Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head `here`.

Math 225- HW 11 Due: Dec 9 by Midnight

Submit the first two problems, along with any three additional problems of your choice.

1. - Prove that if U and T simultaneously diagonalizable then U and T commute. i.e. $UT = TU$

     If U and T are simultaneously diagonalizable, this means that $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. The product of two diagonal matrices is obviously commutative (i.e. if matrices X and Y are diagonal, $XY = YX$). Thus, if U and T are simultaneously diagonalizable then:

     $$[T]_\beta[U]_\beta = [U]_\beta[T]_\beta$$
     $$[TU]_\beta = [UT]_\beta$$
     $$TU = UT$$

   - Conclude that if matrices A,B are simultaneously diagonalizable then A,B commute

     If A and B are simultaneously diagonalizable then we know that $\exists$ Q invertible s.t. $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal and so, using the logic that the product of two diagonal matrices is commutative, we have:

     $$Q^{-1}AQQ^{-1}BQ = Q^{-1}BQQ^{-1}AQ$$
     $$Q^{-1}ABQ = Q^{-1}BAQ$$
     $$AB = BA$$

   - Let T be diagonalizable linear operator on a finite dimensional vector space, then T and T^m are simultaneously diagonalizable for any m positive integer.

     Because T is diagonalizable, $[T]_\beta$ is a diagonal matrix. Thus, $[T^m]_\beta = [\underbrace{T\dots T}_{m \text{ times}}]_\beta = \underbrace{[T]_\beta \dots [T]_\beta}_{m \text{ times}} = \prod_{i=1}^m [T]_\beta$. Because $[T]_\beta$ is diagonal, $[T^m]_\beta = \prod_{i=1}^m [T]_\beta$ is also diagonal $\Rightarrow$ T and T^m are simultaneously diagonalizable.

2. a) For any vector $w \in E_\lambda$, $T(w) = \lambda w \in E_\lambda$. Let us define $u = \lambda w$. Because $T(u) = T(\lambda w) = \lambda T(w) = \lambda^2 w = \lambda u$, $u = \lambda w \in E_\lambda$. Thus we have shown $\forall w \in E_\lambda, T(w) \in E_\lambda \Rightarrow E_\lambda$ is a T-invariant subspace of V.

   b) Let us define this T-cyclic subspace generated by v as $W \leq V$. W can be expressed as $\text{Span}(\{v, T(v), \dots T^n(v)\})$. We now show that $\forall w \in W, T(w) \in W$. By definition, $\forall w \in W$ can be expressed as $w = \Sigma_{i=0}^n c_i T^i(v)$ and so $T(w) = c_n T^{n+1}(v) + \Sigma_{i=1}^n c_i T^i(v)$. Because $T^{n+1}(v)$ can be expressed as a linear combination of $\{v, T(v), \dots T^n(v)\}$, this means that $T(w)$ can be expressed as a linear combination of $\{v, T(v), \dots T^n(v)\} \Rightarrow T(w) \in \text{Span}(\{v, T(v), \dots T^n(v)\}) \Rightarrow T(w) \in W$. Thus, we have shown $\forall w \in W, T(w) \in W \Rightarrow W$ is a T-invariant subspace of V.

   c) The T-cyclilc subspace W can be given by $W = \text{Span}\{v, T(v), \dots T^n(v)\}$. We now prove both directions of this statement:

      1. If $w \in W$, $w = g(T)v$

         If $w \in W$, w can be expressed as $\Sigma_{i=0}^n c_i T^i(v) = U(v)$, where $U = \Sigma_{i=0}^n c_i T^i$ is an operator. Defining $g(x) = \Sigma_{i=0}^n c_i x^i$, $U = g(T)$ and so we have that $w = g(T)v$.

