2K17/EE/099
JISHNU PURKAYASTHA

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ASSIGNMENT 4,5

1. $u_t - \alpha^2 u_{xx} = 0$

$u_x = 0$ for $x = 0, l$

$u = lx - x^2$ for $t = 0$ & $0 \le x \le l$

Substituting $u = X(x)T(t)$.

$XT' = \alpha^2 X'' T$

$\Rightarrow \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -k^2$

$\Rightarrow \frac{\partial^2 X}{\partial x^2} + k^2 X = 0$ | $\frac{\partial T}{\partial t} + k^2\alpha^2 T = 0$ — (1)

$\Rightarrow X = c_1 \cos kx + c_2 \sin kx$ | $T = c_3 e^{-k^2\alpha^2 t}$ — (2)

If $k^2 = -k^2$,
$X = c_4 e^{kx} + c_5 e^{-kx}$ | $T = c_6 e^{k^2\alpha^2 t}$ — (3)

If $k = 0$,
$X = c_7 x + c_8$ | $T = c_9$ — (4).

In (3), for $t \to \infty$, $T \to \infty$
$\Rightarrow u \to \infty$ [but u is not infinite for $t \to \infty$]

$\Rightarrow$ (3) is rejected.

In (4), $\frac{\partial u}{\partial x} = c_7 c_9 = 0 \Rightarrow c_7 = 0$

$\Rightarrow u = c_8 c_9 = a_0$ — (5)

From (2),

$\frac{\partial u}{\partial x} = k\left[-c_1 \sin kx + c_2 \cos kx\right] c_3 e^{-k^2\alpha^2 t}$

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$\because \frac{\partial u}{\partial x} = 0$ for $x = 0$ & $x = l$

$\Rightarrow c_2 = 0$ & $-c_1 \sin kl + c_2 \cos kl = 0$

$\cancel{\Rightarrow \tan kl}$

$\Rightarrow \sin kl = 0$

$\Rightarrow kl = n\pi$

$$\therefore u = c_1 \cos kx \cdot c_3 e^{-k^2\alpha^2 t}$$

$$= a_n \cos\left[\frac{n\pi x}{l}\right] e^{-\frac{n^2\pi^2}{l^2}\alpha^2 t} \quad \text{---} (6)$$

$\therefore$ General solⁿ:

$$u(x,t) = a_0 + \Sigma a_n \cos\left[\frac{n\pi x}{l}\right] e^{-\frac{n^2\pi^2}{l^2}\alpha^2 t} \quad - (7)$$

$\because u = lx - x^2$

$$\Rightarrow lx - x^2 = a_0 + \Sigma a_n \cos\left[\frac{n\pi}{l}x\right] e^{-\frac{n^2\pi^2}{l^2}\alpha^2 t}\Bigg|_{t=0}$$

$$= a_0 + \Sigma a_n \cos\left[\frac{n\pi x}{l}\right]$$

Comparing with Fourier Series,

$$\therefore a_0 = \frac{1}{l}\int_0^l lx - x^2 = \frac{l^2}{6}$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}$$

Even series.

$$= \frac{2}{l}\Bigg[(lx - x^2)\frac{l}{n\pi}\sin\frac{n\pi x}{l} - (l - 2x)\left[-\frac{l^2}{n^2\pi^2}\right]\cos\frac{n\pi x}{l}$$
$$- 2\left[-\frac{l^3}{n^3\pi^3}\right]\sin\frac{n\pi}{l}x\Bigg]_0^l$$

$$= -\frac{4l^2}{n^2\pi^2}$$

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$$\therefore \quad u = \frac{l^2}{6} - \frac{4l^2}{\pi^2} \sum_{n=2,4,6\ldots} \frac{1}{n^2} \cos\left[\frac{n\pi x}{l}\right] e^{-\frac{n^2\pi^2}{l^2}\alpha^2 t}$$

