Discretionary Note

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Math 226: HW 8
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1. a) We can give $R(UT) = \{U(w) : w \in R(T)\}$ and $R(U) = \{U(w) : w \in W\}$. Because $R(T) \subseteq W$, $R(UT) \subseteq R(U)$. This means that all of the vectors that form a basis of $R(UT)$ are in $R(U)$. Thus, the number of vectors that form a basis of $R(UT)$ is at least the number of vectors that form a basis of $R(U) \Rightarrow dim(R(UT)) \leq dim(R(U))$.

   b) As given by Theorem 3.5, for any given $D \in M_{m \times n}(\mathbb{F})$, $rank(D) = rank(L_D)$. Thus, given two linear transformations L_A and L_B we get:

$$dim(R(L_A L_B)) \leq dim(R(L_A))$$
$$rank(L_{AB}) \leq rank(L_A)$$
$$rank(AB) \leq rank(A)$$

   The first statement $dim(R(L_A L_B)) \leq dim(R(L_A))$ was proved in part (a).

   c) We can give $R(UT) = \{U(w) : w \in R(T)\}$ and $R(T) = \{T(v) : v \in V\}$. Let us inspect the case in which U is injective and is not injective.

   (1) **U is injective**
   If U is injective, every pre-image in $R(T)$ maps to a *unique* image in Z through U. Let us define the basis for $R(T)$ as $\beta_{R(T)}$. We consider the two cases where U is bijective and not bijective (i.e. U is not surjective).

   (i) **U is bijective**
   As proved in Question 2 of Homework 4, in the case U is bijective, the set $U(\beta_{R(T)}) = \{U(w) : w \in \beta_{R(T)}\}$ acts as a basis for $R(UT) \Rightarrow |U(\beta_{R(T)})| = |\beta_{R(T)}| \Rightarrow dim(R(UT)) = dim(R(T))$.

   (ii) **U is not surjective**
   For proof by contradiction, let us assume $dim(R(UT)) > dim(R(T))$. Let us define $dim(R(UT)) = b$ and $dim(R(T)) = a$ and $b > a$. This means that there are b linearly independent vectors in $R(UT)$, which are all formed from U applied to b pre-images in $R(T)$. These b pre-images in $R(T)$ are linearly independent[1] and so this means that $dim(R(T)) = b \neq a$. Thus, by proof by contradiction, we have proven $dim(R(UT)) \leq dim(R(T)$.

   Thus, we can see in both cases $dim(R(UT)) \leq dim(R(T))$.

   (2) **U is not injective**
   Let us define a new linear map $U' : R(T) \to R(UT)$ that applies function U to all elements in $R(T)$. From Dimension Theorem, we know that $dim(R(T)) = nullity(U') + rank(U')$. Because U is surjective by definition, $R(U') = R(UT)$ and so $rank(U') = dim(R(U')) = dim(R(UT))$. Thus, we get that $dim(R(T)) = dim(N(U')) + dim(R(UT))$. Because $dim(N(U')) \geq 0$ by definition, $dim(R(T)) \geq dim(R(UT)) \Rightarrow dim(R(UT)) \leq dim(R(T))$.

   Thus, we can see in both cases $dim(R(UT)) \leq dim(R(T))$.

   d) We apply Theorem 3.5 in a similar manner as in part (b). Let us suppose we have two linear transformations L_A and L_B.

[1] Let us define these b linearly independent vectors in $R(UT)$ as $\{z_1, \ldots, z_b\}$. Because each of these vectors has a pre-image in $R(T)$, we can represent these b linearly independent vectors as $\{U(w_1), \ldots, U(w_b)\}$ where $w_i \in R(T)$. Because these vectors are linearly independent, the solution to $\Sigma_{i=1}^{b} a_i z_i = 0$ is for all $a_i = 0$. We can re-express this equation as $U(\Sigma_{i=1}^{b} a_i w_i) = 0$. Because $N(U) = \{0\}$ as U is injective, the solution to $\Sigma_{i=1}^{b} a_i w_i = 0$ is all $a_i = 0 \Rightarrow$ these b pre-images in $R(T)$ given by $\{w_1, \ldots, w_b\}$ are linearly independent.

$$dim[R(L_A L_B)] \leq dim[R(L_B)]$$
$$rank(L_{AB}) \leq rank(L_B)$$
$$rank(AB) \leq rank(B)$$

This first statement $dim[R(L_A L_B)] \leq dim[R(L_B)]$ is proven by part (c).

