

## Discretionary Note

Anish Krishna Lakkapragada

**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

**CONTENT STARTS ON NEXT PAGE.**

To access the general instructions for this repository head [here](#).

# STATS 242 HW 6

March 5, 2025

---

**Number of late days: 0; Collaborators: Matt Sprintson**

1.

We first compute the first moment  $\mu_1$  of  $X \sim \text{Geom}(p)$ , which is given by  $\mu_1 = \mathbb{E}[X] = \frac{1}{p}$ . We now solve for the MoM estimator  $\hat{p}$  where the observed sample moment  $\hat{\mu}_1 = \frac{1}{n}(X_1 + \dots + X_n) = \bar{X}$  is equal to theoretical moment  $\mu_1$  with the estimated  $\hat{p}$  parameter:

$$\begin{aligned}\hat{\mu}_1 = \bar{X} &= \frac{1}{\hat{p}} \\ \hat{p} &= \frac{1}{\bar{X}}\end{aligned}$$

Thus,  $\hat{p} = \frac{1}{\bar{X}}$  is the MoM estimator  $\hat{p}$  for  $p$ . We now compute the MLE of  $p$ . We first start by finding the log-likelihood function  $\ell_n(p)$ :

$$\begin{aligned}\ell_n(p) = \log[\text{lik}(p)] &= \sum_{i=1}^n \log f(X_i|p) = \sum_{i=1}^n \log(p(1-p)^{x_i-1}) = \sum_{i=1}^n \log(p) + (x_i - 1)\log(1-p) \\ &= n\log(p) + \sum_{i=1}^n (x_i - 1)\log(1-p)\end{aligned}$$

We now solve for the MLE  $\hat{p}$  by solving  $\ell'_n(\hat{p}) = 0$ :

$$\begin{aligned}
\ell'_n(\hat{p}) &= 0 \\
(n \log(\hat{p}) + \sum_{i=1}^n (x_i - 1) \log(1 - \hat{p}))' &= 0 \\
\frac{n}{\hat{p}} - \sum_{i=1}^n \frac{(x_i - 1)}{1 - \hat{p}} &= 0 \\
\frac{n}{\hat{p}} - \frac{1}{1 - \hat{p}} (n\bar{X} - n) &= 0 \\
\frac{n(1 - \hat{p}) - \hat{p}(n\bar{X} - n)}{\hat{p}(1 - \hat{p})} &= 0 \\
n(1 - \hat{p}) - \hat{p}(n\bar{X} - n) &= 0 \\
n - n\hat{p} - \hat{p}n\bar{X} + n\hat{p} &= 0 \\
n &= n\hat{p}\bar{X} \\
1 &= \hat{p}\bar{X} \\
\hat{p} &= \frac{1}{\bar{X}}
\end{aligned}$$

Thus the MLE of  $p$  is  $\hat{p} = \frac{1}{\bar{X}}$ . We now find the sampling distribution of the MLE of  $p$ , under large  $n$ . We begin by computing the Fisher Information  $I(p)$ :

$$\begin{aligned}
I(p) &= -\mathbb{E}_p \left[ \frac{\partial^2}{\partial p^2} \log f(X|p) \right] = -\mathbb{E}_p \left[ \frac{\partial^2}{\partial p^2} (\log(p) + (x-1)\log(1-p)) \right] \\
&= -\mathbb{E}_p \left[ \frac{\partial}{\partial p} \left( \frac{1}{p} + (1-x)\frac{1}{1-p} \right) \right] = -\mathbb{E}_p \left[ -\frac{1}{p^2} + \frac{(1-x)}{(1-p)^2} \right] = -\left( -\frac{1}{p^2} + \frac{1}{(1-p)^2} - \frac{\mathbb{E}_p[x]}{(1-p)^2} \right) \\
&= \frac{1}{p^2} - \frac{1}{(1-p)^2} + \frac{1}{p(1-p)^2} = \frac{(1-p)^2 - p^2 + p}{p^2(1-p)^2} = \frac{1-p}{p^2(1-p)^2} = \frac{1}{p^2(1-p)}
\end{aligned}$$

So this means that for the MLE  $\hat{p}$ ,  $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, \frac{1}{I(p)})$  or  $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, p^2(1-p))$  meaning  $\hat{p} \rightarrow \mathcal{N}(p, \frac{p^2(1-p)}{n})$  as  $n \rightarrow \infty$ .

