MATH 255 HW 2

January 30, 2025

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1. **Exercise 2.1 (5 points; Rudin 1.5)**

   We first show that $\inf(A) \geq -\sup(-A)$. By the definition of supremum, $\forall a \in -A, \sup(-A) \geq a \implies \forall a \in A, \sup(-A) \geq -a \implies \forall a \in A, -\sup(-A) \leq a \implies -\sup(-A)$ is a lower bound for A. By definition of infimum, $\inf(A) \geq -\sup(-A)$.

   We now show that $\inf(A) \leq -\sup(-A)$. By the definition of infimum, $\forall a \in A, \inf(A) \leq a \implies \forall a \in -A, \inf(A) \leq -a \implies \forall a \in -A, -\inf(A) \geq a \implies -\inf(A)$ is an upper bound for $-A$. By definition of supremum, $\sup(-A) \leq -\inf(A) \implies \inf(A) \leq -\sup(-A)$.

   Thus because $\inf(A) \geq -\sup(-A)$ and $\inf(A) \leq -\sup(-A)$, we have proven $\inf(A) = -\sup(A)$.

2. **Exercise 2.2 (5 points)**

   We first show that $\inf(A^{-1}) \geq (\sup(A))^{-1}$. By the definition of supremum, $\forall a \in A, \sup(A) \geq a \implies \forall a \in A^{-1}, \sup(A) \geq a^{-1} \implies \forall a \in A^{-1}, \frac{1}{\sup(A)} = (\sup(A))^{-1} \leq a \implies (\sup(A))^{-1}$ is a lower bound for A^{-1}. By definition of infimum, $\inf(A^{-1}) \geq (\sup(A))^{-1}$.

   We now show that $\inf(A^{-1}) \leq (\sup(A))^{-1}$. By the definition of infimum, $\forall a \in A^{-1}, \inf(A^{-1}) \leq a \implies \forall a \in A, \inf(A^{-1}) \leq a^{-1} \implies \forall a \in A, (\inf(A^{-1}))^{-1} \geq a \implies (\inf(A^{-1}))^{-1}$ is an upper bound for A. By definition of supremum, $(\inf(A^{-1}))^{-1} \geq \sup(A) \implies \inf(A^{-1}) \leq (\sup(A))^{-1}$.

   Because we have shown $\inf(A^{-1}) \geq (\sup(A))^{-1}$ and $\inf(A^{-1}) \leq (\sup(A))^{-1}$, we have proven $\inf(A^{-1}) = (\sup(A))^{-1}$.

3. **Exercise 2.3 (5 points)**

   Lemma 0.1 *Note that by definition of supremum,* $\forall a \in A, a \leq sup(A)$ *and* $\forall\ b \in B, b \leq sup(B)$. *Thus,* $\forall a \in A$ *and* $b \in B, a + b \leq sup(A) + b \leq sup(A) + sup(B) \implies \forall a \in A$ *and* $b \in B, a + b \leq sup(A) + sup(B)$.

   To prove $\sup(A + B) = \sup(A) + \sup(B)$ we prove $\sup(A + B) \leq \sup(A) + \sup(B)$ and $\sup(A) + \sup(B) \leq \sup(A + B)$. We prove both directions of this statement below.

1. $\sup(A) + \sup(B) \leq \sup(A + B)$

   By **Lemma 0.1**, $\forall a \in A$ and $b \in B, a+b \leq \sup(A+B) \implies \forall a \in A$ and $b \in B, a \leq \sup(A+B) - b$. Hence, for a given $b \in B$, $\sup(A+B) - b$ is an upper bound for A. Because $\sup(A)$ is a supremum, for a given $b \in B$, $\sup(A) \leq \sup(A+B) - b \implies \forall\, b \in B, b \leq \sup(A+B) - \sup(A) \implies \sup(A+B) - \sup(A)$ is an upper bound for B. Because $\sup(B)$ is the lowest upper bound for B, $\sup(B) \leq \sup(A+B) - \sup(A) \implies \sup(A) + \sup(B) \leq \sup(A+B)$.