2. $u_t = K u_{xx}$

$u_x(0,t) = 0$

$u_x(200,t) = -\frac{h}{K}\left[u(200,t) - 20\right]$

$u(x,0) = \sin \pi x$

Taking Laplace X-form

$$sU(x,s) - \sin\pi x = K\frac{\partial^2 U}{\partial x^2}(x,s)$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} - \frac{s}{K}U = -\frac{\sin\pi x}{K}$$

$$\Rightarrow U_{CF}(x,s) = C_1 e^{\sqrt{\frac{s}{K}}x} + C_2 e^{\sqrt{\frac{s}{K}}x}$$

$$U_{PI}(x,s) = \frac{-\sin\pi x}{K\left[D^2 - \frac{s}{K}\right]} = \frac{+\sin\pi x}{K\left(\pi^2 + \frac{s}{K}\right)} = \frac{\sin\pi x}{s + K\pi^2}$$

$$\therefore U(x,s) = C_1 e^{\sqrt{\frac{s}{K}}x} + C_2 e^{-\sqrt{\frac{s}{K}}x} + \frac{\sin\pi x}{s+K\pi^2} \quad \text{--- (1)}$$

$$\frac{\partial U}{\partial x}(x,s) = \sqrt{\frac{s}{K}}\left[C_1 e^{\sqrt{s/K}\,x} - C_2 e^{-\sqrt{s/K}\,x}\right] + \frac{\pi\cos\pi x}{s+K\pi^2}$$

$$L\left[u_x(0,t)\right] = L[0]$$

$$\Rightarrow \sqrt{\frac{s}{K}}\left[C_1 - C_2\right] + \frac{\pi}{s+K\pi^2} = 0 \quad \text{--- (2)}$$

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$$L\left[u_x(200,t)\right] = L\left[-\frac{h}{K}\left[u(200,t) - 20\right]\right]$$

$$\Rightarrow \sqrt{\frac{s}{K}}\left[C_1 e^{\sqrt{\frac{s}{K}}200} - C_2 e^{-\sqrt{\frac{s}{K}}200}\right] + \frac{\pi}{s+K\pi^2}$$

$$= \left[C_1 e^{\sqrt{\frac{s}{K}}200} + C_2 e^{-\sqrt{\frac{s}{K}}200} - \frac{20}{s}\right]\frac{-h}{K}. \quad — (3)$$

(2) in (3)

$$C_1 e^{\sqrt{\frac{s}{K}}200}\left[\sqrt{\frac{s}{K}} + \frac{h}{K}\right] - C_2 e^{-\sqrt{\frac{s}{K}}200}\left[\sqrt{\frac{s}{K}} - \frac{h}{K}\right] - \sqrt{\frac{s}{K}}\left[C_1 - C_2\right]$$

$$= \frac{20}{s}\frac{h}{K}.$$

$$\Rightarrow C_1 e^{\sqrt{\frac{s}{K}}200}\frac{h}{K} + C_2 e^{-\sqrt{\frac{s}{K}}200}\frac{h}{K} = \frac{20}{s}\frac{h}{K}.$$

$$\Rightarrow C_1 e^{\sqrt{\frac{s}{K}}200} + C_2 e^{-\sqrt{\frac{s}{K}}200} = \frac{20}{s}. \quad — (4)$$

& (2) $\Rightarrow C_1 - C_2 = \frac{-\pi}{s+K\pi^2}\sqrt{\frac{K}{s}}$

Solving ② & ④

$$G_2 = \frac{1}{2\cosh 200\sqrt{\frac{s}{K}}}\left[\frac{20}{s} + \frac{\pi}{s+K\pi^2}\sqrt{\frac{K}{s}}\, e^{\sqrt{\frac{s}{K}}200}\right]$$

$$G_1 = \frac{1}{2\cosh 200\sqrt{\frac{s}{K}}}\left[\frac{20}{s} - \frac{\pi}{s+K\pi^2}\sqrt{\frac{K}{s}}\, e^{-\sqrt{\frac{s}{K}}200}\right]$$

$$\therefore U(x,s) = \frac{1}{2\cosh 200\sqrt{\frac{s}{K}}}\left[\frac{20}{s}\left[e^{\sqrt{\frac{s}{K}}x} + e^{-\sqrt{\frac{s}{K}}x}\right] + \frac{\pi}{s+K\pi^2}\sqrt{\frac{K}{s}}\right] + \frac{\sin \pi x}{s+K\pi^2}$$

