8

Classification of P.D.E. Reduction to Canonical or Normal Forms. Riemann Method

8.1. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER.

Consider a general partial differential equation of second order for a function of two independent variables x and y in the form:

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \qquad ...(1)$$

where R, S and T are continuous functions of x and y only possessing partial derivatives defined in some domain D on the xy-plane. Then (1) is said to be

(i) *Hyperbolic* at a point (x, y) in domain D if $S^2 - 4RT > 0$

(ii) *Parabolic* at a point (x, y) in domain D if $S^2 - 4RT = 0$

(iii) *Elliptic* at a point (x, y) in domain D if $S^2 - 4RT < 0$.

Observe that the type of (1) is determined solely by its principal part ($Rr + Ss + Tt$, which involves the highest order derivatives of z) and that the type will generally change with position in the xy-plane unless R, S and T are constants

Remark. Some authors use u in place of z. Then, we have

$r = \partial^2 u/\partial x^2$, $s = \partial^2 u/\partial x \partial y$ and $t = \partial^2 u/\partial t^2$. etc.

Examples: (i) Consider the one-dimensional wave equation $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ *i.e.* $r - t = 0$.
Comparing it with (1), here $R = 1$, $S = 0$ and $T = -1$.
Hence $S^2 - 4RT = 0 - \{4 \times 1 \times (-1)\} = 4 > 0$ and so the given equation is hyperbolic.

(ii) Consider the one-dimensional diffusion equation $\partial^2 z/\partial x^2 = \partial z/\partial y$ *i.e.* $r - q = 0$.
Comparing it with (1), here $R = 1$ and $S = T = 0$.
Hence $S^2 - 4RT = 0 - (4 \times 1 \times 0) = 0$ and so the given equation is parabolic.

(iii) Consider two dimensional Laplace's equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ *i.e.* $r + t = 0$.
Comparing it with (1), here $R = 1$, $S = 0$ and $T = 1$.
Hence $S^2 - 4RT = 0 - (4 \times 1 \times 1) = -4 < 0$ and so the given equation is elliptic.

Ex. 2. *Classify the following partial differential equations:*

(i) $2(\partial^2 u/\partial x^2) + 4(\partial^2 u/\partial x \partial y) + 3(\partial^2 u/\partial y^2) = 2$ **[Meerut 2006]**

(ii) $\partial^2 u/\partial x^2 + 4(\partial^2 u/\partial x \partial y) + 4(\partial^2 u/\partial y^2) = 0$ **[I.F.S. 2005]**

(iii) $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$ **[Delhi Maths (G) 2006]**

(iv) $x^2(y-1)r - x(y^2 - 1)s + y(y-1)t + xyp - q = 0$ **[Delhi Maths (Prog) 2007]**

(v) $x(xy-1)r - (x^2y^2 - 1)s + y(xy-1)t + xp + yq = 0$ **[Delhi 2008]**

(vi) $(x - y)(xr - xs - ys + yt) = (x + y)(p - q)$ **[Delhi BA (Prog) II 2011]**

Sol. (*i*) Re-writing the given equation, we get $2r + 4s + 3t - 2 = 0$ ...(1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, u, p, q) = 0,$ we get $R = 2, S = 4$ and $T = 3.$ So $S^2 - 4RT = (4)^2 - (4 \times 2 \times 3) = -8 < 0,$ showing that the given equation is elliptic at all points .

(*ii*) Re-writing the given equation, we get $r + 4s + 4t = 0$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, u, p, q) = 0,$ we get $R = 1, S = 4$ and $T = 4.$ So $S^2 - 4RT = (4)^2 - (4 \times 1 \times 4) = 0,$ showing that the given equation is parabolic at all points.

(*iii*) Given $xyr - (x^2 - y^2)s - xyt + py - qx - 2(x^2 - y^2) = 0$ ...(1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0,$ we get $R = xy, S = -(x^2 - y^2)$ and $T = -xy.$ So, here $S^2 - 4RT = (x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2 > 0,$ showing that the given equation is hyperbolic at all points.

(*iv*) Hyperbolic (*v*) Hyperbolic

(*vi*) Hyperbolic

8.2. CLASSIFICATION OF A PARTIAL DIFFERENTIAL EQUATION IN THREE INDEPENDENT VARIABLES.

A linear partial differential differential equation of the second order in 3 independent variables x_1, x_2, x_3 is given by

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu = 0 \quad \text{... (1)}$$

where a_{ij} $(= a_{ji})$, b_i and c are constants or some functions of the independent variables x_1, x_2, x_3 and u is the dependent variable.

Since $a_{ij} = a_{ji}$, $A = [a_{ij}]_{3 \times 3}$ is a real symmetric matrix of order 3 × 3. The eigen values of matrix A are roots of the characteristic equation of A, namely, $|A - \lambda I| = 0$.

With help of matrix A, (1) is classified as follows:

I. If all the eigenvalues of A are non-zero and have the same sign, except precisely one of them, then (1) is known as *hyperbolic type of equation*.

II. If $|A| = 0$, *i.e.,* any one of the eigenvalues of A is zero, then (1) is known as *parabolic type of equation*

III. If all the eigenvalues of A are non-zero and of the same sign, then (1) is known as *elliptic type of equation*.

Note. the matrix A can be remembered as indicated below:

$$A = \begin{bmatrix} \text{Coeff. of } u_{xx} & \text{Coeff. of } u_{xy} & \text{Coeff of } u_{xz} \\ \text{Coeff. of } u_{yx} & \text{Coeff. of } u_{yy} & \text{Coeff of } u_{yz} \\ \text{Coeff. of } u_{zx} & \text{Coeff. of } u_{zy} & \text{Coeff of } u_{zz} \end{bmatrix}$$

8.2.A SOLVED EXAMPLES BASED ON ART. 8.2

Ex. 1. *Classify* $u_{xx} + u_{yy} = u_{zz}$ **[Delhi Maths (H) 2007; Kanpur 2011]**

The matrix A of the given equation is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

The eigenvalues of A are given by $|A - \lambda I| = 0$, *i.e.,*

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad -(1+\lambda)\,(1-\lambda)^2 = 0.$$

Hence $\lambda = -1$, 1, 1, showing that all the eigenvalues are non-zero and have the same sign except one. Hence the given equation is of hyperbolic type.

Ex. 2. *Classify* $u_{xx} + u_{yy} + u_{zz} + u_{yz} + u_{zy} = 0$.