2. If $w = g(T)v$, $w \in W$
We can express polynomial g as $g(x) = \Sigma_{i=0}^{n} c_i x^i$. Thus, we have that $w = g(T)v = \Sigma_{i=0}^{n} c_i T^i(v) \Rightarrow w \in \text{Span}\{v, T(v), \ldots T^n(v)\} \Rightarrow w \in W$.

d) Because V is a T-cyclic subspace of itself, we can express $V = \text{Span}(\{v, T(v), T^2(v), \ldots, T^n(v)\})$. Thus, this means $\forall z \in V$, $z = \Sigma_{i=0}^{n} c_i T^i(v) \Rightarrow$ because $U(v) \in V, U(v)$ can be expressed as a linear combination of $T^i(v)$. Note that if U commutes with T, that means $UT^2 = UTT = TUT = TTU = T^2U$, or more generally $UT^\alpha = T^\alpha U$ for $\alpha \geq 0$. Thus, we have that for $i \geq 0$:

$$UT^i = T^i U$$
$$UT^i(v) = T^i U(v)$$
$$U(T^i(v)) = T^i(\Sigma_{k=0}^{n} c_k T^k(v))$$
$$U(T^i(v)) = \Sigma_{k=0}^{n} c_k T^{i+k}(v)$$

Setting $a = T^i(v)$, we have:

$$U(a) = \Sigma_{k=0}^{n} c_k T^k(a)$$

Thus, we can clearly see that $U = g(T)$, where polynomial g is given by $g(x) = \Sigma_{k=0}^{n} c_k x^k$.

e) There are two cases in this scenario: (1) all vectors in V are eigenvectors or (2) not all vectors in V are eigenvectors. We address both cases below:

1. All vectors in V are eigenvectors
This means that $\forall v \in V, T(v) = \lambda v$ where $\lambda \in \mathbb{F}$. Let us define two vectors $a, b \in V$ and compute $T(a+b)$:

$$T(a+b) = \lambda_{a+b}(a+b)$$

However, by linearity, we also have that $T(a+b) = T(a) + T(b) = \lambda_a a + \lambda_b b$. Thus, we have that:

$$T(a+b) = T(a+b)$$
$$\lambda_{a+b}(a+b) = \lambda_a a + \lambda_b b$$

This means that $\lambda_{a+b} = \lambda_a = \lambda_b \Rightarrow \forall v \in V, T(v) = \lambda_a v \Rightarrow T = cI$ where $c \in \mathbb{F}$.

2. Not all vectors in V are eigenvectors
This means that $\exists\ v \neq 0 \in V$ s.t. $T(v) \neq \lambda v$, $\forall \lambda \in \mathbb{F}$. Consider the set $\{v, T(v)\}$. In order for the set of vectors $\{a, b\}$ to be linearly independent, neither a nor b can be expressed as a scalar multiple of the either vector. Because we know that $\forall \lambda \in \mathbb{F}$, $T(v) \neq \lambda v$, $\{v, T(v)\}$ are a linearly independent set of two vectors $\Rightarrow$ because $\dim(V) = 2$, $\{v, T(v)\}$ serve as a basis for $V \Rightarrow V = \text{Span}(\{v, T(v)\}) \Rightarrow V$ is a T-cyclic subspace of itself.

3. I didn't do this question.

4. (a) We use induction to prove this statement.

1. Base Case: Single element v_1
If $n = 1$, then given $\Sigma_{i=1}^{n} v_i \in W \Rightarrow v_1 \in W$.

2. Inductive Step: Given $v_1, \dots, v_{k-1} \in W$, prove that $v_k \in W$
For proof by contrapositive, let us assume that $v_k \notin W$. Let us define $v = v_1 + \cdots + v_n$. We start with our given:

$$v = v_1 + \cdots + v_n \in W$$
$$v = (v_1 + \cdots + v_{k-1}) + v_k + (v_k + \cdots + v_n)$$
$$v_1 + \cdots + v_{k-1} = v - v_k - (v_k + \cdots + v_n)$$

Because W is a subspace, it is closed under addition. Thus, because $v_k \notin W \Rightarrow v - v_k - (v_k + \cdots + v_n) \notin W \Rightarrow v_1 + \cdots + v_{k-1} \notin W$. By proof by contrapositive, we have proven if $v_1, \dots, v_{k-1}$, then $\in W, v_k \in W$.

