

## Discretionary Note

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# MATH 255 HW 2

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## 1. Exercise 3.1 (5 points)

We prove this statement by contradiction: suppose  $\mathbb{N}$  is bounded above in  $\mathbb{R}$ . This means an upper bound  $\exists m \in \mathbb{R}$  for  $\mathbb{N}$ . Let us choose any arbitrary  $x > 0 \in \mathbb{N}, \mathbb{R}$ . By the Archimedean property of  $\mathbb{R}$ ,  $\exists n \in \mathbb{N}$  s.t.  $nx > m$ . Because  $n, x \in \mathbb{N} \implies nx \in \mathbb{N}$  and so we have shown  $\exists$  an element  $nx \in \mathbb{N}$  that is greater than  $m \implies m$  is not an upper bound for  $\mathbb{N} \implies \mathbb{N}$  is not bounded above in  $\mathbb{R}$ .

## 2. Exercise 3.2 (10 points)

- (1) To show that  $\sqrt{5}$  is algebraic, we show that there exists a polynomial where  $x = \sqrt{5}$  is the solution:

$$\begin{aligned}x &= \sqrt{5} \\x^2 &= 5 \\x^2 - 5 &= 0\end{aligned}$$

This polynomial is given by  $n = 2$  and  $a_0 = 1, a_1 = 0, a_2 = -5$ . Thus, we can conclude  $\sqrt{5}$  is algebraic. We take the same approach to show that  $\sqrt{2 + \sqrt{3}}$  is algebraic:

$$\begin{aligned}x &= \sqrt{2 + \sqrt{3}} \\x^2 &= 2 + \sqrt{3} \\x^2 - 2 &= \sqrt{3} \\(x^2 - 2)^2 &= 3 \\x^4 - 4x^2 + 4 - 3 &= 0 \\x^4 - 4x^2 + 1 &= 0\end{aligned}$$

This polynomial is given by  $n = 4$  and  $a_0 = 1, a_1 = a_2 = 0, a_3 = -4, a_4 = 1$ . Thus, we can conclude  $\sqrt{2 + \sqrt{3}}$  is algebraic.

- (2) Let us pick any given natural  $n \in \mathbb{N}$ . Given this  $n$ , we have to choose coefficients  $a_0, \dots, a_n$  all of which lie in  $\mathbb{Z}$ , a countable set. The union of these  $n$  countable sets to get the arrangement of coefficients  $a_0, \dots, a_n$  is countable. Because each of these arrangement of coefficients (i.e. a  $n$ -length tuple like  $(a_0, \dots, a_n)$ ) maps to at most  $n$  roots (i.e.  $n$  algebraic real numbers), the set of potential algebraic real numbers (i.e. a set of  $(a_0, \dots, a_n, x)$  where  $x$  is the algebraic real number) is countable. Finally, the infinite union of all these countable sets (i.e.  $\forall n \in \mathbb{N}$ ) is countable as well  $\implies$  the set of all algebraic real numbers is countable.
- (3) We prove this by contradiction and assume that all real numbers are algebraic. This means that the real numbers  $\mathbb{R}$  are an infinite subset of the algebraic real numbers, a countable set  $\implies \mathbb{R}$  is countable, which is a contradiction.

### 3. Exercise 3.3 (5 points)

**Note on notation:** Given  $p, q \in \mathbb{R}$ , we define the following notations for this problem:

(i)  $\mathbb{R}_{(p,q)} = \{x \in \mathbb{R} : p < x < q\}$  (ii)  $\mathbb{Q}_{(p,q)} = \{x \in \mathbb{Q} : p < x < q\}$  (iii)  $\mathbb{R} \setminus \mathbb{Q}_{(p,q)} = \{x \in \mathbb{R} / \mathbb{Q} : p < x < q\}$

We first establish the fact that there are uncountably many real numbers on the interval  $(a, b)$ . We have established by Cantor's theorem that there are uncountably many real numbers on the interval  $(0, 1)$ . We can create a trivial bijection  $f : \mathbb{R}_{(0,1)} \rightarrow \mathbb{R}_{(a,b)}$  as such:

$$f(x) = x(b - a) + a$$

To prove that this bijection establishes  $\mathbb{R}_{(a,b)}$  is uncountable, we proceed by contradiction. If  $\mathbb{R}_{(a,b)}$  was countable, that means there exists a bijection (let's call it  $g$ ) from  $\mathbb{N} \rightarrow \mathbb{R}_{(a,b)}$ . But then  $f^{-1} \circ g$  would be a bijection from  $\mathbb{N} \rightarrow \mathbb{R}_{(0,1)} \implies \mathbb{R}_{(0,1)}$  is countable which is a contradiction. Thus we have proved  $\mathbb{R}_{(a,b)}$  is uncountable.

