

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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## Math 226: HW 3

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1. a) The set  $S$  is linearly independent if for  $a_1, a_2, \dots, a_n \in \mathbb{F}$  and  $e_i \in S$ ,  $\sum_{i=1}^N a_i e_i = 0$  when  $a_1, a_2, \dots, a_n$  all equal zero. To solve the equation  $\sum_{i=1}^N a_i e_i = (a_1, a_2, \dots, a_n) = \mathbf{0}^n$ , all elements in the set  $\{a_1, a_2, \dots, a_n\}$  must be equal to zero. Thus we have proved that  $a_1, a_2, \dots, a_n$  all equal zero as the only solution to  $\sum_{i=1}^N a_i e_i = 0$  and so  $S$  is proven to be linearly independent.

We define  $\text{Span}(S) = \{\sum_{i=1}^N a_i e_i : a_1, a_2, \dots, a_n \in \mathbb{F}\} = \{(a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in \mathbb{F}\}$  and define  $\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_1, a_2, \dots, a_n \in \mathbb{F}\}$ . Thus  $\text{Span}(S) = \mathbb{F}^n$  and so we have proven that  $S$  generates  $\mathbb{F}^n$ .

- c) To prove that  $\{u, v\}$  is linearly independent if and only if  $\{u + v, u - v\}$  is linearly independent, we first show (1)  $\{u + v, u - v\}$  is linearly independent if  $\{u, v\}$  is linearly independent and (2)  $\{u, v\}$  is linearly independent if  $\{u + v, u - v\}$  is linearly independent.

① **Given  $\{u, v\}$  is linearly independent, prove  $\{u + v, u - v\}$  is linearly independent**

If  $\{u, v\}$  is linearly independent, for  $a, b \in \mathbb{F}$ , the solution to  $au + bv = \mathbf{0}$  is  $a = b = 0$ . Given  $c, d \in \mathbb{F}$ , let us now re-express  $au + bv = \mathbf{0}$  with  $a = c + d, b = c - d$ .

$$\begin{aligned} au + bv &= \mathbf{0} \\ (c + d)u + (c - d)v &= \mathbf{0} \\ c(u + v) + d(u - v) &= \mathbf{0} \end{aligned}$$

As we are given  $\{u, v\}$  is linearly independent, we know that  $a = b = 0$  and so given  $c + d = c - d = 0$ , we know that  $c = d = 0$ <sup>1</sup>. Thus,  $\{u + v, u - v\}$  is proven to be linearly independent as the only solution to  $c(u + v) + d(u - v) = \mathbf{0}$  is proven to be  $c = d = 0$ .

② **Given  $\{u + v, u - v\}$  is linearly independent, prove  $\{u, v\}$  is linearly independent**

Given  $a, b \in \mathbb{F}$ , if we know  $\{u + v, u - v\}$  is linearly independent, we know that the solution to  $a(u + v) + b(u - v) = \mathbf{0}$  is  $a = b = 0$ . We can also simplify this as:

$$\begin{aligned} a(u + v) + b(u - v) &= \mathbf{0} \\ (a + b)u + (a - b)v &= \mathbf{0} \end{aligned}$$

Given that we know  $a = b = 0$ , if we define  $c = a + b = 0 \in \mathbb{F}$  and  $d = a - b = 0 \in \mathbb{F}$ , we get that

$$cu + dv = \mathbf{0}$$

If  $\{u, v\}$  is linearly independent, the solution for the equation  $eu + fv = \mathbf{0}$  is  $e = f = 0$  for  $e, f \in \mathbb{F}$ . From the above equation, we know that  $e = c = 0$

<sup>1</sup>Note that we can only conclude this because  $\mathbb{F}$  has a characteristic not equal to two. If this was not the case, the condition  $c + d = c - d = 0$  can be met if  $c = d = 1$ .

and  $f = d = 0$  and thus  $e = f = 0 \Rightarrow \{u, v\}$  is linearly independent. Thus we have proven that if  $\{u + v, u - v\}$  is linearly independent,  $\{u, v\}$  is linearly independent.

