Chapter 1

First Order ODEs

1.1 Basic Concepts: Modeling

1. Using the substitution $v = 2\pi x$

$$y' = -2\sin(2\pi x) \qquad dy = -2\int \sin(2\pi x) \quad dx \tag{1.1.1}$$

$$dy = \frac{-1}{\pi}\int \sin(v) \quad dv \qquad y = \frac{\cos(2\pi x)}{\pi} + c \tag{1.1.2}$$

2. Using the substitution $v = -x^2/2$

$$y' = -x\exp(-x^2/2) \qquad dy = -\int x\exp(-x^2/2) \quad dx \tag{1.1.3}$$

$$dy = \int \exp(v) \quad dv \qquad y = \exp(-x^2/2) + c \tag{1.1.4}$$

3. Using the integration over y

$$y' = y \qquad dx = \int \frac{1}{y} \quad dy \tag{1.1.5}$$

$$x = \ln(y) + c \qquad y = C\exp(x) \tag{1.1.6}$$

4. Using the integration over y

$$y' = -1.5y \qquad dx = \int \frac{-2}{3y} \quad dy \tag{1.1.7}$$

$$x = \frac{-2\ln(y)}{3} + c \qquad y = C\exp(-1.5x) \tag{1.1.8}$$

5. Using the substitution $v = 2\pi x$

$$y' = 4e^{-x}\cos(x) \tag{1.1.9}$$

$$\int \quad dy = 4\int e^{-x}\cos(x) \quad dx \tag{1.1.10}$$

$$y = 4e^{-x}\sin(x) + 4\int e^{-x}\sin(x) \quad dx \tag{1.1.11}$$

$$y = 4e^{-x}\sin(x) - 4e^{-x}\cos(x) - 4\int e^{-x}\cos(x) \quad dx \tag{1.1.12}$$

$$y = 2e^{-x}(\sin(x) - \cos(x)) \tag{1.1.13}$$

$$y = -2\sqrt{2}e^{-x}\cos\left(x + \frac{\pi}{4}\right) + c \tag{1.1.14}$$

6. Using the standard result for trigonometric ODEs, (second order ODE will result in two arbitrary constants)

$$y'' = -y \qquad y = c_1\cos(x) + c_2\sin(x) \tag{1.1.15}$$

7. Using the substitution $a = 5.13$

$$y' = \cosh(5.13x) \qquad \int \quad dy = \int \cosh(ax) \quad dx \tag{1.1.16}$$

$$y = \int \frac{e^{ax} + e^{-ax}}{2} \quad dx \qquad y = \frac{\sinh(ax)}{a} + c \tag{1.1.17}$$

8. Using the substitution $a = -0.2$ (Third order ODE results in 3 arbitrary constants)

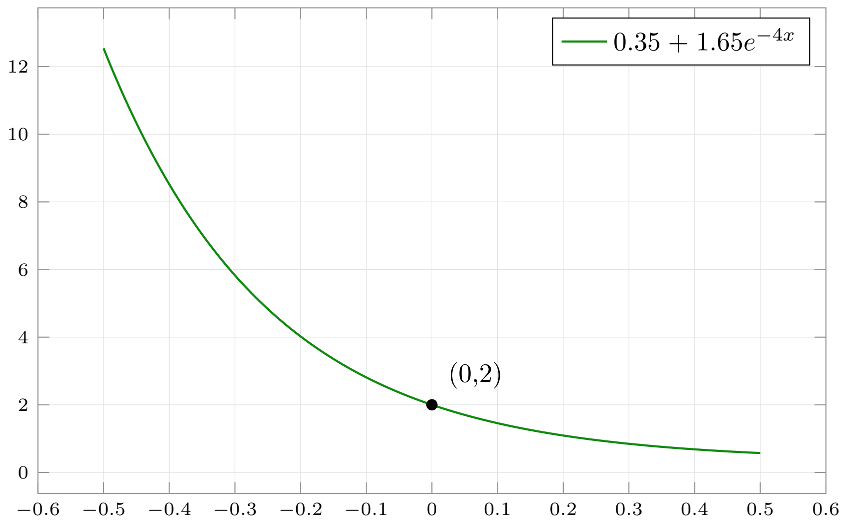
$$y''' = \exp(-0.2x) \qquad y = \frac{\exp(ax)}{a^3} + bx^2 + cx + d \tag{1.1.18}$$

9. IC is $y(0) = 2$

$$4y = 4c\exp(-4x) + 1.4 \qquad y' = -4c\exp(-4x) \tag{1.1.19}$$

$$y' + 4y = 1.4 \qquad y(0) = c + 0.35 = 2 \tag{1.1.20}$$

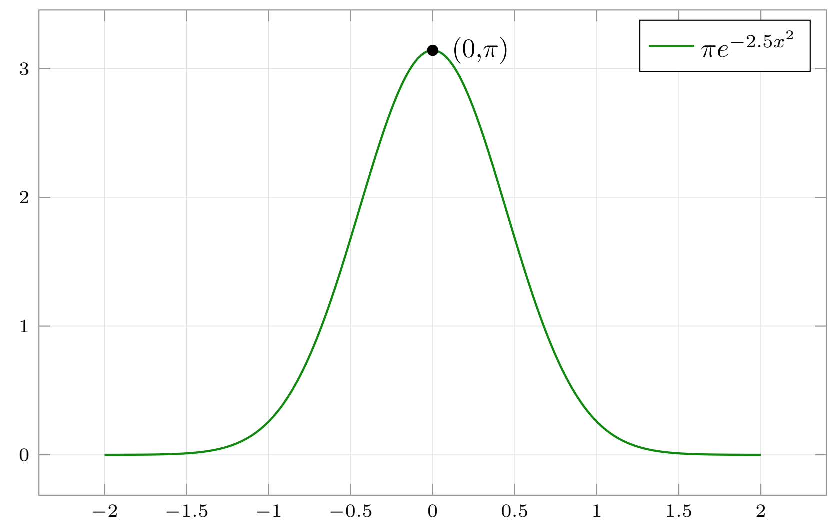
$$c = 1.65 \tag{1.1.21}$$



10. IC is $y(0) = \pi$

$$5xy = (5x)\ ce^{-2.5x^2} \qquad y' = (-5x)\ ce^{-2.5x^2} \tag{1.1.22}$$

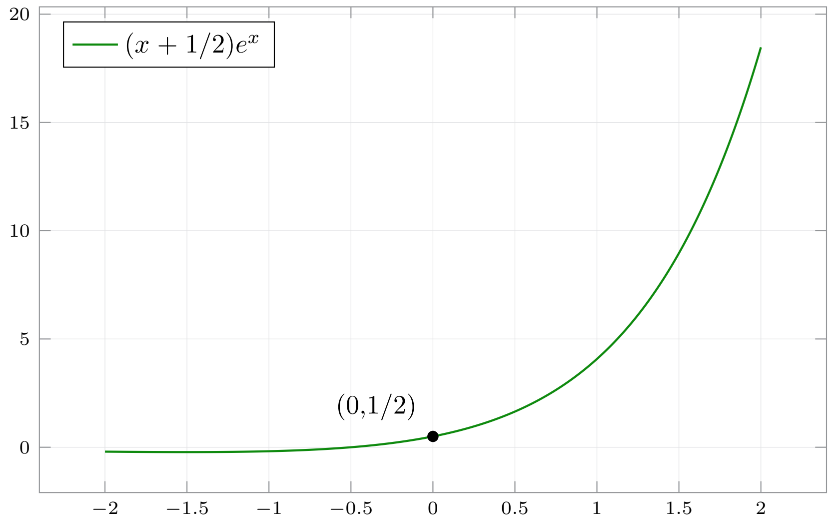
$$y' + 5xy = 0 \qquad y(0) = c = \pi \tag{1.1.23}$$



11. IC is $y(0) = 1/2$

$$y' = (1 + x + c)\, e^x \qquad y' - y = e^x \tag{1.1.24}$$

$$y(0) = c = 1/2 \tag{1.1.25}$$



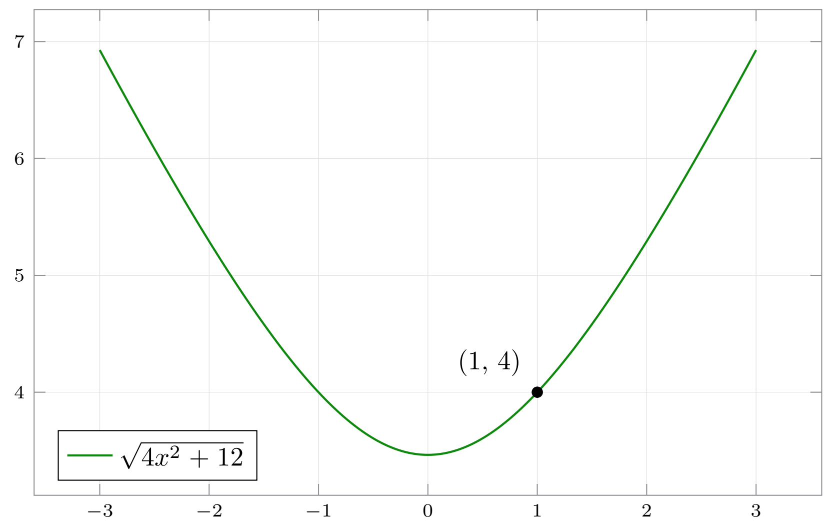
12. IC is $y(1) = 4$

$$2yy' = 8x \qquad y > 0 \tag{1.1.26}$$

$$yy' = 4x \qquad y > 0 \tag{1.1.27}$$

$$y(1) = c + 4 = 16 \tag{1.1.28}$$

$$c = 12 \tag{1.1.29}$$

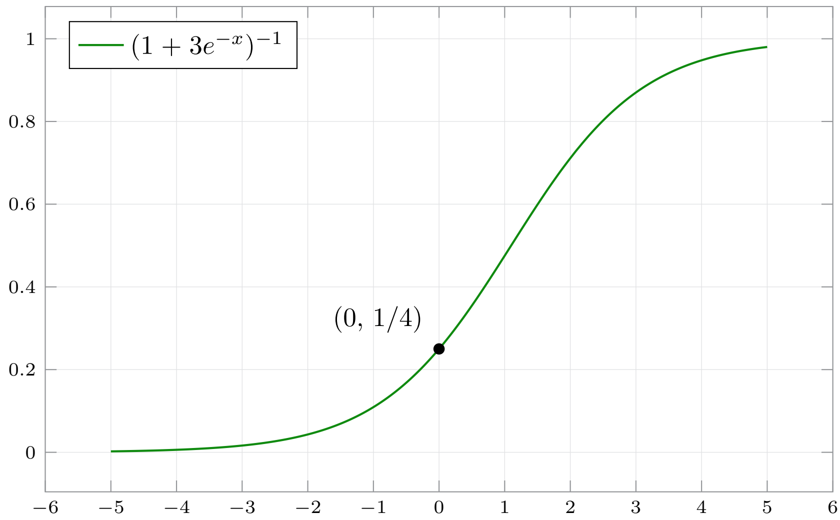


13. IC is $y(0) = 1/4$

$$y' = \frac{ce^{-x}}{(1+ce^{-x})^2} \qquad y - y^2 = \frac{1+ce^{-x}-1}{(1+ce^{-x})^2} \tag{1.1.30}$$

$$y' = y - y^2 \qquad y(0) = \frac{1}{(1+c)} = 1/4 \tag{1.1.31}$$

$$c = 3 \tag{1.1.32}$$

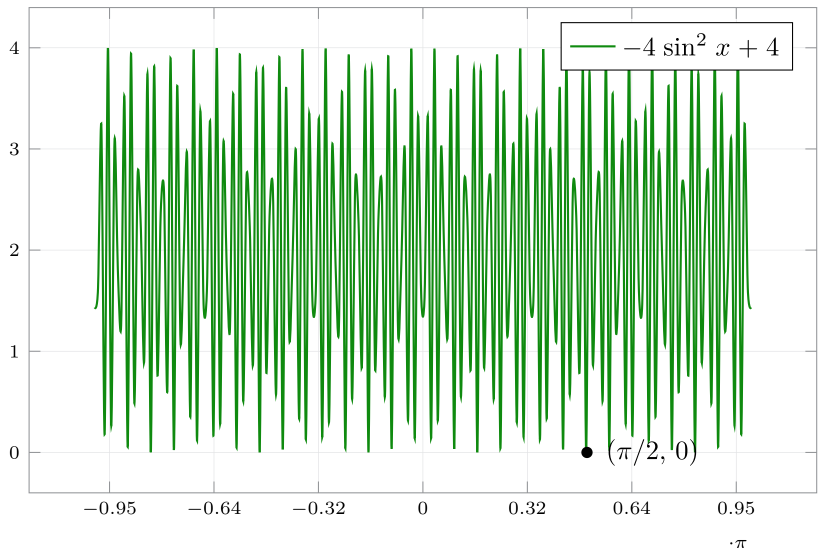


14. IC is $y(\pi/2) = 0$

$$2y - 8 = 2c\ \sin^2 x \qquad y' \tan x = 2c\ \sin x\ \cos x\ \tan x \tag{1.1.33}$$

$$y' \tan x = 2y - 8 \qquad y(\pi/2) = c + 4 = 0 \tag{1.1.34}$$

$$c = -4 \tag{1.1.35}$$



15. By Inspection , the ODE in Problem 13 has the constant solutions

$$y = 1 \qquad y = 0 \tag{1.1.36}$$

16. Verifying the general solution by substitution

$$xy' = cx \qquad y'^2 = c^2 \tag{1.1.37}$$

$$y = cx - c^2 \qquad y = xy' - y'^2 \tag{1.1.38}$$

Verifying the singular solution by substitution

$$xy' = \frac{x^2}{2} \qquad y'^2 = \frac{x^2}{4} \tag{1.1.39}$$

$$y = \frac{x^2}{4} \qquad y = xy' - y'^2 \tag{1.1.40}$$

The parabola happens to have the same derivative for all x as the member of the family of straight lines tangent to it. This makes the parabola a singular solution to the ODE.

17. Given a starting mass of 1 gm, find the time taken to reach a mass of 0.5 gm. This is equal to half-life.

$$\frac{dy}{dt} = -ky \qquad \ln y = -kt + c \tag{1.1.41}$$

$$y = c\, e^{-kt} \qquad y_T = \frac{y_0}{2} \tag{1.1.42}$$

$$e^{-kT} = 1/2 \tag{1.1.43}$$

$$T = \frac{\ln 2}{k} = \frac{ln2}{1.4 \times 10^{-11}} = 1568.89 \text{ years} \tag{1.1.44}$$

18. Given the half life $T = 3.6$ days, in 1 day,

$$y = c\, e^{-kt} \qquad y(t=1) = c\, e^{-k} = y_0 e^{-k} \tag{1.1.45}$$

$$y(t=1) = 1 \text{ g} \times \exp\left(\frac{-\ln 2}{T}\right) \qquad y(t=1) = 0.825 \text{ g} \tag{1.1.46}$$

and for 1 year,

$$y(t=365) = c\, e^{-k} = y_0 e^{-365\ k} \tag{1.1.47}$$

$$y(t=365) = 1 \text{ g} \times \exp\left(\frac{-\ln 2 \times 365}{T}\right) \tag{1.1.48}$$

$$y(t=365) = 0 \text{ g} \tag{1.1.49}$$

19. Given IC is $y(0) = 0$ and $y'(0) = 0$,

$$\frac{d^2y}{dt^2} = g \qquad y = a + bt + \frac{gt^2}{2} \tag{1.1.50}$$

$$y(0) = 0 \qquad \Longrightarrow a = 0 \tag{1.1.51}$$

$$y'(0) = 0 \qquad \Longrightarrow b = 0 \tag{1.1.52}$$

$$y = \frac{gt^2}{2} \tag{1.1.53}$$

20. Given IC is $y(18,000) = 1/2 \times y(0)$ and height t

$$\frac{dy}{dt} = -ky \qquad \ln y = -kt + c \tag{1.1.54}$$

$$y = c\, e^{-kt} \qquad y_T = \frac{y_0}{2} \tag{1.1.55}$$

$$e^{-kT} = 1/2 \qquad k = \frac{\ln 2}{T} = \frac{ln2}{18000}\ \text{ft}^{-1} \tag{1.1.56}$$

$$y(35,000) = y(0) \times 2^{-35/18} \qquad y(35,000) = y(0) \times 0.259 \tag{1.1.57}$$

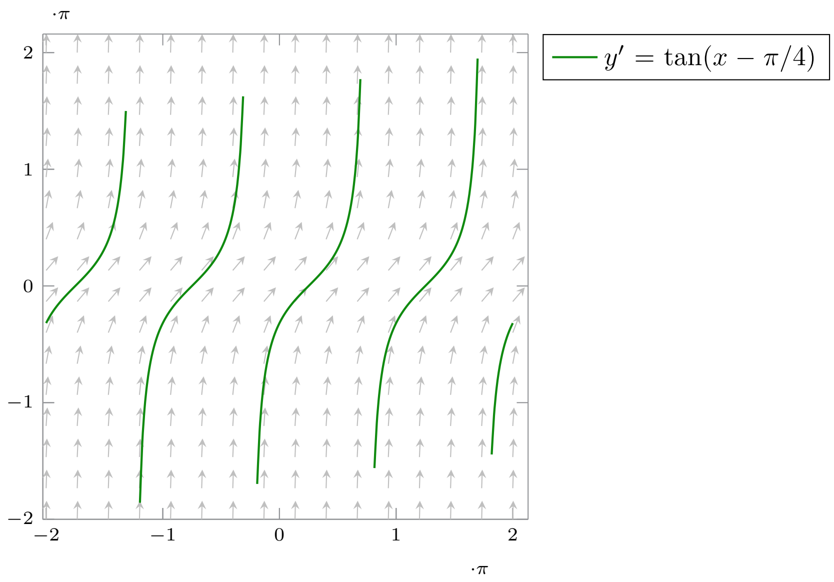
1.2 Geometric Meaning of y' = f(x, y)

1. Plotting direction field and curve passing through $(\pi/4, 0)$

$$y' = 1 + y^2 \qquad \mathrm{d}x = \int \frac{1}{1+y^2}\ \mathrm{d}y \tag{1.2.1}$$

$$x = \arctan y + c \qquad y = \tan(x + c) \tag{1.2.2}$$

$$y(\pi/4) = \tan(\pi/4 + c) = 0 \qquad c = -\pi/4 \tag{1.2.3}$$

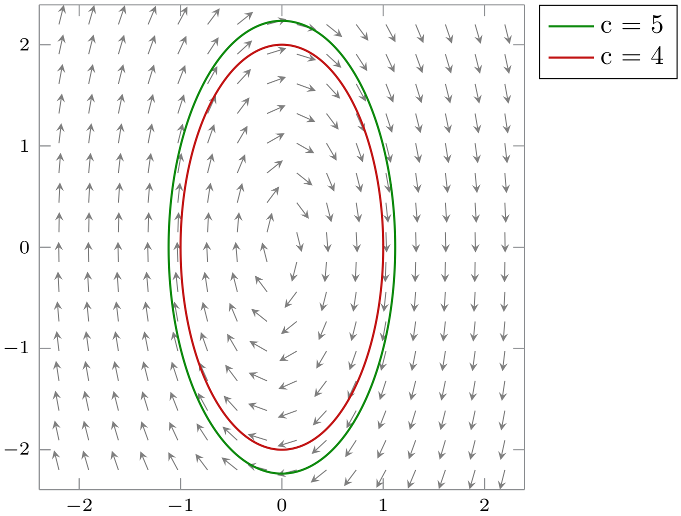


2. Plotting direction field and curve passing through $(1, 1)$ and $(0, 2)$

$$y' = \frac{-4x}{y} \qquad \int -4x \ \ \mathrm{d}x = \int y \ \ \mathrm{d}y \tag{1.2.4}$$

$$\frac{y^2}{2} = -2x^2 + c \qquad y^2 + 4x^2 = c \tag{1.2.5}$$

$$c_1 = 5, \quad c_2 = 4 \tag{1.2.6}$$

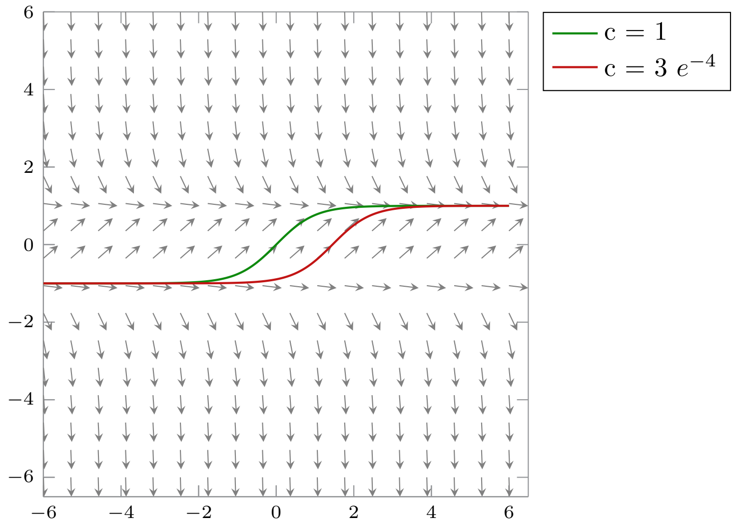


3. Plotting direction field and curve passing through (0, 0) and (2, 1/2)

$$y' = 1 - y^2 \qquad \int \ \mathrm{d}x = \int \frac{1}{1-y^2} \ \ \mathrm{d}y \tag{1.2.7}$$

$$2\int \ \mathrm{d}x = \int \frac{1}{1+y} + \frac{1}{1-y} \ \ \mathrm{d}y \qquad 2x + a = \ln\left(\frac{1+y}{1-y}\right) \tag{1.2.8}$$

$$y = \left(\frac{ce^{2x} - 1}{ce^{2x} + 1}\right) \qquad c_1 = 1, \quad c_2 = \frac{3}{e^4} \tag{1.2.9}$$



4. Plotting direction field and curve passing through (0, 0), (0, 1), (0, 2) and (0, 3)

$$y' = 2y - y^2 \tag{1.2.10}$$

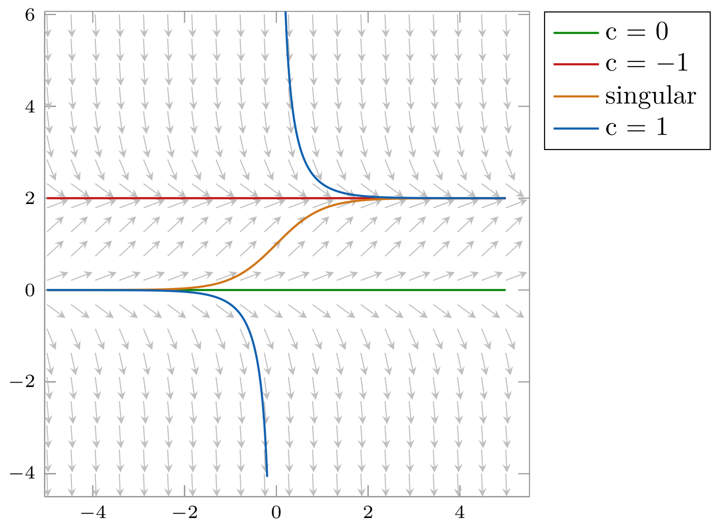
$$\int \; dx = \int \frac{1}{y(2-y)} \; dy \tag{1.2.11}$$

$$\int 2 \; dx = \int \frac{1}{y} - \frac{1}{y-2} \; dy \tag{1.2.12}$$

$$\ln\left(\frac{y}{y-2}\right) = 2x + b \tag{1.2.13}$$

$$y = \frac{2c\, e^{2x}}{c\, e^{2x} - 1} \tag{1.2.14}$$

$$c_1 = 0, \quad c_2 = -1, \quad c_3 = \text{singular}, \quad c_4 = 1 \tag{1.2.15}$$



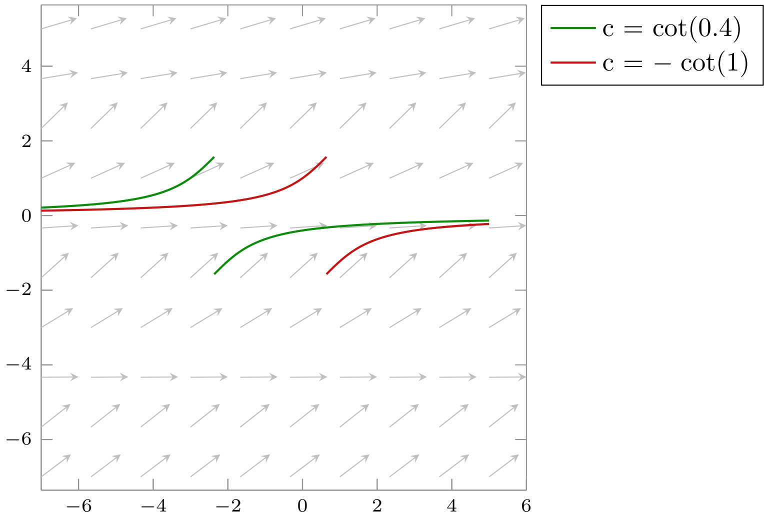
5. Chini's equation

6. Plotting direction field and curve passing through $(0, -0.4)$ and $(0, 1)$

$$y' = \sin^2 y \qquad \int \mathrm{d}x = \int \csc^2 y \ \mathrm{d}y \tag{1.2.16}$$

$$\tan y = \frac{-1}{x+c} \qquad y = \arctan\left(\frac{-1}{x+c}\right) \tag{1.2.17}$$

$$c_1 = \cot(0.4), \quad c_2 = -\cot(1) \tag{1.2.18}$$



7. Plotting direction field and curve passing through $(2, 2)$ and $(3, 3)$, using the substitution $y = vx$

$$y' = e^{y/x} \tag{1.2.19}$$

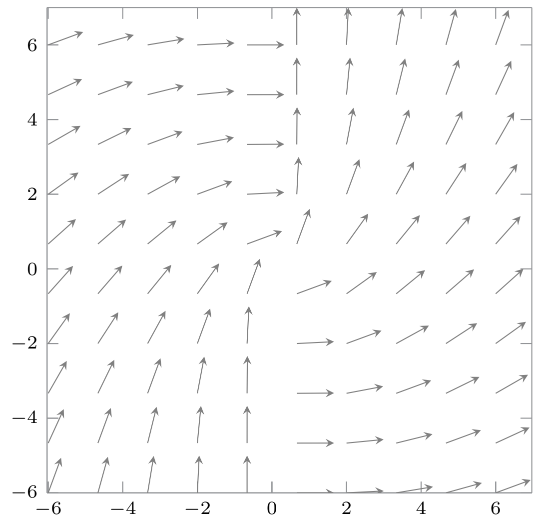
$$y' = e^{v} \qquad dv = -\frac{y}{x^2} \ \mathrm{d}x \tag{1.2.20}$$

$$\int \ \mathrm{d}y = xe^{y/x} - \int \frac{-y \ e^{y/x}}{x} \ \mathrm{d}x \tag{1.2.21}$$

$$y = xe^{y/x} - \int \frac{y \ e^{v}}{v} \ \mathrm{d}v \tag{1.2.22}$$

$$y = xe^{y/x} - y \ \mathrm{Ei}(y/x) + c \tag{1.2.23}$$

$$c_1 = 2\mathrm{Ei}(1) - 2e, \quad c_2 = 3\mathrm{Ei}(1) - 3e \tag{1.2.24}$$

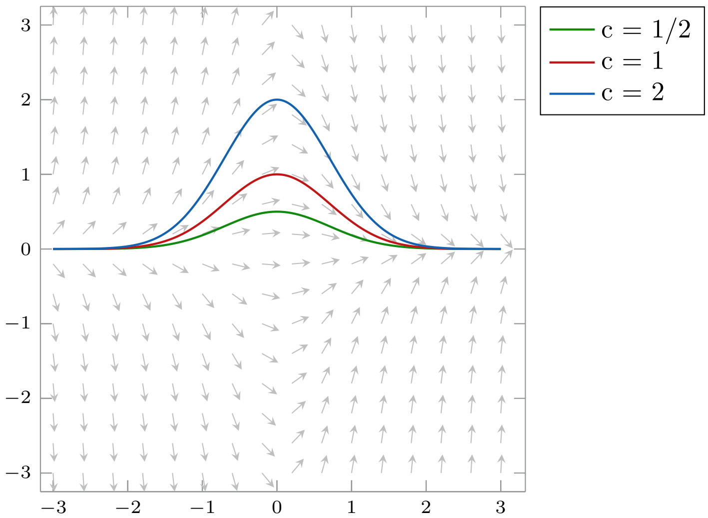


8. Plotting direction field and curve passing through (0, 1/2), (0, 1) and (0, 2)

$$y' = -2xy \qquad \int \frac{1}{y}\ \mathrm{d}y = -2\int x\ \mathrm{d}x \tag{1.2.25}$$

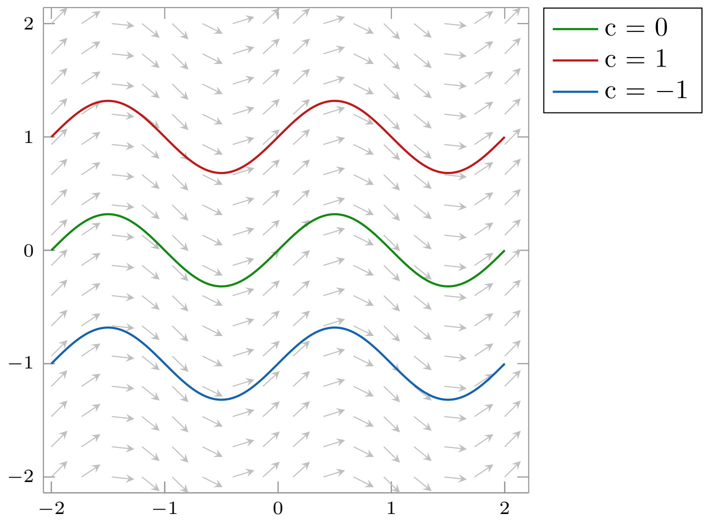
$$\ln y = -x^2 + b \qquad y = c\, e^{-x^2} \tag{1.2.26}$$

$$c_1 = 1/2, \quad c_2 = 1, \quad c_3 = 2 \tag{1.2.27}$$



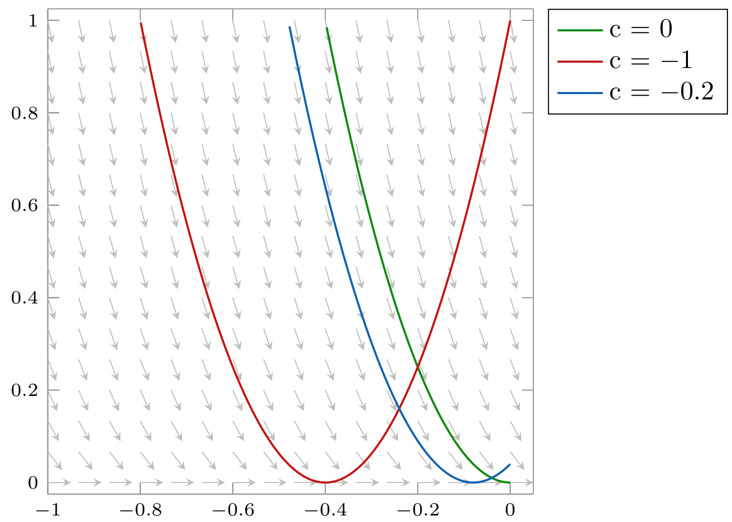
9. Plotting direction field and curve passing through (0, 1/2), (0, 1) and (0, 2)

$$y' = \cos(\pi x) \qquad y = \frac{1}{\pi}\ \sin(\pi x) + c \tag{1.2.28}$$



10. Plotting direction field and curve passing through (0, 1/2), (0, 1) and (0, 2)

$$y' = -5y^{1/2} \qquad \sqrt{y} = -2.5x + c \tag{1.2.29}$$



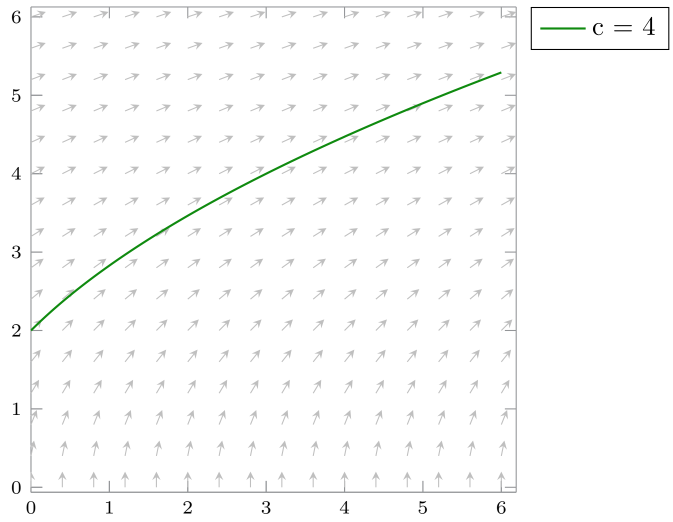
11. Isoclines with an ODE of the form $y' = f(y)$ will be of the form $f(y) = c$. These are straight lines parallel to the x axis.

