MATH 255 PSET 7

March 6, 2025

1.

Before solving this question, note that:

$$\sqrt{n^2+n}-n = (\sqrt{n^2+n}-n)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{(n^2+n)-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$

$$= \frac{n}{\sqrt{n^2+n}+n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \frac{1}{\sqrt{\frac{n^2+n}{n^2}}+1} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

We first calculate $\lim\limits_{n\to\infty}\sqrt{n^2+n}-n$. We use the fact that $\lim\limits_{n\to\infty}\frac{1}{n}=0$.

$$\lim_{n\to\infty}\sqrt{n^2+n}-n = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

We now prove that $\lim\limits_{n\to\infty}\sqrt{n^2+n}-n = \frac{1}{2}$. Pick $\epsilon > 0$. To show $\lim\limits_{n\to\infty}\sqrt{n^2+n}-n = \frac{1}{2}$, we want to find $N \in \mathbb{N}$ s.t. $\forall n \geq N, d(\sqrt{n^2+n}-n, \frac{1}{2}) < \epsilon$ or more simply s.t. $\forall n \geq N, |\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}| < \epsilon$. Thus, we solve the $|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}| < \epsilon$ inequality below:

$$\left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| < \epsilon$$

Note that because $n \in \mathbb{N} \implies \frac{1}{n} > 0 \implies 1+\frac{1}{n} > 1 \implies \sqrt{1+\frac{1}{n}} > 1 \implies 1+\sqrt{1+\frac{1}{n}} > 2 \implies \frac{1}{1+\sqrt{1+\frac{1}{n}}} < \frac{1}{2}$. Thus, $|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}| = \frac{1}{2} - \frac{1}{1+\sqrt{1+\frac{1}{n}}}$:

$$\frac{1}{2} - \frac{1}{1 + \sqrt{1 + \frac{1}{n}}} < \epsilon$$

$$\frac{1}{2} - \epsilon < \frac{1}{1 + \sqrt{1 + \frac{1}{n}}}$$

$$1 > (1 + \sqrt{1 + \frac{1}{n}})(0.5 - \epsilon)$$

$$1 - (0.5 - \epsilon) > (0.5 - \epsilon)\sqrt{1 + \frac{1}{n}}$$

$$\frac{0.5 + \epsilon}{0.5 - \epsilon} > \sqrt{1 + \frac{1}{n}}$$

$$(\frac{0.5 + \epsilon}{0.5 - \epsilon})^2 > 1 + \frac{1}{n}$$

$$\frac{1}{n} < (\frac{0.5 + \epsilon}{0.5 - \epsilon})^2 - 1$$

$$n > \frac{1}{(\frac{0.5+\epsilon}{0.5-\epsilon})^2 - 1}$$

Thus we see $d(\sqrt{n^2 + n} - n, \frac{1}{2}) < \epsilon$ for any $n > \frac{1}{(\frac{0.5+\epsilon}{0.5-\epsilon})^2-1}$. Thus we aim to choose N as any natural number $> \frac{1}{(\frac{0.5+\epsilon}{0.5-\epsilon})^2-1}$. By Archimedian property, $\exists\, m \in \mathbb{N}$ s.t. $m(1) > \frac{1}{(\frac{0.5+\epsilon}{0.5-\epsilon})^2-1}$ and so we can simply choose $N = m$. Thus, we have shown $\forall \epsilon > 0, \exists\, N \in \mathbb{N}$ s.t. $\forall n \geq N, d(\sqrt{n^2 + n} - n, \frac{1}{2}) < \epsilon$ and so we have proven $\lim_{n \to \infty} \sqrt{n^2 + n} - n = \frac{1}{2}$.

2.

