Iterative Water-Filling for Gaussian Vector Multiple-Access Channels

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Iterative Water-filling for Gaussian Vector Multiple Access Channels *

Wei Yu, Wonjong Rhee, Stephen Boyd, and John M. Cioffi
Packard Electrical Engineering Building
350 Serra Mall, Room 360, Stanford University
Stanford, CA 94305-9515, USA
Email: {weiyu,wonjong,boyd,cioffi}@stanford.edu
Phone: (650) 723-2525, FAX: (650) 723-9251

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Abstract

This paper characterizes the capacity region of a Gaussian multiple access channel with vector inputs and a vector output with or without intersymbol interference. The problem of finding the optimal input distribution is shown to be a convex programming problem, and an efficient numerical algorithm is developed to evaluate the optimal transmit spectrum under the maximum sum data rate criterion. The numerical algorithm has an iterative water-filling interpretation. It converges from any starting point and it reaches within $\frac{K-1}{2}$ nats per output dimension per transmission from the K-user multiple access sum capacity after just one iteration. These results are also applicable to vector multiple access fading channels.

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1 Introduction

One of the fundamental results in the multiuser information theory is the single-letter characterization of the capacity region for a multiple access channel. The capacity region of a discrete-time synchronous memoryless multiple access channel with two input terminals $\mathbf{x}_1$ $\mathbf{x}_2$ and an output terminal $\mathbf{y}$ with a joint distribution $p(\mathbf{y}|\mathbf{x}_1, \mathbf{x}_2)p_1(\mathbf{x}_1)p_2(\mathbf{x}_2)$ is a pentagon region represented by:

$$\begin{aligned} R_1 &\leq I(\mathbf{x}_1; \mathbf{y}|\mathbf{x}_2); \\ R_2 &\leq I(\mathbf{x}_2; \mathbf{y}|\mathbf{x}_1); \\ R_1 + R_2 &\leq I(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}), \end{aligned} \tag{1}$$

where the mutual information expressions are computed with the joint distribution [1]. When the input distribution of a multiple access channel is not fixed, but constrained in some ways, the capacity region is the convex hull of the union of all capacity pentagons whose corresponding input distributions, after the convex hull operation, satisfy the input constraint [2] [3]. This is true for Gaussian multiple access channels under power constraints, where different points on the capacity region boundary generally correspond to different input distributions (or different time-sharing of distributions). However, when the Gaussian multiple access channel is memoryless and when it has only scalar inputs, the union and the convex hull operation turns out to be superfluous. The capacity region in this case is the well-known Cover-Wyner pentagon region [4] and every boundary point on the capacity region can be achieved with the same optimal input distribution, namely, a Gaussian distribution whose variance is equal to the power constraint. Such is not the case for the more general Gaussian multiple access channel with vector inputs. Here, the characterization of the capacity region involves an optimization over vector input distributions, and the power constraints become constraints on the trace of the covariance matrices. The Gaussian vector multiple access channel is important in practice because it is often used to model wireless communication channels with multiple transmitter and receiver antennas, and to model channels with memory, or channel with time-varying fading statistics. This paper will focus on the Gaussian vector multiple access channels. Our aim is to characterize the capacity region and to show that the optimal input distribution for vector multiple access channels can be computed efficiently by numerical methods.

Communication situations where the input and output signals are vector-valued and the noise signal is Gaussian can be modeled as a Gaussian vector channel. The single-user vector channel is represented by:

$$\mathbf{y} = H\mathbf{x} + \mathbf{n}, \tag{2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are input and output vector signals, H is the channel represented by a matrix, and $\mathbf{n}$ is the Gaussian noise vector. The capacity for the single-user channel is the maximum mutual information between $\mathbf{x}$ and $\mathbf{y}$ optimized over all input distributions subject to the power constraint. Assuming that the noise vector $\mathbf{n}$ is Gaussian with a covariance equal to an identity matrix, this input optimization problem has a vector-coding solution based on the singular-value decomposition of the matrix channel $H = FSM^T$, where F and M are orthogonal matrices, and S is the diagonal matrix containing the singular values [5]. In this case, the capacity-achieving input distribution is a Gaussian vector with the following two properties. First, the eigenvectors of the input covariance matrix must align with M, the right singular vectors of the channel matrix. By doing so, the matrix channel is decomposed into parallel independent additive white Gaussian (AWGN) sub-channels with gains corresponding to the singular values. Secondly, the amount of power allocated to each sub-channel must be a water-filling allocation with respect to the singular values. This maximizes the total data rate from all sub-channels.

We would like to generalize these ideas to the multiuser case. However, in a vector multiple access channel, each user has a different matrix channel with different eigen-modes. Moreover, because signals from different users interfere into each other, the optimal signaling direction and the optimal power allocation depend not only on each user's individual channel but also on the structure of the interference from all other users. In this light, it is perhaps surprising that the input optimization problem can indeed be solved by a generalization of the traditional water-filling algorithm. As this paper will show, the optimal signaling direction and optimal power allocation that maximize the sum capacity can be found by an iterative water-filling procedure over the users. Such an iterative procedure will be able to find the right compromise among different user's signaling strategies.

Recent interests in multiple access channels have been motivated by the fact that the single-cell wireless communication scenario can be modeled by a multiple access channel [6] [7]. The capacity region of multiple access channels has been studied extensively in the literature. For example, multiple access channels with intersymbol interference was studied by Cheng and Verdu [8], where they characterized the optimal power allocation across the frequencies. The analogous situation in the time domain for i.i.d. fading channels was studied by Knopp and Humblet [9] and Tse and Hanly [10] where the optimal power allocation over time was characterized. Both the scalar frequency selective channel and the scalar i.i.d. fading channel can be thought of as special cases of the vector multiple access channel considered in this paper. In both cases, individual channels can be decomposed into independent sub-channels in a way that is independent of individual channels. For time-invariant frequency selective channels, cyclic prefix can be appended to the input signal so that the channel can be diagonalized in the frequency domain by a discrete Fourier transform, and for the i.i.d fading channel, the independence among the sub-channels in time is explicitly assumed. In both cases, the optimal signaling direction is just the direction of the simultaneous diagonalization, and the input optimization problem is reduced into the power allocation problem among the sub-channels.

The situation becomes considerably more complicated when such simultaneous diagonalization cannot be found. This more general setting corresponds to the wireless multiple access situation where both transmitters and the receiver are equipped with multiple antenna elements. In the spatial domain, signal from each transmit antenna could experience an arbitrary channel gain to each receive antenna, thus creating an arbitrary matrix channel. It is in general not possible to simultaneously decompose an arbitrary set of matrix channels into parallel independent sub-channels. This is so because unlike the stationary ISI channels where the time-invariance property gives the special toeplitz structure to the channel matrix, antenna gain matrix does not follow spatial-invariance. Consequently, the equivalence of cyclic prefix does not exist in the spatial domain, and the transmitter optimization problem becomes a combination of choosing the optimal signaling directions for each user (e.g. "beamforming") and allocating the correct amount of power in each signaling direction. Such joint optimization is considerably more difficult because the optimal solution is a compromise between maximizing each user's data rate and minimizing its interference into other users. In this regard, only asymptotic results have been reported so far [11]. A similar situation exists for CDMA systems, where the matrix channel is created by each user's spreading sequence. In this situation, the aim is not just power allocation, but also the assignment of optimal sequence. Recent results have been obtained in [12].

The rest of the paper is organized as follows. Section 2 provides a general formulation for the Gaussian vector multiple access capacity region problem in the convex programming framework. Section 3 focuses on the rate-sum point, derives the iterative water-filling algorithm, studies its convergence property, and provides a lower and a upper bound on the general capacity region. Section 4 extends the result to the Gaussian vector multiple access channel with intersymbol interference. The results there are also applicable to the multiple access fading channels. Conclusions are drawn in section 5.

2 Vector Gaussian Multiple Access Channel

A memoryless two-user Gaussian vector multiple access channel is shown in Figure 1:

$$\mathbf{y} = H_1\mathbf{x}_1 + H_2\mathbf{x}_2 + \mathbf{n}, \tag{3}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$ are input vector signals with dimensions n_1 and n_2 respectively, $\mathbf{y}$ is the m dimensional output vector signal, $\mathbf{n}$ is the m dimensional vector additive Gaussian noise whose covariance matrix is denoted as Z, and H_1, H_2 are channel matrices of dimensions $m \times n_1$ and $m \times n_2$ respectively. The attention is restricted to the two-user case first for notational convenience, but all development can easily be generalized to cases with more than two users.

