

Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

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MATH 241 PSET 8

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1.

- a) The PDF of $T \sim \text{Expo}(\lambda)$ is given by $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. We define the half-life time as H . The half-life time is given as the time that $P(T \leq H) = 0.5$. We solve for H below:

$$\begin{aligned}P(T \leq H) &= 0.5 \\ \int_0^H f(x) dx &= 0.5 \\ \int_0^H \lambda e^{-\lambda x} dx &= 0.5 \\ \int_0^H e^{-\lambda x} dx &= \frac{1}{2\lambda} \\ -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^H &= \frac{1}{2\lambda} \\ -\frac{1}{\lambda} (e^{-\lambda H} - 1) &= \frac{1}{2\lambda} \\ e^{-\lambda H} - 1 &= -\frac{1}{2} \\ e^{-\lambda H} &= \frac{1}{2} \\ H &= \frac{-\ln(0.5)}{\lambda} = \frac{\ln(2)}{\lambda}\end{aligned}$$

- b) To ease our computations, we first find the CDF of T , $F(x) = \int_{-\infty}^x f(t) dt = \int_0^x \lambda e^{-\lambda t} dt = \lambda \int_0^x e^{-\lambda t} dt = -\frac{\lambda}{\lambda} e^{-\lambda t} \Big|_0^x = -1(e^{-\lambda x} - 1) = 1 - e^{-\lambda x}$ for $x > 0$. The probability a particle decays in the time interval $[t, t + \epsilon]$, given that it has survived since time t , is given by $P(t \leq T \leq t + \epsilon | T > t)$. Using Bayes Rule:

$$\begin{aligned}
P(t \leq T \leq t + \epsilon | T > t) &= \frac{P(t \leq T \leq t + \epsilon \cap T > t)}{P(T > t)} = \frac{P(t \leq T \leq t + \epsilon)}{P(T > t)} = \\
&= \frac{F(t + \epsilon) - F(t)}{1 - F(t)} = \frac{1 - e^{-\lambda(t+\epsilon)} - (1 - e^{-\lambda t})}{1 - (1 - e^{-\lambda t})} = \frac{-(e^{-\lambda t} e^{-\lambda \epsilon}) + e^{-\lambda t}}{e^{-\lambda t}} = 1 - e^{-\lambda \epsilon} = \\
&= 1 - (e^\epsilon)^{-\lambda} \approx 1 - (1 + \epsilon)^{-\lambda}
\end{aligned}$$

Using a first-degree Taylor series expansion of $(1+\epsilon)^{-\lambda}$ about $\epsilon \approx 0$, we get $(1+\epsilon)^{-\lambda} \approx (1+0)^{-\lambda} - \lambda\epsilon(1+0)^{-\lambda-1}$ or that $(1+\epsilon)^{-\lambda} \approx 1 - \lambda\epsilon$. Thus, $P(t \leq T \leq t + \epsilon | T > t) \approx 1 - (1+\epsilon)^{-\lambda} \approx \lambda\epsilon$, and so $P(t \leq T \leq t + \epsilon | T > t)$ is approximately proportional to ϵ . Furthermore, there is no t term present and so we have shown that this probability does not depend on t .

- c) From Example 5.6.3, we know that $L \sim \text{Expo}(n\lambda)$. The CDF of L can be given as, for $x \geq 0$, $F(x) = P(L \leq x) = 1 - P(L > x) = 1 - \prod_{i=1}^n P(T_i \geq x) = 1 - (e^{-\lambda x})^n = 1 - e^{-n\lambda x}$. Furthermore, as we know $L \sim \text{Expo}(n\lambda)$, $E[L] = \frac{1}{n\lambda}$ and $\text{Var}(L) = \frac{1}{(n\lambda)^2}$.
- d) We can model this scenario as $M = Z_1 + Z_2 + \dots + Z_n$, where Z_i is the time for the i th particle to decay. Because the time for the i th particle to decay (i.e. Z_i) is given as the minimum time to decay of all the $n - i + 1$ particles which have not decayed, $Z_i \sim \text{Expo}((n - i + 1)\lambda)$. From my work in part (c), we know that $E[Z_i] = \frac{1}{(n-i+1)\lambda}$. Thus, we can compute the expectation of M below as:

$$\begin{aligned}
M &= \sum_{i=1}^n Z_i \\
E[M] &= \sum_{i=1}^n E[Z_i] \\
E[M] &= \sum_{i=1}^n \frac{1}{(n - i + 1)\lambda} \\
E[M] &= \frac{1}{\lambda} \left(\frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{1} \right)
\end{aligned}$$

Thus, $E[M] = \frac{H_n}{\lambda}$, where H_n is the n th harmonic number.

