Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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MATH 244 PSET 4

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1.

This question asks us to consider how many permutations of $[n]$ have a single cycle. If a permutation of $[n]$ has a single cycle, that means all its n elements are in the same one cycle. Thus, to compute the number of unique permutations of $[n]$ that have a single cycle, we have to just find the number of unique cycles of $[n]$.

Fixing the first element of this cycle to be any arbitrary element of $[n]$, we have $(n-1)!$ different ways to order the remaining $n-1$ elements $\implies$ there are $(n-1)!$ unique cycles of $[n]$ $\implies$ there are $(n-1)!$ unique permutations of $[n]$ with a single cycle.

2.

Lemma 0.1 *Suppose that permutation $p : X \to X$ and X is a finite set. This means that p has finite order.* ***Proof****: Let us define each of the m cycles that p possesses as $c_1, \ldots, c_m$ where the length of each cycle is given by $l_1, \ldots, l_m$. Because X is finite, the number of cycles that p has is finite and each cycle is guaranteed to be of finite length. Because the permutation will return all elements to their original positions after going through each cycle $\implies$ the permutation is the identity function if it is applied n times, where n is a multiple of each cycle's length $\implies$ the order of a permutation is given by $LCM(l_1, \ldots, l_m)$. Because each of the cycle lengths of p is finite and there are finitely many cycles $\implies$ the LCM of these cycle lengths is finite $\implies$ p is of finite order.*

We assume that X is a finite set[1] $\implies$ (**Lemma 0.1**) p has finite order $\implies$ $\exists\ q \in \mathbb{N}$ s.t. $p^q(i) = i$. We first show that $\approx$ is an equivalence relation by showing that it is (i) reflexive (ii) symmetric (iii) transitive below:

(i) Reflexivity

Let $i \in X$. $i \approx i \iff \exists\ q \geq 1$ s.t. $p^q(i) = i$. This is guaranteed by the fact that p is of finite order $\implies \forall i \in X, i \approx i \implies p$ is reflexive.

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[1]Given in #29 on Ed.

(ii) Symmetric

Note that because p is a permutation, p is invertible. Let us define the inverse of p as $p^{-1} : X \to X$ and let us define $p^{-k} = p^{-1} \circ p^{-k+1}$.

Let $i, j \in X$. We WTS if $i \approx j \implies j \approx i$. If $i \approx j \implies \exists\, k \geq 1$ s.t. $p^k(i) = j$. Applying p^{-k} to both sides we get:

$$
\begin{aligned}
p^k(i) &= j \\
p^{-k} \circ p^k(i) &= p^{-k}(j) \\
\mathrm{id}(i) = i &= p^{-k}(j) \\
i &= p^{-k}(j)
\end{aligned}
$$

Recall that because p is of finite order, $\exists\, q \in \mathbb{N}$ s.t. $p^q(i) = i$. By the Archimedean property, $\exists\, n \in \mathbb{Z}$ s.t. $nq > k \implies nq - k > 0 \implies$ (because $n, q, k \in \mathbb{N}$) $nq - k \geq 1$. We apply p^{nq} on both sides of the equation $i = p^{-k}(j)$:

$$
\begin{aligned}
p^{nq}(i) &= p^{nq} \circ p^{-k}(j) \\
p^{nq}(i) = \underbrace{p^q \circ \cdots \circ p^q(i)}_{n \text{ times}} &= \underbrace{\mathrm{id} \circ \cdots \circ \mathrm{id}(i)}_{n \text{ times}} = i = p^{nq-k}(j) \\
i &= p^{nq-k}(j)
\end{aligned}
$$

Thus $\exists\, m = nq - k \geq 1$ such that $p^m(j) = i \implies j \approx i$. This shows that $\forall i, j \in X, i \approx j \iff j \approx i$ and so we can conclude that $\approx$ is symmetric.

(iii) Transitivity

Let $a, b, c \in X$ where $a \approx b$ and $b \approx c$. We WTS $a \approx c$.

If $a \approx b \implies \exists\, k \geq 1$ s.t. $p^k(a) = b$. If $b \approx c \implies \exists\, q \geq 1$ s.t. $p^q(b) = c$. Thus, this means $p^{q+k}(a) = p^q \circ p^k(a) = p^q(b) = c \implies \exists\, z = q + k \geq 1$ s.t. $p^z(a) = c \implies a \approx c$.

Thus, because we have shown these three properties, we can conclude $\approx$ is an equivalence relation. We now show that the classes of $\approx$ are cycles of p: if a and $b \in$ the same class $\iff a \approx b \iff \exists\, k \geq 1$ s.t. $p^k(a) = b \iff a$ and b are in the same cycle of p.

3.

