PSETs Landing Page*

Anish Krishna Lakkapragada

This is the documentation for using my PSET PDFs responsibly. I post these LaTEX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarily, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

Math 226: HW 3
Completed By: Anish Lakkapragada (NETID: al2778)

---

1. a) The set S is linearly independent if for $a_1, a_2, \dots a_n \in \mathbb{F}$ and $e_i \in S$, $\Sigma_{i=1}^{N} a_i e_i = 0$ when $a_1, a_2, \dots a_n$ all equal zero. To solve the equation $\Sigma_{i=1}^{N} a_i e_i = (a_1, a_2, \dots, a_n) = \mathbf{0}^n$, all elements in the set $\{a_1, a_2, \dots, a_n\}$ must be equal to zero. Thus we have proved that $a_1, a_2, \dots a_n$ all equal zero as the only solution to $\Sigma_{i=1}^{N} a_i e_i = 0$ and so S is proven to be linearly independent.

   We define $Span(S) = \{\Sigma_{i=1}^{N} a_i e_i : a_1, a_2, \dots a_n \in \mathbb{F}\} = \{(a_1, a_2, \dots a_n) : a_1, a_2, \dots a_n \in \mathbb{F}\}$ and define $\mathbb{F}^n = \{(a_1, a_2, \dots a_n) : a_1, a_2, \dots a_n \in \mathbb{F}\}$. Thus $Span(S) = \mathbb{F}^n$ and so we have proven that S generates $\mathbb{F}^n$.

   c) To prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent, we first show (1) $\{u + v, u - v\}$ is linearly independent if $\{u, v\}$ is linearly independent and (2) $\{u, v\}$ is linearly independent if $\{u+v, u-v\}$ is linearly independent.

   1 **Given $\{u, v\}$ is linearly independent, prove $\{u + v, u - v\}$ is linearly independent**

   If $\{u, v\}$ is linearly independent, for $a, b \in \mathbb{F}$, the solution to $au + bv = \mathbf{0}$ is $a = b = 0$. Given $c, d \in \mathbb{F}$, let us now re-express $au+bv = \mathbf{0}$ with $a = c+d, b = c-d$.

$$
\begin{aligned}
au + bv &= \mathbf{0} \\
(c + d)u + (c - d)v &= \mathbf{0} \\
c(u + v) + d(u - v) &= \mathbf{0}
\end{aligned}
$$

   As we are given $\{u, v\}$ is linearly independent, we know that $a = b = 0$ and so given $c+d = c-d = 0$, we know that $c = d = 0$[1]. Thus, $\{u+v, u-v\}$ is proven to be linearly independent as the only solution to $c(u + v) + d(u - v) = \mathbf{0}$ is proven to be $c = d = 0$.

   2 **Given $\{u + v, u - v\}$ is linearly independent, prove $\{u, v\}$ is linearly independent**

   Given $a, b \in \mathbb{F}$, if we know $\{u + v, u - v\}$ is linearly independent, we know that the solution to $a(u + v) + b(u - v) = \mathbf{0}$ is $a = b = 0$. We can also simplify this as:

$$
\begin{aligned}
a(u + v) + b(u - v) &= \mathbf{0} \\
(a + b)u + (a - b)v &= \mathbf{0}
\end{aligned}
$$

   Given that we know $a = b = 0$, if we define $c = a + b = 0 \in \mathbb{F}$ and $d = a - b = 0 \in \mathbb{F}$, we get that

$$cu + dv = 0$$

   If $\{u, v\}$ is linearly independent, the solution for the equation $eu + fv = \mathbf{0}$ is $e = f = 0$ for $e, f \in \mathbb{F}$. From the above equation, we know that $e = c = 0$

[1] Note that we can only conclude this because $\mathbb{F}$ has a characteristic not equal to two. If this was not the case, the condition $c + d = c - d = 0$ can be met if $c = d = 1$.

and $f = d = 0$ and thus $e = f = 0 \Rightarrow \{u, v\}$ is linearly independent. Thus we haven proven that if $\{u+v, u-v\}$ is linearly independent, $\{u, v\}$ is linearly independent.

2. a) To prove that for every $x \in V$, $x \in Span(S)$ iff $Span(S) = Span(S \cup \{x\})$ we must show (1) given $x \in Span(S)$, $Span(S) = Span(S \cup \{x\})$ and (2) given $Span(S) = Span(S \cup \{x\})$, $x \in Span(S)$.

