

Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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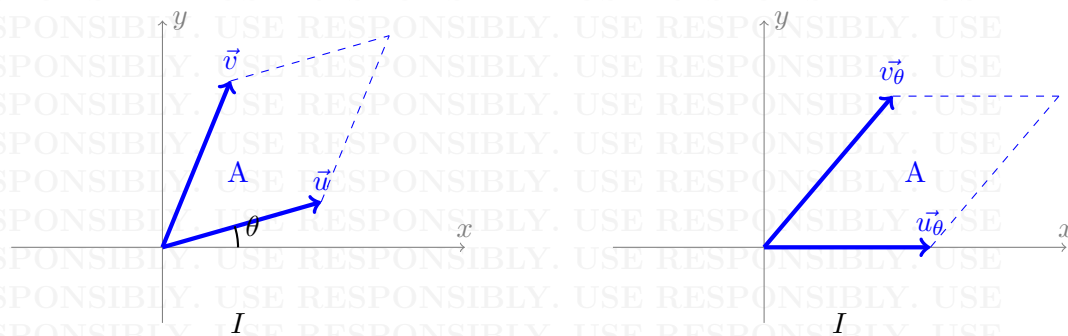
Math 226- HW 8 Due: Nov 5 by Midnight

1. (20) Let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear transformations on finite-dimensional vector spaces V, W, Z . Moreover, let A, B be matrices such that AB is defined.
 - a) (6 points) Prove that $\dim(R(UT)) \leq \dim(R(U))$
 - b) (4 points) Use pat a) to conclude that $\text{rank}(AB) \leq \text{rank}(A)$
 - c) (6 points) Prove that $\dim(R(UT)) \leq \dim(R(T))$
 - d) (4 points) Use pat c) to conclude that $\text{rank}(AB) \leq \text{rank}(B)$
2. (12 points) Suppose that the augmented matrix of a system $Ax = b$ is transformed into a matrix $[A'|b']$ in reduced row echelon form by finite sequence of elementary row operations.
 - a) (10 points) Prove that $\text{rank}(A) \neq \text{rank}[A'|b']$ if and only if $[A'|b']$ contains a row in which the only nonzero entry lies in the last column.
 - b) (2 points) Deduce that $Ax = b$ is consistent if and only if $[A'|b']$ contains no row in which the only nonzero entry lies in the last column.
3. (12 points) Each of the following equations determines a plane in \mathbb{R}^3 .

$$x + 4y + 5z = 1$$

$$2x + 2y - 3z = 4$$

- a) Find the intersection of these two planes, and draw a rough graph of the solution set.
 - b) Find the intersection when the equations are both homogenous, and draw rough graph of the solution set.
 - c) What is the relationship between these two set?
4. (10 points) In this problem you will prove that if u, v are two vectors in \mathbb{R}^2 , then the area of the parallelogram generated by u, v is equal to $|\det(u, v)|$. Let u and v are vectors as in picture I . Let $A_{-\theta} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ be the clockwise rotation by θ degree. One has $A_{-\theta}(v) = v_\theta$, and $A_{-\theta}(u) = u_\theta$, where v_θ and u_θ are as in the picture II



- a) (4 points) Let $u_\theta = (u_1, u_2)$, and $v_\theta = (v_1, v_2)$ in picture II. Calculate the area, A , of the parallelogram generated by u_θ and v_θ using geometry, and show by direct calculation $A = |\det[u_\theta, v_\theta]|$. - Think here u_θ, v_θ , as the column vectors of the matrix $[u_\theta, v_\theta]$.
- b) (2 points) Show that $A_{-\theta}(v) = v_\theta$, $A_{-\theta}(u) = u_\theta$ mean that $A_{-\theta}[u, v] = [u_\theta, v_\theta]$
- c) (4 points) Use part b), and Theorem 4.7 to show that $|\det[u, v]| = A$.

Remark : This is the geometric interpretation of the determinant. In general, if $A \in M_{n \times n}(\mathbb{R})$, and if the columns of A is (a_1, a_2, \dots, a_n) , then $\det(A)$ is the n -dimensional volume of the parallelepiped having the vectors a_1, a_2, \dots, a_n as adjacent sides. You can also think $\det(A)$ as the volume of the parallelepiped having the vectors $A(e_1), A(e_2), \dots, A(e_n)$ as adjacent sides.

5. (15 points) Recall that we defined $\delta : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$ to be n -linear functional if it is linear with respect to each row. Prove that $\delta : M_{2 \times 2}(\mathbb{F}) \rightarrow \mathbb{F}$ is a 2-linear functional if and only if it has the form

$$\delta(A) = Aa_{11}a_{22} + Ba_{11}a_{21} + Ca_{12}a_{22} + Da_{12}a_{21}$$

for any $[a_{ij}]_{2 \times 2}$ matrix and $A, B, C, D \in \mathbb{F}$.