We first transform this into an easier question[2]. Let us define $k_i = f(i+1) - f(i)$ where $0 \leq i \leq n$ and let us artifically define $f(0) = 1$ and $f(n+1) = n$. The desired number of monotonic functions is given by the amount of non-negative integer solutions[3] to the equation $k_0 + k_1 + \cdots + k_n = n - 1$.

[2] Shoutout to the Hint in the Back of The Book.

[3] The solutions must be non-negative as that indicates $k_1 \geq 0 \implies f(i+1) - f(i) \geq 0 \implies f(i+1) \geq f(i)$ which is a required property for f to be monotonic.

This is equivalent to the question of having to place $n-1$ identical balls (in our case the unit 1) into n buckets (bucket i corresponds to the k_i term in this equation).

We now can use a Stars & Bars argument to solve this question. We have $n-1$ balls which can be placed as stars. We then can divide these balls into n different buckets by placing $n-1$ bars. The positioning of these bars will indicate how many balls belong to each bucket (i.e. the multiple of one, or the value itself, stored in bucket k_i). We have # of balls + # of buckets $= n-1+n = 2n-1$ possible locations from which we must place $n-1$ bars. Thus the number of possible arrangements of these buckets is given by $\binom{2n-1}{n-1} = \binom{2n-1}{n} \implies$ we have $\binom{2n-1}{n}$ non-negative integer solutions to $k_0 + k_1 + \cdots + k_n = n-1 \implies$ there are $\binom{2n-1}{n}$ monotonic functions from $[n] \to [n]$.

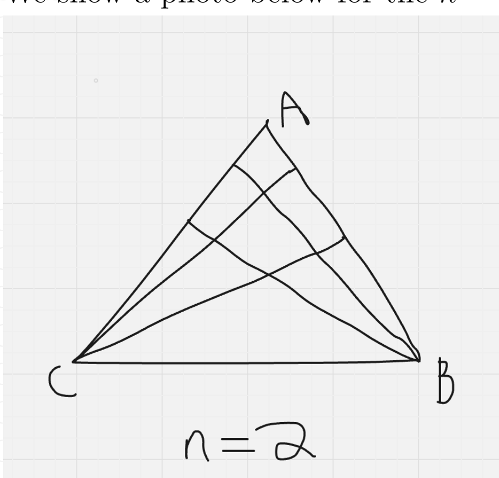
4.

(a) **Step 1: n segments from vertex C to AB.** Drawing n segments from vertex C to AB will divide the triangle into $n+1$ regions. This is obvious and analogous to the reasoning that $r-1$ bars will divide a set of stars into r regions.

Step 2: n segments from vertex B to AC. Each of these n segments from vertex B to AC will intersect (once) each of the n segments from vertex C to AB $\implies$ each of the n segments from vertex B to AC will create n intersections. Note that all these intersections are unique (shown in photo below) $\implies$ each of the n segments from vertex B to AC will create n intersections $\implies$ there are n^2 intersections of the drawn segments.

We now compute how many regions these drawn segments created. As stated before, the already drawn n segments from vertex C to AB divided the plane into $n+1$ regions. The first of the n segments from vertex B to AC will divide all of these $n+1$ regions in half, resulting in $2(n+1)$ regions. Then, each of the $n-1$ remaining line segments from vertex B to AC will divide the $(n+1)$ defined regions from Step 1 (this is essentially adding $n+1$ regions). Thus, the total number of regions is given by $2(n+1)+(n-1)(n+1) = 2n+2+n^2-1 = n^2+2n+1 = (n+1)^2$.

We show a photo below for the $n=2$ case.



(b) We are going to draw n points on the side BC and connect to vertex A, which essentially means that we will draw n line segments from BC to vertex A. Note that we have the constraint that no three of these drawn segments will intersect at a single point. This means that each of these n line segments from BC to vertex A will not intersect with an already defined intersection of two line segments (as this would lead to an intersection of three drawn segments, which is prohibited.)

So far, we have drawn $2n$ line segments with n^2 intersections. Each of these n line segments from BC to vertex A will intersect (at a unique point) with all of the $2n$ line segments already drawn, thus each line segment contributes $2n$ new intersections $\implies$ all n line segments contribute $n(2n) = 2n^2$ new intersections. Thus, the total number of intersections is given by the previous number of intersections, n^2, plus the number of newly formed intersections, $2n^2$, or $3n^2$.

(c) As we have already stated, each of the n drawn line segments from BC to vertex A will intersect with $2n$ other line segments $\implies$ will split up $2n+1$ regions $\implies$ will contribute $2n+1$ new regions. Thus, these n line segments from BC to vertex A will create $n(2n+1)$ new regions.

The total number of regions after these line segments are drawn is given by the previous number of regions, $(n+1)^2$, plus the number of newly formed regions, $n(2n+1)$, or $(n+1)^2 + n(2n+1) = n^2 + 2n + 1 + 2n^2 + n = 3n^2 + 3n + 1$.