Taking inverse laplace transform,

$$u(x,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}\left[\underbrace{\frac{20\cosh\left(\sqrt{\frac{s}{K}}x\right)}{s\cosh\left(200\sqrt{\frac{s}{K}}\right)}}_{\mathrm{I}} + \frac{1}{2(s+K\pi^2)}\left[\underbrace{\pi\sqrt{\frac{K}{s}}}_{\mathrm{II}} + \underbrace{\sin\pi x}_{\mathrm{III}}\right]\right]ds$$

I : Roots lie at $s=0$, & $200\sqrt{\frac{s}{K}} = j(2n+1)\frac{\pi}{2} \Rightarrow s = -\frac{(2n+1)^2\pi^2 K}{16\times10^4}$,

$$\text{Residue}_{s=0} \lim_{s\to 0} \frac{20\cosh\sqrt{\frac{s}{K}}x\, e^{st}}{\cosh\sqrt{\frac{s}{K}}} = \frac{20\times 1}{1} = 20$$

$$\text{Residue}_{s=(\ )} \lim_{s\to(\ )} \frac{\left[s+\frac{(2n+1)^2\pi^2K}{16\times10^4}\right]}{s}\,\frac{\cosh\left(\sqrt{\frac{s}{K}}x\right)e^{st}}{\cosh 200\sqrt{\frac{s}{K}}}$$

$$= \frac{e^{st}}{s}\,\frac{e^{\frac{j(2n+1)\pi/2}{200}} + e^{-\frac{j(2n+1)\pi/2}{200}}}{e^{j(2n+1)\frac{\pi}{2}} - e^{-j(2n+1)\frac{\pi}{2}}} \times \frac{\sqrt{K}}{200} \times 2\sqrt{s}\,.$$

$$= \frac{-1600}{(2n+1)^2\pi^2\sqrt{K}} \frac{\cos\left[\frac{(2n+1)\pi}{400}\right]}{\sin\left[(2n+1)\frac{\pi}{2}\right]} \frac{\sqrt{K}}{200} \int (2n+1)\frac{\pi}{2} \, e^{-\frac{(2n+1)^2\pi^2 K t}{160000}}$$

$$= \frac{-4j}{(2n+1)\pi} \frac{\cos\left((2n+1)\frac{\pi}{400}\right)}{\sin\left((2n+1)\frac{\pi}{2}\right)} e^{(\ )t}$$

$$\therefore I = \frac{10}{\pi j} - \frac{2}{(2n+1)\pi^2} \frac{\cos\left((2n+1)\frac{\pi}{400}\right)}{\sin\left((2n+1)\frac{\pi}{2}\right)} e^{\frac{-(2n+1)^2\pi^2 K t}{16\times10^4}}$$

II: Roots lie at $s=0$, $s=-k\pi^2$

Residue: $\lim_{s\to 0} \frac{\pi}{2} \frac{\sqrt{s}\sqrt{K}}{s+k\pi^2} = 0$
$s=0$

Residue: $\lim_{s\to ?} \frac{\pi}{2}\sqrt{\frac{K}{s}} = \frac{\pi}{2}\sqrt{\frac{K}{-K\pi^2}} = \frac{j}{2}$
$s=-k\pi^2$

$$\therefore II = \frac{1}{2\pi j} \times \frac{j}{2} = \frac{1}{4\pi}$$

$$III = \frac{1}{2\pi j}(\sin \pi x)\frac{e^{-k\pi^2 t}}{2}$$

$$\therefore u(x,t) = 2\pi j\left(I + II + III\right)$$

$$= 20 + \frac{j}{2} + \frac{1 \sin \pi x e^{-k\pi^2 t}}{2} + \sum_{n=0}^{\infty} \frac{(-4j)}{\pi(2n+1)} \frac{\cos\left((2n+1)\frac{\pi}{400}\right)}{\sin\left((2n+1)\frac{\pi}{2}\right)} e^{-\frac{(2n+1)^2\pi^2 K}{16\times10^4}}$$

