Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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MATH 255 HW 2

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1. **Exercise 3.1 (5 points)**

   We prove this statement by contradiction: suppose $\mathbb{N}$ is bounded above in $\mathbb{R}$. This means an upper bound $\exists\ m \in \mathbb{R}$ for $\mathbb{N}$. Let us choose any arbitrary $x > 0 \in \mathbb{N}, \mathbb{R}$. By the Archimedean property of $\mathbb{R}$, $\exists\ n \in \mathbb{N}$ s.t. $nx > m$. Because $n, x \in \mathbb{N} \implies nx \in \mathbb{N}$ and so we have shown $\exists$ an element $nx \in \mathbb{N}$ that is greater than $m \implies m$ is not an upper bound for $\mathbb{N} \implies \mathbb{N}$ is not bounded above in $\mathbb{R}$.

2. **Exercise 3.2 (10 points)**

   (1) To show that $\sqrt{5}$ is algebraic, we show that there exists a polynomial where $x = \sqrt{5}$ is the solution:

   $$\begin{gathered} x = \sqrt{5} \\ x^2 = 5 \\ x^2 - 5 = 0 \end{gathered}$$

   This polynomial is given by $n = 2$ and $a_0 = 1, a_1 = 0, a_2 = -5$. Thus, we can conclude $\sqrt{5}$ is algebraic. We take the same approach to show that $\sqrt{2+\sqrt{3}}$ is algebraic:

   $$\begin{gathered} x = \sqrt{2+\sqrt{3}} \\ x^2 = 2 + \sqrt{3} \\ x^2 - 2 = \sqrt{3} \\ (x^2-2)^2 = 3 \\ x^4 - 4x^2 + 4 - 3 = 0 \\ x^4 - 4x^2 + 1 = 0 \end{gathered}$$

   This polynomial is given by $n = 4$ and $a_0 = 1, a_1 = a_2 = 0, a_3 = -4, a_4 = 1$. Thus, we can conclude $\sqrt{2+\sqrt{3}}$ is algebraic.

(2) Let us pick any given natural $n \in \mathbb{N}$. Given this n, we have to choose coefficients $a_0, \ldots, a_n$ all of which lie in $\mathbb{Z}$, a countable set. The union of these n countable sets to get the arrangement of coefficients $a_0, \ldots, a_n$ is countable. Because each of these arrangement of coefficients (i.e. a n-length tuple like $(a_0, \ldots, a_n)$) maps to at most n roots (i.e. n algebraic real numbers), the set of potential algebraic real numbers (i.e. a set of $(a_0, \ldots, a_n, x)$ where x is the algebraic real number) is countable. Finally, the infinite union of all these countable sets (i.e. $\forall n \in \mathbb{N}$) is countable as well $\implies$ the set of all algebraic real numbers is countable.

(3) We prove this by contradiction and assume that all real numbers are algebraic. This means that the real numbers $\mathbb{R}$ are an infinite subset of the algebraic real numbers, a countable set $\implies$ $\mathbb{R}$ is countable, which is a contradiction.

3. **Exercise 3.3 (5 points)**

Note on notation: *Given $p, q \in \mathbb{R}$, we define the following notations for this problem: (i) $\mathbb{R}_{(p,q)} = \{x \in \mathbb{R} : p < x < q\}$ (ii) $\mathbb{Q}_{(p,q)} = \{x \in \mathbb{Q} : p < x < q\}$ (iii) $\mathbb{R}\backslash\mathbb{Q}_{(p,q)} = \{x \in \mathbb{R}/\mathbb{Q} : p < x < q\}$*

