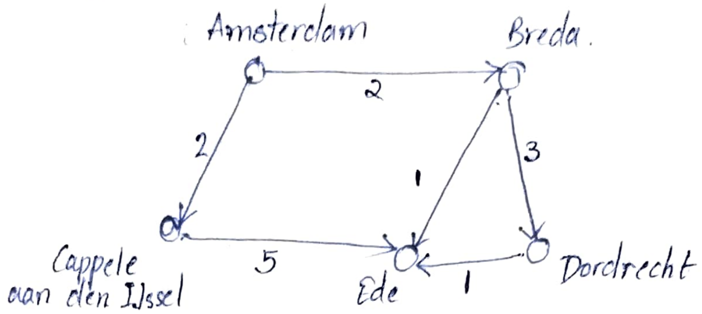
Graph Theory (MC-405)
Assignment-1

DEEPTI SINGH | 2K17-MC-37

1. The given situation can be modelled into the graph given below.



here each node represents a city and each edge $(\vec{u,v})$ represents a highway from city u to v, the weights on the edges represents the no. of lanes on the highway.

2. (i) given series - (6, 6, 5, 4, 3, 3, 2)

$\therefore \Sigma$ degree $= 6+6+5+4+3+3+2$
$\neq$ even

therefore the sequence can't be graphical.

(ii) $\because$ (6, 6, 5, 4, 3, 3, 1)

~~since there's even no. of odd degrees & $\Sigma = 28 < 7(7-1)$ therefore the given sequence can be graphical.~~

using havel hakimi ~~algo~~ theorem,

(6, 6, 5, 4, 3, 3, 1).

↓

(5, 4, 3, 2, 2, 0)

↓

(3, 1, 1, -1)

Since one of the elements becomes negative, we can say that the given deg. sequence is not graphical.

3. The degree sequence (0, 0, 0) is ~~such~~ one such example.

~~Only po~~

4 (a)

$$A = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

since the vertexes (1,4) has more than 1 edge between them therefore the graph is not simple.

(b) degree of vertices

$1 \to 4$
$2 \to 2$
$3 \to 3$
$4 \to 3$
$5 \to 2$

Hence the degree sequence will be ~~2,2,3~~ (4,3,3,2,2)

(c) Some of degree $= 4+2+3+3+2$
$= 14$

$\Rightarrow$ No of edges $=$ ~~do~~ $\dfrac{\text{Total sum of degree}}{2} = \dfrac{14}{2} = 7$

5. Let $d_1, d_2, d_3 \ldots d_n$ be the degree of vertices.

Now we know that $d_1 + d_2 + d_3 + \cdots d_n = 2m$

$\Rightarrow \sum_{i=1}^{N} d_i = 2m$

also $d_1 + d_2 + \cdots d_n \geq \delta + \delta + \cdots \delta$

$\sum_{i=1}^{N} d \geq n\delta.$

$\Rightarrow 2m \geq n\delta$

$\Rightarrow \dfrac{2m}{n} \geq \delta$

or $\delta \leq \dfrac{2m}{n}$ —— (1)

sim, $d_1 + d_2 + \cdots d_n \leq \Delta + \Delta + \cdots \Delta$

$\sum_{i=1}^{N} d. \leq n\Delta$

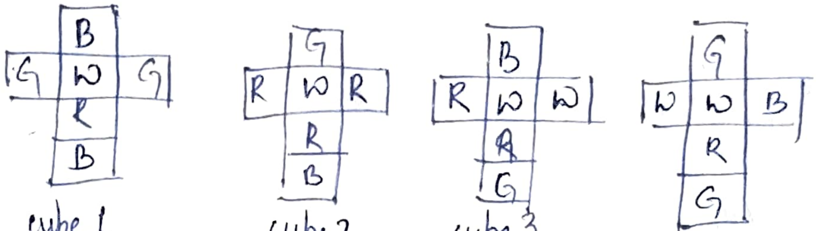
$\Rightarrow n\Delta \geq 2m$

$\Rightarrow \Delta \geq \dfrac{2m}{n}.$ —— (2)

