Keywords are posted here!

Learning under Cryptographic Hardness

By: Anish Jayant

Cryptographic Assumption

Is there an efficient protocol that is hard to break?

$\Downarrow$ [Shannon, 1946] $\qquad\qquad$ $\Downarrow$ “computer science”

Is there a relaxation of security that is hard in poly-time?

For modern cryptography to be possible…

Definition (One-way function): A poly-time computable $f: \mathcal{X} \to \mathcal{Y}$ such that for all $\mathcal{A} \in PPT$,

$$\Pr[f(\mathcal{A}(f(x))) = f(x)] < \frac{1}{poly(|x|)}$$

Throughout the talk, we assume one-way functions exist $\Rightarrow P \neq NP, RP \neq NP$

Cryptographic Assumption

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Definition (One-way function): A poly-time computable $f: \mathcal{X} \to \mathcal{Y}$ such that for all $\mathcal{A} \in PPT$,

$$\Pr\left[f\left(\mathcal{A}\left(f(x)\right)\right) = f(x)\right] < \frac{1}{poly(|x|)}$$

implies robustness to interactive attacker! (think about public-key…)

Has access to one-way f

Alice
$x \sim \mathcal{U}(\mathcal{X})$

Hacker

Has access to any $\mathcal{A} \in PPT$

cipher: $f(x) \in \mathcal{Y}$

Views cipher requests data

request $x_i{}'$

repeat n times
$n = poly(|x|)$

Labels data

return $(x_i', f(x_i'))$

Runs $\mathcal{A}(f(x)|x_i, f(x_i))$

Tries to guess a pre-image

message?: $\hat{x}$

Output grade
$f(x) = f(\hat{x})$

Cryptographic Assumption

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$$\Pr\left[f\left(\mathcal{A}\left(f(x)\right)\right) = f(x)\right] < \frac{1}{poly(|x|)}$$

Conjectured Hard Functions

Claim (Discrete Log. is One-Way): Given $f(x) = x^a \ (mod\ N)$, $\Pr[\mathcal{A}(f(x), x, N) = a]$ is small.

Claim (Factoring is One-Way): Given $N = pq$, $\Pr[\mathcal{A}(N) \in \{p, q\}]$ is small.

Key Takeaways:
- All classes are *finite* thus mathematically possible to break (by brute force)
- Erratic: polytime algorithms can't distinguish a *hard-core bit* from pure randomness!

Attacking as Learning

- Property: Even with *best case* dataset, polytime attacker has no edge on f^{-1}: even worse on average!

Question: What does it mean to learn a set of behaviors ?

Is there a set of behaviors that is hard to learn efficiently,
but easy to learn otherwise

↓

One-Way Functions?

↑

Is there a relaxation of security that is hard in poly-time,
but defeated with more compute

Goal Questions

By the end of the talk, we'd like some insight into

- What does it mean to learn a behavior, be learnable?
  - How does 'learning theory' differ from 'statistics'?

- What does it mean to (run-time) efficiently learn?

- Can one-way functions be efficiently learnable?

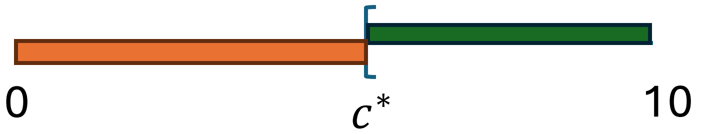
- (**) How do one-way results influence other structures, like Boolean circuits?

Learning a Threshold

Let's look at a simple behavior that *can* be learned…

(English) Setup:
- User adds movies to watchlist if they are *above* some rating
- After observing m many decisions, learn a recommendation

Truth classifier



User Decisions, $m = 6$

Question: Given the decisions, how should we recommend?

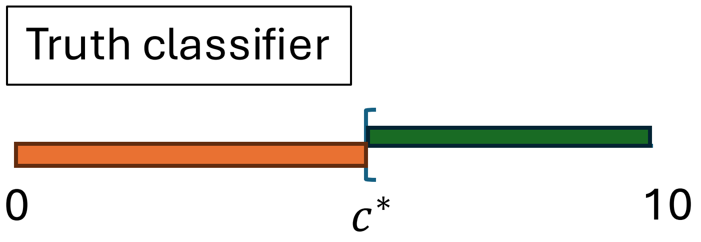
Learning a Threshold

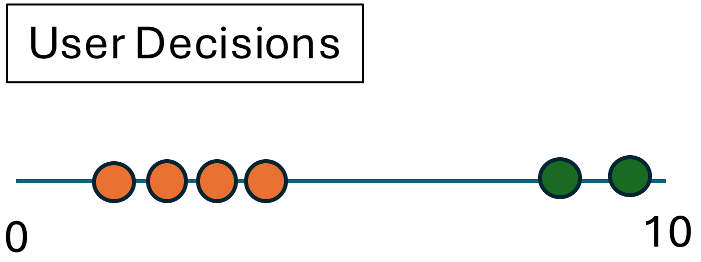
A simple behavior…

(English) Setup:
- User adds movies to watchlist if they are *above* some rating
- After observing m many decisions, learn a recommendation

(Formal) Setup:
- Ratings $\mathcal{X} = [0, 10]$, Outputs $\mathcal{Y} = \{0, 1\}$
- For some c^*, $f(x) = 1(x \geq c^*)$

$$\mathcal{F} = \{ 1(x \geq c) | c \in [0, 10]\}$$





Question: Given the decisions, how should we guess a $f \in \mathcal{F}$?

