

PSETs Landing Page*

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where `TYPE` is `stats` or `math`. Similarly, to access my solution for this PSET you can go to:

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These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

STATS 242 HW 4

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1.

- (a) A pivotal statistic is one where the test statistic null distribution does not change regardless of the parameters of the sample data. Because under H_0 , **(1)** $S_i \sim \text{Bern}(\frac{1}{2})$ as any PDF f (regardless of its parameters) is symmetric around 0 (i.e. $P(X_i > 0) = 50\%$) and **(2)** any given X_i is equally likely to have a rank of 1 to n (i.e. $R_i \sim \text{Unif}(0, 1)$), we can see that $W = \sum_{i=1}^n S_i R_i$ does not depend on any parameters of $X_i \implies W$ is pivotal under H_0 .

If the X_i 's tended to take positive values, then the S_i 's would be more likely to be one than zero and so $W = \sum_{i=1}^n S_i R_i$ would count more of the (strictly positive) R_i 's so W would be larger. To test against a one-sided alternative H_1 that the X_i 's tended to take positive values, I would reject H_0 for large values of W .

- (b) We can represent $W = \sum_{k=1}^n k I_k$, where I_k is one if observation i with rank $R_i = k$ has $S_i = 1$ and zero otherwise. In other words, this summation is essentially an enumeration over ranks $1 \dots n$ that only sums the ranks of data points where $X_i \geq 0 \implies S_i = 1$. Note that under H_0 , $\forall k \in [1, n]$, $I_k \sim \text{Bern}(\frac{1}{2})$ as we are assuming f is symmetric around zero which means there is a 50% chance $X_k \geq 0$. Given this, we compute the expectation of W below¹:

$$\mathbb{E}[W] = \mathbb{E}[\sum_{k=1}^n k I_k] = \sum_{k=1}^n \mathbb{E}[k I_k] = \sum_{k=1}^n k \mathbb{E}[I_k] = \frac{1}{2} \sum_{k=1}^n k = \frac{1}{2} \frac{n(n+1)}{2} = \frac{n(n+1)}{4}$$

We now compute the $\text{Var}(W)$. Note that because each X_i is independent, each I_k is independent and so $\text{Var}(W) = \text{Var}(\sum_{k=1}^n k I_k) = \sum_{k=1}^n \text{Var}(k I_k)$. We compute the variance of W below²:

$$\begin{aligned} \text{Var}(W) &= \sum_{k=1}^n \text{Var}(k I_k) = \sum_{k=1}^n k^2 \text{Var}(I_k) = \text{Var}(I_k) \sum_{k=1}^n k^2 = \frac{1}{2} \left(1 - \frac{1}{2}\right) \sum_{k=1}^n k^2 \\ &= \frac{1}{4} \sum_{k=1}^n k^2 = \frac{1}{4} \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(2n+1)}{24} \end{aligned}$$

¹We use the fact that $\sum_{k=1}^n k = \frac{n(n+1)}{2}$.

²We use the fact that $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.

Let us assume that under large n , W can be approximated by $\mathcal{N}(\frac{n(n+1)}{4}, \frac{n(n+1)(2n+1)}{24})$, which we will refer to by W_n . As stated in part (a), if X_i 's tended to take positive values, we would reject H_0 for large values of W . This means that to perform this test at α significance level, we would first find the upper- α point z^α of W^3 and then if test statistic W for X_1, \dots, X_n is greater than z^α , we reject H_0 .

2.

- (a) Given $|X_1|, \dots, |X_n|$, we have no information on the signed values X_1, \dots, X_n . Given $|X_i|$, there are two possible values of X_i (i.e. $|X_i|$ or $-|X_i|$). Thus, given $|X_1|, \dots, |X_n|$, there are 2^n possible values of the set X_1, \dots, X_n . This means that the distribution of T conditional on $|X_1|, \dots, |X_n|$ can take (at most) 2^n unique values⁴ as there are 2^n possible unique configurations of X_1, \dots, X_n . The probability of any of these unique values of T is given by $\frac{k}{2^n}$ where k is the number of times this value occurs as the evaluation of T across all configurations of X_1, \dots, X_n . The $\frac{1}{2^n}$ term reflects the fact that each of these configurations are equally likely⁵.
- (b) To conduct a level- α test that rejects H_0 for large values of T , we first have to find the null distribution of T . We do this with computer simulation. Because we are not given the PDF f , we cannot just repeatedly sample values from this distribution. However, we are given a set X_1, \dots, X_n that is realized from this unknown distribution. Under H_0 , this set of data X_1, \dots, X_n is just as likely as any set of $\pm X_1, \dots, \pm X_n$. Thus, we can compute T for all sign permutations of X_1, \dots, X_n . From this set of values of T , we have an approximation of the null distribution of T and thus can take the top $(100 - \alpha)$ th percentile of T as the upper- α point of T . If $T(X_1, \dots, X_n) >$ this upper- α point of T , then we will reject H_0 . If Y_1, \dots, Y_n and Z_1, \dots, Z_n are each n IID data points and each X_i is given by $Y_i - Z_i$, then the H_0 that f is symmetric around zero \implies each X_i follows the same distribution as $-X_i = Z_i - Y_i \implies X_i$ has the same distribution regardless of if it is computed on (Y_i, Z_i) or $(Z_i, Y_i) \implies (Y_i, Z_i)$ and (Z_i, Y_i) have the same (bivariate) distribution.

