MATH 241 PSET 7

October 31, 2024

1.

Because F is a continuous function and is strictly increasing (i.e. $\frac{dF}{dx} > 0$) for $x > 0$, this means that $F^{-1}(x)$ exists. Thus, by Universality of the Uniform, $X = F^{-1}(U)$ gives us an r.v. with a CDF of F. We compute $F^{-1}(u)$ below:

$$F(x) = 1 - e^{-x^3}$$
$$u = 1 - e^{-x^3}$$
$$e^{-x^3} = 1 - u$$
$$x^3 = -ln(1-u)$$
$$x = [-ln(1-u)]^{\frac{1}{3}}$$

Thus, $F^{-1}(u) = [-ln(1-u)]^{\frac{1}{3}}$ and so the r.v. $X = F^{-1}(U) = [-ln(1-U)]^{\frac{1}{3}}$ has the CDF F.

2.

The general form of a transformation of the standard normal distribution is $Y = \mu + \sigma Z$, where $Z \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(\mu, \sigma^2)$. Thus, for a distribution Y with $\sigma = \sqrt{4} = 2$ and $\mu = 1$, $Y = 1 + 2Z$.

3.

Let us define ϵ as an r.v. where $\epsilon \sim \mathcal{N}(0, 0.04)$. The probability that the observed distance is within 0.4 meters of the true distance is given by $P(-0.4 \leq \epsilon \leq 0.4) = \Phi(\frac{0.4-0}{\sqrt{0.04}}) - \Phi(\frac{-0.4-0}{\sqrt{0.04}}) = \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2)) = 2\Phi(2) - 1$.

In numerical terms, we can calculate $P(-0.4 \leq \epsilon \leq 0.4)$ by first computing $\Phi(2)$:

$$\Phi(2) = \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi(0.04)}} e^{-\frac{x^2}{2(0.04)}}\, dx \approx 0.977$$

and so, we get that $P(-0.4 \leq \epsilon \leq 0.4) = 2\Phi(2) - 1 = 2(0.977) - 1 = 0.954$.

4.

The distribution $X - Y$ is given by the addition of distributions $X \sim \mathcal{N}(a, b)$ and $-Y \sim \mathcal{N}(-c, d)$. Thus, $X - Y = X + (-Y)$ is distributed by $\mathcal{N}(a - c, b + d)$. The standardized version of $X - Y$ is given by $\frac{X-Y-\mu}{\sigma}$, where $\mu = 0$ and $\sigma = \sqrt{b + d}$. Re-expressed, the standardized version of $X - Y$ is given by the following: $\frac{X-Y}{\sqrt{b+d}}$. Given F as the CDF of $X - Y$, $P(X - Y < 0) = F(0) = \Phi(\frac{0}{\sqrt{b+d}}) = \Phi(0) = 0.5$ as normal distributions (i.e. $X - Y$) are symmetric.

5.

Let us define the r.v. $N \sim Pois(20 * 0.1)$ as the number of emails arrived in the first 0.1 hours. Note that the rate parameter of N is $20 * 0.1 = 2$ emails as we are expecting 20 emails/hour over a duration of 0.1 hours. Using count-time duality:

$$
\begin{aligned}
P(T > 0.1) = P(N < 3) &= P(N = 0) + P(N = 1) + P(N = 3) \\
&= \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!} = e^{-2}[1 + 2 + 2] = 5e^{-2}
\end{aligned}
$$

6. Anish Lakkapragada. I worked independently.