

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

**CONTENT STARTS ON NEXT PAGE.**

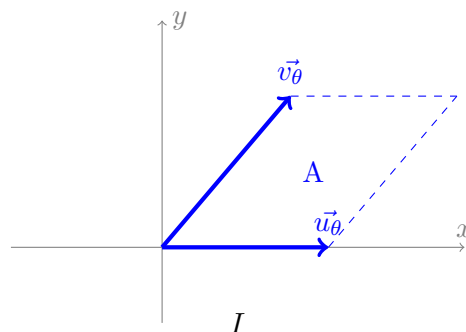
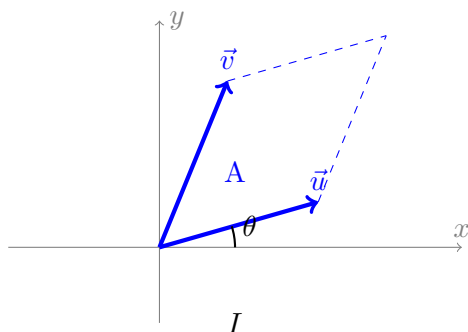
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## Math 226- HW 8 Due: Nov 5 by Midnight

1. (20) Let  $T : V \rightarrow W$  and  $U : W \rightarrow Z$  be linear transformations on finite-dimensional vector spaces  $V, W, Z$ . Moreover, let  $A, B$  be matrices such that  $AB$  is defined.
  - a) ( 6 points) Prove that  $\dim(R(UT)) \leq \dim(R(U))$
  - b) (4 points) Use pat a) to conclude that  $\text{rank}(AB) \leq \text{rank}(A)$
  - c) ( 6 points) Prove that  $\dim(R(UT)) \leq \dim(R(T))$
  - d) (4 points) Use pat c) to conclude that  $\text{rank}(AB) \leq \text{rank}(B)$
2. (12 points) Suppose that the augmented matrix of a system  $Ax = b$  is transformed into a matrix  $[A'|b']$  in reduced row echelon form by finite sequence of elementary row operations.
  - a) (10 points) Prove that  $\text{rank}(A) \neq \text{rank}[A'|b']$  if and only if  $[A'|b']$  contains a row in which the only nonzero entry lies in the last column.
  - b) (2 points) Deduce that  $Ax = b$  is consistent if and only if  $[A'|b']$  contains no row in which the only nonzero entry lies in the last column.
3. (12 points) Each of the following equations determines a plane in  $\mathbb{R}^3$ .

$$\begin{aligned} x + 4y + 5z &= 1 \\ 2x + 2y - 3z &= 4 \end{aligned}$$

- a) Find the intersection of these two planes, and draw a rough graph of the solution set.
  - b) Find the intersection when the equations are both homogenous, and draw rough graph of the solution set.
  - c) What is the relationship between these two set?
4. (10 points) In this problem you will prove that if  $u, v$  are two vectors in  $\mathbb{R}^2$ , then the area of the parallelogram generated by  $u, v$  is equal to  $|\det(u, v)|$ . Let  $u$  and  $v$  are vectors as in picture I . Let  $A_{-\theta} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$  be the clockwise rotation by  $\theta$  degree. One has  $A_{-\theta}(v) = v_{\theta}$ , and  $A_{-\theta}(u) = u_{\theta}$ , where  $v_{\theta}$  and  $u_{\theta}$  are as in the picture II



- a) (4 points) Let  $u_{\theta} = (u_1, u_2)$ , and  $v_{\theta} = (v_1, v_2)$  in picture II. Calculate the area,  $A$ , of the parallelogram generated by  $u_{\theta}$  and  $v_{\theta}$  using geometry, and show by direct calculation  $A = |\det[u_{\theta}, v_{\theta}]|$ . - Think here  $u_{\theta}, v_{\theta}$ , as the column vectors of the matrix  $[u_{\theta}, v_{\theta}]$ .
- b) (2 points) Show that  $A_{-\theta}(v) = v_{\theta}$ ,  $A_{-\theta}(u) = u_{\theta}$  mean that  $A_{-\theta}[u, v] = [u_{\theta}, v_{\theta}]$
- c) (4 points) Use part b), and Theorem 4.7 to show that  $|\det[u, v]| = A$ .

**Remark :** This is the geometric interpretation of the determinant. In general, if  $A \in M_{n \times n}(\mathbb{R})$ , and if the columns of  $A$  is  $(a_1, a_2, \dots, a_n)$ , then  $\det(A)$  is the  $n$ -dimensional volume of the parallelepiped having the vectors  $a_1, a_2, \dots, a_n$  as adjacent sides. You can also think  $\det(A)$  as the volume of the parallelepiped having the vectors  $A(e_1), A(e_2), \dots, A(e_n)$  as adjacent sides.

5. (15 points) Recall that we defined  $\delta : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$  to be  $n$  – linear functional if it is linear with respect to each row. Prove that  $\delta : M_{2 \times 2}(\mathbb{F}) \rightarrow \mathbb{F}$  is a 2 – linear functional if and only if it has the form

$$\delta(A) = Aa_{11}a_{22} + Ba_{11}a_{21} + Ca_{12}a_{22} + Da_{12}a_{21}$$

for any  $[a_{ij}]_{2 \times 2}$  matrix and  $A, B, C, D \in \mathbb{F}$ .