Following the development in [2] (see also [3]), define the directly achievable region of a memoryless synchronous Gaussian vector multiple access channel with power constraints q_1 and q_2 as:

$$\mathcal{A}(q_1, q_2) = \bigcup_{p_1(\mathbf{x}_1)p_2(\mathbf{x}_2)} \left\{ (R_1, R_2) : \begin{array}{c} R_1 \leq I(\mathbf{x}_1; \mathbf{y}|\mathbf{x}_2); \\ R_2 \leq I(\mathbf{x}_2; \mathbf{y}|\mathbf{x}_1); \\ R_1 + R_2 \leq I(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}). \end{array} \right\} \tag{4}$$

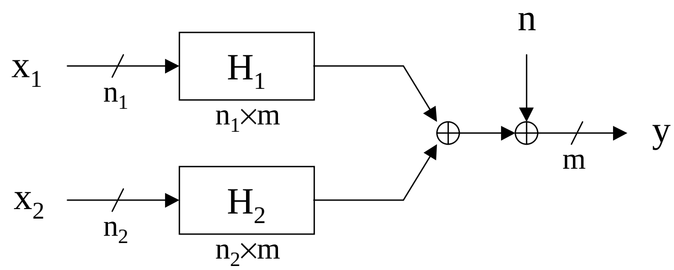

Figure 1: A two-user vector multiple access channel.

where the union is taken over all independent input distributions $p_1(\mathbf{x}_1)p_2(\mathbf{x}_2)$ that satisfy the power constraints:

$$\mathrm{tr}(\mathbf{E}[\mathbf{x}_1\mathbf{x}_1^T]) \leq q_1, \tag{5}$$
$$\mathrm{tr}(\mathbf{E}[\mathbf{x}_2\mathbf{x}_2^T]) \leq q_2, \tag{6}$$

where "tr" denotes the matrix trace operator. For each fixed input distribution $p_1(\mathbf{x}_1)p_2(\mathbf{x}_2)$, let S_1 and S_2 be the covariance matrices of $\mathbf{x}_1$ and $\mathbf{x}_2$ under the respective marginals:

$$S_1 = \mathbf{E}[\mathbf{x}_1\mathbf{x}_1^T], \tag{7}$$
$$S_2 = \mathbf{E}[\mathbf{x}_2\mathbf{x}_2^T]. \tag{8}$$

The first mutual information expression in (4) can be expanded as:

$$\begin{aligned} I(\mathbf{x}_1;\mathbf{y}|\mathbf{x}_2) &= h(\mathbf{y}|\mathbf{x}_2) - h(\mathbf{y}|\mathbf{x}_1,\mathbf{x}_2) \\ &= h(H_1\mathbf{x}_1 + \mathbf{n}) - h(\mathbf{n}) \\ &\leq \frac{1}{2}\log\frac{|H_1S_1H_1^T + Z|}{|Z|}, \end{aligned}$$

where $|\cdot|$ denotes the determinant operator, and the last inequality follows from the fact that Gaussian distribution maximizes entropy for a given covariance. It is then easy to see that, under fixed covariance matrices, the Gaussian distributions $\mathbf{x}_1 \sim \mathcal{N}(0, S_1)$ and $\mathbf{x}_2 \sim \mathcal{N}(0, S_2)$ simultaneously maximize all mutual information bounds in (4).

The mutual information bounds for given covariances S_1 and S_2 can be explictly computed using entropy expression for Gaussian random vectors:

$$C_1(S_1) = \max I(\mathbf{x}_1;\mathbf{y}|\mathbf{x}_2) = \frac{1}{2}\log\frac{|H_1S_1H_1^T + Z|}{|Z|}, \tag{9}$$

$$C_2(S_2) = \max I(\mathbf{x}_2;\mathbf{y}|\mathbf{x}_1) = \frac{1}{2}\log\frac{|H_2S_2H_2^T + Z|}{|Z|}, \tag{10}$$

and

$$C_{12}(S_1, S_2) = \max I(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}) = \frac{1}{2} \log \frac{|H_2 S_2 H_2^T + H_1 S_1 H_1^T + Z|}{|Z|}. \tag{11}$$

Denote the achievable region with covariance matrices (S_1, S_2) as:

$$\mathcal{B}(S_1, S_2) = \left\{ (R_1, R_2) : \begin{array}{c} R_1 \leq C_1(S_1); \\ R_2 \leq C_2(S_2); \\ R_1 + R_2 \leq C_{12}(S_1, S_2). \end{array} \right\}. \tag{12}$$

The directly achievable region $\mathcal{A}(q_1, q_2)$ can then be expressed as,

$$\mathcal{A}(q_1, q_2) = \bigcup_{\substack{\text{tr}(S_1) \leq q_1, \\ \text{tr}(S_2) \leq q_2, \\ S_1, S_2 \geq 0}} \mathcal{B}(S_1, S_2), \tag{13}$$

where $S \geq 0$ means that S is non-negative semidefinite. At this point, it is tempting to jump to the conclusion that the capacity region of a multiple access channel under power constraints P_1 and P_2 is simply $\mathcal{A}(P_1, P_2)$. This would be true if the multiple access channel is totally asynchronous [13]. For synchronous multiple access channels, however, the input terminals have the ability to coordinate the timing, and thus achieving the time-sharing or convex combination of directly achievable rate pairs. For channels with input constraints, such convex hull operation is in general necessary and it must be taken over the constraint set as well as the rate regions themselves. More precisely, as characterized in [2] and [3], the capacity region can be expressed as:

$$\mathcal{C}(P_1, P_2) = \text{closure} \left\{ (R_1, R_2) : \begin{array}{l} ((R_1, R_2), (P_1, P_2)) \in \\ \text{convex} \bigcup_{q_1, q_2 \geq 0} (\mathcal{A}(q_1, q_2), (q_1, q_2)) \end{array} \right\}. \tag{14}$$

It turns out, however, for Gaussian channels under power constraints, the capacity region is just the directly achievable region, and the convex hull operation is not necessary after all. This, we shall prove next.

Theorem 1 *For a Gaussian vector multiple access channel* $\mathbf{y} = H_1 \mathbf{x}_1 + H_2 \mathbf{x}_2 + \mathbf{n}$ *under a power constraint* P_1*,* P_2*, the capacity region* $\mathcal{C}(P_1, P_2)$ *is precisely* $\mathcal{A}(P_1, P_2)$ *without the need of convex hull and union operations. The capacity region is convex and its extreme points may be found by maximizing a weighted sum of data rates* $\mu_1 R_1 + \mu_2 R_2$*, where* $\mu_1 \geq 0$*,* $\mu_2 \geq 0$*, and* $\mu_1 + \mu_2 = 1$*. When* $\mu_1 \leq \mu_2$*, the optimization problem is:*

$$\begin{array}{ll} \textit{maximize} & \mu_1 \cdot \frac{1}{2} \log |H_1 S_1 H_1^T + H_2 S_2 H_2^T + Z| + \\ & (\mu_2 - \mu_1) \cdot \frac{1}{2} \log |H_2 S_2 H_2^T + Z| - \mu_2 \cdot \frac{1}{2} \log |Z| \\ \textit{subject to} & \text{tr}(S_1) \leq P_1, \\ & \text{tr}(S_2) \leq P_2, \\ & S_1, S_2 \geq 0. \end{array} \tag{15}$$

When $\mu_1 \geq \mu_2$, the optimization problem is,

$$\begin{array}{ll} \textit{maximize} & \mu_2 \cdot \frac{1}{2}\log|H_1 S_1 H_1^T + H_2 S_2 H_2^T + Z| + \\ & (\mu_1 - \mu_2)\cdot \frac{1}{2}\log|H_1 S_1 H_1^T + Z| - \mu_1 \cdot \frac{1}{2}\log|Z| \\ \textit{subject to} & \mathrm{tr}(S_1) \leq P_1, \\ & \mathrm{tr}(S_2) \leq P_2, \\ & S_1, S_2 \geq 0. \end{array} \tag{16}$$

Lemma 1 $\log|M|$ *is concave in the space of semidefinite matrices M.*

Proof: See [14, p.466], [17, p.48], or [18]. □

Proof of Theorem 1: The main claim here is that the convex hull operation on the rate region and the constraints is not necessary, and the capacity region $\mathcal{C}(P_1, P_2)$ is just $\mathcal{A}(P_1, P_2)$, a union of pentagons, and it is a convex set without time-sharing. If this is true, the boundary points of the capacity region can then be found by maximizing the weighted sums of the data rates, $\mu_1 R_1 + \mu_2 R_2$. Because $\mathcal{A}(P_1, P_2)$ is the union of pentagons, the maximization can be done in two steps: first for each pentagon, next over all pentagons. Assuming $\mu_1 \leq \mu_2$, the maximizing point in each pentagon is the upper corner point $R_2 = C_2(S_2)$, $R_1 = C_{12}(S_1, S_2) - C_2(S_2)$. Then, the maximization of over all pentagons is just a maximization over upper corner points. Substituting the expression for R_1 and R_2 gives (15). An identical approach gives (16).

Thus it remains to prove that the convex hull operation in (14) may be removed. This is a direct consequence of Lemma 1. First, let's consider the convex combination of two rate-power pairs $((R_1, R_2), q_1, q_2)$ and $((R_1', R_2'), q_1', q_2')$, where $(R_1, R_2) \in \mathcal{A}(q_1, q_2)$ and $(R_1', R_2') \in \mathcal{A}(q_1', q_2')$. Since $\mathcal{A}(q_1, q_2)$ is a union of pentagons, there exist (S_1, S_2) and (S_1', S_2') such that $(\mathrm{tr}(S_1), \mathrm{tr}(S_2)) \leq (q_1, q_2)$, $(\mathrm{tr}(S_1'), \mathrm{tr}(S_2')) \leq (q_1', q_2')$, $(R_1, R_2) \in \mathcal{B}(S_1, S_2)$ and $(R_1', R_2') \in \mathcal{B}(S_1', S_2')$. ("$\leq$" here means less than or equal to in each component.) Now, consider any convex combination of the rate-power pairs:

$$\alpha((R_1, R_2), q_1, q_2) + (1-\alpha)((R_1', R_2'), q_1', q_2'), \tag{17}$$

where $\alpha \geq 0$. For this convex combination to be in $\mathcal{C}(P_1, P_2)$, we need, $\alpha(q_1, q_2) + (1 - \alpha)(q_1', q_2') \leq (P_1, P_2)$. Thus, we have, $\alpha(\mathrm{tr}(S_1), \mathrm{tr}(S_2)) + (1-\alpha)(\mathrm{tr}(S_1'), \mathrm{tr}(S_2')) \leq (P_1, P_2)$. Define $\hat{S}_1 = \alpha S_1 + (1-\alpha)S_1'$, and $\hat{S}_2 = \alpha S_2 + (1-\alpha)S_2'$. We have

$$(\mathrm{tr}(\hat{S}_1), \mathrm{tr}(\hat{S}_2)) \leq (P_1, P_2), \tag{18}$$

and also

$$\begin{array}{rclcl} \alpha R_1 + (1-\alpha)R_1' & \leq & \alpha C_1(S_1) + (1-\alpha)C_1(S_1') & \leq & C_1(\hat{S}_1) \\ \alpha R_2 + (1-\alpha)R_2' & \leq & \alpha C_2(S_2) + (1-\alpha)C_2(S_2') & \leq & C_2(\hat{S}_2) \\ \alpha(R_1 + R_2) + (1-\alpha)(R_1' + R_2') & \leq & \alpha C_{12}(S_1, S_2) + (1-\alpha)C_{12}(S_1', S_2') & \leq & C_{12}(\hat{S}_1, \hat{S}_2). \end{array} \tag{19}$$

