

## Discretionary Note

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## Math 226: HW 4

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1. a) We prove each of the two inequalities below.

①  $\dim(\text{Span}(S)) \leq \dim(\text{Span}(S \cup \{x\}))$

Let us specify the basis for  $S$  as  $\beta_S$  and basis for  $S \cup \{x\}$  as  $\beta_{S \cup \{x\}}$ . Thus  $\dim(\text{Span}(S)) = |\beta_S|$ . Let us explore two cases: (1)  $x \in \text{Span}(S)$  and (2)  $x \notin \text{Span}(S)$ . In case (1),  $\beta_{S \cup \{x\}} = \beta_S$  and so  $\dim(\text{Span}(S \cup \{x\})) = |\beta_{S \cup \{x\}}| = |\beta_S| \geq |\beta_S|$ . In case (2),  $\beta_{S \cup \{x\}} = \beta_S \cup \{x\}$  and so  $\dim(\text{Span}(S \cup \{x\})) = |\beta_{S \cup \{x\}}| = |\beta_S| + 1 \geq |\beta_S|$ . In either case, ① is true.

②  $\dim(\text{Span}(S \cup \{x\})) \leq \dim(\text{Span}(S)) + 1$

From ①, we know that  $\dim(\text{Span}(S \cup \{x\}))$  is equal to either  $|\beta_S| + 1$  or  $|\beta_S|$ . Because  $\dim(\text{Span}(S)) + 1 = |\beta_S| + 1$ ,  $\dim(\text{Span}(S \cup \{x\})) \leq \dim(\text{Span}(S)) + 1$  regardless of the particular value of  $\dim(\text{Span}(S \cup \{x\}))$ .

- b) Let us define the basis of  $U \cap W$  as  $\beta_{U \cap W} = \{v_1, \dots, v_n\}$  where  $n \in \mathbb{Z}$ . Let us also define  $\beta_U$  and  $\beta_W$  as the basis for  $U$  and  $W$ , respectively.

Using Steinitz Exchange Lemma, we can extend  $\beta_{U \cap W}$  to  $\beta_U$  as  $\beta_U = \beta_{U \cap W} \cup \{u_1, \dots, u_m\}$  where  $u_i \in U$ . Similarly, we can extend  $\beta_{U \cap W}$  to  $\beta_W$  as  $\beta_W = \beta_{U \cap W} \cup \{w_1, \dots, w_k\}$  where  $w_i \in W$ .

We now evaluate the statement  $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$ :

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

$$\dim(U + W) = (n + m) + (n + k) - n$$

$$\dim(U + W) = n + m + k$$

We now compute  $\dim(U + W)$  by trying to find the basis of  $U + W$ ,  $\beta_{U+W}$ . Any given element in  $U + W$  is given by  $u + w$  where  $u \in U, w \in W$ . We can write  $u$  and  $w$  as linear combinations of  $\beta_U$  and  $\beta_W$  respectively. Let us define the notation  $\beta_K[i]$  as giving the  $i$ th element of the basis for vector space  $K$ . Then given  $a_i, b_i \in \mathbb{F}$ :

$$u + w = \sum_i^{n+m} a_i \beta_U[i] + \sum_i^{n+k} b_i \beta_W[i]$$

$$u + w = \sum_i^n a_i v_i + \sum_i^m a_{n+i} u_i + \sum_i^n b_i v_i + \sum_i^k b_{n+i} w_i$$

$$u + w = \sum_i^n (a_i + b_i) v_i + \sum_i^m a_{n+i} u_i + \sum_i^k b_{n+i} w_i$$