12. Plotting direction field and curve passing through (0, 2)

$$vy = 2 \qquad \int y \ \mathrm{d}y = 2\int \ \mathrm{d}t \tag{1.2.30}$$

$$y^2 = 4t + c \qquad y(0) = \sqrt{c} = 2 \tag{1.2.31}$$

$$y = \sqrt{4t + 4} \tag{1.2.32}$$

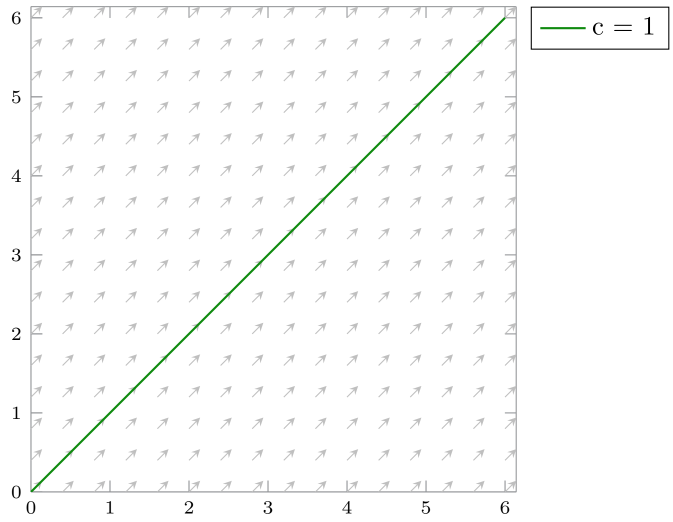


13. Plotting direction field and curve passing through (1, 1)

$$y = vt \qquad \int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{t}\ \mathrm{d}t \tag{1.2.33}$$

$$\ln y = \ln t + b \qquad y = ct \tag{1.2.34}$$

$$c = 1 \tag{1.2.35}$$

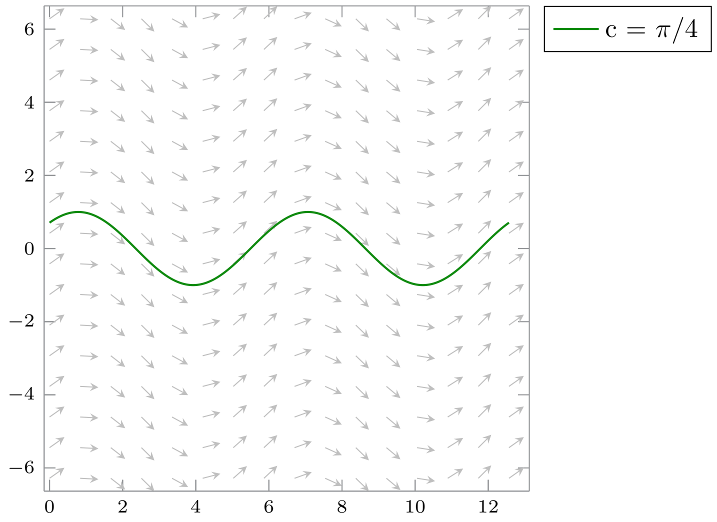


14. Plotting direction field and curve passing through $(0, 1/\sqrt{2})$

$$y^2 + v^2 = 1 \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{1 - y^2} \tag{1.2.36}$$

$$\int \frac{1}{\sqrt{1 - y^2}} \ \mathrm{d}y = \int \ \mathrm{d}t \qquad \arcsin y = t + c \tag{1.2.37}$$

$$y = \sin(t + c) \qquad c = \frac{\pi}{4} \tag{1.2.38}$$



15. Plotting direction field given $m = k = 1$ and $v_0 = 10$ and drag proportional to v^2. Terminal velocity is $v^T = \sqrt{g} = 3.13\ m/s^2$

$$my'' = mv' = mg - kv^2 \tag{1.2.39}$$

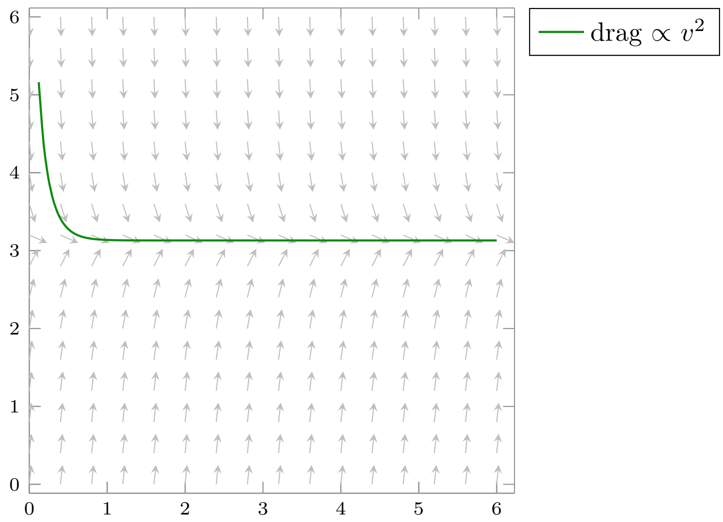
$$\int \frac{1}{g - v^2} \ \mathrm{d}v = \int \ \mathrm{d}t \tag{1.2.40}$$

$$\frac{1}{2\sqrt{g}} \int \frac{1}{\sqrt{g} - v} + \frac{1}{\sqrt{g} + v} \ \mathrm{d}v = \int \ \mathrm{d}t \tag{1.2.41}$$

$$\ln\left(\frac{v + \sqrt{g}}{v - \sqrt{g}}\right) = 2\sqrt{g}t + b \tag{1.2.42}$$

$$v = \sqrt{g}\, \frac{c\ \exp(2\sqrt{g}t) + 1}{c\ \exp(2\sqrt{g}t) - 1} \tag{1.2.43}$$

$$c = \frac{10 + \sqrt{g}}{10 - \sqrt{g}} \tag{1.2.44}$$



Plotting direction field given $m = k = 1$ and $v_0 = 10$ and drag proportional to v.
Terminal velocity is $v^T = g = 9.8\ m/s^2$

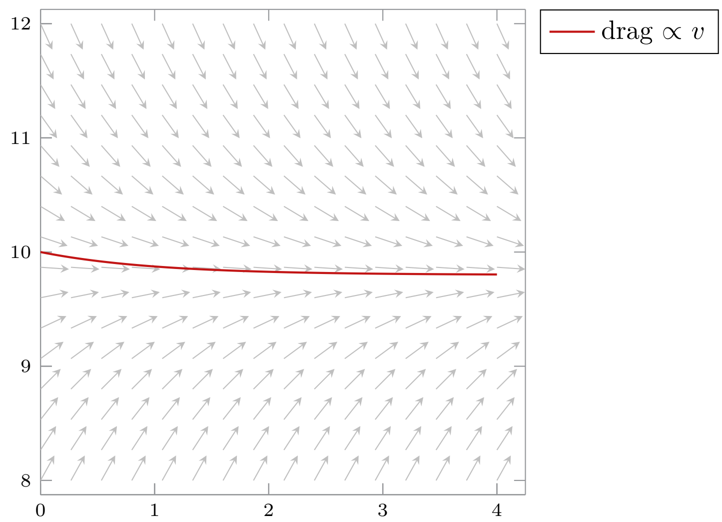
$$my'' = mv' = mg - kv \tag{1.2.45}$$

$$\int \frac{1}{g - v}\ \mathrm{d}v = \int\ \mathrm{d}t \tag{1.2.46}$$

$$\ln\left(\frac{1}{v - g}\right) = t + b \tag{1.2.47}$$

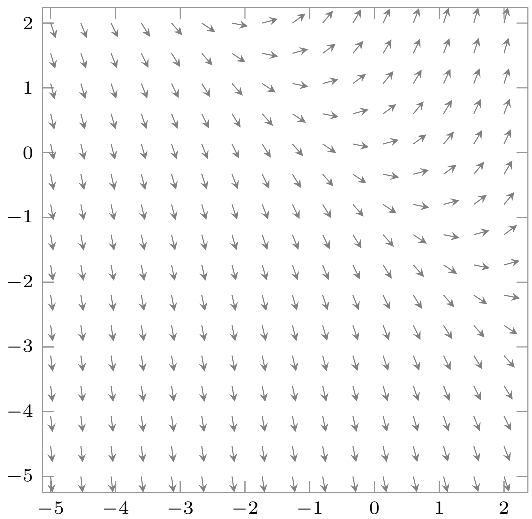
$$v = g + ce^{-t} \tag{1.2.48}$$

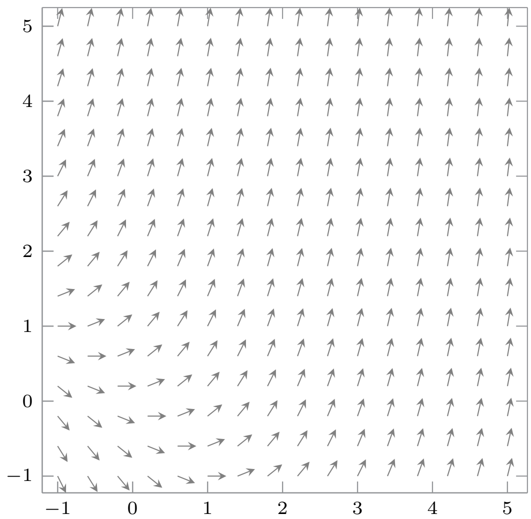
$$c = v_0 - g = 0.2 \tag{1.2.49}$$



16. CAS Project using the ODE $y' = y + x$

(a) Zooming in the regions $x \in [-5, 2]$ shows the upper part of the solutions which increase for large x
Zooming in the regions $y \in [-1, 5]$ shows the lower part of the solutions which decreases for large x, as well as the straight line unstable equilibrium solution





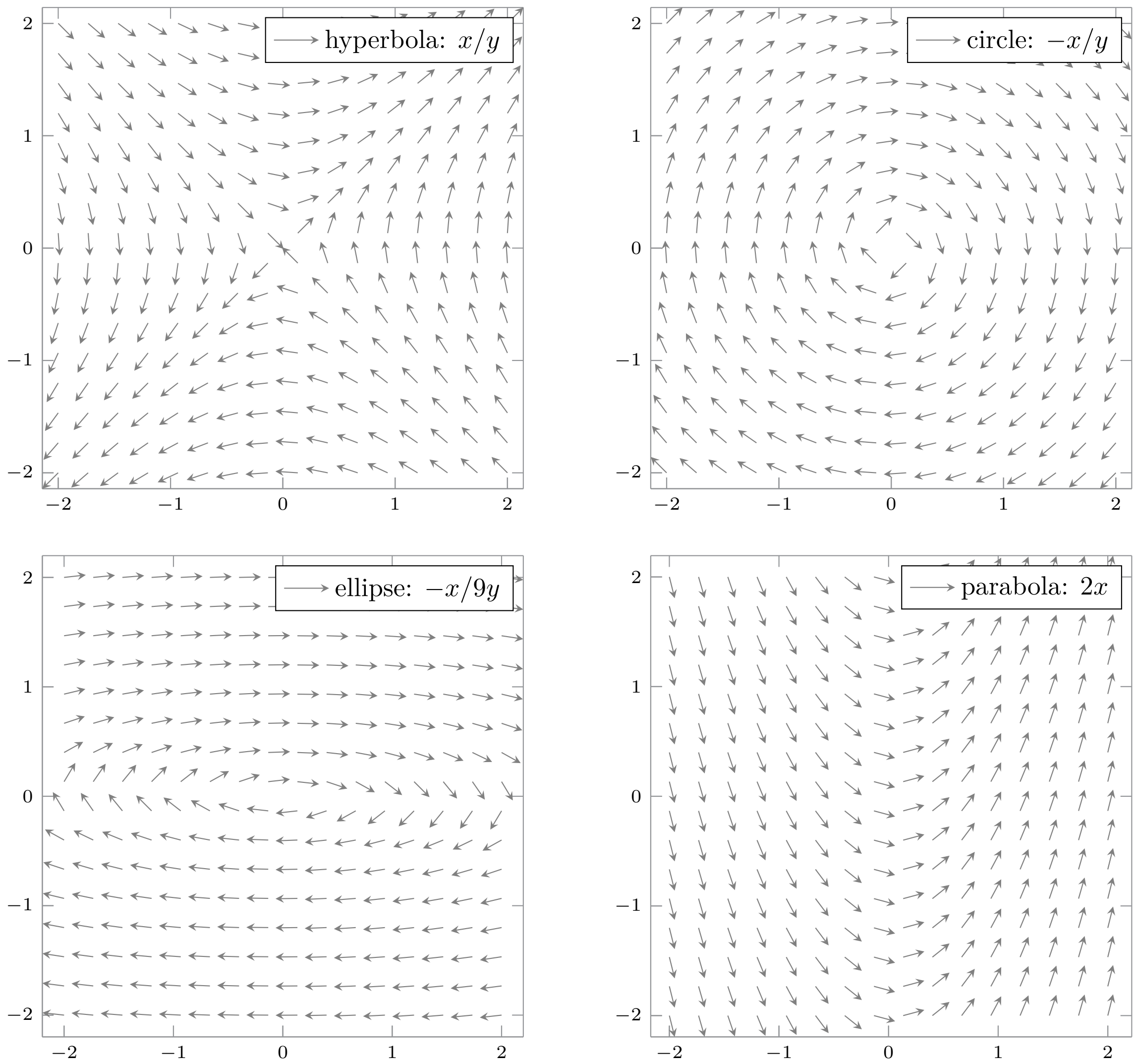
(b) Implicit differentiation gives the ODE

$$x^2 + 9y^2 = c \quad (y > 0) \tag{1.2.50}$$

$$x + 9yy' = 0 \tag{1.2.51}$$

$$y' = \frac{-x}{9y} \quad (y > 0) \tag{1.2.52}$$

The sign of the RHS determine whether the direction field is an ellipse or a hyperbola. A special case of the RHS being $-x/y$ gives a circle (a special ellipse)



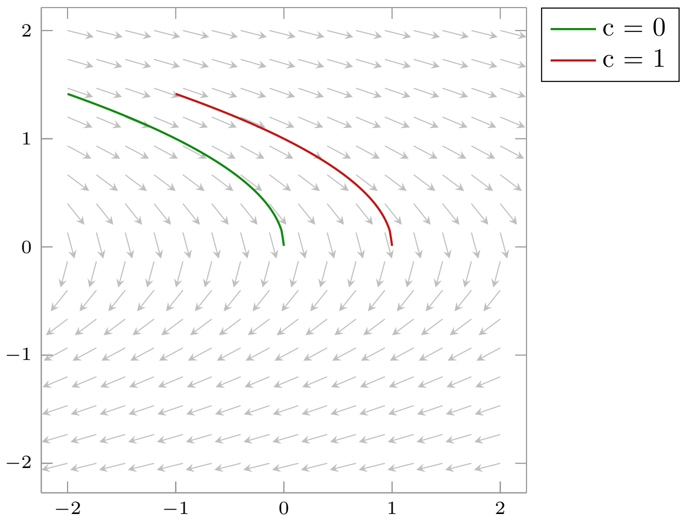
(c) From the figure above, $y' = -x/y$ produces a circle

(d) For the ODE $y' = -y/2$

$$y' = \frac{-1}{2y} \tag{1.2.53}$$

$$y^2 = -x + c \tag{1.2.54}$$

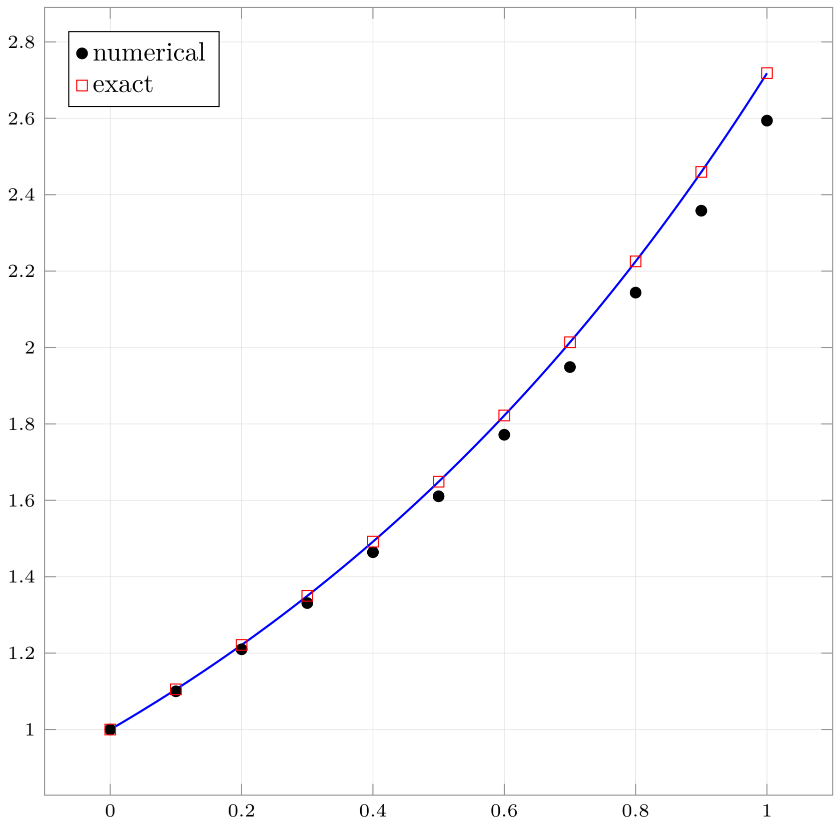
For $y > 0$, the solutions decrease because the ODE is monotonically negative for all $y > 0$.



17. Euler's method with $y(0) = 1$ and step size $h = 0.1$

$$y' = y \qquad y = ce^x \tag{1.2.55}$$

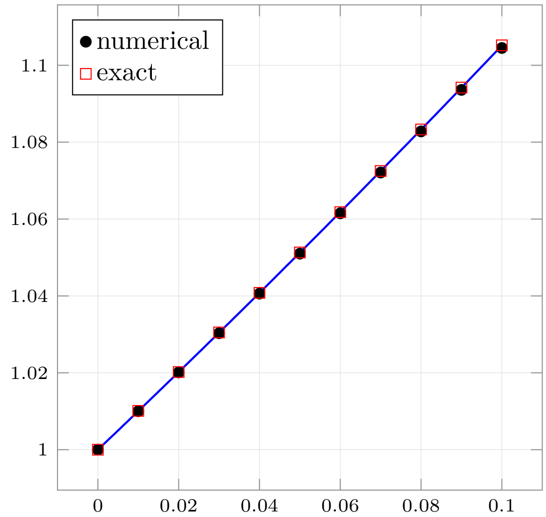
$$c = 1 \tag{1.2.56}$$



18. Euler's method with $y(0) = 1$ and step size $h = 0.01$

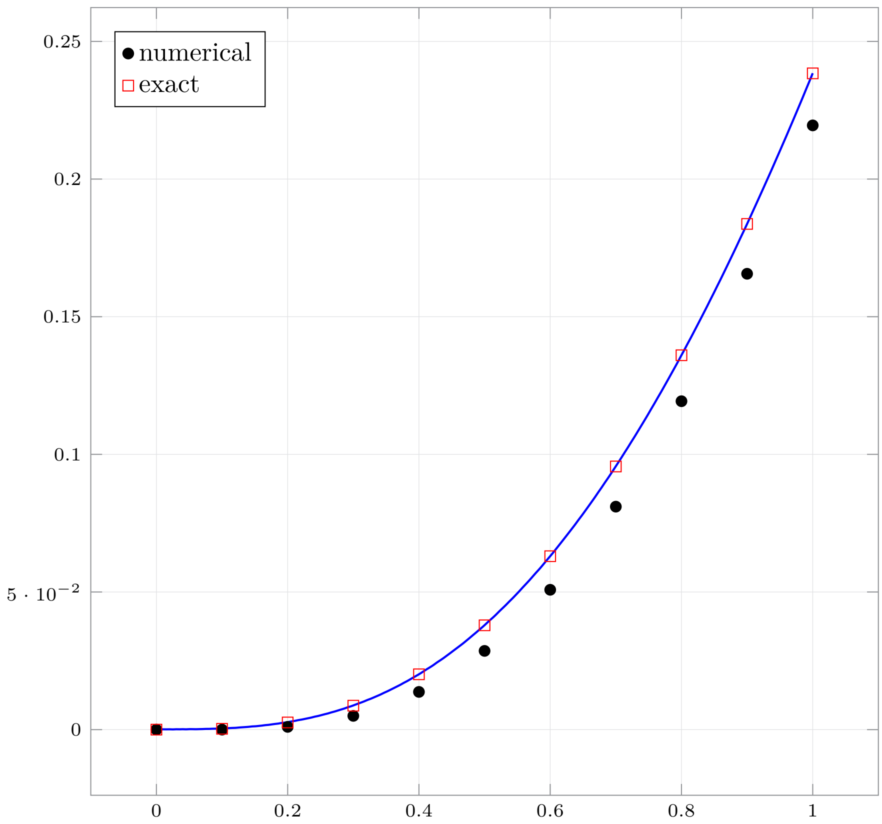
$$y' = y \qquad\qquad y = ce^x \tag{1.2.57}$$

$$c = 1 \tag{1.2.58}$$



19. Euler's method with $y(0) = 0$ and step size $h = 0.1$

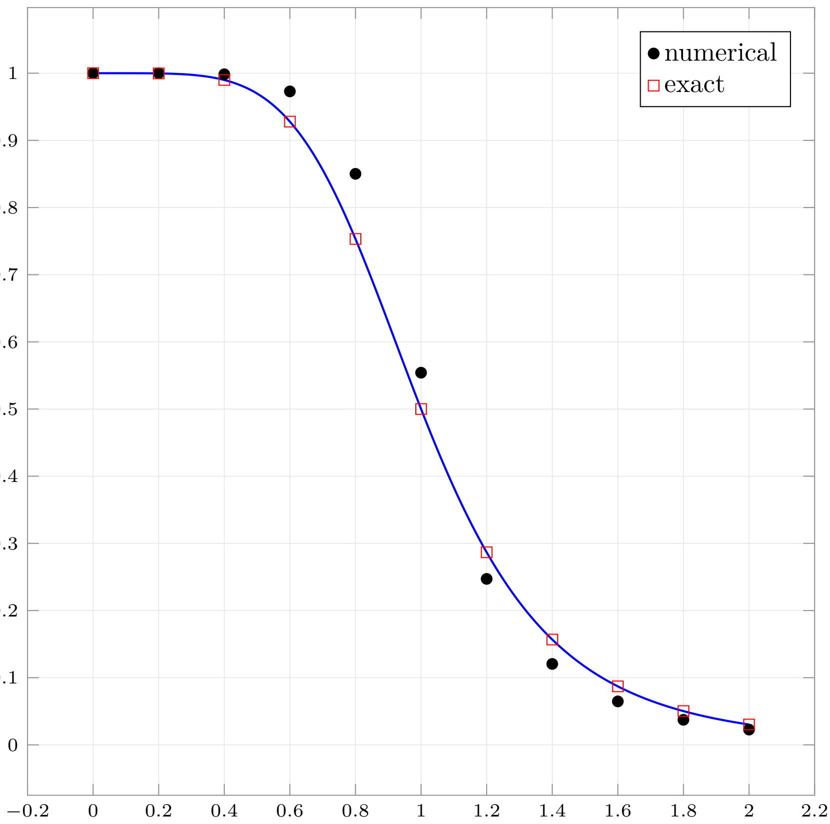
$$y' = (y - x)^2 \qquad\qquad y = x - \tanh x \tag{1.2.59}$$



20. Euler's method with $y(0) = 1$ and step size $h = 0.2$

$$y' = -5x^4y^2 \qquad y = \frac{1}{(c + x^5)} \tag{1.2.60}$$

$$c = 1 \tag{1.2.61}$$



1.3 Separable ODEs. Modeling

1. Adding the constant of integration later might result in a vastly different general solution which is almost always wrong.

2. Using $u = y/x$. Then substituting $(1 + u^4) = m$

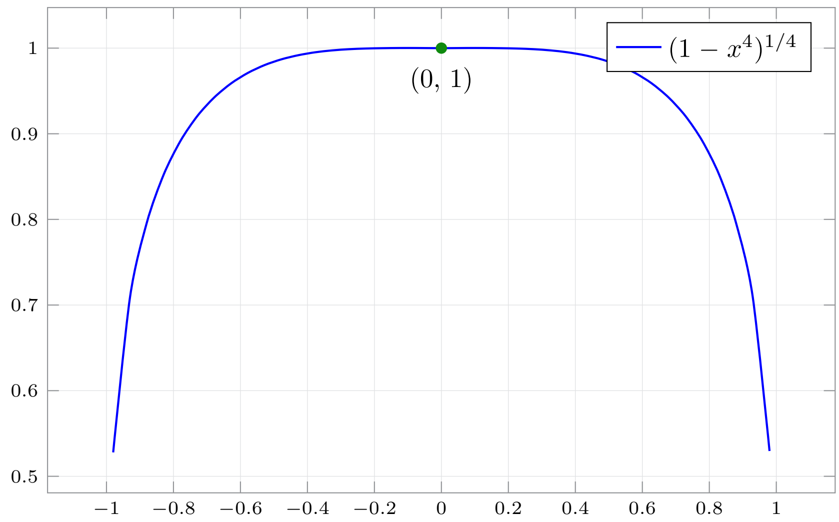
$$y' = \frac{-x^3}{y^3} = \frac{-1}{u^3} \tag{1.3.1}$$

$$\frac{\mathrm{d}u}{f(u) - u} = \frac{\mathrm{d}x}{x} = \frac{-u^3 \ \mathrm{d}u}{1 + u^4} \tag{1.3.2}$$

$$\ln(|x|) + b = \int \frac{-\,\mathrm{d}m}{4m} = \frac{-1}{4} \ln m \tag{1.3.3}$$

$$1 + \left(\frac{y}{x}\right)^4 = \frac{c}{x^4} \tag{1.3.4}$$

$$y^4 = c - x^4 \tag{1.3.5}$$



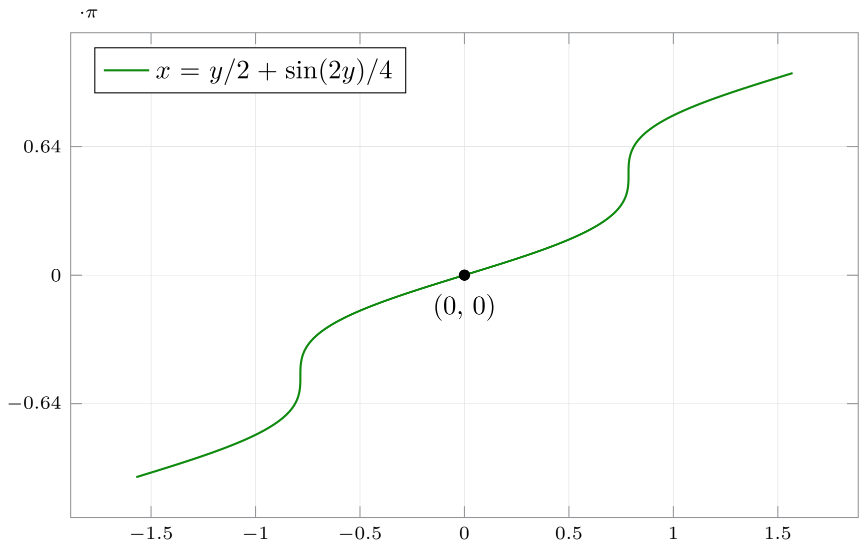
Checking by differentiation and substitution,

$$4y^3 \ \ \mathrm{d}y = -4x^3 dx \qquad\qquad y^3 \ y' + x^3 = 0 \tag{1.3.6}$$

3. Separating variables,

$$\int \cos^2 y \ \ \mathrm{d}y = \int \ \ \mathrm{d}x \qquad\qquad \int \frac{1}{2} + \frac{\cos(2y)}{2} \ \ \mathrm{d}y = \int \ \ \mathrm{d}x \tag{1.3.7}$$

$$\frac{y}{2} + \frac{\sin(2y)}{4} = x + c \tag{1.3.8}$$



Checking by differentiation and substitution,

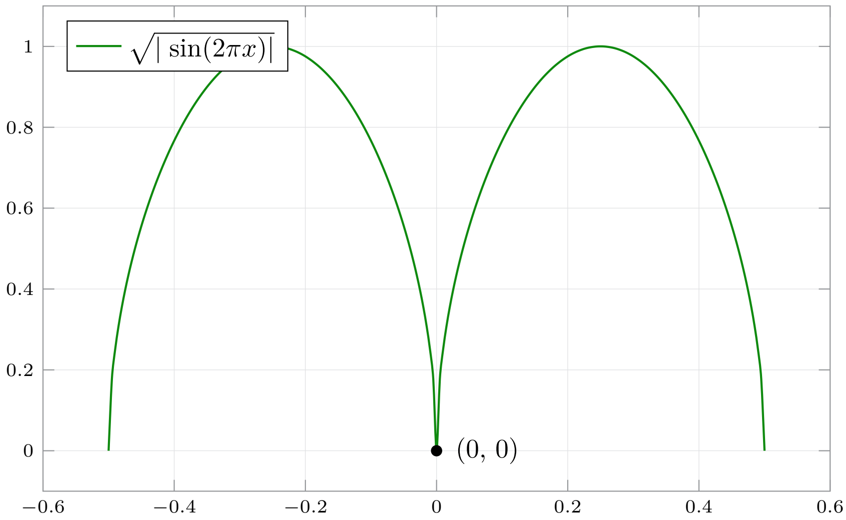
$$dx = \frac{(1 + \cos(2y)) \ dy}{2} \qquad\qquad y' = \sec^2 y \tag{1.3.9}$$

4. Separating variables, with $u = \sin(2\pi x)$

$$\int \frac{1}{y}\ \mathrm{d}y = \pi \int \frac{\cos(2\pi x)}{\sin(2\pi x)}\ \mathrm{d}x \qquad \mathrm{d}u = 2\pi\ \cos(2\pi x)\ \ \mathrm{d}x \tag{1.3.10}$$

$$\int \frac{2}{y}\ \mathrm{d}y = \int \frac{1}{u}\ \mathrm{d}u \qquad 2\ \ln|y| = \ln|u| + b \tag{1.3.11}$$

$$y^2 = c\ |\sin(2\pi x)| \tag{1.3.12}$$

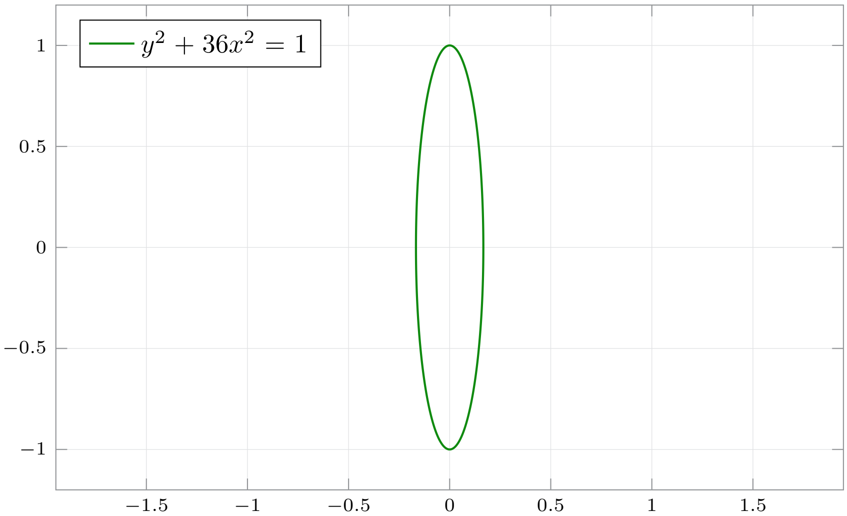


Checking by differentiation and substitution,

$$y' = \begin{cases} \dfrac{\pi\ \ \cos(2\pi x)}{\sqrt{\sin(2\pi x)}} = \dfrac{\pi y\ \ \cos(2\pi x)}{\sin(2\pi x)} & \text{for} \quad x >= 0 \\[2ex] \dfrac{-\pi\ \ \cos(-2\pi x)}{\sqrt{\sin(-2\pi x)}} = \dfrac{-\pi y\ \ \cos(2\pi x)}{\sin(-2\pi x)} & \text{for} \quad x < 0 \end{cases} \tag{1.3.13}$$

5. Separating variables,

$$\int y\ \ \mathrm{d}y = -36 \int x\ \ \mathrm{d}x \qquad y^2 + 36x^2 = c \tag{1.3.14}$$



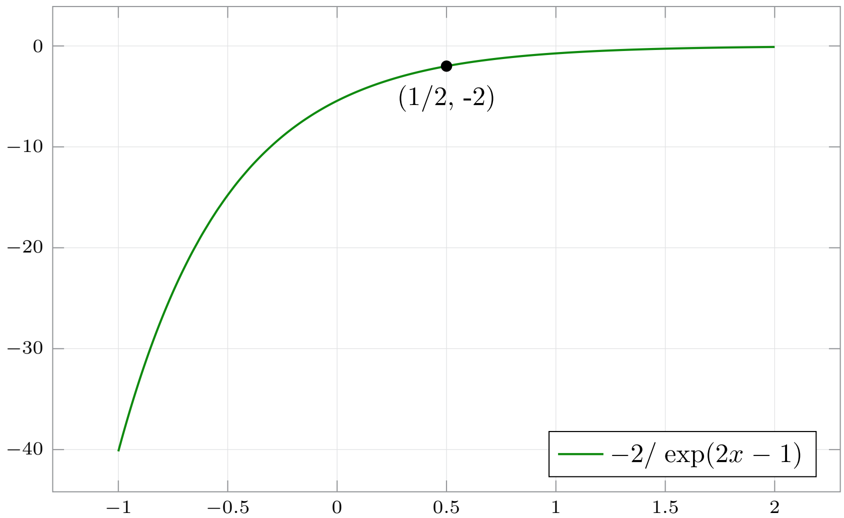
Checking by differentiation and substitution,

$$2y\ y' = -72x\ \ \mathrm{d}x \qquad\qquad y\ y' + 36x = 0 \tag{1.3.15}$$

6. Separating variables,

$$\int \frac{1}{y^2}\ \ \mathrm{d}y = \int \exp(2x-1)\ \ \mathrm{d}x \qquad\qquad \frac{-1}{y} = \frac{\exp(2x-1)}{2} + b \tag{1.3.16}$$

$$y = \frac{-2}{c + e^{(2x-1)}} \tag{1.3.17}$$



Checking by differentiation and substitution,

$$y' = \frac{4\ e^{(2x-1)}}{(c + e^{(2x-1)})^2} \qquad\qquad y' = y^2\ e^{(2x-1)} \tag{1.3.18}$$

7. Using $u = y/x$.

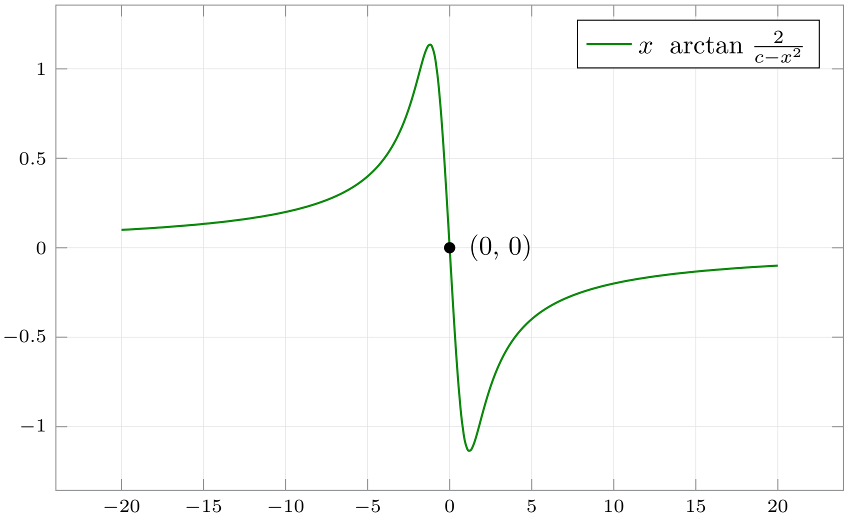
$$y' = u + \frac{2y^2}{u^2} \sin^2 u \tag{1.3.19}$$

$$x\ du + u\ \ \mathrm{d}x = (u + x^2 \sin^2 u)\ \ \mathrm{d}x \tag{1.3.20}$$

$$\int \frac{1}{\sin^2 u}\ du = \int x\ \ \mathrm{d}x \tag{1.3.21}$$

$$-\cot u = \frac{x^2}{2} + b \tag{1.3.22}$$

$$y = x\ \ \arctan\left(\frac{2}{c - x^2}\right) \tag{1.3.23}$$



Checking by differentiation and substitution,

$$y' = \arctan\left(\frac{2}{c - x^2}\right) + \frac{x\ (c - x^2)^2}{4 + (c - x^2)^2} \times \frac{4x}{(c - x^2)^2} \tag{1.3.24}$$

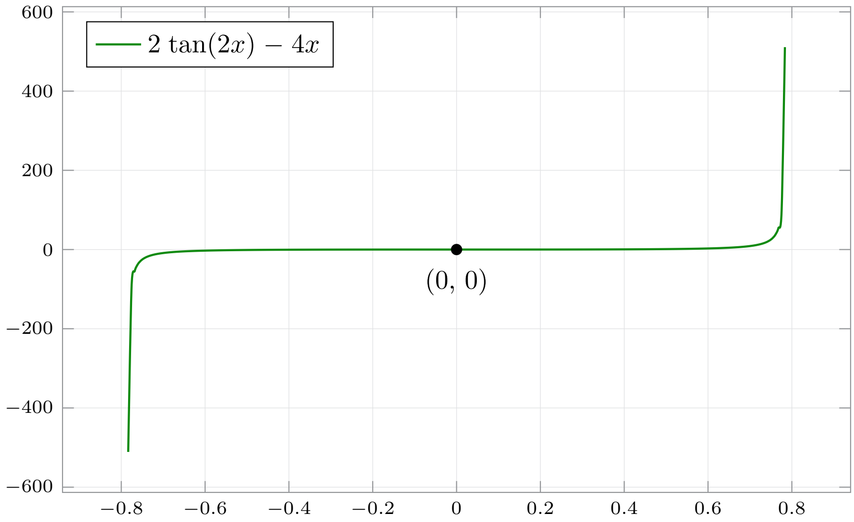
$$y' = \frac{y}{x} + x^2\ \frac{4}{4 + (c - x^2)^2} \tag{1.3.25}$$

$$y' = \frac{y}{x} + x^2\ \ \sin^2\left[\arctan\left(\frac{2}{c - x^2}\right)\right] = \frac{y}{x} + x^2\ \ \sin^2\left(\frac{y}{x}\right) \tag{1.3.26}$$

8. Using $u = y + 4x$.

$$u' - 4 = u^2 \qquad\qquad \int \frac{1}{4 + u^2}\ du = \int\ \ \mathrm{d}x \tag{1.3.27}$$

$$\frac{1}{2}\ \arctan\left(\frac{u}{2}\right) = x + b \qquad\qquad y = 2\tan(2x + c) - 4x \tag{1.3.28}$$



Checking by differentiation and substitution,

$$y' = -4 + \frac{4}{\cos^2(2x+c)} \tag{1.3.29}$$

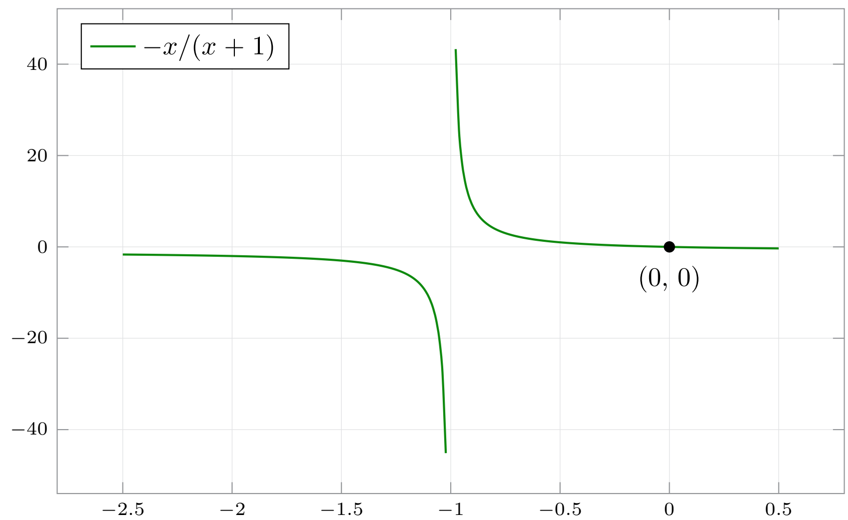
$$y' = 4\frac{\sin^2(2x+c)}{\cos^2(2x+c)} = (2\tan(2x+c))^2 \tag{1.3.30}$$

$$y' = (y+4x)^2 \tag{1.3.31}$$

9. Using $u = y/x$.

$$y' = xu^2 + u = u + xu' \qquad \int \frac{1}{u^2}\, du = \int \; dx \tag{1.3.32}$$

$$\frac{-1}{u} = x + b = \frac{-x}{y} \qquad y = \frac{-x}{x+c} = \frac{-1}{1+c/x} \tag{1.3.33}$$