Sol. The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + u_{yz} + 0 \cdot u_{zx} + u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$, using properties of determinents

Since $|A| = 0$, the given equation is of parabolic type.

Ex. 3. *Classify* $u_{xx} + u_{yy} + u_{zz} = 0$ **[Meerut 2007, 08; Kanpur 2011]**

Sol. The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + 0 \cdot u_{yz} + 0 \cdot u_{zx} + 0 \cdot u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The eigen values of A are given by $|A - \lambda I| = 0$,

i.e. $$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad (1-\lambda)^3 = 0 \quad \text{giving} \quad \lambda = 1, 1, 1.$$

Since all eigenvalues are non-zero and of the same sign, the given equation is of parabolic type.

Ex. 4. *Classify the following equations:*

(*i*) $u_{xx} + u_{yy} = u_z$ **[Kanpur 2011]** (*ii*) $u_{xx} + 2\,u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$. **[Delhi 2008]**

Sol. Try yourself **Ans.** (*i*) parabolic (*ii*) parabolic

8.3. Cauchy's problem for second order partial differential equation. Characteristic equation and characteristic curves (or simply characteristics) of the second order partial differential equations. (Delhi Maths (H) 2001)

Cauchy' problem. Consider the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \text{... (1)}$$

in which R, S and T are functions of x and y only. The Cauchy's problem consists of the problem of determining the solution of (1) such that on a given space curve C it takes on prescribed values of z and $\partial z / \partial n$, where n is the distance measured along the normal to the curve.

As an example of Cauchy's problem for the second order partial differential equation, consider the following problem :

To determine solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ with the following data prescribed on the x-axis: z (x, 0) = f (x), z_y (x, 0) = g (x). Observe that y-axis is the normal to the given curve (x-axis here)

Characteristic equations and characteristic curves.

Corresponding to (1), consider the λ-quadratic

$$R\lambda^2 + S\lambda + T = 0 \qquad \text{... (2)}$$

where $S^2 - 4RT \geq 0$, (2) has real roots. Then, the ordinary differential equations

$$(dy/dx) + \lambda(x, y) = 0 \qquad \text{... (3)}$$

are called the *characteristic equations*.

The solutions of (3) are knwon as *characteristic curves* or simply the *characteristics* of the second order partial differential equation (1).

Now, consider the following three cases:

Case (*i*) If $S^2 - 4RT > 0$ (*i.e.,* if (1) is hyperbolic), then (2) has two distinct real roots λ_1, λ_2 say so that we have two characteristic equations $(dy/dx) + \lambda_1(x, y) = 0$ and $(dy/dx) + \lambda_2(x, y) = 0$.

Solving these we get two distinct families of characteristics.

Case (*ii*). If $S^2 - 4RT = 0$ (*i.e.* (1) is parabolic), then (2) has two equal real roots λ, λ so that we get only one characteristic equation (3). Solving it, we get only one family of characteristics.

Case (*iii*) If $S^2 - 4RT < 0$ (*i.e.* (1) is elliptic), then (2) has complex roots. Hence there are no real characteristics. Thus we get two families of complex characteristics when (1) is elliptic

8.4 ILLUSTRATIVE SOLVED EXAMPLES BASED ON ART. 8.3

Ex. 1. *Find the characteristics of $y^2 r - x^2 t = 0$* **[I.A.S. 2009]**

Sol. Given $$y^2 r - x^2 t = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = 0$ and $T = -x^2$. Then $S^2 - 4RT = 0 - 4 \times y^2 \times (-x^2) = 4x^2y^2 > 0$ and hence (1) is hyperbolaic everywhere except on the coordinate axes $x = 0$ and $y = 0$.

The λ-quadratic is $R\lambda^2 + S\lambda + T = 0$ or $y^2\lambda^2 - x^2 = 0$ ... (2)

Solving (2), $\lambda = x/y, -x/y$ (two distinct real roots). Corresponding characteristic equations are

$(dy/dx) + (x/y) = 0$ and $(dy/dx) - (x/y) = 0$

or $x\,dx + y\,dx = 0$ and $x dx - y\,dy = 0$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$, which are the required families of characteristics. Here these are families of circles and hyperbolas respectively.

Ex. 2. *Find the characteristics of $x^2 r + 2xys + y^2 t = 0$.*

Sol. Given $$x^2 r + 2xys + y^2 t = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$. Then, $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$ and hence (1) is parabolic everywhere.

The λ-quadratic is $R\lambda^2 + S\lambda + T = 0$ or $x^2\lambda^2 + 2xy\lambda + y^2 = 0$ ... (2)

Solving (2), $(x\lambda + y)^2 = 0$ so that $\lambda = -y/x, -y/x$ (equal roots). The characteristic equation is $(dy/dx) - (y/x) = 0$ or $(1/y)\,dy - (1/x)\,dx = 0$ giving $y/x = c_1$ or $y = c_1 x$, which is the required family of characteristics. Here it represents a family of straight lines passing through the origin.

Ex. 3. *Find the characteristics of $4r + 5s + t + p + q - 2 = 0$.*

Sol. Try yourself. **Ans.** $y - x = c_1$ and $y - (x/y) = c_2$.

Ex. 4. *Find the characteristics of $(\sin^2 x)\, r + (2 \cos x)\, s - t = 0$*

Sol. Try yourself **Ans.** $y + \operatorname{cosec} x - \cot x = c_1$, $y + \operatorname{cosec} x + \cot x = c_2$

8.5. Laplace transformation. Reduction to Canonical (or normal) forms.

[Himanchal 2007; Avadh 2001; Delhi Maths (H) 2004, 09]

Consider partial differential equation of the type $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, ...(1)

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary. Laplace transformation on (1) consists of changing the independent variables x, y to new set of continuously differentiable independent variables u, v where

$$u = u(x, y) \qquad \text{and} \qquad v = v(x, y) \qquad \text{...(2)}$$

are to be chosen so that the resulting equation in independent variables u, v is transformed into one of three canonical forms, which are easily integrable. From (2), we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \text{and} \qquad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \qquad \text{...(3)}$$

$$(3) \Rightarrow \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}. \qquad \text{...(4)}$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right), \text{ by (3) and (4)}$$

$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right), \text{by (3) and (4)}$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x}$$

and
$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right), \text{ by (3) and (4)}$$

$$= \frac{\partial u}{\partial y}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}.$$

Putting the above values of p, q, r, s, t, in (1) and simplifying, we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial v^2} + F(u, v, z, \partial z/\partial u, \partial z/\partial v) = 0, \qquad \text{...(5)}$$

where
$$A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2, \qquad \text{...(6)}$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}, \qquad \text{...(7)}$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \qquad \text{...(8)}$$

and $F(u, v, z, \partial z/\partial u, \partial z/\partial v)$ is the transformed form of $f(x, y, z, p, q)$.

