

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# MATH 255 PSET 7

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1.

Before solving this question, note that:

$$\begin{aligned}\sqrt{n^2+n}-n &= (\sqrt{n^2+n}-n) \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{(n^2+n)-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} \\ &= \frac{n}{\sqrt{n^2+n}+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{1}{\sqrt{\frac{n^2+n}{n^2}}+1} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}\end{aligned}$$

We first calculate  $\lim_{n \rightarrow \infty} \sqrt{n^2+n}-n$ . We use the fact that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

$$\lim_{n \rightarrow \infty} \sqrt{n^2+n}-n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

We now prove that  $\lim_{n \rightarrow \infty} \sqrt{n^2+n}-n = \frac{1}{2}$ . Pick  $\epsilon > 0$ . To show  $\lim_{n \rightarrow \infty} \sqrt{n^2+n}-n = \frac{1}{2}$ , we want to find  $N \in \mathbb{N}$  s.t.  $\forall n \geq N, d(\sqrt{n^2+n}-n, \frac{1}{2}) < \epsilon$  or more simply s.t.  $\forall n \geq N, |\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}| < \epsilon$ . Thus, we solve the  $|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}| < \epsilon$  inequality below:

$$|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}| < \epsilon$$

Note that because  $n \in \mathbb{N} \implies \frac{1}{n} > 0 \implies 1 + \frac{1}{n} > 1 \implies \sqrt{1+\frac{1}{n}} > 1 \implies 1 + \sqrt{1+\frac{1}{n}} > 2 \implies \frac{1}{1+\sqrt{1+\frac{1}{n}}} < \frac{1}{2}$ . Thus,  $|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}| = \frac{1}{2} - \frac{1}{1+\sqrt{1+\frac{1}{n}}}$ :

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$$\frac{1}{2} - \frac{1}{1 + \sqrt{1 + \frac{1}{n}}} < \epsilon$$

$$\frac{1}{2} - \epsilon < \frac{1}{1 + \sqrt{1 + \frac{1}{n}}}$$

$$1 > (1 + \sqrt{1 + \frac{1}{n}})(0.5 - \epsilon)$$

$$1 - (0.5 - \epsilon) > (0.5 - \epsilon)\sqrt{1 + \frac{1}{n}}$$

$$\frac{0.5 + \epsilon}{0.5 - \epsilon} > \sqrt{1 + \frac{1}{n}}$$

$$\left(\frac{0.5 + \epsilon}{0.5 - \epsilon}\right)^2 > 1 + \frac{1}{n}$$

$$\frac{1}{n} < \left(\frac{0.5 + \epsilon}{0.5 - \epsilon}\right)^2 - 1$$

$$n > \frac{1}{\left(\frac{0.5 + \epsilon}{0.5 - \epsilon}\right)^2 - 1}$$

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Thus we see  $d(\sqrt{n^2 + n} - n, \frac{1}{2}) < \epsilon$  for any  $n > \frac{1}{(\frac{0.5 + \epsilon}{0.5 - \epsilon})^2 - 1}$ . Thus we aim to choose  $N$  as any natural number  $> \frac{1}{(\frac{0.5 + \epsilon}{0.5 - \epsilon})^2 - 1}$ . By Archimedian property,  $\exists m \in \mathbb{N}$  s.t.  $m(1) > \frac{1}{(\frac{0.5 + \epsilon}{0.5 - \epsilon})^2 - 1}$  and so we can simply choose  $N = m$ . Thus, we have shown  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N, d(\sqrt{n^2 + n} - n, \frac{1}{2}) < \epsilon$  and so we have proven  $\lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n = \frac{1}{2}$ .

2.

**Lemma 0.1** Given  $(a_n), (b_n)$  as bounded sequences in  $\mathbb{R}$ , we prove if some subsequence of  $(a_n + b_n)$  converges to  $x \implies$  the limit of some subsequence of  $a_n$  plus the limit of some subsequence of  $b_n$  is equal to  $x$ .

