1)^{1/4}\, e^{\mathrm{i}k\pi/2} \tag{14.2.38}$$

$$I = \oint_C \frac{1}{z^4 - 1.1} \, \mathrm{d}z \qquad = 0 \tag{14.2.39}$$

14. Cauchy's theorem is not applicable. Using the standard result for the integral of z^n around the unit circle,

$$\bar{z} = 0 \qquad \Longrightarrow \qquad z = 0 + 0\,\mathrm{i} \tag{14.2.40}$$

$$I = \oint_C \frac{z}{|z|^2} \, \mathrm{d}z \qquad = 0 \tag{14.2.41}$$

15. Cauchy's theorem is not applicable, since the function is not analytic anywhere.

$$z(t) = \cos t + \mathrm{i}\, \sin t \qquad t \in [0, 2\pi] \tag{14.2.42}$$

$$I = \oint_C f(z) \, \mathrm{d}z \qquad = \int_0^{2\pi} (\sin t)\, (-\sin t + \mathrm{i}\, \cos t) \, \mathrm{d}t \tag{14.2.43}$$

$$= \int_0^{2\pi} \frac{[\cos(2t) - 1]}{2} \, \mathrm{d}t \qquad = -\pi \tag{14.2.44}$$

16. Cauchy's theorem is not applicable. Using the standard result for the integral of z^n around the unit circle,

$$\pi z - 1 = 0 \qquad \Longrightarrow \qquad z = \frac{1}{\pi} + 0\,\mathrm{i} \tag{14.2.45}$$

$$I = \oint_C \frac{(1/\pi)}{z - (1/\pi)} \, \mathrm{d}z \qquad = 2\mathrm{i} \tag{14.2.46}$$

17. Cauchy's theorem is not applicable. Using the standard result for the integral of z^n around the unit circle,

$$|z|^2 = 0 \qquad \Longrightarrow \qquad z = 0 + 0\,\mathrm{i} \tag{14.2.47}$$

$$z(t) = \cos t + \mathrm{i}\ \sin t \qquad t \in [0,\, 2\pi] \tag{14.2.48}$$

$$f[z(t)] = 1 \qquad I = \oint_C \mathrm{i}\ e^{\mathrm{i}t}\ \mathrm{d}t = 0 \tag{14.2.49}$$

18. Cauchy's theorem is not applicable. Using the standard result for the integral of z^n around the unit circle,

$$4z - 3 = 0 \qquad \Longrightarrow \qquad z = \frac{3}{4} + 0\,\mathrm{i} \tag{14.2.50}$$

$$I = \oint_C \frac{0.25}{z - 0.75}\ \mathrm{d}z \qquad = \frac{\pi}{2}\ \mathrm{i} \tag{14.2.51}$$

19. Cauchy's theorem is applicable.

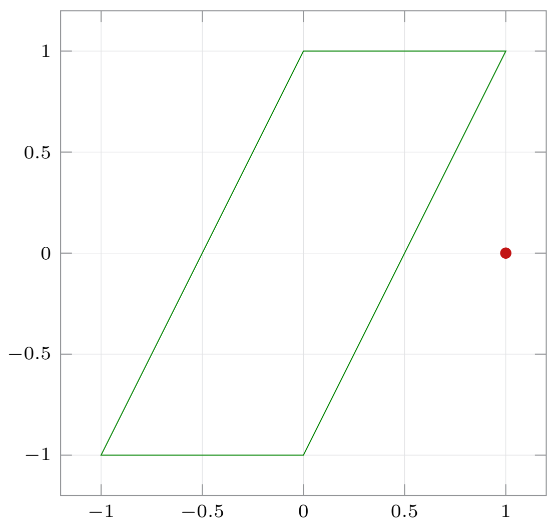
$$\cot(z) = 0 \qquad \Longrightarrow \qquad z = n\pi \tag{14.2.52}$$

$$I = \oint_C z^3\ \cot(z)\ \mathrm{d}z \qquad = 0 \tag{14.2.53}$$

20. Cauchy's theorem is applicable.

$$\mathrm{Ln}(1 - z) = 0 \qquad \Longrightarrow \qquad z = 1 + 0\,\mathrm{i} \tag{14.2.54}$$

$$I = \oint_C \mathrm{Ln}(1 - z)\ \mathrm{d}z \qquad = 0 \tag{14.2.55}$$



21. Cauchy's theorem is not applicable. Using the standard result for the integral of z^n around the unit circle,

$$z - 3\mathrm{i} = 0 \qquad \Longrightarrow \qquad z = 0 + 3\,\mathrm{i} \tag{14.2.56}$$

$$z(t) = \pi\ e^{\mathrm{i}t} \qquad t \in [0,\, 2\pi] \tag{14.2.57}$$

$$I = \oint_C \frac{1}{z - 3\mathrm{i}}\ \mathrm{d}z \qquad = 2\pi\ \mathrm{i} \tag{14.2.58}$$

22. Cauchy's theorem is not applicable. Using the standard result for the integral of z^n around the unit circle,

$$f[z(t)] = \cos t \qquad z(t) = e^{\mathrm{i}t} \qquad t \in [0,\, \pi] \tag{14.2.59}$$

$$\dot{z}(t) = \mathrm{i}e^{\mathrm{i}t} \tag{14.2.60}$$

$$I_1 = \int_0^{\pi} (\mathrm{i}\cos t - \sin t)\ \ \cos t\ \ \mathrm{d}t \qquad = \frac{\pi}{2}\ \mathrm{i} \tag{14.2.61}$$

For the straight line path,

$$f[z(t)] = t \qquad t \in [-1,\, 1] \tag{14.2.62}$$

$$z(t) = t + 0\ \mathrm{i} \tag{14.2.63}$$

$$I_2 = \int_{-1}^{1} (1)(t)\ \ \mathrm{d}t = 0 \qquad I = I_1 + I_2 = 0 \tag{14.2.64}$$

23. Cauchy's theorem is not applicable

Resolving using partial fractions,

$$g(z) = \frac{2z - 1}{z(z-1)} \qquad = \frac{a}{z} + \frac{b}{z - 1} \tag{14.2.65}$$

$$a = 1 \qquad b = 1 \tag{14.2.66}$$

$$\tag{14.2.67}$$

Since the path encloses both points $(0,\, 0)$ and $(1,\, 0)$, the standard result for z^n gives,

$$I = \oint_C \left[\frac{1}{z} + \frac{1}{z-1}\right]\ \mathrm{d}z = 4\pi\ \mathrm{i} \tag{14.2.68}$$

24. Cauchy's theorem is not applicable.

Let the path be split into two parts, the clockwise circle centered on $(-1,\, 0)$ called C_1 and the counterclockwise circle centered on $(1,\, 0)$ called C_2.

$$g(z) = \frac{1}{z^2 - 1} \qquad = \frac{0.5}{z-1} - \frac{0.5}{z+1} \tag{14.2.69}$$

$$I_1 = \oint_C \frac{0.5}{z-1} \qquad = \oint_{C_1} \frac{0.5}{z-1} + \oint_{C_2} \frac{0.5}{z-1} \tag{14.2.70}$$

$$= 0 + \pi\,\mathrm{i} \tag{14.2.71}$$

$$I_2 = \oint_C \frac{0.5}{z+1} \qquad = \oint_{C_1} \frac{0.5}{z+1} + \oint_{C_2} \frac{0.5}{z+1} \tag{14.2.72}$$

$$= -\pi\,\mathrm{i} + 0 \tag{14.2.73}$$

$$I = I_1 - I_2 = 2\pi\,\mathrm{i} \tag{14.2.74}$$

Since the path is the union of two simple closed paths, each of the paths can be integrated over separately.

