MATH 255 PSET 5

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1.

To prove that E is open, we WTS $\forall (a,b) \in E$, (a,b) is an interior point of E. Pick $(a,b) \in E$. Because $(a,b) \in E \implies a < b$ and so we can define $h = b - a > 0$. We now need to show that we can create a neighborhood around (a,b) that is $\subset E$. Let us define $\epsilon = \frac{h}{2} > 0$ and create the following neighborhood:

$$N_\epsilon((a,b)) = \{(x,y) : \sqrt{(a-x)^2 + (y-b)^2} < \epsilon\}$$

We now need to show that $N_\epsilon((a,b)) \subset E \implies \forall (x,y) \in N_\epsilon((a,b)), (x,y) \in E$. Pick $(x,y) \in N_\epsilon((a,b))$. This implies the following two statements: (i) $(a-x)^2 < \epsilon^2 \implies |a-x| < \epsilon$ and (ii) $(b-y)^2 < \epsilon^2 \implies |b-y| < \epsilon$. Note that $|a-x| < \epsilon \implies -\epsilon < a - x < \epsilon \implies a - \epsilon < x < a + \epsilon$ and with the same logic $b - \epsilon < y < b + \epsilon$. Substituting ϵ for $0.5h$, we know the following: $x < a + \epsilon \implies x < a + \frac{h}{2}$ and $y > b + \epsilon \implies y > b - \frac{h}{2}$. Note that $b = a + h$ and so $y > b - \frac{h}{2} \implies y > a + \frac{h}{2} > x \implies y > x \implies x < y \implies (x,y) \in E$. Thus, we have shown $N_\epsilon((a,b)) \subset E$ and so we have proven $\forall (a,b) \in E, \exists\, \epsilon > 0$ s.t. $N_\epsilon((a,b)) \subset E \implies E$ is open.

2.

Let us define $C_1, \ldots, C_k$ to be k compact sets. Let us define set $C = \bigcup_{i=1}^k C_i$. We WTS that C is compact, or that any open cover of C has a finite subcover. Let $\{S_j\} \supset C$ be an open cover of C. Because $\forall\ 1 \le i \le k, \{S_j\} \supset C \supset C_i \implies \{S_j\} \supset C_i$, $\{S_j\}$ serves as an open cover for each C_i. Because each C_i is compact, any open cover of C_i has a finite subcover. Thus, for each C_i, its open cover $\{S_j\}$ has a finite subcover $\{F_z^{(i)}\} \supset C_i$ where $\{F_z^{(i)}\} \subset \{S_j\}$.

Let us define $F = \bigcup_{i=1}^k \{F_z^{(i)}\}$ to be the union of all these finite subcovers of C_i[1]. We now WTS that F is a finite subcover of C. To do so, we need to show the following:

1. **F is an open cover of C**

   Because $\forall x \in C, x \in$ some $C_i \implies x \in$ some $\{F_z^{(i)}\} \implies x \in F$, we have $C \subset F$, meaning that F is a (finite) open cover of C.

[1] Because each finite subcover is finite, a union of these finite sets (i.e. F) will also be finite.

2. **F is a finite subcover of open cover $\{S_j\}$ of C**

   Because $\forall \{F_z^{(i)}\} \in F, \{F_z^{(i)}\} \in \{S_j\}$, F is a subcover of open cover $\{S_j\}$ of C.

Thus, we have proven any open cover of C, a union of finitely many compact sets, has a finite subcover $\implies$ C is compact.

3. An open cover of $(0,1) \subset \mathbb{R}$ can be given by $\mathbb{R} \supset \{G_\alpha : \alpha \in \mathbb{N}\} \supset (0,1)$, where open set $G_\alpha = (\frac{1}{\alpha}, 1)$. We WTS $(0,1) \subset \mathbb{R}$ is not compact by showing that this open cover does not have a finite subcover (as this implies that not all open covers of $(0,1) \subset \mathbb{R}$ have a finite subcover $\implies$ $(0,1) \subset \mathbb{R}$ is not compact).

