MATH 255 HW 1

January 23, 2025

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1. **Exercise 1.1 (5 points)**

   (1) We prove this by contrapositive and thus assume f is not injective $\Rightarrow \exists\ x, x' \in A$ s.t. $x \neq x'$ and $f(x) = f(x')$. Let us define $u = f(x) = f(x')$. Then, h is not injective as $h(x) = g(f(x)) = g(u)$ and $h(x') = g(f(x')) = g(u)$ and so $\exists\ x, x' \in A$ s.t. $x \neq x'$ and $h(x) = h(x')$.

   (2) We prove this by contrapositive and thus assume that g is not surjective $\Rightarrow \exists\ c \in C$ s.t. $\nexists\ b \in B$ where $g(b) = c$. Because no element in B maps to C by g, this means $\forall x \in A, h(x) = g(f(x)) \neq c$ and so because $\nexists\ x \in A$ s.t. $h(x) = c$, h is not surjective.

2. **Exercise 1.2 (5 points; Rudin 1.1)**

   (1) Prove $r + x \notin \mathbb{Q}$

   Let us prove this by contradiction and assume that $r + x \in \mathbb{Q}$. By negation rule, $r \in \mathbb{Q} \Rightarrow -r \in \mathbb{Q}$. By addition rule, $(-r) + (r + x) \in \mathbb{Q}$ and so this means $(-r) + (r + x) = (-r + r) + x = 0 + x = x \in \mathbb{Q}$, which is a contradiction.

   (2) Prove $rx \notin \mathbb{Q}$

   Let us prove this by contradiction and assume that $rx \in \mathbb{Q}$. Because $r \neq 0$, by the inversion rule we have that r has an inverse r^{-1}. By multiplication rule, $(r^{-1}) \cdot rx \in \mathbb{Q}$ or $(r^{-1}) \cdot rx = (r^{-1} \cdot r)x = 1 \cdot x = x \in \mathbb{Q}$, which is a contradiction.

3. **Exercise 1.3 (10 points; Rudin 1.3)**

   (1) Because $x \neq 0$, by the inversion rule we know $\exists\ x^{-1}$ s.t. $x^{-1} \cdot x = 1$. Thus:

$$
\begin{aligned}
xy &= xz \\
x^{-1}xy &= x^{-1}xz \\
1 \cdot y &= 1 \cdot z \\
y &= z
\end{aligned}
$$

(2) Because $x \neq 0$, we can apply the inversion rule again:

$$
\begin{aligned}
xy &= x \\
x^{-1}xy &= x^{-1}x \\
1 \cdot y &= 1 \\
y &= 1
\end{aligned}
$$

(3) We first prove that $0 \cdot y = 0$:

$$
\begin{aligned}
0 \cdot y + 0 \cdot y = (0+0) \cdot y = 0 \cdot y \\
0 \cdot y + 0 \cdot y = 0 \cdot y
\end{aligned}
$$

Because $0 \cdot y \in F$, its additive inverse is given by $-0 \cdot y$:

$$
\begin{aligned}
0 \cdot y + 0 \cdot y - 0 \cdot y &= 0 \cdot y - 0 \cdot y \\
0 \cdot y + (0 \cdot y - 0 \cdot y) &= 0 \\
0 \cdot y + 0 &= 0 \\
0 \cdot y &= 0
\end{aligned}
$$

To prove that if $xy = 1 \implies x \neq 0$ we proceed by contradiction. If $x = 0$, then $xy = 0 \cdot y = 0 \neq 1$. We now show $y = x^{-1}$. Thus, $x \neq 0$. Because $x \neq 0$, we can apply the inversion rule again:

$$
\begin{aligned}
xy &= 1 \\
x^{-1}xy &= x^{-1} \\
1 \cdot y &= x^{-1} \\
y &= x^{-1}
\end{aligned}
$$

(4) Let us define inverse of x^{-1} to be u, where by definition $x^{-1} \cdot u = 1$. Because $x^{-1} \cdot x = 1$ by definition, then we have that $u = x$, or that the inverse of x^{-1} is x. Expressed as an equation, we have shown: $(x^{-1})^{-1} = x$.

