# Simulation of quantum trajectories and measurement induced entanglement transition

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Non Equillibrium Quantum Many Body Dynamics
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# Abstract

Quantum trajectory dynamics for a many-body state are investigated under the influence of continuous measurements. A chain of hard-core bosons is subjected to continuous measurements of occupation numbers, exploring both integrable and non-integrable models. With increasing measurement strength, a transition in entanglement is observed, shifting from a volume-law regime to an area-law regime. The critical point of this transition is identified by the peak in mutual information. At the transition point, the von Neumann entanglement entropy exhibits a logarithmic scaling with system size.
Lindbland master equation leads to mixed state density matrix of equillibrium state ( infinite temperature) resulting in featureless steady state properties. For the Quantum trajectory dynamics, the U(1) symmetry in addition with quick relaxation of particle density results in close resemblence with unitary circuit models with random projective measurement dynamics.

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# Introduction

Quantum dynamics of open system cannot be represented in terms of unitary time evolution. Instead, the dynamics of an open system can be approximately formulated by equation of motion for density matrix, a quantum master equation.

## 1.1 Closed and open quantum systems

### 1.1.1 The Liouville - von Neumann equation

The time evolution of the state vector $|\psi\rangle$ according to the Schrodinger equation is given by,

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{1.1}$$

where H(t) is the Hamiltonian of the system and Planck's constant $\hbar$ is set equal to 1. The evolution of the state using unitary time evolution from time $t_0$ to $t$ is given by,

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \tag{1.2}$$

Substituting Eq. (1.2) in Eq. (1.1), subjected to the initial condition ,we get,

$$U(t_0,t_0) = I \tag{1.3}$$

$$i\frac{\partial}{\partial t}U(t,t_0) = H(t)U(t,t_0) \tag{1.4}$$

For closed, isolated physical system H is time independent and Eq. (1.4) integrated to give,

$$U(t,t_0) = \exp[-iH(t-t_0)] \tag{1.5}$$

For a time dependent Hamiltoninan H(t), the system is closed . The solution of Eq. (1.4) with respect to Eq. (1.3) is represented as time -ordered exponential,

$$U(t,t_0) = T_{\leftarrow}\exp\left[-i\int_{t_0}^{t} dsH(s)\right] \tag{1.6}$$

, where $T_{\leftarrow}$denotes chronological time ordering operator.
In case the system is in a mixed state , the quantum statistical ensemble corresponding to the system under consideration is characterised with the help of $\rho$. Let us assume that at initial time $t_0$, the state of the system is given by the density matrix

$$\rho(t_0) = \sum_\alpha w_\alpha \left|\psi_\alpha(t_0)\right\rangle \left\langle\psi_\alpha(t_0)\right| \tag{1.7}$$

where $w_\alpha$ are the positive weights and the $|\psi_\alpha(t_0)\rangle$ are normalised state vectors. The state of the system at time $t$ is given by using Eq. (1.2)

$$\rho(t) = \sum_\alpha w_\alpha U(t,t_0) \left|\psi_\alpha(t_0)\right\rangle \left\langle\psi_\alpha(t_0)\right| U^\dagger(t,t_0), \tag{1.8}$$

concisely it can be written as,

$$\rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0) \tag{1.9}$$

Differentiating the equation wrt time we get the equation ,

$$\frac{d}{dt}\rho(t) = -i[H(t),\rho(t)], \tag{1.10}$$

**The analogy of von Neumann equation with corresponding equation of motion for probability density in classical statistical mechanics can be stressed as follows**

$$\frac{d}{dt}\rho(t) = \mathcal{L}(t)\rho(t) \tag{1.11}$$

where $\mathcal{L}$ is the *Liouville Operator* and is defined as

$$\mathcal{L}(t)\rho(t) = -i[H(t),\rho(t)]$$

$\mathcal{L}$ is usually called a Liouville *super operator* For the simple case of time independent hamiltonian the liouville operator is also time independent, and we get :

$$\rho(t) = \exp[\mathcal{L}(t-t_0)]\rho(t_0) \tag{1.12}$$

### 1.1.2 Heisenberg and interaction picture

Schrodinger and Heisenberg picture operators are related though canonical transformation:

$$A_H(t) = U^\dagger(t,t_0)A(t)U(t,t_0) \tag{1.13}$$

where $A(t)$ explicitly depends on time. The equation of motion for a Heisenberg operator can be obtained by differentiating both side of (1.13),

$$\frac{d}{dt}A_H(t) = i[H_H(t),A_H(t)] + \frac{\partial A_H(t)}{\partial t} \tag{1.14}$$

where $H_H(t)$ denotes Hamiltonian in the Heisenberg picture,

$$H_H(t) = U^\dagger(t,t_0)H(t)U(t,t_0) \tag{1.15}$$

The Hamiltonian of the system consists of two parts $H(t) = H_0 + \hat{H}_I(t)$ For the interaction picture , we introduce the unitary time evolution operators:

$$U_0(t, t_0) \equiv \exp[-iH_0(t - t_0)] \tag{1.16}$$

and

$$U_I(t, t_0) \equiv U_0^\dagger(t, t_0)U(t, t_0) \tag{1.17}$$

The interaction picture $A_I(t)$ and $\rho_I(t)$ are defined as follows,

$$\begin{aligned} A_I(t) &\equiv U_0^\dagger(t, t_0)A(t)U_0(t, t_0) \\ \rho_I(t) &\equiv U_I(t, t_0)\rho(t_0)U_I^\dagger(t, t_) \end{aligned} \tag{1.18}$$

The corresponding von Neumann equation in the interaction picture takes the form

$$\frac{d}{dt}\rho_I(t) = -i[H_I(t), \rho_I(t)] \tag{1.19}$$

where $H_I(t) \equiv U_0^\dagger(t, t_0)\hat{H}_I(t)U_0(t, t_0)$

### 1.1.3 Dynamics of Open system

In general words, an open quantum system is a quantum system $S$ coupled to another quantum system $B$ called the environment.Let us denote $\mathcal{H}_S$ as the Hilbert space of the system and $\mathcal{H}_B$, the Hilbert space of the environment. For the total system $S + B$, the Hilbert Space is given by $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$, the Hamiltonian takes the following form:

$$H(t) = H_S \otimes I_B + I_S \otimes H_B + \hat{H}_I(t) \tag{1.20}$$

For the state of the system described by a density matrix $\rho$, the expectation values of all observables acting on open system's Hilbert space is given by

$$\langle A \rangle = tr_S\{A\rho_S\} \tag{1.21}$$

where $\rho_S = tr_B\rho$ is the reduced density matrix of open quantum system S. The reduced density matrix $\rho_S(t)$ and the equation of motion for reduced density matrix is given by

$$\begin{aligned} \rho_S(t) &= tr_B\{U(t, t_0)\rho(t_0)U^\dagger(t, t_0)\} \\ \frac{d}{dt}\rho_S(t) &= -itr_B[H(t), \rho(t)] \end{aligned} \tag{1.22}$$

## 1.2 Quantum Markov Processes

The extension of semigroup property of Classical Homogeneous Markov Process leads to the concept of quantum dynamical semigroup and quantum Markov process.

### 1.2.1 Quantum dynamical semigroup

The dynamics of the reduced system given by (1.22) are involved. Under short environmental correlation times, one formulates quantum dynamics using the quantum dynamical subgroup. We first introduce the dynamical map.
Let us suppose that we prepared the state at t=0 for the system as $\rho(0) = \rho_S(0) \otimes \rho_B$ where $\rho_S(0)$ is the initial state of the reduced system S and $\rho_B$ is the environment. The transformation of $\rho_S(0) \to \rho_S(t)$ is of the form

$$\rho_S(0) \to \rho_S(t) = V(t)\rho_S(0) \equiv tr_B\{U(t,0)[\rho_S \otimes \rho_B]U^\dagger(t,0)\} \tag{1.23}$$

Considering that $\rho_B$ and final time t is fixed, the above relation defines the following map:

$$V(t) : \mathcal{S}(\mathcal{H}_\mathcal{S}) \to \mathcal{S}(\mathcal{H}_\mathcal{S}) \tag{1.24}$$

This map is known as the dynamical map i.e. the state change of the open system over time t. The map is characterised completely by operators pertaining to open system's hilbert space $\mathcal{H}_\mathcal{S}$. Using spectral decomposition of the density matrix of the environment $\rho_B = \sum_\alpha \lambda_\alpha |\varphi_\alpha\rangle \langle\varphi_\alpha|$and putting it in Eq. (1.23) , we get

$$V(t)\rho_S = \sum_{\alpha,\beta} W_{\alpha\beta}(t)\rho_S W^\dagger_{\alpha\beta}(t), \tag{1.25}$$

where the $W_{\alpha\beta}$ are defined as

$$W_{\alpha\beta} = \sqrt{\lambda_\beta}\, \langle\varphi_\alpha| U(t,0) |\varphi_\beta\rangle \tag{1.26}$$

