Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head here.

MATH 255 HW 2

January 30, 2025

---

1. **Exercise 2.1 (5 points; Rudin 1.5)**

   We first show that $\inf(A) \geq -\sup(-A)$. By the definition of supremum, $\forall a \in -A, \sup(-A) \geq a \implies \forall a \in A, \sup(-A) \geq -a \implies \forall a \in A, -\sup(-A) \leq a \implies -\sup(-A)$ is a lower bound for A. By definition of infimum, $\inf(A) \geq -\sup(-A)$.

   We now show that $\inf(A) \leq -\sup(-A)$. By the definition of infimum, $\forall a \in A, \inf(A) \leq a \implies \forall a \in -A, \inf(A) \leq -a \implies \forall a \in -A, -\inf(A) \geq a \implies -\inf(A)$ is an upper bound for $-A$. By definition of supremum, $\sup(-A) \leq -\inf(A) \implies \inf(A) \leq -\sup(-A)$.

   Thus because $\inf(A) \geq -\sup(-A)$ and $\inf(A) \leq -\sup(-A)$, we have proven $\inf(A) = -\sup(A)$.

2. **Exercise 2.2 (5 points)**

   We first show that $\inf(A^{-1}) \geq (\sup(A))^{-1}$. By the definition of supremum, $\forall a \in A, \sup(A) \geq a \implies \forall a \in A^{-1}, \sup(A) \geq a^{-1} \implies \forall a \in A^{-1}, \frac{1}{\sup(A)} = (\sup(A))^{-1} \leq a \implies (\sup(A))^{-1}$ is a lower bound for A^{-1}. By definition of infimum, $\inf(A^{-1}) \geq (\sup(A))^{-1}$.

   We now show that $\inf(A^{-1}) \leq (\sup(A))^{-1}$. By the definition of infimum, $\forall a \in A^{-1}, \inf(A^{-1}) \leq a \implies \forall a \in A, \inf(A^{-1}) \leq a^{-1} \implies \forall a \in A, (\inf(A^{-1}))^{-1} \geq a \implies (\inf(A^{-1}))^{-1}$ is an upper bound for A. By definition of supremum, $(\inf(A^{-1}))^{-1} \geq \sup(A) \implies \inf(A^{-1}) \leq (\sup(A))^{-1}$.

   Because we have shown $\inf(A^{-1}) \geq (\sup(A))^{-1}$ and $\inf(A^{-1}) \leq (\sup(A))^{-1}$, we have proven $\inf(A^{-1}) = (\sup(A))^{-1}$.

3. **Exercise 2.3 (5 points)**

   Lemma 0.1 *Note that by definition of supremum, $\forall a \in A, a \leq sup(A)$ and $\forall\ b \in B, b \leq sup(B)$. Thus, $\forall a \in A$ and $b \in B, a + b \leq sup(A) + b \leq sup(A) + sup(B) \implies \forall a \in A$ and $b \in B, a + b \leq sup(A) + sup(B)$.*

   To prove $\sup(A + B) = \sup(A) + \sup(B)$ we prove $\sup(A + B) \leq \sup(A) + \sup(B)$ and $\sup(A) + \sup(B) \leq \sup(A + B)$. We prove both directions of this statement below.

1. $\sup(A) + \sup(B) \leq \sup(A + B)$

   By **Lemma 0.1**, $\forall a \in A$ and $b \in B, a+b \leq \sup(A+B) \implies \forall a \in A$ and $b \in B, a \leq \sup(A + B) - b$. Hence, for a given $b \in B$, $\sup(A + B) - b$ is an upper bound for A. Because $\sup(A)$ is a supremum, for a given $b \in B$, $\sup(A) \leq \sup(A + B) - b \implies \forall\, b \in B, b \leq \sup(A+B) - \sup(A) \implies \sup(A+B) - \sup(A)$ is an upper bound for B. Because $\sup(B)$ is the lowest upper bound for B, $\sup(B) \leq \sup(A+B) - \sup(A) \implies \sup(A) + \sup(B) \leq \sup(A + B)$.

