2

Linear Partial differential equations of order one

2.1. LAGRANGE'S EQUATION

A quasi–linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z. Such a partial differential equation is known as *Lagrange equation.*

For Example $xyp + yzq = zx$ is a Lagrange equation.

2.2. Lagrange's method of solving *Pp* + *Qq* = *R*, when *P*, *Q* and *R* are functions of *x*, *y*, *z*

(Delhi Maths (H) 2009; Meerut 2003; Poona 2003, 10; Lucknow 2010)

Theorem. *The general solution of Lagrange equation*

$$Pp + Qq = R, \qquad \text{... (1)}$$

is

$$\phi(u, v) = 0 \qquad \text{... (2)}$$

where ϕ *is an arbitrary function and*

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \qquad \text{... (3)}$$

are two independent solutions of

$$(dx)/P = (dy)/Q = (dz)/R \qquad \text{... (4)}$$

Here, c_1 *and* c_2 *are arbitrary constants and at least one of u, v must contain z. Also recall that u and v are said to be independent if u/v is not merely a constant.*

Proof. Differentiating (2) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \text{... (5)}$$

and

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \text{... (6)}$$

Eliminating $\partial\phi/\partial u$ and $\partial\phi/\partial v$ between (5) and (6), we have

$$\begin{vmatrix} \partial u/\partial x + p(\partial u/\partial z) & \partial v/\partial x + p(\partial v/\partial z) \\ \partial u/\partial y + q(\partial u/\partial z) & \partial v/\partial y + q(\partial v/\partial z) \end{vmatrix} = 0$$

or

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

or

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0$$

$$\therefore \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{... (7)}$$

Hence (2) is a solution of the equation (7)

Taking the differentials of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, we get

$$(\partial u/\partial x)dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz = 0 \qquad \text{... (8)}$$

and $$(\partial v/\partial x)dx + (\partial v/\partial y)dy + (\partial v/\partial z)dz = 0 \qquad ...(9)$$

Since u and v are independent functions, solving (8) and (9) for the ratios $dx : dy : dz$, gives

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \qquad ...(10)$$

Comparing (4) and (10), we obtain

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P} = \frac{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q} = \frac{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}{R} = k, \text{ say}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = kP, \qquad \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = kQ \qquad \text{and} \qquad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = kR$$

Substituting these values in (7), we get $k(Pp + Qq) = kR$ or $Pp + Qq = R$, which is the given equation (1).

Therefore, if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of the system of differential equations $(dx)/P = (dy)/Q = (dz)/R$, then $\phi(u, v) = 0$ is a solution of $Pp + Qq = R$, ϕ being an arbitrary function. This is what we wished to prove.

Note. Equations (4) are called *Lagrange's auxillary* (*or subsidiary*) equations for (1).

2.3. Working Rule for solving *Pp + Qq = R* by Lagrange's method.

[Delhi Maths Hons. 1998]

Step 1. Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R. \qquad ...(1)$$

Step 2. Write down Lagrange's auxiliary equations for (1) namely,

$$(dx)/P = (dy)/Q = (dz)/R \qquad ...(2)$$

Step 3. Solve (2) by using the well known methods (refer Art. 2.5, 2.7, 2.9 and 2.11). Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms :

$\phi(u, v) = 0$, $\quad u = \phi(v)$ $\quad$ or $\quad v = \phi(u)$, ϕ being an arbitrary function.

2.4. Examples based on working rule 2.3. In what follows we shall discuss four rules for getting two independent solutions of $(dx)/P = (dy)/Q = (dz)/R$. Accordingly, we have four types of problems based on $Pp + Qq = R$.

2.5. Type 1 based on Rule I for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of given equations (1).

2.6. SOLVED EXAMPLES BASED ON ART. 2.5

Ex. 1. *Solve* $(y^2z/x)p + xzq = y^2$. **[Indore 2004; Sagar 1994]**

Sol. Given $$(y^2z/x)p + xzq = y^2. \qquad ...(1)$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{(y^2z/x)} = \frac{dy}{xz} = \frac{dz}{y^2}. \qquad ..(2)$$

Taking the first two fractions of (2), we have

$$x^2z dx = y^2z dy \qquad \text{or} \qquad 3x^2dx - 3y^2dy = 0, \qquad ...(3)$$

Integrating (3), $x^3 - y^3 = c_1$, c_1 being an arbitrary constant ...(4)

Next, taking the first and the last fractions of (2), we get

$xy^2 dx = y^2 z dz$ or $2xdx - 2zdz = 0.$ ...(5)

Integrating (5), $x^2 - z^2 = c_2$, c_2 being an arbitrary constant ...(6)

From (4) and (6), the required general integral is

$$\phi(x^3 - y^3, x^2 - z^2) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* (*i*) $a(p + q) = z$. **[Bangalore 1997]** (*ii*) $2p + 3q = 1$. **[Bangalore 1995]**

Sol. (*i*) Given $ap + aq = z.$ ...(1)

The Lagrange's auxiliary equation for (1) are $(dx)/a = (dy)/a = (dz)/1.$ ...(2)

Taking the first two members of (1), $dx - dy = 0.$ ...(3)

Integrating (3), $x - y = c_1$, c_1 being an arbitrary constant ...(4)

Taking the last two members of (1), $dy - adz = 0.$ ...(5)

Integrating (5), $y - az = c_2$, c_2 being an arbitrary constant. ...(6)

From (4) and (6), the required solution is given by

$$\phi(x - y, y - az) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $p \tan x + q \tan y = \tan z$. **[Madras 2005 ; Kanpur 2007]**

Sol. Given $(\tan x)p + (\tan y)q = \tan z.$ ...(1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}.$ ...(2)

Taking the first two fractions of (2), $\cot x \, dx - \cot y \, dy = 0.$

Integrating, $\log \sin x - \log \sin y = \log c_1$ or $(\sin x)/(\sin y) = c_1.$ ...(3)

Taking the last two fractions of (2), $\cot y \, dy - \cot z \, dz = 0.$

Integrating, $\log \sin y - \log \sin z = \log c_2$ or $(\sin y)/(\sin z) = c_2.$ ...(4)

From (3) and (4), the required general solution is

$$\sin x/\sin y = \phi(\sin y/\sin z), \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $zp = -x$.

Sol. Given $zp + 0.q = -x.$ ...(1)

The Lagrange's subsidiary equations for (1) are $(dx)/z = (dy)/0 = (dz)/(-x)$ ...(2)

Taking the first and the last members of (2), we get

$-xdx = zdz$ or $2xdx + 2zdz = 0.$ ...(3)

Integrating (3), $x^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Next, the second fraction of (2) implies that $dy = 0$ giving $y = c_2$ ...(3)

From (4) and (5), the required solution is $x^2 + z^2 = \phi(y)$, ϕ being an arbitrary function.

Ex. 5. *Solve* $y^2 p - xyq = x(z - 2y)$ **[Delhi Maths Hons. 1995, Delhi Meths(G) 2006]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{y^2} = \dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)}.$ ...(1)

Taking the first two fractions of (1) and re–writing, we get

$2xdx + 2ydy = 0$ so that $x^2 + y^2 = c_1.$ ...(2)

Now, taking the last two fractions of (1) and re–writing, we get

$\dfrac{dz}{dy} = -\dfrac{z-2y}{y}$ or $\dfrac{dz}{dy} + \dfrac{1}{y} z = 2$ ...(3)

which is linear in z and y. Its I.F. $= e^{\int (1/y) dy} = e^{\log y} = y$. Hence solution of (3) is

$z \cdot y = \int 2y dy + c_2$ or $zy - y^2 = c_2$. ...(4)

Hence $\phi(x^2 + y^2, zy - y^2) = 0$ is the desired solution, where ϕ is an arbitrary function.

Ex. 6. *Solve* $(x^2 + 2y^2)p - xyq = xz$ **[K.U. Kurukshetra 2005]**

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{x^2 + 2y^2} = \frac{dy}{-xy} = \frac{dz}{xz} \quad \text{... (1)}$$

Taking the last two fractions of (2) and re–writing, we get

$(1/y)\, dy + (1/z) dz = 0$ so that $\log y + \log z = \log c_1$ or $yz = c_1$ ... (2)

Taking the first two fractions of (1), we have

$$\frac{dx}{dy} = \frac{x^2 + 2y^2}{-xy} \quad \text{or} \quad 2x\frac{dx}{dy} + \left(\frac{2}{y^2}\right)x^2 = -4y \quad \text{... (3)}$$

Putting $x^2 = v$ and $2x(dx/dy) = dv/dx$, (3) yields

$dv/dx + (2/y)\, v = -4y$, which is a linear equation.

Its integrating factor $= e^{\int (2/y) dy} = e^{2 \log y} = y^2$ and hence its solution is

$$yv^2 = \int \left\{(-4y)xy^2\right\} dy + c_2 \quad \text{or} \quad y^2x^2 + y^4 = c_2 \quad \text{... (4)}$$

From (2) and (4), the required solution is $\phi(yz, y^2x^2 + y^4) = 0$, ϕ being an arbitrary function.

EXERCISE 2 (A)

Solve the following partial differential equations

1. $(-a + x)p + (-b + y)q = (-c + z)$. **Ans.** $\phi\{(x - a)/(y - b), (y - b)/(z - c)\} = 0$
2. $xp + yq = z$ **(Kanpur 2011)** **Ans.** $\phi(x/z, y/z) = 0$
3. $p + q = 1$ **Ans.** $\phi(x - y, x - z) = 0$
4. $x^2p + y^2p = z^2$ **[Bilaspur 2001, Jabalpur 2000, Sagar 2000, Vikram 1999]** **Ans.** $\phi(1/x - 1/y, 1/y - 1/z) = 0$
5. $x^2p + y^2q + z^2 = 0$ **Ans.** $\phi(1/x - 1/y, 1/y + 1/z) = 0$
6. $\partial z/\partial x + \partial z/\partial y = \sin x$ **[Meerut 1995]** **Ans.** $\phi(x - y, z + \cos x) = 0$
7. $yzp + 2xq = xy$ **[Nagpur 1996]** **Ans.** $\phi(x^2 - z^2, y^2 - 4z) = 0$
8. $xp + yq = z$ **[Bangalore 1995]** **Ans.** $\phi(x/y, x/z) = 0$
9. $yzp + zxq = xy$ **[M.S. Univ. T.N. 2007, Lucknow 2010, Revishankar 2004]** **Ans.** $\phi(x^2 - y^2, x^2 - z^2) = 0$
10. $zp = x$ **Ans.** $\phi(y, x^2 - z^2) = 0$
11. $y^2p^2 + x^2q^2 = x^2y^2z^2$ **Ans.** $\phi(x^3 - y^3, y^3 + 3z^{-1}) = 0$

2.7. Type 2 based on Rule II for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one integral of (1) is known by using rule I explained in Art 2.5 and suppose also that another integral cannot be obtained by using rule I of Art. 2.5. Then one integral known to us is used to find another integral as shown in the following solved examples. Note that in the second integral, the constant of integration of first integral should be removed later on.

2.8. SOLVED EXAMPLES BASED ON ART. 2.7

Ex. 1. *Solve* $p + 3q = 5z + \tan(y - 3x)$.

[Agra 2006; Meerut 2003; Indore 2002; Ravishankar 2003]

Sol. Given $$p + 3q = 5z + \tan(y - 3x). \qquad ...(1)$$

The Lagrange's subsidary equations for (1) are $$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}. \qquad ...(2)$$

Taking the first two fractions, $$dy - 3dx = 0. \qquad ...(3)$$

Integrating (3), $y - 3x = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), from (2) we get $$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}. \qquad ...(5)$$

Integrating (5), $x - (1/5) \times \log(5z + \tan c_1) = (1/5) \times c_2$, c_2 being an arbitrary constant.

or $$5x - \log[5z + \tan(y - 3x)] = c_2, \text{ using (4)} \qquad ...(6)$$

From (4) and (6), the required general integral is

$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x)$, where ϕ is an arbitrary function.

Ex. 2. *Solve* $z(z^2 + xy)(px - qy) = x^4$.

