Math 226- HW 10 Due: November 22 by Midnight

1. a) By definition, $[T]_\beta$ and $[T]_\gamma$ represent the same linear operator T across different bases. Thus, that means that we can establish the following relationship between $[T]_\beta$ and $[T]_\gamma$, where $Q = [I_V]_\beta^\gamma$:

$$[T]_\beta = Q^{-1}[T]_\gamma Q$$

Applying the determinant on each side we get:

$$\begin{aligned} det([T]_\beta) &= det(Q^{-1}[T]_\gamma Q) \\ det([T]_\beta) &= det(Q^{-1})det([T]_\gamma)det(Q) \\ det([T]_\beta) &= det(Q^{-1})det(Q)det([T]_\gamma) \end{aligned}$$

Because Q and Q^{-1} are inverses, $QQ^{-1} = I \Rightarrow det(QQ^{-1}) = det(I) \Rightarrow det(Q)det(Q^-1) = 1$. Thus,

$$det([T]_\beta) = det([T]_\gamma)$$

b) We reuse the relationship $[T]_\beta = Q^{-1}[T]_\gamma Q$ from part (a). We compute $det([T]_\beta - \lambda I)$ below:

$$det([T]_\beta - \lambda I) = det(Q^{-1}[T]_\gamma Q - \lambda I)$$

Note that $I = Q^{-1}Q = Q^{-1}IQ$ and so we have that $\lambda I = \lambda(Q^{-1}IQ) = Q^{-1}\lambda IQ$. Thus, we have that:

$$det([T]_\beta - \lambda I) = det(Q^{-1}[T]_\gamma Q - Q^{-1}\lambda IQ) = det(Q^{-1}([T]_\gamma - \lambda I)Q) =$$
$$det(Q^{-1})det([T]_\gamma - \lambda I)det(Q) = det(Q)det(Q^{-1})det([T]_\gamma - \lambda I) = det([T]_\gamma - \lambda I)$$

c) Two matrices $A, B \in M_{n\times n}(\mathbb{R})$ are similar if $\exists\ P \in M_{n\times n}(\mathbb{R})$ s.t. $PAP^{-1} = B$. For example, $[T]_\gamma$ and $[T]_\beta$ are similar matrices as $\exists\ Q$ (defined in part (a)) s.t. $[T]_\beta = Q^{-1}[T]_\gamma Q$. We can generalize the proof in (b) from two matrices representing the same transformation in different bases to two similar matrices. Thus, we have that for two similar matrices A and B, $det(A - \lambda I) = det(B - \lambda I) \Rightarrow$ two similar matrices A and B have the same characteristic polynomial.

d) We first define $g(t)$ as an nth degree polynomial given by $g(t) = \Sigma_{i=0}^n a_i t^n$, where coefficient $a_i \in \mathbb{R}$. This question asks us to prove that if x is an eigenvector for T with corresponding eigenvalue λ (i.e. $Tx = \lambda x$), then x is an eigenvector for $g(T)$ with corresponding eigenvalue $g(\lambda)$ (i.e. $g(T)x = g(\lambda)x$). To prove that $g(T)x = g(\lambda)x$, we compute LHS and RHS and show that they are equal. We first compute the LHS:

$$g(T)x = (\Sigma_{i=0}^n a_i T^i)x = \Sigma_{i=0}^n a_i T^i(x)$$

To understand the value of $T^i(x)$ we show an example with $i = 2$: $T^2(x) = T(T(x)) = T(\lambda x) = \lambda T(x) = \lambda^2 x$. Thus, we can see that $T^i(x)$ represents applying T i times, sequentially, on x and so $T^i(x) = \lambda^i x$. Thus, we can simplify the LHS as:

$$g(T)x = \Sigma_{i=0}^{n} a_i T^i(x) = \Sigma_{i=0}^{n} a_i \lambda^i x = (\Sigma_{i=0}^{n} a_i \lambda^i)x$$

We now compute the RHS:

$$g(\lambda)x = (\Sigma_{i=0}^{n} a_i \lambda^i)x$$

Thus, we can clearly see that the LHS = RHS and so we have proven that if $Tx = \lambda x$, then $g(T)x = g(\lambda)x$.

