Math 226: HW 5
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1. a) We can define $[T]_\alpha$ below as:

$$T_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4x4}$$

b) **Answers for** $[U]_\beta^\gamma, [T]_\beta, [UT]_\beta^\gamma$

$$[U]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}_{3x3} \quad [T]_\beta = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}_{3x3} \quad [UT]_\beta^\gamma = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}_{3x3}$$

Answers for $[h(x)]_\beta$ **and** $[U(h(x))]_\gamma$

$$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} [U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

c)

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{n \ x \ n}$$

2. a) For T^{-1} to be a linear operator, then $T^{-1}(cx + y) = cT^{-1}(x) + T^{-1}(y)$ where $c \in \mathbb{F}$ and $x, y \in V$.

Let us define $u_1, u_2 \in U$ and $v_1 = T(u_1), v_2 = T(u_2) \in V$. We also define $c \in \mathbb{F}$. We evaluate $T^{-1}(cv_1 + v_2)$ below:

$$\begin{aligned} T^{-1}(cv_1 + v_2) &= T^{-1}(cT(u_1) + T(u_2)) \\ T^{-1}(cv_1 + v_2) &= T^{-1}(T(cu_1 + u_2)) \\ T^{-1}(cv_1 + v_2) &= cu_1 + u_2 \end{aligned}$$

We evaluate $cT^{-1}(v_1) + T^{-1}(v_2)$:

$$cT^{-1}(v_1) + T^{-1}(v_2) = cu_1 + u_2$$

Because we have shown $T^{-1}(cv_1 + v_2) = cT^{-1}(v_1) + T^{-1}(v_2) = cu_1 + u_2$, we have shown that T^{-1} is a linear operator.

b) Let us define $(x, y) = xe_1 + ye_2 \in \mathbb{R}^2$, where $\beta_2 = \{e_1, e_2\}$ is the standard ordered basis for $\mathbb{R}^2$. We will also define the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ through the following matrix:

$$[T]_{\beta_2}^{\beta_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a, b, c, d \in \mathbb{R}$. We inspect the value of $T(x, y)$ below.

$$\begin{aligned} T(x, y) &= [T]_{\beta_2}^{\beta_2} \begin{bmatrix} x \\ y \end{bmatrix} \\ T(x, y) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ T(x, y) &= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \end{aligned}$$

Thus, we get that $T(x, y) = (ax + by, cx + dy) \in \mathbb{R}^2$.
We now prove by contradiction that for T to be a valid isomorphism, and thus bijective, $ad - bc \neq 0$. Suppose $ad - bc = 0$. This means that $\frac{b}{a} = \frac{d}{c}$. Because any input with the form $(x, -\frac{a}{b}x)$ will be part of $N(T)$, we have found multiple pre-images in U to the same image (i.e. $\mathbf{0}^2$) in $V \Rightarrow T$ is not injective $\Rightarrow T$ is not bijective $\Rightarrow T$ is not a valid isomorphism. Thus, we have proven that for T to be a valid isomorphism, $ad - bc \neq 0$ so that $N(T) = \{0\} \Rightarrow T$ is injective $\Rightarrow T$ is surjective $\Rightarrow T$ is invertible $\Rightarrow T$ isomorphism.

3. a) For proof by contrapositive, let us assume T is not injective. This means $\exists\, x, y \in V$ s.t. $T(x) = T(y)$ and $x \neq y$. Let us define $x \neq y$, $x' = T(x) \in W$, and $y' = T(y) \in W$ where $x' = y'$. We prove that $UT(x) = UT(y)$ below.

$$\begin{aligned} UT(x) &= UT(y) \\ U(T(x)) &= U(T(y)) \\ U(x') &= U(y') \\ U(x') &= U(x') \\ 0 &= 0 \end{aligned}$$

Thus, we have shown $\exists\ x, y \in V$ s.t. $UT(x) = UT(y)$ and $x \neq y \Rightarrow UT$ is not injective. Thus, we have shown if T is not injective, UT is not injective. By proof by contrapositive, we have proved if UT is injective, T is injective.

Does U have to be injective if $\{UT, T\}$ are injective?

Let us define $x, y \in V$ and $w_1 = T(x), w_2 = T(y) \in R(T) \leq W$. Suppose $w_1 \neq w_2$. Let us assume U is not injective. If U is not injective, then it is possible for $U(w_1) = U(w_2)$ even if $w_1 \neq w_2$. This would mean that $U(T(x)) = U(T(y)) \Rightarrow UT(x) = UT(y)$. Because UT is given as injective, if $UT(x) = UT(y)$, then we know that $x = y$. However, because we are considering the case for which $w_1 \neq w_2$ or $T(x) \neq T(y)$, we know that $x \neq y$. Thus, if U is not injective, the property $UT(x) = UT(y) \Rightarrow x = y$ is not necessarily true and so this contradicts our given that UT is injective. Note, however, that it only matters that for U is injective for input $w \in R(T)$. This is because ensuring the injectivity of UT means that U has to be injective only for outputs of T (i.e. $R(T)$). By proof by contradiction, we have proven that if $\{UT, T\}$ are injective, then U must be injective as well over $R(T)$. *However, U does not need to be fully injective.*

b) For proof by contrapositive, let us assume U is not surjective. This means $\exists z \in Z$ s.t. $\nexists\ w \in W$ s.t. $U(w) = z$.[1] If UT is surjective, this means that $\exists v \in V$ s.t. $UT(v) = z$. In other words, this means that $U(T(v)) = z$, or that $\exists\ w \in R(T) \leq W$ s.t. $U(w) = z$. However, because we are assuming U is not surjective, $\nexists\ w \in W$ s.t. $U(w) = z \Rightarrow \nexists\ v \in V$ s.t. $UT(v) = z \Rightarrow UT$ is not surjective. Thus, we have shown that if U is not surjective, UT is not surjective. By proof by contrapositive, we have shown that if UT is surjective, then U is surjective.

