13. Features of Membership Function

Kernel or Core of a fuzzy set $\tilde{A}$ consists of elements of $\tilde{A}$ whose membership grade is one. In other words

$$\mathrm{Ker}(\tilde{A}) = \{x \in \tilde{A} \mid \mu_A(x) = 1\}$$

Elements of $\tilde{A}$ that have non-zero membership grades not equal to one are said to constitute **boundaries** of the fuzzy set $\tilde{A}$.

Elements with membership grade $\mu_A(x) = 1$ which constitute the core of the fuzzy set $\tilde{A}$ are said to have complete membership whereas elements which belong to the boundaries of $\tilde{A}$ are said to have partial membership of $\tilde{A}$. Even memberships of fuzzy sets representing the same concept may vary considerably. In such a case however, they also have to be similar in some key features. As an example, let us consider fuzzy sets whose membership functions are to express a class of real numbers that are close to 2. In spite of their differences.

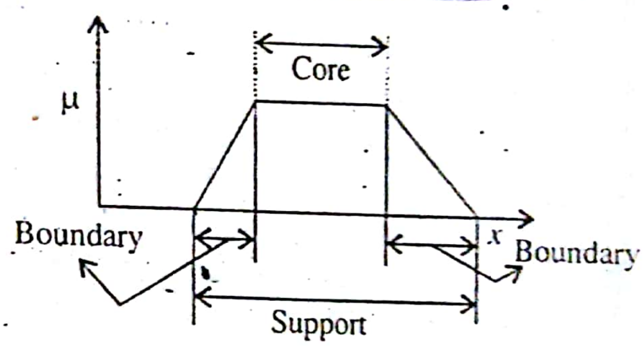

Fig: 5. Features of membership function.

These fuzzy sets have to be similar in the sense that the following properties have to be possessed by each of these:

(i) $\mu_A(2) = 1$ and $\mu_A(x) < 1$ for all $x \neq 2$;

(ii) μ_A should be symmetric with respect to $x = 2$, that is $\mu_A(2 + x) = \mu_A(2 - x)$ for all $x \in R$;

(iii) $\mu_A(x)$ should decrease monotonically from 1 to 0 with the increasing difference $| 2 - x |$.

These properties are necessary in order to properly represent the given concep. Any fuzzy set attempting to represent the same concep would have to possess them.

DEFUZZIFICATION TO CRISP SETS

We begin by considering a fuzzy set $\underset{\sim}{A}$, then define a lambda-cut set, A_λ, where $0 \le \lambda \le 1$. The set A_λ is a crisp set called the *lambda* (λ)*-cut* (or *alpha-cut*) set of the fuzzy set $\underset{\sim}{A}$, where $A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) \ge \lambda\}$. Note that the λ-cut set A_λ does not have a tilde underscore; it is a crisp set derived from its parent fuzzy set, $\underset{\sim}{A}$. Any particular fuzzy set $\underset{\sim}{A}$ can be transformed into an infinite number of λ-cut sets, because there are an infinite number of values λ on the interval $[0, 1]$.

Any element $x \in A_\lambda$ belongs to $\underset{\sim}{A}$ with a grade of membership that is greater than or equal to the value λ. The following example illustrates this idea.

Example 4.1. Let us consider the discrete fuzzy set, using Zadeh's notation, defined on universe $X = \{a, b, c, d, e, f\}$,

$$\underset{\sim}{A} = \left\{\frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f}\right\}.$$

This fuzzy set is shown schematically in Figure 4.8. We can reduce this fuzzy set into several λ-cut sets, all of which are crisp. For example, we can define λ-cut sets for the values of $\lambda = 1, 0.9, 0.6, 0.3, 0^+$, and 0.

