# Stochastic Calculus

Let the time interval be $[0,T]$ and its ~~partition~~ one of the partition is

$$\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\} . \quad \text{— (1)}$$

Now $\Pi$ is collection of all such partition i.e. $\pi \in \Pi$.

Now $\|\pi\| = \max_{0 \le k \le n-1} (t_{k+1} - t_k)$ where $\pi$ is an arbitrary partition. — (2)

The quadratic variation for B.M. $\{W(t) : t \ge 0\}$ over the interval $[0,T]$ is denoted by $[W,W](T)$ and is given by

$$[W,W](T) = \lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))^2 = \lim_{\|\pi\| \to 0} Q_\pi \quad \text{— (3)}$$

where $$Q_\pi = \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))^2 . \quad \text{— (4)}$$

(note: as $n \to \infty \Rightarrow \|\pi\| \to 0$ )

*Let $\{X_n, n \ge 1\}$ and $X$ be random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. We say that $X_n$ converges to $X$ in mean square sense (in $L^2$ sense) if*

$$\lim_{n\to\infty} E(|X_n - X|^2) = 0 .$$

**Theorem 9.3.1** *Let $Q_\pi$ be defined as in (9.3). Then*

*(i)* $E(Q_\pi) = T$,

*(ii)* $Var(Q_\pi) \le 2 \parallel \pi \parallel T$ .

**Proof.**

(i) We have

$$E(Q_\pi) = \sum_{i=0}^{n-1} E\left(W(t_{i+1}) - W(t_i)\right)^2 . \tag{9.5}$$

Since, for fixed $i$, $W(t_{i+1}) - W(t_i)$ has normal distribution with mean zero and variance $(t_{i+1} - t_i)$, equation (9.5) gives

$$E(Q_\pi) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) = T .$$

(ii) Next we have

$$Var(Q_\pi) = \sum_{i=0}^{n-1} Var(W(t_{i+1}) - W(t_i))^2 . \tag{9.6}$$

But

$$Var(W(t_{i+1}) - W(t_i))^2 = E(W(t_{i+1}) - W(t_i))^4 - 2E(W(t_{i+1}) - W(t_i))^2 (t_{i+1} - t_i) + (t_{i+1} - t_i)^2 . \tag{9.7}$$

$Var(X) = E(X - E(X))^2$

Since the fourth order moment of normal distribution with mean zero and variance $(t_{i+1} - t_i)$ is $3(t_{i+1} - t_i)^2$ (see Exercise 7.12 in Chapter 7), we get from (9.7)

$$\begin{aligned} Var(W(t_{i+1}) - W(t_i))^2 &= 3(t_{i+1} - t_i)^2 - 2(t_{i+1} - t_i)^2 + (t_{i+1} - t_i)^2 \\ &= 2(t_{i+1} - t_i)^2 . \end{aligned} \tag{9.8}$$

Substituting (9.8) in (9.6), we get

$$Var(Q_\pi) = \sum_{i=0}^{n-1} 2(t_{i+1} - t_i)^2$$

$$\leq \sum_{i=0}^{n-1} 2 \parallel \pi \parallel (t_{i+1} - t_i) = 2 \parallel \pi \parallel T \, . \tag{9.9}$$

□

**Remark 9.3.1** *The above theorem tells that for the Brownian motion* $\{W(t), t \geq 0\}$, $[W,W](T) = T$ *for all* $T \geq 0$ *and almost surely. This is because* $Var(Q_\pi) = E(Q_\pi - E(Q_\pi))^2 = E(Q_\pi - T)^2$, *which from (9.9) gives*

$$\lim_{\|\pi\| \to 0} E(Q_\pi - T)^2 = \lim_{\|\pi\| \to 0} Var(Q_\pi) = 0 \, .$$

*Therefore*

$$[W,W](T) = \lim_{\|\pi\| \to 0} Q_\pi = T \, ,$$

*where the limit is understood in the mean square sense because*

$$\lim_{\|\pi\| \to 0} E(|Q_\pi - T|^2) = 0 \, .$$

*Hence we write* $[W,W](T) = T$ *almost surely. Here the terminology almost surely means that there can be some paths of the Brownian motion for which the assertion* $[W,W](T) = T$ *is not true. But the 'the set of all such paths' has zero probability. Though we write* $[W,W](T) = T$, *we must realise that it is to be understood in the sense as described above.*