4. **Exercise 1.4 (10 points)**

(1) Let us suppose $x = \frac{p}{q} \in \mathbb{Q}$, where $p, q \in \mathbb{Z}$ and $\frac{p}{q}$ are in lowest terms. For proof by contradiction, we assume $x^2 = 3$. This means $\frac{p^2}{q^2} = 3$ or $p^2 = 3q^2$ and thus p^2 has a factor of three.

We now prove that p has a factor of three. Because p^2 has a factor of three, the prime factorization of p^2 can be given by $3^\alpha \ldots$ where $\alpha \geq 1 \in \mathbb{Z}$. Let us define the prime factorization of $p = 3^\beta \ldots$ where $\beta \geq 0 \in \mathbb{Z}$. Note that since $p^2 = (3^\beta \ldots)^2$,

$\alpha = 2\beta$. The lowest possible integer value of α s.t. $\alpha \geq 1$ and $\beta \in \mathbb{Z}$ is then $\alpha = 2$ and $\beta = 1$, so we are guaranteed that p has a factor of three. This means we can express $p = 3k$, where $k \in \mathbb{Z}$ and so $p^2 = (3k)^2 = 9k^2 = 3q^2$ or $q^2 = 3k^2$. Using the same logic as before because q^2 has a factor of three, so does q. Thus, p and q both have a factor of three and so this contradicts the assumption that $\frac{p}{q}$ are in lowest terms.

(2) We show that these provided operations define $\mathbb{Q}(\sqrt{3})$ as a field:

1. **Zero & One Element**
   Our zero and one element in $\mathbb{Q}(\sqrt{3})$ are given by $0+0\sqrt{3}$ and $1+0\sqrt{3}$ respectively. Written as ordered pairs, they are given by $(0,0)$ and $(1,0)$ respectively.
2. **Negation Law**
   For an element $u = a + b\sqrt{3} \in \mathbb{Q}(\sqrt{3})$, $-u = (-a) + (-b)\sqrt{3} \in \mathbb{Q}(\sqrt{3})$.
3. **Inversion Law**
   For an element $u = a + b\sqrt{3} \in \mathbb{Q}$, u^{-1} is given by:

$$uu^{-1} = 1_{\mathbb{Q}\sqrt{3}} = 1 + 0\sqrt{3}$$
$$u^{-1} = \frac{1+0\sqrt{3}}{u} = \frac{1+0\sqrt{3}}{a+b\sqrt{3}} = \frac{(1+0\sqrt{3})(a-b\sqrt{3})}{(a+b\sqrt{3})(a-b\sqrt{3})} = \frac{(1+0\sqrt{3})(a-b\sqrt{3})}{(a^2-3b^2)+(-ab+ba)\sqrt{3}}$$
$$= \frac{a-b\sqrt{3}}{a^2-3b^2} = \frac{a}{a^2-3b} + (\frac{-b}{a^2-3b^2})\sqrt{3}$$

4. $\forall x, y \in \mathbb{Q}\sqrt{3}, x + y = y + x$
   Define $x = a + b\sqrt{3}, y = a' + b'\sqrt{3} \in \mathbb{Q}\sqrt{3}$. $x + y$ is given by:

$$x + y = (a + b\sqrt{3}) + (a' + b'\sqrt{3}) = (a + a') + (b + b')\sqrt{3}$$

   and $y + x$ is given by:

$$y + x = (a' + b'\sqrt{3}) + (a + b\sqrt{3}) = (a' + a) + (b' + b)\sqrt{3} = (a' + a) + (b + b')\sqrt{3}$$

   and so $x + y = y + x$.
5. $\forall x, y, z \in \mathbb{Q}\sqrt{3}, (x + y) + z = x + (y + z)$
   We define x, y the same as above. We define $z = a'' + b''\sqrt{3} \in \mathbb{Q}\sqrt{3}$. Then $(x + y) + z$ is given by:

$$(x+y)+z = [(a+a')+(b+b')\sqrt{3}] + z = ((a+a')+(b+b')\sqrt{3}) + (a''+b''\sqrt{3})$$
$$= ((a+a')+a'') + ((b+b')+b'')\sqrt{3} = (a+a'+a'') + (b+b'+b'')\sqrt{3}$$

   and $x + (y + z)$ is given by:

$$x+(y+z)=x+((a'+b'\sqrt{3})+(a''+b''\sqrt{3}))=x+((a'+a'')+(b'+b'')\sqrt{3})$$
$$=(a+b\sqrt{3})+((a'+a'')+(b'+b'')\sqrt{3})=(a+a'+a'')+(b+b'+b'')\sqrt{3}$$

and so $(x+y)+z=x+(y+z)$.

