

## Discretionary Note

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# MATH 244 HW 2

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## 1. Section 1.5, Question 5

We first prove if  $R \circ (S \circ T)$  is well defined  $\implies (R \circ S) \circ T$  is well-defined. If  $R \circ (S \circ T)$  is well-defined we can define sets  $A, B, C, D$ , where  $R \subseteq A \times B, S \subseteq B \times C$ , and  $T \subseteq C \times D$ . This means  $S \circ T \subseteq B \times D$  and so  $R \circ (S \circ T) \subseteq A \times D$ , and so  $R \circ (S \circ T)$  is well-defined<sup>1</sup>. We continue with these sets  $A, B, C, D$  for the rest of this problem.

To prove  $R \circ (S \circ T) = (R \circ S) \circ T$ , we prove  $R \circ (S \circ T) \subseteq (R \circ S) \circ T$  and  $(R \circ S) \circ T \subseteq R \circ (S \circ T)$ .

### 1. $R \circ (S \circ T) \subseteq (R \circ S) \circ T$

Let us pick  $(x, y) \in R \circ (S \circ T)$ . Let us define arbitrary elements  $b \in B, c \in C$ . Given  $(x, y) \in R \circ (S \circ T)$ , we know  $\exists (x, b) \in R$  and  $(b, y) \in S \circ T$ . Furthermore, if  $(b, y) \in S \circ T$ , we know  $\exists (b, c) \in S$  and  $(c, y) \in T$ . Because  $(x, b) \in R$  and  $(b, c) \in S \implies (x, c) \in R \circ S$ . Furthermore,  $(c, y) \in T$ , so  $(R \circ S) \circ T$  will contain  $(x, y) \implies \forall (x, y) \in R \circ (S \circ T), (x, y) \in (R \circ S) \circ T \implies R \circ (S \circ T) \subseteq (R \circ S) \circ T$ .

### 2. $(R \circ S) \circ T \subseteq R \circ (S \circ T)$

Let us pick  $(x, y) \in (R \circ S) \circ T$ . Let us define arbitrary elements  $c \in C, b \in B$ . Given  $(x, y) \in (R \circ S) \circ T$ , we know  $\exists (x, c) \in R \circ S$  and  $(c, y) \in T$ . If  $(x, c) \in R \circ S$ , we know  $\exists (x, b) \in R$  and  $(b, c) \in S$ . We now show  $(x, y) \in R \circ (S \circ T)$ . Because  $(b, c) \in S$  and  $(c, y) \in T$ ,  $(b, y) \in S \circ T$ . Furthermore, because  $(x, b) \in R$ ,  $(x, y) \in R \circ (S \circ T)$ . Thus, we have shown,  $\forall (x, y) \in (R \circ S) \circ T, (x, y) \in R \circ (S \circ T) \implies (R \circ S) \circ T \subseteq R \circ (S \circ T)$ .

## 2. Section 1.6, Question 3

We prove both directions of this statement below.

### 1. If $R \circ R \subseteq R \implies R$ is transitive

$R \circ R$  contains a pair  $(x, z)$  if  $(x, y)$  and  $(y, z)$  are both in  $R$ . If  $R \circ R \subseteq R$ , then this means if  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R \circ R \implies (x, z) \in R$ . This satisfies the definition of transitivity, which is given as follows: if  $(x, y) \in R$  and  $(y, z) \in R \implies (x, z) \in R$ . Thus,  $R$  is transitive.

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<sup>1</sup>Put in other words,  $R \circ (S \circ T)$  is well-defined because the codomain of  $R$  (set  $B$ ) is the domain of  $S \circ T$  and the codomain of  $S$  (set  $C$ ) is the domain of  $T$ .

