

# Solutions to Cornell/Bard “Introduction to the Lebesgue Integral” Notes

These are some solutions I have written to exercises from these [notes](#) from Cornell University / Bard College’s [course on measure theory taught by Dr. Jim Belk](#). I found the notes and exercises to be very helpful.

Please email [anish.lakapragada@yale.edu](mailto:anish.lakapragada@yale.edu) for any questions or errors. **Please note for the following exercises that  $(X, \mathcal{M}, \mu)$  is the assumed measure space.**

## Exercise 1

Prove that if  $f$  is a measurable function on  $X$ , then the set

$$f^{-1}(\infty) = \{x \in X \mid f(x) = \infty\}$$

is measurable.

Observe

$$f^{-1}(\infty) = \bigcap_{a \in \mathbb{N}} f^{-1}((a, \infty])$$

where each set in the intersection is measurable.

## Exercise 2

Let  $f$  and  $g$  be measurable functions on  $X$ , and suppose that  $f + g$  is everywhere defined. Prove directly from definition that  $f + g$  is measurable.

Let us define  $h := f + g$ . We will take advantage of the fact that  $\mathbb{Q}$  is countable:

$$\forall a \in \mathbb{R}, h^{-1}((a, \infty]) = \{x \in X \mid f(x) + g(x) > a\} = \bigcup_{q \in \mathbb{Q}} f^{-1}((q, \infty]) \cup g^{-1}((a - q, \infty])$$

Because both  $f^{-1}((q, \infty])$  and  $g^{-1}((a - q, \infty])$  are measurable  $\implies h^{-1}((a, \infty])$  is measurable by a countable union  $\implies h$  is a measurable function.

## Exercise 3

Let  $f : X \rightarrow [-\infty, \infty]$  be a measurable function. Prove directly from the definition that  $-f$  is measurable.

Define  $g := -f$ , then:

$$\forall a \in \mathbb{R}, g^{-1}((a, \infty]) = f^{-1}([-\infty, -a)) = (f^{-1}([a, \infty]))^c = \left( \bigcap_{n \in \mathbb{N}} f^{-1}((a - \frac{1}{n}, \infty]) \right)^c$$

#### Exercise 4

Prove that if  $S \subseteq X$ , then  $\chi_S$  is a measurable function if and only if  $S$  is a measurable set.

Note the following:

$$\forall a \in \mathbb{R}, \chi_S^{-1}((a, \infty]) = \begin{cases} \emptyset & \text{if } a \geq 1 \\ S & \text{if } 0 \leq a < 1 \\ X & \text{if } a < 0 \end{cases}$$

$\emptyset$  and  $X$  are measurable. So  $\chi_S^{-1}((a, \infty])$  is measurable  $\iff S$  is measurable.

#### Exercise 5

Let  $f$  and  $g$  be measurable functions on  $X$ , and let  $E \subseteq X$  be a measurable set, and define a function  $h : X \rightarrow [-\infty, \infty]$  by

$$h(x) = \begin{cases} f(x) & \text{if } x \in E \\ g(x) & \text{if } x \in E^c \end{cases}$$

Prove that  $h$  is measurable.

Note that  $E^c$  is also measurable. As shown in the notes,  $f\chi_E$  is measurable, and thus so is  $g\chi_{E^c}$ . So by Exercise 2 we have that  $h = f\chi_E + g\chi_{E^c}$  is measurable.

#### Exercise 6

Let  $f$  be a Lebesgue integrable function on  $X$ . Use the positive and negative parts of  $f$  to prove that

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

Observe that because  $|f| = f^+ + f^-$ , have:

$$\left| \int_X f d\mu \right| = \left| \int_X f^+ d\mu - \int_X f^- d\mu \right| \leq \int_X f^+ d\mu + \int_X f^- d\mu = \int_X |f| d\mu$$

This is a simple example of  $\forall a, b \in \mathbb{R}, |a - b| \leq |a| + |b|$ .

#### Exercise 7

Let  $f$  be a non-negative measurable function on  $X$  and suppose that  $f \leq M$  for some constant  $M$ . Prove that

$$\int_E f d\mu \leq M\mu(E)$$

for any measurable set  $E$ .

