Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head here.

MATH 255 PSET 4

February 16, 2025

1.

a) We prove that X with this distance function d is a metric space by showing that d obeys all the required properties:

   1. $\forall x, y \in X,$ if $x \neq y, d(x,y) = 1 > 0$
   2. $\forall x \in X, d(x,x) = 0$[1].
   3. We show that $\forall x, y \in X, d(x,y) = d(y,x)$ with casework:
      a) **Case One:** $x = y$
      Then $d(x,y) = 0 = d(y,x) \implies d(x,y) = d(y,x)$.
      b) **Case Two:** $x \neq y$
      Then $d(x,y) = 1$ and $d(y,x) = 1 \implies d(x,y) = 1 = d(y,x) \implies d(x,y) = d(y,x)$
   4. Given $x, y, r \in X$, we show $d(x,y) \leq d(x,r) + d(r,y)$ with casework:
      (a) **Case One:** $x = y$
      If $x = y$, then $d(x,y) = 0$. Because $d(x,r)$ and $d(r,y)$ are strictly ≥ 0, then $d(x,r) + d(r,y) \geq 0$ and so $d(x,r) + d(r,y) \geq d(x,y) \implies d(x,y) \leq d(x,r) + d(r,y)$.
      (b) **Case Two:** $x \neq y$
      If $x \neq y$, then $d(x,y) = 1$. Consider the following two (sub)cases: (i) $r = x$ and (ii) $r \neq x$. In case (i), $d(x,r) = 0$ and because $r = x \implies r \neq y \implies d(r,y) = 1$. So $d(x,r) + d(r,y) = 1 \implies d(x,y) = 1 \leq d(x,r) + d(r,y) \implies d(x,y) \leq d(x,r) + d(r,y)$.

      In case (ii), $d(x,r) = 1$ and we know by properties (1) and (2) that $d(r,y) \geq 0$. Thus, $d(x,r) + d(r,y) \geq 1 \implies d(x,r) + d(r,y) \geq d(x,y) \implies d(x,y) \leq d(x,r) + d(r,y)$.

b) We consider values of ϵ below:

   1. $\epsilon = 0.5$
   For $\epsilon = 0.5$, $N_\epsilon(x) = \{y \in X : d(x,y) < 0.5\}$. Because $\forall x, y \in X, d(x,y) < 0.5 \iff d(x,y) = 0 \iff x = y$, $N_\epsilon(x) = \{x\}$.

[1] This is given by the $d(x,y) = 0$ if $x = y$ piecewise case of d.

2. $\epsilon = 1$
   For $\epsilon = 1$, $N_\epsilon(x) = \{y \in X : d(x,y) < 1\}$. $\forall x, y \in X, d(x,y) < 1 \iff d(x,y) = 0 \iff x = y \implies N_\epsilon(x) = \{x\}$.
3. $\epsilon = 2$
   For $\epsilon = 2$, $N_\epsilon(x) = \{y \in X : d(x,y) < 2\}$. Note that $\forall x, y \in X, d(x,y) \leq 1 \implies \forall x, y \in X, d(x,y) < 2 \implies N_\epsilon(x) = X$.

c) **Open subsets of X**: A subset $E \subset X$ is open if all points in E are interior points of E. This means that $\forall x \in E, \exists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E$. As shown in part (b), for $\epsilon = 1 > 0$, $\forall x \in X, N_\epsilon(x) = \{x\} \subset E \implies \forall x \in E, \exists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E \implies \forall x \in E, x$ is an interior point of $E \implies \forall E \subset X, E$ is open $\implies$ any subset of X is open.

   Closed subsets of X: A subset $E \subset X$ is closed if E contains its limit points. A limit point p is one where every neighborhood contains some $q \in X$ where $q \neq p$. Note this is for every neighborhood (i.e. $\forall\ \epsilon > 0$) - as shown in part (b), $\exists\ \epsilon > 0$ such as 0.5 or 1 where $N_\epsilon(p)$ contains no points other than p. Thus, no limit points exist for $X \implies$ any subset of X is vacuously closed as it has no limit points to contain.

2.

A particular set $S \subset \mathbb{R}$ with exactly three limit points can be given by:

$$S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{3 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{5 - \frac{1}{n} : n \in \mathbb{N}\}$$

The bounds of S are 0 and 5 for the lower and upper bound, respectively. The limit points of S are given by $0, 3, 5$.

3.

1. To prove that E° is open, we prove $(E^\circ)^c$ is closed, meaning that it contains all its limit points.
   Let us define x as a limit point of $(E^\circ)^c$. We WTS $x \in (E^\circ)^c$. Because x is a limit point of $(E^\circ)^c \implies \forall \epsilon > 0, N_\epsilon(x)$ contains some $q \neq x$ s.t. $q \in (E^\circ)^c$. Note that because $q \notin E^\circ \implies q$ is not an interior point of $E \implies$ all neighborhoods of q will contain some element not in E. Thus, defining h as any value $\leq \epsilon - d(q,x)$, $N_h(q)$ contains some element $\notin E$. Because[2] $N_h(q) \subset N_\epsilon(x) \implies N_\epsilon(x)$ contains some element not in $E \implies \nexists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E \implies x$ is not an interior point of $E \implies x \notin E^\circ \implies x \in (E^\circ)^c$. Thus, $(E^\circ)^c$ contains all its limit points $\implies (E^\circ)^c$ is closed $\implies E^\circ$ is open.
2. We prove both directions of this statement below:
   1. **If $E^\circ = E \implies E$ is open**
      E is open if all points of E are interior points. If $E = E^\circ \implies \forall x \in E, x \in E^\circ \implies \forall x \in E, x$ is an interior point of E. Thus, E is open.