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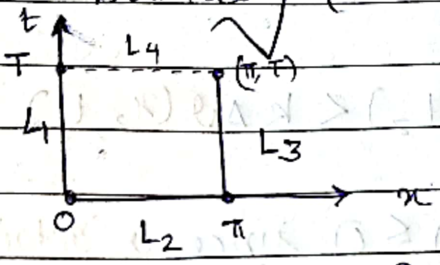
3. Let $u \in C_H$ be a sol^n of the heat eq^n. Then max. value of u on $\bar{R}$ is achieved on the parabolic boundary $\partial_p R$.

Let $M : \max\limits_{\bar{R}} u$ & $m : \max\limits_{\partial_p u} u$.

To prove the max value principle, we must show $m < M$ is not possible.

Where R is boundary (rectangular) formed by L_1, L_2, L_3, L_4.



The parabolic boundary is $\partial_p R$ defined by

$$\partial_p R = L_1 \cup L_2 \cup L_3$$

Let $(x_1, t_1) \in R \cup L_4$ be such that $u(x_1, t_1) = M$

Let $v : \bar{R} \to R$ be defined by

$$v(x,t) = u(x,t) + \frac{M-m}{4\pi^2}(x-x_1)^2 \quad \text{—} (1)$$

For $(x,t) \in \partial_p R$, we have

$$v(x,t) \leq m + \frac{M-m}{4\pi^2}\pi^2 = m + \frac{M-m}{4} < M \quad \text{—} (2)$$

Further, $v(x_1, t_1) = u(x_1, t_1) = M$. Thus, the function v assumes its max. value namely M on $R \cup L_4$.

Let $(x_2, t_2) \in R \cup L_4$ be such that $v(x_2, t_2) = M$.

Note that • $0 < x_2 < \pi$

• If $(x_2, t_2) \in R$, then $v_t(x_2, t_2) = 0$

• If $(x_2, t_2) \in L_4$, then $v_t(x_2, t_2) \geq 0$

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Thus, we have $v_t(x_2, t_2) \geq 0$.
In view of the relations,

$$v_t(x_2, t_2) = u_t(x_2, t_2)$$

$$= k \, \Delta u(x_2, t_2)$$

[Δu = laplacian]

$$= k \left[\Delta v(x_2, t_2) - \frac{M-m}{2\pi^2} \right]$$

we get

$$0 \leq v_t(x_2, t_2) < k \, \Delta v(x_2, t_2) \quad \text{--- (3)}$$

However, $\Delta v(x_2, t_2) \leq 0$ since v attains a max. at (x_2, t_2) which contradicts (3)
Thus $m < M$ is not possible.

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4. $$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \sin\pi x \\ u(x,0) &= 1\end{aligned}\right\} \quad \text{— (1)}$$

Taking Laplace Transform of (1),

$$sU(x,s) - u(x,0) = \frac{\partial^2 U(x,s)}{\partial x^2} + \frac{\sin\pi x}{s}$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} - sU = \frac{\sin\pi x}{s} + 1$$

$$\Rightarrow U_{CF} = C_1 e^{\sqrt{s}\,x} + C_2 e^{-\sqrt{s}\,x}. \quad \text{— (2)}$$

$$U_{PI} = \frac{\sin\pi x}{s\,[D^2 - s]} + \frac{1}{D^2 - s} \quad \text{— (3)}$$

$$= \frac{-\sin\pi x}{s(\pi^2 + s)} - \frac{1}{s}.$$

Taking Inverse laplace of (2),

$$u_{CF} = C_1\left[\frac{-x e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t^3}}\right] + C_2\left[\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t^3}}\right] = \frac{C x e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t^3}}$$

