MATH 241 PSET 3

September 16, 2024

1.

a) We can compute the probability Fred completes the project on time, given that he completes his first milestone on time as:

$$P(A_3|A_1) = P(A_3|A_2, A_1)P(A_2|A_1) + P(A_3|A_2^c, A_1)P(A_2^c|A_1)$$

Because A_3 and A_1 are conditionally independent given A_2 and A_2^c, $P(A_3|A_2, A_1) = P(A_3|A_2) = 0.8$ and $P(A_3|A_2^c, A_1) = P(A_3|A_2^c) = 0.3$. We are also know that $P(A_2|A_1) = 0.8$ and $P(A_2^c|A_1) = 1 - P(A_2^c|A_1) = 0.2$. Thus, $P(A_3|A_1) = 0.8(0.8) + 0.3(0.2) = 0.7$.

We now compute the probability Fred completes the project on time, given that he completes his first milestone late as:

$$P(A_3|A_1^c) = P(A_3|A_2, A_1^c)P(A_2|A_1^c) + P(A_3|A_2^c, A_1^c)P(A_2^c|A_1^c)$$

Given $P(A_3|A_2, A_1^c) = P(A_3|A_2) = 0.8$, $P(A_3|A_2^c, A_1^c) = P(A_3|A_2^c) = 0.3$, $P(A_2|A_1^c) = 0.3$ and $P(A_2^c|A_1^c) = 1 - P(A_2|A_1^c) = 0.7$, we get that $P(A_3|A_1^c) = 0.8(0.3) + 0.3(0.7) = 0.45$.

b) The probability Fred will finish his project on time is given by $P(A_3)$. Given $P(A_1) = 0.75$ and $P(A_3|A_1) = 0.7$, $P(A_3|A_1^c) = 0.45$ from (a), we can compute $P(A_3)$ as:

$$\begin{aligned}
P(A_3) &= P(A_3|A_1)P(A_1) + P(A_3|A_1^c)P(A_1^c) \\
P(A_3) &= 0.7(0.75) + 0.45(1 - P(A_1)) \\
P(A_3) &= 0.7(0.75) + 0.45(0.25) \\
P(A_3) &= 0.6375
\end{aligned}$$

2.

We are given $P(A) = 1$ and we have to prove that for any B where $P(B) > 0$, $P(A|B) = 1$. The proof is below.

$$P(B) = P(B \cap A) + P(B \cap A^c)$$

Note that $P(B \cap A) = P(B|A)P(A) = P(B)$ as event A is guaranteed to happen. Thus:

$$P(B) = P(B) + P(B \cap A^c)$$
$$P(B \cap A^c) = 0$$

Because $P(A^c|B) = \frac{P(B \cap A^c)}{P(B)}$ and we know $P(B) > 0$, $P(A^c|B) = 0$. Thus, we have proven $P(A|B) = 1 - P(A^c|B) = 1$ given $P(A) = 1$ and for any B where $P(B) > 0$.

3.

b) We compute $P(G|A^c)$ below.

$$P(G|A^c) = \frac{P(A^c|G)P(G)}{P(A^c)} = \frac{(1 - P(A|G))g}{P(A^c|G)P(G) + P(A^c|G^c)P(G^c)}$$
$$P(G|A^c) = \frac{(1 - p_1)g}{(1 - P(A|G))g + (1 - P(A|G^c))(1 - g)} = \frac{(1 - p_1)g}{(1 - p_1)g + (1 - p_2)(1 - g)}$$

c) Note that because A and B are conditionally independent given G or G^c, $P(B|A^c, G) = P(B|G)$ and $P(B|A^c, G^c) = P(B|G^c)$. We compute $P(B|A^c)$ below.

$$P(B|A^c) = P(B|A^c, G)P(G|A^c) + P(B|A^c, G^c)P(G^c|A^c)$$
$$P(B|A^c) = P(B|G)P(G|A^c) + P(B|G^c)(1 - P(G|A^c))$$
$$P(B|A^c) = p_1 P(G|A^c) + p_2(1 - P(G|A^c)$$
$$P(B|A^c) = p_2 + \frac{(p_1 - p_2)(1 - p_1)g}{(1 - p_1)g + (1 - p_2)(1 - g)}$$

4.