25. Cauchy's theorem is not applicable.

Since the function is not analytic only at the origin, path deformation theorem ensures that the closed integral over any path encircling the origin remains the same.

$$I_1 : |z| = 1 \qquad \text{clockwise} \tag{14.2.75}$$

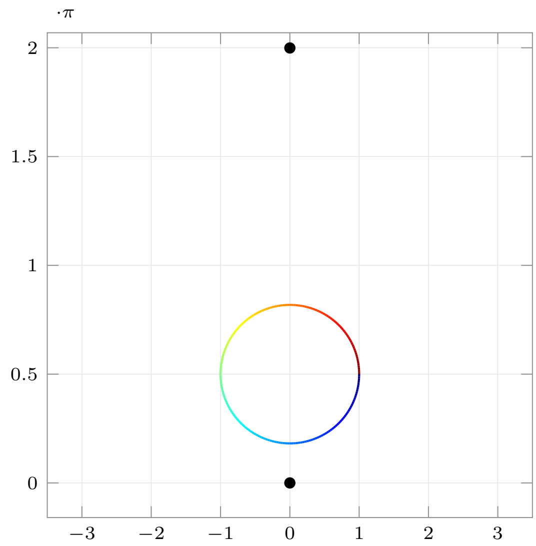
$$I_2 : |z| = 2 \qquad \text{counterclockwise} \tag{14.2.76}$$

$$I_2 = -I_1 \qquad \Longrightarrow \qquad I = 0 \tag{14.2.77}$$

26. Cauchy's theorem is not applicable.

$$\sinh(z/2) = 0 \qquad z = 2n\pi\,\mathrm{i} \tag{14.2.78}$$

$$I = \oint_C f(z)\ \mathrm{d}z = 0 \tag{14.2.79}$$



27. Cauchy's theorem is not applicable.

Since the function is not analytic only at the origin, path deformation theorem ensures that the closed integral over any path encircling the origin remains the same.

$$I_1 : |z| = 3 \qquad \text{clockwise} \tag{14.2.80}$$

$$I_2 : |z| = 1 \qquad \text{counterclockwise} \tag{14.2.81}$$

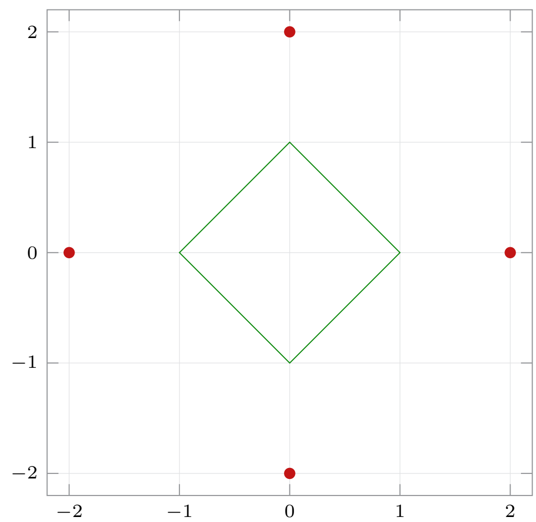
$$I_2 = -I_1 \qquad \Longrightarrow \qquad I = 0 \tag{14.2.82}$$

28. Cauchy's theorem is not applicable.

The function is analytic at all enclosed by the path of integration.

$$z^4 - 16 = 0 \qquad z = \{\pm 2, \pm 2\mathrm{i}\} \tag{14.2.83}$$

$$I = \oint_C f(z)\ \ \mathrm{d}z = 0 \tag{14.2.84}$$



29. Cauchy's theorem is not applicable since the contour includes $z = 0$. Using path deformation to convert the path into the unit circle,

$$f(z) = \frac{\sin z}{z} \qquad = \frac{\sin(e^{-\mathrm{i}t})}{e^{-\mathrm{i}t}} \tag{14.2.85}$$

$$z(t) = e^{-\mathrm{i}t} \qquad t \in [0, 2\pi] \tag{14.2.86}$$

$$\dot{z}(t) = -\mathrm{i}\ e^{-\mathrm{i}t} \qquad I = \int_0^{2\pi} \sin(e^{-\mathrm{i}t})(-\mathrm{i})\ \mathrm{d}t \tag{14.2.87}$$

$$I_1 = \int_0^{\pi} \sin\Big[\cos t - \mathrm{i}\ \sin t\Big]\ \mathrm{d}t \qquad I_2 = \int_\pi^{2\pi} \sin\Big[\cos t - \mathrm{i}\ \sin t\Big]\ \mathrm{d}t \tag{14.2.88}$$

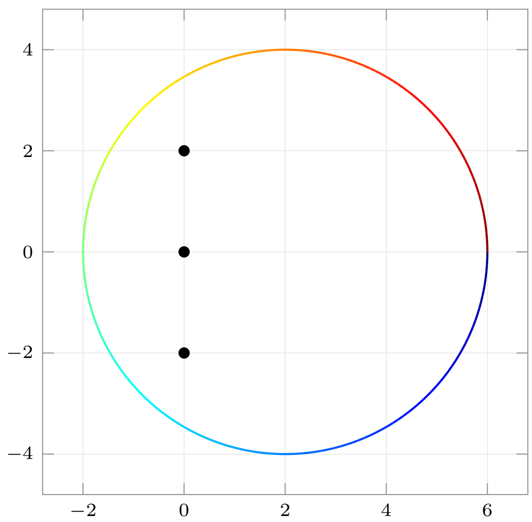
$$u = t - \pi \qquad \mathrm{d}u = \mathrm{d}t \tag{14.2.89}$$

$$I_2 = \int_0^{\pi} \sin\Big[\cos(u+\pi) - \mathrm{i}\sin(u+\pi)\Big]\,\mathrm{d}u \qquad I_2 = -I_1 \tag{14.2.90}$$

$$I_1 + I_2 = 0 \tag{14.2.91}$$

30. Resolving into partial fractions,

$$f(z) = \frac{2z^3 + z^2 + 4}{z^2(z^2+4)} = \frac{1}{z+2\mathrm{i}} + \frac{1}{z-2\mathrm{i}} + \frac{1}{z^2} \tag{14.2.92}$$



Using the standard result for z^n, and the fact that all 3 poles are contained within the contour,

$$I = -2\pi\ \mathrm{i} - 2\pi\ \mathrm{i} - 0 = -4\pi\ \mathrm{i} \tag{14.2.93}$$

14.3 Cauchy's Integral Formula

1. Plotting the contour and the poles of the function,

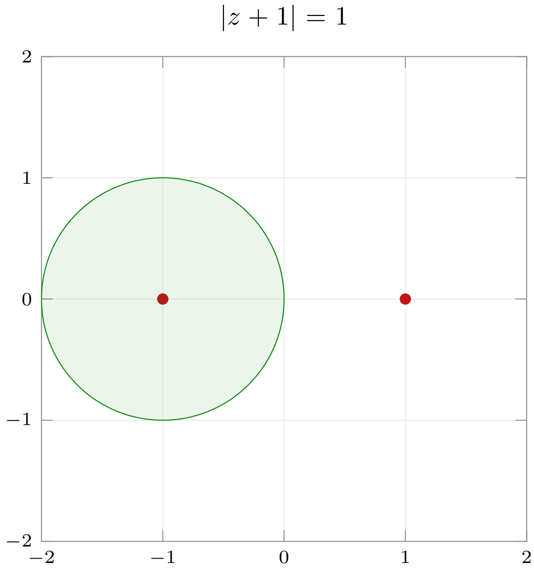
$$I = \oint_C \frac{z^2}{z^2 - 1} \, \mathrm{d}z \qquad C : |z + 1| = 1 \tag{14.3.1}$$

