

Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head [here](#).

STATS 242 HW 3

February 4, 2025

Number of late days: 0; Collaborators: Alex Wa, Derek Gao

1.

a) We detail our devised two possible test statistics:

(1) **Test Statistic T_1 : Z-Statistic**

For this test statistic, we want to check whether the expected number of male children in each family is actually $12(0.5) = 6$, what we would expect under H_0 if $X_i \sim \text{Bin}(12, 0.5)$. In other words, $H_0 : \mu = 6$ and $H_1 : \mu \neq 6$. Because the variance of each X_i is known under H_0 , we can use the Z -statistic as our test statistic:

$$T_1 = \frac{\sqrt{6115} \bar{X}}{\sigma}$$

In my code, we compute the T_1 statistic for this data to be **260.48**.

(2) **Test Statistic T_2 : Sample Variance**

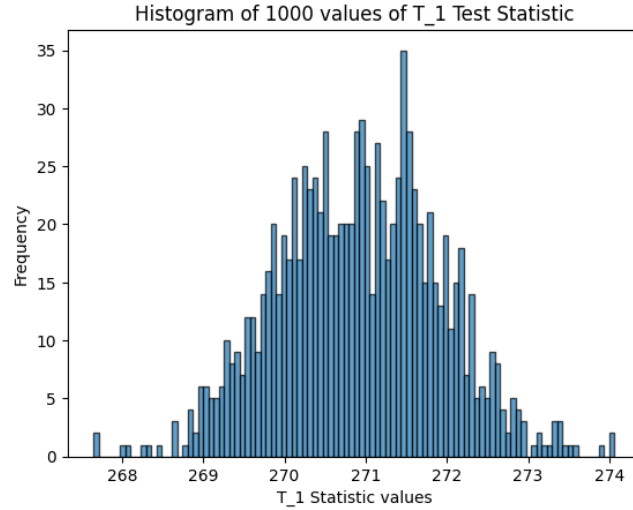
For this test statistic, we test if sample variance is actually equal to $12(0.5)(0.5) = 3$, as it would be under H_0 where the observed male child frequencies follow a $\text{Bin}(12, 0.5)$ distribution. Our test statistic is the sample variance:

$$T_2 = \frac{\sum_{i=1}^{6115} (X_i - \bar{X})^2}{6115 - 1}$$

In my code, we compute the T_2 statistic for this data to be **3.490**.

b) (1) **Test Statistic T_1**

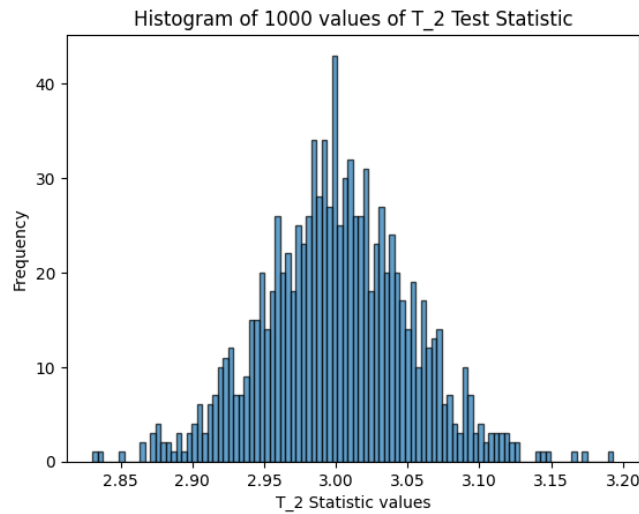
Below is our histogram for the null distribution of the T_1 test statistic.



For this test statistic at the $\alpha=0.05$ significance level, the upper- α point is **268.60** and the lower- α point which is **268.03**, which is less than the observed T_1 statistic for this data (**260.48**). Thus, we reject H_0 .

(2) **Test Statistic T_2**

Below is our histogram for the null distribution of the T_2 test statistic.