We can redefine  $c_i = a_i + b_i \in \mathbb{F}$ :

$$u + w = \sum_i^n c_i v_i + \sum_i^m a_{n+i} u_i + \sum_i^k b_{n+i} w_i$$

Thus, we have shown every element in  $U + W$  can be written as a linear combination of  $\beta_{U \cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$ . We must show now that  $\beta_{U \cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$  is linearly independent to show that  $\beta_{U+W} = \beta_{U \cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$ . Because basis  $\beta_U = \beta_{U \cap W} \cup \{u_1, \dots, u_m\}$  and basis  $\beta_W = \beta_{U \cap W} \cup \{w_1, \dots, w_k\}$ , we know that  $\beta_{U \cap W} \cup \{u_1, \dots, u_m\}$  and  $\beta_{U \cap W} \cup \{w_1, \dots, w_k\}$  are linearly independent. Thus to show  $\beta_{U+W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$

is linearly independent, we must show that  $\{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$  is linearly independent.

Let us define this set as  $K = \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$  and let us assume that it is linearly dependent. This means that there exists a linear combination of  $\{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$  that exists in  $U$  and  $W$  (i.e. in  $U \cap W$ ). By our construction of  $\{u_1, \dots, u_m\}$  and  $\{w_1, \dots, w_k\}$  through Steinitz Exchange Lemma, we know that all  $u_i, w_i \notin \beta_{U \cap W}$  as they are the set of vectors required to extend  $\beta_{U \cap W}$  to  $\beta_U$  or  $\beta_W$ . This would mean that we have found some element  $\in U \cap W$  that cannot be represented as a linear combination of  $\beta_{U \cap W}$ . This violates the definition of  $\text{Span}(\beta_{U \cap W})$  and thus by proof by contradiction, we have shown  $\{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$  is linearly independent  $\Rightarrow \beta_{U \cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$  is linearly independent.

Thus,  $\beta_{U+W} = \beta_{U \cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$ . This means:

$$\dim(U + W) = n + m + k$$

$$|\beta_{U+W}| = n + m + k$$

$$n + m + k = n + m + k$$

$$0 = 0$$

Thus, we have proven  $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$ .

- c) We use the notation of  $\beta_{U \cap W}$  and  $\beta_{U+W}$  from part (b). We are given the following statement:

$$\dim(U + W) = 1 + \dim(U \cap W)$$

This statement implies that one element must be unioned to  $\beta_{U \cap W}$  to form  $\beta_{U+W}$  as  $U \cap W \subseteq U + W$ . This statement implies that  $|\beta_{U+W}| = n + m + k = 1 + |\beta_{U \cap W}| = 1 + n \Rightarrow m + k = 1$ . Because  $m, k \in \mathbb{Z}$  and  $m \geq 0, n \geq 0$ , we know that either  $m = 1$  or  $k = 1$ . We investigate these two cases below.

①  $m = 1$

If  $m = 1$ ,  $\exists u \in \{u_1, \dots, u_m\}$  s.t.  $\beta_{U+W} = \{u\} \cup \beta_{U \cap W}$ .  $W$  is given by  $\text{Span}(\beta_W)$ . Because  $m = 1 \Rightarrow k = 0$ ,  $\beta_{U \cap W} = \beta_W \Rightarrow U \cap W = W$ . Because this means every element in  $W$  exists in  $U$ ,  $U + W = \{u + w : u \in U, w \in W\} = \{u + w : u \in U, w \in U\}$ . Given  $U$  is closed under addition because it is a vector subspace, we know that  $U + W = U$ . Thus we have shown that in this case  $U + W = U$  and  $U \cap W = W$ .

②  $k = 1$

If  $k = 1$ ,  $\exists w \in \{w_1, \dots, w_k\}$  s.t.  $\beta_{U+W} = \{w\} \cup \beta_{U \cap W}$ .  $W$  is given by  $\text{Span}(\beta_W)$ . Because  $k = 1 \Rightarrow m = 0$ ,  $\beta_{U \cap W} = \beta_U \Rightarrow U \cap W = U$ . Because this means every element in  $U$  exists in  $W$ ,  $U + W = \{u + w : u \in U, w \in W\} = \{u + w : u \in W, w \in W\}$ . Given  $W$  is closed under addition because it is a vector subspace, we know that  $U + W = W$ . Thus we have shown that in this case  $U + W = W$  and  $U \cap W = U$ .

