Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head here.

STATS 242 HW 2

January 29, 2025

Number of late days: 0; Collaborators: None.

1.

For $1 \leq i \leq n$, let us define r.v. $B_i \sim Bern(p)$, where for $i \neq j$, B_i and B_j are independent. Then, we have that $X = \Sigma_{i=1}^{n} B_i$. Because all B_is are independent, we have that:

$$M_X(t) = M_{\Sigma_{i=1}^{n} B_i}(t) = \prod_{i=1}^{n} M_{B_i}(t)$$

The MGF of r.v. B_i can be computed as: $M_{B_i}(t) = \mathbb{E}[e^{tB_i}] = e^{t(1)}P(B_i = 1) + e^{t(0)}P(B_i = 0) = pe^t + (1-p) = 1 + p(e^t - 1)$. Thus, we have $M_X(t)$ as:

$$M_X(t) = \prod_{i=1}^{n} M_{B_i}(t) = \prod_{i=1}^{n} 1 + p(e^t - 1) = [1 + p(e^t - 1)]^n$$

2.

We first start by computing the distributions of X_1 and X_2. Note that because Z_1 and Z_2 are independent normal distributions, their sum forms a normal distribution as well.

1. **Distribution of X_1**

   Since $c_1 Z_1 \sim \mathcal{N}(0, c_1^2)$ and $d_1 Z_1 \sim \mathcal{N}(0, d_1^2)$, $c_1 Z_1 + d_1 Z_2 \sim \mathcal{N}(0, c_1^2 + d_1^2)$. Finally, e_1 is just a constant and so it doesn't affect the variance so $X_1 = c_1 Z_1 + d_1 Z_2 + e_1 \sim \mathcal{N}(e_1, c_1^2 + d_1^2)$. Therefore we can set $e_1 = \mu_1$, the mean of X_1. Furthermore, the variance σ_1^2 of X_1 is equal to $c_1^2 + d_1^2$.

2. **Distribution of X_2**

   We use identical reasoning as with before. $c_2 Z_1 \sim \mathcal{N}(0, c_2^2)$ and $d_2 Z_2 \sim \mathcal{N}(0, d_2^2)$, so we have $c_2 Z_2 + d_2 Z_2 \sim \mathcal{N}(0, c_2^2 + d_2^2)$. Thus, $X_2 = c_2 Z_2 + d_2 Z_2 + e_2 \sim \mathcal{N}(e_2, c_2^2 + d_2^2)$. Therefore we can set $e_2 = \mu_2$, the mean of X_2. Furthermore, the variance σ_2^2 of X_2 is equal to $c_2^2 + d_2^2$.

We now have c_1, c_2, d_1, d_2 remaining to assign. We compute the correlation ρ between X_1 and X_2, starting by computing the covariance between X_1 and X_2:

$$\text{Cov}(X_1, X_2) = \text{Cov}(c_1 Z_1 + d_1 Z_2 + e_1, c_2 Z_1 + d_2 Z_2 + e_2)$$
$$= c_1 \text{Cov}(Z_1, c_2 Z_1 + d_2 Z_2 + e_2) + d_1 \text{Cov}(Z_2, c_2 Z_1 + d_2 Z_2, e_2) + \text{Cov}(e_1, c_2 Z_1 + d_2 Z_2 + e_2)$$

Note that because e_1 is a fixed constant, $\text{Cov}(e_1, c_2 Z_1 + d_2 Z_2 + e_2) = 0$. Also note that because Z_1 and Z_2 are independent, $\text{Cov}(Z_1, Z_2) = 0$.

$$\text{Cov}(X_1, X_2) = c_1 [c_2 \text{Cov}(Z_1, Z_1) + d_2 \text{Cov}(Z_1, Z_2) + \text{Cov}(Z_1, e_2)] + d_1 \text{Cov}(Z_2, c_2 Z_1 + d_2 Z_2, e_2)$$
$$= c_1 [c_2 + 0 + 0] + d_1 [c_2 \text{Cov}(Z_2, Z_1) + d_2 \text{Cov}(Z_2, Z_2) + \text{Cov}(Z_2, e_2)] = c_1 c_2 + d_1 [0 + d_2 + 0]$$
$$= c_1 c_2 + d_1 d_2$$

and now we compute the correlation ρ between X_1 and X_2 as:

$$\rho = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} = \frac{c_1 c_2 + d_1 d_2}{\sigma_1 \sigma_2}$$