1 **Given $x \in Span(S)$, $Span(S) = Span(S \cup \{x\})$**
Let us define $N = |S|$. $Span(S) = \{\Sigma_{i=1}^{N} a_i v_i : a_i \in \mathbb{F}, v_i \in V\}$ and $Span(S \cup \{x\}) = \{cx + \Sigma_{i=1}^{N} b_i v_i : b_i, c \in \mathbb{F}, v_i \in V\}$. If $x \in Span(S)$, x can be expressed as $\Sigma_{i=1}^{N} a_i v_i$ for some set of values $a_i \in \mathbb{F}$. We can re-express our d $Span(S \cup \{x\})$ as $\{\Sigma_{i=1}^{N} c a_i v_i + \Sigma_{i=1}^{N} b_i v_i : b_i, c \in \mathbb{F}, v_i \in V\} = \{\Sigma_{i=1}^{N}(c a_i + b_i) v_i : b_i, c \in \mathbb{F}, v_i \in V\}$. Because $c a_i + b_i \in \mathbb{F}$, $Span(S \cup \{x\}) = \{\Sigma_{i=1}^{N} d_i v_i : d_i \in \mathbb{F}, v_i \in V\} = Span(S)$. Thus, we have proved given $x \in Span(S)$, $Span(S) = Span(S \cup \{x\})$.

2 **Given $Span(S) = Span(S \cup \{x\})$, $x \in Span(S)$**
We use the definitions provided from 1. Because we know $x \in S \cup \{x\}$, we know that $x \in Span(S \cup \{x\})$. Since we are given $Span(S) = Span(S \cup \{x\})$, if $x \in Span(S \cup \{x\}) \Rightarrow x \in Span(S)$. Thus, we have proved if $Span(S) = Span(S \cup \{x\})$, $x \in Span(S)$.

b) We are given that S is linearly independent. To prove that $S \cup \{w\}$ is linearly independent iff $w \notin Span(S)$, we first prove (1) if $w \notin Span(S)$, $S \cup \{w\}$ is linearly independent and then prove (2) if $S \cup \{w\}$ is linearly independent, $w \notin Span(S)$.

1 **Given $w \notin Span(S)$, $S \cup \{w\}$ is linearly independent**
We define $n = |S|$.
For proof by contrapositive, let us assume that $S \cup \{w\}$ is linearly dependent. That means that for the solution to the equation $cw + \Sigma_{i=1}^{n} a_i v_i = 0$, where $c, a_1, a_2, \ldots a_n \in \mathbb{F}$, there exists at least one nonzero element of $\{c, a_1, a_2, \ldots a_n\}$. For this solution $c \neq 0$, as if $c = 0$, then the equation would simply be $\Sigma_{i=1}^{n} a_i v_i = 0$. Because S is linearly independent, the only solution to this equation is all $a_i = 0$. However, this would violate the condition that there exists at least one nonzero element in set $\{c, a_1, a_2, \ldots a_n\}$ because we are assuming $S \cup \{w\}$ is linearly dependent. Thus, we know $c \neq 0$ and so we can re-express w as:

$$w = -\frac{\Sigma_{i=1}^{n} a_i v_i}{c} = \Sigma_{i=1}^{n} \frac{-a_i}{c} v_i$$

This expression of w is an expression of w as a linear combination of S. Thus, $w \in Span(S)$ if $S \cup \{w\}$ is linearly dependent. By proof by contrapositive, we haven proven if $w \notin Span(S)$, $S \cup \{w\}$ is linearly independent.

2 **Given $S \cup \{w\}$ is linearly independent, $w \notin Span(S)$**
We use the same definition for n provided in 1.
If $S \cup \{w\}$ is linearly independent, it means that the solution to the equation $cw + \Sigma_{i=1}^{n} a_i v_i = 0$, where $c, a_1, a_2, \ldots a_n \in \mathbb{F}$, is that all elements of the set $\{c, a_1, a_2, \ldots a_n\}$ must be zero. We now try to express w as some linear combination of S.

$$cw + \Sigma_{i=1}^{n} a_i v_i = 0$$
$$cw = -\Sigma_{i=1}^{n} a_i v_i$$

Note that because $S \cup \{w\}$ is linearly independent, $c = 0$. Because we cannot divide the above equation by c on both sides, there does not exist any linear combination of S that is equal to w. Because the $Span(S)$ represents all possible linear combinations of elements in S, we have proven $w \notin Span(S)$.

c) Given $S = \{u_1, u_2, u_3, \dots u_k\}$, to prove iff Condition M (defined below)

$$M : \{0\} \subsetneq Span(\{u_1\}) \subsetneq Span(\{u_1, u_2\}) \subsetneq Span(\{u_1, u_2, u_3\}) \subsetneq Span(\{u_1, \dots u_k\})$$

then S is linearly independent, we must first prove (1) if condition M holds, then S is linearly independent and (2) if S is linearly independent, then condition M holds.