Now we shall find out u and v so that (5) reduces to simplest possible form. The method of evaluation of desired values of u and v becomes easy when the discriminant $S^2 - 4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad \text{...(9)}$$

is everywhere either positive, negative or zero, and now we shall present these three cases separately.

Case I. Let $S^2 - 4RT > 0$. When this condition is satisfied, then the roots λ_1, λ_2 of the equation (9) are real and distinct. The coefficients of $\partial^2 z/\partial u^2$ and $\partial^2 z/\partial v^2$ in the equation (5) will vanish if we choose u and v such that

$$\partial u/\partial x = \lambda_1 (\partial u/\partial y) \qquad \text{...(10)}$$

and
$$\partial v/\partial x = \lambda_2 (\partial v/\partial y). \qquad \text{...(11)}$$

Since λ_1 is a root of (9), we have $\quad R\lambda_1^2 + S\lambda_1 + T = 0.$ ...(12)

Using (10), (6) gives $\quad A = (R\lambda_1^2 + S\lambda_1 + T)(\partial u/\partial y)^2 = 0$, by (12) ...(13)

Again, since λ_2 is a root of (9), we have $\quad R\lambda_2^2 + S\lambda_2 + T = 0$ ...(14)

Using (11), (8) gives $\quad C = (R\lambda_2^2 + S\lambda_2 + T)(\partial v/\partial y)^2 = 0$, by (14) ...(15)

Re-writing (10), we have $\quad (\partial u/\partial x) - \lambda_1(\partial u/\partial y) = 0.$ ...(16)

Lagrange's auxiliary equation for (16) are $\quad dx/1 = dy/(-\lambda_1) = du/0$ ...(17)

Taking third fraction of (17), $du = 0$ so that $u = c_1$, c_1 being an arbitrary constant ...(18)

Taking first and second fractions of (17), we get $\quad (dy/dx) + \lambda_1 = 0$ ...(19)

Let the solution of (19) be $\quad f_1(x, y) = c_2$, c_2 being an arbitrary constant ...(20)

From (18) and (20), the general solution of (16) [*i.e.* (10)] is $\quad u = f_1(x, y).$ ...(21)

Similarly, the general solution of (11) can be taken as $\quad v = f_2(x, y).$ ...(22)

Here f_1 and f_2 are arbitrary function

We can easily verify that $\quad AC - B^2 = \dfrac{1}{4}(4RT - S^2)\left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right)^2$

or $$B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2, \text{ as } A = C = 0. \qquad \text{...(23)}$$

Let the Jacobian J of u and v be non-zero, i.e., let

$$J = \partial(u, v)/\partial(x, y) = (\partial u/\partial x)(\partial v/\partial y) - (\partial u/\partial y)(\partial v/\partial x) \neq 0$$

Since $S^2 - 4RT > 0$, (23) shows that $B^2 > 0$. Hence we may divide both sides of (5) by B^2. Then noting that $A = C = 0$, (5) transforms to the form $\quad \partial^2 z/\partial u\,\partial v = \phi\left(u, v, z, \partial z/\partial u, \partial z/\partial v\right)$,

...(24)

which is the canonical form of (1) in this case.

Case II. Let $S^2 - 4RT = 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are real and equal. We now take u exactly as in case I and take v to be any function of x, y which is independent of u. We have, as in case I, $A = 0$. Also, since $S^2 - 4RT = 0$, (23) shows that $B^2 = 0$ so that $B = 0$.

Moreover in this case $C \neq 0$, otherwise v would be a function of u and consequently v would not be independent of u as already assumed.

Putting $A = 0$, $B = 0$ and dividing by C, (5) transforms to the form

$$\partial^2 z/\partial v^2 = \phi(u, v, z, \partial z/\partial u, \partial z/\partial v). \qquad \text{...(25)}$$

which is the canonical form of (1) in this case.

Case III. Let $S^2 - 4RT < 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are complex. Hence this case III is formally the same as case I. Therefore, proceeding as in case I, we find that (1) reduces to (24) but that the variables u, v instead of being real are now complex conjugates. To obtain a real canonical form we make further transformation $u = \alpha + i\beta$ and $v = \alpha - i\beta$ so that

$\alpha = (u + v)/2$, $\quad$ and $\quad \beta = i(v - u)/2.$ ...(26)

Now, $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right), \text{ by (26)} \qquad \text{...(27)}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right), \text{ by (26)} \qquad \text{...(28)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{dv}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right) \times \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right), \text{ by (27) and (28)}$$

$$= \frac{1}{4}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) - i\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)\right] = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + i\frac{\partial^2 z}{\partial \alpha \partial \beta} - i\frac{\partial^2 z}{\partial \beta \partial \alpha} + i\frac{\partial^2 z}{\partial \beta^2}\right)$$

or $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right), \qquad \text{as} \qquad \frac{\partial^2 z}{\partial \alpha \partial \beta} = \frac{\partial^2 z}{\partial \beta \partial \alpha} \qquad ...(29)$$

Putting $u = \alpha + i\beta$, $v = \alpha - i\beta$ and using (27), (28) and (29), (24) reduces to

$$(\partial^2 z/\partial \alpha^2) + (\partial^2 z/\partial \beta^2) = \psi(\alpha, \beta, z, \partial z/\partial \alpha, \partial z/\partial \beta), \qquad ...(30)$$

which is the canonical form of (1) in this case.

8.6 Working rule for reducing a hyperbolic equation to its canonical form

Step 1. Let the given equation $\quad Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad$ ... (1)

be hyperbolic so that $\quad S^2 - 4RT > 0.$

Step 2. Write λ-quadratic equation $\quad R\lambda^2 + S\lambda + T = 0 \quad$ ... (2)

Let λ_1 and λ_2 be its two distinct roots of (2).

Step 3. Then corresponding characteristic equations are

$$(dy/dx) + \lambda_1 = 0 \qquad \text{and} \qquad (dy/dx) + \lambda_2 = 0$$

Solving these, we get $f_1(x, y) = c_1$ and $f_2(x, y) = c_2$ ... (3)

Step 4. We select u, v such that $u = f_1(x, y)$ and $v = f_2(x, y)$ ... (4)

Step 5. Using relations (4), find p, q, r, s and t in terms of u and v as shown in Art. 8.5.