   We now prove that there is no finite subcover of $\{G_\alpha\}$. We prove this by contradiction and assume that there is a finite subcover of $\{G_\alpha\}$, given by $\{G_k\} = \bigcup_{i=1}^{k}(\frac{1}{n_i}, 1) \subset \{G_\alpha\}$ where $n_1, \ldots, n_k \in \mathbb{N}$. Because $n_1, \ldots, n_k$ form a (finite) subset of $\mathbb{N}$, they have a minimum which we can call $n' = \min(n_1, \ldots, n_k)$. This means that the interval $\{G_k\} = \bigcup_{i=1}^{k}(\frac{1}{n_i}, 1)$ can be simplified to $(\frac{1}{n'}, 1)$. Because $\exists\ n'' > n'$ where $(\frac{1}{n''}, 1) \subset (0,1)$ but $\not\subset \{G_k\} = (\frac{1}{n'}, 1)$, G_k is not an open cover of $(0,1)$ $\implies$ $\{G_k\}$ is not a finite subcover of $\{G_\alpha\}$. Thus, we have proved by contradiction that the open cover $\{G_\alpha\}$ has no finite subcover $\implies$ $(0,1) \subset \mathbb{R}$ is not compact.

4. (1) If A and B are disjoint sets then $A \cap B = \emptyset$. Furthermore, if A and B are closed that means $A = \bar{A}$ and $B = \bar{B}$. Thus $A \cap \bar{B} = A \cap B = \emptyset$ and $\bar{A} \cap B = A \cap B = \emptyset$, and so we know A and B are separated.

   (2) Let us define A, B as two disjoint open sets. We WTS that $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$. To do so, we prove that no limit points of B are in A and no limit points of A are in B.

   WLOG, let us prove why no limit points of B are in A. We prove this by contradiction and assume that for a limit point x of B, $x \in A$. This means $\forall \epsilon > 0, N_\epsilon(x)$ contains some $b \neq x$ s.t. $b \in B$. However, because $x \in A$ and A is open $\implies$ x is an interior point $\implies \exists\ \epsilon > 0$ s.t. $N_\epsilon(x) \subset A$. Thus, this means that $N_\epsilon(x)$, which will contain some $b \neq x \in B$ by virtue of x being a limit point of B, is fully contained in A $\implies \exists\ b \in B$ and A $\implies$ $A \cap B \neq \emptyset$, which is a contradiction of A and B being disjoint. Thus, we have proven that no limit points of B are in A and that no limit points of A are in B.

   Let us define A' and B' to be the limit points of A and B, respectively. Based on our proof above we know $B' \cap A = \emptyset$ and $A' \cap B = \emptyset$. Thus, we have:

$$A \cap \bar{B} = A \cap (B \cup B') = (A \cap B) \cup (A \cap B') = \emptyset \cup \emptyset = \emptyset$$
$$\bar{A} \cap B = (A \cup A') \cap B = (A \cap B) \cup (A' \cap B) = \emptyset \cup \emptyset = \emptyset$$

   Thus, $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset \implies A$ and B are separated.

   (3) $\forall x \in A, d(p,x) < \delta \implies d(p,x) \not> \delta \implies x \notin B$. The same applies in the other direction to show $\forall x \in B, x \notin A$ and so we have $A \cap B = \emptyset \implies A$ and B are disjoint.

We now prove that A and B are open. We first prove A is open. Note that A is essentially $N_\delta(p)$. As we have proved, any neighborhood of a point is open $\implies A = N_\delta(p)$ is open.

