Math 244 - Problem Set 3

due Monday, February 10, 2025, at 11:59pm

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Section 2.3

1. A linear extension $\leq$ of poset $\mathcal{B}_2$ is a total ordering where $\forall x, y \in \mathcal{B}_2, x \subseteq y \implies x \leq y$. In such a total ordering, all elements should be comparable. This means that we could write them all out in a sorted list, according to our linear extension. We try to create a total ordering for $\mathcal{B}_2$ by trying to assemble a sorted list of $\mathcal{B}_2$. First, because $\forall A \neq \emptyset \in \mathcal{B}_2, \emptyset \subset A$, the $\emptyset$ must be our lowest element in this ordering. Similarily, because $\forall A \neq \{1,2\} \in \mathcal{B}_\in, A \subset \{1,2\}$, $\{1,2\}$ must be our greatest element in this ordering. Thus, we are left with two remaining elements, $\{1\}$ and $\{2\}$, with two remaining positions. Because there are $2! = 2$ ways to order two elements, we have that two unique listings (i.e. two unique total orderings) are possible for $\mathcal{B}_2 \implies \mathcal{B}_2$ has two linear extensions.

   We now proceed in the same fashion to find all possible linear extensions of $\mathcal{B}_3$: we find all possible sorted orderings of all elements in $\mathcal{B}_3$. Identical to our reasoning above, $\emptyset$ must be the smallest element and $\{1,2,3\}$ must be the largest element in the list. Thus the remaining elements are $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$, and $\{1,2,3\}$.

   There are now two possible options for how these elements can be ordered. Consider the distinct numbers $a, b, c \in \{1,2,3\}$. The following two orderings work:

$$\textbf{Order 1}: \emptyset, \underbrace{\{a\}, \{b\}, \{c\}}_{\textbf{Seq 1.1}}, \underbrace{\{a,b\}, \{a,c\}, \{b,c\}}_{\textbf{Seq 1.2}}, \{a,b,c\}$$

$$\textbf{Order 2}: \emptyset, \underbrace{\{a\}, \{b\}}_{\textbf{Seq 2.1}}, \{a,b\}, \{c\}, \underbrace{\{a,c\}, \{b,c\}}_{\textbf{Seq 2.2}}, \{a,b,c\}$$

The total number of possible total orderings for $\mathcal{B}_3$ is the sum of possible orderings for order 1 and order 2. We compute the number of orderings possible for both:

1. **Order 1**

   In this ordering, there are $3! = 6$ different ways to order $\{1,2,3\} \implies$ there are 6 unique ways to order **Seq 1.1**. Similarily, there are $3! = 6$ different ways to order **Seq 1.2**, given by $\{a,b\}, \{a,c\}, \{b,c\}$[1]. Thus we have $6 \times 6 = 36$ total orderings of $\mathcal{B}_3$ for Order 1.

2. **Order 2**

   We first look at **Seq 2.1**. We have ${}_3P_2 = 6$ unique orderings of 2 elements selected from 3 elements $\implies$ **Seq 2.1** has 6 possible orderings. Note that the selection of a, b in this sequence will naturally lead to only one possible option for the next elements in Order 2: $\{a,b\}$ and $\{c\}$. Next we look at the possible orderings for **Seq 2.2**. For this sequence, either $\{a,c\}$ or $\{b,c\}$ can be placed first. Thus, there are 2 possible orderings for **Seq 2.2**. So in total, there are $6 \times 2 = 12$ total orderings of $\mathcal{B}_3$ for Order 2.

Thus, for both Order 1 and Order 2, we have $36 + 12 = 48$ unique total orderings of $\mathcal{B}_3 \implies \mathcal{B}_3$ has 48 linear extensions.

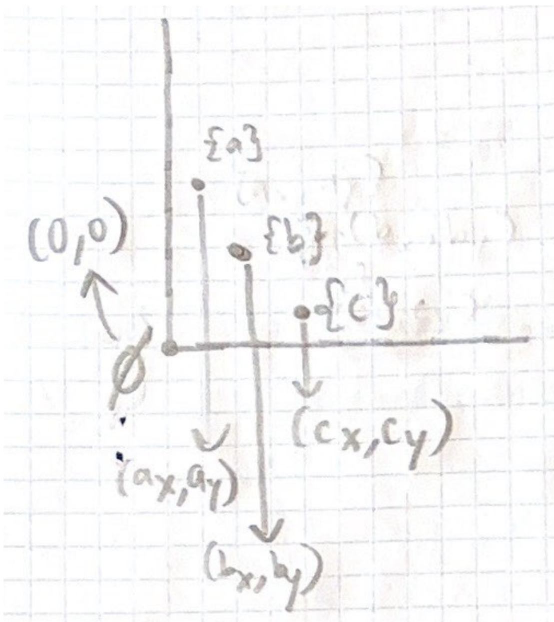
5. To show that not every finite poset admits an embedding into the poset $(\mathbb{N}^2, \preceq)$, we demonstrate that for the finite poset $\mathcal{B}_3$, such an embedding is not possible.

