

# Chapter 5

## Predicate Logic

In this chapter, we consider predicate logic, with functions and quantifiers.<sup>1</sup> The discussion is broken up into syntax, semantics, and proofs.

Predicate logic is a richer system than sentential logic and allows us to move closer to the kind of system we need for natural language semantics. It allows us to introduce *model theory* at an introductory level, a set-theoretic characterization of predicate logic semantics.

### 5.1 Syntax

The syntax of predicate logic can be broken up into a vocabulary and a set of rules for constructing formulas out of that vocabulary. Basically, the primitives are constants, variables, predicates, connectives, quantifiers, and delimiters. Constants and variables correspond to objects in our world. Constants are like names, e.g. *Ernie* or *Hortence*. Variables are more like pronouns, e.g. *they* or *it*. Predicates allow us to describe properties of objects and sets of objects. Quantifiers allow us to refer to sets of things. The connectives and delimiters are the same from sentential logic.

These are the elements of the vocabulary of first-order predicate logic:

**Constants**  $a, b, c, \dots$ ; with or without superscripts. The superscripts allow us to convert a finite set of letters into an infinite set of constants.

**Variables**  $x, y, z, \dots$ ; with or without superscripts. Again, the superscripts allow us to convert a finite set of letters into an infinite set of variables.

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<sup>1</sup>We do not consider identity or function terms.

**Terms** Constants  $\cup$  Variables = Terms. That is, all the constants and variables grouped together constitute the *terms* of predicate logic.

**Predicates**  $F, G, H, \dots$  Each takes a fixed number of terms. These too can be superscripted to get an infinite set of predicates.

**Connectives** The usual suspects:  $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ . We use the same connectives as in sentential logic.

**Quantifiers**  $\forall$  and  $\exists$ . The first is the *universal quantifier* and the second is the *existential quantifier*.

**Delimiters** Parentheses.

With superscripts, there are an infinite number of constants, variables, and predicates.

The vocabulary is used to define the set of WFFs recursively.

1. If  $P$  is an  $n$ -ary predicate and  $t_1, \dots, t_n$  are terms, then  $P(t_1, \dots, t_n)$  is a WFF.
2. If  $\varphi$  and  $\psi$  are WFFs, then  $\neg\varphi$ ,  $(\varphi \wedge \psi)$ ,  $(\varphi \vee \psi)$ ,  $(\varphi \rightarrow \psi)$ , and  $(\varphi \leftrightarrow \psi)$  are WFFs.
3. If  $\varphi$  is a WFF and  $x$  is a variable, then  $(\forall x)\varphi$  and  $(\exists x)\varphi$  are WFFs. (The *scope* of the quantifiers in these WFFs is  $\varphi$ .)
4. That's it.

Notice that there is *no* requirement that the variable next to the quantifier be paired with the same variable in the formula in its scope. Thus  $(\exists x)F(x)$  is as well-formed a formula as  $(\exists x)F(y)$ . In fact,  $(\exists x)(\forall y)(\forall y)F(z)$  is just as well-formed.

Notice too that we've defined a notion *scope*. For example, in a WFF like  $(\forall x)(\forall y)G(x, y)$ , the scope of  $(\forall y)$  is  $G(x, y)$  and the scope of  $(\forall x)$  is  $(\forall y)G(x, y)$ . Notice too that the scope of a quantifier is not simply everything to the right. In a WFF like  $((\exists x)G(y) \wedge F(a))$ , the scope of  $(\exists x)$  is only  $G(y)$ . On the other hand, in a WFF like  $(\forall x)(F(a) \wedge G(b))$ , the scope of  $(\forall x)$  is  $(F(a) \wedge G(b))$ . Notice how important parentheses and the syntax of a WFF is to determining scope. We will make use of scope in the semantics of predicate logic.

## 5.2 Semantics

The semantics of predicate logic can be understood in terms of set theory. First, we define the notion of *model*.

**Definition 6 (Model)** *A model is a set  $D$  and a function  $f$ .*

1.  *$f$  assigns each constant to a member of  $D$ .*
2.  *$f$  assigns each one-place predicate to a subset of  $D$ .*
3.  *$f$  assigns each two-place predicate to a subset of  $D \times D$ .*
4. *etc.*

The basic idea is that a set of constants and predicates are paired with elements from the set of elements provided by the model. We can think of those elements as things in the world. Each constant can be paired directly with an element of the model. Thus if our model includes the individual *Ernie*, we might pair the constant  $a$  with that individual.

