Math 225- HW 9 Due: December 8 by Midnight

1. (13 points) Let $T : V \to V$, be a linear operator and V finite dimensional vector space. Recall that $\det(T) = \det[T]_\beta$ for some β ordered basis for V. Prove that

   a) (4 points) The definition of $\det(T)$ is well-defined, i.e., if γ is another ordered basis for V then $\det[T]_\beta = \det[T]_\gamma$.

   b) (2 points) Show that $\det([T]_\gamma - \lambda I) = \det([T]_\beta - \lambda I)$.

   c) (2 points) Use part *b*) to deduce that similar matrices have the same characteristic polynomial. (Definition of similar matrices is given in the remark)

   d) (5 points) If $g(t)$ be polynomial with coefficient from $\mathbb{R}$, then if x is an eigenvector for T with corresponding eigenvalue λ, then x is an eigenvector for $g(T)$ with corresponding eigenvalue $g(\lambda)$. - Use definition of eigenvalue.

2. (10 points) Let A be an upper(lower) triangular matrix, and has the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ with corresponding multiplicities $m_1, m_2, ..., m_k$.

   a) Prove that $\mathrm{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i$

   b) Prove that $\det(A) = \prod_{i=1}^{k} (\lambda_i)^{m_i}$.

   Remark: We say $A, B \in M_{n\times n)}(\mathbb{R})$ are *similar matrices* if there exist $P \in M_{n\times n)}(\mathbb{R})$ invertible such that $PAP^{-1} = B$. Recall that if A is similar to B then $\mathrm{tr}(A) = \mathrm{tr}(B)$, and $\det(A) = \det(B)$. Therefore, the statement of this problem is true for any matrix that is similar to upper (lower) triangle matrix.

3. (31 points) Consider the following matrices $A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

   a) (9 points) Decide if they are diagonalizable in $\mathbb{R}$.

   b) (2 points) Decide if they are diagonalizable in $\mathbb{C}$.

   c) (5 points) Find Q and D matrix such that $A = Q^{-1}DQ$.

   d) (5 points) Use part *c*) to find A^k for $k = 0, 1, 2, ...$
   Hint: $(Q^{-1}DQ)^k = \underbrace{(Q^{-1}DQ)(Q^{-1}DQ), ...(Q^{-1}DQ)}_{\text{k times}}$.

   e) (10 points) Find e^A. Hint: Use the Taylor expansion of $e^x = \sum_{n=0}^{\infty}(x^n/n!)$ and part *d*).

   I think it is so cool to be able to define exponential of a matrix. You can do it for any function that has Taylor expansion in its radius of convergence. We will define the norm of a matrix in the coming weeks.

4. (23 points)

   a) (5 points) Let A be a matrix whose characteristic polynomial split over its field $\mathbb{F}$. Prove that the determinant of A is the product of its eigenvalues, each counted with its multiplicity. (that is if the algebraic multiplicity of an eigenvalue if m then it is multiplied m times.)

   b) (3 points) Use part a to conclude that if A is defined over $\mathbb{C}$, the complex numbers, then the determinant of A is always the product of its eigenvalues, each counted with its multiplicity.

c) (15 points) Suppose A is a real $n \times n$ matrix which satisfies $A^3 = A + I_n$. Show that A has a positive determinant.

Hint: Even though A is real valued you can consider its eigenvalues in $\mathbb{C}$. So, try to find an equation that the eigenvalues satisfy. Here the fact that A is real must give you hint about complex eigenvalues.