

Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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MATH 255 PSET 6

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1.

Pick $n \in \mathbb{N}$. We can construct $S'_n = \{N_{\frac{1}{n}}(x) : x \in K\} \supset K$ as an open cover of K . Because K is compact, every open cover of K has a finite subcover. So open cover S'_n has a subcover which we can define as: $S_n = \{N_{\frac{1}{n}}(x_i^{(n)}) : i = 1 \dots m_n\} \subset S'_n$ where m_n is the number of points (in K) required for S_n to be an open cover of K . We can then define the subset $C' = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{m_n} x_j^{(n)}$ which is the union of all the required points for each finite subcover S_n to cover K . Because C' is a countable union of finite sets¹, C' is at most countable. Furthermore, $\forall x \in C', x \in K \implies C' \subset K$.

We now show that C' is a *dense* subset of K . To do so, we show $\forall x \in K, x$ is either in C' or x is a limit point of C' . We prove this occurs with casework:

1. **Case One:** $x \in C'$

In this case, our job is done.

2. **Case Two:** $x \notin C'$

In this case, we WTS x is a limit point of C' or that $\forall \epsilon > 0, \exists p \in N_{\epsilon}(x)$ s.t. $p \neq x$ and $p \in C'$. Pick $\epsilon > 0$. Note that $\forall n \in \mathbb{N}, C'$ contains all the points which neighborhoods with size $\frac{1}{n}$ will cover K . By the Archimidean property, $\exists n \in \mathbb{N}$ s.t. $n(1/n) = n > \frac{1}{\epsilon} \implies \exists n$ s.t. $\frac{1}{n} < \epsilon$. We proceed with this value of n . Because S_n is an open cover of K , $x \in K \implies x$ is contained in some set² of $S_n \implies \exists 1 \leq k \leq m_n$ s.t. $x \in N_{\frac{1}{n}}(x_k^{(n)})$ where $x_k^{(n)} \in C'$ and $x \neq x_k^{(n)}$ (given by $x \notin C'$). Thus, $d(x, x_k^{(n)}) < \frac{1}{n} \implies N_{\frac{1}{n}}(x)$ contains some $x_k^{(n)} \in C'$. Because $\frac{1}{n} < \epsilon$, $N_{\frac{1}{n}}(x) \subset N_{\epsilon}(x)$ and so $N_{\epsilon}(x)$ contains some $x_k^{(n)} \in C'$ where $x \neq x_k^{(n)}$. Thus, $\forall \epsilon > 0, N_{\epsilon}(x)$ contains some $p \in C'$ s.t. $p \neq x$. So x is a limit point of C' .

2.

Let $\{G_i\}$ be an open cover of $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$. This means that \exists some open set $G_j \in \{G_i\}$ s.t. $x \in G_j$. Because G_j is open, all points of G_j are interior points of $G_j \implies x$ is an interior point of $G_j \implies \exists \epsilon > 0$ s.t. $N_{\epsilon}(x) \subset G_j$. Because $(x_n) \rightarrow x \implies \exists N$

¹For clarity, the n th finite set is given by $\{x_1^{(n)}, \dots, x_{m_n}^{(n)}\}$.

²i.e. a neighborhood

s.t. $\forall n \geq N, d(x_n, x) < \epsilon$. Thus, this means that $N_\epsilon(x)$ will contain x and x_N, x_{N+1}, \dots . Because $N_\epsilon(x) \subset G_j$, this means x and x_N, x_{N+1}, \dots are contained in G_j . Now for each of the finitely many points x_1, \dots, x_{N-1} (all of which are contained in $\{G_i\}$), we can pick a given set in $\{G_i\}$ which contains this point. Let $G_{n_k} \in \{G_i\}$ be the set which contains the k th point x_k where $1 \leq k \leq N-1$. Then $G' = G_j \cup \bigcup_{i=1}^{N-1} G_{n_k}$ covers $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$. Because $G' \subset \{G_i\} \implies G'$ is a finite subcover of $\{G_i\}$. Thus we have shown all open covers of $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ have a finite subcover $\implies \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is compact.

3.

