PSETs Landing Page*

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This is the documentation for using my PSET PDFs responsibly. I post these LaTEX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

```
https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf
```

where `TYPE` is `stats` or `math`. Similarily, to access my solution for this PSET you can go to:

```
https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf
```

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email `anish.lakkapragada@yale.edu`.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

Problem set 1
Due Thursday January 23 at 11pm

Exercise 1.1 (5 points). Suppose A, B, C are sets and $f : A \to B$, $g : B \to C$ are functions. Let $h = g \circ f : A \to C$. Show that

(1) If h is injective, then f is injective.

(2) If h is surjective, then g is surjective.

Exercise 1.2 (5 points; Rudin 1.1). This exercise uses the real numbers $\mathbb{R}$, which we have not defined yet; but, for this exercise, all you need to know about $\mathbb{R}$ is that it is a field containing the field $\mathbb{Q}$ of rational numbers.

Suppose $r \in \mathbb{Q}$, $r \neq 0$, and $x \in \mathbb{R}$, $x \notin \mathbb{Q}$. Prove that $r + x \notin \mathbb{Q}$ and $rx \notin \mathbb{Q}$.

Exercise 1.3 (10 points; Rudin 1.3). Suppose F is a field with $x, y, z \in F$. Prove carefully from the field axioms:

(1) If $x \neq 0$ and $xy = xz$, then $y = z$.

(2) If $x \neq 0$ and $xy = x$, then $y = 1$.

(3) If $xy = 1$, then $x \neq 0$ and $y = x^{-1}$.

(4) If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Exercise 1.4 (10 points). This exercise concerns the field $\mathbb{Q}(\sqrt{3})$.

(1) Prove that $\mathbb{Q}$ does not contain any x with $x^2 = 3$.

(2) Consider the set $\mathbb{Q}(\sqrt{3})$, defined as the set of expressions "$a + b\sqrt{3}$" with $a, b \in \mathbb{Q}$. (More formally, we could think of $\mathbb{Q}(\sqrt{3})$ as the set of ordered pairs (a, b) with $a, b \in \mathbb{Q}$.) We can equip $\mathbb{Q}(\sqrt{3})$ with addition and product laws by

$$(a + b\sqrt{3}) + (a' + b'\sqrt{3}) = (a + a') + (b + b')\sqrt{3}\,,$$
$$(a + b\sqrt{3})(a' + b'\sqrt{3}) = (aa' + 3bb') + (ab' + ba')\sqrt{3}\,.$$

Show that $\mathbb{Q}(\sqrt{3})$ can be made into a field, with these addition and product laws. (This means saying carefully what are the negation law, inversion law, 0 and 1 elements, and then checking that all the axioms of a field are satisfied.)

(3) We could similarly consider the set $\mathbb{Z}(\sqrt{3})$, defined as above except that now we require $a, b \in \mathbb{Z}$, with the same addition and product laws. Show that $\mathbb{Z}(\sqrt{3})$ cannot be made into a field with these addition and product laws.

Exercise 1.5 (5 points). As we have stated, the usual order relation on $\mathbb{Q}$ can be expressed as:

$$p/q < p'/q' \quad \Longleftrightarrow \quad pq' < p'q$$

when q and q' are positive.

We can define a different relation $\prec$, as follows: assume that all fractions are written in lowest terms and with positive denominators. Define

$$p/q \prec p'/q' \quad \Longleftrightarrow \quad pq < p'q'\,.$$

Prove that the relation $\prec$ does *not* make $\mathbb{Q}$ into an ordered set.

Exercise 1.6 (5 points). Let A be a nonempty subset of an ordered set S. If there exists an element $\alpha \in S$ such that $\alpha \leq x$ for all $x \in A$, then we call α a *lower bound* for A. Similarly, if there exists $\beta \in S$ such that $x \leq \beta$ for all $x \in A$, we call β an *upper bound* for A. Suppose α is a lower bound of A and β is an upper bound of A. Prove that $\alpha \leq \beta$.