

## Discretionary Note

Anish Krishna Lakkapragada

**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

**CONTENT STARTS ON NEXT PAGE.**

To access the general instructions for this repository head [here](#).

# Problem set 7

**Exercise 7.1 (10 points; Rudin 3.2, modified).** Calculate  $\lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n$ , and prove that your answer is correct. (Hint: first show that  $\sqrt{n^2 + n} - n = \frac{n}{\sqrt{n^2 + n} + n}$ .)

**Exercise 7.2 (10 points).** For any two bounded real sequences  $(a_n), (b_n)$  in  $\mathbb{R}$  prove that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

Give an example where this  $\leq$  is  $<$ , and an example where it is  $=$ .

**Exercise 7.3 (10 points; Rudin 3.24).** Suppose  $(p_n)$  and  $(q_n)$  are Cauchy sequences in a metric space  $X$ . Prove that the sequence  $(d(p_n, q_n))$  in  $\mathbb{R}$  has a limit.

**Exercise 7.4 (10 points).** Suppose  $(x_n)$  is a sequence in  $\mathbb{R}$ . We say  $a \in \mathbb{R}$  is an *essential upper bound* for  $(x_n)$  if there exists some  $N$  such that, for all  $n \geq N$ ,  $x_n \leq a$ .

Prove that

$$\limsup_{n \rightarrow \infty} x_n = \inf \{a \in \mathbb{R} \mid a \text{ is an essential upper bound for } (x_n)\}.$$

**Exercise 7.5 (not for credit; Rudin 3.25, in part).** Let  $X$  be a metric space.  $X$  might or might not be complete. If  $X$  is not complete, it would be nice to know how to “fill in the holes” to make it complete. This exercise explains a way of doing so: it constructs a new complete metric space  $X^*$  which has  $X$  as a subset.

We call two Cauchy sequences  $(p_n), (q_n)$  in  $X$  *equivalent* if  $d(p_n, q_n) \rightarrow 0$ . We write this relation as  $(p_n) \sim (q_n)$ .

(1) Prove that this is an equivalence relation, i.e.

- (a) Any Cauchy sequence  $(p_n)$  has  $(p_n) \sim (p_n)$ ,
- (b) If  $(p_n) \sim (q_n)$  then  $(q_n) \sim (p_n)$ ,
- (c) If  $(p_n) \sim (q_n)$  and  $(q_n) \sim (r_n)$ , then  $(p_n) \sim (r_n)$ .

(2) If  $(p_n) \sim (p'_n)$  and  $(q_n) \sim (q'_n)$ , prove that

$$\lim_{n \rightarrow \infty} d(p_n, q_n) = \lim_{n \rightarrow \infty} d(p'_n, q'_n).$$

(Note that the limit does exist, by the result of the previous exercise.)

Now we divide the set of Cauchy sequences in  $X$  into *equivalence classes*: any two elements of a given class  $P$  are equivalent, and elements of different classes  $P, Q$  are not equivalent. Let  $X^*$  be the set of all equivalence classes of Cauchy sequences in  $X$ .

Then, define a distance function  $\Delta$  on  $X^*$  as follows: if  $(p_n)$  is in the class  $P$ , and  $(q_n)$  is in the class  $Q$ , then

$$\Delta(P, Q) = \lim_{n \rightarrow \infty} d(p_n, q_n).$$

The previous parts show that this distance function is well defined.

(3) Prove that the distance function  $\Delta$  makes  $X^*$  into a metric space.

(4) Prove that  $X^*$  with this distance function is complete.

(5) Consider the map  $\phi : X \rightarrow X^*$  which maps any  $x \in X$  to the class of the Cauchy sequence  $(x, x, x, \dots)$ . Prove that  $\phi$  is injective and  $\Delta(\phi(x), \phi(y)) = d(x, y)$ .

**Exercise 7.6 (not for credit).** Suppose  $X$  is any metric space, with a distance function  $d$ . Then define a new distance function  $d'$  on  $X$  by

$$d'(x, y) = \min\{d(x, y), 1\}.$$

(1) Prove that  $d'$  indeed makes  $X$  into a metric space.

(2) Prove that a sequence is Cauchy for  $d$  if and only if it is Cauchy for  $d'$ .

(3) Prove that a sequence is convergent for  $d$  if and only if it is convergent for  $d'$ .

**Exercise 7.7 (not for credit).** Suppose  $X$  is any metric space. We say  $X$  is *totally bounded* if, for every  $\epsilon > 0$ ,  $X$  can be covered by finitely many neighborhoods  $N_\epsilon(x)$ . Prove that a subset  $E \subset X$  is compact if and only if  $E$  is closed and totally bounded.