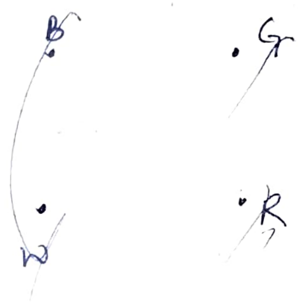
Hence from (1) & (2),

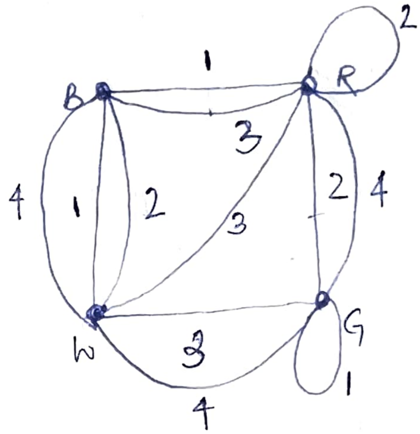
$\delta \leq \dfrac{2m}{n} \leq \Delta$

Sol 7. let's draw a graph with 4 vertices R, W, B & G, where edge u–v represents color u is opposite to v, & the edge label i, representing the ith cube ~~number~~.

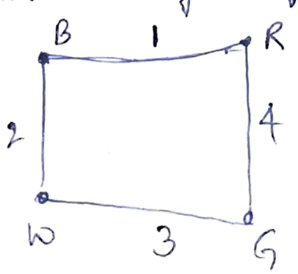


cube 1 cube 2 cube 3 cube 4

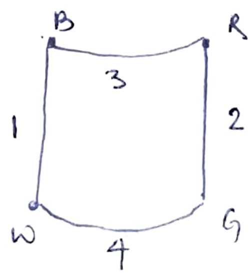




Now in order to have a ~~pile~~ stack of 4 cubes with no two colours appearing twice on either side, we should be able to find two edge disjoint subgraphs of the above graph, where each subgraph contains each of the 4 vertices with different edge labels, also the degree of each vertex should be 2.



(1)



(2)

since (1) & (2) are two such ~~sub~~ ~~di~~ edge disjoint subgraphs, ~~therefore~~ hence the solution.

8. Let the number of vertices in each of the k components of a Graph G be $n_1, n_2 \ldots n_k$, Thus we have -

$$n_1 + n_2 + n_3 + \ldots n_k = n$$

$$n_i \geq 1$$

also
$$\sum_{i=1}^{k} n_i = n.$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1) = n - k$$

squaring both sides.

$$\left(\sum_{i=1}^{K} (n_i - 1)\right)^2 = n^2 + k^2 - 2nk$$

$$\Rightarrow \sum (n_i^2 - 2n_i) + K + \text{non negative cross terms} = n^2 + k^2 - 2nk$$

$$\Rightarrow \sum n_i^2 \leq n^2 + k^2 - 2nk - k + 2n$$

$$\sum n_i^2 \leq n^2 - (k-1)(2n-k) \quad \text{--- (1)}$$

Maximum no. of edges in ith component of $G = \frac{1}{2} n_i (n_i - 1)$

$\therefore$ Maximum no. of edges in $G = \frac{1}{2} \sum_{i=1}^{k} (n_i - 1) n_i$

$\Rightarrow$

$$= \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2}$$

$$\leq \frac{1}{2}\left[n^2 - (k-1)(2n-k)\right] - \frac{n}{2}$$

$$= \frac{(n-k)(n-k+1)}{2}$$

9. Let's say that the ~~best~~ bipartite graph could be divided into two pieces having p & q vertices,

now every vertex from set 1 can have at most q edges.

$\Rightarrow$ sum of degree in set 1 $= p * q = pq$.

simq, sum of degree in set 2 $= qp$

$\Rightarrow$ Total degree $= 2qp$.

