Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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MATH 244 PSET 5

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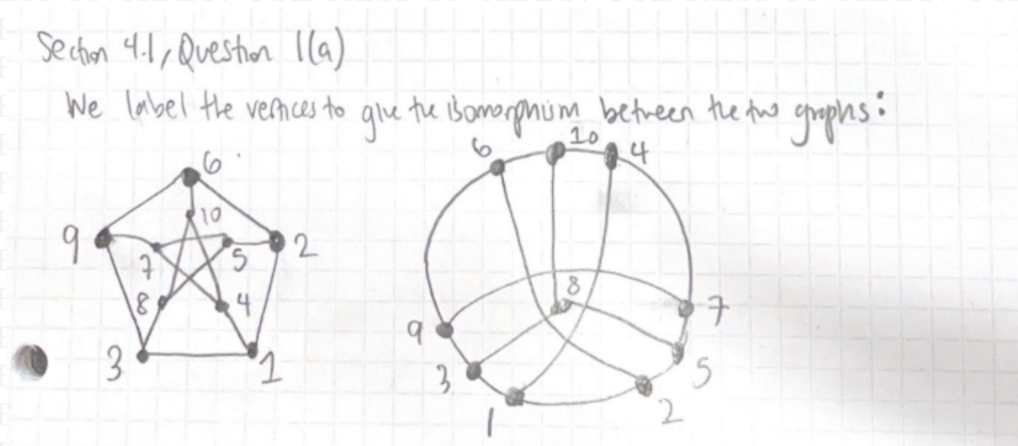
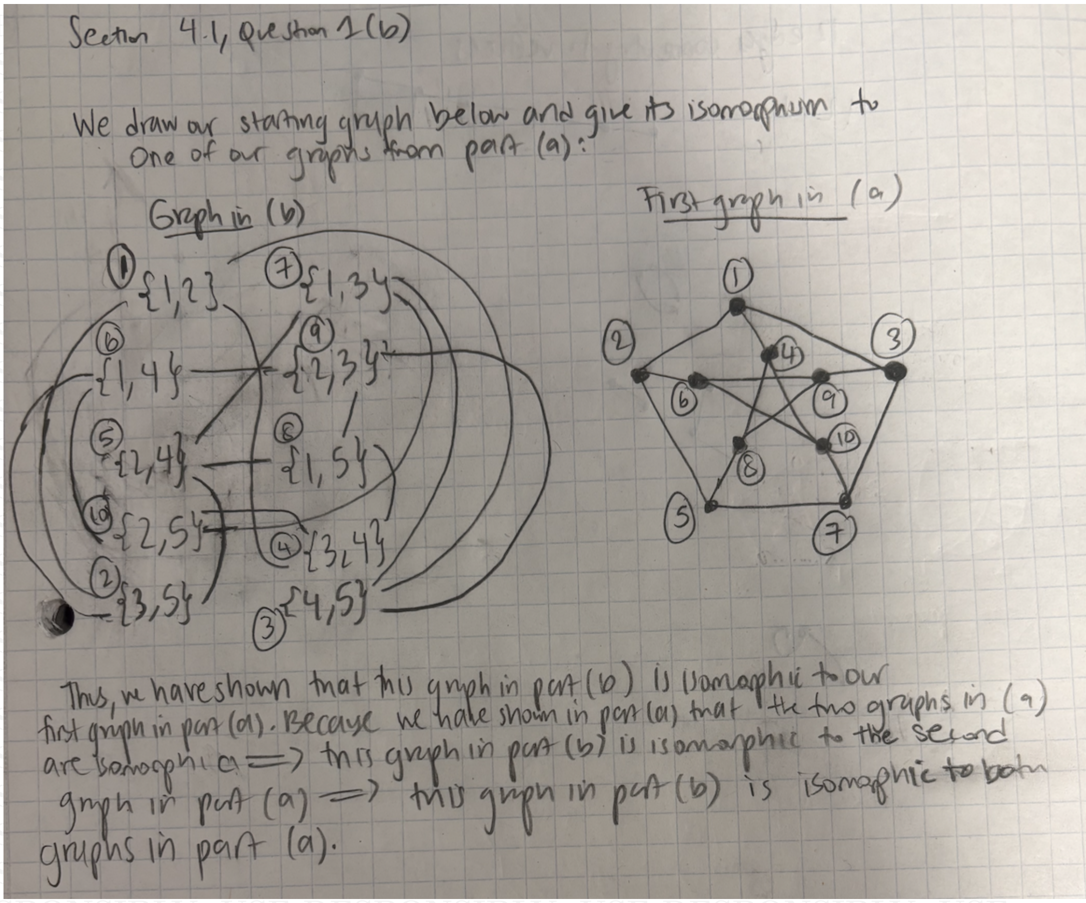
Number of late days: 0; Collaborators: Matt Sprintson, Christina Xu

