

PSETs Landing Page*

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The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarly, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

STATS 242 HW 3

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Number of late days: 0; Collaborators: Alex Wa, Derek Gao

1.

a) We detail our devised two possible test statistics:

(1) **Test Statistic T_1 : Z-Statistic**

For this test statistic, we want to check whether the expected number of male children in each family is actually $12(0.5) = 6$, what we would expect under H_0 if $X_i \sim \text{Bin}(12, 0.5)$. In other words, $H_0 : \mu = 6$ and $H_1 : \mu \neq 6$. Because the variance of each X_i is known under H_0 , we can use the Z -statistic as our test statistic:

$$T_1 = \frac{\sqrt{6115} \bar{X}}{\sigma}$$

In my code, we compute the T_1 statistic for this data to be **260.48**.

(2) **Test Statistic T_2 : Sample Variance**

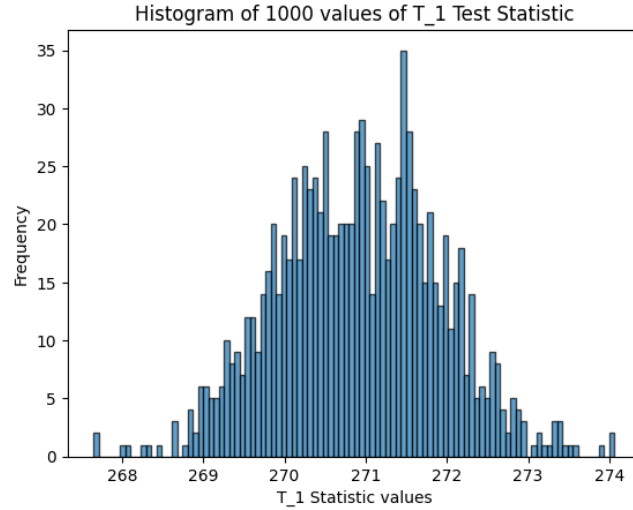
For this test statistic, we test if sample variance is actually equal to $12(0.5)(0.5) = 3$, as it would be under H_0 where the observed male child frequencies follow a $\text{Bin}(12, 0.5)$ distribution. Our test statistic is the sample variance:

$$T_2 = \frac{\sum_{i=1}^{6115} (X_i - \bar{X})^2}{6115 - 1}$$

In my code, we compute the T_2 statistic for this data to be **3.490**.

b) (1) **Test Statistic T_1**

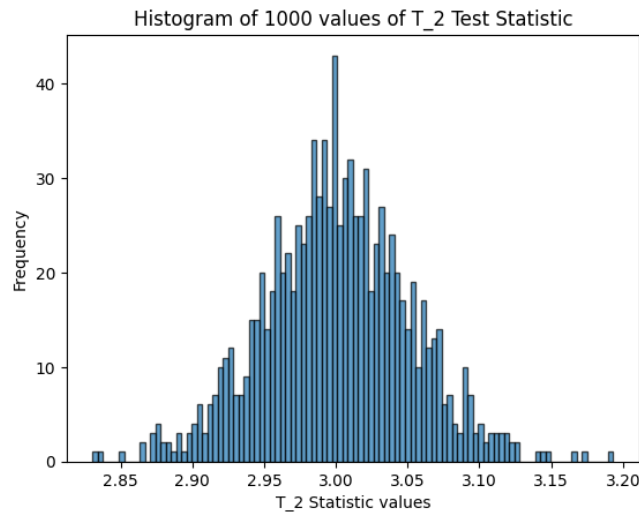
Below is our histogram for the null distribution of the T_1 test statistic.



For this test statistic at the $\alpha=0.05$ significance level, the upper- α point is **268.60** and the lower- α point which is **268.03**, which is less than the observed T_1 statistic for this data (**260.48**). Thus, we reject H_0 .

(2) **Test Statistic T_2**

Below is our histogram for the null distribution of the T_2 test statistic.



