Math 226: HW 9
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1. a) We prove this statement via induction below.

   (1) **Base Case**

   For a given upper triangular matrix $A \in M_{1\times 1}(\mathbb{F})$, $det(A) = a_{11} = \prod_{i=1}^{1} a_{ii}$.

   (2) **Inductive Step**

   We now prove that, if $det(A) = \prod_{i=1}^{m} A_{ii}$ for an upper triangular matrix $A \in M_{m\times m}(\mathbb{F})$, the determinant of upper triangular matrix $B \in M_{m+1\times m+1}(\mathbb{F})$ is given by $det(B) = \prod_{i=1}^{m+1} A_{ii}$.

   We can evaluate $det(B)$ through cofactor expansion along row $m+1$ as so:

$$det(B) = \Sigma_{j=1}^{m+1}(-1)^{m+1+j}B_{(m+1)j}det(\tilde{B}_{(m+1)j})$$

   Because B is an upper triangular matrix, the only element in its last $m+1$th row that is nonzero is in the $m+1$th column. Furthermore, because matrix B with the $m+1$th row and $m+1$th column removed is an upper triangular matrix, $det(\tilde{B}_{(m+1)j}) = \prod_{i=1}^{m} B_{ii}$. Thus, the above computation simplifies to:

$$det(B) = B_{(m+1)(m+1)}det(\tilde{B}_{(m+1)j}) = B_{(m+1)(m+1)}\prod_{i=1}^{m} B_{ii}$$

$$det(B) = \prod_{i=1}^{m+1} B_{ii}$$

   Thus, by induction we have proven that the determinant of all upper triangular matrices is equal to the product of their main diagonal.

   Because all lower triangular matrices are just the transpose of corresponding upper triangular matrices, and (1) we have proved that the determinant of all upper triangular matrices is the product of their main diagonal and (2) $det(A^T) = det(A)$, the determinant of all lower triangular matrices is the product of their main diagonal.

   b) Please see the end of this PDF for the solution.

2. a) For $i, n \in \mathbb{R}$, we define a_i as any vector belonging to $\mathbb{R}^n$. Let us use a matrix defined by $[\dots, a_k, \dots, a_s, \dots]^T$ where rows a_k and a_s are the two rows of interest. We compute $\delta([\dots, a_k, \dots, a_s, \dots]^T)$ below:

$$\delta([\dots, a_k, \dots, a_s, \dots]^T) = \delta([\dots, a_s + (a_k - a_s), \dots, a_k + (a_s - a_k), \dots]^T)$$
$$= \delta([\dots, a_s, \dots, a_k + (a_s - a_k), \dots]^T) + \delta([\dots, a_k - a_s, \dots, a_k + (a_s - a_k), \dots]^T)$$

   For brevity, I now *remove* the $\dots$ present above. Note that this does not at all affect the proof.

$$= \delta([a_s, a_k]^T) + \delta([a_s, a_s - a_k]^T) + \delta([a_k - a_s, a_k]^T) + \delta([a_k - a_s, a_s - a_k]^T)$$
$$= \delta([a_s, a_k]^T) + \delta([a_s, a_s]^T) - \delta([a_s, a_k]^T)$$
$$+\delta([a_k, a_k]^T) - \delta([a_s, a_k]^T) - \delta([a_s - a_k, a_s - a_k]^T)$$
$$= -\delta([a_s, a_k]^T)$$

Note that all the other terms were zero because there were two identical rows in the matrix. Thus, we have proven $\delta([\dots, a_k, \dots, a_s, \dots]^T) = -\delta([\dots, a_s, \dots, a_k, \dots]^T)$.

b) We prove that for all three types of elementary row operation matrices, $\delta(\mathcal{E}) = \delta(I)det(\mathcal{E})$ below.

1. $\mathcal{E}$: Interchanging two rows
Because $\mathcal{E}$ is the identity matrix with any two rows interchanged, as proved in part (a), $\delta(\mathcal{E}) = -\delta(I)$. The determinant of $\mathcal{E}$ is given by -1 and so $\delta(\mathcal{E}) = -\delta(I) = \delta(I)det(\mathcal{E})$.

2. $\mathcal{E}$: Multiplying some row of I by scalar $k \neq 0$
$\mathcal{E}$ is the identity matrix with any given row multiplied by k. Thus, by n-linearity, $\delta(\mathcal{E}) = k\delta(I)$. Because the determinant is a n-linear function, $det(\mathcal{E}) = kdet(I) = k$. Thus, $\delta(\mathcal{E}) = \delta(I)det(\mathcal{E}) = k$.