1. **Section 4.1, Question 1**

   Note that for both parts, to show an isomorphism between two graphs we simply label the vertices on each of the graphs.



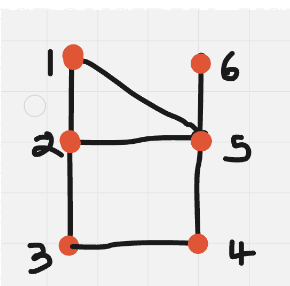

Section 4.1, Question 1(b)

We draw our starting graph below and give its isomorphism to one of our graphs from part (a):

This, we have shown that this graph in part (b) is isomorphic to our first graph in part (a). Because we have shown in part (a) that the two graphs in (a) are isomorphic $\Longrightarrow$ this graph in part (b) is isomorphic to the second graph in part (a) $\Longrightarrow$ this graph in part (b) is isomorphic to both graphs in part (a).

2. **Section 4.1, Question 3(a)**

Below is an image of an asymmetric graph with six vertices.



We now explain why this graph is asymmetric. A graph is symmetric if we can map it onto itself with a non-trivial automorphism (e.g. not the identity function), which means that this graph can be either rotated or reflected while maintaining the same structure. We can see for our above graph that it is impossible to reflect/rotate and maintain the same graph structure $\implies$ the above graph is not symmetric $\implies$ this graph is asymmetric.

3. **Section 4.1, Question 3(b)**

Lemma 0.1 *A graph G is symmetric $\iff$ the complement of G is also symmetric.*
Proof: Let us define the complement of G as $G^c = (V, \binom{V}{2} \backslash E)$. Given G is symmetric $\implies$

$\exists$ a non-trivial automorphism $f: V \to V$ s.t. $\{v_i, v_j\} \in E \iff \{f(v_i), f(v_j)\} \in E$. We know that for any two vertices v_i, v_j: $\{v_i, v_j\} \notin E \iff \{f(v_i), f(v_j)\} \notin E$. By definition of a graph complement $\{v_i, v_j\} \notin E \iff \{v_i, v_j\} \in E^c$ and $\{f(v_i), f(v_j)\} \notin E \iff \{f(v_i), f(v_j)\} \in E^c$ and so $\{v_i, v_j\} \notin E \iff \{f(v_i), f(v_j)\} \notin E$ implies $\{v_i, v_j\} \in E^c \iff \{f(v_i), f(v_j)\}$. This proves that there exists a non-trivial automorphism, in our case f, for graph $G^c \implies G^c$ is symmetric. Note that because $(G^c)^c = G$, we could have gone in the reverse direction to show if G^c is symmetric $\implies G$ is symmetric. So we have proven G is symmetric $\iff G^c$ is symmetric.

Let us define a connected graph $G = (V, E)$. We first explain that if for each vertex, its degree is ≤ 2, it will not be asymmetric. This is because if for each vertex its degree is ≤ 2, we will either have a ring graph or a path graph both of which are trivially symmetric (i.e. we can clearly permute the vertices to create an identical graph structure). Note that we can generalize G beyond connected graphs to all graphs – disconnected graphs will be split into connected subgraphs with degree ≤ 2 and thus will also be symmetrical. Thus, a graph must have at least one vertex with a degree of ≥ 3 to be symmetric $\implies$ a graph must have at least four vertices (with at least one vertex of degree ≥ 3) to be asymmetric. For clarity, this means we have shown that no asymmetric graph G exists for $|V(G)| < 4$.