$\Rightarrow$ The graph can have at most qp edges.

also since $q + p = n$

$\Rightarrow$ maximum edges $= p(n-p)$

using calculus we can deduce that
$f(p) = p(n-p)$ ~~will be maximu~~ on $[0,n]$ will be maximum
at $p = \frac{n}{2}$ (as $f'(p) = 0$ at $p = n/2$)

$$\Rightarrow \quad f(p)_{max} = \frac{n}{2} \times \frac{n}{2} = \frac{n^2}{4}$$

$$\Rightarrow \quad (m)_{max} = \frac{n^2}{4}$$

$$\Rightarrow \quad m \leq \frac{n^2}{4}$$

10. Let $\{v_1, v_2, v_3 \cdots v_n\}$ ~~be the vert~~, & $\{e_1, e_2 \cdots e_m\}$ be the ~~edge~~ ~~vert~~ vertex set & edge set of the graph G_1

Let G_2 be a complete graph having $\{v_1', v_2', v_3', \cdots v_n'\}$

, ~~since G_2 is a co~~

Let f be a function such that

$$f(v_1) = v_1'$$
$$f(v_2) = v_2'$$
$$\vdots$$
$$f(v_n) = v_n'$$

now, let H be a subgraph of G_2 such that it contains all ~~them~~ the vertices from G_2.

Now in H, we only take edge ~~$\{v_j', v_k'\}$~~ ~~if $\{f^{-1}(v_1'), f^{-1}(v_2')\}$~~ $\{v_j', v_k'\}$ from G_2 is edge $\{v_j, v_k\}$ exists. (where $(i,j) \in [1,n]$)

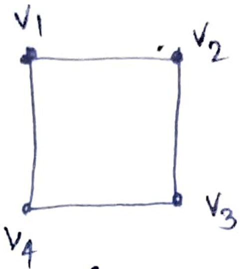
$\Rightarrow$ in H, ~~for each~~ each edge $\{v_j', v_k'\}$ corresponds to $\{v_j, v_k\}$

~~sim $\{v_j, v_k\}$~~ = ~~$\{$~~

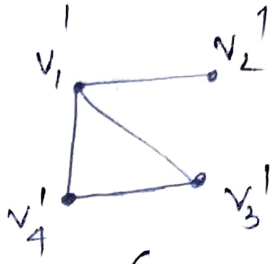
Hence H is isomorphic to G_1

$\Rightarrow$

11. consider the following graph G_1 & G_2



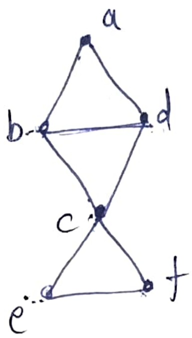
G_1



G_2

Here both G_1 & G_2 have 4 vertices & 4 edges, but they're not isomorphic as ~~their~~ ~~the~~ one of vertex in G_2 is of degree 3 (i.e. v_1') but there's no vertex in G_1 having degree 2.

12.



i) abcefcbd.
walk

b) abcefed
walk, path

e) abcefcdba
walk, closed walk

d) bcefcdb
walk, closed walk, circuit

e) bcdb
walk, closed walk, path, cycle, circuit

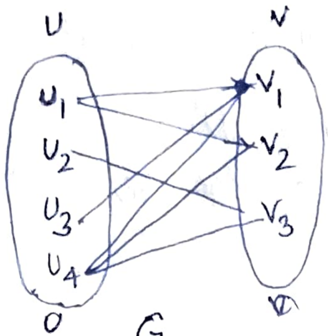
f) abefcd
None.

13. Let's assume that if a knows b then b also knows a.
Now, let's mode the given situation using a graph.
Let the vertex set {1, 2, 3, 4, 5, 6, 7} represent each of the ~~friend~~ person, & edges (u,v) will represent u & v know each other.
Now since each ~~friend knows~~ person knows exactly 3 persons in the group therefore, total degree $= 7 \times 3$
$= 21$

Now, from handshaking lemma, such a graph is not possible & therefore the given situation is not possible.

~~14 Let G be a bip bipartite graph simple graph~~

14 Consider the following simple graph

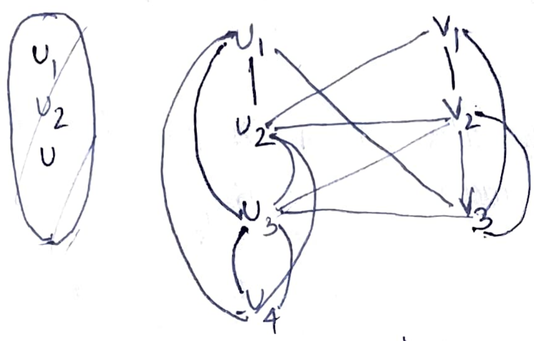


G

here vertex set of $G = \{U\} \cup \{V\}$,

Clearly G is a bipartite graph.

