PARTIAL DIFFERENTIAL EQUATIONS

MC – 406

ASSIGNMENT – 3

ANISH SACHDEVA
DTU/2K16/MC/013

Q1) Obtain a Fourier Series solution for the wave equation

$u_{tt} - c^2 u_{xx} = 0$ such that

$$\begin{cases} u(0,t) = 0 \quad , u(L,t) = 0 \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) \end{cases}$$

We have

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{——} \quad (1)$$

Now, let

$$u_x(x,t) = T(t) X(x) \quad \text{——} \quad (2)$$

From the boundary conditions we have :-

$X(0) = 0$, $X(l) = 0$

From (2), $u_{tt} = T''(t) X(x)$ and $u_{xx} = T(t) X''(x)$ —(3)

From (1) and (3)

$$T''(t) X(x) = c^2 T(t) X''(x)$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda \text{ (say)}$$

$$T''(t) + \lambda c^2 T(t) = 0 \quad \text{——} \quad (4)$$

$$X''(t) + \cancel{} \lambda X(x) = 0 \quad \text{——} \quad (5)$$

$X(0) = X(l) = 0 \rightarrow$ We know

<u>Case I</u> :

$\lambda = -\mu^2 < 0$ where $\mu > 0$

Scanned with CamScanner

The solution for

$$X''(x) - \mu^2 X(x) = 0 \text{ will be}$$

$$X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$$

$$X(0) = 0 \Rightarrow C_1 + C_2 = 0$$

$$X(l) = 0$$

$$C_1 e^{l\mu} + C_2 e^{-\mu l} = 0$$

Hence, we do not have any non-trivial solution.

<u>Case II:</u>

$$\lambda = 0$$

Given solution to $X'' = 0$

$$X(x) = C_1 x + C_2$$

$$X(0) = 0$$

$$C_2 = 0$$

$$X(l) = 0$$

$$C_1 l = 0$$

$$C_1 = 0$$

This is a non trivial solution

<u>Case III:</u>

$$\lambda = \mu^2 > 0, \mu > 0$$

$$X'' + \mu^2 X = 0$$

Solving, we get

$$X(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x)$$

$$X(0) = 0$$

$$C_2 \cos(\mu \cdot 0) = \boxed{C_2 = 0}$$

$X(l) = 0$

$C_1 \sin(\mu l) = 0$

$\sin \mu l = 0$

$\mu_k l = k\pi$

$\mu_k = \frac{k\pi}{l} \quad \forall\, k \in \mathbb{N}$

The eigenvalues of eigen functions will be :-

$\lambda_k = \left(\frac{k\pi}{l}\right)^2, \; X_k(x) = \sin\left(\frac{k\pi x}{l}\right), \; k \in \mathbb{N}$

solving this equation for $T(t)$ with $\lambda = \lambda_k$ we get

$T_k(t) = A_k \cos\left(\frac{k\pi ct}{l}\right) + B_t \sin\left(\frac{k\pi ct}{l}\right)$

$\Rightarrow u_k(x,t) = \left[A_k \cos\left(\frac{k\pi ct}{l}\right) + B_k \sin\left(\frac{k\pi ct}{l}\right)\right] \sin\left(\frac{k\pi x}{l}\right)$

$k \in \mathbb{N}$

u_k can be written as a infinite series of u_k i.e.

$u(x,t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{k\pi ct}{l}\right) + B_k \sin\left(\frac{k\pi ct}{l}\right)\right] \sin\left(\frac{k\pi x}{l}\right)$

From initial conditions

$u(x,0) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{l}\right) = f(x)$

$u_t(x,0) = \sum_{k=1}^{\infty} \left(\frac{k\pi c}{l}\right) B_k \sin\left(\frac{k\pi x}{l}\right) = g(x)$

Using the formulae in the fourier coefficient for the sine series expansion of f and g in $[0,l]$

We get:-

$$A_k = \frac{2}{l}\int_0^l f(y)\sin\left(\frac{k\pi y}{l}\right)dy$$

$$B_k = \frac{2}{k\pi c}\int_0^l g(y)\sin\left(\frac{k\pi y}{l}\right)dy$$

Q2) Derive the formula for the D'Alembert solution of the wave equation on a hay line.

