STATS 242 HW 5

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1.

(a) Under H_0, the median of f is zero $\implies$ each sample X_i has a 50% chance of being above zero. Let us define r.v. $I_i \sim \text{Bern}(0.5)$ to represent if $X_i \geq 0$. Because $S = \Sigma_{i=1}^n I_i, S \sim \text{Bin}(n, 0.5)$. Furthermore, with a large n, the CLT enables us to approximate Binomial distributions with normal distributions[1] and thus $S \sim \mathcal{N}(n\mathbb{E}[I_i], n\text{Var}[I_i])$ or $S \sim \mathcal{N}(0.5n, 0.25n)$. This means that the distribution of $S - \frac{n}{2} \sim \mathcal{N}(0, 0.25n)$ and the distribution of $T = \sqrt{\frac{4}{n}}(S - \frac{n}{2}) \sim \mathcal{N}(0,1)$. Thus $T \sim \mathcal{N}(0,1)$.

To test H_0 vs. H_1 at the significance level α, we would compute T for a sample of data and if it is above the upper-α point of $\mathcal{N}(0,1)$, we will reject H_0.

(b) Note that because under H_1', $X_i \sim \mathcal{N}(\frac{h}{\sqrt{n}}, 1)$ this means that $X_i - \frac{h}{\sqrt{n}} \sim \mathcal{N}(0,1)$.

$$\mathbb{P}_{H_1'}[X_i > 0] = \mathbb{P}_{H_1'}[X_i - \frac{h}{\sqrt{n}} > -\frac{h}{\sqrt{n}}] = 1 - \Phi(-\frac{h}{\sqrt{n}}) = \Phi(\frac{h}{\sqrt{n}})$$

Assuming that h is a small fixed value and n is large, then $\frac{h}{\sqrt{n}}$ is close to zero. This means we can approximate $\mathbb{P}_{H_1'}[X_i > 0] = \Phi(\frac{h}{\sqrt{n}})$ at zero with the following first-degree Taylor Series approximation:

$$\mathbb{P}_{H_1'}[X_i > 0] = \Phi(\frac{h}{\sqrt{n}})$$
$$\approx \Phi(0) + \Phi'(\frac{h}{\sqrt{n}}) \cdot \frac{h}{\sqrt{n}} = \frac{1}{2} + \phi(\frac{h}{\sqrt{n}}) \cdot \frac{h}{\sqrt{n}} = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{h^2}{2n}} \cdot \frac{h}{\sqrt{n}}$$

where ϕ is the standard normal PDF. Note that because $h \ll n$, $\frac{h^2}{2n} \approx 0$ and so:

$$\mathbb{P}_{H_1'}[X_i > 0] = \Phi(\frac{h}{\sqrt{n}}) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^0 \cdot \frac{h}{\sqrt{n}}$$
$$\mathbb{P}_{H_1'}[X_i > 0] \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \cdot \frac{h}{\sqrt{n}}$$

[1] This is because the Binomial Distribution is given by $n(\bar{I})$ where $\bar{I}$ is the mean of n i.i.d Bernoulli Variables.

(c) Under H_1', the previously defined r.v. $I_i \sim \text{Bern}(\mathbb{P}_{H_1'}[X_i > 0])$) or $I_i \sim \text{Bern}(\Phi(\frac{h}{\sqrt{n}}))$. As stated in part (a), r.v. S is given by a Binomial distribution but for large n can be approximated by $\mathcal{N}(n\mathbb{E}[I_i], n\text{Var}[I_i])$. As such, we first compute $n\mathbb{E}[I_i]$:

$$n\mathbb{E}[I_i] = n\mathbb{P}_{H_1'}[X_i > 0] = n\Phi(\frac{h}{\sqrt{n}}) \approx n(\frac{1}{2} + \frac{h}{\sqrt{2\pi n}}) = \frac{n}{2} + \frac{h\sqrt{n}}{\sqrt{2\pi}}$$
$$n\mathbb{E}[I_i] \approx \frac{n}{2} + \frac{h\sqrt{n}}{\sqrt{2\pi}}$$

and then $n\text{Var}(I_i)$:

$$n\text{Var}(I_i) = n(\mathbb{P}_{H_1'}[X_i > 0])(1 - \mathbb{P}_{H_1'}[X_i > 0]) = n(\Phi(\frac{h}{\sqrt{n}}))(1 - \Phi(\frac{h}{\sqrt{n}}))$$
$$\approx n(\frac{1}{2} + \frac{h}{\sqrt{2\pi n}})(1 - (\frac{1}{2} + \frac{h}{\sqrt{2\pi n}}) = n(\frac{1}{2} + \frac{h}{\sqrt{2\pi n}})(\frac{1}{2} - \frac{h}{\sqrt{2\pi n}}) = n(\frac{1}{4} - \frac{h^2}{2\pi n}) = \frac{n}{4} - \frac{h^2}{2\pi}$$
$$n\text{Var}(I_i) \approx \frac{n}{4} - \frac{h^2}{2\pi}$$

