Gröbner Basis

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Polynomial Division

- Leading monomial: Monomial of the polynomial with the highest degree.
- Leading coefficient: Coefficient of the leading monomial.
- Leading term: Product of *a* and *b*.
- Degree of polynomial: Degree of the leading term.
- *Degree of remainder always lesser than that of the divisor.*

Monomial Orderings

- Lexicographic ordering or Lex

  For $\alpha, \beta \in Z^n_{\geq 0}$, then $\alpha \geq_{lex} \beta$ if in $\alpha - \beta \in Z^n_{\geq 0}$, the leftmost non-zero entry is positive.

  For eg:

  $f = x^2 + xy + y^2 \Rightarrow$

  $x^2 >_{lex} xy$ because $(2,0) >_{lex} (1,1)$

  and

  $x^2 >_{lex} y^2$ because $(2,0) >_{lex} (0,2)$

  ordering: $\therefore$ Leading term is x^2
- Graded lexicographic ordering:

Let $\alpha, \beta \in Z^n_{\geq 0}$ and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad |\beta| = \sum_{i=1}^{n} \beta_i$$

then $\alpha \geq_{glex} \beta$ iff:

$|\alpha| \geq |\beta|$ and $\alpha \geq_{glex} \beta$

For eg:

$f = x + y^4 \Rightarrow$

$\therefore$ Leading term is y^4

Multi-variate Polynomial Division

- Dividing 2 polynomials:

$f = x^2 + xy + 1$

$g = xy - x$

Divide f by g using $glex$ ordering

$\Rightarrow$

$Lt(f) = x^2$ and $Lt(g) = xy$

both are not divisible. In one variable case we can stop here.

$$\begin{array}{r|l} & q \Rightarrow 1 \\ xy - x & x^2 + xy + 1 \\ & \quad\; xy - x \\ & r \Rightarrow x^2 + x + 1 \end{array}$$

Here $Lt(r) >_{glex} Lt(g)$

```
clear all;clc
syms x y
f = x^2+x*y+1;
d = x*y-x;
[r,q] = polynomialReduce(f,d)
```

r = $x^2 + x + 1$

q = 1

- Dividing by a system of polynomials:

Let $F = \{f_1, \dots, f_n\}$

We divide f by F to write $f = \sum_i^n q_i f_i + r$.

If $r = 0$ then F is the roots of f.

For eg:

$F = \{f_1, f_2\}$ where $f_1 = xy - 1$ and $f_2 = y^2 - 1$.

Here $f = xy^2 - y^3 + x^2 - 1$.

Dividing f by f_1 then f_2 w.r.t *glex* ordering gives:

$q_1 \Rightarrow y$ and $q_2 \Rightarrow -y$

$r \Rightarrow x^2 - 1$

If we do the reverse, divide f by f_2 then f_1 w.r.t *glex* ordering gives:

$q_1 \Rightarrow 0$ and $q_2 \Rightarrow x - y$

$r \Rightarrow x^2 + x - y - 1$

$\therefore$ *to get a unique remainder for a given monomial ordering we use Gröbner bases.*

```
syms x y
f = x*y^2-y^3+x^2-1;
d1 = [x*y - 1, y^2 - 1];
d2 = [y^2 - 1, x*y - 1];
[r1,q1] = polynomialReduce(f,d1)
```

r1 = $x^2 - 1$

q1 = $\begin{pmatrix} y & -y \end{pmatrix}$

```
[r2,q2] = polynomialReduce(f,d2)
```

r2 = $x^2 + x - y - 1$

q2 = $\begin{pmatrix} x - y & 0 \end{pmatrix}$

Rings

A ring is a non-empty set equipped with 2 operations that satisfy the following axioms:

- It is closed under addition: $if \;\; a \in R \;\; and \;\; b \in R \;\; then \;\; a + b \in R$
- Associative
- Commutative
- Should have additive identity: $i.e \;\; 0_R \in a \;\; such \;\; that \;\; a + 0_R = a = 0_R + a \;\; \forall a \in R$
- Should have additive inverse.
- Closed under multiplication: $if \;\; a \in R \;\; and \;\; b \in R \;\; then \;\; a \bullet b \in R$
- Distributive and assosiative.
- *A set of **Z**, **Q** and **C** are rings. Set of even numbers form a ring but set of odd nos don't.*

Ideals & Fields

- A subring *I* of a ring *R* is an **ideal** *iff* when $r \in R$ and $a \in I$, then $r.a \in I$ and $a.r \in I$. For eg. a set of even integers is an ideal of the ring ***Z*** and $\{0_R\}$ and ***R*** are ideals for every ring ***R***.

