Chapter 25

Mathematical Statistics

25.1 Introduction, Random Sampling

1. No problem set in this section

25.2 Point Estimation of Parameters

1. The normal distribution with $\mu = 0$, is

$$f(x) = \frac{1}{\theta\,\sqrt{2\pi}} \ \exp\left[-\frac{x^2}{2\theta^2}\right] \tag{25.2.1}$$

$$l(\theta) = \left(\frac{1}{\theta\,\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{2\theta^2}\ \sum_j x_j^2\right] \tag{25.2.2}$$

$$\ln(l) = -n\,\ln(\theta\sqrt{2\pi}) - \frac{1}{2\theta^2}\ \sum_j x_j^2 \tag{25.2.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\ln(l) = -\frac{n}{\theta} + \frac{1}{\theta^3}\ \sum_j x_j^2 = 0 \tag{25.2.4}$$

$$\theta^* = \sqrt{\frac{\sum_j x_j^2}{n}} = \widetilde{\sigma} \tag{25.2.5}$$

The other parameter is the sample standard deviation with a change in the pre-factor from $(n-1)$ to (n).

2. Known variance is $\sigma^2 = 16$.

$$f(x) = \frac{1}{4\sqrt{2\pi}} \exp\left[-\frac{(x-\theta)^2}{32}\right] \tag{25.2.6}$$

$$l(\theta) = \left(\frac{1}{4\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{32}\sum_j (x_j-\theta)^2\right] \tag{25.2.7}$$

$$\ln(l) = -n\ln(4\sqrt{2\pi}) - \frac{1}{32}\sum_j (x_j-\theta)^2 \tag{25.2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\ln(l) = \frac{1}{16}\sum_j (x_j-\theta) = 0 \tag{25.2.9}$$

$$\theta^* = \frac{1}{n}\sum_{j=1}^{n} x_j = \bar{x} \tag{25.2.10}$$

The MLE estimator is the sample mean.

3. For a Poisson distribution with parameter θ,

$$l = \prod_{j=1}^{n} e^{-\theta}\frac{\theta^{x_j}}{x_j!} = e^{-n\theta}\frac{\theta^{x_1+\cdots+x_n}}{x_1!\,x_2!\,\cdots x_n!} \tag{25.2.11}$$

$$\ln(l) = -n\theta + \left[\sum_j x_j\right]\ln(\theta) - \ln(x_1!\,x_2!\,\cdots\,x_n!) \tag{25.2.12}$$

$$\frac{\partial}{\partial\theta}\ln(l) = -n + \frac{1}{\theta}\sum_j x_j = 0 \tag{25.2.13}$$

$$\theta^* = \frac{1}{n}\sum_j x_j = \bar{x} \tag{25.2.14}$$

The MLE estimate of the Poisson parameter is the sample mean.

4. For the uniform distribution with parameters a, b,

$$f(x) = \frac{1}{(b-a)} \quad \forall \quad x \in [a, b] \tag{25.2.15}$$

$$l = (b-a)^{-n}, \qquad \ln(l) = -n\ln(b-a) \tag{25.2.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}\ln(l) = \frac{n}{(b-a)} = 0 \tag{25.2.17}$$

This has no solution for nonzero n. This means the MLE does not exist for the parameters a, b. Instead, look at the smallest and largest sample value x_1, x_n. These are intuitively, the estimates of the smallest and largest possible values of x in the population.

$$\widetilde{a} = \min_j x_j \qquad \widetilde{b} = \max_j x_j \tag{25.2.18}$$

As the sample size approaches the population size, it is more and more probable that $x_1 \to a$ and $x_n \to b$ asymptotically.

5. Using the MLE method for a sample of size n drawn from a Bernoulli PDF, of which x are successes,

$$l = P(X = x) = p^x \; q^{n-x} \qquad \ln(l) = x \; \ln(p) + (n - x) \; \ln(q) \tag{25.2.19}$$

$$\frac{\partial l}{\partial \theta} = \frac{x}{\theta} - \frac{n - x}{1 - \theta} \qquad 0 = x(1 - \theta) + (x - n)\theta \tag{25.2.20}$$

$$\theta^* = \frac{x}{n} \tag{25.2.21}$$

The MLE estimator of the binomial parameter p is the fraction of successes in the sample, x/n

6. Extending the experiment to m times the earlier experiment,

$$l = p^{x_1 + \dots + x_m} \cdot q^{n - x_1 + n - x_2 + \dots + n - x_m} \tag{25.2.22}$$

$$\ln(l) = \sum_j x_j \cdot \ln(p) + \Big(nm - \sum_j x_j\Big) \cdot \ln(q) \tag{25.2.23}$$

$$\frac{\partial}{\partial \theta} \; \ln(l) = \frac{m\bar{x}}{\theta} - \frac{m(n - \bar{x})}{1 - \theta} = 0 \tag{25.2.24}$$

$$\theta^* = \frac{1}{mn} \sum_{j=1}^{m} x_j \tag{25.2.25}$$

7. Using the result from Problem 6,

$$\theta^* = \frac{7}{12} \tag{25.2.26}$$

8. The event $A^{\complement}$ occurs $(x - 1)$ times and then the event A occurs for a trial to have $X = x$,

$$P(X = x) = f(x) \qquad f(x) = (1 - p)^{x-1} \cdot p = q^{x-1} \cdot p \tag{25.2.27}$$

for some positive integer x.

For a sample of size n and with observed values $\{x_1, \ldots, x_n\}$,

$$P(\{x_j\}) = q^{x_1-1+x_2-1+\ldots x_n-1} \cdot p^n = l \tag{25.2.28}$$

$$\ln(l) = n \ln(p) + \left[\sum_{j=1}^{n} x_j - n \right] \ln(1-p) \tag{25.2.29}$$

$$\frac{\partial}{\partial\theta} \ln(l) = \frac{n}{\theta} - \frac{(\sum x_j - n)}{1-\theta} = 0 \tag{25.2.30}$$

$$\theta^* = \frac{n}{\sum_j x_j} = \frac{1}{\bar{x}} \tag{25.2.31}$$

The reciprocal of the sample mean is the MLE of the parameter.

9. Using the sum of a geometric progression,

$$\sum_{j=1}^{\infty} f(X=j) = p + qp + q^2p + \ldots \tag{25.2.32}$$

$$= p(1 + q + q^2 + \ldots) = \frac{p}{1-q} = 1 \tag{25.2.33}$$

Thus, the normalization condition holds.

Using the MLE method on a single observed value $X = x$,

$$l = q^{x-1}p \qquad \ln(l) = (x-1)\ln(q) + \ln(p) \tag{25.2.34}$$

$$\frac{\partial}{\partial\theta} \ln(l) = \frac{1-x}{1-\theta} + \frac{1}{\theta} \qquad \theta^* = \frac{1}{x} \tag{25.2.35}$$

10. Using the result from Problem 8,

$$\theta_1 = \frac{1}{6.5} = \frac{2}{13} \tag{25.2.36}$$

11. Using the MLE method, for a sample of size n,

$$P(\{x_j\}) = l = \theta^n \, e^{-\theta \sum x_j} \qquad \ln(l) = n\ln(\theta) - \theta \sum x_j \tag{25.2.37}$$

$$\frac{\partial}{\partial\theta} \ln(l) = \frac{n}{\theta} - \sum x_j = 0 \qquad \theta^* = \frac{n}{\sum x_j} = \frac{1}{\bar{x}} \tag{25.2.38}$$

The reciprocal of the sample mean is the MLE of the parameter.

12. Finding the mean directly,

$$\mathbb{E}[X] = \int_0^\infty x\theta\ e^{-\theta x}\ \mathrm{d}x \qquad = \Big[-(x + 1/\theta)\ e^{-\theta x} \Big]_0^\infty \tag{25.2.39}$$

$$= \frac{1}{\theta} \tag{25.2.40}$$

Substituting into the original PDF,

$$f(x) = \frac{e^{-x/\mu}}{\mu} \quad \forall \quad x \geq 0 \tag{25.2.41}$$

Using the same procedure as in Problem 11,

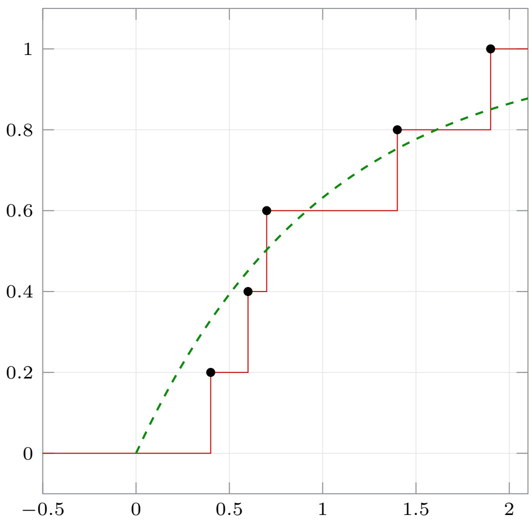
$$\hat{\mu} = \bar{x} \qquad\qquad \hat{\theta} = \frac{1}{\bar{x}} \tag{25.2.42}$$

13. Computing the sample statistic,

$$\hat{\theta} = \frac{1}{\bar{x}} = \frac{5}{1.9 + 0.4 + 0.7 + 0.6 + 1.4} = 1 \tag{25.2.43}$$

$$F(x) = \int_0^x e^{-x}\ \mathrm{d}x = 1 - e^{-x} \tag{25.2.44}$$

Discrete vs Continuous CDF

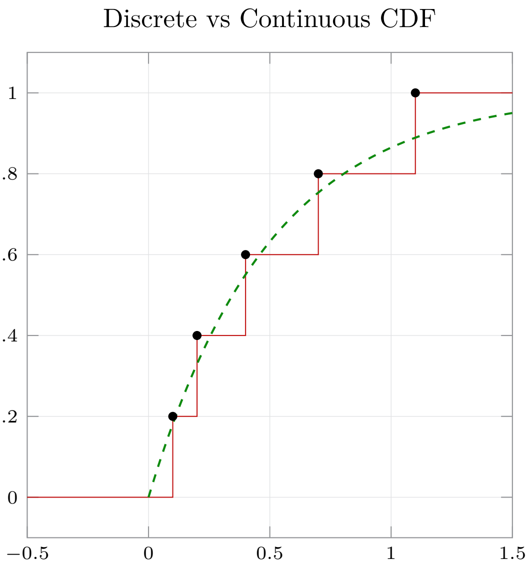


Even this small sample is a good approximation of the underlying CDF,

14. Computing the sample statistic,

$$\hat{\theta} = \frac{1}{\bar{x}} = \frac{5}{0.4 + 0.7 + 0.2 + 1.1 + 0.1} = 2 \tag{25.2.45}$$

$$F(x) = \int_0^x 2e^{-2x} \ \mathrm{d}x = 1 - e^{-2x} \tag{25.2.46}$$



15. Looking at the sample mean and variance as a function of the size of each sample, keeping the number of samples m fixed.

n	Sample Mean		Sample Variance	
	Mean	Variance	Mean	Variance
10	0.005	0.096	0.881	0.166
100	0.00234	0.0100	0.9873	0.0210
1000	0.00138	0.0009297	1.0005	0.002
10000	-0.000197	0.0000103	0.9997	0.0002

As the sample size increases, the MLE estimators of the population mean and variance become better point estimates, as evidenced by each of these point estimates getting more tightly distributed about their true values.

25.3 Confidence Intervals

1. Point estimates offer no indication of their relation to the true value, which confidence intervals are able to provide through their confidence value γ.

2. The known variance is $\sigma^2 = 16$. When increasing the confidence value to 0.99, the interval is 1.3 times larger.

