Chapter 12

Partial Differential Equations

12.1 Basic Concepts of PDEs

1. Writing a general second order ODE in two variables $u(x, y)$,

$$\mathcal{P}[u] = A\,\frac{\partial^2 u}{\partial x^2} + B\,\frac{\partial^2 u}{\partial y^2} + C\,\frac{\partial^2 u}{\partial x\,\partial y} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + F\,u = 0 \tag{12.1.1}$$

Using the linearity of partial differentiation,

$$\mathcal{L}[u_1] = 0 \qquad \mathcal{L}[u_2] = 0 \tag{12.1.2}$$

$$\implies \mathcal{L}[u] = \mathcal{L}[c_1 u_1 + c_2 u_2] \qquad c_1 \mathcal{L}[u_1] + c_2 \mathcal{L}[u_2] = 0 \tag{12.1.3}$$

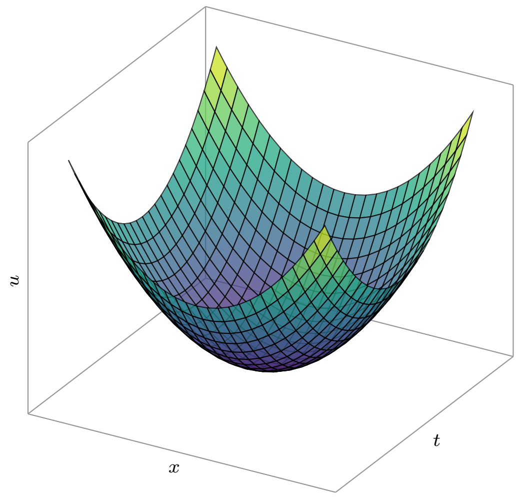
A similar proof follows for $v(x, y, z)$ which whose differential operator will contain more terms, but still be linear. So, the proof remains the same.

2. Verifying that the given function solves the one dimensional wave equation,

$$u(x, t) = x^2 + t^2 \qquad \frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2} \tag{12.1.4}$$

$$2 = c^2\ (2) \qquad c^2 = 1 \tag{12.1.5}$$

$u = x^2 + t^2$

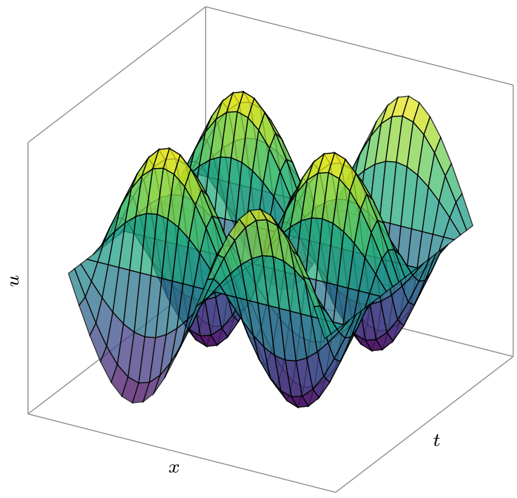


3. Verifying that the given function solves the one dimensional wave equation,

$$u(x,t) = \cos(4t)\ \ \sin(2x) \qquad \frac{\partial^2 u}{\partial t^2} = c^2\ \frac{\partial^2 u}{\partial x^2} \tag{12.1.6}$$

$$\sin(2x)[-16\cos(4t)] = c^2\ \ \cos(4t)\ [-4\sin(2x)] \qquad c^2 = 4 \tag{12.1.7}$$

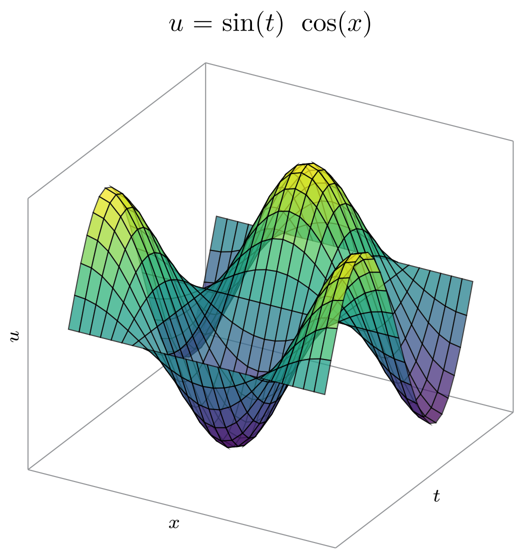
$u = \cos(4t)\ \ \sin(2x)$



4. Verifying that the given function solves the one dimensional wave equation,

$$u(x,t) = \sin(kct)\ \ \cos(kx) \qquad \frac{\partial^2 u}{\partial t^2} = c^2\ \frac{\partial^2 u}{\partial x^2} \tag{12.1.8}$$

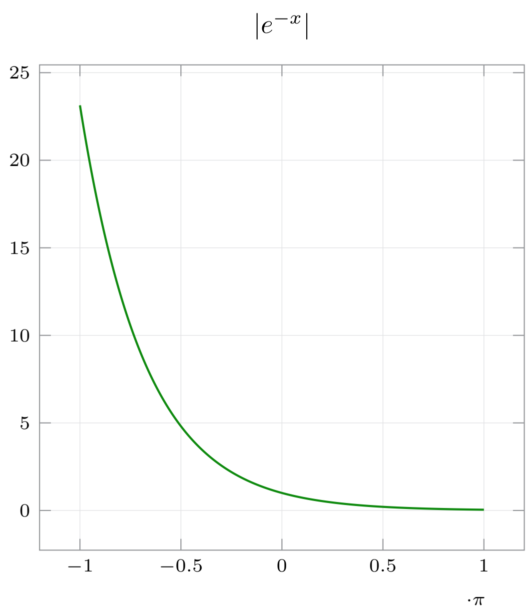
$$\cos(kx)[-k^2c^2\sin(kct)] = c^2\ \ \sin(kct)\ [-k^2\cos(kx)] \qquad c = \text{free} \tag{12.1.9}$$



5. Verifying that the given function solves the one dimensional wave equation,

$$u(x,t) = \sin(at)\ \sin(bx) \qquad \frac{\partial^2 u}{\partial t^2} = c^2\ \frac{\partial^2 u}{\partial x^2} \tag{12.1.10}$$

$$\sin(bx)[-a^2\ \sin(at)] = c^2\ \ \sin(at)\ [-b^2\ \sin(bx)] \qquad c = \frac{a^2}{b^2} \tag{12.1.11}$$

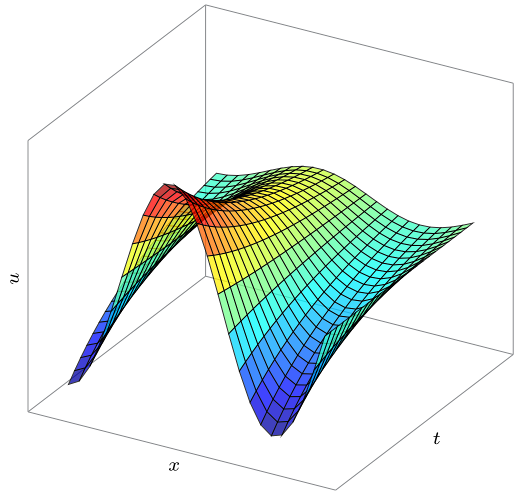


6. Verifying that the given function solves the one dimensional heat equation,

$$u(x,t) = e^{-t}\ \ \sin(x) \qquad \frac{\partial u}{\partial t} = c^2\ \frac{\partial^2 u}{\partial x^2} \tag{12.1.12}$$

$$[-e^{-t}]\ \ \sin(x) = c^2\ e^{-t}\ [-\sin(x)] \qquad c^2 = 1 \tag{12.1.13}$$

$u = e^{-t} \ \sin x$

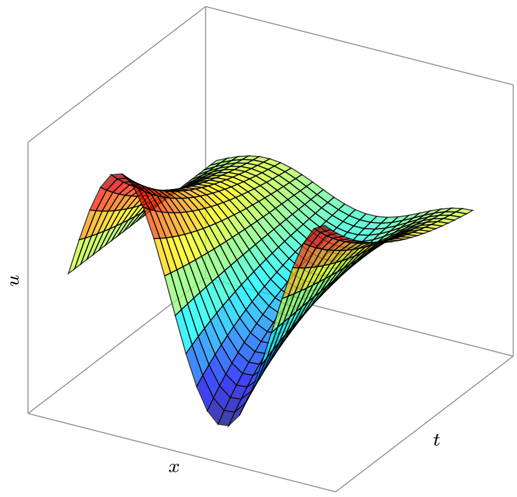


7. Verifying that the given function solves the one dimensional heat equation,

$$u(x,\,t) = e^{-\omega^2 c^2 t} \ \cos(\omega x) \qquad \frac{\partial u}{\partial t} = c^2 \ \frac{\partial^2 u}{\partial x^2} \tag{12.1.14}$$

$$[-\omega^2 c^2 \ e^{-\omega^2 c^2 t}] \ \cos(\omega x) = c^2 \ e^{-\omega^2 c^2 t} \ [-\omega^2 \ \cos(\omega x)] \qquad c = \text{free} \tag{12.1.15}$$

$u = e^{-\omega^2 c^2 t} \ \cos(\omega x)$

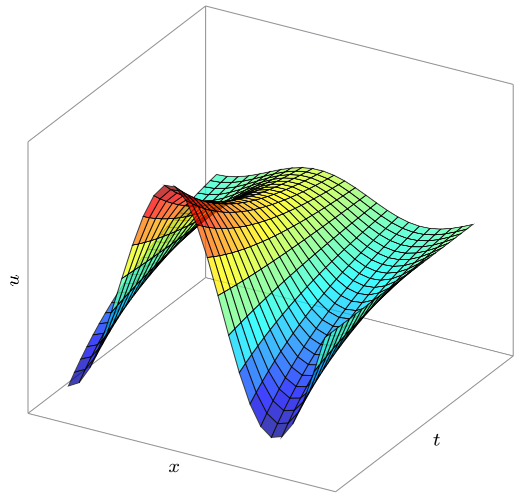


8. Verifying that the given function solves the one dimensional heat equation,

$$u(x,\,t) = e^{-9t} \ \sin(\omega x) \qquad \frac{\partial u}{\partial t} = c^2 \ \frac{\partial^2 u}{\partial x^2} \tag{12.1.16}$$

$$[-9 \ e^{-9t}] \ \sin(\omega x) = c^2 \ e^{-9t} \ [-\omega^2 \ \sin(\omega x)] \qquad c^2 = \frac{9}{\omega^2} \tag{12.1.17}$$

$$u = e^{-9t} \ \sin(\omega x)$$

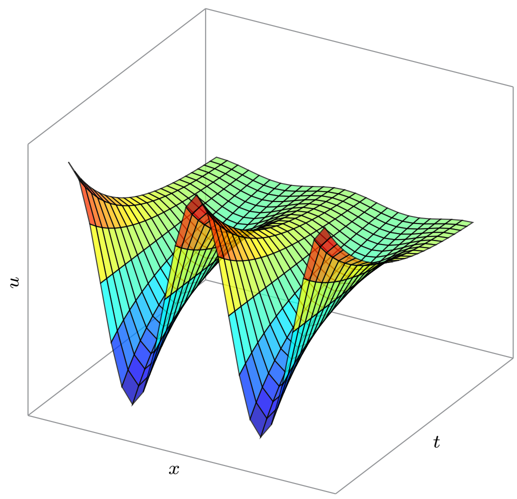


9. Verifying that the given function solves the one dimensional heat equation,

$$u(x,\, t) = e^{-\pi^2 t} \ \cos(25x) \qquad \frac{\partial u}{\partial t} = c^2 \ \frac{\partial^2 u}{\partial x^2} \tag{12.1.18}$$

$$[-\pi^2 \ e^{-\pi^2 t}] \ \cos(25x) = c^2 \ e^{-\pi^2 t} \ [-25^2 \ \cos(25x)] \qquad c^2 = \frac{\pi^2}{25^2} \tag{12.1.19}$$

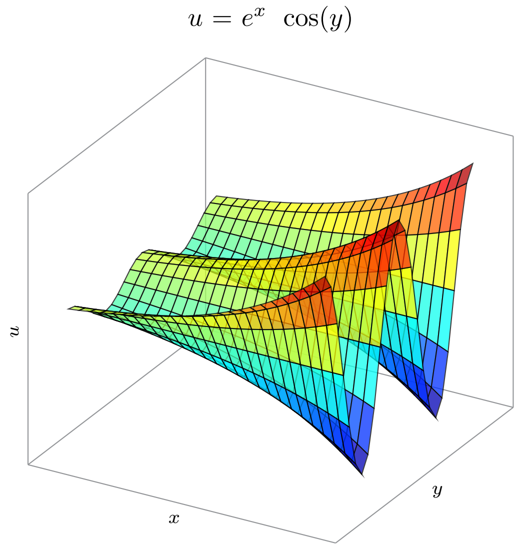
$$u = e^{-\pi^2 t} \ \cos(25x)$$



10. Verifying that the given function solves the two dimensional Laplace equation,

$$u(x,\, t) = e^{x} \ \cos(y) \qquad 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{12.1.20}$$

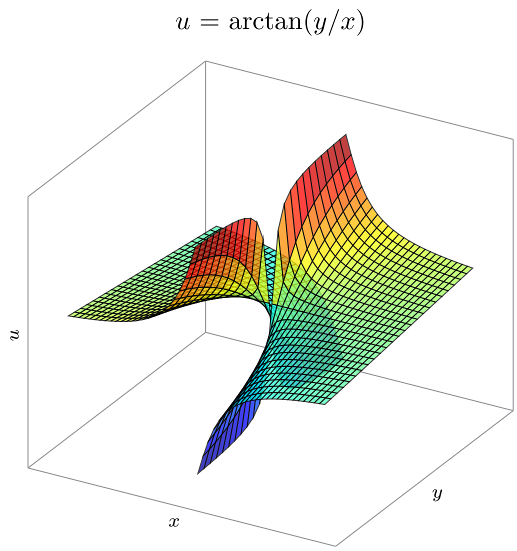
$$0 = [e^x] \cos(y) + e^x [-\cos(y)] \tag{12.1.21}$$



11. Verifying that the given function solves the two dimensional Laplace equation,

$$u(x,t) = \arctan(y/x) \qquad\qquad 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{12.1.22}$$

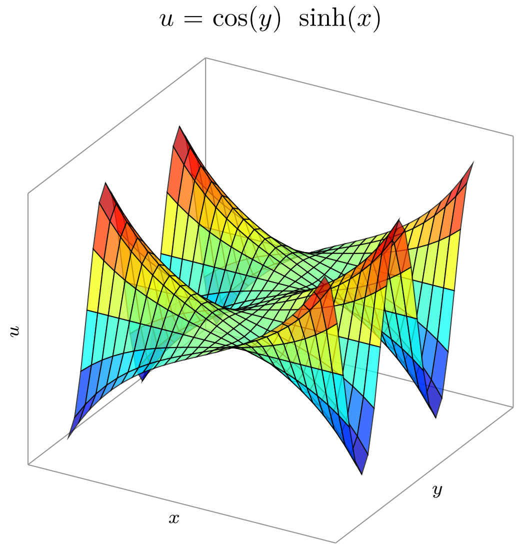
$$\frac{\partial^2 u}{\partial x^2} = \frac{y(2x)}{(x^2+y^2)^2} \qquad\qquad \frac{\partial^2 u}{\partial y^2} = \frac{-x(2y)}{(x^2+y^2)^2} \tag{12.1.23}$$



12. Verifying that the given function solves the two dimensional Laplace equation,

$$u(x,t) = \cos(y) \ \sinh(x) \qquad\qquad 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{12.1.24}$$

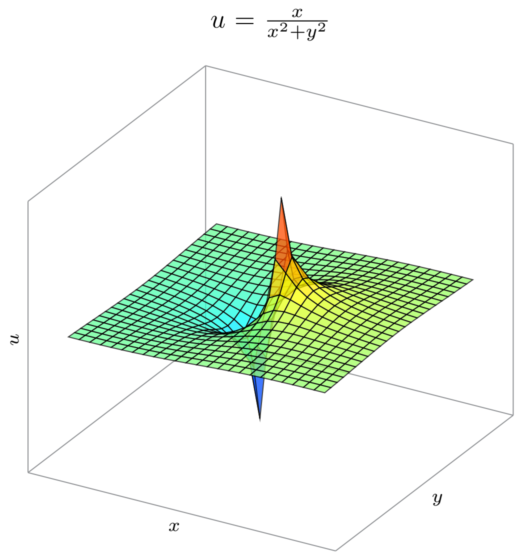
$$\frac{\partial^2 u}{\partial x^2} = \cos(y) \ [\sinh(x)] \qquad\qquad \frac{\partial^2 u}{\partial y^2} = [-\cos(y)] \ \sinh(x) \tag{12.1.25}$$



13. Verifying that the given function solves the two dimensional Laplace equation,

$$u(x,t) = \frac{x}{x^2+y^2} \qquad 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{12.1.26}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2x\ (x^2-3y^2)}{(x^2+y^2)^3} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{2x\ (3y^2-x^2)}{(x^2+y^2)^3} \tag{12.1.27}$$



14. Verifying special forms of solutions,

(a) For the wave equation,

$$u(x,\, t) = v(x + ct) + w(x - ct) \qquad \frac{\partial^2 u}{\partial t^2} = c^2\, \frac{\partial^2 u}{\partial x^2} \tag{12.1.28}$$

$$\frac{\partial^2 v}{\partial t^2}\, c^2 = \frac{\partial^2 v}{\partial x^2} \qquad \frac{\partial^2 w}{\partial t^2}\, c^2 = \frac{\partial^2 w}{\partial x^2} \tag{12.1.29}$$

$$c^2\, \frac{\partial^2 (v + w)}{\partial t^2} = \frac{\partial^2 (v + w)}{\partial x^2} \tag{12.1.30}$$

This uses the transformation of variables,

$$\frac{\partial v}{\partial (ct)} \cdot \frac{\partial (ct)}{\partial t} = \frac{\partial v}{\partial t} \tag{12.1.31}$$

This function does satisfy the given PDE.

(b) Verifying the given functions against Laplace's equation,
Yes, No, Yes,

$$u(x,\, y) = \frac{y}{x} \qquad f = \frac{2y}{x^3} \tag{12.1.32}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2y}{x^3} \qquad \frac{\partial^2 u}{\partial y^2} = 0 \tag{12.1.33}$$

$$u(x,\, y) = \sin(xy) \qquad f = (x^2 + y^2)\ \sin(xy) \tag{12.1.34}$$

$$\frac{\partial^2 u}{\partial x^2} = -y^2\ \sin(xy) \qquad \frac{\partial^2 u}{\partial y^2} = -x^2\ \sin(xy) \tag{12.1.35}$$

$$u(x,\, y) = e^{x^2 - y^2} \qquad f = 4(x^2 + y^2)\ e^{x^2 - y^2} \tag{12.1.36}$$

$$\frac{\partial^2 u}{\partial x^2} = (2 + 4x^2)\ e^{x^2 - y^2} \qquad \frac{\partial^2 u}{\partial y^2} = (-2 + 4y^2)\ e^{x^2 - y^2} \tag{12.1.37}$$

$$u(x,\, y) = \frac{1}{\sqrt{x^2 + y^2}} \qquad f = (x^2 + y^2)^{-3/2} \tag{12.1.38}$$

$$\frac{\partial u}{\partial x} = \frac{-x}{(x^2 + y^2)^{3/2}} \qquad \frac{\partial u}{\partial y} = \frac{-y}{(x^2 + y^2)^{3/2}} \tag{12.1.39}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-r^3 + 3x^2 r}{r^6} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{-r^3 + 3y^2 r}{r^6} \tag{12.1.40}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^3} \qquad = \frac{1}{(x^2 + y^2)^{3/2}} \tag{12.1.41}$$

(c) $3d$ Laplace equation satisfied the given function, Yes

$$u = \frac{1}{r} \qquad r^2 = x^2 + y^2 + z^2 \tag{12.1.42}$$

$$\partial_x u = \frac{-x}{r^3} \qquad \partial_x^2 u = \frac{-r^3 + (x)\ (3xr)}{r^6} \tag{12.1.43}$$

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = \frac{-3r^3 + 3r(x^2 + y^2 + z^2)}{r^6} \qquad = 0 \tag{12.1.44}$$

$2d$ Laplace equation satisfied by the given function, Yes

$$u = \ln(x^2 + y^2) \qquad r^2 = x^2 + y^2 \tag{12.1.45}$$

$$\partial_x u = \frac{2}{r} \cdot \frac{x}{r} = \frac{2x}{r^2} \qquad \partial_x^2 u = \frac{2r^2 - 4x^2}{r^4} \tag{12.1.46}$$

$$\partial_x^2 u + \partial_y^2 u = \frac{4r^2 - 4(x^2 + y^2)}{r^4} \qquad = 0 \tag{12.1.47}$$

From part b, the fourth Poisson equation is the $f(x, y)$ needed on the right hand side. This also means that it does not satisfy the $2d$ Poisson equation

(d) Verifying whether the given functions satisfy the PDEs,
Yes, Yes, Yes

$$u = v(x) + w(y) \qquad \partial_x \partial_y u = 0 \tag{12.1.48}$$

$$\partial_x u = \partial_x v + 0 \qquad \partial_x \partial_y u = 0 + 0 \tag{12.1.49}$$

$$u = v(x) \cdot w(y) \qquad u \cdot \partial_x \partial_y u = \partial_x u \cdot \partial_y u \tag{12.1.50}$$

$$\partial_x \partial_y u = \partial_x v \cdot \partial_y w \qquad \partial_x u \cdot \partial_y u = [w \cdot \partial_x v] \cdot [v\ \partial_y w] \tag{12.1.51}$$

$$= [v \cdot w]\ [\partial_x v \cdot \partial_y w] \tag{12.1.52}$$

$$u = v(x+2t) + w(x-2t) \qquad \partial_t^2 u = 4\,\partial_x^2 u \tag{12.1.53}$$

$$\partial_t^2 u = 4\,\partial_t^2 v + 4\,\partial_t^2 w \qquad \partial_x^2 u = \partial_x^2 v + \partial_x^2 w \tag{12.1.54}$$

15. Checking if the given function satisfies Laplace's equation,

$$u(x,y) = a\,\ln(x^2+y^2) + b \qquad \partial_x^2 u + \partial_y^2 u = 0 \tag{12.1.55}$$

$$\partial_x^2 u = 2a\,\frac{y^2 - x^2}{(x^2+y^2)^2} \qquad \partial_y^2 u = 2a\,\frac{x^2 - y^2}{(x^2+y^2)^2} \tag{12.1.56}$$

$$\tag{12.1.57}$$

Using the boundary conditions,

$$x^2 + y^2 = 1 \qquad \implies 110 = b \tag{12.1.58}$$

$$x^2 + y^2 = 100 \qquad \implies 0 = a\,\ln(100) + b \tag{12.1.59}$$

$$b = 110 \qquad a = \frac{-110}{\ln(100)} \tag{12.1.60}$$

16. Solving using ODE methods,

$$\partial_y^2 u = 0 \qquad \partial_y u = f(x) \tag{12.1.61}$$

$$u = y \cdot f(x) + g(x) \tag{12.1.62}$$

17. Solving using ODE methods,

$$\partial_x^2 u = -16\pi^2\,u \qquad u = f(y)\cdot\cos(4\pi x) + g(y)\cdot\sin(4\pi x) \tag{12.1.63}$$

18. Solving using ODE methods,

$$\partial_y^2 u = \frac{4u}{25} \qquad u = f(x)\cdot e^{-2y/5} + g(x)\cdot e^{2y/5} \tag{12.1.64}$$

19. Solving using ODE methods,

$$\frac{\partial u}{\partial y} = -y^2\,u \qquad \ln(u) = \frac{-y^3}{3} + f(x) \tag{12.1.65}$$

$$u = g(x)\cdot e^{-y^3/3} \tag{12.1.66}$$

20. Solving the homogeneous ODE in x,

$$2\,\frac{\partial^2 u}{\partial x^2} + 9\,\frac{\partial u}{\partial x} + 4u = -3\,\cos\,x - 29\,\sin\,x \tag{12.1.67}$$

$$2u'' + 9u' + 4u = 0 \tag{12.1.68}$$

$$\lambda = \frac{-9 \pm 7}{4} = \{-4,\, -1/2\} \tag{12.1.69}$$

$$u_h = c_1(y)e^{-4x} + c_2(y)e^{-x/2} \tag{12.1.70}$$

Solving the non-homogeneous ODE in x,

$$u_p = K\,\cos(x) + M\,\sin(x) \tag{12.1.71}$$

$$-3 = -2K + 4K + 9M \qquad\qquad -29 = -2M + 4M - 9K \tag{12.1.72}$$

$$u_p = 3\,\cos\,x - \sin\,x \tag{12.1.73}$$

Combining the two parts of the ODE solution,

$$u(x,\,y) = c_1(y)e^{-4x} + c_2(y)e^{-x/2} + 3\,\cos\,x - \sin\,x \tag{12.1.74}$$

21. Solving the homogeneous ODE in y,

$$\frac{\partial^2 u}{\partial y^2} + 6\,\frac{\partial u}{\partial y} + 13u = 4e^{3y} \tag{12.1.75}$$

$$u'' + 6u' + 13u = 0 \tag{12.1.76}$$

$$\lambda = \frac{-6 \pm 4i}{2} = \{-3 \pm 2i\} \tag{12.1.77}$$

$$u_h = e^{-3y}\,\left[c_1(x)\,\cos(2y) + c_2(x)\,\sin(2y)\right] \tag{12.1.78}$$

Solving the non-homogeneous ODE in y,

$$u_p = K\,e^{3y} \tag{12.1.79}$$

$$4 = 9K + 18K + 13K \qquad\qquad K = 0.1 \tag{12.1.80}$$

Combining the two parts of the ODE solution,

$$u(x,\,y) = e^{-3y}\,\left[c_1(x)\,\cos(2y) + c_2(x)\,\sin(2y)\right] + 0.1\,e^{3y} \tag{12.1.81}$$

22. Solving the homogeneous ODE in y,

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial u}{\partial x} \qquad \frac{\partial u}{\partial y} = u + f(y) \tag{12.1.82}$$

$$\ln[u + f(y)] = y + c_2 \qquad u = Ae^y + B(y) \tag{12.1.83}$$

23. Solving the homogeneous ODE in y,

$$x^2\,\frac{\partial^2 u}{\partial x^2} + 2x\,\frac{\partial u}{\partial x} - 2u = 0 \qquad x^2\,u'' + 2x\,6u - 2u = 0 \tag{12.1.84}$$

$$m^2 + (2-1)m - 2 = 0 \qquad m = \frac{-1 \pm 3}{2} = \{-2, 1\} \tag{12.1.85}$$

$$u_h = f(y)\,x^{-2} + g(y)x \tag{12.1.86}$$

24. Using the given equation and cylindrical coordinates,

$$y\,\partial_x z = x\,\partial_y z \qquad x = r\cos\theta \qquad y = r\sin\theta \tag{12.1.87}$$

$$\frac{\partial z}{\partial \theta} = \partial_x z \cdot \partial_\theta x + \partial_y z \cdot \partial_\theta y \qquad \partial_\theta z = \Big[-r\sin\theta \cdot \cot\theta + r\cos\theta\Big]\,\partial_y z \tag{12.1.88}$$

$$\partial_\theta z = 0 \tag{12.1.89}$$

Since z is independent of θ, the surface is symmetric about the polar axis which makes it a surface of revolution about the z axis.

25. Solving the system of ODEs,

$$\partial_x^2 u = 0 \qquad \partial_y^2 u = 0 \tag{12.1.90}$$

$$u = f(y)\cdot(c_1 x + c_2) \qquad u = g(x)\cdot(b_1 y + b_2) \tag{12.1.91}$$

$$u = (c_1 x + c_2)(b_1 y + b_2) \tag{12.1.92}$$

12.2 Modeling: Vibrating String, Wave Equation

1. No problem set in this section.

12.3 Solution by Separating Variables, Use of Fourier Series

1. Fundamental frequency is,

$$\lambda_n = \frac{cn\pi}{L} \qquad f_n = \frac{\lambda_n}{2\pi} = \frac{cn}{2L} \tag{12.3.1}$$

$$f_1 = \frac{c}{2L} = \frac{1}{2L} \cdot \sqrt{\frac{T}{\rho}} \tag{12.3.2}$$

$$f_1 \propto \sqrt{T} \qquad f_1 \propto \frac{1}{\sqrt{\rho}} \tag{12.3.3}$$

$$T \to 2T \qquad \implies \quad f \to 1.414\ f \tag{12.3.4}$$

2.
- The motion is no longer strictly vertical.
  Then the net force in the vertical direction is no longer zero and the PDE cannot be simplified using Newton's second law.
- Gravitational force is no longer negligible.
  This means that there is a gravitational term in the PDE

$$T_2 \sin\beta - T_1 \sin\alpha - (\rho\Delta x)g = (\rho\Delta x)\ \frac{\partial^2 u}{\partial t^2} \tag{12.3.5}$$

  The resulting PDE is much less tractable.
- The string is no longer perfectly elastic.
  This means that the motion is a damped oscillation, and the ODE becomes,

$$\frac{\partial^2 u}{\partial t^2} + \mu\ \frac{\partial u}{\partial t} = c^2\ \frac{\partial^2 u}{\partial x^2} \tag{12.3.6}$$

  $\mu(x)$ here is a constant function.
- The string is no longer homogeneous.
  The parameter c^2 in the PDE is now dependent on x, since $c^2 = T/\rho$.

3. Derivation for special case $L = \pi$,

$$F_n(x) = \sin(nx) \qquad \lambda_n = cn \tag{12.3.7}$$

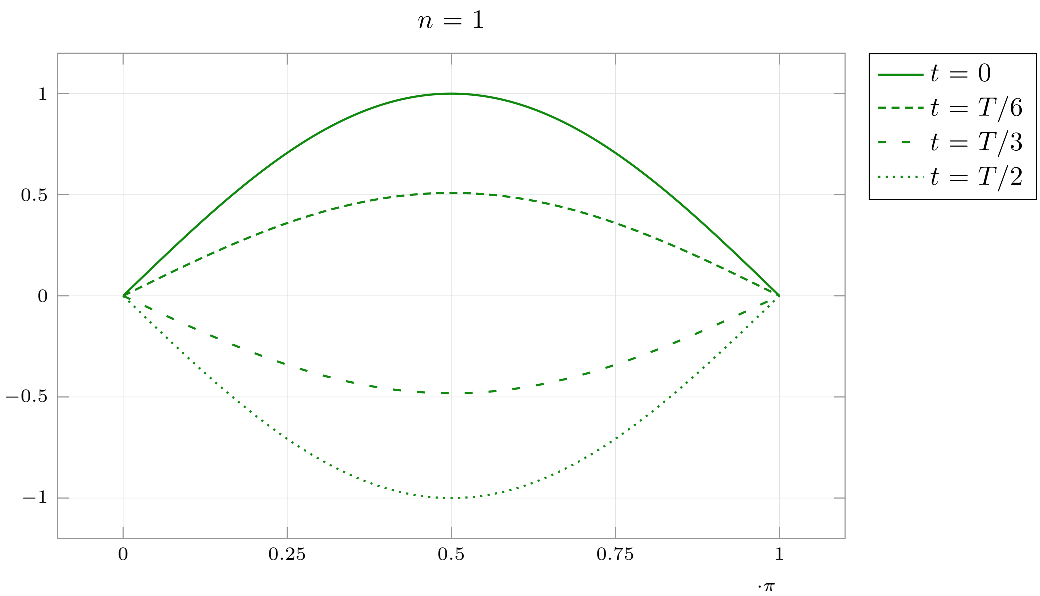
$$G_n(t) = B_n \cos(cn\ t) + B_n^* \sin(cn\ t) \tag{12.3.8}$$

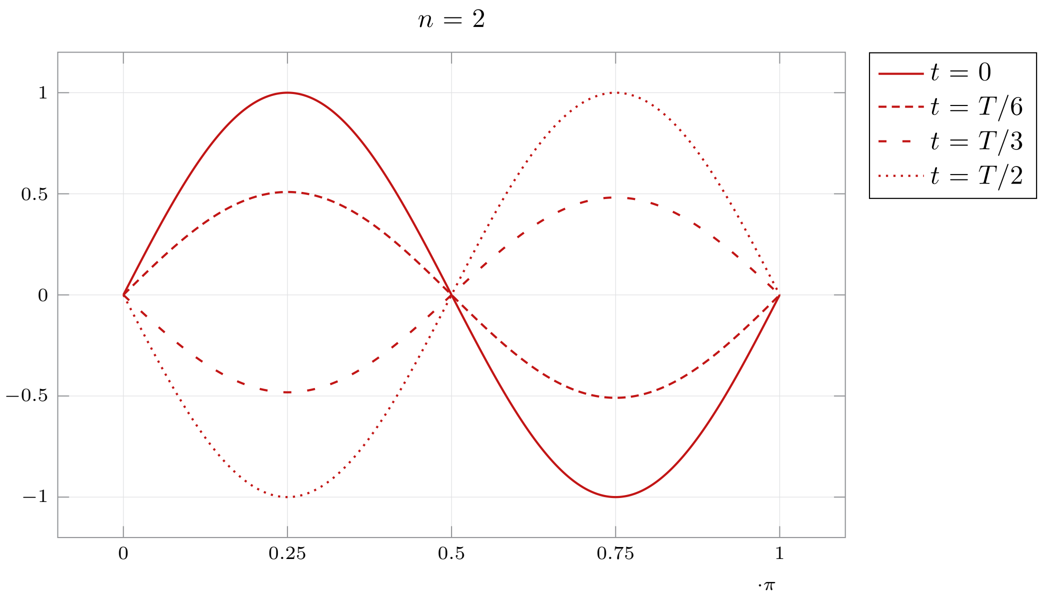
Given the initial deflection $f(x)$ and initial velocity $g(x)$,

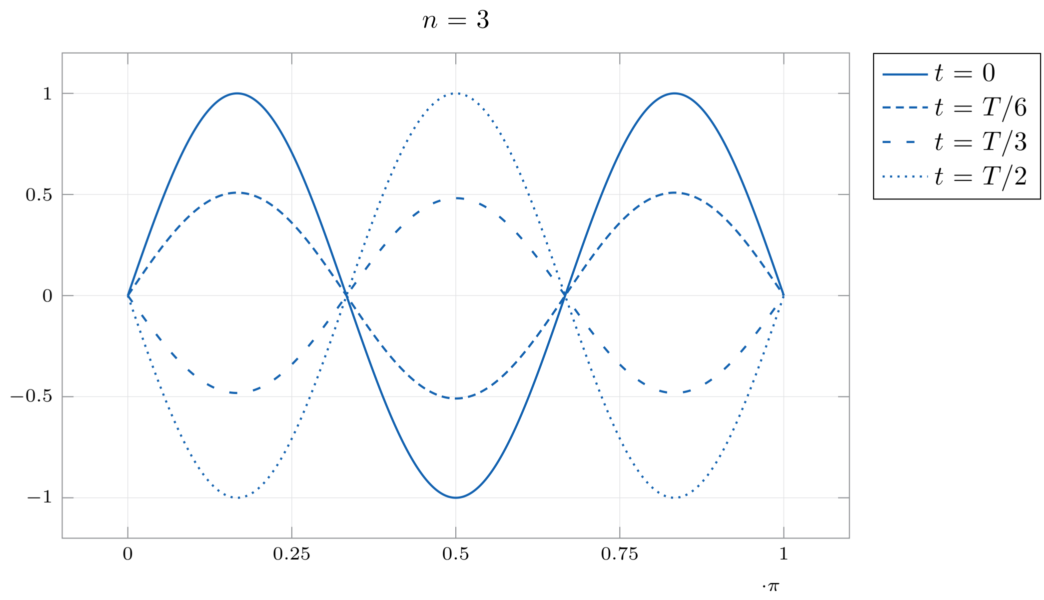
$$B_n = \frac{2}{\pi} \int_0^\pi f(x)\ \sin(nx)\ \mathrm{d}x \qquad B_n^* = \frac{2}{cn\pi} \int_0^\pi g(x)\ \sin(nx)\ \mathrm{d}x \tag{12.3.9}$$

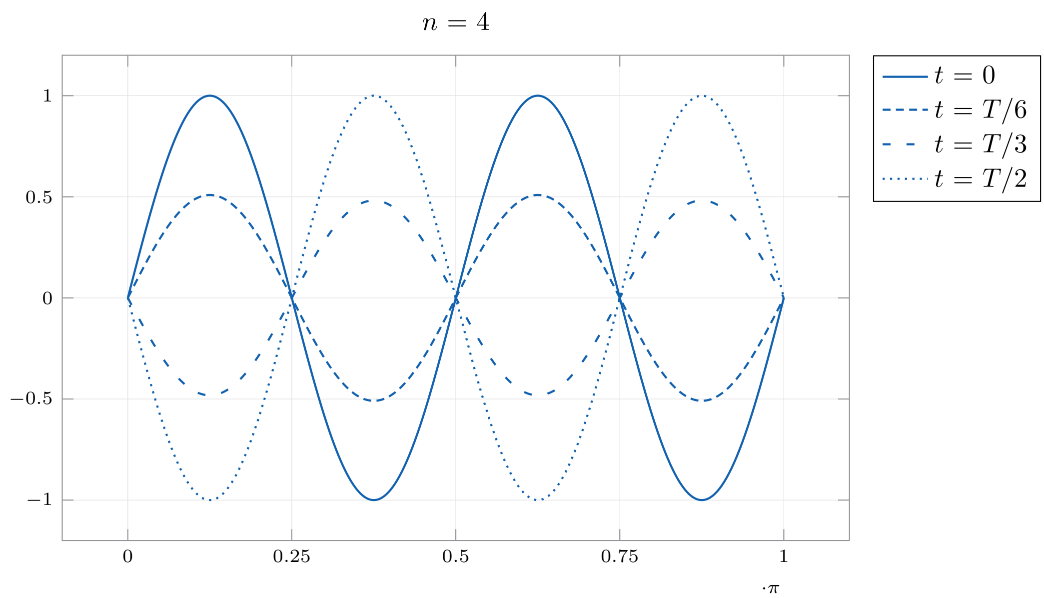
It is much easier to see the significance of the Fourier series expansion in solving this PDE.

