

Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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Math 226: HW 3

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1. a) The set S is linearly independent if for $a_1, a_2, \dots, a_n \in \mathbb{F}$ and $e_i \in S$, $\sum_{i=1}^N a_i e_i = 0$ when a_1, a_2, \dots, a_n all equal zero. To solve the equation $\sum_{i=1}^N a_i e_i = (a_1, a_2, \dots, a_n) = \mathbf{0}^n$, all elements in the set $\{a_1, a_2, \dots, a_n\}$ must be equal to zero. Thus we have proved that a_1, a_2, \dots, a_n all equal zero as the only solution to $\sum_{i=1}^N a_i e_i = 0$ and so S is proven to be linearly independent.

We define $\text{Span}(S) = \{\sum_{i=1}^N a_i e_i : a_1, a_2, \dots, a_n \in \mathbb{F}\} = \{(a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in \mathbb{F}\}$ and define $\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in \mathbb{F}\}$. Thus $\text{Span}(S) = \mathbb{F}^n$ and so we have proven that S generates \mathbb{F}^n .

- c) To prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent, we first show (1) $\{u + v, u - v\}$ is linearly independent if $\{u, v\}$ is linearly independent and (2) $\{u, v\}$ is linearly independent if $\{u + v, u - v\}$ is linearly independent.

① **Given $\{u, v\}$ is linearly independent, prove $\{u + v, u - v\}$ is linearly independent**

If $\{u, v\}$ is linearly independent, for $a, b \in \mathbb{F}$, the solution to $au + bv = \mathbf{0}$ is $a = b = 0$. Given $c, d \in \mathbb{F}$, let us now re-express $au + bv = \mathbf{0}$ with $a = c + d, b = c - d$.

$$\begin{aligned} au + bv &= \mathbf{0} \\ (c + d)u + (c - d)v &= \mathbf{0} \\ c(u + v) + d(u - v) &= \mathbf{0} \end{aligned}$$

As we are given $\{u, v\}$ is linearly independent, we know that $a = b = 0$ and so given $c + d = c - d = 0$, we know that $c = d = 0$ ¹. Thus, $\{u + v, u - v\}$ is proven to be linearly independent as the only solution to $c(u + v) + d(u - v) = \mathbf{0}$ is proven to be $c = d = 0$.

② **Given $\{u + v, u - v\}$ is linearly independent, prove $\{u, v\}$ is linearly independent**

Given $a, b \in \mathbb{F}$, if we know $\{u + v, u - v\}$ is linearly independent, we know that the solution to $a(u + v) + b(u - v) = \mathbf{0}$ is $a = b = 0$. We can also simplify this as:

$$\begin{aligned} a(u + v) + b(u - v) &= \mathbf{0} \\ (a + b)u + (a - b)v &= \mathbf{0} \end{aligned}$$

Given that we know $a = b = 0$, if we define $c = a + b = 0 \in \mathbb{F}$ and $d = a - b = 0 \in \mathbb{F}$, we get that

$$cu + dv = 0$$

If $\{u, v\}$ is linearly independent, the solution for the equation $eu + fv = \mathbf{0}$ is $e = f = 0$ for $e, f \in \mathbb{F}$. From the above equation, we know that $e = c = 0$

¹Note that we can only conclude this because \mathbb{F} has a characteristic not equal to two. If this was not the case, the condition $c + d = c - d = 0$ can be met if $c = d = 1$.

and $f = d = 0$ and thus $e = f = 0 \Rightarrow \{u, v\}$ is linearly independent. Thus we have proven that if $\{u + v, u - v\}$ is linearly independent, $\{u, v\}$ is linearly independent.

2. a) To prove that for every $x \in V$, $x \in \text{Span}(S)$ iff $\text{Span}(S) = \text{Span}(S \cup \{x\})$ we must show (1) given $x \in \text{Span}(S)$, $\text{Span}(S) = \text{Span}(S \cup \{x\})$ and (2) given $\text{Span}(S) = \text{Span}(S \cup \{x\})$, $x \in \text{Span}(S)$.