For this test statistic at the $\alpha=0.05$ significance level, the upper- α point and lower- α point are given by **2.875** and **2.834** respectively. Because the observed T_2 statistic for this data (**3.490**) is greater than the upper- α point, we reject H_0 .

```
1 # %%
2 """Run all imports"""
```

```

3 import numpy as np
4 import matplotlib.pyplot as plt
5 import math
6
7 NUM_MALES = np.array([i for i in range(0, 12 + 1)])
8 NUM_FAMILIES = np.array([7, 45, 181, 478, 829, 1112, 1343, 1033, 670,
    ↪ 286, 104, 24, 3])
9 VARIANCE = 12 * 0.5 * (1 - 0.5)
10
11 assert np.sum(NUM_FAMILIES).item() == 6115
12
13 def t1_statistic(num_males, num_families):
14     mean_num_males = np.sum(num_families * num_males) /
    ↪ np.sum(num_families)
15     return np.sqrt(6115 / VARIANCE) * mean_num_males
16
17 def t2_statistic(num_males, num_families):
18     mean_num_males = np.sum(num_families * num_males) /
    ↪ np.sum(num_families)
19     return np.sum(num_families * (num_males - mean_num_males) ** 2) /
    ↪ (6115 - 1)
20
21 print(f"On this data, our T_1 statistic is {t1_statistic(NUM_MALES,
    ↪ NUM_FAMILIES)}")
22 print(f"On this data, our T_2 statistic is {t2_statistic(NUM_MALES,
    ↪ NUM_FAMILIES)}")
23 # %%
24 def get_data_for_one_simulation():
25     # for 6115 families, sample the number of children they have based
    ↪ on Bin(12, 0.5)
26     num_males = NUM_MALES
27     num_families = np.zeros_like(NUM_FAMILIES)
28     num_males_6115 = np.random.binomial(12, 0.5, 6115)
29     for i, num_male in enumerate(num_males):
30         num_families[i] = np.sum(num_males_6115 == num_male)
31     assert np.sum(num_families) == 6115
32     return num_males, num_families
33
34
35 N_SIMULATIONS = 1000
36
37 def get_statistics_simulation(statistic_func):
38     test_statistics = []
39     for _ in range(N_SIMULATIONS):
40         num_males, num_families = get_data_for_one_simulation()
41         test_statistics.append(statistic_func(num_males,
    ↪ num_families))

```

```

42     return np.array(test_statistics)
43
44     """
45     Run 1000 simulations to approximate null distribution of T_1 and T_2
46     ↪ Statistics.
47     """
48
49     t1_statistics = get_statistics_simulation(t1_statistic)
50     t2_statistics = get_statistics_simulation(t2_statistic)
51     """
52     Plot histogram of these statistics.
53     """
54
55     plt.hist(t1_statistics, bins=100, edgecolor='black', alpha=0.7)
56
57     # Add labels and title
58     plt.xlabel('T_1 Statistic values')
59     plt.ylabel('Frequency')
60     plt.title('Histogram of 1000 values of T_1 Test Statistic')
61     plt.show()
62
63     plt.hist(t2_statistics, bins=100, edgecolor='black', alpha=0.7)
64     plt.xlabel('T_2 Statistic values')
65     plt.ylabel('Frequency')
66     plt.title('Histogram of 1000 values of T_2 Test Statistic')
67     # %%
68     """
69     For the T_1 Statistic, can we reject?
70     """
71
72     SIGNIFICANCE_LEVEL = 0.05
73     OBSERVED_T1_STATISTIC = t1_statistic(NUM_MALES, NUM_FAMILIES)
74     t1_critical_value_upper_end = np.percentile(t1_statistics, 0.95)
75     t1_critical_value_lower_end = np.percentile(t1_statistics, 0.05)
76     if (t1_critical_value_upper_end <= OBSERVED_T1_STATISTIC): print("Can
77     ↪ reject null hypothesis for T_1 test statistic.")
78     if (t1_critical_value_lower_end >= OBSERVED_T1_STATISTIC): print("Can
79     ↪ reject null hypothesis for T_1 test statistic.")
80
81     """
82     For the T_2 Statistic, can we reject?
83     """
84
85     SIGNIFICANCE_LEVEL = 0.05
86     OBSERVED_T2_STATISTIC = t2_statistic(NUM_MALES, NUM_FAMILIES)
87     t2_critical_value_upper_end = np.percentile(t2_statistics, 0.95)
88     t2_critical_value_lower_end = np.percentile(t2_statistics, 0.05)
89     if (t2_critical_value_upper_end <= OBSERVED_T2_STATISTIC): print("Can
90     ↪ reject null hypothesis for T_2 test statistic.")

