

# PSETs Landing Page\*

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This is the documentation for using my PSET PDFs responsibly. I post these LaTeX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarly, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

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\*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

# S&DS 242/542: Homework 6

Due Wednesday, March 5, at 1PM

1. **The geometric model.** Suppose  $X_1, \dots, X_n \stackrel{IID}{\sim} \text{Geometric}(p)$ , where  $\text{Geometric}(p)$  is the geometric distribution on the positive integers  $\{1, 2, 3, \dots\}$  defined by the PMF

$$f(x | p) = p(1 - p)^{x-1}$$

with a single parameter  $p \in [0, 1]$ . You may use without proof that this distribution has mean  $1/p$  and variance  $(1 - p)/p^2$ .

Compute the method-of-moments estimate of  $p$ , as well as the MLE of  $p$ . For large  $n$ , what approximately is the sampling distribution of the MLE?

2. **The negative binomial model.** Suppose  $X_1, \dots, X_n \stackrel{IID}{\sim} \text{NegBinom}(r, p)$ , where  $\text{NegBinom}(r, p)$  is the negative binomial distribution on  $\{0, 1, 2, 3, \dots\}$  defined by the PMF

$$f(x | p) = \binom{x + r - 1}{x} (1 - p)^r p^x.$$

Here  $r > 0$  is a fixed and known positive integer, and  $p \in [0, 1]$  is the unknown parameter. You may use without proof that this distribution has mean  $pr/(1-p)$  and variance  $pr/(1-p)^2$ .

Compute the method-of-moments estimate of  $p$ , as well as the MLE of  $p$ . For large  $n$ , what approximately is the sampling distribution of the MLE?

### 3. Generalized method-of-moments and the MLE.

Consider a parametric model  $f(x | \theta)$  with parameter  $\theta \in \mathbb{R}$ , whose PDF takes a form

$$f(x | \theta) = e^{\theta T(x) - A(\theta)} h(x) \text{ for } x \in \mathcal{X} \quad (*)$$

where  $\mathcal{X}$  is the range of possible data values.

(a) Show that the model  $\text{Pareto}(\theta, 1)$  is of this form, where  $\mathcal{X} = [1, \infty)$ . What are the functions  $T(x)$ ,  $A(\theta)$ , and  $h(x)$  for this Pareto model?

(b) For any model of the form  $(*)$ , differentiate the identity

$$1 = \int_{\mathcal{X}} e^{\theta T(x) - A(\theta)} h(x) dx$$

with respect to  $\theta$  on both sides, to obtain a formula for  $\mathbb{E}_\theta[T(X)]$  in terms of  $A(\theta)$ . Verify that your formula is correct for the Pareto model in part (a).

[You may use  $\frac{d}{d\theta} \int_{\mathcal{X}} e^{\theta T(x) - A(\theta)} h(x) dx = \int_{\mathcal{X}} \frac{d}{d\theta} [e^{\theta T(x) - A(\theta)} h(x)] dx$  without justifying this exchange of differentiation in  $\theta$  and integration in  $x$ .]

(c) Let  $X_1, \dots, X_n \stackrel{IID}{\sim} f(x | \theta)$  where  $f(x | \theta)$  is of the form  $(*)$ , and consider the generalized method-of-moments estimator  $\hat{\theta}$  based on  $T(x)$ , i.e.  $\hat{\theta}$  is the value of  $\theta$  for which

$$\mathbb{E}_\theta[T(X)] = \frac{1}{n} \sum_{i=1}^n T(X_i).$$

If the MLE is the unique solution to the equation  $0 = \ell'_n(\theta)$  where  $\ell_n(\theta)$  is the log-likelihood, show that this generalized method-of-moments estimator is the same as the MLE.

Use this to explain why the generalized method-of-moments estimator based on  $T(x) = \log x$  in the Pareto( $\theta, 1$ ) model coincides with the MLE.

#### 4. Confidence intervals for a binomial proportion.

Let  $X_1, \dots, X_n \stackrel{IID}{\sim} \text{Bernoulli}(p)$ , and let  $\hat{p} = \bar{X}$ . We compare two different ways to construct a 95% confidence interval for  $p$ , both based on the Central Limit Theorem result

$$\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, p(1 - p)). \quad (**)$$

(a) Use the plugin estimate  $\hat{p}(1 - \hat{p})$  for the variance  $p(1 - p)$  to write down a 95% confidence interval for  $p$ . This is the approach discussed in Lecture 13.

(b) Instead of using this plugin estimate, note that equation  $(**)$  implies, for large  $n$ ,

$$\mathbb{P} \left[ -\sqrt{p(1 - p)} z^{(\alpha/2)} \leq \sqrt{n}(\hat{p} - p) \leq \sqrt{p(1 - p)} z^{(\alpha/2)} \right] \approx 1 - \alpha.$$

Solve the two equations  $\sqrt{n}(\hat{p} - p) = \pm \sqrt{p(1 - p)} z^{(\alpha/2)}$  for  $p$  in terms of  $\hat{p}$ , to obtain a different 95% confidence interval for  $p$ .

(c) Perform a simulation study to determine the true probability that the confidence intervals in parts (a) and (b) cover  $p$ , for the 9 combinations of sample sizes  $n = 10, 40, 100$  and true parameters  $p = 0.1, 0.3, 0.5$ . Report the simulated coverage probabilities in two tables. Which interval construction yields true coverage closer to 95% for small values of  $n$ ?

[For each combination of  $n$  and  $p$ , it may be helpful to perform at least 100,000 simulations. In R, you may simulate  $\hat{p}$  directly as `phat = rbinom(1,n,p)/n`.]