

## Discretionary Note

Anish Krishna Lakkapragada

**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# MATH 226 - HW 1

Anish Lakkapragada – `anish.lakkapragada@yale.edu` – al2778

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1.

(a)

Given that image  $b$  is the output of  $f(a)$  where pre-image  $a \in A$ ,  $g(b)$  will always equal  $a$  as it is guaranteed that there exists a pre-image  $a$  which by function  $f$  will map to  $b$ . Thus, for all  $a \in A$ ,  $g \circ f(a) = a$  and so  $g$  is a left inverse of  $f$ .

(b)

For  $g$  to be the right inverse of  $f$ ,  $f \circ g(b) = b$  where we assume  $b \in B$ . Because  $f$  is an injective function, it is not guaranteed that every image in set  $B$  has a corresponding pre-image in set  $A$  as defined by function  $f$ . If the given aforementioned  $b$  does not have a pre-image in set  $A$  as defined by function  $f$ ,  $g(b) = a_0$ . And because it is not guaranteed  $f(a_0)$  equals  $b$ , it is not guaranteed  $f(g(b)) = b$ . Thus  $g$  may not be the right inverse of  $f$ .

$g$  would be the right inverse of  $f$  if there was a one-to-one correspondence between each pre-image in  $A$  and image  $B$  through function  $f$ . This would occur if  $f$  was a bijective function.

(c)

Given the surjective function  $f : A \rightarrow B$ , the right inverse is given by  $g(b)$  where  $f \circ g(b) = b$ . The set of pre-images in  $A$  that map by function  $f$  to image  $b \in B$  is given by the set  $P = \{x | f(x) = b\}$ . Because  $f$  is surjective,  $|P| \geq 1$ . In cases where  $|P| > 1$ ,  $g(b)$  must be able to choose one pre-image from  $P$ . I define  $g(b)$  as taking any arbitrary pre-image from set  $P$ . Because  $f$  is surjective,  $g(b)$  is guaranteed to return a pre-image that maps by function  $f$  to  $b$ . Thus the right inverse condition  $f \circ g(b) = b$  is upheld, and so  $g(b)$  is proved as the right inverse of  $f$ .

2.

(a)

We are given that the composition of  $g$  with  $f$ ,  $g \circ f$ , maps set  $A$  to  $C$ . This composition  $g \circ f$  is injective if any image in  $C$  has at most one pre-image in set  $A$ . If we assume  $f$  is an injective function, we know that only certain pre-images in set  $A$  will map to unique elements in set  $B$ . Similarly, if we assume  $g$  is an injective function, we know that only certain pre-images in set  $B$  will map to unique elements

in set  $C$ . Because we know a subset of pre-images in  $A$  will map to a subset of unique images in  $B$ , and a subset of those images (now pre-images) in  $B$  will map to unique images in  $C$ , we know the composition function  $g \circ f$  will map a subset of the pre-images in  $A$  to unique images in  $C$ . By definition,  $g \circ f$  is an injective function if  $f$  and  $g$  are both injective.

(b)

If we know  $g$  is a surjective function, we know that for each image in  $C$ , there exists at least one pre-image in  $B$ . Similarly, if we know that  $f$  is a surjective function, we know that for each of these pre-images that exist in  $B$  that map to images in  $C$ , there exists at least one pre-image in  $A$ . Because every element in  $C$  is guaranteed to have at least one pre-image in  $A$  by the function  $g \circ f$ , by definition,  $g \circ f$  is a surjection if  $f$  and  $g$  are both surjective.

(c)

If functions  $f$  and  $g$  are both bijective, both functions map all pre-images in  $A$  and  $B$  respectively to unique images in  $B$  and  $C$  respectively. Because  $f$  provides a one-to-one correspondence between elements in  $A$  and  $B$  and  $g$  provides a one-to-one correspondence between elements in  $B$  and  $C$ , a one-to-one correspondence is maintained between each pre-image in  $A$  and image in  $C$  through the function  $g(f(a))$  or  $g \circ f$ . Thus  $g \circ f$  is a bijective function if  $f$  and  $g$  are both bijective.

3.

(a)

Let us define  $x = m_1 + n_1\sqrt{2}$ , where  $m_1, n_1 \in \mathbb{Z}$  and  $y = m_2 + n_2\sqrt{2}$ , where  $m_2, n_2 \in \mathbb{Z}$ . Given these definitions,  $x + y = (m_1 + m_2) + (n_1 + n_2)\sqrt{2}$ . Because  $(m_1 + m_2), (n_1 + n_2) \in \mathbb{Z} \Rightarrow x + y \in B$  if  $x, y \in B$ .