2.

We first compute the MoM estimate of  $p$ . To do so, we compute the first moment  $\mu_1$  of  $X \sim \text{NegBinom}(r, p)$  which is given by  $\mu_1 = \mathbb{E}[X] = \frac{pr}{1-p}$ . We now solve for the MoM estimator  $\hat{p}$  where the observed sample moment  $\hat{\mu}_1 = \frac{1}{n}(X_1 + \dots + X_n) = \bar{X}$  is equal to theoretical moment  $\mu_1$  with the estimated  $\hat{p}$  parameter:

$$\begin{aligned}
\hat{\mu}_1 &= \bar{X} = \frac{\hat{p}r}{1 - \hat{p}} \\
\bar{X}(1 - \hat{p}) &= \hat{p}r \\
\bar{X} &= \hat{p}\bar{X} + \hat{p}r \\
\bar{X} &= \hat{p}(\bar{X} + r) \\
\hat{p} &= \frac{\bar{X}}{\bar{X} + r}
\end{aligned}$$

Thus the MoM estimator of  $p$  is given by  $\hat{p} = \frac{\bar{X}}{\bar{X} + r}$ . We now compute the MLE of  $p$ . We first start by finding the log-likelihood function  $\ell_n(p)$ :

$$\begin{aligned}
\ell_n(p) &= \log[\text{lik}(p)] = \sum_{i=1}^n \log f(x_i|p) = \sum_{i=1}^n \log \left( \binom{x_i + r - 1}{x_i} (1-p)^r p^{x_i} \right) \\
&= \sum_{i=1}^n \log \left( \binom{x_i + r - 1}{x_i} \right) + r \log(1-p) + x_i \log(p) = nr \log(1-p) + \sum_{i=1}^n \log \left( \binom{x_i + r - 1}{x_i} \right) + x_i \log(p)
\end{aligned}$$

We now solve for the MLE  $\hat{p}$  by solving  $\ell'_n(\hat{p}) = 0$ :

$$\begin{aligned}
\ell'_n(\hat{p}) &= 0 \\
(nr \log(1 - \hat{p}) + \sum_{i=1}^n \log \left( \binom{x_i + r - 1}{x_i} \right) + x_i \log(\hat{p}))' &= 0 \\
\frac{-nr}{1 - \hat{p}} + \sum_{i=1}^n \frac{x_i}{\hat{p}} &= 0 \\
-\frac{nr\hat{p}}{\hat{p}(1 - \hat{p})} + \frac{n\bar{X}(1 - \hat{p})}{\hat{p}(1 - \hat{p})} &= 0 \\
\frac{n\bar{X}(1 - \hat{p}) - nr\hat{p}}{\hat{p}(1 - \hat{p})} &= 0 \\
n\bar{X}(1 - \hat{p}) - nr\hat{p} &= 0 \\
\bar{X}(1 - \hat{p}) - r\hat{p} &= 0 \\
\bar{X} - \hat{p}\bar{X} - \hat{p}r &= 0 \\
\bar{X} &= \hat{p}(\bar{X} + r) \\
\hat{p} &= \frac{\bar{X}}{\bar{X} + r}
\end{aligned}$$

Thus the MLE of  $p$  is given by  $\hat{p} = \frac{\bar{X}}{\bar{X} + r}$ . We now find the sampling distribution of the MLE of  $p$ , under large  $n$ . We begin by computing the Fisher Information  $I(p)$ :

$$\begin{aligned}
I(p) &= \text{Var}_p\left[\frac{\partial}{\partial p} \log f(x|p)\right] = \text{Var}_p\left[\frac{\partial}{\partial p} \left[\log\left(\binom{x+r-1}{x}\right) + r\log(1-p) + x\log(p)\right]\right] \\
&= \text{Var}_p\left[\frac{-r}{1-p} + \frac{x}{p}\right] = \frac{1}{p^2} \text{Var}_p[x] = \frac{r}{p(1-p)^2}
\end{aligned}$$

So this means that for the MLE  $\hat{p}$ ,  $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, \frac{1}{I(p)})$  or  $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, \frac{p(1-p)^2}{r})$  so  $\hat{p} \rightarrow \mathcal{N}(p, \frac{p(1-p)^2}{nr})$ .

