Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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MATH 255 PSET 7

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1.

Before solving this question, note that:

$$\sqrt{n^2+n}-n=(\sqrt{n^2+n}-n)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}=\frac{(n^2+n)-n^2}{\sqrt{n^2+n}+n}=\frac{n}{\sqrt{n^2+n}+n}$$
$$=\frac{n}{\sqrt{n^2+n}+n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}=\frac{1}{\sqrt{\frac{n^2+n}{n^2}}+1}=\frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

We first calculate $\lim_{n\to\infty}\sqrt{n^2+n}-n$. We use the fact that $\lim_{n\to\infty}\frac{1}{n}=0$.

$$\lim_{n\to\infty}\sqrt{n^2+n}-n=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{1+1}=\frac{1}{2}$$

We now prove that $\lim_{n\to\infty}\sqrt{n^2+n}-n=\frac{1}{2}$. Pick $\epsilon>0$. To show $\lim_{n\to\infty}\sqrt{n^2+n}-n=\frac{1}{2}$, we want to find $N\in\mathbb{N}$ s.t. $\forall n\geq N, d(\sqrt{n^2+n}-n,\frac{1}{2})<\epsilon$ or more simply s.t. $\forall n\geq N, |\frac{1}{\sqrt{1+\frac{1}{n}}+1}-\frac{1}{2}|<\epsilon$. Thus, we solve the $|\frac{1}{\sqrt{1+\frac{1}{n}}+1}-\frac{1}{2}|<\epsilon$ inequality below:

$$|\frac{1}{\sqrt{1+\frac{1}{n}}+1}-\frac{1}{2}|<\epsilon$$

Note that because $n\in\mathbb{N}\implies\frac{1}{n}>0\implies 1+\frac{1}{n}>1\implies\sqrt{1+\frac{1}{n}}>1\implies 1+\sqrt{1+\frac{1}{n}}>2\implies\frac{1}{1+\sqrt{1+\frac{1}{n}}}<\frac{1}{2}$. Thus, $|\frac{1}{\sqrt{1+\frac{1}{n}}+1}-\frac{1}{2}|=\frac{1}{2}-\frac{1}{1+\sqrt{1+\frac{1}{n}}}$:

$$\frac{1}{2} - \frac{1}{1+\sqrt{1+\frac{1}{n}}} < \epsilon$$
$$\frac{1}{2} - \epsilon < \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$
$$1 > (1+\sqrt{1+\frac{1}{n}})(0.5-\epsilon)$$
$$1-(0.5-\epsilon) > (0.5-\epsilon)\sqrt{1+\frac{1}{n}}$$
$$\frac{0.5+\epsilon}{0.5-\epsilon} > \sqrt{1+\frac{1}{n}}$$
$$(\frac{0.5+\epsilon}{0.5-\epsilon})^2 > 1+\frac{1}{n}$$
$$\frac{1}{n} < (\frac{0.5+\epsilon}{0.5-\epsilon})^2 - 1$$
$$n > \frac{1}{(\frac{0.5+\epsilon}{0.5-\epsilon})^2 - 1}$$

Thus we see $d(\sqrt{n^2+n}-n, \frac{1}{2}) < \epsilon$ for any $n > \frac{1}{(\frac{0.5+\epsilon}{0.5-\epsilon})^2-1}$. Thus we aim to choose N as any natural number $> \frac{1}{(\frac{0.5+\epsilon}{0.5-\epsilon})^2-1}$. By Archimedian property, $\exists\ m \in \mathbb{N}$ s.t. $m(1) > \frac{1}{(\frac{0.5+\epsilon}{0.5-\epsilon})^2-1}$ and so we can simply choose $N = m$. Thus, we have shown $\forall \epsilon > 0, \exists\ N \in \mathbb{N}$ s.t. $\forall n \geq N, d(\sqrt{n^2+n}-n, \frac{1}{2}) < \epsilon$ and so we have proven $\lim_{n\to\infty} \sqrt{n^2+n}-n = \frac{1}{2}$.

2.

Lemma 0.1 *Given $(a_n), (b_n)$ as bounded sequences in $\mathbb{R}$, we prove if some subsequence of (a_n+b_n) converges to $x \implies$ the limit of some subsequence of a_n plus the limit of some subsequence of b_n is equal to x.*

Proof: Let us define this subsequence of (a_n+b_n) that converges to x as a sequence given by $a_{n_1}+b_{n_1}, a_{n_2}+b_{n_2}, \ldots$ where $n_1 < n_2 < \ldots$. In other words, this subsequence is the sequence (a_n+b_n) indexed by the monotonically increasing sequence (n_k). Given that this sequence $a_{n_k}+b_{n_k} \to x$, we know:

$$\lim_{k\to\infty}(a_{n_k}+b_{n_k}) = \lim_{k\to\infty} a_{n_k} + \lim_{k\to\infty} b_{n_k}$$
$$x = \lim_{k\to\infty} a_{n_k} + \lim_{k\to\infty} b_{n_k}$$

Thus, we have shown that the limit of some subsequence of a_n, given by $\lim_{k\to\infty} a_{n_k}$, plus the limit of some subsequence of b_n, given by $\lim_{k\to\infty} b_{n_k}$, is equal to x.