2. **If  $R$  is transitive  $\implies R \circ R \subseteq R$**

If  $R$  is transitive, this means if  $(x, y)$  and  $(y, z) \in R \implies (x, z) \in R$ .  $R \circ R$  contains a pair  $(x, z)$  if  $(x, y)$  and  $(y, z)$  are both in  $R$ . However, because  $R$  is transitive, it is guaranteed by definition that  $(x, z) \in R$  if  $(x, y)$  and  $(y, z) \in R$ . Thus, if  $R$  is transitive,  $\forall (x, z) \in R \circ R, (x, z) \in R \implies R \circ R \subseteq R$ .

3. **Section 1.6, Question 6**

A relation  $R$  on  $X$  is an equivalence relation if it is reflexive, transitive, and anti-symmetric. A relation is an ordering relation if it is reflexive, transitive, and symmetric. So a relation  $R$  is both an equivalence and ordering relation if it is reflexive, transitive, and both anti-symmetric & symmetric.  $R$  is anti-symmetric and symmetric if (i) if  $\forall x, y \in X$ , if  $(x, y) \in R$ , then  $(y, x) \in R \iff y = x$  and (ii)  $\forall x, y \in X, (x, y) \in R \implies (y, x) \in R$ .

A relation  $R$  which is reflexive, transitive, antisymmetric, & symmetric will have the following property:  $(x, y) \in R \iff x = y$ . We prove this below:

1.  $(x, y) \in R \implies x = y$

Because  $R$  is symmetric,  $(x, y) \in R \implies (y, x) \in R$ . However, because  $R$  is anti-symmetric,  $(y, x) \in R \implies y = x$  or  $x = y$ .

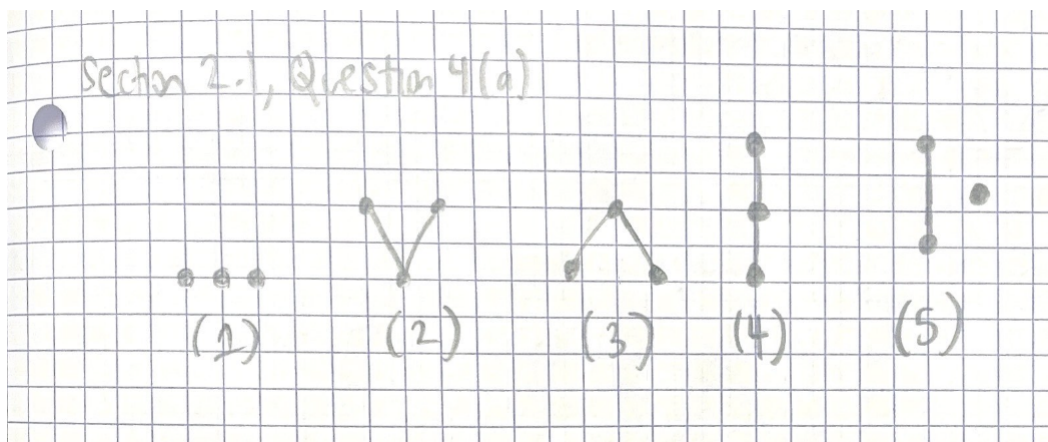
2.  $x = y \implies (x, y) \in R$

This property is satisfied by the fact that  $R$  is reflexive.

Thus, the relations on the set  $X$  that are both equivalences and orderings are identity relations given by  $R = \{(x, x) : x \in X\}$ .

4. **Section 2.1, Question 4**

a) We show all possible non-isomorphic 3-element posets below:



We enumerate over all possibilities of a non-isomorphic three element poset to clearly show we have classified all possible non-isomorphic three-element posets:

1. **All elements in the poset are not comparable to each other.**

This is case (1).

2. **One minimal element in the poset. The other two elements are incomparable.**

This is case (2).

3. **One maximal element in the poset. The other two elements are incomparable.**

This is case (3).

4. **Every pair of elements is comparable,**

This is case (4).

5. **One pair of elements in the poset are comparable to each other, and the remaining element is incomparable to all other elements.**

This is case (5).