2. $\sup(A) + \sup(B) \geq \sup(A + B)$

   By **Lemma 0.1**, $\sup(A) + \sup(B)$ is an upper bound for $A + B$. Because $\sup(A+B)$ is the lowest upper bound of $A + B$, $\sup(A) + \sup(B) \geq \sup(A + B)$.

4. **Exercise 2.4 (15 points)**

   Note that if a set S has a defined maximum element, then its supremum is its maximum element.

   (1) $A = \{2, 3\}$ is clearly bounded above as it has an upper bound (e.g. 4) $\in \mathbb{Z}$. A also has a maximum element, 3, as $2 \leq 3$ and $3 \leq 3 \implies \forall x \in A, x \leq 3$. Because A's maximum element is defined as 3, its supremum is also 3.

   (2) The set A is given as $A = \{-\frac{2}{5}, -\frac{4}{5}, -\frac{6}{5}, \dots\}$. A is bounded above as $\exists$ an upper bound (e.g. 0) that exists in $\mathbb{Q}$. The maximum element of A is $-\frac{2}{5}$. Because the maximum element of A is defined, this maximum element also is the supremum of A.

   (3) The set A is given as $A = \{-\frac{1}{1}, -\frac{1}{2}, -\frac{1}{3}, \dots\}$. A is bounded above as $\exists$ an upper bound $\in \mathbb{Q}$ (e.g. $0 \in \mathbb{Q}$). We discuss maximum element & supremum below:

   (a) A **has no maximum element**

   We prove this by statement by contradiction. Let us define $m = -\frac{1}{n} \in A$, where $n \in \mathbb{N}$, to be the maximum element in A. Because $n \in \mathbb{N}$, then $n + 1 > n \in \mathbb{N}$ and so we have[1]:

$$
\begin{gathered}
n + 1 > n \\
(n + 1)^{-1} \cdot (n + 1) = 1 > (n + 1)^{-1} \cdot n \\
n^{-1} \cdot 1 > (n + 1)^{-1} \cdot n^{-1} \cdot n \\
n^{-1} > (n + 1)^{-1} \\
\frac{1}{n} > \frac{1}{n + 1} \\
-\frac{1}{n + 1} > -\frac{1}{n} = m
\end{gathered}
$$

   and so because $\exists\ -\frac{1}{n+1} \in A$ where $-\frac{1}{n+1} > m \implies m$ is not the maximum element of $A \implies A$ has no maximum element.

[1] Because $n, n + 1 \in \mathbb{Q}$, $n \neq 0$ and $n + 1 \neq 0$ and so they both have well-defined inverses.

(b) **The supremum of A is zero.**
We first prove that zero is an upper bound of A. Because $\forall a \in A, a < 0 \implies 0$ is an upper bound of A. We now prove by contradiction that there does not exist any upper bound for A lower than 0 (i.e. the supremum of A is zero). Let us define a supremum of A, as $s < 0$ where $s \in \mathbb{Q}$. Because $s \in \mathbb{Q}$, we can define $s = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Thus, we can re-express $s = \frac{1}{\frac{q}{p}}$ and because $\exists\, n \in \mathbb{N}$ s.t. $n > |\frac{q}{p}| \implies \exists\, a = \frac{1}{n} \in A$ s.t. $a > s \implies s$ is not an upper bound $\implies s$ is not a supremum. Thus, by proof by contradiction we have proven that there does not exist any upper bound for A lower than 0 $\implies$ the supremum of A is zero.

(4) The set A is given by $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. A is bounded above as $\exists$ an upper bound $\in \mathbb{Q}$ (e.g. 2). A also has a maximum element, 1, which also serves as its supremum.