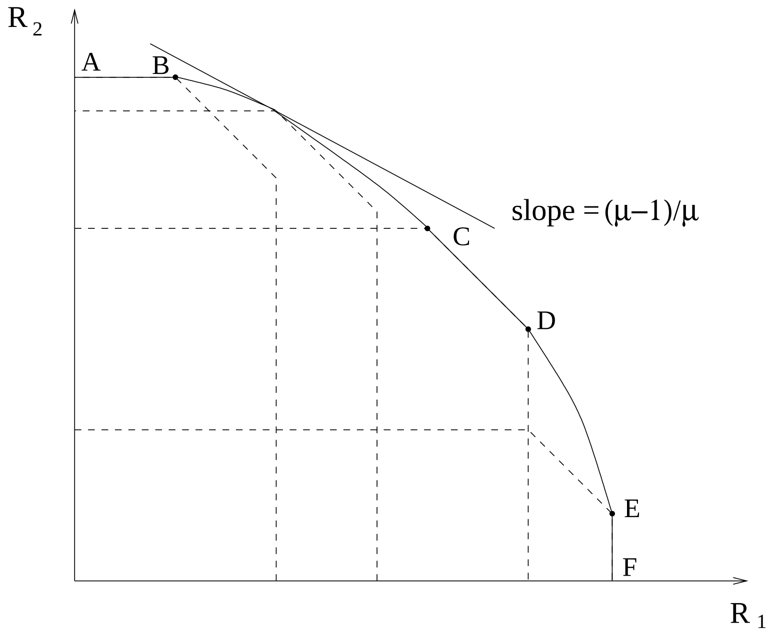

Figure 2: Capacity region of a two-user vector multiple access channel.

This follows from the fact that C_1, C_2 and C_{12} are all concave functions by lemma. Therefore, any rate-power pair that can be achieved with the convex combination of two other rate-power pairs may also be achieved without the need for the convex combination. This argument works for arbitrary convex combinations, so the convex hull operation in (14) may be removed. Finally, since $\mathcal{A}(q_1, q_2) \subseteq \mathcal{A}(q_1', q_2')$ whenever $(q_1, q_2) \leq (q_1', q_2')$, the union operation also simplifies, and the capacity region $\mathcal{C}(P_1, P_2)$ is just $\mathcal{A}(P_1, P_2)$. This region itself is also convex since this is a special case of the argument made before. This proves the main claim. $\square$

Figure 2 shows a typical capacity region of a Gaussian vector multiple access channel as a union of pentagons. Each pentagon corresponds to a transmit covariance matrix pair. Fixing the covariance matrices, different rate-pairs within the pentagon can be achieved with time-sharing.

Aspects of Theorem 1 have been observed in the literature by several authors in many different contexts. Most notably, the first part of this theorem is a special case of the Theorem 1 in [8], where multiple access channels with memory is treated. Here, we have chosen not to use general results from channels with memory [15], and we deal with input constraints explicitly. Similar results have also appeared in [10], [12] and [11] where the single-antenna fading channel, CDMA channel, and vector fading channel, are treated respectively. The concavity of the logdet function was previous observed in [16] and [11] for sum capacity. The connection between concavity and the ability to remove the convex hull operation is shown explicitly here.

Concavity is a key observation not only in simplifying the capacity expression but also in providing computationally efficient algorithms to numerically compute the capacity. The optimization problem in Theorem 1 belongs to the class of convex programming problem for which the global optimum can be found efficiently [17] [18]. In fact, the classical water-filling and the multiuser water-filling algorithm in [8] can be thought of as special purpose convex optimization algorithms.

For the sake of completeness, we state the analogous result for the general K-user multiple access channel.

Theorem 2 *The capacity region for a K-user multiple access channel $\mathbf{y} = \sum_{i=1}^{K} H_i \mathbf{x}_i + n$ with power constraints $\{P_i\}_{i=1}^{K}$ is:*

$$\mathbf{C} = \bigcup_{\substack{\mathrm{tr}(S_i) \leq P_i, \\ S_i \geq 0}} \left\{ (R_1, \cdots, R_K) : \sum_{i \in \mathbf{I}} R_i \leq \frac{1}{2} \log \frac{\left| \sum_{i \in \mathbf{I}} H_i S_i H_i^T + Z \right|}{|Z|}, \forall \mathbf{I} \subseteq \{1, \cdots, K\} \right\}. \tag{20}$$

The capacity region is convex. Its extreme points are achieved with Gaussian input distributions with covariance matrices $\{S_i\}_{i=1}^{K}$, where S_i may be found by maximizing a weighted rate sum $\sum_{i=1}^{K} \mu_i R_i$, with $\mu_i \geq 0$ and $\sum_{i=1}^{K} \mu_i = 1$. Without loss of generality, assume $\mu_K \geq \mu_{K-1} \geq \cdots \geq \mu_1$. In this case, the maximization problem is the following convex programming problem:

$$\begin{array}{lll} \textit{maximize} & \mu_1 \cdot \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^T + Z \right| - \mu_K \cdot \frac{1}{2} \log |Z| + & \\ & \sum_{j=2}^{K} (\mu_j - \mu_{j-1}) \cdot \frac{1}{2} \log \left| \sum_{i=j}^{K} H_i S_i H_i^T + Z \right| & \\ \textit{subject to} & \mathrm{tr}(S_i) \leq P_i, & i = 1, \cdots, K \\ & S_i \geq 0. & i = 1, \cdots, K \end{array} \tag{21}$$

The proof is an easy generalization of the two-user case, and is omitted here.

3 Sum Capacity

We have shown that the capacity for a vector Gaussian multiple access channel may be computed via a convex programming problem. So, in theory, the optimization can be done efficiently, and in practice, general purpose convex programming routines such as interior point methods [16] can be used to solve such problems. For large dimension problems, however, the optimization is computationally intensive because the optimization of $\{S_1, S_2\}$

is performed in the space of non-negative semidefinite matrices, so the number of scalar variables increases quadratically with the number of input dimensions. In the single-user case, the transmitter optimization problem has a well-known water-filling solution. The water-filling algorithm takes advantage of the problem structure by decomposing the channel into orthogonal modes, which greatly reduces the computational complexity. It turns out that this idea may be extended to the multiuser case under the objective of maximizing the sum-rate.

3.1 Single-User Water-filling

Before going into multiuser water-filling, let us first cast the single-user water-filling into the convex programming framework. In the single-user case, the mutual information maximization problem is the following:

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2}\log|HSH^T + Z| - \frac{1}{2}\log|Z| \\ \text{subject to} \quad & \mathrm{tr}(S) \le P, \\ & S \ge 0. \end{aligned} \tag{22}$$

The analytical solution to this problem involves two steps. First, since Z is a symmetric positive definite matrix, it has an orthogonal factorization $Z = Q\Delta Q^T$, where Q is an orthogonal matrix $QQ^T = I$, and Δ is a diagonal matrix of eigenvalues $\mathrm{diag}\{\delta_1, \cdots, \delta_m\}$. Defining $\hat{H} = \Delta^{-\frac{1}{2}}Q^T H$, the objective can then be re-written as

$$\text{maximize} \quad \frac{1}{2}\log|\hat{H}S\hat{H}^T + I|. \tag{23}$$

Next, let $\hat{H} = F\Sigma M^T$ be the singular-value decomposition of $\hat{H}$, where F and M are orthogonal matrices, and Σ is a diagonal matrix of singular values $\mathrm{diag}(\sigma_1, \sigma_2, \cdots \sigma_r)$, where r is the rank of $\hat{H}$. Consider $\hat{S} = M^T S M$ as the new optimization variable, the problem is then transformed into,

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2}\log|\Sigma\hat{S}\Sigma^T + I| \\ \text{subject to} \quad & \mathrm{tr}(\hat{S}) \le P, \\ & \hat{S} \ge 0. \end{aligned} \tag{24}$$

Using Hadamard's inequality [4], it is easy to show that the solution is the well-known water-filling algorithm. The optimal $\hat{S}$ is a diagonal matrix, $\mathrm{diag}(p_1, p_2, \cdots p_r)$, such that,

$$p_i + 1/\sigma_i^2 = K, \quad \text{if} \quad 1/\sigma_i^2 < K, \tag{25}$$

$$p_i = 0, \quad \text{if} \quad 1/\sigma_i^2 \ge K, \tag{26}$$

where K is a constant chosen so that $\sum_i p_i = P$. Therefore, to achieve the single-user capacity, first, the transmit directions need to align with the right singular-vectors of the effective channel. Secondly, the amount of energy in each direction depends on the noise to channel gain ratio in that direction in a water-filling fashion. Solving the single-user input optimization via water-filling is much more efficient than general purpose convex programming algorithms, because water-filling takes advantage of the problem structure by decomposing the equivalent channel into its eigen-modes.

3.2 Simultaneous Water-filling

In a vector Gaussian multiple access channel, if the objective is to maximize the sum of the data rates, the optimal transmit covariance matrices satisfy a multiuser water-filling condition.

Theorem 3 *In a K-user multiple access channel, $\{S_i\}$ is an optimal solution to the rate-sum maximization problem*

$$\begin{array}{lll} \textit{maximize} & \frac{1}{2}\log\left|\sum_{i=1}^{K} H_i S_i H_i^T + Z\right| - \frac{1}{2}\log|Z| & \\ \textit{subject to} & \mathrm{tr}(S_i) \le P_i, & i = 1, \dots, K \\ & S_i \ge 0, & i = 1, \dots, K \end{array} \tag{27}$$

if and only if S_i is the single-user water-filling covariance matrix of the channel H_i with $Z + \sum_{j=1, j\neq i}^{K} H_j S_j H_j^T$ as noise, for all $i = 1, 2, \cdots K$.

