STATS 242 HW 2

January 29, 2025

Number of late days: 0; Collaborators: None.

1.

For $1 \leq i \leq n$, let us define r.v. $B_i \sim Bern(p)$, where for $i \neq j$, B_i and B_j are independent. Then, we have that $X = \Sigma_{i=1}^{n} B_i$. Because all B_is are independent, we have that:

$$M_X(t) = M_{\Sigma_{i=1}^{n} B_i}(t) = \prod_{i=1}^{n} M_{B_i}(t)$$

The MGF of r.v. B_i can be computed as: $M_{B_i}(t) = \mathbb{E}[e^{tB_i}] = e^{t(1)}P(B_i = 1) + e^{t(0)}P(B_i = 0) = pe^t + (1-p) = 1 + p(e^t - 1)$. Thus, we have $M_X(t)$ as:

$$M_X(t) = \prod_{i=1}^{n} M_{B_i}(t) = \prod_{i=1}^{n} 1 + p(e^t - 1) = [1 + p(e^t - 1)]^n$$

2.

We first start by computing the distributions of X_1 and X_2. Note that because Z_1 and Z_2 are independent normal distributions, their sum forms a normal distribution as well.

1. **Distribution of X_1**
   Since $c_1 Z_1 \sim \mathcal{N}(0, c_1^2)$ and $d_1 Z_1 \sim \mathcal{N}(0, d_1^2)$, $c_1 Z_1 + d_1 Z_2 \sim \mathcal{N}(0, c_1^2 + d_1^2)$. Finally, e_1 is just a constant and so it doesn't affect the variance so $X_1 = c_1 Z_1 + d_1 Z_2 + e_1 \sim \mathcal{N}(e_1, c_1^2 + d_1^2)$. Therefore we can set $e_1 = \mu_1$, the mean of X_1. Furthermore, the variance σ_1^2 of X_1 is equal to $c_1^2 + d_1^2$.
2. **Distribution of X_2**
   We use identical reasoning as with before. $c_2 Z_1 \sim \mathcal{N}(0, c_2^2)$ and $d_2 Z_2 \sim \mathcal{N}(0, d_2^2)$, so we have $c_2 Z_2 + d_2 Z_2 \sim \mathcal{N}(0, c_2^2 + d_2^2)$. Thus, $X_2 = c_2 Z_2 + d_2 Z_2 + e_2 \sim \mathcal{N}(e_2, c_2^2 + d_2^2)$. Therefore we can set $e_2 = \mu_2$, the mean of X_2. Furthermore, the variance σ_2^2 of X_2 is equal to $c_2^2 + d_2^2$.

We now have c_1, c_2, d_1, d_2 remaining to assign. We compute the correlation ρ between X_1 and X_2, starting by computing the covariance between X_1 and X_2:

$$\text{Cov}(X_1, X_2) = \text{Cov}(c_1 Z_1 + d_1 Z_2 + e_1, c_2 Z_1 + d_2 Z_2 + e_2)$$
$$= c_1 \text{Cov}(Z_1, c_2 Z_1 + d_2 Z_2 + e_2) + d_1 \text{Cov}(Z_2, c_2 Z_1 + d_2 Z_2, e_2) + \text{Cov}(e_1, c_2 Z_1 + d_2 Z_2 + e_2)$$

Note that because e_1 is a fixed constant, $\text{Cov}(e_1, c_2 Z_1 + d_2 Z_2 + e_2) = 0$. Also note that because Z_1 and Z_2 are independent, $\text{Cov}(Z_1, Z_2) = 0$.

$$\text{Cov}(X_1, X_2) = c_1[c_2 \text{Cov}(Z_1, Z_1) + d_2 \text{Cov}(Z_1, Z_2) + \text{Cov}(Z_1, e_2)] + d_1 \text{Cov}(Z_2, c_2 Z_1 + d_2 Z_2, e_2)$$
$$= c_1[c_2 + 0 + 0] + d_1[c_2 \text{Cov}(Z_2, Z_1) + d_2 \text{Cov}(Z_2, Z_2) + \text{Cov}(Z_2, e_2)] = c_1 c_2 + d_1[0 + d_2 + 0]$$
$$= c_1 c_2 + d_1 d_2$$

and now we compute the correlation ρ between X_1 and X_2 as:

$$\rho = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} = \frac{c_1 c_2 + d_1 d_2}{\sigma_1 \sigma_2}$$

