

# On the Achievable Sum-rate of the RIS-aided MIMO Broadcast Channel

*Invited Paper*

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**Abstract**—Reconfigurable intelligent surfaces (RISs) represent a new technology that can shape the radio wave propagation and thus offers a great variety of possible performance and implementation gains. Motivated by this, we investigate the achievable sum-rate optimization in a broadcast channel (BC) that is equipped with an RIS. We exploit the well-known duality between the Gaussian multiple-input multiple-output (MIMO) BC and multiple-access channel (MAC) to derive an alternating optimization (AO) algorithm which jointly optimizes the users' covariance matrices and the RIS phase shifts in the dual MAC. The optimal users' covariance matrices are obtained by a dual decomposition method in which each iteration is solved in closed-form. The optimal RIS phase shifts are also computed using a derived closed-form expression. Furthermore, we present a computational complexity analysis for the proposed AO algorithm. Simulation results show that the proposed AO algorithm can provide significant achievable sum-rate gains in a BC.

**Index Terms**—Achievable sum-rate, alternating optimization (AO), broadcast channel (BC), multiple-access channel (MAC), reconfigurable intelligent surface (RIS).

## I. INTRODUCTION

The need to satisfy constantly increasing data rate demands in wireless communication networks motivates the development of new technology solutions such as reconfigurable intelligent surfaces (RISs). An RIS is a thin metasurface that consists of a large number of small, low-cost, and passive elements. Since each of these elements can reflect the incident signal with an adjustable phase shift, an RIS can effectively shape the propagation of the impinging wave [1], [2]. Therefore, the introduction of RISs offers a wide variety of possible implementation gains and potentially presents a new milestone in wireless communications.

In order to fully exploit the gains that arise from the use of RISs, we need to obtain a deep understanding of different aspects of RIS-assisted wireless communication systems. Probably the most important aspect concerns the optimal design of the RIS phase shifts, so that the incoming radio wave is altered in a way that maximizes the aforementioned gains. In this regard, the development of algorithms for the achievable rate optimization is of particular interest for RIS-aided communications. A significant body of research work in this

area concentrates on the achievable rate optimization for point-to-point communications with continuous [3]–[5] and discrete [6] signaling. Another equally important part of this research work has been dedicated to the achievable sum-rate optimization for multi-user systems. In [7], the authors studied the capacity/achievable rate region for the multiple-access channel (MAC) and for the broadcast channel (BC) by using the well-known BC-MAC duality; however, the analysis was limited to single-antenna user terminals and a single-antenna base station (BS). Methods for optimization of the achievable sum-rate for multi-user multiple-input single-output (MISO) communication systems equipped with RISs were introduced in [8], [9]. The use of an RIS in multi-cell multiple-input multiple-output (MIMO) systems was investigated in [10], where the aim was to improve downlink transmission to cell-edge users by employing an RIS which increases the weighted sum-rate in the considered communication system. An extension of this work to the case of an RIS-aided MIMO system performing simultaneous wireless information and power transfer was presented in [11].

Against this background, the contributions of this paper are listed as follows:

- We exploit the Gaussian MIMO BC-MAC duality to maximize the achievable sum-rate of a MIMO system equipped with an RIS communicating over a BC, and formulate a joint optimization problem of the users' covariance matrices and the RIS elements' phase shifts. To solve this problem, we propose an iterative algorithm which operates in the dual MAC, and optimizes the users' covariance matrices and the RIS elements' phase shifts in an alternating manner. The optimal users' covariance matrices are obtained by a dual decomposition method, while the optimal RIS phase shifts are computed by a derived closed-form expression.
- For the proposed alternating optimization (AO) algorithm, we derive an expression for the computational complexity in terms of the number of complex multiplications.
- Simulation results show that the AO can provide significant achievable sum-rate gains. These gains increase with

the number of users and the number of transmit antennas, especially when the direct links are present in the BC.

*Notation:* Bold lower and upper case letters represent vectors and matrices, respectively.  $\mathbb{C}^{m \times n}$  denotes the space of  $m \times n$  complex matrices.  $\mathbf{H}^T$  and  $\mathbf{H}^\dagger$  denote the transpose and Hermitian transpose of  $\mathbf{H}$ , respectively;  $|\mathbf{H}|$  is the determinant of  $\mathbf{H}$ .  $\text{Tr}(\mathbf{H})$  stands for the trace of  $\mathbf{H}$  and  $\text{rank}(\mathbf{H})$  denotes the rank of  $\mathbf{H}$ .  $\log_2(\cdot)$  is the binary logarithm,  $\ln(\cdot)$  is the natural logarithm and  $(x)_+$  denotes  $\max(0, x)$ .  $\mathbb{E}\{\cdot\}$  stands for the expectation operator and  $(\cdot)^*$  denotes the complex conjugate. The notation  $\mathbf{A} \succeq (\succ) \mathbf{B}$  means that  $\mathbf{A} - \mathbf{B}$  is positive semidefinite (definite).  $\mathbf{I}$  represents an identity matrix whose size should be clear from the context. For a vector  $\mathbf{x}$ ,  $\text{diag}(\mathbf{x})$  denotes a diagonal matrix with the elements of  $\mathbf{x}$  on the diagonal.  $\mathcal{CN}(\mu, \sigma^2)$  denotes a circularly symmetric complex Gaussian random variable of mean  $\mu$  and variance  $\sigma^2$ .

