Math 226- HW 9 Due: November 12 by Midnight

1. (16 points) Let $A \in M_{n\times n}(\mathbb{R})$

   a) (8 points) We call a matrix upper triangular if all the entries below the main diagonal are zero, and lower triangular if all the entries above the main diagonal are zero. Show that if A is upper triangular then $det(A) = \prod_{i=1}^{n} a_{ii}$. Use the fact that $\det(A) = \det(A^T)$ to conclude the same for lower triangular matrices.
   Hint: Use induction with the determinant formula expanded wrt. n-th row.

   b) (8 points) Use elementary row operations to transform the following matrix to a triangle matrix, and use it to show $\det(A) = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix}$$

2. (18 points) Let $\delta : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be a function that is n-linear and alternating.

   - n-linear: if δ is a linear function of each row of an $n \times n$ matrix when the remaining $n-1$ rows are held fixed.
   - alternating: if A has two identical rows, then $\delta(A) = 0$.

   Show that

   a) (5 points) If A is a matrix obtained from B by interchanging any two rows, then $\delta(A) = -\delta(B)$.

   b) (5 points) Let $\mathcal{E}$ be any row operation in pg. 223 of the book. Show that $\delta(\mathcal{E}) = \delta(I)\det(\mathcal{E})$. (Check out Corollary 3 in section 4.5)

   c) (8 points) Use part b), to show that for all $A \in M_{n\times n}(\mathbb{F})$, $\delta(A) = k\det(A)$. (As always consider two cases, if A is not invertible and if A is invertible. In the former case both of the quantities will be zero. In the latter case A is product of elementary matrices.)

3. (14 points) Let A_n be a $n-1 \times n-1$ matrix given with the formula

$$A_n = \begin{pmatrix} 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & 1 & \dots & 1 \\ 1 & 1 & 5 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & n+1 \end{pmatrix}$$

   Show that the set $\{\frac{det(A_n)}{n!}, n \geq 2\}$ is unbounded.

   Hint: Use elementary operations and try to turn A_n to an upper triangle matrix-as much as you can.

4. (22 points) Let $A \in M_{n\times n}(\mathbb{F})$

   a) (6 points) Prove that if $A = \begin{pmatrix} B & C \\ 0 & I_k \end{pmatrix}$ for some $k \leq n$ then $\det(A) = \det(B)$.

   b) (10 points) Use part a) to prove if $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ for some $D \in M_{k\times k}(\mathbb{F})$, $k \leq n$, $\det(A) = \det(B)\det(D)$.

   c) (5 points) A matrix $m \in M_{n\times n}(\mathbb{C})$ is called skew-symmetric if $M^t = -M$. Prove that if M is skew symmetric and n is odd, then M is not invertible.

d) (1 points) What can you say if n is even?
Remark: In fact if M is skew symmetric then $\det(M) \geq 0$. To understand the proof read this very short paper -link only works in the pdf file in canvas not in gradescope

Practice Problems: Sec 4.1 : 1-11, Sec 4.2: 1-30, Sec 4.3: 1-25, Sec 4.4: 1-5