For this test statistic at the $\alpha=0.05$ significance level, the upper- α point and lower- α point are given by **2.875** and **2.834** respectively. Because the observed T_2 statistic for this data (**3.490**) is greater than the upper- α point, we reject H_0 .

```
1 # %%
2 """Run all imports"""
```

```

3 import numpy as np
4 import matplotlib.pyplot as plt
5 import math
6
7 NUM_MALES = np.array([i for i in range(0, 12 + 1)])
8 NUM_FAMILIES = np.array([7, 45, 181, 478, 829, 1112, 1343, 1033, 670,
    ↪ 286, 104, 24, 3])
9 VARIANCE = 12 * 0.5 * (1 - 0.5)
10
11 assert np.sum(NUM_FAMILIES).item() == 6115
12
13 def t1_statistic(num_males, num_families):
14     mean_num_males = np.sum(num_families * num_males) /
    ↪ np.sum(num_families)
15     return np.sqrt(6115 / VARIANCE) * mean_num_males
16
17 def t2_statistic(num_males, num_families):
18     mean_num_males = np.sum(num_families * num_males) /
    ↪ np.sum(num_families)
19     return np.sum(num_families * (num_males - mean_num_males) ** 2) /
    ↪ (6115 - 1)
20
21 print(f"On this data, our T_1 statistic is {t1_statistic(NUM_MALES,
    ↪ NUM_FAMILIES)}")
22 print(f"On this data, our T_2 statistic is {t2_statistic(NUM_MALES,
    ↪ NUM_FAMILIES)}")
23 # %%
24 def get_data_for_one_simulation():
25     # for 6115 families, sample the number of children they have based
    ↪ on Bin(12, 0.5)
26     num_males = NUM_MALES
27     num_families = np.zeros_like(NUM_FAMILIES)
28     num_males_6115 = np.random.binomial(12, 0.5, 6115)
29     for i, num_male in enumerate(num_males):
30         num_families[i] = np.sum(num_males_6115 == num_male)
31     assert np.sum(num_families) == 6115
32     return num_males, num_families
33
34
35 N_SIMULATIONS = 1000
36
37 def get_statistics_simulation(statistic_func):
38     test_statistics = []
39     for _ in range(N_SIMULATIONS):
40         num_males, num_families = get_data_for_one_simulation()
41         test_statistics.append(statistic_func(num_males,
    ↪ num_families))

```

```

42     return np.array(test_statistics)
43
44     """
45     Run 1000 simulations to approximate null distribution of T_1 and T_2
46     ↪ Statistics.
47     """
48
49     t1_statistics = get_statistics_simulation(t1_statistic)
50     t2_statistics = get_statistics_simulation(t2_statistic)
51     """
52     Plot histogram of these statistics.
53     """
54
55     plt.hist(t1_statistics, bins=100, edgecolor='black', alpha=0.7)
56
57     # Add labels and title
58     plt.xlabel('T_1 Statistic values')
59     plt.ylabel('Frequency')
60     plt.title('Histogram of 1000 values of T_1 Test Statistic')
61     plt.show()
62
63     plt.hist(t2_statistics, bins=100, edgecolor='black', alpha=0.7)
64     plt.xlabel('T_2 Statistic values')
65     plt.ylabel('Frequency')
66     plt.title('Histogram of 1000 values of T_2 Test Statistic')
67     # %%
68     """
69     For the T_1 Statistic, can we reject?
70     """
71
72     SIGNIFICANCE_LEVEL = 0.05
73     OBSERVED_T1_STATISTIC = t1_statistic(NUM_MALES, NUM_FAMILIES)
74     t1_critical_value_upper_end = np.percentile(t1_statistics, 0.95)
75     t1_critical_value_lower_end = np.percentile(t1_statistics, 0.05)
76     if (t1_critical_value_upper_end <= OBSERVED_T1_STATISTIC): print("Can
77     ↪ reject null hypothesis for T_1 test statistic.")
78     if (t1_critical_value_lower_end >= OBSERVED_T1_STATISTIC): print("Can
79     ↪ reject null hypothesis for T_1 test statistic.")
80
81     """
82
83     For the T_2 Statistic, can we reject?
84     """
85
86     SIGNIFICANCE_LEVEL = 0.05
87     OBSERVED_T2_STATISTIC = t2_statistic(NUM_MALES, NUM_FAMILIES)
88     t2_critical_value_upper_end = np.percentile(t2_statistics, 0.95)
89     t2_critical_value_lower_end = np.percentile(t2_statistics, 0.05)
90     if (t2_critical_value_upper_end <= OBSERVED_T2_STATISTIC): print("Can
91     ↪ reject null hypothesis for T_2 test statistic.")

