Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head here.

Question 1

(a) $T_1 = \dfrac{(40+30+T_3+T_2)}{4} \rightarrow 4T_1 - T_3 - T_2 = 70$

$T_2 = \dfrac{(10+30+T_4+T_1)}{4} \rightarrow 4T_2 - T_4 - T_1 = 40$

$T_3 = \dfrac{(20+40+T_4+T_1)}{4} \rightarrow 4T_3 - T_4 - T_1 = 60$

$T_4 = \dfrac{(20+10+T_2+T_3)}{4} \rightarrow 4T_4 - T_3 - T_2 = 30$

(b) $A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$

Applying G.E.:

$$\left[\begin{array}{cccc|cccc} 4 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = \frac{R_1}{4}} \left[\begin{array}{cccc|cccc} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 + R_1} \left[\begin{array}{cccc|cccc} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{15}{4} & -\frac{1}{4} & -1 & \frac{1}{4} & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{15}{4} & -1 & \frac{1}{4} & 0 & 1 & 0 \\ 0 & -1 & -1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\frac{4}{15}R_2 \rightarrow R_2$

$$\left[\begin{array}{cccc|cccc} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} & \frac{1}{15} & \frac{4}{15} & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{15}{4} & -1 & \frac{1}{4} & 0 & 1 & 0 \\ 0 & -1 & -1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 + \frac{R_2}{4} \rightarrow R_1$

$R_3 + \frac{R_2}{4} \rightarrow R_3$

$R_4 + R_2 \rightarrow R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & \frac{1}{15} & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} & \frac{1}{15} & \frac{4}{15} & 0 & 0 \\ 0 & 0 & \frac{56}{15} & -\frac{16}{15} & \frac{4}{15} & \frac{1}{15} & 1 & 0 \\ 0 & 0 & -\frac{16}{15} & \frac{56}{15} & \frac{1}{15} & \frac{4}{15} & 0 & 1 \end{array}\right]$$

$\frac{15}{56}R_3 \rightarrow R_3$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & \frac{1}{15} & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} & \frac{1}{15} & \frac{4}{15} & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{1}{14} & \frac{1}{56} & \frac{15}{56} & 0 \\ 0 & 0 & -\frac{16}{15} & \frac{56}{15} & \frac{1}{15} & \frac{4}{15} & 0 & 1 \end{array}\right]$$

$R_2 + \frac{R_3}{15} \rightarrow R_2$

$R_1 + \frac{4R_3}{15} \rightarrow R_1$

$R_4 + \frac{16}{5}R_3 \rightarrow R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{1}{14} & \frac{1}{14} & 0 \\ 0 & 1 & 0 & -\frac{2}{7} & \frac{1}{14} & \frac{15}{56} & \frac{1}{56} & 0 \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{1}{14} & \frac{1}{56} & \frac{15}{56} & 0 \\ 0 & 0 & 0 & \frac{24}{7} & \frac{1}{7} & \frac{2}{7} & \frac{2}{7} & 1 \end{array}\right]$$

$R_4 = \frac{7R_4}{24}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{1}{14} & \frac{1}{14} & 0 \\ 0 & 1 & 0 & -\frac{2}{7} & \frac{1}{14} & \frac{15}{56} & \frac{1}{56} & 0 \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{1}{14} & \frac{1}{56} & \frac{15}{56} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{24} & \frac{1}{12} & \frac{1}{12} & \frac{7}{24} \end{array}\right]$$

$R_1 + \frac{R_4}{7} \rightarrow R_1$

$R_2 + \frac{2R_4}{7} \rightarrow R_2$

$R_3 + \frac{2R_3}{7} \rightarrow R_3$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{7}{24} & \frac{1}{12} & \frac{1}{12} & \frac{1}{24} \\ 0 & 1 & 0 & 0 & \frac{1}{12} & \frac{7}{24} & \frac{1}{24} & \frac{1}{12} \\ 0 & 0 & 1 & 0 & \frac{1}{12} & \frac{1}{24} & \frac{7}{24} & \frac{1}{12} \\ 0 & 0 & 0 & 1 & \frac{1}{24} & \frac{1}{12} & \frac{1}{12} & \frac{7}{24} \end{array}\right]$$