Checking by differentiation and substitution,

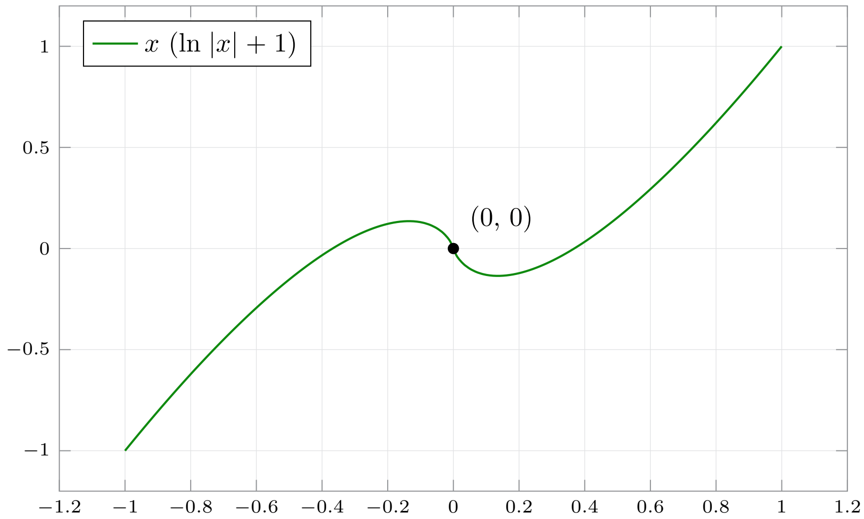
$$xy' = \frac{-cx}{(x+c)^2} \qquad y + y^2 = \frac{-x^2 + x^2 - cx}{(x+c)^2} \tag{1.3.34}$$

$$xy' = y + y^2 \tag{1.3.35}$$

10. Using $u = y/x$.

$$y' = 1 + u = u + xu' \qquad \int du = \int \frac{1}{x} dx \tag{1.3.36}$$

$$u = \ln|x| + b \qquad y = x\ln|x| + cx \tag{1.3.37}$$



Checking by differentiation and substitution,

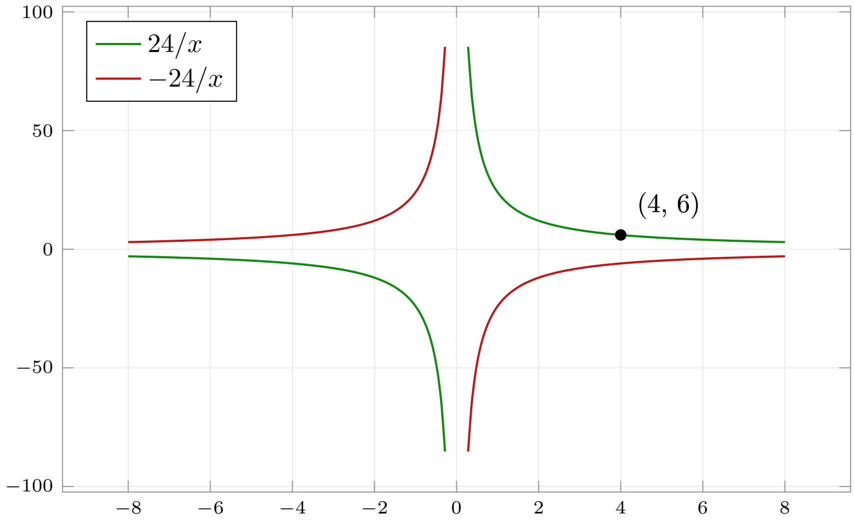
$$xy' = xc + x\ln|x| + x \qquad x + y = xy' \tag{1.3.38}$$

11. Given IC is $y(4) = 6$

$$y' = \frac{-y}{x} \qquad \int \frac{-1}{y} dy = \int \frac{1}{x} dx \tag{1.3.39}$$

$$\ln|y| = -\ln|x| - b \qquad |y| = \frac{c}{|x|} \tag{1.3.40}$$

$$c = 24 \qquad \implies y = \frac{24}{x} \tag{1.3.41}$$



Checking by differentiation and substitution,

$$y = \frac{24}{x} \qquad y' = \frac{-24}{x^2} \tag{1.3.42}$$

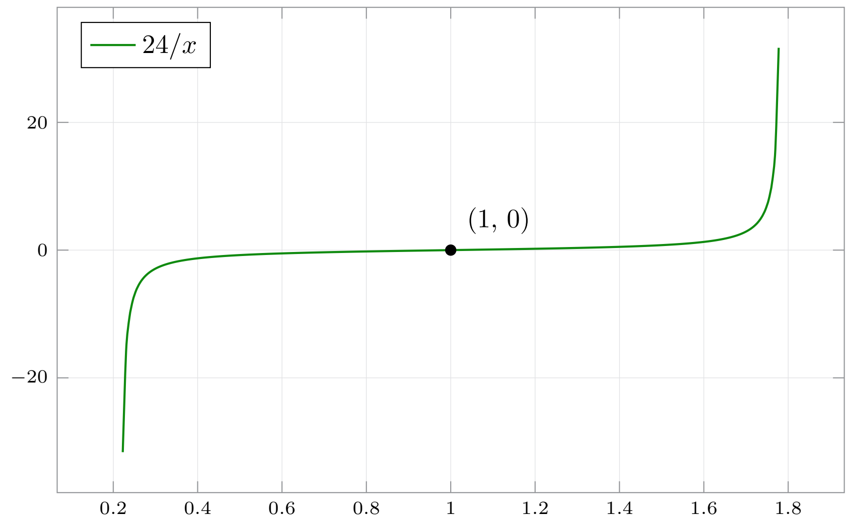
$$\frac{-y}{x} = \frac{-24}{x^2} \tag{1.3.43}$$

12. Given IC is $y(1) = 0$

$$y' = 1 + 4y^2 \qquad \int \frac{1}{y^2 + 1/4} \ \mathrm{d}y = \int 4 \ \mathrm{d}x \tag{1.3.44}$$

$$2\arctan(2y) = 4x + c \qquad c = -4 \tag{1.3.45}$$

$$y = \frac{\tan(2x-2)}{2} \tag{1.3.46}$$



Checking by differentiation and substitution,

$$1 + 4y^2 = 1 + \tan^2(2x - 2) = \sec^2(2x - 2) \tag{1.3.47}$$

$$y' = \sec^2(2x - 2) \tag{1.3.48}$$

13. Given IC is $y(0) = \pi/2$

$$\int \frac{1}{\sin^2 y} \ \mathrm{d}y = \int \frac{1}{\cosh^2 x} \ \mathrm{d}x \tag{1.3.49}$$

$$-\cot y = \tanh x + c \tag{1.3.50}$$

$$y = \arctan\left(\frac{-1}{\tanh x + c}\right) \tag{1.3.51}$$

$$c = 0 \tag{1.3.52}$$

$$y = \arctan\left(\frac{-1}{\tanh x}\right) \tag{1.3.53}$$

Checking by differentiation and substitution,

$$y' = \frac{\tanh^2 x}{1 + \tanh^2 x} \times \frac{1}{\tanh^2 x} \times \frac{1}{\cosh^2 x} \tag{1.3.54}$$

$$y' \cosh^2(x) = \frac{1}{1 + \tanh^2 x} \tag{1.3.55}$$

$$= \frac{(-1/\tanh x)^2}{1 + (-1/\tanh x)^2} = \sin^2\left[\arctan\left(\frac{-1}{\tanh x}\right)\right] \tag{1.3.56}$$

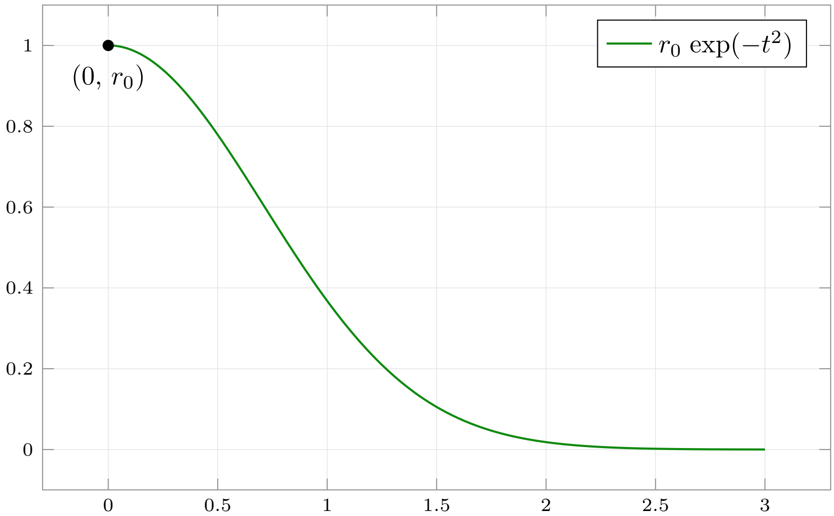
$$= \sin^2 y \tag{1.3.57}$$

14. Given IC is $r(0) = r_0$

$$\mathrm{d}r = -2tr \ \mathrm{d}t \qquad \int \frac{1}{r} \ \mathrm{d}r = \int -2t \ \mathrm{d}t \tag{1.3.58}$$

$$\ln|r| = -t^2 + b \qquad |r| = c\exp(-t^2) \tag{1.3.59}$$

$$c = r_0 \qquad r = r_0 \exp(-t^2) \tag{1.3.60}$$



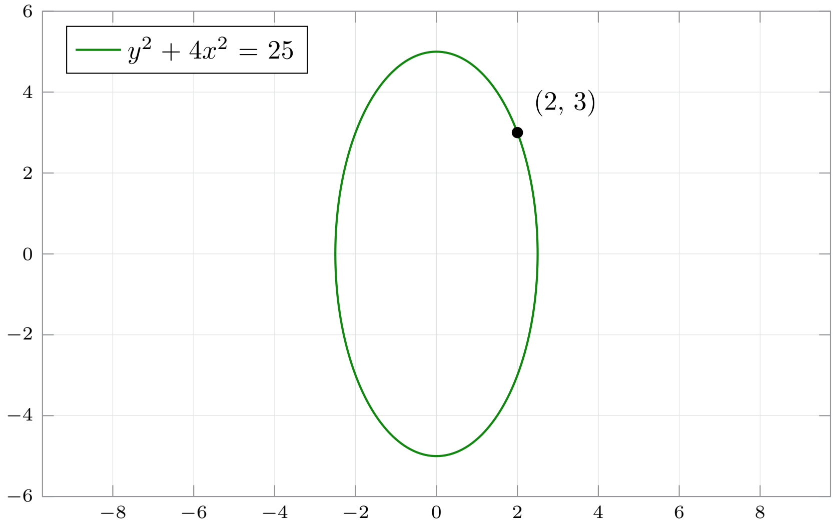
Checking by differentiation and substitution,

$$r' = c\,\exp(-t^2) \times (-2t) \qquad -2tr = (-2t) \times c\,\exp(-2t^2) \tag{1.3.61}$$

15. Given IC is $y(2) = 3$

$$\int y \;\mathrm{d}y = \int -4x \;\mathrm{d}x \qquad \frac{y^2}{2} = -2x^2 + b \tag{1.3.62}$$

$$y^2 + 4x^2 = c \qquad c = 25 \tag{1.3.63}$$



Checking by differentiation and substitution,

$$2yy' + 8x = 0 \qquad y' = \frac{-4x}{y} \tag{1.3.64}$$

16. Given IC is $y(0) = 2$, and substituting $u = x + y - 2$

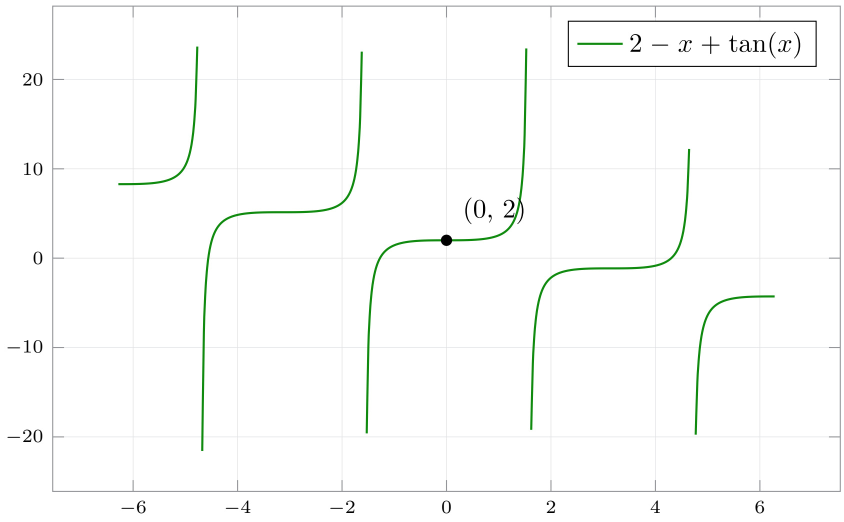
$$y' = u^2 = u' - 1 \tag{1.3.65}$$

$$\int \frac{1}{1+u^2} \ \mathrm{d}u = \int \ \mathrm{d}x \tag{1.3.66}$$

$$\arctan(u) = x + c \tag{1.3.67}$$

$$y = 2 - x + \tan(x + c) \tag{1.3.68}$$

$$c = 0 \tag{1.3.69}$$



Checking by differentiation and substitution,

$$y' = -1 + \sec^2(x + c) = \tan^2(x + c) \tag{1.3.70}$$

$$(x + y - 2)^2 = \tan^2(x + c) \tag{1.3.71}$$

17. Given IC is $y(1) = 0$, and substituting $u = y/x$

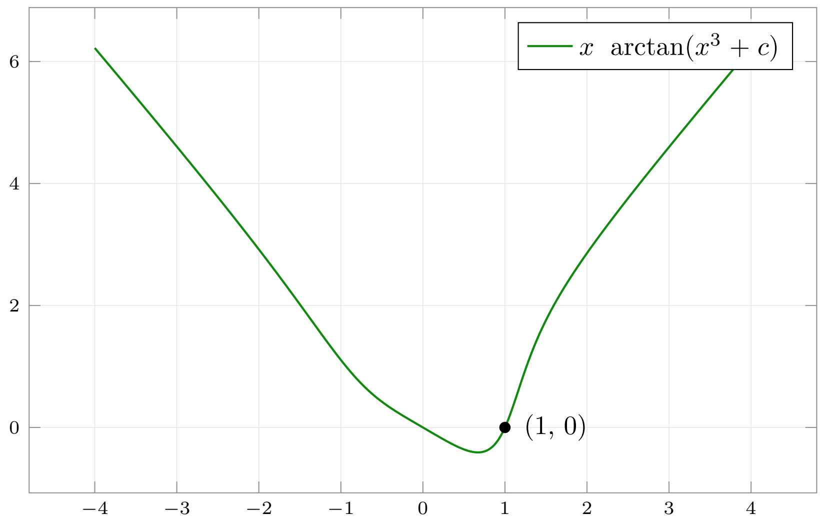
$$y' = u + 3x^3 \cos^2 u = u + xu' \tag{1.3.72}$$

$$\int \sec^2 u \ \mathrm{d}u = \int 3x^2 \ \mathrm{d}x \tag{1.3.73}$$

$$\tan u = x^3 + c \tag{1.3.74}$$

$$y = x \ \arctan(x^3 + c) \tag{1.3.75}$$

$$c = -1 \tag{1.3.76}$$



Checking by differentiation and substitution,

$$y' = \arctan(x^3 + c) + \frac{3x^3}{1 + (x^3 + c)^2} \tag{1.3.77}$$

$$xy' - y = \frac{3x^4}{1 + (x^3 + c)^2} \tag{1.3.78}$$

$$3x^4 \times \cos^2(y/x) = 3x^4 \times \frac{1}{1 + (x^3 + c)^2} \tag{1.3.79}$$

18. Introducing limits into the equation

$$\int g(y) \ \mathrm{d}y = \int f(x) \ \mathrm{d}x + c \tag{1.3.80}$$

$$y(x_0) = y_0 \tag{1.3.81}$$

$$\int_{x_0}^{x} f(x) \ \mathrm{d}x + c = \int_{x_0}^{x} g(y)y' \ \mathrm{d}x \tag{1.3.82}$$

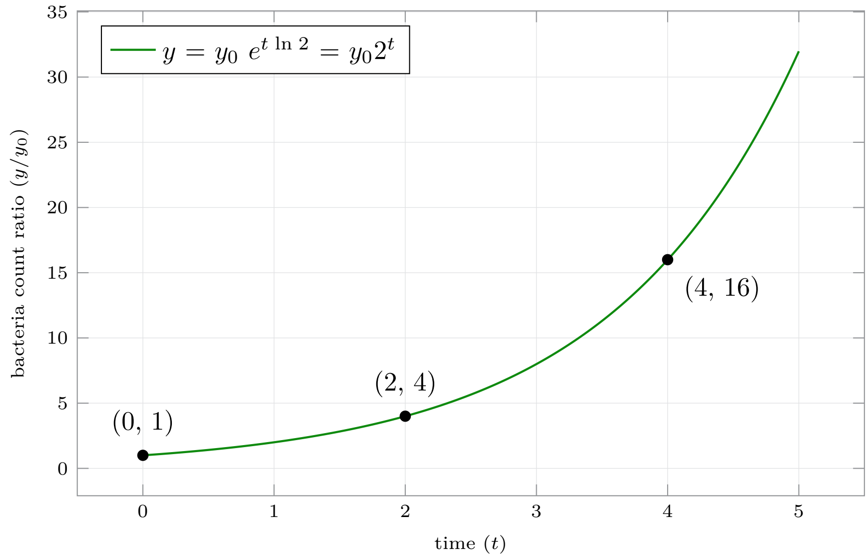
$$= \int_{y(x_0)}^{y(x)} g(y) \ \mathrm{d}y \tag{1.3.83}$$

$$= \int_{y_0}^{y} g(y) \ dy \tag{1.3.84}$$

19. Given IC is $y(t = 0) = y_0$ where t is the time in weeks, and y is the number of bacteria.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky \qquad \ln |y| = kt + b \tag{1.3.85}$$

$$y = ce^{kt} \qquad y > 0 \qquad c = y_0 \tag{1.3.86}$$



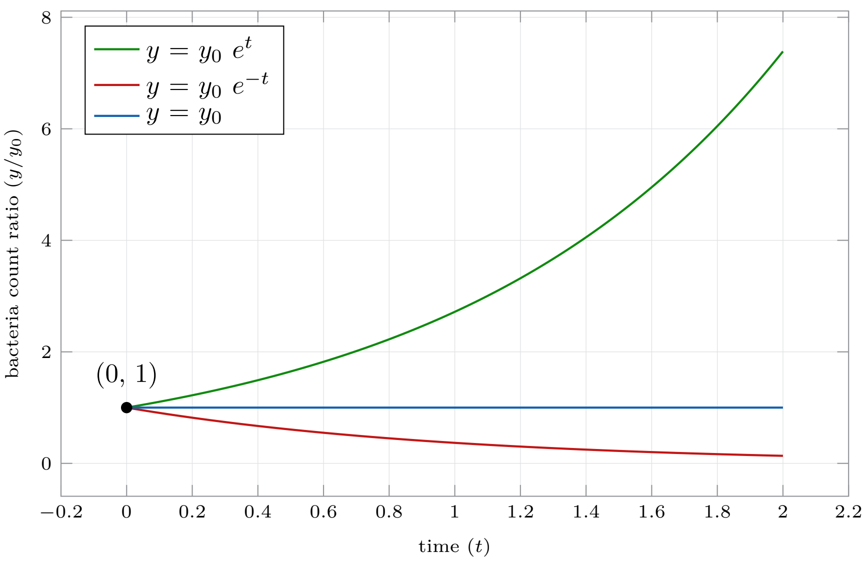
Inserting into ODE solution,

$$y(t=2) = 2y_0 \qquad\qquad y(t=4) = 16y_0 \tag{1.3.87}$$

20. Given IC is $y(t=0) = y_0$ where t is the time in weeks, and y is the number of bacteria, b is the birth rate and k is the death rate

$$\frac{\mathrm{d}y}{\mathrm{d}t} = by - ky \qquad\qquad \ln|y| = (b-k)t + c \tag{1.3.88}$$

$$y = ce^{(b-k)t} \qquad y > 0 \qquad\qquad c = y_0 \tag{1.3.89}$$



Interpreting the results,

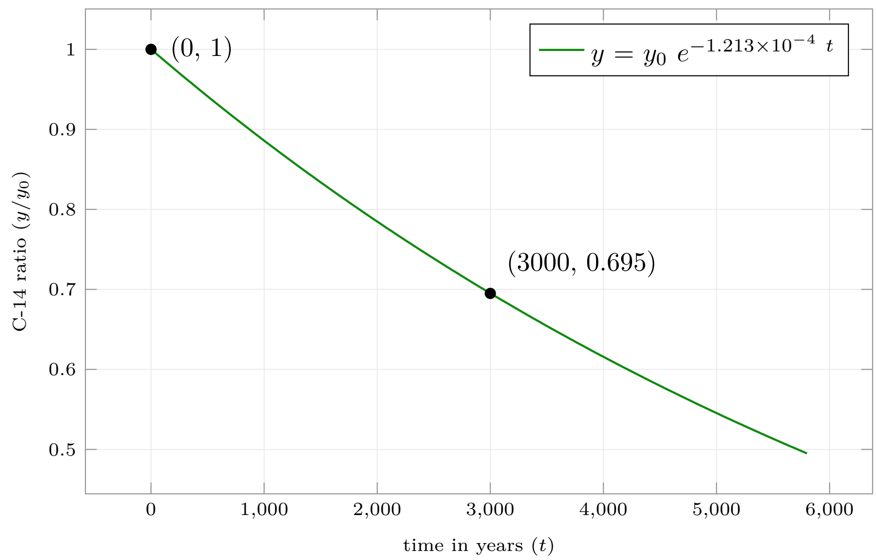
$$\lim_{t\to\infty} y = \begin{cases} \infty & \text{if} \quad b > k \\ 0 & \text{if} \quad b < k \\ y_0 & \text{if} \quad b = k \end{cases} \tag{1.3.90}$$

21. Given IC is $y(t = 0) = y_0$ where t is the time in years, and y is the quantity of C-14. Half life of C-14 is $T_{1/2} = 5715$ years.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky \qquad\qquad \ln|y| = -kt + c \tag{1.3.91}$$

$$y = ce^{-kt} \qquad y > 0 \qquad\qquad c = y_0 \tag{1.3.92}$$

$$k = \frac{\ln 2}{T_{1/2}} \tag{1.3.93}$$



After 3000 years the C-14 content is 69.5% of y_0.

22. For the particle, let v_0 be initial velocity, $v(t)$ be the velocity at time t in seconds.

$$\frac{dv}{dt} = v' = k \tag{1.3.94}$$

$$v = kt + c \tag{1.3.95}$$

$$c = v_0 \tag{1.3.96}$$

$$k = \frac{v - v_0}{t} = \frac{9 \times 10^3}{1 \times 10^{-3}} = 9 \times 10^6 \text{ m s}^{-2} \tag{1.3.97}$$

In order to find the distance traveled s in time t,

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = kt + v_0 \qquad s = \frac{kt^2}{2} + v_0 t + b \tag{1.3.98}$$

$$b = 0 \qquad s = 4.5 + 1 = 5.5 \text{ m} \tag{1.3.99}$$

23. At constant temperature, pressure p and volume V are related by,

$$\frac{dV}{dp} = \frac{-V}{p} \tag{1.3.100}$$

$$\int \frac{-1}{V}\, dV = \int \frac{1}{p}\, dp \tag{1.3.101}$$

$$\ln\left(\frac{1}{V}\right) = \ln p + b \tag{1.3.102}$$

$$Vp = c \tag{1.3.103}$$

24. Standard mixing problem with brine y in lb and time t in minutes

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0 - \frac{2y}{400} \tag{1.3.104}$$

$$\int \frac{1}{y}\ \mathrm{d}y = \frac{-1}{200}\int\ \mathrm{d}t \tag{1.3.105}$$

$$y = ce^{-t/200} \tag{1.3.106}$$

$$y(t=0) = 100 \qquad c = 100 \tag{1.3.107}$$

$$y(t=60) = 100 \times \exp\left(\frac{-60}{200}\right) = 74.08 \text{ lb} \tag{1.3.108}$$

After 1 hour, 74.08 lb of salt remains in the tank

25. Newton's law of cooling problem, with temperature y in celsius and time t in minutes

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(y - y_A) \tag{1.3.109}$$

$$\int \frac{1}{y - y_A} \, \mathrm{d}y = k \int \, \mathrm{d}t \tag{1.3.110}$$

$$y = y_A + ce^{kt} \tag{1.3.111}$$

$$y_A = 22 \qquad c = -17 \tag{1.3.112}$$

$$y(t = 1) = 22 - 17e^k = 12 \qquad k = \ln\left(\frac{10}{17}\right) \tag{1.3.113}$$

$$21.9 = 22 - 17e^{kt} \qquad t = 9.68 \tag{1.3.114}$$

For the temperature to be 21.9 °C, the time taken is 9.68 min

26. Gompertz growth model for tumours, with time t and mass of tumour y,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -Ay \ln y \qquad A > 0 \tag{1.3.115}$$

$$y' = 0 \qquad \implies y = 0 \quad \text{or} \quad y = 1 \tag{1.3.116}$$

$$y'' = -A(1 + \ln y) \tag{1.3.117}$$

$$\implies y''(y = 0) > 0 \qquad y''(y = 1) < 0 \tag{1.3.118}$$

$y = 0$ is the unstable equilibrium solution of no tumour. $y = 1$ is the stable equilibrium solution. On solving the ODE explicitly,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -Ay \ln y \tag{1.3.119}$$

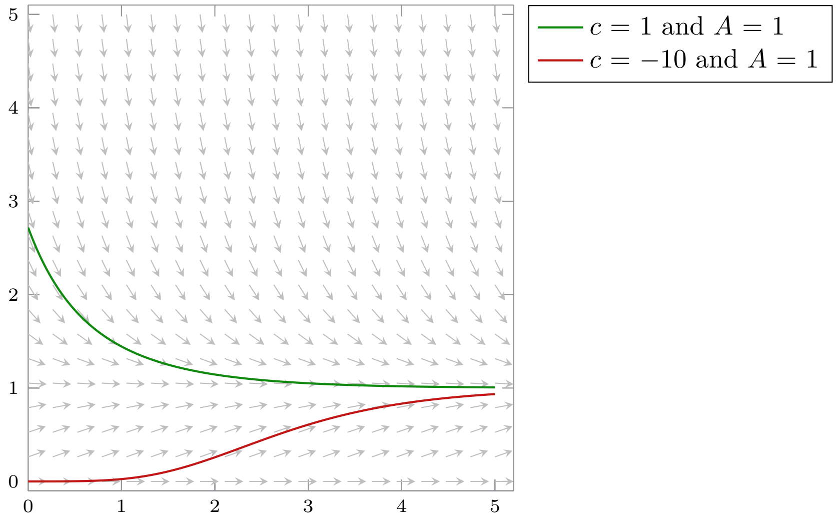
$$\int \frac{1}{y \ln y} \, \mathrm{d}y = -A \int \, \mathrm{d}t \tag{1.3.120}$$

$$u = \ln y \qquad du = \frac{\mathrm{d}y}{y} \tag{1.3.121}$$

$$\int \frac{\mathrm{d}u}{u} = -At + b \qquad \ln u = -At + b \tag{1.3.122}$$

$$\ln y = ce^{-At} \qquad y = \exp(ce^{-At}) \tag{1.3.123}$$

$$y(t = 0) = e^c \qquad y_0 > 1 \implies c > 0 \tag{1.3.124}$$



Solutions with $y_0 > 1$ decay to the steady state solution $y = 1$, whereas $y_0 < 1$ increases sigmoidally to the same stable equilibrium value.

27. Let y be the moisture content and t be time in minutes.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky \qquad 1.3.125$$

$$y = ce^{-kt} \qquad c = y_0 \qquad 1.3.126$$

$$y(t = 10) = y_0 e^{-10k} = y_0/2 \qquad k = \frac{\ln 2}{10} \qquad 1.3.127$$

$$y_0/100 = y_0 e^{-kt} \qquad t = 66.4 \text{ min} \qquad 1.3.128$$

28. It will take between 60 and 70 minutes since $64 < 100 < 128$ and thus $2^6 < 2^{t*} < 2^7$. $t* = 6.64$.

29. Using Newton's Law of cooling, consider the temperature y in Fahrenheit and time t in minutes

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(y - y_A) \qquad 1.3.130$$

$$\int \frac{1}{y - y_A}\,\mathrm{d}y = k\int\,\mathrm{d}t \qquad 1.3.131$$

$$y = y_A + ce^{kt} \qquad 1.3.132$$

$$y(t = 30) = y_A + ce^{30k} \qquad y(t = 40) = y_A + ce^{40k} \qquad 1.3.133$$

$$190 - 110 = 80 = c(e^{30k} - e^{40k}) \qquad 1.3.134$$

Since Newton's law of cooling specifies that the rate of cooling slows down with time, $y_{20} - y_{30} > y_{30} - y_{40}$. This means $y_{20} > 270$ F. Since this is above boiling point of water, Jack can only have been inside the bar for about 6 minutes at most.

30. In the first powered stage, the distance travelled by the rocket is,

$$y'' = 7t \tag{1.3.135}$$

$$y = \frac{7t^3}{6} + bt + c \tag{1.3.136}$$

$$y(t = 0) = c = 0 \qquad\qquad y'(t = 0) = b = 0 \tag{1.3.137}$$

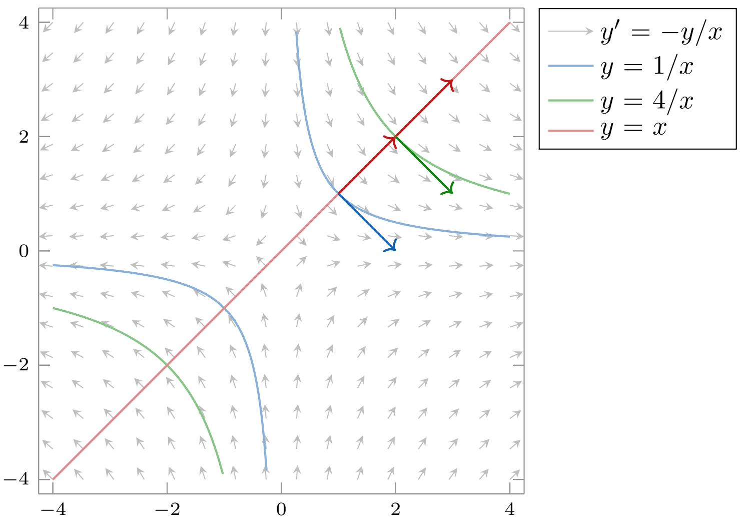
$$y(t = 10) = \frac{7000}{6} \qquad\qquad y'(t = 10) = 350 \tag{1.3.138}$$

In the next stage, under the influence of only gravity ($g = 10$ m s^{-2}), the distance traveled is,

$$y_f = \frac{0^2 - 350^2}{-2g} = 6125 \text{ m} \tag{1.3.139}$$

The maximum height reached by the rocket is thus, 7291.7 m

31. Non-vertical straight line through the origin is of the form $y = mx$ for finite m.



Consider the set of solutions to the ODE $y' = g(y/x)$ intersecting the line $y = mx$. $y' = g(m)$ by definition, which means that the tangents to the family of solutions to the ODE are all parallel and aligned in the direction $g(m)$. This means that the line $y = mx$ intersects all of the solutions of the ODE at the angle between the directions m and $g(m)$.

32. Resolving the gravitational force on the block into two components along and perpendicular to the inclined plane,

$$N = mg \ \cos\alpha \qquad \text{perpendicular to plane} \tag{1.3.140}$$

$$ma = mg \ \sin\alpha - f \qquad \text{along plane} \tag{1.3.141}$$

$$f = \mu N = \mu mg \ \cos\alpha \qquad \text{kinetic friction} \tag{1.3.142}$$

$$a = g(\sin\alpha - \mu\cos\alpha) \qquad = 10\left(\frac{1}{2} - \frac{0.2\sqrt{3}}{2}\right) \tag{1.3.143}$$

Now, given the initial velocity $u = 0$ m s^{-1}, and the distance to slide along the plane $s = 10$ m,

$$v^2 = u^2 + 2as \tag{1.3.144}$$

$$= (1 - 0.2\sqrt{3}) \times 100 \tag{1.3.145}$$

$$v = 10 \times \sqrt{1 - 0.2\sqrt{3}} = 8.08 \text{ m s}^{-1} \tag{1.3.146}$$

33. Given force S and angle ϕ,

$$\Delta S = kS\ \Delta\phi \qquad \int \frac{1}{S}\, dS = k \int \ d\phi \tag{1.3.147}$$

$$S = ce^{k\phi} \qquad c = S_0 \tag{1.3.148}$$

$$\phi = \frac{\ln(1000)}{0.15} \qquad \phi = 2\pi \times 7.34 \tag{1.3.149}$$

Thus, 7.5 revolutions of the rope around the bollard is enough to withstand a force 1000 times larger on the other end of the rope (because of the nature of exponential increase).

34. For conic sections,

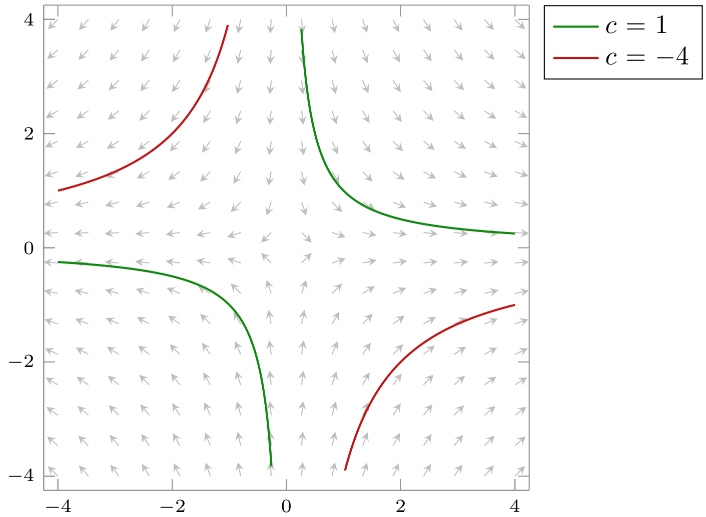
(a) For cirlces with center as the origin, the equation is,

$$y^2 + x^2 = r^2 \qquad 2yy' + 2x = 0 \tag{1.3.150}$$

$$y' = \frac{-x}{y} \tag{1.3.151}$$

(b) For the family of parabolas $xy = c$,

$$xy' + y = 0 \qquad y' = \frac{-y}{x} \tag{1.3.152}$$



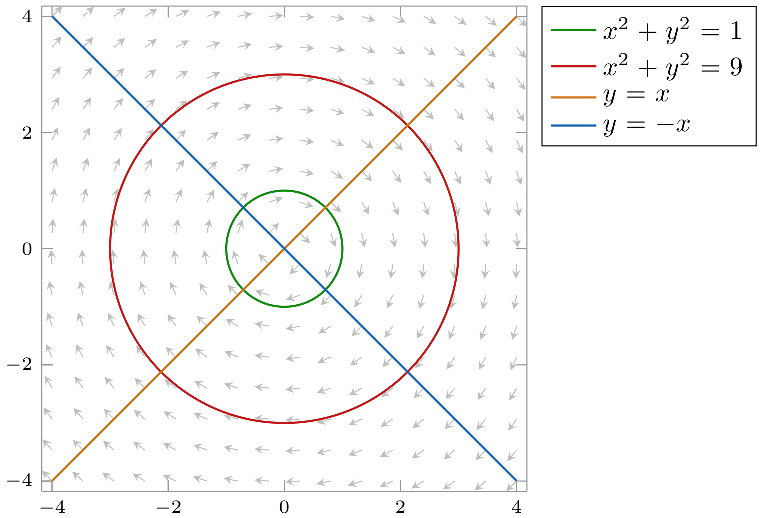
(c) For the family of straight lines passing through the origin, $y = mx$,

$$y' = m = \frac{y}{x} \tag{1.3.153}$$

(d) The product of the RHS of these two families of curves is,

$$\frac{y}{x} \times \frac{-x}{y} = -1 \qquad m_1 \times m_2 = -1 \tag{1.3.154}$$

By definition, the product of slopes being -1 is the condition for the two curves to be orthogonal. This also extends to two families of curves being orthogonal.



