Math 226- HW 8 Due: Nov 5 by Midnight

1. (20) Let $T : V \to W$ and $U : W \to Z$ be linear transformations on finite-dimensional vector spaces V, W, Z. Moreover, let A, B be matrices such that AB is defined.

   a) (6 points) Prove that $\dim(R(UT)) \leq \dim(R(U))$

   b) (4 points) Use pat $a)$ to conclude that $\text{rank}(AB) \leq \text{rank}(A)$

   c) (6 points) Prove that $\dim(R(UT)) \leq \dim(R(T))$

   d) (4 points) Use pat $c)$ to conclude that $\text{rank}(AB) \leq \text{rank}(B)$

2. (12 points) Suppose that the augmented matrix of a system $Ax = b$ is transformed into a matrix $[A'|b']$ in reduced row echolon form by finite sequence of elementary row operations.

   a) (10 points) Prove that $\text{rank}(A) \neq \text{rank}[A'|b']$ if and only if $[A'|b']$ contains a row in which the only nonzero entry lies in the last column.

   b) (2 points) Deduce that $Ax = b$ s consistent if and only if $[A'|b']$ contains no row in which the only nonzero entry lies in the last column.

3. (12 points) Each of the following equations determines a plane in $\mathbb{R}^3$.

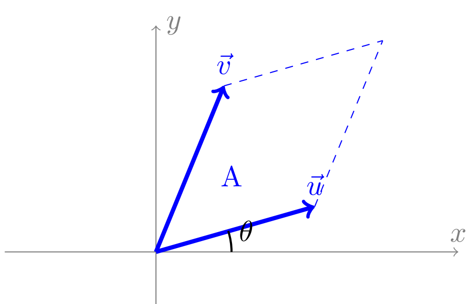
$$x + 4y + 5z = 1$$
$$2x + 2y - 3z = 4$$

   a) Find the intersection of these two planes, and draw a rough graph of the solution set.

   b) Find the intersection when the equations are both homogenous, and draw rough graph of the solution set.

   c) What is the relationship between these two set?

4. (10 points) In this problem you will prove that if u, v are two vectors in $\mathbb{R}^2$, then the area of the parallelogram generated by u, v is equal to $|\det(u, v)|$. Let u and v are vectors as in picture I . Let $A_{-\theta} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ be the clockwise rotation by θ degree. One has $A_{-\theta}(v) = v_\theta$, and $A_{-\theta}(u) = u_\theta$, where v_θ and u_θ are as in the picture II



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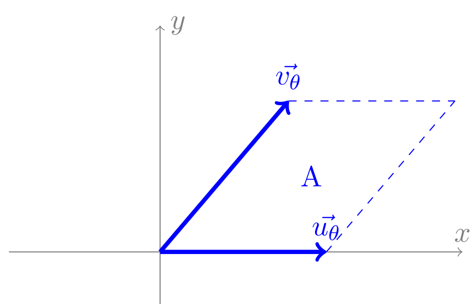

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   a) (4 points) Let $u_\theta = (u_1, u_2)$, and $v_\theta = (v_1, v_2)$ in picture II. Calculate the area, A, of the parallelogram generated by u_θ and v_θ using geometry, and show by direct calculation $A = |det[u_\theta, v_\theta]|$. - Think here u_θ, v_θ, as the column vectors of the matrix $[u_\theta, v_\theta]$.

   b) (2 points) Show that $A_{-\theta}(v) = v_\theta$, $A_{-\theta}(u) = u_\theta$ mean that $A_{-\theta}[u, v] = [u_\theta, v_\theta]$

   c) (4 points) Use part $b)$, and Theorem 4.7 to show that $|det[u, v]| = A$.

Remark : This is the geometric interpretation of the determinant. In general, if $A \in M_{n\times n}(\mathbb{R})$, and if the columns of A is $(a_1, a_2, ..., a_n)$, then $\det(A)$ is the n-dimensional volume of the parallelepiped having the vectors $a_1, a_2, .., a_3$ as adjacent sides. You can also think $\det(A)$ as the volume of the parallelepiped having the vectors $A(e_1), A(e_2), ..., A(e_n)$ as adjacent sides.

5. (15 points) Recall that we defined $\delta : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ to be $n-$ linear functional if it is linear with respect to each row. Prove that $\delta : M_{2\times 2}(\mathbb{F}) \to \mathbb{F}$ is a $2-$ linear functional if and only if it has the form

$$\delta(A) = Aa_{11}a_{22} + Ba_{11}a_{21} + Ca_{12}a_{22} + Da_{12}a_{21}$$

for any $[a_{ij}]_{2\times 2}$ matrix and $A, B, C, D \in \mathbb{F}$.