Step 6. Substituting the values of p, q, r, s, t obtained in step 4 in (1) and simplifying we shall get the following canonical form of (1):

$$\partial^2 z / \partial u \partial v = \phi\,(u, v, z, \ \partial z/\partial u, \ \partial z/\partial u).$$

8.7. SOLVED EXAMPLES BASED ON ART. 8.6

Ex.1. (*a*) *Write canonical form of* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$. **[Sagar 2004; Delhi Maths (H) 2002]**

(*b*) Reduc $3(\partial^2 z/\partial x^2) + 10\,(\partial^2 z/\partial x \partial y) + 3\,(\partial^2 z/\partial y^2) = 0$ to canonical form and hence solve it **(Himanchal 2008)**

Sol. (*a*) Re-writing the given equation, we get $\quad r - t = 0 \quad$ ... (1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$ so that $S^2 - 4RT = 4 > 0$, showing that (1) is hyperbolic

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\quad \lambda^2 - 1 = 0$

Hence $\lambda = 1, -1$. So $\lambda_1 = 1, \lambda_2 = -1$ (Real and distinct roots).

Then the characteristic equations $dy/dx + \lambda_1 = 0$, $dy/dx + \lambda_2 = 0$ reduces to

$(dy/dx) + 1 = 0$ and $(dy/dx) - 1 = 0.$

Integrating these, $\quad y + x = c_1 \quad$ and $\quad y - x = c_2.$

In order to reduce (1) to its canonical form, we choose

$$u = y + x \qquad \text{and} \qquad v = y - x \qquad ... (2)$$

$$\therefore \qquad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (3)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (4)$$

From (3) and (4), $$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \qquad ... (5)$$

$$\therefore \qquad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

or $$r = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad ... (6)$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

or $$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial u \partial v} = 0.$$

(b) $\partial^2 z / \partial u \partial v = 0; \; z = f(y - 3x) + g(3y - x)$

Ex. 2. *Reduce* $\partial^2 z / \partial x^2 = (1+y)^2 (\partial^2 z / \partial y^2)$ *to canonical form*

Sol. Re-writing the given equation, $r - (1 + y^2)\, t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$, and $T = -(1 + y)^2$ so that $S^2 - 4RT = (1 + y^2) > 0$ for $y \neq -1$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - (1+y)^2 = 0$ so that $\lambda = 1 + y, -(1 + y)$. Hence the corresponding characteristic equations are given by

$$(dy/dx) + (1 + y) = 0 \qquad \text{and} \qquad (dy/dx) - (1 + y) = 0$$

Integrating these, $\log(1 + y) + x = C_1$ and $\log(1 + y) - x = C_2$.

In order to reduce (1) to its canonical form, we choose

$$u = \log(1 + y) + x \qquad \text{and} \qquad v = \log(1 + y) - x \qquad \text{... (2)}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (3)}$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{1+y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \qquad \text{... (4)}$$

From (3) $$\partial / \partial x \equiv \partial / \partial u - \partial / \partial v \qquad \text{... (5)}$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

or $$r = \partial^2 z / \partial u^2 - 2\,(\partial^2 z / \partial u\, \partial v) + \partial^2 z / \partial v^2 \qquad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{\frac{1}{1+y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\} = -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1+y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4)}$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1+y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1+y}\left[\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right)\frac{1}{y+1} + \left(\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}\right)\frac{1}{1+y}\right], \text{ by (2)}$$

or $$t = \frac{1}{(1+y)^2}\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \qquad \text{... (7)}$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0 \qquad \text{or} \qquad 4\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}..$$

Ex. 3. *Reduce the differential equation* $t - s + p - q(1 + 1/x) + (z/x) = 0$ *to canonical form.*

[Delhi Maths (H) 2004]

Sol. Given $$0 \cdot r - s + t + p - q\,(1 + 1/x) + (z/x) = 0 \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 0$, $S = -1$ and $T = 1$. Hence $S^2 - 4RT = 1 > 0$, showing that the given equation is hyperbolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $-\lambda + 1 = 0$ giving $\lambda = 1$. Hence the corresponding characteristic equation $dy/dx + \lambda = 0$ yields $dy/dx + 1 = 0$ or $dx + dy = 0$

Integrating it, $x + y = c$, c being an arbitrary constant

Choose $u = x + y$ and $v = x$, ...(2)

where we have chosen $v = x$ in such a manner that u and v are independent as verified below:

Jacobian of u and $v = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow$ u and v are independent functions.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}, \text{ using (2)} \qquad ...(4)$$

From (4), we have $$\partial/\partial y \equiv \partial/\partial u \qquad ...(5)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)} \qquad ...(6)$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right), \text{ using (5)}$$

or $$t = \partial^2 z/\partial u^2. \qquad ...(6)$$

Using (2), (3), (4), (6) and (7), (1) reduces to

$$-\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\left(1 + \frac{1}{v}\right) + \frac{z}{v} = 0$$

or $\partial^2 z/\partial u \partial v - (\partial z/\partial v) + (1/v) \times (\partial z/\partial u) - (z/v) = 0$, which is the required canonical form.

Ex. 4. *Reduce the equation $yr + (x + y)\,s + xt = 0$ to canonical form and hence find its general solution.* **(Delhi Maths (Hons) 2007)**

Sol. Given $$yr + (x + y)\,s + xt = 0 \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y$, $S = (x + y)$ and $T = x$ so that $S^2 - 4RT = (x + y)^2 - 4xy = (x - y)^2 > 0$ for $x \neq y$ and so (1) is hyperbolic. Its λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $y\lambda^2 + (x + y)\,\lambda + x = 0$ or $(y\lambda + x)\,(\lambda + 1) = 0$ so that $\lambda = -1$, $-x/y$. Then the corresponding characteristic equations are given by

$(dy/dx) - 1 = 0$ and $(dy/dx) - (x/y) = 0$

Integrating these, $y - x = c_1$ and $y^2/2 - x^2/2 = c_2$

In order to reduce (1) to its canonical form, we choose

$u = y - x$ and $v = y^2/2 - x^2/2$ ... (2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}, \text{ using (2)} \qquad ...(4)$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right), \text{ using (3)}$$

$$=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\right]=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)-\frac{\partial z}{\partial v}$$

$$=-\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]-x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]-\frac{\partial z}{\partial v}$$

$$=-\left(-\frac{\partial^2 z}{\partial u^2}-x\frac{\partial^2 z}{\partial v\partial u}\right)-x\left(-\frac{\partial^2 z}{\partial u\partial v}-x\frac{\partial^2 z}{\partial v^2}\right)-\frac{\partial z}{\partial v}, \text{ using (2)}$$

$$\therefore \qquad r=\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial v} \qquad \text{... (5)}$$

