Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head here.

Math 226: HW 5
Completed By: Anish Lakkapragada (NETID: al2778)

1. a) We can define $[T]_\alpha$ below as:

$$T_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4x4}$$

b) **Answers for** $[U]_\beta^\gamma, [T]_\beta, [UT]_\beta^\gamma$

$$[U]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}_{3x3} \quad [T]_\beta = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}_{3x3} \quad [UT]_\beta^\gamma = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}_{3x3}$$

Answers for $[h(x)]_\beta$ **and** $[U(h(x))]_\gamma$

$$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} [U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

c)

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{n \ x \ n}$$

2. a) For T^{-1} to be a linear operator, then $T^{-1}(cx + y) = cT^{-1}(x) + T^{-1}(y)$ where $c \in \mathbb{F}$ and $x, y \in V$.

Let us define $u_1, u_2 \in U$ and $v_1 = T(u_1), v_2 = T(u_2) \in V$. We also define $c \in \mathbb{F}$. We evaluate $T^{-1}(cv_1 + v_2)$ below:

$$T^{-1}(cv_1 + v_2) = T^{-1}(cT(u_1) + T(u_2))$$
$$T^{-1}(cv_1 + v_2) = T^{-1}(T(cu_1 + u_2))$$
$$T^{-1}(cv_1 + v_2) = cu_1 + u_2$$

We evaluate $cT^{-1}(v_1) + T^{-1}(v_2)$:

$$cT^{-1}(v_1) + T^{-1}(v_2) = cu_1 + u_2$$

Because we have shown $T^{-1}(cv_1 + v_2) = cT^{-1}(v_1) + T^{-1}(v_2) = cu_1 + u_2$, we have shown that T^{-1} is a linear operator.

b) Let us define $(x, y) = xe_1 + ye_2 \in \mathbb{R}^2$, where $\beta_2 = \{e_1, e_2\}$ is the standard ordered basis for $\mathbb{R}^2$. We will also define the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ through the following matrix:

$$[T]_{\beta_2}^{\beta_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a, b, c, d \in \mathbb{R}$. We inspect the value of $T(x, y)$ below.

$$T(x, y) = [T]_{\beta_2}^{\beta_2} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$T(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$T(x, y) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Thus, we get that $T(x, y) = (ax + by, cx + dy) \in \mathbb{R}^2$.

We now prove by contradiction that for T to be a valid isomorphism, and thus bijective, $ad - bc \neq 0$. Suppose $ad - bc = 0$. This means that $\frac{b}{a} = \frac{d}{c}$. Because any input with the form $(x, -\frac{a}{b}x)$ will be part of $N(T)$, we have found multiple pre-images in U to the same image (i.e. $\mathbf{0}^2$) in $V \Rightarrow T$ is not injective $\Rightarrow T$ is not bijective $\Rightarrow T$ is not a valid isomorphism. Thus, we have proven that for T to be a valid isomorphism, $ad - bc \neq 0$ so that $N(T) = \{0\} \Rightarrow T$ is injective $\Rightarrow T$ is surjective $\Rightarrow T$ is invertible $\Rightarrow T$ isomorphism.

3. a) For proof by contrapositive, let us assume T is not injective. This means $\exists\, x, y \in V$ s.t. $T(x) = T(y)$ and $x \neq y$. Let us define $x \neq y$, $x' = T(x) \in W$, and $y' = T(y) \in W$ where $x' = y'$. We prove that $UT(x) = UT(y)$ below.

$$UT(x) = UT(y)$$
$$U(T(x)) = U(T(y))$$
$$U(x') = U(y')$$
$$U(x') = U(x')$$
$$0 = 0$$

Thus, we have shown $\exists\ x, y \in V$ s.t. $UT(x) = UT(y)$ and $x \neq y \Rightarrow UT$ is not injective. Thus, we have shown if T is not injective, UT is not injective. By proof by contrapositive, we have proved if UT is injective, T is injective.

Does U have to be injective if $\{UT, T\}$ are injective?

Let us define $x, y \in V$ and $w_1 = T(x), w_2 = T(y) \in R(T) \leq W$. Suppose $w_1 \neq w_2$. Let us assume U is not injective. If U is not injective, then it is possible for $U(w_1) = U(w_2)$ even if $w_1 \neq w_2$. This would mean that $U(T(x)) = U(T(y)) \Rightarrow UT(x) = UT(y)$. Because UT is given as injective, if $UT(x) = UT(y)$, then we know that $x = y$. However, because we are considering the case for which $w_1 \neq w_2$ or $T(x) \neq T(y)$, we know that $x \neq y$. Thus, if U is not injective, the property $UT(x) = UT(y) \Rightarrow x = y$ is not necessarily true and so this contradicts our given that UT is injective. Note, however, that it only matters that for U is injective for input $w \in R(T)$. This is because ensuring the injectivity of UT means that U has to be injective only for outputs of T (i.e. $R(T)$). By proof by contradiction, we have proven that if $\{UT, T\}$ are injective, then U must be injective as well over $R(T)$. *However, U does not need to be fully injective.*

b) For proof by contrapositive, let us assume U is not surjective. This means $\exists z \in Z$ s.t. $\nexists\ w \in W$ s.t. $U(w) = z$.[1] If UT is surjective, this means that $\exists v \in V$ s.t. $UT(v) = z$. In other words, this means that $U(T(v)) = z$, or that $\exists\ w \in R(T) \leq W$ s.t. $U(w) = z$. However, because we are assuming U is not surjective, $\nexists\ w \in W$ s.t. $U(w) = z \Rightarrow \nexists\ v \in V$ s.t. $UT(v) = z \Rightarrow UT$ is not surjective. Thus, we have shown that if U is not surjective, UT is not surjective. By proof by contrapositive, we have shown that if UT is surjective, then U is surjective.