We now prove there are uncountably many irrationals on the interval  $(a, b)$ . For proof by contradiction, let us now assume there are countably many irrationals on the interval  $(a, b)$ . We can call this set  $\mathbb{R} \setminus \mathbb{Q}_{(a,b)}$  where  $\mathbb{R}_{(a,b)}$  is given by  $\mathbb{R} \setminus \mathbb{Q}_{(a,b)} \cup \mathbb{Q}_{(a,b)}$ . Because  $\mathbb{Q}$  is countable, infinite subset  $\mathbb{Q}_{(a,b)} \subset \mathbb{Q}$ , is also countable. Because the union of two countable sets is countable,  $\mathbb{R} \setminus \mathbb{Q}_{(a,b)} \cup \mathbb{Q}_{(a,b)} = \mathbb{R}_{(a,b)}$  is countable which is a contradiction. Thus, we have proven  $\mathbb{R} \setminus \mathbb{Q}_{(a,b)}$  is uncountable.

### 4. Exercise 3.4 (10 points)

- (1) Let us define the set  $S_n$  where  $n \in \mathbb{N}$  to be set of all finite subsets of  $\mathbb{N}$  with size  $n$ . We first demonstrate that  $\forall n \in \mathbb{N}, S_n$  is countable.

For  $n = 1$ , we have  $S_1 = \{\{1\}, \{2\}, \dots\}$ . We can easily create a bijection  $f$  between  $\mathbb{N} \rightarrow S_1$ :  $f$  takes  $z \in \mathbb{N}$  and maps it to a set  $\{z\}$ . Because we can create such a bijection between  $\mathbb{N}$  to  $S_1 \implies S_1$  is countable.

For  $n = 2$ , we have  $S_2 = \{\{1, 2\}, \{1, 3\}, \dots, \{2, 3\}, \{2, 4\}, \dots\}$ . Note that this can be simplified as  $S_2 = \mathbb{N} \times \mathbb{N}$ . Because  $S_2$  is given by the Cartesian product of two countable sets ( $\mathbb{N}$ ), it is also countable.

We can see now that for  $n = 3$ ,  $S_3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ , and more generally for  $k \in \mathbb{N}$ ,  $S_k = \underbrace{\mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N}}_{k \text{ times}}$ . Because the Cartesian product of countably many countable sets is countable<sup>1</sup>, we can conclude that  $\forall k \in \mathbb{N}$ ,  $S_k$  is countable.

The set of all finite subsets of  $\mathbb{N}$  can be given as  $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$ . Because the union of countably many countable sets is countable,  $\bigcup_{k=1}^{\infty} S_k$  is countable. Furthermore, because the union of two sets that are at most countable,  $\emptyset$  (finite) and  $\bigcup_{k=1}^{\infty} S_k$  (countable), is at most countable, we can conclude  $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$  is countable<sup>2</sup>. Thus we have proved the set of all finite subsets of  $\mathbb{N}$ ,  $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$ , is countable.

- (2) For proof by contradiction, let us assume that the set of all subsets of  $\mathbb{N}$  are countable. This means that we can list out the set of subsets of  $\mathbb{N}$  as such:

1.  $\emptyset$
2.  $\{1\}$
3.  $\{2\}$
4.  $\{1, 2\}$
- ...

We now try to assemble a new set not in this list. For the  $i$ th set in this list, if the set does not contain 1, we add 1 to this new set. If the  $i$ th set does contain 1, then we do not add 1 to this new set. This means that for all subsets of  $\mathbb{N}$  listed above, none can equal our new set as we have constructed them to be different by either inclusion/exclusion of 1  $\implies$  we can find a subset of  $\mathbb{N}$  not in the above list  $\implies \mathbb{N}$  is uncountable.

