Almgren-Chriss Extension: Ensemble and Self-Impact Effects

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Abstract

We present an extension of the classical Almgren-Chriss optimal execution framework incorporating two key elements: self-alpha feedback and ensemble coupling. Self-alpha captures the dynamic adjustment of a trader's perceived alpha as a function of their own execution trajectory, while ensemble coupling introduces mean-field–type interactions reflecting the impact of other market participants' concurrent trades. Drawing on analogies from statistical physics—particularly interacting particle systems and mean-field models—we formalize this extended setup as a convex optimization problem that remains tractable and interpretable.

We validate the model through both synthetic data experiments using a structured agent count approximation and real-market applications based on intraday AAPL equity data. Across both settings, we observe consistent cost-reduction behavior as ensemble coupling strength increases, while self-alpha strength primarily governs the structure and stability of optimal trade paths. The results highlight both the conceptual value and practical limitations of mean-field extensions to standard execution models, opening avenues for further research in integrating market microstructure effects and machine learning–enhanced alpha signals.

1 Introduction

Convex optimization plays a foundational role in quantitative finance, serving as the mathematical backbone for tasks such as portfolio allocation, risk management, derivatives pricing, and trading execution [5]. Among these applications, optimizing execution in the presence of market impact is particularly vital for large institutional trades where order sizes influence market prices significantly.

The Almgren-Chriss model [2] provides one of the most widely cited frameworks for tackling the optimal trade execution problem. It formulates execution cost as a convex optimization problem balancing expected trading cost against market risk, captured via a mean-variance trade-off. This model introduced clarity and structure to a domain that previously relied heavily on heuristics, and it remains a cornerstone of both academic research and practical trading systems.

Despite its utility, the classical Almgren-Chriss model makes simplifying assumptions that limit its realism in today's complex market environments. Specifically:

- It models a single isolated agent, ignoring interactions with other market participants.

- It assumes a static alpha signal, not accounting for feedback effects where executing one's order may alter perceived opportunities or slippage expectations.

In real-world markets, traders operate within a shared limit order book, where ensemble behavior emerges as a result of many agents acting simultaneously. Additionally, agents frequently adjust their trading aggressiveness based on ongoing execution feedback. These effects suggest that an extended model incorporating both ensemble impact and self-alpha dynamics is necessary.

In this paper, we propose such an extension. Drawing inspiration from statistical physics, particularly interacting particle systems and mean-field approximations [4, 8, 13], we introduce two new components into the Almgren-Chriss framework:

1. **Ensemble Coupling:** Modeling cross-agent interactions by coupling execution trajectories through an ensemble impact function.
2. **Self-Alpha Feedback:** Allowing alpha to dynamically evolve as a function of execution progress and market response.

We believe this work holds value for:

- Quantitative researchers aiming to refine execution algorithms.
- Algorithmic trading practitioners seeking more realistic execution models.
- Academics exploring market microstructure and agent-based modeling.

Paper Layout:

- **Section 2:** Reviews the classical Almgren-Chriss model.
- **Section 3:** Introduces our ensemble and self-impact extensions.
- **Section 4:** Presents implementation details and experiments using synthetic and real market data.
- **Section 5:** Discusses results and implications.
- **Section 6:** Concludes with directions for future work.

2 The Classical Almgren-Chriss Model

The Almgren-Chriss (AC) model [2] formulates the optimal execution problem as a trade-off between minimizing expected execution cost and controlling risk from price uncertainty. It models trading over a fixed horizon $[0, T]$, discretized into N intervals of length $\Delta t = T/N$.

2.1 Model Setup

Let:

- $X(t)$: Remaining inventory at time t.
- $x_k = X(t_k) - X(t_{k+1})$: Quantity traded in interval k.
- $S(t)$: Asset mid-price.

The mid-price evolves as:

$$S(t_{k+1}) = S(t_k) + \sigma\sqrt{\Delta t}\,\epsilon_k + \theta x_k$$

where:

- σ: Asset volatility.
- $\epsilon_k \sim \mathcal{N}(0,1)$: i.i.d. Gaussian shocks.
- θ: Permanent impact parameter.

The effective execution price is given by:

$$\tilde{S}(t_k) = S(t_k) - \eta\frac{x_k}{\Delta t}$$

where η is the temporary impact parameter.