## II. SYSTEM MODEL

We consider a BC in which one BS simultaneously serves  $K$  users. Both the BS and the users are equipped with multiple antennas, such that the BS and the  $k$ -th user have  $N_t$  and  $n_k$  antennas, respectively. The BS antennas are placed in a uniform linear array (ULA) with inter-antenna separation  $s_t$ . In a similar manner, all the antennas of a single user are placed in a ULA with inter-antenna separation  $s_r$ . In order to improve the system performance, an RIS is also present in the considered communication environment. It consists of  $N_{\text{ris}}$  reflecting elements which are placed in a uniform rectangular array (URA), so that the separation between the centers of adjacent RIS elements in both dimensions is  $s_{\text{ris}}$ .

The received signal at the  $k$ -th user is given by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \sum_{j=1, j \neq k}^K \mathbf{H}_k \mathbf{x}_j + \mathbf{n}_k \quad (1)$$

where  $\mathbf{H}_k \in \mathbb{C}^{n_k \times N_t}$  is the channel matrix for the  $k$ -th user,  $\mathbf{x}_k \in \mathbb{C}^{N_t \times 1}$  is the transmitted signal intended for the  $k$ -th user, and  $\mathbf{x}_j \in \mathbb{C}^{N_t \times 1}$  for  $j \neq k$  are the transmitted signals intended for other users, which act as interference for the detection of  $\mathbf{x}_k$ . The noise vector  $\mathbf{n}_k \in \mathbb{C}^{n_k \times 1}$  consists of independent and identically distributed (i.i.d.) elements that are distributed according to  $\mathcal{CN}(0, N_0)$ , where  $N_0$  is the noise variance.

Due to the presence of the RIS, the channel matrix  $\mathbf{H}_k$  can be expressed as

$$\mathbf{H}_k = \mathbf{D}_k + \mathbf{G}_k \mathbf{F}(\boldsymbol{\theta}) \mathbf{U} \quad (2)$$

where  $\mathbf{D}_k \in \mathbb{C}^{n_k \times N_t}$  is the direct link channel matrix between the BS and the  $k$ -th user,  $\mathbf{U} \in \mathbb{C}^{N_{\text{ris}} \times N_t}$  is the channel matrix between the BS and the RIS, and  $\mathbf{G}_k \in \mathbb{C}^{n_k \times N_{\text{ris}}}$  is the channel matrix between the RIS and the  $k$ -th user. Signal reflection from the RIS is modeled by  $\mathbf{F}(\boldsymbol{\theta}) = \text{diag}(\boldsymbol{\theta}) \in \mathbb{C}^{N_{\text{ris}} \times N_{\text{ris}}}$ , where  $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_{N_{\text{ris}}}]^T \in \mathbb{C}^{N_{\text{ris}} \times 1}$ . We assume that the signal reflection from any RIS element is ideal (i.e., without any power loss) and therefore we may write  $\theta_l = e^{j\phi_l}$  for  $l = 1, 2, \dots, N_{\text{ris}}$ , where  $\phi_l$  is the phase shift induced by the  $l$ -th RIS element. Equivalently, this can be written as

$$|\theta_l| = 1, \quad l = 1, 2, \dots, N_{\text{ris}}. \quad (3)$$

## III. PROBLEM FORMULATION

In this paper, we are interested in maximizing the achievable sum-rate of the considered RIS-assisted wireless communication system. To accomplish this, we exploit the fact that the achievable rate region of a Gaussian MIMO BC can be achieved by dirty paper coding (DPC) [12]. DPC enables us to perfectly eliminate the interference term  $\sum_{j < k} \mathbf{H}_k \mathbf{x}_j$  for the  $k$ -th user, assuming that the BS has full (non-causal) knowledge of this interference term. In this regard, the ordering of the users clearly matters. Let  $\pi$  be an ordering of users, i.e., a permutation of the set  $\{1, 2, \dots, K\}$ . Then for this ordering, the achievable rate for the  $k$ -th user can be computed as [13, Eq. (3)]

$$R_{\pi(k)} = \log_2 \left| \frac{\mathbf{I} + \mathbf{H}_{\pi(k)} (\sum_{j \geq k} \mathbf{S}_{\pi(j)}) \mathbf{H}_{\pi(k)}^\dagger}{\mathbf{I} + \mathbf{H}_{\pi(k)} (\sum_{j > k} \mathbf{S}_{\pi(j)}) \mathbf{H}_{\pi(k)}^\dagger} \right|, \quad k = 1, 2, \dots, K \quad (4)$$