2. a) We prove both directions of this if and only if statement below.

   1 **If $rank(A) \neq rank([A'|b'])$, $[A'|b']$ contains a row in which the only nonzero entry is in the last column.**
   For proof by contrapositive, let us assume that $[A'|b']$ does not contain any rows in which the only nonzero entry is in the last column. This means that a valid solution to the equation $Ax = b$ exists $\Rightarrow b \in R(L_A) \Rightarrow b$ is in the span of the columns of $A \Rightarrow rank(A) = rank([A|b])$. Because $rank([A|b]) = rank([A'|b'])$, this means that $rank(A) = rank([A'|b'])$. Thus, by proof by contrapositive, we have proved that if $rank(A) \neq rank([A'|b'])$, $[A'|b']$ contains a row in which the only nonzero entry is in the last column.

   2 **If $[A'|b']$ contains a row in which the only nonzero entry is in the last column, $rank(A) \neq rank([A'|b'])$.**
   If there is a row in which the only nonzero entry is in the last column, this means that there is no solution to the equation $Ax = b$ as zero multiplied by each index of x cannot be nonzero. This means that $b \notin R(L_A) \Rightarrow b$ is not in the span of the columns of $A \Rightarrow b$ is linearly indepdent from the columns of $A \Rightarrow rank(A) \neq rank([A|b])$. Because $rank([A|b]) = rank([A'|b'])$, this means $rank(A) \neq rank([A'|b'])$ if there is a row in which the only nonzero entry is in the last column.

   b) If the equation $Ax = b$ is consistent, this means that there is a solution to this system. This means that $b \in R(L_A) \Rightarrow b \in$ the span of the columns of $A \Rightarrow rank(A) = rank([A|b]) \Rightarrow rank(A) = rank([A'|b'])$. Thus, this statement is essentially: $rank(A) = rank([A'|b']) \Leftrightarrow [A'|b']$ contains no row in which the only nonzero entry is in the last column. This statement is the negative (on both sides of the if and only if) of the statement proved in part (a). Thus, this statement is true.

3. Answer on paper. Please go to the end of this PDF.

4. a) The area of a parallelogram is given by the product of the base and the height. The length of the base can be given by $\|u_\theta\|$ and the height can be given as $\|v_\theta\| sin(\theta)$. Thus, the area A is given as $\|u_\theta\|\|v_\theta\| sin(\theta)$, which we compute below:

$$A = \|u_\theta\|\|v_\theta\| sin(\theta) = \|u_\theta\|\|v_\theta\| \sqrt{1 - (cos(\theta))^2} = \|u_\theta\|\|v_\theta\| \sqrt{1 - (\frac{u_\theta \cdot v_\theta}{\|u_\theta\|\|v_\theta\|})^2} =$$
$$\|u_\theta\|\|v_\theta\| \sqrt{\frac{\|u_\theta\|^2\|v_\theta\|^2 - (u_\theta \cdot v_\theta)^2}{(\|u_\theta\|\|v_\theta\|)^2}} = \sqrt{\|u_\theta\|^2\|v_\theta\|^2 - (u_\theta \cdot v_\theta)^2} =$$
$$\sqrt{(u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1 v_1 + u_2 v_2)^2} = \sqrt{u_1^2 v_2^2 - 2u_1 v_1 u_2 v_2 + u_2^2 v_1^2} = \sqrt{(u_1 v_2 - u_2 v_1)^2} =$$
$$u_1 v_2 - u_2 v_1$$

Thus, we have computed $A = u_1 v_2 - u_2 v_1 \geq 0$ due to the square root function. We now show that $A = |det[u_\theta, v_\theta]|$. The determinant of a 2×2 matrix is $ad - bc$ and the matrix $[u_\theta, v_\theta]$ is given by:

