MATH 241 PSET 4

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1.

Let us define the r.v. X as the number of games ending in draws. Because n games are being played independently, each with a 60% change of ending in a tie, $X \sim Bin(n, 0.6)$ and so the PMF of X is given by $P(X = k) = \binom{n}{k}(0.6)^k(0.4)^{n-k}$ for $0 \leq k \leq n$. The support of X is given by the set $\{0, 1, \ldots, n\}$.

Because each of the players play only one game, the r.v. $Y = 2X$ defines the number of players whose games end in draws. This means that $P(Y = 2k) = P(X = k)$. Thus, for $0 \leq k \leq 2n$, the PMF for Y can be given as:

$$P(Y = k) = \begin{cases} 0, & \text{if } k \text{ odd} \\ \binom{n}{0.5k}(0.6)^{0.5k}(0.4)^{n-0.5k}, & \text{otherwise} \end{cases}$$

The support of Y is twice the support of X and is thus given by the set $\{0, 2, \ldots, 2n\}$.

2.

a) Let us define r.v. X as the number of winning tickets out of 3 selected tickets. $X \sim Bin(3, p)$.

b) **Using Inclusion-Exclusion Principle**

Let us define events X_1, X_2, X_3 as the events that the first, second, and third tickets are winning tickets, respectively. Note that $X_1, X_2, X_3 \sim Bern(p)$. The probability that at least 1 of the 3 tickets is a winning ticket can be given as $P(X_1 \cup X_2 \cup X_3)$.

$$P(X_1 \cup X_2 \cup X_3) = P(X_1) + P(X_2) + P(X_3)$$
$$-P(X_1 \cap X_2) - P(X_1 \cap X_3) - P(X_2 \cap X_3) + P(X_1 \cap X_2 \cap X_3)$$

Given $P(X_1) = P(X_2) = P(X_3) = p$ and $P(X_1 \cap X_2) = P(X_1 \cap X_3) = P(X_2 \cap X_3) = p^2$ and $P(X_1 \cap X_2 \cap X_3) = p^3$, we get that:

$$P(X_1 \cup X_2 \cup X_3) = p + p + p - p^2 - p^2 - p^2 + p^3$$
$$P(X_1 \cup X_2 \cup X_3) = 3p - 3p^2 + p^3$$

Using Complement Rule

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \binom{3}{0} p^0 (1-p)^3 \\ &= 1 - (1-p)^3 = 3p - 3p^2 + p^3 \end{aligned}$$

3.

a) Let us define event C_1 and C_2 as the events that the first and second coin was chosen, respectively. We are given $P(C_1) = P(C_2) = 0.5$. Using LOTP to compute the PMF $P(X = k)$:

$$P(X = k) = P(X = k|C_1)P(C_1) + P(X = k|C_2)P(C_2)$$
$$P(X = k) = \frac{P(X = k|C_1) + P(X = k|C_2)}{2}$$
$$P(X = k) = \frac{\binom{n}{k} p_1^k (1-p_1)^{n-k} + \binom{n}{k} p_2^k (1-p_2)^{n-k}}{2}$$
$$P(X = k) = \frac{\binom{n}{k} [p_1^k (1-p_1)^{n-k} + p_2^k (1-p_2)^{n-k}]}{2}$$

The support for X for $n \geq 2$ flips is given by the set $\{0, \ldots, n\}$.

b) If $p_1 = p_2$, $X \sim Bin(n, p_1)$.

4.

a) Let us define r.v. W as the number of women who get promoted. $W \sim HGeom(n, m, t)$.

b) Let us define r.v. W as the number of women promoted, r.v. W' as the number of women who are not promoted, and r.v. T as the number of employees who are promoted. Their distributions can be given as $W \sim Bin(n, p)$, $W' \sim Bin(n, 1-p)$, and $T \sim Bin(n + m, p)$.

5.

a) We evaluate the probability $P(X \oplus Y = 1)$ below. Note that events X and Y are independent.

$$P(X \oplus Y = 1) = P(X = 1 \cap Y = 0) + P(X = 0 \cap Y = 1)$$
$$P(X \oplus Y = 1) = P(X = 1)P(Y = 0) + P(X = 0)P(Y = 1)$$
$$P(X \oplus Y = 1) = p(1 - \frac{1}{2}) + \frac{1-p}{2}$$
$$P(X \oplus Y = 1) = \frac{1}{2}$$

We now evaluate the probability $P(X \oplus Y = 0)$ below.

$$P(X \oplus Y = 0) = P(X = 1 \cap Y = 1) + P(X = 0 \cap Y = 0)$$
$$P(X \oplus Y = 0) = P(X = 1)P(Y = 1) + P(X = 0)P(Y = 0)$$
$$P(X \oplus Y = 0) = \frac{p}{2} + (1-p)(1 - \frac{1}{2})$$
$$P(X \oplus Y = 0) = \frac{1}{2}$$

Because the support of $X \oplus Y$ is $\{0, 1\}$ and $P(X \oplus Y = 0) = P(X \oplus Y = 1) = \frac{1}{2}$, $X \oplus Y$ is a Bernoulli trial with a 50% chance of success. $X \oplus Y \sim Bern(\frac{1}{2})$.

b) **In the case $p = \frac{1}{2}$.**
We have established in part (a) that regardless of the value of p, $X \oplus Y \sim Bern(\frac{1}{2})$. Because Bernoulli trials have the same probability of success each time, X or Y does not affect the outcome of $X \oplus Y$. Thus, $X \oplus Y$ is independent to X and Y.
In the case $p \neq \frac{1}{2}$.
We first test if $X \oplus Y$ and X are independent. If this was the case then:

$$P(X \oplus Y = 1 | X = k) = P(X \oplus Y = 1)$$

where $k \in \{0, 1\}$[1]. We know from part (a) that regardless of the value of p, $P(X \oplus Y = 1) = \frac{1}{2}$. Let us define $f(k)$ as a function that performs $k + 1$ modulo 2. Thus in order for $P(X \oplus Y = 1|X = k) = \frac{1}{2}$, $P(Y = f(k)) = \frac{1}{2}$. Because $Y \sim Bern(\frac{1}{2})$, we know this statement must be true and so $X \oplus Y$ and X are independent.
We now test if $X \oplus Y$ and Y are independent. If this was the case then:

$$P(X \oplus Y = 1 | Y = k) = P(X \oplus Y = 1) = \frac{1}{2}$$

where $k \in \{0, 1\}$. We use the same definition of $f(k)$ as before. In order for $P(X \oplus Y = 1|Y = k) = \frac{1}{2}$, $P(X = f(k)) = \frac{1}{2}$. $P(X = f(k))$ is either equal to p or $1 - p$. Because $p \neq \frac{1}{2}$, $P(X = f(k)) \neq \frac{1}{2}$ and so $P(X \oplus Y = 1|Y = k) \neq \frac{1}{2} \neq P(X \oplus Y = 1)$. Thus $X \oplus Y$ and Y are not independent.

6. Anish Lakkapragada. I worked independently.

[1] Note that we do not need to check if $P(X \oplus Y = 0|X = k) = P(X \oplus Y = 0)$ to test for independence. This is because if $P(X \oplus Y = 1|X = k) = P(X \oplus Y = 1) \Rightarrow 1 - P(X \oplus Y = 1|X = k) = 1 - P(X \oplus Y = 1) \Rightarrow P(X \oplus Y = 0|X = k) = P(X \oplus Y = 0)$.