Does T have to be surjective if $\{UT, U\}$ are surjective?

If T is surjective, that means that $R(T) = W$. For UT and U to be surjective, that means that there needs to exist pre-images in V and W, respectively, that map to every element in Z. However, for transformation U, if every single pre-image of Z in W exists in the subspace $R(T) \leq W$, then surjectivity for U is maintained. Note that this does not affect the surjectivity of UT as the pre-images of UT in V will all be mapped by T to $R(T) \leq W$ by the definition of a range of a transformation. Thus, T only must be surjective for pre-images in $R(T)$ but not every element in W; *T does not have to be fully surjective.*

c) We are given that the matrix AB is invertible $\Rightarrow$ transformation L_{AB} is invertible. We define L_{AB} as the composition of transformations L_A and L_B. Additionally, a transformation is only invertible if it is bijective. Thus, we know that L_{AB} is bijective and so L_{AB} is both surjective and injective. Finally, because we are given the matrix representations for A and B, we can trivially assume L_A and L_B are linear transformations.

We prove below that matrices A and B are invertible.

1 **A is invertible**

From part (b), we know that if transformation $L_{AB} = L_A L_B$ is surjective, then transformation L_A is surjective. From Theorem 2.5, we know that given a linear

[1] We reference an example of this element of Z with this property as z in the remainder of this proof.

transformation T with input and output vector spaces of equal dimensionality, if T is surjective, then T is also injective. Because L_A is a square matrix, we know that it has input and output vector spaces of equal dimensionality. Thus, we know that given L_A is surjective, L_A is also injective. Because L_A is surjective and injective, it is bijective and so L_A is invertible $\Rightarrow$ the corresponding matrix A is invertible.

2 **B is invertible**
From part (a), we know that if transformation $L_{AB} = L_A L_B$ is injective, then transformation L_B is injective. From Theorem 2.5, we know that given a linear transformation T with input and output vector spaces of equal dimensionality, if T is injective, then T is also surjective. Because L_B is a square matrix, we know that it has input and output vector spaces of equal dimensionality. Thus, we know that given L_B is injective, L_B is also surjective. Because L_B is surjective and injective, it is bijective and so L_B is invertible $\Rightarrow$ the corresponding matrix B is invertible.

d) We are given that matrix $AB = I_n$. Because the inverse of the identity matrix is itself, $(AB)^{-1} = I_n$ and so AB is invertible. From part (c), we have proven that if AB is invertible, then *matrix A is invertible* and matrix B is invertible.
Let us define the inverse of A as A^{-1}. Because A is a square $n\ x\ n$ matrix, $AA^{-1} = I_n$ which is also equal to AB.
We show this below.

$$AA^{-1} = I_n = AB$$
$$AA^{-1} = AB$$

Because A is invertible and appears in the same position on both sides of the equation, we can remove A to get:

$$A^{-1} = B$$

Thus we have proven $A^{-1} = B$.

4. For proof by contrapositive, we will assume that $\{T, U\}$ are linearly dependent subsets of $\mathcal{L}(V, W)$. This means that given $c_1, c_2 \in \mathbb{F}$, there exists a solution to the equation $c_1 T + c_2 U = 0$ where at least one of $\{c_1, c_2\}$ is not equal to zero. This means that we can re-express T or U as a scalar multiplied by the other linear operator. In other words, given $k \in \mathbb{F}$, $T = kU$.

   Consider $w \neq 0 \in R(U)$.[2] This means that $\exists\ v \in V$ s.t. $U(v) = w$. Because we know $T = kU$, $T(v) = kU(v) = kw \Rightarrow kw \in R(T)$. We now show that $kw \in R(U)$. Because V is a vector space, we know that it is closed under scalar multiplication $\Rightarrow kv \in V$. Because U is linear, $U(kv) = kU(v) = kw \Rightarrow kw \in R(U)$. Because $w \neq 0$, $kw \neq 0 \in R(U), R(T) \Rightarrow R(T) \cap R(U) \neq \{0\}$. Thus we have shown that if $\{T, U\}$ are not linearly independent subsets of $\mathcal{L}(V, W)$, then $R(T) \cap R(U) \neq \{0\}$. Using proof by contrapositive, we have proved if $R(T) \cap R(U) = \{0\}$, then $\{T, U\}$ are linearly independent subsets of $\mathcal{L}(V, W)$.

[2] Because $U \in \mathcal{L}(V, W)$, we know that U is a nonzero operator. Thus, $\exists\ w \neq 0 \in R(U)$.