Taking Inverse laplace of (3),

$$u_{PI} = L^{-1}\left[\frac{(-\sin\pi x)}{\pi^2}\left(\frac{1}{s} - \frac{1}{s+\pi^2}\right)\right] + L^{-1}\left[\frac{1}{s}\right].$$

$$= \frac{\sin\pi x}{\pi^2}\left[e^{-\pi^2 t} - 1\right] + 1$$

$$\therefore u(x,t) = \frac{C x e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t^3}} + \frac{\sin\pi x}{\pi^2}\left[e^{-\pi^2 t} - 1\right] + 1$$

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5. $u_t = u_{xx}$ ——— (1).

$u(1,t) = \sin t$

Taking laplace Transform of (1),

$sU = \dfrac{\partial^2 U}{\partial x^2}$ [Assuming 0 initial conditions]

$\Rightarrow U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$.

$\because L[u(0,t)] = L[0]$

$\Rightarrow c_1 = -c_2$

& $L[u(1,t)] = L[\sin t]$

$\Rightarrow c_1 \left[e^{\sqrt{s}} - e^{-\sqrt{s}}\right] = \dfrac{1}{s^2+1}$

$\Rightarrow c_1 = \dfrac{1}{(s^2+1)\, 2\sinh\sqrt{s}}$.

$\therefore U(x,s) = \dfrac{1}{(s^2+1)\sinh\sqrt{s}} \left[\dfrac{e^{\sqrt{s}x} + e^{-\sqrt{s}x}}{2}\right]$

$U(x,s) = \dfrac{1}{(s^2+1)} \dfrac{\sinh(\sqrt{s}x)}{\sinh\sqrt{s}}$ ——— (2)

Taking inverse laplace of (2),

$u(x,t) = \dfrac{1}{2\pi j} \displaystyle\int_{\sigma-j\infty}^{\sigma+j\infty} \dfrac{\sinh(\sqrt{s}x)}{(s^2+1)\sinh\sqrt{s}} e^{st}\, ds$

Roots lie at $s = \pm j$, $s = -n^2\pi^2$

$$\underset{s=+j}{\text{Residue}} = \lim_{s\to j} \frac{1}{(s+j)} \frac{\sinh\sqrt{s}x}{\sinh\sqrt{s}} e^{jt} = \frac{1}{2j} \frac{\sin \pi/_4 x}{\sin \pi/_4} e^{jt} \quad -(3)$$

$$\text{IIly } \underset{s=-j}{\text{Residue}} = \frac{-1}{2j} \frac{\sin \pi/_4 x}{\sin \pi/_4} e^{-jt} \quad -(4)$$

Combining (3) & (4),

$$\underset{s=\pm j}{\text{Residue}} \left[\frac{\sin \pi/_4 x}{\sin \pi/_4}\right] \sin t \quad -(5)$$

$$\& \underset{s=-n^2\pi^2}{\text{Residue}} \lim_{s\to -n^2\pi^2} \frac{s+n^2\pi^2}{s^2+1} \frac{\sinh\sqrt{s}x}{\sinh\sqrt{s}} e^{st}$$

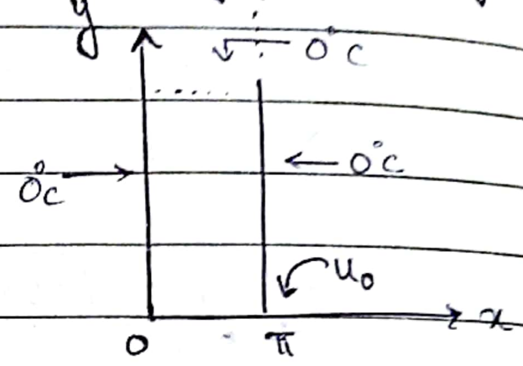
$$= \lim_{s\to -n^2\pi^2} \frac{1}{s^2+1} \frac{\sinh\sqrt{s}x}{\cosh\sqrt{s}} 2\sqrt{s}\, e^{st}.$$

$$= \frac{(-1)^{n+1}}{n^4\pi^4+1} 2n\pi j \cdot \sin(n\pi x) \cdot e^{-n^2\pi^2 t}. \quad -(6)$$

