MATH 241 PSET 8

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1.

a) The PDF of $T \sim Expo(\lambda)$ is given by $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. We define the half-life time as H. The half-life time is given as the time that $P(T \leq H) = 0.5$. We solve for H below:

$$
\begin{aligned}
P(T \leq H) &= 0.5 \\
\int_0^H f(x)dx &= 0.5 \\
\int_0^H \lambda e^{-\lambda x} dx &= 0.5 \\
\int_0^H e^{-\lambda x} dx &= \frac{1}{2\lambda} \\
-\frac{1}{\lambda} e^{-\lambda x} \Big|_0^H &= \frac{1}{2\lambda} \\
-\frac{1}{\lambda}(e^{-\lambda H} - 1) &= \frac{1}{2\lambda} \\
e^{-\lambda H} - 1 &= -\frac{1}{2} \\
e^{-\lambda H} &= \frac{1}{2} \\
H = \frac{-ln(0.5)}{\lambda} &= \frac{ln(2)}{\lambda}
\end{aligned}
$$

b) To ease our computations, we first find the CDF of T, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_0^x \lambda e^{-\lambda t} dt = \lambda \int_0^x e^{-\lambda t} dt = -\frac{\lambda}{\lambda} e^{-\lambda x} \Big|_0^x = -1(e^{-\lambda x} - 1) = 1 - e^{-\lambda x}$ for $x > 0$. The probability a particle decays in the time interval $[t, t+\epsilon]$, given that it has survived since time t, is given by $P(t \leq T \leq t + \epsilon | T > t)$. Using Bayes Rule:

$$P(t \leq T \leq t+\epsilon | T > t) = \frac{P(t \leq T \leq t+\epsilon \cap T > t)}{P(T > t)} = \frac{P(t \leq T \leq t+\epsilon)}{P(T>t)} =$$
$$= \frac{F(t+\epsilon) - F(t)}{1 - F(t)} = \frac{1 - e^{-\lambda(t+\epsilon)} - (1 - e^{-\lambda t})}{1 - (1 - e^{-\lambda t})} = \frac{-(e^{-\lambda t} e^{-\lambda \epsilon}) + e^{-\lambda t}}{e^{-\lambda t}} = 1 - e^{-\lambda \epsilon} =$$
$$= 1 - (e^{\epsilon})^{-\lambda} \approx 1 - (1+\epsilon)^{-\lambda}$$

Using a first-degree Taylor series expansion of $(1+\epsilon)^{-\lambda}$ about $\epsilon \approx 0$, we get $(1+\epsilon)^{-\lambda} \approx (1+0)^{-\lambda} - \lambda\epsilon(1+0)^{-\lambda-1}$ or that $(1+\epsilon)^{-\lambda} \approx 1 - \lambda\epsilon$. Thus, $P(t \leq T \leq t+\epsilon | T > t) \approx 1 - (1+\epsilon)^{-\lambda} \approx \lambda\epsilon$, and so $P(t \leq T \leq t+\epsilon | T > t)$ is approximately proportional to ϵ. Furthermore, there is no t term present and so we have shown that this probability does not depend on t.

c) From Example 5.6.3, we know that $L \sim Expo(n\lambda)$. The CDF of L can be given as, for $x \geq 0$, $F(x) = P(L \leq x) = 1 - P(L > x) = 1 - \prod_{i=1}^{n} P(T_i \geq x) = 1 - (e^{-\lambda x})^n = 1 - e^{-n\lambda x}$. Furthermore, as we know $L \sim Expo(n\lambda)$, $E[L] = \frac{1}{n\lambda}$ and $Var(L) = \frac{1}{(n\lambda)^2}$.