We now prove that B is open, or that all its points are interior points of B. Pick $x \in B$. To show x is an interior point, we must find some $\epsilon > 0$ s.t. $N_\epsilon(x) \subset B$. Note that because $x \in B \implies d(p,x) > \delta$. Consider $\epsilon = d(p,x) - \delta > 0$. We now aim to show that $\forall z \in N_\epsilon(x), z \in B$. Pick $z \in N_\epsilon(x)$. By Triangle Inequality we have:

$$\begin{aligned} d(p,x) &\leq d(p,z) + d(z,x) \\ d(p,z) &\geq d(p,x) - d(z,x) \end{aligned}$$

Because $z \in N_\epsilon(x), d(z,x) < \epsilon \implies d(z,x) < d(p,x) - \delta \implies -d(z,x) > \delta - d(p,x)$. Thus, we get:

$$\begin{gathered} d(p,z) \geq d(p,x) - d(z,x) \\ d(p,z) \geq d(p,x) - d(z,x) > d(p,x) + \delta - d(p,x) \\ d(p,z) > d(p,x) + \delta - d(p,x) \\ d(p,z) > \delta \end{gathered}$$

Thus, $d(p,z) > \delta \implies z \in B \implies \forall z \in N_\epsilon(x), z \in B \implies N_\epsilon(x) \subset B \implies$ all points of B are interior points $\implies B$ is open.

Thus, we have proven that A and B are disjoint open sets. By our proof in part (2), this means that A and B are separated.

(4) We prove this statement by contradiction and thus assume that this connected metric space X with at least two points is not uncountable $\implies X$ is at most countable. Let us define set $D = \{d((p,q)) : (p,q) \in X \times X\} \subset \mathbb{R}^+$. Because X is at most countable $\implies X \times X$ is at most countable $\implies D$ is at most countable[2]. We aim to find a $\delta > 0 \notin D$ where $\exists\, m \in D$ s.t. $m > \delta$. We proceed with casework on D's cardinality:

(i) **Case One: If D is finite**

Let us fix points $p, p' \in X$. D is guaranteed to contain these two elements: $d(p,p) = d(p',p') = 0$ and $d(p,p')$. Listing all elements of D in increasing order as so: $d_1, \ldots, d_n$, we can select i from 1 to $n-1$ and choose $\delta = \frac{d_i + d_{i+1}}{2}$. We are guaranteed this element does not exist in the finitely many elements of D by virtue of it existing in between two consecutive elements d_i and d_{i+1} in D. Because this element is not the maximum of D (i.e. $\delta < d_n$) $\implies \exists\, m \in D$ s.t. $m > \delta$. Furthermore, δ is an average of two non-negative numbers, where only one can be zero[3] $\implies \delta > 0$.

[2]This is because set D cannot have more elements than $X \times X$ as it is simply applying the distance function d to every element of $X \times X$.

[3]This is because D is a set and thus there are no repeat elements.

(ii) **Case Two: If D is countable**
Here, we use a familiar intervals argument to find δ. Fix $p, p' \in X$ and define $a_1 = 0$ and $b_1 = d(p, p')$. Because D is countable, we can write a sequence (q_n) that defines every element of D. Let us first define interval $I_1 = [a_1, b_1]$. Then for $q_2, q_3, \ldots,$ we can construct closed interval $I_i = [a_i, b_i]$ with nonzero length where $I_{i+1} \subset I_i$ and $q_i \notin I_i$.
Defining $\delta = \sup(\{a_i : i \in \mathbb{N}\})$, δ exists in all intervals I_i but does not exist in D. Furthermore, because $\forall i, \delta \in I_i$ we know the following two things: (i) $\delta < b_1 = d(p, q) \implies \exists\, m \in D$ s.t. $m > \delta$ and (ii) $\delta > a_i \implies \delta > 0$.

Because $\delta \notin D \implies \forall p, q \in X, d(p, q) \neq \delta \implies X = \{q \in X : d(p, q) < \delta\} \cup \{q \in X : d(p, q) > \delta\}$. Let us define set $A = \{q \in X : d(p, q) < \delta\}$ and set $B = \{q \in X : d(p, q) > \delta\}$ where, as per our previous sentence, $X = A \cup B$. Note that because $\exists\, m \in D$ s.t. $m > \delta \implies \exists\, q \in X$ s.t. $d(p, q) > \delta \implies q \in B \implies B$ is non-empty. Also note A is guaranteed to be non-empty as $d(p, p) = 0 < \delta \implies p \in A$.