(e) Does every one-parameter family of curves lead to a first order ODE?

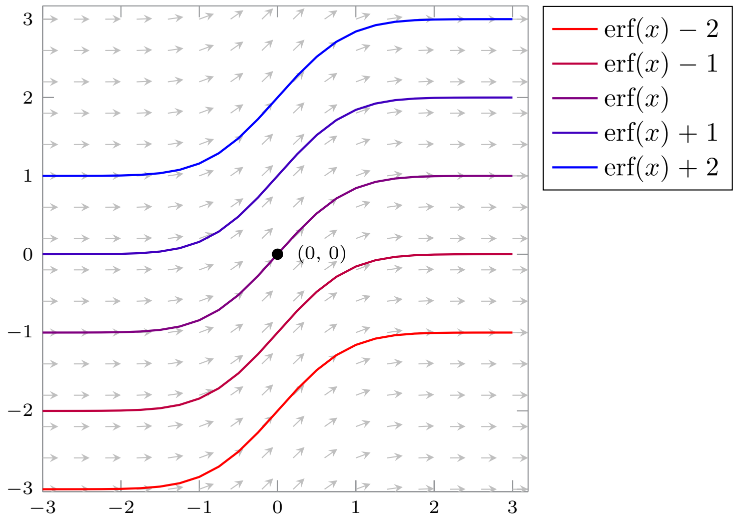
$$f(x, y, c) = 0 \qquad \text{parameter c} \tag{1.3.155}$$

$$g\left(x, y, \frac{dy}{dx}, c\right) = 0 \qquad \text{differentiating wrt } x \tag{1.3.156}$$

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \text{eliminating c from the above two expressions} \tag{1.3.157}$$

The last step is the definition of a first order ODE.

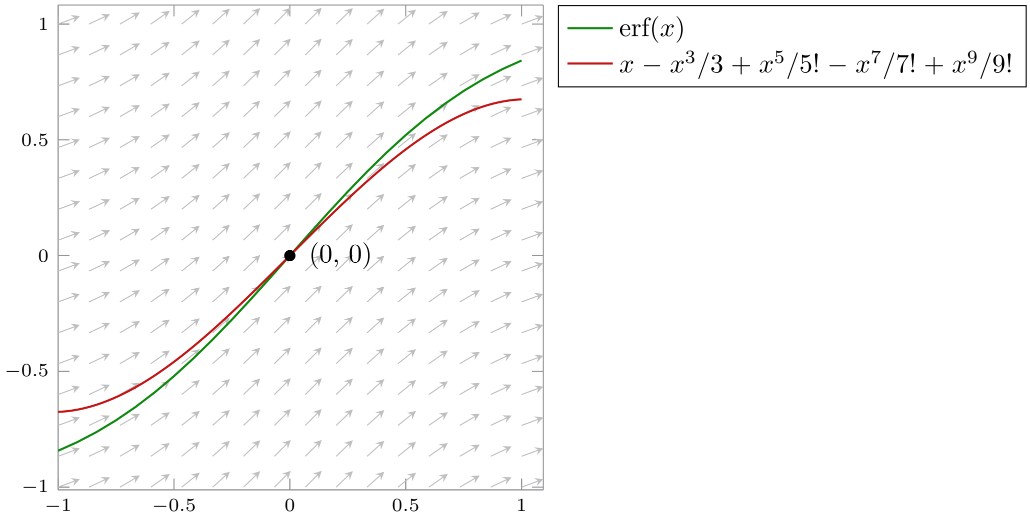
35. (a) Plotting the direction field for the given ODE,



(b) Using the series expansion of $y' = e^{-x^2}$, and integrating the terms in the expansion individually, an approximate solution to the ODE can be plotted

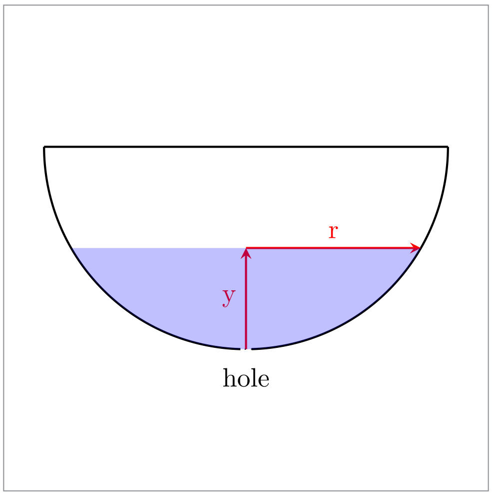
$$y' = e^{-x^2} = 1 - x^2 + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \tag{1.3.158}$$

$$y = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 4!} - \frac{x^7}{7 \cdot 6!} + \ldots \tag{1.3.159}$$



(c) TBC.

36. Torricelli's law for a hemispherical tank (concave up). Same method as in Example 7,



With height y, volume V, radius of hemisphere R, area of the hole A, and area of the water's top surface B,

$$v(t) = 0.6\sqrt{2gy(t)} \tag{1.3.160}$$

$$\Delta V = Av\Delta t \qquad \text{water outflow from hole} \tag{1.3.161}$$

$$\Delta V^* = -B\Delta y \qquad \text{decrease in water volume in vessel} \tag{1.3.162}$$

$$\Delta V^* = -\pi r^2 \Delta y \qquad = -\pi\left[R^2 - (R-y)^2\right]\Delta y \tag{1.3.163}$$

$$= \pi y(y - 2R)\Delta y \tag{1.3.164}$$

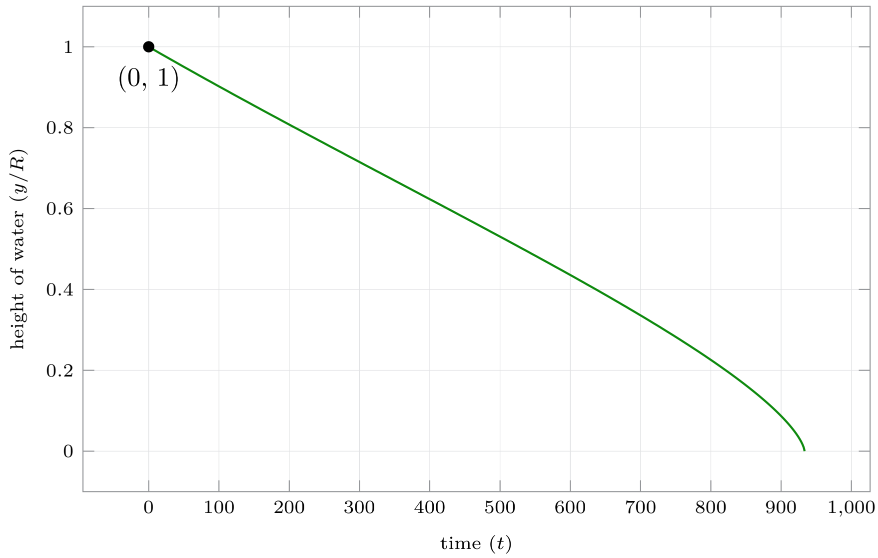
$$0.6A\sqrt{2g}\sqrt{y}\ \ \mathrm{d}t = \pi y(y-2R)\ \ \mathrm{d}y \tag{1.3.165}$$

$$\lambda\ \ \mathrm{d}t = \sqrt{y}(y-2R)\ \ \mathrm{d}y \tag{1.3.166}$$

For simplicity, set $y(t=0)=R$, which means the tank is full initially, and set $\lambda = 1\times 10^{-3}$ so that $A \ll R$

$$\lambda t + c = \frac{2y^{5/2}}{5} - \frac{4Ry^{3/2}}{3} \tag{1.3.167}$$

$$c = \frac{-14R^{5/2}}{15} \qquad \lambda = \frac{0.6A\sqrt{2g}}{\pi} \tag{1.3.168}$$



Time taken to empty the tank is $t^* = \dfrac{14}{15\lambda}$

1.4 Exact ODEs, Integrating Factors

1. Test for exactness passed.

$$\mathrm{d}x + x^2 \ \mathrm{d}y = 0 \tag{1.4.1}$$

$$M = 2xy \qquad N = x^2 \tag{1.4.2}$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x \tag{1.4.3}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.4}$$

Solving ODE,

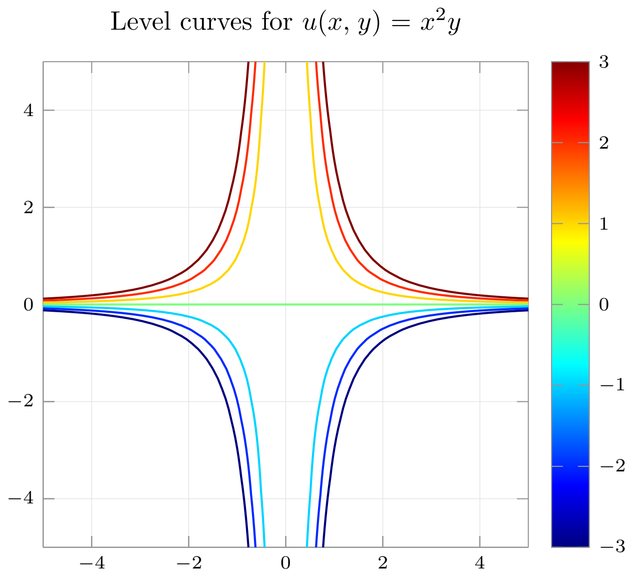
$$u = \int M \ \mathrm{d}x + k(y) \qquad = 2y \int x \ \mathrm{d}x + k(y) \tag{1.4.5}$$

$$u = yx^2 + k(y) \tag{1.4.6}$$

$$\frac{\partial u}{\partial y} = N \qquad x^2 + \frac{\mathrm{d}k}{\mathrm{d}y} = x^2 \tag{1.4.7}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = 0 \qquad k = b \tag{1.4.8}$$

$$u(x, y) = x^2 y \tag{1.4.9}$$



2. Test for exactness passed.

$$x^3 \ \mathrm{d}x + y^3 \ \mathrm{d}y = 0 \tag{1.4.10}$$

$$M = x^3 \qquad N = y^3 \tag{1.4.11}$$

$$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0 \tag{1.4.12}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.13}$$

Solving ODE,

$$u = \int M \;\; \mathrm{d}x + k(y) \qquad = \int \; x^3 \;\; \mathrm{d}x + k(y) \tag{1.4.14}$$

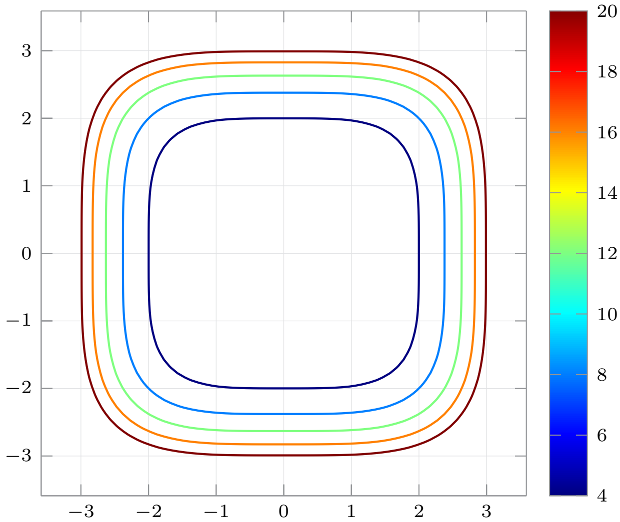
$$u = \frac{x^4}{4} + k(y) \tag{1.4.15}$$

$$\frac{\partial u}{\partial y} = N \qquad \frac{\mathrm{d}k}{\mathrm{d}y} = y^3 \tag{1.4.16}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = y^3 \qquad k = \frac{y^4}{4} \tag{1.4.17}$$

$$u(x,\, y) = \frac{x^4 + y^4}{4} \tag{1.4.18}$$

Level curves for $u(x,\, y) = \dfrac{x^4 + y^4}{4}$



3. Test for exactness passed.

$$0 = \sin\, x \,\cos\, y \;\; \mathrm{d}x + \cos\, x \,\sin\, y \;\; \mathrm{d}y \tag{1.4.19}$$

$$M = \sin\, x \,\cos\, y \qquad N = \cos\, x \,\sin\, y \tag{1.4.20}$$

$$\frac{\partial M}{\partial y} = -\,\sin\, x \,\sin\, y \qquad \frac{\partial N}{\partial x} = -\,\sin\, x \,\sin\, y \tag{1.4.21}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.22}$$

Solving ODE,

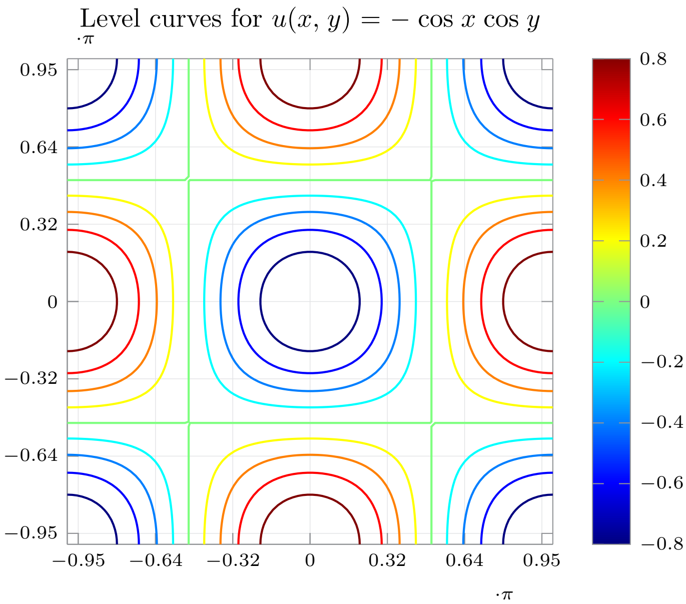
$$u = \int M \ \ \mathrm{d}x + k(y) \qquad = \cos y \int \ \sin x \ \ \mathrm{d}x + k(y) \tag{1.4.23}$$

$$u = -\cos x \cos y + k(y) \tag{1.4.24}$$

$$\frac{\partial u}{\partial y} = N \qquad \frac{\mathrm{d}k}{\mathrm{d}y} + \cos x \sin y = \cos x \sin y \tag{1.4.25}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = 0 \qquad k = b \tag{1.4.26}$$

$$u(x, y) = -\cos x \cos y \tag{1.4.27}$$



4. Test for exactness passed.

$$0 = e^{3\theta} \ \ \mathrm{d}r + 3re^{3\theta} d\theta \tag{1.4.28}$$

$$M = e^{3\theta} \qquad N = 3re^{3\theta} \tag{1.4.29}$$

$$\frac{\partial M}{\partial \theta} = 3e^{3\theta} \qquad \frac{\partial N}{\partial r} = 3e^{3\theta} \tag{1.4.30}$$

$$\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r} \tag{1.4.31}$$

Solving ODE,

$$u = \int M \;\; \mathrm{d}r + k(\theta) \qquad = e^{3\theta} \int \;\; \mathrm{d}r + k(\theta) \tag{1.4.32}$$

$$u = re^{3\theta} + k(\theta) \tag{1.4.33}$$

$$\frac{\partial u}{\partial \theta} = N \qquad \frac{\mathrm{d}k}{\mathrm{d}\theta} + 3re^{3\theta} = 3re^{3\theta} \tag{1.4.34}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = 0 \qquad k = b \tag{1.4.35}$$

$$u(r,\theta) = re^{3\theta} \tag{1.4.36}$$

5. Test for exactness failed.

$$0 = (x^2 + y^2) \;\; \mathrm{d}x - 2xy \;\; \mathrm{d}y \tag{1.4.37}$$

$$P = x^2 + y^2 \qquad Q = -2xy \tag{1.4.38}$$

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = -2y \tag{1.4.39}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.40}$$

Finding the integrating factor,

$$R^* = \frac{1}{P}\left(Q_x - P_y\right) = \frac{-4y}{x^2 + y^2} \tag{1.4.41}$$

$$R = \frac{-1}{Q}\left(Q_x - P_y\right) = \frac{-4y}{2xy} = \frac{-2}{x} = R(x) \tag{1.4.42}$$

$$F = \exp\left(\int \;\; R(x) \;\; \mathrm{d}x\right) = x^{-2} \tag{1.4.43}$$

Checking the integrating factor $F = x^{-2}$,

$$0 = 1 + \frac{y^2}{x^2}\ \mathrm{d}x - \frac{2y}{x}\ \mathrm{d}y \tag{1.4.44}$$

$$M = 1 + \frac{y^2}{x^2} \qquad N = \frac{-2y}{x} \tag{1.4.45}$$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2} \tag{1.4.46}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.47}$$

Solving ODE,

$$u = \int N\ \mathrm{d}y + k(x) \qquad = \frac{-2}{x}\int y\ \mathrm{d}y + k(x) \tag{1.4.48}$$

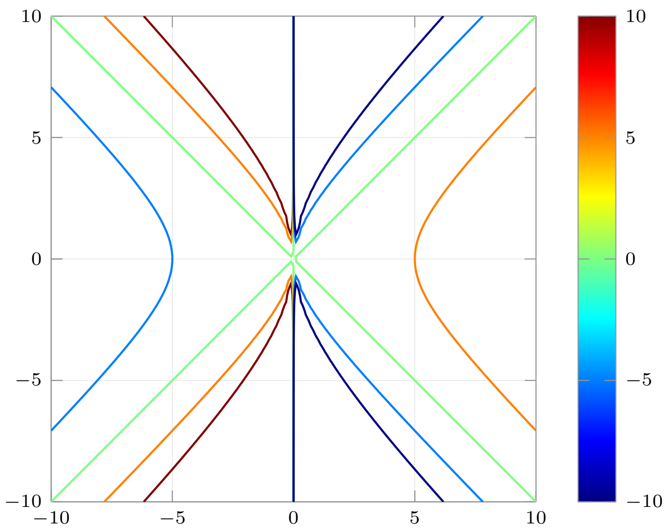
$$u = \frac{-y^2}{x} + k(x) \tag{1.4.49}$$

$$\frac{\partial u}{\partial x} = M \qquad \frac{\mathrm{d}k}{\mathrm{d}x} + \frac{y^2}{x^2} = 1 + \frac{y^2}{x^2} \tag{1.4.50}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = 1 \qquad k = x \tag{1.4.51}$$

$$u(x, y) = x - \frac{y^2}{x} \tag{1.4.52}$$

Level curves for $u(x, y) = x - \dfrac{y^2}{x}$



6. Test for exactness failed. Using the integration factor given $F = (y + 1)x^{-4}$

$$0 = 3(y+1)\ \mathrm{d}x - 2x\ \mathrm{d}y \tag{1.4.53}$$

$$M = 3y + 3 \qquad N = -2x \tag{1.4.54}$$

$$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = -2 \tag{1.4.55}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \tag{1.4.56}$$

Checking the integrating factor,

$$0 = 3(y+1)^2 x^{-4}\ \mathrm{d}x - 2x^{-3}(y+1)\ \mathrm{d}y \tag{1.4.57}$$

$$P = 3x^{-4}(y+1)^2 \qquad Q = -2x^{-3}(y+1) \tag{1.4.58}$$

$$\frac{\partial P}{\partial y} = 6x^{-4}(y+1) \qquad \frac{\partial Q}{\partial x} = 6x^{-4}(y+1) \tag{1.4.59}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1.4.60}$$

Performing the integration,

$$u = \int P\ \mathrm{d}x + k(y) \tag{1.4.61}$$

$$= 3(y+1)^2 \int x^{-4}\ \mathrm{d}x + k(y) \tag{1.4.62}$$

$$u = -(y+1)^2 x^{-3} + k(y) \tag{1.4.63}$$

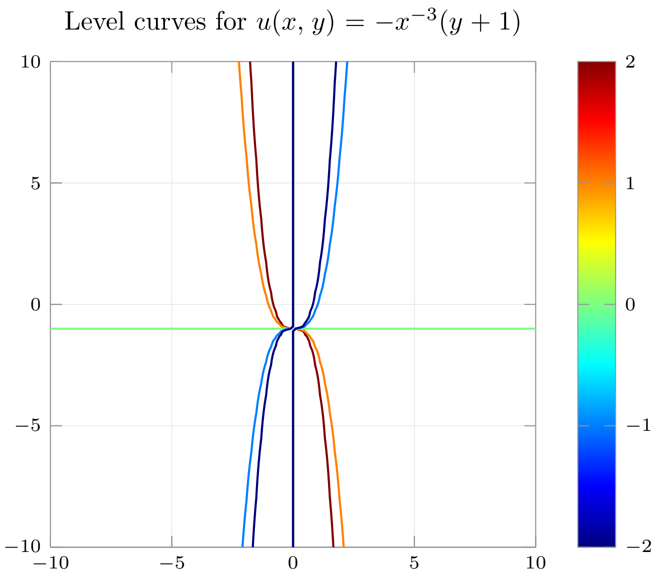
$$\frac{\partial u}{\partial y} = Q \tag{1.4.64}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} - 2x^{-3}(y+1) = -2x^{-3}(y+1) \tag{1.4.65}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = 0 \tag{1.4.66}$$

$$k = b \tag{1.4.67}$$

$$u(x, y) = -x^{-3}(y+1) \tag{1.4.68}$$



7. Test for exactness failed.

$$0 = 2x \tan y \ \ \mathrm{d}x + \sec^2 y \ \ \mathrm{d}y \tag{1.4.69}$$

$$P = 2x \tan y \qquad\qquad Q = \sec^2 y \tag{1.4.70}$$

$$\frac{\partial P}{\partial y} = 2x \sec^2 y \qquad\qquad \frac{\partial Q}{\partial x} = 0 \tag{1.4.71}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.72}$$

Finding the integrating factor,

$$R^* = \frac{1}{P}\left(Q_x - P_y\right) = \frac{-2x \sec^2 y}{2x \tan y} \tag{1.4.73}$$

$$R = \frac{-1}{Q}\left(Q_x - P_y\right) = \frac{-2x \sec^2 y}{-\sec^2 y} = R(x) \tag{1.4.74}$$

$$F = \exp\left(\int R(x) \ \ \mathrm{d}x\right) = e^{x^2} \tag{1.4.75}$$

Checking the integrating factor $F = e^{x^2}$

$$0 = 2xe^{x^2} \tan y \ \ \mathrm{d}x + e^{x^2} \sec^2 y \ \ \mathrm{d}y \tag{1.4.76}$$

$$M = 2xe^{x^2} \tan y \qquad N = e^{x^2} \sec^2 y \tag{1.4.77}$$

$$\frac{\partial M}{\partial y} = 2xe^{x^2} \sec^2 y \qquad \frac{\partial N}{\partial x} = 2xe^{x^2} \sec^2 y \tag{1.4.78}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.79}$$

Solving the ODE

$$u = \int N \ \ \mathrm{d}y + k(x) \tag{1.4.80}$$

$$= e^{x^2} \int \sec^2 y \ \ \mathrm{d}y + k(x) \tag{1.4.81}$$

$$u = e^{x^2} \tan y + k(x) \tag{1.4.82}$$

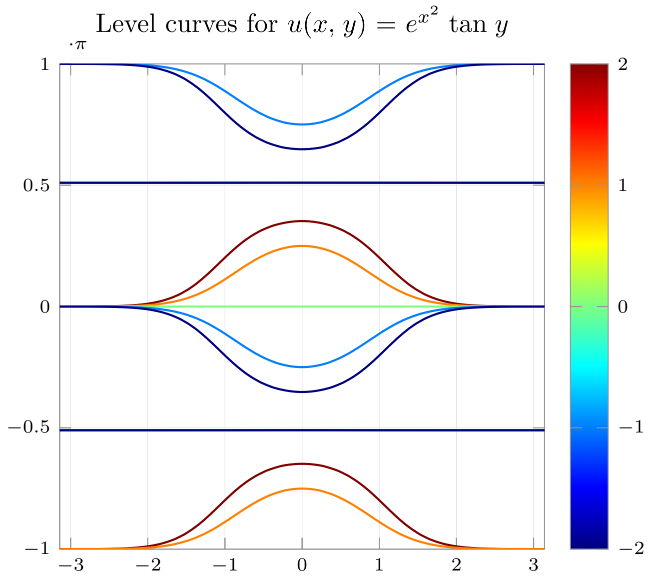
$$\frac{\partial u}{\partial x} = M \tag{1.4.83}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} + 2xe^{x^2} \tan y = 2xe^{x^2} \tan y \tag{1.4.84}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = 0 \tag{1.4.85}$$

$$k = b \tag{1.4.86}$$

$$u(x, y) = e^{x^2} \tan y \tag{1.4.87}$$



8. Test for exactness passed.

$$0 = e^x \cos y \ \ \mathrm{d}x - e^x \sin y \ \ \mathrm{d}y \tag{1.4.88}$$

$$P = e^x \cos y \qquad\qquad Q = -e^x \sin y \tag{1.4.89}$$

$$\frac{\partial P}{\partial y} = -e^x \sin y \qquad\qquad \frac{\partial Q}{\partial x} = -e^x \sin y \tag{1.4.90}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.91}$$

Solving the ODE, settings $(P, Q) \to (M, N)$

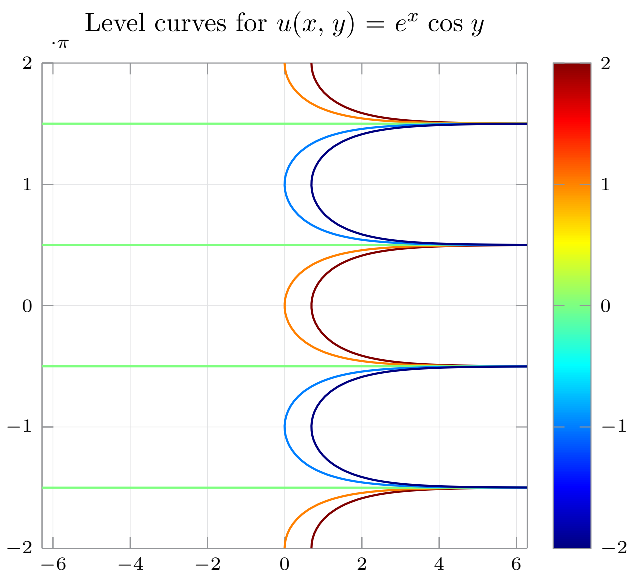
$$u = \int N \ \ \mathrm{d}y + k(x) \qquad\qquad = -e^x \int \sin y \ \ \mathrm{d}y + k(x) \tag{1.4.92}$$

$$u = e^x \cos y + k(x) \tag{1.4.93}$$

$$\frac{\partial u}{\partial x} = M \qquad\qquad \frac{\mathrm{d}k}{\mathrm{d}x} + e^x \cos y = e^x \cos y \tag{1.4.94}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = 0 \qquad\qquad k = b \tag{1.4.95}$$

$$u(x, y) = e^x \cos y \tag{1.4.96}$$



9. Test for exactness passed.

$$0 = e^{2x}(2 \cos y) \;\; \mathrm{d}x - e^{2x}(\sin y) \;\; \mathrm{d}y \tag{1.4.97}$$

$$P = e^{2x}(2 \cos y) \qquad\qquad Q = -e^{2x}(\sin y) \tag{1.4.98}$$

$$\frac{\partial P}{\partial y} = e^{2x}(-2 \sin y) \qquad\qquad \frac{\partial Q}{\partial x} = -e^{2x}(2 \sin y) \tag{1.4.99}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1.4.100}$$

Solving the ODE, setting $(P, Q) \to (M, N)$, and the IC $y(0) = 0$

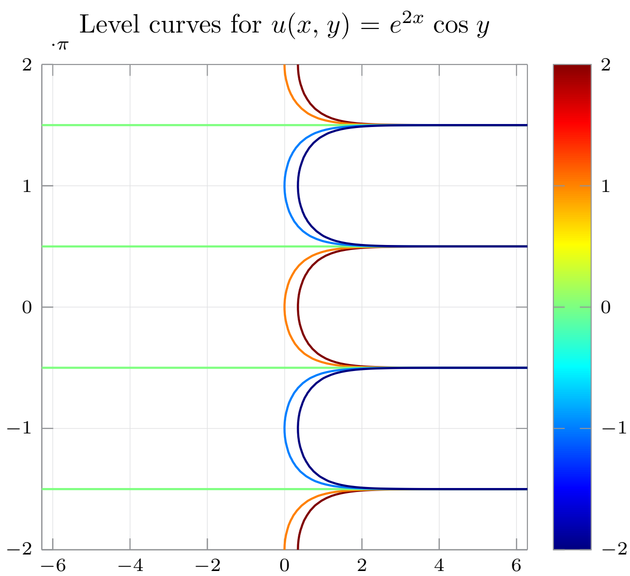
$$u = \int N \;\; \mathrm{d}y + k(x) \qquad\qquad = -e^{2x} \int \sin y \;\; \mathrm{d}y + k(x) \tag{1.4.101}$$

$$u = e^{2x} \cos y + k(x) \qquad\qquad \frac{\partial u}{\partial x} = M \tag{1.4.102}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} + 2e^{2x} \cos y = e^{2x}(2 \cos y) \qquad\qquad \frac{\mathrm{d}k}{\mathrm{d}x} = 0 \tag{1.4.103}$$

$$k = b \qquad\qquad u(x, y) = e^{2x} \cos y + c \tag{1.4.104}$$

$$\text{particular solution} \qquad\qquad e^{2x} \cos y = 1 \tag{1.4.105}$$



10. Test for exactness failed. Using the integration factor given $F = \cos(x + y)$

$$y \;\; \mathrm{d}x + [y + \tan(x + y)] \;\; \mathrm{d}y = 0 \tag{1.4.106}$$

$$M = y \qquad\qquad N = [y + \tan(x + y)] \tag{1.4.107}$$

$$\frac{\partial M}{\partial y} = 1 \qquad\qquad \frac{\partial N}{\partial x} = \sec^2(x + y) \tag{1.4.108}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \tag{1.4.109}$$

Checking the integrating factor,

$$0 = y \cos(x + y) \;\; \mathrm{d}x + [y \cos(x + y) + \sin(x + y)] \;\; \mathrm{d}y \tag{1.4.110}$$

$$P = y \cos(x + y) \tag{1.4.111}$$

$$Q = [y \cos(x + y) + \sin(x + y)] \tag{1.4.112}$$

$$\frac{\partial P}{\partial y} = \cos(x + y) - y \sin(x + y) \tag{1.4.113}$$

$$\frac{\partial Q}{\partial x} = -y \sin(x + y) + \cos(x + y) \tag{1.4.114}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1.4.115}$$

Solving the ODE,

$$u = \int P \ \mathrm{d}x + k(y) \tag{1.4.116}$$

$$= y \int \ \cos(x+y) \ \mathrm{d}x + k(y) \tag{1.4.117}$$

$$u = y \sin(x+y) + k(y) \tag{1.4.118}$$

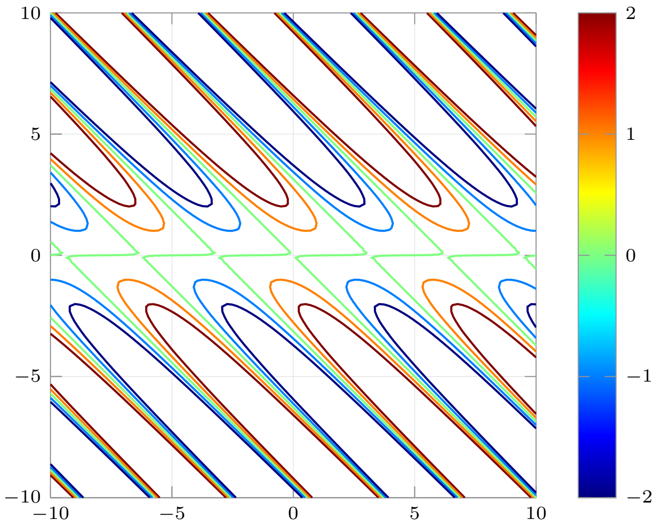
$$\frac{\partial u}{\partial y} = Q \tag{1.4.119}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} + y\cos(x+y) + \sin(x+y) = [y\cos(x+y) + \sin(x+y)] \tag{1.4.120}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = 0 \qquad\qquad k = b \tag{1.4.121}$$

$$u(x,\, y) = y \sin(x+y) \tag{1.4.122}$$

Level curves for $u(x,\, y) = y\sin(x+y)$



11. Test for exactness failed.

$$0 = 2\cosh x \cos y \ \ \mathrm{d}x - \sinh x \sin y \ \ \mathrm{d}y \tag{1.4.123}$$

$$P = 2\cosh x \cos y \qquad\qquad Q = -\sinh x \sin y \tag{1.4.124}$$

$$\frac{\partial P}{\partial y} = -2\cosh x \sin y \qquad\qquad \frac{\partial Q}{\partial x} = -\cosh x \sin y \tag{1.4.125}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.126}$$

Finding the integrating factor,

$$R^* = \frac{1}{P}\left(Q_x - P_y\right) = \frac{\cosh x \sin y}{2 \cosh x \cos y} = R^*(y) \tag{1.4.127}$$

$$R = \frac{-1}{Q}\left(Q_x - P_y\right) = \frac{\cosh x \sin y}{\sinh x \sin y} = R(x) \tag{1.4.128}$$

$$F = \exp\left(\int R(x)\ \mathrm{d}x\right) = \sinh|x| \tag{1.4.129}$$

Checking the integrating factor $F = \sinh|x|$

$$0 = 2 \cosh x \sinh|x| \cos y\ \ \mathrm{d}x - \sinh x \sinh|x| \sin y\ \ \mathrm{d}y \tag{1.4.130}$$

$$M = 2 \cosh x \sinh|x| \cos y \tag{1.4.131}$$

$$N = -\sinh x \sinh|x| \sin y \tag{1.4.132}$$

$$\frac{\partial M}{\partial y} = -2 \cosh x \sinh|x| \sin y \tag{1.4.133}$$

$$\frac{\partial N}{\partial x} = -\sin y \cosh x \sinh|x| - \sin y \sinh x \cosh x \frac{x}{|x|} \tag{1.4.134}$$

$$\text{using} \qquad \sinh(-x) = -\sinh x \qquad \text{and} \qquad \sinh x \frac{x}{|x|} = \sinh|x| \tag{1.4.135}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.136}$$

Solving the ODE,

$$u = \int N \quad \mathrm{d}y + k(x) \tag{1.4.137}$$

$$= -\sinh x \sinh |x| \int \sin(y) \quad \mathrm{d}y + k(x) \tag{1.4.138}$$

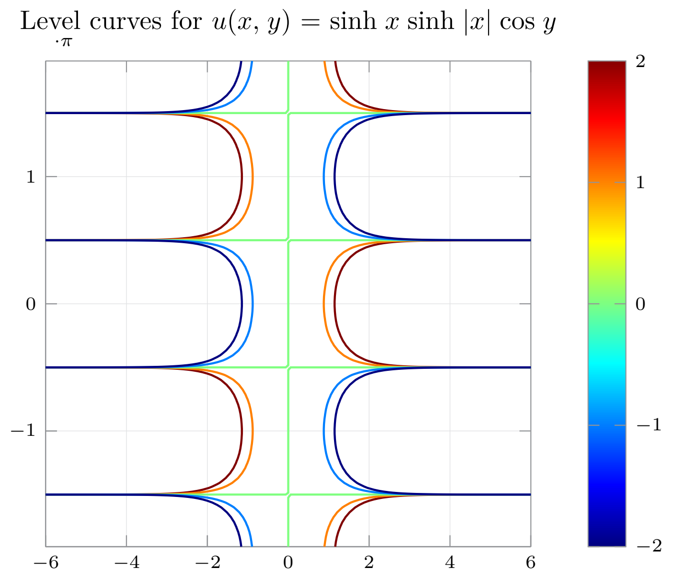
$$u = \sinh x \sinh |x| \cos y + k(x) \tag{1.4.139}$$

$$\frac{\partial u}{\partial x} = M \tag{1.4.140}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} + \cos y \, (2 \sinh |x| \cosh x) = 2 \cosh x \sinh |x| \cos y \tag{1.4.141}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = 0 \qquad\qquad k = b \tag{1.4.142}$$

$$u(x, y) = \sinh x \sinh |x| \cos y \tag{1.4.143}$$



12. Test for exactness passed.

$$0 = 2xy \, e^{x^2} \quad \mathrm{d}x + e^{x^2} \quad \mathrm{d}y \tag{1.4.144}$$

$$P = 2xy \, e^{x^2} \qquad\qquad Q = e^{x^2} \tag{1.4.145}$$

$$\frac{\partial P}{\partial y} = 2x \, e^{x^2} \qquad\qquad \frac{\partial Q}{\partial x} = 2x \, e^{x^2} \tag{1.4.146}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.147}$$