2. a) We first define $A \in M_{n\times n}(\mathbb{R})$. We first compute the eigenvalues of A by solving $det(A - \lambda I) = p(\lambda) = 0$. Because A is an upper (lower) triangular matrix, $A - \lambda I$ is also an upper (lower) triangular matrix and so its determinant is given by the product across the main diagonal. This means that the characteristic polynomial of A is given by:

$$p(\lambda) = det(A - \lambda I) = \prod_{i}^{n} (A_{ii} - \lambda)$$

and so the eigenvalues of A (i.e. the solutions to $p(\lambda) = 0$) are given by the values on the diagonal of A. In other words, λ_j, the jth eigenvalue of A, is given by A_{jj}. Thus, we compute $tr(A)$ as:

$$tr(A) = \Sigma_{j=0}^{n} A_{jj} = \Sigma_{j=0}^{N} \lambda_j$$

Because the ith distinct eigenvalue λ_i has a multiplicity of m_i, λ_i is present m_i times across the diagonal of $A \Rightarrow tr(A) = \Sigma_{i=0}^{k} m_i \lambda_i$.

b) Because A is an upper (lower) triangular matrix, $det(A)$ is given as the product of A along its main diagonal. Thus, we can compute $det(A)$ as:

$$det(A) = \prod_{j=1}^{n} a_{jj} = \prod_{j=1}^{n} \lambda_j$$

Because the ith distinct eigenvalue λ_i has a multiplicity of m_i, λ_i is present m_i times across the diagonal of $A \Rightarrow det(A) = \prod_{i}^{k} (\lambda_i)^{m_i}$.

3. a) Please see end of this PDF for solution to 3(a).

b) Please see end of this PDF for solution to 3(b).

c) Please see end of this PDF for solution to 3(c).

d) We compute A^k below:

$$A^k = \underbrace{(Q^{-1}DQ)(Q^{-1}DQ), ...(Q^{-1}DQ)}_{\text{k times}}$$
$$A^k = Q^{-1}\underbrace{D, \dots, D}_{\text{k times}}Q$$
$$A^k = Q^{-1}D^kQ$$
$$A^k = Q^{-1}\begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}Q$$

where eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 2$ as computed in part(a).

e) We compute e^A below.

$$e^A = \Sigma_{n=0}^{\infty}\frac{A^n}{n!} = \Sigma_{n=0}^{\infty}\frac{Q^{-1}D^nQ}{n!} = Q^{-1}(\Sigma_{n=0}^{\infty}\frac{D^n}{n!})Q =$$
$$Q^{-1}(\Sigma_{n=0}^{\infty}\frac{1}{n!}\begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix})Q = Q^{-1}(\Sigma_{n=0}^{\infty}\begin{bmatrix} \frac{1}{n!} & 0 & 0 \\ 0 & \frac{2^n}{n!} & 0 \\ 0 & 0 & \frac{2^n}{n!} \end{bmatrix})Q = Q^{-1}\begin{bmatrix} e & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^2 \end{bmatrix}Q$$

4. a) Let us define the characteristic polynomial of A as $det(A - \lambda I) = p(\lambda)$. We define this characteristic polynomial $p(\lambda) = (\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_k)^{m_k}$, where k is the number of distinct eigenvalues of A and m_i is the multiplicity of the ith distinct eigenvalue. Thus, we can show that:

$$det(A) = det(A - 0I) = p(0) = (\lambda_1 - 0)^{m_1} \dots (\lambda_k - 0)^{m_k} = \prod_{i=1}^{k} \lambda_i^{m_i}$$

b) By the Fundamental Theorem of Algebra, if A is defined in $\mathbb{C}$, then the characteristic polynomial $p(\lambda)$ of A with complex-valued coefficients will split into a product of linear factors, where each linear factor will have the form $(\lambda_i - \lambda)$. Thus, our proof in part (a) applies to show that $det(A) = \prod_{i=1}^{k} \lambda_i^{m_i}$, even if A is defined in $\mathbb{C}$.

c) Let us define λ as any given eigenvalue of A corresponding to eigenvector v. Thus, we can show:

$$A^3 = A + I_n$$
$$A^3v = (A + I_n)v = Av + I_nv$$
$$\lambda^3v - \lambda v - v = 0$$
$$(\lambda^3 - \lambda - 1)v = 0$$

