GRAPH THEORY

ASSIGNMENT - III

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Q1) Prove that for a graph G with n vertices and e edges,

$$\text{vertex connectivity} \leq \text{edge connectivity} \leq \frac{2e}{n}$$

Let α be the edge connectivity of graph G. $\therefore \exists$ a cutset S in G with α edges. Let S partition the vertices of G into V_1 & V_2. Now, by removing at most α vertices from V_1 or V_2 on which edges in S are incident we can effect the removal of S from G.

$\therefore$ Vertex connectivity $\leq$ Edge connectivity

We know that the graph will have 2e degrees which is divided among the n vertices, so there must be at least one vertex in G whose degree $\leq \frac{2e}{n}$

And since edge connectivity $\leq$ smallest degree in G

$$\Rightarrow \text{edge connectivity} \leq \frac{2e}{n}$$

$$\Rightarrow \text{vertex connectivity} \leq \text{edge connectivity} \leq \frac{2e}{n}$$

Hence proved ■

Q2) Define a seperable graph. Prove that in a non-seperable graph G set of edges incident on each vertex of G is a cut-set.

A connected graph is said to seperable if its vertex connectivity is one.

Let G be a non seperable graph, let v be a some vertex in G, let edges $\{e_1, e_2 \cdots e_j\}$ are incident on v, now if all the edges are removed the graph will be disconnected as their will be two components $G\{v\}$ & ~~G~~ $\{v\}$

let $\{e_j, e_{j+1} \dots e_i\}$ be a subset of $\{e_1, e_2, \dots e_j\}$, now let $\{e_j, e_{j+1} \dots e_i\}$ also be a cut set of G_2

⇒ the block having vertex v has vertex $\{v_1, v_2 \dots v_i\}$ s.t. an edge ~~e_j~~ e_i, from v to u exists

⇒ removal of v from G should disconnect graph but once G is non separable, hence its a contradiction

⇒ $\{e_1, e_2 \dots e_i\}$ can't have a subset which is a cut set of ~~g~~ G

⇒ $\{e_1, e_2 \dots e_i\}$ is a cut set.

⇒ Set of edges incident on each vertex of G is a cutset.

Q3) Define the capacity of a cut-set. Prove that the maximum flow possible between two vertices a and b in a network is equal to the minimum of capacities of all cut-sets with respect to a and b.

Seperable Graph

A graph G is said to be seperable if it is either disconnected or can be disconnected by ~~one~~ removing one vertex, called the cut vertex. A graph that is not seperable is said to be biconnected (or nonseperable)

Maxflow Mincut Theorem

In any source network with source s and target t, the value of the maximum (s,t)-flow is equal to the capacity of the minimum (s,t)-cut.

Ford and Fulkerson proved this theorem as follows. Fix a graph G, vertices s and t and a capacity function $c: E \rightarrow \mathbb{R}_{\geq 0}$. We assume that the capacity function is reduced. For any vertices u and v, either $c(u \rightarrow v) = 0$ or $c(v \rightarrow u) = 0$, or equivalently, if an edge appears in G, then it's reversal does not. This assumption is easy to enforce. Whenever an edge $u \rightarrow v$ and it's reversal $v \rightarrow u$ are both graphs, replace the edge $u \rightarrow v$ with a path $u \rightarrow x \rightarrow v$ of length 2, where x is new vertex and $c(u \rightarrow x) = c(x \rightarrow v) = c(u \rightarrow v)$. The modified graph has the same maximum flow value and minimum cut ~~vertex~~ capacity as the original graph.

Let f be a feasible flow. We define a new capacity function $c_f : V \times V \to \mathbb{R}$, called the residual capacity as follows

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{otherwise} \end{cases}$$

Since $f \geq 0$ and $f \leq c$, the residual capacities are always non-negative. It is possible to have $c_f(u \to v) > 0$ even if $u \to v$ is not an edge in the original graph G. Thus, we define the residual graph $G_f = (V, E_f)$ where E_f is the set of Edges whose residual capacity is positive. ~~Notice~~

Suppose there is no path from the source s to target t in the residual graph G_f. Let S be the set of vertices that are reachable from s in G_f and let $T = V \setminus S$. The partition (S, T) is clearly an (s,t)-cut. For every vertex $u \in S$ and $v \in T$, we have

$$c_f(u \to v) = [c(u \to v) - f(u \to v)] + f(v \to u) = 0$$

which implies that $c(u \to v) - f(u \to v) = 0$ and $f(v \to u) = 0$. In other words, our flow f saturates every edge from S to T. It follows that $f = \|S, T\|$. Moreover, f is maximum flow and (S, T) is a minimum cut.

