

Discretionary Note

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STATS 242 HW 5

February 18, 2025

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1.

- (a) Under H_0 , the median of f is zero \implies each sample X_i has a 50% chance of being above zero. Let us define r.v. $I_i \sim \text{Bern}(0.5)$ to represent if $X_i \geq 0$. Because $S = \sum_{i=1}^n I_i$, $S \sim \text{Bin}(n, 0.5)$. Furthermore, with a large n , the CLT enables us to approximate Binomial distributions with normal distributions¹ and thus $S \sim \mathcal{N}(n\mathbb{E}[I_i], n\text{Var}[I_i])$ or $S \sim \mathcal{N}(0.5n, 0.25n)$. This means that the distribution of $S - \frac{n}{2} \sim \mathcal{N}(0, 0.25n)$ and the distribution of $T = \sqrt{\frac{4}{n}}(S - \frac{n}{2}) \sim \mathcal{N}(0, 1)$. Thus $T \sim \mathcal{N}(0, 1)$.

To test H_0 vs. H_1 at the significance level α , we would compute T for a sample of data and if it is above the upper- α point of $\mathcal{N}(0, 1)$, we will reject H_0 .

- (b) Note that because under H'_1 , $X_i \sim \mathcal{N}(\frac{h}{\sqrt{n}}, 1)$ this means that $X_i - \frac{h}{\sqrt{n}} \sim \mathcal{N}(0, 1)$.

$$\mathbb{P}_{H'_1}[X_i > 0] = \mathbb{P}_{H'_1}[X_i - \frac{h}{\sqrt{n}} > -\frac{h}{\sqrt{n}}] = 1 - \Phi(-\frac{h}{\sqrt{n}}) = \Phi(\frac{h}{\sqrt{n}})$$

Assuming that h is a small fixed value and n is large, then $\frac{h}{\sqrt{n}}$ is close to zero. This means we can approximate $\mathbb{P}_{H'_1}[X_i > 0] = \Phi(\frac{h}{\sqrt{n}})$ at zero with the following first-degree Taylor Series approximation:

$$\begin{aligned}\mathbb{P}_{H'_1}[X_i > 0] &= \Phi(\frac{h}{\sqrt{n}}) \\ &\approx \Phi(0) + \Phi'(\frac{h}{\sqrt{n}}) \cdot \frac{h}{\sqrt{n}} = \frac{1}{2} + \phi(\frac{h}{\sqrt{n}}) \cdot \frac{h}{\sqrt{n}} = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{h^2}{2n}} \cdot \frac{h}{\sqrt{n}}\end{aligned}$$

where ϕ is the standard normal PDF. Note that because $h \ll n$, $\frac{h^2}{2n} \approx 0$ and so:

$$\begin{aligned}\mathbb{P}_{H'_1}[X_i > 0] &= \Phi(\frac{h}{\sqrt{n}}) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^0 \cdot \frac{h}{\sqrt{n}} \\ \mathbb{P}_{H'_1}[X_i > 0] &\approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \cdot \frac{h}{\sqrt{n}}\end{aligned}$$

¹This is because the Binomial Distribution is given by $n(\bar{I})$ where \bar{I} is the mean of n i.i.d Bernoulli Variables.

- (c) Under H'_1 , the previously defined r.v. $I_i \sim \text{Bern}(\mathbb{P}_{H'_1}[X_i > 0])$ or $I_i \sim \text{Bern}(\Phi(\frac{h}{\sqrt{n}}))$. As stated in part (a), r.v. S is given by a Binomial distribution but for large n can be approximated by $\mathcal{N}(n\mathbb{E}[I_i], n\text{Var}[I_i])$. As such, we first compute $n\mathbb{E}[I_i]$:

$$\begin{aligned} n\mathbb{E}[I_i] &= n\mathbb{P}_{H'_1}[X_i > 0] = n\Phi\left(\frac{h}{\sqrt{n}}\right) \approx n\left(\frac{1}{2} + \frac{h}{\sqrt{2\pi n}}\right) = \frac{n}{2} + \frac{h\sqrt{n}}{\sqrt{2\pi}} \\ n\mathbb{E}[I_i] &\approx \frac{n}{2} + \frac{h\sqrt{n}}{\sqrt{2\pi}} \end{aligned}$$

and then $n\text{Var}(I_i)$:

$$\begin{aligned} n\text{Var}(I_i) &= n(\mathbb{P}_{H'_1}[X_i > 0])(1 - \mathbb{P}_{H'_1}[X_i > 0]) = n(\Phi(\frac{h}{\sqrt{n}}))(1 - \Phi(\frac{h}{\sqrt{n}})) \\ &\approx n\left(\frac{1}{2} + \frac{h}{\sqrt{2\pi n}}\right)\left(1 - \left(\frac{1}{2} + \frac{h}{\sqrt{2\pi n}}\right)\right) = n\left(\frac{1}{2} + \frac{h}{\sqrt{2\pi n}}\right)\left(\frac{1}{2} - \frac{h}{\sqrt{2\pi n}}\right) = n\left(\frac{1}{4} - \frac{h^2}{2\pi n}\right) = \frac{n}{4} - \frac{h^2}{2\pi} \\ n\text{Var}(I_i) &\approx \frac{n}{4} - \frac{h^2}{2\pi} \end{aligned}$$