From (5) & (6).

$$u(x,t) = \left[\frac{\sin(\pi/_4 x)}{\sin(\pi/_4)}\right] \sin t + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4\pi^4+1} (n) \sin(n\pi x) e^{-n^2\pi^2 t}$$

6. Let $u(x,y)$ be the temperature at any point (x,y) on the plate, such that



$$u_{xx} + u_{yy} = 0 \quad \text{—— (1)}$$

Sol^n: $u = X(x)Y(y)$

Let the sol^n of (1) be:

$$u(x,y) = [A\cos\lambda x + B\sin\lambda x][Ce^{\lambda y} + De^{-\lambda y}]$$

$\because u(0,y) = 0$

$$\Rightarrow 0 = A[Ce^{\lambda y} + De^{-\lambda y}]$$

$$\Rightarrow A = 0$$

$\because u(\pi,y) = 0$

$$\Rightarrow 0 = B\sin\lambda\pi\,[Ce^{\lambda y} + De^{-\lambda y}]$$

$$\Rightarrow \lambda\pi = n\pi$$

$$\Rightarrow \lambda = n$$

$$\therefore u(x,y) = B\sin(nx)\left[Ce^{ny} + De^{-ny}\right] \quad \text{—— (2)}$$

$\because u(x,\infty) = 0$

$$\Rightarrow 0 = [B\sin nx]\,[Ce^{\infty} + 0]$$

$$\Rightarrow C = 0$$

$$\therefore u(x,y) = C_n \sin(nx)\, e^{-ny} \quad \text{--- (3)}$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} C_n \sin(nx)\, e^{-ny} \quad \text{--- (4)}$$

$$\because u(x,0) = u_0 = \sum_{n=1}^{\infty} C_n \sin(nx)$$

$$\Rightarrow C_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx \, \partial x.$$

$$= \frac{2u_0}{\pi}\,\frac{1}{n}\left[1-(-1)^{n+1}\right] = \begin{cases} \frac{4u_0}{\pi n} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

$$\therefore u(x,y) = \sum_{n=1,3,5,7\ldots} \frac{2u_0}{\pi}\,\frac{1}{n}\,[2]\,\sin nx$$

$$= \sum_{n=1,3,5,\ldots} \frac{4u_0}{n\pi} \sin(nx)$$

7. $u_{xx} + u_{yy} = 0$ — (1)

$u_y(x,0) = g(x)$ — (2)

$u(x,\infty) = 0$ — (3)

$u(\pm\infty, y) = 0$ — (4)

$u_x(\pm\infty, y) = 0$ — (5)

Let $u(x,y) = X(x)Y(y)$

$\Rightarrow X''(x)Y(y) + Y''(y)X(x) = 0.$

$\Rightarrow \dfrac{X''(x)}{X(x)} = \dfrac{-Y''(y)}{Y(y)} = -k^2$ (let)

$\Rightarrow X''(x) + k^2X(x) = 0$	$Y''(y) - k^2Y(y) = 0.$
If $k=0$, $X(x) = C_0x + C_1$	$Y(y) = C_2y + C_3$
Violates (3) & (4).	
If $k^2 = +k^2$, $X(x) = C_4\sin kx + C_5\cos kx$	$Y(y) = C_6e^{ky} + C_7e^{-ky}$ — (6)
If $k^2 = -k^2$, $X(x) = C_8e^{kx} + C_9e^{-kx}$	$Y(y) = C_{10}\sin ky + C_{11}\cos ky$ — (7)

From (7)

