Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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Math 226 - HW 1 Due: Sep 5 by 10:30 p.m.

1. (14 points) Let A and B be two non-empty sets, $f : A \to B$ be an injective function, and a_0 is a fixed element of A. Define $g : B \to A$ such that

$$g(b) = \begin{cases} a & \text{if } f(a) = b \\ a_0 & \text{otherwise} \end{cases}$$

   a) (3 points) Show that g is a left inverse of f. That is for all $a \in A$, $g \circ f(a) = a$

   b) (3 points) Explain why g might not be the right inverse of f. Under which conditions g can be a right inverse of f?

   c) (8 points) If $f : A \to B$ be surjective then it would have right inverse. Define the right inverse and validate that it is indeed right inverse.

2. (10 points) Given two functions $f : A \to B$ and $g : B \to C$, the *composition of g with f*, denoted $g \circ f$ is the map $g \circ f : A \to C$ given by $(g \circ f)(a) = g(f(a))$. Show that the set of bijection functions are closed under $\circ$. That is

   (a) (4 points) show if f and g are injections, then $g \circ f$ is an injection. Hint: Recall the injective and left inverse relation.

   (b) (4 points) show if f and g are surjections, then $g \circ f$ is a surjection. Hint: Use the definition of surjectivety first for g and then for f.

   c) (2 points) conclude that if f and g are bijections, then $g \circ f$ is a bijection.

3. (10 points) Let $B = \{m + n\sqrt{2} \ : \ m, n \in \mathbb{Z}\}$.

   a) (2 points) Show that B is <u>*closed under addition*</u>. In other words, prove "if $x \in B$ and $y \in B$, then their sum $x + y$ must be in B as well.

   b) (3 points) Show that B is <u>*closed under multiplication*</u>. In other words, prove "if $x \in B$ and $y \in B$, then their product xy must be in B as well.

   c) (5 points) For every integer $k \geq 1$, prove that $(-1 + \sqrt{2})^k \in B$ - Try to use an inductive argument for this problem.

4. (20 points) We say a function $T : \mathbb{R} \to \mathbb{R}$ is <u>*additive*</u> iff for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$, we have that $T(x + y) = T(x) + T(y)$. If T is additive, show that

   a) (5 points) for all $x \in \mathbb{R}$ and for all integers $n \geq 1$, we have $T(nx) = nT(x)$.

   b) (3 points) $T(0) = 0$.

   c) (3 points) for all $x \in \mathbb{R}$ we have $T(-x) = -T(x)$.

   d) (5 points) for all $x \in \mathbb{R}$ and for all rational numbers $r \in \mathbb{Q}$, we have $T(rx) = rT(x)$.

   e) (4 points) Is it true that if $r \in \mathbb{R}$ the $T(rx) = rT(x)$ for all $x \in \mathbb{R}$? Why or why not?

5. (16 points) Define $\mathbb{Z}\{\sqrt{3}\} := \{a + b\sqrt{3}, \text{ where } a, b \in \mathbb{Z}\}$. Below we will validate that $(\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$, with usual addition and multiplication does not define a field.

   a) (4 points) Show that if $a^2 - 3b^2 = \pm 1$, then $a + b\sqrt{3}$ has a multiplicative inverse. Hint: Check out the definition of the multiplicative inverse, you can directly give it in each case.

b) (10 points) Show that the above statement is iff, i.e., if $a+b\sqrt{3}$ has a a multiplicative inverse then $a^2-3b^2=\pm 1$ Hint: Solve the problem first for the case when a,b are prime to each other. That will give you the understanding of the rest.

c) (2 points) Use a) and b) to conclude that $(\mathbb{Z}\{\sqrt{3}\},+,\cdot)$, with usual addition and multiplication does not define a field.

- (optional) Prove that the set $\mathbb{Q}[\sqrt{2}]=\{a+b\sqrt{2}\ :\ a,b\in\mathbb{Q}\}$ is indeed a field. Notice that you don't need the conditions like in b,c here.