MATH 244 HW 2

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1. **Section 1.5, Question 5**

   We first prove if $R \circ (S \circ T)$ is well defined $\implies$ $(R \circ S) \circ T$ is well-defined. If $R \circ (S \circ T)$ is well-defined we can define sets A, B, C, D, where $R \subseteq A \times B, S \subseteq B \times C$, and $T \subseteq C \times D$. This means $S \circ T \subseteq B \times D$ and so $R \circ (S \circ T) \subseteq A \times D$, and so $R \circ (S \circ T)$ is well-defined[1]. We continue with these sets A, B, C, D for the rest of this problem.

   To prove $R \circ (S \circ T) = (R \circ S) \circ T$, we prove $R \circ (S \circ T) \subseteq (R \circ S) \circ T$ and $(R \circ S) \circ T \subseteq R \circ (S \circ T)$.

   1. $R \circ (S \circ T) \subseteq (R \circ S) \circ T$
      Let us pick $(x, y) \in R \circ (S \circ T)$. Let us define arbitrary elements $b \in B, c \in C$. Given $(x, y) \in R \circ (S \circ T)$, we know $\exists\ (x, b) \in R$ and $(b, y) \in S \circ T$. Furthermore, if $(b, y) \in S \circ T$, we know $\exists\ (b, c) \in S$ and $(c, y) \in T$. Because $(x, b) \in R$ and $(b, c) \in S \implies (x, c) \in R \circ S$. Furthermore, $(c, y) \in T$, so $(R \circ S) \circ T$ will contain $(x, y) \implies \forall (x, y) \in R \circ (S \circ T), (x, y) \in (R \circ S) \circ T \implies R \circ (S \circ T) \subseteq (R \circ S) \circ T$.
   2. $(R \circ S) \circ T \subseteq R \circ (S \circ T)$
      Let us pick $(x, y) \in (R \circ S) \circ T$. Let us define arbitrary elements $c \in C, b \in B$. Given $(x, y) \in (R \circ S) \circ T$, we know $\exists\ (x, c) \in R \circ S$ and $(c, y) \in T$. If $(x, c) \in R \circ S$, we know $\exists\ (x, b) \in R$ and $(b, c) \in S$. We now show $(x, y) \in R \circ (S \circ T)$. Because $(b, c) \in S$ and $(c, y) \in T$, $(b, y) \in S \circ T$. Furthermore, because $(x, b) \in R, (x, y) \in R \circ (S \circ T)$. Thus, we have shown, $\forall (x, y) \in (R \circ S) \circ T, (x, y) \in R \circ (S \circ T) \implies (R \circ S) \circ T \subseteq R \circ (S \circ T)$.

2. **Section 1.6, Question 3**

   We prove both directions of this statement below.

   1. **If $R \circ R \subseteq R \implies R$ is transitive**
      $R \circ R$ contains a pair (x, z) if (x, y) and (y, z) are both in R. If $R \circ R \subseteq R$, then this means if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R \circ R \implies (x, z) \in R$. This satisfies the definition of transitivity, which is given as follows: if $(x, y) \in R$ and $(y, z) \in R \implies (x, z) \in R$. Thus, R is transitive.

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[1] Put in other words, $R \circ (S \circ T)$ is well-defined because the codomain of R (set B) is the domain of $S \circ T$ and the codomain of S (set C) is the domain of T.

2. **If R is transitive $\implies R \circ R \subseteq R$**
   If R is transitive, this means if (x, y) and $(y, z) \in R \implies (x, z) \in R$. $R \circ R$ contains a pair (x, z) if (x, y) and (y, z) are both in R. However, because R is transitive, it is guaranteed by definition that $(x, z) \in R$ if (x, y) and $(y, z) \in R$. Thus, if R is transitive, $\forall (x, z) \in R \circ R, (x, z) \in R \implies R \circ R \subseteq R$.

3. **Section 1.6, Question 6**

A relation R on X is an equivalence relation if it is reflexive, transitive, and anti-symmetric. A relation is an ordering relation if it is reflexive, transitive, and symmetric. So a relation R is both an equivalence and ordering relation if it is reflexive, transitive, and both anti-symmetric & symmetric. R is anti-symmetric and symmetric if (i) if $\forall x, y \in X, \text{if } (x, y) \in R$, then $(y, x) \in R \iff y = x$ and (ii) $\forall x, y \in X, (x, y) \in R \implies (y, x) \in R$.