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \text{———} \quad (1)$$

$$y(x,0) = f(x) \qquad \frac{\partial y}{\partial t}(t,0) = g(x)$$

Let us use the variables

$$u = x + ct \quad \text{———} \quad (2)$$
$$v = x - ct \quad \text{———} \quad (3)$$

$$\frac{\partial u}{\partial t} = c, \quad \frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial t} = -c \quad \text{———} \quad (4)$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \quad \text{———} \quad (5)$$

Differentiating partially w.r.t x

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

Scanned with CamScanner

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} + 2\frac{\partial^2 y}{\partial u \partial v} \quad \text{—— (6)}$$

Also,

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

$$= c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \quad \text{——}$$

$$\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \quad \text{—— (7)}$$

Differentiating (7) partially once with t

$$\frac{\partial^2 y}{\partial t^2} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$c\left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} - 2\frac{\partial^2 y}{\partial u \partial v}\right) \quad \text{—— (8)}$$

From (7), (8) and (1) we have

$$\frac{\partial^2 y}{\partial u \partial v} = 0$$

Integrating w.r.t u

$$\frac{\partial y}{\partial v} = \bar{\phi}(u) \quad \text{—— (9)}$$

Integrating again w.r.t μ

$$y = \int \bar{\phi}(u)\, du + \psi(v)$$

$$y = \phi(u) + \psi(v)$$

or $\boxed{y = \phi(x+ct) + \psi(x-ct)}$

Using the boundary conditions, we have

$$y(x,0) = f(x) = \phi(x) + \psi(x) \quad \text{——} \quad (9)$$

$$\frac{\partial y}{\partial t} = y_t(x,0) = g(x) = c\phi'(x) - c\psi'(x) \quad \text{——} \quad (10)$$

Integrating (10) on both sides from x_0 to x

$$c\phi(x) - c\psi(x) = \int_{x_0}^{x} y(ct)\,dx + A \quad \text{——} \quad (11)$$

From (9) we have:

$$\phi(x) + \psi(x) = f(x) \quad \text{——} \quad (12)$$

Adding (11) and (12)

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(z)\,dz + \frac{A}{2} \quad \text{——} \quad (13)$$

For $\psi(x)$ substituting value from (13) and (9)

We get $\psi(x) = f(x) - \phi(x) = f(x) - \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(z)\,dz - \frac{A}{2}$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(z)\,dz - \frac{A}{2}$$

$$y(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(z)\,dz + \frac{A}{2} + \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_{x_0}^{x_0-ct} g(z)\,dz - \frac{A}{2}$$

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

Q3) Solve the following using Duhamel's principle:

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) \; ; \; x \in \mathbb{R} \;\; t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \quad \text{——— (1)}$$

$f(x) = 0, \; g(x) = 0$

Let $g(x,t) = h(x,t) + v(x,t)$

where u solves $\left.\begin{array}{l} u_{tt} = c^2 u_{xx} \quad x \in \mathbb{R}, \; t > 0 \\ u(x,0) = 0 \quad u_t(x,0) = 0 \end{array}\right]$ ——— (2)

and v solves

$$\left.\begin{array}{l} v_{tt} = c^2 v_{xx} + h(x,t) \\ v(x,0) = 0 \;, \; v_t(x,0) = 0 \end{array}\right] \text{——— (3)}$$

verification

$$\frac{\partial^2}{\partial t^2}(u+v) = u_{tt} + v_{tt} = c^2 u_{xx} + c^2 v_{xx} + h(x,t)$$

$$= c^2 \frac{\partial^2}{\partial x^2}(u+v) + h(x,t)$$

$$(u+v)(x,0) = u(x,0) + v(x,0) = f(x,0) + 0 = f(x,0)$$

$$= \frac{\partial}{\partial t}(u+v)(x,0) = u_t(x,0) + v_t(x,0)$$

$$= g(x) + 0$$

$$= 0$$

$(u+v)$ satisfies (1) if v solves (2) and v solves (3)

