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Sec-A
MC/044

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Unit I

Wealth & Asset

- Asset
  - Risky
    - Share.
    - Gold, oil, petroleum, forex.
  - Non-Risky (Risk free)
    - Govt bonds.
    - Debentures.

↳ $S(t)$ is price of one share of the stock and $S(0)$ is the price of the share at $t=0$ i.e, it is the time when we have made the investment. For simplicity, we will consider $t=1$ as the Time of selling of stock. The value of share price may go up or down at $t=1$. We will consider the difference of $S(1)-S(0)$ to determine the profit/loss.

Rate of return (K_s) $= \dfrac{S(1)-S(0)}{S(0)}$.

Rate of return is a random quantity because it depends on $S(1)$ which is a random quality

$S(0)$ is a deterministic quantity.

Risk free assets are denoted by A.

$$K_A = \frac{A(1)-A(0)}{A(0)}$$ — {Deterministic Quantity}

(~~Rate~~ return of an asset) K_B & K_S.

Now, to build the mathematical financial securities, we have to make some assumptions.

(1) Randomness: The future value of stock price $S(t)$ is a random variables with atleast two different values while $A(t)$ is {future bond price} a known number.

(2) Positivity of prices: All stocks and bond prices are strictly positive i.e

$S(t)$ & $A(t) > 0$ for $t = 0, 1, \ldots$

The total wealth of the investor holding x shares & y bonds at $t = 0, 1$. will be:

$$V(0) = x S(0) + y A(0)$$
$$V(1) = x S(1) + y (A(1))$$

The ordered pair (x, y) is called as portfolio.

x - Risky assets / y - Risk free assets.

$$K_V = \frac{V(1) - V(0)}{V(0)}.$$

(3) Divisibility, liquidity & short-selling: Since $x, y \in R$, one can hold a fraction of a share or bond is referred to as divisibility,

Since x, y are unbounded, means any asset can be bought or sold on demand at market price in arbitrary quantities, that is liquidity

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The number of securities of a particular kind held in portfolios is positive then investor is said to have long position otherwise we say that the investor has short position or the investor has shorted assets.

(4) Solvency: The wealth of an investor must be non-negative at all times i.e,

$V(t) \geq 0$ where $t = 0, 1$.

A portfolio satisfying this condition is called admissible.

(5) The discrete unit price: The future price $S(1)$ of a share of a stock is a random variable taking only finite values.

(6) No arbitrage Principle: The market doesn't allow risk free profits with no initial investment. This is the foundation for all computations. There is no admissible portfolio with initial value :-

$V(0) = 0$ such that $V(t) > 0$ (here $t=1$).

That means there is no possibility of an investor making a profits without a risk & no investment. If a portfolio violating this principle did exist then we would say that arbitrage opportunity was available.

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Risk & Return -

Instead of using historical data for calculating return & risk, we may use the forcasted data. The price of a share of a particular company depends on many factors which are performance of the company, economic scenarios & other factors like national & international politics.

Eg: Four equally likely possible state of economic condition & performance:

(1) High Growth (3) Stagnation

(2) Expansion (4) Decline.

Suppose the price of share of the company today is \$ 261.25 & above under 4 states you expect it to be

(1) \$ 303.5 (2) \$ 282.5 (3) \$ 263.75 (4) 243.5 \$.

↳ We will calculate return for all cases as follows:

$$K_s = \frac{S(1) - S(0)}{S(0)} \times 100.$$

(1) 18.4 % (2) 10.5 % (3) 1 % (4) - 6 %.

Expected return $(K) = E(K_s) = \frac{18.4 + 10.5 + 1 - 6}{4}$

$= 6\%$.

Since the phases were equally likely to occur i.e $P_i = 1/4$, the possibility of return in various phases is the basis for judgement for investment

Variance of data $\sigma^2 = \sum_{i=1}^{4} (x-\mu)^2 P(x_i)$.

$$= \frac{\quad}{4} = 86.375\%.$$

$$\sigma = 9.29\%.$$

(Represents risk)

4/8 Zero Coupon Bond

↳ Intermediate payments are called coupons.

↳ It is one of the risk free assets in which the payment can be guarenteed to the investors by bond issuing.

(Bonds have a fixed round figure maturity value. Based on that initial value is calculated)

The zero coupon bond involves a single payment.

The issuing agency promises to pay a fixed money called F (face value) in exchange of the bond on a given day T called the maturity date.

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Consider $t = 1$ year (r% SI).

$$P(1+r) = F.$$

$$P = V(0) = \frac{F}{1+r}$$
(Wealth at $t=0$)

Q If F for a bond is 100, with $r = 12\%$ compounded annually. Find $V(0)$

Ans
$$\frac{100}{1.12} = 89.29$$

$$\boxed{r = \frac{F}{V(0)} - 1}$$ – This is called implied rate of interest

For simplicity, consider the face value of the bond to be 1 in any currency. It is called unit bond.

In practice, the bonds are freely traded in the market & their prices are guarenteed by market ~~alone~~ force

Market force governs risk free interest rate

A bond can be sold any time prior to maturity.

This price at t is denoted by $B(t,T)$

$B(0,T)$ – current time price.

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$t=0$ $\quad t=t \quad$ $t=T$

110 $\qquad\qquad$ F(110)

$r = 10\%$

$$B(t,T) = e^{-(1-t).r}$$

Period compounding	Continous compounding
$\left(1+r/m\right)^{mt}$	$\left(\left(1+r/m\right)^{m/r}\right)^{r \cdot t}$
monthly $(m=12)$ daily $(m=365)$	

Continuous $\rightarrow$ $B(t,T) = e^{-(T-t)r}$

Periodic $= \left(1+r/m\right)^{-m(T-t)}$

If F - face value, C - coupon price.
time to maturity - n years.

$$V(0) = ce^{-r} + ce^{-2r} + \dots + e^{-nr}(c+F)$$

- <u>One step Binomial model</u>

$t=0$ $\qquad$ $t=1$

S_d - down value of stock
S_u - up value of stock.

Risky asset (eg: stock) can take two values.

If $S(0) = A(0)$
Then, $S^d < A(1) < S^u$.

$$S(1) \begin{cases} \xrightarrow{p} S^u \\ \xrightarrow{1-p} S^d \end{cases}$$

else an arbitrage would arise

Proof: Case ①: $A(1) \le S_d < S^u$

Liability $\to$ Borrow $V(0)$.

Borrow $A(0)$

Buy stock at $S(0)$

$V(0) = S(0) - A(0)$
$= 0$

$V(1) = 1 \cdot S(1) - A(1)$

$= \begin{cases} S^u - A(1) \\ S^d - A(1) \end{cases}$

Profit.

Case ②: $A(1) \ge S^u > S^d$.

Both cases are not possible

Derivative security

L It's value depends upon
- forward contract - future contract
- options - swaps.

· Forward contract $\to$ It is an agreement to buy or sell a risky asset at a specified future time known as delivery date.

$\to$ An investor who agrees to buy the asset is said to enter into a long forward contract or take a long forward position.

The exercise of the contract/honor of the contract is mandatory for both the parties

The asset price on delivery date $(t=1)$ is $S(1)$

↳ The fixed price on data of contract is F.
$t=0 \Rightarrow F$ / $t=1 \Rightarrow S(1)$

↳ Pay-off for long forward position holder $= S(1) - F$.

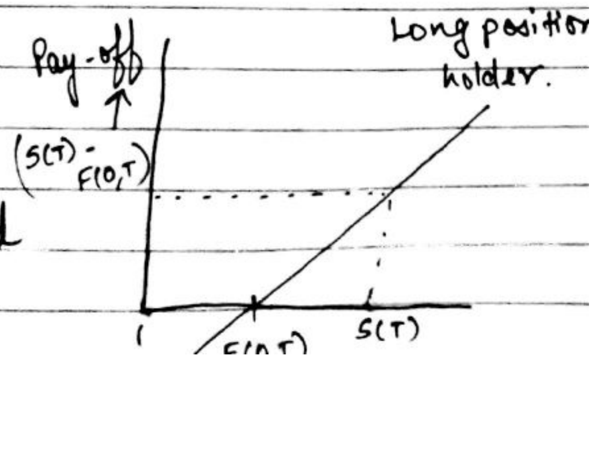
Pay off for short forward position holder $= F - S(1)$.

The value of $S(1) - F$ can be positive / negative / zero.

↳ Apart from bonds & stocks, a portfolio held by an investor may contain forward contracts and it will be described by a triple (x, y, z).
(z - no. of forward contract (+ve / -ve)
as no payment is due when a forward contract is exchanged)

$V(0) = S(0)\cdot x + y \cdot A(0)$.
At the time of delivery, $t=1$.
$V(1) = xS(1) + y\,A(1) + z\,(S(1) - F)$.
We will represent the agreed ~~formula~~ forward price as $F(0,T)$.

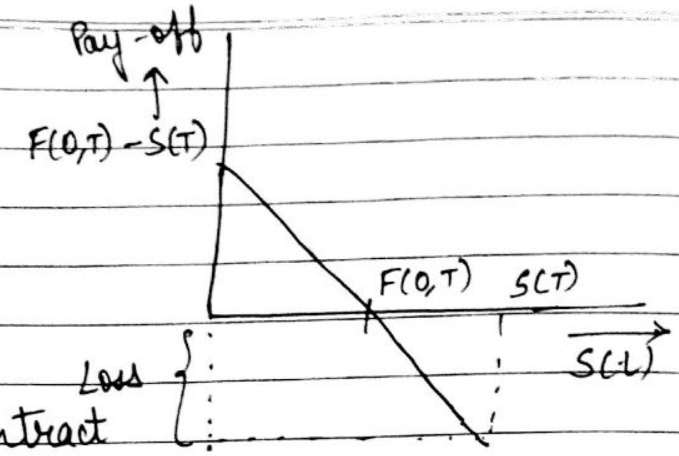
Case 1: In a forward contract b/w two parties at the time for delivery, one has to bear the loss
(1) $F(0,T) < S(T)$ (2) $F(0,T) > S(T)$.
↳ The party having long position will gain.



Asset will be bought at the price of $F(0,T)$ and will be sold in the market at $F(1,T)$.

In case (2)

Short position holder



→ Sometimes, the contract may be initiated at $t < T$. Then, the pay-off will be given by

$\underbrace{S(T) - F(t,T)}_{L.P.H}$ & $\underbrace{F(t,T) - S(T)}_{S.P.H.}$

$F(t,T)$ - forward price at time T

Thm Let the price of an asset at $t=0$ be $S(0)$. Then, the forward price $F(0,T) = \frac{S(0)}{d(0,T)}$.

$d(0,T)$ - discount factor b/w $t=0$ & $t=T$.

Proof: ① Let $F(0,T) > \frac{S(0)}{d(0,T)}$

At $t=0$, construct a portfolio:

Ⓐ Borrow the amount $S(0)$ for time T.

Ⓑ Buying an underline stock at $S(0)$

Ⓒ Take a short forward position with delivery time T with the forward price $F(0,T)$

↳ At $t=0$, value of portfolio is 0 $(1,-1,0)$

↳ At $t=T$, return the appreciated amount against borrowed money $(S(0)/d(0,T))$

↳ Close the short position by selling asset at $F(0,T)$.

Value of portfolio → $F(0,T) - \frac{S(0)}{d(0,T)} > 0$ — (I)

∴ No arbitration is violated.
Hence, $F(0,T)$ is not greater than $S(0)/d(0,T)$.

(2) Let $F(0,T) < S(0)/d(0,T)$.
At $t=0$,
↳ Shortsell 1 unit of underlying asset for $S(0)$.
↳ Invest in risk free for time T.
↳ Enter into a forward contract with long position with Time T & price $F(0,T)$.

At $t=T$,
(1) Close the long forward position for price $F(0,T)$.
Encash the risk free amount $S(0)/d(0,T)$.
(2) Close short sell position by returning the underlying stock.

Value of portfolio at time T.

$$\frac{S(0)}{d(0,T)} - F(0,T) > 0 \quad \text{—(2)}$$

⇒ Violating the no. of arbitrage principle.

From (1) & (2), $\dfrac{S(0)}{d(0,T)} = F(0,T)$.

- Direct results

↳ (1) If the constant interest rate is compounded continuously $\phi(0,T) = e^{-rT}$

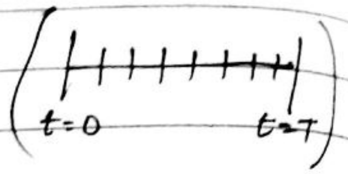
$$F(0,T) = S(0)\cdot e^{-rT}.$$

(2) If contract is initiated at intermediate time $(0<t<T)$

$$d(t,T) = e^{-r(T-t)}$$

Forward price $= F(t,T) = e^{-r(T-t)}\cdot S(0)$.

③ Let an asset carry a holding cost of $C(i)$ per unit in period $(i = 0,1,2,\ldots\ldots n-1)$
Also at $t=0$, the price of the asset be $S(0)$ and short selling be allowed.
Then $F(0,T) = \dfrac{F(0,T)}{d(0,T)} + \sum_{i=0}^{n-1} \dfrac{C(i)}{d(i,n)}$



④ Let an asset be stored at zero cost and also sold short.
Let the price of asset at $t=0$, be $S(0)$ & a dividend of rupees (div), we paid at time $t=T$. in b/w $(0,T)$.
Then, $F(0,T) = S(0) - div(d(0,T)) / d(0,T)$.
(If asset pay (div) continuously at r)
continuous compounding
$\hookrightarrow [S(0) - div(d(0,T))]e^{rT}$.

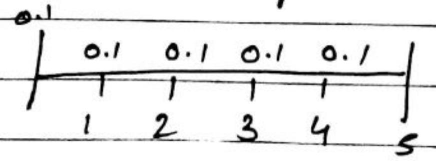
$$F(0,T) = S(0) \cdot e^{(r - r_{div})T}.$$

Q Non-divident stock ($r = 8\%$ pa)
cont. compounding $S(0) = ₹ 100$. (7 months)

Ans $F(0, 7/12) = 100 \times e^{7/12 \times 8/100}$
$= ₹ 104.78$

Q sugar – Rs 60/kg, storage cost → Rs 0.1/kg/month at beginning of each month. ($r = 9$ pa) (5 months)

Ans



$= 60(1.0075)^5 + 0.1(1.0075)\dfrac{(1.0075^{5-1})}{0.0075}$

$=$ Rs 62.79.

Value of forward contract

↳ Let a forward contract initiated at $t=0$ with delivery $t=T$.
$F(0,T)$ — forward price.
Consider an intermediate time $(0<\tau<T)$
Let $F(\tau,T)$ be the forward price of the contract initiated at $t=\tau$ with delivery time (T).

↳ We have two forward contracts, one initiated at $t=0$, another at $t=\tau$. Both with delivery date (T) with forward price $F(0,T)$ & $F(\tau,T)$ respectively

As the time passes by, the value of the forward contract initiated at $t=0$ will change.

Let $f(\tau)$ is the value of forward contract at $t=\tau$.

Thm $f(\tau) = [F(\tau,T) - F(0,T)]\, d(\tau,T)$.
$d(\tau,T)$ - risk free discount factor over the period (τ,T)

Proof I Let $f(\tau) < [F(\tau,T) - F(0,T)]\, d(\tau,T)$.
At $t=\tau$, construct a portfolio (p) is following
① Borrow the amount $f(\tau)$
② Enter into a long forward contract with forward price $F(0,T)$.
③ Enter into a short forward position ~~into a forward position~~ with forward price $F(\tau,T)$.

Value of portfolio at $t=\tau$ is 0. $(V_\tau(P)=0)$

④ Closing forward contract by collecting (paying the amount) $(S(T) - F(0,T))$ for long position & $(F(\tau,T) - S(T))$ for short position.

⑤ Pay back the loan amount with interest that is $f(\tau)/d(\tau,T)$.

$$V_T(P) = F(\tau,T) - F(0,T) - \frac{f(\tau)}{d(\tau,T)} > 0.$$

$\therefore$ Violates no arbitrage principle

II Let $f(\tau) > \{F(\tau,T) - F(0,T)\}\, d(\tau,T)$.
(Construct portfolio at $t=\tau$)

① Sell short the forward contract for the amount $f(\tau)$.
② Invest $f(\tau)$ in risk free for $t=\tau$ to $t=T$.
③ Enter into a long forward contract with $t=T$ at price $F(\tau,T)$.

$F(0,T) - S(T)$.
$S(T) - F(\tau,T)$
$\frac{f(t)}{d(\tau,T)}$

$$V_T(P) = \frac{f(\tau)}{d(\tau,T)} + F(0,T) - F(\tau,T) > 0.$$

$\therefore$ No arbitrage is violated.

