Name: Athawa Shende
Roll No: 2K17/MC/028

DATE / /
PAGE

ASSIGNMENT - 1
(Partial Differential Equations)

[MC - 406]
2K17/MC/028

Q1

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol → Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

differentiating partially w.r.t x & y

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \Rightarrow c^2x + a^2z\frac{\partial z}{\partial x} = 0 \longrightarrow (1)$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \Rightarrow c^2y + b^2z\frac{\partial z}{\partial y} = 0 \longrightarrow (2)$$

differentiating (1) partially w.r.t x & (2) w.r.t to y.

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \longrightarrow (3)$$

$$c^2 + b^2\left(\frac{\partial z}{\partial x}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0 \longrightarrow (4)$$

from (1) we get $\boxed{c^2 = -(a^2 z/x)(\partial z/\partial x)} \longrightarrow (5)$

substituting (5) in (3) and dividing by a^2

$$\Rightarrow -\left(\frac{a^2 z}{x}\right)\left(\frac{\partial z}{\partial x}\right) + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow -\left(\frac{z}{x}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \longrightarrow (6)$$

Raj

DATE / /

PAGE

similarly we can use value of c^2 & equation ③ & ⑤ to get

$$zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0 \longrightarrow (7)$$

similarly we can differentiate partially ① w.r.t y to get

$$0+a^2\left\{\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)+z\left(\frac{\partial^2 z}{\partial x\partial y}\right)\right\}=0$$

$$\left\{\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)+z\left(\frac{\partial^2 z}{\partial x\partial y}\right)\right\}=0 \longrightarrow (8)$$

Equation ⑥, ⑦, ⑧ are 3 possible forms of p.d.e

b) $z=(x^2+a)(y^2+b)$

$p=\frac{\partial z}{\partial x}=2x(y^2+b) \Rightarrow p/2x=(y^2+b^2) \rightarrow (1)$

$q=2y(x^2+a) \Rightarrow q/2y=(x^2+a^2) \rightarrow (2)$

from ① & ②

$z=pq/4xy$

$$\boxed{4xyz=pq}$$

Q2

(a) $x+y+z = f(x^2+y^2+z^2) \longrightarrow (1)$

Differentiating partially w.r.t 'x' & 'y' (1) gives

$$1+p = f'(x^2+y^2+z^2)(2x+2zp) \longrightarrow (2)$$

$$1+q = f'(x^2+y^2+z^2)(2y+2zq) \longrightarrow (3)$$

eliminating $f'(x^2+y^2+z^2)$

$$\frac{1+p}{\cancel{2}(x+zp)} = \frac{1+q}{\cancel{2}(y+zq)}$$

$$(1+p)(y+zq) = (1+q)(x+zp)$$

$$\Rightarrow \boxed{(y-z)p+(z-x)q = x-y}$$

(b) $y = f(x-ct)+g(x+ct) \longrightarrow (1)$

$$\frac{\partial y}{\partial x} = f'(x-ct)+g'(x+ct)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x-ct)+g''(x+ct) \longrightarrow (2)$$

$$\frac{\partial y}{\partial t} = f'(x-ct)\cdot(-c)+g'(x+ct)\cdot c$$

$$\frac{\partial^2 y}{\partial t^2} = f''(x-ct)\cdot c^2+g''(x+ct)\cdot c^2 \longrightarrow (3)$$

Raj

DATE / /

PAGE

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(f''(x-ct) + g''(x+ct)\right)$$

using value of $f''(x-ct) + g''(x+ct)$ from ② in ③

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2}\right)}$$

Q3

ⓐ $y^2 p - xqy = x(z-2y)$

Using Lagrange's Method

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \longrightarrow ①$$

taking 1st 2 fractions

$$\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow -\int x\,dx = \int y\,dy \Rightarrow -\frac{x^2}{2} = \frac{y^2}{2} + c$$

$$\Rightarrow x^2 + y^2 = c_1 \longrightarrow ②$$

taking last 2 fractions

$$\frac{dz}{dy} = \frac{-(z-2y)}{y} \Rightarrow \frac{dz}{dy} + \frac{1}{y}z = 2$$

$$IF = e^{\int 1/y\,dy} = y$$

$$z \cdot y = \int 2y\,dy + c_2 \Rightarrow zy - y^2 = c_2$$

Hence $\phi(x^2+y^2, zy-y^2) = 0$ is the solution & ϕ is an arbitary function.