2. $\sup(A) + \sup(B) \geq \sup(A + B)$

   By **Lemma 0.1**, $\sup(A) + \sup(B)$ is an upper bound for $A+B$. Because $\sup(A+B)$ is the lowest upper bound of $A+B$, $\sup(A) + \sup(B) \geq \sup(A+B)$.

4. **Exercise 2.4 (15 points)**

Note that if a set S has a defined maximum element, then its supremum is its maximum element.

(1) $A = \{2, 3\}$ is clearly bounded above as it has an upper bound (e.g. 4) $\in \mathbb{Z}$. A also has a maximum element, 3, as $2 \leq 3$ and $3 \leq 3 \implies \forall x \in A, x \leq 3$. Because A's maximum element is defined as 3, its supremum is also 3.

(2) The set A is given as $A = \{-\frac{2}{5}, -\frac{4}{5}, -\frac{6}{5}, \dots\}$. A is bounded above as $\exists$ an upper bound (e.g. 0) that exists in $\mathbb{Q}$. The maximum element of A is $-\frac{2}{5}$. Because the maximum element of A is defined, this maximum element also is the supremum of A.

(3) The set A is given as $A = \{-\frac{1}{1}, -\frac{1}{2}, -\frac{1}{3}, \dots\}$. A is bounded above as $\exists$ an upper bound $\in \mathbb{Q}$ (e.g. $0 \in \mathbb{Q}$). We discuss maximum element & supremum below:

   (a) **A has no maximum element**

   We prove this by statement by contradiction. Let us define $m = -\frac{1}{n} \in A$, where $n \in \mathbb{N}$, to be the maximum element in A. Because $n \in \mathbb{N}$, then $n+1 > n \in \mathbb{N}$ and so we have[1]:

$$
\begin{gathered}
n+1 > n \\
(n+1)^{-1} \cdot (n+1) = 1 > (n+1)^{-1} \cdot n \\
n^{-1} \cdot 1 > (n+1)^{-1} \cdot n^{-1} \cdot n \\
n^{-1} > (n+1)^{-1} \\
\frac{1}{n} > \frac{1}{n+1} \\
-\frac{1}{n+1} > -\frac{1}{n} = m
\end{gathered}
$$

   and so because $\exists\ -\frac{1}{n+1} \in A$ where $-\frac{1}{n+1} > m \implies m$ is not the maximum element of $A \implies A$ has no maximum element.

[1] Because $n, n+1 \in \mathbb{Q}$, $n \neq 0$ and $n+1 \neq 0$ and so they both have well-defined inverses.

(b) **The supremum of A is zero.**
We first prove that zero is an upper bound of A. Because $\forall a \in A, a < 0 \implies 0$ is an upper bound of A. We now prove by contradiction that there does not exist any upper bound for A lower than 0 (i.e. the supremum of A is zero). Let us define a supremum of A, as $s < 0$ where $s \in \mathbb{Q}$. Because $s \in \mathbb{Q}$, we can define $s = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Thus, we can re-express $s = \frac{1}{\frac{q}{p}}$ and because $\exists\, n \in \mathbb{N}$ s.t. $n > |\frac{q}{p}| \implies \exists\, a = \frac{1}{n} \in A$ s.t. $a > s \implies s$ is not an upper bound $\implies s$ is not a supremum. Thus, by proof by contradiction we have proven that there does not exist any upper bound for A lower than $0 \implies$ the supremum of A is zero.

(4) The set A is given by $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. A is bounded above as $\exists$ an upper bound $\in \mathbb{Q}$ (e.g. 2). A also has a maximum element, 1, which also serves as its supremum.

(5) The set A is bounded above as $\exists$ an upper bound $\in \mathbb{Q}$ (e.g. 2). The set A also has a maximum element, one, which also serves as its supremum.