**Remark 9.3.3** *We know that for the given Brownian motion* $\{W(t), t \geq 0\}$, $[W,W](T) = T$, *i.e.*

$$\lim_{\|\pi\| \to 0} \left( \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))^2 \right) = T \, . \tag{9.10}$$

*Also for* $0 < T_1 < T_2$, $[W,W](T_2) - [W,W](T_1) = T_2 - T_1$, *the Brownian motion accumulates* $(T_2 - T_1)$ *units of quadratic variation over the interval* $[T_1, T_2]$. *Since this is true for every interval, we infer that the Brownian motion accumulates quadratic variation at rate one per unit time. This last statement we write informally as*

$$dW(t) \, dW(t) = dt \, , \tag{9.11}$$

**Theorem 9.3.2** *Let $\{W(t), t \geq 0\}$ be the given Brownian motion and $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ be a partition of $[0, T]$. Then*

*(i)*

$$\lim_{\|\pi\| \to 0} \left( \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))(t_{i+1} - t_i) \right) = 0 ,$$

*(ii)*

$$\lim_{\|\pi\| \to 0} \left( \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \right) = 0 .$$

**Proof.**

(i) We observe that

$$|(W(t_{i+1}) - W(t_i))(t_{i+1} - t_i)| \leq \operatorname*{Max}_{0 \leq i < n} |W(t_{i+1}) - W(t_i)|\, (t_{i+1} - t_i) .$$

Therefore

$$\left| \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))(t_{i+1} - t_i) \right| \leq \operatorname*{Max}_{0 \leq k \leq n} |W(t_{k+1}) - W(t_k)| . T . \tag{9.12}$$

Since $W(t)$ is continuous, the R.H.S of (9.12) goes to zero as $\| \pi \| \to 0$.

(ii) We note that

$$\sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \leq \left( \operatorname*{Max}_{0 \leq k \leq n-1} (t_{i+1} - t_i) \right) . \sum_{i=0}^{n-1} (t_{i+1} - t_i) = \| \pi \| T ,$$

which goes to zero as $\| \pi \| \to 0$.

**Remark 9.3.4** *In view of Theorem 9.3.2 and in analogy with (9.10) and (9.11), we can informally write*

$$dW(t)\, dt = 0 , \quad dt\, dt = 0 .$$

## Stochastic Integral

Let $\{X(t), t \geq 0\}$ be a stochastic process which is adapted to the natural filtration $\{\mathcal{F}_t, t \geq 0\}$ of Wiener process $\{W(t), t \geq 0\}$, i.e. $X(t)$ is $\mathcal{F}_t$-measurable. Let us assume $E\left( \int_0^T X^2(t)\, dt \right) < \infty$.

We next consider a partition $\pi$ of $[0, T]$ where $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$, and form the sum $\sum_{i=0}^{n-1} X(t_i)(W(t_{i+1}) - W(t_i))$. Now if we take the limit of this sum as $\| \pi \| \to 0$, then in analogy with the procedure discussed in Section 9.4 we can write

$$I(T) = \int_0^T X(s)\, dW(s) = \lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} X(t_i)(W(t_{i+1}) - W(t_i)) . \tag{9.17}$$

as the definition of *Stochastic integral* or *Ito integral* of the stochastic process $\{X(t), t \geq 0\}$ with respect to the Brownian motion $\{W(t), t \geq 0\}$.

