PSETs Landing Page*

Anish Krishna Lakkapragada

This is the documentation for using my PSET PDFs responsibly. I post these LaTEX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarily, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

Math 226- HW 9 Due: November 12 by Midnight

1. (16 points) Let $A \in M_{n\times n}(\mathbb{R})$

   a) (8 points) We call a matrix upper triangular if all the entries below the main diagonal are zero, and lower triangular if all the entries above the main diagonal are zero. Show that if A is upper triangular then $det(A) = \prod_{i=1}^{n} a_{ii}$. Use the fact that $\det(A) = \det(A^T)$ to conclude the same for lower triangular matrices.
   Hint: Use induction with the determinant formula expanded wrt. n-th row.

   b) (8 points) Use elementary row operations to transform the following matrix to a triangle matrix, and use it to show $\det(A) = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix}$$

2. (18 points) Let $\delta : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be a function that is n-linear and alternating.

   - n-linear: if δ is a linear function of each row of an $n \times n$ matrix when the remaining $n-1$ rows are held fixed.
   - alternating: if A has two identical rows, then $\delta(A) = 0$.

   Show that

   a) (5 points) If A is a matrix obtained from B by interchanging any two rows, then $\delta(A) = -\delta(B)$.

   b) (5 points) Let $\mathcal{E}$ be any row operation in pg. 223 of the book. Show that $\delta(\mathcal{E}) = \delta(I)\det(\mathcal{E})$. (Check out Corollary 3 in section 4.5)

   c) (8 points) Use part b), to show that for all $A \in M_{n\times n}(\mathbb{F})$, $\delta(A) = k\det(A)$. (As always consider two cases, if A is not invertible and if A is invertible. In the former case both of the quantities will be zero. In the latter case A is product of elementary matrices.)

3. (14 points) Let A_n be a $n-1 \times n-1$ matrix given with the formula

$$A_n = \begin{pmatrix} 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & 1 & \dots & 1 \\ 1 & 1 & 5 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & n+1 \end{pmatrix}$$

   Show that the set $\{\frac{det(A_n)}{n!}, n \geq 2\}$ is unbounded.

   Hint: Use elementary operations and try to turn A_n to an upper triangle matrix-as much as you can.

4. (22 points) Let $A \in M_{n\times n}(\mathbb{F})$

   a) (6 points) Prove that if $A = \begin{pmatrix} B & C \\ 0 & I_k \end{pmatrix}$ for some $k \leq n$ then $\det(A) = \det(B)$.

   b) (10 points) Use part a) to prove if $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ for some $D \in M_{k\times k}(\mathbb{F})$, $k \leq n$, $\det(A) = \det(B)\det(D)$.

   c) (5 points) A matrix $m \in M_{n\times n}(\mathbb{C})$ is called skew-symmetric if $M^t = -M$. Prove that if M is skew symmetric and n is odd, then M is not invertible.

d) (1 points) What can you say if n is even?
Remark: In fact if M is skew symmetric then $\det(M) \geq 0$. To understand the proof read this very short paper -link only works in the pdf file in canvas not in gradescope

Practice Problems: Sec 4.1 : 1-11, Sec 4.2: 1-30, Sec 4.3: 1-25, Sec 4.4: 1-5