The dynamical map $V(t)$ described in the Eq. (1.25) is of the form $\Phi_m(\rho) = \sum_k \Omega_{mk}\rho\Omega^\dagger_{mk}$ of an operation $\Phi_m$ describing a **generalised quantum measurement**. The operator $W_{\alpha\beta}(t)$ satisfies the condition

$$\sum_{\alpha\beta} W^\dagger_{\alpha\beta}(t)W_{\alpha\beta}(t) = I_S \implies tr_S\{V(t)\rho_S\} = tr_S\rho_S = 1 \tag{1.27}$$

**Thus, dynamical map V(t) represents a convex-linear, completely positive and trace preserving quantum operation.**

### 1.2.2 The Markovian quantum master equation

For a given quantum dynamical subgroup there exists a linear map $\mathcal{L}$, generator of the semigroups, which can be used to represent

$$V(t) = \exp\{\mathcal{L}t\} \tag{1.28}$$

This representation leads to the differential equation for reduced density matrix

$$\frac{d}{dt}\rho_s(t) = \mathcal{L}\rho_S(t) \tag{1.29}$$

called as the Markovian quantum master equation. The generator $\mathcal{L}$ is introduced in the Eq. (1.11).
To construct the most general form for $\mathcal{L}$ of the subgroup. First consider the case of finite dimensional Hilbert Space $\mathcal{H}_\mathcal{S}$, dim $\mathcal{H}_\mathcal{S} = N$. Notation : $(A,B) \equiv tr\{A^\dagger B\}$ . For an orthonormal basis $\{B_i\}$, Orthonormal and completeness condition is given as $(B_i, B_j) = \delta_{ij}$ and $A = \sum_i B_i(B_i, A)$.
The corresponding liouville space is a complex space of dimension $N^2$ and we choose a complete basis of operators $F_i, i = 1, 2, \cdots, N^2$ such that

$$(F_i, F_j) \equiv tr_S\{F_i^\dagger F_j\} = \delta_{ij} \tag{1.30}$$

For convenience , one of the operator proportional to identity $F_{N^2} = (1/N)^{1/2} I_S$ and other operators being traceless i.e. $tr_S F_i = 0$ for $i = 1, 2, \cdots, N^2 - 1$. Applying the relation for each $W_{\alpha\beta}(t)$in Eq. (1.26), we get

$$W_{\alpha\beta}(t) = \sum_{i=1}^{N^2} F_i (F_i, W_{\alpha\beta}(t)) \tag{1.31}$$

Using the representation Eq. (1.25), dynamic map $V(t)$ can be written as

$$V(t)\rho_S = \sum_{i,j=1}^{N^2} c_{ij} F_i \rho_s F_j^\dagger \tag{1.32}$$

where

$$c_{ij}(t) \equiv \sum_{\alpha\beta} (F_i, W_{\alpha\beta}(t))(F_i, W_{\alpha\beta}(t))^* \tag{1.33}$$

The coefficient matrix $c = (c_{ij})$ is Hermitian and positive. For $N^2$ dimensional complex vector $v$ we have

$$\sum_{ij} c_{ij} v_i^* v_j = \sum_{\alpha\beta} |(\sum_i v_i F_i, W_{\alpha\beta}(t))^2| \geq 0 \tag{1.34}$$

which proves that $c \geq 0$.
From the definition Eq. (1.28) for the generator along with Eq. (1.32), we have

$$\begin{aligned} \mathcal{L}\rho_S &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\{V(\epsilon)\rho_S - \rho_S\} \\ &= \lim_{\epsilon\to 0} \left\{ \frac{1}{N} \frac{c_{N^2N^2(\epsilon)} - N}{\epsilon} \rho_S + \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} \left( \frac{c_{iN^2}(\epsilon)}{\epsilon} F_i \rho_S + \frac{c_{n^2 i}(\epsilon)}{\epsilon} \rho_S F_i^\dagger \right) \right. \\ &\left. + \sum_{i,j=1}^{N^2-1} \frac{c_{ij}(\epsilon)}{\epsilon} F_i \rho_s F_j^\dagger . \right. \end{aligned} \tag{1.35}$$

The coefficients $a_{ij}$ can be defined as

$$\begin{aligned} a_{N^2N^2} &= \lim_{\epsilon\to 0} \frac{c_{N^2N^2} - N}{\epsilon} \\ a_{iN^2} &= \lim_{\epsilon\to 0} \frac{c_{iN^2}}{\epsilon}, \qquad i = 1, \cdots, N^2 - 1, \\ a_{ij} &= \lim_{\epsilon\to 0} \frac{c_{ij}}{\epsilon}, \qquad i, j = 1, \cdots, N^2 - 1 \end{aligned} \tag{1.36}$$

we introduce the following quantities

$$F = \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} a_{iN^2} F_i \tag{1.37}$$

and

$$G = \frac{1}{2N} a_{N^2N^2} I_S + \frac{1}{2}(F^\dagger + F), \tag{1.38}$$

as well as the Hermitian operator

$$H = \frac{1}{2i}(F^\dagger - F) \tag{1.39}$$

The $a_ij, i, k = 1, 2, \cdots, N^2 - 1$, matrix of the coefficient is Hermitian and positive. Using the definition we get

$$\mathcal{L}\rho_s = -i[H, \rho_s] + \{G, \rho_S\} + \sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho_S F_j^\dagger \tag{1.40}$$

The semigroup is trace preserving for all density matrices gives us

$$0 = tr_S\{\mathcal{L}\rho_S\} = tr_S \left\{ \left( 2G + \sum_{i,j=1}^{N^2-1} a_{ij} F_j^\dagger F_i \right) \right\} \tag{1.41}$$

from which we can deduce that

$$G = -\frac{1}{2} \sum_{i,j=1}^{N^2-1} a_{ij} F_j^\dagger F_i \tag{1.42}$$

Hence, we get the form of the generator as

$$\mathcal{L}\rho_s = -i[H, \rho_s] + \sum_{i,j=1}^{N^2-1} a_{ij} \left( F_i \rho_S F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_i, \rho_S\} \right) \tag{1.43}$$

The coefficient matrix is positive and it can be diagonalized with the help of a unitary transformation u

$$uau^\dagger = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \gamma_{N^2-1} \end{pmatrix} \tag{1.44}$$

where $\gamma_i$ are non negative eigenvalues. Introducing new set of operators $A_k$ as

$$F_i = \sum_{k=1}^{N^2-1} u_{ki} A_k \tag{1.45}$$

the diagonal form of the generator is obtained

$$\mathcal{L}\rho_s = -i[H, \rho_s] + \sum_{i,j=1}^{N^2-1} \gamma_k \left( A_k \rho_S A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, \rho_S\} \right) \tag{1.46}$$

The first term represents unitary part of the dynamics generated by Hamiltonian H. Operators $A_k$ are referred to as Lindblad operators and the equation is known as Lindblad equation.
The constant $\gamma_k$ controls the strength of the measurement at site j when it comes to continuous measurement of local occupation numbers when we choose $A_k = n_k$,and $1/\gamma_k$ is a characteristic time scale called the Zeno time.

### 1.2.3 General Form of stochastic master equation : Simplified derivation

Under unitary dynamics, for short time interval $dt$, state transforms as

$$|\psi\rangle \to |\psi\rangle + d\,|\psi\rangle = \left(1 - i\frac{H}{\hbar}dt\right)|\psi\rangle \tag{1.47}$$

Similarly, density operator gives the Von Neumann equation of motion:

$$\rho + d\rho = \left(1 - i\frac{H}{\hbar}dt\right)\rho\left(1 + i\frac{H}{\hbar}dt\right) = \rho - \frac{i}{\hbar}[H,\rho]dt \tag{1.48}$$

Any transformation of the density operator as long as it is completely positive (which guarantees no non negative eigenvalues of the operator) is sensible. It turns out the most general form of such transformation is

$$\rho \to \sum_n A_n \rho A_n^\dagger \tag{1.49}$$

where the operators $A_n$ are arbitrary. The transformation for a general **stochastic** operator is of the form

$$A = 1 - i\frac{H}{\hbar}dt + bdt + cdW \tag{1.50}$$

It is assumed that $b$ is Hermitian , putting into the transformation Eq. (1.49), we get

$$d\rho = -i\frac{i}{\hbar}[H,\rho]dt + \{b,\rho\}dt + c\rho c^\dagger dt + (c\rho + \rho c^\dagger)dw \tag{1.51}$$

Taking average over all the possible Wiener processes, which we denote by $\langle\langle\rangle\rangle$ and using the fact that $\langle\langle \rho dW\rangle\rangle = 0$ in Itô calculus, we get:

$$d\langle\langle\rho\rangle\rangle = -\frac{i}{\hbar}[H,\langle\langle\rho\rangle\rangle]dt + \{b,\langle\langle\rho\rangle\rangle\}dt + c\langle\langle\rho\rangle\rangle c^\dagger dt \tag{1.52}$$