- b) Let us define  $(X, \leq)$  and  $(Y, \preceq)$  to be linearly ordered sets with  $|X| = |Y| = n$ . This means that every single element in them can be compared with each other and so we can produce the following ordered<sup>2</sup> sequences  $L_X$  and  $L_Y$  of  $X$  and  $Y$  respectively:

$$L_X = x_1 \leq x_2 \leq \cdots \leq x_n$$

$$L_Y = y_1 \preceq y_2 \preceq \cdots \preceq y_n$$

Let us define the following map  $f : X \rightarrow Y$  that where  $f(x_i) = y_i$ . Because  $\forall y_i \in Y, \exists x_i$  s.t.  $f(x_i) = y_i$ , we can conclude  $f$  is surjective. Because  $\forall 1 \leq i < j \leq n, f(x_i) \neq f(x_j)$ , we also have that  $f$  is one-to-one or injective. Because  $f$  is injective and surjective  $\implies f$  is bijective.

We now must show that  $f$  is order-preserving, or that  $\forall x, x' \in X, x \leq x' \implies f(x) \preceq f(x')$ . Note that because of the way  $L_X$  and  $L_Y$  are ordered (shown above), given  $1 \leq i, j \leq n$ , if  $i < j$  then we have that  $x_i \leq x_j$  and  $f(x_i) \preceq f(x_j)$ . Furthermore, we can see that given  $x_i \leq x_j$ , then  $i < j$  and similarly given  $f(x_i) \preceq f(x_j)$ , then  $i < j$ . This means that we have shown both of these statements: (i) if  $x_i \leq x_j \implies i < j \implies f(x_i) \preceq f(x_j)$  and (ii) if  $f(x_i) \preceq f(x_j) \implies i < j \implies x_i \leq x_j$ . Thus we have shown that  $\forall x, x' \in X, x \leq x' \iff f(x) \preceq f(x')$ . Because we have shown for any two  $n$ -element linearly ordered sets we can create a bijection  $f : X \rightarrow Y$  where  $\forall x, y \in X, x \leq y \iff f(x) \preceq f(y)$ , we have proven that any two  $n$ -element linearly ordered sets are isomorphic.

## 5. Section 2.2, Question 2

- a) We first define  $<$  to be the divisibility relation  $|$ , set  $B = \{1, 2, \dots, n\}$ , and the longest possible subset of  $B$  linearly ordered by  $|$  as set  $A$ , where  $m = |A|$ . Because  $A$  is linearly ordered, this means that  $\forall x, y \in A$ , either  $x \leq y$  or  $y \leq x$ . This means that if set  $A$ , when ordered from least to greatest, is given as  $a_1, a_2, \dots, a_k$  then we must have that  $\forall 2 \leq i \leq k, a_{i-1}$  must be able to divide  $a_i$ . Note that this guarantees  $\forall 1 \leq j < i \leq k, a_i$  can divide any  $a_j$  because:

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<sup>2</sup>in increasing order

$$\frac{a_i}{a_j} = \frac{a_i}{a_{i-1}} \frac{a_{i-1}}{a_{i-2}} \cdots \frac{a_{j+1}}{a_j}$$

which is just a product of natural numbers and thus is a natural number  $\implies a_i$  can divide  $a_j$ . Our goal now is to construct  $A \subseteq B$ . To start, one must be in  $A$  (as its minimal element) as every natural number can be divided by  $A$ . Next to try to include in  $A$  as many elements of  $B$  as possible, we should aim for the  $\forall 1 \leq j < i \leq n, \frac{a_i}{a_j} \in \mathbb{N}$  ratio to be as small as possible. This ratio cannot be 1, as each number appears only once in  $B$ . Thus, the next greatest natural number this ratio can be is two. This means that  $2^{m-1} = |B| = n$  or  $m = \log_2(n) + 1$ . Note that the  $+1$  term here is to account for the inclusion of 1 into set  $A$ .