3.

- (a) The PDF of  $\text{Pareto}(\alpha, x_m)$  is given by  $f(x|\alpha, x_m) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$ . Thus, the PDF of  $\text{Pareto}(\theta, 1)$  is given by:

$$f(x|\theta, 1) = \frac{\theta \cdot 1^\theta}{x^{\theta+1}} = \frac{\theta}{x^{\theta+1}}$$

Expressed differently, we can set functions  $T(x) = \log(\frac{1}{x})$ ,  $A(\theta) = \log(\frac{1}{\theta})$ ,  $h(x) = \frac{1}{x}$  so the PDF

$$f(x|\theta) = e^{\theta T(x) - A(\theta)} h(x) = \frac{e^{\theta T(x)}}{x e^{A(\theta)}} = \frac{(e^{\log(\frac{1}{x})})^\theta}{x e^{\log \frac{1}{\theta}}} = \frac{1}{x^\theta \cdot x \cdot \frac{1}{\theta}} = \frac{\theta}{x^{\theta+1}}$$

is identical to the aforementioned derived  $\text{Pareto}(\theta, 1)$  PDF.

- (b) We differentiate both sides of the identity  $1 = \int_{\mathcal{X}} f(x|\theta) dx$  to compute  $\mathbb{E}_\theta[T(X)]$ :

$$\begin{aligned}
&\int_{\mathcal{X}} f(x|\theta) dx = 1 \\
\frac{d}{d\theta} \int_{\mathcal{X}} f(x|\theta) dx &= \int_{\mathcal{X}} \frac{d}{d\theta} f(x|\theta) dx = \frac{d}{d\theta}(1) = 0 \\
&\int_{\mathcal{X}} \frac{d}{d\theta} [e^{\theta T(x) - A(\theta)} h(x)] dx = 0 \\
\int_{\mathcal{X}} [(e^{\theta T(x) - A(\theta)})' h(x) + h'(x) e^{\theta T(x) - A(\theta)}] dx &= \int_{\mathcal{X}} [(e^{\theta T(x) - A(\theta)})(T(x) - A'(\theta)) h(x) + 0] dx = 0 \\
&\int_{\mathcal{X}} [e^{\theta T(x) - A(\theta)} h(x)] (T(x) - A'(\theta)) dx = 0 \\
\int_{\mathcal{X}} f(x|\theta) (T(x) - A'(\theta)) dx &= \mathbb{E}_\theta[T(x) - A'(\theta)] = 0 \\
\mathbb{E}_\theta[T(x)] &= \mathbb{E}_\theta[A'(\theta)] = A'(\theta)
\end{aligned}$$

So we have  $\mathbb{E}_\theta[T(x)] = A'(\theta)$ . We now verify this is the case for our Pareto( $\theta, 1$ ) model with its defined functions  $T(x) = \log(\frac{1}{x})$ ,  $A(\theta) = \log(\frac{1}{\theta})$ ,  $h(x) = \frac{1}{x}$  in part (a). We first compute  $\mathbb{E}_\theta[T(x)]$ :

$$\mathbb{E}_\theta[T(x)] = \mathbb{E}_\theta[\log(\frac{1}{x})] = \int_{-\infty}^{\infty} f(x|\theta) \log(\frac{1}{x}) dx = \int_1^{\infty} \frac{\theta}{x^{\theta+1}} \log(\frac{1}{x}) dx$$

We can use a u-substitution of  $u = \frac{1}{x}$  so  $-u^{\theta-1} du = -\frac{x}{x^\theta} \cdot -\frac{1}{x^2} dx = \frac{1}{x^{\theta+1}} dx$ . Thus, this integral can be re-written as:

$$\int_1^0 \log(u) \cdot \theta \cdot (-u^{\theta-1}) du = \theta \int_0^1 \log(u) \cdot u^{\theta-1} du$$

