Solutions to Cornell/Bard "Introduction to the Lebesgue Integral" Notes

These are some solutions I have written to exercises from these notes from Cornell University / Bard College's course on measure theory taught by Dr. Jim Belk. I found the notes and exercises to be very helpful.

Please email anish.lakkapragada@yale.edu for any questions or errors. **Please note for the following exercises that $(X, \mathcal{M}, \mu)$ is the assumed measure space.**

Exercise 1

Prove that if f is a measurable function on X, then the set

$$f^{-1}(\infty) = \{x \in X \mid f(x) = \infty\}$$

is measurable.

Observe

$$f^{-1}(\infty) = \bigcap_{a \in \mathbb{N}} f^{-1}((a, \infty])$$

where each set in the intersection is measurable.

Exercise 2

Let f and g be measurable functions on X, and suppose that $f + g$ is everywhere defined. Prove directly from definition that $f + g$ is measurable.

Let us define $h := f + g$. We will take advantage of the fact that $\mathbb{Q}$ is countable:

$$\forall a \in \mathbb{R}, h^{-1}((a, \infty]) = \{x \in X \mid f(x) + g(x) > a\} = \bigcup_{q \in \mathbb{Q}} f^{-1}((q, \infty]) \cup g^{-1}((a - q, \infty])$$

Because both $f^{-1}((q, \infty])$ and $g^{-1}((a - q, \infty])$ are measurable $\implies$ $h^{-1}((a, \infty])$ is measurable by a countable union $\implies$ h is a measurable function.

Exercise 3

Let $f : X \to [-\infty, \infty]$ be a measurable function. Prove directly from the definition that $-f$ is measurable.

Define $g := -f$, then:

$$\forall a \in \mathbb{R}, g^{-1}((a, \infty]) = f^{-1}([-\infty, -a)) = \left(f^{-1}([a, \infty])\right)^c = \Big(\bigcap_{n \in \mathbb{N}} f^{-1}((a - \frac{1}{n}, \infty])\Big)^c$$

Exercise 4

Prove that if $S \subseteq X$, then χ_S is a measurable function if and only if S is a measurable set.

Note the following:

$$\forall a \in \mathbb{R}, \chi_s^{-1}((a, \infty]) = \begin{cases} \emptyset & \text{if } a \geq 1 \\ S & \text{if } 0 \leq a < 1 \\ X & \text{if } a < 0 \end{cases}$$

$\emptyset$ and X are measurable. So $\chi_s^{-1}((a, \infty])$ is measurable $\iff$ S is measurable.

Exercise 5

Let f and g be measurable functions on X, and let $E \subseteq X$ be a measurable set, and define a function $h : X \to [-\infty, \infty]$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in E \\ g(x) & \text{if } x \in E^c \end{cases}$$

Prove that h is measurable.

Note that E^c is also measurable. As shown in the notes, $f\chi_E$ is measurable, and thus so is $g\chi_{E^c}$. So by Exercise 2 we have that $h = f\chi_E + g\chi_{E^c}$ is measurable.

Exercise 6

Let f be a Lebesgue integrable function on X. Use the positive and negative parts of f to prove that

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Observe that because $|f| = f^+ + f^-$, have:

$$\left| \int_X f \, d\mu \right| = \left| \int_X f^+ d\mu - \int_X f^- d\mu \right| \leq \int_X f^+ d\mu + \int_X f^- d\mu = \int_X |f| d\mu$$

This is a simple example of $\forall a, b \in \mathbb{R}, |a - b| \leq |a| + |b|$.

Exercise 7

Let f be a non-negative measurable function on X and suppose that $f \leq M$ for some constant M. Prove that

$$\int_E f d\mu \leq M\mu(E)$$

for any measurable set E.