Predicates are a little more complex. We can think of each predicate as holding either for a set of individual elements of the model or a set of ordered tuples defined in the model. Each predicate thus defines a set of elements or tuples of elements. Thus, if the predicate  $G$  is taken to define the set of individuals that might be in the kitchen at some particular time, we might take  $G$  to be defined as follows:  $f(G) = \{\text{Ernie}, \text{Hortence}\}$ .

Predicates with more than one argument are mapped to a set of tuples. For example, if our model is the set of sounds of English and  $F$  is defined as the set of consonants in English that are paired for voicing, we would have  $f(F) = \{\langle p, b \rangle, \langle t, d \rangle, \langle k, g \rangle, \dots\}$ .<sup>2</sup>

Notice that there is no requirement that there be a single model. A logical system can be paired with any number of models. Thus a predicate  $G$  could be paired with a model of individuals and rooms and define a set of individuals that are in the kitchen or it could be paired with a model of sounds in English and define the set of nasal consonants.

We can use model theory to understand how predicate logic formulas are evaluated. As with simple sentential logic, predicate logic formulas evaluate

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<sup>2</sup>Voicing refers to vibration of the vocal folds.

to one of two values: **T** or **F**. The logical connectives have their usual function, but we must now include a mechanism for understanding predicates and quantifiers.

Consider predicates first. An expression like  $G(a)$  is true just in case  $f(a)$  is in the subset of  $D$  that  $f$  assigns  $G$  to. For example, if  $a$  is paired with *Ernie* and *Ernie* is in the set of  $D$  that  $G$  is paired with,  $f(a) = \text{Ernie}$  and  $\text{Ernie} \in f(G)$ , then  $G(a)$  is true.

Likewise,  $H(a, b)$  is true just in case  $\langle a, b \rangle$  is in the subset of  $D \times D$  that  $f$  assigns  $H$  to. For example, if we take  $D$  to be the set of words of English and we take  $H$  to be the relation ‘has fewer letters than’, then  $H(a, b)$  is true just in case the elements we pair  $a$  and  $b$  with are in the set of ordered pairs defined by  $f(H)$ . For example, if  $f(a) = \text{hat}$  and  $f(b) = \text{chair}$ , then  $\langle \text{hat}, \text{chair} \rangle \in f(H)$  and  $H(a, b)$  is true.

Quantified variables are then understood as follows. Variables range in value over the members of  $D$ . An expression quantified with  $\forall$  is true just in case the expression is true of *all* members of the model  $D$ ; an expression quantified with  $\exists$  is true just in case the expression is true of at least *one* member of the model  $D$ . Let’s go through some examples of this. Assume for these examples, that  $D = \{m, n, \eta\}$ .

For example, an expression like  $(\forall x)G(x)$  is true just in case every member of  $D$  is in the set  $f$  assigns to  $G$ . In the case at hand, we might think of  $G$  as ‘is nasal’, in which case,  $f(G) = \{m, n, \eta\}$ . Since  $f(G) = D$ ,  $(\forall x)G(x)$  is true.<sup>3</sup> On the other hand, if we interpret  $G$  as ‘is coronal’, then  $f(G) = \{n\}$  and  $(\forall x)G(x)$  is false, since  $f(G) \subset D$ .<sup>4</sup> Likewise,  $(\exists x)G(x)$  is true just in case the subset of  $D$  that  $f$  assigns to  $G$  has at least one member. If we interpret  $G$  as ‘is coronal’, then this is true, since  $f$  maps  $G$  to the subset  $\{n\}$ , which has one member, i.e.  $|f(G)| \geq 1$ .

More complex expressions naturally get trickier. Combining quantified variables and constants is straightforward. For example,  $(\forall x)H(a, x)$  is true just in case every member of  $D$  can be the second member of each tuple in the set of ordered pairs that  $f$  assigns to  $H$ , when  $a$  is the first member. Thus, if  $f(a) = m$ , then  $H(a, x) = \{\langle m, m \rangle, \langle m, n \rangle, \langle m, \eta \rangle\}$ .

