

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

**CONTENT STARTS ON NEXT PAGE.**

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**Problem set 1**  
**Due Thursday January 23 at 11pm**

**Exercise 1.1 (5 points).** Suppose  $A, B, C$  are sets and  $f : A \rightarrow B, g : B \rightarrow C$  are functions. Let  $h = g \circ f : A \rightarrow C$ . Show that

- (1) If  $h$  is injective, then  $f$  is injective.
- (2) If  $h$  is surjective, then  $g$  is surjective.

**Exercise 1.2 (5 points; Rudin 1.1).** This exercise uses the real numbers  $\mathbb{R}$ , which we have not defined yet; but, for this exercise, all you need to know about  $\mathbb{R}$  is that it is a field containing the field  $\mathbb{Q}$  of rational numbers.

Suppose  $r \in \mathbb{Q}, r \neq 0$ , and  $x \in \mathbb{R}, x \notin \mathbb{Q}$ . Prove that  $r + x \notin \mathbb{Q}$  and  $rx \notin \mathbb{Q}$ .

**Exercise 1.3 (10 points; Rudin 1.3).** Suppose  $F$  is a field with  $x, y, z \in F$ . Prove carefully from the field axioms:

- (1) If  $x \neq 0$  and  $xy = xz$ , then  $y = z$ .
- (2) If  $x \neq 0$  and  $xy = x$ , then  $y = 1$ .
- (3) If  $xy = 1$ , then  $x \neq 0$  and  $y = x^{-1}$ .
- (4) If  $x \neq 0$ , then  $(x^{-1})^{-1} = x$ .

**Exercise 1.4 (10 points).** This exercise concerns the field  $\mathbb{Q}(\sqrt{3})$ .

- (1) Prove that  $\mathbb{Q}$  does not contain any  $x$  with  $x^2 = 3$ .
- (2) Consider the set  $\mathbb{Q}(\sqrt{3})$ , defined as the set of expressions " $a + b\sqrt{3}$ " with  $a, b \in \mathbb{Q}$ . (More formally, we could think of  $\mathbb{Q}(\sqrt{3})$  as the set of ordered pairs  $(a, b)$  with  $a, b \in \mathbb{Q}$ .) We can equip  $\mathbb{Q}(\sqrt{3})$  with addition and product laws by

$$\begin{aligned}(a + b\sqrt{3}) + (a' + b'\sqrt{3}) &= (a + a') + (b + b')\sqrt{3}, \\ (a + b\sqrt{3})(a' + b'\sqrt{3}) &= (aa' + 3bb') + (ab' + ba')\sqrt{3}.\end{aligned}$$

Show that  $\mathbb{Q}(\sqrt{3})$  can be made into a field, with these addition and product laws. (This means saying carefully what are the negation law, inversion law, 0 and 1 elements, and then checking that all the axioms of a field are satisfied.)

- (3) We could similarly consider the set  $\mathbb{Z}(\sqrt{3})$ , defined as above except that now we require  $a, b \in \mathbb{Z}$ , with the same addition and product laws. Show that  $\mathbb{Z}(\sqrt{3})$  cannot be made into a field with these addition and product laws.

**Exercise 1.5 (5 points).** As we have stated, the usual order relation on  $\mathbb{Q}$  can be expressed as:

$$p/q < p'/q' \iff pq' < p'q$$

when  $q$  and  $q'$  are positive.

We can define a different relation  $\prec$ , as follows: assume that all fractions are written in lowest terms and with positive denominators. Define

$$p/q \prec p'/q' \iff pq < p'q'.$$

Prove that the relation  $\prec$  does *not* make  $\mathbb{Q}$  into an ordered set.