$$A_1 = \{a\}, \quad A_{0.9} = \{a, b\},$$

$$A_{0.6} = \{a, b, c\}, \quad A_{0.3} = \{a, b, c, d\},$$

$$A_{0^+} = \{a, b, c, d, e\}, \quad A_0 = X.$$

The quantity $\lambda = 0^+$ is defined as a small "ε" value >0, that is, a value just greater than zero. By definition, $\lambda = 0$ produces the universe X, since all elements in the universe have at least a 0 membership value in any set on the universe. Since all A_λ are crisp sets, all the elements just shown in the example λ-cut sets have unit membership in the particular λ-cut set. For example, for $\lambda = 0.3$, the elements a, b, c, and d of the universe have a membership of 1 in the λ-cut set, $A_{0.3}$, and the elements e and f of the universe have a membership of 0 in the λ-cut set $A_{0.3}$. Figure 4.9 shows schematically the crisp λ-cut sets for the values $\lambda = 1, 0.9, 0.6, 0.3, 0^+$, and 0. Note, in these plots of membership value versus the universe X, that the effect of a λ-cut is to rescale the membership values to one for all elements of the fuzzy set $\underset{\sim}{A}$ having membership values greater than or equal to λ and to zero for all elements of the fuzzy set $\underset{\sim}{A}$ having membership values less than λ.

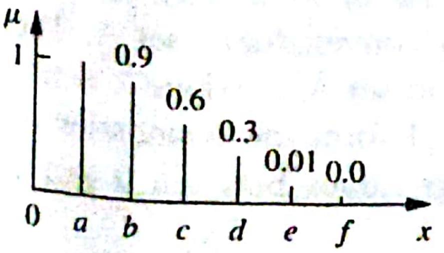

FIGURE 4.8
A discrete fuzzy set $\underset{\sim}{A}$.

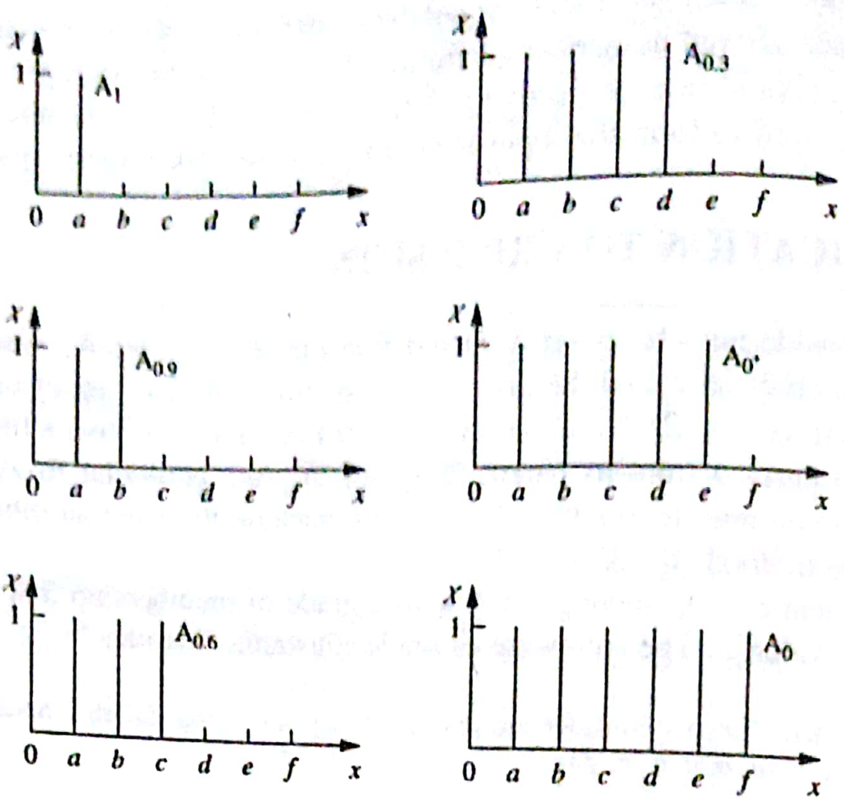

FIGURE 4.9
Lambda-cut sets for $\lambda = 1, 0.9, 0.6, 0.3, 0^+, 0$.