For a ring R and $F = \{f_1, \dots, f_s\}$; the set $I = \left\{ \sum_{i=1}^{s} a_i.f_i : a_i \in R \right\}$ is an ideal.

I is called the ideal generated by F and is denoted by $I = \langle f_1, \dots, f_s \rangle$.

- The zero ideal is generated by a single element set; $I = \langle 0_R \rangle = \{0_R\}$ $\forall$ ring ***R***.
- Ideals can have different set of generators.
- Noetherian Rings: A ring ***R*** is a Noetherian ring if every ideal *I* of ***R*** is finitely generated. And, according to Hilbert's Basis Theorem if *R* is a Noetherian ring then so is the polynomial ring *R[x]*.

Gröbner Basis

(LM(*f*) denotes fixed monomial ordering and LT(*f*) denotes the leading term.)

- A **Gröbner basis** is a set of multivariate nonlinear polynomials enjoying certain properties that allow simple algorithmic solutions for many fundamental problems in mathematics and natural and technical sciences.
- An arbitrary set of polynomials is in general not a Gröbner basis. However, for every basis F there exists a Gröbner basis G which is equivalent to F in the sense that the linear combinations of elements of F are precisely the same as the linear combinations of elements of G. For F and G equivalent in this sense, the congruence of polynomials modulo F agrees with the congruence modulo G.
- Given an arbitrary basis F, an equivalent Gröbner basis G can be found constructively by Buchberger's Algorithm.
- A Gröbner basis of an ideal *I* is a set of generators of

  Let $I \subset k[x_1, \dots, x_n]$ be an ideal other than $\{0\}$ then;

  $LT(I) = \{cx^\alpha : \exists f \in I, LT(f) = cx^\alpha\}$

  thus $\langle LT(I) \rangle$ is the ideal generated by the elements of I.

  I.

- For a fixed monomial ordering a finite subset $G = \{g_1, \dots, g_t\}$ of an ideal *I* is said to be a Gröbner basis if: $< LT(g_1), \dots, LT(g_t) > = < LT(I) >$
- Reduced Gröbner basis: A RGb for a set of polynomials *F* is a Gröbner basis *G* of *F* such that: 1. LC is 1 $\forall p \in G$ and for all $p \in G$, none of the terms of *p* is divisible by the $LT(q)\ \forall\ q \in G - \{p\}$. That is a basis is a Gröbner basis if and only if all S-polynomials of any two basis elements reduce to zero.
- For a linear system of equations Grobner Basis is basically the Row Echelon form. For a univariate system of equations it is the GCD($f_1, \dots, f_m$). In general for an ideal **I** the Grobner basis consists of all the "smallest polynomials" in **I** for a given monomial ordering.
- S-polynomials:

For polynomials f and g and:

$$LM(f)=\prod_{i=1}^{n} x_i^{\alpha_i} \text{ and } LM(g)=\prod_{i=1}^{n} x_i^{\beta_i}$$

then we call x^{γ} as the LCM of the $LM(f)$ and $LM(g)$, if $\gamma=(\gamma_1,\ldots,\gamma_n)$

such that $\gamma_i=\max(\alpha_i,\beta_i)\,\forall i$.

and

The s-polynomial of f and g is the combination of:

$$S(f,g)=\frac{x^{\gamma}}{LT(f)}\cdot f-\frac{x^{\gamma}}{LT(g)}\cdot g$$

- For example:

Let $f_1=xy-1$ and $f_2=y^2-1$, considering the *glex* ordering.

