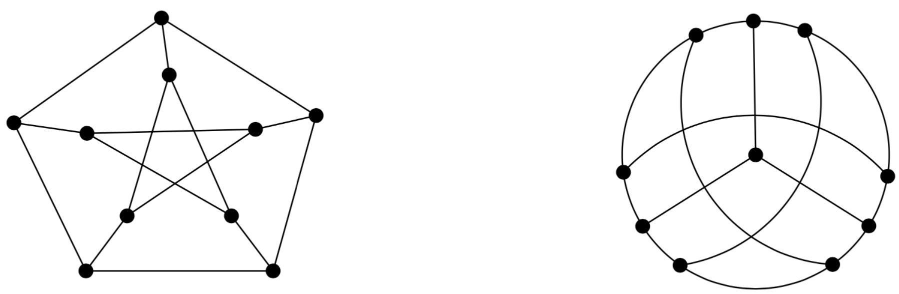
Math 244 - Problem Set 5

due Friday, March 7, 2025, at 11:59pm

Section 4.1

1. (optional bonus problem)

   (a) Find an isomorphism of the following graphs:



   (b) Show that both the graphs above are isomorphic to the following graph: the vertex set is $\binom{\{1,2,\ldots,5\}}{2}$ (unordered pairs of numbers), and two vertices $\{i, j\}$ and $\{k, \ell\}$ ($i, j, k, \ell \in \{1, 2, \ldots, 5\}$) form an edge if and only if $\{i, j\} \cap \{k, \ell\} = \emptyset$.

3. An *automorphism* of a graph $G = (V, E)$ is any isomorphism of G and G, i.e., any bijection $f : V \to V$ such that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E$. A graph is called *asymmetric* if its only automorphism is the identity mapping (each vertex is mapped to itself).

   (a) Find an example of an asymmetric graph with at least two vertices.

   (b) Show that no asymmetric graph G exists with $1 < |V(G)| \leq 5$.

Section 4.2

1. Prove that the complement of a disconnected graph G is connected. (The *complement* of a graph $G = (V, E)$ is the graph $(V, \binom{V}{2} \setminus E)$.)

10. Show that a graph G contains a triangle (i.e., a K_3) if and only if there exist indices i and j such that both the matrices A_G and A_G^2 have the entry (i, j) nonzero, where A_G is the adjacency matrix of G.

Section 4.3

5. Draw all nonisomorphic graphs with score $(6, 3, 3, 3, 3, 3, 3)$. Prove that none was left out!

12. A graph G is called k-regular if all its vertices have degree exactly k. Determine all (k, n) such that there exists a k-regular graph on n vertices. *The textbook has a hint to this problem in the back.*