(b) $xzp + \cancel{yzp}\ yzq = xy$

$$xzp + yzq = xy \rightarrow (1)$$

The Lagrange auxillary eqn for given eqn is

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

$$\frac{dx}{xz} = \frac{dy}{yz} \Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y} \Rightarrow \log\left(\frac{y}{x}\right) = c_0$$

$$\boxed{y/x = c_1} \longrightarrow (2)$$

$$\frac{dx}{x\cancel{z}} = \frac{dy}{y\cancel{z}} \Rightarrow y\,dx = x\,dy \longrightarrow (3)$$

Last two fractions

$$\frac{dy}{yz} = \frac{dz}{x\cancel{y}} \Rightarrow x\,dy = z\,dz$$

from (2) we can write

$$x\,dy + y\,dx = 2z\,dz$$

$$\int d(xy) - \int 2z\,dz = \int 0$$

Raj

DATE / /

PAGE

$$xy - z^2 = c_2 \longrightarrow (4)$$

from (2) & (4) we get solution of given equation as

$\boxed{\phi(y/x,\ xy - z^2)}$ where ϕ is an arbitrary function

(c) $z(x+y)p + z(x-y)q = x^2 + y^2$

sol^n → using Lagrange's subsidiary equations

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{(x^2+y^2)} \longrightarrow (1)$$

using multiplier $x, -y, -z$ for (1)

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2+y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

$$\int 2x\,dx - \int 2y\,dy - \int 2z\,dz = \int 0$$

$$x^2 - y^2 - z^2 = c_1 \longrightarrow (2)$$

Again using l, m, n as $(y, x, -z)$.

$$\frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}$$

$y\,dx + x\,dy - z\,dz = 0$

$2\int d(xy) - 2\int z\,dz = 0$

$2xy - z^2 = c_2 \longrightarrow (3)$

from (2) & (3) $\phi(x^2-y^2-z^2, 2xy-z^2) = 0$ is general solution, ϕ being the arbitrary f^n.

Q4

$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$

Lagrange's auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

using $1/x$, $1/y$, $1/z$ as multipliers

$$= \frac{(1/x)\,dx + 1/y\,dy + 1/z\,dz}{y^2+z-(x^2+z)+x^2-y^2} = \frac{(1/x)dx + 1/y\,dy + 1/z\,dz}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$\log x + \log y + \log z = \log c_1$

$\boxed{xyz = c_1} \longrightarrow (1)$

Raj

DATE / /
PAGE

Again $x, y, -1$ as multiplier.

$$= \frac{xdx + ydy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{xdx + ydy - dz}{0}$$

$$\int xdx + ydy - dz = 0$$

$$\boxed{x^2 + y^2 - 2z = c_2} \longrightarrow (2)$$

taking t as parameter, the given equation of the straight line $x+y=0$, $z=1$ can be put in parametric form

$$x = t, \; y = -t \quad z = 1 \longrightarrow (3)$$

using (1), (2), (3), (1) & (2) can be written as

$$-t^2 = c_1 \quad \& \quad 2t^2 - 2 = c_2 \longrightarrow (4)$$

eliminating t from (4) we have

$$2(-c_1) - 2 = c_2 \quad \text{or} \quad 2c_1 + c_2 + 2 = 0$$

putting value of c_1 & c_2 from (1) & (2), the desired integral surface is

$$\boxed{2xyz + x^2 + y^2 - 2z + 2 = 0}$$

Raj

DATE / /
PAGE

Q5

Sol^n Given equation is $pq = px + qy$

$$f(x,y,z,p,q) = pq - px - qy = 0 \longrightarrow (1)$$

Charpit auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \longrightarrow (2)$$

i) finding first complete integral, taking 1st two fractions of fraction below

$$\frac{dp}{-p} = \frac{dq}{-q} = \frac{dz}{-p(q-x)-p(p-y)} = \frac{dx}{-(q-x)} = \frac{dy}{-(p-y)}$$

$$\frac{dp}{-p} = \frac{dq}{-q} \Rightarrow \log p = \log q + \log a \quad \text{or} \quad p = aq \rightarrow (3)$$

using (3) in (1)

