

PSETs Landing Page*

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This is the documentation for using my PSET PDFs responsibly. I post these LaTeX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarly, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

Math 225- HW 9 Due: December 8 by Midnight

1. (13 points) Let $T : V \rightarrow V$, be a linear operator and V finite dimensional vector space. Recall that $\det(T) = \det[T]_\beta$ for some β ordered basis for V . Prove that
 - a) (4 points) The definition of $\det(T)$ is well-defined, i.e., if γ is another ordered basis for V then $\det[T]_\beta = \det[T]_\gamma$.
 - b) (2 points) Show that $\det([T]_\gamma - \lambda I) = \det([T]_\beta - \lambda I)$.
 - c) (2 points) Use part b) to deduce that similar matrices have the same characteristic polynomial. (Definition of similar matrices is given in the remark)
 - d) (5 points) If $g(t)$ be polynomial with coefficient from \mathbb{R} , then if x is an eigenvector for T with corresponding eigenvalue λ , then x is an eigenvector for $g(T)$ with corresponding eigenvalue $g(\lambda)$. - Use definition of eigenvalue.
2. (10 points) Let A be an upper(lower) triangular matrix, and has the distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ with corresponding multiplicities m_1, m_2, \dots, m_k .
 - a) Prove that $\text{tr}(A) = \sum_{i=1}^k m_i \lambda_i$
 - b) Prove that $\det(A) = \prod_{i=1}^k (\lambda_i)^{m_i}$.

Remark: We say $A, B \in M_{n \times n}(\mathbb{R})$ are *similar matrices* if there exist $P \in M_{n \times n}(\mathbb{R})$ invertible such that $PAP^{-1} = B$. Recall that if A is similar to B then $\text{tr}(A) = \text{tr}(B)$, and $\det(A) = \det(B)$. Therefore, the statement of this problem is true for any matrix that is similar to upper (lower) triangle matrix.

3. (31 points) Consider the following matrices $A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$
 - a) (9 points) Decide if they are diagonalizable in \mathbb{R} .
 - b) (2 points) Decide if they are diagonalizable in \mathbb{C} .
 - c) (5 points) Find Q and D matrix such that $A = Q^{-1}DQ$.
 - d) (5 points) Use part c) to find A^k for $k = 0, 1, 2, \dots$
 Hint: $(Q^{-1}DQ)^k = \underbrace{(Q^{-1}DQ)(Q^{-1}DQ), \dots (Q^{-1}DQ)}_{k \text{ times}}$.
 - e) (10 points) Find e^A . Hint: Use the Taylor expansion of $e^x = \sum_{n=0}^{\infty} (x^n/n!)$ and part d).

I think it is so cool to be able to define exponential of a matrix. You can do it for any function that has Taylor expansion in its radius of convergence. We will define the norm of a matrix in the coming weeks.

4. (23 points)
 - a) (5 points) Let A be a matrix whose characteristic polynomial split over its field \mathbb{F} . Prove that the determinant of A is the product of its eigenvalues, each counted with its multiplicity. (that is if the algebraic multiplicity of an eigenvalue if m then it is multiplied m times.)
 - b) (3 points) Use part a to conclude that if A is defined over \mathbb{C} , the complex numbers, then the determinant of A is always the product of its eigenvalues, each counted with its multiplicity.

- c) (15 points) Suppose A is a real $n \times n$ matrix which satisfies $A^3 = A + I_n$. Show that A has a positive determinant.

Hint: Even though A is real valued you can consider its eigenvalues in \mathbb{C} . So, try to find an equation that the eigenvalues satisfy. Here the fact that A is real must give you hint about complex eigenvalues.