

# PSETs Landing Page\*

Anish Krishna Lakkapragada

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\*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

# MATH 241 PSET 8

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1.

- a) The PDF of  $T \sim \text{Expo}(\lambda)$  is given by  $f(x) = \lambda e^{-\lambda x}$  for  $x > 0$ . We define the half-life time as  $H$ . The half-life time is given as the time that  $P(T \leq H) = 0.5$ . We solve for  $H$  below:

$$\begin{aligned}P(T \leq H) &= 0.5 \\ \int_0^H f(x) dx &= 0.5 \\ \int_0^H \lambda e^{-\lambda x} dx &= 0.5 \\ \int_0^H e^{-\lambda x} dx &= \frac{1}{2\lambda} \\ -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^H &= \frac{1}{2\lambda} \\ -\frac{1}{\lambda} (e^{-\lambda H} - 1) &= \frac{1}{2\lambda} \\ e^{-\lambda H} - 1 &= -\frac{1}{2} \\ e^{-\lambda H} &= \frac{1}{2} \\ H &= \frac{-\ln(0.5)}{\lambda} = \frac{\ln(2)}{\lambda}\end{aligned}$$

- b) To ease our computations, we first find the CDF of  $T$ ,  $F(x) = \int_{-\infty}^x f(t) dt = \int_0^x \lambda e^{-\lambda t} dt = \lambda \int_0^x e^{-\lambda t} dt = -\frac{\lambda}{\lambda} e^{-\lambda x} \Big|_0^x = -1(e^{-\lambda x} - 1) = 1 - e^{-\lambda x}$  for  $x > 0$ . The probability a particle decays in the time interval  $[t, t + \epsilon]$ , given that it has survived since time  $t$ , is given by  $P(t \leq T \leq t + \epsilon | T > t)$ . Using Bayes Rule:

$$\begin{aligned}
P(t \leq T \leq t + \epsilon | T > t) &= \frac{P(t \leq T \leq t + \epsilon \cap T > t)}{P(T > t)} = \frac{P(t \leq T \leq t + \epsilon)}{P(T > t)} = \\
&= \frac{F(t + \epsilon) - F(t)}{1 - F(t)} = \frac{1 - e^{-\lambda(t+\epsilon)} - (1 - e^{-\lambda t})}{1 - (1 - e^{-\lambda t})} = \frac{-(e^{-\lambda t} e^{-\lambda \epsilon}) + e^{-\lambda t}}{e^{-\lambda t}} = 1 - e^{-\lambda \epsilon} = \\
&= 1 - (e^\epsilon)^{-\lambda} \approx 1 - (1 + \epsilon)^{-\lambda}
\end{aligned}$$

Using a first-degree Taylor series expansion of  $(1+\epsilon)^{-\lambda}$  about  $\epsilon \approx 0$ , we get  $(1+\epsilon)^{-\lambda} \approx (1+0)^{-\lambda} - \lambda\epsilon(1+0)^{-\lambda-1}$  or that  $(1+\epsilon)^{-\lambda} \approx 1 - \lambda\epsilon$ . Thus,  $P(t \leq T \leq t + \epsilon | T > t) \approx 1 - (1+\epsilon)^{-\lambda} \approx \lambda\epsilon$ , and so  $P(t \leq T \leq t + \epsilon | T > t)$  is approximately proportional to  $\epsilon$ . Furthermore, there is no  $t$  term present and so we have shown that this probability does not depend on  $t$ .

- c) From Example 5.6.3, we know that  $L \sim \text{Expo}(n\lambda)$ . The CDF of  $L$  can be given as, for  $x \geq 0$ ,  $F(x) = P(L \leq x) = 1 - P(L > x) = 1 - \prod_{i=1}^n P(T_i \geq x) = 1 - (e^{-\lambda x})^n = 1 - e^{-n\lambda x}$ . Furthermore, as we know  $L \sim \text{Expo}(n\lambda)$ ,  $E[L] = \frac{1}{n\lambda}$  and  $\text{Var}(L) = \frac{1}{(n\lambda)^2}$ .
- d) We can model this scenario as  $M = Z_1 + Z_2 + \dots + Z_n$ , where  $Z_i$  is the time for the  $i$ th particle to decay. Because the time for the  $i$ th particle to decay (i.e.  $Z_i$ ) is given as the minimum time to decay of all the  $n - i + 1$  particles which have not decayed,  $Z_i \sim \text{Expo}((n - i + 1)\lambda)$ . From my work in part (c), we know that  $E[Z_i] = \frac{1}{(n-i+1)\lambda}$ . Thus, we can compute the expectation of  $M$  below as:

$$\begin{aligned}
M &= \sum_{i=1}^n Z_i \\
E[M] &= \sum_{i=1}^n E[Z_i] \\
E[M] &= \sum_{i=1}^n \frac{1}{(n-i+1)\lambda} \\
E[M] &= \frac{1}{\lambda} \left( \frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{1} \right)
\end{aligned}$$