- b) We define the set  $B = 2^{\{1,2,\dots,n\}}$  and the longest possible subset of  $B$  linearly ordered by  $\subseteq$  as set  $A$ , where  $m = |A|$ . Because  $A$  is linearly ordered, this means that  $\forall x, y \in A$ , either  $x \subseteq y$  or  $y \subseteq x$ . This means that if set  $A$ , when ordered from least to greatest, is given as  $a_1, a_2, \dots, a_k$  then we must have that  $\forall 2 \leq i \leq k, a_{i-1} \subseteq a_i$ . This guarantees  $\forall 1 \leq j < i \leq k, a_j \subseteq a_i$  as:

$$a_j \subseteq a_{j+1} \subseteq a_{j+2} \cdots \subseteq a_i$$

Our goal now is to construct  $A \subseteq B$ . To start,  $\emptyset$  must be in  $A$  (as its minimal element) as any  $\emptyset \subseteq$  any set. Next to try to include in  $A$  as many of elements of  $B$ , we should aim for  $\forall 2 \leq i \leq n, a_i/a_{i-1}$  to be as small as possible. We can make  $|a_i/a_{i-1}| = 1$  if every consecutive element includes one more element than the last. It will take  $n$  elements to go from a set with only one element to one with all  $n$  elements (i.e. set  $\{1, 2, \dots, n\}$ ). Therefore the maximum number of elements required to make this chain is  $n + 1$ . I show a visualization below of these  $n + 1$  elements that can build this set  $A$ :

$$A = \{\emptyset, \underbrace{\{1\}, \{1, 2\}, \dots, \{1, 2, \dots, n\}}_{n+1 \text{ elements}}\}$$

and so  $m = |A| = n + 1$ .

## 6. Section 2.2, Question 3: Optional Bonus Problem

- a) We prove both directions of this statement:

### 1. $\text{le}(X, \preceq) = 1 \implies (X, \preceq)$ is a linear ordering

We prove this statement by contrapositive and assume that  $(X, \preceq)$  is not a linear ordering  $\implies \exists x_i, x_j \in X$  s.t.  $x_i \not\preceq x_j$  and  $x_j \not\preceq x_i$ . This means that we can assemble at least two linear orderings for  $(X, \preceq)$ : one in which  $x_i \preceq x_j$  and one in which  $x_j \preceq x_i$ . This means  $\text{le}(X, \preceq) \neq 1$ . Thus, we have proved this statement by contrapositive.

2.  $(X, \preceq)$  is a linear ordering  $\implies \text{le}(X, \preceq) = 1$

Let us define a linear extension  $<$  of  $\preceq$ : this means  $\forall x, x' \in X$ , if  $x \preceq x' \implies x \leq x'$ . Because  $\preceq$  is already a linear ordering, this means  $\forall x, x' \in X$ ,  $x \preceq x'$  or  $x' \preceq x$ . This means this total ordering  $<$  must have the following property:  $\forall x, x' \in X$ ,  $x \leq x'$  if  $x \preceq x'$  or  $x' \leq x$  if  $x' \preceq x$ . Thus,  $<$  is the same ordering as  $\preceq$  and so the number of linear extensions possible for  $\preceq$  is only one (i.e.  $\text{le}(X, \preceq) = 1$ .)

- b) The partial ordering  $\preceq$  which can have the most possible linear extensions is one which imposes the least constraints. Such an ordering would be one where  $\forall x, x' \in X$ ,  $x$  and  $x'$  are not comparable to each other (i.e.  $x \not\preceq x'$  and  $x' \not\preceq x$ ). This is because for such an ordering, a linear extension of this ordering can choose to order the  $n$  elements of  $X$  in any possible way. Because there are  $n!$  ways to order  $n$  elements (i.e.  $X$ ), this means for this such ordering (let's call it  $\preceq$ ),  $\text{le}(X, \preceq) = n!$ . Note that for any other partial ordering it is guaranteed  $\exists x, y \in X$  s.t.  $x \preceq y$  and so any linear extension of this ordering must be compatible to this constraint  $\implies$  this linear extension has  $< n!$  ways to order  $X \implies$  the number of linear extensions of this ordering is  $< n!$ . As such, for any partial ordering  $\preceq$ ,  $\text{le}(X, \preceq) \leq n!$ .