a) Let us define events A and B as the event that the sample goes to labs A and B, respectively. Let us define C as the event in which the patient has the disease conditionitis. and the events $+$ and $-$ as the events in which the patient tested positive and negative, respectively.
Given these definitions, $P(C) = p$, $P(A) = P(B) = \frac{1}{2}$, $P(+|C, A) = a_1$, $P(-|C^c, A) = a_2$, $P(+|C, B) = b_1$, and $P(-|C^c, B) = b_2$. We compute $P(C|+)$ below.

$$P(C|+) = \frac{P(+|C)P(C)}{P(+)} = p\frac{P(+|C, A)P(A) + P(+|C, B)P(B)}{P(+)} = p\frac{(a_1 + b_1)}{2P(+)}$$

We now compute $P(+)$ below. Note that events (C, A) and (C, B) are entirely independent as the patient having conditionitis has no relation to which lab their sample is tested at.

$$\begin{aligned}
P(+) &= P(+|C,A)P(C \cap A) + P(+|C^c,A)P(C^c \cap A) + P(+|C,B)P(C \cap B) + P(+|C^c,B)P(C^c \cap B) \\
&= a_1 P(A)P(C) + (1-a_2)P(C^c)P(A) + b_1 P(C)P(B) + (1-b_2)P(C^c)P(B) \\
&= \frac{p(a_1+b_1)}{2} + \frac{(1-a_2)(1-p)}{2} + \frac{(1-b_2)(1-p)}{2}
\end{aligned}$$

As such, given $P(+)$, we can compute $P(C|+)$ as:

$$P(C|+) = \frac{p(a_1+b_1)}{p(a_1+b_2) + (1-p)[2-a_2-b_2]}$$

b) We compute $P(A|+)$ below:

$$\begin{aligned}
P(A|+) &= \frac{P(+|A)P(A)}{P(+)} \\
&= \frac{P(+|A,C)P(C) + P(+|A,C^c)P(C^c)}{2P(+)} \\
&= \frac{pa_1 + (1-a_2)(1-p)}{2P(+)}
\end{aligned}$$

Using our calculation for $P(+)$ in (a), we get our final answer:

$$P(A|+) = \frac{pa_1 + (1-a_2)(1-p)}{p(a_1+b_1) + (1-p)[2-a_2-b_2]}$$

5.

a) Let us define M as the event that the mother has the disease and C_1, C_2 as the events that the first and second child have the disease, respectively. Given these definitions, $P(M) = \frac{1}{3}$, $P(C_1|M) = P(C_2|M) = \frac{1}{2}$, and $P(C_1|M^c) = P(C_2|M^c) = 0$. We compute the probability neither children has the condition given by $P(C_1^c \cap C_2^c)$ below:

$$P(C_1^c \cap C_2^c) = P(C_1^c \cap C_2^c|M)P(M) + P(C_1^c \cap C_2^c|M^c)P(M^c)$$

Note that C_1 and C_2 are conditionally independent given M. As such,

$$
\begin{aligned}
P(C_1^c \cap C_2^c) = \frac{P(C_1^c|M)P(C_2^c|M)}{3} + P(C_1^c \cap C_2^c|M^c)P(M^c) \\
= \frac{(1 - P(C_1|M))(1 - P(C_2|M))}{3} + P(C_1^c \cap C_2^c|M^c)P(M^c) \\
= \frac{1}{12} + P(C_1^c \cap C_2^c|M^c)P(M^c)
\end{aligned}
$$

We also know that events C_1, C_2 will not occur given M^c. Thus $P(C_1^c \cap C_2^c|M^c) = 1$:

$$
\begin{aligned}
P(C_1^c \cap C_2^c) = \frac{1}{12} + (1 - P(M)) \\
P(C_1^c \cap C_2^c) = \frac{1}{12} + \frac{2}{3} \\
P(C_1^c \cap C_2^c) = \frac{3}{4}
\end{aligned}
$$

c) We compute $P(M|C_2^c \cap C_1^c)$ below, given that $P(C_1^c \cap C_2^c) = \frac{3}{4}$ from (a). Note again that C_1 and C_2 are conditionally independent given M.

$$
\begin{aligned}
P(M|C_2^c \cap C_1^c) = \frac{P(C_2^c \cap C_1^c|M)P(M)}{P(C_1^c \cap C_2^c)} \\
= \frac{4(1 - P(C_1|M))(1 - P(C_2|M))}{9} = \frac{4}{2 * 2 * 9} \\
P(M|C_2^c \cap C_1^c) = \frac{1}{9}
\end{aligned}
$$

6. Anish Lakkapragada. I worked independently.