

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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## Math 226: HW 6

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1. a) We evaluate  $(AB)^t$  first. Defining  $AB = [c_{ij}]_{m \times k} \in M_{m \times k}(\mathbb{F})$ , we get that  $c_{ij} = \sum_{l=1}^N \alpha_{il} \beta_{lj}$  through the definition of matrix multiplication. Transposing this matrix, we get that  $(AB)^t = [d_{ij}]_{k \times m} \in M_{k \times m}(\mathbb{F})$ , where:

$$d_{ij} = c_{ji} = \sum_{l=1}^n \alpha_{jl} \beta_{li}$$

We now compute  $B^t A^t$ .  $B^t = [\beta'_{ij}]_{k \times n} \in M_{k \times n}(\mathbb{F})$  is given by  $\beta'_{ij} = \beta_{ji}$  through definition of the transpose operation. Similarly,  $A^t = [\alpha'_{ij}]_{n \times m} \in M_{n \times m}(\mathbb{F})$  is given by  $\alpha'_{ij} = \alpha_{ji}$ . The matrix multiplication of  $B^t$  and  $A^t$  is given by  $B^t A^t = [c'_{ij}]_{k \times m} \in M_{k \times m}(\mathbb{F})$ , where:

$$c'_{ij} = \sum_{l=1}^n \beta'_{il} \alpha'_{lj} = \sum_{l=1}^n \alpha_{jl} \beta_{li}$$

Because we have shown that  $(AB)^t = [d_{ij}]_{k \times m}$  and  $B^t A^t = [c'_{ij}]_{k \times m}$  are of the same dimension and that  $d_{ij} = c'_{ij}$ , we have shown that  $(AB)^t = B^t A^t$ .

- b) (1) **Prove if  $P$  is invertible, then  $P^t$  is invertible.**

If matrix  $P$  is invertible, there exists some  $n \times n$  matrix  $A$  s.t.  $PA = AP = I$ , where  $I$  is the identity matrix. Applying the transpose operation on both sides of the equation  $PA = I$ , we get that  $(PA)^t = I^t = I$ . Using part (a), this means that  $A^t P^t = I$ . Similarly, applying the tranpose operation on both sides of the equation  $AP = I$  gives us  $(AP)^t = I \Rightarrow P^t A^t = I$ . Thus, for matrix  $P^t$  we have found a matrix  $A^t$  s.t.  $A^t P^t = P^t A^t = I \Rightarrow P^t$  has a matrix inverse  $A^t \Rightarrow P^t$  is invertible.

- (2) **Prove  $(P^t)^{-1} = (P^{-1})^t$ .** This is shown in (1). The inverse of matrix  $P^t$  is  $A^t$ , which is equal to the inverse of  $P$ , matrix  $A$ , tranposed. Thus  $(P^t)^{-1} = A^t = (P^{-1})^t$ .

- c) Because matrices  $P$  and  $Q$  are both  $n \times n$  matrices,  $PQ$  is also an  $n \times n$  matrix. In part (b) we proved that if an  $n \times n$  matrix is invertible, its transpose is invertible. Applying part (b), we know that if matrix  $PQ$  is invertible, then  $(PQ)^t$  is invertible. In part (b), we also proved that if an  $n \times n$  matrix was invertible, the inverse of its transpose is equal to the transpose of its inverse:  $(P^t)^{-1} = (P^{-1})^t$ . Replacing  $P$  for matrix  $PQ$  in this equation, we get that:

$$\begin{aligned} ((PQ)^t)^{-1} &= ((PQ)^{-1})^t \\ ((PQ)^t)^{-1} &= (Q^{-1}P^{-1})^t \\ ((PQ)^t)^{-1} &= (P^{-1})^t(Q^{-1})^t \end{aligned}$$

2. a) By the definition of matrix multiplication,  $AB = [c_{ij}]_{n \times n}$  where  $c_{ij} = \sum_{l=1}^N \alpha_{il} \beta_{lj}$  and  $BA = [d_{ij}]_{n \times n}$  where  $d_{ij} = \sum_{l=1}^N \beta_{il} \alpha_{lj}$ .

We now compute  $Tr(AB)$  and  $Tr(BA)$  below:

$$\begin{aligned} Tr(AB) &= \sum_{i=1}^N c_{ii} = \sum_{i=1}^N \sum_{l=1}^N \alpha_{il} \beta_{li} \\ Tr(BA) &= \sum_{i=1}^N d_{ii} = \sum_{i=1}^N \sum_{l=1}^N \beta_{il} \alpha_{li} \end{aligned}$$

Note that for our calculation of  $Tr(BA)$  above, if we rename index  $i$  as  $l$  and vice versa, we get  $Tr(BA) = \sum_{l=1}^N \sum_{i=1}^N \alpha_{il} \beta_{li} = \sum_{i=1}^N \sum_{l=1}^N \alpha_{il} \beta_{li} = Tr(AB)$ .

