Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head `here`.

Math 226: HW 2
Completed By: Anish Lakkapragada (NETID: al2778)

1. a) We demonstrate that $L^2(\mathbb{R})$ has the identity element $f(x) = 0$, is closed under addition, and closed under scalar multiplication to prove that $L^2(\mathbb{R})$ is a vector space.

1 Existence of Additive Identity Element in $L^2(\mathbb{R})$
The function $f(x) = 0 \in L^2(\mathbb{R})$ as $f(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $\int_{-\infty}^{\infty} f(x)dx = 0 < \infty$. $f(x) = 0$ is the additive identify element $L^2(\mathbb{R})$ as $\forall g(x) \in L^2(\mathbb{R})$, $f(x) + g(x) = g(x)$.

2 Closed Under Addition
Given $a, b \in \mathbb{R}$:

$$(a - b)^2 \geq 0$$
$$a^2 + b^2 \geq 2ab$$
$$2a^2 + 2b^2 \geq (a + b)^2$$

If we switch sides and then substitute $a = f(x) \in L^2(\mathbb{R})$ and $b = g(x) \in L^2(\mathbb{R})$, we get:

$$(f(x) + g(x))^2 \leq 2[f(x)]^2 + 2[g(x)]^2$$
$$|f(x) + g(x)|^2 \leq 2|f(x)|^2 + 2|g(x)|^2$$

We now integrate from $-\infty$ to ∞ on both sides:

$$\int_{-\infty}^{\infty} |f(x) + g(x)|^2 dx \leq 2 \int_{-\infty}^{\infty} |f(x)|^2 dx + 2 \int_{-\infty}^{\infty} |g(x)|^2 dx$$

Because $f(x), g(x) \in L^2(\mathbb{R})$, we know that $\int_{-\infty}^{\infty} |f(x)|^2 < \infty$ and $\int_{-\infty}^{\infty} |g(x)|^2 < \infty$. Thus $2\int_{-\infty}^{\infty} |f(x)|^2 dx + 2\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$ as well, and so we know that:

$$\int_{-\infty}^{\infty} |f(x) + g(x)|^2 dx < \infty \quad (1)$$

Given that $f(x), g(x) \in L^2(\mathbb{R})$, we know that $f(x) + g(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ as $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space and thus is closed under addition. Thus, because we have proved Equation 1 and $f(x) + g(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, $f(x) + g(x) \in L^2(\mathbb{R})$ and so $L^2(\mathbb{R})$ is closed under addition.

3 Closed Under Scalar Multiplication
Consider for $x \in \mathbb{R}$ a function $f(x) \in L^2(\mathbb{R})$. Given $c \in \mathbb{R}$, let us define $g(x) = cf(x)$. Because $f(x) \in \mathbb{R}$ and $c \in \mathbb{R}$, $g(x) = cf(x) \in \mathbb{R}$. If $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$, $k\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ for $k \in \mathbb{R}$. Thus $\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |cf(x)|^2 dx = |c|^2 \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ as $|c|^2 \in \mathbb{R}$ and so it is proven $\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$. Thus $g(x) \in L^2(\mathbb{R})$ and so $L^2(\mathbb{R})$ is proven to be closed under scalar multiplication.

Because we have demonstrated 1, 2, and 3, we have demonstrated $L^2(\mathbb{R})$ is a vector space.

b) In order for a set V to define a vector space over field $\mathbb{R}$, the vector $\mathbf{0} \in V$ s.t. $\forall v \in V, v + \mathbf{0} = v$. For set V, this vector $\mathbf{0} = (0,1)$ as $\forall v = (a_1, b_1) \in V, v + \mathbf{0} = (a_1, b_1) + (0,1) = (a_1, b_1) = v$.
Another property for a set V to define a vector space is that $\forall v \in V, \exists -v \in V$ s.t. $v + (-v) = \mathbf{0} = (0,1)$. Let us define $v = (a_1, b_1) \in V$ and vector $-v = (a_2, b_2)$. For $v + (-v) = (0,1)$, $a_2 = -a_1$ and $b_2(b_1) = 1$. In the case where $b_1 = 0$, $b_2(b_1) \neq 1$ and thus $\forall v \in V$ it is not guaranteed $\exists -v \in V$ s.t. $v + (-v) = \mathbf{0} = (0,1)$. Because this condition is not met, V does not define a valid vector space over $\mathbb{R}$.

2. a) We go through the three conditions of testing if vector space W_1 and W_2 are subspaces of $\mathbb{F}^n$.