The operator $\langle\langle\rho\rangle\rangle$ is a density operator and satisfies $\mathrm{Tr}[\langle\langle\rho\rangle\rangle] = 1$. We get $d\,\mathrm{Tr}[\langle\langle\rho\rangle\rangle] = \mathrm{Tr}[d\langle\langle\rho\rangle\rangle]] = 0$. Using the cyclic property we get :

$$\mathrm{Tr}\big[\langle\langle\rho\rangle\rangle(2b + c^\dagger c)\big] = 0 \tag{1.53}$$

Hence , we obtain the dependence of operator as

$$b = -\frac{c^\dagger c}{2} \tag{1.54}$$

This gives us the **Lindblad form of the master equation** of the master equation

$$d\langle\langle\rho\rangle\rangle = -\frac{i}{\hbar}[H,\langle\langle\rho\rangle\rangle]dt + D[c]\langle\langle\rho\rangle\rangle dt \tag{1.55}$$

where we define Lindblad superoperator:

$$D[c]\rho := c\rho c^\dagger - \frac{1}{2}(c^\dagger c\rho + \rho c^\dagger c) \tag{1.56}$$

This equation is for single dissipation process and is unconditioned master equation. So the full transformation becomes:

$$d\rho = -\frac{i}{\hbar}[H,\rho]dt + D[c]\rho dt + (c\rho + \rho c^\dagger)dW \tag{1.57}$$

### 1.2.4 The Liouville quantum master and stochastic differential equation

For an open system S, as described above that dynamics of $\rho_S(t)$ can be captured by Lindblad form of the Markovian master equation Eq. (1.46) . **The dynamics given by the equation can be represented by (PDP)piecewise deterministic process $\psi(t)$ in the Hilbert space of the open system S.** Time evolution equation for its probability density functional $P[\psi,t]$ is required to define such a process.

The functional at time $t$ is related to the same at time $t_0$ by

$$P[\psi,t] = \int D\tilde{\psi} D\tilde{\psi}^* T[\psi,t[\tilde{\psi},t_0]]P[\tilde{\psi},t_0] \tag{1.58}$$

where $T[\psi,t[\tilde{\psi},t_0]]$ is the conditional transition probability of the process.
A PDP can be defined in terms of a Liouville master equation for its probability density [BP95a; BP95b]. Liouville master equation for a PDP in Hilbert Space can be considered as

$$\begin{aligned}\frac{\partial}{\partial t}P[\psi,t] = i\int dx \left\{\frac{\delta}{\delta\psi(x)}G(\psi)(x) - \frac{\delta}{\delta\psi^*(x)}G(\psi)^*(x)\right\}P[\psi,t] \\ + \int D\tilde{\psi}D\tilde{\psi}^*\left\{W[\psi|\tilde{\psi}]P[\tilde{\psi},t] - W[\tilde{\psi}|\psi]P[\psi,t]\right\}\end{aligned} \tag{1.59}$$

First term provides the Liouville part and it represents the generator corresponding to

$$\frac{d}{dt}\psi(t) = -iG(\psi(t)) \tag{1.60}$$

where $G(\psi) = \hat{H}\psi + \frac{i}{2}\sum_i \gamma_i \|A_i\psi\|^2\psi$ and $\hat{H} = H - \frac{i}{2}\sum_i \gamma_i A_i^\dagger A_i$.
The second part of the Eq. (1.59) describes the discontinuous jumps of the wave function. The gain term and loss terms gives the total rate of all transitions from one to another state. The transition rate is defined as

$$W[\psi|\tilde{\psi}] = \sum_i \gamma_i \left\|A_i\tilde{\psi}\right\|^2 \delta\left\{\frac{A_i\tilde{\psi}}{\|A_i\tilde{\psi}\|} - \psi\right\} \tag{1.61}$$

**A PDP defined by Liouvile master equation of the form (1.59) formulated in terms of the stochastic differential equation is given by**

$$d\psi(t) = -iG(\psi(t))dt + \sum_i \left(\frac{A_i\psi(t)}{\|A_i\psi(t)\| - \psi(t)}\right)dN_i(t) \tag{1.62}$$

where the $dN_i(t)$ satisfy:

$$\begin{aligned} dN_i(t)dN_j(t) &= \delta_{ij}dN_j(t) \\ E[dN_i(t)] &= \gamma_i\|A_i\psi(t)\|^2 dt\end{aligned} \tag{1.63}$$

### 1.2.5 Quantum Trajectory Dynamics: A Marked-Point Process Formulation

We describe the quantum trajectory dynamics based on the Schrödinger equation with a non-Hermitian Hamiltonian. Starting from an initial state $|\psi(0)$ at $t = 0$, the process proceeds as follows:

#### Non-Hermitian Schrödinger Evolution

The state evolves according to the Schrödinger equation with the effective non-Hermitian Hamiltonian $H_{\text{eff}}$:

$$\frac{d}{dt}|\psi(t)\rangle = -iH_{\text{eff}}|\psi(t)\rangle, \tag{1.64}$$

where $H_{\text{eff}}$ is defined later.

### Waiting Time Determination

The evolution continues until a waiting time $t = \tau$, determined by the following condition:

$$e^{-iH_{\text{eff}}\tau}|\psi(t)\rangle = \eta, \tag{1.65}$$

where $\eta$ is randomly chosen from a uniform distribution in the interval $[0, 1]$.

### Quantum Jump Process

At $t = \tau$, a quantum jump occurs, governed by the Lindblad operator $L_j$. The probability $p_j$ of a jump at site $j$ is given by:

$$p_j = \frac{\gamma_j \|L_j|\psi(\tau)\rangle\|^2}{\sum_k \gamma_k \|L_k|\psi(\tau)\rangle\|^2}. \tag{1.66}$$

Following the jump, the state is updated as:

$$|\psi(\tau)\rangle \to \frac{L_j|\psi(\tau)\rangle}{\|L_j|\psi(\tau)\rangle\|}. \tag{1.67}$$

This process is then repeated, with the updated state from Eq. (1.67) replacing the initial state.

### Waiting Time Distribution

From Eq. (1.65), the distribution of waiting times for quantum jumps (or projective measurements) is determined by the norm of the state under nonunitary evolution. This norm does not generally ensure a uniform distribution.

### Continuous Measurement in Space

When the measurement rate $\gamma_j$ is uniform across all sites, i.e., $\gamma_j = \gamma$, the non-Hermitian Hamiltonian takes the form:

$$H_{\text{eff}} = H - i\frac{\gamma}{2}\sum_{j=1}^{L} n_j, \tag{1.68}$$

where $n_j$ is the particle number operator at site $j$.

Upon normalization of the state norm, this evolution becomes equivalent to unitary dynamics with an effective Hermitian Hamiltonian:

$$H_{\text{eff}} = H - iL\frac{\gamma\nu}{2}, \tag{1.69}$$

where $\nu = \frac{n_{\text{tot}}}{L}$ is the filling factor, and $n_{\text{tot}} = \sum_{j=1}^{L} n_j$ is the conserved total particle number.

### Constant Measurement Rate

Using Eq. (1.65) and Eq. (1.69), the waiting time for projective measurements is determined to follow:

$$\tau = -\frac{2\ln\eta}{L\gamma\nu}, \tag{1.70}$$

showing that projective measurements occur at a constant rate.

# Chapter 2

# Hamiltonian

## 2.1 Analysis of the Hamiltonian

The Hamiltonian under consideration is given by:

$$H = \sum_{j=1}^{L} \left[ \frac{J}{2} \left( b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right) + V n_j n_{j+1} + \frac{J'}{2} \left( b_j^\dagger b_{j+2} + b_{j+2}^\dagger b_j \right) \right], \tag{2.1}$$

where:

- $b_j$ ($b_j^\dagger$) are the bosonic annihilation (creation) operators acting on site $j$,
- $n_j = b_j^\dagger b_j$ is the number operator,
- $J$ and $J'$ represent the nearest-neighbor and next-nearest-neighbor hopping amplitudes, respectively,
- $V$ is the strength of the nearest-neighbor interaction, and
- $L$ is the system size under periodic boundary conditions: $b_{L+j} \equiv b_j$.

The bosonic operators are subject to the hard-core constraint:

$$(b_j)^2 = 0, \quad (b_j^\dagger)^2 = 0, \tag{2.2}$$

which ensures that each site can be occupied by at most one particle. This constraint effectively maps the bosonic operators to spin-1/2 operators or fermionic operators under certain transformations.

### 2.1.1 Special Cases of the Hamiltonian

Nearest-Neighbor Hopping Only ($J' = 0$):

$$H = \sum_{j=1}^{L} \left[ \frac{J}{2} \left( b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right) + V n_j n_{j+1} \right]. \tag{2.3}$$

This case corresponds to the well-known integrable XXZ chain.