This gives us three equations:

$$\rho \sigma_1 \sigma_2 = c_1 c_2 + d_1 d_2$$
$$c_1^2 + d_1^2 = \sigma_1^2$$
$$c_2^2 + d_2^2 = \sigma_2^2$$

to solve for four variables. As such, we arbitrarily choose to set $c_1 = 0$, giving us $d_1 = \sigma_1$ and $\rho \sigma_1 \sigma_2 = d_1 d_2 = \sigma_1 d_2 \implies d_2 = \rho \sigma_2$. Finally, we can solve for $c_2^2 = \sigma_2^2 - d_2^2 = \sigma_2^2 - \rho^2 \sigma_2^2 = \sigma_2^2(1 - \rho^2) \implies c_2 = \sigma_2 \sqrt{1 - \rho^2}$.
As a summary of my answer,

$$e_1 = \mu_1$$
$$e_2 = \mu_2$$
$$c_1 = 0$$
$$c_2 = \sigma_2 \sqrt{1 - \rho^2}$$
$$d_1 = \sigma_1$$
$$d_2 = \rho \sigma_2$$

3.

(a) We first show $\mathbb{E}[\hat{I}_n(f)] = I(f)$. Note that $\frac{f(X_i)}{g(X_i)}$ is a random variable and so its expectation is given to us by $\mathbb{E}[\frac{f(X_i)}{g(X_i)}] = \int_{-\infty}^{\infty} \frac{f(u)}{g(u)} g(u)du = \int_a^b \frac{f(u)}{g(u)} g(u)du = \int_a^b f(u)du = I(f)$. Thus:

$$\mathbb{E}[\hat{I}_n(f)] = \mathbb{E}[\frac{1}{n}\Sigma_{i=1}^n \frac{f(X_i)}{g(X_i)}] = \frac{1}{n} \cdot n \cdot \mathbb{E}[\frac{f(X_i)}{g(X_i)}] = I(f)$$

Notice that $\hat{I}_n(f)$ is essentially an average of n random variables, each of with an expectation of $I(f)$. Thus, $\hat{I}_n(f) \to I(f)$ in probability as $n \to \infty$ due to the (Weak) Law of Large Numbers.

b) We first compute $\mathrm{Var}[\hat{I}_n(f)]$. We begin by computing the variance of random variable $\frac{f(X_i)}{g(X_i)}$:

$$\mathrm{Var}[\frac{f(X_i)}{g(X_i)}] = \mathbb{E}[(\frac{f(X_i)}{g(X_i)})^2] - \mathbb{E}[\frac{f(X_i)}{g(X_i)}]^2 = \int_a^b \frac{f^2(u)}{g^2(u)} g(u)du - I^2(f) = \int_a^b \frac{f^2(u)}{g(u)} du - I^2(f)$$

Let us call define this quantity to be $\sigma^2 = \int_a^b \frac{f^2(u)}{g(u)} g(u)du - I^2(f) \in \mathbb{R}$.
and so we have:

$$\mathrm{Var}[\hat{I}_n(f)] = \mathrm{Var}[\frac{1}{n}\Sigma_{i=1}^n \frac{f(X_i)}{g(X_i)}] = \frac{1}{n^2}\Sigma_{i=1}^n \mathrm{Var}[\frac{f(X_i)}{g(X_i)}] = \frac{1}{n}\sigma^2$$

Let us define $c_n = \frac{\sqrt{n}}{\sigma} \in \mathbb{R}$. Thus, by the Central Limit Theorem, we have that:

$$c_n(\hat{I}_n(f) - I(f)) \to \mathcal{N}(0,1) \text{ as } n \to \infty$$

c) In this problem, $a = 0$ and $b = 1$, $f(x) = \cos(2\pi x)$, and $g(x) = 1$ if $x \in [0,1]$ and 0 otherwise. We first compute $I(f)$:

$$I(f) = \int_0^1 \cos(2\pi x)dx = \sin(2\pi x)\Big|_0^1 = \sin(2\pi) - \sin(0) = 0 - 0 = 0$$

and then σ^2:

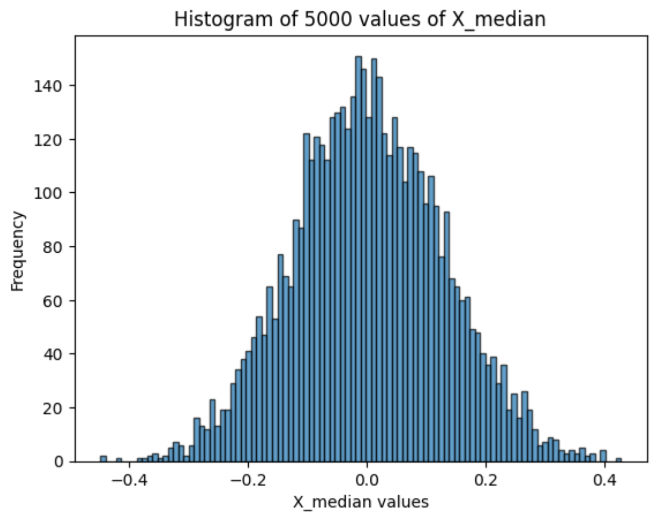
$$\sigma^2 = \int_0^1 \frac{\cos^2(2\pi x)}{1} dx - I^2(f)dx = \int_0^1 [\frac{1}{2} + \frac{\cos(4\pi x)}{2}]dx = [\frac{x}{2} + \frac{\sin(4\pi x)}{8}]\Big|_0^1$$
$$= \frac{1}{2} + \frac{1}{8}(\sin(4\pi) - \sin(0)) = \frac{1}{2}$$

and so c_n is given by:

$$c_n = \frac{\sqrt{n}}{\sigma} = \sqrt{\frac{n}{\sigma^2}} = \sqrt{2n}$$

4.

From my simulation, the mean and standard deviation of X_{median} are given by 0.00178 and 0.1265 respectively. Below is the histogram of the 5000 values of X_{median} from my simulation:



Based on the above histogram, we can see that the sampling distribution of X_{median} follows a normal distribution. We now derive the standard deviation of the sample mean $\bar{X}$. Note that 99 observations is enough for us to apply the Central Limit Theorem with confidence. Thus[1], $\bar{X} = \frac{X_1+\cdots+X_{99}}{99} \sim \mathcal{N}(\mathbb{E}[X_1], \frac{\text{Var}(X_1)}{99})$ or $\bar{X} \sim \mathcal{N}(0, \frac{1}{99})$. Thus the analytically-computed standard deviation of sample mean $\bar{X}$ is $\sqrt{\frac{1}{99}}$ or 0.1005, which is less than my calculated simulated standard deviation for X_{median} (0.1265). According to my simulation, X_{median} is more variable than $\bar{X}$.

```python
# %%

# Run all imports first and then write helper functions.

import numpy as np
import math
import matplotlib.pyplot as plt

def get_N_obs_iid_standard_normal(N):
    return np.random.normal(0, 1, N)

def compute_median(arr):
    # assume arr is numpy
    return np.median(arr)
```

[1]Note that $X_1 \dots X_{99}$ are identical distributions and so they all have the same mean and variances.

```python
# %%
N_SIMULATIONS = 5000
X_medians = []
for _ in range(N_SIMULATIONS):
    samples = get_N_obs_iid_standard_normal(99)
    X_median_curr = compute_median(samples)
    X_medians.append(X_median_curr)

X_medians = np.array(X_medians)
X_median_mean = np.mean(X_medians)
X_median_std = np.std(X_medians)

print(f"X_median mean: {X_median_mean} and std: {X_median_std}")
# %%

"""
Plot a histogram.
"""

plt.hist(X_medians, bins=100, edgecolor='black', alpha=0.7)

# Add labels and title
plt.xlabel('X_median values')
plt.ylabel('Frequency')
plt.title('Histogram of 5000 values of X_median')

# %%
```