To get this set of ordered pairs, we might interpret  $H(a, b)$  as saying that  $a$  is at least as anterior as  $b$ , where anteriority refers to how far forward in the mouth the primary constriction for the sound is. On this model,

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<sup>3</sup>Recall that nasal sounds are produced with air flowing through the nose.

<sup>4</sup>Coronal sounds are produced with the tip of the tongue.

$$f(H) = \{\langle m, m \rangle, \langle m, n \rangle, \langle m, \eta \rangle, \langle n, n \rangle, \langle n, \eta \rangle, \langle \eta, \eta \rangle\}.$$

We can use this latter interpretation of  $H$  to treat another predicate logic formula:  $(\forall x)H(x, x)$ . Here there is still only one quantifier and no connectives, but there is more than one quantified variable. The interpretation is that both arguments must be the same. This expression is true if  $H$  can pair all elements of  $D$  with themselves. This is true in the just preceding case since  $\{\langle m, m \rangle, \langle n, n \rangle, \langle \eta, \eta \rangle\} \subseteq f(H)$ . That is, every sound in the set  $D$  is at least as anterior as itself!

Let's now consider how to interpret quantifiers and connectives in formulas together. The simplest case is where some connective occurs 'outside' any quantifier, e.g.  $((\forall x)G(x) \wedge (\forall y)H(y))$ . This is true just in case  $f(G) = D$  and  $f(H) = D$ , that is, if  $G$  is true of all the members of  $D$  and  $H$  is true of all the members of  $D$ , e.g.  $(f(G) = D \wedge f(H) = D)$ .

If the universal quantifier  $\forall$  is 'outside' the connective,  $(\forall x)(G(x) \wedge H(x))$ , the formula ends up having the same truth value, but the interpretation is a little different. This latter formula is true on the model where  $G$  and  $H$  apply to every member of  $D$ . Here, we interpret the conjunction within the scope of the universal quantifier in its set-theoretic form: intersection. We then intersect  $f(G)$  and  $f(H)$ . If this intersection is  $D$ , then the original expression is true. Thus  $(\forall x)(G(x) \wedge H(x))$  is true just in case  $(f(G) \cap f(H)) = D$  is true.<sup>5</sup> What this means is that a universal quantifier can be raised from or lowered into a conjunction changing the scope of the quantifier with no difference in truth value.

With the existential quantifier, scope forces different interpretations. The expression with the existential quantifier inside conjunction:

$$((\exists x)G(x) \wedge (\exists y)H(y))$$

does not have the same value as the expression with the existential quantifier outside the conjunction:

$$(\exists x)(G(x) \wedge H(x)).$$

The first is true just in case there is some element of  $D$  that  $G$  holds of and there is some element of  $D$  that  $H$  holds of; the two elements need not

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<sup>5</sup>It is a theorem of set theory that if  $A \cap B = U$ , then  $A = B = U$ .

be the same. In formal terms:  $|f(G)| \geq 1$  and  $|f(H)| \geq 1$ , e.g.  $(|f(G)| \geq 1 \wedge |f(H)| \geq 1)$ . The two sets,  $f(G)$  and  $f(H)$ , need not have any elements in common. The second is true *only* if there is some element of  $D$  that both  $G$  and  $H$  hold of, that is in  $f(G)$  and in  $f(H)$ ; that is, they *must* have at least one element in common:  $|(f(G) \cap f(H))| \geq 1$ .

The point of these latter two examples is twofold. First, nesting connectives within the scope of quantifiers requires that we convert logical connectives to set-theoretic operations. Second, nesting quantifiers and connectives can result in different interpretations depending on the quantifier and depending on the connective.

The relative scope of more than one quantifier also matters. Consider the following four formulas.

- (5.1) a.  $(\forall x)(\forall y)G(x, y)$   
 b.  $(\exists x)(\exists y)G(x, y)$   
 c.  $(\forall x)(\exists y)G(x, y)$   
 d.  $(\exists x)(\forall y)G(x, y)$

When the quantifiers are the same, their relative nesting is irrelevant. Thus (5.1a) is true just in case every element can be paired with every element, i.e.  $f(G) = D \times D$ . Reversing the universal quantifiers gives exactly the same interpretation:  $(\forall y)(\forall x)G(x, y)$ . Likewise, (5.1b) is true just in case there's at least one pair of elements in  $D$  that  $G$  holds of, i.e.  $|f(G)| \geq 1$ . Reordering the existential quantifiers does not change this interpretation:  $(\exists y)(\exists x)G(x, y)$ .