$LM(f_1)=xy$ and $LM(f_2)=y^2$

$x^{\gamma}=xy^2$

$$\therefore S(f_1,f_2)=\frac{xy^2}{xy}\cdot(xy-1)-\frac{xy^2}{y^2}\cdot(y^2-1)$$

$$S(f_1,f_2)=xy^2-y-xy^2+x=x-y$$

Buchberger's Algorithm

A variety of frequently arising questions about sets of polynomial equations can be answered easily when the sets are "Gröbner bases" while they are not easy to answer for an arbitrary set of polynomials. Buchberger's algorithm provides a means to transform an arbitrary set of polynomials into an equivalent Gröbner basis. Therefore, questions about some set of polynomials arising in an application can be answered by first using Buchberger's algorithm to compute an equivalent Gröbner basis and then answering the question for the Gröbner basis.

Gröbner basis is a reduction step of a polynomial *p* with respect to a fixed set *B* of polynomials consists of subtracting some multiple of an element of *B* from *p* in such a way that some power product in *p* is replaced by power products which are smaller (w.r.t. a fixed order imposed on the power products).

Repeated reduction steps, after finitely many steps, will always lead to some polynomial which cannot be reduced any further. Such a polynomial is called a *reduced form* of *p* with respect to *B* . Reduced forms are in general not unique. However, one can give an algorithm which for given *p* and *B* computes *some* reduced form of *p* with respect to *B* . By RED we denote such a function, i.e., RED(*p*,*B*) is a reduced form of *p* with respect to *B* obtained by repeated reduction steps.

We also recall from the Gröbner basis article that the S-polynomial of two polynomials *p* and *q* is defined as:

$$S(f,g)=LCM(LT(f),LT(g))\left(\frac{f}{LM(f)}-\frac{g}{LM(g)}\right)$$

where the symbols LT and LM refer to the leading power product and the leading monomial (w.r.t. a fixed order imposed on the power products), and LCM to the least common multiple.

Termination of the algorithm is shown by considering the leading power products of the polynomials in *G* . A leading power product of some new polynomial *h* cannot be a multiple of a leading power product of a polynomial in *G* , because the polynomials *h* are completely reduced with respect to *G* . If on some input the algorithm did not terminate, this would give rise to an infinite sequence of power products in which none is a multiple of any power product appearing earlier in the sequence.

Rough algorithm:

- Chose the monomial ordering.
- Start with *G := F*.
- Repeat G' := G until G = G'

For example:

To compute the RGb of the I generated by $F = \{f_1, f_2\}$ where:

$f_1 = xy - 1$ *and* $f_2 = y^2 - 1$; w.r.t *glex* ordering.

$\Rightarrow$

Initially take G=F

1. On computation we will get $S(f_1, f_2) = x - y$.

Let $f_3 = S(f_1, f_2)$. Since f_3 wil not divide G, the remainder is f_3 and since $f_3 \neq 0$,, we add f_3 to G.

$\therefore G = \{f_1, f_2, f_3\}$

Since the remainder is 0 when f_1 is divided by $\{f_2, f_3\}$, $G = \{f_2, f_3\}$.

Since no more polynomials can be eliminated from G we go back to the begining with $G = \{f_2, f_3\}$.

Now $f_4 = S(f_2, f_3) = y^3 - x$

Its remainder is 0 when we divide it by G. Hence we can exit the loop

and say the RGb w.r.t *glex* ordering is:

$G = \{x - y, y^2 - 1\}$

```
syms x y
r = [x*y-1, y^2-1];
G = gbasis(r)
```

G = $\begin{pmatrix} y^2 - 1 & x - y \end{pmatrix}$

Proper algorithm:

Input: A finite set *F* of polynomials.

Output: A finite Gröbner basis *G* equivalent to *F*

1. $G := F$
2. $C := G \times G$

3. while $C \neq \varnothing$ do
4. choose a pair (f, g) from C
5. $C := C/\{(f, g)\}$
6. $h := RED(S(f, g), G)$
7. if $h \neq 0$ then
8. $C := C \cup (G \times \{h\})$
9. $G := G \cup \{h\}$
10. return G

Unique Remainders

- For example:

For $F = \{f_1, f_2\}$; where $f_1 = xy - 1$, $f_2 = y^2 - 1$ and

$f = xy^2 - y^3 + x^2 - 1$

It is known from the previous example that the RGb w.r.t $glex$ ordering of the

$I =< F >$ is $G = \{g_1 = x - y, g_2 = y^2 - 1\}$

Unlike previous divisions the order of division does not change the r

When f is divided by G, g_2 then g_1:

$q_1 \Rightarrow x + y + 1$

$q_2 \Rightarrow x - y + 1$

$r \Rightarrow 0$

When f is divided by G, g_1 then g_2:

$q_1 \Rightarrow y^2 + x + y$

$q_2 \Rightarrow 1$

$r \Rightarrow 0$

$\therefore f = q_1 \cdot f_3 + f_2 = \left(y^2 + x + y\right) f_3 + f_2 \quad \cdots(1)$

Since $f_3 = S\left(f_1, f_2\right) = yf_1 - xf_2$

$\therefore f = q_1 f_1 + q_2 f_2 + 0$, where: $\quad \cdots(2)$

$q_1 = y\left(y^2 + x + y\right)$

$q_2 = -x\left(y^2 - x - y\right) + 1$

$0 = r$

```
syms x y
f = x*y^2-y^3+x^2-1;
d = [x*y-1,y^2-1];
G = gbasis(d);
Gd = [G(1,2) G(1,1)];
[r1,q1] = polynomialReduce(f,G)
```

```
r1 = 0
q1 = (x - y + 1   x + y + 1)
```

```
[r2,q2] = polynomialReduce(f,Gd)
```

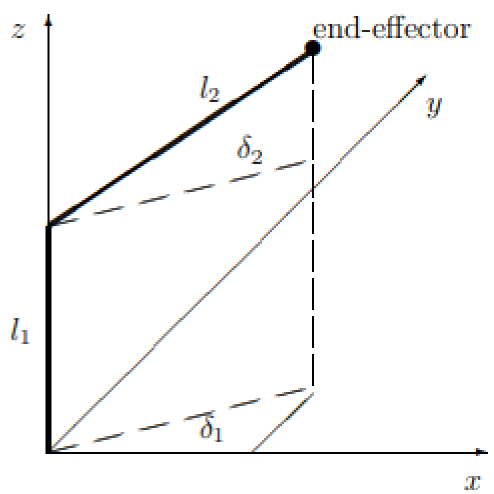
```
r2 = 0
q2 = (y^2 + y + x   1)
```

A zero remainder implies that the solutions of *F* are the roots of *f*. In conclusion, to divide a polynomial *f* by a set of polynomials *F* to get a unique remainder the following steps are to be followed:

- Compute Gröbner basis $G = \{g_1, ..., g_t\}$ of the ideal *I=<F>.*
- Divide *f* by *G* to get a unique remainder *r*.
- Trace the quotients $q_i,\ i = 1\ \ to\ \ n$ from the S-polynomials to write $f = q_1 f_1 + ... + q_n f_n + r$

Applications in Inverse Kinematics

Consider a robot with 2 revolute joints.



here we have 9 parameters; $l_1, l_2, p_x, p_y, p_z, \delta_1, \delta_2, s_1, s_2, c_1, c_2$ where l_1, l_2 are the link lengths; p_x, p_y, p_z are the end-effector x-, y- and z- coordinates; δ_1, δ_2 are the angles describing the revolutions of the revolute joints; and s_1, s_2, c_1, c_2 are the sines and cosines of $\delta_1\ \ and\ \ \delta_2$.

So with these we consider a system of equations:

given $l_1, l_2, p_x, p_z,$

solve for s_1, s_2, c_1, c_2, p_y

```
syms l1 l2 px pz s1 s2 c1 c2 py
```

Using these we can get 5 equations:

$l_2.c_1.c_2 - p_x = 0,$

$l_2.s_1.c_2 - p_y = 0,$

$l_2.s_2 + l_1 - p_z = 0,$

$c_1^2 + s_1^2 - 1 = 0$

$c_2^2 + s_2^2 - 1 = 0$

```
F = [l2*c1*c2 - px, l2*s1*c2-py, l2*s2+l1-pz, c1^2+s1^2-1, c2^2+s2^2-1];
G = gbasis(F,[c1 c2 s1 s2 py],'MonomialOrder','lexicographic');
siz = size(G);
for i=1:siz(2)
    disp(G(i))
end
```

$$c_1 + \frac{\mathrm{px}\ \mathrm{py}\ s_1}{{l_1}^2 - 2\,l_1\ \mathrm{pz} - {l_2}^2 + \mathrm{px}^2 + \mathrm{pz}^2}$$

