

# MATH 255 PSET 4

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1.

a) We prove that  $X$  with this distance function  $d$  is a metric space by showing that  $d$  obeys all the required properties:

1.  $\forall x, y \in X$ , if  $x \neq y$ ,  $d(x, y) = 1 > 0$

2.  $\forall x \in X$ ,  $d(x, x) = 0$ <sup>1</sup>.

3. We show that  $\forall x, y \in X$ ,  $d(x, y) = d(y, x)$  with casework:

a) **Case One:**  $x = y$

Then  $d(x, y) = 0 = d(y, x) \implies d(x, y) = d(y, x)$ .

b) **Case Two:**  $x \neq y$

Then  $d(x, y) = 1$  and  $d(y, x) = 1 \implies d(x, y) = 1 = d(y, x) \implies d(x, y) = d(y, x)$

4. Given  $x, y, r \in X$ , we show  $d(x, y) \leq d(x, r) + d(r, y)$  with casework:

(a) **Case One:**  $x = y$

If  $x = y$ , then  $d(x, y) = 0$ . Because  $d(x, r)$  and  $d(r, y)$  are strictly  $\geq 0$ , then  $d(x, r) + d(r, y) \geq 0$  and so  $d(x, r) + d(r, y) \geq d(x, y) \implies d(x, y) \leq d(x, r) + d(r, y)$ .

(b) **Case Two:**  $x \neq y$

If  $x \neq y$ , then  $d(x, y) = 1$ . Consider the following two (sub)cases: (i)  $r = x$  and (ii)  $r \neq x$ . In case (i),  $d(x, r) = 0$  and because  $r = x \implies r \neq y \implies d(r, y) = 1$ . So  $d(x, r) + d(r, y) = 1 \implies d(x, y) = 1 \leq d(x, r) + d(r, y) \implies d(x, y) \leq d(x, r) + d(r, y)$ .

In case (ii),  $d(x, r) = 1$  and we know by properties (1) and (2) that  $d(r, y) \geq 0$ . Thus,  $d(x, r) + d(r, y) \geq 1 \implies d(x, r) + d(r, y) \geq d(x, y) \implies d(x, y) \leq d(x, r) + d(r, y)$ .

b) We consider values of  $\epsilon$  below:

1.  $\epsilon = 0.5$

For  $\epsilon = 0.5$ ,  $N_\epsilon(x) = \{y \in X : d(x, y) < 0.5\}$ . Because  $\forall x, y \in X$ ,  $d(x, y) < 0.5 \iff d(x, y) = 0 \iff x = y$ ,  $N_\epsilon(x) = \{x\}$ .

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<sup>1</sup>This is given by the  $d(x, y) = 0$  if  $x = y$  piecewise case of  $d$ .

2.  $\epsilon = 1$

For  $\epsilon = 1$ ,  $N_\epsilon(x) = \{y \in X : d(x, y) < 1\}$ .  $\forall x, y \in X, d(x, y) < 1 \iff d(x, y) = 0 \iff x = y \implies N_\epsilon(x) = \{x\}$ .

3.  $\epsilon = 2$

For  $\epsilon = 2$ ,  $N_\epsilon(x) = \{y \in X : d(x, y) < 2\}$ . Note that  $\forall x, y \in X, d(x, y) \leq 1 \implies \forall x, y \in X, d(x, y) < 2 \implies N_\epsilon(x) = X$ .

c) **Open subsets of  $X$ :** A subset  $E \subset X$  is open if all points in  $E$  are interior points of  $E$ . This means that  $\forall x \in E, \exists \epsilon > 0$  s.t.  $N_\epsilon(x) \subset E$ . As shown in part (b), for  $\epsilon = 1 > 0$ ,  $\forall x \in X, N_\epsilon(x) = \{x\} \subset E \implies \forall x \in E, \exists \epsilon > 0$  s.t.  $N_\epsilon(x) \subset E \implies \forall x \in E, x$  is an interior point of  $E \implies \forall E \subset X, E$  is open  $\implies$  any subset of  $X$  is open.

