Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

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MATH 241 PSET 10

December 5, 2024

1.

Let us first define the PDF of r.v. X as $f_X(x) = e^{-x}$ and r.v. $Y = g(X)$ where $g(x) = e^{-x}$. Because g is differentiable and strictly decreasing, we can compute the PDF of Y as:

$$f_Y(y) = f_X(x)|\frac{dx}{dy}|$$

where $x = g^{-1}(y)$, or $x = -ln(y)$ for $0 < y \leq 1$. Thus, we have that:

$$f_Y(y) = f_X(-ln(y))|\frac{1}{\frac{dy}{dx}}|$$
$$f_Y(y) = f_X(-ln(y))|\frac{1}{-e^{-x}}|$$
$$f_Y(y) = f_X(-ln(y))| - e^{-ln(y)}|$$
$$f_Y(y) = y\frac{1}{y} = 1$$

Thus, we have that the PDF of e^{-X} can be given by 1 for $0 < y \leq 1$.

2.

We compute the joint PDF $f_{T,W}(t,w)$ for random variables T and W. To do so, we first compute the absolute value of the Jacobian matrix $\frac{\partial(t,w)}{\partial(x,y)}$, which is given by:

$$\frac{\partial(t,w)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Thus, we get that $|\frac{\partial(t,w)}{\partial(x,y)}| = |1(-1) - 1(1)| = |-2| = 2$. From this, using the Change of Variables Theorem, we can compute $f_{T,W}(t,w)$ as:

$$f_{T,W}(t,w) = f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(t,w)}\right| = f_{X,Y}(x,y)\left|\frac{\partial(t,w)}{\partial(x,y)}\right|^{-1} = \frac{f_{X,Y}(x,y)}{2}$$

where $f_{X,Y}(x,y)$ is the joint PDF of random variables X and Y. Note that because X and Y are independent, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, where $f_X(x)$ and $f_Y(y)$ are the PDFs for X and Y, respectively. Thus, we have that:

$$f_{T,W}(t,w) = \frac{f_X(x)f_Y(y)}{2}$$

Note that because $T = X + Y$ and $W = X - Y$, we can express x in the above equation as $\frac{t+w}{2}$ and y as $\frac{t-w}{2}$. Thus, we have:

$$f_{T,W}(t,w) = \frac{f_X(\frac{t+w}{2})f_Y(\frac{t-w}{2})}{2} = \frac{1}{2}\frac{e^{-\frac{1}{2}(\frac{t+w}{2})^2}}{\sqrt{(2\pi)}}\frac{e^{-\frac{1}{2}(\frac{t-w}{2})^2}}{\sqrt{2\pi}} = \frac{1}{4\pi}e^{-\frac{1}{8}[-(t+w)^2-(t-w)^2]}$$

$$= \frac{1}{4\pi}e^{-2(t^2+w^2)} = \frac{1}{4\pi}e^{-2t^2}e^{-2w^2}$$

Thus, because we can factor the joint PDF $f_{T,W}$ into a function of t times a function of w, we can conclude that random variables T and W are independent.

3.

Let us define r.v. $T = U + X$. Let us define the PDFs of random variables U and X as $f_U(u) = \frac{1}{1-0} = 1$ for $0 \le u \le 1$ and $f_X(x) = e^{-x}$ for $x \ge 0$. We compute the PDF $f_T(t)$ of r.v. T below:

$$f_T(t) = \int_{-\infty}^{\infty} f_U(t-x)f_X(x)dx$$

Because $f_U(u) = 0$ for $u \notin [0,1]$ and $f_X(x) = 0$ for $x < 0$, we must restrict this integral to $0 \le t - x \le 1 \Rightarrow x \le t \le 1 + x$ and $x \ge 0$. Upon inspection, we can see that the bounds for x vary according to the value of t: when $0 \le t \le 1$, x is constrained to $(0, t)$ and when $t > 1$, x is constrained to $(t-1, t)$. Thus, the PDF of T can be given as a piecewise function:

$$f_T(t) = \begin{cases} \int_0^t f_U(t-x)f_x(x)dx = \int_0^t e^{-x}dx = -[e^{-t}-1] = 1 - e^{-t} & \text{for } 0 \le t \le \\ \int_{t-1}^t f_U(t-x)f_x(x)dx = \int_{t-1}^t e^{-x}dx = -[e^{-t}-e^{1-t}] = -[e^{-t}-ee^{-t}] = (e-1)e^{-t}) & \text{for } t > 1 \end{cases}$$

2

4.

Let us denote the number of ticket sold for movie i of the year as $T_i \sim Pois(\lambda_2)$. Thus, the number of movie tickets sold next year can be given as: $T = \Sigma_{i=1}^{N} T_i$. We compute $\mathbb{E}[T]$ below:

$$\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T|N]] = \mathbb{E}[\mathbb{E}[\Sigma_{i=1}^{N} T_i|N]] = \mathbb{E}[\Sigma_{i=1}^{N} \mathbb{E}[T_i|N]]$$

Because the number of tickets sold for a given movie, T_i, is independent of N, $\mathbb{E}[T_i|N] = \mathbb{E}[T_i]$ and $Var(T_i|N) = Var(T_i)$. Thus, we get that:

$$\mathbb{E}[T] = \mathbb{E}[\Sigma_{i=1}^{N} \mathbb{E}[T_i]] = \mathbb{E}[N\lambda_2] = \lambda_2 \mathbb{E}[N] = \lambda_1 \lambda_2$$

Note that in the above computations, we have computed $\mathbb{E}[T|N] = N\lambda_2$. We can compute the $Var(T)$ as such:

$$Var(T) = \mathbb{E}[Var(T|N)] + Var(\mathbb{E}[T|N])$$
$$Var(T) = \mathbb{E}[Var(\Sigma_{i=1}^{N} T_i|N)] + Var(\mathbb{E}[T|N])$$
$$Var(T) = \mathbb{E}[\Sigma_{i=1}^{N} Var(T_i|N)] + Var(N\lambda_2)$$
$$Var(T) = \mathbb{E}[\Sigma_{i=1}^{N} Var(T_i)] + \lambda_2^2 Var(N)$$
$$Var(T) = \mathbb{E}[N\lambda_2] + \lambda_2^2 Var(N)$$
$$Var(T) = \lambda_2 \mathbb{E}[N] + \lambda_2^2 Var(N)$$
$$Var(T) = \lambda_2 \lambda_1 + \lambda_2^2 \lambda_1 = \lambda_1 \lambda_2 (1 + \lambda_2)$$

5. We compute $\mathbb{E}[Y]$ through Adam's Law as:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[c] = c$$

We compute $\mathbb{E}[XY]$ through Adam's Law as:

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\mathbb{E}[Y|X]] = \mathbb{E}[cX] = c\mathbb{E}[X]$$

Using the formula of covariance, we can compute the $Cov(X, Y)$ as:

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = c\mathbb{E}[X] - c\mathbb{E}[X] = 0$$

Because $Cov(X, Y) = 0$, $Corr(X, Y) = 0$ and so we can conclude that X and Y are uncorrelated.

6. Anish Lakkapragada. I worked independently.