(4) $\Rightarrow u(\infty, y) = 0 \Rightarrow C_8 = 0$

$u(-\infty, y) = 0 \Rightarrow C_9 = 0$

$\therefore u(x,y) = [C_4\sin kx + C_5\cos kx][C_6e^{ky} + C_7e^{-ky}]$

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(3) $\Rightarrow u(x,\infty) = 0 \Rightarrow C_6 = 0$

$$\Rightarrow u(x,y) = \left[A_1 \sin kx + A_2 \cos kx\right] e^{-ky}.$$

$\boxed{A_1 = C_4 C_7}$
$\boxed{A_2 = C_5 C_7}$

(4) $\Rightarrow u(\infty,y) = \lim_{l\to\infty}\left[A_1 \sin kl + A_2 \cos kl\right] = 0$ — (8)

(5) $\Rightarrow u_x(\infty,y) = k\left[A_1 \cos kx - A_2 \sin kx\right] = 0$ — (9)

$\because \int_{-\infty}^{\infty} g(x) = 0 \Rightarrow$ odd function or +ve/-ve funcⁿ

$\Rightarrow u(x,y)$ must be a funcⁿ of sine.

$\therefore$ from (8) & (9) $\Rightarrow A_2 = 0$

$\Rightarrow$ From (8), $\lim_{l\to\infty} \sin kl = 0$

$$\Rightarrow k = \lim_{l\to\infty} \frac{n\pi}{l}$$

$$\therefore u(x,y) = A_1 \sin kx\, e^{-ky}.$$

$$u_y(x,0) = -kA_1 \sin kx = g(x)$$

$$\therefore A_1 = \lim_{l\to\infty} \frac{2}{l}\int_0^l \frac{-g(x)\, l}{n\pi} \sin\left(\frac{n\pi}{l}\right)x \, dx.$$

$$\Rightarrow A_1 = \lim_{l\to\infty} \frac{-2}{n\pi}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

8. $u_{xx} + u_{yy} = 0$ ⟵ (1)

$u(0,y) = 0$

$u(l,y) = 0$

$u(x,0) = 0$

$u(x,a) = \sin \frac{n\pi x}{l}$

Let $u(x,y) = X(x)\,Y(y)$.

$\Rightarrow \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -k^2$

$\Rightarrow X''(x) + k^2 X(x) = 0$	$Y''(y) - k^2 Y(y) = 0$.
$\Rightarrow X(x) = c_1 \sin kx + c_2 \cos kx$	$Y(y) = c_3 e^{ky} + c_4 e^{-ky}$ ⟵ (2)
If $k = 0$, $X(x) = c_5 x + c_6$	$Y(y) = c_7 y + c_8$ — (3)
If $k = -k^2$ $X(x) = c_9 e^{kx} + c_{10} e^{-kx}$	$Y(y) = c_{11} \sin ky + c_{12} \cos ky$ — (4)

(3) $\Rightarrow u(0,y) = c_6 [c_7 y + c_8] = 0$

$\Rightarrow c_6 = 0$

$u(x,0) = 0 \Rightarrow c_8 = 0$

$u(l,y) = 0 \Rightarrow c_5 = 0$

$\Rightarrow u = 0$

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④ $\Rightarrow u(0,y)=0 \Rightarrow C_9+C_{10}=0$

$$C_9 e^{kl} + C_{10} e^{-kl} = 0$$

$$C_9\left[e^{kl} - e^{-kl}\right] = 0$$

$$\Rightarrow C_9 = 0$$

$$\Rightarrow C_{10} = 0.$$

$\Rightarrow u = 0.$

② $\Rightarrow u(0,y)=0 \Rightarrow C_2 = 0$

$u(l,y)=0 \Rightarrow \sin kl = 0$

$$\Rightarrow k = \frac{n\pi}{l}$$

$u(x,0)=0 \Rightarrow C_3+C_4=0$

$\Rightarrow C_3 = -C_4.$

$$\therefore u(x,y) = C_3C_1 \sin\left(\frac{n\pi}{l}x\right)\left[e^{\frac{n\pi}{l}y} - e^{\frac{n\pi}{l}y}\right]$$