- (3) **Lemma 0.1** *Let  $A, B$  be sets where  $A$  is uncountable and  $A \subset B$ . We prove that  $B$  is uncountable. We proceed by proof by contradiction and assume that  $B$  is countable. Because  $A$  is then a subset of a countable set  $\implies A$  is at most countable, which is a contradiction. Thus, we have proved that a superset of an uncountable set must be uncountable.*

Let us define a given polynomial  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  as  $f(x) = \sum_{i=0}^{\infty} a_i x^i$  where  $\forall i \in \mathbb{Z}$ ,  $a_i \in \mathbb{Q}$ . We can represent polynomial  $f$  as a set given by  $\{a_0, a_1, a_2, a_3, \dots\}$ . Note that this representation is bijective, meaning each distinct polynomial has only one unique representation.

We approach this proof by showing that the set of all polynomials from  $\mathbb{Q} \rightarrow \mathbb{Q}$  are uncountable. For proof by contradiction, let us assume that the set of all polynomials from  $\mathbb{Q} \rightarrow \mathbb{Q}$  are countable. This would mean that we could list them all out as such:

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<sup>1</sup>We would prove this simply through induction on the fact that the Cartesian product of two countable sets is countable.

<sup>2</sup>For a short proof on how  $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$  is most countable  $\implies$  it is countable, we can assume for contradiction that it is finite which would imply that  $\bigcup_{k=1}^{\infty} S_k$  is finite, which is a contradiction as we have proved it is countable.

1.  $\{1, 0, 0, 0, \dots\}$
2.  $\{0, 1, 0, 0, \dots\}$
3.  $\{0, 0, 1, 0, \dots\}$
- ...

We now try to assemble a new polynomial  $g$  not in this list. For the  $i$ th polynomial in this list, if the polynomial has  $a_i = 0$ , we set  $a_i = 1$  for  $g$  and if the polynomial has  $a_i \neq 0$ , we set  $a_i = 0$  for  $g$ . This means that for all the polynomials from  $\mathbb{Q} \rightarrow \mathbb{Q}$  listed above, none can equal our new polynomial  $g$  as we have constructed  $g$  to be different to each listed polynomial by at least one coefficient  $\implies$  we can find a polynomial  $g$  not in the above list  $\implies$  the set of polynomials from  $\mathbb{Q} \rightarrow \mathbb{Q}$  is uncountable.

The set of polynomials from  $\mathbb{Q} \rightarrow \mathbb{Q}$  is a subset of the set of all functions from  $\mathbb{Q} \rightarrow \mathbb{Q}$ . By **Lemma 0.1**, a superset of an uncountable set is uncountable, and so because the set of all polynomials from  $\mathbb{Q} \rightarrow \mathbb{Q}$  is uncountable  $\implies$  the set of all functions from  $\mathbb{Q} \rightarrow \mathbb{Q}$  is uncountable.

### 5. Exercise 3.5 (10 points)

**Lemma 0.2** *By the triangle inequality, if  $x, y \in \mathbb{R}^n$  for  $n \in \mathbb{N}$ ,  $\|x + y\| \leq \|x\| + \|y\|$ . Defining  $a, b, c \in \mathbb{R}^n$ , we can set  $x = a - c$ ,  $y = c - b$  and yield the following result:*

$$\begin{aligned}\|a - c + c - b\| &\leq \|a - c\| + \|c - b\| \\ \|a - b\| &\leq \|a - c\| + \|c - b\|\end{aligned}$$

In  $\mathbb{R}$ , this can be given as  $|a - b| \leq |a - c| + |c - b|$ .

