

## Discretionary Note

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**CONTENT STARTS ON NEXT PAGE.**

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**Math 226: HW 2**
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1. a) We demonstrate that  $L^2(\mathbb{R})$  has the identity element  $f(x) = 0$ , is closed under addition, and closed under scalar multiplication to prove that  $L^2(\mathbb{R})$  is a vector space.

① Existence of Additive Identity Element in  $L^2(\mathbb{R})$

The function  $f(x) = 0 \in L^2(\mathbb{R})$  as  $f(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$  and  $\int_{-\infty}^{\infty} f(x)dx = 0 < \infty$ .  $f(x) = 0$  is the additive identity element  $L^2(\mathbb{R})$  as  $\forall g(x) \in L^2(\mathbb{R})$ ,  $f(x) + g(x) = g(x)$ .

② Closed Under Addition

Given  $a, b \in \mathbb{R}$ :

$$\begin{aligned}(a - b)^2 &\geq 0 \\ a^2 + b^2 &\geq 2ab \\ 2a^2 + 2b^2 &\geq (a + b)^2\end{aligned}$$

If we switch sides and then substitute  $a = f(x) \in L^2(\mathbb{R})$  and  $b = g(x) \in L^2(\mathbb{R})$ , we get:

$$\begin{aligned}(f(x) + g(x))^2 &\leq 2[f(x)]^2 + 2[g(x)]^2 \\ |f(x) + g(x)|^2 &\leq 2|f(x)|^2 + 2|g(x)|^2\end{aligned}$$

We now integrate from  $-\infty$  to  $\infty$  on both sides:

$$\int_{-\infty}^{\infty} |f(x) + g(x)|^2 dx \leq 2 \int_{-\infty}^{\infty} |f(x)|^2 dx + 2 \int_{-\infty}^{\infty} |g(x)|^2 dx$$

Because  $f(x), g(x) \in L^2(\mathbb{R})$ , we know that  $\int_{-\infty}^{\infty} |f(x)|^2 < \infty$  and  $\int_{-\infty}^{\infty} |g(x)|^2 < \infty$ . Thus  $2 \int_{-\infty}^{\infty} |f(x)|^2 dx + 2 \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$  as well, and so we know that:

$$\int_{-\infty}^{\infty} |f(x) + g(x)|^2 dx < \infty \quad (1)$$

Given that  $f(x), g(x) \in L^2(\mathbb{R})$ , we know that  $f(x) + g(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$  as  $\mathcal{F}(\mathbb{R}, \mathbb{R})$  is a vector space and thus is closed under addition. Thus, because we have proved Equation 1 and  $f(x) + g(x) \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ ,  $f(x) + g(x) \in L^2(\mathbb{R})$  and so  $L^2(\mathbb{R})$  is closed under addition.

③ Closed Under Scalar Multiplication

Consider for  $x \in \mathbb{R}$  a function  $f(x) \in L^2(\mathbb{R})$ . Given  $c \in \mathbb{R}$ , let us define  $g(x) = cf(x)$ . Because  $f(x) \in \mathbb{R}$  and  $c \in \mathbb{R}$ ,  $g(x) = cf(x) \in \mathbb{R}$ . If  $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ ,  $k \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$  for  $k \in \mathbb{R}$ . Thus  $\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |cf(x)|^2 dx = |c|^2 \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$  as  $|c|^2 \in \mathbb{R}$  and so it is proven  $\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$ . Thus  $g(x) \in L^2(\mathbb{R})$  and so  $L^2(\mathbb{R})$  is proven to be closed under scalar multiplication.

Because we have demonstrated ①, ②, and ③, we have demonstrated  $L^2(\mathbb{R})$  is a vector space.

- b) In order for a set  $V$  to define a vector space over field  $\mathbb{R}$ , the vector  $\mathbf{0} \in V$  s.t.  $\forall v \in V, v + \mathbf{0} = v$ . For set  $V$ , this vector  $\mathbf{0} = (0, 1)$  as  $\forall v = (a_1, b_1) \in V, v + \mathbf{0} = (a_1, b_1) + (0, 1) = (a_1, b_1) = v$ .