d) We can model this scenario as $M = Z_1 + Z_2 + \cdots + Z_n$, where Z_i is the time for the ith particle to decay. Because the time for the ith particle to decay (i.e. Z_i) is given as the minimum time to decay of all the $n - i + 1$ particles which have not decayed, $Z_i \sim Expo((n-i+1)\lambda)$. From my work in part (c), we know that $E[Z_i] = \frac{1}{(n-i+1)\lambda}$. Thus, we can compute the expectation of M below as:

$$M = \Sigma_{i=1}^{n} Z_i$$
$$E[M] = \Sigma_{i=1}^{n} E[Z_i]$$
$$E[M] = \Sigma_{i=1}^{n} \frac{1}{(n-i+1)\lambda}$$
$$E[M] = \frac{1}{\lambda}(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1})$$

Thus, $E[M] = \frac{H_n}{\lambda}$, where H_n is the nth harmonic number.
Due to the memoryless property, $Z_1, \ldots, Z_n$ are all independent. From my work in part (c), we know $Var(Z_i) = \frac{1}{(n-i+1)^2\lambda^2}$ Thus, we can compute the $Var(M)$ as such:

$$M = \Sigma_{i=1}^{n} Z_i$$
$$Var(M) = \Sigma_{i=1}^{n} Var(Z_i)$$
$$Var(M) = \Sigma_{i=1}^{N} \frac{1}{(n-i+1)^2\lambda^2}$$
$$Var(M) = \frac{1}{\lambda^2}(\frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + 1)$$

2.

a) Because X, Y are independent and identically distributed, distributions X^2 and Y^2 are as well and thus have the same MGF. Thus, we can compute the MGF of $W = X^2 + Y^2$ as so:

$$M_W(t) = M_{X^2+Y^2}(t) = M_{X^2}(t) \cdot M_{Y^2}(t) = ((1-2t)^{-\frac{1}{2}})^2 = \frac{1}{1-2t} = \frac{0.5}{0.5-t}$$

b) The distribution W has an MGF of the form of an Exponential Distribution MGF with $\lambda = 0.5$. Thus, $W \sim Expo(0.5)$.

3.

The MGF of the Geometric distribution is given by $M(t) = \frac{p}{1-qe^t}$. We first compute $E[X]$ of this distribution below by applying the formula $E[X^n] = M^{(n)}(0)$.

$$E[X^1] = M^{(1)}(0)$$
$$E[X] = (\frac{p}{1-qe^t})'(0)$$
$$E[X] = (-\frac{p}{(1-qe^t)^2} \cdot (-qe^t))(0)$$
$$E[X] = (\frac{pqe^t}{(1-qe^t)^2})(0)$$
$$E[X] = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

Thus, we get that the mean of the distribution $E[X] = \frac{q}{p}$. We now use this same formula $E[X^n] = M^{(n)}(0)$ to compute $E[X^2]$:

$$E[X^2] = M^{(2)}(0)$$
$$E[X^2] = (\frac{p}{1-qe^t})''(0)$$
$$E[X^2] = (\frac{pqe^t}{(1-qe^t)^2})'(0)$$
$$E[X^2] = (pq(\frac{e^t(1-qe^t)^2 + 2qe^{2t}(1-qe^t)}{(1-qe^t)^4}))(0)$$
$$E[X^2] = pq\frac{(1-q)^2 + 2q(1-q)}{(1-q)^4} = \frac{pq}{(1-q)^2} + \frac{2p^2q^2}{(1-q)^4} = \frac{pq}{p^2} + \frac{2p^2q^2}{p^4} = \frac{q}{p} + \frac{2q^2}{p^2}$$

Given, $E[X^2] = \frac{q}{p} + \frac{2q^2}{p^2}$ and $E[X] = \frac{q}{p}$, we now compute $Var(X) = E[X^2] - (E[X])^2$ as:

$$Var(X) = E[X^2] - (E[X]^2)$$
$$Var(X) = \frac{q}{p} + \frac{2q^2}{p^2} - (\frac{q}{p})^2 = \frac{q}{p} + \frac{q^2}{p^2} = \frac{q}{p} + \frac{q(1-p)}{p^2} = \frac{q}{p} + \frac{q}{p^2} - \frac{qp}{p^2} = \frac{q}{p} + \frac{q}{p^2} - \frac{q}{p} = \frac{q}{p^2}$$

4.