We can express λ-cut sets using Zadeh's notation. For example, λ-cut sets for the valu[es] $\lambda = 0.9$ and 0.25 are given here:

$$A_{0.9} = \left\{\frac{1}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0}{d} + \frac{0}{e} + \frac{0}{f}\right\} \quad \text{and} \quad A_{0.25} = \left\{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{0}{e} + \frac{0}{f}\right\}$$

λ-cut sets obey the following four very special properties:

1. $(\underset{\sim}{A} \cup \underset{\sim}{B})_\lambda = A_\lambda \cup B_\lambda$ (4.1a)
2. $(\underset{\sim}{A} \cap \underset{\sim}{B})_\lambda = A_\lambda \cap B_\lambda$ (4.1b)
3. $(\overline{\underset{\sim}{A}})_\lambda \neq \overline{A}_\lambda$ except for a value of $\lambda = 0.5$ (4.1c)
4. For any $\lambda \leq \alpha$, where $0 \leq \alpha \leq 1$, it is true that $A_\alpha \subseteq A_\lambda$, where $A_0 = X$ (4.1d)

These properties show that λ-cuts on standard operations on fuzzy sets are equivalent with standard set operations on λ-cut sets. The last operation, Equation (4.1d), can be shown more conveniently using graphics. Figure 4.10 shows a continuous-valued fuzzy set with two λ-cut values. Notice in the graphic that for $\lambda = 0.3$ and $\alpha = 0.6$, $A_{0.3}$ has a greater domain than $A_{0.6}$, that is, for $\lambda \leq \alpha (0.3 \leq 0.6)$, $A_{0.6} \subseteq A_{0.3}$.

In this chapter, various definitions of a membership function are discussed and illustrated. Many of these same definitions arise through the use of λ-cut sets. As seen in Figure 4.1, we can provide the following definitions for a convex fuzzy set $\underset{\sim}{A}$. The core of $\underset{\sim}{A}$ is the $\lambda = 1$ cut set, A_1. The support of $\underset{\sim}{A}$ is the λ-cut set A_{0^+}, where $\lambda = 0^+$, or symbolically, $A_{0^+} = \{x \mid \mu_{\underset{\sim}{A}}(x) > 0\}$. The intervals $[A_{0^+}, A_1]$ form the boundaries of the fuzzy set $\underset{\sim}{A}$, that is, those regions that have membership values between 0 and 1 (exclusive of 0 and 1): that is, for $0 < \lambda < 1$.

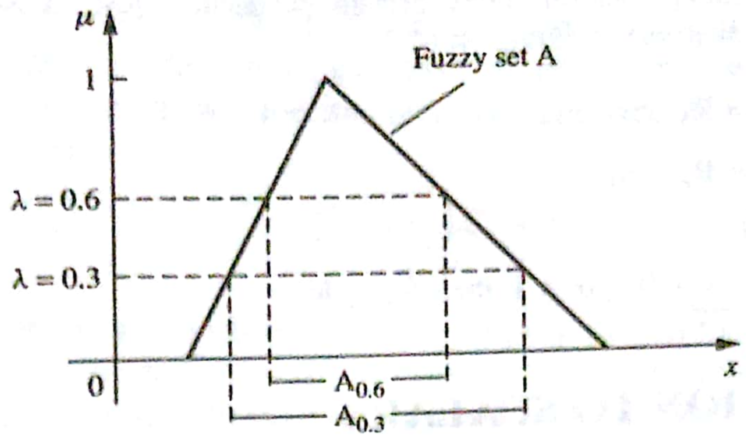

FIGURE 4.10
Two different λ-cut sets for a continuous-valued fuzzy set.