Quantity	Value
$\bar{x}$	49.67
γ	0.95
c	1.96
k	3.20
Lower limit	46.47
Upper limit	52.87

Quantity	Value
$\bar{x}$	49.67
γ	0.99
c	2.58
k	4.21
Lower limit	45.46
Upper limit	53.87

3. Using the relation,

$$k \propto \frac{1}{\sqrt{n}} \qquad n \to 2n \implies k \to \frac{k}{\sqrt{2}} \tag{25.3.1}$$

4. The known variance is $\sigma^2 = 16$.

Quantity	Value
$\bar{x}$	74.81
γ	0.95
c	1.96
k	0.55
Lower limit	74.26
Upper limit	75.36

5. Using the formula for the confidence interval length,

$$k = \frac{c\sigma}{\sqrt{n}} = \sigma \qquad n = c^2 = 1.96^2 \tag{25.3.2}$$

$$n_1 = 4 \tag{25.3.3}$$

$$k = \frac{c\sigma}{\sqrt{n}} = \frac{\sigma}{2} \qquad n = c^2 = 4 \cdot 1.96^2 \tag{25.3.4}$$

$$n_2 = 16 \tag{25.3.5}$$

6. By direct calculation,

$$k = \frac{c\sigma}{\sqrt{n}} \qquad 1 = \frac{2.576 \cdot 5}{\sqrt{n}} \tag{25.3.6}$$

$$n = 165.89 \qquad n^* = 166 \tag{25.3.7}$$

From the figure, this number should lie in the range $150 - 170$, which matches the true value of n^*.

7. Since this is a binomial distribution, with

$$\bar{p} = \frac{126}{800} \qquad \bar{x} = n\bar{p} \tag{25.3.8}$$

$$s = \sqrt{n\bar{p}(1-\bar{p})} \tag{25.3.9}$$

Quantity	Value
$\bar{p}$	0.1575
γ	0.95
c	1.9629
k	0.71504
Lower limit p	15.66 %
Upper limit p	15.84 %

8. Since this is a binomial distribution, with

$$\bar{p} = \frac{12012}{24000} \qquad \bar{x} = n\bar{p} \tag{25.3.10}$$

$$s = \sqrt{n\bar{p}(1-\bar{p})} \tag{25.3.11}$$

Quantity	Value
$\bar{p}$	0.5005
γ	0.99
c	2.576
k	1.288
Lower limit p	50.045 %
Upper limit p	50.055 %

9. Using the student's t distribution since the variance is unknown,

10. Using the student's t distribution since the variance is unknown,

Quantity	Value
$\bar{x}$	65.0000
γ	0.9900
c	3.2498
k	1.2818
Lower limit	63.7182
Upper limit	66.2818

Quantity	Value
$\bar{x}$	661.2000
γ	0.9900
c	4.6041
k	9.8101
Lower limit	651.3899
Upper limit	671.0101

11. Using the student's t distribution since the variance is unknown,

Quantity	Value
$\bar{x}$	9533.3333
γ	0.9900
c	4.0321
k	366.8536
Lower limit	9166.4797
Upper limit	9900.1870

12. Using the student's χ^2 distribution for the variance,

13. Using the student's χ^2 distribution for the variance,

Quantity	Value
γ	0.9500
c_1	8.9065
c_2	32.8523
Lower limit	0.0231
s	0.0400
Upper limit	0.0853

Quantity	Value
γ	0.9500
c_1	1.6899
c_2	16.0128
Lower limit	0.0461
s	0.1055
Upper limit	0.4372

14. Using the student's χ^2 distribution for the variance,

15. Using the student's χ^2 distribution for the variance, textbook uses $n - 1 = 100$ instead of $n = 100$.

Quantity	Value
γ	0.9500
c_1	1.6899
c_2	16.0128
Lower limit	0.0461
s	0.1055
Upper limit	0.4372

Quantity	Value
γ	0.9500
c_1	73.3611
c_2	128.4220
Lower limit	7.1693
s	9.3000
Upper limit	12.5503

16. Using the student's χ^2 distribution for the variance,

17. Using the student's χ^2 distribution for the variance, with data from Problem 9,

Quantity	Value
γ	0.9500
c_1	2.7004
c_2	19.0228
Lower limit	2.8650
s	6.0556
Upper limit	20.1823

Quantity	Value
γ	0.9500
c_1	2.7004
c_2	19.0228
Lower limit	0.7360
s	1.5556
Upper limit	5.1844

18. The difference of two normal RVs is,

$$Y = 3X_1 - X_2 \qquad \mu_Y = 3^2\mu_1 - \mu_2 = 34 \tag{25.3.12}$$

$$\sigma_y^2 = 3^2\sigma_1^2 + \sigma_2^2 = 23 \tag{25.3.13}$$

19. Using the sum of normal RVs,

$$\mu = 100 + 5 = 105 \qquad \sigma = \sqrt{1^2 + 0.5^2} = \frac{\sqrt{5}}{2} \tag{25.3.14}$$

$$P(104 < Y < 106) = P\left(\frac{-2}{\sqrt{5}} < Z < \frac{2}{\sqrt{5}}\right) \qquad = 0.6289 \tag{25.3.15}$$

20. Using the sum of normal RVs, $Y = nX$,

$$\mu = 40n \qquad \sigma = \sqrt{4n} \tag{25.3.16}$$

$$P(Y > 2000) = P\left(Z > \frac{2000 - 40n}{2\sqrt{n}}\right) \qquad = 0.05 \tag{25.3.17}$$

$$\frac{1000}{\sqrt{n}} - 20\sqrt{n} = 1.64485 \qquad n = (7.03)^2 = 49.421 \tag{25.3.18}$$

$$n^* \geq 50 \tag{25.3.19}$$

25.4 Testing of Hypotheses. Decisions

1. Examples, for the two sided test, the right sided and left sided tests,

$$\text{null: } \theta = a \qquad \text{alternative: } \theta \neq a \tag{25.4.1}$$

$$\text{null: } \theta \leq b \qquad \text{alternative: } \theta > b \tag{25.4.2}$$

$$\text{null: } \theta \geq c \qquad \text{alternative: } \theta < c \tag{25.4.3}$$

2. The tests are,

- Mean of a normal RV with known variance. Uses the standard normal PDF
- Mean of a normal RV with unknown variance. Uses Student's t-RV
- Variance of a normal RV. Uses the χ^2 RV
- Comparison of the means of two normal RVs with paired datasets, using the standard normal RV
- Comparison of the means of two normal RVs, using a t-RV and the pooled sample variance.
- Comparison of the variances of two normal RVs, using the F-distribution

3. Using `numpy` to do the right-sided test, the null hypothesis is accepted.

Quantity	Value
α	0.0500
$\bar{x}$	0.2857
s	4.3095
n	7
Statistic	0.1754
Limit	1.9432

4. Using `numpy` to do the two-sided test, the null hypothesis is accepted.

Quantity	Value
α	0.0500
$\bar{x}$	0.5069
s	0.5000
n	4040
Statistic	0.0139
Lower Limit	-1.9600
Upper Limit	1.9600

5. Using `numpy` to do the two-sided test, the null hypothesis is accepted.

Quantity	Value
α	0.0500
$\bar{x}$	0.5016
s	0.5000
n	12000
Statistic	0.0032
Lower Limit	-1.9600
Upper Limit	1.9600

6. Using `numpy` to do the left-sided test, the null hypothesis is rejected.

Quantity	Value
α	0.0500
$\bar{x}$	58.5000
σ^2	9.0000
n	20.0000
Statistic	-2.2361
Limit	-1.6449

7. Using `numpy` to do the left-sided test, the null hypothesis is accepted.

Quantity	Value
α	0.0500
$\bar{x}$	58.5000
σ^2	9.0000
n	5.0000
Statistic	-1.1180
Limit	-1.6449

8. The value of c is

$$c = \Phi(0.05) \cdot \frac{\sigma}{\sqrt{n}} + \mu_0 \qquad = 58.8966 \tag{25.4.4}$$

$$P(X > c)_{\mu=\mu_1} = 1 - \Phi\left(\frac{c - \mu_1}{\sigma/\sqrt{n}}\right) \qquad \beta = 0.00235 \tag{25.4.5}$$

$$\eta = 1 - \beta = 99.76\ \% \tag{25.4.6}$$

9. The upper and lower limits are,

$$l_1 = \Phi^{-1}(0.025) \cdot \frac{\sigma}{\sqrt{n}} + \mu_0 = 58.685 \tag{25.4.7}$$

$$l_2 = \Phi^{-1}(0.975) \cdot \frac{\sigma}{\sqrt{n}} + \mu_0 = 61.315 \tag{25.4.8}$$

Where Φ^{-1} is the inverse of the standard normal distribution, also called the percent point function `PPF`.

10. Using the right sided test for m samples each of size $n = 10$, at $\alpha = 0.1$,

m	Fraction Accepted
100	0.91
1000	0.907
10000	0.9022
100000	0.89894

Performing a similar test for the variance,

m	Fraction Accepted
100	0.93
1000	0.897
10000	0.9033
100000	0.90124

The fraction of random samples that fail the test asymptotically converges to the false negative rate $\alpha = 0.1$.

11. Performing the two sided $t-$ test using `numpy`, the hypothesis is rejected.

Quantity	Value
α	0.9500
$\bar{x}$	4990
s	20
n	50
Statistic	-3.5355
Lower Limit	-2.0096
Upper Limit	2.0096

12. Performing the two sided $t-$ test using `numpy`, the hypothesis is accepted.

Quantity	Value
α	0.9500
$\bar{x}$	37000
s	5000
n	25
Statistic	2.0000
Limit	1.7109

13. Performing the two sided $t-$ test using `numpy`, the hypothesis is accepted.

Quantity	Value
α	0.9500
$\bar{x}$	0.5500
s	0.7387
n	8
Statistic	2.1058
Lower Limit	-2.3646
Upper Limit	2.3646

14. Performing the two sided $z-$ test using `numpy`, the hypothesis is accepted.

Quantity	Value
α	0.05
$\bar{x}$	0.7750
σ	75
n	400
Statistic	0.0577
Lower Limit	-1.9600
Upper Limit	1.9600

15. Performing the right-sided χ^2- test using `numpy`, the hypothesis is accepted.

Quantity	Value
α	0.0500
σ	0.8000
s	1.0000
n	20
Statistic	29.6875
Limit	30.1435

16. Performing the right-sided χ^2- test using `numpy`, the hypothesis is accepted.

Quantity	Value
α	0.0500
σ	5.0000
s	3.5000
n	28
Statistic	13.2300
Limit	40.1133

17. Performing the right-sided two-sample paired $t-$ test using `numpy`, the hypothesis that the mileage of B is significantly greater is accepted.

Quantity	Value
α	0.05
$\bar{x}$	-0.6
s_x	0.4
s_y	0.6
n_1	16
n_2	16
Statistic	-3.3282
Limit	1.6973

18. Performing the right-sided two-sample paired $F-$ test using `numpy`, the hypothesis that the variance of the first sample is smaller is rejected.

Quantity	Value
α	0.0500
s_x	18.7083
s_y	7.8680
n_1	6
n_2	7
Statistic	5.6538
Limit	4.3874

19. Let c be some limit dependent on the confidence value γ. For Type I errors, with $\mu_1 > \mu_0$,

$$\alpha = P\left(\frac{z - \mu_0}{\sigma/\sqrt{n}} > c\right) \tag{25.4.9}$$

This quantity goes up with increasing n, which means α goes down with increasing n keeping all other elements fixed.

For Type II errors, once again with $\mu_1 > \mu_0$

$$\beta = P\left(z + \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} > c\right) \tag{25.4.10}$$

This value can also be made arbitrarily small by increasing n.

The fact that normal distributions span the entire real line means that these values can never be zero.

By a similar procedure, the case $\mu_1 < \mu_0$ can also be proved.

20. Performing the right-sided two-sample generalized $t-$ test using `numpy`, the hypothesis that the means of the two samples are equal is rejected.