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

Thus, the determinant of this matrix is given by $u_1v_2 - u_2v_1$. As we saw before, this value is ≥ 0 and so the absolute value of this determinant is $u_1v_2 - u_2v_1 = A$.

b) If $A_{-\theta}(v) = v_\theta$, this means that $v_\theta = (v_1cos(\theta) + v_2sin(\theta), -v_1sin(\theta) + v_2cos(\theta))$. Similarily, if $A_{-\theta}(u) = u_\theta$, then $u_\theta = (u_1cos(\theta) + u_2sin(\theta), -u_1sin(\theta) + u_2cos(\theta))$. We compute $A_{-\theta}[u, v]$ below:

$$A_{-\theta}[u,v] = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} u_1cos(\theta) + u_2sin(\theta) & v_1cos(\theta) + v_2sin(\theta) \\ -u_1sin(\theta) + u_2cos(\theta) & -v_1sin(\theta) + v_2cos(\theta) \end{bmatrix} =$$

$$\begin{bmatrix} u_\theta & v_\theta \end{bmatrix}$$

where u_θ and v_θ act as column vectors.

c) By Theorem 4.7, we have $det(A_{-\theta}[u, v]) = det(A_{-\theta})det([u, v])$. As shown in (b), $A_\theta[u, v] = [u_\theta, v_\theta]$ and so this simplifies to:

$$det([u_\theta, v_\theta]) = det(A_{-\theta})det([u, v]).$$

As shown in part (a), $det([u_\theta, v_\theta]) = A$. We can also compute $det(A_{-\theta}) = cos^2(\theta) + sin^2(\theta) = 1$. Thus, we get:

$$A = 1 * det([u, v])$$
$$A = det([u, v])$$

Futhermore, in part (a) we have shown that $A \geq 0$. Thus, $|det([u, v])| = |A| = A$.

5. We prove both directions of this if and only if statement below.

(1) **If δ has the form $\delta(A) = \alpha a_{11}a_{22} + Ba_{11}a_{21} + Ca_{12}a_{22} + Da_{12}a_{21}$ where $\alpha, B, C, D \in \mathbb{F}$, δ is a 2-linear functional.**

We prove that δ is a 2-linear functional by proving, without loss of generality, that δ is a 2-linear functional if it's second row is $a + cb$. Note that this proof can be done identically for the first row. We define $c \in \mathbb{F}$.

$$\delta(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} + cb_{21} & a_{22} + cb_{22} \end{bmatrix}) = \delta(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) + c\delta(\begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix})$$

The LHS of this equation can be given as $\alpha a_{11}(a_{22} + cb_{22}) + Ba_{11}(a_{21} + cb_{21}) + Ca_{12}(a_{22} + cb_{22}) + Da_{12}(a_{21} + cb_{21})$. We now compute the RHS:

$$\delta(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) + c\delta(\begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}) = \alpha a_{11}a_{22} + Ba_{11}a_{21} + Ca_{21}a_{22} + Da_{12}a_{21}a_{21} +$$

$$c[\alpha a_{11}b_{22} + Ba_{11}b_{21} + Ca_{12}b_{22} + Da_{12}b_{21}] =$$

$$\alpha a_{11}(a_{22} + cb_{22}) + Ba_{11}(a_{21} + cb_{21}) + Ca_{12}(a_{22} + cb_{22}) + Da_{12}(a_{21} + cb_{21})$$

Thus, we have proven that LHS = RHS and so we have proven that if δ has this form, it is a 2-linear functional.