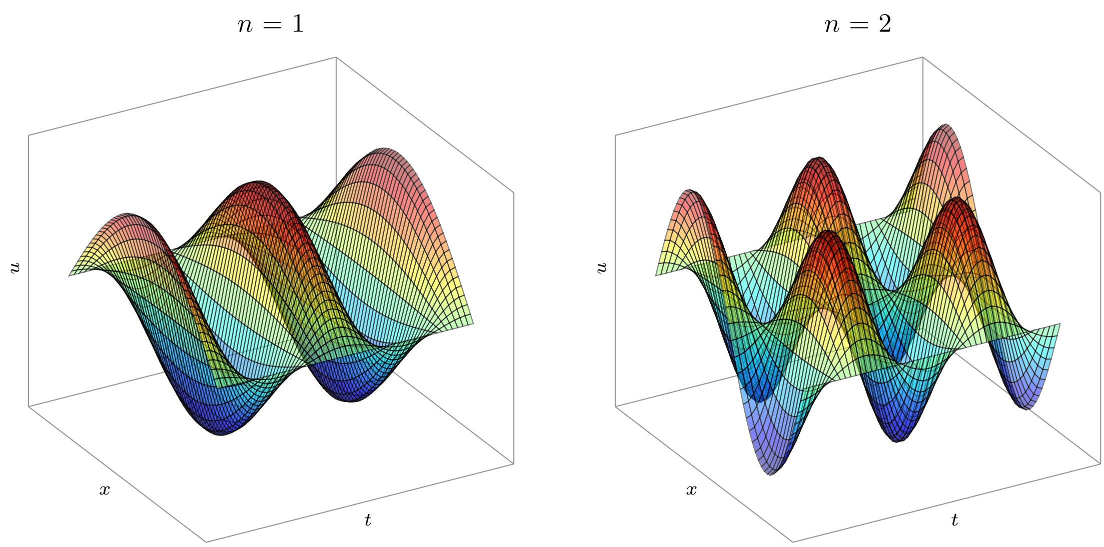
4. Graphing the first few normal modes as a function of time, evaluated at $t = 0, T/6, T/3, T/2$ where the time period is T. For simplicity, the amplitude is normalized to 1.

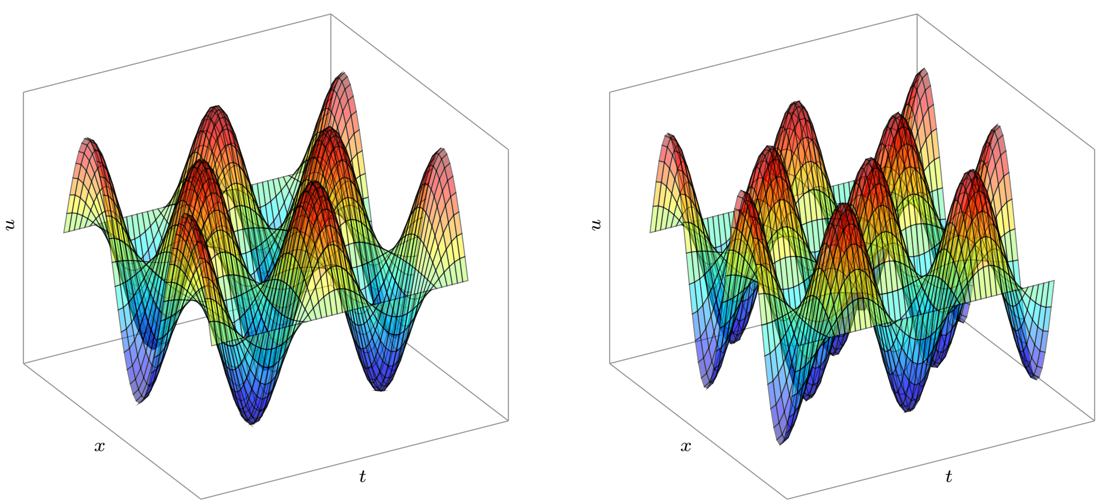

$n = 1$
1
0.5
0
-0.5
-1
0
0.25
0.5
0.75
1
$\cdot\pi$
$t = 0$
$t = T/6$
$t = T/3$
$t = T/2$


$n = 2$
1
0.5
0
-0.5
-1
0
0.25
0.5
0.75
1
$\cdot\pi$
$t = 0$
$t = T/6$
$t = T/3$
$t = T/2$


$n = 3$
1
0.5
0
−0.5
−1
0
0.25
0.5
0.75
1
$\cdot\pi$
$t = 0$
$t = T/6$
$t = T/3$
$t = T/2$


$n = 4$
1
0.5
0
−0.5
−1
0
0.25
0.5
0.75
1
$\cdot\pi$
$t = 0$
$t = T/6$
$t = T/3$
$t = T/2$


$n = 1$
u
x
t
$n = 2$
u
x
t



The sinusoidal nature of the surface along both the x and t axes is apparent from the $u_n(x, t)$ plots.

5. Finding the general solution of the PDE, with $L = c^2 = 1$

$$B_n = 2 \int_0^1 f(x) \sin(n\pi x) \ \mathrm{d}x \qquad = 2 \int_0^1 k \ \sin(3\pi x) \ \sin(n\pi x) \ \mathrm{d}x \tag{12.3.10}$$

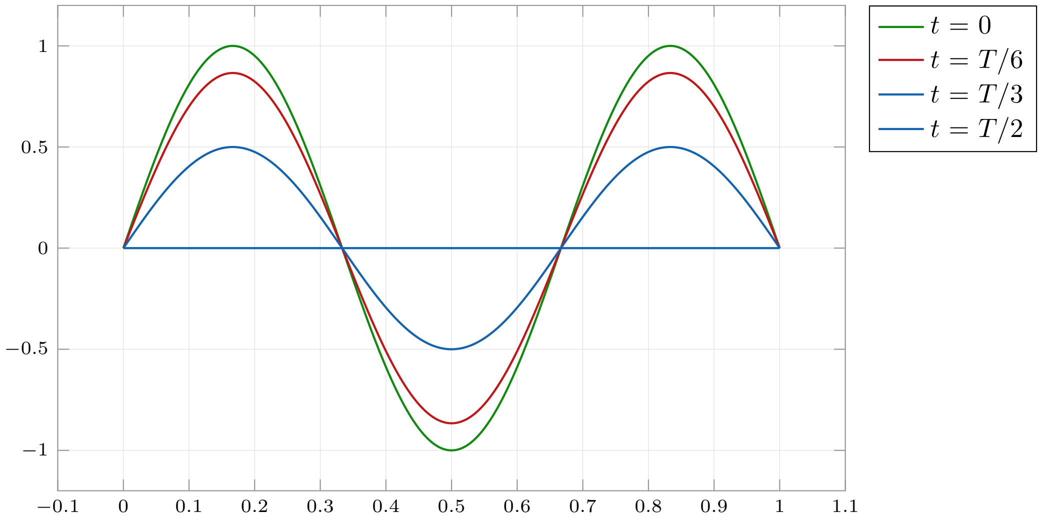
$$B_n = 0 \ \forall \ n \neq 3 \qquad B_3 = k \left[x - \cos(2\pi x) \right]_0^1 = k \tag{12.3.11}$$

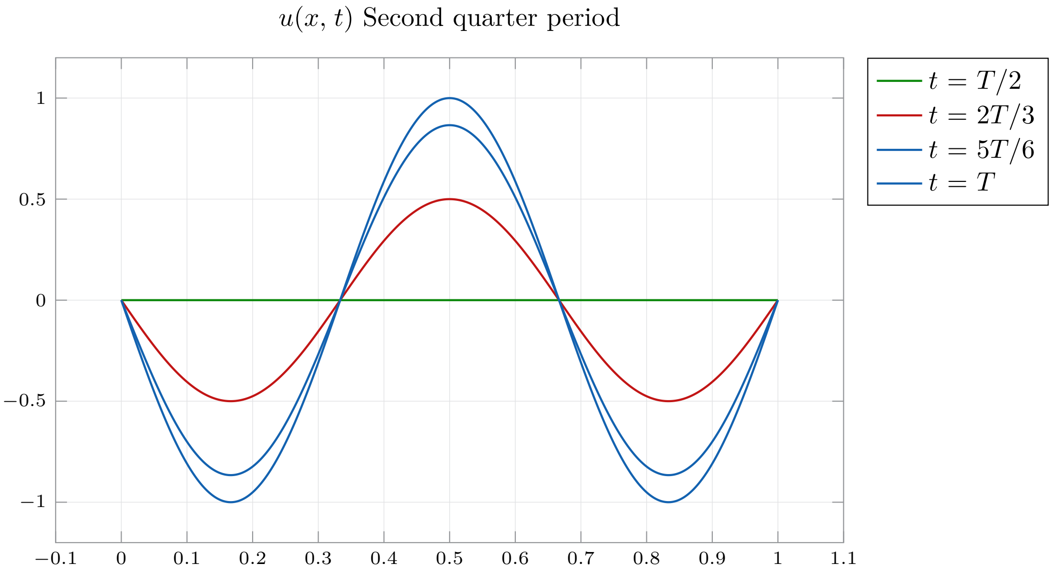
Substituting into the general solution,

$$\lambda_n = \frac{cn\pi}{L} = n\pi \qquad B_n^* = 0 \tag{12.3.12}$$

$$u_n(x, t) = B_n \ \cos(n\pi \ t) \ \sin(n\pi \ x) \qquad u = u_3 = k \ \cos(3\pi t) \ \sin(3\pi x) \tag{12.3.13}$$

$u(x, t)$ First quarter period





6. Finding the general solution of the PDE, with $L = c^2 = 1$

$$B_n = 2 \int_0^1 f(x) \sin(n\pi x) \, dx \tag{12.3.14}$$

$$= 2 \int_0^1 k \left[\sin(\pi x) - 0.5 \sin(2\pi x) \right] \sin(n\pi x) \, dx \tag{12.3.15}$$

$$B_n = 0 \; \forall \; n \neq \{1, 2\} \tag{12.3.16}$$

$$B_1 = k \left[x - \cos(2\pi x) \right]_0^1 = k \tag{12.3.17}$$

$$B_2 = -0.5k \left[x - \cos(4\pi x) \right]_0^1 = -0.5k \tag{12.3.18}$$

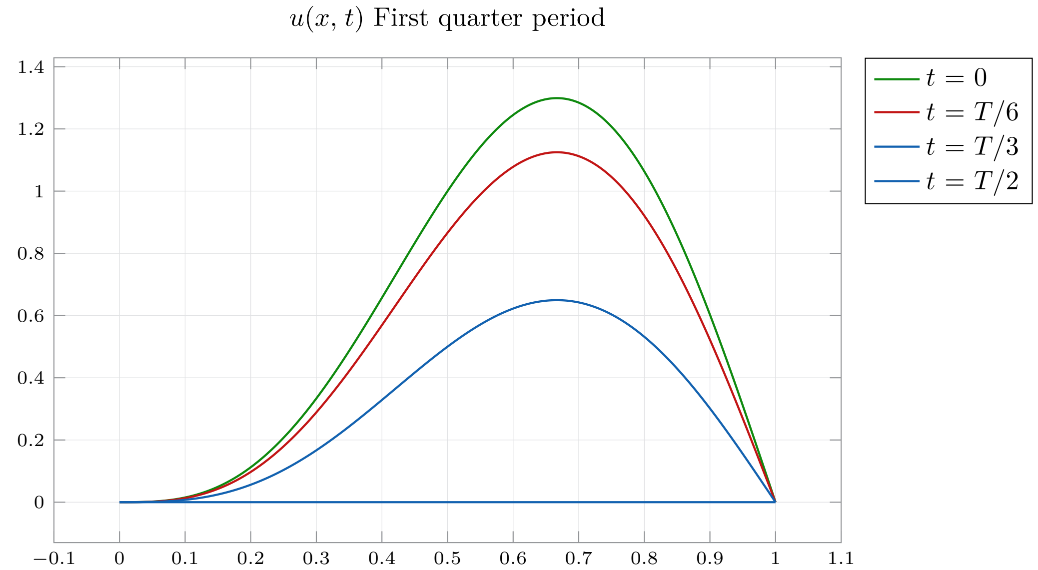
Substituting into the general solution,

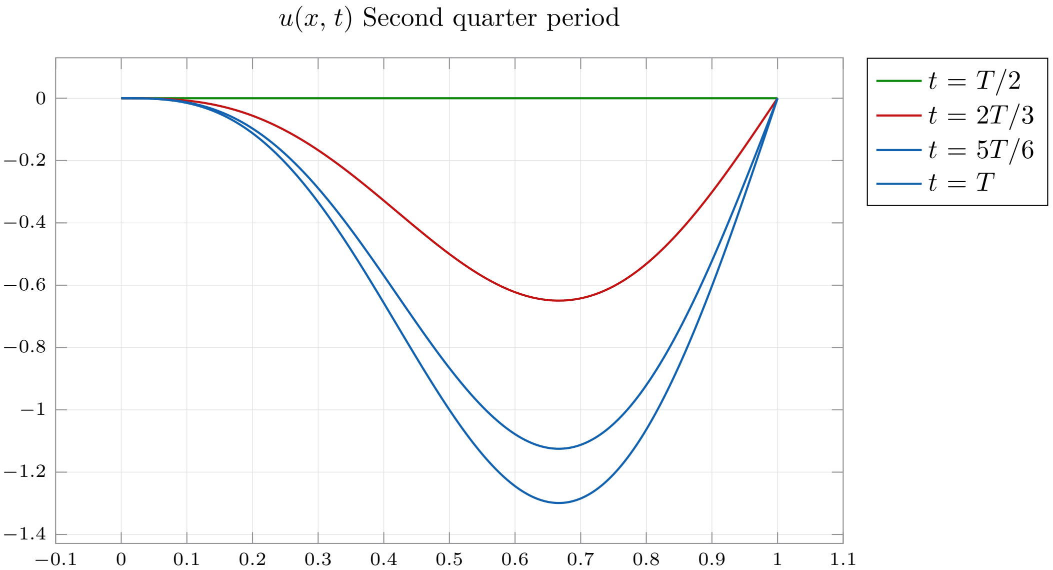
$$\lambda_n = \frac{cn\pi}{L} = n\pi \tag{12.3.19}$$

$$B_n^* = 0 \tag{12.3.20}$$

$$u_n(x, t) = \sum_{n=1}^{\infty} B_n \, \cos(n\pi \, t) \, \sin(n\pi \, x) \tag{12.3.21}$$

$$u = k \, \cos(\pi t) \, \sin(\pi x) - 0.5k \, \cos(2\pi \, t) \, \sin(2\pi \, x) \tag{12.3.22}$$





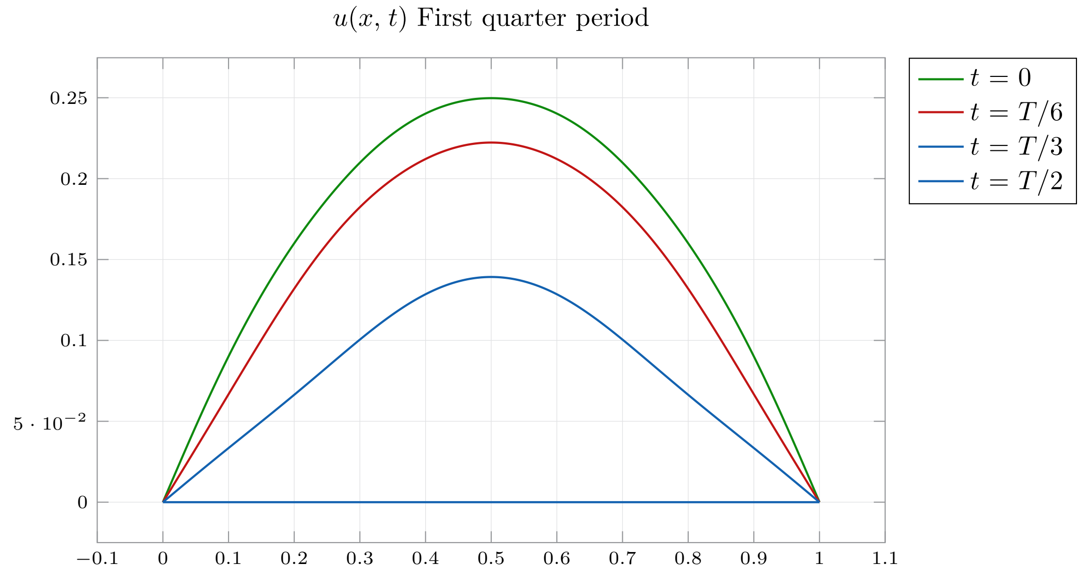
7. Finding the general solution of the PDE, with $L = c^2 = 1$

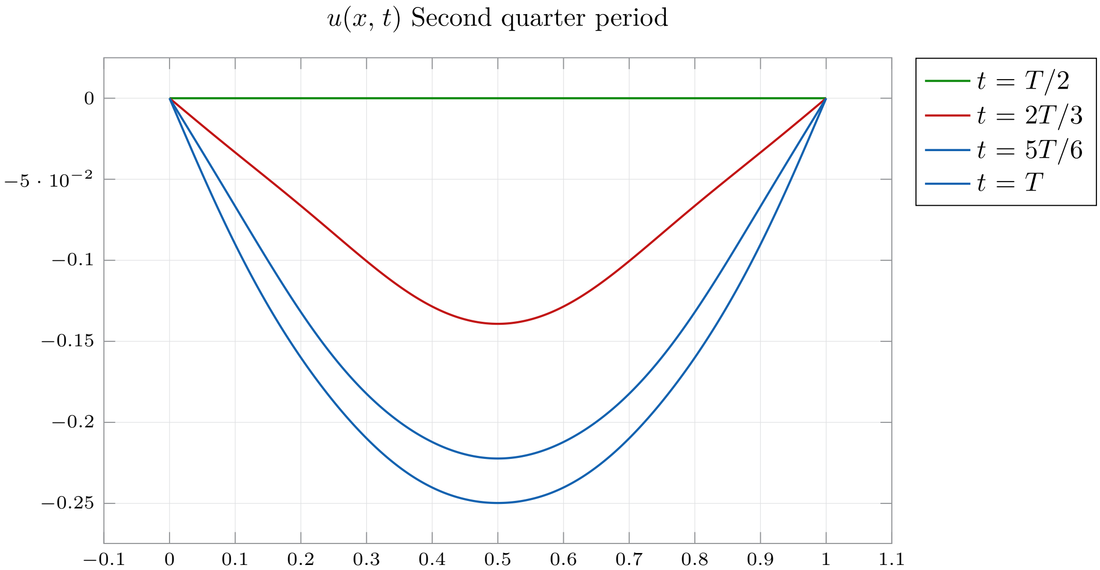
$$B_n = 2 \int_0^1 f(x)\sin(n\pi x)\ \mathrm{d}x \tag{12.3.23}$$

$$= 2 \int_0^1 kx(1-x)\ \sin(n\pi x)\ \mathrm{d}x \tag{12.3.24}$$

$$B_n = 2k\left[\frac{(1-2x)}{n^2\pi^2}\ \sin(n\pi x) + \frac{x(x-1)n^2\pi^2 - 2}{\pi^3 n^3}\ \cos(n\pi x)\right]_0^1 \tag{12.3.25}$$

$$= \frac{4k}{\pi^3 n^3}\left[1 - \cos(n\pi)\right] \tag{12.3.26}$$





8. Finding the general solution of the PDE, with $L = c^2 = 1$

$$B_n = 2 \int_0^1 f(x) \sin(n\pi x) \; \mathrm{d}x \tag{12.3.27}$$

$$= 2 \int_0^1 kx^2(1-x) \; \sin(n\pi x) \; \mathrm{d}x \tag{12.3.28}$$

$$B_n = 2k \left[f(x) \; \sin(n\pi x) + \frac{x^2(x-1)n^3\pi^3 - 6n\pi x + 2n\pi}{\pi^4 n^4} \; \cos(n\pi x) \right]_0^1 \tag{12.3.29}$$

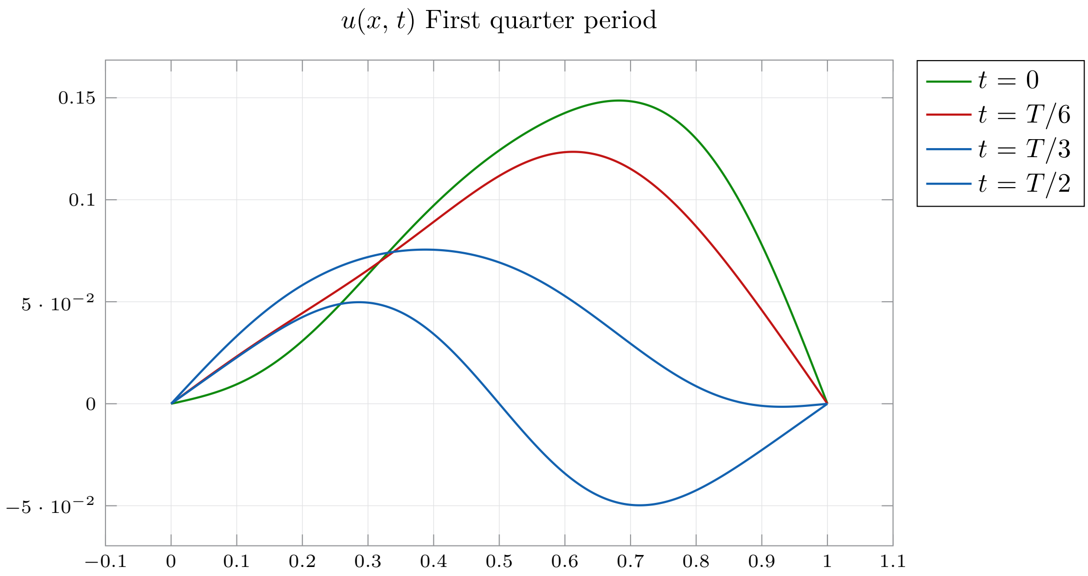
$$= \frac{-4k}{\pi^3 n^3} \left[1 + 2\cos(n\pi)\right] \tag{12.3.30}$$

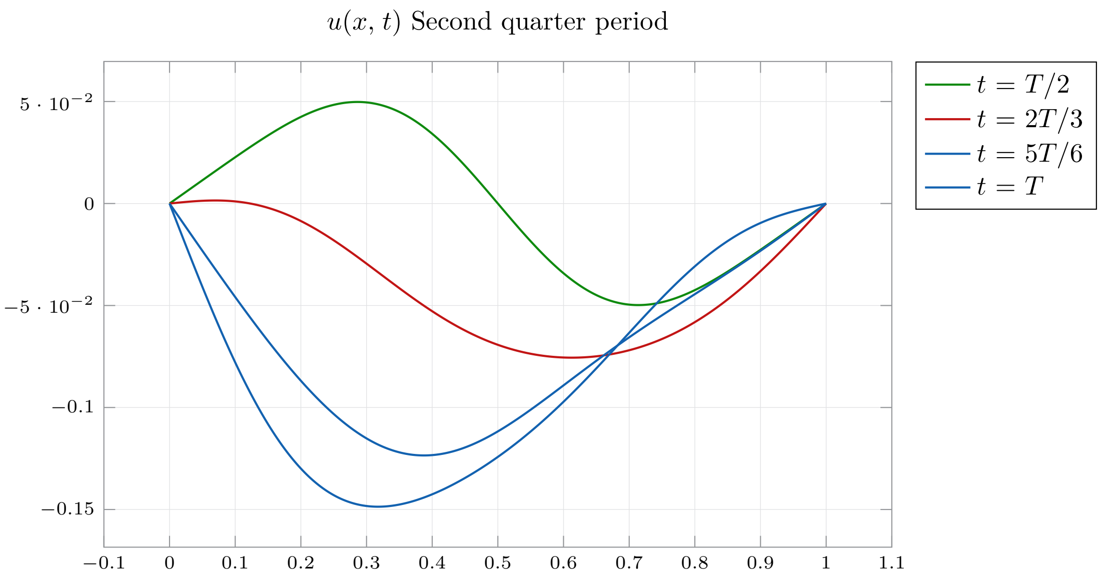
Substituting into the general solution,

$$\lambda_n = \frac{cn\pi}{L} = n\pi \quad 12.3.31$$

$$B_n^* = 0 \quad 12.3.32$$

$$u_n(x,t) = \sum_{n=1}^{\infty} B_n \ \cos(n\pi\ t)\ \ \sin(n\pi\ x)\sin(2\pi\ x) \quad 12.3.33$$





9. Finding the general solution of the PDE, with $L = c^2 = 1$

$$B_n = 2 \int_0^1 f(x) \sin(n\pi x) \; dx \tag{12.3.34}$$

$$= 0.4 \int_0^{0.5} (x) \; \sin(n\pi x) \; dx + 0.4 \int_{0.5}^1 (1-x) \; \sin(n\pi x) \; dx \tag{12.3.35}$$

$$B_n = 0.4 \left[\frac{\sin(n\pi x) - n\pi x \; \cos(n\pi x)}{\pi^2 n^2} \right]_0^{0.5} \tag{12.3.36}$$

$$+ 0.4 \left[\frac{\sin(n\pi x) - (x-1)n\pi \; \cos(n\pi x)}{\pi^2 n^2} \right]_1^{0.5} \tag{12.3.37}$$

$$= \frac{0.8}{n^2 \pi^2} \; \sin\left(\frac{n\pi}{2}\right) \tag{12.3.38}$$

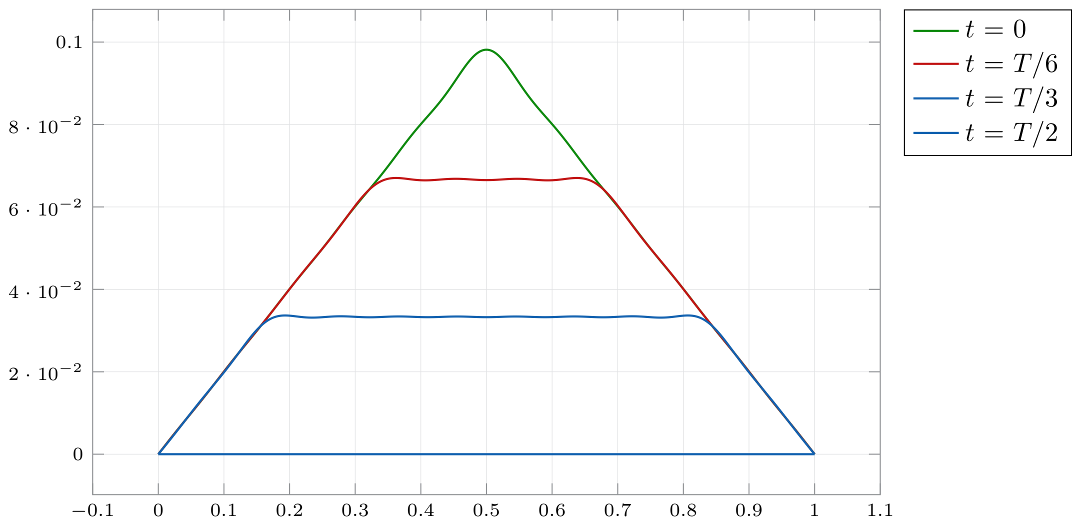
Substituting into the general solution,

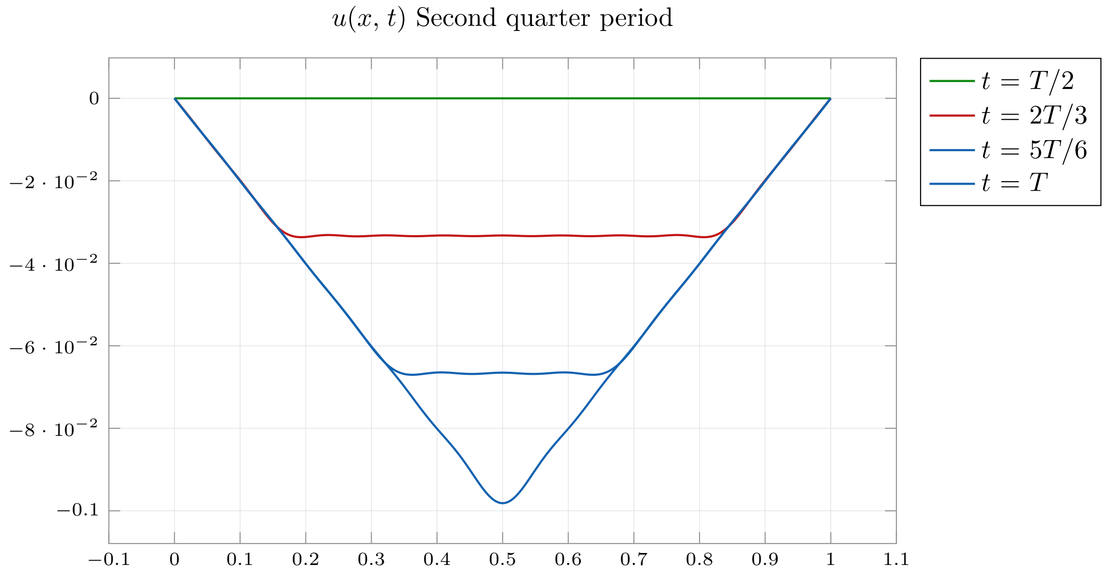
$$\lambda_n = \frac{cn\pi}{L} = n\pi \tag{12.3.39}$$

$$B_n^* = 0 \tag{12.3.40}$$

$$u_n(x, t) = \sum_{n=1}^{\infty} B_n \; \cos(n\pi \; t) \; \sin(n\pi \; x) \tag{12.3.41}$$

$u(x, t)$ First quarter period





10. Finding the general solution of the PDE, with $L = c^2 = 1$

$$B_n = 2 \int_0^1 f(x) \sin(n\pi x) \; \mathrm{d}x \tag{12.3.42}$$

$$= 2 \int_0^{0.25} (x) \; \sin(n\pi x) \; \mathrm{d}x + 2 \int_{0.25}^{0.75} (0.5 - x) \; \sin(n\pi x) \; \mathrm{d}x \tag{12.3.43}$$

$$+ 2 \int_{0.75}^{1} (x - 1) \; \sin(n\pi x) \; \mathrm{d}x \tag{12.3.44}$$

$$B_n = 2 \left[\frac{\sin(n\pi x) - n\pi x \; \cos(n\pi x)}{\pi^2 n^2} \right]_0^{0.25} \tag{12.3.45}$$

$$+ 2 \left[\frac{-\sin(n\pi x) + n\pi(x - 0.5) \; \cos(n\pi x)}{\pi^2 n^2} \right]_{0.25}^{0.75} \tag{12.3.46}$$

$$+ 2 \left[\frac{\sin(n\pi x) + n\pi(1 - x) \; \cos(n\pi x)}{\pi^2 n^2} \right]_{0.75}^{1} \tag{12.3.47}$$

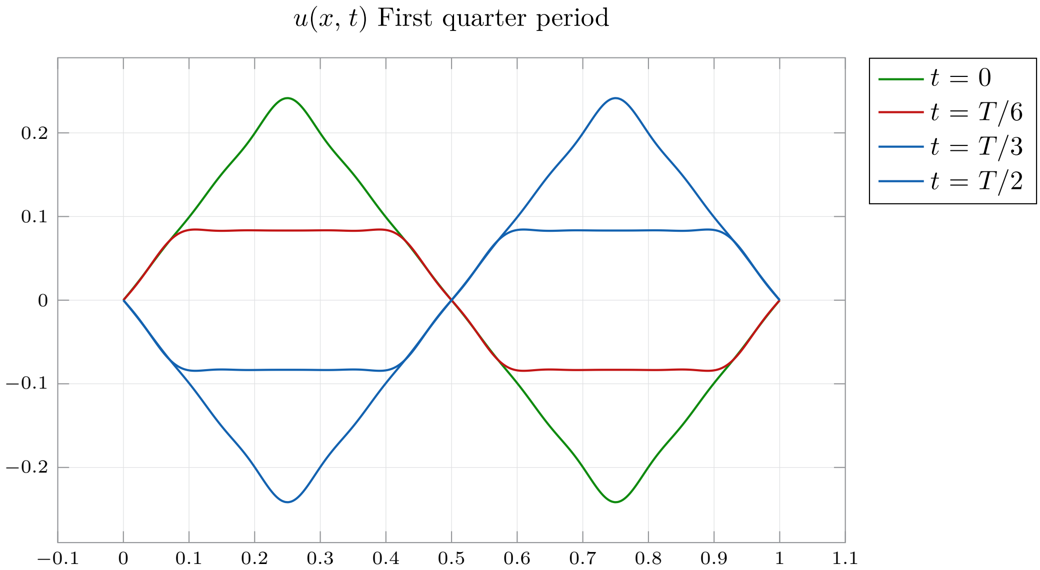
$$= \frac{4}{n^2\pi^2} \left[\sin(n\pi/4) - \sin(3n\pi/4) \right] \tag{12.3.48}$$

Substituting into the general solution,

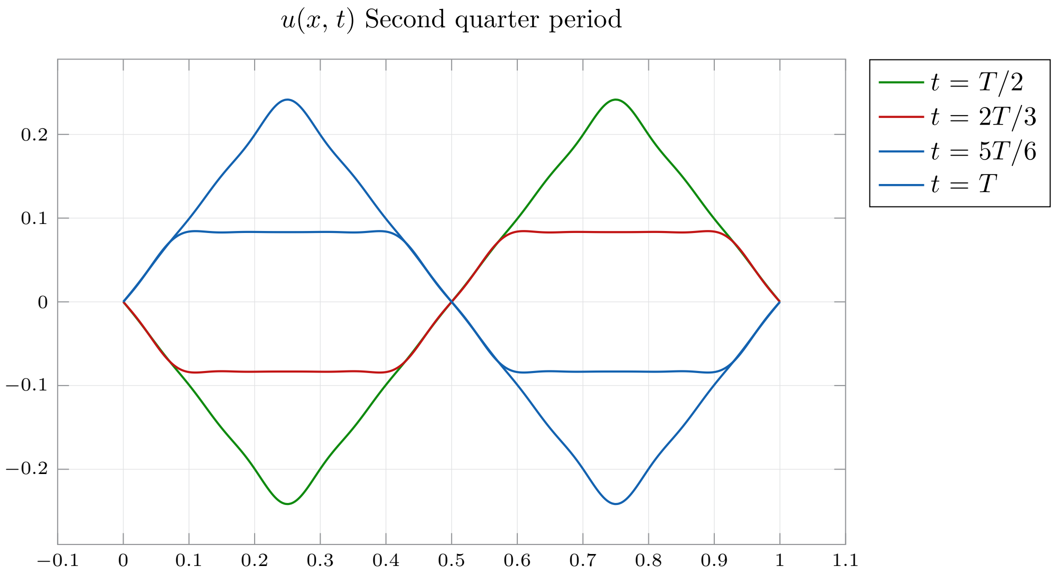
$$\lambda_n = \frac{cn\pi}{L} = n\pi \tag{12.3.49}$$

$$B_n^* = 0 \tag{12.3.50}$$

$$u_n(x, t) = \sum_{n=1}^{\infty} B_n \; \cos(n\pi \; t) \; \sin(n\pi \; x) \tag{12.3.51}$$



$u(x,t)$ First quarter period
$t = 0$
$t = T/6$
$t = T/3$
$t = T/2$
0.2
0.1
0
−0.1
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1



$u(x,t)$ Second quarter period
$t = T/2$
$t = 2T/3$
$t = 5T/6$
$t = T$
0.2
0.1
0
−0.1
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1

11. Finding the general solution of the PDE, with $L = c^2 = 1$

$$B_n = 2 \int_0^1 f(x) \sin(n\pi x) \ \mathrm{d}x \tag{12.3.52}$$

$$= 2 \int_{0.25}^{0.5} (x - 0.25) \ \sin(n\pi x) \ \mathrm{d}x + 2 \int_{0.5}^{0.75} (0.75 - x) \ \sin(n\pi x) \ \mathrm{d}x \tag{12.3.53}$$

$$B_n = 2 \left[\frac{\sin(n\pi x) - n\pi(x - 0.25) \ \cos(n\pi x)}{\pi^2 n^2} \right]_{0.25}^{0.5} \tag{12.3.54}$$

$$+ 2 \left[\frac{\sin(n\pi x) + (0.75 - x) n\pi \ \cos(n\pi x)}{\pi^2 n^2} \right]_{0.75}^{0.5} \tag{12.3.55}$$

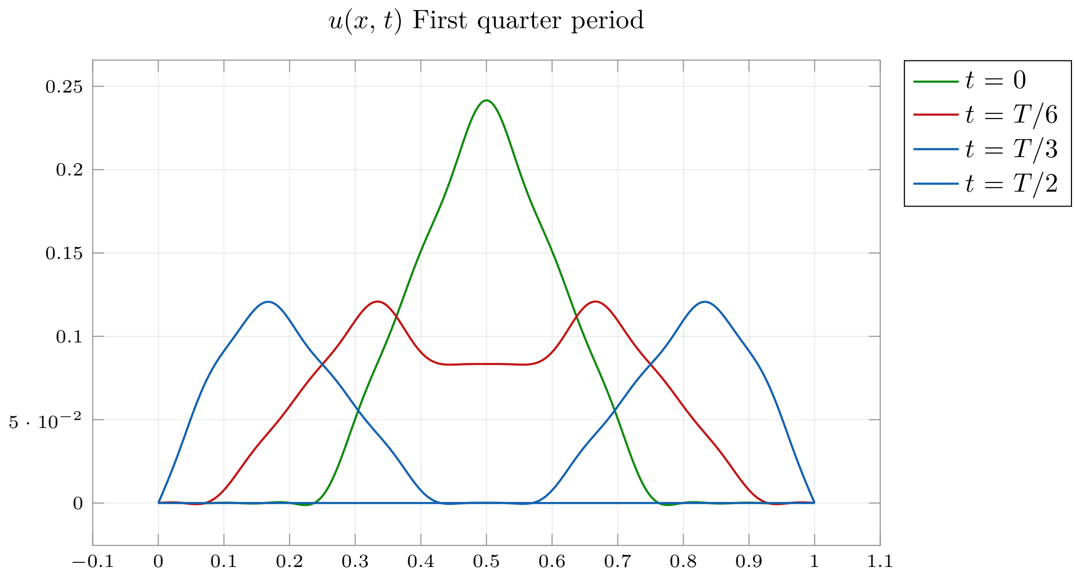
$$= \frac{2}{n^2\pi^2} \left[2 \sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{3n\pi}{4}\right) \right] \tag{12.3.56}$$

