Math and probability

S320/520

Trosset chs. 2 & 3

Note: These are part of my personal notes for the lectures. They are *not* complete.
Math you need to do probability:

- Sets
- Counting
- Functions
- (Limits and calculus)

Sets

- **Sample space**, S: The set of *possible outcomes.* Exactly one of the possible outcomes happens.

- **Event**, E: A subset of the sample space. It may consist of some, all, or none of the possible outcomes.

- **Countable set:** A set that can be placed in a counting order.

  e.g. The set {red, white, blue} is countable.

  e.g. The integers are countable: We can order them as $\{0, 1, -1, 2, -2, \ldots\}$ and we'll eventually reach any integer you specify.

  e.g. The real numbers are *not* countable. There's no way of putting them into one-to-one correspondence with the counting numbers.

- **Intersection:** The intersection of sets A and B, which we write $A \cap B$, is the set of outcomes that are in both A and B.

  Disjoint: Sets are disjoint if their intersection is empty. A collection of sets is *pairwise disjoint* if any pair of the sets is disjoint.

- **Union:** The union of sets A and B, which we write $A \cup B$, is the set of outcomes that are in either A or B (or both.)

- **Complement:** The complement of set A, which we write A^c, is the set of all outcomes in the sample space that are *not* in A.

The three axioms of probability

Mathematical probability is a way of attaching numbers to sets that is consistent with certain axioms.

Let $P(E)$ be the probability that event E happens.

Kolmogorov's axioms:[1]

1. If E is an event, then $0 \leq P(E) \geq 1$.
2. $P(S) = 1$.
3. If $\{E_1, E_2, E_3 \ldots\}$ is a countable collection of pairwise disjoint events,
$$P(\cup E_i) = \sum P(E_i).$$

Some more rules quickly follow from these axioms:

$$\begin{aligned} P(A) + P(A^c) &= P(A \cup A^c) = P(S) = 1 \\ \implies P(A^c) &= 1 - P(A). \end{aligned}$$

This is the *complement rule* or *subtraction rule.*

$$\begin{aligned} P(A) &= P(A \cap B^c) + P(A \cap B) \\ P(B) &= P(B \cap A^c) + P(A \cap B) \\ P(A) + P(B) &= P(A \cap B^c) + P(A \cap B) + P(B \cap A^c) + P(A \cap B) \\ &= P(A \cup B) + P(A \cap B) \\ P(A \cup B) &= P(A) + P(B) - P(B \cap A^c) \end{aligned}$$

Addition rule: The probability of A or B is the probability of A plus the probability of B minus the probability of both.

How to come up with probabilities

- *Classical probability: Equally likely outcomes*

  Roll a six-sided die. Assuming all six sides are equally likely, the probability of a three is 1/6.

- *Frequentist probability: Repeated experiments*

  Roll the the die a thousand times. The probability of a three is the proportion of times a three comes up.

- *Subjective probability: Represent your strength of belief with a number*

  I believe the following bet is fair: I give you a sixth of a dollar, then you give a dollar back if I roll a three.

[1] In a slightly different form from the original.

Classical probability and counting

We can solve many probability problems by correctly identifying the equally likely outcomes. Then we reduce them to counting problems:

$$P(A) = \frac{\#(A)}{\#(S)}$$

where # means we count the number of outcomes in that set.

Example. I toss two coins. What are the equally likely outcomes? How many are there?

- *Wrong answer:* Three: two heads, two tails, one head one tail.
- *Right answer:* Four: $\{H, H\}$, $\{H, T\}$, $\{T, H\}$, $\{T, T\}$.

If you don't believe me, toss a pair of coins a hundred times.

We can use the **multiplication principle** to help our counting:

> Suppose we do two experiments. If the first experiment has n_1 equally likely outcomes, and for each of those outcomes, the second experiment has n_2 equally likely outcomes, then the two experiments considered together have $n_1 \times n_2$ equally likely outcomes.

Example. If I roll two six-sided dice, there are 36 equally likely outcomes, each with probability $1/36$. If I roll ten six-sided dice, there are 6^{10} equally likely outcomes, each with probability $1/6^{10}$.

Example. There are 3 starter Pokemon: Bulbasaur, Squirtle, and Charmander. I choose one starter at random, and my opponent chooses one at random. What is the probability we both choose Squirtle?

Factorial notation

I have all three starter Pokemon and need to put them in an order. If I order them randomly, what's the probability the order is 1. Bulbasaur 2. Charmander 3. Squirtle?

Permutations

There are 150 Pokemon. I choose six *different* Pokemon, in order and at random, for my team. How many equally likely outcomes are there, and what's the probability of each of them?

Combinations

There are 150 Pokemon. I choose six different Pokemon, but order doesn't matter. How many equally likely outcomes are there, and what's the probability of each of them?

More examples

Next: Probability trees, conditional probability, independence, random variables.