Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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MATH 241 PSET 5

October 2, 2024

1.

a) Let us define r.v. B as the birth rank of a child. Let us define the events f_1, f_2, f_3 as the events that a family with one/two/three children are chosen, respectively. Given this,

$$P(B = n) = P(B = n|f_1)P(f_1) + P(B = n|f_2)P(f_2) + P(B = n|f_3)P(f_3)$$
$$P(B = n) = 0.3P(B = n|f_1) + 0.5P(B = n|f_2) + 0.2P(B = n|f_3)$$

Note that for $k \leq n$, $P(B = k|f_n) = \frac{1}{n}$ as if a family has n children, it has exactly one child with a birth rank from $\{1, \ldots, n\}$. Using this, we get the PMF of B as:

$$P(B = n) = \begin{cases} 0.6167, & n = 1 \\ 0.3167, & n = 2 \\ 0.0667, & n = 3 \\ 0, & \text{else} \end{cases}$$

The expectation and variance of B are given below.

$$\mathbb{E}[B] = \Sigma_{n=1}^{3} nP(B = n) = 1(0.6167) + 2(0.3167) + 3(0.0667) = 1.4502$$
$$Var(B) = \mathbb{E}[B^2] - (\mathbb{E}[B])^2 = \Sigma_{n=1}^{3} n^2 P(B = n) - 1.4502^2$$
$$Var(B) = 1(0.6167) + 4(0.3167) + 9(0.0667) - 1.4502^2 = 0.381$$

b) In the set of 30 families with only one child, there are 30 children with birth rank one. In the set of 50 families with two children, there are 50 children with a birth rank of one and 50 children with a birth rank of two. In the set of 20 families with 3 children, there are 20 children with birth rank one, 20 children with birth rank two, and 20 children with birth rank three. Adding it all together, across all 100 families, there are $30 + 50 + 20 = 100$ children with birth rank one, $50 + 20 = 70$ children with birth rank two, and 20 children with birth rank three.

Let r.v. B be the birth rank of a random child chosen in the town. The PMF of B is:

$$P(B=n) = \begin{cases} \frac{10}{19}, & n=1 \\ \frac{7}{19}, & n=2 \\ \frac{2}{19}, & n=3 \\ 0, & \text{else} \end{cases}$$

The expectation and variance of B are given below.

$$\mathbb{E}[B] = \Sigma_{n=1}^{3} nP(B=n) = 1(\frac{10}{19}) + 2(\frac{7}{19}) + 3(\frac{2}{19}) = 1.5789$$
$$Var(B) = \mathbb{E}[B^2] - (\mathbb{E}[B])^2 = \Sigma_{n=1}^{3} n^2 P(B=n) - 1.5789^2$$
$$Var(B) = 1(\frac{10}{19}) + 4(\frac{7}{19}) + 9(\frac{2}{19}) - 1.5789^2 = 0.454$$

2.

a) Let r.v. K be the number of children in a family selected at random in a given era. We compute $\mathbb{E}[K]$ below[1].

$$\mathbb{E}[K] = \Sigma_{k=0}^{\infty} kP(K=k)$$
$$\mathbb{E}[K] = \Sigma_{k=0}^{\infty} k\frac{n_k}{\Sigma_{k=0}^{\infty} n_k}$$
$$\mathbb{E}[K] = \Sigma_{k=0}^{\infty} k\frac{n_k}{m_0}$$
$$\mathbb{E}[K] = \frac{\Sigma_{k=0}^{\infty} kn_k}{m_0}$$
$$\mathbb{E}[K] = \frac{m_1}{m_0}$$

b) Let r.v. C be the number of children in a child's family for a child selected at random in a given era. We compute $\mathbb{E}[C]$ below[2].

$$\mathbb{E}[C] = \Sigma_{k=0}^{\infty} kP(C=k)$$
$$\mathbb{E}[C] = \Sigma_{k=0}^{\infty} k\frac{kn_k}{\Sigma_{k=0}^{\infty} kn_k}$$
$$\mathbb{E}[C] = \Sigma_{k=0}^{\infty} \frac{k^2 n_k}{m_1}$$
$$\mathbb{E}[C] = \frac{\Sigma_{k=0}^{\infty} k^2 n_k}{m_1}$$
$$\mathbb{E}[C] = \frac{m_2}{m_1}$$

[1] In this case, $P(K=k)$ gives the probability that a family selected at random has k children.
[2] In this case, $P(C=k)$ gives the probability that a child selected at random belongs to a family of k children.