Lemma 0.2 *Let us define sets X, Y where $X \subset Y$. Then sup$(X) \leq$ sup(Y).*
Proof: sup(Y) is the lowest upper bound of Y and because $X \subset Y \implies$ sup(Y) is an upper bound of X. However, sup(X) is the lowest upper bound of X and so sup$(X) \leq$ sup(Y).

Let us define sets A and B below as such:

$$A = \{x \in \mathbb{R}_{\text{ext}} \mid \text{some subseq of } (a_n) \text{ converges to } x\}$$
$$B = \{x \in \mathbb{R}_{\text{ext}} \mid \text{some subseq of } (b_n) \text{ converges to } x\}$$

We are given that $(a_n), (b_n)$ are bounded real sequences $\implies$ sequences $(a_n), (b_n)$ are bounded above and below $\implies$ sequences $(a_n), (b_n)$ and their subsequences cannot converge to $\pm\infty \implies A \subset \mathbb{R}$ and $B \subset \mathbb{R}$. Furthermore, this means the sequence $(a_n + b_n)$ is bounded[1] and so we can define the following set:

$$A + B = \{x + y \in \mathbb{R} \mid \text{some subseq of } (a_n) \to x \text{ and some subseq of } (b_n) \to y\} \subset \mathbb{R}$$

As proved in Homework 2, because $A, B, A + B \subset \mathbb{R}$, $\sup(A + B) = \sup(A) + \sup(B) = \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n)$. Now consider the set below:

$$C = \{x \in \mathbb{R} \mid \text{some subseq of } (a_n + b_n) \to x\}$$

As proved in **Lemma 0.1**, if some subseq of $(a_n + b_n) \to x \implies$ the limit of some subsequence of a_n plus the limit of some subsequence of b_n is equal to x. Thus, $C \subset A + B \implies$ (**using Lemma 0.2**) $\sup(C) \leq \sup(A + B) = \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n) \implies \sup(C) \leq \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n)$. Note that $\sup(C) = \lim_{n\to\infty} \sup(a_n + b_n)$ and so we have proven:

$$\lim_{n\to\infty} \sup(a_n + b_n) \leq \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n)$$

Example of $\lim_{n\to\infty} \sup(a_n + b_n) = \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n)$

We can just set $(a_n) = (b_n) = 1, 1, 1, 1 \ldots$. In this case $\lim_{n\to\infty} \sup(a_n + b_n) = 2$ and $\lim_{n\to\infty} \sup(a_n) = \lim_{n\to\infty} \sup(b_n) = 1$. So $\lim_{n\to\infty} \sup(a_n + b_n) = 2 = \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n)$.

Example of $\lim_{n\to\infty} \sup(a_n + b_n) < \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n)$

[1] Proof: WLOG, we show $(a_n + b_n)$ is bounded above. If (a_n) is bounded $\implies$ (a_n) is bounded above $\implies \exists\ z_a \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, z_a \geq a_n$. We can define $z_b \in \mathbb{R}$ similarly as an upper bound for (b_n). Thus, $\forall n \in \mathbb{N}, a_n + b_n \leq z_a + z_b \implies z_a + z_b \in \mathbb{R}$ is an upper bound for $(a_n + b_n) \implies (a_n + b_n)$ is bounded above. This shows $(a_n + b_n)$ is bounded above (and with identical logic bounded below) $\implies (a_n + b_n)$ is bounded.

We can define sequence (a_n) where $a_n = \frac{1+(-1)^n}{2}$ and sequence (b_n) where $b_n = \frac{-1-(-1)^n}{2}$. Writing out the elements of (a_n) and (b_n) out we get:

$$(a_n) = 0, 1, 0, 1, \ldots$$
$$(b_n) = 0, -1, 0, -1, \ldots$$

So sequence $(a_n + b_n) = 0, 0, 0, 0, \ldots$ and so $\limsup_{n\to\infty}(a_n + b_n) = 0$. However, there exists a subsequence of (a_n) given by $1, 1, 1, \ldots$ and so $\limsup_{n\to\infty}(a_n) = 1$. There also exists a subsequence of (b_n) given by $0, 0, 0, \ldots$ and so $\limsup_{n\to\infty}(b_n) = 0$. Thus, $\limsup_{n\to\infty}(a_n + b_n) = 0 < \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n) = 1 + 0 = 1$.