where  $\mathbf{S}_k = \mathbb{E}\{\mathbf{x}_k \mathbf{x}_k^\dagger\} \succeq \mathbf{0}$  is the input covariance matrix of user  $k$ . In this paper, we consider a sum power constraint at the BS, i.e.,

$$\sum_{k=1}^K \text{Tr}(\mathbf{S}_k) \leq P \quad (5)$$

where  $P$  is the maximum total power at the BS. Therefore, the achievable rate optimization problem for the RIS-assisted MIMO BC can be expressed as

$$\underset{\{\mathbf{S}_k \succeq \mathbf{0}\}, \boldsymbol{\theta}}{\text{maximize}} \quad \sum_{k=1}^K \log_2 \left| \frac{\mathbf{I} + \mathbf{H}_{\pi(k)} (\sum_{j \geq k} \mathbf{S}_{\pi(j)}) \mathbf{H}_{\pi(k)}^\dagger}{\mathbf{I} + \mathbf{H}_{\pi(k)} (\sum_{j > k} \mathbf{S}_{\pi(j)}) \mathbf{H}_{\pi(k)}^\dagger} \right| \quad (6a)$$

$$\text{subject to} \quad \sum_{k=1}^K \text{Tr}(\mathbf{S}_k) \leq P \quad (6b)$$

$$|\theta_l| = 1, \quad l = 1, 2, \dots, N_{\text{ris}}. \quad (6c)$$

It is worth mentioning that the achievable sum-rate in (6) is independent of the ordering of users  $\pi$  [13]. We remark that the objective function of the above problem is neither convex nor concave with the input covariance matrices and the phase shifts, and thus directly solving (6) is difficult. In [13], Vishwanath *et al.* established what is now well-known as the *BC-MAC duality*, and showed that the achievable sum-rate of the MIMO BC equals the achievable rate of the dual Gaussian MIMO MAC. As a result, (6) is equivalent to

$$\underset{\{\bar{\mathbf{S}}_k \succeq \mathbf{0}\}, \boldsymbol{\theta}}{\text{maximize}} \quad \log_2 \left| \mathbf{I} + \sum_{k=1}^K \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k \right| \quad (7a)$$

$$\text{subject to} \quad \sum_{k=1}^K \text{Tr}(\bar{\mathbf{S}}_k) \leq P \quad (7b)$$

$$|\theta_l| = 1, \quad l = 1, 2, \dots, N_{\text{ris}} \quad (7c)$$

where  $\mathbf{H}_k^\dagger$  is referred to as the dual MAC corresponding to  $\mathbf{H}_k$  and  $\bar{\mathbf{S}}_k \in \mathbb{C}^{n_k \times n_k}$  is the input covariance matrix of user  $k$  in the dual MAC.

## IV. ALTERNATING OPTIMIZATION (AO)

To solve (7), we propose an efficient AO method, which adjusts the covariance matrices and the RIS element phase shifts in an alternating fashion. First, we propose an iterative approach which optimizes all of the covariance matrices in the dual MAC in a successive manner. Next, the optimal phase shift value for each RIS element is obtained using a derived closed-form expression, similar to [4]. We will analyze the computational complexity of the proposed AO method in Subsection IV-D.

### A. Covariance Matrix Optimization

For a given  $\theta$ , the achievable rate optimization problem in (7) can be simplified as

$$\underset{\{\bar{\mathbf{S}}_k \succeq \mathbf{0}\}}{\text{maximize}} \quad \log_2 \left| \mathbf{I} + \sum_{k=1}^K \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k \right| \quad (8a)$$

$$\text{subject to} \quad \sum_{k=1}^K \text{Tr}(\bar{\mathbf{S}}_k) \leq P. \quad (8b)$$

This redefined optimization problem is convex and thus it can be solved by off-the-shelf convex solvers. However, we apply instead the dual decomposition method [14] to solve (8), which is more efficient and is described in Algorithm 1.

The partial Lagrangian function of (8) is

$$L(\mu, \{\bar{\mathbf{S}}_k\}) = \ln \left| \mathbf{I} + \sum_{k=1}^K \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k \right| - \mu \left[ \sum_{k=1}^K \text{Tr}(\bar{\mathbf{S}}_k) - P \right] \quad (9)$$

where  $\mu$  is the Lagrangian multiplier for the constraint (8b). For mathematical convenience, we use the natural logarithm in the previous expression without affecting the optimality of (8). For a given  $\mu$ , the dual objective is given as

$$g(\mu) = \max \{ L(\mu, \{\bar{\mathbf{S}}_k\}) \mid \{\bar{\mathbf{S}}_k\} \succeq \mathbf{0} \} \quad (10)$$