Now complement of G will be -

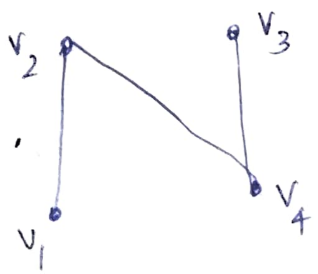


G'

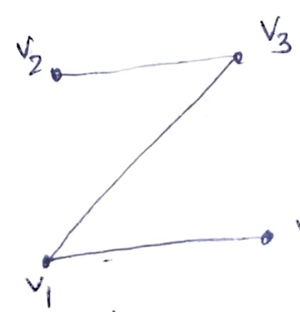
Here no partition of vertex set is possible such that G' is bipartite.

Hence complement of bipartite graph need not be a bipartite graph.

15. Consider the following graph G



G' will be



Now we define a mapping f such that

$f(v_1) = v_2$, $f(v_2) = v_3$, $f(v_3) = v_4$, $f(v_4) = v_1$

Now

for edge $\{v_1, v_2\}$ in G $\rightarrow$ $\{f(v_1), f(v_2)\} = \{v_2, v_3\}$ in G' exist

" $\{v_2, v_4\} \rightarrow \{f(v_2), f(v_4)\} = \{v_3, v_1\}$ "

$\{v_4, v_3\} \rightarrow \{f(v_4), f(v_3)\} = \{v_1, v_4\}$ "

since for each $\{v_i, v_j\}$ in G' $\{f(v_i), f(v_j)\}$ exists

& we can show that for $\forall$ ~~$\{u_i, u_j\}$~~ edge $\{u_i, u_j\}$ in G'

$\{f^{-1}(u_i), f(u_j)\}$ exists

therefor G & G' are isomorphic

Hence G is, G' are self complementary graph.

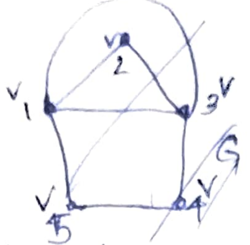
16.

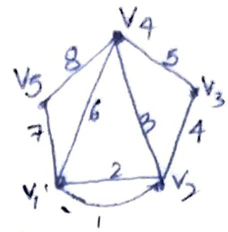
A given connected graph G is Euler graph if and only if all vertices of G are of even degree.

Proof- Suppose G is a Euler Graph, therefore it contains a euler line. So while tracing this walk we observe that ~~ev~~ for every vertex v encountered in the walk, we enter through one edge & exit through the other (even for terminal vertex) therefore every vertex must have even degree.

For sufficiency of the condition → Lets suppose ~~every~~ all vertices of G is are of even degree, so let's start traversing from vertex v & since every vertex is of even degree we can exit from every vertex we enter, so the path will eventually end at v. Now let if this closed walk is h, let say h' is set of remaining edges. & since both G & h have even degree vertices h' also has even degree ~~vertex~~ vertices.

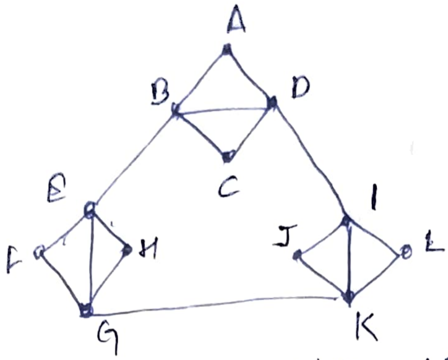
Also h' must touch ~~at~~ h at ~~point a~~ vertex a, since G is connected & so this walk h can be combined with h', we repeat this process until we obtain a close walk that traverse all the edges of G & hence G is a euler graph.





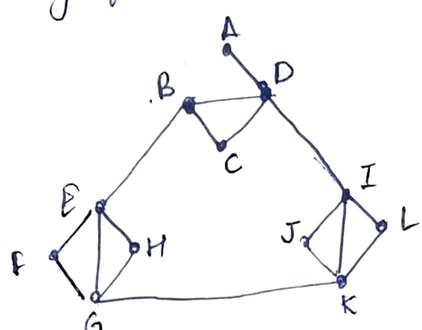
the above graph G, is a euler graph with 5 vertices & 8 edges.

17.