Hence, $f(\tau) = (F(\tau,T) - F(0,T))\, d(\tau,T)$.

Q $S(0) = 45$, $r = 6\%$. Find the forward price if $T = 1$ year. Find value after 9 months if stock price at that time is 49. (cont. compounding)

Ans
$$F(0,T) = \frac{S(0)}{d(0,T)} = 45e^{0.06\times 1} = 47.78.$$

$$f(\tau = 9/12) = (49.74 - 47.78)\, e^{-0.06(1-9/12)} = 1.93.$$

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• Options

↳ They are instruments that give the right to the holder to trade without obligation to buy or sell an asset at an agreed price (on or before the specified period of time)

↳ Two types of options :-

(1) Call option. — Right to buy an asset at a specified price

(2) Put option. — Right to sell an asset at a specified price.

↳ In neither case, does the holder have an obligation to buy or sell.

The specified price at which the option holder has the right to trade is known as strike price / exercise price.

The asset on which the call or put option is created is referred to as underlying asset.

↳ The option doesn't come free. Option premium is the price that the holder of an option has to pay to the seller of the option.

The exercise of the option results into 3 possibilities :-

(i) In the money - if call/put option is advantageous to the investor to exercise

(ii) Out of the money - Disadvantages to the investor.

(iii) At the money - situation of no profit / no loss.

↳ Selling - writing the option

Buyers - ~~have~~ referred to having long position.

Sellers - referred to having short position.

Pay-off of a call option

↳ **Buyer**

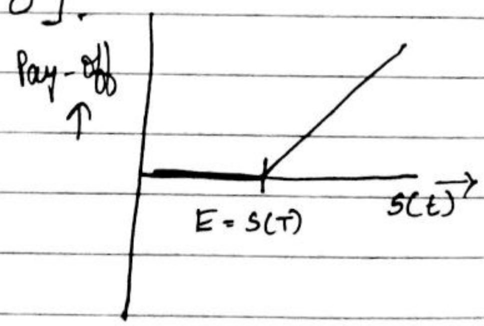
Let S be the current price of the asset which becomes $S(T)$ at maturity date
(E - exercise price)

~~① S(T)~~ ① $S(T) < E$
Call option should not be exercised as it will attract loss. Hence, pay off is zero.
(Not including commissions or premium)

② $S(T) > E$
The call option should be exercised.
Pay-off is $S(T) - E$.

③ $S(T) = E$
Option will not be exercised, payoff will be zero.

Pay off $C(T) = \max[S(T) - E, 0]$.

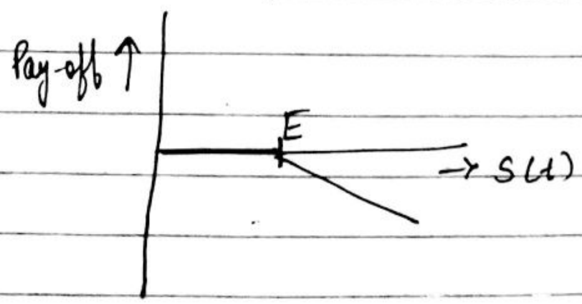


• **Seller**

(i) $S(T) > E$
Option will be exercised

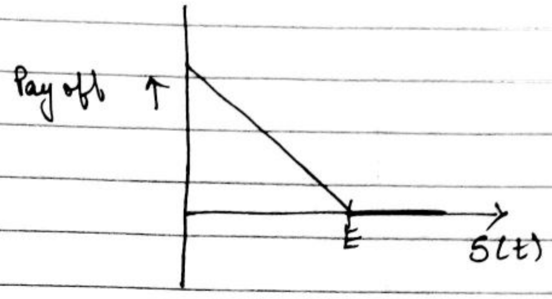
(ii) $S(T) \leq E$
Pay off will be zero.

$$C(T) = \min[E - S(T), 0]$$



Pay-off of a put option

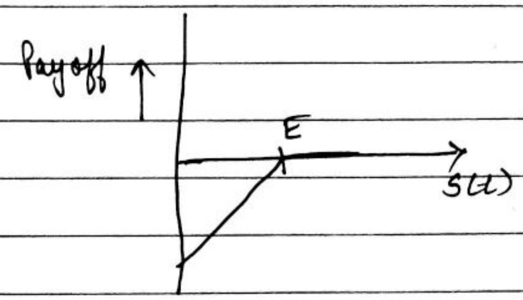
- **Buyer**



(i) $S(T) < E$
Option will be exercised.
Pay off is $E - S(T)$.

(ii) $S(T) \geq E$
Option will not be exercised.
Pay-off will be 0.

- **Seller.**

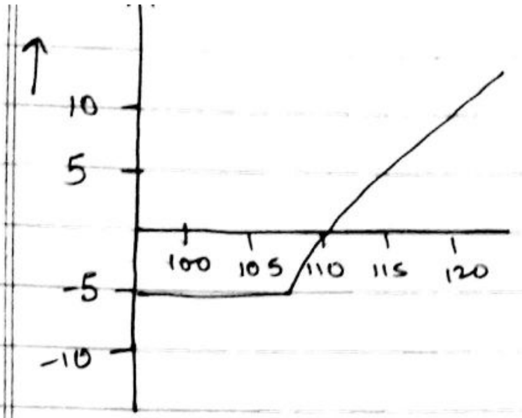


Q. The share of Telco is selling for Rs 104.
Rajan buys a 3 month call option at a premium of Rs 5.
The ex. price is Rs 105. What is Rajan's pay-off.
if the share price is 100, 105, 110, 115, 120. at the time option is exercised.

Ans:

Share price	100	105	110	115	120
Buyer's inflow { → Sale of share }	–	–	110	115	120
Buyer's outflow { → Premium.	5	5	5	5	5.
→ Purchase of share. }	–	–	105	105	105
Net pay off	−5	−5	0	5	10.

Pay-off



Share price →	100	105	110	115	120
Seller inflow	-	-	105	105	105.
Seller of share premium	5	5	5	5	5.
Seller outflow purchase of share	-	-	110	115	120
Net pay off:	5	5	0	-5	-10.

- Pricing of the option

Q $S(0) = 100$, $A(0) = 100$, $A(t) = 110$.

$S(T) \begin{cases} 120 \ (p) \\ 80. \ (1-p). \end{cases}$ $(X = 100)$ $C(1) \begin{cases} 20 \\ 0. \end{cases}$

The writer of the option created a portfolio. (x, y) such that

$c(0) = x S(0) + y A(0).$

$x S(T) + y A(t) = 20.$

$x \cdot 120 + y \cdot 110 = 20$ / $x \cdot 80 + y \cdot 110 = 0$

$x = 1/2$, $y = -4/11$

$c(0) = \frac{1}{2} \times 100 + \left(-\frac{4}{11}\right) 100 = 13.6364.$

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To represent the pay-off, we will use the notation.

$$x^+ = \begin{cases} x & , \text{ if } x > 0 \\ 0 & , \text{ otherwise} \end{cases}.$$

$$\max(S(T) - X, 0) = (S(T) - X)^+ .$$

The gain of the option buyer (seller) is the pay off modified by the premium

C^E or p^E paid (received)

$(S(T) - X)^+ - C^E e^{rt}$, r - risk free interest rate.

In case of put options,

$$(X - S(T))^+ - p^E e^{rT}.$$

• Put-call Parity

↳ For the stock paying no dividend, the following relation holds b/w price of European call & put option, both with exercise price X and ex. time T.

$$C^E - p^E = S(0) - X e^{-rt}.$$

Proof: ① Let $C^E - p^E > S(0) - X e^{-rt}$.

Construct a portfolio as following

(1) Buy one share for $S(0)$

(2) Buy a put option for p^E.

③ Write and sell a call option for C^E

④ Invest or borrow $[C^E - p^E - S(0)]$ for risk free for the period T at interest rate (r).

(5) The balance is 0, i.e, portfolio value at $t=0$ is 0.

(6) At $t=T$,
(i) Close the market position collecting (paying)
$[c^E - p^E - S(0)]e^{rT}$
(ii) Sell the share for X. to close the
either by exercise of put if $S(T) \leq X$ or settling the short position if $S(T) > X$.

$[c^E - p^E - S(0)]e^{rT} + X > 0$

which violates the no arbitrage principle.

(ii) $c^E - p^E < S(0) - Xe^{-rT}$.
Make the following portfolio.-
(1) sell short one share for $S(0)$
(2) Write & sell one put option for p^E.
(3) Buy one call option for c^E.
(4) Invest $S(0) + p^E - c^E$ for time T at interest rate (r) in the money market.
(5) $V(0) = 0$

At $V(T)$, close the money market position by recieving or paying $(S(0) + p^E - c^E)e^{rT}$.

(6) Buy one share for X by exercising call or selling put option

If $S(T) \leq X \rightarrow$ put option

$S(T) \geq X \rightarrow$ call option and close the short position of X.

$(S(0) + p^E - c^E)e^{rT} - X > 0$
which violates the no arbitrage principle.

$\therefore \; c^E - p^E = S(0) - Xe^{-rT}$.

Bounds on the option price.

$c^E < S(0)$.

If the inequality is reversed, i.e $c^E \geq S(0)$, then we could write and sell the option and buy the stock investing the balance on the money market. On the exercise date, T we could sell ~~the~~ the stock for minimum of $\min\{S(T), X\}$.

The net profit will be $(c^E - S(0))e^{rT} + \min[S(T); X]$. which is a violation of no arbitrage principle

$c^E \geq S(0) - Xe^{-rT}$

$\left(p^E \geq Xe^{-rT} - S(0)\right)$.

$p^E < Xe^{-rT}$

- Dependency

Consider option on the same underlying asset and same exercise time T but different values of the strike price X and underlying asset price S(0) will be kept fixed.

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If $X_1 < X_2$, then

$$c^E(X_1) > c^E(X_2)$$
$$p^E(X_1) < p^E(X_2)$$

This means that $c^E(X)$ is strictly decreasing and $p^E(X)$ is strictly increasing.

<u>Note</u>: If $X_1 < X_2$, $c^E(X_1) - c^E(X_2) < e^{-rT}(X_2 - X_1)$

$$p^E(X_2) - p^E(X_1) < e^{-rT}(X_2 - X_1)$$

<u>Proof</u>:

Using call put parity

$$c^E(X_1) - p^E(X_1) = S(0) - X_1 e^{-rT}$$
$$c^E(X_2) - p^E(X_2) = S(0) - X_2 e^{-rT}.$$

Subtracting them,

$$(c^E(X_1) - c^E(X_2)) + (p^E(X_2) - p^E(X_1)) = (X_2 - X_1)e^{-rT}.$$

<u>Depending on the underlying asset price</u>

Assuming that all remaining variables are fixed, the current price of the underlying asset $(S(0))$ is given by the market and cannot be altered.

We can consider an option on a portfolio considering x shares
$S = x S(0)$. with strike price X.

On such portfolio, at expiry time T, the payoff will be $- [x S(T) - S(0), 0]$.
$\equiv [x S(T) - X]^+$ for call option
and $[X - x S(T)]^+$ for put option.

If $S_1 < S_2$ $(X_1 < X_2)$

$c^E(S_1) < c^E(S_2)$
$p^E(S_1) > p^E(S_2)$

Proof: suppose that $c^E(S_1) \geq C^E(S_2)$
for some $S_1 < S_2$, we can write and sell a call option on the portfolio with x_1 ~~and~~ share and buy a call option on the portfolio with x_2 share

Having the same exercise price, invest $c^E(S_1) - c^E(S_2) \geq 0$ in risk free.
Hence, the pay off will be $[x_1 S(T) - X]^+ \leq [x_2 (S(T)) - X]$
(as $x_2 > x$

with the strict inequality when $X < x_2 S(T)$.

After covering our liability, we are left with arbitrage profit and thus, our assumption is wrong.

6/9 **Unit 2: Binomial Model**

- **Assumptions**

$$-1 < d < u.$$

$$\begin{tikzcd} & (1+u)S(0) \\ S(0) \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & (1+d)S(0) \end{tikzcd}$$

↳ If r is the risk free interest rate for one period, then $d < r < u$.

$$\begin{tikzcd} & (1+r)A(0) \\ A(0) \arrow[ur, "p"] \arrow[dr, "(1-p)"'] & \\ & (1+r)A(0) \end{tikzcd}$$

$(1+u), (1+d), (1+r)$ is the growth factor
Now if u & d represent the growth factor directly.

$$\begin{tikzcd} & uS(0) \\ S(0) \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & dS(0) \end{tikzcd}$$

Expected return $E(K(1)) = pu + (1-p)d$.

Now if for some probability (p^*), the average return on a risky asset is equal to risk free return then such a probability is known as risk neutral probability

$$E(K(1)) = r$$

$$up^* + (1-p^*)d = r. \Rightarrow p^* = \frac{r-d}{u-d}$$

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$$E(S(1)) = p\,S(0)(1+u) + (1-p)\,S(0)(1+d)$$
$$= S(0)\{\cancel{p}+pu+1-pd-\cancel{p}+d\} = S(0)\{1+pu+(1-p)d\}$$
$$= S(0)\{1+E(K(1))\}.$$

<u>Thm</u> Expected stock price for $n = 0, 1, 2, \ldots$
$$E(S(n)) = S(0)\{1+E(K(1))\}^n.$$

<u>Proof</u>: Since the step returns $K(1), K(2), \ldots$ are independent and so are the random variables $((1+K(1)), (1+K(2)) \ldots)$

$$= S(0)\,E(1+K(1))\,E(1+K(2)) \cdots E(1+K(n))$$

$\therefore E(K(1)) = E(K(2)) \ldots = E(K(n))$.
$$= S(0)(1+E(K(1)))(1+E(K(1))) \ldots (1+E(K(1)))$$
$$= S(0)(1+E(K(1)))^n.$$

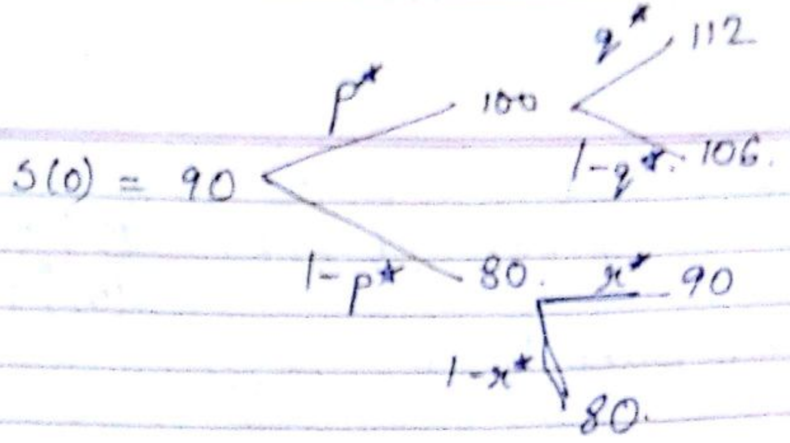
~~Imp~~ In case of risk neutral probability expected price after n^{th} internal will be $S(0)(1+E(K(1)))^n$.

Extending the result of two state model to n times steps, stock price becomes $S(n)$ after n time steps. In the risk neutral probabity, ~~expected~~ ~~price~~ expectation of the price $S(n+1)$ is

$$E^*(S(n+1)/S(n)) = S(n)(1+r).$$

<u>Q</u> Let $A(0) = 100$, $A(1) = 110$, $A(2) = 121$.
The stock price can follow four possible scenarios.

Scenario.	$S(0)$	$S(1)$	$S(2)$
ω_1	90	100	112
ω_2	90	100	106.
ω_3	90	80	90
ω_4	90	80	80.



p^*, q^*, r^* are risk-neutral-probability-major (rnpm)

The discounted stock price at each node is given as.

$$\left.\begin{aligned} q^*\frac{(112)}{121} + (1-q^*)\frac{106}{121} &= \frac{100}{110}. \\ r^*\frac{(90)}{121} + \frac{(1-r^*)80}{121} &= \frac{80}{110}. \end{aligned}\right\} \text{Using } A(1) \,\&\, A(2)$$

$$\left. p^*\frac{(100)}{110} + (1-p^*)\frac{80}{110} = \frac{90}{100}. \right\} \text{Using } A(0) \,\&\, A(1)$$

$$p^* = \frac{19}{20}, \quad q^* = \frac{2}{3}, \quad r^* = \frac{4}{5}.$$

$$P^*(\omega_1) = p^* \cdot q^* = \frac{19}{30}.$$

$$P^*(\omega_2) = p^*(1-q^*) = \frac{19}{60}.$$

$$P^*(\omega_3) = (1-p^*)\, r^* = \cancel{\tfrac{19}{20}}\ \frac{4}{100} = \frac{2}{50}$$

$$P^*(\omega_4) = (1-p^*).(1-r^*) = \frac{1}{100}.$$

t=1

Q2 Consider a two step binomial tree model.