and its optimization can be performed by cyclically optimizing each  $\bar{\mathbf{S}}_k$  in turn while keeping the other  $\bar{\mathbf{S}}_j$  ( $j \neq k$ ) fixed. To this end, let us consider the optimization of (10) over  $\bar{\mathbf{S}}_k$ , which is expressed as

$$\underset{\bar{\mathbf{S}}_k \succeq \mathbf{0}}{\text{maximize}} \quad \ln \left| \mathbf{I} + \bar{\mathbf{H}}_k^{-1/2} \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k \bar{\mathbf{H}}_k^{-1/2} \right| - \mu \text{Tr}(\bar{\mathbf{S}}_k) \quad (11)$$

where

$$\bar{\mathbf{H}}_k = \mathbf{I} + \sum_{j=1, j \neq k}^K \mathbf{H}_j^\dagger \bar{\mathbf{S}}_j \mathbf{H}_j. \quad (12)$$

It is easy to see that the optimal solution to (11) is given by

$$\bar{\mathbf{S}}_k^* = \mathbf{V}_k \text{diag} \left( \left[ \left( \frac{1}{\mu} - \frac{1}{\sigma_1} \right)_+, \left( \frac{1}{\mu} - \frac{1}{\sigma_2} \right)_+, \dots, \left( \frac{1}{\mu} - \frac{1}{\sigma_r} \right)_+ \right]^T \right) \mathbf{V}_k^\dagger \quad (13)$$

where  $\mathbf{H}_k \bar{\mathbf{H}}_k^{-1} \mathbf{H}_k^\dagger = \mathbf{V}_k \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r) \mathbf{V}_k^\dagger$  is the eigenvalue decomposition (EVD) of  $\mathbf{H}_k \bar{\mathbf{H}}_k^{-1} \mathbf{H}_k^\dagger$  and  $r = \text{rank}(\mathbf{H}_k) \leq \min(N_t, n_k)$ . Let  $\{\bar{\mathbf{S}}_k^*\}_{k=1}^K$  be the optimal solution of (10). Next, the dual problem is

$$\text{minimize} \quad \{g(\mu) \mid \mu \geq 0\}. \quad (14)$$

Since  $P - \sum_{k=1}^K \text{Tr}(\bar{\mathbf{S}}_k^*)$  is a subgradient of  $g(\mu)$ , the dual problem (14) can be efficiently solved by a bisection search as outlined in Algorithm 1. In particular, increase  $\mu_{\min}$  if  $P - \sum_{k=1}^K \text{Tr}(\bar{\mathbf{S}}_k^*) < 0$  and decrease  $\mu_{\max}$  otherwise.

A possible upper limit of the bisection search for Algorithm 1 can be found as follows. From the Karush-Kuhn-Tucker (KKT) condition of (11) we have

$\mathbf{H}_k (\bar{\mathbf{H}}_k^{-1/2})^\dagger (\mathbf{I} + \bar{\mathbf{H}}_k^{-1/2} \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k \bar{\mathbf{H}}_k^{-1/2})^{-1} \bar{\mathbf{H}}_k^{-1/2} \mathbf{H}_k^\dagger + \mathbf{M}_k = \mu \mathbf{I}$  where  $\mathbf{M}_k \succeq \mathbf{0}$  is the Lagrangian multiplier of the constraints  $\bar{\mathbf{S}}_k \succeq \mathbf{0}$ . Further, this yields

$$\mathbf{H}_k (\bar{\mathbf{H}}_k^{-1/2})^\dagger (\mathbf{I} + \bar{\mathbf{H}}_k^{-1/2} \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k \bar{\mathbf{H}}_k^{-1/2})^{-1} \bar{\mathbf{H}}_k^{-1/2} \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k = \mu \bar{\mathbf{S}}_k$$

and thus

$$\text{Tr}((\mathbf{I} + \bar{\mathbf{H}}_k^{-1/2} \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k \bar{\mathbf{H}}_k^{-1/2})^{-1} \bar{\mathbf{H}}_k^{-1/2} \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k (\bar{\mathbf{H}}_k^{-1/2})^\dagger) = \mu \text{Tr}(\bar{\mathbf{S}}_k). \quad (15)$$

Note that  $\text{Tr}((\mathbf{I} + \mathbf{A})^{-1} \mathbf{A}) = \text{Tr}((\mathbf{I} + \mathbf{A}^{-1})^{-1}) \leq N_t$  and thus the above equality implies  $\mu \text{Tr}(\bar{\mathbf{S}}_k) \leq N_t$ . Combining this inequality for all users, we have  $\mu \leq KN_t/P$ . Hence, setting  $\mu_{\max} = KN_t/P$  in Algorithm 1 guarantees finding the optimal solution to (8).

