

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# MATH 244 HW 1

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## 1. Section 1.2, Question 5

If  $X \times Y = X \times Z$ , this implies  $X \times Y \subseteq X \times Z$  and  $X \times Z \subseteq X \times Y$ . We look at the implications of these two facts below:

①  $X \times Y \subseteq X \times Z$

If  $X \times Y \subseteq X \times Z$ , this means that every element in the set  $\{(x, y) : x \in X, y \in Y\}$  belongs to the set  $X \times Z = \{(x, z) : x \in X, z \in Z\}$ . Because the first element in each of these ordered products is drawn from the same set  $X$ , this means that  $\forall y \in Y, y \in Z \Rightarrow Y \subseteq Z$ .

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Because we have proven  $Y \subseteq Z$  and  $Z \subseteq Y$ , we can conclude  $Y = Z$  if  $X \times Y = X \times Z$ .

## 2. Section 1.2, Question 6

To prove  $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ , we prove  $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$  and  $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ .

1.  $\forall x \in (A \setminus B) \cup (B \setminus A), x \in (A \cup B) \setminus (A \cap B)$

**Let us consider the case in which  $x \in A$ .** If  $x \in A$ , we are guaranteed that  $x \in (A \setminus B) \Rightarrow x \notin B$ . Furthermore, because  $x \in A \Rightarrow x \in (A \cup B)$  and since  $x \notin B \Rightarrow x \notin (A \cap B)$ . Thus,  $x \in (A \cup B) \setminus (A \cap B)$ .

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2.  $\forall x \in (A \cup B) \setminus (A \cap B), x \in (A \setminus B) \cup (B \setminus A)$

**Let us consider the case in which  $x \in A$ .** Because  $x \notin (A \cap B) \Rightarrow x \notin B$ . Thus, since  $x \in A$  and  $x \notin B$ , then  $x \in (A \setminus B) \Rightarrow x \in (A \setminus B) \cup (B \setminus A)$ .

**Let us consider the case in which  $x \in B$ .** Because  $x \notin (A \cap B) \Rightarrow x \notin A$ . Thus, since  $x \in B$  and  $x \notin A$ , then  $x \in (B \setminus A) \Rightarrow x \in (A \setminus B) \cup (B \setminus A)$ .

### 3. Section 1.3, Question 2

Let us define  $s = \frac{1+\sqrt{5}}{2} \in \mathbb{R}$ . To prove this statement, we use induction:

- ① Base cases:  $n = 0$  and  $n = 1$

We first show that  $F_n \leq s^{n-1}$  holds for  $n = 0$  and  $n = 1$ . For the  $n = 0$  case,  $F_0 = 0 \leq \frac{1}{s}$ . For the  $n = 1$  case,  $F_1 = 1 \leq s^0$ .

- ② Inductive step: Show  $F_n \leq s^{n-1}$  for  $n \geq 2$

Our inductive hypothesis for this case is that both  $F_{n-1} \leq s^{n-2}$  and  $F_{n-2} \leq s^{n-3}$ , and we must now show that  $F_n \leq s^{n-1}$ . We first investigate the value of  $F_n$  below:

$$F_n = F_{n-1} + F_{n-2} \leq s^{n-2} + s^{n-3}$$

Note that  $s^{n-2} + s^{n-3} = s^{n-1}(\frac{1}{s} + \frac{1}{s^2})$ . Because  $\frac{1}{s} + \frac{1}{s^2} = 1$ , we know that  $s^{n-2} + s^{n-3} = s^{n-1}$ . Thus, we can restate the previous inequality as:

$$F_n \leq s^{n-1}$$

and so we have proven this step.

### 4. Section 1.4, Question 2

- a)  $f(x) = x^2$   
b)  $f(x) = |x - 2| + 1$

### 5. Section 1.4, Question 6

To prove that statements (i) and (ii) are equivalent, we prove the following statements:

- ① If (i), then (ii)

$g_1$  and  $g_2$  have the same domain and co-domain. However, because they are distinct functions, this means  $\exists z \in Z$  s.t.  $g_1(z) \neq g_2(z)$ . For this  $z$ ,  $f \circ g_1(z) \neq f \circ g_2(z)$ . This is because inputs  $g_1(z) \neq g_2(z)$  and so because  $f$  is injective,  $f(g_1(z)) \neq f(g_2(z))$  or  $f \circ g_1(z) \neq f \circ g_2(z)$ . Thus,  $\exists z \in Z$  s.t.  $f \circ g_1(z) \neq f \circ g_2(z)$  and so we can conclude that  $f \circ g_1$  and  $f \circ g_2$  are distinct.

- ② If (ii), then (i)

To prove that  $f$  is injective, we will proceed by contradiction and assume that  $f$  is not injective. This means  $\exists x, x' \in X$  s.t.  $f(x) = f(x')$  and  $x \neq x'$ . Let us define set  $Z = \{u\}$  and distinct functions  $g_1, g_2 : Z \rightarrow X$  where  $g_1(u) = x$  and  $g_2(u) = x'$ . By assumption (ii), this means that  $f \circ g_1, f \circ g_2 : Z \rightarrow Y$  are distinct. In order for these two functions to be distinct,  $f(g_1(u)) \neq f(g_2(u))$  or  $f(x) \neq f(x')$ . Thus, our assumption that  $f$  is not injective is contradicted, and so we have proven that if (ii), then  $f$  is injective.