Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head here.

Math 225- HW 11 Due: Dec 9 by Midnight

Submit the first two problems, along with any three additional problems of your choice.

1. Two linear operators U and T on a finite dimensional vector space are called simultaneously diagonalizable if there exist an ordered basis β such that both $[T]_\beta$ and $[U]_\beta$ are diagonal. Similarly A,B are simultaneously diagonalizable if there exist Q invertible such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal.

   - Prove that if U and T simultaneously diagonalizable then U and T commute. i.e. $UT = TU$
   - Conclude that A,B are simultaneously diagonalizable then A,B commute
   - Let T be diagonalizable linear operator on a finite dimensional vector space, then T and T^m are simultaneously diagonalizable for any m positive integer.

2. Let T, U be a linear operator on a vector space V, and let v be a non zero vector in V.

   a) Show that E_λ for any eigenvalue λ of T is a T-invariant subspace of V.
   b) Show that T-cyclic subspace generated by v is a T-invariant subspace of V.
   c) Let W be the T-cyclic subspace generated by v. Then for any $w \in V$, $w \in W$ iff $w = g(T)v$ for some polynomial g.
   d) Let V be T- cyclic subspace of itself. Show that if U commutes with T then $U = g(T)$ for some polynomial g.
   e) If V is two dimensional then either V is T-cyclic subspace of itself or $T = cI$.

3. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, ..., \lambda_k$. Prove that

$$\text{span}(\{x \in V : \ x \text{ is an eigenvector of } T\}) = E_{\lambda_1} \bigoplus E_{\lambda_2} \bigoplus ... \bigoplus E_{\lambda_k}$$

4. Let T be a linear operator on a finite dimensional vector space V, and W be an invariant subspace of V. Suppose that $v_1, v_2, ..., v_n$ are eigenvectors of T corresponding to distinct eigenvalues.

   (a) Prove that if $v_1 + v_2 + ... + v_n$ is in W, then v_i is in W for all i. (Use induction)
   (b) Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable. Hint: Use the fact that any element of the T-invariant subspace is a linear combination of some eigenvalues, and part a).
   (c) Us part a) to show that V is a T-cyclic subspace of itself. Hint: Pick a vector that gives a basis to V

5. Let T be a linear operator on a finite dimensional vector space V.

   (a) Prove that T is diagonalizable if and only if V is the direct sum of one-dimensional T-invariant subspaces.
   b) Let $V = W_1 \bigoplus W_2 \bigoplus ... \bigoplus W_k$ where $W_1, W_2, ...W_k$ are T-invariant subspaces. Prove that

$$\det(T) = \det(T_{W_1}) \cdot \det(T_{W_2}) \cdot ... \cdot \det(T_{W_k})$$

6. Prove the parallelogram law on an inner product space V;

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \text{for all } x, y \in V$$

7. Let V be a finite dimensional inner product space over $\mathbb{F}$ and let $S = \{v_1, v_2, ...v_n\}$ be an orthanormal subset of V. Show that if If S is a basis for V then for any $x, y \in V$ one has

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}$$

(This is called Parseval's equality)

8. Let $V = C[0,1]$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$. Let $W = \text{Span}\{t, \sqrt{t}\}$.

   a) Find an orthonormal basis for W. (I suggest you to practice Gram-Schmidt process -problem 2 of Section 6.2 till you feel comfortable)

   b) Let $h(t) = t^2$. Use the orthogonal basis obtained in part a) to obtain the closest approximation of h in W. Use Theorem 6.6

   c) Let $V = C([-1.1])$ Let W_e denote the subspace of V that includes all even functions. Find $W_e^{\perp}$.