MATH 241 PSET 10

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1.

Let us first define the PDF of r.v. X as $f_X(x) = e^{-x}$ and r.v. $Y = g(X)$ where $g(x) = e^{-x}$. Because g is differentiable and strictly decreasing, we can compute the PDF of Y as:

$$f_Y(y) = f_X(x)|\frac{dx}{dy}|$$

where $x = g^{-1}(y)$, or $x = -ln(y)$ for $0 < y \leq 1$. Thus, we have that:

$$f_Y(y) = f_X(-ln(y))|\frac{1}{\frac{dy}{dx}}|$$
$$f_Y(y) = f_X(-ln(y))|\frac{1}{-e^{-x}}|$$
$$f_Y(y) = f_X(-ln(y))| - e^{-ln(y)}|$$
$$f_Y(y) = y\frac{1}{y} = 1$$

Thus, we have that the PDF of e^{-X} can be given by 1 for $0 < y \leq 1$.

2.

We compute the joint PDF $f_{T,W}(t, w)$ for random variables T and W. To do so, we first compute the absolute value of the Jacobian matrix $\frac{\partial(t,w)}{\partial(x,y)}$, which is given by:

$$\frac{\partial(t, w)}{\partial(x, y)} = \begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Thus, we get that $|\frac{\partial(t,w)}{\partial(x,y)}| = |1(-1) - 1(1)| = | - 2| = 2$. From this, using the Change of Variables Theorem, we can compute $f_{T,W}(t, w)$ as:

$$f_{T,W}(t,w) = f_{X,Y}(x,y)|\frac{\partial(x,y)}{\partial(t,w)}| = f_{X,Y}(x,y)|\frac{\partial(t,w)}{\partial(x,y)}|^{-1} = \frac{f_{X,Y}(x,y)}{2}$$

where $f_{X,Y}(x,y)$ is the joint PDF of random variables X and Y. Note that because X and Y are independent, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, where $f_X(x)$ and $f_Y(y)$ are the PDFs for X and Y, respectively. Thus, we have that:

$$f_{T,W}(t,w) = \frac{f_X(x)f_Y(y)}{2}$$

Note that because $T = X + Y$ and $W = X - Y$, we can express x in the above equation as $\frac{t+w}{2}$ and y as $\frac{t-w}{2}$. Thus, we have:

$$\begin{aligned} f_{T,W}(t,w) = \frac{f_X(\frac{t+w}{2})f_Y(\frac{t-w}{2})}{2} &= \frac{1}{2}\frac{e^{-\frac{1}{2}(\frac{t+w}{2})^2}}{\sqrt{(2\pi)}}\frac{e^{-\frac{1}{2}(\frac{t-w}{2})^2}}{\sqrt{2\pi}} = \frac{1}{4\pi}e^{-\frac{1}{8}[-(t+w)^2-(t-w)^2]} \\ &= \frac{1}{4\pi}e^{-2(t^2+w^2)} = \frac{1}{4\pi}e^{-2t^2}e^{-2w^2} \end{aligned}$$

Thus, because we can factor the joint PDF $f_{T,W}$ into a function of t times a function of w, we can conclude that random variables T and W are independent.

3.

Let us define r.v. $T = U + X$. Let us define the PDFs of random variables U and X as $f_U(u) = \frac{1}{1-0} = 1$ for $0 \leq u \leq 1$ and $f_X(x) = e^{-x}$ for $x \geq 0$. We compute the PDF $f_T(t)$ of r.v. T below:

$$f_T(t) = \int_{-\infty}^{\infty} f_U(t-x)f_X(x)dx$$

Because $f_U(u) = 0$ for $u \notin [0,1]$ and $f_X(x) = 0$ for $x < 0$, we must restrict this integral to $0 \leq t - x \leq 1 \Rightarrow x \leq t \leq 1 + x$ and $x \geq 0$. Upon inspection, we can see that the bounds for x vary according to the value of t: when $0 \leq t \leq 1$, x is constrained to $(0,t)$ and when $t > 1$, x is constrained to $(t-1,t)$. Thus, the PDF of T can be given as a piecewise function:

$$f_T(t) = \begin{cases} \int_0^t f_U(t-x)f_x(x)dx = \int_0^t e^{-x}dx = -[e^{-t}-1] = 1 - e^{-t} & \text{for } 0 \leq t \leq \\ \int_{t-1}^t f_U(t-x)f_x(x)dx = \int_{t-1}^t e^{-x}dx = -[e^{-t} - e^{1-t}] = -[e^{-t} - ee^{-t}] = (e-1)e^{-t)} & \text{for } t > 1 \end{cases}$$

4.

Let us denote the number of ticket sold for movie i of the year as $T_i \sim Pois(\lambda_2)$. Thus, the number of movie tickets sold next year can be given as: $T = \Sigma_{i=1}^{N} T_i$. We compute $\mathbb{E}[T]$ below:

$$\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T|N]] = \mathbb{E}[\mathbb{E}[\Sigma_{i=1}^{N} T_i|N]] = \mathbb{E}[\Sigma_{i=1}^{N} \mathbb{E}[T_i|N]]$$

Because the number of tickets sold for a given movie, T_i, is independent of N, $\mathbb{E}[T_i|N] = \mathbb{E}[T_i]$ and $Var(T_i|N) = Var(T_i)$. Thus, we get that:

$$\mathbb{E}[T] = \mathbb{E}[\Sigma_{i=1}^{N} \mathbb{E}[T_i]] = \mathbb{E}[N\lambda_2] = \lambda_2 \mathbb{E}[N] = \lambda_1 \lambda_2$$

Note that in the above computations, we have computed $\mathbb{E}[T|N] = N\lambda_2$. We can compute the $Var(T)$ as such:

$$\begin{aligned}
Var(T) &= \mathbb{E}[Var(T|N)] + Var(\mathbb{E}[T|N]) \\
Var(T) &= \mathbb{E}[Var(\Sigma_{i=1}^{N} T_i|N)] + Var(\mathbb{E}[T|N]) \\
Var(T) &= \mathbb{E}[\Sigma_{i=1}^{N} Var(T_i|N)] + Var(N\lambda_2) \\
Var(T) &= \mathbb{E}[\Sigma_{i=1}^{N} Var(T_i)] + \lambda_2^2 Var(N) \\
Var(T) &= \mathbb{E}[N\lambda_2] + \lambda_2^2 Var(N) \\
Var(T) &= \lambda_2 \mathbb{E}[N] + \lambda_2^2 Var(N) \\
Var(T) &= \lambda_2 \lambda_1 + \lambda_2^2 \lambda_1 = \lambda_1 \lambda_2 (1 + \lambda_2)
\end{aligned}$$

5. We compute $\mathbb{E}[Y]$ through Adam's Law as:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[c] = c$$

We compute $\mathbb{E}[XY]$ through Adam's Law as:

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\mathbb{E}[Y|X]] = \mathbb{E}[cX] = c\mathbb{E}[X]$$

Using the formula of covariance, we can compute the $Cov(X, Y)$ as:

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = c\mathbb{E}[X] - c\mathbb{E}[X] = 0$$

Because $Cov(X, Y) = 0$, $Corr(X, Y) = 0$ and so we can conclude that X and Y are uncorrelated.

6. Anish Lakkapragada. I worked independently.