Math 226: HW 4
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1. a) We prove each of the two inequalities below.

   1 $dim(Span(S)) \leq dim(Span(S \cup \{x\}))$
   Let us specify the basis for S as β_S and basis for $S \cup \{x\}$ as $\beta_{S\cup\{x\}}$. Thus $dim(Span(S)) = |\beta_S|$. Let us explore two cases: (1) $x \in Span(S)$ and (2) $x \notin Span(S)$. In case (1), $\beta_{S\cup\{x\}} = \beta_S$ and so $dim(Span(S \cup \{x\})) = |\beta_{S\cup\{x\}}| = |\beta_S| \geq |\beta_S|$. In case (2), $\beta_{S\cup\{x\}} = \beta_S \cup \{x\}$ and so $dim(Span(S \cup \{x\})) = |\beta_{S\cup\{x\}}| = |\beta_S| + 1 \geq |\beta_S|$. In either case, **1** is true.

   2 $dim(Span(S \cup \{x\})) \leq dim(Span(S)) + 1$
   From **1**, we know that $dim(Span(S \cup \{x\}))$ is equal to either $|\beta_S| + 1$ or $|\beta_S|$. Because $dim(Span(S)) + 1 = |\beta_S| + 1$, $dim(Span(S \cup \{x\})) \leq dim(Span(S)) + 1$ regardless of the particular value of $dim(Span(S \cup \{x\}))$.

   b) Let us define the basis of $U \cap W$ as $\beta_{U\cap W} = \{v_1, \dots, v_n\}$ where $n \in \mathbb{Z}$. Let us also define β_U and β_W as the basis for U and W, respectively.
   Using Steinz Exchange Lemma, we can extend $\beta_{U\cap W}$ to β_U as $\beta_U = \beta_{U\cap W} \cup \{u_1, \dots u_m\}$ where $u_i \in U$. Similarily, we can extend $\beta_{U\cap W}$ to β_W as $\beta_W = \beta_{U\cap W} \cup \{w_1, \dots, w_k\}$ where $w_i \in W$.
   We now evaluate the statement $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$:

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$
$$dim(U + W) = (n + m) + (n + k) - n$$
$$dim(U + W) = n + m + k$$

   We now compute $dim(U + W)$ by trying to find the basis of $U + W$, β_{U+W}. Any given element in $U + W$ is given by $u + w$ where $u \in U, w \in W$. We can write u and w as linear combinations of β_U and β_W respectively. Let us define the notation $\beta_K[i]$ as giving the ith element of the basis for vector space K. Then given $a_i, b_i \in \mathbb{F}$:

$$u + w = \Sigma_i^{n+m} a_i \beta_U[i] + \Sigma_i^{n+k} b_i \beta_W[i]$$
$$u + w = \Sigma_i^n a_i v_i + \Sigma_i^m a_{n+i} u_i + \Sigma_i^n b_i v_i + \Sigma_i^k b_{n+i} w_i$$
$$u + w = \Sigma_i^n (a_i + b_i) v_i + \Sigma_i^m a_{n+i} u_i + \Sigma_i^k b_{n+i} w_i$$

   We can redefine $c_i = a_i + b_i \in \mathbb{F}$:

$$u + w = \Sigma_i^n c_i v_i + \Sigma_i^m a_{n+i} u_i + \Sigma_i^k b_{n+i} w_i$$

   Thus, we have shown every element in $U + W$ can be written as a linear combination of $\beta_{U\cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$. We must show now that $\beta_{U\cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$ is linearly independent to show that $\beta_{U+W} = \beta_{U\cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$. Because basis $\beta_U = \beta_{U\cap W} \cup \{u_1, \dots, u_m\}$ and basis $\beta_W = \beta_{U\cap W} \cup \{w_1, \dots, w_k\}$, we know that $\beta_{U\cap W} \cup \{u_1, \dots, u_m\}$ and $\beta_{U\cap W} \cup \{w_1, \dots, w_k\}$ are linearly independent. Thus to show $\beta_{U\cap W} \cup \{u_1, \dots, u_m\} \cup \{w_1, \dots, w_k\}$

is linearly independent, we must show that $\{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ is linearly independent.

