Problem set 1
Due Thursday January 23 at 11pm

Exercise 1.1 (5 points). Suppose A, B, C are sets and $f : A \to B$, $g : B \to C$ are functions. Let $h = g \circ f : A \to C$. Show that

(1) If h is injective, then f is injective.

(2) If h is surjective, then g is surjective.

Exercise 1.2 (5 points; Rudin 1.1). This exercise uses the real numbers $\mathbb{R}$, which we have not defined yet; but, for this exercise, all you need to know about $\mathbb{R}$ is that it is a field containing the field $\mathbb{Q}$ of rational numbers.

Suppose $r \in \mathbb{Q}$, $r \neq 0$, and $x \in \mathbb{R}$, $x \notin \mathbb{Q}$. Prove that $r + x \notin \mathbb{Q}$ and $rx \notin \mathbb{Q}$.

Exercise 1.3 (10 points; Rudin 1.3). Suppose F is a field with $x, y, z \in F$. Prove carefully from the field axioms:

(1) If $x \neq 0$ and $xy = xz$, then $y = z$.

(2) If $x \neq 0$ and $xy = x$, then $y = 1$.

(3) If $xy = 1$, then $x \neq 0$ and $y = x^{-1}$.

(4) If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Exercise 1.4 (10 points). This exercise concerns the field $\mathbb{Q}(\sqrt{3})$.

(1) Prove that $\mathbb{Q}$ does not contain any x with $x^2 = 3$.

(2) Consider the set $\mathbb{Q}(\sqrt{3})$, defined as the set of expressions "$a + b\sqrt{3}$" with $a, b \in \mathbb{Q}$. (More formally, we could think of $\mathbb{Q}(\sqrt{3})$ as the set of ordered pairs (a, b) with $a, b \in \mathbb{Q}$.) We can equip $\mathbb{Q}(\sqrt{3})$ with addition and product laws by

$$(a + b\sqrt{3}) + (a' + b'\sqrt{3}) = (a + a') + (b + b')\sqrt{3}\,,$$
$$(a + b\sqrt{3})(a' + b'\sqrt{3}) = (aa' + 3bb') + (ab' + ba')\sqrt{3}\,.$$

Show that $\mathbb{Q}(\sqrt{3})$ can be made into a field, with these addition and product laws. (This means saying carefully what are the negation law, inversion law, 0 and 1 elements, and then checking that all the axioms of a field are satisfied.)

(3) We could similarly consider the set $\mathbb{Z}(\sqrt{3})$, defined as above except that now we require $a, b \in \mathbb{Z}$, with the same addition and product laws. Show that $\mathbb{Z}(\sqrt{3})$ cannot be made into a field with these addition and product laws.

Exercise 1.5 (5 points). As we have stated, the usual order relation on $\mathbb{Q}$ can be expressed as:

$$p/q < p'/q' \quad \iff \quad pq' < p'q$$

when q and q' are positive.

We can define a different relation $\prec$, as follows: assume that all fractions are written in lowest terms and with positive denominators. Define

$$p/q \prec p'/q' \quad \iff \quad pq < p'q'\,.$$

Prove that the relation $\prec$ does *not* make $\mathbb{Q}$ into an ordered set.

Exercise 1.6 (5 points). Let A be a nonempty subset of an ordered set S. If there exists an element $\alpha \in S$ such that $\alpha \leq x$ for all $x \in A$, then we call α a *lower bound* for A. Similarly, if there exists $\beta \in S$ such that $x \leq \beta$ for all $x \in A$, we call β an *upper bound* for A. Suppose α is a lower bound of A and β is an upper bound of A. Prove that $\alpha \leq \beta$.