3.

a) Because Penny and Nick's coinflips are independent, $P(X_n = 1 = Y_n) = P(X_n = 1 \cap Y_n = 1) = P(X_n = 1)P(Y_n = 1) = p_1p_2$. We define the r.v. $Z_n \sim Bern(p_1p_2)$ as a Bernoulli trial which is successful if both Penny and Nick's coinflips land heads. Thus, the distribution of the first time $Z_n = 1$ (i.e. the smallest n s.t. $X_n = Y_n = 1$), is given by $FS(p_1p_2)$. The expected value of this distribution is $\frac{1}{p_1p_2}$.

b) The probability that at least one of the coinflips is heads is given by $P(X_n = 1 \cup Y_n = 1) = P(X_n = 1) + P(Y_n = 1) - P(X_n = 1 \cap Y_n = 1) = p_1 + p_2 - p_1p_2$. We define r.v. $Z_n \sim Bern(p_1 + p_2 - p_1p_2)$ as a Bernoulli trial which is successful if at least one of Penny and Nick's coinflips land heads. Thus, the distribution of the first time $Z_n = 1$ is given by $FS(p_1 + p_2 - p_1p_2)$. The expected value of this distribution is $\frac{1}{p_1+p_2-p_1p_2}$.

4.

Let us define the indicator variable I_k as whether the kth person to draw a name receives their name. In order for the kth person to pick their name (i.e. $I_k = 1$), two conditions must be met: (1) all those who drew names prior must have not chosen this person's name and (2) this person must have chosen their name. We show this below for computing $P(I_3 = 1)$:

$$P(I_3 = 1) = \frac{N-1}{N}\frac{N-2}{N-1}(\frac{1}{N-2}) = \frac{1}{n}$$

Generalizing to $P(I_k = 1)$, we get:

$$P(I_k = 1) = \frac{\frac{(N-1)!}{(N-k)!}}{\frac{N!}{(N-k+1)!}}(\frac{1}{N-k+1})$$
$$P(I_k = 1) = \frac{(N-1)!(N-k+1)!}{N!(N-k)!}(\frac{1}{N-k+1})$$
$$P(I_k = 1) = \frac{N-k+1}{N}(\frac{1}{N-k+1})$$
$$P(I_k = 1) = \frac{1}{n}$$

Thus, because $P(I_k = 1)$ has a constant probability, $I_k \sim Bern(\frac{1}{p})$. Let us define the r.v. $X = \Sigma_{k=1}^{n} I_k$ as the number of people who have selected their names. We can compute $\mathbb{E}[X] = n\mathbb{E}[I_k] = \frac{n}{n} = 1$.

5.

Let us define the indicator variable I_k as whether the kth pair of cards (the kth card and $(k+1)$th card) are both red. The probability $I_k = 1$ is given by the probability that

the first card is red ($\frac{1}{2}$) multiplied by the chance that the second card is red ($\frac{26-1}{52-1} = \frac{25}{51}$): $P(I_k = 1) = \frac{25}{2(52)} = \frac{25}{102}$. Thus $I_k \sim Bern(\frac{25}{102})$.

The number of pairs of adjacent cards in a card deck with N cards is given by $N - 1$. Thus, a regular card deck will have 51 adjacent pairs. Let us define the r.v. $X = \Sigma_k^{51} I_k$ as the number of adjacent pairs where both cards are red. We can compute $\mathbb{E}[X] = 51\mathbb{E}[I_k] = 51\frac{25}{102} = \frac{25}{2} = 12.5$. On average, there will be 12.5 pairs of adjacent cards where both cards are red.

6. Anish Lakkapragada. I worked independently.