# Math 226: HW 3
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1. a) The set $S$ is linearly independent if for $a_1, a_2, \ldots a_n \in \mathbb{F}$ and $e_i \in S$, $\Sigma_{i=1}^{N} a_i e_i = 0$ when $a_1, a_2, \ldots a_n$ all equal zero. To solve the equation $\Sigma_{i=1}^{N} a_i e_i = (a_1, a_2, \ldots, a_n) = \mathbf{0}^n$, all elements in the set $\{a_1, a_2, \ldots, a_n\}$ must be equal to zero. Thus we have proved that $a_1, a_2, \ldots a_n$ all equal zero as the only solution to $\Sigma_{i=1}^{N} a_i e_i = 0$ and so $S$ is proven to be linearly independent.

   We define $Span(S) = \{\Sigma_{i=1}^{N} a_i e_i : a_1, a_2, \ldots a_n \in \mathbb{F}\} = \{(a_1, a_2, \ldots a_n) : a_1, a_2, \ldots a_n \in \mathbb{F}\}$ and define $\mathbb{F}^n = \{(a_1, a_2, \ldots a_n) : a_1, a_2, \ldots a_n \in \mathbb{F}\}$. Thus $Span(S) = \mathbb{F}^n$ and so we have proven that $S$ generates $\mathbb{F}^n$.

   c) To prove that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent, we first show (1) $\{u+v, u-v\}$ is linearly independent if $\{u, v\}$ is linearly independent and (2) $\{u, v\}$ is linearly independent if $\{u+v, u-v\}$ is linearly independent.

   1 **Given $\{u, v\}$ is linearly independent, prove $\{u+v, u-v\}$ is linearly independent**

   If $\{u, v\}$ is linearly independent, for $a, b \in \mathbb{F}$, the solution to $au + bv = \mathbf{0}$ is $a = b = 0$. Given $c, d \in \mathbb{F}$, let us now re-express $au + bv = \mathbf{0}$ with $a = c+d, b = c-d$.

$$
\begin{aligned}
au + bv &= \mathbf{0} \\
(c+d)u + (c-d)v &= \mathbf{0} \\
c(u+v) + d(u-v) &= \mathbf{0}
\end{aligned}
$$

   As we are given $\{u, v\}$ is linearly independent, we know that $a = b = 0$ and so given $c + d = c - d = 0$, we know that $c = d = 0$[1]. Thus, $\{u+v, u-v\}$ is proven to be linearly independent as the only solution to $c(u+v) + d(u-v) = \mathbf{0}$ is proven to be $c = d = 0$.

   2 **Given $\{u+v, u-v\}$ is linearly independent, prove $\{u, v\}$ is linearly independent**

   Given $a, b \in \mathbb{F}$, if we know $\{u+v, u-v\}$ is linearly independent, we know that the solution to $a(u+v) + b(u-v) = \mathbf{0}$ is $a = b = 0$. We can also simplify this as:

$$
\begin{aligned}
a(u+v) + b(u-v) &= \mathbf{0} \\
(a+b)u + (a-b)v &= \mathbf{0}
\end{aligned}
$$

   Given that we know $a = b = 0$, if we define $c = a + b = 0 \in \mathbb{F}$ and $d = a - b = 0 \in \mathbb{F}$, we get that

$$cu + dv = 0$$

   If $\{u, v\}$ is linearly independent, the solution for the equation $eu + fv = \mathbf{0}$ is $e = f = 0$ for $e, f \in \mathbb{F}$. From the above equation, we know that $e = c = 0$

[1] Note that we can only conclude this because $\mathbb{F}$ has a characteristic not equal to two. If this was not the case, the condition $c + d = c - d = 0$ can be met if $c = d = 1$.

and $f = d = 0$ and thus $e = f = 0 \Rightarrow \{u, v\}$ is linearly independent. Thus we haven proven that if $\{u+v, u-v\}$ is linearly independent, $\{u, v\}$ is linearly independent.

2. a) To prove that for every $x \in V$, $x \in Span(S)$ iff $Span(S) = Span(S \cup \{x\})$ we must show (1) given $x \in Span(S)$, $Span(S) = Span(S \cup \{x\})$ and (2) given $Span(S) = Span(S \cup \{x\})$, $x \in Span(S)$.

