

# PSETs Landing Page\*

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**Math 226: HW 5**
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1. a) We can define  $[T]_\alpha$  below as:

$$T_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4 \times 4}$$

- b) **Answers for**  $[U]_\beta^\gamma, [T]_\beta, [UT]_\beta^\gamma$

$$[U]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}_{3 \times 3} \quad [T]_\beta = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}_{3 \times 3} \quad [UT]_\beta^\gamma = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}_{3 \times 3}$$

**Answers for**  $[h(x)]_\beta$  **and**  $[U(h(x))]_\gamma$

$$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad [U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

- c)

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{n \times n}$$

2. a) For  $T^{-1}$  to be a linear operator, then  $T^{-1}(cx + y) = cT^{-1}(x) + T^{-1}(y)$  where  $c \in \mathbb{F}$  and  $x, y \in V$ .

Let us define  $u_1, u_2 \in U$  and  $v_1 = T(u_1), v_2 = T(u_2) \in V$ . We also define  $c \in \mathbb{F}$ . We evaluate  $T^{-1}(cv_1 + v_2)$  below:

$$\begin{aligned} T^{-1}(cv_1 + v_2) &= T^{-1}(cT(u_1) + T(u_2)) \\ T^{-1}(cv_1 + v_2) &= T^{-1}(T(cu_1 + u_2)) \\ T^{-1}(cv_1 + v_2) &= cu_1 + u_2 \end{aligned}$$

We evaluate  $cT^{-1}(v_1) + T^{-1}(v_2)$ :

$$cT^{-1}(v_1) + T^{-1}(v_2) = cu_1 + u_2$$

Because we have shown  $T^{-1}(cv_1 + v_2) = cT^{-1}(v_1) + T^{-1}(v_2) = cu_1 + u_2$ , we have shown that  $T^{-1}$  is a linear operator.

- b) Let us define  $(x, y) = xe_1 + ye_2 \in \mathbb{R}^2$ , where  $\beta_2 = \{e_1, e_2\}$  is the standard ordered basis for  $\mathbb{R}^2$ . We will also define the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  through the following matrix:

$$[T]_{\beta_2}^{\beta_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where  $a, b, c, d \in \mathbb{R}$ . We inspect the value of  $T(x, y)$  below.

$$\begin{aligned} T(x, y) &= [T]_{\beta_2}^{\beta_2} \begin{bmatrix} x \\ y \end{bmatrix} \\ T(x, y) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ T(x, y) &= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \end{aligned}$$

Thus, we get that  $T(x, y) = (ax + by, cx + dy) \in \mathbb{R}^2$ .

We now prove by contradiction that for  $T$  to be a valid isomorphism, and thus bijective,  $ad - bc \neq 0$ . Suppose  $ad - bc = 0$ . This means that  $\frac{b}{a} = \frac{d}{c}$ . Because any input with the form  $(x, -\frac{a}{b}x)$  will be part of  $N(T)$ , we have found multiple pre-images in  $U$  to the same image (i.e.  $\mathbf{0}^2$ ) in  $V \Rightarrow T$  is not injective  $\Rightarrow T$  is not bijective  $\Rightarrow T$  is not a valid isomorphism. Thus, we have proven that for  $T$  to be a valid isomorphism,  $ad - bc \neq 0$  so that  $N(T) = \{0\} \Rightarrow T$  is injective  $\Rightarrow T$  is surjective  $\Rightarrow T$  is invertible  $\Rightarrow T$  isomorphism.

3. a) For proof by contrapositive, let us assume  $T$  is not injective. This means  $\exists x, y \in V$  s.t.  $T(x) = T(y)$  and  $x \neq y$ . Let us define  $x' = T(x) \in W$ , and  $y' = T(y) \in W$  where  $x' = y'$ . We prove that  $UT(x) = UT(y)$  below.

$$\begin{aligned}
UT(x) &= UT(y) \\
U(T(x)) &= U(T(y)) \\
U(x') &= U(y') \\
U(x') &= U(x') \\
0 &= 0
\end{aligned}$$

Thus, we have shown  $\exists x, y \in V$  s.t.  $UT(x) = UT(y)$  and  $x \neq y \Rightarrow UT$  is not injective. Thus, we have shown if  $T$  is not injective,  $UT$  is not injective. By proof by contrapositive, we have proved if  $UT$  is injective,  $T$  is injective.

