PSETs Landing Page*

Anish Krishna Lakkapragada

This is the documentation for using my PSET PDFs responsibly. I post these LaTEX'd PSETs (1) as an education resource for friends at other universities, fellow Yalies, and all those interested and (2) for quick reference. These PSETs are not to be used irresponsibly; only look at the solution after giving each problem an honest attempt. **If YOU USE THESE PSETS TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

The general format for accessing the (one-indexed) Nth assigned PSET PDF of a Yale course with course number CODE is:

https://anish.lakkapragada.com/notes/TYPE-CODE/psets/N.pdf

where TYPE is stats or math. Similarily, to access my solution for this PSET you can go to:

https://anish.lakkapragada.com/notes/TYPE-CODE/sols/N.pdf

These PSETs and associated solution PDFs are synchronized daily at 4:20AM with my computer files through a Cronjob Shell Script. If you want to contribute any corrections, please email anish.lakkapragada@yale.edu.

---

*Note that PDF here is referring to Portable Document Format, not to be confused with the veritable Probability Density Function.

Math 226: HW 4
Completed By: Anish Lakkapragada (NETID: al2778)

1. a) We prove each of the two inequalities below.

   (1) $dim(Span(S)) \leq dim(Span(S \cup \{x\}))$

   Let us specify the basis for S as β_S and basis for $S \cup \{x\}$ as $\beta_{S \cup \{x\}}$. Thus $dim(Span(S)) = |\beta_S|$. Let us explore two cases: (1) $x \in Span(S)$ and (2) $x \notin Span(S)$. In case (1), $\beta_{S \cup \{x\}} = \beta_S$ and so $dim(Span(S \cup \{x\})) = |\beta_{S \cup \{x\}}| = |\beta_S| \geq |\beta_S|$. In case (2), $\beta_{S \cup \{x\}} = \beta_S \cup \{x\}$ and so $dim(Span(S \cup \{x\})) = |\beta_{S \cup \{x\}}| = |\beta_S| + 1 \geq |\beta_S|$. In either case, **(1)** is true.

   (2) $dim(Span(S \cup \{x\})) \leq dim(Span(S)) + 1$

   From **(1)**, we know that $dim(Span(S \cup \{x\}))$ is equal to either $|\beta_S| + 1$ or $|\beta_S|$. Because $dim(Span(S)) + 1 = |\beta_S| + 1$, $dim(Span(S \cup \{x\})) \leq dim(Span(S)) + 1$ regardless of the particular value of $dim(Span(S \cup \{x\}))$.

   b) Let us define the basis of $U \cap W$ as $\beta_{U \cap W} = \{v_1, \ldots, v_n\}$ where $n \in \mathbb{Z}$. Let us also define β_U and β_W as the basis for U and W, respectively.

   Using Steinz Exchange Lemma, we can extend $\beta_{U \cap W}$ to β_U as $\beta_U = \beta_{U \cap W} \cup \{u_1, \ldots u_m\}$ where $u_i \in U$. Similarily, we can extend $\beta_{U \cap W}$ to β_W as $\beta_W = \beta_{U \cap W} \cup \{w_1, \ldots, w_k\}$ where $w_i \in W$.

   We now evaluate the statement $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$:

   $$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$
   $$dim(U + W) = (n + m) + (n + k) - n$$
   $$dim(U + W) = n + m + k$$

   We now compute $dim(U + W)$ by trying to find the basis of $U + W$, β_{U+W}. Any given element in $U + W$ is given by $u + w$ where $u \in U, w \in W$. We can write u and w as linear combinations of β_U and β_W respectively. Let us define the notation $\beta_K[i]$ as giving the ith element of the basis for vector space K. Then given $a_i, b_i \in \mathbb{F}$:

   $$u + w = \Sigma_i^{n+m} a_i \beta_U[i] + \Sigma_i^{n+k} b_i \beta_W[i]$$
   $$u + w = \Sigma_i^n a_i v_i + \Sigma_i^m a_{n+i} u_i + \Sigma_i^n b_i v_i + \Sigma_i^k b_{n+i} w_i$$
   $$u + w = \Sigma_i^n (a_i + b_i) v_i + \Sigma_i^m a_{n+i} u_i + \Sigma_i^k b_{n+i} w_i$$

   We can redefine $c_i = a_i + b_i \in \mathbb{F}$:

   $$u + w = \Sigma_i^n c_i v_i + \Sigma_i^m a_{n+i} u_i + \Sigma_i^k b_{n+i} w_i$$

   Thus, we have shown every element in $U + W$ can be written as a linear combination of $\beta_{U \cap W} \cup \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$. We must show now that $\beta_{U \cap W} \cup \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ is linearly independent to show that $\beta_{U+W} = \beta_{U \cap W} \cup \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$. Because basis $\beta_U = \beta_{U \cap W} \cup \{u_1, \ldots, u_m\}$ and basis $\beta_W = \beta_{U \cap W} \cup \{w_1, \ldots, w_k\}$, we know that $\beta_{U \cap W} \cup \{u_1, \ldots, u_m\}$ and $\beta_{U \cap W} \cup \{w_1, \ldots, w_k\}$ are linearly independent. Thus to show $\beta_{U \cap W} \cup \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$

is linearly independent, we must show that $\{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ is linearly independent.