① **Given** $x \in \text{Span}(S)$, $\text{Span}(S) = \text{Span}(S \cup \{x\})$

Let us define $N = |S|$. $\text{Span}(S) = \{\sum_{i=1}^N a_i v_i : a_i \in \mathbb{F}, v_i \in V\}$ and $\text{Span}(S \cup \{x\}) = \{cx + \sum_{i=1}^N b_i v_i : b_i, c \in \mathbb{F}, v_i \in V\}$. If $x \in \text{Span}(S)$, x can be expressed as $\sum_{i=1}^N a_i v_i$ for some set of values $a_i \in \mathbb{F}$. We can re-express our $\text{Span}(S \cup \{x\})$ as $\{\sum_{i=1}^N ca_i v_i + \sum_{i=1}^N b_i v_i : b_i, c \in \mathbb{F}, v_i \in V\} = \{\sum_{i=1}^N (ca_i + b_i) v_i : b_i, c \in \mathbb{F}, v_i \in V\}$. Because $ca_i + b_i \in \mathbb{F}$, $\text{Span}(S \cup \{x\}) = \{\sum_{i=1}^N d_i v_i : d_i \in \mathbb{F}, v_i \in V\} = \text{Span}(S)$. Thus, we have proved given $x \in \text{Span}(S)$, $\text{Span}(S) = \text{Span}(S \cup \{x\})$.

② **Given** $\text{Span}(S) = \text{Span}(S \cup \{x\})$, $x \in \text{Span}(S)$

We use the definitions provided from ①. Because we know $x \in S \cup \{x\}$, we know that $x \in \text{Span}(S \cup \{x\})$. Since we are given $\text{Span}(S) = \text{Span}(S \cup \{x\})$, if $x \in \text{Span}(S \cup \{x\}) \Rightarrow x \in \text{Span}(S)$. Thus, we have proved if $\text{Span}(S) = \text{Span}(S \cup \{x\})$, $x \in \text{Span}(S)$.

- b) We are given that S is linearly independent. To prove that $S \cup \{w\}$ is linearly independent iff $w \notin \text{Span}(S)$, we first prove (1) if $w \notin \text{Span}(S)$, $S \cup \{w\}$ is linearly independent and then prove (2) if $S \cup \{w\}$ is linearly independent, $w \notin \text{Span}(S)$.

① **Given** $w \notin \text{Span}(S)$, $S \cup \{w\}$ is linearly independent

We define $n = |S|$.

For proof by contrapositive, let us assume that $S \cup \{w\}$ is linearly dependent. That means that for the solution to the equation $cw + \sum_{i=1}^n a_i v_i = 0$, where $c, a_1, a_2, \dots, a_n \in \mathbb{F}$, there exists at least one nonzero element of $\{c, a_1, a_2, \dots, a_n\}$. For this solution $c \neq 0$, as if $c = 0$, then the equation would simply be $\sum_{i=1}^n a_i v_i = 0$. Because S is linearly independent, the only solution to this equation is all $a_i = 0$. However, this would violate the condition that there exists at least one nonzero element in set $\{c, a_1, a_2, \dots, a_n\}$ because we are assuming $S \cup \{w\}$ is linearly dependent. Thus, we know $c \neq 0$ and so we can re-express w as:

$$w = -\frac{\sum_{i=1}^n a_i v_i}{c} = \sum_{i=1}^n \frac{-a_i}{c} v_i$$

This expression of w is an expression of w as a linear combination of S . Thus, $w \in \text{Span}(S)$ if $S \cup \{w\}$ is linearly dependent. By proof by contrapositive, we have proven if $w \notin \text{Span}(S)$, $S \cup \{w\}$ is linearly independent.

② **Given** $S \cup \{w\}$ is linearly independent, $w \notin \text{Span}(S)$

We use the same definition for n provided in ①.

If $S \cup \{w\}$ is linearly independent, it means that the solution to the equation $cw + \sum_{i=1}^n a_i v_i = 0$, where $c, a_1, a_2, \dots, a_n \in \mathbb{F}$, is that all elements of the set $\{c, a_1, a_2, \dots, a_n\}$ must be zero. We now try to express w as some linear combination of S .

$$\begin{aligned} cw + \sum_{i=1}^n a_i v_i &= 0 \\ cw &= -\sum_{i=1}^n a_i v_i \end{aligned}$$

Note that because $S \cup \{w\}$ is linearly independent, $c = 0$. Because we cannot divide the above equation by c on both sides, there does not exist any linear

combination of S that is equal to w . Because the $\text{Span}(S)$ represents all possible linear combinations of elements in S , we have proven $w \notin \text{Span}(S)$.

c) Given $S = \{u_1, u_2, u_3, \dots, u_k\}$, to prove iff Condition M (defined below)

$$M : \{0\} \subsetneq \text{Span}(\{u_1\}) \subsetneq \text{Span}(\{u_1, u_2\}) \subsetneq \text{Span}(\{u_1, u_2, u_3\}) \subsetneq \text{Span}(\{u_1, \dots, u_k\})$$

then S is linearly independent, we must first prove (1) if condition M holds, then S is linearly independent and (2) if S is linearly independent, then condition M holds.