Proof: We prove the *only if* part first. Suppose that at the rate-sum optimum, there is an S_i which does not satisfy the single-user water-filling condition. Fix all other covariance matrices, and water-fill S_i regarding other users' signal as noise. Since the single-user optimization problem in S_i differs from the rate-sum optimization problem only by a constant when all other covariances are fixed, the water-filling adjustment on S_i will strictly increase the rate-sum objective. This contradicts the optimality of S_i. Thus, at the optimum, all S_i's must satisfy the single-user water-filling condition.

The *if* part also holds. To prove this, however, we will need some general results from convex analysis. We will come back to the proof after the detour.

3.3 Convex Optimization

We briefly review convex optimization in this section. A general convex optimization problem is of the form:

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0 & i = 1, \dots, K, \end{array} \tag{28}$$

where $x \in \mathbb{R}^n$ is the optimization variable, and $f_0, \cdots, f_K$ are convex functions. We call the original problem the "primal" problem, and associate a dual variable λ_i with each primal constraint $f_i(x) \le 0$. The dual variable belongs to the dual space of the constraint space, and each dual variable defines a linear functional (or an inner product) from the constraint space to the real line. For example, when the constraint space is a real line, the dual variable is also real, and the inner product is just the usual product. When the constraint space is the set of non-negative semidefinite matrices, the dual space (more precisely, the dual cone) is the set of non-negative semidefinite matrices also, and the inner product in this case is the trace of the matrix product. The dual variables always take on non-negative values.

The Lagrangian of an optimization problem is a linear combination of the primal objective and the inner product defined by the dual variables:

$$L(x, \lambda) = f_0(x) + \sum_i \langle \lambda_i, f_i(x) \rangle, \tag{29}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. The dual objective is defined to be

$$g(\lambda) = \inf_x L(x, \lambda). \tag{30}$$

It is easy to see that $g(\lambda)$ is a lower bound on the optimal $f_0(x)$:

$$\begin{aligned} f_0(x) &\geq f_0(x) + \sum_i \langle \lambda_i, f_i(x) \rangle && (31) \\ &\geq \inf_z \left(f_0(z) + \sum_i \langle \lambda_i, f_i(z) \rangle \right) && (32) \\ &\geq g(\lambda). && (33) \end{aligned}$$

So,

$$\max_\lambda g(\lambda) \leq \min_x f_0(x), \tag{34}$$

where the maximization is over all non-negative λ_i's, and the minimization is over the original constraint set. The difference between the primal objective $f_0(x)$ and the dual objective $g(\lambda)$ is called the duality-gap. A central result in convex analysis [17] is that when $f_0, \cdots f_K$ are convex, under some technical conditions (called constraint qualifications) [19], the duality gap reduces to zero at the optimal, i.e. (34) is achieved with equality for some x^*, λ^*. Thus, one way to solve the original problem is to solve its dual problem.

Let x^* and λ_i^* be the primal and dual variables at the optimum. If we substitute them in the chain of inequalities (31) to (33), we see that each of the inequalities must be satisfied with equality. Since $\lambda_i \geq 0$ and $f_i(x) \leq 0$, the inner products are all less than or equal to zero. Thus to have equality in (31), we must have $\langle \lambda_i^*, f_i(x^*) \rangle = 0$. This is the so-called complementary slackness condition. Moreover, the inequality in (32) is also satisfied with equality, so the infimum is achieved at x^*. When the functions $f_0, \cdots f_K$'s are differentiable, the gradient of the Lagrangian with respect to x, $\nabla L(x, \lambda^*)$, must be zero at x^*. These two facts, together with the constraints on the primal and dual variables form the Karush-Kuhn-

Tucker (KKT) conditions:

$$
\begin{aligned}
f_i(x^*) &\leq 0 && (35)\\
\lambda_i^* &\geq 0 && (36)\\
\nabla f_0(x^*) + \nabla\langle \lambda_i^*, f_i(x^*)\rangle &= 0 && (37)\\
\langle \lambda_i^*, f_i(x^*)\rangle &= 0. && (38)
\end{aligned}
$$

Under some technical conditions, the KKT conditions are necessary and sufficient for optimality. One simple version of the technical condition is Slater's condition, which is satisfied when there exists x such that $f_i(x) < 0$, $i = 1, \cdots, K$ [19] [17].

3.4 Proof of Theorem 3

We now prove the *if* part of theorem 3. First, (27) can be reformulated into the following equivalent form:

$$
\begin{array}{lll}
\text{minimize} & -\log|T| & \\
\text{subject to} & T \leq \sum_{i=1}^{K} H_i S_i H_i^T + Z & \\
 & \operatorname{tr}(S_i) \leq P_i, & i = 1, \dots, K \\
 & S_i \geq 0, & i = 1, \dots, K
\end{array}
\tag{39}
$$

The coefficient 1/2 and the constant $\log|Z|$ are omitted for simplicity. Associate dual variables Γ, $\{\lambda_i\}$, $\{\Psi_i\}$ to each of the constraints. Note that the first and the third constraints are matrix inequalities, so the dual variables Γ and $\{\Psi_i\}$ are matrices and the inner product is the trace of the matrix product. The power constraint is a constraint on real numbers, so its associated dual variable $\{\lambda_i\}$ is real. The Lagrangian of the optimization problem is:

$$
\begin{aligned}
& L(\{S_i\}, T, \Gamma, \{\lambda_i\}, \{\Psi_i\}) \\
&= -\log|T| + \operatorname{tr}\left[\Gamma\left(T - \sum_{i=1}^{K} H_i S_i H_i^T - Z\right)\right] + \sum_{i=1}^{K} \lambda_i(\operatorname{tr}(S_i) - P_i) - \sum_{i=1}^{K} \operatorname{tr}(\Psi_i S_i) \\
&= -\log|T| + \operatorname{tr}(\Gamma T) - \operatorname{tr}(\Gamma Z) - \sum_{i=1}^{K} \lambda_i P_i + \sum_{i=1}^{K} \operatorname{tr}[(\lambda_i I - H_i^T \Gamma H_i - \Psi_i) S_i]
\end{aligned}
\tag{40}
$$

where the fact $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ is used. The objective of the dual program is

$$
g(\Gamma, \{\lambda_i\}, \{\Psi_i\}) = \inf_{\{S_i\}, T} L(\{S_i\}, T, \Gamma, \{\lambda_i\}, \{\Psi_i\}). \tag{41}
$$

At the infimum, $\partial L / \partial S_i$ must be zero. This leads to:

$$
\lambda_i I = H_i^T \Gamma H_i + \Psi_i, \quad i = 1, 2, \cdots, K. \tag{42}
$$

Further, the gradient with respect to T must also be zero:

$$\frac{\partial}{\partial T}(-\log|T| + \operatorname{tr}(\Gamma T)) = 0, \tag{43}$$

which implies that $\operatorname{tr}(T^{-1}M) = \operatorname{tr}(\Gamma M), \forall M$. So,

$$T^{-1} = \Gamma. \tag{44}$$

Therefore, $g(\Gamma, \{\lambda_i\}, \{\Psi_i\}) = \log|\Gamma| + m - \operatorname{tr}(\Gamma Z) - \sum_{i=1}^{K} \lambda_i P_i$, where m is the number of output dimensions. The dual problem of (27) is then,

$$\begin{array}{lll} \text{maximize} & \log|\Gamma| + m - \operatorname{tr}(\Gamma Z) - \sum_{i=1}^{K} \lambda_i P_i & \\ \text{subject to} & \lambda_i I \geq H_i^T \Gamma H_i, & i = 1, \ldots, K \\ & \Gamma \geq 0. & \end{array} \tag{45}$$

Note, the only constraints on $\{\Psi_i\}$ are non-negative semi-definite constraints, so (42) is equivalent to the inequality in (45). Because the primal program is convex, the dual problem achieves a maximum at the minimum value of the primal objective.

The primal constraints are such that the Slater's condition is satisfied, so the KKT condition is sufficient and necessary. The KKT conditions include the stationarity conditions on the Lagrangian (42) and (44), as well as the complementary slackness conditions:

$$\operatorname{tr}\left[\Gamma\left(T - \sum_{i=1}^{K} H_i S_i H_i^T - Z\right)\right] = 0, \tag{46}$$

$$\lambda_i(\operatorname{tr}(S_i) - P_i) = 0, \qquad i = 1, \cdots K \tag{47}$$

$$\operatorname{tr}(\Psi_i S_i) = 0, \qquad i = 1, \cdots K \tag{48}$$

Consider the original optimization problem. Observe that at the optimum, we must have $T = \sum_{i=1}^{K} H_i S_i H_i^T + Z$, and $\operatorname{tr}(S_i) = P_i, i = 1, \ldots, K$ (otherwise sum rate can be increased). So, only the last complementary slackness condition (48) is useful. Because the stationary and complementary slackness conditions, together with primal and dual constraints, are necessary and sufficient, the optimization problem can be transformed into the problem of finding primal variables $\{S_i\}$, T, and dual variables Γ, $\{\Psi_i\}$, $\{\lambda_i\}$ that satisfy:

$$\begin{gathered} \lambda_i I = H_i^T \left(\sum_{j=1}^{K} H_j S_j H_j^T + Z\right)^{-1} H_i + \Psi_i, \\ \operatorname{tr}(S_i) = P_i, \\ \operatorname{tr}(\Psi_i S_i) = 0, \\ \Psi_i, S_i, \lambda_i \geq 0, \end{gathered} \tag{49}$$

for all $i = 1, \ldots, K$.