```

```

84 if (t2_critical_value_lower_end >= OBSERVED_T2_STATISTIC): print("Can
    ↪ reject null hypothesis for T_2 test statistic.")
85
86 # %%

```

2.

- a) Note that because H_0 is true, then we can assume the true distribution of to be $Z \sim \mathcal{N}(0, 1)$. To find the distribution of $P = 1 - \Phi(Z)$, we compute its CDF below:

$$P(P \leq p) = P(1 - \Phi(Z) \leq p)$$

Note that by the Universality of the Uniform, $\Phi(Z) \sim \text{Unif}(0, 1)$. Therefore, $1 - \Phi(Z) = 1 - \text{Unif}(0, 1) = \text{Unif}(-1, 0) + 1 = \text{Unif}(0, 1)$. Let us define $U \sim \text{Unif}(0, 1)$. Putting this all together, we have:

$$P(P \leq p) = P(1 - \Phi(Z) \leq p) = P(U \leq p) = F_U(p)$$

where F_U is the CDF of U . Because r.v. P has the same CDF as $U \sim \text{Unif}(0, 1)$, we can conclude $P \sim \text{Unif}(0, 1)$ if H_0 is true.

- b) If H_1 was true, then we would expect to observe greater values of test statistic $Z \implies \Phi(Z)$, which is strictly non-decreasing, would be greater $\implies P = 1 - \Phi(Z)$ would be smaller. Thus, if H_1 was true, we would expect to observe smaller p-values. As shown in part (a), $P \sim \text{Unif}(0, 1)$ so $P_{H_0}(P \leq \alpha) = F_U(\alpha) = \alpha$. Therefore, if we were to use P as the test statistic, we would reject H_0 when $P \leq \alpha$.

3.

- a) Let us define Z_1, \dots, Z_n to all be i.i.d $\mathcal{N}(0, 1)$ distributions. We can define distribution $U_n = \frac{Z_1^2 + \dots + Z_n^2}{n}$, or equivalently that $U_n \sim \frac{1}{n} \cdot \chi_n^2$. We first compute $\mathbb{E}[Z_i]^2 = \text{Var}[Z_i] + \mathbb{E}[Z_i] = 1 + 0 = 1$. By the Weak Law of Large Numbers, since $\mathbb{E}[Z_i] = 1$ and Z_i has finite variance, $U_n \rightarrow \mathbb{E}[Z_i]$ or $U_n \rightarrow 1$ in probability as $n \rightarrow \infty$.

Let us now define function $g(z) = \sqrt{z}$, where g is continuous for $(0, \infty)$. By the Continuous Mapping Theorem, distribution $g(U_n) = \sqrt{U_n} \rightarrow \sqrt{1}$, or $\sqrt{U_n} \rightarrow 1$ in probability as $n \rightarrow \infty$.

- b) The t -distribution with n degrees of freedom is given by:

$$T_n = \frac{\frac{\sqrt{n} \bar{X}}{\sigma}}{\sqrt{\frac{1}{n} \chi_n^2}}$$

where $\bar{X} = \frac{X_1 + \dots + X_n}{n}$ and $\sigma = \sqrt{\text{Var}(X_i)}$. As shown in part (a), $\sqrt{\frac{1}{n}\chi_n^2} \rightarrow 1$ in probability as $n \rightarrow \infty$. Furthermore, by CLT as $n \rightarrow \infty$, $\bar{X} \sim \mathcal{N}(0, \sigma^2)$ and thus $\frac{\sqrt{n}}{\sigma}\bar{X} \rightarrow \mathcal{N}(0, 1)$. Because $\sqrt{\frac{1}{n}\chi_n^2} \rightarrow 1 \neq 0$, we can apply Slutsky's Lemma. By Slutsky's Lemma, $T_n = \frac{\frac{\sqrt{n}}{\sigma}\bar{X}}{\sqrt{\frac{1}{n}\chi_n^2}} \rightarrow \frac{\mathcal{N}(0,1)}{1}$ or $T_n \rightarrow \mathcal{N}(0, 1)$ as $n \rightarrow \infty$.