This gives us three equations:

$$\rho \sigma_1 \sigma_2 = c_1 c_2 + d_1 d_2$$
$$c_1^2 + d_1^2 = \sigma_1^2$$
$$c_2^2 + d_2^2 = \sigma_2^2$$

to solve for four variables. As such, we arbitrarily choose to set $c_1 = 0$, giving us $d_1 = \sigma_1$ and $\rho \sigma_1 \sigma_2 = d_1 d_2 = \sigma_1 d_2 \implies d_2 = \rho \sigma_2$. Finally, we can solve for $c_2^2 = \sigma_2^2 - d_2^2 = \sigma_2^2 - \rho^2 \sigma_2^2 = \sigma_2^2(1 - \rho^2) \implies c_2 = \sigma_2\sqrt{1 - \rho^2}$.

As a summary of my answer,

$$e_1 = \mu_1$$
$$e_2 = \mu_2$$
$$c_1 = 0$$
$$c_2 = \sigma_2\sqrt{1 - \rho^2}$$
$$d_1 = \sigma_1$$
$$d_2 = \rho \sigma_2$$

2

3.

(a) We first show $\mathbb{E}[\hat{I}_n(f)] = I(f)$. Note that $\frac{f(X_i)}{g(X_i)}$ is a random variable and so its expectation is given to us by $\mathbb{E}[\frac{f(X_i)}{g(X_i)}] = \int_{-\infty}^{\infty} \frac{f(u)}{g(u)} g(u) du = \int_a^b \frac{f(u)}{g(u)} g(u) du = \int_a^b f(u) du = I(f)$. Thus:

$$\mathbb{E}[\hat{I}_n(f)] = \mathbb{E}[\frac{1}{n}\Sigma_{i=1}^n \frac{f(X_i)}{g(X_i)}] = \frac{1}{n} \cdot n \cdot \mathbb{E}[\frac{f(X_i)}{g(X_i)}] = I(f)$$

Notice that $\hat{I}_n(f)$ is essentially an average of n random variables, each of with an expectation of $I(f)$. Thus, $\hat{I}_n(f) \to I(f)$ in probability as $n \to \infty$ due to the (Weak) Law of Large Numbers.

b) We first compute $\mathrm{Var}[\hat{I}_n(f)]$. We begin by computing the variance of random variable $\frac{f(X_i)}{g(X_i)}$:

$$\mathrm{Var}[\frac{f(X_i)}{g(X_i)}] = \mathbb{E}[(\frac{f(X_i)}{g(X_i)})^2] - \mathbb{E}[\frac{f(X_i)}{g(X_i)}]^2 = \int_a^b \frac{f^2(u)}{g^2(u)} g(u) du - I^2(f) = \int_a^b \frac{f^2(u)}{g(u)} du - I^2(f)$$

Let us call define this quantity to be $\sigma^2 = \int_a^b \frac{f^2(u)}{g(u)} g(u) du - I^2(f) \in \mathbb{R}$.
and so we have:

$$\mathrm{Var}[\hat{I}_n(f)] = \mathrm{Var}[\frac{1}{n}\Sigma_{i=1}^n \frac{f(X_i)}{g(X_i)}] = \frac{1}{n^2}\Sigma_{i=1}^n \mathrm{Var}[\frac{f(X_i)}{g(X_i)}] = \frac{1}{n}\sigma^2$$