Sol. Given $$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4. \qquad ...(1)$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}. \qquad ...(2)$$

Cancelling $z(z^2 + xy)$, the first two fractions give

$$(1/x)\,dx = -(1/y)\,dy \quad \text{or} \quad (1/x)\,dx + (1/y)\,dy = 0. \qquad ...(3)$$

Integrating (3), $\log x + \log y = \log c_1$ or $xy = c_1$. ...(4)

Using (4), from (2) we get $$\frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4}$$

or $$x^3 dx = z(z^2 + c_1)dz \quad \text{or} \quad x^3 dx - (z^3 + c_1 z)dz = 0. \qquad ...(5)$$

Integrating (5), $x^4/4 - z^4/4 - (c_1 z^2)/2 = c_2/4$ or $x^4 - z^4 - 2c_1 z^2 = c_2$

or $$x^4 - z^4 - 2xy\,z^2 = c_2, \text{ using (4)} \qquad ...(6)$$

From (4) and (6), the required general integral is

$\phi(xy, x^4 - z^4 - 2xy\,z^2) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $xyp + y^2 q = zxy - 2x^2$. **[Garhwal 2005]**

Sol. Given $$xyp + y^2 q = zxy - 2x^2. \qquad ...(1)$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}. \qquad ...(2)$$

Taking the first two fractions of (2), we have

$$(dx)/xy = (dy)/y^2 \quad \text{or} \quad (1/x)dx - (1/y)dy = 0 \qquad ...(3)$$

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1 y$. Hence from second and third fractions of (2), we get

$$\frac{dy}{y^2} = \frac{dz}{c_1 zy^2 - 2c_1^2 y^2} \quad \text{or} \quad c_1 dy - \frac{dz}{z - 2c_1^2} = 0. \qquad ...(5)$$

Integrating (5), $c_1 y - \log(z - 2c_1^2) = c_2$ or $x - \log[z - 2(x^2/y^2)] = c_2$, using (4). ...(6)

From (4) and (6), the required general solution is

$x - \log[z - 2(x^2/y^2)] = \phi(x/y)$, ϕ being an arbitrary function.

Ex. 4. *Solve* $xzp + yzq = xy$. **[Bhopal 1996; Jabalpur 1999; Jiwaji 2000; Punjab 2005; Agra 2007; Ravishanker 1996; Vikram 2000]**

Sol. Given $$xzp + yzq = xy. \qquad ...(1)$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}. \qquad ...(2)$$

Taking the first two fractions of (2), $(1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1 y$. Hence, from second and third fractions of (2), we get

$(1/yz)dy = (1/c_1 y^2)dz$ or $2c_1 y\, dy - 2z\, dz = 0$. ...(5)

Integrating (5), $c_1 y^2 - z^2 = c_2$ or $xy - z^2 = c_2$, using (4). ...(6)

From (4) and (6), the required solution is $\phi(xy - z^2, x/y) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $py + qx = xyz^2 (x^2 - y^2)$.

Sol. Given $$py + qx = xyz^2 (x^2 - y^2). \qquad ...(1)$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}. \qquad ...(2)$$

Taking the first two fractions of (2), $2xdx - 2ydy = 0$. ...(3)

Integrating. $x^2 - y^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), the last two fractions of (2) give

$(dy)/x = (dz)/(xyz^2 c_1)$ or $2c_1 y\, dy - 2z^{-2} dz = 0$. ...(5)

Integrating (5), $c_1 y^2 + (2/z) = c_2$, c_2 being an arbitrary constant.

or $$y^2 (x^2 - y^2) + (2/z) = c_2, \text{ using (4).} \qquad ...(6)$$

From (4) and (6), the required general solution is

$$y^2 (x^2 - y^2) + (2/z) = \phi(x^2 - y^2), \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 6. *Solve* $xp - yq = xy$ **[Madras 2005]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$(dx)/x = (dy)/(-y) = (dz)/(xy) \qquad ... (1)$$

Taking the first two fractions of (1), $(1/x)dx + (1/y)dy = 0$

Intergrating, $\log x + \log y = c_1$ so that $xy = c_1$ ... (2)

Using (2), (1) yields $(1/x)dx = (1/c_1)\, dz$ so that $\log x - \log c_2 = z/c_1$

or $\log (x/c_2) = z/c_1$ or $\log (x/c_2) = z/(xy)$, by (2)

Thus, $x/c_2 = e^{z/(xy)}$ or $xe^{-z/(xy)} = c_2$, c_2 being an arbitrary constant. ... (3)

From (2) and (3), the required solution is $x\, e^{-z/(xy)} = \phi(xy)$, ϕ being an arbitrary function

Ex. 7. *Solve* $p + 3q = z + \cot (y - 3x)$. **[M.D.U Rohtak 2006]**

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{z + \cot (y - 3x)} \qquad ... (1)$$

Taking the first two fractions of (1), $dy - 3\, dx = 0$ so that $y - 3x = c_1$ ... (2)

Taking the first and last fraction of (1), we have

$$dx = \frac{dz}{z + \cot (y - 3x)} \quad \text{or} \quad dx = \frac{dz}{z + \cot c_1}, \text{ using (2)}$$

Intergrating, $x = \log | z + \cot c_1| + c_2$, c_1 and c_2 being an arbitrary constants.

or $$x - \log|z + \cot(y - 3x)| = c_2, \text{ using (2)} \qquad \text{... (3)}$$

From (2) and (3), the required general solution is

$$x - \log|z + \cot(y - 3x)| = \phi(y - 3x), \quad \phi \text{ being an arbitrary function.}$$

Ex. 8. *Solve* $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$ **[Delhi B.Sc. II 2008; Delhi B.A. II 2010]**

Sol. Re–writing the given equation, we have

$$x(z - 2y^2)\,p + y\,(z - y^2 - 2x^3)\,q = z(z - y^2 - 2x^3) \qquad \text{... (1)}$$

The Lagrange's subsidiary equations for (1) are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \qquad \text{... (2)}$$

Taking the last two fraction, we get $(1/z)dz = (1/y)dy$

Integrating, $\log z = \log y + \log a$ or $z/y = a$ ... (3)

where a is an arbitrary constant. Using (3), (2) yields

$$\frac{dx}{x(ay-2y^2)} = \frac{dy}{y(ay-y^2-2x^3)} \quad \text{so that} \quad (ay - y^2 - 2x^3)\,dx + x\,(2y - a)\,dy = 0 \qquad \text{... (4)}$$

Comparing (4) with $Mdx + Ndy = 0$, here $M = ay - y^2 - 2x^3$ and $N = x\,(2y - a)$. Then $\partial M/\partial y = a - 2y$ and $\partial N/\partial x = 2y - a$. Now, we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x(2y-a)} \times 2(a-2y) = -\frac{2}{x}, \text{ which is a function of } x \text{ alone.}$$

Hence, by usual rule, integrating factor of (1) $= e^{\int(-2/x)dx} = e^{-2\log x} = e^{x^{-2}} = x^{-2}$

Multiplying (4) by x^{-2}, we get exact equation $(ayx^{-2} - y^2x^{-2} - 2x)dx + x^{-1}(2y - a)\,dy = 0$

By the usual rule of solving an exact equation, its solution is

$$\int\left\{(ay-y^2)x^{-2} - 2x\right\}dx + \int x^{-1}(2y-a)dy = b$$

(Treating y as constant) (Integrating terms free from x)

or $(ay - y^2) \times (-1/x) - x^2 = b$ or $(y^2 - ax)/x - x^2 = b$

or $(y^2 - ax - x^3)/x = b$, where b is an arbitrary constant. ... (5)

From (3) and (5), required solution is $(y^2 - ax - x^3)/x = \phi(z/y)$, ϕ being an arbitrary function

EXERCISE 2 (B)

Solved the following differential equations:

1. $p - 2q = 3x^2 \sin(y + 2x)$. **Ans.** $x^2 \sin(y+2x) - z = \phi(y+2x)$
2. $p - q = z/(x + y)$. **Ans.** $x - (x+y)\log z = \phi(x+y)$
3. $xy^2p - y^3q + axz = 0$. **Ans.** $\log z + (ax/3y^2) = \phi(xy)$
4. $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. **Ans.** $(x^2 + y^2 + z^2)/z = \phi(y/z)$
5. (*a*) $z(p - q) = z^2 + (x + y)^2$. **(Meerut 2011)** **Ans.** $e^{2y}[z^2 + (x+y)^2] = \phi(x+y)$

   (*b*) $z(p+q) = z^2 + (x-y)^2$ **Ans.** $e^{2y}[z^2 + (x-y)^2] = \phi(x-y)$
6. $p - 2q = 3x^2 \sin(y + 2x)$. **Ans.** $x^3 \sin(y+2x) - z = \phi(y+2x)$
7. $p - q = z/(x + y)$. **Ans.** $x - (x+y)\log z = \phi(x+y)$

8. $zp - zq = x + y$. **Ans.** $2x(x+y) - z^2 = \phi(x+y)$

9. $xyp + y^2q + 2x^2 - xyz = 0$. **Ans.** $x - \log|z - (2x/y)| = \phi(x/y)$

2.9. Type 3 based on Rule III for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction in (1) will be equal to $(P_1dx + Q_1dy + R_1dz)/(P_1P + Q_1Q + R_1R)$. ...(2)

If $P_1P + Q_1Q + R_1R = 0$, then we know that the numerator of (2) is also zero. This gives $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible by use of multipliers. In such cases second integral should be obtained by using rule I of Art. 2.5 or rule II of Art. 2.7 as the case may be.

2.10. SOLVED EXAMPLES BASED ON ART. 2.9

Ex.1. *Solve* $\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$.

Sol. Given $\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$. ...(1)

The Lagrange's subsidiary equations of (1) are $\dfrac{a\,dx}{(b-c)yz} = \dfrac{b\,dy}{(c-a)zx} = \dfrac{c\,dz}{(a-b)xy}$. ...(2)

Choosing x, y, z as multipliers, each fraction for (2)

$$= \frac{a\,xdx + by\,dy + cz\,dz}{xyz\,[(b-c)+(c-a)+(a-b)]} = \frac{ax\,dx + by\,dy + cz\,dz}{0}.$$

$\therefore$ $ax\,dx + by\,dy + cz\,dz = 0$ or $2axdx + 2bydy + 2czdz = 0$.

Integrating, $ax^2 + by^2 + cz^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing ax, by, cz as multipliers, each fraction of (2)

$$= \frac{a^2xdx + b^2ydy + c^2\,zdz}{xyz\,[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2xdx + b^2ydy + c^2\,zdz}{0}.$$

$\therefore a^2xdx + b^2ydy + c^2zdz = 0$ or $2a^2xdx + 2b^2ydy + 2c^2zdz = 0$.

Integrating, $a^2x^2 + b^2y^2 + c^2z^2 = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$$\phi\,(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2) = 0,$$ where ϕ is an arbitrary function.

Ex. 2. *Solve* $z(x+y)p + z(x-y)q = x^2 + y^2$.

Sol. Given $z(x+y)p + z(x-y)q = x^2 + y^2$. ...(1)

The Langrange's subsidiary equations for (1) are $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$. ...(2)

Choosing x, $-y$, $-z$, as multipliers, each fraction

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}.$$

$\therefore$ $x\,dx - y\,dy - z\,dz$ or $2x\,dx - 2y\,dy - 2z\,dz = 0$.

Integrating, $x^2 - y^2 - z^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing y, x, $-z$ as multipliers, each fraction

$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}.$$

$\therefore$ $y\,dx + x\,dy - z\,dz = 0$ or $2d(xy) - 2zdz = 0$.

Integrating, $2xy - z^2 = c_2$, c2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$$\phi\,(x^2 - y^2 - z^2,\ 2xy - z^2) = 0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $(mz - ny)p + (nx - lz)q = ly - mx$. **[Patna 2003; Madras 2005; Delhi Maths Hons. 1991; Bhopal 2004; Meerut 2008, 10; Sagar 2002; I.A.S. 1977; Kanpur 2005, 06]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \qquad ...(1)$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0$

Integrating, $x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Again, choosing l, m, n as multipliers, each fraction of (1)

$$= \frac{ldx + mdy + n\,dz}{l(mx - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + n\,dz}{0}.$$

$\therefore\ ldx + mdy + n\,dz = 0$ so that $l\,x + m\,y + n\,z = c_2$. ...(3)

From (2) and (3), the required general solution is given by

$$\phi\,(x^2 + y^2 + z^2,\ l\,x + m\,y + n\,z) = 0,\ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $x(y^2 - z^2)q - y(z^2 + x^2)q = z(x^2 + y^2)$.

Sol. The lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}. \qquad ...(1)$$

Choosing x, y, z, as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow\ xdx + ydy + zdz = 0$ so that $x^2 + y^2 + z^2 = c_1$. ...(2)

Choosing $1/x, -1/y, -1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx - (1/y)dy - (1/z)dz}{y^2 - z^2 + z^2 + x^2 - (x^2 + y^2)} = \frac{(1/x)dx - (1/y)dy - (1/z)dz}{0}$$

$\Rightarrow\ (1/x)dx - (1/y)dy - (1/z)dz = 0$ so that $\log x - \log y - \log z = \log c_2$

$\Rightarrow\ \log\{x/(yz)\} = \log c_2 \quad \Rightarrow \quad x/yz = c_2$. ...(3)

$\therefore$ The required solution is $\phi\,(x^2 + y^2 + z^2,\ x/yz) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $(y - zx)p + (x + yz)q = x^2 + y^2$.

