Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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Math 226- HW 3 Due: Sep 19 by Midnight

1. (10 points)

   a) (6 points) In $\mathbb{F}^n$ (Take $\mathbb{F}=\mathbb{R}$ first if $\mathbb{F}$ confuses you). Let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $S:=\{e_1,e_2,...,e_n\}$ is linearly independent set and generates $\mathbb{F}^n$. **Remark:** The vectors $e_i : i=1,2,..,n$ are called standard vectors. In $\mathbb{R}^n$, we use these frequently and call the set of standard vectors "the standard bases".

   c) (4 points) Let V be vector space over a field $\mathbb{F}$ with characteristic not equal to two. (This means that if $x\neq 0$, then $x+x\neq 0$ for any $x\in\mathbb{F}$.) Let u and v be distinct vectors in V. Prove that $\{u,v\}$ is linearly independent if and only if $\{u+v,u-v\}$ is linearly independent.

      To see why, characteristic causes a trouble, note that you can gain zero while adding and subtracting the coefficients.

2. (22 pts) Let V be a vector space over $\mathbb{F}$ and $S=\{v_1,v_2,...,v_k\}\subseteq V$.

   (a) (6 points) For every $x\in V$, prove that $x\in\text{span}(S)$ iff $\text{span}(S)=\text{span}(S\cup\{x\})$.

   (b) (6 points) Suppose that $w\in V$ but $w\notin S$. *Further suppose that S is linearly independent.* Prove that $S\cup\{w\}$ is linearly independent iff $w\notin\text{span}(S)$.

   (c) (10 points) We write $A\subsetneqq B$ to mean $A\subseteq B$ and $A\neq B$. Prove that a set $S=\{u_1,u_2,\ldots,u_k\}$ is linearly independent iff

   $$\{0\}\subsetneqq\text{span}(\{u_1\})\subsetneqq\text{span}(\{u_1,u_2\})\subsetneqq\text{span}(\{u_1,u_2,u_3\})\subsetneqq\cdots\subsetneqq\text{span}(\{u_1,...,u_k\}).$$

   Remark: Another insignificant looking, but <u>fundamental</u> result. With this problem, we say that a set is linearly independent iff the span gets bigger with addition of each vector vector.

3. (14 points) Suppose U and W are subspaces of V. Recall we define the set $U+W=\{u+w : u\in U, w\in W\}$. We say $U+W$ is a direct sum if and only if $U\cap W=\{0\}$. In this case we represent the sum as $U\bigoplus W$. For example, if $U=\text{Span}((1,0))$ and $W=\text{Span}((0,1))$ then $\mathbb{R}^2$ =U $\bigoplus W$ since $\mathbb{R}^2=U+W$ and $U\cap W=\{(0,0)\}$

   a) (3 points) Let U be the subspace of $\mathbb{R}^5$ defined by $U:=\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5 : x_1=3x_2, x_3=7x_4\}$. Find a basis of U. (Recall we did a simpler version of that in the class.)

   - (2 points) Extend the basis in the first part to a basis of $\mathbb{R}^5$
   - (2 points) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5=U\bigoplus W$. Validate your answer.

   b) (3 points) Let U be the subspace of $P_4(\mathbb{R})$ defined by $U=:\{f\in P_4(\mathbb{R}) : f'(1)=0\}$. Find a basis of U.

   - (2 points) Extend the basis in the first part to a basis of $P_4(\mathbb{R})$.
   - (2 points) Find a subspace W of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R})=U\bigoplus W$. Validate your answer.

4. (7 points) We define $M_{n\times n}(\mathbb{F})$ as the matrices defined over $\mathbb{F}$ with size $n\times n$. We also define trace of $A=[a_{ij}]_{n\times n}\in M_{n\times n}(\mathbb{F})$ as

$$Tr(A)=\sum_{i=1}^{n}a_{ii}$$

Find a basis for matrices in $M_{3\times 3}(\mathbb{R})$ with trace zero.

Practice Problems : Sec 1.4 : 8-16 , Sec 1.5 2-7, 9, 14, 15, 17, 18, 20