   Let us define distinct numbers $a, b, c \in \{1,2,3\}$ and define $a_x, a_y, b_x, b_y, c_x, c_y \in \mathbb{N}$. WLOG[2], let us define $a_x < b_x < c_x$. Then, we can provide

[1] Note that any possible ordering works as they are all after $\{a\}, \{b\}, \{c\}$ (**Seq 1.1**) in the sequence.

[2] We are using the generic variables a, b, c to show that this can occur for any ordering of numbers in [3].

the following diagram to show that after embedding four elements $\emptyset, \{a\}, \{b\}, \{c\} \in \mathcal{B}_3$ at respective locations $(0,0), (a_x, a_y), (b_x, b_y), (c_x, c_y) \in (\mathbb{N}^2, \preceq)$, then it will be impossible for us to embed $\{a, c\}$ that respects the embedding relationship[3]. Here is our diagram of these four elements embedded in $(\mathbb{N}^2, \preceq)$:



Let us define $d_x, d_y \in \mathbb{N}$, where we want to embed $\{a, c\}$ at (d_x, d_y). Because $\{a\} \subset \{a, c\} \implies (a_x, a_y) \preceq (d_x, d_y) \implies a_x \leq d_x$ and $a_y \leq d_y$. Similarily, because $\{c\} \subset \{a, c\} \implies (c_x, c_y) \preceq (d_x, d_y) \implies c_x \leq d_x$ and $c_y \leq d_y$.

As shown in our diagram, we had to place $c_x > b_x$ so that the following would not be met: $c_x \leq b_x$ and $c_y \leq b_y \implies (c_x, c_y) \preceq (b_x, b_y) \implies \{c\} \subset \{b\}$, which is a contradiction. By this same argument, we required that $a_y > b_y$ so that $\{b\} \not\preceq \{a\}$.

Thus we require that $d_x \geq c_x > b_x \implies b_x < d_x$ and $d_y \geq a_y > b_y \implies b_y < d_y$. These two necessary conditions $b_x < d_x$ and $b_y < d_y$ for placing $\{a, c\}$ force $(b_x, b_y) \preceq (d_x, d_y) \implies \{b\} \subset \{a, c\}$, which is a contradiction. Thus, we have shown there is no place for us to embed

[3] The $\emptyset$ must map on $\mathbb{N}^2$ to an element that is smaller (by the $\preceq$ relationship) to all elements in $\mathbb{N}^2$. For simplicity, we choose $(0,0)$ for this proof.

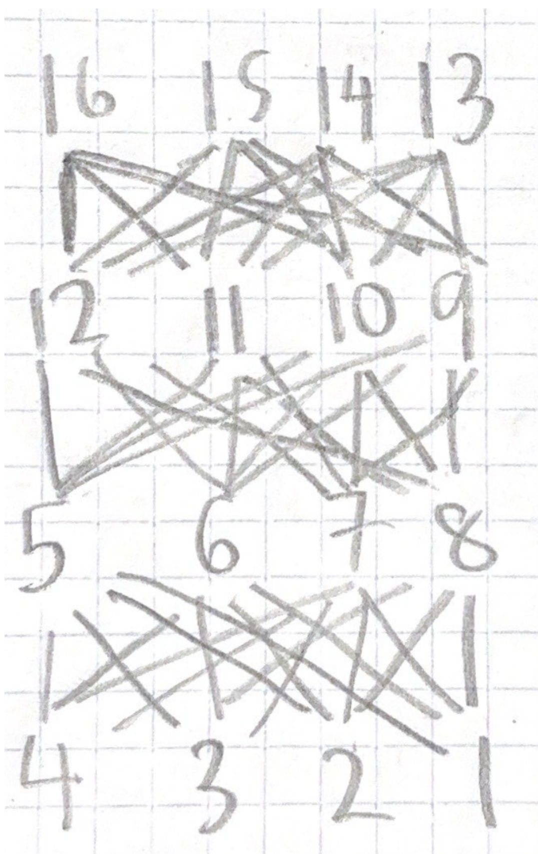
$\{a, c\}$ in $(\mathbb{N}^2, \preceq)$ in a way that respects the embedding relationship $\implies$ $\mathcal{B}_3$ does not have an embedding into $(\mathbb{N}^2, \preceq)$ $\implies$ not every finite poset admits an embedding into $(\mathbb{N}^2, \preceq)$.