**Does  $U$  have to be injective if  $\{UT, T\}$  are injective?**

Let us define  $x, y \in V$  and  $w_1 = T(x), w_2 = T(y) \in R(T) \leq W$ . Suppose  $w_1 \neq w_2$ . Let us assume  $U$  is not injective. If  $U$  is not injective, then it is possible for  $U(w_1) = U(w_2)$  even if  $w_1 \neq w_2$ . This would mean that  $U(T(x)) = U(T(y)) \Rightarrow UT(x) = UT(y)$ . Because  $UT$  is given as injective, if  $UT(x) = UT(y)$ , then we know that  $x = y$ . However, because we are considering the case for which  $w_1 \neq w_2$  or  $T(x) \neq T(y)$ , we know that  $x \neq y$ . Thus, if  $U$  is not injective, the property  $UT(x) = UT(y) \Rightarrow x = y$  is not necessarily true and so this contradicts our given that  $UT$  is injective. Note, however, that it only matters that for  $U$  is injective for input  $w \in R(T)$ . This is because ensuring the injectivity of  $UT$  means that  $U$  has to be injective only for outputs of  $T$  (i.e.  $R(T)$ ). By proof by contradiction, we have proven that if  $\{UT, T\}$  are injective, then  $U$  must be injective as well over  $R(T)$ . *However,  $U$  does not need to be fully injective.*

- b) For proof by contrapositive, let us assume  $U$  is not surjective. This means  $\exists z \in Z$  s.t.  $\nexists w \in W$  s.t.  $U(w) = z$ .<sup>1</sup> If  $UT$  is surjective, this means that  $\exists v \in V$  s.t.  $UT(v) = z$ . In other words, this means that  $U(T(v)) = z$ , or that  $\exists w \in R(T) \leq W$  s.t.  $U(w) = z$ . However, because we are assuming  $U$  is not surjective,  $\nexists w \in W$  s.t.  $U(w) = z \Rightarrow \nexists v \in V$  s.t.  $UT(v) = z \Rightarrow UT$  is not surjective. Thus, we have shown that if  $U$  is not surjective,  $UT$  is not surjective. By proof by contrapositive, we have shown that if  $UT$  is surjective, then  $U$  is surjective.

**Does  $T$  have to be surjective if  $\{UT, U\}$  are surjective?**

If  $T$  is surjective, that means that  $R(T) = W$ . For  $UT$  and  $U$  to be surjective, that means that there needs to exist pre-images in  $V$  and  $W$ , respectively, that map to every element in  $Z$ . However, for transformation  $U$ , if every single pre-image of  $Z$  in  $W$  exists in the subspace  $R(T) \leq W$ , then surjectivity for  $U$  is maintained. Note that this does not affect the surjectivity of  $UT$  as the pre-images of  $UT$  in  $V$  will all be mapped by  $T$  to  $R(T) \leq W$  by the definition of a range of a transformation. Thus,  $T$  only must be surjective for pre-images in  $R(T)$  but not every element in  $W$ ;  *$T$  does not have to be fully surjective.*

- c) We are given that the matrix  $AB$  is invertible  $\Rightarrow$  transformation  $L_{AB}$  is invertible. We define  $L_{AB}$  as the composition of transformations  $L_A$  and  $L_B$ . Additionally, a transformation is only invertible if it is bijective. Thus, we know that  $L_{AB}$  is bijective and so  $L_{AB}$  is both surjective and injective. Finally, because we are given the matrix representations for  $A$  and  $B$ , we can trivially assume  $L_A$  and  $L_B$  are linear transformations.

We prove below that matrices  $A$  and  $B$  are invertible.