2. a) To prove that for every  $x \in V$ ,  $x \in \text{Span}(S)$  iff  $\text{Span}(S) = \text{Span}(S \cup \{x\})$  we must show (1) given  $x \in \text{Span}(S)$ ,  $\text{Span}(S) = \text{Span}(S \cup \{x\})$  and (2) given  $\text{Span}(S) = \text{Span}(S \cup \{x\})$ ,  $x \in \text{Span}(S)$ .

① **Given**  $x \in \text{Span}(S)$ ,  $\text{Span}(S) = \text{Span}(S \cup \{x\})$

Let us define  $N = |S|$ .  $\text{Span}(S) = \{\sum_{i=1}^N a_i v_i : a_i \in \mathbb{F}, v_i \in V\}$  and  $\text{Span}(S \cup \{x\}) = \{cx + \sum_{i=1}^N b_i v_i : b_i, c \in \mathbb{F}, v_i \in V\}$ . If  $x \in \text{Span}(S)$ ,  $x$  can be expressed as  $\sum_{i=1}^N a_i v_i$  for some set of values  $a_i \in \mathbb{F}$ . We can re-express our  $\text{Span}(S \cup \{x\})$  as  $\{\sum_{i=1}^N ca_i v_i + \sum_{i=1}^N b_i v_i : b_i, c \in \mathbb{F}, v_i \in V\} = \{\sum_{i=1}^N (ca_i + b_i) v_i : b_i, c \in \mathbb{F}, v_i \in V\}$ . Because  $ca_i + b_i \in \mathbb{F}$ ,  $\text{Span}(S \cup \{x\}) = \{\sum_{i=1}^N d_i v_i : d_i \in \mathbb{F}, v_i \in V\} = \text{Span}(S)$ . Thus, we have proved given  $x \in \text{Span}(S)$ ,  $\text{Span}(S) = \text{Span}(S \cup \{x\})$ .

② **Given**  $\text{Span}(S) = \text{Span}(S \cup \{x\})$ ,  $x \in \text{Span}(S)$

We use the definitions provided from ①. Because we know  $x \in S \cup \{x\}$ , we know that  $x \in \text{Span}(S \cup \{x\})$ . Since we are given  $\text{Span}(S) = \text{Span}(S \cup \{x\})$ , if  $x \in \text{Span}(S \cup \{x\}) \Rightarrow x \in \text{Span}(S)$ . Thus, we have proved if  $\text{Span}(S) = \text{Span}(S \cup \{x\})$ ,  $x \in \text{Span}(S)$ .

- b) We are given that  $S$  is linearly independent. To prove that  $S \cup \{w\}$  is linearly independent iff  $w \notin \text{Span}(S)$ , we first prove (1) if  $w \notin \text{Span}(S)$ ,  $S \cup \{w\}$  is linearly independent and then prove (2) if  $S \cup \{w\}$  is linearly independent,  $w \notin \text{Span}(S)$ .

① **Given**  $w \notin \text{Span}(S)$ ,  $S \cup \{w\}$  is linearly independent

We define  $n = |S|$ .

For proof by contrapositive, let us assume that  $S \cup \{w\}$  is linearly dependent. That means that for the solution to the equation  $cw + \sum_{i=1}^n a_i v_i = 0$ , where  $c, a_1, a_2, \dots, a_n \in \mathbb{F}$ , there exists at least one nonzero element of  $\{c, a_1, a_2, \dots, a_n\}$ . For this solution  $c \neq 0$ , as if  $c = 0$ , then the equation would simply be  $\sum_{i=1}^n a_i v_i = 0$ . Because  $S$  is linearly independent, the only solution to this equation is all  $a_i = 0$ . However, this would violate the condition that there exists at least one nonzero element in set  $\{c, a_1, a_2, \dots, a_n\}$  because we are assuming  $S \cup \{w\}$  is linearly dependent. Thus, we know  $c \neq 0$  and so we can re-express  $w$  as:

$$w = -\frac{\sum_{i=1}^n a_i v_i}{c} = \sum_{i=1}^n \frac{-a_i}{c} v_i$$

This expression of  $w$  is an expression of  $w$  as a linear combination of  $S$ . Thus,  $w \in \text{Span}(S)$  if  $S \cup \{w\}$  is linearly dependent. By proof by contrapositive, we have proven if  $w \notin \text{Span}(S)$ ,  $S \cup \{w\}$  is linearly independent.