5.

Note on Notation. Let set A_k denote the set of all the multiples of $k \in \mathbb{N}$ in $[1000]$. Thus, the number of multiples of k in $[1000]$ can be given by $|A_k|$.

Note on the cardinality of intersections of A_k sets. Let us define $n_1, n_2 \ldots, n_k \in \mathbb{N}$. We denote here the procedure for computing $|\bigcap_{i=1}^{k} A_{n_i}|$, or the size of the number of elements in $[1000]$ that are multiples of $n_1, \ldots, n_k$. A number is a multiple of $n_1, \ldots, n_k$ if it is a multiple of $\gamma = \mathrm{LCM}(n_1, \ldots, n_k)$. Thus, every γ-th number in $[1000]$ is a multiple of $n_1, \ldots, n_k \implies |\bigcap_{i=1}^{k} A_{n_i}| = \mathrm{flr}(\frac{1000}{\mathrm{LCM}(n_1,\ldots,n_k)})$ where $\mathrm{flr}(\cdot)$ is the floor function.

Solving This Problem We first aim to compute the number of elements of $[1000]$ that are multiples of at least one of the following: $2, 3, 5$, or 7. This can be given by $|A_2 \cup A_3 \cup A_5 \cup A_7|$. By the Inclusion-Exclusion Principle, we have:

$$\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= |A_2| + |A_3| + |A_5| + |A_7| \\
&-(|A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7|) \\
&+(|A_2 \cap A_3 \cap A_5| + |A_2 \cap A_3 \cap A_7| + |A_2 \cap A_5 \cap A_7| + |A_3 \cap A_5 \cap A_7|) \\
&-|A_2 \cap A_3 \cap A_5 \cap A_7|
\end{aligned}$$

We use the aforementioned procedure to compute the cardinality of each intersection of A_k sets. This gives us:

$$
\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= 500 + 333 + 200 + 142 \\
&-(166 + 100 + 71 + 66 + 47 + 28) \\
&+(33 + 23 + 14 + 9) \\
&-4 = 772
\end{aligned}
$$

The number of elements of [1000] that are remaining after removing all elements that are multiples of $2, 3, 5$, or 7 is given by $1000 - 772 = \mathbf{228}$.

6.

$n!$ gives the number of ways to give unique orderings, or permutations, of n elements. Each permutation can be classified according to this property: the number of *fixed points* that stay the same before and after the permutation. The number of fixed points is strictly less than the number of elements getting permuted (i.e. n). Thus we can informally write:

$$\# \text{ of permutations of } n \text{ elements} = n! = \Sigma_{k=0}^{n} \# \text{ of permutations of } n \text{ elements with } k \text{ fixed points}$$

We now try to see for k fixed points, how many unique permutations of n elements we have. We go through a few cases before generalizing our argument:

Zero Fixed Points. If there are zero fixed points, the number of possible orderings is given by $D(n)$, where $D(n)$ represents the number of possible dearrangements for n elements.

One Fixed Point. We now try to understand the number of orderings of n elements with one fixed point. We have $\binom{n}{1} = n$ choices on where to place this fixed point. Fixing that position, there are $D(n-1)$ ways to de-arrange (i.e. have 0 fixed points) for the remaining $n-1$ elements. Thus, we have $nD(n-1)$ unique orderings of n elements with one fixed point.

Two Fixed Points. We have $\binom{n}{2}$ choices on where to place the two fixed points. For the remaining $n-2$ elements, we can arrange them in $D(n-2)$ ways to maintain their order. Thus there are $\binom{n}{2}D(n-2)$ unique orderings of n elements with one fixed point.

Thus, we can see for $0 \leq k \leq n$ fixed points, we have $\binom{n}{k}D(n-k)$ unique orderings. Re-using our expression from above, we have:

$$n! = \Sigma_{k=0}^{n} \binom{n}{k} D(n-k)$$

$$n! = D(n) + nD(n-1) + \binom{n}{2} D(n-2) + \cdots + \binom{n}{n-1} D(n-(n-1)) + \binom{n}{n} D(n-n)$$

$$D(n) = n! - nD(n-1) - \binom{n}{2} D(n-2) - \cdots - \binom{n}{n-1} D(1) - (1)D(0)$$

We now compute $D(0)$. Because there is only one permutation of $\emptyset$ (i.e. $0! = 1$) and any permutation of $\emptyset$ has no fixed points as there are no points of $\emptyset$ to be fixed $\implies$ the one possible permutation of $\emptyset$ is vacuously de-arranged $\implies D(0) = 1$.

$$D(n) = n! - nD(n-1) - \binom{n}{2} D(n-2) - \cdots - \binom{n}{n-1} D(1) - 1$$