*Proof:* Let us define this subsequence of  $(a_n + b_n)$  that converges to  $x$  as a sequence given by  $a_{n_1} + b_{n_1}, a_{n_2} + b_{n_2}, \dots$  where  $n_1 < n_2 < \dots$ . In other words, this subsequence is the sequence  $(a_n + b_n)$  indexed by the monotonically increasing sequence  $(n_k)$ . Given that this sequence  $a_{n_k} + b_{n_k} \rightarrow x$ , we know:

$$\lim_{k \rightarrow \infty} (a_{n_k} + b_{n_k}) = \lim_{k \rightarrow \infty} a_{n_k} + \lim_{k \rightarrow \infty} b_{n_k}$$

$$x = \lim_{k \rightarrow \infty} a_{n_k} + \lim_{k \rightarrow \infty} b_{n_k}$$

Thus, we have shown that the limit of some subsequence of  $a_n$ , given by  $\lim_{k \rightarrow \infty} a_{n_k}$ , plus the limit of some subsequence of  $b_n$ , given by  $\lim_{k \rightarrow \infty} b_{n_k}$ , is equal to  $x$ .

**Lemma 0.2** Let us define sets  $X, Y$  where  $X \subset Y$ . Then  $\sup(X) \leq \sup(Y)$ .

*Proof:*  $\sup(Y)$  is the lowest upper bound of  $Y$  and because  $X \subset Y \implies \sup(Y)$  is an upper bound of  $X$ . However,  $\sup(X)$  is the lowest upper bound of  $X$  and so  $\sup(X) \leq \sup(Y)$ .

Let us define sets  $A$  and  $B$  below as such:

$$A = \{x \in \mathbb{R}_{\text{ext}} \mid \text{some subseq of } (a_n) \text{ converges to } x\}$$

$$B = \{x \in \mathbb{R}_{\text{ext}} \mid \text{some subseq of } (b_n) \text{ converges to } x\}$$

We are given that  $(a_n), (b_n)$  are bounded real sequences  $\implies$  sequences  $(a_n), (b_n)$  are bounded above and below  $\implies$  sequences  $(a_n), (b_n)$  and their subsequences cannot converge to  $\pm\infty \implies A \subset \mathbb{R}$  and  $B \subset \mathbb{R}$ . Furthermore, this means the sequence  $(a_n + b_n)$  is bounded<sup>1</sup> and so we can define the following set:

$$A + B = \{x + y \in \mathbb{R} \mid \text{some subseq of } (a_n) \rightarrow x \text{ and some subseq of } (b_n) \rightarrow y\} \subset \mathbb{R}$$

As proved in Homework 2, because  $A, B, A + B \subset \mathbb{R}$ ,  $\sup(A + B) = \sup(A) + \sup(B) = \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n)$ . Now consider the set below:

$$C = \{x \in \mathbb{R} \mid \text{some subseq of } (a_n + b_n) \rightarrow x\}$$

As proved in **Lemma 0.1**, if some subseq of  $(a_n + b_n) \rightarrow x \implies$  the limit of some subsequence of  $a_n$  plus the limit of some subsequence of  $b_n$  is equal to  $x$ . Thus,  $C \subset A + B \implies$  (using **Lemma 0.2**)  $\sup(C) \leq \sup(A + B) = \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n) \implies \sup(C) \leq \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n)$ . Note that  $\sup(C) = \limsup_{n \rightarrow \infty} (a_n + b_n)$  and so we have proven:

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n)$$

**Example of**  $\limsup_{n \rightarrow \infty} (a_n + b_n) = \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n)$

We can just set  $(a_n) = (b_n) = 1, 1, 1, 1, \dots$ . In this case  $\limsup_{n \rightarrow \infty} (a_n + b_n) = 2$  and  $\limsup_{n \rightarrow \infty} (a_n) = \limsup_{n \rightarrow \infty} (b_n) = 1$ . So  $\limsup_{n \rightarrow \infty} (a_n + b_n) = 2 = \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n)$ .