So, for large n we have that S can be approximated by $\mathcal{N}(\frac{n}{2} + \frac{h\sqrt{n}}{\sqrt{2\pi}}, \frac{n}{4} - \frac{h^2}{2\pi})$. This means that the distribution $S - \frac{n}{2}$ is given by approximately $\mathcal{N}(\frac{h\sqrt{n}}{\sqrt{2\pi}}, \frac{n}{4} - \frac{h^2}{2\pi})$ and so the distribution of $T = \sqrt{\frac{4}{n}}(S - \frac{n}{2})$ is approximately $\mathcal{N}(\sqrt{\frac{4}{n}} \cdot \frac{h\sqrt{n}}{\sqrt{2\pi}}, \frac{4}{n}(\frac{n}{4} - \frac{h^2}{2\pi}))$ or $\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1 - \frac{2h^2}{\pi n})$. Note that under our assumption $h \ll n$, we can drop the $\frac{2h^2}{\pi n}$ term in the variance. Thus we can give the following normal approximation for T that only relies on h and not n: $\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1)$.

(d) As stated in part (a), we would reject H_0 if the computed test statistic T is above the upper-α point of $\mathcal{N}(0,1)$ (given by $z^{(\alpha)}$). The power of a test is given by $\mathbb{P}_{H_1'}[\text{reject } H_0] = \mathbb{P}_{H_1'}[T > z^{(a)}]$. As shown in part (c), under H_1', T can be approximated by $\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1)$. Thus, we can compute the power:

$$\mathbb{P}_{H_1'}[\text{reject } H_0] = \mathbb{P}_{H_1'}[T > z^{(a)}] \approx \mathbb{P}[\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1) > z^{(a)}]$$
$$= \mathbb{P}[\mathcal{N}(\sqrt{\frac{2}{\pi}}h, 1) - \sqrt{\frac{2}{\pi}}h > z^{(a)} - \sqrt{\frac{2}{\pi}}h] = \mathbb{P}[\mathcal{N}(0,1) > z^{(a)} - \sqrt{\frac{2}{\pi}}h]$$
$$= 1 - \mathbb{P}[\mathcal{N}(0,1) \leq z^{(a)} - \sqrt{\frac{2}{\pi}}h] = 1 - \Phi(z^{(a)} - \sqrt{\frac{2}{\pi}}h) = \Phi(\sqrt{\frac{2}{\pi}}h - z^{(a)})$$

and so the power of this test against alternative H_1' is approximately $\Phi(\sqrt{\frac{2}{\pi}}h - z^{(a)})$.

2.

(a) The simulated probability of a Type I Error for the Sign Test was 0.04. The simulated probability of a Type I Error for the t-test was 0.0522. We report the simulated power against each alternative for both tests in the table below:

	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$
Sign test	0.1754	0.4509	0.7529	0.9287
t-test	0.1629	0.5011	0.8408	0.9771

(b) The power of the one-sample z-test is given by $\Phi(\sqrt{n}\mu - z^{(\alpha)})$. Comparing that to the simulated power of the one-sample t-test, we find that the z-test, across all alternatives, has a consistently greater power.

In problem 1(d), we found that the power of the sign test was approximately $\Phi(\sqrt{\frac{2}{\pi}} \cdot \sqrt{n}\mu - z^{(\alpha)})$. Comparing that to the simulated power of the sign test, we find that this approximation, across alternatives, yields a consistently greater power. Furthermore, our simulated power of the sign test is generally lower than our simulated power of the t-test and considerably lower than our power derivation for a one-sample z-test. We report the analytically computed powers for the z-test and sign test below:

	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$
Sign test: $\approx \Phi(\sqrt{\frac{2}{\pi}} \cdot \sqrt{n}\mu - z^{(\alpha)})$	0.1985	0.4804	0.7730	0.9390
z-test: $\Phi(\sqrt{n}\mu - z^{(\alpha)})$	0.2595	0.6388	0.9123	0.9907

```python
# %%
import numpy as np
import math
from scipy import stats

NUM_SAMPLES = 10000
SIG_LEVEL = 0.05

upper_alpha = stats.norm.ppf(1 - SIG_LEVEL, loc=0, scale=1)

def get_N_normal_observations(mu, N=100):
    return np.random.normal(mu, 1, N)

def reject_ttest(samples):
    t_stat, p_value = stats.ttest_1samp(samples, popmean=0)
    return 1 if p_value <= SIG_LEVEL else 0

def compute_sign_statistic(samples):
    n = len(samples)
    return math.sqrt(4 / n) * (np.sum(samples > 0) - 0.5 * n)

def reject_sign_test(samples):
```

```python
    sign_statistic = compute_sign_statistic(samples)
    return 1 if sign_statistic >= upper_alpha else 0
# %%
MEANS = [0, 0.1, 0.2, 0.3, 0.4]
REJ_TTEST_COUNTS = [0] * len(MEANS)
REJ_SGN_COUNTS = [0] * len(MEANS)

for m_i, mean in enumerate(MEANS):
    for _ in range(NUM_SAMPLES):
        samples = get_N_normal_observations(mean)
        rej_ttest = reject_ttest(samples)
        rej_sign_test = reject_sign_test(samples)
        REJ_TTEST_COUNTS[m_i] += rej_ttest
        REJ_SGN_COUNTS[m_i] += rej_sign_test


# %%
"""Get the simulated Type I Error."""
print(f"Type I Error for Sign Statistic: {REJ_SGN_COUNTS[0] / NUM_SAMPLES}")
print(f"Type I Error for T-Test Statistic: {REJ_TTEST_COUNTS[0] / NUM_SAMPLES}")# %%
"""Get the Power Against Each Alternative"""
for i, mean in enumerate(MEANS):
    if mean == 0: continue
    print(f"Power for sign test @ {mean} mean: {REJ_SGN_COUNTS[i] / NUM_SAMPLES}")

for i, mean in enumerate(MEANS):
    if mean == 0: continue
    print(f"Power for t-test @ {mean} mean: {REJ_TTEST_COUNTS[i] / NUM_SAMPLES}")

"""Compare to z-test"""

for i, mean in enumerate(MEANS):
    if mean == 0: continue
    print(f"Estimated power for z-test @ {mean} mean: {stats.norm.cdf(math.sqrt(100) * mean - upper_alpha)}")
# %%

"""Compare to sign test"""
for i, mean in enumerate(MEANS):
    if mean == 0: continue
    print(f"Estimated power for sign-test  @ {mean} mean: {stats.norm.cdf(math.sqrt(2 / math.pi) * math.sqrt(100) * mean - upper_alpha)}")
```

3.

(a) We are given that the FWER is controlled at level $\alpha \implies \mathbb{P}[\text{reject any true } H_0] \leq \alpha$. Let us define V and R as the number of true null hypotheses rejected and the number of total null hypotheses rejected, respectively. The FDR is controlled at level α if $\mathbb{E}[\frac{V}{R}] \leq \alpha$. Using LOTE, we can write the FDR as the following:

$$\mathbb{E}[\frac{V}{R}] = \mathbb{E}[\frac{V}{R}|\frac{V}{R} = 0]P(\frac{V}{R} = 0) + \mathbb{E}[\frac{V}{R}|\frac{V}{R} \neq 0]P(\frac{V}{R} \neq 0)$$
$$\mathbb{E}[\frac{V}{R}] = (0)P(\frac{V}{R} = 0) + \mathbb{E}[\frac{V}{R}|\frac{V}{R} \neq 0]P(\frac{V}{R} \neq 0) = \mathbb{E}[\frac{V}{R}|\frac{V}{R} \neq 0]P(\frac{V}{R} \neq 0)$$

We first compute $P(\frac{V}{R} \neq 0) = P(V \neq 0) = \mathbb{P}[\text{reject any true } H_0]$. Note that this last probability is guaranteed to be $\leq \alpha$ by the FWER and so $P(\frac{V}{R} \neq 0) \leq \alpha$. Given this, we can create the following inequality for the FDR:

$$\mathbb{E}[\frac{V}{R}] \leq \mathbb{E}[\frac{V}{R}|\frac{V}{R} \neq 0]\alpha$$

This question asks us to consider if the FDR is necessarily controlled at level α, given the FWER is. Note that V is strictly less than R, and so $\frac{V}{R} \leq 1 \implies \mathbb{E}[\frac{V}{R}|\frac{V}{R} \neq 0] \leq 1 \implies \text{FDR} = \mathbb{E}[\frac{V}{R} \neq 0]\alpha \leq \alpha \implies$ the FDR is controlled at level α.