Solving the ODE, setting $(P, Q) \to (M, N)$, and the IC $y(0) = 2$

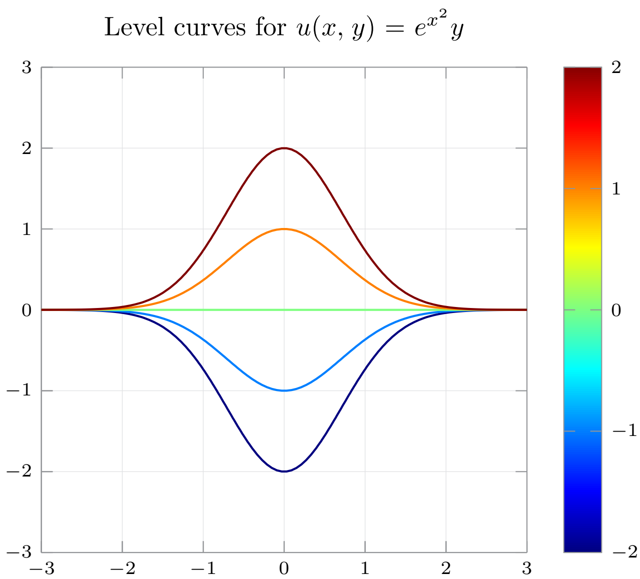
$$u = \int M \; \mathrm{d}x + k(y) \qquad\qquad = y \int 2x\, e^{x^2} \; \mathrm{d}x + k(y) \tag{1.4.148}$$

$$u = e^{x^2} y + k(y) \qquad\qquad \frac{\partial u}{\partial y} = N \tag{1.4.149}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} + e^{x^2} = e^{x^2} \tag{1.4.150}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = 0 \qquad\qquad k = b \tag{1.4.151}$$

$$u(x, y) = e^{x^2} y + c \qquad\qquad c = -2 \tag{1.4.152}$$



13. Test for exactness failed. Using the integration factor given $F = \exp(x + y)$

$$0 = e^{-y} \; \mathrm{d}x + e^{-x}[1 - e^{-y}] \; \mathrm{d}y \tag{1.4.153}$$

$$M = e^{-y} \qquad\qquad N = e^{-x}[1 - e^{-y}] \tag{1.4.154}$$

$$\frac{\partial M}{\partial y} = -e^{-y} \qquad\qquad \frac{\partial N}{\partial x} = -e^{-x}[1 - e^{-y}] \tag{1.4.155}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \tag{1.4.156}$$

Checking the integrating factor,

$$0 = e^x \ \ \mathrm{d}x + (e^y - 1) \ \ \mathrm{d}y \qquad P = e^x \tag{1.4.157}$$

$$Q = e^y - 1 \qquad \frac{\partial P}{\partial y} = 0 \tag{1.4.158}$$

$$\frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1.4.159}$$

Solving the ODE,

$$u = \int P \ \ \mathrm{d}x + k(y) \qquad = \int e^x \ \ \mathrm{d}x + k(y) \tag{1.4.160}$$

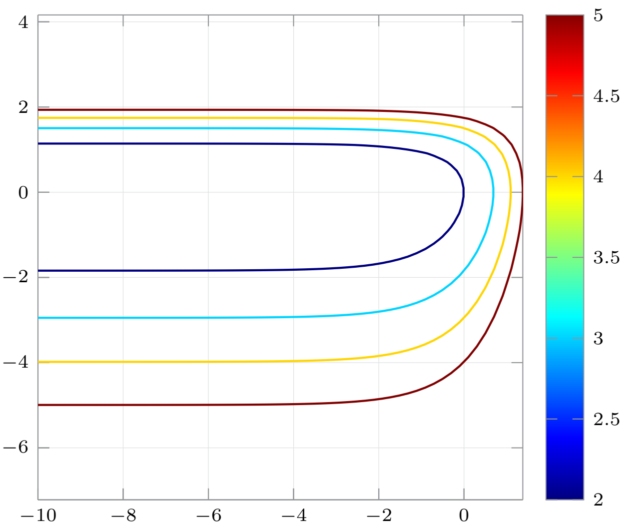
$$u = e^x + k(y) \qquad \frac{\partial u}{\partial y} = Q \tag{1.4.161}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} + 0 = e^y - 1 \tag{1.4.162}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = e^y - 1 \qquad k = e^y - y \tag{1.4.163}$$

$$u(x, y) = e^x + e^y - y \tag{1.4.164}$$

Level curves for $u(x, y) = e^x + e^y - y$



14. Test for exactness failed. Using the integration factor given $F = x^a y^b$

$$0 = (a+1)y \ \ \mathrm{d}x + (b+1)x \ \ \mathrm{d}y \tag{1.4.165}$$

$$M = (a+1)y \qquad\qquad N = (b+1)x \tag{1.4.166}$$

$$\frac{\partial M}{\partial y} = (a+1) \qquad\qquad \frac{\partial N}{\partial x} = (b+1) \tag{1.4.167}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \tag{1.4.168}$$

Checking the integrating factor,

$$0 = (a+1)x^a y^{(b+1)} \ \ \mathrm{d}x + (b+1)x^{(a+1)} y^b \ \ \mathrm{d}y \tag{1.4.169}$$

$$P = (a+1)x^a y^{(b+1)} \tag{1.4.170}$$

$$Q = (b+1)x^{(a+1)} y^b \tag{1.4.171}$$

$$\frac{\partial P}{\partial y} = (a+1)(b+1)x^a y^b \tag{1.4.172}$$

$$\frac{\partial Q}{\partial x} = (b+1)(a+1)x^a y^b \tag{1.4.173}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1.4.174}$$

Solving the ODE,

$$u = \int P \ \ \mathrm{d}x + k(y) \tag{1.4.175}$$

$$= (a+1)y^{(b+1)} \int \ x^a \ \ \mathrm{d}x + k(y) \tag{1.4.176}$$

$$u = x^{(a+1)} y^{(b+1)} + k(y) \tag{1.4.177}$$

$$\frac{\partial u}{\partial y} = Q \tag{1.4.178}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} + (b+1)x^{(a+1)} y^b = (b+1)x^{(a+1)} y^b \tag{1.4.179}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = 0 \qquad\qquad k = b \tag{1.4.180}$$

$$u(x, y) = x^{(a+1)} y^{(b+1)} \tag{1.4.181}$$

15. Test for exactness passes if $b = k$.

$$0 = (ax + by)\ \mathrm{d}x + (kx + ly)\ \mathrm{d}y \tag{1.4.182}$$

$$M = ax + by \qquad N = kx + ly \tag{1.4.183}$$

$$\frac{\partial M}{\partial y} = b \qquad \frac{\partial N}{\partial x} = k \tag{1.4.184}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{if} \quad b = k \tag{1.4.185}$$

Solving the ODE, after setting $k = b$, and $(M, N) \to (P, Q)$

$$u = \int P\ \mathrm{d}x + g(y) \tag{1.4.186}$$

$$= \int (ax + by)\ \mathrm{d}x + g(y) \tag{1.4.187}$$

$$u = \frac{ax^2}{2} + bxy + g(y) \tag{1.4.188}$$

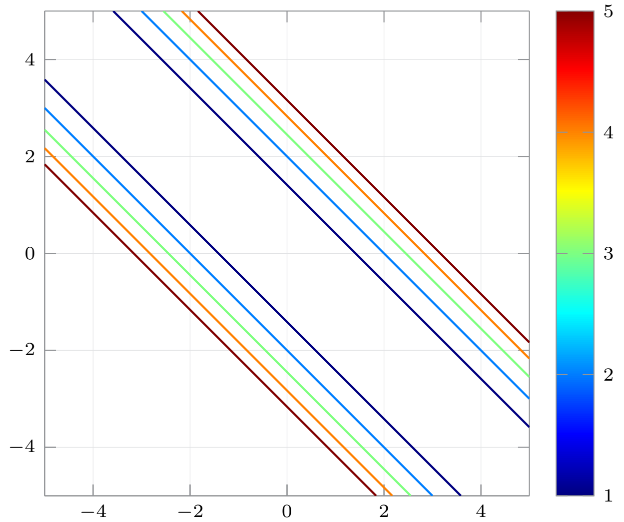
$$\frac{\partial u}{\partial y} = Q \tag{1.4.189}$$

$$\frac{\mathrm{d}g}{\mathrm{d}y} + bx = bx + ly \tag{1.4.190}$$

$$\frac{\mathrm{d}g}{\mathrm{d}y} = ly \qquad g(y) = \frac{ly^2}{2} \tag{1.4.191}$$

$$u(x, y) = \frac{ax^2}{2} + bxy + \frac{ly^2}{2} \tag{1.4.192}$$

Level curves for $u(x, y) = \dfrac{x^2}{2} + xy + \dfrac{y^2}{2}$



16. **(a)** Solving as an exact ODE,

$$0 = e^y \sinh x \;\; \mathrm{d}x + e^y \cosh x \;\; \mathrm{d}y \tag{1.4.193}$$

$$M = e^y \sinh x \qquad\qquad N = e^y \cosh x \tag{1.4.194}$$

$$\frac{\partial M}{\partial y} = e^y \sinh x \qquad\qquad \frac{\partial N}{\partial x} = e^y \sinh x \tag{1.4.195}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.196}$$

Solving the ODE, after setting $k = b$, and $(M, N) \to (P, Q)$

$$u = \int P \;\; \mathrm{d}x + g(y) \tag{1.4.197}$$

$$= e^y \int \sinh x \;\; \mathrm{d}x + g(y) \tag{1.4.198}$$

$$u = e^y \cosh x + g(y) \tag{1.4.199}$$

$$\frac{\partial u}{\partial y} = Q \tag{1.4.200}$$

$$\frac{\mathrm{d}g}{\mathrm{d}y} + e^y \cosh x = e^y \cosh x \tag{1.4.201}$$

$$\frac{\mathrm{d}g}{\mathrm{d}y} = 0 \qquad\qquad g(y) = b \tag{1.4.202}$$

$$u(x, y) = e^y \cosh x \tag{1.4.203}$$

Solving as a separable ODE,

$$\int \tanh x \; \mathrm{d}x = -\int \; \mathrm{d}y \tag{1.4.204}$$

$$\ln(\cosh x) = -y + b \tag{1.4.205}$$

$$\cosh x = ce^{-y} \tag{1.4.206}$$

$$e^y \cosh x = c \tag{1.4.207}$$

Both methods match.

(b) Test for exactness failed.

$$0 = (1+2x)\cos y \; \mathrm{d}x + \frac{1}{\cos y} \; \mathrm{d}y \tag{1.4.208}$$

$$P = (1+2x)\cos y \qquad Q = \sec y \tag{1.4.209}$$

$$\frac{\partial P}{\partial y} = -(1+2x)\sin y \qquad \frac{\partial Q}{\partial x} = 0 \tag{1.4.210}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.211}$$

Finding the integrating factor,

$$R^* = \frac{1}{P}\,(Q_x - P_y) = \frac{(1+2x)\sin y}{(1+2x)\cos y} = R^*(y) \tag{1.4.212}$$

$$R = \frac{-1}{Q}\,(Q_x - P_y) = \frac{(1+2x)\sin y}{-\sec y} \tag{1.4.213}$$

$$F = \exp\left(\int R^*(y) \; \mathrm{d}y\right) = \exp(-\ln|\cos y|) = |\sec y| \tag{1.4.214}$$

Checking the integrating factor $F = |\sec y|$

$$0 = (1+2x)\frac{\cos y}{|\cos y|} \;\; \mathrm{d}x + \frac{1}{\cos y \, |\cos y|} \;\; \mathrm{d}y \tag{1.4.215}$$

$$M = (1+2x)\frac{\cos y}{|\cos y|} \tag{1.4.216}$$

$$N = \frac{1}{\cos y \, |\cos y|} \tag{1.4.217}$$

$$\frac{\partial M}{\partial y} = 0 \tag{1.4.218}$$

$$\frac{\partial N}{\partial x} = 0 \tag{1.4.219}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.220}$$

Solving the ODE,

$$u = \int M \;\; \mathrm{d}x + k(y) \tag{1.4.221}$$

$$= \frac{\cos y}{|\cos y|} \int (1+2x) \;\; \mathrm{d}x + k(y) \tag{1.4.222}$$

$$u = \frac{\cos y}{|\cos y|}(x + x^2) + k(y) \tag{1.4.223}$$

$$\frac{\partial u}{\partial y} = N \tag{1.4.224}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} + 0 = \frac{1}{\cos y \, |\cos y|} \tag{1.4.225}$$

$$\frac{\mathrm{d}k}{\mathrm{d}y} = \frac{1}{\cos y \, |\cos y|} \qquad k = \frac{\sin y}{|\cos y|} \tag{1.4.226}$$

$$u(x, y) = \frac{(x+x^2)\cos y + \sin y}{|\cos y|} \tag{1.4.227}$$

Solving by separation,

$$\int (1+2x) \;\; \mathrm{d}x = -\int \sec^2 y \;\; \mathrm{d}y \tag{1.4.228}$$

$$x + x^2 + \tan y = c \tag{1.4.229}$$

Both answers match, given $\cos y > 0$.

(c) Test for exactness failed.

$$0 = (x^2 + y^2)\ \mathrm{d}x - 2xy\ \mathrm{d}y \tag{1.4.230}$$

$$P = x^2 + y^2 \qquad Q = -2xy \tag{1.4.231}$$

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = -2y \tag{1.4.232}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.233}$$

Finding the integrating factor,

$$R^* = \frac{1}{P}\left(Q_x - P_y\right) = \frac{-4y}{x^2 + y^2} \tag{1.4.234}$$

$$R = \frac{-1}{Q}\left(Q_x - P_y\right) = \frac{-4y}{2xy} = R(x) \tag{1.4.235}$$

$$F = \exp\left(\int R(x)\ \mathrm{d}x\right) = \exp(-2\ln x) = x^{-2} \tag{1.4.236}$$

Checking the integrating factor $F = x^{-2}$

$$0 = \left(1 + \frac{y^2}{x^2}\right)\ \mathrm{d}x - \frac{2y}{x}\ \mathrm{d}y \qquad M = 1 + \frac{y^2}{x^2} \tag{1.4.237}$$

$$N = -\frac{2y}{x} \qquad \frac{\partial M}{\partial y} = \frac{2y}{x^2} \tag{1.4.238}$$

$$\frac{\partial N}{\partial x} = \frac{2y}{x^2} \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.239}$$

Solving the ODE,

$$u = \int N\ \mathrm{d}y + k(x) \qquad = \int \frac{-2y}{x}\ \mathrm{d}y + k(x) \tag{1.4.240}$$

$$u = \frac{-y^2}{x} + k(x) \qquad \frac{\partial u}{\partial x} = M \tag{1.4.241}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} + \frac{y^2}{x^2} = 1 + \frac{y^2}{x^2} \tag{1.4.242}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = 1 \qquad k = x \tag{1.4.243}$$

$$u(x, y) = x - \frac{-y^2}{x} \tag{1.4.244}$$

Solving by separation, with $v = y/x$

$$y' = \frac{x^2 + y^2}{2xy} \qquad = \frac{1}{2v} + \frac{v}{2} \tag{1.4.245}$$

$$v = \frac{y}{x} \qquad y' = xv' + v \tag{1.4.246}$$

$$xv' = \frac{1 - v^2}{2v} \qquad \frac{2v}{1 - v^2}\ dv = \frac{1}{x}\ \mathrm{d}x \tag{1.4.247}$$

$$\ln\left(\frac{1}{1 - v^2}\right) = \ln x + b \tag{1.4.248}$$

$$\frac{1}{1 - v^2} = cx \qquad \frac{x^2}{x^2 - y^2} = cx \tag{1.4.249}$$

$$\frac{1}{c} = x - \frac{y^2}{x} \tag{1.4.250}$$

Both answers match.

(d) Test for exactness failed.

$$0 = 3x^2y\ \ \mathrm{d}x + 4x^3\ \ \mathrm{d}y \tag{1.4.251}$$

$$P = 3x^2y \qquad Q = 4x^3 \tag{1.4.252}$$

$$\frac{\partial P}{\partial y} = 3x^2 \qquad \frac{\partial Q}{\partial x} = 12x^2 \tag{1.4.253}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.254}$$

Finding the integrating factor,

$$R^* = \frac{1}{P}\left(Q_x - P_y\right) = \frac{9x^2}{3x^2y} = R^*(y) \tag{1.4.255}$$

$$R = \frac{-1}{Q}\left(Q_x - P_y\right) = \frac{9x^2}{-4x^3} = R(x) \tag{1.4.256}$$

$$F = \exp\left(\int R^*(y)\ \ \mathrm{d}y\right) = \exp(3\ln y) = y^3 \tag{1.4.257}$$

Checking the integrating factor $F = y^3$

$$0 = 3x^2y^4 \ \mathrm{d}x + 4x^3y^3 \ \mathrm{d}y \qquad M = 3x^2y^4 \tag{1.4.258}$$

$$N = 4x^3y^3 \qquad \frac{\partial M}{\partial y} = 12x^2y^3 \tag{1.4.259}$$

$$\frac{\partial N}{\partial x} = 12x^2y^3 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.4.260}$$

Solving the ODE,

$$u = \int N \ \mathrm{d}y + k(x) \qquad = \int 4x^3y^3 \ \mathrm{d}y + k(x) \tag{1.4.261}$$

$$u = x^3y^4 + k(x) \qquad \frac{\partial u}{\partial x} = M \tag{1.4.262}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} + 3x^2y^4 = 3x^2y^4 \tag{1.4.263}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = 0 \qquad k = b \tag{1.4.264}$$

$$u(x, y) = x^3y^4 \tag{1.4.265}$$

Solving by separation,

$$3y \ \mathrm{d}x = -4x \ dy \qquad \int \frac{-4}{y} \ \mathrm{d}y = \int \frac{3}{x} \ \mathrm{d}x \tag{1.4.266}$$

$$\ln y^{-4} = \ln x^3 + b \qquad x^3y^4 = c \tag{1.4.267}$$

Both answers match.

(e) Wherever possible, separation of variables is a far shorter method. TBC.

17. Starting with $u(x, y) = x^2 \cos y$,

$$du = 0 = 2x \cos y \ \ \mathrm{d}x - x^2 \sin y \ \ \mathrm{d}y \tag{1.4.268}$$

$$\text{dividing by } x \text{ to destroy exactness,} \tag{1.4.269}$$

$$0 = 2 \cos y \ \ \mathrm{d}x - x \sin y \ dy \tag{1.4.270}$$

The above ODE is still solvable by separation, after destroying exactness. More complex integrating factors can be divided out from the ODE if needed.

18. **(a)** Test for exactness failed.

$$0 = \mathrm{d}y - y^2 \sin x \;\; \mathrm{d}x \tag{1.4.271}$$

$$P = -y^2 \sin x \qquad Q = 1 \tag{1.4.272}$$

$$\frac{\partial P}{\partial y} = -2y \sin x \qquad \frac{\partial Q}{\partial x} = 0 \tag{1.4.273}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \tag{1.4.274}$$

Finding the integrating factor,

$$R^* = \frac{1}{P}\left(Q_x - P_y\right) = \frac{2y \sin x}{-y^2 \sin x} = R^*(y) \tag{1.4.275}$$

$$R = \frac{-1}{Q}\left(Q_x - P_y\right) = \frac{2y \sin x}{-1} \tag{1.4.276}$$

$$F = \exp\left(\int R^*(y) \;\; \mathrm{d}y\right) = y^{-2} \tag{1.4.277}$$

Solving the ODE, using $(P, Q) \to (M, N)$

$$0 = y^{-2} \;\; \mathrm{d}y - \sin x \;\; \mathrm{d}x \qquad u = \int N \;\; \mathrm{d}y + k(x) \tag{1.4.278}$$

$$= \int y^{-2} \;\; \mathrm{d}y + k(x) \qquad u = -1/y + k(x) \tag{1.4.279}$$

$$\frac{\partial u}{\partial x} = M \qquad \frac{\mathrm{d}k}{\mathrm{d}x} + 0 = -\sin x \tag{1.4.280}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = -\sin x \qquad k = \cos x \tag{1.4.281}$$

$$u(x, y) = \frac{-1}{y} + \cos x \tag{1.4.282}$$

(b) Solving by separation of variables,

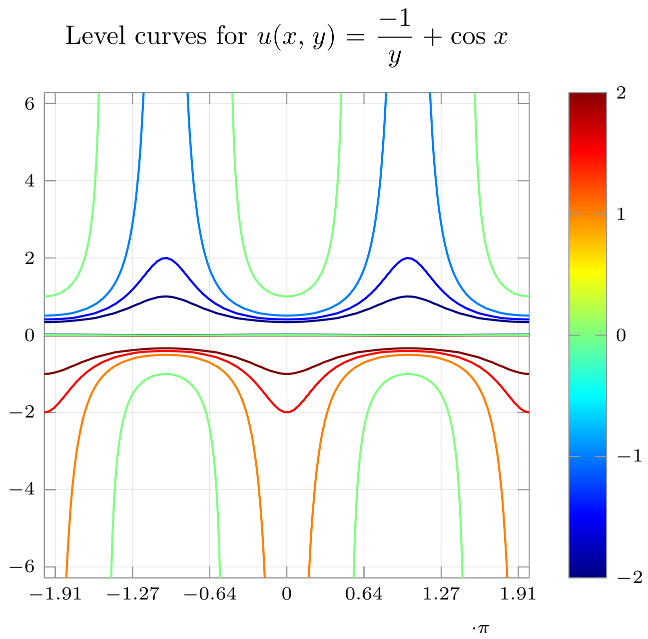
$$\frac{\mathrm{d}y}{y^2} = \sin x \;\; \mathrm{d}x \tag{1.4.283}$$

$$\int \frac{1}{y^2} \;\; \mathrm{d}y = \int \sin x \;\; \mathrm{d}x \tag{1.4.284}$$

$$\frac{-1}{y} + \cos x = c \tag{1.4.285}$$

Both methods match, but this method is simpler.

(c) Graphing the particular solutions given, $c = 0, -2, +2, +1.5, -1.5, +1, -1$



(d) The solutions $y = 0$ and $x = n\pi + \frac{\pi}{2}$ do not show up in the general solution.

1.5 Linear ODEs, Bernoulli equation, Population Dynamics

1. To prove the two relations,

$$\exp(-\ln x) = x^{-\ln e} = x^{-1} = \frac{1}{x} \tag{1.5.1}$$

$$\exp(-\ln \sec x) = \sec x^{-\ln e} = \sec x^{-1} \qquad = \cos x \tag{1.5.2}$$

2. Suppose $c \neq 0$,

$$h = \int p(x) \, \mathrm{d}x + c \tag{1.5.3}$$

$$y = e^{-c} \, e^{-h} \int e^{c} \, e^{h} r(x) \, \mathrm{d}x + b \tag{1.5.4}$$

$$y = e^{-h} \int e^{h} r(x) \, \mathrm{d}x + b \tag{1.5.5}$$

It is evident that the choice of c did not affect the final expression. So, the easy way out is to set $c = 0$

3. Solving ODE and graphing,

$$y' - y = 5.2 \tag{1.5.6}$$

$$p(x) = -1 \qquad r(x) = 5.2 \tag{1.5.7}$$

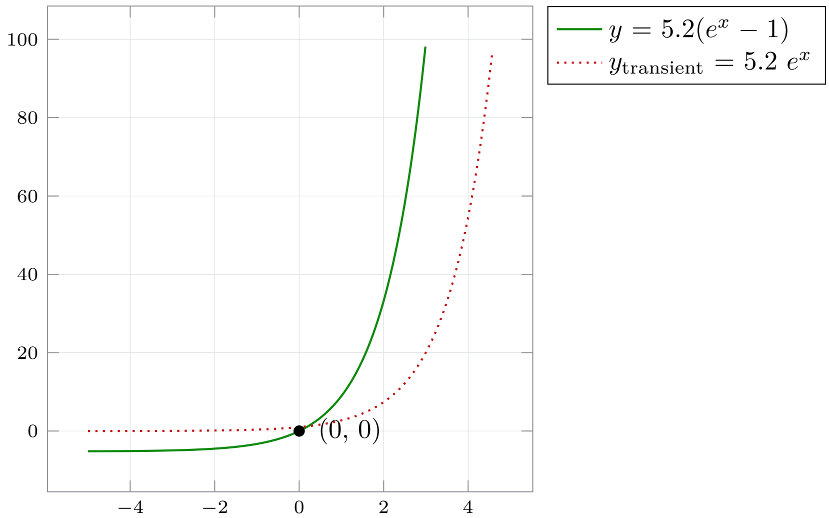
$$h = \int p(x) \ \mathrm{d}x \qquad = \int -1 \ \mathrm{d}x \tag{1.5.8}$$

$$= -x \tag{1.5.9}$$

$$y = e^{-h} \int e^{h} r(x) \ \mathrm{d}x + c \tag{1.5.10}$$

$$= e^{x} \left[\int e^{-x}(5.2) \ \mathrm{d}x + c \right] \tag{1.5.11}$$

$$y = -5.2 + ce^{x} \qquad c = 5.2 \tag{1.5.12}$$



4. Solving ODE and graphing,

$$y' - 2y = -4x \tag{1.5.13}$$

$$p(x) = -2 \qquad\qquad r(x) = -4x \tag{1.5.14}$$

$$h = \int p(x)\ \mathrm{d}x \qquad\qquad = \int -2\ \mathrm{d}x \tag{1.5.15}$$

$$= -2x \tag{1.5.16}$$

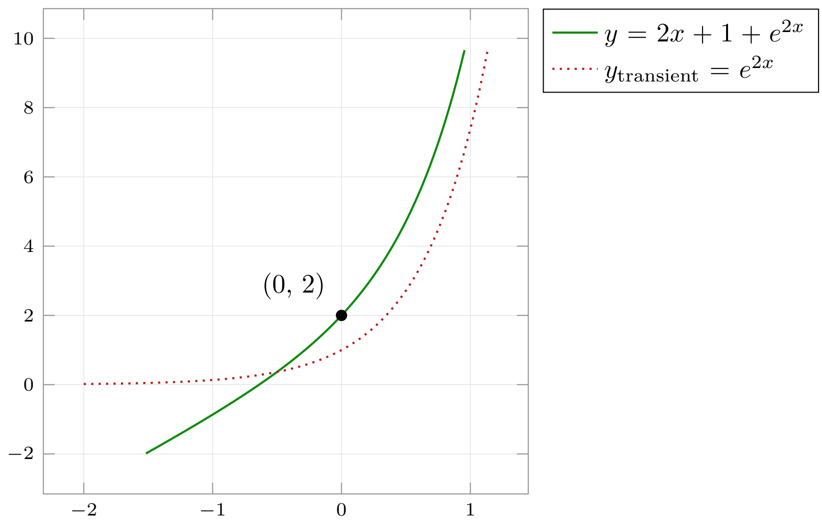
$$y = e^{-h} \int e^h r(x)\ \mathrm{d}x + c \tag{1.5.17}$$

$$= e^{2x} \left[\int e^{-2x}(-4x)\ \mathrm{d}x + c \right] \tag{1.5.18}$$

$$\int e^{-2x}(-4x)\ \mathrm{d}x = 2xe^{-2x} - \int 2e^{-2x}\ \mathrm{d}x \tag{1.5.19}$$

$$= e^{-2x}(2x+1) \tag{1.5.20}$$

$$y = (2x+1) + ce^{2x} \qquad\qquad c = 1 \tag{1.5.21}$$



5. Solving ODE and graphing,

$$y' + ky = e^{-kx} \tag{1.5.22}$$

$$p(x) = k \qquad r(x) = e^{-kx} \tag{1.5.23}$$

$$h = \int p(x)\ \mathrm{d}x \qquad = \int k\ \mathrm{d}x \tag{1.5.24}$$

$$= kx \tag{1.5.25}$$

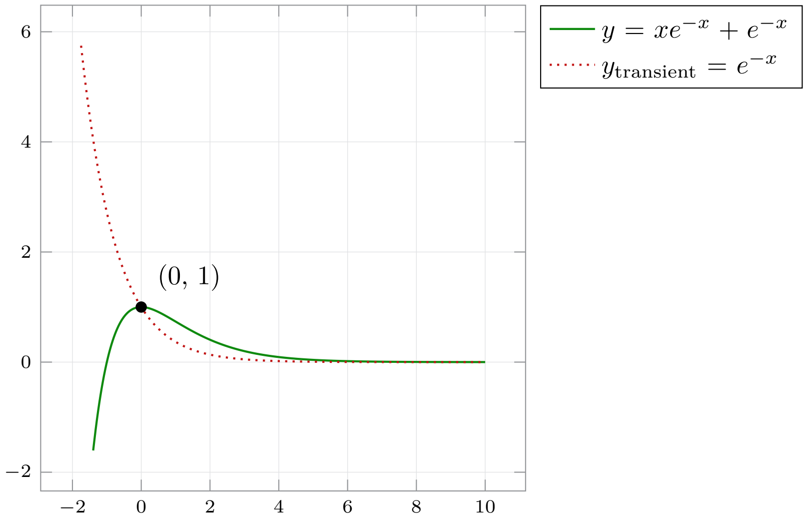
$$y = e^{-h} \int e^{h} r(x)\ \mathrm{d}x + c \tag{1.5.26}$$

$$= e^{-kx} \left[\int e^{kx} e^{-kx}\ \mathrm{d}x + c \right] \tag{1.5.27}$$

$$\int e^{kx} e^{-kx}\ \mathrm{d}x = x \tag{1.5.28}$$

$$= e^{-kx}(x + c) \tag{1.5.29}$$

$$y = xe^{-kx} + ce^{-kx} \qquad c = k = 1 \tag{1.5.30}$$



6. Solving ODE and graphing, using the IC $y(\pi/4) = 3$

$$y' + 2y = 4\cos 2x \tag{1.5.31}$$

$$p(x) = 2 \qquad r(x) = 4\cos 2x \tag{1.5.32}$$

$$h = \int p(x)\ \mathrm{d}x \qquad = \int 2\ \mathrm{d}x \tag{1.5.33}$$

$$= 2x \tag{1.5.34}$$

$$y = e^{-h} \int e^h r(x)\ \mathrm{d}x + c \tag{1.5.35}$$

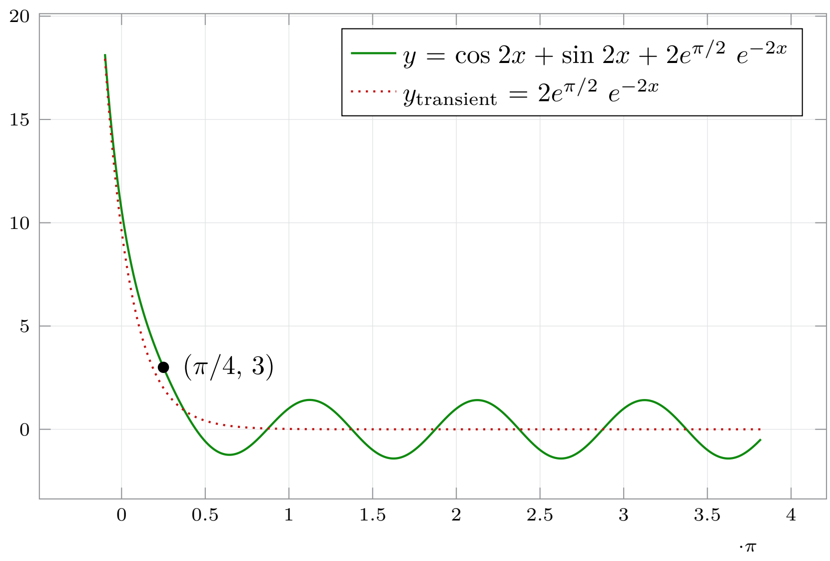
$$= e^{-2x} \left[\int e^{2x}\ 4\cos 2x\ \ \mathrm{d}x + c\right] \tag{1.5.36}$$

$$4 \int e^{2x} \cos 2x\ \ \mathrm{d}x = \frac{4e^{2x}}{2^2 + 2^2}\ (2\cos 2x + 2\sin 2x) \tag{1.5.37}$$

$$= e^{2x}(\cos 2x + \sin 2x) \tag{1.5.38}$$

$$\tag{1.5.39}$$

$$y = \cos 2x + \sin 2x + ce^{-2x} \qquad c = \frac{2}{e^{-\pi/2}} \tag{1.5.40}$$



7. Solving ODE and graphing, using the IC $y(\pi/4) = 3$

$$xy' - 2y = x^3 e^x \tag{1.5.41}$$

$$p(x) = \frac{-2}{x} \qquad r(x) = x^2 e^x \tag{1.5.42}$$

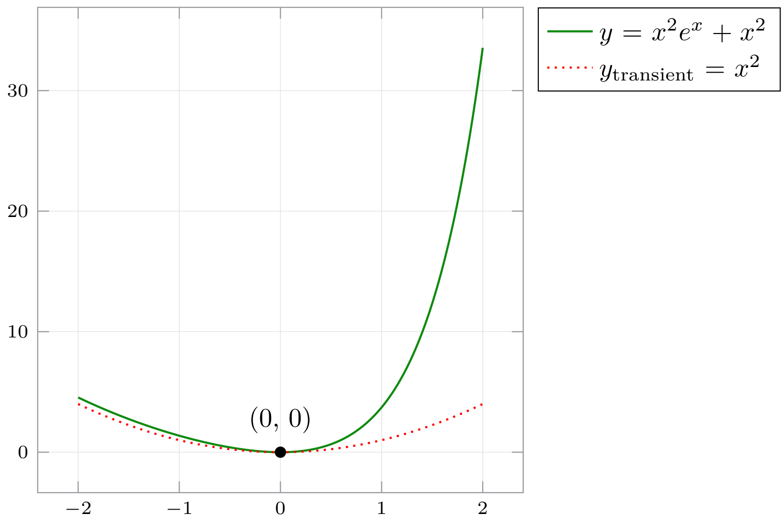
$$h = \int p(x)\ \mathrm{d}x \qquad = \int \frac{-2}{x}\ \mathrm{d}x \tag{1.5.43}$$

$$= -2\ \ln(x) \tag{1.5.44}$$

$$y = e^{-h} \int e^h r(x)\ \mathrm{d}x + c \tag{1.5.45}$$

$$= x^2 \left[\int x^{-2}\ (x^2 e^x)\ \mathrm{d}x + c \right] \tag{1.5.46}$$

$$y = x^2 e^x + cx^2 \qquad c = 1 \tag{1.5.47}$$



8. Solving ODE and graphing, using the IC $y(0) = 0$

$$y' + y \tan x = e^{-0.01x} \cos x \quad (1.5.48)$$

$$p(x) = \tan x \qquad r(x) = e^{-0.01x} \cos x \quad (1.5.49)$$

$$h = \int p(x) \ \mathrm{d}x \qquad = \int \tan x \ \mathrm{d}x \quad (1.5.50)$$