① If condition M holds, then S is linearly independent

For proof by contrapositive, let us assume S is linearly dependent. Thus, there exists some $m < k \in \mathbb{Z}$ where subset $D = \{u_1, \dots, u_m\} \subsetneq S$ is linearly independent and subset $K = \{u_1, \dots, u_m, u_{m+1}\} \subseteq S$ is linearly dependent. Let us define $a_1, a_2, \dots, a_{m+1} \in \mathbb{F}$ and $u_i \in K$. Because K is linearly dependent, we know that for the solution to the equation $\sum_{i=1}^{m+1} a_i u_i = 0$, there exists at least one nonzero element in the set $\{a_1, a_2, \dots, a_{m+1}\}$. We can further develop this equation as:

$$a_{m+1}u_{m+1} + \sum_{i=1}^m a_i u_i = 0$$

Let us consider the case in which $a_{m+1} = 0$. This would leave us with the equation $\sum_{i=1}^m a_i u_i = 0$. Because D is linearly independent, we know that the only solution to this equation is $\forall a_i \in \{a_1, a_2, \dots, a_m\}, a_i = 0$. However, because this solution violates the condition that there exists at least one nonzero element in the set $\{a_1, a_2, \dots, a_{m+1}\}$, we know that $a_{m+1} \neq 0$. Thus, $u_{m+1} = \sum_{i=1}^m (-\frac{a_i}{a_{m+1}})u_i$ and so because we can express u_{m+1} as linear combination of D , $u_{m+1} \in \text{Span}(D)$.

We now compute $\text{Span}(K) = \{cu_{m+1} + \sum_{i=1}^m b_i u_i : c, b_i \in \mathbb{F}\} = \{c \sum_{i=1}^m (-\frac{a_i}{a_{m+1}})u_i + \sum_{i=1}^m b_i u_i : c, b_i \in \mathbb{F}\} = \{\sum_{i=1}^m (-\frac{ca_i}{a_{m+1}} + b_i)u_i : c, b_i \in \mathbb{F}\}$. Because $-\frac{ca_i}{a_{m+1}} + b_i \in \mathbb{F}$, $\text{Span}(K) = \{\sum_{i=1}^m d_i u_i : d_i \in \mathbb{F}\} = \text{Span}(D)$. Because $\text{Span}(D) = \text{Span}(K)$, condition M does not hold if S is linearly dependent. Thus, we have proven by contrapositive that if condition M holds, then S is linearly independent.

② If S is linearly independent, then condition M holds

We use the definitions of m, k, D, K from ①.

For proof by contrapositive, let us assume that condition M does not hold² and so for some $m < k \in \mathbb{R}$, $\text{Span}(D) = \text{Span}(K)$. Because $u_{m+1} \in \text{Span}(K) \Rightarrow u_{m+1} \in \text{Span}(D)$, we know that u_{m+1} can be written as $u_{m+1} = \sum_{i=1}^m a_i u_i$ for $a_i \in \mathbb{F}, u_i \in D$.

The set S is linearly independent if given $b_i \in \mathbb{F}$ and $u_i \in S$, the only solution to the equation $\sum_{i=1}^k b_i u_i = 0$ is $\forall b_i \in \{b_1, b_2, \dots, b_k\}, b_i = 0$. We can re-express this equation below as:

$$\begin{aligned} \sum_{i=1}^k b_i u_i &= 0 \\ b_{m+1}u_{m+1} + \sum_{i=m+2}^k b_i u_i + \sum_{i=1}^m b_i u_i &= 0 \end{aligned}$$

²There is only one case in which condition M does not hold. Condition M would not hold if $\exists j < k \in \mathbb{Z}$ s.t. $\text{Span}(\{u_1, \dots, u_j\}) = \text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$ or $\text{Span}(\{u_1, \dots, u_j\}) \subsetneq \text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$. All elements in $\text{Span}(\{u_1, \dots, u_j\})$ exist in $\text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$ as the last element in the set u_{j+1} can always be ignored in a linear combination of $\{u_1, \dots, u_j, u_{j+1}\}$ by setting its coefficient to zero. Thus $\text{Span}(\{u_1, \dots, u_j\}) \subseteq \text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$ and so condition M can only be violated if $\text{Span}(\{u_1, \dots, u_j\}) = \text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$.