Let us define this set as $K = \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ and let us assume that it is linearly dependent. This means that there exists a linear combination of $\{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ that exists in U and W (i.e. in $U \cap W$). By our construction of $\{u_1, \ldots, u_m\}$ and $\{w_1, \ldots, w_k\}$ through Steinitz Exchange Lemma, we know that all $u_i, w_i \notin \beta_{U \cap W}$ as they are the set of vectors required to extend $\beta_{U \cap W}$ to β_U or β_W. This would mean that we have found some element $\in U \cap W$ that cannot be represented as a linear combination of $\beta_{U \cap W}$. This violates the definition of $Span(\beta_{U \cap W})$ and thus by proof by contradiction, we have shown $\{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ is linearly independent $\Rightarrow \beta_{U \cap W} \cup \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$ is linearly independent.

Thus, $\beta_{U+W} = \beta_{U \cap W} \cup \{u_1, \ldots, u_m\} \cup \{w_1, \ldots, w_k\}$. This means:

$$dim(U + W) = n + m + k$$
$$|\beta_{U+W}| = n + m + k$$
$$n + m + k = n + m + k$$
$$0 = 0$$

Thus, we have proven $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.

c) We use the notation of $\beta_{U \cap W}$ and β_{U+W} from part (b). We are given the following statement:

$$dim(U + W) = 1 + dim(U \cap W)$$

This statement implies that one element must be unioned to $\beta_{U \cap W}$ to form β_{U+W} as $U \cap W \subseteq U + W$. This statement implies that $|\beta_{U+W}| = n + m + k = 1 + |\beta_{U \cap W}| = 1 + n \Rightarrow m + k = 1$. Because $m, k \in \mathbb{Z}$ and $m \geq 0, n \geq 0$, we know that either $m = 1$ or $k = 1$. We investigate these two cases below.

(1) $m = 1$

If $m = 1$, $\exists\, u \in \{u_1, \ldots, u_m\}$ s.t. $\beta_{U+W} = \{u\} \cup \beta_{U \cap W}$. W is given by $Span(\beta_W)$. Because $m = 1 \Rightarrow k = 0$, $\beta_{U \cap W} = \beta_W \Rightarrow U \cap W = W$. Because this means every element in W exists in U, $U + W = \{u + w : u \in U, w \in W\} = \{u + w : u \in U, w \in U\}$. Given U is closed under addition because it is a vector subspace, we know that $U + W = U$. Thus we have shown that in this case $U + W = U$ and $U \cap W = W$.

(2) $k = 1$

If $k = 1$, $\exists\, w \in \{w_1, \ldots, w_k\}$ s.t. $\beta_{U+W} = \{w\} \cup \beta_{U \cap W}$. W is given by $Span(\beta_W)$. Because $k = 1 \Rightarrow m = 0$, $\beta_{U \cap W} = \beta_U \Rightarrow U \cap W = U$. Because this means every element in U exists in W, $U + W = \{u + w : u \in U, w \in W\} = \{u + w : u \in W, w \in W\}$. Given W is closed under addition because it is a vector subspace, we know that $U + W = W$. Thus we have shown that in this case $U + W = W$ and $U \cap W = U$.

2. a) We show the two parts of the problem below.

(1) If T is injective, $T(B)$ is linearly independent

Let us define $B = \{u_1, \ldots, u_{|B|}\}$. To show that $T(B)$ is linearly independent, we need to show that given $a_i \in \mathbb{F}$, the solution to $\Sigma_i^{|B|} a_i T(u_i) = 0$ is for all $a_i = 0$. We inspect this below:

$$\Sigma_i^{|B|} a_i T(u_i) = 0$$
$$T(\Sigma_i^{|B|} a_i u_i) = 0$$

Because we know T is injective, if $T(x) = 0 \Rightarrow x = 0$. Thus, we know that $\Sigma_i^{|B|} a_i u_i = 0$. Because $B = \{u_1, \ldots, u_{|B|}\}$ is a basis and thus is linearly independent, the solution to $\Sigma_i^{|B|} a_i u_i = 0$ is for all $a_i = 0$. Thus, the solution to $\Sigma_i^{|B|} a_i T(u_i) = 0$ is for all $a_i = 0 \Rightarrow T(B)$ is linearly independent.