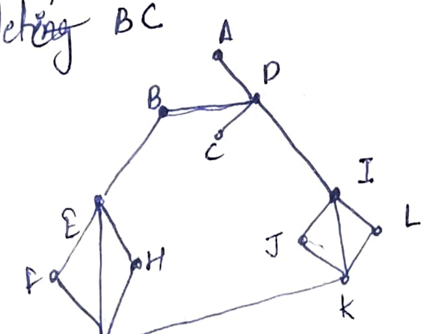
starting from A, deleting AB.

~~path~~ circuit so far : AB



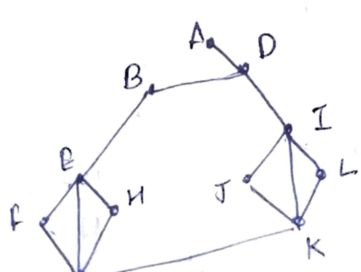
circuit so far : ABC

deleting BC



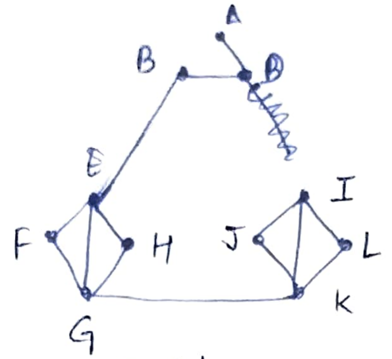
circuit so far : ABCD

delete BCD



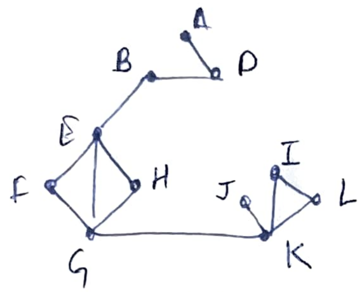
circuit - ABCDI.

delete DI



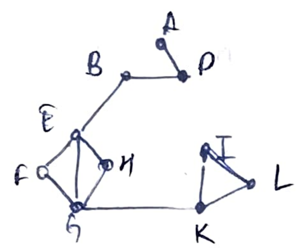
Delete IJ

circuit → ABCDIJ



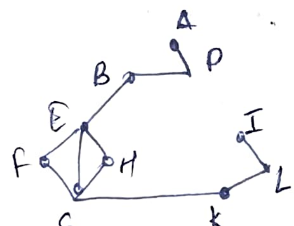
circuit → ABCDIJK

Delete JK



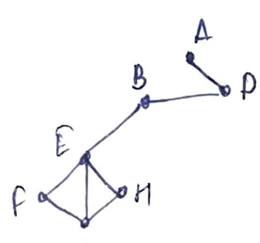
circuil → ABCDIJKI

Delete KI (I<L)



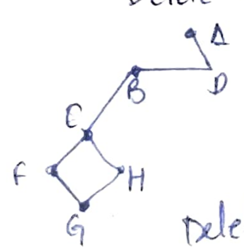
circuit → ABCDIJKILKG

Delete IL &, LK & then KG

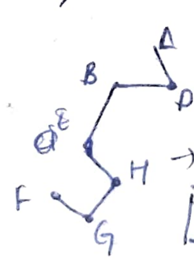


circuit → ABCDIJKILKGE

Delete GE & (E<H, E<F)



→



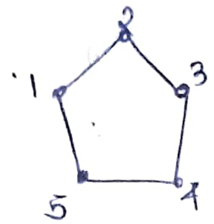
Delete EF

→ now only one way possible to complete the circuit

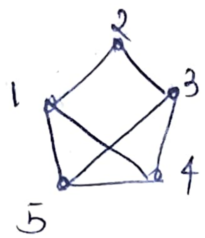
$\boxed{\text{ABCDIJKILKGEFGHEBDA}}$

circuit → ABCDIJKILKGEF

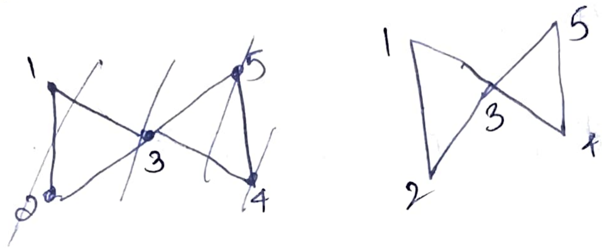
18. (a) Hamilton & Eulerian.