3.

Fix $\epsilon > 0$ and define $\epsilon' = \frac{\epsilon}{2} > 0$. Given $(p_n), (q_n)$ are Cauchy sequences, $\exists\ N_p \in \mathbb{N}$ s.t. $\forall n, m \geq N_p, d(p_n, p_m) < \epsilon'$ and $\exists\ N_q \in \mathbb{N}$ s.t. $\forall n, m \geq N_q, d(p_n, p_m) < \epsilon'$. Set $N = \max(N_p, N_q) \in \mathbb{N}$. Then, $\forall n, m \geq N, d(p_n, p_m) < \epsilon'$ and $d(q_n, q_m) < \epsilon'$. Thus, applying Triangle Inequality, $\forall n, m \geq N$:

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n)$$

By Triangle Inequality, $d(p_m, q_n) \leq d(p_m, q_m) + d(q_m, q_n)$ and so:

$$\begin{aligned} d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n) &\leq d(p_n, q_m) + d(p_m, q_m) + d(q_m, q_n) \\ d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ d(p_n, q_n) - d(p_m, q_m) &\leq d(p_n, p_m) + d(q_m, q_n) \end{aligned}$$

We know $d(p_n, p_m) < \epsilon'$ and $d(q_m, q_n) = d(q_n, q_m) < \epsilon'$ and so:

$$d(p_n, q_n) - d(p_m, q_m) < \epsilon' + \epsilon' = \epsilon$$

Note that we could also start with: $d(p_m, q_m) \leq d(p_m, p_n) + d(p_n, q_m)$ to show through identical logic $d(p_m, q_m) - d(p_n, q_n) < \epsilon$. Given $d(p_n, q_n) - d(p_m, q_m) < \epsilon$ and $d(p_m, q_m) - d(p_n, q_n) < \epsilon \implies |d(p_n, q_n) - d(p_m, q_m)| < \epsilon$.

Thus, we have shown $\forall \epsilon > 0, \exists\ N' \in \mathbb{N}$ s.t. $\forall n, m \geq N', d_{\mathbb{R}}(d(p_n, q_n), d(p_m, q_m)) < \epsilon$ where $d_{\mathbb{R}}$ is the standard distance function in metric space $\mathbb{R}$ (i.e. $d_{\mathbb{R}}(x, y) = |x - y|$). This proves that the sequence $(d(p_n, q_n))$ in $\mathbb{R}$ is a Cauchy Sequence. Because all Cauchy Sequences in $\mathbb{R}$ converge, this proves that the sequence $(d(p_n, q_n))$ converges $\implies$ $(d(p_n, q_n))$ has a limit.

4.

Lemma 0.3 *We prove that if $b \in \mathbb{R}$ is an effective upper bound (EUB) of (x_n), $b \geq \limsup_{n\to\infty} x_n$.*

Proof: Suppose we have a subsequence of (x_n) that converges to x. Then $\forall \epsilon > 0, \exists\ N_x \in \mathbb{N}$ s.t. $\forall n \geq N_x, d(x_n, x) = |x_n - x| < \epsilon \implies x_n - x > -\epsilon \implies x < x_n + \epsilon$. Now consider a given EUB $b \in \mathbb{R}$ of (x_n): by definition of EUB, $\exists\ N_b \in \mathbb{N}$ s.t. $\forall n \geq N_b, x_n \leq b$. Thus $\forall \epsilon > 0, \exists\ M = max(N_x, N_b)$ s.t. $\forall n \geq M, x < x_n + \epsilon$ and $x_n \leq b$. Thus this means $\forall \epsilon > 0, \exists\ M \in \mathbb{N}$ s.t. $\forall n \geq M, x < x_n + \epsilon \leq b + \epsilon$. Note that x and b are constants (independent of n) and so this proof shows $\implies \forall \epsilon > 0, x < b + \epsilon \implies x \leq b$ (see footnote[2]).

Let us define the set $A = \{x \in \mathbb{R}_{ext} \mid \text{some subsequence of } (x_n) \text{ converges to } x\}$. We show in the following two cases, $\limsup_{n\to\infty} x_n = sup(A) \leq b$.

1. ***Case One:*** $A = \emptyset$
   If $A = \emptyset \implies \forall x \in \mathbb{R}, x$ is vacuously an upper bound of A. So EUB $b \in \mathbb{R}$ is an upper bound of A. However, because $\limsup_{n\to\infty} x_n = sup(A)$ is the lowest upper bound of $A \implies \limsup_{n\to\infty} x_n \leq b$.
2. ***Case Two:*** $A \neq \emptyset$
   As previously established, the limit of any convergent subsequence of (x_n) is $\leq b$. So, $\forall a \in A, a \leq b$. Thus, b is an upper bound of A. However by definition $\limsup_{n\to\infty} x_n = sup(A)$ is the lowest upper bound of $A \implies \limsup_{n\to\infty} x_n \leq b$.