3.

Because both the null and alternative distributions, given by $f_0(x)$ and $f_1(x)$, are fully specified (i.e. no unknown parameters), they are both simple hypotheses and so we can apply the Neyman-Pearson Lemma. The Neyman-Pearson Lemma tells us that if we can find a c such that the Type I error probability is equal to $\alpha = 0.10$, the likelihood ratio test⁶ is guaranteed to be the test with the highest power. Thus, we first solve for c , which is the upper- α point of the likelihood ratio test statistic $L(x) = \frac{f_1(x)}{f_0(x)} = 2x$ where $x \in [0, 1]$ and $L(x) \in [0, 2]$:

³More formally, z^α is given by $\int_{z^\alpha}^\infty f_{W_n}(w)dw = \alpha$, where f_{W_n} is the PDF of W_n .

⁴Expressed differently, the 2^n values T can take are $T(\pm X_1, \pm X_2, \dots, \pm X_n)$.

⁵This is for two reasons: **(1)** all X_i are independent and **(2)** under H_0 , f is symmetric around zero and so X_i is equally likely to be positive or negative.

⁶using c to define the rejection region

$$\begin{aligned}
\mathbb{P}[\text{Type I Error}] &= \mathbb{P}_{H_0}[\text{reject } H_0] = \mathbb{P}_{H_0}[L(x) > c] = \alpha = 0.10 \\
\mathbb{P}_{H_0}[L(x) > c] &= \mathbb{P}_{H_0}[2x > c] = \mathbb{P}_{H_0}[x > \frac{c}{2}] = \int_{0.5c}^1 f_0(x)dx = 1 - 0.5c = 0.10 \\
c &= 1.8
\end{aligned}$$

Thus, we will reject any sample x when $L(x) > c$. Given this threshold, we can compute the power of the test, which is given by $\mathbb{P}_{H_1}[\text{reject } H_0] = \mathbb{P}_{H_1}[L(X) > c]$:

$$\begin{aligned}
\mathbb{P}_{H_1}[L(X) > c] &= \mathbb{P}_{H_1}[2X > c] = \mathbb{P}_{H_1}[X > \frac{c}{2}] = \int_{0.5c}^1 f_1(x)dx \\
&= \int_{0.5c}^1 2x dx = x^2 \Big|_{0.5c}^1 = 1 - 0.25c^2 = 1 - 1.8^2(0.25) = 0.19
\end{aligned}$$

Thus the maximum power of a test with these hypotheses at the significance level $\alpha = 0.1$ significance level is 19%.

4.

- a) Because both σ_0^2 and σ_1^2 are known, H_0 and H_1 are simple hypotheses. So the Neyman-Pearson Lemma applies in this scenario and guarantees that the likelihood ratio test is the most powerful test. Let us define $f_0(x)$ be the PDF of $X \sim \mathcal{N}(0, \sigma_0^2)$ under H_0 and $f_1(x)$ be the PDF of $X \sim \mathcal{N}(0, \sigma_1^2)$ under H_1 . Furthermore, let vector $\mathbf{x} = (X_1, \dots, X_n)$. Then the likelihood ratio test statistic on X_1, \dots, X_n is given by $L(\mathbf{x}) = \frac{f_1(\mathbf{x})}{f_0(\mathbf{x})}$. We compute this below:

