

Discretionary Note

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IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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Math 226- HW 4 Due: Sep 26 by Midnight

1. (21 points) Let V be a vector space over a field \mathbb{F} , $S \subseteq V$ is finite, and U and W be finite dimensional subspaces of V .

(a) (6 points) Let $\vec{x} \in V$. Show that

$$\dim(\text{Span}(S)) \leq \dim(\text{Span}(S \cup \{\vec{x}\})) \leq \dim(\text{Span}(S)) + 1.$$

b) (10 points) Prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Hint: Start with a basis for $\dim(U \cap W)$, then extend it to W , and U . Finally do some counting.

(c) (5 points) Prove that if

$$\dim(U + W) = 1 + \dim(U \cap W).$$

Then the sum of $U + W$ is equal to one of these spaces (either U or W) and the intersection $U \cap W$ is equal to the other one

2. (22 points) Let U, V be vector spaces over \mathbb{F} , and $T : U \rightarrow V$ be linear operator. Let B be basis of U and define $T(B) := \{T(u) : u \in B\}$.

(a) (10 points) Show that $T(B)$ is linearly independent if T is injective. Moreover, prove that if $T(B)$ is linearly independent and $\infty > |T(B)| \geq |B|$, then T is injective.

(b) (10 points) Show that T is surjective iff $T(B)$ spans V .

(c) (2 points) Prove that $T(B)$ is a basis for V if T is bijective. Moreover, if $T(B)$ is a basis and $\infty > |T(B)| \geq |B|$ then T is bijective.

3. (19 points) Recall that $C^1(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} : \text{such that } f'(x) \text{ exist for all } x \in \mathbb{R}\}$. Define T as

$$\begin{aligned} T : C^1(\mathbb{R}) &\rightarrow \mathbb{R}^2 \\ f(x) &\mapsto T(f) := (f'(3), f(3)) \end{aligned}$$

(a) (4 points) Show that T is a linear transformation.

(b) (8 points) Let $H := \{(x - 3)^2 g(x) : g(x) \in C^1(\mathbb{R})\}$, and $V := \{f : T(f) = (1, 2)\}$. Show that $H + V = V$.

Remark: This statement is true for any $V := \{f : T(f) = (a, b)\}$ where (a, b) is a fixed vector in \mathbb{R}^2 .

(c) (7 points) Deduce from part (b) that T is not injective. Is T surjective? Validate your answer.

4. (8 points) Let \mathbb{F} be a field with regular addition and multiplication.

a) (6 points) Let $T : \mathbb{F}^5 \rightarrow \mathbb{F}^4$ with

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3)$$

– Find a basis, and the dimension of $N(T)$. (Don't use elementary operation method for matrices)

- Find a basis, and the dimension of $R(T)$. (Don't use elementary operation method for matrices)

b) (2 points) Show that no linear map $T : \mathbb{F}^5 \rightarrow \mathbb{F}^2$ can have as its null space the set

$$N(T) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 7x_3, x_2 = x_4 = x_5\}$$

Practice Problems

(a) Suppose $S \subseteq V$ and $|S| = \dim(V)$. Prove that the followings are equivalent

- (i) S is a basis for V
- (ii) S spans V
- (iii) S is linearly independent

Hint: Use Exchange Theorem. To show a set of statements are equivalent you need to prove the statements in a cycle, i.e., (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).

(b) Recall $P(\mathbb{R}) = \{p(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n; n \in \mathbb{N}, c_i \in \mathbb{R}\}$. Prove that $P(\mathbb{R})$ is infinite dimensional.

Sec 1.6: 1-28