(b) The Bonferroni method applied to control FWER $\leq \alpha$ will reject any null hypotheses that has a p-value $\leq \frac{\alpha}{n}$. The BH procedure will reject hypotheses where their p-value is less than a multiple (i.e. their rank $r \in \mathbb{N}$) of $\frac{\alpha}{n}$: the BH procedure rejects hypotheses with p-values $\leq \frac{\alpha r}{n}$. Let us say that hypothesis H_k with p-value P_k. Let us also suppose that this hypothesis has a rank r_k when compared to all other hypotheses' p-values in this multiple hypotheses testing experiment. If H_k was rejected by the Bonferroni method $\implies P_k \leq \frac{\alpha}{n} \leq \frac{\alpha r_k}{n} \implies P_k \leq \frac{\alpha r_k}{n} \implies P_k$ will be rejected by the BH procedure. Thus, all hypotheses rejected by the Bonferroni method will be rejected by the BH procedure.

4.

(a) We compute this below:

$$\begin{aligned}\mathbb{P}[\text{reject any true null hypothesis}] &= 1 - \mathbb{P}[\text{reject no true null hypotheses}] \\ &= 1 - \prod_{i=1}^{n_0} \mathbb{P}[\text{accept this null hypothesis}]\end{aligned}$$

The probability of rejecting any true null hypothesis is given by the probability $P_i \leq t$. Because this null hypothesis is true, $P_i \sim \text{Unif}(0,1) \implies \mathbb{P}[P_i \leq t] = t$. The probability of accepting any true null hypothesis is the complement of this probability, $1-t$. Thus we have:

$$\mathbb{P}[\text{reject any true null hypothesis}] = 1 - \prod_{i=1}^{n_0}(1-t) = 1-(1-t)^{n_0}$$

(b) The FWER is given by $\mathbb{P}$[reject any true hypothesis]. As computed in (a), for a given cutoff t, this probability is given by $1-(1-t)^{n_0}$. Setting $t = 1-(1-\alpha)^{\frac{1}{n}}$, we have that:

$$\mathbb{P}[\text{reject any true null hypothesis}] = 1-(1-t)^{n_0} = 1-(1-(1-(1-\alpha)^{\frac{1}{n}}))^{n_0} = 1-(1-\alpha)^{\frac{n_0}{n}}$$

Because $n_0 \leq n$ and $1-\alpha \leq 1$, $(1-\alpha)^{\frac{n_0}{n}} \geq (1-\alpha) \implies -(1-\alpha)^{\frac{n_0}{n}} \leq \alpha - 1$ and so:

$$\mathbb{P}[\text{reject any true null hypothesis}] = 1-(1-\alpha)^{\frac{n_0}{n}} \leq 1 + (\alpha - 1)$$
$$\mathbb{P}[\text{reject any true null hypothesis}] \leq \alpha$$

and so we have shown for $t = 1-(1-\alpha)^{\frac{1}{n}}$, the FWER $\leq \alpha$, meaning that the FWER is controlled at level α.

We now compare if this choice of $t = 1-(1-\alpha)^{\frac{1}{n}}$ or $t = \frac{\alpha}{n}$ will reject more hypotheses. Because whichever choice of t is greater will reject *more* hypotheses[2], we aim to find which choice of t is greater. We first assume that n, the number of hypotheses tests, is ≥ 1 and that $0 \leq \alpha \leq 1$. Note by Bernoulli's inequality that for $0 \leq r \leq 1$ and $x \geq -1$, $(1+x)^r \leq 1 + rx$. Because $0 \leq \frac{1}{n} \leq 1$ and $-\alpha \geq -1$, we can apply Bernoulli's inequality and so $(1-\alpha)^{\frac{1}{n}} \leq 1 - \frac{\alpha}{n} \implies -(1-\alpha)^{\frac{1}{n}} \geq \frac{\alpha}{n} - 1$. So the choice of $t = 1-(1-\alpha)^{\frac{1}{n}} \geq 1 + \frac{\alpha}{n} - 1 \implies t = 1-(1-\alpha)^{\frac{1}{n}} \geq \frac{\alpha}{n}$. Thus choosing $t = 1-(1-\alpha)^{\frac{1}{n}}$ will reject more hypotheses as it is a greater threshold.

w The procedure of choosing $t = 1-(1-\alpha)^{\frac{1}{n}}$ differs from the Bonferroni correction because it assumes independence between all n hypothesis tests, which the Bonferroni correction does not assume. This makes sense as this stronger assumption allows this choice of t to reject more hypotheses when compared to the Bonferroni correction.

[2] this means that more computed p-values will meet this threshold $\implies$ more hypotheses will be rejected