1 **If condition M holds, then S is linearly independent**
For proof by contrapositive, let us assume S is linearly dependent. Thus, there exists some $m < k \in \mathbb{Z}$ where subset $D = \{u_1, \dots u_m\} \subsetneq S$ is linearly independent and subset $K = \{u_1, \dots u_m, u_{m+1}\} \subseteq S$ is linearly dependent. Let us define $a_1, a_2, \dots a_{m+1} \in \mathbb{F}$ and $u_i \in K$. Because K is linearly dependent, we know that for the solution to the equation $\Sigma_{i=1}^{m+1} a_i u_i = 0$. there exists at least one nonzero element in the set $\{a_1, a_2, \dots a_{m+1}\}$. We can further develop this equation as:

$$a_{m+1}u_{m+1} + \Sigma_{i=1}^{m} a_i u_i = 0$$

Let us consider the case in which $a_{m+1} = 0$. This would leave us with the equation $\Sigma_{i=1}^{m} a_i u_i = 0$. Because D is linearly independent, we know that the only solution to this equation is $\forall a_i \in \{a_1, a_2, \dots a_m\}, a_i = 0$. However, because this solution violates the condition that there exists at least one nonzero element in the set $\{a_1, a_2, \dots a_{m+1}\}$, we know that $a_{m+1} \neq 0$. Thus, $u_{m+1} = \Sigma_{i=1}^{m}(-\frac{a_i}{a_{m+1}})u_i$ and so because we can express u_{m+1} as linear combination of D, $u_{m+1} \in Span(D)$.
We now compute $Span(K) = \{cu_{m+1} + \Sigma_{i=1}^{m} b_i u_i : c, b_i \in \mathbb{F}\} = \{c\Sigma_{i=1}^{m}(-\frac{a_i}{a_{m+1}})u_i + \Sigma_{i=1}^{m} b_i u_i : c, b_i \in \mathbb{F}\} = \{\Sigma_{i=1}^{m}(\frac{-ca_i}{a_{m+1}} + b_i)u_i : c, b_i \in \mathbb{F}\}$. Because $\frac{-ca_i}{a_{m+1}} + b_i \in \mathbb{F}$, $Span(K) = \{\Sigma_{i=1}^{m} d_i u_i : d_i \in \mathbb{F}\} = Span(D)$. Because $Span(D) = Span(K)$, condition M does not hold if S is linearly dependent. Thus, we have proven by contrapositive that if condition M holds, then S is linearly independent.

2 **If S is linearly independent, then condition M holds**
We use the definitions of m, k, D, K from 1.
For proof by contrapositive, let us assume that condition M does not hold[2] and so for some $m < k \in \mathbb{R}$, $Span(D) = Span(K)$ Because $u_{m+1} \in Span(K) \Rightarrow u_{m+1} \in Span(D)$, we know that u_{m+1} can be written as $u_{m+1} = \Sigma_{i=1}^{m} a_i u_i$ for $a_i \in \mathbb{F}$, $v_i \in D$.
The set S is linearly independent if given $b_i \in \mathbb{F}$ and $u_i \in S$, the only solution to the equation $\Sigma_{i=1}^{k} b_i u_i = 0$ is $\forall b_i \in \{b_1, b_2, \dots b_k\}, b_i = 0$. We can re-express this equation below as:

$$\Sigma_{i=1}^{k} b_i u_i = 0$$
$$b_{m+1}u_{m+1} + \Sigma_{i=m+2}^{k} b_i u_i + \Sigma_{i=1}^{m} b_i u_i = 0$$

[2]There is only one case in which condition M does not hold. Condition M would not hold if $\exists\, j < k \in \mathbb{Z}$ s.t. $Span(\{u_1, \dots, u_j\}) = Span(\{u_1, \dots, u_j, u_{j+1}\})$ or $Span(\{u_1, \dots u_j\}) \not\subseteq Span(\{u_1, \dots, u_j, u_{j+1}\})$. All elements in $Span(\{u_1, \dots u_j\})$ exist in $Span(\{u_1, \dots u_j, u_{j+1}\})$ as the last element in the set u_{j+1} can always be ignored in a linear combination of $\{u_1, \dots, u_j, u_{j+1}\}$ by setting its coefficient to zero. Thus $Span(\{u_1, \dots, u_j\}) \subseteq Span(\{u_1, \dots, u_j, u_{j+1}\})$ and so condition M can only be violated if $Span(\{u_1, \dots, u_j\}) = Span(\{u_1, \dots, u_j, u_{j+1}\})$.

