

## Solutions to Cornell/Bard “Lebesgue Measure” Notes

These are some solutions I have written to exercises from these [notes](#) from Cornell University / Bard College’s [course on measure theory taught by Dr. Jim Belk](#). I found the notes and exercises to be very helpful.

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### Exercise 1

If  $\{E_n\}$  is a sequence of measurable sets, prove that the intersection  $\bigcap_{n \in \mathbb{N}} E_n$  is measurable.

$\bigcap_{n \in \mathbb{N}} E_n = (\bigcup_{n \in \mathbb{N}} E_n^c)^c$ . Since all  $\{E_n^c\}$  measurable,  $\bigcup_{n \in \mathbb{N}} E_n^c$  measurable  $\implies (\bigcup_{n \in \mathbb{N}} E_n^c)^c$  is measurable.

### Exercise 2

Prove that if  $A \subseteq \mathbb{R}$  and  $m^*(A) = 0$ , then  $A$  is measurable.

We are given  $m^*(A) = 0$ , and now want to show that  $A$  is measurable. BWOC, assume  $A$  not measurable  $\implies \exists C \subseteq \mathbb{R}$  such that:

$$m^*(C) > m^*(C \cap A) + m^*(C \cap A^c)$$

First observe here that  $C \cap A \subseteq A \implies 0 \leq m^*(C \cap A) \leq m^*(A) \leq 0 \implies m^*(C \cap A) = 0$  and so this means:

$$m^*(C) > m^*(C \cap A^c)$$

But by countable subadditivity,  $C = (C \cap A) \cup (C \cap A^c) \implies m^*(C) \leq m^*(C \cap A) + m^*(C \cap A^c) = m^*(C \cap A^c)$ , which is a contradiction. Thus,  $A$  must be measurable.

### Exercise 3

- If  $E \subseteq F$  are measurable sets, prove that  $F - E$  is measurable.
- Prove that if  $m(E) < \infty$  then  $m(F - E) = m(F) - m(E)$ .

Part (a).  $F - E = F \cap E^c$ , which is measurable.

Part (b). Define  $\{A_n\}$  to be a sequence of pairwise disjoint measurable subsets of  $\mathbb{R}$  where  $A_1 = F - E$ ,  $A_2 = E$  and  $\forall k \geq 3, A_k = \emptyset$ . Note that  $F = \biguplus_{k \in \mathbb{N}} A_n$  and so:

$$m(F) = m(\biguplus_{k \in \mathbb{N}} A_n) = m(F - E) + m(E) + 0 \implies m(F - E) = m(F) - m(E)$$

**Exercise 4**

If  $E$  and  $F$  are measurable sets with finite measure, prove that

$$m(E \cup F) = m(E) + m(F) - m(E \cap F)$$

$$E \cap (F \cap E^c) = \emptyset \implies m(E \cup F) = m(E \cup (F \cap E^c)) = m(E) + m(F \cap E^c) = m(E) + m(F) - m(E \cap F).$$

**Exercise 5**

Suppose that  $E \subseteq S \subseteq F$ , where  $E$  and  $F$  are measurable. Prove that if  $m(E) = m(F)$  and this measure is finite, then  $S$  is measurable as well.

Note that because  $E \subseteq S \subseteq F \implies m^*(E) \leq m^*(S) \leq m^*(F)$ . But since  $m^*(E) = m^*(F) \implies m^*(S) = m^*(E) = m^*(F)$ . Now pick test subset  $T \subseteq \mathbb{R}$ . Note the following two equations:

$$m^*(E \cap T) \leq m^*(S \cap T) \leq m^*(F \cap T)$$

and similarly:

$$m^*(E \cap T^c) \leq m^*(S \cap T^c) \leq m^*(F \cap T^c)$$

Adding them together, we have  $m^*(E) \leq m^*(S \cap T) + m^*(S \cap T^c) \leq m^*(F) \implies m^*(S \cap T) + m^*(S \cap T^c) = m^*(S) \implies S$  is measurable.

**Exercise 6**

Prove that every countable subset of  $\mathbb{R}$  is measurable and has measure zero.

As a hint, note that this statement is true for every finite subset of  $\mathbb{R}$ . Pick any countable subset  $S \subseteq \mathbb{R}$ . In view of Exercise 2, it is sufficient to show  $m^*(S) = 0$ . Pick any  $\epsilon > 0$ . We now define a collection of open intervals  $\mathcal{C}$  to cover  $S$  where for the  $k$ th point in  $S$  (denoted by  $S_k$ ), the  $k$ th (open) interval in  $\mathcal{C}$  is given by  $(S_k - \frac{\epsilon}{2^{k+1}}, S_k + \frac{\epsilon}{2^{k+1}})$ . Then we have:

$$\sum_{I \in \mathcal{C}} \ell(I) = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$$

But because  $\epsilon$  was arbitrary we have  $m^*(S) \leq 0$ . But  $m^*(S) \geq 0 \implies m^*(S) = 0$ , and so we are finished.

