Solutions to Cornell/Bard "Lebesgue Measure" Notes

These are some solutions I have written to exercises from these notes from Cornell University / Bard College's course on measure theory taught by Dr. Jim Belk. I found the notes and exercises to be very helpful.

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Exercise 1

If $\{E_n\}$ is a sequence of measurable sets, prove that the intersection $\bigcap_{n\in\mathbb{N}} E_n$ is measurable.

$\bigcap_{n\in\mathbb{N}} E_n = (\bigcup_{n\in\mathbb{N}} E_n^c)^c$. Since all $\{E_n^c\}$ measurable, $\bigcup_{n\in\mathbb{N}} E_n^c$ measurable $\implies (\bigcup_{n\in\mathbb{N}} E_n^c)^c$ is measurable.

Exercise 2

Prove that if $A \subseteq \mathbb{R}$ and $m^*(A) = 0$, then A is measurable.

We are given $m^*(A) = 0$, and now want to show that A is measurable. Pick arbitrary $C \subseteq \mathbb{R}$. Observe that $C \cap A \subseteq A \implies 0 \leq m^*(C \cap A) \leq m^*(A) = 0 \implies m^*(C \cap A) = 0$. Thus:

$$m^*(C \cap A) + m^*(C \cap A^c) = m^*(C \cap A^c) \leq m^*(C)$$

as $C \cap A^c \subseteq C$ so we can use monotonicity. But by countable subadditivity, $C = (C \cap A) \cup (C \cap A^c) \implies m^*(C) \leq m^*(C \cap A) + m^*(C \cap A^c) = m^*(C \cap A^c)$. So $m^*(C) = m^*(C \cap A) + m^*(C \cap A^c) = m^*(C \cap A^c)$ Thus, A must be measurable.

Exercise 3

a) *If $E \subseteq F$ are measurable sets, prove that $F - E$ is measurable.*

b) *Prove that if $m(E) < \infty$ then $m(F - E) = m(F) - m(E)$.*

Part (a). $F - E = F \cap E^c$, which is measurable.
Part (b). Define $\{A_n\}$ to be a sequence of pairwise disjoint measurable subsets of $\mathbb{R}$ where $A_1 = F - E, A_2 = E$ and $\forall k \geq 3, A_k = \emptyset$. Note that $F = \biguplus_{k\in\mathbb{N}} A_n$ and so:

$$m(F) = m(\biguplus_{k\in\mathbb{N}} A_n) = m(F - E) + m(E) + 0 \implies m(F - E) = m(F) - m(E)$$

Exercise 4

If E and F are measurable sets with finite measure, prove that

$$m(E \cup F) = m(E) + m(F) - m(E \cap F)$$

$E \cap (F \cap E^c) = \emptyset \implies m(E \cup F) = m(E \cup (F \cap E^c)) = m(E) + m(F \cap E^c) = m(E) + m(F) - m(E \cap F)$.

Exercise 5

Suppose that $E \subseteq S \subseteq F$, where E and F are measurable. Prove that if $m(E) = m(F)$ and this measure is finite, then S is measurable as well.

Lemma. If $E \subseteq F$ and E is measurable with $m^*(E) = m^*(F) \implies m^*(F \setminus E) = 0$.

Proof. Observe here that by measurability of E:

$$m^*(F) = m^*(E \cap F) + m^*(E^c \cap F) = m^*(E) + m^*(E^c \cap F)$$
$$\implies m^*(F) - m^*(E) = 0 = m^*(E^c \cap F)$$

but $E^c \cap F = F \setminus E$. □

We will use the monotonicity $E \subseteq S \subseteq F$ as it relates to Lebesgue measure. Now pick test subset $T \subseteq \mathbb{R}$. Note the following two equations:

$$m^*(E \cap T) \leq m^*(S \cap T) \leq m^*(F \cap T)$$

and similarly:

$$m^*(F^c \cap T) \leq m^*(S^c \cap T) \leq m^*(E^c \cap T)$$

and so adding them together we have:

$$m^*(E \cap T) + m^*(F^c \cap T) \leq m^*(S \cap T) + m^*(S^c \cap T) \leq m^*(F \cap T) + m^*(E^c \cap T)$$

Now applying our lemma we have for the right side:

$$m^*(F \cap T) + m^*(E^c \cap T) = m^*(F \cap T) + [m^*(E^c \cap T) + m^*(E \cap T)] - m^*(E \cap T)$$
$$= m^*(T) + [m^*(F \cap T) - m^*(E \cap T)] \leq m^*(T) + m^*((F \setminus E) \cap T)$$
$$\leq m^*(T) + m^*(F \setminus E) = m^*(T)$$

and applying the same for the left side with a countable subadditivity argument, we have:

$$m^*(E \cap T) + m^*(F^c \cap T) \geq [m^*(F \cap T) - m^*((F \setminus E) \cap T)] + m^*(F^c \cap T) = m^*(T)$$

so we can immediately establish:

$$m^*(T) \leq m^*(S \cap T) + m^*(S^c \cap T) \leq m^*(T)$$
$$\implies m^*(T) = m^*(S \cap T) + m^*(S^c \cap T)$$

and so S is measurable as well.

Exercise 6

Prove that every countable subset of $\mathbb{R}$ is measurable and has measure zero.