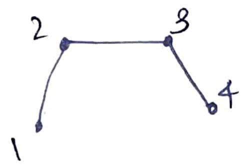
b) Hamilton & non Eulerian



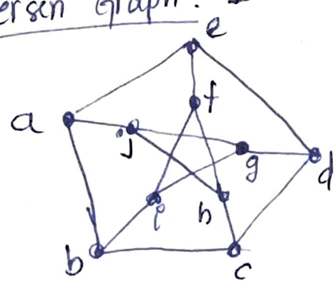
c) Non hamiton & Eulerian



d) Non hamiton & non eulerian.



19. Petersen Graph :-



The petersen graph is an undirected graph having 10 vertices & 15 edges, usually drawn as a pentagram within a pentagon attatched to each corresponding vertices others.

The graph doesn't have a hamiltonian cycle, but it has a hamittonian path, one such path is b a e d c h j i g f

~~22~~ ~~205~~ a simple connected graph with n vertices $n \geq 3$, is Hamitonian if ~~$\deg(v) \geq n/2 \ \forall v \in V(G)$~~ $d(v) + d(w) \geq n$ where v & w are not adjacent.

Proof. Conside a longes path in G: $v_1, v_2, \ldots v_k$.

Suppose for a contradiction, that $k < n$, so there is some vertex w adjacent to one of $v_2, v_3, \ldots, v_{k-1}$ say to v_i. If v_1 is adjacent to v_k then $w, v_i, v_{i+1}, \ldots v_k, v_1, v_2 \ldots v_{i-1}$ is a path of length $k+1$, a contradiction.. Hence, v_1 is not adjacent to v_k & so $d(v_1) + d(v_k) \geq n$. The neighbors of v_1 are among $\{v_2, v_3 \ldots v_{k-1}\}$ as are the neighbors of v_k Consider the vertices

$$w = \{v_{l+1} \mid v_l \text{ is a neighbor of } v_k\}$$

<u>20</u> A simple graph with n vertices & m edges is Hamiltonian if $m \geq (n^2 - 3n + 6)/2$.

Proof.

Let G be a simple graph with n vertices & $m = \cancel{\frac{1}{2}(n-1)(n-2)+2}$ $\frac{n^2-3n+6}{2}$ edges. Let u, v be two non-adjocent vertices of G.

Then, Consider the subgraph H of G induced by the vertices $v_1, v_2, \ldots v_{n-2}$. This is the subgrap containg all edges of G with both endpoints from the set $\{v_1, \ldots v_{n-2}\}$. Since the number of edges in H is at most ${}^{n-2}C_2 = \frac{1}{2}(n-2)(n-3)$

$\Rightarrow$ There's at least $m - {}^{n-2}C_2$ edges in G that ~~are~~ havent endpoints in u or v

$$\Rightarrow \deg(u) + \deg(v) \geq m - {}^{n-2}C_2$$

$$\geq \cancel{\tfrac{1}{2}(n-1)(n-2)+2} \ \frac{n^2-3n+6}{2} - \frac{(n-2)(n-3)}{2}$$

$$\geq \frac{n^2-3n+6}{2} - \frac{n^2 - \cancel{6n}\ 5n + 6}{2}$$

$$\deg(u) + \deg(v) \geq \frac{2n}{2} = n$$

$\Rightarrow$ Hence from Ore's thm G is Hamiltonian

Q1. A simple graph G is Hamiltonian if $deg(v) \geq (n/2)$ $\forall$ $v \in V(G)$ where n is the no. of vertices in G & $n \geq 3$.

Proof - Let u & v be two non adjacent vertices, ~~& G be of~~ a ~~E~~ simple graph G having n vertices where $deg(v) \geq (n/2)$ $\forall$ $v \in V(G)$

now, $deg(u) \geq (n/2)$
$deg(v) \geq (n/2)$

$\Rightarrow deg(u) + deg(v) \geq n$

~~The~~, Hence from Ore's theorem the given graph will be Hamiltonian.