**Example 9.5.1** *Evaluate the Ito integral*

$$\int_0^T W(s)\,dW(s)\,.$$

**Solution** Let $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ be an arbitrary partition of $[0, T]$. We have

$$\int_0^T W(s)\,dW(s) = \lim_{\|\pi\|\to 0} \sum_{i=0}^{n-1} W(t_i)\left(W(t_{i+1}) - W(t_i)\right)\,. \tag{9.18}$$

But, for each $i$, $W(t_i)$ and $W(t_{i+1}) - W(t_i)$ are independent random variables and are having normal distributions. Hence, the right hand side terms within the summation are nothing but the sum of independent random variables. Hence, the integral is nothing but the limit of sum of such random variables. Now, we have

$$\begin{aligned} Q_\pi &= \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))^2 \\ &= \sum_{i=0}^{n-1} \left(W^2(t_{i+1}) - W(t_i)^2 - 2W(t_i)(W(t_{i+1}) - W(t_i))\right) \\ &= W^2(T) - W^2(0) - 2\sum_{i=0}^{n-1} W(t_i)(W(t_{i+1}) - W(t_i))\,, \end{aligned}$$

i.e.

$$\sum_{i=0}^{n-1} W(t_i)\left(W(t_{i+1}) - W(t_i)\right) = \frac{1}{2}\left[W^2(T) - W^2(0) - Q_\pi\right]. \tag{9.19}$$

Now taking limit as $\| \pi \| \to 0$ and using $E(Q_\pi) = T$, we get

$$\int_0^T W(s)\,dW(s) = \frac{W^2(T) - T}{2}\,.$$

**Example 9.5.2** *Evaluate*

$$\int_0^t W(1)\,dW(s), \quad 0 \le t \le 1\,.$$

Imp

**Solution** Note that, $W(1)$ is not adapted to the filtration $\sigma\{W(s), 0 < s \le t\}$, $0 \le t \le 1$, because it depends on future events. Hence, this Ito integral does not exist. This example shows that, assumption of the integrand adapted to the filtration $\{\mathcal{F}_t, t \ge 0\}$ is needed to have existence of the Ito integral.

## Ito-Doeblin Formula for Brownian Motion: First Version

Let $f$ be at least twice continuously differentiable function of $x$ and $\{W(t), t \geq 0\}$ be a Weiner process. Then

$$df(W(t)) = f'(W(t))\, dW(t) + \frac{1}{2} f''(W(t))\, dt\,, \tag{9.20}$$

or equivalently

$$f(W(t)) = f(W(0)) + \int_0^t f'(W(u))\, dW(u) + \frac{1}{2}\int_0^t f''(W(u))\, du\,. \tag{9.21}$$

Here the first integral in (9.21) is an Ito integral whereas the second integral in (9.21) is a Reimann integral.

**Example 9.6.1.** Find $\int_0^T W(t)\, dW(t)$ using the Ito-Doeblin formula.

**Solution** Consider the first version of the Ito-Doeblin formula

$$df(W(t)) = f'(W(t))\, dW(t) + \frac{1}{2} f''(W(t))\, dt\,,$$

or equivalently

$$f(W(T)) - f(W(0)) = \int_0^T f'(W(t))\, dW(t) + \frac{1}{2}\int_0^T f''(W(t))\, dt\,.$$

Taking motivation from the fact that $\int_0^t x\, dx = t^2/2$, we choose $f(x) = x^2/2$. This gives $f'(x) = x$ and $f''(x) = 1$. Hence the above Ito-Doeblin formula gives

$$\frac{W^2(T)}{2} - 0 = \int_0^T W(t)\, dW(t) + \frac{1}{2}\int_0^T dt\,.$$

Therefore

$$\int_0^T W(t)\, dW(t) = \frac{W^2(T) - T}{2}\,. \tag{9.22}$$

***Check the solution we have obtained using quadratic variation (ex. 9.5.1)***

**Ito-Doeblin Formula for Brownian Motion: Second Version**

Let $f(t,x)$ have continuous partial derivatives of at least second order and $\{W(t), t \geq 0\}$ be the given Weiner process. Then

$$df(t, W(t)) = f_t(t, W(t))\, dt + f_x(t, W(t))\, dW(t) + \frac{1}{2} f_{xx}(t, W(t))\, dt\,,$$

or equivalently where $x = W(t)$

$$f(t, W(t)) - f(0, W(0)) = \int_0^t \left[ f_t(u, W(u)) + \frac{1}{2} f_{xx}(u, W(u)) \right] du + \int_0^t f_x(u, W(u))\, dW(u)\,.$$