Let us define this set as $K = \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ and let us assume that it is linearly dependent. This means that there exists a linear combination of $\{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ that exists in U and W (i.e. in $U \cap W$). By our construction of $\{u_1, \ldots, u_m\}$ and $\{w_1, \ldots, w_k\}$ through Steinitz Exchange Lemma, we know that all $u_i, w_i \notin \beta_{U \cap W}$ as they are the set of vectors required to extend $\beta_{U \cap W}$ to β_U or β_W. This would mean that we have found some element $\in U \cap W$ that cannot be represented as a linear combination of $\beta_{U \cap W}$. This violates the definition of $Span(\beta_{U \cap W})$ and thus by proof by contradiction, we have shown $\{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ is linearly independent $\Rightarrow \beta_{U \cap W} \cup \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ is linearly independent.

Thus, $\beta_{U+W} = \beta_{U \cap W} \cup \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$. This means:

$$
\begin{aligned}
dim(U + W) &= n + m + k \\
|\beta_{U+W}| &= n + m + k \\
n + m + k &= n + m + k \\
0 &= 0
\end{aligned}
$$

Thus, we have proven $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.

c) We use the notation of $\beta_{U \cap W}$ and β_{U+W} from part (b). We are given the following statement:

$$dim(U + W) = 1 + dim(U \cap W)$$

This statement implies that one element must be unioned to $\beta_{U \cap W}$ to form β_{U+W} as $U \cap W \subseteq U + W$. This statement implies that $|\beta_{U+W}| = n + m + k = 1 + |\beta_{U \cap W}| = 1 + n \Rightarrow m + k = 1$. Because $m, k \in \mathbb{Z}$ and $m \geq 0, n \geq 0$, we know that either $m = 1$ or $k = 1$. We investigate these two cases below.

(1) $m = 1$

If $m = 1$, $\exists\ u \in \{u_1, \ldots, u_m\}$ s.t. $\beta_{U+W} = \{u\} \cup \beta_{U \cap W}$. W is given by $Span(\beta_W)$. Because $m = 1 \Rightarrow k = 0$, $\beta_{U \cap W} = \beta_W \Rightarrow U \cap W = W$. Because this means every element in W exists in U, $U + W = \{u + w : u \in U, w \in W\} = \{u + w : u \in U, w \in U\}$. Given U is closed under addition because it is a vector subspace, we know that $U + W = U$. Thus we have shown that in this case $U + W = U$ and $U \cap W = W$.

(2) $k = 1$

If $k = 1$, $\exists\ w \in \{w_1, \ldots, w_k\}$ s.t. $\beta_{U+W} = \{w\} \cup \beta_{U \cap W}$. W is given by $Span(\beta_W)$. Because $k = 1 \Rightarrow m = 0$, $\beta_{U \cap W} = \beta_U \Rightarrow U \cap W = U$. Because this means every element in U exists in W, $U + W = \{u + w : u \in U, w \in W\} = \{u + w : u \in W, w \in W\}$. Given W is closed under addition because it is a vector subspace, we know that $U + W = W$. Thus we have shown that in this case $U + W = W$ and $U \cap W = U$.

2. a) We show the two parts of the problem below.

(1) **If T is injective, $T(B)$ is linearly independent**

Let us define $B = \{u_1, \ldots, u_{|B|}\}$. To show that $T(B)$ is linearly independent, we need to show that given $a_i \in \mathbb{F}$, the solution to $\Sigma_i^{|B|} a_i T(u_i) = 0$ is for all $a_i = 0$. We inspect this below:

$$\Sigma_i^{|B|} a_i T(u_i) = 0$$
$$T(\Sigma_i^{|B|} a_i u_i) = 0$$

Because we know T is injective, if $T(x) = 0 \Rightarrow x = 0$. Thus, we know that $\Sigma_i^{|B|} a_i u_i = 0$. Because $B = \{u_1, \ldots, u_{|B|}\}$ is a basis and thus is linearly independent, the solution to $\Sigma_i^{|B|} a_i u_i = 0$ is for all $a_i = 0$. Thus, the solution to $\Sigma_i^{|B|} a_i T(u_i) = 0$ is for all $a_i = 0 \Rightarrow T(B)$ is linearly independent.

