

MATH 255 HW 2

February 7, 2025

1. Exercise 3.1 (5 points)

We prove this statement by contradiction: suppose \mathbb{N} is bounded above in \mathbb{R} . This means an upper bound $\exists m \in \mathbb{R}$ for \mathbb{N} . Let us choose any arbitrary $x > 0 \in \mathbb{N}, \mathbb{R}$. By the Archimedean property of \mathbb{R} , $\exists n \in \mathbb{N}$ s.t. $nx > m$. Because $n, x \in \mathbb{N} \implies nx \in \mathbb{N}$ and so we have shown \exists an element $nx \in \mathbb{N}$ that is greater than $m \implies m$ is not an upper bound for $\mathbb{N} \implies \mathbb{N}$ is not bounded above in \mathbb{R} .

2. Exercise 3.2 (10 points)

- (1) To show that $\sqrt{5}$ is algebraic, we show that there exists a polynomial where $x = \sqrt{5}$ is the solution:

$$\begin{aligned}x &= \sqrt{5} \\x^2 &= 5 \\x^2 - 5 &= 0\end{aligned}$$

This polynomial is given by $n = 2$ and $a_0 = 1, a_1 = 0, a_2 = -5$. Thus, we can conclude $\sqrt{5}$ is algebraic. We take the same approach to show that $\sqrt{2 + \sqrt{3}}$ is algebraic:

$$\begin{aligned}x &= \sqrt{2 + \sqrt{3}} \\x^2 &= 2 + \sqrt{3} \\x^2 - 2 &= \sqrt{3} \\(x^2 - 2)^2 &= 3 \\x^4 - 4x^2 + 4 - 3 &= 0 \\x^4 - 4x^2 + 1 &= 0\end{aligned}$$

This polynomial is given by $n = 4$ and $a_0 = 1, a_1 = a_2 = 0, a_3 = -4, a_4 = 1$. Thus, we can conclude $\sqrt{2 + \sqrt{3}}$ is algebraic.

- (2) Let us pick any given natural $n \in \mathbb{N}$. Given this n , we have to choose coefficients a_0, \dots, a_n all of which lie in \mathbb{Z} , a countable set. The union of these n countable sets to get the arrangement of coefficients a_0, \dots, a_n is countable. Because each of these arrangement of coefficients (i.e. a n -length tuple like (a_0, \dots, a_n)) maps to at most n roots (i.e. n algebraic real numbers), the set of potential algebraic real numbers (i.e. a set of (a_0, \dots, a_n, x) where x is the algebraic real number) is countable. Finally, the infinite union of all these countable sets (i.e. $\forall n \in \mathbb{N}$) is countable as well \implies the set of all algebraic real numbers is countable.
- (3) We prove this by contradiction and assume that all real numbers are algebraic. This means that the real numbers \mathbb{R} are an infinite subset of the algebraic real numbers, a countable set $\implies \mathbb{R}$ is countable, which is a contradiction.

3. Exercise 3.3 (5 points)

Note on notation: Given $p, q \in \mathbb{R}$, we define the following notations for this problem:

(i) $\mathbb{R}_{(p,q)} = \{x \in \mathbb{R} : p < x < q\}$ (ii) $\mathbb{Q}_{(p,q)} = \{x \in \mathbb{Q} : p < x < q\}$ (iii) $\mathbb{R} \setminus \mathbb{Q}_{(p,q)} = \{x \in \mathbb{R} / \mathbb{Q} : p < x < q\}$

We first establish the fact that there are uncountably many real numbers on the interval (a, b) . We have established by Cantor's theorem that there are uncountably many real numbers on the interval $(0, 1)$. We can create a trivial bijection $f : \mathbb{R}_{(0,1)} \rightarrow \mathbb{R}_{(a,b)}$ as such:

$$f(x) = x(b - a) + a$$

To prove that this bijection establishes $\mathbb{R}_{(a,b)}$ is uncountable, we proceed by contradiction. If $\mathbb{R}_{(a,b)}$ was countable, that means there exists a bijection (let's call it g) from $\mathbb{N} \rightarrow \mathbb{R}_{(a,b)}$. But then $f^{-1} \circ g$ would be a bijection from $\mathbb{N} \rightarrow \mathbb{R}_{(0,1)} \implies \mathbb{R}_{(0,1)}$ is countable which is a contradiction. Thus we have proved $\mathbb{R}_{(a,b)}$ is uncountable.