(2) **If $T(B)$ is linearly independent and $\infty > |T(B)| \geq |B|$, T is injective**
To show that T is injective, we need to show that given $x, y \in U$ s.t. $T(x) = T(y)$, $x = y$. We prove this below. Note that because B is a basis for U, x and y can be represented as a linear combination of B. Thus given $c_i, d_i \in \mathbb{F}$, $x = \Sigma_i^{|B|} c_i B_i$ and $y = \Sigma_i^{|B|} d_i B_i$.

$$T(x) = T(y)$$
$$T(\Sigma_i^{|B|} c_i B_i) = T(\Sigma_i^{|B|} d_i B_i)$$
$$\Sigma_i^{|B|} c_i T(B_i) = \Sigma_i^{|B|} d_i T(B_i)$$
$$\Sigma_i^{|B|} (c_i - d_i) T(B_i) = 0$$

Because we are given $T(B)$ is linearly independent, we know that the only solution to the equation $\Sigma_i^{|B|} a_i T(B_i) = 0$ is for all $a_i = 0$. Thus, we know that in the above equation, $c_i - d_i = 0 \Rightarrow c_i = d_i$. Thus we have proved $x = y$ if we know $T(x) = T(y) \Rightarrow T$ is injective.

b) In order to show that T is surjective iff $Span(T(B)) = V$, we must show (1) if $Span(T(B)) = V$, T is surjective and (2) if T is surjective, $Span(T(B)) = V$.

(1) **If $Span(T(B)) = V$, T is surjective**
If $Span(T(B)) = V$, $T(B)$ is a basis for V. This means $\forall v \in V$, v can be expressed as a linear combination of $T(B)$. Given $B = \{b_1, \ldots, b_{|B|}\}$ and $a_i \in \mathbb{F}$:

$$v = \Sigma_i^{|B|} a_i T(b_i)$$

This is equivalent to:

$$v = T(\Sigma_i^{|B|} a_i b_i)$$

Because $\Sigma_i^{|B|} a_i b_i$ is a linear combination of B, $\Sigma_i^{|B|} a_i b_i \in U$. Defining $w = \Sigma_i^{|B|} a_i b_i$, we have shown $\forall v \in V, \exists\, w \in U$ s.t. $T(w) = v$. Thus T is proven to be surjective.

(2) **If T is surjective, $Span(T(B)) = V$**
If T is surjective, $\forall v \in V, \exists\, w \in U$ s.t. $T(w) = v$. Because B is a basis for U, w can be expressed as a linear combination of B. Given $B = \{b_1, \ldots, b_{|B|}\}$ and $a_i \in \mathbb{F}$:

$$T(w) = v$$
$$T(\Sigma_i^{|B|} a_i b_i) = v$$

Applying T to each element in the summation and switching sides:

$$v = \Sigma_i^{|B|} a_i T(b_i)$$

This shows that $\forall v \in V$, v can be expressed as a linear combination of $T(B)$. Furthermore, this also shows that all linear combinations of $T(B)$ are elements of V. Thus, we know that $Span(T(B)) = V$.

c) We prove both parts of this question below.

    (1) **If T is bijective, $T(B)$ is a basis for V**
From part (b), we have proved if $Span(T(B)) = V$, T is surjective. From part (a), we have proved if $T(B)$ is linearly independent if T is injective. If T is bijective, T is surjective and injective, meaning that $T(B)$ is linearly independent and generates V. By the definition of a basis, $T(B)$ is a basis for V.

    (2) **If $T(B)$ is a basis and $\infty > |T(B)| \geq |B|$, then T is bijective**
We assume $T(B)$ is a basis for V, which means that $Span(T(B)) = V$ and $T(B)$ is linearly independent. From part (b), we have proved if $T(B)$ spans V, T is surjective. From part (a), we have proved if $T(B)$ is linearly independent and $\infty > |T(B)| \geq |B|$, T is injective. Thus if $T(B)$ is a basis and $\infty > |T(B)| \geq |B|$, T is injective and surjective $\Rightarrow T$ is bijective.