Another property for a set  $V$  to define a vector space is that  $\forall v \in V, \exists -v \in V$  s.t.  $v + (-v) = \mathbf{0} = (0, 1)$ . Let us define  $v = (a_1, b_1) \in V$  and vector  $-v = (a_2, b_2)$ . For  $v + (-v) = (0, 1)$ ,  $a_2 = -a_1$  and  $b_2(b_1) = 1$ . In the case where  $b_1 = 0$ ,  $b_2(b_1) \neq 1$  and thus  $\forall v \in V$  it is not guaranteed  $\exists -v \in V$  s.t.  $v + (-v) = \mathbf{0} = (0, 1)$ . Because this condition is not met,  $V$  does not define a valid vector space over  $\mathbb{R}$ .

2. a) We go through the three conditions of testing if vector space  $W_1$  and  $W_2$  are subspaces of  $\mathbb{F}^n$ .

① Closed Under Scalar Multiplication

- a)  $W = W_1$

For a given  $x = (a_1, a_2, \dots, a_n) \in W_1$  and  $c \in \mathbb{F}$ ,  $cx = (ca_1, ca_2, \dots, ca_n)$ . Because  $ca_1, ca_2, \dots, ca_n \in \mathbb{F}^n$ , and  $c \sum_{i=1}^N a_i = 0$  given  $\sum_{i=1}^N a_i = 0$ ,  $W_1$  meets this condition to be a subspace of  $\mathbb{F}^n$ .

- b)  $W = W_2$

$cx = (ca_1, ca_2, \dots, ca_n)$ . Given  $\sum_{i=1}^N a_i = 1$ ,  $c \sum_{i=1}^N a_i \neq 1$  and thus  $cx \notin W_2$ . Thus  $W_2$  does not meet this condition to be a subspace of  $\mathbb{F}^n$ . *Because  $W_2$  does not meet this condition to be a subspace, we do not need to check if it meets any of the other conditions.*

② Closed Under Addition

- a)  $W = W_1$

Given  $x = (a_1, a_2, \dots, a_n) \in W_1, y = (b_1, b_2, \dots, b_n) \in W_1$ ,  $x + y = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ .  $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 0 \Rightarrow \sum_{i=1}^N a_i + b_i = 0 \Rightarrow x + y \in W_1$ . Thus  $W_1$  meets this condition to be a subspace of  $\mathbb{F}^n$ .

③  $\exists \mathbf{0} \in W$

- a)  $W = W_1$

For  $x = \mathbf{0}$ ,  $x_i = 0 \Rightarrow \sum_{i=1}^N x_i = 0$ . Thus,  $\mathbf{0} \in W_1$  and so  $W_1$  meets this condition to be a subspace of  $\mathbb{F}^n$ .

Because  $W_1$  meets all the conditions to be a subspace whereas  $W_2$  does not,  $W_1$  is a subspace of  $\mathbb{F}^n$ .

- b) We define the subset of  $\mathbb{Z}_2^n$  with even  $E_n$  as  $Q^n = \{v \in \mathbb{Z}_2^n : E_n(v) \in 2\mathbb{Z}\}$ . We now assess if  $Q^n \leq \mathbb{Z}_2^n$  by checking if  $Q^n$  meets the following three conditions.

①  $\exists \mathbf{0} \in Q^n$

Let us define the zero vector as  $z = \mathbf{0} \in \mathbb{Z}_2^n$ . Because  $E_n(z) = 0 \in 2\mathbb{Z}$ ,  $z = \mathbf{0} \in Q^n$ .

② Closed Under Addition

Let us define two vectors  $x, y \in Q^n$ . Because  $\mathbb{Z}_2^n$  is a vector space and thus closed under addition,  $x + y \in \mathbb{Z}_2^n$ . The number of nonzero components of  $x + y$  is given by  $E_n(x + y) = E_n(x) + E_n(y) - 2k$ , where  $k \in \mathbb{Z}$  is given by the number of indices where  $x$  and  $y$  have the same value. Because  $E_n(x), E_n(y) \in 2\mathbb{Z}$  as  $x, y \in \mathbb{Z}_2^n$ ,  $E_n(x) + E_n(y) \in 2\mathbb{Z}$ . Because  $k \in \mathbb{Z}$ ,  $2k \in 2\mathbb{Z}$  and so  $E_n(x + y) = E_n(x) + E_n(y) - 2k \in 2\mathbb{Z}$ . Because  $E_n(x + y) \in 2\mathbb{Z}$  and  $x + y \in \mathbb{Z}_2^n \Rightarrow x + y \in Q^n$ . Thus  $Q^n$  is closed under addition.