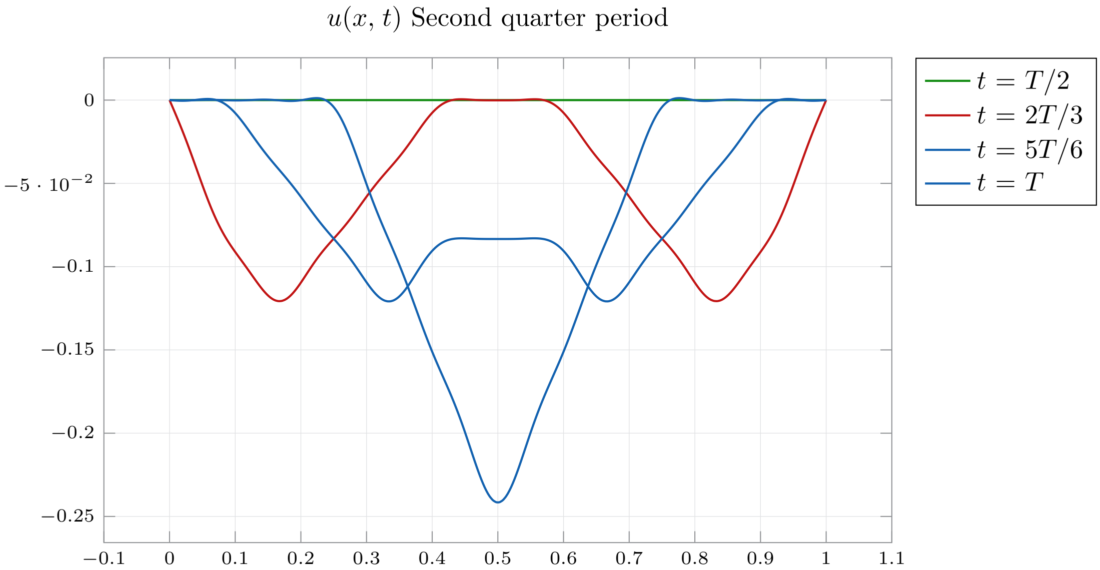
Substituting into the general solution,

$$\lambda_n = \frac{cn\pi}{L} = n\pi \tag{12.3.57}$$

$$B_n^* = 0 \tag{12.3.58}$$

$$u_n(x, t) = \sum_{n=1}^{\infty} B_n \ \cos(n\pi \ t) \ \sin(n\pi \ x) \tag{12.3.59}$$





12. Finding the general solution of the PDE, with $L = c^2 = 1$

$$B_n = 2\int_0^1 f(x)\sin(n\pi x)\ \mathrm{d}x \tag{12.3.60}$$

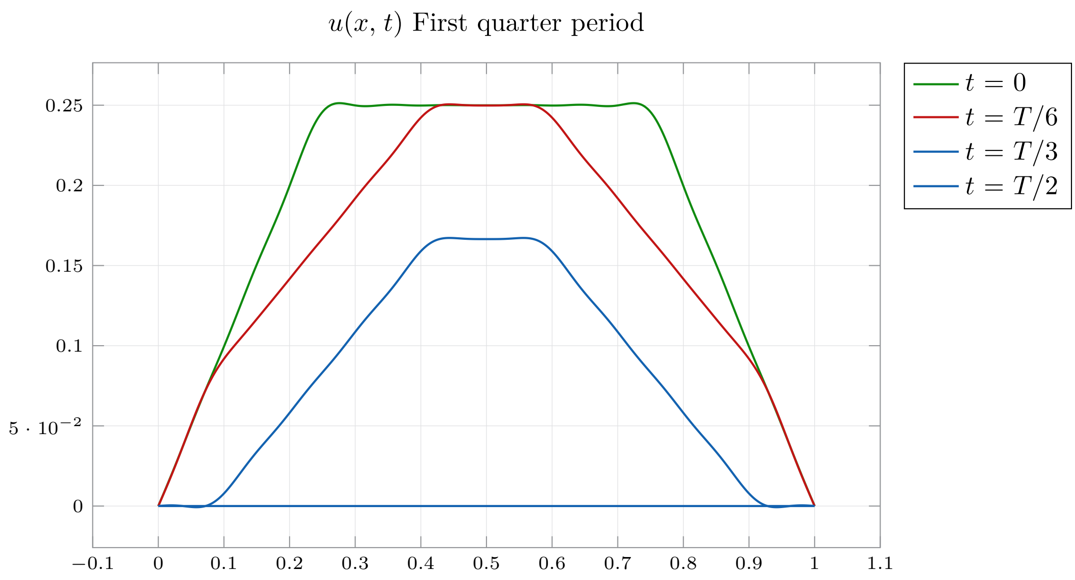
$$= 2\int_0^{0.25} (x)\ \sin(n\pi x)\ \mathrm{d}x + 2\int_{0.25}^{0.75} (0.25)\ \sin(n\pi x)\ \mathrm{d}x \tag{12.3.61}$$

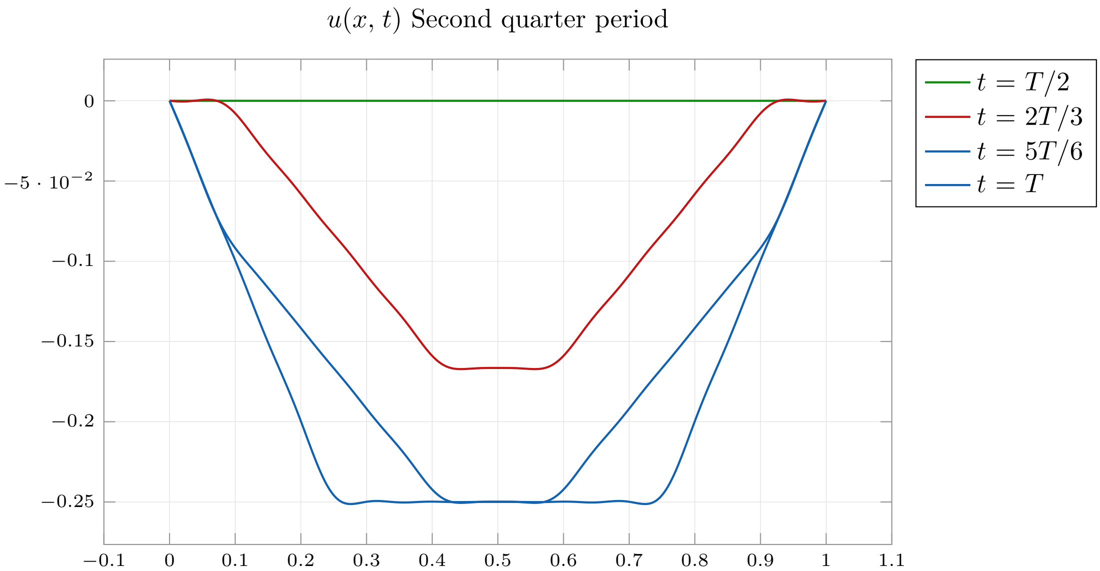
$$+ 2\int_{0.75}^{1} (1-x)\ \sin(n\pi x)\ \mathrm{d}x \tag{12.3.62}$$

$$B_n = 2\left[\frac{\sin(n\pi x) - n\pi x\ \cos(n\pi x)}{\pi^2 n^2}\right]_0^{0.25} + 2\left[\frac{-0.25\cos(n\pi x)}{n\pi}\right]_{0.25}^{0.75} \tag{12.3.63}$$

$$+ 2\left[\frac{\sin(n\pi x) + n\pi(1-x)\ \cos(n\pi x)}{\pi^2 n^2}\right]_1^{0.75} \tag{12.3.64}$$

$$= \frac{2}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)\right] \tag{12.3.65}$$





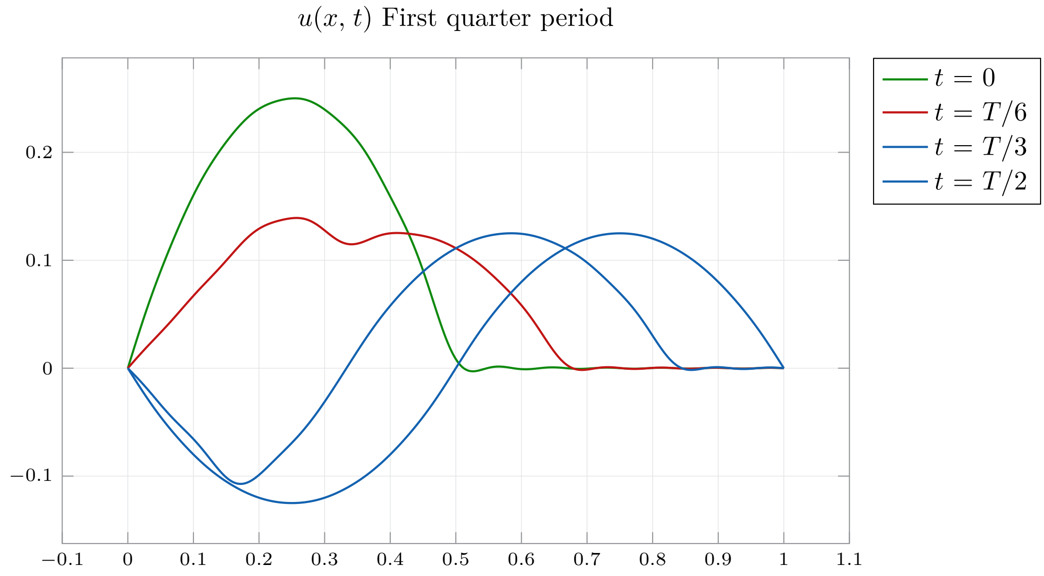
13. Finding the general solution of the PDE, with $L = c^2 = 1$

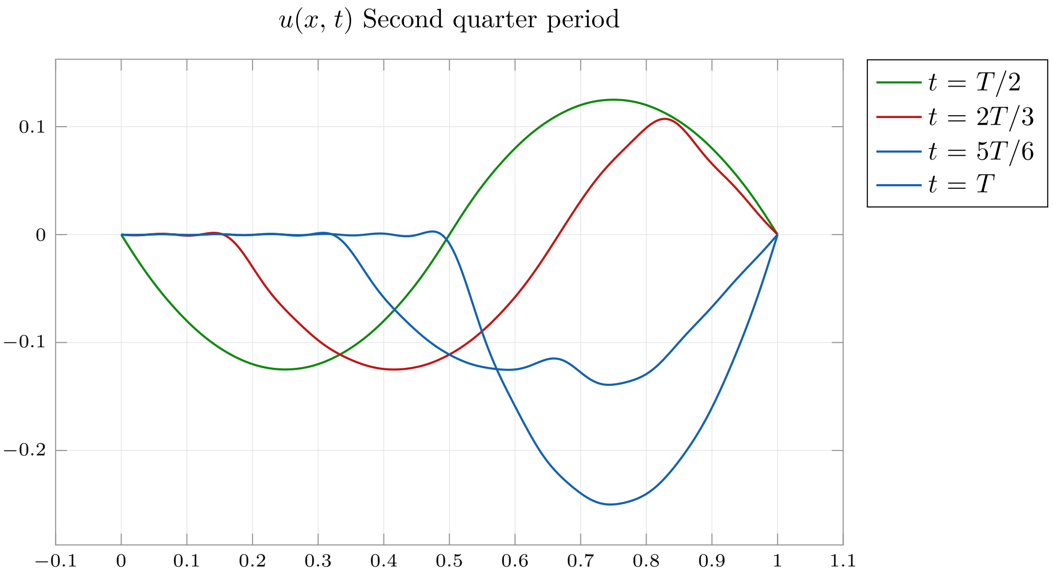
$$B_n = 2 \int_0^1 f(x) \sin(n\pi x) \ \mathrm{d}x \tag{12.3.66}$$

$$= 2 \int_0^{0.5} (2x)(1 - 2x) \ \sin(n\pi x) \ \mathrm{d}x \tag{12.3.67}$$

$$B_n = 4 \Bigg[\frac{n\pi(1 - 4x)}{n^3\pi^3} \ \sin(n\pi x) - \frac{n^2\pi^2 x(1 - 2x) + 4}{n^3\pi^3} \ \cos(n\pi x) \Bigg]_0^{0.5} \tag{12.3.68}$$

$$= \frac{1}{n^3\pi^3} \Bigg[- 4n\pi \sin\left(\frac{n\pi}{2}\right) - 16 \cos\left(\frac{n\pi}{2}\right) + 16 \Bigg] \tag{12.3.69}$$





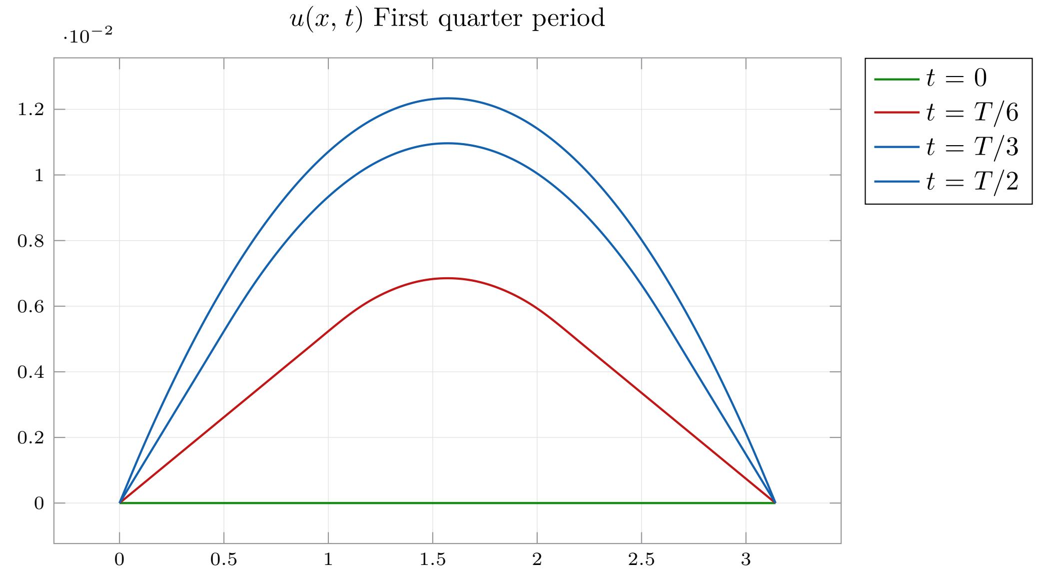
14. Finding the general solution of the PDE, with $L = c^2 = 1$

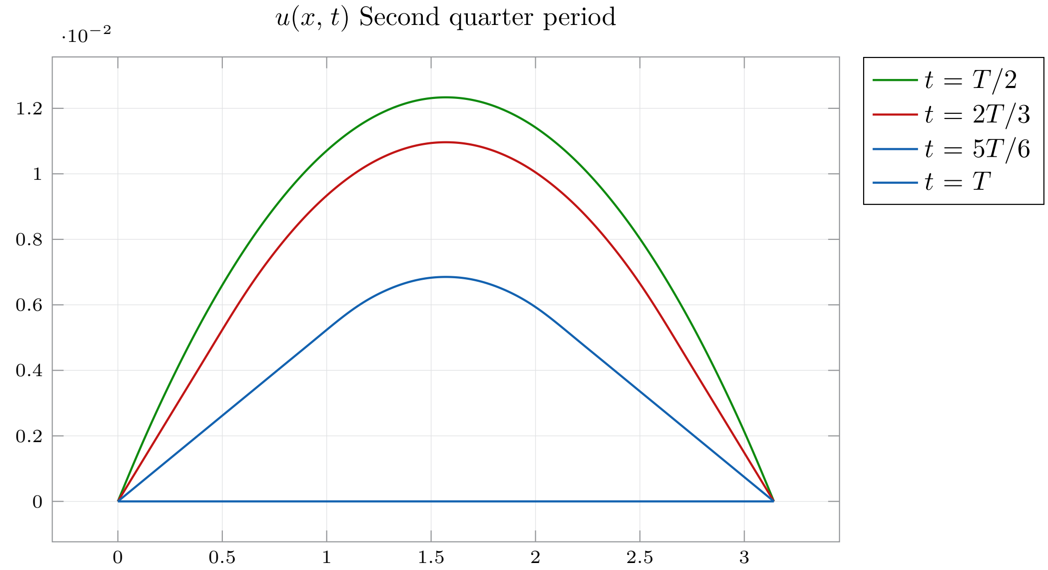
$$B_n^* = \frac{2}{n\pi} \int_0^{\pi} g(x) \sin(nx) \, \mathrm{d}x \tag{12.3.70}$$

$$= \frac{0.02}{n\pi} \left[\int_0^{\pi/2} (x) \sin(nx) \, \mathrm{d}x + \int_{\pi/2}^{\pi} (\pi - x) \, \sin(nx) \, \mathrm{d}x \right] \tag{12.3.71}$$

$$= \frac{0.02}{n\pi} \left[\frac{\sin(nx) - nx\cos(nx)}{n^2} \right]_0^{\pi/2} + \frac{0.02}{n\pi} \left[\frac{\sin(nx) + n(\pi - x)\cos(nx)}{n^2} \right]_{\pi}^{\pi/2} \tag{12.3.72}$$

$$B_n^* = \frac{0.04}{n^3\pi} \sin\left(\frac{n\pi}{2}\right) \tag{12.3.73}$$





15. Elastic beam PDE, Substituting,

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \qquad u(x, t) = F(x) \cdot G(t) \tag{12.3.74}$$

$$F \cdot \ddot{G} = -c^2 \; G \cdot F^{(4)} \qquad \frac{F^{(4)}}{F} = \frac{-\ddot{G}}{c^2 \cdot G} = \beta^4 \tag{12.3.75}$$

Solving the ODE in time,

$$\ddot{G} = -(c\beta^2)^2 \cdot G \qquad G(t) = a \; \cos(c\beta^2 \; t) + b \; \sin(c\beta^2 \; t) \tag{12.3.76}$$

Solving the ODE in space,

$$\frac{\mathrm{d}^4 F}{\mathrm{d}x^4} = \beta^4 \ F(x) \tag{12.3.77}$$

$$\lambda = \{\pm 1, \pm i\} \tag{12.3.78}$$

$$F(x) = A \ \cos(\beta x) + B \ \sin(\beta x) + C \ \cosh(\beta x) + D \ \sinh(\beta x) \tag{12.3.79}$$

16. For a simply supported beam,

$$u(0, t) = 0 \qquad (A + C) \cdot G(t) = 0 \tag{12.3.80}$$

$$A = -C \tag{12.3.81}$$

$$u(L, t) = 0 \qquad A\ [\cos(\beta L) - \cosh(\beta L)] + B \ \sin(\beta L) + D \ \sinh(\beta L) = 0 \tag{12.3.82}$$

Finding the second partial derivative in x,

$$\frac{\partial^2 u}{\partial x^2} = -\beta^2 \Big[A \ \cos(\beta x) + B \ \sin(\beta x) + A \ \cosh(\beta x) - D \ \sinh(\beta x)\Big] \tag{12.3.83}$$

$$\partial_x^2 u\Big|_{(0,t)} = 0 \tag{12.3.84}$$

$$A = 0, \qquad C = 0 \tag{12.3.85}$$

$$\partial_x^2 u\Big|_{(L,t)} = 0 \tag{12.3.86}$$

Solving the linear system in B, D gives,

$$B \ \sin(\beta L) + D \ \sinh(\beta L) = 0 \tag{12.3.87}$$

$$B \ \sin(\beta L) - D \ \sinh(\beta L) = 0 \tag{12.3.88}$$

$$2D \ \sinh(\beta L) = 0 \implies D = 0 \tag{12.3.89}$$

$$2B \ \sin(\beta L) = 0 \implies \beta L = 0 \tag{12.3.90}$$

Since $B = 0$ would lead to a trivial solution, the infinite set of eigenvalues λ and corresponding eigenfunctions are,

$$\beta_n = \frac{n\pi}{L} \tag{12.3.91}$$

$$u_n(x,t) = \Big[a_n \ \cos(c\beta_n^2 \ t) + b_n \ \sin(c\beta_n^2 \ t)\Big] \ \sin\left(\frac{n\pi}{L} \ x\right) \tag{12.3.92}$$

$$\frac{\partial u}{\partial t} = (c\beta^2)\Big[-a_n \ \sin(c\beta^2 \ t) + b_n \ \cos(c\beta^2 \ t)\Big] \ \sin\left(\frac{n\pi}{L} \ x\right) \tag{12.3.93}$$

$$\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = 0 \implies b_n = 0 \tag{12.3.94}$$

$$u(x,t) = a_n \ \cos(c\beta_n^2 \ t) \cdot \sin(\beta_n \ x) \tag{12.3.95}$$

17. Using the result of Problem 17, and the half-range Fourier series

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = x(L-x) \tag{12.3.96}$$

$$a_n = \frac{2}{L}\int_0^L f(x) \ \sin\left(\frac{n\pi x}{L}\right) \ \mathrm{d}x \tag{12.3.97}$$

$$= 2\left[g(x) \cdot \sin\left(\frac{n\pi x}{L}\right) + \frac{n^2\pi^2 \ x(x-L) - 2L^2}{n^3\pi^3} \ \cos\left(\frac{n\pi x}{L}\right)\right]_0^L \tag{12.3.98}$$

$$= \frac{4L^2}{n^3\pi^3} \ [1 - \cos(n\pi)] \tag{12.3.99}$$

$$u(x,t) = a_n \ \cos(c\beta_n^2 \ t) \cdot \sin(\beta_n \ x) \tag{12.3.100}$$

$$\beta_n = \frac{n\pi}{L} \tag{12.3.101}$$

18. Compared to Problem 7, the argument of $G(t)$ has a square term (in n) in the beam, and only a linear term in the string.

19. The endpoints of the beam have zero deflection and zero velocity for all t.

$$u(0,t) = 0 \qquad u(L,t) = 0 \tag{12.3.102}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,t)} = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{(L,t)} = 0 \tag{12.3.103}$$

Using these conditions,

$$F(0) \cdot G(t) = (A + C) \cdot G(t) = 0 \tag{12.3.104}$$

$$A = -C \tag{12.3.105}$$

$$F(L) \cdot G(t) = \Big[A \cos(\beta L) - A \cosh(\beta L) + B \sin(\beta L) + D \sinh(\beta L)\Big] \cdot G(t) = 0 \tag{12.3.106}$$

$$\frac{\partial u}{\partial x} = \beta \Big[- A \sin(\beta x) + B \cos(\beta x) - A \sinh(\beta x) + D \cosh(\beta x)\Big] \cdot G(t) \tag{12.3.107}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,t)} = 0 \qquad \Longrightarrow \qquad B = -D \tag{12.3.108}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(L,t)} = 0 = -A\Big[\sin(\beta L) + \sinh(\beta L)\Big] + B\Big[\cos(\beta L) - \cosh(\beta L)\Big] \tag{12.3.109}$$

This is a system of two linear equations in A, B. The Cramer determinant has to be zero to ensure a nontrivial solution. With $z = \beta L$ for convenience,

$$M = \begin{bmatrix} (\cos z - \cosh z) & (\sin z - \sinh z) \\ (-\sin z - \sinh z) & (\cos z - \cosh z) \end{bmatrix} \tag{12.3.110}$$

$$\det M = \cos^2 z + \cosh^2 z - 2 \cos z \cosh z + \sin^2 z - \sinh^2 z \tag{12.3.111}$$

$$= 2(1 - \cos z \cosh z) \tag{12.3.112}$$

From reading the graphs of these two functions, solutions are,

$$\cos(\theta) \cosh(\theta) = 1 \tag{12.3.113}$$

$$\theta = \{4.73, 7.85, 10.99, \dots\} \tag{12.3.114}$$

There are infinitely many solutions.

20. Using the given boundary conditions, and $z = \beta L$

$$F(x) = A \ \cos(\beta x) + B \ \sin(\beta x) + C \ \cosh(\beta x) + D \ \sinh(\beta x) \tag{12.3.115}$$

$$F(0) = A + C = 0 \tag{12.3.116}$$

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \beta\Big[- A \sin(\beta x) + B \cos(\beta x) + C \sinh(\beta x) + D \cosh(\beta x)\Big] \tag{12.3.117}$$

$$\left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=0} = B + D = 0 \tag{12.3.118}$$

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = \beta^2\Big[- A \cos(\beta x) - B \sin(\beta x) + C \cosh(\beta x) + D \sinh(\beta x)\Big] \tag{12.3.119}$$

$$\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right|_{x=L} = -A\Big[\cos z + \cosh z\Big] - B\Big[\sin z + \sinh z\Big] = 0 \tag{12.3.120}$$

$$\frac{\mathrm{d}^3F}{\mathrm{d}x^3} = \beta^3\Big[A \sin(\beta x) - B \cos(\beta x) + C \sinh(\beta x) + D \cosh(\beta x)\Big] \tag{12.3.121}$$

$$\left.\frac{\mathrm{d}^3F}{\mathrm{d}x^3}\right|_{x=L} = A\Big[\sin z - \sinh z\Big] - B\Big[\cos z + \cosh z\Big] = 0 \tag{12.3.122}$$

This is a system of linear equations in A, B which needs a zero Cramer determinant to ensure a nontrivial solution.

$$M = \begin{bmatrix} (\cos z + \cosh z) & (\sin z + \sinh z) \\ (\sin z - \sinh z) & (\cos z + \cosh z) \end{bmatrix} \tag{12.3.123}$$

$$\det M = \cos^2 z + \cosh^2 z + 2 \cos z \cosh z + \sin^2 z - \sinh^2 z \tag{12.3.124}$$

$$= 2(1 + \cos z \cosh z) \tag{12.3.125}$$

From reading the graphs of these two functions, solutions are,

$$\cos(\theta) \cosh(\theta) = -1 \tag{12.3.126}$$

$$\theta = \{4.73, 7.85, 10.99, \dots\} \tag{12.3.127}$$

There are infinitely many solutions.

12.4 D'Alembert's Solution of the Wave Equation

1. Speed is defined as the distance traveled per unit time. Consider the change in argument in going from $t = t_1$ to $t = t_2$,

$$(x + ct_2) - (x + ct_1) = c(t_2 - t_1) \qquad s = \frac{\delta x}{\delta t} = c \tag{12.4.1}$$

$$(x - ct_2) - (x - ct_1) = c(t_1 - t_2) \qquad s = \frac{\delta x}{\delta t} = -c \tag{12.4.2}$$

Thus, the waveforms ϕ, ψ are moving in opposite directions with the same speed c.

2. The boundary conditions are,

$$u(0, t) = 0 \quad \forall \quad t \geq 0 \qquad u(L, t) = 0 \quad \forall \quad t \geq 0 \tag{12.4.3}$$

The effect on the solution is now,

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} \tag{12.4.4}$$

$$u(0, t) = 0 = \frac{f(ct) + f(-ct)}{2} \qquad f(-ct) = -f(ct) \tag{12.4.5}$$

This makes $f(z)$ an odd function.

3. From the text,

$$c^2 = \frac{T}{\rho} = \frac{T \cdot Lg}{W} \qquad = \frac{300 \cdot 2 \cdot 9.8}{0.9} \tag{12.4.6}$$

$$c = 80.83 \text{ m s}^{-1} \tag{12.4.7}$$

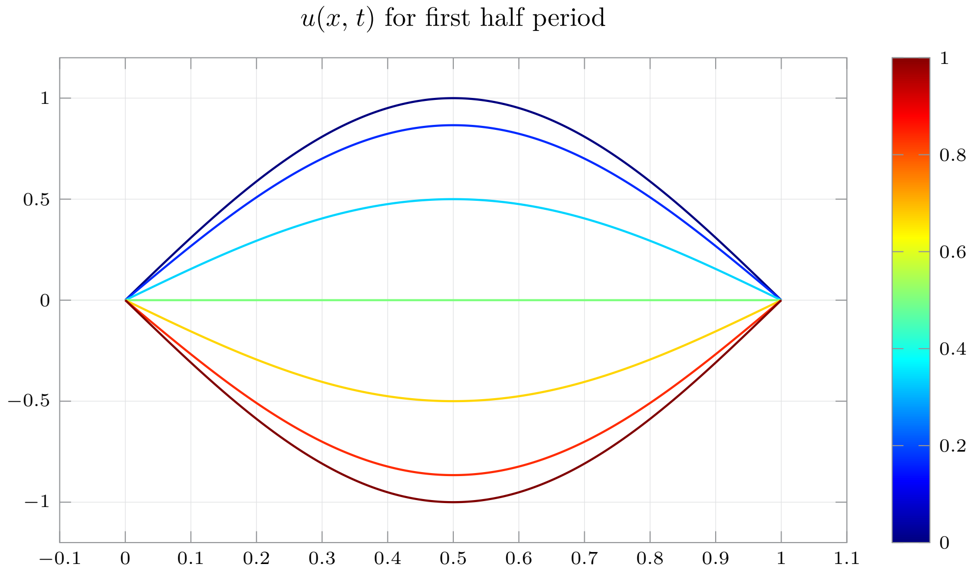
4. From the text,

$$\lambda_n = \frac{c_n \pi}{L} = \frac{80.83 \cdot \pi}{L} n \qquad \lambda_n = 127n \tag{12.4.8}$$

5. Using the direct result for $u(x, t)$ given zero initial velocity,

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} \qquad f(x) = k \sin(\pi x) \tag{12.4.9}$$

$$u(x, t) = \frac{k}{2}\Big[\sin(\pi x + \pi ct) + \sin(\pi x - \pi ct)\Big] \qquad u(x, t) = k \sin(\pi x) \cos(\pi ct) \tag{12.4.10}$$

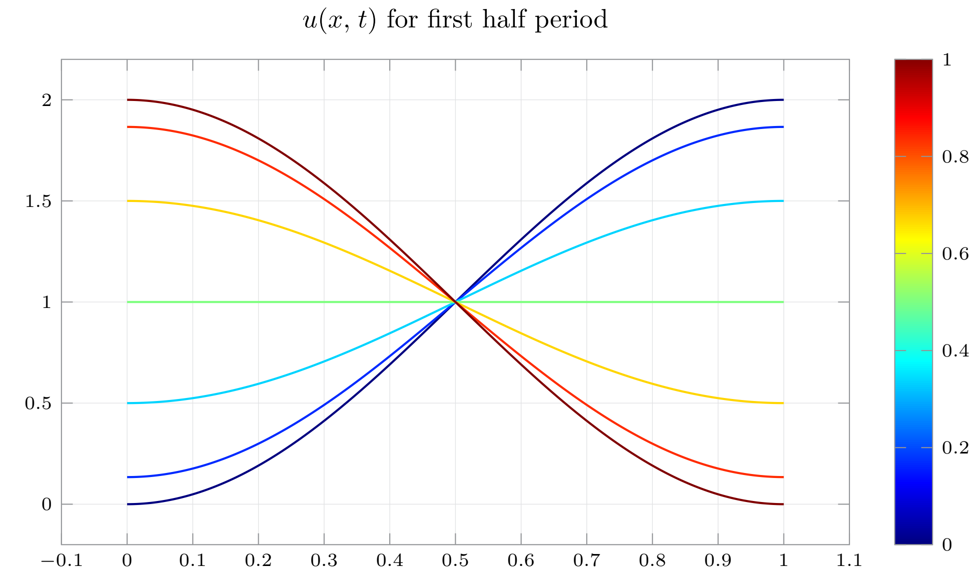


6. Using the direct result for $u(x, t)$ given zero initial velocity,

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} \qquad f(x) = k[1 - \cos(\pi x)] \tag{12.4.11}$$

$$u(x, t) = \frac{k}{2}\Big[1 - \cos(\pi x + \pi ct) + 1 - \cos(\pi x - \pi ct)\Big] \tag{12.4.12}$$

$$u(x, t) = k[1 - \cos(\pi x)\cos(\pi ct)] \tag{12.4.13}$$



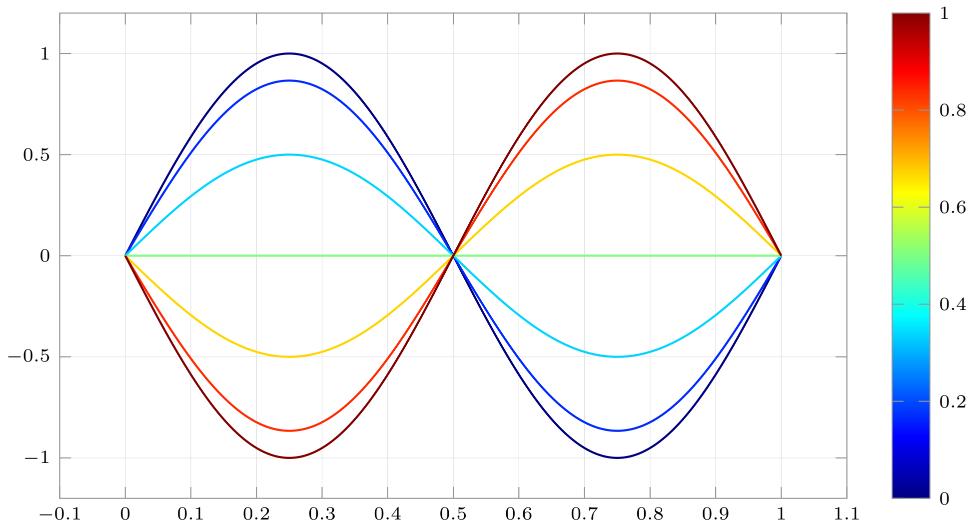
7. Using the direct result for $u(x,t)$ given zero initial velocity,

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} \qquad f(x) = k[\sin(2\pi x)] \tag{12.4.14}$$

$$u(x,t) = \frac{k}{2}\Big[\sin(2\pi x + 2\pi ct) + \sin(2\pi x - 2\pi ct)\Big] \tag{12.4.15}$$

$$u(x,t) = k[\sin(2\pi x)\cos(2\pi ct)] \tag{12.4.16}$$

$u(x,t)$ for first half period

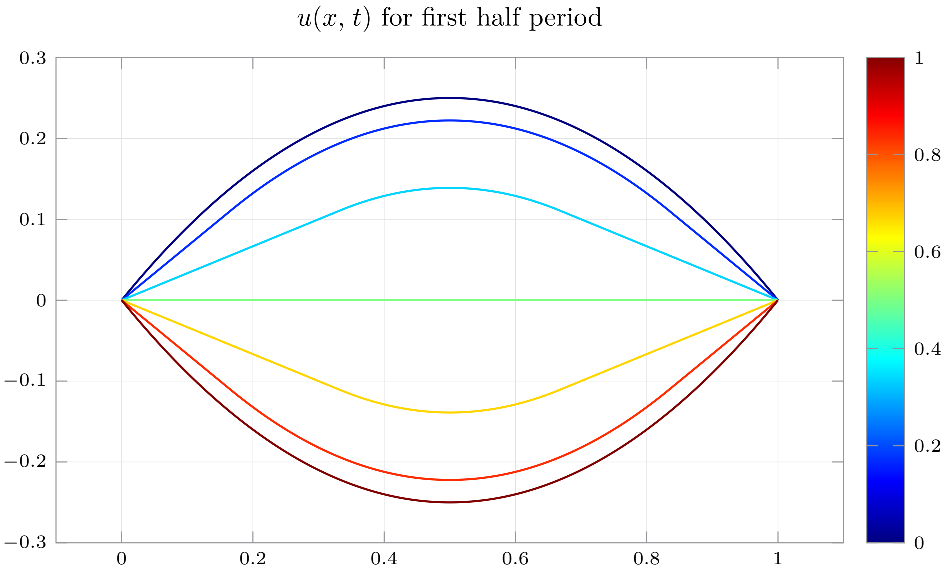


8. Using the direct result for $u(x,t)$ given zero initial velocity,

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} \tag{12.4.17}$$

$$f(x) = kx(1-x) \tag{12.4.18}$$

$$f^*(x) = \begin{cases} kx(1+x) & x \in [-1, 0] \\ kx(1-x) & x \in [0, 1] \end{cases} \tag{12.4.19}$$



In the above plot, the odd periodic extension of f is found using

$$f(-x) = -f(x) \tag{12.4.20}$$

$$F(x) \text{ defined in } [-L, L] \tag{12.4.21}$$

$$u(x, t) = \frac{F\Big(\text{modulo}(x + ct, 2L) - L\Big) + F\Big(\text{modulo}(x - ct, 2L) - L\Big)}{2} \tag{12.4.22}$$

This becomes an infitely repeating periodic odd function with period $2L$.

