

Question 1

$$(a) \quad T_1 = \frac{(40 + 30 + T_3 + T_2)}{4} \rightarrow 4T_1 - T_3 - T_2 = 70$$

$$T_2 = (10 + 30 + T_4 + T_1) \rightarrow 4T_2 - T_4 - T_1 = 140$$

$$T_3 = \left(\frac{20 + 40 + T_4 + T_1}{4} \right) \rightarrow 4T_3 - T_4 - T_1 = 60$$

$$T_4 = \frac{(20 + 10 + T_2 + T_3)}{4} \rightarrow 4T_4 - T_3 - T_2 = 30$$

(b) $A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$

Applying G.E.

$$\begin{aligned} R_1 &= \frac{R_1}{4} \rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & | & \frac{1}{4} & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & | & 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & -1 & | & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & | & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} R_2 &= R_2 + R_1 \\ R_3 &= R_3 + R_1 \end{aligned} \rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & | & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{15}{4} & -\frac{1}{4} & -1 & | & \frac{1}{4} & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{15}{4} & -1 & | & \frac{1}{4} & 0 & 1 & 0 \\ 0 & -1 & -1 & 4 & | & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & \frac{1}{15} & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} & \frac{1}{15} & -\frac{4}{15} & 0 & 0 \\ 0 & 0 & \frac{56}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & 1 & 0 \\ 0 & 0 & -\frac{16}{15} & \frac{56}{15} & \frac{1}{15} & \frac{4}{15} & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + \frac{R_2}{4} \rightarrow R_1 \\ R_3 + \frac{R_2}{4} \rightarrow R_3 \\ R_4 + R_2 \rightarrow R_4 \end{array}$$

$$\begin{aligned} & \left(\frac{15}{15} R_3 \Rightarrow R_3 \right. \\ & \left. \begin{array}{c} R_2 + \frac{R_3}{15} \rightarrow R_2 \\ R_1 + 4 \frac{R_3}{15} \rightarrow R_1 \\ R_4 + \frac{16}{5} R_3 \rightarrow R_4 \end{array} \right. \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{4}{15} & \frac{1}{15} & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & \frac{1}{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{15} & \frac{1}{15} & -\frac{5}{6} & 0 & 1 \\ 0 & 0 & -\frac{16}{15} & \frac{5}{15} & 0 & 1 & 0 & 0 \end{array} \right] \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{24} & \frac{1}{12} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{12} & \frac{7}{24} & \frac{1}{24} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{24} & \frac{7}{24} & \frac{1}{12} \\ 1 & \frac{1}{24} & \frac{1}{12} & \frac{7}{24} \end{bmatrix} \begin{matrix} \leftarrow R_1 + R_4 \rightarrow R_1 \\ \frac{7}{24} \\ R_2 + 2R_4 \end{matrix}$$

$$V(A^{-1})$$

$$R_3 + 2R_2 \rightarrow R_3$$

Question 1 continued

$$(c) \quad AT = X$$

$$T = A^{-1}X$$

$$T = \begin{bmatrix} \frac{7}{24} & \frac{1}{12} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{12} & \frac{7}{24} & \frac{1}{24} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{24} & \frac{7}{24} & \frac{1}{12} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{12} & \frac{7}{24} \end{bmatrix} \begin{bmatrix} 30 \\ 40 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 30 \\ 22.5 \\ 27.5 \\ 20 \end{bmatrix}$$