2.2 Optimization Objective

We define:

- $\mathbb{E}[\cdot]$: Expectation operator with respect to the distribution of price paths under stochastic price evolution.
- $\mathrm{Var}[\cdot]$: Variance operator quantifying execution risk due to price uncertainty.

The trader's objective is to minimize:

$$\min_{\{x_k\}} \; \mathbb{E}\left[\sum_{k=1}^{N} x_k \tilde{S}(t_k)\right] + \lambda \cdot \mathrm{Var}\left[\sum_{k=1}^{N} x_k \tilde{S}(t_k)\right]$$

where λ is the risk aversion parameter balancing cost versus risk.

Under quadratic approximation and simplification, this becomes:

$$\min_{\{x_k\}} \sum_{k=1}^{N} \left(\eta\frac{x_k^2}{\Delta t} + \frac{\lambda}{2}\sigma^2 X(t_k)^2 \Delta t\right)$$

2.3 Solution Structure

The optimal execution path $\{X(t_k)\}$ is derived by solving the Euler-Lagrange equations for the discretized system. In continuous-time limit, the optimal trading trajectory follows:

$$\frac{dX(t)}{dt} = -\kappa X(t)$$

where κ depends on λ, η, σ, and T. Specifically:

$$\kappa = \frac{\sqrt{\lambda}\,\sigma}{\eta}$$

The solution yields an exponentially decaying inventory trajectory:

$$X(t) = X(0) \cdot e^{-\kappa t}$$

This trajectory balances the competing objectives of minimizing market impact and controlling risk from price volatility.

3 Extensions: Ensemble and Self-Impact Effects

In this section, we extend the classical Almgren-Chriss model by introducing two new components:

1. **Ensemble Coupling:** Capturing the impact of other market participants executing trades concurrently.

2. **Self-Alpha Feedback:** Allowing the trader's perceived alpha to evolve dynamically as a function of their own execution trajectory.

3.1 Motivation

The classical Almgren-Chriss model assumes a single isolated agent and neglects informational and behavioral feedback effects. In reality:

- Traders interact through a shared order book, influencing each other's execution costs.

- Executing large orders alters a trader's own estimate of alpha as slippage and market conditions evolve.

These observations motivate the introduction of interaction terms analogous to mean-field and self-interaction effects in statistical physics [13, 4].

3.2 Extended Model Setup

We modify the mid-price dynamics as follows:

$$S(t_{k+1}) = S(t_k) + \alpha(t_k)\Delta t + \sigma\sqrt{\Delta t}\,\epsilon_k + \theta x_k$$

where:

- $\alpha(t_k)$ is the time-varying alpha incorporating both exogenous signals and self-feedback.

  We model $\alpha(t_k)$ as:

$$\alpha(t_k) = \alpha_0 + \eta_{\text{self}} \cdot g(X(t_k)) + \eta_{\text{ens}} \cdot h(\bar{x}_k)$$

with the following explicit forms:

- Self-impact function:

$$g(X(t_k)) = -X(t_k)$$

  This models diminishing alpha as the remaining inventory $X(t_k)$ decreases, consistent with informational decay or execution fatigue.

- Ensemble impact function:

$$h(\bar{x}_k) = \bar{x}_k$$

  where $\bar{x}_k$ represents the average execution rate of other agents in the same interval. This captures collective market pressure.

3.2.1 Convexity Consideration

We note that the extended optimization problem retains convexity under the proposed self-alpha and ensemble coupling extensions, provided that both η_{self} and η_{ens} terms scale linearly with $X(t_k)$ or x_k. This ensures the problem remains tractable via standard convex solvers such as CVXPY.

3.3 Extended Optimization Objective

The trader now minimizes:

$$\min_{\{x_k\}} \mathbb{E}\left[\sum_{k=1}^{N} x_k \tilde{S}(t_k) - \sum_{k=1}^{N} \alpha(t_k) \cdot X(t_k)\Delta t\right] + \lambda \cdot \text{Var}\left[\sum_{k=1}^{N} x_k \tilde{S}(t_k)\right]$$

3.4 Remarks

- When $\eta_{\text{self}} = \eta_{\text{ens}} = 0$, the model reduces to the classical Almgren-Chriss formulation.
- The choice of $g(X(t_k)) = -X(t_k)$ ensures convexity in $X(t_k)$. Similarly, $h(\bar{x}_k) = \bar{x}_k$ introduces linear ensemble coupling, preserving tractability.
- This extension incorporates statistical physics analogies: self-interacting systems and mean-field effects.

