PSETs Landing Page*

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*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

MATH 255 PSET 4

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1.

a) We prove that X with this distance function d is a metric space by showing that d obeys all the required properties:

1. $\forall x, y \in X,$ if $x \neq y, d(x,y) = 1 > 0$
2. $\forall x \in X, d(x,x) = 0$[1].
3. We show that $\forall x, y \in X, d(x,y) = d(y,x)$ with casework:
   a) **Case One:** $x = y$
   Then $d(x,y) = 0 = d(y,x) \implies d(x,y) = d(y,x)$.
   b) **Case Two:** $x \neq y$
   Then $d(x,y) = 1$ and $d(y,x) = 1 \implies d(x,y) = 1 = d(y,x) \implies d(x,y) = d(y,x)$
4. Given $x, y, r \in X$, we show $d(x,y) \leq d(x,r) + d(r,y)$ with casework:
   (a) **Case One:** $x = y$
   If $x = y$, then $d(x,y) = 0$. Because $d(x,r)$ and $d(r,y)$ are strictly ≥ 0, then $d(x,r) + d(r,y) \geq 0$ and so $d(x,r) + d(r,y) \geq d(x,y) \implies d(x,y) \leq d(x,r) + d(r,y)$.
   (b) **Case Two:** $x \neq y$
   If $x \neq y$, then $d(x,y) = 1$. Consider the following two (sub)cases: (i) $r = x$ and (ii) $r \neq x$. In case (i), $d(x,r) = 0$ and because $r = x \implies r \neq y \implies d(r,y) = 1$. So $d(x,r) + d(r,y) = 1 \implies d(x,y) = 1 \leq d(x,r) + d(r,y) \implies d(x,y) \leq d(x,r) + d(r,y)$.

   In case (ii), $d(x,r) = 1$ and we know by properties (1) and (2) that $d(r,y) \geq 0$. Thus, $d(x,r) + d(r,y) \geq 1 \implies d(x,r) + d(r,y) \geq d(x,y) \implies d(x,y) \leq d(x,r) + d(r,y)$.

b) We consider values of ϵ below:

1. $\epsilon = 0.5$
   For $\epsilon = 0.5$, $N_\epsilon(x) = \{y \in X : d(x,y) < 0.5\}$. Because $\forall x, y \in X, d(x,y) < 0.5 \iff d(x,y) = 0 \iff x = y$, $N_\epsilon(x) = \{x\}$.

[1] This is given by the $d(x,y) = 0$ if $x = y$ piecewise case of d.

2. $\epsilon = 1$
   For $\epsilon = 1$, $N_\epsilon(x) = \{y \in X : d(x,y) < 1\}$. $\forall x, y \in X, d(x,y) < 1 \iff d(x,y) = 0 \iff x = y \implies N_\epsilon(x) = \{x\}$.
3. $\epsilon = 2$
   For $\epsilon = 2$, $N_\epsilon(x) = \{y \in X : d(x,y) < 2\}$. Note that $\forall x, y \in X, d(x,y) \leq 1 \implies \forall x, y \in X, d(x,y) < 2 \implies N_\epsilon(x) = X$.

c) **Open subsets of X**: A subset $E \subset X$ is open if all points in E are interior points of E. This means that $\forall x \in E, \exists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E$. As shown in part (b), for $\epsilon = 1 > 0$, $\forall x \in X, N_\epsilon(x) = \{x\} \subset E \implies \forall x \in E, \exists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E \implies \forall x \in E, x$ is an interior point of $E \implies \forall E \subset X, E$ is open $\implies$ any subset of X is open.

   Closed subsets of X: A subset $E \subset X$ is closed if E contains its limit points. A limit point p is one where every neighborhood contains some $q \in X$ where $q \neq p$. Note this is for every neighborhood (i.e. $\forall\ \epsilon > 0$) - as shown in part (b), $\exists\ \epsilon > 0$ such as 0.5 or 1 where $N_\epsilon(p)$ contains no points other than p. Thus, no limit points exist for $X \implies$ any subset of X is vacuously closed as it has no limit points to contain.

2.

A particular set $S \subset \mathbb{R}$ with exactly three limit points can be given by:

$$S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{3 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{5 - \frac{1}{n} : n \in \mathbb{N}\}$$

The bounds of S are 0 and 5 for the lower and upper bound, respectively. The limit points of S are given by $0, 3, 5$.

3.

1. To prove that E° is open, we prove $(E^\circ)^c$ is closed, meaning that it contains all its limit points.

   Let us define x as a limit point of $(E^\circ)^c$. We WTS $x \in (E^\circ)^c$. Because x is a limit point of $(E^\circ)^c \implies \forall \epsilon > 0, N_\epsilon(x)$ contains some $q \neq x$ s.t. $q \in (E^\circ)^c$. Note that because $q \notin E^\circ \implies q$ is not an interior point of $E \implies$ all neighborhoods of q will contain some element not in E. Thus, defining h as any value $\leq \epsilon - d(q,x)$, $N_h(q)$ contains some element $\notin E$. Because[2] $N_h(q) \subset N_\epsilon(x) \implies N_\epsilon(x)$ contains some element not in $E \implies \nexists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E \implies x$ is not an interior point of $E \implies x \notin E^\circ \implies x \in (E^\circ)^c$. Thus, $(E^\circ)^c$ contains all its limit points $\implies (E^\circ)^c$ is closed $\implies E^\circ$ is open.
2. We prove both directions of this statement below:
   1. **If $E^\circ = E \implies E$ is open**
      E is open if all points of E are interior points. If $E = E^\circ \implies \forall x \in E, x \in E^\circ \implies \forall x \in E, x$ is an interior point of E. Thus, E is open.