② **Given**  $S \cup \{w\}$  is linearly independent,  $w \notin \text{Span}(S)$

We use the same definition for  $n$  provided in ①.

If  $S \cup \{w\}$  is linearly independent, it means that the solution to the equation  $cw + \sum_{i=1}^n a_i v_i = 0$ , where  $c, a_1, a_2, \dots, a_n \in \mathbb{F}$ , is that all elements of the set  $\{c, a_1, a_2, \dots, a_n\}$  must be zero. We now try to express  $w$  as some linear combination of  $S$ .

$$\begin{aligned} cw + \sum_{i=1}^n a_i v_i &= 0 \\ cw &= -\sum_{i=1}^n a_i v_i \end{aligned}$$

Note that because  $S \cup \{w\}$  is linearly independent,  $c = 0$ . Because we cannot divide the above equation by  $c$  on both sides, there does not exist any linear

combination of  $S$  that is equal to  $w$ . Because the  $\text{Span}(S)$  represents all possible linear combinations of elements in  $S$ , we have proven  $w \notin \text{Span}(S)$ .

c) Given  $S = \{u_1, u_2, u_3, \dots, u_k\}$ , to prove iff Condition  $M$  (defined below)

$$M : \{0\} \subsetneq \text{Span}(\{u_1\}) \subsetneq \text{Span}(\{u_1, u_2\}) \subsetneq \text{Span}(\{u_1, u_2, u_3\}) \subsetneq \text{Span}(\{u_1, \dots, u_k\})$$

then  $S$  is linearly independent, we must first prove (1) if condition  $M$  holds, then  $S$  is linearly independent and (2) if  $S$  is linearly independent, then condition  $M$  holds.

**① If condition  $M$  holds, then  $S$  is linearly independent**

For proof by contrapositive, let us assume  $S$  is linearly dependent. Thus, there exists some  $m < k \in \mathbb{Z}$  where subset  $D = \{u_1, \dots, u_m\} \subsetneq S$  is linearly independent and subset  $K = \{u_1, \dots, u_m, u_{m+1}\} \subseteq S$  is linearly dependent. Let us define  $a_1, a_2, \dots, a_{m+1} \in \mathbb{F}$  and  $u_i \in K$ . Because  $K$  is linearly dependent, we know that for the solution to the equation  $\sum_{i=1}^{m+1} a_i u_i = 0$ , there exists at least one nonzero element in the set  $\{a_1, a_2, \dots, a_{m+1}\}$ . We can further develop this equation as:

$$a_{m+1}u_{m+1} + \sum_{i=1}^m a_i u_i = 0$$

Let us consider the case in which  $a_{m+1} = 0$ . This would leave us with the equation  $\sum_{i=1}^m a_i u_i = 0$ . Because  $D$  is linearly independent, we know that the only solution to this equation is  $\forall a_i \in \{a_1, a_2, \dots, a_m\}, a_i = 0$ . However, because this solution violates the condition that there exists at least one nonzero element in the set  $\{a_1, a_2, \dots, a_{m+1}\}$ , we know that  $a_{m+1} \neq 0$ . Thus,  $u_{m+1} = \sum_{i=1}^m (-\frac{a_i}{a_{m+1}})u_i$  and so because we can express  $u_{m+1}$  as linear combination of  $D$ ,  $u_{m+1} \in \text{Span}(D)$ .

