Problem set 2
Due Thursday January 30 at 11pm

Exercise 2.1 (5 points; Rudin 1.5). Let $A \subset \mathbb{R}$ be nonempty and bounded below. Define

$$-A = \{-x \mid x \in A\}.$$

Prove that $\inf A = -\sup(-A)$.

Exercise 2.2 (5 points). Let $A \subset \{x \in \mathbb{R} \mid x > 0\}$ be nonempty and bounded above. Define

$$A^{-1} = \{x^{-1} \mid x \in A\}.$$

Prove that $\inf(A^{-1}) = (\sup A)^{-1}$.

Exercise 2.3 (5 points). Suppose $A, B \subset \mathbb{R}$ are both nonempty and bounded above. Define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Prove that $\sup(A + B) = \sup A + \sup B$.

Exercise 2.4 (15 points). In each of the following, S is an ordered set, and $A \subset S$. Answer the following in each case, and prove your answers:

- Is A bounded above?
- Does A have a maximum element, and if so, what is it?
- Does A have a supremum in S, and if so, what is it?

(1) $S = \mathbb{Z}$, $A = \{2, 3\}$.

(2) $S = \mathbb{Q}$, $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\}$.

(3) $S = \mathbb{Q}$, $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$.

(4) $S = \mathbb{Q}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

(5) $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$.

(6) $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.

(7) $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x^3 < 2\}$.

Exercise 2.5 (10 points). Suppose F is an ordered field and $x, y, z \in F$.

(1) Prove that $0 < 1$.

(2) Prove that if $x > 0$, then $x^{-1} > 0$.

(3) Prove that if $x > 0$, then $y > z$ if and only if $xy > xz$.

(4) Recall we defined the field $\mathbb{F}_3 = \{0, 1, 2\}$ on class. Prove that there does not exist an order on $\mathbb{F}_3$ such that it is an ordered field.

Exercise 2.6 (optional, not for credit; Rudin 1.8 and 1.9). Define the complex numbers as the set

$$\mathbb{C} = \{a + b\mathrm{i} \mid a, b \in \mathbb{R}\}$$

with the addition and multiplication rules given by

- $(a + b\mathrm{i}) + (a' + b'\mathrm{i}) = (a + a') + (b + b')\mathrm{i}$,
- $(a + b\mathrm{i})(a' + b'\mathrm{i}) = (aa' - bb') + (ab' + a'b)\mathrm{i}$.

(1) Prove that $\mathbb{C}$ with these operations (and an appropriate 0, 1, negation, inversion) is a field.

(2) Prove that $\mathbb{C}$ with these operations cannot be made into an ordered field.

(3) Define the *lexicographic order* on $\mathbb{C}$ by defining

$$a + b\mathrm{i} < c + d\mathrm{i} \qquad \Longleftrightarrow \qquad a < c \ \text{ or } \ (a = c \text{ and } b < d).$$

Prove that this makes $\mathbb{C}$ into an ordered *set* (though not an ordered *field*).

(4) Does the lexicographic order on $\mathbb{C}$ have the least upper bound property?