Let us set $b_i = 0$ for $m + 2 \leq i \leq k$, $b_i = -a_i$ for $1 \leq i \leq m$, and $b_{m+1} = 1$. This gives us:

$$u_{m+1} = \sum_{i=1}^m b_i u_i = \sum_{i=1}^m a_i u_i$$

As shown before, we know this statement is true. Thus, we have found a solution to the equation $\sum_{i=1}^k b_i u_i = 0$ with at least one $b_i \neq 0$. This means S is linearly dependent. Thus, if condition M does not hold, we have shown S is linearly dependent. By proof by contrapositive, we have proven that if S is linearly independent, then condition M holds.

3. a) Given $U = \{(x_1, \frac{x_1}{3}, x_3, \frac{x_3}{7}, x_5) \in \mathbb{R}^5; x_1, x_3, x_5 \in \mathbb{F}\}$, a basis of U , β_U , can be given by:

$$\beta_U = \{(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1)\}$$

- i) Extending β_U to \mathbb{R}^5

$$\beta_{\mathbb{R}^5} = \{(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0)\}$$

- ii) Find a subspace W of \mathbb{R}^5 such that $\mathbb{R}^5 = U \oplus W$

$$W = \{(0, a_1, 0, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$$

We now validate (1) that $U + W = U \oplus W$ and (2) $U \oplus W = \mathbb{R}^5$.

- ① $U + W = U \oplus W$

Let us define $z_1, z_2, z_3, z_4, z_5 \in \mathbb{R}$. Given $x = (z_1, \frac{z_1}{3}, z_3, \frac{z_3}{7}, z_5) \in U$ and $y = (0, z_2, 0, z_4, 0) \in W$, we can find $U \cap W$ as the set of solutions to $x = y$. This would be defined as the solution to the system of equations below.

$$z_1 = 0$$

$$\frac{z_1}{3} = z_2$$

$$z_3 = 0$$

$$\frac{z_3}{7} = z_4$$

$$z_5 = 0$$

The solution to this system is $z_1 = z_2 = z_3 = z_4 = z_5 = 0$. Thus $U \cap W = \{\mathbf{0}^5\} \Rightarrow U + W = U \oplus W$.

- ② $U \oplus W = \mathbb{R}^5$

We use the same definitions of $x, y, z_1, z_2, z_3, z_4, z_5$ from ①. We compute $x + y = (z_1, z_2 + \frac{z_1}{3}, z_3, z_4 + \frac{z_3}{7}, z_5) \in U \oplus W$. Because $z_1, z_2 + \frac{z_1}{3}, z_3, z_4 + \frac{z_3}{7}, z_5 \in \mathbb{R}$, $x + y \in \mathbb{R}^5$ and so $U \oplus W \subseteq \mathbb{R}^5$. Because $\mathbf{0}^5 \in U \oplus W$ and $U \oplus W$ can be trivially shown to be closed under addition and scalar multiplication, $U \oplus W \leq \mathbb{R}^5$.

We now prove that $U \oplus W = \mathbb{R}^5$ by showing that $\dim(U \oplus W) = \dim(\mathbb{R}^5) = 5$.

The basis of U , β_U , is given above and we give the basis of W as $\beta_W = \{(0, 1, 0, 0, 0), (0, 0, 0, 1, 0)\}$. We now compute $\text{Span}(\beta_U \cup \beta_W)$ as given $v_i \in \beta_U \cup \beta_W$ and $a_i \in \mathbb{R}$, $\text{Span}(\beta_U \cup \beta_W) = \Sigma_{i=1}^3 a_i v_i + \Sigma_{i=5}^4 a_i v_i$. Because all vectors in U can be expressed as $\Sigma_{i=1}^3 a_i v_i$ (a linear combination of β_U) and all vectors in W can be expressed as $\Sigma_{i=5}^4 a_i v_i$ (a linear combination of β_W), $\text{Span}(\beta_U \cup \beta_W) = \{u + w : u \in U, w \in W\} = U \oplus W \Rightarrow \beta_U \cup \beta_W$ is the basis for $U \oplus W$. Thus $\dim(U \oplus W) = |\beta_U \cup \beta_W| = 5 = \dim(\mathbb{R}^5)$. Thus, we have proved $U \oplus W = \mathbb{R}^5$.

