Math 226 - HW 1 Due: Sep 5 by 10:30 p.m.

1. (14 points) Let A and B be two non-empty sets, $f : A \to B$ be an injective function, and a_0 is a fixed element of A. Define $g : B \to A$ such that

$$g(b) = \begin{cases} a & \text{if } f(a) = b \\ a_0 & \text{otherwise} \end{cases}$$

   a) (3 points) Show that g is a left inverse of f. That is for all $a \in A$, $g \circ f(a) = a$
   b) (3 points) Explain why g might not be the right inverse of f. Under which conditions g can be a right inverse of f?
   c) (8 points) If $f : A \to B$ be surjective then it would have right inverse. Define the right inverse and validate that it is indeed right inverse.

2. (10 points) Given two functions $f : A \to B$ and $g : B \to C$, the *composition of g with f*, denoted $g \circ f$ is the map $g \circ f : A \to C$ given by $(g \circ f)(a) = g(f(a))$. Show that the set of bijection functions are closed under $\circ$. That is

   (a) (4 points) show if f and g are injections, then $g \circ f$ is an injection. Hint: Recall the injective and left inverse relation.
   (b) (4 points) show if f and g are surjections, then $g \circ f$ is a surjection. Hint: Use the definition of surjectivety first for g and then for f.
   c) (2 points) conclude that if f and g are bijections, then $g \circ f$ is a bijection.

3. (10 points) Let $B = \{m + n\sqrt{2} \ : \ m, n \in \mathbb{Z}\}$.

   a) (2 points) Show that B is <u>*closed under addition*</u>. In other words, prove "if $x \in B$ and $y \in B$, then their sum $x + y$ must be in B as well.
   b) (3 points) Show that B is <u>*closed under multiplication*</u>. In other words, prove "if $x \in B$ and $y \in B$, then their product xy must be in B as well.
   c) (5 points) For every integer $k \geq 1$, prove that $(-1 + \sqrt{2})^k \in B$ - Try to use an inductive argument for this problem.

4. (20 points) We say a function $T : \mathbb{R} \to \mathbb{R}$ is <u>*additive*</u> iff for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$, we have that $T(x + y) = T(x) + T(y)$. If T is additive, show that

   a) (5 points) for all $x \in \mathbb{R}$ and for all integers $n \geq 1$, we have $T(nx) = nT(x)$.
   b) (3 points) $T(0) = 0$.
   c) (3 points) for all $x \in \mathbb{R}$ we have $T(-x) = -T(x)$.
   d) (5 points) for all $x \in \mathbb{R}$ and for all rational numbers $r \in \mathbb{Q}$, we have $T(rx) = rT(x)$.
   e) (4 points) Is it true that if $r \in \mathbb{R}$ the $T(rx) = rT(x)$ for all $x \in \mathbb{R}$? Why or why not?

5. (16 points) Define $\mathbb{Z}\{\sqrt{3}\} := \{a + b\sqrt{3}, \text{ where } a, b \in \mathbb{Z}\}$. Below we will validate that $(\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$, with usual addition and multiplication does not define a field.

   a) (4 points) Show that if $a^2 - 3b^2 = \pm 1$, then $a + b\sqrt{3}$ has a multiplicative inverse. Hint: Check out the definition of the multiplicative inverse, you can directly give it in each case.

b) (10 points) Show that the above statement is iff, i.e., if $a+b\sqrt{3}$ has a a multiplicative inverse then $a^2 - 3b^2 = \pm 1$ Hint: Solve the problem first for the case when a, b are prime to each other. That will give you the understanding of the rest.

c) (2 points) Use a) and b) to conclude that $(\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$, with usual addition and multiplication does not define a field.

- (optional) Prove that the set $\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} \ : \ a, b \in \mathbb{Q}\}$ is indeed a field. Notice that you don't need the conditions like in b, c here.