MATH 244 HW 1

January 24, 2025

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1. **Section 1.2, Question 5**

   If $X \times Y = X \times Z$, this implies $X \times Y \subseteq X \times Z$ and $X \times Z \subseteq X \times Y$. We look at the implications of these two facts below:

   1 $X \times Y \subseteq X \times Z$
   If $X \times Y \subseteq X \times Z$, this means that every element in the set $\{(x, y) : x \in X, y \in Y\}$ belongs to the set $X \times Z = \{(x, z) : x \in X, z \in Z\}$. Because the first element in each of these ordered products is drawn from the same set X, this means that $\forall y \in Y, y \in Z \Rightarrow Y \subseteq Z$.

   2 $X \times Z \subseteq X \times Y$
   If $X \times Z \subseteq X \times Y$, this means that every element in the set $\{(x, z) : x \in X, z \in Z\}$ belongs to the set $X \times Y = \{(x, y) : x \in X, y \in Y\}$. Because the first element in each of these ordered products is drawn from the same set X, this means that $\forall z \in Z, z \in Y \Rightarrow Z \subseteq Y$.

   Because we have proven $Y \subseteq Z$ and $Z \subseteq Y$, we can conclude $Y = Z$ if $X \times Y = X \times Z$.

2. **Section 1.2, Question 6**

   To prove $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$, we prove $(A \backslash B) \cup (B \backslash A) \subseteq (A \cup B) \backslash (A \cap B)$ and $(A \cup B) \backslash (A \cap B) \subseteq (A \backslash B) \cup (B \backslash A)$.

   1. $\forall x \in (A \backslash B) \cup (B \backslash A), x \in (A \cup B) \backslash (A \cap B)$
      Let us consider the case in which $x \in A$. If $x \in A$, we are guaranteed that $x \in (A \backslash B) \Rightarrow x \notin B$. Furthermore, because $x \in A \Rightarrow x \in (A \cup B)$ and since $x \notin B \Rightarrow x \notin (A \cap B)$. Thus, $x \in (A \cup B) \backslash (A \cap B)$.
      Let us consider the case in which $x \in B$. If $x \in B$, we are guaranteed that $x \in (B \backslash A) \Rightarrow x \notin A$. Furthermore, because $x \in B \Rightarrow x \in (A \cup B)$ and since $x \notin A \Rightarrow x \notin (A \cap B)$. Thus, $x \in (A \cup B) \backslash (A \cap B)$.
   2. $\forall x \in (A \cup B) \backslash (A \cap B), x \in (A \backslash B) \cup (B \backslash A)$
      Let us consider the case in which $x \in A$. Because $x \notin (A \cap B) \Rightarrow x \notin B$. Thus, since $x \in A$ and $x \notin B$, then $x \in (A \backslash B) \Rightarrow x \in (A \backslash B) \cup (B \backslash A)$.
      Let us consider the case in which $x \in B$. Because $x \notin (A \cap B) \Rightarrow x \notin A$. Thus, since $x \in B$ and $x \notin A$, then $x \in (B \backslash A) \Rightarrow x \in (A \backslash B) \cup (B \backslash A)$.

3. **Section 1.3, Question 2**

Let us define $s = \frac{1+\sqrt{5}}{2} \in \mathbb{R}$. To prove this statement, we use induction:

1 Base cases: $n = 0$ and $n = 1$

We first show that $F_n \leq s^{n-1}$ holds for $n = 0$ and $n = 1$. For the $n = 0$ case, $F_0 = 0 \leq \frac{1}{s}$. For the $n = 1$ case, $F_1 = 1 \leq s^0$.

2 Inductive step: Show $F_n \leq s^{n-1}$ for $n \geq 2$

Our inductive hypothesis for this case is that both $F_{n-1} \leq s^{n-2}$ and $F_{n-2} \leq s^{n-3}$, and we must now show that $F_n \leq s^{n-1}$. We first investigate the value of F_n below:

$$F_n = F_{n-1} + F_{n-2} \leq s^{n-2} + s^{n-3}$$

Note that $s^{n-2}+s^{n-3} = s^{n-1}(\frac{1}{s}+\frac{1}{s^2})$. Because $\frac{1}{s}+\frac{1}{s^2} = 1$, we know that $s^{n-2}+s^{n-3} = s^{n-1}$. Thus, we can restate the previous inequality as:

$$F_n \leq s^{n-1}$$

and so we have proven this step.

4. **Section 1.4, Question 2**

a) $f(x) = x^2$

b) $f(x) = |x-2| + 1$

5. **Section 1.4, Question 6**

To prove that statements (i) and (ii) are equivalent, we prove the following statements:

1 If (i), then (ii)

g_1 and g_2 have the same domain and co-domain. However, because they are distinct functions, this means $\exists z \in Z$ s.t. $g_1(z) \neq g_2(z)$. For this z, $f \circ g_1(z) \neq f \circ g_2(z)$. This is because inputs $g_1(z) \neq g_2(z)$ and so because f is injective, $f(g_1(z)) \neq f(g_2(z))$ or $f \circ g_1(z) \neq f \circ g_2(z)$. Thus, $\exists z \in Z$ s.t. $f \circ g_1(z) \neq f \circ g_2(z)$ and so we can conclude that $f \circ g_1$ and $f \circ g_2$ are distinct.

2 If (ii), then (i)

To prove that f is injective, we will proceed by contradiction and assume that f is not injective. Ths means $\exists\ x, x' \in X$ s.t. $f(x) = f(x')$ and $x \neq x'$. Let us define set $Z = \{u\}$ and distinct functions $g_1, g_2 : Z \to X$ where $g_1(u) = x$ and $g_2(u) = x'$. By assumption (ii), this means that $f \circ g_1, f \circ g_2 : Z \to Y$ are distinct. In order for these two functions to be distinct, $f(g_1(u)) \neq f(g_2(u))$ or $f(x) \neq f(x')$. Thus, our assumption that f is not injective is contradicted, and so we have proven that if (ii), then f is injective.