- b) If  $A$  and  $B$  are similar matrices, this means that there exists an invertible matrix  $Q$  s.t.  $B = Q^{-1}AQ$ . We prove  $Tr(A) = Tr(B)$  below.

$$\begin{aligned} Tr(A) &= Tr(B) \\ Tr(A) &= Tr(Q^{-1}AQ) \end{aligned}$$

By definition of matrix inverse,  $QQ^{-1} = I$ . Thus, we re-express  $A$  as  $A = AI = AQQ^{-1}$ . Using this, we get:

$$Tr(AQQ^{-1}) = Tr(Q^{-1}AQ)$$

From part (a), we proved that if  $C, D$  are  $n \times n$  matrices, then  $Tr(CD) = Tr(DC)$ . Defining matrix  $C = AQ \in M_{n \times n}(\mathbb{F})$  and  $D = Q^{-1} \in M_{n \times n}(\mathbb{F})$ , we know from part (a)  $Tr(CD) = Tr(DC) \Rightarrow Tr(AQQ^{-1}) = Tr(Q^{-1}AQ)$ .

3. **① Show that if  $A$  is a  $n \times 1$  matrix and  $B$  is a  $n \times 1$  matrix,  $AB$  has a rank of at most 1.**

We inspect the value of  $AB$  below:

$$\begin{aligned} AB &= AB \\ AB &= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & \dots & a_1 b_n \\ \vdots & \ddots & \vdots \\ a_n b_1 & \dots & a_n b_n \end{bmatrix} \end{aligned}$$

Note that every single column vector of  $AB$  is given by  $b_j(a_1, \dots, a_n)$ , where  $j$  is the column position of this column vector. Thus, all column vectors of  $AB$  are scalar multiples of each other and so there can be at most one linearly independent column vector in  $AB$  (i.e. we can treat one arbitrary column vector of  $AB$  as linear independent and the rest as scalar multiples of it). Because the rank of a matrix is given by the number of linearly independent columns it possesses, the rank of  $AB$  is at most one.

- ② If  $C$  is any  $n \times n$  matrix having rank 1, then there exist  $n \times 1$  matrix  $A$ , and  $1 \times n$  matrix  $B$  such that  $C = AB$ .**

If matrix  $C = [c_{ij}]_{n \times n}$  has a rank 1, that means that there is only one column vector which can be considered linearly independent (i.e. non-zero) and that the rest of the column vectors can be represented as scalar multiples of this linearly independent column vector<sup>1</sup>. Let us arbitrarily choose the single linearly independent column vector in  $C$  as  $c_1$ , the first column of  $C$ <sup>2</sup>. A given column vector  $c_j$  in the  $j$ th column of  $C$  can be expressed as a scalar multiple of  $c_1$ :  $c_j = \lambda_j c_1$ . Note that for  $j = 1$ ,  $\lambda_j = 1$ .

The  $n \times 1$  matrix  $A$  can be given by  $c_1$  and the  $1 \times n$  matrix  $B$  can be given by the values  $\lambda_1, \lambda_2, \dots, \lambda_n$ . We show below that  $C = AB$ :

<sup>1</sup>In other words, the dimension of span of all the column vectors is one.

<sup>2</sup>Note that our column choice for this vector does not matter as the rank of a matrix is unaffected by the order of its column vectors. We choose  $c_1$  in this proof for convenience.

$$C = AB$$

$$C = c_1 \begin{bmatrix} \lambda_1 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} c_{11}\lambda_1 & \dots & c_{11}\lambda_n \\ c_{21}\lambda_1 & \dots & c_{21}\lambda_n \\ \vdots & \ddots & \vdots \\ c_{n1}\lambda_1 & \dots & c_{n1}\lambda_n \end{bmatrix}$$

As we can see above,  $C$  expresses a general form for any  $n \times n$  matrix which has a rank of 1 (i.e.  $C$  has only one linearly independent column  $c_1$  and the remaining column vectors can be composed as scalar multiples of  $c_1$ ). As shown above, such a matrix  $C$  can be decomposed into a matrix multiplication of matrices  $A \in M_{n \times 1}(\mathbb{F})$  and  $B \in M_{1 \times n}(\mathbb{F})$ .

4. a)  $v_1 \in \mathbb{R}^2$  s.t.  $T(v_1) = v_1$  can be given by  $v_1 = (1, m)$ .  $v_2 \in \mathbb{R}^2$  s.t.  $T(v_2) = -v_2$  can be given by  $v_2 = (-m, 1) \in \mathbb{R}^2$ .  
b)

$$[T]_{\beta'} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

c)

$$Q = [I_{\mathbb{R}^2}]_{\beta}^{\beta'} = \begin{bmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{-m}{m^2+1} & \frac{1}{m^2+1} \end{bmatrix}; Q^{-1} = [I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}$$

d) We first compute  $[T]_{\beta} = Q^{-1}[T]_{\beta'}Q$  below:

$$[T]_{\beta} = Q^{-1}[T]_{\beta'}Q$$

$$[T]_{\beta} = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{-m}{m^2+1} & \frac{1}{m^2+1} \end{bmatrix}$$

$$[T]_{\beta} = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$$

Given  $v = (x, y) \in \mathbb{R}^2$ ,  $T(v) = [T]_{\beta}[v]_{\beta}$ . Thus, we get the expression  $T(v)$  as:

$$T(v) = [T]_{\beta}[v]_{\beta} = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \frac{1-m^2}{m^2+1} + \frac{2my}{m^2+1} \\ \frac{2mx}{m^2+1} + y \frac{m^2-1}{m^2+1} \end{bmatrix}$$