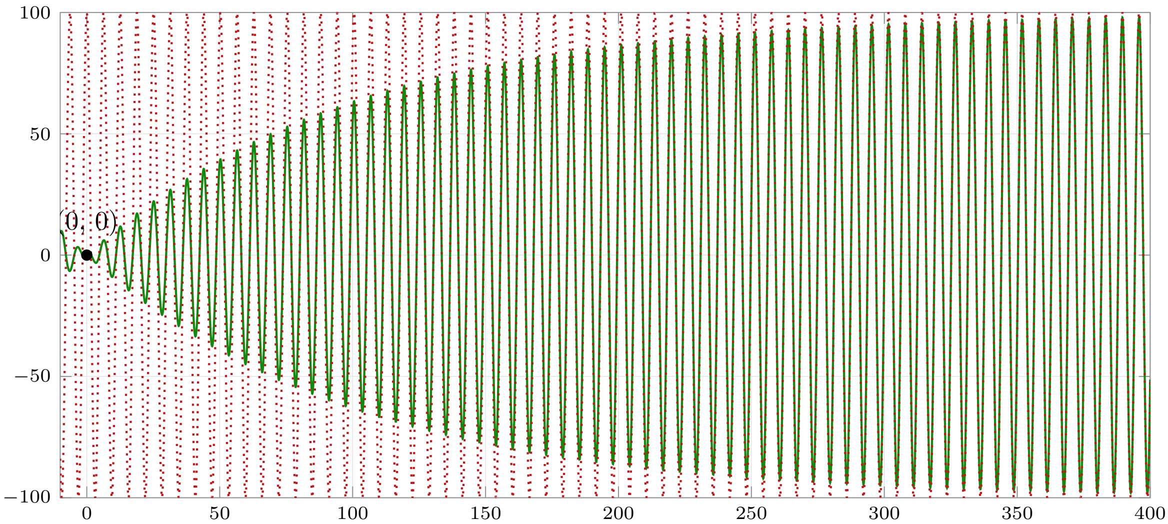
$$= \ln |\sec x| \quad (1.5.51)$$

$$y = e^{-h} \int e^{h} r(x) \ \mathrm{d}x + c \quad (1.5.52)$$

$$= \cos x \left[\int \sec x \ \left(e^{-0.01x} \cos x\right) \ \mathrm{d}x + c \right] \quad (1.5.53)$$

$$y = -100 \ \cos x \ e^{-0.01x} + c \ \cos x \qquad c = 100 \quad (1.5.54)$$

$$y = -100 \ \cos x \ \left(e^{-0.01x} - 1\right) \quad (1.5.55)$$



9. Solving ODE and graphing, using the IC $y(0) = -2.5$

$$y' + y \sin x = e^{\cos x} \tag{1.5.56}$$

$$p(x) = \sin x \qquad r(x) = e^{\cos x} \tag{1.5.57}$$

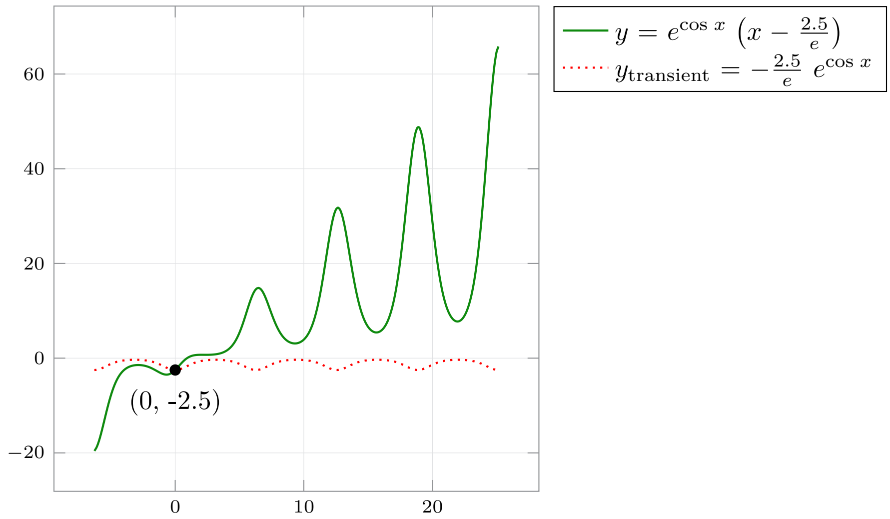
$$h = \int p(x) \ \mathrm{d}x \qquad = \int \sin x \ \mathrm{d}x \tag{1.5.58}$$

$$= - \cos x \tag{1.5.59}$$

$$y = e^{-h} \int e^h r(x) \ \mathrm{d}x + c \tag{1.5.60}$$

$$= e^{\cos x} \left[\int e^{- \cos x} \left(e^{\cos x} \right) \ \mathrm{d}x + c \right] \tag{1.5.61}$$

$$y = e^{\cos x}(x + c) \qquad c = \frac{-2.5}{e} \tag{1.5.62}$$



10. Solving ODE and graphing, using the IC $y(\pi/4) = 4/3$

$$y' \cos x = (1 - 3y) \sec x \tag{1.5.63}$$

$$y' + 3 \sec^2 x \; y = \sec^2 x \tag{1.5.64}$$

$$p(x) = 3 \sec^2 x \tag{1.5.65}$$

$$r(x) = \sec^2 x \tag{1.5.66}$$

$$h = \int p(x) \; \mathrm{d}x \tag{1.5.67}$$

$$= 3 \int \sec^2 x \; \mathrm{d}x = 3 \tan x \tag{1.5.68}$$

$$y = e^{-h} \int e^h r(x) \; \mathrm{d}x + c \tag{1.5.69}$$

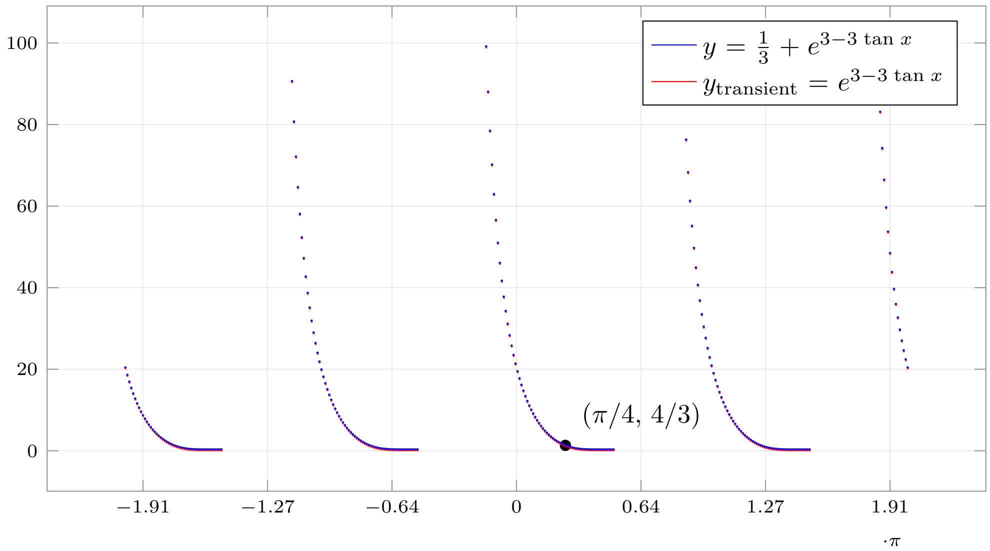
$$= e^{-3 \tan x} \left[\int e^{3 \tan x} \left(\sec^2 x\right) \; \mathrm{d}x + c \right] \tag{1.5.70}$$

$$\int e^{3 \tan x} \left(\sec^2 x\right) \; \mathrm{d}x = \int e^{3u} \; \mathrm{d}u \tag{1.5.71}$$

$$= \frac{e^{3u}}{3} = \frac{e^{3 \tan x}}{3} \tag{1.5.72}$$

$$y = \frac{1}{3} + c e^{-3 \tan x} \tag{1.5.73}$$

$$c = e^3 \tag{1.5.74}$$



11. Solving ODE and graphing, using the IC $y(0) = -2.5$

$$y' = (y-2)\cot x \qquad \text{1.5.75}$$

$$y' - \cot x\; y = -2\cot x \qquad \text{1.5.76}$$

$$p(x) = -\cot x \qquad r(x) = -2\cot x \qquad \text{1.5.77}$$

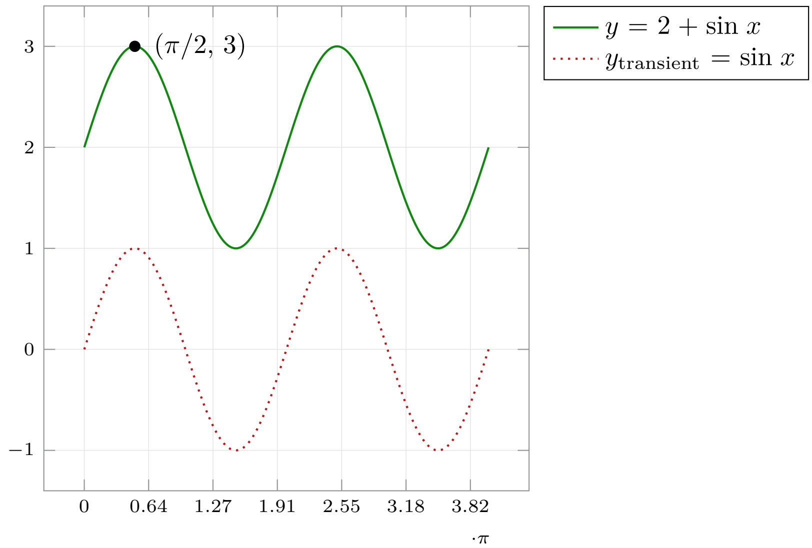
$$h = \int p(x)\ \mathrm{d}x \qquad = -\int \cot x\ \mathrm{d}x \qquad \text{1.5.78}$$

$$= -\ln|\sin x| \qquad \text{1.5.79}$$

$$y = e^{-h}\int e^{h} r(x)\ \mathrm{d}x + c \qquad \text{1.5.80}$$

$$= \sin x\left[\int \csc x\,(-2\cot x)\ \mathrm{d}x + c\right] \qquad \text{1.5.81}$$

$$y = 2 + c\sin x \qquad c = 1 \qquad \text{1.5.82}$$



12. Solving ODE and graphing, using the IC $y(1) = 2$

$$xy' + 4y = 8x^4 \tag{1.5.83}$$

$$p(x) = 4/x \qquad r(x) = 8x^3 \tag{1.5.84}$$

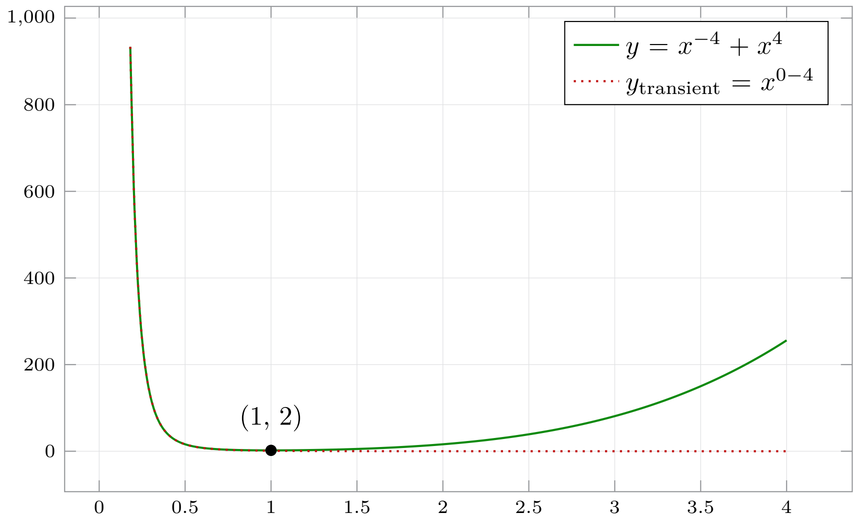
$$h = \int p(x)\ \mathrm{d}x \qquad = \int \frac{4}{x}\ \mathrm{d}x \tag{1.5.85}$$

$$= 4\ln x \tag{1.5.86}$$

$$y = e^{-h} \int e^h r(x)\ \mathrm{d}x + c \tag{1.5.87}$$

$$= x^{-4}\left[\int x^4\ (8x^3)\ \mathrm{d}x + c\right] \tag{1.5.88}$$

$$y = cx^{-4} + x^4 \qquad c = 1 \tag{1.5.89}$$



13. Solving ODE by separation of variables and graphing

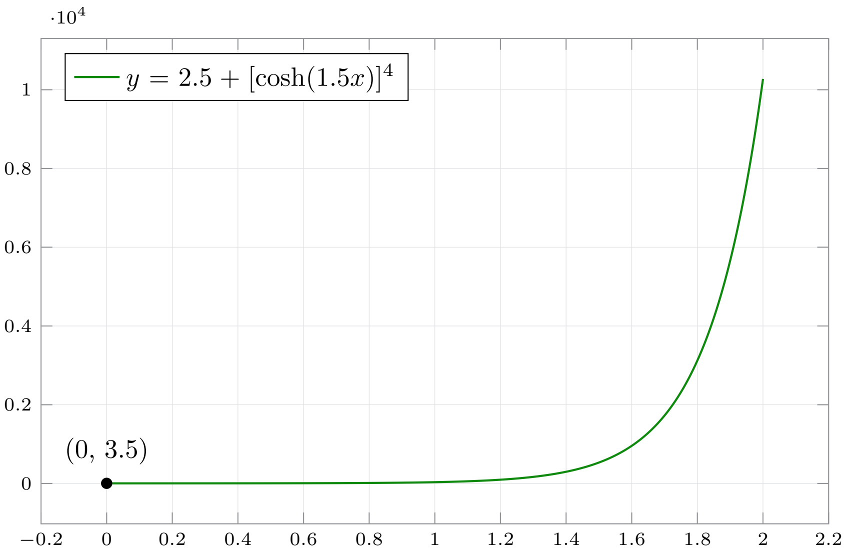
$$y' = (6y - 15)\tanh(1.5x) \tag{1.5.90}$$

$$\int \frac{1}{6y-15}\ \mathrm{d}y = \int \tanh(1.5x)\ \mathrm{d}x \tag{1.5.91}$$

$$\frac{1}{6}\ \ln\left(\frac{1}{6y-15}\right) = \frac{2}{3}\ \ln[\cosh(1.5x)] \tag{1.5.92}$$

$$6y - 15 = b[\cosh(1.5x)]^{-4} \tag{1.5.93}$$

$$y = 2.5 + c\ [\cosh(1.5x)]^4 \tag{1.5.94}$$



14. Solving,

(a) Solving ODE

$$y' - \frac{y}{x} = \frac{-1}{x} \cos(1/x) \tag{1.5.95}$$

$$p(x) = -1/x \qquad r(x) = \frac{-1}{x} \cos(1/x) \tag{1.5.96}$$

$$h = \int p(x) \, dx \qquad = \int \frac{-1}{x} \, dx \tag{1.5.97}$$

$$= -\ln x \tag{1.5.98}$$

$$y = e^{-h} \int e^h r(x) \, dx + c \tag{1.5.99}$$

$$= x \left[\int \frac{-1}{x^2} \cos(1/x) \, dx + c \right] \tag{1.5.100}$$

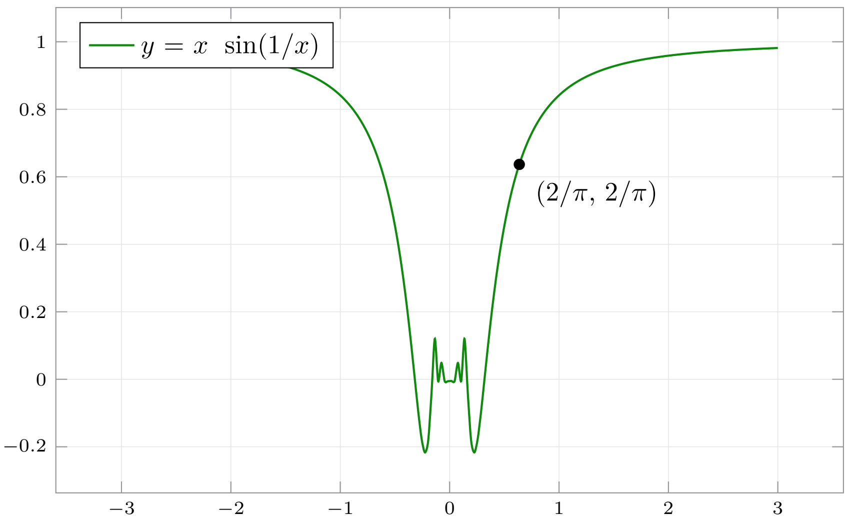
$$I = \int \frac{-1}{x^2} \cos(1/x) \, dx \tag{1.5.101}$$

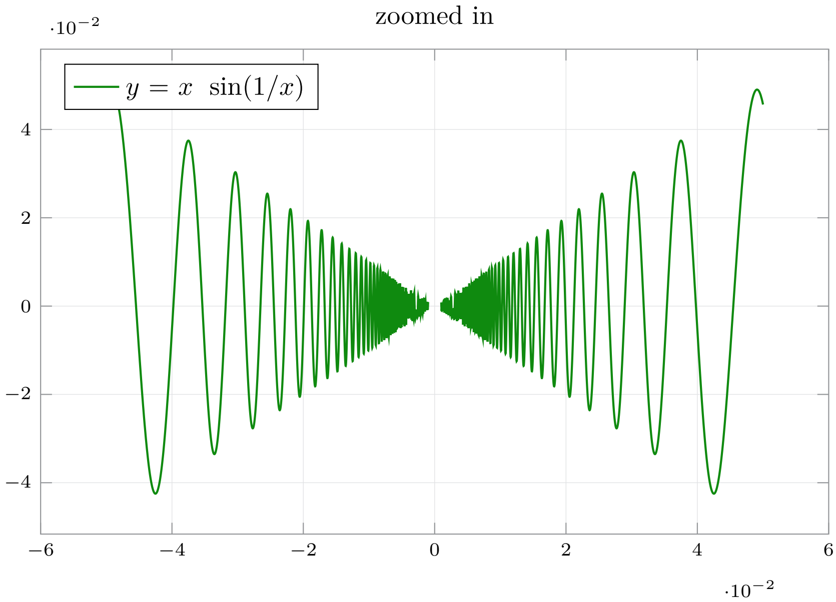
$$= \int \cos(u) \, du \qquad u = \frac{1}{x} \quad du = \frac{-1}{x^2} dx \tag{1.5.102}$$

$$= \sin u = \sin(1/x) \tag{1.5.103}$$

$$y = cx + x \sin(1/x) \tag{1.5.104}$$

For $c = 0$, an IC is $(2/\pi, 2/\pi)$





(b) Solving ODE for general n

$$y' - \frac{ny}{x} = -x^{n-2} \cos(1/x) \quad 1.5.105$$

$$p(x) = -n/x \qquad r(x) = -x^{n-2} \cos(1/x) \quad 1.5.106$$

$$h = \int p(x)\ \mathrm{d}x \qquad = \int \frac{-n}{x}\ \mathrm{d}x \quad 1.5.107$$

$$= -n \ln x \quad 1.5.108$$

$$y = e^{-h} \int e^h r(x)\ \mathrm{d}x + c \quad 1.5.109$$

$$= x^n \left[\int -x^{-2} \cos(1/x)\ \mathrm{d}x + c\right] \quad 1.5.110$$

$$I = \int \frac{-1}{x^2} \cos(1/x)\ \mathrm{d}x \quad 1.5.111$$

$$= \int \cos(u)\ du \qquad u = \frac{1}{x} \quad du = \frac{-1}{x^2}dx \quad 1.5.112$$

$$= \sin u = \sin(1/x) \quad 1.5.113$$

$$y = cx^n + x^n \sin(1/x) \quad 1.5.114$$

The graph is untractable close to $x = 0$. For large positive powers, the expression is dominated by the polynomial term. TBC.

15. y_1 and y_2 are solutions of the homogenous ODE, then

$$\text{Let} \quad y_1' + py_1 = 0 \qquad y_2' + py_2 = 0 \quad 1.5.115$$

$$y_1' + y_2' + (y_1 + y_2)p = 0 \qquad \text{summing} \quad 1.5.116$$

$$y_3' + py_3 = 0 \qquad y_1 + y_2 = y_3 \quad 1.5.117$$

$y_1 + y_2$ is also a solution to the homogenous ODE. This does not hold true for the nonhomogenous ODE as the RHS is $2r$ instead of r.

$$\text{Let} \quad y_1' + py_1 = r \qquad y_2' + py_2 = r \quad 1.5.118$$

$$y_1' + y_2' + (y_1 + y_2)p = 2r \qquad \text{summing} \quad 1.5.119$$

$$y_3' + py_3 \neq r \qquad y_1 + y_2 = y_3 \quad 1.5.120$$

For a scalar multiple a, y_4 is also a solution to the homogenous ODE. This does not hold for the inhomogenous ODE.

$$
\begin{aligned}
\text{Let}\quad y_1' + py_1 &= 0 & ay_1' + apy_1 &= 0 && \text{1.5.121}\\
y_4' + py_4 &= 0 & y_4 &= ay_1 && \text{1.5.122}\\
\text{Let}\quad y_1' + py_1 &= r & ay_1' + apy_1 &= ar && \text{1.5.123}\\
y_4' + py_4 &\neq r & y_4 &= ay_1 && \text{1.5.124}
\end{aligned}
$$

16. The trivial solution $y_t(x) \equiv 0 \; \forall \; x$ is a solution of every homogenous ODE, but not every nonhomogenous ODE.

$$
\begin{aligned}
\text{Let}\quad y_1' + py_1 &= 0 && \text{1.5.125}\\
y_t' + py_t &= 0 && \text{1.5.126}\\
0 &= 0 && \text{1.5.127}\\
\text{Let}\quad y_1' + py_1 &= r && \text{1.5.128}\\
y_t' + py_t &= r && \text{1.5.129}\\
0 &\neq r && \text{1.5.130}
\end{aligned}
$$

17. Let y_n and y_h solve the nonhomogenous and homogenous ODE respectively.

$$
\begin{aligned}
\text{Let}\quad y_h' + py_h &= 0 & y_n' + py_n &= r && \text{1.5.131}\\
y_h' + y_n' + p(y_h + y_n) &= 0 + r & & && \text{1.5.132}\\
y_c' + py_c &= r & y_h + y_n &= y_c && \text{1.5.133}
\end{aligned}
$$

Their sum is also a solution to the nonhomogenous ODE.

18. Let y_1 and y_2 solve the nonhomogenous ODE,

$$
\begin{aligned}
\text{Let}\quad y_1' + py_1 &= r & y_2' + py_2 &= r && \text{1.5.134}\\
y_1' - y_2' + (y_1 - y_2)p &= 0 & &\text{difference} && \text{1.5.135}\\
y_3' + py_3 &= 0 & y_1 - y_2 &= y_3 && \text{1.5.136}
\end{aligned}
$$

Their difference solves the homogenous ODE.

19. Let y_1 solve the nonhomogenous ODE and $y_2 = cy_1$,

$$\text{Let} \quad y_1' + py_1 = r \qquad cy_1' + cpy_1 = cr \tag{1.5.137}$$

$$y_2' + py_2 = cr \tag{1.5.138}$$

cy_1 solves a different nonhomogenous ODE with RHS being cr.

20. Both ODEs have the same p and differ in their RHS.

$$\text{Let} \quad y_1' + py_1 = r_1 \qquad y_2' + py_2 = r_2 \tag{1.5.139}$$

$$y_1' + y_2' + (y_1 + y_2)p = r_1 + r_2 \qquad \text{summing} \tag{1.5.140}$$

$$y_3' + py_3 = r_3 \qquad r_1 + r_2 = r_3 \tag{1.5.141}$$

$y_1 + y_2$ solves the ODE with the same p and RHS $r_1 + r_2$.

21. Method of variation,

$$y = c \ \exp\left(-\int \ p(x) \ \mathrm{d}x\right) = cy_* \tag{1.5.142}$$

$$y_* \quad \text{solves the ODE} \quad y' + py = 0 \tag{1.5.143}$$

$$y_*' + py_* = 0 \tag{1.5.144}$$

uy_* is a solution of the nonhomogenous ODE $y' + py = r$,

$$y = u(x) \ \exp\left(-\int \ p(x) \ \mathrm{d}x\right) \qquad = u(x) \ e^{-h} \tag{1.5.145}$$

$$r = u'y_* + uy_*' + p(uy_*) \qquad r = u'y_* + u(y_*' + py_*) \tag{1.5.146}$$

$$u' = \frac{r}{y_*} \qquad h = \int \ p(x) \ \mathrm{d}x \tag{1.5.147}$$

$$u' = re^h \tag{1.5.148}$$

$$u = \int \ e^h \ r(x) \ \mathrm{d}x + c \tag{1.5.149}$$

$$y = u(x) \ e^{-h} \qquad = e^{-h} \left[\int \ e^h \ r(x) \ \mathrm{d}x + c\right] \tag{1.5.150}$$

which matches the earlier result.

22. Solving ODE by separation of variables and graphing, given the IC $y(0) = -1/3$

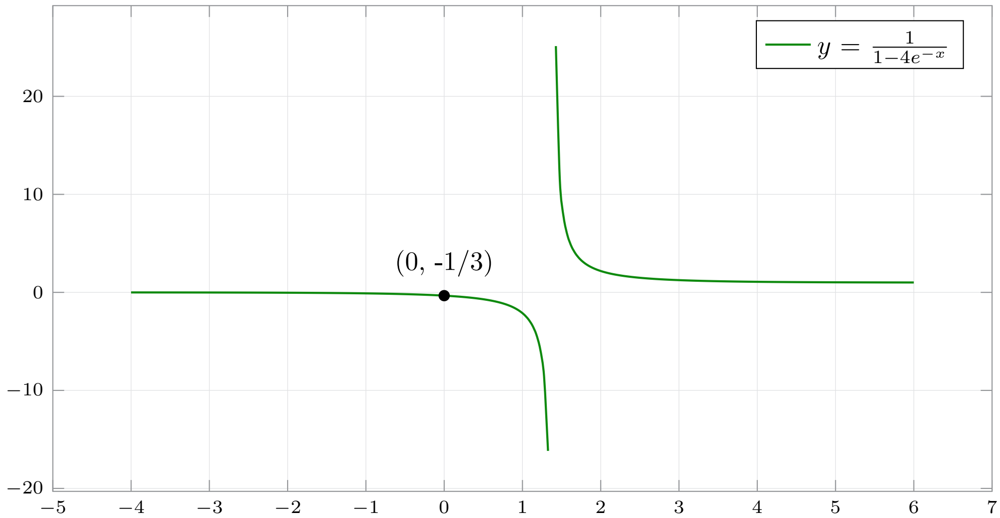
$$y' + y = y^2 \tag{1.5.151}$$

$$\int \frac{1}{y(y-1)} \; \mathrm{d}y = \int \; \mathrm{d}x \tag{1.5.152}$$

$$\ln y - \ln(y-1) = x + b \tag{1.5.153}$$

$$\frac{y}{y-1} = ce^x \tag{1.5.154}$$

$$y = \frac{1}{1-ce^{-x}} \qquad\qquad c = 4 \tag{1.5.155}$$



23. Solving ODE by separation of variables and graphing, given the IC $y(0) = 3$

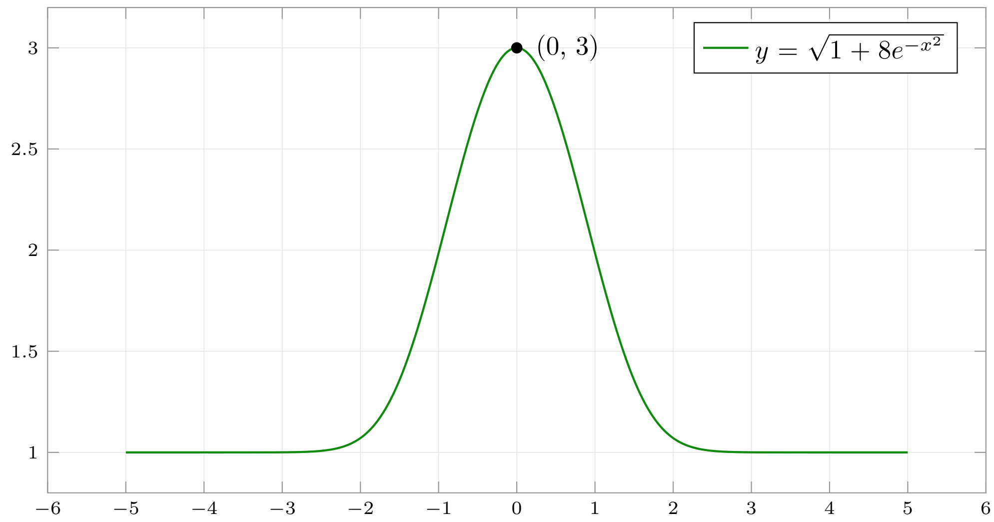
$$y' + xy = xy^{-1} \tag{1.5.156}$$

$$\int \frac{y}{(1-y^2)} \; \mathrm{d}y = \int \; x \; \mathrm{d}x \tag{1.5.157}$$

$$\ln(1-y^2) = -x^2 + b \tag{1.5.158}$$

$$(1-y^2) = ce^{-x^2} \tag{1.5.159}$$

$$y = \sqrt{1-ce^{-x^2}} \qquad\qquad c = -8 \tag{1.5.160}$$



24. Solving ODE and graphing,

$$y' + y = -xy^{-1} \tag{1.5.161}$$

$$y' + py = gy^{a} \qquad a = -1 \tag{1.5.162}$$

$$p(x) = 1 \qquad g(x) = -x \tag{1.5.163}$$

$$u' + (1-a)pu = (1-a)g \qquad u = y^{(1-a)} \tag{1.5.164}$$

$$u' + 2pu = 2g \tag{1.5.165}$$

$$h = \int 2p(x) \ \mathrm{d}x \qquad = \int 2 \ \mathrm{d}x \tag{1.5.166}$$

$$= 2x \tag{1.5.167}$$

$$y = e^{-h} \int e^{h}\{2g(x)\} \ \mathrm{d}x + c \tag{1.5.168}$$

$$= e^{-2x}\left[\int e^{2x} \ \{-2x\} \ \mathrm{d}x + c\right] \tag{1.5.169}$$

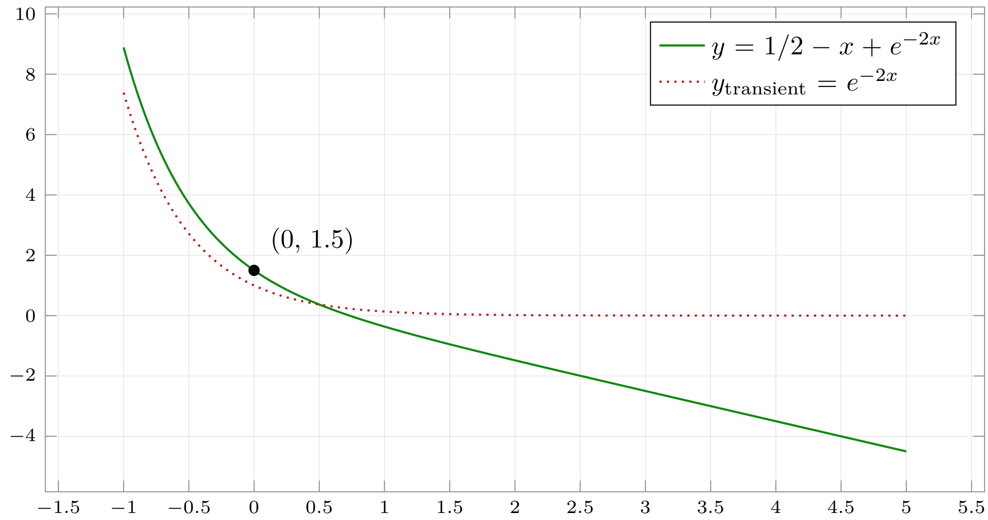
$$I = \int e^{2x} \ \{2x\} \ \mathrm{d}x \tag{1.5.170}$$

$$= xe^{2x} - \int e^{2x} \ \mathrm{d}x \tag{1.5.171}$$

$$= \left(x - \frac{1}{2}\right) e^{2x} \tag{1.5.172}$$

$$= \sin u = \sin(1/x) \tag{1.5.173}$$

$$y = \frac{1}{2} - x + ce^{-2x} \qquad c = 1 \tag{1.5.174}$$



25. Solving ODE by separation and graphing,

$$y' = 3.2y - 10y^2 \tag{1.5.175}$$

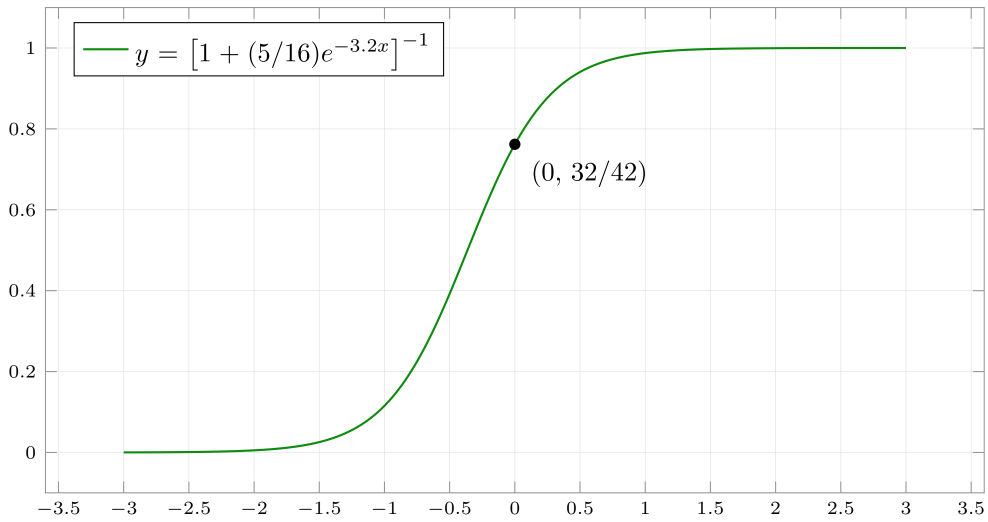
$$\int \frac{1}{y(3.2 - 10y)}\ \mathrm{d}y = \int\ \mathrm{d}x \tag{1.5.176}$$

$$3.2A - 10Ay + By = 1 \qquad A = \frac{1}{3.2},\ \ B = \frac{10}{3.2} \tag{1.5.177}$$

$$\ln\left(\frac{y}{3.2 - 10y}\right) = 3.2x + b \tag{1.5.178}$$

$$\frac{y}{3.2 - 10y} = ce^{3.2x} \tag{1.5.179}$$

$$y = \frac{3.2e^{3.2x}}{1 + 10ce^{3.2x}} \qquad c = 0.32 \tag{1.5.180}$$



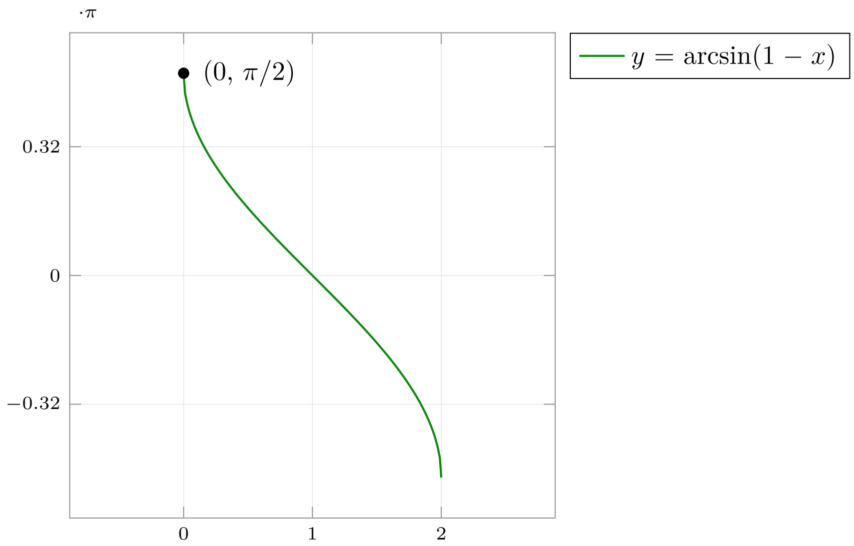
26. Solving ODE by separation and graphing,

$$y' = \frac{\tan y}{x-1} \tag{1.5.181}$$

$$\int \cot y \ \ \mathrm{d}y = \int \frac{1}{x-1} \ \ \mathrm{d}x \tag{1.5.182}$$

$$\ln|\sin y| = \ln(x-1) + b \tag{1.5.183}$$

$$y = \arcsin[c \cdot (x-1)] \qquad\qquad c = -1 \tag{1.5.184}$$



27. Solving ODE by separation and graphing, setting $c = 1$,

$$y' = \frac{1}{6e^y - 2x} \quad 1.5.185$$

$$\frac{dx}{dy} = 6e^y - 2x \quad 1.5.186$$

$$y' + 2y = 6e^x \qquad x \rightleftharpoons y \quad 1.5.187$$

$$p(x) = 2 \qquad r(x) = 6e^x \quad 1.5.188$$

$$h = \int p(x) \ \mathrm{d}x \qquad = \int 2 \ \mathrm{d}x \quad 1.5.189$$

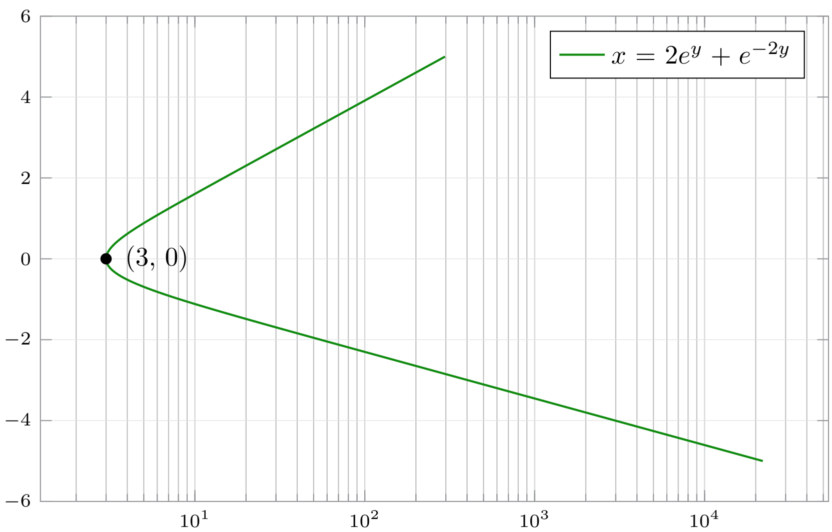
$$= 2x \quad 1.5.190$$

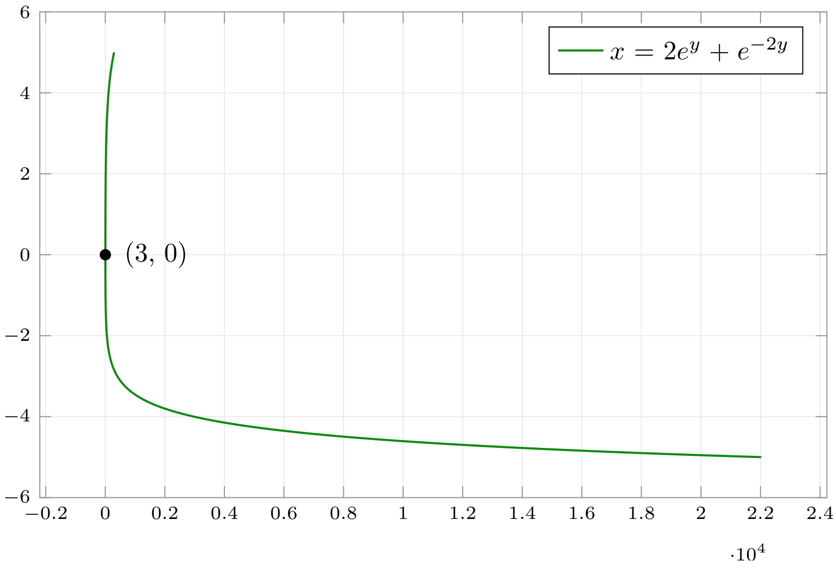
$$y = e^{-h} \int e^h \ r(x) \ \mathrm{d}x + c \quad 1.5.191$$

$$= e^{-2x} \left[\int e^{2x} \ \{6e^x\} \ \mathrm{d}x + c \right] \quad 1.5.192$$

$$y = 2e^x + ce^{-2x} \quad 1.5.193$$

$$x = 2e^y + ce^{-2y} \qquad y \rightleftharpoons x \quad 1.5.194$$





28. Solving ODE by separation and graphing,

$$2xyy' + (x-1)y^2 = x^2 e^x \tag{1.5.195}$$

$$y^2 = z \qquad 2yy' = z' \tag{1.5.196}$$

$$z' + \frac{x-1}{x} z = xe^x \tag{1.5.197}$$

$$p(x) = 1 - 1/x \qquad r(x) = xe^x \tag{1.5.198}$$

$$h = \int p(x) \ \mathrm{d}x \qquad = \int (1 - 1/x) \ \mathrm{d}x \tag{1.5.199}$$

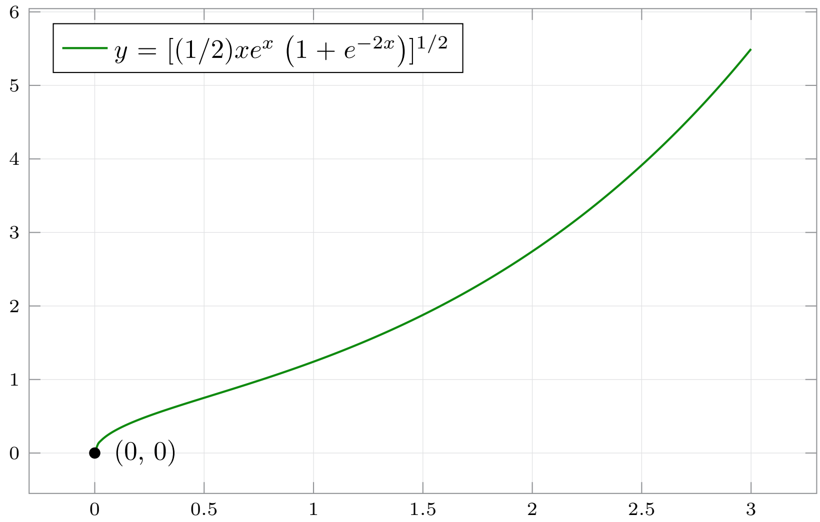
$$= x - \ln x \tag{1.5.200}$$

$$z = e^{-h} \int e^h \, r(x) \ \mathrm{d}x + c \tag{1.5.201}$$

$$= xe^{-x} \left[\int \frac{e^x}{x} \{xe^x\} \ \mathrm{d}x + c \right] \tag{1.5.202}$$

$$z = cxe^{-x} + \frac{xe^x}{2} \tag{1.5.203}$$

$$y = \sqrt{\frac{xe^x}{2} (1 + e^{-2x})} \qquad c = 1 \tag{1.5.204}$$



29. Examples are in the exercises above.

(a) Separable ODEs are of the form,

$$\frac{dy}{dx} = \frac{g(x)}{f(y)} \tag{1.5.205}$$

$$\int f(y)\ \mathrm{d}y = \int g(x)\ \mathrm{d}x \tag{1.5.206}$$

(b) Exact ODEs are of the form,

$$M(x, y)\ \mathrm{d}x + N(x, y)\ \mathrm{d}y = 0 \tag{1.5.207}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.5.208}$$

They are useful when separation of variables is not possible.