Does T have to be surjective if $\{UT, U\}$ are surjective?

If T is surjective, that means that $R(T) = W$. For UT and U to be surjective, that means that there needs to exist pre-images in V and W, respectively, that map to every element in Z. However, for transformation U, if every single pre-image of Z in W exists in the subspace $R(T) \leq W$, then surjectivity for U is maintained. Note that this does not affect the surjectivity of UT as the pre-images of UT in V will all be mapped by T to $R(T) \leq W$ by the definition of a range of a transformation. Thus, T only must be surjective for pre-images in $R(T)$ but not every element in W; *T does not have to be fully surjective.*

c) We are given that the matrix AB is invertible $\Rightarrow$ transformation L_{AB} is invertible. We define L_{AB} as the composition of transformations L_A and L_B. Additionally, a transformation is only invertible if it is bijective. Thus, we know that L_{AB} is bijective and so L_{AB} is both surjective and injective. Finally, because we are given the matrix representations for A and B, we can trivially assume L_A and L_B are linear transformations.

We prove below that matrices A and B are invertible.

1 **A is invertible**

From part (b), we know that if transformation $L_{AB} = L_A L_B$ is surjective, then transformation L_A is surjective. From Theorem 2.5, we know that given a linear

[1] We reference an example of this element of Z with this property as z in the remainder of this proof.

transformation T with input and output vector spaces of equal dimensionality, if T is surjective, then T is also injective. Because L_A is a square matrix, we know that it has input and output vector spaces of equal dimensionality. Thus, we know that given L_A is surjective, L_A is also injective. Because L_A is surjective and injective, it is bijective and so L_A is invertible $\Rightarrow$ the corresponding matrix A is invertible.

2 B **is invertible**
From part (a), we know that if transformation $L_{AB} = L_A L_B$ is injective, then transformation L_B is injective. From Theorem 2.5, we know that given a linear transformation T with input and output vector spaces of equal dimensionality, if T is injective, then T is also surjective. Because L_B is a square matrix, we know that it has input and output vector spaces of equal dimensionality. Thus, we know that given L_B is injective, L_B is also surjective. Because L_B is surjective and injective, it is bijective and so L_B is invertible $\Rightarrow$ the corresponding matrix B is invertible.

d) We are given that matrix $AB = I_n$. Because the inverse of the identity matrix is itself, $(AB)^{-1} = I_n$ and so AB is invertible. From part (c), we have proven that if AB is invertible, then *matrix* A *is invertible* and matrix B is invertible.
Let us define the inverse of A as A^{-1}. Because A is a square $n\ x\ n$ matrix, $AA^{-1} = I_n$ which is also equal to AB.
We show this below.

$$AA^{-1} = I_n = AB$$
$$AA^{-1} = AB$$

Because A is invertible and appears in the same position on both sides of the equation, we can remove A to get:

$$A^{-1} = B$$

Thus we have proven $A^{-1} = B$.

4. For proof by contrapositive, we will assume that $\{T, U\}$ are linearly dependent subsets of $\mathcal{L}(V, W)$. This means that given $c_1, c_2 \in \mathbb{F}$, there exists a solution to the equation $c_1 T + c_2 U = 0$ where at least one of $\{c_1, c_2\}$ is not equal to zero. This means that we can re-express T or U as a scalar multiplied by the other linear operator. In other words, given $k \in \mathbb{F}$, $T = kU$.

   Consider $w \neq 0 \in R(U)$.[2] This means that $\exists\ v \in V$ s.t. $U(v) = w$. Because we know $T = kU$, $T(v) = kU(v) = kw \Rightarrow kw \in R(T)$. We now show that $kw \in R(U)$. Because V is a vector space, we know that it is closed under scalar multiplication $\Rightarrow kv \in V$. Because U is linear, $U(kv) = kU(v) = kw \Rightarrow kw \in R(U)$. Because $w \neq 0$, $kw \neq 0 \in R(U), R(T) \Rightarrow R(T) \cap R(U) \neq \{0\}$. Thus we have shown that if $\{T, U\}$ are not linearly independent subsets of $\mathcal{L}(V, W)$, then $R(T) \cap R(U) \neq \{0\}$. Using proof by contrapositive, we have proved if $R(T) \cap R(U) = \{0\}$, then $\{T, U\}$ are linearly independent subsets of $\mathcal{L}(V, W)$.

[2] Because $U \in \mathcal{L}(V, W)$, we know that U is a nonzero operator. Thus, $\exists\ w \neq 0 \in R(U)$.