③ Closed Under Scalar Multiplication

Let us consider a scalar  $c \in \mathbb{Z}_2$  and  $v \in Q^n$ .  $c$  can either equal zero or one. If  $c = 0$ ,  $cv = \mathbf{0} \in Q^n$ . If  $c = 1$ ,  $cv = v \in Q^n$ . Thus,  $cv \in Q^n$  for any  $c \in \mathbb{Z}_2$  and so  $Q^n$  is closed under scalar multiplication.

Because  $Q^n$  meets all the three conditions to be a subspace to  $\mathbb{Z}_2^n$ ,  $Q^n \leq \mathbb{Z}_2^n$ .

- c) The general form for function  $f \in P_3(\mathbb{R})$  is given by  $f(x) = c_1 + c_2x + c_3x^2 + c_4x^3$  where  $c_1, c_2, c_3, c_4 \in \mathbb{R}$ . Given the constraints  $f(0) = f'(0)$  and  $f(1) = 0$ , the form for any function  $f \in W$  is given by:

$$f(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3$$

We now test if  $W$  defines a subspace of  $P_3(\mathbb{R})$ .

- ①  $\exists \mathbf{0} \in W$

Because when  $f(x) = 0$  when  $c_1 = c_3 = 0$  and  $x = 0$ , the zero polynomial is defined in  $W$ .

- ② Closed Under Addition

Given  $p(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3 \in W$  and  $q(x) = b_1 + b_1x + b_3x^2 + (-2b_1 - b_3)x^3 \in W$ ,  $p(x) + q(x) = (c_1 + b_1) + (c_1 + b_1)x + (c_3 + b_3)x^2 + (-2c_1 - c_3 - 2b_1 - b_3)x^3 \in P_3(\mathbb{R})$ . Defining  $z(x) = p(x) + q(x)$ ,  $z(0) = z'(0) = c_1 + b_1$  and  $z(1) = p(1) + q(1) = 0 + 0 = 0$ . Thus  $p(x) + q(x) \in W$  and thus  $W$  is proven to be closed under addition.

- ③ Closed Under Scalar Multiplication

Given  $p(x) = c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3 \in W$  and  $k \in \mathbb{R}$ ,  $kp(x) = k(c_1 + c_1x + c_3x^2 + (-2c_1 - c_3)x^3)$ . Defining  $g(x) = kp(x)$ ,  $g(0) = g'(0) = kc_1$  and  $g(1) = k(p(1)) = k(0) = 0$ . Thus  $kp(x) \in W$  and  $W$  is proven to be closed under scalar multiplication.

Because I have shown  $W$  contains the zero polynomial and is closed under addition and scalar multiplication,  $W \leq P_3(\mathbb{R})$ .

Because  $P_k(\mathbb{R})$  is the set of polynomials that have a degree of *at most*  $k$ ,  $P_n(\mathbb{R}) \leq P_k(\mathbb{R})$  where  $n \leq k$ . Thus  $P_3(\mathbb{R}) \leq P_4(\mathbb{R})$ . Because  $W \leq P_3(\mathbb{R}) \leq P_4(\mathbb{R})$ ,  $W \leq P_4(\mathbb{R})$ .

- d) Because this statement is an *if and only if*, we must show (1) that if these two conditions are met,  $W \leq V$  and (2) that if  $W \leq V$ , these two conditions are met. We show (1) and (2) below.

- ① If Condition 1 and Condition 2 are met,  $W \leq V$

- ⓐ Condition 1:  $W \neq \emptyset$

If  $W = \emptyset$ , the standard condition to define a subspace for  $\mathbf{0} \in W$  cannot be met because there are no elements in  $W$ . Note  $W \neq \emptyset \not\Rightarrow \mathbf{0} \in W$ .