Quantity	Value
α	0.0500
$\bar{x} - \bar{y}$	-4
s_x	3
s_y	3
n_1	12
n_2	18
Statistic	-0.2520
Lower Limit	-2.0484
Upper Limit	2.0484

25.5 Quality Control

1. The control limits for the mean are,

Quantity	Value
`LCL` (1%)	0.9485
`UCL` (1%)	1.0515

2. For integer multiples of σ, with the pre-factors k_1 and k_2 attached to the terms $\sigma/\sqrt{n}$ to cross check,

Quantity	Value
`LCL` (0.3%)	0.9703
k_1	2.97
`UCL` (0.3%)	1.0297
k_2	-2.97

3. Using the formula for the range,

$$Y = \text{UCL} - \text{LCL} \qquad Y = 2 \cdot \frac{2.58\sigma}{\sqrt{n}} \tag{25.5.1}$$

$$n = \left(\frac{2 \cdot 2.58 \cdot 0.02}{0.02} \right)^2 = 26.6256 \qquad n \geq 27 \tag{25.5.2}$$

4. $n \to 2n$ makes the ctrol limit span go from $Y \to Y/\sqrt{2}$

$$\alpha = 1\% \qquad k_1 = 2.58 \tag{25.5.3}$$

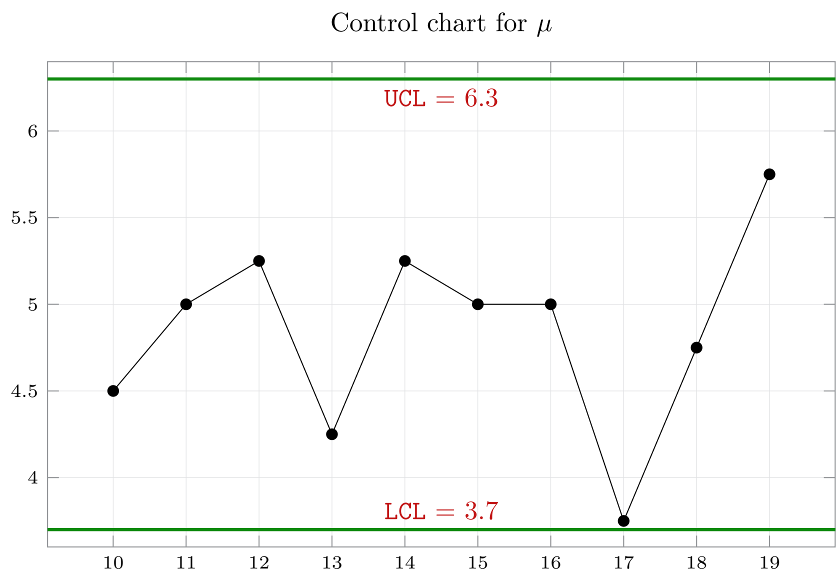
$$\alpha = 5\% \qquad k_1 = 1.96 \tag{25.5.4}$$

$$k_1 \to k_2 \qquad Y \to 1.316\ Y \tag{25.5.5}$$

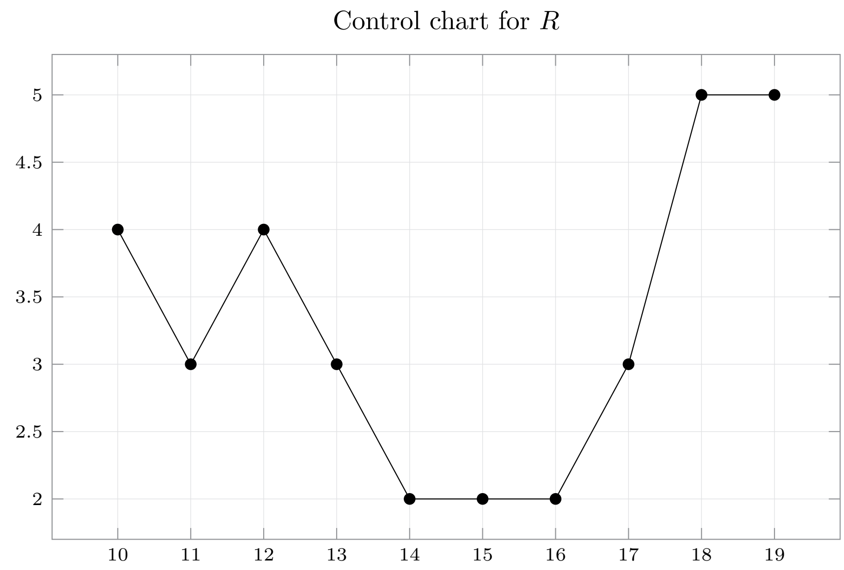
5. Using the relation,

$$Y = \frac{2k_\alpha \cdot \sigma}{\sqrt{n}} \qquad Y \to \frac{Y}{2} \implies n \to 4n \tag{25.5.6}$$

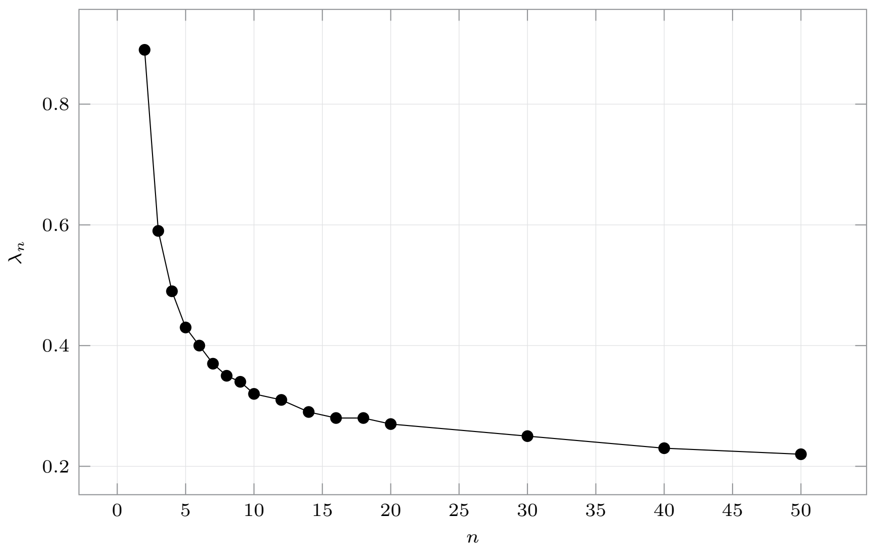
6. The control chart for the mean is,



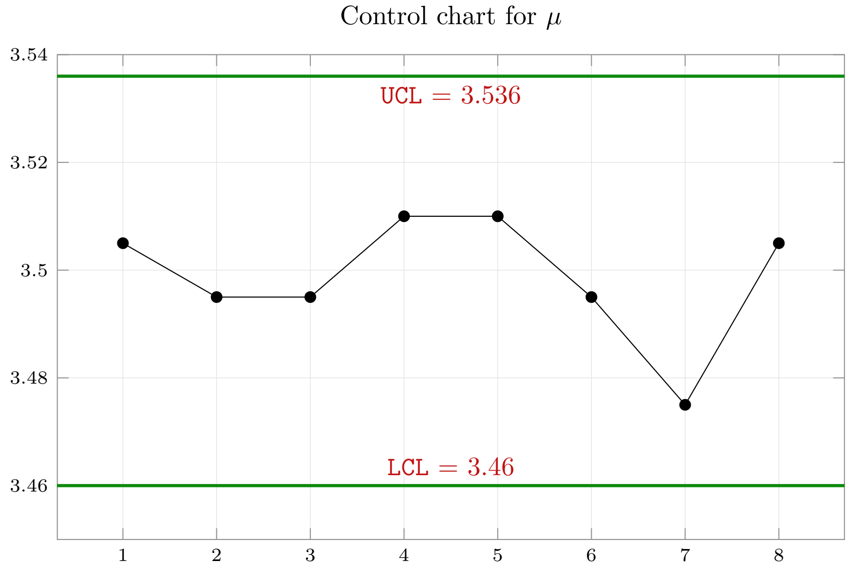
7. The control chart for the ranges is,



8. The function is a monotone decreasing function, since an increase in sample size leads to a decrease in standard error in the mean and therefore in the interval.



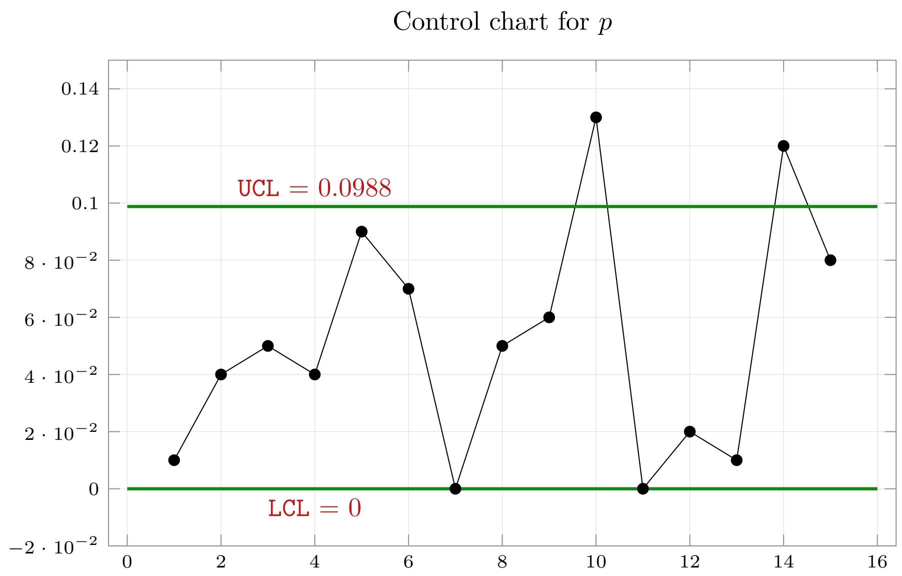
9. The control chart for the mean is,



10. The control chart for the fraction defective is,

$$p = 0.04 \qquad \sigma^2 = pq = 0.0384 \tag{25.5.7}$$

$$\sigma = 0.196 \qquad \text{UCL} = p + \frac{3\sigma}{\sqrt{n}} = 0.0988 \tag{25.5.8}$$



11. The control chart for the fraction defective is,

$$\text{UCL} = p + \frac{3\sigma}{\sqrt{n}} \qquad \text{LCL} = p - \frac{3\sigma}{\sqrt{n}} \tag{25.5.9}$$

$$\text{CL} = p \tag{25.5.10}$$

Now, for the number defective per sample,

$$\text{UCL} = p + 3\sqrt{n}\,\sigma \qquad \text{LCL} = p - 3\sqrt{n}\,\sigma \tag{25.5.11}$$

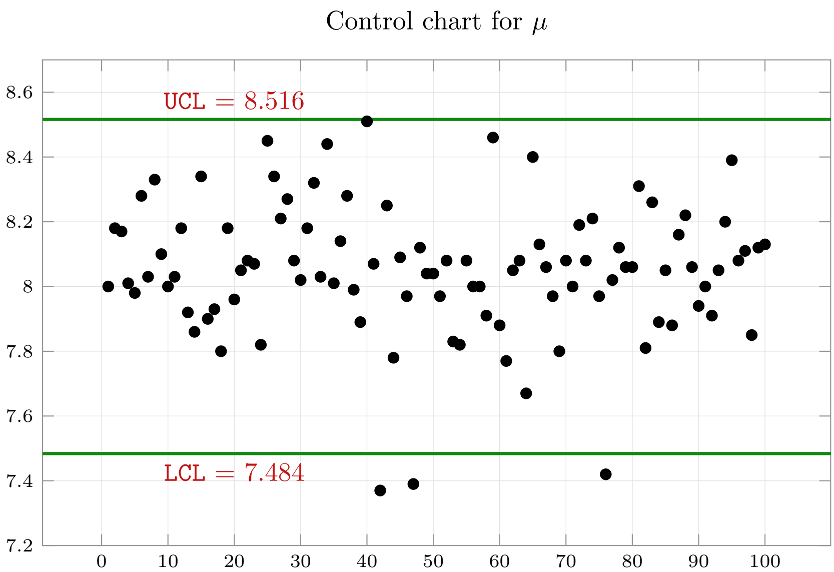
$$\text{CL} = \mu = np \qquad \sigma = \sqrt{p(1-p)} \tag{25.5.12}$$

12. Control charts

(a) Random numbers generated in `numpy`

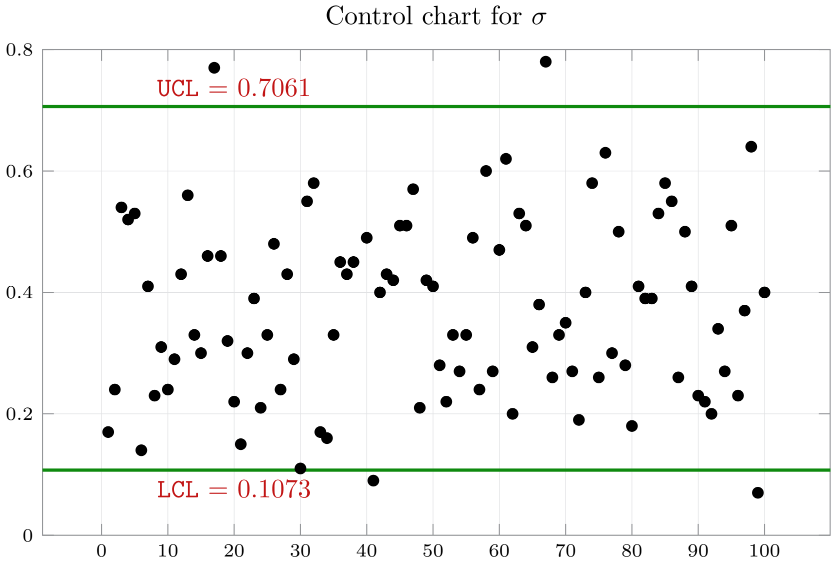
(b) The control charts for the mean,

$$\text{LCL} = \mu_0 - 2.58\frac{\sigma}{\sqrt{n}} = 7.484 \qquad \text{UCL} = \mu_0 + 2.58\frac{\sigma}{\sqrt{n}} = 8.516 \tag{25.5.13}$$



