

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# STATS 242 HW 3

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Number of late days: 0; Collaborators: Alex Wa, Derek Gao

1.

a) We detail our devised two possible test statistics:

(1) **Test Statistic  $T_1$ : Z-Statistic**

For this test statistic, we want to check whether the expected number of male children in each family is actually  $12(0.5) = 6$ , what we would expect under  $H_0$  if  $X_i \sim \text{Bin}(12, 0.5)$ . In other words,  $H_0 : \mu = 6$  and  $H_1 : \mu \neq 6$ . Because the variance of each  $X_i$  is known under  $H_0$ , we can use the  $Z$ -statistic as our test statistic:

$$T_1 = \frac{\sqrt{6115} \bar{X}}{\sigma}$$

In my code, we compute the  $T_1$  statistic for this data to be **260.48**.

(2) **Test Statistic  $T_2$ : Sample Variance**

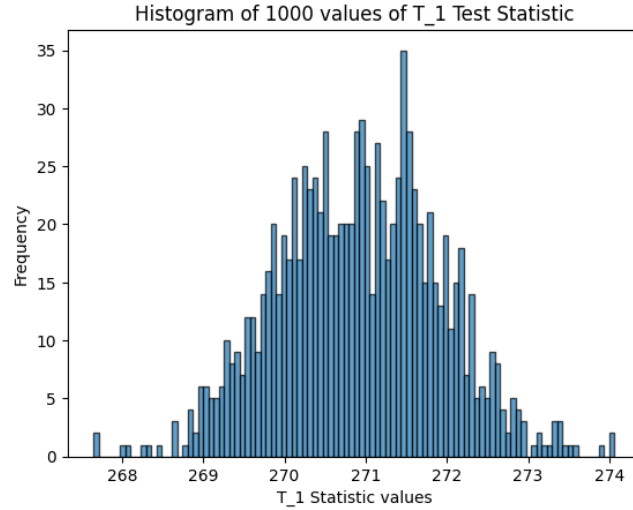
For this test statistic, we test if sample variance is actually equal to  $12(0.5)(0.5) = 3$ , as it would be under  $H_0$  where the observed male child frequencies follow a  $\text{Bin}(12, 0.5)$  distribution. Our test statistic is the sample variance:

$$T_2 = \frac{\sum_{i=1}^{6115} (X_i - \bar{X})^2}{6115 - 1}$$

In my code, we compute the  $T_2$  statistic for this data to be **3.490**.

b) (1) **Test Statistic  $T_1$**

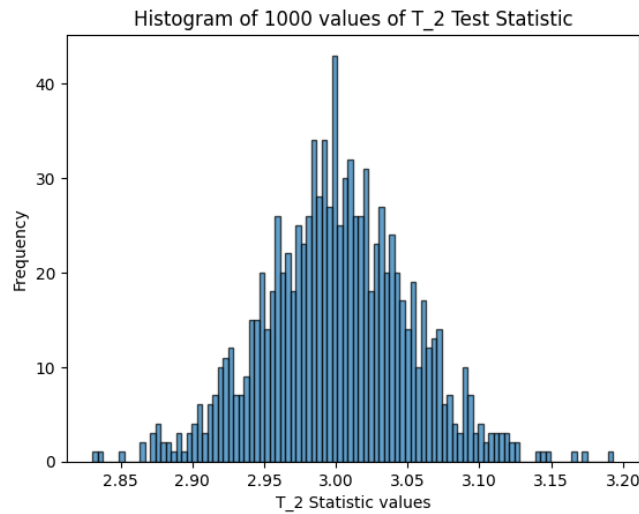
Below is our histogram for the null distribution of the  $T_1$  test statistic.



For this test statistic at the  $\alpha=0.05$  significance level, the upper- $\alpha$  point is **268.60** and the lower- $\alpha$  point which is **268.03**, which is less than the observed  $T_1$  statistic for this data (**260.48**). Thus, we reject  $H_0$ .

(2) **Test Statistic  $T_2$**

Below is our histogram for the null distribution of the  $T_2$  test statistic.