As a hint, note that this statement is true for every finite subset of $\mathbb{R}$. Pick any countable subset $S \subseteq \mathbb{R}$. In view of Exercise 2, it is sufficient to show $m^*(S) = 0$. Pick any $\epsilon > 0$. We now define a collection of open intervals $\mathcal{C}$ to cover S where for the kth point in S (denoted by S_k), the kth (open) interval in $\mathcal{C}$ is given by $(S_k - \frac{\epsilon}{2^{k+1}}, S_k + \frac{\epsilon}{2^{k+1}})$. Then we have:

$$\sum_{I \in \mathcal{C}} \ell(I) = \sum_{k=1} \frac{\epsilon}{2^k} = \epsilon$$

But because ϵ was arbitrary we have $m^*(S) \leq 0$. But $m^*(S) \geq 0 \implies m^*(S) = 0$, and so we are finished.

Exercise 7

Given a nested sequence $E_1 \subseteq E_2 \subseteq \dots$ of measurable sets, prove that

$$m(\bigcup_{n \in \mathbb{N}} E_n) = \sup_{n \in \mathbb{N}} m(E_n)$$

Note that this proof is really the general case proof of measures being continuous from above/below. Also note that $\uparrow$ means convergence from above.

Because $\{E_n\}$ is a non-decreasing sequence of sets, $\{m(E_n)\}$ is also a non-decreasing sequence $\implies m(\bigcup_{n \in \mathbb{N}} E_n) = \sup_{n \in \mathbb{N}} m(E_n) \iff m(E_n) \uparrow m(\bigcup_{n \in \mathbb{N}} E_n) \iff \forall \epsilon > 0, \exists\ k$ s.t. $m(E_k) > m(\bigcup_{n \in \mathbb{N}} E_n) - \epsilon$. We prove this last statement.

We first start by defining the following:

$$A_1 := E_1, \quad A_{n+1} := E_{n+1} - E_n$$

where $\{A_n\}$ is clearly a sequence of pairwise disjoint measurable sets and $E_k = \bigcup_{n=1}^k A_n$ and $\bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} A_n$. For convenience, we can define $E := \bigcup_{n \in \mathbb{N}} E_n$. We proceed by defining the "partial sum of measurable sets" $S_k := \sum_{n=1}^k m(A_n) = m(E_k)$. Note $\{S_k\}$ and $\{m(E_n)\}$ are non-decreasing sequences (so their limits are equal to their supremum) and thus:

$$m(E) = \sum_{n=1}^{\infty} m(A_n) = \lim_{n \to \infty} S_n = \lim_{n \to \infty} m(E_n) = \sup_{n \in \mathbb{N}} m(E_n)$$

Exercise 8

a) *Let* $E_1 \supseteq E_2 \supseteq \dots$ *be a nested sequence of measurable sets with*

$$\bigcap_{n\in\mathbb{N}} E_n = \emptyset$$

Prove that if $m(E_1) < \infty$, *then* $m(E_n) \to 0$ *as* $n \to \infty$.

b) *Let* $E_1 \supseteq E_2 \supseteq \dots$ *be a nested sequence of measurable sets, and suppose that* $m(E_1) < \infty$. *Prove that*

$$m(\bigcap_{n\in\mathbb{N}} E_n) = \inf_{n\in\mathbb{N}} m(E_n)$$

c) *Give an example of a nested sequence* $E_1 \supseteq E_2 \supseteq \dots$ *of measurable sets such that* $m(E_n) = \infty$ *for all* n *but*

$$m(\bigcap_{n\in\mathbb{N}} E_n) < \infty$$

Part (a). Note that $E_1 - E_n \uparrow E_1$ and so by Exercise 7, $m(E_1 - E_n) \uparrow m(E_1)$. Note that because $\forall n, E_1 \supseteq E_n \implies m(E_1 - E_n) = m(E_1) - m(E_n)$ (see Exercise 3b, note $E_n \subseteq E_1 \implies m(E_n) \le m(E_1) < \infty$). Thus we have:

$$m(E_1 - E_n) \uparrow m(E_1) \implies m(E_1) - m(E_n) \uparrow m(E_1) \implies m(E_n) \downarrow 0$$

Part (b). Note that $\bigcap_{n\in\mathbb{N}} E_n = E_1 - \bigcup_{n\in\mathbb{N}}[E_1 - E_n]$. Using Exercise 7 and Exercise 3(b), we have:

$$\begin{aligned} m(\bigcap_{n\in\mathbb{N}} E_n) &= m(E_1 - \bigcup_{n\in\mathbb{N}}[E_1 - E_n]) = m(E_1) - m(\bigcup_{n\in\mathbb{N}}[E_1 - E_n]) \\ &= m(E_1) - \sup_{n\in\mathbb{N}}[m(E_1) - m(E_n)] = -\sup_{n\in\mathbb{N}}[-m(E_n)] = \inf_{n\in\mathbb{N}} m(E_n) \end{aligned}$$

Part (c). Define each $E_n = (n, \infty)$. While each $m(E_n) = \infty$, $\bigcap_{n\in\mathbb{N}} E_n = \emptyset \implies m(\bigcap_{n\in\mathbb{N}} E_n) = m(\emptyset) = 0 < \infty$.