### Algorithm 1: Dual decomposition for solving (8).

**Input:**  $\mu_{\min} = 0, \mu_{\max} > 0, \epsilon > 0$ : desired accuracy.  
**1 repeat**  
 2   Set  $\mu = \frac{\mu_{\max} + \mu_{\min}}{2}$  and  $k = 0$   
 3   **repeat**  
 4     Set  $k \leftarrow (k \bmod K) + 1$   
 5     Compute  $\bar{\mathbf{S}}_k^*$  according to (13)  
 6     **until convergence of** (9)  
 7     **if**  $P < \sum_{k=1}^K \text{Tr}(\bar{\mathbf{S}}_k^*)$  **then** Set  $\mu_{\min} = \mu$   
 8     **else** Set  $\mu_{\max} = \mu$   
 9   **until**  $\mu_{\max} - \mu_{\min} < \epsilon$

### B. RIS Optimization

For fixed  $\{\bar{\mathbf{S}}_k\}_{k=1}^K$  and  $\{\theta_m, m \neq l\}_{m=1}^{N_{\text{ris}}}$ , the optimization problem in (7) with respect to  $\theta_l$  can be explicitly written as

$$\underset{\theta_l}{\text{maximize}} \quad \log_2 \left| \mathbf{I} + \sum_{k=1}^K \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k \right| \quad (16a)$$

$$\text{subject to} \quad |\theta_l| = 1. \quad (16b)$$

To proceed further, we present the objective of (16) as  $\log_2 |\mathbf{A}_l + \theta_l \mathbf{B}_l + \theta_l^* \mathbf{B}_l^\dagger|$ , where

$$\begin{aligned} \mathbf{A}_l &= \mathbf{I} + \sum_{k=1}^K (\mathbf{D}_k^\dagger + \sum_{m=1, m \neq l}^{N_{\text{ris}}} \theta_m^* \mathbf{u}_m^\dagger \mathbf{g}_{k,m}^\dagger) \bar{\mathbf{S}}_k \\ &\times (\mathbf{D}_k + \sum_{n=1, n \neq l}^{N_{\text{ris}}} \theta_n \mathbf{g}_{k,n} \mathbf{u}_n) + \sum_{k=1}^K \mathbf{u}_l^\dagger \mathbf{g}_{k,l}^\dagger \bar{\mathbf{S}}_k \mathbf{g}_{k,l} \mathbf{u}_l, \end{aligned} \quad (17)$$

$$\mathbf{B}_l = \left[ \sum_{k=1}^K (\mathbf{D}_k^\dagger + \sum_{m=1, m \neq l}^{N_{\text{ris}}} \theta_m^* \mathbf{u}_m^\dagger \mathbf{g}_{k,m}^\dagger) \bar{\mathbf{S}}_k \mathbf{g}_{k,l} \right] \mathbf{u}_l, \quad (18)$$

$\mathbf{U} = [\mathbf{u}_1^T \mathbf{u}_2^T \dots \mathbf{u}_{N_{\text{ris}}}^T]^T$  and  $\mathbf{G}_k = [\mathbf{g}_{k,1} \mathbf{g}_{k,2} \dots \mathbf{g}_{k,N_{\text{ris}}}]$ .

The optimal solution to (16) is then given by [4]

$$\theta_l^* = \exp(-j \arg(\sigma_l)), \quad (19)$$

where  $\sigma_l$  is the only non-zero eigenvalue of  $\mathbf{A}_l^{-1} \mathbf{B}_l$  (it can be observed from (18) that the rank of  $\mathbf{B}_l$  is equal to 1).

### C. Overall AO Method

The overall AO algorithm description is given in Algorithm 2. At first, we compute the optimal covariance matrices for all users,  $\{\bar{\mathbf{S}}_k\}_{k=1}^K$ . Next, we sequentially obtain the optimal phase shift value for each RIS element. These two optimization steps constitute one *outer iteration* of Algorithm 2.

It is obvious that each iteration of the AO algorithm increases the achievable sum-rate. Also, the solution in each iteration of the AO method is unique and the feasible set is compact. Thus, the convergence of the AO method to a stationary solution is guaranteed. However, since the problem (6) is non-convex, we cannot claim that the obtained solution is globally optimal.

### D. Computational Complexity

In this subsection, the computational complexity is obtained by counting the required number of complex multiplications. The complexity of the AO is determined by the computation of the covariance matrices  $\{\bar{\mathbf{S}}_k\}_{k=1}^K$  and the RIS phase shifts  $\{\theta_m\}_{m=1}^{N_{\text{ris}}}$  in Algorithm 2. In the following complexity derivation, for ease of exposition we assume that all the users have the same number of antennas, i.e.,  $n_k = N_r$  for all  $k = 1, 2, \dots, K$ . At first, we need to compute all the users' channel matrices. To compute  $\mathbf{F}(\theta) \mathbf{U}$  requires  $N_{\text{ris}} N_t$  multiplications and it is common for all users. To form  $\mathbf{G}_k \mathbf{F}(\theta) \mathbf{U}$ ,