Thus,  $E[M] = \frac{H_n}{\lambda}$ , where  $H_n$  is the  $n$ th harmonic number.

Due to the memoryless property,  $Z_1, \dots, Z_n$  are all independent. From my work in part (c), we know  $\text{Var}(Z_i) = \frac{1}{(n-i+1)^2\lambda^2}$ . Thus, we can compute the  $\text{Var}(M)$  as such:

$$\begin{aligned}
M &= \sum_{i=1}^n Z_i \\
\text{Var}(M) &= \sum_{i=1}^n \text{Var}(Z_i) \\
\text{Var}(M) &= \sum_{i=1}^n \frac{1}{(n-i+1)^2\lambda^2} \\
\text{Var}(M) &= \frac{1}{\lambda^2} \left( \frac{1}{n^2} + \frac{1}{(n-1)^2} + \dots + 1 \right)
\end{aligned}$$

- a) Because  $X, Y$  are independent and identically distributed, distributions  $X^2$  and  $Y^2$  are as well and thus have the same MGF. Thus, we can compute the MGF of  $W = X^2 + Y^2$  as so:

$$M_W(t) = M_{X^2+Y^2}(t) = M_{X^2}(t) \cdot M_{Y^2}(t) = ((1-2t)^{-\frac{1}{2}})^2 = \frac{1}{1-2t} = \frac{0.5}{0.5-t}$$

- b) The distribution  $W$  has an MGF of the form of an Exponential Distribution MGF with  $\lambda = 0.5$ . Thus,  $W \sim Expo(0.5)$ .

3.

The MGF of the Geometric distribution is given by  $M(t) = \frac{p}{1-qe^t}$ . We first compute  $E[X]$  of this distribution below by applying the formula  $E[X^n] = M^{(n)}(0)$ .

$$\begin{aligned} E[X^1] &= M^{(1)}(0) \\ E[X] &= \left(\frac{p}{1-qe^t}\right)'(0) \\ E[X] &= \left(-\frac{p}{(1-qe^t)^2} \cdot (-qe^t)\right)(0) \\ E[X] &= \left(\frac{pqe^t}{(1-qe^t)^2}\right)(0) \\ E[X] &= \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p} \end{aligned}$$

Thus, we get that the mean of the distribution  $E[X] = \frac{q}{p}$ . We now use this same formula  $E[X^n] = M^{(n)}(0)$  to compute  $E[X^2]$ :

$$\begin{aligned} E[X^2] &= M^{(2)}(0) \\ E[X^2] &= \left(\frac{p}{1-qe^t}\right)''(0) \\ E[X^2] &= \left(\frac{pqe^t}{(1-qe^t)^2}\right)'(0) \\ E[X^2] &= \left(pq \left(\frac{e^t(1-qe^t)^2 + 2qe^{2t}(1-qe^t)}{(1-qe^t)^4}\right)\right)(0) \\ E[X^2] &= pq \frac{(1-q)^2 + 2q(1-q)}{(1-q)^4} = \frac{pq}{(1-q)^2} + \frac{2p^2q^2}{(1-q)^4} = \frac{pq}{p^2} + \frac{2p^2q^2}{p^4} = \frac{q}{p} + \frac{2q^2}{p^2} \end{aligned}$$

Given,  $E[X^2] = \frac{q}{p} + \frac{2q^2}{p^2}$  and  $E[X] = \frac{q}{p}$ , we now compute  $Var(X) = E[X^2] - (E[X])^2$  as:

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ Var(X) &= \frac{q}{p} + \frac{2q^2}{p^2} - \left(\frac{q}{p}\right)^2 = \frac{q}{p} + \frac{q^2}{p^2} = \frac{q}{p} + \frac{q(1-p)}{p^2} = \frac{q}{p} + \frac{q}{p^2} - \frac{qp}{p^2} = \frac{q}{p} + \frac{q}{p^2} - \frac{q}{p} = \frac{q}{p^2} \end{aligned}$$

4.

