

## Discretionary Note

Anish Krishna Lakkapragada

**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

**CONTENT STARTS ON NEXT PAGE.**

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## Math 226 - HW 1 Due: Sep 5 by 10:30 p.m.

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1. (14 points) Let  $A$  and  $B$  be two non-empty sets,  $f : A \rightarrow B$  be an injective function, and  $a_0$  is a fixed element of  $A$ . Define  $g : B \rightarrow A$  such that
 
$$g(b) = \begin{cases} a & \text{if } f(a) = b \\ a_0 & \text{otherwise} \end{cases}$$
  - a) (3 points) Show that  $g$  is a left inverse of  $f$ . That is for all  $a \in A$ ,  $g \circ f(a) = a$
  - b) (3 points) Explain why  $g$  might not be the right inverse of  $f$ . Under which conditions  $g$  can be a right inverse of  $f$ ?
  - c) (8 points) If  $f : A \rightarrow B$  be surjective then it would have right inverse. Define the right inverse and validate that it is indeed right inverse.
2. (10 points) Given two functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , the *composition of  $g$  with  $f$* , denoted  $g \circ f$  is the map  $g \circ f : A \rightarrow C$  given by  $(g \circ f)(a) = g(f(a))$ . Show that the set of bijection functions are closed under  $\circ$ . That is
  - a) (4 points) show if  $f$  and  $g$  are injections, then  $g \circ f$  is an injection. Hint: Recall the injective and left inverse relation.
  - b) (4 points) show if  $f$  and  $g$  are surjections, then  $g \circ f$  is a surjection. Hint: Use the definition of surjectivity first for  $g$  and then for  $f$ .
  - c) (2 points) conclude that if  $f$  and  $g$  are bijections, then  $g \circ f$  is a bijection.
3. (10 points) Let  $B = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$ .
  - a) (2 points) Show that  $B$  is closed under addition. In other words, prove “if  $x \in B$  and  $y \in B$ , then their sum  $x + y$  must be in  $B$  as well.
  - b) (3 points) Show that  $B$  is closed under multiplication. In other words, prove “if  $x \in B$  and  $y \in B$ , then their product  $xy$  must be in  $B$  as well.
  - c) (5 points) For every integer  $k \geq 1$ , prove that  $(-1 + \sqrt{2})^k \in B$  - Try to use an inductive argument for this problem.
4. (20 points) We say a function  $T : \mathbb{R} \rightarrow \mathbb{R}$  is additive iff for all  $x \in \mathbb{R}$  and for all  $y \in \mathbb{R}$ , we have that  $T(x + y) = T(x) + T(y)$ . If  $T$  is additive, show that
  - a) (5 points) for all  $x \in \mathbb{R}$  and for all integers  $n \geq 1$ , we have  $T(nx) = nT(x)$ .
  - b) (3 points)  $T(0) = 0$ .
  - c) (3 points) for all  $x \in \mathbb{R}$  we have  $T(-x) = -T(x)$ .
  - d) (5 points) for all  $x \in \mathbb{R}$  and for all rational numbers  $r \in \mathbb{Q}$ , we have  $T(rx) = rT(x)$ .
  - e) (4 points) Is it true that if  $r \in \mathbb{R}$  the  $T(rx) = rT(x)$  for all  $x \in \mathbb{R}$ ? Why or why not?
5. (16 points) Define  $\mathbb{Z}\{\sqrt{3}\} := \{a + b\sqrt{3}, \text{ where } a, b \in \mathbb{Z}\}$ . Below we will validate that  $(\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$ , with usual addition and multiplication does not define a field.
  - a) (4 points) Show that if  $a^2 - 3b^2 = \pm 1$ , then  $a + b\sqrt{3}$  has a multiplicative inverse. Hint: Check out the definition of the multiplicative inverse, you can directly give it in each case.

- b) (10 points) Show that the above statement is iff, i.e., if  $a+b\sqrt{3}$  has a multiplicative inverse then  $a^2 - 3b^2 = \pm 1$  Hint: Solve the problem first for the case when  $a, b$  are prime to each other. That will give you the understanding of the rest.
- c) (2 points) Use a) and b) to conclude that  $(\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$ , with usual addition and multiplication does not define a field.
- (optional) Prove that the set  $\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$  is indeed a field. Notice that you don't need the conditions like in b, c here.