**Example of**  $\limsup_{n \rightarrow \infty} (a_n + b_n) < \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n)$

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<sup>1</sup>Proof: WLOG, we show  $(a_n + b_n)$  is bounded above. If  $(a_n)$  is bounded  $\implies (a_n)$  is bounded above  $\implies \exists z_a \in \mathbb{R}$  s.t.  $\forall n \in \mathbb{N}, z_a \geq a_n$ . We can define  $z_b \in \mathbb{R}$  similarly as an upper bound for  $(b_n)$ . Thus,  $\forall n \in \mathbb{N}, a_n + b_n \leq z_a + z_b \implies z_a + z_b \in \mathbb{R}$  is an upper bound for  $(a_n + b_n) \implies (a_n + b_n)$  is bounded above. This shows  $(a_n + b_n)$  is bounded above (and with identical logic bounded below)  $\implies (a_n + b_n)$  is bounded.

We can define sequence  $(a_n)$  where  $a_n = \frac{1+(-1)^n}{2}$  and sequence  $(b_n)$  where  $b_n = \frac{-1-(-1)^n}{2}$ . Writing out the elements of  $(a_n)$  and  $(b_n)$  out we get:

$$\begin{aligned}(a_n) &= 0, 1, 0, 1, \dots \\ (b_n) &= 0, -1, 0, -1, \dots\end{aligned}$$

So sequence  $(a_n + b_n) = 0, 0, 0, 0, \dots$  and so  $\limsup_{n \rightarrow \infty} (a_n + b_n) = 0$ . However, there exists a subsequence of  $(a_n)$  given by  $1, 1, 1, \dots$  and so  $\limsup_{n \rightarrow \infty} (a_n) = 1$ . There also exists a subsequence of  $(b_n)$  given by  $0, 0, 0, \dots$  and so  $\limsup_{n \rightarrow \infty} (b_n) = 0$ . Thus,  $\limsup_{n \rightarrow \infty} (a_n + b_n) = 0 < \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n) = 1 + 0 = 1$ .

3.

Fix  $\epsilon > 0$  and define  $\epsilon' = \frac{\epsilon}{2} > 0$ . Given  $(p_n), (q_n)$  are Cauchy sequences,  $\exists N_p \in \mathbb{N}$  s.t.  $\forall n, m \geq N_p, d(p_n, p_m) < \epsilon'$  and  $\exists N_q \in \mathbb{N}$  s.t.  $\forall n, m \geq N_q, d(q_n, q_m) < \epsilon'$ . Set  $N = \max(N_p, N_q) \in \mathbb{N}$ . Then,  $\forall n, m \geq N, d(p_n, p_m) < \epsilon'$  and  $d(q_n, q_m) < \epsilon'$ . Thus, applying Triangle Inequality,  $\forall n, m \geq N$ :

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n)$$

By Triangle Inequality,  $d(p_m, q_n) \leq d(p_m, q_m) + d(q_m, q_n)$  and so:

$$\begin{aligned}d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ d(p_n, q_n) - d(p_m, q_m) &\leq d(p_n, p_m) + d(q_m, q_n)\end{aligned}$$

We know  $d(p_n, p_m) < \epsilon'$  and  $d(q_m, q_n) = d(q_n, q_m) < \epsilon'$  and so:

$$d(p_n, q_n) - d(p_m, q_m) < \epsilon' + \epsilon' = \epsilon$$

Note that we could also start with:  $d(p_m, q_m) \leq d(p_m, p_n) + d(p_n, q_m)$  to show through identical logic  $d(p_m, q_m) - d(p_n, q_n) < \epsilon$ . Given  $d(p_n, q_n) - d(p_m, q_m) < \epsilon$  and  $d(p_m, q_m) - d(p_n, q_n) < \epsilon \implies |d(p_n, q_n) - d(p_m, q_m)| < \epsilon$ .

