Problem set 6
Due Friday February 27 at 11pm

Exercise 6.1 (10 points; Rudin 2.25, modified). Suppose that K is a compact metric space. Prove that K has a dense subset which is at most countable. (Hint: first show that for every $n \in \mathbb{N}$, there are finitely many neighborhoods of radius $1/n$ whose union covers K.)

Exercise 6.2 (10 points). Suppose (x_n) is a sequence in a metric space X that converges to some $x \in X$. Show that the set $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is compact.

Exercise 6.3 (10 points). Use the definition of convergence to show that $\lim_{n\to\infty} \dfrac{2n+1}{3n-1} = \dfrac{2}{3}$.

Exercise 6.4 (10 points). Let $\{x_n\}$ be a sequence of real numbers converging to $x \in \mathbb{R}$. Show that $|x_n| \to |x|$. Is the converse true?

Exercise 6.5 (not for credit). Show that the set $\{p \in \mathbb{Q} \mid 2 < p^2 < 3\}$ is closed and bounded in $\mathbb{Q}$ (with the usual metric), but that it is not compact. (Note: This implies that the Heine-Borel theorem does not hold for $\mathbb{Q}$!)

Exercise 6.6 (not for credit). Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers. Prove that if $x_n \to x$ and $y_n \to y$, then $\lim_{n\to\infty}(x_n + y_n) = x + y$.