MATH 255 PSET 6

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1.

Pick $n \in \mathbb{N}$. We can construct $S'_n = \{N_{\frac{1}{n}}(x) : x \in K\} \supset K$ as an open cover of K. Because K is compact, every open cover of K has a finite subcover. So open cover S'_n has a subcover which we can define as: $S_n = \{N_{\frac{1}{n}}(x_i^{(n)}) : i = 1 \dots m_n\} \subset S'_n$ where m_n is the number of points (in K) required for S_n to be an open cover of K. We can then define the subset $C' = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{m_n} x_j^{(n)}$ which is the union of all the required points for each finite subcover S_n to cover K. Because C' is a countable union of finite sets[1], C' is at most countable. Furthermore, $\forall x \in C', x \in K \implies C' \subset K$.

We now show that C' is a *dense* subset of K. To do so, we show $\forall x \in K, x$ is either in C' or x is a limit point of C'. We prove this occurs with casework:

1. **Case One:** $x \in C'$

   In this case, our job is done.

2. **Case Two:** $x \notin C'$

   In this case, we WTS x is a limit point of C' or that $\forall\ \epsilon > 0, \exists\ p \in N_\epsilon(x)$ s.t. $p \neq x$ and $p \in C'$. Pick $\epsilon > 0$. Note that $\forall n \in \mathbb{N}, C'$ contains all the points which neighborhoods with size $\frac{1}{n}$ will cover K. By the Archmidean property, $\exists\ n \in \mathbb{N}$ s.t. $n(1) = n > \frac{1}{\epsilon} \implies \exists\ n$ s.t. $\frac{1}{n} < \epsilon$. We proceed with this value of n. Because S_n is an open cover of K, $x \in K \implies$ x is contained in some set[2] of $S_n \implies \exists\ 1 \leq k \leq m_n$ s.t. $x \in N_{\frac{1}{n}}(x_k^{(n)})$ where $x_k^{(n)} \in C'$ and $x \neq x_k^{(n)}$ (given by $x \notin C'$). Thus, $d(x, x_k^{(n)}) < \frac{1}{n} \implies N_{\frac{1}{n}}(x)$ contains some $x_k^{(n)} \in C'$. Because $\frac{1}{n} < \epsilon, N_{\frac{1}{n}}(x) \subset N_\epsilon(x)$ and so $N_\epsilon(x)$ contains some $x_k^{(n)} \in C'$ where $x \neq x_k^{(n)}$. Thus, $\forall\ \epsilon > 0, N_\epsilon(x)$ contains some $p \in C'$ s.t. $p \neq x$. So x is a limit point of C'.

2.

Let $\{G_i\}$ be an open cover of $\{x_n\ | n \in \mathbb{N}\} \cup \{x\}$. This means that $\exists$ some open set $G_j \in \{G_i\}$ s.t. $x \in G_j$. Because G_j is open, all points of G_j are interior points of $G_j \implies x$ is an interior point of $G_j \implies \exists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset G_j$. Because $(x_n) \to x \implies \exists\ N$

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[1] For clarity, the nth finite set is given by $\{x_1^{(n)}, \dots, x_{m_n}^{(n)}\}$.

[2] i.e. a neighborhood

s.t. $\forall n \geq N, d(x_n, x) < \epsilon$. Thus, this means that $N_\epsilon(x)$ will contain x and $x_N, x_{N+1}, \ldots$. Because $N_\epsilon(x) \subset G_j$, this means x and $x_N, x_{N+1}, \ldots$ are contained in G_j. Now for each of the finitely many points $x_1, \ldots, x_{N-1}$ (all of which are contained in $\{G_i\}$), we can pick a given set in $\{G_i\}$ which contains this point. Let $G_{n_k} \in \{G_i\}$ be the set which contains the kth point x_k where $1 \leq k \leq N-1$. Then $G' = G_j \cup \bigcup_{i=1}^{N-1} G_{n_k}$ covers $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$. Because $G' \subset \{G_i\} \implies G'$ is a finite subcover of $\{G_i\}$. Thus we have shown all open covers of $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ have a finite subcover $\implies \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is compact.

3.