Since (2) is homogenous equation its solution ~~is~~ is:—

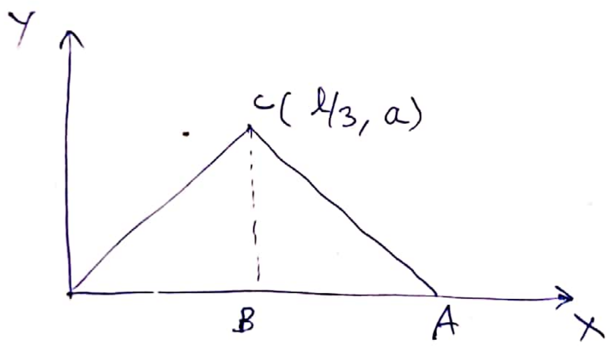
$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz = 0$$

$$v(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(z,s)\,dz\,ds$$

$$y(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(z,s)\,dz\,ds$$

Q4) A tightly stretched violin string of length l fixed at both ends is placed at $x = l/3$ and assumes the shape of a triangle of height a initially. Find the displacement of the string if it is released from rest.



Wave Equation can be represented by:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——— (1)}$$

Equation Line:

$$y - 0 = \frac{a-0}{l/3 - 0}(x-0)$$

$$y = \frac{3ax}{l} \quad \text{——— (2)}$$

Equation Line CA

$$y - a = -\frac{a}{2l/3}(x - l/3)$$

$$y = \frac{3a}{2}\left(1 - \frac{x}{l}\right) \quad \text{——} \quad (3)$$

Hence the boundary conditions are

$$\left.\begin{aligned} y(0,t) &= 0 \\ y(l,t) &= 0 \end{aligned}\right] \quad \text{——} \quad (4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{——} \quad (5)$$

$$y(x,0) = \begin{cases} \frac{3ax}{l} & 0 < x < l/3 \\ \frac{3a}{2}\left(1 - \frac{x}{l}\right) & \frac{l}{3} < x < l \end{cases} \quad \text{——} \quad (6)$$

Solution of (1) :-

$$y(x,t) = [c_1 \cos(cpt) + c_2 \sin(cpt)] / [c_3 \cos(px) + c_4 \sin(px)]$$

$$y(0,t) = 0$$

$$[c_1 \cos(cpt) + c_2 \sin(cpt)]\, c_3 = 0$$

$$\boxed{c_3 = 0}$$

$$y(x,t) = [c_1 \cos(cpt) + c_2 \sin(cpt)]\, c_4 \sin(px)$$

$$y(l,t) = 0 = [c_1 \cos(cpt) + c_2 \sin(cpt)]\, c_4 \sin(pl)$$

$$\sin(pl) = 0$$

$$pl = k\pi, \quad k \in N$$

$$p = \frac{k\pi}{l} \quad \forall k \in N$$

$$y(x,t) = \left[C_1 \cos\left(\frac{n\pi ct}{l}\right) + C_2 \sin\left(\frac{k\pi ct}{l}\right)\right] C_4 \sin\left(\frac{k\pi x}{l}\right)$$

$$\frac{\partial y}{\partial t} = \frac{k\pi c}{l}\left[-C_1 \sin\left(\frac{k\pi ct}{l}\right) + C_2 \cos\left(\frac{k\pi ct}{l}\right)\right] C_4 \sin\left(\frac{k\pi x}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{k\pi c}{l}\left[C_2 C_4 \sin\left(\frac{k\pi x}{l}\right)\right]$$

$$\Rightarrow C_2 = 0$$

$$y(x,t) = C_1 b_n \cos\left(\frac{k\pi ct}{l}\right) \sin\left(\frac{k\pi ct}{l}\right) \quad \text{where } b_n = C_1 C_4$$