Lemma 0.1 *Given $(a_n), (b_n)$ as bounded sequences in $\mathbb{R}$, we prove if some subsequence of $(a_n + b_n)$ converges to $x \implies$ the limit of some subsequence of a_n plus the limit of some subsequence of b_n is equal to x.*
Proof: Let us define this subsequence of $(a_n + b_n)$ that converges to x as a sequence given by $a_{n_1} + b_{n_1}, a_{n_2} + b_{n_2}, \ldots$ where $n_1 < n_2 < \ldots$. In other words, this subsequence is the sequence $(a_n + b_n)$ indexed by the monotonically increasing sequence (n_k). Given that this sequence $a_{n_k} + b_{n_k} \to x$, we know:

$$\lim_{k \to \infty}(a_{n_k} + b_{n_k}) = \lim_{k \to \infty} a_{n_k} + \lim_{k \to \infty} b_{n_k}$$
$$x = \lim_{k \to \infty} a_{n_k} + \lim_{k \to \infty} b_{n_k}$$

Thus, we have shown that the limit of some subsequence of a_n, given by $\lim_{k \to \infty} a_{n_k}$, plus the limit of some subsequence of b_n, given by $\lim_{k \to \infty} b_{n_k}$, is equal to x.

2

Lemma 0.2 *Let us define sets X, Y where $X \subset Y$. Then $\sup(X) \leq \sup(Y)$.*
Proof: $\sup(Y)$ is the lowest upper bound of Y and because $X \subset Y \implies \sup(Y)$ is an upper bound of X. However, $\sup(X)$ is the lowest upper bound of X and so $\sup(X) \leq \sup(Y)$.

Let us define sets A and B below as such:

$$A = \{x \in \mathbb{R}_{\text{ext}} \mid \text{some subseq of } (a_n) \text{ converges to } x\}$$
$$B = \{x \in \mathbb{R}_{\text{ext}} \mid \text{some subseq of } (b_n) \text{ converges to } x\}$$

We are given that $(a_n), (b_n)$ are bounded real sequences $\implies$ sequences $(a_n), (b_n)$ are bounded above and below $\implies$ sequences $(a_n), (b_n)$ and their subsequences cannot converge to $\pm\infty \implies A \subset \mathbb{R}$ and $B \subset \mathbb{R}$. Furthermore, this means the sequence $(a_n + b_n)$ is bounded[1] and so we can define the following set:

$$A + B = \{x + y \in \mathbb{R} \mid \text{some subseq of } (a_n) \to x \text{ and some subseq of } (b_n) \to y\} \subset \mathbb{R}$$

As proved in Homework 2, because $A, B, A + B \subset \mathbb{R}$, $\sup(A + B) = \sup(A) + \sup(B) = \limsup\limits_{n \to \infty}(a_n) + \limsup\limits_{n \to \infty}(b_n)$. Now consider the set below:

$$C = \{x \in \mathbb{R} \mid \text{some subseq of } (a_n + b_n) \to x\}$$

As proved in **Lemma 0.1**, if some subseq of $(a_n + b_n) \to x \implies$ the limit of some subsequence of a_n plus the limit of some subsequence of b_n is equal to x. Thus, $C \subset A + B \implies$ (**using Lemma 0.2**) $\sup(C) \leq \sup(A + B) = \limsup\limits_{n \to \infty}(a_n) + \limsup\limits_{n \to \infty}(b_n) \implies \sup(C) \leq \limsup\limits_{n \to \infty}(a_n) + \limsup\limits_{n \to \infty}(b_n)$. Note that $\sup(C) = \limsup\limits_{n \to \infty}(a_n + b_n)$ and so we have proven:

$$\limsup\limits_{n \to \infty}(a_n + b_n) \leq \limsup\limits_{n \to \infty}(a_n) + \limsup\limits_{n \to \infty}(b_n)$$

Example of $\limsup\limits_{n \to \infty}(a_n + b_n) = \limsup\limits_{n \to \infty}(a_n) + \limsup\limits_{n \to \infty}(b_n)$

We can just set $(a_n) = (b_n) = 1, 1, 1, 1 \ldots$. In this case $\limsup\limits_{n \to \infty}(a_n + b_n) = 2$ and $\limsup\limits_{n \to \infty}(a_n) = \limsup\limits_{n \to \infty}(b_n) = 1$. So $\limsup\limits_{n \to \infty}(a_n + b_n) = 2 = \limsup\limits_{n \to \infty}(a_n) + \limsup\limits_{n \to \infty}(b_n)$.

