

PSETs Landing Page*

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The general format for accessing the (one-indexed) `N`th assigned PSET PDF of a Yale course with course number `CODE` is:

`https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf`

where `TYPE` is `stats` or `math`. Similarly, to access my solution for this PSET you can go to:

`https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf`

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email anish.lakkapragada@yale.edu.

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

MATH 241 PSET 3

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1.

- a) We can compute the probability Fred completes the project on time, given that he completes his first milestone on time as:

$$P(A_3|A_1) = P(A_3|A_2, A_1)P(A_2|A_1) + P(A_3|A_2^c, A_1)P(A_2^c|A_1)$$

Because A_3 and A_1 are conditionally independent given A_2 and A_2^c , $P(A_3|A_2, A_1) = P(A_3|A_2) = 0.8$ and $P(A_3|A_2^c, A_1) = P(A_3|A_2^c) = 0.3$. We are also know that $P(A_2|A_1) = 0.8$ and $P(A_2^c|A_1) = 1 - P(A_2|A_1) = 0.2$. Thus, $P(A_3|A_1) = 0.8(0.8) + 0.3(0.2) = 0.7$.

We now compute the probability Fred completes the project on time, given that he completes his first milestone late as:

$$P(A_3|A_1^c) = P(A_3|A_2, A_1^c)P(A_2|A_1^c) + P(A_3|A_2^c, A_1^c)P(A_2^c|A_1^c)$$

Given $P(A_3|A_2, A_1^c) = P(A_3|A_2) = 0.8$, $P(A_3|A_2^c, A_1^c) = P(A_3|A_2^c) = 0.3$, $P(A_2|A_1^c) = 0.3$ and $P(A_2^c|A_1^c) = 1 - P(A_2|A_1^c) = 0.7$, we get that $P(A_3|A_1^c) = 0.8(0.3) + 0.3(0.7) = 0.45$.

- b) The probability Fred will finish his project on time is given by $P(A_3)$. Given $P(A_1) = 0.75$ and $P(A_3|A_1) = 0.7$, $P(A_3|A_1^c) = 0.45$ from (a), we can compute $P(A_3)$ as:

$$P(A_3) = P(A_3|A_1)P(A_1) + P(A_3|A_1^c)P(A_1^c)$$

$$P(A_3) = 0.7(0.75) + 0.45(1 - P(A_1))$$

$$P(A_3) = 0.7(0.75) + 0.45(0.25)$$

$$P(A_3) = 0.6375$$

2.

We are given $P(A) = 1$ and we have to prove that for any B where $P(B) > 0$, $P(A|B) = 1$. The proof is below.

$$P(B) = P(B \cap A) + P(B \cap A^c)$$

Note that $P(B \cap A) = P(B|A)P(A) = P(B)$ as event A is guaranteed to happen. Thus:

$$\begin{aligned} P(B) &= P(B) + P(B \cap A^c) \\ P(B \cap A^c) &= 0 \end{aligned}$$

Because $P(A^c|B) = \frac{P(B \cap A^c)}{P(B)}$ and we know $P(B) > 0$, $P(A^c|B) = 0$. Thus, we have proven $P(A|B) = 1 - P(A^c|B) = 1$ given $P(A) = 1$ and for any B where $P(B) > 0$.

3.

b) We compute $P(G|A^c)$ below.

$$\begin{aligned} P(G|A^c) &= \frac{P(A^c|G)P(G)}{P(A^c)} = \frac{(1 - P(A|G))g}{P(A^c|G)P(G) + P(A^c|G^c)P(G^c)} \\ P(G|A^c) &= \frac{(1 - p_1)g}{(1 - P(A|G))g + (1 - P(A|G^c))(1 - g)} = \frac{(1 - p_1)g}{(1 - p_1)g + (1 - p_2)(1 - g)} \end{aligned}$$

c) Note that because A and B are conditionally independent given G or G^c , $P(B|A^c, G) = P(B|G)$ and $P(B|A^c, G^c) = P(B|G^c)$. We compute $P(B|A^c)$ below.

$$\begin{aligned} P(B|A^c) &= P(B|A^c, G)P(G|A^c) + P(B|A^c, G^c)P(G^c|A^c) \\ P(B|A^c) &= P(B|G)P(G|A^c) + P(B|G^c)(1 - P(G|A^c)) \\ P(B|A^c) &= p_1P(G|A^c) + p_2(1 - P(G|A^c)) \\ P(B|A^c) &= p_2 + \frac{(p_1 - p_2)(1 - p_1)g}{(1 - p_1)g + (1 - p_2)(1 - g)} \end{aligned}$$

4.

a) Let us define events A and B as the event that the sample goes to labs A and B, respectively. Let us define C as the event in which the patient has the disease conditionitis. and the events $+$ and $-$ as the events in which the patient tested positive and negative, respectively.