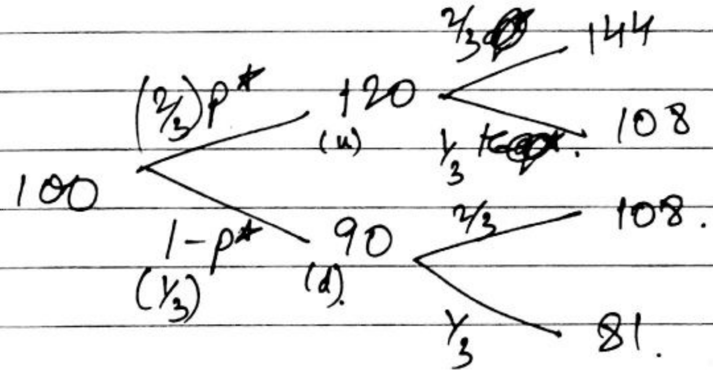
$S(0) = \$100, \quad u = 0.2, \quad d = -0.1, \quad r = +0.1.$

Ans $p^* = \dfrac{r-d}{u-d} = 2/3.$

Assumption
$(p^* = q^* = r^*)$

$(1-p^*) = 1/3.$

$$E^*(S(2)) = S(0)(1+r)^2$$
$$= 100(1.1)^2$$
$$= \$121$$



Case 1: suppose price goes up.

$$E^*(S(2)) = 2/3 \times 144 + 1/3 \times 108.$$
$$= \$132.$$

Case 2: suppose price goes down.

$$E^*(S(2)) = 2/3 \times 108 + 1/3 \times 81.$$
$$= \$99.$$

- In case 1, it can written using conditional expectation of $S(2)$. $E^*(S(2)/S(1))$

$$E^*(S(n+1)/S(n)) = S(n)(1+r)$$
$$= \$132$$

- $\underline{\text{Case 2}}$: $E^*(S(2)/S(1)=90) = 90 \times 1.1$
$$= \$99$$

$$E(S(2)) = \frac{2}{3} \times 132 + \frac{1}{3} \times 99$$
$$= \$121.$$

- $\underline{\text{Option Pricing}}$

We will consider the pricing of general European derivatives security in particular, will be option.

Pay-off function (f) with stock S as the underlying asset

$D(T)$ is a random variable.

$$D(T) = f(S(T)).$$

In particular for call option
$f(S) = (S-X)^+$

for put option, $f(S) = (X-S)^+$.

Stock with current price $S(0)$ after time (1) is $S(1)$.

$$S(0) \begin{cases} \xrightarrow{p} S(0)(1+u) = S^u \text{ with prob} = p \\ \xrightarrow{1-p} S(0)(1+d) = S^d \text{ with prob} = 1-p \end{cases}$$

To replicate a general derivative security which pay-off 'f', we have to solve the system of equations.

$$x(1)\, S^u + y(1)\,(1+r) = f(S^u)$$

$\{A(0) = 1\}$ - value of bond initially.

$$x(1)\, S^d + y(1)\,(1+r) = f(S^d).$$

This gives $x(1) = \dfrac{f(S^u) - f(S^d)}{S^u - S^d}$
(Delta)

Position of shares in a replicating portfolio.

$$y(1) = -\left(\frac{(1+d)\, f(S^u) + (1+u)\, f(S^d)}{(u-d)(1+r)}\right)$$

Initial value of replication portfolio. $= x(1) \cdot S(0) + y(1) = D(0)$.

$\{A(0) = 1\}$ (Pay off at $t = 0$)

$$D(0) = \frac{f(S^u) - f(S^d)}{u-d} - \frac{((1+d) f(S^u) - (1+u) f(S^d))}{(u-d)(1+r)}$$

Theorem The expectation of discounted pay off computed with respect to risk neutral ~~property~~ probability is equal to the present value of European derivative security (The premium).

$$E^*(f(S(1))/(1+r)) = D(0).$$

Proof: $$D(0) = \frac{1}{(1+r)} \left(\frac{(r-d) f(S^u)}{(u-d)} + \frac{(u-r) f(S^d)}{(u-d)} \right)$$

$$= \frac{1}{1+r} \left(\frac{(r-d)}{u-d} f(S^u) + \frac{(u-r)}{(u-d)} f(S^d) \right)$$

$$p^* = \frac{(r-d)}{(u-d)}, \quad 1 - p^* = \frac{(u-r)}{(u-d)}$$

$$\boxed{D(0) = \frac{1}{1+r} E^*(f(S(1)))}$$

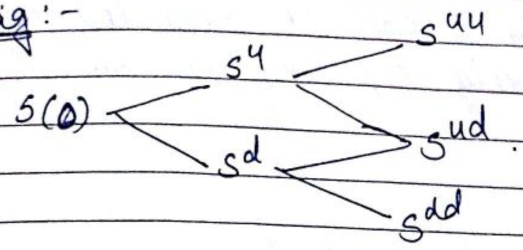
$$\boxed{= E^*\left(\frac{f(S(1))}{1+r}\right)}$$

For a 2-period model

S(2) has 3 ~~probabi~~ possibilities

(S^{uu}, S^{dd}, S^{ud})

Diag :-



↳ At time $(t=2)$, the derivative security is represented by

$D(2) = f(S(2))$ which has 3 possible values.

The derivative security price $D(1)$ has two values.

↑ ① Price goes up $D(1) = \frac{1}{1+r}\left(f(S^{uu})p^* + (1-p^*)f(S^{ud})\right)$.

↓ ② Price goes down $D(1) = \frac{1}{1+r}\left(p^* f(S^{ud}) + (1-p^*)f(S^{dd})\right)$.

$$D(1) = \frac{1}{1+r}\left(p^* f(S(1)\cdot(1+u)) + (1-p^*)f(S(1)(1+d))\right).$$

$(S(1) = S^u$ or $S(1) = S^d)$.

$\boxed{D(1) = \frac{1}{1+r}\, g(S(1)).}$ where $g(x)$.

is equal to. $\frac{1}{1+r}\left(p^* f(x(1+u)) + (1-p^*)f(x(1+d))\right)$.

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As a result, $D(1)$ can be regarded as derivative security expiring at time $(t=1)$ with payoff $g(1)$.

($f(S(T))$ — payoff at time $t=T$)

This means that, one step procedure can be applied once again to the single subtree at the root of the tree.

$$D(0) = \frac{1}{1+r}\Big(p^* g(S(0)(1+u)) + (1-p^*)\big(g(S(0)(1+d))\big)\Big)$$

$$= \frac{1}{(1+r)^2}\Big((p^*)^2 f(S^{uu}) + (1-p^*)^2 f(S^{dd}) + 2p^*(1-p^*) f(S^{ud})\Big).$$

(Time $(t=3)$)

$$D(0) = \frac{1}{(1+r)^3}\Big((p^*)^3 f(S^{uuu}) + (1-p^*)^3 f(S^{ddd}) + (p^*)^2(1-p^*) f(S^{uud}) + p^*(1-p^*)^2 f(S^{udd})\Big).$$

(Time $(t=n)$)

$$\cancel{D(0) = \frac{1}{(1+r)^n}\left((p^*)^n f(\right.}$$

$$\boxed{D(0) = \frac{1}{(1+r)^n}\sum_{i=0}^{n} {}^nC_i (p^*)^i (1-p^*)^{n-i} f(S^{u^i \cdot d^{n-i}})}$$

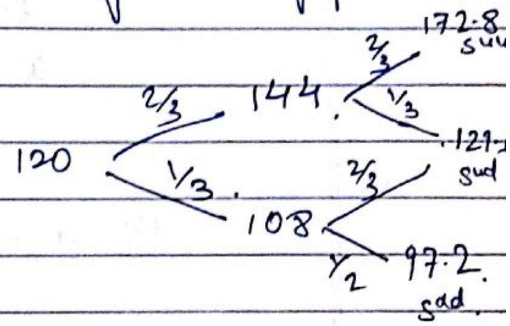
- The value of the European derivative security with payoff in n-step binomial model.

— is the expectation of discounted payoff under risk neutral probability.

$$\boxed{D(0) = E^{*}\left(\frac{f(S(N))}{(1+r)^{N}}\right).}$$

Imp Q $S(0) = \$120, \quad u = 0.2, \quad d = -0.1, \quad r = 0.1.$
$(p^{*} = 2/3)$. Consider a call option with strike price = \$120. & $T = 2$.
Find the option price & replicating strategy?

Ans $p^{*} = \frac{r-d}{u-d} = 2/3.$



$f(S^{uu}) = \$52.8 \quad (X-S)^{+}$
(pay-off)

$f(S^{ud}) = \$9.6$

$f(S^{dd}) = 0$ (option not exercised as strike price is greater)

$$D(0) = \frac{1}{(1.1)^2}\left(\left(\frac{2}{3}\right)^2 52.8 + 2\times\frac{2}{3}\times\frac{1}{3}\times 9.6\right).$$

$$= \cancel{\frac{1}{(1.1)^2}} \quad \frac{\$249.6}{10.89} = \$22.92.$$

Replicating strategy.

↳ $f(S(1))$ for two trees in second period

$f(S(1)) = \left(\frac{2}{3} \times 172.8 + \frac{1}{3} \times 129.6.\right) \frac{1}{1.1}$
$(f(S^u))$

$f(S^u) = \cancel{\$\,38}\ \frac{384}{11}$

$f(S^d) = \$\ \frac{64}{11}.$

Let the portfolio be (a, b)

$a \cdot S^u + b^{(1+r)} = f(S^u)$
$a \cdot S^d + \underset{(1+r)}{b} = f(S^d).$

Value of bond
at $t=0 \to 1$
$t=1 \to (1+r) \times 1$

$144a + 1.1b = \frac{384}{11}$

$108a + 1.1b = \frac{64}{11}$

$9\cancel{36}a = \frac{\cancel{320}^{80}}{11}$

$a = \frac{80}{99} = 0.8081$

$b = -74.05$

$D(0) = 120a + b \times 1.$
$= \cancel{120}^{40} \times \frac{80}{\cancel{99}_{33}} - 74.05$
$\simeq \$32.$

- Cox - Ross - Rubeinstein (CRR).

↳ The assumption in the financial market made for single period binomial model are carried forward as following.

① The underlying stock on which the option is written is perfectly divisible

② ~~If~~ The underlying stock pays no dividend

③ There is no Transaction cost in buying or selling & no taxes

④ Short selling allowed

⑤ The risk-less interest is known & constant till time of expiration of the option

⑥ No arbitration principle holds.

↳ We divide the time period in equal intervals
$[0, T] \rightarrow \Delta t = T/n$.

and assume that in each sub-interval, stock price changes like one-period binomial case.

We define,

$$E_k = \begin{cases} u, & \text{with propability } p. \\ d, & \text{with probability } (1-p). \end{cases} \quad \text{--- (1)}$$

E_k - random bernoulli variable.

At $t = T$, stock price $S(T)$ is

$$S(T) = S(0)\, E_1 E_2 \cdots E_n$$

$$S(T) = S(0)\, e^H$$

$$\left(e^H = E_1 E_2 \ldots E_n \right)$$

$$\Rightarrow H = \sum_{i=1}^{n} \ln(E_i).$$

$$\ln(S(T)) = \ln(S(0)) + H. \quad \text{--- (2)}$$

H - represents the logarithmic growth of the stock price

$$H = \ln\left(\frac{S(T)}{S(0)}\right).$$

(3) (a) ~~$E\left[\ln\left[\frac{S(T)}{}\right.\right.$~~ $E(\ln(E_k)) = p\ln(u) + (1-p)\ln(d).$

(b) $\mathrm{Var}(\ln(E_k)) = p(1-p)\left[\ln(u) - \ln(d)\right]^2$

We now introduce parameters (μ, σ^2)

$$\left.\begin{aligned} \mu \cdot \Delta t &= E(\ln(E_k)) \\ \sigma^2 \Delta t &= Var(\ln(E_k)) \end{aligned}\right\} \text{— } 4.$$

(μ – Drift / σ – Volatility)

$$X_k = \frac{\ln(E_k) - E(\ln(E_k))}{\sqrt{Var(\ln(E_k))}} \quad - 5$$

(Random variable)

Using 3 in 5,

$$\cancel{X_k = \ln(E_k)}$$

$$X_k = \begin{cases} \dfrac{(1-p)}{\sqrt{p(1-p)}} & \text{with probability } p. \\ \dfrac{-p}{\sqrt{p(1-p)}} & \text{with probability } (1-p) \end{cases}$$

$(k = 1, 2 \ldots n)$

For each k,

$$E(X_k) = 0.$$

$$Var(X_k) = 1.$$

Now, $\quad H = \sum_{k=1}^{n} \ln(E_k).$ $\quad$ (Using 4 & 5)

$$= \sum_{k=1}^{n} (\mu \Delta t + \sigma \sqrt{\Delta t} \cdot X_k).$$

$$= \cancel{\mu T + \sigma \sqrt{T} \sum_{k=1}^{n} X_k}$$

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$= \mu T + \sigma\sqrt{\Delta t}\, Y.$ — 6

(where $Y = \sum_{k=1}^{n} X_k$) — 7

└ a random walk.

→ A random walk is a stochastic process $\{ S_n : n = 0, 1, 2 \ldots \}$ with ~~S(0)=0~~. $S_0 = 0$.

$$S_n = \sum_{k=1}^{n} X_k$$

where $\{X_k\}$ are independent and identically distributed random variables.

The random walk is simple if for each k $k = 1, 2, \ldots, n$, X_k takes values from ~~{a, b}~~ $\{a, b\}$. where a, b are real constants with

$$P(X_k = a) = p \;\; \& \;\; P(X_k = b) = 1 - p.$$

$$S_n = S_{n-1} + X_n.$$

↳ For CRR models with probability of uptick 'u' equals p and probability of downtick 'd' equals $(1-p)$ lifetime T and time increment $\Delta t = T/n$ then, the stock price is given by

$$S(T) = S(0) \exp\left(\mu T + \sigma\sqrt{\Delta t} \cdot Y \right). \quad — 8.$$

μ - drift, σ - volatility defined by 4 and Y - simple random walk given by 7.

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Now, $E(H) = E(\mu T + \sigma\sqrt{\Delta t}\cdot Y)$.

$= \mu T + \sigma\sqrt{\Delta t}\cdot E(Y)$.

$= \mu T + \sigma\sqrt{\Delta t}\, E(\sum_{k=1}^{n} X_k)$.

since $E(X_k) = 0$ — (9).

$\Rightarrow E(H) = \mu T$.

↳ $Var\left(\ln \frac{S(T)}{S(0)}\right) = Var(H)$

$= Var(\mu T + \sigma\sqrt{\Delta t}\cdot Y)$.

$= \sigma^2(\Delta t)\, Var(Y)$

$= \sigma^2(\Delta t)\, Var(\sum_{k=1}^{n} X_k)$.

$= \sigma^2(\Delta t) \cancel{\times} \sum_{k=1}^{n} Var(X_k)$.

$= \sigma^2(n)(\Delta t) = \sigma^2 T$.

From (8) & (9), μ — expectation of logarithmic return

Volatility (σ) is the S.D of log. return

Matching of CRR model with multi-period binomial model

To determine ~~B~~ 3 parameters (p, u, d) using (8), assuming that σ is known from the past data.
Since, we need to solve 3 variables with 2 eq^n,

$U = \ln(u)$ and $D = \ln(d)$.

~~and assume $D = -U$~~ $\Rightarrow d = 1/u$.

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Now (iv) becomes.

$$(2p-1)U = \mu \Delta t \quad \text{— (10) A.}$$
$$4p(1-p)U^2 = \sigma^2 \Delta t \quad \text{— (10) B.}$$

Squaring (10).A & adding to (10) B.

$$\Rightarrow (2p-1)^2 U^2 + 4p(1-p)U^2 = \mu^2 \Delta t^2 + \sigma^2 \Delta t \quad \text{— (11)}$$

$$\Rightarrow U = \sqrt{\mu^2 \Delta t^2 + \sigma^2 \Delta t} \quad \text{— A}$$
$$U = -D.$$

solving (10) A & (11).

$$p = \frac{1}{2} + \frac{\mu \Delta t}{2\sqrt{\mu^2 \Delta t^2 + \sigma^2 \Delta t}} \quad \text{— (11) A.}$$

Let us take n sufficiently large i.e
$\Delta t = T/n$. As n increases, Δt decreases.

Then Δt^2 can be neglected & thus.