(b)

Let us define  $x$  and  $y$  the same as in the above part (a). Given these definitions,  $xy = m_1m_2 + m_1n_2\sqrt{2} + m_2n_1\sqrt{2} + 2n_1n_2 = (2n_1n_2 + m_1m_2) + (m_1n_2 + m_2n_1)\sqrt{2}$ . Because  $(2n_1n_2 + m_1m_2), (m_1n_2 + m_2n_1) \in \mathbb{Z} \Rightarrow xy \in B$  if  $x, y \in B$ .

(c)

For the base case  $k = 1$ ,  $(-1 + \sqrt{2})^k = (-1 + \sqrt{2}) \in B$ . Given integer  $k \geq 1$  and  $(-1 + \sqrt{2})^k \in B$ ,  $(-1 + \sqrt{2})^{k+1} \in B$ . This is because  $(-1 + \sqrt{2})^{k+1} = (-1 + \sqrt{2})^k * (-1 + \sqrt{2})$  and both factors  $(-1 + \sqrt{2})^k, (-1 + \sqrt{2}) \in B$ . As proven in (b),  $B$  is closed under multiplication and so when both factors are in  $B$ , their product will be in  $B$ . Thus, as proven by induction, for all integers  $k \geq 1$ ,  $(-1 + \sqrt{2})^k \in B$ .

4.

(a)

Note that for any set  $C$  with  $N$  items,  $T(\Sigma_{i=1}^N C_i)$  will equal  $\Sigma_{i=1}^N T(C_i)$  due to  $T$  being an additive function.

For all given integers  $n \geq 1$ :

$$T(\Sigma_{i=1}^n x) = \Sigma_{i=1}^n T(x)$$

$$T(nx) = nT(x)$$

(b)

$$T(x + y) = T(x) + T(y)$$

$$T(0 + 0) = T(0) + T(0)$$

$$T(0) = 2T(0)$$

$$T(0) = 0$$

(c)

$$T(x + y) = T(x) + T(y)$$

$$T(x + (-x)) = T(x) + T(-x)$$

$$T(0) = T(x) + T(-x)$$

$$0 = T(x) + T(-x)$$

$$T(x) = -T(-x)$$

(d) For all integers  $n$  and all integers  $k \neq 0$ ,

$$T((\frac{n}{k} * k)x) = \Sigma_{i=1}^k T(\frac{n}{k}x) = kT(\frac{n}{k}x)$$

If we define  $r$  to be the fraction  $\frac{n}{k}$ , by definition  $r \in \mathbb{Q}$  as  $n$  and  $k$  are both integers where denominator  $k \neq 0$ . Using  $r$ , we can simplify the above expression further:

$$T(nx) = \frac{n}{r}T(rx)$$

$$rT(nx) = nT(rx)$$

As  $n$  is defined as  $n \in \mathbb{Z}$ , we first generalize our proof in (a) that  $T(nx) = nT(x)$  from all integers  $n \geq 1$  to  $n \in \mathbb{Z}$ . Given an integer  $n < 0$ , if  $u = nx$ , we can use our proof in (c) that for  $u \in \mathbb{R}$ ,  $T(u) = -T(-u)$  and thus since  $T(-u) = T(|n|x) = |n|T(x)$ , we can conclude that in cases where  $n < 0$ ,  $T(u) = T(nx) = -|n|T(x) = nT(x)$  or more simply,  $T(nx) = nT(x)$ . And in cases where  $n = 0$ ,  $T(nx) = nT(x)$  as  $T(0) = 0$  as proven in (b). Thus our proof in (a) is generalized to  $n \in \mathbb{Z}$ . Using this result, we can continue simplifying our above expressions.

$$r(nT(x)) = nT(rx)$$

$$rT(x) = T(rx)$$

$$T(rx) = rT(x)$$

for all rational numbers  $r \in \mathbb{Q}$ .

(e)

Let us define  $T(x)$ :

$$T(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

Defining  $r = \sqrt{2}, x = 1$ :

$$T(\sqrt{2}x) = \sqrt{2}T(x)$$

$$T(\sqrt{2}) = \sqrt{2} * T(1)$$

$$0 \neq 1$$

$$T(rx) \neq rT(x)$$

As proven by contradiction,  $T(rx) \neq rT(x)$  for all reals  $r \in \mathbb{R}$ .