Define function $g : X \to [-\infty, \infty]$ where $g(x) = M$. The main part of this question is proving that g is Lebesgue integrable. To start, we first show it is measurable:

$$\forall a \in \mathbb{R}, g^{-1}((a, \infty]) = \begin{cases} \emptyset & \text{if } a \geq M \\ X & \text{if } a < M \end{cases}$$

Furthermore, note that either g^+ or g^- is equal to the zero function (e.g. if $M \geq 0 \implies g^- = 0$) and so either $\int_X g^+ d\mu < \infty$ or $\int_X g^- d\mu < \infty$. So g is Lebesgue integrable. Then note that $f\chi_E \leq g\chi_E$, so we have:

$$\int_E f d\mu = \int_X f\chi_E d\mu \leq \int_X g\chi_E d\mu = \int_X M\chi_E d\mu = M\mu(E)$$

Exercise 8

Prove that if $f : X \to [-\infty, \infty]$ is Lebesgue integrable on X, then $f\chi_E$ is Lebesgue integrable for every measurable set $E \subset X$, and hence all of the integrals

$$\int_E f d\mu$$

are defined.

Because the notes have already shown that $f\chi_E$ is a measurable function, the only task for us is to show that either of the two cases holds:

$$\int_X (f\chi_E)^+ d\mu < \infty \quad \text{or} \quad \int_X (f\chi_E)^- d\mu < \infty$$

Because f is Lebesgue integrable, WLOG let us assume that $\int_X f^+ d\mu < \infty$. Note that $(f\chi_E)^+ \leq f^+$ and so:

$$\int_X (f\chi_E)^+ d\mu \leq \int_X f^+ d\mu < \infty$$

and so we are finished.

Exercise 9

Prove that if "$f = g$ almost everywhere" is an equivalence relation for measurable functions on X.

The reflexive and symmetric properties of this equivalence relation are trivial to show. We thus only show the transitivity property of this equivalence relation. Suppose f, g, h are all measurable functions on X and we have "$f = g$ almost everywhere" and "$g = h$ almost everywhere". Then we have measure zero sets A and B such that $\forall x \in X - A, f(x) = g(x)$ and similarly $\forall x \in X - B, g(x) = h(x)$. Defining measure set[a] $C := A \cup B$, we have $\forall x \in X - C = (X - A) \cap (X - B)$, $f(x) = g(x) = h(x)$. So we are finished.

[a] Note that $\mu(C) = \mu(A) + \mu(B) - \mu(A \cap B) = 0 - \mu(A \cap B)$. But $\mu(A \cap B) \leq \mu(A) = 0 \implies \mu(A \cap B) = 0$.

Exercise 10

Let $f : X \to [-\infty, \infty]$ be a Lebesgue integrable function, and let $E, F \subseteq X$ be disjoint measurable sets. Prove that

$$\int_{E \cup F} f d\mu = \int_E f d\mu + \int_F f d\mu$$

Note that because E and F are disjoint, $f\chi_{E\cup F} = f\chi_E + f\chi_F$. Then we have:

$$\int_{E\cup F} f d\mu = \int_X f\chi_{E\cup F} d\mu = \int_X f\chi_E d\mu + \int_X f\chi_F d\mu = \int_E f d\mu + \int_F f d\mu$$

Exercise 11

Let $\{f_n\}$ be a sequence of non-negative measurable functions on X. Prove that $\sum_{n\in\mathbb{N}} f_n$ is measurable, and that

$$\int_X \sum_{n\in\mathbb{N}} f_n d\mu = \sum_{n\in\mathbb{N}} \int_X f_n d\mu$$

We define the partial sums of these functions by $g_n = \sum_{i=1}^n f_i$. Note that $\{g_n\}$ is a sequence of increasing measurable functions on X where $g_n \uparrow \sum_{n\in\mathbb{N}} f_n$ as each $f_n \geq 0$ (so $\sum_{n\in\mathbb{N}} f_n$ is not an alternating series.) Then by Lebesgue's Monotone Convergence Theorem we have:

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X \lim_{n\to\infty} g_n d\mu = \int_X \sum_{n\in\mathbb{N}} f_n d\mu$$

But

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \lim_{n\to\infty} \int_X (\sum_{i=1}^n f_i) d\mu = \lim_{n\to\infty} \sum_{i=1}^n \int_X f_i d\mu = \sum_{n\in\mathbb{N}} \int_X f_n d\mu$$

where the last limit is established because each $\int_X f_i d\mu \geq 0$.

