

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# MATH 255 HW 1

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## 1. Exercise 1.1 (5 points)

- (1) We prove this by contrapositive and thus assume  $f$  is not injective  $\Rightarrow \exists x, x' \in A$  s.t.  $x \neq x'$  and  $f(x) = f(x')$ . Let us define  $u = f(x) = f(x')$ . Then,  $h$  is not injective as  $h(x) = g(f(x)) = g(u)$  and  $h(x') = g(f(x')) = g(u)$  and so  $\exists x, x' \in A$  s.t.  $x \neq x'$  and  $h(x) = h(x')$ .
- (2) We prove this by contrapositive and thus assume that  $g$  is not surjective  $\Rightarrow \exists c \in C$  s.t.  $\nexists b \in B$  where  $g(b) = c$ . Because no element in  $B$  maps to  $C$  by  $g$ , this means  $\forall x \in A, h(x) = g(f(x)) \neq c$  and so because  $\nexists x \in A$  s.t.  $h(x) = c$ ,  $h$  is not surjective.

## 2. Exercise 1.2 (5 points; Rudin 1.1)

- (1) Prove  $r + x \notin \mathbb{Q}$   
Let us prove this by contradiction and assume that  $r + x \in \mathbb{Q}$ . By negation rule,  $r \in \mathbb{Q} \Rightarrow -r \in \mathbb{Q}$ . By addition rule,  $(-r) + (r + x) \in \mathbb{Q}$  and so this means  $(-r) + (r + x) = (-r + r) + x = 0 + x = x \in \mathbb{Q}$ , which is a contradiction.
- (2) Prove  $rx \notin \mathbb{Q}$   
Let us prove this by contradiction and assume that  $rx \in \mathbb{Q}$ . Because  $r \neq 0$ , by the inversion rule we have that  $r$  has an inverse  $r^{-1}$ . By multiplication rule,  $(r^{-1}) \cdot rx \in \mathbb{Q}$  or  $(r^{-1}) \cdot rx = (r^{-1} \cdot r)x = 1 \cdot x = x \in \mathbb{Q}$ , which is a contradiction.

## 3. Exercise 1.3 (10 points; Rudin 1.3)

- (1) Because  $x \neq 0$ , by the inversion rule we know  $\exists x^{-1}$  s.t.  $x^{-1} \cdot x = 1$ . Thus:

$$\begin{aligned} xy &= xz \\ x^{-1}xy &= x^{-1}xz \\ 1 \cdot y &= 1 \cdot z \\ y &= z \end{aligned}$$

(2) Because  $x \neq 0$ , we can apply the inversion rule again:

$$\begin{aligned} xy &= x \\ x^{-1}xy &= x^{-1}x \\ 1 \cdot y &= 1 \\ y &= 1 \end{aligned}$$

(3) We first prove that  $0 \cdot y = 0$ :

$$\begin{aligned} 0 \cdot y + 0 \cdot y &= (0 + 0) \cdot y = 0 \cdot y \\ 0 \cdot y + 0 \cdot y &= 0 \cdot y \end{aligned}$$

Because  $0 \cdot y \in F$ , its additive inverse is given by  $-0 \cdot y$ :

$$\begin{aligned} 0 \cdot y + 0 \cdot y - 0 \cdot y &= 0 \cdot y - 0 \cdot y \\ 0 \cdot y + (0 \cdot y - 0 \cdot y) &= 0 \\ 0 \cdot y + 0 &= 0 \\ 0 \cdot y &= 0 \end{aligned}$$

To prove that if  $xy = 1 \implies x \neq 0$  we proceed by contradiction. If  $x = 0$ , then  $xy = 0 \cdot y = 0 \neq 1$ . We now show  $y = x^{-1}$ . Thus,  $x \neq 0$ . Because  $x \neq 0$ , we can apply the inversion rule again:

$$\begin{aligned} xy &= 1 \\ x^{-1}xy &= x^{-1} \\ 1 \cdot y &= x^{-1} \\ y &= x^{-1} \end{aligned}$$

(4) Let us define inverse of  $x^{-1}$  to be  $u$ , where by definition  $x^{-1} \cdot u = 1$ . Because  $x^{-1} \cdot x = 1$  by definition, then we have that  $u = x$ , or that the inverse of  $x^{-1}$  is  $x$ . Expressed as an equation, we have shown:  $(x^{-1})^{-1} = x$ .