2 **If δ is a 2-linear functional, it has the form $\delta(A) = \alpha a_{11}a_{22} + Ba_{11}a_{21} + Ca_{12}a_{22} + Da_{12}a_{21}$ where $\alpha, B, C, D \in \mathbb{F}$**

To prove this statement, we inspect the value of $\delta(A)$, where $A \in M_{2\times 2}(\mathbb{F})$ and δ is a 2-linear functional:

$$\delta(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = \delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ a_{21}e_1 + a_{22}e_2 \end{bmatrix}) = \delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ a_{21}e_1 \end{bmatrix}) + a_{22}\delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ e_2 \end{bmatrix})$$

We refer to the first and second element in this summation as i and ii, respectively. We compute i below[2]:

$$\delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ a_{21}e_1 \end{bmatrix}) = \delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ 0 \end{bmatrix}) + a_{21}\delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ e_1 \end{bmatrix})$$

$$= \delta(\begin{bmatrix} a_{11}e_1 \\ 0 \end{bmatrix}) + a_{12}\delta(\begin{bmatrix} e_2 \\ 0 \end{bmatrix}) + a_{21}\delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ e_1 \end{bmatrix})$$

$$= \delta(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) + a_{11}\delta(\begin{bmatrix} e_1 \\ 0 \end{bmatrix}) + a_{12}\delta(\begin{bmatrix} e_2 \\ 0 \end{bmatrix}) + a_{21}\delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ e_1 \end{bmatrix})$$

$$= a_{11}\delta(\begin{bmatrix} e_1 \\ 0 \end{bmatrix}) + a_{12}\delta(\begin{bmatrix} e_2 \\ 0 \end{bmatrix}) + a_{21}\delta(\begin{bmatrix} a_{11}e_1 \\ e_1 \end{bmatrix}) + a_{21}a_{12}\delta(\begin{bmatrix} e_2 \\ e_1 \end{bmatrix})$$

$$= a_{11}\delta(\begin{bmatrix} e_1 \\ 0 \end{bmatrix}) + a_{12}\delta(\begin{bmatrix} e_2 \\ 0 \end{bmatrix}) + a_{21}\delta(\begin{bmatrix} 0 \\ e_1 \end{bmatrix}) + a_{21}a_{11}\delta(\begin{bmatrix} e_1 \\ e_1 \end{bmatrix}) + a_{21}a_{12}\delta(\begin{bmatrix} e_2 \\ e_1 \end{bmatrix})$$

Note that for any $i \in \mathbb{R}$, $\delta([e_i, 0]^T) = 0$. We show this below (for any vector $\vec{k}$):

[2]Note that because δ is a linear functional, $\delta(\mathbf{0}) = 0$. This is proven by the fact that, for any vector $\vec{k}$, $\delta(\mathbf{0}) = \delta([0, \vec{k} - \vec{k}]^T) = \delta([0, \vec{k}]^T) - \delta([0, \vec{k}]^T) = 0$.

$$\delta([e_i, \vec{k}]^T) = \delta([e_i, 0]^T) + \delta([e_i, \vec{k}]^T)$$
$$\delta([e_i, 0]^T) = 0$$

Similarily, for any $i \in \mathbb{R}$, $\delta([0, e_i]^T) = 0$. We show this for any vector $\vec{k}$:

$$\delta([\vec{k}, e_i]^T) = \delta([0, e_i]^T) + \delta([\vec{k}, e_i]^T)$$
$$\delta([0, e_i]^T) = 0$$

Because we cannot express $\delta([e_1, e_1]^T)$ and $\delta([e_2, e_1]^T)$ any further, we define constants $B = \delta([e_1, e_1]^T)$ and $D = \delta([e_2, e_1]^T)$. Thus, we can express i as:

$$Ba_{11}a_{21} + Da_{12}a_{21}$$

We now compute ii as:

$$a_{22}\delta(\begin{bmatrix} a_{11}e_1 + a_{12}e_2 \\ e_2 \end{bmatrix}) = a_{22}[\delta(\begin{bmatrix} a_{11}e_1 \\ e_2 \end{bmatrix}) + a_{12}\delta(\begin{bmatrix} e_2 \\ e_2 \end{bmatrix})] = a_{22}\delta(\begin{bmatrix} a_{11}e_1 \\ e_2 \end{bmatrix}) + a_{22}a_{12}\delta(\begin{bmatrix} e_2 \\ e_2 \end{bmatrix})$$
$$= a_{22}\delta(\begin{bmatrix} 0 \\ e_2 \end{bmatrix}) + a_{22}a_{11}\delta(\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}) + a_{22}a_{12}\delta(\begin{bmatrix} e_2 \\ e_2 \end{bmatrix}) = a_{22}a_{11}\delta(\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}) + a_{22}a_{12}\delta(\begin{bmatrix} e_2 \\ e_2 \end{bmatrix})$$