(i) Triangular Membership Function

A triangular membership function is one of the most commonly used membership functions. It is specified by three parameters (a, b, c) such that the value of the membership function is zero for $x \le a$ and $x \ge c$. It increases linearly from value zero at $x = a$ to value 1 at $x = b$ and then decreases linearly from value 1 at $x = b$ to value 0 to $x = c$ as shown in Fig. 6(a). Mathematically the membership function $\mu_A(x)$ of such a triangular fuzzy set may be expressed as

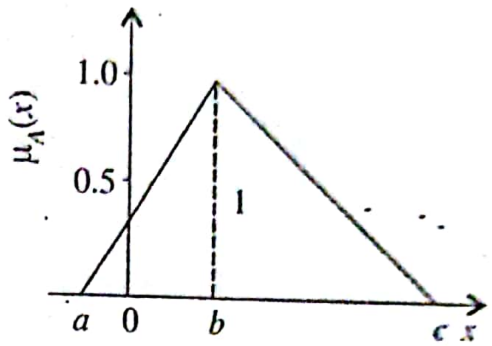

Fig. 6(a). Triangular membership function.

$$\mu_A(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a), & a \le x \le b \\ (c-x)/(c-b), & b \le x \le c \\ 0 & x > c \end{cases}$$

The precise appearance of the membership function is determined by the choice of parameters a, b, c $(a < b < c)$. The name comes from its figure which is a triangle. For instance in order to specify real numbers close to 10, we could have used a triangular fuzzy set (8, 10, 12) where membership function is given by

$$\mu_A(x) = \begin{cases} 0 & x \le 8 \\ (x-8)/2 & 8 \le x \le 10 \\ (12-x)/2 & 10 \le x \le 12 \\ 0 & x \ge 12 \end{cases}$$

The membership function of this number is shown diagrammatically in Fig. 6(b).

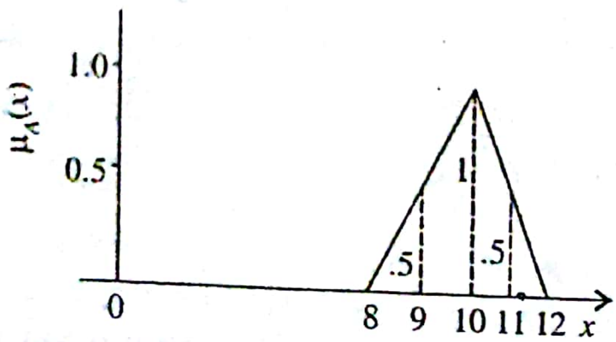

Fig. 6(b). Triangular membership function for real numbers close to 10.

In this case we assume that the real numbers less than 8 or greater than 12 are not acceptable whereas membership grade increases linearly from 0 to 1 as x increases from 8 to 10 and then decreases linearly from 1 to 0 as x increases from 10 to 12. Membership function considered in Example 2 for numbers close to 10 assumed a nonlinear variation, which gradually approaches 0 as x deviates from 10 on either side. However, in the present case numbers less than 8 or greater than 10 are completely excluded from being considered as numbers close to 10.

Some authors also use the notation (a, a_1, a_2) to represent a triangular fuzzy number, a being the middle value where membership function value is one and a_1 and a_2 are the spreads on left and right of a in which membership value is non-zero. Triangular fuzzy number (8, 10, 14) in this notation becomes (10, 2, 4).

(ii) Trapezoidal Membership Function

A trapezoidal membership function is specified by four parameters, a, b, c and d $(a < b < c < d)$ with membership function given as

$$\mu_A(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a), & a \le x \le b \\ 1 & b \le x \le c \\ (d-x)/(d-c), & c \le x \le d \\ 0 & x > d \end{cases}$$

Geometrically it is shown in Fig. 7.