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y - zx} = \frac{dy}{x + yz} = \frac{dz}{x^2 + y^2}. \qquad ...(1)$$

Choosing $x, -y, z$ as multipliers, each fraction of (1)

$$= \frac{xdx - ydy + zdz}{x(y-zx) - y(x+yz) + z(x^2+y^2)} = \frac{xdx - ydy + zdz}{0}$$

$\Rightarrow$ $2xdx - 2ydy + 2zdz = 0$ so that $x^2 - y^2 + z^2 = c_1$. ...(2)

Choosing $y, x, -1$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - dz}{y(y-zx) + x(x+yz) - (x^2+y^2)} = \frac{d(xy) - dz}{0}$$

$\Rightarrow$ $d(xy) - dz = 0$ so that $xy - z = c_2$. ...(3)

$\therefore$ From (2) and (3) solution is $\phi(x^2 - y^2 + z^2, xy - z) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve* $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$. **[I.A.S. 2004; Agra 2005 ; Delhi Maths (H) 2006; M.S. Univ. T.N. 2007; Indore 2003; Meerut 2009; Purvanchal 2007]**

Sol. Here Lagrange's subsidiary equations for given equation are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}. \qquad ...(1)$$

Choosing $1/x, 1/y, 1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2 + z - (x^2+z) + x^2 - y^2} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\Rightarrow$ $\log(xyz) = \log c_1$ $\Rightarrow$ $xyz = c_1$. ...(2)

Choosing $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{xdx + ydy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{xdx + ydy - dz}{0}$$

$\Rightarrow$ $x\,dx + y\,dy - z\,dz = 0$ so that $x^2 + y^2 - 2z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi(x^2 + y^2 - 2z, xyz) = 0$, ϕ is being an arbitrary function.

Ex. 7. *Solve* $(x + 2z)q + (4zx - y)q = 2x^2 + y$. **[Meerut 2005]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{x+2z} = \dfrac{dy}{4zx-y} = \dfrac{dz}{2x^2+y}$. ...(1)

Choosing $y, x, -2z$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - 2zdz}{y(x+2z) + x(4zx-y) - 2z(2x^2+y)} = \frac{d(xy) - 2zdz}{0}$$

$\Rightarrow$ $d(xy) - 2zdz = 0$ so that $xy - z^2 = c_1$. ...(2)

Choosing $2x, -1, -1$ as multipliers, each fraction of (1)

$$= \frac{2xdx - dy - dz}{2x(x+2z) - (4zx-y) - (2x^2+y)} = \frac{2xdx - dy - dz}{0}$$

$\Rightarrow$ $2xdx - dy - dz = 0$ so that $x^2 - y - z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi(xy - z^2, x^2 - y - z) = 0$, ϕ being an arbitrary function.

Ex. 8. *Solve* $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. **[Ranchi 2010; Meerut 1994]**

If the solution of the above equation represents a sphere, what will be the coordinates of its centre.

Sol. Here Lagrange's auxiliary equations for given equation are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}. \qquad ...(1)$$

Taking the last two fractions of (1), we have

$(y - z)dy = (y + z)dz$ or $2ydy - 2zdz - 2(zdy + ydz) = 0.$

Integrating, $y^2 - z^2 - 2yz = c_1$, c_1 being an arbitrary constant. ...(2)

Choosing x, y, z as multipliers, each fractrion of (1)

$$= \frac{xdx + ydy + zdz}{x(z^2 - 2yz - y^2) + xy(y+z) + xz(y-z)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow$ $2xdx + 2ydy + 2zdz = 0$ so that $x^2 + y^2 + z^2 = c_2$. ...(3)

From (2) and (3), solution is $\phi\,(y^2 - z^2 - 2yz,\, x^2 + y^2 + z^2) = 0$, ϕ being an arbitrary function.

From the solution of the given equation, it follows that if it represents a sphere, then its centre must be at (0,0,0), *i.e.*, origin.

Ex. 9. *Solve* $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^2 - y^3)$. **[Jabalpur 2004; M.S. Univ. T.N. 2007]**

Sol. Here Lagrange's auxiliary equations for the given equation are given by

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}. \qquad ...(1)$$

Taking first two fractions of (1), we have $(2y^4 - x^3y)dx = (y^3x - 2x^4)dy$

Dividing both sides by x^3y^3 gives $\left(\frac{2y}{x^3} - \frac{1}{y^2}\right)dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right)dy$

or $\left(\frac{1}{x^2}dy - \frac{2y}{x^3}dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0$ or $d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0.$

Integrating, $(y/x^2) + (x/y^2) = c_1$, c_1 being an arbitrary constant. ...(2)

Choosing $1/x$, $1/y$, $1/3z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)} = \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{0}$$

$\Rightarrow$ $(1/x)dx + (1/y)dy + (1/3z)dz = 0$ so that $\log x + \log y + (1/3) \times \log z = \log c_2$

$\Rightarrow$ $\log (xy\, z^{1/3}) = \log c_2$ $\Rightarrow$ $xyz^{1/3} = c_2$. ...(3)

From (2) and (3) solution is $\phi\,(xyz^{1/3},\, y/x^2 + x/y^2) = 0$, ϕ being an arbitrary function.

Ex. 10. *Solve* $x^2p + y^2q = nxy$. **[Ravishankar 1998; Bhopal 1998; Jabalpur 2002]**

Sol. Here Lagrange's auxiliary equations are $(dx)/x^2 = (dy)/y^2 = (dz)/nxy$ ... (1)

Taking the first two fractions of (1), we get $x^{-2}dx - y^{-2}dy = 0.$

Integrating, $-1/x + 1/y = -c_1$ so that $(y - x)/xy = c_1$. ...(2)

Choosing $1/x$, $-1/y$, c_1/n as multipliers, each fraction of (2)

$$= \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + c_1xy} = \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + y - x}, \text{ by (2)}$$

$$= \frac{(1/x)dx + (1/y)dy + (c_1/n)dz}{0} \quad \text{so that} \quad \frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz = 0.$$

Integrating, $\log x - \log y + (c_1/n)\, z = (c_1/n)c_2$, c_2 being an arbitrary constant.

or $z - (n/c_1)\,(\log y - \log x) = c_2$ or $z - (n/c_1)\log (y/x) = c_2$

or $$z - \frac{nxy}{y-x}\log\frac{y}{x} = c_2, \text{ using (2).} \qquad ...(3)$$

From (2) and (3), the required general solution is

$$\phi\left(\frac{y-x}{xy}, z-\frac{nxy}{y-x}\log\frac{y}{x}\right)=0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 11. *Solve* $(x-y)p+(x+y)q=2xz$.

Sol. Here the Lagrange's subsidiary equations are $\quad \dfrac{dx}{x-y}=\dfrac{dy}{x+y}=\dfrac{dz}{2xz}$. ...(1)

Taking the first two fractions of (1), $\quad \dfrac{dy}{dx}=\dfrac{x+y}{x-y}=\dfrac{1+(y/x)}{1-(y/x)}$. ...(2)

Let $y/x=v$ *i.e.*, $y=xv$. ...(3)

From (3), $(dy/dx)=v+x(dv/dx)$. ...(4)

Using (3) and (4), (2) gives $\quad v+x\dfrac{dv}{dx}=\dfrac{1+v}{1-v}$

or $$x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v-v(1-v)}{1-v}=\frac{1+v^2}{1-v}$$

or $\dfrac{1-v}{1+v^2}\,dv=\dfrac{dx}{x}$ or $\left(\dfrac{2}{1+v^2}-\dfrac{2v}{1+v^2}\right)dv=\dfrac{2dx}{x}$

Integrating, $2\tan^{-1}v-\log(1+v^2)=2\log x-\log c_1$

or $\log x^2-\log(1+v^2)-\log c_1=2\tan^{-1}v$

or $\log\{x^2(1+v^2)/c_1\}=2\tan^{-1}v$ or $x^2(1+v^2)=c_1e^{2\tan^{-1}v}$

or $x^2[1+(y^2/x^2)]=c_1e^{2\tan^{-1}(y/x)}$, as $v=y/x$ by (3)

or $(x^2+y^2)\,e^{-2\tan^{-1}(y/x)}=c_1$, c_1 being an arbitrary constant. ...(5)

Choosing $1, 1, -1/z$ as multipliers, each fraction of (1)

$$=\frac{dx+dy-(1/z)dz}{(x-y)+(x+y)-(1/z)\times(2xz)}=\frac{dx+dy-(1/z)dz}{0}$$

$\Rightarrow dx+dy-(1/z)dz=0$ so that $x+y-\log z=c_2$. ...(6)

From (5) and (6), the required general solution is

$$\phi\,(x+y-\log z,\ (x^2+y^2)\,e^{-2\tan^{-1}(y/x)})=0, \quad \text{where } \phi \text{ is an arbitrary function.}$$

Ex. 12. *Solve* $y^2p+x^2q=x^2y^2z^2$.

Sol. Here Lagrange's auxiliary equations are $\quad (dx)/y^2=(dy)/x^2=(dz)/x^2y^2z^2$. ...(1)

Taking the first two fractions of (1), we have

$3x^2dx-3y^2dy=0$ so that $x^3-y^3=c_1$. ...(2)

Choosing $x^2, y^2, -2/z^2$ as multipliers, each fraction of (1) $=\{x^2dx+y^2dy-(2/z^2)dz\}/0$

so that $3x^2dx+3y^2dy-(6/z^2)dz=0$.

Integrating, $x^3+y^3+(6/z)=c_2$, c_2 being an arbitrary constant. ...(3)

From (2) and (3), the required general solution is

$$\phi\,[x^3-y^3,\ x^3+y^3+(6/z)]=0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 13. *Solve* $(3x+y-z)p+(x+y-z)q=2(z-y)$. **[Bangalore 1992]**

Sol. Here Lagrange's auxiliary equations are $\quad \dfrac{dx}{3x+y-z}=\dfrac{dy}{x+y-z}=\dfrac{dz}{2(z-y)}$ ...(1)

Choosing 1, –3, 1 as multipliers, each ratio of (1) $= \{dx - 3dy - dz\}/0$

so that $$dx - 3dy - dz = 0.$$

Integrating, $x - 3y - z = c_1$, c_1 being an arbitrary constant. ...(2)

From (2), $$z = c_1 - x + 3y. \qquad ...(3)$$

Substituting the above value of z, the first two fractions of (2) reduce to

$$\frac{dx}{3x+y-(c_1-x+3y)} = \frac{dy}{x+y-(c_1-x+3y)} \quad \text{or} \quad \frac{dx}{2x+4y+c_1} = \frac{dy}{4y+c_1}. \qquad ...(3)$$

Let $u = 4y + c_1$ so that $dy = (1/4) \times du$. ...(4)

Then, (3) $\Rightarrow$ $\dfrac{dx}{2x+u} = \dfrac{(1/4)du}{u}$ or $\dfrac{dx}{du} = \dfrac{1}{4}\dfrac{2x+u}{u}$ or $\dfrac{dx}{du} - \dfrac{1}{2u}x = \dfrac{1}{4}$, which is linear. ...(5)

Integrating factor of (5) $= e^{-\int (1/2u)du} = e^{-(1/2)\log u} = e^{\log (u)^{-1/2}} = u^{-1/2} = 1/\sqrt{u}$.