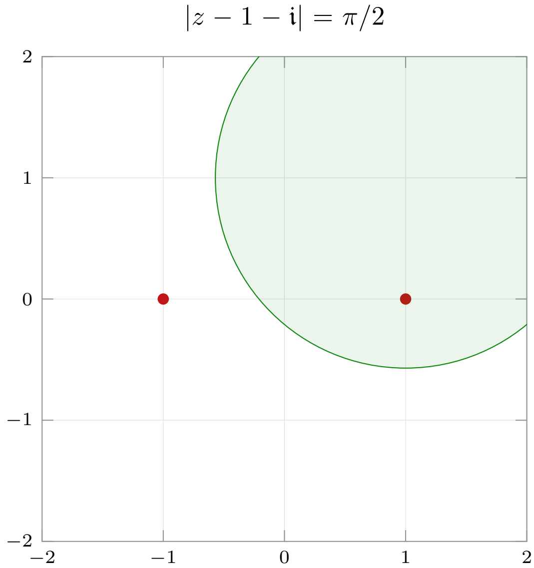
$$g(z) = \frac{z^2}{z - 1} \cdot \frac{1}{z + 1} \qquad I = 2\pi\mathrm{i} \left[\frac{z^2}{z - 1}\right]_{(-1)} = -\pi \ \mathrm{i} \tag{14.3.2}$$

2. Plotting the contour and the poles of the function,

$$I = \oint_C \frac{z^2}{z^2 - 1} \, \mathrm{d}z \qquad C : |z - 1 - \mathrm{i}| = \pi/2 \tag{14.3.3}$$

$$g(z) = \frac{z^2}{z + 1} \cdot \frac{1}{z - 1} \qquad I = 2\pi\mathrm{i} \left[\frac{z^2}{z + 1}\right]_{(1)} = \pi \ \mathrm{i} \tag{14.3.4}$$





3. The contour does not contain either pole of the function since $1.4 < \sqrt{2}$.

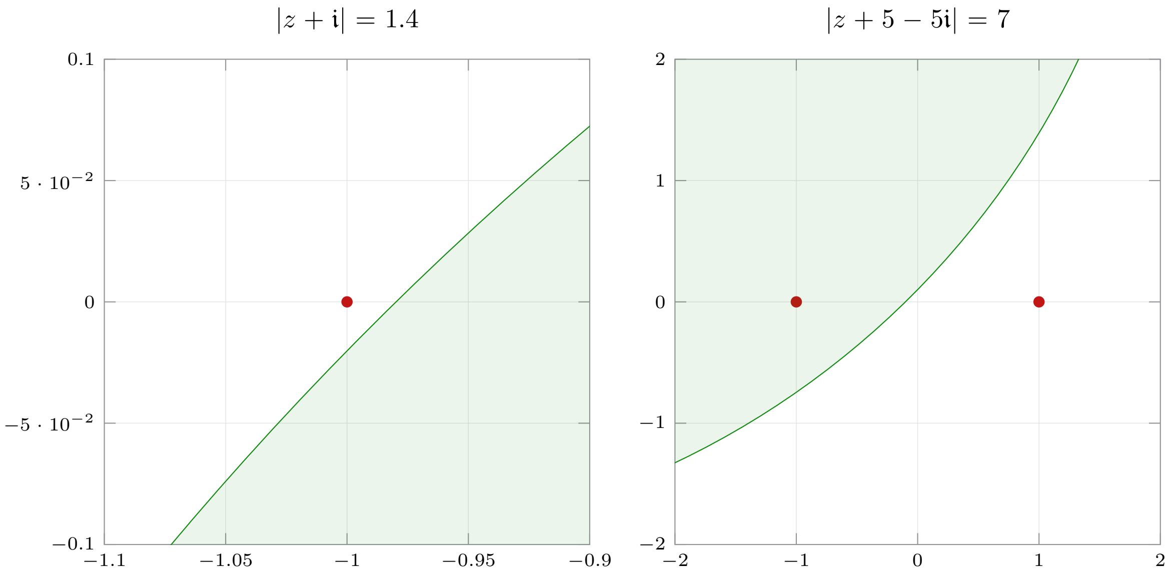
$$I = \oint_C \frac{z^2}{z^2 - 1} \, \mathrm{d}z \qquad C : |z + \mathrm{i}| = 1.4 \tag{14.3.5}$$

$$I = 0 \tag{14.3.6}$$

4. Plotting the contour and the poles of the function,

$$I = \oint_C \frac{z^2}{z^2 - 1} \ \mathrm{d}z \qquad C : |z + 5 - 5\mathrm{i}| = 7 \tag{14.3.7}$$

$$g(z) = \frac{z^2}{z-1} \cdot \frac{1}{z+1} \qquad I = 2\pi\mathrm{i} \left[\frac{z^2}{z-1}\right]_{(-1)} = -\pi \ \mathrm{i} \tag{14.3.8}$$



5. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{\cos(3z)}{6} \cdot \frac{1}{z} \qquad I = 2\pi\mathrm{i} \left[\frac{\cos(3z)}{6}\right]_0 \tag{14.3.9}$$

$$I = \frac{\pi}{3} \ \mathrm{i} \tag{14.3.10}$$

6. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{\cos(e^{2z})}{\pi z - \mathrm{i}} \qquad = \frac{e^{2z}}{\pi} \cdot \frac{1}{z - (1/\pi)\mathrm{i}} \tag{14.3.11}$$

$$I = 2\pi\mathrm{i} \left[\frac{e^{2z}}{\pi}\right]_{z_0} \qquad I = 2\mathrm{i} \left[\cos(2/\pi) + \mathrm{i} \ \sin(2/\pi)\right] \tag{14.3.12}$$

$$I = -2\sin(2/\pi) + 2\mathrm{i} \ \cos(2/\pi) \tag{14.3.13}$$

7. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{z^3}{2z - \mathrm{i}} \qquad = \frac{z^3}{2} \cdot \frac{1}{z - 0.5\mathrm{i}} \tag{14.3.14}$$

$$I = 2\pi\mathrm{i} \left[\frac{z^3}{2}\right]_{z_0} \qquad I = \frac{\pi}{8} \tag{14.3.15}$$

8. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{z^2 \sin z}{4z - 1} \qquad = \frac{z^2 \sin z}{4} \cdot \frac{1}{z - 0.25} \tag{14.3.16}$$

$$I = 2\pi\mathrm{i} \left[\frac{z^2 \sin z}{4}\right]_{z_0} \qquad I = \frac{\pi \sin(1/4)}{32}\ \mathrm{i} \tag{14.3.17}$$

9. Program written in `sympy` to detect whether the contour includes poles and adjust the integration method accordingly.

10. Using the contour method,

   (a) For the given function, the path encloses $z_0 = 0.5\mathrm{i}$

$$I = \oint_C \frac{z^3 - 6}{(z - 0.5\mathrm{i})}\ \mathrm{d}z \tag{14.3.18}$$

$$= \oint_C \frac{(-\mathrm{i}/8 - 6) + (z^3 - 6 + \mathrm{i}/8 + 6)}{z - 0.5\mathrm{i}}\ \mathrm{d}z \tag{14.3.19}$$

$$= -(\mathrm{i}/8 + 6)\ 2\pi\ \mathrm{i} + \oint_C \frac{z^3 - (0.5\mathrm{i})^3}{z - 0.5\mathrm{i}}\ \mathrm{d}z \tag{14.3.20}$$

$$= -(\mathrm{i}/8 + 6)\ 2\pi\ \mathrm{i} + \oint_C \left[z^2 - \frac{1}{4} + \frac{z\mathrm{i}}{2}\right]\ \mathrm{d}z \tag{14.3.21}$$

$$= -(\mathrm{i}/8 + 6)\ 2\pi\ \mathrm{i} + 0 = \frac{\pi}{4} - 12\pi\ \mathrm{i} \tag{14.3.22}$$

   The second term consists of entire functions in the integrand which have to integrate to zero over any closed simple path.