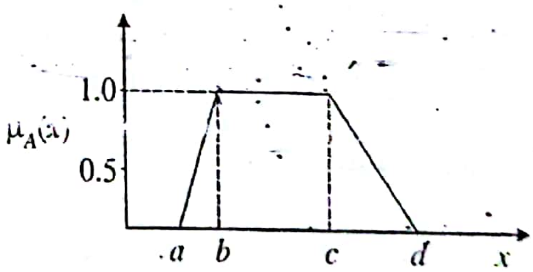

Fig. 7. Trapezoidal shaped membership function.

The name trapezoidal comes from its figure, which is a trapezium. In fact a triangular fuzzy membership function is a special case of a trapezoidal fuzzy membership function when $b = c$. Whereas in a fuzzy set with triangular membership function there is only one complete member $x = b$ with full membership function value $\mu_A(b) = 1$, in a trapezoidal fuzzy set all members between $x = b$ and $x = c$ are full members. The members from a to b and then from c to d being partial members with membership value increasing linearly from 0 to 1 from $x = a$ to $x = b$ and decreasing linearly from 1 to 0 from $x = c$ to $x = d$.

In case membership function is to taper only on one side, we may treat it as a special case of trapezoidal fuzzy membership by taking $d = c$. In this case the membership value increases from 0 and 1 from $x = a$ to $x = b$ and then remains 1 thereafter (Fig. 7(a)). On the contrary, if we set $a = b$, membership value is 1 up to $x = c$ and then tapers off to 0 from c to d (Fig. 7(b)).

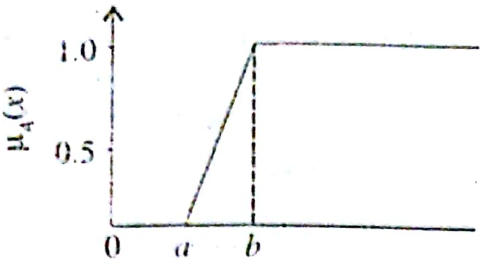

Fig. 7(a). Trapezoidal membership function opening to the right ($c = d$).

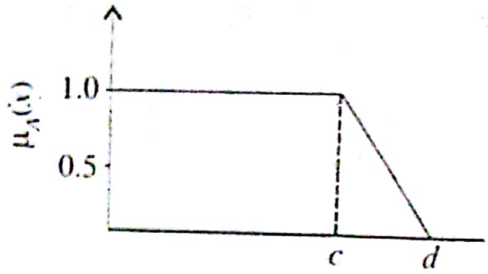

Fig. 7(b). Trapezoidal membership function opening to the left ($a = b$).

Membership grades of these two types of functions are

$$\mu_A(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a), & a \le x \le b \\ 1 & x > b \end{cases}$$

in the first case (Fig. 7(a)) and

$$u_A(x) = \begin{cases} 1 & x < c \\ (d-x)/(d-c), & c \le x \le d \\ 0 & x < c \end{cases}$$

in the second case (Fig. 7(b)). A trapezoidal fuzzy number is often used to represent an interval (a, b) where the values of the end points a and b are determined experimentaly and have some margin of error on either side. It can also be used to represent middle-aged persons, or middle income group. Trapezoidal fuzzy numbers in which there is tapering only on the left side (Fig. 7(a)) may be used to represent high income group, or old persons or large numbers. Similarly, trapezoidal fuzzy numbers which taper only on the right may be used to represent low income group or infants or small numbers.

(iii) Bell-shaped, Gaussian and Sigmoidal Membership Functions

Triangular and trapezoidal membership functions assume linear rise and fall in membership values between 0 and 1. In order to account for nonlinear variation in the values of membership functions Bell-shaped, Gaussian and Sigmoidal types of membership functions have been used in literature

