

Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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STATS 242 HW 6

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1.

We first compute the first moment μ_1 of $X \sim \text{Geom}(p)$, which is given by $\mu_1 = \mathbb{E}[X] = \frac{1}{p}$. We now solve for the MoM estimator \hat{p} where the observed sample moment $\hat{\mu}_1 = \frac{1}{n}(X_1 + \dots + X_n) = \bar{X}$ is equal to theoretical moment μ_1 with the estimated \hat{p} parameter:

$$\begin{aligned}\hat{\mu}_1 = \bar{X} &= \frac{1}{\hat{p}} \\ \hat{p} &= \frac{1}{\bar{X}}\end{aligned}$$

Thus, $\hat{p} = \frac{1}{\bar{X}}$ is the MoM estimator \hat{p} for p . We now compute the MLE of p . We first start by finding the log-likelihood function $\ell_n(p)$:

$$\begin{aligned}\ell_n(p) = \log[\text{lik}(p)] &= \sum_{i=1}^n \log f(X_i|p) = \sum_{i=1}^n \log(p(1-p)^{x_i-1}) = \sum_{i=1}^n \log(p) + (x_i - 1)\log(1-p) \\ &= n\log(p) + \sum_{i=1}^n (x_i - 1)\log(1-p)\end{aligned}$$

We now solve for the MLE \hat{p} by solving $\ell'_n(\hat{p}) = 0$:

$$\begin{aligned}
\ell'_n(\hat{p}) &= 0 \\
(n \log(\hat{p}) + \sum_{i=1}^n (x_i - 1) \log(1 - \hat{p}))' &= 0 \\
\frac{n}{\hat{p}} - \sum_{i=1}^n \frac{(x_i - 1)}{1 - \hat{p}} &= 0 \\
\frac{n}{\hat{p}} - \frac{1}{1 - \hat{p}} (n\bar{X} - n) &= 0 \\
\frac{n(1 - \hat{p}) - \hat{p}(n\bar{X} - n)}{\hat{p}(1 - \hat{p})} &= 0 \\
n(1 - \hat{p}) - \hat{p}(n\bar{X} - n) &= 0 \\
n - n\hat{p} - \hat{p}n\bar{X} + n\hat{p} &= 0 \\
n &= n\hat{p}\bar{X} \\
1 &= \hat{p}\bar{X} \\
\hat{p} &= \frac{1}{\bar{X}}
\end{aligned}$$

Thus the MLE of p is $\hat{p} = \frac{1}{\bar{X}}$. We now find the sampling distribution of the MLE of p , under large n . We begin by computing the Fisher Information $I(p)$:

$$\begin{aligned}
I(p) &= -\mathbb{E}_p \left[\frac{\partial^2}{\partial p^2} \log f(X|p) \right] = -\mathbb{E}_p \left[\frac{\partial^2}{\partial p^2} (\log(p) + (x-1)\log(1-p)) \right] \\
&= -\mathbb{E}_p \left[\frac{\partial}{\partial p} \left(\frac{1}{p} + (1-x)\frac{1}{1-p} \right) \right] = -\mathbb{E}_p \left[-\frac{1}{p^2} + \frac{(1-x)}{(1-p)^2} \right] = -\left(-\frac{1}{p^2} + \frac{1}{(1-p)^2} - \frac{\mathbb{E}_p[x]}{(1-p)^2} \right) \\
&= \frac{1}{p^2} - \frac{1}{(1-p)^2} + \frac{1}{p(1-p)^2} = \frac{(1-p)^2 - p^2 + p}{p^2(1-p)^2} = \frac{1-p}{p^2(1-p)^2} = \frac{1}{p^2(1-p)}
\end{aligned}$$

So this means that for the MLE \hat{p} , $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, \frac{1}{I(p)})$ or $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, p^2(1-p))$ meaning $\hat{p} \rightarrow \mathcal{N}(p, \frac{p^2(1-p)}{n})$ as $n \rightarrow \infty$.

2.