We first establish the fact that their are uncountably many real numbers on the interval (a, b). We have established by Cantor's theorem that there are uncountably many real numbers on the interval $(0, 1)$. We can create a trivial bijection $f : \mathbb{R}_{(0,1)} \to \mathbb{R}_{(a,b)}$ as such:

$$f(x) = x(b - a) + a$$

To prove that this bijection establishes $\mathbb{R}_{(a,b)}$ is uncountable, we proceed by contradiction. If $\mathbb{R}_{(a,b)}$ was countable, that means there exists a bijection (let's call it g) from $\mathbb{N} \to \mathbb{R}_{(a,b)}$. But then $f^{-1} \circ g$ would be a bijection from $\mathbb{N} \to \mathbb{R}_{(0,1)} \implies \mathbb{R}_{(0,1)}$ is countable which is a contradiction. Thus we have proved $\mathbb{R}_{(a,b)}$ is uncountable.

We now prove there are uncountably many irrationals on the interval (a, b). For proof by contradiction, let us now assume there are countably many irrationals on the interval (a, b). We can call this set $\mathbb{R}\backslash\mathbb{Q}_{(a,b)}$ where $\mathbb{R}_{(a,b)}$ is given by $\mathbb{R}\backslash\mathbb{Q}_{(a,b)} \cup \mathbb{Q}_{(a,b)}$. Because $\mathbb{Q}$ is countable, infinite subset $\mathbb{Q}_{(a,b)} \subset \mathbb{Q}$, is also countable. Because the union of two countable sets is countable, $\mathbb{R}\backslash\mathbb{Q}_{(a,b)} \cup \mathbb{Q}_{(a,b)} = \mathbb{R}_{(a,b)}$ is countable which is a contradiction. Thus, we have proven $\mathbb{R}\backslash\mathbb{Q}_{(a,b)}$ is uncountable.

4. **Exercise 3.4 (10 points)**

(1) Let us define the set S_n where $n \in \mathbb{N}$ to be set of all finite subsets of $\mathbb{N}$ with size n. We first demonstrate that $\forall n \in \mathbb{N}, S_n$ is countable.

For $n = 1$, we have $S_1 = \{\{1\}, \{2\}, \ldots\}$. We can easily create a bijection f between $\mathbb{N} \to S_1$: f takes $z \in \mathbb{N}$ and maps it to a set $\{z\}$. Because we can create such a bijection between $\mathbb{N}$ to $S_1 \implies S_1$ is countable.

For $n = 2$, we have $S_2 = \{\{1, 2\}, \{1, 3\}, \ldots, \{2, 3\}, \{2, 4\}, \ldots\}$. Note that this can be simplified as $S_2 = \mathbb{N} \times \mathbb{N}$. Because S_2 is given by the Cartesian product of two countable sets ($\mathbb{N}$), it is also countable.

We can see now that for $n = 3$, $S_3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and more generally for $k \in \mathbb{N}, S_k = \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \mathbb{N}}_{k \text{ times}}$. Because the Cartesian product of countably many countable sets is countable[1], we can conclude that $\forall k \in \mathbb{N}, S_k$ is countable.

The set of all finite subsets of $\mathbb{N}$ can be given as $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$. Because the union of countably many countable sets is countable, $\bigcup_{k=1}^{\infty} S_k$ is countable. Furthermore, because the union of two sets that are at most countable, $\emptyset$ (finite) and $\bigcup_{k=1}^{\infty} S_k$ (countable), is at most countable, we can conclude $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$ is countable[2]. Thus we have proved the set of all finite subsets of $\mathbb{N}$, $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$, is countable.

(2) For proof by contradiction, let us assume that the set of all subsets of $\mathbb{N}$ are countable. This means that we can list out the set of subsets of $\mathbb{N}$ as such:

$$
\begin{aligned}
&1.\emptyset \\
&2.\{1\} \\
&3.\{2\} \\
&4.\{1, 2\} \\
&\ldots
\end{aligned}
$$

We now try to assemble a new set not in this list. For the ith set in this list, if the set does not contain 1, we add 1 to this new set. If the ith set does contain 1, then we do not add 1 to this new set. This means that for all subsets of $\mathbb{N}$ listed above, none can equal our new set as we have constructed them to be different by either inclusion/exclusion of 1 $\implies$ we can find a subset of $\mathbb{N}$ not in the above list $\implies$ $\mathbb{N}$ is uncountable.