**Algorithm 2:** AO algorithm for solving (7).

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**Input:**  $|\theta_l| = 1, l = 1, 2, \dots, N_{\text{ris}}$

1 **repeat**

2   Solve for  $\{\bar{\mathbf{S}}_k^*\}_{k=1}^K$  using **Algorithm 1**

3   **for**  $l = 1, 2, \dots, N_{\text{ris}}$  **do**

4     Solve for  $\theta_l^*$  using (19)

5   **end**

6 **until** convergence

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we require  $N_{\text{ris}}N_tN_r$  further multiplications per user, so the complexity of calculating all of the user's channel matrices is  $\mathcal{O}(KN_{\text{ris}}N_tN_r)$ . To reduce the complexity of computing  $\bar{\mathbf{H}}_k$ , instead of following (12) we compute and store  $\mathbf{H}_{\text{sum}} = \mathbf{I} + \sum_{j=1}^K \mathbf{H}_j^\dagger \bar{\mathbf{S}}_j \mathbf{H}_j$ , which requires  $\mathcal{O}(KN_tN_r^2 + KN_t^2N_r)$  multiplications. Then  $\bar{\mathbf{H}}_k^{-1} = (\mathbf{H}_{\text{sum}} - \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k)^{-1}$  has a complexity of  $\mathcal{O}(N_t^3)$  and  $\mathbf{H}_k \bar{\mathbf{H}}_k^{-1} \mathbf{H}_k^\dagger$  has a complexity of  $\mathcal{O}(N_tN_r^2 + N_t^2N_r)$ . The EVD of  $\mathbf{H}_k \bar{\mathbf{H}}_k^{-1} \mathbf{H}_k^\dagger$  requires  $\mathcal{O}(N_r^3)$  multiplications, while the complexity of computing  $\bar{\mathbf{S}}_k^*$  is  $\mathcal{O}(N_r^3)$ . To recalculate  $\mathbf{H}_{\text{sum}} = \bar{\mathbf{H}}_k + \mathbf{H}_k^\dagger \bar{\mathbf{S}}_k \mathbf{H}_k$ , we need  $\mathcal{O}(N_tN_r^2 + N_t^2N_r)$  multiplications. Finally, the complexity of calculating  $\{\bar{\mathbf{S}}_k^*\}_{k=1}^K$  may be expressed as  $\mathcal{O}(KN_{\text{ris}}N_tN_r + KN_tN_r^2 + KN_t^2N_r + LI(N_t^3 + 2N_tN_r^2 + 2N_t^2N_r + 2N_r^3)) = \mathcal{O}(KN_{\text{ris}}N_tN_r + LI(N_t^3 + N_tN_r^2 + N_t^2N_r + N_r^3))$ , where  $L$  is the required number of outer iterations (i.e., lines 1-9) in Algorithm 1, and  $I$  is the average number of iterations required for the optimization of the covariance matrices (i.e., rows 3 to 6) in Algorithm 1. In our case,  $L$  is the smallest integer that satisfies  $\mu_{\text{max}}/2^L < \epsilon$ . From numerical experiments, we have observed that  $I < 2K$  is usually sufficient to attain a difference between two consecutive values of (9) that is lower than  $10^{-6}$ .

The complexity of computing the optimal RIS phase shifts is primarily dependent on (17) and (18). Let us define  $\mathbf{C}_k = \mathbf{H}_k - \theta_l \mathbf{g}_{k,l} \mathbf{u}_l$  to simplify the complexity derivation. It is easy to see that we need  $\mathcal{O}(N_tN_r)$  multiplications to obtain  $\mathbf{C}_k$  from  $\mathbf{H}_k$ . The complexity of computing the matrix product  $\mathbf{C}_k \bar{\mathbf{S}}_k \mathbf{C}_k$  is  $\mathcal{O}(N_tN_r^2 + N_t^2N_r)$ . In a similar manner, the complexity of  $\mathbf{u}_l^\dagger \mathbf{g}_{k,l}^\dagger \bar{\mathbf{S}}_k \mathbf{g}_{k,l} \mathbf{u}_l$  is equal to  $\mathcal{O}(N_tN_r^2 + N_t^2N_r)$ . Hence, the complexity of computing  $\mathbf{A}_l$  in (17) is  $\mathcal{O}(KN_tN_r^2 + KN_t^2N_r)$ . Also, we need  $\mathcal{O}(KN_t^2N_r)$  more multiplications to obtain  $\mathbf{B}_l$  in (18). Inverting  $\mathbf{A}_l$  requires  $\mathcal{O}(N_t^3)$  multiplications. The same complexity is required for computing  $\mathbf{A}_l^{-1} \mathbf{B}_l$  and for obtaining the EVD of that product. The complexity of computing a single RIS phase shift is  $\mathcal{O}(KN_tN_r^2 + KN_t^2N_r + N_t^3)$ , which gives a total of  $\mathcal{O}(KN_{\text{ris}}N_tN_r^2 + KN_{\text{ris}}N_t^2N_r + N_{\text{ris}}N_t^3)$  for the whole RIS.