Learning a Threshold

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- Ratings $\mathcal{X} = [0, 10]$, Outputs $\mathcal{Y} = \{0, 1\}$
- For some c^*, $f(x) = 1(x \geq c^*)$

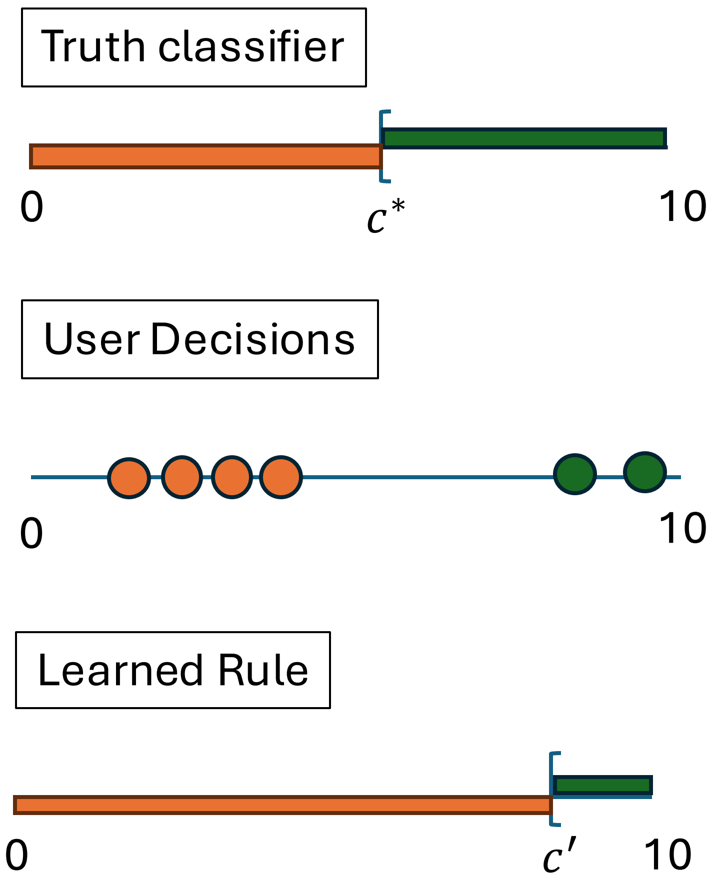
$$\mathcal{F} = \{ 1(x \geq c) | c \in [0, 10]\}$$

Answer: Find the worst watchable movie!

Algorithm

$c' \leftarrow 10$
for $x \in S$:
    if x watched, set $c' \leftarrow x$
return c'

Question
- what is the runtime of ***Algorithm***?
- what is the accuracy on S?
- what is the relationship between c' and c^*?



Learning a Threshold

Question: How does our guess perform?

Algorithm

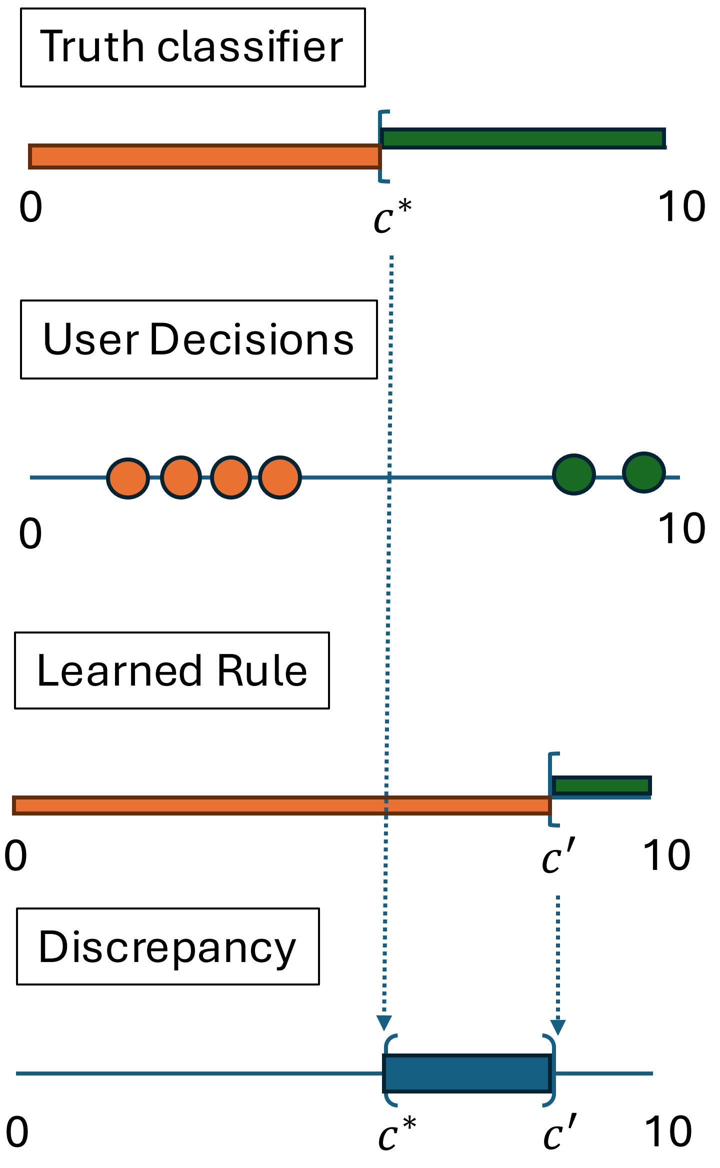
$c' \leftarrow 10$
for $x \in S$:
    if $x \geq c^*$, set $c' \leftarrow x$
return c'

Troublesome region: $[c^*, c']$...

Idea: If $m = |S|$ grows, $[c^*, c']$ becomes less significant...

$$\Pr\left[x \in [c^*, c']\right] \to 0$$

Ideologically, this means we learn!



