Conditional Expectation

If X & Y are discrete r.v., the conditional probability mass fn. of X, given $Y=y$ is defined for all y s.t. $P(Y=y)>0$ as

$$P(X=x/Y=y) = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} \quad \text{——— (1)}$$

The conditional distribution fn. of X given $Y=y$ is defined by

$$F(x|y) = P\{X \leq x | Y=y\} \quad \text{——— (2)}$$

and the conditional expectation of X given $Y=y$, as

$$E\{X|Y=y\} = \int x \, dF(x|y) = \sum x P\{X=x|Y=y\} \quad \text{— (3)}$$

If X & Y have a joint pdf $f(x,y)$ then conditional pdf of X for $Y=y$ is defined for all y s.t. $f_Y(y)>0$ by

$$f(x/y) = \frac{f(x,y)}{f_Y(y)} \quad \text{——— (4)}$$

and the conditional probability distribution fn. of X for $Y=y$ by

$$F(x|y) = P\{X \leq x|Y=y\} = \int_{-\infty}^{x} f(x|y)\,dx \quad \text{——— (5)}$$

The conditional expectation of X given $Y=y$ is defined by

$$E(X|Y=y) = \int_{-\infty}^{\infty} x f(x|y)\cdot dx.$$

If X & Y are r.v.

$$E(X) = \sum_y E(X|Y=y)P(Y=y)$$ in case X & Y are discrete. For continuous also similar result holds.

Proof: $\sum_y E(X|Y=y)P(Y=y) = \sum_y \sum_x x P\{X=x|Y=y\}P(Y=y)$

$$= \sum_y \sum_x x P\{X=x, Y=y\}$$

$$= \sum_x x \sum_y P\{X=x, Y=y\}$$

$$= \sum_x x P\{X=x\}$$

$$= E[X].$$

Prob. Two refills for a ballpoint pen are selected at random from a box containing 3-blue, 2-red, & 3-green refills. X is no. of blue refills & Y is no. of ~~green~~ red refills selected. find

(a) j.p.d. (b) $P[(X,Y)\in A]$ where A is the region $\{x+y\le 1\}$.

(d) $E(X|Y=y), y=1$ (c) Conditional distribution of X given that Y=1.

Sol $R_x = \{0,1,2\}$, $R_y = \{0,1,2\}$; possible pairs (x,y) (0,0), (0,1), (0,2), (1,0), (1,1), (2,0)

(a)

$x \backslash y$	0	1	2	$h(x)$
0	3/28	6/28	1/28	10/28
1	9/28	6/28	0	15/28
2	3/28	0	0	3/28
$g(y)$	15/28	12/28	1/28	

(b) $P[(0,0),(0,1),(1,0)] = f(0,0)+f(0,1)+f(1,0)$

$= \frac{3}{28}+\frac{6}{28}+\frac{9}{28} = \frac{18}{28}$

(c) Conditional distribution is given as

$$f(X|Y=y) = \frac{f(x,y)}{g(y)} \quad \text{or} \quad f(Y|X=x) = \frac{f(x,y)}{h(x)}$$

hence $g(1) = 12/28 = 3/7 = \sum_{x=0}^{2} f(x,1) = f(0,1)+f(1,1)+f(2,1) = \frac{6}{28}+\frac{6}{28}+0$

$f(x|1) = \frac{f(x,1)}{g(1)} = \frac{7}{3} f(x,1)$ ✓

therefore $f(0|1) = \frac{7}{3}\times\frac{6}{28} = \frac{1}{2}$

$f(1|1) = \frac{7}{3}\times\frac{6}{28} = \frac{1}{2}$

$f(2|1) = 7/3 \times 0 = 0$

(d) $E(X|Y=1) = \sum_{x=0}^{2} x.P(X|Y=1) = 0.P(0|1)+1\,P(1|1)+2P(2|1)$

$= 0 + 1.\frac{1}{2} + 2.0 = \frac{1}{2}$.