Because we cannot express $\delta([e_1, e_2]^T)$ and $\delta([e_2, e_2]^T)$ any further, we define constants $\alpha = \delta([e_1, e_2]^T)$ and $C = \delta([e_2, e_2]^T)$. Thus, we can express ii as:

$$\alpha a_{11}a_{22} + Ca_{12}a_{22}$$

Thus, we get that $\delta(A)$, where $A \in M_{2x2}(\mathbb{F})$, can be given by i + ii, or:

$$\delta(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = \alpha a_{11}a_{22} + Ca_{12}a_{22} + Ba_{11}a_{21} + Da_{12}a_{21}$$
$$\delta(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = \alpha a_{11}a_{22} + Ba_{11}a_{21} + Ca_{12}a_{22} + Da_{12}a_{21}$$

Thus we have proved that if δ is a 2-linear functional, it has the form $\delta(A) = \alpha a_{11}a_{22} + Ba_{11}a_{21} + Ca_{12}a_{22} + Da_{12}a_{21}$ where $\alpha, B, C, D \in \mathbb{F}$.

Question 3

(a) I first compute the solution to this system of equations.

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 2 & 2 & -3 & 4 \end{array}\right] \xrightarrow[R_2 - 2R_1 \to R_2]{} \left[\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 0 & -6 & -13 & 2 \end{array}\right] \xrightarrow[3R_1 + 2R_2 \to R_1]{} \left[\begin{array}{ccc|c} 3 & 0 & -11 & 7 \\ 0 & -6 & -13 & 2 \end{array}\right]$$

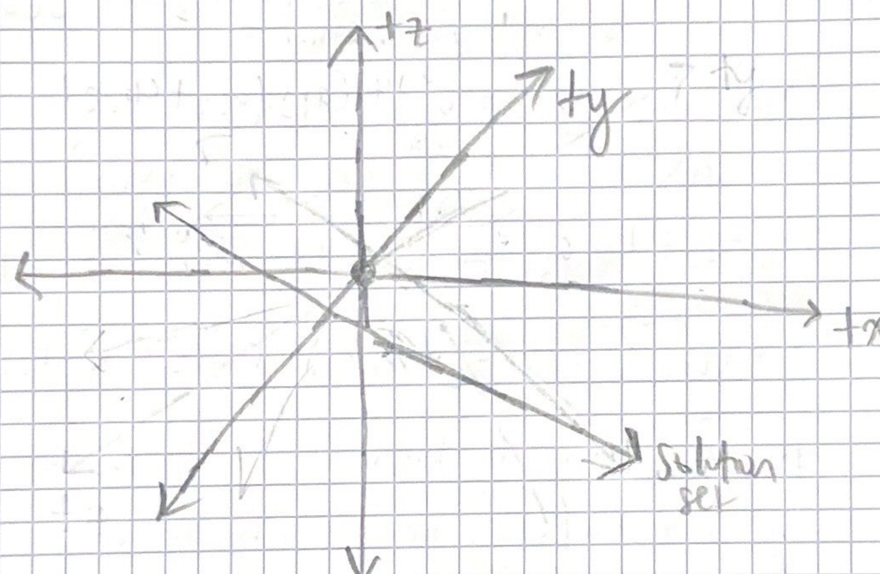
$$\xrightarrow[\frac{R_1}{3} \to R_1,\ -\frac{R_2}{6} \to R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{11}{3} & 7/3 \\ 0 & 1 & \frac{13}{6} & -\frac{1}{3} \end{array}\right]$$

$$x_1 - \frac{11}{3}x_3 = \frac{7}{3} \Rightarrow x_1 = \frac{11}{3}t_3 + \frac{7}{3}$$

$$x_2 + \frac{13}{6}x_3 = -\frac{1}{3} \Rightarrow x_2 = -\frac{13}{6}t_3 - \frac{1}{3}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{11}{3}t_3 + \frac{7}{3} \\ -\frac{13}{6}t_3 - \frac{1}{3} \\ t_3 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 11/3 \\ -13/6 \\ 1 \end{pmatrix} \text{ where } t_3 \in \mathbb{R}.$$

The solution is given by $\begin{pmatrix} 7/3 \\ -1/3 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 11/3 \\ -13/6 \\ 1 \end{pmatrix}$ where $t_3 \in \mathbb{R}$.