$$c_2 + \frac{\mathrm{py}\ s_1\ \left(-{l_1}^2 + 2\,l_1\ \mathrm{pz} + {l_2}^2 - \mathrm{pz}^2\right)}{l_2\ \left({l_1}^2 - 2\,l_1\ \mathrm{pz} - {l_2}^2 + \mathrm{px}^2 + \mathrm{pz}^2\right)}$$

$$\frac{{l_1}^2 - 2\,l_1\ \mathrm{pz} - {l_2}^2 + \mathrm{px}^2 + \mathrm{pz}^2}{-{l_1}^2 + 2\,l_1\ \mathrm{pz} + {l_2}^2 - \mathrm{pz}^2} + {s_1}^2$$

$$s_2 + \frac{l_1 - \mathrm{pz}}{l_2}$$

$${l_1}^2 - 2\,l_1\ \mathrm{pz} - {l_2}^2 + \mathrm{px}^2 + \mathrm{py}^2 + \mathrm{pz}^2$$

These can be transformed into a Grobner Basis in the polynomial ring $\mathbb{Q}(l_1, l_2, p_x, p_z)[c_1, c_2, s_1, s_2, p_y]$:

This gives us 5 equations and 5 unknowns and thus solves the problem. Let:

$l_1 = 30$

$l_2 = 45$

$p_x = \frac{45}{4}\sqrt{6}$

$p_z = \frac{45}{2}\sqrt{2} + 30$

```
L1 = 30;L2 = 45; PX = 45*sqrt(6)/4; PZ = 45*sqrt(2)/2+30;
```

```
subs1 = subs(G(5), [l1, l2, px, pz], [L1, L2, PX, PZ]);
sol1 = solve(subs1 , py,'ReturnConditions',true);
disp('py=')
```

py=

```
disp(vpa(sol1.py,6))
```

$$\begin{pmatrix} -15.9099 \\ 15.9099 \end{pmatrix}$$

```
siz = size(sol1.py);
if(siz(1) > 1)
    PY = vpa(sol1.py(2),6);
else
    PY = vpa(sol1.py,6);
end
```

Here we get py= $\begin{pmatrix} -15.9099 \\ 15.9099 \end{pmatrix}$. We take py=15.9099

```
subs2 = subs(G(4), [l1, l2, pz], [L1, L2, PZ]);
sol2 = solve(subs2 , s2,'ReturnConditions',true);
disp('s2=')
```

```
s2=
```

```
disp(vpa(sol2.s2,6))
```

$$0.707107$$

```
siz = size(sol2.s2);
if(siz(1) > 1)
    S2_1 = vpa(sol2.s2(2),6);
else
    S2_1 = vpa(sol2.s2,6);
end
```

s2=0.707107

```
subs3 = subs(G(3), [l1, l2, px, pz], [L1, L2, PX, PZ]);
sol3 = solve(subs3 , s1,'ReturnConditions',true);
disp('s1=')
```

```
s1=
```

```
disp(vpa(sol3.s1,6))
```

$$\begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}$$

```
siz = size(sol3.s1);
if(siz(1) > 1)
    S1_1 = vpa(sol3.s1(2),6);
else
    S1_1 = vpa(sol3.s1,6);
end
```

s1= $\begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}$. We take s1=0.5

```
subs4 = subs(G(2), [s1, l1, l2, px, py, pz], [S1_1, L1, L2, PX, PY, PZ]);
sol4 = solve(subs4 , c2,'ReturnConditions',true);
disp('c2=')
```

```
c2=
```

```
disp(vpa(sol4.c2,6))
```

0.707107

```
siz = size(sol4.c2);
if(siz(1) > 1)
    C2_1 = vpa(sol4.c2(2),6);
else
    C2_1 = vpa(sol4.c2,6);
end
```

c2=0.707107

```
subs5 = subs(G(1), [s1, l1, l2, px, py, pz], [S1_1, L1, L2, PX, PY, PZ]);
sol5 = solve(subs5 , c1,'ReturnConditions',true);
disp('c1=')
```

```
c1=
```

```
disp(vpa(sol5.c1,6))
```