Given these definitions, $P(C) = p$, $P(A) = P(B) = \frac{1}{2}$, $P(+|C, A) = a_1$, $P(-|C^c, A) = a_2$, $P(+|C, B) = b_1$, and $P(-|C^c, B) = b_2$. We compute $P(C|+)$ below.

$$P(C|+) = \frac{P(+|C)P(C)}{P(+)} = p \frac{P(+|C, A)P(A) + P(+|C, B)P(B)}{P(+)} = p \frac{(a_1 + b_1)}{2P(+)}$$

We now compute $P(+)$ below. Note that events (C, A) and (C, B) are entirely independent as the patient having conditionitis has no relation to which lab their sample is tested at.

$$\begin{aligned} P(+) &= P(+|C, A)P(C \cap A) + P(+|C^c, A)P(C^c \cap A) + P(+|C, B)P(C \cap B) + P(+|C^c, B)P(C^c \cap B) \\ &= a_1P(A)P(C) + (1 - a_2)P(C^c)P(A) + b_1P(C)P(B) + (1 - b_2)P(C^c)P(B) \\ &= \frac{p(a_1 + b_1)}{2} + \frac{(1 - a_2)(1 - p)}{2} + \frac{(1 - b_2)(1 - p)}{2} \end{aligned}$$

As such, given $P(+)$, we can compute $P(C|+)$ as:

$$P(C|+) = \frac{p(a_1 + b_1)}{p(a_1 + b_1) + (1 - p)[2 - a_2 - b_2]}$$

b) We compute $P(A|+)$ below:

$$\begin{aligned} P(A|+) &= \frac{P(+|A)P(A)}{P(+)} \\ &= \frac{P(+|A, C)P(C) + P(+|A, C^c)P(C^c)}{2P(+)} \\ &= \frac{pa_1 + (1 - a_2)(1 - p)}{2P(+)} \end{aligned}$$

Using our calculation for $P(+)$ in (a), we get our final answer:

$$P(A|+) = \frac{pa_1 + (1 - a_2)(1 - p)}{p(a_1 + b_1) + (1 - p)[2 - a_2 - b_2]}$$

5.

a) Let us define M as the event that the mother has the disease and C_1, C_2 as the events that the first and second child have the disease, respectively. Given these definitions, $P(M) = \frac{1}{3}$, $P(C_1|M) = P(C_2|M) = \frac{1}{2}$, and $P(C_1|M^c) = P(C_2|M^c) = 0$. We compute the probability neither children has the condition given by $P(C_1^c \cap C_2^c)$ below:

$$P(C_1^c \cap C_2^c) = P(C_1^c \cap C_2^c|M)P(M) + P(C_1^c \cap C_2^c|M^c)P(M^c)$$

Note that C_1 and C_2 are conditionally independent given M . As such,

$$\begin{aligned}
P(C_1^c \cap C_2^c) &= \frac{P(C_1^c|M)P(C_2^c|M)}{3} + P(C_1^c \cap C_2^c|M^c)P(M^c) \\
&= \frac{(1 - P(C_1|M))(1 - P(C_2|M))}{3} + P(C_1^c \cap C_2^c|M^c)P(M^c) \\
&= \frac{1}{12} + P(C_1^c \cap C_2^c|M^c)P(M^c)
\end{aligned}$$

We also know that events C_1, C_2 will not occur given M^c . Thus $P(C_1^c \cap C_2^c|M^c) = 1$:

$$\begin{aligned}
P(C_1^c \cap C_2^c) &= \frac{1}{12} + (1 - P(M)) \\
P(C_1^c \cap C_2^c) &= \frac{1}{12} + \frac{2}{3} \\
P(C_1^c \cap C_2^c) &= \frac{3}{4}
\end{aligned}$$

- c) We compute $P(M|C_2^c \cap C_1^c)$ below, given that $P(C_1^c \cap C_2^c) = \frac{3}{4}$ from (a). Note again that C_1 and C_2 are conditionally independent given M .

$$\begin{aligned}
P(M|C_2^c \cap C_1^c) &= \frac{P(C_2^c \cap C_1^c|M)P(M)}{P(C_1^c \cap C_2^c)} \\
&= \frac{4(1 - P(C_1|M))(1 - P(C_2|M))}{9} = \frac{4}{2 * 2 * 9} \\
P(M|C_2^c \cap C_1^c) &= \frac{1}{9}
\end{aligned}$$

6. Anish Lakapragada. I worked independently.