**①  $A$  is invertible**

From part (b), we know that if transformation  $L_{AB} = L_A L_B$  is surjective, then transformation  $L_A$  is surjective. From Theorem 2.5, we know that given a linear

<sup>1</sup>We reference an example of this element of  $Z$  with this property as  $z$  in the remainder of this proof.

transformation  $T$  with input and output vector spaces of equal dimensionality, if  $T$  is surjective, then  $T$  is also injective. Because  $L_A$  is a square matrix, we know that it has input and output vector spaces of equal dimensionality. Thus, we know that given  $L_A$  is surjective,  $L_A$  is also injective. Because  $L_A$  is surjective and injective, it is bijective and so  $L_A$  is invertible  $\Rightarrow$  the corresponding matrix  $A$  is invertible.

②  **$B$  is invertible**

From part (a), we know that if transformation  $L_{AB} = L_A L_B$  is injective, then transformation  $L_B$  is injective. From Theorem 2.5, we know that given a linear transformation  $T$  with input and output vector spaces of equal dimensionality, if  $T$  is injective, then  $T$  is also surjective. Because  $L_B$  is a square matrix, we know that it has input and output vector spaces of equal dimensionality. Thus, we know that given  $L_B$  is injective,  $L_B$  is also surjective. Because  $L_B$  is surjective and injective, it is bijective and so  $L_B$  is invertible  $\Rightarrow$  the corresponding matrix  $B$  is invertible.

- d) We are given that matrix  $AB = I_n$ . Because the inverse of the identity matrix is itself,  $(AB)^{-1} = I_n$  and so  $AB$  is invertible. From part (c), we have proven that if  $AB$  is invertible, then *matrix  $A$  is invertible* and matrix  $B$  is invertible.

Let us define the inverse of  $A$  as  $A^{-1}$ . Because  $A$  is a square  $n \times n$  matrix,  $AA^{-1} = I_n$  which is also equal to  $AB$ .

We show this below.

$$\begin{aligned} AA^{-1} &= I_n = AB \\ AA^{-1} &= AB \end{aligned}$$

Because  $A$  is invertible and appears in the same position on both sides of the equation, we can remove  $A$  to get:

$$A^{-1} = B$$

Thus we have proven  $A^{-1} = B$ .

4. For proof by contrapositive, we will assume that  $\{T, U\}$  are linearly dependent subsets of  $\mathcal{L}(V, W)$ . This means that given  $c_1, c_2 \in \mathbb{F}$ , there exists a solution to the equation  $c_1 T + c_2 U = 0$  where at least one of  $\{c_1, c_2\}$  is not equal to zero. This means that we can re-express  $T$  or  $U$  as a scalar multiplied by the other linear operator. In other words, given  $k \in \mathbb{F}$ ,  $T = kU$ .

Consider  $w \neq 0 \in R(U)$ .<sup>2</sup> This means that  $\exists v \in V$  s.t.  $U(v) = w$ . Because we know  $T = kU$ ,  $T(v) = kU(v) = kw \Rightarrow kw \in R(T)$ . We now show that  $kw \in R(U)$ . Because  $V$  is a vector space, we know that it is closed under scalar multiplication  $\Rightarrow kv \in V$ . Because  $U$  is linear,  $U(kv) = kU(v) = kw \Rightarrow kw \in R(U)$ . Because  $w \neq 0$ ,  $kw \neq 0 \in R(U)$ ,  $R(T) \Rightarrow R(T) \cap R(U) \neq \{0\}$ . Thus we have shown that if  $\{T, U\}$  are not linearly independent subsets of  $\mathcal{L}(V, W)$ , then  $R(T) \cap R(U) \neq \{0\}$ . Using proof by contrapositive, we have proved if  $R(T) \cap R(U) = \{0\}$ , then  $\{T, U\}$  are linearly independent subsets of  $\mathcal{L}(V, W)$ .

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<sup>2</sup>Because  $U \in \mathcal{L}(V, W)$ , we know that  $U$  is a nonzero operator. Thus,  $\exists w \neq 0 \in R(U)$ .