Bell-shaped Function

A bell-shaped membership function (Fig. 8(a)) is specified by three parameters a, b and c and has the membership function value

$$\boxed{\mu_A(x) = \frac{1}{1+\left(\frac{x-c}{a}\right)^{2b}}}$$

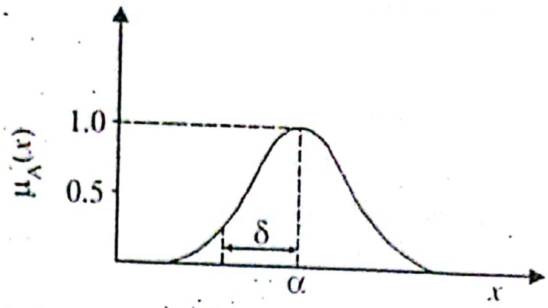

Fig. 8(a). Bell-shaped membership function.

The name comes from the fact that the graph of this function is bell-shaped. A desired bell-shaped membership function can be obtained by proper selection of the values of the parameters a, b and c. Parameter b is usually taken positive. Value of the parameter c determines the centre and parameters a and b help in controlling the width and shape of the bell. In the case of real numbers close to 10 in Example 2, we have used bell-shaped membership function with $c = 10$, $a = 1$ and $b = 1$.

Gaussian Membership Function

A Gaussian membership function is specified by two parameters m and σ as follows:

$$\boxed{\mu_A(x) = e^{-\left(\frac{x-m}{\sigma}\right)^2}}$$

Its shape is shown in Fig. 8 (b). Its value is 1 at $x = m$ and tapers nonlinearly to 0 on both sides as x deviates from m. The value of σ determines the rate of tapering of its value. Graph of this function spreads symmetrically on both sides of m. Theoretically $\mu_A(x) \to 0$ as $x \to \infty$ or $x \to -\infty$.

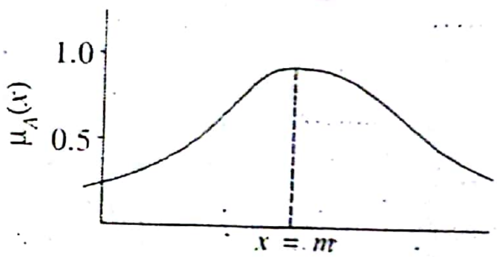

Fig. 8(b). Gaussian membership function

This membership function could have also been used to denote real numbers close to 10, with $m = 10$ and σ suitably chosen.

Sigmoidal Membership Function

This membership function is used in situations where membership value is expected to increase nonlinearly to 1. Sigmoidal membership function is

$$\boxed{\mu_A(x) = \frac{1}{1+e^{-a(x-c)}}}$$

It is shown graphically in Fig. 9.

In this function $\mu_A(c) = 0.5$, and as the value of parameter a increases the transition in value from 0 to 1 becomes sharper.

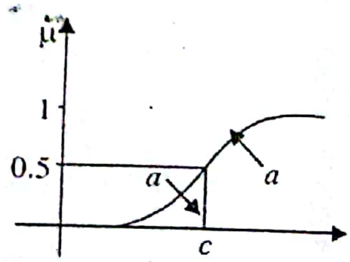

Fig. 9. Sigmoidal membership function.

(iv) Some Other Commonly Used Membership Functions

The membership functions introduced so far are available in the membership function editor of the Fuzzy Logic Tool Box for MATLAB. Some other membership functions which have also been frequently used to incorporate the nonlinearity in variation of membership function are as follows:

***S*-Membership Function**

S-Membership function is a smooth membership function with two parameters a and b whose membership function value is given as

$$\mu_A(x) = \begin{cases} 0 & x < a \\ 2\left(\dfrac{x-a}{b-a}\right)^2, & a \le x \le \dfrac{a+b}{2} \\ 1-2\left(\dfrac{x-b}{b-a}\right)^2, & \dfrac{a+b}{2} \le x < b \\ 1 & x \ge b \end{cases}$$

Shape of this function is as shown in Fig. (10).