(5) The set A is bounded above as $\exists$ an upper bound $\in \mathbb{Q}$ (e.g. 2). The set A also has a maximum element, one, which also serves as its supremum.

(6) The set A is bounded above as $\exists$ an upper bound $\in \mathbb{Q}$ (e.g. 1). We now discuss maximum element & supremum:

(a) A **has no maximum element**
We prove this by contradiction; consider $m \in A$ to be the maximum element of A. Because $m \in A \implies m < 1$.We can construct a number $m' = m + \frac{1-m}{2}$. Note that because $m < 1, \frac{1-m}{2} > 0$ and so $m' > m$. Furthermore, $m' < m + (1 - m) \implies m' < 1 \implies m' \in A$. Thus, we have that $\forall a \in A, m' > m \geq a \implies \forall a \in A, m' > a \implies m'$ is a maximum element for $A \implies A$ has no maximum element.

(b) **The supremum of A is one.**
The supremum of A is its lowest upper bound of A. One is an upper bound for A as $\forall a \in A, a < 1$. We now prove by contradiction that there does not exist any upper bound for A lower than 1 (i.e. the supremum of A is one). Let us define a supremum $s < 1$ where $s \in \mathbb{Q}$. Because s is a supremum, $\forall a \in A, s > a > 0 \implies s > 0$. Therefore, because $s \in \mathbb{Q}$ and $0 < s < 1$, $s \in A$. As we have proven above, A does not have a maximum element and so s is not an upper bound for $A \implies s$ is not a supremum $\implies \nexists$ an upper bound for A less than one.

(7) A is bounded above as $\exists$ an upper bound for $A \in \mathbb{Q}$ (e.g. 2). We now discuss maximum element & supremum:

(a) A **has no maximum element**
We prove this by contradiction; consider $m \in A$ to be the maximum element of A. Note that because $m \in A \implies m^3 < 2$. We now try to find a new maximum element of A. Let us define a tiny rational $\epsilon > 0$. Our new maximum element can be given as $m' = m + \epsilon > m$. We now solve for ϵ such that $(m')^3 < 2$ (so $m' \in A$)[2]:

[2]Note that all the numbers involved below are rationals $\implies$ the solution of ϵ is also a rational $\implies m' = m + \epsilon \in \mathbb{Q}$.

$$(m')^3 < 2$$
$$(m+\epsilon)^3 < 2$$
$$m^3 + 3m^2\epsilon + 3m\epsilon^2 + \epsilon^3 < 2$$
$$3m^2\epsilon + 3m\epsilon^2 + \epsilon^3 < 2 - m^3$$

Let us add another constraint that $\epsilon < 1$ to help remove the ϵ^2 and ϵ^3 terms:

$$3m^2\epsilon < 1 - m^3 - 3m$$
$$\epsilon < \frac{1 - m^3 - 3m}{3m^2}$$

Thus, a solution for ϵ can be given by:

$$0 < \epsilon < \min(\frac{1 - m^3 - 3m}{3m^2}, 1)$$

which means that $\exists\ m' > m$ s.t. $m \in A \implies A$ has no maximum element.

(a) A **has no supremum**

Lemma 0.2 *We first prove that if given $y \in \mathbb{Q}$ where $y > 0$ and $y^3 > 2$, then y is an upper bound of A. Consider any $x \in A$. We have $y^3 > 2 > x^3 \implies y^3 > x^3$. If $x > y$ then $x^3 > x^2y$ and $xy^2 > y^3 \implies x^3 > xy^2 > y^3 \implies x^3 > y^3$, which contradicts $y^3 > x^3$. Thus $\forall x \in A, x \leq y \implies y$ is an upper bound of A.*

