discussion on this important topic because it requires a high degree of mathematical maturity. Interested readers may refer to Shreve [122] and Karatzas and Shreve [75] in this regard.

- In many economic processes, volatility of the stock may itself be a stochastic process changing randomly over time. This flexibility produces more realistic models for pricing options. To study the models with stochastic volatility, interested readers may refer to Hull and White [66] and Heston [62].

9.10 Exercises

Exercise 9.1 *Let $f : [-1,1] \to \mathbf{R}$ be given by*

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0\,. \end{cases}$$

Is f continuous? Is f of bounded variations? What will be your answer if for $x \neq 0$, $f(x)$ is taken as $x^2 \sin\left(\dfrac{1}{x}\right)$?

Exercise 9.2 *Let $Y_1, Y_2, \ldots$ be independent random variables each taking two values +1 and -1 with equal probabilities. Define $X(0) = 0$ and $X(n) = \sum_{j=1}^{n} Y_j$, $(n = 1,2,\ldots)$. This stochastic process $\{X(n), n = 0,1,2\ldots\}$ is a symmetric random walk. Show that, the quadratic variation of $X(n)$ up to k is k, i.e. $[X(n), X(n)](k) = k$.*

Exercise 9.3 *For a Poisson process $\{X(t), t \geq 0\}$ with rate 1, find*

(i) $E\left[\int_0^t X(s)\, dW(s)\right]$

(ii) $Var\left[\int_0^t X(s)\, dW(s)\right]$.

Exercise 9.4 *Find the stochastic differentials of $\sin(W(t))$ and $\cos(W(t))$.*

Exercise 9.5 *Show that the process $\{X(t), t \geq 0\}$ given by*

$$X(t) = -1 + e^{W(t)} - \frac{1}{2} \int_0^t e^{W(s)}\, ds$$

is a martingale.

Exercise 9.6 *Let $h(s)$ be a real-valued function which is differentiable and such that $\int_0^t h^2(s) ds < \infty$.*

(i) Show that $h(s)$ is Ito-integrable.
(ii) Use Ito formula to prove the identity

$$\int_0^t h(s)\, dW(s) = h(t)\, W(t) - \int_0^t h'(s)\, W(s)\, ds\,.$$

(iii) Find the distribution of $\int_0^t h(s)\, dW(s)$.

Exercise 9.7 *Consider the SDE of the form*

$$dX(t) = \mu\, dt + \sigma\, dW(t), \quad X(0) = x\,.$$

Find a deterministic function $A(t)$ such that $\exp(X(t) + A(t))$ is a martingale.

Exercise 9.8 *Consider the SDE of the form*

$$dX(t) = -\mu X(t)\, dt + \sigma\, dW(t)\,,$$

where $X(0)$, μ and $\sigma > 0$ are constants. Find the strong solution of the above SDE. Also, find the distribution of $X(t)$.

Exercise 9.9 *Prove that*

$$I(t) = \int_0^t X(s)\, dW(s)$$

is a martingale.

Exercise 9.10 *Prove that*

$$W(T) = \int_0^T dW(t)$$

is an Ito process.

Exercise 9.11 *Consider the SDE of the form $dX(t) = X(t)\, dW(t)$ with $X(0) = 1$. Prove that its solution $X(t) = e^{W(t)-\frac{t}{2}}$ is an Ito process.*

Exercise 9.12 *Find the stochastic differential of $W^2(t)$ and show that $W^2(t)$ is an Ito process.*

Exercise 9.13 *Using the first version of Ito-Doeblin formula, to evaluate $\int_0^T W^2(t)dW(t)$.*

Exercise 9.14 *Are the random variables* $\int_0^T t\, dW(t)$ *and* $\int_0^T W(t)\, dt$ *independent? Also, find the mean and variance of these random variables.*

Exercise 9.15 *An option is called digital option if the pay-off is 1 for* $S(T) > S(0)$ *at the time of exercise* T*, and zero otherwise. Find the arbitrage free price of a digital option (European) with strike price* $K = S(0)$*. You may assume that the stock price follows the SDE.*

$$dS(t) = r\, S(t)\, dt + \sigma\, S(t)\, dW(t)\ ,$$

where r *is the interest rate and* $W(t)$ *is the Brownian motion under risk neutral probability measure.*

Exercise 9.16 *Consider the SDE*

$$dX(t) = c(t)\, X(t)\, dt + \sigma(t)\, X(t)\, dW(t),\quad t \in [0, T]\ .$$

Using the second version of Ito-Doeblin formula, prove that, the solution is

$$X(t) = X(0)\ \exp\left\{\int_0^t \left(c(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_0^t \sigma(s)\, dW(s)\right\},\quad t \in [0, T]\ .$$

Exercise 9.17 *Consider the SDE*

$$dX(t) = A(t)\, dt + B(t)\, dW(t),\quad X(0) = x\ ,$$

where $A(t)$ *and* $B(t)$ *are two time-dependent functions. Find* $A(t)$ *such that* $Z(t) = \exp(X(t))$ *is an exponential martingale?*

Exercise 9.18 *Let* $\mu_n(t)$ *be the* n*-th order moment about the origin for the Brownian motion* $\{W(t), t \geq 0\}$*. Using Ito-Doeblin formula, prove that*

$$\mu_n(t) = \frac{1}{2}n(n-1)\int_0^t \mu_{n-2}(t),\quad (n = 2, 3, \ldots)\ .$$

Also, deduce that $\mu_4(t) = 3t^2$.

Exercise 9.19 *Consider the SDE*

$$dX(t) = -\frac{X(t)}{1-t}\, dt + dW(t),\quad 0 \leq t < 1\ ,$$

with $X(0) = 0$*. Prove that it's solution*

$$X(t) = (1-t)\int_0^t \frac{1}{1-s}\, dW(s),\quad 0 \leq t < 1$$

is a Brownian bridge, between time 0 and time 1.

Exercise 9.20 *Consider the SDE*

$$X(t) = x_0 + \int_0^t sgn(X(s))\, dW(s)\,.$$

Prove that, it has a weak solution, but does not have a strong solution. Further, prove that, when $x_0 = 0$, the weak solution of $X(t)$ is implicitly given by

$$W(t) = \int_0^t sgn(X(s))\, dX(s)\,.$$

Exercise 9.21 *Consider the SDE*

$$dX(t) = X(t)dt + dW(t)\,,$$

with initial condition $X(o) = c$. Obtain the strong solution of $X(t)$. Prove that,

$$X(t) = c\, e^t + e^t \int_0^t e^{-s}\, dW(t)\,.$$

Exercise 9.22 *Let $Q(t) = \ln S(t)$ and $dQ(t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW(t)$. Find $dS(t)$.*