

PSETs Landing Page*

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This is the documentation for using my PSET PDFs responsibly. I post these LaTeX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarly, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

Math 226- HW 4 Due: Sep 26 by Midnight

1. (21 points) Let V be a vector space over a field \mathbb{F} , $S \subseteq V$ is finite, and U and W be finite dimensional subspaces of V .

(a) (6 points) Let $\vec{x} \in V$. Show that

$$\dim(\text{Span}(S)) \leq \dim(\text{Span}(S \cup \{\vec{x}\})) \leq \dim(\text{Span}(S)) + 1.$$

b) (10 points) Prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Hint: Start with a basis for $\dim(U \cap W)$, then extend it to W , and U . Finally do some counting.

(c) (5 points) Prove that if

$$\dim(U + W) = 1 + \dim(U \cap W).$$

Then the sum of $U + W$ is equal to one of these spaces (either U or W) and the intersection $U \cap W$ is equal to the other one

2. (22 points) Let U, V be vector spaces over \mathbb{F} , and $T : U \rightarrow V$ be linear operator. Let B be basis of U and define $T(B) := \{T(u) : u \in B\}$.

(a) (10 points) Show that $T(B)$ is linearly independent if T is injective. Moreover, prove that if $T(B)$ is linearly independent and $\infty > |T(B)| \geq |B|$, then T is injective.

(b) (10 points) Show that T is surjective iff $T(B)$ spans V .

(c) (2 points) Prove that $T(B)$ is a basis for V if T is bijective. Moreover, if $T(B)$ is a basis and $\infty > |T(B)| \geq |B|$ then T is bijective.

3. (19 points) Recall that $C^1(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} : \text{such that } f'(x) \text{ exist for all } x \in \mathbb{R}\}$. Define T as

$$\begin{aligned} T : C^1(\mathbb{R}) &\rightarrow \mathbb{R}^2 \\ f(x) &\mapsto T(f) := (f'(3), f(3)) \end{aligned}$$

(a) (4 points) Show that T is a linear transformation.

(b) (8 points) Let $H := \{(x - 3)^2 g(x) : g(x) \in C^1(\mathbb{R})\}$, and $V := \{f : T(f) = (1, 2)\}$. Show that $H + V = V$.

Remark: This statement is true for any $V := \{f : T(f) = (a, b)\}$ where (a, b) is a fixed vector in \mathbb{R}^2 .

(c) (7 points) Deduce from part (b) that T is not injective. Is T surjective? Validate your answer.

4. (8 points) Let \mathbb{F} be a field with regular addition and multiplication.

a) (6 points) Let $T : \mathbb{F}^5 \rightarrow \mathbb{F}^4$ with

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3)$$

– Find a basis, and the dimension of $N(T)$. (Don't use elementary operation method for matrices)

- Find a basis, and the dimension of $R(T)$. (Don't use elementary operation method for matrices)

b) (2 points) Show that no linear map $T : \mathbb{F}^5 \rightarrow \mathbb{F}^2$ can have as its null space the set

$$N(T) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 7x_3, x_2 = x_4 = x_5\}$$

Practice Problems

(a) Suppose $S \subseteq V$ and $|S| = \dim(V)$. Prove that the followings are equivalent

- (i) S is a basis for V
- (ii) S spans V
- (iii) S is linearly independent

Hint: Use Exchange Theorem. To show a set of statements are equivalent you need to prove the statements in a cycle, i.e., (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).

(b) Recall $P(\mathbb{R}) = \{p(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n; n \in \mathbb{N}, c_i \in \mathbb{R}\}$. Prove that $P(\mathbb{R})$ is infinite dimensional.

Sec 1.6: 1-28