We prove that $\lim_{n \rightarrow \infty} \frac{2n+1}{3n-1} = \frac{2}{3}$ by showing that the sequence $(p_n) \rightarrow \frac{2}{3}$ in metric space \mathbb{R} where $p_n = \frac{2n+1}{3n-1}$. Pick $\epsilon > 0$. We now aim to find $N \in \mathbb{N}$ s.t. $\forall n \geq N, d(p_n, \frac{2}{3}) < \epsilon$ or expressed more simply³, we aim to find $N \in \mathbb{N}$ s.t. $\forall n \geq N, d(p_n, \frac{2}{3}) = d(\frac{2n+1}{3n-1}, \frac{2}{3}) = |\frac{2n+1}{3n-1} - \frac{2}{3}| = |\frac{3(2n+1)-2(3n-1)}{3(3n-1)}| = |\frac{5}{3(3n-1)}| < \epsilon$. We solve the $|\frac{5}{3(3n-1)}| < \epsilon$ inequality for n below:

$$|\frac{5}{3(3n-1)}| < \epsilon$$

Because $n \in \mathbb{N} \implies n \geq 1 \implies \frac{5}{3(3n-1)} > 0 \implies |\frac{5}{3(3n-1)}| = \frac{5}{3(3n-1)}$ and so we can proceed removing the absolute value term:

$$\begin{aligned} \frac{5}{3(3n-1)} &< \epsilon \\ 5 &< \epsilon(9n-3) \\ \frac{5}{\epsilon} &< 9n-3 \\ n &> \frac{1}{9}(\frac{5}{\epsilon} + 3) \end{aligned}$$

Thus, we see $d(p_n, \frac{2}{3}) < \epsilon$ for any $n > \frac{1}{9}(\frac{5}{\epsilon} + 3)$. Thus, we aim to choose for N any natural number $> \frac{1}{9}(\frac{5}{\epsilon} + 3)$. By Archimidean property, $\exists m \in \mathbb{N}$ s.t. $m(1) > \frac{1}{9}(\frac{5}{\epsilon} + 3)$ and so we can simply take this m to be our choice of N . Thus we have shown $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, d(p_n, \frac{2}{3}) < \epsilon \implies (p_n) \rightarrow \frac{2}{3} \implies \lim_{n \rightarrow \infty} \frac{2n+1}{3n-1} = \frac{2}{3}$.

4.

Lemma 0.1 *Let $x, y \in \mathbb{R}$. We will prove $||x| - |y|| \leq |x - y|$. By Triangle Inequality, we know $|x + y| \leq |x| + |y|$ and thus we can show these two facts:*

³Because we are operating in the metric space \mathbb{R} with the standard distance function, $d(x, y) = |x - y|$.

1. By Triangle Inequality, we know $|x| + |y - x| \geq |x + y - x|$ and so we have:

$$\begin{aligned} |x| + |y - x| &\geq |x + y - x| \\ |y - x| &\geq |y| - |x| \\ |x - y| &\geq |y| - |x| \end{aligned}$$

2. By Triangle Inequality, we know $|y| + |x - y| \geq |y + x - y|$ and so we have:

$$\begin{aligned} |y| + |x - y| &\geq |y + x - y| \\ |x - y| &\geq |x| - |y| \end{aligned}$$

Thus, we know the two facts: $|x - y| \geq |y| - |x|$ and $|x - y| \geq |x| - |y|$ which together imply $|x - y| \geq \pm(|x| - |y|) \implies |x - y| \geq ||x| - |y||$.

We WTS sequence $|x_n|$ will converge to $|x|$. Pick $\epsilon > 0$. To show $(|x_n|) \rightarrow |x|$, we must find some $N \in \mathbb{N}$ s.t. $\forall n \geq N, d(|x_n|, |x|) = ||x_n| - |x|| < \epsilon$. Because $(x_n) \rightarrow x \implies \exists M \in \mathbb{N}$ s.t. $\forall n \geq M, d(x_n, x) < \epsilon \implies \forall n \geq M, |x_n - x| < \epsilon \implies$ by **(Lemma 0.1)** $\forall n \geq M, ||x_n| - |x|| \leq |x_n - x| < \epsilon \implies \forall n \geq M, d(|x_n|, |x|) = ||x_n| - |x|| < \epsilon$. Thus, we can simply set $N = M$ and so we have shown $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, d(|x_n|, |x|) < \epsilon$. This proves $|x_n| \rightarrow |x|$.

We now show that the converse is not true. Let us define sequence (x_n) in metric space \mathbb{R} where $x_n = -1$. Because every element in this sequence is equal to -1 , $(x_n) \rightarrow -1$. We can now define sequence (y_n) where $y_n = |x_n| = |-1| = 1$. Because every element in (y_n) is equal to 1 , $(y_n) \rightarrow 1$. Expressed differently, $y_n = |x_n| \rightarrow |1|$. So we have found a case where $|x_n| \rightarrow |1|$ but $x_n \not\rightarrow 1$ and thus we have disproved the converse of this statement.