(6) The set A is bounded above as $\exists$ an upper bound $\in \mathbb{Q}$ (e.g. 1). We now discuss maximum element & supremum:

(a) **A has no maximum element**
We prove this by contradiction; consider $m \in A$ to be the maximum element of A. Because $m \in A \implies m < 1$.We can construct a number $m' = m + \frac{1-m}{2}$. Note that because $m < 1$, $\frac{1-m}{2} > 0$ and so $m' > m$. Furthermore, $m' < m + (1 - m) \implies m' < 1 \implies m' \in A$. Thus, we have that $\forall a \in A, m' > m \geq a \implies \forall a \in A, m' > a \implies m'$ is a maximum element for $A \implies A$ has no maximum element.

(b) **The supremum of A is one.**
The supremum of A is its lowest upper bound of A. One is an upper bound for A as $\forall a \in A, a < 1$. We now prove by contradiction that there does not exist any upper bound for A lower than 1 (i.e. the supremum of A is one). Let us define a supremum $s < 1$ where $s \in \mathbb{Q}$. Because s is a supremum, $\forall a \in A, s > a > 0 \implies s > 0$. Therefore, because $s \in \mathbb{Q}$ and $0 < s < 1$, $s \in A$. As we have proven above, A does not have a maximum element and so s is not an upper bound for $A \implies s$ is not a supremum $\implies \nexists$ an upper bound for A less than one.

(7) A is bounded above as $\exists$ an upper bound for $A \in \mathbb{Q}$ (e.g. 2). We now discuss maximum element & supremum:

(a) **A has no maximum element**
We prove this by contradiction; consider $m \in A$ to be the maximum element of A. Note that because $m \in A \implies m^3 < 2$. We now try to find a new maximum element of A. Let us define a tiny rational $\epsilon > 0$. Our new maximum element can be given as $m' = m + \epsilon > m$. We now solve for ϵ such that $(m')^3 < 2$ (so $m' \in A)^2$:

[2]Note that all the numbers involved below are rationals $\implies$ the solution of ϵ is also a rational $\implies m' = m + \epsilon \in \mathbb{Q}$.

$$(m')^3 < 2$$
$$(m+\epsilon)^3 < 2$$
$$m^3 + 3m^2\epsilon + 3m\epsilon^2 + \epsilon^3 < 2$$
$$3m^2\epsilon + 3m\epsilon^2 + \epsilon^3 < 2 - m^3$$

Let us add another constraint that $\epsilon < 1$ to help remove the ϵ^2 and ϵ^3 terms:

$$3m^2\epsilon < 1 - m^3 - 3m$$
$$\epsilon < \frac{1 - m^3 - 3m}{3m^2}$$

Thus, a solution for ϵ can be given by:

$$0 < \epsilon < \min(\frac{1 - m^3 - 3m}{3m^2}, 1)$$

which means that $\exists\ m' > m$ s.t. $m \in A \implies A$ has no maximum element.

(a) A **has no supremum**

Lemma 0.2 *We first prove that if given $y \in \mathbb{Q}$ where $y > 0$ and $y^3 > 2$, then y is an upper bound of A. Consider any $x \in A$. We have $y^3 > 2 > x^3 \implies y^3 > x^3$. If $x > y$ then $x^3 > x^2y$ and $xy^2 > y^3 \implies x^3 > xy^2 > y^3 \implies x^3 > y^3$, which contradicts $y^3 > x^3$. Thus $\forall x \in A, x \leq y \implies y$ is an upper bound of A.*