Let us set $b_i = 0$ for $m+2 \leq i \leq k$, $b_i = -a_i$ for $1 \leq i \leq m$, and $b_{m+1} = 1$. This gives us:

$$u_{m+1} = \Sigma_{i=1}^{m} b_i u_i = \Sigma_{i=1}^{m} a_i u_i$$

As shown before, we know this statement is true. Thus, we have found a solution to the equation $\Sigma_{i=1}^{k} b_i u_i = 0$ with at least one $b_i \neq 0$. This means S is linearly dependent. Thus, if condition M does not hold, we have shown S is linearly dependent. By proof by contrapositive, we have proven that if S is linearly independent, then condition M holds.

3. a) Given $U = \{(x_1, \frac{x_1}{3}, x_3, \frac{x_3}{7}, x_5) \in \mathbb{R}^5; x_1, x_3, x_5 \in \mathbb{F}\}$, a basis of U, β_U, can be given by:

$$\beta_U = \{(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1)\}$$

i) Extending β_U to $\mathbb{R}^5$

$$\beta_{\mathbb{R}^5} = \{(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0)\}$$

ii) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \bigoplus W$

$$W = \{(0, a_1, 0, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$$

We now validate (1) that $U + W = U \bigoplus W$ and (2) $U \bigoplus W = \mathbb{R}^5$.

1 $U + W = U \bigoplus W$

Let us define $z_1, z_2, z_3, z_4, z_5 \in \mathbb{R}$. Given $x = (z_1, \frac{z_1}{3}, z_3, \frac{z_3}{7}, z_5) \in U$ and $y = (0, z_2, 0, z_4, 0) \in W$, we can find $U \cap W$ as the set of solutions to $x = y$. This would be defined as the solution to the system of equations below.

$$\begin{aligned} z_1 &= 0 \\ \frac{z_1}{3} &= z_2 \\ z_3 &= 0 \\ \frac{z_3}{7} &= z_4 \\ z_5 &= 0 \end{aligned}$$

The solution to this system is $z_1 = z_2 = z_3 = z_4 = z_5 = 0$. Thus $U \cap W = \{\mathbf{0}^5\} \Rightarrow U + W = U \bigoplus W$.

2 $U \bigoplus W = \mathbb{R}^5$

We use the same definitions of $x, y, z_1, z_2, z_3, z_4, z_5$ from **1**. We compute $x + y = (z_1, z_2 + \frac{z_1}{3}, z_3, z_4 + \frac{z_3}{7}, z_5) \in U \bigoplus W$. Because $z_1, z_2 + \frac{z_1}{3}, z_3, z_4 + \frac{z_3}{7}, z_5 \in \mathbb{R}$, $x + y \in \mathbb{R}^5$ and so $U \bigoplus W \subseteq \mathbb{R}^5$. Because $\mathbf{0}^5 \in U \bigoplus W$ and $U \bigoplus W$ can be trivially shown to be closed under addition and scalar multiplication, $U \bigoplus W \leq \mathbb{R}^5$.

We now prove that $U \bigoplus W = \mathbb{R}^5$ by showing that $dim(U \bigoplus W) = dim(\mathbb{R}^5) = 5$.

The basis of U, β_U, is given above and we give the basis of W as $\beta_W = \{(0,1,0,0,0),(0,0,0,1,0)\}$. We now compute $Span(\beta_U \cup \beta_W)$ as given $v_i \in \beta_U \cup \beta_W$ and $a_i \in \mathbb{R}, Span(\beta_U \cup \beta_W) = \Sigma_{i=1}^{3} a_i v_i + \Sigma_{i=5}^{4} a_i v_i$. Because all vectors in U can be expressed as $\Sigma_{i=1}^{3} a_i v_i$ (a linear combination of β_U) and all vectors in W can be expressed as $\Sigma_{i=5}^{4} a_i v_i$ (a linear combination of β_W), $Span(\beta_U \cup \beta_W) = \{u + w : u \in U, w \in W\} = U \bigoplus W \Rightarrow \beta_U \cup \beta_W$ is the basis for $U \bigoplus W$. Thus $dim(U \bigoplus W) = |\beta_U \cup \beta_W| = 5 = dim(\mathbb{R}^5)$. Thus, we have proved $U \bigoplus W = \mathbb{R}^5$.