Nearest-Neighbor Interaction Only ($J = J' = 0$): The system reduces to a classical nearest-neighbor interaction model:

$$H = \sum_{j=1}^{L} V n_j n_{j+1}. \tag{2.4}$$

No Interaction ($V = 0$): For $V = 0$, the Hamiltonian reduces to a free hopping model with nearest and next-nearest-neighbor hopping:

$$H = \sum_{j=1}^{L} \left[ \frac{J}{2} \left( b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right) + \frac{J'}{2} \left( b_j^\dagger b_{j+2} + b_{j+2}^\dagger b_j \right) \right]. \tag{2.5}$$

### 2.1.2 Symmetries of the Model

The Hamiltonian possesses several important symmetries, which we detail below.

#### Particle Number Conservation

The Hamiltonian conserves the total particle number:

$$N = \sum_{j=1}^{L} n_j, \tag{2.6}$$

due to the commutation relation:

$$[H, N] = 0. \tag{2.7}$$

This arises because each term in $H$ either involves hopping processes that conserve particle number or interactions dependent on $n_j$, which does not change the particle count.

#### Translational Symmetry

Under periodic boundary conditions ($b_{L+j} \equiv b_j$), the Hamiltonian is invariant under a translation of all indices by one site:

$$j \to j + 1 \mod L. \tag{2.8}$$

This implies the system is translationally symmetric, with the generator of translations being the shift operator:

$$T : b_j \to b_{j+1}. \tag{2.9}$$

The commutation relation $[H, T] = 0$ holds, indicating that momentum is a conserved quantity in the system.

#### Parity Symmetry

For $J = J'$, the Hamiltonian is invariant under spatial inversion (parity symmetry), where site indices are reversed:

$$j \to L - j + 1. \tag{2.10}$$

Mathematically, parity symmetry means:

$$H = PHP^{-1}, \quad Pb_jP^{-1} = b_{L-j+1}. \tag{2.11}$$

### Time-Reversal Symmetry

The Hamiltonian is time-reversal symmetric if there are no complex coefficients in the hopping amplitudes $(J, J')$ or interaction strength $(V)$. Time-reversal symmetry implies:

$$H = \mathcal{T}H\mathcal{T}^{-1}, \tag{2.12}$$

where $\mathcal{T}$ is the time-reversal operator that complex conjugates the Hamiltonian.

### Integrability and Non-Integrability

For $J' = 0$, the Hamiltonian reduces to the XXZ chain, which is integrable. A nonzero $J'$ introduces next-nearest-neighbor hopping, breaking integrability.

## 2.1.3 Mapping the Hard-Core Bosonic Hamiltonian to the XXZ Chain

We start with the hard-core bosonic Hamiltonian for the nearest-neighbor case ($J' = 0$):

$$H = \sum_{j=1}^{L} \left[ \frac{J}{2} \left( b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right) + V n_j n_{j+1} \right], \tag{2.13}$$

where $b_j$ ($b_j^\dagger$) are bosonic annihilation (creation) operators subject to the **hard-core constraint**:

$$b_j^2 = 0, \quad (b_j^\dagger)^2 = 0. \tag{2.14}$$

The number operator is defined as:

$$n_j = b_j^\dagger b_j. \tag{2.15}$$

The goal is to map this Hamiltonian to a spin-1/2 system, specifically the **XXZ chain**. This is achieved by identifying the hard-core bosonic operators with spin-1/2 operators.

—

### Spin-1/2 Representation of Hard-Core Bosons

For hard-core bosons, we introduce the following correspondences to spin-1/2 operators:

$$b_j \to S_j^-, \quad b_j^\dagger \to S_j^+, \quad n_j \to S_j^z + \frac{1}{2}, \tag{2.16}$$

where:

- $S_j^+$ ($S_j^-$) are the spin raising (lowering) operators defined as:

$$S_j^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_j^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.17}$$

- $S_j^z$ is the spin $z$-component operator defined as:

$$S_j^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.18}$$

The commutation relations of these spin operators are:

$$[S_j^z, S_k^\pm] = \pm\delta_{jk} S_k^\pm, \quad [S_j^+, S_k^-] = 2\delta_{jk} S_j^z. \tag{2.19}$$

The hard-core bosonic constraint $b_j^2 = 0$ translates naturally to the spin operators since:

$$(S_j^-)^2 = (S_j^+)^2 = 0. \tag{2.20}$$

—

## Mapping the Hamiltonian Terms

### (a) Hopping Terms

The hopping terms in the Hamiltonian are:

$$\frac{J}{2}\left(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j\right). \tag{2.21}$$

Using the correspondence $b_j^\dagger \to S_j^+$ and $b_j \to S_j^-$, this becomes:

$$\frac{J}{2}\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right). \tag{2.22}$$

The operators $S_j^+ S_{j+1}^-$ and $S_j^- S_{j+1}^+$ correspond to spin-flip processes between neighboring sites $j$ and $j+1$. These terms collectively represent the $XY$-coupling in the spin language.

### (b) Interaction Terms

The interaction terms in the Hamiltonian are:

$$V n_j n_{j+1}. \tag{2.23}$$

Substituting $n_j \to S_j^z + \frac{1}{2}$, this becomes:

$$n_j n_{j+1} \to \left(S_j^z + \frac{1}{2}\right)\left(S_{j+1}^z + \frac{1}{2}\right). \tag{2.24}$$

Expanding the product:

$$n_j n_{j+1} \to S_j^z S_{j+1}^z + \frac{1}{2}S_j^z + \frac{1}{2}S_{j+1}^z + \frac{1}{4}. \tag{2.25}$$

The constant term $\frac{1}{4}$ contributes an overall energy shift and can be ignored for simplicity. Thus, the

interaction term becomes:

$$Vn_jn_{j+1} \to VS_j^zS_{j+1}^z + \frac{V}{2}\left(S_j^z + S_{j+1}^z\right). \tag{2.26}$$

—

### Combined Hamiltonian in Spin-1/2 Representation

Combining the hopping and interaction terms, the Hamiltonian becomes:

$$H = \sum_{j=1}^{L}\left[\frac{J}{2}\left(S_j^+S_{j+1}^- + S_j^-S_{j+1}^+\right) + VS_j^zS_{j+1}^z + \frac{V}{2}\left(S_j^z + S_{j+1}^z\right)\right]. \tag{2.27}$$

The term $\frac{V}{2}\left(S_j^z + S_{j+1}^z\right)$ represents a boundary term and can often be ignored for periodic boundary conditions, reducing the Hamiltonian to:

$$H = \sum_{j=1}^{L}\left[\frac{J}{2}\left(S_j^+S_{j+1}^- + S_j^-S_{j+1}^+\right) + VS_j^zS_{j+1}^z\right]. \tag{2.28}$$

The hard-core bosonic operators can be mapped to spin-1/2 operators via the following correspondence: We map the hard-core bosonic operators $b_j$ and $b_j^\dagger$ to spin-1/2 operators using the following correspondences:

$$b_j \to S_j^-, \quad b_j^\dagger \to S_j^+, \quad n_j \to S_j^z + \frac{1}{2}, \tag{2.29}$$

where $S_j^+$, $S_j^-$, and $S_j^z$ are the spin-1/2 raising, lowering, and $z$-component operators, respectively. Substituting these into the Hamiltonian (for $J' = 0$), we get:

$$H_{\text{XXZ}} = \sum_{j=1}^{L}\left[J\left(S_j^xS_{j+1}^x + S_j^yS_{j+1}^y\right) + VS_j^zS_{j+1}^z\right], \tag{2.30}$$

where $S_j^x$ and $S_j^y$ are the spin-1/2 $x$- and $y$-direction operators:

$$S_j^x = \frac{1}{2}\left(S_j^+ + S_j^-\right), \quad S_j^y = \frac{i}{2}\left(S_j^- - S_j^+\right). \tag{2.31}$$

This is the standard form of the XXZ chain, characterized by:

- Nearest-neighbor $XY$-type spin exchange,
- Nearest-neighbor $z$-axis spin interaction proportional to $V$.

### 2.1.4 Jordan-Wigner Transformation and Mapping to Fermions

For $V = 0$, the hard-core bosons can be mapped to fermions via the Jordan-Wigner transformation:

$$b_j \to c_j\prod_{k=1}^{j-1}e^{-i\pi n_k}, \quad b_j^\dagger \to c_j^\dagger\prod_{k=1}^{j-1}e^{i\pi n_k}, \tag{2.32}$$

$$n_j \to c_j^\dagger c_j, \tag{2.33}$$

where $c_j$ and $c_j^\dagger$ are fermionic annihilation and creation operators satisfying:

$$\{c_j, c_k^\dagger\} = \delta_{jk}, \quad \{c_j, c_k\} = 0. \tag{2.34}$$

Applying the Jordan-Wigner transformation to the hopping terms in the Hamiltonian:

$$b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \to c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j, \tag{2.35}$$

we map the Hamiltonian to a free fermion model:

$$H = \sum_{j=1}^{L} \left[ \frac{J}{2} \left( c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + \frac{J'}{2} \left( c_j^\dagger c_{j+2} + c_{j+2}^\dagger c_j \right) \right]. \tag{2.36}$$

### 2.1.5 Boundary Conditions and Fermionic Phase Factor

The Jordan-Wigner string introduces a fermionic phase factor for periodic boundary conditions:

$$c_L^\dagger c_1 + c_1^\dagger c_L \to (-1)^N \left( c_L^\dagger c_1 + c_1^\dagger c_L \right), \tag{2.37}$$

where $N$ is the total number of particles. For an even number of particles, the boundary conditions remain periodic; for an odd number, they become anti-periodic.