3.   a) T is a linear transformation if given $c \in R$ and $f, g \in C^1(\mathbb{R})$, $T(cf + g) = cT(f) + T(g)$. We evaluate $T(cf + g)$ below as:

$$T(cf + g) = ((cf + g)'(3), (cf + g)(3)) = ((cf' + g')(3), (cf + g)(3)) = (cf'(3) + g'(3), cf(3) + g(3))$$

We now compute $cT(f) + T(g)$ below.

$$cT(f) + T(g) = c(f'(3), f(3)) + (g'(3), g(3)) = (cf'(3), cf(3)) + (g'(3), g(3)) = (cf'(3) + g'(3), cf(3) + g(3))$$

Because the two above expressions are equivalent we know that $T(cf+g) = cT(f) + T(g) \Rightarrow T$ is a linear transformation.

  b) If $H + V = V$, this means that $\forall h \in H$ and $\forall v \in V$, $h + v = v$. We prove this below.

$$h + v = v$$
$$T(h + v) = T(v)$$
$$((h + v)'(3), (h + v)(3)) = (1, 2)$$
$$(h'(3) + v'(3), h(3) + v(3)) = (1, 2)$$

Because $h = (x - 3)^2 g(x)$ where $g(x) \in C^1(\mathbb{R})$, $h'(x) = 2(x - 3)g(x) + (x - 3)^2 g'(x)$. Thus, $h(3) = 0 = h'(3)$. Given this,

$$(v'(3), v(3)) = (1, 2)$$
$$T(v) = (1, 2)$$

Thus we have proven $\forall h \in H$ and $\forall v \in V$, $h + v = v$. This proves that $H + V = V$.

c) Let us define $x, y \in C^1(\mathbb{R})$. For T to be injective, if $T(x) = T(y)$ then $x = y$. From part (b), we see that for two pre-images $h + v, v \in C^1(\mathbb{R})$ where $T(h + v) = T(v)$ but $h + v \neq v$. Thus, we have shown that by definition T is not injective.

In order for T to be surjective, $\forall v \in \mathbb{R}^2, \exists\ f(x) \in C^1(\mathbb{R})$ s.t. $T(f(x)) = v$. Let us define a function $f(x) \in C^1(\mathbb{R})$ and $v = (a_1, a_2) \in \mathbb{R}^2$. The function $g(x) = a_1 x + a_2 - 3a_1 \in C^1(\mathbb{R})$ and has the property $T(g) = v$. Thus, $\forall v = (a_1, a_2) \in \mathbb{R}^2, \exists\ g(x) = a_1 x + a_2 - 3a_1 \in C^1(\mathbb{R})$ s.t. $T(g) = v$. Thus, T is surjective.

4. a) **Computing basis and dimension for $N(T)$**

$N(T) = \{v \in \mathbb{F}^5 : T(v) = \mathbf{0}^4\}$. $T(v) = \mathbf{0}^4$ when for a given $v = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5$, $T(v) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3) = \mathbf{0}^4$. This occurs under the following conditions:

$$x_2 = -x_1$$
$$x_3 = 0$$
$$x_5 = -\frac{x_4}{3}$$

Thus, $N(T) = \{(x_1, -x_1, 0, x_4, -\frac{x_4}{3}) : x_1, x_4 \in \mathbb{F}\}$. Thus the basis for $N(T)$ can be given as $\beta_{N(T)}$:

$$\beta_{N(T)} = \{(1, -1, 0, 0, 0), (0, 0, 0, 1, -\frac{1}{3})\}$$

So we get $dim(N(T)) = |\beta_{N(T)}| = 2$.

Computing basis and dimension for $R(T)$

$R(T) = \{w \in \mathbb{F}^4 : \exists\ v \in \mathbb{F}^5 \text{ s.t. } T(v) = w\}$. If we redefine $z_1 = x_1 + x_2$ and $z_2 = x_4 + 3x_5$, we get that $T(v) = (x_1 + x_2, x_3, x_4 + 3x_5, x_3) = (z_1, x_3, z_2, x_3)$ where $z_1, x_3, z_2, \in \mathbb{F}$. Thus the basis for $R(T)$ given as $\beta_{R(T)}$:

$$\beta_{R(T)} = \{(1, 0, 0, 0), (0, 1, 0, 1), (0, 0, 1, 0)\}$$

So we get $dim(R(T)) = |\beta_{R(T)}| = 3$.

b) From Dimension Theorem, we know that if T is linear, $dim(N(T)) + dim(R(T)) = dim(\mathbb{R}^5) = 5$. Because the output of T is in $\mathbb{R}^2$, $dim(R(T)) \leq 2$. Thus, $dim(N(T)) \geq 3$ if T is linear.

Expressed differently, $N(T) = \{(x_1, x_2, \frac{x_1}{7}, x_2, x_2) : x_1, x_2 \in \mathbb{F}\}$. Thus the basis of $N(T)$ is given by $\beta_{N(T)} = \{(1, 0, \frac{1}{7}, 0, 0, 0), (0, 1, 0, 1, 1)\}$ and so $dim(N(T)) = |\beta_{N(T)}| = 2$. Because $dim(N(T)) = 2 \not\geq 3$, T cannot be linear for this given null space $N(T)$.