

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# MATH 255 PSET 5

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1.

To prove that  $E$  is open, we WTS  $\forall (a, b) \in E$ ,  $(a, b)$  is an interior point of  $E$ . Pick  $(a, b) \in E$ . Because  $(a, b) \in E \implies a < b$  and so we can define  $h = b - a > 0$ . We now need to show that we can create a neighborhood around  $(a, b)$  that is  $\subset E$ . Let us define  $\epsilon = \frac{h}{2} > 0$  and create the following neighborhood:

$$N_\epsilon((a, b)) = \{(x, y) : \sqrt{(a - x)^2 + (y - b)^2} < \epsilon\}$$

We now need to show that  $N_\epsilon((a, b)) \subset E \implies \forall (x, y) \in N_\epsilon((a, b)), (x, y) \in E$ . Pick  $(x, y) \in N_\epsilon((a, b))$ . This implies the following two statements: (i)  $(a - x)^2 < \epsilon^2 \implies |a - x| < \epsilon$  and (ii)  $(b - y)^2 < \epsilon^2 \implies |b - y| < \epsilon$ . Note that  $|a - x| < \epsilon \implies -\epsilon < a - x < \epsilon \implies a - \epsilon < x < a + \epsilon$  and with the same logic  $b - \epsilon < y < b + \epsilon$ . Substituting  $\epsilon$  for  $0.5h$ , we know the following:  $x < a + \epsilon \implies x < a + \frac{h}{2}$  and  $y > b - \epsilon \implies y > b - \frac{h}{2}$ . Note that  $b = a + h$  and so  $y > b - \frac{h}{2} \implies y > a + \frac{h}{2} > x \implies y > x \implies x < y \implies (x, y) \in E$ . Thus, we have shown  $N_\epsilon((a, b)) \subset E$  and so we have proven  $\forall (a, b) \in E, \exists \epsilon > 0$  s.t.  $N_\epsilon((a, b)) \subset E \implies E$  is open.

2.

Let us define  $C_1, \dots, C_k$  to be  $k$  compact sets. Let us define set  $C = \bigcup_{i=1}^k C_i$ . We WTS that  $C$  is compact, or that any open cover of  $C$  has a finite subcover. Let  $\{S_j\} \supset C$  be an open cover of  $C$ . Because  $\forall 1 \leq i \leq k, \{S_j\} \supset C \supset C_i \implies \{S_j\} \supset C_i$ ,  $\{S_j\}$  serves as an open cover for each  $C_i$ . Because each  $C_i$  is compact, any open cover of  $C_i$  has a finite subcover. Thus, for each  $C_i$ , its open cover  $\{S_j\}$  has a finite subcover  $\{F_z^{(i)}\} \supset C_i$  where  $\{F_z^{(i)}\} \subset \{S_j\}$ .

Let us define  $F = \bigcup_{i=1}^k \{F_z^{(i)}\}$  to be the union of all these finite subcovers of  $C_i$ <sup>1</sup>. We now WTS that  $F$  is a finite subcover of  $C$ . To do so, we need to show the following:

## 1. $F$ is an open cover of $C$

Because  $\forall x \in C, x \in \text{some } C_i \implies x \in \text{some } \{F_z^{(i)}\} \implies x \in F$ , we have  $C \subset F$ , meaning that  $F$  is a (finite) open cover of  $C$ .

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<sup>1</sup>Because each finite subcover is finite, a union of these finite sets (i.e.  $F$ ) will also be finite.

## 2. $F$ is a finite subcover of open cover $\{S_j\}$ of $C$

Because  $\forall \{F_z^{(i)}\} \in F, \{F_z^{(i)}\} \in \{S_j\}$ ,  $F$  is a subcover of open cover  $\{S_j\}$  of  $C$ .

Thus, we have proven any open cover of  $C$ , a union of finitely many compact sets, has a finite subcover  $\implies C$  is compact.

3. An open cover of  $(0, 1) \subset \mathbb{R}$  can be given by  $\mathbb{R} \supset \{G_\alpha : \alpha \in \mathbb{N}\} \supset (0, 1)$ , where open set  $G_\alpha = (\frac{1}{\alpha}, 1)$ . We WTS  $(0, 1) \subset \mathbb{R}$  is not compact by showing that this open cover does not have a finite subcover (as this implies that not all open covers of  $(0, 1) \subset \mathbb{R}$  have a finite subcover  $\implies (0, 1) \subset \mathbb{R}$  is not compact).