   (1) **Given** $x \in Span(S)$**,** $Span(S) = Span(S \cup \{x\})$
   Let us define $N = |S|$. $Span(S) = \{\Sigma_{i=1}^{N} a_i v_i : a_i \in \mathbb{F}, v_i \in V\}$ and $Span(S \cup \{x\}) = \{cx + \Sigma_{i=1}^{N} b_i v_i : b_i, c \in \mathbb{F}, v_i \in V\}$. If $x \in Span(S)$, $x$ can be expressed as $\Sigma_{i=1}^{N} a_i v_i$ for some set of values $a_i \in \mathbb{F}$. We can re-express our d $Span(S \cup \{x\})$ as $\{\Sigma_{i=1}^{N} ca_i v_i + \Sigma_{i=1}^{N} b_i v_i : b_i, c \in \mathbb{F}, v_i \in V\} = \{\Sigma_{i=1}^{N} (ca_i + b_i) v_i : b_i, c \in \mathbb{F}, v_i \in V\}$. Because $ca_i + b_i \in \mathbb{F}$, $Span(S \cup \{x\}) = \{\Sigma_{i=1}^{N} d_i v_i : d_i \in \mathbb{F}, v_i \in V\} = Span(S)$. Thus, we have proved given $x \in Span(S)$, $Span(S) = Span(S \cup \{x\})$.

   (2) **Given** $Span(S) = Span(S \cup \{x\})$**,** $x \in Span(S)$
   We use the definitions provided from (1). Because we know $x \in S \cup \{x\}$, we know that $x \in Span(S \cup \{x\})$. Since we are given $Span(S) = Span(S \cup \{x\})$, if $x \in Span(S \cup \{x\}) \Rightarrow x \in Span(S)$. Thus, we have proved if $Span(S) = Span(S \cup \{x\})$, $x \in Span(S)$.

   b) We are given that $S$ is linearly independent. To prove that $S \cup \{w\}$ is linearly independent iff $w \notin Span(S)$, we first prove (1) if $w \notin Span(S)$, $S \cup \{w\}$ is linearly independent and then prove (2) if $S \cup \{w\}$ is linearly independent, $w \notin Span(S)$.

   (1) **Given** $w \notin Span(S)$**,** $S \cup \{w\}$ **is linearly independent**
   We define $n = |S|$.
   For proof by contrapositive, let us assume that $S \cup \{w\}$ is linearly dependent. That means that for the solution to the equation $cw + \Sigma_{i=1}^{n} a_i v_i = 0$, where $c, a_1, a_2, \ldots a_n \in \mathbb{F}$, there exists at least one nonzero element of $\{c, a_1, a_2, \ldots a_n\}$. For this solution $c \neq 0$, as if $c = 0$, then the equation would simply be $\Sigma_{i=1}^{n} a_i v_i = 0$. Because $S$ is linearly independent, the only solution to this equation is all $a_i = 0$. However, this would violate the condition that there exists at least one nonzero element in set $\{c, a_1, a_2, \ldots a_n\}$ because we are assuming $S \cup \{w\}$ is linearly dependent. Thus, we know $c \neq 0$ and so we can re-express $w$ as:

$$w = -\frac{\Sigma_{i=1}^{n} a_i v_i}{c} = \Sigma_{i=1}^{n} \frac{-a_i}{c} v_i$$

   This expression of $w$ is an expression of $w$ as a linear combination of $S$. Thus, $w \in Span(S)$ if $S \cup \{w\}$ is linearly dependent. By proof by contrapositive, we haven proven if $w \notin Span(S)$, $S \cup \{w\}$ is linearly independent.

   (2) **Given** $S \cup \{w\}$ **is linearly independent,** $w \notin Span(S)$
   We use the same definition for $n$ provided in (1).
   If $S \cup \{w\}$ is linearly independent, it means that the solution to the equation $cw + \Sigma_{i=1}^{n} a_i v_i = 0$, where $c, a_1, a_2, \ldots a_n \in \mathbb{F}$, is that all elements of the set $\{c, a_1, a_2, \ldots a_n\}$ must be zero. We now try to express $w$ as some linear combination of $S$.

$$cw + \Sigma_{i=1}^{n} a_i v_i = 0$$
$$cw = -\Sigma_{i=1}^{n} a_i v_i$$

   Note that because $S \cup \{w\}$ is linearly independent, $c = 0$. Because we cannot divide the above equation by $c$ on both sides, there does not exist any linear combination of $S$ that is equal to $w$. Because the $Span(S)$ represents all possible linear combinations of elements in $S$, we have proven $w \notin Span(S)$.

c) Given $S = \{u_1, u_2, u_3, \dots u_k\}$, to prove iff Condition $M$ (defined below)

$$M : \{0\} \subsetneq Span(\{u_1\}) \subsetneq Span(\{u_1, u_2\}) \subsetneq Span(\{u_1, u_2, u_3\}) \subsetneq Span(\{u_1, \dots u_k\})$$

then $S$ is linearly independent, we must first prove (1) if condition $M$ holds, then $S$ is linearly independent and (2) if $S$ is linearly independent, then condition $M$ holds.