(b) Let us define U as a non-trivial T-invariant subspace of V. If T is a diagonalizable linear operator, that means its eigenvectors $v_1, v_2, \dots, v_n$ form a basis for V. Because U is a non-trivial subspace, $\exists\ v \neq 0 \in U$. Furthermore, given that $\forall v \neq 0 \in U \leq V$, v can be written as a linear combination of $\{v_1, \dots, v_n\}$, we can define the nonempty set of eigenvectors which all elements of U are a linear combination of as $\{u_1, \dots, u_k\} \Rightarrow \text{Span}(\{u_1, \dots, u_k\}) = U$. Note that $\{u_1, \dots, u_k\}$ are all part of the basis $\{v_1, \dots, v_n\}$ for V and so they are are all linearly independent. Thus, we can conclude the linearly independent and generating set of eigenvectors $\{u_1, \dots, u_k\}$ forms a basis for U and so $T|_U$ is diagonalizable.

(c) Because $v_1, v_2, \dots, v_n \in V$ all correspond to distinct eigenvalues, they are all linearly independent. Given these n linearly independent vectors and that $\dim(V) = n$, we can conclude that the eigenvectors $v_1, v_2, \dots, v_n$ form a basis for V. This means that $V = \text{Span}(\{v_1, v_2, \dots, v_n\})$. Let us define vector $v = v_1 + v_2 + \cdots + v_n$. Note that $\text{Span}(\{v, T(v), \dots, T^n(v)\}) = \text{Span}(\{\Sigma_{i=1}^n v_i, \Sigma_{i=1}^n \lambda_i v_i, \dots, \Sigma_{i=1}^n \lambda_i^n v_i\})$. We can write out this transformation from the eigenvectors $v_1, \dots, v_n$ to $\Sigma_{i=1}^n v_i, \Sigma_{i=1}^n \lambda_i v_i, \dots \Sigma_{i=1}^n \lambda_i^n v_i$ as such:

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_1^n & \lambda_2^n & \dots & \lambda_n^n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \Sigma_{i=1}^n v_i \\ \Sigma_{i=1}^n \lambda_i v_i \\ \vdots \\ \Sigma_{i=1}^n \lambda_i^n v_i \end{bmatrix}$$

Note that the leftmost matrix above, which I refer to as V, is the Vandermonde matrix (pg 230.) Because all $\forall 0 \leq i < j \leq, \lambda_i \neq \lambda_j$, $\det(V) = \prod_{0 \leq i < j \leq n}(\lambda_j - \lambda_i) \neq 0 \Rightarrow V$ is invertible $\Rightarrow$ because $\{v_1, v_2, \dots, v_n\}$ serve as a basis for V, so does $\{\Sigma_{i=1}^n v_i, \Sigma_{i=1}^n \lambda_i v_i, \dots, \Sigma_{i=1}^n \lambda_i^n v_i\} \Rightarrow V = \text{Span}(\{v, T(v), \dots, T^n(v)\}) \Rightarrow V$ is a T-cyclic subspace of itself.

5. (a) We prove both directions of this statement below:

1. If T is diagonalizable, V is the direct sum of one-dimensional T-invariant subspaces
If T is diagonalizable, that means that eigenvectors $v_1, v_2, \dots v_n \in V$ serve as a basis for V. This means $V = \text{Span}(\{v_1, \dots, v_n\}) = \{\Sigma_{i=1}^n c_i v_i : c_i \in \mathbb{R}\} = \{\Sigma_{i=1}^n T(\frac{c_i}{\lambda_i} v_i)\}$. Let us define the set $W_j = \{cv_j : c \in \mathbb{F}\}$ for a given eigenvector v_j. Note that W_j is a one-dimensional subspace as W_j is composed of scalar multiples of one unique vector, v_j. Furthermore, W_j is a T-invariant subspace

as $\forall w \in W_j, T(w)$ is equal to a scalar multiple of $v_j \Rightarrow w \in W_j$. Furthermore, because $\{v_1, \ldots, v_n\}$, serve as a basis, that means that all the eigenvectors are linearly independent $\Rightarrow$ for $0 \le i < j \le n$, $W_i \cap W_j = \emptyset$ and so we have that $V = \{\Sigma_{i=1}^n c_i v_i : c_i \in \mathbb{R}\} = W_1 \bigoplus W_2 \cdots \bigoplus W_n$.

