

Discretionary Note

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CONTENT STARTS ON NEXT PAGE.

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MATH 244 HW 2

February 4, 2025

1. Section 1.5, Question 5

We first prove if $R \circ (S \circ T)$ is well defined $\implies (R \circ S) \circ T$ is well-defined. If $R \circ (S \circ T)$ is well-defined we can define sets A, B, C, D , where $R \subseteq A \times B, S \subseteq B \times C$, and $T \subseteq C \times D$. This means $S \circ T \subseteq B \times D$ and so $R \circ (S \circ T) \subseteq A \times D$, and so $R \circ (S \circ T)$ is well-defined¹. We continue with these sets A, B, C, D for the rest of this problem.

To prove $R \circ (S \circ T) = (R \circ S) \circ T$, we prove $R \circ (S \circ T) \subseteq (R \circ S) \circ T$ and $(R \circ S) \circ T \subseteq R \circ (S \circ T)$.

1. $R \circ (S \circ T) \subseteq (R \circ S) \circ T$

Let us pick $(x, y) \in R \circ (S \circ T)$. Let us define arbitrary elements $b \in B, c \in C$. Given $(x, y) \in R \circ (S \circ T)$, we know $\exists (x, b) \in R$ and $(b, y) \in S \circ T$. Furthermore, if $(b, y) \in S \circ T$, we know $\exists (b, c) \in S$ and $(c, y) \in T$. Because $(x, b) \in R$ and $(b, c) \in S \implies (x, c) \in R \circ S$. Furthermore, $(c, y) \in T$, so $(R \circ S) \circ T$ will contain $(x, y) \implies \forall (x, y) \in R \circ (S \circ T), (x, y) \in (R \circ S) \circ T \implies R \circ (S \circ T) \subseteq (R \circ S) \circ T$.

2. $(R \circ S) \circ T \subseteq R \circ (S \circ T)$

Let us pick $(x, y) \in (R \circ S) \circ T$. Let us define arbitrary elements $c \in C, b \in B$. Given $(x, y) \in (R \circ S) \circ T$, we know $\exists (x, c) \in R \circ S$ and $(c, y) \in T$. If $(x, c) \in R \circ S$, we know $\exists (x, b) \in R$ and $(b, c) \in S$. We now show $(x, y) \in R \circ (S \circ T)$. Because $(b, c) \in S$ and $(c, y) \in T$, $(b, y) \in S \circ T$. Furthermore, because $(x, b) \in R$, $(x, y) \in R \circ (S \circ T)$. Thus, we have shown, $\forall (x, y) \in (R \circ S) \circ T, (x, y) \in R \circ (S \circ T) \implies (R \circ S) \circ T \subseteq R \circ (S \circ T)$.

2. Section 1.6, Question 3

We prove both directions of this statement below.

1. If $R \circ R \subseteq R \implies R$ is transitive

$R \circ R$ contains a pair (x, z) if (x, y) and (y, z) are both in R . If $R \circ R \subseteq R$, then this means if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R \circ R \implies (x, z) \in R$. This satisfies the definition of transitivity, which is given as follows: if $(x, y) \in R$ and $(y, z) \in R \implies (x, z) \in R$. Thus, R is transitive.

¹Put in other words, $R \circ (S \circ T)$ is well-defined because the codomain of R (set B) is the domain of $S \circ T$ and the codomain of S (set C) is the domain of T .

2. **If R is transitive $\implies R \circ R \subseteq R$**

If R is transitive, this means if (x, y) and $(y, z) \in R \implies (x, z) \in R$. $R \circ R$ contains a pair (x, z) if (x, y) and (y, z) are both in R . However, because R is transitive, it is guaranteed by definition that $(x, z) \in R$ if (x, y) and $(y, z) \in R$. Thus, if R is transitive, $\forall (x, z) \in R \circ R, (x, z) \in R \implies R \circ R \subseteq R$.

3. **Section 1.6, Question 6**

A relation R on X is an equivalence relation if it is reflexive, transitive, and anti-symmetric. A relation is an ordering relation if it is reflexive, transitive, and symmetric. So a relation R is both an equivalence and ordering relation if it is reflexive, transitive, and both anti-symmetric & symmetric. R is anti-symmetric and symmetric if (i) if $\forall x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R \iff y = x$ and (ii) $\forall x, y \in X, (x, y) \in R \implies (y, x) \in R$.

A relation R which is reflexive, transitive, antisymmetric, & symmetric will have the following property: $(x, y) \in R \iff x = y$. We prove this below:

1. $(x, y) \in R \implies x = y$

Because R is symmetric, $(x, y) \in R \implies (y, x) \in R$. However, because R is anti-symmetric, $(y, x) \in R \implies y = x$ or $x = y$.