```

```

84 if (t2_critical_value_lower_end >= OBSERVED_T2_STATISTIC): print("Can
    ↪ reject null hypothesis for T_2 test statistic.")
85
86 # %%

```

2.

- a) Note that because H_0 is true, then we can assume the true distribution of to be $Z \sim \mathcal{N}(0, 1)$. To find the distribution of $P = 1 - \Phi(z)$, we compute its CDF below:

$$P(P \leq p) = P(1 - \Phi(Z) \leq p)$$

Note that by the Universality of the Uniform, $\Phi(Z) \sim \text{Unif}(0, 1)$. Therefore, $1 - \Phi(Z) = 1 - \text{Unif}(0, 1) = \text{Unif}(-1, 0) + 1 = \text{Unif}(0, 1)$. Let us define $U \sim \text{Unif}(0, 1)$. Putting this all together, we have:

$$P(P \leq p) = P(1 - \Phi(Z) \leq p) = P(U \leq p) = F_U(p)$$

where F_U is the CDF of U . Because r.v. P has the same CDF as $U \sim \text{Unif}(0, 1)$, we can conclude $P \sim \text{Unif}(0, 1)$ if H_0 is true.

- b) If H_1 was true, then we would expect to observe greater values of test statistic $Z \implies \Phi(Z)$, which is strictly non-decreasing, would be greater $\implies P = 1 - \Phi(Z)$ would be smaller. Thus, if H_1 was true, we would expect to observe smaller p-values. As shown in part (a), $P \sim \text{Unif}(0, 1)$ so $P_{H_0}(P \leq \alpha) = F_U(\alpha) = \alpha$. Therefore, if we were to use P as the test statistic, we would reject H_0 when $P \leq \alpha$.

3.

- a) Let us define Z_1, \dots, Z_n to all be i.i.d $\mathcal{N}(0, 1)$ distributions. We can define distribution $U_n = \frac{Z_1^2 + \dots + Z_n^2}{n}$, or equivalently that $U_n \sim \frac{1}{n} \cdot \chi_n^2$. We first compute $\mathbb{E}[Z_i]^2 = \text{Var}[Z_i] + \mathbb{E}[Z_i] = 1 + 0 = 1$. By the Weak Law of Large Numbers, since $\mathbb{E}[Z_i] = 1$ and Z_i has finite variance, $U_n \rightarrow \mathbb{E}[Z_i]$ or $U_n \rightarrow 1$ in probability as $n \rightarrow \infty$.

Let us now define function $g(z) = \sqrt{z}$, where g is continuous for $(0, \infty)$. By the Continuous Mapping Theorem, distribution $g(U_n) = \sqrt{U_n} \rightarrow \sqrt{1}$, or $\sqrt{U_n} \rightarrow 1$ in probability as $n \rightarrow \infty$.

- b) The t -distribution with n degrees of freedom is given by:

$$T_n = \frac{\frac{\sqrt{n} \bar{X}}{\sigma}}{\sqrt{\frac{1}{n} \chi_n^2}}$$

where $\bar{X} = \frac{X_1 + \dots + X_n}{n}$ and $\sigma = \sqrt{\text{Var}(X_i)}$. As shown in part (a), $\sqrt{\frac{1}{n}\chi_n^2} \rightarrow 1$ in probability as $n \rightarrow \infty$. Furthermore, by CLT as $n \rightarrow \infty$, $\bar{X} \sim \mathcal{N}(0, \sigma^2)$ and thus $\frac{\sqrt{n}}{\sigma}\bar{X} \rightarrow \mathcal{N}(0, 1)$. Because $\sqrt{\frac{1}{n}\chi_n^2} \rightarrow 1 \neq 0$, we can apply Slutsky's Lemma. By Slutsky's Lemma, $T_n = \frac{\frac{\sqrt{n}}{\sigma}\bar{X}}{\sqrt{\frac{1}{n}\chi_n^2}} \rightarrow \frac{\mathcal{N}(0,1)}{1}$ or $T_n \rightarrow \mathcal{N}(0, 1)$ as $n \rightarrow \infty$.