9. Transforming to normal form,

$$u_{xx} + 4u_{yy} = 0 \qquad A = 1, \; B = 0, \; C = 4 \tag{12.4.23}$$

$$AC - B^2 > 0 \qquad \text{Elliptic} \tag{12.4.24}$$

$$y'^2 + 4 = 0 \qquad y' = \{-2i, 2i\} \tag{12.4.25}$$

$$\psi(x, y) = v = y + 2ix \qquad \phi(x, y) = w = y - 2ix \tag{12.4.26}$$

Expressing the PDE in terms of (v, w),

$$u_x = u_v \, v_x + u_w \, w_x \qquad = 2iu_v - 2iu_w \tag{12.4.27}$$

$$u_{xx} = 2i[u_{vv} - u_{wv}](2i) - 2i[u_{vw} - u_{ww}](-2i) \qquad = -4[u_{vv} + u_{ww}] \tag{12.4.28}$$

$$u_y = u_v \, v_y + u_w \, w_y \qquad = u_v + u_w \tag{12.4.29}$$

$$u_{yy} = u_{vv} + 2u_{vw} + u_{ww} \tag{12.4.30}$$

The PDE in standard form for some sufficiently differentiable functions f, g is,

$$u_{xx} + 4u_{yy} = 0 \qquad \Longrightarrow \qquad u_{vw} = 0 \tag{12.4.31}$$

$$u(v, w) = f(v) + g(w) \qquad u(x, y) = f(y + 2ix) + g(y - 2ix) \tag{12.4.32}$$

10. Transforming to normal form,

$$u_{xx} - 16u_{yy} = 0 \qquad A = 1,\ \ B = 0,\ \ C = -16 \tag{12.4.33}$$

$$AC - B^2 < 0 \qquad \text{Hyperbolic} \tag{12.4.34}$$

$$y'^2 - 16 = 0 \qquad y' = \{-4, 4\} \tag{12.4.35}$$

$$\psi(x, y) = v = y + 4x \qquad \phi(x, y) = w = y - 4x \tag{12.4.36}$$

Expressing the PDE in terms of (v, w),

$$u_x = u_v\ v_x + u_w\ w_x \qquad = 4u_v - 4u_w \tag{12.4.37}$$

$$u_{xx} = 4[u_{vv} - u_{wv}](4) + 4[u_{vw} - u_{ww}](-4) \qquad = 16[u_{vv} + u_{ww} - 2u_{vw}] \tag{12.4.38}$$

$$u_y = u_v\ v_y + u_w\ w_y \qquad = u_v + u_w \tag{12.4.39}$$

$$u_{yy} = u_{vv} + 2u_{vw} + u_{ww} \tag{12.4.40}$$

The PDE in standard form for some sufficiently differentiable functions f, g is,

$$u_{xx} - 16u_{yy} = 0 \qquad \Longrightarrow \qquad u_{vw} = 0 \tag{12.4.41}$$

$$u(v, w) = f(v) + g(w) \qquad u(x, y) = f(y + 4x) + g(y - 4x) \tag{12.4.42}$$

11. Transforming to normal form,

$$u_{xx} + 2u_{xy} + u_{yy} = 0 \qquad A = 1,\ \ B = 1,\ \ C = 1 \tag{12.4.43}$$

$$AC - B^2 = 0 \qquad \text{Parabolic} \tag{12.4.44}$$

$$y'^2 - 2y' + 1 = 0 \qquad y' = \{1, 1\} \tag{12.4.45}$$

$$\psi(x, y) = v = x \qquad \phi(x, y) = w = y - x \tag{12.4.46}$$

Expressing the PDE in terms of (v, w),

$$u_x = u_v\ v_x + u_w\ w_x \qquad = u_v - u_w \tag{12.4.47}$$

$$u_{xx} = [u_{vv} - u_{wv}](1) + [u_{vw} - u_{ww}](-1) \qquad = u_{vv} + u_{ww} - 2u_{vw} \tag{12.4.48}$$

$$u_y = u_v\ v_y + u_w\ w_y \qquad = u_w \tag{12.4.49}$$

$$u_{yy} = u_{ww} \tag{12.4.50}$$

$$u_{xy} = u_{ww}(-1) + u_{wv}(1) \qquad = -u_{ww} + u_{vw} \tag{12.4.51}$$

The PDE in standard form for some sufficiently differentiable functions f, g is,

$$u_{xx} + 2u_{xy} + u_{yy} = 0 \qquad u_{vv} = 0 \tag{12.4.52}$$

$$u(v, w) = v \cdot f(w) + g(w) \qquad u(x, y) = x \cdot f(y - x) + g(y - x) \tag{12.4.53}$$

12. Transforming to normal form,

$$u_{xx} - 2u_{xy} + u_{yy} = 0 \qquad A = 1,\ B = -1,\ C = 1 \tag{12.4.54}$$

$$AC - B^2 = 0 \qquad \text{Parabolic} \tag{12.4.55}$$

$$y'^2 + 2y' + 1 = 0 \qquad y' = \{-1, -1\} \tag{12.4.56}$$

$$\psi(x, y) = v = x \qquad \phi(x, y) = w = y + x \tag{12.4.57}$$

Expressing the PDE in terms of (v, w),

$$u_x = u_v\ v_x + u_w\ w_x \qquad = u_v + u_w \tag{12.4.58}$$

$$u_{xx} = [u_{vv} + u_{wv}](1) + [u_{vw} + u_{ww}](1) \qquad = u_{vv} + u_{ww} + 2u_{vw} \tag{12.4.59}$$

$$u_y = u_v\ v_y + u_w\ w_y \qquad = u_w \tag{12.4.60}$$

$$u_{yy} = u_{ww} \tag{12.4.61}$$

$$u_{xy} = u_{wv}(1) + u_{ww}(1) \qquad = u_{ww} + u_{vw} \tag{12.4.62}$$

The PDE in standard form for some sufficiently differentiable functions f, g is,

$$u_{xx} - 2u_{xy} + u_{yy} = 0 \qquad u_{vv} = 0 \tag{12.4.63}$$

$$u(v, w) = v \cdot f(w) + g(w) \qquad u(x, y) = x \cdot f(y + x) + g(y + x) \tag{12.4.64}$$

13. Transforming to normal form,

$$u_{xx} + 5u_{xy} + 4u_{yy} = 0 \qquad A = 1, \;\; B = 2.5, \;\; C = 4 \tag{12.4.65}$$

$$AC - B^2 < 0 \qquad \text{Hyperbolic} \tag{12.4.66}$$

$$y'^2 - 5y' + 4 = 0 \qquad y' = \{4, 1\} \tag{12.4.67}$$

$$\psi(x, y) = v = y - 4x \qquad \phi(x, y) = w = y - x \tag{12.4.68}$$

Expressing the PDE in terms of (v, w),

$$u_x = u_v \; v_x + u_w \; w_x \qquad = -4u_v - u_w \tag{12.4.69}$$

$$u_{xx} = 4 \left[4u_{vv} + u_{wv}\right] + \left[4u_{vw} + u_{ww}\right] \qquad = 16u_{vv} + u_{ww} + 8u_{vw} \tag{12.4.70}$$

$$u_y = u_v \; v_y + u_w \; w_y \qquad = u_v + u_w \tag{12.4.71}$$

$$u_{yy} = u_{vv} + 2u_{wv} + u_{ww} \tag{12.4.72}$$

$$u_{xy} = (-4)(u_{vv} + u_{wv}) + (-1)(u_{vw} + u_{ww}) \qquad = -4u_{vv} - u_{ww} - 5u_{vw} \tag{12.4.73}$$

The PDE in standard form for some sufficiently differentiable functions f, g is,

$$u_{xx} + 5u_{xy} + 4u_{yy} = 0 \qquad u_{vw} = 0 \tag{12.4.74}$$

$$u(v, w) = f(v) + g(w) \qquad u(x, y) = f(y - x) + g(y - 4x) \tag{12.4.75}$$

14. Transforming to normal form,

$$x \; u_{xy} - y \; u_{yy} = 0 \qquad A = 0, \;\; B = x/2, \;\; C = -y \tag{12.4.76}$$

$$AC - B^2 = -\frac{x^2}{4} < 0 \qquad \text{Hyperbolic} \tag{12.4.77}$$

$$-x \; y' - y = 0 \qquad y = \frac{c}{x} \tag{12.4.78}$$

$$\psi(x, y) = v = x \qquad \phi(x, y) = w = yx \tag{12.4.79}$$

Expressing the PDE in terms of (v, w),

$$u_x = u_v \; v_x + u_w \; w_x \qquad = u_v + y \; u_w \tag{12.4.80}$$

$$u_{xy} = (u_{vw} + y \; u_{ww})(x) + u_w \qquad = x \; u_{vw} + xy \; u_{ww} + u_w \tag{12.4.81}$$

$$u_y = u_v \; v_y + u_w \; w_y \qquad = x \; u_w \tag{12.4.82}$$

$$u_{yy} = x^2 \; u_{ww} \tag{12.4.83}$$

The PDE in standard form for some sufficiently differentiable functions f, g, h is,

$$x\ u_{xy} - y\ u_{yy} = 0 \qquad x^2\ u_{vw} + x\ u_w = 0 \tag{12.4.84}$$

$$u_w \equiv z \qquad z + v\ z_v = 0 \tag{12.4.85}$$

$$\ln(z) = \ln(-v) + f(w) \qquad z = u_w = \frac{f(w)}{v} \tag{12.4.86}$$

$$u(v,\, w) = \frac{g(w)}{v} + h(v) \qquad u(x,\, y) = \frac{g(xy)}{x} + h(x) \tag{12.4.87}$$

15. Transforming to normal form,

$$x\ u_{xx} - y\ u_{xy} = 0 \qquad A = x,\ \ B = -y/2,\ \ C = 0 \tag{12.4.88}$$

$$AC - B^2 = \frac{y^2}{4} > 0 \qquad \text{Elliptic} \tag{12.4.89}$$

$$x\ y'^2 + y\ y' = 0 \qquad y'\ \Big[xy' + y\Big] = 0 \tag{12.4.90}$$

$$\psi(x,\, y) = v = y \qquad \phi(x,\, y) = w = xy \tag{12.4.91}$$

Expressing the PDE in terms of $(v,\, w)$,

$$u_x = u_v\ v_x + u_w\ w_x \qquad = y\ u_w \tag{12.4.92}$$

$$u_{xy} = (u_{wv} + x\ u_{ww})(y) + u_w \qquad = y\ u_{vw} + xy\ u_{ww} + u_w \tag{12.4.93}$$

$$u_{xx} = y^2\ u_{ww} \tag{12.4.94}$$

The PDE in standard form for some sufficiently differentiable functions f, g, h is,

$$x\ u_{xx} - y\ u_{xy} = 0 \qquad -y^2\ u_{vw} - y\ u_w = 0 \tag{12.4.95}$$

$$u_w \equiv z \qquad z + v\ z_v = 0 \tag{12.4.96}$$

$$\ln(z) = \ln(-v) + f(w) \qquad z = u_w = \frac{f(w)}{v} \tag{12.4.97}$$

$$u(v,\, w) = \frac{g(w)}{v} + h(v) \qquad u(x,\, y) = \frac{g(xy)}{y} + h(y) \tag{12.4.98}$$

16. Transforming to normal form,

$$u_{xx} + 2u_{xy} + 10u_{yy} = 0 \qquad A = 1,\ B = 1,\ C = 10 \tag{12.4.99}$$

$$AC - B^2 > 0 \qquad \text{Elliptic} \tag{12.4.100}$$

$$y'^2 - 2y' + 10 = 0 \qquad y' = \{1 \pm 3i\} \tag{12.4.101}$$

$$\psi(x, y) = v = y - (1 + 3i)x \qquad \phi(x, y) = w = y - (1 - 3i)x \tag{12.4.102}$$

Expressing the PDE in terms of (v, w), with z_1, z_2 for the complex numbers.

$$u_x = u_v\ v_x + u_w\ w_x \tag{12.4.103}$$

$$= -\lambda_1 u_v - \lambda_2 u_w \tag{12.4.104}$$

$$u_{xx} = (-\lambda_1)\ [-\lambda_1 u_{vv} - \lambda_2 u_{wv}] + (-\lambda_2)\ [-\lambda_1 u_{vw} - \lambda_2 u_{ww}] \tag{12.4.105}$$

$$= \lambda_1^2\ u_{vv} + \lambda_2^2\ u_{ww} + 2\lambda_1\lambda_2\ u_{vw} \tag{12.4.106}$$

$$u_y = u_v\ v_y + u_w\ w_y \tag{12.4.107}$$

$$= u_v + u_w \tag{12.4.108}$$

$$u_{yy} = u_{vv} + 2u_{wv} + u_{ww} \tag{12.4.109}$$

$$u_{xy} = -\lambda_1 u_{vv} - (\lambda_2 + \lambda_1)u_{wv} - \lambda_2 u_{ww} \tag{12.4.110}$$

The PDE in standard form for some sufficiently differentiable functions f, g is,

$$u_{vw} = 0 \tag{12.4.111}$$

$$u(v, w) = f(v) + g(w) \tag{12.4.112}$$

$$u(x, y) = f(y) + g(3x) \tag{12.4.113}$$

17. Transforming to normal form,

$$u_{xx} - 4u_{xy} + 5u_{yy} = 0 \qquad A = 1,\ B = -2,\ C = 5 \tag{12.4.114}$$

$$AC - B^2 > 0 \qquad \text{Elliptic} \tag{12.4.115}$$

$$y'^2 + 4y' + 5 = 0 \qquad y' = \{-2 \pm i\} \tag{12.4.116}$$

$$\psi(x, y) = v = y - (2 + i)x \qquad \phi(x, y) = w = y - (2 - i)x \tag{12.4.117}$$

Expressing the PDE in terms of (v, w), follows the exact procedure as in Problem 16.

The PDE in standard form for some sufficiently differentiable functions f, g is,

$$u_{vw} = 0 \tag{12.4.118}$$

$$u(v, w) = f(v) + g(w) \tag{12.4.119}$$

$$u(x, y) = f[y - (2+i)x] + g[y - (2-i)x] \tag{12.4.120}$$

18. Transforming to normal form,

$$u_{xx} - 6u_{xy} + 9u_{yy} = 0 \qquad A = 1,\ B = -3,\ C = 9 \tag{12.4.121}$$

$$AC - B^2 = 0 \qquad \text{Parabolic} \tag{12.4.122}$$

$$y'^2 + 6y' + 9 = 0 \qquad y' = \{-3, -3\} \tag{12.4.123}$$

$$\psi(x, y) = v = x \qquad \phi(x, y) = w = y + 3x \tag{12.4.124}$$

Expressing the PDE in terms of (v, w), follows the exact procedure as in Problem 12. The PDE in standard form for some sufficiently differentiable functions f, g is,

$$u_{vv} = 0 \tag{12.4.125}$$

$$u(v, w) = vf(w) + g(w) \tag{12.4.126}$$

$$u(x, y) = x \cdot f(y + 3x) + g(y + 3x) \tag{12.4.127}$$

19. The PDE is the same as the transverse vibrations in a string, whose general solution is,

$$\partial_t^2 u = c^2\ \partial_x^2 u \tag{12.4.128}$$

$$u(x, t) = F(x) \cdot G(t) \tag{12.4.129}$$

$$F_n(x) = a_n \cos(p_n x) + b_n \sin(p_n x) \tag{12.4.130}$$

$$G_n(t) = A_n\ \cos(p_n\ ct) + A_n^*\ \sin(p_n\ ct) \qquad \lambda_n = \frac{cn\pi}{L} \tag{12.4.131}$$

Using the initial conditions provided,

$$F(0) = 0 \qquad \implies a_n = 0 \tag{12.4.132}$$

$$F_x(L) = 0 \qquad \implies \cos(p_n\ L) = 0 \tag{12.4.133}$$

$$p_n = \frac{(2n+1)\pi}{2L} \tag{12.4.134}$$

$$u_t(x,\, 0) = 0 \qquad \implies A_n^* = 0 \tag{12.4.135}$$

$$u(x,\, 0) = f(x) \qquad \implies f(x) = \sum_{n=0}^{\infty} A_n\ b_n\ \ \sin(p_n\ x) \tag{12.4.136}$$

Since this is simply a Fourier sine series for $f(x)$, the Fourier sine coefficients are equal to $A_n b_n = C_n$,

$$C_n = \frac{2}{L} \int_0^L f(x)\ \sin(p_n\ x)\ \mathrm{d}x \tag{12.4.137}$$

$$u(x,\, t) = \sum_{n=0}^{\infty} (C_n)\ \sin(p_n\ x)\ \cos(p_n\ ct) \tag{12.4.138}$$

20. Tricomi equation,

$$y\ u_{xx} + u_{yy} = 0 \qquad A = y,\ \ B = 0,\ \ C = 1 \tag{12.4.139}$$

$$AC - B^2 = y \tag{12.4.140}$$

Mixed type simply means that $AC - B^2$ is of varying sign depending on the point in xy space, which is evident above. Using separation of variables,

$$u(x,\, y) = F(x) \cdot G(y) \tag{12.4.141}$$

$$G \cdot y\ F'' + F \cdot \ddot{G} = 0 \tag{12.4.142}$$

Here, the primes and dots are differentiation w.r.t x and y respectively. Equating both to the same constant,

$$-\frac{F''}{F} = \frac{\ddot{G}}{yG} = \alpha \tag{12.4.143}$$

$$\ddot{G} - \alpha\ yG = 0 \tag{12.4.144}$$

For simplicity, setting $\alpha = 1$ gives the Airy equation.

12.5 Modeling: Heat Flow from a Body in Space. Heat Equation

1. No problem set in this section.

12.6 Heat Equation: Solution by Fourier Series, Dirichlet Problem

1. The rate of decay is,

$$G(t) = \exp(-\lambda_n^2 t) \qquad \lambda^2 \propto c^2 \tag{12.6.1}$$

$$c^2 = \frac{K}{\rho\sigma} \tag{12.6.2}$$

K is the thermal conductivity, σ is the specific heat and ρ is the density.

2. For the first eigenfunction, $n = 1$. This gives,

$$\exp(-\lambda_1^2\, T) = 0.5 \qquad \lambda_1^2\, T = \ln(2) \tag{12.6.3}$$

$$\lambda_1^2 = \frac{\ln(2)}{T} \qquad \lambda_1 = \frac{c\pi}{L} \tag{12.6.4}$$

$$c^2 = \frac{L^2}{\pi^2} \cdot \frac{\ln(2)}{T} \tag{12.6.5}$$

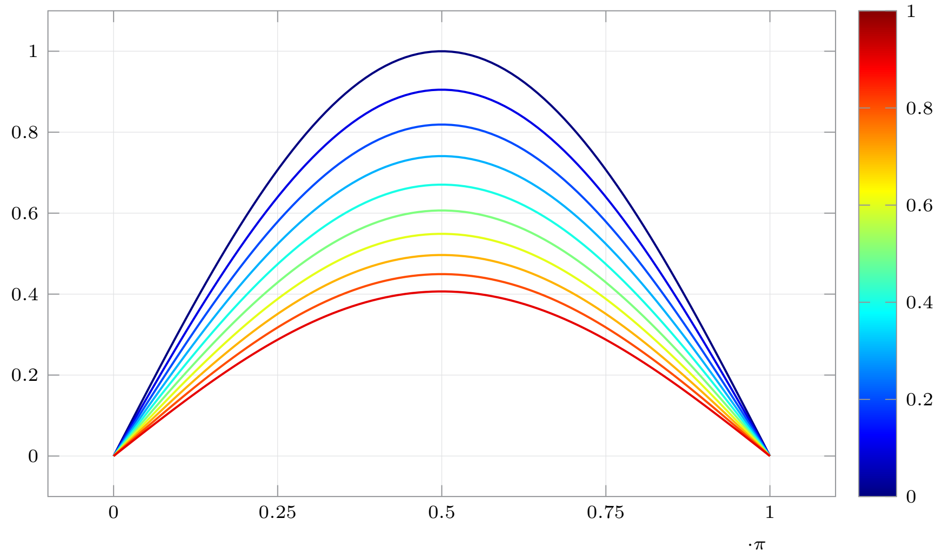
3. The eigenfunctions are,

$$u_n(x, t) = B_n \;\sin\left(\frac{n\pi x}{L}\right)\; \exp(-\lambda_n^2\; t) \tag{12.6.6}$$

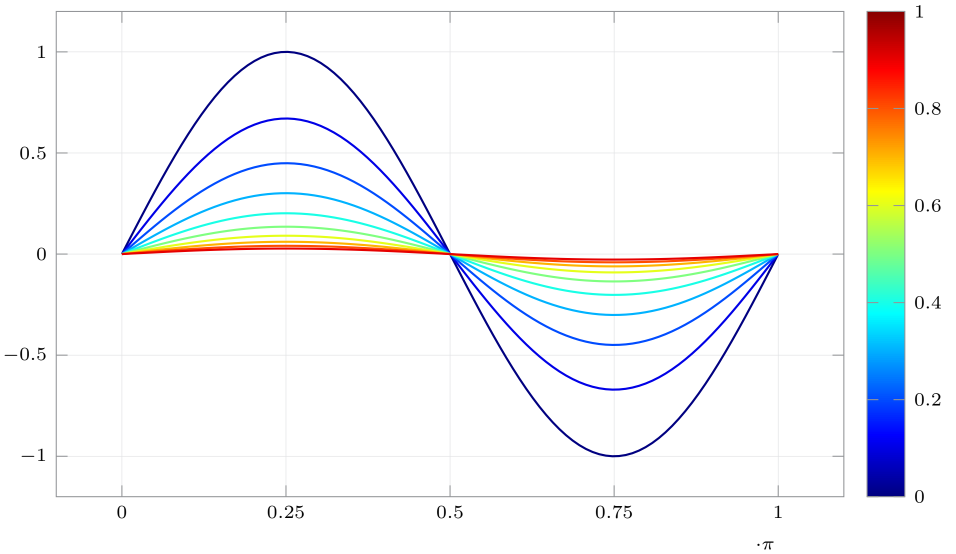
$$\lambda_n = n \qquad B_n = 1 \tag{12.6.7}$$

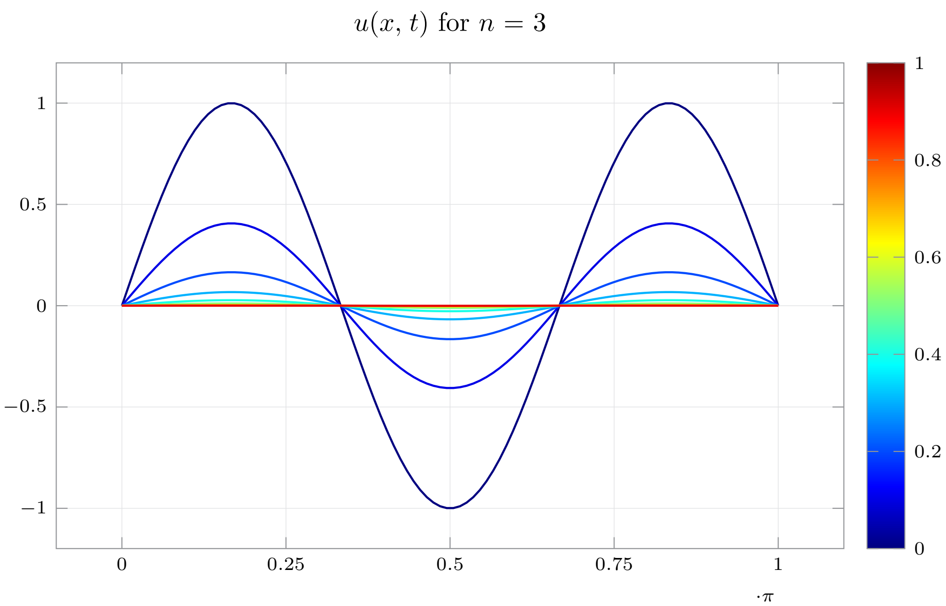
$$u_n(x, t) = \sin(nx)\;\; \exp(-n^2 t) \tag{12.6.8}$$

$u(x, t)$ for $n = 1$



$u(x, t)$ for $n = 2$





4. TBC. Refer notes. The important difference is in the time component of the solution, which happens to be an exponential term in the heat equation, instead of the sinusoidal term in the wave equation.

5. Using the constants given, the thermal diffusivity is,

$$c^2 = 1.75 \qquad c = 1.32 \tag{12.6.9}$$

$$\lambda_n = \frac{cn\pi}{L} = 0.4156n \tag{12.6.10}$$

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad G_n(t) = B_n \exp(-\lambda_n^2 t) \tag{12.6.11}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\ \sin\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \tag{12.6.12}$$

Using the given function to find B_n,

$$f(x) = \sin(0.1\pi x) \qquad B_1 = 1 \tag{12.6.13}$$

$$B_n = 0 \;\forall\; n > 1 \tag{12.6.14}$$

$$u(x, t) = \sin\left(\frac{\pi x}{10}\right)\ \exp(-0.173t) \tag{12.6.15}$$

6. From Problem 5, and using the given function B_n,

$$f(x) = 4 - 0.8\,|x-5| = \begin{cases} 0.8x & x \in [0,\,5] \\ -0.8x + 8 & x \in [5,\,10] \end{cases} \qquad \text{12.6.16}$$

$$B_n = \frac{2}{10} \int_0^{10} f(x)\ \sin(0.1\pi\ nx)\ \mathrm{d}x \qquad \text{12.6.17}$$

$$= \frac{4}{25} \int_0^{5} (x) \sin(0.1\pi\ nx)\ \mathrm{d}x + \frac{4}{25} \int_5^{10} (-x+10) \sin(0.1\pi\ nx)\ \mathrm{d}x \qquad \text{12.6.18}$$

$$= 0.16 \left[\frac{\sin(0.1\pi\ nx) - (0.1\pi\ nx)\cos(0.1\pi\ nx)}{(0.1n\pi)^2} \right]_0^5 \qquad \text{12.6.19}$$

$$+ 0.16 \left[\frac{\sin(0.1\pi\ nx) + (0.1\pi\ n)(10-x)\cos(0.1\pi\ nx)}{(0.1n\pi)^2} \right]_{10}^5 \qquad \text{12.6.20}$$

$$B_n = \frac{32}{n^2\pi^2}\ \sin\left(\frac{n\pi}{2}\right) \qquad \text{12.6.21}$$

$$u_n(x,\,t) = B_n\ \sin\left(\frac{n\pi x}{L}\right)\ \exp(-0.173 n^2\ t) \qquad \text{12.6.22}$$

7. From Problem 5, and using the given function B_n,

$$f(x) = 4 - 0.8\,|x-5| \qquad \text{12.6.23}$$

$$B_n = \frac{2}{10} \int_0^{10} f(x)\ \sin(0.1\pi\ nx)\ \mathrm{d}x \qquad \text{12.6.24}$$

$$= 0.2 \int_0^{5} (x)(10-x) \sin(0.1\pi\ nx)\ \mathrm{d}x \qquad \text{12.6.25}$$

$$= 0.2 \left[\frac{(10-2x)}{(0.1n\pi)^2}\ \sin(0.1\pi\ nx) + \frac{(0.1n\pi)^2(x)(x-10) - 2}{(0.1n\pi)^3}\ \cos(0.1\pi\ nx) \right]_0^{10} \qquad \text{12.6.26}$$

$$B_n = \frac{400}{n^3\pi^3}\ [1 - \cos(n\pi)] \qquad \text{12.6.27}$$

$$u_n(x,\,t) = B_n\ \sin\left(\frac{n\pi x}{L}\right)\ \exp(-0.173 n^2\ t) \qquad \text{12.6.28}$$

8. Guess: The temperature is a linear gradient in the bar from U_1 to U_2 in the x direction. After a very long time, the bar does not lose any heat.

$$\frac{\partial H}{\partial t} = 0 \qquad \frac{\partial^2 u}{\partial x^2} = 0 \tag{12.6.29}$$

$$u(x) = c_1 x + c_2 \tag{12.6.30}$$

$$u(0) = U_1 \implies c_2 = U_1 \qquad u(L) = U_2 \implies c_1 = \frac{U_2 - U_1}{L} \tag{12.6.31}$$

Since the problem is time independent, it reduces to Laplace's equation in $1d$ with Dirichlet B.C.

9. For the transient solution, using the result from Problem 8.

$$g(x) = U_1 + \frac{U_2 - U_1}{L}\, x \qquad g(0) = U_1 = 100 \qquad g(L) = U_2 = 0 \tag{12.6.32}$$

$$u(x,\, t) = F(x) \cdot G(t) \qquad u(0,\, t) = U_1 \qquad u(L,\, t) = U_2 \tag{12.6.33}$$

$$v(x,\, t) = u(x,\, t) - g(x) \qquad v(0,\, t) = 0 \qquad v(L,\, t) = 0 \tag{12.6.34}$$

Now, $v(t)$ satisfies the Dirichlet conditions solved in the text, where both ends are kept at zero temperature.

$$\partial_t v = \partial_t u \qquad \partial_t^2 v = \partial_t^2 u \qquad \partial_t v = c^2\, \partial_x^2 v \tag{12.6.35}$$

$$v_n(x,\, t) = B_n\, \sin\left(\frac{n\pi x}{L}\right)\, \exp(-\lambda_n^2 t) \tag{12.6.36}$$

$$B_n = \frac{2}{L} \int_0^L [f(x) - g(x)]\, \sin\left(\frac{n\pi x}{L}\right)\, dx \tag{12.6.37}$$

10. After a long time, the entire bar is at the same temperature. Using the result from Problem 9,

$$f(x) = 100 \qquad g(x) = 100 - 10x \tag{12.6.38}$$

$$u(0,\, t) = 0 \qquad u(L,\, t) = 100 \tag{12.6.39}$$

$$B_n = 2 \int_0^{10} (x)\, \sin\left(\frac{n\pi x}{10}\right)\, dx \tag{12.6.40}$$

$$= 2 \left[\frac{\sin(0.1n\pi\, x) - (0.1n\pi)x\cos(0.1n\pi\, x)}{(0.1n\pi)^2}\right]_0^{10} \tag{12.6.41}$$

$$B_n = \frac{-200}{n\pi}\, \cos(n\pi) \tag{12.6.42}$$

From Problem 5,

$$\lambda_n = 0.4156n \tag{12.6.43}$$

$$u(x,\, t) = [100 - 10x] + B_n \sin(0.1n\pi\; x) \exp(-\lambda_n^2 t) \tag{12.6.44}$$

$$u_n(5,\, t) = 50 - \sum_{n=1}^{\infty} \frac{200 \cos(n\pi) \sin(n\pi/2)}{n\pi} \exp(-0.173n^2\; t) \tag{12.6.45}$$

$$u(5,\, 1) = 99.2 \qquad u(5,\, 2) = 94.1 \tag{12.6.46}$$

$$u(5,\, 3) = 87.68 \qquad u(5,\, 10) = 61.28 \tag{12.6.47}$$

$$u(5,\, 50) = 50.01 \tag{12.6.48}$$

11. Heat flux is proportional to $\partial_x u$. Since adiabatic means that heat cannot flow from the ends to the environment, and there are no sources or sinks of heat anywhere in the bar,

$$u_x(0,\, t) = u_x(L,\, t) = 0 \tag{12.6.49}$$

Using the general solution to the heat equation in the text,

$$F(x) = a\;\cos(px) + b\;\sin(px) \qquad F'(x) = -pa\;\sin(px) + pb\;\cos(px) \tag{12.6.50}$$

$$F'(x=0) = 0 \qquad \implies b = 0 \tag{12.6.51}$$

$$F'(x=L) = 0 \qquad \implies \sin(pL) = 0, \qquad p_n = \frac{n\pi}{L} \tag{12.6.52}$$

$$\tag{12.6.53}$$

The time dependent part of $u(x,\, t)$ is,

$$G_n(t) = A_n \exp(-c^2 p^2\; t) \qquad \lambda_n = cp_n = \frac{cn\pi}{L} \tag{12.6.54}$$

$$u_n(x,\, t) = A_n\;\cos(p_n\; x)\;\exp(-\lambda_n^2\; t) \tag{12.6.55}$$

Using the initial condition $u(x,\, 0) = f(x)$, and its Fourier cosine series expansion,

$$u(x,\, t) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(px)\;\exp(-\lambda_n^2 t) \tag{12.6.56}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\;\mathrm{d}x \tag{12.6.57}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos(px)\;\mathrm{d}x \tag{12.6.58}$$

12. Using the general solution from Problem 11, with $L = \pi$, $c = 1$

$$f(x) = x \qquad p = n, \qquad \lambda_n = n \tag{12.6.59}$$

$$A_0 = \frac{1}{L} \int_0^L f(x) \ \mathrm{d}x \qquad = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^\pi = \frac{\pi}{2} \tag{12.6.60}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \ \cos(px) \ \mathrm{d}x \qquad = \frac{2}{\pi} \left[\frac{nx \sin(nx) + \cos(nx)}{n^2} \right]_0^\pi \tag{12.6.61}$$

$$= \frac{2}{\pi n^2} \left[\cos(n\pi) - 1 \right] \tag{12.6.62}$$

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) \exp(-n^2 t) \tag{12.6.63}$$

13. Using the general solution from Problem 11, with $L = \pi$, $c = 1$

$$f(x) = 1 \qquad p = n, \qquad \lambda_n = n \tag{12.6.64}$$

$$A_0 = \frac{1}{L} \int_0^L f(x) \ \mathrm{d}x \qquad = \frac{1}{\pi} \Big[x \Big]_0^\pi = 1 \tag{12.6.65}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \ \cos(px) \ \mathrm{d}x \qquad = \frac{2}{\pi} \left[\frac{\sin(nx)}{n} \right]_0^\pi = 0 \tag{12.6.66}$$

$$u(x, t) = 1 \tag{12.6.67}$$

14. Using the general solution from Problem 11, with $L = \pi$, $c = 1$

$$f(x) = x \qquad p = n, \qquad \lambda_n = n \tag{12.6.68}$$

$$A_0 = \frac{1}{L} \int_0^L f(x) \ \mathrm{d}x \qquad = \frac{1}{\pi} \left[\frac{\sin(2x)}{2} \right]_0^\pi = 0 \tag{12.6.69}$$

$$A_n = 0 \qquad \forall\, n \neq 1 \qquad A_1 = 1 \tag{12.6.70}$$

$$u(x, t) = \cos(2x) \ \exp(-4t) \tag{12.6.71}$$

15. Using the general solution from Problem 11, with $L = \pi$, $c = 1$

$$f(x) = 1 - \frac{x}{\pi} \qquad p = n, \qquad \lambda_n = n \tag{12.6.72}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\ \mathrm{d}x \qquad = \frac{1}{\pi}\left[x - \frac{x^2}{2\pi}\right]_0^\pi = \frac{1}{2} \tag{12.6.73}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\ \cos(px)\ \mathrm{d}x \qquad = \frac{2}{\pi}\int_0^\pi \left[1 - \frac{x}{\pi}\right] \cos(nx)\ \mathrm{d}x \tag{12.6.74}$$

$$= \frac{2}{\pi}\left[\frac{n(\pi - x)\sin(nx) - \cos(nx)}{\pi n^2}\right]_0^\pi \qquad = \frac{-2}{n^2\pi^2}\left[\cos(n\pi) - 1\right] \tag{12.6.75}$$

$$u(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[1 - \cos(n\pi)]}{n^2\pi^2} \cos(nx)e^{-n^2 t} \tag{12.6.76}$$

16. Heat is generated in the rod at constant rate $H > 0$,

$$\partial_t u = \partial_x^2 u + H \tag{12.6.77}$$

$$c^2\ v_{xx} = c^2\ u_{xx} + H \qquad c^2\ v_x = c^2\ u_x + Hx + c_1 \tag{12.6.78}$$

$$v = u + \frac{1}{c^2}\left[\frac{Hx^2}{2} + b_1 x + b_2\right] \tag{12.6.79}$$

Ensuring that $v(x, t)$ satisfies the Dirichlet zero boundary conditions,

$$v(0, t) = u(0, t) + b_2 \qquad b_2 = 0 \tag{12.6.80}$$

$$v(\pi, t) = u(\pi, t) + \frac{H\pi^2}{2} + b_1\pi = 0 \qquad b_1 = -\frac{H\pi}{2} \tag{12.6.81}$$

$$v(x, t) = u(x, t) + \frac{Hx(x - \pi)}{2c^2} \tag{12.6.82}$$

By construction, $v(x, t)$ has the solution from the text,

$$\partial_t v = c^2\ \partial_x^2 v \tag{12.6.83}$$

$$v(0, t) = v(\pi, t) = 0 \tag{12.6.84}$$

$$v(x, t) = \sum_{n=1}^{\infty} B_n\ \sin(nx)\ e^{-c^2 n^2\ t} \tag{12.6.85}$$

$$B_n = \frac{2}{\pi}\int_0^\pi \left[f(x) + \frac{Hx(x - \pi)}{2c^2}\right] \sin(nx)\ \mathrm{d}x \tag{12.6.86}$$

17. From equation 9 in the text,

$$u_x(x, t) = \sum_{n=1}^{\infty} pB_n \ \cos(px) \ \exp(-c^2 p^2 \ t) \tag{12.6.87}$$

$$\phi(t) = -K u_x(0, t) = -K \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \exp(-\lambda_n^2 t) \tag{12.6.88}$$

$$p = \frac{n\pi}{L} \qquad \lambda_n = \frac{cn\pi}{L} \tag{12.6.89}$$

$$\phi(t) = -\frac{K\pi}{L} \sum_{n=1}^{\infty} nB_n \exp(-\lambda_n^2 t) \tag{12.6.90}$$

18. Solving the two dimensional Laplace equation given, $x \in [0, 20]$ and $y \in [0, 40]$

$$u(x, 40) = 110 \qquad u(x, 0) = 0 \tag{12.6.91}$$

$$u(0, y) = 0 \qquad u(20, y) = 0 \tag{12.6.92}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x, y) = F(x) \cdot G(y) \tag{12.6.93}$$

Separating variables and solving the ODE in x

$$\frac{1}{F} \cdot \frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = -k \qquad F(x) = A \cos(\sqrt{k}x) + B \sin(\sqrt{k}x) \tag{12.6.94}$$

$$F(0) = 0 \qquad \implies B = 0 \tag{12.6.95}$$

$$F(20) = 0 \qquad \implies \sqrt{k} = \frac{n\pi}{20} \tag{12.6.96}$$

$$F(x) = A \sin\left(\frac{n\pi x}{20}\right) \tag{12.6.97}$$

Separating variables and solving the ODE in y

$$\frac{1}{G} \cdot \frac{\mathrm{d}^2 G}{\mathrm{d}y^2} = k \qquad G(y) = c_1 \cosh(\sqrt{k}y) + c_2 \sinh(\sqrt{k}y) \tag{12.6.98}$$

$$G(0) = 0 \qquad \implies c_1 = 0 \tag{12.6.99}$$

$$G(y) = c_2 \sinh\left(\frac{n\pi y}{20}\right) \tag{12.6.100}$$

Satifsying the nonzero boundary condition $u(x, 40)$,

$$u(x, 40) = 110 = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{20}\right) \sinh(2n\pi) \tag{12.6.101}$$

$$A_n^* \sinh(2n\pi) = \frac{2}{20} \int_0^{20} 110 \sin\left(\frac{n\pi x}{20}\right) \, dx \tag{12.6.102}$$

$$= \frac{220}{n\pi} \left[\cos\left(\frac{n\pi x}{20}\right)\right]_{20}^{0} \tag{12.6.103}$$

$$A_n^* = \frac{220}{n\pi \, \sinh(2n\pi)} \left[1 - \cos(n\pi)\right] \tag{12.6.104}$$

$$u(x, y) = \frac{220}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - \cos(n\pi)}{n \, \sinh(2n\pi)}\right] \sin\left(\frac{n\pi x}{20}\right) \sinh\left(\frac{n\pi y}{20}\right) \tag{12.6.105}$$

19. Solving the two dimensional Laplace equation given, $x \in [0, 2]$ and $y \in [0, 2]$

$$u(x, 2) = 1000 \sin\left(\frac{\pi x}{2}\right) \qquad u(x, 0) = 0 \tag{12.6.106}$$

$$u(0, y) = 0 \qquad u(2, y) = 0 \tag{12.6.107}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x, y) = F(x) \cdot G(y) \tag{12.6.108}$$

Separating variables and solving the ODE in x

$$\frac{1}{F} \cdot \frac{d^2 F}{dx^2} = -k \qquad F(x) = A \cos(\sqrt{k}x) + B \sin(\sqrt{k}x) \tag{12.6.109}$$

$$F(0) = 0 \qquad \implies B = 0 \tag{12.6.110}$$

$$F(2) = 0 \qquad \implies \sqrt{k} = \frac{n\pi}{2} \tag{12.6.111}$$

$$F(x) = A \sin\left(\frac{n\pi x}{2}\right) \tag{12.6.112}$$

Separating variables and solving the ODE in y

$$\frac{1}{G} \cdot \frac{\mathrm{d}^2 G}{\mathrm{d}y^2} = k \qquad G(y) = c_1 \cosh(\sqrt{k}y) + c_2 \sinh(\sqrt{k}y) \tag{12.6.113}$$

$$G(0) = 0 \qquad \Longrightarrow \quad c_1 = 0 \tag{12.6.114}$$

$$G(y) = c_2 \sinh\left(\frac{n\pi y}{2}\right) \tag{12.6.115}$$

Satifsying the nonzero boundary condition $u(x, 2)$,

$$u(x, 2) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{2}\right) \sinh(n\pi) \tag{12.6.116}$$

$$A_n^* \sinh(n\pi) = \frac{2}{2} \int_0^2 1000 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{\pi x}{2}\right) \, \mathrm{d}x \tag{12.6.117}$$

$$A_n^* = \begin{cases} \dfrac{1000}{\sinh(n\pi)} & n = 1 \\ \\ 0 & \text{otherwise} \end{cases} \tag{12.6.118}$$

$$u(x, y) = \frac{1000}{\sinh(\pi)} \sin\left(\frac{\pi x}{2}\right) \sinh\left(\frac{\pi y}{2}\right) \tag{12.6.119}$$

20. Graphing the isotherms,

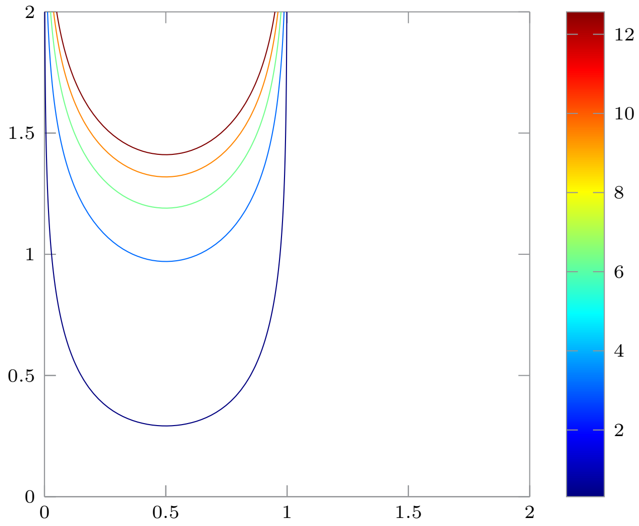
(a) Satifsying the nonzero boundary condition $u(x, 2)$,

$$u(x, 2) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{2}\right) \sinh(n\pi) \tag{12.6.120}$$

$$A_n^* \sinh(n\pi) = \frac{2}{2} \int_0^2 80 \sin(\pi x) \sin\left(\frac{n\pi x}{2}\right) \, \mathrm{d}x \tag{12.6.121}$$

$$A_n^* = \begin{cases} \dfrac{80}{\sinh(n\pi)} & n = 2 \\ \\ 0 & \text{otherwise} \end{cases} \tag{12.6.122}$$

$$u(x, y) = \frac{80}{\sinh(2\pi)} \sin(\pi x) \sinh(\pi y) \tag{12.6.123}$$



(b) Mixed boundary conditions are

$$u_y(x, 2) = 0 \qquad u_y(x, 0) = 0 \tag{12.6.124}$$

$$u(0, y) = 0 \qquad u(2, y) = 0 \tag{12.6.125}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x, y) = F(x) \cdot G(y) \tag{12.6.126}$$

Separating variables and solving the ODE in x

$$\frac{1}{F} \cdot \frac{d^2 F}{dx^2} = -k \qquad F(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x) \tag{12.6.127}$$

$$F(0) = 0 \qquad \implies B = 0 \tag{12.6.128}$$

$$F(2) = 0 \qquad \implies \sqrt{k} = \frac{n\pi}{2} \tag{12.6.129}$$

$$F(x) = A\sin\left(\frac{n\pi x}{2}\right) \tag{12.6.130}$$

Separating variables and solving the ODE in y

$$\frac{1}{G} \cdot \frac{d^2 G}{dy^2} = k \qquad G(y) = c_1\cosh(\sqrt{k}y) + c_2\sinh(\sqrt{k}y) \tag{12.6.131}$$

$$G_y(0) = 0 \qquad \implies c_2 = 0 \tag{12.6.132}$$

$$G_y(2) = 0 \qquad \implies c_2 = 0 \tag{12.6.133}$$

$$G(y) = 0 \qquad u(x, y) = 0 \tag{12.6.134}$$

The isotherm is the entire surface since the temperature is identically zero.