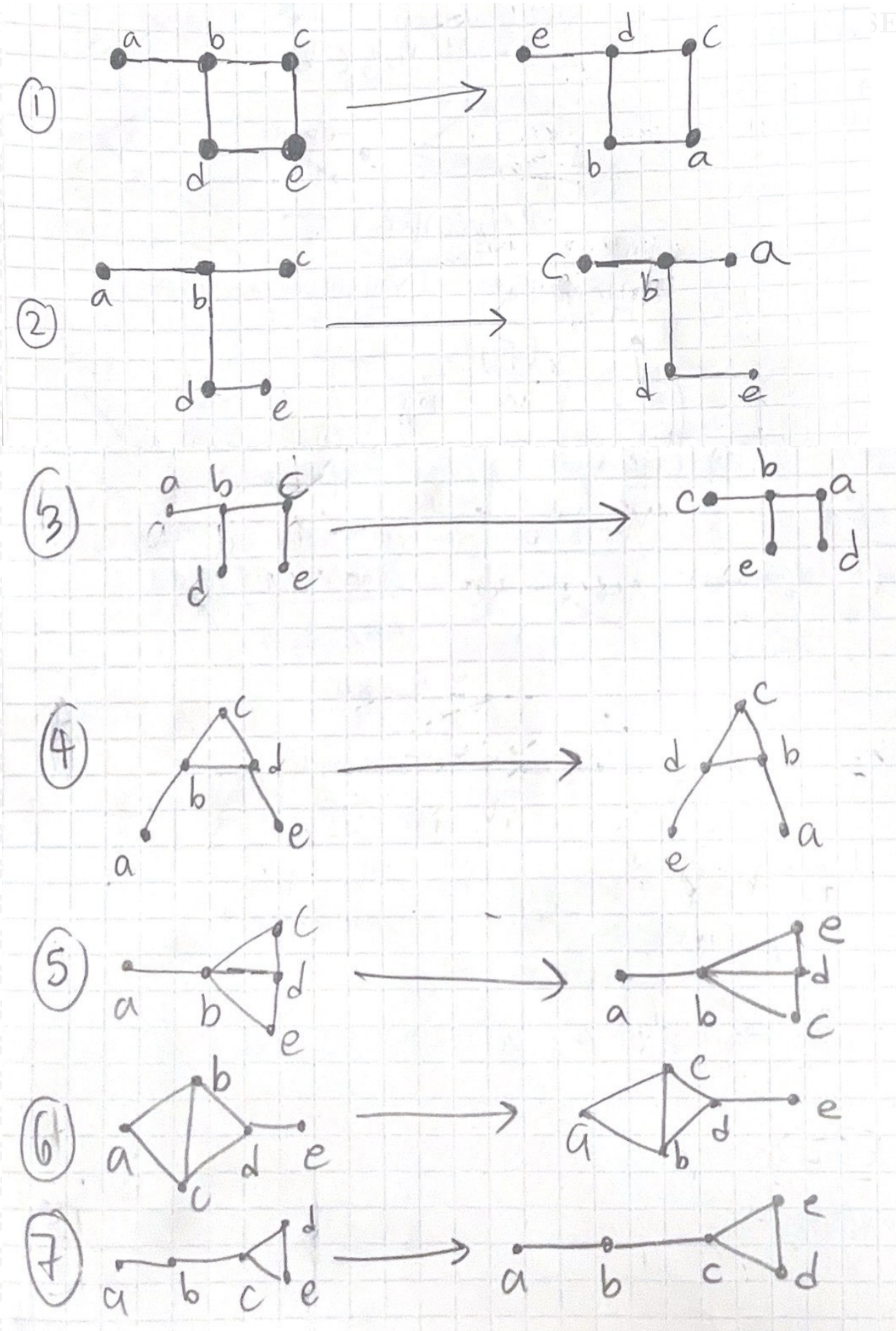
We now consider the below cases to show no asymmetric graph G exists for $|V(G)| \leq 5$:

1. $|V(G)| = 4$
   We now consider all graphs with four vertices, at least one of which has a degree ≥ 3. In the complement of such a graph, the vertex with degree ≥ 3 will be guaranteed to have a degree of zero $\implies$ the complement of such a graph will be disconnected. A disconnected graph of $|V(G)| = 4$ will have two subgraphs each of which have vertices of degrees at most 2 $\implies$ this disconnected graph is symmetric $\implies$ through **Lemma 0.1** all graphs with four vertices and at least one vertex with degree ≥ 3 will be symmetric.

2. $|V(G)| = 5$
   We break down all graphs with $|V(G)| = 5$ and at least one vertex of degree ≥ 3 through casework:

   (a) **Case One: Graphs with at least one vertex of degree ≥ 4**
   Let us define the set of graphs in this case as G'.
   The complement of all graphs in G' have a vertex with degree zero $\implies$ the complement of all graphs in G' are disconnected with subgraphs of ≤ 4 vertices $\implies$ the complement of all graphs in G' lead to symmetrical (sub)graphs $\implies$ the complement of all graphs in G' are symmetric $\implies$ through **Lemma 0.1** all graphs in set G' are symmetric.

   (b) **Case Two: Graphs with no vertex of degree ≥ 4 but one vertex with at least degree 3**
   We show all seven possible graphs meeting this criteria are symmetric below by first drawing the graph and then drawing its symmetrical graph:



Thus no graph with $|V(G)| = 5$ is asymmetric.

Thus we conclude for $1 < |V(G)| \leq 5$, no asymmetric graph exists.

4. **Section 4.2, Question 1**

Lemma 0.2 *Let us define a graph $G = (V, E)$. The relation $\approx$ on set V defined as:*

$v_i \approx v_j \iff \exists$ a path between v_i and v_j is an equivalence relation.
Proof: To show $\approx$ is an equivalence relation, we show $\approx$ is reflexive, symmetric, and transitive: ***(1) Reflexive:*** *There exists a trivial path between every vertex and itself (just the vertex itself) and so $\forall v \in V, v \approx v \implies \approx$ is reflexive.* ***(2) Symmetric****: If $v_i \approx v_j \implies \exists$ a path from v_i to $v_j \implies \exists$ a path from v_j to v_i (i.e. this original path reversed) $\implies v_j \approx v_i$. Thus this shows $v_i \approx v_j \implies v_j \approx v_i$ which demonstrates that $\approx$ is symmetric.* ***(3) Transitive****: We are given $v_i \approx v_j, v_j \approx v_k$ and WTS $v_i \approx v_k$. If $v_i \approx v_j \implies \exists$ a path, which we can call P_{ij}, from v_i to v_j. Similarly if $v_j \approx v_k \implies \exists$ a path, which we can call P_{jk}, from v_j to v_k. We can concatenate P_{ij} with P_{jk} to create a path from v_i to $v_k \implies v_i \approx v_k$. This proves $\approx$ is transitive.*

We are given some $G = (V, E)$ that is disconnected and WTS G^c is connected. Given G is disconnected, we can use the $\approx$ equivalence relation to create equivalence classes of the vertices $V_1, V_2, \ldots V_k$ where $V_i \subset V$. Put differently, these equivalence classes of the vertices V give the vertices V in each connected component of G. Note that because G is disconnected, $\nexists \{v_i, v_j\} \in E$ where v_i and v_j do not belong to the same equivalence class as this would contradict disconnected components not being connected. Thus, by by definition of graph complement this means $\forall v_i, v_j \in V$ s.t. v_i and v_j are in different equivalence classes, $\exists \{v_i, v_j\} \in$ the edges of G^c.