3.5 Implementation Note

For practical convex optimization implementation, we simplify the self-alpha feedback term by directly coupling it to the trade size variable x_k rather than cumulative inventory $X(t_k)$. Specifically, we substitute $\eta_{\text{self}} \cdot g(X(t_k))$ with a linear term proportional to x_k in the objective function:

$$-\alpha_{\text{self}} \cdot x_k$$

This maintains the convex structure of the problem while capturing the essential self-impact behavior. Similarly, ensemble coupling is implemented using a pre-computed signal $\bar{x}_k$ (denoted as `ensemble_alpha` in code), applied linearly to x_k.

These simplifications preserve conceptual alignment with the extended Almgren-Chriss framework while ensuring numerical tractability and interpretability in applied settings.

4 Synthetic Data Experiments

We validate the extended Almgren-Chriss model using synthetic experiments designed to isolate and study the impact of self-alpha feedback and ensemble coupling under controlled conditions. Two distinct setups are explored:

- **Fixed Self-Alpha with Ensemble Count Approximation:** Self-alpha is held constant, while ensemble alpha is approximated via stochastic agent count dynamics. This setup highlights how ensemble coupling strength γ affects execution cost and trade path shape when internal feedback is stable.
- **Time-Varying Self-Alpha and Ensemble Alpha:** Both self-alpha and ensemble alpha are defined as structured sinusoidal and cosine functions, allowing us to observe how dynamic alpha signals interact with ensemble coupling strength to shape execution behavior.

In both cases, we solve the convex optimization problem defined in Section 3 using CVXPY, and systematically explore parameter grids for γ and self-alpha magnitude.

4.1 Experimental Setup

We define the following parameters:

- Time horizon $T = 20$ steps.
- Total shares to execute: 1000.
- Number of agents $N = 100$.
- Base parameters: $\alpha_0 = 5.0$, $\kappa = 1.0$, $\lambda = 0.1$, $\sigma^2 = 1.0$.
- Ensemble alpha $\bar{x}_k$: Modeled through momentum and mean-reversion agent counts.
- Self-alpha: Constant per experiment, varying across magnitudes.

4.2 Ensemble Alpha Approximation

To approximate ensemble behavior, we partition N synthetic agents into momentum-type and mean-reversion-type at each time step k. Their counts are updated according to a random walk process:

- momentum_count_k and $\text{mean\_reversion\_count}_k = N - \text{momentum\_count}_k$.
- $\bar{x}_k = (\text{momentum\_count}_k - \text{mean\_reversion\_count}_k)/N \times \alpha_0$.

This creates a discrete approximation to the mean-field term using a simplified interacting agent system rather than a continuous signal.

4.3 Optimization Problem

We solve:

$$\min_{\{x_k\}} \sum_{k=1}^{T} \left(\kappa \cdot x_k^2 + \lambda \cdot \sigma^2 \cdot X_k^2 - \alpha_{\text{self}} \cdot x_k - \gamma \cdot \bar{x}_k \cdot x_k \right)$$

subject to:

$$\sum_{k=1}^{T} x_k = \text{total shares}$$

where $X_k = \sum_{i=1}^{k} x_i$.

4.4 Parameter Grids

We perform grid searches over:

- $\gamma \in \{0, 0.1, 0.5, 1.0, 2.0\}$.
- Self-alpha strength $\in \{0.01, 1.0\}$.

4.5 Implementation Note

This setup directly mirrors interacting agent systems in statistical physics where aggregate market behavior is approximated using counts of discrete states (momentum or mean-reverting agents) instead of full continuous simulations. The random walk adjustment of counts preserves stochasticity while remaining tractable.

The equality constraint on total inventory is enforced explicitly using CVXPY rather than through a penalty term in this version of the experiment.

4.6 Model Approximation Note

While the extended Almgren-Chriss model introduced in Section 3 defines ensemble impact $\bar{x}_k$ abstractly as a mean-field term averaging over n_{agents} trading rates, direct simulation of all individual agent trajectories is computationally intensive and outside the present scope.