This formula can again be justified by considering the classical Taylor's expansion for a function of two variables. In particular we may take

$$\begin{aligned} f(t + \Delta t, W(t + \Delta t)) - f(t, W(t)) = &\left[ \frac{\partial f(t, W(t))}{\partial t} dt + \frac{\partial f(t, W(t))}{\partial x} dW(t) \right] \\ &+ \frac{1}{2}\left[ \frac{\partial^2 f(t, W(t))}{\partial t^2} (dt)^2 + 2\frac{\partial^2 f(t, W(t))}{\partial t \partial x} dt\, dW(t) \right. \\ &\left. + \frac{\partial^2 f(t, W(t))}{\partial x^2} dW(t)\, dW(t) \right] + \dots \end{aligned}$$

$$+\frac{\partial^2 f(t,W(t))}{\partial x^2}\, dW(t)\, dW(t)\Big] + \dots$$

But since we have $dW(t)\,dW(t) = dt$, $dW(t)\,dt = 0$ and $dt\,dt = 0$, therefore,

$$f(t+\Delta t, W(t+\Delta t)) - f(t,W(t)) = \left[\frac{\partial f(t,W(t))}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t,W(t))}{\partial x^2}\right] dt + \frac{\partial f(t,W(t))}{\partial x}\, dW(t),$$

where

$$\frac{\partial f(t,W(t))}{\partial x} = \left.\frac{\partial f(t,x)}{\partial x}\right|_{x=W(t)}, \frac{\partial f(t,W(t))}{\partial t} = \left.\frac{\partial f(t,x)}{\partial t}\right|_{x=W(t)}, \frac{\partial^2 f(t,W(t))}{\partial x^2} = \left.\frac{\partial^2 f(t,x)}{\partial^2 x}\right|_{x=W(t)}$$

**Example 9.6.2** *Using second version of the Ito-Doeblin formula, find* $\int_0^T W(t)\, dW(t)$.

**Solution** For any $T > 0$, we have

$$f(T,W(T)) - f(0,W(0)) = \int_0^T \left[f_t(u,W(u)) + \frac{1}{2} f_{xx}(u,W(u))\right] du + \int_0^T f_x(u,W(u))\, dW(u)\,.$$

Choose $f(t,x) = \frac{x^2}{2}$. Then $f_x(t,x) = x$, $f_t(t,x) = 0$ and $f_{xx}(t,x) = 1$. Substituting, we get

$$\frac{W^2(T)}{2} - 0 = \int_0^T \left(0 + \frac{1}{2}\right) du + \int_0^T W(u)\, dW(u)$$

$$\int_0^T W(t)\, dW(t) = \frac{W^2(T)}{2} - \frac{T}{2}.$$

## Stochastic Differential Equation

Let us consider the initial value problem

$$\frac{dx(t)}{dt} = f(t,x(t)), \quad t \in [0,T], \quad x(0) = x_0\,, \qquad (9.23)$$

where $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is a continuous function.

This ODE possess a solution

$$x(t) = x_0 + \int_0^t f(s,x(s))\, ds\,.$$

Provided Lipschitz condition is satisfied by '$f$' i.e, if there exist a $k > 0$, such that

$$|\, f(t,x) - f(t,y)\,| \;\le\; k\,|\,x - y\,| \quad \text{for all } t \in [0,T], \quad x,y \in \mathbf{R}\,.$$

Now suppose $x_0$ or $f$ is random then solution is not unique rather it will depend on the value $\omega \in \Omega$ (sample space)

i.e $\{X(t,\omega(t)) ; \omega \in \Omega ; t \in [0,T]\}$ which becomes an S.P. and such d.e is called random differential equation.

Adding an uncertainties by way differential of B.M..

We get

$$\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t)) \frac{dW(t)}{dt}, (0 \le t \le T),$$

where $b : [0, T] \times \mathbf{R} \to \mathbf{R}$ and $\sigma : [0, T] \times \mathbf{R} \to \mathbf{R}$ are two given functions. The above equation can also be symbolically written

$$dX(t) = b(t, X(t))\, dt + \sigma(t, X(t))\, dW(t) . \tag{9.25}$$

Equation 9.25 is Stochastic Differential Equation. It can equivalently be written as

$$X(t) = X(0) + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s), (0 < t \le T) . \tag{9.26}$$

It is called Stochastic Integral Equation.