- (1) For  $d$  to be a metric space,  $\forall x, y \in X, d(x, y) > 0$  if  $x \neq y$ . Let us define  $x \in X$  and  $y = -x$  so that  $y \neq x$ .  $d(x, y) = |x^2 - y^2| = |x^2 - (-x)^2| = |x^2 - x^2| = |0| = 0$  and so this property is violated  $\implies d$  is not a metric space.
- (2) For  $d$  to be a metric space,  $\forall x \in X, d(x, x) = 0$ . Because  $\forall x \neq 0 \in X, d(x, x) = |x - 2x| = |-x| = |x| \neq 0$ , this means that it is not guaranteed  $\forall x \in X, d(x, x) = 0 \implies d$  is not a metric space.
- (3) We show  $d$  is a metric space below by showing  $d$  satisfies the four properties of metric spaces:
  - (i)  $\forall x, y \in X$ , if  $x \neq y \implies x - y \neq 0 \implies |x - y| > 0 \implies d(x, y) = \frac{|x - y|}{1 + |x - y|} > 0$ .
  - (ii)  $\forall x \in X, x - x = 0 \implies |x - x| = 0 \implies d(x, x) = \frac{|x - x|}{1 + |x - x|} = \frac{0}{1 + 0} = 0$
  - (iii)  $\forall x, y \in X, d(x, y) = \frac{|x - y|}{1 + |x - y|} = \frac{|y - x|}{1 + |y - x|} = d(y, x)$

- (iv) By **Lemma 0.2**, we have  $\forall x, y, r \in X = \mathbb{R}, |x - y| \leq |x - r| + |r - y|$ . Note that because  $|x - r|$  and  $|x - y|$  are  $\geq 0$ ,  $1 + |x - r|$  and  $1 + |x - y|$  are  $\geq 1$ . Thus we have  $|x - r| \geq \frac{|x - r|}{1 + |x - r|}$ ,  $|x - y| \geq \frac{|x - y|}{1 + |x - y|}$ , and  $|r - y| \geq \frac{|r - y|}{1 + |r - y|}$ . We apply these inequalities to show  $d$  obeys the Triangle Inequality:

$$\begin{aligned} |x - y| &\leq |x - r| + |r - y| \\ \frac{|x - y|}{1 + |x - y|} &\leq |x - y| \leq \frac{|x - r|}{1 + |x - r|} + \frac{|r - y|}{1 + |r - y|} \leq |x - r| + |r - y| \end{aligned}$$

By transitivity,

$$\begin{aligned} \frac{|x - y|}{1 + |x - y|} &\leq \frac{|x - r|}{1 + |x - r|} + \frac{|r - y|}{1 + |r - y|} \\ d(x, y) &\leq d(x, r) + d(r, y) \end{aligned}$$

- (4) We show  $d$  is a metric space below by showing  $d$  satisfies the four properties of metric spaces:
- (i)  $\forall x, y \in X$ , if  $x \neq y \implies$  at least one of the following is true: (1)  $x_1 \neq y_1$   
(2)  $x_2 \neq y_2 \implies$  at least one of the following is true: (i)  $|x_1 - y_1| > 0$  (ii)  $|x_2 - y_2| > 0 \implies d(x, y) = |x_1 - y_1| + |x_2 - y_2| > 0$ .
  - (ii)  $\forall x \in X, d(x, x) = |x_1 - x_1| + |x_2 - x_2| = |0| + |0| = 0 + 0 = 0$
  - (iii)  $\forall x, y \in X, d(x, y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y, x)$
  - (iv) Let us define  $x, y, r \in X$ . By **Lemma 0.2**, we have the following statements:

$$\begin{aligned} |x_1 - y_1| &\leq |x_1 - r_1| + |r_1 - y_1| \\ |x_2 - y_2| &\leq |x_2 - r_2| + |r_2 - y_2| \end{aligned}$$

Adding these inequalities together we have:

$$\begin{aligned} |x_1 - y_1| + |x_2 - y_2| &\leq |x_1 - r_1| + |r_1 - y_1| + |x_2 - r_2| + |r_2 - y_2| \\ d(x, y) &\leq |x_1 - r_1| + |x_2 - r_2| + |r_1 - y_1| + |r_2 - y_2| \\ d(x, y) &\leq d(x, r) + d(r, y) \end{aligned}$$

and so we have proven the Triangle Inequality for  $d$ .

- (5) For  $d$  to be a metric space,  $\forall x \in X, d(x, x) = 0$ . Consider  $x = (x_1, x_2) \in \mathbb{R}^2$  where  $x_1 \neq x_2$ . Then  $d(x, x) = |x_1 - x_2| + |x_2 - x_1| = 2|x_1 - x_2|$ . Because  $x_1 \neq x_2, x_1 - x_2 \neq 0 \implies 2|x_1 - x_2| \neq 0$  and so it is not guaranteed  $\forall x \in X, d(x, x) = 0 \implies d$  is not a metric space.