(b) For the given function, the path encloses $z_0 = 0.5\pi$

$$I = \oint_C \frac{\sin z}{(z - 0.5\pi)} \, \mathrm{d}z \tag{14.3.23}$$

$$= \oint_C \frac{(1) + (\sin z - 1)}{z - 0.5\pi} \, \mathrm{d}z \tag{14.3.24}$$

$$= 2\pi \, \mathrm{i} + \oint_C \frac{\sin z - 1}{z - 0.5\pi} \, \mathrm{d}z \tag{14.3.25}$$

$$= 2\pi \, \mathrm{i} + \oint_{C^*} \frac{\cos w - 1}{w} \, \mathrm{d}w \tag{14.3.26}$$

$$= 2\pi \, \mathrm{i} + \oint_{C^*} \left[-\frac{z}{2!} + \frac{z^4}{4!} - \dots \right] \, \mathrm{d}w \tag{14.3.27}$$

$$= 2\pi \, \mathrm{i} + 0 = 2\pi \, \mathrm{i} \tag{14.3.28}$$

The substitution $w = z - 0.5\pi$ makes the contour a circle centered around $w = 0$.
The second term consists of entire functions in the integrand which have to integrate to zero over any closed simple path.

(c) TBC.

11. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{1}{z^2 + 4} \qquad = \frac{1}{(z + 2\mathrm{i})(z - 2\mathrm{i})} \tag{14.3.29}$$

$$z(t) = \cos t + \mathrm{i} \, (2 \sin t + 2) \qquad t \in [0, 2\pi] \tag{14.3.30}$$

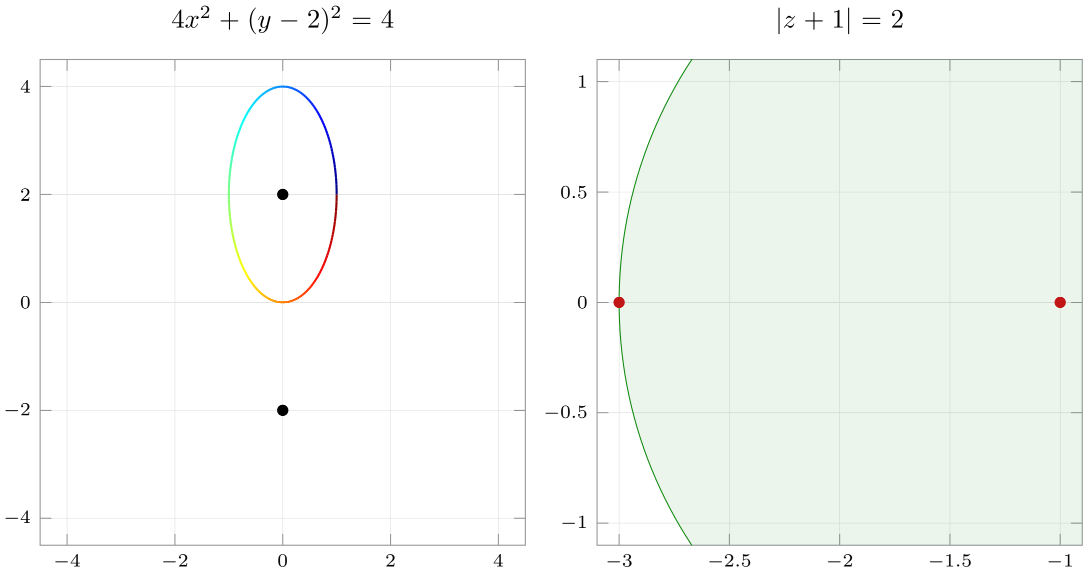
$$I = 2\pi\mathrm{i} \cdot \left[\frac{1}{z + 2\mathrm{i}} \right]_{z_0} \qquad I = \frac{\pi}{2} \tag{14.3.31}$$

12. Integrating around the unit circle anticlockwise, using residue theorem for poles lying on top of the contour.

$$f(z) = \frac{z}{z^2 + 4z + 3} \qquad = \frac{z}{(z + 1)(z + 3)} \tag{14.3.32}$$

$$C : |z + 1| = 2 \qquad I = 2\pi\mathrm{i} \cdot \left[\frac{z}{z + 3} \right]_{-1} + \pi\mathrm{i} \left[\frac{z}{z + 1} \right]_{-3} \tag{14.3.33}$$

$$I = \frac{\pi}{2} \, \mathrm{i} \tag{14.3.34}$$



13. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{z+1}{z-2} \qquad C : |z-1| = 2 \tag{14.3.35}$$

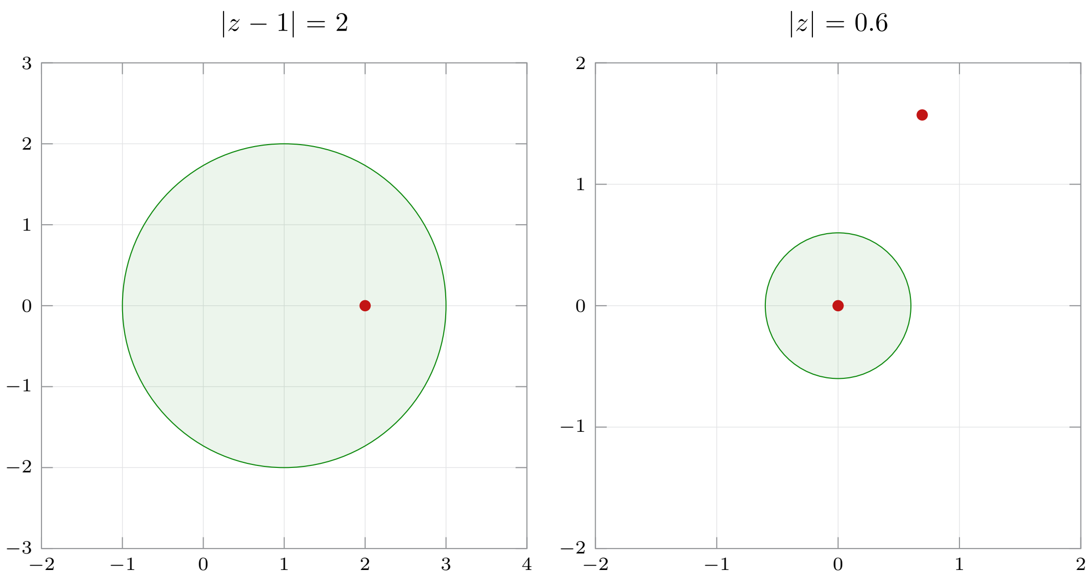
$$I = 2\pi \mathrm{i} \cdot \Big[(z+2)\Big]_2 \qquad I = 8\pi\ \mathrm{i} \tag{14.3.36}$$

14. Integrating around the unit circle anticlockwise.

$$f(z) = \frac{e^z}{z(e^z - 2\mathrm{i})} \qquad z_1 = \ln(2\mathrm{i}) = \ln(2) + \frac{\pi}{2}\ \mathrm{i} \tag{14.3.37}$$

$$C : |z+1| = 2 \qquad I = 2\pi \mathrm{i} \cdot \left[\frac{e^z}{e^z - 2\mathrm{i}}\right]_0 \tag{14.3.38}$$

$$I = \frac{2\pi}{5}\ (-2 + \mathrm{i}) \tag{14.3.39}$$



15. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{\cosh(z^2 - \pi\mathrm{i})}{z - \pi\mathrm{i}} \qquad C : \text{square with vertices } \{\pm 2, \pm 2 + 4\mathrm{i}\} \tag{14.3.40}$$

$$I = 2\pi\mathrm{i} \cdot \Big[\cosh(z^2 - \pi\mathrm{i})\Big]_{\pi\mathrm{i}} \qquad I = 2\pi\mathrm{i}\ \cosh(\pi^2 + \pi\mathrm{i}) \tag{14.3.41}$$