1 **If condition $M$ holds, then $S$ is linearly independent**
For proof by contrapositive, let us assume $S$ is linearly dependent. Thus, there exists some $m < k \in \mathbb{Z}$ where subset $D = \{u_1, \dots u_m\} \subsetneq S$ is linearly independent and subset $K = \{u_1, \dots u_m, u_{m+1}\} \subseteq S$ is linearly dependent. Let us define $a_1, a_2, \dots a_{m+1} \in \mathbb{F}$ and $u_i \in K$. Because $K$ is linearly dependent, we know that for the solution to the equation $\Sigma_{i=1}^{m+1} a_i u_i = 0$. there exists at least one nonzero element in the set $\{a_1, a_2, \dots a_{m+1}\}$. We can further develop this equation as:

$$a_{m+1}u_{m+1} + \Sigma_{i=1}^{m} a_i u_i = 0$$

Let us consider the case in which $a_{m+1} = 0$. This would leave us with the equation $\Sigma_{i=1}^{m} a_i u_i = 0$. Because $D$ is linearly independent, we know that the only solution to this equation is $\forall a_i \in \{a_1, a_2, \dots a_m\}, a_i = 0$. However, because this solution violates the condition that there exists at least one nonzero element in the set $\{a_1, a_2, \dots a_{m+1}\}$, we know that $a_{m+1} \neq 0$. Thus, $u_{m+1} = \Sigma_{i=1}^{m}(-\frac{a_i}{a_{m+1}})u_i$ and so because we can express $u_{m+1}$ as linear combination of $D$, $u_{m+1} \in Span(D)$.
We now compute $Span(K) = \{cu_{m+1} + \Sigma_{i=1}^{m} b_i u_i : c, b_i \in \mathbb{F}\} = \{c\Sigma_{i=1}^{m}(-\frac{a_i}{a_{m+1}})u_i + \Sigma_{i=1}^{m} b_i u_i : c, b_i \in \mathbb{F}\} = \{\Sigma_{i=1}^{m}(\frac{-ca_i}{a_{m+1}} + b_i)u_i : c, b_i \in \mathbb{F}\}$. Because $\frac{-ca_i}{a_{m+1}} + b_i \in \mathbb{F}$, $Span(K) = \{\Sigma_{i=1}^{m} d_i u_i : d_i \in \mathbb{F}\} = Span(D)$. Because $Span(D) = Span(K)$, condition $M$ does not hold if $S$ is linearly dependent. Thus, we have proven by contrapositive that if condition $M$ holds, then $S$ is linearly independent.

2 **If $S$ is linearly independent, then condition $M$ holds**
We use the definitions of $m, k, D, K$ from 1.
For proof by contrapositive, let us assume that condition $M$ does not hold[2] and so for some $m < k \in \mathbb{R}$, $Span(D) = Span(K)$ Because $u_{m+1} \in Span(K) \Rightarrow u_{m+1} \in Span(D)$, we know that $u_{m+1}$ can be written as $u_{m+1} = \Sigma_{i=1}^{m} a_i u_i$ for $a_i \in \mathbb{F}$, $v_i \in D$.
The set $S$ is linearly independent if given $b_i \in \mathbb{F}$ and $u_i \in S$, the only solution to the equation $\Sigma_{i=1}^{k} b_i u_i = 0$ is $\forall b_i \in \{b_1, b_2, \dots b_k\}, b_i = 0$. We can re-express this equation below as:

$$\Sigma_{i=1}^{k} b_i u_i = 0$$
$$b_{m+1}u_{m+1} + \Sigma_{i=m+2}^{k} b_i u_i + \Sigma_{i=1}^{m} b_i u_i = 0$$

[2] There is only one case in which condition $M$ does not hold. Condition $M$ would not hold if $\exists\ j < k \in \mathbb{Z}$ s.t. $Span(\{u_1, \dots, u_j\}) = Span(\{u_1, \dots, u_j, u_{j+1}\})$ or $Span(\{u_1, \dots u_j\}) \not\subseteq Span(\{u_1, \dots, u_j, u_{j+1}\})$. All elements in $Span(\{u_1, \dots u_j\})$ exist in $Span(\{u_1, \dots u_j, u_{j+1}\})$ as the last element in the set $u_{j+1}$ can always be ignored in a linear combination of $\{u_1, \dots, u_j, u_{j+1}\}$ by setting its coefficient to zero. Thus $Span(\{u_1, \dots, u_j\}) \subseteq Span(\{u_1, \dots, u_j, u_{j+1}\})$ and so condition $M$ can only be violated if $Span(\{u_1, \dots, u_j\}) = Span(\{u_1, \dots, u_j, u_{j+1}\})$.