To approximate this multi-agent structure in a tractable manner, we adopt a discrete state-count approach inspired by mean-field statistical physics models:

- At each time step k, N agents are partitioned into momentum-type and mean-reversion-type based on a random walk adjustment.
- Ensemble alpha is calculated as:

$$\bar{x}_k = \frac{\text{momentum count}_k - \text{mean reversion count}_k}{N} \times \alpha_0$$

- This approximation captures key market regime effects—momentum and mean reversion—while preserving analytical and computational tractability.

This setup mirrors interacting particle system logic, where the macroscopic field experienced by an individual agent is determined by aggregate system state rather than through exogenous signal specification. While this count-based approach captures directional market pressure through aggregate momentum versus mean-reversion states, it does not attempt to reproduce full heterogeneity in agent behavior or strategy beyond this binary classification.

4.7 Time-Varying Self-Alpha and Ensemble Alpha Experiment

In addition to fixed self-alpha strength experiments, we also consider a more general setup where both self-alpha and ensemble alpha vary over time in structured patterns.

- **Self-Alpha:** Defined as a time-varying cosine function:

$$\alpha_{\text{self}}(t_k) = A \cdot \cos\left(\frac{3\pi k}{T}\right)$$

  where A controls the self-alpha magnitude.

- **Ensemble Alpha:** Defined as a time-varying sine function:

$$\bar{x}_k = \alpha_0 \cdot \sin\left(\frac{3\pi k}{T}\right)$$

  where α_0 controls ensemble alpha magnitude.

- **Optimization Structure:** The convex optimization problem remains the same as described in Section 4.3, with pre-computed time-varying $\alpha_{\text{self}}(t_k)$ and $\bar{x}_k$ values replacing fixed or count-based approximations.

This setup provides a clearer demonstration of how dynamic alpha signals interact with ensemble coupling strength γ in shaping trade path structures, beyond simply affecting execution cost.

4.8 Results and Observations

We present selected plots from the synthetic data experiments using the agent count–based ensemble approximation.

4.8.1 Interpretation and Discussion

- **Effect of Ensemble Coupling on Execution Cost:** As shown in Figure 1, execution cost decreases monotonically as ensemble coupling strength γ increases. This holds consistently across both low and high self-alpha settings. The mechanism parallels mean-field systems: increasing γ aligns execution behavior more closely with collective flow, reducing adverse selection and slippage costs through coordination effects.

- **Trade Path Adjustment and Self-Alpha Influence:** Figure 2 shows that ensemble coupling introduces measurable but limited adjustments in trade path shape. The overall trajectory remains similar—especially when self-alpha strength is held constant—highlighting that absolute self-alpha magnitude, rather than its dynamic variation, primarily governs path structure. This reflects a balance between internal bias (self-alpha as an internal field) and external coupling (ensemble alpha as a mean field).

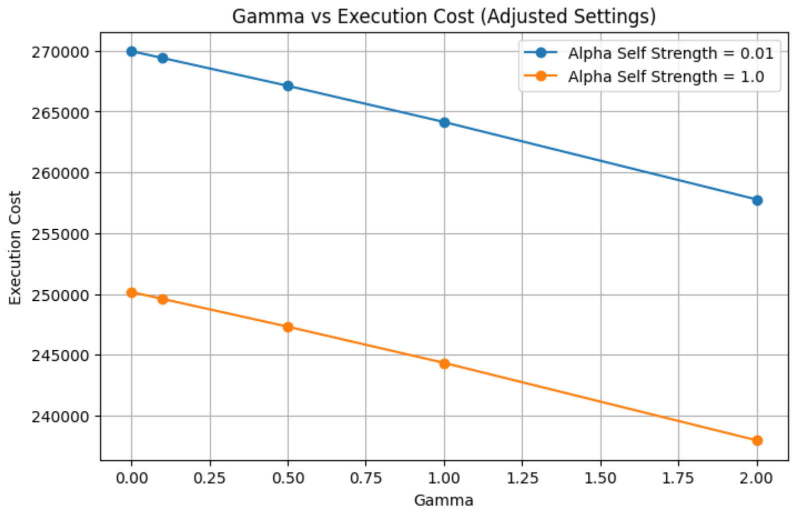


Figure 1: Execution Cost as a Function of Ensemble Coupling γ for Different Self-Alpha Strengths.