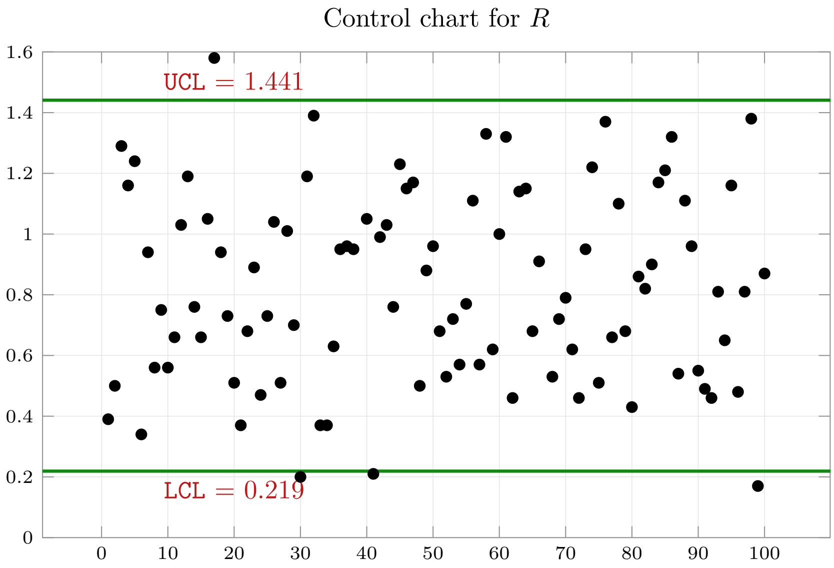
(c) The control charts for the standard deviation,

$$\text{LCL} = \sqrt{\frac{\sigma^2}{(n-1)}\,c_1} = 0.1073 \qquad \text{UCL} = \sqrt{\frac{\sigma^2}{(n-1)}\,c_2} = 0.7061 \tag{25.5.14}$$



(d) The control charts for the range,

$$\text{LCL} = \frac{\sigma_L}{\lambda_4} = 0.219 \qquad \text{UCL} = \frac{\sigma_U}{\lambda_4} = 1.441 \tag{25.5.15}$$



(e) In correspondence with $\alpha = 5\%$, a small fraction of the samples are outside control limits in each of the three control charts.

13. The fraction of all data points lying outside $\mu \pm k\sigma$ for a normal RV is,

$$1 - P(\mu - \sigma < X < \mu + \sigma) = 31.73\% \tag{25.5.16}$$

$$1 - P(\mu - 2\sigma < X < \mu + 2\sigma) = 4.55\% \tag{25.5.17}$$

$$1 - P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.27\% \tag{25.5.18}$$

14. The new control limits are,

$$\bar{x} \to \sum_j x_j \qquad \mu \to n\mu \tag{25.5.19}$$

$$\sigma \to n\sigma \qquad \text{LCL/UCL} = n\mu \pm k_\alpha\, \sigma\sqrt{n} \tag{25.5.20}$$

The mean and variance are both multiplied by a factor n. Here, k is a function of the confidence level.

15. For a Poisson process with parameter λ,

$$\text{LCL} = \mu - 3\sigma = \lambda - 3\sqrt{\lambda} \tag{25.5.21}$$

$$\text{UCL} = \mu + 3\sigma = \lambda + 3\sqrt{\lambda} \tag{25.5.22}$$

For the specific case of $\lambda = 3.6$,

$$\text{LCL} = 3.6 - 3\sqrt{3.6} = -2.09 \qquad \text{LCL} = 3.6 + 3\sqrt{3.6} = 9.29 \tag{25.5.23}$$

$$\text{CL} = \lambda = 3.6 \tag{25.5.24}$$

Since a Poisson process has to be non-negative, the `LCL` is set to zero whenever the formula yields a negative lower limit.

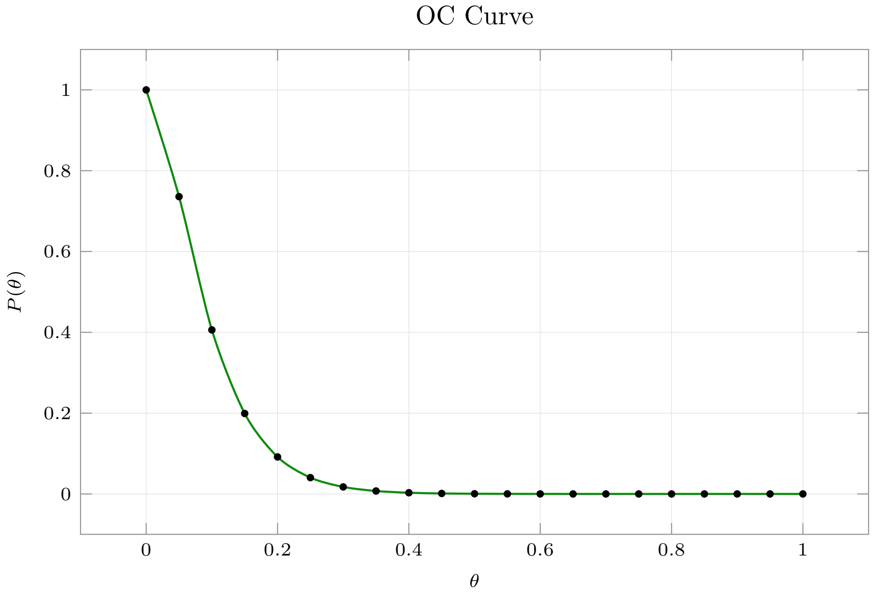
25.6 Acceptance Sampling

1. The specific values are,

$$\theta = 0.01 \qquad P(A\ ;\ \theta) = 0.9825 \tag{25.6.1}$$

$$\theta = 0.02 \qquad P(A\ ;\ \theta) = 0.9384 \tag{25.6.2}$$

$$\theta = 0.1 \qquad P(A\ ;\ \theta) = 0.4060 \tag{25.6.3}$$

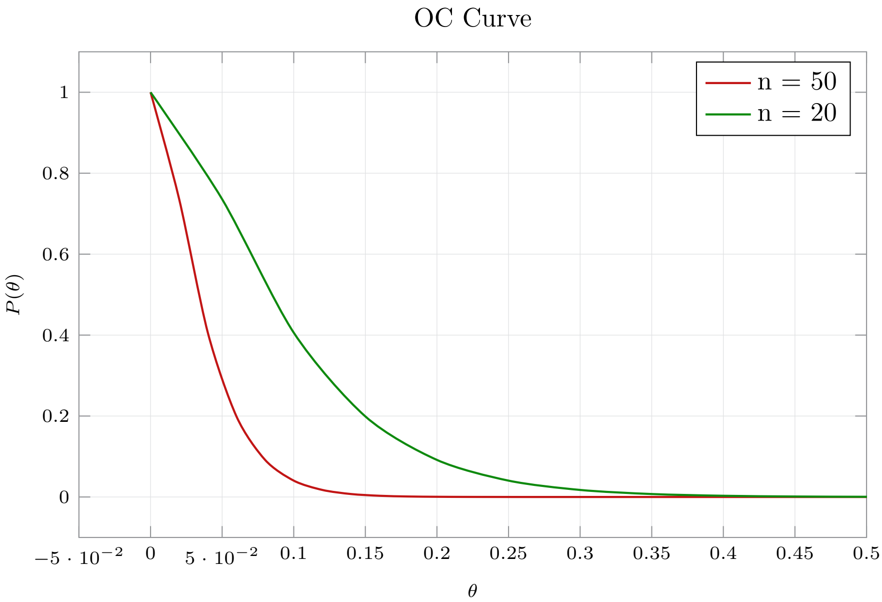


2. Changing the sample size to $n = 50$,

$$\theta = 0.01 \qquad P(A\ ;\ \theta) = 0.9098 \tag{25.6.4}$$

$$\theta = 0.02 \qquad P(A\ ;\ \theta) = 0.7357 \tag{25.6.5}$$

$$\theta = 0.1 \qquad P(A\ ;\ \theta) = 0.0404 \tag{25.6.6}$$



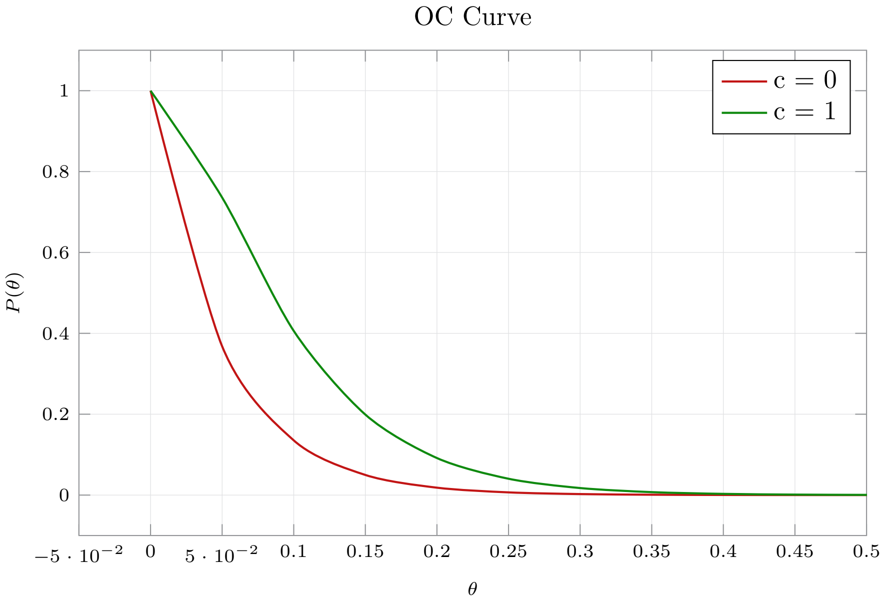
The effect is to make the OC curve decay to zero much faster. This is reasonable, since it is much more probable that a sample contains a defective item, with larger n, keeping θ constant.

3. Changing the acceptance limit to $c = 0$,

$$\theta = 0.01 \qquad P(A\ ;\ \theta) = 0.8187 \tag{25.6.7}$$

$$\theta = 0.02 \qquad P(A\ ;\ \theta) = 0.6703 \tag{25.6.8}$$

$$\theta = 0.1 \qquad P(A\ ;\ \theta) = 0.1353 \tag{25.6.9}$$



The effect is to make the OC curve decay to zero much faster. This is reasonable, since it is much more probable that a sample contains one defective item than zero, keeping θ constant.