0.866025

```
siz = size(sol5.c1);
if(siz(1) > 1)
    C1_1 = vpa(sol5.c1(2),6);
else
    C1_1 = vpa(sol5.c1,6);
end
```

and c1=0.866025

Thus the problem is solved with:

$l_1 = 30; l_2 = 45,$

$p_x = \frac{45}{4}\sqrt{6}$

$p_z = \frac{45}{2}\sqrt{2} + 30$

$\textit{and the solved variables are}:$

$p_y = 15.9099$

$s_2 = 0.707107$

$s_1 = 0.5$

$c_2 = 0.707107$

$c_1 = 0.866025$

and the angles are:

```
del1 = vpa(rad2deg(acos(C1_1)),6);
del2 = vpa(rad2deg(acos(C2_1)),6);
disp('1=')
```

1=

```
disp(del1);
```

30.0

```
disp('2=')
```

2=

```
disp(del2);
```

45.0

Now considering a 3-DOF 5R Robot.

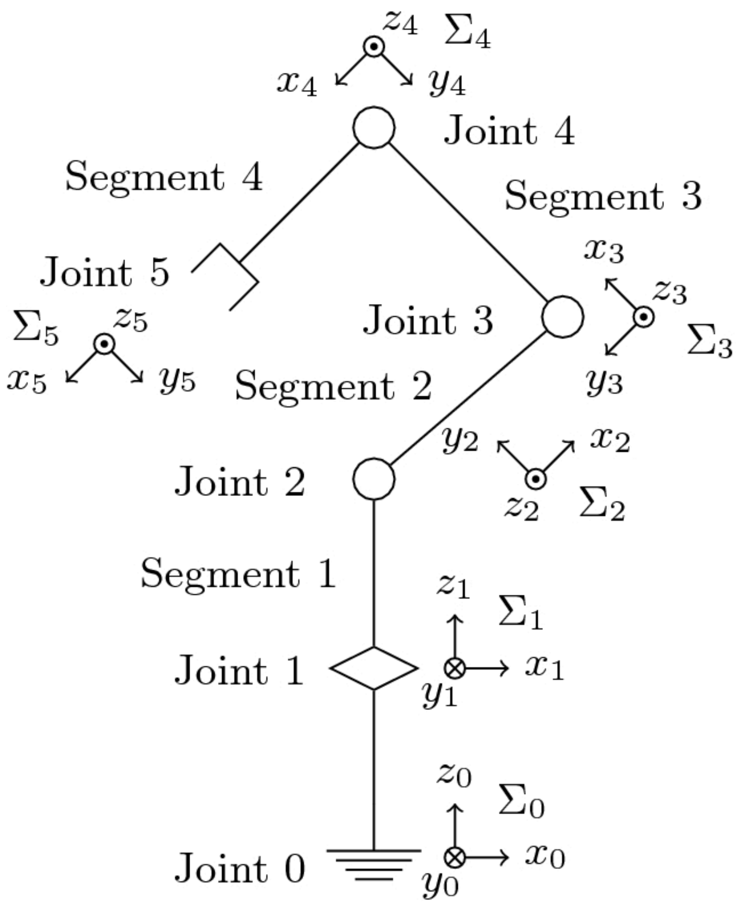


Table 1.
Joint parameters for EV3.

i	a_i (cm)	α_i	d_i (cm)	θ_i
1	0	0	8	θ_1
2	0	$\pi/2$	0	$\pi/4$
3	16	0	0	θ_2
4	16	0	0	θ_3
5	16	0	0	0

- where EV3 is the manipulator being used in this example.

DH convention to express transformation from $(i-1)^{th}$ system to i^{th} system.

$$T_i^{i-1} = \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\alpha_i) & -sin(\alpha_i) & 0 \\ 0 & sin(\alpha_i) & cos(\alpha_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta_i) & -sin(\theta_i) & 0 & 0 \\ sin(\theta_i) & cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} cos(\theta_i) & -sin(\theta_i) & 0 & a_i \\ cos(\alpha_i)sin(\theta_i) & cos(\alpha_i)cos(\theta_i) & -sin(\alpha_i) & -d_i\ sin(\alpha_i) \\ sin(\alpha_i)sin(\theta_i) & sin(\alpha_i)cos(\theta_i) & cos(\alpha_i) & d_i\ cos(\alpha_i) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where a_i is the distance beween z_{i-1} and z_i, α_i is the angle between axes z_{i-1} and z_i with respect to x_i axis, d_i is the distance between x_{i-1} and x_i, and θ_i the angle between axes x_{i-1} and x_i with respect to z_i axis.