# Chapter 3

# Numerical Simulation, entanglement behaviour and algorithms

Quantum trajectory approach has been used for continuous measurements in the system described by the Hamiltonian for boson chain. The behaviour of entanglement and correlation is captured using the numerical simulation of the model. The numerical results focuses on the spatiotemporal evolution of particle density, entanglement entropy profiles, mutual information, correlation functions, and scaling behaviors. These findings provide insights into the critical dynamics and potential implications for quantum error correction thresholds.

## 3.1 Introduction

Measurement-induced entanglement transitions have gained significant attention due to their implications for quantum information dynamics and quantum error correction. In this study, we analyze a hard core boson chain under periodic boundary conditions with the following Hamiltonian:

$$H = J\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} + \text{h.c.}\right) + J'\sum_{j=1}^{L}\left(c_j^\dagger c_{j+2} + \text{h.c.}\right) + V\sum_{j=1}^{L} n_j n_{j+1}, \tag{3.1}$$

where $J$ and $J'$ are the hopping amplitude, $V$ is the interaction strength, $c_j^\dagger(c_j)$ are the fermionic creation (annihilation) operators, and $n_j = c_j^\dagger c_j$ is the number operator.

The system is initialized in the Néel state:

$$|\psi(0)\rangle = |0101\ldots01\rangle, \tag{3.2}$$

corresponding to half-filling $\nu = \frac{1}{2}$. Non-unitary time evolution follows projective measurements with a strength $\gamma$. This work probes critical dynamics in both integrable ($J' = 0$) and non-integrable ($V = 0$) regimes.

# Algorithm: Non-Hermitian Time Evolution with Quantum Jumps

This algorithm describes the non-Hermitian evolution of a quantum state on a lattice, incorporating both coherent evolution and quantum jumps.

**Algorithm 1** Time Evolution with Quantum Jumps

**Input:** Lattice size $L$, Hamiltonian parameters $(J, J', V, \gamma)$, initial state $\psi_0$, final time $t_{\text{final}}$, time step $\Delta t$.
Generate basis states $\mathcal{B}$ for the system with specified filling fraction.
Construct Hamiltonian $\hat{H}$:

- Add interaction terms $(V\hat{n}_j\hat{n}_{j+1})$.
- Add nearest-neighbor hopping terms $(J/2)$.
- Add next-nearest-neighbor hopping terms $(J'/2)$.

Construct the effective non-Hermitian Hamiltonian:

$$\hat{H}_{\text{eff}} = \hat{H} - i\frac{\gamma}{2}\sum_j \hat{n}_j$$

Initialize wavefunction $\psi = \psi_0$, time $t = 0$.
Sample first quantum jump time $\tau = -2\ln(r)/(L\gamma\nu)$, where $r$ is a uniform random number, and $\nu$ is the filling fraction.
**while** $t < t_{\text{final}}$ **do**
    **Non-unitary evolution:**

- Update $\psi$ using RK4 integration for $\frac{d\psi}{dt} = -i\hat{H}_{\text{eff}}\psi$.
- Increment time $t \leftarrow t + \Delta t$.

    **if** $t \geq \tau$ **then** ▷ Check for quantum jump
        Normalize $\psi$.
        Compute jump probabilities $\{p_j\}$ based on $\langle\psi|\hat{n}_j|\psi\rangle$.
        Choose jump site $j$ using probabilities $\{p_j\}$.
        Apply jump operator at site $j$: $\psi \leftarrow \hat{L}_j\psi$, renormalize $\psi$.
        Sample next jump time $\tau \leftarrow t + -2\ln(r')/(L\gamma\nu)$.
    **end if**
    Save $\psi$ and $t$.
**end while**
**Output:** Wavefunction evolution $\psi(t)$ and corresponding times $t$.

# Algorithm: Calculation of Entanglement Entropy

This algorithm calculates the von Neumann entropy for a subsystem in a quantum many-body system by computing the reduced density matrix.

**Algorithm 2** Calculation of von Neumann Entropy

**Input:** Wavefunction evolution $\{\psi_t\}$, basis states $\mathcal{B}$, subsystem sites $S$.
**for** each wavefunction $\psi \in \{\psi_t\}$ **do**
    **Step 1: Compute reduced density matrix $\rho_A$.**

- Extract subsystem basis states $\mathcal{B}_S$ by retaining only the sites in $S$ for each state in $\mathcal{B}$.
- Construct a mapping from subsystem basis states to their indices.
- Initialize $\rho_A = 0$, a matrix of size $|\mathcal{B}_S| \times |\mathcal{B}_S|$.
- For each pair of basis states $(\phi_i, \phi_j) \in \mathcal{B} \times \mathcal{B}$:
  - Extract reduced states $\phi_i^{(S)}$ and $\phi_j^{(S)}$ by retaining only subsystem sites.
  - If $\phi_i^{(S)} = \phi_j^{(S)}$, update $\rho_A$:

$$\rho_A[\phi_i^{(S)}, \phi_j^{(S)}] \mathrel{+}= \psi[i] \cdot \psi^*[j]$$

    **Step 2: Compute von Neumann entropy.**

- Compute eigenvalues $\{\lambda_k\}$ of $\rho_A$.
- Exclude eigenvalues below a small threshold $\epsilon$ to avoid numerical errors.
- Compute entropy:

$$S = -\sum_k \lambda_k \log \lambda_k$$

    Append $S$ to the list of entropies.
**end for**
**Output:** Array of entropies $\{S_t\}$ for each time step $t$.

## 3.2 Benchmarking with the Hermitian Case

The $\langle n_1(t) \rangle$ is measured with the hamiltonian with unitary evolution using terrorization for the hermitian case and RK4 for the $\gamma \to 0$ non-hermitian case. Simulation is carried out for $t = 100$ for each case with time slice of $\Delta t = 0.01$ for RK4 for 100 trajectories.