$$\begin{aligned}
f_0(\mathbf{x}) &= \prod_{i=1}^n f_0(x_i) = \prod_{i=1}^n \frac{\exp(\frac{-x_i^2}{2\sigma_0^2})}{\sqrt{2\pi}\sigma_0} = (\frac{1}{\sqrt{2\pi}\sigma_0})^n \exp(\frac{-1}{2\sigma_0^2}[x_1^2 + \dots + x_n^2]) \\
f_1(\mathbf{x}) &= \prod_{i=1}^n f_1(x_i) = \prod_{i=1}^n \frac{\exp(\frac{-x_i^2}{2\sigma_1^2})}{\sqrt{2\pi}\sigma_1} = (\frac{1}{\sqrt{2\pi}\sigma_1})^n \exp(\frac{-1}{2\sigma_1^2}[x_1^2 + \dots + x_n^2]) \\
L(\mathbf{x}) &= \frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} = (\frac{\sigma_1}{\sigma_0})^n \exp([\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}][x_1^2 + \dots + x_n^2]) \\
&= (\frac{\sigma_1}{\sigma_0})^n \exp(\frac{2\sigma_1^2 - 2\sigma_0^2}{4\sigma_0^2\sigma_1^2}[x_1^2 + \dots + x_n^2]) = (\frac{\sigma_1}{\sigma_0})^n \exp(\frac{1}{2} \frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2\sigma_1^2}[x_1^2 + \dots + x_n^2])
\end{aligned}$$

We can observe that for $\sigma_1^2 > \sigma_0^2 \implies \frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2} > 0$, the test statistic $L(\mathbf{x})$ is thus an increasing function of $x_1^2 + \dots + x_n^2$. Under H_0 , each $x_i \sim \mathcal{N}(0, \sigma_0^2)$, and so to standardize $x_1^2 + \dots + x_n^2$, we can say that $L(\mathbf{X})$ is an increasing function of $\frac{1}{\sigma_0^2}(x_1^2 + \dots + x_n^2)$, which we can define as X_n^2 . Note that because each $\frac{X_i}{\sigma_0} \sim \mathcal{N}(0, 1)$,

$X_n^2 \sim \chi_n^2$. Because $L(\mathbf{X})$ is an increasing function of X_n^2 , this means the rejection event $L(\mathbf{x}) > \text{upper-}\alpha \text{ point of } L(\mathbf{X}) \text{ null distribution}$ is *equivalent* to the rejection event $X_n^2 > \text{upper-}\alpha \text{ point of its distribution, } \chi_n^2$. This upper- α point is given to us by $\chi_n^2(\alpha)$.

Given this, we can define a test statistic $T(\mathbf{x})$:

$$T(\mathbf{x}) = \frac{x_1^2 + \cdots + x_n^2}{\sigma_0^2}$$

and the rejection region \mathcal{R} for this test can be defined as:

$$\mathcal{R} = \{x : T(x) > \chi_n^2(\alpha)\}$$

- b) Under the alternative hypothesis H_1 , each $X_i \sim \mathcal{N}(0, \sigma_1^2)$. This means that $\frac{x_1^2 + \cdots + x_n^2}{\sigma_1^2}$ follows a χ_n^2 distribution, and so:

$$T(\mathbf{x}) = \frac{x_1^2 + \cdots + x_n^2}{\sigma_0^2} = \frac{x_1^2 + \cdots + x_n^2}{\sigma_1^2} \cdot \frac{\sigma_1^2}{\sigma_0^2} \sim \frac{\sigma_1^2}{\sigma_0^2} \chi_n^2$$

We now solve for the power of this test, which is given by $\mathbb{P}_{H_1}[\text{reject } H_0]$:

$$\begin{aligned} \mathbb{P}_{H_1}[\text{reject } H_0] &= \mathbb{P}_{H_1}[T(\mathbf{x}) > \chi_n^2(\alpha)] = \mathbb{P}_{H_1}\left[\frac{\sigma_1^2}{\sigma_0^2} \chi_n^2 > \chi_n^2(\alpha)\right] = \mathbb{P}_{H_1}\left[\chi_n^2 > \frac{\sigma_0^2}{\sigma_1^2} \chi_n^2(\alpha)\right] \\ &= 1 - F\left(\frac{\sigma_0^2}{\sigma_1^2} \chi_n^2(\alpha)\right) \end{aligned}$$

Keeping σ_0^2 and α fixed, we can see that as $\sigma_1^2 \rightarrow \infty$, $\frac{\sigma_0^2}{\sigma_1^2} \chi_n^2(\alpha)$ goes closer to zero and so $F\left(\frac{\sigma_0^2}{\sigma_1^2} \chi_n^2(\alpha)\right)$ also goes closer to zero which means the power given by $1 - F\left(\frac{\sigma_0^2}{\sigma_1^2} \chi_n^2(\alpha)\right)$ approaches one.