Now, the above KKT condition is also valid for the single-user water-filling problem when K is set to 1. In this case, it is easy to verify that the single-user solution (25) - (26) satisfies the condition exactly. But, for each user i, the multiuser KKT condition and the single-user KKT condition differ only by the additional noise term $\sum_{j=1,j\neq i}^{K} H_j S_j H_j^T$. So, if each S_i satisfies the single-user condition while regarding other users' signals as additional noise, then collectively, the set of $\{S_i\}$ must also satisfy the multiuser KKT condition. By the sufficiency of the KKT condition, $\{S_i\}$ must then be the optimal covariance for the multiuser problem. This proves the *if* part of the theorem. □

3.5 Iterative Water-filling

Since at the optimum, each user's covariance is a water-filling of noise and all other users' interference combined, we might expect that the rate-sum optimal covariance may be found with an iterative algorithm.

Algorithm 1 *Iterative water-filling algorithm for a vector Gaussian multiple access channel:*

initialize $S_i = 0$, $i = 1, \ldots K$.

repeat

for i=1 to K

$$N = \sum_{j=1,j\neq i}^{K} H_j S_j H_j^T + Z;$$

$$S_i = \arg\max_S \frac{1}{2} \log |H_i S H_i^T + N|;$$

end

until the desired accuracy is reached.

Theorem 4 *The iterative water-filling algorithm converges to a limit point $\{\hat{S}_i\}$ from any initial assignment of $\{S_i\}$. The limit point maximizes the rate-sum of a K-user Gaussian vector multiple access channel.*

Proof: At each step, the iterative water-filling algorithm finds the single-user water-filling covariance matrix for each user while regarding all other user's signal as additional noise. Since the single-user rate objective differs from the multiuser rate-sum objective by only a constant, the rate-sum objective is non-decreasing after each water-filling step. The rate-sum objective is bounded above, so the rate-sum converges to a limit. At the rate-sum limit, because each single-user water-filling gives an unique covariance, the covariance matrices also converges to a limit $\{\hat{S}_i\}$. The limit is a fixed point of the algorithm, and at the limit, all S_i's are simultaneously the single-user water-filling covariance matrices of user i while regarding all the other users as additional noise. Then, by Theorem 3, the limit must be rate-sum optimal.

The above proof does not depend on the initial value. So the algorithm always converges, and it converges to an optimum $\{\hat{S}_i\}$ from any starting point. □

This result may appear counter-intuitive at first. A multiple access channel capacity may be achieved with superposition coding at each transmitter and interference subtraction at the receiver. A coding strategy where each user regards all other users as noise does not achieve the capacity. Yet, it turns out that the iterative procedure where each water-filling regards all other users as noise is precisely the one that converges to an optimal set of covariance matrices.

The rate-sum optimal covariance matrices may not be unique. Depending on the initial value, the iterative water-filling algorithm may converge to two different sets of covariance matrices both giving the optimal sum rate. The following is an example of when this happens. Let $H_1 = H_2 = Z = I_{2\times 2}$, and $P_1 = P_2 = 2$. Then, $S_1 = S_2 = I_{2\times 2}$, and $S_1' = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $S_2' = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ both achieve the same maximum rate-sum.

Figure 3 gives a graphical interpretation of the algorithm. The capacity region of a two-user vector multiple access channel is shown in Figure 3(a). The sum rate $R_1 + R_2$ reaches the maximum on the line between C and D. Initially, the covariance matrices for the two users, $S_1^{(0)}$ and $S_2^{(0)}$, are zero matrices.

1. The first iteration is shown in Figure 3(b). After a single-user water-filling for $S_1^{(1)}$, the rate pair (R_1, R_2) is at point 'F'. Then, treating $S_1^{(1)}$ as noise, a single-user water-filling for $S_2^{(1)}$ moves the rate pair to point 'E'.

2. The second iteration is shown in Figure 3(c). First, note that fixing covariance matrices $S_1^{(1)}$ and $S_2^{(1)}$, the capacity region is the pentagon 'abEFO'. So, by changing the decoding order of user 1 and 2, we can move rate pair to point 'b' without affecting the rate sum. Once at point 'b', we can then water-fill $S_1^{(1)}$ treating $S_2^{(1)}$ as noise to get $S_1^{(2)}$. This would increase $I(\mathbf{x}_1; \mathbf{y})$, while keeping $I(\mathbf{x}_2; \mathbf{y}|\mathbf{x}_1)$ fixed, thus moving the rate pair to point 'c'.

3. The capacity pentagon with $(S_1^{(2)}, S_2^{(1)})$ is now represented by 'acdeO'. So, we can again interchange the decoding order to get to the point 'd', and perform another single-user water-filling treating $S_2^{(1)}$ as additional noise. This gives us $S_1^{(2)}$, and the corresponding rate-pair point 'f' in Figure 3(d). The process continues until it converges to points 'C' and 'D'.

Note that in every step, each user negotiates for itself the best signaling direction as well as the optimal power allocation while regarding the interference generated by all other users as noise. The convergence happens in the space of all possible covariance matrices. The iterative water-filling algorithm is more efficient than solving general-purpose convex programming routines because in each step, the algorithm takes advantage of the problem structure by doing an eigen-mode decompositions and water-filling. In fact, the convergence is very fast.

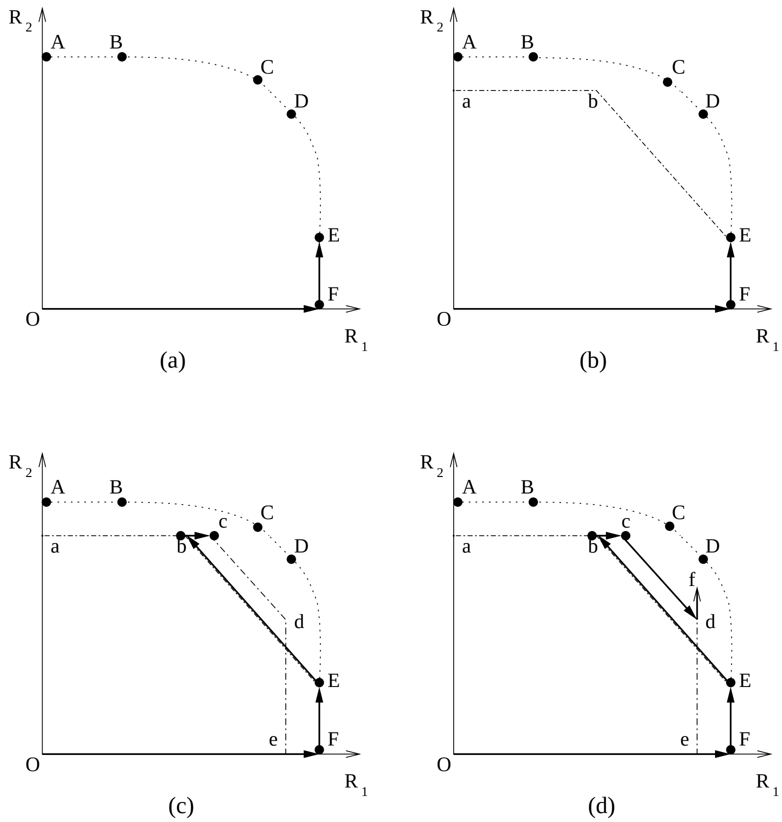

Figure 3: First two iterations of iterative water-filling algorithm

3.6 Convergence Properties

Depending on the order of water-filling, the iterative procedure arrives at a corner point of some pentagon after the first iteration. As the following theorem shows, this corner point is only 1/2 nats per user per output dimension away from the sum capacity. In other words, the iterative water-filling algorithm is close to the sum capacity after just one iteration.

Theorem 5 *After one iteration of the iterative water-filling algorithm, $\{S_i\}$ achieves a total data rate $\sum_{i=1}^{K} R_i$ that is at most $(K-1)m/2$ nats away from the sum capacity.*

Lemma 2 *Let X and Y be positive semidefinite matrices. The followings are true:*

1. *if $X \geq Y$, then $\mathrm{tr}(X) \geq \mathrm{tr}(Y)$;*
2. *$\mathrm{tr}(XY) \geq 0$;*
3. *if $X \geq Y$, then $\max\mathrm{eig}(X) \geq \max\mathrm{eig}(Y)$.*

Proof: The trace of a matrix is the sum of eigenvalues. Eigenvalues of a positive semidefinite matrix are non-negative, so its trace is non-negative. If $X \geq Y$, then $X - Y \geq 0$. So, $\mathrm{tr}(X-Y) \geq 0$, and $\mathrm{tr}(X) \geq \mathrm{tr}(Y)$. Further, positive semidefinite matrices may be represented by their square roots: $X = AA^T$ and $Y = BB^T$. So, $\mathrm{tr}(XY) = \mathrm{tr}(AA^TBB^T) = \mathrm{tr}((B^TA)(A^TB)) \geq 0$. Lastly, if $X \geq Y$, then $v^TXv \geq v^TYv$ for all unit vectors v. So, $\max\mathrm{eig}(X) = \max v^TXv \geq \max v^TYv = \max\mathrm{eig}(Y)$, where the middle two maximizations are over all unit vectors v. □

Proof of Theorem 5: The idea is to use the fact that the dual objective is always a bound on the primal objective (c.f. equation (34)), thus the difference between the primal and dual objectives, the so-called "duality gap" is a upper bound on how far away the true optimum is from the present primal objective.

Starting with $S_i = 0$. The first iteration of the algorithm consists of K water-fillings: S_1 is the single-user water-filling covariance of noise Z alone, S_2 is the water-filling of noise plus interference from S_1, and so on. S_K is the water-filling of noise plus interference from all other users. For this set of primal feasible $\{S_i\}$, the difference between the primal problem (39) and the dual problem (45), which we denote as γ, is:

$$\gamma = \mathrm{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + Z\right)^{-1} Z\right] + \sum_{i=1}^{K} \lambda_i P_i - m. \tag{50}$$

The bound holds for all dual feasible λ_i's. (It can be used, for example, as a stopping criterion in the iterative algorithm.) The bound is tightest when λ_i is chosen to be the smallest non-negative value satisfying the dual constraints in (45):

$$\lambda_i = \max\mathrm{eig}\left[H_i^T \left(\sum_{j=1}^{K} H_j S_j H_j^T + Z\right)^{-1} H_i\right], \quad i = 1, \cdots, K. \tag{51}$$

In fact, the duality gap reduces to zero if the primal feasible S_i is the optimal S_i^*, and the dual feasible λ_i's are chosen in the above fashion.