[2]We have shown $N_h(q) \subset N_\epsilon(x)$ in our proof that neighborhoods are open.

2. **If E is open $\implies E^\circ = E$**
   If E is open, that means $\forall x \in E$, x is an interior point of E. The set E° contains all interior points of E. Because, $\forall x \in E, x$ is an interior point $\implies \forall x \in E, x \in E^\circ \implies E \subset E^\circ$. Furthermore, because E° only contains points in E (by definition of an interior point) we know that $E^\circ \subset E$. $E \subset E^\circ$ and $E^\circ \subset E \implies E^\circ = E$.

3. Because G is open, $\forall x \in G$, x is an interior point of $G \implies \forall x \in G, \exists\ \epsilon > 0$ such that $N_\epsilon(x) \subset G \subset E \implies \forall x \in G, \exists\ \epsilon > 0$ such that $N_\epsilon(x) \subset E \implies \forall x \in G$, x is an interior point of $E \implies \forall x \in G, x \in E^\circ \implies G \subset E^\circ$.

4. In this question, we are asked to prove $(E^\circ)^c = \bar{E^c}$. To do so, we prove both directions of this statement.
   (i) **Case One:** $(E^\circ)^c \subset \bar{E^c}$
   Pick $x \in (E^\circ)^c$. There are two cases for x, that (a) $x \in E^c$ or that (b) x in E. We consider both cases below:
   (a) $x \in E^c$
   If $x \in E^c \implies x \in \bar{E^c}$.
   (b) $x \in E$
   Because $x \in (E^\circ)^c \implies x$ is not an interior point of E $\implies \nexists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E \implies \forall \epsilon > 0, N_\epsilon(x) \not\subset E \implies \forall \epsilon > 0, \exists\ q \in N_\epsilon(x)$ s.t. $q \notin E$ or expressed differently, $q \in E^c$. Note that because $x \in E$, we can be guaranteed that $q \neq x$. Thus, this statement can be written as $\forall \epsilon > 0, \exists\ q \in N_\epsilon(x)$ s.t. $q \neq x$ and $q \in E^c \implies x$ is a limit point of $E^c \implies x \in \bar{E^c}$.
   Thus, in both cases, $x \in \bar{E^c}$. Thus, we have shown $\forall x \in (E^\circ)^c, x \in \bar{E^c} \implies (E^\circ)^c \subset \bar{E^c}$.
   (ii) **Case Two:** $\bar{E^c} \subset (E^\circ)^c$
   Pick $x \in \bar{E^c}$. At least one of the two cases is true: (a) $x \in E^c$ and (b) x is a limit point of E^c. We consider both cases below:
   (a) $x \in E^c$
   If $x \in E^c \implies x \notin E$. Because $E^\circ \subset E$, $x \notin E \implies x \notin E^\circ \implies x \in (E^\circ)^c$.
   (b) x is a limit point of E^c
   If x is a limit point of $E^c \implies \forall \epsilon > 0, N_\epsilon(x)$ contains some $p \in E^c$ (or $p \notin E$) s.t. $p \neq x \implies \nexists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E \implies x$ is not an interior point of $E \implies x \notin E^\circ \implies x \in (E^\circ)^c$.
   Thus, in both cases, $x \in (E^\circ)^c$. Thus, we have shown $\forall x \in \bar{E^c}, x \in (E^\circ)^c \implies \bar{E^c} \subset (E^\circ)^c$

5. Let us define $E = (-\infty, 0) \cup (0, \infty)$ on the standard metric space $\mathbb{R}$. The closure of E is given by $\bar{E} = (-\infty, \infty) = \mathbb{R}$. Because $\mathbb{R}$ is open $\implies$ every point of $\mathbb{R}$ is an interior point of $\mathbb{R}$, $\bar{E}^\circ = \mathbb{R}^\circ = \mathbb{R}$.
   We now look at the interior of E. All points in $(-\infty, 0)$ and $(0, \infty)$ are interior points. However, 0 is not an interior point of E as it is not in E. Thus, $E^\circ = E = (-\infty, 0) \cup (0, \infty) \neq \bar{E}^\circ \implies E$ and $\bar{E}$ do not always have the same interiors.

6. We inspect the set $E = \{\frac{1}{n} : n \in \mathbb{N}\}$ defined on the standard metric space $\mathbb{R}$. E has no interior points ($\forall x \in E, \nexists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E$) and so $E^\circ = \emptyset$. The emptyset trivially has no limit points and so the closure of E° is just $\bar{E}^\circ = \bar{\emptyset} = \emptyset \cup \emptyset = \emptyset$.

   We now consider the closure of E, $\bar{E}$. The only limit point of E is zero, and so $\bar{E} = E \cup \{0\}$. Because $\bar{E} \neq \bar{E}^\circ \implies E$ and E° do not always have the same closures.