On the other hand, when the quantifiers are different, the interpretation changes depending on which quantifier comes first. Example (5.1c) is true just in case every member of  $D$  can occur as the first member of at least one of the ordered pairs of  $f(G)$ . Reversing the quantifiers produces a *different* interpretation. Thus  $(\exists y)(\forall x)G(x, y)$  means that there is at least one element  $y$  that can occur as the second member of a pair with *all* elements.

These different interpretations can be depicted below. The required interpretation for (5.1c) is this:

$$(5.2) \quad G(m, ?) \quad G(n, ?) \quad G(\eta, ?)$$

It doesn't matter what the second member of each of these pairs is, as long as  $f(G)$  includes at least three with these first elements.

When we reverse the quantifiers, we must have one of the following three situations: a, b, or c.

- (5.3)   a.    $G(m, m)$     $G(n, m)$     $G(\eta, m)$   
           b.    $G(m, n)$     $G(n, n)$     $G(\eta, n)$   
           c.    $G(m, \eta)$     $G(n, \eta)$     $G(\eta, \eta)$

All members of  $D$  must be paired with some unique member of  $D$ .

The interpretation of (5.1d) is similar to the interpretation of (5.1c) with reversed quantifiers; it is true just in case there is some element that can be paired with every element of  $D$  as the second member of the ordered pair. We must have  $m$ ,  $n$ , and  $\eta$  as the second member of the ordered pairs, but *the first member must be the same across all three*.

- (5.4)   a.    $G(m, m)$     $G(m, n)$     $G(m, \eta)$   
           b.    $G(n, m)$     $G(n, n)$     $G(n, \eta)$   
           c.    $G(\eta, m)$     $G(\eta, n)$     $G(\eta, \eta)$

Some unique member of  $D$  must be paired with all members of  $D$ .

Reversing the quantifiers in (5.1d),  $(\forall y)(\exists x)G(x, y)$ , is interpreted like this.

- (5.5)    $G(?, m)$     $G(?, n)$     $G(?, \eta)$

Every member of  $D$  must occur as the second member of some pair in  $f(G)$ .

The interpretation of quantifier scope can get quite tricky in more complex WFFs.

Finally, notice that this system provides no interpretation for unquantified or *free* variables. Thus an expression like  $G(x)$  has no interpretation. This would seem to be as it should be.

## 5.3 Laws and Rules

We can reason over formulas with quantifiers, but we need some additional Laws and Rules of Inference.

### 5.3.1 Laws

As we've already seen, quantifiers can distribute over logical connectives in various ways. The Laws of Quantifier Distribution capture the relationships between the quantifiers and conjunction and disjunction.

#### (5.6) Laws of Quantifier Distribution

Law 1: QD1  $\neg(\forall x)\varphi(x) \iff (\exists x)\neg\varphi(x)$

Law 2: QD2  $(\forall x)(\varphi(x) \wedge \psi(x)) \iff ((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$

Law 3: QD3  $(\exists x)(\varphi(x) \vee \psi(x)) \iff ((\exists x)\varphi(x) \vee (\exists x)\psi(x))$

Law 4: QD4  $((\forall x)\varphi(x) \vee (\forall x)\psi(x)) \implies (\forall x)(\varphi(x) \vee \psi(x))$

Law 5: QD5  $(\exists x)(\varphi(x) \wedge \psi(x)) \implies ((\exists x)\varphi(x) \wedge (\exists x)\psi(x))$

Consider first Law 1 (QD1). This says that that if  $\varphi(x)$  is not universally true, then there must be at least one element for which  $\varphi(x)$  is not true. This actually follows directly from what we have said above and the fact that the set-theoretic equivalent of negation is complement. The set-theoretic translation of  $\neg(\forall x)\varphi(x)$  is  $\neg(f(\varphi) = D)$ . If this is true, then it follows that the complement of  $f(\varphi)$  contains at least one element. The model-theoretic interpretation of the right side of the first law says just this:  $f(\varphi)' \geq 1$ .

