Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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Math 226: HW 2
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1. a) We demonstrate that $L^2(\mathbb{R})$ has the identity element $f(x) = 0$, is closed under addition, and closed under scalar multiplication to prove that $L^2(\mathbb{R})$ is a vector space.

   1 Existence of Additive Identity Element in $L^2(\mathbb{R})$
   The function $f(x) = 0 \in L^2(\mathbb{R})$ as $f(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $\int_{-\infty}^{\infty} f(x)dx = 0 < \infty$. $f(x) = 0$ is the additive identify element $L^2(\mathbb{R})$ as $\forall g(x) \in L^2(\mathbb{R})$, $f(x)+g(x) = g(x)$.

   2 Closed Under Addition
   Given $a, b \in \mathbb{R}$:

$$
\begin{aligned}
(a-b)^2 &\geq 0 \\
a^2 + b^2 &\geq 2ab \\
2a^2 + 2b^2 &\geq (a+b)^2
\end{aligned}
$$

   If we switch sides and then substitute $a = f(x) \in L^2(\mathbb{R})$ and $b = g(x) \in L^2(\mathbb{R})$, we get:

$$
\begin{aligned}
(f(x) + g(x))^2 &\leq 2[f(x)]^2 + 2[g(x)]^2 \\
|f(x) + g(x)|^2 &\leq 2|f(x)|^2 + 2|g(x)|^2
\end{aligned}
$$

   We now integrate from $-\infty$ to ∞ on both sides:

$$\int_{-\infty}^{\infty} |f(x) + g(x)|^2 dx \leq 2\int_{-\infty}^{\infty} |f(x)|^2 dx + 2\int_{-\infty}^{\infty} |g(x)|^2 dx$$

   Because $f(x), g(x) \in L^2(\mathbb{R})$, we know that $\int_{-\infty}^{\infty} |f(x)|^2 < \infty$ and $\int_{-\infty}^{\infty} |g(x)|^2 < \infty$. Thus $2\int_{-\infty}^{\infty} |f(x)|^2 dx + 2\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$ as well, and so we know that:

$$\int_{-\infty}^{\infty} |f(x) + g(x)|^2 dx < \infty \tag{1}$$

   Given that $f(x), g(x) \in L^2(\mathbb{R})$, we know that $f(x)+g(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ as $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space and thus is closed under addition. Thus, because we have proved Equation 1 and $f(x) + g(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, $f(x) + g(x) \in L^2(\mathbb{R})$ and so $L^2(\mathbb{R})$ is closed under addition.

   3 Closed Under Scalar Multiplication
   Consider for $x \in \mathbb{R}$ a function $f(x) \in L^2(\mathbb{R})$. Given $c \in \mathbb{R}$, let us define $g(x) = cf(x)$. Because $f(x) \in \mathbb{R}$ and $c \in \mathbb{R}$, $g(x) = cf(x) \in \mathbb{R}$. If $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$, $k\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ for $k \in \mathbb{R}$. Thus $\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |cf(x)|^2 dx = |c|^2 \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ as $|c|^2 \in \mathbb{R}$ and so it is proven $\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$. Thus $g(x) \in L^2(\mathbb{R})$ and so $L^2(\mathbb{R})$ is proven to be closed under scalar multiplication.

   Because we have demonstrated 1, 2, and 3, we have demonstrated $L^2(\mathbb{R})$ is a vector space.

b) In order for a set V to define a vector space over field $\mathbb{R}$, the vector $\mathbf{0} \in V$ s.t. $\forall v \in V, v + \mathbf{0} = v$. For set V, this vector $\mathbf{0} = (0, 1)$ as $\forall v = (a_1, b_1) \in V, v + \mathbf{0} = (a_1, b_1) + (0, 1) = (a_1, b_1) = v$.

Another property for a set V to define a vector space is that $\forall v \in V, \exists -v \in V$ s.t. $v + (-v) = \mathbf{0} = (0, 1)$. Let us define $v = (a_1, b_1) \in V$ and vector $-v = (a_2, b_2)$. For $v + (-v) = (0, 1)$, $a_2 = -a_1$ and $b_2(b_1) = 1$. In the case where $b_1 = 0$, $b_2(b_1) \neq 1$ and thus $\forall v \in V$ it is not guaranteed $\exists -v \in V$ s.t. $v + (-v) = \mathbf{0} = (0, 1)$. Because this condition is not met, V does not define a valid vector space over $\mathbb{R}$.

2. a) We go through the three conditions of testing if vector space W_1 and W_2 are subspaces of $\mathbb{F}^n$.