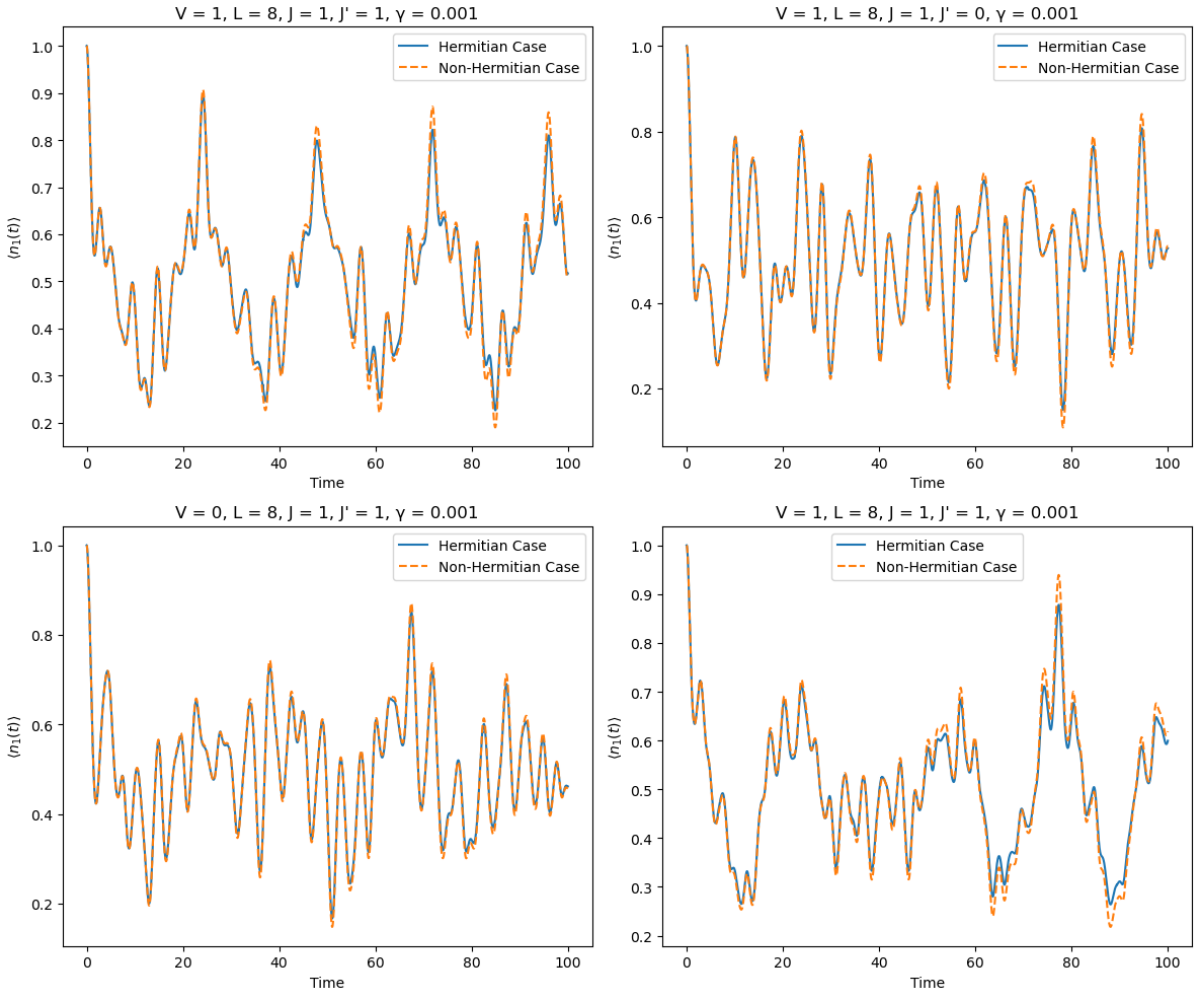


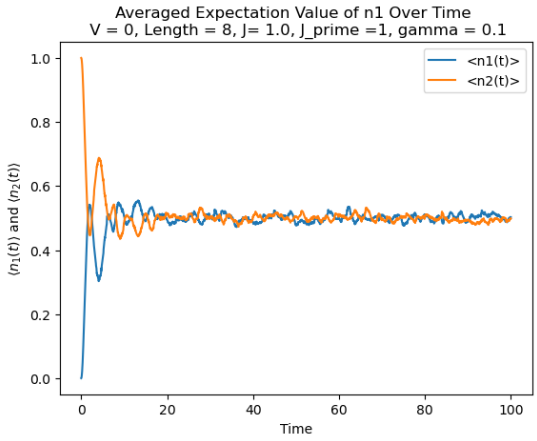

Figure 3.1: Hermitian Benchmarking

## 3.3 Numerical Methods and Results

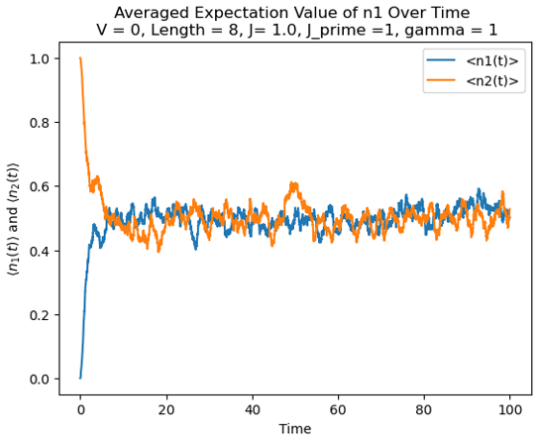
Time evolution is implemented using the fourth-order Runge-Kutta (RK4) method with a time slice $\Delta t = 0.01$. Simulations are carried out for up to $t = 100$ in nonintegrable cases and $t = 200$ in integrable cases. The results are averaged over 400–800 realizations of quantum trajectories.

Key observables include:

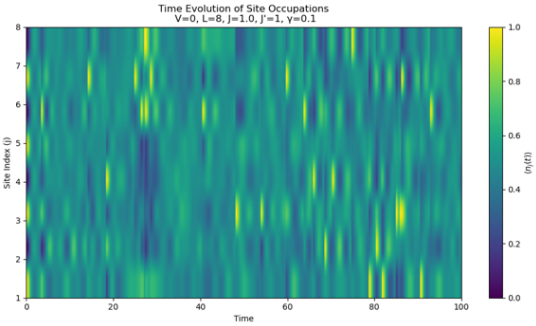
1. **Particle Density Profiles:** Evolution of $n_j(t) = \langle \psi(t)|n_j|\psi(t)\rangle$. The spatiotemporal evolution of particle density $n_j(t)$ shows relaxation to $\nu = \frac{1}{2}$ under weak measurements ($\gamma$). At larger $\gamma$, localization effects dominate, affecting density profiles across the chain. The system is simulated for both integrable and non-integrable regime. Particle densities clearly averaged over quantum trajectories converges to 1/2 for a late time, independent of the initial value.



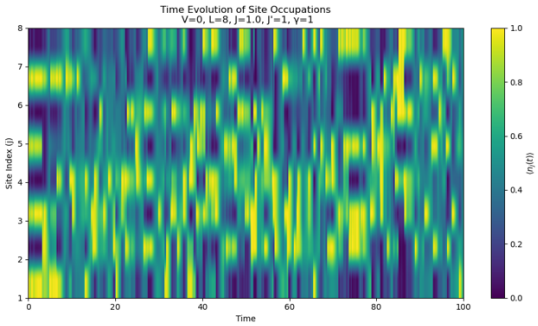

(a) Time evolution over 100 trajectories of site 1 and 2 for $\gamma = 0.1$




(b) Time evolution over 100 trajectories of site 1 and 2 for $\gamma = 1$

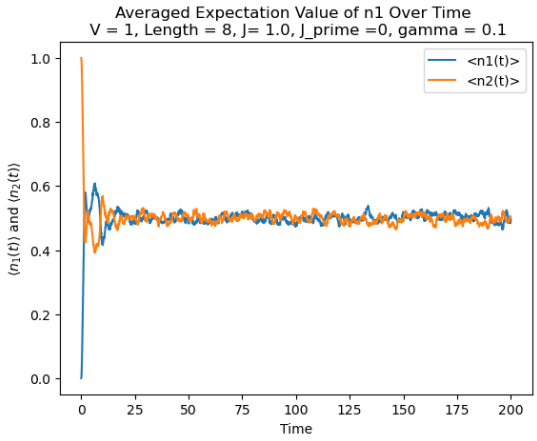



(c) Time evolution of single trajectory for $\gamma = 0.1$

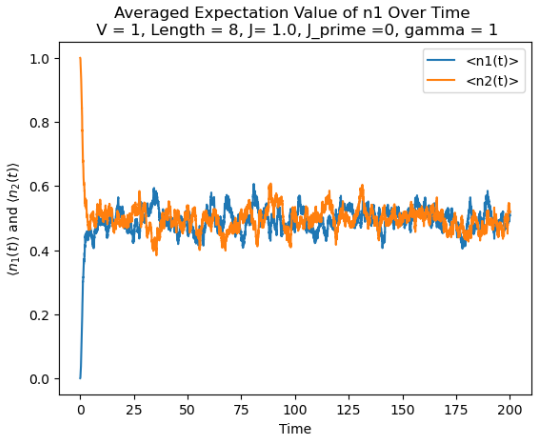



(d) Time evolution of single trajectory for $\gamma = 1$

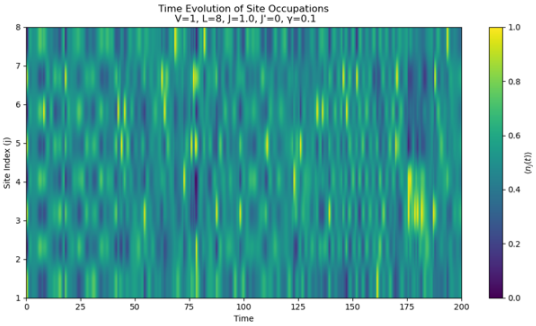
Figure 3.2: Time evolution for NonIntegrable Model $V = 0$ and $J' = 1$ of local particle density for L = 8.



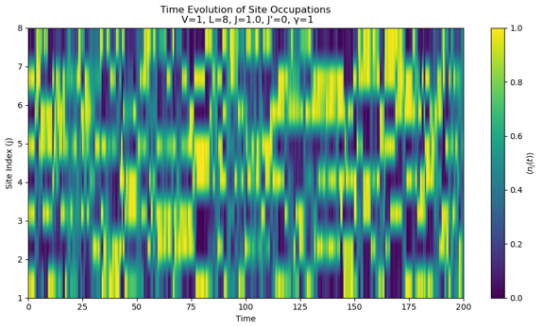

(a) Time evolution over 100 trajectories of site 1 and 2 for $\gamma = 0.1$




(b) Time evolution over 100 trajectories of site 1 and 2 for $\gamma = 1$




(c) Time evolution of single trajectory for $\gamma = 0.1$




(d) Time evolution of single trajectory for $\gamma = 1$

Figure 3.3: Time evolution for Integrable Model $V = 1$ and $J' = 0$ of local particle density for L = 8.

The steady state values of $\sqrt{\langle n_j \rangle}$ over appears to saturate around 0.7 for our case. The steady value is same irrespective of the particle position j, this indicates that there is no memory of the initial Neel state in the configuration now. Thus, the probability distribution of the projective measurement is uniform in space for an averaged late time of the system.