2. a) We show the two parts of the problem below.

① If  $T$  is injective,  $T(B)$  is linearly independent

Let us define  $B = \{u_1, \dots, u_{|B|}\}$ . To show that  $T(B)$  is linearly independent, we need to show that given  $a_i \in \mathbb{F}$ , the solution to  $\sum_i^{|B|} a_i T(u_i) = 0$  is for all  $a_i = 0$ . We inspect this below:

$$\begin{aligned}\sum_i^{|B|} a_i T(u_i) &= 0 \\ T(\sum_i^{|B|} a_i u_i) &= 0\end{aligned}$$

Because we know  $T$  is injective, if  $T(x) = 0 \Rightarrow x = 0$ . Thus, we know that  $\sum_i^{|B|} a_i u_i = 0$ . Because  $B = \{u_1, \dots, u_{|B|}\}$  is a basis and thus is linearly independent, the solution to  $\sum_i^{|B|} a_i u_i = 0$  is for all  $a_i = 0$ . Thus, the solution to  $\sum_i^{|B|} a_i T(u_i) = 0$  is for all  $a_i = 0 \Rightarrow T(B)$  is linearly independent.

**② If  $T(B)$  is linearly independent and  $\infty > |T(B)| \geq |B|$ ,  $T$  is injective**

To show that  $T$  is injective, we need to show that given  $x, y \in U$  s.t.  $T(x) = T(y)$ ,  $x = y$ . We prove this below. Note that because  $B$  is a basis for  $U$ ,  $x$  and  $y$  can be represented as a linear combination of  $B$ . Thus given  $c_i, d_i \in \mathbb{F}$ ,  $x = \sum_i^{|B|} c_i B_i$  and  $y = \sum_i^{|B|} d_i B_i$ .

$$\begin{aligned}T(x) &= T(y) \\ T(\sum_i^{|B|} c_i B_i) &= T(\sum_i^{|B|} d_i B_i) \\ \sum_i^{|B|} c_i T(B_i) &= \sum_i^{|B|} d_i T(B_i) \\ \sum_i^{|B|} (c_i - d_i) T(B_i) &= 0\end{aligned}$$

Because we are given  $T(B)$  is linearly independent, we know that the only solution to the equation  $\sum_i^{|B|} a_i T(B_i) = 0$  is for all  $a_i = 0$ . Thus, we know that in the above equation,  $c_i - d_i = 0 \Rightarrow c_i = d_i$ . Thus we have proved  $x = y$  if we know  $T(x) = T(y) \Rightarrow T$  is injective.

b) In order to show that  $T$  is surjective iff  $\text{Span}(T(B)) = V$ , we must show (1) if  $\text{Span}(T(B)) = V$ ,  $T$  is surjective and (2) if  $T$  is surjective,  $\text{Span}(T(B)) = V$ .

**① If  $\text{Span}(T(B)) = V$ ,  $T$  is surjective**

If  $\text{Span}(T(B)) = V$ ,  $T(B)$  is a basis for  $V$ . This means  $\forall v \in V$ ,  $v$  can be expressed as a linear combination of  $T(B)$ . Given  $B = \{b_1, \dots, b_{|B|}\}$  and  $a_i \in \mathbb{F}$ :

$$v = \sum_i^{|B|} a_i T(b_i)$$

This is equivalent to:

$$v = T(\sum_i^{|B|} a_i b_i)$$

Because  $\sum_i^{|B|} a_i b_i$  is a linear combination of  $B$ ,  $\sum_i^{|B|} a_i b_i \in U$ . Defining  $w = \sum_i^{|B|} a_i b_i$ , we have shown  $\forall v \in V, \exists w \in U$  s.t.  $T(w) = v$ . Thus  $T$  is proven to be surjective.

