Discretionary Note

Anish Krishna Lakkapragada

IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH. Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

CONTENT STARTS ON NEXT PAGE.

To access the general instructions for this repository head `here`.

Problem set 2
Due Thursday January 30 at 11pm

Exercise 2.1 (5 points; Rudin 1.5). Let $A \subset \mathbb{R}$ be nonempty and bounded below. Define
$$-A = \{-x \mid x \in A\}.$$
Prove that $\inf A = -\sup(-A)$.

Exercise 2.2 (5 points). Let $A \subset \{x \in \mathbb{R} \mid x > 0\}$ be nonempty and bounded above. Define
$$A^{-1} = \{x^{-1} \mid x \in A\}.$$
Prove that $\inf(A^{-1}) = (\sup A)^{-1}$.

Exercise 2.3 (5 points). Suppose $A, B \subset \mathbb{R}$ are both nonempty and bounded above. Define
$$A + B = \{a + b \mid a \in A, b \in B\}.$$
Prove that $\sup(A + B) = \sup A + \sup B$.

Exercise 2.4 (15 points). In each of the following, S is an ordered set, and $A \subset S$. Answer the following in each case, and prove your answers:

- Is A bounded above?
- Does A have a maximum element, and if so, what is it?
- Does A have a supremum in S, and if so, what is it?

(1) $S = \mathbb{Z}$, $A = \{2, 3\}$.

(2) $S = \mathbb{Q}$, $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\}$.

(3) $S = \mathbb{Q}$, $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$.

(4) $S = \mathbb{Q}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

(5) $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$.

(6) $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.

(7) $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x^3 < 2\}$.

Exercise 2.5 (10 points). Suppose F is an ordered field and $x, y, z \in F$.

(1) Prove that $0 < 1$.

(2) Prove that if $x > 0$, then $x^{-1} > 0$.

(3) Prove that if $x > 0$, then $y > z$ if and only if $xy > xz$.

(4) Recall we defined the field $\mathbb{F}_3 = \{0, 1, 2\}$ on class. Prove that there does not exist an order on $\mathbb{F}_3$ such that it is an ordered field.

Exercise 2.6 (optional, not for credit; Rudin 1.8 and 1.9). Define the complex numbers as the set
$$\mathbb{C} = \{a + b\mathrm{i} \mid a, b \in \mathbb{R}\}$$
with the addition and multiplication rules given by

- $(a + b\mathrm{i}) + (a' + b'\mathrm{i}) = (a + a') + (b + b')\mathrm{i}$,
- $(a + b\mathrm{i})(a' + b'\mathrm{i}) = (aa' - bb') + (ab' + a'b)\mathrm{i}$.

(1) Prove that $\mathbb{C}$ with these operations (and an appropriate 0, 1, negation, inversion) is a field.

(2) Prove that $\mathbb{C}$ with these operations cannot be made into an ordered field.

(3) Define the *lexicographic order* on $\mathbb{C}$ by defining

$$a + b\mathrm{i} < c + d\mathrm{i} \quad \Longleftrightarrow \quad a < c \text{ or } (a = c \text{ and } b < d).$$

Prove that this makes $\mathbb{C}$ into an ordered *set* (though not an ordered *field*).

(4) Does the lexicographic order on $\mathbb{C}$ have the least upper bound property?