Exercise 12

Let $f : X \to [0, \infty)$ be a measurable function, let $\{E_n\}$ be a sequence of pairwise disjoint, measurable subsets of X, and let $E = \biguplus_{n\in\mathbb{N}} E_n$. Prove that

$$\int_E f d\mu = \sum_{n\in\mathbb{N}} \int_{E_n} f d\mu$$

Hint: See previous exercise. We can define our sequence $\{f_n\}$ of non-negative measurable functions on X as $\{f\chi_{E_n}\}$. Then $f = \sum_{n\in\mathbb{N}} f_n$ and each $\int_X f_n d\mu = \int_X f\chi_{E_n} d\mu = \int_{E_n} f d\mu$. Using the previous exercise, we are finished.

Exercise 13

Prove that

$$\lim_{n\to\infty}\int_0^1 x^n dx = 0.$$

This question can be solved with an application of Lebesgue's Dominated Convergence Theorem. To do so rigorously, we define our (Lebesgue) measure space $([0,1], \mathcal{M}|_{[0,1]}, m|_{[0,1]})$ where $\mathcal{M}$ is the Lebesgue measurable sets. Let us define our pointwise convergent sequence $\{f_n\}$ of measurable functions on $[0,1]$ to be $\{x_n\}$ (note that $\forall x \in [0,1], x^n \to 0$ as $n \to \infty$). We define continuous and measurable[a] constant function $g : [0,1] \to [0,\infty]$ as $g(x) = 1$. Then note that:

$$\int_{[0,1]} g\ dm = \int_0^1 1dx = 1 < \infty$$

and also that $\forall n, |f_n| = |x^n| \leq 1$. So by applying Lebesgue's Dominated Convergence Theorem, we have:

$$\lim_{n\to\infty}\int_{[0,1]} x^n dm = \int_{[0,1]}\lim_{n\to\infty} x^n dm = \int_{[0,1]} 0\ dm = 0$$

But because each $f_n = x_n$ is continuous, $\int_{[0,1]} x^n dm = \int_0^1 x^n dx$ and so:

$$\lim_{n\to\infty}\int_{[0,1]} x^n dm = \lim_{n\to\infty}\int_0^1 x^n dx$$

Thus we are finished.

[a] See Exercise 7 for justification.

Exercise 14

Prove that

$$\lim_{n\to\infty}\int_0^1 \tan^{-1}(nx)dx = \frac{\pi}{2}$$

Hint: See answer to last exercise. We use the same measure space as in the last exercise and define our pointwise convergent sequence $\{f_n\}$ of measurable functions on $[0,1]$ to be $\{\tan^{-1}(nx)\}$ (note that $\forall x \in [0,1], \tan^{-1}(nx) \to \frac{\pi}{2}$). We define continuous and measurable constant function $g : [0,1] \to [0,\infty]$ as $g(x) = 2$ where:

$$\int_{[0,1]} g\ dm = \int_0^1 2dx = 2 < \infty$$

and $\forall n, |f_n| = |\tan^{-1}(nx)| \leq 2$. So by Lebesgue's Dominated Convergence Theorem and the fact that each $f_n = \tan^{-1}(nx)$ is continuous we have:

$$\lim_{n\to\infty}\int_0^1 \tan^{-1}(nx)dx = \lim_{n\to\infty}\int_{[0,1]} \tan^{-1}(nx)\ dm = \int_{[0,1]} \lim_{n\to\infty} \tan^{-1}(nx)\ dm = \int_0^1 \frac{\pi}{2}dx = \frac{\pi}{2}$$