(3) **Lemma 0.1** *Let A, B be sets where A is uncountable and $A \subset B$. We prove that B is uncountable. We proceed by proof by contradiction and assume that B is countable. Because A is then a subset of a countable set $\implies$ A is at most countable, which is a contradiction. Thus, we have proved that a superset of an uncountable set must be uncountable.*

Let us define a given polynomial $f : \mathbb{Q} \to \mathbb{Q}$ as $f(x) = \Sigma_{i=0}^{\infty} a_i x^i$ where $\forall i \in \mathbb{Z}, a_i \in \mathbb{Q}$. We can represent polynomial f as a set given by $\{a_0, a_1, a_2, a_3, \ldots\}$. Note that this representation is bijective, meaning each distinct polynomial has only one unique representation.

We approach this proof by showing that the set of all polynomials from $\mathbb{Q} \to \mathbb{Q}$ are uncountable. For proof by contradiction, let us assume that the set of all polynomials from $\mathbb{Q} \to \mathbb{Q}$ are countable. This would mean that we could list them all out as such:

[1] We would prove this simply through induction on the fact that the Cartesian product of two countable sets is countable.

[2] For a short proof on how $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$ is most countable $\implies$ it is countable, we can assume for contradiction that it is finite which would imply that $\bigcup_{k=1}^{\infty} S_k$ is finite, which is a contradiction as we have proved it is countable.

$$1.\{1,0,0,0\dots\}$$
$$2.\{0,1,0,0,\dots\}$$
$$3.\{0,0,1,0,\dots\}$$
$$\dots$$

We now try to assemble a new polynomial g not in this list. For the ith polynomial in this list, if the polynomial has $a_i = 0$, we set $a_i = 1$ for g and if the polynomial has $a_i \neq 0$, we set $a_i = 0$ for g. This means that for all the polynomials from $\mathbb{Q} \to \mathbb{Q}$ listed above, none can equal our new polynomial g as we have constructed g to different to each listed polynomial by at least one coefficient $\implies$ we can find a polynomial g not in the above list $\implies$ the set of polynomials from $\mathbb{Q} \to \mathbb{Q}$ is uncountable.

The set of polynomials from $\mathbb{Q} \to \mathbb{Q}$ is a subset of the set of all functions from $\mathbb{Q} \to \mathbb{Q}$. By **Lemma 0.1**, a superset of an uncountable set is uncountable, and so because the set of all polynomials from $\mathbb{Q} \to \mathbb{Q}$ is uncountable $\implies$ the set of all functions from $\mathbb{Q} \to \mathbb{Q}$ is uncountable.

5. **Exercise 3.5 (10 points)**

Lemma 0.2 *By the triangle inequality, if $x, y \in \mathbb{R}^n$ for $n \in \mathbb{N}$, $||x+y|| \leq ||x|| + ||y||$. Defining $a, b, c \in \mathbb{R}^n$, we can set $x = a - c, y = c - b$ and yield the following result:*

$$||a-c+c-b|| \leq ||a-c|| + ||c-b||$$
$$||a-b|| \leq ||a-c|| + ||c-b||$$

In $\mathbb{R}$, this can be given as $|a-b| \leq |a-c| + |c-b|$.

(1) For d to be a metric space, $\forall x, y \in X, d(x,y) > 0$ if $x \neq y$. Let us define $x \in X$ and $y = -x$ so that $y \neq x$. $d(x,y) = |x^2 - y^2| = |x^2 - (-x)^2| = |x^2 - x^2| = |0| = 0$ and so this property is violated $\implies$ d is not a metric space.