Note that because v is an eigenvector, $v \neq 0$. Thus we have that any eigenvalue λ of A must satisfy:

$$\lambda^3 - \lambda - 1 = 0$$

From part (a) and (b), we know that $det(A) = \prod_i^k (\lambda_i)^{m_i}$. We show below that for the following cases that cover the entirety of the possible values of λ:

- (1) eigenvalue λ is complex-valued
- (2) eigenvalue λ is real-valued and positive
- (3) eigenvalue λ is equal to zero
- (4) eigenvalue λ is real-valued and negative

the only possible cases are (1) and (2) and so we are guaranteed that $det(A) = \prod_i^k \lambda_i^{m_i} > 0$.

Case 1 Eigenvalue λ is complex-valued

By the Conjugate Zeroes Theorem, complex roots (e.g. eigenvalues) of polynomials with real coefficients (i.e. $\lambda^3 - \lambda - 1$) always come in pairs. Let us define two unique complex eigenvalue roots as Z and $\bar{Z}$ with multiplicities of m_Z and $m_{\bar{Z}}$ respectively. Because complex eigenvalue roots always come in pairs, $m_Z = m_{\bar{Z}}$. Thus, in the computation of $det(A) = \prod_i^k \lambda_i^{m_i}$, we will have $Z^{m_Z} \bar{Z}^{m_{\bar{Z}}} = (Z\bar{Z})^{2m_Z}$. Because the product of two complex conjugates is always a positive real number, $(Z\bar{Z})^{2m_Z}$ is equal to a positive number raised to $2m_Z$th power and thus this resulting quantity is positive. This means that we have shown that if a given eigenvalue is complex-valued, it is guaranteed to have a *positive* contribution to $det(A)$.

Case 2 Eigenvalue λ is real-valued and positive

If this given eigenvalue is positive and real-valued, its contribution to the product of eigenvalues (i.e. $det(A)$) is positive.

Case 3 Eigenvalue λ is equal to zero

This is an impossible case. $\lambda = 0$ is not a solution to $\lambda^3 - \lambda - 1 = 0$ as $0^3 - 0 - 1 = -1 \neq 0$.

Case 4 Eigenvalue λ is real-valued and negative

We consider three subcases that cover all the possible values of λ for this case: (a) $-1 < \lambda < 0$; (b) $\lambda = -1$; (c) $\lambda < -1$.

For subcase (a), $\lambda^3 - \lambda - 1 = 0$ can be expressed as $\lambda^3 - c = 0$ where $c > 0$. This gives us that $\lambda^3 = c$, however this is not possible as the cube of a negative number cannot be positive. Thus λ cannot be from $(-1, 0]$.

For subcase (b), the equation $\lambda^3 - \lambda - 1 = 0$ simplifies to $(-1)^3 - (-1) - 1 = -1 \neq 0$. Thus, λ cannot be equal to -1.

For subcase (c), the equation $\lambda^3 - \lambda - 1 = 0$ can be expressed as $\lambda^3 + c = 0$, where $c = -\lambda - 1 > 0$. This means that $\lambda^3 = -c$. Note that there is no solution to this equation because λ^3 grows in magnitude at a much faster rate than c, which is linear to λ. Thus, λ cannot be from $(-\infty, 1)$.

Thus, we have shown that any eigenvalue λ can only be complex-valued or real-valued & positive. Because we have shown in both of these cases that the contribution to $det(A)$ is positive, we can guarantee that $det(A)$ is overall positive as well.

Question 3

(a) $\det(A-\lambda I)=0$

$$\det\begin{pmatrix}-1-\lambda & 3 & -1\\ -3 & 5-\lambda & -1\\ -3 & 3 & 1-\lambda\end{pmatrix}$$

$$=(-1-\lambda)\det\begin{pmatrix}5-\lambda & -1\\ 3 & 1-\lambda\end{pmatrix}-3\det\begin{pmatrix}-3 & -1\\ -3 & 1-\lambda\end{pmatrix}$$

$$-\det\begin{pmatrix}-3 & 5-\lambda\\ -3 & 3\end{pmatrix}=(-1-\lambda)[(5-\lambda)(1-\lambda)+3]$$

$$-3[-3(1-\lambda)-3]-[-9+3(5-\lambda)]$$

$$=(-1-\lambda)(\lambda^2-6\lambda+8)-6\lambda+12=-\lambda^3+5\lambda^2-8\lambda+4$$

$$=-(\lambda-1)(\lambda-2)^2$$

We now look at the eigenspaces:

$\underline{\lambda=1}$

$$\begin{pmatrix}-2 & 3 & -1\\ -3 & 4 & -1\\ -3 & 3 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$-2a+3b-c=0$ $\quad c=3b-2a$

$-3a+4b-c=0$ $\quad =a=b$

$-3a+3b=0 \Rightarrow a=b$

$$E_{\lambda=1}=\operatorname{span}\left(\left\{\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}\right\}\right)$$

G.M. $=1=$ A.M

$\underline{\lambda=2}$

$$\begin{pmatrix}-3 & 3 & -1\\ -3 & 3 & -1\\ -3 & 3 & -1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$-3a+3b-c=0$

$c=-3a+3b$

$$E_{\lambda=2}=\operatorname{span}\left(\left\{\begin{pmatrix}1\\ 1\\ 0\end{pmatrix},\begin{pmatrix}-1\\ 0\\ 3\end{pmatrix}\right\}\right)$$

G.M $=2=$ A.M.

Because for each $\lambda\in\mathbb{R}$, the geometric multiplicity is equal to the algebraic multiplicity, A is diagonalizable in $\mathbb{R}$.

Matrix B

$$\det(B-\lambda I)=0$$

$$\det\begin{pmatrix} 4-\lambda & 7 & -5 \\ -4 & 5-\lambda & 0 \\ 1 & 9 & -4-\lambda \end{pmatrix}=0$$

$$(4-\lambda)\det\begin{pmatrix} 5-\lambda & 0 \\ 9 & -4-\lambda \end{pmatrix} - 7\det\begin{pmatrix} -4 & 0 \\ 1 & -4-\lambda \end{pmatrix}$$

$$+5\det\begin{pmatrix} -4 & 5-\lambda \\ 1 & 9 \end{pmatrix}=0$$

$$=(4-\lambda)[(5-\lambda)(-4-\lambda)]-7[(-4)(-4-\lambda)]$$

$$+5[-36-(5-\lambda)]=0$$

$$-\lambda^3+5\lambda^2+16\lambda-80-112-28\lambda+205-5\lambda=0$$

$$-\lambda^3+5\lambda^2-17\lambda+13=0$$

$$-(\lambda-1)(\lambda^2-4\lambda+13)=0$$

Because this matrix has complex eigenvalues that do not belong in $\mathbb{R}$, ~~this~~ matrix is not diagonalizable in $\mathbb{C}$.

Matrix C

$$\det(C-\lambda I)=0$$

$$\det\begin{pmatrix}1-\lambda & 1 & 0\\ 0 & 1-\lambda & 2\\ 0 & 0 & 3-\lambda\end{pmatrix}=0$$

$$(1-\lambda)\det\begin{pmatrix}1-\lambda & 2\\ 0 & 3-\lambda\end{pmatrix}-\det\begin{pmatrix}0 & 2\\ 0 & 3-\lambda\end{pmatrix}=0$$

$$(1-\lambda)\left[(1-\lambda)(3-\lambda)\right]=0$$

$$(1-\lambda)^2(3-\lambda)=0$$

We now look at the eigenspaces

$\lambda=1$ (A.M=2)

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$b=0$

$2c=0 \Rightarrow c=0$

$$E_{\lambda=1}=\text{span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}\right)$$

$\downarrow$

the dimension of $E_{\lambda=1}$ is 1

so the geometric multiplicity is 1.

because the G.M=1 $\neq$ A.M,

where the A.M.=2, we can conclude that matrix C is not diagonalizable in $\mathbb{R}$.

$\lambda=3$ (A.M=1)

$$\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$-2a+b=0 \Rightarrow b=2a$

$-2b+2c=0 \Rightarrow b=c$

$$E_{\lambda=2}=\text{Span}\left(\left\{\begin{pmatrix}1/2\\1\\1\end{pmatrix}\right\}\right)$$

Question 3(b)

Matrix A : Because matrix A is diagonalizable in $\mathbb{R}$, and $\mathbb{R}$ is a subset of $\mathbb{C}$, A is diagonalizable in $\mathbb{C}$.