6. $\forall x\in\mathbb{Q}\sqrt{3}, 0+x=x$
   Using the previous definition of x:

$$0_{\mathbb{Q}\sqrt{3}}+x=(0+0\sqrt{3})+(a+b\sqrt{3})=(0+a)+(0+b)\sqrt{3}=a+b\sqrt{3}=x$$

7. $\forall x\in\mathbb{Q}\sqrt{3}, -x+x=0$.
   Using the previous definition of x:

$$-x+x=((-a)+(-b)\sqrt{3})+(a+b\sqrt{3})=((-a+a)+(-b+b)\sqrt{3})=0+0\sqrt{3}=0_{\mathbb{Q}\sqrt{3}}$$

8. $\forall x,y\in\mathbb{Q}\sqrt{3}, xy=yx$
   We define x,y the same as before. This gives us xy as:

$$xy=(a+b\sqrt{3})(a'+b'\sqrt{3})=(aa'+3bb')+(ab'+ba')\sqrt{3}$$

and yx as:

$$yx=(a'+b'\sqrt{3})(a+b\sqrt{3})=(a'a+3b'b)+(a'b+b'a)\sqrt{3}=(aa'+3bb')+(ba'+ab')\sqrt{3}$$
$$=(aa'+3bb')+(ab'+ba')\sqrt{3}$$

and so $xy=yx$.

9. $\forall x,y,z\in\mathbb{Q}\sqrt{3}, (xy)z=x(yz)$
   We use the previous definitions of x,y,z as before. This gives us $(xy)z$ as:

$$(xy)z=((aa'+3bb')+(ab'+ba')\sqrt{3})(z)=((aa'+3bb')+(ab'+ba')\sqrt{3})(a''+b''\sqrt{3})$$
$$=(a''(aa'+3bb')+3(ab'+ba')b''))+((aa'+3bb')b''+(ab'+ba')a'')\sqrt{3}$$

and $x(yz)$ as:

$$x(yz)=x((a'+b'\sqrt{3})(a''+b''\sqrt{3}))=x((a'a''+3b'b'')+(a'b''+b'a'')\sqrt{3})$$
$$=(a+b\sqrt{3})((a'a''+3b'b'')+(a'b''+b'a'')\sqrt{3})$$
$$=(a(a'a''+3b'b'')+3b(a'b''+b'a''))+(a(a'b''+b'a'')+b(a'a''+3b'b''))\sqrt{3}$$
$$=(a''(aa'+3bb')+3(ab'+ba')b''))+((aa'+3bb')b''+(ab'+ba')a'')\sqrt{3}$$

and so $(xy)z=x(yz)$.

10. $\forall x \in \mathbb{Q}, 1 \cdot x = x$
    Using previous definition of x:

$$1_{\mathbb{Q}\sqrt{3}} \cdot x = (1 + 0\sqrt{3})(a + b\sqrt{3}) = (1 \cdot a + 3 \cdot 0 \cdot b) + (1 \cdot b + 0 \cdot a)\sqrt{3} = a + b\sqrt{3} = x$$

11. $\forall x \in \mathbb{Q}\sqrt{3}$ with $x \neq 0, x \cdot x^{-1} = 1$
    Using the previous definition of x:

$$\begin{aligned} x \cdot x^{-1} = (a + b\sqrt{3})(\frac{a}{a^2 - 3b^2} + (\frac{-b}{a^2 - 3b^2})\sqrt{3}) = \\ (\frac{a^2}{a^2 - 3b^2} - \frac{3b^2}{a^2 - 3b^2}) + (\frac{-ab}{a^2 - 3b^2} + \frac{ba}{a^2 - 3b^2})\sqrt{3} \\ = \frac{a^2 - 3b}{a^2 - 3b} + (\frac{ab - ab}{a^2 - 3b})\sqrt{3} = 1 + 0\sqrt{3} = 1_{\mathbb{Q}\sqrt{3}} \end{aligned}$$