We compute the MGF of $X \sim Expo(1)$ as $M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}e^{-x}dx = \frac{1}{1-t}$ for $t < 1$. We can now compute the MGF of distribution $-Y = -X$ as $M_{-Y}(t) = E[e^{t(-Y)}] = E[e^{-tY}] = E[e^{-tX}] = M_X(-t)$ for $-t < 1 \Rightarrow t > -1$.

Because we are given that X and Y are independent, X and $-Y$ are independent. Thus, the MGF of $L = X + (-Y)$ is given by $M_L(t) = M_X(t) \cdot M_{-Y}(t)$ for $-1 < t < 1$. We compute $M_L(t)$ below for $-1 < t < 1$

$$M_L(t) = M_X(t) \cdot M_{-Y}(t) = M_X(t) \cdot M_X(-t)$$
$$M_L(t) = \frac{1}{1-t} \cdot \frac{1}{1+t} = \frac{1}{1-t^2}$$

To show that L has the Laplace distribution, we compute the MGF $M_W(t)$ of the Laplace Distribution for $-1 < t < 1$[1]:

$$M_W(t) = E[e^{tW}] = \int_{-\infty}^{\infty} e^{tw} f(w)dw = \frac{1}{2}\int_{-\infty}^{\infty} e^{tw-|w|}dw =$$
$$\frac{1}{2}[\int_{-\infty}^{0} e^{w(t+1)}dw + \int_{0}^{\infty} e^{w(t-1)}dw] = \frac{1}{2}[\frac{e^{w(t+1)}}{t+1}\Big|_{-\infty}^{0} + \frac{e^{w(t-1)}}{t-1}\Big|_{0}^{\infty}]$$
$$= \frac{1}{2}[\frac{1}{t+1} - \frac{1}{t-1}] = \frac{-2}{2(t^2-1)} = \frac{1}{1-t^2}$$

Thus, because $M_L(t) = M_W(t)$, we have shown that distribution L is a Laplace distribution as it has the identical MGF (i.e. $M_L(t)$) as the Laplace distribution MGF (i.e. $M_W(t)$).

5.

a) The MGF of a $Bin(n, p)$ r.v. is $M(t) = (pe^t + q)^n$. So, the MGFs of distributions X_1 and X_2 are $(pe^t + q)^{n_1}$ and $(pe^t + q)^{n_2}$, respectively. Because X_1 and X_2 are independent, the distribution $X_1 + X_2$ has the MGF $M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = (pe^t + q)^{n_1+n_2}$. Because the distribution $X_1 + X_2$ has the MGF of the distribution $Bin(n_1 + n_2, p)$, $X_1 + X_2 \sim Bin(n_1 + n_2, p)$.

b) The MGF of a $Expo(\lambda)$ r.v. is given by $M(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$. So, the MGFs of distributions Y_1 and Y_2 are given by $\frac{\lambda_1}{\lambda_1 - t}$ for $t < \lambda_1$ and $\frac{\lambda_2}{\lambda_2 - t}$ for $t < \lambda_2$, respectively. Let us define $\lambda_s = min(\lambda_1, \lambda_2)$. Because Y_1 and Y_2 are independent, the distribution $Y_1 + Y_2$ has the MGF $M_{Y_1+Y_2}(t) = M_{Y_1}(t) \cdot M_{Y_2}(t) = \frac{\lambda_1\lambda_2}{(\lambda_1 - t)(\lambda_2 - t)}$ for $t < \lambda_s$. As we can see, the MGF of $Y_1 + Y_2$ does not have the form of the MGF of an Exponential distribution, and thus $Y_1 + Y_2$ does not follow an Exponential distribution.

6. Anish Lakkapragada. I worked independently.

[1] Note that this range of $|t| < 1$ is required to ensure the below integrals in deriving the MGF of the Laplace Distribution converge.