**② If  $T$  is surjective,  $\text{Span}(T(B)) = V$**

If  $T$  is surjective,  $\forall v \in V, \exists w \in U$  s.t.  $T(w) = v$ . Because  $B$  is a basis for  $U$ ,  $w$  can be expressed as a linear combination of  $B$ . Given  $B = \{b_1, \dots, b_{|B|}\}$  and  $a_i \in \mathbb{F}$ :

$$\begin{aligned}T(w) &= v \\ T(\sum_i^{|B|} a_i b_i) &= v\end{aligned}$$

Applying  $T$  to each element in the summation and switching sides:

$$v = \sum_i^{|B|} a_i T(b_i)$$

This shows that  $\forall v \in V$ ,  $v$  can be expressed as a linear combination of  $T(B)$ . Furthermore, this also shows that all linear combinations of  $T(B)$  are elements of  $V$ . Thus, we know that  $\text{Span}(T(B)) = V$ .

c) We prove both parts of this question below.

**① If  $T$  is bijective,  $T(B)$  is a basis for  $V$**

From part (b), we have proved if  $\text{Span}(T(B)) = V$ ,  $T$  is surjective. From part (a), we have proved if  $T(B)$  is linearly independent if  $T$  is injective. If  $T$  is bijective,  $T$  is surjective and injective, meaning that  $T(B)$  is linearly independent and generates  $V$ . By the definition of a basis,  $T(B)$  is a basis for  $V$ .

**② If  $T(B)$  is a basis and  $\infty > |T(B)| \geq |B|$ , then  $T$  is bijective**

We assume  $T(B)$  is a basis for  $V$ , which means that  $\text{Span}(T(B)) = V$  and  $T(B)$  is linearly independent. From part (b), we have proved if  $T(B)$  spans  $V$ ,  $T$  is surjective. From part (a), we have proved if  $T(B)$  is linearly independent and  $\infty > |T(B)| \geq |B|$ ,  $T$  is injective. Thus if  $T(B)$  is a basis and  $\infty > |T(B)| \geq |B|$ ,  $T$  is injective and surjective  $\Rightarrow T$  is bijective.

3. a)  $T$  is a linear transformation if given  $c \in \mathbb{R}$  and  $f, g \in C^1(\mathbb{R})$ ,  $T(cf + g) = cT(f) + T(g)$ . We evaluate  $T(cf + g)$  below as:

$$T(cf + g) = ((cf + g)'(3), (cf + g)(3)) = ((cf' + g')(3), (cf + g)(3)) = (cf'(3) + g'(3), cf(3) + g(3))$$

We now compute  $cT(f) + T(g)$  below.

$$cT(f) + T(g) = c(f'(3), f(3)) + (g'(3), g(3)) = (cf'(3), cf(3)) + (g'(3), g(3)) = (cf'(3) + g'(3), cf(3) + g(3))$$

Because the two above expressions are equivalent we know that  $T(cf + g) = cT(f) + T(g) \Rightarrow T$  is a linear transformation.

- b) If  $H + V = V$ , this means that  $\forall h \in H$  and  $\forall v \in V$ ,  $h + v = v$ . We prove this below.

$$h + v = v$$

$$T(h + v) = T(v)$$

$$((h + v)'(3), (h + v)(3)) = (1, 2)$$

$$(h'(3) + v'(3), h(3) + v(3)) = (1, 2)$$

Because  $h = (x - 3)^2 g(x)$  where  $g(x) \in C^1(\mathbb{R})$ ,  $h'(x) = 2(x - 3)g(x) + (x - 3)^2 g'(x)$ . Thus,  $h(3) = 0 = h'(3)$ . Given this,

$$(v'(3), v(3)) = (1, 2)$$

$$T(v) = (1, 2)$$

Thus we have proven  $\forall h \in H$  and  $\forall v \in V$ ,  $h + v = v$ . This proves that  $H + V = V$ .

