Problem set 7

Exercise 7.1 (10 points; Rudin 3.2, modified). Calculate $\lim_{n\to\infty} \sqrt{n^2+n} - n$, and prove that your answer is correct. (Hint: first show that $\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}$.)

Exercise 7.2 (10 points). For any two bounded real sequences (a_n), (b_n) in $\mathbb{R}$ prove that

$$\lim_{n\to\infty} \sup(a_n + b_n) \le \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n.$$

Give an example where this $\le$ is $<$, and an example where it is $=$.

Exercise 7.3 (10 points; Rudin 3.24). Suppose (p_n) and (q_n) are Cauchy sequences in a metric space X. Prove that the sequence $(d(p_n, q_n))$ in $\mathbb{R}$ has a limit.

Exercise 7.4 (10 points). Suppose (x_n) is a sequence in $\mathbb{R}$. We say $a \in \mathbb{R}$ is an *essential upper bound* for (x_n) if there exists some N such that, for all $n \ge N$, $x_n \le a$.

Prove that

$$\lim_{n\to\infty} \sup x_n \;=\; \inf\{a \in \mathbb{R} \mid a \text{ is an essential upper bound for } (x_n)\}.$$

Exercise 7.5 (not for credit; Rudin 3.25, in part). Let X be a metric space. X might or might not be complete. If X is not complete, it would be nice to know how to "fill in the holes" to make it complete. This exercise explains a way of doing so: it constructs a new complete metric space X^* which has X as a subset.

We call two Cauchy sequences (p_n), (q_n) in X *equivalent* if $d(p_n, q_n) \to 0$. We write this relation as $(p_n) \sim (q_n)$.

(1) Prove that this is an equivalence relation, i.e.
   (a) Any Cauchy sequence (p_n) has $(p_n) \sim (p_n)$,
   (b) If $(p_n) \sim (q_n)$ then $(q_n) \sim (p_n)$,
   (c) If $(p_n) \sim (q_n)$ and $(q_n) \sim (r_n)$, then $(p_n) \sim (r_n)$.

(2) If $(p_n) \sim (p'_n)$ and $(q_n) \sim (q'_n)$, prove that

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n).$$

(Note that the limit does exist, by the result of the previous exercise.)

Now we divide the set of Cauchy sequences in X into *equivalence classes*: any two elements of a given class P are equivalent, and elements of different classes P, Q are not equivalent. Let X^* be the set of all equivalence classes of Cauchy sequences in X.

Then, define a distance function Δ on X^* as follows: if (p_n) is in the class P, and (q_n) is in the class Q, then

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n).$$

The previous parts show that this distance function is well defined.

(3) Prove that the distance function Δ makes X^* into a metric space.

(4) Prove that X^* with this distance function is complete.

(5) Consider the map $\phi : X \to X^*$ which maps any $x \in X$ to the class of the Cauchy sequence $(x, x, x, \dots)$. Prove that ϕ is injective and $\Delta(\phi(x), \phi(y)) = d(x, y)$.

Exercise 7.6 (not for credit). Suppose X is any metric space, with a distance function d. Then define a new distance function d' on X by

$$d'(x, y) = \min\{d(x, y), 1\}\,.$$

(1) Prove that d' indeed makes X into a metric space.

(2) Prove that a sequence is Cauchy for d if and only if it is Cauchy for d'.

(3) Prove that a sequence is convergent for d if and only if it is convergent for d'.

Exercise 7.7 (not for credit). Suppose X is any metric space. We say X is *totally bounded* if, for every $\epsilon > 0$, X can be covered by finitely many neighborhoods $N_\epsilon(x)$. Prove that a subset $E \subset X$ is compact if and only if E is closed and totally bounded.