(c) TBC.

21. Solving the two dimensional Laplace equation given, $x \in [0, 24]$ and $y \in [0, 24]$

$$u(x, 24) = 25 \qquad u(x, 0) = 0 \tag{12.6.135}$$

$$u(0, y) = 0 \qquad u(24, y) = 0 \tag{12.6.136}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x, y) = F(x) \cdot G(y) \tag{12.6.137}$$

Separating variables and solving the ODE in x

$$\frac{1}{F} \cdot \frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = -k \qquad F(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x) \tag{12.6.138}$$

$$F(0) = 0 \qquad \implies B = 0 \tag{12.6.139}$$

$$F(24) = 0 \qquad \implies \sqrt{k} = \frac{n\pi}{24} \tag{12.6.140}$$

$$F(x) = A\sin\left(\frac{n\pi x}{24}\right) \tag{12.6.141}$$

Separating variables and solving the ODE in y

$$\frac{1}{G} \cdot \frac{\mathrm{d}^2 G}{\mathrm{d}y^2} = k \qquad G(y) = c_1\cosh(\sqrt{k}y) + c_2\sinh(\sqrt{k}y) \tag{12.6.142}$$

$$G(0) = 0 \qquad \implies c_1 = 0 \tag{12.6.143}$$

$$G(y) = c_2\sinh\left(\frac{n\pi y}{24}\right) \tag{12.6.144}$$

Satifsying the nonzero boundary condition $u(x, 24)$,

$$u(x, 24) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{2}\right)\sinh(n\pi) \tag{12.6.145}$$

$$A_n^* \sinh(n\pi) = \frac{2}{24}\int_0^{24} 25\sin\left(\frac{n\pi x}{24}\right)\,\mathrm{d}x \tag{12.6.146}$$

$$= \frac{50}{n\pi}\left[\cos\left(\frac{n\pi x}{24}\right)\right]_{24}^{0} \tag{12.6.147}$$

$$A_n^* = \frac{[1-\cos(n\pi)]\,50}{n\pi\sinh(n\pi)} \tag{12.6.148}$$

$$u(x, y) = \frac{50}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-\cos(n\pi)}{n\sinh(n\pi)}\right]\sin\left(\frac{n\pi x}{24}\right)\sinh\left(\frac{n\pi y}{24}\right) \tag{12.6.149}$$

22. Using the result from Problem 21,

$$v(x,\,0) = v(0,\,y) = v(24,\,y) = 0 \qquad\qquad v(x,\,24) = U_2 \tag{12.6.150}$$

$$w(x,\,24) = w(0,\,y) = w(24,\,y) = 0 \qquad\qquad w(x,\,0) = U_1 \tag{12.6.151}$$

Separating variables and solving the ODE in y for the first half v

$$\frac{1}{G}\cdot\frac{\mathrm{d}^2 G}{\mathrm{d}y^2} = k \qquad\qquad G(y) = c_1\cosh(\sqrt{k}y) + c_2\sinh(\sqrt{k}y) \tag{12.6.152}$$

$$G(0) = 0 \qquad\qquad \implies\; c_1 = 0 \tag{12.6.153}$$

$$G(y) = c_2\sinh\left(\frac{n\pi y}{24}\right) \tag{12.6.154}$$

Satifsying the nonzero boundary condition $v(x,\,24)$,

$$v(x,\,24) = \sum_{n=1}^{\infty} A_n^*\sin\left(\frac{n\pi x}{24}\right)\sinh(n\pi) \tag{12.6.155}$$

$$A_n^*\sinh(n\pi) = \frac{2}{24}\int_0^{24}(U_2)\sin\left(\frac{n\pi x}{24}\right)\,\mathrm{d}x \tag{12.6.156}$$

$$= \frac{2U_2}{n\pi}\left[\cos\left(\frac{n\pi x}{24}\right)\right]_{24}^{0} \tag{12.6.157}$$

$$A_n^* = \frac{[1-\cos(n\pi)]\;2U_2}{n\pi\,\sinh(n\pi)} \tag{12.6.158}$$

$$u(x,\,y) = \frac{2U_2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-\cos(n\pi)}{n\,\sinh(n\pi)}\right]\sin\left(\frac{n\pi x}{24}\right)\;\sinh\left(\frac{n\pi y}{24}\right) \tag{12.6.159}$$

Separating variables and solving the ODE in y for the second half w

$$\frac{1}{G}\cdot\frac{\mathrm{d}^2 G}{\mathrm{d}y^2} = k \qquad\qquad G(y) = c_1\cosh(\sqrt{k}y) + c_2\sinh(\sqrt{k}y) \tag{12.6.160}$$

$$G(24) = 0 \qquad\qquad \implies\; 0 = c_1\cosh(n\pi) + c_2\sinh(n\pi) \tag{12.6.161}$$

$$G(0) = U_1 \qquad\qquad \implies\; c_1 = U_1 \tag{12.6.162}$$

$$c_2 = -U_1\coth(n\pi) \tag{12.6.163}$$

Satifsying the nonzero boundary condition $w(x,0)$,

$$w(x,0) = \sum_{n=1}^{\infty} A_n^* \; U_1 \;\; \sin\left(\frac{n\pi x}{24}\right) \tag{12.6.164}$$

$$A_n^* = \frac{2}{24}\int_0^{24} \sin\left(\frac{n\pi x}{24}\right) \; \mathrm{d}x \tag{12.6.165}$$

$$= \frac{2}{n\pi}\left[\cos\left(\frac{n\pi x}{24}\right)\right]_{24}^{0} \tag{12.6.166}$$

$$A_n^* = \frac{[1-\cos(n\pi)]\;2}{n\pi} \tag{12.6.167}$$

$$w(x,y) = \frac{2U_1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-\cos(n\pi)}{n}\right]\sin\left(\frac{n\pi x}{24}\right)\cdot G_n(y) \tag{12.6.168}$$

$$G_n(y) = \cosh\left(\frac{n\pi y}{24}\right) - \coth(n\pi)\sinh\left(\frac{n\pi y}{24}\right) \tag{12.6.169}$$

23. Mixed boundary conditions are,

$$u(0,y) = 0 \qquad\qquad u(24,y) = h(y) \tag{12.6.170}$$

$$u_y(x,0) = 0 \qquad\qquad u_y(x,24) = 0 \tag{12.6.171}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad\qquad u(x,y) = F(x)\cdot G(y) \tag{12.6.172}$$

Separating variables and solving the ODE in y

$$\frac{1}{G}\cdot\frac{\mathrm{d}^2 G}{\mathrm{d}y^2} = -k \qquad\qquad G_y(y) = \sqrt{k}\left[-A\sin(\sqrt{k}y) + B\cos(\sqrt{k}y)\right] \tag{12.6.173}$$

$$G_y(0) = 0 \qquad\qquad \implies B = 0 \tag{12.6.174}$$

$$G_y(24) = 0 \qquad\qquad \implies \sqrt{k} = \frac{n\pi}{24} \tag{12.6.175}$$

$$G(y) = A\cos\left(\frac{n\pi y}{24}\right) \tag{12.6.176}$$

Separating variables and solving the ODE in x with $n > 0$,

$$\frac{1}{F} \cdot \frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = k \qquad F(x) = c_1 \cosh(\sqrt{k}x) + c_2 \sinh(\sqrt{k}x) \tag{12.6.177}$$

$$F(0) = 0 \qquad \implies c_1 = 0 \tag{12.6.178}$$

$$F(24) = 1 \qquad \implies c_2 = \frac{1}{\sinh(n\pi)} \tag{12.6.179}$$

$$F(x) = \frac{1}{\sinh n\pi} \sinh\left(\frac{n\pi x}{24}\right) \tag{12.6.180}$$

Using the Fourier cosine series to find the full result,

$$u(x, y) = \sum_{n=1}^{\infty} A_n^* \frac{1}{\sinh(n\pi)} \sinh\left(\frac{n\pi x}{24}\right) \cos\left(\frac{n\pi y}{24}\right) \tag{12.6.181}$$

$$u(24, y) = \sum_{n=1}^{\infty} A_n^* \cos\left(\frac{n\pi y}{24}\right) = h(y) \tag{12.6.182}$$

$$A_n^* = \frac{2}{24} \int_0^{24} h(y) \cos\left(\frac{n\pi y}{24}\right) \mathrm{d}y \tag{12.6.183}$$

For the special case when the temperature is independent of y, and

$$\frac{\partial u}{\partial y} = 0 \qquad u(x, y) = u_0(x) \tag{12.6.184}$$

$$u_0(0, y) = 0 \qquad u_0(24, y) = h(y) \tag{12.6.185}$$

$$u_0(x) = \left[\frac{1}{24} \int_0^{24} h(y) \mathrm{d}y\right] \frac{x}{24} \tag{12.6.186}$$

24. For radiation, mixed boundary conditions are,

$$u_x(0, y) = 0 \qquad u_y(x, b) = 0 \tag{12.6.187}$$

$$u(x, 0) = v(x) \qquad u_x(a, y) = -hu(a, y) \tag{12.6.188}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x, y) = F(x) \cdot G(y) \tag{12.6.189}$$

Separating variables and solving the ODE in x

$$\frac{1}{F} \cdot \frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = -k \qquad F(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x) \tag{12.6.190}$$

$$F_x(0) = 0 \qquad \implies B = 0 \tag{12.6.191}$$

$$F_x(a) = -hF(a) \qquad \implies \sqrt{k} = h\cot(\sqrt{k}a) \tag{12.6.192}$$

This is a transcendental equation with an infinite set of increasing solutions. The solution is now,

$$F_n(x) = A\cos\left(\sqrt{k_n}x\right) \qquad \sqrt{k_n} = h\cot\left(\sqrt{k_n}a\right) \tag{12.6.193}$$

Separating variables and solving the ODE in y

$$\frac{1}{G} \cdot \frac{\mathrm{d}^2 G}{\mathrm{d}y^2} = k \qquad G(y) = c_1\cosh(\sqrt{k}y) + c_2\sinh(\sqrt{k}y) \tag{12.6.194}$$

$$G_y(b) = 0 \qquad \implies c_2 = -c_1\tanh(\sqrt{k_n}b) \tag{12.6.195}$$

$$G(0) = v(x) \qquad \implies c_1 = v(x) \tag{12.6.196}$$

The general solution in y is now,

$$G_n(y) = \left[\cosh\left(\sqrt{k_n}y\right) - \tanh\left(\sqrt{k_n}b\right)\sinh\left(\sqrt{k_n}y\right)\right]v(x) \tag{12.6.197}$$

25. Similar to the example in the text, the boundary conditions are,

$$u(0,y) = 0 \qquad u(a,y) = 0 \tag{12.6.198}$$

$$u(x,0) = f(x) \qquad u(x,b) = 0 \tag{12.6.199}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,y) = F(x)\cdot G(y) \tag{12.6.200}$$

Separating variables and solving the ODE in x

$$\frac{1}{F} \cdot \frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = -k \qquad F(x) = A\sin(\sqrt{k}x) + B\cos(\sqrt{k}x) \tag{12.6.201}$$

$$F(0) = 0 \qquad \implies B = 0 \tag{12.6.202}$$

$$F(a) = 0 \qquad \implies \sqrt{k} = \frac{n\pi}{a} \tag{12.6.203}$$

$$F(x) = A\sin\left(\frac{n\pi x}{a}\right) \tag{12.6.204}$$

Separating variables and solving the ODE in y,

$$\frac{1}{F} \cdot \frac{\mathrm{d}^2 G}{\mathrm{d}y^2} = k \tag{12.6.205}$$

$$G(y) = c_1 \cosh(\sqrt{k}y) + c_2 \sinh(\sqrt{k}y) \tag{12.6.206}$$

$$G(b) = 0 \tag{12.6.207}$$

$$c_2 = -c_1 \coth(\sqrt{k}b) \tag{12.6.208}$$

$$G(y) = c_1 \Big[\cosh(\sqrt{k}y) - \coth(\sqrt{k}b) \sinh(\sqrt{k}y) \Big] \tag{12.6.209}$$

$$= c_1 \frac{\sinh(\sqrt{k}b - \sqrt{k}y)}{\sinh(\sqrt{k}b)} \tag{12.6.210}$$

Solving the nonzero boundary condition $u(x, 0)$,

$$u(x, y) = \sum_{n=1}^{\infty} A_n^* \frac{\sinh[\sqrt{k}(b-y)]}{\sinh(\sqrt{k}b)} \sin\left(\frac{n\pi x}{a}\right) \tag{12.6.211}$$

$$u(x, 0) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{a}\right) = f(x) \tag{12.6.212}$$

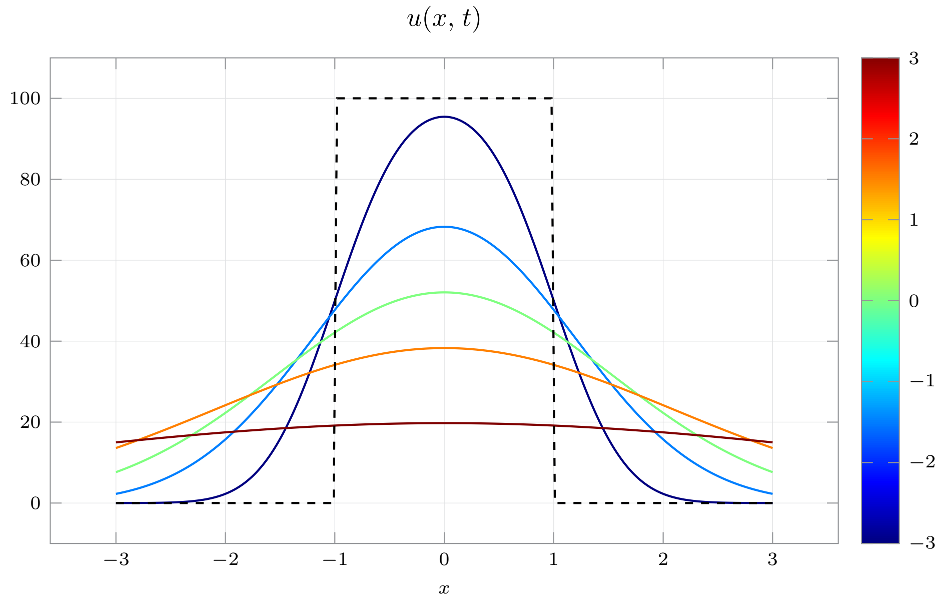
$$A_n^* = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x \tag{12.6.213}$$

In the text, the factor $\sinh(\sqrt{k}b)$ in the denominator is folded into the parameter A_n^*

12.7 Heat Equation: Modeling Very Long Bars, Solution using Fourier transforms

1. Plotting the error function using `gnuplot`,

- **(a)** Refer part b
- **(b)** USing a sequence of plots to visualize time advancing,



(c) Since $u(x, t)$ is independent of y the surface plot is simply the same plot from part b extended in the y direction.
It would resemble a sheet which flattens over time to the xy plane.

2. The solution in integral form is,

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{otherwise} \end{cases} \tag{12.7.1}$$

$$A(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos(pv) \ \mathrm{d}v = \frac{2}{\pi} \int_0^a \cos(pv) \ \mathrm{d}v \tag{12.7.2}$$

$$= \frac{2}{p\pi} \left[\sin(px) \right]_0^a = \frac{2 \sin(pa)}{p\pi} \tag{12.7.3}$$

$$B(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(pv) \ \mathrm{d}v = \frac{1}{\pi} \int_{-a}^{a} \sin(pv) \ \mathrm{d}v = 0 \tag{12.7.4}$$

Using the Fourier integrals, the general solution is,

$$u(x, t) = \int_0^\infty \frac{2 \sin(pa) \cos(px)}{p\pi} \exp(-c^2 p^2 t) \ \mathrm{d}p \tag{12.7.5}$$

3. The solution in integral form is,

$$f(x) = \frac{1}{1+x^2} \tag{12.7.6}$$

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(pv)\ \mathrm{d}v = \frac{2}{\pi}\int_{0}^{a} \frac{\cos(pv)}{1+v^2}\ \mathrm{d}v \tag{12.7.7}$$

$$= e^{-p} \tag{12.7.8}$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(pv)\ \mathrm{d}v = \frac{1}{\pi}\int_{-a}^{a} \sin(pv)\ \mathrm{d}v = 0 \tag{12.7.9}$$

Using the Fourier integrals, the general solution is,

$$u(x,t) = \int_{0}^{\infty} e^{-p}\cos(px)\ \exp(-c^2p^2t)\ \mathrm{d}p \tag{12.7.10}$$

4. The solution in integral form is,

$$f(x) = \exp(-|x|) \tag{12.7.11}$$

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(pv)\ \mathrm{d}v = \frac{2}{\pi}\int_{0}^{\infty} e^{-v}\ \cos(pv)\ \mathrm{d}v \tag{12.7.12}$$

$$= \frac{2}{\pi}\left[\frac{e^{-v}}{1+p^2}\left[-\cos(pv) + p\sin(pv)\right]\right]_0^{\infty} = \frac{2}{\pi}\cdot\frac{1}{1+p^2} \tag{12.7.13}$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(pv)\ \mathrm{d}v = 0 \tag{12.7.14}$$

Using the Fourier integrals, the general solution is,

$$u(x,t) = \int_{0}^{\infty} \frac{2}{\pi(1+p^2)}\cos(px)\ \exp(-c^2p^2t)\ \mathrm{d}p \tag{12.7.15}$$

5. The solution in integral form is,

$$f(x) = \begin{cases} |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad 12.7.16$$

$$A(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos(pv) \; \mathrm{d}v = \frac{2}{\pi} \int_0^1 v \; \cos(pv) \; \mathrm{d}v \qquad 12.7.17$$

$$= \frac{2}{\pi} \left[\frac{pv \sin(pv) + \cos(pv)}{p^2} \right]_0^1 = \frac{2}{\pi} \cdot \frac{p \sin(p) + \cos(p) - 1}{p^2} \qquad 12.7.18$$

$$B(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(pv) \; \mathrm{d}v = 0 \qquad 12.7.19$$

Using the Fourier integrals, the general solution is,

$$u(x,\, t) = \int_0^{\infty} \frac{2(p \sin p + \cos p - 1)}{\pi p^2} \; \cos(px) \; \exp(-c^2 p^2 t) \; \mathrm{d}p \qquad 12.7.20$$

6. The solution in integral form is,

$$f(x) = \begin{cases} x & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad 12.7.21$$

$$B(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(pv) \; \mathrm{d}v = \frac{2}{\pi} \int_0^1 v \; \sin(pv) \; \mathrm{d}v \qquad 12.7.22$$

$$= \frac{2}{\pi} \left[\frac{\sin(pv) - pv \cos(pv)}{p^2} \right]_0^1 = \frac{2}{\pi} \cdot \frac{\sin p - p \cos p}{p^2} \qquad 12.7.23$$

$$A(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos(pv) \; \mathrm{d}v = 0 \qquad 12.7.24$$

Using the Fourier integrals, the general solution is,

$$u(x,\, t) = \int_0^{\infty} \frac{2(\sin p - p \cos p)}{\pi p^2} \sin(px) \; \exp(-c^2 p^2 t) \; \mathrm{d}p \qquad 12.7.25$$

7. The solution in integral form is,

$$f(x) = \frac{\sin(x)}{x} \tag{12.7.26}$$

$$A(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos(pv) \ \mathrm{d}v = \frac{2}{\pi} \int_0^{\infty} \frac{\sin v}{v} \ \cos(pv) \ \mathrm{d}v \tag{12.7.27}$$

$$= \frac{-2}{\pi} \int_0^{\infty} \frac{1 - \cos(pv) - 1}{v} \ \sin(v) \ \mathrm{d}v \tag{12.7.28}$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin(v)}{v} \ \mathrm{d}v - \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos(pv)}{v} \ \sin(v) \ \mathrm{d}v \tag{12.7.29}$$

$$w = \frac{pv}{\pi} \qquad \mathrm{d}w = \frac{p}{\pi} \ \mathrm{d}v \tag{12.7.30}$$

$$A(p) = 1 - \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos(\pi w)}{w} \ \sin\left(\frac{\pi w}{p}\right) \ \mathrm{d}w \tag{12.7.31}$$

$$= \begin{cases} 0 & p > 1 \\ 1 & 0 < p < 1 \end{cases} \tag{12.7.32}$$

$$B(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(pv) \ \mathrm{d}v = 0 \tag{12.7.33}$$

Using the Fourier integrals, the general solution is,

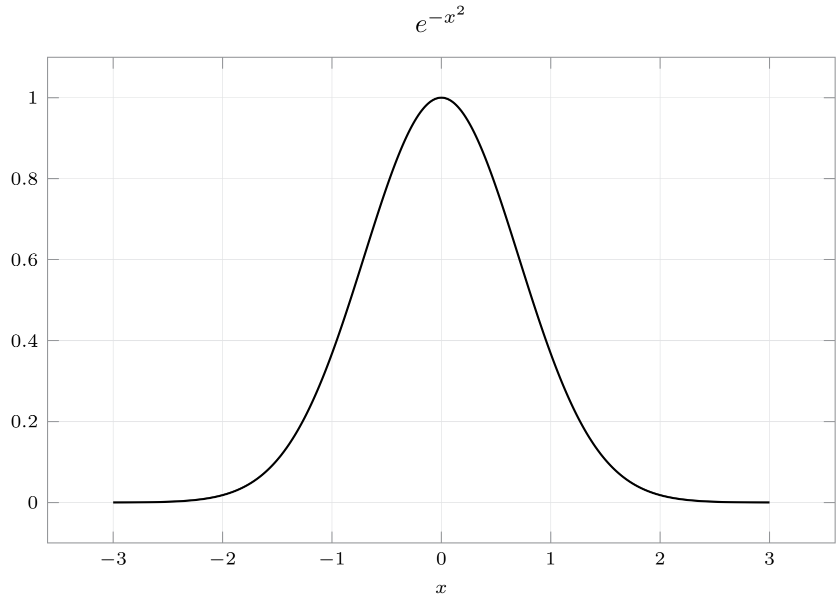
$$u(x, t) = \int_0^1 \ \cos(px) \ \exp(-c^2 p^2 t) \ \mathrm{d}p \tag{12.7.34}$$

8. Checking the solution of Problem 7 against the initial conditions,

$$u(x, 0) = \int_0^1 \ \cos(px) \ \mathrm{d}p \tag{12.7.35}$$

$$= \left[\frac{\sin(px)}{x}\right]_0^1 = \frac{\sin(x)}{x} \tag{12.7.36}$$

9. Graphing the integrand of the error function,



Checking if the error function is odd, using the fact that the integrand is even,

$$\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-w^2} \ \mathrm{d}w = -\frac{2}{\sqrt{\pi}} \int_{-x}^{0} e^{-w^2} \ \mathrm{d}w \tag{12.7.37}$$

$$= -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-w^2} \ \mathrm{d}w = -\operatorname{erf}(x) \tag{12.7.38}$$

Proving the relations,

$$\int_a^b e^{-w^2} \ \mathrm{d}w = \int_0^b e^{-w^2} \ \mathrm{d}w - \int_0^a e^{-w^2} \ \mathrm{d}w \tag{12.7.39}$$

$$= \frac{\sqrt{\pi}}{2}(\operatorname{erf} b - \operatorname{erf} a) \tag{12.7.40}$$

Using $a = -b$ in the above result, and the odd nature of $\operatorname{erf}(x)$,

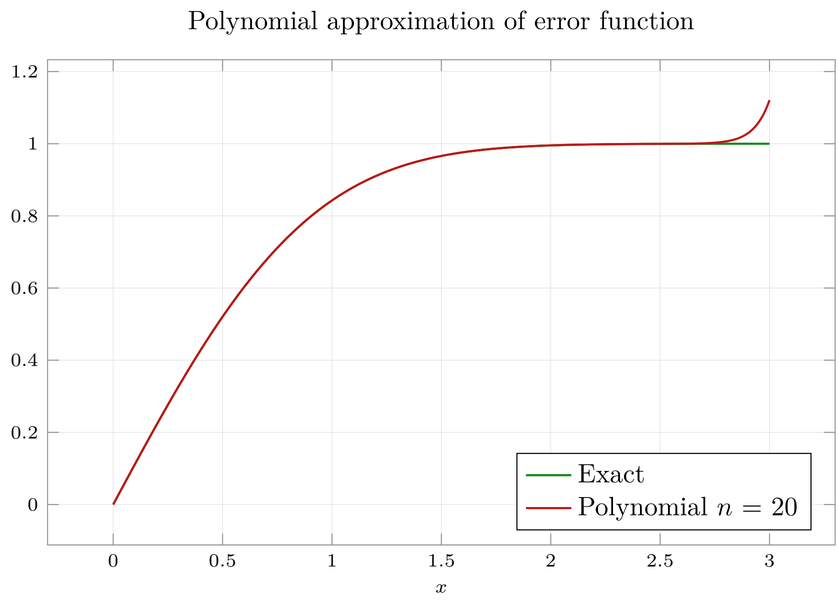
$$\int_{-b}^b e^{-w^2} \ \mathrm{d}w = \frac{\sqrt{\pi}}{2}[\operatorname{erf}(b) - \operatorname{erf}(-b)] = \sqrt{\pi}\operatorname{erf}(b) \tag{12.7.41}$$

10. From the Maclaurin series of the integrand,

$$e^{-w^2} = 1 - \frac{w^2}{1!} + \frac{w^4}{2!} - \frac{w^6}{3!} + \dots \tag{12.7.42}$$

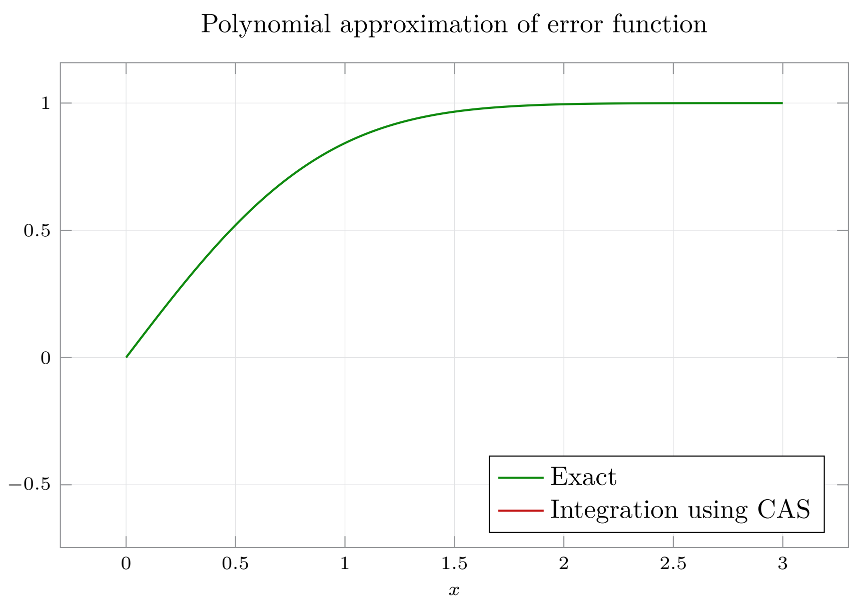
$$\int_0^x e^{-w^2} \ \mathrm{d}w = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots \tag{12.7.43}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots\right] \tag{12.7.44}$$

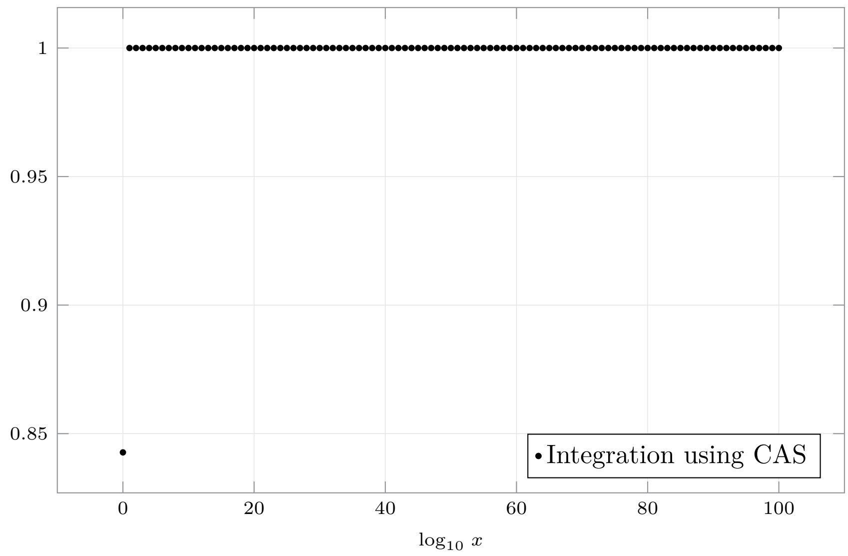


11. Using `sympy` as a CAS to perform the integration yields much more accurate results.

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2} \, \mathrm{d}w \tag{12.7.45}$$



12. Using `sympy` as a CAS to perform the integration for large values of x.



13. Using the result from Problem 12,

$$u(x,\,t) = \frac{1}{2c\sqrt{\pi t}} \int_0^\infty \exp\left[-\frac{(x-w)^2}{4c^2 t}\right] \,\mathrm{d}w \tag{12.7.46}$$

$$z = \frac{w-x}{2c\sqrt{t}} \tag{12.7.47}$$

$$z^- = \frac{-x}{2c\sqrt{t}} \qquad\qquad z^+ = \infty \tag{12.7.48}$$

$$u(x,\,t) = \frac{1}{\sqrt{\pi}} \int_{z^-}^{z^+} e^{-z^2} \,\mathrm{d}z = \frac{1}{2}\left[\operatorname{erf}(z^+) - \operatorname{erf}(z^-)\right] \tag{12.7.49}$$

$$= \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{-x}{2c\sqrt{t}}\right)\right] \tag{12.7.50}$$

14. Using the limits of integration z^+ and z^- as in the text,

$$u(x,\,t) = \frac{U_0}{2}\left[\operatorname{erf}(z^+) - \operatorname{erf}(z^-)\right] \tag{12.7.51}$$

$$z^+ = \frac{1-x}{2c\sqrt{t}} \qquad\qquad z^- = \frac{-1-x}{2c\sqrt{t}} \tag{12.7.52}$$

15. Expressing the given function in terms of the error function,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \ \mathrm{d}s \qquad\qquad w = \frac{s}{\sqrt{2}} \qquad \mathrm{d}w = \frac{\mathrm{d}s}{\sqrt{2}} \tag{12.7.53}$$

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{2}} e^{-w^2} \ \mathrm{d}w \qquad\qquad = \frac{\mathrm{erf}(x/\sqrt{2}) - \mathrm{erf}(-\infty)}{2} \tag{12.7.54}$$

$$= \frac{\mathrm{erf}(x/\sqrt{2}) + \mathrm{erf}(\infty)}{2} \qquad\qquad = \frac{\mathrm{erf}(x/\sqrt{2}) + 1}{2} \tag{12.7.55}$$

12.8 Modeling: Membrane, Two-Dimensional Wave Equation

1. No problem set in this section.

12.9 Rectangular Membrane. Double Fourier Series

1. Frequency of the eigenfunction u_{mn} is given by

$$f_{mn} = \frac{\lambda_{mn}}{2\pi} \qquad\qquad \lambda_{mn} = \sqrt{\frac{T\pi^2}{\rho}} \cdot \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{12.9.1}$$

$$T \to 2T \qquad\qquad \implies f \to \sqrt{2} f \tag{12.9.2}$$

$$\rho \to \rho/2 \qquad\qquad \implies f \to \sqrt{2} f \tag{12.9.3}$$

$$a,\, b \to 2a,\, 2b \qquad\qquad \implies f \to f/2 \tag{12.9.4}$$

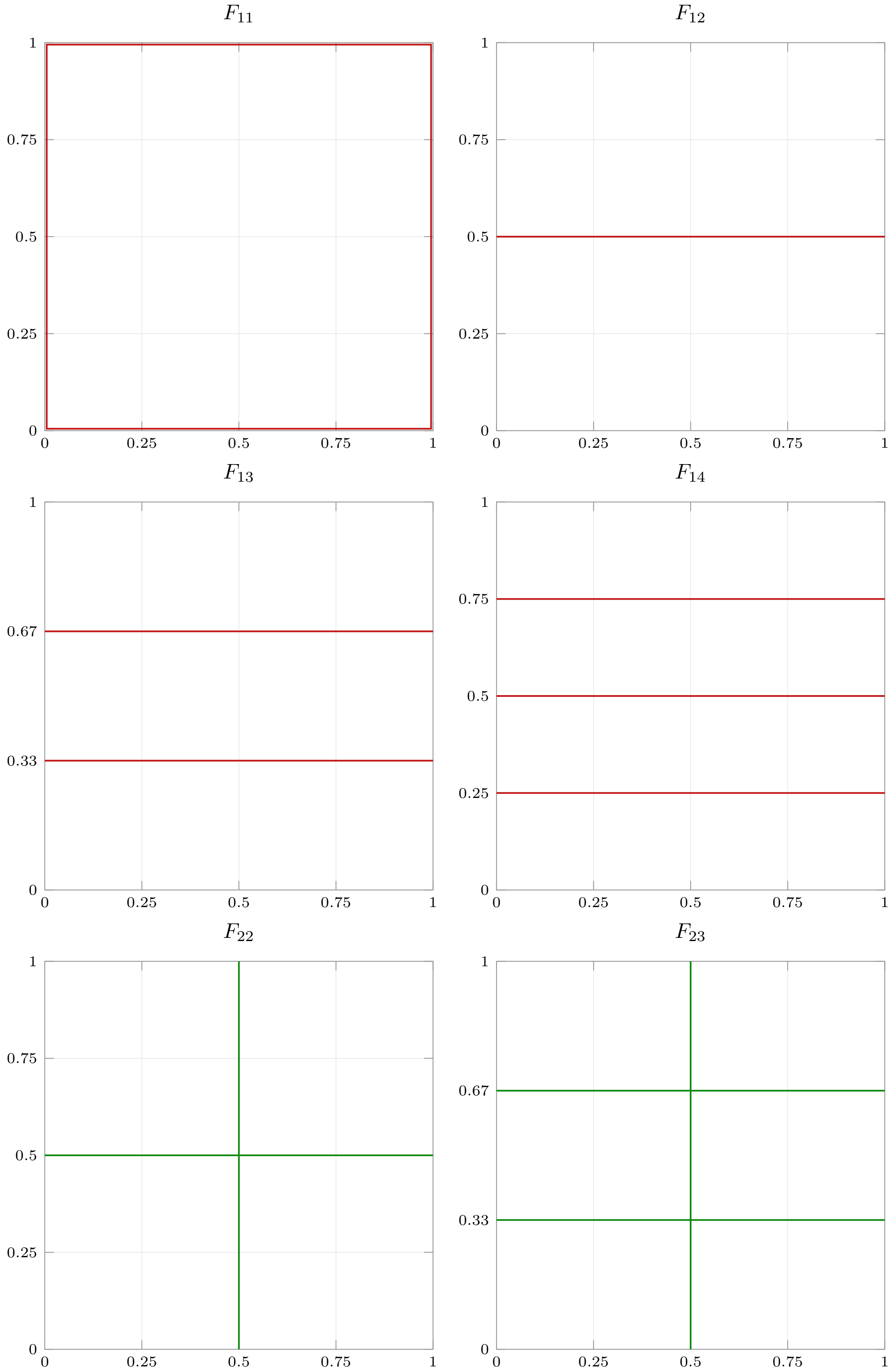
2. The small angle assumption enables the following approximation,

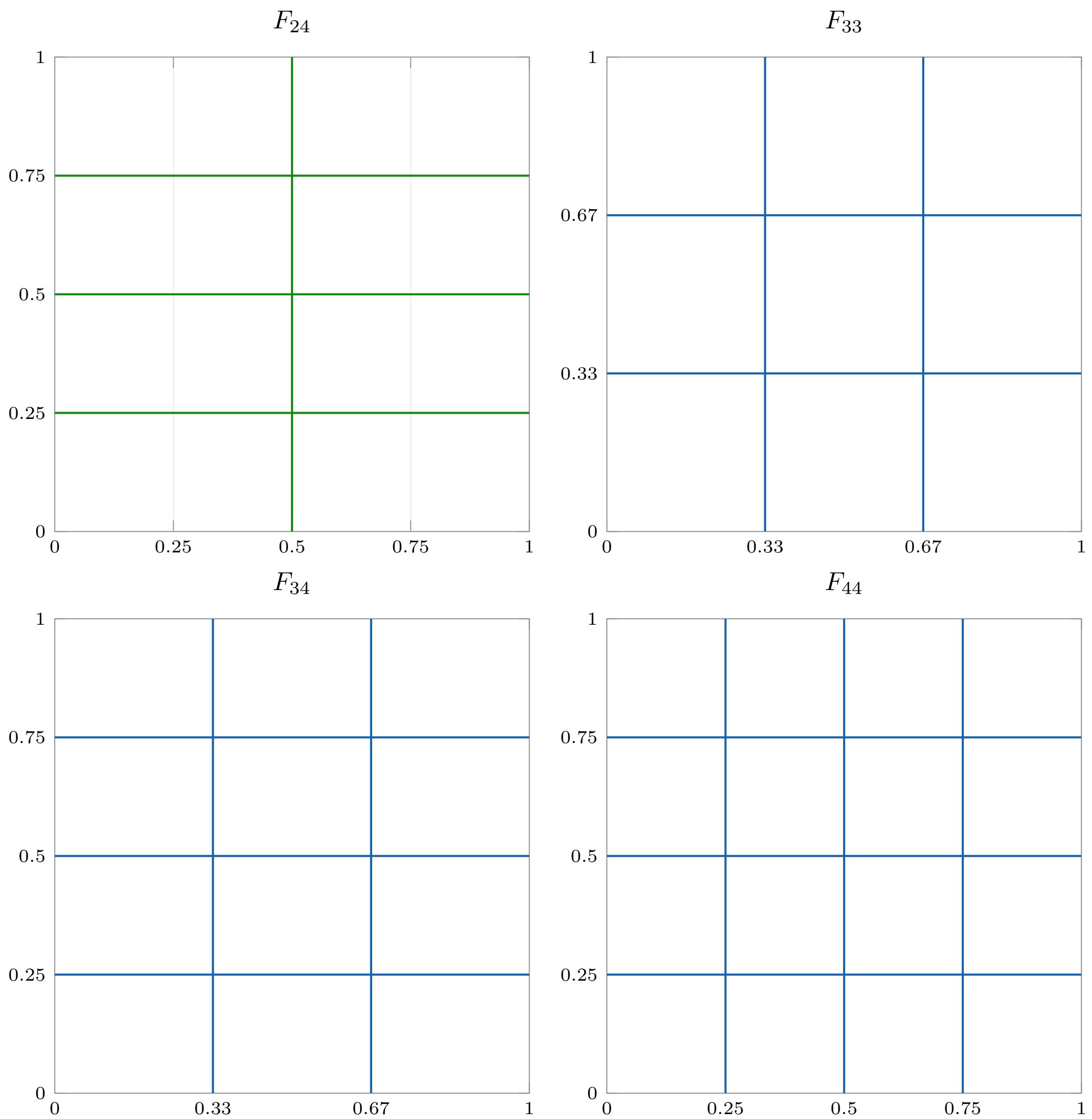
$$\sin(\theta) \approxeq \theta \approxeq \tan(\theta) \tag{12.9.5}$$

The assumption that the tension in the membrane is constant at all points in space and through time is unrealistic.