We prove this statement through casework on $\limsup_{n\to\infty} x_n$:

1. **Case One:** $\limsup_{n\to\infty} x_n \in \mathbb{R}$
   We first show that $\forall \epsilon > 0$, $\limsup_{n\to\infty} x_n + \epsilon$ is an upper bound. This follows directly from Proposition 6.37: $\limsup_{n\to\infty} x_n < \limsup_{n\to\infty} x_n + \epsilon \implies \exists\ N \in \mathbb{N}$ s.t. $\forall n \geq N, x_n < \limsup_{n\to\infty} x_n + \epsilon \implies \limsup_{n\to\infty} x_n + \epsilon$ is an EUB of (x_n). Furthermore, note that by implication of **Lemma 0.3**, $\nexists$ a real-valued EUB $< \limsup_{n\to\infty} x_n$. So, the set $\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}$ can be given by the interval $(\limsup_{n\to\infty} x_n, \infty) \subset \mathbb{R}$ or alternatively the set $\{\limsup_{n\to\infty} x_n + \epsilon : \forall \epsilon > 0\}$.
   The greatest lower bound of this set[3][4] is $\limsup_{n\to\infty} x_n$ and so we have proven $\limsup_{n\to\infty} x_n = \inf\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}$.

[2]For more clarity on the statement $\forall \epsilon > 0, x < b + \epsilon \implies x \leq b$, we can think of the maximum possible value of x as $\inf\{b + \epsilon : \epsilon > 0\}$. The infimum of this set is obviously b (b is a lower bound of this set and any real number greater than b fails to be a lower bound of this set so b is the greatest lower bound) and so we have that the largest value x can hold is $b \implies x \leq b$.

[3]Note that it does not matter if $\limsup_{n\to\infty} x_n$ is included in this set of real-valued EUBs of (x_n). This is because the supremum of this set would have been $\limsup_{n\to\infty} x_n$ regardless.

[4]The proof for this is trivial: $\limsup_{n\to\infty} x_n$ is a lower bound of this set and any number slightly greater than

2. **Case Two:** $\limsup_{n\to\infty} x_n \notin \mathbb{R}$

   By definition $\limsup_{n\to\infty} x_n \in \mathbb{R}_{\text{ext}}$. Given $\limsup_{n\to\infty} x_n \notin \mathbb{R} \implies \limsup_{n\to\infty} x_n = \pm\infty$. We consider each case below:

   (a) $\limsup_{n\to\infty} x_n = \infty$

   As proven in **Lemma 0.3**, all EUBs must be $\geq \limsup_{n\to\infty} x_n$. Thus $\limsup_{n\to\infty} x_n = \infty \implies (x_n)$ cannot have any EUBs in $\mathbb{R}$ as $\nexists\ x \in \mathbb{R}$ s.t. $x \geq \limsup_{n\to\infty} x_n = \infty$. Thus, the set $\{a \in \mathbb{R}\ |a$ is an EUB for $(x_n)\} = \emptyset$ and because[5] $\inf(\emptyset) = \infty = \limsup_{n\to\infty} x_n$ we have proven:

   $$\limsup_{n\to\infty} x_n = \inf\{a \in \mathbb{R}\ |a \text{ is an EUB for } (x_n)\}$$

   (b) $\limsup_{n\to\infty} x_n = -\infty$

   Because $\limsup_{n\to\infty} x_n = -\infty, \forall x \in \mathbb{R}, x > \limsup_{n\to\infty} x_n \implies$ (through Proposition 6.37) $\forall x \in \mathbb{R}, x$ is an EUB for (x_n). Thus the set $\{a \in \mathbb{R}\ |a$ is an EUB for $(x_n)\} = \mathbb{R}$ and so the only lower bound (and only greatest lower bound) of this set is $-\infty \implies \inf(\{a \in \mathbb{R}\ |a$ is an EUB for $(x_n)\}) = -\infty = \limsup_{n\to\infty} x_n$. So we have again proven:

   $$\limsup_{n\to\infty} x_n = \inf\{a \in \mathbb{R}\ |a \text{ is an EUB for } (x_n)\}$$

it (can be given as $\limsup_{n\to\infty} x_n + \epsilon$ where $\epsilon > 0$) fails to be a lower bound for this set $\implies \limsup_{n\to\infty} x_n$ is the greatest lower bound of this set.

[5]The following statement reflects the fact that the infimum is the greatest lower bound. Because $\forall x \in \mathbb{R}_{\text{ext}}$, x is a lower bound for $\emptyset$, the greatest lower bound of $\emptyset$ is the greatest value in $\mathbb{R}_{\text{ext}}$ or ∞.