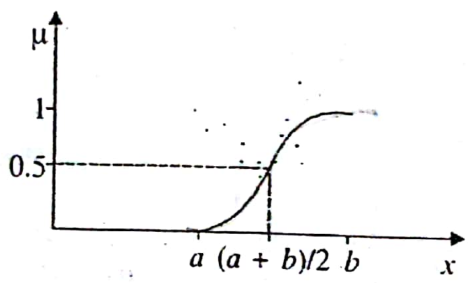

Fig. 10. *S*-membership function.

Membership value is 0 for $x < a$, 1 for $x > b$, and 0.5 for $x = (a + b)/2$. The name of the function comes from the shape of its graph which resembles *S*.

***π*-Membership Function**

There are two types of π membership functions. The first π membership function has two parameters a and b and is given as

$$\boxed{\mu_{\pi_1}(x) = \frac{1}{1+\left(\dfrac{x-a}{b}\right)^2}}$$

It is also called π_1 membership function. Shape of its graph is shown in Fig. 11(a). In fact this is a particular case of the bell-shaped function. Function has membership value 1 at $x = a$ and 0.5 at $x = a - b$ and $a + b$. Value of the function decreases asymptotically to 0 as we move away from a on either side.

The other π membership function (also called π_2 membership function) is

$$\mu_{\pi_2}(x) = \begin{cases} \dfrac{a}{a+b-x}, & x < b \\ 1 & b \le x \le c \\ \dfrac{d}{x-c+d}, & x > c \end{cases}$$

The shape of this membership function is shown in Fig. 11(b). It is primarily a nonlinear generalization of trapezoidal membership function.

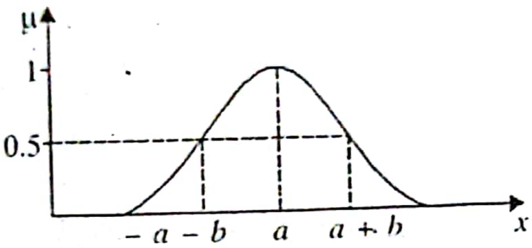

Fig. 11(a). π_1 membership function.

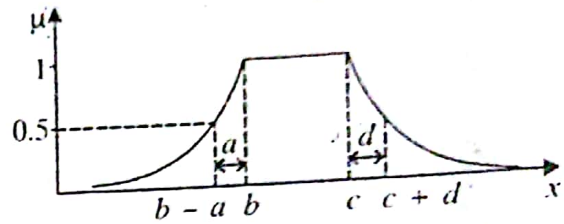

Fig. 11(b). π_2 membership function.

We have listed here some of the important types of fuzzy membership functions which have been frequently used in literature. These can take care of a large variety of real life situations. Majority of the users prefer to use linear membership functions in view of the ease with which mathematical operations can be carried out on these. Non-linear membership functions have also been used when the practical situation so demands and the linear approximation is not expected to produce the desired results. The list of membership functions given above is in no case exhaustive. As mentioned earlier, if need be a user can design his/her own membership function to take care of specific requirements of his/her problems. However, the membership function thus defined should not in any way violate any of the basic requirements listed in the definition of fuzzy membership function.

Short Answer Review Questions

1. Mark the following statements as true or false. Also correct the false statements meaningfully wherever possible.

    (i) The support of every fuzzy set is a crisp set.
    (ii) The law of excluded middle does not apply to fuzzy sets.
    (iii) A crisp set can be regarded as a fuzzy set whose membership value is either 0 or 1.
    (iv) The value of membership function of a fuzzy set can be any real number positive, zero or negative.
    (v) At least one member of a fuzzy set must have membership value equal to one.
    (vi) The fuzzier a set more similar it is to its compliment.
    (vii) Membership function graph of a fuzzy set must always be symmetric.
    (viii) Every fuzzy set is a convex set.
    (ix) Fuzzy sets formed over the same universe of discourse should not overlap.
    (x) Cardinality of a fuzzy set can be more than one.
    (xi) The cardinality of every fuzzy set can be defined.