Now, since S_1 is a single-user water-filling, the duality gap for this single-user water-filling must be zero. Thus,

$$\operatorname{tr}[(H_1 S_1 H_1^T + Z)^{-1} Z] + \lambda_1' P_1 - m = 0, \tag{52}$$

where

$$\lambda_1' = \max \operatorname{eig}[H_1^T (H_1 S_1 H_1^T + Z)^{-1} H_1]. \tag{53}$$

More generally, S_i is the single-user water-filling regarding $\sum_{j=1}^{i-1} H_j S_j H_j^T + Z$ is regarded as noise. Thus,

$$\operatorname{tr}\left[\left(\sum_{j=1}^{i} H_j S_j H_j^T + Z\right)^{-1} \left(\sum_{j=1}^{i-1} H_j S_j H_j^T + Z\right)\right] + \lambda_i' P_i - m = 0, \tag{54}$$

where

$$\lambda_i' = \max \operatorname{eig}\left[H_i^T \left(\sum_{j=1}^{i} H_j S_j H_j^T + Z\right)^{-1} H_i\right]. \tag{55}$$

We now use Lemma 2 to prove the following three facts. First

$$\operatorname{tr}\left(\sum_{j=1}^{K} H_j S_j H_j^T + I\right)^{-1} \leq \operatorname{tr}(H_1 S_1 H_1^T + I)^{-1} \tag{56}$$

This is a straightforward consequence of lemma 2.1. Secondly,

$$\lambda_i \leq \lambda_i'. \tag{57}$$

This follows from their definitions (51) and (55). Since

$$H_i^T \left(\sum_{j=1}^{K} H_j S_j H_j^T + Z\right)^{-1} H_i \leq H_i^T \left(\sum_{j=1}^{i} H_j S_j H_j^T + Z\right)^{-1} H_i, \tag{58}$$

their respective maximum eigenvalues follow the same relation by lemma 2.3. Thirdly,

$$\lambda_i' P_i \leq m. \tag{59}$$

This follows from (54). The two matrices in the trace expression are both positive semidefinite, so lemma 2.2 implies that $\lambda_i' P_i \leq m$.

Now, putting everything together,

$$\begin{aligned}
\gamma &= \operatorname{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + Z\right)^{-1} Z\right] + \sum_{i=1}^{K} \lambda_i P_i - m &(60)\\
&\leq \operatorname{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + Z\right)^{-1} Z\right] + \sum_{i=1}^{K} \lambda_i' P_i - m &(61)\\
&= \operatorname{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + Z\right)^{-1} Z\right] + \lambda_1' P_1 - m + \sum_{i=2}^{K} \lambda_i' P_i &(62)\\
&\leq \sum_{i=2}^{K} \lambda_i' P_i &(63)\\
&\leq (K-1)m, &(64)
\end{aligned}$$

where the first inequalities follows from (57), the second inequality follows from (56) and (52), and the last inequality follows from (59). Recall that a factor of $\frac{1}{2}$ was omitted in our statement of the primal and dual problems: (39) and (45). Therefore the true duality gap is $(K-1)m/2$ nats. $\square$

The capacity region of a K-user multiple access channel with a fixed input covariance is a polytope. Depending on the order of water-filling, after the first iteration, the iterative water-filling algorithm reaches one of the $K!$ corner points of the capacity polytope. The above theorem asserts that none of these corner points is more than $(K-1)m/2$ nats away from the capacity sum, where K is the number of users, and m is the number of output dimensions. This result roughly states that the capacity loss per user per output dimension is at most $\frac{1}{2}$ nats after just one iteration. This bound is rather general. It works for arbitrary channel matrices, arbitrary power constraints, and for arbitrary input dimensions. Numerical simulation on realistic channels suggests that in most cases the actual difference from the capacity is even smaller.

A system with $K = 10$ users is simulated below. Each user has 100 dimensions, and the receiver also has 100 dimensions (so $n = m = 100$). The channel matrix is chosen to be block-diagonal, where each block is of size 10×10. The block matrix entries are randomly generated from an i.i.d. Gaussian distribution with mean zero and unit variance. The total power constraint for each user is chosen so that the total power to noise variance ratio is 10dB.

Figure 4 shows the percentage difference from the sum capacity for the iterative water-filling algorithm. The percentage difference is defined as $(C_{sum} - R_{sum}^{(n)})/C_{sum}$, where C_{sum} is the sum capacity, and $R_{sum}^{(n)}$ is the sum rate achieved after n iterations. Both the actual difference from the algorithm and the gap bound derived from the duality theory are plotted against the number of iterations. Observe that the algorithm is able to approach to the correct value with very few iterations, and the convergence is exponentially fast asymptotically.

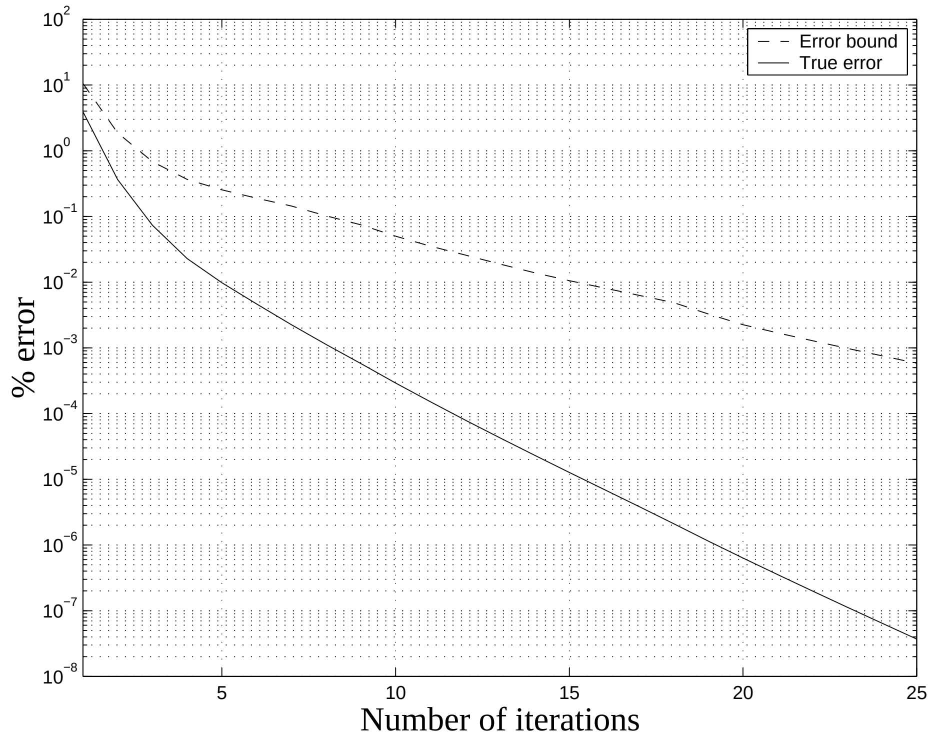

Figure 4: Percentage difference from the sum capacity in the iterative water-filling algorithm. Dashed line is the gap bound. Solid line is the actual percentage difference.

3.7 Bounds on Capacity Region

As pointed out before, after one iteration, the iterative water-filling algorithm reaches one of $K!$ corner points in the capacity polytope. In addition, when the algorithm has converged, the optimal covariance matrices give another set of $K!$ corner points all of which achieve the maximum sum data rate. Although it is more difficult to compute the capacity points between these two sets of corner points, bounds can be obtained relatively easily.

We will use the 2-user multiple access channel as an example. In figure 5, the points B and E can be found after one iteration of water-filling. Let their respective input covariance matrices be $S_B = (S_{B,1}, S_{B,2})$ and $S_E = (S_{E,1}, S_{E,2})$. Also, the sum capacity points C and D are found when iterative water-filling has converged. Denote the sum-capacity achieving covariance matrix as $S_{CD} = (S_{CD,1}, S_{CD,2})$. Note that the portion of boundary points between C and D is linear, unless the optimal sum-capacity covariance happen to be orthogonal, in which case points C and D collapse to the same point.

A lower bound for the region between B and C (or D and E) can be found based on the linear combination of covariance matrices S_B and S_{CD} (or S_E and S_{CD} respectively). Consider the data rates associated with the covariance matrices $\alpha S_B + (1-\alpha)S_{CD}$ with user 1 decoded first (or $\beta S_E + (1-\beta)S_{CD}$ with user 2 decoded first), where α (or β) ranges from 0 to 1. These rates are achievable, so they are lower bounds. Because the objective is concave as a function of the covariance matrices, this lower bound is better than the time-sharing of data rates associated with B and C (or D and E). Since the corner points after one iteration (i.e. B and E) are at most $(K-1)m/2$ nats away from the sum-capacity, the lower bound is a close approximation of the capacity region. A typical example is shown in figure 5. Extensive numerical simulations show that the bound is tight. An upper bound is also plotted there. The upper bound is obtained by extending the line segments AB, CD, and EF. This is an upper bound because of the convexity of the capacity region.

