

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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## Math 226: HW 2, Due Sep 12 by Midnight

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1. (9 points) Part *a*) requires familiarity with the vector space  $\mathcal{F}(S, \mathbb{F})$ . Check out Example 3 of Section 1.2 which is discussed in ULA section.

- a) (6 points) We define the set of  $L^2(\mathbb{R})$  functions as

$$L^2(\mathbb{R}) := \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}$$

Prove that the set of  $L^2(\mathbb{R})$  functions defined on the real line with the operations of addition, scalar multiplication (as in the Example 3) is a vector space.

- b) (3 points) (Problem 13 in the book) Let  $V := \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$ . If  $(a_1, a_2)$  and  $(b_1, b_2)$  are elements on  $V$  and  $c \in \mathbb{R}$ , define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \quad c(a_1, a_2) = (ca_1, ca_2)$$

Is  $V$  a vector space over  $\mathbb{R}$  with these operations? Justify your answer.

### Practice Problems : Sec 1.2 : 10 - 22

2. (26 points)

- a) (7 points) (P.10, Sec 1.3) Prove that  $W_1 = \{(a_1, a_2, \dots, a_n) \in \mathbb{F}^n : a_1 + a_2 + \dots + a_n = 0\}$  is a subspace of  $\mathbb{F}^n$ , but  $W_2 = \{(a_1, a_2, \dots, a_n) \in \mathbb{F}^n : a_1 + a_2 + \dots + a_n = 1\}$  is not.

- **Remark:** This problem tells you that if  $V$  is any vector space, then any hyper-plane of  $V$  is a vector space if it passes through the origin. Take for example  $W_1$  as  $x + y + z = 0$  and  $W_2$  as  $x + y + z = 1$  as a subspace of  $\mathbb{R}^3$ .  $x + y + z = 0$ ,  $x + y + z = 1$  are the hyper-planes of  $\mathbb{R}^3$ . One is a subspace the other one is not. (Check out the Problems 11, 13, 14 in Section 1.3)

- b) (7 points) We define

$$\mathbb{Z}_2^n := \{v = (a_1, a_2, \dots, a_n) : a_i \in \mathbb{Z}_2\}$$

Recall that  $\mathbb{Z}_2^n$  defines a vector space over  $\mathbb{Z}_2$  (with addition and multiplication we discussed in class). Let  $E_n(v)$  define the number of nonzero components of  $v$ . For example if  $v = (1, 0, 1, 0, 0, 1)$  then  $E_6(v) = 3$ . Prove that the subset of  $\mathbb{Z}_2^n$  with even  $E_n$  defines a subspace of  $\mathbb{Z}_2^n$ .

Hint:  $E_n(u + v)$  can be formulated in terms of  $E_n(u)$ ,  $E_n(v)$ , and the common non-zero components of  $u, v$ .

- c) (6 points) Let  $W := \{f \in P_3(\mathbb{R}) : f(0) = f'(0), f(1) = 0\}$ . How does the elements of  $W$  look like? Show that  $W$  is a subspace of  $P_3(\mathbb{R})$ . Is it a subspace of  $P_4(\mathbb{R})$ ?
- d) (6 points) (P.18, Sec 1.3) Prove that a subset  $W$  of a vector space  $V$  is a subspace of  $V$  if and only if  $W \neq \emptyset$ , and whenever  $a \in \mathbb{F}$  and  $x, y \in W$ , then  $ax + y \in W$ .
3. (16 points) Let  $V$  be a vector space over a field  $\mathbb{F}$ , and suppose  $U \leq V$  and  $W \leq V$ . That is suppose that  $U$  and  $W$  are subspaces of  $V$ .
- a) (6 points) Prove that  $U \cap W \leq V$ .
- b) (6 points) We define  $U + W = \{u + w : u \in U, w \in W\}$ . Prove that  $U + W \leq V$ .

- c) (4 points) Find two subspaces of  $\mathbb{R}^2$  whose union is *not* a subspace of  $\mathbb{R}^2$ . Make sure to validate your answer.

• **Remark:** This shows some important things. The intersection of subspaces is a subspace, but the union of subspaces (usually) isn't. The latter one is a subspace if and only if one of the subspaces is the subset of the other one. You can prove it as an exercise. ( It is problem 19 of Sec 1.2)

**Practice Problems :** Sec 1.3 : 1, 2, 3, 8, 9, 11,12, 13, 14, 17, 19

4. (13 points)

Let  $S := \{1 + x, 1 - x^2, x^3, x^4\}$  be a subset of  $P_4(\mathbb{R})$ .

- a) (2 points) How does the elements of  $\text{Span}(S)$  look like. Write down a clear definition for  $\text{Span}(S)$ .
- b) (5 points) Find a polynomial in  $P_4(\mathbb{R})$  that can not be written as a linear combination of the elements of  $S$ . Validate your answer. Deduce that  $S$  does not generate  $P_4(\mathbb{R})$ .
- c) (6 points) Prove that  $\text{Span}(S) = \{f \in P_4(\mathbb{R}) : 2f(0) = 2f'(0) - f''(0)\}$ . Hint: Let  $A, B$  are subsets of the same set. Then to show  $A = B$ , one needs to show  $A \subseteq B$ , and  $B \subseteq A$ .