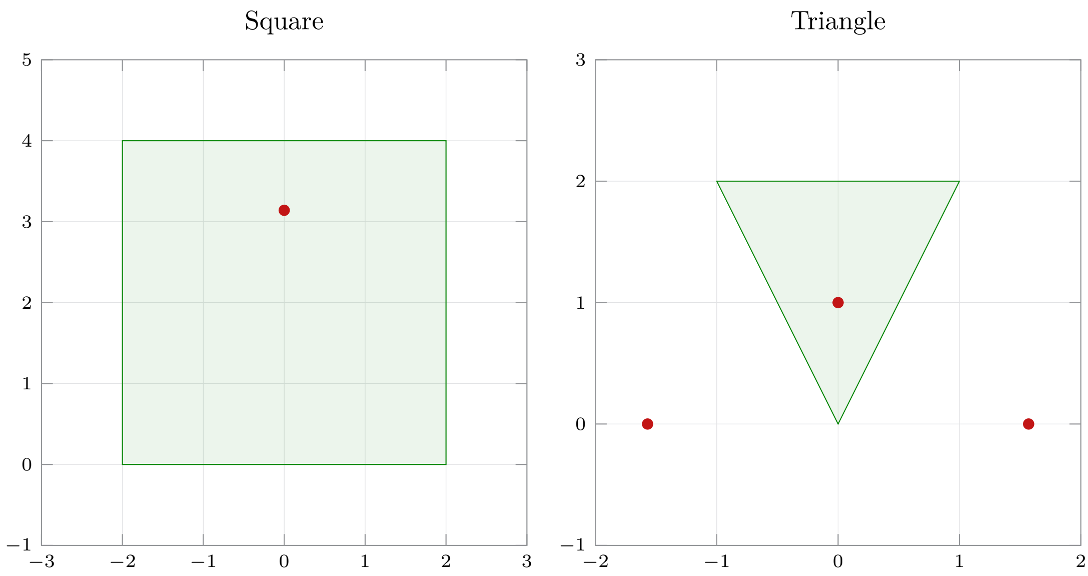
$$I = -2\pi\mathrm{i}\ \cos(\pi^2\mathrm{i}) \qquad = -2\pi\mathrm{i}\ \cosh(\pi^2) \tag{14.3.42}$$

16. Integrating around the unit circle anticlockwise.

$$f(z) = \frac{\tan z}{(z - \mathrm{i})} \qquad z_1 = \frac{(2n+1)\pi}{2} + 0\ \mathrm{i} \tag{14.3.43}$$

$$C : \text{triangle with vertices } \{0, \pm 1 + 2\mathrm{i}\} \qquad I = 2\pi\mathrm{i} \cdot \Big[\tan z\Big]_{(\mathrm{i})} \tag{14.3.44}$$

$$I = -2\pi\ \tanh(1) \tag{14.3.45}$$



17. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{\mathrm{Ln}(z+1)}{z^2+1} \qquad C : |z - \mathfrak{i}| = 1.4 \tag{14.3.46}$$

$$I = 2\pi\mathfrak{i} \cdot \left[\frac{\mathrm{Ln}(z+1)}{z+\mathfrak{i}}\right]_{(\mathfrak{i})} \qquad I = \frac{\pi}{2}\ \ln(2) + \frac{\pi^2}{4}\ \mathfrak{i} \tag{14.3.47}$$

$$I = -2\pi\mathfrak{i}\ \cos(\pi^2\mathfrak{i}) \qquad = -2\pi\mathfrak{i}\ \cosh(\pi^2) \tag{14.3.48}$$

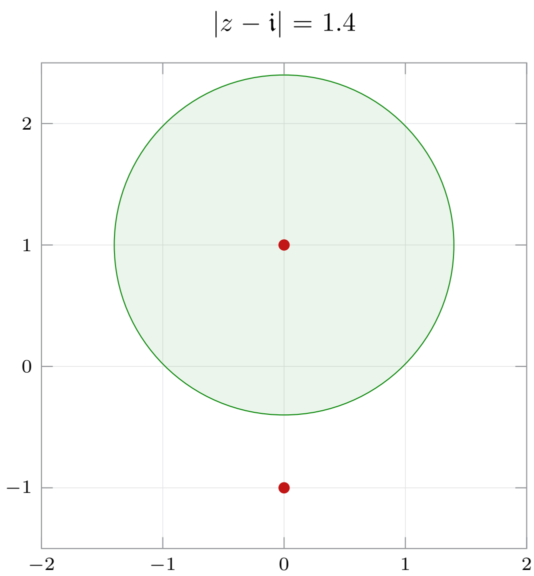
18. Integrating around the unit circle anticlockwise.

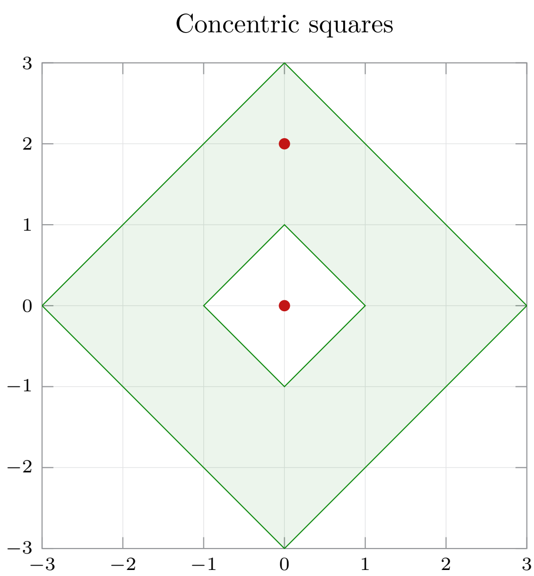
$$f(z) = \frac{\sin z}{4z(z-2\mathfrak{i})} \tag{14.3.49}$$

$$C_1 = \text{outer square ccl} \qquad C_2 = \text{inner square cl} \tag{14.3.50}$$

$$I_1 + I_2 = 2\pi\mathfrak{i} \cdot \left[\frac{\sin z}{4z}\right]_{(2\mathfrak{i})} \tag{14.3.51}$$

$$I = \frac{\pi}{4}\ \sin(2\mathfrak{i}) \qquad I = \frac{\pi \sinh(2)}{4}\ \mathfrak{i} \tag{14.3.52}$$



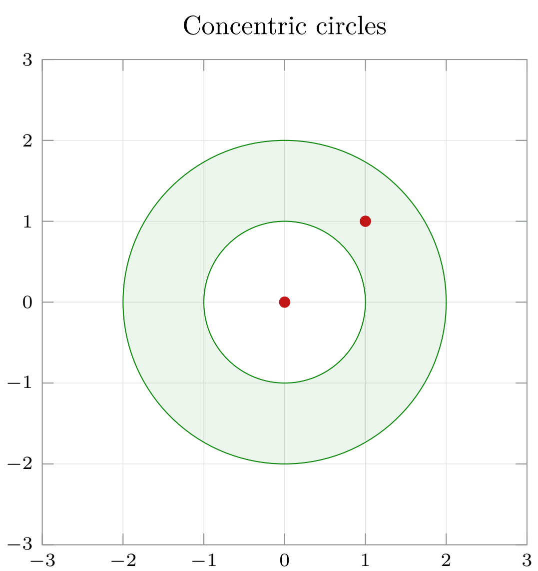


19. Integrating around the unit circle anticlockwise,

$$f(z) = \frac{\exp(z^2)}{z^2\ (z - 1 - \mathrm{i})} \tag{14.3.53}$$

$$C_1 = \text{outer circle ccl} \qquad C_2 = \text{inner circle cl} \tag{14.3.54}$$

$$I_1 + I_2 = 2\pi\mathrm{i} \cdot \left[\frac{\exp(z^2)}{z^2}\right]_{(1+\mathrm{i})} \qquad I = \pi\ \exp(2\mathrm{i}) \tag{14.3.55}$$