Now, $t=\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}+y\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}\right)+\dfrac{\partial}{\partial y}\left(y\dfrac{\partial z}{\partial v}\right)$, using (4)

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}+\frac{\partial z}{\partial v}$$

$$\therefore\ t=\frac{\partial^2 z}{\partial u^2}+y\frac{\partial^2 z}{\partial u\partial v}+y\left(\frac{\partial^2 z}{\partial u\partial v}+y\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial v}=\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+y^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v} \qquad \text{...(6)}$$

Also, $s=\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}+y\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right)+\dfrac{\partial}{\partial x}\left(y\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right)+y\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial v}\right)$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}=-\frac{\partial^2 z}{\partial u^2}-x\frac{\partial^2 z}{\partial v\partial u}-y\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2}, \text{ using (2)}$$

$$\therefore \qquad s=-\frac{\partial^2 z}{\partial u^2}-(x+y)\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

Using (5) (6) and (7) in (1), we get $\quad y\left(\dfrac{\partial^2 z}{\partial u^2}+2x\dfrac{\partial^2 z}{\partial u\partial v}+x^2\dfrac{\partial^2 z}{\partial v^2}-\dfrac{\partial z}{\partial v}\right)$

$$+(x+y)\left\{-\frac{\partial^2 z}{\partial u^2}-(x+y)\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2}\right\}+x\left(\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+y^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v}\right)=0$$

or $\quad \{4xy-(x+y)^2\}\dfrac{\partial^2 z}{\partial u\partial v}-y\dfrac{\partial z}{\partial v}+x\dfrac{\partial z}{\partial v}=0 \qquad$ or $\qquad (y-x)^2\dfrac{\partial^2 z}{\partial u\partial v}+(y-x)\dfrac{\partial z}{\partial v}=0$

or $\quad u^2\dfrac{\partial^2 z}{\partial u\partial v}+u\dfrac{\partial z}{\partial v}=0$, by (2) $\qquad$ or $\qquad u\dfrac{\partial^2 z}{\partial v\partial v}+\dfrac{\partial z}{\partial v}=0$, as $u\neq 0 \quad$ ... (8)

(8) is the required canonical form of (1).

Solution of (8). Multiplying both sides of (8) by v, we get

$uv\,(\partial^2 z/\partial u\partial v)+v(\partial z/\partial v)=0 \qquad$ or $\qquad (uv\ DD'+vD')z=0 \qquad$ ... (9)

where $D\equiv\partial/\partial u$ and $D'\equiv\partial/\partial v$. To reduce (9) into linear equation with constant coefficients, we take new variables X and Y as follows. For details refer Art. 6.3.

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $Y = \log v$ ... (10)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$. Then (9) reduces to

$$(D_1 D_1' + D_1')z = 0 \quad \text{or} \quad D_1'(D_1 + 1)z = 0$$

Its general solution is $z = e^{-X}\phi_1(Y) + \phi_2(X) = u^{-1}\phi_1(\log v) + \phi_2(\log u)$ [See Art. 5.6]

or $z = u^{-1}\psi_1(v) + \psi_2(u) = (y-x)^{-1}\psi_1(y^2 - x^2) + \psi_2(y-x)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex.5. *Reduce the equation* $r - (2\sin x)s - (\cos^2 x)t - (\cos x)q = 0$ *to canonical form and hence solve it.* **(Himanchal 2008)**

Sol. Given $$r - (2\sin x)\, s - (\cos^2 x)t - (\cos x)q = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = -2\sin x$ and $T = -\cos^2 x$ so that $S^2 - 4RT = 4(\sin^2 x + \cos^2 x) = 4 > 0$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - (2\sin x)\lambda - \cos^2 x = 0$ so that $\lambda = \sin x + 1, \ \sin x - 1$. Hence the corresponding characteristic equations become

$dy/dx + \sin x + 1 = 0$ and $dy/dx + \sin x - 1 = 0$

Integrating these, $y - \cos x + x = c_1$ and $y - \cos x - x = c_2$

Choose $u = y - \cos x + x$ and $v = y - \cos x - x$ ... (2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (1 + \sin x)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}, \text{ by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{... (4)}$$

From (4), we have $$\partial/\partial y = \partial/\partial u + \partial/\partial v \quad \text{... (5)}$$

$$\therefore \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (4) and (5)}$$

or $$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{... (6)}$$

Now, $$s = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left\{(1 + \sin x)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}\right\}, \text{ by (3)}$$

$$= (\sin x + 1)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + (\sin x - 1)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$

$$= (\sin x + 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right\} + (\sin x - 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}$$

$$= (\sin x + 1)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + (\sin x - 1)\left(\frac{\partial^2 z}{\partial u \partial x} + \frac{\partial^2 z}{\partial v^2}\right)$$

or $$s = \sin x\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} \quad \text{... (7)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{(\sin x + 1)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}\right\} = \cos x\frac{\partial z}{\partial u} + (\sin x + 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \cos x\frac{\partial z}{\partial v} + (\sin x - 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(\sin x+1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\}+(\sin x-1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$= \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(\sin x+1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u^2}+(\sin x-1)\frac{\partial^2 z}{\partial v^2}\right\}+(\sin x-1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u\partial v}+(\sin x-1)\frac{\partial^2 z}{\partial v^2}\right\}$$

$$\therefore \qquad r = \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(1+\sin x)^2\frac{\partial^2 z}{\partial u^2}+(\sin x-1)^2\frac{\partial^2 z}{\partial v^2}-2\cos^2 x\frac{\partial^2 z}{\partial u\partial v} \qquad \text{... (8)}$$

Using (4) (6), (7) and (8) in (1), we get

$$\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(1+2\sin x+\sin^2 x)\frac{\partial^2 x}{\partial u^2}+(\sin^2 x+1-2\sin x)\frac{\partial^2 z}{\partial v^2}-2\cos^2 x\frac{\partial^2 z}{\partial u\partial v}$$

$$-2\sin x\left\{\sin x\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}\right\}-\cos^2 x\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=0$$

or $\qquad (1+2\sin x+\sin^2 x-2\sin^2 x-2\sin x-\cos^2 x)\times(\partial^2 z/\partial u^2)+(\sin^2 x+1-2\sin x-2\sin^2 x$

$$+2\sin x-\cos^2 x)\times(\partial^2 z/\partial v^2)-(2\cos^2 x+4\sin^2 x+2\cos^2 x)\times(\partial^2 z/\partial u\,\partial v)=0$$

or $\qquad \partial^2 z/\partial u\,\partial v=0$, on simplification. ... (9)

(9) is the required canonical form of (1).