b) Given $U = \{f(x) = c_1 + (-2c_3 - 3c_4 - 4c_5)x + c_3x^2 + c_4x^3 + c_5x^4 \in P_4(\mathbb{R}) : c_1, c_3, c_4, c_5 \in \mathbb{R}\}$, a basis of U can be given by:

$$\beta_U = \{1, -2x + x^2, -3x + x^3, -4x + x^4\}$$

i) Extending β_U to $P_4(\mathbb{R})$

$$\beta_{P_4(\mathbb{R})} = \{1, x, -2x + x^2, -3x + x^3, -4x + x^4\}$$

ii) Find a subspace W of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R}) = U \bigoplus W$

$$W = \{f(x) = a_1x \in P_4(\mathbb{R}) : a_1 \in \mathbb{R}\}$$

We now validate (1) that $U + W = U \bigoplus W$ and (2) $U \bigoplus W = P_4(\mathbb{R})$.

1 $U + W = U \bigoplus W$

Let us define $z_1, z_2, z_3, z_4, z_5 \in \mathbb{F}$. Given $u = z_1 + (-2z_3 - 3z_4 - 4z_5)x + z_3x^2 + z_4x^3 + z_5x^4 \in U$ and $w = z_2x \in W$, we find $U \cap W$ as the set of solutions to $u = w$. This would be defined as the solution to the system of equations below.

$$\begin{aligned} z_1 &= 0 \\ -2z_3 - 3z_4 - 4z_5 &= z_2 \\ z_3 &= 0 \\ z_4 &= 0 \\ z_5 &= 0 \end{aligned}$$

The solution to this system is $z_1 = z_2 = z_3 = z_4 = z_5 = 0$. Thus since $U \cap W = \{f(x) = 0 \in P_4(\mathbb{R})\} \Rightarrow U + W = U \bigoplus W$.

2 $U \bigoplus W = P_4(\mathbb{R})$

We use the same definitions of $u, w, z_1, z_2, z_3, z_4, z_5$ from 1. We compute $u + w = z_1 + (z_2 - 2z_3 - 3z_4 - 4z_5)x + z_3x^2 + z_4x^3 + z_5x^4 \in U \bigoplus W$. Because $z_1, z_2 - 2z_3 - 3z_4 - 4z_5, z_3, z_4, z_5 \in \mathbb{R}$, $u + w \in P_4(\mathbb{R})$ and so $U \bigoplus W \subseteq P_4(\mathbb{R})$. Because $0 \in U \bigoplus W$ and $U \bigoplus W$ can be trivially shown to be closed under addition and scalar multiplication, $U \bigoplus W \leq P_4(\mathbb{R})$. We now prove that $U \bigoplus W = P_4(\mathbb{R})$ by showing that $dim(U \bigoplus W) = dim(P_4(\mathbb{R}) = 5$.

The basis of U, β_U, is given above and we give the basis of W as $\beta_W = \{x\}$. We now compute $Span(\beta_U \cup \beta_W)$ as given $f_i(x) \in \beta_U \cup \beta_W$ and $a_i \in \mathbb{R}, Span(\beta_U \cup \beta_W) = \Sigma_{i=1}^{3} a_i f_i(x) + \Sigma_{i=5}^{4} a_i f_i(x)$. Because all functions in U can be expressed as $\Sigma_{i=1}^{3} a_i f_i(x)$ (a linear combination of β_U) and all functions in W can be expressed as $\Sigma_{i=5}^{4} a_i f_i(x)$ (a linear combination of

β_W), $Span(\beta_U \cup \beta_W) = \{u + w : u \in U, w \in W\} = U \bigoplus W \Rightarrow \beta_U \cup \beta_W$ is the basis for $U \bigoplus W$. Thus $dim(U \bigoplus W) = |\beta_U \cup \beta_W| = 5 = dim(P_4(\mathbb{R}))$. Thus, we have proved $U \bigoplus W = P_4(\mathbb{R})$.

4. Let us define a matrix E_{ij} as a 3x3 matrix with all zeros except in the ith row and jth column, where there is a one.

   The basis for $M_{3x3}(\mathbb{R}) = \{\bigcup_{\substack{i,j=1 \\ i \neq j}}^{3} E_{ij}\} \cup \{E_{00} - E_{33}, E_{11} - E_{33}\}$.