20. Integrating around the unit circle anticlockwise.

$$f(z) = \frac{1}{(z - z_1)(z - z_2)} \qquad f(z) = \frac{1}{(z_1 - z_2)} \left[\frac{1}{z - z_1} - \frac{1}{z - z_2} \right] \tag{14.3.56}$$

$$I_1 = \frac{1}{z_1 - z_2} \oint_C \frac{1}{z - z_1} \, \mathrm{d}z \qquad = \frac{2\pi \, \mathrm{i}}{z_1 - z_2} \tag{14.3.57}$$

$$I_2 = \frac{-1}{z_1 - z_2} \oint_C \frac{1}{z - z_2} \, \mathrm{d}z \qquad = \frac{-2\pi \, \mathrm{i}}{z_1 - z_2} \tag{14.3.58}$$

$$I_1 + I_2 = 0 \tag{14.3.59}$$

14.4 Derivatives of Analytic Functions

1. Integrating,

$$I = \oint_C \frac{\sin z}{z^4} \, \mathrm{d}z \qquad f^{(3)}(z_0) = \Big[-\cos z \Big]_{(0)} = -1 \tag{14.4.1}$$

$$I = (-1) \, \frac{2\pi \mathrm{i}}{3!} \qquad = \frac{-\pi}{3} \, \mathrm{i} \tag{14.4.2}$$

2. Integrating,

$$I = \oint_C \frac{z^6}{(2z - 1)^6} \, \mathrm{d}z \qquad = \oint_C \frac{(z/2)^6}{(z - 1/2)^6} \, \mathrm{d}z \tag{14.4.3}$$

$$f^{(5)}(z_0) = \left[\frac{6! \, z}{64} \right]_{(0.5)} = \frac{45}{8} \tag{14.4.4}$$

$$I = \frac{45}{8} \, \frac{2\pi \mathrm{i}}{5!} \qquad = \frac{3\pi}{32} \, \mathrm{i} \tag{14.4.5}$$

3. Integrating, with non-negative integer n,

$$I = \oint_C \frac{e^z}{z^n} \, \mathrm{d}z \qquad f^{(n-1)}(z_0) = \Big[e^z \Big]_{(0)} = 1 \tag{14.4.6}$$

$$I = \frac{2\pi \mathrm{i}}{(n - 1)!} \tag{14.4.7}$$

4. Integrating,

$$I = \oint_C \frac{e^z \cos z}{(z - 0.25\pi)^3} \, \mathrm{d}z \tag{14.4.8}$$

$$f^{(2)}(z_0) = \Big[-2e^z \sin z \Big]_{(0.25\pi)} \qquad = -\sqrt{2} e^{\pi/4} \tag{14.4.9}$$

$$I = -\sqrt{2} e^{\pi/4} \, \frac{2\pi \mathrm{i}}{2!} \qquad = -\sqrt{2} \pi e^{\pi/4} \, \mathrm{i} \tag{14.4.10}$$

5. Integrating,

$$I = \oint_C \frac{\cos 2z}{(z - 0.5)^4} \, \mathrm{d}z \tag{14.4.11}$$

$$f^{(3)}(z_0) = \Big[8 \sinh 2z \Big]_{(0.5)} \qquad = 8 \sinh(1) \tag{14.4.12}$$

$$I = 8 \sinh(1) \, \frac{2\pi \mathrm{i}}{3!} \qquad = \frac{8\pi}{3} \sinh(1) \, \mathrm{i} \tag{14.4.13}$$

6. Integrating,

$$I = \oint_C \frac{1}{(z - 0.5\mathrm{i})^2 (z - 2\mathrm{i})^2} \, \mathrm{d}z \qquad f(z) = \frac{1}{(z - 2\mathrm{i})^2} \tag{14.4.14}$$

$$f'(z_0) = \left[\frac{-2}{(z - 2\mathrm{i})^3} \right]_{(0.5\mathrm{i})} \qquad = \frac{16}{27} \, \mathrm{i} \tag{14.4.15}$$

$$I = \frac{16 \, \mathrm{i}}{27} \, \frac{2\pi \mathrm{i}}{1!} \qquad = \frac{-32\pi}{27} \tag{14.4.16}$$

7. Integrating,

$$I = \oint_C \frac{cos z}{z^{2n+1}} \, \mathrm{d}z \qquad f(z) = \cos z \tag{14.4.17}$$

$$f^{(2n)}(z_0) = \Big[(-1)^n \cos z \Big]_{(0)} \qquad = (-1)^n \tag{14.4.18}$$

$$I = (-1)^n \, \frac{2\pi \mathrm{i}}{(2n)!} \tag{14.4.19}$$

8. Integrating,

$$I = \oint_C \frac{z^3 + \sin z}{(z - \mathrm{i})^3} \; \mathrm{d}z \qquad f(z) = z^3 + \sin z \tag{14.4.20}$$

$$f''(z_0) = \bigg[6z - \sin z\bigg]_{(\mathrm{i})} \qquad = [6 - \sinh(1)]\; \mathrm{i} \tag{14.4.21}$$

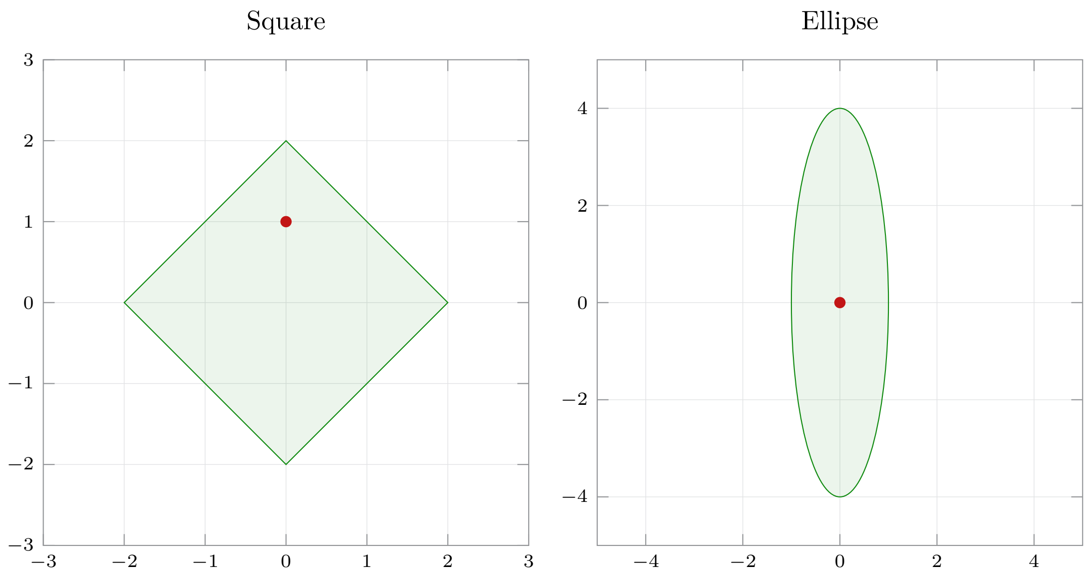
$$I = \pi(\sinh 1 - 6) \tag{14.4.22}$$

9. Integrating, with a minus sign to accommodate the clockwise orientation,

$$I = -\oint_C \frac{\tan(\pi z)}{z^2} \; \mathrm{d}z \qquad f(z) = \tan(\pi z) \tag{14.4.23}$$

$$f'(z_0) = \bigg[\pi \sec^2(z)\bigg]_{(0)} \qquad = \pi \tag{14.4.24}$$

$$I = -2\pi^2 \mathrm{i} \tag{14.4.25}$$



10. Integrating, with the outer circle being ccw, and the inner circle being cw,

$$I = \oint_C \frac{4z^3 - 6}{z(z - 1 - \mathrm{i})^2} \; \mathrm{d}z \qquad f(z) = 4z^2 - \frac{6}{z} \tag{14.4.26}$$

$$f'(z_0) = \bigg[8z + \frac{6}{z^2}\bigg]_{(1+\mathrm{i})} \qquad = 8 + 5\mathrm{i} \tag{14.4.27}$$