Solution of (9). Integrating (9) w.r.t. 'u', $\partial z/\partial v=\phi(v), \phi$ being an arbitrary function ... (10)

Integrating (10) w.r.t. 'v', $\qquad z=\int \phi(v)\,dv+F(u)=G(v)+F(u)$,

where $G(v)=\int \phi(v)\,dv$, F and G are arbitrary functions.

$\therefore \quad z=G(y-\cos x-x)+F(y-\cos x+x)$ is the required solution.

Ex. 6. *Reduce* $\partial^2 z/\partial x^2 = x^2(\partial^2 z/\partial y^2)$ *to canonical form.*

[Agra 2005; Himanchal 2005; Delhi B.Sc. (Prog) II 2002, 07; Kurukshetra 2004; Ravishankar 2004; Nagpur 2010, Kanpur 2011]

Sol. Re-writing the given equation becomes $\qquad r-x^2t=0.$ ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have $R=1,\ S=0,\ T=-x^2$.

Now, the λ-quadratic $R\lambda^2+S\lambda+T=0$ gives $\lambda^2-x^2=0$ so that $\lambda=\pm x$.

$\therefore$ Here $\quad \lambda_1=x \quad$ and $\quad \lambda_2=-x \quad$ (Real and disticnt roots)

Hence characterstic equations $\quad dy/dx+\lambda_1=0 \quad$ and $\quad dy/dx+\lambda_2=0$

become $\quad dy/dx+x=0 \quad$ and $\quad dy/dx-x=0.$

Integrating these, $\quad y+(x^2/2)=c_1 \quad$ and $\quad y-(x^2/2)=c_2.$

Hence in order to reduce (1) to canonical form, we change x, y, to u, v by taking

$u=y+(x^2/2) \quad$ and $\quad v=y-(x^2/2)$ ...(2)

Now, $\qquad p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=x\dfrac{\partial z}{\partial u}-x\dfrac{\partial z}{\partial v}$, using (2) ...(3)

$\qquad q=\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}=\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$, using (2) ...(4)

$$\therefore\ r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right\}=x\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+1.\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right), \text{ using (3)}$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\ =x^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial^2 v^2}\right)+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

and $\quad t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}\right) = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right) = \dfrac{\partial^2 z}{\partial u^2} + 2\dfrac{\partial^2 z}{\partial u \partial v} + \dfrac{\partial^2 z}{\partial v^2}$, using (4)

Putting the above values of r and t in (1), we get

$$x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $\quad \dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4x^2}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right) \quad$ or $\quad \dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4(u-v)}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$, by (2)

which is the required canonical form of the given equation.

Ex. 7. *Reduce the equation* $(n-1)^2(\partial^2 z/\partial x^2) - y^{2n}(\partial^2 z/\partial y^2) = n y^{2n-1}(\partial z/\partial y)$ *to canonical form, and find its general solution*. **[Delhi Maths. (H) 2000, 01, 05; Himanchal 2004; Ravishankar 2004]**

Sol. Given $\quad (n-1)^2 r - y^{2n}\, t - n\, y^{2n-1}\, q = 0.$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = (n-1)^2$, $S = 0$, $T = -y^{2n}$.

Now, the λ—quadratic $R\lambda^2 + S\lambda + T = 0$ gives

$(n-1)^2\lambda^2 - y^{2n} = 0$ so that $\lambda = \pm (n-1)^{-1} y^n$.

$\therefore$ Here $\lambda_1 = (n-1)^{-1} y^n$ and $\lambda_2 = -(n-1)^{-1} y^n$.

Hence, characterstic equations $dy/dx + \lambda_1 = 0$ and $dy/dx + \lambda_2 = 0$

become $dy/dx + (n-1)^{-1} y^n = 0$ and $dy/dx - (n-1)^{-1} y^n = 0$.

Integrating these, $x - y^{-n+1} = c_1$ and $x + y^{-n+1} = c_2$.

Hence in order to reduce (1) to canonical form, we change x, y to u, v by taking

$u = x - y^{-n+1}$ and $v = x + y^{-n+1}$. ...(2)

$\therefore\ p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$ so that $\dfrac{\partial}{\partial x} \equiv \dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}$,

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = (n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right),$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{(n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)^2 y^{-2n}\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

Substituting the above values of r, t, q in (1) and simplifying, we obtain

$$\partial^2 z / \partial u \partial v = 0, \qquad \text{...(3)}$$

which is the required canonical form of the given equation

Integrating (3) w.r.t. 'v', $\quad \partial z/\partial u = F(u)$, where $F(u)$ is an arbitrary function of u, ...(4)

Integrating (4) w.r.t. 'u', $\quad z = G(u) + H(v)$,

where $G(u) = \int F(u)\, du$ and $G(u)$, $H(v)$ are arbitrary functions

Using (2), the solution of the given equation is $\quad z = G(x - y^{-n+1}) + H(x + y^{-n+1})$.

Ex. 8. *Reduce the equation* $(y-1)\,r-(y^2-1)s+y(y-1)t+p-q=2ye^{2x}(1-y)^3$ to *canonical form and hence solve it.* **[Delhi B.Sc. (Hons) III 2008; Rohilkhand 1992]**

Sol. Given $$(y-1)\,r-(y^2-1)s+y(y-1)t+p-q-2ye^{2x}(1-y)^3=0. \qquad ...(1)$$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$$R=y-1, \qquad S=-(y^2-1) \qquad \text{and} \qquad T=y(y-1). \qquad ...(2)$$

$\therefore$ The λ–quardratic $R\lambda^2-S\lambda+T=0$ gives

$(y-1)\lambda^2-(y^2-1)\lambda+y(y-1)=0 \quad \Rightarrow \quad \lambda_1=1$ and $\lambda_2=y$ (real and distinct roots)

Hence characterstic equations $(dy/dx)+\lambda_1=0$ and $(dy/dx)+\lambda_2=0$ become

$(dy/dx)+1=0$ and $(dy/dx)+y=0$.