**Closed subsets of  $X$ :** A subset  $E \subset X$  is closed if  $E$  contains its limit points. A limit point  $p$  is one where every neighborhood contains some  $q \in X$  where  $q \neq p$ . Note this is for every neighborhood (i.e.  $\forall \epsilon > 0$ ) - as shown in part (b),  $\exists \epsilon > 0$  such as 0.5 or 1 where  $N_\epsilon(p)$  contains no points other than  $p$ . Thus, no limit points exist for  $X \implies$  any subset of  $X$  is vacuously closed as it has no limit points to contain.

2.

A particular set  $S \subset \mathbb{R}$  with exactly three limit points can be given by:

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ 3 - \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ 5 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

The bounds of  $S$  are 0 and 5 for the lower and upper bound, respectively. The limit points of  $S$  are given by 0, 3, 5.

3.

1. To prove that  $E^\circ$  is open, we prove  $(E^\circ)^c$  is closed, meaning that it contains all its limit points.

Let us define  $x$  as a limit point of  $(E^\circ)^c$ . We WTS  $x \in (E^\circ)^c$ . Because  $x$  is a limit point of  $(E^\circ)^c \implies \forall \epsilon > 0, N_\epsilon(x)$  contains some  $q \neq x$  s.t.  $q \in (E^\circ)^c$ . Note that because  $q \notin E^\circ \implies q$  is not an interior point of  $E \implies$  all neighborhoods of  $q$  will contain some element not in  $E$ . Thus, defining  $h$  as any value  $\leq \epsilon - d(q, x)$ ,  $N_h(q)$  contains some element  $\notin E$ . Because<sup>2</sup>  $N_h(q) \subset N_\epsilon(x) \implies N_\epsilon(x)$  contains some element not in  $E \implies \nexists \epsilon > 0$  s.t.  $N_\epsilon(x) \subset E \implies x$  is not an interior point of  $E \implies x \notin E^\circ \implies x \in (E^\circ)^c$ . Thus,  $(E^\circ)^c$  contains all its limit points  $\implies (E^\circ)^c$  is closed  $\implies E^\circ$  is open.

2. We prove both directions of this statement below:

1. **If  $E^\circ = E \implies E$  is open**

$E$  is open if all points of  $E$  are interior points. If  $E = E^\circ \implies \forall x \in E, x \in E^\circ \implies \forall x \in E, x$  is an interior point of  $E$ . Thus,  $E$  is open.

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<sup>2</sup>We have shown  $N_h(q) \subset N_\epsilon(x)$  in our proof that neighborhoods are open.

2. **If  $E$  is open  $\implies E^\circ = E$**

If  $E$  is open, that means  $\forall x \in E$ ,  $x$  is an interior point of  $E$ . The set  $E^\circ$  contains all interior points of  $E$ . Because,  $\forall x \in E$ ,  $x$  is an interior point  $\implies \forall x \in E, x \in E^\circ \implies E \subset E^\circ$ . Furthermore, because  $E^\circ$  only contains points in  $E$  (by definition of an interior point) we know that  $E^\circ \subset E$ .  $E \subset E^\circ$  and  $E^\circ \subset E \implies E^\circ = E$ .

3. Because  $G$  is open,  $\forall x \in G$ ,  $x$  is an interior point of  $G \implies \forall x \in G, \exists \epsilon > 0$  such that  $N_\epsilon(x) \subset G \subset E \implies \forall x \in G, \exists \epsilon > 0$  such that  $N_\epsilon(x) \subset E \implies \forall x \in G$ ,  $x$  is an interior point of  $E \implies \forall x \in G, x \in E^\circ \implies G \subset E^\circ$ .

4. In this question, we are asked to prove  $(E^\circ)^c = \bar{E}^c$ . To do so, we prove both directions of this statement.

(i) **Case One:**  $(E^\circ)^c \subset \bar{E}^c$

Pick  $x \in (E^\circ)^c$ . There are two cases for  $x$ , that (a)  $x \in E^c$  or that (b)  $x$  in  $E$ . We consider both cases below:

(a)  $x \in E^c$

If  $x \in E^c \implies x \in \bar{E}^c$ .