Learning a Threshold

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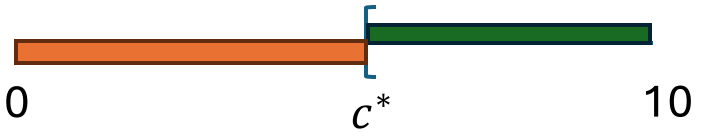
Conjecture: If $m = |S|$ grows, $[c^*, c']$ becomes less significant…

$$\Pr\left[x \in [c^*, c']\right] \to 0$$

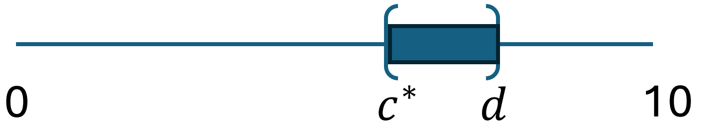
Let d be such that $\Pr\left[x \in [c^*, d]\right] \leq \epsilon$

Claim 1: If we receive *any* sample in $[c^*, d]$, then $\Pr\left[x \in [c^*, c']\right] \leq \epsilon$

Truth classifier



ϵ-tolerance



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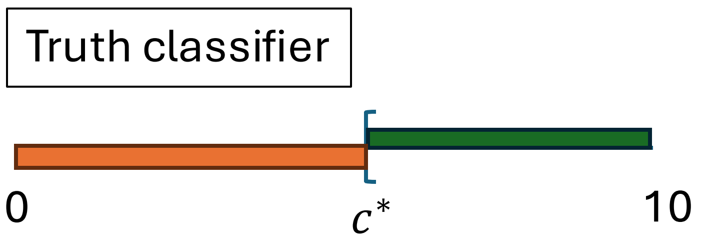
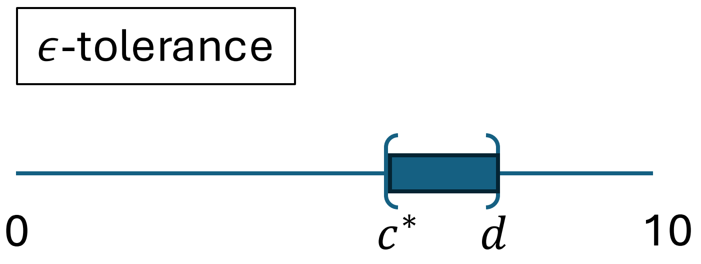
...then $c' \leq d$, so $\Pr\left[x \in [c^*, c']\right] \leq \Pr\left[x \in [c^*, d]\right] \leq \epsilon$





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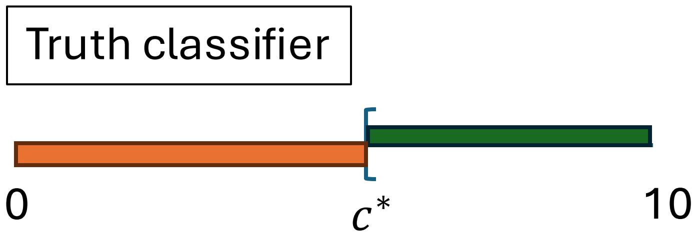
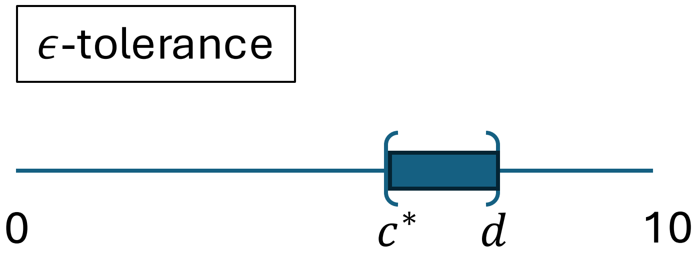
Claim 2: All samples miss with probability $(1-\epsilon)^m \leq e^{-\epsilon m}$

… since samples are i.i.d., the misses compound *exponentially!*





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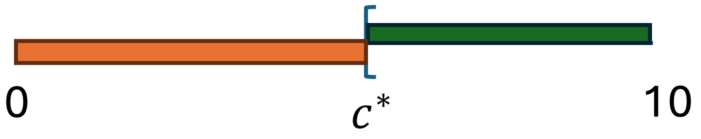
Claim 3: If $m \geq \frac{\log(\frac{1}{\delta})}{\epsilon}$, then we get a bad sample with prob. $\leq \delta$

… just by solving $e^{-\epsilon m} \leq \delta$

Truth classifier



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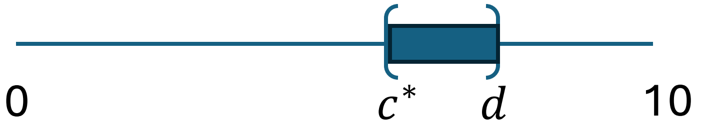

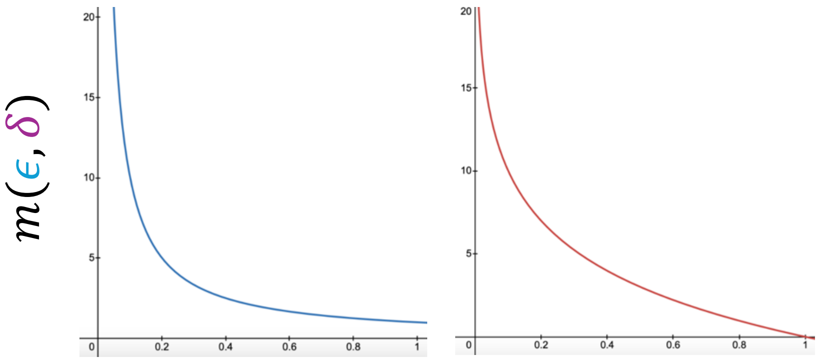

$\epsilon \in [0, 1]$

$\delta = 0.1$

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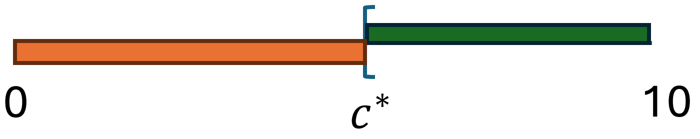
Claim 3: If $m \geq \frac{\log(\frac{1}{\delta})}{\epsilon}$, then we get a bad sample with prob. $\leq \delta$