3. $\mathcal{E}$: Adding a multiple of some row of I to another row
Let us define a given identity matrix as $[\dots, a_k, \dots, a_s, \dots]^T$ where a_k and a_s represent the kth and sth rows of the identity matrix, respectively. $\mathcal{E}$ can be given as $[\dots, a_k + ca_s, \dots, a_s, \dots]^T$ where $c \in \mathbb{F}$. Thus, by n-linearity, $\delta(\mathcal{E})$ is given as:

$$\delta(\mathcal{E}) = \delta([\dots, a_k, \dots, a_s, \dots]^T) + c\delta([\dots, a_s, \dots, a_s, \dots]^T)$$
$$\delta(\mathcal{E}) = \delta([\dots, a_k, \dots, a_s, \dots]^T)$$
$$\delta(\mathcal{E}) = \delta(I)$$

Because $det(\mathcal{E}) = 1$, we have proven that $\delta(\mathcal{E}) = \delta(I)det(\mathcal{E}) = \delta(I)$.

c) We first prove that given $Z \in M_{2\times 2}(\mathbb{F})$ and $\mathcal{E}$ as any elementary matrix row operation, $\delta(\mathcal{E}Z) = \det(\mathcal{E})\delta(Z)$.

1. $\mathcal{E}$: Interchanging two rows
As proved in part (a), $\delta(\mathcal{E}Z) = -\delta(Z) = det(\mathcal{E})\delta(Z)$.

2. $\mathcal{E}$: Multiplying some row by scalar $k \neq 0$
Let us define matrix Z as $[\dots, a_s, \dots]^T$ where a_s is a vector of interest. $\mathcal{E}Z = [\dots, ka_s, \dots]^T$ and so, by n-linearity, we have that $\delta(\mathcal{E}Z) = \delta([\dots, ka_s, \dots]^T) = k\delta(Z) = det(\mathcal{E})\delta(Z)$. Thus, we have proven $\delta(\mathcal{E}Z) = k\delta(Z) = det(\mathcal{E})\delta(Z)$.

3. $\mathcal{E}$: Adding a multiple of some row to another row
Let us define $c \in \mathbb{F}$ and matrix Z as $[\dots, a_k, \dots, a_s, \dots]^T$, where a_k and a_s are vectors of interests. Thus, we have:

$$\delta(\mathcal{E}Z) = \delta([\dots, a_k + ka_s, \dots, a_s, \dots]^T)$$
$$= \delta([\dots, a_k, \dots, a_s, \dots]^T) + k\delta([\dots, a_s, \dots, a_s, \dots]^T)$$
$$= \delta([\dots, a_k, \dots, a_s, \dots]^T) = \delta(Z)$$

Thus we have proven $\delta(\mathcal{E}Z) = \delta(Z) = det(\mathcal{E})\delta(Z)$.

We now show that $\delta(A) = kdet(A)$. We prove this in the case that A is invertible or not, below.

1. **If A is not invertible**
If A is not invertible, it means that its rows are linearly dependent. This means that through a given set of elementary matrix row operations $\mathcal{E}_m, \dots, \mathcal{E}_1$, we can create a matrix $Z = \mathcal{E}_m \dots \mathcal{E}_1 A$ with two identical rows. We compute $\delta(A)$ as:

$$Z = \mathcal{E}_m \dots \mathcal{E}_1 A$$
$$\delta(Z) = \delta(\mathcal{E}_m \dots \mathcal{E}_1 A)$$
$$\delta(Z) = det(\mathcal{E}_m)\delta(\mathcal{E}_{m-1} \dots \mathcal{E}_1 A)$$
$$\delta(Z) = \prod_{i=1}^{m} det(\mathcal{E}_i)\delta(A)$$

Because Z has two identical rows, $\delta(Z) = 0$. Furthermore, because $\forall i, det(\mathcal{E}_i) \neq 0 \Rightarrow \prod_{i=1}^{m} det(\mathcal{E}_i) \neq 0$, we know that $\delta(A) = 0$.
Because A is not invertible, we know that $det(A) = 0$. Thus, we have proven $\delta(A) = kdet(A) = 0$ if A is not invertible.