$$aq^2 = q(ax+y) \Rightarrow q = (ax+y)/a \rightarrow (4)$$

Hence from (3) we have $p = ax+y \longrightarrow (5)$

$$\therefore dz = p\,dx + q\,dy$$

$$= (ax+y)\,dx + \left(\frac{(ax+y)}{a}\right)dy$$

$$= \left(\frac{1}{a}\right)(ax+y)(a\,dx + dy)$$

DATE / /
PAGE

putting $ax+y=t$

$dt = adx + dy$

$$\int dz = \int \left(\frac{1}{a}\right) \times t \times dt \Rightarrow \frac{1}{2a} t^2 + b$$

$$\boxed{z = \frac{1}{2a}(ax+y)^2 + b}$$

ii) To find second complete integral → taking second & 4th ratios in (2) we get

$$\frac{dx}{(q-x)} = \frac{dq}{q} \rightarrow q\,dx + x\,dq = q\,dq$$

integrating we get

$$qx = \frac{q^2}{2} + \frac{a}{2} \quad \text{or} \quad q^2 - 2xq + a = 0$$

$$\therefore q = \frac{1}{2}\left(2x \pm 2(x^2-a)^{1/2}\right) \Rightarrow q = x + (x^2-a)^{1/2} \rightarrow (6)$$

$$(1) \Rightarrow p\left(x + (x^2-a)^{1/2}\right) - px - y\left(x + (x^2-a)\right) = 0$$

$$p = \{1 + x/(x^2-a)^{1/2}\}y \longrightarrow (7)$$

$$dz = p\,dx + q\,dy = y\left(1 + \frac{x}{(x^2-a)^{1/2}}\right)dx + \left(x + (x^2-a)^{1/2}\right)dy$$

$$dz = (y\,dx + x\,dy) + \left[\frac{xy\,dy}{(x^2-a)^{1/2}} + (x^2-a)^{1/2}dy\right]$$

Raj

DATE / /

PAGE

$$dz = y\,dx + x\,dy + \left[\frac{xy\,dy}{(x^2-a)^{1/2}} + (x^2-a)^{1/2}dy\right]$$

$$dz = d(xy) + d\left(y(x^2-a)^{1/2}\right)$$

integrating

$$\boxed{z = xy + y(x^2-a)^{1/2} + b}$$ a, b are arbitrary constants

(iii) To find the 3rd complete integral we use first & fifth ratios of ②

$$\frac{dp}{-p} = \frac{dy}{-(p-y)} \Rightarrow (p-y)\,dp = p\,dy$$

$$p\,dp - y\,dp = p\,dy$$

$$p\,dp = p\,dy + y\,dp \longrightarrow ①$$

Integrating we get

$$py = \frac{p^2}{2} + a \Rightarrow \boxed{p^2 - 2py + a = 0}$$

$$p = \frac{1}{2}\left[2y \pm 2(y^2-a)^{1/2}\right] \Rightarrow p = \left(y + (y^2-a)^{1/2}\right) \longrightarrow ②$$

using ② in given equation

DATE / /
PAGE

$$q\left(y+(y^2-a)^{1/2}\right)-\left(y+(y^2-a)^{1/2}\right)x-qy=0$$

$$\cancel{q\left(y+(y^2-a)^{1/2}\right)(}$$

$$q(y^2-a)^{1/2}-x(y^2-a)^{1/2}-yx=0$$

$$q=\frac{yx}{(y^2-a)^{1/2}}+x \Rightarrow \cancel{q=\{1+x/(x^2-a)^{1/2}\}y}$$

$$q=\{1+y/(y^2-a)^{1/2}\}x \rightarrow (3)$$

$$\Rightarrow dz=p\,dx+q\,dy$$

$$\Rightarrow dz=\left(y+(y^2-a)^{1/2}\right)dx+\left(1+\frac{y}{(y^2-a)^{1/2}}\right)x\,dy$$

$$\Rightarrow dz=(y\,dx+x\,dy)+(y^2-a)^{1/2}dx+\frac{xy\,dy}{(y^2-a)^{1/2}}$$

$$\Rightarrow dz=d(xy)+d\left(x(y^2-a)^{1/2}\right)$$

$$\Rightarrow dz=d(xy)+d\left(x(y^2-a)^{1/2}\right)$$

$$\Rightarrow \boxed{z=xy+x(y^2-a)^{1/2}+b}$$ a, b are arbitary constants

Raj

DATE / /

PAGE

Q6

a) $(p+q)(px+qy)=1$

$p^2x+pqy+pqx+q^2y-1=0 = f(x,y,z,p,q)$ (1)