Law 2 allows us to move a universal quantifier down into a conjunction. The logic is that if something true for a whole set of predicates, then it is true for each individual predicate. Law 3 allows us to move an existential quantifier down into a disjunction. The logic is that if something is true for at least one of a set of predicates, then it is true for at least one of them, each considered independently. Both Law 1 and Law 2 should be expected given the example we went through in the previous section to explain the relationship of quantifiers and connectives. In general, the relationships above can be made sense of if we think of the universal quantifier as a conjunction of all the elements of the model  $D$  and the existential quantifier as a disjunction of all the elements of the model  $D$ . Thus:

$$(\forall x)F(x) = \bigwedge_{i=1}^{|D|} F(x_i)$$

The big wedge symbol is interpreted as indicating that every element  $x$  in the superscripted set  $D$  is conjoined together. In other words,  $(\forall x)F(x)$  is



true just in case we apply  $F$  to every member of  $D$  and conjoin the values. The same is true for the existential quantifier:

$$(\exists x)F(x) = \bigvee_{i=1}^{|D|} F(x_i)$$

The big ‘ $\vee$ ’ symbol is interpreted as indicating that every element  $x$  in the superscripted set  $D$  is disjoined together. In other words,  $(\exists x)F(x)$  is true just in case we apply  $F$  to every member of  $D$  and disjoin the values.

Given the associativity of disjunction and conjunction (4.33), it follows that the universal quantifier can raise and lower into a conjunction and the existential quantifier can raise and lower into a disjunction.

For example, imagine that our universe  $D$  is composed of only two elements  $a$  and  $b$ . It would then follow that an expression like

$$(\forall x)(F(x) \wedge G(x))$$

is equivalent to

$$((F(a) \wedge G(a)) \wedge (F(b) \wedge G(b)))$$

Using Associativity, we can move terms around to produce

$$((F(a) \wedge F(b)) \wedge (G(a) \wedge G(b)))$$

Translating each conjunct back, this is equivalent to

$$((\forall x)F(x) \wedge (\forall x)G(x))$$

Law 1 can also be cast in these terms, given DeMorgan’s Law (4.38). Thus

$$\neg(\forall x)F(x) \iff (\exists x)\neg F(x)$$

is really the same thing as:

$$\neg \bigwedge_{i=1}^{|D|} F(x_i) \iff \bigvee_{i=1}^{|D|} \neg F(x_i)$$

The only difference is that the quantifiers range over a whole set of values from  $D$ , not just a pair of values.

Law 4 says that a universal quantifier can be raised out of a disjunction. This is a logical consequence, not a logical equivalence. Thus, if we know that  $((\forall x)G(x) \vee (\forall x)H(x))$ , then we know  $(\forall x)(G(x) \vee H(x))$ , but not vice versa. For example, if we know that everybody in the room either all likes logic or all likes rock climbing, then we know that everybody in the room either likes logic or likes rock climbing. However, if we know the latter, we *cannot* conclude the former. The latter is consistent with a situation where some people like logic, but other people like rock climbing. The former does not have this interpretation. Casting this in model-theoretic terms, if we have that  $(f(G) = D \vee f(H) = D)$ , then we have  $(f(G) \cup f(H)) = D$ .

Law 5 says that an existential quantifier can be lowered into a conjunction. As with Law 4, this is a logical consequence, not a logical equivalence. Thus, if we know  $(\exists x)(G(x) \wedge H(x))$ , then we know  $((\exists x)G(x) \wedge (\exists x)H(x))$ , but not vice versa. For example, if we know that there is at least one person in the room that either likes logic or likes rock climbing, then we know that at least one person in the room likes logic or at least one person in the room likes rock climbing. The converse implication does not hold. From the fact that somebody likes logic or somebody likes rock climbing, it does not follow that that is the *same* somebody. Casting this in model-theoretic terms, if we have that  $(f(G) \cap f(H)) \geq 1$ , then we have  $(f(G) \geq 1 \wedge f(H) \geq 1)$ .