Using integration by parts, we can set  $a = \log(u)$ ,  $da = \frac{1}{u} du$ ,  $db = u^{\theta-1} du$ ,  $b = \frac{1}{\theta} u^\theta$ . As such, we get:

$$\begin{aligned} \theta \int_0^1 \log(u) \cdot u^{\theta-1} du &= \theta [ab \Big|_{u=0}^{u=1} - \int_0^1 b da] \\ &= \theta \left[ \frac{\log(u) \cdot u^\theta}{\theta} \right] \Big|_0^1 - \theta \int_0^1 \frac{1}{\theta} u^{\theta-1} du = 0 - \int_0^1 u^{\theta-1} du = -\frac{1}{\theta} u^\theta \Big|_0^1 = -\frac{1}{\theta} + 0 = -\frac{1}{\theta} \end{aligned}$$

So  $\mathbb{E}_\theta[T(x)] = -\frac{1}{\theta}$ . We now compute  $A'(\theta)$ :

$$A'(\theta) = \frac{1}{\frac{1}{\theta}} \cdot \frac{-1}{\theta^2} = \frac{-\theta}{\theta^2} = -\frac{1}{\theta}$$

So because  $\mathbb{E}_\theta[T(x)] = -\frac{1}{\theta} = A'(\theta)$ , we have verified for our Pareto model in part (a) that  $\mathbb{E}_\theta[T(x)] = A'(\theta)$ .

(c) We first find the log-likelihood function  $\ell_n(\theta)$ :

$$\begin{aligned} \ell_n(\theta) &= \log[\text{lik}(\theta)] = \sum_{i=1}^n \log f(x_i|\theta) = \sum_{i=1}^n \log[e^{\theta T(x_i) - A(\theta)} h(x_i)] = \\ &= \sum_{i=1}^n \log[e^{\theta T(x_i)}] - \log[e^{A(\theta)}] + \log[h(x_i)] = -nA(\theta) + \sum_{i=1}^n \theta T(x_i) + \log[h(x_i)] \end{aligned}$$

To get the MLE of  $\theta$  we solve for the solution to  $0 = \ell'_n(\theta)$ :

$$\begin{aligned} \ell'_n(\theta) &= 0 \\ (-nA(\theta) + \sum_{i=1}^n \theta T(x_i) + \log[h(x_i)])' &= 0 \\ -nA'(\theta) + \sum_{i=1}^n T(x_i) &= 0 \\ nA'(\theta) &= \sum_{i=1}^n T(x_i) \\ A'(\theta) &= \frac{1}{n} \sum_{i=1}^n T(x_i) \\ \mathbb{E}_\theta[T(x)] = A'(\theta) &= \frac{1}{n} \sum_{i=1}^n T(x_i) \end{aligned}$$

So therefore the MLE of  $\theta$  is the unique solution to  $\mathbb{E}_\theta[T(x)] = \frac{1}{n}\sum_{i=1}^n T(x_i)$ . Because the generalized MoM estimator of  $\theta$  is also the solution to  $\mathbb{E}_\theta[T(x)] = \frac{1}{n}\sum_{i=1}^n T(x_i)$ , and we are guaranteed this solution is unique, the generalized MoM estimator of  $\theta$  is equivalent to the MLE of  $\theta$ .

An example of this occurring would be the generalized MoM estimator based on  $T(x) = \log(x)$  for the  $\text{Pareto}(\theta, 1)$  model coinciding with the MLE.

4.

- (a) By the CLT,  $\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, p(1 - p))$ . Informally, this gives us that  $\text{Var}[\sqrt{n}(\hat{p} - p)] \approx \hat{p}(1 - \hat{p}) \implies n\text{Var}(\hat{p} - p) \approx \hat{p}(1 - \hat{p}) \implies \text{Var}(p) \approx \frac{\hat{p}(1 - \hat{p})}{n}$ . Using this approximation for the variance of  $p$ , we can compute a 95% confidence interval for  $p$  as  $\hat{p} \pm z^{\frac{0.05}{2}} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$  where  $z^{\frac{0.05}{2}}$  is the upper- $\frac{0.05}{2}$  point of the standard normal distribution.
- (b) We solve  $\sqrt{n}(\hat{p} - p) = \pm \sqrt{p(1 - p)}z^{(\alpha/2)}$  below to create a confidence interval for  $p$ :