3. Nodal lines are points on the membrane that do not move. Since one half of the nodal lines are the same as the other half with x, y interchanged, only the nodal lines for $n \geq m$ are depicted here.

$$F_{mn} = \sin(m\pi x) \ \ \sin(n\pi y) \qquad\qquad a = b = 1 \tag{12.9.6}$$


F_{11}
1
0.75
0.5
0.25
0
0 0.25 0.5 0.75 1
F_{12}
1
0.75
0.5
0.25
0
0 0.25 0.5 0.75 1
F_{13}
1
0.67
0.33
0
0 0.25 0.5 0.75 1
F_{14}
1
0.75
0.5
0.25
0
0 0.25 0.5 0.75 1
F_{22}
1
0.75
0.5
0.25
0
0 0.25 0.5 0.75 1
F_{23}
1
0.67
0.33
0
0 0.25 0.5 0.75 1



4. Representing the given initial displacement as a double Fourier series,

$$f(x,\, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \;\, \sin(m\pi x) \;\, \sin(n\pi y) \tag{12.9.7}$$

$$K_m(y) = 2 \int_0^1 f(x,\, y) \;\, \sin(m\pi x) \;\, \mathrm{d}x \tag{12.9.8}$$

$$= \left[\frac{2}{m\pi} \;\, \cos(m\pi x) \right]_1^0 = \frac{2[1 - \cos(m\pi)]}{m\pi} \tag{12.9.9}$$

$$B_{mn} = 2 \int_0^1 K_m(y) \;\, \sin(n\pi y) \;\, \mathrm{d}y \tag{12.9.10}$$

$$= \frac{4}{mn\pi^2} \; [1 - \cos(m\pi)][1 - \cos(n\pi)] \tag{12.9.11}$$

5. Representing the given initial displacement as a double Fourier series,

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \ \sin(m\pi x) \ \sin(n\pi y) \tag{12.9.12}$$

$$K_m(y) = 2 \int_0^1 (y) \ \sin(m\pi x) \ \mathrm{d}x \tag{12.9.13}$$

$$= \left[\frac{2y}{m\pi} \ \cos(m\pi x) \right]_1^0 = \frac{2y}{m\pi} \ [1 - \cos(m\pi)] \tag{12.9.14}$$

$$B_{mn} = 2 \int_0^1 K_m(y) \ \sin(n\pi y) \ \mathrm{d}y \tag{12.9.15}$$

$$= \frac{4[1 - \cos(m\pi)]}{m\pi} \int_0^1 (y) \ \sin(n\pi y) \ \mathrm{d}y \tag{12.9.16}$$

$$= \frac{4[1 - \cos(m\pi)]}{m\pi} \left[\frac{\sin(n\pi y) - n\pi y \cos(n\pi y)}{n^2\pi^2} \right]_0^1 \tag{12.9.17}$$

$$= \frac{-4}{mn\pi^2} \ [1 - \cos(m\pi)][\cos(n\pi)] \tag{12.9.18}$$

6. Representing the given initial displacement as a double Fourier series,

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \ \sin(m\pi x) \ \sin(n\pi y) \tag{12.9.19}$$

$$K_m(y) = 2 \int_0^1 (x) \ \sin(m\pi x) \ \mathrm{d}x \tag{12.9.20}$$

$$= 2 \left[\frac{\sin(m\pi x) - m\pi x \cos(m\pi x)}{m^2\pi^2} \right]_0^1 = \frac{-2 \cos(m\pi)}{m\pi} \tag{12.9.21}$$

$$B_{mn} = 2 \int_0^1 K_m(y) \ \sin(n\pi y) \ \mathrm{d}y \tag{12.9.22}$$

$$= \frac{-4 \cos(m\pi)}{m\pi} \int_0^1 \sin(n\pi y) \ \mathrm{d}y \tag{12.9.23}$$

$$= \frac{-4 \cos(m\pi)}{m\pi} \left[\frac{\cos(n\pi y)}{n\pi} \right]_1^0 \tag{12.9.24}$$

$$= \frac{-4}{mn\pi^2} \ [1 - \cos(n\pi)][\cos(m\pi)] \tag{12.9.25}$$

7. Representing the given initial displacement as a double Fourier series,

$$f(x,\,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\ \sin\left(\frac{m\pi x}{a}\right)\ \sin\left(\frac{n\pi y}{b}\right) \tag{12.9.26}$$

$$K_m(y) = \frac{2}{a}\int_0^a (xy)\ \sin\left(\frac{m\pi x}{a}\right)\ \mathrm{d}x \tag{12.9.27}$$

$$= \frac{2y}{m^2\pi^2}\left[a\ \sin\left(\frac{m\pi x}{a}\right) + (m\pi x)\ \cos\left(\frac{m\pi x}{a}\right)\right]_0^a = \frac{2ay\ \cos(m\pi)}{m\pi} \tag{12.9.28}$$

$$B_{mn} = \frac{2}{b}\int_0^b K_m(y)\ \sin\left(\frac{n\pi y}{b}\right)\ \mathrm{d}y \tag{12.9.29}$$

$$= \frac{4a\ \cos(m\pi)}{bm\pi}\int_0^b (y)\ \sin\left(\frac{n\pi y}{b}\right)\ \mathrm{d}y \tag{12.9.30}$$

$$= \frac{4a\ \cos(m\pi)}{m\pi\ (n^2\pi^2)}\left[b\ \sin\left(\frac{n\pi y}{b}\right) + (n\pi y)\ \cos\left(\frac{n\pi y}{b}\right)\right]_0^b \tag{12.9.31}$$

$$= \frac{4ab}{mn\pi^2}\ [\cos(n\pi)\ \ \cos(m\pi)] \tag{12.9.32}$$

8. Representing the given initial displacement as a double Fourier series,

$$f(x,\,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\ \sin\left(\frac{m\pi x}{a}\right)\ \sin\left(\frac{n\pi y}{b}\right) \tag{12.9.33}$$

$$K_m(y) = \frac{2}{a}\int_0^a (xy)(a-x)(b-y)\ \sin\left(\frac{m\pi x}{a}\right)\ \mathrm{d}x \tag{12.9.34}$$

$$= \frac{2y(b-y)}{am^3\pi^3}\left[h(x)\ \sin\left(\frac{m\pi x}{a}\right) + [(m^2\pi^2 x)(x-a) - 2a^2]\ \cos\left(\frac{m\pi x}{a}\right)\right]_0^a \tag{12.9.35}$$

$$= \frac{4ay(b-y)}{m^3\pi^3}\ [1-\cos(m\pi)] \tag{12.9.36}$$

$$B_{mn} = \frac{2}{b}\int_0^b K_m(y)\ \sin\left(\frac{n\pi y}{b}\right)\ \mathrm{d}y \tag{12.9.37}$$

$$= \frac{8a[1-\cos(m\pi)]}{bm^3\pi^3}\int_0^b (y)(b-y)\ \sin\left(\frac{n\pi y}{b}\right)\ \mathrm{d}y \tag{12.9.38}$$

$$= \frac{8a[1-\cos(m\pi)]}{bm^3n^3\pi^6}\left[h(y)\ \sin\left(\frac{n\pi y}{b}\right) + [(n^2\pi^2 y)(y-b) - 2b^2]\ \cos\left(\frac{n\pi y}{b}\right)\right]_0^b \tag{12.9.39}$$

$$= \frac{16ab}{m^3n^3\pi^6}\ [1-\cos(n\pi)]\ [1-\cos(m\pi)] \tag{12.9.40}$$

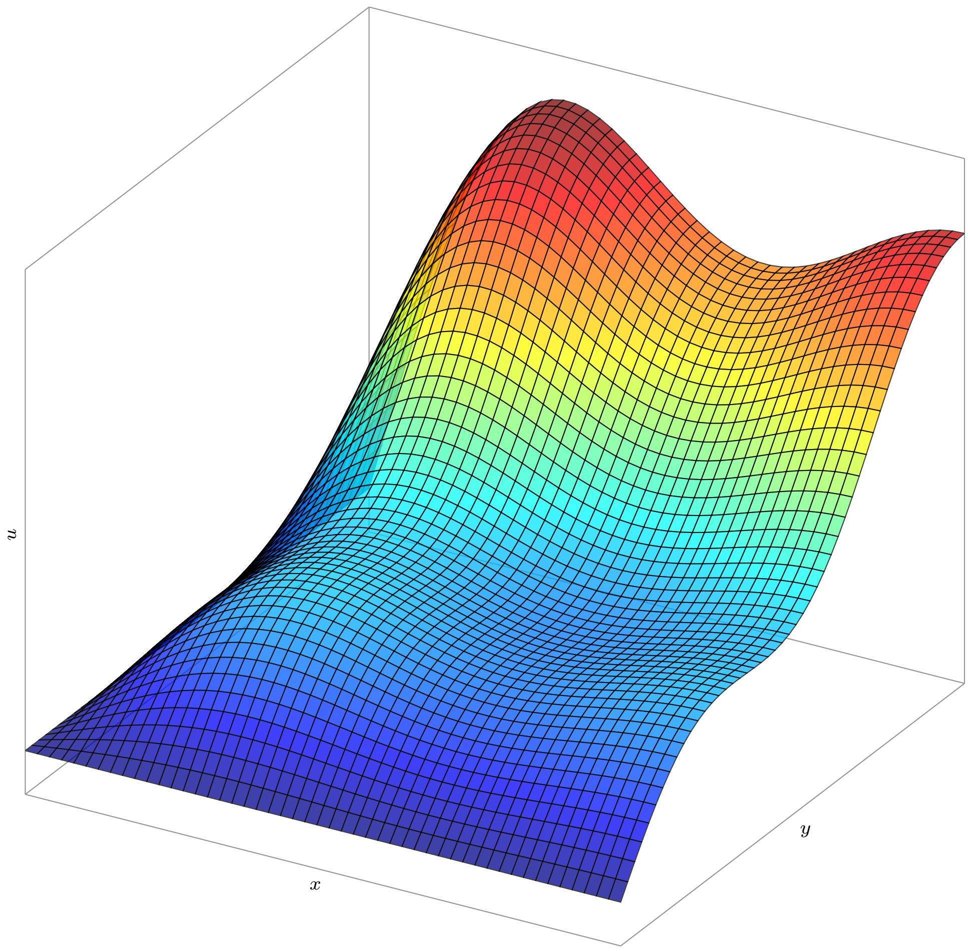
9. Using `gnuplot` to plot the series sum,

(a) For problem 5,

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \ \sin(m\pi x) \ \sin(n\pi y) = y \tag{12.9.41}$$

$$B_{mn} = \frac{-4}{mn\pi^2} \ [1 - \cos(m\pi)][\cos(n\pi)] \tag{12.9.42}$$

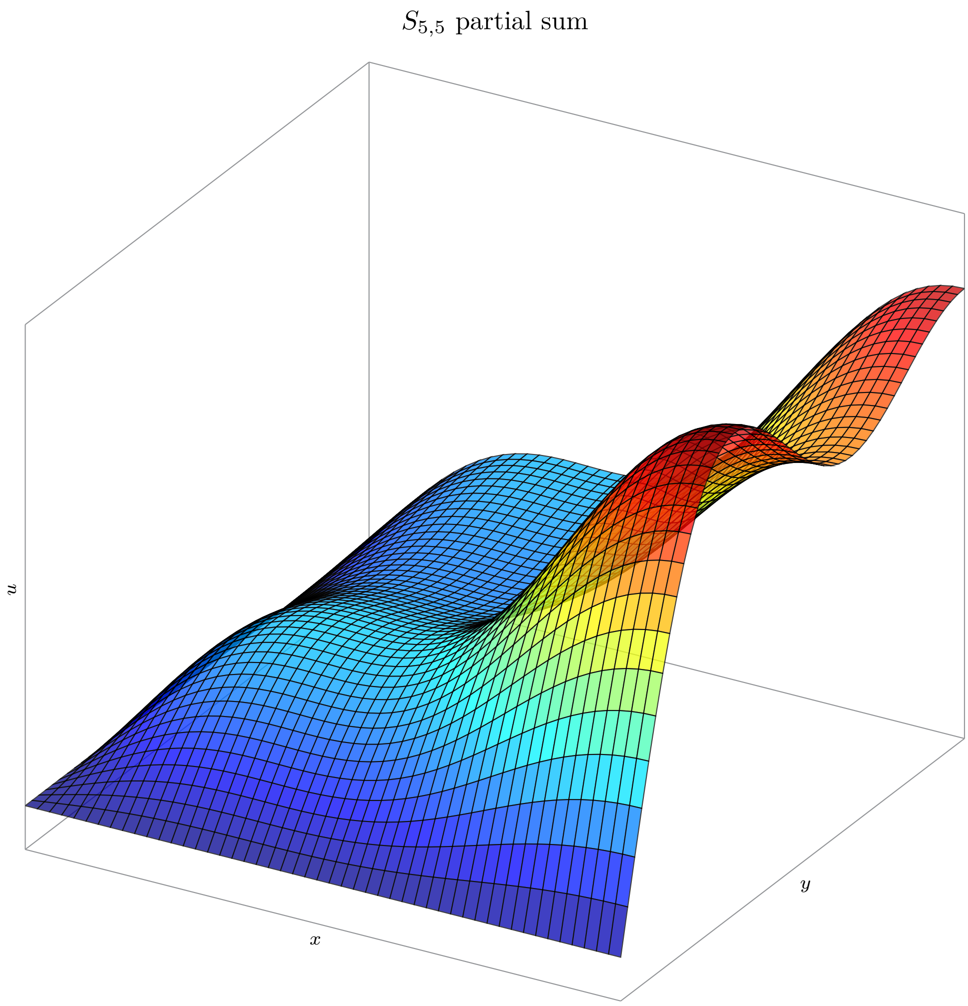
$S_{5,5}$ partial sum



For problem 6,

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \ \sin(m\pi x) \ \sin(n\pi y) = y \tag{12.9.43}$$

$$B_{mn} = \frac{-4}{mn\pi^2} \ [1 - \cos(n\pi)][\cos(m\pi)] \tag{12.9.44}$$

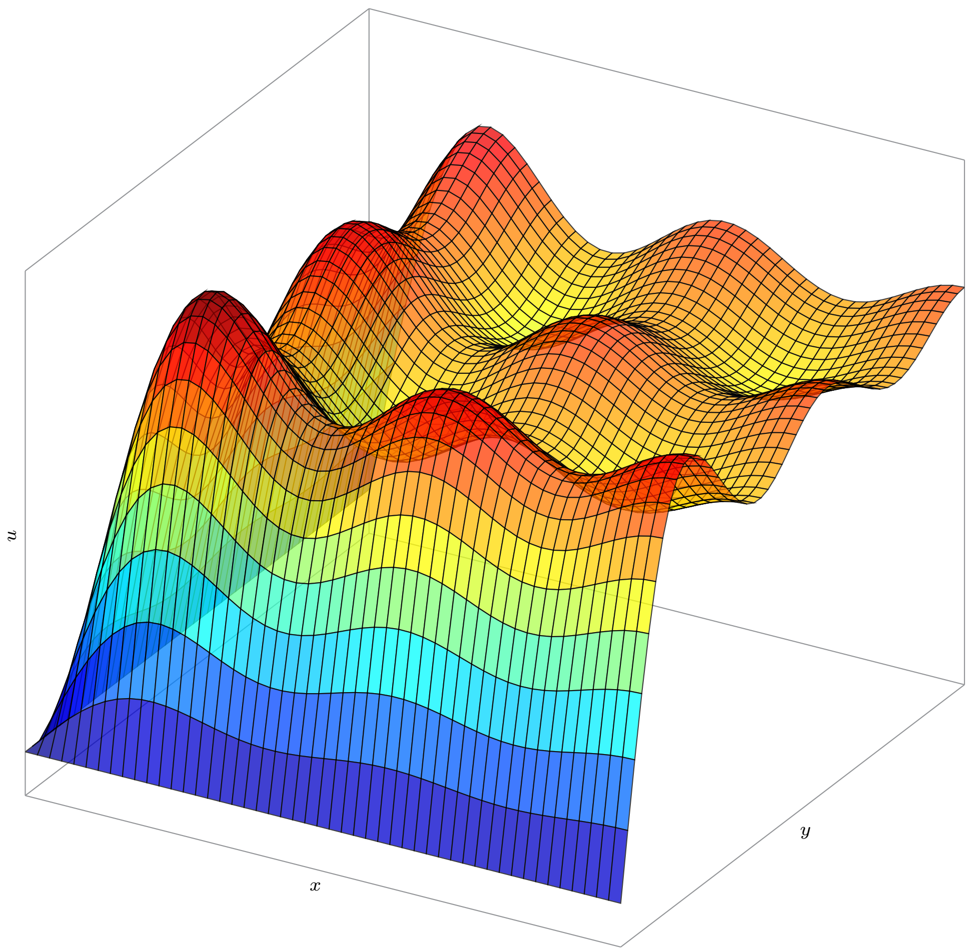


(b) For problem 4,

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \ \sin(m\pi x) \ \sin(n\pi y) = y \qquad 12.9.45$$

$$B_{mn} = \frac{-4}{mn\pi^2} \ [1 - \cos(n\pi)][\cos(m\pi)] \qquad 12.9.46$$

$S_{9,9}$ partial sum



(c) TBC.

10. The same number has to be decomposed into the sums of squares of integers in two different ways. This is achieved using the Brahmagupta-Fibonacci identity. Using `python`,

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2 \tag{12.9.47}$$

$$= (ac+bd)^2 + (ad-bc)^2 \tag{12.9.48}$$

$$65 = (1^2+2^2)(2^2+3^2) = (4)^2 + (7)^2 \tag{12.9.49}$$

$$= (8)^2 + (1)^2 \tag{12.9.50}$$

Brute force algorithm which fits integers a, b, c, d into the above formula such that the right hand side does not have zero terms.

Further, impose the constraint $(ad+bc) \neq (ac+bd)$. Nodal lines TBC.

11. Given the side length $a = b = \pi$ and $c^2 = 1$, and zero initial velocity,

$$f(x,\, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \;\sin(mx)\; \sin(ny) \tag{12.9.51}$$

$$K_m(y) = \frac{2}{\pi} \int_0^{\pi} (0.1 \sin(2x) \sin(4y)) \;\sin(mx)\; \mathrm{d}x \tag{12.9.52}$$

$$= \begin{cases} 0.1 \sin(4y) & m = 2 \\ 0 & \text{otherwise} \end{cases} \tag{12.9.53}$$

$$B_{mn} = \frac{2}{\pi} \int_0^{\pi} K_m(y) \;\sin(ny)\; \mathrm{d}y \tag{12.9.54}$$

$$= \frac{2}{\pi} \int_0^{\pi} (0.1 \sin 4y) \;\sin(ny)\; \mathrm{d}y \tag{12.9.55}$$

$$= \begin{cases} 0.1 & m = 2,\, n = 4 \\ 0 & \text{otherwise} \end{cases} \tag{12.9.56}$$

$$u(x,\, y,\, t) = 0.1 \cos(\sqrt{20}t) \;\sin(2x)\; \sin(4y) \tag{12.9.57}$$

12. Using the fact that $f(x)$ is already a term in the double Fourier series,

$$u(x,\, y,\, t) = 0.01 \cos(\sqrt{2}t) \;\sin(x)\; \sin(y) \tag{12.9.58}$$

13. Using the result from Problem 8,

$$f(x,\, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \;\sin(mx)\; \sin(ny) \tag{12.9.59}$$

$$K_m(y) = \frac{2}{\pi} \int_0^{\pi} (xy)(\pi - x)(\pi - y) \;\sin(mx)\; \mathrm{d}x \tag{12.9.60}$$

$$= \frac{2y(\pi - y)}{m^3\pi} \left[h(x) \;\sin(mx) + [(m^2x)(x - \pi) - 2] \;\cos(mx) \right]_0^{\pi} \tag{12.9.61}$$

$$= \frac{4y(\pi - y)}{m^3\pi} \;[1 - \cos(m\pi)] \tag{12.9.62}$$

$$B_{mn} = \frac{2}{b} \int_0^{b} K_m(y) \;\sin\left(\frac{n\pi y}{b}\right)\; \mathrm{d}y \tag{12.9.63}$$

$$= \frac{8[1 - \cos(m\pi)]}{m^3\pi^2} \int_0^{b} (y)(b - y) \;\sin(ny)\; \mathrm{d}y \tag{12.9.64}$$

$$= \frac{16}{m^3 n^3 \pi^2} \;[1 - \cos(n\pi)]\; [1 - \cos(m\pi)] \tag{12.9.65}$$

$$u(x,\, y,\, t) = 0.1 B_{mn} \;\sin(mx)\; \sin(ny) \tag{12.9.66}$$

14. The nodal lines are located when,

$$\sin\left(\frac{m\pi x}{4}\right) = 0 \qquad \text{or} \qquad \sin\left(\frac{n\pi y}{2}\right) = 0 \tag{12.9.67}$$

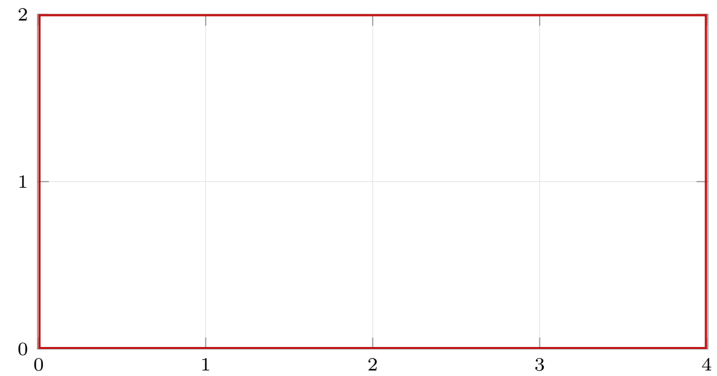
$$\tag{12.9.68}$$

The amplitude of each term varies as $(mn)^{-3}$. Clearly, this decays very quickly in m, n, which means that the first term dominates the series sum.

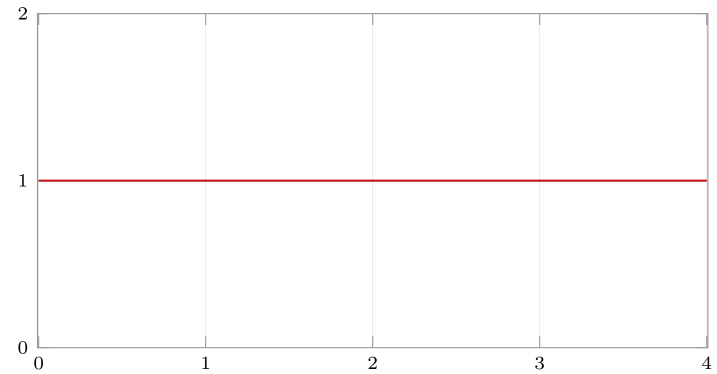
15. Nodal lines are points on the membrane that do not move. Since one half of the nodal lines are the same as the other half with x, y interchanged, only the nodal lines for $n \geq m$ are depicted here.

$$F_{mn} = \sin(m\pi x)\ \sin(n\pi y) \qquad a = 4 \qquad b = 2 \tag{12.9.69}$$

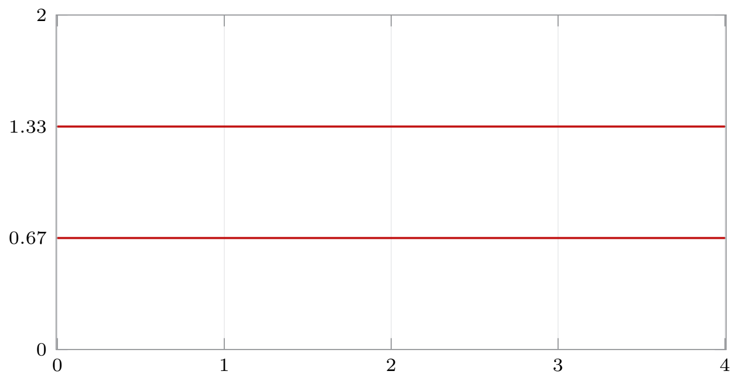
F_{11}



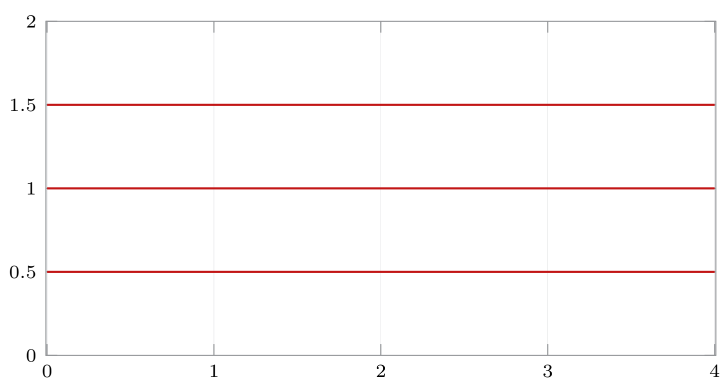
F_{12}



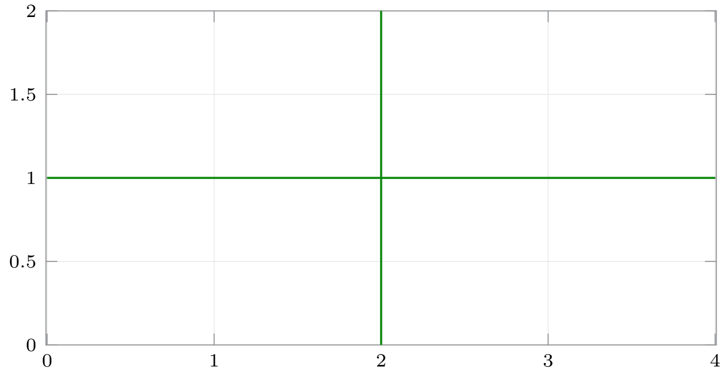
F_{13}



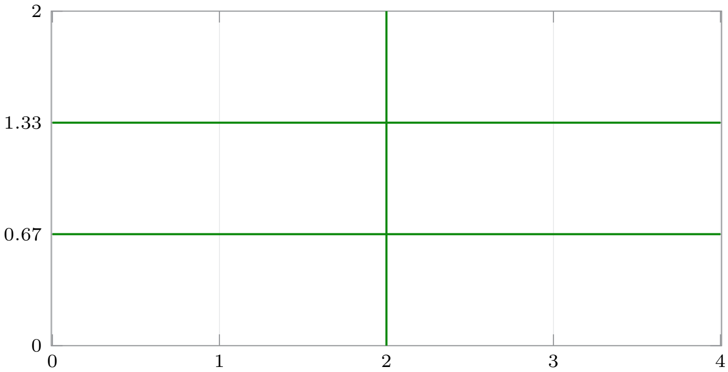
F_{14}



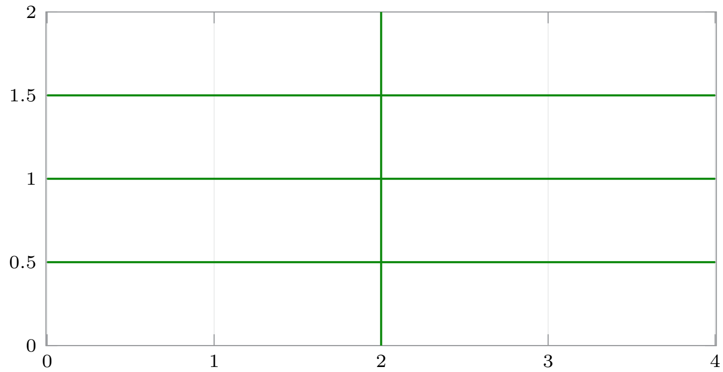
F_{22}



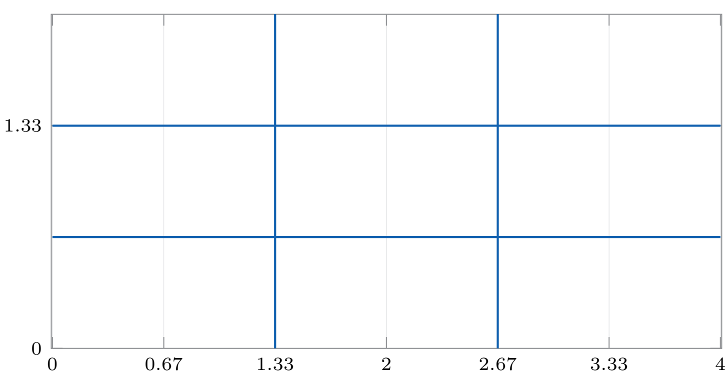
F_{23}



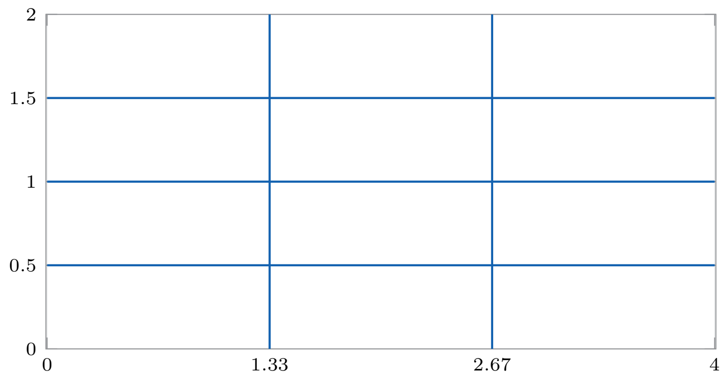
F_{24}



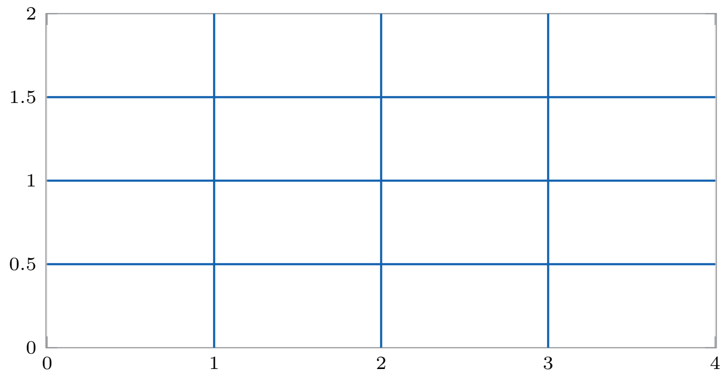
F_{33}



F_{34}



F_{44}



16. Using integration by parts,

$$I_1 = \int_0^4 (4x - x^2) \sin\left(\frac{m\pi x}{4}\right) \, \mathrm{d}x \tag{12.9.70}$$

$$= \left[\frac{4(x^2 - 4x)}{m\pi} \cos\left(\frac{m\pi x}{4}\right)\right]_0^4 + \int_0^4 \frac{4(4 - 2x)}{m\pi} \cos\left(\frac{m\pi x}{4}\right) \, \mathrm{d}x \tag{12.9.71}$$

$$= \left[\frac{16(4 - 2x)}{m^2\pi^2} \sin\left(\frac{m\pi x}{4}\right)\right]_0^4 + \int_0^4 \frac{32}{m^2\pi^2} \sin\left(\frac{m\pi x}{4}\right) \, \mathrm{d}x \tag{12.9.72}$$

$$= \left[\frac{128}{m^3\pi^3} \cos\left(\frac{m\pi x}{4}\right)\right]_4^0 = \frac{128}{m^3\pi^3} [1 - \cos(m\pi)] \tag{12.9.73}$$

For the integral in y,

$$I_2 = \int_0^2 (2y - y^2) \sin\left(\frac{n\pi y}{2}\right) \, \mathrm{d}y \tag{12.9.74}$$

$$= \left[\frac{2(y^2 - 2y)}{n\pi} \cos\left(\frac{n\pi y}{2}\right)\right]_0^2 + \int_0^2 \frac{2(2 - 2y)}{n\pi} \cos\left(\frac{n\pi y}{2}\right) \, \mathrm{d}y \tag{12.9.75}$$

$$= \left[\frac{4(2 - 2y)}{n^2\pi^2} \sin\left(\frac{n\pi y}{2}\right)\right]_0^2 + \int_0^2 \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi y}{2}\right) \, \mathrm{d}y \tag{12.9.76}$$

$$= \left[\frac{16}{n^3\pi^3} \cos\left(\frac{n\pi y}{2}\right)\right]_2^0 = \frac{16}{n^3\pi^3} \left[1 - \cos(n\pi)\right] \tag{12.9.77}$$

17. Using the code form Problem 10, and taking a simple example,

$$145 = 8^2 + 9^2 \qquad = \left(\frac{m_1}{2}\right)^2 + n_1^2 \tag{12.9.78}$$

$$= 12^2 + 1^2 \qquad = \left(\frac{m_2}{2}\right)^2 + n_2^2 \tag{12.9.79}$$

$$m_1, n_1 = 16, 3 \qquad m_2, n_2 = 24, 1 \tag{12.9.80}$$

18. Minimizing the frequency with respect to the ratio of the sides of the rectangle,

$$\lambda_{11} = c\pi\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} \qquad b = \frac{K}{a} \tag{12.9.81}$$

$$\frac{\mathrm{d}\lambda_{11}}{\mathrm{d}a} = 0 \qquad \implies 0 = -2a^{-3} + \frac{2a}{K^2} \tag{12.9.82}$$

$$K^2 = a^4 \qquad a = \sqrt{K} = b \tag{12.9.83}$$

Since both sides are equal, the minimum is achieved for a square membrane.