4.

- (a) $T \sim t_1$ is given by $\frac{X}{\sqrt{\frac{1}{1}\chi_1^2}} = \frac{X}{\sqrt{Y^2}}$, where X, Y are i.i.d $\mathcal{N}(0, 1)$. By the Continuous Mapping Theorem, given $g(x) = \sqrt{x^2} = |x|$ then $g(Y)$ will converge to $|Y|$ and so T is given by $\frac{X}{|Y|}$.

We can define $\frac{X}{|Y|}$ as the following:

$$\frac{X}{|Y|} = \begin{cases} \frac{X}{Y} & \text{if } Y > 0 \\ -\frac{X}{Y} = \frac{X}{-Y} & \text{if } Y < 0 \end{cases}$$

Because Y and $-Y$ have the same distributions (because Y is symmetric about the origin), both the $\frac{X}{Y}$ and $\frac{X}{-Y}$ have the same distribution (which we can call U). Because then we have $\frac{X}{|Y|} = U \cdot P(Y > 0) + U \cdot P(Y < 0) = \frac{U+U}{2} = U$, we can conclude $\frac{X}{|Y|} \sim U$ or $\frac{X}{|Y|} \sim \frac{X}{Y}$ ¹

- b) Let us define $g(x, y) = (\frac{x}{y}, y)$. Because g is bijective, its inverse can be given by $g^{-1}(t, u) = (tu, u)$. We can use the change-of-variables formula to compute the joint PDF $f_{T,U}(t, u)$ ²:

$$f_{T,U}(t, u) = f_{X,Y}(g^{-1}(t, u)) \cdot \left| \det \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = f_{X,Y}(tu, u) \cdot \left| \det \begin{pmatrix} \frac{\partial(tu)}{\partial t} & \frac{\partial(tu)}{\partial u} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial u} \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = f_{X,Y}(tu, u) \cdot \left| \det \begin{pmatrix} u & t \\ 0 & 1 \end{pmatrix} \right|$$

$$f_{T,U}(t, u) = |u|f_{X,Y}(tu, u) = |u|f_X(tu)f_Y(u) = |u|\frac{1}{2\pi}e^{-\frac{[(tu)^2+u^2]}{2}} = |u|\frac{1}{2\pi}e^{-\frac{u^2(t^2+1)}{2}}$$

We now compute f_T :

$$f_T(t) = \int_{-\infty}^{\infty} f_{T,U}(t, u) du = \int_{-\infty}^{\infty} \frac{|u|}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du = \int_{-\infty}^0 \frac{-u}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du + \int_0^{\infty} \frac{u}{2\pi} e^{-\frac{u^2(t^2+1)}{2}} du$$

¹Note that $P(Y < 0) = P(Y > 0) = 0.5$ because Y is a normal distribution and is therefore symmetric about the origin.

²Note that because X and Y are independent, $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.

Note that these two integrals are equivalent (we can use a change of variables $dv = -u$ in the first one) and so we can combine them:

$$f_T(t) = \frac{1}{\pi} \left[\int_0^\infty u e^{-\frac{u^2(t^2+1)}{2}} du \right]$$

Using a u-substitution for $v = \frac{u^2(t^2+1)}{2}$, we have:

$$f_T(t) = \frac{1}{\pi} \int_0^\infty \frac{1}{t^2+1} e^{-v} dv = -\frac{1}{\pi(t^2+1)} [e^{-v}] \Big|_0^\infty = -\frac{1}{\pi(t^2+1)} [0 - 1] = \frac{1}{\pi(t^2+1)}$$

Given the PDF of T we now can compute its expectation:

$$\mathbb{E}[T^2] = \int_{-\infty}^\infty \frac{t^2}{\pi(t^2+1)} dt = \frac{1}{\pi} \left[\int_{-\infty}^\infty 1 - \frac{1}{t^2+1} dt \right] = \frac{1}{\pi} [t - \arctan(t)] \Big|_{-\infty}^\infty$$

which diverges to ∞ . Thus we have $\mathbb{E}[T^2] = \infty$.