Q2. Let G be a connected graph with n vertices, $n > 2$ & no loops or multiple edges, i.e. G is a simple graph. (1)

To show G is hamiltonian if $deg(u) + deg(v) \geq n$ & u, v are non adjacent it is sufficient to show that every non Hamitonian graph G does not obey the given condn.

Suppose G be a non Hamiltonian graph. & Let H be formed from G by adding edges one at a time that do not create a Hamitonian cycle, until no mored edges can be addes.

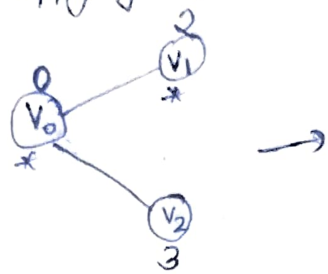
Let $x, y \in V(H)$ & non-adjacent ver'

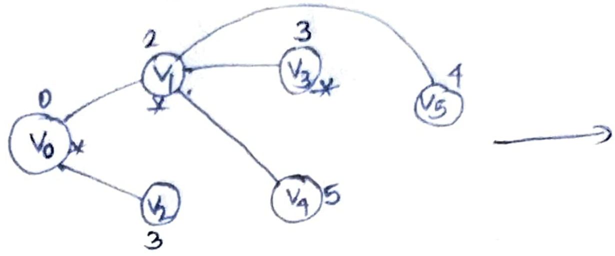
$\Rightarrow$ adding xy would add atleast one Hamiltonian cycle.

so edges other than xy must form a Hamitonian path in H. with $x = v_1$ & $y = v_n$.

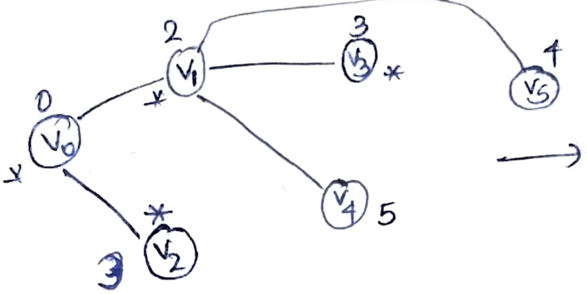
say $v_1 v_2 \ldots v_n$ in H. with $x = v_1$ & $y = v_n$. consider two possible edges in H from v_1 to v_i & $\forall$ $i \in [2, n]$. consider two possible edges in H from v_{i-1} to v_n, at most one of these edge could be present in H, otherwise the cycle $v_1 v_2 \ldots v_{i-1} v_n v_{n-1} \ldots v_i$ would be Hamiltonian cycle. Thus the edges incident to either v_1 or v_n is at most equal to the number of choices of i which is n-1. $\therefore$ H doesn't obey the given condn (1). Since vertex degrees in G are at most equal to H, $\Rightarrow$ G doesn't obey (1).

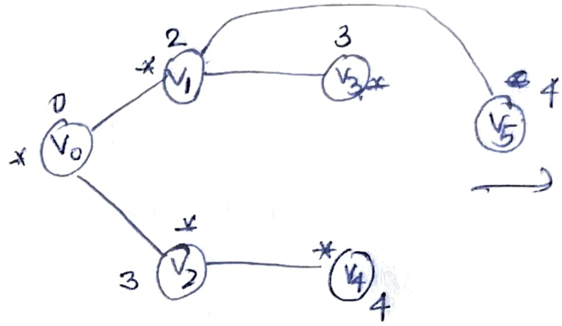
23 Applying dijkstra's algo on the given grap to the vertex V_0

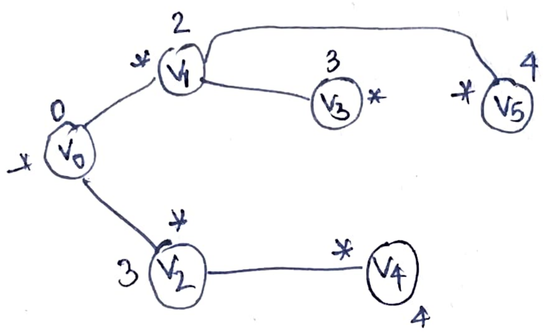




1st picking vertex with minm distance at every point & updating distance, (*) asterisk represents the vertex has been included in the spanning tree.







vertex	minm distance from V_0
V_0	0
V_1	2
V_2	3
V_3	3
V_4	4
V_5	4