

## Discretionary Note

Anish Krishna Lakkapragada

**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# MATH 241 PSET 6

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1.

- a) The PMF of r.v.  $X$  given by  $P(X = k)$  can be divided into two cases: (1)  $k = 0$  and (2)  $k > 0$ . In the case that  $k = 0$ ,  $P(X = k)$  is given by the probability that either of the two following events occurred: (1) that there was either a structural zero (i.e. the coin landed heads with probability  $p$ ) or (2) the coin landed tails and the Poisson r.v. turned out to be zero anyways. For event (2), the subevent that the coins turned out to be tails (given by probability  $1 - p$ ) and the subevent the Poisson r.v. turned out to be zero (given by probability  $\frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$ ) are both independent and so the probability both occur is given by the product of their probabilities,  $(1 - p)e^{-\lambda}$ . In short, the probability event (2) occurs is given by  $(1 - p)e^{-\lambda}$ . Because events (1) and (2) are mutually exclusive, the probability either occurs is given by the sum of their probabilities:  $P(X = 0) = p + (1 - p)e^{-\lambda}$ .

For the case  $k > 0$ ,  $P(X = k)$  is given by the probability that two independent events both occurred: (1) that the coin landed tails (given by probability  $1 - p$ ) and (2) the Poisson r.v. materialized as  $k$  (given by probability  $\frac{e^{-\lambda}\lambda^k}{k!}$ ). Because these two events are independent, the probability both occur is given by the product of their probabilities:  $P(X = k) = (1 - p)\frac{e^{-\lambda}\lambda^k}{k!}$  for  $k > 0$ .

Thus, we get that  $P(X = k)$  is given by:

$$P(X = k) = \begin{cases} p + (1 - p)e^{-\lambda} & k = 0 \\ (1 - p)\frac{e^{-\lambda}\lambda^k}{k!} & k > 0 \end{cases}$$

2.

- a) We compute  $P(1 < X < 3) = \int_1^3 f(x)dx$  below:

$$\begin{aligned} P(1 < X < 3) &= \int_1^3 f(x)dx = \int_1^3 xe^{-\frac{x^2}{2}} dx \\ &= [-e^{-\frac{x^2}{2}}]_1^3 = e^{-0.5} - e^{-4.5} \end{aligned}$$

b) The CDF of  $X$  is given by  $F(x) = \int_{-\infty}^x f(t)dt = \int_{-\infty}^x te^{-\frac{t^2}{2}}dt = [-e^{-\frac{t^2}{2}}]_{-\infty}^x = 1 - e^{-\frac{x^2}{2}}$ .

Because  $P(X \leq q_j) = F(q_j)$  by definition of CDF, we can compute the quartiles asked by finding the function  $q_j(j)$ . We find this function through manipulating the equation  $F(q_j) = 1 - e^{-\frac{q_j^2}{2}} = \frac{j}{4}$ . We solve the equation below:

$$\begin{aligned} P(X \leq q_j) &= F(q_j) = \frac{j}{4} \\ 1 - e^{-\frac{q_j^2}{2}} &= \frac{j}{4} \\ e^{-\frac{q_j^2}{2}} &= 1 - \frac{j}{4} \\ -\frac{q_j^2}{2} &= \ln(1 - \frac{j}{4}) \\ q_j &= \sqrt{-2\ln(1 - \frac{j}{4})} \end{aligned}$$

For  $j = 1, 2, 3$ , we get:

$$\begin{aligned} q_1 &= \sqrt{-2\ln(\frac{3}{4})} \\ q_2 &= \sqrt{-2\ln(\frac{1}{2})} \\ q_3 &= \sqrt{-2\ln(\frac{1}{4})} \end{aligned}$$

3.

a) Given r.v.  $R$ , we are asked to compute the mean and variance of r.v.  $A = \pi R^2$ . We denote  $f(x) = \frac{1}{x}$  as the PDF for  $R \sim Unif(1, 0)$ . We compute mean  $\mathbb{E}[A]$  below.

$$\begin{aligned} \mathbb{E}[A] &= \mathbb{E}[\pi R^2] = \pi \mathbb{E}[R^2] = \\ &= \pi \int_{-\infty}^{\infty} x^2 f(x) dx = \pi \int_0^1 x^2 f(x) dx = \pi \int_0^1 x^2 dx = \pi [\frac{x^3}{3}]_0^1 = \frac{\pi}{3} \end{aligned}$$

We compute  $Var(A)$  below. Note that above we have computed  $\mathbb{E}[R^2] = \frac{1}{3}$ .

$$\begin{aligned}
Var(A) &= Var(\pi R^2) = \pi^2 Var(R^2) = \\
\pi^2 [\mathbb{E}[R^4] - \mathbb{E}[R^2]^2] &= \pi^2 \left[ \int_0^1 x^4 f(x) dx - \frac{1}{9} \right] = \pi^2 \left[ \int_0^1 x^4 dx - \frac{1}{9} \right] = \pi^2 \left[ \left[ \frac{x^5}{5} \right]_0^1 - \frac{1}{9} \right] = \\
\pi^2 \left[ \frac{1}{5} - \frac{1}{9} \right] &= \frac{4\pi^2}{45}
\end{aligned}$$