Due to the memoryless property, Z_1, \dots, Z_n are all independent. From my work in part (c), we know $\text{Var}(Z_i) = \frac{1}{(n-i+1)^2\lambda^2}$. Thus, we can compute the $\text{Var}(M)$ as such:

$$\begin{aligned}
M &= \sum_{i=1}^n Z_i \\
\text{Var}(M) &= \sum_{i=1}^n \text{Var}(Z_i) \\
\text{Var}(M) &= \sum_{i=1}^n \frac{1}{(n - i + 1)^2\lambda^2} \\
\text{Var}(M) &= \frac{1}{\lambda^2} \left(\frac{1}{n^2} + \frac{1}{(n-1)^2} + \dots + 1 \right)
\end{aligned}$$

- a) Because X, Y are independent and identically distributed, distributions X^2 and Y^2 are as well and thus have the same MGF. Thus, we can compute the MGF of $W = X^2 + Y^2$ as so:

$$M_W(t) = M_{X^2+Y^2}(t) = M_{X^2}(t) \cdot M_{Y^2}(t) = ((1-2t)^{-\frac{1}{2}})^2 = \frac{1}{1-2t} = \frac{0.5}{0.5-t}$$

- b) The distribution W has an MGF of the form of an Exponential Distribution MGF with $\lambda = 0.5$. Thus, $W \sim Expo(0.5)$.

3.

The MGF of the Geometric distribution is given by $M(t) = \frac{p}{1-qe^t}$. We first compute $E[X]$ of this distribution below by applying the formula $E[X^n] = M^{(n)}(0)$.

$$\begin{aligned} E[X^1] &= M^{(1)}(0) \\ E[X] &= \left(\frac{p}{1-qe^t}\right)'(0) \\ E[X] &= \left(-\frac{p}{(1-qe^t)^2} \cdot (-qe^t)\right)(0) \\ E[X] &= \left(\frac{pqe^t}{(1-qe^t)^2}\right)(0) \\ E[X] &= \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p} \end{aligned}$$

Thus, we get that the mean of the distribution $E[X] = \frac{q}{p}$. We now use this same formula $E[X^n] = M^{(n)}(0)$ to compute $E[X^2]$:

$$\begin{aligned} E[X^2] &= M^{(2)}(0) \\ E[X^2] &= \left(\frac{p}{1-qe^t}\right)''(0) \\ E[X^2] &= \left(\frac{pqe^t}{(1-qe^t)^2}\right)'(0) \\ E[X^2] &= \left(pq \left(\frac{e^t(1-qe^t)^2 + 2qe^{2t}(1-qe^t)}{(1-qe^t)^4}\right)\right)(0) \\ E[X^2] &= pq \frac{(1-q)^2 + 2q(1-q)}{(1-q)^4} = \frac{pq}{(1-q)^2} + \frac{2p^2q^2}{(1-q)^4} = \frac{pq}{p^2} + \frac{2p^2q^2}{p^4} = \frac{q}{p} + \frac{2q^2}{p^2} \end{aligned}$$

Given, $E[X^2] = \frac{q}{p} + \frac{2q^2}{p^2}$ and $E[X] = \frac{q}{p}$, we now compute $Var(X) = E[X^2] - (E[X])^2$ as:

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ Var(X) &= \frac{q}{p} + \frac{2q^2}{p^2} - \left(\frac{q}{p}\right)^2 = \frac{q}{p} + \frac{q^2}{p^2} = \frac{q}{p} + \frac{q(1-p)}{p^2} = \frac{q}{p} + \frac{q}{p^2} - \frac{qp}{p^2} = \frac{q}{p} + \frac{q}{p^2} - \frac{q}{p} = \frac{q}{p^2} \end{aligned}$$

4.