We now prove that there is no finite subcover of  $\{G_\alpha\}$ . We prove this by contradiction and assume that there is a finite subcover of  $\{G_\alpha\}$ , given by  $\{G_k\} = \bigcup_{i=1}^k (\frac{1}{n_i}, 1) \subset \{G_\alpha\}$  where  $n_1, \dots, n_k \in \mathbb{N}$ . Because  $n_1, \dots, n_k$  form a (finite) subset of  $\mathbb{N}$ , they have a minimum which we can call  $n' = \min(n_1, \dots, n_k)$ . This means that the interval  $\{G_k\} = \bigcup_{i=1}^k (\frac{1}{n_i}, 1)$  can be simplified to  $(\frac{1}{n'}, 1)$ . Because  $\exists n'' > n'$  where  $(\frac{1}{n''}, 1) \subset (0, 1)$  but  $\not\subset \{G_k\} = (\frac{1}{n'}, 1)$ ,  $G_k$  is not an open cover of  $(0, 1) \implies \{G_k\}$  is not a finite subcover of  $\{G_\alpha\}$ . Thus, we have proved by contradiction that the open cover  $\{G_\alpha\}$  has no finite subcover  $\implies (0, 1) \subset \mathbb{R}$  is not compact.

4. (1) If  $A$  and  $B$  are disjoint sets then  $A \cap B = \emptyset$ . Furthermore, if  $A$  and  $B$  are closed that means  $A = \bar{A}$  and  $B = \bar{B}$ . Thus  $A \cap \bar{B} = A \cap B = \emptyset$  and  $\bar{A} \cap B = A \cap B = \emptyset$ , and so we know  $A$  and  $B$  are separated.
- (2) Let us define  $A, B$  as two disjoint open sets. We WTS that  $A \cap \bar{B} = \emptyset$  and  $\bar{A} \cap B = \emptyset$ . To do so, we prove that no limit points of  $B$  are in  $A$  and no limit points of  $A$  are in  $B$ .

WLOG, let us prove why no limit points of  $B$  are in  $A$ . We prove this by contradiction and assume that for a limit point  $x$  of  $B$ ,  $x \in A$ . This means  $\forall \epsilon > 0, N_\epsilon(x)$  contains some  $b \neq x$  s.t.  $b \in B$ . However, because  $x \in A$  and  $A$  is open  $\implies x$  is an interior point  $\implies \exists \epsilon > 0$  s.t.  $N_\epsilon(x) \subset A$ . Thus, this means that  $N_\epsilon(x)$ , which will contain some  $b \neq x \in B$  by virtue of  $x$  being a limit point of  $B$ , is fully contained in  $A \implies \exists b \in B$  and  $A \implies A \cap B \neq \emptyset$ , which is a contradiction of  $A$  and  $B$  being disjoint. Thus, we have proven that no limit points of  $B$  are in  $A$  and that no limit points of  $A$  are in  $B$ .

Let us define  $A'$  and  $B'$  to be the limit points of  $A$  and  $B$ , respectively. Based on our proof above we know  $B' \cap A = \emptyset$  and  $A' \cap B = \emptyset$ . Thus, we have:

$$\begin{aligned} A \cap \bar{B} &= A \cap (B \cup B') = (A \cap B) \cup (A \cap B') = \emptyset \cup \emptyset = \emptyset \\ \bar{A} \cap B &= (A \cup A') \cap B = (A \cap B) \cup (A' \cap B) = \emptyset \cup \emptyset = \emptyset \end{aligned}$$

Thus,  $A \cap \bar{B} = \emptyset$  and  $\bar{A} \cap B = \emptyset \implies A$  and  $B$  are separated.

- (3)  $\forall x \in A, d(p, x) < \delta \implies d(p, x) \not\geq \delta \implies x \notin B$ . The same applies in the other direction to show  $\forall x \in B, x \notin A$  and so we have  $A \cap B = \emptyset \implies A$  and  $B$  are disjoint.

We now prove that  $A$  and  $B$  are open. We first prove  $A$  is open. Note that  $A$  is essentially  $N_\delta(p)$ . As we have proved, any neighborhood of a point is open  $\implies A = N_\delta(p)$  is open.