19. Using the standard result, and the fact that the initial deflection is already a term in the double Fourier series,

$$B_{mn} = \begin{cases} 1 & m = 6, n = 2 \\ 0 & \text{otherwise} \end{cases} \tag{12.9.84}$$

$$u(x, y, t) = \cos\left(\sqrt{\left(\frac{6}{a}\right)^2 + \left(\frac{2}{b}\right)^2} \, \pi t\right) \sin\left(\frac{6\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right) \tag{12.9.85}$$

20. Using Newton's second law,

$$u_{tt} = \frac{T}{\rho}\ \nabla^2 u + \frac{F_{\text{ext}}}{\rho \Delta A} \qquad\qquad P = \frac{F}{\Delta A} \tag{12.9.86}$$

$$u_{tt} = c^2\ \nabla^2 u + \frac{P}{\rho} \tag{12.9.87}$$

12.10 Circular Membrane, Fourier-Bessel Series

1. Polar coordinates are necessary to simplify the analysis of circular membranes , especially when the displacement is radially symmetric.

2. Using radial symmetry from the very beginning, which makes $u_\theta = 0$,

$$x = r\cos\theta \qquad\qquad y = r\sin\theta \tag{12.10.1}$$

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \arctan(y/x) \tag{12.10.2}$$

$$u_x = u_r\ r_x + u_\theta\ \theta_x \qquad\qquad u_x = u_r\ \frac{x}{\sqrt{x^2+y^2}} \tag{12.10.3}$$

$$u_{xx} = (u_r\ r_x)_x = (u_r)_x\ r_x + u_r\ r_{xx} \qquad\qquad = u_{rr}\ (r_x)^2 + u_r\ r_{xx} \tag{12.10.4}$$

$$u_{xx} = u_{rr}\ \frac{x^2}{r^2} + u_r\ \frac{y^2}{r^3} \tag{12.10.5}$$

Since the result for u_{yy} is similar, the Laplacian becomes,

$$\nabla^2 u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}\ u_r \tag{12.10.6}$$

3. Rewriting the Laplacian using the chain rule,

$$\nabla^2 u = u_{rr} + \frac{1}{r}\ u_r + \frac{1}{r^2}\ u_{\theta\theta} \qquad\qquad (r\ u_r)_r = r\ u_{rr} + u_r \tag{12.10.7}$$

$$\nabla^2 u = \frac{1}{r}\ (r\ u_r)_r + \frac{1}{r^2}\ u_{\theta\theta} \tag{12.10.8}$$

4. Looking at the given functions,

(a) Checking if the given functions satisfy Laplace's equation in polar coordinates,

$$u = r^n\cos(n\theta) \qquad\qquad u_r = n(r^{n-1})\cos(n\theta) \tag{12.10.9}$$

$$u_{rr} = n(n-1)r^{n-2}\cos(n\theta) \qquad\qquad u_{\theta\theta} = -n^2 r^n\cos(n\theta) \tag{12.10.10}$$

Substituting into the equation,

$$u_{rr} + \frac{1}{r}\, u_r + \frac{1}{r^2}\, u_{\theta\theta} = \left[n(n-1) + n - n^2\right] r^{n-2} \cos(n\theta) = 0 \tag{12.10.11}$$

$v_n = r^n \sin(\theta)$ also satisfies this equation similarly.
For small values of n,

$$u_0 = 1 \qquad v_0 = 0 \tag{12.10.12}$$

$$u_1 = x \qquad v_1 = y \tag{12.10.13}$$

$$u_2 = x^2 - y^2 \qquad v_2 = 2xy \tag{12.10.14}$$

$$u_3 = x^3 - 3xy^2 \qquad v_3 = 3x^2y - y^3 \tag{12.10.15}$$

(b) Checking if the given function satisfies Laplace's equation,

$$u_r = \sum_{n=1}^{\infty} \frac{na_n}{R} \left(\frac{r}{R}\right)^{n-1} \cos(n\theta) + \frac{nb_n}{R} \left(\frac{r}{R}\right)^{n-1} \sin(n\theta) \tag{12.10.16}$$

$$u_{rr} = \sum_{n=1}^{\infty} \frac{n(n-1)a_n}{R^2} \left(\frac{r}{R}\right)^{n-2} \cos(n\theta) + \frac{n(n-1)b_n}{R^2} \left(\frac{r}{R}\right)^{n-2} \sin(n\theta) \tag{12.10.17}$$

$$u_{\theta\theta} = \sum_{n=1}^{\infty} -n^2 a_n \left(\frac{r}{R}\right)^{n} \cos(n\theta) - n^2 b_n \left(\frac{r}{R}\right)^{n} \sin(n\theta) \tag{12.10.18}$$

Substituting into Laplace's equation,

$$u_{rr} + \frac{1}{r}\, u_r + \frac{1}{r^2}\, u_{\theta\theta} = \left(\frac{r}{R}\right)^{n-2} \cos(n\theta) \left[\frac{a_n}{R^2}\, [n^2 - n + n - n^2]\right] \tag{12.10.19}$$

$$+ \left(\frac{r}{R}\right)^{n-2} \sin(n\theta) \left[\frac{b_n}{R^2}\, [n^2 - n + n - n^2]\right] \tag{12.10.20}$$

$$= 0 + 0 \tag{12.10.21}$$

Satisfying the boundary conditions,

$$u(R, \theta) = f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n\ \cos(n\theta) + b_n \sin(n\theta) \tag{12.10.22}$$

This form of series expansion is satisfied by the set of coefficients being the Fourier coefficients of $f(\theta)$.

(c) Given the boundary condition, the Fourier coefficients are,

$$f(\theta) = \begin{cases} -100 & \theta \in [-\pi, 0] \\ 100 & \theta \in [0, \pi] \end{cases} \tag{12.10.23}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \ \mathrm{d}\theta = \frac{1}{2\pi} \Big[-100\theta \Big]_{-\pi}^{0} + \frac{1}{2\pi} \Big[100\theta \Big]_{0}^{\pi} = 0 \tag{12.10.24}$$

$$a_n = 0 \tag{12.10.25}$$

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} (100) \ \sin(n\theta) \ \mathrm{d}\theta = \frac{200}{n\pi} \ [1 - \cos(n\pi)] \tag{12.10.26}$$

The general solution using part b is,

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n \ r^n \ \sin(n\theta) \tag{12.10.27}$$

(d) For the Neumann boundary conditions,

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} a_n \ r^n \cos(n\theta) + b_n \ r^n \sin(n\theta) \tag{12.10.28}$$

$$u_r(R, \theta) = g(\theta) = \sum_{n=1}^{\infty} n a_n R^{n-1} \cos(n\theta) + n b_n R^{n-1} \sin(n\theta) \tag{12.10.29}$$

$$n a_n R^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \ \cos(n\theta) \ \mathrm{d}\theta \tag{12.10.30}$$

$$n b_n R^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \ \sin(n\theta) \ \mathrm{d}\theta \tag{12.10.31}$$

There is no condition on a_0 since it gets deleted in the differentiation.

(e) From Section 10.4,

$$\iint_R \nabla^2 u \ \ \mathrm{d}x \ \mathrm{d}y = \oint_C u_r(R, \theta) \ R \ \ \mathrm{d}\theta \tag{12.10.32}$$

$$\nabla^2 u = 0 \quad \forall \quad r < R \tag{12.10.33}$$

$$g(\theta) = u_r(R, \theta) \tag{12.10.34}$$

The fact that u solves Laplace's equation makes the LHS zero. Thus, the RHS is also zero.

$$\oint_C u_r(R, \theta) \ \ \mathrm{d}\theta = \int_{-\pi}^{\pi} g(\theta) \ \ \mathrm{d}\theta = 0 \tag{12.10.35}$$

(f) Solving Laplace's equation in polar coordinates,

$$u(r,\theta) = F(r)\cdot G(\theta) \tag{12.10.36}$$

$$0 = G\cdot\frac{\partial^2 F}{\partial r^2} + \frac{G}{r}\,\frac{\partial F}{\partial r} + \frac{F}{r^2}\cdot\frac{\partial^2 G}{\partial\theta^2} \tag{12.10.37}$$

$$-\frac{1}{G}\,\frac{\partial^2 G}{\partial\theta^2} = \frac{1}{F}\left[r^2F'' + rF'\right] = k \tag{12.10.38}$$

For $k = 0$, Solving the ODE in θ

$$(F')' = \frac{-F'}{r} \qquad \ln(F') = -\ln(r) + a_1 \tag{12.10.39}$$

$$F' = \frac{a_1}{r} \qquad F(r) = a_1\ln(r) + a_2 \tag{12.10.40}$$

$$G(\theta) = b_1\theta + b_2 \qquad G(-\pi) = G(\pi) \tag{12.10.41}$$

$$b_1 = 0 \quad b_2 = 1 \tag{12.10.42}$$

For $k < 0$, no periodic solutions exist for $G(\theta)$, which means this case can be ignored.
For $k = m^2 > 0$,

$$G(\theta) = h_m\cos(m\theta) + j_m\sin(m\theta) \tag{12.10.43}$$

$$r^2\;F'' + r\;F' - m^2\;F = 0 \tag{12.10.44}$$

$$F(r) = p_m r^m + q_m r^{-m} \tag{12.10.45}$$

Using the periodic nature of $G(\theta)$,

$$G(-\pi) = G(\pi) \qquad \Longrightarrow\; \sin(m\pi) = 0 \tag{12.10.46}$$

$$m = (\text{integer}) \tag{12.10.47}$$

Since there are infinite solutions (one for each integer,) the series sum is the full solution,

$$u(r,\theta) = F_0\cdot G_0 + \sum_{m=1}^{\infty} F_m\cdot G_m \tag{12.10.48}$$

$$= a_1\ln(r) + a_2 + \sum_{m=1}^{\infty}(p_m r^m + q_m r^{-m})\cdot\left[h_m\cos(m\theta) + j_m\sin(m\theta)\right] \tag{12.10.49}$$

Applying the boundary conditions at either edge of the annulus, and absorbing the coefficients

h_m, j_m into p_m, q_m,

$$u_r(1, \theta) = \sin\theta = a_1 + \sum_{m=1}^{\infty} m(p_m - q_m)\ G(\theta) \tag{12.10.50}$$

$$u_r(3, \theta) = 0 = \frac{a_1}{3} + \sum_{m=1}^{\infty} (mp_m 3^{m-1} - mq_m 3^{-m-1}) \cdot G(\theta) \tag{12.10.51}$$

Since $\sin\theta$ is already a term in the Fourier expansion, all terms $m \neq 1$ are zero.

$$p_1 - q_1 = 1 \qquad\qquad p_1 - \frac{q_1}{9} = 0 \tag{12.10.52}$$

$$a_1 = 0 \qquad\qquad p_1, q_1 = -\frac{1}{8}, \ -\frac{9}{8} \tag{12.10.53}$$

$$u(r, \theta) = -\frac{\sin\theta}{8}\left[r + \frac{9}{r}\right] \tag{12.10.54}$$

5. Finding the potential using equation 20,

$$u(1, \theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\theta) + b_m \sin(m\theta) \tag{12.10.55}$$

$$u(1, \theta) = f(\theta) = \begin{cases} 220 & \theta \in (-\pi/2, \pi/2) \\ 0 & \text{otherwise} \end{cases} \tag{12.10.56}$$

$$b_m = 0 \tag{12.10.57}$$

$$a_m = \frac{2}{\pi}\int_0^{\pi/2} 220\cos(m\theta)\ \mathrm{d}\theta \tag{12.10.58}$$

$$= \frac{440}{m\pi}\Big[\sin(m\theta)\Big]_0^{\pi/2} = \frac{440}{m\pi}\ \sin(m\pi/2) \tag{12.10.59}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi/2} 220\ \mathrm{d}\theta = 110 \tag{12.10.60}$$

$$u(r, \theta) = 110 + \frac{440}{\pi}\sum_{m=1}^{\infty} \frac{\sin(m\pi/2)}{m}\ r^m\ \cos(m\theta) \tag{12.10.61}$$

6. Finding the potential using equation 20,

$$u(1,\theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\theta) + b_m \sin(m\theta) \tag{12.10.62}$$

$$u(1,\theta) = f(\theta) = 400\cos^3\theta = 100\left[\cos(3\theta) + 3\cos(\theta)\right] \tag{12.10.63}$$

$$b_m = 0 \tag{12.10.64}$$

$$a_m = 0 \quad \forall\ m \neq 1, 3 \tag{12.10.65}$$

$$a_1, a_3 = 100, 300 \tag{12.10.66}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi/2} 220 \ \mathrm{d}\theta = 110 \tag{12.10.67}$$

$$u(r,\theta) = 300\ r\cos(\theta) + 100\ r^3\cos(3\theta) \tag{12.10.68}$$

7. Finding the potential using equation 20,

$$u(1,\theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\theta) + b_m \sin(m\theta) \tag{12.10.69}$$

$$u(1,\theta) = f(\theta) = 110\ |\theta| \tag{12.10.70}$$

$$b_m = 0 \tag{12.10.71}$$

$$a_m = \frac{2}{\pi}\int_0^{\pi} 110\theta\cos(m\theta) \ \mathrm{d}\theta \tag{12.10.72}$$

$$= \frac{220}{\pi}\left[\frac{m\theta\sin(m\theta) + \cos(m\theta)}{m^2}\right]_0^{\pi} = \frac{-220}{\pi m^2}\left[1 - \cos(m\pi)\right] \tag{12.10.73}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} 110\ \theta\ \mathrm{d}\theta = 55\pi \tag{12.10.74}$$

$$u(r,\theta) = 55\pi - \frac{220}{\pi}\sum_{m=1}^{\infty}\frac{1-\cos(m\pi)}{m^2}\ r^m\ cos(m\theta) \tag{12.10.75}$$

8. Finding the potential using equation 20,

$$u(1,\theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\theta) + b_m \sin(m\theta) \tag{12.10.76}$$

$$u(1,\theta) = f(\theta) = \begin{cases} \theta & \theta \in (-\pi/2, \pi/2) \\ 0 & \text{otherwise} \end{cases} \tag{12.10.77}$$

$$a_m = a_0 = 0 \tag{12.10.78}$$

$$b_m = \frac{2}{\pi} \int_0^{\pi/2} \theta \sin(m\theta) \ d\theta \tag{12.10.79}$$

$$= \frac{2}{\pi} \left[\frac{\sin(m\theta) - m\theta \cos(m\theta)}{m^2} \right]_0^{\pi/2} \tag{12.10.80}$$

$$= \frac{2}{m^2 \pi} \ \sin(m\pi/2) - \frac{1}{m} \ \cos(m\pi/2) \tag{12.10.81}$$

9. TBC

10. Finding the potential using equation 20,

$$u(1,\theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\theta) + b_m \sin(m\theta) \tag{12.10.82}$$

$$u(r,0) = 0 = a_0 + \sum_{m=1}^{\infty} a_m \ r^m \tag{12.10.83}$$

$$u(r,\pi) = 0 = a_0 + \sum_{m=1}^{\infty} a_m \ r^m \ \cos(m\pi) \tag{12.10.84}$$

$$u(1,\theta) = f(\theta) = \begin{cases} 110 \ \theta(\pi - \theta) & \theta \in (0, \pi) \\ 0 & \text{otherwise} \end{cases} \tag{12.10.85}$$

$$a_m = a_0 = 0 \tag{12.10.86}$$

$$b_m = \frac{220}{\pi} \int_0^{\pi} \theta(\pi - \theta) \sin(m\theta) \ d\theta \tag{12.10.87}$$

$$= \frac{220}{\pi} \left[h(\theta) \ \sin(m\theta) + \frac{m^2 \theta(\theta - \pi) - 2}{m^3} \ \cos(m\theta) \right]_0^{\pi} \tag{12.10.88}$$

$$= \frac{440}{m^3 \pi} \left[1 - \cos(m\pi) \right] \tag{12.10.89}$$

11. Finding the potential using equation 20,

$$u(1,\,\theta) = a_0 + \sum_{m=1}^{\infty} a_m\,\cos(m\theta) + b_m\,\sin(m\theta) \tag{12.10.90}$$

$$u(r,\,0) = 0 = a_0 + \sum_{m=1}^{\infty} a_m\,r^m \tag{12.10.91}$$

$$u(r,\,\pi) = 0 = a_0 + \sum_{m=1}^{\infty} a_m\,r^m\;\cos(m\pi) \tag{12.10.92}$$

$$u(1,\,\theta) = f(\theta) = \begin{cases} u_0 & \theta \in (0,\,\pi) \\ 0 & \text{otherwise} \end{cases} \tag{12.10.93}$$

$$a_m = a_0 = 0 \tag{12.10.94}$$

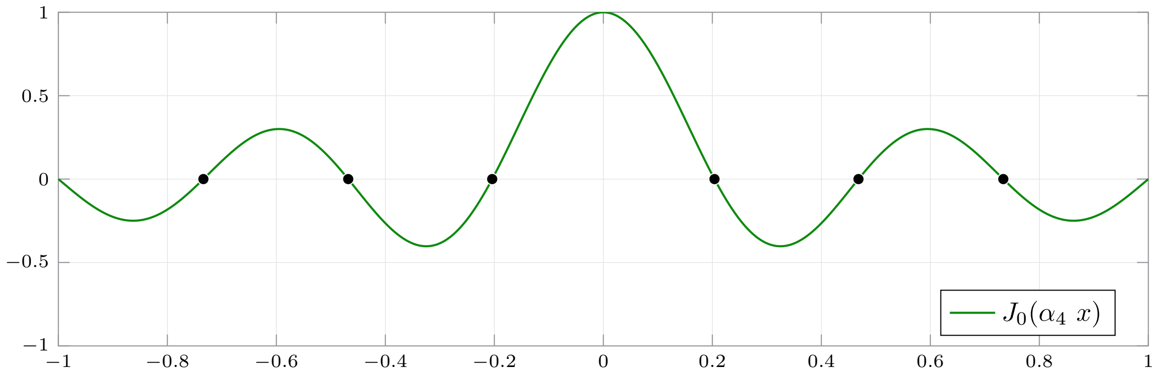
$$b_m = \frac{2}{\pi}\int_0^{\pi} u_0\,\sin(m\theta)\;\;d\theta \tag{12.10.95}$$

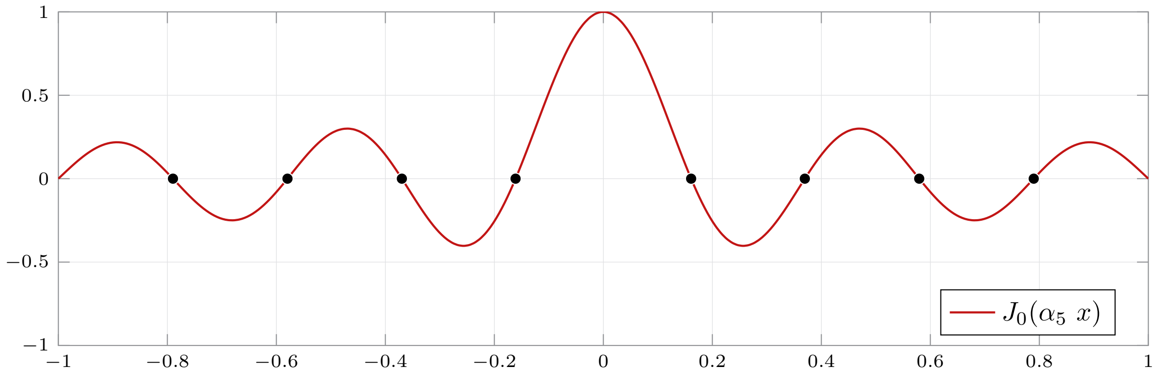
$$= \frac{2u_0}{m\pi}\,\Big[1 - \cos(m\pi)\Big] \tag{12.10.96}$$

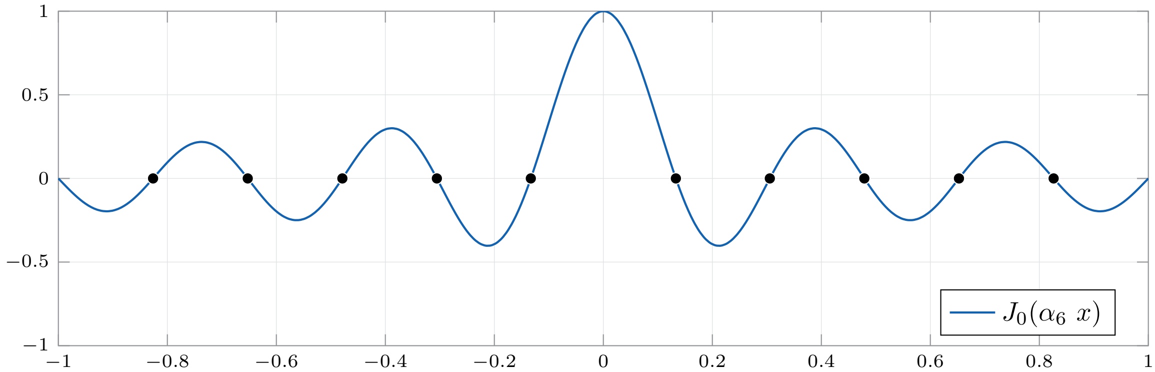
$$u(r,\,\theta) = \frac{2u_0}{\pi}\sum_{m=1}^{\infty} \frac{1-\cos(m\pi)}{m}\,\frac{r^m}{a^m}\;\sin(m\theta) \tag{12.10.97}$$

12. Graphing normal modes,

(a) Typo in question. Referring to Figure 309 in the text,



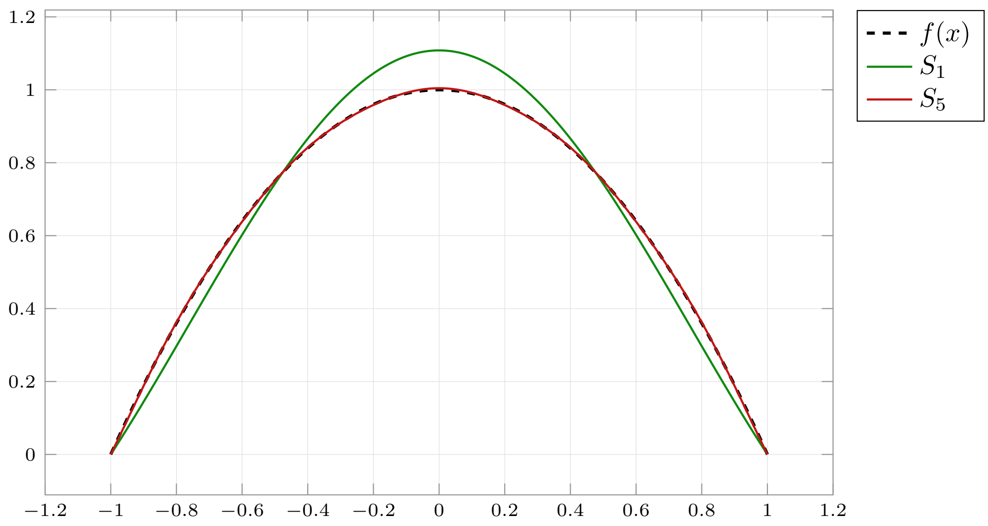




(b) Tabulating the requisite items,

m	$\alpha_{\mathbf{m}}$	$\mathbf{A_m}$	$(\mathbf{m}-\mathbf{0.25})\pi$	**Error**
1	2.40483	1.1080	2.3562	-0.048631
2	5.52008	-0.13978	5.4978	-0.022291
3	8.65373	0.045476	8.6394	-0.014348
4	11.79153	-0.020991	11.781	-0.010562
5	14.93092	0.011636	14.923	-0.0083526
6	18.07106	-0.0072212	18.064	-0.0069062
7	21.21164	0.0048379	21.206	-0.0058862
8	24.35247	-0.0034257	24.347	-0.0051285
9	27.49348	0.0025295	27.489	-0.0045434
10	30.63461	-0.0019301	30.631	-0.0040781
11	33.77582	0.0015122	33.772	-0.0036992
12	36.9171	-0.0012108	36.914	-0.0033847
13	40.05843	0.00098719	40.055	-0.0031194
14	43.19979	-0.00081739	43.197	-0.0028927
15	46.34119	0.00068583	46.339	-0.0026967

(c) Plotting the first few partial sums,



The convergence is very fast since $f(x)$ is already shaped like a Bessel function.

(d) The ratio of the nodal lines is a simple fraction of the form $x = r/n$ where n is the order of the normal mode and r is every positive integer less than n

Since the zeros of Bessel function are not evenly spaced, this no longer holds true for the nodal lines of a circular membrane.

The raadius of each nodal line of the m^{th} normal mode is simply α_r/α_m for all positive integers $r < m$.

13. Doubling the tension,

$$c^2 = \frac{T}{\rho} \qquad \lambda \propto c \tag{12.10.98}$$

$$F \to 2T \qquad \implies f \to \sqrt{2}F \tag{12.10.99}$$

14. Looking at the fundamental frequency f_1,

$$f_1 = \frac{\lambda_1}{2\pi} \qquad \lambda_1 = \frac{c\alpha_1}{R} \tag{12.10.100}$$

$$R \downarrow \implies f_1 \uparrow \tag{12.10.101}$$

15. Targeting a fundamental frequency, using tension as the variable,

$$f_1 = c\,\frac{\alpha_1}{2\pi R} \qquad f_1 = \sqrt{T}\,\frac{\alpha_1}{\sqrt{\rho}\,2\pi R} \tag{12.10.102}$$

$$T = \rho R^2 f_1^2 \left(\frac{2\pi}{\alpha_1}\right)^2 \qquad T = 6.826 \cdot \rho R^2 f_1^2 \tag{12.10.103}$$

16. Using $1r = 0$ in Example 1, all the coefficients have to sum to 1.

$$J_0(0) = 1 \tag{12.10.104}$$

$$f(0) = 1 = \sum_{m=1}^{\infty} A_m \ J_0(0) = \sum_{m=1}^{\infty} A_m \tag{12.10.105}$$

The larger the number of terms needed to achieve high accuracy, the larger the problem of overtones polluting the pure fundamental frequency in a musical instrument.

17. The eigenvalues and eigenfunctions are,

$$\lambda_m = \frac{c}{R} \ \alpha_m \tag{12.10.106}$$

$$u_m = \Big[A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t)\Big] \cdot \Big[J_0(\lambda_m R/c)\Big] \tag{12.10.107}$$

Clearly, there is a one to one correspondence. No, two or more u_m cannot correspond to the same λ_m

18. Given an initial veclocity $g(r)$,

$$u_t(r, 0) = \sum_{m=1}^{\infty} (\lambda_m B_m) \ J_0 \left(\frac{\alpha_m r}{R}\right) \tag{12.10.108}$$

$$B_m = \frac{2}{(c\alpha_m R) \ J_1^2(\alpha_m)} \int_0^R r \ g(r) \ J_0 \left(\frac{\alpha_m r}{R}\right) \ \mathrm{d}r \tag{12.10.109}$$

19. Using separation of variables,

$$u_{tt} = c^2 \ \nabla^2 u \qquad\qquad u = F(r, \theta) \cdot G(t) \tag{12.10.110}$$

$$F \cdot \ddot{G} = G \cdot c^2 \left[F'' + \frac{F'}{r} + \frac{1}{r^2} \ \frac{\partial^2 F}{\partial \theta^2}\right] \tag{12.10.111}$$

Equating both sides to the same constant $-k^2$ and with $\lambda = ck$,

$$\ddot{G} + \lambda^2 G = 0 \tag{12.10.112}$$

$$F'' + \frac{F'}{r} + \frac{1}{r^2} \ \frac{\partial^2 F}{\partial \theta^2} + k^2 F = 0 \tag{12.10.113}$$

Further separating $F(r, \theta) = W(r) \cdot Q(\theta)$,

$$Q\left[W'' + \frac{W'}{r} + k^2 W\right] = -\frac{W}{r^2}\frac{\partial^2 Q}{\partial \theta^2} \tag{12.10.114}$$

$$\frac{1}{W}\left[r^2 W'' + rW' + r^2 k^2 W\right] = -\frac{1}{Q}\frac{\partial^2 Q}{\partial \theta^2} \tag{12.10.115}$$

$$\tag{12.10.116}$$

Equating both sides of the above to n^2,

$$\frac{\partial^2 Q}{\partial \theta^2} + n^2 Q = 0 \tag{12.10.117}$$

$$r^2 W'' + rW' + (r^2 k^2 - n^2)W = 0 \tag{12.10.118}$$

20. Since the circular membrane has domain $\theta \in [-\pi, \pi]$,

$$u(r, \theta) = u(r, \theta + 2\pi) \qquad \Longrightarrow \qquad P = 2\pi \tag{12.10.119}$$

$$Q'' = -n^2 Q \qquad\qquad Q = A_n \cos(n\theta) + B_n \sin(n\theta) \tag{12.10.120}$$

Imposing the periodicity condition on $Q(\theta)$,

$$Q(\theta + 2\pi) = A_n \cos(n\theta + 2n\pi) + B_n \sin(n\theta + 2n\pi) \tag{12.10.121}$$

This means that n is the set of non-negative integers. Substituting this set of values of n into the ODE for W, yields bessel functions in kr as the result.

$$W_n = J_n(kr) \qquad\qquad n \in \{0, 1, 2, \dots\} \tag{12.10.122}$$

21. The boundary condition yields,

$$u_{mn}(R, \theta, t) = 0 \qquad\qquad W(R) \cdot Q(\theta) \cdot G(t) = 0 \tag{12.10.123}$$

$$J_n(kR) = 0 \qquad \Longrightarrow \qquad k_{mn} = \frac{\alpha_{mn}}{R} \tag{12.10.124}$$

22. Consolidating G, W, Q into the full solution $u(r, \theta, t)$,

$$\lambda_{mn} = c k_{mn} \tag{12.10.125}$$

$$u_{mn} = J_n(k_{mn}\, r) \cdot \cos(n\theta) \cdot \Big[A_{mn} \cos(\lambda_{mn} t) + B_{mn} \sin(\lambda_{mn} t)\Big] \tag{12.10.126}$$

$$u^*_{mn} = J_n(k_{mn}\, r) \cdot \sin(n\theta) \cdot \Big[A^*_{mn} \cos(\lambda_{mn} t) + B^*_{mn} \sin(\lambda_{mn} t)\Big] \tag{12.10.127}$$

23. The initial condition on the velocity gives,

$$u_t = F(r,\theta)\cdot\lambda_{mn}\Big[-A_{mn}\sin(\lambda_{mn}t)+B_{mn}\cos(\lambda_{mn}t)\Big] \quad 12.10.128$$

$$u_t(r,\theta,0)=0 \implies B_{mn}=0 \quad 12.10.129$$

This includes $B^*_{mn}=0$ as well, since $G(\theta)$ includes both sine and cosine terms in θ.

24. $n=0$ makes the $\sin(n\theta)$ term zero for all m.

The other kind of solution reduces to,

$$u_{m0} = J_0(k_m\, r)\cdot\Big[A_m\cos(\lambda_m t)+B_m\sin(\lambda_m t)\Big] \quad 12.10.130$$

This is the same as equation 16 in the text.

25. Setting $R=c^2=1$, and extending the semicircular membrane to be a full circle, the nodal line has to be the diameter separating the two halves.

$$\cos(\theta)=0 \qquad \implies \qquad \theta=\pm\pi/2 \quad 12.10.131$$

This fixes $n=1$ and thus the order of the Bessel function.

Now, looking at the fact that the number of nodal circles is zero, the value of $m=1$ is also determined.

The fact that u_{mn} solves the given PDE and the given boundary conditions has already been established.

12.11 Laplace's Equation in Cylindrical and Spherical Coordinates

1. Deriving the Laplacian in spherical coordinates from the Laplacian in cylindrical coordinates,

$$x = s\cos\theta \qquad y = s\sin\theta \quad 12.11.1$$

$$z = r\cos\phi \qquad s = r\sin\phi = \sqrt{x^2+y^2} \quad 12.11.2$$

Consider the xy plane with fixed z and then the sz plane with fixed θ.

$$\nabla^2 u = u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\theta\theta} + u_{zz} \quad 12.11.3$$

$$u_{xx}+u_{yy} = u_{ss} + \frac{1}{s}\,u_s + \frac{1}{s^2}\,u_{\theta\theta} \quad 12.11.4$$

$$u_{ss}+u_{zz} = u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\phi\phi} \quad 12.11.5$$

In the xy plane the magnitute is $\sqrt{x^2+y^2}=s$ and direction is θ. However, in the sz plane, the magnitude is now $\sqrt{s^2+z^2}=r$ and direction is ϕ.

Summing the above terms and eliminating s in favour of the spherical coordinates.

$$\nabla^2 u = u_{rr} + \frac{1}{r}\, u_r + \frac{1}{r^2}\, u_{\phi\phi} + \frac{1}{s}\, u_s + \frac{1}{s^2}\, u_{\theta\theta} \tag{12.11.6}$$

$$= u_{rr} + \frac{1}{r}\, u_r + \frac{1}{r^2}\, u_{\phi\phi} + \frac{1}{r^2 \sin^2 \phi}\, u_{\theta\theta} + \frac{1}{s}\, u_s \tag{12.11.7}$$

$$u_s = u_r\, r_s + u_\theta\, \theta_s + u_\phi\, \phi_s \tag{12.11.8}$$

$$= u_r\, \frac{\partial}{\partial s}\left[\sqrt{s^2 + z^2}\right] + u_\theta\, (0) + u_\phi\, \frac{\partial}{\partial s}\left[\arcsin(s/r)\right] \tag{12.11.9}$$

$$= u_r\, \frac{s}{r} + u_\phi\, \frac{1}{\sqrt{r^2 - s^2}} = u_r\, \frac{s}{r} + u_\phi\, \frac{\cos \phi}{r} \tag{12.11.10}$$

Substituting into the above expression,

$$\frac{1}{s}\, u_s = \frac{1}{r}\, u_r + \frac{\cot \phi}{r^2}\, u_\phi \tag{12.11.11}$$

$$\nabla^2 u = u_{rr} + \frac{2}{r}\, u_r + \frac{1}{r^2}\left[u_{\phi\phi} + \frac{1}{\sin^2 \phi}\, u_{\theta\theta} + \cot \phi\, u_\phi\right] \tag{12.11.12}$$

2. Converting the Laplacian in cylindrical coordinates back to Carteisan coordinates,

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \tag{12.11.13}$$

$$x = r \cos \theta \qquad y = r \sin \theta \tag{12.11.14}$$

$$u_r = u_x\, x_r + u_y\, y_r \qquad = u_x \cos \theta + u_y \sin \theta \tag{12.11.15}$$

$$u_r = u_x\, \frac{x}{r} + u_y\, \frac{y}{r} \tag{12.11.16}$$

$$u_\theta = u_x\, x_\theta + u_y\, y_\theta \qquad = -u_x\, r \sin \theta + u_y\, r \cos \theta \tag{12.11.17}$$

$$u_\theta = -u_x\, y + u_y\, x \tag{12.11.18}$$

To find the second derivative,

$$u_{rr} = \frac{x}{r}\left[u_{xx}\, \frac{x}{r} + u_x\, \frac{y^2}{r^3} + u_{yx}\, \frac{y}{r} - u_y\, \frac{xy}{r^3}\right] \tag{12.11.19}$$

$$+ \frac{y}{r}\left[u_{xy}\, \frac{x}{r} - u_x\, \frac{xy}{r^3} + u_{yy}\, \frac{y}{r} + u_y\, \frac{x^2}{r^3}\right] \tag{12.11.20}$$

$$u_{\theta\theta} = (-y)[-u_{xx}\, y + u_{yx}\, x + u_y] + (x)[-u_{xy}\, y - u_x + u_{yy}\, x] \tag{12.11.21}$$

$$= y^2\, u_{xx} - x\, u_x + x^2\, u_{yy} - y\, u_y - 2xy\, u_{xy} \tag{12.11.22}$$

Consolidating the three terms,

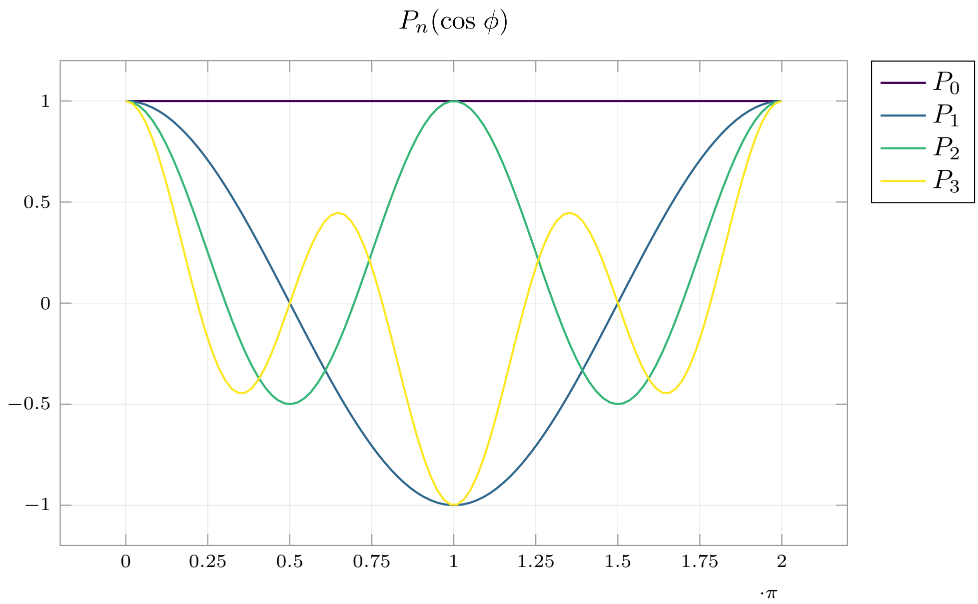
$$u_{\theta\theta} + r^2\ u_{rr} + r\ u_r = u_{xx}\Big[y^2 + x^2\Big] + u_{yy}\Big[x^2 + y^2\Big] \tag{12.11.23}$$

$$+ u_x\Big[-x + x\Big] + u_y\Big[-y + y\Big] + u_{xy}\Big[-2xy + 2xy\Big] \tag{12.11.24}$$

$$= r^2\ (u_{xx} + u_{yy}) \tag{12.11.25}$$

$$\frac{1}{r^2}\ u_{\theta\theta} + \frac{1}{r}\ u_r + u_{rr} + u_{zz} = u_{xx} + u_{yy} + u_{zz} \tag{12.11.26}$$

3. Plotting the first few Legendre polynomials in $\cos\phi$,



4. The zero surfaces correspond to the Legendre polynomials $P_n(\cos\phi) = 0$.

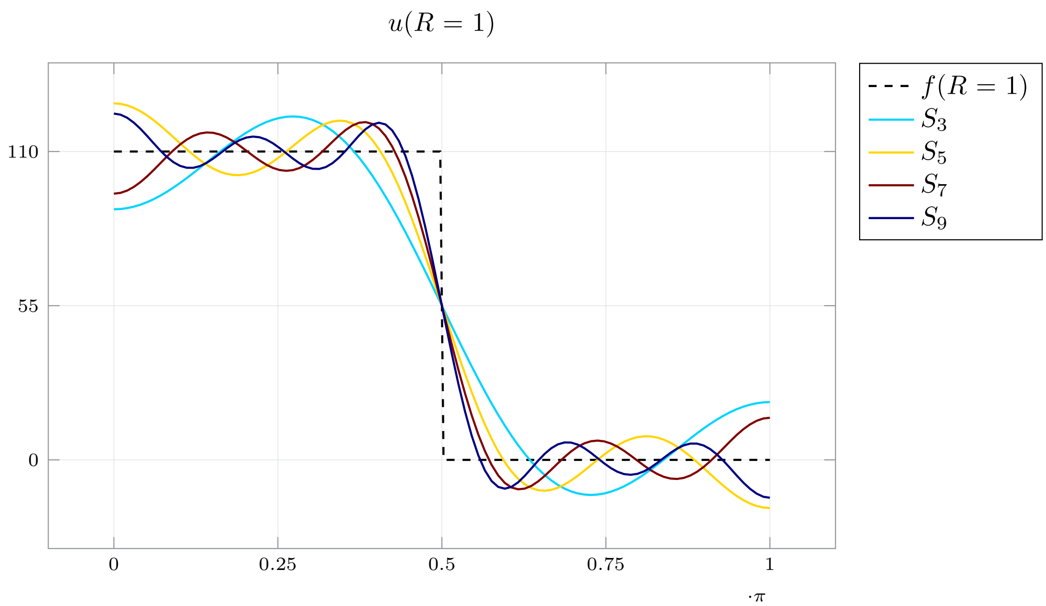
$$P_1(\cos\phi) = \cos\phi = 0 \qquad \implies \phi = \frac{\pi}{2} \tag{12.11.27}$$

$$P_2(\cos\phi) = 3\cos^2\phi - 1 = 0 \qquad \implies \phi = \frac{\pi}{6}, \frac{5\pi}{6} \tag{12.11.28}$$

$$P_3(\cos\phi) = 5\cos^3\phi - 3\cos\phi = 0 \qquad \implies \phi = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \tag{12.11.29}$$

Geometrically these are the xy plane, the double cones with angle 30°, and their union.