We compute the MGF of  $X \sim Expo(1)$  as  $M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} e^{-x} dx = \frac{1}{1-t}$  for  $t < 1$ . We can now compute the MGF of distribution  $-Y = -X$  as  $M_{-Y}(t) = E[e^{t(-Y)}] = E[e^{-tY}] = E[e^{-tX}] = M_X(-t)$  for  $-t < 1 \Rightarrow t > -1$ .

Because we are given that  $X$  and  $Y$  are independent,  $X$  and  $-Y$  are independent. Thus, the MGF of  $L = X + (-Y)$  is given by  $M_L(t) = M_X(t) \cdot M_{-Y}(t)$  for  $-1 < t < 1$ . We compute  $M_L(t)$  below for  $-1 < t < 1$

$$\begin{aligned} M_L(t) &= M_X(t) \cdot M_{-Y}(t) = M_X(t) \cdot M_X(-t) \\ M_L(t) &= \frac{1}{1-t} \cdot \frac{1}{1+t} = \frac{1}{1-t^2} \end{aligned}$$

To show that  $L$  has the Laplace distribution, we compute the MGF  $M_W(t)$  of the Laplace Distribution for  $-1 < t < 1$ <sup>1</sup>:

$$\begin{aligned} M_W(t) &= E[e^{tW}] = \int_{-\infty}^{\infty} e^{tw} f(w) dw = \frac{1}{2} \int_{-\infty}^{\infty} e^{tw-|w|} dw = \\ &= \frac{1}{2} \left[ \int_{-\infty}^0 e^{w(t+1)} dw + \int_0^{\infty} e^{w(t-1)} dw \right] = \frac{1}{2} \left[ \frac{e^{w(t+1)}}{t+1} \Big|_{-\infty}^0 + \frac{e^{w(t-1)}}{t-1} \Big|_0^{\infty} \right] \\ &= \frac{1}{2} \left[ \frac{1}{t+1} - \frac{1}{t-1} \right] = \frac{-2}{2(t^2-1)} = \frac{1}{1-t^2} \end{aligned}$$

Thus, because  $M_L(t) = M_W(t)$ , we have shown that distribution  $L$  is a Laplace distribution as it has the identical MGF (i.e.  $M_L(t)$ ) as the Laplace distribution MGF (i.e.  $M_W(t)$ ).

5.

- a) The MGF of a  $Bin(n, p)$  r.v. is  $M(t) = (pe^t + q)^n$ . So, the MGFs of distributions  $X_1$  and  $X_2$  are  $(pe^t + q)^{n_1}$  and  $(pe^t + q)^{n_2}$ , respectively. Because  $X_1$  and  $X_2$  are independent, the distribution  $X_1 + X_2$  has the MGF  $M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = (pe^t + q)^{n_1+n_2}$ . Because the distribution  $X_1 + X_2$  has the MGF of the distribution  $Bin(n_1 + n_2, p)$ ,  $X_1 + X_2 \sim Bin(n_1 + n_2, p)$ .
- b) The MGF of a  $Expo(\lambda)$  r.v. is given by  $M(t) = \frac{\lambda}{\lambda-t}$  for  $t < \lambda$ . So, the MGFs of distributions  $Y_1$  and  $Y_2$  are given by  $\frac{\lambda_1}{\lambda_1-t}$  for  $t < \lambda_1$  and  $\frac{\lambda_2}{\lambda_2-t}$  for  $t < \lambda_2$ , respectively. Let us define  $\lambda_s = \min(\lambda_1, \lambda_2)$ . Because  $Y_1$  and  $Y_2$  are independent, the distribution  $Y_1 + Y_2$  has the MGF  $M_{Y_1+Y_2}(t) = M_{Y_1}(t) \cdot M_{Y_2}(t) = \frac{\lambda_1 \lambda_2}{(\lambda_1-t)(\lambda_2-t)}$  for  $t < \lambda_s$ . As we can see, the MGF of  $Y_1 + Y_2$  does not have the form of the MGF of an Exponential distribution, and thus  $Y_1 + Y_2$  does not follow an Exponential distribution.

6. Anish Lakkapragada. I worked independently.

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<sup>1</sup>Note that this range of  $|t| < 1$  is required to ensure the below integrals in deriving the MGF of the Laplace Distribution converge.