Our proof in part (c) applies and so we get that A and B are separated $\implies \exists$ non-empty sets A, B s.t. $X = A \cup B$ where $\bar{A} \cap B = A \cap \bar{B} = \emptyset \implies X$ is disconnected, which is a contradiction to our given that X is connected.

5. To prove that $\mathbb{Q}$ is dense in $\mathbb{R}$, we aim to prove that $\bar{\mathbb{Q}} = \mathbb{R}$. We prove both directions of this statement below:

(a) $\bar{\mathbb{Q}} \subset \mathbb{R}$
Pick $x \in \bar{\mathbb{Q}}$. This means that at least one of the two cases is true:

1. **Case One:** $x \in \mathbb{Q}$
If $x \in \mathbb{Q} \implies x \in \mathbb{R}$.
2. **Case Two:** x **is a limit point of** $\mathbb{Q}$
If x is a limit point of $\mathbb{Q}$, that means $\forall \epsilon > 0, N_\epsilon(x)$ contains some $q \neq x$ s.t. $q \in \mathbb{Q}$. Because $\mathbb{Q} \subset \mathbb{R}$, this means that $\forall \epsilon > 0, N_\epsilon(x)$ contains some $q \neq x$ s.t. $q \in \mathbb{R} \implies x$ is a limit point of $\mathbb{R}$. Because $\mathbb{R}$ is closed, this means that $x \in \mathbb{R}$.

Thus we have shown in both cases that $x \in \mathbb{R}$ and so we have shown $\forall x \in \bar{\mathbb{Q}}, x \in \mathbb{R} \implies \bar{\mathbb{Q}} \subset \mathbb{R}$.

(b) $\mathbb{R} \subset \bar{\mathbb{Q}}$
Pick $x \in \mathbb{R}$. We perform casework:

1. **Case One:** $x \in \mathbb{Q}$
If $x \in \mathbb{Q} \implies x \in \bar{\mathbb{Q}}$.
2. **Case Two:** $x \notin \mathbb{Q}$
We know that $\forall x \in \mathbb{R}, x$ is a limit point of $\mathbb{R}$. Proof[4]. Thus, $\forall x \in \mathbb{R}$ we know that $\forall \epsilon > 0, N_\epsilon(x)$ contains some $y \neq x$ s.t. $y \in \mathbb{R}$. Because $x \neq y \implies$ either $x < y$ or $x > y$ because $\mathbb{R}$ is an ordered field $\implies \min(x, y) \neq \max(x, y) \implies \min(x, y) < \max(x, y)$.

[4]Pick $x \in \mathbb{R}$ and define $\epsilon > 0 \in \mathbb{R}$. Then $N_e(x) = (x - \epsilon, x + \epsilon)$, which contains $x + 0.5\epsilon$. Because x and 0.5ϵ exist in $\mathbb{R}$ and $\mathbb{R}$ is closed under addition because it is a field $\implies x + 0.5\epsilon \in \mathbb{R} \implies \forall \epsilon > 0, N_e(x)$ contains some $x' \neq x$ s.t. $x' \in \mathbb{R} \implies \forall x \in \mathbb{R}, x$ is a limit point of $\mathbb{R}$.

Let us define $\alpha = \min(x, y)$ and $\beta = \max(x, y)$. Because $x, y \in \mathbb{R} \implies \alpha, \beta \in \mathbb{R}$. Thus, the density of rationals proof (Prop 3.45) applies: we know there exists some rational $r \in \mathbb{Q}$ s.t. $\alpha < r < \beta$. Thus, this means $\forall \epsilon > 0, N_\epsilon(x)$ contains some $r \neq x$ s.t. $r \in \mathbb{Q}$. This implies x is a limit point of $\mathbb{Q} \implies x \in \bar{\mathbb{Q}}$.

Thus, we have shown in either case, $x \in \bar{\mathbb{Q}} \implies \forall x \in \mathbb{R}, x \in \bar{\mathbb{Q}} \implies \mathbb{R} \subset \bar{\mathbb{Q}}$