Suppose that there is a path $s = v_0 v_1 v_2 \dots v_r = t$ in G_f. We refer to $v_0 v_1 v_2 \dots v_r$ as an augmenting path. Let $F = \min_i c_f(v_i \to v_{i+1})$ denote the maximum amount of flow that we can push through the augmenting path in G_f. We define a new flow function $f' : E \to \mathbb{R}$ as follows:

$$f'(u \to v) = \begin{cases} f(u \to v) + F & \text{if } u \to v \text{ is in the augmenting path} \\ f(u \to v) - F & \text{if } v \to u \text{ is the augmenting path} \\ f(u \to v) & \text{otherwise} \end{cases}$$

To prove that flow f' is feasible with respect to the original capacities c, we need ~~to~~ to verify that $f' \geq 0$ and $f' \leq c$. Consider an edge $u \to v$ in G. If $u \to v$ is in the augmenting path, then

$f'(u \to v) > f(u \to v) \geq 0$ and

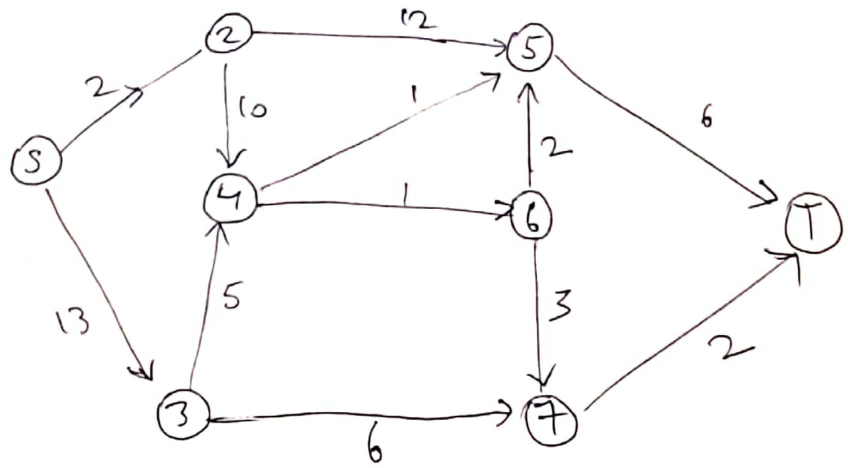
$$\begin{aligned} f'(u \to v) = f(u \to v) + F &\leq f(u \to v) + c_f(u \to v) \\ &= f(u \to v) + c(u \to v) - f(u \to v) \\ &= c(u \to v) \end{aligned}$$

On the other hand reversal $v \to u$ is in the augmenting path we get $f'(u \to v) = 0$

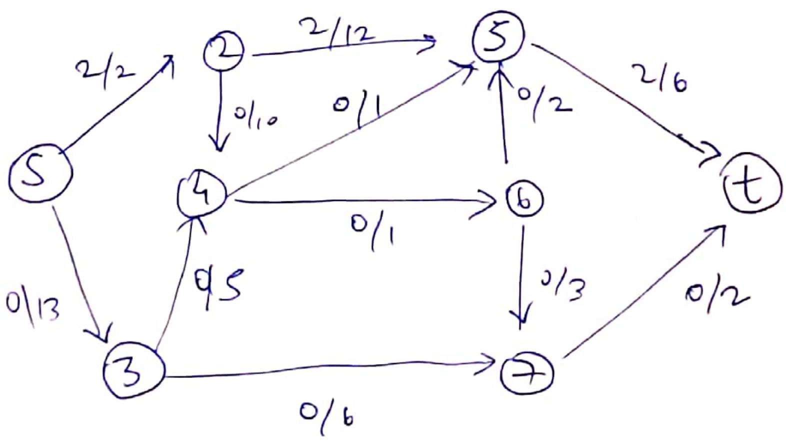
Finally, we observe that WLG only the first edge in the augmenting path leaves s, so $|f'| = |f| + F > 0$. In other words, f is not a maximum flow.