2. If V is the direct sum of one-dimensional T-invariant subspaces, T is diagonalizable

Let us define $V = W_1 \bigoplus W_2 \cdots \bigoplus W_k$ and the one basis vector for the jth subspace W_j as v_j. Because W_j is one-dimensional and a subspace (thus closed under addition and scalar multiplication), $W_j = \{cv_j : c \in \mathbb{F}\}$. Furthermore, because W_j is T-invariant and $v_j \in W$, $T(v_j) \in W_j \Rightarrow T(v_j) \in \{cv_j : c \in \mathbb{F}\} \Rightarrow v_j$ is an eigenvector of T_{W_j} as $T(v_j) = kv_j$ where $k \in \mathbb{F}$. Because V is a *direct* sum of $W_1, \ldots W_k$, the individual basis vector v_j for each subspace is linearly independent from all of the vectors in $\{v_1, \ldots, v_{j-1}\} \cup \{v_{j+1}, \ldots, v_k\} \Rightarrow \{v_1, \ldots, v_k\}$ are linearly independent[1]. Furthermore, $V = W_1 \bigoplus W_2 \cdots \bigoplus W_k$ means that V contains all possible linear combinations of $\{v_1, \ldots v_k\} \Rightarrow V = \text{Span}(\{v_1, \ldots, v_k\}$. Thus we can conclude that the linearly independent and generating eigenvectors $\{v_1, \ldots, v_k\}$ forms a basis for V and so T is diagonalizable.

b) Let us define the unordered basis for the T-invariant subspace W_j as β_j. This means that the ordered basis γ for vector space V can be given as $\gamma = \beta_1 \cup \beta_2 \cdots \cup \beta_k$. We now try to understand what the matrix $[T]_\beta$ looks like. Note that $\forall v \in \beta_j, v \in W_j$ and so $T(v) = T_{W_j}(v) \in W_j \Rightarrow T(v)$ can be expressed as a linear combination of β_j. Thus, $[T]_\beta$ will be given as a collection of block matrices $[T_{W_j}]_{\beta_j}$ along the diagonal:

$$[T]_\beta = \begin{bmatrix} [T_{W_1}]_{\beta_1} & 0 & \cdots & 0 \\ 0 & [T_{W_2}]_{\beta_2} & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & [T_{W_k}]_{\beta_k} \end{bmatrix}$$

From this matrix, it is obvious that:

$$\det(T) = \det([T]_\beta) = \prod_{i=1}^{k} \det([T_{W_i}]_{\beta_i}) = \prod_{i=1}^{k} \det(T_{W_i})$$
$$\det(T) = \prod_{i=1}^{k} \det(T_{W_i})$$

6. To prove this law, we compare the LHS with the RHS. The LHS can be given as:

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle - \langle y, x-y\rangle$$
$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle = 2\langle x,x\rangle + 2\langle y,y\rangle$$

We now compare this with the RHS:

[1] This can be trivially proven by induction by the following proof: $\{v_1\}$ is a linearly independent set, $\{v_1, v_2\}$ is a linearly independent set, $\{v_1, v_2, v_3\}$ is a linearly independent set, and so on until $\{v_1, \ldots, v_n\}$ is a linearly independent set. I believe we did this proof in class.

$$2\|x\|^2 + 2\|y\|^2 = 2\langle x, x\rangle + 2\langle y, y\rangle$$

Thus, we can clearly see that the LHS = RHS and so we have proven this law.