We first compute the MoM estimate of p . To do so, we compute the first moment μ_1 of $X \sim \text{NegBinom}(r, p)$ which is given by $\mu_1 = \mathbb{E}[X] = \frac{pr}{1-p}$. We now solve for the MoM estimator \hat{p} where the observed sample moment $\hat{\mu}_1 = \frac{1}{n}(X_1 + \dots + X_n) = \bar{X}$ is equal to theoretical moment μ_1 with the estimated \hat{p} parameter:

$$\begin{aligned}
\hat{\mu}_1 &= \bar{X} = \frac{\hat{p}r}{1 - \hat{p}} \\
\bar{X}(1 - \hat{p}) &= \hat{p}r \\
\bar{X} &= \hat{p}\bar{X} + \hat{p}r \\
\bar{X} &= \hat{p}(\bar{X} + r) \\
\hat{p} &= \frac{\bar{X}}{\bar{X} + r}
\end{aligned}$$

Thus the MoM estimator of p is given by $\hat{p} = \frac{\bar{X}}{\bar{X} + r}$. We now compute the MLE of p . We first start by finding the log-likelihood function $\ell_n(p)$:

$$\begin{aligned}
\ell_n(p) &= \log[\text{lik}(p)] = \sum_{i=1}^n \log f(x_i|p) = \sum_{i=1}^n \log \left(\binom{x_i + r - 1}{x_i} (1-p)^r p^{x_i} \right) \\
&= \sum_{i=1}^n \log \left(\binom{x_i + r - 1}{x_i} \right) + r \log(1-p) + x_i \log(p) = nr \log(1-p) + \sum_{i=1}^n \log \left(\binom{x_i + r - 1}{x_i} \right) + x_i \log(p)
\end{aligned}$$

We now solve for the MLE \hat{p} by solving $\ell'_n(\hat{p}) = 0$:

$$\begin{aligned}
\ell'_n(\hat{p}) &= 0 \\
(nr \log(1 - \hat{p}) + \sum_{i=1}^n \log \left(\binom{x_i + r - 1}{x_i} \right) + x_i \log(\hat{p}))' &= 0 \\
\frac{-nr}{1 - \hat{p}} + \sum_{i=1}^n \frac{x_i}{\hat{p}} &= 0 \\
-\frac{nr\hat{p}}{\hat{p}(1 - \hat{p})} + \frac{n\bar{X}(1 - \hat{p})}{\hat{p}(1 - \hat{p})} &= 0 \\
\frac{n\bar{X}(1 - \hat{p}) - nr\hat{p}}{\hat{p}(1 - \hat{p})} &= 0 \\
n\bar{X}(1 - \hat{p}) - nr\hat{p} &= 0 \\
\bar{X}(1 - \hat{p}) - r\hat{p} &= 0 \\
\bar{X} - \hat{p}\bar{X} - \hat{p}r &= 0 \\
\bar{X} &= \hat{p}(\bar{X} + r) \\
\hat{p} &= \frac{\bar{X}}{\bar{X} + r}
\end{aligned}$$

Thus the MLE of p is given by $\hat{p} = \frac{\bar{X}}{\bar{X} + r}$. We now find the sampling distribution of the MLE of p , under large n . We begin by computing the Fisher Information $I(p)$:

$$\begin{aligned}
I(p) &= \text{Var}_p\left[\frac{\partial}{\partial p} \log f(x|p)\right] = \text{Var}_p\left[\frac{\partial}{\partial p} \left[\log\left(\binom{x+r-1}{x}\right) + r\log(1-p) + x\log(p)\right]\right] \\
&= \text{Var}_p\left[\frac{-r}{1-p} + \frac{x}{p}\right] = \frac{1}{p^2} \text{Var}_p[x] = \frac{r}{p(1-p)^2}
\end{aligned}$$

So this means that for the MLE \hat{p} , $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, \frac{1}{I(p)})$ or $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, \frac{p(1-p)^2}{r})$ so $\hat{p} \rightarrow \mathcal{N}(p, \frac{p(1-p)^2}{nr})$.

3.

