

# PSETs Landing Page\*

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This is the documentation for using my PSET PDFs responsibly. I post these LaTeX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarly, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

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\*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

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## Math 225- HW 11 Due: Dec 9 by Midnight

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Submit the first two problems, along with any three additional problems of your choice.

- Two linear operators  $U$  and  $T$  on a finite dimensional vector space are called simultaneously diagonalizable if there exist an ordered basis  $\beta$  such that both  $[T]_\beta$  and  $[U]_\beta$  are diagonal. Similarly  $A, B$  are simultaneously diagonalizable if there exist  $Q$  invertible such that both  $Q^{-1}AQ$  and  $Q^{-1}BQ$  are diagonal.

- Prove that if  $U$  and  $T$  simultaneously diagonalizable then  $U$  and  $T$  commute. i.e.  $UT = TU$
- Conclude that  $A, B$  are simultaneously diagonalizable then  $A, B$  commute
- Let  $T$  be diagonalizable linear operator on a finite dimensional vector space, then  $T$  and  $T^m$  are simultaneously diagonalizable for any  $m$  positive integer.

- Let  $T, U$  be a linear operator on a vector space  $V$ , and let  $v$  be a non zero vector in  $V$ .

- Show that  $E_\lambda$  for any eigenvalue  $\lambda$  of  $T$  is a  $T$ -invariant subspace of  $V$ .
- Show that  $T$ -cyclic subspace generated by  $v$  is a  $T$ -invariant subspace of  $V$ .
- Let  $W$  be the  $T$ -cyclic subspace generated by  $v$ . Then for any  $w \in V$ ,  $w \in W$  iff  $w = g(T)v$  for some polynomial  $g$ .
- Let  $V$  be  $T$ -cyclic subspace of itself. Show that if  $U$  commutes with  $T$  then  $U = g(T)$  for some polynomial  $g$ .
- If  $V$  is two dimensional then either  $V$  is  $T$ -cyclic subspace of itself or  $T = cI$ .

- Let  $T$  be a linear operator on a finite-dimensional vector space  $V$ , and suppose that the distinct eigenvalues of  $T$  are  $\lambda_1, \lambda_2, \dots, \lambda_k$ . Prove that

$$\text{span}(\{x \in V : x \text{ is an eigenvector of } T\}) = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots \oplus E_{\lambda_k}$$

- Let  $T$  be a linear operator on a finite dimensional vector space  $V$ , and  $W$  be an invariant subspace of  $V$ . Suppose that  $v_1, v_2, \dots, v_n$  are eigenvectors of  $T$  corresponding to distinct eigenvalues.

- Prove that if  $v_1 + v_2 + \dots + v_n$  is in  $W$ , then  $v_i$  is in  $W$  for all  $i$ . (Use induction)
- Prove that the restriction of a diagonalizable linear operator  $T$  to any nontrivial  $T$ -invariant subspace is also diagonalizable. Hint: Use the fact that any element of the  $T$ -invariant subspace is a linear combination of some eigenvalues, and part *a*).
- Use part *a*) to show that  $V$  is a  $T$ -cyclic subspace of itself. Hint: Pick a vector that gives a basis to  $V$

- Let  $T$  be a linear operator on a finite dimensional vector space  $V$ .

- Prove that  $T$  is diagonalizable if and only if  $V$  is the direct sum of one-dimensional  $T$ -invariant subspaces.
- Let  $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$  where  $W_1, W_2, \dots, W_k$  are  $T$ -invariant subspaces. Prove that

$$\det(T) = \det(T_{W_1}) \cdot \det(T_{W_2}) \cdot \dots \cdot \det(T_{W_k})$$

6. Prove the parallelogram law on an inner product space  $V$ ;

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \text{for all } x, y \in V$$

7. Let  $V$  be a finite dimensional inner product space over  $\mathbb{F}$  and let  $S = \{v_1, v_2, \dots, v_n\}$  be an orthonormal subset of  $V$ . Show that if  $S$  is a basis for  $V$  then for any  $x, y \in V$  one has

$$\langle x, y \rangle = \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$$

(This is called Parseval's equality)

8. Let  $V = C[0, 1]$  with the inner product  $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ . Let  $W = \text{Span}\{t, \sqrt{t}\}$ .
- Find an orthonormal basis for  $W$ . ( I suggest you to practice Gram-Schmidt process -problem 2 of Section 6.2 till you feel comfortable)
  - Let  $h(t) = t^2$ . Use the orthogonal basis obtained in part *a*) to obtain the closest approximation of  $h$  in  $W$ . Use Theorem 6.6
  - Let  $V = C([-1, 1])$  Let  $W_e$  denote the subspace of  $V$  that includes all even functions. Find  $W_e^\perp$ .