Let us set $b_i = 0$ for $m+2 \leq i \leq k$, $b_i = -a_i$ for $1 \leq i \leq m$, and $b_{m+1} = 1$. This gives us:

$$u_{m+1} = \Sigma_{i=1}^{m} b_i u_i = \Sigma_{i=1}^{m} a_i u_i$$

As shown before, we know this statement is true. Thus, we have found a solution to the equation $\Sigma_{i=1}^{k} b_i u_i = 0$ with at least one $b_i \neq 0$. This means $S$ is linearly dependent. Thus, if condition $M$ does not hold, we have shown $S$ is linearly dependent. By proof by contrapositive, we have proven that if $S$ is linearly independent, then condition $M$ holds.

3. a) Given $U = \{(x_1, \frac{x_1}{3}, x_3, \frac{x_3}{7}, x_5) \in \mathbb{R}^5; x_1, x_3, x_5 \in \mathbb{F}\}$, a basis of U, $\beta_U$, can be given by:

$$\beta_U = \{(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1)\}$$

i) Extending $\beta_U$ to $\mathbb{R}^5$

$$\beta_{\mathbb{R}^5} = \{(1, \frac{1}{3}, 0, 0, 0), (0, 0, 1, \frac{1}{7}, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0)\}$$

ii) Find a subspace $W$ of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \bigoplus W$

$$W = \{(0, a_1, 0, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$$

We now validate (1) that $U + W = U \bigoplus W$ and (2) $U \bigoplus W = \mathbb{R}^5$.

**1** $U + W = U \bigoplus W$

Let us define $z_1, z_2, z_3, z_4, z_5 \in \mathbb{R}$. Given $x = (z_1, \frac{z_1}{3}, z_3, \frac{z_3}{7}, z_5) \in U$ and $y = (0, z_2, 0, z_4, 0) \in W$, we can find $U \cap W$ as the set of solutions to $x = y$. This would be defined as the solution to the system of equations below.

$$\begin{aligned} z_1 &= 0 \\ \frac{z_1}{3} &= z_2 \\ z_3 &= 0 \\ \frac{z_3}{7} &= z_4 \\ z_5 &= 0 \end{aligned}$$

The solution to this system is $z_1 = z_2 = z_3 = z_4 = z_5 = 0$. Thus $U \cap W = \{\mathbf{0}^5\} \Rightarrow U + W = U \bigoplus W$.

**2** $U \bigoplus W = \mathbb{R}^5$

We use the same definitions of $x, y, z_1, z_2, z_3, z_4, z_5$ from **1**. We compute $x + y = (z_1, z_2 + \frac{z_1}{3}, z_3, z_4 + \frac{z_3}{7}, z_5) \in U \bigoplus W$. Because $z_1, z_2 + \frac{z_1}{3}, z_3, z_4 + \frac{z_3}{7}, z_5 \in \mathbb{R}$, $x + y \in \mathbb{R}^5$ and so $U \bigoplus W \subseteq \mathbb{R}^5$. Because $\mathbf{0}^5 \in U \bigoplus W$ and $U \bigoplus W$ can be trivially shown to be closed under addition and scalar multiplication, $U \bigoplus W \leq \mathbb{R}^5$.

We now prove that $U \bigoplus W = \mathbb{R}^5$ by showing that $dim(U \bigoplus W) = dim(\mathbb{R}^5) = 5$.

The basis of $U$, $\beta_U$, is given above and we give the basis of W as $\beta_W = \{(0,1,0,0,0),(0,0,0,1,0)\}$. We now compute $Span(\beta_U \cup \beta_W)$ as given $v_i \in \beta_U \cup \beta_W$ and $a_i \in \mathbb{R}, Span(\beta_U \cup \beta_W) = \Sigma_{i=1}^{3} a_i v_i + \Sigma_{i=5}^{4} a_i v_i$. Because all vectors in $U$ can be expressed as $\Sigma_{i=1}^{3} a_i v_i$ (a linear combination of $\beta_U$) and all vectors in $W$ can be expressed as $\Sigma_{i=5}^{4} a_i v_i$ (a linear combination of $\beta_W$), $Span(\beta_U \cup \beta_W) = \{u + w : u \in U, w \in W\} = U \bigoplus W \Rightarrow \beta_U \cup \beta_W$ is the basis for $U \bigoplus W$. Thus $dim(U \bigoplus W) = |\beta_U \cup \beta_W| = 5 = dim(\mathbb{R}^5)$. Thus, we have proved $U \bigoplus W = \mathbb{R}^5$.