Section 2.4

3. We give the sequence below:

$$4, 3, 2, 1, 8, 7, 6, 5, 12, 11, 10, 9, 16, 15, 14, 13$$

Using the relation $\preceq$ defined in Theorem 2.4.6, we can draw the following poset from this sequence:



Because the height and width of this poset is four, there are no monotone subsequences greater than five in this sequence.

4. Let us define this sequence of real numbers as poset $(X, \preceq)$. Given that this sequence is of length $kl+1$, we know from Theorem 2.4.5 that $\alpha(X, \preceq)\omega(X, \preceq) \geq kl+1$, where $\alpha(X, \preceq)$ is the length of the largest antichain of X and $\omega(X, \preceq)$ is the length of the largest chain of X.

   From Theorem 2.4.6, we know that $\alpha(X, \preceq)$ corresponds to the largest possible decreasing subsequence of X and $\omega(X, \preceq)$ corresponds to the the largest possible nondecreasing subsequence of X. We now prove this statement by contradiction and thus assume $\alpha(X, \preceq) < l+1 \implies \alpha(X, \preceq) \leq l$ and $\omega(X, \preceq) < k+1 \implies \omega(X, \preceq) \leq k$. This means $\alpha(X, \preceq)\omega(X, \preceq) \leq kl \implies \alpha(X, \preceq)\omega(X, \preceq) \not\geq kl+1$, which is a contradiction. Thus, we have proven this statement with contradiction.

Section 3.1

2. This question asks us to consider all the unique pairings of (A, B). Note that sets A and B are determined by which elements of $\{1, 2, \ldots, n\}$ are in them.

   $\forall x \in \{1, 2, \ldots, n\}$, there are three possibilities: **(i)** $x \in A \implies x \in B$ **(ii)** $x \notin A, x \in B$ **(iii)** $x \notin B \implies x \notin A$. Note that whichever of these choices one element in $\{1, 2, \ldots, n\}$ will take will have no affect on the choices possible for any of the other elements in $\{1, 2, \ldots, n\}$. Thus, we have three choices for all n different elements and so we can assemble 3^n different arrangements of sets A and $B \implies \exists\, 3^n$ ordered pairs of (A, B).

6. We prove both directions of this statement below:

   1. **If $\sqrt{n} \in \mathbb{Z} \implies n$ has an odd number of divisors**
      Let us define all the product pairings of n as the set $P = \{(a, b) \mid a, b \in \mathbb{Z},\ ab = n,\ a \leq b\}$. Note that in each of these pairs we mandate $a \leq b$ to prevent any redundant pairs. The number of distinct divisors of n is the number of distinct numbers in the pairs of P.

      We can now go through each of these (a, b) to investigate two cases:

      (i) **Case One:** $a \neq b$
      If $a \neq b \implies$ both a and b are distinct divisors of n.

(ii) **Case Two:** $a = b$

In this case, $a = b \implies ab = a^2 = n \implies a = \sqrt{n} \in \mathbb{Z}$ as a is a divisor. Note that we are guaranteed for this case to occur if $\sqrt{n} \in \mathbb{Z}$ as the pairing $(\sqrt{n}, \sqrt{n})$ will appear in P as $\sqrt{n} \in \mathbb{Z}, \sqrt{n} \times \sqrt{n} = n$, and $\sqrt{n} \leq \sqrt{n}$. In this case, there is only one unique divisor, $\sqrt{n}$, of n. Furthermore, note if $\sqrt{n} \in \mathbb{Z}$, this case will only occur once as the square root is unique.

Across all enumerations of the pairings of P, let us define the number of times we encounter case one as k. As stated before, if $\sqrt{n} \in \mathbb{Z}$, we are guaranteed to arrive at Case Two is only once. Thus, the number of distinct divisors is given by $2k + 1$ which is odd $\implies n$ has an odd number of (distinct) divisors.

2. **If n has an odd number of divisors $\implies \sqrt{n} \in \mathbb{Z}$**

We prove this statement by contrapositive and thus assume $\sqrt{n} \notin \mathbb{Z}$. We now define D as the set of all divisors of n. Note that divisors come in pairs, if a is a divisor of $n \implies \exists\ b \in [n]$ s.t. $ab = n$. Furthermore, we can be guaranteed $a \neq b$ because if $a = b$ that would imply that $a = \sqrt{n} \in D \implies \sqrt{n} \in \mathbb{Z}$ which is a contradiction.

We can denote the number of these pairs of divisors of n as k. Because both of the two elements for each of these pairs exist in D, $|D| = 2k$, which is even. This means that n has an even number of divisors. Thus we have proved this statement with contrapositive.