4 Multiple Access Channels with ISI

The iterative water-filling algorithm has a natural extension to vector multiple access channels with intersymbol interference. Assuming that the ISI spans a finite duration, the multiple access channel inputs and output can be treated in a block-by-block basis with an appropriate guard periods, thus reducing the ISI channel into a special case of vector channel. Moreover, if the channel is assumed stationary, toeplitz structure of the channel can be exploited to simplify the computation of the capacity region. Because of time-invariance, interference in the time-domain enjoys special properties that do not have a counter-part for interference in the space domain. The basic idea is to recognize that by adding a cyclic prefix, the Toeplitz matrix channel becomes circulant, whose eigen-decomposition is independent of the channel. Moreover, the eigen-decomposition is a particularly simple one: it is the Discrete Fourier Transform (DFT). Therefore its computational complexity can be reduced by using a Fast Fourier Transform (FFT). This idea has been exploited for scalar single-user channels [20],

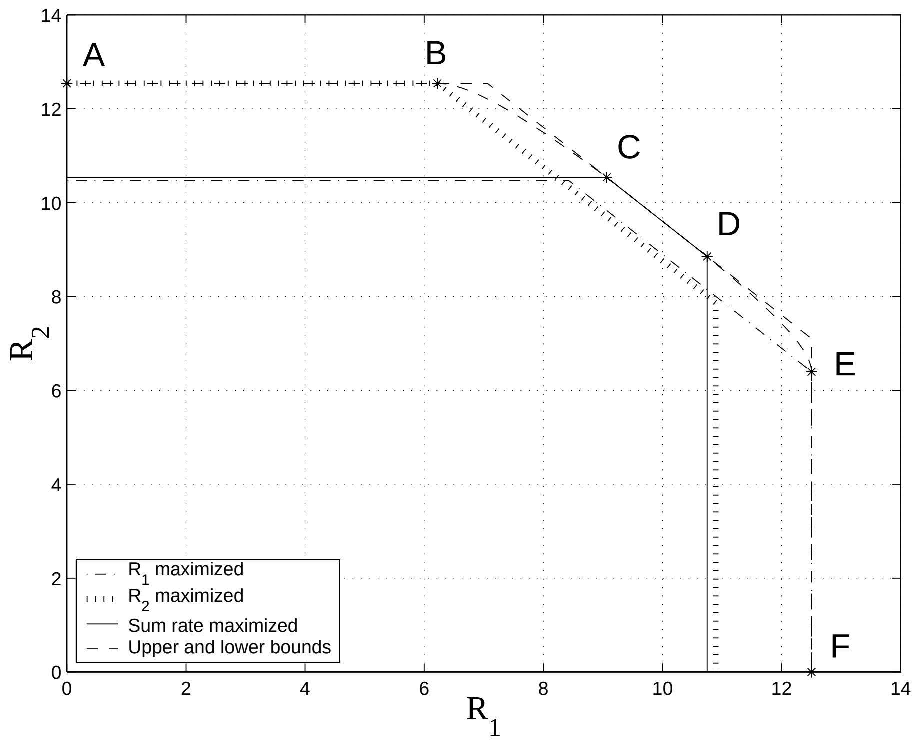

Figure 5: A lower bound and an upper bound of a typical capacity region. The channel elements are generated from an i.i.d Gaussian distribution with zero mean and unit variance. $n_1 = n_2 = m = 7$, $SNR_1 = SNR_2$=10dB. ($SNR_i \equiv \frac{P_i}{N_0}$, N_0 is noise power of each receive antenna.)

vector single user channels [21], and multiple access channels [8].

A Gaussian vector multiple access channel with finite ISI can be modeled as follows:

$$\mathbf{y}_k = \left(\sum_{i=1}^{K} \sum_{d=0}^{\nu} H_d^{(i)} \mathbf{x}_{k-d}^{(i)} \right) + \mathbf{n}_k, \tag{65}$$

where ν is the length of channel memory, the superscript represents user index, subscript in H_d represents the ISI at delay d, and subscripts of $\mathbf{y}$, $\mathbf{x}$, and $\mathbf{n}$ represent the time index. We assume that $\mathbf{n}_k$ is a memoryless Gaussian random process with covariance matrix $\mathbf{E}[\mathbf{n}_k \mathbf{n}_k^T] = Z$. Define the Fourier transform

$$\mathsf{H}^{(i)}(\omega) = \sum_{d=0}^{\nu} H_d^{(i)} e^{-jd\omega}. \tag{66}$$

The capacity region for the Gaussian vector multiple access channel with ISI has characterized in [8], and capacity can be expressed as an optimization problem in the following.

Theorem 6 *For a Gaussian vector multiple access channel with finite ISI, under a power constraint P_1, P_2, the capacity region can be characterized by maximizing a weighted sum of data rates $\mu_1 R_1 + \mu_2 R_2$, where $\mu_1 \geq 0$, $\mu_2 \geq 0$, and $\mu_1 + \mu_2 = 1$. When $\mu_1 \geq \mu_2$, the optimization problem is,*

$$\begin{array}{lc} \textit{maximize} & \mu_2 \cdot \frac{1}{2\pi} \int_0^\pi \log |\mathsf{H}^{(1)}(\omega) S^{(1)}(\omega) \mathsf{H}^{(1)}(\omega)^* + \mathsf{H}^{(2)}(\omega) S^{(2)}(\omega) \mathsf{H}^{(2)}(\omega)^* + Z| + \\ & (\mu_1 - \mu_2) \cdot \frac{1}{2\pi} \int_0^\pi \log |\mathsf{H}^{(1)}(\omega) S^{(1)}(\omega) \mathsf{H}^{(1)}(\omega)^* + Z| d\omega - \mu_1 \cdot \frac{1}{2\pi} \int_0^\pi \log |Z| d\omega \\ \textit{subject to} & \frac{1}{\pi} \int_0^\pi \mathrm{tr}(S^{(1)}(\omega)) d\omega \leq P_1, \\ & \frac{1}{\pi} \int_0^\pi \mathrm{tr}(S^{(2)}(\omega)) d\omega \leq P_2, \\ & S^{(1)}(\omega), S^{(2)}(\omega) \geq 0. \end{array} \tag{67}$$

The case when $\mu_1 \leq \mu_2$ is similar.

The rate-sum maximization problem again has a simultaneous water-filling interpretation, and an optimal input spectrum can be found via iterative water-filling.

Theorem 7 *In a K-user Gaussian vector multiple access channel with finite ISI, $\{S^{(i)}(\omega)\}$ is an optimal solution to the rate-sum maximization problem if and only if $S^{(i)}(\omega)$ is the single-user water-filling of the channel $\mathsf{H}^{(i)}(\omega)$ with $Z + \sum_{j=1, j \neq i}^{K} \mathsf{H}^{(j)}(\omega) S^{(j)}(\omega) \mathsf{H}^{(j)}(\omega)^*$ as noise for all $i = 1, 2, \cdots K$. Moreover, an optimal set of $\{S^{(i)}(\omega)\}$ may be found by the iterative water-filling algorithm. The algorithm converges from any initial assignment, and the limit maximizes the rate-sum. Also, the covariance matrices after one iteration of iterative water-filling achieves a total data rate $\sum_{i=1}^{K} R_i$ that is at most $(K-1)m/2$ per transmission away from the sum capacity, where m is the number of receive antennas.*

The proof of the above theorem follows the exact same way as in the memoryless case. However, the optimization variables are now functions of real variables. So, a rigorous argument involves the generalized Kuhn-Tucker condition [22], and the notion of differentiation also needs to be appropriately generalized.

The results presented in this section have an identical counter-part for multiple access vector fading channels with i.i.d. fading statistics. The above equations remains valid if ω is interpreted as the random variable representing the fading distribution.

As mentioned before, the capacity of the ISI channel is derived using a cyclic prefix, which allows channel independent diagonalization by a discrete Fourier transform, and whose effect becomes negligible as the block size goes to infinity. The use of discrete Fourier transform also reduces computational complexity as the following example illustrates.

Example 1. We consider a two-user multiple access channel with $n_1 = n_2 = m = 2$, and $\nu = 1$.

$$\mathbf{y}_k = \sum_{d=0}^{1} H_d^{(1)} \mathbf{x}_{k-d}^{(1)} + \sum_{i=0}^{1} H_d^{(2)} \mathbf{x}_{k-d}^{(2)} + \mathbf{n}_k. \tag{68}$$

Consider the example with a block size $N = 3$. An extra sample at the output need to be discarded to eliminate inter-block interference. Let $h_{d,rc}^{(i)}$ be the (r, c) element of $H_d^{(i)}$, $x_{k,n}$ be n'th element of $\mathbf{x}_k$, and $y_{k,n}$ be n'th element of $\mathbf{y}_k$. The channel model is as follows:

$$\begin{bmatrix} y_{3,1} \\ y_{3,2} \\ y_{2,1} \\ y_{2,2} \\ y_{1,1} \\ y_{1,2} \end{bmatrix} = \sum_{i=1}^{2} \begin{bmatrix} h_{1,11}^{(i)} & h_{1,12}^{(i)} & h_{2,11}^{(i)} & h_{2,12}^{(i)} & 0 & 0 & 0 & 0 \\ h_{1,21}^{(i)} & h_{1,22}^{(i)} & h_{2,21}^{(i)} & h_{2,22}^{(i)} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{1,11}^{(i)} & h_{1,12}^{(i)} & h_{2,11}^{(i)} & h_{2,12}^{(i)} & 0 & 0 \\ 0 & 0 & h_{1,21}^{(i)} & h_{1,22}^{(i)} & h_{2,21}^{(i)} & h_{2,22}^{(i)} & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{1,11}^{(i)} & h_{1,12}^{(i)} & h_{2,11}^{(i)} & h_{2,12}^{(i)} \\ 0 & 0 & 0 & 0 & h_{1,21}^{(i)} & h_{1,22}^{(i)} & h_{2,21}^{(i)} & h_{2,22}^{(i)} \end{bmatrix} \begin{bmatrix} x_{3,1}^{(i)} \\ x_{3,2}^{(i)} \\ x_{2,1}^{(i)} \\ x_{2,2}^{(i)} \\ x_{1,1}^{(i)} \\ x_{1,2}^{(i)} \\ x_{0,1}^{(i)} \\ x_{0,2}^{(i)} \end{bmatrix} + \begin{bmatrix} n_{3,1} \\ n_{3,2} \\ n_{2,1} \\ n_{2,2} \\ n_{1,1} \\ n_{1,2} \end{bmatrix} \tag{69}$$