(From A)

$$U = \ln(u) = \sigma\sqrt{\Delta t}$$
$$\Rightarrow u = e^{\sigma\sqrt{\Delta t}}$$
$$d = e^{-\sigma\sqrt{\Delta t}}$$
$$p = \frac{1}{2} + \frac{\mu\sqrt{\Delta t}}{2\sigma}$$
— (12)

A non-dividend paying stock is selling at Rs. 100. annual volatility (20%). Continuously compounded (risk free interest rate 5%). 2 period CRR ~~model~~. binomial option pricing model. to find the price of European call option $X = Rs\ 80$., time to expiration is 4 years.

$S(0) = 100$, $\sigma = 0.2$, $r = 0.05$, $T = 4\ yr$, $n = 2$, $X = 80$

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$= e^{0.2\sqrt{2}}$$
$$u = 1.3269$$
$$d = 0.7536.$$

$$S^{uu} = 100(1.3269)^2 = 176.0664$$
$$S^{ud} = 100(1.3269)(0.7536) = 100.$$
$$S^{dd} = 100(0.7536)(0.7536) = 56.7913.$$

$$f(S^{uu}) = 96.0664 \qquad f(S^{dd}) = 0.$$
$$f(S^{ud}) = 20$$

$$D(0) = \frac{1}{(e^{r\Delta t})^2}\left(p^* f(S^{uu}) + 2p^*(1-p^*) f(S^{ud}) + (1-p^*)^2 f(S^{dd})\right)$$

$$p^* = \frac{e^{r\Delta t} - d}{u - d} = 0.6132.$$

$$\Rightarrow D(0) = Rs\ 37.34.$$

Black Scholes Formula (from CRR Model)

→ Now, we define a counter on uptick and downtick movements on stock price at time $k\Delta t$ $(k = 1, 2, \dots n)$ as a bernoulli random variable.

$$Y_k = \begin{cases} 1 & \text{, with prob } (p) \quad (\text{stock } \uparrow) \\ 0 & \text{, with prob } (1-p) \quad (\text{stock } \downarrow) \end{cases}$$

Then, $S(n\Delta t) = S(T) = S(0)\, u^{\sum_{k=1} Y_k (n - \sum_{k=1}^{n} Y_k)}$.

$$\Rightarrow S(T) = S(0) \times d^n \left(\frac{u}{d}\right)^{\sum_{k=1}^{n} Y_k}.$$

$$\Rightarrow \frac{S(T)}{S(0)} = d^{T/\Delta t} \left(\frac{u}{d}\right)^{\sum_{k=1}^{n} Y_k} \quad \text{— } (13).$$

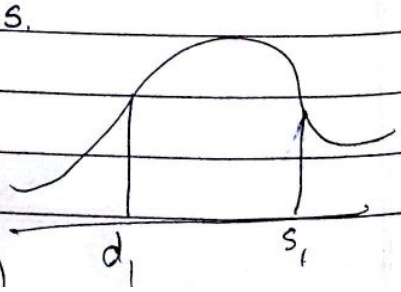
Derivation from book.

$$c(0) = S(0) . \phi(d_1) - K e^{-rT} \phi(d_2).$$

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

$$\phi(d_1) = \phi(-s) = \int_{s > s_1}^{\infty} e^{-s^2/2}\, ds.$$



$(d_1 = -s)$

In case, the european call option price is to be computed ~~at time~~ at $(0 < t < T)$ ~~during~~ at time ~~interval~~ t

Black Scholes formula reduces to $\rightarrow$

$$d_1 \cancel{C(t)} = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$C(t) = S(t)\cdot\phi(d_1) - K e^{-r(T-t)}\phi(d_2)$$

Value of put option $P(t) = K e^{-r(T-t)}\phi(-d_2) - S(t)\phi(-d_1)$

$\hookrightarrow$ In case, the stock is paying dividend, then the adjusted ~~obtained~~ stock price can be obtained by reducing for the discounted amount of the dividend upto time $t=0$

$$\boxed{S_{eff}(0) = S(0) - d e^{-r t_{div}}}$$

$t=0$ — t_{div} — $t=T$.

Q. Non paying dividend stock $(S(0)) = 100$
Volatility $(\sigma) = 0.24$. European call option offered
$r = 0.05$ (continuous compounding) $t = 3$ months $= 1/4$ yr
Strike price $\rightarrow K = X = 125$.
Find the option price using B.S.F?

Ans

$$d_1 = \frac{\ln\left(\frac{100}{125}\right) + \left(0.05 + \frac{0.24^2}{2}\right)\frac{1}{4}}{0.24\sqrt{1/4}}$$

$$\boxed{d_1 = -1.6954}$$

$d_2 = d_1 - \sigma\sqrt{T}$
$= -1.6954 - 0.24 \times \frac{1}{2}$
$= -1.8154.$

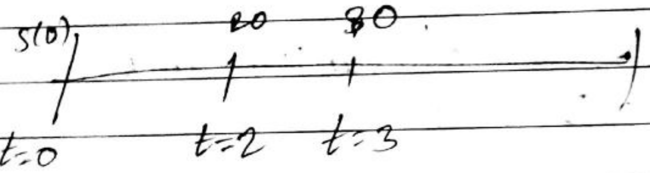
$\phi(d_1) = 0.0466.$, $\phi(d_2) = 0.0344.$

$C(0) = 0.2134.$

For 100 units $\to$ $C(0) = 21.34.$

Q $S(0) = 100$, $\sigma = 0.3$, $r = 0.05$, $T = 5yr.$
$K = 80.$
$d = Rs\ 20$ (2 years) , $d = Rs\ 30$ (3 years)

Ans



$S'(0) = S(0) - 20 \cdot e^{-0.05 \cdot 2} - 30\, e^{-0.15}.$
$= S(0) - 10\, e^{-0.1}\left(2 + 3e^{-0.05}\right).$
$= 56.082.$

(Rest as above).

$d_1 = 0.1786. = 0.18$
$d_2 = 0 - 0.4922.. = -0.49$

$\phi(d_1) = 0.5 + P(d_1) = 0.5714$
$\phi(d_2) = 0.5 - P(d_2)$
$= 0.3121.$

(Pg 127 → Ex 4.2)

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$C(0) = 12.68.$
(Use $S(0)$)

Q. $S(0) = 260$, $T = \frac{1}{2}$ yr. $K = X = 256.$
~~Black~~ (BSF) (C.C) $r = 0.04$, no dividend
$\sigma = 0.25.$ ($S(0)$ & $P(0) = ?$)

Ans $d_1 = 0.29$
(Using formula)

$d_2 = 0.11.$

$\phi(d_1) = 0.5 + P(0.29)$
$= 0.6141$
$\phi(d_2) = 0.5 + P(0.11)$
$= 0.5438.$

$\therefore C(0) = 23.21.$

$P(0) = 14.14.$

$\phi(-d_1) = 0.3859$
$\phi(-d_2) = 0.4562.$

σ - Implied volatility (When we change $C(0)$ & $P(0)$).

(Option Premium
||
Price of option)

CRR formula for Binomial approx.

The pay off for a call option with the strike price (X) satisfies

(x - payoff)

[current price at time = t]

$$f(x) = 0 \quad , \quad x \leq X.$$

(As option won't be exercised. so $f(x) = 0$)

The summation starts with least 'm'. such that

$$S(0)(1+u)^{m}(1+d)^{N-m} > X.$$

$$C(0) = (1+r)^{-N}\left(\sum_{k=m}^{N} {}^{N}C_{k}\,(p^*)^{k}(1-p^*)^{N-k} f\left(S(0)(1+u)^{k}(1+d)^{N-k}\right)\right)$$

$$= (1+r)^{-N}\left(\sum_{k=m}^{N} {}^{N}C_{k}\,(p^*)^{k}(1-p^*)^{N-k}\left[S(0)(1+u)^{k}(1+d)^{N-k} - X\right]\right)$$

XXXXX

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Stochastic Processes

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↳ A σ - field F is a family of subset of Ω (sample space) satisfying the following properties

① $\phi \in F$
② If $A \in F \Rightarrow A^c \in F$
③ If $A_i \in F \Rightarrow \bigcup_i A_i \in F$.

Eg 1 : $\Omega = \{ HHH, HHT, \cancel{HHH} HTH, HTT, THH, THT$
$\cancel{TH}\ TTH, TTT \}$

$A_1 = \{ HHH, HHT, HTH, HTT \}$

$A_2 = \{ THH, THT, TTH, TTT \}$.

$F = \{ \phi, A_1, A_2, \Omega \}$.

↳ Thus, σ field.

Def: If Ω is the sample space and P be the probability measure defined on F, then collection of random variable $\{ X_t, t \in T \}$ where T is the index set (time) defined on the probability space (Ω, F, P) is called a Stochastic Process (also random process)

$$X_t = X_t(\omega) \quad , \omega \in \Omega .$$

$$\{ X_t, t \in T \} = \{ \cancel{8}\ x_t(\omega) : \omega \in \Omega, t \in T \}$$

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$X: T \times \Omega \to R$.

└ represented as $X(t, \omega)$.

\# For any stochastic process $\{X_t, t \in T\}$. $\{t \in T\}$ parameter state & X_t for $t \in T$ is called state space (represented by S).

$X(t, \omega)$ has 4 possibilities.

(ω, t - discrete or continuous).

(time) (space)

① t - continuous & ω - discrete

↳ at any time, no. of stocks held by investor.

② t - continuous & ω - continuous.

↳ price of stock at any time

↳ humidity of room at any time.

- Independent Increment

↳ If for all n, $t_1 < t_2 < \cdots < t_n$, then the random variable $X_{t_2} - X_{t_1}$, $X(t_3) - X(t_2)$, …. $X(t_n) - X(t_{n-1})$ are independent random variable, then the process is said to have independent increment.

Strict sense is also called strong stationary

(SP) (S.P)

↳ The S.P ~~is~~ $\{X_t, t \geq 0\}$ is called strict sense stationary process if for arbitrary $t_n, t_{n-1}, \dots t_2, t_1 > 0$ the finite dimensional random vector.

$\{X(t_1), X(t_2) \dots\dots X(t_n)\}$ & $\{X(t_1+h), X(t_2+h) \dots\dots X(t_n+h)\}$

have the same joint distribution for all $h > 0$ & all $t_1, t_2 \dots t_n$.

- Wide sense stationary S.P

↳ A stochastic process $\{X_t, t \geq 0\}$ is a wide sense SSP if it satisfies the following conditions.

(i) $E(X(t))$ ~~= $\mu(t)$~~ is independent of t.

(ii) $\mathrm{Cov}(X(t), X(s))$ depends only on time difference $(t-s)$ for all t & s.

(iii) $E(X^2(t)) < \infty$ (should be finite)

Wide sense SSP ≡ Covariant stationary ≡ Weak stationary ≡ second order SSP.

Eg: $X(t) = A\cos(\omega t) + B\sin(\omega t)$

A, B - uncorrelated with expectation 0 & variance 1.

ω - positive constant.

Show that $X(t)$ is ~~weak~~ wide sense S.P.

Ans (1) $E(x(t)) = E(A\cos(\omega t) + B\sin(\omega t))$.

$= 0$ (finite & independent of t)

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(2) $Cov(X(t), X(s)) \quad (s<t)$.

$= E[X(t) \cdot X(s)] - E(X(t))E(X(s))$.

$= E(X(t) \cdot X(s))$.

$\Rightarrow$

$E(A^2) = 1, \quad E(B^2) = 1, \quad E(AB) = 0$

(as variance is 1) $\quad$ (as $E(A)E(B) = 0$).

$= \cos(\omega t)\cos(\omega s)E(A^2) + \sin(\omega t)\sin(\omega s)E(B^2)$
$+ (\cos(\omega t)\sin(\omega s) + \sin(\omega t)\cos(\omega s))E(AB)$

$= \cos(\omega(t-s))$.

(3) $E(X^2(t)) < \infty$.

$\therefore$ It is wide sense S.P.

Markov's Property

↳ A given stochastic process $\{X_t (t \in T)\}$ is said to have markov's property if for all n and for all $0 < t_1 < t_2 \dots < t_n < t$, then (cumulative density function) satisfies

$P(X(t) \leq x \mid X(0) = x_0, X(t_1) = x_1, \dots, X(t_n) = x_n\}$

$= P(X(t) \leq x \mid X(t_n) = x_n)$.

i.e, future prediction depends only on the current state of stochastic process & does not depend upon the past information

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Def Random Walk.

Symm Random walk

↳ Consider a random experiment for tossing a fair coin ~~~~ infinitely many times & let successive outcomes be denoted by.

$$\omega = \{\omega_1, \omega_2, \dots\}, \quad X_j = \begin{cases} 1 & \text{if } \omega_j = H \\ -1 & \text{if } \omega_j = T \end{cases}$$

then $P(X_j = 1) = \frac{1}{2}$, $P(X_j = -1) = \frac{1}{2}$.

Let $S_0 = 0$ and $S_k = \sum_{j=1}^{k} X_j$

(final displacement from mean position)

$\{S_n ; n = 0, 1, 2 \dots\}$ – symmetric random walk.

Thm: If $\{S_k ; k = 0, 1, 2 \dots\}$ be a symm. random walk.

(i) $E(S_k) = 0$ for each k

(ii) $Var(S_k) = k \quad \forall k$.

(iii) It has independent increment

(iv) It has stationary increment

(v) It is a markov's process.

Proof: ① Since in a random walk.

$$E(X_j) = 0, \quad Var(X_j) = 1.$$

$$E(S_k) = E\left(\sum_{j=1}^{n} X_j\right)$$

$$= E(X_1 + X_2 + \dots X_k) = \cancel{E(X_1}$$

$$+ E(X_n)$$

$$= E(X_1) + E(X_2) + \cdots$$

$$= \sum_{j=1}^{n} E(X_j)$$

$$= 0.$$

(2) Since $Cov(X_j)$ is $0 \Rightarrow$ all X_j are independent of each other.

$$Var(S_K) = Var\left(\sum_{1}^{n} X_j\right)$$

$$= \sum_{j=1}^{n} (Var(X_j))$$

$$= K.$$

(3) Choose an arbitrary non-negative positive integer

$$0 = K_0 < K_1 < \cdots < K_n.$$

$$S_{K_{i+1}} - S_K = \sum_{j=K_i+1}^{K(n)} (X_j)$$

Since X_j are independent and identically distributed random variables having Bernoulli's distribution therefore

$S_{K_1} - S_{K_0}$, $S_{K_2} - S_{K_1}$ and so on. are mutually independent variables & hence S.P defined has independent increment.

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④ Stationary increment

↳ Choose non-negative integer, $k_1 < k_2$

$$S_{k_2} - S_{k_1} = \sum_{j=k_1+1}^{k_{(n)}} X_j$$

Since X_j are independent and identically distributed random variable having Bernoulli distribution = $S_{k_2} - S_{k_1}$ has same distribution of $S_{k_2-k_1} - S_0$.

Therefore, they have stationary increment as they are time independent.

⑤ Markov's process

$$S_k = S_{k-1} + X_k \quad \text{for } k = 1, 2 \ldots$$

$$P(S_k \le x \mid S_{k-1} = X_{k+1}, S_{k_2} = X_{k_2} \ldots\ldots, S_1 = X_1\}$$

$$= \frac{P(S_k \le x,\ S_{k-1} = X_{k-1} \ldots\ S_1 = X_1)}{P(S_{k-1} = X_{k-1} \ldots\ldots\ S_1 = X_1)}.$$

Cancelling numerator & denominator.

$$= P(S_k \le X \mid S_{k-1} = X_{k-1}).$$

∴ Symm R.W is a markov's process.

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Poisson Process

↳ A stochastic process $\{X_t, t > 0\}$ is said to be a poisson's process with intensity or rate (parameter) $\lambda > 0$ if it satisfies the following

(i) It starts from 0. $(X_0 = 0)$.

(ii) For all n & for all $0 < t_0 < t_1 < \dots < t_n$, increments $S(t_i) - S(t_{i-1})$ $(i = 1, 2 \dots n)$ are independent & stationary.

(iii) For $0 \le s < t$, $S(t) - S(s)$ is a poisson distributed r.v with parameter $\lambda(t-s)$ i.e,

$$P(S(t) - S(s) = n) = \frac{\lambda(t-s)^n e^{-\lambda(t-s)}}{n!}$$

$(n = 0, 1, 2 \dots \dots)$.

↳ Memory less distribution

↳ By second property, it is a markov's process.

Brownian Motion (Weiner Process).

