Math 225- HW 11 Due: Dec 9 by Midnight

Submit the first two problems, along with any three additional problems of your choice.

1. Two linear operators U and T on a finite dimensional vector space are called simultaneously diagonalizable if there exist an ordered basis β such that both $[T]_\beta$ and $[U]_\beta$ are diagonal. Similarly A,B are simultaneously diagonalizable if there exist Q invertible such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal.

   - Prove that if U and T simultaneously diagonalizable then U and T commute. i.e. $UT = TU$
   - Conclude that A,B are simultaneously diagonalizable then A,B commute
   - Let T be diagonalizable linear operator on a finite dimensional vector space, then T and T^m are simultaneously diagonalizable for any m positive integer.

2. Let T, U be a linear operator on a vector space V, and let v be a non zero vector in V.

   a) Show that E_λ for any eigenvalue λ of T is a T-invariant subspace of V.
   b) Show that T-cyclic subspace generated by v is a T-invariant subspace of V.
   c) Let W be the T-cyclic subspace generated by v. Then for any $w \in V$, $w \in W$ iff $w = g(T)v$ for some polynomial g.
   d) Let V be T- cyclic subspace of itself. Show that if U commutes with T then $U = g(T)$ for some polynomial g.
   e) If V is two dimensional then either V is T-cyclic subspace of itself or $T = cI$.

3. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, ..., \lambda_k$. Prove that

$$\text{span}(\{x \in V : \ x \text{ is an eigenvector of } T\}) = E_{\lambda_1} \bigoplus E_{\lambda_2} \bigoplus ... \bigoplus E_{\lambda_k}$$

4. Let T be a linear operator on a finite dimensional vector space V, and W be an invariant subspace of V. Suppose that $v_1, v_2, ..., v_n$ are eigenvectors of T corresponding to distinct eigenvalues.

   (a) Prove that if $v_1 + v_2 + ... + v_n$ is in W, then v_i is in W for all i. (Use induction)
   (b) Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable. Hint: Use the fact that any element of the T-invariant subspace is a linear combination of some eigenvalues, and part a).
   (c) Us part a) to show that V is a T-cyclic subspace of itself. Hint: Pick a vector that gives a basis to V

5. Let T be a linear operator on a finite dimensional vector space V.

   (a) Prove that T is diagonalizable if and only if V is the direct sum of one-dimensional T-invariant subspaces.
   b) Let $V = W_1 \bigoplus W_2 \bigoplus ... \bigoplus W_k$ where $W_1, W_2, ...W_k$ are T-invariant subspaces. Prove that

$$\det(T) = \det(T_{W_1}) \cdot \det(T_{W_2}) \cdot ... \cdot \det(T_{W_k})$$

6. Prove the parallelogram law on an inner product space V;

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \text{for all } x, y \in V$$

7. Let V be a finite dimensional inner product space over $\mathbb{F}$ and let $S = \{v_1, v_2, ...v_n\}$ be an orthanormal subset of V. Show that if If S is a basis for V then for any $x, y \in V$ one has

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$$

   (This is called Parseval's equality)

8. Let $V = C[0,1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Let $W = \text{Span}\{t, \sqrt{t}\}$.

   a) Find an orthonormal basis for W. (I suggest you to practice Gram-Schmidt process -problem 2 of Section 6.2 till you feel comfortable)

   b) Let $h(t) = t^2$. Use the orthogonal basis obtained in part a) to obtain the closest approximation of h in W. Use Theorem 6.6

   c) Let $V = C([-1.1])$ Let W_e denote the subspace of V that includes all even functions. Find $W_e^{\perp}$.