

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# MATH 241 PSET 10

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1.

Let us first define the PDF of r.v.  $X$  as  $f_X(x) = e^{-x}$  and r.v.  $Y = g(X)$  where  $g(x) = e^{-x}$ . Because  $g$  is differentiable and strictly decreasing, we can compute the PDF of  $Y$  as:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

where  $x = g^{-1}(y)$ , or  $x = -\ln(y)$  for  $0 < y \leq 1$ . Thus, we have that:

$$f_Y(y) = f_X(-\ln(y)) \left| \frac{1}{\frac{dy}{dx}} \right|$$

$$f_Y(y) = f_X(-\ln(y)) \left| \frac{1}{-e^{-x}} \right|$$

$$f_Y(y) = f_X(-\ln(y)) | -e^{-\ln(y)} |$$

$$f_Y(y) = y \frac{1}{y} = 1$$

Thus, we have that the PDF of  $e^{-X}$  can be given by 1 for  $0 < y \leq 1$ .

2.

We compute the joint PDF  $f_{T,W}(t, w)$  for random variables  $T$  and  $W$ . To do so, we first compute the absolute value of the Jacobian matrix  $\frac{\partial(t,w)}{\partial(x,y)}$ , which is given by:

$$\frac{\partial(t,w)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Thus, we get that  $|\frac{\partial(t,w)}{\partial(x,y)}| = |1(-1) - 1(1)| = |-2| = 2$ . From this, using the Change of Variables Theorem, we can compute  $f_{T,W}(t, w)$  as:

$$f_{T,W}(t, w) = f_{X,Y}(x, y) \left| \frac{\partial(x, y)}{\partial(t, w)} \right| = f_{X,Y}(x, y) \left| \frac{\partial(t, w)}{\partial(x, y)} \right|^{-1} = \frac{f_{X,Y}(x, y)}{2}$$

where  $f_{X,Y}(x, y)$  is the joint PDF of random variables  $X$  and  $Y$ . Note that because  $X$  and  $Y$  are independent,  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ , where  $f_X(x)$  and  $f_Y(y)$  are the PDFs for  $X$  and  $Y$ , respectively. Thus, we have that:

$$f_{T,W}(t, w) = \frac{f_X(x)f_Y(y)}{2}$$

Note that because  $T = X + Y$  and  $W = X - Y$ , we can express  $x$  in the above equation as  $\frac{t+w}{2}$  and  $y$  as  $\frac{t-w}{2}$ . Thus, we have:

$$\begin{aligned} f_{T,W}(t, w) &= \frac{f_X(\frac{t+w}{2})f_Y(\frac{t-w}{2})}{2} = \frac{1}{2} \frac{e^{-\frac{1}{2}(\frac{t+w}{2})^2}}{\sqrt{(2\pi)}} \frac{e^{-\frac{1}{2}(\frac{t-w}{2})^2}}{\sqrt{2\pi}} = \frac{1}{4\pi} e^{-\frac{1}{8}[-(t+w)^2 - (t-w)^2]} \\ &= \frac{1}{4\pi} e^{-2(t^2+w^2)} = \frac{1}{4\pi} e^{-2t^2} e^{-2w^2} \end{aligned}$$

Thus, because we can factor the joint PDF  $f_{T,W}$  into a function of  $t$  times a function of  $w$ , we can conclude that random variables  $T$  and  $W$  are independent.

3.

Let us define r.v.  $T = U + X$ . Let us define the PDFs of random variables  $U$  and  $X$  as  $f_U(u) = \frac{1}{1-0} = 1$  for  $0 \leq u \leq 1$  and  $f_X(x) = e^{-x}$  for  $x \geq 0$ . We compute the PDF  $f_T(t)$  of r.v.  $T$  below:

$$f_T(t) = \int_{-\infty}^{\infty} f_U(t-x)f_X(x)dx$$

Because  $f_U(u) = 0$  for  $u \notin [0, 1]$  and  $f_X(x) = 0$  for  $x < 0$ , we must restrict this integral to  $0 \leq t-x \leq 1 \Rightarrow x \leq t \leq 1+x$  and  $x \geq 0$ . Upon inspection, we can see that the bounds for  $x$  vary according to the value of  $t$ : when  $0 \leq t \leq 1$ ,  $x$  is constrained to  $(0, t)$  and when  $t > 1$ ,  $x$  is constrained to  $(t-1, t)$ . Thus, the PDF of  $T$  can be given as a piecewise function:

$$f_T(t) = \begin{cases} \int_0^t f_U(t-x)f_X(x)dx = \int_0^t e^{-x}dx = -[e^{-t} - 1] = 1 - e^{-t} & \text{for } 0 \leq t \leq 1 \\ \int_{t-1}^t f_U(t-x)f_X(x)dx = \int_{t-1}^t e^{-x}dx = -[e^{-t} - e^{1-t}] = -(e^{-t} - ee^{-t}) = (e-1)e^{-t} & \text{for } t > 1 \end{cases}$$

4.

Let us denote the number of ticket sold for movie  $i$  of the year as  $T_i \sim Pois(\lambda_2)$ . Thus, the number of movie tickets sold next year can be given as:  $T = \sum_{i=1}^N T_i$ . We compute  $\mathbb{E}[T]$  below:

$$\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T|N]] = \mathbb{E}[\mathbb{E}[\sum_{i=1}^N T_i|N]] = \mathbb{E}[\sum_{i=1}^N \mathbb{E}[T_i|N]]$$

Because the number of tickets sold for a given movie,  $T_i$ , is independent of  $N$ ,  $\mathbb{E}[T_i|N] = \mathbb{E}[T_i]$  and  $Var(T_i|N) = Var(T_i)$ . Thus, we get that:

$$\mathbb{E}[T] = \mathbb{E}[\sum_{i=1}^N \mathbb{E}[T_i]] = \mathbb{E}[N\lambda_2] = \lambda_2 \mathbb{E}[N] = \lambda_1 \lambda_2$$

Note that in the above computations, we have computed  $\mathbb{E}[T|N] = N\lambda_2$ . We can compute the  $Var(T)$  as such:

$$Var(T) = \mathbb{E}[Var(T|N)] + Var(\mathbb{E}[T|N])$$

$$Var(T) = \mathbb{E}[Var(\sum_{i=1}^N T_i|N)] + Var(\mathbb{E}[T|N])$$

$$Var(T) = \mathbb{E}[\sum_{i=1}^N Var(T_i|N)] + Var(N\lambda_2)$$

$$Var(T) = \mathbb{E}[\sum_{i=1}^N Var(T_i)] + \lambda_2^2 Var(N)$$

$$Var(T) = \mathbb{E}[N\lambda_2] + \lambda_2^2 Var(N)$$

$$Var(T) = \lambda_2 \mathbb{E}[N] + \lambda_2^2 Var(N)$$

$$Var(T) = \lambda_2 \lambda_1 + \lambda_2^2 \lambda_1 = \lambda_1 \lambda_2 (1 + \lambda_2)$$

5. We compute  $\mathbb{E}[Y]$  through Adam's Law as:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[c] = c$$

We compute  $\mathbb{E}[XY]$  through Adam's Law as:

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\mathbb{E}[Y|X]] = \mathbb{E}[cX] = c\mathbb{E}[X]$$

Using the formula of covariance, we can compute the  $Cov(X, Y)$  as:

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = c\mathbb{E}[X] - c\mathbb{E}[X] = 0$$

Because  $Cov(X, Y) = 0$ ,  $Corr(X, Y) = 0$  and so we can conclude that  $X$  and  $Y$  are uncorrelated.

6. Anish Lakkapragada. I worked independently.