We now compute  $\text{Span}(K) = \{cu_{m+1} + \sum_{i=1}^m b_i u_i : c, b_i \in \mathbb{F}\} = \{c \sum_{i=1}^m (-\frac{a_i}{a_{m+1}})u_i + \sum_{i=1}^m b_i u_i : c, b_i \in \mathbb{F}\} = \{\sum_{i=1}^m (-\frac{ca_i}{a_{m+1}} + b_i)u_i : c, b_i \in \mathbb{F}\}$ . Because  $-\frac{ca_i}{a_{m+1}} + b_i \in \mathbb{F}$ ,  $\text{Span}(K) = \{\sum_{i=1}^m d_i u_i : d_i \in \mathbb{F}\} = \text{Span}(D)$ . Because  $\text{Span}(D) = \text{Span}(K)$ , condition  $M$  does not hold if  $S$  is linearly dependent. Thus, we have proven by contrapositive that if condition  $M$  holds, then  $S$  is linearly independent.

**② If  $S$  is linearly independent, then condition  $M$  holds**

We use the definitions of  $m, k, D, K$  from ①.

For proof by contrapositive, let us assume that condition  $M$  does not hold<sup>2</sup> and so for some  $m < k \in \mathbb{R}$ ,  $\text{Span}(D) = \text{Span}(K)$ . Because  $u_{m+1} \in \text{Span}(K) \Rightarrow u_{m+1} \in \text{Span}(D)$ , we know that  $u_{m+1}$  can be written as  $u_{m+1} = \sum_{i=1}^m a_i u_i$  for  $a_i \in \mathbb{F}, u_i \in D$ .

The set  $S$  is linearly independent if given  $b_i \in \mathbb{F}$  and  $u_i \in S$ , the only solution to the equation  $\sum_{i=1}^k b_i u_i = 0$  is  $\forall b_i \in \{b_1, b_2, \dots, b_k\}, b_i = 0$ . We can re-express this equation below as:

$$\begin{aligned} \sum_{i=1}^k b_i u_i &= 0 \\ b_{m+1}u_{m+1} + \sum_{i=m+2}^k b_i u_i + \sum_{i=1}^m b_i u_i &= 0 \end{aligned}$$

<sup>2</sup>There is only one case in which condition  $M$  does not hold. Condition  $M$  would not hold if  $\exists j < k \in \mathbb{Z}$  s.t.  $\text{Span}(\{u_1, \dots, u_j\}) = \text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$  or  $\text{Span}(\{u_1, \dots, u_j\}) \subsetneq \text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$ . All elements in  $\text{Span}(\{u_1, \dots, u_j\})$  exist in  $\text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$  as the last element in the set  $u_{j+1}$  can always be ignored in a linear combination of  $\{u_1, \dots, u_j, u_{j+1}\}$  by setting its coefficient to zero. Thus  $\text{Span}(\{u_1, \dots, u_j\}) \subseteq \text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$  and so condition  $M$  can only be violated if  $\text{Span}(\{u_1, \dots, u_j\}) = \text{Span}(\{u_1, \dots, u_j, u_{j+1}\})$ .

Let us set  $b_i = 0$  for  $m + 2 \leq i \leq k$ ,  $b_i = -a_i$  for  $1 \leq i \leq m$ , and  $b_{m+1} = 1$ . This gives us:

$$u_{m+1} = \sum_{i=1}^m b_i u_i = \sum_{i=1}^m a_i u_i$$

As shown before, we know this statement is true. Thus, we have found a solution to the equation  $\sum_{i=1}^k b_i u_i = 0$  with at least one  $b_i \neq 0$ . This means  $S$  is linearly dependent. Thus, if condition  $M$  does not hold, we have shown  $S$  is linearly dependent. By proof by contrapositive, we have proven that if  $S$  is linearly independent, then condition  $M$  holds.

3. a) Given  $U = \{(x_1, \frac{x_1}{3}, x_3, \frac{x_3}{7}, x_5) \in \mathbb{R}^5; x_1, x_3, x_5 \in \mathbb{F}\}$ , a basis of  $U$ ,  $\beta_U$ , can be given by:

$$\beta_U = \{(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1)\}$$

- i) Extending  $\beta_U$  to  $\mathbb{R}^5$

$$\beta_{\mathbb{R}^5} = \{(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0)\}$$

- ii) Find a subspace  $W$  of  $\mathbb{R}^5$  such that  $\mathbb{R}^5 = U \oplus W$

$$W = \{(0, a_1, 0, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$$

We now validate (1) that  $U + W = U \oplus W$  and (2)  $U \oplus W = \mathbb{R}^5$ .