- ⓑ Condition 2: for  $a \in \mathbb{F}$  and  $x, y \in W$ ,  $\exists ax + y \in W$ .

Given  $a = -1$  and  $x = y$ , if  $W$  meets Condition 2, it is guaranteed that  $-x + y = \mathbf{0} \in W$ . Thus, the standard condition of existence of a zero vector in a subspace is met if Condition 2 is met.

In the case  $y = \mathbf{0} \in W$ , if  $W$  meets Condition 2,  $ax \in W$  for  $a \in \mathbb{F}$  and  $x \in W$ . Thus, closure under scalar multiplication is met if Condition 2 is met.

Let us define  $z = ax \in W$ . Then, if Condition 2 is met, we know given  $z, y \in W$ ,  $z + y \in W$ . Thus, closure under addition is met if Condition 2 is met.

Thus, we have shown that if Condition 1 and Condition 2 are met,  $W$  meets the three properties to be defined as a subspace and so  $W \leq V$ .

- ② If  $W \leq V$ , Condition 1 and Condition 2 are met

We discuss below the implications of the properties of  $W$  we know given  $W \leq V$ .

- ⓐ  $\exists \mathbf{0} \in W$

If  $\exists \mathbf{0} \in W$ ,  $|W| \geq 1$  and so  $W \neq \emptyset$ . Thus, Condition 1 is met.

- ⓑ  $W$  is closed under addition and scalar multiplication

Let us define  $a \in \mathbb{F}$  and  $x, y \in W$ . If  $W$  is closed under scalar multiplication,  $ax \in W$ . If  $W$  is closed under addition,  $ax + y \in W$ . Thus, Condition 2 is met.

Thus, we have shown that if  $W \leq V$ , Condition 1 and Condition 2 are met.

Because we have proven both ① and ②, we have shown that if and only if Condition 1 and Condition 2 are met for a given subset  $W$  of a vector space  $V$  will  $W \leq V$ .

3. a) We test if  $U \cap W$  is a subspace of  $V$  below.

①  $\mathbf{0} \in U \cap W$

Because both  $U$  and  $W$  are valid subspaces,  $\exists \mathbf{0} \in U$  and  $\exists \mathbf{0} \in W$ . Thus  $\mathbf{0} \in U \cap W$ .

② Closed Under Addition

Let us consider  $x, y \in U \cap W$ . Because  $U$  and  $W$  are valid subspaces,  $U$  and  $W$  are closed under addition. Thus  $x + y \in U$  and  $x + y \in W \Rightarrow x + y \in U \cap W$ .

③ Closed Under Scalar Multiplication

Let us consider  $c \in \mathbb{F}$  and  $x \in U \cap W$ . Because  $U$  and  $W$  are valid subspaces,  $U$  and  $W$  are closed under scalar multiplication. Thus  $cx \in U$  and  $cx \in W \Rightarrow cx \in U \cap W$ .

Thus we have proven  $U \cap W \leq V$ .

- b) We test if  $U + W$  is a subspace of  $V$  below.

①  $\mathbf{0} \in U + W$

Because  $U$  and  $W$  are both valid subspaces,  $\mathbf{0} \in U, W$ . Thus, for  $u = \mathbf{0} \in U$  and  $w = \mathbf{0} \in W$ ,  $u + w = \mathbf{0} \in U + W$ .

② Closed Under Addition

Let us consider  $u_1, u_2 \in U$  and  $w_1, w_2 \in W$ . Let us define elements  $x = u_1 + w_1 \in U + W$  and  $y = u_2 + w_2 \in U + W$ .  $x + y = u_1 + w_1 + u_2 + w_2 \rightarrow (u_1 + u_2) + (w_1 + w_2)$ . Because  $U$  and  $W$  are valid subspaces, they are both closed under addition and thus  $z_1 = u_1 + u_2 \in U$  and  $z_2 = w_1 + w_2 \in W$ . As such,  $x + y = z_1 + z_2 \in U + W$  and so  $U + W$  is proven to be closed under addition.