In summary, the computational complexity of one outer iteration (i.e., lines 1 to 6 in Algorithm 2) of the AO algorithm is given by

$$C_{\text{AO}} = \mathcal{O}(KN_{\text{ris}}N_tN_r^2 + KN_{\text{ris}}N_t^2N_r + N_{\text{ris}}N_t^3 + LI(N_t^3 + N_tN_r^2 + N_t^2N_r + N_r^3)). \quad (20)$$

## V. SIMULATION RESULTS

In this section, we evaluate the achievable rate of the proposed AO algorithm with the aid of Monte Carlo simulations. The study is conducted for a typical multi-user propagation environment in three different scenarios: (i) where only the

direct link (i.e., the first term in (2)) is present; (ii) where only the link via the RIS (i.e., the second term in (2)) is present; and (iii) where both of these links are present. In order to better quantify the gains of the proposed AO method, we present the achievable sum-rate results for different numbers of users and for different numbers of transmit antennas.

The positions of the BS, the RIS and the users are specified by a three-dimensional (3D) Cartesian coordinate system. The BS ULA is placed parallel to the  $y$ -axis and the position of its midpoint is set as  $(0, l_t, h_t)$ . The RIS is located in the  $xz$ -plane and the position of its midpoint is  $(d_{\text{ris}}, 0, h_{\text{ris}})$ . For simplicity, we assume that all of the users' ULAs are parallel to the  $y$ -axis and the midpoint of the  $k$ -th user's ULA is located at  $(d_k, l_k, h_k)$ . For the considered system geometry, the distance between the midpoint of the BS ULA and the midpoint of the RIS is  $d_{t,\text{ris}} = \sqrt{d_{\text{ris}}^2 + l_t^2 + (h_t - h_{\text{ris}})^2}$ , the distance between the midpoint of the RIS and the midpoint of the  $k$ -th user's ULA is  $d_{\text{ris},k} = \sqrt{(d_{\text{ris}} - d_k)^2 + l_k^2 + (h_{\text{ris}} - h_k)^2}$ , and the distance between the midpoint of the BS ULA and the midpoint of the  $k$ -th user's ULA is  $d_{t,k} = \sqrt{d_k^2 + (l_t - l_k)^2 + (h_t - h_k)^2}$ .

In the following simulations, all of the channel matrices are modeled according to the Rician fading channel model with Rician factor equal to 1, as specified in [3]. Also, we neglect spatial correlation among the elements of matrices  $\mathbf{U}$  and  $\mathbf{G}_k$ . The distance-dependent path loss for the direct link of the  $k$ -th user is  $\beta_{\text{DIR},k} = (4\pi/\lambda)^2 d_{t,k}^{\alpha_{\text{DIR}}}$ , where  $\alpha_{\text{DIR}}$  denotes the path loss exponent of the direct link. The far-field free space path loss (FSPL) for the RIS link of the  $k$ -th user  $\beta_{\text{RIS},k}$  is equal to  $\beta_{\text{RIS},k}^{-1} = G_t G_r \lambda^4 \cos \gamma_t \cos \gamma_r / (256\pi^2 d_{t,\text{ris}}^2 d_{\text{ris},k}^2)$ , where  $\gamma_t$  is the angle between the incident wave propagation direction and the normal to the RIS, and  $\gamma_r$  is the angle between the normal to the RIS and the reflected wave propagation direction [15, Eq. (7), (9)]. Hence, we have  $\cos \gamma_t = l_t/d_{t,\text{ris}}$  and  $\cos \gamma_r = l_k/d_{\text{ris},k}$ . Here  $G_t$  and  $G_r$  represent the transmit and receive antenna gains respectively; these values are both set to 2, since we assume that these antennas radiate/sense signals to/from the relevant half space [15]. In this paper,  $\sqrt{\beta_{\text{DIR},k}^{-1}/N_0}$  and  $\sqrt{\beta_{\text{RIS},k}^{-1}/N_0}$  are embedded as scaling factors in  $\mathbf{D}_k$  and  $\mathbf{G}_k$ , respectively.

In the following simulation setup, the parameters are  $f = 2$  GHz (i.e.,  $\lambda = 15$  cm),  $s_t = s_r = s_{\text{ris}} = \lambda/2 = 7.5$  cm,  $l_t = 20$  m,  $h_t = 10$  m,  $d_{\text{ris}} = 30$  m,  $h_{\text{ris}} = 5$  m,  $N_t = 8$ ,  $\alpha_{\text{DIR}} = 3$ ,  $P = 1$  W, and  $N_0 = -110$  dB. The RIS consists of  $N_{\text{ris}} = 225$  elements placed in a  $15 \times 15$  square formation. As in the previous section, we assume that all users are equipped with  $N_r = 2$  antennas. The users' coordinates are randomly selected such that  $d_k$  is chosen from a uniform distribution between 200 m to 500 m with a resolution of 2 m,  $l_k$  is chosen from a uniform distribution between 0 to 70 m with a resolution of 1 m, and  $h_k$  is chosen from a uniform distribution between 1.5 m to 2 m with a resolution of 1 cm. All results are averaged over 1000 independent channel realizations.