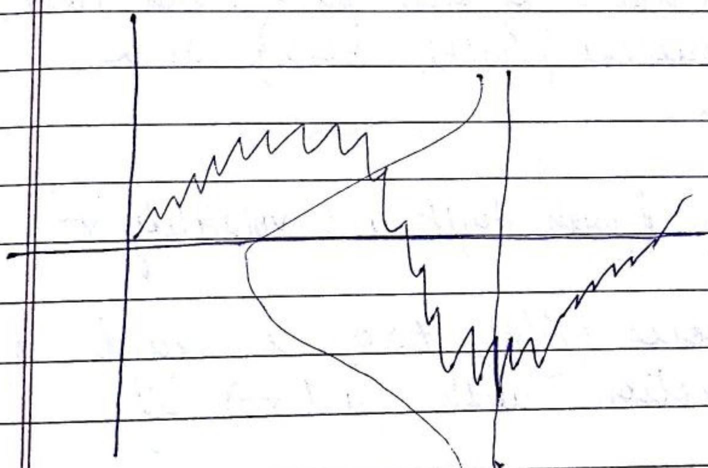
→ A S.P $\{w_t, t \geq 0\}$ is said to be brownian motion if it satisfies.

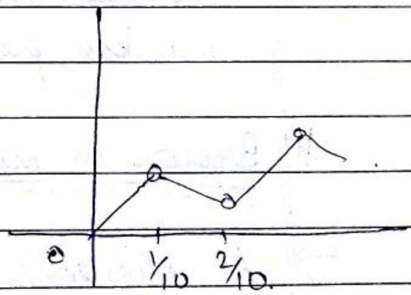
① $w(0) = 0$

② for. $t > 0$, sample path of w_t is continuous

③ $\{w(t), t \geq 0\}$ has stationary & ~~independent~~ independent increments.

④ For $0 \leq s \leq t < \infty$.
$w(t) - w(s)$ is a normally distributed R.V with mean 0 & variance $(t-s)$.





→ Path is always continuous but it is nowhere differentiable.

→ A weiner's process is not wide sense stationary. because the covariance of $w(t), w(s)$ for $0 \leq s < t$ is not a function of $(t-s)$.

$$Cov(w(t), w(s)) = \min(t, s).$$

$$E(w(t) - w(0)) = 0 = E(w(0) - E(s))$$

↳ For brownian motion. use normal distribution

$$\text{Cov}(w(t), w(s)) = E\{(w(t) - E(w(t)))(w(s) - E(w(s)))\}$$

$$= E(w(t) \cdot w(s))$$

$$= E((w(t) - w(s) + w(s))\, w(s))$$

$$= E((w(t) - w(s))\, w(s)) + E(w^2(s))$$

$$= E(w(t) - w(s))\, E(w(s)) + E(\underbrace{w^2(s)}_{(w(s) - w(0))})$$

$$= s$$

$$= \min(s, t)$$

↳ Given $w(t)$, future $w(t+h)$ $h > 0$ only depends on the increment $(w(t+h) - w(t))$ which is independent of the past. and hence, the stochastic process $\{w(t), t \geq 0\}$ is a markov process.

Brownian motion (with drift u & volatility σ)

↳ A stochastic process $(X(t), t \geq 0)$ is said to be a brownian motion with (u & σ) if

$$X(t) = ut + \sigma(w(t)) \text{ where}$$

(1) $w(t)$ is a standard brownian motion.
(2) $-\infty < u < \infty$ is a constant
(3) $\sigma > 0$ is a constant.

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Geometric B.M

→ $X(t) \rightarrow$ Stochastic process
$X(t) = X(0)\, e^{W(t)} \rightarrow$ Standard BM

$$\log\left(\frac{X(t)}{X(0)}\right) = W(t)$$

- Independent increments

- Markov Property

$$X(t+h) = X(0)\, e^{W(t+h)} \qquad (h>0)$$

$$W(t+h) - W(t) = X(0)\, e^{W(t+h) - W(t) + W(t)}$$
$$= X(0)\, e^{W(t)} \cdot e^{W(t+h) - W(t)}$$
$$= X(t) \cdot e^{W(t+h) - W(t)}$$

→ $S(t) = S(0) \cdot e^{H(t)}$ where $H(t) = ut + \sigma W(t)$

$E(\cancel{S(t)})\; H(t) = \ln\left(\frac{S(t)}{S(0)}\right)$

$H(t) \sim N(ut, \sigma^2 t)$

$E(S(t)) \rightarrow e^{ut}$

$Var(S(t)) \rightarrow e^{ut + \frac{1}{2}\sigma^2 t}$

$\frac{S(t)}{S(t)}$ is log normally distributed

$$M_x(\theta) = E(e^{\theta x}) = e^{u\theta + \frac{1}{2}\sigma^2\theta^2}$$

$$E(ax) = a\,E(x)$$

$$E(S(t)) = E\left(S(0)e^{W(t)}\right)$$

$$= S(0)\,E\left(e^{W(t)}\right)$$

$$E(S(0)) = S(0)\cdot e^{ut + \frac{1}{2}\sigma^2 t}$$

$$Var(S(t)) \to E(S(t)^2) - \left(E(S(t))\right)^2$$

$$= (S(0))^2\left(e^{2ut + \frac{1}{2}\sigma^2 tu}\right) - (S(0))^2 e^{2(ut + \frac{1}{2}\sigma^2 t)}$$

$$= \left[S(0)e^{(ut + \frac{1}{2}\sigma^2 t)}\right]^2\left(e^{\sigma^2 t} - 1\right) = S(t) = S(0)e^{W(t)}$$

↳ Filteration: $\Omega = \{H, T\}$

$$P(\Omega) = \{\phi, \{H\}, \{T\}, \{H, T\}\}$$

$$F_0 = \{\phi, \Omega\}$$

$$F = \{\phi, \{H\}, \{T\}, \Omega\}$$

↳ Filteration (in discrete ~~terms~~ time)

Let Ω be a sample space, then filteration in discrete time is an increasing sequence

$F_0 \subset F_1 \subset F_2 \subset \dots$ of σ field one per time instant.

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↳ In Continuous terms (filteration)

↳ Let Ω be a sample space.
$T \to$ fixed.

$t \in [0,T]$.

Assume that, if $s \le t$ then $F_s \subset F_t$

Then, collection of σ fields $\{F_t : 0 \le t \le T\}$ is filteration in continuous terms.

↳ Filteration is used to model ~~filteration~~ flow of information over time.

- σ field generated by a S.P

↳ Let $\{y(t), t \in T\}$ be a given S.P, then the σ field generated by the S.P is the smallest σ field containing all the sets of the form.

$\{\omega :$ the sample path, $(y(t), t \in T)$ belongs to $C\}$ for all suitable sets (C) of functions on T.

Eg: σ field generated by brownian motion.

$W = \{w(s) : 0 \le s \le t\}$. be the given brownian motion on $[0,t]$ then σ field generated by all sets of the form.

$A_{t_1, t_2, \ldots t_n} = \{\omega \in \Omega : w(t_1, \omega); w(t_2 : \omega), \ldots w(t_n, \omega) \in$

for any n-dimensional Borel set C. and for any choice of $t_i \in [0,t]$ is called

σ field generated by brownian motion (ω).

X_t is F_t - measurable.

↳ Let F_t be a σ field, of subsets of Ω, then a r.v X_t is F_t measurable if every set in $\sigma(X_t)$ is also present in F_t. then, X_t will be called F_t measurable.

A r.v is F_t measureable iff the information in F_t is sufficient to determine value of X_t.

- Conditional Expectation

→ If X & Y are distinct r.v, then the conditional probability mass function of X for given Y = y is defined for all y such that

$$\cancel{E(Y}\ P(Y=y) > 0 \text{ as } P(X=x/Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} \quad \text{— (1)}$$

The conditional distribution fn of X for given Y = y is defined as.

$$F(X/Y=y) = P(X \leq x \mid Y=y).$$

The condition expectation of X for given Y = y is defined as

$$E(X/Y=y) = \begin{cases} \int_{-\infty}^{\infty} x \, d(F(X/Y=y)) \, dx, & \text{continuous} \\ \sum_x x \, P(X/Y=y), & \text{discrete.} \end{cases}$$

→ $E(X) = \sum_y E(X/Y=y) \cdot P(Y=y)$.

$$= \sum_x \sum_y x \, P(X/Y=y) \, P(Y=y) = \sum_x x \sum_y P(X/Y=y) \, P(Y=y)$$

$$= \sum_x x \, P(x)$$

$E(X)$

Q. 2 Refills for a ball point are selected ~~at~~ from a bag which has 3 blue, 2 red, 3 green.

(1) X - no. of blue refills.

(2) Y - no. of red refills.

(i) Find Joint P. Distribution

(ii) $P(X, Y) \in A$ $A = x + y \le 1$.

(iii) $E(X / Y = y)_{y=1}$

(iv) Conditional distribution of X for $Y = y$.

Ans (i) $R_x = \{0, 1, 2\}$, $R_y = \{0, 1, 2\}$.

$\frac{3\times 7}{8\times 7}$ 4

X \ Y	0	1	2	Marginal h(x)
0	$\frac{^3C_2}{^8C_2}$	$^3C_1 \times {^2C_1}$	$^2C_2 / {^8C_2}$	$10/28$
1	$\frac{^3C_1 \times {^3C_1}}{^8C_2}$	$\frac{^2C_1 \times {^3C_1}}{^8C_2}$	0	$15/28$
2	$^3C_2 / {^8C_2}$	0	0	$3/28$
Marginal y g(y)	$15/28$	$12/28$	$1/28$	1

(ii) $x + y \le 1$

$(0,0), (1,0), (0,1)$

$$P(X,Y) \in A = P(0,0) + P(0,1) + P(1,0)$$
$$= 9/14.$$

(iii) ~~E E(X|Y=1)~~ =

(iii) $f(X/Y=y) = \frac{f(x,y)}{g(y)}$.

For $Y=1 \rightarrow f(X/Y=1) = \frac{f(x,y)}{3/7} = \frac{7f(x,y=1)}{3}$.

$f(0/1) = 1/2$, $f(1,1) = 1/2$, $f(2,1) = 0$.

Conditional distribution

(iv) $E(X/Y=1) = \cancel{\frac{6}{28}\times 0 + \frac{6}{28}\times 1 + 0\times 0}$.

$= \cancel{\frac{6}{28} = \frac{3}{14}} = 0f(0/1) + 1f(1/1) + 2f(2/1)$
$= 1/2$.

Martingales

$\hookrightarrow$ ~~Let~~

Let (Ω, F, P) be a probability space and $\{X_n, n=0,1,2,\dots\}$ be the S.P and.

$\{F_n, n=0,1,2\dots\}$ be the filteration.

Then, this S.P is said to be martingale corresponding to the filtration if it satisfies the following conditions.

(i) For every n, $E(X_n)$ exists

(ii) Each x_n is F_n - measurable

(iii) For every n, $E(X_{n+1}/F_n) = X_n$.

Note: $E(E(X/Y)) = E(X)$.

Observing the definition of martingale,

$E(X_{n+1}) = E(X_n) \quad \forall n$.

$\Rightarrow E(X_n) = $ Constant.

(1) $E(X_{n+1}/F_n) > X_n$ then S.P $\{X_n, n \in I\}$ is sub-martingale

(2) $E(X_{n+1}/F_n) < X_n$ then S.P $\{X_n, n \in I\}$ is super-martingale

Q Let $X_1, X_2 \ldots$, be sequence of iid (identically independent distributed) r.v each taking value $(+1, -1)$ with equal probabilities.
Let us define $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$

Thus this discrete time S.P $\{S_n, n = 0, 1, 2 \ldots\}$ is a symmetric random walk.

Prove that $\{S_n, n = 0, 1, 2 \ldots\}$ is a martingale w.r.t $\{X_n, n = 0, 1, 2 \ldots\}$.

Proof: $E(|S_n|) = E(|\sum_{i=1}^{n} X_i|) \leq E(\sum_{i=1}^{n} |X_i|) \leq \sum_{i=1}^{n} E(|X_i|) < \infty$

All info of S_n is through X_n. i.e, they are finite.

$\therefore S_n$ is X_n - measurable.

$\therefore E(X_n)$ exists.

$E(S_{n+1} / X_1, X_2, \ldots X_n)$.

$= E(S_n + X_{n+1} / X_1, X_2, \ldots X_n)$.

$= E(S_n / X_1, X_2 \ldots X_n) + E(X_{n+1} / X_1, X_2 \ldots X_n)$.

$= S_n + \underbrace{E(X_{n+1})}_{\to 0 \text{ (iid)}}$.

$= S_n$.

$\therefore$ The S.P (S_n) is a martingale.

Ques: Pg-261. (Q 8.6.3)

Prob 8.6.5

Q A fair coin tossed infinitely many times (coin = Rs 1). Person get rupees 2 for heads & nothing for tails.

Let Y_n be his fortune at the n^{th} toss. Prove that Y_n is a martingale.

Ans Let $X_1, X_2 \dots X_n$ be iid.

$$X_i = \begin{cases} 2, & p = 0.5 \\ 0, & p = 0.5 \end{cases}$$

$Y_n = X_1 \cdot X_2 \dots X_n$, $n = 1, 2 \dots$

Let F_n be the σ field generated by $X_1, X_2 \dots X_n$

$(0 \leq y \leq 2^n)$

$E(X_i) = 1$

First two conditions are satisfied.

$E(Y_{n+1}/F_n) = E(Y_n \cdot X_{n+1}/F_n)$.

$= Y_n \, E(X_{n+1}/F_n)$.

$= Y_n E(X_{n+1})$

Since X_i are iid r.v

$= Y_n \times 1$

$= Y_n$.

$\therefore Y_n$ is a martingale.

Q Consider a binomial lattice model, let S_n be stock price at period n. ~~and S_{n+1} is equal to uS_n~~

$$S_{n+1} = \begin{cases} uS_n, & p \\ dS_n, & (1-p) \end{cases}$$

Define a related process $R_n = \ln(S_n) - n(p\ln(u) + (1-p)\ln(d))$

Prove that $\{\ln(S_n);\ n = 1, 2 \ldots\}$ is not a martingale whereas $\{R_n : n = 1, 2\}$ is a martingale w.r.t $\{S_n : n = 1, 2 \ldots\}$

Also, prove that discounted stock process $\{S_0, e^{-r}S_1, e^{-2r}S_2 \ldots\}$ is a martingale only if $\left(p = \dfrac{e^r - d}{u-d}\right)$. where r - nominal interest rate.

Proof (Ans) In this binomial lattice model. $\{S_0, S_1, \ldots\}$ with the natural filteration $\{F_0, F_1 \ldots\}$ we have,

$$\cancel{P(S_{n+1})} \cdot \underset{(uS_n)}{P(S_{n+1} / F_n)} = \underset{(1-p)}{1} - P(S_{n+1} = dS_n / F_n). \quad \text{--- (1)}$$

$\therefore$ The ~~expectation of S_{n+1}, i.e~~

$$E(S_{n+1} / F_n) = puS_n + (1-p)dS_n \quad \text{--- (2)}$$

We consider $\ln(S_n)$.

$$E\left(\ln\left(\frac{S_n}{S_{n-1}}\right) / S_0, S_1 \ldots S_{n-1}\right)$$

$$= p\ln(u) + (1-p)\ln(d). \quad -(2)$$

$\hookrightarrow E(\ln(S_n) - \ln(S_{n-1}) / S_0, S_1 \ldots S_{n-1})$

$= E(\ln(S_n) / S_0, S_1, S_2 \ldots S_{n-1}) - \underbrace{E(\ln(S_{n-1}) / S_0, S_1, S_2 \ldots S_{n-1})}_{= \ln(S_{n-1})}$

$\Rightarrow E(\ln(S_n) / S_0, S_1 \ldots S_{n-1}) = \ln(S_{n-1}) + p\ln(u) + (1-p)\ln(d)$

(Logarithmic price of stock is not a martingale) $\neq \ln(S_{n-1})$.

- $E(R_n / R_0, R_1, \ldots R_{n-1}) = R_{n-1}$ for martingale

~~$E(R_{n+1} / R_0, R_1 \ldots R_n) = E(\ln(S_{n+1}) - (n+1)$~~

(As first two conditions are satisfied as it only depends of S_n which itself satisfies them).

$E(\ln(S_n) - n(p\ln(u) + (1-p)\ln(d)) / R_0, R_1 \ldots R_{n-1})$.

$= E(\ln(S_n) / R_0 \ldots R_{n-1}) - E(n p\ln(u) + n(1-p)\ln(d)) / R_0 \ldots R_{n-1})$

(R_i can be replaced by S_i)

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$= E(\ln(S_n) / S_0 \ldots S_{n-1}) - n(p\ln(u) + (1-p)\ln(d))$

$= \ln(S_{n-1}) - (n-1)(p\ln(u) + (1-p)\ln(d))$

$= R_{n-1}$.

Hence, R_n is a martingale.

