Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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## Problem set 7

**Exercise 7.1 (10 points; Rudin 3.2, modified).** Calculate $\lim_{n\to\infty} \sqrt{n^2+n} - n$, and prove that your answer is correct. (Hint: first show that $\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}$.)

**Exercise 7.2 (10 points).** For any two bounded real sequences $(a_n)$, $(b_n)$ in $\mathbb{R}$ prove that

$$\lim_{n\to\infty} \sup(a_n + b_n) \le \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n.$$

Give an example where this $\le$ is $<$, and an example where it is $=$.

**Exercise 7.3 (10 points; Rudin 3.24).** Suppose $(p_n)$ and $(q_n)$ are Cauchy sequences in a metric space $X$. Prove that the sequence $(d(p_n, q_n))$ in $\mathbb{R}$ has a limit.

**Exercise 7.4 (10 points).** Suppose $(x_n)$ is a sequence in $\mathbb{R}$. We say $a \in \mathbb{R}$ is an *essential upper bound* for $(x_n)$ if there exists some $N$ such that, for all $n \ge N$, $x_n \le a$.

Prove that

$$\lim_{n\to\infty} \sup x_n \;=\; \inf\{a \in \mathbb{R} \mid a \text{ is an essential upper bound for } (x_n)\}\,.$$

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**Exercise 7.5 (not for credit; Rudin 3.25, in part).** Let $X$ be a metric space. $X$ might or might not be complete. If $X$ is not complete, it would be nice to know how to "fill in the holes" to make it complete. This exercise explains a way of doing so: it constructs a new complete metric space $X^*$ which has $X$ as a subset.

We call two Cauchy sequences $(p_n)$, $(q_n)$ in $X$ *equivalent* if $d(p_n, q_n) \to 0$. We write this relation as $(p_n) \sim (q_n)$.

(1) Prove that this is an equivalence relation, i.e.
   (a) Any Cauchy sequence $(p_n)$ has $(p_n) \sim (p_n)$,
   (b) If $(p_n) \sim (q_n)$ then $(q_n) \sim (p_n)$,
   (c) If $(p_n) \sim (q_n)$ and $(q_n) \sim (r_n)$, then $(p_n) \sim (r_n)$.

(2) If $(p_n) \sim (p'_n)$ and $(q_n) \sim (q'_n)$, prove that

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)\,.$$

(Note that the limit does exist, by the result of the previous exercise.)

Now we divide the set of Cauchy sequences in $X$ into *equivalence classes*: any two elements of a given class $P$ are equivalent, and elements of different classes $P$, $Q$ are not equivalent. Let $X^*$ be the set of all equivalence classes of Cauchy sequences in $X$.

Then, define a distance function $\Delta$ on $X^*$ as follows: if $(p_n)$ is in the class $P$, and $(q_n)$ is in the class $Q$, then

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)\,.$$

The previous parts show that this distance function is well defined.

(3) Prove that the distance function $\Delta$ makes $X^*$ into a metric space.

(4) Prove that $X^*$ with this distance function is complete.

(5) Consider the map $\phi : X \to X^*$ which maps any $x \in X$ to the class of the Cauchy sequence $(x, x, x, \dots)$. Prove that $\phi$ is injective and $\Delta(\phi(x), \phi(y)) = d(x, y)$.

**Exercise 7.6 (not for credit).** Suppose $X$ is any metric space, with a distance function $d$. Then define a new distance function $d'$ on $X$ by

$$d'(x, y) = \min\{d(x, y), 1\}\,.$$

(1) Prove that $d'$ indeed makes $X$ into a metric space.

(2) Prove that a sequence is Cauchy for $d$ if and only if it is Cauchy for $d'$.

(3) Prove that a sequence is convergent for $d$ if and only if it is convergent for $d'$.

**Exercise 7.7 (not for credit).** Suppose $X$ is any metric space. We say $X$ is *totally bounded* if, for every $\epsilon > 0$, $X$ can be covered by finitely many neighborhoods $N_\epsilon(x)$. Prove that a subset $E \subset X$ is compact if and only if $E$ is closed and totally bounded.