We now prove by contradiction that A has no supremum. Let us suppose $s \in \mathbb{Q}$ is a supremum of A. Let us define a tiny rational $\epsilon > 0$. We now find a new supremum of A, $s' = s - \epsilon < s$. Applying **Lemma 0.2**, if $s' > 0$ and $s^3 \geq 2$, s' is an upper bound for A. Thus, we solve for ϵ s.t. $(s')^3 > 2$ and $s' > 0$ (i.e. $e < s$):

$$(s')^3 > 2$$
$$(s-\epsilon)^3 > 2$$
$$s^3 - 3s^2\epsilon + 3s\epsilon^2 - \epsilon^3 > 2$$

We add another constraint that $\epsilon < 1$ constraint to help handle e^2 and e^3 terms:

$$s^3 - 3s^2\epsilon + 3s - 1 > 2$$
$$-3s^2\epsilon < 3 - s^3$$
$$\epsilon < \frac{s^3 - 3}{3s^2}$$

Thus, a solution for ϵ can be given by:

$$0 < \epsilon < \min(\frac{s^3 - 3}{3s^2}, s)$$

which means that $s' < s$ is an upper bound for $A \implies A$ has no supremum.

5. **Exercise 2.5 (10 points)**

(1) For proof by contradiction, let us assume $0 \geq 1$. We find contradictions for the $0 = 1$ and $0 > 1$ cases below:

1. $0 = 1$
   Pick $a \in F$. Then $a \cdot 1 = a \cdot 0 = 0 \neq a$, which is a contradiction to the multiplicative identity field axiom.
2. $0 > 1$
   If $0 > 1$, then this means that:

$$\begin{aligned} 1 + (-1) &> 1 + 0 \\ -1 &> 0 \end{aligned}$$

Because $-1 > 0 \implies (-1)^2 > 0 \cdot 1 \implies 1 > 0$, so we have a contradiction. Thus we have proved $0 < 1$.

(2) We prove this with contradiction, and thus assume $x^{-1} \leq 0$.
If $x^{-1} = 0$, then $x \cdot x^{-1} = x \cdot 0 = 0 \neq 1$, and thus this is a contradiction with the definition of inverses in fields.
If $x^{-1} < 0$, then we can multiply both sides of the inequality $x^{-1} < 0$ with x and so the inequality sign will not change because $x > 0$. Thus:

$$\begin{aligned} x^{-1} &< 0 \\ x \cdot x^{-1} &< x \cdot 0 \\ 1 &< 0 \end{aligned}$$

which is a contradiction to our proof in (1).

(3) We prove both directions of this statement.

1 $xy > xz \implies y > z$
We are given $x > 0$, and so from part (2) we know that $x^{-1} > 0$. Therefore:

$$\begin{aligned} xy &> xz \\ x^{-1} \cdot xy &> x^{-1} \cdot xz \\ (x^{-1} \cdot x)y &> (x^{-1} \cdot x)z \\ y &> z \end{aligned}$$

2 $y > z \implies xy > xz$

$$y > z \tag{1}$$
$$y - z > 0 \tag{2}$$
$$\tag{3}$$

Because $y - z > 0$ and $x > 0$, $x(y-z) > 0$ so:

$$x(y-z) > 0$$
$$xy - xz > 0$$
$$xy > xz$$

(4) Because $0 < 1$, there are three possible orderings. We show that all three of these orderings violate the ordered field axioms:

1. **Ordering #1:** $0 < 1 < 2$
   Under this ordering, we have $1 + 2 = 0 < 2$ and thus:

   $$1 + 2 < 1 + 1$$
   $$1 + 2 < 1 + 1$$
   $$2 < 1$$

   which is a contradiction.
2. **Ordering #2:** $0 < 2 < 1$
   Under this ordering, we have $1 + 2 = 0 < 2$

   $$1 + 2 < 0 + 2$$
   $$1 < 0$$

   which is a contradiction with our proof in (1).
3. **Ordering #3:** $2 < 0 < 1$
   Under this ordering, we have that $2 < 1$ and so:

   $$2 < 1$$
   $$2 + 1 < 1 + 1$$
   $$0 < 2$$

   which is a contradiction.