(c) Linear ODEs are of the form,

$$y' + p(x)y = r(x) \tag{1.5.209}$$

$$y = e^{-h} \int e^{h}\ r(x)\ \mathrm{d}x + c \tag{1.5.210}$$

$$h = \int p(x)\ \mathrm{d}x \tag{1.5.211}$$

This form helps separate the transient response (term with c) from the steady state response (term without c).

30. Solving,

(a) Riccati's equation, (with p, g, h being functions of x)

$$y' + py = gy^2 + h \tag{1.5.212}$$

$$Y' + pY = gY^2 + h \tag{1.5.213}$$

$$Y \text{ solves the ODE} \tag{1.5.214}$$

$$\left(Y + \frac{1}{u}\right)' + p\left(Y + \frac{1}{u}\right) = g\left(Y + \frac{1}{u}\right)^2 + h \tag{1.5.215}$$

$$y = Y + \frac{1}{u} \tag{1.5.216}$$

$$Y' + pY - \frac{u'}{u^2} + \frac{p}{u} = gY^2 + h + \frac{g}{u^2} + \frac{2Yg}{u} \tag{1.5.217}$$

$$u' - pu = -g - 2Ygu \tag{1.5.218}$$

$$u' + (2Yg - p)\,u = -g \tag{1.5.219}$$

(b) $Y = x$ is a solution of the ODE,

$$y' - (2x^3 + 1)y = -x^2y^2 - x^4 - x + 1 \tag{1.5.220}$$

$$y' + py = gy^2 + h \tag{1.5.221}$$

$$p(x) = -(2x^3 + 1) \qquad g(x) = -x^2 \tag{1.5.222}$$

$$h(x) = 1 - x - x^4 \tag{1.5.223}$$

Checking if the given Y is a solution,

$$1 - (2x^3 + 1)x + (x^4 + x - 1) + x^4 = 0 \tag{1.5.224}$$

Solving the Riccati equation,

$$u' + (2Yg - p)\ u = -g \tag{1.5.225}$$

$$u' + (-2x^3 + 2x^3 + 1)\ u = x^2 \tag{1.5.226}$$

$$u' + u = x^2 \tag{1.5.227}$$

$$u = e^{-x}\ \left[\int\ e^x\ x^2\ \ \mathrm{d}x + c\right] \tag{1.5.228}$$

$$I = \int\ e^x\ x^2\ \ \mathrm{d}x \tag{1.5.229}$$

$$= x^2e^x - \int\ 2xe^x\ \ \mathrm{d}x \tag{1.5.230}$$

$$= x^2e^x - 2xe^x + \int\ 2e^x\ \ \mathrm{d}x \tag{1.5.231}$$

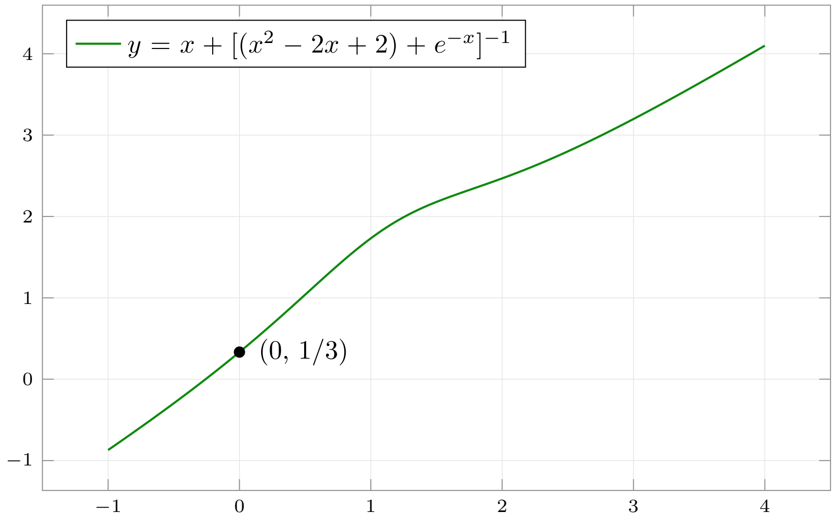
$$= (x^2 - 2x + 2)e^x \tag{1.5.232}$$

$$u = (x^2 - 2x + 2) + ce^{-x} \tag{1.5.233}$$

$$= \frac{1}{y - x} \tag{1.5.234}$$

$$y = x + \frac{1}{(x^2 - 2x + 2) + ce^{-x}} \qquad c = 1 \tag{1.5.235}$$

This graph passes through (0, 1/3).



(c) Clairaut equation,

$$y'^2 - xy' + y = 0 \quad \text{1.5.236}$$

$$2y'y'' - xy'' - y' + y' = 0 \qquad \text{differentiating} \quad \text{1.5.237}$$

$$y''\,(2y' - x) = 0 \quad \text{1.5.238}$$

$$y'' = 0 \qquad y = ax + b \quad \text{1.5.239}$$

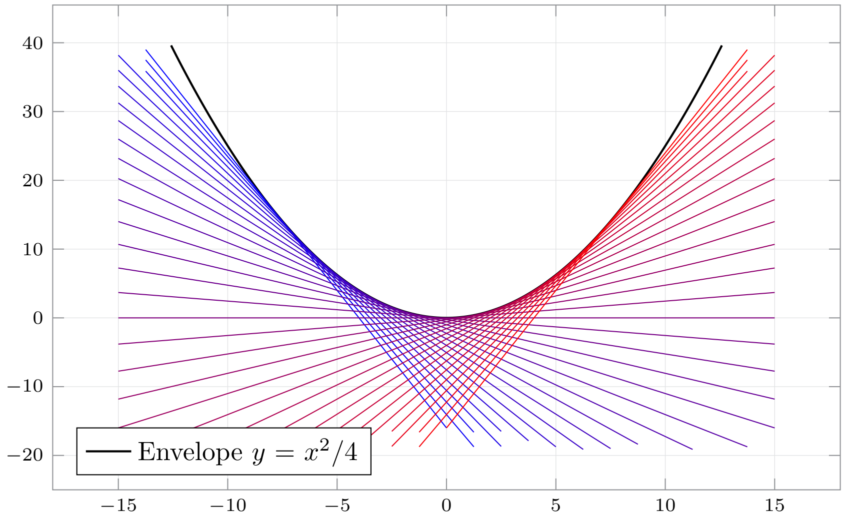
$$y' = x/2 \qquad y = x^2/4 + c \quad \text{1.5.240}$$

$$Y = ax + b \quad \text{1.5.241}$$

$$Y'^2 - xY' + Y = a^2 - ax + ax + b \qquad = 0 \quad \text{1.5.242}$$

$$-a^2 = b \qquad c = 0 \quad \text{1.5.243}$$

General solution is $Y = ax - a^2$. A singular solution $y = x^2/4$ which cannot be obtained from the general solution also exists.



(d) The general Clairaut equation is,

$$y = xy' + g(y') \quad \text{1.5.244}$$

$$Y = cx + g(c) \qquad Y' = c \quad \text{1.5.245}$$

$$xY' + g(Y') = xc + g(c) = Y \quad \text{1.5.246}$$

This straight line family $Y = xc + g(c)$ is thus a solution to the ODE.

$$g'(y') = -x \qquad g(y') = \frac{-x^2}{2} \tag{1.5.247}$$

$$y' = xy'' + y' + g'(y')y'' \qquad 0 = xy'' + g'(y')y'' \tag{1.5.248}$$

$$-x = g'(y') \tag{1.5.249}$$

$g'(y') = -x$ is thus a solution to the ODE.

31. Newton's law of cooling, with temperature y and time t,

$$y' = k(y - y_a) \tag{1.5.250}$$

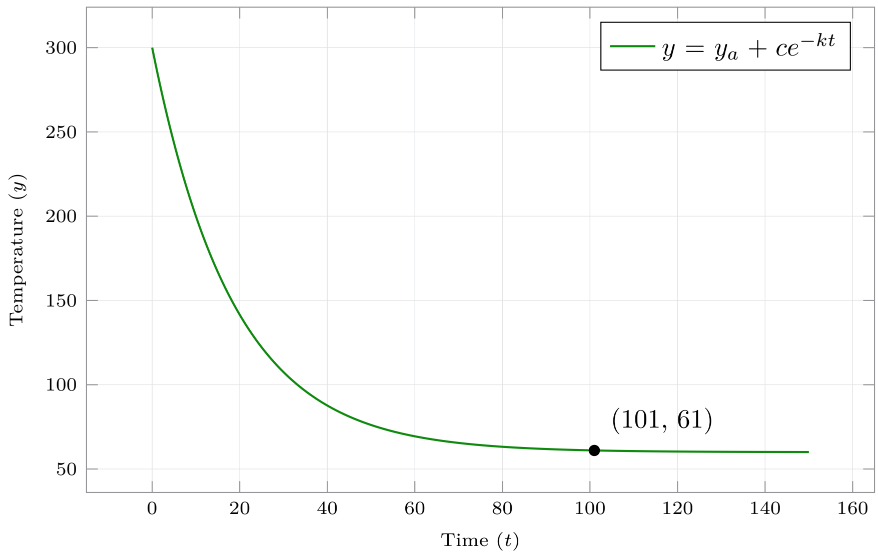
$$\ln(y - y_a) = kt + b \qquad y = y_a + ce^{kt} \tag{1.5.251}$$

$$y(0) = 300 \qquad c = 240 \tag{1.5.252}$$

$$y(10) = 200 \qquad k = \frac{1}{10} \ln(7/12) \tag{1.5.253}$$

$$y(t^*) = 61 \qquad t^* = 10 \frac{\ln(240)}{\ln(12/7)} \tag{1.5.254}$$

$$t^* = 101.68 \text{ min} \tag{1.5.255}$$



32. Newton's law of cooling, with temperature y and time x,

$$y' = k_1(y - y_a) + k_2(y - y_\omega) + P \tag{1.5.256}$$

$$y' - y\ (k_1 + k_2) = P - k_1 y_a - k_2 y_\omega \tag{1.5.257}$$

$$= [P - k_2 y_\omega - k_1 A] + k_1 C \cos(\lambda x) \tag{1.5.258}$$

$$p(x) = -m \tag{1.5.259}$$

$$r(x) = a + b \cos(\lambda x) \tag{1.5.260}$$

$$h = \int p(x)\ \mathrm{d}x \tag{1.5.261}$$

$$= \int -m\ \mathrm{d}x \tag{1.5.262}$$

$$= -mx \tag{1.5.263}$$

$$y = e^{-h} \int e^h\ r(x)\ \mathrm{d}x + c \tag{1.5.264}$$

$$= e^{mx} \left[\int e^{-mx}\ \{a + b \cos(\lambda x)\}\ \ \mathrm{d}x + c \right] \tag{1.5.265}$$

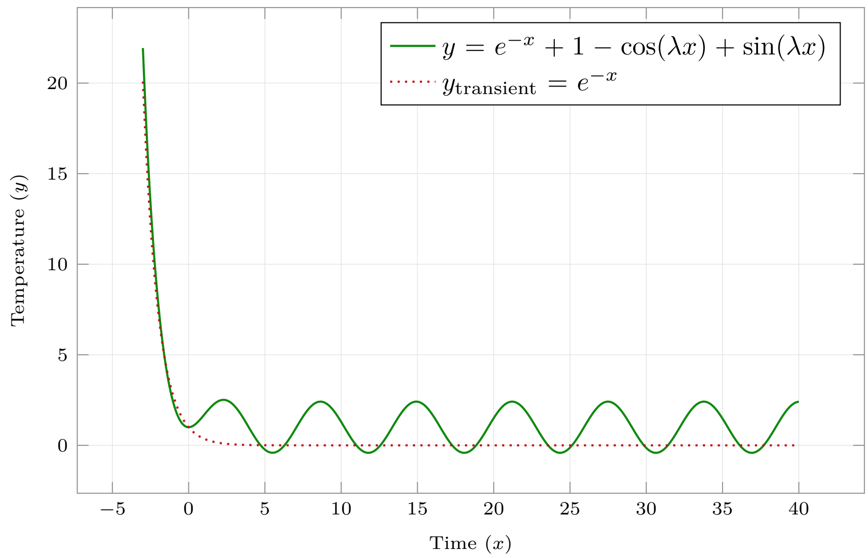
$$y = ce^{mx} - \frac{a}{m} + \left[\frac{b}{m^2 + \lambda^2} \right] (-m \cos(\lambda x) + \lambda \sin(\lambda x)) \tag{1.5.266}$$

$$= ce^{(k_1 + k_2)x} - \frac{P - k_1 y_a - k_2 y_\omega}{k_1 + k_2} \tag{1.5.267}$$

$$+ \frac{k_1 C}{(k_1 + k_2)^2 + \lambda^2}\ [-(k_1 + k_2) \cos(\lambda x) + \lambda \sin(\lambda x)] \tag{1.5.268}$$

$$y = ce^{-\mu x} + \nu + \alpha \cos(\lambda x) + \beta \sin(\lambda x) \tag{1.5.269}$$

The solution contains a transient term (dependent on the IC), a constant term to vertically shift the solution, and the sinusoidal terms with their own respective coefficients. The two sinusoidal terms, share the same time period.



33. Drug injection model, with drug quantity y and time x,

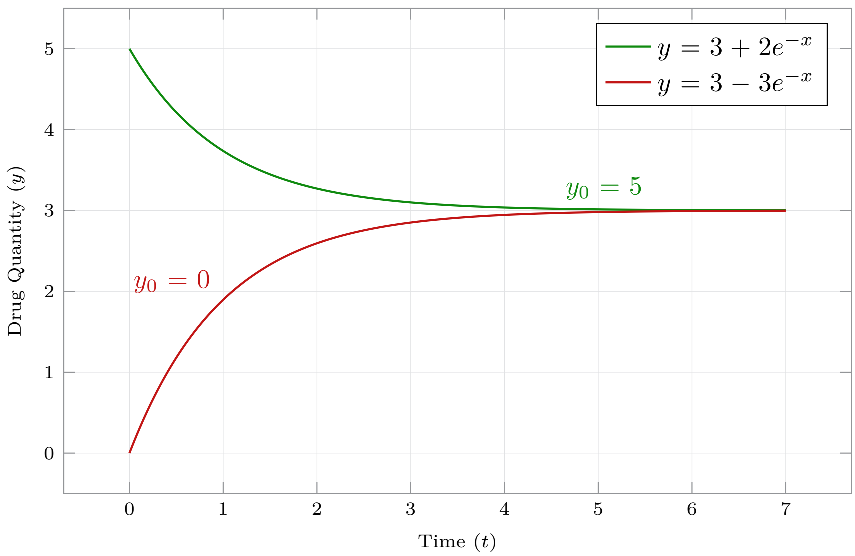
$$y' = A - ky \tag{1.5.270}$$

$$\ln(A - ky) = -kx + b \tag{1.5.271}$$

$$y = \frac{A}{k} - ce^{-kx} \tag{1.5.272}$$

$$y(0) = 0 \qquad c = \frac{A}{k} \tag{1.5.273}$$

$$A = 3 \qquad k = 1 \tag{1.5.274}$$



34. Epidemic model. Let the fraction diseased be y, and the healthy fraction be $1 - y$. Every diseased individual has contact with every healthy individual.

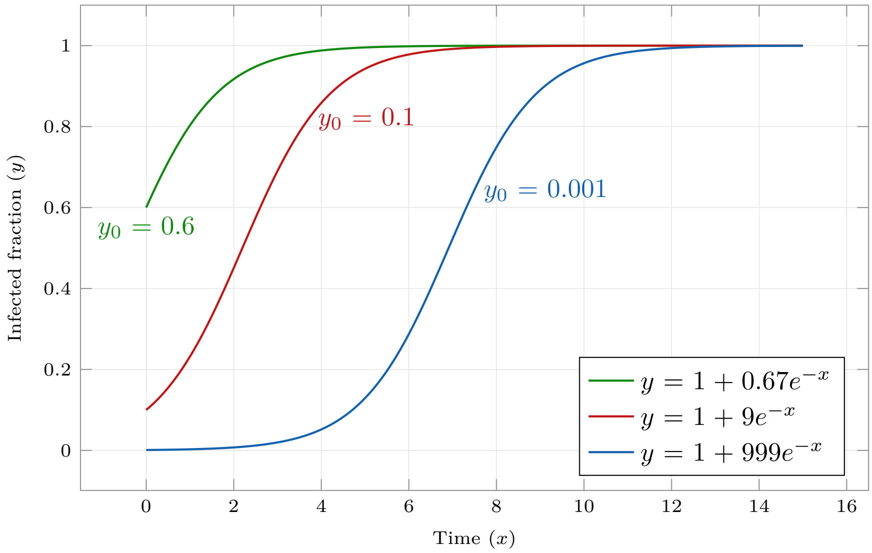
$$y' = Ky(1-y) \tag{1.5.275}$$

$$\int \frac{1}{y(1-y)}\ \mathrm{d}y = K\int\ \mathrm{d}x \tag{1.5.276}$$

$$\ln y - \ln(y-1) = Kx + b \tag{1.5.277}$$

$$\frac{y}{y-1} = ce^{Kx} \tag{1.5.278}$$

$$y = \frac{1}{1-c^{-Kx}} \tag{1.5.279}$$



35. Water inflow and outflow rate is λ, and total volume of the lake is V, with pollutant concentration y and time x,

$$y' = \frac{\lambda}{V}\left(\frac{p}{4} - y\right) \tag{1.5.280}$$

$$y = \frac{p}{4} + ce^{-\lambda x/V} \tag{1.5.281}$$

$$y(0) = \frac{p}{4} + c = p \qquad c = \frac{3p}{4} \tag{1.5.282}$$

$$y = \frac{p}{4}\left(1 + 3e^{-\lambda x/V}\right) \tag{1.5.283}$$

$$y(t^*) = \frac{p}{2} \qquad t^* = \frac{\ln 3}{\lambda/V} \tag{1.5.284}$$

Time needed is $t^* = 2.825$ years.

36. Schaefer model, based on logistic equation, with quantity of fish y, time x,

$$y' = (A - H)y - By^2 \qquad H < A \tag{1.5.285}$$

$$\int \frac{\lambda}{y} + \frac{\mu}{A - H - By} \ \mathrm{d}y = \int \ \mathrm{d}x \tag{1.5.286}$$

$$\lambda(A - H) + y(\mu - B\lambda) = 1 \tag{1.5.287}$$

$$\lambda = \frac{1}{A - H} \qquad \mu = \frac{B}{A - H} \tag{1.5.288}$$

$$\ln y - \ln(A - H - By) = (A - H)x + D \tag{1.5.289}$$

$$y = \frac{(A - H)}{B + ce^{-(A-H)x}} \tag{1.5.290}$$

To find the equilibrium solutions,

$$y' = 0 \tag{1.5.291}$$

$$y_1 = 0 \qquad y_2 = \frac{A - H}{B} > 0 \tag{1.5.292}$$

$$y'(y_2) = Ay_2 - By_2^2 - Hy_2 = 0 \tag{1.5.293}$$

At $y = y_2$, the harvesting term $-Hy_2$ is equal to the natural change term $Ay_2 - By_2^2$. This results in a steady state of the fish quantity in spite of a non-zero harvesting of fish.

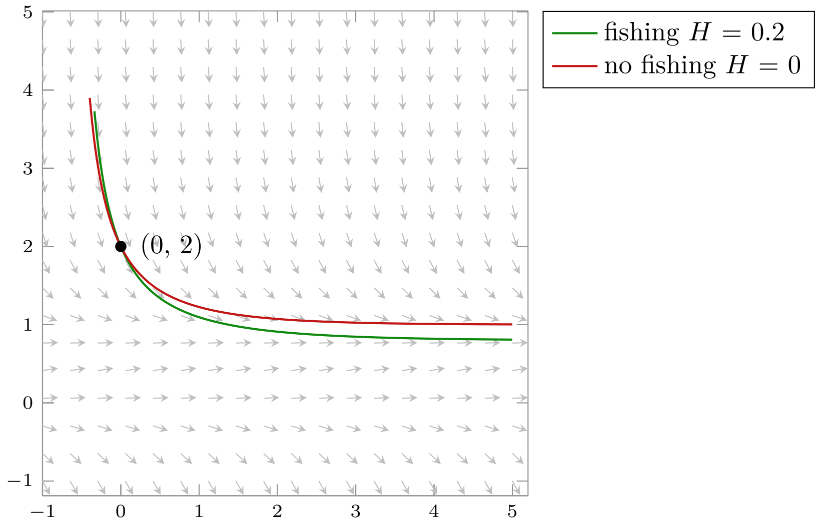
37. $A = B = 1$, $H = 0.2$, $y(0) = 2$,

$$y' = 0.8y - y^2 \tag{1.5.294}$$

$$y = \frac{0.8}{1 + ce^{-0.8x}} \tag{1.5.295}$$

$$y(0) = 2 \qquad c = -0.6 \tag{1.5.296}$$

From the direction field, y_1 is an unstable equilibrium and y_2 is a stable equilibrium, and is the asymptotic value of y.



In the absence of fishing the asymptotic value is $A = 1$.

38. Same equation as last problem with intermittient fishing every 3 years.

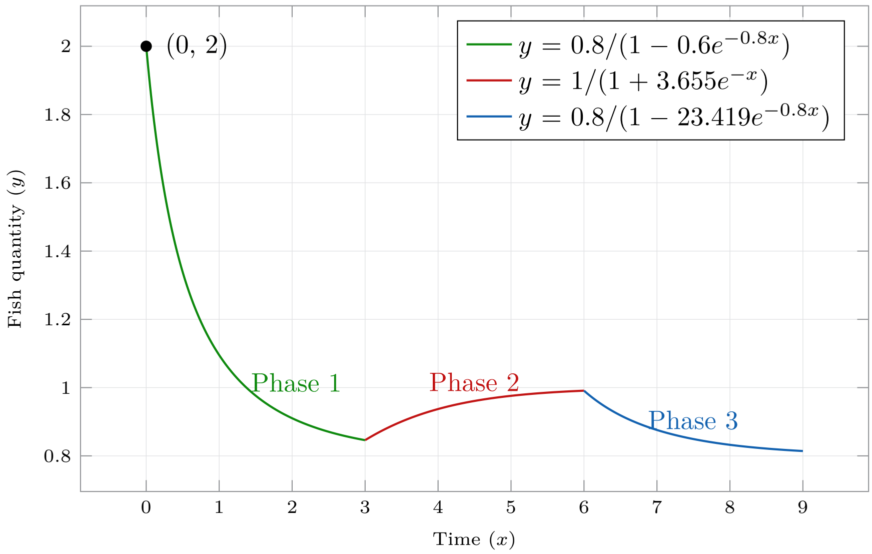
$$y = \frac{0.8}{1 - 0.6e^{-0.8x}} \qquad y(0) = 2 \tag{1.5.297}$$

$$y(3) = 1.0835 \tag{1.5.298}$$

$$z = \frac{1}{1 - c_2 e^{-x}} \qquad c_2 = -3.655 \tag{1.5.299}$$

$$z(6) = 0.991 \tag{1.5.300}$$

$$w = \frac{0.8}{1 - c_3 e^{-0.8x}} \qquad c_3 = 23.419 \tag{1.5.301}$$



39. Death rate B, Birth rate A, with population y and time x,

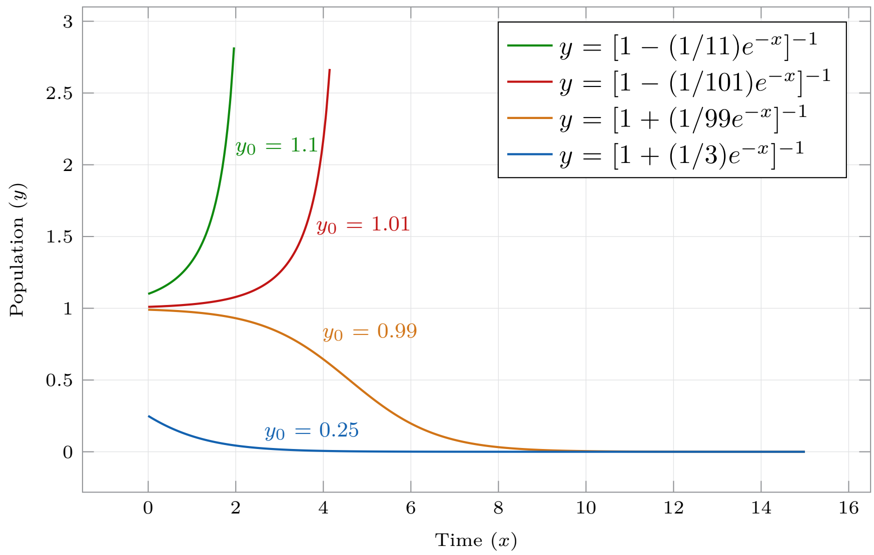
$$y' = Ay^2 - By \qquad = Ay(y - B/A) \tag{1.5.302}$$

$$y_1 = 0 \qquad y_2 = B/A \tag{1.5.303}$$

$$\int \frac{1}{y} - \frac{1}{y - B/A} \;\mathrm{d}x = -B \int \;\mathrm{d}x \tag{1.5.304}$$

$$\frac{y}{y - B/A} = ce^{-Bx} \tag{1.5.305}$$

$$y = \frac{(B/A)}{1 - ce^{Bx}} \tag{1.5.306}$$



If $y_0 < B/A$, then $y_0' < 0$ and extinction happens.
If $y_0 > B/A$, then $y_0' > 0$ and exponential growth happens.

40. Air inflow and outflow rate is λ, and total volume of the room is V, with fresh air y and time x,

$$y' = \lambda \left(1 - \frac{y}{V}\right) \tag{1.5.307}$$

$$\ln(1 - y/V) = \frac{-\lambda}{V} x + b \tag{1.5.308}$$

$$1 - \frac{y}{V} = ce^{-\lambda x/V} \tag{1.5.309}$$

$$\frac{y}{V} = 1 - ce^{-\lambda x/V} \qquad c = 1 \tag{1.5.310}$$

$$y(t^*) = 0.9V \qquad t^* = \frac{\ln 0.1}{-\lambda/V} \tag{1.5.311}$$

Time needed is $t^* = 76.753$ min.

1.6 Orthogonal Trajectories

1. All ellipses with foci -3 and +3 on the x axis, will have eccentricity $c = 3$,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 3^2} = 1 \tag{1.6.1}$$

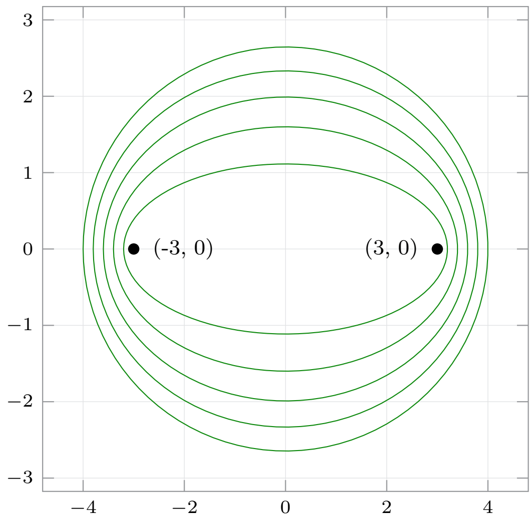
$$a^2 yy' + x(a^2 - 9) = 0 \tag{1.6.2}$$

$$a^2 = \frac{9x}{yy' + x} \tag{1.6.3}$$

$$\frac{x^2(x + yy')}{x} - \frac{y^2(x + yy')}{yy'} = 9 \tag{1.6.4}$$

$$x(x + yy')(yy') - y^2(x + yy') = 9yy' \tag{1.6.5}$$

$$y'^2(xy^2) + y'(x^2y - y^3 - 9y) - xy^2 = 0 \tag{1.6.6}$$



2. All circles passing through origin and having centre on $y = x^3$

$$(x-a)^2 + (y-a^3)^2 = c^2 \tag{1.6.7}$$

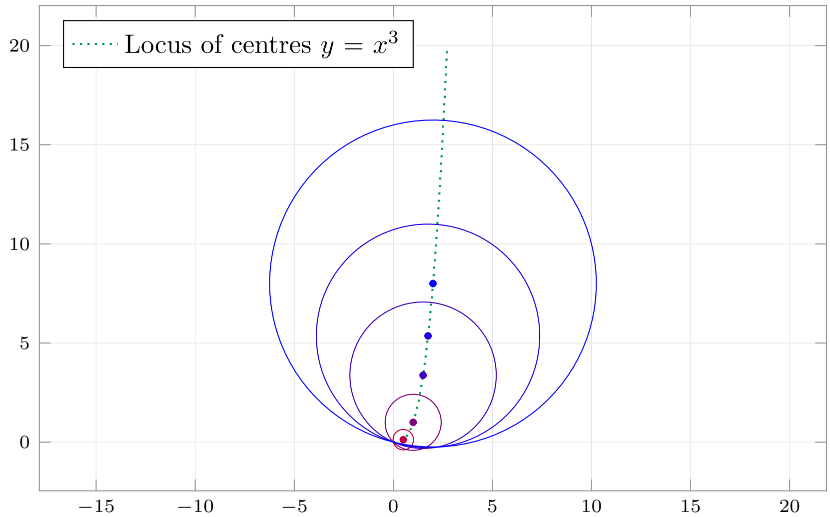
$$a^2 + a^6 = c^2 \tag{1.6.8}$$

$$(x-a)^2 + (y-a^3)^2 = a^2 + a^6 \tag{1.6.9}$$

$$\frac{x^2(x+yy')}{x} - \frac{y^2(x+yy')}{yy'} = 9 \tag{1.6.10}$$

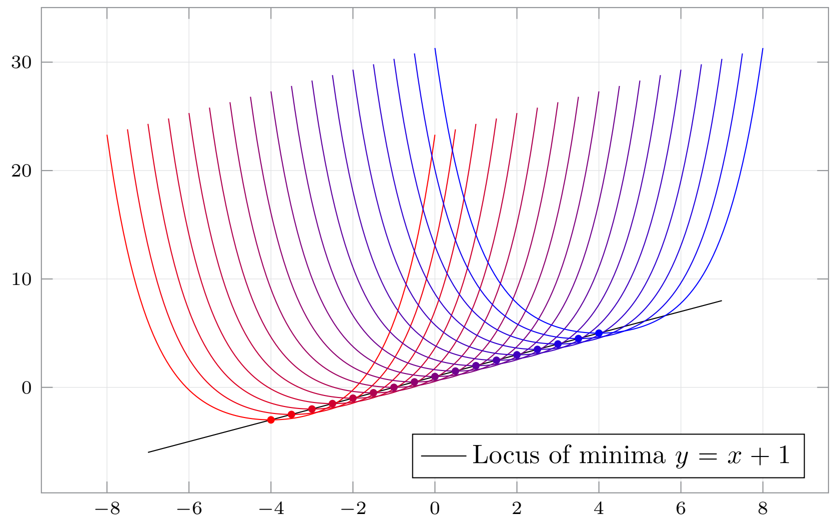
$$x(x+yy')(yy') - y^2(x+yy') = 9yy' \tag{1.6.11}$$