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)\sinh\left(\frac{n\pi}{l}y\right)$$

$$\because u(x,a) = \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)\sinh\left(\frac{n\pi}{l}a\right)$$

$$\Rightarrow A_n = \frac{2}{l}\int_0^l \frac{1}{\sinh\left(\frac{n\pi}{l}a\right)} \sin^2\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{l\sinh\left(\frac{n\pi a}{l}\right)}.l$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} \frac{1}{\sinh\left[\frac{n\pi a}{l}\right]}\sin\left(\frac{n\pi}{l}x\right)\sinh\left[\frac{n\pi}{l}y\right]$$

9 $\Delta u = 0$

$u[\text{boundary}] = f(\theta)$. [Spherical coordinates]

Let the soln be :

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} C_n r^{|n|} e^{in\theta} \quad \text{--- (1)}$$

In such a case,

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) e^{-in\alpha} \partial\alpha. \quad \text{--- (2)}$$

$$\Rightarrow u(r,\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) e^{-in\alpha} d\alpha \cdot r^{|n|} \cdot e^{in\theta}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-\alpha)} d\alpha.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) \left[\sum_{n=0}^{\infty} r^n e^{in(\theta-\alpha)} + \sum_{n=0}^{\infty} r^n e^{-in(\theta-\alpha)} - 1\right] d\alpha.$$

$$\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\Rightarrow u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) \left[\frac{1}{1-re^{i(\theta-\alpha)}} + \frac{1}{1-re^{-i(\theta-\alpha)}} - 1\right] d\alpha \quad \text{--- (3)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) \left[\frac{2-2r\cos(\theta-\alpha)}{1-2r\cos(\theta-\alpha)+r^2} - 1\right] d\alpha.$$

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$$\therefore u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,\frac{(1-r^2)}{1-2r\cos(\theta-\alpha)+r^2}\,d\alpha.$$

At centre, $r=0$

$$\Rightarrow u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,d\alpha.$$

$$= \text{Avg}[f(\alpha)].$$

10. $\Delta^2 u = 0$

$$u[1,\theta] = \underset{\mathrm{I}}{1} + \underset{\mathrm{II}}{\sin\theta} + \underset{\mathrm{III}}{\frac{1}{2}\sin 3\theta} + \underset{\mathrm{IV}}{\cos 4\theta} = f(\alpha).$$

From Poisson's formula,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,\frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}\,d\alpha.$$

$$\because \frac{1}{2} + \sum_{n=1}^{\infty} a^n \cos nx = \frac{1-a^2}{2(1+a^2-2a\cos x)}$$

$$\Rightarrow u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\left[1 + 2\sum_{n=1}^{\infty} r^n \cos n(\theta-\alpha)\right] d\alpha$$

$$\Rightarrow u(1,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\left[1 + \sum_{n=1}^{\infty} \cos(n(\theta-\alpha))\right] d\alpha$$

$$\mathrm{I}: \quad u_{\mathrm{I}} = \frac{1}{2\pi}\,2\pi = 1$$

$$\mathrm{II}: \quad u_{\mathrm{II}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cancel{f(\alpha)}\left[1 + \sum_{n=1}^{\infty} \cos n(\theta-\alpha)\right] \sin\alpha\, d\alpha.$$

$$= \frac{1}{2\pi}\,\frac{2}{n-1}\cdot \sin[n\theta]\,\sin\left[\frac{(n-1)\pi}{2}\right]$$

$$= \frac{1}{2}\sin(n\theta)\,\frac{\sin\frac{\pi}{2}(n-1)}{\frac{\pi}{2}(n-1)}$$

$$\mathrm{III}: u_{\mathrm{III}} = \frac{1}{2}\sin(n\theta)\,\frac{\sin\frac{\pi}{2}(n-3)}{(n-3)\frac{\pi}{2}}$$

$$\mathrm{IV}: \quad u_{\mathrm{IV}} = 0$$

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$$\therefore u(r,\theta) = 1 + \frac{1}{2}\sum_{n=1}^{\infty} r^n \sin(n\theta)\left[\frac{\sin(n-1)\pi/2}{(n-1)\pi/2} + \frac{\sin(n-3)\pi/2}{(n-3)\pi/2}\right]$$