- c) Let us define  $x, y \in C^1(\mathbb{R})$ . For  $T$  to be injective, if  $T(x) = T(y)$  then  $x = y$ . From part (b), we see that for two pre-images  $h + v, v \in C^1(\mathbb{R})$  where  $T(h + v) = T(v)$  but  $h + v \neq v$ . Thus, we have shown that by definition  $T$  is not injective.

In order for  $T$  to be surjective,  $\forall v \in \mathbb{R}^2, \exists f(x) \in C^1(\mathbb{R})$  s.t.  $T(f(x)) = v$ . Let us define a function  $f(x) \in C^1(\mathbb{R})$  and  $v = (a_1, a_2) \in \mathbb{R}^2$ . The function  $g(x) = a_1x + a_2 - 3a_1 \in C^1(\mathbb{R})$  and has the property  $T(g) = v$ . Thus,  $\forall v = (a_1, a_2) \in \mathbb{R}^2, \exists g(x) = a_1x + a_2 - 3a_1 \in C^1(\mathbb{R})$  s.t.  $T(g) = v$ . Thus,  $T$  is surjective.

4. a) **Computing basis and dimension for  $N(T)$**

$N(T) = \{v \in \mathbb{F}^5 : T(v) = \mathbf{0}^4\}$ .  $T(v) = \mathbf{0}^4$  when for a given  $v = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5$ ,  $T(v) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3) = \mathbf{0}^4$ . This occurs under the following conditions:

$$\begin{aligned} x_2 &= -x_1 \\ x_3 &= 0 \\ x_5 &= -\frac{x_4}{3} \end{aligned}$$

Thus,  $N(T) = \{(x_1, -x_1, 0, x_4, -\frac{x_4}{3}) : x_1, x_4 \in \mathbb{F}\}$ . Thus the basis for  $N(T)$  can be given as  $\beta_{N(T)}$ :

$$\beta_{N(T)} = \{(1, -1, 0, 0, 0), (0, 0, 0, 1, -\frac{1}{3})\}$$

So we get  $\dim(N(T)) = |\beta_{N(T)}| = 2$ .

**Computing basis and dimension for  $R(T)$**

$R(T) = \{w \in \mathbb{F}^4 : \exists v \in \mathbb{F}^5 \text{ s.t. } T(v) = w\}$ . If we redefine  $z_1 = x_1 + x_2$  and  $z_2 = x_4 + 3x_5$ , we get that  $T(v) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3) = (z_1, x_3, z_2, x_3)$  where  $z_1, x_3, z_2 \in \mathbb{F}$ . Thus the basis for  $R(T)$  given as  $\beta_{R(T)}$ :

$$\beta_{R(T)} = \{(1, 0, 0, 0), (0, 1, 0, 1), (0, 0, 1, 0)\}$$

So we get  $\dim(R(T)) = |\beta_{R(T)}| = 3$ .

- b) From Dimension Theorem, we know that if  $T$  is linear,  $\dim(N(T)) + \dim(R(T)) = \dim(\mathbb{R}^5) = 5$ . Because the output of  $T$  is in  $\mathbb{R}^2$ ,  $\dim(R(T)) \leq 2$ . Thus,  $\dim(N(T)) \geq 3$  if  $T$  is linear.

Expressed differently,  $N(T) = \{(x_1, x_2, \frac{x_1}{7}, x_2, x_2) : x_1, x_2 \in \mathbb{F}\}$ . Thus the basis of  $N(T)$  is given by  $\beta_{N(T)} = \{(1, 0, \frac{1}{7}, 0, 0), (0, 1, 0, 1, 1)\}$  and so  $\dim(N(T)) = |\beta_{N(T)}| = 2$ . Because  $\dim(N(T)) = 2 \not\geq 3$ ,  $T$  cannot be linear for this given null space  $N(T)$ .