We prove that $\lim\limits_{n\to\infty} \frac{2n+1}{3n-1} = \frac{2}{3}$ by showing that the sequence $(p_n) \to \frac{2}{3}$ in metric space $\mathbb{R}$ where $p_n = \frac{2n+1}{3n-1}$. Pick $\epsilon > 0$. We now aim to find $N \in \mathbb{N}$ s.t. $\forall n \geq N, d(p_n, \frac{2}{3}) < \epsilon$ or expressed more simply[3], we aim to find $N \in \mathbb{N}$ s.t. $\forall n \geq N, d(p_n, \frac{2}{3}) = d(\frac{2n+1}{3n-1}, \frac{2}{3}) = |\frac{2n+1}{3n-1} - \frac{2}{3}| = |\frac{3(2n+1)-2(3n-1)}{3(3n-1)}| = |\frac{5}{3(3n-1)}| < \epsilon$. We solve the $|\frac{5}{3(3n-1)}| < \epsilon$ inequality for n below:

$$|\frac{5}{3(3n-1)}| < \epsilon$$

Because $n \in \mathbb{N} \implies n \geq 1 \implies \frac{5}{3(3n-1)} > 0 \implies |\frac{5}{3(3n-1)}| = \frac{5}{3(3n-1)}$ and so we can proceed removing the absolute value term:

$$\begin{aligned}
\frac{5}{3(3n-1)} &< \epsilon \\
5 &< \epsilon(9n-3) \\
\frac{5}{\epsilon} &< 9n-3 \\
n &> \frac{1}{9}(\frac{5}{\epsilon}+3)
\end{aligned}$$

Thus, we see $d(p_n, \frac{2}{3}) < \epsilon$ for any $n > \frac{1}{9}(\frac{5}{\epsilon}+3)$. Thus, we aim to choose for N any nautral number $> \frac{1}{9}(\frac{5}{\epsilon}+3)$. By Archmidean property, $\exists\, m \in \mathbb{N}$ s.t. $m(1) > \frac{1}{9}(\frac{5}{\epsilon}+3)$ and so we can simply take this m to be our choice of N. Thus we have shown $\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\forall n \geq N, d(p_n, \frac{2}{3}) < \epsilon \implies (p_n) \to \frac{2}{3} \implies \lim\limits_{n\to\infty} \frac{2n+1}{3n-1} = \frac{2}{3}$.

4.

Lemma 0.1 *Let $x, y \in \mathbb{R}$. We will prove $||x| - |y|| \leq |x-y|$. By Triangle Inequality, we know $|x+y| \leq |x| + |y|$ and thus we can show these two facts:*

[3]Because we are operating in the metric space $\mathbb{R}$ with the standard distance function, $d(x,y) = |x-y|$.

1. *By Triangle Inequality, we know* $|x| + |y - x| \geq |x + y - x|$ *and so we have:*

$$|x| + |y - x| \geq |x + y - x|$$
$$|y - x| \geq |y| - |x|$$
$$|x - y| \geq |y| - |x|$$

2. *By Triangle Inequality, we know* $|y| + |x - y| \geq |y + x - y|$ *and so we have:*

$$|y| + |x - y| \geq |y + x - y|$$
$$|x - y| \geq |x| - |y|$$

Thus, we know the two facts: $|x - y| \geq |y| - |x|$ *and* $|x - y| \geq |x| - |y|$ *which together imply* $|x - y| \geq \pm(|x| - |y|) \implies |x - y| \geq ||x| - |y||$.

We WTS sequence $|x_n|$ will converge to $|x|$. Pick $\epsilon > 0$. To show $(|x_n|) \to |x|$, we must find some $N \in \mathbb{N}$ s.t. $\forall n \geq N, d(|x_n|, |x|) = ||x_n| - |x|| < \epsilon$. Because $(x_n) \to x \implies \exists\ M \in \mathbb{N}$ s.t. $\forall n \geq M, d(x_n, x) < \epsilon \implies \forall n \geq M, |x_n - x| < \epsilon \implies$ by (**Lemma 0.1**) $\forall n \geq M, ||x_n| - |x|| \leq |x_n - x| < \epsilon \implies \forall n \geq M, d(|x_n|, |x|) = ||x_n| - |x|| < \epsilon$. Thus, we can simply set $N = M$ and so we have shown $\forall \epsilon > 0, \exists\ N \in \mathbb{N}$ s.t. $\forall n \geq N, d(|x_n|, |x|) < \epsilon$. This proves $|x_n| \to |x|$.

We now show that the converse is not true. Let us define sequence (x_n) in metric space $\mathbb{R}$ where $x_n = -1$. Because every element in this sequence is equal to -1, $(x_n) \to -1$. We can now define sequence (y_n) where $y_n = |x_n| = |-1| = 1$. Because every element in (y_n) is equal to $1, (y_n) \to 1$. Expressed differently, $y_n = |x_n| \to |1|$. So we have found a case where $|x_n| \to |1|$ but $x_n \not\to 1$ and thus we have disproved the converse of this statement.