So, for large n we have that S can be approximated by $\mathcal{N}(\frac{n}{2} + \frac{h\sqrt{n}}{\sqrt{2\pi}}, \frac{n}{4} - \frac{h^2}{2\pi})$. This means that the distribution $S - \frac{n}{2}$ is given by approximately $\mathcal{N}(\frac{h\sqrt{n}}{\sqrt{2\pi}}, \frac{n}{4} - \frac{h^2}{2\pi})$ and so the distribution of $T = \sqrt{\frac{4}{n}}(S - \frac{n}{2})$ is approximately $\mathcal{N}(\sqrt{\frac{4}{n}} \cdot \frac{h\sqrt{n}}{\sqrt{2\pi}}, \frac{4}{n}(\frac{n}{4} - \frac{h^2}{2\pi}))$ or $\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1 - \frac{2h^2}{\pi n})$. Note that under our assumption $h \ll n$, we can drop the $\frac{2h^2}{\pi n}$ term in the variance. Thus we can give the following normal approximation for T that only relies on h and not n : $\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1)$.

- (d) As stated in part (a), we would reject H_0 if the computed test statistic T is above the upper- α point of $\mathcal{N}(0, 1)$ (given by $z^{(a)}$). The power of a test is given by $\mathbb{P}_{H'_1}[\text{reject } H_0] = \mathbb{P}_{H'_1}[T > z^{(a)}]$. As shown in part (c), under H'_1 , T can be approximated by $\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1)$. Thus, we can compute the power:

$$\begin{aligned} \mathbb{P}_{H'_1}[\text{reject } H_0] &= \mathbb{P}_{H'_1}[T > z^{(a)}] \approx \mathbb{P}[\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1) > z^{(a)}] \\ &= \mathbb{P}[\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1) - \sqrt{\frac{2}{\pi}}h > z^{(a)} - \sqrt{\frac{2}{\pi}}h] = \mathbb{P}[\mathcal{N}(0, 1) > z^{(a)} - \sqrt{\frac{2}{\pi}}h] \\ &= 1 - \mathbb{P}[\mathcal{N}(0, 1) \leq z^{(a)} - \sqrt{\frac{2}{\pi}}h] = 1 - \Phi(z^{(a)} - \sqrt{\frac{2}{\pi}}h) = \Phi(\sqrt{\frac{2}{\pi}}h - z^{(a)}) \end{aligned}$$

and so the power of this test against alternative H'_1 is approximately $\Phi(\sqrt{\frac{2}{\pi}}h - z^{(a)})$.

- (a) The simulated probability of a Type I Error for the Sign Test was 0.04. The simulated probability of a Type I Error for the t-test was 0.0522. We report the simulated power against each alternative for both tests in the table below:

	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$
Sign test	0.1754	0.4509	0.7529	0.9287
t-test	0.1629	0.5011	0.8408	0.9771

- (b) The power of the one-sample z-test is given by $\Phi(\sqrt{n}\mu - z^{(\alpha)})$. Comparing that to the simulated power of the one-sample t-test, we find that the z-test, across all alternatives, has a consistently greater power.

In problem 1(d), we found that the power of the sign test was approximately $\Phi(\sqrt{\frac{2}{\pi}} \cdot \sqrt{n}\mu - z^{(\alpha)})$. Comparing that to the simulated power of the sign test, we find that this approximation, across alternatives, yields a consistently greater power. Furthermore, our simulated power of the sign test is generally lower than our simulated power of the t-test and considerably lower than our power derivation for a one-sample z-test. We report the analytically computed powers for the z-test and sign test below:

	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$
Sign test: $\approx \Phi(\sqrt{\frac{2}{\pi}} \cdot \sqrt{n}\mu - z^{(\alpha)})$	0.1985	0.4804	0.7730	0.9390
z-test: $\Phi(\sqrt{n}\mu - z^{(\alpha)})$	0.2595	0.6388	0.9123	0.9907

```

1  # %%
2  import numpy as np
3  import math
4  from scipy import stats
5
6  NUM_SAMPLES = 10000
7  SIG_LEVEL = 0.05
8
9  upper_alpha = stats.norm.ppf(1 - SIG_LEVEL, loc=0, scale=1)
10
11 def get_N_normal_observations(mu, N=100):
12     return np.random.normal(mu, 1, N)
13
14 def reject_ttest(samples):
15     t_stat, p_value = stats.ttest_1samp(samples, popmean=0)
16     return 1 if p_value <= SIG_LEVEL else 0
17
18 def compute_sign_statistic(samples):
19     n = len(samples)
20     return math.sqrt(4 / n) * (np.sum(samples > 0) - 0.5 * n)
21
22 def reject_sign_test(samples):

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23     sign_statistic = compute_sign_statistic(samples)
24     return 1 if sign_statistic >= upper_alpha else 0
25 # %%
26 MEANS = [0, 0.1, 0.2, 0.3, 0.4]
27 REJ_TTEST_COUNTS = [0] * len(MEANS)
28 REJ_SGN_COUNTS = [0] * len(MEANS)
29
30 for m_i, mean in enumerate(MEANS):
31     for _ in range(NUM_SAMPLES):
32         samples = get_N_normal_observations(mean)
33         rej_ttest = reject_ttest(samples)
34         rej_sign_test = reject_sign_test(samples)
35         REJ_TTEST_COUNTS[m_i] += rej_ttest
36         REJ_SGN_COUNTS[m_i] += rej_sign_test
37
38 # %%
39 """Get the simulated Type I Error."""
40 print(f"Type I Error for Sign Statistic: {REJ_SGN_COUNTS[0] /
41     ↪ NUM_SAMPLES}")
42 print(f"Type I Error for T-Test Statistic: {REJ_TTEST_COUNTS[0] /
43     ↪ NUM_SAMPLES}")# %%
44 """Get the Power Against Each Alternative"""
45 for i, mean in enumerate(MEANS):
46     if mean == 0: continue
47     print(f"Power for sign test @ {mean} mean: {REJ_SGN_COUNTS[i] /
48         ↪ NUM_SAMPLES}")
49
50 for i, mean in enumerate(MEANS):
51     if mean == 0: continue
52     print(f"Power for t-test @ {mean} mean: {REJ_TTEST_COUNTS[i] /
53         ↪ NUM_SAMPLES}")
54
55 """Compare to z-test"""
56 for i, mean in enumerate(MEANS):
57     if mean == 0: continue
58     print(f"Estimated power for z-test @ {mean} mean:
59         ↪ {stats.norm.cdf(math.sqrt(100) * mean - upper_alpha)}")
60 # %%
61 """Compare to sign test"""
62 for i, mean in enumerate(MEANS):
63     if mean == 0: continue
64     print(f"Estimated power for sign-test @ {mean} mean:
65         ↪ {stats.norm.cdf(math.sqrt(2 / math.pi) * math.sqrt(100) * mean
66         ↪ - upper_alpha)}")

```

3.

- (a) We are given that the FWER is controlled at level $\alpha \implies \mathbb{P}[\text{reject any true } H_0] \leq \alpha$. Let us define V and R as the number of true null hypotheses rejected and the number of total null hypotheses rejected, respectively. The FDR is controlled at level α if $\mathbb{E}[\frac{V}{R}] \leq \alpha$. Using LOTE, we can write the FDR as the following:

$$\begin{aligned}\mathbb{E}[\frac{V}{R}] &= \mathbb{E}[\frac{V}{R} | \frac{V}{R} = 0]P(\frac{V}{R} = 0) + \mathbb{E}[\frac{V}{R} | \frac{V}{R} \neq 0]P(\frac{V}{R} \neq 0) \\ \mathbb{E}[\frac{V}{R}] &= (0)P(\frac{V}{R} = 0) + \mathbb{E}[\frac{V}{R} | \frac{V}{R} \neq 0]P(\frac{V}{R} \neq 0) = \mathbb{E}[\frac{V}{R} | \frac{V}{R} \neq 0]P(\frac{V}{R} \neq 0)\end{aligned}$$

We first compute $P(\frac{V}{R} \neq 0) = P(V \neq 0) = \mathbb{P}[\text{reject any true } H_0]$. Note that this last probability is guaranteed to be $\leq \alpha$ by the FWER and so $P(\frac{V}{R} \neq 0) \leq \alpha$. Given this, we can create the following inequality for the FDR:

$$\mathbb{E}[\frac{V}{R}] \leq \mathbb{E}[\frac{V}{R} | \frac{V}{R} \neq 0]\alpha$$

This question asks us to consider if the FDR is necessarily controlled at level α , given the FWER is. Note that V is strictly less than R , and so $\frac{V}{R} \leq 1 \implies \mathbb{E}[\frac{V}{R} | \frac{V}{R} \neq 0] \leq 1 \implies \text{FDR} = \mathbb{E}[\frac{V}{R} \neq 0]\alpha \leq \alpha \implies$ the FDR is controlled at level α .