Now any two nodes v_i and v_j in the same equivalence class. Because this graph is disconnected $\implies$ there exists more than one connected component $\implies \exists v_k \in V$ not in the same equivalence class as v_i and v_j. Because v_i and v_j are in different equivalence classes than $v_k \implies \exists \{v_i, v_k\}, \{v_j, v_k\} \in$ the edges of the graph complement. So for graph G^c, $v_i \approx v_k$ and $v_j \approx v_k \implies v_i \approx v_j$ by transitivity. Thus, in G^c, (1) all vertices from the same equivalence class of G are connected (i.e. related by $\approx$) and (2) all vertices from different equivalence classes of G are connected as they have an edge to each other $\implies$ all vertices in G^c are connected to each other $\implies G^c$ is connected.

5. **Section 4.2, Question 10**

A note on notation: identical to Proposition 4.2.4, we let $a_{ij}^{(k)}$ denote the element of matrix A_G^k at entry (i, j).

We prove both directions of this statement below:

1. **If $G = (V, E)$ contains a triangle $\implies \exists\ i, j$ s.t. $a_{ij} \neq 0$ and $a_{ij}^{(2)} \neq 0$.**

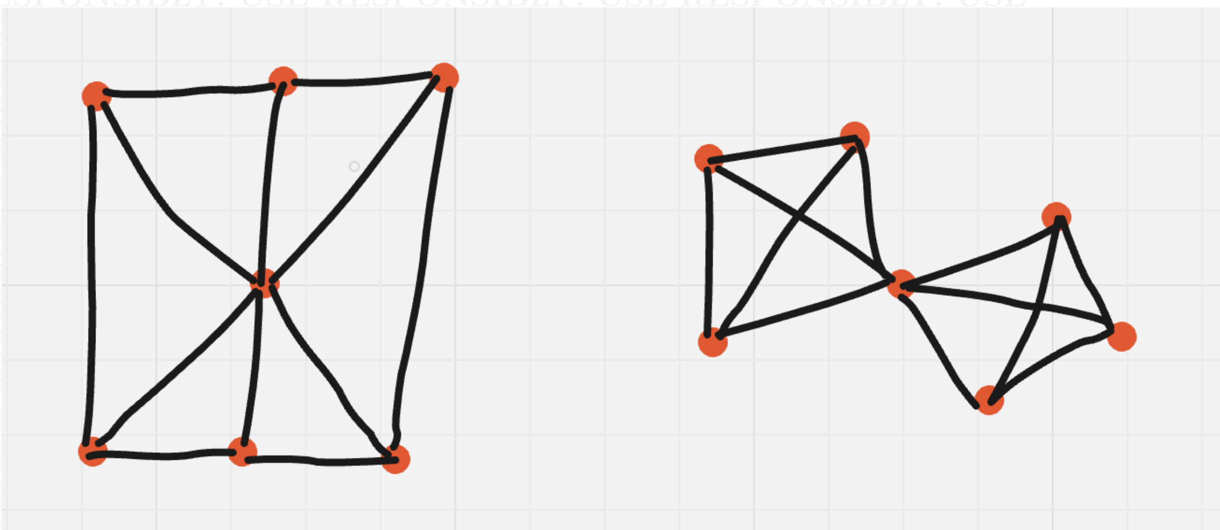
   If $G = (V, E)$ contains a triangle, there exists three vertices that form a triangle $\implies \exists\ v_i, v_j, v_k \in V$ s.t. (1) v_i is connected by an edge to v_j (2) v_i is connected by an edge to v_k (3) v_j is connected by an edge to v_k. Thus, $a_{ij} = 1 \neq 0$ as v_i is connected to v_j and so we have proven $\exists\ i, j$ s.t. $a_{ij} \neq 0$. By Proposition 4.2.4, $a_{ij}^{(2)}$ tells us the number of walks of length k from v_i to v_j. One such walk is given by $v_i, \{v_i, v_k\}, v_k, \{v_k, v_j\}, v_j$ which goes through the two edges $\{v_i, v_k\}$and $\{v_k, v_j\}$ and so has a length of two. Thus, $a_{ij}^{(2)} \geq 1 \implies a_{ij}^{(2)} \neq 0$.

