

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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**Problem set 2**  
**Due Thursday January 30 at 11pm**

**Exercise 2.1 (5 points; Rudin 1.5).** Let  $A \subset \mathbb{R}$  be nonempty and bounded below. Define

$$-A = \{-x \mid x \in A\}.$$

Prove that  $\inf A = -\sup(-A)$ .

**Exercise 2.2 (5 points).** Let  $A \subset \{x \in \mathbb{R} \mid x > 0\}$  be nonempty and bounded above. Define

$$A^{-1} = \{x^{-1} \mid x \in A\}.$$

Prove that  $\inf(A^{-1}) = (\sup A)^{-1}$ .

**Exercise 2.3 (5 points).** Suppose  $A, B \subset \mathbb{R}$  are both nonempty and bounded above. Define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Prove that  $\sup(A + B) = \sup A + \sup B$ .

**Exercise 2.4 (15 points).** In each of the following,  $S$  is an ordered set, and  $A \subset S$ . Answer the following in each case, and prove your answers:

- Is  $A$  bounded above?
- Does  $A$  have a maximum element, and if so, what is it?
- Does  $A$  have a supremum in  $S$ , and if so, what is it?

(1)  $S = \mathbb{Z}$ ,  $A = \{2, 3\}$ .

(2)  $S = \mathbb{Q}$ ,  $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\}$ .

(3)  $S = \mathbb{Q}$ ,  $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$ .

(4)  $S = \mathbb{Q}$ ,  $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ .

(5)  $S = \mathbb{Q}$ ,  $A = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$ .

(6)  $S = \mathbb{Q}$ ,  $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$ .

(7)  $S = \mathbb{Q}$ ,  $A = \{x \in \mathbb{Q} \mid 0 < x^3 < 2\}$ .

**Exercise 2.5 (10 points).** Suppose  $F$  is an ordered field and  $x, y, z \in F$ .

- (1) Prove that  $0 < 1$ .
- (2) Prove that if  $x > 0$ , then  $x^{-1} > 0$ .
- (3) Prove that if  $x > 0$ , then  $y > z$  if and only if  $xy > xz$ .
- (4) Recall we defined the field  $\mathbb{F}_3 = \{0, 1, 2\}$  on class. Prove that there does not exist an order on  $\mathbb{F}_3$  such that it is an ordered field.

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**Exercise 2.6 (optional, not for credit; Rudin 1.8 and 1.9).** Define the complex numbers as the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

with the addition and multiplication rules given by

- $(a + bi) + (a' + b'i) = (a + a') + (b + b')i$ ,
- $(a + bi)(a' + b'i) = (aa' - bb') + (ab' + a'b)i$ .

- (1) Prove that  $\mathbb{C}$  with these operations (and an appropriate 0, 1, negation, inversion) is a field.
- (2) Prove that  $\mathbb{C}$  with these operations cannot be made into an ordered field.
- (3) Define the *lexicographic order* on  $\mathbb{C}$  by defining

$$a + bi < c + di \iff a < c \text{ or } (a = c \text{ and } b < d).$$

Prove that this makes  $\mathbb{C}$  into an ordered *set* (though not an ordered *field*).

- (4) Does the lexicographic order on  $\mathbb{C}$  have the least upper bound property?