(1) Closed Under Scalar Multiplication

a) $W = W_1$

For a given $x = (a_1, a_2, \dots a_n) \in W_1$ and $c \in \mathbb{F}$, $cx = (ca_1, ca_2, \dots ca_n)$. Because $ca_1, ca_2, \dots ca_n \in \mathbb{F}^n$, and $c\Sigma_{i=1}^{N} a_i = 0$ given $\Sigma_{i=1}^{N} a_i = 0$, W_1 meets this condition to be a subspace of $\mathbb{F}^n$.

b) $W = W_2$

$cx = (ca_1, ca_2, \dots ca_n)$. Given $\Sigma_{i=1}^{N} a_i = 1$, $c\Sigma_{i=1}^{N} a_i \neq 1$ and thus $cx \notin W_2$. Thus W_2 does not meet this condition to be a subspace of $\mathbb{F}^n$. *Because W_2 does not meet this condition to be a subspace, we do not need to check if it meets any of the other conditions.*

(2) Closed Under Addition

a) $W = W_1$

Given $x = (a_1, a_2, \dots a_n) \in W_1, y = (b_1, b_2, \dots b_n) \in W_1$, $x + y = (a_1 + b_1, a_2 + b_2, \dots a_n + b_n)$. $\Sigma_{i=1}^{N} a_i = \Sigma_{i=1}^{N} b_i = 0 \Rightarrow \Sigma_{i=1}^{N} a_i + b_i = 0 \Rightarrow x + y \in W_1$. Thus W_1 meets this condition to be a subspace of $\mathbb{F}^n$.

(3) $\exists\, \mathbf{0} \in W$

a) $W = W_1$

For $x = \mathbf{0}$, $x_i = 0 \Rightarrow \Sigma_{i=1}^{N} x_i = 0$. Thus, $\mathbf{0} \in W_1$ and so W_1 meets this condition to be a subspace of $\mathbb{F}^n$.

Because W_1 meets all the conditions to be a subspace whereas W_2 does not, W_1 is a subspace of $\mathbb{F}^n$.

b) We define the subset of $\mathbb{Z}_2^n$ with even E_n as $Q^n = \{v \in \mathbb{Z}_2^n : E_n(v) \in 2\mathbb{Z}\}$. We now assess if $Q^n \leq \mathbb{Z}_2^n$ by checking if Q^n meets the following three conditions.

(1) $\exists\, \mathbf{0} \in Q^n$

Let us define the zero vector as $z = \mathbf{0} \in \mathbb{Z}_2^n$. Because $E_n(z) = 0 \in 2\mathbb{Z}$, $z = \mathbf{0} \in Q^n$.

(2) Closed Under Addition

Let us define two vectors $x, y \in Q^n$. Because $\mathbb{Z}_2^n$ is a vector space and thus closed under addition, $x + y \in \mathbb{Z}_2^n$. The number of nonzero components of $x + y$ is given by $E_n(x+y) = E_n(x) + E_n(y) - 2k$, where $k \in \mathbb{Z}$ is given by the number of indices where x and y have the same value. Because $E_n(x), E_n(y) \in 2\mathbb{Z}$ as $x, y \in \mathbb{Z}_2^n$, $E_n(x) + E_n(y) \in 2\mathbb{Z}$. Because $k \in \mathbb{Z}$, $2k \in 2\mathbb{Z}$ and so $E_n(x + y) = E_n(x) + E_n(y) - 2k \in 2\mathbb{Z}$. Because $E_n(x+y) \in 2\mathbb{Z}$ and $x + y \in \mathbb{Z}_2^n \Rightarrow x + y \in Q^n$. Thus Q^n is closed under addition.

(3) Closed Under Scalar Multiplication

Let us consider a scalar $c \in \mathbb{Z}_2$ and $v \in Q^n$. c can either equal zero or one. If $c = 0$, $cv = \mathbf{0} \in Q^n$. If $c = 1$, $cv = v \in Q^n$. Thus, $cv \in Q^n$ for any $c \in \mathbb{Z}_2$ and so Q^n is closed under scalar multiplication.

Because Q^n meets all the three conditions to be a subspace to $\mathbb{Z}_2^n$, $Q^n \leq \mathbb{Z}_2^n$.

c) The general form for function $f \in P_3(\mathbb{R})$ is given by $f(x) = c_1 + c_2x + c_3x^2 + c_4x^3$ where $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Given the constraints $f(0) = f'(0)$ and $f(1) = 0$, the form for any function $f \in W$ is given by:

$$f(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3$$

We now test if W defines a subspace of $P_3(\mathbb{R})$.

1 $\exists\, \mathbf{0} \in W$
Because when $f(x) = 0$ when $c_1 = c_3 = 0$ and $x = 0$, the zero polynomial is defined in W.

