

PSETs Landing Page*

Anish Krishna Lakkapragada

This is the documentation for using my PSET PDFs responsibly. I post these LaTeX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarly, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email anish.lakkapragada@yale.edu.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

MATH 241 PSET 4

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1.

Let us define the r.v. X as the number of games ending in draws. Because n games are being played independently, each with a 60% chance of ending in a tie, $X \sim \text{Bin}(n, 0.6)$ and so the PMF of X is given by $P(X = k) = \binom{n}{k}(0.6)^k(0.4)^{n-k}$ for $0 \leq k \leq n$. The support of X is given by the set $\{0, 1, \dots, n\}$.

Because each of the players play only one game, the r.v. $Y = 2X$ defines the number of players whose games end in draws. This means that $P(Y = 2k) = P(X = k)$. Thus, for $0 \leq k \leq 2n$, the PMF for Y can be given as:

$$P(Y = k) = \begin{cases} 0, & \text{if } k \text{ odd} \\ \binom{n}{0.5k}(0.6)^{0.5k}(0.4)^{n-0.5k}, & \text{otherwise} \end{cases}$$

The support of Y is twice the support of X and is thus given by the set $\{0, 2, \dots, 2n\}$.

2.

a) Let us define r.v. X as the number of winning tickets out of 3 selected tickets. $X \sim \text{Bin}(3, p)$.

b) **Using Inclusion-Exclusion Principle**

Let us define events X_1, X_2, X_3 as the events that the first, second, and third tickets are winning tickets, respectively. Note that $X_1, X_2, X_3 \sim \text{Bern}(p)$. The probability that at least 1 of the 3 tickets is a winning ticket can be given as $P(X_1 \cup X_2 \cup X_3)$.

$$\begin{aligned} P(X_1 \cup X_2 \cup X_3) &= P(X_1) + P(X_2) + P(X_3) \\ &\quad - P(X_1 \cap X_2) - P(X_1 \cap X_3) - P(X_2 \cap X_3) + P(X_1 \cap X_2 \cap X_3) \end{aligned}$$

Given $P(X_1) = P(X_2) = P(X_3) = p$ and $P(X_1 \cap X_2) = P(X_1 \cap X_3) = P(X_2 \cap X_3) = p^2$ and $P(X_1 \cap X_2 \cap X_3) = p^3$, we get that:

$$P(X_1 \cup X_2 \cup X_3) = p + p + p - p^2 - p^2 - p^2 + p^3$$

$$P(X_1 \cup X_2 \cup X_3) = 3p - 3p^2 + p^3$$

Using Complement Rule

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - \binom{3}{0} p^0 (1-p)^3$$

$$= 1 - (1-p)^3 = 3p - 3p^2 + p^3$$

3.

- a) Let us define event C_1 and C_2 as the events that the first and second coin was chosen, respectively. We are given $P(C_1) = P(C_2) = 0.5$. Using LOTP to compute the PMF $P(X = k)$:

$$P(X = k) = P(X = k|C_1)P(C_1) + P(X = k|C_2)P(C_2)$$

$$P(X = k) = \frac{P(X = k|C_1) + P(X = k|C_2)}{2}$$

$$P(X = k) = \frac{\binom{n}{k} p_1^k (1-p_1)^{n-k} + \binom{n}{k} p_2^k (1-p_2)^{n-k}}{2}$$

$$P(X = k) = \frac{\binom{n}{k} [p_1^k (1-p_1)^{n-k} + p_2^k (1-p_2)^{n-k}]}{2}$$

The support for X for $n \geq 2$ flips is given by the set $\{0, \dots, n\}$.

- b) If $p_1 = p_2$, $X \sim \text{Bin}(n, p_1)$.

4.

- a) Let us define r.v. W as the number of women who get promoted. $W \sim HGeom(n, m, t)$.
- b) Let us define r.v. W as the number of women promoted, r.v. W' as the number of women who are not promoted, and r.v. T as the number of employees who are promoted. Their distributions can be given as $W \sim \text{Bin}(n, p)$, $W' \sim \text{Bin}(n, 1-p)$, and $T \sim \text{Bin}(n+m, p)$.