2 **If $T(B)$ is linearly independent and $\infty > |T(B)| \geq |B|$, T is injective**
To show that T is injective, we need to show that given $x, y \in U$ s.t. $T(x) = T(y)$, $x = y$. We prove this below. Note that because B is a basis for U, x and y can be represented as a linear combination of B. Thus given $c_i, d_i \in \mathbb{F}$, $x = \Sigma_i^{|B|} c_i B_i$ and $y = \Sigma_i^{|B|} d_i B_i$.

$$T(x) = T(y)$$
$$T(\Sigma_i^{|B|} c_i B_i) = T(\Sigma_i^{|B|} d_i B_i)$$
$$\Sigma_i^{|B|} c_i T(B_i) = \Sigma_i^{|B|} d_i T(B_i)$$
$$\Sigma_i^{|B|} (c_i - d_i) T(B_i) = 0$$

Because we are given $T(B)$ is linearly independent, we know that the only solution to the equation $\Sigma_i^{|B|} a_i T(B_i) = 0$ is for all $a_i = 0$. Thus, we know that in the above equation, $c_i - d_i = 0 \Rightarrow c_i = d_i$. Thus we have proved $x = y$ if we know $T(x) = T(y) \Rightarrow T$ is injective.

b) In order to show that T is surjective iff $Span(T(B)) = V$, we must show (1) if $Span(T(B)) = V$, T is surjective and (2) if T is surjective, $Span(T(B)) = V$.

1 **If $Span(T(B)) = V$, T is surjective**
If $Span(T(B)) = V$, $T(B)$ is a basis for V. This means $\forall v \in V$, v can be expressed as a linear combination of $T(B)$. Given $B = \{b_1, \ldots, b_{|B|}\}$ and $a_i \in \mathbb{F}$:

$$v = \Sigma_i^{|B|} a_i T(b_i)$$

This is equivalent to:

$$v = T(\Sigma_i^{|B|} a_i b_i)$$

Because $\Sigma_i^{|B|} a_i b_i$ is a linear combination of B, $\Sigma_i^{|B|} a_i b_i \in U$. Defining $w = \Sigma_i^{|B|} a_i b_i$, we have shown $\forall v \in V, \exists\ w \in U$ s.t. $T(w) = v$. Thus T is proven to be surjective.

2 **If T is surjective, $Span(T(B)) = V$**
If T is surjective, $\forall v \in V, \exists\ w \in U$ s.t. $T(w) = v$. Because B is a basis for U, w can be expressed as a linear combination of B. Given $B = \{b_1, \ldots, b_{|B|}\}$ and $a_i \in \mathbb{F}$:

$$T(w) = v$$
$$T(\Sigma_i^{|B|} a_i b_i) = v$$

Applying T to each element in the summation and switching sides:

$$v = \Sigma_i^{|B|} a_i T(b_i)$$

This shows that $\forall v \in V$, v can be expressed as a linear combination of $T(B)$. Furthermore, this also shows that all linear combinations of $T(B)$ are elements of V. Thus, we know that $Span(T(B)) = V$.

c) We prove both parts of this question below.

1 **If T is bijective, $T(B)$ is a basis for V**
From part (b), we have proved if $Span(T(B)) = V$, T is surjective. From part (a), we have proved if $T(B)$ is linearly independent if T is injective. If T is bijective, T is surjective and injective, meaning that $T(B)$ is linearly independent and generates V. By the definition of a basis, $T(B)$ is a basis for V.

2 **If $T(B)$ is a basis and $\infty > |T(B)| \geq |B|$, then T is bijective**
We assume $T(B)$ is a basis for V, which means that $Span(T(B)) = V$ and $T(B)$ is linearly independent. From part (b), we have proved if $T(B)$ spans V, T is surjective. From part (a), we have proved if $T(B)$ is linearly independent and $\infty > |T(B)| \geq |B|$, T is injective. Thus if $T(B)$ is a basis and $\infty > |T(B)| \geq |B|$, T is injective and surjective $\Rightarrow$ T is bijective.

3. a) T is a linear transformation if given $c \in R$ and $f, g \in C^1(\mathbb{R})$, $T(cf + g) = cT(f) + T(g)$. We evaluate $T(cf + g)$ below as:

$$T(cf+g) = ((cf+g)'(3), (cf+g)(3)) = ((cf'+g')(3), (cf+g)(3)) = (cf'(3)+g'(3), cf(3)+g(3))$$

We now compute $cT(f) + T(g)$ below.

$$cT(f)+T(g) = c(f'(3), f(3)) + (g'(3), g(3)) = (cf'(3), cf(3)) + (g'(3), g(3)) = (cf'(3)+g'(3), cf(3)+g(3))$$

Because the two above expressions are equivalent we know that $T(cf+g) = cT(f)+T(g) \Rightarrow T$ is a linear transformation.