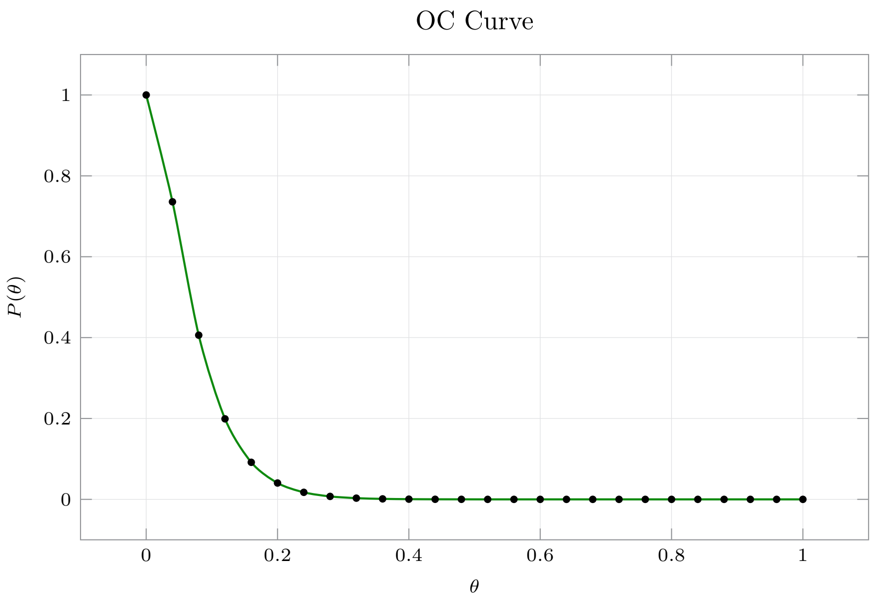
4. From the definition of AQL and RQL, the given defective fractions are θ_0 and θ_1 respectively.

$$\text{Risk}_p = 1 - P(A\ ;\ \theta_0) = 0.0616 \tag{25.6.10}$$

$$\text{Risk}_c = P(A\ ;\ \theta_1) = 0.1991 \tag{25.6.11}$$

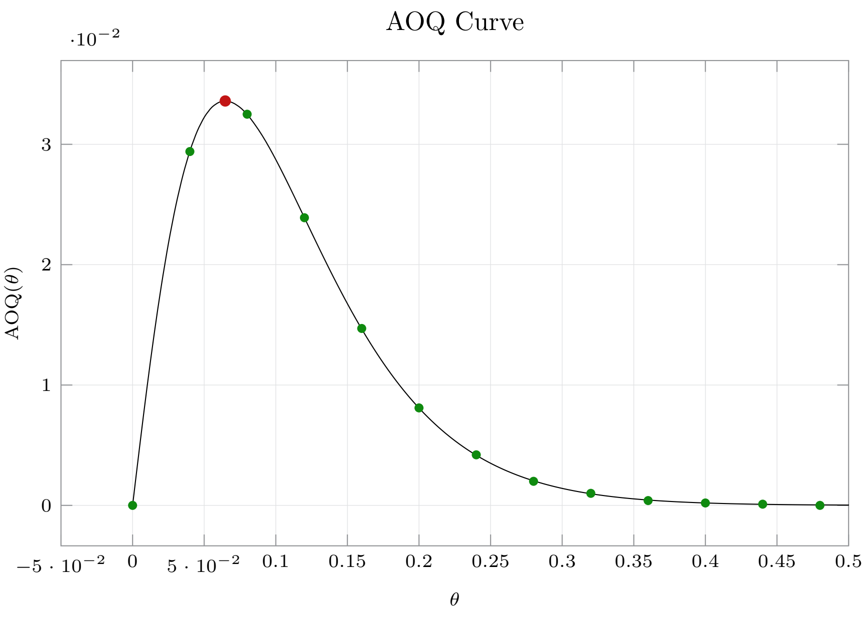
5. Given $n = 25$ and $c = 1$,

$$\text{Risk}_p = 1 - P(A\ ;\ \texttt{AQL}) \qquad = 0.055 \tag{25.6.12}$$



6. Given $n = 25$ and $c = 1$,

$$\text{Risk}_p = 1 - P(A\ ;\ \texttt{AQL}) \qquad = 0.055 \tag{25.6.13}$$



Since c is small, the analytical maximum can be found using differentiation

$$f(x) = (25x) \cdot e^{-25x}(1 + 25x) \tag{25.6.14}$$

$$x^* = 0.0647 \qquad f_{\max} = \text{AOQL} = 0.0336 \tag{25.6.15}$$

7. From the definition of AQL and RQL, the given defective fractions are θ_0 and θ_1 respectively.

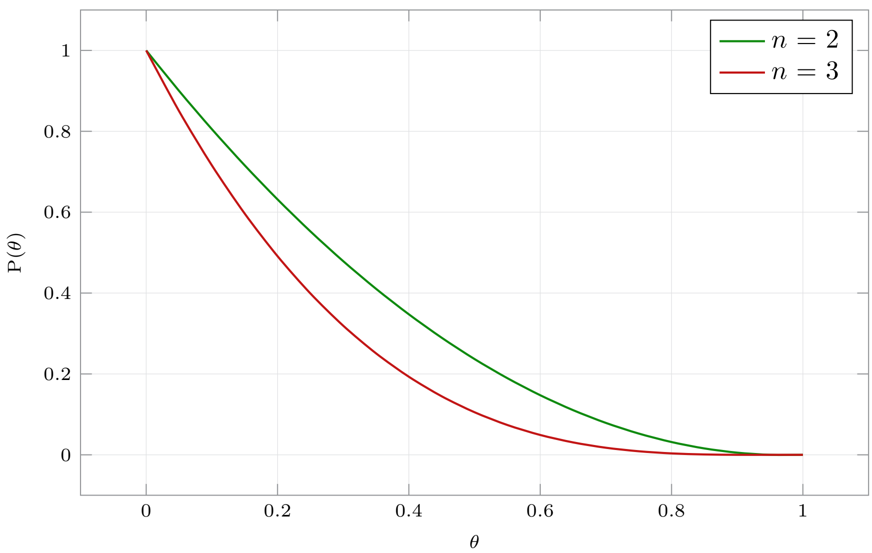
$$\text{Risk}_p = 1 - P(A\ ;\ \texttt{AQL}) = 1 - \frac{\binom{2}{0}\binom{18}{2}}{\binom{20}{2}} = 0.1947 \tag{25.6.16}$$

$$\text{Risk}_c = P(A\ ;\ \texttt{RQL}) = \frac{\binom{12}{0}\binom{8}{2}}{\binom{20}{2}} = 0.1474 \tag{25.6.17}$$

8. Comparing $c = 3$ and $c = 2$, the probabilties go down as the number of objects being sampled goes up, for the same θ.

θ	$c = 2$	$c = 3$
0.1	0.8053	0.7158
0.2	0.6316	0.4912

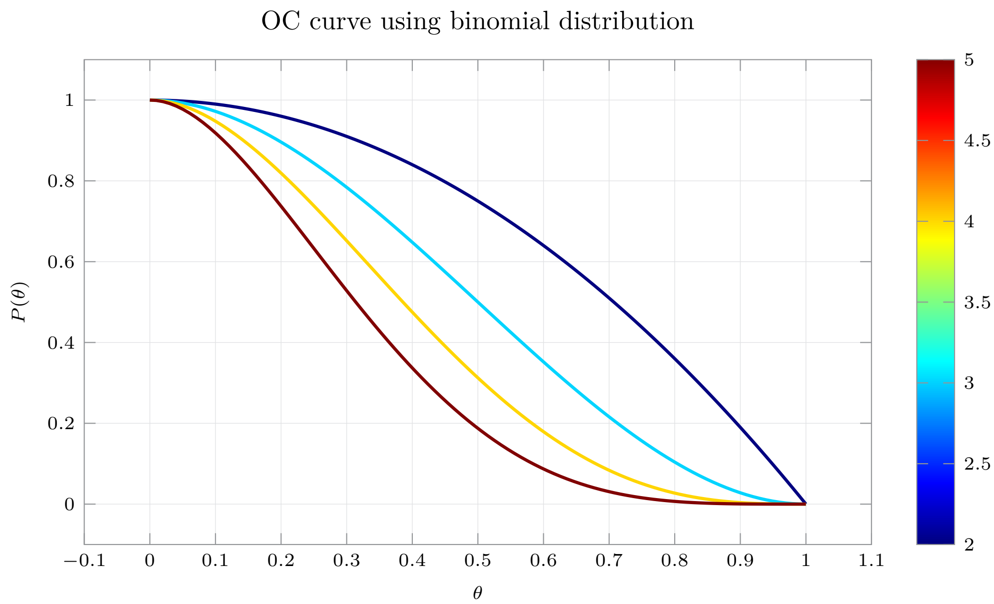
OC Curve



9. Using the binomial distribution with the defective probability being θ,

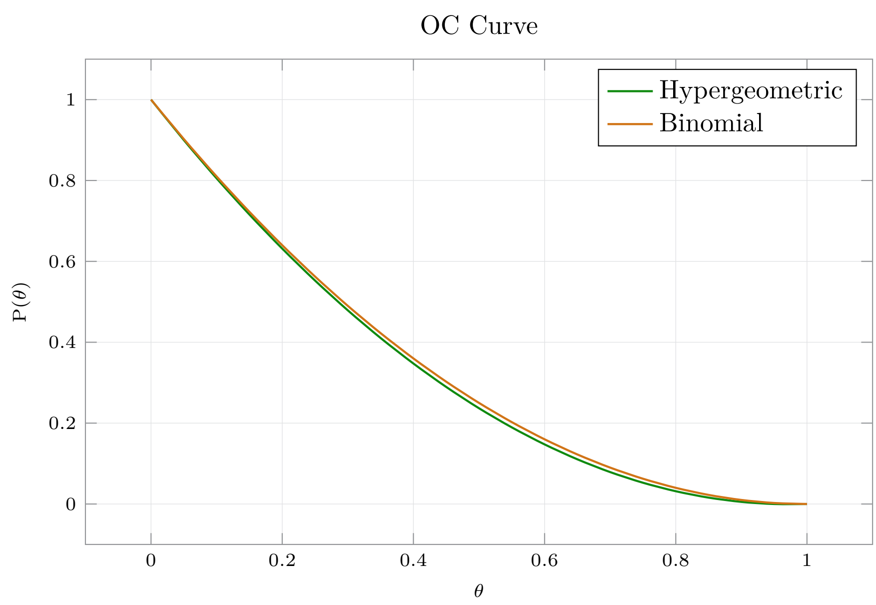
$$P(A\ ;\ \theta) = \sum_{x=0}^{1} \binom{n}{x} (1-\theta)^{n-x} \cdot \theta^x \tag{25.6.18}$$

$$= (1-\theta)^n + n\theta\,(1-\theta)^{n-1} \tag{25.6.19}$$



10. Using the formula in Problem 9, with $n = 2$

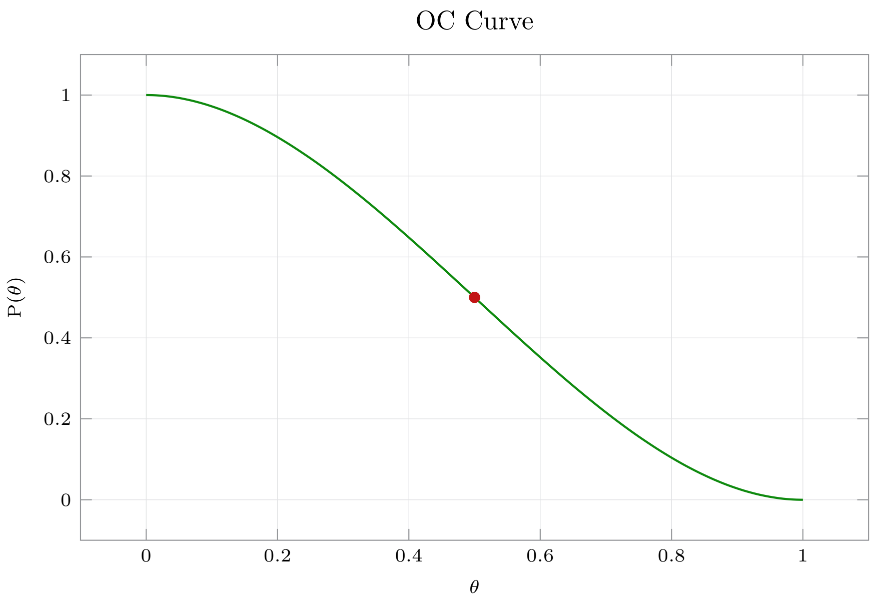
$$P(A\ ;\ \theta) = \sum_{x=0}^{0} \binom{n}{x} (1-\theta)^{n-x} \cdot \theta^x \qquad = (1-\theta)^2 \tag{25.6.20}$$



The approximation is very close to the accurate values from the hypergeometric PDF.