Hence solution of (5) is $$x \times \frac{1}{\sqrt{u}} = \int \frac{1}{4}\frac{1}{\sqrt{u}}\,du + c = \frac{1}{2}\sqrt{u} + c_2$$

or $$\frac{2x-u}{\sqrt{u}} = c_2 \quad \text{or} \quad \frac{2x-(4y+c_1)}{\sqrt{4y+c_1}} = c_2, \text{ by (4)}$$

or $$\frac{2x-4y-(x-3y-z)}{\sqrt{4y+x-3y-z}} = c_2, \text{ using (2)} \quad \text{or} \quad \frac{x-y+z}{\sqrt{x+y-z}} = c_2 \qquad ...(6)$$

From (2) and (6), the required general solution is

$$\phi\left(x-3y-z, (x-y+z)/\sqrt{x+y-z}\right) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 14. *Solve* $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$. **[Delhi Maths Hons 95, 2000]**

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(x^2+3y^2)} = \frac{dy}{-y(3x^2+y^2)} = \frac{dz}{2z(y^2-x^2)}. \qquad ...(1)$$

Choosing $1/x$, $1/y$, $-1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0} \quad \text{so that} \quad \frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0.$$

Integrating, $\log x + \log y - \log z = \log c_1$ so that $(xy)/z = c_1$. ...(2)

Taking the first two ratios of (1), $$\frac{dy}{dx} = -\frac{y(3x^2+y^2)}{x(x^2+3y^2)} = -\left(\frac{y}{x}\right)\frac{3+(y/x)^2}{1+3(y/x)^2}. \qquad ...(3)$$

Put $y/x = v$ or $y = xv$ so that $(dy/dx) = v + x(dv/dx)$. ...(4)

Using (4), (3) reduces to $$v + x\frac{dv}{dx} = -v\,\frac{3+v^2}{1+3v^2} \quad \text{or} \quad x\frac{dv}{dx} = -v\left[\frac{3+v^2}{1+3v^2}+1\right]$$

or $$x\frac{dv}{dx} = -\frac{4(1+v^2)v}{1+3v^2} \quad \text{or} \quad 4\frac{dx}{x} + \frac{1+3v^2}{v(1+v^2)}dv = 0$$

or $$4\,\frac{dx}{x} + \left(\frac{1}{v} + \frac{2v}{1+v^2}\right)dv, \text{ on resolving into partial fractions}$$

Integrating, $4\log x + \log v + \log(1+v^2)$ or $x^4 v(1+v^2) = c_2'$

$$x^4(y/x)[1+(y/x)^2]=c_2' \quad \text{or} \quad xy(x^2+y^2)=c_2' \quad \text{or} \quad c_1 z(x^2+y^2)=c_2', \text{ by (2)}$$

or $\quad z(x^2+y^2)=c_2'/c_1 \quad$ or $\quad z(x^2+y^2)=c_2$, where $c_2=c_2'/c_1$. ... (5)

$\therefore$ From (2) and (5) solution is $\phi(z(x^2+y^2), xy/z)=0$, ϕ being an arbitrary function.

Ex. 15. *Solve* $(y-z)p+(z-x)q=x-y$. **[Agra 2010; Delhi Maths Hons. 1992]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{y-z}=\dfrac{dy}{z-x}=\dfrac{dz}{x-y}$. ...(1)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$=\frac{dx+dy+dz}{(y-z)+(z-x)+(x-y)}=\frac{dx+dy+dz}{0}.$$

$\therefore$ $dx+dy+dz=0$ so that $x+y+z=c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$=\frac{x\,dx+y\,dy+z\,dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\therefore$ $2x\,dx+2y\,dy+2z\,dz=0$ so that $x^2+y^2+z^2=c_2$ ...(3)

$\therefore$ From (2) and (3) solution is $\phi(x+y+z, x^2+y^2+z^2)=0$, ϕ being an arbitrary of function.

Ex. 16. *Solve the general solution of the equation* $(y+zx)p-(x+yz)q+y^2-x^2=0$.
[Delhi B.Sc. (Prog) II 2011; GATE 2001; Delhi Math Hons. 1997, 98]

Sol. Given $(y+zx)p-(x+yz)q=x^2-y^2$. ...(1)

Here the Lagrange's auxiliary equations are $\dfrac{dx}{y+zx}=\dfrac{dy}{-(x+yz)}=\dfrac{dz}{x^2-y^2}$. ...(2)

Choosing $x, y, -z$ as multipliers, each fraction of (2)

$$=\frac{xdx+ydy-zdz}{x(y+zx)-y(x+yz)-z(x^2-y^2)}=\frac{xdx+ydy-zdz}{0}$$

$\therefore$ $xdx+ydy-zdz=0$ so that $2xdx+2ydy-2zdz=0$.

Integrating, $x^2+y^2-z^2=c_1$, c_1 being an arbitrary constant. ...(3)

Choosing $y, x, 1$ as multipliers, each fraction of (2)

$$=\frac{ydx+xdy+dz}{y(y+zx)-x(x+yz)+x^2-y^2}=\frac{ydx+xdy+dz}{0}$$

$\therefore$ $ydx+xdy+dz=0$ or $d(xy)+dz=0$.

Integrating, $xy+z=c_2$, c_2 being an arbitrary constant. ...(4)

$\therefore$ The required solution is $\phi(x^2+y^2-z^2, xy+z)=0$, ϕ being an arbitrary function.

Ex. 17. *Solve* $x(y-z)p+y(z-x)q=z(x-y)$, *i.e.*, $\{(y-z)/(yz)\}p+\{(z-x)/(zx)\}q=(x-y)/(xy)$. **[Delhi B.A (Prog) II 2010; I.A.S. 2005, M.S. Univ. T.N. 2007; Vikram 2003]**

Sol. Given $x(y-z)p+y(z-x)q=z(x-y)$ ... (1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}$ ... (2)

Choosing $1/x$, $1/y$, $1/z$ as multipliers each fraction of (1)

$$=\frac{(1/x)dx+(1/y)dy+(1/z)dz}{(y-z)+(z-x)+(x-y)}=\frac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$$

$\Rightarrow$ $(1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\therefore$ $\log (xyz) = c_1$ or $xyz = c_1$ ... (3)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(xy - xz) + (yz - yx) + (zx - zy)} = \frac{dx + dy + dz}{0}$$

$\Rightarrow$ $dx + dy + dz = 0$ so that $x + y + z = c_2$ ... (4)

From (3) and (4), solution is $\phi(x + y + z, xyz) = 0$, ϕ being an arbitrary function.

Ex. 18. *Solve* $2y(z - 3)p + (2x - z)q = y(2x - 3)$ **[Delhi Math (H) 1999]**

Sol. The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x - z} = \frac{dz}{y(2x-3)} \qquad \text{... (1)}$$

Taking the first and third fractions, $(2x - 3)dx = 2(z - 3)dz$.

Integrating, $x^2 - 3x = z^2 - 6z + C_1$ or $x^2 - 3x - z^2 + 6z = C_1$ ... (2)

Choosing 1, 2y, –2 as multipliers, each fraction of (1)

$$= \frac{dx + 2ydy - 2dz}{2y(z-3) + 2y(2x - z) - 2y(2x - 3)} = \frac{dx + 2ydy - 2dz}{0}$$

$\therefore$ $dx + 2ydy - 2dz = 0$ so that $x + y^2 - 2z = C_2$ ... (3)

From (2) and (3), solution is $\phi(x^2 - 3x - z^2 + 6z, x + y^2 - 2z) = 0$, ϕ being an arbitrary function.

Ex. 19. *Solve* $x^2(\partial z/\partial x) + y^2(\partial z/\partial y) = (x + y)z.$ **[Delhi Maths (H) 2001]**

Sol. Re-writing the given equation $x^2p + y^2q = (x + y)z$ ... (1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{(x+y)z}$ ... (2)

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0.$

Integrating, $-(1/x) + (1/y) = C_1$ or $1/y - 1/x = C_1$ ... (3)

Choosing $1/x$, $1/y$, $-1/z$ as multipliers, each fraction of (2)

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{x + y - (x + y)} = \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0}$$

$\therefore$ $(1/x)dx + (1/y)dy - (1/z)dz = 0$ so that $xy/z = C_2$ ... (4)

From (3) and (4), solution is $\Phi(1/y - 1/x, xy/z) = 0$, Φ being an arbitrary function.

Ex. 20. *Solve* $z(x + 2y)\, p - z(y + 2x)\, q = y^2 - x^2$ **[Vikram 1999]**

Sol. The Lagrange's subsidiary equations are $\dfrac{dx}{z(x+2y)} = \dfrac{dy}{-z(y+2x)} = \dfrac{dz}{y^2 - x^2}$ ... (1)

Taking the first two fraction of (1), we have

$(y + 2x)\, dx + (x + 2y)\, dy = 0$ or $2xdx + 2y\, dy + d\,(xy) = 0$

Integrating, $x^2 + y^2 + xy = C_1$, C_1 being an arbitrary constant ... (2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{(x^2z + 2xyz) - (y^2z + 2xyz) + (zy^2 - zx^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow\ 2x\,dx + 2y\,dy + 2z\,dz = 0$ so that $x^2 + y^2 + z^2 = C_2$ ... (3)

From (2) and (3), solution is $\phi(x^2+y^2+z^2,\ x^2+y^2+xy)=0$, ϕ being an arbitrary function

EXERCISE 2(C)

Solve the following partial differential equations:

1. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ **Ans.** $\phi(x^2+y^2+z^2,\ xyz)=0$

[Mysore 2004, Delhi B.Sc. (Prog). II 2007, M.S. Unit. T.N. 2007]

2. $z(xp - yq) = y^2 - x^2$ **Ans.** $\phi(x^2+y^2+z^2,\ xy)=0$

3. $(y^2+z^2)p - xyq + xz = 0$ **[I.A.S. 1990]** **Ans.** $\phi(x^2+y^2+z^2,\ y/z)=0$

4. $yp - xq = 2x - 3y$ **[M.S. Univ. T.N. 2007]** **Ans.** $\phi(x^2+y^2,\ 3x+2y+z)=0$

5. $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$ **Ans.** $\phi(xyz,\ 1/x+1/y+1/z)=0$

[Meerut 2007, Bilaspur 2004, Rewa 2003]

2.11. Type 4 based on Rule IV for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction of (1) will be equal to $(P_1dx+Q_1dy+R_1dz)/(P_1P+Q_1Q+R_1R)$. ...(2)

Suppose the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$(P_2dx+Q_2dy+R_2dz)/(P_2P+Q_2Q+R_2R) \qquad ...(3)$$

is such that its numerrator is exact differential of denominator. Fractions (2) and (3) are then combined to given an integral. This method may be repeated in some problems to get another integral. Sometimes only one integral is possible by using the above rule IV. In such cases second integral should be obtained by using rule 1 of Art. 2.5 or rule 2 of Art. 2.7 or rule 3 of Art. 2.9.

2.12. SOLVED EXAMPLES BASED IN ART. 2.11

Ex. 1. *Solve* $(y + z)p + (z + x)q = x + y$. **[Indore 2000; Jabalpur 2000, Jiwaji 2002, Kanpur 2008; Purvanchal 2007, Ravishankar 2002, 2005; Delhi BA (Prog.) II 2011]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$. ...(1)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx-dy}{(y+z)-(z+x)} = \dfrac{d(x-y)}{-(x-y)}$. ...(2)

Again, choosing 0, 1, –1 as multipliers, each fraction of (1) $= \dfrac{dy-dz}{(z+x)-(x+y)} = \dfrac{d(y-z)}{-(y-z)}$. ...(3)

Finally, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)}. \qquad ...(4)$$

(2), (3) and (4) $\Rightarrow$ $\dfrac{d(x-y)}{-(x-y)} = \dfrac{d(y-z)}{-(y-z)} = \dfrac{d(x+y+z)}{2\,(x+y+z)}$. ...(5)

Taking the first two fractions of (5), $\dfrac{d(x-y)}{x-y} = \dfrac{d(y-z)}{y-z}$.

Integrating, $\log(x-y) = \log(y-z) + \log c_1$, c_1 being an arbitrary constant.

or $\quad \log\{(x-y)/(y-z)\} = \log c_1$ or $(x-y)/(y-z) = c_1$. ...(6)

Taking the first and the third fractions of (5), $\quad 2\,\dfrac{d(x-y)}{(x-y)} + \dfrac{d(x+y+z)}{x+y+z} = 0$

Integrating, $2\log(x-y) + \log(x+y+z) = \log c_2$ or $(x-y)^2(x+y+z) = c_2$. ...(7)

From (6) and (7), the required general solution is

$$\phi\,[(x-y)^2(x+y+z),\ (x-y)/(y-z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$ **[Delhi Maths Hons 1997; Nagpur 2010]**

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}. \qquad ...(1)$$

Taking the first two fractions of (1), $\quad x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0$.

Integrating, $x^3 + y^3 = c_1$, c_1 being an arbitrary as constant. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)}. \qquad ...(3)$$

Combining the third fraction of (1) with fraction (3), we get

$$\frac{dx-dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

Integrating, $\quad \log(x-y) - \log z = \log c_2$ or $(x-y)/z = c_2$. ...(4)

From (3) and (4), solution is $\phi\,(x^3+y^3,\ (x-y)/z) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $(x^2-y^2-z^2)p + 2xyq = 2xz$ or $(y^2+z^2-x^2)p - 2xyq = -2xz$.

[Bangalore 1993, I.A.S. 1973; P.C.S. (U.P.) 1991; Bhopal 2010]

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}. \qquad ...(1)$$

Taking the last two fractions of (1), we have

$(1/y)dy = (1/z)dz$ so that $(1/y)dy - (1/z)dz = 0$.