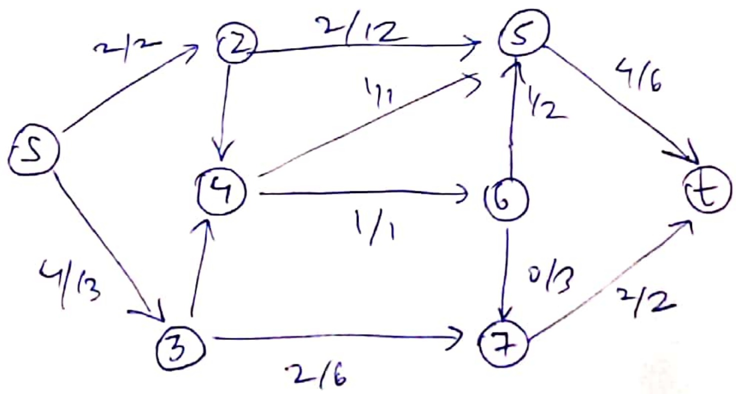
So, for 2 vertices a and b, the maximum flow between them is the minimum of all cut sets between a and b.

Q.4) Describe Ford-Fulkerson Algorithm for maximum flow and hence find maximum flow for network given below.



Augmenting path	Bottleneck capacity
$s \to 2 \to 5 \to t$	2
$s \to 3 \to 7 \to t$	2
$s \to 3 \to 4 \to 6 \to 5 \to t$	1
$s \to 3 \to 4 \to 5 \to t$	1

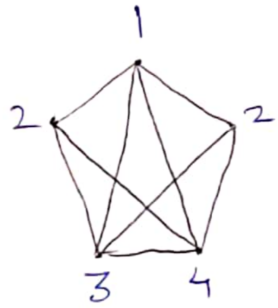




Total Flow = 2+4 = 6

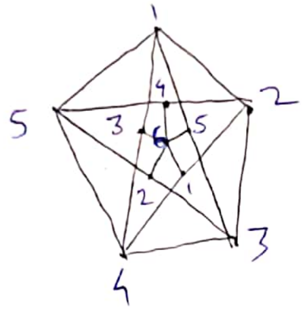
Q5) Find the Chromatic Number of each of the graph given below.

a)



$\chi = 4$ colors needed

b)



$\chi = 6$ colors needed

Q6) Prove that the non-empty graph G is bicolorable iff G is bipartite.

Let G be a bipartite graph i.e. the vertex set of G can be partitioned into 2 sets V_1 & V_2 such that $V_1 \cup V_2 = V(G)$

Since in V_1 for any edge u,v u–v doesn't exist, we can

~~$\forall u \in V_1$ $v \in V_2$ ed~~

assign color 1 to V_1 and 2 to V_2.

Conversely let G be bicolorable. Let V_1 = set of vertices having color 1 and similarly ~~$\forall v \in V_2$ have color~~ V_2 have color 2. No 2 vertices in the set V_1 are adjacent and this applies for V_2 as well. So, if a non-empty graph G is bicolorable, it can be divided (partitioned) into 2 sets and is hence bipartite, where & the 2 sets individually have no edges. Hence proved.

Q7) Define complete matching in a graph. Find the number of complete matchings in $K_{n,n}$; a complete bipartite graph with n vertices in each subset.

Complete matching in a graph

A matching in a graph G is said to be perfect if every vertex is connected to exactly one edge.

A complete bipartite graph $K_{n,n}$ with n vertices in each set.

Let V_1 and V_2 be the 2 subsets.

Let $\{v_1, v_2 \ldots v_n\}$ denote the vertices in V_1 and

let $\{u_1 u_2 \ldots u_n\}$ denote the vertices in V_2

For vertex v_1 we have n options,

for vertex v_2 we have n-1 options

⋮

for vertex v_n we have 1 option

Therefore total no. of matchings $= n(n-1)(n-2) \ldots 2.1$
$= n!$

Q8) Define Perfect matching in graph. Define the number of perfect matchings in:

a) K_{2n}, a complete graph with 2n vertices

A matching M is said to be perfect if vertex of graph is incident to an edge in matching.

For 1st vertex we have 2n-1 choices

For 2nd vertex we have .. 2n-3 choices

For 3rd vertex we have 2n-5 choices

For last vertex (n^{th}) we have one choice.

No. of complete mappings $= (2n-1)(2n-3)(2n-5)\ldots 3.1$

$$= \frac{(2n-1)(2n-2)(2n-3)(2n-5)\ldots 3.2.1}{(2n-2)(2n-4)(2n-6)\ldots 2}$$

$$\frac{(2n-1)(2n-2)(2n-3)\ldots 3.2.1}{2^n . n!}$$

$$\frac{(2n)!}{2^n n!}$$

b) $C_{2n} \rightarrow$ cycle with $2n$ vertices

Starting from v_1, we have 2 options $\rightarrow$ edge a & b, after choosing either of these, we can only choose alternate edges e.g. $\{(v_1 v_2)(v_3 v_4)\ldots\ldots\}$

No. of perfect matching $= 2$