↳ Assume; discounted stock price is martingale, solve to get value of p.

$$E(e^{-(n+1)r} S_{n+1} / F_0, \ldots F_n) = e^{-nr} S_n.$$

• Wealth process. (Do from book)

Continuous time

↳ Let (Ω, F, P) be the probability space. Let $\{X(t), t \geq 0\}$ be a S.P. and $\{F_t, t \geq 0\}$ be filteration.

The S.P $X(t)$ is said to be martingale corresponding to filteration F_t if it satisfies the following conditions:

(i) $E(X(t))$ exist

(ii) $X(t)$ is F_t measurable

(iii) $0 < s < t$, $E(X(t) | F_s) = X_s$.

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Q Prove that $\{w(t), t \geq 0\}$ ~~w~~ - is a martingale (w - Brownian motion)

Ans (i) $E(w(t))$ exist
(ii) F_t measurable
(iii) $E(w(t)/F_s) = E(w(t) - w(s) + w(s)/F_s)$
$= E(w(t) - w(s)/F_s) + E(w(s)/F_s)$

(For any s, $0 \leq s \leq t$)

$\overset{\shortparallel}{0}$ (normally distributed).

Info till current ~~state is~~ known
$w(s)$ - constant

$\therefore = w(s)$

$\therefore$ It is a martingale

Imp
One of
Q

Q Show that $\exp(w(t) - t/2)$ is a martingale. $(t \geq 0)$

Ans For any s, $0 \leq s \leq t$, $\{F(t), t \geq 0\}$ is a natural filteration.

$w(t) - t/2 \to -t/2$ is not an r.v.

So, we consider.

$$E(e^{w(t)}/F(s)) = E(e^{w(t) - w(s) + w(s)}/F_s)$$

$$= e^{w(s)} E(e^{w(t) - w(s)}/F_s).$$

$$= e^{w(s)} E(e^{w(t) - w(s)})$$

Ans

$$\psi_X(\theta) = E(e^{\theta X}) = \exp(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2)$$

$$= e^{w(s)} e^{\frac{(t-s)}{2}}.$$

$$E(\cancel{w} e^{w(t) - t/2} / F_s) = e^{-t/2} E(e^{w(t)}/F_s)$$

$$= e^{-t/2} e^{w(s)} e^{t - s/2}$$

$$= e^{w(s) - s/2}.$$

$\therefore$ It is a martingale.

Imp ~~Prob~~ 8.9, 11, 12, 22, 23 — Assignment

one of these

Q $\{S_n, n = 0, 1, 2 \dots\}$. $F_n \rightarrow$ Filt, $Y_n = (-1)^n \cos(\pi S_n)$.
$\{N(t), t \geq 0\} \rightarrow$ P.P with parameter λ.

P.T $\{N(t) - \lambda t, t \geq 0\}$ is martingale.

Ans (1) $E(N(t) - \lambda(t))$
$= E(N(t)) - \lambda t$
$= \lambda t - \lambda t$
$= 0$ (Expectation exists at all times)

(2) F_t is natural ~~ac~~ filtration.
For any s, $0 < s < t$.

$N(t) - \lambda t$ is F_t measurable.

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(iii) For any s, $0 < s < t$.

$$E(N(t) - \lambda(t) / F_s)$$

$$= E(N(t) - N(s) + N(s) - \lambda(t) / F_s).$$

$$= E(N(t) - N(s) / F_s) + E(N(s) - \lambda(t) / F_s).$$

(memoryless distribution)

$$= E(N(t) - N(s)) + N(s) - \lambda(t).$$

$$= \lambda(t-s) + N(s) - \lambda(t)$$

$$= N(s) - \lambda(s)$$

$\therefore$ It is a martingale.

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19/10 Stochastic Calculus

↳ Let the time interval be $[0, T]$. And it's one of the partition is (π).

$$\pi = [0 = t_0 < t_1 < \dots < t_n = T\}. \quad — (1)$$

(small pie)

$\pi \in \Pi$ = Collection of all partitions

$$\|\pi\| = \max_{0 \le k \le n-1} (t_{k+1} - t_k) \quad — (2)$$

The quadratic variation for brownian motion $\{w(t), t \ge 0\}$ over the interval is denoted by:-

$$[w, w](T) = \lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} (w(t_{i+1}) - w(t_i))^2.$$

$$= \lim_{\|\pi\| \to 0} Q_\pi \quad — (3)$$

S.P on index (n).

$\{X_n, n \ge 1\}$ — Let X be a r.v defined on the normal probability space (Ω, F, P).

We say that, X_n converges to X in mean square sense if $\lim_{n \to \infty} E(|X_n - X|^2) = 0 \quad — (4)$

Let $\{X_n, n \ge 1\}$ be a SP and X be a random variable defined on common probability space (Ω, F, P). We say that X_n converges to X in mean square sense if $\lim_{n \to \infty} E(|X_n - X|)^2 = 0$

— (5)

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Th: Let Q_π be as defined in (iv), then:-

(i) $E(Q_\pi) = T$
(ii) $Var(Q_\pi) \leq 2\|\pi\| T$

Proof: i) $E(Q_\pi) = E\left(\sum_{i=0}^{n-1} W(t_{i+1}) - W(t_i)\right)^2$.

Since $W(t_{i+1}) - W(t_i)$ is normally distributed with mean 0 and variance $t_{i+1} - t_i$.

$$\Rightarrow E(Q_\pi) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) = T \quad \text{--- (6)}$$

~~QP~~.

(ii) $$Var(Q_\pi) = \sum_{i=0}^{n-1} Var\left(W(t_{i+1}) - W(t_i)\right)^2 \quad \text{--- (7)}$$

$$Var\left[\left(W(t_{i+1}) - W(t_i)\right]^2 \cancel{(t_{i+1} - t_i)}\right. = E\left[W(t_{i+1}) - W(t_i)\right]^4$$

$$- 2E\left[\left\{W(t_{i+1}) - W(t_i)\right\}^2 (t_{i+1} - t_i)\right]$$

$$+ [t_{i+1} - t_i]^2 \quad \text{--- (8)}$$

↳ Since the fourth order normal distribution, mean 0 and variance $t_{i+1} - t_i$ is $3(t_{i+1} - t_i)^2$.

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$\Rightarrow 3(t_{i+1}-t_i)^2 - 2(t_{i+1}-t_i)^2 + (t_{i+1}-t_i)^2$

$= 2(t_{i+1}-t_i)^2$ — (9)

~~Var~~

$$\hookrightarrow \mathrm{Var}(Q_\pi) = \sum_{i=0}^{n-1} 2(t_{i+1}-t_i)^2$$

$$= \sum_{i=0}^{n-1} 2(t_{i+1}-t_i)(t_{i+1}-t_i)$$

$$\leq 2\|\pi\| \sum_{i=0}^{n-1} (t_{i+1}-t_i)$$

$$= 2\|\pi\| T$$

— (10)

uted

$)^4]$

Note We have with the discussion since variance of $Q_\pi = E(Q_\pi - T)^2$.

Using (10),

$$\lim_{\|\pi\|\to 0} E(Q_\pi - T)^2 = \lim_{\|\pi\|\to 0} Var(Q_\pi) = \lim_{\|\pi\|\to 0} 2\|\pi\| T$$

$$= 0 \quad \text{(As it cannot be} < 0)$$

$\therefore$ The quadratic variation $w(t)$ will be equal to. T.

$$\left([w,w](T) = \lim_{\|\pi\|\to 0} Q_\pi = T \right)$$

→ Since the quadratic variation of $w(t)$ on interval $[0,T]$ is T.

$$\lim_{\|\pi\|\to 0} \sum_{i=0}^{n-1} [w(t_{i+1}) - w(t_i)]^2 = T \quad -(11).$$

Also, For $0 < T_1 < T_2$.

Quadratic variation of T_2 minus that of T_1 is $T_2 - T_1$

The Brownian motion accumulates the length of the interval $[T_1, T_2]$.

Since, it is true for every interval, we infer that Brownian motion accumulates the quadratic variation at the rate ~~1/per~~ 1 per unit time & Thus, can be written as

$$\left(\lim_{h \to 0} \frac{w(t_{i+1}) - w(t_i)}{h = t_{i+1} - t_i} \right)$$

$$(dw \cdot dw) = dt. \quad — (12)$$

Thm Let $\{w(t), t \geq 0\}$ be the given brownian motion and $\pi = \{0 = t_0 < t_1 < t_2 \dots < t_n = T\}$ be a partition of $[0, T]$. Then:-

$$(1) \lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} (w(t_{i+1}) - w(t_i))(t_{i+1} - t_i) = 0. \quad — (13)$$

$$(2) \lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = 0 \quad — (14)$$

Proof: We consider $\sum_{i=0}^{n-1} |(w(t_{i+1}) - w(t_i))(t_{i+1} - t_i)|$.

$$\leq \sum_{i=0}^{n-1} \left\{ \max_{0 \leq i \leq n-1} |(w(t_{i+1}) - w(t_i))| \right\} (t_{i+1} - t_i).$$

$$\leq \max_{0 \leq i \leq n-1} |(w(t_{i+1}) - w(t_i))| \times T.$$

$$\therefore \lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} (w(t_{i+1}) - w(t_i))(t_{i+1} - t_i) = 0. \qquad \left(\begin{array}{c} \|\pi\| \to 0 \\ \text{means } \Delta t \to 0 \end{array} \right)$$

Similarly,

$$\lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = 0.$$

$$= \sum_{i=0}^{n-1} (t_{i+1} - t_i)(t_{i+1} - t_i) = \sum_{i=0} \left\{ \max_i (t_{i+1} - t_i) \right\} (t_{i+1}$$

$$= \sum_{i=0}^{n-1} \|\pi\| (t_{i+1} - t_i) = \|\pi\| T. = 0$$

$$\left\{ \lim_{\|\pi\| \to 0} \|\pi\| T = 0 \right\}$$

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↳ $dw\,dt = 0$ — (13) — (12)(A)
$dt \cdot dt = 0.$ — (14) — (12)(B)

- **Stochastic Integral**

↳ Let $\{X_t : t \geq 0\} \rightarrow$ S.P adopted to the natural filtration $\{F_t : t \geq 0\}$. of brownian motion of $\{w(t);\ t \geq 0\}$.

Consider a partition (π) of $[0, T]$ interval.
$\pi = \{0 \leq t_0 < t_1 < t_2 < \ldots < t_n = T\}$.

We form the sum, under the limit $\|\pi\| \rightarrow 0$.

$$\lim_{\|\pi\| \rightarrow 0} \sum X(t_i)\{w(t_{i+1}) - w(t_i)\}$$
$$= \int_0^T X(s)\, d(w(s)) = I(T). \quad -(15)$$

defining the stochastic integral, the convergence is used as mean square convergence.
(refer to eq (3)).

For ito integral ($I(T)$, $t \geq 0$)
↳ (i) $E(I(T)) = 0$.
(ii) $E\left(\int_0^t X(s)\, dw(s)\right)^2 = E\left[\int_0^t X^2(s)\, dw(s)\right]$.

(iii) The process $I(T)$ has continuous sample path.

(iv) $I(t)$, for each t. $(0 \leq t \leq T)$
$I(t)$ — is F_t measurable.

Q
Ans

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Q Find value of the integral $\int_0^T w(s)\, d\,w(s)$.

Ans Consider π — an arbitrary partition. $(0 = t_0 < t_1 < t_2 \cdots < t_n = T)$

$$\int_0^T w(s)\, d\,w(s) = \lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} w(t_i)\left\{ w(t_{i+1}) - w(t_i) \right\}$$

For each i, $w(t_i)$ and $(w(t_{i+1}) - w(t_i))$ are independent random variable having normal distribution

$$Q_\pi = \sum_{i=0}^{n-1} \left(w(t_{i+1}) - w(t_i) \right)^2$$

$$= \sum_{i=0}^{n-1} \left\{ w^2(t_{i+1}) + w^2(t_i) - 2\,w(t_{i+1})\,w(t_i) + 2\,w^2(t_i) - 2w^2(t_i) \right\}.$$

$$= \sum_{i=0}^{n-1} \left\{ w^2(t_{i+1}) - w^2(t_i) + 2w(t_i)(w(t_{i+1}) - w(t_i)) \right\}$$

$$= w^2(T) - w^2(0) + 2\sum_{i=0}^{n-1} w(t_i)\left\{ w(t_{i+1}) - w(t_i) \right\}.$$

$$\lim_{\|\pi\| \to 0} \sum_{i=0}^{n-1} w(t_i)\left(w(t_{i+1}) - w(t_i) \right) = \frac{w^2(T) - \lim_{\|\pi\| \to 0} Q_\pi}{2}.$$

$$= \frac{w^2(T) - T}{2}.$$

Q $0 \le t \le 1$, $\int_0^t w(1)\, dw(s)$

↳ $w(1)$ is not F_t measureable
∴ It is not an adapted process.

Thus, the integral does not exist.

• Ito - Doeblin formula for B.M (first version)

↳ Let f be ~~to~~ atleast twice continuously differentiable function of t.

$\{w(t), t \ge 0\} \rightarrow$ Weiner process or B.M.

$$d(f(w(t))) = f'(w(t))\, dw(t) + \frac{1}{2} f''(w(t))\, dt. \quad \text{--- (1)}$$

$$f(w(T)) = f(w(0)) + \int_0^T f'(w(t))\, dw(t) + \frac{1}{2}\int_0^T f''(w(t))$$

Let $f(x) = \frac{x^2}{2}$, $f'(x) = x$, $f''(x) = 1$

$(x = w(t))$.

$$f(w(T)) = \underbrace{f(w(0))}_{=0} + \int_0^T w(s)\, d\, w(s) + \frac{1}{2}\int_0^T 1 \cdot dt.$$

$$\frac{w^2(T)}{2} = 0 + I + \frac{T}{2} \Rightarrow \boxed{\frac{w^2(T) - T}{2} = I}$$

• Ito - Doeblin formula for B.M (second version)

↳ Let f ~~it~~ be a function of t & x. and have continuous partial derivative of second order.

$\{w(t), t \geq 0\} \longrightarrow$ W.P or BM.
Then, we have

$$df(t, w(t)) = f_t(t, w(t))dt + f_x(t, w(t))\,d(w(t)) + \frac{1}{2} f_{xx}(t, w(t))\,dt. \quad -③$$

$(dw \cdot dw = dt)$

$$f(t, w(t)) = f(0, w(0)) + \int_0^T \left(f_t(s, w(s)) + f_{xx}(s, w(s))\right) ds$$
$$\cancel{+ \frac{1}{2}\int_0^T f_{xx}(s, w(s))\,ds.}$$
$$+ \int_0^t f_x(s, w(s))\,d(w(s)).$$

Let $f(t, x) = x^2/2$, $f_x = x$, $f_{xx} = 1$, $f_t = 0$.

$$f(T, w(T)) = f(0, w(0)) + \int_0^T \left(0 + \frac{1}{2} \cdot 1\right) ds$$
$$+ \int_0^T \left(f_x(s, w(s)) = w(s)\right) d\,w(s).$$

$$\frac{w^2(T)}{2} = 0 + T/2 + I. \Rightarrow \boxed{I = \frac{w^2(T) - T}{2}}$$

Q $I = \int_0^T w^2(t)\, d(w(t))$.

Ans $f(x) = x^3/3$, $x = w(t)$.

$f'(x) = x^2$, $f''(x) = 2x$.

First version

$$f(w(T)) = f(w(0)) + \int_0^T w^2(s)\, d(w(s)) + \frac{2}{2}\int_0^T w(s)\, ds$$

$$\frac{w^3(T)}{3} = 0 + I + \cancel{\frac{w^2(T) - T}{2}} \int_0^T w(s) \cdot ds.$$

$$\cancel{I = \frac{2w^3(T) - 3(w^2(T) - T)}{6}}.$$

$$I = \frac{w^3(T)}{3} - \int_0^T w(s) \cdot ds.$$

- Stochastic D.E

$$\frac{d(x(t))}{dt} = f(t, x(t)) \qquad \begin{cases} t \in [0, T] \\ x(0) = x_0 \end{cases}$$

$f : [0, T] \times R \to R$.

$$X_t = X_0 + \int_0^t f(u, x(u))\, du$$

Provided the Libschiltz condition exists ie,
$\exists\, k > 0$ such that

$$|f(t,x) - f(t,y)| \le k\,|x-y|.$$

for all values of $t \in [0,T]$ & $x, y \in R$.

If X_0 & f is random, then solⁿ is not unique and will depend on the value of $\omega \in \Omega$ (sample space).

X - which is now a function ω & t

$\{X(\omega,t),\ t \in [0,T],\ \omega \in \Omega\}$ — Stochastic process

└ Such diff. equation is called Random D.E.