- b) We denote the CDF and PDF of  $A$  as  $F_A(x)$  and  $f_A(x)$ , respectively. Note that for a circle with given area  $a \in [0, \pi]$ , the radius for this circle can be computed as  $\sqrt{\frac{a}{\pi}}$ . A circle with an area outside of these bounds is impossible as the radius  $R$  is constrained from 0 to 1.

$$\begin{aligned}
F_A(x) = P(A \leq x) &= \begin{cases} 0 & x < 0 \\ P(R \leq \sqrt{\frac{x}{\pi}}) & 0 \leq x \leq \pi \\ 1 & x > \pi \end{cases} \\
F_A(x) = \begin{cases} 0 & x < 0 \\ \frac{\sqrt{\frac{x}{\pi}} - 0}{1-0} & 0 \leq x \leq \pi \\ 1 & x > \pi \end{cases} &\Rightarrow F_A(x) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{x}{\pi}} & 0 \leq x \leq \pi \\ 1 & x > \pi \end{cases}
\end{aligned}$$

Because PDF  $f_A(x)$  is given as the derivative of  $F_A(x)$ , we get:

$$f_a(x) = F'_A(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2\sqrt{\pi x}} & 0 \leq x \leq \pi \\ 0 & x > \pi \end{cases}$$

4.

- a) We denote the CDF of  $X$  as  $F(x)$ . We compute  $F(x)$  below, given  $0 < x < 1$ .

$$\begin{aligned}
F(x) &= \int_{-\infty}^x f(t) dt = \int_{-\infty}^0 0 dt + \int_0^x f(t) dt = \int_0^x 12t^2(1-t) dt = \int_0^x 12t^2 - 12t^3 dt = \\
12 \left[ \frac{t^3}{3} - \frac{t^4}{4} \right]_0^x &= 12 \left( \frac{x^3}{3} - \frac{x^4}{4} \right) = 4x^3 - 3x^4
\end{aligned}$$

- b)  $P(0 < X < \frac{1}{2}) = F(\frac{1}{2}) - F(0) = \frac{4}{8} - \frac{3}{16} = \frac{5}{16}$ .

- c) We compute  $\mathbb{E}[X]$  below. Note we can ignore all intervals outside of  $0 < x < 1$  as we assume  $f(x) = 0$  for  $x \notin [0, 1]$ .

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 (12x^3 - 12x^4)dx = [3x^4 - \frac{12x^5}{5}]_0^1 = 3 - \frac{12}{5} = \frac{3}{5}$$

We now compute  $Var(X)$ .

$$\begin{aligned} Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \int_0^1 x^2 f(x)dx - \mathbb{E}[X]^2 = \int_0^1 12x^4(1-x)dx - \mathbb{E}[X]^2 = \\ &= \int_0^1 12x^4 - 12x^5 dx - \mathbb{E}[X]^2 = 12[\frac{x^5}{5} - \frac{x^6}{6}]_0^1 - \mathbb{E}[X]^2 = \frac{12}{5} - 2 - (\frac{3}{5})^2 = 0.4 - 0.36 = 0.04 \end{aligned}$$

5.

For values  $0 \leq x \leq 1$ , we denote the CDF of  $X$  as  $F(x)$ . By the definition of a CDF,  $F(x) = P(X \leq x)$ , which represents the probability the max of  $U_1, \dots, U_n$  is equal to  $x$ . This is the same as the probability that  $U_1, \dots, U_n$  are all  $\leq x$ . Because  $U_1, \dots, U_n$  are independent random variables described by the distribution  $Unif(0, 1)$ , the probability all  $U_1, \dots, U_n$  are  $\leq x$  (i.e.  $P(X \leq x)$ ) is given by  $\prod_{i=1}^n P(U_i \leq x) = \prod_{i=1}^n \frac{x-0}{1-0} = \prod_{i=1}^n x = x^n$ . Thus, we get that the CDF for  $X$  is given by  $F(x) = x^n$ . Because the PDF  $f(x)$  for  $X$  is given by the derivative of the CDF  $F(x)$ , we get the PDF  $\mathbf{f}(\mathbf{x}) = \mathbf{F}'(\mathbf{x}) = \mathbf{n}\mathbf{x}^{n-1}$  for  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}$ .

We compute  $\mathbb{E}[X]$  below. Once again, we ignore all values outside  $0 \leq x \leq 1$  as they are impossible (a maximum of values from  $[0, 1]$  cannot be outside the interval  $[0, 1]$ .)

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 nx^n dx = n[\frac{x^{n+1}}{n+1}]_0^1 = \frac{n}{n+1}$$

6. Anish Lakkapragada. I worked independently.