MATH 241 PSET 6

October 10, 2024

1.

a) The PMF of r.v. X given by $P(X = k)$ can be divided into two cases: (1) $k = 0$ and (2) $k > 0$. In the case that $k = 0$, $P(X = k)$ is given by the probability that either of the two following events occurred: (1) that there was either a structural zero (i.e. the coin landed heads with probability p) or (2) the coin landed tails and the Poisson r.v. turned out to be zero anyways. For event (2), the subevent that the coins turned out to be tails (given by probability $1 - p$) and the subevent the Poisson r.v. turned out to be zero (given by probability $\frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$) are both independent and so the probability both occur is given by the product of their probabilities, $(1-p)e^{-\lambda}$. In short, the probability event (2) occurs is given by $(1-p)e^{-\lambda}$. Because events (1) and (2) are mutually exclusive, the probability either occurs is given by the sum of their probabilities: $P(X = 0) = p + (1 - p)e^{-}\lambda$.

For the case $k > 0$, $P(X = k)$ is given by the probability that two independent events both occurred: (1) that the coin landed tails (given by probability $1 - p$) and (2) the Poisson r.v. materialized as k (given by probability $\frac{e^{-\lambda}\lambda^k}{k!}$). Because these two events are independent, the probability both occur is given by the product of their probabilities: $P(X = k) = (1 - p)\frac{e^{-\lambda}\lambda^k}{k!}$ for $k > 0$.

Thus, we get that $P(X = k)$ is given by:

$$P(X = k) = \begin{cases} p + (1 - p)e^{-\lambda} & k = 0 \\ (1 - p)\frac{e^{-\lambda}\lambda^k}{k!} & k > 0 \end{cases}$$

2.

a) We compute $P(1 < X < 3) = \int_1^3 f(x)dx$ below:

$$P(1 < X < 3) = \int_1^3 f(x)dx = \int_1^3 xe^{-\frac{x^2}{2}}dx$$
$$= [-e^{-\frac{x^2}{2}}]_1^3 = e^{-0.5} - e^{-4.5}$$

b) The CDF of X is given by $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} te^{-\frac{t^2}{2}}dt = [-e^{-\frac{t^2}{2}}]_{-\infty}^{x} = 1 - e^{-\frac{x^2}{2}}$. Because $P(X \leq q_j) = F(q_j)$ by definition of CDF, we can compute the quartiles asked by finding the function $q_j(j)$. We find this function through manipulating the equation $F(q_j) = 1 - e^{-\frac{q_j^2}{2}} = \frac{j}{4}$. We solve the equation below:

$$
\begin{aligned}
P(X \leq q_j) = F(q_j) &= \frac{j}{4} \\
1 - e^{-\frac{q_j^2}{2}} &= \frac{j}{4} \\
e^{-\frac{q_j^2}{2}} &= 1 - \frac{j}{4} \\
-\frac{q_j^2}{2} &= ln(1 - \frac{j}{4}) \\
q_j &= \sqrt{-2ln(1 - \frac{j}{4})}
\end{aligned}
$$

For $j = 1, 2, 3$, we get:

$$
\begin{aligned}
q_1 &= \sqrt{-2ln(\frac{3}{4})} \\
q_2 &= \sqrt{-2ln(\frac{1}{2})} \\
q_3 &= \sqrt{-2ln(\frac{1}{4})}
\end{aligned}
$$

3.

a) Given r.v. R, we are asked to compute the mean and variance of r.v. $A = \pi R^2$. We denote $f(x) = \frac{1}{1} = 1$ as the PDF for $R \sim Unif(1, 0)$. We compute mean $\mathbb{E}[A]$ below.

$$
\begin{aligned}
\mathbb{E}[A] = \mathbb{E}[\pi R^2] = \pi \mathbb{E}[R^2] = \\
= \pi \int_{-\infty}^{\infty} x^2 f(x)dx = \pi \int_{0}^{1} x^2 f(x)dx = \pi \int_{0}^{1} x^2 dx = \pi[\frac{x^3}{3}]_0^1 = \frac{\pi}{3}
\end{aligned}
$$

We compute $Var(A)$ below. Note that above we have computed $\mathbb{E}[R^2] = \frac{1}{3}$.