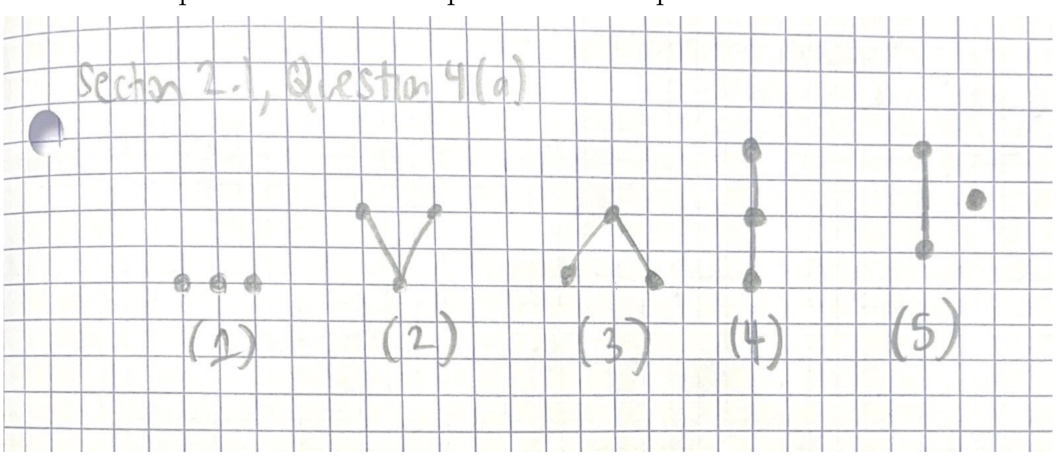
A relation R which is reflexive, transitive, antisymmetric, & symmetric will have the following property: $(x, y) \in R \iff x = y$. We prove this below:

1. $(x, y) \in R \implies x = y$
   Because R is symmetric, $(x, y) \in R \implies (y, x) \in R$. However, because R is anti-symmetric, $(y, x) \in R \implies y = x$ or $x = y$.
2. $x = y \implies (x, y) \in R$
   This property is satisfied by the fact that R is reflexive.

Thus, the relations on the set X that are both equivalences and orderings are identity relations given by $R = \{(x, x) : x \in X\}$.

4. **Section 2.1, Question 4**

a) We show all possible non-isomorphic 3-element posets below:



We enumerate over all possibilities of a non-isomorphic three element poset to clearly show we have classified all possible non-isomorphic three-element posets:

1. **All elements in the poset are not comparable to each other.**
   This is case (1).

2. **One minimal element in the poset. The other two elements are incomparable.**
   This is case (2).
3. **One maximal element in the poset. The other two elements are incomparable.**
   This is case (3).
4. **Every pair of elements is comparable,**
   This is case (4).
5. **One pair of elements in the poset are comparable to each other, and the remaining element is incomparable to all other elements.**
   This is case (5).

b) Let us define $(X, \leq)$ and $(Y, \preceq)$ to be linearly ordered sets with $|X| = |Y| = n$. This means that every single element in them can be compared with each other and so we can produce the following ordered[2] sequences L_X and L_Y of X and Y respectively:

$$L_X = x_1 \leq x_2 \leq \cdots \leq x_n$$
$$L_Y = y_1 \preceq y_2 \preceq \cdots \preceq y_n$$

Let us define the following map $f : X \to Y$ that where $f(x_i) = y_i$. Because $\forall y_i \in Y, \exists\ x_i$ s.t. $f(x_i) = y_i$, we can conclude f is surjective. Because $\forall\ 1 \leq i < j \leq n, f(x_i) \neq f(x_j)$, we also have that f is one-to-one or injective. Because f is injective and surjective $\implies f$ is bijective.

We now must show that f is order-preserving, or that $\forall x, x' \in X, x \leq x' \implies f(x) \preceq f(x')$. Note that because of the way L_X and L_Y are ordered (shown above), given $1 \leq i, j \leq n$, if $i < j$ then we have that $x_i \leq x_j$ and $f(x_i) \preceq f(x_j)$. Furthermore, we can see that given $x_i \leq x_j$, then $i < j$ and similarily given $f(x_i) \preceq f(x_j)$, then $i < j$. This means that we have shown both of these statements: (i) if $x_i \leq x_j \implies i < j \implies f(x_i) \preceq f(x_j)$ and (ii) if $f(x_i) \preceq f(x_j) \implies i < j \implies x_i \leq x_j$. Thus we have shown that $\forall x, x' \in X, x \leq x' \iff f(x) \preceq f(x')$. Because we have shown for any two n-element linearly ordered sets we can create a bijection $f : X \to Y$ where $\forall x, y \in X, x \leq y \iff f(x) \preceq f(y)$, we have proven that any two n-element linearly ordered sets are isomorphic.