Conclusion: With $m \geq \log\left(\frac{1}{\delta}\right)/\epsilon$ samples, our $f \in \mathcal{F}$ has error at most ϵ with probability at least $1 - \delta$!

Truth classifier



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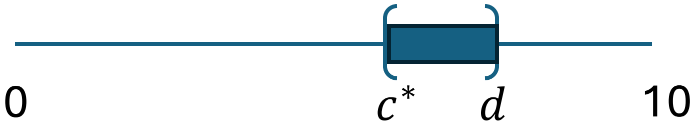


(Proper) Learnability

Setup: input domain $\mathcal{X}$, output $\mathcal{Y} = \{0, 1\}$, **function class** $\boldsymbol{\mathcal{F}} \subset \{\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \{\mathbf{0}, \mathbf{1}\}\}$

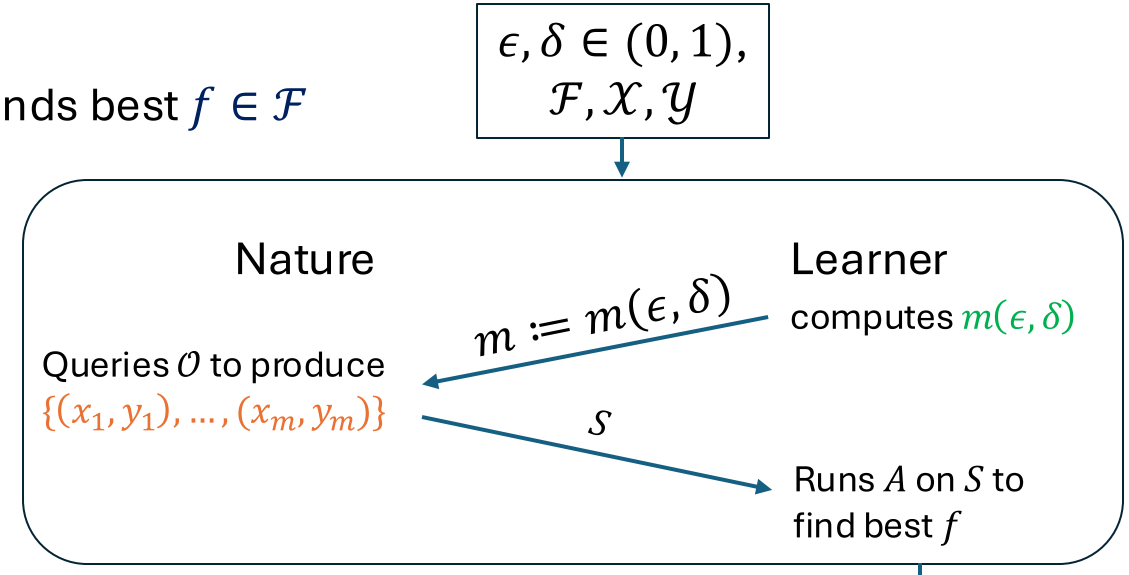
Learner: Computes $m(\epsilon, \delta)$, requests a dataset of that size, and finds best $f \in \mathcal{F}$ for the sample.

Definition (Learnable Class): If, for any $\epsilon, \delta \in (0, 1)$ and any distribution $\mathcal{D}: \mathcal{X} \to [0, 1]$ there exists $m(\epsilon, \delta)$ such that protocol output f_{ERM} satisfies

$$\Pr_{x \sim \mathcal{D}}[f_{ERM}(x) \neq y] \leq \epsilon$$

with probability at least $1 - \delta$, then $\mathcal{F}$ is learnable.

- Distribution independent
- Non-asymptotic (finite samples)
- Provides a learning scheme
- Makes a statement about a whole function class



Theorem: The class of thresholds is learnable!