Thus auxiliary equations are

$f_x = p^2+pq = p(p+q)$ | $f_z = 0$

$f_y = q^2+pq$

$f_p = 2xp+qy+qx$ $\quad f_q = 2yq+py+px$

$$\frac{dp}{p(p+q)} = \frac{dq}{q(p+q)} = \frac{dx}{-(2px+qy+qx)} = \frac{dy}{-(px+py+2qy)} = \frac{-dz}{}$$

taking 1st two fractions we get

$p = aq \rightarrow$ in (1)

$(aq)^2 + aq^2y + aq^2x + q^2y = 1$

$$q^2 = \frac{1}{(a+1)(ax+y)} \longrightarrow (2)$$

similarly

$$p^2 = \frac{a^2}{(a+1)(ax+y)} \longrightarrow (3)$$

thus

$$p = \frac{a}{\sqrt{(a+1)(ax+y)}} \qquad q = \frac{1}{\sqrt{(a+1)(ax+y)}}$$

DATE / /
PAGE

$$dz = p\,dx + q\,dy \rightarrow$$

$$dz = \frac{a}{\sqrt{(a+1)(ax+y)}}dx + \frac{1}{\sqrt{(a+1)(ax+y)}}dy$$

$$(a+1)^{1/2}dz = \frac{1}{\sqrt{(ax+y)}}d(ax+y)$$

setting $t = ax+y$

$dt = a\,dx + dy$

$$(a+1)^{1/2}\int dz = \int \frac{dt}{\sqrt{t}}$$

$$\boxed{(a+1)^{1/2} z = 2(ax+b)^{1/2} + b}$$

b) $(y-x)(qy-px) = (p-q)^2$

Sol^n let $(x+y) = u$ & $xy = v$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \left(\frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot y\right) \quad \text{as } \frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = y$$

similarly

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \left(\frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot x\right) \quad \text{as } \frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial y} = x$$

Raj

DATE / /

PAGE

Now substituting $\partial z/\partial x$ & $\frac{\partial z}{\partial y}$ in the given equation we get

$$\Rightarrow (x-y)\left(x\left(\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right)-y\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right)\right)$$
$$=\left[\left(\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right)-\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right)\right]^2$$

$$\Rightarrow (x-y)\left(x\frac{\partial z}{\partial u}+\cancel{xy\frac{\partial z}{\partial v}}-y\frac{\partial z}{\partial u}-\cancel{xy\frac{\partial z}{\partial v}}\right)=\left[\left(y\frac{\partial z}{\partial v}-x\frac{\partial z}{\partial v}\right)\right]$$

$$\Rightarrow (x-y)(x-y)\frac{\partial z}{\partial u}=(x-y)^2\left(\frac{\partial z}{\partial v}\right)^2$$

$$\Rightarrow \frac{\partial z}{\partial u}=\left(\frac{\partial z}{\partial v}\right)^2$$

$P-Q^2=0$ where $P=\frac{\partial z}{\partial u}$, $Q=\frac{\partial z}{\partial v}$

now the equation is of form $f(p,q)=0$
here we can replace $p=a$ & $q=b$ i.e

$a-b^2=0$ or $a=b^2$ hence

~~dz = adx +~~ ~~dz~~ = ~~pdx + qdy~~

~~dz~~ = ~~adx~~ $f(a,b)=0$

Raj

DATE / /

PAGE

$$z = au + bv + c$$
$$= a(x+y) + bxy + c$$

$$\boxed{z = b^2(x+y) + b(xy) + c}$$

c) $pz = 1 + q^2$

The given equation is of form $f(p, q, z) = 0$ so we use $\boxed{u = ax + y}$

→ a here is arbitrary constant

→ Replacing p by $\partial z/\partial u$ and q by $a\,\partial z/\partial u$

$\cancel{p = \frac{\partial z}{\partial u},\ q = \frac{\partial z}{\partial u}}$

$$z\frac{dz}{du} = 1 + a^2\left(\frac{dz}{du}\right)^2 \Rightarrow a^2\left(\frac{dz}{du}\right)^2 - z\frac{dz}{du} + 1 = 0$$

$$\therefore \frac{dz}{du} = \frac{z \pm (z^2 - 4a^2)^{1/2}}{2a^2} \rightarrow \frac{dz}{du} = \frac{z \pm (z^2 - 4a^2)^{1/2}}{2a^2}$$