Let us define $c_n = \frac{\sqrt{n}}{\sigma} \in \mathbb{R}$. Thus, by the Central Limit Theorem, we have that:

$$c_n(\hat{I}_n(f) - I(f)) \to \mathcal{N}(0, 1) \text{ as } n \to \infty$$

c) In this problem, $a = 0$ and $b = 1$, $f(x) = \cos(2\pi x)$, and $g(x) = 1$ if $x \in [0, 1]$ and 0 otherwise. We first compute $I(f)$:

$$I(f) = \int_0^1 \cos(2\pi x) dx = \sin(2\pi x) \Big|_0^1 = \sin(2\pi) - \sin(0) = 0 - 0 = 0$$

and then σ^2:
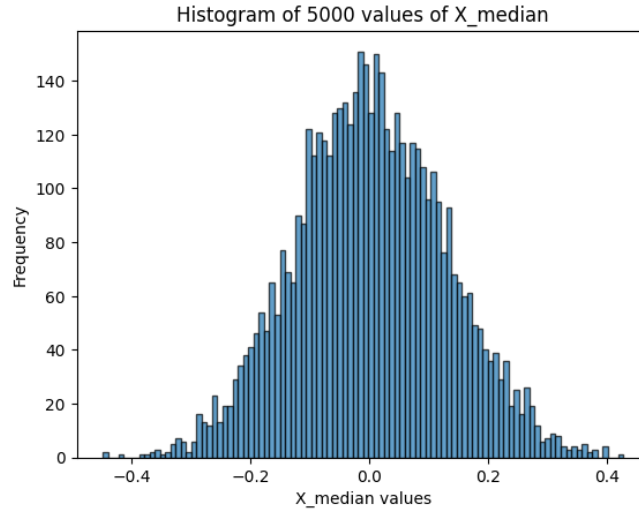
$$\sigma^2 = \int_0^1 \frac{\cos^2(2\pi x)}{1} dx - I^2(f) dx = \int_0^1 [\frac{1}{2} + \frac{\cos(4\pi x)}{2}] dx = [\frac{x}{2} + \frac{\sin(4\pi x)}{8}] \Big|_0^1$$
$$= \frac{1}{2} + \frac{1}{8}(\sin(4\pi) - \sin(0)) = \frac{1}{2}$$

and so c_n is given by:

$$c_n = \frac{\sqrt{n}}{\sigma} = \sqrt{\frac{n}{\sigma^2}} = \sqrt{2n}$$

4.

From my simulation, the mean and standard deviation of X_{median} are given by 0.00178 and 0.1265 respectively. Below is the histogram of the 5000 values of X_{median} from my simulation:

Histogram of 5000 values of X_median

Based on the above histogram, we can see that the sampling distribution of X_{median} follows a normal distribution. We now derive the standard deviation of the sample mean $\bar{X}$. Note that 99 observations is enough for us to apply the Central Limit Theorem with confidence. Thus[1], $\bar{X} = \frac{X_1 + \cdots + X_{99}}{99} \sim \mathcal{N}(\mathbb{E}[X_1], \frac{\text{Var}(X_1)}{99})$ or $\bar{X} \sim \mathcal{N}(0, \frac{1}{99})$. Thus the analytically-computed standard deviation of sample mean $\bar{X}$ is $\sqrt{\frac{1}{99}}$ or 0.1005, which is less than my calculated simulated standard deviation for X_{median} (0.1265). According to my simulation, X_{median} is more variable than $\bar{X}$.

```
1   # %%
2
3   # Run all imports first and then write helper functions.
4
5   import numpy as np
6   import math
7   import matplotlib.pyplot as plt
8
9   def get_N_obs_iid_standard_normal(N):
10      return np.random.normal(0, 1, N)
11
12  def compute_median(arr):
13      # assume arr is numpy
14      return np.median(arr)
```

---

[1]Note that $X_1 \ldots X_{99}$ are identical distributions and so they all have the same mean and variances.

```python
15  # %%
16  N_SIMULATIONS = 5000
17  X_medians = []
18  for _ in range(N_SIMULATIONS):
19      samples = get_N_obs_iid_standard_normal(99)
20      X_median_curr = compute_median(samples)
21      X_medians.append(X_median_curr)
22
23  X_medians = np.array(X_medians)
24  X_median_mean = np.mean(X_medians)
25  X_median_std = np.std(X_medians)
26
27  print(f"X_median mean: {X_median_mean} and std: {X_median_std}")
28  # %%
29
30  """
31  Plot a histogram.
32  """
33
34  plt.hist(X_medians, bins=100, edgecolor='black', alpha=0.7)
35
36  # Add labels and title
37  plt.xlabel('X_median values')
38  plt.ylabel('Frequency')
39  plt.title('Histogram of 5000 values of X_median')
40
41  # %%
```