

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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# S&DS 242/542: Homework 6

Due Wednesday, March 5, at 1PM

1. **The geometric model.** Suppose  $X_1, \dots, X_n \stackrel{IID}{\sim} \text{Geometric}(p)$ , where  $\text{Geometric}(p)$  is the geometric distribution on the positive integers  $\{1, 2, 3, \dots\}$  defined by the PMF

$$f(x | p) = p(1 - p)^{x-1}$$

with a single parameter  $p \in [0, 1]$ . You may use without proof that this distribution has mean  $1/p$  and variance  $(1 - p)/p^2$ .

Compute the method-of-moments estimate of  $p$ , as well as the MLE of  $p$ . For large  $n$ , what approximately is the sampling distribution of the MLE?

2. **The negative binomial model.** Suppose  $X_1, \dots, X_n \stackrel{IID}{\sim} \text{NegBinom}(r, p)$ , where  $\text{NegBinom}(r, p)$  is the negative binomial distribution on  $\{0, 1, 2, 3, \dots\}$  defined by the PMF

$$f(x | p) = \binom{x + r - 1}{x} (1 - p)^r p^x.$$

Here  $r > 0$  is a fixed and known positive integer, and  $p \in [0, 1]$  is the unknown parameter. You may use without proof that this distribution has mean  $pr/(1 - p)$  and variance  $pr/(1 - p)^2$ .

Compute the method-of-moments estimate of  $p$ , as well as the MLE of  $p$ . For large  $n$ , what approximately is the sampling distribution of the MLE?

### 3. Generalized method-of-moments and the MLE.

Consider a parametric model  $f(x | \theta)$  with parameter  $\theta \in \mathbb{R}$ , whose PDF takes a form

$$f(x | \theta) = e^{\theta T(x) - A(\theta)} h(x) \text{ for } x \in \mathcal{X} \quad (*)$$

where  $\mathcal{X}$  is the range of possible data values.

(a) Show that the model  $\text{Pareto}(\theta, 1)$  is of this form, where  $\mathcal{X} = [1, \infty)$ . What are the functions  $T(x)$ ,  $A(\theta)$ , and  $h(x)$  for this Pareto model?

(b) For any model of the form  $(*)$ , differentiate the identity

$$1 = \int_{\mathcal{X}} e^{\theta T(x) - A(\theta)} h(x) dx$$

with respect to  $\theta$  on both sides, to obtain a formula for  $\mathbb{E}_\theta[T(X)]$  in terms of  $A(\theta)$ . Verify that your formula is correct for the Pareto model in part (a).

[You may use  $\frac{d}{d\theta} \int_{\mathcal{X}} e^{\theta T(x) - A(\theta)} h(x) dx = \int_{\mathcal{X}} \frac{d}{d\theta} [e^{\theta T(x) - A(\theta)} h(x)] dx$  without justifying this exchange of differentiation in  $\theta$  and integration in  $x$ .]

(c) Let  $X_1, \dots, X_n \stackrel{IID}{\sim} f(x | \theta)$  where  $f(x | \theta)$  is of the form  $(*)$ , and consider the generalized method-of-moments estimator  $\hat{\theta}$  based on  $T(x)$ , i.e.  $\hat{\theta}$  is the value of  $\theta$  for which

$$\mathbb{E}_\theta[T(X)] = \frac{1}{n} \sum_{i=1}^n T(X_i).$$

If the MLE is the unique solution to the equation  $0 = \ell'_n(\theta)$  where  $\ell_n(\theta)$  is the log-likelihood, show that this generalized method-of-moments estimator is the same as the MLE.

Use this to explain why the generalized method-of-moments estimator based on  $T(x) = \log x$  in the Pareto( $\theta, 1$ ) model coincides with the MLE.

#### 4. Confidence intervals for a binomial proportion.

Let  $X_1, \dots, X_n \stackrel{IID}{\sim} \text{Bernoulli}(p)$ , and let  $\hat{p} = \bar{X}$ . We compare two different ways to construct a 95% confidence interval for  $p$ , both based on the Central Limit Theorem result

$$\sqrt{n}(\hat{p} - p) \rightarrow \mathcal{N}(0, p(1 - p)). \quad (**)$$

(a) Use the plugin estimate  $\hat{p}(1 - \hat{p})$  for the variance  $p(1 - p)$  to write down a 95% confidence interval for  $p$ . This is the approach discussed in Lecture 13.

(b) Instead of using this plugin estimate, note that equation  $(**)$  implies, for large  $n$ ,

$$\mathbb{P} \left[ -\sqrt{p(1 - p)} z^{(\alpha/2)} \leq \sqrt{n}(\hat{p} - p) \leq \sqrt{p(1 - p)} z^{(\alpha/2)} \right] \approx 1 - \alpha.$$

Solve the two equations  $\sqrt{n}(\hat{p} - p) = \pm \sqrt{p(1 - p)} z^{(\alpha/2)}$  for  $p$  in terms of  $\hat{p}$ , to obtain a different 95% confidence interval for  $p$ .

(c) Perform a simulation study to determine the true probability that the confidence intervals in parts (a) and (b) cover  $p$ , for the 9 combinations of sample sizes  $n = 10, 40, 100$  and true parameters  $p = 0.1, 0.3, 0.5$ . Report the simulated coverage probabilities in two tables. Which interval construction yields true coverage closer to 95% for small values of  $n$ ?

[For each combination of  $n$  and  $p$ , it may be helpful to perform at least 100,000 simulations. In R, you may simulate  $\hat{p}$  directly as `phat = rbinom(1,n,p)/n`.]