(1) Closed Under Scalar Multiplication

a) $W = W_1$
For a given $x = (a_1, a_2, \dots a_n) \in W_1$ and $c \in \mathbb{F}$, $cx = (ca_1, ca_2, \dots ca_n)$. Because $ca_1, ca_2, \dots ca_n \in \mathbb{F}^n$, and $c\Sigma_{i=1}^N a_i = 0$ given $\Sigma_{i=1}^N a_i = 0$, W_1 meets this condition to be a subspace of $\mathbb{F}^n$.

b) $W = W_2$
$cx = (ca_1, ca_2, \dots ca_n)$. Given $\Sigma_{i=1}^N a_i = 1$, $c\Sigma_{i=1}^N a_i \neq 1$ and thus $cx \notin W_2$. Thus W_2 does not meet this condition to be a subspace of $\mathbb{F}^n$. *Because W_2 does not meet this condition to be a subspace, we do not need to check if it meets any of the other conditions.*

(2) Closed Under Addition

a) $W = W_1$
Given $x = (a_1, a_2, \dots a_n) \in W_1, y = (b_1, b_2, \dots b_n) \in W_1$, $x + y = (a_1 + b_1, a_2 + b_2, \dots a_n + b_n)$. $\Sigma_{i=1}^N a_i = \Sigma_{i=1}^N b_i = 0 \Rightarrow \Sigma_{i=1}^N a_i + b_i = 0 \Rightarrow x + y \in W_1$. Thus W_1 meets this condition to be a subspace of $\mathbb{F}^n$.

(3) $\exists\, \mathbf{0} \in W$

a) $W = W_1$
For $x = \mathbf{0}$, $x_i = 0 \Rightarrow \Sigma_{i=1}^N x_i = 0$. Thus, $\mathbf{0} \in W_1$ and so W_1 meets this condition to be a subspace of $\mathbb{F}^n$.

Because W_1 meets all the conditions to be a subspace whereas W_2 does not, W_1 is a subspace of $\mathbb{F}^n$.

b) We define the subset of $\mathbb{Z}_2^n$ with even E_n as $Q^n = \{v \in \mathbb{Z}_2^n : E_n(v) \in 2\mathbb{Z}\}$. We now assess if $Q^n \leq \mathbb{Z}_2^n$ by checking if Q^n meets the following three conditions.

(1) $\exists\, \mathbf{0} \in Q^n$
Let us define the zero vector as $z = \mathbf{0} \in \mathbb{Z}_2^n$. Because $E_n(z) = 0 \in 2\mathbb{Z}$, $z = \mathbf{0} \in Q^n$.

(2) Closed Under Addition
Let us define two vectors $x, y \in Q^n$. Because $\mathbb{Z}_2^n$ is a vector space and thus closed under addition, $x + y \in \mathbb{Z}_2^n$. The number of nonzero components of $x + y$ is given by $E_n(x+y) = E_n(x) + E_n(y) - 2k$, where $k \in \mathbb{Z}$ is given by the number of indices where x and y have the same value. Because $E_n(x), E_n(y) \in 2\mathbb{Z}$ as $x, y \in \mathbb{Z}_2^n$, $E_n(x) + E_n(y) \in 2\mathbb{Z}$. Because $k \in \mathbb{Z}$, $2k \in 2\mathbb{Z}$ and so $E_n(x + y) = E_n(x) + E_n(y) - 2k \in 2\mathbb{Z}$. Because $E_n(x+y) \in 2\mathbb{Z}$ and $x + y \in \mathbb{Z}_2^n \Rightarrow x + y \in Q^n$. Thus Q^n is closed under addition.

(3) Closed Under Scalar Multiplication
Let us consider a scalar $c \in \mathbb{Z}_2$ and $v \in Q^n$. c can either equal zero or one. If $c = 0$, $cv = \mathbf{0} \in Q^n$. If $c = 1$, $cv = v \in Q^n$. Thus, $cv \in Q^n$ for any $c \in \mathbb{Z}_2$ and so Q^n is closed under scalar multiplication.

Because Q^n meets all the three conditions to be a subspace to $\mathbb{Z}_2^n$, $Q^n \leq \mathbb{Z}_2^n$.

c) The general form for function $f \in P_3(\mathbb{R})$ is given by $f(x) = c_1 + c_2x + c_3x^2 + c_4x^3$ where $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Given the constraints $f(0) = f'(0)$ and $f(1) = 0$, the form for any function $f \in W$ is given by:

$$f(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3$$

We now test if W defines a subspace of $P_3(\mathbb{R})$.

(1) $\exists\ \mathbf{0} \in W$
Because when $f(x) = 0$ when $c_1 = c_3 = 0$ and $x = 0$, the zero polynomial is defined in W.