Define function  $g : X \rightarrow [-\infty, \infty]$  where  $g(x) = M$ . The main part of this question is proving that  $g$  is Lebesgue integrable. To start, we first show it is measurable:

$$\forall a \in \mathbb{R}, g^{-1}((a, \infty]) = \begin{cases} \emptyset & \text{if } a \geq M \\ X & \text{if } a < M \end{cases}$$

Furthermore, note that either  $g^+$  or  $g^-$  is equal to the zero function (e.g. if  $M \geq 0 \implies g^- = 0$ ) and so either  $\int_X g^+ d\mu < \infty$  or  $\int_X g^- d\mu < \infty$ . So  $g$  is Lebesgue integrable. Then note that  $f\chi_E \leq g\chi_E$ , so we have:

$$\int_E f d\mu = \int_X f\chi_E d\mu \leq \int_X g\chi_E d\mu = \int_X M\chi_E d\mu = M\mu(E)$$

### Exercise 8

*Prove that if  $f : X \rightarrow [-\infty, \infty]$  is Lebesgue integrable on  $X$ , then  $f\chi_E$  is Lebesgue integrable for every measurable set  $E \subset X$ , and hence all of the integrals*

$$\int_E f d\mu$$

*are defined.*

Because the notes have already shown that  $f\chi_E$  is a measurable function, the only task for us is to show that either of the two cases holds:

$$\int_X (f\chi_E)^+ d\mu < \infty \quad \text{or} \quad \int_X (f\chi_E)^- d\mu < \infty$$

Because  $f$  is Lebesgue integrable, WLOG let us assume that  $\int_X f^+ d\mu < \infty$ . Note that  $(f\chi_E)^+ \leq f^+$  and so:

$$\int_X (f\chi_E)^+ d\mu \leq \int_X f^+ d\mu < \infty$$

and so we are finished.

### Exercise 9

*Prove that if “ $f = g$  almost everywhere” is an equivalence relation for measurable functions on  $X$ .*

The reflexive and symmetric properties of this equivalence relation are trivial to show. We thus only show the transitivity property of this equivalence relation. Suppose  $f, g, h$  are all measurable functions on  $X$  and we have “ $f = g$  almost everywhere” and “ $g = h$  almost everywhere”. Then we have countable sets  $A$  and  $B$  such that  $\forall x \in X - A, f(x) = g(x)$  and similarly  $\forall x \in X - B, g(x) = h(x)$ . Defining countable set  $C := A \cup B$ , we have  $\forall x \in X - C = (X - A) \cap (X - B), f(x) = g(x) = h(x)$ . So we are finished.

### Exercise 10

Let  $f : X \rightarrow [-\infty, \infty]$  be a Lebesgue integrable function, and let  $E, F \subseteq X$  be disjoint measurable sets. Prove that

$$\int_{E \cup F} f d\mu = \int_E f d\mu + \int_F f d\mu$$

Note that because  $E$  and  $F$  are disjoint,  $f\chi_{E \cup F} = f\chi_E + f\chi_F$ . Then we have:

$$\int_{E \cup F} f d\mu = \int_X f\chi_{E \cup F} d\mu = \int_X f\chi_E d\mu + \int_X f\chi_F d\mu = \int_E f d\mu + \int_F f d\mu$$

### Exercise 11

Let  $\{f_n\}$  be a sequence of non-negative measurable functions on  $X$ . Prove that  $\sum_{n \in \mathbb{N}} f_n$  is measurable, and that

$$\int_X \sum_{n \in \mathbb{N}} f_n d\mu = \sum_{n \in \mathbb{N}} \int_X f_n d\mu$$

We define the partial sums of these functions by  $g_n = \sum_{i=1}^n f_i$ . Note that  $\{g_n\}$  is a sequence of increasing measurable functions on  $X$  where  $g_n \uparrow \sum_{n \in \mathbb{N}} f_n$  as each  $f_n \geq 0$  (so  $\sum_{n \in \mathbb{N}} f_n$  is not an alternating series.) Then by Lebesgue's Monotone Convergence Theorem we have:

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X \lim_{n \rightarrow \infty} g_n d\mu = \int_X \sum_{n \in \mathbb{N}} f_n d\mu$$

But

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \lim_{n \rightarrow \infty} \int_X \left( \sum_{i=1}^n f_i \right) d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_X f_i d\mu = \sum_{n \in \mathbb{N}} \int_X f_n d\mu$$

where the last limit is established because each  $\int_X f_i d\mu \geq 0$ .

