Math 226: HW 6
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1. a) We evaluate $(AB)^t$ first. Defining $AB = [c_{ij}]_{m\times k} \in M_{m\times k}(\mathbb{F})$, we get that $c_{ij} = \Sigma_{l=1}^{N}\alpha_{il}\beta_{lj}$ through the definition of matrix multiplication. Transposing this matrix, we get that $(AB)^t = [d_{ij}]_{k\times m} \in M_{k\times m}(\mathbb{F})$, where:

$$d_{ij} = c_{ji} = \Sigma_{l=1}^{n}\alpha_{jl}\beta_{li}$$

We now compute B^tA^t. $B^t = [\beta'_{ij}]_{k\times n} \in M_{k\times n}(\mathbb{F})$ is given by $\beta'_{ij} = \beta_{ji}$ through definition of the transpose operation. Similarily, $A^t = [\alpha'_{ij}]_{n\times m} \in M_{n\times m}(\mathbb{F})$ is given by $\alpha'_{ij} = \alpha_{ji}$. The matrix multiplication of B^t and A^t is given by $B^tA^t = [c'_{ij}]_{k\times m} \in M_{k\times m}(\mathbb{F})$, where:

$$c'_{ij} = \Sigma_{l=1}^{n}\beta'_{il}\alpha'_{lj} = \Sigma_{l=1}^{n}\alpha_{jl}\beta_{li}$$

Because we have shown that $(AB)^t = [d_{ij}]_{k\times m}$ and $B^tA^t = [c'_{ij}]_{k\times m}$ are of the same dimension and that $d_{ij} = c'_{ij}$, we have shown that $(AB)^t = B^tA^t$.

b) 1 **Prove if P is invertible, then P^t is invertible.**
If matrix P is invertible, there exists some $n \times n$ matrix A s.t. $PA = AP = I$, where I is the identity matrix. Applying the transpose operation on both sides of the equation $PA = I$, we get that $(PA)^t = I^t = I$. Using part (a), this means that $A^tP^t = I$. Similarily, applying the tranpose operation on both sides of the equation $AP = I$ gives us $(AP)^t = I \Rightarrow P^tA^t = I$. Thus, for matrix P^t we have found a matrix A^t s.t. $A^tP^t = P^tA^t = I \Rightarrow P^t$ has a matrix inverse $A^t \Rightarrow P^t$ is invertible.

2 **Prove $(P^t)^{-1} = (P^{-1})^t$.** This is shown in 1. The inverse of matrix P^t is A^t, which is equal to the inverse of P, matrix A, tranposed. Thus $(P^t)^{-1} = A^t = (P^{-1})^t$.

c) Because matrices P and Q are both $n \times n$ matrices, PQ is also an $n \times n$ matrix. In part (b) we proved that if an $n \times n$ matrix is invertible, its transpose is invertible. Applying part (b), we know that if matrix PQ is invertible, then $(PQ)^t$ is invertible. In part (b), we also proved that if an $n \times n$ matrix was invertible, the inverse of its transpose is equal to the transpose of its inverse: $(P^t)^{-1} = (P^{-1})^t$. Replacing P for matrix PQ in this equation, we get that:

$$((PQ)^t)^{-1} = ((PQ)^{-1})^t$$
$$((PQ)^t)^{-1} = (Q^{-1}P^{-1})^t$$
$$((PQ)^t)^{-1} = (P^{-1})^t(Q^{-1})^t$$

2. a) By the definition of matrix multiplication, $AB = [c_{ij}]_{n\times n}$ where $c_{ij} = \Sigma_{l=1}^{N}\alpha_{il}\beta_{lj}$ and $BA = [d_{ij}]_{n\times n}$ where $d_{ij} = \Sigma_{l=1}^{N}\beta_{il}\alpha_{lj}$.
We now compute $Tr(AB)$ and $Tr(BA)$ below:

$$Tr(AB) = \Sigma_{i=1}^{N}c_{ii} = \Sigma_{i=1}^{N}\Sigma_{l=1}^{N}\alpha_{il}\beta_{li}$$
$$Tr(BA) = \Sigma_{i=1}^{N}d_{ii} = \Sigma_{i=1}^{N}\Sigma_{l=1}^{N}\beta_{il}\alpha_{li}$$

Note that for our calculation of $Tr(BA)$ above, if we rename index i as l and vice versa, we get $Tr(BA) = \Sigma_{l=1}^{N}\Sigma_{i=1}^{N}\alpha_{il}\beta_{li} = \Sigma_{i=1}^{N}\Sigma_{l=1}^{N}\alpha_{il}\beta_{li} = Tr(AB)$.

b) If A and B are similar matrices, this means that there exists an invertible matrix Q s.t. $B = Q^{-1}AQ$. We prove $Tr(A) = Tr(B)$ below.