- **Interpretation in Market Microstructure Terms:** In practical terms, traders operating with strong and stable private signals (modeled via fixed self-alpha) exhibit less sensitivity to ensemble effects. Conversely, when self-alpha is weak or varies, responsiveness to market-wide behavior plays a larger role in optimizing execution cost.

4.9 Additional Results: Time-Varying Self-Alpha and Ensemble Alpha

To further investigate the effect of dynamic alpha signals on execution behavior, we present results from the structured time-varying alpha experiment described in Section 4.6.

4.9.1 Interpretation

- **Trade Path Shape Sensitivity:** Unlike in the constant self-alpha setup, introducing structured time-varying self-alpha and ensemble alpha visibly alters the shape of optimal trade paths as ensemble coupling strength γ changes. This effect is especially clear when self-alpha magnitude is moderate (e.g., magnitude = 1.0).

- **Self-Alpha Magnitude Effects:** At higher self-alpha magnitudes (e.g., magnitude = 10.0), ensemble coupling still introduces measurable adjustments, but the overall path structure becomes dominated by the self-alpha term. This behavior is consistent with mean-field models where internal field strength overwhelms external coupling.

- **Theoretical Alignment:** These results further validate the statistical physics analogy underpinning our model. When both self and ensemble terms vary dynamically, the system exhibits behavior akin to coupled oscillator models, where both internal and external fields actively shape state evolution.

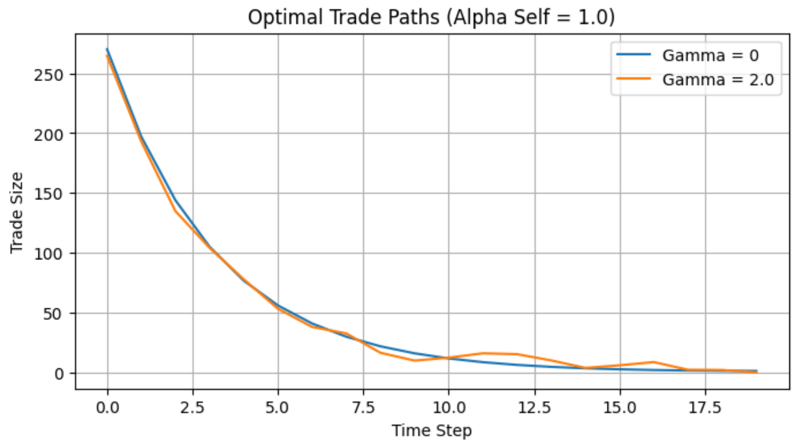
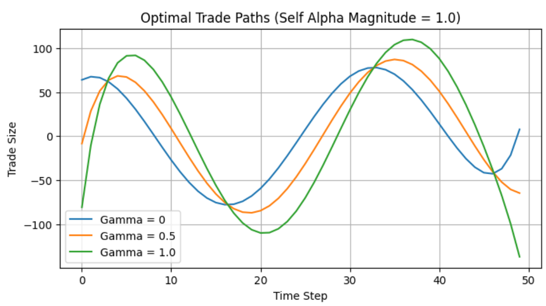
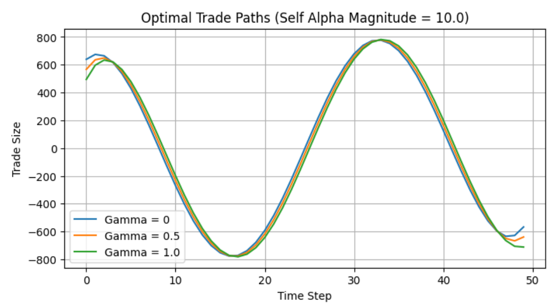


Figure 2: Optimal Trade Paths for Self-Alpha Strength = 1.0 under Varying γ.



Trade Paths: Self-Alpha Magnitude = 1.0, Time-Varying Cosine Pattern



Trade Paths: Self-Alpha Magnitude = 10.0, Time-Varying Cosine Pattern

5 Real Market Data Experiments

In addition to synthetic experiments, we apply the extended Almgren-Chriss model to real-world intraday equity data to evaluate practical relevance and behavior under live market conditions.

5.1 Data Source and Preprocessing

We use 5-minute interval data for AAPL downloaded via Yahoo Finance over a 5-day window. The data includes:

- Open, High, Low, Close prices.
- Volume traded per interval.

