

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

**CONTENT STARTS ON NEXT PAGE.**

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# MATH 255 PSET 6

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1.

Pick  $n \in \mathbb{N}$ . We can construct  $S'_n = \{N_{\frac{1}{n}}(x) : x \in K\} \supset K$  as an open cover of  $K$ . Because  $K$  is compact, every open cover of  $K$  has a finite subcover. So open cover  $S'_n$  has a subcover which we can define as:  $S_n = \{N_{\frac{1}{n}}(x_i^{(n)}) : i = 1 \dots m_n\} \subset S'_n$  where  $m_n$  is the number of points (in  $K$ ) required for  $S_n$  to be an open cover of  $K$ . We can then define the subset  $C' = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{m_n} x_j^{(n)}$  which is the union of all the required points for each finite subcover  $S_n$  to cover  $K$ . Because  $C'$  is a countable union of finite sets<sup>1</sup>,  $C'$  is at most countable. Furthermore,  $\forall x \in C', x \in K \implies C' \subset K$ .

We now show that  $C'$  is a *dense* subset of  $K$ . To do so, we show  $\forall x \in K, x$  is either in  $C'$  or  $x$  is a limit point of  $C'$ . We prove this occurs with casework:

1. **Case One:**  $x \in C'$

In this case, our job is done.

2. **Case Two:**  $x \notin C'$

In this case, we WTS  $x$  is a limit point of  $C'$  or that  $\forall \epsilon > 0, \exists p \in N_{\epsilon}(x)$  s.t.  $p \neq x$  and  $p \in C'$ . Pick  $\epsilon > 0$ . Note that  $\forall n \in \mathbb{N}, C'$  contains all the points which neighborhoods with size  $\frac{1}{n}$  will cover  $K$ . By the Archimidean property,  $\exists n \in \mathbb{N}$  s.t.  $n(1/n) = n > \frac{1}{\epsilon} \implies \exists n$  s.t.  $\frac{1}{n} < \epsilon$ . We proceed with this value of  $n$ . Because  $S_n$  is an open cover of  $K$ ,  $x \in K \implies x$  is contained in some set<sup>2</sup> of  $S_n \implies \exists 1 \leq k \leq m_n$  s.t.  $x \in N_{\frac{1}{n}}(x_k^{(n)})$  where  $x_k^{(n)} \in C'$  and  $x \neq x_k^{(n)}$  (given by  $x \notin C'$ ). Thus,  $d(x, x_k^{(n)}) < \frac{1}{n} \implies N_{\frac{1}{n}}(x)$  contains some  $x_k^{(n)} \in C'$ . Because  $\frac{1}{n} < \epsilon$ ,  $N_{\frac{1}{n}}(x) \subset N_{\epsilon}(x)$  and so  $N_{\epsilon}(x)$  contains some  $x_k^{(n)} \in C'$  where  $x \neq x_k^{(n)}$ . Thus,  $\forall \epsilon > 0, N_{\epsilon}(x)$  contains some  $p \in C'$  s.t.  $p \neq x$ . So  $x$  is a limit point of  $C'$ .

2.

Let  $\{G_i\}$  be an open cover of  $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ . This means that  $\exists$  some open set  $G_j \in \{G_i\}$  s.t.  $x \in G_j$ . Because  $G_j$  is open, all points of  $G_j$  are interior points of  $G_j \implies x$  is an interior point of  $G_j \implies \exists \epsilon > 0$  s.t.  $N_{\epsilon}(x) \subset G_j$ . Because  $(x_n) \rightarrow x \implies \exists N$

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<sup>1</sup>For clarity, the  $n$ th finite set is given by  $\{x_1^{(n)}, \dots, x_{m_n}^{(n)}\}$ .

<sup>2</sup>i.e. a neighborhood

s.t.  $\forall n \geq N, d(x_n, x) < \epsilon$ . Thus, this means that  $N_\epsilon(x)$  will contain  $x$  and  $x_N, x_{N+1}, \dots$ . Because  $N_\epsilon(x) \subset G_j$ , this means  $x$  and  $x_N, x_{N+1}, \dots$  are contained in  $G_j$ . Now for each of the finitely many points  $x_1, \dots, x_{N-1}$  (all of which are contained in  $\{G_i\}$ ), we can pick a given set in  $\{G_i\}$  which contains this point. Let  $G_{n_k} \in \{G_i\}$  be the set which contains the  $k$ th point  $x_k$  where  $1 \leq k \leq N-1$ . Then  $G' = G_j \cup \bigcup_{i=1}^{N-1} G_{n_k}$  covers  $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ . Because  $G' \subset \{G_i\} \implies G'$  is a finite subcover of  $\{G_i\}$ . Thus we have shown all open covers of  $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$  have a finite subcover  $\implies \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$  is compact.

3.