- ①  $U + W = U \oplus W$

Let us define  $z_1, z_2, z_3, z_4, z_5 \in \mathbb{R}$ . Given  $x = (z_1, \frac{z_1}{3}, z_3, \frac{z_3}{7}, z_5) \in U$  and  $y = (0, z_2, 0, z_4, 0) \in W$ , we can find  $U \cap W$  as the set of solutions to  $x = y$ . This would be defined as the solution to the system of equations below.

$$z_1 = 0$$

$$\frac{z_1}{3} = z_2$$

$$z_3 = 0$$

$$\frac{z_3}{7} = z_4$$

$$z_5 = 0$$

The solution to this system is  $z_1 = z_2 = z_3 = z_4 = z_5 = 0$ . Thus  $U \cap W = \{\mathbf{0}^5\} \Rightarrow U + W = U \oplus W$ .

- ②  $U \oplus W = \mathbb{R}^5$

We use the same definitions of  $x, y, z_1, z_2, z_3, z_4, z_5$  from ①. We compute  $x + y = (z_1, z_2 + \frac{z_1}{3}, z_3, z_4 + \frac{z_3}{7}, z_5) \in U \oplus W$ . Because  $z_1, z_2 + \frac{z_1}{3}, z_3, z_4 + \frac{z_3}{7}, z_5 \in \mathbb{R}$ ,  $x + y \in \mathbb{R}^5$  and so  $U \oplus W \subseteq \mathbb{R}^5$ . Because  $\mathbf{0}^5 \in U \oplus W$  and  $U \oplus W$  can be trivially shown to be closed under addition and scalar multiplication,  $U \oplus W \leq \mathbb{R}^5$ .

We now prove that  $U \oplus W = \mathbb{R}^5$  by showing that  $\dim(U \oplus W) = \dim(\mathbb{R}^5) = 5$ .

The basis of  $U$ ,  $\beta_U$ , is given above and we give the basis of  $W$  as  $\beta_W = \{(0, 1, 0, 0, 0), (0, 0, 0, 1, 0)\}$ . We now compute  $\text{Span}(\beta_U \cup \beta_W)$  as given  $v_i \in \beta_U \cup \beta_W$  and  $a_i \in \mathbb{R}$ ,  $\text{Span}(\beta_U \cup \beta_W) = \sum_{i=1}^3 a_i v_i + \sum_{i=5}^4 a_i v_i$ . Because all vectors in  $U$  can be expressed as  $\sum_{i=1}^3 a_i v_i$  (a linear combination of  $\beta_U$ ) and all vectors in  $W$  can be expressed as  $\sum_{i=5}^4 a_i v_i$  (a linear combination of  $\beta_W$ ),  $\text{Span}(\beta_U \cup \beta_W) = \{u + w : u \in U, w \in W\} = U \oplus W \Rightarrow \beta_U \cup \beta_W$  is the basis for  $U \oplus W$ . Thus  $\dim(U \oplus W) = |\beta_U \cup \beta_W| = 5 = \dim(\mathbb{R}^5)$ . Thus, we have proved  $U \oplus W = \mathbb{R}^5$ .

- b) Given  $U = \{f(x) = c_1 + (-2c_3 - 3c_4 - 4c_5)x + c_3x^2 + c_4x^3 + c_5x^4 \in P_4(\mathbb{R}) : c_1, c_3, c_4, c_5 \in \mathbb{R}\}$ , a basis of  $U$  can be given by:

$$\beta_U = \{1, -2x + x^2, -3x + x^3, -4x + x^4\}$$

- i) Extending  $\beta_U$  to  $P_4(\mathbb{R})$

$$\beta_{P_4(\mathbb{R})} = \{1, x, -2x + x^2, -3x + x^3, -4x + x^4\}$$

- ii) Find a subspace  $W$  of  $P_4(\mathbb{R})$  such that  $P_4(\mathbb{R}) = U \oplus W$

$$W = \{f(x) = a_1x \in P_4(\mathbb{R}) : a_1 \in \mathbb{R}\}$$

We now validate (1) that  $U + W = U \oplus W$  and (2)  $U \oplus W = P_4(\mathbb{R})$ .