Thus we can get $T_5^0 = T_1^0 T_2^1 T_3^2 T_4^3 T_5^4$ and the end effector position is the translation part of the matrix.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t_{14} \\ t_{24} \\ t_{34} \end{pmatrix}$$

```
syms th1 th2 th3
the = [th1, pi/4, th2, th3, 0];
d = [8, 0, 0, 0, 0];
alp = [0, pi/2, 0, 0, 0];
a = [0, 0, 16, 16, 16];
T0_1 = getMat(a(1), alp(1),d(1),the(1));
T1_2 = getMat(a(2), alp(2),d(2),the(2));
T2_3 = getMat(a(3), alp(3),d(3),the(3));
T3_4 = getMat(a(4), alp(4),d(4),the(4));
T4_5 = getMat(a(5), alp(5),d(5),the(5));
T0_5 = T0_1*T1_2*T2_3*T3_4*T4_5;
% simplify(vpa(T0_5,2))
x = 12;
y = 9;
z = 4;
EP = [x ; y ; z]
```

```
EP = 3×1
    12
     9
     4
```

```
syms c1 s1 c2 s2 c3 s3
f1 = 8*sqrt(2)*c1*(c2+c3*(c2-s2)-s2-s3*(c2+s2)+1) - x;
f2 = 8*sqrt(2)*s1*(c2+c3*(c2-s2)-s2-s3*(c2+s2)+1) - y;
f3 = 8*sqrt(2)*(c2+c3*(c2+s2)+s2+s3*(c2-s2)+1) + 8 - z;
f4 = s1^2 + c1^2 - 1;
f5 = s2^2 + c2^2 - 1;
f6 = s3^2 + c3^2 - 1;

F = [f1,f2,f3,f4,f5,f6];
```

```
G = gbasis(F,[c1 c2 s1 s2 c3 s3],'MonomialOrder','lexicographic');

[a1,a2,a3,a4,a5,a6] = solve(G,[c1 s1 c2 s2 c3 s3]);
C1_1 = double(vpa(a1(1),3));
S1_1 = double(vpa(a2(1),3));
C2_1 = double(vpa(a3(1),3));
S2_1 = double(vpa(a4(1),3));
C3_1 = double(vpa(a5(1),3));
S3_1 = double(vpa(a6(1),3));

C1_2 = double(vpa(a1(2),3));
S1_2 = double(vpa(a2(2),3));
C2_2 = double(vpa(a3(2),3));
S2_2 = double(vpa(a4(2),3));
C3_2 = double(vpa(a5(2),3));
S3_2 = double(vpa(a6(2),3));

theta1_1 = rad2deg(acos(C1_1));
theta2_1 = rad2deg(acos(C2_1));
theta3_1 = rad2deg(acos(C3_1));

theta1_2 = rad2deg(acos(C1_2));
theta2_2 = rad2deg(acos(C2_2));
theta3_2 = rad2deg(acos(C3_2));

disp("Orientation 1:")
```

```
Orientation 1:
```

```
disp("θ1="+theta1_1)
```

```
θ1=143.1301
```

```
disp("θ2="+theta2_1)
```

```
θ2=147.2651
```

```
disp("θ3="+theta3_1)
```

```
θ3=35.8657
```

```

disp("Orientation 2:")
```

```
Orientation 2:
```

```
disp("θ1="+theta1_2)
```

```
θ1=143.1301
```

```
disp("θ2="+theta2_2)
```

```
θ2=176.8692
```

```
disp("θ3="+theta3_2)
```

```
θ3=35.8657
```

```
function T = getMat(a, alp, d, the)
T = [cos(the), -sin(the), 0, a;
     cos(alp)*sin(the), cos(alp)*cos(the), -sin(alp), -d*sin(alp);
     sin(alp)*sin(the), sin(alp)*cos(the), cos(alp), d*cos(alp);
     0, 0, 0, 1];
end
```