Note: the solution set is <u>increasing</u> in height from left to right.

(b) We solve the below equation.

$$x + 4y + 5z = 0$$

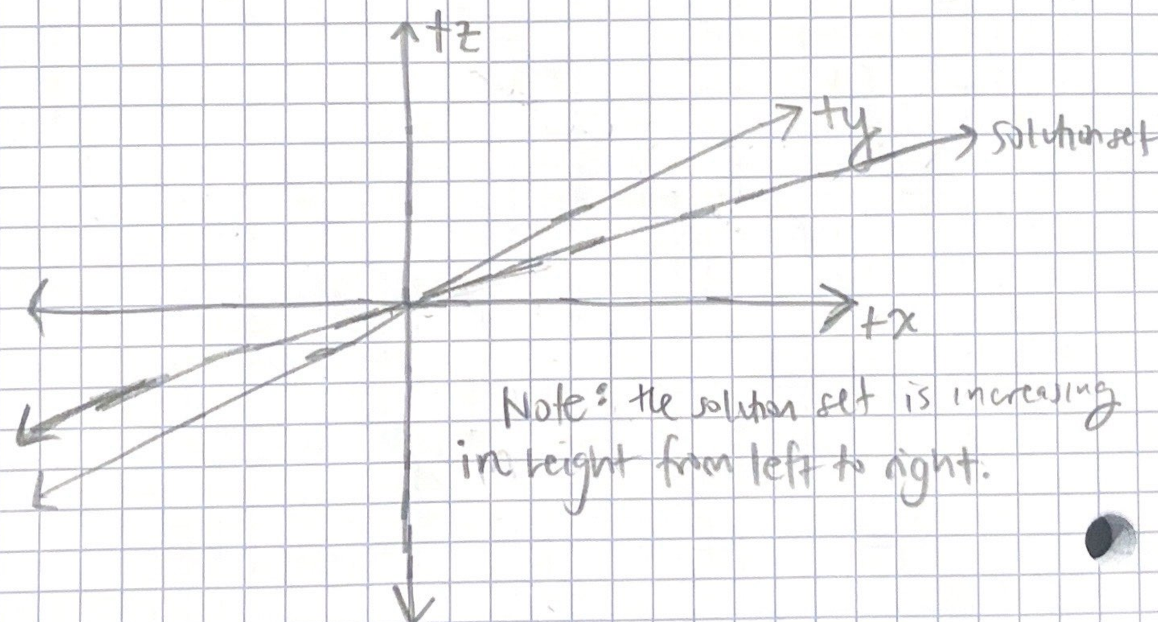
$$2x + 2y - 3z = 0$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right] \xrightarrow[\text{in part (a)}]{\text{the same sequence of elementary row operations as}} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{11}{3} & 0 \\ 0 & 1 & \frac{13}{6} & 0 \end{array}\right]$$

$$x_1 - \frac{11}{3}x_3 = 0 \Rightarrow x_1 = \frac{11}{3}x_3$$

$$x_2 + \frac{13}{6}x_3 = 0 \Rightarrow x_2 = \frac{-13}{6}x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11t_3/3 \\ -13t_3/6 \\ t_3 \end{pmatrix} = \underbrace{t_3\begin{pmatrix} 11/3 \\ -13/6 \\ 1 \end{pmatrix}, t_3 \in \mathbb{R}.}_{\text{homogenous solution sets}}$$



Note: the solution set is increasing in height from left to right.

(d) The solution set to the non-homogenous equation is equal to all elements in the solution set to the homogenous equation plus

$$\begin{pmatrix} 7/3 \\ -1/3 \\ 0 \end{pmatrix}.$$