Math 226- HW 3 Due: Sep 19 by Midnight

1. (10 points)

   a) (6 points) In $\mathbb{F}^n$ (Take $\mathbb{F} = \mathbb{R}$ first if $\mathbb{F}$ confuses you). Let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $S := \{e_1, e_2, ..., e_n\}$ is linearly independent set and generates $\mathbb{F}^n$. **Remark:** The vectors $e_i : i = 1, 2, .., n$ are called standard vectors. In $\mathbb{R}^n$, we use these frequently and call the set of standard vectors "the standard bases".

   c) (4 points) Let V be vector space over a field $\mathbb{F}$ with characteristic not equal to two. (This means that if $x \neq 0$, then $x + x \neq 0$ for any $x \in \mathbb{F}$.) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

   To see why, characteristic causes a trouble, note that you can gain zero while adding and subtracting the coefficients.

2. (22 pts) Let V be a vector space over $\mathbb{F}$ and $S = \{v_1, v_2, ..., v_k\} \subseteq V$.

   (a) (6 points) For every $x \in V$, prove that $x \in \text{span}(S)$ iff $\text{span}(S) = \text{span}(S \cup \{x\})$.

   (b) (6 points) Suppose that $w \in V$ but $w \notin S$. *Further suppose that S is linearly independent.* Prove that $S \cup \{w\}$ is linearly independent iff $w \notin \text{span}(S)$.

   (c) (10 points) We write $A \subsetneqq B$ to mean $A \subseteq B$ and $A \neq B$. Prove that a set $S = \{u_1, u_2, \ldots, u_k\}$ is linearly independent iff

   $$\{0\} \subsetneqq \text{span}(\{u_1\}) \subsetneqq \text{span}(\{u_1, u_2\}) \subsetneqq \text{span}(\{u_1, u_2, u_3\}) \subsetneqq \cdots \subsetneqq \text{span}(\{u_1, ..., u_k\}).$$

   Remark: Another insignificant looking, but <u>fundamental</u> result. With this problem, we say that a set is linearly independent iff the span gets bigger with addition of each vector vector.

3. (14 points) Suppose U and W are subspaces of V. Recall we define the set $U + W = \{u + w : u \in U, w \in W\}$. We say $U + W$ is a direct sum if and only if $U \cap W = \{0\}$. In this case we represent the sum as $U \bigoplus W$. For example, if $U = \text{Span}((1, 0))$ and $W = \text{Span}((0, 1))$ then $\mathbb{R}^2$ =U $\bigoplus W$ since $\mathbb{R}^2 = U + W$ and $U \cap W = \{(0, 0)\}$

   a) (3 points) Let U be the subspace of $\mathbb{R}^5$ defined by $U := \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, x_3 = 7x_4\}$. Find a basis of U. (Recall we did a simpler version of that in the class.)

   - (2 points) Extend the basis in the first part to a basis of $\mathbb{R}^5$
   - (2 points) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \bigoplus W$. Validate your answer.

   b) (3 points) Let U be the subspace of $P_4(\mathbb{R})$ defined by $U =: \{f \in P_4(\mathbb{R}) : f'(1) = 0\}$. Find a basis of U.

   - (2 points) Extend the basis in the first part to a basis of $P_4(\mathbb{R})$.
   - (2 points) Find a subspace W of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R}) = U \bigoplus W$. Validate your answer.

4. (7 points) We define $M_{n\times n}(\mathbb{F})$ as the matrices defined over $\mathbb{F}$ with size $n \times n$. We also define trace of $A = [a_{ij}]_{n\times n} \in M_{n\times n}(\mathbb{F})$ as

   $$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

Find a basis for matrices in $M_{3\times 3}(\mathbb{R})$ with trace zero.

Practice Problems : Sec 1.4 : 8-16 , Sec 1.5 2-7, 9, 14, 15, 17, 18, 20