We prove that  $\lim_{n \rightarrow \infty} \frac{2n+1}{3n-1} = \frac{2}{3}$  by showing that the sequence  $(p_n) \rightarrow \frac{2}{3}$  in metric space  $\mathbb{R}$  where  $p_n = \frac{2n+1}{3n-1}$ . Pick  $\epsilon > 0$ . We now aim to find  $N \in \mathbb{N}$  s.t.  $\forall n \geq N, d(p_n, \frac{2}{3}) < \epsilon$  or expressed more simply<sup>3</sup>, we aim to find  $N \in \mathbb{N}$  s.t.  $\forall n \geq N, d(p_n, \frac{2}{3}) = d(\frac{2n+1}{3n-1}, \frac{2}{3}) = |\frac{2n+1}{3n-1} - \frac{2}{3}| = |\frac{3(2n+1)-2(3n-1)}{3(3n-1)}| = |\frac{5}{3(3n-1)}| < \epsilon$ . We solve the  $|\frac{5}{3(3n-1)}| < \epsilon$  inequality for  $n$  below:

$$|\frac{5}{3(3n-1)}| < \epsilon$$

Because  $n \in \mathbb{N} \implies n \geq 1 \implies \frac{5}{3(3n-1)} > 0 \implies |\frac{5}{3(3n-1)}| = \frac{5}{3(3n-1)}$  and so we can proceed removing the absolute value term:

$$\begin{aligned} \frac{5}{3(3n-1)} &< \epsilon \\ 5 &< \epsilon(9n-3) \\ \frac{5}{\epsilon} &< 9n-3 \\ n &> \frac{1}{9}(\frac{5}{\epsilon} + 3) \end{aligned}$$

Thus, we see  $d(p_n, \frac{2}{3}) < \epsilon$  for any  $n > \frac{1}{9}(\frac{5}{\epsilon} + 3)$ . Thus, we aim to choose for  $N$  any natural number  $> \frac{1}{9}(\frac{5}{\epsilon} + 3)$ . By Archimidean property,  $\exists m \in \mathbb{N}$  s.t.  $m(1) > \frac{1}{9}(\frac{5}{\epsilon} + 3)$  and so we can simply take this  $m$  to be our choice of  $N$ . Thus we have shown  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N, d(p_n, \frac{2}{3}) < \epsilon \implies (p_n) \rightarrow \frac{2}{3} \implies \lim_{n \rightarrow \infty} \frac{2n+1}{3n-1} = \frac{2}{3}$ .

4.

**Lemma 0.1** *Let  $x, y \in \mathbb{R}$ . We will prove  $||x| - |y|| \leq |x - y|$ . By Triangle Inequality, we know  $|x + y| \leq |x| + |y|$  and thus we can show these two facts:*

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<sup>3</sup>Because we are operating in the metric space  $\mathbb{R}$  with the standard distance function,  $d(x, y) = |x - y|$ .

1. By Triangle Inequality, we know  $|x| + |y - x| \geq |x + y - x|$  and so we have:

$$\begin{aligned} |x| + |y - x| &\geq |x + y - x| \\ |y - x| &\geq |y| - |x| \\ |x - y| &\geq |y| - |x| \end{aligned}$$

2. By Triangle Inequality, we know  $|y| + |x - y| \geq |y + x - y|$  and so we have:

$$\begin{aligned} |y| + |x - y| &\geq |y + x - y| \\ |x - y| &\geq |x| - |y| \end{aligned}$$

Thus, we know the two facts:  $|x - y| \geq |y| - |x|$  and  $|x - y| \geq |x| - |y|$  which together imply  $|x - y| \geq \pm(|x| - |y|) \implies |x - y| \geq ||x| - |y||$ .

We WTS sequence  $|x_n|$  will converge to  $|x|$ . Pick  $\epsilon > 0$ . To show  $(|x_n|) \rightarrow |x|$ , we must find some  $N \in \mathbb{N}$  s.t.  $\forall n \geq N, d(|x_n|, |x|) = ||x_n| - |x|| < \epsilon$ . Because  $(x_n) \rightarrow x \implies \exists M \in \mathbb{N}$  s.t.  $\forall n \geq M, d(x_n, x) < \epsilon \implies \forall n \geq M, |x_n - x| < \epsilon \implies$  by **(Lemma 0.1)**  $\forall n \geq M, ||x_n| - |x|| \leq |x_n - x| < \epsilon \implies \forall n \geq M, d(|x_n|, |x|) = ||x_n| - |x|| < \epsilon$ . Thus, we can simply set  $N = M$  and so we have shown  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N, d(|x_n|, |x|) < \epsilon$ . This proves  $|x_n| \rightarrow |x|$ .

We now show that the converse is not true. Let us define sequence  $(x_n)$  in metric space  $\mathbb{R}$  where  $x_n = -1$ . Because every element in this sequence is equal to  $-1$ ,  $(x_n) \rightarrow -1$ . We can now define sequence  $(y_n)$  where  $y_n = |x_n| = |-1| = 1$ . Because every element in  $(y_n)$  is equal to  $1$ ,  $(y_n) \rightarrow 1$ . Expressed differently,  $y_n = |x_n| \rightarrow |1|$ . So we have found a case where  $|x_n| \rightarrow |1|$  but  $x_n \not\rightarrow 1$  and thus we have disproved the converse of this statement.