Example of $\limsup\limits_{n \to \infty}(a_n + b_n) < \limsup\limits_{n \to \infty}(a_n) + \limsup\limits_{n \to \infty}(b_n)$

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[1]Proof: WLOG, we show $(a_n + b_n)$ is bounded above. If (a_n) is bounded $\implies$ (a_n) is bounded above $\implies$ $\exists\, z_a \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, z_a \geq a_n$. We can define $z_b \in \mathbb{R}$ similarly as an upper bound for (b_n). Thus, $\forall n \in \mathbb{N}, a_n + b_n \leq z_a + z_b \implies z_a + z_b \in \mathbb{R}$ is an upper bound for $(a_n + b_n) \implies (a_n + b_n)$ is bounded above. This shows $(a_n + b_n)$ is bounded above (and with identical logic bounded below) $\implies (a_n + b_n)$ is bounded.

We can define sequence (a_n) where $a_n = \frac{1+(-1)^n}{2}$ and sequence (b_n) where $b_n = \frac{-1-(-1)^n}{2}$. Writing out the elements of (a_n) and (b_n) out we get:

$$(a_n) = 0, 1, 0, 1, \dots$$
$$(b_n) = 0, -1, 0, -1, \dots$$

So sequence $(a_n + b_n) = 0, 0, 0, 0, \dots$ and so $\limsup_{n \to \infty}(a_n + b_n) = 0$. However, there exists a subsequence of (a_n) given by $1, 1, 1, \dots$ and so $\limsup_{n \to \infty}(a_n) = 1$. There also exists a subsequence of (b_n) given by $0, 0, 0, \dots$ and so $\limsup_{n \to \infty}(b_n) = 0$. Thus, $\limsup_{n \to \infty}(a_n + b_n) = 0 < \limsup_{n \to \infty}(a_n) + \limsup_{n \to \infty}(b_n) = 1 + 0 = 1$.

3.

Fix $\epsilon > 0$ and define $\epsilon' = \frac{\epsilon}{2} > 0$. Given $(p_n), (q_n)$ are Cauchy sequences, $\exists \, N_p \in \mathbb{N}$ s.t. $\forall n, m \geq N_p, d(p_n, p_m) < \epsilon'$ and $\exists \, N_q \in \mathbb{N}$ s.t. $\forall n, m \geq N_q, d(p_n, p_m) < \epsilon'$. Set $N = \max(N_p, N_q) \in \mathbb{N}$. Then, $\forall n, m \geq N, d(p_n, p_m) < \epsilon'$ and $d(q_n, q_m) < \epsilon'$. Thus, applying Triangle Inequality, $\forall n, m \geq N$:

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n)$$

By Triangle Inequality, $d(p_m, q_n) \leq d(p_m, q_m) + d(q_m, q_n)$ and so:

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n) \leq d(p_n, q_m) + d(p_m, q_m) + d(q_m, q_n)$$
$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$
$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n)$$

We know $d(p_n, p_m) < \epsilon'$ and $d(q_m, q_n) = d(q_n, q_m) < \epsilon'$ and so:

$$d(p_n, q_n) - d(p_m, q_m) < \epsilon' + \epsilon' = \epsilon$$

Note that we could also start with: $d(p_m, q_m) \leq d(p_m, p_n) + d(p_n, q_m)$ to show through identical logic $d(p_m, q_m) - d(p_n, q_n) < \epsilon$. Given $d(p_n, q_n) - d(p_m, q_m) < \epsilon$ and $d(p_m, q_m) - d(p_n, q_n) < \epsilon \implies |d(p_n, q_n) - d(p_m, q_m)| < \epsilon$.