It is now possible to apply the iterative water-filling algorithm on this model to obtain the maximum rate-sum and the optimal power allocation. However, each iteration in the algorithm involves a singular-value decomposition (SVD), which is computationally intensive on large matrices. So instead, we insert a cyclic prefix by letting $x_{0,1} = x_{3,1}$, $x_{0,2} = x_{3,2}$. The channel model then becomes block circulant:

$$\begin{bmatrix} y_{3,1} \\ y_{3,2} \\ y_{2,1} \\ y_{2,2} \\ y_{1,1} \\ y_{1,2} \end{bmatrix} = \sum_{i=1}^{2} \begin{bmatrix} h_{1,11}^{(i)} & h_{1,12}^{(i)} & h_{2,11}^{(i)} & h_{2,12}^{(i)} & 0 & 0 \\ h_{1,21}^{(i)} & h_{1,22}^{(i)} & h_{2,21}^{(i)} & h_{2,22}^{(i)} & 0 & 0 \\ 0 & 0 & h_{1,11}^{(i)} & h_{1,12}^{(i)} & h_{2,11}^{(i)} & h_{2,12}^{(i)} \\ 0 & 0 & h_{1,21}^{(i)} & h_{1,22}^{(i)} & h_{2,21}^{(i)} & h_{2,22}^{(i)} \\ h_{2,11}^{(i)} & h_{2,12}^{(i)} & 0 & 0 & h_{1,11}^{(i)} & h_{1,12}^{(i)} \\ h_{2,21}^{(i)} & h_{2,22}^{(i)} & 0 & 0 & h_{1,21}^{(i)} & h_{1,22}^{(i)} \end{bmatrix} \begin{bmatrix} x_{3,1}^{(i)} \\ x_{3,2}^{(i)} \\ x_{2,1}^{(i)} \\ x_{2,2}^{(i)} \\ x_{1,1}^{(i)} \\ x_{1,2}^{(i)} \end{bmatrix} + \begin{bmatrix} n_{3,1} \\ n_{3,2} \\ n_{2,1} \\ n_{2,2} \\ n_{1,1} \\ n_{1,2} \end{bmatrix}. \tag{70}$$

After re-arranging the matrix indices, the above channel model may be re-written as:

$$\begin{bmatrix} y_{3,1} \\ y_{2,1} \\ y_{1,1} \\ y_{3,2} \\ y_{2,2} \\ y_{1,2} \end{bmatrix} = \sum_{i=1}^{2} \begin{bmatrix} h_{1,11}^{(i)} & h_{2,11}^{(i)} & 0 & h_{1,12}^{(i)} & h_{2,12}^{(i)} & 0 \\ 0 & h_{1,11}^{(i)} & h_{2,11}^{(i)} & 0 & h_{1,12}^{(i)} & h_{2,12}^{(i)} \\ h_{2,11}^{(i)} & 0 & h_{1,11}^{(i)} & h_{2,12}^{(i)} & 0 & h_{1,12}^{(i)} \\ h_{1,21}^{(i)} & h_{2,21}^{(i)} & 0 & h_{1,22}^{(i)} & h_{2,22}^{(i)} & 0 \\ 0 & h_{1,21}^{(i)} & h_{2,21}^{(i)} & 0 & h_{1,22}^{(i)} & h_{2,22}^{(i)} \\ h_{2,21}^{(i)} & 0 & h_{1,21}^{(i)} & h_{2,22}^{(i)} & 0 & h_{1,22}^{(i)} \end{bmatrix} \begin{bmatrix} x_{3,1}^{(i)} \\ x_{2,1}^{(i)} \\ x_{1,1}^{(i)} \\ x_{3,2}^{(i)} \\ x_{2,2}^{(i)} \\ x_{1,2}^{(i)} \end{bmatrix} + \begin{bmatrix} n_{3,1} \\ n_{2,1} \\ n_{1,1} \\ n_{3,2} \\ n_{2,2} \\ n_{1,2} \end{bmatrix}, \qquad (71)$$

or more compactly, $\hat{\mathbf{y}} = \sum_{i=1}^{2} \hat{H}^{(i)}\hat{\mathbf{x}}^{(i)} + \hat{\mathbf{n}}$. Note that each of the four 3×3 submatrices now has a circulant structure. So they can be simultaneously diagonalized by a DFT matrix. Denote the (r, c)-submatrix of $\hat{H}^{(i)}$ by $\hat{H}_{rc}^{(i)}$. It can be diagonalized as $\hat{H}_{rc}^{(i)} = F^* \Sigma_{rc}^{(i)} F$ where $\Sigma_{rc}^{(i)}$ is diagonal and F is the 3×3 DFT matrix. Define $Q = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$. Let $\tilde{\mathbf{y}} = Q\hat{\mathbf{y}}$, $\tilde{\mathbf{x}} = Q\hat{\mathbf{x}}$, and $\tilde{\mathbf{n}} = Q\hat{\mathbf{n}}$. Then,

$$\begin{bmatrix} \tilde{y}_{3,1} \\ \tilde{y}_{2,1} \\ \tilde{y}_{1,1} \\ \tilde{y}_{3,2} \\ \tilde{y}_{2,2} \\ \tilde{y}_{1,2} \end{bmatrix} = \sum_{i=1}^{2} \begin{bmatrix} \sigma_{1,11}^{(i)} & 0 & 0 & \sigma_{1,12}^{(i)} & 0 & 0 & 0 \\ 0 & \sigma_{2,11}^{(i)} & 0 & 0 & \sigma_{2,12}^{(i)} & 0 & 0 \\ 0 & 0 & \sigma_{3,11}^{(i)} & 0 & 0 & 0 & \sigma_{3,12}^{(i)} \\ \sigma_{1,21}^{(i)} & 0 & 0 & \sigma_{1,22}^{(i)} & 0 & 0 & 0 \\ 0 & \sigma_{2,21}^{(i)} & 0 & 0 & \sigma_{2,22}^{(i)} & 0 & 0 \\ 0 & 0 & \sigma_{3,21}^{(i)} & 0 & 0 & 0 & \sigma_{3,22}^{(i)} \end{bmatrix} \begin{bmatrix} \tilde{x}_{3,1}^{(i)} \\ \tilde{x}_{2,1}^{(i)} \\ \tilde{x}_{1,1}^{(i)} \\ \tilde{x}_{3,2}^{(i)} \\ \tilde{x}_{2,2}^{(i)} \\ \tilde{x}_{1,2}^{(i)} \end{bmatrix} + \begin{bmatrix} \tilde{n}_{3,1} \\ \tilde{n}_{2,1} \\ \tilde{n}_{1,1} \\ \tilde{n}_{3,2} \\ \tilde{n}_{2,2} \\ \tilde{n}_{1,2} \end{bmatrix}. \qquad (72)$$

Now, re-arrange the matrix index back to the original order:

$$\begin{bmatrix} \tilde{y}_{3,1} \\ \tilde{y}_{3,2} \\ \tilde{y}_{2,1} \\ \tilde{y}_{2,2} \\ \tilde{y}_{1,1} \\ \tilde{y}_{1,2} \end{bmatrix} = \sum_{i=1}^{2} \begin{bmatrix} \sigma_{1,11}^{(i)} & \sigma_{1,12}^{(i)} & 0 & 0 & 0 & 0 \\ \sigma_{1,21}^{(i)} & \sigma_{1,22}^{(i)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{2,11}^{(i)} & \sigma_{2,12}^{(i)} & 0 & 0 \\ 0 & 0 & \sigma_{2,21}^{(i)} & \sigma_{2,22}^{(i)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{3,11}^{(i)} & \sigma_{3,12}^{(i)} \\ 0 & 0 & 0 & 0 & \sigma_{3,21}^{(i)} & \sigma_{3,22}^{(i)} \end{bmatrix} \begin{bmatrix} \tilde{x}_{3,1}^{(i)} \\ \tilde{x}_{3,2}^{(i)} \\ \tilde{x}_{2,1}^{(i)} \\ \tilde{x}_{2,2}^{(i)} \\ \tilde{x}_{1,1}^{(i)} \\ \tilde{x}_{1,2}^{(i)} \end{bmatrix} + \begin{bmatrix} \tilde{n}_{3,1} \\ \tilde{n}_{3,2} \\ \tilde{n}_{2,1} \\ \tilde{n}_{2,2} \\ \tilde{n}_{1,1} \\ \tilde{n}_{1,2} \end{bmatrix}. \qquad (73)$$

At a cost of four DFT's, each of which can be implemented in $N \log N$ times for an $N \times N$ matrix, the iterative water-filling algorithm on the above matrix representation of the channel can now be run much more efficiently. Singular-value decomposition is now performed on four submatrices of smaller size, instead of on one large matrix. Since the computational complexity of singular-value decomposition is in the order of N^3 for an $N \times N$ matrix, this represents significant computational saving. Although there is a capacity loss due to the use of cyclic prefix, the effect becomes negligible when the block size is large.

5 Conclusions

This paper addresses the problem of finding the optimal transmitter covariance for Gaussian multiple access channels with vector inputs and outputs. The capacity region under an input power constraint is explicitly characterized. The computation of the capacity region is formulated in the convex optimization framework, and a simultaneous water-filling condition is found for achieving the sum capacity in multiple access channels. We then proposed an iterative water-filling algorithm to numerically compute the optimal covariance matrix for maximizing the sum rate. Such an iterative algorithm finds the correct compromise among the users by finding the best signaling direction and the optimal power allocation for each user. The iterative water-filling algorithm is shown to converge to the sum capacity from any starting point, and a general error bound after one iteration is found. Finally, we turned to multiple access channels with ISI, where analogous results are derived. These results are also applicable to fading channels with i.i.d. fading statistics.

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