

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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## Math 226- HW 9 Due: November 12 by Midnight

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1. (16 points) Let  $A \in M_{n \times n}(\mathbb{R})$ 
  - a) (8 points) We call a matrix upper triangular if all the entries below the main diagonal are zero, and lower triangular if all the entries above the main diagonal are zero. Show that if  $A$  is upper triangular then  $\det(A) = \prod_{i=1}^n a_{ii}$ . Use the fact that  $\det(A) = \det(A^T)$  to conclude the same for lower triangular matrices.  
Hint: Use induction with the determinant formula expanded wrt.  $n$ -th row.
  - b) (8 points) Use elementary row operations to transform the following matrix to a triangle matrix, and use it to show  $\det(A) = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ .

$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix}$$

2. (18 points) Let  $\delta : M_{n \times n}(\mathbb{F}) \rightarrow \mathbb{F}$  be a function that is  $n$ -linear and alternating.
  - $n$ -linear: if  $\delta$  is a linear function of each row of an  $n \times n$  matrix when the remaining  $n - 1$  rows are held fixed.
  - alternating: if  $A$  has two identical rows, then  $\delta(A) = 0$ .

Show that

- a) (5 points) If  $A$  is a matrix obtained from  $B$  by interchanging any two rows, then  $\delta(A) = -\delta(B)$ .
  - b) (5 points) Let  $\mathcal{E}$  be any row operation in pg. 223 of the book. Show that  $\delta(\mathcal{E}) = \delta(I)\det(\mathcal{E})$ . (Check out Corollary 3 in section 4.5)
  - c) (8 points) Use part *b*), to show that for all  $A \in M_{n \times n}(\mathbb{F})$ ,  $\delta(A) = k\det(A)$ . (As always consider two cases, if  $A$  is not invertible and if  $A$  is invertible. In the former case both of the quantities will be zero. In the latter case  $A$  is product of elementary matrices.)
3. (14 points) Let  $A_n$  be a  $(n-1) \times (n-1)$  matrix given with the formula

$$A_n = \begin{pmatrix} 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & 1 & \dots & 1 \\ 1 & 1 & 5 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & n+1 \end{pmatrix}$$

Show that the set  $\{\frac{\det(A_n)}{n!}, n \geq 2\}$  is unbounded.

Hint: Use elementary operations and try to turn  $A_n$  to an upper triangle matrix-as much as you can.

4. (22 points) Let  $A \in M_{n \times n}(\mathbb{F})$ 
  - a) (6 points) Prove that if  $A = \begin{pmatrix} B & C \\ 0 & I_k \end{pmatrix}$  for some  $k \leq n$  then  $\det(A) = \det(B)$ .
  - b) (10 points) Use part *a*) to prove if  $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$  for some  $D \in M_{k \times k}(\mathbb{F})$ ,  $k \leq n$ ,  $\det(A) = \det(B)\det(D)$ .
  - c) (5 points) A matrix  $m \in M_{n \times n}(\mathbb{C})$  is called skew-symmetric if  $M^t = -M$ . Prove that if  $M$  is skew symmetric and  $n$  is odd, then  $M$  is not invertible.

d) (1 points) What can you say if  $n$  is even?

Remark: In fact if  $M$  is skew symmetric then  $\det(M) \geq 0$ . To understand the proof read this very short paper -link only works in the pdf file in canvas not in gradescope

**Practice Problems:** Sec 4.1 : 1-11, Sec 4.2: 1-30, Sec 4.3: 1-25, Sec 4.4: 1-5