A^{-1}

Question 1 continued

(c) $AT = X$

$T = A^{-1}X$

$$T = \begin{bmatrix} \frac{7}{24} & \frac{1}{12} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{12} & \frac{7}{24} & \frac{1}{24} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{24} & \frac{7}{24} & \frac{1}{12} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{12} & \frac{7}{24} \end{bmatrix} \begin{bmatrix} 70 \\ 40 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 30 \\ 22.5 \\ 27.5 \\ 20 \end{bmatrix}$$

Question 2

To find the basis of $N(L_A)$, we solve $AX=0$

$$\left(\begin{array}{cccc|c} 3 & 5 & -4 & 2 & 0 \\ 2 & 4 & -6 & 3 & 0 \\ 11 & 17 & -8 & 4 & 0 \end{array}\right) \xrightarrow[3R_3-11R_1\to R_3]{3R_2-2R_1\to R_2} \left(\begin{array}{cccc|c} 3 & 5 & -4 & 2 & 0 \\ 0 & 2 & -10 & 5 & 0 \\ 0 & -4 & 20 & -10 & 0 \end{array}\right) \xrightarrow[2R_1-5R_2\to R_1]{2R_2+R_3\to R_3} \left(\begin{array}{cccc|c} 6 & 0 & 42 & -21 & 0 \\ 0 & 2 & -10 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{R_1}{6}\to R_1} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 7 & -3.5 & 0 \\ 0 & 2 & -10 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$x_1+7x_3-3.5x_4=0$

$2x_2-10x_3+5x_4=0$

$x_1 = 3.5x_4 - 7x_3$

$x_2 = 5x_3 - 2.5x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3.5t_4 - 7t_3 \\ 5t_3 - 2.5t_4 \\ t_3 \\ t_4 \end{pmatrix} = t_3\begin{pmatrix} -7 \\ 5 \\ 1 \\ 0 \end{pmatrix} + t_4\begin{pmatrix} 3.5 \\ -2.5 \\ 0 \\ 1 \end{pmatrix}$$

Thus, the basis for $N(L_A)$ can be given by $\left\{\begin{pmatrix} -7 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3.5 \\ -2.5 \\ 0 \\ 1 \end{pmatrix}\right\}$.

The set of solutions to the given system is given by the span of this basis.

Question 3

$$Ax = b$$
$$\downarrow$$
$$A'x = b'$$

$$[A|B] = \left(\begin{array}{cccc|c} 1 & -3 & -3 & -14 & 8 \\ 2 & -6 & -3 & -1 & -5 \\ 3 & -9 & -5 & -6 & -4 \end{array}\right) \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left(\begin{array}{cccc|c} 1 & -3 & -3 & -14 & 8 \\ 0 & 0 & 3 & 27 & -21 \\ 0 & 0 & 4 & 36 & -28 \end{array}\right)$$

$$\xrightarrow{R_1 + R_2 \to R_1} \left(\begin{array}{cccc|c} 1 & -3 & 0 & 13 & -13 \\ 0 & 0 & 3 & 27 & -21 \\ 0 & 0 & 4 & 36 & -28 \end{array}\right) \xrightarrow[3R_3 - 4R_2 \to R_3]{\frac{R_2}{3} \to R_2} \left(\begin{array}{cccc|c} 1 & -3 & 0 & 13 & -13 \\ 0 & 0 & 1 & 9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - 3x_2 + 13x_4 = -13 \Rightarrow x_1 = -13 + 3x_2 - 13x_4 = -13 + 3t_2 - 13t_4$$
$$x_3 + 9x_4 = -7 \Rightarrow x_3 = -7 - 9t_4$$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -13 + 3t_2 - 13t_4 \\ t_2 \\ -7 - 9t_4 \\ +t_4 \end{pmatrix} = \begin{pmatrix} -13 \\ 0 \\ -7 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} -13 \\ 0 \\ -9 \\ 1 \end{pmatrix}}$$

vectors in $\mathbb{R}^4$ whose image is equal to $b \in \mathbb{R}^3$ under linear map given in A.

The basis of $N(L_A)$ can be given by $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -13 \\ 0 \\ 9 \\ 1 \end{pmatrix} \right\}$.