Integrating, $\quad \log y - \log z = \log c_1$ or $y/z = c_1$. ...(2)

Choosing x, y, z as mnultipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}. \qquad ...(3)$$

Combining the third fraction of (1) with fraction (3), we have

$$\frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)} = \frac{dz}{-2xz} \quad \text{or} \quad \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} - \frac{dz}{z} = 0.$$

Integrating, $\log(x^2+y^2+z^2) - \log z = \log c_2$ or $(x^2+y^2+z^2)/z = c_2$. ... (4)

From (2) and (4) solution is $\phi\,(y/z,\ (x^2+y^2+z^2)/z) = 0$, ϕ being an arbitrary function.

Ex. 4. *Solve* $(1+y)p + (1+x)q = z$. **[M.S. Univ. T.N. 2007; Kanpur 2011]**

Sol. Here the Lagrange's auxiliary equations are $\quad \dfrac{dx}{1+y} = \dfrac{dy}{1+x} = \dfrac{dz}{z}$. ...(1)

Taking the first two fractions of (1), we have

$(1+x)dx = (1+y)dy$ or $2(1+x)dx - 2(1+y)dy = 0$.

Integrating, $(1 + x)^2 - (1 + y)^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Taking 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx+dy}{1+y+1+x} = \dfrac{d(2+x+y)}{2+x+y}$. ...(3)

Combining the last fraction of (1) with fraction (3), we get

$$\frac{d(2+x+y)}{2+x+y} = \frac{dz}{z} \qquad \text{or} \qquad \frac{d(2+x+y)}{2+x+y} - \frac{dz}{z} = 0.$$

Integrating, $\log (2 + x + y) - \log z = \log c_2$ or $(2 + x + y)/z = c_2$. ...(4)

From (2) and (4), the required general solution is given by

$$\phi\,[(1 + x)^2 - (1 + y^2), (2 + x + y)/z] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 5. *Find the general integral of $xzp + yzq = xy$.*

Sol. Here the Lagrange's auxiliary equations are $(dx)/xz = (dy)/yz = (dz)/xy$ ... (1)

From the first two fractions of (1), $(1/x)dx = (1/y)dy$.

Integrating, $\log x = \log y + \log c_1$ or $x/y = c_1$. ...(2)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1) $= \dfrac{(1/x)dx + (1/y)dy}{(1/x)xz + (1/y)yz} = \dfrac{ydx + xdy}{2xyz}$ ...(3)

Combining the last fraction of (1) with fraction (3), we have

$$\frac{ydx+xdy}{2xyz} = \frac{dz}{xy} \quad \text{or} \quad ydx + xdy = 2zdz \quad \text{or} \quad d(xy) = 2zdz \quad \text{or} \quad d(xy) - 2zdz = 0$$

Integrating, $xy - z^2 = c_2$, c_2 being an arbitrary constant. ...(4)

From (2) and (4) solution is $\phi\,(x/y, xy - z^2) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.* **Delhi Math (H) 2005, 11, M.D.U. Rohtak 2005; Agra 2008, 09; Guwahati 2007; Meerut 2006; Sagar 2000; Ravishankar 2000; Lucknow 2010]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$. ...(1)

Choosing 1, –1, 0 and 0, 1, –1 as multipliers in turn, each fraction of (1)

$$= \frac{dx-dy}{x^2 - y^2 + z(x-y)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

so that $$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0.$$

Integrating, $\log (x - y) - \log (y - z) = \log c_2$ or $(x - y)/(y - z) = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}. \qquad ...(3)$$

Again, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}. \qquad ...(4)$$

From (3) and (4), $$\frac{xdx+ydy+zdz}{x+y+z} = dx + dy + dz$$

or $$2(x + y + z)\, d(x + y + z) - (2xdx + 2ydy + 2zdz) = 0.$$

Integrating, $$(x + y + z)^2 - (x^2 + y^2 + z^2) = 2c_2$$

or $$(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) - (x^2 + y^2 + z^2) = 2c_2$$

or $xy + yz + zx = c_2$, c_2 being an arbitrary constant. ...(5)

From (2) and (5), the required general solution is given by

$$\phi\,[xy + yz + zx, (x - y)/(y - z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 7. *Solve* $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$.

Sol. Here Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x-y)} \qquad ...(1)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 - y^2 - yz) - (x^2 - y^2 - zx)} = \frac{dx - dy}{z(x-y)}. \qquad ...(2)$$

Choosing x, $-y$, 0 as multipliers each fraction of (1)

$$= \frac{xdx - ydy}{x(x^2 - y^2 - yz) - y(x^2 - y^2 - zx)} = \frac{xdx - ydy}{(x-y)(x^2 - y^2)}. \qquad ...(3)$$

From (1), (2), (3) we have

$$\frac{dz}{z(x-y)} = \frac{dx - dy}{z(x-y)} = \frac{xdx - ydy}{(x-y)(x^2 - y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx - dy}{z} = \frac{2xdx - 2ydy}{2(x^2 - y^2)}. \qquad ...(4)$$

Taking the first two fractions of (4), we have

$dz = dx - dy$ so that $z - x + y = c_1$ ...(5)

Again , taking the first and third fractions of (4), $d(x^2 - y^2)/(x^2 - y^2) - (2/z)\,dz = 0$

Integrating, $\log (x^2 - y^2) - 2 \log z = c_2$ or $(x^2 - y^2)/z^2 = c_2$. ...(6)

From (5) and (6), solution is $\phi\,(z - x + y, (x^2 + y^2)/z^2) = 0$, ϕ being an arbitrary function.

Ex. 8. *Solve* $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)$.

Sol. Here the Lagrange's, auxiliary equations are $\dfrac{dx}{x^2 + y^2 + yz} = \dfrac{dy}{x^2 + y^2 - xz} = \dfrac{dz}{z(x+y)}$. ...(1)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 + y^2 + yz) - (x^2 + y^2 - xz)} = \frac{dx - dy}{z(x+y)}. \qquad ...(2)$$

Choosing x, y, 0 as multipliers, each fraction of (1)

$$= \frac{xdx + ydy}{x(x^2 + y^2 + yz) + y(x^2 + y^2 - xz)} = \frac{xdx + ydy}{(x+y)(x^2 + y^2)}. \qquad ...(3)$$

From (1), (2) and (3), we have

$$\frac{dz}{z(x+y)} = \frac{dx - dy}{z(x+y)} = \frac{xdx + ydy}{(x+y)(x^2 + y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx - dy}{z} = \frac{xdx + ydy}{x^2 + y^2}. \qquad ...(4)$$

Taking the first two fractions of (4), we have

$dz = dx - dy$ or $dz - dx + dy = 0$.

Integrating, $z - x + y = c_1$, c_1 being an arbitrary constant. ...(5)

Taking the first and third fractions of (4), we have

$$\frac{2xdx + 2ydy}{x^2 + y^2} = 2\frac{dz}{z} \quad \text{or} \quad \frac{d(x^2 + y^2)}{x^2 + y^2} - 2\frac{dz}{z} = 0.$$

Integrating, $\log (x^2 + y^2) - 2\log z = \log c_2$ or $(x^2 + y^2)/z^2 = c_2$. ...(6)

From (5) and (6), solution is $\phi\,(z - x + y, (x^2 + y^2)/z^2) = 0$, ϕ being an arbitrary function.

Ex. 9. *Solve* $\cos(x+y)p + \sin(x+y)q = z$. **[Garhwal 2010, Vikram 1998; Meerut 2007; Delhi Maths (H) 2007; Rajasthan 1994; Delhi B.A./B.Sc. (Prog.) Maths 2007]**

Sol. Here the Lagrange's auxiliary equations are $$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}. \qquad ...(1)$$

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \qquad ...(2)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1) $$= \frac{dx-dy}{\cos(x+y)-\sin(x+y)}. \qquad ...(3)$$

From (1), (2) and (3), $$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}. \qquad ...(4)$$

Taking the first two fractions of (4), $$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \qquad ...(5)$$

Putting $x+y=t$ so that $d(x+y)=dt$, (5) reduces to

$$\frac{dz}{z} = \frac{dt}{\cos t+\sin t} = \frac{dt}{\sqrt{2}\left\{(1/\sqrt{2})\cos t+(1/\sqrt{2})\sin t\right\}} = \frac{dt}{\sqrt{2}\{\sin(\pi/4)\cos t+\cos(\pi/4)\sin t\}} = \frac{dt}{\sqrt{2}\sin(t+\pi/4)}$$

Thus, $$(\sqrt{2}/z)dz = \operatorname{cosec}(t+\pi/4)\,dt.$$

Integrating, $\sqrt{2}\log z = \log\tan\frac{1}{2}\left(t+\frac{\pi}{4}\right) + \log c_1$, or $z^{\sqrt{2}} = c_1\tan\left(\frac{t}{2}+\frac{\pi}{8}\right)$

or $$z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = c_1. \text{ as } t = x+y \qquad ...(6)$$

Taking the last two fraction of (4), $$dx-dy = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y). \qquad ...(7)$$

On R.H.S. of (7), putting $x+y=t$, so that $d(x+y)=dt$, (7) reduces to

$dx-dy = \dfrac{\cos t-\sin t}{\cos t+\sin t}dt.$ so that $x-y = \log(\sin t+\cos t) - \log c_2$

or $(\sin t+\cos t)/c_2 = e^{x-y}$ or $e^{-(x-y)}(\sin t+\cos t) = c_2$

or $$e^{y-x}[\sin(x+y)+\cos(x+y)] = c_2, \quad \text{as } t = x+y. \qquad ...(8)$$

From (6) and (8), the required general solution is

$$\phi\left[z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right), e^{y-x}\{\sin(x+y)+\cos(x+y)\}\right] = 0, \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 10. *Solve* $\cos(x+y)p + \sin(x+y)q = z + (1/z)$. **[Delhi B.A. (Prog.) 2011]**

Sol. Do like Ex. 9. **Ans.** $\phi\left[(z^2+1)^{1/\sqrt{2}}\tan\left(\frac{3\pi}{8}-\frac{x+y}{2}\right), e^{y-x}\{\cos(x+y)+\sin(x+y)\}\right] = 0$

Ex. 11. *Solve* $xp + yq = z - a\sqrt{(x^2+y^2+z^2)}$. **[Meerut 1997; Jiwaji 1997; Rawa 1999]**

Sol. Here the lagrange's auxiliary equations are $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}. \qquad ...(1)$$

Taking the first two fractions of (1), we have

$(1/x)dx = (1/y)dy$ or $(1/x)dx - (1/y)dy = 0.$

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1) $= \dfrac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}$ ...(3)

Combining first and third fractions of (1) with fraction (3), we get

$$\frac{dx}{x} = \frac{dz}{z - a\sqrt{(x^2 + y^2 + z^2)}} = \frac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}. \quad ...(4)$$

Putting $x^2 + y^2 + z^2 = t^2$ so that $xdx + ydy + zdz = tdt$, (4) gives

$$\frac{dx}{x} = \frac{dz}{z - at} = \frac{tdt}{t^2 - azt} \quad \text{or} \quad \frac{dx}{x} = \frac{dz}{z - at} = \frac{dt}{t - az}. \quad ...(5)$$

Choosing 0, 1, 1 as multipliers, each fraction of (5) $= \dfrac{dz + dt}{(z + t) - a(t + z)} = \dfrac{d(z + t)}{(1 - a)(z + t)}$. ...(6)

Combining the first fraction of (5) with fraction (6), we get

$$\frac{dx}{x} = \frac{d(z + t)}{(1 - a)(z + t)} \quad \text{or} \quad (1 - a)\frac{dx}{x} - \frac{d(z + t)}{z + t} = 0.$$

Integrating, $(1 - a) \log x - \log (z + t) = \log c_2$, c_2 being an arbitrary constant.

or $$\frac{x^{a-1}}{z + t} = c_2 \quad \text{or} \quad \frac{x^{a-1}}{z + \sqrt{(x^2 + y^2 + z^2)}} = c_2, \quad \text{as} \quad t = (x^2 + y^2 + z^2)^{1/2} \quad ...(7)$$

From (2) and (7), the required general solution is

$$\phi\,[x^{a-1}/\{z + \sqrt{(x^2 + y^2 + z^2)}\},\, x/y] = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 12. *Solve* $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2 + y^2)$. **[I.A.S. 1993]**

Sol. Here the Lagrange's subsidiary equations are $\dfrac{dx}{x^3 + 3xy^2} = \dfrac{dy}{y^3 + 3x^2y} = \dfrac{dz}{2z(x^2 + y^2)}$. ...(1)

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx + dy}{x^3 + 3xy^2 + 3x^2y + y^3} = \dfrac{d(x + y)}{(x + y)^3}$. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx - dy}{x^3 + 3xy^2 - y^3 - 3x^2y} = \dfrac{d(x - y)}{(x - y)^3}$. ...(3)

From (2) and (3), $(x + y)^{-3}\, d(x + y) = (x - y)^{-3}\, d(x - y)$

or $u^{-3}du - v^{-3}dv = 0$, on putting $u = x + y$ and $v = x - y$.

