Problem set 3
Due Thursday February 6 at 11pm

Exercise 3.1 (5 points). Prove that $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Exercise 3.2 (10 points; Rudin 2.2+2.3, modified). A real number x is called *algebraic* if there exist $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{Z}$, with $a_0 \neq 0$, such that

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0.$$

(1) Prove that $\sqrt{5}$ and $\sqrt{2+\sqrt{3}}$ are algebraic.

(2) Prove that the set of all algebraic real numbers is countable. (You may use without proof the fact that a polynomial of degree n has at most n roots.)

(3) Prove that there exist real numbers which are not algebraic.

Exercise 3.3 (5 points). Suppose $a, b \in \mathbb{R}$ with $a < b$. Prove that there are uncountably many irrational numbers in the interval (a, b).

Exercise 3.4 (10 points). Are the following sets finite, countable or uncountable? Prove your answers.

(1) The set of all finite subsets of $\mathbb{N}$.

(2) The set of all subsets of $\mathbb{N}$.

(3) The set of all functions $f : \mathbb{Q} \to \mathbb{Q}$.

Exercise 3.5 (10 points). For each of the following, determine whether X with the distance function d is a metric space, and prove your answer.

(1) $X = \mathbb{R}$, $d(x, y) = |x^2 - y^2|$

(2) $X = \mathbb{R}$, $d(x, y) = |x - 2y|$

(3) $X = \mathbb{R}$, $d(x, y) = \frac{|x-y|}{1+|x-y|}$

(4) $X = \mathbb{R}^2$, $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ [where $x = (x_1, x_2)$ and $y = (y_1, y_2)$]

(5) $X = \mathbb{R}^2$, $d(x, y) = |x_1 - y_2| + |x_2 - y_1|$ [where $x = (x_1, x_2)$ and $y = (y_1, y_2)$]