5. **Section 2.2, Question 2**

a) We first define $<$ to be the divisibility relation $|$, set $B = \{1, 2, \ldots, n\}$, and the longest possible subset of B linearly ordered by $|$ as set A, where $m = |A|$. Because A is linearly ordered, this means that $\forall x, y \in A$, either $x \leq y$ or $y \leq x$. This means that if set A, when ordered from least to greatest, is given as $a_1, a_2, \ldots, a_k$ then we must have that $\forall\ 2 \leq i \leq k$, a_{i-1} must be able to divide a_i. Note that this guarantees $\forall 1 \leq j < i \leq k, a_i$ can divide any a_j because:

[2] in increasing order

$$\frac{a_i}{a_j} = \frac{a_i}{a_{i-1}} \frac{a_{i-1}}{a_{i-2}} \cdots \frac{a_{j+1}}{a_j}$$

which is just a product of natural numbers and thus is a natural number $\implies a_i$ can divide a_j. Our goal now is to construct $A \subseteq B$. To start, one must be in A (as its minimal element) as every natural number can be divided by A. Next to try to include in A as many elements of B as possible, we should aim for the $\forall\ 1 \leq j < i \leq n, \frac{a_i}{a_j} \in \mathbb{N}$ ratio to be as small as possible. This ratio cannot be 1, as each number appears only once in B. Thus, the next greatest natural number this ratio can be is two. This means that $2^{m-1} = |B| = n$ or $m = \log_2(n) + 1$. Note that the $+1$ term here is to account for the inclusion of 1 into set A.

b) We define the set $B = 2^{\{1,2,\dots n\}}$ and the longest possible subset of B linearly ordered by $\subseteq$ as set A, where $m = |A|$. Because A is linearly ordered, this means that $\forall x, y \in A$, either $x \subseteq y$ or $y \subseteq x$. This means that if set A, when ordered from least to greatest, is given as $a_1, a_2, \dots, a_k$ then we must have that $\forall\ 2 \leq i \leq k, a_{i-1} \subseteq a_i$. This guarantees $\forall 1 \leq j < i \leq k, a_j \subseteq a_i$ as:

$$a_j \subseteq a_{j+1} \subseteq a_{j+2} \cdots \subseteq a_i$$

Our goal now is to construct $A \subseteq B$. To start, $\emptyset$ must be in A (as its minimal element) as any $\emptyset \subseteq$ any set. Next to try to include in A as many of elements of B, we should aim for $\forall\ 2 \leq i \leq n$, a_i/a_{i-1} to be as small as possible. We can make $|a_i/a_{i-1}| = 1$ if every consecutive element includes one more element than the last. It will take n elements to go from a set with only one element to one with all n elements (i.e. set $\{1, 2, \dots, n\}$). Therefore the maximum number of elements required to make this chain is $n + 1$. I show a visualization below of these $n + 1$ elements that can build this set A:

$$A = \{\underbrace{\emptyset, \{1\}, \{1, 2\}, \dots, \{1, 2, \dots, n\}}_{n+1 \text{ elements}}\}$$

and so $m = |A| = n + 1$.

6. **Section 2.2, Question 3: Optional Bonus Problem**

a) We prove both directions of this statement:

1. **le**$(X, \preceq) = 1 \implies (X, \preceq)$ **is a linear ordering**
We prove this statement by contrapositive and assume that $(X, \preceq)$ is not a linear ordering $\implies \exists\ x_i, x_j \in X$ s.t. $x_i \npreceq x_j$ and $x_j \npreceq x_i$. This means that we can assemble at least two linear orderings for $(X, \preceq)$: one in which $x_i \preceq x_j$ and one in which $x_j \preceq x_i$. This means $\text{le}(X, \preceq) \neq 1$. Thus, we have proved this statement by contrapositive.

2. $(X, \preceq)$ **is a linear ordering** $\implies$ $\mathbf{le}(X, \preceq) = 1$
   Let us define a linear extension $<$ of $\preceq$: this means $\forall x, x' \in X, \text{if } x \preceq x' \implies x \leq x'$. Because $\preceq$ is already a linear ordering, this means $\forall x, x' \in X, x \preceq x'$ or $x' \preceq x$. This means this total ordering $<$ must have the following property: $\forall x, x' \in X, x \leq x'$ if $x \preceq x'$ or $x' \leq x$ if $x' \preceq x$. Thus, $<$ is the same ordering as $\preceq$ and so the number of linear extensions possible for $\preceq$ is only one (i.e. $\text{le}(X, \preceq) = 1$.)

b) The partial ordering $\preceq$ which can have the most possible linear extensions is one which imposes the least constraints. Such an ordering would be one where $\forall x, x' \in X$, x and x' are not comparable to each other (i.e. $x \not\preceq x'$ and $x' \not\preceq x$). This is because for such an ordering, a linear extension of this ordering can choose to order the n elements of X in any possible way. Because there are $n!$ ways to order n elements (i.e. X), this means for this such ordering (let's call it $\preceq$), $\text{le}(X, \preceq) = n!$. Note that for any other partial ordering it is guaranteed $\exists\, x, y \in X$ s.t. $x \preceq y$ and so any linear extension of this ordering must be compatible to this constraint $\implies$ this linear extension has $< n!$ ways to order X $\implies$ the number of linear extensions of this ordering is $< n!$. As such, for any partial ordering $\preceq$, $\text{le}(X, \preceq) \leq n!$.