Integrating, $u^{-2}/(-2) - v^{-2}/(-2) = c_1/2$ or $v^{-2} - u^{-2} = c_1$

or $(x - y)^{-2} - (x + y)^{-2} = c_1$, as $u = x + y$ and $v = x - y$. ...(4)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)\,dx + (1/y)\,dy}{(1/x) \times (x^3 + 3xy^2) + (1/y) \times (y^3 + 3x^2y)} = \frac{(1/x)dx + (1/y)dy}{4(x^2 + y^2)}. \quad ...(5)$$

Combining the last fraction of (1) with fraction (5), we have

$$\frac{dz}{2z(x^2 + y^2)} = \frac{(1/x)dx + (1/y)dy}{4(x^2 + y^2)} \quad \text{or} \quad \frac{dx}{x} + \frac{dy}{y} - 2\frac{dz}{z} = 0.$$

Integrating, $\log x + \log y - 2 \log z = \log c_2$ or $(xy)/z^2 = c_2$. ...(6)

From (4) and (6), the required general solution is given by

$$\phi\,[(x-y)^{-2}-(x+y)^{-2},\ (xy)/z^2]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 13. *Solve* $p+q=x+y+z$. **[Bhopal 2010, Bilaspur 2000, 02; I.A.S. 1975; Gulberge 2005]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{1}=\dfrac{dy}{1}=\dfrac{dz}{x+y+z}$. ...(1)

Taking the first two fractions of (1), $dx-dy=0$ so that $x-y=c_1$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction of (1) $=\dfrac{dx+dy+dz}{1+1+(x+y+z)}=\dfrac{d(2+x+y+z)}{2+x+y+z}$ ... (3)

Combining the first fraction of (1) with fraction (3), $d(2+x+y+z)/(2+x+y+z)=dx$.

Integrating, $\log(2+x+y+z)-\log c_2=x$ or $(2+x+y+z)/c_2=e^x$

or $e^{-x}(2+x+y+z)=c_2$, c_2 being arbitrary function ...(4)

From (2) and (4), the required general solution is

$$\phi[x-y,\ e^{-x}(2+x+y+z)]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 14. *Solve* $(2x^2+y^2+z^2-2yz-zx-xy)p+(x^2+2y^2+z^2-yz-2zx-xy)q=x^2+y^2+2z^2-yz-zx-2xy$. **[Meerut 1996 ; I.A.S. 1992]**

Sol. Here Lagrange's auxiliary equations are

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy}=\frac{dy}{x^2+2y^2+z^2-yz-2zx-xy}=\frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}. \quad ...(1)$$

Choosing 1, –1, 0 ; 0, 1, –1 and –1, 0, 1 as multipliers in turn, each fraction of (1)

$$=\frac{dx-dy}{x^2-y^2-yz+zx}=\frac{dy-dz}{y^2-z^2-zx+xy}=\frac{dz-dx}{z^2-x^2-xy+yz}$$

$$\therefore\quad \frac{dx-dy}{(x-y)\,(x+y+z)}=\frac{dy-dz}{(y-z)\,(x+y+z)}=\frac{dz-dx}{(z-x)\,(x+y+z)}. \quad ...(2)$$

Taking the first two fractions of (2), we have

$$(dx-dy)/(x-y)-(dy-dz)/(y-z)=0.$$

Integrating, $\log(x-y)-\log(y-z)=\log c_1$ or $(x-y)/(y-z)=c_1$. ...(3)

Taking the last two fractions of (2), $(dy-dz)/(y-z)-(dz-dx)/(z-x)=0$.

Integrating, $\log(y-z)-\log(z-x)=\log c_2$ or $(y-z)/(z-x)=c_2$. ...(4)

From (3) and (4), the required general solution is

$$\phi[(x-y)/(y-z),\ (y-z)/(z-x)]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 15. *Find the general solution of the partial differential equation* $px(x+y)-qy(x+y)+(x-y)(2x+2y+z)=0$. **[Delhi B.Sc. II (Prog) 2009; Delhi Maths Hons. 2006, 09, 11]**

Sol. Given $x(x+y)p-y(x+y)q=-(x-y)(2x+2y+z)$. ...(1)

Lagrange's auxiliary equations are $\dfrac{dx}{x\,(x+y)}=\dfrac{dy}{-y\,(x+y)}=\dfrac{dz}{-(x-y)\,(2x+2y+z)}$. ...(2)

Taking the first two fractions, $(1/x)dx=-(1/y)dy$ or $(1/x)dx+(1/y)dy=0$.

Integrating, $\log x+\log y=\log c_1$ or $xy=c_1$. ...(3)

Again, each fraction of (2)

$$=\frac{dx+dy}{x(x+y)-y(x+y)}=\frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)}$$

$$=\frac{dx+dy}{(x-y)(x+y)}=\frac{dx+dy+dz}{(x-y)(x+y)-(x-y)(2x+2y+z)}$$

Thus, $$\frac{dx+dy}{(x+y)} = \frac{dx+dy+dz}{x+y-(2x+2y+z)} = -\frac{dx+dy+dz}{x+y+z}$$

Thus, $\dfrac{dx+dy}{x+y} + \dfrac{dx+dy+dz}{x+y+z} = 0$, so that $\log(x+y) + \log(x+y+z) = \log c_2$

or $(x+y)(x+y+z) = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), solution is $\phi[xy, (x+y)(x+y+z)] = 0$, ϕ being an arbitrary function.

Ex. 16. *Solve* $\{my(x+y)-nz^2\}(\partial z/\partial x) - \{lx(x+y)-nz^2\}(\partial z/\partial y) = (lx-my)z$ **[I.A.S. 2001]**

Sol. Re-writing the given equation, $\{my(x+y)-nz^2\}\,p - \{lx(x+y)-nz^2\}q = (lx-my)z$ ...(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{my(x+y)-nz^2} = \dfrac{dy}{-lx(x+y)+nz^2} = \dfrac{dz}{(lx-my)z}$ ...(2)

Each fraction of (2) = $\dfrac{dx+dy}{(my-lx)(x+y)} = \dfrac{dz}{-(my-lx)z}$ so that $\dfrac{d(x+y)}{x+y} = -\dfrac{dz}{z}$

Integrating, $\log(x+y) = -\log z + \log C_1$ or $(x+y)z = C_1$ ... (3)

Taking lx, my, nz as multipliers, each fraction of (2)

$$= \frac{lxdx+mydy+nzdz}{lx\,my(x+y)-lx\,nz^2-mylx(x+y)+mynz^2+nz^2(lx-my)} = \frac{lx\,dx+my\,dy+nz\,dz}{0}$$

$\therefore\ 2lx\,dx + 2my\,dy + 2nzdz = 0$ so that $lx^2+my^2+nz^2 = C_2$ ...(4)

From (3) and (4), solution is $\Phi(xz+yz,\, lx^2+my^2+nz^2) = 0$, Φ being an arbitrary function.

Ex. 17. *Solve* $px(z-2y^2) = (z-qy)(z-y^2-2x^2)$. **[Delhi Maths (H) 2002]**

Sol. Re-writing the given equation $x(z-2y^2)p + y(z-y^2-2x^2)q = z(z-y^2-2x^2)$ ... (1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x(z-2y^2)} = \dfrac{dy}{y(z-y^2-2x^2)} = \dfrac{dz}{z(z-y^2-2x^2)}$ ...(2)

Taking the last two fractions, $(1/y)\,dy - (1/z)\,dz = 0$ so that $y/z = C_1$ ...(3)

Taking 0, $-2y$, 1 as multipliers, each fraction of (2)

$$= \frac{-2ydy+dz}{-2y^2(z-y^2-2x^2)+z(z-y^2-2x^2)} = \frac{d(z-y^2)}{(z-2y^2)(z-y^2-2x^2)} \quad \text{... (4)}$$

Combining fraction (4) with first fraction of (2), we get

$$\frac{dx}{x(z-2y^2)} = \frac{d(z-y^2)}{(z-2y^2)(z-y^2-2x^2)} \quad \text{or} \quad \frac{d(z-y^2)}{dx} = \frac{z-y^2-2x^2}{x}$$

or $du/dx = (u-2x^2)/x$, taking $z-y^2 = u$ ... (5)

or $(du/dx) - (1/x)u = -2x$ which is an ordinary linear differential equation

whose I.F. $= e^{-\int (1/x)dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$ and solution is

$$u.\frac{1}{x} = \int (-2x)\left(\frac{1}{x}\right)dx + C_2 \quad \text{or} \quad \frac{z-y^2}{x} = -2x + C_2, \text{ using (5)}$$

or $(z-y^2)/x+2x=C_2$ or $(z-y^2+2x^2)/x=C_2$ ... (6)

From (3) and (6), the required general solution of (1)

$\Phi(y/z,\ (z-y^2-2x^2)/x)=0$, Φ being an arbitrary function.

Ex. 18. *Solve* $px(z-2y^2)=(z-qy)\ (z-y^2-2x^3)$. **[I.A.S. 2006]**

Sol. Do like Ex. 17, **Ans.** $\Phi(y/z,\ (z-y^2+x^3)/x)=0$

For another method of solution, refer solved Ex. 8 of Art. 2.8.

Ex. 19. *Solve* $x(z+2a)p+(xz+2yz+2ay)q=z(z+a)$.

Sol. The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{x(z+2a)}=\frac{dy}{xz+2yz+2ay}=\frac{dz}{z(z+a)} \quad \text{... (1)}$$

Each fraction of (1) = $\dfrac{dx+dy}{2(x+y)(z+a)}=\dfrac{dz}{z(z+a)}$ or $\dfrac{d(x+y)}{x+y}=\dfrac{2}{z}dz$

Integrating, log $(x+y)$ = 2 log z + log C_1 or $(x+y)/z^2=C_1$ ...(2)

Taking the first and third ratios of (4), $\dfrac{dx}{x}=\dfrac{z+2a}{z(z+a)}dz$ or $\dfrac{dx}{x}=\left(\dfrac{2}{z}-\dfrac{1}{z+a}\right)dz$

Integrating, log x = 2 log z – log $(z+a)$ + log C_2 or $x(z+a)/z^2=C_2$ ... (3)

From (2) and (3), solution is $\Phi\ \{(x+y)/z^2,\ x(z+a)/z^2)\}=0$. ϕ being an arbitrary function.

Ex. 20. *Solve* $2x\ (y+z^2)p+y(2y+z^2)q=z^3$ **[Delhi Maths (Hans.) 2007]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{2x(y+z^2)}=\frac{dy}{y(2y+z^2)}=\frac{dz}{z^3} \quad \text{...(1)}$$

Each fraction of (1) = $\dfrac{dx}{2x(y+z^2)}=\dfrac{z\,dy+y\,dz}{2yz(y+z^2)}=\dfrac{d(yz)}{2yz(y+z^2)}$

$\therefore$ $(1/x)\ dx+(1/yz)\ d(yz)=0$ so that $x/(yz)=C_1$ ...(2)

From the last two fractions of (1), $\dfrac{dy}{dz}=\dfrac{y(2y+z^2)}{z^3}=\dfrac{2y^2}{z^3}+\dfrac{y}{z}$ or $y^{-2}\dfrac{dy}{dz}-\dfrac{1}{z}y^{-1}=\dfrac{2}{z^3}$ ...(3)

Putting $-y^{-1}=u$ and $(1/y^2)\times(dy/dz)=du/dz$ in (3), we get

$(du/dz)+(1/z)\ u=2/z^3$, which is an ordinary linear equation.

Its I.F. = $e^{\int(1/z)dz}=e^{\log z}=z$ and solution is $u\,z=\int(2/z^3)\ z\ dz-C_2=-2\ z^{-1}-C_2$

or $-y^{-1}z-2z^{-1}=-C_2$ or $z/y-2/z=C_2$ ... (4)

From (3) and (4), solution is $\Phi\ (x/yz,\ z/y-2/z)=0$, ϕ being arbitrary function.

