MATH 241 PSET 1

September 11, 2024

1.

We prove the following three inequalities and for each provide a case when the inequality would become an equality.

(1) $P(A) + P(B) - 1 \leq P(A \cap B)$

$$\begin{gathered} P(A) + P(B) - 1 \leq P(A \cap B) \\ P(A) + P(B) - P(A \cap B) \leq 1 \\ P(A \cup B) \leq 1 \end{gathered}$$

Because we know $P(A \cup B) \leq 1$ by axioms of probability, the inequality in (1) is proven. We now investigate the equality case in which $P(A) + P(B) - 1 = P(A \cap B)$

$$\begin{gathered} P(A) + P(B) - 1 = P(A \cap B) \\ P(A) + P(B) - P(A \cap B) = 1 \\ P(A \cup B) = 1 \end{gathered}$$

Because $P(A \cup A^c) = P(S) = 1$, **the inequality would be an equality if $B = A^c$.**

(2) $P(A \cap B) \leq P(A \cup B)$

$$\begin{gathered} P(A \cap B) \leq P(A \cup B) \\ P(A \cap B) \leq P(A) + P(B) - P(A \cap B) \\ 2P(A \cap B) \leq P(A) + P(B) \\ 2P(A \cap B) \leq P(A \cap B) + P(A \cap B^c) + P(B \cap A) + P(B \cap A^c) \\ P(A \cap B^c) + P(B \cap A^c) \geq 0 \end{gathered}$$

Because we know $P(A \cap B^c) \geq 0$ and $P(B \cap A^c) \geq 0$ by axioms of probability, the inequality in (2) is proven. We now investigate the equality case in which $P(A \cap B) = P(A \cup B)$.

$$\begin{gathered} P(A \cap B) = P(A \cup B) \\ P(A \cap B) = P(A) + P(B) - P(A \cap B) \\ P(A) + P(B) = 2P(A \cap B) \end{gathered}$$

The inequality would be an equality if $A = B$. If $A = B$, $P(A) = P(B)$ and $A \cap B = A$ so $P(A \cap B) = P(A)$. Thus $P(A) + P(B) = 2P(A) = 2P(A \cap B)$.

(3) $P(A \cup B) \leq P(A) + P(B)$

$$\begin{gathered} P(A \cup B) \leq P(A) + P(B) \\ P(A) + P(B) - P(A \cap B) \leq P(A) + P(B) \\ -P(A \cap B) \leq 0 \\ P(A \cap B) \geq 0 \end{gathered}$$

Because we know $P(A \cap B) \geq 0$ by axioms of probability, the inequality in (3) is proven. We now investigate the equality case in which $P(A \cup B) = P(A) + P(B)$.

$$\begin{gathered} P(A \cup B) = P(A) + P(B) \\ P(A) + P(B) - P(A \cap B) = P(A) + P(B) \\ P(A \cap B) = 0 \end{gathered}$$

When $P(A \cap B) = 0$, the equality case holds. **The inequality would be an equality if** A **and** B **are disjoint events.**

2.

Given events A and B where $A \subseteq B$, the set of elements of B not in A is given by $B - A = A^c \cap B$.

$$\begin{gathered} P(B - A) = P(A^c \cap B) = P(A^c) + P(B) - P(A^c \cup B) \\ P(B - A) = P(A^c \cap B) = 1 - P(A) + P(B) - P(A^c \cup B) \end{gathered}$$

Because $P(A^c \cup B) = 1$ as $A^c \cup B = S$ and $P(S) = 1$, we know $P(B - A) = P(A^c \cap B) = 1 - P(A) + P(B) - 1$. Thus, $P(B - A) = P(A^c \cap B) = P(B) - P(A)$.

3.

$$P(S|LC) = \frac{P(LC \cap S)}{P(LC)} = \frac{P(S) \cdot P(LC|S)}{P(S)P(LC|S) + P(S')P(LC|S')}$$

Given $P(S) = 0.216, P(S') = 1 - 0.216 = 0.784, P(LC|S) = 23P(LC|S')$, we can re-express $P(S|LC)$ as:

$$P(S|LC) = \frac{23P(S)P(LC|S')}{23P(S)P(LC|S') + P(S')P(LC|S')} = \frac{23P(S)}{23P(S) + P(S')}$$

This gives us $P(S|LC) = 0.8637$.

4.

a)

$$P(K|R) = \frac{P(K \cap R)}{P(R)} = \frac{P(R|K) \cdot P(K)}{P(R)} = \frac{P(R|K) \cdot P(K)}{P(R|K)P(K) + P(R|K')P(K')}$$

Given $P(R|K) = 1$, $P(K) = p$, $P(K') = 1 - P(K) = 1 - p$, and $P(R|K') = \frac{1}{n}$ as the answer is chosen randomly in the case Fred doesn't know the answer, $P(K|R) = \frac{p}{p + \frac{1-p}{n}}$.

b) We show $P(K|R) \geq p$ below.

$$\begin{gathered}
P(K|R) \geq p \\
\frac{p}{p + \frac{1-p}{n}} \geq p \\
p + \frac{1-p}{n} \leq 1 \\
\frac{1-p}{n} \leq 1 - p \\
\frac{1-p}{1-p} \leq n \\
n \geq 1
\end{gathered}$$

Because all tests have at least one question, $n \geq 1$, and thus $P(K|R) \geq p$. This makes sense as $P(K|R)$ refers to the probability that Fred knows the answer to a question given that he got the question right. Intuitively, $P(K|R) \geq p$ as we would expect a drastically higher percentage of the questions he gets correct to be because he knew the answer to them as opposed to choosing the question's correct answer by luck.

The only case in which $P(K|R) = p$ is when $n = 1$.

5.

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D|A) \cdot P(A)}{P(D)} = \frac{P(D|A) \cdot P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)}$$

Given that $P(A) = 0.5, P(B) = 0.3, P(C) = 0.2$ and $P(D|A) = 0.01, P(D|B) = 0.02, P(D|C) = 0.03$, $P(A|D) = 0.294$.

6. Anish Lakkapragada. I worked independently.