**1. Strong solution**

A strong solution to the SDE (9.25) is a stochastic process $\{X(t); t \in [0, T]\}$ which satisfies the following

(i) $\{X(t), t \in [0, T]\}$ is adapted to the Brownian motion, i.e. at time $t$ it is a function of $W(s)$, $s \le t$.

(ii) The integrals in (9.26) are well defined and $\{X(t); t \in [0, T]\}$ satisfies the same.

(iii) $\{X(t); t \in [0, T]\}$ is a function of the underlying Brownian sample path and of the coefficients $b(t, x)$ and $\sigma(t, x)$.

Thus a strong solution is an explicit function $f$ such that $X(t) = f(t, W(s) : s \le t)$.

A strong solution to (9.26) is based on the path of the underlying Brownian motion. The solution $\{X(t); t \in [0, T]\}$ is said to be unique strong solution if given any other solution $\{Y(t), t \in [0, T]\}$, $P(X(t) = Y(t)) = 1$ for all $t \in [0, T]$

**2. Weak solution**

For a weak solution, the path behaviour is not essential. Hence we are only interested in the distribution of $X$. Thus weak solutions are sufficient to determine the expectation, variance and covariance functions of the process. In this case we do not have to know the sample paths of $X$.

A strong or weak solution $X$ of the given SDE is called a *diffusion*. We may note that Brownian motion is also a diffusion process because in (9.26) we can take $b(t, x) = 0$ and $\sigma(t, x) = 1$.

**Theorem 9.7.1 (Existence Theorem)** *Let $E(X^2(0)) < \infty$ and $X(0)$ be independent of $\{W(t), t \ge 0\}$. Let for all $t \in [0, T]$ and $x, y \in \mathbf{R}$, $b(t, x)$ and $\sigma(t, x)$ be continuous and satisfy Lipschitz condition with respect to second variable, i.e.*

$$| b(t, x) - b(t, y) | + | \sigma(t, x) - \sigma(t, y) | \le K | x - y | .$$

**Example 9.7.1** *Consider the SDE*

$$dX(t) = X(t)\ dW(t), \ \text{with}\ X(0) = 1\ .$$

*Find the strong solution using Ito-Doeblin formula.*

**Solution** We have $b(t,x) = 0$ and $\sigma(t,x) = x$. Hence, Lipschitz condition is satisfied. Now we use the second version of Ito-Doeblin formula with $f(t,x) = e^{(x-\frac{t}{2})}$. This gives

$$\frac{\partial f(t,x)}{\partial t} = \frac{-1}{2}\ e^{x-\frac{t}{2}},\ \frac{\partial f(t,x)}{\partial x} = e^{x-\frac{t}{2}}\ \text{ and }\ \frac{\partial^2 f(t,x)}{\partial x^2} = e^{x-\frac{t}{2}}.$$

On substituting these expressions in the second version of Ito-Doeblin formula, we get

$$\begin{aligned} dX(t) &= df(t, W(t)) \\ &= f_t(t, W(t))\ dt + f_x(t, W(t))\ dW(t) + \frac{1}{2} f_{xx}(t, W(t))\ dt \\ &= \left(-\frac{1}{2}X(t) + \frac{1}{2}X(t)\right)\ dt + X(t)\ dW(t) \\ &= X(t)\ dW(t)\ . \end{aligned}$$

Hence, the required strong solution is $X(t) = e^{W(t)-\frac{t}{2}}$. $\square$

**Example 9.7.2** *A stochastic process* $\{S(t), t \geq 0\}$ *is governed by*

$$dS(t) = a\ S(t)\ dt + bS(t)\ dW(t)$$

*where a and b are constants. Find the SDE of*

(i) $\sqrt{S(t)}$.

(ii) $ln(S(t))$.