2 Closed Under Addition
Given $p(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3 \in W$ and $q(x) = b_1 + b_1x + b_3x^2 + (-2b_1 - b_3)x^3 \in W$, $p(x) + q(x) = (c_1 + b_1) + (c_1 + b_1)x + (c_3 + b_3)x^2 + (-2c_1 - c_3 - 2b_1 - b_3)x^3 \in P_3(\mathbb{R})$. Defining $z(x) = p(x) + q(x)$, $z(0) = z'(0) = c_1 + b_1$ and $z(1) = p(1) + q(1) = 0 + 0 = 0$. Thus $p(x) + q(x) \in W$ and thus W is proven to be closed under addition.

3 Closed Under Scalar Multiplication
Given $p(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3 \in W$ and $k \in \mathbb{R}$, $kp(x) = k(c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3)$. Defining $g(x) = kp(x)$, $g(0) = g'(0) = kc_1)$ and $g(1) = k(p(1)) = k(0) = 0$. Thus $kp(x) \in W$ and W is proven to be closed under scalar multiplication.

Because I have shown W contains the zero polynomial and is closed under addition and scalar multiplication, $W \leq P_3(\mathbb{R})$.

Because $P_k(\mathbb{R})$ is the set of polynomials that have a degree of *at most* k, $P_n(\mathbb{R}) \leq P_k(\mathbb{R})$ where $n \leq k$. Thus $P_3(\mathbb{R}) \leq P_4(\mathbb{R})$. Because $W \leq P_3(\mathbb{R}) \leq P_4(\mathbb{R})$, $W \leq P_4(\mathbb{R})$.

d) Because this statement is an *if and only if*, we must show (1) that if these two conditions are met, $W \leq V$ and (2) that if $W \leq V$, these two conditions are met. We show (1) and (2) below.

1 If Condition 1 and Condition 2 are met, $W \leq V$

a Condition 1: $W \neq \emptyset$
If $W = \emptyset$, the standard condition to define a subspace for $\mathbf{0} \in W$ cannot be met because there are no elements in W. Note $W \neq \emptyset \not\Longrightarrow \mathbf{0} \in W$.

b Condition 2: for $a \in \mathbb{F}$ and $x, y \in W$, $\exists\, ax + y \in W$.
Given $a = -1$ and $x = y$, if W meets Condition 2, it is guaranteed that $-x + y = \mathbf{0} \in W$. Thus, the standard condition of existence of a zero vector in a subspace is met if Condition 2 is met.
In the case $y = \mathbf{0} \in W$, if W meets Condition 2, $ax \in W$ for $a \in \mathbb{F}$ and $x \in W$. Thus, closure under scalar multiplication is met if Condition 2 is met.
Let us define $z = ax \in W$. Then, if Condition 2 is met, we know given $z, y \in W$, $z + y \in W$. Thus, closure under addition is met if Condition 2 is met.

Thus, we have shown that if Condition 1 and Condition 2 are met, W meets the three properties to be defined as a subspace and so $W \leq V$.

2 If $W \leq V$, Condition 1 and Condition 2 are met
We discuss below the implications of the properties of W we know given $W \leq V$.

a $\exists\, \mathbf{0} \in W$
If $\exists\, \mathbf{0} \in W$, $|W| \geq 1$ and so $W \neq \emptyset$. Thus, Condition 1 is met.

b W is closed under addition and scalar multiplication

Let us define $a \in \mathbb{F}$ and $x, y \in W$. If W is closed under scalar multiplication, $ax \in W$. If W is closed under addition, $ax + y \in W$. Thus, Condition 2 is met.

Thus, we have shown that if $W \leq V$, Condition 1 and Condition 2 are met.

Because we have proven both **1** and **2**, we have shown that if and only if Condition 1 and Condition 2 are met for a given subset W of a vector space V will $W \leq V$.

3. a) We test if $U \cap W$ is a subspace of V below.

1 $\mathbf{0} \in U \cap W$

Because both U and W are valid subspaces, $\exists\ \mathbf{0} \in U$ and $\exists\ \mathbf{0} \in W$. Thus $\mathbf{0} \in U \cap W$.

2 Closed Under Addition

Let us consider $x, y \in U \cap W$. Because U and W are valid subspaces, U and W are closed under addition. Thus $x + y \in U$ and $x + y \in W \Rightarrow x + y \in U \cap W$.

3 Closed Under Scalar Multiplication

Let us consider $c \in \mathbb{F}$ and $x \in U \cap W$. Because U and W are valid subspaces, U and W are closed under scalar multiplication. Thus $cx \in U$ and $cx \in W \Rightarrow cx \in U \cap W$.

Thus we have proven $U \cap W \leq V$.

b) We test if $U + W$ is a subspace of V below.