#### 4. Exercise 1.4 (10 points)

(1) Let us suppose  $x = \frac{p}{q} \in \mathbb{Q}$ , where  $p, q \in \mathbb{Z}$  and  $\frac{p}{q}$  are in lowest terms. For proof by contradiction, we assume  $x^2 = 3$ . This means  $\frac{p^2}{q^2} = 3$  or  $p^2 = 3q^2$  and thus  $p^2$  has a factor of three.

We now prove that  $p$  has a factor of three. Because  $p^2$  has a factor of three, the prime factorization of  $p^2$  can be given by  $3^\alpha \dots$  where  $\alpha \geq 1 \in \mathbb{Z}$ . Let us define the prime factorization of  $p = 3^\beta \dots$  where  $\beta \geq 0 \in \mathbb{Z}$ . Note that since  $p^2 = (3^\beta \dots)^2$ ,

$\alpha = 2\beta$ . The lowest possible integer value of  $\alpha$  s.t.  $\alpha \geq 1$  and  $\beta \in \mathbb{Z}$  is then  $\alpha = 2$  and  $\beta = 1$ , so we are guaranteed that  $p$  has a factor of three. This means we can express  $p = 3k$ , where  $k \in \mathbb{Z}$  and so  $p^2 = (3k)^2 = 9k^2 = 3q^2$  or  $q^2 = 3k^2$ . Using the same logic as before because  $q^2$  has a factor of three, so does  $q$ . Thus,  $p$  and  $q$  both have a factor of three and so this contradicts the assumption that  $\frac{p}{q}$  are in lowest terms.

(2) We show that these provided operations define  $\mathbb{Q}(\sqrt{3})$  as a field:

**1. Zero & One Element**

Our zero and one element in  $\mathbb{Q}(\sqrt{3})$  are given by  $0+0\sqrt{3}$  and  $1+0\sqrt{3}$  respectively. Written as ordered pairs, they are given by  $(0,0)$  and  $(1,0)$  respectively.

**2. Negation Law**

For an element  $u = a + b\sqrt{3} \in \mathbb{Q}(\sqrt{3})$ ,  $-u = (-a) + (-b)\sqrt{3} \in \mathbb{Q}(\sqrt{3})$ .

**3. Inversion Law**

For an element  $u = a + b\sqrt{3} \in \mathbb{Q}$ ,  $u^{-1}$  is given by:

$$\begin{aligned} uu^{-1} &= 1_{\mathbb{Q}\sqrt{3}} = 1 + 0\sqrt{3} \\ u^{-1} &= \frac{1 + 0\sqrt{3}}{u} = \frac{1 + 0\sqrt{3}}{a + b\sqrt{3}} = \frac{(1 + 0\sqrt{3})(a - b\sqrt{3})}{(a + b\sqrt{3})(a - b\sqrt{3})} = \frac{(1 + 0\sqrt{3})(a - b\sqrt{3})}{(a^2 - 3b^2) + (-ab + ba)\sqrt{3}} \\ &= \frac{a - b\sqrt{3}}{a^2 - 3b^2} = \frac{a}{a^2 - 3b^2} + \left(\frac{-b}{a^2 - 3b^2}\right)\sqrt{3} \end{aligned}$$

**4.  $\forall x, y \in \mathbb{Q}\sqrt{3}, x + y = y + x$**

Define  $x = a + b\sqrt{3}, y = a' + b'\sqrt{3} \in \mathbb{Q}\sqrt{3}$ .  $x + y$  is given by:

$$x + y = (a + b\sqrt{3}) + (a' + b'\sqrt{3}) = (a + a') + (b + b')\sqrt{3}$$

and  $y + x$  is given by:

$$y + x = (a' + b'\sqrt{3}) + (a + b\sqrt{3}) = (a' + a) + (b' + b)\sqrt{3} = (a' + a) + (b + b')\sqrt{3}$$

and so  $x + y = y + x$ .

**5.  $\forall x, y, z \in \mathbb{Q}\sqrt{3}, (x + y) + z = x + (y + z)$**

We define  $x, y$  the same as above. We define  $z = a'' + b''\sqrt{3} \in \mathbb{Q}\sqrt{3}$ . Then  $(x + y) + z$  is given by:

$$\begin{aligned} (x + y) + z &= [(a + a') + (b + b')\sqrt{3}] + z = ((a + a') + (b + b')\sqrt{3}) + (a'' + b''\sqrt{3}) \\ &= ((a + a') + a'') + ((b + b') + b'')\sqrt{3} = (a + a' + a'') + (b + b' + b'')\sqrt{3} \end{aligned}$$

and  $x + (y + z)$  is given by:

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$$x + (y + z) = x + ((a' + b'\sqrt{3}) + (a'' + b''\sqrt{3})) = x + ((a' + a'') + (b' + b'')\sqrt{3})$$

$$= (a + b\sqrt{3}) + ((a' + a'') + (b' + b'')\sqrt{3}) = (a + a' + a'') + (b + b' + b'')\sqrt{3}$$

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and so  $(x + y) + z = x + (y + z)$ .