**Solution**

(i) Choose $f(x) = x^{1/2}$, then

$$f_t = 0;\ \ f_x = \frac{1}{2\sqrt{x}};\ \ f_{xx} = -\frac{1}{4x^{3/2}}\ .$$

now applying Ito-Doeblin formula, we get

$$d(\sqrt{S(t)}) = f_t\,dt + f_x\,dS(t) + \frac{1}{2}f_{xx}\,dS(t)\,dS(t)$$

$$= 0 + \frac{1}{2\sqrt{S(t)}}dS(t) - \frac{1}{8\sqrt{x^{3/2}}}dS(t)\,dS(t)$$

$$= \frac{1}{2\sqrt{S(t)}}[a\,S(t)\,dt + bS(t)\,dW(t)] - \frac{1}{8\sqrt{(S(t))^{3/2}}}b^2(S(t))^2dt$$

$$= \left(\frac{a}{2} - \frac{b^2}{8}\right)\sqrt{S(t)}\,dt + \frac{b}{2}\sqrt{S(t)}\,dW(t)\,.$$

(ii) Choose $f(x) = ln(x)$, then

$$f_t = 0;\quad f_x = \frac{1}{x};\quad f_{xx} = -\frac{1}{x^2}\,.$$

Apply Ito-Doeblin formula, we get

$$d(ln(S(t))) = f_t\,dt + f_x\,dS_t + \frac{1}{2}f_{xx}\,dS_t\,dS_t$$

$$= 0 + \frac{1}{S(t)}dS(t) - \frac{1}{2(S(t))^2}dS(t)\,dS(t)$$

$$= \frac{1}{S(t)}[a\,S(t)\,dt + bS(t)\,dW(t)] - \frac{1}{2(S(t))^2}b^2(S(t))^2dt$$

$$= \left(a - \frac{b^2}{2}\right)dt + b\,dW(t)\,.$$

## SDE of G.B.M.(Geometric Brownian Motion)

Let $S(t)$ be the stock price at time $t$. Let $-\infty < \mu < \infty$ be the constant growth rate of the stock and $\sigma > 0$ be the volatility. Let us consider the SDE

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t),\quad S(0)\ \text{is known.}$$

We are interested in finding the strong solution of $S(t)$, if it exists. For this we

The condition of existence theorem is verified as μ, σ are constant.

$\mu$ and $\sigma$ are constants. Now we assume that $S(t) = f(t, W(t))$ and make use of the second version of Ito-Doeblin formula. This gives

$$df(t) = f_t\,dt + f_x\,dW(t) + \frac{1}{2}f_{xx}\,dt\,,$$

where $f_t = \dfrac{\partial f(t,x)}{\partial t}$, $f_x = \dfrac{\partial f(t,x)}{\partial x}$ and $f_{xx} = \dfrac{\partial^2 f(t,x)}{\partial x^2}$. Now, on comparing with the given SDE, we get

$$f_x = \sigma f \tag{9.30}$$

$$f_t + \frac{1}{2} f_{xx} = \mu f \ . \tag{9.31}$$

Now solving equation (9.30), we get $f(t,x) = e^{\sigma x} k(t)$ , for some function $k(t)$. From here we get $f_t = k'(t)\ e^{\sigma x}$ and $f_{xx} = \sigma^2 e^{\sigma x}\ k(t)$ which on substituting in equation (9.31), we get

$$k'(t)\ e^{\sigma x} = \left(\mu - \frac{\sigma^2}{2}\right) e^{\sigma x}\ k(t)\ .$$

Solving the above equation, we get

$$k(t) = S(0)\ e^{\left(\mu - \frac{\sigma^2}{2}\right) t} .$$

Therefore, the required solution is

$$S(t) = S(0)\ e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\ W(t)} .$$

Here we may observe that for fixed $t$, $S(t)$ follows lognormal distribution. Hence, it can be verified that

(i) $E(S(t)) = E(S(0))\ e^{\mu t}$.
(ii) $E((S(t))^2) = E(S^2(0))\ e^{(2\mu + \sigma^2)t}$.

$Var(S_T) = E\left({S_T}^2\right) - \{E(S_T)\}^2 = {S_0}^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$