We now prove there are uncountably many irrationals on the interval (a, b) . For proof by contradiction, let us now assume there are countably many irrationals on the interval (a, b) . We can call this set $\mathbb{R} \setminus \mathbb{Q}_{(a,b)}$ where $\mathbb{R}_{(a,b)}$ is given by $\mathbb{R} \setminus \mathbb{Q}_{(a,b)} \cup \mathbb{Q}_{(a,b)}$. Because \mathbb{Q} is countable, infinite subset $\mathbb{Q}_{(a,b)} \subset \mathbb{Q}$, is also countable. Because the union of two countable sets is countable, $\mathbb{R} \setminus \mathbb{Q}_{(a,b)} \cup \mathbb{Q}_{(a,b)} = \mathbb{R}_{(a,b)}$ is countable which is a contradiction. Thus, we have proven $\mathbb{R} \setminus \mathbb{Q}_{(a,b)}$ is uncountable.

4. Exercise 3.4 (10 points)

- (1) Let us define the set S_n where $n \in \mathbb{N}$ to be set of all finite subsets of \mathbb{N} with size n . We first demonstrate that $\forall n \in \mathbb{N}, S_n$ is countable.
- For $n = 1$, we have $S_1 = \{\{1\}, \{2\}, \dots\}$. We can easily create a bijection f between $\mathbb{N} \rightarrow S_1$: f takes $z \in \mathbb{N}$ and maps it to a set $\{z\}$. Because we can create such a bijection between \mathbb{N} to $S_1 \implies S_1$ is countable.
- For $n = 2$, we have $S_2 = \{\{1, 2\}, \{1, 3\}, \dots, \{2, 3\}, \{2, 4\}, \dots\}$. Note that this can be simplified as $S_2 = \mathbb{N} \times \mathbb{N}$. Because S_2 is given by the Cartesian product of two countable sets (\mathbb{N}), it is also countable.

We can see now that for $n = 3$, $S_3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and more generally for $k \in \mathbb{N}$, $S_k = \underbrace{\mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N}}_{k \text{ times}}$. Because the Cartesian product of countably many countable sets is countable¹, we can conclude that $\forall k \in \mathbb{N}$, S_k is countable.

The set of all finite subsets of \mathbb{N} can be given as $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$. Because the union of countably many countable sets is countable, $\bigcup_{k=1}^{\infty} S_k$ is countable. Furthermore, because the union of two sets that are at most countable, \emptyset (finite) and $\bigcup_{k=1}^{\infty} S_k$ (countable), is at most countable, we can conclude $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$ is countable². Thus we have proved the set of all finite subsets of \mathbb{N} , $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$, is countable.

- (2) For proof by contradiction, let us assume that the set of all subsets of \mathbb{N} are countable. This means that we can list out the set of subsets of \mathbb{N} as such:

1. \emptyset
2. $\{1\}$
3. $\{2\}$
4. $\{1, 2\}$
- ...

We now try to assemble a new set not in this list. For the i th set in this list, if the set does not contain 1, we add 1 to this new set. If the i th set does contain 1, then we do not add 1 to this new set. This means that for all subsets of \mathbb{N} listed above, none can equal our new set as we have constructed them to be different by either inclusion/exclusion of 1 \implies we can find a subset of \mathbb{N} not in the above list $\implies \mathbb{N}$ is uncountable.