From this raw data, we construct self-alpha and ensemble alpha signals as follows:

- **Ensemble Alpha Approximation:**

$$\bar{x}_k = \text{sign}(\text{Close}_k - \text{Open}_k) \times \text{Volume}_k$$

The series is normalized and smoothed using a rolling mean window. This provides a practical proxy for directional market pressure.

- **Self-Alpha Approximation:**

$$\alpha_{\text{self}}(t_k) = \text{MA}_{\text{short}}(\text{Close}) - \text{MA}_{\text{long}}(\text{Close})$$

Using 3-period and 10-period moving averages respectively. The resulting series is normalized and smoothed similarly to ensemble alpha.

5.2 Optimization Problem

We solve the same convex optimization form as defined in Section 4:

$$\min_{\{x_k\}} \sum_{k=1}^{T} \left(\kappa \cdot x_k^2 + \lambda \cdot X_k^2 - \alpha_{\text{self}}(t_k) \cdot x_k - \gamma \cdot \bar{x}_k \cdot x_k\right) + \text{Penalty} \left(\sum_{k=1}^{T} x_k - \text{total shares}\right)^2$$

Where $X_k = \sum_{i=1}^{k} x_i$. Pre-computed self-alpha and ensemble alpha values from market data replace synthetic signal generation.

5.3 Parameter Grids

We perform grid searches over:

- $\gamma \in \{0, 0.5, 1.0\}$.
- Self-alpha is defined directly from observed price dynamics; no synthetic scaling is applied.

5.4 Results and Observations

5.4.1 Interpretation and Discussion

- **Execution Cost Behavior:** As shown in Figure 3, execution cost decreases notably as ensemble coupling strength γ increases. The reduction is more pronounced between $\gamma = 0$ and $\gamma = 1.0$ compared to intermediate steps. This confirms that ensemble feedback improves execution efficiency, though its marginal benefit diminishes beyond a certain threshold.
- **Trade Path Variability:** Figure 4 reveals that unlike synthetic experiments, trade paths under real data conditions exhibit higher variability and less smooth, decaying structure. This reflects the noisier and less regular nature of real-market self-alpha and ensemble alpha signals. Nevertheless, increasing γ introduces a mild smoothing and centering effect, distributing trades more evenly rather than clustering them.
- **Comparison with Synthetic Results:** While the cost-reduction trend with increasing γ is consistent with synthetic experiments, the noisier trade path structure underscores the importance of alpha signal quality in real-world execution modeling. In practice, self-alpha and ensemble alpha would likely require more sophisticated filtering or feature engineering to fully leverage the benefits demonstrated in controlled synthetic settings.

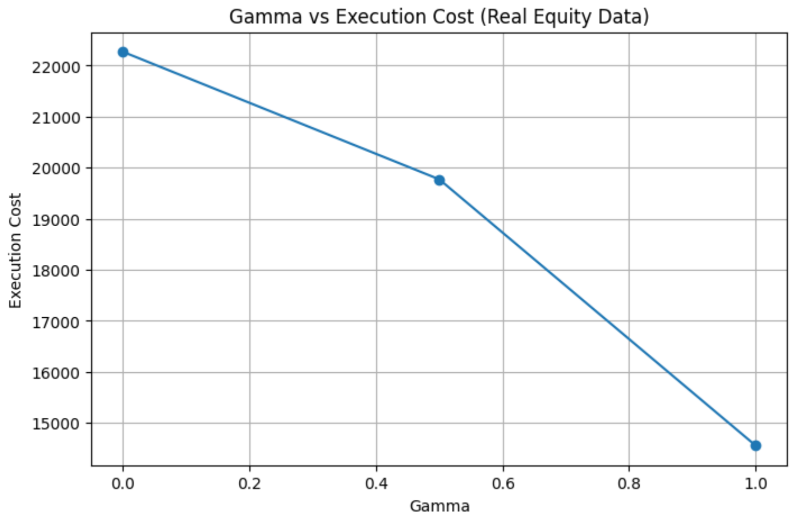


Figure 3: Execution Cost as a Function of Ensemble Coupling γ Using Real Equity Data.