$f_{ERM} \in \mathcal{F}$

(Proper) Learnability

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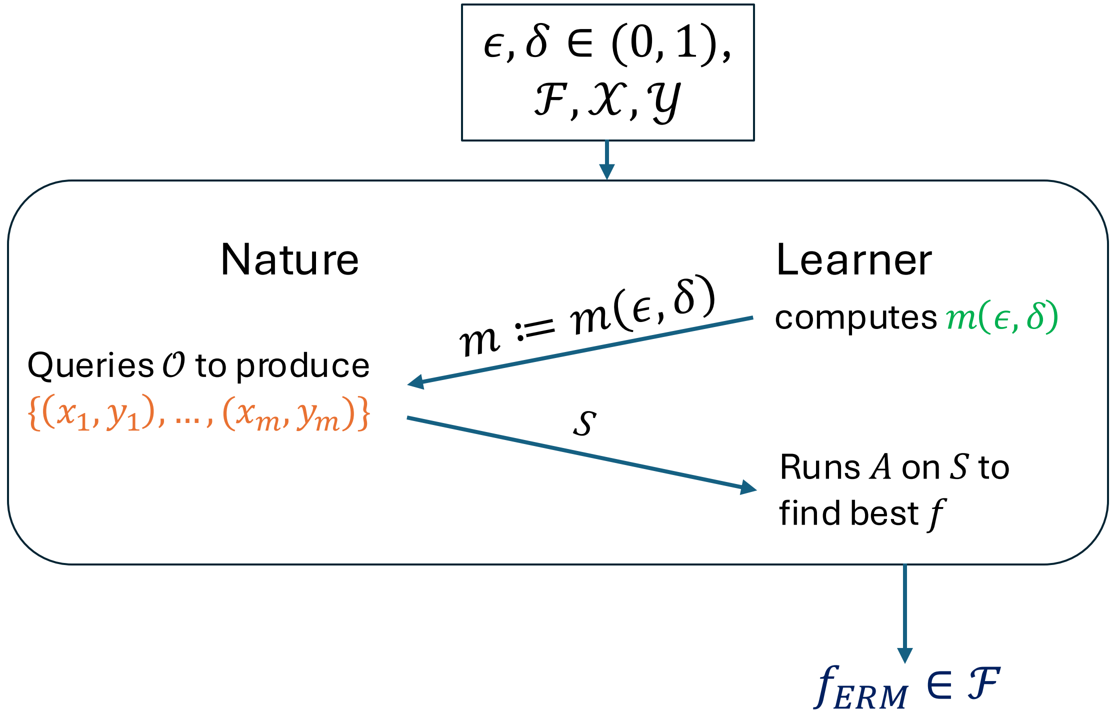


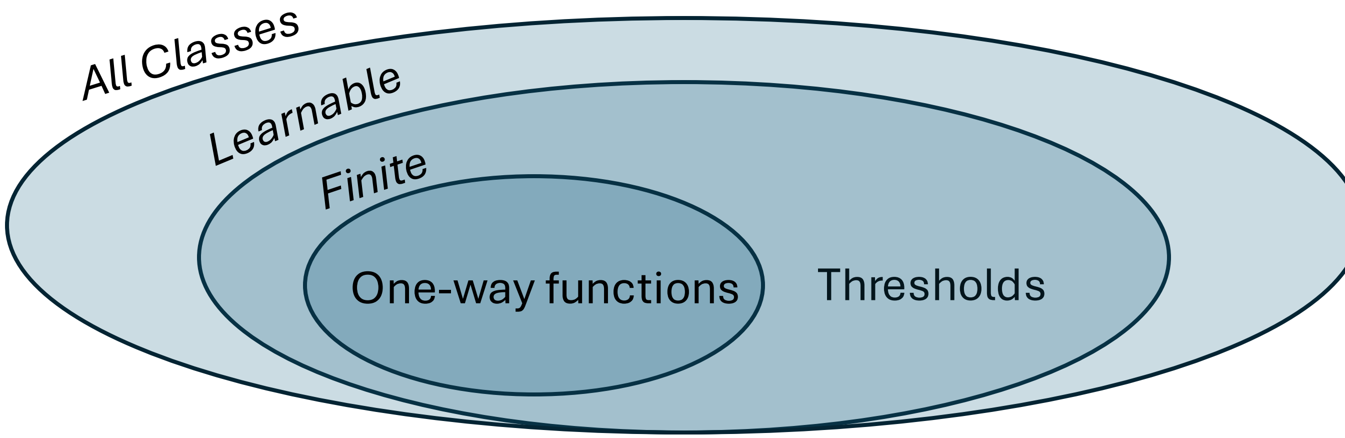
We've shown…

Theorem: The class of thresholds is learnable!

It's also true:

Theorem: Any finite hypothesis class is learnable.
(In general, $m \geq \log\left(\frac{|\mathcal{F}|}{\delta}\right)/\epsilon \ldots$)

Corollary: One-way functions are learnable



Efficiently Learnable

Definition (Efficient Learnability): A class that is learnable by an algorithm with time complexity polynomial in $\log(\frac{1}{\delta})$ and $1/\epsilon$

We've shown…

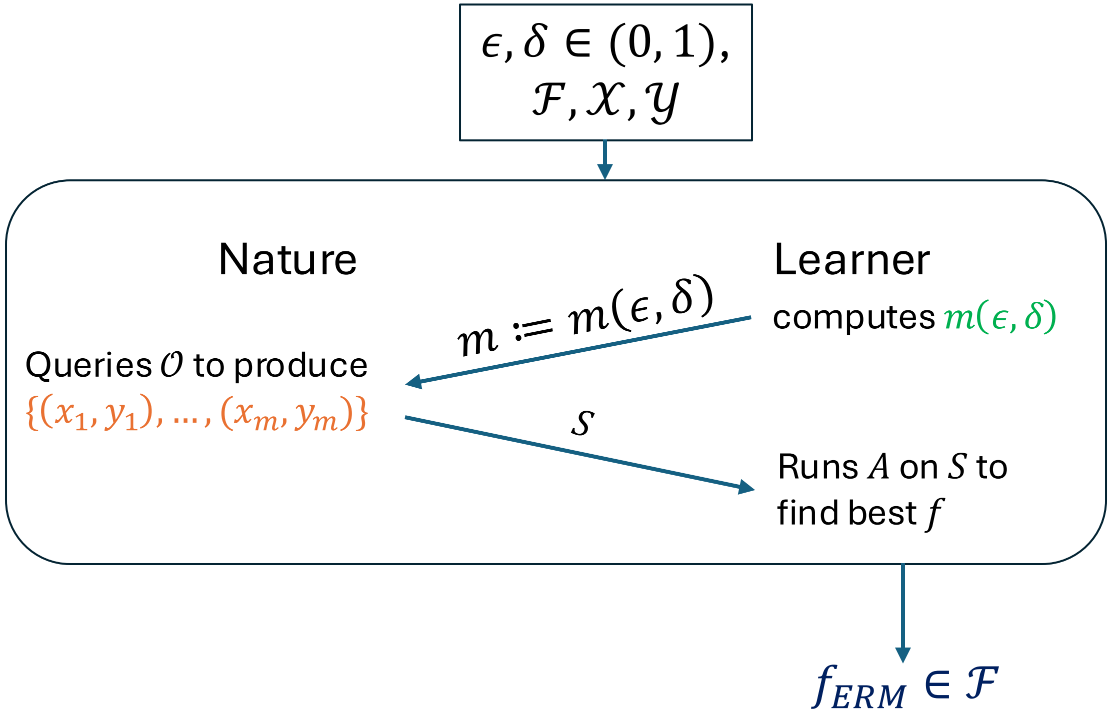
> **Theorem**: The class of thresholds is learnable efficiently!

