

Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head [here](#).

Math 226- HW 3 Due: Sep 19 by Midnight

1. (10 points)

a) (6 points) In \mathbb{F}^n (Take $\mathbb{F} = \mathbb{R}$ first if \mathbb{F} confuses you). Let e_j denote the vector whose j th coordinate is 1 and whose other coordinates are 0. Prove that $S := \{e_1, e_2, \dots, e_n\}$ is linearly independent set and generates \mathbb{F}^n . **Remark:** The vectors $e_i : i = 1, 2, \dots, n$ are called standard vectors. In \mathbb{R}^n , we use these frequently and call the set of standard vectors “the standard bases”.

c) (4 points) Let V be vector space over a field \mathbb{F} with characteristic not equal to two. (This means that if $x \neq 0$, then $x + x \neq 0$ for any $x \in \mathbb{F}$.) Let u and v be distinct vectors in V . Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

To see why, characteristic causes a trouble, note that you can gain zero while adding and subtracting the coefficients.

2. (22 pts) Let V be a vector space over \mathbb{F} and $S = \{v_1, v_2, \dots, v_k\} \subseteq V$.

(a) (6 points) For every $x \in V$, prove that $x \in \text{span}(S)$ iff $\text{span}(S) = \text{span}(S \cup \{x\})$.

(b) (6 points) Suppose that $w \in V$ but $w \notin S$. Further suppose that S is linearly independent. Prove that $S \cup \{w\}$ is linearly independent iff $w \notin \text{span}(S)$.

(c) (10 points) We write $A \subsetneq B$ to mean $A \subseteq B$ and $A \neq B$. Prove that a set $S = \{u_1, u_2, \dots, u_k\}$ is linearly independent iff

$$\{0\} \subsetneq \text{span}(\{u_1\}) \subsetneq \text{span}(\{u_1, u_2\}) \subsetneq \text{span}(\{u_1, u_2, u_3\}) \subsetneq \dots \subsetneq \text{span}(\{u_1, \dots, u_k\}).$$

Remark: Another insignificant looking, but fundamental result. With this problem, we say that a set is linearly independent iff the span gets bigger with addition of each vector vector.

3. (14 points) Suppose U and W are subspaces of V . Recall we define the set $U + W = \{u + w : u \in U, w \in W\}$. We say $U + W$ is a direct sum if and only if $U \cap W = \{0\}$. In this case we represent the sum as $U \oplus W$. For example, if $U = \text{Span}((1, 0))$ and $W = \text{Span}((0, 1))$ then $\mathbb{R}^2 = U \oplus W$ since $\mathbb{R}^2 = U + W$ and $U \cap W = \{(0, 0)\}$

a) (3 points) Let U be the subspace of \mathbb{R}^5 defined by $U := \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, x_3 = 7x_4\}$. Find a basis of U . (Recall we did a simpler version of that in the class.)

- (2 points) Extend the basis in the first part to a basis of \mathbb{R}^5

- (2 points) Find a subspace W of \mathbb{R}^5 such that $\mathbb{R}^5 = U \oplus W$. Validate your answer.

b) (3 points) Let U be the subspace of $P_4(\mathbb{R})$ defined by $U = \{f \in P_4(\mathbb{R}) : f'(1) = 0\}$. Find a basis of U .

- (2 points) Extend the basis in the first part to a basis of $P_4(\mathbb{R})$.

- (2 points) Find a subspace W of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R}) = U \oplus W$. Validate your answer.

4. (7 points) We define $M_{n \times n}(\mathbb{F})$ as the matrices defined over \mathbb{F} with size $n \times n$. We also define trace of $A = [a_{ij}]_{n \times n} \in M_{n \times n}(\mathbb{F})$ as

$$\text{Tr}(A) = \sum_{i=1}^n a_{ii}$$