For this test statistic at the  $\alpha=0.05$  significance level, the upper- $\alpha$  point and lower- $\alpha$  point are given by **2.875** and **2.834** respectively. Because the observed  $T_2$  statistic for this data (**3.490**) is greater than the upper- $\alpha$  point, we reject  $H_0$ .

```
1 # %%
2 """Run all imports"""
```

```

3 import numpy as np
4 import matplotlib.pyplot as plt
5 import math
6
7 NUM_MALES = np.array([i for i in range(0, 12 + 1)])
8 NUM_FAMILIES = np.array([7, 45, 181, 478, 829, 1112, 1343, 1033, 670,
    ↪ 286, 104, 24, 3])
9 VARIANCE = 12 * 0.5 * (1 - 0.5)
10
11 assert np.sum(NUM_FAMILIES).item() == 6115
12
13 def t1_statistic(num_males, num_families):
14     mean_num_males = np.sum(num_families * num_males) /
    ↪ np.sum(num_families)
15     return np.sqrt(6115 / VARIANCE) * mean_num_males
16
17 def t2_statistic(num_males, num_families):
18     mean_num_males = np.sum(num_families * num_males) /
    ↪ np.sum(num_families)
19     return np.sum(num_families * (num_males - mean_num_males) ** 2) /
    ↪ (6115 - 1)
20
21 print(f"On this data, our T_1 statistic is {t1_statistic(NUM_MALES,
    ↪ NUM_FAMILIES)}")
22 print(f"On this data, our T_2 statistic is {t2_statistic(NUM_MALES,
    ↪ NUM_FAMILIES)}")
23 # %%
24 def get_data_for_one_simulation():
25     # for 6115 families, sample the number of children they have based
    ↪ on Bin(12, 0.5)
26     num_males = NUM_MALES
27     num_families = np.zeros_like(NUM_FAMILIES)
28     num_males_6115 = np.random.binomial(12, 0.5, 6115)
29     for i, num_male in enumerate(num_males):
30         num_families[i] = np.sum(num_males_6115 == num_male)
31     assert np.sum(num_families) == 6115
32     return num_males, num_families
33
34
35 N_SIMULATIONS = 1000
36
37 def get_statistics_simulation(statistic_func):
38     test_statistics = []
39     for _ in range(N_SIMULATIONS):
40         num_males, num_families = get_data_for_one_simulation()
41         test_statistics.append(statistic_func(num_males,
    ↪ num_families))

```

```

42     return np.array(test_statistics)
43
44     """
45     Run 1000 simulations to approximate null distribution of T_1 and T_2
46     ↪ Statistics.
47     """
48
49     t1_statistics = get_statistics_simulation(t1_statistic)
50     t2_statistics = get_statistics_simulation(t2_statistic)
51     """
52     Plot histogram of these statistics.
53     """
54
55     plt.hist(t1_statistics, bins=100, edgecolor='black', alpha=0.7)
56
57     # Add labels and title
58     plt.xlabel('T_1 Statistic values')
59     plt.ylabel('Frequency')
60     plt.title('Histogram of 1000 values of T_1 Test Statistic')
61     plt.show()
62
63     plt.hist(t2_statistics, bins=100, edgecolor='black', alpha=0.7)
64     plt.xlabel('T_2 Statistic values')
65     plt.ylabel('Frequency')
66     plt.title('Histogram of 1000 values of T_2 Test Statistic')
67     # %%
68     """
69     For the T_1 Statistic, can we reject?
70     """
71
72     SIGNIFICANCE_LEVEL = 0.05
73     OBSERVED_T1_STATISTIC = t1_statistic(NUM_MALES, NUM_FAMILIES)
74     t1_critical_value_upper_end = np.percentile(t1_statistics, 0.95)
75     t1_critical_value_lower_end = np.percentile(t1_statistics, 0.05)
76     if (t1_critical_value_upper_end <= OBSERVED_T1_STATISTIC): print("Can
77     ↪ reject null hypothesis for T_1 test statistic.")
78     if (t1_critical_value_lower_end >= OBSERVED_T1_STATISTIC): print("Can
79     ↪ reject null hypothesis for T_1 test statistic.")
80
81     """
82     For the T_2 Statistic, can we reject?
83     """
84
85     SIGNIFICANCE_LEVEL = 0.05
86     OBSERVED_T2_STATISTIC = t2_statistic(NUM_MALES, NUM_FAMILIES)
87     t2_critical_value_upper_end = np.percentile(t2_statistics, 0.95)
88     t2_critical_value_lower_end = np.percentile(t2_statistics, 0.05)
89     if (t2_critical_value_upper_end <= OBSERVED_T2_STATISTIC): print("Can
90     ↪ reject null hypothesis for T_2 test statistic.")

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```

84 if (t2_critical_value_lower_end >= OBSERVED_T2_STATISTIC): print("Can
    ↪ reject null hypothesis for T_2 test statistic.")
85
86 # %%

```

2.