- (3) **Lemma 0.1** *Let A, B be sets where A is uncountable and $A \subset B$. We prove that B is uncountable. We proceed by proof by contradiction and assume that B is countable. Because A is then a subset of a countable set $\implies A$ is at most countable, which is a contradiction. Thus, we have proved that a superset of an uncountable set must be uncountable.*

Let us define a given polynomial $f : \mathbb{Q} \rightarrow \mathbb{Q}$ as $f(x) = \sum_{i=0}^{\infty} a_i x^i$ where $\forall i \in \mathbb{Z}$, $a_i \in \mathbb{Q}$. We can represent polynomial f as a set given by $\{a_0, a_1, a_2, a_3, \dots\}$. Note that this representation is bijective, meaning each distinct polynomial has only one unique representation.

We approach this proof by showing that the set of all polynomials from $\mathbb{Q} \rightarrow \mathbb{Q}$ are uncountable. For proof by contradiction, let us assume that the set of all polynomials from $\mathbb{Q} \rightarrow \mathbb{Q}$ are countable. This would mean that we could list them all out as such:

¹We would prove this simply through induction on the fact that the Cartesian product of two countable sets is countable.

²For a short proof on how $\{\emptyset\} \cup \bigcup_{k=1}^{\infty} S_k$ is most countable \implies it is countable, we can assume for contradiction that it is finite which would imply that $\bigcup_{k=1}^{\infty} S_k$ is finite, which is a contradiction as we have proved it is countable.

1. $\{1, 0, 0, 0, \dots\}$
2. $\{0, 1, 0, 0, \dots\}$
3. $\{0, 0, 1, 0, \dots\}$
- ...

We now try to assemble a new polynomial g not in this list. For the i th polynomial in this list, if the polynomial has $a_i = 0$, we set $a_i = 1$ for g and if the polynomial has $a_i \neq 0$, we set $a_i = 0$ for g . This means that for all the polynomials from $\mathbb{Q} \rightarrow \mathbb{Q}$ listed above, none can equal our new polynomial g as we have constructed g to be different to each listed polynomial by at least one coefficient \implies we can find a polynomial g not in the above list \implies the set of polynomials from $\mathbb{Q} \rightarrow \mathbb{Q}$ is uncountable.

The set of polynomials from $\mathbb{Q} \rightarrow \mathbb{Q}$ is a subset of the set of all functions from $\mathbb{Q} \rightarrow \mathbb{Q}$. By **Lemma 0.1**, a superset of an uncountable set is uncountable, and so because the set of all polynomials from $\mathbb{Q} \rightarrow \mathbb{Q}$ is uncountable \implies the set of all functions from $\mathbb{Q} \rightarrow \mathbb{Q}$ is uncountable.

5. Exercise 3.5 (10 points)

Lemma 0.2 *By the triangle inequality, if $x, y \in \mathbb{R}^n$ for $n \in \mathbb{N}$, $\|x + y\| \leq \|x\| + \|y\|$. Defining $a, b, c \in \mathbb{R}^n$, we can set $x = a - c$, $y = c - b$ and yield the following result:*

$$\begin{aligned}\|a - c + c - b\| &\leq \|a - c\| + \|c - b\| \\ \|a - b\| &\leq \|a - c\| + \|c - b\|\end{aligned}$$

In \mathbb{R} , this can be given as $|a - b| \leq |a - c| + |c - b|$.

- (1) For d to be a metric space, $\forall x, y \in X, d(x, y) > 0$ if $x \neq y$. Let us define $x \in X$ and $y = -x$ so that $y \neq x$. $d(x, y) = |x^2 - y^2| = |x^2 - (-x)^2| = |x^2 - x^2| = |0| = 0$ and so this property is violated $\implies d$ is not a metric space.
- (2) For d to be a metric space, $\forall x \in X, d(x, x) = 0$. Because $\forall x \neq 0 \in X, d(x, x) = |x - 2x| = |-x| = |x| \neq 0$, this means that it is not guaranteed $\forall x \in X, d(x, x) = 0 \implies d$ is not a metric space.
- (3) We show d is a metric space below by showing d satisfies the four properties of metric spaces:
 - (i) $\forall x, y \in X$, if $x \neq y \implies x - y \neq 0 \implies |x - y| > 0 \implies d(x, y) = \frac{|x - y|}{1 + |x - y|} > 0$.
 - (ii) $\forall x \in X, x - x = 0 \implies |x - x| = 0 \implies d(x, x) = \frac{|x - x|}{1 + |x - x|} = \frac{0}{1 + 0} = 0$
 - (iii) $\forall x, y \in X, d(x, y) = \frac{|x - y|}{1 + |x - y|} = \frac{|y - x|}{1 + |y - x|} = d(y, x)$