General solution will be

$$y(x,t) = \sum_{1}^{\infty} b_n \cos\left(\frac{k\pi ct}{l}\right) \sin\left(\frac{k\pi ct}{l}\right) \quad \text{——} \quad (7)$$

$$y(x,0) = \sum_{k=1}^{\infty} b_n \sin\left(\frac{k\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l y(x,0) \sin\left(\frac{k\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin\left(\frac{k\pi x}{l}\right) dx + \int_{l/3}^{l} \frac{3a}{2}\left(1-\frac{x}{2}\right) \sin\left(\frac{k\pi x}{l}\right) dx\right]$$

$$\frac{2}{l}\left[\frac{3a}{l}\int_0^{l/3} \sin\left(\frac{k\pi x}{l}\right) dx + \frac{3a}{2}\int_{l/3}^{l}\left(1-\frac{x}{l}\right) \sin\left(\frac{k\pi x}{l}\right) dx\right]$$

$$\frac{6a}{l^2}\left[\left(x\left[-\frac{\cos\left(\frac{\pi x k}{l}\right)}{(k\pi/l)}\right]\right)^{l/3} - \int_0^{l/3} -\frac{\cos\left(\frac{k\pi x}{l}\right)}{k\pi/l} dx\right.$$

$$\left. + \frac{3a}{l}\left[\left(1-\frac{x}{l}\right)\left(-\frac{\cos\left(\frac{k\pi x}{l}\right)}{k\pi/l}\right)\right]_{l/3}^{l}\right.$$

Scanned with CamScanner

$$= \frac{6a}{l^2}\left[-\frac{l^2}{3k\pi}\cos\left(\frac{k\pi}{3}\right) + \frac{l^2}{k^2\pi^2}\sin\left(\frac{k\pi}{3}\right)\right] + \frac{3a}{l}\left[\frac{2l}{3\pi k}\cos\frac{k\pi}{3} - \frac{l}{k^2\pi^2}\left(-\sin\left(\frac{k\pi}{3}\right)\right)\right]$$

$$= \frac{9a}{k^2\pi^2}\sin\left(\frac{k\pi}{3}\right)$$

$$y(x,t) = \frac{9a}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\sin\left(\frac{k\pi}{3}\right)\cos\left(\frac{k\pi c t}{l}\right)\sin\left(\frac{k\pi x}{l}\right)\right]$$

Q5) Solve

$$\begin{cases} u_{tt} - 9u_{xx} = 2\sinh x & x \in \mathbb{R},\ t>0 \\ u(x,0) = x \\ u_t(x,0) = \sin x. \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} h(z,s)\,dz\,ds$$

i) $c^2 = 9,\ c = \pm 3$

Let $c = 3$

ii) $f(x) = x$

$f(x+ct) = x+3t, \quad f(x-ct) = x-3t$

iii) $g(z) = \sin z$

iv) $h(z,s) = 2\sinh(z)$

$$u(x,t) = \frac{1}{2}\left[x+3t+x-3t\right] + \frac{1}{2(3)}\int_{x-3t}^{x+3t} 2\sinh z\,dz + \frac{1}{2(3)}\int_0^t\int_{x-3(t-s)}^{x+3(t-s)} 2\sinh z\,dz\,ds$$

$$= x + \int_0^t \cosh(x+3(t-s)) - \cosh(x-3(t-s))\,ds$$

$$= x + \frac{1}{3} \sin(x) \sin(3t) + \frac{1}{3}\left[-\frac{1}{3} \sinh[x + 3(t-s)] - \frac{1}{3} \sinh(x - 3(t-s))\right]_0^t$$

$$= x + \frac{1}{3} \sin x \sin 3t - \frac{2}{9} \sinh(x) + \frac{2}{9} \sinh(x) \cosh(3t)$$