Integrating these, $x+y=c_1$ and $y\,e^x=c_2$.

To reduce (1) to canonical form, we change the independent variables x, y, to new independent variables u, v by taking

$$u=x+y \qquad \text{and} \qquad v=y\,e^x. \qquad ...(3)$$

$$\therefore \qquad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v}=\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}, \text{ by (3)} \qquad ...(4)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}, \text{ by (3)} \qquad ...(5)$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2v\frac{\partial^2 z}{\partial u\partial v}+v^2\frac{\partial^2 z}{\partial v^2}+v\frac{\partial z}{\partial v}, \text{ by (4)}$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)+e^x\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}=\frac{\partial^2 z}{\partial u^2}+(e^x+v)\frac{\partial^2 z}{\partial u\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial z}{\partial y}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Substituting the above values in (1) and simplifying, we have

$$(1-y)^3e^x\frac{\partial^2 z}{\partial u\partial v}=2y\,e^{2x}(1-y)^3 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial u\partial v}=2v, \qquad ...(6)$$

which is the canonical form of (1).

Integrating (6) w.r.t. 'v', $\quad \partial z/\partial u=v^2+\phi(u)$, $\phi(u)$ being an arbitrary function $\qquad ...(7)$

Integrating (7) w.r.t. 'u', $\quad z=uv^2+\phi_1(u)+\phi_2(v)$, where $\phi_1(u)=\int\phi(u)\,du$

$\therefore$ Using (3) $\quad z=(x+y)y^2e^{2x}+\phi_1(x+y)+\phi_2(ye^x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 9. *Solve* $x^2(y-1)\,r-x\,(y^2-1)\,s+y(y-1)\,t+xyp-q=0$.

Sol. Given $$x^2(y-1)\,r-x(y^2-1)\,s+y(y-1)t+xyp-q=0. \qquad ...(1)$$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$$R=x^2(y-1), \qquad S=-x\,(y^2-1) \qquad \text{and} \qquad T=y\,(y-1).$$

$\therefore$ λ–quadratic $R\lambda^2+S\lambda+T=0$ reduces to

$x^2(y-1)\,\lambda^2-x(y^2-1)\,\lambda+y(y-1)=0 \quad \Rightarrow \quad \lambda_1=y/x$ and $\lambda_2=1/x$ (real and distinct)

So characterstic equations $(dy/dx)+\lambda_1=0$ and $(dy/dx)+\lambda_2=0$ become

$$(dy/dx)+(y/x)=0 \quad \text{and} \quad (dy/dx)+(1/x)=0$$

Integrating these, $xy=c_1$ and $xe^y=c_2$ so for canonical form, we take

$$u=xy \quad \text{and} \quad v=xe^y. \qquad ...(2)$$

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=y\frac{\partial z}{\partial u}+e^y\frac{\partial z}{\partial v}, \text{ by (2)} \qquad ...(3)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v}, \text{ by (2)} \qquad ...(4)$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u}+e^y\frac{\partial z}{\partial v}\right)=y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$=y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]=y^2\frac{\partial^2 z}{\partial u^2}+2ye^x\frac{\partial^2 z}{\partial u\partial v}+e^{2y}\frac{\partial^2 z}{\partial y^2},$$

$$s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v}\right)=\frac{\partial z}{\partial u}+x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^y\frac{\partial z}{\partial v}+xe^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial z}{\partial u}+e^y\frac{\partial z}{\partial v}+x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=\frac{\partial z}{\partial u}+e^y\frac{\partial z}{\partial v}+xy\frac{\partial^2 z}{\partial u^2}+(yxe^y+e^yx)\frac{\partial^2 z}{\partial u\partial v}+xe^{2y}\frac{\partial^2 z}{\partial v^2}$$

and
$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v}\right)=x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+xe^y\frac{\partial z}{\partial v}+x\,e^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+xe^y\frac{\partial z}{\partial v}+xe^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=x^2\frac{\partial^2 z}{\partial u^2}+2x^2e^y\frac{\partial^2 z}{\partial u\partial v}+x^2e^{2y}\frac{\partial^2 z}{\partial y^2}+xe^y\frac{\partial z}{\partial v}.$$

Substituting the above values in (1) and simplifying, we get $\partial^2 z/\partial u\partial v=0,$ ... (5)

which is canonical form of (1).

Integrating (5) w.r.t. 'u', $\partial z/\partial v=\phi(v)$, $\phi(v)$ being an arbitrary function.

Integrating it w.r.t. 'v', $z=\phi_1(v)+\phi_2(u)$, where $\phi_1(v)=\int\phi(v)\,dv$.

$\therefore$ $z=\phi_1(xe^y)+\phi_2(xy)$, by (2). This is the required solution, ϕ_1,ϕ_2 being arbitrary functions

Ex. 10. *Solve* (*i*) $xyr-(x^2-y^2)s-xyt+py-qx=2(x^2-y^2)$. **[Delhi Maths (H) 2006]**
(*ii*) $x(y-x)r-(y^2-x^2)s+y(y-x)t+(y+x)(p-x)=2x+2y+2$.

Sol. (*i*) Given $xyr-(x^2-y^2)s-xyt+py-qx-2(x^2-y^2)=0$ ...(1)

Comparing (*i*) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have

$$R=xy, \quad S=-(x^2-y^2) \quad \text{and} \quad T=-xy.$$

So λ–quadratic $R\lambda^2+S\lambda+T=0$ becomes $xy\lambda^2-(x^2-y^2)\lambda-xy=0$ giving $\lambda=-y/x, x/y$.

$$\therefore \quad \frac{dy}{dx}+\lambda_1=0 \quad \text{and} \quad \frac{dy}{dx}+\lambda_2=0 \quad \Rightarrow \quad \frac{dy}{dx}-\frac{y}{x}=0 \quad \text{and} \quad \frac{dy}{dx}+\frac{y}{x}=0.$$

