Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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Problem set 5
Due Friday February 20 at 11pm

Exercise 5.1 (5 points). Show that the subset of $\mathbb{R}^2$ given by

$$E = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$$

is open.

Exercise 5.2 (5 points). Show that the union of a finite number of compact sets is compact.

Exercise 5.3 (5 points; Rudin 2.14). Prove directly that the interval $(0, 1) \subset \mathbb{R}$ is not compact, by giving an example of an open cover of $(0, 1)$ which has no finite subcover. Include a proof that your cover has no finite subcover.

Exercise 5.4 (20 points; Rudin 2.19). If A and B are subsets of a metric space X, we say A and B are *separated* if $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$. (We used this notion when we defined connectedness.)

(1) If A and B are disjoint closed sets in some metric space X, prove that A and B are separated.

(2) Prove the same for disjoint open sets.

(3) Fix $p \in X$ and $\delta > 0$. Define $A = \{q \in X \mid d(p, q) < \delta\}$. Define $B = \{q \in X \mid d(p, q) > \delta\}$. Prove that A and B are separated.

(4) Prove that every connected metric space with at least two points is uncountable. (Hint: use the previous part.)

Exercise 5.5 (5 points; Rudin 2.22, modified). Given a metric space X and a set $E \subset X$, we say E is *dense* in X if $\overline{E} = X$. Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 5.6 (not for credit; Rudin 2.8). Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.