Thus, we have shown $\forall \epsilon > 0, \exists \, N' \in \mathbb{N}$ s.t. $\forall n, m \geq N', d_{\mathbb{R}}(d(p_n, q_n), d(p_m, q_m)) < \epsilon$ where $d_{\mathbb{R}}$ is the standard distance function in metric space $\mathbb{R}$ (i.e. $d_{\mathbb{R}}(x, y) = |x - y|$). This proves that the sequence $(d(p_n, q_n))$ in $\mathbb{R}$ is a Cauchy Sequence. Because all Cauchy Sequences in $\mathbb{R}$ converge, this proves that the sequence $(d(p_n, q_n))$ converges $\implies (d(p_n, q_n))$ has a limit.

4.

Lemma 0.3 *We prove that if $b \in \mathbb{R}$ is an effective upper bound (EUB) of (x_n), $b \geq \limsup\limits_{n \to \infty} x_n$.*

Proof: Suppose we have a subsequence of (x_n) that converges to x. Then $\forall \epsilon > 0, \exists N_x \in \mathbb{N}$ s.t. $\forall n \geq N_x, d(x_n, x) = |x_n - x| < \epsilon \implies x_n - x > -\epsilon \implies x < x_n + \epsilon$. Now consider a given EUB $b \in \mathbb{R}$ of (x_n): by definition of EUB, $\exists N_b \in \mathbb{N}$ s.t. $\forall n \geq N_b, x_n \leq b$. Thus $\forall \epsilon > 0, \exists M = max(N_x, N_b)$ s.t. $\forall n \geq M, x < x_n + \epsilon$ and $x_n \leq b$. Thus this means $\forall \epsilon > 0, \exists M \in \mathbb{N}$ s.t. $\forall n \geq M, x < x_n + \epsilon \leq b + \epsilon$. Note that x and b are constants (independent of n) and so this proof shows $\implies \forall \epsilon > 0, x < b + \epsilon \implies x \leq b$ (see footnote[2]).

Let us define the set $A = \{x \in \mathbb{R}_{ext} \mid$ some subsequence of (x_n) converges to $x\}$. We show in the following two cases, $\limsup\limits_{n \to \infty} x_n = sup(A) \leq b$.

1. ***Case One:*** $A = \emptyset$

   If $A = \emptyset \implies \forall x \in \mathbb{R}, x$ is vacuously an upper bound of A. So EUB $b \in \mathbb{R}$ is an upper bound of A. However, because $\limsup\limits_{n \to \infty} x_n = sup(A)$ is the lowest upper bound of $A \implies \limsup\limits_{n \to \infty} x_n \leq b$.

2. ***Case Two:*** $A \neq \emptyset$

   As previously established, the limit of any convergent subsequence of (x_n) is $\leq b$. So, $\forall a \in A, a \leq b$. Thus, b is an upper bound of A. However by definition $\limsup\limits_{n \to \infty} x_n = sup(A)$ is the lowest upper bound of $A \implies \limsup\limits_{n \to \infty} x_n \leq b$.

We prove this statement through casework on $\limsup\limits_{n \to \infty} x_n$:

1. **Case One: $\limsup\limits_{n \to \infty} x_n \in \mathbb{R}$**

   We first show that $\forall \epsilon > 0, \limsup\limits_{n \to \infty} x_n + \epsilon$ is an upper bound. This follows directly from Proposition 6.37: $\limsup\limits_{n \to \infty} x_n < \limsup\limits_{n \to \infty} x_n + \epsilon \implies \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, x_n < \limsup\limits_{n \to \infty} x_n + \epsilon \implies \limsup\limits_{n \to \infty} x_n + \epsilon$ is an EUB of (x_n). Furthermore, note that by implication of **Lemma 0.3**, $\nexists$ a real-valued EUB $< \limsup\limits_{n \to \infty} x_n$. So, the set $\{a \in \mathbb{R} \mid a$ is an EUB for $(x_n)\}$ can be given by the interval $(\limsup\limits_{n \to \infty} x_n, \infty) \subset \mathbb{R}$ or alternatively the set $\{\limsup\limits_{n \to \infty} x_n + \epsilon : \forall \epsilon > 0\}$.