b) Given $U = \{f(x) = c_1 + (-2c_3 - 3c_4 - 4c_5)x + c_3x^2 + c_4x^3 + c_5x^4 \in P_4(\mathbb{R}) : c_1, c_3, c_4, c_5 \in \mathbb{R}\}$, a basis of $U$ can be given by:

$$\beta_U = \{1, -2x + x^2, -3x + x^3, -4x + x^4\}$$

i) Extending $\beta_U$ to $P_4(\mathbb{R})$

$$\beta_{P_4(\mathbb{R})} = \{1, x, -2x + x^2, -3x + x^3, -4x + x^4\}$$

ii) Find a subspace $W$ of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R}) = U \bigoplus W$

$$W = \{f(x) = a_1 x \in P_4(\mathbb{R}) : a_1 \in \mathbb{R}\}$$

We now validate (1) that $U + W = U \bigoplus W$ and (2) $U \bigoplus W = P_4(\mathbb{R})$.

1 $U + W = U \bigoplus W$

Let us define $z_1, z_2, z_3, z_4, z_5 \in \mathbb{F}$. Given $u = z_1 + (-2z_3 - 3z_4 - 4z_5)x + z_3x^2 + z_4x^3 + z_5x^4 \in U$ and $w = z_2 x \in W$, we find $U \cap W$ as the set of solutions to $u = w$. This would be defined as the solution to the system of equations below.

$$\begin{aligned} z_1 &= 0 \\ -2z_3 - 3z_4 - 4z_5 &= z_2 \\ z_3 &= 0 \\ z_4 &= 0 \\ z_5 &= 0 \end{aligned}$$

The solution to this system is $z_1 = z_2 = z_3 = z_4 = z_5 = 0$. Thus since $U \cap W = \{f(x) = 0 \in P_4(\mathbb{R})\} \Rightarrow U + W = U \bigoplus W$.

2 $U \bigoplus W = P_4(\mathbb{R})$

We use the same definitions of $u, w, z_1, z_2, z_3, z_4, z_5$ from 1. We compute $u + w = z_1 + (z_2 - 2z_3 - 3z_4 - 4z_5)x + z_3x^2 + z_4x^3 + z_5x^4 \in U \bigoplus W$. Because $z_1, z_2 - 2z_3 - 3z_4 - 4z_5, z_3, z_4, z_5 \in \mathbb{R}$, $u + w \in P_4(\mathbb{R})$ and so $U \bigoplus W \subseteq P_4(\mathbb{R})$. Because $0 \in U \bigoplus W$ and $U \bigoplus W$ can be trivially shown to be closed under addition and scalar multiplication, $U \bigoplus W \leq P_4(\mathbb{R})$. We now prove that $U \bigoplus W = P_4(\mathbb{R})$ by showing that $dim(U \bigoplus W) = dim(P_4(\mathbb{R}) = 5$.

The basis of $U$, $\beta_U$, is given above and we give the basis of W as $\beta_W = \{x\}$. We now compute $Span(\beta_U \cup \beta_W)$ as given $f_i(x) \in \beta_U \cup \beta_W$ and $a_i \in \mathbb{R}, Span(\beta_U \cup \beta_W) = \Sigma_{i=1}^{3} a_i f_i(x) + \Sigma_{i=5}^{4} a_i f_i(x)$. Because all functions in $U$ can be expressed as $\Sigma_{i=1}^{3} a_i f_i(x)$ (a linear combination of $\beta_U$) and all functions in $W$ can be expressed as $\Sigma_{i=5}^{4} a_i f_i(x)$ (a linear combination of

$\beta_W$), $Span(\beta_U \cup \beta_W) = \{u + w : u \in U, w \in W\} = U \bigoplus W \Rightarrow \beta_U \cup \beta_W$ is the basis for $U \bigoplus W$. Thus $dim(U \bigoplus W) = |\beta_U \cup \beta_W| = 5 = dim(P_4(\mathbb{R}))$. Thus, we have proved $U \bigoplus W = P_4(\mathbb{R})$.

4. Let us define a matrix $E_{ij}$ as a 3x3 matrix with all zeros except in the $i$th row and $j$th column, where there is a one.

   The basis for $M_{3x3}(\mathbb{R}) = \{\bigcup_{\substack{i,j=1 \\ i \neq j}}^{3} E_{ij}\} \cup \{E_{00} - E_{33}, E_{11} - E_{33}\}$.