We now prove that  $B$  is open, or that all its points are interior points of  $B$ . Pick  $x \in B$ . To show  $x$  is an interior point, we must find some  $\epsilon > 0$  s.t.  $N_\epsilon(x) \subset B$ . Note that because  $x \in B \implies d(p, x) > \delta$ . Consider  $\epsilon = d(p, x) - \delta > 0$ . We now aim to show that  $\forall z \in N_\epsilon(x), z \in B$ . Pick  $z \in N_\epsilon(x)$ . By Triangle Inequality we have:

$$\begin{aligned} d(p, x) &\leq d(p, z) + d(z, x) \\ d(p, z) &\geq d(p, x) - d(z, x) \end{aligned}$$

Because  $z \in N_\epsilon(x), d(z, x) < \epsilon \implies d(z, x) < d(p, x) - \delta \implies -d(z, x) > \delta - d(p, x)$ . Thus, we get:

$$\begin{aligned} d(p, z) &\geq d(p, x) - d(z, x) \\ d(p, z) &\geq d(p, x) - d(z, x) > d(p, x) + \delta - d(p, x) \\ d(p, z) &> d(p, x) + \delta - d(p, x) \\ d(p, z) &> \delta \end{aligned}$$

Thus,  $d(p, z) > \delta \implies z \in B \implies \forall z \in N_\epsilon(x), z \in B \implies N_\epsilon(x) \subset B \implies$  all points of  $B$  are interior points  $\implies B$  is open.

Thus, we have proven that  $A$  and  $B$  are disjoint open sets. By our proof in part (2), this means that  $A$  and  $B$  are separated.

- (4) We prove this statement by contradiction and thus assume that this connected metric space  $X$  with at least two points is not uncountable  $\implies X$  is at most countable. Let us define set  $D = \{d((p, q)) : (p, q) \in X \times X\} \subset \mathbb{R}^+$ . Because  $X$  is at most countable  $\implies X \times X$  is at most countable  $\implies D$  is at most countable<sup>2</sup>. We aim to find a  $\delta > 0 \notin D$  where  $\exists m \in D$  s.t.  $m > \delta$ . We proceed with casework on  $D$ 's cardinality:

(i) **Case One: If  $D$  is finite**

Let us fix points  $p, p' \in X$ .  $D$  is guaranteed to contain these two elements:  $d(p, p) = d(p', p') = 0$  and  $d(p, p')$ . Listing all elements of  $D$  in increasing order as so:  $d_1, \dots, d_n$ , we can select  $i$  from 1 to  $n - 1$  and choose  $\delta = \frac{d_i + d_{i+1}}{2}$ . We are guaranteed this element does not exist in the finitely many elements of  $D$  by virtue of it existing in between two consecutive elements  $d_i$  and  $d_{i+1}$  in  $D$ . Because this element is not the maximum of  $D$  (i.e.  $\delta < d_n$ )  $\implies \exists m \in D$  s.t.  $m > \delta$ . Furthermore,  $\delta$  is an average of two non-negative numbers, where only one can be zero<sup>3</sup>  $\implies \delta > 0$ .

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<sup>2</sup>This is because set  $D$  cannot have more elements than  $X \times X$  as it is simply applying the distance function  $d$  to every element of  $X \times X$ .

<sup>3</sup>This is because  $D$  is a set and thus there are no repeat elements.

(ii) **Case Two: If  $D$  is countable**

Here, we use a familiar intervals argument to find  $\delta$ . Fix  $p, p' \in X$  and define  $a_1 = 0$  and  $b_1 = d(p, p')$ . Because  $D$  is countable, we can write a sequence  $(q_n)$  that defines every element of  $D$ . Let us first define interval  $I_1 = [a_1, b_1]$ . Then for  $q_2, q_3, \dots$ , we can construct closed interval  $I_i = [a_i, b_i]$  with nonzero length where  $I_{i+1} \subset I_i$  and  $q_i \notin I_i$ .

Defining  $\delta = \sup(\{a_i : i \in \mathbb{N}\})$ ,  $\delta$  exists in all intervals  $I_i$  but does not exist in  $D$ . Furthermore, because  $\forall i, \delta \in I_i$  we know the following two things: (i)  $\delta < b_1 = d(p, q) \implies \exists m \in D$  s.t.  $m > \delta$  and (ii)  $\delta > a_i \implies \delta > 0$ .

Because  $\delta \notin D \implies \forall p, q \in X, d(p, q) \neq \delta \implies X = \{q \in X : d(p, q) < \delta\} \cup \{q \in X : d(p, q) > \delta\}$ . Let us define set  $A = \{q \in X : d(p, q) < \delta\}$  and set  $B = \{q \in X : d(p, q) > \delta\}$  where, as per our previous sentence,  $X = A \cup B$ . Note that because  $\exists m \in D$  s.t.  $m > \delta \implies \exists q \in X$  s.t.  $d(p, q) > \delta \implies q \in B \implies B$  is non-empty. Also note  $A$  is guaranteed to be non-empty as  $d(p, p) = 0 < \delta \implies p \in A$ .