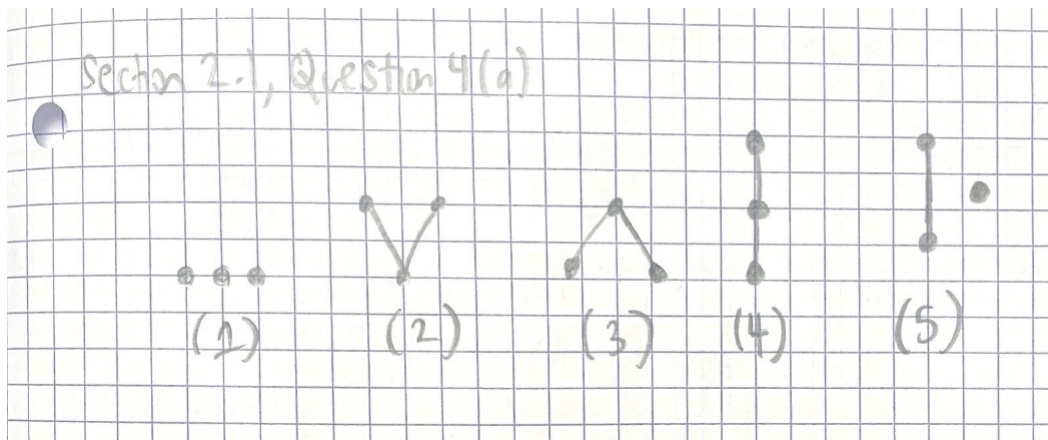
2. $x = y \implies (x, y) \in R$

This property is satisfied by the fact that R is reflexive.

Thus, the relations on the set X that are both equivalences and orderings are identity relations given by $R = \{(x, x) : x \in X\}$.

4. **Section 2.1, Question 4**

a) We show all possible non-isomorphic 3-element posets below:



We enumerate over all possibilities of a non-isomorphic three element poset to clearly show we have classified all possible non-isomorphic three-element posets:

1. **All elements in the poset are not comparable to each other.**

This is case (1).

2. **One minimal element in the poset. The other two elements are incomparable.**

This is case (2).

3. **One maximal element in the poset. The other two elements are incomparable.**

This is case (3).

4. **Every pair of elements is comparable,**

This is case (4).

5. **One pair of elements in the poset are comparable to each other, and the remaining element is incomparable to all other elements.**

This is case (5).

b) Let us define (X, \leq) and (Y, \preceq) to be linearly ordered sets with $|X| = |Y| = n$. This means that every single element in them can be compared with each other and so we can produce the following ordered² sequences L_X and L_Y of X and Y respectively:

$$\begin{aligned} L_X &= x_1 \leq x_2 \leq \cdots \leq x_n \\ L_Y &= y_1 \preceq y_2 \preceq \cdots \preceq y_n \end{aligned}$$

Let us define the following map $f : X \rightarrow Y$ that where $f(x_i) = y_i$. Because $\forall y_i \in Y, \exists x_i$ s.t. $f(x_i) = y_i$, we can conclude f is surjective. Because $\forall 1 \leq i < j \leq n, f(x_i) \neq f(x_j)$, we also have that f is one-to-one or injective. Because f is injective and surjective $\implies f$ is bijective.

We now must show that f is order-preserving, or that $\forall x, x' \in X, x \leq x' \implies f(x) \preceq f(x')$. Note that because of the way L_X and L_Y are ordered (shown above), given $1 \leq i, j \leq n$, if $i < j$ then we have that $x_i \leq x_j$ and $f(x_i) \preceq f(x_j)$. Furthermore, we can see that given $x_i \leq x_j$, then $i < j$ and similarly given $f(x_i) \preceq f(x_j)$, then $i < j$. This means that we have shown both of these statements: (i) if $x_i \leq x_j \implies i < j \implies f(x_i) \preceq f(x_j)$ and (ii) if $f(x_i) \preceq f(x_j) \implies i < j \implies x_i \leq x_j$. Thus we have shown that $\forall x, x' \in X, x \leq x' \iff f(x) \preceq f(x')$. Because we have shown for any two n -element linearly ordered sets we can create a bijection $f : X \rightarrow Y$ where $\forall x, y \in X, x \leq y \iff f(x) \preceq f(y)$, we have proven that any two n -element linearly ordered sets are isomorphic.