Now consider the Laws of Quantifier Scope. These govern when quantifier scope is and is not relevant.

(5.7) Laws of Quantifier Scope

$$\text{Law 6: QS6} \quad (\forall x)(\forall y)\varphi(x, y) \iff (\forall y)(\forall x)\varphi(x, y)$$

$$\text{Law 7: QS7} \quad (\exists x)(\exists y)\varphi(x, y) \iff (\exists y)(\exists x)\varphi(x, y)$$

$$\text{Law 8: QS8} \quad (\exists x)(\forall y)\varphi(x, y) \implies (\forall y)(\exists x)\varphi(x, y)$$

The first two are straightforward. The first says the relative scope of two universal quantifiers is irrelevant. The second says the relative scope of two existential quantifiers is irrelevant.

The last reflects an implicational relationship between the antecedent and the consequent. The antecedent is true just in case there is some  $x$  that bears  $\varphi$  to every  $y$ . The consequent is true just in case for every  $y$  there is at least one  $x$ , not necessarily the same one, that it bears  $\varphi$  to.

### 5.3.2 Rules of Inference

There are four Rules of Inference for adding and removing quantifiers. You can use these to convert quantified expressions into simpler expressions with constants, that our existing Laws and Rules will apply to, and then convert them back. These are thus *extremely* useful.

Universal Instantiation (U.I.) allows us to replace a universal quantifier with *any* arbitrary constant.

(5.8) Universal Instantiation (U.I.)

$$\frac{(\forall x)\varphi(x)}{\therefore \varphi(c)}$$

The intuition is that if  $\varphi$  is true of everything, then it is true of any individual thing we might cite.

Universal Generalization (U.G.) allows us to assume an arbitrary individual  $v$  and establish some fact about it. If something is true of  $v$ , then it must be true of anything. Hence,  $v$  can be replaced with a universal quantifier.

(5.9) Universal Generalization (U.G.)

$$\frac{\varphi(v)}{\therefore (\forall x)\varphi(x)}$$

The constant  $v$  is special; only it can be replaced with the universal quantifier. The intuition here is that if we establish some property holds of an arbitrary individual, then it must hold of *all* individuals. Notice that Universal Generalization (5.9) can proceed *only* from  $v$ , but Universal Instantiation (5.8) can instantiate to *any* constant, including  $v$ .

Existential Generalization (E.G.) allows us to proceed from *any* constant to an existential quantifier.

(5.10) Existential Generalization (E.G.)

$$\therefore \frac{\varphi(c)}{(\exists x)\varphi(x)}$$

Thus if some property holds of some specific individual, we can conclude that it holds of at least one individual.

Finally, Existential Instantiation (E.I.) allows us to go from an existential quantifier to a constant, as long as the constant has not been used yet in the proof. It must be a new constant.

(5.11) Existential Instantiation (E.I.)

$$\therefore \frac{(\exists x)\varphi(x)}{\varphi(w)}$$

where  $w$  is a new constant

This one is a bit tricky to state in intuitive terms. The basic idea is that if we know that some property holds of at least one individual, we can name that individual (as long as we don't use a name we already know).

## 5.4 Proofs

Let's now look at some simple proofs using this new machinery. First, we consider an example of Universal Instantiation. We prove  $H(a)$  from  $G(a)$  and  $(\forall x)(G(x) \rightarrow H(x))$ .

(5.12)	1	$(\forall x)(G(x) \rightarrow H(x))$	Given
	2	$G(a)$	Given
	3	$G(a) \rightarrow H(a)$	1, U.I.
	4	$H(a)$	2,3, M.P.

First, we remove the universal quantifier with Universal Instantiation and then use *Modus Ponens* to get the desired conclusion.

Next, we have an example of Universal Generalization. We try to prove  $(\forall x)(R(x) \rightarrow W(x))$  from  $(\forall x)(R(x) \rightarrow Q(x))$  and  $(\forall x)(Q(x) \rightarrow W(x))$ .

(5.13)	1	$(\forall x)(R(x) \rightarrow Q(x))$	Given
	2	$(\forall x)(Q(x) \rightarrow W(x))$	Given
	3	$(R(v) \rightarrow Q(v))$	1, U.I.
	4	$(Q(v) \rightarrow W(v))$	2, U.I.
	5	$(R(v) \rightarrow W(v))$	3,4 H.S.
	6	$(\forall x)(R(x) \rightarrow W(x))$	5 U.G.