Ex. 21. $xp+zq+y=0$. **[M.D.U. Rohtak 2004]**

Sol. Given equation is $xp+zq=-y$

Its Lagrange's auxiliary equation are $$\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{-y} \qquad \text{... (1)}$$

Taking the last two fractions of (2), $2ydy + 2zdz = 0$ so that $y^2 + z^2 = C_1$ ... (2)
Choosing 0, z, $-y$ as multipliers, each fraction of (1)

$$= \frac{zdy - ydz}{z^2 + y^2} = \frac{(1/z)dy - (y/z^2)dz}{1 + (y/z)^2} = \frac{d(y/z)}{1 + (y/z)^2} \qquad \text{... (3)}$$

Combining the first faction of (1) with fraction (3), we get

$$\frac{dx}{x} = \frac{d(y/z)}{1 + (y/z)^2} \qquad \text{or} \qquad \frac{dx}{x} - d\left(\tan^{-1}\frac{y}{z}\right) = 0$$

Integrating, $\log|x| - \tan^{-1}(y/z) = C_2$, c_2 being an arbitrary constant. ... (4)
From (2) and (4), the required general solution is

$$\log|x| - \tan^{-1}(y/z) = \phi(y^2 + z^2), \ \phi \text{ being an arbitrary function.}$$

Ex. 22. *Find the general solution of the differential equation* $x^2(\partial z/\partial x) + y^2(\partial z/\partial y) = (x + y)z.$ **[Delhi B.A./B.Sc. (Prog.) Maths 2007)**

Sol. Let $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Then, the given equation takes the form

$$x^2 p = y^2 q = z(x + y) \qquad \text{... (1)}$$

The Lagrange's auxiliary equations for (1) are

$$(dx)/x^2 = (dy)/y^2 = (dz)/z(x + y) \qquad \text{... (2)}$$

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0$
Integrating, $-(1/x) + (1/y) = c_1$ or $(x - y)/xy = c_1$ ... (3)

Chossing 1, –1, 0 as multipliers, each fraction of (2) $= \dfrac{dx - dy}{x^2 - y^2}$ ... (4)

Combining the last fraction of (2) with fraction (4), we have

$$\frac{dx - dy}{(x - y)(x + y)} = \frac{dz}{z(x + y)} \qquad \text{or} \qquad \frac{dx - dy}{x - y} - \frac{dz}{z} = 0$$

Integrating, $\log(x - y) - \log z = \sin c^2$ or $(x - y)/z = c^2$ ... (5)
From (5), $x - y = c_2 z$ ... (6)
using (6), (3) becomes $(c_2 z)/xy = a$ or $(xy)/z = c_2/c_1 = c_3$ say ... (7)
From (5) and (7), the required solution is $\phi\big((x, y)/z, (x - y)/z\big) = 0.$

EXERCISE 2(D)

Solve the following partial differential equations:

1. $(x^2 + y^2)p + 2xy\, q = z(x + y)$ **Ans.** $(x + y)/z = \phi\big(y/(x^2 - y^2)\big)$
2. $\{y(x + y) + az\}\, p + \{x(x + y) - az\}q = z(x + y)$ **Ans.** $(x + y)/z = \phi(x^2 - y^2 - 2az)$
3. $(y^2 + yz + z^2)p + (z^2 + zx + x^2)q = x^2 + xy + y^2$ **Ans.** $\phi\left(\dfrac{y - z}{x - y}, \dfrac{x - z}{x - y}\right) = 0$

2.13. Miscellaneous Examples on $Pp + Qq = R$

Ex. 1. *Solve* $(x + y - z)(p - q) + a(px - qy + x - y) = 0.$

Sol. Let $u = x + y$ and $v = x - y.$ ...(1)

Then $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (1)} \qquad ...(2)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (1)} \qquad ...(3)$$

From (2) and (3), we get $$p - q = 2(\partial z/\partial v). \qquad ...(4)$$

and $$px - qy = x\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} - y\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}$$

or $$px - qy = (x - y)\frac{\partial z}{\partial u} + (x + y)\frac{\partial z}{\partial v} = v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v}, \text{using (1)} \qquad ...(5)$$

Using (1), (4) and (5), the given equation reduces to

$$2(u - z)\frac{\partial z}{\partial v} + a\left(v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v} + v\right) = 0$$

or $$av(\partial z/\partial u) + (2u - 2z + au)(\partial z/\partial v) = -av, \qquad ...(6)$$

which is Lagrange's linear equation. Its Lagrange's auxiliary equations are

$$\frac{du}{av} = \frac{dv}{2u - 2z + au} = \frac{dz}{-av}. \qquad ...(7)$$

Taking the first and third fractions of (7), we have

$du + dz = 0$ so that $u + z = c_1.$ ...(8)

Considering the first two fractions of (7) and eliminating z with help of (8), we have

$$\frac{du}{av} = \frac{dv}{2u - 2(c_1 - u) + au} \quad \text{or} \quad avdv = (4u - 2c_1 + au)du.$$

Integrating, $(1/2) \times av^2 = 2u^2 - 2c_1 u + (1/2) \times au^2 + c_2/2$

$av^2 = 4u^2 - 4u(u + z) + au^2 + c_2,$ or $av^2 + 4uz - au^2 = c_2$ (using (8) ....(9)

From (8) and (9), the required general solution is given by

$\phi(u + z, av^2 + 4uz - au^2) = 0,$ where ϕ is an arbitrary function and u and v are given by (1).

Ex. 2 (*a*). *Find the surface whose tangent planes cut off an intercept of constant length k from the axis of z.*

(*b*) *Formulate partial differential equation for surfaces whose tangent planes form a tetrahedrom of constant volume with the coordinate planes.* **[I.A.S. 2005]**

Sol. (*a*) We know that the equation of the tangent plane at point (x, y, z) to a surface is given by $$p(X - x) + q(Y - y) = Z - z, \qquad ...(1)$$
where X, Y, Z denote current coodrinates of any point on the plane (1). Since (1) cuts an intercept k on the z–axis, it follows that (1) must pass through the point $(0, 0, k)$. Hence putting $X = 0$, $Y = 0$ and $Z = k$ in (1), we obtain

$$px + qy = z - k, \qquad ...(2)$$

which is well known Lagrange's linear equation. For (2), the Lagrange's auxiliary equations are
$$(dx)/x = (dy)/y = (dz)/(z - x). \qquad ...(3)$$

Taking the first two fractions of (3), $(1/x)dx - (1/y)dy = 0.$ so that $x/y = c_1.$ ...(4)

Again, taking the first and third fraction of (3), $[1/(z - k)]dz - (1/x)dx = 0$

Integrating, $\log(z - k) - \log x = \log c_2$ or $(z - k)/x = c_2.$ ...(5)

From (4) and (5), the required surface (solution) is given by

$$\phi[y/x, (z - k)/x] = 0, \quad \phi \text{ being an arbitrary function.}$$

(*b*) Left as an exercise.

EXERCISE 2 (E)

Solve the following partial differential equations :

1. $p - qy \log y = z \log y$. **Ans.** $\phi(yz, e^x \log y) = 0$

2. $(p + q)(x + y) = 1$. **Ans.** $\phi(y - x, e^{-2z}y + x) = 0$

3. $x^2p + y^2q = x + y$. **Ans.** $\phi[(1/y) - (1/x), e^{-z}(x - y)] = 0$

4. $(x^2 + 2y^2)p - xyq = xz$. **Ans.** $\phi(x^2y^2 + y^4, yz) = 0$

5. $px - qy = (z - xy)^2$. **Ans.** $\phi[xy, xe^{1/(z-xy)}] = 0$

6. $zp + zq = z^2 + (x - y)^2$. **Ans.** $\log [z^2 + (x - y)^2] - 2x = \phi(x - y)$.

7. $x(y^n - z^n)p + y(z^n - x^n)q = z(x^n - y^n)$. **Ans.** $x^n + y^n + z^n = \phi(xyz)$.

8. $(xz + y^2)p + (yz - 2x^2)q + 2xy + z^2 = 0$. **Ans.** $\phi(yz + x^2, 2xz - y^2) = 0$.

9. $xyp + y(2x - y)q = 2xz$. **Ans.** $\phi(xy - x^2, z/xy) = 0$.

2.14. Integral surfaces passing through a given curve. In the last article we obtained general integral of $Pp + Qq = R$. We shall now present two methods of using such a general solution for getting the integral surface which passes through a given curve.

Method I. Let
$$Pp + Qq = R \qquad ...(1)$$
be the given equation. Let its auxiliary equations give the following two independent solutions
$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2. \qquad ...(2)$$
Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by
$$x = x(t), \qquad y = (t), \qquad z = z(t), \qquad ...(3)$$
where t is a parameter. Then (2) may be expressed as
$$u[x(t), y(t), z(t)] = c_1 \quad \text{and} \quad v[x(t), y(t), z(t)] = c_2. \qquad ...(4)$$
We eliminate single parameter t from the equations of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

2.15. SOLVED EXAMPLES BASED ON ART. 2.14.

Ex. 1. *Find the integral surface of the linear partial differential equation* $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ *which contains the straight line* $x + y = 0$, $z = 1$. **[Delhi 2008; Pune 2010]**

Sol. Given
$$x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z. \qquad ...(1)$$
Lagrange's auxiliary equations of (1) are
$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}. \qquad ...(2)$$
Proceed as in solved Ex. 6, Art. 2.10 and show that
$$xyz = c_1 \quad \text{and} \quad x^2 + y^2 - 2z = c_2. \qquad ...(3)$$
Taking t as parameter, the given equation of the straight line $x + y = 0$, $z = 1$ can be put in parametric form
$$x = t, \qquad y = -t, \qquad z = 1. \qquad ...(4)$$
Using (4), (3) may be re–written as
$$-t^2 = c_1 \quad \text{and} \quad 2t^2 - 2 = c_2. \qquad ...(5)$$
Eliminating t from the equations of (5), we have
$$2(-c_1) - 2 = c_2 \quad \text{or} \quad 2c_1 + c_2 + 2 = 0. \qquad ...(6)$$
Putting values of c_1 and c_2 from (3) in (6), the desired integral surface is
$$2xyz + x^2 + y^2 - 2z + 2 = 0.$$

Ex. 2. *Find the equation of the integral surface of the differential equation* $2y(z - 3)p + (2x - z)q = y(2x - 3)$, *which pass through the circle* $z = 0$, $x^2 + y^2 = 2x$. **[Meerut 2007]**

Sol. Given equation is
$$2y(z - 3)p + (2x - z)q = y(2x - 3). \qquad ...(1)$$
Given circle is
$$x^2 + y^2 = 2x, \qquad z = 0. \qquad ...(2)$$
Lagrange's auxiliary equations for (1) are
$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}. \qquad ...(3)$$

Taking the first and third fractions of (3), $(2x-3)dx - 2(z-3)dz = 0.$

Integrating, $x^2 - 3x - z^2 + 6z = c_1$, c_1 being an arbitrary constant. ...(4)

Choosing 1/2, y, -1 as multipliers, each fraction of (3)

$$= \frac{(1/2)dx + ydy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{(1/2)dx + ydy - dz}{0}$$

Hence $(1/2)dx + ydy - dz = 0$ or $dx + 2ydy - 2dz = 0.$

Integrating, $x + y^2 - 2z = c_2$, c2 being an arbitrary constant. ...(5)

Now, the parametric equations of given circle (2) are $x = t$, $y = (2t - t^2)^{1/2}$, $z = 0$. ...(6)

Substituting these values in (4) and (5), we have

$$t^2 - 3t = c_1 \quad \text{and} \quad 3t - t^2 = c_2. \qquad ...(7)$$

Eliminating t from the above equations (7), we have $c_1 + c_2 = 0.$ ...(8)

Substituting the values of c_1 and c_2 from (4) and (5) in (8), the desired integral surface is

$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$ or $x^2 + y^2 - z^2 - 2x + 4z = 0.$

Method II. Let $Pp + Qq = R$ ...(1)

be the given equation. Let is Lagrange's auxiliary equations give the following two independent integrals $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. ...(2)

Suppose we wish to obtain the integral surface passing though the curve which is determined by the following two equations

$$\phi(x, y, z) = 0 \quad \text{and} \quad \psi(x, y, z) = 0. \qquad ...(3)$$

We eliminate x, y, z from four equations of (2) and (3) and obtain a relation between c_1 and c_2. Finally, replace c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and obtain the desired integral surface.