$$Tr(A) = Tr(B)$$
$$Tr(A) = Tr(Q^{-1}AQ)$$

By definition of matrix inverse, $QQ^{-1} = I$. Thus, we re-express A as $A = AI = AQQ^{-1}$. Using this, we get:

$$Tr(AQQ^{-1}) = Tr(Q^{-1}AQ)$$

From part (a), we proved that if C, D are $n \times n$ matrices, then $Tr(CD) = Tr(DC)$. Defining matrix $C = AQ \in M_{n\times n}(\mathbb{F})$ and $D = Q^{-1} \in M_{n\times n}(\mathbb{F})$, we know from part (a) $T(CD) = T(DC) \Rightarrow Tr(AQQ^{-1}) = Tr(Q^{-1}AQ)$.

3. 1 **Show that if A is a $n \times 1$ matrix and B is a $n \times 1$ matrix, AB has a rank of at most 1.**

We inspect the value of AB below:

$$AB = AB$$
$$AB = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & \dots & a_1b_n \\ \vdots & \ddots & \vdots \\ a_nb_1 & \dots & a_nb_n \end{bmatrix}$$

Note that every single column vector of AB is given by $b_j(a_1, \dots, a_n)$, where j is the column position of this column vector. Thus, all column vectors of AB are scalar multiples of each other and so there can be at most one linearly independent column vector in AB (i.e. we can treat one arbitrary column vector of AB as linear independent and the rest as scalar multiples of it). Because the rank of a matrix is given by the number of linearly independent columns it possesses, the rank of AB is at most one.

2 **If C is any $n \times n$ matrix having rank 1, then there exist $n \times 1$ matrix A, and $1 \times n$ matrix B such that $C = AB$.**

If matrix $C = [c_{ij}]_{n\times n}$ has a rank 1, that means that there is only one column vector which can be considered linearly independent (i.e. non-zero) and that the rest of the column vectors can be represented as scalar multiples of this linearly independent column vector[1]. Let us arbitrarily choose the single linearly independent column vector in C as c_1, the first column of C[2]. A given column vector c_j in the jth column of C can be expressed as a scalar multiple of c_1: $c_j = \lambda_j c_1$. Note that for $j = 1$, $\lambda_j = 1$.

The $n \times 1$ matrix A can be given by c_1 and the $1 \times n$ matrix B can be given by the values $\lambda_1, \lambda_2, \dots, \lambda_n$. We show below that $C = AB$:

[1] In other words, the dimension of span of all the column vectors is one.

[2] Note that our column choice for this vector does not matter as the rank of a matrix is unaffected by the order of its column vectors. We choose c_1 in this proof for convenience.

$$C = AB$$

$$C = c_1 \begin{bmatrix} \lambda_1 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & \lambda_n \end{bmatrix} = \begin{bmatrix} c_{11}\lambda_1 & \dots & c_{11}\lambda_n \\ c_{21}\lambda_1 & \dots & c_{21}\lambda_n \\ \vdots & \ddots & \vdots \\ c_{n1}\lambda_1 & \dots & c_{n1}\lambda_n \end{bmatrix}$$

As we can see above, C expresses a general form for any $n \times n$ matrix which has a rank of 1 (i.e. C has only one linearly independent column c_1 and the remaining column vectors can be composed as scalar multiples of c_1). As shown above, such a matrix C can be decomposed into a matrix multiplication of matrices $A \in M_{n\times 1}(\mathbb{F})$ and $B \in M_{1\times n}(\mathbb{F})$.

4. a) $v_1 \in \mathbb{R}^2$ s.t. $T(v_1) = v_1$ can be given by $v_1 = (1, m)$. $v_2 \in \mathbb{R}^2$ s.t. $T(v_2) = -v_2$ can be given by $v_2 = (-m, 1) \in \mathbb{R}^2$.

   b)

$$[T]_{\beta'} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   c)

$$Q = [I_{\mathbb{R}^2}]_{\beta}^{\beta'} = \begin{bmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{-m}{m^2+1} & \frac{1}{m^2+1} \end{bmatrix} ; Q^{-1} = [I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}$$

   d) We first compute $[T]_\beta = Q^{-1}[T]_{\beta'}Q$ below:

$$[T]_\beta = Q^{-1}[T]_{\beta'}Q$$

$$[T]_\beta = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{-m}{m^2+1} & \frac{1}{m^2+1} \end{bmatrix}$$

$$[T]_\beta = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$$

Given $v = (x, y) \in \mathbb{R}^2$, $T(v) = [T]_\beta [v]_\beta$. Thus, we get the expression $T(v)$ as:

$$T(v) = [T]_\beta [v]_\beta = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\frac{1-m^2}{m^2+1} + \frac{2my}{m^2+1} \\ \frac{2mx}{m^2+1} + y\frac{m^2-1}{m^2+1} \end{bmatrix}$$