b) If $H + V = V$, this means that $\forall h \in H$ and $\forall v \in V$, $h + v = v$. We prove this below.

$$\begin{gathered} h + v = v \\ T(h+v) = T(v) \\ ((h+v)'(3), (h+v)(3)) = (1,2) \\ (h'(3) + v'(3), h(3) + v(3)) = (1,2) \end{gathered}$$

Because $h = (x-3)^2 g(x)$ where $g(x) \in C^1(\mathbb{R})$, $h'(x) = 2(x-3)g(x) + (x-3)^2 g'(x)$. Thus, $h(3) = 0 = h'(3)$. Given this,

$$\begin{gathered} (v'(3), v(3)) = (1,2) \\ T(v) = (1,2) \end{gathered}$$

Thus we have proven $\forall h \in H$ and $\forall v \in V$, $h + v = v$. This proves that $H + V = V$.

c) Let us define $x, y \in C^1(\mathbb{R})$. For T to be injective, if $T(x) = T(y)$ then $x = y$. From part (b), we see that for two pre-images $h + v, v \in C^1(\mathbb{R})$ where $T(h + v) = T(v)$ but $h + v \neq v$. Thus, we have shown that by definition T is not injective.

In order for T to be surjective, $\forall v \in \mathbb{R}^2, \exists\ f(x) \in C^1(\mathbb{R})$ s.t. $T(f(x)) = v$. Let us define a function $f(x) \in C^1(\mathbb{R})$ and $v = (a_1, a_2) \in \mathbb{R}^2$. The function $g(x) = a_1 x + a_2 - 3a_1 \in C^1(\mathbb{R})$ and has the property $T(g) = v$. Thus, $\forall v = (a_1, a_2) \in \mathbb{R}^2, \exists\ g(x) = a_1 x + a_2 - 3a_1 \in C^1(\mathbb{R})$ s.t. $T(g) = v$. Thus, T is surjective.

4. a) **Computing basis and dimension for $N(T)$**

$N(T) = \{v \in \mathbb{F}^5 : T(v) = \mathbf{0}^4\}$. $T(v) = \mathbf{0}^4$ when for a given $v = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5$, $T(v) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3) = \mathbf{0}^4$. This occurs under the following conditions:

$$
\begin{aligned}
x_2 &= -x_1 \\
x_3 &= 0 \\
x_5 &= -\frac{x_4}{3}
\end{aligned}
$$

Thus, $N(T) = \{(x_1, -x_1, 0, x_4, -\frac{x_4}{3}) : x_1, x_4 \in \mathbb{F}\}$. Thus the basis for $N(T)$ can be given as $\beta_{N(T)}$:

$$\beta_{N(T)} = \{(1, -1, 0, 0, 0), (0, 0, 0, 1, -\frac{1}{3})\}$$

So we get $dim(N(T)) = |\beta_{N(T)}| = 2$.

Computing basis and dimension for $R(T)$

$R(T) = \{w \in \mathbb{F}^4 : \exists\ v \in \mathbb{F}^5 \text{ s.t. } T(v) = w\}$. If we redefine $z_1 = x_1 + x_2$ and $z_2 = x_4 + 3x_5$, we get that $T(v) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3) = (z_1, x_3, z_2, x_3)$ where $z_1, x_3, z_2, \in \mathbb{F}$. Thus the basis for $R(T)$ given as $\beta_{R(T)}$:

$$\beta_{R(T)} = \{(1, 0, 0, 0), (0, 1, 0, 1), (0, 0, 1, 0)\}$$

So we get $dim(R(T)) = |\beta_{R(T)}| = 3$.

b) From Dimension Theorem, we know that if T is linear, $dim(N(T)) + dim(R(T)) = dim(\mathbb{R}^5) = 5$. Because the output of T is in $\mathbb{R}^2$, $dim(R(T)) \leq 2$. Thus, $dim(N(T)) \geq 3$ if T is linear.

Expressed differently, $N(T) = \{(x_1, x_2, \frac{x_1}{7}, x_2, x_2) : x_1, x_2 \in \mathbb{F}\}$. Thus the basis of $N(T)$ is given by $\beta_{N(T)} = \{(1, 0, \frac{1}{7}, 0, 0, 0), (0, 1, 0, 1, 1)\}$ and so $dim(N(T)) = |\beta_{N(T)}| = 2$. Because $dim(N(T)) = 2 \not\geq 3$, T cannot be linear for this given null space $N(T)$.