Ex. 3. *Find the integral surface of the partial differential equation* $(x - y)p + (y - x - z)q = z$ through the circle $z = 1$, $x^2 + y^2 = 1$. **(Nagpur 2002)**

Sol. Given $(x - y)p + (y - x - z)q = z.$ ...(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x-y} = \dfrac{dy}{y-x-z} = \dfrac{dz}{z}.$ ...(2)

Choosing 1, 1, 1 as multipliers, each fraction on (2) = $(dx + dy + dz)/0$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_1.$ ...(3)

Taking the last two fractions of (2) and using (3) we get

$$\frac{dy}{y-(c_1-y)} = \frac{dz}{z} \quad \text{or} \quad \frac{2dy}{2y-c_1} - \frac{2dz}{z} = 0.$$

Integrating it, $\log\ (2y - c_1) - 2\log z = \log c_2$ or $(2y - c_1)/z^2 = c_2$

or $(2y - x - y - z)/z^2 = c_2$ or $(y - x - z)/z^2 = c_2$. ....(4)

The given curve is given by $z = 1$ and $x^2 + y^2 = 1.$ ...(5)

Putting $z = 1$ in (3) and (4), we get $x + y = c_1 - 1$ and $y - x = c_2 + 1.$ ...(6)

But $2(x^2 + y^2) = (x + y)^2 + (y - x)^2.$ ...(7)

Using (5) and (6), (7) becomes

$2 = (c_1 - 1)^2 + (c_2 + 1)^2$ or $c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0.$ ...(8)

Putting the values of c_1 and c_2 from (3) and (4) in (8), required integral surface is

$$(x + y + z)^2 + (y - x - z)^2/z^4 - 2(x + y + z) + 2(y - x - z)/z^2 = 0$$

or $$z^4(x + y + z)^2 + (y - x - z)^2 - 2z^4(x + y + z) + 2z^2(y - x - z) = 0.$$

Ex. 4. *Find the equation of the integral surface of the differential equation* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ *which passes through the line* $x = 1, y = 0$.

Sol. Given $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$ ...(1)

Proceed as in solved Ex. 6, Art. 2.12 and show that

$(x - y)/(y - z) = c_1$ ...(2)

and $xy + yz + zx = c_2.$ ...(3)

The given curve is represented by $x = 1$ and $y = 0.$ ...(4)

Using (4) in (2) and (3), we obtain $-1/z = c_1$ and $z = c_2$

so that $(-1/z) \times z = c_1c_2$ or $c_1c_2 + 1 = 0.$ ...(5)

Putting the values of c_1 and c_2 from (2) and (3) in (5), the required integral surface is

$[(x - y)/(y - z)]\,(xy + yz + zx) + 1 = 0$ or $(x - y)\,(xy + yz + zx) + y - z = 0$

Ex. 5. *Find the equation of surface satisfying* $4yzp + q + 2y = 0$ *and passing through* $y^2 + z^2 = 1, x + z = 2.$ **[I.A.S. 1997]**

Sol. Given $4yzp + q = -2y.$ ...(1)

Given curve is given by $y^2 + z^2 = 1,$ and $x + z = 2.$ ...(2)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{4yz} = \dfrac{dy}{1} = \dfrac{dz}{-2y}.$ ...(3)

Taking the first and third fractions of (3), $dx + 2zdz = 0$ so that $x + z^2 = c_1.$ ...(4)

Taking the last two fractions of (3), $dz + 2ydy = 0$ so that $z + y^2 = c_2.$ ...(5)

Adding (4) and (5), $(y^2 + z^2) + (x + z) = c_1 + c_2$

or $1 + 2 = c_1 + c_2,$ using (2) ...(6)

Putting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required suface is given by $3 = x + z^2 + z + y^2$ or $y^2 + z^2 + x + z - 3 = 0.$

Ex. 6. *Find the general integral of the partial differential equation* $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ *and also the particular integral which passes through the line* $x = 1, y = 0$**. [I.A.S. 2008]**

Sol. Given $(2xy - 1)p + (z - 2x^2)q = 2(x - yz).$ ...(1)

Given line is given by $x = 1$ and $y = 0.$ ...(2)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{2xy-1} = \dfrac{dy}{z-2x^2} = \dfrac{dz}{2x-2yz}.$ ...(3)

Taking z, 1, x as multipliers, each fraction of (3) $= (zdx + dy + x\,dz)/0$

so that $zdx + dy + xdz = 0$ or $d(xz) + dy = 0$

Integrating, $xz + y = c_1.$ ...(4)

Again, taking x, y, 1/2 as multipliers, each fraction of (3) $= \{xdx + ydy + (1/2)dz\}/0$

so that $x\,dx + ydy + (1/2) \times dz = 0$ or $2xdx + 2ydy + dz = 0$

Integrating, $x^2 + y^2 + z = c_2.$ ...(5)

Since the required curve given by (4) and (5) passes through the line (2), so putting $x = 1$ and $y = 0$ in (4) and (5), we get

$z = c_1$ and $1 + z = c_2$ so that $1 + c_1 = c_2.$ ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), the eqution of the required surface is given by

$1 + xz + y = x^2 + y^2 + z$ or $x^2 + y^2 + z - xz - y = 1.$

Ex. 7. *Find the integral surface of* $x^2p + y^2q + z^2 = 0$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ *which passes through the hyperbola* $xy = x + y$, $z = 1.$ **[I.A.S. 1994, 2009]**

Sol. Given $x^2p + y^2q + z^2 = 0$ or $x^2p + y^2q = -z^2.$ ...(1)

Given curve is given by $xy = x + y$ and $z = 1.$ ...(2)

Here Lagrange's auxiliary equations for (1) are $(dx)/x^2 = (dy)/y^2 = (dz)/(-z^2).$ ...(3)

Taking the first and third fractions of (1), $x^{-2}dx + z^{-2}dz = 0.$

Integrating, $-(1/x) - (1/z) = -c_1$ or $1/x + 1/z = c_1.$ ...(4)

Taking the second and third fractions of (1), $y^{-2}dy + z^{-2}dz = 0.$

Integrating, $-(1/y) - (1/z) = -c_2$ or $1/y + 1/z = c_2.$ ...(5)

Adding (4) and (5), $\frac{1}{x}+\frac{1}{y}+\frac{2}{z} = c_1 + c_2$ or $\frac{x+y}{xy}+\frac{2}{z} = c_1 + c_2$

or $(xy)/(xy) + 2 = c_1 + c_2$, using (2) or $c_1 + c_2 = 3.$ ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we get

$1/x + 1/z + 1/y + 1/z = 3$ or $yz + 2xy + xz = 3xyz.$

Ex. 6. *Find the integral surface of the linear first order partial differential equation* $yp + xq = z - 1$ *which passes through the curve* $z = x^2 + y^2 + z,\ y = 2x$

Sol. Given equation is $yp + xq = z - 1$ ... (1)

and the given curve is given by $z = x^2 + y^2 + 1$ and $y = 2x$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}$ ... (3)

Taking the first two fractions, $2ydy - 2xdx = 0$

Integrating, it, $y^2 - x^2 = C_1$, C_1 being an arbitrary constant ... (4)

Taking the first and the last fractions of (3) and using (4), we get

$\frac{dx}{(x^2+C_1)^{1/2}} = \frac{dz}{z-1}$ so that $\log(z-1) - \log\{x + (x^2 + C_1)^{1/2}\} = \log C_2$

or $\log(z - 1) - \log(x + y) = \log C_2$, by (4) or $(z - 1)/(x + y) = C_2$ ... (5)

The parametric form of the given curve (2) is $x = t,$ $y = 2t,$ $z = 5t^2 + 1$ ... (6)

Substituting these values in (4) and (5), we get $3t^2 = C_1$ and $5t/3 = C_2$ ... (7)

Eliminating t from the above equations (7), we get $5\sqrt{C_1}/3\sqrt{3} = C_2$ ... (8)

Substituting the values of C_1 and C_2 from (4) and (5) in (8), the required surface is given by

$$5(y^2 - x^2)^{1/2}/3\sqrt{3} = (z-1)/(x+y).$$

Ex. 7. *Find the integral surface of the partial differential equation* $(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$ *passing through the curve* $xz = a^3,\ y = 0$.

Sol. Given equation is $(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z$ ... (1)

and the given curve is given by $xz = a^3$ and $y = 0$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$ ... (3)

Each fraction of (3) = $\frac{dx - dy}{(x-y)\,(y^2+x^2)} = \frac{dz}{(x^2+y^2)z}$ so that $\frac{d(x-y)}{x-y} - \frac{dz}{z} = 0$

Integrating it, $(x-y)/z = C_1$, C_1 being an arbitrary contant ... (4)

Taking the first two fractions, $3x^2dx + 3y^2dy = 0$

Integrating it, $x^3 + y^3 = C_2$, C^2 being an arbitrary constant. ... (5)

The parameteric form of the given curve (2) is $z = t,$ $x = a^3/t,$ $y = 0$ ... (6)

Substituting these values in (4) and (5), we get

$$a^3/t^2 = C_1 \quad \text{so that} \quad t^2 = a^3/C_1 \qquad \text{... (7)}$$

and $$(a^3/t)^3 = C_2 \quad \text{so that} \quad t^3 = a^9/C_2 \qquad \text{... (8)}$$

Squaring both sides of (8), $t^6 = a^{18}/C_2^2$ or $(t^2)^3 = a^{18}/C_2^2$

or $(a^3/C_1)^3 = a^{18}/C_2^2$, since $t^2 = a^3/C_1$, by (7)

or $$a^9/C_1^3 = a^{18}/C_2^2, \quad \text{or} \quad C_2^2 = a^9\ C_1^3 \qquad \text{... (9)}$$

Substituting the values of C_1 and C_2 from (4) and (5) in (9), the required integral surface of (1) is given by

$$(x^3+y^3)^2 = a^9\ (x-y)^3/z^3 \quad \text{or} \quad z^3(x^3+y^3)^2 = a^9\ (x-y)^3.$$

EXERCISE 2(F)

1. *Find particular integrals of the following partial differential equations to represent surfaces passing through the given curves :*

(*i*) $p + q = 1$; $x = 0$, $y^2 = z$. **Ans.** $(y - x)^2 = z - x$.

(*ii*) $xp + yq = z$; $x + y = 1$, $yz = 1$. **Ans.** $yz = (x + y)^2$.

(*iii*) $(y - z)p + (z - x)q = x - y$; $z = 0$, $y = 2x$ **Ans.** $5(x + y + z)^2 = 9(x^2 + y^2 + z^2)$.

(iv) $x(y - z)p + y(z - x)q = z(x - y)$; $x = y$; $x = y = z$. **Ans.** $(x + y + z)^3 = 27xyz$.

(*v*) $yp - 2xyq = 2xz$; $x = t$, $y = t^2$, $z = t^3$. **Ans.** $(x^2 + y^2)^5 = 32y^2z^2$.

(*vi*) $(y - z)\ [2xyp + (x^2 - y^2)q] + z(x^2 - y^2) = 0$; $x = t^2$, $y = 0$, $z = t^3$.

Ans. $x^3 - 3xy^2 = z^2 - 2yz$.

2. Find the general solution of the equation $2x(y + z^2)p + y(2y + z^2)q = z^2$ and deduce that $yz(z^2 + yz - 2y) = x^2$ is a solution.

3. Find the general solution of $x(z + 2a)p + (xz + 2yz + 2ay)q = z(z + a)$.

Find also the integral surfaces which pass through the curves :

(*i*) $y = 0$, $z^2 = 4ax$. (*ii*) $y = 0$, $z^3 + x(z + a)^2 = 0$.

4. Solve $xp + yq = z$. Find a solution representing a surface meeting the parabola

$y^2 = 4x$, $z = 1$. **Ans.** General solution $\phi\ (x/2, y/2) = 0$; surface $y^2 = 4xz$.

2.16. SURFACES ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES

Let $$f(x, y, z) = C \qquad \text{...(1)}$$

represents a system of surfaces where C is parameter. Suppose we wish to obtain a system of surfaces which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1) which passes through that point are $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

Let the surface $$z = \phi(x, y) \qquad \text{...(2)}$$

cuts each surface of (1) at right angles. Then the normal at (x, y, z) to (2) has direction ratios $\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1. Since normals at (x, y, z) to (1) and (2) are at right angles, we have

$$p(\partial f/\partial x) + q(\partial f/\partial y) - (\partial f/\partial z) = 0 \quad \text{or} \quad p(\partial f/\partial x) + q(\partial f/\partial y) = \partial f/\partial z \qquad \text{...(3)}$$

which is of the form $Pp + Qq = R$.

Conversely, we easily verify that any solution of (3) is orthogonal to every surface of (1).

2.17. SOLVED EXAMPLES BASED ON ART. 2.16.

Ex. 1. *Find the surface which intersects the surfaces of the system* $z(x + y) = c(3z + 1)$ *orthogonally and which passes through the circle* $x^2 + y^2 = 1$, $z = 1$. **[I.A.S. 1999]**

Sol. The given system of surfaces is $$f(x, y, z) \equiv \{z(x+y\}/(3z+1) = C. \qquad \text{...(1)}$$