5. Using `sympy` to evaluate the terms upto A_{10} and plotting the partial sums at $r = R$,



The partial sums seem to get better at approximating the boundary condition.

6. Refer to plot in Problem 5. The oscillations move closer to the jump discontinuity as the number of terms in the Fourier series increases.

7. Verifying that the solutions u_n satisfy Laplace's equation.

$$u(r, \phi) = A_n r^n P_n(\cos\phi) \tag{12.11.30}$$

$$u_r = n\ A_n r^{n-1} P_n(\cos\phi) \tag{12.11.31}$$

$$\frac{\partial}{\partial r}\,(r^2 u_r) = n(n+1)\ A_n r^n P_n(\cos\phi) \tag{12.11.32}$$

Differentiating w.r.t. ϕ,

$$u_\phi = A_n r^n\ P_n'(\cos\phi) \tag{12.11.33}$$

$$\frac{1}{\sin\phi}\,\frac{\partial}{\partial\phi}\,(\sin\phi\ u_\phi) = A_n r^n \left[\cot\phi\ P_n'(\cos\phi) + P_n''(\cos\phi)\right] \tag{12.11.34}$$

Using the Legendre ODE, and abbreviating $P_n(\cos\phi)$ to P, it solves this ODE by definition,

$$\frac{\mathrm{d}^2 P}{\mathrm{d}\phi^2} + \cot\phi\,\frac{\mathrm{d}P}{\mathrm{d}\phi} + n(n+1)P = 0 \tag{12.11.35}$$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\,(r^2\ u_r) + \frac{1}{\sin\phi}\,\frac{\partial}{\partial\phi}\,(\sin\phi\ u_\phi)\right] = \frac{A_n r^n}{r^2}\ (0) = 0 \tag{12.11.36}$$

Verifying that the solutions u_n^* satisfy Laplace's equation.

$$u^*(r,\phi) = B_n r^{-n-1} P_n(\cos\phi) \tag{12.11.37}$$

$$u_r^* = -(n+1)\ B_n r^{-n-2} P_n(\cos\phi) \tag{12.11.38}$$

$$\frac{\partial}{\partial r}\ (r^2 u_r) = n(n+1)\ B_n r^{-n-1} P_n(\cos\phi) \tag{12.11.39}$$

Differentiating w.r.t. ϕ,

$$u_\phi = B_n r^{-n-1}\ P_n'(\cos\phi) \tag{12.11.40}$$

$$\frac{1}{\sin\phi}\ \frac{\partial}{\partial\phi}\ (\sin\phi\ u_\phi) = B_n r^{-n-1}\left[\cot\phi\ P_n'(\cos\phi) + P_n''(\cos\phi)\right] \tag{12.11.41}$$

Using the Legendre ODE, and abbreviating $P_n(\cos\phi)$ to P, it solves this ODE by definition,

$$\frac{\mathrm{d}^2 P}{\mathrm{d}\phi^2} + \cot\phi\ \frac{\mathrm{d}P}{\mathrm{d}\phi} + n(n+1)P = 0 \tag{12.11.42}$$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\ (r^2\ u_r) + \frac{1}{\sin\phi}\ \frac{\partial}{\partial\phi}\ (\sin\phi\ u_\phi)\right] = \frac{A_n r^{-n-1}}{r^2}\ (0) = 0 \tag{12.11.43}$$

8. Since the potential depends only on r,

$$u(r) = \frac{c}{r} \qquad \nabla^2 u = u_{rr} + \frac{2}{r}\ u_r \tag{12.11.44}$$

$$\nabla^2 u = \frac{2c}{r^3} - \frac{2}{r}\cdot\frac{c}{r^2} = 0 \tag{12.11.45}$$

9. Since the potential depends only on r,

$$\nabla^2 u = u_{rr} + \frac{2}{r}\ u_r \qquad \nabla^2 u = \frac{1}{r^2}\ \frac{\partial}{\partial r}\ (r^2 u_r) = 0 \tag{12.11.46}$$

$$r^2\frac{\partial u}{\partial r} = c_1 \qquad \frac{\partial u}{\partial r} = \frac{k_1}{r^2} \tag{12.11.47}$$

$$u = \frac{k_1}{r} + k_2 \tag{12.11.48}$$

This solution is unique due to the uniqueness of general solutions of ODEs.

10. Since the potential depends only on r,

$$\nabla^2 u = u_{rr} + \frac{1}{r}\, u_r = 0 \qquad r^2\, u_{rr} + r\, u_r = 0 \tag{12.11.49}$$

$$\lambda^2 = 0 \qquad \lambda = \{0, 0\} \tag{12.11.50}$$

$$u = k_1 + k_2 \ln(r) \tag{12.11.51}$$

This solution is unique due to the uniqueness of general solutions of ODEs.

11. Substituting into the Laplace equation in Cartesian coordinates,

$$u = \frac{c_1}{\sqrt{x^2 + y^2 + z^2}} + c_2 \qquad \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \tag{12.11.52}$$

$$u_x = \frac{-c_1 x}{(x^2 + y^2 + z^2)^{3/2}} \qquad u_{xx} = -c_1\, \frac{r^3 - 3x^2 r}{r^6} \tag{12.11.53}$$

$$\nabla^2 u = \frac{-c_1}{r^6}\left[3r^3 - 3r(x^2 + y^2 + z^2)\right] \qquad \nabla^2 u = 0 \tag{12.11.54}$$

Substituting into the Laplace equation in Spherical coordinates,

$$u = u'' + \frac{2u'}{r} \qquad u' = \frac{-c_1}{r^2} \tag{12.11.55}$$

$$u'' = \frac{2c_1}{r^3} \qquad u'' + \frac{2u'}{r} = 0 \tag{12.11.56}$$

12. From Problem 10, the potential depending only on r is,

$$u = c_1 \ln(r) + c_2 \tag{12.11.57}$$

$$u(2) = c_1 \ln(2) + c_2 = 220 \qquad u(4) = c_1 \ln(4) + c_2 = 140 \tag{12.11.58}$$

$$c_1 = \frac{-80}{\ln(2)} \qquad c_2 = 300 \tag{12.11.59}$$

$$u = -\frac{80}{\ln(2)}\, \ln(r) + 300 \tag{12.11.60}$$

13. From Problem 9, the potential depending only on r is,

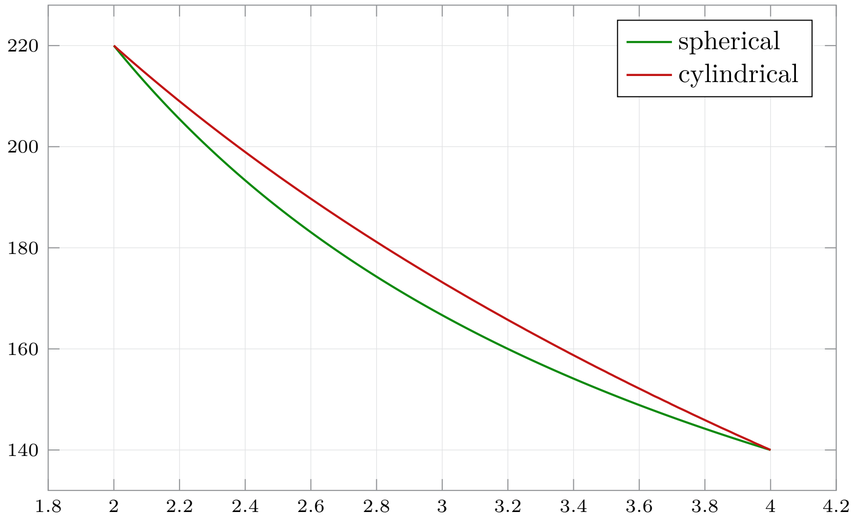
$$u = \frac{c_1}{r} + c_2 \tag{12.11.61}$$

$$u(2) = \frac{c_1}{2} + c_2 = 220 \qquad u(4) = \frac{c_1}{4} + c_2 = 140 \tag{12.11.62}$$

$$c_1 = 320 \qquad c_2 = 60 \tag{12.11.63}$$

$$u = \frac{320}{r} + 60 \tag{12.11.64}$$

Plotting the two potentials in the domain $r \in [2, 4]$, the cylindrical potential is larger than the spherical potential



The equipotential lines are cylinders and spheres respectively. These would look like circular cross sections in the xy plane. The above plot is more useful in comparing the two potentials.

14. Using the Laplacian for spherical coordinates when the potential is dependent only on r,

$$\nabla^2 u = u_{rr} + \frac{2}{r}\, u_r \qquad c^2 \left[u_{rr} + \frac{2}{r}\, u_r \right] = u_t \tag{12.11.65}$$

The initial condition is $u(r, 0) = f(r)$ and the boundary condition is $u(R, t) = 0$.

Introducing the new variable $v = ur$,

$$v_t = r\, u_t \qquad v_r = r\, u_r + u \tag{12.11.66}$$

$$v_{rr} = r\, u_{rr} + 2u_r \qquad \frac{v_t}{r} = c^2 \left[\frac{v_{rr}}{r} \right] \tag{12.11.67}$$

$$v_t = c^2\, v_{rr} \tag{12.11.68}$$

The constraint on v in order for the temperature to be bounded at $r = 0$ is that $v(0, t) = 0$.

The new I.C. $v(r, 0) = r f(r)$ and the new B.C. is $v(R, t) = 0$.

$$v(r, t) = A(r) \cdot B(t) \qquad A \cdot \dot{B} = c^2\ B \cdot A'' \tag{12.11.69}$$

$$\frac{1}{c^2 B} \frac{\mathrm{d}B}{\mathrm{d}t} = \frac{1}{A} \frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = -k^2 \tag{12.11.70}$$

$$\tag{12.11.71}$$

Solving the two ODEs in t, r separately,

$$\frac{\mathrm{d}B}{\mathrm{d}t} = -(ck)^2\ B \qquad B = p_1 \exp(-\lambda^2 t) \tag{12.11.72}$$

$$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = -k^2 A \qquad A = q_1 \cos(kr) + q_2 \sin(kr) \tag{12.11.73}$$

Applying the boundary conditions on $A(r)$,

$$A(0) = 0 \qquad \implies q_1 = 0 \tag{12.11.74}$$

$$A(R) = 0 \qquad \implies \sin(kR) = 0 \tag{12.11.75}$$

$$k = \frac{n\pi}{R} \qquad \lambda = \frac{cn\pi}{R} \tag{12.11.76}$$

Consolidating the two functions into $v(r, t)$, and combining the two constants p, q

$$v_n(r, t) = d_n\ \sin\left(\frac{n\pi r}{R}\right) \cdot \exp(-\lambda_n^2 t) \tag{12.11.77}$$

$$v(r, 0) = \sum_{n=1}^{\infty} v_n(r, 0) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi r}{R}\right) = r f(r) \tag{12.11.78}$$

Since this is simply the Fourier sine expansion of $f(r)$,

$$d_n = \frac{2}{R} \int_0^R r f(r)\ \sin\left(\frac{n\pi r}{R}\right)\ \mathrm{d}r \tag{12.11.79}$$

15. The analog of Problem 12 is hot and cold concentric cylinders. For Problem 13, it is concentric hot and cold spheres.

16. Decomposing the boundary function into a linear superposition of Legendre polynomials in $\cos\phi$,

$$f(\phi) = \cos\phi = P_1(\cos\phi) \qquad u(R = 1) = \sum_{n=0}^{\infty} A_n\ P_n(\cos\phi) \tag{12.11.80}$$

$$A_n = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases} \qquad u_r = r\ \cos\phi \tag{12.11.81}$$

17. Decomposing the boundary function into a linear superposition of Legendre polynomials in $\cos\phi$,

$$f(\phi) = 1 = P_0(\cos\phi) \qquad u(R=1) = \sum_{n=0}^{\infty} A_n\ P_n(\cos\phi) \tag{12.11.82}$$

$$A_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \qquad u_r = 1 \tag{12.11.83}$$

18. Decomposing the boundary function into a linear superposition of Legendre polynomials in $\cos\phi$,

$$f(\phi) = 1 - \cos^2\phi = \frac{-2P_2 + 2P_0}{3} \qquad u(R=1) = \sum_{n=0}^{\infty} A_n\ P_n(\cos\phi) \tag{12.11.84}$$

$$A_n = \begin{cases} 2/3 & n = 0 \\ -2/3 & n = 2 \\ 0 & \text{otherwise} \end{cases} \qquad u_r = \frac{2}{3} - \frac{(3\cos^2\phi - 1)}{3}\ r^2 \tag{12.11.85}$$

19. Decomposing the boundary function into a linear superposition of Legendre polynomials in $\cos\phi$,

$$f(\phi) = \cos(2\phi) = \frac{4P_2 - P_0}{3} \qquad u(R=1) = \sum_{n=0}^{\infty} A_n\ P_n(\cos\phi) \tag{12.11.86}$$

$$A_n = \begin{cases} -1/3 & n = 0 \\ 4/3 & n = 2 \\ 0 & \text{otherwise} \end{cases} \qquad u_r = -\frac{1}{3} + \frac{(6\cos^2\phi - 2)}{3}\ r^2 \tag{12.11.87}$$

20. Decomposing the boundary function into a linear superposition of Legendre polynomials in $\cos\phi$,

$$f(\phi) = \cos(2\phi) = 4P_3 - 2P_2 + P_1 - 2P_0 \tag{12.11.88}$$

$$u(R=1) = \sum_{n=0}^{\infty} A_n\ P_n(\cos\phi) \tag{12.11.89}$$

$$A_n = \begin{cases} -1 & n = 0 \\ 2 & n = 1 \\ -2 & n = 2 \\ 4 & n = 3 \\ 0 & \text{otherwise} \end{cases} \tag{12.11.90}$$

$$u_r = -1 + 2r\ P_1(\cos\phi) - 2r^2\ P_2(\cos\phi) + 4r^3\ P_3(\cos\phi) \tag{12.11.91}$$

21. From the boundary condition in Problem 17, the potential outside the sphere is,

$$u(r, \phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}}\, P_n(\cos\phi) \qquad\qquad f(\phi) = P_0 = \sum_{n=0}^{\infty} B_n\, P_n(\cos\phi) \tag{12.11.92}$$

$$u(r, \phi) = \frac{1}{r} \tag{12.11.93}$$

This happens to be the same as the potential due to a point charge at the origin.

22. Using the same Fourier-Legendre coefficients, in Problem 16,

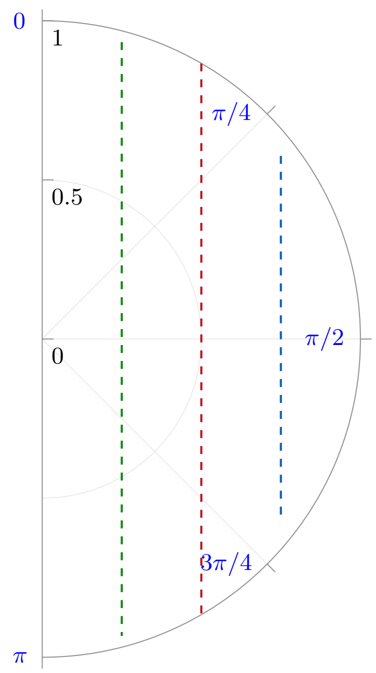
$$u^*(r, \phi) = \frac{1}{r^2}\, P_1(\cos\phi) \tag{12.11.94}$$

Using the same Fourier-Legendre coefficients, in Problem 19,

$$u^*(r, \phi) = -\frac{1}{3}\,\frac{P_0(\cos\phi)}{r} + \frac{4}{3}\,\frac{P_2(\cos\phi)}{r^3} \tag{12.11.95}$$

23. Plotting the xz plane cross sections of equipotential surfaces,

$$u = r\ \cos\phi \qquad\qquad r = \frac{k}{\cos\phi} \tag{12.11.96}$$



These are planes perpendicular to the x axis, which show up as lines parallel to the z axis in the plot.

24. Transmission line,

(a) Consider a small segment of cable at position x and width Δx,

$$u_B - u_A = Rj + L\,\frac{\partial j}{\partial t} \qquad\qquad Rj + L\,\frac{\partial j}{\partial t} = -\frac{\partial u}{\partial x} \tag{12.11.97}$$

Here the PDE comes from the infinitesimal limit of the wire segment. The negative sign comes from the fact that potential is decreasing left to right.

(b) Using Kirchoff's current law, on a small segment of wire,

$$j_B - j_A = \Delta j_C + \Delta j_G \qquad\qquad -\frac{\partial j}{\partial x} = C\,\frac{\partial u}{\partial t} + Gu \tag{12.11.98}$$

Here, G is the reciprocal of the resistance from the wire to the ground, which is different from R, the resistance per unit length of the wire itself.

(c) Eliminating current j,

$$-j_x = C\,u_t + Gu \qquad\qquad -j_{xt} = C\,u_{tt} + G\,u_t \tag{12.11.99}$$

$$j_t = \frac{-1}{L}\,u_x - \frac{R}{L}\,j \qquad\qquad j_{tx} = \frac{-1}{L}\,u_{xx} - \frac{R}{L}\,j_x \tag{12.11.100}$$

$$u_{xx} = LC\,u_{tt} + (RC + LG)\,u_t + RG\,u \tag{12.11.101}$$

Eliminating potential u,

$$-u_x = Rj + L\,j_t \qquad\qquad -u_{xt} = R\,j_t + L\,j_{tt} \tag{12.11.102}$$

$$u_t = \frac{-1}{C}\,j_x - \frac{G}{C}\,u \qquad\qquad u_{tx} = \frac{-1}{C}\,j_{xx} - \frac{G}{C}\,u_x \tag{12.11.103}$$

$$j_{xx} = LC\,j_{tt} + (RC + LG)\,j_t + RG\,j \tag{12.11.104}$$

(d) For the special case where $G \to 0$ and frequency is negligible, leading to $L \to 0$,

$$u_{xx} = RC\,u_t \qquad\qquad j_{xx} = RC\,j_t \tag{12.11.105}$$

Since this is the heat equation with B.C. $u(0, t) = u(L, t) = 0$.

Further, the I.C is $u(x, 0) = U_0$. Decomposing the function $u(x, t)$ into $V(x) \cdot W(t)$,

$$c^2 = \frac{1}{RC} \tag{12.11.106}$$

$$V(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{12.11.107}$$

$$W(t) = \exp\left[-\left(\frac{n\pi}{L}\right)^2 \cdot \frac{t}{RC}\right] \tag{12.11.108}$$

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \tag{12.11.109}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \ \sin\left(\frac{n\pi x}{L}\right) \ dx \tag{12.11.110}$$

Applying the given boundary condition to the general solution above,

$$A_n = \frac{2U_0}{L}\int_0^L \ \sin\left(\frac{n\pi x}{L}\right) \ dx \tag{12.11.111}$$

$$= \frac{2U_0}{n\pi}\left[\cos\left(\frac{n\pi x}{L}\right)\right]_L^0 = \frac{2U_0}{n\pi}\left[1 - \cos(n\pi)\right] \tag{12.11.112}$$

$$u(x, t) = \sum_{n=1}^{\infty} A_n \ V_n(x) \ W_n(t) \tag{12.11.113}$$

(e) For the special case where $L \gg R$ and frequency is negligible, leading to $C \gg G$,

$$u_{xx} = LC \ u_{tt} \qquad\qquad j_{xx} = LC \ j_{tt} \tag{12.11.114}$$

Since this is the wave equation with B.C. $u(0, t) = u(L, t) = 0$ and
Further, the I.C is $u(x, 0) = U_0 \sin(\pi x/l)$ and $u_t(x, 0) = 0$. Decomposing the function $u(x, t)$ into $V(x) \cdot W(t)$,

$$c^2 = \frac{1}{LC}, \qquad \lambda_n = \frac{cn\pi}{l} \qquad\qquad V(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{12.11.115}$$

$$W(t) = \cos(\lambda_n t) \qquad\qquad u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \tag{12.11.116}$$

Applying the given boundary condition to the general solution above,

$$A_n = \begin{cases} U_0 & n = 1 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad u(x, t) = U_0 \ \cos(\lambda_1 t) \ \sin\left(\frac{\pi x}{l}\right) \tag{12.11.117}$$

25. Transforming $u \to v$, to reflect on the unit sphere,

$$W(r,\theta,\phi) = \frac{U(1/r,\theta,\phi)}{r} \qquad r \cdot W(r,\theta,\phi) = U(\rho,\theta,\phi) \tag{12.11.118}$$

$$U_\theta = r \cdot W_\theta \qquad U_{\theta\theta} = r \cdot W_{\theta\theta} \tag{12.11.119}$$

$$U_\phi = r \cdot W_\phi \qquad U_{\phi\phi} = r \cdot W_{\phi\phi} \tag{12.11.120}$$

Relating U_ρ to W_r,

$$U_\rho = (W + rW_r) \cdot \frac{\partial r}{\partial \rho} = \left(\frac{-1}{\rho^2}\right)[W + rW_r] \tag{12.11.121}$$

$$U_{\rho\rho} = \left(\frac{1}{\rho^4}\right)[rW_{rr} + 2W_r] + \left(\frac{2}{\rho^3}\right)[W + rW_r] \tag{12.11.122}$$

$$U_{\rho\rho} + \frac{2}{\rho}U_\rho = r^5\left[W_{rr} + \frac{2}{r}W_r\right] \tag{12.11.123}$$

$$\frac{1}{\rho^2}U_{\phi\phi} + \frac{\cot\phi}{\rho^2}U_\phi = r^3 W_{\phi\phi} + r^3 \cot\phi\, W_\phi \tag{12.11.124}$$

$$\frac{1}{\rho^2\sin^2\phi}U_{\theta\theta} = \frac{r^3}{\sin^2\phi}W_{\theta\theta} \tag{12.11.125}$$

$$\nabla^2 U(\rho,\theta,\phi) = r^5\left[\nabla^2 W(r,\theta,\phi)\right] \tag{12.11.126}$$

Since U satisfies Laplace's equation, so does W.

12.12 Solution of PDEs by Laplace Transforms

1. Verifying the solution,

$$w(x,t) = \sin\left(t - \frac{x}{c}\right) \cdot u\left(t - \frac{x}{c}\right) \tag{12.12.1}$$

$$w(0,t) = \begin{cases} \sin(t) & t \in [0, 2\pi] \\ 0 & \text{otherwise} \end{cases} \tag{12.12.2}$$

The boundary condition on the $x = 0$ is satisfied.

$$\lim_{x\to\infty} w(x,t) = f(t - x/c) \cdot 0 = 0 \tag{12.12.3}$$

Since the step function is never turned on at infinite distance, the boundary condition on large x is also true.

$$w(x, 0) = \sin(-x/c) \cdot u(-x/c) = 0 \quad \forall \quad x > 0,\ c > 0 \tag{12.12.4}$$

The initial deflection is satisfied.

$$w(x, t) = \begin{cases} \sin\left(t - \dfrac{x}{c}\right) & t \in \left[\dfrac{x}{c}, \dfrac{x}{c} + 2\pi\right] \\ 0 & \text{otherwise} \end{cases} \tag{12.12.5}$$

$$w_t(x, t) = \begin{cases} \cos\left(t - \dfrac{x}{c}\right) & t \in \left[\dfrac{x}{c}, \dfrac{x}{c} + 2\pi\right] \\ 0 & \text{otherwise} \end{cases} \tag{12.12.6}$$

$$w_t(x, 0) = \begin{cases} \cos\left(\dfrac{x}{c}\right) & x \in [-2\pi c, 0] \\ 0 & \text{otherwise} \end{cases} \tag{12.12.7}$$

Clearly $w_t(x, 0)$ is zero everywhere on the positive x axis, which satisfies the initial condition on the velocity.

Substituting the new function into $w(x, t)$,

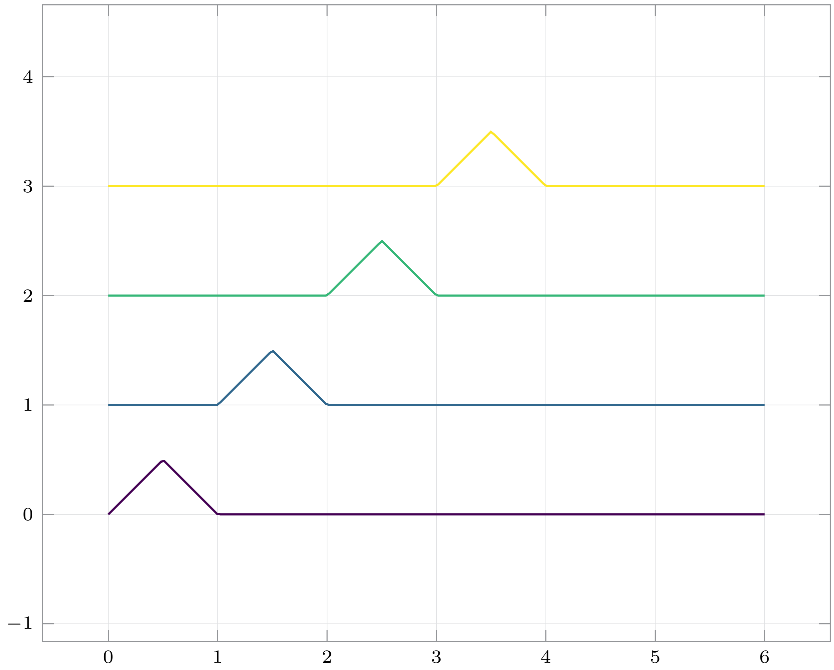
$$f(t) = \begin{cases} \sin(t) & t > 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{12.12.8}$$

$$u(x, t) = \begin{cases} \sin\left(t - \dfrac{x}{c}\right) & t > \dfrac{x}{c} + 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{12.12.9}$$

2. A triangular wave whose defined by,

$$f(x) = \begin{cases} x & x \in [0, 1/2) \\ 1 - x & x \in [1/2, 1] \\ 0 & \text{otherwise} \end{cases} \tag{12.12.10}$$

Plotting this function moving to the right at speed c,



3. The speed is related to c using,

$$v = c = \sqrt{\frac{T}{\rho}} \tag{12.12.11}$$

where the tension in the string is T and its mass per unit length is ρ.

4. Solving the PDE using Laplace transforms w.r.t. t,

$$\frac{\partial w}{\partial x} + x\,\frac{\partial w}{\partial t} = x \qquad\qquad w(x,0) = 1 \qquad w(0,t) = 1 \tag{12.12.12}$$

$$\frac{\partial W}{\partial x} + x\left[sW - 1\right] = \frac{x}{s} \qquad\qquad \frac{\partial W}{\partial x} = x\,(1/s + 1 - sW) \tag{12.12.13}$$

$$\ln(1/s + 1 - sW) = \frac{-sx^2}{2} + \alpha^*(s) \qquad\qquad \frac{1}{s} + 1 - sW = \alpha^*(s) \cdot e^{-sx^2/2} \tag{12.12.14}$$

$$W = \frac{1}{s} + \frac{1}{s^2} - \alpha(s)\,e^{-sx^2/2} \tag{12.12.15}$$

Recovering the solution using the inverse Laplace transform,

$$W(0,s) = \mathscr{L}\{w(0,t)\} = \mathscr{L}\{(1)\} = \frac{1}{s} \qquad\qquad \alpha(s) = \frac{1}{s^2} \tag{12.12.16}$$

$$W(x,s) = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-sx^2/2}}{s^2} \tag{12.12.17}$$

$$w(x,t) = (1+t) - \left(t - \frac{x^2}{2}\right) \cdot u\left[t - \frac{x^2}{2}\right] \tag{12.12.18}$$

Expressing the solution in piecewise form,

$$u(x,\, t) = \begin{cases} 1 + t & t < x^2/2 \\ 1 + x^2/2 & t > x^2/2 \end{cases} \tag{12.12.19}$$

5. Solving the PDE using Laplace transforms w.r.t. t,

$$x\,\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} = xt \tag{12.12.20}$$

$$x\,\frac{\partial W}{\partial x} + \Big[sW\Big] = \frac{x}{s^2} \qquad \frac{\partial W}{\partial x} + \frac{s}{x}\cdot W = \frac{1}{s^2} \tag{12.12.21}$$

$$\text{I.F.} = \exp\left(\int (s/x)\ \mathrm{d}x\right) \qquad \text{I.F.} = x^s \tag{12.12.22}$$

$$x^s\ W = \int \frac{x^s}{s^2}\ \mathrm{d}x \qquad W = \frac{x}{s^2(s+1)} + \frac{\alpha(s)}{x^s} \tag{12.12.23}$$

Applying the B.C. and I.C. to $W(x,\, s)$,

$$W(0,\, s) = \mathscr{L}\{w(0,\, t)\} = 0 \qquad \alpha(s) = 0 \tag{12.12.24}$$

$$W(x,\, s) = x\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right] \qquad w(x,\, t) = x\,\Big[-1 + t + e^{-t}\Big] \tag{12.12.25}$$

6. Solving the PDE using Laplace transforms w.r.t. t,

$$\frac{\partial w}{\partial x} + 2x\,\frac{\partial w}{\partial t} = 2x \qquad w(x,\, 0) = 1 \qquad w(0,\, t) = 1 \tag{12.12.26}$$

$$\frac{\partial W}{\partial x} + 2x\,\Big[sW - 1\Big] = \frac{2x}{s} \qquad \frac{\partial W}{\partial x} = 2x\ (1/s + 1 - sW) \tag{12.12.27}$$

$$\ln(1/s + 1 - sW) = -sx^2 + \alpha^*(s) \qquad \frac{1}{s} + 1 - sW = \alpha^*(s)\cdot e^{-sx^2} \tag{12.12.28}$$

$$W = \frac{1}{s} + \frac{1}{s^2} - \alpha(s)\ e^{-sx^2} \tag{12.12.29}$$

Recovering the solution using the inverse Laplace transform,

$$W(0,\, s) = \mathscr{L}\{w(0,\, t)\} = \mathscr{L}\{(1)\} = \frac{1}{s} \qquad \alpha(s) = \frac{1}{s^2} \tag{12.12.30}$$

$$W(x,\, s) = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-sx^2}}{s^2} \tag{12.12.31}$$

$$w(x,\, t) = (1 + t) - \Big(t - x^2\Big)\cdot u\Big[t - x^2\Big] \tag{12.12.32}$$

Expressing the solution in piecewise form,

$$u(x,\,t) = \begin{cases} 1+t & t < x^2 \\ 1+x^2 & t > x^2 \end{cases} \tag{12.12.33}$$

7. TBC

8. Solving the PDE using Laplace transforms w.r.t. t,

$$\frac{\partial^2 w}{\partial x^2} = 100\,\frac{\partial^2 w}{\partial t^2} + 100\,\frac{\partial w}{\partial t} + 25w \tag{12.12.34}$$

$$w(x,\,0) = 0 \qquad w(0,\,t) = \sin(t) \qquad w_t(x,\,0) = 0 \tag{12.12.35}$$

$$\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} = (10s+5)^2\;W \tag{12.12.36}$$

$$W = A(s)\,\exp[(10s+5)x] + B(s)\,\exp[-(10s+5)x] + C(s) \tag{12.12.37}$$

Applying the initial conditions,

$$\lim_{x\to\infty} w(x,\,t) = 0 \qquad\qquad \lim_{x\to\infty} W(x,\,s) = 0 \tag{12.12.38}$$

$$A(s) = C(s) = 0 \tag{12.12.39}$$

$$W(0,\,s) = \mathscr{L}\{w(0,\,t)\} = \mathscr{L}\{\sin(t)\} = \frac{1}{s} \qquad\qquad B(s) = \frac{1}{1+s^2} \tag{12.12.40}$$

$$W(x,\,s) = \frac{e^{-(10s+5)x}}{1+s^2} \tag{12.12.41}$$

$$w(x,\,t) = e^{-5x}\;\sin(t-10x)\;u\Big[t-10x\Big] \tag{12.12.42}$$

Expressing the solution in piecewise form,

$$u(x,\,t) = \begin{cases} 0 & t < 10x \\ e^{-5x}\;\sin(t-10x) & t > 10x \end{cases} \tag{12.12.43}$$

9. Starting with the heat equation and taking the Laplace transform,

$$w_t = c^2\;w_{xx} \qquad\qquad w(x,\,0) = 0 \tag{12.12.44}$$

$$sW = c^2\,\frac{\partial^2 W}{\partial x^2} \qquad\qquad W = A(s)\;e^{\sqrt{s}x/c} + B(s)\;e^{-\sqrt{s}x/c} \tag{12.12.45}$$

$$\lim_{x\to\infty} w = 0 \qquad\qquad \implies A(s) = 0 \tag{12.12.46}$$

$$W(0,\,s) = \mathscr{L}\{w(0,\,t)\} \qquad\qquad \mathscr{L}\{f(t)\} = F(s) \tag{12.12.47}$$

Using the initial conditions, and referring to the table of Laplace transforms,

$$W(0,\, s) = F(s) = B(s) \qquad\qquad W(x,\, s) = F(s)\; e^{-\sqrt{s}x/c} \tag{12.12.48}$$

$$G(s) = e^{-\sqrt{s}x/c} \qquad\qquad g(t) = \frac{x}{2c\sqrt{\pi t^3}}\; \exp\left[-\,\frac{x^2}{4c^2t}\right] \tag{12.12.49}$$

10. Substituting the functional form of $g(t)$ into the convolution integral,

$$w(x,\, t) = \frac{x}{2c\sqrt{\pi}} \int_0^t \frac{\exp[-x^2/(4c^2\tau)]}{\tau^{3/2}}\; f(t-\tau)\;\; \mathrm{d}\tau \tag{12.12.50}$$

11. Since the limits of integration mean that $t \geq \tau$, the step function $u(t-\tau)$ is always on.

$$w_0(x,\, t) = \frac{x}{2c\sqrt{\pi}} \int_0^t \frac{\exp[-x^2/(4c^2\tau)]}{\tau^{3/2}}\;\; \mathrm{d}\tau \tag{12.12.51}$$

$$y = \frac{x}{2c\sqrt{\tau}} \qquad\qquad \mathrm{d}y = \frac{-x}{4c\;\tau^{3/2}}\;\; \mathrm{d}\tau \tag{12.12.52}$$

$$w_0(x,\, t) = \frac{-2}{\sqrt{\pi}} \int_\infty^{x/(2c\sqrt{t})} e^{-y^2}\;\; \mathrm{d}y \tag{12.12.53}$$

$$= \mathrm{erf}(\infty) - \mathrm{erf}\left(\frac{x}{2c\sqrt{t}}\right) \tag{12.12.54}$$

$$= 1 - \mathrm{erf}\left(\frac{x}{2c\sqrt{t}}\right) \tag{12.12.55}$$

12. Using the form of $w_0(x,\, t)$ from Problem 11,

$$W_0(x,\, s) = F(s) \cdot e^{-x\sqrt{s}/c} = \frac{e^{-x\sqrt{s}/c}}{s} \tag{12.12.56}$$

$$\mathscr{L}\{u[t-a]\} = \frac{e^{-as}}{s} \tag{12.12.57}$$

$$\frac{\partial w_0}{\partial t} = \frac{x}{2c\sqrt{\pi}}\; \exp\left[-\,\frac{x^2}{4c^2\; t}\right] t^{-3/2} \tag{12.12.58}$$

$$w(x,\, t) = \int_0^t \frac{\partial}{\partial \tau}\left[w_0(\tau)\right]\, f(t-\tau)\;\; \mathrm{d}\tau \tag{12.12.59}$$

The last step directly uses the result of Problem 10.