### Exercise 7

Given a nested sequence  $E_1 \subseteq E_2 \subseteq \dots$  of measurable sets, prove that

$$m\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sup_{n \in \mathbb{N}} m(E_n)$$

Note that this proof is really the general case proof of measures being continuous from above/below. Also note that  $\uparrow$  means convergence from above.

Because  $\{E_n\}$  is a non-decreasing sequence of sets,  $\{m(E_n)\}$  is also a non-decreasing sequence  $\implies m\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sup_{n \in \mathbb{N}} m(E_n) \iff m(E_n) \uparrow m\left(\bigcup_{n \in \mathbb{N}} E_n\right) \iff \forall \epsilon > 0, \exists k \text{ s.t. } m(E_k) > m\left(\bigcup_{n \in \mathbb{N}} E_n\right) - \epsilon$ . We prove this last statement.

We first start by defining the following:

$$A_1 := E_1, \quad A_{n+1} := E_{n+1} - E_n$$

where  $\{A_n\}$  is clearly a sequence of pairwise disjoint measurable sets and  $E_k = \bigcup_{n=1}^k A_n$  and  $\bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} A_n$ . For convenience, we can define  $E := \bigcup_{n \in \mathbb{N}} E_n$ . We proceed by defining the “partial sum of measurable sets”  $S_k := \sum_{n=1}^k m(A_n) = m(E_k)$ . Note  $\{S_k\}$  and  $\{m(E_n)\}$  are non-decreasing sequences (so their limits are equal to their supremum) and thus:

$$m(E) = \sum_{n=1}^{\infty} m(A_n) = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} m(E_n) = \sup_{n \in \mathbb{N}} m(E_n)$$

### Exercise 8

a) Let  $E_1 \supseteq E_2 \supseteq \dots$  be a nested sequence of measurable sets with

$$\bigcap_{n \in \mathbb{N}} E_n = \emptyset$$

Prove that if  $m(E_1) < \infty$ , then  $m(E_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

b) Let  $E_1 \supseteq E_2 \supseteq \dots$  be a nested sequence of measurable sets, and suppose that  $m(E_1) < \infty$ . Prove that

$$m\left(\bigcap_{n \in \mathbb{N}} E_n\right) = \inf_{n \in \mathbb{N}} m(E_n)$$

c) Give an example of a nested sequence  $E_1 \supseteq E_2 \supseteq \dots$  of measurable sets such that  $m(E_n) = \infty$  for all  $n$  but

$$m\left(\bigcap_{n \in \mathbb{N}} E_n\right) < \infty$$

Part (a). Note that  $E_1 - E_n \uparrow E_1$  and so by Exercise 7,  $m(E_1 - E_n) \uparrow m(E_1)$ . Note that because  $\forall n, E_1 \supseteq E_n \implies m(E_1 - E_n) = m(E_1) - m(E_n)$  (see Exercise 3b, note  $E_n \subseteq E_1 \implies m(E_n) \leq m(E_1) < \infty$ ). Thus we have:

$$m(E_1 - E_n) \uparrow m(E_1) \implies m(E_1) - m(E_n) \uparrow m(E_1) \implies m(E_n) \downarrow 0$$

Part (b). Note that  $\bigcap_{n \in \mathbb{N}} E_n = E_1 - \bigcup_{n \in \mathbb{N}} [E_1 - E_n]$ . Using Exercise 7 and Exercise 3(b), we have:

$$\begin{aligned} m\left(\bigcap_{n \in \mathbb{N}} E_n\right) &= m(E_1 - \bigcup_{n \in \mathbb{N}} [E_1 - E_n]) = m(E_1) - m\left(\bigcup_{n \in \mathbb{N}} [E_1 - E_n]\right) \\ &= m(E_1) - \sup_{n \in \mathbb{N}} [m(E_1) - m(E_n)] = -\sup_{n \in \mathbb{N}} [-m(E_n)] = \inf_{n \in \mathbb{N}} m(E_n) \end{aligned}$$

Part (c). Define each  $E_n = (n, \infty)$ . While each  $m(E_n) = \infty$ ,  $\bigcap_{n \in \mathbb{N}} E_n = \emptyset \implies m(\bigcap_{n \in \mathbb{N}} E_n) = m(\emptyset) = 0 < \infty$ .