4.

- (a) $T \sim t_1$ is given by $\frac{X}{\sqrt{\frac{1}{1}\chi_1^2}} = \frac{X}{\sqrt{Y^2}}$, where X, Y are i.i.d $\mathcal{N}(0, 1)$. By the Continuous Mapping Theorem, given $g(x) = \sqrt{x^2} = |x|$ then $g(Y)$ will converge to $|Y|$ and so T is given by $\frac{X}{|Y|}$.

We can define $\frac{X}{|Y|}$ as the following:

$$\frac{X}{|Y|} = \begin{cases} \frac{X}{Y} & \text{if } Y > 0 \\ -\frac{X}{Y} = \frac{X}{-Y} & \text{if } Y < 0 \end{cases}$$

Because Y and $-Y$ have the same distributions (because Y is symmetric about the origin), both the $\frac{X}{Y}$ and $\frac{X}{-Y}$ have the same distribution (which we can call U). Because then we have $\frac{X}{|Y|} = U \cdot P(Y > 0) + U \cdot P(Y < 0) = \frac{U+U}{2} = U$, we can conclude $\frac{X}{|Y|} \sim U$ or $\frac{X}{|Y|} \sim \frac{X}{Y}$ ¹

- b) Let us define $g(x, y) = (\frac{x}{y}, y)$. Because g is bijective, its inverse can be given by $g^{-1}(t, u) = (tu, u)$. We can use the change-of-variables formula to compute the joint PDF $f_{T,U}(t, u)$ ²:

$$f_{T,U}(t, u) = f_{X,Y}(g^{-1}(t, u)) \cdot \left| \det \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = f_{X,Y}(tu, u) \cdot \left| \det \begin{pmatrix} \frac{\partial(tu)}{\partial t} & \frac{\partial(tu)}{\partial u} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial u} \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = f_{X,Y}(tu, u) \cdot \left| \det \begin{pmatrix} u & t \\ 0 & 1 \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = |u|f_{X,Y}(tu, u) = |u|f_X(tu)f_Y(u) = |u|\frac{1}{2\pi}e^{-\frac{[(tu)^2+u^2]}{2}} = |u|\frac{1}{2\pi}e^{-\frac{u^2(t^2+1)}{2}}$$

We now compute f_T :

$$f_T(t) = \int_{-\infty}^{\infty} f_{T,U}(t, u) du = \int_{-\infty}^{\infty} \frac{|u|}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du = \int_{-\infty}^0 \frac{-u}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du + \int_0^{\infty} \frac{u}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du$$

¹Note that $P(Y < 0) = P(Y > 0) = 0.5$ because Y is a normal distribution and is therefore symmetric about the origin.

²Note that because X and Y are independent, $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.

Note that these two integrals are equivalent (we can use a change of variables $dv = -u$ in the first one) and so we can combine them:

$$f_T(t) = \frac{1}{\pi} \left[\int_0^\infty u e^{-\frac{u^2(t^2+1)}{2}} du \right]$$

Using a u-substitution for $v = \frac{u^2(t^2+1)}{2}$, we have:

$$f_T(t) = \frac{1}{\pi} \int_0^\infty \frac{1}{t^2+1} e^{-v} dv = -\frac{1}{\pi(t^2+1)} [e^{-v}] \Big|_0^\infty = -\frac{1}{\pi(t^2+1)} [0 - 1] = \frac{1}{\pi(t^2+1)}$$

Given the PDF of T we now can compute its expectation:

$$\mathbb{E}[T^2] = \int_{-\infty}^\infty \frac{t^2}{\pi(t^2+1)} dt = \frac{1}{\pi} \left[\int_{-\infty}^\infty 1 - \frac{1}{t^2+1} dt \right] = \frac{1}{\pi} [t - \arctan(t)] \Big|_{-\infty}^\infty$$

which diverges to ∞ . Thus we have $\mathbb{E}[T^2] = \infty$.