- (a) The PDF of $\text{Pareto}(\alpha, x_m)$ is given by $f(x|\alpha, x_m) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$. Thus, the PDF of $\text{Pareto}(\theta, 1)$ is given by:

$$f(x|\theta, 1) = \frac{\theta \cdot 1^\theta}{x^{\theta+1}} = \frac{\theta}{x^{\theta+1}}$$

Expressed differently, we can set functions $T(x) = \log(\frac{1}{x})$, $A(\theta) = \log(\frac{1}{\theta})$, $h(x) = \frac{1}{x}$ so the PDF

$$f(x|\theta) = e^{\theta T(x) - A(\theta)} h(x) = \frac{e^{\theta T(x)}}{x e^{A(\theta)}} = \frac{(e^{\log(\frac{1}{x})})^\theta}{x e^{\log \frac{1}{\theta}}} = \frac{1}{x^\theta \cdot x \cdot \frac{1}{\theta}} = \frac{\theta}{x^{\theta+1}}$$

is identical to the aforementioned derived $\text{Pareto}(\theta, 1)$ PDF.

- (b) We differentiate both sides of the identity $1 = \int_{\mathcal{X}} f(x|\theta) dx$ to compute $\mathbb{E}_\theta[T(X)]$:

$$\begin{aligned}
\int_{\mathcal{X}} f(x|\theta) dx &= 1 \\
\frac{d}{d\theta} \int_{\mathcal{X}} f(x|\theta) dx &= \int_{\mathcal{X}} \frac{d}{d\theta} f(x|\theta) dx = \frac{d}{d\theta}(1) = 0 \\
\int_{\mathcal{X}} \frac{d}{d\theta} [e^{\theta T(x) - A(\theta)} h(x)] dx &= 0 \\
\int_{\mathcal{X}} [(e^{\theta T(x) - A(\theta)})' h(x) + h'(x) e^{\theta T(x) - A(\theta)}] dx &= \int_{\mathcal{X}} [(e^{\theta T(x) - A(\theta)})(T(x) - A'(\theta)) h(x) + 0] dx = 0 \\
\int_{\mathcal{X}} [e^{\theta T(x) - A(\theta)} h(x)] (T(x) - A'(\theta)) dx &= 0 \\
\int_{\mathcal{X}} f(x|\theta) (T(x) - A'(\theta)) dx &= \mathbb{E}_\theta[T(x) - A'(\theta)] = 0 \\
\mathbb{E}_\theta[T(x)] &= \mathbb{E}_\theta[A'(\theta)] = A'(\theta)
\end{aligned}$$

So we have $\mathbb{E}_\theta[T(x)] = A'(\theta)$. We now verify this is the case for our Pareto($\theta, 1$) model with its defined functions $T(x) = \log(\frac{1}{x})$, $A(\theta) = \log(\frac{1}{\theta})$, $h(x) = \frac{1}{x}$ in part (a). We first compute $\mathbb{E}_\theta[T(x)]$:

$$\mathbb{E}_\theta[T(x)] = \mathbb{E}_\theta[\log(\frac{1}{x})] = \int_{-\infty}^{\infty} f(x|\theta) \log(\frac{1}{x}) dx = \int_1^{\infty} \frac{\theta}{x^{\theta+1}} \log(\frac{1}{x}) dx$$

We can use a u-substitution of $u = \frac{1}{x}$ so $-u^{\theta-1} du = -\frac{x}{x^\theta} \cdot -\frac{1}{x^2} dx = \frac{1}{x^{\theta+1}} dx$. Thus, this integral can be re-written as:

$$\int_1^0 \log(u) \cdot \theta \cdot (-u^{\theta-1}) du = \theta \int_0^1 \log(u) \cdot u^{\theta-1} du$$

Using integration by parts, we can set $a = \log(u)$, $da = \frac{1}{u} du$, $db = u^{\theta-1} du$, $b = \frac{1}{\theta} u^\theta$. As such, we get:

$$\begin{aligned} \theta \int_0^1 \log(u) \cdot u^{\theta-1} du &= \theta [ab \Big|_{u=0}^{u=1} - \int_0^1 b da] \\ &= \theta \left[\frac{\log(u) \cdot u^\theta}{\theta} \right] \Big|_0^1 - \theta \int_0^1 \frac{1}{\theta} u^{\theta-1} du = 0 - \int_0^1 u^{\theta-1} du = -\frac{1}{\theta} u^\theta \Big|_0^1 = -\frac{1}{\theta} + 0 = -\frac{1}{\theta} \end{aligned}$$