We now prove by contradiction that A has no supremum. Let us suppose $s \in \mathbb{Q}$ is a supremum of A. Let us define a tiny rational $\epsilon > 0$. We now find a new supremum of A, $s' = s - \epsilon < s$. Applying **Lemma 0.2**, if $s' > 0$ and $s^3 \geq 2$, s' is an upper bound for A. Thus, we solve for ϵ s.t. $(s')^3 > 2$ and $s' > 0$ (i.e. $e < s$):

$$(s')^3 > 2$$
$$(s-\epsilon)^3 > 2$$
$$s^3 - 3s^2\epsilon + 3s\epsilon^2 - \epsilon^3 > 2$$

We add another constraint that $\epsilon < 1$ constraint to help handle e^2 and e^3 terms:

$$s^3 - 3s^2\epsilon + 3s - 1 > 2$$
$$-3s^2\epsilon < 3 - s^3$$
$$\epsilon < \frac{s^3 - 3}{3s^2}$$

Thus, a solution for ϵ can be given by:

$$0 < \epsilon < \min(\frac{s^3 - 3}{3s^2}, s)$$

which means that $s' < s$ is an upper bound for $A \implies A$ has no supremum.

5. **Exercise 2.5 (10 points)**

(1) For proof by contradiction, let us assume $0 \geq 1$. We find contradictions for the $0 = 1$ and $0 > 1$ cases below:

1. $0 = 1$
   Pick $a \in F$. Then $a \cdot 1 = a \cdot 0 = 0 \neq a$, which is a contradiction to the multiplicative identity field axiom.
2. $0 > 1$
   If $0 > 1$, then this means that:

$$\begin{aligned} 1 + (-1) &> 1 + 0 \\ -1 &> 0 \end{aligned}$$

Because $-1 > 0 \implies (-1)^2 > 0 \cdot 1 \implies 1 > 0$, so we have a contradiction. Thus we have proved $0 < 1$.

(2) We prove this with contradiction, and thus assume $x^{-1} \leq 0$.
If $x^{-1} = 0$, then $x \cdot x^{-1} = x \cdot 0 = 0 \neq 1$, and thus this is a contradiction with the definition of inverses in fields.
If $x^{-1} < 0$, then we can multiply both sides of the inequality $x^{-1} < 0$ with x and so the inequality sign will not change because $x > 0$. Thus:

$$\begin{aligned} x^{-1} &< 0 \\ x \cdot x^{-1} &< x \cdot 0 \\ 1 &< 0 \end{aligned}$$

which is a contradiction to our proof in (1).

(3) We prove both directions of this statement.

1 $xy > xz \implies y > z$
We are given $x > 0$, and so from part (2) we know that $x^{-1} > 0$. Therefore:

$$\begin{aligned} xy &> xz \\ x^{-1} \cdot xy &> x^{-1} \cdot xz \\ (x^{-1} \cdot x)y &> (x^{-1} \cdot x)z \\ y &> z \end{aligned}$$

2 $y > z \implies xy > xz$

$$y > z \tag{1}$$
$$y - z > 0 \tag{2}$$
$$\tag{3}$$

Because $y - z > 0$ and $x > 0$, $x(y - z) > 0$ so:

$$x(y - z) > 0$$
$$xy - xz > 0$$
$$xy > xz$$

(4) Because $0 < 1$, there are three possible orderings. We show that all three of these orderings violate the ordered field axioms:

1. **Ordering #1:** $0 < 1 < 2$
   Under this ordering, we have $1 + 2 = 0 < 2$ and thus:

   $$1 + 2 < 1 + 1$$
   $$1 + 2 < 1 + 1$$
   $$2 < 1$$

   which is a contradiction.
2. **Ordering #2:** $0 < 2 < 1$
   Under this ordering, we have $1 + 2 = 0 < 2$

   $$1 + 2 < 0 + 2$$
   $$1 < 0$$

   which is a contradiction with our proof in (1).
3. **Ordering #3:** $2 < 0 < 1$
   Under this ordering, we have that $2 < 1$ and so:

   $$2 < 1$$
   $$2 + 1 < 1 + 1$$
   $$0 < 2$$

   which is a contradiction.