5.

(a)

Let us define field  $\mathbb{F} = (\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$  and the multiplicative inverse of  $a + b\sqrt{3}$  as  $z$  where  $(a + b\sqrt{3})z = 1$ . Given that  $a^2 - 3b^2 \neq 0$ ,  $z$  is given by  $\frac{1}{a+b\sqrt{3}} = \frac{a-b\sqrt{3}}{a^2-3b^2}$ . In the case where  $a^2 - 3b^2 = 1$ ,  $z = a - b\sqrt{3}$ . Because  $a - b\sqrt{3} \in \mathbb{F}$ ,  $a + b\sqrt{3}$  has a multiplicative inverse in this case. Similarly, in the case where  $a^2 - 3b^2 = -1$ ,  $z = -a + b\sqrt{3}$ . Because  $-a + b\sqrt{3} \in \mathbb{F}$ ,  $a + b\sqrt{3}$  has a multiplicative inverse in this case as well.

(b)

Given that  $a + b\sqrt{3} \in \mathbb{F}$  has a multiplicative inverse, let us define this multiplicative inverse as  $c + d\sqrt{3} \in \mathbb{F}$  where  $c, d \in \mathbb{Z}$  and  $(a + b\sqrt{3})(c + d\sqrt{3}) = 1$ . Let us also define the greatest common divisor of  $a$  and  $b$  as  $k = \gcd(a, b) \in \mathbb{Z}$  where  $a = ka'$  and  $b = kb'$  and  $a', b' \in \mathbb{Z}$  are coprime. We inspect the possible values of  $k$  below.

$$(a + b\sqrt{3}) * (c + d\sqrt{3}) = 1$$

$$(ka' + kb'\sqrt{3}) * (c + d\sqrt{3}) = 1$$

$$ka'c + ka'd\sqrt{3} + kb'c\sqrt{3} + 3kb'd = 1 + 0\sqrt{3}$$

$$ka'c + 3kb'd = 1$$

$$k(a'c + 3b'd) = 1$$

$$(a'c + 3b'd) = \frac{1}{k}$$

Because  $a'c + 3b'd \in \mathbb{Z} \Rightarrow k \leq 1$ . Because  $k \in \mathbb{Z} \Rightarrow k = 1$ . We now define the values of  $c, d$  in terms of  $a, b$ . Defining  $z = c + d\sqrt{3}$ ,  $z = \frac{1}{a+b\sqrt{3}} = \frac{a-b\sqrt{3}}{a^2-3b^2} \Rightarrow c = \frac{a}{a^2-3b^2}, d = \frac{-b}{a^2-3b^2}$ . Because the largest possible value that can divide two integers,  $a, b$ , into integers is given by  $k = \gcd(a, b)$ , the denominator in  $c, d$  of  $a^2 - 3b^2$  must equal  $k = 1$  in order to ensure  $c, d \in \mathbb{Z}$  so that  $c + d\sqrt{3} \in \mathbb{F}$ . Note that  $a^2 - 3b^2 = -1$  also ensures the multiplicative inverse  $c + d\sqrt{3} \in \mathbb{F}$  as  $c, d \in \mathbb{Z}$  because  $\pm a, \pm b \in \mathbb{Z}$ . Thus, if  $a + b\sqrt{3} \in \mathbb{F}$  has a multiplicative inverse, we know that  $|a^2 - 3b^2| = 1$ . If there is no multiplicative inverse in  $\mathbb{F}$ , that is because  $c \notin \mathbb{Z}$  or  $d \notin \mathbb{Z}$ , which would happen only if the denominator  $|a^2 - 3b^2| \neq \gcd(a, b)$  or  $|a^2 - 3b^2| \neq 1$ . Thus, if  $a + b\sqrt{3} \in \mathbb{F}$  has a multiplicative inverse  $\iff |a^2 - 3b^2| = 1$ .

(c)

In order for  $\mathbb{F} = (\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$  to define a field,  $\forall m \in \mathbb{F}, \exists n \in \mathbb{F}$  such that  $m \cdot n = 1$ . However, as shown in (b),  $a + b\sqrt{3} \in \mathbb{F}$  will only have a guaranteed multiplicative inverse in the special case that  $|a^2 - 3b^2| = 1$ . Thus,  $\mathbb{F}$  fails to meet the multiplicative inverse condition to be defined as a valid field.