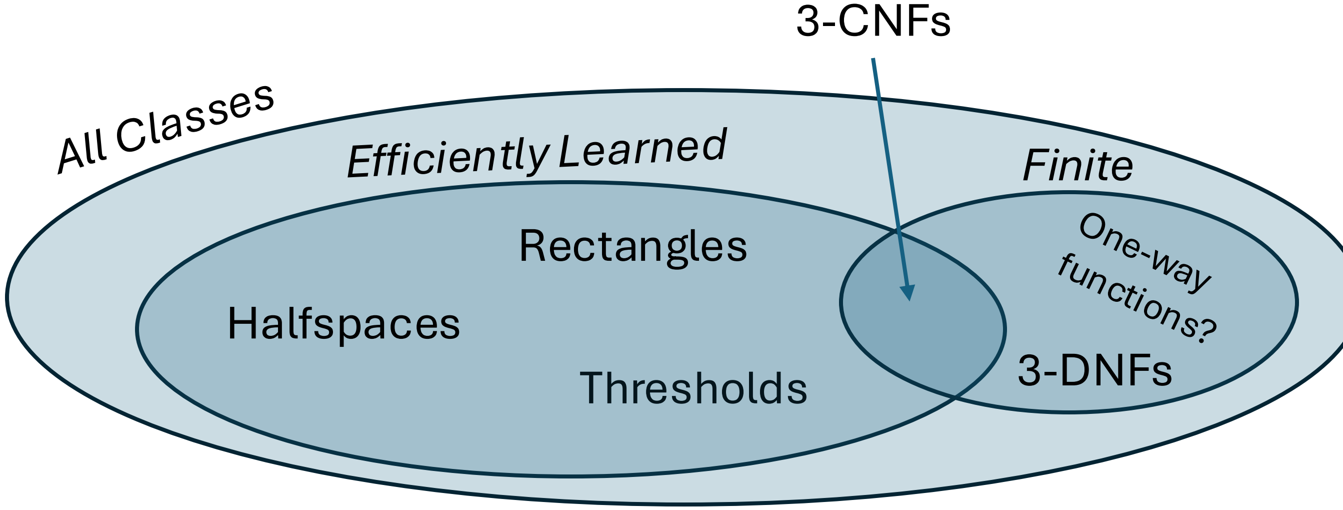
Proof: Recall worst-watchable ***Algorithm***

- Learnable with $m = O\left(\frac{\log\left(\frac{1}{\delta}\right)}{\epsilon}\right)$ samples
- Runs in $O(m)$

> **Theorem** (rest of this talk): There exists a finite class that is *not* learnable efficiently!

Proof sketch: Suppose we could learn each digit in f^{-1} efficiently (binary class!), then we can reconstruct $f^{-1}(x)$ efficiently and with good probability, thus violating hardness.





Discrete Cube Root Assumption

Special case RSA function…

$f_N(x) = x^3 (mod\ N)$ where $N = pq$ is n digits, and $3 \nmid (p-1)(q-1)$

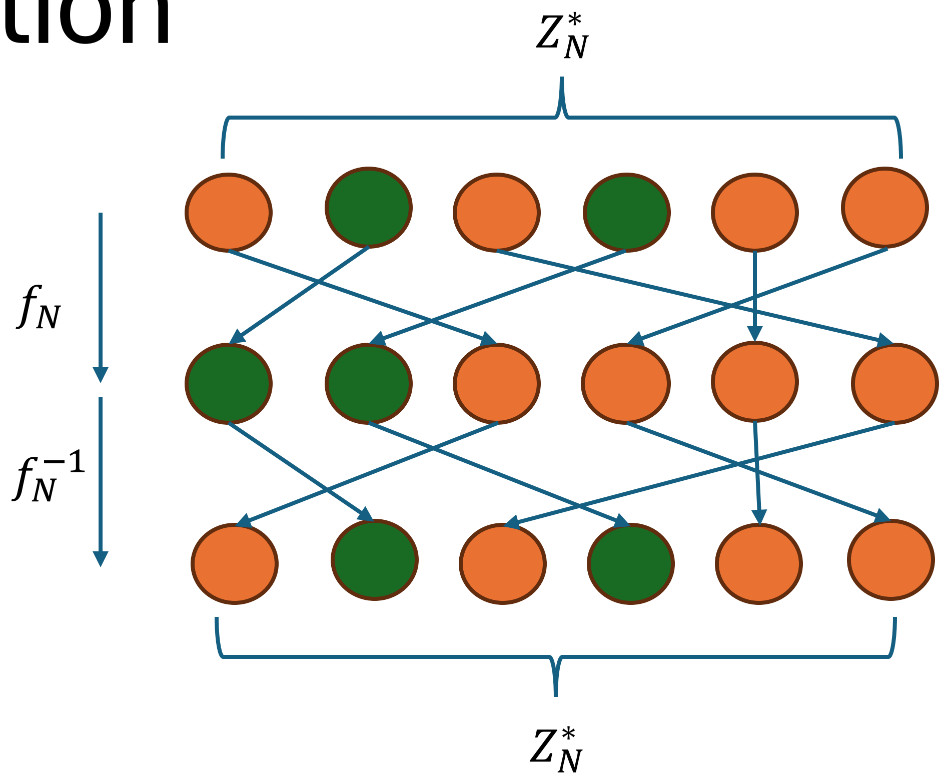
Theorem (RSA Correctness): f_N permutes Z_N^*