- b) Given $U = \{f(x) = c_1 + (-2c_3 - 3c_4 - 4c_5)x + c_3x^2 + c_4x^3 + c_5x^4 \in P_4(\mathbb{R}) : c_1, c_3, c_4, c_5 \in \mathbb{R}\}$, a basis of U can be given by:

$$\beta_U = \{1, -2x + x^2, -3x + x^3, -4x + x^4\}$$

- i) Extending β_U to $P_4(\mathbb{R})$

$$\beta_{P_4(\mathbb{R})} = \{1, x, -2x + x^2, -3x + x^3, -4x + x^4\}$$

- ii) Find a subspace W of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R}) = U \oplus W$

$$W = \{f(x) = a_1x \in P_4(\mathbb{R}) : a_1 \in \mathbb{R}\}$$

We now validate (1) that $U + W = U \oplus W$ and (2) $U \oplus W = P_4(\mathbb{R})$.

① $U + W = U \oplus W$

Let us define $z_1, z_2, z_3, z_4, z_5 \in \mathbb{F}$. Given $u = z_1 + (-2z_3 - 3z_4 - 4z_5)x + z_3x^2 + z_4x^3 + z_5x^4 \in U$ and $w = z_2x \in W$, we find $U \cap W$ as the set of solutions to $u = w$. This would be defined as the solution to the system of equations below.

$$\begin{aligned} z_1 &= 0 \\ -2z_3 - 3z_4 - 4z_5 &= z_2 \\ z_3 &= 0 \\ z_4 &= 0 \\ z_5 &= 0 \end{aligned}$$

The solution to this system is $z_1 = z_2 = z_3 = z_4 = z_5 = 0$. Thus since $U \cap W = \{f(x) = 0 \in P_4(\mathbb{R})\} \Rightarrow U + W = U \oplus W$.

② $U \oplus W = P_4(\mathbb{R})$

We use the same definitions of $u, w, z_1, z_2, z_3, z_4, z_5$ from ①. We compute $u + w = z_1 + (z_2 - 2z_3 - 3z_4 - 4z_5)x + z_3x^2 + z_4x^3 + z_5x^4 \in U \oplus W$. Because $z_1, z_2 - 2z_3 - 3z_4 - 4z_5, z_3, z_4, z_5 \in \mathbb{R}$, $u + w \in P_4(\mathbb{R})$ and so $U \oplus W \subseteq P_4(\mathbb{R})$. Because $0 \in U \oplus W$ and $U \oplus W$ can be trivially shown to be closed under addition and scalar multiplication, $U \oplus W \leq P_4(\mathbb{R})$. We now prove that $U \oplus W = P_4(\mathbb{R})$ by showing that $\dim(U \oplus W) = \dim(P_4(\mathbb{R})) = 5$.

The basis of U , β_U , is given above and we give the basis of W as $\beta_W = \{x\}$. We now compute $\text{Span}(\beta_U \cup \beta_W)$ as given $f_i(x) \in \beta_U \cup \beta_W$ and $a_i \in \mathbb{R}$, $\text{Span}(\beta_U \cup \beta_W) = \Sigma_{i=1}^3 a_i f_i(x) + \Sigma_{i=5}^4 a_i f_i(x)$. Because all functions in U can be expressed as $\Sigma_{i=1}^3 a_i f_i(x)$ (a linear combination of β_U) and all functions in W can be expressed as $\Sigma_{i=5}^4 a_i f_i(x)$ (a linear combination of

β_W), $\text{Span}(\beta_U \cup \beta_W) = \{u + w : u \in U, w \in W\} = U \oplus W \Rightarrow \beta_U \cup \beta_W$ is the basis for $U \oplus W$. Thus $\dim(U \oplus W) = |\beta_U \cup \beta_W| = 5 = \dim(P_4(\mathbb{R}))$. Thus, we have proved $U \oplus W = P_4(\mathbb{R})$.

4. Let us define a matrix E_{ij} as a 3x3 matrix with all zeros except in the i th row and j th column, where there is a one.

The basis for $M_{3 \times 3}(\mathbb{R}) = \{\bigcup_{\substack{i,j=1 \\ i \neq j}}^3 E_{ij}\} \cup \{E_{00} - E_{33}, E_{11} - E_{33}\}$.