- a) Note that because  $H_0$  is true, then we can assume the true distribution of to be  $Z \sim \mathcal{N}(0, 1)$ . To find the distribution of  $P = 1 - \Phi(Z)$ , we compute its CDF below:

$$P(P \leq p) = P(1 - \Phi(Z) \leq p)$$

Note that by the Universality of the Uniform,  $\Phi(Z) \sim \text{Unif}(0, 1)$ . Therefore,  $1 - \Phi(Z) = 1 - \text{Unif}(0, 1) = \text{Unif}(-1, 0) + 1 = \text{Unif}(0, 1)$ . Let us define  $U \sim \text{Unif}(0, 1)$ . Putting this all together, we have:

$$P(P \leq p) = P(1 - \Phi(Z) \leq p) = P(U \leq p) = F_U(p)$$

where  $F_U$  is the CDF of  $U$ . Because r.v.  $P$  has the same CDF as  $U \sim \text{Unif}(0, 1)$ , we can conclude  $P \sim \text{Unif}(0, 1)$  if  $H_0$  is true.

- b) If  $H_1$  was true, then we would expect to observe greater values of test statistic  $Z \implies \Phi(Z)$ , which is strictly non-decreasing, would be greater  $\implies P = 1 - \Phi(Z)$  would be smaller. Thus, if  $H_1$  was true, we would expect to observe smaller p-values. As shown in part (a),  $P \sim \text{Unif}(0, 1)$  so  $P_{H_0}(P \leq \alpha) = F_U(\alpha) = \alpha$ . Therefore, if we were to use  $P$  as the test statistic, we would reject  $H_0$  when  $P \leq \alpha$ .

3.

- a) Let us define  $Z_1, \dots, Z_n$  to all be i.i.d  $\mathcal{N}(0, 1)$  distributions. We can define distribution  $U_n = \frac{Z_1^2 + \dots + Z_n^2}{n}$ , or equivalently that  $U_n \sim \frac{1}{n} \cdot \chi_n^2$ . We first compute  $\mathbb{E}[Z_i]^2 = \text{Var}[Z_i] + \mathbb{E}[Z_i] = 1 + 0 = 1$ . By the Weak Law of Large Numbers, since  $\mathbb{E}[Z_i] = 1$  and  $Z_i$  has finite variance,  $U_n \rightarrow \mathbb{E}[Z_i]$  or  $U_n \rightarrow 1$  in probability as  $n \rightarrow \infty$ .

Let us now define function  $g(z) = \sqrt{z}$ , where  $g$  is continuous for  $(0, \infty)$ . By the Continuous Mapping Theorem, distribution  $g(U_n) = \sqrt{U_n} \rightarrow \sqrt{1}$ , or  $\sqrt{U_n} \rightarrow 1$  in probability as  $n \rightarrow \infty$ .

- b) The  $t$ -distribution with  $n$  degrees of freedom is given by:

$$T_n = \frac{\frac{\sqrt{n} \bar{X}}{\sigma}}{\sqrt{\frac{1}{n} \chi_n^2}}$$

where  $\bar{X} = \frac{X_1 + \dots + X_n}{n}$  and  $\sigma = \sqrt{\text{Var}(X_i)}$ . As shown in part (a),  $\sqrt{\frac{1}{n}\chi_n^2} \rightarrow 1$  in probability as  $n \rightarrow \infty$ . Furthermore, by CLT as  $n \rightarrow \infty$ ,  $\bar{X} \sim \mathcal{N}(0, \sigma^2)$  and thus  $\frac{\sqrt{n}}{\sigma}\bar{X} \rightarrow \mathcal{N}(0, 1)$ . Because  $\sqrt{\frac{1}{n}\chi_n^2} \rightarrow 1 \neq 0$ , we can apply Slutsky's Lemma. By Slutsky's Lemma,  $T_n = \frac{\frac{\sqrt{n}}{\sigma}\bar{X}}{\sqrt{\frac{1}{n}\chi_n^2}} \rightarrow \frac{\mathcal{N}(0,1)}{1}$  or  $T_n \rightarrow \mathcal{N}(0, 1)$  as  $n \rightarrow \infty$ .