1 $\mathbf{0} \in U + W$

Because U and W are both valid subspaces, $\mathbf{0} \in U, W$. Thus, for $u = \mathbf{0} \in U$ and $w = \mathbf{0} \in W$, $u + w = \mathbf{0} \in U + W$.

2 Closed Under Addition

Let us consider $u_1, u_2 \in U$ and $w_1, w_2 \in W$. Let us define elements $x = u_1+w_1 \in U+W$ and $y = u_2+w_2 \in U+W$. $x+y = u_1+w_1+u_2+w_2 \rightarrow (u_1+u_2)+(w_1+w_2)$. Because U and W are valid subspaces, they are both closed under addition and thus $z_1 = u_1+u_2 \in U$ and $z_2 = w_1+w_2 \in W$. As such, $x+y = z_1+z_2 \in U+W$ and so $U + W$ is proven to be closed under addition.

3 Closed Under Scalar Multiplication

Let us define $u \in U, w \in W, x = u_1 + w_1 \in U$. Given $c \in \mathbb{F}$, $cx = cu + cw$. Because U and W are valid subspaces, U and W are both closed under scalar multiplication and so $cu \in U$ and $cw \in W$. Thus, $cx = cu + cw \in U + W$ and so $U + W$ is proven to be closed under scalar multiplication.

Thus we have proven $U + W \leq V$.

c) Two subspaces of $\mathbb{R}^2$ whose union is not a subspace of $\mathbb{R}^2$ is $\mathbb{Q}^2$ and $W = \{(a_1, a_2) \in \mathbb{F}^2 : a_1 + a_2 = 0\}$ where field $\mathbb{F}^2 = (\mathbb{R}^2, +, \cdot)$.

An example proving $\mathbb{Q}^2 \cup W$ is not a subspace of $\mathbb{R}^2$ is choosing $x = (1.5, 0) \in \mathbb{Q}^2 \cup W$ and $y = (-\sqrt{2}, \sqrt{2}) \in \mathbb{Q}^2 \cup W$. $x + y = (1.5 - \sqrt{2}, \sqrt{2}) \notin \mathbb{Q}^2 \cup W$ and so $\mathbb{Q}^2 \cup W$ does not define a valid subspace as it is not closed under addition.

4. a)

$$Span(S) = \{f(x) = (c_1 - c_2) + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4; c_1, c_2, c_3, c_4 \in \mathbb{R}\}$$

b) A polynomial $p(x) \in P_4(\mathbb{R})$ that cannot be written as a linear combination of S (i.e. $p(x) \notin Span(S)$) is $p(x) = 5 - 2x^2$.

Let us try to see if $p(x) = 5 - 2x^2 \in Span(S)$. Because in the general form of a function $f \in Span(S)$ the only coefficient affecting the x^2 term is $-c_2$, $c_2 = 2$. Because the constant term 5 is given by $c_1 - c_2$, $c_1 = 7$ for $p(x) = 5 - 2x^2 \in Span(S)$. However, because this leads to a nonzero x term as $c_1 \neq 0$, $p(x) = 5 - 2x^2 \notin Span(S)$. Because $p(x) \notin Span(S)$ and $p(x) \in P_4(\mathbb{R})$, S does not generate $P_4(\mathbb{R})$.

c) The general form of the function $f \in P_4(\mathbb{R})$ is given by $f(x) = k_1 + k_2x + k_3x^2 + k_4x^3 + k_5x^4$ where $k_1, k_2, k_3, k_4, k_5 \in \mathbb{R}$. Through simple differentiation, we find that $f'(0) = k_2, f''(0) = 2k_3$. The set $\{f \in P_4(\mathbb{R}) : 2f(0) = 2f'(0) - f''(0)\}$ is equal to the set of all functions $f \in P_4(\mathbb{R})$ where:

$$\begin{aligned} 2f(0) &= 2f'(0) - f''(0) \\ 2k_1 &= 2k_2 - 2k_3 \\ k_1 &= k_2 - k_3 \end{aligned}$$

Re-expressing $f(x)$ with $k_1 = k_2 - k_3$ we get:

$$f(x) = (k_2 - k_3) + k_2x + k_3x^2 + k_4x^3 + k_5x^4$$

If we re-express our function above with $c_1 = k_2, c_2 = k_3, c_3 = k_4, c_4 = k_5$, we see that $f(x) = (c_1 - c_2) + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ where $c_1, c_2, c_3, c_4 \in \mathbb{R}$. This is the same form of a function $g \in Span(S)$. Thus, we have shown that the set $\{f \in P_4(\mathbb{R}) : 2f(0) = 2f'(0) - f''(0)\} = Span(S)$.