Thus, we have shown  $\forall \epsilon > 0, \exists N' \in \mathbb{N}$  s.t.  $\forall n, m \geq N', d_{\mathbb{R}}(d(p_n, q_n), d(p_m, q_m)) < \epsilon$  where  $d_{\mathbb{R}}$  is the standard distance function in metric space  $\mathbb{R}$  (i.e.  $d_{\mathbb{R}}(x, y) = |x - y|$ ). This proves that the sequence  $(d(p_n, q_n))$  in  $\mathbb{R}$  is a Cauchy Sequence. Because all Cauchy Sequences in  $\mathbb{R}$  converge, this proves that the sequence  $(d(p_n, q_n))$  converges  $\implies (d(p_n, q_n))$  has a limit.

4.

**Lemma 0.3** We prove that if  $b \in \mathbb{R}$  is an effective upper bound (EUB) of  $(x_n)$ ,  $b \geq \limsup_{n \rightarrow \infty} x_n$ .

*Proof:* Suppose we have a subsequence of  $(x_n)$  that converges to  $x$ . Then  $\forall \epsilon > 0, \exists N_x \in \mathbb{N}$  s.t.  $\forall n \geq N_x, d(x_n, x) = |x_n - x| < \epsilon \implies x_n - x > -\epsilon \implies x < x_n + \epsilon$ . Now consider a given EUB  $b \in \mathbb{R}$  of  $(x_n)$ : by definition of EUB,  $\exists N_b \in \mathbb{N}$  s.t.  $\forall n \geq N_b, x_n \leq b$ . Thus  $\forall \epsilon > 0, \exists M = \max(N_x, N_b)$  s.t.  $\forall n \geq M, x < x_n + \epsilon$  and  $x_n \leq b$ . Thus this means  $\forall \epsilon > 0, \exists M \in \mathbb{N}$  s.t.  $\forall n \geq M, x < x_n + \epsilon \leq b + \epsilon$ . Note that  $x$  and  $b$  are constants (independent of  $n$ ) and so this proof shows  $\implies \forall \epsilon > 0, x < b + \epsilon \implies x \leq b$  (see footnote<sup>2</sup>).

Let us define the set  $A = \{x \in \mathbb{R}_{\text{ext}} \mid \text{some subsequence of } (x_n) \text{ converges to } x\}$ . We show in the following two cases,  $\limsup_{n \rightarrow \infty} x_n = \sup(A) \leq b$ .

1. **Case One:**  $A = \emptyset$

If  $A = \emptyset \implies \forall x \in \mathbb{R}, x$  is vacuously an upper bound of  $A$ . So EUB  $b \in \mathbb{R}$  is an upper bound of  $A$ . However, because  $\limsup_{n \rightarrow \infty} x_n = \sup(A)$  is the lowest upper bound of  $A \implies \limsup_{n \rightarrow \infty} x_n \leq b$ .

2. **Case Two:**  $A \neq \emptyset$

As previously established, the limit of any convergent subsequence of  $(x_n)$  is  $\leq b$ . So,  $\forall a \in A, a \leq b$ . Thus,  $b$  is an upper bound of  $A$ . However by definition  $\limsup_{n \rightarrow \infty} x_n = \sup(A)$  is the lowest upper bound of  $A \implies \limsup_{n \rightarrow \infty} x_n \leq b$ .

We prove this statement through casework on  $\limsup_{n \rightarrow \infty} x_n$ :

1. **Case One:**  $\limsup_{n \rightarrow \infty} x_n \in \mathbb{R}$

We first show that  $\forall \epsilon > 0, \limsup_{n \rightarrow \infty} x_n + \epsilon$  is an upper bound. This follows directly from Proposition 6.37:  $\limsup_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n + \epsilon \implies \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N, x_n < \limsup_{n \rightarrow \infty} x_n + \epsilon \implies \limsup_{n \rightarrow \infty} x_n + \epsilon$  is an EUB of  $(x_n)$ . Furthermore, note that by implication of **Lemma 0.3**,  $\nexists$  a real-valued EUB  $< \limsup_{n \rightarrow \infty} x_n$ . So, the set  $\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}$  can be given by the interval  $(\limsup_{n \rightarrow \infty} x_n, \infty) \subset \mathbb{R}$  or alternatively the set  $\{\limsup_{n \rightarrow \infty} x_n + \epsilon : \forall \epsilon > 0\}$ .