So $\mathbb{E}_\theta[T(x)] = -\frac{1}{\theta}$. We now compute $A'(\theta)$:

$$A'(\theta) = \frac{1}{\frac{1}{\theta}} \cdot \frac{-1}{\theta^2} = \frac{-\theta}{\theta^2} = -\frac{1}{\theta}$$

So because $\mathbb{E}_\theta[T(x)] = -\frac{1}{\theta} = A'(\theta)$, we have verified for our Pareto model in part (a) that $\mathbb{E}_\theta[T(x)] = A'(\theta)$.

(c) We first find the log-likelihood function $\ell_n(\theta)$:

$$\begin{aligned} \ell_n(\theta) &= \log[\text{lik}(\theta)] = \sum_{i=1}^n \log f(x_i|\theta) = \sum_{i=1}^n \log[e^{\theta T(x_i) - A(\theta)} h(x_i)] = \\ &= \sum_{i=1}^n \log[e^{\theta T(x_i)}] - \log[e^{A(\theta)}] + \log[h(x_i)] = -nA(\theta) + \sum_{i=1}^n \theta T(x_i) + \log[h(x_i)] \end{aligned}$$

To get the MLE of θ we solve for the solution to $0 = \ell'_n(\theta)$:

$$\begin{aligned} \ell'_n(\theta) &= 0 \\ (-nA(\theta) + \sum_{i=1}^n \theta T(x_i) + \log[h(x_i)])' &= 0 \\ -nA'(\theta) + \sum_{i=1}^n T(x_i) &= 0 \\ nA'(\theta) &= \sum_{i=1}^n T(x_i) \\ A'(\theta) &= \frac{1}{n} \sum_{i=1}^n T(x_i) \\ \mathbb{E}_\theta[T(x)] &= A'(\theta) = \frac{1}{n} \sum_{i=1}^n T(x_i) \end{aligned}$$

So therefore the MLE of θ is the unique solution to $\mathbb{E}_\theta[T(x)] = \frac{1}{n}\sum_{i=1}^n T(x_i)$. Because the generalized MoM estimator of θ is also the solution to $\mathbb{E}_\theta[T(x)] = \frac{1}{n}\sum_{i=1}^n T(x_i)$, and we are guaranteed this solution is unique, the generalized MoM estimator of θ is equivalent to the MLE of θ .

An example of this occurring would be the generalized MoM estimator based on $T(x) = \log(x)$ for the $\text{Pareto}(\theta, 1)$ model coinciding with the MLE.

4.

- (a) By the CLT, $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, p(1 - p))$. Informally, this gives us that $\text{Var}[\sqrt{n}(\hat{p} - p)] \approx \hat{p}(1 - \hat{p}) \implies n\text{Var}(\hat{p} - p) \approx \hat{p}(1 - \hat{p}) \implies \text{Var}(p) \approx \frac{\hat{p}(1 - \hat{p})}{n}$. Using this approximation for the variance of p , we can compute a 95% confidence interval for p as $\hat{p} \pm z^{\frac{0.05}{2}} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$ where $z^{\frac{0.05}{2}}$ is the upper- $\frac{0.05}{2}$ point of the standard normal distribution.
- (b) We solve $\sqrt{n}(\hat{p} - p) = \pm \sqrt{p(1 - p)}z^{(\alpha/2)}$ below to create a confidence interval for p :

$$\begin{aligned}\sqrt{n}(\hat{p} - p) &= \pm \sqrt{p(1 - p)}z^{(\alpha/2)} \\ n(\hat{p} - p)^2 &= p(1 - p)(z^{(\alpha/2)})^2 \\ n\hat{p}^2 - 2n\hat{p}p + np^2 &= (z^{(\alpha/2)})^2 p - (z^{(\alpha/2)})^2 p^2 \\ (n + (z^{(\alpha/2)})^2)p^2 - (2n\hat{p} + (z^{(\alpha/2)})^2)p + n\hat{p}^2 &= 0\end{aligned}$$