   The greatest lower bound of this set[34] is $\limsup\limits_{n \to \infty} x_n$ and so we have proven $\limsup\limits_{n \to \infty} x_n = \inf\{a \in \mathbb{R} \mid a$ is an EUB for $(x_n)\}$.

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[2]For more clarity on the statement $\forall \epsilon > 0, x < b + \epsilon \implies x \leq b$, we can think of the maximum possible value of x as $\inf\{b + \epsilon : \epsilon > 0\}$. The infimum of this set is obviously b (b is a lower bound of this set and any real number greater than b fails to be a lower bound of this set so b is the greatest lower bound) and so we have that the largest value x can hold is $b \implies x \leq b$.

[3]Note that it does not matter if $\limsup\limits_{n \to \infty} x_n$ is included in this set of real-valued EUBs of (x_n). This is because the supremum of this set would have been $\limsup\limits_{n \to \infty} x_n$ regardless.

[4]The proof for this is trivial: $\limsup\limits_{n \to \infty} x_n$ is a lower bound of this set and any number slightly greater than

2. **Case Two: $\limsup\limits_{n\to\infty} x_n \notin \mathbb{R}$**

   By definition $\limsup\limits_{n\to\infty} x_n \in \mathbb{R}_{\text{ext}}$. Given $\limsup\limits_{n\to\infty} x_n \notin \mathbb{R} \implies \limsup\limits_{n\to\infty} x_n = \pm\infty$. We consider each case below:

   (a) $\limsup\limits_{n\to\infty} x_n = \infty$

   As proven in **Lemma 0.3**, all EUBs must be $\geq \limsup\limits_{n\to\infty} x_n$. Thus $\limsup\limits_{n\to\infty} x_n = \infty \implies (x_n)$ cannot have any EUBs in $\mathbb{R}$ as $\nexists\ x \in \mathbb{R}$ s.t. $x \geq \limsup\limits_{n\to\infty} x_n = \infty$. Thus, the set $\{a \in \mathbb{R} \mid a$ is an EUB for $(x_n)\} = \emptyset$ and because[5] $\inf(\emptyset) = \infty = \limsup\limits_{n\to\infty} x_n$ we have proven:

   $$\limsup\limits_{n\to\infty} x_n = \inf\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}$$

   (b) $\limsup\limits_{n\to\infty} x_n = -\infty$

   Because $\limsup\limits_{n\to\infty} x_n = -\infty, \forall x \in \mathbb{R}, x > \limsup\limits_{n\to\infty} x_n \implies$ (through Proposition 6.37) $\forall x \in \mathbb{R}, x$ is an EUB for (x_n). Thus the set $\{a \in \mathbb{R} \mid a$ is an EUB for $(x_n)\} = \mathbb{R}$ and so the only lower bound (and only greatest lower bound) of this set is $-\infty \implies \inf(\{a \in \mathbb{R} \mid a$ is an EUB for $(x_n)\}) = -\infty = \limsup\limits_{n\to\infty} x_n$. So we have again proven:

   $$\limsup\limits_{n\to\infty} x_n = \inf\{a \in \mathbb{R} \mid a \text{ is an EUB for } (x_n)\}$$

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it (can be given as $\limsup\limits_{n\to\infty} x_n + \epsilon$ where $\epsilon > 0$) fails to be a lower bound for this set $\implies \limsup\limits_{n\to\infty} x_n$ is the greatest lower bound of this set.

[5]The following statement reflects the fact that the infimum is the greatest lower bound. Because $\forall x \in \mathbb{R}_{\text{ext}}$, x is a lower bound for $\emptyset$, the greatest lower bound of $\emptyset$ is the greatest value in $\mathbb{R}_{\text{ext}}$ or ∞.