4.

- (a)  $T \sim t_1$  is given by  $\frac{X}{\sqrt{\frac{1}{1}\chi_1^2}} = \frac{X}{\sqrt{Y^2}}$ , where  $X, Y$  are i.i.d  $\mathcal{N}(0, 1)$ . By the Continuous Mapping Theorem, given  $g(x) = \sqrt{x^2} = |x|$  then  $g(Y)$  will converge to  $|Y|$  and so  $T$  is given by  $\frac{X}{|Y|}$ .

We can define  $\frac{X}{|Y|}$  as the following:

$$\frac{X}{|Y|} = \begin{cases} \frac{X}{Y} & \text{if } Y > 0 \\ -\frac{X}{Y} = \frac{X}{-Y} & \text{if } Y < 0 \end{cases}$$

Because  $Y$  and  $-Y$  have the same distributions (because  $Y$  is symmetric about the origin), both the  $\frac{X}{Y}$  and  $\frac{X}{-Y}$  have the same distribution (which we can call  $U$ ). Because then we have  $\frac{X}{|Y|} = U \cdot P(Y > 0) + U \cdot P(Y < 0) = \frac{U+U}{2} = U$ , we can conclude  $\frac{X}{|Y|} \sim U$  or  $\frac{X}{|Y|} \sim \frac{X}{Y}$ <sup>1</sup>

- b) Let us define  $g(x, y) = (\frac{x}{y}, y)$ . Because  $g$  is bijective, its inverse can be given by  $g^{-1}(t, u) = (tu, u)$ . We can use the change-of-variables formula to compute the joint PDF  $f_{T,U}(t, u)$ <sup>2</sup>:

$$f_{T,U}(t, u) = f_{X,Y}(g^{-1}(t, u)) \cdot \left| \det \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = f_{X,Y}(tu, u) \cdot \left| \det \begin{pmatrix} \frac{\partial(tu)}{\partial t} & \frac{\partial(tu)}{\partial u} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial u} \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = f_{X,Y}(tu, u) \cdot \left| \det \begin{pmatrix} u & t \\ 0 & 1 \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = |u|f_{X,Y}(tu, u) = |u|f_X(tu)f_Y(u) = |u|\frac{1}{2\pi}e^{-\frac{[(tu)^2+u^2]}{2}} = |u|\frac{1}{2\pi}e^{-\frac{u^2(t^2+1)}{2}}$$

We now compute  $f_T$ :

$$f_T(t) = \int_{-\infty}^{\infty} f_{T,U}(t, u) du = \int_{-\infty}^{\infty} \frac{|u|}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du = \int_{-\infty}^0 \frac{-u}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du + \int_0^{\infty} \frac{u}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du$$

<sup>1</sup>Note that  $P(Y < 0) = P(Y > 0) = 0.5$  because  $Y$  is a normal distribution and is therefore symmetric about the origin.

<sup>2</sup>Note that because  $X$  and  $Y$  are independent,  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ .

Note that these two integrals are equivalent (we can use a change of variables  $dv = -u$  in the first one) and so we can combine them:

$$f_T(t) = \frac{1}{\pi} \left[ \int_0^\infty u e^{-\frac{u^2(t^2+1)}{2}} du \right]$$

Using a u-substitution for  $v = \frac{u^2(t^2+1)}{2}$ , we have:

$$f_T(t) = \frac{1}{\pi} \int_0^\infty \frac{1}{t^2+1} e^{-v} dv = -\frac{1}{\pi(t^2+1)} [e^{-v}] \Big|_0^\infty = -\frac{1}{\pi(t^2+1)} [0 - 1] = \frac{1}{\pi(t^2+1)}$$

Given the PDF of  $T$  we now can compute its expectation:

$$\mathbb{E}[T^2] = \int_{-\infty}^\infty \frac{t^2}{\pi(t^2+1)} dt = \frac{1}{\pi} \left[ \int_{-\infty}^\infty 1 - \frac{1}{t^2+1} dt \right] = \frac{1}{\pi} [t - \arctan(t)] \Big|_{-\infty}^\infty$$

which diverges to  $\infty$ . Thus we have  $\mathbb{E}[T^2] = \infty$ .