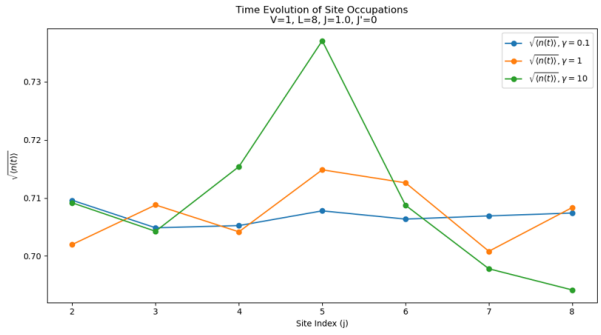



Figure 3.4: Steady state averaged $\sqrt{\langle n_j \rangle}$ for L = 8

The system described by random projective measurement is considered not very far away from unitary dynamics. This marks the **spatial translatin invariance** of avearaged-steady state quantities.

2. **Entanglement Entropy:** The von Neumann entropy $S_A(t)$ of a subsystem $A$.

The von Neumann entanglement entropy:

$$S_A(t) = -\mathrm{Tr}_A[\rho_A(t) \ln \rho_A(t)], \tag{3.3}$$

quantifies quantum correlations in subsystem $A$. Steady-state entropy transitions from volume-law scaling ($\gamma < \gamma_c$) to area-law scaling ($\gamma > \gamma_c$), highlighting the measurement-induced entanglement transition.

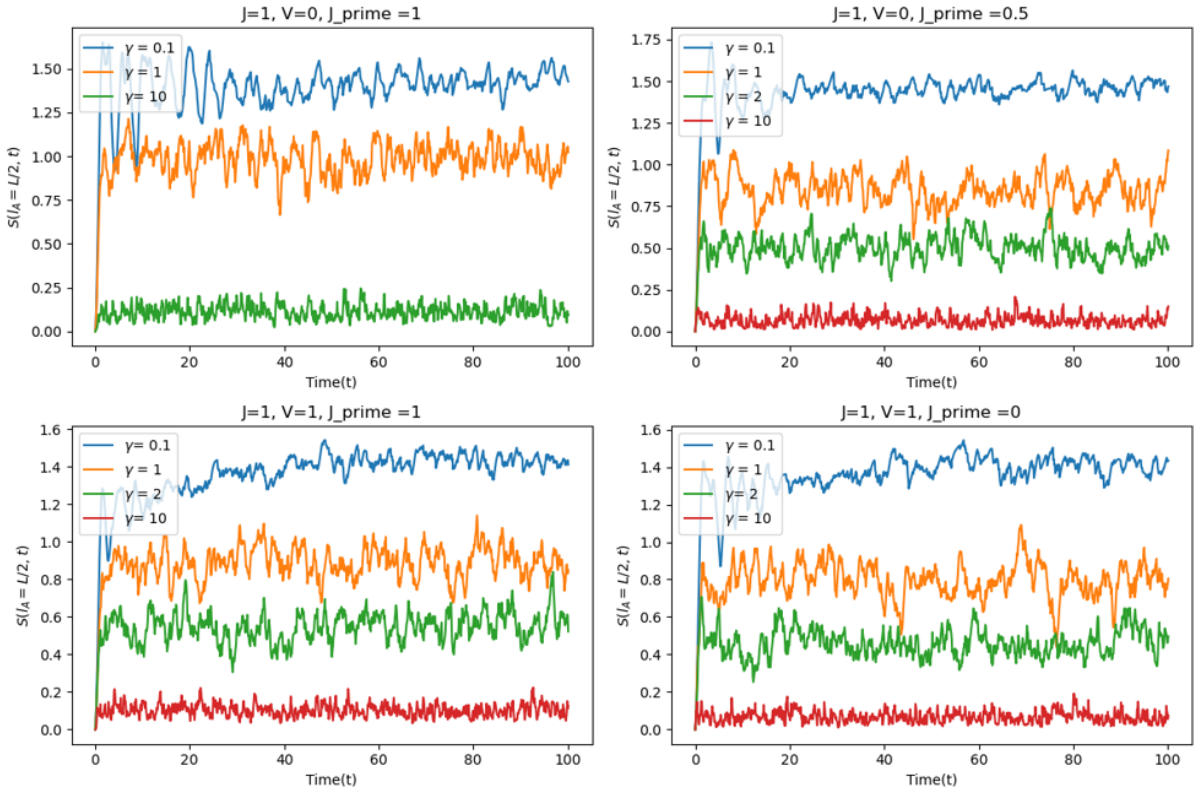



Figure 3.5: Time evolution of von Neumann entanglement entropies over 25 trajectories for various $\gamma$ values of length 6

Due to translational invariance of $S_A(t)$ over the trajectories, length of $A$ can be chosen without loss of generality. The entanglement entropy is zero to start with , the averaged value increases linearly with time and saturates to steady value in late time. It also decreases with increasing measurement strength $\gamma$ for both integrable and non integrable model.

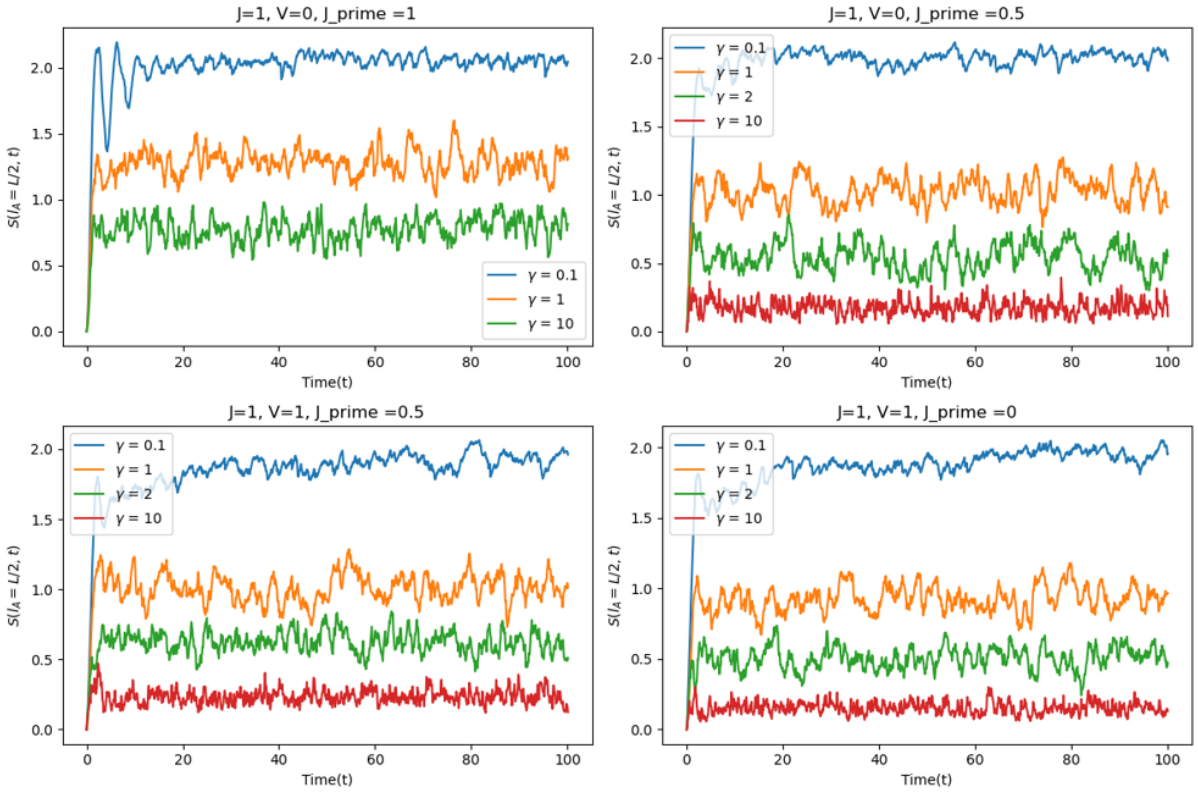
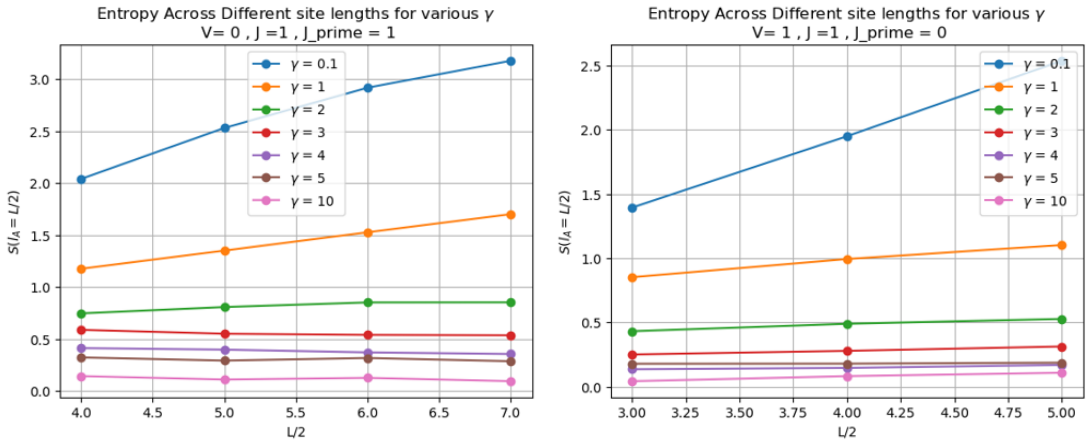



Figure 3.6: Time evolution of von Neumann entanglement entropies over 25 trajectories for various $\gamma$ values of length 8




(a) For the non-integrable model $V = 0\,,\ J' = 1$

(b) For the integrable model $V = 1\,,\ J' = 0$

Figure 3.7: Steady state values of von Neumann entanglement entropies$S(l_A = L/2)$ for half the system size for various $\gamma$ strength.

