

## Discretionary Note

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**IF YOU USE THIS FILE TO CHEAT, YOU ARE NOT ONLY STUPID BUT YOU ARE CHEATING YOURSELF OUT OF THE ABILITY TO FALL IN LOVE WITH MATH.** Furthermore, I am not smarter than you and my solutions did not always get a perfect score.

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**Problem set 5**  
**Due Friday February 20 at 11pm**

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**Exercise 5.1 (5 points).** Show that the subset of  $\mathbb{R}^2$  given by

$$E = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$$

is open.

**Exercise 5.2 (5 points).** Show that the union of a finite number of compact sets is compact.

**Exercise 5.3 (5 points; Rudin 2.14).** Prove directly that the interval  $(0, 1) \subset \mathbb{R}$  is not compact, by giving an example of an open cover of  $(0, 1)$  which has no finite subcover. Include a proof that your cover has no finite subcover.

**Exercise 5.4 (20 points; Rudin 2.19).** If  $A$  and  $B$  are subsets of a metric space  $X$ , we say  $A$  and  $B$  are *separated* if  $A \cap \overline{B} = \emptyset$  and  $\overline{A} \cap B = \emptyset$ . (We used this notion when we defined connectedness.)

- (1) If  $A$  and  $B$  are disjoint closed sets in some metric space  $X$ , prove that  $A$  and  $B$  are separated.
- (2) Prove the same for disjoint open sets.
- (3) Fix  $p \in X$  and  $\delta > 0$ . Define  $A = \{q \in X \mid d(p, q) < \delta\}$ . Define  $B = \{q \in X \mid d(p, q) > \delta\}$ . Prove that  $A$  and  $B$  are separated.
- (4) Prove that every connected metric space with at least two points is uncountable. (Hint: use the previous part.)

**Exercise 5.5 (5 points; Rudin 2.22, modified).** Given a metric space  $X$  and a set  $E \subset X$ , we say  $E$  is *dense* in  $X$  if  $\overline{E} = X$ . Prove that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

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**Exercise 5.6 (not for credit; Rudin 2.8).** Is every point of every open set  $E \subset \mathbb{R}^2$  a limit point of  $E$ ? Answer the same question for closed sets in  $\mathbb{R}^2$ .