(2) Closed Under Addition
Given $p(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3 \in W$ and $q(x) = b_1 + b_1x + b_3x^2 + (-2b_1 - b_3)x^3 \in W$, $p(x) + q(x) = (c_1 + b_1) + (c_1 + b_1)x + (c_3 + b_3)x^2 + (-2c_1 - c_3 - 2b_1 - b_3)x^3 \in P_3(\mathbb{R})$. Defining $z(x) = p(x) + q(x)$, $z(0) = z'(0) = c_1 + b_1$ and $z(1) = p(1) + q(1) = 0 + 0 = 0$. Thus $p(x) + q(x) \in W$ and thus W is proven to be closed under addition.

(3) Closed Under Scalar Multiplication
Given $p(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3 \in W$ and $k \in \mathbb{R}$, $kp(x) = k(c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3)$. Defining $g(x) = kp(x)$, $g(0) = g'(0) = kc_1)$ and $g(1) = k(p(1)) = k(0) = 0$. Thus $kp(x) \in W$ and W is proven to be closed under scalar multiplication.

Because I have shown W contains the zero polynomial and is closed under addition and scalar multiplication, $W \leq P_3(\mathbb{R})$.

Because $P_k(\mathbb{R})$ is the set of polynomials that have a degree of *at most* k, $P_n(\mathbb{R}) \leq P_k(\mathbb{R})$ where $n \leq k$. Thus $P_3(\mathbb{R}) \leq P_4(\mathbb{R})$. Because $W \leq P_3(\mathbb{R}) \leq P_4(\mathbb{R})$, $W \leq P_4(\mathbb{R})$.

d) Because this statement is an *if and only if*, we must show (1) that if these two conditions are met, $W \leq V$ and (2) that if $W \leq V$, these two conditions are met. We show (1) and (2) below.

(1) If Condition 1 and Condition 2 are met, $W \leq V$

(a) Condition 1: $W \neq \emptyset$
If $W = \emptyset$, the standard condition to define a subspace for $\mathbf{0} \in W$ cannot be met because there are no elements in W. Note $W \neq \emptyset \not\Rightarrow \mathbf{0} \in W$.

(b) Condition 2: for $a \in \mathbb{F}$ and $x, y \in W$, $\exists\ ax + y \in W$.
Given $a = -1$ and $x = y$, if W meets Condition 2, it is guaranteed that $-x + y = \mathbf{0} \in W$. Thus, the standard condition of existence of a zero vector in a subspace is met if Condition 2 is met.
In the case $y = \mathbf{0} \in W$, if W meets Condition 2, $ax \in W$ for $a \in \mathbb{F}$ and $x \in W$. Thus, closure under scalar multiplication is met if Condition 2 is met.
Let us define $z = ax \in W$. Then, if Condition 2 is met, we know given $z, y \in W$, $z + y \in W$. Thus, closure under addition is met if Condition 2 is met.

Thus, we have shown that if Condition 1 and Condition 2 are met, W meets the three properties to be defined as a subspace and so $W \leq V$.

(2) If $W \leq V$, Condition 1 and Condition 2 are met
We discuss below the implications of the properties of W we know given $W \leq V$.

(a) $\exists\ \mathbf{0} \in W$
If $\exists\ \mathbf{0} \in W$, $|W| \geq 1$ and so $W \neq \emptyset$. Thus, Condition 1 is met.

(b) W is closed under addition and scalar multiplication

Let us define $a \in \mathbb{F}$ and $x, y \in W$. If W is closed under scalar multiplication, $ax \in W$. If W is closed under addition, $ax + y \in W$. Thus, Condition 2 is met.

Thus, we have shown that if $W \leq V$, Condition 1 and Condition 2 are met.

Because we have proven both (1) and (2), we have shown that if and only if Condition 1 and Condition 2 are met for a given subset W of a vector space V will $W \leq V$.

3. a) We test if $U \cap W$ is a subspace of V below.

(1) $\mathbf{0} \in U \cap W$

Because both U and W are valid subspaces, $\exists\ \mathbf{0} \in U$ and $\exists\ \mathbf{0} \in W$. Thus $\mathbf{0} \in U \cap W$.

(2) Closed Under Addition

Let us consider $x, y \in U \cap W$. Because U and W are valid subspaces, U and W are closed under addition. Thus $x + y \in U$ and $x + y \in W \Rightarrow x + y \in U \cap W$.

(3) Closed Under Scalar Multiplication

Let us consider $c \in \mathbb{F}$ and $x \in U \cap W$. Because U and W are valid subspaces, U and W are closed under scalar multiplication. Thus $cx \in U$ and $cx \in W \Rightarrow cx \in U \cap W$.

Thus we have proven $U \cap W \leq V$.

b) We test if $U + W$ is a subspace of V below.

