

Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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Problem set 1
Due Thursday January 23 at 11pm

Exercise 1.1 (5 points). Suppose A, B, C are sets and $f : A \rightarrow B, g : B \rightarrow C$ are functions. Let $h = g \circ f : A \rightarrow C$. Show that

- (1) If h is injective, then f is injective.
- (2) If h is surjective, then g is surjective.

Exercise 1.2 (5 points; Rudin 1.1). This exercise uses the real numbers \mathbb{R} , which we have not defined yet; but, for this exercise, all you need to know about \mathbb{R} is that it is a field containing the field \mathbb{Q} of rational numbers.

Suppose $r \in \mathbb{Q}, r \neq 0$, and $x \in \mathbb{R}, x \notin \mathbb{Q}$. Prove that $r + x \notin \mathbb{Q}$ and $rx \notin \mathbb{Q}$.

Exercise 1.3 (10 points; Rudin 1.3). Suppose F is a field with $x, y, z \in F$. Prove carefully from the field axioms:

- (1) If $x \neq 0$ and $xy = xz$, then $y = z$.
- (2) If $x \neq 0$ and $xy = x$, then $y = 1$.
- (3) If $xy = 1$, then $x \neq 0$ and $y = x^{-1}$.
- (4) If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Exercise 1.4 (10 points). This exercise concerns the field $\mathbb{Q}(\sqrt{3})$.

- (1) Prove that \mathbb{Q} does not contain any x with $x^2 = 3$.
- (2) Consider the set $\mathbb{Q}(\sqrt{3})$, defined as the set of expressions “ $a + b\sqrt{3}$ ” with $a, b \in \mathbb{Q}$. (More formally, we could think of $\mathbb{Q}(\sqrt{3})$ as the set of ordered pairs (a, b) with $a, b \in \mathbb{Q}$.) We can equip $\mathbb{Q}(\sqrt{3})$ with addition and product laws by

$$(a + b\sqrt{3}) + (a' + b'\sqrt{3}) = (a + a') + (b + b')\sqrt{3},$$

$$(a + b\sqrt{3})(a' + b'\sqrt{3}) = (aa' + 3bb') + (ab' + ba')\sqrt{3}.$$

Show that $\mathbb{Q}(\sqrt{3})$ can be made into a field, with these addition and product laws. (This means saying carefully what are the negation law, inversion law, 0 and 1 elements, and then checking that all the axioms of a field are satisfied.)

- (3) We could similarly consider the set $\mathbb{Z}(\sqrt{3})$, defined as above except that now we require $a, b \in \mathbb{Z}$, with the same addition and product laws. Show that $\mathbb{Z}(\sqrt{3})$ cannot be made into a field with these addition and product laws.

Exercise 1.5 (5 points). As we have stated, the usual order relation on \mathbb{Q} can be expressed as:

$$p/q < p'/q' \iff pq' < p'q$$

when q and q' are positive.

We can define a different relation \prec , as follows: assume that all fractions are written in lowest terms and with positive denominators. Define

$$p/q \prec p'/q' \iff pq < p'q'.$$

Prove that the relation \prec does *not* make \mathbb{Q} into an ordered set.

Exercise 1.6 (5 points). Let A be a nonempty subset of an ordered set S . If there exists an element $\alpha \in S$ such that $\alpha \leq x$ for all $x \in A$, then we call α a *lower bound* for A . Similarly, if there exists $\beta \in S$ such that $x \leq \beta$ for all $x \in A$, we call β an *upper bound* for A . Suppose α is a lower bound of A and β is an upper bound of A . Prove that $\alpha \leq \beta$.