[2]We have shown $N_h(q) \subset N_\epsilon(x)$ in our proof that neighborhoods are open.

2. **If E is open $\implies E^\circ = E$**
   If E is open, that means $\forall x \in E$, x is an interior point of E. The set E° contains all interior points of E. Because, $\forall x \in E, x$ is an interior point $\implies$ $\forall x \in E, x \in E^\circ \implies E \subset E^\circ$. Furthermore, because E° only contains points in E (by definition of an interior point) we know that $E^\circ \subset E$. $E \subset E^\circ$ and $E^\circ \subset E \implies E^\circ = E$.

3. Because G is open, $\forall x \in G$, x is an interior point of $G \implies \forall x \in G, \exists\ \epsilon > 0$ such that $N_\epsilon(x) \subset G \subset E \implies \forall x \in G, \exists\ \epsilon > 0$ such that $N_\epsilon(x) \subset E \implies \forall x \in G$, x is an interior point of $E \implies \forall x \in G, x \in E^\circ \implies G \subset E^\circ$.

4. In this question, we are asked to prove $(E^\circ)^c = \bar{E^c}$. To do so, we prove both directions of this statement.

   (i) **Case One:** $(E^\circ)^c \subset \bar{E^c}$
   Pick $x \in (E^\circ)^c$. There are two cases for x, that (a) $x \in E^c$ or that (b) x in E. We consider both cases below:

   (a) $x \in E^c$
   If $x \in E^c \implies x \in \bar{E^c}$.

   (b) $x \in E$
   Because $x \in (E^\circ)^c \implies x$ is not an interior point of E $\implies \nexists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E \implies \forall \epsilon > 0, N_\epsilon(x) \not\subset E \implies \forall \epsilon > 0, \exists\ q \in N_\epsilon(x)$ s.t. $q \notin E$ or expressed differently, $q \in E^c$. Note that because $x \in E$, we can be guaranteed that $q \neq x$. Thus, this statement can be written as $\forall \epsilon > 0, \exists\ q \in N_\epsilon(x)$ s.t. $q \neq x$ and $q \in E^c \implies x$ is a limit point of $E^c \implies x \in \bar{E^c}$.

   Thus, in both cases, $x \in \bar{E^c}$. Thus, we have shown $\forall x \in (E^\circ)^c, x \in \bar{E^c} \implies (E^\circ)^c \subset \bar{E^c}$.

   (ii) **Case Two:** $\bar{E^c} \subset (E^\circ)^c$
   Pick $x \in \bar{E^c}$. At least one of the two cases is true: (a) $x \in E^c$ and (b) x is a limit point of E^c. We consider both cases below:

   (a) $x \in E^c$
   If $x \in E^c \implies x \notin E$. Because $E^\circ \subset E$, $x \notin E \implies x \notin E^\circ \implies x \in (E^\circ)^c$.

   (b) x is a limit point of E^c
   If x is a limit point of $E^c \implies \forall \epsilon > 0, N_\epsilon(x)$ contains some $p \in E^c$ (or $p \notin E$) s.t. $p \neq x \implies \nexists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E \implies x$ is not an interior point of $E \implies x \notin E^\circ \implies x \in (E^\circ)^c$.

   Thus, in both cases, $x \in (E^\circ)^c$. Thus, we have shown $\forall x \in \bar{E^c}, x \in (E^\circ)^c \implies \bar{E^c} \subset (E^\circ)^c$

5. Let us define $E = (-\infty, 0) \cup (0, \infty)$ on the standard metric space $\mathbb{R}$. The closure of E is given by $\bar{E} = (-\infty, \infty) = \mathbb{R}$. Because $\mathbb{R}$ is open $\implies$ every point of $\mathbb{R}$ is an interior point of $\mathbb{R}$, $\bar{E}^\circ = \mathbb{R}^\circ = \mathbb{R}$.
   We now look at the interior of E. All points in $(-\infty, 0)$ and $(0, \infty)$ are interior points. However, 0 is not an interior point of E as it is not in E. Thus, $E^\circ = E = (-\infty, 0) \cup (0, \infty) \neq \bar{E}^\circ \implies E$ and $\bar{E}$ do not always have the same interiors.

6. We inspect the set $E = \{\frac{1}{n} : n \in \mathbb{N}\}$ defined on the standard metric space $\mathbb{R}$. E has no interior points ($\forall x \in E, \nexists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset E$) and so $E^\circ = \emptyset$. The emptyset trivially has no limit points and so the closure of E° is just $\bar{E}^\circ = \bar{\emptyset} = \emptyset \cup \emptyset = \emptyset$.

   We now consider the closure of E, $\bar{E}$. The only limit point of E is zero, and so $\bar{E} = E \cup \{0\}$. Because $\bar{E} \neq \bar{E}^\circ \implies E$ and E° do not always have the same closures.