$$Var(A) = Var(\pi R^2) = \pi^2 Var(R^2) =$$
$$\pi^2[\mathbb{E}[R^4] - \mathbb{E}[R^2]^2] = \pi^2[\int_0^1 x^4 f(x)dx - \frac{1}{9}] = \pi^2[\int_0^1 x^4 dx - \frac{1}{9}] = \pi^2[[\frac{x^5}{5}]_0^1 - \frac{1}{9}] =$$
$$\pi^2[\frac{1}{5} - \frac{1}{9}] = \frac{4\pi^2}{45}$$

b) We denote the CDF and PDF of A as $F_A(x)$ and $f_A(x)$, respectively. Note that for a circle with given area $a \in [0, \pi]$, the radius for this circle can be computed as $\sqrt{\frac{a}{\pi}}$. A circle with an area outside of these bounds is impossible as the radius R is constrained from 0 to 1.

$$F_A(x) = P(A \leq x) = \begin{cases} 0 & x < 0 \\ P(R \leq \sqrt{\frac{x}{\pi}}) & 0 \leq x \leq \pi \\ 1 & x > \pi \end{cases}$$

$$F_A(x) = \begin{cases} 0 & x < 0 \\ \frac{\sqrt{\frac{x}{\pi}} - 0}{1 - 0} & 0 \leq x \leq \pi \\ 1 & x > \pi \end{cases} \Rightarrow F_A(x) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{x}{\pi}} & 0 \leq x \leq \pi \\ 1 & x > \pi \end{cases}$$

Because PDF $f_A(x)$ is given as the derivative of $F_A(x)$, we get:

$$f_a(x) = F_A^{'}(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2\sqrt{\pi x}} & 0 \leq x \leq \pi \\ 0 & x > \pi \end{cases}$$

4.

a) We denote the CDF of X as $F(x)$. We compute $F(x)$ below, given $0 < x < 1$.

$$F(x) = \int_{-\infty}^x f(t)dt = \int_{-\infty}^0 0dt + \int_0^x f(t)dt = \int_0^x 12t^2(1-t)dt = \int_0^x 12t^2 - 12t^3 dt =$$
$$12[\frac{t^3}{3} - \frac{t^4}{4}]_0^x = 12(\frac{x^3}{3} - \frac{x^4}{4}) = 4x^3 - 3x^4$$

b) $P(0 < X < \frac{1}{2}) = F(\frac{1}{2}) - F(0) = \frac{4}{8} - \frac{3}{16} = \frac{5}{16}$.

c) We compute $\mathbb{E}[X]$ below. Note we can ignore all intervals outside of $0 < x < 1$ as we assume $f(x) = 0$ for $x \notin [0, 1]$.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 (12x^3 - 12x^4)dx = [3x^4 - \frac{12x^5}{5}]_0^1 = 3 - \frac{12}{5} = \frac{3}{5}$$

We now compute $Var(X)$.

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \int_0^1 x^2 f(x)dx - \mathbb{E}[X]^2 = \int_0^1 12x^4(1-x)dx - \mathbb{E}[X]^2 =$$
$$\int_0^1 12x^4 - 12x^5 dx - \mathbb{E}[X]^2 = 12[\frac{x^5}{5} - \frac{x^6}{6}]_0^1 - \mathbb{E}[X]^2 = \frac{12}{5} - 2 - (\frac{3}{5})^2 = 0.4 - 0.36 = 0.04$$

5.

For values $0 \leq x \leq 1$, we denote the CDF of X as $F(x)$. By the definition of a CDF, $F(x) = P(X \leq x)$, which represents the probability the max of $U_1, \ldots, U_n$ is equal to x. This is the same as the probability that $U_1, \ldots, U_n$ are all $\leq x$. Because $U_1, \ldots, U_n$ are independent random variables described by the distribution $Unif(0, 1)$, the probability all $U_1, \ldots, U_n$ are $\leq x$ (i.e. $P(X \leq x)$) is given by $\prod_{i=1}^n P(U_i \leq x) = \prod_{i=1}^n \frac{x-0}{1-0} = \prod_{i=1}^n x = x^n$. Thus, we get that the CDF for X is given by $F(x) = x^n$. Because the PDF $f(x)$ for X is given by the derivative of the CDF $F(x)$, we get the PDF $\mathbf{f(x) = F'(x) = nx^{n-1}}$ **for** $\mathbf{0 \leq x \leq 1}$.

We compute $\mathbb{E}[X]$ below. Once again, we ignore all values outside $0 \leq x \leq 1$ as they are impossible (a maximum of values from $[0, 1]$ cannot be outside the interval $[0, 1]$.)

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 nx^n dx = n[\frac{x^{n+1}}{n+1}]_0^1 = \frac{n}{n+1}$$

6. Anish Lakkapragada. I worked independently.