11. Given $c = 1,\ n = 3$ and using the binomial distribution,

$$P(A\ ;\ 0.5) = \binom{3}{0} \cdot (1-0.5)^3 \cdot 0.5^0 + \binom{3}{1} \cdot (1-0.5)^2 \cdot 0.5^1 = \frac{1}{2} \tag{25.6.21}$$



12. Using the binomial distribution, with $n = 100,\ p = 0.05$

$$\text{Risk}_p = 1 - P(A\ ;\ \texttt{AQL}) = 1 - \sum_{x=0}^{c} \binom{n}{x} q^{n-x}\ p^x \tag{25.6.22}$$

$$1 - 0.02 = \sum_{x=0}^{c} \binom{100}{x} (0.95)^{100-x}\ (0.05)^x = 0.98 \tag{25.6.23}$$

Using the normal approximation to the binomial distribution,

$$Z \sim \frac{X - np}{\sqrt{npq}} \tag{25.6.24}$$

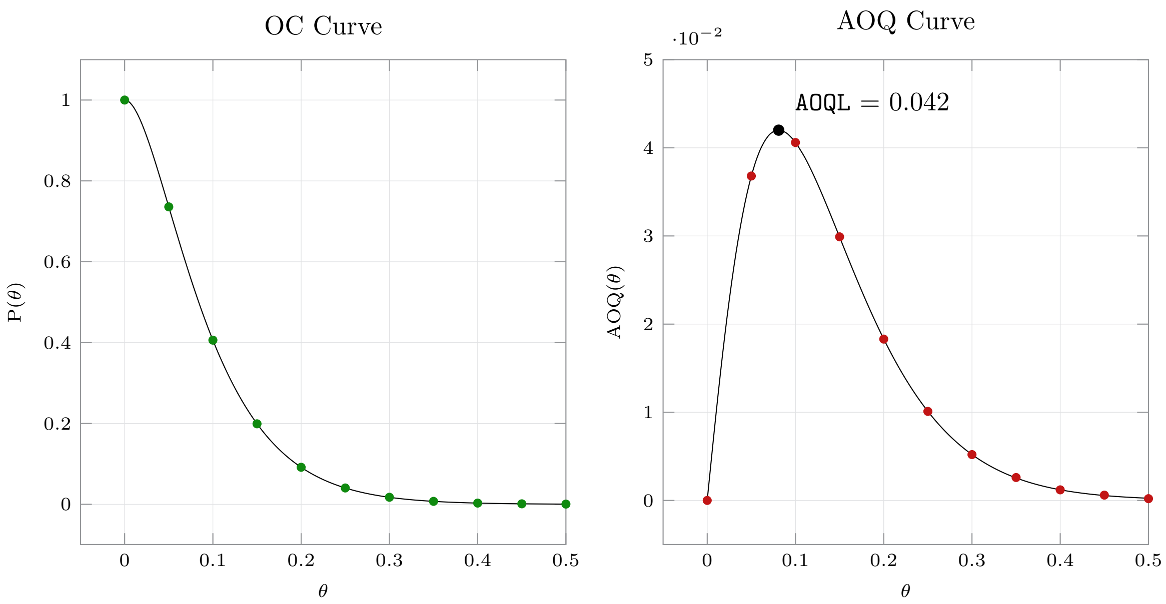
Since the producer's risk has to be 2% or larger, the smallest possible value of c is 9

13. The consumer's risk formula for the data in Problem 12 with $\texttt{RQL} = 0.12$ is,

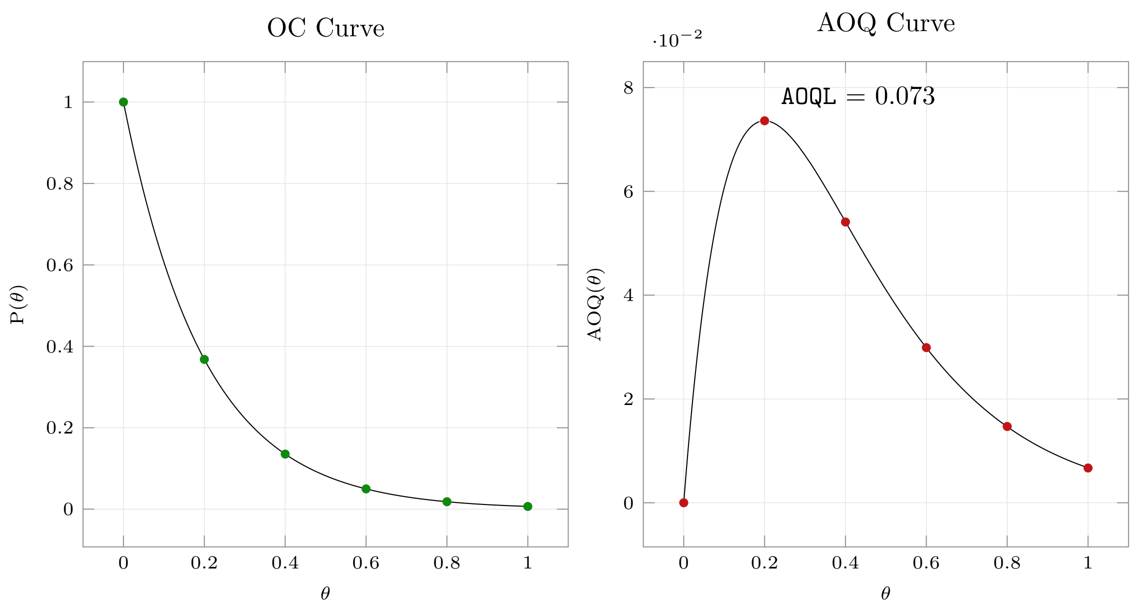
$$\text{Risk}_c = P(A\ ;\ \texttt{RQL}) = \sum_{x=0}^{9} \binom{100}{x} (0.12)^x\ (0.88)^{100-x} \tag{25.6.25}$$

$$= 0.2256 \tag{25.6.26}$$

14. Given $c = 1,\ n = 20$ and using the Poisson distribution,



15. Given $c = 0,\ n = 5$ and using the Poisson distribution,



25.7 Goodness of Fit, Chi-squared Test

1. Performing the χ^2 goodness of fit test, the sample being normally distributed is accepted.

Quantity	Value
α	0.05
K	10
$\widetilde{\mu}$	364.7
$\widetilde{\sigma}$	26.7
Statistic	2.81
Limit	14.07

2. Using the binomial distribution, the sample having a binomial distribution with $p = 0.25$ is accepted.

$$\chi^2 = \frac{(27 - 0.25 \cdot 80)^2}{0.25 \cdot 80} + \frac{(53 - 0.75 \cdot 80)^2}{0.75 \cdot 80} = 3.27 \tag{25.7.1}$$

$$\text{limit} = \chi_1^2(0.05) = 3.8414 \tag{25.7.2}$$

3. The sample having a binomial distribution with $p = 0.5$ is rejected.

$$\chi^2 = \frac{(40 - 0.5 \cdot 100)^2}{0.5 \cdot 100} + \frac{(60 - 0.5 \cdot 100)^2}{0.5 \cdot 100} = 4 \tag{25.7.3}$$

$$\text{limit} = \chi_1^2(0.05) = 3.8414 \tag{25.7.4}$$

4. The sample having a uniform distribution with $K = 2$ is accepted.

$$\chi^2 = \frac{(4 - 0.5 \cdot 10)^2}{0.5 \cdot 10} + \frac{(6 - 0.5 \cdot 10)^2}{0.5 \cdot 10} = 0.4 \tag{25.7.5}$$

$$\text{limit} = \chi_1^2(0.05) = 3.8414 \tag{25.7.6}$$

This is a much less egregious result under the assumption of fairness, and the hypothesis is much less likely to be rejected.

5. The sample having a uniform distribution with $K = 6$ is accepted.

$$\chi^2 = \frac{(10 - 10)^2}{10} + \frac{(13 - 10)^2}{10} + \frac{(9 - 10)^2}{10} \tag{25.7.7}$$

$$+ \frac{(11 - 10)^2}{10} + \frac{(9 - 10)^2}{10} + \frac{(8 - 10)^2}{10} = 1.6 \tag{25.7.8}$$

$$\text{limit} = \chi_5^2(0.05) = 11.07 \tag{25.7.9}$$

6. The sample having a uniform distribution with $K = 6$ is accepted.

$$\chi^2 = \frac{(33-30)^2}{30} + \frac{(27-30)^2}{30} + \frac{(29-30)^2}{30} \tag{25.7.10}$$

$$+ \frac{(35-30)^2}{30} + \frac{(25-30)^2}{30} + \frac{(31-30)^2}{30} = \frac{7}{3} \tag{25.7.11}$$

$$\text{limit} = \chi^2_5(0.05) = 11.07 \tag{25.7.12}$$

7. The sample having a uniform distribution with $K = 6$ is accepted.

$$\widetilde{x} = \frac{60+49+56+46+68+39}{6} = 53 \tag{25.7.13}$$

$$\chi^2 = \frac{(60-53)^2}{53} + \frac{(49-53)^2}{53} + \frac{(56-53)^2}{53} \tag{25.7.14}$$

$$+ \frac{(46-53)^2}{53} + \frac{(68-53)^2}{53} + \frac{(39-53)^2}{53} = 10.26 \tag{25.7.15}$$

$$\text{limit} = \chi^2_5(0.05) = 11.07 \tag{25.7.16}$$

8. The sample having a binomial distribution with $p = 0.025$ is rejected.

$$\chi^2 = \frac{(17 - 0.025 \cdot 400)^2}{0.025 \cdot 400} + \frac{(383 - 0.975 \cdot 400)^2}{0.975 \cdot 400} = 5.026 \tag{25.7.17}$$

$$\text{limit} = \chi^2_1(0.05) = 3.8414 \tag{25.7.18}$$

9. The sample having a uniform distribution with $K = 2$ is accepted.

$$\chi^2 = \frac{(42 - 0.5 \cdot 90)^2}{0.5 \cdot 90} + \frac{(48 - 0.5 \cdot 90)^2}{0.5 \cdot 90} = 0.4 \tag{25.7.19}$$

$$\text{limit} = \chi^2_1(0.05) = 3.8414 \tag{25.7.20}$$

10. Choosing random numbers

(a) The sample having a uniform distribution with $K = 20$ is accepted.

$$\chi^2 = \sum_{j=1}^{20} \frac{(x_j - 11.55)^2}{11.55} = 24.32 \tag{25.7.21}$$

$$\text{limit} = \chi^2_{19}(0.05) = 30.14 \tag{25.7.22}$$

(b) The sample having a uniform distribution with $K = 2$ is rejected.

$$\chi^2 = \frac{(143 - 115.5)^2}{115.5} + \frac{(88 - 115.5)^2}{115.5} = 13.1 \tag{25.7.23}$$

$$\text{limit} = \chi_1^2(0.05) = 3.84 \tag{25.7.24}$$

(c) The sample having a binomial distribution with $K = 2$ and $p = 0.3$ is rejected.

$$\chi^2 = \frac{(96 - 0.3 * 231)^2}{0.3 * 231} + \frac{(135 - 0.7 * 231)^2}{0.7 * 231} = 14.69 \tag{25.7.25}$$

$$\text{limit} = \chi_1^2(0.05) = 3.84 \tag{25.7.26}$$

11. Running n experiments with a die rolled 300 times and checking for fairness using the χ^2 goodness of fit test on the uniform distribution,

n	Number rejected at $\alpha = 0.05$
10	1
100	7
1000	45
10000	465
100000	4931

Similar tests TBC.

12. Performing the χ^2 goodness of fit test, the sample being normally distributed is accepted.

Quantity	Value
α	0.01
K	4.00
$\widetilde{\mu}$	59.87
$\widetilde{\sigma}$	1.50
Statistic	3.31
Limit	6.63

This uses 5 bins by combining the first 2 and last 3 bins together. The resulting limit is $\chi_2^2(0.01)$

13. Using the binomial distribution, the sample having a binomial distribution with $p = 0.25$ is accepted.

$$\chi^2 = \frac{(123 - 0.25 \cdot 478)^2}{0.25 \cdot 478} + \frac{(355 - 0.75 \cdot 478)^2}{0.75 \cdot 478} = 0.1367 \tag{25.7.27}$$

$$\text{limit} = \chi_1^2(0.05) = 3.8414 \tag{25.7.28}$$

14. The values of e_j are computed first, using $\widetilde{x} = 23/50 = 0.46$

$$\{e_0, e_1, e_2\} = \{0.6313, 0.2904, 0.06679\} \tag{25.7.29}$$

$$\chi^2 = \frac{(33-31.56)^2}{31.56} + \frac{(11-14.52)^2}{14.52} + \frac{(6-3.34)^2}{3.34} \tag{25.7.30}$$

$$= 3.037 \tag{25.7.31}$$

$$\text{limit} = \chi_1^2(0.95) = 3.84 \tag{25.7.32}$$

The sample having a Poisson distribution with is accepted.

