Math 226- HW 4 Due: Sep 26 by Midnight

1. (21 points) Let V be a vector space over a field $\mathbb{F}$, $S \subseteq V$ is finite, and U and W be finite dimensional subspaces of V.

   (a) (6 points) Let $\vec{x} \in V$. Show that

   $$\dim\Big(\mathrm{Span}(S)\Big) \le \dim\Big(\mathrm{Span}(S \cup \{\vec{x}\})\Big) \le \dim\Big(\mathrm{Span}(S)\Big) + 1.$$

   b) (10 points) Prove that

   $$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

   Hint: Start with a basis for $\dim(U \cap W)$, then extend it to W, and U. Finally do some counting.

   (c) (5 points) Prove that if

   $$\dim(U + W) = 1 + \dim(U \cap W).$$

   Then the sum of $U + W$ is equal to one of these spaces (either U or W) and the intersection $U \cap W$ is equal to the other one

2. (22 points) Let U, V be vector spaces over $\mathbb{F}$, and $T : U \to V$ be linear operator. Let B be basis of U and define $T(B) := \{T(u) : u \in B\}$.

   (a) (10 points)Show that $T(B)$ is linearly independent if T is injective. Moreover, prove that if $T(B)$ is linearly independent and $\infty > |T(B)| \ge |B|$, then T is injective.

   (b) (10 points) Show that T is surjective iff $T(B)$ spans V.

   (c) (2 points) Prove that $T(B)$ is a basis for V if T is bijective. Moreover, if $T(B)$ is a basis and $\infty > |T(B)| \ge |B|$ then T is bijective.

3. (19 points) Recall that $C^1(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \text{such that } f'(x) \text{ exist for all } x \in \mathbb{R}\}$. Define T as

   $$\begin{aligned} T :& C^1(\mathbb{R}) \to \mathbb{R}^2 \\ & f(x) \to T(f) := (f'(3), f(3)) \end{aligned}$$

   (a) (4 points) Show that T is a linear transformation.

   (b) (8 points) Let $H := \{(x-3)^2 g(x) : g(x) \in C^1(\mathbb{R})\}$, and $V := \{f : T(f) = (1, 2)\}$. Show that $H + V = V$.
   Remark: This statement is true for any $V := \{f : T(f) = (a, b)\}$ where (a, b) is a fixed vector in $\mathbb{R}^2$.

   (c) (7 points) Deduce from part (b) that T is not injective. Is T surjective? Validate your answer.

4. (8 points) Let $\mathbb{F}$ be a field with regular addition and multiplication.

   a) (6 points) Let $T : \mathbb{F}^5 \to \mathbb{F}^4$ with

   $$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3)$$

   – Find a basis, and the dimension of $N(T)$. (Don't use elementary operation method for matrices)

- Find a basis, and the dimension of $R(T)$. (Don't use elementary operation method for matrices)

b) (2 points)Show that no linear map $T : \mathbb{F}^5 \to \mathbb{F}^2$ can have as its null space the set

$$N(T) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 7x_3, x_2 = x_4 = x_5\}$$

Practice Problems

(a) Suppose $S \subseteq V$ and $|S| = \dim(V)$. Prove that the followings are equivalent

(i) S is a basis for V

(ii) S spans V

(iii) S is linearly independent

Hint: Use Exchange Theorem. To show a set of statements are equivalent you need to prove the statements in a cycle, i.e., (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(b) Recall $P(\mathbb{R}) = \{p(x) = c_0 + c_1x + c_2x^2 + ... + c_nx^n; n \in \mathbb{N}, c_i \in \mathbb{R}\}$. Prove that $P(\mathbb{R})$ is infinite dimensional.

Sec 1.6: 1-28