

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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## Math 225- HW 11 Due: Dec 9 by Midnight

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Submit the first two problems, along with any three additional problems of your choice.

1. Two linear operators  $U$  and  $T$  on a finite dimensional vector space are called simultaneously diagonalizable if there exist an ordered basis  $\beta$  such that both  $[T]_\beta$  and  $[U]_\beta$  are diagonal. Similarly  $A, B$  are simultaneously diagonalizable if there exist  $Q$  invertible such that both  $Q^{-1}AQ$  and  $Q^{-1}BQ$  are diagonal.
  - Prove that if  $U$  and  $T$  simultaneously diagonalizable then  $U$  and  $T$  commute. i.e.  $UT = TU$
  - Conclude that  $A, B$  are simultaneously diagonalizable then  $A, B$  commute
  - Let  $T$  be diagonalizable linear operator on a finite dimensional vector space, then  $T$  and  $T^m$  are simultaneously diagonalizable for any  $m$  positive integer.
2. Let  $T, U$  be a linear operator on a vector space  $V$ , and let  $v$  be a non zero vector in  $V$ .
  - a) Show that  $E_\lambda$  for any eigenvalue  $\lambda$  of  $T$  is a  $T$ -invariant subspace of  $V$ .
  - b) Show that  $T$ -cyclic subspace generated by  $v$  is a  $T$ -invariant subspace of  $V$ .
  - c) Let  $W$  be the  $T$ -cyclic subspace generated by  $v$ . Then for any  $w \in V$ ,  $w \in W$  iff  $w = g(T)v$  for some polynomial  $g$ .
  - d) Let  $V$  be  $T$ -cyclic subspace of itself. Show that if  $U$  commutes with  $T$  then  $U = g(T)$  for some polynomial  $g$ .
  - e) If  $V$  is two dimensional then either  $V$  is  $T$ -cyclic subspace of itself or  $T = cI$ .
3. Let  $T$  be a linear operator on a finite-dimensional vector space  $V$ , and suppose that the distinct eigenvalues of  $T$  are  $\lambda_1, \lambda_2, \dots, \lambda_k$ . Prove that

$$\text{span}(\{x \in V : x \text{ is an eigenvector of } T\}) = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots \oplus E_{\lambda_k}$$

4. Let  $T$  be a linear operator on a finite dimensional vector space  $V$ , and  $W$  be an invariant subspace of  $V$ . Suppose that  $v_1, v_2, \dots, v_n$  are eigenvectors of  $T$  corresponding to distinct eigenvalues.
  - (a) Prove that if  $v_1 + v_2 + \dots + v_n$  is in  $W$ , then  $v_i$  is in  $W$  for all  $i$ . (Use induction)
  - (b) Prove that the restriction of a diagonalizable linear operator  $T$  to any nontrivial  $T$ -invariant subspace is also diagonalizable. Hint: Use the fact that any element of the  $T$ -invariant subspace is a linear combination of some eigenvalues, and part a).
  - (c) Use part a) to show that  $V$  is a  $T$ -cyclic subspace of itself. Hint: Pick a vector that gives a basis to  $V$
5. Let  $T$  be a linear operator on a finite dimensional vector space  $V$ .
  - (a) Prove that  $T$  is diagonalizable if and only if  $V$  is the direct sum of one-dimensional  $T$ -invariant subspaces.
  - b) Let  $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$  where  $W_1, W_2, \dots, W_k$  are  $T$ -invariant subspaces. Prove that

$$\det(T) = \det(T_{W_1}) \cdot \det(T_{W_2}) \cdot \dots \cdot \det(T_{W_k})$$

6. Prove the parallelogram law on an inner product space  $V$ ;

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \text{for all } x, y \in V$$

7. Let  $V$  be a finite dimensional inner product space over  $\mathbb{F}$  and let  $S = \{v_1, v_2, \dots, v_n\}$  be an orthonormal subset of  $V$ . Show that if  $S$  is a basis for  $V$  then for any  $x, y \in V$  one has

$$\langle x, y \rangle = \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$$

(This is called Parseval's equality)

8. Let  $V = C[0, 1]$  with the inner product  $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ . Let  $W = \text{Span}\{t, \sqrt{t}\}$ .
- a) Find an orthonormal basis for  $W$ . ( I suggest you to practice Gram-Schmidt process -problem 2 of Section 6.2 till you feel comfortable)
  - b) Let  $h(t) = t^2$ . Use the orthogonal basis obtained in part a) to obtain the closest approximation of  $h$  in  $W$ . Use Theorem 6.6
  - c) Let  $V = C([-1, 1])$  Let  $W_e$  denote the subspace of  $V$  that includes all even functions. Find  $W_e^\perp$ .