③ Closed Under Scalar Multiplication

Let us define  $u \in U, w \in W, x = u_1 + w_1 \in U + W$ . Given  $c \in \mathbb{F}$ ,  $cx = cu + cw$ . Because  $U$  and  $W$  are valid subspaces,  $U$  and  $W$  are both closed under scalar multiplication and so  $cu \in U$  and  $cw \in W$ . Thus,  $cx = cu + cw \in U + W$  and so  $U + W$  is proven to be closed under scalar multiplication.

Thus we have proven  $U + W \leq V$ .

- c) Two subspaces of  $\mathbb{R}^2$  whose union is not a subspace of  $\mathbb{R}^2$  is  $\mathbb{Q}^2$  and  $W = \{(a_1, a_2) \in \mathbb{F}^2 : a_1 + a_2 = 0\}$  where field  $\mathbb{F}^2 = (\mathbb{R}^2, +, \cdot)$ .

An example proving  $\mathbb{Q}^2 \cup W$  is not a subspace of  $\mathbb{R}^2$  is choosing  $x = (1.5, 0) \in \mathbb{Q}^2 \cup W$  and  $y = (-\sqrt{2}, \sqrt{2}) \in \mathbb{Q}^2 \cup W$ .  $x + y = (1.5 - \sqrt{2}, \sqrt{2}) \notin \mathbb{Q}^2 \cup W$  and so  $\mathbb{Q}^2 \cup W$  does not define a valid subspace as it is not closed under addition.

4. a)

$$\text{Span}(S) = \{f(x) = (c_1 - c_2) + c_1x + c_2x^2 + c_3x^3 + c_4x^4; c_1, c_2, c_3, c_4 \in \mathbb{R}\}$$

- b) A polynomial  $p(x) \in P_4(\mathbb{R})$  that cannot be written as a linear combination of  $S$  (i.e.  $p(x) \notin \text{Span}(S)$ ) is  $p(x) = 5 - 2x^2$ .

Let us try to see if  $p(x) = 5 - 2x^2 \in \text{Span}(S)$ . Because in the general form of a function  $f \in \text{Span}(S)$  the only coefficient affecting the  $x^2$  term is  $-c_2$ ,  $c_2 = 2$ . Because the constant term 5 is given by  $c_1 - c_2$ ,  $c_1 = 7$  for  $p(x) = 5 - 2x^2 \in \text{Span}(S)$ . However, because this leads to a nonzero  $x$  term as  $c_1 \neq 0$ ,  $p(x) = 5 - 2x^2 \notin \text{Span}(S)$ . Because  $p(x) \notin \text{Span}(S)$  and  $p(x) \in P_4(\mathbb{R})$ ,  $S$  does not generate  $P_4(\mathbb{R})$ .

- c) The general form of the function  $f \in P_4(\mathbb{R})$  is given by  $f(x) = k_1 + k_2x + k_3x^2 + k_4x^3 + k_5x^4$  where  $k_1, k_2, k_3, k_4, k_5 \in \mathbb{R}$ . Through simple differentiation, we find that  $f'(0) = k_2, f''(0) = 2k_3$ . The set  $\{f \in P_4(\mathbb{R}) : 2f(0) = 2f'(0) - f''(0)\}$  is equal to the set of all functions  $f \in P_4(\mathbb{R})$  where:

$$2f(0) = 2f'(0) - f''(0)$$

$$2k_1 = 2k_2 - 2k_3$$

$$k_1 = k_2 - k_3$$

Re-expressing  $f(x)$  with  $k_1 = k_2 - k_3$  we get:

$$f(x) = (k_2 - k_3) + k_2x + k_3x^2 + k_4x^3 + k_5x^4$$

If we re-express our function above with  $c_1 = k_2, c_2 = k_3, c_3 = k_4, c_4 = k_5$ , we see that  $f(x) = (c_1 - c_2) + c_1x + c_2x^2 + c_3x^3 + c_4x^4$  where  $c_1, c_2, c_3, c_4 \in \mathbb{R}$ . This is the same form of a function  $g \in \text{Span}(S)$ . Thus, we have shown that the set  $\{f \in P_4(\mathbb{R}) : 2f(0) = 2f'(0) - f''(0)\} = \text{Span}(S)$ .