- (iv) By **Lemma 0.2**, we have $\forall x, y, r \in X = \mathbb{R}, |x - y| \leq |x - r| + |r - y|$. Note that because $|x - r|$ and $|x - y|$ are ≥ 0 , $1 + |x - r|$ and $1 + |x - y|$ are ≥ 1 . Thus we have $|x - r| \geq \frac{|x - r|}{1 + |x - r|}$, $|x - y| \geq \frac{|x - y|}{1 + |x - y|}$, and $|r - y| \geq \frac{|r - y|}{1 + |r - y|}$. We apply these inequalities to show d obeys the Triangle Inequality:

$$\begin{aligned} |x - y| &\leq |x - r| + |r - y| \\ \frac{|x - y|}{1 + |x - y|} &\leq |x - y| \leq \frac{|x - r|}{1 + |x - r|} + \frac{|r - y|}{1 + |r - y|} \leq |x - r| + |r - y| \end{aligned}$$

By transitivity,

$$\begin{aligned} \frac{|x - y|}{1 + |x - y|} &\leq \frac{|x - r|}{1 + |x - r|} + \frac{|r - y|}{1 + |r - y|} \\ d(x, y) &\leq d(x, r) + d(r, y) \end{aligned}$$

- (4) We show d is a metric space below by showing d satisfies the four properties of metric spaces:
- (i) $\forall x, y \in X$, if $x \neq y \implies$ at least one of the following is true: (1) $x_1 \neq y_1$
 (2) $x_2 \neq y_2 \implies$ at least one of the following is true: (i) $|x_1 - y_1| > 0$ (ii) $|x_2 - y_2| > 0 \implies d(x, y) = |x_1 - y_1| + |x_2 - y_2| > 0$.
 - (ii) $\forall x \in X, d(x, x) = |x_1 - x_1| + |x_2 - x_2| = |0| + |0| = 0 + 0 = 0$
 - (iii) $\forall x, y \in X, d(x, y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y, x)$
 - (iv) Let us define $x, y, r \in X$. By **Lemma 0.2**, we have the following statements:

$$\begin{aligned} |x_1 - y_1| &\leq |x_1 - r_1| + |r_1 - y_1| \\ |x_2 - y_2| &\leq |x_2 - r_2| + |r_2 - y_2| \end{aligned}$$

Adding these inequalities together we have:

$$\begin{aligned} |x_1 - y_1| + |x_2 - y_2| &\leq |x_1 - r_1| + |r_1 - y_1| + |x_2 - r_2| + |r_2 - y_2| \\ d(x, y) &\leq |x_1 - r_1| + |x_2 - r_2| + |r_1 - y_1| + |r_2 - y_2| \\ d(x, y) &\leq d(x, r) + d(r, y) \end{aligned}$$

and so we have proven the Triangle Inequality for d .

- (5) For d to be a metric space, $\forall x \in X, d(x, x) = 0$. Consider $x = (x_1, x_2) \in \mathbb{R}^2$ where $x_1 \neq x_2$. Then $d(x, x) = |x_1 - x_2| + |x_2 - x_1| = 2|x_1 - x_2|$. Because $x_1 \neq x_2, x_1 - x_2 \neq 0 \implies 2|x_1 - x_2| \neq 0$ and so it is not guaranteed $\forall x \in X, d(x, x) = 0 \implies d$ is not a metric space.