We compute the MGF of $X \sim Expo(1)$ as $M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} e^{-x} dx = \frac{1}{1-t}$ for $t < 1$. We can now compute the MGF of distribution $-Y = -X$ as $M_{-Y}(t) = E[e^{t(-Y)}] = E[e^{-tY}] = E[e^{-tX}] = M_X(-t)$ for $-t < 1 \Rightarrow t > -1$.

Because we are given that X and Y are independent, X and $-Y$ are independent. Thus, the MGF of $L = X + (-Y)$ is given by $M_L(t) = M_X(t) \cdot M_{-Y}(t)$ for $-1 < t < 1$. We compute $M_L(t)$ below for $-1 < t < 1$

$$\begin{aligned} M_L(t) &= M_X(t) \cdot M_{-Y}(t) = M_X(t) \cdot M_X(-t) \\ M_L(t) &= \frac{1}{1-t} \cdot \frac{1}{1+t} = \frac{1}{1-t^2} \end{aligned}$$

To show that L has the Laplace distribution, we compute the MGF $M_W(t)$ of the Laplace Distribution for $-1 < t < 1$ ¹:

$$\begin{aligned} M_W(t) &= E[e^{tW}] = \int_{-\infty}^{\infty} e^{tw} f(w) dw = \frac{1}{2} \int_{-\infty}^{\infty} e^{tw-|w|} dw = \\ &= \frac{1}{2} \left[\int_{-\infty}^0 e^{w(t+1)} dw + \int_0^{\infty} e^{w(t-1)} dw \right] = \frac{1}{2} \left[\frac{e^{w(t+1)}}{t+1} \Big|_{-\infty}^0 + \frac{e^{w(t-1)}}{t-1} \Big|_0^{\infty} \right] \\ &= \frac{1}{2} \left[\frac{1}{t+1} - \frac{1}{t-1} \right] = \frac{-2}{2(t^2-1)} = \frac{1}{1-t^2} \end{aligned}$$

Thus, because $M_L(t) = M_W(t)$, we have shown that distribution L is a Laplace distribution as it has the identical MGF (i.e. $M_L(t)$) as the Laplace distribution MGF (i.e. $M_W(t)$).

5.

- a) The MGF of a $Bin(n, p)$ r.v. is $M(t) = (pe^t + q)^n$. So, the MGFs of distributions X_1 and X_2 are $(pe^t + q)^{n_1}$ and $(pe^t + q)^{n_2}$, respectively. Because X_1 and X_2 are independent, the distribution $X_1 + X_2$ has the MGF $M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = (pe^t + q)^{n_1+n_2}$. Because the distribution $X_1 + X_2$ has the MGF of the distribution $Bin(n_1 + n_2, p)$, $X_1 + X_2 \sim Bin(n_1 + n_2, p)$.
- b) The MGF of a $Expo(\lambda)$ r.v. is given by $M(t) = \frac{\lambda}{\lambda-t}$ for $t < \lambda$. So, the MGFs of distributions Y_1 and Y_2 are given by $\frac{\lambda_1}{\lambda_1-t}$ for $t < \lambda_1$ and $\frac{\lambda_2}{\lambda_2-t}$ for $t < \lambda_2$, respectively. Let us define $\lambda_s = \min(\lambda_1, \lambda_2)$. Because Y_1 and Y_2 are independent, the distribution $Y_1 + Y_2$ has the MGF $M_{Y_1+Y_2}(t) = M_{Y_1}(t) \cdot M_{Y_2}(t) = \frac{\lambda_1 \lambda_2}{(\lambda_1-t)(\lambda_2-t)}$ for $t < \lambda_s$. As we can see, the MGF of $Y_1 + Y_2$ does not have the form of the MGF of an Exponential distribution, and thus $Y_1 + Y_2$ does not follow an Exponential distribution.

6. Anish Lakkapragada. I worked independently.

¹Note that this range of $|t| < 1$ is required to ensure the below integrals in deriving the MGF of the Laplace Distribution converge.