6 Conclusions, Limitations, and Future Work

6.1 Conclusions

This paper introduced an extension of the classical Almgren-Chriss optimal execution model by incorporating two statistically motivated components:

1. **Self-alpha feedback**, modeled as a function of remaining inventory or trade path history, including both constant and time-varying specifications.
2. **Ensemble coupling**, modeled via mean-field–type interactions using either synthetic agent counts or empirical market proxies.

Through both synthetic experiments (using fixed and time-varying alpha signals) and real-market applications using intraday AAPL equity data, we observed consistent patterns:

- Increasing ensemble coupling strength γ reliably reduced execution cost across settings, consistent with theoretical expectations from mean-field models.
- Under constant self-alpha, trade path structure remained stable, with ensemble coupling primarily affecting cost rather than path shape.
- Under time-varying self-alpha and ensemble alpha, ensemble coupling introduced significant trade path adjustments, highlighting the model's flexibility in capturing more dynamic execution scenarios.
- Real market data introduced additional noise and irregularity, underscoring the difference between idealized and empirical signal environments.

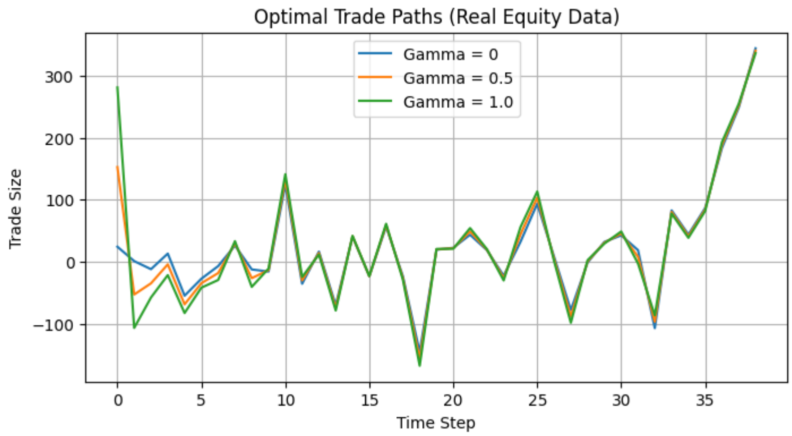


Figure 4: Optimal Trade Paths Under Varying γ Using Real Equity Data.

6.2 Limitations

The following modeling and practical limitations should be noted:

- **Simplified Alpha Signal Construction:** Self-alpha and ensemble alpha in the real data experiment were derived using basic rolling window methods and signed volume heuristics. More sophisticated alpha models, such as those based on order flow imbalance [6], Hawkes process intensities [3], or machine-learned predictors [11], could refine signal quality.
- **Static Parameter Grids:** Parameters such as κ, λ, and γ were explored using static grids rather than adaptive calibration. In practical deployment, online learning or Bayesian updating methods would likely be required [7].
- **No Explicit Market Impact Feedback Loops:** The current model assumes market impact parameters remain static. Extensions incorporating dynamic impact estimation [9], or feedback-aware control mechanisms, would enhance realism.
- **Discrete Time and No Latency Effects:** The model operates in discrete, uniform time steps and ignores latency or asynchronous trading effects. Continuous-time approximations [6] or stochastic control formulations could address this gap.
- **Agent Simplification:** Ensemble effects were approximated via simple state counts or smoothed real-market proxies. Full agent-based simulations or mean-field game formulations [10] would provide a richer, though computationally heavier, framework.

6.3 Future Work

Building on these foundations, several research directions are worth pursuing:

- **Integration with Market Microstructure Models:** Embedding self-alpha and ensemble coupling into limit order book (LOB)–aware frameworks [1], incorporating queue position and execution priority effects.

- **Machine Learning Enhanced Alpha Signals:** Exploring the integration of deep learning methods for alpha estimation, such as recurrent neural networks (RNNs) or transformer architectures applied to order book and news data streams [14, 12].
- **Robust Control Extensions:** Investigating robust optimization formulations that explicitly handle parameter uncertainty, including model misspecification in self-alpha and ensemble alpha dynamics.
- **Empirical Backtesting and Live Trading Tests:** Deploying the extended model in realistic backtesting environments with transaction cost analysis (TCA) metrics and slippage benchmarking.
- **Generalized Mean-Field Games:** Extending the theoretical formulation toward generalized mean-field game structures as described in [10], potentially combining continuous-time dynamics with risk-sensitive control.

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