Math 226: HW 2, Due Sep 12 by Midnight

1. (9 points) Part a) requires familiarity with the vector space $\mathcal{F}(S, \mathbb{F})$. Check out Example 3 of Section 1.2 which is discussed in ULA section.

   a) (6 points) We define the set of $L^2(\mathbb{R})$ functions as

   $$L^2(\mathbb{R}) := \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}$$

   Prove that the set of $L^2(\mathbb{R})$ functions defined on the real line with the operations of addition, scalar multiplication (as in the Example 3) is a vector space.

   b) (3 points) (Problem 13 in the book) Let $V := \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. If (a_1, a_2) and (b_1, b_2) are elements on V and $c \in \mathbb{R}$, define

   $$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad c(a_1, a_2) = (ca_1, a_2)$$

   Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

   Practice Problems : Sec 1.2 : 10 - 22

2. (26 points)

   a) (7 points) (P.10, Sec 1.3) Prove that $W_1 = \{(a_1, a_2, ..., a_n) \in \mathbb{F}^n : a_1 + a_2 + ... + a_n = 0\}$ is a subspace of $\mathbb{F}^n$, but $W_2 = \{(a_1, a_2, ..., a_n) \in \mathbb{F}^n : a_1 + a_2 + ... + a_n = 1\}$ is not.

   - **Remark:** This problem tells you that if V is any vector space, then any hyper-plane of V is a vector space if it passes through the origin. Take for example W_1 as $x + y + z = 0$ and W_2 as $x + y + z = 1$ as a subspace of $\mathbb{R}^3$. $x + y + z = 0$, $x + y + z = 1$ are the hyper-planes of $\mathbb{R}^3$. One is a subspace the other one is not. (Check out the Problems 11, 13, 14 in Section 1.3)

   b) (7 points) We define

   $$\mathbb{Z}_2^n := \{v = (a_1, a_2, ..., a_n) : a_i \in \mathbb{Z}_2\}$$

   Recall that $\mathbb{Z}_2^n$ defines a vector space over $\mathbb{Z}_2$ (with addition and multiplication we discussed in class). Let $E_n(v)$ define the number of nonzero components of v. For example if $v = (1, 0, 1, 0, 0, 1)$ then $E_6(v) = 3$. Prove that the subset of $\mathbb{Z}_2^n$ with even E_n defines a subspace of $\mathbb{Z}_2^n$.
   Hint: $E_n(u + v)$ can be formulated in terms of $E_n(u)$, $E_n(v)$, and the common non-zero components of u, v.

   c) (6 points) Let $W := \{f \in P_3(\mathbb{R}) : f(0) = f'(0), f(1) = 0\}$. How does the elements of W look like? Show that W is a subspace of $P_3(\mathbb{R})$. Is it a subspace of $P_4(\mathbb{R})$?

   d) (6 points) (P.18, Sec 1.3) Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \emptyset$, and whenever $a \in \mathbb{F}$ and $x, y \in W$, then $ax + y \in W$.

3. (16 points) Let V be a vector space over a field $\mathbb{F}$, and suppose $U \leq V$ and $W \leq V$. That is suppose that U and W are subspaces of V.

   a) (6 points) Prove that $U \cap W \leq V$.

   b) (6 points) We define $U + W = \{u + w \ : \ u \in U, \ w \in W\}$. Prove that $U + W \leq V$.

c) (4 points)Find two subspaces of $\mathbb{R}^2$ whose union is *not* a subspace of $\mathbb{R}^2$. Make sure to validate your answer.

- **Remark:** This shows some important things. The intersection of subspaces is a subspace, but the union of subspaces (usually) isn't. The latter one is a subspace if and only if one of the subspaces is the subset of the other one. You can prove it as an exercise. (It is problem 19 of Sec 1.2)

Practice Problems : Sec 1.3 : 1, 2, 3, 8, 9, 11,12, 13, 14, 17, 19

4. (13 points)

Let $S := \{1 + x, 1 - x^2, x^3, x^4\}$ be a subset of $P_4(\mathbb{R})$.

a) (2 points) How does the elements of Span(S) look like. Write down a clear definition for Span(S).

b) (5 points) Find a polynomial in $P_4(\mathbb{R})$ that can not be written as a linear combination of the elements of S. Validate your answer. Deduce that S does not generate $P_4(\mathbb{R})$.

c) (6 points) Prove that Span(S) $= \{f \in P_4(\mathbb{R}) : 2f(0) = 2f'(0) - f''(0)\}$. Hint: Let A,B are subsets of the same set. Then to show $A = B$, one needs to show $A \subseteq B$, and $B \subseteq A$.