12. $\forall x, y, z \in \mathbb{Q}\sqrt{3}, x(y + z) = xy + xz$
    We use the previous definitions for $x, y, z \in \mathbb{Q}\sqrt{3}$. This gives us $x(y + z)$ as:

$$\begin{aligned} x(y + z) = x((a' + b'\sqrt{3}) + (a'' + b''\sqrt{3})) = x((a' + a'') + (b' + b'')\sqrt{3}) \\ = (a + b\sqrt{3})((a' + a'') + (b' + b'')\sqrt{3}) \\ = (a(a' + a'') + 3b(b' + b'')) + (a(b' + b'') + b(a' + a''))\sqrt{3} \end{aligned}$$

    and $xy + xz$ as:

$$\begin{aligned} xy + xz = ((aa' + 3bb') + (ab' + ba')\sqrt{3}) + xz \\ = ((aa' + 3bb') + (ab' + ba')\sqrt{3}) + ((a + b\sqrt{3})(a'' + b''\sqrt{3})) \\ = ((aa' + 3bb') + (ab' + ba')\sqrt{3}) + ((aa'' + 3bb'') + (ab'' + ba'')\sqrt{3}) \\ = ((aa' + 3bb' + aa'' + 3bb'') + (ab' + ba' + ab'' + ba'')\sqrt{3}) \\ = (a(a' + a'') + 3b(b' + b'')) + (a(b' + b'') + b(a' + a''))\sqrt{3} \end{aligned}$$

    and so $x(y + z) = xy + xz$.

(3) Given these addition and product laws, the inversion rule we defined in (2) for $\mathbb{Q}\sqrt{3}$ looks like:

$$(a + b\sqrt{3})^{-1} = \frac{a}{a^2 - 3b} + (\frac{-b}{a^2 - 3b^2})\sqrt{3}$$

and the zero element was given by $0 + 0\sqrt{3}$.
Now consider element $x = 2 + 0\sqrt{3} \neq 0 + 0\sqrt{3}$, where $x \in \mathbb{Z}\sqrt{3}$. The inverse of x is given by $x^{-1} = \frac{2}{4} + 0\sqrt{3}$. Because $\frac{2}{4} \notin \mathbb{Z}, x^{-1} \notin \mathbb{Z}$ and so the inversion rule does not apply for $\mathbb{Z}\sqrt{3}$ with the provided addition and product rules.

5. **Exercise 1.5 (5 points)**

   We prove that $\prec$ does not make $\mathbb{Q}$ into an ordered set by a contradictory example. Consider $x = \frac{1}{6}$ and $y = \frac{2}{3}$. For these values, $x \not\prec y$ and $y \not\prec x$. Furthermore, $x \neq y$ because they are not the same element in the set $\mathbb{Q}$ (this set is reduced to lowest terms, so two elements are equivalent only if their numerator and denominator are the same.) Thus, we have shown for $x, y \in \mathbb{Q}$, none of the following statements are true: $x \prec y, y \prec x, x = y$ and so $\prec$ does not make $\mathbb{Q}$ an ordered set.

6. **Exercise 1.6 (5 points)**

   Theorem 0.1 *If S is an ordered set with elements $x, y, z \in S$, $x \leq y$ and $y \leq z \implies x \leq z$.*

   Proof: We do casework:

   1. **Case 1:** $x < y$

      We investigate the two subcases:

      1. **Subcase 1:** $y < z$
         By the transitivity property of ordered sets, $x < y$ and $y < z \implies x < z \implies x \leq z$.
      2. **Subcase 2:** $y = z$
         We are given $x < y$. Because $y = z$, $x < z \implies x \leq z$.

   2. **Case 2:** $x = y$

      We are given $y \leq z$. Because $x = y$, we can conclude $x \leq z$.

   Because A is a non-empty set, $\exists\, x \in A$. Pick any $x \in A$. Because α is a lower bound and β is an upper bound $\alpha \leq x \leq \beta \implies \alpha \leq \beta$ by **Theorem 0.1**.