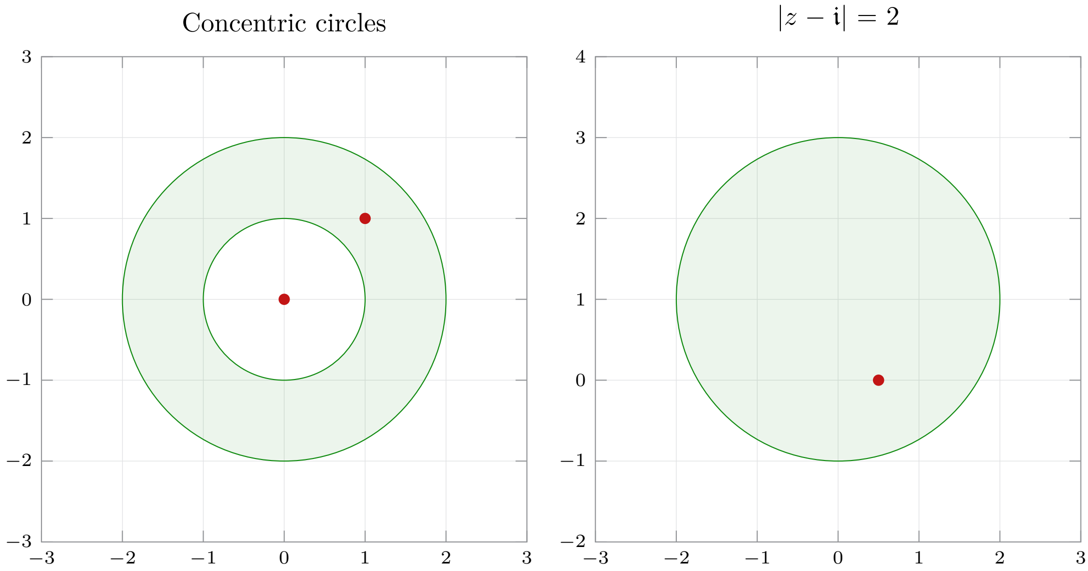
$$I = (8 + 5\mathrm{i})\; \frac{2\pi \mathrm{i}}{1!} \qquad = 2\pi\;(-5 + 8\mathrm{i}) \tag{14.4.28}$$

11. Integrating,

$$I = -\oint_C \frac{(1+z)\sin z}{(2z-1)^2}\ \mathrm{d}z \qquad f(z) = \frac{(1+z)\sin z}{4} \tag{14.4.29}$$

$$f'(z_0) = \left[\frac{(1+z)\cos z + \sin z}{4}\right]_{(0.5)} \qquad = \frac{1.5\cos(1/2) + \sin(1/2)}{4} \tag{14.4.30}$$

$$I = \Big[1.5\cos(1/2) + \sin(1/2)\Big]\ \frac{\pi}{2}\ \mathrm{i} \tag{14.4.31}$$



12. Integrating, with a minus sign to accommodate the clockwise orientation,

$$I = -\oint_C \frac{e^{z^2}}{z(z-2\mathrm{i})^2}\ \mathrm{d}z \qquad f(z) = \frac{\exp(z^2)}{z} \tag{14.4.32}$$

$$f'(z_0) = \left[\frac{(2z^2-1)\exp(z^2)}{z^2}\right]_{(2\mathrm{i})} \qquad = \frac{9e^{-4}}{4} \tag{14.4.33}$$

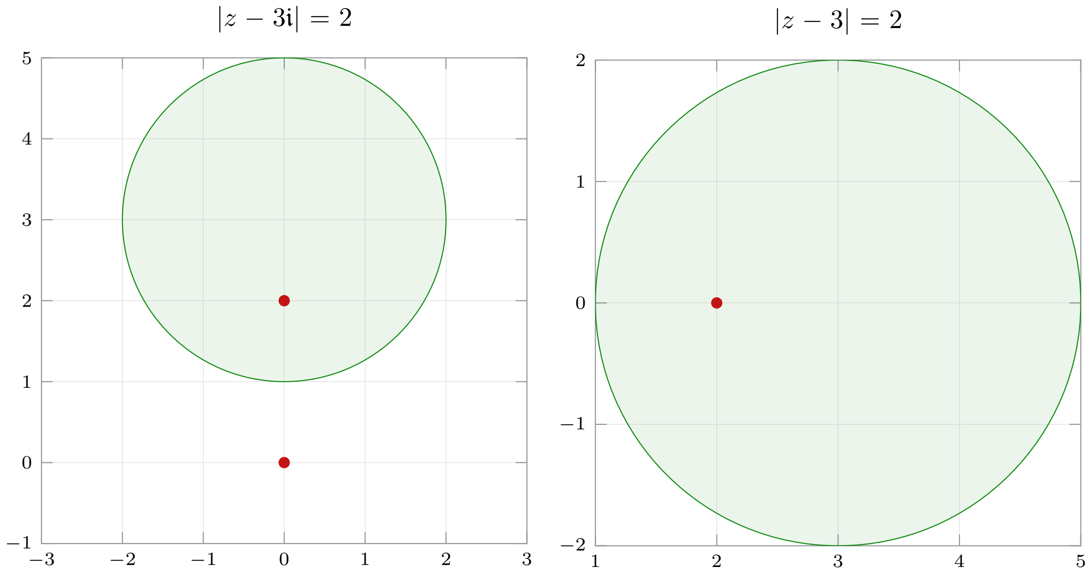
$$I = -\frac{9e^{-4}}{4}\ \frac{2\pi\mathrm{i}}{1!} \qquad = -\frac{9e^{-4}}{2}\ \pi\ \mathrm{i} \tag{14.4.34}$$

13. Integrating,

$$I = -\oint_C \frac{\mathrm{Ln}\ z}{(z-2)^2}\ \mathrm{d}z \qquad f(z) = \mathrm{Ln}\ z \tag{14.4.35}$$

$$f'(z_0) = \left[\frac{1}{z}\right]_{(2)} \qquad = 0.5 \tag{14.4.36}$$

$$I = \pi\mathrm{i} \tag{14.4.37}$$



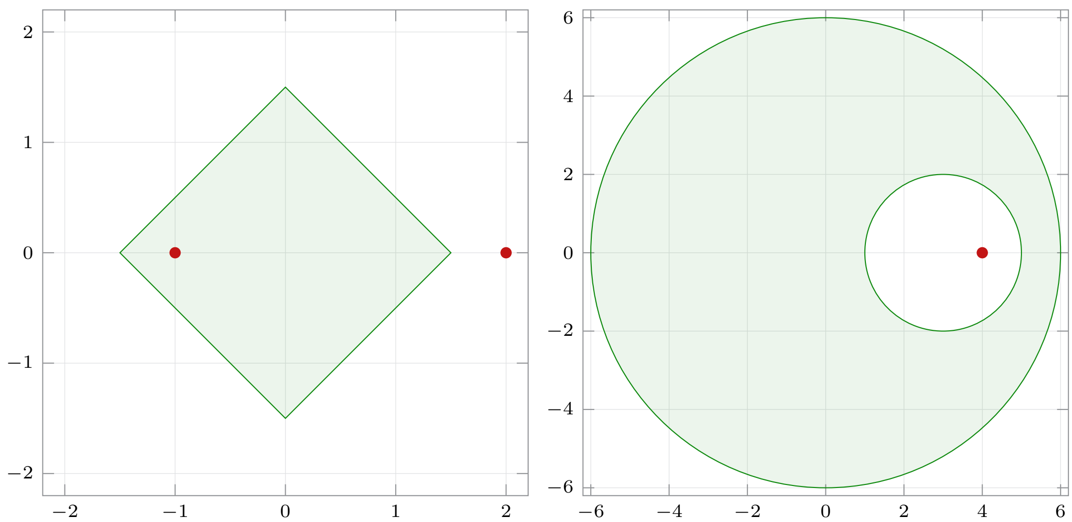
14. Integrating, with a minus sign to accommodate the clockwise orientation,

$$I = \oint_C \frac{\mathrm{Ln}\,(z+3)}{(z-2)(z+1)^2}\ \mathrm{d}z \qquad f(z) = \frac{\mathrm{Ln}\,(z+3)}{(z-2)} \tag{14.4.38}$$

$$f'(z_0) = \left[\frac{1}{(z-2)(z+3)} - \frac{\mathrm{Ln}\,(z+3)}{(z-2)^2}\right]_{(-1)} = -\frac{1}{6} - \frac{\ln(2)}{9} \tag{14.4.39}$$

$$I = \left[\frac{-3-\ln(4)}{18}\right]\frac{2\pi\mathfrak{i}}{1!} = \frac{-3-\ln 4}{9}\ \pi\mathfrak{i} \tag{14.4.40}$$

15. Integrating results in zero, since the integrand is analytic in the entire region between the curves (shaded). Path deformation and the fact that the orientation of the curves are opposite means that the integrals are equal and opposite.