Since only $K = 3$ bins are used, the d.o.f. of the χ^2 distribution is 1.

15. Performing the χ^2 goodness of fit test, the sample being Poisson distributed is accepted.

Quantity	Value
α	0.05
K	13
Statistic	13.36
Limit	19.68

25.8 Nonparametric Tests

1. For $n = 6$, the binomial distribution gives,

$$P(X=5) = \binom{6}{5} \cdot (0.5)^5 \cdot (0.5)^1 = 0.09375 P(X=4) \quad = \binom{6}{4} \cdot (0.5)^4 \cdot (0.5)^2 = 0.234375 \tag{25.8.1}$$

Using $\alpha = 0.05$, the hypothesis would be rejected with less than 6 positive values.

2. For $n = 6$, the binomial distribution gives,

$$P(X=4) = \binom{6}{4} \cdot (0.5)^4 \cdot (0.5)^2 = 0.234375 \tag{25.8.2}$$

Using $\alpha = 0.05$, the hypothesis would be rejected.

3. For $n = 8$, the binomial distribution gives,

$$P(X=7) = \binom{8}{7} \cdot (0.5)^7 \cdot (0.5)^1 = 0.03125 \tag{25.8.3}$$

Using $\alpha = 0.05$, the hypothesis would be accepted.

4. For $n = 18$, and the normal approximation to the binomial distribution,

$$p = 0.5 \qquad \mu = np = 9 \tag{25.8.4}$$

$$\sigma = \sqrt{np(1-p)} = \frac{3}{\sqrt{2}} \tag{25.8.5}$$

$$P(X \geq 15) = 1 - P(X \leq 15) \qquad = 1 - P\left(Z \leq \frac{15 - 9 + 0.5}{3/\sqrt{2}}\right) \tag{25.8.6}$$

$$= 1.09\ \% \tag{25.8.7}$$

Using $\alpha = 0.05$, the hypothesis would be rejected.

5. Repeating Problem 4 using the binomial distribution directly,

$$P(X = 15) = \sum_{x=15}^{18} \binom{18}{x} \cdot (0.5)^x \cdot (0.5)^{18-x} = 0.377\ \% \tag{25.8.8}$$

Using $\alpha = 0.05$, the hypothesis would be rejected.

6. The set of differences in scores $(B - A)$ are,

$$\Delta s = \{5, 15, 5, -5, 20, 25, 30, 15, -5, 5, 10, 15, 10, -15, 15\} \tag{25.8.9}$$

For $n = 15$, the binomial distribution gives,

$$P(X = 12) = \binom{15}{12} \cdot (0.5)^12 \cdot (0.5)^3 = 0.0139 \tag{25.8.10}$$

Using $\alpha = 0.05$, the hypothesis would be rejected.

7. Using a test for the mean of a normal distribution (hypothesized to be zero) with unknown variance, the hypothesis would be rejected

Quantity	Value
α	0.05
$\bar{x}$	9.6667
s	11.8723
n	15
Statistic	3.1535
Limit	1.7613

8. Using the non-parametric sign test, with $n = 9$

$$\Delta s = \begin{bmatrix} 1.2 & 2.4 & 1.3 & 1.3 & 0.0 & 1.0 & 1.8 & 0.8 & 4.6 & 1.4 \end{bmatrix} \tag{25.8.11}$$

$$P(X = 9) = \binom{9}{9} \cdot (0.5)^9 \cdot (0.5)^0 = 0.002 \tag{25.8.12}$$

Using $\alpha = 0.05$, the hypothesis would be rejected.

9. Using a test for the mean of a normal distribution (hypothesized to be zero) with unknown variance, the hypothesis would be rejected and the effect on sleeping time is positive.

Quantity	Value
α	0.05
$\bar{x}$	1.58
s	1.23
n	10
Statistic	4.0621
Limit	1.8331

10. Using the non-parametric sign test, with $n = 9$

$$P(X \geq 8) = \binom{9}{8} \cdot (0.5)^8 \cdot (0.5)^1 + \binom{9}{9} \cdot (0.5)^9 \cdot (0.5)^0 = 0.02 \qquad 25.8.13$$

Using $\alpha = 0.05$, the hypothesis would be rejected, and the thermostat setting is too low.

11. Instead of checking whether the numbers are positive or negative, check if they are larger or smaller than μ_0

12. Using $n = 5$ and $T = 3$,

$$P(T \leq 3) = 0.242 \qquad 25.8.14$$

Using $\alpha = 0.05$, the hypothesis would be rejected, and the trend is increasing.

13. Using $n = 8$ and $T = 4$,

$$P(T \leq 4) = 0.007 \qquad 25.8.15$$

Using $\alpha = 0.05$, the hypothesis would be rejected, and the trend is increasing.

14. After sorting by x, and using $n = 10$ and $T = 10$,

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 91 & 92 & 95 & 100 & 102 & 112 & 120 & 121 & 123 & 140 \\ 418 & 301 & 352 & 395 & 375 & 388 & 465 & 521 & 455 & 490 \end{bmatrix} \qquad 25.8.16$$

$$P(T \leq 10) = 0.014 \qquad 25.8.17$$

Using $\alpha = 0.05$, the hypothesis would be rejected, and the trend is increasing.

15. Using $n = 6$ and $T = 2$,

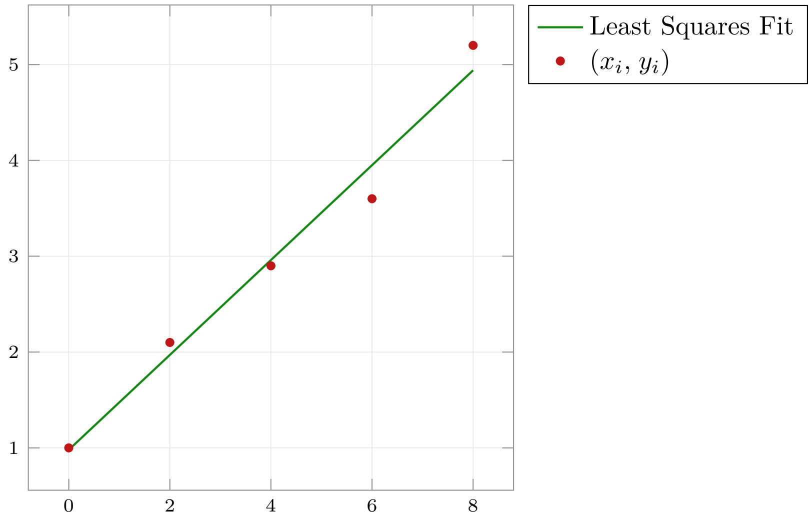
$$P(T \leq 2) = 0.028 \qquad 25.8.18$$

Using $\alpha = 0.05$, the hypothesis would be rejected, and the trend is increasing.