Adding uncertainities by way of ~~differentiation~~ differential of B.M.

$$\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, w(t))\,\frac{d(w(t))}{dt}. \quad \text{—} (1)$$

$$\left\{ \begin{aligned} \text{where } b &: [0,T] \times R \to R. \\ \sigma &: [0,T] \times R \to R. \end{aligned} \right\}$$

b & σ - are two given functions defined by the above conditions

$$dX(t) = b(t, X(t))\,dt + \sigma(t, w(t))\,d(w(t)). \quad \text{—} (2)$$

(eq(1) × dt)

└ Stochastic D.E

Equivalently, it can be written

$$X(t) = X(0) + \int_0^t b(s, X(s))\,ds + \int_0^t \sigma(s, w(s))\,dw \quad — ③$$

$$(0 \le t \le T)$$

Stochastic I.E.

<u>Strong Solution</u>

> Strong solⁿ of SDE is the stochastic process of $\{X(t), t \in [0,T]\}$ which satisfies the following

(1) $\{X(t), t \in [0,T]\}$ is adopted to brownian motion.

(2) The integral in eq ③ is well-defined and satisfied by $X(t)$

(3) $\{X(t), t \in [0,T]\}$ is function of underlying B.M sample path & the co-efficients ~~of~~ (b & σ).

For a weak sol^n, we are only interested in the distribution of $X(t)$.

<u>Diffusion</u>: Sol^n of a S.D.E is called diffusion. (W.P. a B.M is a diffusion process. as $X(t) = w(t)$ for $b=0, \sigma=1$)

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· SDE for geometric B.M

Let $S(t)$ be the stock price at time t.

$-\infty < \mu < \infty$ (growth rate of stock) and $\sigma > 0$ be the volatility.

Consider the stochastic D.E

$$dS(t) = \mu S(t)dt + \sigma S(t)\, dW(t). \quad — (1).$$

($S(0)$ - present stock price)

The condition of existance is verified as μ & σ are constant.

We assume that, $S(t) = f(t, W(t))$.
and using the second version of Ito - Doeblin formula.

$$df(t, W(t)) \cancel{d f(t)} = f_t\, dt + f_x\, dW(t) + \frac{1}{2} f_{xx}\, dt. \quad — (2).$$

$$f_x = \sigma S(t) = \sigma f. \quad — (3)$$

$$\mu f = \mu S(t) = f_t + \frac{1}{2} f_{xx}. \quad — (4).$$

Eq (1) & (2) are same.

Solving (3), we get.

$$f = k(t) e^{\sigma x}.$$

$$f_t = k'(t) e^{\sigma x}. \qquad f_{xx} = \sigma^2 k(t) e^{\sigma x}.$$

$$\mu\, k(t)\, \cancel{e^{\sigma x}} = k'(t)\, \cancel{e^{\sigma x}} + \frac{1}{2}\sigma^2 k(t)\, \cancel{e^{\sigma x}}$$

$$\left(\mu - \frac{\sigma^2}{2}\right) = \frac{k'(t)}{k(t)}$$

$$k(t) = c\, e^{(\mu - \frac{\sigma^2}{2})t} \qquad (c = S(0))$$

$$\boxed{f = S(0)\, e^{\sigma x + (\mu - \frac{\sigma^2}{2})t}}$$

$$\boxed{f(t, w(t)) = S(0)\, e^{\sigma w(t) + (\mu - \frac{\sigma^2}{2})t}}$$

H.W $\rightarrow$ $E(S(t)) =$

$Var(S(t)) =$

Q Find the stochastic differential of $\sin(w(t))$

Ans $f(x) = \sin x = f(x, t)$

$$f_x = \frac{df}{dx} = \cos x$$

$$f_{xx} = \frac{d^2 f}{dx^2} = -\sin x.$$

$$\therefore\ d(\sin(w(t))) = \cos w(t) \cdot d\,w(t) + \frac{1}{2}(-\sin(w(t)))$$

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- Discounted Portfolio Process

→ Let a stock having a price $S(t)$ per unit follow a generalized geometric BM with constant mean return (μ) and constant volatility $\sigma > 0$ be governed by an SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)\, d(W(t)) \quad -(1)$$

where $t \in [0, T]$.

Also, let $\beta(t)$ be the risk free asset which satisfies the ODE.

$$d\beta(t) = r\beta(t)dt \quad -(2)$$

where r is the constant risk free interest rate

Suppose at time t, we take a portfolio consisting of $a(t)$ shares of such stock and $b(t)$ shares of risk free.

$$V(t) = a(t)S(t) + b(t)\cdot\beta(t) \quad -(3)$$

→ $d(V(t)) = a(t)\cdot dS(t) + b(t)\, d\beta(t) \quad -(4)$

Now, discounted price of one shares of stock is

$$\tilde{S}(t) = e^{-rt}S(t), \quad t \in [0, T] \quad -(5)$$

~~Ap~~ Applying Ito-Doblin formula of second version,

$$d\tilde{S}(t) = -re^{-rt}S(t)dt + e^{-rt}d(S(t)) \quad -(6)$$

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$$f(x,t) = e^{-rt} \cdot x$$
$$f_t = -re^{-rt} \cdot x, \quad f_x = e^{-rt}, \quad f_{xx} = 0.$$

substituting (i) in (vi).

$$d\tilde{S}(t) = \tilde{S}(t)\left[(\mu - r)dt + \sigma d(W(t))\right]$$
$$= \sigma \tilde{S}(t)\, d\tilde{W}(t).$$

where $\tilde{W}(t) = \left(\frac{\mu - r}{\sigma}\right) t + W(t), \quad t \in [0,T]$

$\mu - r$, is called the risk premium.

$\frac{\mu - r}{\sigma}$ is risk premium per unit risk (also called market price of the risk)

PORTFOLIO OPTIMISATION

↳ The portfolio is a collection of two or more assets.
$(a_1, a_2 \ldots \ldots a_n)$, $\theta = (x_1, x_2 \ldots x_n)$.

(We consider a single period model)
$V_i(0)$ - value of i^{th} asset at $t=0$.
$V_i(T)$ — value of i^{th} asset at $t = T$

↳ The total value of the portfolio will be represented by V_0
$$V_0(0) = \sum_{i=1}^{n} V_i(0) \cdot x_i, \quad V_0(T) = \sum_{i=1}^{n} V_i(T) \cdot x_i.$$

$$r_0(T) \text{ return} = \frac{V_0(T) - V_0(0)}{V_0(0)}.$$

↳ The weight w_i of the asset a_i is the proportion of the value of the asset in the portfolio

$$w_i = \frac{x_i V_i(0)}{V_0(0)} = \frac{x_i V_i(0)}{\sum_{i=1}^{n} x_i V_i(0)}.$$

$$\sum_{i=1}^{n} w_i = 1 = w_1 + w_2 + \cdots + w_n.$$ (w_i - can be negative)

↳ **Mean of portfolio return**

↳ Let $(w_1, w_2 \ldots w_n)$ is the weight of $(a_1, a_2 \ldots a_n)$ assets & r_i $(i = 1, 2, \ldots n)$ be the return of asset a_i. Also. $E(r_i) = \mu_i$ $(i = 1 \text{ to } n)$

Then, mean of portfolio.

$$\mu = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i E(r_i). = \sum_{i=1}^{n} w_i \mu_i.$$

Variance of portfolio:- $\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}.$

where. $\sigma_{ij} = Cov(r_i, r_j)$, $\sigma_i^2 = Var(r_i)$

If ρ_{ij} is the correlation coefficient b/w r_i & r_j., then.

$$\rho_{ij} = \frac{Cov(r_i \cdot r_j)}{\sigma_i \cdot \sigma_j} \Rightarrow \sigma_{ij} = \sigma_i \sigma_j \rho_{ij}.$$

$$\sigma^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij}.$$

We plot (σ_a, μ_a) of a given portfolio on the (σ, μ) plane. Then the portfolio optimisation is referring to

(i) Minimise the risk

Minimise $\left(\sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij}\right)$

(ii) Max $\sum w_i r_i$ such that $\sum w_i = 1$

Two ~~great~~ asset portfolio optimization

$\hookrightarrow$ (a_1, a_2), (r_1, r_2) return, (σ_1, σ_2) risk, (w_1, w_2)

$$\mu = E(w_1 r_1 + w_2 r_2) = w_1 \mu_1 + w_2 \mu_2 \quad \text{—} (1)$$

$$\sigma^2 = Var(w_1 r_1 + w_2 r_2) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 (\rho) \quad \text{—} (2)$$

(ρ - coefficient of correlation b/w r_1 & r_2)
($\rho \in [-1, 1]$)

$\llcorner$ Provides the measure of diversification of portfolio ~~as~~ so as to reduce the risk

$\hookrightarrow$ There can be short selling, so that weight can be negative.

$$w_1 + w_2 = 1$$

Let $w_2 = s \Rightarrow w_1 = 1 - s$, $s \in \mathbb{R}$.

$$\mu = (1-s)\mu_1 + s\mu_2 \quad \text{—} (3)$$

$$\sigma^2 = (1-s)^2 \sigma_1^2 + s^2 \sigma_2^2 + 2s(1-s)\sigma_1 \sigma_2 \rho \quad \text{—} (4)$$

$$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) s^2 + 2(\sigma_1\sigma_2\rho - \sigma_1^2) s + \sigma_1^2$$

(Let $0 \le \sigma_1 \le \sigma_2$)

$\llcorner$ Case 1: $\rho = \pm 1$

$\llcorner$ Case 2: $\rho \in (-1, 1)$

Case 1: $u = (1-s)u_1 + s u_2$

$\sigma = |(1-s)\sigma_1 \pm 1\,\sigma_2|$

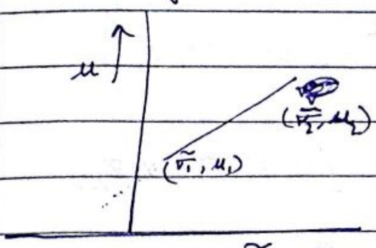
For $s \in (0,1)$. — not considering short-selling.

u - same as above

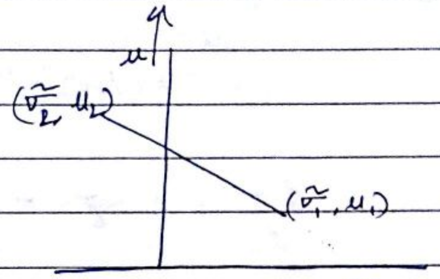
$$\tilde{\sigma} = \begin{cases} (1-s)\sigma_1 + s\sigma_2, & \rho = 1 \\ (1-s)\sigma_1 - s\sigma_2, & \rho = -1 \end{cases}$$

2)

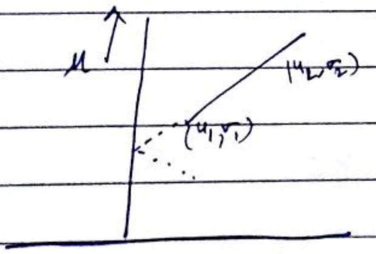
Plotting $\tilde{\sigma}$ vs u,



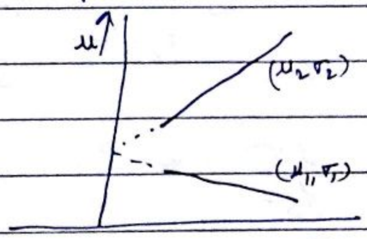
($\rho=1$)



($\rho=-1$)



($\rho=1$)



($\rho=-1$)

$\hookrightarrow$ For $\rho = 1$, $\dfrac{d\sigma^2}{ds} = 0 = 2(\sigma_2 - \sigma_1)(\sigma_1 - s(\sigma_1 - \sigma_2))$.

$\therefore \boxed{s = \dfrac{-\sigma_1}{\sigma_2 - \sigma_1}}$ — It is negative.

$\dfrac{d^2\sigma^2}{ds^2} = (\sigma_2 - \sigma_1)^2 > 0$ — Delivers σ^2 minimum.

(3)

$$\boxed{S_{min} = \frac{-\sigma_1}{\sigma_2 - \sigma_1}} \quad (w_2) \qquad , \qquad 1 - S_{min} = \frac{\sigma_2}{\sigma_2 - \sigma_1} \quad (w_1)$$

$$\left.\begin{aligned} &\mu_{\text{(with min}^m\text{ risk)}} = \frac{(\sigma_2 \mu_1 - \sigma_1 \mu_2)}{\sigma_2 - \sigma_1} \\ &\cancel{\text{Min}^n}\ \text{Min}^m \underset{\text{(Risk)}}{(\sigma^2)} = 0 \end{aligned}\right\} \text{ — (4)}$$

Case 2 : $\rho = -1$

$$\mu = (1-s)\mu_1 + s\mu_2$$

$$\sigma^2 = (1-s)^2 \sigma_1^2 + s^2 \sigma_2^2 - 2s(1-s)\sigma_1 \sigma_2$$

$$\frac{d\sigma^2}{ds} = 0 \Rightarrow -2(\sigma_1 + \sigma_2)\left[(1-s)\sigma_1 - s\sigma_2\right]$$

$$\boxed{S_{min} = \frac{\sigma_1}{\sigma_1 + \sigma_2} > 0} \text{ — (5)}$$

$$\frac{d\sigma^2}{ds^2} = 2(\sigma_1 + \sigma_2)^2 > 0$$

$\therefore$ s is min^m of fn. σ^2.

$$\left(1 - S_{min} = \frac{\sigma_2}{\sigma_1 + \sigma_2}\right)$$

$$\left.\begin{aligned} &\mu_{\text{(with min risk)}} = \frac{\mu_1 \sigma_2 + \mu_2 \sigma_1}{\sigma_1 + \sigma_2} \\ &\sigma^2_{min} = 0 \end{aligned}\right\} \text{ — (6)}$$

~~∴ As w_1, w_2~~ Risk can be eliminated without short-selling ($s \in (0,1)$)

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Case 3: $-1 < \rho < 1$.

From above, $\mu = (1-s)\mu_1 + s\mu_2$

$$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)s^2 + 2(\sigma_1\sigma_2\rho - \sigma_1^2)s + \sigma_1^2.$$

↳ The graph (σ^2, μ) is a parabola.

$$\frac{d\sigma^2}{ds} = 0 \Rightarrow \boxed{s = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}$$

$$\frac{d^2\sigma^2}{ds^2} = 2(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) > 0.$$

$\therefore$ s is $\min^m$. ~~min~~

$$\boxed{s_{min} = \frac{(\sigma_1^2 - \rho\sigma_1\sigma_2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}} \quad \text{— (7)}$$

$$\mu_{\substack{min\\risk}} = (\mu_2 - \mu_1)s_{min} + \mu_1.$$

$$\sigma^2_{min} = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad \text{— (8)}$$

↳ If $-1 < \rho < \frac{\sigma_1}{\sigma_2} < 1$, then

$s_{min} > 0$ — No short selling. (from eq (7))

~~If~~ $\rho =$

↳ If $\rho = \frac{\sigma_1}{\sigma_2}$ then

$s_{min} = 0$.

$\sigma^2_{min} = \sigma_1^2$.

↳ If $\rho > \frac{\sigma_1}{\sigma_2}$. (but $\rho < 1$)

↳ So, min^m risk will ~~need~~ be attained by short selling

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Q: $S_1(0) = 10$, $x_1 = 50$, $S_1(T) = 12$
$S_2(0) = 20$, $x_2 = 25$, $S_2(T) = 22$

$V(0) = 10 \times 50 + 20 \times 25$
$= 1000$

$V(T) = 12 \times 50 + 22 \times 25$.
$= 1150$

$w_1(0) = \frac{500}{1000} = 0.5$, $w_1(T) = \frac{600}{1150}$.

$w_2(0) = \frac{500}{1000} = 0.5$ $w_2(T) = \frac{600}{1150}$.

Q Compute the value $V(1)$ of the portfolio worth initially $V(0) = 100$; consists of 2 assets with $w_1 = 25\%$, $w_2 = 75\%$

$S_1(0) = 45$, $S_1(1) = 48$
$S_2(0) = 33$, $S_2(1) = 32$.

Find value of portfolio $V_1(T)$.

Ans $x_1 = ?$
$x_2 = ?$, $w_i = \frac{x_i V_i(0)}{\sum V_i(0)}$

$$w_1 = \frac{x_1 \times S_1(0)}{V(0)}$$

$$0.25 = \frac{x_1 \times 45}{100}, \quad x_1 = \frac{25}{45} = 0.55$$

$$x_2 = \frac{100 \times 0.75}{33} = 2.27$$

$$V(1) = x_1 S_1(1) + x_2 S_2(1)$$

$$= \frac{0.25 \times 100}{45} \times 48 + \frac{0.75 \times 100}{33} \times 32$$

$$= 99.39$$

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Thm: → The variance σ^2 of the portfolio cannot exceed the greater of the variance (σ_1^2 & σ_2^2) of the components if short sell is not allowed.

$$\sigma^2 \leq \max(\sigma_1^2, \sigma_2^2)$$

Proof: Let us assume that $\sigma_1^2 < \sigma_2^2$.

