Problem set 5
Due Friday February 20 at 11pm

Exercise 5.1 (5 points). Show that the subset of $\mathbb{R}^2$ given by

$$E = \{(x,y) \in \mathbb{R}^2 \mid x < y\}$$

is open.

Exercise 5.2 (5 points). Show that the union of a finite number of compact sets is compact.

Exercise 5.3 (5 points; Rudin 2.14). Prove directly that the interval $(0,1) \subset \mathbb{R}$ is not compact, by giving an example of an open cover of $(0,1)$ which has no finite subcover. Include a proof that your cover has no finite subcover.

Exercise 5.4 (20 points; Rudin 2.19). If A and B are subsets of a metric space X, we say A and B are *separated* if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$. (We used this notion when we defined connectedness.)

(1) If A and B are disjoint closed sets in some metric space X, prove that A and B are separated.
(2) Prove the same for disjoint open sets.
(3) Fix $p \in X$ and $\delta > 0$. Define $A = \{q \in X \mid d(p,q) < \delta\}$. Define $B = \{q \in X \mid d(p,q) > \delta\}$. Prove that A and B are separated.
(4) Prove that every connected metric space with at least two points is uncountable. (Hint: use the previous part.)

Exercise 5.5 (5 points; Rudin 2.22, modified). Given a metric space X and a set $E \subset X$, we say E is *dense* in X if $\overline{E} = X$. Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 5.6 (not for credit; Rudin 2.8). Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.