$$\begin{aligned}\sqrt{n}(\hat{p} - p) &= \pm \sqrt{p(1 - p)}z^{(\alpha/2)} \\ n(\hat{p} - p)^2 &= p(1 - p)(z^{(\alpha/2)})^2 \\ n\hat{p}^2 - 2n\hat{p}p + np^2 &= (z^{(\alpha/2)})^2 p - (z^{(\alpha/2)})^2 p^2 \\ (n + (z^{(\alpha/2)})^2)p^2 - (2n\hat{p} + (z^{(\alpha/2)})^2)p + n\hat{p}^2 &= 0\end{aligned}$$

Using the quadratic formula, we construct the following confidence interval for  $p$ :

$$p = \frac{(2n\hat{p} + (z^{(\alpha/2)})^2) \pm \sqrt{(2n\hat{p} + (z^{(\alpha/2)})^2)^2 - 4n\hat{p}^2(n + (z^{(\alpha/2)})^2)}}{2(n + (z^{(\alpha/2)})^2)}$$

- (c) For our confidence interval in part (a), we report the simulated coverage probabilities below:

	$n = 10$	$n = 40$	$n = 100$
$p = 0.1$	0.65061	0.91435	0.93267
$p = 0.3$	0.83926	0.92955	0.94984
$p = 0.5$	0.8922	0.91919	0.94205

We similarly report the simulated coverage probabilities for our confidence interval in part (b):

	$n = 10$	$n = 40$	$n = 100$
$p = 0.1$	0.93053	0.94406	0.9375
$p = 0.3$	0.92463	0.94432	0.93659
$p = 0.5$	0.97861	0.961	0.94316

We can clearly see from comparing these two tables that our interval construction in part (b) leads to generally much higher simulated coverage probabilities.

```

1      # %%
2  import numpy as np
3  from scipy.stats import norm
4

```



```

5 z_alpha = np.abs(norm.ppf(0.05/2))
6 z_alpha_squared = z_alpha ** 2
7
8 NUM_SIMULATIONS = 100 * 1000
9 SAMPLE_SIZES = [10, 40, 100]
10 TRUE_PARAMS = [0.1, 0.3, 0.5]
11
12 def generate_data(n,p):
13     return np.random.binomial(n=n, p=p) # number of successes
14
15 def part_a_interval(num_successes, n):
16     p_hat = num_successes / n
17     delta = z_alpha * np.sqrt(p_hat * (1 - p_hat) / n)
18     return p_hat - delta, p_hat + delta
19
20 def part_b_interval(num_successes, n):
21     p_hat = num_successes / n
22     denominator = 2 * (n + z_alpha_squared)
23     center = (2 * n * p_hat + z_alpha_squared) / denominator
24     delta = np.sqrt((2 * n * p_hat + z_alpha_squared) ** 2 - 4 * n *
25     ↪ (p_hat ** 2) * (n + z_alpha_squared))
26     delta /= denominator
27     return center - delta, center + delta
28
29 # %%
30
31 for n in SAMPLE_SIZES:
32     for p in TRUE_PARAMS:
33         NUM_COVERAGE_A = 0
34         for _ in range(NUM_SIMULATIONS):
35             n_successes = generate_data(n, p)
36             lower, upper = part_a_interval(n_successes, n)
37
38             if p >= lower and p <= upper: NUM_COVERAGE_A += 1
39
40     print(f"[PART (A)] For sample size n={n} and p={p}, simulated
41     ↪ coverage probability: {NUM_COVERAGE_A / NUM_SIMULATIONS}")
42
43
44 for n in SAMPLE_SIZES:
45     for p in TRUE_PARAMS:
46         NUM_COVERAGE_B = 0
47         for _ in range(NUM_SIMULATIONS):
48             n_successes = generate_data(n, p)
49             lower, upper = part_b_interval(n_successes, n)

```

```
50         if p >= lower and p <= upper: NUM_COVERAGE_B += 1
51
52
53     print(f"[PART (B)] For sample size n={n} and p={p}, simulated
54           ↪ coverage probability: {NUM_COVERAGE_B / NUM_SIMULATIONS}")
55
56 # %%
```