Using the quadratic formula, we construct the following confidence interval for p :

$$p = \frac{(2n\hat{p} + (z^{(\alpha/2)})^2) \pm \sqrt{(2n\hat{p} + (z^{(\alpha/2)})^2)^2 - 4n\hat{p}^2(n + (z^{(\alpha/2)})^2)}}{2(n + (z^{(\alpha/2)})^2)}$$

- (c) For our confidence interval in part (a), we report the simulated coverage probabilities below:

| | $n = 10$ | $n = 40$ | $n = 100$ |
|-----------|----------|----------|-----------|
| $p = 0.1$ | 0.65061 | 0.91435 | 0.93267 |
| $p = 0.3$ | 0.83926 | 0.92955 | 0.94984 |
| $p = 0.5$ | 0.8922 | 0.91919 | 0.94205 |

We similarly report the simulated coverage probabilities for our confidence interval in part (b):

| | $n = 10$ | $n = 40$ | $n = 100$ |
|-----------|----------|----------|-----------|
| $p = 0.1$ | 0.93053 | 0.94406 | 0.9375 |
| $p = 0.3$ | 0.92463 | 0.94432 | 0.93659 |
| $p = 0.5$ | 0.97861 | 0.961 | 0.94316 |

We can clearly see from comparing these two tables that our interval construction in part (b) leads to generally much higher simulated coverage probabilities.

```

1      # %%
2  import numpy as np
3  from scipy.stats import norm
4

```



```

5 z_alpha = np.abs(norm.ppf(0.05/2))
6 z_alpha_squared = z_alpha ** 2
7
8 NUM_SIMULATIONS = 100 * 1000
9 SAMPLE_SIZES = [10, 40, 100]
10 TRUE_PARAMS = [0.1, 0.3, 0.5]
11
12 def generate_data(n,p):
13     return np.random.binomial(n=n, p=p) # number of successes
14
15 def part_a_interval(num_successes, n):
16     p_hat = num_successes / n
17     delta = z_alpha * np.sqrt(p_hat * (1 - p_hat) / n)
18     return p_hat - delta, p_hat + delta
19
20 def part_b_interval(num_successes, n):
21     p_hat = num_successes / n
22     denominator = 2 * (n + z_alpha_squared)
23     center = (2 * n * p_hat + z_alpha_squared) / denominator
24     delta = np.sqrt((2 * n * p_hat + z_alpha_squared) ** 2 - 4 * n *
25     ↪ (p_hat ** 2) * (n + z_alpha_squared))
26     delta /= denominator
27     return center - delta, center + delta
28
29 # %%
30
31 for n in SAMPLE_SIZES:
32     for p in TRUE_PARAMS:
33         NUM_COVERAGE_A = 0
34         for _ in range(NUM_SIMULATIONS):
35             n_successes = generate_data(n, p)
36             lower, upper = part_a_interval(n_successes, n)
37
38             if p >= lower and p <= upper: NUM_COVERAGE_A += 1
39
40     print(f"[PART (A)] For sample size n={n} and p={p}, simulated
41     ↪ coverage probability: {NUM_COVERAGE_A / NUM_SIMULATIONS}")
42
43
44 for n in SAMPLE_SIZES:
45     for p in TRUE_PARAMS:
46         NUM_COVERAGE_B = 0
47         for _ in range(NUM_SIMULATIONS):
48             n_successes = generate_data(n, p)
49             lower, upper = part_b_interval(n_successes, n)

```

```
50         if p >= lower and p <= upper: NUM_COVERAGE_B += 1
51
52
53     print(f"[PART (B)] For sample size n={n} and p={p}, simulated
54           ↪ coverage probability: {NUM_COVERAGE_B / NUM_SIMULATIONS}")
55
56 # %%
```