Matrix B : We compare the algebraic and geometric multiplicities of each eigenvalue

$$p(\lambda) = -(\lambda-1)(\lambda^2-4\lambda+13)=0$$

$$\lambda = \{1, 2 \pm 3i\}$$

$\lambda = 1$ (AM = 1)

$$\begin{pmatrix} 3 & 7 & -5 \\ -4 & 4 & 0 \\ 1 & 9 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$3a+7b-5c=0$ $\quad 10a-5c=0$

$\quad c=2a$

$-4a+4b=0 \Rightarrow a=b$

$a+9b-5c=0$

$$E_{\lambda=1} = \text{span}\left(\left\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}\right)$$

$G.M. = \dim(E_{\lambda=1}) = 1 = A.M$ ✓

$\lambda = 2+3i$ (AM = 1)

$$\begin{pmatrix} 2-3i & 7 & -5 \\ -4 & 3-3i & 0 \\ 1 & +9 & -6-3i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(2-3i)a+7b-5c=0$

$-4a+(3-3i)b=0 \Rightarrow a = \left(\frac{3-3i}{4}\right)b$

$a+9b+(-6-3i)c=0$

$$\left[\frac{(3-3i)}{4}+9\right]b = (6+3i)c$$

$$\left[\frac{3-3i+36}{4}\right]b = (6+3i)c$$

$$\left(\frac{-3i+39}{4}\right)b = (6+3i)c$$

$$b = \frac{4(6+3i)}{(-3i+39)}c = \frac{4(2+i)}{(-i+13)}c = \left(\frac{10}{17}+\frac{6}{17}i\right)$$

$$E_{\lambda=2+3i} = \text{span}\left(\left\{\begin{pmatrix} -\frac{3i}{17}+\frac{12}{17} \\ \frac{10}{17}+\frac{6i}{17} \\ 1 \end{pmatrix}\right\}\right)$$

$G.M. = \dim(E_{\lambda=2+3i}) = 1 = A.M$ ✓

Question 3b, Matrix B continued

$\lambda = 2-3i$ (A.M=1)

$$\begin{pmatrix} 2+3i & 7 & -5 \\ -4 & 3+3i & 0 \\ 1 & 9 & -6+3i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(2+3i)a + 7b - 5c = 0$

$-4a + (3+3i)b = 0 \Rightarrow a = \frac{(3+3i)}{4}b$

$a + 9b + (-6+3i)c = 0$

$\left[\frac{(3+3i)}{4} + 9\right]b = (6-3i)c$

$\left[\frac{3+3i+36}{4}\right]b = (6-3i)c$

$b = \frac{4(6-3i)}{[39+3i]}c = \left(\frac{10}{17} - \frac{6i}{17}\right)c$

$$E_{\lambda=2-3i} = \text{span}\left(\left\{\begin{pmatrix} \frac{12}{17} + \frac{3i}{17} \\ \frac{10}{17} - \frac{6i}{17} \\ 1 \end{pmatrix}\right\}\right)$$

$$\downarrow$$

$G.M. = \dim(E_{\lambda=2-3i}) = 1 = A.M.$

Because each eigenvalue exists in $\mathbb{C}$ and has an equal geometric multiplicity and algebraic multiplicity, matrix B is diagonalizable in $\mathbb{C}$.

Matrix C

Because all eigenvalues of matrix C existed in $\mathbb{C}$ and the algebraic multiplicity of them was not always equal to their geometric multiplicities, matrix C is not diagonalizable in $\mathbb{C}$.

(c) We define matrix D as having the eigenvalues of matrix A on its diagonal.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Because we are trying to find Q s.t. $A = Q^{-1}DQ$, we are trying to find Q s.t. $D = QAQ^{-1}$. Here, Q^{-1} would be given as the $n \times n$ matrix where the jth column is eigenvector v_j. Thus, to compute Q we find the inverse of Q^{-1} below.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 4 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 + R_3 \to R_1]{R_2 - R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & -1 \\ 0 & -1 & 4 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & -3 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 + 3R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

Thus, $Q = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 4 & -1 \\ -1 & 0 & 0 \end{bmatrix}$