(1) $\mathbf{0} \in U + W$

Because U and W are both valid subspaces, $\mathbf{0} \in U, W$. Thus, for $u = \mathbf{0} \in U$ and $w = \mathbf{0} \in W$, $u + w = \mathbf{0} \in U + W$.

(2) Closed Under Addition

Let us consider $u_1, u_2 \in U$ and $w_1, w_2 \in W$. Let us define elements $x = u_1 + w_1 \in U + W$ and $y = u_2 + w_2 \in U + W$. $x + y = u_1 + w_1 + u_2 + w_2 \rightarrow (u_1 + u_2) + (w_1 + w_2)$. Because U and W are valid subspaces, they are both closed under addition and thus $z_1 = u_1 + u_2 \in U$ and $z_2 = w_1 + w_2 \in W$. As such, $x + y = z_1 + z_2 \in U + W$ and so $U + W$ is proven to be closed under addition.

(3) Closed Under Scalar Multiplication

Let us define $u \in U, w \in W, x = u_1 + w_1 \in U$. Given $c \in \mathbb{F}$, $cx = cu + cw$. Because U and W are valid subspaces, U and W are both closed under scalar multiplication and so $cu \in U$ and $cw \in W$. Thus, $cx = cu + cw \in U + W$ and so $U + W$ is proven to be closed under scalar multiplication.

Thus we have proven $U + W \leq V$.

c) Two subspaces of $\mathbb{R}^2$ whose union is not a subspace of $\mathbb{R}^2$ is $\mathbb{Q}^2$ and $W = \{(a_1, a_2) \in \mathbb{F}^2 : a_1 + a_2 = 0\}$ where field $\mathbb{F}^2 = (\mathbb{R}^2, +, \cdot)$.

An example proving $\mathbb{Q}^2 \cup W$ is not a subspace of $\mathbb{R}^2$ is choosing $x = (1.5, 0) \in \mathbb{Q}^2 \cup W$ and $y = (-\sqrt{2}, \sqrt{2}) \in \mathbb{Q}^2 \cup W$. $x + y = (1.5 - \sqrt{2}, \sqrt{2}) \notin \mathbb{Q}^2 \cup W$ and so $\mathbb{Q}^2 \cup W$ does not define a valid subspace as it is not closed under addition.

4. a)

$$Span(S) = \{f(x) = (c_1 - c_2) + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4; c_1, c_2, c_3, c_4 \in \mathbb{R}\}$$

b) A polynomial $p(x) \in P_4(\mathbb{R})$ that cannot be written as a linear combination of S (i.e. $p(x) \notin Span(S)$) is $p(x) = 5 - 2x^2$.

Let us try to see if $p(x) = 5 - 2x^2 \in Span(S)$. Because in the general form of a function $f \in Span(S)$ the only coefficient affecting the x^2 term is $-c_2$, $c_2 = 2$. Because the constant term 5 is given by $c_1 - c_2$, $c_1 = 7$ for $p(x) = 5 - 2x^2 \in Span(S)$. However, because this leads to a nonzero x term as $c_1 \neq 0$, $p(x) = 5 - 2x^2 \notin Span(S)$. Because $p(x) \notin Span(S)$ and $p(x) \in P_4(\mathbb{R})$, S does not generate $P_4(\mathbb{R})$.

c) The general form of the function $f \in P_4(\mathbb{R})$ is given by $f(x) = k_1 + k_2x + k_3x^2 + k_4x^3 + k_5x^4$ where $k_1, k_2, k_3, k_4, k_5 \in \mathbb{R}$. Through simple differentiation, we find that $f'(0) = k_2, f''(0) = 2k_3$. The set $\{f \in P_4(\mathbb{R}) : 2f(0) = 2f'(0) - f''(0)\}$ is equal to the set of all functions $f \in P_4(\mathbb{R})$ where:

$$2f(0) = 2f'(0) - f''(0)$$
$$2k_1 = 2k_2 - 2k_3$$
$$k_1 = k_2 - k_3$$

Re-expressing $f(x)$ with $k_1 = k_2 - k_3$ we get:

$$f(x) = (k_2 - k_3) + k_2x + k_3x^2 + k_4x^3 + k_5x^4$$

If we re-express our function above with $c_1 = k_2, c_2 = k_3, c_3 = k_4, c_4 = k_5$, we see that $f(x) = (c_1 - c_2) + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ where $c_1, c_2, c_3, c_4 \in \mathbb{R}$. This is the same form of a function $g \in Span(S)$. Thus, we have shown that the set $\{f \in P_4(\mathbb{R}) : 2f(0) = 2f'(0) - f''(0)\} = Span(S)$.