Proof: $3d \equiv 1\ (mod\ \phi(N))$ exists uniquely by construction.
Inverse mapping $f_N^{-1}(y) = y^d\ \big(mod\ \phi(N)\big)$ by Euler Theorem.

Theorem (RSA Decoding): f_N^{-1} is computable efficiently, given d

Proof: Recall the squaring trick.

Assumption (RSA Security): f_N^{-1} is hard to find given only N. In particular, f_N is one-way



From the sample set, you can't find permutation structure

Suspicion: $\Pi = \{\pi: Z_N^* \to Z_N^*\}$ is hard to learn

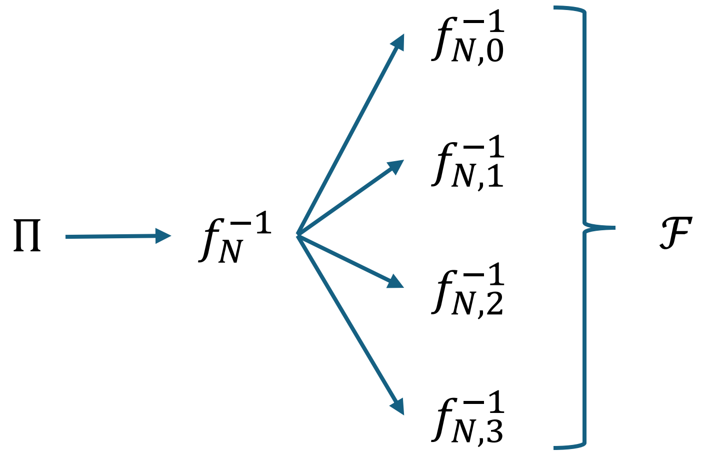
Bitwise Dissection

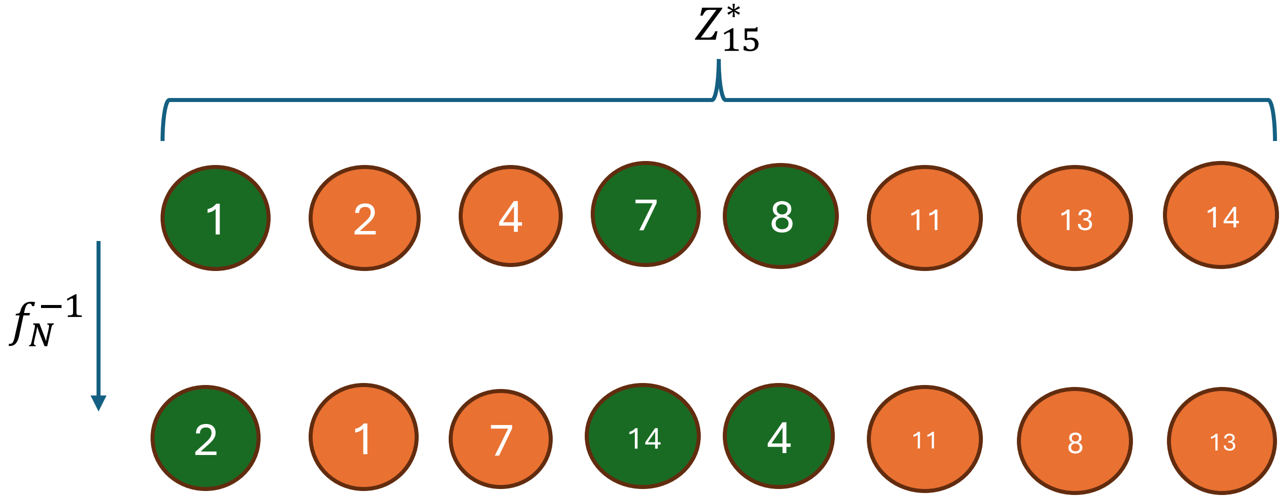
Special case RSA function…

$f_N(x) = x^3 (mod\ N)$ where $N = pq$ is n digits, and $3 \nmid (p-1)(q-1)$

Suspicion: $\Pi = \{\pi : Z_N^* \to Z_N^*\}$ is hard to learn

Convert to a hard *binary* class – isolate f_N^{-1}digit by digit!





y	1	2	4	7	8	11	13	14
$f_{N,0}^{-1}$	0	1	1	0	0	1	0	1

$f_{N,i}^{-1}(y) :=$ i-th digit $f_N^{-1}(y)$

Observation: Learning $\mathcal{F}$ means learning a digit in the inverse

$\mathcal{F}$ cannot be efficiently learned

Recall $\mathcal{F}$ from the previous slide, $\{f_{N,i}^{-1}(y) \mid 1 \le i \le n\}$

Observation: $\mathcal{F}$ has finite size

Proof: $N!$ permutations n digit functions each.