5.

- a) We evaluate the probability $P(X \oplus Y = 1)$ below. Note that events X and Y are independent.

$$\begin{aligned}
P(X \oplus Y = 1) &= P(X = 1 \cap Y = 0) + P(X = 0 \cap Y = 1) \\
P(X \oplus Y = 1) &= P(X = 1)P(Y = 0) + P(X = 0)P(Y = 1) \\
P(X \oplus Y = 1) &= p(1 - \frac{1}{2}) + \frac{1-p}{2} \\
P(X \oplus Y = 1) &= \frac{1}{2}
\end{aligned}$$

We now evaluate the probability $P(X \oplus Y = 0)$ below.

$$\begin{aligned}
P(X \oplus Y = 0) &= P(X = 1 \cap Y = 1) + P(X = 0 \cap Y = 0) \\
P(X \oplus Y = 0) &= P(X = 1)P(Y = 1) + P(X = 0)P(Y = 0) \\
P(X \oplus Y = 0) &= \frac{p}{2} + (1-p)(1 - \frac{1}{2}) \\
P(X \oplus Y = 0) &= \frac{1}{2}
\end{aligned}$$

Because the support of $X \oplus Y$ is $\{0, 1\}$ and $P(X \oplus Y = 0) = P(X \oplus Y = 1) = \frac{1}{2}$, $X \oplus Y$ is a Bernoulli trial with a 50% chance of success. $X \oplus Y \sim \text{Bern}(\frac{1}{2})$.

b) **In the case $p = \frac{1}{2}$.**

We have established in part (a) that regardless of the value of p , $X \oplus Y \sim \text{Bern}(\frac{1}{2})$. Because Bernoulli trials have the same probability of success each time, X or Y does not affect the outcome of $X \oplus Y$. Thus, $X \oplus Y$ is independent to X and Y .

In the case $p \neq \frac{1}{2}$.

We first test if $X \oplus Y$ and X are independent. If this was the case then:

$$P(X \oplus Y = 1|X = k) = P(X \oplus Y = 1)$$

where $k \in \{0, 1\}$ ¹. We know from part (a) that regardless of the value of p , $P(X \oplus Y = 1) = \frac{1}{2}$. Let us define $f(k)$ as a function that performs $k + 1$ modulo 2. Thus in order for $P(X \oplus Y = 1|X = k) = \frac{1}{2}$, $P(Y = f(k)) = \frac{1}{2}$. Because $Y \sim \text{Bern}(\frac{1}{2})$, we know this statement must be true and so $X \oplus Y$ and X are independent.

We now test if $X \oplus Y$ and Y are independent. If this was the case then:

$$P(X \oplus Y = 1|Y = k) = P(X \oplus Y = 1) = \frac{1}{2}$$

where $k \in \{0, 1\}$. We use the same definition of $f(k)$ as before. In order for $P(X \oplus Y = 1|Y = k) = \frac{1}{2}$, $P(X = f(k)) = \frac{1}{2}$. $P(X = f(k))$ is either equal to p or $1 - p$. Because $p \neq \frac{1}{2}$, $P(X = f(k)) \neq \frac{1}{2}$ and so $P(X \oplus Y = 1|Y = k) \neq \frac{1}{2} \neq P(X \oplus Y = 1)$. Thus $X \oplus Y$ and Y are not independent.

6. Anish Lakkapragada. I worked independently.

¹Note that we do not need to check if $P(X \oplus Y = 0|X = k) = P(X \oplus Y = 0)$ to test for independence. This is because if $P(X \oplus Y = 1|X = k) = P(X \oplus Y = 1) \Rightarrow 1 - P(X \oplus Y = 1|X = k) = 1 - P(X \oplus Y = 1) \Rightarrow P(X \oplus Y = 0|X = k) = P(X \oplus Y = 0)$.