Question 2

To find the basis of $N(L_A)$, we solve $AX=0$

$$\left(\begin{array}{cccc|c} 3 & 5 & -4 & 2 & 0 \\ 2 & 4 & -6 & 3 & 0 \\ 11 & 17 & -8 & 9 & 0 \end{array}\right)$$

$$\begin{array}{l} 3R_2 - 2R_1 \rightarrow R_2 \\ 3R_3 - 11R_1 \rightarrow R_3 \end{array}$$

$$\left(\begin{array}{cccc|c} 3 & 5 & -4 & 2 & 0 \\ 0 & 2 & -10 & 5 & 0 \\ 0 & -4 & 20 & -10 & 0 \end{array}\right)$$

$$\begin{array}{l} 2R_2 + R_3 \rightarrow R_3 \\ 2R_1 - 5R_2 \rightarrow R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 6 & 0 & 42 & -21 & 0 \\ 0 & 2 & -10 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + 7x_3 - 3.5x_4 = 0$$

$$2x_2 - 10x_3 + 5x_4 = 0$$

$$x_1 = 3.5x_4 - 7x_3$$

$$x_2 = 5x_3 - 2.5x_4$$

$$\begin{array}{c} x_1 \ x_2 \ x_3 \ x_4 \\ \left(\begin{array}{cccc|c} 1 & 0 & 7 & -3.5 & 0 \\ 0 & 2 & -10 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$$\frac{R_1}{6} \rightarrow R_1$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3.5t_4 - 7t_3 \\ 5t_3 - 2.5t_4 \\ t_3 \\ t_4 \end{pmatrix} = t_3 \begin{pmatrix} -7 \\ 5 \\ 1 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} 3.5 \\ -2.5 \\ 0 \\ 1 \end{pmatrix}$$

Thus, the basis for $N(L_A)$ can be given by $\left\{ \begin{pmatrix} -7 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3.5 \\ -2.5 \\ 0 \\ 1 \end{pmatrix} \right\}$.

The set of solutions to the given system is given by the span of this basis.

Question 3

$$AX=b$$

$$A^T X = b$$

$$[A|b] = \left(\begin{array}{cccc|c} 1 & -3 & -3 & -14 & 8 \\ 2 & -6 & -3 & -1 & -5 \\ 3 & -9 & -5 & -6 & -4 \end{array} \right) \xrightarrow{\substack{R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3}} \left(\begin{array}{cccc|c} 1 & -3 & -3 & -14 & 8 \\ 0 & 0 & 3 & 27 & -21 \\ 0 & 0 & 4 & 36 & -28 \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 1 & -3 & 0 & 13 & -13 \\ 0 & 0 & 1 & 9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \xleftarrow{\substack{R_2 \rightarrow R_2 \\ \frac{R_2}{3} \rightarrow R_2}} \left(\begin{array}{cccc|c} 1 & -3 & 0 & 13 & -13 \\ 0 & 0 & 3 & 27 & -21 \\ 0 & 0 & 4 & 36 & -28 \end{array} \right) \xleftarrow{R_1 + R_2 \rightarrow R_1}$$

$$3R_3 - 4R_2 \rightarrow R_3$$

$$\begin{cases} x_1 - 3x_2 + 13x_4 = -13 \Rightarrow x_1 = -13 + 3x_2 - 13x_4 = -13 + 3t_2 - 13t_4 \\ x_3 + 9x_4 = -7 \Rightarrow x_3 = -7 - 9t_4 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -13 + 3t_2 - 13t_4 \\ t_2 \\ -7 - 9t_4 \\ t_4 \end{pmatrix} = \begin{pmatrix} -13 \\ 0 \\ -7 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} -13 \\ 0 \\ -9 \\ 1 \end{pmatrix}$$

Vectors in \mathbb{R}^4 whose image is equal to $b \in \mathbb{R}^3$ under linear map given in A.

The basis of $N(A)$ can be given by $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -13 \\ 0 \\ 9 \\ 1 \end{pmatrix} \right\}$.