5. Section 2.2, Question 2

a) We first define $<$ to be the divisibility relation $|$, set $B = \{1, 2, \dots, n\}$, and the longest possible subset of B linearly ordered by $|$ as set A , where $m = |A|$. Because A is linearly ordered, this means that $\forall x, y \in A$, either $x \leq y$ or $y \leq x$. This means that if set A , when ordered from least to greatest, is given as a_1, a_2, \dots, a_k then we must have that $\forall 2 \leq i \leq k, a_{i-1}$ must be able to divide a_i . Note that this guarantees $\forall 1 \leq j < i \leq k, a_i$ can divide any a_j because:

²in increasing order

$$\frac{a_i}{a_j} = \frac{a_i}{a_{i-1}} \frac{a_{i-1}}{a_{i-2}} \cdots \frac{a_{j+1}}{a_j}$$

which is just a product of natural numbers and thus is a natural number $\implies a_i$ can divide a_j . Our goal now is to construct $A \subseteq B$. To start, one must be in A (as its minimal element) as every natural number can be divided by A . Next to try to include in A as many elements of B as possible, we should aim for the $\forall 1 \leq j < i \leq n, \frac{a_i}{a_j} \in \mathbb{N}$ ratio to be as small as possible. This ratio cannot be 1, as each number appears only once in B . Thus, the next greatest natural number this ratio can be is two. This means that $2^{m-1} = |B| = n$ or $m = \log_2(n) + 1$. Note that the $+1$ term here is to account for the inclusion of 1 into set A .

- b) We define the set $B = 2^{\{1,2,\dots,n\}}$ and the longest possible subset of B linearly ordered by \subseteq as set A , where $m = |A|$. Because A is linearly ordered, this means that $\forall x, y \in A$, either $x \subseteq y$ or $y \subseteq x$. This means that if set A , when ordered from least to greatest, is given as a_1, a_2, \dots, a_k then we must have that $\forall 2 \leq i \leq k, a_{i-1} \subseteq a_i$. This guarantees $\forall 1 \leq j < i \leq k, a_j \subseteq a_i$ as:

$$a_j \subseteq a_{j+1} \subseteq a_{j+2} \cdots \subseteq a_i$$

Our goal now is to construct $A \subseteq B$. To start, \emptyset must be in A (as its minimal element) as any $\emptyset \subseteq$ any set. Next to try to include in A as many of elements of B , we should aim for $\forall 2 \leq i \leq n, a_i/a_{i-1}$ to be as small as possible. We can make $|a_i/a_{i-1}| = 1$ if every consecutive element includes one more element than the last. It will take n elements to go from a set with only one element to one with all n elements (i.e. set $\{1, 2, \dots, n\}$). Therefore the maximum number of elements required to make this chain is $n + 1$. I show a visualization below of these $n + 1$ elements that can build this set A :

$$A = \{\emptyset, \underbrace{\{1\}, \{1, 2\}, \dots, \{1, 2, \dots, n\}}_{n+1 \text{ elements}}\}$$

and so $m = |A| = n + 1$.

6. Section 2.2, Question 3: Optional Bonus Problem

- a) We prove both directions of this statement:

1. $\text{le}(X, \preceq) = 1 \implies (X, \preceq)$ is a linear ordering

We prove this statement by contrapositive and assume that (X, \preceq) is not a linear ordering $\implies \exists x_i, x_j \in X$ s.t. $x_i \not\preceq x_j$ and $x_j \not\preceq x_i$. This means that we can assemble at least two linear orderings for (X, \preceq) : one in which $x_i \preceq x_j$ and one in which $x_j \preceq x_i$. This means $\text{le}(X, \preceq) \neq 1$. Thus, we have proved this statement by contrapositive.

2. (X, \preceq) is a linear ordering $\implies \text{le}(X, \preceq) = 1$

Let us define a linear extension $<$ of \preceq : this means $\forall x, x' \in X$, if $x \preceq x' \implies x \leq x'$. Because \preceq is already a linear ordering, this means $\forall x, x' \in X$, $x \preceq x'$ or $x' \preceq x$. This means this total ordering $<$ must have the following property: $\forall x, x' \in X$, $x \leq x'$ if $x \preceq x'$ or $x' \leq x$ if $x' \preceq x$. Thus, $<$ is the same ordering as \preceq and so the number of linear extensions possible for \preceq is only one (i.e. $\text{le}(X, \preceq) = 1$.)

- b) The partial ordering \preceq which can have the most possible linear extensions is one which imposes the least constraints. Such an ordering would be one where $\forall x, x' \in X$, x and x' are not comparable to each other (i.e. $x \not\preceq x'$ and $x' \not\preceq x$). This is because for such an ordering, a linear extension of this ordering can choose to order the n elements of X in any possible way. Because there are $n!$ ways to order n elements (i.e. X), this means for this such ordering (let's call it \preceq), $\text{le}(X, \preceq) = n!$. Note that for any other partial ordering it is guaranteed $\exists x, y \in X$ s.t. $x \preceq y$ and so any linear extension of this ordering must be compatible to this constraint \implies this linear extension has $< n!$ ways to order $X \implies$ the number of linear extensions of this ordering is $< n!$. As such, for any partial ordering \preceq , $\text{le}(X, \preceq) \leq n!$.