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$$6. \forall x \in \mathbb{Q}\sqrt{3}, 0 + x = x$$

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$$0_{\mathbb{Q}\sqrt{3}} + x = (0 + 0\sqrt{3}) + (a + b\sqrt{3}) = (0 + a) + (0 + b)\sqrt{3} = a + b\sqrt{3} = x$$

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$$7. \forall x \in \mathbb{Q}\sqrt{3}, -x + x = 0.$$

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Using the previous definition of  $x$ :

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$$-x + x = ((-a) + (-b)\sqrt{3}) + (a + b\sqrt{3}) = ((-a + a) + (-b + b)\sqrt{3}) = 0 + 0\sqrt{3} = 0_{\mathbb{Q}\sqrt{3}}$$

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$$8. \forall x, y \in \mathbb{Q}\sqrt{3}, xy = yx$$

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We define  $x, y$  the same as before. This gives us  $xy$  as:

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$$xy = (a + b\sqrt{3})(a' + b'\sqrt{3}) = (aa' + 3bb') + (ab' + ba')\sqrt{3}$$

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and  $yx$  as:

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$$yx = (a' + b'\sqrt{3})(a + b\sqrt{3}) = (a'a + 3b'b) + (a'b + b'a)\sqrt{3} = (aa' + 3bb') + (ba' + ab')\sqrt{3}$$

$$= (aa' + 3bb') + (ab' + ba')\sqrt{3}$$

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and so  $xy = yx$ .

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$$9. \forall x, y, z \in \mathbb{Q}\sqrt{3}, (xy)z = x(yz)$$

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We use the previous definitions of  $x, y, z$  as before. This gives us  $(xy)z$  as:

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$$(xy)z = ((aa' + 3bb') + (ab' + ba')\sqrt{3})(z) = ((aa' + 3bb') + (ab' + ba')\sqrt{3})(a'' + b''\sqrt{3})$$

$$= (a''(aa' + 3bb') + 3(ab' + ba')b'') + ((aa' + 3bb')b'' + (ab' + ba')a'')\sqrt{3}$$

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and  $x(yz)$  as:

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$$x(yz) = x((a' + b'\sqrt{3})(a'' + b''\sqrt{3})) = x((a'a'' + 3b'b'') + (a'b'' + b'a'')\sqrt{3})$$

$$= (a + b\sqrt{3})((a'a'' + 3b'b'') + (a'b'' + b'a'')\sqrt{3})$$

$$= (a(a'a'' + 3b'b'') + 3b(a'b'' + b'a'')) + (a(a'b'' + b'a'') + b(a'a'' + 3b'b''))\sqrt{3}$$

$$= (a''(aa' + 3bb') + 3(ab' + ba')b'') + ((aa' + 3bb')b'' + (ab' + ba')a'')\sqrt{3}$$

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RESPONSIBLY. USE RESPONSIBLY. USE RESPONSIBLY. USE

and so  $(xy)z = x(yz)$ .

10.  $\forall x \in \mathbb{Q}, 1 \cdot x = x$

Using previous definition of  $x$ :

$$1_{\mathbb{Q}\sqrt{3}} \cdot x = (1 + 0\sqrt{3})(a + b\sqrt{3}) = (1 \cdot a + 3 \cdot 0 \cdot b) + (1 \cdot b + 0 \cdot a)\sqrt{3} = a + b\sqrt{3} = x$$

11.  $\forall x \in \mathbb{Q}\sqrt{3}$  with  $x \neq 0, x \cdot x^{-1} = 1$

Using the previous definition of  $x$ :

$$\begin{aligned} x \cdot x^{-1} &= (a + b\sqrt{3})\left(\frac{a}{a^2 - 3b^2} + \left(\frac{-b}{a^2 - 3b^2}\right)\sqrt{3}\right) = \\ &= \left(\frac{a^2}{a^2 - 3b^2} - \frac{3b^2}{a^2 - 3b^2}\right) + \left(\frac{-ab}{a^2 - 3b^2} + \frac{ba}{a^2 - 3b^2}\right)\sqrt{3} \\ &= \frac{a^2 - 3b^2}{a^2 - 3b^2} + \left(\frac{ab - ab}{a^2 - 3b^2}\right)\sqrt{3} = 1 + 0\sqrt{3} = 1_{\mathbb{Q}\sqrt{3}} \end{aligned}$$