(b)  $x \in E$

Because  $x \in (E^\circ)^c \implies x$  is not an interior point of  $E \implies \nexists \epsilon > 0$  s.t.  $N_\epsilon(x) \subset E \implies \forall \epsilon > 0, N_\epsilon(x) \not\subset E \implies \forall \epsilon > 0, \exists q \in N_\epsilon(x)$  s.t.  $q \notin E$  or expressed differently,  $q \in E^c$ . Note that because  $x \in E$ , we can be guaranteed that  $q \neq x$ . Thus, this statement can be written as  $\forall \epsilon > 0, \exists q \in N_\epsilon(x)$  s.t.  $q \neq x$  and  $q \in E^c \implies x$  is a limit point of  $E^c \implies x \in \bar{E}^c$ .

Thus, in both cases,  $x \in \bar{E}^c$ . Thus, we have shown  $\forall x \in (E^\circ)^c, x \in \bar{E}^c \implies (E^\circ)^c \subset \bar{E}^c$ .

(ii) **Case Two:**  $\bar{E}^c \subset (E^\circ)^c$

Pick  $x \in \bar{E}^c$ . At least one of the two cases is true: (a)  $x \in E^c$  and (b)  $x$  is a limit point of  $E^c$ . We consider both cases below:

(a)  $x \in E^c$

If  $x \in E^c \implies x \notin E$ . Because  $E^\circ \subset E$ ,  $x \notin E \implies x \notin E^\circ \implies x \in (E^\circ)^c$ .

(b)  $x$  is a limit point of  $E^c$

If  $x$  is a limit point of  $E^c \implies \forall \epsilon > 0, N_\epsilon(x)$  contains some  $p \in E^c$  (or  $p \notin E$ ) s.t.  $p \neq x \implies \nexists \epsilon > 0$  s.t.  $N_\epsilon(x) \subset E \implies x$  is not an interior point of  $E \implies x \notin E^\circ \implies x \in (E^\circ)^c$ .

Thus, in both cases,  $x \in (E^\circ)^c$ . Thus, we have shown  $\forall x \in \bar{E}^c, x \in (E^\circ)^c \implies \bar{E}^c \subset (E^\circ)^c$

5. Let us define  $E = (-\infty, 0) \cup (0, \infty)$  on the standard metric space  $\mathbb{R}$ . The closure of  $E$  is given by  $\bar{E} = (-\infty, \infty) = \mathbb{R}$ . Because  $\mathbb{R}$  is open  $\implies$  every point of  $\mathbb{R}$  is an interior point of  $\mathbb{R}$ ,  $\bar{E}^\circ = \mathbb{R}^\circ = \mathbb{R}$ .

We now look at the interior of  $E$ . All points in  $(-\infty, 0)$  and  $(0, \infty)$  are interior points. However,  $0$  is not an interior point of  $E$  as it is not in  $E$ . Thus,  $E^\circ = E = (-\infty, 0) \cup (0, \infty) \neq \bar{E}^\circ \implies E$  and  $\bar{E}$  do not always have the same interiors.

6. We inspect the set  $E = \{\frac{1}{n} : n \in \mathbb{N}\}$  defined on the standard metric space  $\mathbb{R}$ .  $E$  has no interior points ( $\forall x \in E, \nexists \epsilon > 0$  s.t.  $N_\epsilon(x) \subset E$ ) and so  $E^\circ = \emptyset$ . The empty set trivially has no limit points and so the closure of  $E^\circ$  is just  $\bar{E}^\circ = \bar{\emptyset} = \emptyset \cup \emptyset = \emptyset$ .

We now consider the closure of  $E$ ,  $\bar{E}$ . The only limit point of  $E$  is zero, and so  $\bar{E} = E \cup \{0\}$ . Because  $\bar{E} \neq \bar{E}^\circ \implies E$  and  $E^\circ$  do not always have the same closures.