Our proof in part (c) applies and so we get that  $A$  and  $B$  are separated  $\implies \exists$  non-empty sets  $A, B$  s.t.  $X = A \cup B$  where  $\bar{A} \cap B = A \cap \bar{B} = \emptyset \implies X$  is disconnected, which is a contradiction to our given that  $X$  is connected.

5. To prove that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , we aim to prove that  $\bar{\mathbb{Q}} = \mathbb{R}$ . We prove both directions of this statement below:

(a)  $\bar{\mathbb{Q}} \subset \mathbb{R}$

Pick  $x \in \bar{\mathbb{Q}}$ . This means that at least one of the two cases is true:

1. **Case One:**  $x \in \mathbb{Q}$

If  $x \in \mathbb{Q} \implies x \in \mathbb{R}$ .

2. **Case Two:**  $x$  is a limit point of  $\mathbb{Q}$

If  $x$  is a limit point of  $\mathbb{Q}$ , that means  $\forall \epsilon > 0, N_\epsilon(x)$  contains some  $q \neq x$  s.t.  $q \in \mathbb{Q}$ . Because  $\mathbb{Q} \subset \mathbb{R}$ , this means that  $\forall \epsilon > 0, N_\epsilon(x)$  contains some  $q \neq x$  s.t.  $q \in \mathbb{R} \implies x$  is a limit point of  $\mathbb{R}$ . Because  $\mathbb{R}$  is closed, this means that  $x \in \mathbb{R}$ .

Thus we have shown in both cases that  $x \in \mathbb{R}$  and so we have shown  $\forall x \in \bar{\mathbb{Q}}, x \in \mathbb{R} \implies \bar{\mathbb{Q}} \subset \mathbb{R}$ .

(b)  $\mathbb{R} \subset \bar{\mathbb{Q}}$

Pick  $x \in \mathbb{R}$ . We perform casework:

1. **Case One:**  $x \in \mathbb{Q}$

If  $x \in \mathbb{Q} \implies x \in \bar{\mathbb{Q}}$ .

2. **Case Two:**  $x \notin \mathbb{Q}$

We know that  $\forall x \in \mathbb{R}, x$  is a limit point of  $\mathbb{R}$ . Proof<sup>4</sup>. Thus,  $\forall x \in \mathbb{R}$  we know that  $\forall \epsilon > 0, N_\epsilon(x)$  contains some  $y \neq x$  s.t.  $y \in \mathbb{R}$ . Because  $x \neq y \implies$  either  $x < y$  or  $x > y$  because  $\mathbb{R}$  is an ordered field  $\implies \min(x, y) \neq \max(x, y) \implies \min(x, y) < \max(x, y)$ .

<sup>4</sup>Pick  $x \in \mathbb{R}$  and define  $\epsilon > 0 \in \mathbb{R}$ . Then  $N_\epsilon(x) = (x - \epsilon, x + \epsilon)$ , which contains  $x + 0.5\epsilon$ . Because  $x$  and  $0.5\epsilon$  exist in  $\mathbb{R}$  and  $\mathbb{R}$  is closed under addition because it is a field  $\implies x + 0.5\epsilon \in \mathbb{R} \implies \forall \epsilon > 0, N_\epsilon(x)$  contains some  $x' \neq x$  s.t.  $x' \in \mathbb{R} \implies \forall x \in \mathbb{R}, x$  is a limit point of  $\mathbb{R}$ .

Let us define  $\alpha = \min(x, y)$  and  $\beta = \max(x, y)$ . Because  $x, y \in \mathbb{R} \implies \alpha, \beta \in \mathbb{R}$ . Thus, the density of rationals proof (Prop 3.45) applies: we know there exists some rational  $r \in \mathbb{Q}$  s.t.  $\alpha < r < \beta$ . Thus, this means  $\forall \epsilon > 0, N_\epsilon(x)$  contains some  $r \neq x$  s.t.  $r \in \mathbb{Q}$ . This implies  $x$  is a limit point of  $\mathbb{Q} \implies x \in \bar{\mathbb{Q}}$ .

Thus, we have shown in either case,  $x \in \bar{\mathbb{Q}} \implies \forall x \in \mathbb{R}, x \in \bar{\mathbb{Q}} \implies \mathbb{R} \subset \bar{\mathbb{Q}}$