Math 244 - Problem Set 2

due Monday, February 3, 2025, at 11:59pm

Section 1.5

5. Prove the associativity of composing relations: if R, S, T are relations such that $(R \circ S) \circ T$ is well-defined, then $R \circ (S \circ T)$ is also well-defined and equals $(R \circ S) \circ T$.

Section 1.6

3. Prove that a relation R is transitive if and only if $R \circ R \subseteq R$.

6. Describe all relations on a set X that are equivalences and orderings at the same time.

Section 2.1

4. Let $(X, \leq), (Y, \preceq)$ be ordered sets. We say that they are *isomorphic* (meaning that they "look the same" from the point of view of ordering) if there exists a bijection $f : X \to Y$ such that for every $x, y \in X$, we have $x \leq y$ if and only if $f(x) \preceq f(y)$. *Note from Prof. Hall: In the book this problem has four parts, but you are only being asked to do parts (a) and (b).*

   (a) Draw Hasse diagrams for all non-isomorphic 3-element posets.

   (b) Prove that any two n-element linearly ordered sets are isomorphic. *The textbook has a hint to this problem in the back.*

Section 2.2

2. (a) Consider the set $\{1, 2, \ldots n\}$ ordered by the divisibility relation $|$ (see Example 2.1.2). What is the maximum possible number of elements of a set $X \subseteq \{1, 2, \ldots n\}$ that is ordered linearly by the relation $|$ (such a set X is called a *chain*)?

   (b) Solve the same question for the set $2^{\{1,2,\ldots n\}}$ ordered by the relation $\subseteq$ (see Example 2.1.3).

3. (optional bonus problem) Let $\mathrm{le}(X, \preceq)$ denote the number of linear extensions of a partially ordered set $(X, \preceq)$. Prove:

   (a) $\mathrm{le}(X, \preceq) = 1$ if and only if $(X, \preceq)$ is a linear ordering;

   (b) $\mathrm{le}(X, \preceq) \leq n!$, where $n = |X|$ (you may want to read Chapter 3 first).