①  $U + W = U \oplus W$

Let us define  $z_1, z_2, z_3, z_4, z_5 \in \mathbb{F}$ . Given  $u = z_1 + (-2z_3 - 3z_4 - 4z_5)x + z_3x^2 + z_4x^3 + z_5x^4 \in U$  and  $w = z_2x \in W$ , we find  $U \cap W$  as the set of solutions to  $u = w$ . This would be defined as the solution to the system of equations below.

$$\begin{aligned} z_1 &= 0 \\ -2z_3 - 3z_4 - 4z_5 &= z_2 \\ z_3 &= 0 \\ z_4 &= 0 \\ z_5 &= 0 \end{aligned}$$

The solution to this system is  $z_1 = z_2 = z_3 = z_4 = z_5 = 0$ . Thus since  $U \cap W = \{f(x) = 0 \in P_4(\mathbb{R})\} \Rightarrow U + W = U \oplus W$ .

②  $U \oplus W = P_4(\mathbb{R})$

We use the same definitions of  $u, w, z_1, z_2, z_3, z_4, z_5$  from ①. We compute  $u + w = z_1 + (z_2 - 2z_3 - 3z_4 - 4z_5)x + z_3x^2 + z_4x^3 + z_5x^4 \in U \oplus W$ . Because  $z_1, z_2 - 2z_3 - 3z_4 - 4z_5, z_3, z_4, z_5 \in \mathbb{R}$ ,  $u + w \in P_4(\mathbb{R})$  and so  $U \oplus W \subseteq P_4(\mathbb{R})$ . Because  $0 \in U \oplus W$  and  $U \oplus W$  can be trivially shown to be closed under addition and scalar multiplication,  $U \oplus W \leq P_4(\mathbb{R})$ . We now prove that  $U \oplus W = P_4(\mathbb{R})$  by showing that  $\dim(U \oplus W) = \dim(P_4(\mathbb{R})) = 5$ .

The basis of  $U$ ,  $\beta_U$ , is given above and we give the basis of  $W$  as  $\beta_W = \{x\}$ . We now compute  $\text{Span}(\beta_U \cup \beta_W)$  as given  $f_i(x) \in \beta_U \cup \beta_W$  and  $a_i \in \mathbb{R}$ ,  $\text{Span}(\beta_U \cup \beta_W) = \sum_{i=1}^3 a_i f_i(x) + \sum_{i=5}^4 a_i f_i(x)$ . Because all functions in  $U$  can be expressed as  $\sum_{i=1}^3 a_i f_i(x)$  (a linear combination of  $\beta_U$ ) and all functions in  $W$  can be expressed as  $\sum_{i=5}^4 a_i f_i(x)$  (a linear combination of

$\beta_W$ ),  $\text{Span}(\beta_U \cup \beta_W) = \{u + w : u \in U, w \in W\} = U \oplus W \Rightarrow \beta_U \cup \beta_W$  is the basis for  $U \oplus W$ . Thus  $\dim(U \oplus W) = |\beta_U \cup \beta_W| = 5 = \dim(P_4(\mathbb{R}))$ . Thus, we have proved  $U \oplus W = P_4(\mathbb{R})$ .

4. Let us define a matrix  $E_{ij}$  as a 3x3 matrix with all zeros except in the  $i$ th row and  $j$ th column, where there is a one.

The basis for  $M_{3 \times 3}(\mathbb{R}) = \{\bigcup_{\substack{i,j=1 \\ i \neq j}}^3 E_{ij}\} \cup \{E_{00} - E_{33}, E_{11} - E_{33}\}$ .