- (b) The Bonferroni method applied to control $\text{FWER} \leq \alpha$ will reject any null hypotheses that has a p-value $\leq \frac{\alpha}{n}$. The BH procedure will reject hypotheses where their p-value is less than a multiple (i.e. their rank $r \in \mathbb{N}$) of $\frac{\alpha}{n}$: the BH procedure rejects hypotheses with p-values $\leq \frac{\alpha r}{n}$. Let us say that hypothesis H_k with p-value P_k . Let us also suppose that this hypothesis has a rank r_k when compared to all other hypotheses' p-values in this multiple hypotheses testing experiment. If H_k was rejected by the Bonferroni method $\implies P_k \leq \frac{\alpha}{n} \leq \frac{\alpha r_k}{n} \implies P_k \leq \frac{\alpha r_k}{n} \implies P_k$ will be rejected by the BH procedure. Thus, all hypotheses rejected by the Bonferroni method will be rejected by the BH procedure.

4.

- (a) We compute this below:

$$\begin{aligned}\mathbb{P}[\text{reject any true null hypothesis}] &= 1 - \mathbb{P}[\text{reject no true null hypotheses}] \\ &= 1 - \prod_{i=1}^{n_0} \mathbb{P}[\text{accept this null hypothesis}]\end{aligned}$$

The probability of rejecting any true null hypothesis is given by the probability $P_i \leq t$. Because this null hypothesis is true, $P_i \sim \text{Unif}(0, 1) \implies \mathbb{P}[P_i \leq t] = t$. The probability of accepting any true null hypothesis is the complement of this probability, $1 - t$. Thus we have:

$$\mathbb{P}[\text{reject any true null hypothesis}] = 1 - \prod_{i=1}^{n_0} (1 - t) = 1 - (1 - t)^{n_0}$$

- (b) The FWER is given by $\mathbb{P}[\text{reject any true hypothesis}]$. As computed in (a), for a given cutoff t , this probability is given by $1 - (1 - t)^{n_0}$. Setting $t = 1 - (1 - \alpha)^{\frac{1}{n}}$, we have that:

$$\mathbb{P}[\text{reject any true null hypothesis}] = 1 - (1 - t)^{n_0} = 1 - (1 - (1 - (1 - \alpha)^{\frac{1}{n}}))^{n_0} = 1 - (1 - \alpha)^{\frac{n_0}{n}}$$

Because $n_0 \leq n$ and $1 - \alpha \leq 1$, $(1 - \alpha)^{\frac{n_0}{n}} \geq (1 - \alpha) \implies -(1 - \alpha)^{\frac{n_0}{n}} \leq \alpha - 1$ and so:

$$\begin{aligned} \mathbb{P}[\text{reject any true null hypothesis}] &= 1 - (1 - \alpha)^{\frac{n_0}{n}} \leq 1 + (\alpha - 1) \\ \mathbb{P}[\text{reject any true null hypothesis}] &\leq \alpha \end{aligned}$$

and so we have shown for $t = 1 - (1 - \alpha)^{\frac{1}{n}}$, the FWER $\leq \alpha$, meaning that the FWER is controlled at level α .

We now compare if this choice of $t = 1 - (1 - \alpha)^{\frac{1}{n}}$ or $t = \frac{\alpha}{n}$ will reject more hypotheses. Because whichever choice of t is greater will reject *more* hypotheses², we aim to find which choice of t is greater. We first assume that n , the number of hypotheses tests, is ≥ 1 and that $0 \leq \alpha \leq 1$. Note by Bernoulli's inequality that for $0 \leq r \leq 1$ and $x \geq -1$, $(1 + x)^r \leq 1 + rx$. Because $0 \leq \frac{1}{n} \leq 1$ and $-\alpha \geq -1$, we can apply Bernoulli's inequality and so $(1 - \alpha)^{\frac{1}{n}} \leq 1 - \frac{\alpha}{n} \implies -(1 - \alpha)^{\frac{1}{n}} \geq \frac{\alpha}{n} - 1$. So the choice of $t = 1 - (1 - \alpha)^{\frac{1}{n}} \geq 1 + \frac{\alpha}{n} - 1 \implies t = 1 - (1 - \alpha)^{\frac{1}{n}} \geq \frac{\alpha}{n}$. Thus choosing $t = 1 - (1 - \alpha)^{\frac{1}{n}}$ will reject more hypotheses as it is a greater threshold.

w The procedure of choosing $t = 1 - (1 - \alpha)^{\frac{1}{n}}$ differs from the Bonferroni correction because it assumes independence between all n hypothesis tests, which the Bonferroni correction does not assume. This makes sense as this stronger assumption allows this choice of t to reject more hypotheses when compared to the Bonferroni correction.

²this means that more computed p-values will meet this threshold \implies more hypotheses will be rejected