   So we have proven this statement entirely.

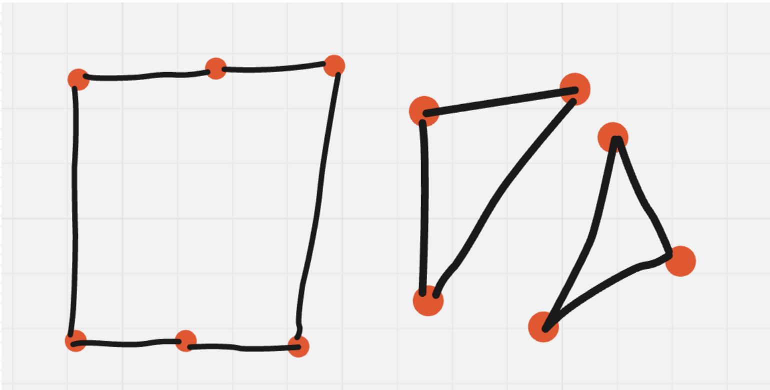
2. **If $\exists\ i, j$ s.t. $a_{ij} \neq 0$ and $a_{ij}^{(2)} \neq 0 \implies G = (V, E)$ contains a triangle.**
   By the definition of an adjacency matrix, a_{ij} is zero if there is no edge from v_i to v_j and one otherwise. Thus if given $a_{ij} \neq 0 \implies a_{ij} = 1 \implies \exists\ \{v_i, v_j\} \in E$. Note an adjacency matrix's entries by definition are always stricly non-negative $\implies \forall i, j \leq |V(G)|, a_{ij}^{(2)} \geq 0$. Thus given $a_{ij}^{(2)} \neq 0 \implies a_{ij}^{(2)} > 0 \implies$ (by Proposition 4.2.4) there is at least one walk from v_i to v_j of length 2. For such a walk to exist, this means that there are two edges that take v_i to v_j. Note that we cannot use our edge $\{v_i, v_j\} \in E$ as this takes v_i to v_j in only one edge (i.e. this is a walk of length one, not two). Thus, we can express the path of this walk of length two as $v_i, \{v_i, v_k\}, v_k, \{v_k, v_j\}, v_j$ where $k \notin \{i, j\}$ (i.e. v_k is distinct from v_i and v_j) and $v_k \in V$. Thus, the three following edges exist in E: $\{v_i, v_j\}, \{v_i, v_k\}, \{v_k, v_j\}$ which means that G contains a triangle with vertices v_i, v_j, v_k.

6. **Section 4.3, Question 5**

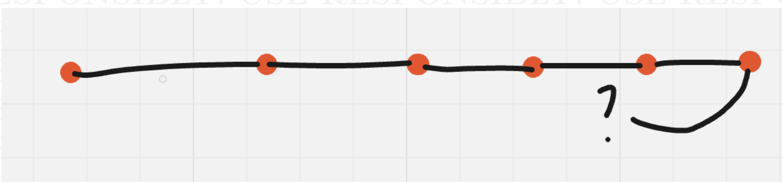
The below image shows two non-isomorphic graphs with a degree sequence $(6, 3, 3, 3, 3, 3, 3)$.



We now show that no other graphs with a degree sequence of $(6, 3, 3, 3, 3, 3, 3)$ exist. Note that our first vertex is connected to all other six vertices. So the problem of finding all non-isomorphic graphs with degree sequence $(6, 3, 3, 3, 3, 3, 3)$ is equivalent to finding all non-isomorphic graphs with degree sequence $(2, 2, 2, 2, 2, 2)$. We draw two non-isomorphic graphs with this degree sequence:



We now explain why these are the only non-isomorphic graphs with degree sequence $(2,2,2,2,2,2)$. We first show that our graph must have a cycle: if we were to draw a line graph such that each of the six vertices were connected by one edge, after adding a new edge (which cannot be a self-loop) we will have formed a cycle. Picture explaining this below:



So we have established that our graph with degree sequence $(2,2,2,2,2,2)$ must have a cycle. Note that the minimum length possible for such a cycle is three[1] and so our graph with six vertices can either have a cycle of six vertices or two disjoint cycles with three vertices. These are exactly the two graphs of degree sequence $(2,2,2,2,2,2)$ we have given above. Note that each of these two graphs clearly correspond to degree-sequence $(6,3,3,3,3,3,3)$ graphs we drew originally (i.e. we can add a new vertex and connect to all other vertices) $\implies$ we have drawn all non-isomorphic graphs of $(6,3,3,3,3,3,3)$.