Since short sell is not allowed.

ω_1 & $\omega_2 > 0$.

$$\omega_1\sigma_1 + \omega_2\sigma_2 \leq (\omega_1 + \omega_2)\sigma_2 = \sigma_2$$

$$(\because \omega_1 + \omega_2 = 1)$$

Since, $-1 \leq \rho_{12} \leq 1$.

So variance of portfolio is.

$$\sigma^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\sigma_1\sigma_2\omega_1\omega_2\rho_{12}.$$

$$\leq (\omega_1\sigma_1 + \omega_2\sigma_2)^2.$$

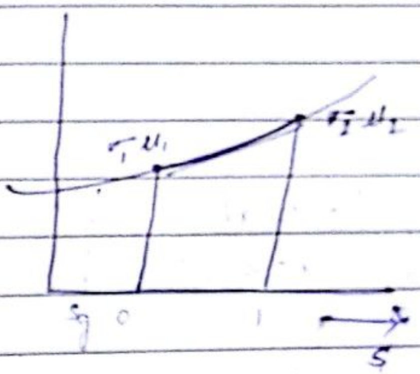
$$\leq \sigma_2^2,$$

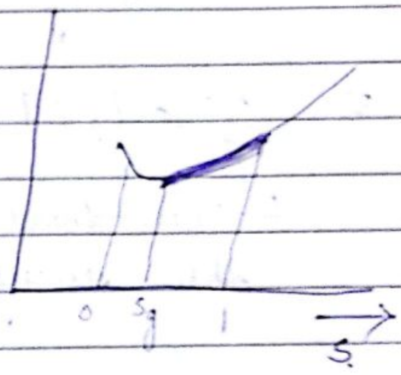
<u>Thm</u>: $$s_g = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$$

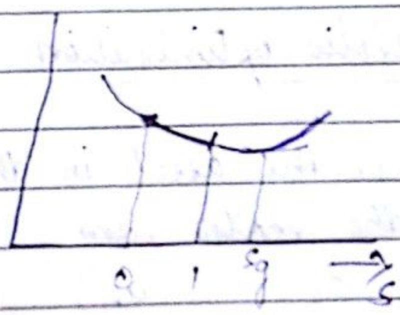
$$s_{min} = \begin{cases} 0, & s_g < 0 \\ s_g, & 0 \le s_g \le 1 \\ 1, & s_g > 1 \end{cases}$$

<u>Proof</u>: We plot (s vs σ^2) graph.

If no short sell is allowed & since expression for σ^2 is quadratic in s







↳ The bold part corresponds to portfolio with no short selling.

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Thm: For $-1 \le \rho_{12} \le 1$, we have the following 3 possibilities

(1) If $-1 \le \rho_{12} \le \frac{\sigma_1}{\sigma_2}$ where $\sigma_2 > \sigma_1$ for each portfolio then there is a portfolio without short selling such that $\sigma < \sigma_1$.

(2) $\rho_{12} = \frac{\sigma_1}{\sigma_2}$.

Then, $\sigma \ge \sigma_1$ for each portfolio (as $s_g = 0$)

(3) $\frac{\sigma_1}{\sigma_2} < \rho_{12} \le 1$.

↳ Then there is a portfolio with short selling such that $\sigma < \sigma_1$.

Multi-asset portfolio optimisation

↳ The weight for the asset in the portfolio is written in the vector form.

$(a_1\ a_2\ a_3 \ldots a_n)$

$$w^T = \{w_1, w_2, w_3 \ldots w_n\}$$

$$e^T = [1, 1, 1 \ldots] \in R^n$$

$$\sum_{i=1}^{n} w_i = 1$$

$\rightarrow \boxed{e^T w = 1}$

We define $m^T = [\mu_1\ \mu_2\ \dots\ \mu_n]$.

μ_i - expected return on that asset.
$\mu_i = E(r_i)$

The matrix $[c_{ij}]_{n \times n}$ — denotes variance & covariance matrix.

$c_{ij} = Cov(r_i, r_j)$ (when $i=j$ ∟ variance)

$[C_{ij}]_{n \times n}$ — $(c_{ji} = c_{ij})$ & symmetric matrix.
— Positive definite.

∟ We assume that $[c_{ij}]$ has ~~a~~ ~~relation~~ an inverse.

∟ The expected return of the portfolio (μ) is given by
$\mu = E\left(\sum_{i=1}^{n} w_i r_i\right)$
— r_i - is a R.V

$$\mu = \sum_{i=1}^{n} w_i E(r_i)$$

$$= \sum_{i=1}^{n} w_i \mu_i = m^T w$$

$$\boxed{\mu = m^T w}$$

∟ Variance of the portfolio (σ^2)

$$\sigma^2 = Var\left(\sum w_i r_i\right) = w^T C w.$$

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$u^T = [u_1, u_2, u_3 \ldots u_n]$.

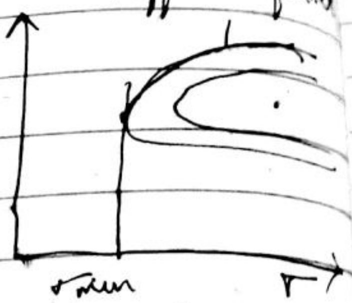
$[\sigma, u]$ - plane.

$e^T w = 1$.

$w_1 + w_2 + \ldots + w_n = 1$.

$\sigma^2 = (\alpha u + \beta)^2 + \delta(\alpha u + \beta) + \eta$.

$\alpha = [u^T s]$, $\beta = -(u^T b)(u^T s)^T$.



(Solid ~~surface~~ region.)

Markowitz curve & region)

Thm: $\sigma^2 = w^T C w$ (minimize)

$e^T w = 1$. (subject to this)

Portfolio with min^m risk has ~~to~~ weights given by $w = \dfrac{C^{-1} e}{e^T C^{-1} e}$.

Proof: Using the method of lagrange's multiplier

$L(w, \lambda) = w^T C w + \lambda(1 - e^T w)$.

$= \sum_{ij} w_i w_j \sigma_{ij} + \lambda(1 - e^T w)$.

Diff w.r.t w. & equating to zero.

$\Rightarrow 2 w^T C - \lambda e^T = 0$

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$$\Rightarrow \quad w = \frac{\lambda}{2} C^{-1} e. \quad — (2)$$

substituting in $e^T w = 1$

$$e^T\left(\frac{\lambda}{2} C^{-1} e\right) = 1.$$

$$\Rightarrow \frac{\lambda}{2} = \frac{1}{e^T C^{-1} e.} \quad — (3)$$

Putting (3) in (2),

$$\boxed{w = \frac{C^{-1} e}{e^T C^{-1} e}}$$

Thm: For a given expected return μ, a portfolio with the min^m risk has weights given by

$$w = \frac{\det\begin{pmatrix} \mu & w^T C^{-1} e \\ 1 & e^T C^{-1} e \end{pmatrix} C^{-1} m + \det\begin{pmatrix} w^T C^{-1} m & \mu \\ e^T C^{-1} m & 1 \end{pmatrix} C^{-1} e}{\det\begin{pmatrix} w^T C^{-1} m & m^T C^{-1} e \\ e^T C^{-1} m & e^T C^{-1} e \end{pmatrix}}.$$

Proof: Let $\sigma^2 = \frac{1}{2} w^T C w$ subject to
(minimize)

$$m^T w = \mu$$
$$\& \ e^T w = 1$$

$$L(w, \alpha, \beta) = \frac{1}{2} w^T C w \cancel{+ \alpha m^T} + \alpha(\mu - m^T w) + \beta(1 - e^T w).$$

$$\frac{\partial L}{\partial w} = 0 \Rightarrow w^T C - \alpha m^T - \beta e^T = 0.$$

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$$w = c^{-1}(\alpha m + \beta e) \quad - (4).$$

substituting.

$$\left.\begin{aligned} m^T c^{-1}(\alpha m + \beta e) &= \mu \\ e^T c^{-1}(\alpha m + \beta e) &= 1. \end{aligned}\right\} - (5).$$

(Cramers rule gives w)

- Capital-Asset Pricing Model

↳ Consider a portfolio with n risky assets $(a_1, a_2, \ldots a_n)$ with weights $(w_1, w_2 \ldots w_n)$ and one risk-free asset (a_{rf}) with weight (w_{rf}).

Total weight $= 1 \Rightarrow w_{risky} + w_{rf} = 1.$

$$\Rightarrow \sum_{i=1}^{n} w_i + w_{rf} = 1. \quad - (1).$$

$$\text{Expected return } (\mu) = \underbrace{\sum_{i=1}^{n} w_i \mu_i}_{\mu_{risky}} + \mu_{rf}. \quad - (2)$$

$$\text{Variance } (\sigma^2) = \mathrm{Var}\left[\sum_{i=1}^{n} w_i r_i + \mu_{rf} w_{rf}\right].$$

$$= \mathrm{Var}\left[\sum_{i=1}^{n} w_i r_i\right].$$

$$= \sum_{i=1}^{n} \mathrm{Var}(w_i r_i). = \sigma^2_{risky}$$

↳ If we remove the risk-free asset from the portfolio and adjust the weights of the risky asset so that their sum becomes one.

The resultant portfolio so obtained is referred as Derived Risky Portfolio. $(\mu_{der}, \sigma^2_{der})$.

$$\mu = w_{risky} \sum \frac{w_i \mu_i}{w_{risky}} + w_{rf}\,\mu_{rf}$$

$$= w_{risky}\,\mu_{der} + \mu_{rf}\,w_{rf}.$$

$$= w_{risky}\,\mu_{der} + \mu_{rf}(1 - w_{risky}).$$

$$= w_{risky}(\mu_{der} - \mu_{rf}) + \mu_{rf}. \quad - (4).$$

$$\sigma^2 = Var\left[w_{risky} \sum_{i=1}^{n} \frac{w_i r_i}{w_{risky}} + \mu_{rf}\,w_{rf}\right]$$

$$= w^2_{risky} \sum Var\left(\frac{w_i r_i}{w_{risky}}\right)$$

$$= w^2_{risky}\,\sigma^2_{der}.$$

$$w_{risky} = \frac{\sigma}{\sigma_{der}} \quad - (5).$$

Putting in (4),

$$\mu = \mu_{rf} + \frac{(\mu_{der} - \mu_{rf})}{\sigma_{der}}\sigma \quad - (6).$$

↳ It is an eqⁿ of line in $[\sigma, \mu]$ plane joining two points

$(0, \mu_{rf})$. $(\sigma_{der}, \mu_{der})$

Now, for given risk σ, we choose various weight combinations of risky & risk-free assets. for which (sum of weights = 1) holds.

We generate different lines, represented by ⓒ.

The line that produces the point with highest expected return for a given risk is tangent to the upper portion of the Markowitz bullet & is called Capital Market Line and the point on the.

Markowitz bullet where the CML is tangential is said to represent the Markowitz portfolio.

Thm ↳ For any expected risk free return, μ_{rf}, the weight vector w_m of the market portfolio is given by

$$w_m = \frac{C^{-1}(m - \mu_{rf}e)}{e^T C^{-1}(m - \mu_{rf}e)}.$$

(matrix (cannot be cancelled))

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↳ We have for any point (σ, μ) in the markowitz bullet, the slope of lines joining $(0, \mu_{rf})$ & (σ, μ) will be given as.

$$\text{slope} = \frac{\mu - \mu_{rf}}{\sigma} = \frac{\sum w_i \mu_i - \mu_{rf}}{\left[\sum_{i,j} w_i w_j c_{ij}\right]^{1/2}}$$

Now, for the line joining these points to become tangent to ~~mo~~ Markowitz bullet.

$$\left.\begin{array}{l} \left(\dfrac{m^T w - \mu_{rf}}{(w^T c w)^{1/2}}\right) \text{ should be maximum} \\ \\ \text{subject to condition } e^T w = 1. \end{array}\right\} \quad - (1)$$

(Using legrange's method)

$$L(w, \lambda) = \frac{(\overset{\mu}{m^T w} - \mu_{rf})}{(w^T c w)^{1/2}} + \lambda(1 - e^T w)$$

$$0 = L_w(w, \lambda) = \frac{1}{(w^T c w)}\left((w^T c w)^{1/2} m - (m^T w - \mu_{rf})(w^T c w)^{-1/2} c w\right) - \lambda e^T$$

$$\Rightarrow \sigma m - (\mu - \mu_{rf}) \frac{1}{\sigma} c w = \lambda \sigma^2 e.$$

$$\Rightarrow \sigma^2 m - (\mu - \mu_{rf}) c w = \lambda \sigma^3 e. \quad - (2)$$

$$\Rightarrow \sigma^2 w^T m - (\mu - \mu_{rf}) w^T c w = \lambda w^T \sigma^3 e.$$

$$\Rightarrow \sigma^2 \mu - (\mu - \mu_{rf}) \sigma^2 = \lambda \sigma^3.$$

$$\Rightarrow \boxed{\lambda = \frac{\mu_{rf}}{\sigma}} \quad - (3).$$

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<u>Thm</u>: Suppose the market portfolio is (σ_m, μ_m), the expected return of the asset a_i is given by

$$\mu_i = \mu_{rf} + \beta_i(\mu_m - \mu_{rf})$$

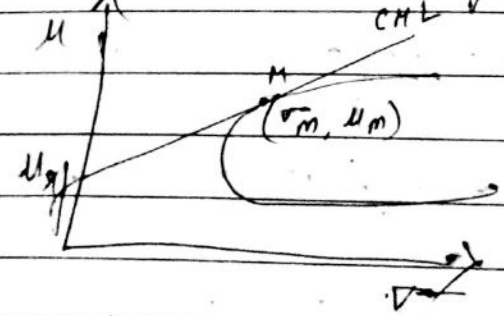
$$\left(\beta_i = \frac{\text{Cov}(r_i, r_m)}{\sigma_m^2}\right)$$

<u>Thm</u>: Suppose an investor's portfolio comprises of asset (a_i), with weight w & market portfolio m with the weight $(1-w)$.
(M.P - treated as an asset)

$$\left.\begin{aligned} \text{Expected return}\ (\mu) &= w\mu_i + (1-w)\mu_m \\ \text{Variance}\ (\sigma^2) &= w^2\sigma_i^2 + (1-w)^2\sigma_m^2 \\ &\quad + 2w(1-w)\rho\,\sigma_i\sigma_m \end{aligned}\right\} - (1)$$

ρ - coefficient of correlation b/w return of asset a_i & mp

↳ As w varies, then, (σ, μ) graph.



↳ When $w=0$, CML becomes tagent at point M

$$\text{Slope}_{\text{curve}} = \text{Slope}_{\substack{\text{CML}\\ \text{at } M}} = \left.\frac{d\mu}{d\sigma}\right|_{w=0} = \left.\frac{d\mu}{dw}\cdot\frac{dw}{d\sigma}\right|_{w=0}$$

$$= (\mu_i - \mu_m) \left.\frac{dw}{d\sigma}\right|_{w=0}$$

From ①, $\frac{dw}{d\sigma} = \frac{1}{d\sigma/dw}$.

$$\frac{dw}{d\sigma} = \frac{\rho\sigma_i\sigma_m - \sigma_m^2}{\sigma_m}$$
(on solving ①)

$$\Rightarrow \frac{(\mu_i - \mu_m)\sigma_m}{\rho\sigma_i\sigma_m - \sigma_m^2} = \frac{\mu_m - \mu_{rf}}{\sigma_m}$$

(Solving) $\Rightarrow$ $$\mu_i = \mu_{rf} + \frac{(\mu_m - \mu_{rf})}{\sigma_m^2}(\rho\sigma_i\sigma_m)$$

$$\mu_i = \mu_{rf} + \beta_i(\mu_m - \mu_{rf}).$$

↳ If $\beta_i = 0$:- Asset is totally uncorrelated to portfolio.
($\mu_{ri} = \mu_{rm}$).
showing however large is the σ_i, return will always be limited to risk free.
or we can say, no premium for risk.
(MP - 1 risk free & 1 risky asset,
on adding, subsequent risky assets, no premium is made)

↳ If $\beta_i < 0$, then ~~$\beta_i < \rho_{rf}$ ... $\mu_m - \mu_{rf}$~~ $\mu_i < \mu_{rf}$.

Hence, ~~we~~ it can used to reduce overall risk of the portfolio when other assets are not doing well.

(For this region, it is called Insurance)

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→ The overall β of the portfolio is defined as

$$\beta = \sum_{i=1}^{n} w_i \beta_i .$$

× × × × ×