

Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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Math 225- HW 9 Due: December 8 by Midnight

1. (13 points) Let $T : V \rightarrow V$, be a linear operator and V finite dimensional vector space. Recall that $\det(T) = \det[T]_\beta$ for some β ordered basis for V . Prove that
 - a) (4 points) The definition of $\det(T)$ is well-defined, i.e., if γ is another ordered basis for V then $\det[T]_\beta = \det[T]_\gamma$.
 - b) (2 points) Show that $\det([T]_\gamma - \lambda I) = \det([T]_\beta - \lambda I)$.
 - c) (2 points) Use part b) to deduce that similar matrices have the same characteristic polynomial. (Definition of similar matrices is given in the remark)
 - d) (5 points) If $g(t)$ be polynomial with coefficient from \mathbb{R} , then if x is an eigenvector for T with corresponding eigenvalue λ , then x is an eigenvector for $g(T)$ with corresponding eigenvalue $g(\lambda)$. - Use definition of eigenvalue.
2. (10 points) Let A be an upper(lower) triangular matrix, and has the distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ with corresponding multiplicities m_1, m_2, \dots, m_k .
 - a) Prove that $\text{tr}(A) = \sum_{i=1}^k m_i \lambda_i$
 - b) Prove that $\det(A) = \prod_{i=1}^k (\lambda_i)^{m_i}$.

Remark: We say $A, B \in M_{n \times n}(\mathbb{R})$ are *similar matrices* if there exist $P \in M_{n \times n}(\mathbb{R})$ invertible such that $PAP^{-1} = B$. Recall that if A is similar to B then $\text{tr}(A) = \text{tr}(B)$, and $\det(A) = \det(B)$. Therefore, the statement of this problem is true for any matrix that is similar to upper (lower) triangle matrix.

3. (31 points) Consider the following matrices $A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$
 - a) (9 points) Decide if they are diagonalizable in \mathbb{R} .
 - b) (2 points) Decide if they are diagonalizable in \mathbb{C} .
 - c) (5 points) Find Q and D matrix such that $A = Q^{-1}DQ$.
 - d) (5 points) Use part c) to find A^k for $k = 0, 1, 2, \dots$
 Hint: $(Q^{-1}DQ)^k = \underbrace{(Q^{-1}DQ)(Q^{-1}DQ), \dots (Q^{-1}DQ)}_{k \text{ times}}$.
 - e) (10 points) Find e^A . Hint: Use the Taylor expansion of $e^x = \sum_{n=0}^{\infty} (x^n/n!)$ and part d).

I think it is so cool to be able to define exponential of a matrix. You can do it for any function that has Taylor expansion in its radius of convergence. We will define the norm of a matrix in the coming weeks.

4. (23 points)
 - a) (5 points) Let A be a matrix whose characteristic polynomial split over its field \mathbb{F} . Prove that the determinant of A is the product of its eigenvalues, each counted with its multiplicity. (that is if the algebraic multiplicity of an eigenvalue if m then it is multiplied m times.)
 - b) (3 points) Use part a to conclude that if A is defined over \mathbb{C} , the complex numbers, then the determinant of A is always the product of its eigenvalues, each counted with its multiplicity.

- c) (15 points) Suppose A is a real $n \times n$ matrix which satisfies $A^3 = A + I_n$. Show that A has a positive determinant.

Hint: Even though A is real valued you can consider its eigenvalues in \mathbb{C} . So, try to find an equation that the eigenvalues satisfy. Here the fact that A is real must give you hint about complex eigenvalues.