25.9 Regression. Fitting Straight Lines. Correlation

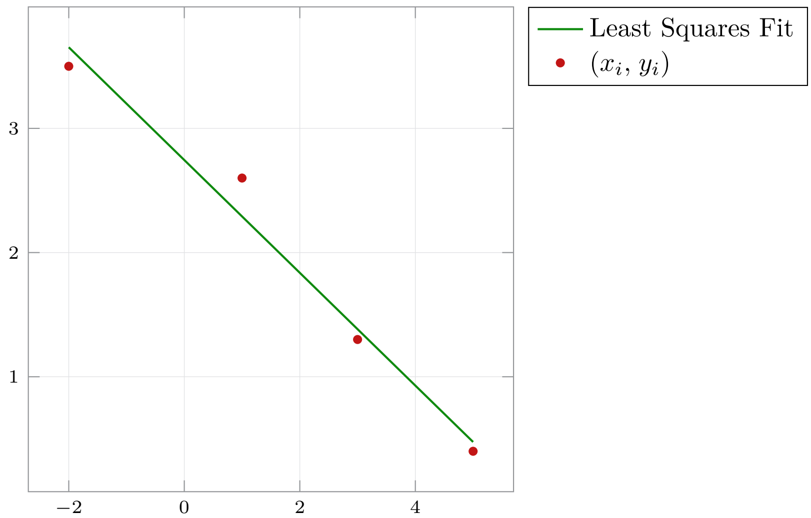
1. Graphing the regression line,

$$k_0 = 0.98, \quad k_1 = 0.495$$

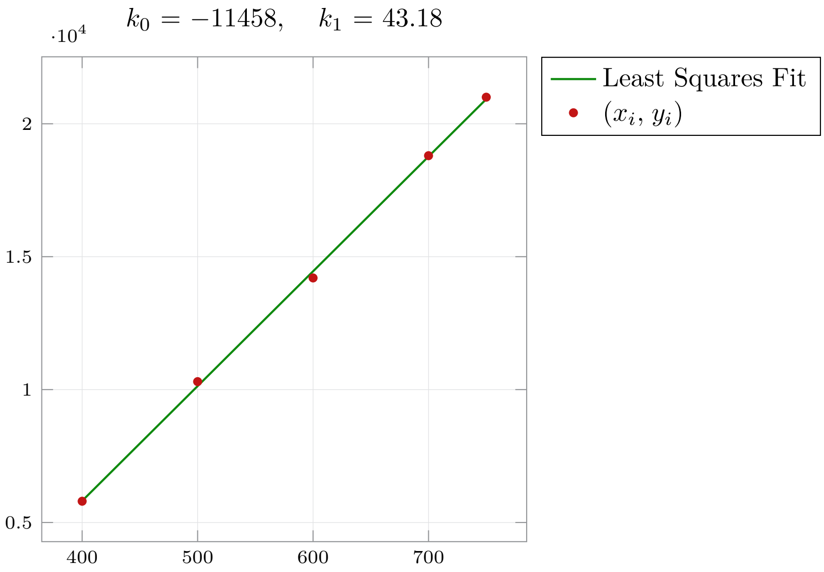


2. Graphing the regression line,

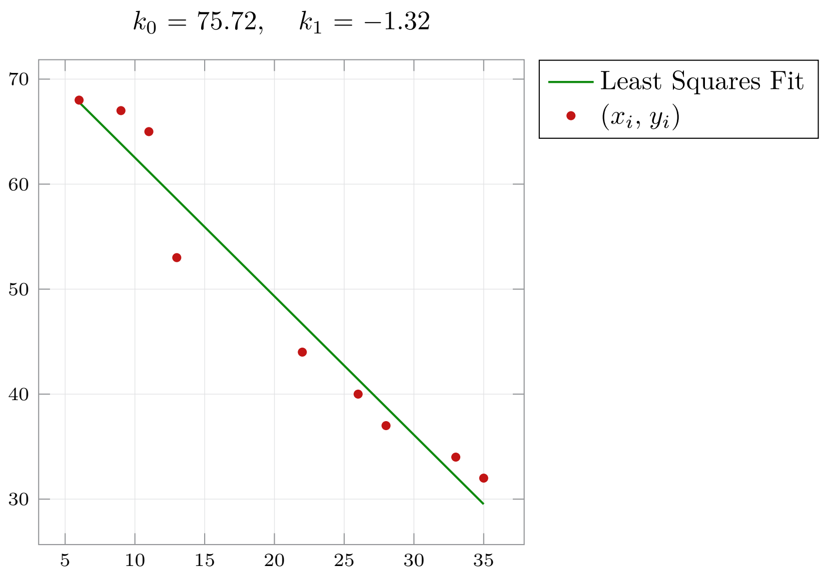
$$k_0 = 2.745, \quad k_1 = -0.454$$



3. Graphing the regression line,

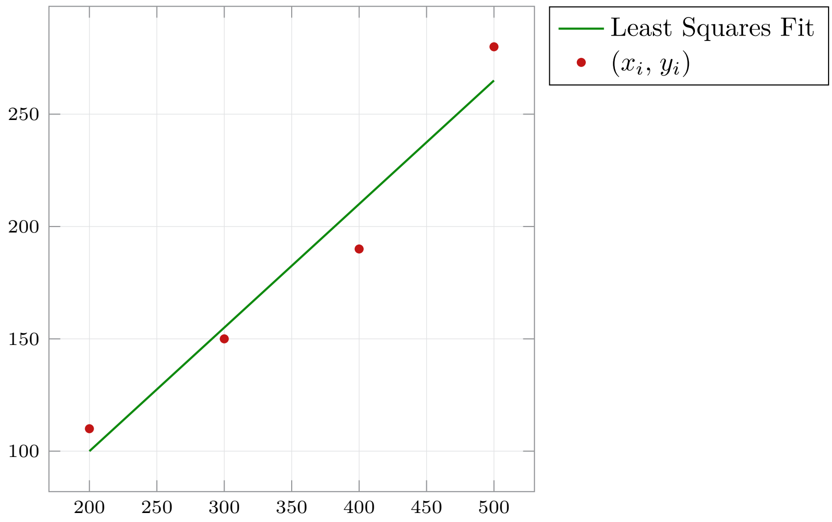


4. Graphing the regression line,



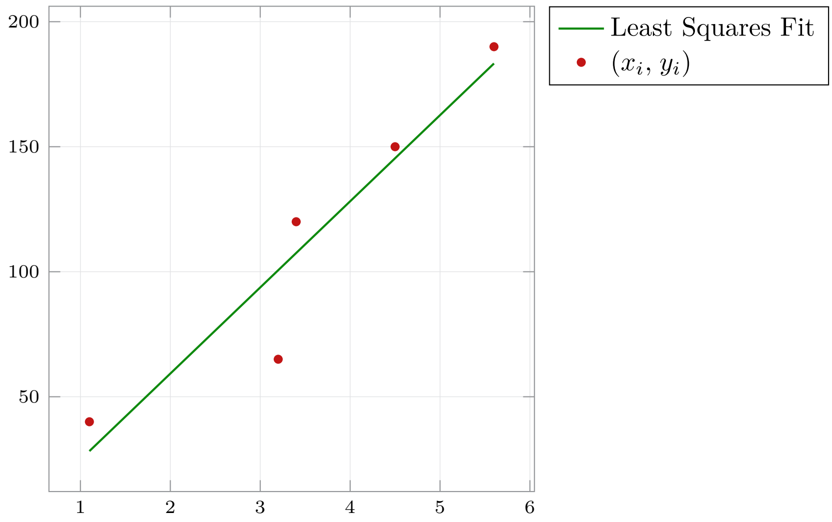
5. Graphing the regression line,

$$k_0 = -10, \quad k_1 = 0.55$$



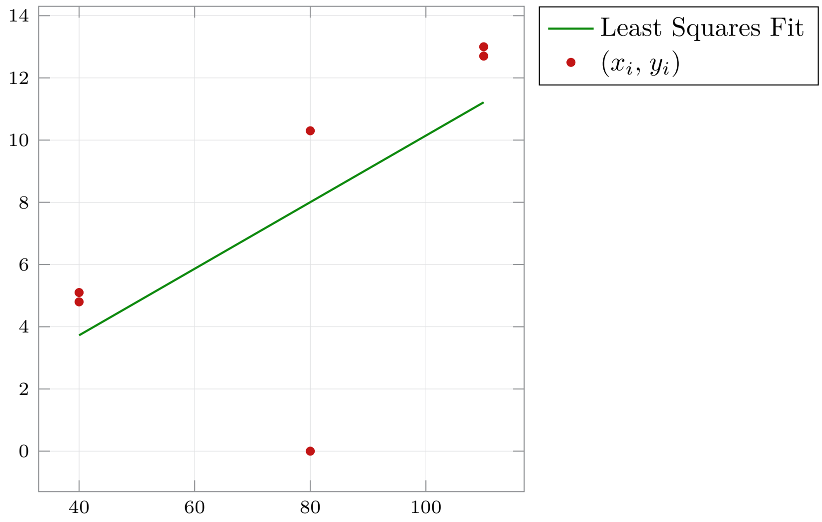
6. Graphing the regression line,

$$k_0 = -9.632, \quad k_1 = 34.45$$



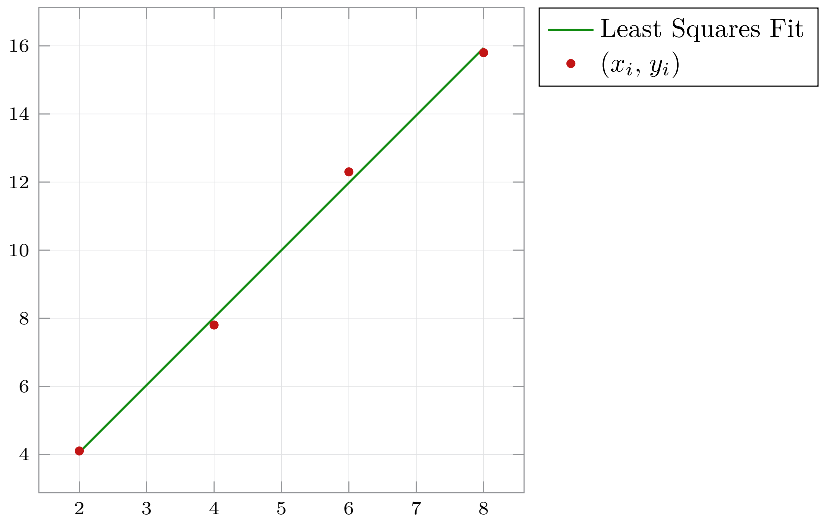
7. Graphing the regression line, and all 6 points, instead of just 3 of the points as used in the textbook solutions. $R = 1/k_1 = 9.34\ \Omega$

$k_0 = -0.555, \quad k_1 = 0.107$



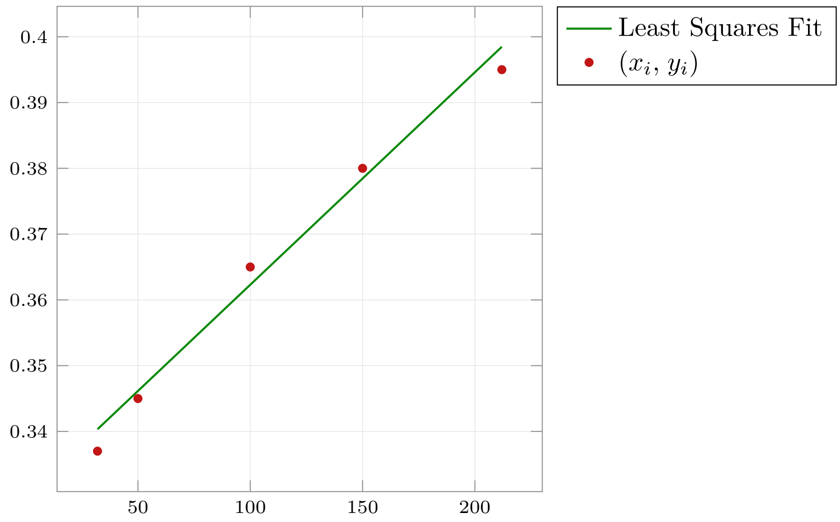
8. The spring modulus is $M = 1/k_1 = 0.505$

$k_0 = 0.1, \quad k_1 = 1.98$



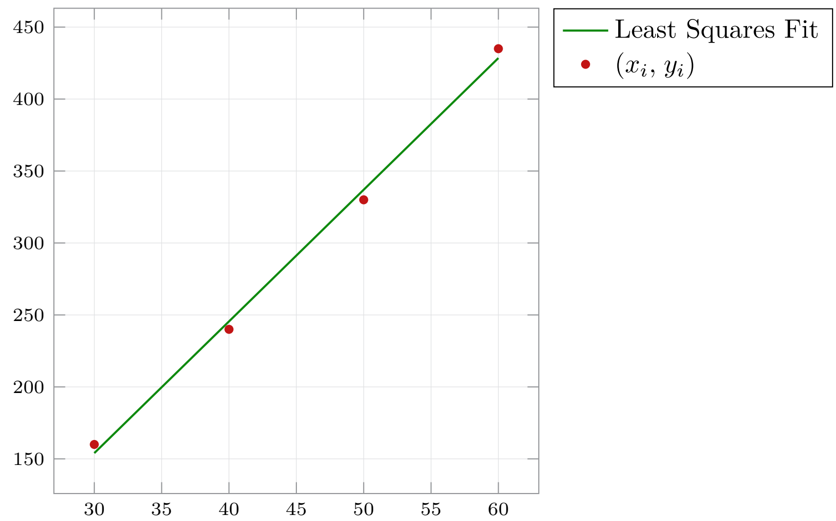
9. At room temperature, $x = 66 \implies y = 0.351$

$$k_0 = 0.33, \quad k_1 = 0.000323$$

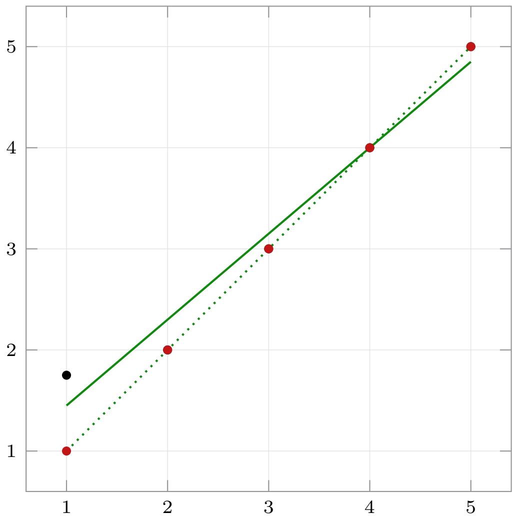


10. At room temperature, $x = 35 \implies y = 199.75$

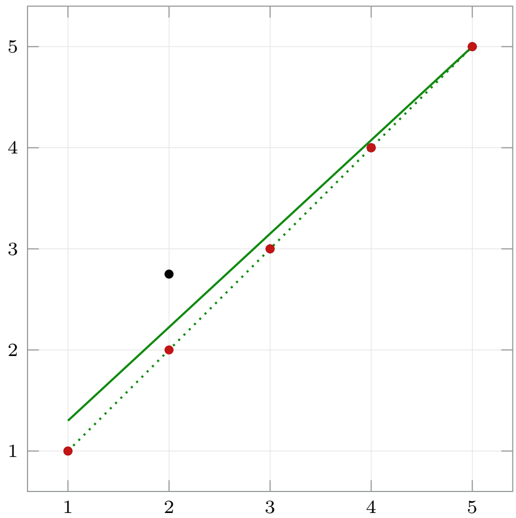
$$k_0 = -120.5, \quad k_1 = 9.15$$



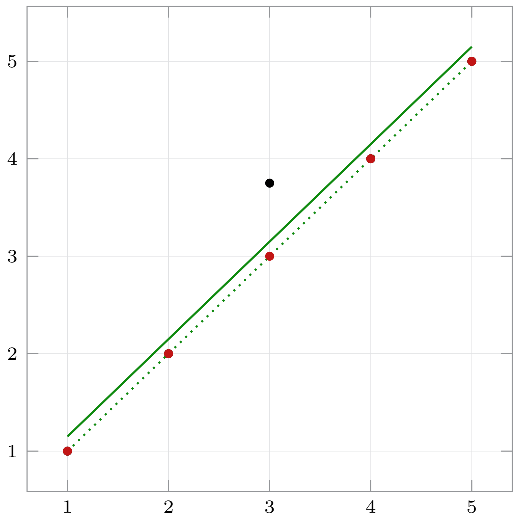
11. Looking at the effect of perturbations, Perturbed y_1



Perturbed y_2



Perturbed y_3



The perturbation with the greatest effect is at the edges of the domain spanned by the sample.

12. Using the data in Problem 2,

Quantity	Value
γ	0.95
k_1	−0.4542
q_0	0.1314
n	4
c	4.3027
Lower Limit	−0.6674
Upper Limit	−0.2410

13. Using the data in Problem 3,

Quantity	Value
γ	0.95
k_1	43.1829
q_0	97256.0976
n	5
c	3.1824
Lower Limit	41.1819
Upper Limit	45.1839

14. Using the data in Problem 4,

Quantity	Value
γ	0.95
k_1	-1.3196
q_0	77.1543
n	9
c	2.3646
Lower Limit	-1.5751
Upper Limit	-1.0641

15. Using the given data,

Quantity	Value
γ	0.95
k_1	0.0670
q_0	0.0230
n	4
c	4.3027
Lower Limit	0.0464
Upper Limit	0.0876