The achievable sum-rate for the proposed AO method for the three cases where the channel consists of the direct link

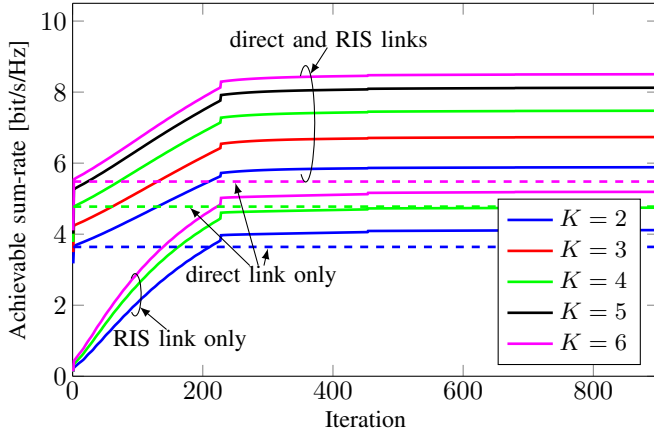


Fig. 1. Achievable sum-rate for the proposed AO method with direct and RIS links, direct link only, and RIS link only.

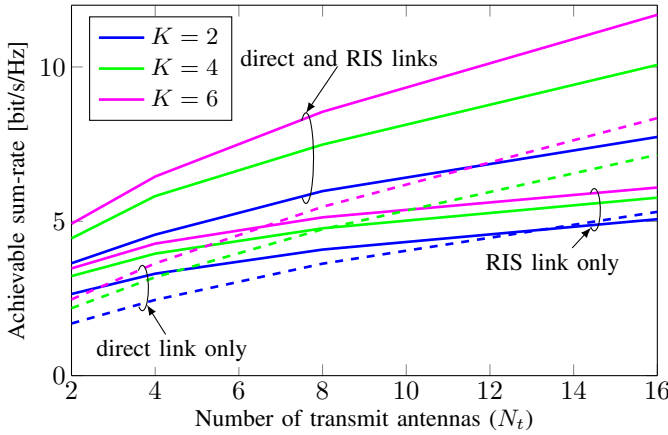


Fig. 2. Achievable sum-rate versus the number of transmit antennas ( $N_t$ ). only, the RIS link only, and both the direct and RIS links, are shown in Fig. 1. As expected, the achievable sum-rate in multi-user communications is higher when the direct link is present, similar to observations regarding the achievable rate in point-to-point communications which were reported in [3]. Moreover, the achievable sum-rate increases with the number of users  $K$ , and this increase is more substantial when the direct link is present. It seems that the lack of amplitude adjustment capability prevents the RIS from achieving a significant suppression of the multi-user interference. Therefore, higher achievable sum-rate gains can be expected when a part of a signal is transmitted via the direct link, since the BS with its amplitude adjustment capabilities is better equipped to suppress the aforementioned interference. In addition, the achievable sum-rate increase starts to noticeably decline when the number of users is greater than four. The reason for this is that the channel matrix of the aggregate BC between the BS and all users has the maximum rank when the number of users  $K$  is greater than or equal to four. In other words, the rank of the channel matrix (i.e., number of degrees of freedom) does not change if  $K \geq 4$  and this leads to a reduced gain in the achievable sum-rate when increasing  $K$  beyond 4.

In Fig. 2, we show the achievable sum-rate for the proposed

AO method versus the number of transmit antennas  $N_t$ . The achievable sum-rate curves have an approximately logarithmic shape. Also, it can be observed that the achievable sum-rate increases with the number of users. However, it seems that this increase gradually declines with the increase of the number of users. At the same time, the achievable sum-rate increases with the number of transmit antennas. For example, for 6 users and 2 transmit antennas, a 99 % increase in the achievable sum-rate is obtained by adding the RIS to the multi-user system.

## VI. CONCLUSION

In this paper, we proposed an AO algorithm for the achievable sum-rate optimization in a multi-user BC that is equipped with an RIS. The algorithm is based on the well-known BC-MAC duality for multi-user systems. The users' covariance matrices were optimized by a dual decomposition method, while the optimal RIS phase shifts were computed by using a derived closed-form expression. Also, we presented a computation complexity analysis for the proposed AO algorithm. Simulation results show that adding the RIS can significantly improve the achievable sum-rate in a BC.

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