$$y'^2(xy^2) + y'(x^2y - y^3 - 9y) - xy^2 = 0 \tag{1.6.12}$$



3. The catenary is $y = \cosh x$,

$$y - a = \cosh(x-a) \tag{1.6.13}$$



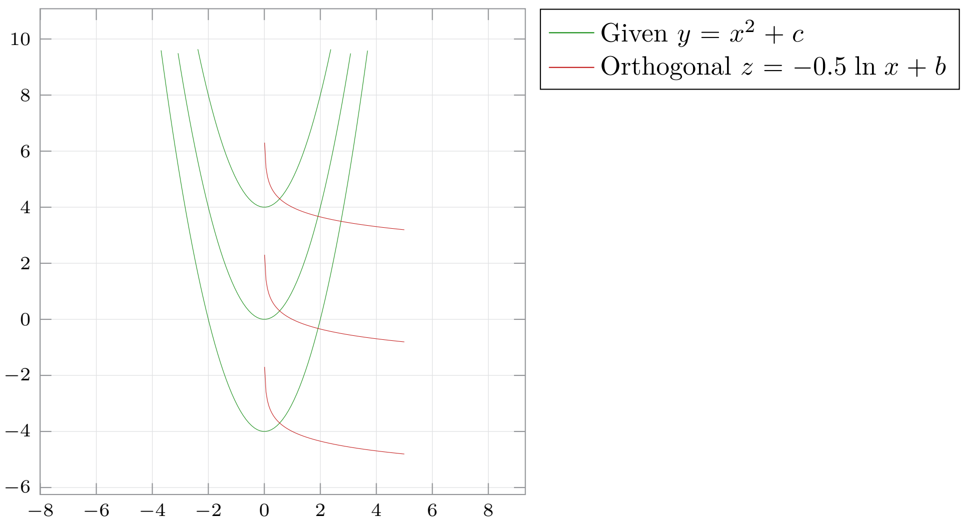
4. Finding OT for given family of curves $z(x; b)$,

$$y = x^2 + c \tag{1.6.14}$$

$$y' = 2x \tag{1.6.15}$$

$$y \rightarrow z \qquad y' \rightarrow \frac{-1}{z'} \tag{1.6.16}$$

$$z' = \frac{-1}{2x} \qquad z = -0.5 \ln x + b \tag{1.6.17}$$



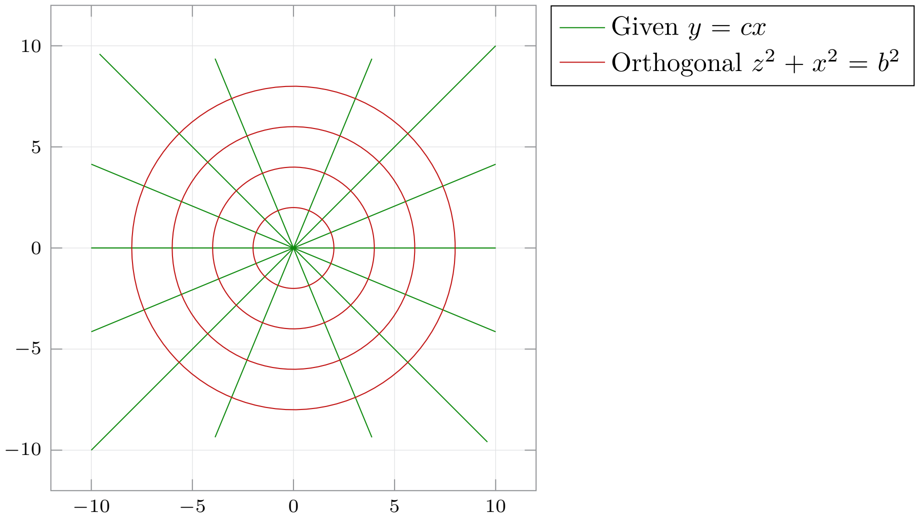
5. Finding OT for given family of curves $z(x; b)$,

$$y = cx \tag{1.6.18}$$

$$\frac{y'}{x} - \frac{y}{x^2} = 0 \qquad y' = \frac{y}{x} \tag{1.6.19}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.20}$$

$$z' = \frac{-x}{z} \qquad z^2 + x^2 = b^2 \tag{1.6.21}$$



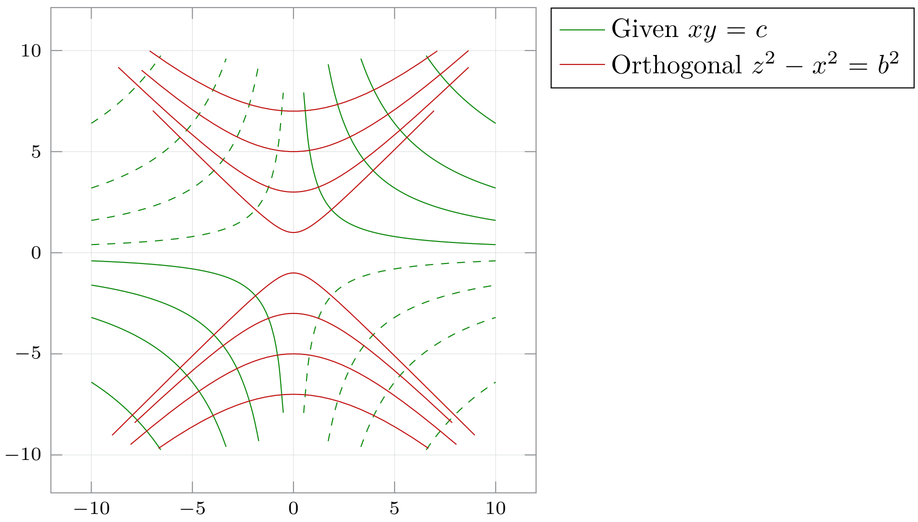
6. Finding OT for given family of curves $z(x; b)$,

$$xy = c \tag{1.6.22}$$

$$xy' + y = 0 \qquad y' = \frac{-y}{x} \tag{1.6.23}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.24}$$

$$z' = \frac{x}{z} \qquad z^2 - x^2 = b \tag{1.6.25}$$



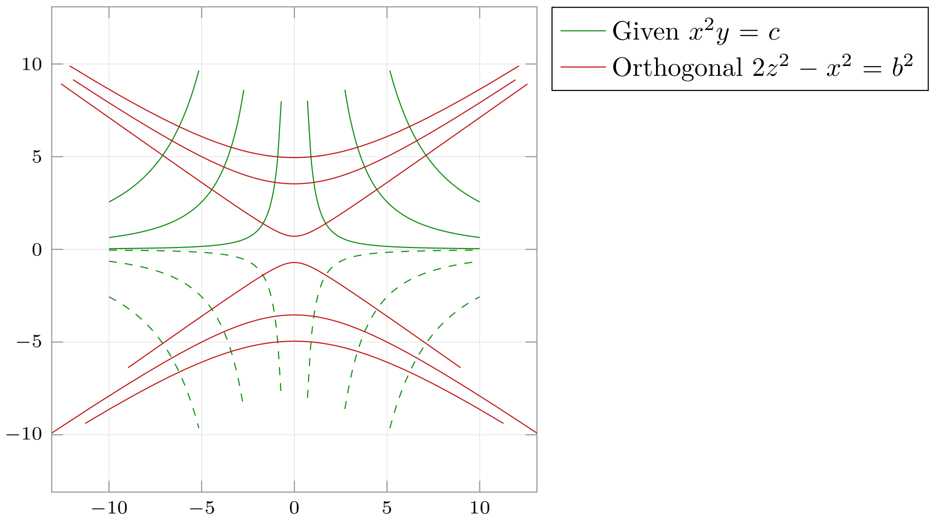
7. Finding OT for given family of curves $z(x; b)$,

$$y = \frac{c}{x^2} \tag{1.6.26}$$

$$x^2 y' + 2xy = 0 \qquad y' = \frac{-2y}{x} \tag{1.6.27}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.28}$$

$$z' = \frac{x}{2z} \qquad 2z^2 - x^2 = b^2 \tag{1.6.29}$$



8. Finding OT for given family of curves $z(x; b)$,

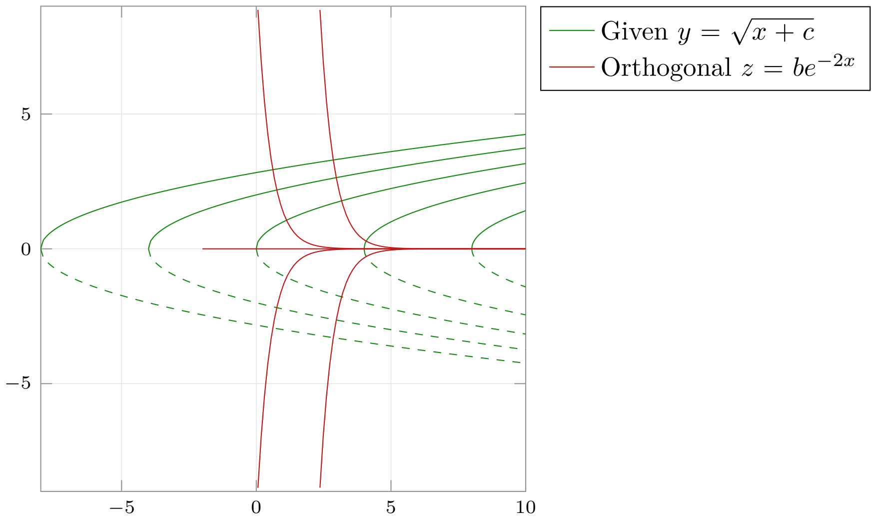
$$y = \sqrt{x+c} \tag{1.6.30}$$

$$2yy' - 1 = 0 \qquad y' = \frac{1}{2y} \tag{1.6.31}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.32}$$

$$z' = -2z \qquad \ln z = -2x + b^* \tag{1.6.33}$$

$$z = be^{-2x} \tag{1.6.34}$$



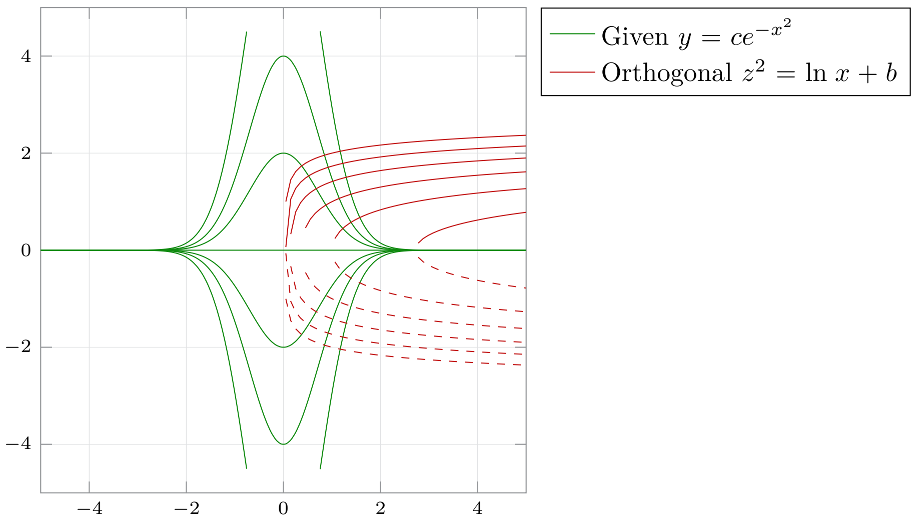
9. Finding OT for given family of curves $z(x; b)$,

$$y = ce^{-x^2} \tag{1.6.35}$$

$$e^{x^2}y' + 2xe^{x^2}y = 0 \qquad y' = -2xy \tag{1.6.36}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.37}$$

$$z' = \frac{1}{2xz} \qquad z^2 = \ln x + b \tag{1.6.38}$$



10. Finding OT for given family of curves $z(x; b)$,

$$x^2 + (y - c)^2 = c^2 \tag{1.6.39}$$

$$x + (y - c)y' = 0 \qquad c = y + \frac{x}{y'} \tag{1.6.40}$$

$$x^2 + \frac{x^2}{y'^2} = y^2 + \frac{x^2}{y'^2} + \frac{2xy}{y'} \tag{1.6.41}$$

$$y' = \frac{2xy}{x^2 - y^2} \tag{1.6.42}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.43}$$

$$z' = \frac{z^2 - x^2}{2xz} \qquad \left(x^2 - z^2\right)\ \mathrm{d}x + 2xz\ \mathrm{d}z = 0 \tag{1.6.44}$$

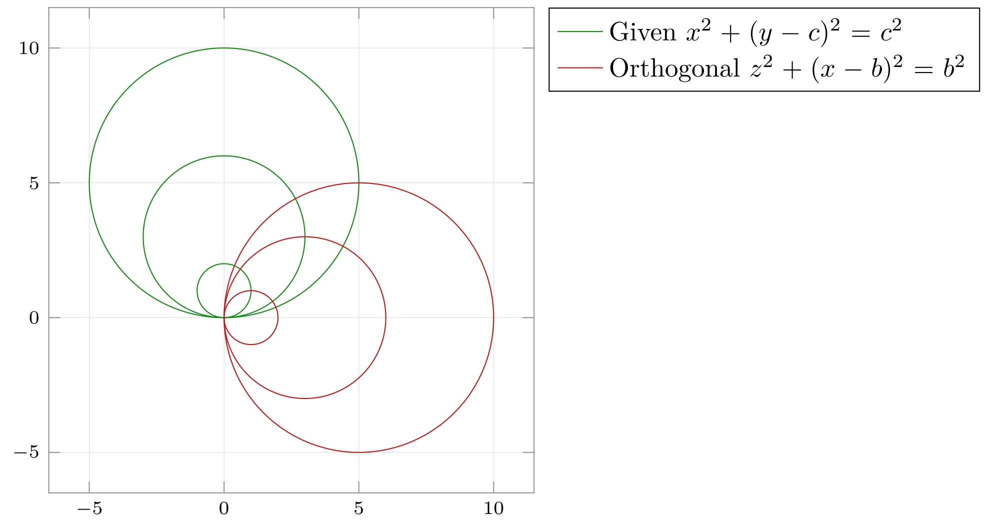
$$F = e^{-2\ln x} \qquad = x^{-2} \tag{1.6.45}$$

$$\left(1 - \frac{z^2}{x^2}\right)\ \mathrm{d}x + \frac{2z}{x}\ \mathrm{d}z = 0 \tag{1.6.46}$$

$$\frac{z^2}{x} + l(x) = u(x, z) \qquad 1 = \frac{\mathrm{d}l(x)}{\mathrm{d}x} \tag{1.6.47}$$

$$u(x, y) = \frac{z^2}{x} + x \qquad z^2 + x^2 = 2bx \tag{1.6.48}$$

$$z^2 + (x - b)^2 = b^2 \tag{1.6.49}$$



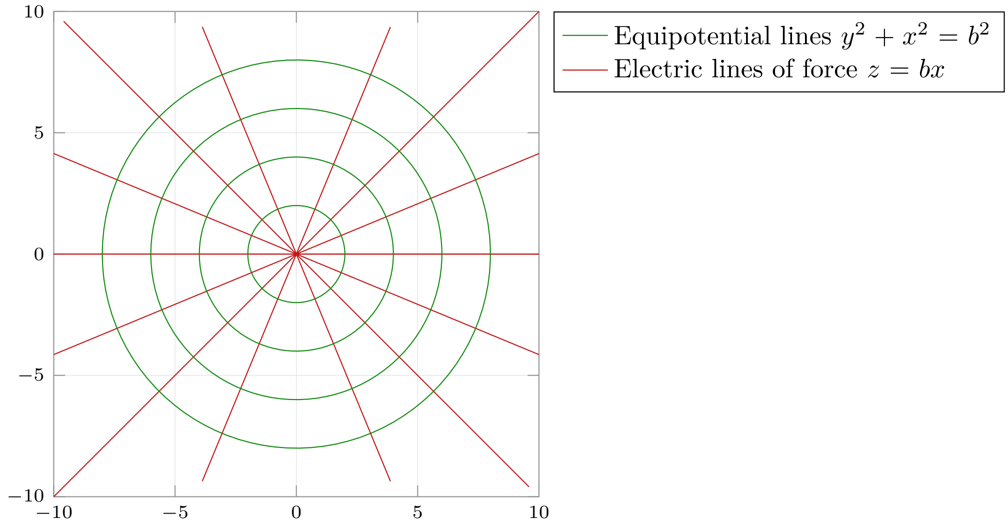
11. Finding OT for given family of curves $z(x; b)$,

$$x^2 + y^2 = c \tag{1.6.50}$$

$$2x + 2yy' = 0 \qquad y' = \frac{-x}{y} \tag{1.6.51}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.52}$$

$$z' = \frac{z}{x} \qquad z = bx \tag{1.6.53}$$



12. Locus of circles passing through $(1, 0)$ and $(-1, 0)$,

$$x^2 + (y - c)^2 = r^2 \qquad r^2 = 1 + c^2 \tag{1.6.54}$$

$$x + (y - c)y' = 0 \qquad c = y + \frac{x}{y'} \tag{1.6.55}$$

$$x^2 + \frac{x^2}{y'^2} = y^2 + \frac{x^2}{y'^2} + \frac{2xy}{y'} + 1 \tag{1.6.56}$$

$$y' = \frac{2xy}{x^2 - y^2 - 1} \tag{1.6.57}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.58}$$

$$z' = \frac{1 + z^2 - x^2}{2xz} \tag{1.6.59}$$

$$0 = (x^2 - z^2 - 1)\ \mathrm{d}x + 2xz\ \mathrm{d}z \tag{1.6.60}$$

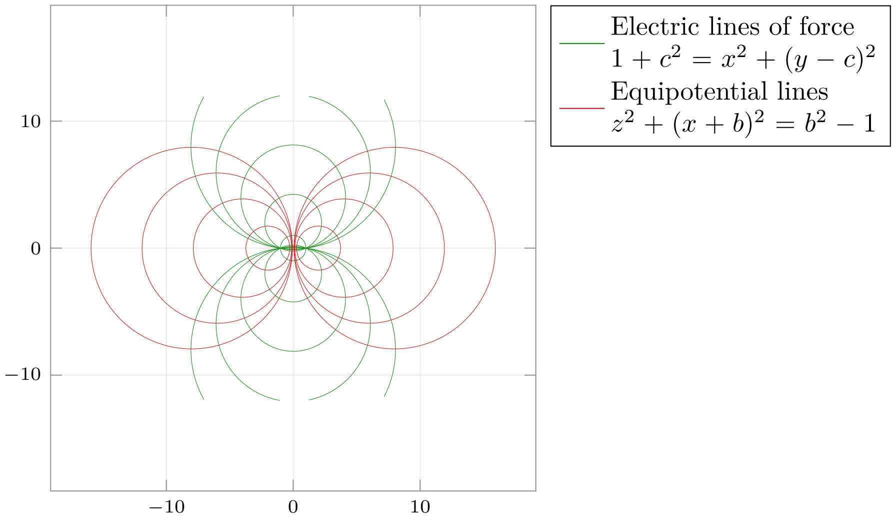
$$F = e^{-2\ln x} \qquad = x^{-2} \tag{1.6.61}$$

$$0 = \left(1 - \frac{z^2}{x^2} - \frac{1}{x^2}\right)\ \mathrm{d}x + \frac{2z}{x}\ \mathrm{d}z \tag{1.6.62}$$

$$\frac{z^2}{x} + l(x) = u(x, z) \qquad 1 - \frac{1}{x^2} = \frac{\mathrm{d}l(x)}{\mathrm{d}x} \tag{1.6.63}$$

$$u(x, z) = \frac{z^2}{x} + x + \frac{1}{x} \qquad z^2 + x^2 + 1 = -2bx \tag{1.6.64}$$

$$z^2 + (x + b)^2 = b^2 - 1 \tag{1.6.65}$$



13. Finding OT of given family $g(z, x; b)$,

$$4x^2 + 9y^2 = c \tag{1.6.66}$$

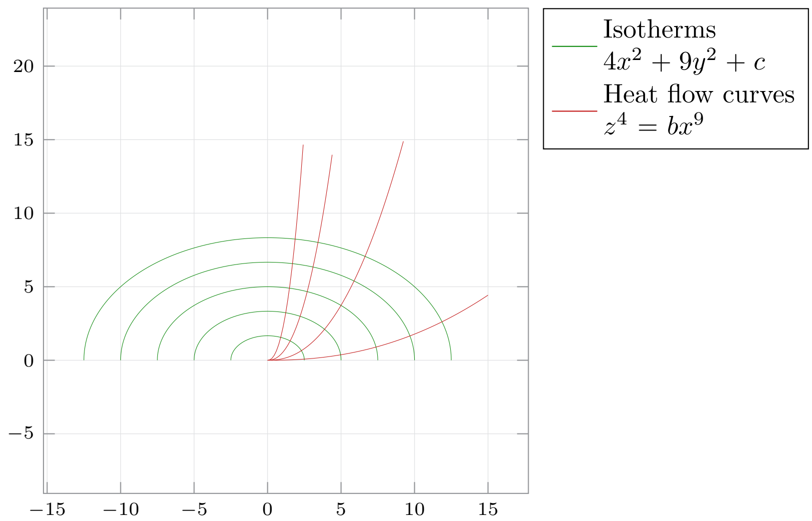
$$4x + 9yy' = 0 \tag{1.6.67}$$

$$y' = \frac{-4x}{9y} \tag{1.6.68}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.69}$$

$$z' = \frac{9z}{4x} \tag{1.6.70}$$

$$4 \ln z = 9 \ln x + b^* \qquad z = bx^{9/4} \tag{1.6.71}$$



14. Finding condition for OT of family of ellipses $g(z, x; b)$, being a conic section

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c^2 \tag{1.6.72}$$

$$\frac{x}{a^2} + \frac{yy'}{b^2} = 0 \tag{1.6.73}$$

$$y' = \frac{b^2}{a^2}\,\frac{-x}{y} \tag{1.6.74}$$

$$y \to z \qquad y' \to \frac{-1}{z'} \tag{1.6.75}$$

$$z' = \frac{b^2}{a^2}\,\frac{z}{x} \tag{1.6.76}$$

$$a^2 \ln z = b^2 \ln x + m^* \qquad z = mx^{b^2/a^2} \tag{1.6.77}$$

For OT to be a straight line, $b^2 = a^2$. For a family of ellipses, $b, a > 0$. This rules out $b = 0$. A parabola requires $b^2 = 2a^2$ or $2b^2 = a^2$. These are the only conic section OT possible.
$a \to 0$ makes the ellipse a straight line segment along the x-axis. (Analogous for $b \to 0$). TBC

15. Cauchy-Riemann equations,

$$u(x, y) = c \qquad \frac{\partial u}{\partial x}\ \mathrm{d}x + \frac{\partial u}{\partial y}\ \mathrm{d}y = 0 \tag{1.6.78}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\partial u/\partial x}{\partial u/\partial y} = m_1 \qquad \frac{-1}{\mathrm{d}y/\mathrm{d}x} = \frac{-1}{m_1} = \frac{\partial u/\partial y}{\partial u/\partial x} \tag{1.6.79}$$

$$v(x, y) = b \qquad \frac{\partial v}{\partial x}\ \mathrm{d}x + \frac{\partial v}{\partial y}\ \mathrm{d}y = 0 \tag{1.6.80}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\partial v/\partial x}{\partial v/\partial y} \qquad \frac{-1}{\mathrm{d}y/\mathrm{d}x} = m_2 \tag{1.6.81}$$

$$m_2 = \frac{-1}{m_1} \qquad \text{if} \quad u_x = v_y \quad \text{and} \quad u_y = -v_x \tag{1.6.82}$$

This is only the forward proof, since no complex differentiation is involved.

To find the OT of the given curves,

$$e^x \sin y = c \tag{1.6.83}$$

$$v_y = e^x \sin y \qquad v_x = -e^x \cos y \tag{1.6.84}$$

$$-\cos y\ \ \mathrm{d}x + \sin y\ \ \mathrm{d}y = 0 \tag{1.6.85}$$

$$\ln|\cos y| + x = b \tag{1.6.86}$$

16. Consider the family of curves whose direction field is given by $(1, f(x))$. Since the direction field of every point with the same y-coordinate is identical,

$$y' = f(x) \qquad y = \int f(x)\ \ \mathrm{d}x = g(x) + c \tag{1.6.87}$$

$$z' = \frac{-1}{f(x)} = h(x) \qquad z = \int h(x)\ \ \mathrm{d}x = j(x) + b \tag{1.6.88}$$

Two members of the family of curves $y(x; c)$ cannot intersect at the same x value, as they will always be separated by an additive non-zero constant $c_1 - c_2$.

$y_1(x) = y_2(x)$ automatically means that $z_1(x) = z_2(x)$, where $z(x; b)$ is the OT family.

y' being independent of y means that z' is also independent of z. So the same congruence condition holds for the OT family.

1.7 Existence and Uniqueness of Solutions for Initial Value Problems

1. p and r are continuous

$$p,\, q \text{ are continuous} \qquad \forall\, x \in |x - x_0| \le a \tag{1.7.1}$$

$$y' = f(x, y) = r(x) - yp(x) \tag{1.7.2}$$

$$\frac{\partial f}{\partial y} = -p(x) \tag{1.7.3}$$

The fact than r, p are continuous means that they are bounded in that interval. Thus, f and f_y are bounded in the interval.

This means a unique solution exists for an IVP over this interval.

2. Checking for existence of solution,

$$(x - 2)y' = y \qquad y(2) = 1 \tag{1.7.4}$$

$$y' = \frac{y}{x - 2} \qquad y = c(x - 2) \tag{1.7.5}$$

y' is not bounded in any region including $x = 2$. So the IVP above has no solution.By explicitly solving, the solution cannot pass through $(2, 1)$ for any finite c.

3. Solution exists in the smaller region among

$$|x - x_0| \le a \qquad |x - x_0| \le b/K \tag{1.7.6}$$

For very large b, $a < b/K$ and thus the region in which a solution exists is $|x - x_0| \le a$.

4. $k = 0$ gives infinitely many solutions.
$k \neq 0$ gives no solution for finite k, as evident from the explicit solution to the ODE shown above $y = c(x - 2)$.

5. Using the existence theorem,

$$y' = 2y^2 \qquad y(1) = 1 \tag{1.7.7}$$

$$f(x, y) = 2y^2 \qquad f_y = 4y \tag{1.7.8}$$

In the rectangle of length a and height b centered on $(1, 1)$,

$$|f| \le K \qquad |f_y| \le M \tag{1.7.9}$$

$$\left| 2\left(1 + \frac{b}{2}\right)^2 \right| \le K \tag{1.7.10}$$

$$\left| 2 + 2b + \frac{b^2}{2} \right| \le K \tag{1.7.11}$$

$\alpha = b/K$ is the subinterval of the rectangle in which the solution exists. Maximizing α,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}b} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}b}\frac{2b}{(b+2)^2} = 0 \tag{1.7.12}$$

$$\frac{(b+2)\cdot(2b+4-4b)}{(b+2)^4} = 0 \qquad \frac{b-2}{(b+2)^3} = 0 \tag{1.7.13}$$

The optimal value of $b = 2$ and $\alpha = 1/4$, which is the largest possible rectangle in which a solution is guaranteed to exist.

An explicit solution to the ODE is

$$y = \frac{1}{3-2x} \tag{1.7.14}$$

6. (a) Picard Iteration,

$$\int_y^{y_0} \mathrm{d}y = \int_x^{x_0} f(x, y)\ \mathrm{d}x \tag{1.7.15}$$

$$y - y_0 = \int_{x_0}^{x} f(t, y(t))\ \mathrm{d}t \tag{1.7.16}$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\ \mathrm{d}t \tag{1.7.17}$$

(b) By the iterative method,

$$y' = x + y \qquad y(0) = 0 \tag{1.7.18}$$

$$y_1 = y_0 + \int_{x_0}^{x} (t + y_0(t))\ \mathrm{d}t \qquad y_1 = \left.\frac{t^2}{2}\right|_0^x \tag{1.7.19}$$

$$= \frac{x^2}{2} \tag{1.7.20}$$

$$y_2 = y_0 + \int_{x_0}^{x} (t + y_1(t))\ \mathrm{d}t \qquad y_2 = \left.\frac{t^2}{2} + \frac{t^3}{3!}\right|_0^x \tag{1.7.21}$$

$$y_n = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} \tag{1.7.22}$$

Solving exactly,

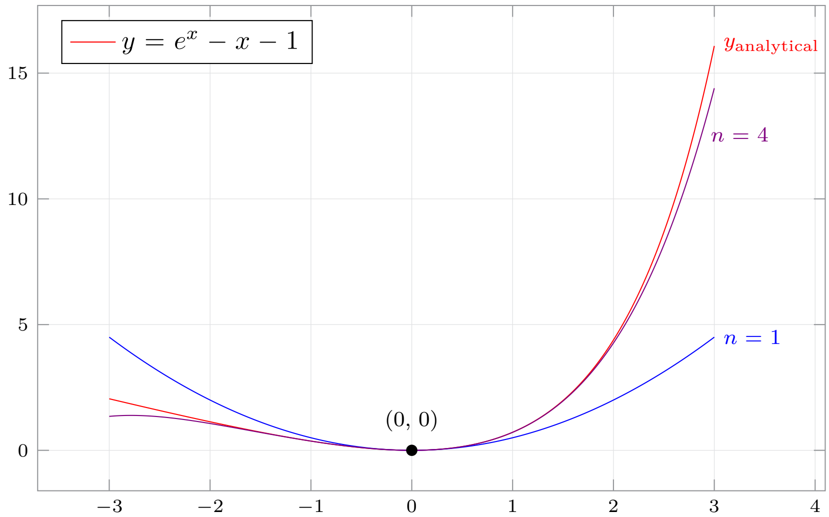
$$y' = x + y \qquad y' - y = x \tag{1.7.23}$$

$$h = \int -1 \ \mathrm{d}x \qquad = -x \tag{1.7.24}$$

$$y = e^{-h}\left[\int e^{h}\{r(x)\} \ \mathrm{d}x + c\right] \tag{1.7.25}$$

$$\int xe^{-x} \ \mathrm{d}x = I = -xe^{-x} + \int e^{-x} \ \mathrm{d}x \tag{1.7.26}$$

$$y = -(x+1) + ce^{x} \qquad c = 1 \tag{1.7.27}$$



(c) By the iterative method,

$$y' = 2y^2 \qquad y(0) = 1 \tag{1.7.28}$$

$$y_1 = y_0 + \int_{x_0}^{x} (2y_0^2) \ \mathrm{d}t \qquad y_1 = 1 + 2t\Big|_0^x \tag{1.7.29}$$

$$= 1 + 2x \tag{1.7.30}$$

$$y_2 = y_0 + \int_{x_0}^{x} 2y_1^2(t) \ \mathrm{d}t \tag{1.7.31}$$

$$y_2 = 1 + 2x + 4x^2 + \frac{8x^3}{3} \tag{1.7.32}$$

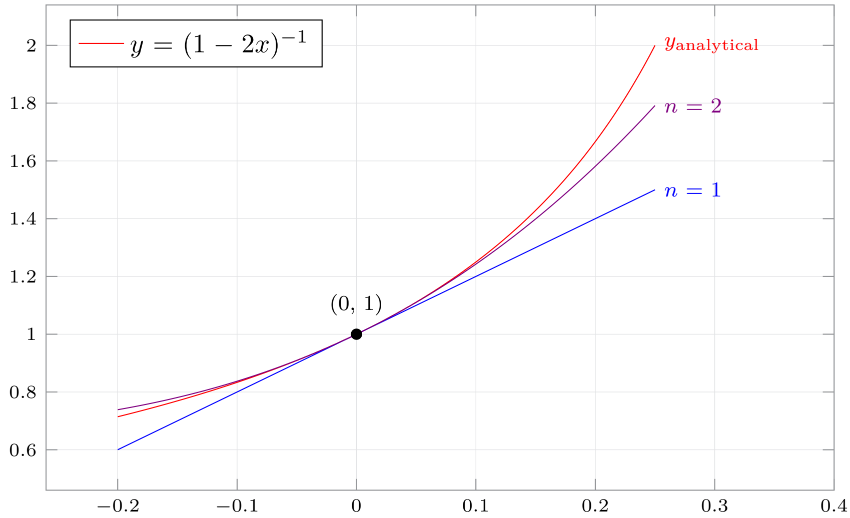
Solving exactly,

$$y' = 2y^2 \tag{1.7.33}$$

$$\frac{-1}{y} = 2x + c \tag{1.7.34}$$

$$y = \frac{-1}{2x + c} \qquad c = -1 \tag{1.7.35}$$

$$y = \frac{1}{1 - 2x} \tag{1.7.36}$$



(d) By the iterative method,

$$y' = 2y^{1/2} \qquad y(1) = 0 \tag{1.7.37}$$

$$y_1 = y_0 + \int_{x_0}^{x} \left(2y_0^{1/2}\right) \, \mathrm{d}t \qquad y_1 = 0\Big|_1^x \tag{1.7.38}$$

$$= 0 \tag{1.7.39}$$

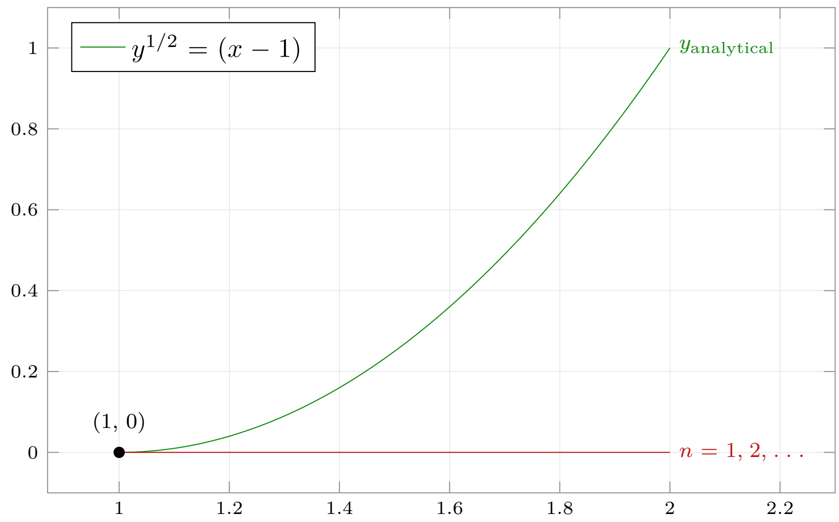
$$y_2 = y_3 = \cdots = y_n = 0 \tag{1.7.40}$$

Solving exactly,

$$y' = 2y^{1/2} \tag{1.7.41}$$

$$y^{1/2} = x + c \qquad c = -1 \tag{1.7.42}$$

$$y = (x - 1)^2 \qquad x - 1 > 0 \tag{1.7.43}$$



Picard's iteration approximates the trivial solution, not the parabolic one.

(e) TBC. Proof requires more real analysis than I know.

7. From example, rectangle $\mathcal{R}$ has length $2a$ and height $2b$.

$$y' = 1 + y^2 \qquad y(0) = 0 \tag{1.7.44}$$

$$f(x, y) = 1 + y^2 \qquad f_y = 2y \tag{1.7.45}$$

$$|f| \leq K \qquad \implies |1 + (0+b)^2| \leq K \tag{1.7.46}$$

$$\tag{1.7.47}$$

$\alpha = b/K$ is the subinterval of the rectangle in which the solution exists. Maximizing α,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}b} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}b}\frac{b}{1+b^2} = 0 \tag{1.7.48}$$

$$\frac{(1+b^2) - b(2b)}{(1+b^2)^2} = 0 \qquad \frac{1-b^2}{(1+b^2)^2} = 0 \tag{1.7.49}$$

The optimal value of $b = 1$ and $\alpha = 1/2$, which is the largest possible rectangle in which a solution is guaranteed to exist.

An explicit solution to the ODE is

$$y = \tan x \tag{1.7.50}$$

8. Showing that a Lipschitz condition holds for a linear ODE, with p, r continuous in $|x - x_0| \leq a$. From the mean value theorem of differential calculus,

$$y' = f(x, y) = r(x) - yp(x) \tag{1.7.51}$$

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\left(\frac{\partial f}{\partial y}\right)_{y=\tilde{y}} \tag{1.7.52}$$

$$= -(y_2 - y_1)\, p(x)\Big|_{y=\tilde{y}} \tag{1.7.53}$$

$$\text{if} \quad |p(x)| \le M \tag{1.7.54}$$

$$\text{then} \quad |f(x, y_2) - f(x, y_1)| \le M|y_2 - y_1| \tag{1.7.55}$$

$$|p(x)| \cdot |y_2 - y_1| \le M|y_2 - y_1| \tag{1.7.56}$$

$p(x)$is continuous in the rectangle is thus bounded in the rectangle. Let this bound be MThis proves the existence of a Lipschitz condition for linear first order ODEs.

Additionally, the continuity of $f(x, y)$ automatically guarantees the uniqueness of a solution to these IVPs. (using the integrating factor method)

9. If the existence and uniqueness theorems are satisfied in the rectangle $\mathcal{R}$, then the point of intersection of two distinct solutions to the ODE would satisfy the IVP for both of the IVPs.

This directly violates the uniqueness theorem.

10. All three possible kinds of IVP needed.

$$(x^2 - x)y' = (2x - 1)y \tag{1.7.57}$$

$$\ln y = \ln(x^2 - x) + b \tag{1.7.58}$$

$$y = cx(x - 1) \tag{1.7.59}$$

(a) for infinitely many solutions, $y(0) = 0$ and $y(1) = 0$

(b) for no solution, $y(0) = k$ and $y(1) = k$ with $k \neq 0$

(c) for a unique solution, $y(\alpha) = \beta$ with $\alpha \in \mathbb{R} - \{0, 1\}$ and $\beta \in \mathbb{R}$