First, we use Universal Instantiation on the two initial assumptions, mas-saging them into a form appropriate for Hypothetical Syllogism. We then use Universal Generalization to convert that result back into a universally quantified expression. Notice how we judiciously chose to instantiate to  $v$ , anticipating that we would be using Universal Generalization later.

Finally, we consider a case of Existential Instantiation (E.I.). We prove  $((\exists x)P(x) \wedge (\exists x)Q(x))$  from  $(\exists x)(P(x) \wedge Q(x))$ .

(5.14)	1	$(\exists x)(P(x) \wedge Q(x))$	Given
	2	$(P(w) \wedge Q(w))$	1 E.I.
	3	$P(w)$	2 Simp.
	4	$(\exists x)P(x)$	3 E.G.
	5	$Q(w)$	2 Simp.
	6	$(\exists x)Q(x)$	5 E.G.
	7	$((\exists x)P(x) \wedge (\exists x)Q(x))$	4,6 Conj.

First, we use Existential Instantiation to strip the quantifier and replace the variable with a *new* constant. We then split off the conjuncts with Simplification and use Existential Generalization on each to add separate new existential quantifiers. We then conjoin the results with Conjunction.

Notice that the basic strategy in most proofs is fairly clear. Simplify the initial formulas so that quantifiers can be removed. Manipulate the instantiated formulas using the Laws and Rules from the preceding chapter. Finally, generalize to appropriate quantifiers.

### 5.4.1 Indirect Proof

We can use our other proof techniques with predicate logic too. Here we show how to use indirect proof. We prove  $((\forall x)G(x) \rightarrow (\exists x)G(x))$  from no

assumptions.

(5.15)	1	$\neg((\forall x)G(x) \rightarrow (\exists x)G(x))$	Auxiliary Premise
	2	$\neg\neg((\forall x)G(x) \wedge \neg(\exists x)G(x))$	1 Cond.
	3	$((\forall x)G(x) \wedge \neg(\exists x)G(x))$	2 Compl.
	4	$(\forall x)G(x)$	3 Simp.
	5	$\neg(\exists x)G(x)$	3 Simp.
	6	$(\forall x)\neg G(x)$	5 Law 1
	7	$G(a)$	4 U.I.
	8	$\neg G(a)$	6 U.I.
	9	$(G(a) \wedge \neg G(a))$	7,8 Add.
	10	$((\forall x)G(x) \rightarrow (\exists x)G(x))$	1–8 Indirect Proof

We start off by negating our conclusion and then attempting to produce a contradiction. The basic idea is to convert the negated conditional into a conjunction. We then extract the conjuncts. We use Law 1 to shift the negation inward with the second conjunct and then we instantiate both. We conjoin them into a single contradiction and that completes the indirect proof.

### 5.4.2 Conditional Proof

We can prove the same thing by Conditional Proof. We assume the antecedent  $(\forall x)G(x)$  and then attempt to prove the consequent  $(\exists x)G(x)$  from it.

(5.16)	1	$(\forall x)G(x)$	Auxiliary Premise
	2	$G(a)$	1 U.I.
	3	$(\exists x)G(x)$	2 E.G.
	4	$((\forall x)G(x) \rightarrow (\exists x)G(x))$	1–3 Conditional Proof

We begin by assuming the antecedent. From that assumption, we can instantiate to a constant and then generalize to an existential quantifier. That completes the proof.

## 5.5 Summary

In this chapter, we have treated the basics of predicate logic, covering syntax, semantics, and proof mechanisms.

We began with an introduction of the basic syntax of the system. Well-formed formulas of predicate logic (WFFs) are built on well-formed atomic statements. Atomic statements are built up from a finite alphabet of (lower-case) letters and a potentially infinite number of primes, e.g.  $p, q, r, p', q', q''$ , etc.

These, in turn, are combined via a restricted set of connectives into well-formed formulas. There are three important differences with respect to simple sentential logic. First, we have predicate symbols, e.g.  $F(a)$  or  $G(b, c)$ , etc. In addition, we have the universal and existential quantifiers:  $\forall$  and  $\exists$ . Finally, we have a notion of *scope* with respect to quantifiers which is important in the semantics of predicate logic.