7. **Section 4.3, Question 12**

To give all the (k,n) where there exists a k-regular graph on n vertices, we prove that we can construct a k-regular graph on n vertices $\iff$ k or n is even and $k < n$. We prove both directions of this statement below:

1. **If we can construct a k-regular graph on n-vertices $\implies$ k or n is even and $k < n$**

   A k-regular graph has all vertices be of degree k. The degree of a given vertex can be at most $n-1$ (as a given node cannot have an edge to itself), and so $k \leq n-1 \implies k < n$.

   We now show that k or n must be even. Given a k-regular graph $G = (V,E)$, $\Sigma_{v \in V}\deg(v) = nk = 2|E|$. Note that $|E| \in \mathbb{N}$ and so nk is even $\implies$ n or k is even.

[1] This makes sense from the picture above, however another explanation is that (1) a cycle of length one is just an edge and therefore not a cycle and (2) a cycle of length two must involve either two edges from the same vertex or a self-loop.

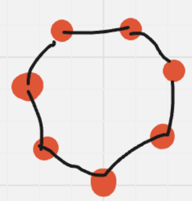
2. **If k or n is even and $k < n \implies$ we can construct a k-regular graph on n vertices**

We show that we can construct cases by breaking this down into cases:

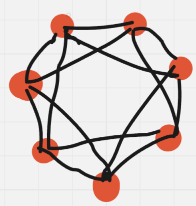
1. **Case One: If k is even**

   In the case $k = 0$, then our 0-regular graph we can construct is just the n vertices with zero edges. Now that we have considered the $k = 0$ case, we show how we can struct a k-regular graph for $k \geq 2$.

   We can first start by placing an edge between each vertex (e.g. creating a ring graph). Call this Stage 1. Then in Stage 2, we can connect each vertex to its two closest vertices it does not already have an edge with. We repeat Stage 2 until every vertex has a degree of k (i.e. the graph is k-regular). For an example, we show Stage 1 and Stage 2 for constructing a 4-regular graph with $n = 7$ vertices:

   Stage 1

   Stage 2

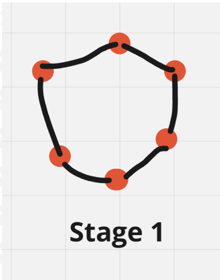
   Thus we have shown for all even k s.t. $k < n$, we can construct a k-regular graph on n vertices.

2. **Case Two: If k is not even**

   If k is not even, by our given that n or k is even $\implies$ n must be even. Note that k not being even $\implies$ k is odd and since $n > k \implies n \geq 1$.

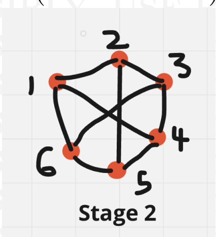
   To construct a k-regular graph we use the identical aforementioned Stage 1 of first forming a ring graph. Then, we connect each vertex to its opposite vertex[2] to create a k-regular graph. We can again call this procedure Stage 2. We show this process below for constructing a 3-regular graph with $n = 6$ vertices:

[2]For a more rigorous explanation of how this can be calculated, if we were to label each vertex $1 \ldots n$, then the opposite vertex can be given by $(k + \frac{n}{2}) \bmod n$.



Stage 1

We label the vertices to be make more clear what each vertex's opposite vertex is (see footnote):



Stage 2

Thus we have shown for all odd k and even n s.t. $k < n$, we can construct a k-regular graph on n vertices.