Question 4

(i) $[T]_\gamma^\beta = \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$

$T(1) = (1,1,1,1)$
$T(x) = (-2,-1,1,2)$
$T(x^2) = (4,1,1,4)$
$T(x^3) = (-8,-1,1,8)$

(ii) $Tx = \begin{bmatrix} 1 \\ 3 \\ 13 \\ 33 \end{bmatrix}$

$$\left[T \,\middle|\, \begin{matrix} 1 \\ 3 \\ 13 \\ 33 \end{matrix}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & 1 \\ 1 & -1 & 1 & -1 & 3 \\ 1 & 1 & 1 & 1 & 13 \\ 1 & 2 & 4 & 8 & 33 \end{array}\right] \xrightarrow[\substack{R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ R_4 - R_1 \to R_4}]{} \left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & 1 \\ 0 & 1 & -3 & 7 & 2 \\ 0 & 3 & -3 & 9 & 12 \\ 0 & 4 & 0 & 16 & 32 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 - 3R_2 \to R_3 \\ R_4 - 4R_2 \to R_4 \\ 2R_2 + R_1 \to R_1}]{} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 6 & 5 \\ 0 & 1 & -3 & 7 & 2 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 12 & -12 & 24 \end{array}\right] \xrightarrow[\substack{R_4 - 2R_3 \to R_4 \\ 2R_2 + R_3 \to R_2 \\ 3R_1 + R_3 \to R_1}]{} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 6 & 21 \\ 0 & 2 & 0 & 2 & 10 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right]$$

$$\xrightarrow[\substack{2R_1 - R_4 \to R_1 \\ 6R_2 - R_4 \to R_2 \\ R_3 + R_4 \to R_3}]{} \left[\begin{array}{cccc|c} 6 & 0 & 0 & 0 & 30 \\ 0 & 12 & 0 & 0 & 48 \\ 0 & 0 & 6 & 0 & 18 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right] \xrightarrow[\substack{\frac{R_2}{12} \to R_2 \\ \frac{R_3}{6} \to R_3 \\ \frac{R_4}{12} \to 12}]{\frac{R_1}{6} \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, we get this polynomial as:

$$f(x) = 5 + 4x + 3x^2 + x^3$$

Question 5

$$V = \{(x_1, x_2, x_3, x_4, x_5) : x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0\}$$

is trivially a subspace of $\mathbb{R}^5$.

We first find the basis of V:

$$x_1 = 2x_2 - 3x_3 + x_4 - 2x_5$$

$$(x_1, x_2, x_3, x_4, x_5) = (2t_2 - 3t_3 + t_4 - 2t_5, t_2, t_3, t_4, t_5)$$

$$= t_2(2,1,0,0,0) + t_3(-3,0,1,0,0) + t_4(1,0,0,1,0) + t_5(-2,0,0,0,1)$$

Thus, the basis for V, β_V is given by

$$\beta_V = \{(2,1,0,0,0), (-3,0,1,0,0), (1,0,0,1,0), (-2,0,0,0,1)\}$$

Given set $S = \{v\} = \{(1,2,1,0,0)\}$:

$$\begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[2R_1 - R_2 \to R_2]{R_3 - R_1 \to R_3} \begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 0 & 3 & -6 & 2 & -4 \\ 0 & -2 & 4 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[3R_3 + 2R_2 \to R_3]{3R_1 - 2R_2 \to R_1} \begin{pmatrix} 3 & 0 & 3 & -1 & 2 \\ 0 & 3 & -6 & 2 & -4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 - R_4 \to R_4,\ R_3 + R_1 \to R_1,\ R_2 - 2R_3 \to R_2} \begin{pmatrix} 3 & 0 & 3 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 + 2R_5 \to R_3,\ 2R_5 + R_4 \to R_5} \begin{pmatrix} 3 & 0 & 3 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{R_1}{3} \to R_1,\ \frac{R_2}{3} \to R_2,\ \frac{R_4}{-2} \to R_4} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 5 continued

Thus,

$$\{(1,2,1,0,0),(2,1,0,0,0),(1,0,0,1,0),(-2,0,0,0,1)\}$$

is a basis for V that includes v.