2 **If A is invertible**
In the case that A is invertible, $A = \mathcal{E}_m\mathcal{E}_{m-1} \dots \mathcal{E}_1$, where $\mathcal{E}_i$ denotes an elementary row/column operation matrix. Thus, $kdet(A) = k\prod_{i=1}^{m} det(\mathcal{E}_i)$. We compute $\delta(A)$ below:

$$\delta(A) = \delta(\mathcal{E}_m \dots \mathcal{E}_1)$$
$$= det(\mathcal{E}_m)\delta(\mathcal{E}_{m-1} \dots \mathcal{E}_1)$$
$$= \prod_{i=1}^{m} det(\mathcal{E}_i)$$

Thus, we have that $\delta(A) = kdet(A) = \prod_{i=1}^{m} det(\mathcal{E}_i)$, where constant $k = 1$.

3. Please go to the end of the PDF to see the solution to this question.

4. a) We can evaluate $det(A)$ through cofactor expansion along the last (nth) row:

$$det(A) = \Sigma_{j=1}^{n}(-1)^{n+j}A_{nj}det(\tilde{A}_{nj})$$

As we can see, this formula goes up the rows of matrix A. For the first k rows, the only non-zero element will be in the last row and column and will be equal to one. Note that these last row and column values will always be one, as they are tracing upward the diagonal of the I_k identity matrix. After k recursions, this formula will compute, via cofactor expansion along the last row, the determinant of B as it has reduced (i.e. removed rows & columns) the matrix A just to matrix B. Thus, $det(A) = (1)^k det(B) = det(B)$.

b) Through a series of type elementary row operations $\mathcal{E}_1, \dots, \mathcal{E}_m$ we can convert matrix A to matrix Z, which resembles matrix A except for the fact that the bottom right $k \times k$ corner is no longer but D but an upper triangular matrix[1]. We refer to this upper triangular matrix in Z as D', where $D' = \mathcal{E}_m \dots \mathcal{E}_1 D$. We show $det(Z) = det(A)$ below:

$$Z = \mathcal{E}_m \dots \mathcal{E}_1 A$$
$$det(Z) = \prod_{i=1}^{m} \mathcal{E}_i det(A)$$
$$det(Z) = (1)^m det(A)$$
$$det(Z) = det(A)$$

[1]Note that this means that all the elementary row operations $\mathcal{E}_1, \dots, \mathcal{E}_m$ apply to the last k rows of A.

By the identical reasoning, $det(D') = det(D)$.

Thus, to compute $det(A)$ we have to compute $det(Z)$. We show $det(Z) = det(A) = det(B)det(D') = det(B)det(D)$ for all possible cases:

1. **D' contains a zero row**
   If D' contains a zero row, then Z contains a zero row $\Rightarrow det(Z) = det(A) = 0$. Furthermore, $det(D') = det(D) = 0$. Thus, $det(A) = 0 = det(B)det(D)$.

2. **Otherwise**
   We compute $det(Z)$ first. Similar to part (a), we can compute $det(Z)$ using cofactor expansion along the last row. In this case again, the only non-zero element is in the last row and column values. So this means that we are again going upward the diagonal of the D' matrix until arriving at B and computing its determinant. Thus, we get that $det(Z) = det(A) = \prod_{i=1}^{k} D'_{ii}det(B)$. Because the determinant of an upper triangular matrix (i.e. D') is the product of its main diagonal, we have $det(Z) = det(A) = \prod_{i=1}^{k} D'_{ii}det(B) = det(D')det(B) = det(D)det(B)$. Or in short, we have proved $det(A) = det(B)det(D)$.

c) We show $det(M) = 0$ below. Let us first define $\mathcal{E}_1, \ldots, \mathcal{E}_n$ as all type 2 elementary row operations in matrix form that for a given matrix $\mathcal{E}_i$ multiplies row i by negative one. Note that $\forall i, det(\mathcal{E}_i) = -1$.

$$
\begin{aligned}
M^t &= -M \\
M^t &= \prod_{i=1}^{n} \mathcal{E}_i M \\
det(M^t) &= \prod_{i=1}^{n} det(\mathcal{E}_i) det(M) \\
det(M) &= (-1)^n det(M)
\end{aligned}
$$

Because n is odd, $(-1)^n = -1$. Thus:

$$
\begin{aligned}
det(M) &= -det(M) \\
2det(M) &= 0 \\
det(M) &= 0
\end{aligned}
$$

Because $det(M) = 0$, M is not invertible.

d) If n is even, we have that $det(M) = (-1)^n det(M) \Rightarrow det(M) = det(M)$. Thus, we have no information on the determinant of M.