The greatest lower bound of this set<sup>34</sup> is  $\limsup_{n \rightarrow \infty} x_n$  and so we have proven  $\limsup_{n \rightarrow \infty} x_n = \inf\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}$ .

<sup>2</sup>For more clarity on the statement  $\forall \epsilon > 0, x < b + \epsilon \implies x \leq b$ , we can think of the maximum possible value of  $x$  as  $\inf\{b + \epsilon : \epsilon > 0\}$ . The infimum of this set is obviously  $b$  ( $b$  is a lower bound of this set and any real number greater than  $b$  fails to be a lower bound of this set so  $b$  is the greatest lower bound) and so we have that the largest value  $x$  can hold is  $b \implies x \leq b$ .

<sup>3</sup>Note that it does not matter if  $\limsup_{n \rightarrow \infty} x_n$  is included in this set of real-valued EUBs of  $(x_n)$ . This is because the supremum of this set would have been  $\limsup_{n \rightarrow \infty} x_n$  regardless.

<sup>4</sup>The proof for this is trivial:  $\limsup_{n \rightarrow \infty} x_n$  is a lower bound of this set and any number slightly greater than

## 2. Case Two: $\limsup_{n \rightarrow \infty} x_n \notin \mathbb{R}$

By definition  $\limsup_{n \rightarrow \infty} x_n \in \mathbb{R}_{\text{ext}}$ . Given  $\limsup_{n \rightarrow \infty} x_n \notin \mathbb{R} \implies \limsup_{n \rightarrow \infty} x_n = \pm\infty$ .

We consider each case below:

(a)  $\limsup_{n \rightarrow \infty} x_n = \infty$

As proven in **Lemma 0.3**, all EUBs must be  $\geq \limsup_{n \rightarrow \infty} x_n$ . Thus  $\limsup_{n \rightarrow \infty} x_n = \infty \implies (x_n)$  cannot have any EUBs in  $\mathbb{R}$  as  $\nexists x \in \mathbb{R}$  s.t.  $x \geq \limsup_{n \rightarrow \infty} x_n = \infty$ .

Thus, the set  $\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\} = \emptyset$  and because<sup>5</sup>  $\inf(\emptyset) = \infty = \limsup_{n \rightarrow \infty} x_n$  we have proven:

$$\limsup_{n \rightarrow \infty} x_n = \inf\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}$$

(b)  $\limsup_{n \rightarrow \infty} x_n = -\infty$

Because  $\limsup_{n \rightarrow \infty} x_n = -\infty, \forall x \in \mathbb{R}, x > \limsup_{n \rightarrow \infty} x_n \implies$  (through Proposition 6.37)  $\forall x \in \mathbb{R}, x$  is an EUB for  $(x_n)$ . Thus the set  $\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\} = \mathbb{R}$  and so the only lower bound (and only greatest lower bound) of this set is  $-\infty \implies \inf(\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}) = -\infty = \limsup_{n \rightarrow \infty} x_n$ . So we have again proven:

$$\limsup_{n \rightarrow \infty} x_n = \inf\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}$$

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it (can be given as  $\limsup_{n \rightarrow \infty} x_n + \epsilon$  where  $\epsilon > 0$ ) fails to be a lower bound for this set  $\implies \limsup_{n \rightarrow \infty} x_n$  is the greatest lower bound of this set.

<sup>5</sup>The following statement reflects the fact that the infimum is the greatest lower bound. Because  $\forall x \in \mathbb{R}_{\text{ext}}$ ,  $x$  is a lower bound for  $\emptyset$ , the greatest lower bound of  $\emptyset$  is the greatest value in  $\mathbb{R}_{\text{ext}}$  or  $\infty$ .