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Period 5
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HW # 10.4: 4-9, 20, 23, 26, 32

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$

let $b_n = \frac{n^4}{n^3+1}$

i. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^4}{n^3+1}$

$= \lim_{n\to\infty} \frac{n}{\frac{1}{n}+\frac{1}{n^4}} = \infty$

$\therefore$ The given series diverges by the nth term test since the limit is infinite.

5. $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(1.1)^n}$

let $b_n = \frac{1}{(1.1)^n}$

i. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(1.1)^n} = 0$ ✓

ii. $b_{n+1} \leq b_n \ \forall \ n \geq 2$ ✓ $\therefore \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(1.1)^n}$ converges by the alternating series test

$\sum_{n=0}^{\infty} \left| \frac{(-1)^{n-1}}{(1.1)^n} \right|$

$= \sum_{n=0}^{\infty} \frac{1}{(1.1)^n} = \sum_{n=0}^{\infty} \left(\frac{1}{1.1}\right)^n$, $\sum_{n=0}^{\infty} \left(\frac{1}{1.1}\right)^n$ converges by the geometric series test since $|r| = \left|\frac{1}{1.1}\right| = \frac{1}{1.1} < 1$

$\therefore$ The series $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(1.1)^n}$ converges absolutely.

6. $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi n}{4}\right)}{n^2}$

The given series resembles $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test as $p = 2 > 1$.

Since $\frac{\sin\left(\frac{\pi n}{4}\right)}{n^2} \leq \frac{1}{n^2}$ $\forall$ $n \geq 1$, then $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi n}{4}\right)}{n^2}$ converges by direct comparison.

$$\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{\pi n}{4}\right)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{\left|\sin\left(\frac{\pi n}{4}\right)\right|}{n^2}$$

The given series resembles $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test as $p = 2 > 1$. Since $\frac{\left|\sin\left(\frac{\pi n}{4}\right)\right|}{n^2} \leq \frac{1}{n^2}$ $\forall n \geq 1$, then $\sum_{n=1}^{\infty} \frac{\left|\sin\left(\frac{\pi n}{4}\right)\right|}{n^2}$ converges by direct comparison.

$\therefore$ $\boxed{\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi n}{4}\right)}{n^2} \text{ converges absolutely}}$

7. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

let $b_n = \frac{1}{n \ln n}$

i. $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n \ln n} = 0$

ii. $b_{n+1} \leq b_n$ $\forall$ $n \geq 2$ $\therefore$ $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the alternating series test

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$, let $f(x) = \frac{1}{x \ln x}$

Since f is positive, continuous, and decreasing, then the integral test applies

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$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln x}$$

let $J = \int \frac{dx}{x\ln x}$, let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\ln x| + C$$

$$\therefore \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln x} = \lim_{b\to\infty} \ln|\ln x| \Big|_2^b$$

$$= \lim_{b\to\infty} \left(\ln|\ln b| - \ln|\ln 2|\right)$$

$$= \infty$$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges by the integral test

$\therefore$ $\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n} \text{ converges conditionally}}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\frac{1}{n}}$

let $b_n = \frac{1}{1+\frac{1}{n}}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= 1$$

$\therefore$ $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\frac{1}{n}} \text{ diverges by the nth term test}}$

9. $\sum_{n=2}^{\infty} \frac{\cos \pi n}{(\ln n)^2}$

$= \frac{1}{(\ln 2)^2} - \frac{1}{(\ln 3)^2} + \frac{1}{(\ln 4)^2} - \dots$

$= \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^2}$

let $b_n = \frac{1}{(\ln n)^2}$

i. $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{(\ln n)^2} = 0$

ii. $b_{n+1} \leq b_n \ \forall \ n \geq 2$

$\therefore \sum_{n=2}^{\infty} \frac{\cos \pi n}{(\ln n)^2}$ converges by the alternating series test

$\sum_{n=2}^{\infty} \left| \frac{\cos \pi n}{(\ln n)^2} \right| = \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{(\ln n)^2} \right|$

$= \sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

The series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ resembles $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$.

The series $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series test since $p = \frac{1}{2} \leq 1$. Since $\frac{1}{(\ln n)^2} \geq \frac{1}{n^{1/2}} \ \forall$ $n \geq 95000$, then $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ diverges by direct comparison.

$\therefore$ $\boxed{\sum_{n=2}^{\infty} \frac{\cos \pi n}{(\ln n)^2} \text{ converges conditionally.}}$

20. $\sum_{n=2}^{\infty} \frac{n}{n^2-n}$

The series $\sum_{n=2}^{\infty} \frac{n}{n^2-n}$ resembles $\sum_{n=2}^{\infty} \frac{1}{n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by p-series test, since $p=1$.

$$\lim_{n\to\infty} \frac{\frac{n}{n^2-n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2-n}$$
$$= \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}}$$
$$= 1$$

Since the limit is positive and finite, both series diverge by the limit comparison test.

23. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

The series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2+1}}$ resembles $\sum_{n=2}^{\infty} \frac{1}{n}$.

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-series test, since $p=1$.

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$$
$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2\left(1+\frac{1}{n^2}\right)}}$$
$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n^2}}} = 1$$

$\therefore$ Since the limit is positive and finite, both series diverge by the limit comparison test.

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26. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$

let $b_n = \frac{1}{(2n+1)!}$

i. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$

ii. $b_{n+1} \le b_n \ \forall \ n \ge 1$

$\therefore$ the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$ converges by the alternating series test

32. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

The series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ resembles $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$.

The series $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series test since $p = \frac{1}{2} \le 1$. Since $\frac{1}{(\ln n)^2} \ge \frac{1}{n^{1/2}} \ \forall$ $n \ge 95000$, then $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ diverges by direct comparison.