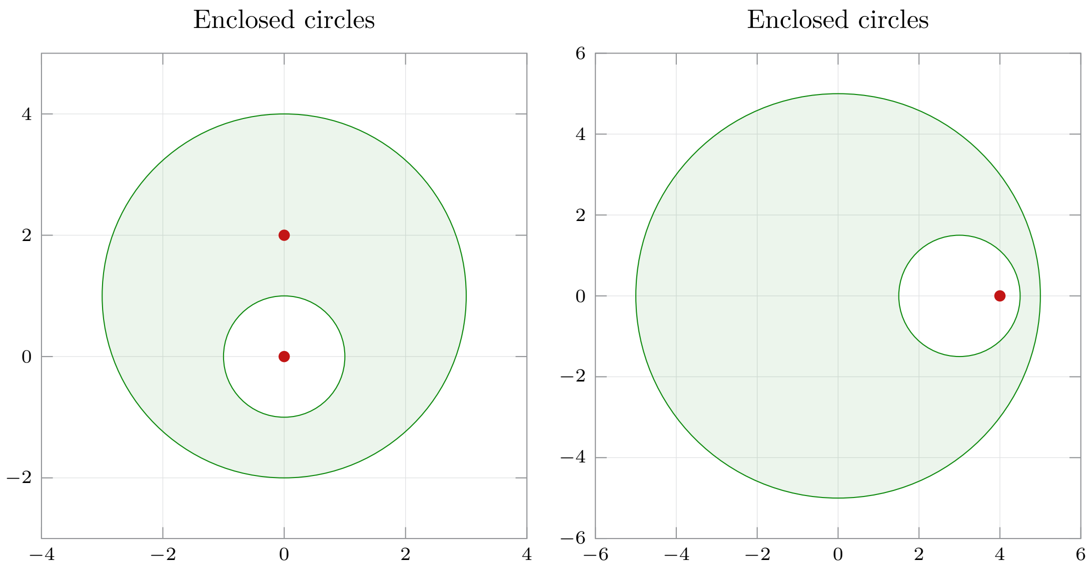
16. Integrating, with a minus sign to accommodate the clockwise orientation,

$$I = -\oint_C \frac{e^{4z}}{z(z-2\mathrm{i})^2}\ \mathrm{d}z \qquad f(z) = \frac{\exp(4z)}{z} \tag{14.4.41}$$

$$f'(z_0) = \left[\frac{(4z-1)\exp(4z)}{z^2}\right]_{(2\mathrm{i})} \qquad = \frac{(1-8\mathrm{i})e^{8\mathrm{i}}}{4} \tag{14.4.42}$$

$$I = \frac{(1-8\mathrm{i})e^{8\mathrm{i}}}{4}\ \frac{2\pi\mathrm{i}}{1!} \qquad = \frac{(\mathrm{i}+8)\pi}{2}\ e^{8\mathrm{i}} \tag{14.4.43}$$

17. Integrating results in zero, since the integrand is analytic in the entire region between the curves (shaded). Path deformation and the fact that the orientation of the curves are opposite means that the integrals are equal and opposite.



18. Integrating, with integer $n > 0$,

$$I = -\oint_C \frac{\sinh z}{z^n}\ \mathrm{d}z \qquad f(z) = \sinh z \tag{14.4.44}$$

$$f^{(n-1)}(z_0) = \begin{cases} 1 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \tag{14.4.45}$$

$$I = \frac{2\pi\mathrm{i}}{(n-1)!}\ f^{(n-1)}(0) \tag{14.4.46}$$

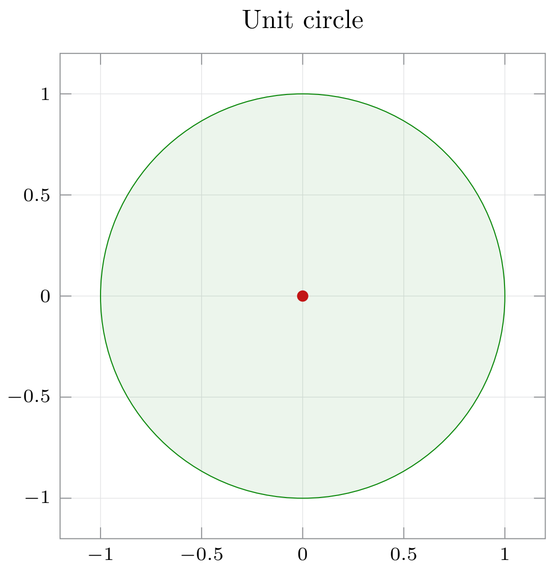
For the case where, $n \leq 0$, the answer is zero, since the integrand becomes analytic everywhere.

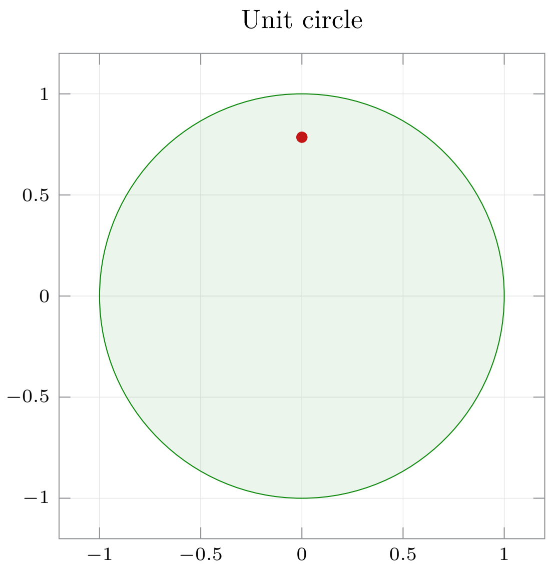
19. Integrating,

$$I = -\oint_C \frac{e^{3z}}{(4z-\pi\mathrm{i})^3} \, \mathrm{d}z \qquad f(z) = \frac{e^{3z}}{64} \tag{14.4.47}$$

$$f'(z_0) = \left[\frac{9e^{3z}}{64}\right]_{(\pi/4\ \mathrm{i})} \qquad = \frac{9}{64\sqrt{2}} (-1+\mathrm{i}) \tag{14.4.48}$$

$$I = \frac{9}{64\sqrt{2}} (-1+\mathrm{i}) \frac{2\pi\mathrm{i}}{2!} \qquad I = \frac{9\pi}{64\sqrt{2}} (-1-\mathrm{i}) \tag{14.4.49}$$





20. Growth

(a) Suppose $f(z)$ is bounded,

$$|f(z)| \leq M \qquad \forall \quad z \in \mathcal{C} \tag{14.4.50}$$

$$\tag{14.4.51}$$

By Lioville's theorem, this function is constant. This contradicts the assumption that the function is not constant. R.A.A.

(b) $f(z)$ is a polynomial of positive degree. It is a non constant entire function. From part a, it is unbounded.

(c) $f(z) = e^x$. This is a real function of a real variable. It is a non-constant entire function and thus obeys the relation in part a.
Fix an M_0 such that $R_0 = \ln(M_0)$. Now,

$$|f(z)| > M_0 \qquad \forall \quad \text{Re}\,(z) > R_0 \tag{14.4.52}$$

A complex number can have a modulus larger than R_0 while still having a real part less than R_0. So, the relation in part b is violated.

(d) If $f(z)$ is never zero, then it has either a lower or upper bound equal to zero, by definition. This violates result a, and makes the function constant, which is a contradiction.