$$\frac{dz}{(z \pm (z^2 - 4a^2)^{1/2})} = \frac{du}{2a^2}$$

$$\frac{dz \cdot [z \mp (z^2 - 4a^2)^{1/2}]}{[z \pm (z^2 - 4a^2)^{1/2}][z \mp (z^2 - 4a^2)^{1/2}]} = \frac{du}{2a^2}$$

$$\frac{\left[z \mp (z^2-4a^2)^{1/2}\right]dz}{4\cancel{a^2}} = \frac{du}{2\cancel{a^2}}$$

$$\int \left(z \mp (z^2-4a^2)^{1/2}\right)dz = \int 2\,du$$

$$\frac{z^2}{2} \mp \left[\frac{z}{2}(z^2-4a^2)^{1/2} - \frac{4a^2}{2}\log\left\{z + (z^2-4a^2)^{1/2}\right\}\right]$$

$$= 2u + \frac{b}{2}$$

$$\boxed{z^2 \mp \left[z(z^2-4a^2)^{1/2} - 4a^2\log\left\{z + (z^2-4a^2)^{1/2}\right\}\right] = 4(x+ay) + b}$$

Raj

DATE / /

PAGE

Que 7

Soln $p_1^3 + p_2^2 + p_3 - 1 = 0 \longrightarrow (1) = f(x_1, x_2, x_3, p_1, p_2, p_3)$

The given partial differential equation of order one with dependent variable z and 3 independent variables x_1, x_2 & x_3 in its standard form.

Jacobi's auxiliary equations for (1)

$$\frac{dp_1}{\left(\frac{\partial f}{\partial x_1}\right)} = \frac{dx_1}{-\left(\frac{\partial f}{\partial p_1}\right)} = \frac{dp_2}{\left(\frac{\partial f}{\partial x_2}\right)} = \frac{dx_2}{-\left(\frac{\partial f}{\partial p_2}\right)} = \frac{dp_3}{\left(\frac{\partial f}{\partial x_3}\right)} = \frac{dx_3}{-\left(\frac{\partial f}{\partial p_3}\right)}$$

$$\frac{dp}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{0} = \frac{dx_3}{-1} \longrightarrow (2)$$

taking first 2 fractions of (2)

$dp_1 = 0$ & $dp_2 = 0$ so that $p_1 = c_1$ & $p_2 = c_2$

where c_1 & c_2 are arbitrary constants

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = c_1 \rightarrow (3)$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_2) = p_2 = c_2 \longrightarrow (4)$$

Now verifying the relation between F_1 & F_2.

Raj

DATE / /
PAGE

$$(F_1, F_2) = \sum_{r=1}^{3} \left(\frac{\partial f_1}{\partial x_r} \cdot \frac{\partial f_2}{\partial p_r} - \frac{\partial f_1}{\partial p_r} \frac{\partial f_2}{\partial x_r} \right)$$

$$\Rightarrow \left(\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial p_1} - \frac{\partial f_1}{\partial p_1} \frac{\partial f_2}{\partial x_1} \right) + \left(\frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial p_2} - \frac{\partial f_1}{\partial p_2} \frac{\partial f_2}{\partial x_2} \right)$$

$$+ \left(\frac{\partial f_1}{\partial x_3} \frac{\partial f_2}{\partial p_3} - \frac{\partial f_1}{p_3} \frac{\partial f_2}{\partial x_3} \right)$$

$$0 \cdot 0 - (1)(0) + (0)(1) - (0)(0) + (0)(0) = 0$$

Thus the relation $(F_1, F_2) = 0$ is verified for ③ & ④
Hence equation ③ & ④ can be used for additional equations

Now solving ①, ③, ④ for p_1, p_2, p_3

$$p_1 = c_1, \quad p_2 = c_2 \quad \& \quad p_3 = 1 - c_1^3 - c_2^2 \longrightarrow ⑤$$

substituting the values of p_1, p_2 & p_3 from ⑤ in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

$$\int dz = \int c_1 dx_1 + \int c_2 dx_2 + \int (1 - c_1^3 - c_2^2) dx_3$$

$$\boxed{z = c_1 x_1 + c_2 x_2 + (1 - c_1^3 - c_2^2) x_3 + c_3}$$

c_1, c_2, c_3 are arbitrary constants.

complete integral of PDE