(2) For d to be a metric space, $\forall x \in X, d(x,x) = 0$. Because $\forall x \neq 0 \in X, d(x,x) = |x - 2x| = |-x| = |x| \neq 0$, this means that it is not guaranteed $\forall x \in X, d(x,x) = 0 \implies d$ is not a metric space.

(3) We show d is a metric space below by showing d satisfies the four properties of metric spaces:

(i) $\forall x, y \in X,$ if $x \neq y \implies x - y \neq 0 \implies |x-y| > 0 \implies d(x,y) = \frac{|x-y|}{1+|x-y|} > 0.$

(ii) $\forall x \in X, x - x = 0 \implies |x-x| = 0 \implies d(x,x) = \frac{|x-x|}{1+|x-x|} = \frac{0}{1+0} = 0$

(iii) $\forall x, y \in X, d(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|} = d(y,x)$

(iv) By **Lemma 0.2**, we have $\forall x, y, r \in X = \mathbb{R}, |x-y| \leq |x-r| + |r-y|$. Note that because $|x-r|$ and $|x-y|$ are ≥ 0, $1+|x-r|$ and $1+|x-y|$ are ≥ 1. Thus we have $|x-r| \geq \frac{|x-r|}{1+|x-r|}$, $|x-y| \geq \frac{|x-y|}{1+|x-y|}$, and $|r-y| \geq \frac{|r-y|}{1+|r-y|}$. We apply these inequalities to show d obeys the Triangle Inequality:

$$|x-y| \leq |x-r| + |r-y|$$
$$\frac{|x-y|}{1+|x-y|} \leq |x-y| \leq \frac{|x-r|}{1+|x-r|} + \frac{|r-y|}{1+|r-y|} \leq |x-r| + |r-y|$$

By transitivity,

$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-r|}{1+|x-r|} + \frac{|r-y|}{1+|r-y|}$$
$$d(x,y) \leq d(x,r) + d(r,y)$$

(4) We show d is a metric space below by showing d satisfies the four properties of metric spaces:

(i) $\forall x, y \in X$, if $x \neq y \implies$ at least one of the following is true: (1) $x_1 \neq y_1$ (2) $x_2 \neq y_2 \implies$ at least one of the following is true: (i) $|x_1 - y_1| > 0$ (ii) $|x_2 - y_2| > 0 \implies d(x,y) = |x_1 - y_1| + |x_2 - y_2| > 0$.

(ii) $\forall x \in X, d(x,x) = |x_1 - x_1| + |x_2 - x_2| = |0| + |0| = 0 + 0 = 0$

(iii) $\forall x, y \in X, d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y,x)$

(iv) Let us define $x, y, r \in X$. By **Lemma 0.2**, we have the following statements:

$$|x_1 - y_1| \leq |x_1 - r_1| + |r_1 - y_1|$$
$$|x_2 - y_2| \leq |x_2 - r_2| + |r_2 - y_2|$$

Adding these inequalities together we have:

$$|x_1 - y_1| + |x_2 - y_2| \leq |x_1 - r_1| + |r_1 - y_1| + |x_2 - r_2| + |r_2 - y_2|$$
$$d(x,y) \leq |x_1 - r_1| + |x_2 - r_2| + |r_1 - y_1| + |r_2 - y_2|$$
$$d(x,y) \leq d(x,r) + d(r,y)$$

and so we have proven the Triangle Inequality for d.

(5) For d to be a metric space, $\forall x \in X, d(x,x) = 0$. Consider $x = (x_1, x_2) \in \mathbb{R}^2$ where $x_1 \neq x_2$. Then $d(x,x) = |x_1 - x_2| + |x_2 - x_1| = 2|x_1 - x_2|$. Because $x_1 \neq x_2, x_1 - x_2 \neq 0 \implies 2|x_1 - x_2| \neq 0$ and so it is not guaranteed $\forall x \in X, d(x,x) = 0 \implies$ d is not a metric space.