## Solutions to Cornell/Bard "Introduction to Measure-Theoretic Probability" Notes

These are some solutions I have written to exercises from these notes from Cornell University / Bard College's course on measure theory taught by Dr. Jim Belk. I found the notes (particularly their presentation of examples) and exercises to be very helpful.

Please email anish.lakkapragada@yale.edu for any questions or errors. **Please note for the following exercises that $(X, \mathcal{M}, \mu)$ is the assumed measure space.**

### Exercise 1

*Let $X : \Omega \to (0, 1]$ have the uniform distribution, and let $Y = 1/X$.*

a) *Find the probability density function $f_Y : [1, \infty) \to [0, \infty]$ for $Y$.*

b) *What is the expected value of $Y$?*

Part (a). First note that for any interval $[a, b] \subset (0, 1]$, we have:

$$Y \in [a, b] \iff X \in [\frac{1}{b}, \frac{1}{a}]$$

So therefore, for $x \in [1, \infty)$ we have:

$$f_Y(x) = \lim_{h \to 0^+} \frac{P_Y([x-h, x+h])}{2h} = \lim_{h \to 0^+} \frac{P_X([\frac{1}{x+h}, \frac{1}{x-h}])}{2h} = \lim_{h \to 0^+} \frac{\frac{1}{x-h} - \frac{1}{x+h}}{2h}$$
$$= \lim_{h \to 0^+} \frac{1}{(x-h)(x+h)} = \lim_{h \to 0^+} \frac{1}{x^2 - h^2} = \frac{1}{x^2}$$

where the last limit is justified by the fact $x^2 - h^2 \to x^2$ and $x \neq 0$.

Part (b). Applying the weighted integration formula:

$$\mathbb{E}[Y] = \int_{(0,1]} x dP_x(x) = \int_{(0,1]} x f_Y(x) \, dm(x) = \int_{(0,1]} \frac{1}{x} dm(x) = \int_0^1 \frac{dx}{x} = \ln(x)\Big|_0^1 = \infty$$

### Exercise 2

*Let $X : \Omega \to [0, 1]$ have the uniform distribution and let $Y = \sin(8X)$. Use the integration formula for distributions (Proposition 1) to compute $\mathbb{E}[Y]$.*

Directly applying Proposition 1 with $S = [0, 1]$ and $g(x) = \sin(8x)$, we get:

$$\mathbb{E}[Y] = \mathbb{E}[g(X)] = \int_\Omega g(X) \, dP = \int_{[0,1]} g \, dP_X = \int_{[0,1]} g(x) f_X(x) \, dm(x)$$
$$= \int_0^1 \sin(8x) dx = -\frac{1}{8}[\cos(8) - \cos(0)] = \frac{1}{8}[1 - \cos(8)]$$

### Exercise 3

*Let $X$ and $Y$ be the value of two die rolls, and let $Z = \max(X, Y)$.*

a) *Find $P_Z(\{n\})$ for $n \in \{1, 2, 3, 4, 5, 6\}$.*

b) *Determine $\mathbb{E}[Z]$, and find the standard deviation for $Z$.*

Part (a). These probabilities can be found by simply making a grid of outcomes:

$$P_Z(\{1\}) = \frac{1}{36}, \quad P_Z(\{2\}) = \frac{3}{36}, \quad P_Z(\{3\}) = \frac{5}{36}, \quad P_Z(\{4\}) = \frac{7}{36},$$
$$P_Z(\{5\}) = \frac{9}{36}, P_Z(\{6\}) = \frac{11}{36}$$

Part (b). Using the formula directly from Example 11 (note $[6] = [1, 6] \cap \mathbb{N}$),

$$\mathbb{E}[Z] = \sum_{x \in [6]} x P_Z(\{x\}) = \frac{1 + 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11}{36} = \frac{161}{36} \approx 4.47$$

Similarly, we can compute $\mathrm{Var}(Z)$:

$$\mathrm{Var}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \sum_{x \in [6]} x^2 P_Z(\{x\}) - \frac{161^2}{36^2}$$
$$= \frac{1^2 \cdot 1 + 2^2 \cdot 3 + 3^2 \cdot 5 + 4^2 \cdot 7 + 5^2 \cdot 9 + 6^2 \cdot 11}{36} - \frac{161^2}{36^2} = \frac{2555}{1296}$$

so we have a standard deviation of $\sqrt{\frac{2555}{1296}} \approx 0.443$.

### Exercise 4

*Let $X : \Omega \to [1, \infty)$ be a continuous random variable with probability density function*

$$f_X(x) = \frac{3}{x^4}$$

*Compute $\mathbb{E}[X]$ and Var$(X)$.*

We start with the expectation:

$$\mathbb{E}[X] = \int_{[1,\infty]} x \; dP_x(x) = \int_1^\infty x f_X(x) dx = \int_1^\infty \frac{3}{x^3} dx = -\frac{3}{2} x^{-2} \Big|_1^\infty = \frac{3}{2}$$

We now compute $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \int_{[1,\infty)} x^2 dP_X = \int_{[1,\infty]} x^2 f_X(x) dm(x) = \int_1^\infty \frac{3}{x^2} dx = -\frac{3}{x} \Big|_1^\infty = 3$$

and so:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 3 - \frac{9}{4} = 0.75$$