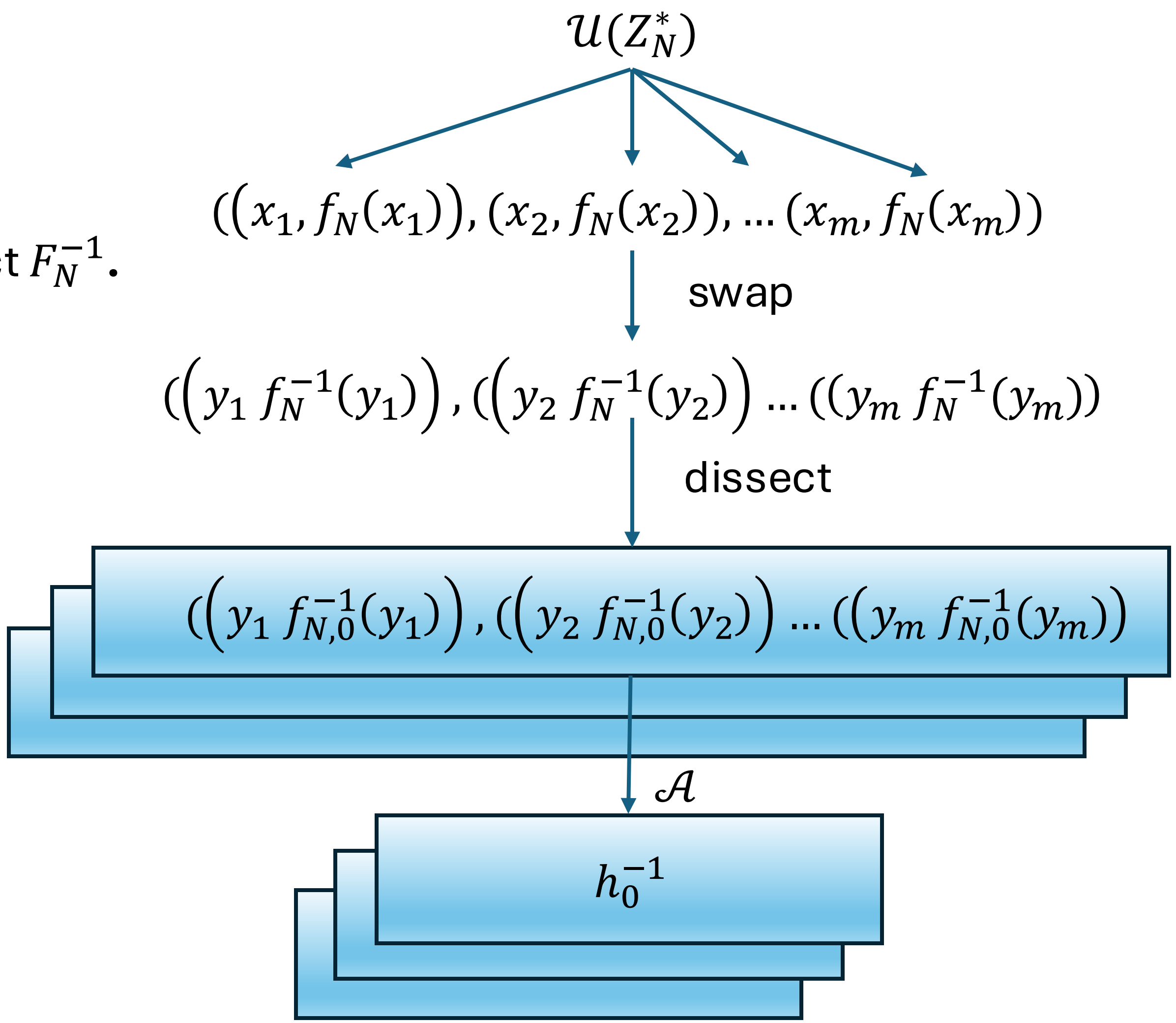
Theorem: If $\mathcal{F}$ is efficiently learnable, then we can reconstruct F_N^{-1}.

Proof: Algorithm $\mathcal{A}$ that (ϵ, δ)-learns any digit function with $m(\epsilon, \delta)$ many samples

<u>Algorithm</u>

set $\epsilon \leftarrow \frac{1}{n^2}, \delta \leftarrow O(\frac{1}{n})$
generate $m(\epsilon, \delta)$ many samples $(y, f_N^{-1}(y))$
parse each sample into $\{f_{N,0}^{-1}(y), f_{N,1}^{-1}(y), \dots, f_{N,n-1}^{-1}(y)\}$
for $i \in [n]$:
 run subroutine $\mathcal{A}$ to learn $(y, f_{N,i}^{-1}(y))$ and get h_i^{-1}
return $(h_1^{-1}, h_2^{-1}, \dots, h_n^{-1})$

All steps are efficient!



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for $i \in [n]$:
    run subroutine $\mathcal{A}$ to learn $(y, f_{N,i}^{-1}(y))$ and get h_i^{-1}
return $(h_1^{-1}, h_2^{-1}, \ldots, h_n^{-1})$

For each i, $\Pr\left[h_i^{-1}(y') \neq f_{N,i}^{-1}(y')\right] \leq \epsilon = 1/n^2$

$$\Pr[h_1^{-1}(y')||h_2^{-1}(y')|| \cdots ||h_{n-1}^{-1}(y') \neq F_N^{-1}(y)] \leq n \cdot \epsilon = \frac{1}{n}$$

So, we have learned a one-way function...

Conclusion

Recall $\mathcal{F}$ from the previous slide, $\{f_{N,i}^{-1}(y) \big| 1 \leq i \leq n\}$

Theorem: If $\mathcal{F}$ is efficiently learnable, then we can reconstruct F_N^{-1}.

Assumption: It's not possible to reconstruct F_N^{-1} efficiently

Conclusion: Under DCRA, the hypothesis class $\mathcal{F}$ is not efficiently learnable