Math 244 - Problem Set 4

due Friday, February 21, 2025, at 11:59pm

Section 3.2

1. How many permutations of $\{1, 2, \ldots, n\}$ have a single cycle?

2. For a permutation $p : X \to X$, let p^k denote the permutation arising by a k-fold composition of p, i.e., $p^1 = p$ and $p^k = p \circ p^{k-1}$. Define a relation $\approx$ on the set X as follows: $i \approx j$ if and only if there exists a $k \geq 1$ such that $p^k(i) = j$. Prove that $\approx$ is an equivalence relation on X, and that its classes are the cycles of p.

Section 3.3

7. How many functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there that are *monotonic*; that is, for $i < j$ we have $f(i) \leq f(j)$? *The textbook has a hint to this problem in the back.*

21. (optional bonus problem) Draw a triangle ABC. Draw n points lying on the side AB (but different from A and B) and connect them all by segments to the vertex C. Similarly, draw n points on the side AC and connect them to B.

    (a) How many intersections of the drawn segments are there? Into how many regions is the triangle ABC partitioned by the drawn segments?

    (b) Draw n points on the side BC and connect them to A. Assume that no 3 of the drawn segments intersect at a single point. How many intersections are there now?

    (c) How many regions are there in the situation of (b)?

Section 3.7

3. (Sieve of Eratosthenes) How many numbers are left in the set $\{1, 2, \ldots 1000\}$ after all multiples of $2, 3, 5$, and 7 are crossed out?

Section 3.8

4. Prove the equation

$$D(n) = n! - nD(n-1) - \binom{n}{2} D(n-2) - \ldots - \binom{n}{n-1} D(1) - 1.$$