For both integrable and non integrable system, the entanglement entropy increases with system size linearly for small values of measurement strength $\gamma$. Frequent measurement for large $\gamma$ supresses the volume-law entanglement expected for thermalising state to area - law entanglement. It is difficult to estimate critial $\gamma$, mutual information can be used to estimate to quantify the critical strength of measurement. The result however indicates entanglement entropy scales logarithmically as $\ln L$.

3. **Mutual Information:** Quantifies correlations between antipodal subsystems.

   The von Neumann mutual information is:

$$I_{AB}(t) = S_A(t) + S_B(t) - S_{A\cup B}(t), \tag{3.4}$$

   which measures correlations between antipodal subsystems. Peaks in mutual information indicate criticality associated with the entanglement transition.

4. **Scaling Behavior:** Analyzes steady-state entanglement entropy as a function of subsystem size.

The steady-state entanglement entropy $S(l_A)$ exhibits logarithmic scaling:

$$S(l_A) = \alpha_S \ln x_A + \beta_S, \tag{3.5}$$

where $x_A = \frac{L}{\pi} \sin\left(\frac{\pi l_A}{L}\right)$ is the chord length. This behavior confirms conformal invariance at the critical point.

# Chapter 4

# Appendix

## 4.1 Python Code for Hamiltonian, Quantum Jump Dynamics, and Entropy Calculation

In this appendix, we provide the Python code used for constructing the Hamiltonian, simulating quantum jump dynamics, and calculating the reduced density matrix and entropy. The code is written in Python and is designed for a system with periodic boundary conditions.

### 4.1.1 Hamiltonian Construction

The following code constructs the Hamiltonian for a one-dimensional system with nearest-neighbor and next-nearest-neighbor interactions at half-filling.

```python
import numpy as np
from scipy.sparse import lil_matrix
from itertools import combinations

def basis_states(L, filling_fraction=0.5):
    """Generate all basis states with the specified filling fraction."""
    num_particles = int(L * filling_fraction)  # Half filling
    states = []
    for particle_sites in combinations(range(L), num_particles):
        state = np.zeros(L, dtype=int)
        state[list(particle_sites)] = 1
        states.append(state)
    return np.array(states)

def index_of_state(state, basis):
    """Find the index of a state in the basis array."""
    for idx, basis_state in enumerate(basis):
        if np.array_equal(state, basis_state):
            return idx
    return None

def construct_hamiltonian(L, J, V, J_prime, filling_fraction=0.5):
```

```
    """Construct the Hamiltonian matrix."""
    basis = basis_states(L, filling_fraction)
    num_states = len(basis)
    H = lil_matrix((num_states, num_states), dtype=complex)

    for i, state in enumerate(basis):
        for j in range(L):  # On-site interaction
            n_j = state[j]
            n_j1 = state[(j + 1) % L]  # Periodic boundary condition
            H[i, i] += V * n_j * n_j1

        for j in range(L):  # Nearest-neighbor hopping
            if state[j] == 1 and state[(j + 1) % L] == 0:
                new_state = state.copy()
                new_state[j] = 0
                new_state[(j + 1) % L] = 1
                new_index = index_of_state(new_state, basis)
                if new_index is not None:
                    H[i, new_index] += J / 2
                    H[new_index, i] += J / 2  # Hermitian conjugate

        for j in range(L):  # Next-nearest-neighbor hopping
            if state[j] == 1 and state[(j + 2) % L] == 0:
                new_state = state.copy()
                new_state[j] = 0
                new_state[(j + 2) % L] = 1
                new_index = index_of_state(new_state, basis)
                if new_index is not None:
                    H[i, new_index] += J_prime / 2
                    H[new_index, i] += J_prime / 2  # Hermitian conjugate

    return H.tocsr(), basis
```

Listing 4.1: Hamiltonian Construction

### 4.1.2 Quantum Jump Dynamics

This section describes the quantum jump dynamics, including time evolution with an effective non-Hermitian Hamiltonian and stochastic quantum jumps.

```
from scipy.sparse import csr_matrix
import numpy as np

def construct_effective_hamiltonian(H, L, gamma, filling_fraction=0.5):
    """Construct the effective non-Hermitian Hamiltonian H_eff."""
    nu = filling_fraction  # Filling fraction
    decay_term = -1j * gamma * L * nu / 2  # Non-Hermitian decay term
    H_eff = H + decay_term * csr_matrix(np.identity(H.shape[0], dtype=complex))
    return H_eff

def schrodinger_rhs(psi, H_eff):
```

```
    """Calculate the RHS of the Schr dinger equation with H_eff."""
    return -1j * H_eff.dot(psi)

def rk4_step(psi, H_eff, dt):
    """Perform a single RK4 step for non-unitary evolution."""
    k1 = dt * schrodinger_rhs(psi, H_eff)
    k2 = dt * schrodinger_rhs(psi + 0.5 * k1, H_eff)
    k3 = dt * schrodinger_rhs(psi + 0.5 * k2, H_eff)
    k4 = dt * schrodinger_rhs(psi + k3, H_eff)
    return psi + (k1 + 2 * k2 + 2 * k3 + k4) / 6

def quantum_jump(psi, basis, gamma, L):
    """Perform a quantum jump."""
    jump_probabilities = np.zeros(L)
    for j in range(L):
        n_j = np.array([state[j] for state in basis])
        jump_probabilities[j] = np.abs(psi @ n_j) ** 2

    total_prob = np.sum(jump_probabilities)
    if total_prob > 0:
        jump_probabilities /= total_prob
        jump_site = np.random.choice(range(L), p=jump_probabilities)
        psi = np.array([basis[i][jump_site] for i in range(len(basis))])
        psi /= np.linalg.norm(psi)
    return psi

def solve_schrodinger_with_jumps(psi0, H_eff, t_final, dt, gamma, L):
    """Solve the Schr dinger equation with quantum jumps."""
    psi = psi0
    times = [0]
    psi_t = [psi0]
    t = 0
    while t < t_final:
        psi = rk4_step(psi, H_eff, dt)
        if np.random.rand() < gamma * dt:
            psi = quantum_jump(psi, basis_states(L))
        psi_t.append(psi)
        times.append(t)
        t += dt
    return np.array(times), np.array(psi_t)
```

Listing 4.2: Quantum Jump Dynamics

### 4.1.3 Reduced Density Matrix and Entropy Calculation

The reduced density matrix and the von Neumann entropy for a subsystem are calculated as follows:

```
from scipy.linalg import logm

def reduced_density_matrix(psi, basis, subsystem_sites):
    """Calculate the reduced density matrix for a subsystem."""
```

```
    dim_sub = 2 ** len(subsystem_sites)
    rho_A = np.zeros((dim_sub, dim_sub), dtype=complex)
    for i, state_i in enumerate(basis):
        for j, state_j in enumerate(basis):
            if np.all(state_i[subsystem_sites] == state_j[subsystem_sites]):
                idx_i = int("".join(map(str, state_i[subsystem_sites])), 2)
                idx_j = int("".join(map(str, state_j[subsystem_sites])), 2)
                rho_A[idx_i, idx_j] += psi[i] * np.conj(psi[j])
    return rho_A

def von_neumann_entropy(rho):
    """Calculate the von Neumann entropy."""
    eigenvalues = np.linalg.eigvalsh(rho)
    eigenvalues = eigenvalues[eigenvalues > 1e-10]  # Exclude small eigenvalues
    return -np.sum(eigenvalues * np.log(eigenvalues))
```

Listing 4.3: Reduced Density Matrix and Entropy Calculation

# Bibliography

[BP95a] Heinz-Peter Breuer and Francesco Petruccione. “On a Liouville-master equation formulation of open quantum systems”. In: *Zeitschrift für Physik B Condensed Matter* 98 (1995), pp. 139–145.

[BP95b] Heinz-Peter Breuer and Francesco Petruccione. “Stochastic dynamics of quantum jumps”. In: *Physical Review E* 52.1 (1995), p. 428.