12.  $\forall x, y, z \in \mathbb{Q}\sqrt{3}, x(y + z) = xy + xz$

We use the previous definitions for  $x, y, z \in \mathbb{Q}\sqrt{3}$ . This gives us  $x(y + z)$  as:

$$\begin{aligned} x(y + z) &= x((a' + b'\sqrt{3}) + (a'' + b''\sqrt{3})) = x((a' + a'') + (b' + b'')\sqrt{3}) \\ &= (a + b\sqrt{3})((a' + a'') + (b' + b'')\sqrt{3}) \\ &= (a(a' + a'') + 3b(b' + b'')) + (a(b' + b'') + b(a' + a''))\sqrt{3} \end{aligned}$$

and  $xy + xz$  as:

$$\begin{aligned} xy + xz &= ((aa' + 3bb') + (ab' + ba')\sqrt{3}) + xz \\ &= ((aa' + 3bb') + (ab' + ba')\sqrt{3}) + ((a + b\sqrt{3})(a'' + b''\sqrt{3})) \\ &= ((aa' + 3bb') + (ab' + ba')\sqrt{3}) + ((aa'' + 3bb'') + (ab'' + ba'')\sqrt{3}) \\ &= ((aa' + 3bb' + aa'' + 3bb'') + (ab' + ba' + ab'' + ba'')\sqrt{3}) \\ &= (a(a' + a'') + 3b(b' + b'')) + (a(b' + b'') + b(a' + a''))\sqrt{3} \end{aligned}$$

and so  $x(y + z) = xy + xz$ .

(3) Given these addition and product laws, the inversion rule we defined in (2) for  $\mathbb{Q}\sqrt{3}$  looks like:

$$(a + b\sqrt{3})^{-1} = \frac{a}{a^2 - 3b^2} + \left(\frac{-b}{a^2 - 3b^2}\right)\sqrt{3}$$

and the zero element was given by  $0 + 0\sqrt{3}$ .

Now consider element  $x = 2 + 0\sqrt{3} \neq 0 + 0\sqrt{3}$ , where  $x \in \mathbb{Z}\sqrt{3}$ . The inverse of  $x$  is given by  $x^{-1} = \frac{2}{4} + 0\sqrt{3}$ . Because  $\frac{2}{4} \notin \mathbb{Z}$ ,  $x^{-1} \notin \mathbb{Z}$  and so the inversion rule does not apply for  $\mathbb{Z}\sqrt{3}$  with the provided addition and product rules.

5. **Exercise 1.5 (5 points)**

We prove that  $\prec$  does not make  $\mathbb{Q}$  into an ordered set by a contradictory example. Consider  $x = \frac{1}{6}$  and  $y = \frac{2}{3}$ . For these values,  $x \not\prec y$  and  $y \not\prec x$ . Furthermore,  $x \neq y$  because they are not the same element in the set  $\mathbb{Q}$  (this set is reduced to lowest terms, so two elements are equivalent only if their numerator and denominator are the same.) Thus, we have shown for  $x, y \in \mathbb{Q}$ , none of the following statements are true:  $x \prec y, y \prec x, x = y$  and so  $\prec$  does not make  $\mathbb{Q}$  an ordered set.

6. **Exercise 1.6 (5 points)**

**Theorem 0.1** *If  $S$  is an ordered set with elements  $x, y, z \in S$ ,  $x \leq y$  and  $y \leq z \implies x \leq z$ .*

*Proof:* We do casework:

1. **Case 1:**  $x < y$

We investigate the two subcases:

1. **Subcase 1:**  $y < z$

By the transitivity property of ordered sets,  $x < y$  and  $y < z \implies x < z \implies x \leq z$ .

2. **Subcase 2:**  $y = z$

We are given  $x < y$ . Because  $y = z$ ,  $x < z \implies x \leq z$ .

2. **Case 2:**  $x = y$

We are given  $y \leq z$ . Because  $x = y$ , we can conclude  $x \leq z$ .

Because  $A$  is a non-empty set,  $\exists x \in A$ . Pick any  $x \in A$ . Because  $\alpha$  is a lower bound and  $\beta$  is an upper bound  $\alpha \leq x \leq \beta \implies \alpha \leq \beta$  by **Theorem 0.1**.