The semantics of predicate logic is more complex than that of sentential logic. Specifically, formulas are true or false with respect to a model, where a model is a set of individuals and a mapping from elements of the syntax to sets of elements drawn from the set of elements in the model.

Quantifiers control how many individuals must be in the range of the mapping. For example,  $(\exists x)F(x)$  is true only if the predicate  $F$  is mapped to at least one individual;  $(\forall x)G(x)$  is true only if the predicate  $G$  is mapped to all individuals. These restrictions hold for any predicate in the scope of the quantifier with an as yet free variable.

All the Laws and Rules of Inference of sentential logic apply to predicate logic as well. However, there are additional Laws and Rules that govern quantifiers. The Laws of Quantifier Distribution and Scope govern the relations between quantifiers and between quantifiers and connectives. The Rules of Instantiation and Generalization govern how quantifiers can be added to or removed from formulas.

The chapter concluded with demonstrations of these various Laws and Rules in proofs. We also showed how Conditional Proof and Indirect Proof techniques are applicable in predicate logic.

## 5.6 Exercises

1. Identify the errors in the following WFFs:

- (a)  $(\forall x)G(y) \rightarrow H(x)$
- (b)  $(\forall z)(F(z) \iff G(z))$
- (c)  $(F(x) \wedge G(y) \wedge H(z))$
- (d)  $F(X')$
- (e)  $\neg(\neg(\forall z)(F(x) \vee K(w)))$
- (f)  $(F(x) \leftarrow \neg F(x))$

2. In the following WFFs, mark the scope of each quantifier with labelled underlining.

- (a)  $(\forall x)(\forall y)(\forall z)F(a)$
- (b)  $((\forall x)F(y) \wedge (\exists y)G(x))$
- (c)  $(\exists x)((\forall y)F(x) \leftrightarrow F(y))$
- (d)  $(\exists z)(F(a) \wedge (F(b) \wedge (G(c) \rightarrow F(z))))$
- (e)  $\neg(\exists y)\neg\neg F(y)$

3. For the following questions, assume this model:

$$\begin{aligned}
 D &= \{canto, cantas, canta, cantamos, cantan\} \\
 f(\text{FIRST}) &= \{canto, cantamos\} \\
 f(\text{SECOND}) &= \{cantas\} \\
 f(\text{THIRD}) &= \{canta, cantan\} \\
 f(\text{SG}) &= \{canto, cantas, canta\} \\
 f(\text{PL}) &= \{cantamos, cantan\}
 \end{aligned}$$

For each of the following WFFs, indicate whether it is true or false with respect to this model.

- (a)  $(\exists x)\text{FIRST}(x)$
- (b)  $(\forall x)\text{FIRST}(x)$
- (c)  $(\forall x)(\text{SECOND}(x) \vee \text{PL}(x))$
- (d)  $(\forall y)(\text{SG}(x) \vee \text{PL}(x))$



(e)  $(\exists z)(\text{SG}(z) \wedge \text{SECOND}(z))$

4. Prove  $\neg(\forall z)(F(z) \wedge G(z))$  from  $(\neg F(a) \vee \neg G(a))$ .
5. Prove  $G(a)$  from  $(\exists x)(G(x) \wedge F(x))$ .
6. Prove  $(\forall x)(\exists y)G(x, y)$  from  $(\forall z)(\forall x)G(x, z)$ .
7. Prove that  $(\forall x)(G(x) \vee \neg G(x))$  is a tautology.
8. Prove that  $((\forall x)F(x) \rightarrow (\exists x)F(x))$  is a tautology.
9. Prove  $F(a)$  from  $(\forall x)F(x)$  using Indirect Proof.
10. Prove  $\neg(\forall x)F(x)$  from  $\neg F(a)$  using Indirect Proof.
11. Use Conditional Proof to prove  $((\exists x)F(x) \rightarrow (\exists x)G(x))$  from the WFFs  $((\exists x)F(x) \rightarrow (\forall z)H(z))$  and  $(H(a) \rightarrow G(b))$ .
12. Construct a set of predicates and a model for a *small* area of language.