Question 4

(i)

$$[T]^\beta = \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$$

$$T(1) = (1, 1, 1, 1)$$

$$T(x) = (-2, -1, 1, 2)$$

$$T(x^2) = (4, 1, 1, 4)$$

$$T(x^3) = (-8, -1, 1, 8)$$

(ii)

$$Tx = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 33 \end{bmatrix}$$

$$\left[T \mid \begin{bmatrix} 1 \\ 3 \\ 3 \\ 33 \end{bmatrix} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & 1 \\ 1 & -1 & 1 & -1 & 3 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & 33 \end{array} \right] \xrightarrow{\substack{R_2 - R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \\ R_4 - R_1 \rightarrow R_4}} \left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & 1 \\ 0 & 1 & -3 & 7 & 2 \\ 0 & 3 & -3 & 9 & 12 \\ 0 & 4 & 0 & 16 & 32 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & 6 & 21 \\ 0 & 2 & 0 & 2 & 10 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 12 & 12 \end{array} \right]$$

$$\begin{aligned} R_4 - 2R_3 &\rightarrow R_4 \\ 2R_2 + R_3 &\rightarrow R_2 \\ 3R_1 + R_3 &\rightarrow R_1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 6 & 5 \\ 0 & 1 & -3 & 7 & 2 \\ 0 & 0 & 6 & -12 & 18 \\ 0 & 0 & 12 & -12 & 24 \end{array} \right]$$

$$\begin{aligned} R_3 - 3R_2 &\rightarrow R_3 \\ R_4 - 4R_2 &\rightarrow R_4 \\ 2R_2 + R_1 &\rightarrow R_1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 6 & 0 & 0 & 0 & 30 \\ 0 & 12 & 0 & 0 & 48 \\ 0 & 0 & 6 & 6 & 18 \\ 0 & 0 & 0 & 12 & 12 \end{array} \right] \xrightarrow{\substack{R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 \\ R_3 \rightarrow R_3 \\ R_4 \rightarrow R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

Thus, we get this polynomial as:

$$f(x) = 5 + 4x + 3x^2 + x^3$$

Question 5

$$V = \{ (x_1, x_2, x_3, x_4, x_5) : x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0 \}$$

is finally a subspace of \mathbb{R}^5 .

We first find the basis of V :

$$x_1 = 2x_2 - 3x_3 + x_4 - 2x_5$$

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (2t_2 - 3t_3 + t_4 - 2t_5, t_2, t_3, t_4, t_5) \\ &= t_2(2, 1, 0, 0, 0) + t_3(-3, 0, 1, 0, 0) + t_4(1, 0, 0, 1, 0) + t_5(-2, 0, 0, 0, 1) \end{aligned}$$

Thus, the basis for V , B_V is given by

$$B_V = \{ (2, 1, 0, 0, 0), (-3, 0, 1, 0, 0), (1, 0, 0, 1, 0), (-2, 0, 0, 0, 1) \}$$

Given set $S = \{v\} = \{ (1, 2, 1, 0, 0) \}$:

$$\begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{R_3 - R_1 \rightarrow R_3 \\ 2R_1 - R_2 \rightarrow R_2}} \begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 0 & 3 & -6 & 2 & -4 \\ 0 & -2 & 4 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} &\xrightarrow{\substack{R_3 + 2R_5 \rightarrow R_3 \\ 2R_5 + R_4 \rightarrow R_5}} \begin{pmatrix} 3 & 0 & 3 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{R_3 - R_4 \rightarrow R_3 \\ R_3 + R_1 \rightarrow R_1 \\ R_2 - 2R_3 \rightarrow R_2}} \begin{pmatrix} 3 & 0 & 3 & -1 & 2 \\ 0 & 3 & -6 & 2 & -4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{3R_1 - 2R_2 \rightarrow R_1 \\ 3R_3 + 2R_2 \rightarrow R_3}} \\ &\xrightarrow{\substack{R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 \\ R_4 \rightarrow R_4}} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Question 5 continued

Thus,

$$\{(1, 2, 1, 0, 0), (2, 1, 0, 0, 0), (1, 0, 0, 1, 0), (-2, 0, 0, 0, 1)\}$$

is a basis for V that includes v .