### Exercise 12

Let  $f : X \rightarrow [0, \infty)$  be a measurable function, let  $\{E_n\}$  be a sequence of pairwise disjoint, measurable subsets of  $X$ , and let  $E = \biguplus_{n \in \mathbb{N}} E_n$ . Prove that

$$\int_E f d\mu = \sum_{n \in \mathbb{N}} \int_{E_n} f d\mu$$

*Hint:* See previous exercise. We can define our sequence  $\{f_n\}$  of non-negative measurable functions on  $X$  as  $\{f\chi_{E_n}\}$ . Then  $f = \sum_{n \in \mathbb{N}} f_n$  and each  $\int_X f_n d\mu = \int_X f\chi_{E_n} d\mu = \int_{E_n} f d\mu$ . Using the previous exercise, we are finished.

### Exercise 13

Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 x^n dx = 0.$$

This question can be solved with an application of Lebesgue's Dominated Convergence Theorem. To do so rigorously, we define our (Lebesgue) measure space  $([0, 1], \mathcal{M}|_{[0,1]}, m|_{[0,1]})$  where  $\mathcal{M}$  is the Lebesgue measurable sets. Let us define our pointwise convergent sequence  $\{f_n\}$  of measurable functions on  $[0, 1]$  to be  $\{x_n\}$  (note that  $\forall x \in [0, 1], x^n \rightarrow 0$  as  $n \rightarrow \infty$ ). We define continuous and measurable<sup>a</sup> constant function  $g : [0, 1] \rightarrow [0, \infty]$  as  $g(x) = 1$ . Then note that:

$$\int_{[0,1]} g \, dm = \int_0^1 1 dx = 1 < \infty$$

and also that  $\forall n, |f_n| = |x^n| \leq 1$ . So by applying Lebesgue's Dominated Convergence Theorem, we have:

$$\lim_{n \rightarrow \infty} \int_{[0,1]} x^n dm = \int_{[0,1]} \lim_{n \rightarrow \infty} x^n dm = \int_{[0,1]} 0 \, dm = 0$$

But because each  $f_n = x_n$  is continuous,  $\int_{[0,1]} x^n dm = \int_0^1 x^n dx$  and so:

$$\lim_{n \rightarrow \infty} \int_{[0,1]} x^n dm = \lim_{n \rightarrow \infty} \int_0^1 x^n dx$$

Thus we are finished.

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<sup>a</sup>See Exercise 7 for justification.

### Exercise 14

Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 \tan^{-1}(nx) dx = \frac{\pi}{2}$$

*Hint: See answer to last exercise.* We use the same measure space as in the last exercise and define our pointwise convergent sequence  $\{f_n\}$  of measurable functions on  $[0, 1]$  to be  $\{\tan^{-1}(nx)\}$  (note that  $\forall x \in [0, 1], \tan^{-1}(nx) \rightarrow \frac{\pi}{2}$ ). We define continuous and measurable constant function  $g : [0, 1] \rightarrow [0, \infty]$  as  $g(x) = 2$  where:

$$\int_{[0,1]} g \, dm = \int_0^1 2 dx = 2 < \infty$$

and  $\forall n, |f_n| = |\tan^{-1}(nx)| \leq 2$ . So by Lebesgue's Dominated Convergence Theorem and the fact that each  $f_n = \tan^{-1}(nx)$  is continuous we have:

$$\lim_{n \rightarrow \infty} \int_0^1 \tan^{-1}(nx) dx = \lim_{n \rightarrow \infty} \int_{[0,1]} \tan^{-1}(nx) \, dm = \int_{[0,1]} \lim_{n \rightarrow \infty} \tan^{-1}(nx) \, dm = \int_0^1 \frac{\pi}{2} dx = \frac{\pi}{2}$$