Integrating, $y/x=c_1$, and $x^2+y^2=c_2$. So, we take

$$u=y/x \quad \text{and} \quad v=x^2+y^2. \qquad ...(2)$$

$\therefore$ Proceeding as usual, we obtain

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \left(-\frac{y}{x^2}\right)\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}, \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{x}\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v},$$

$$r = \left(-\frac{y}{x^2}\right)^2 \frac{\partial^2 z}{\partial u^2} + 2\times(2x)\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial v \partial u} + 4x^2\frac{\partial^2 z}{\partial v^2} + \frac{2y}{x^3}\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}$$

$$s = \left(-\frac{y}{x^2}\right)\left(\frac{1}{x}\right)\frac{\partial^2 z}{\partial u^2} + \left\{2y\left(-\frac{y}{x^2}\right) + 2x \times \frac{1}{x}\right\}\frac{\partial^2 z}{\partial u \partial v} + 4xy\frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2}\frac{\partial z}{\partial u}$$

and
$$t = \left(\frac{1}{x}\right)^2 \frac{\partial^2 z}{\partial u^2} + 2\times\frac{1}{x}\times(2y)\frac{\partial^2 z}{\partial u \partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial v}.$$

Substituting these in (1) we get

$$(x^2+y^2)^2\frac{\partial^2 z}{\partial u \partial v} = (y^2-x^2)x^2 \quad \text{or} \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{(y^2-x^2)x^2}{(x^2+y^2)^2} = \frac{u^2-1}{(u^2+1)^2}, \text{ by (2)} \quad ...(3)$$

Integrating (3) w.r.t. 'u', we have

$$\frac{\partial z}{\partial v} = \int \frac{u^2-1}{(u^2+1)^2}du + \phi(v) = \int \frac{du}{u^2+1} - 2\int \frac{du}{(u^2+1)^2} + \phi(v) \qquad ...(4)$$

We have,
$$\int 1\cdot\frac{1}{u^2+1}du = u\times\frac{1}{u^2+1} - \int u \times\left(\frac{-2u}{(u^2+1)^2}\right)du, \text{ integrating by parts}$$

or
$$\int \frac{du}{u^2+1} = \frac{u}{u^2+1} + 2\int \frac{(u^2+1)-1}{(u^2+1)^2}du = \frac{u}{u^2+1} + 2\int\frac{du}{u^2+1} - 2\int\frac{du}{(u^2+1)^2}$$

Then,
$$\int \frac{du}{u^2+1} - 2\int\frac{du}{(u^2+1)^2} = -\frac{u}{u^2+1} \qquad ...(4)$$

Using (5), (4) gives $\quad \partial z/\partial v = -u/(u^2+1) + \phi(v)$, $\phi(v)$ being an arbitrary function ...(6)

Integrating (6) w.r.t. v, $\quad z = -(uv)/(u^2+v^2) + \phi_1(v) + \phi_2(u)$, where $\phi_1(v) =$

$\int \phi(v)\,dv$

$\therefore$ Using (2), $z = -xy + \phi_1(x^2+y^2) + \phi_2(y/x)$, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) **Hint.** Since $R = x(y-x)$, $S = -(y^2-x^2)$, $T = y(y-x)$, so here $\lambda_1 = y/x$, $\lambda_2 = 1$.

So we get $(dy/dx) + (y/x) = 0$ and $(dy/dx) + 1 = 0$ as characteristic equations

These give $\quad xy = c_1 \quad$ and $\quad x + y = c_2$. Hence take

$u = xy \quad$ and $\quad v = x + y$. ...(1)

As usual, $\quad p = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad$ and $\quad q = x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$,

$$r = y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}, \quad t = x^2\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}, \quad s = xy\frac{\partial^2 z}{\partial u^2} + (x+y)\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u}.$$

$\therefore$ Given equation becomes
$$-(y-x)^3\frac{\partial^2 z}{\partial v \partial u} = 2x + 2y + 2 \qquad ...(2)$$

or $$\frac{\partial^2 z}{\partial v \partial u} = -\frac{2(x+y+1)}{(y-x)^3} = -\frac{2(x+y+1)}{[(y+x)^2-4xy]^{3/2}} = \frac{2(v+1)}{(v^2-4u)^{3/2}}, \text{ by (1)}$$

Integrating (2) w.r.t. 'u', we get $$\frac{\partial z}{\partial v} = \frac{v+1}{\sqrt{(v^2-4u)}} + \phi(v). \qquad \text{... (3)}$$

Integrating, (3) w.r.t. v, $$z = \sqrt{(v^2-4u)} + \log\,[v + \sqrt{(v^2-4u)}\,] + \phi_1(v) + \phi_2(u)$$

or $z = x - y + \log(2x) + \phi_1(x+y) + \phi_2(xy)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 11. *Solve* (*i*) $y(x+y)(r-s) - xp - yq - z = 0$ **[Delhi Maths (H) 1998]**

(*ii*) $xys - x^2 r - px - qy + z = -2xy^2 y$.

Sol. (*i*) Given $$y(x+y)\,r - y\,(x+y)s - xp - yq - z = 0. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, $R = y\,(x+y)$, $S = -y\,(x+y)$, $T = 0$.

So, the λ–quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$y(x+y)\lambda^2 - y(x+y)\lambda = 0$, giving $\lambda = 0, 1$. Thus $\lambda_1 = 1$ and $\lambda_2 = 0$ and so

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} + 1 = 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Integrating these, $x + y = c_1$, and $y = c_2$.

So we take $u = x + y$ and $v = y$ ...(2)

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}, \text{ by (2)} \qquad \text{...(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{....(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2}, \text{ by (3)} \qquad \text{...(5)}$$

$$s = \frac{\partial^2 z}{\partial v \partial u} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}, \text{ using (3) and (4)} \qquad \text{...(6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values in (1), we have

$$y(x+y)\left(-\frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial z}{\partial u} - y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0 \quad \text{or} \quad uv\frac{\partial^2 z}{\partial v \partial u} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} + z = 0.$$

or $$\frac{\partial^2 z}{\partial v \partial u} + \frac{1}{v}\frac{\partial z}{\partial u} + \frac{1}{u}\frac{\partial z}{\partial v} + \frac{1}{uv}z = 0 \quad \text{or} \quad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0. \quad \text{... (7)}$$

Let $\partial z/\partial v + (z/v) = w$. ...(8)

Then, the above equation (7) becomes $\partial w/\partial u + w/u = 0$.

Integrating, $wu = \phi(v)$ or $w = (1/u) \times \phi(v)$.

Substituting this value of w in (8), we have $$\frac{\partial z}{\partial v} + \frac{1}{v}z = \frac{1}{u}\phi(v), \qquad \text{...(9)}$$

I.F. of (9) $= e^{\int (1/v)dv} = v$ and solution of (9) is

$$zv = \frac{1}{u}\int \phi(v)dv + \phi_2(u) \quad \text{or} \quad z = \frac{1}{uv}\phi_1(v) + \frac{1}{v}\phi_2(u), \text{ where } \phi_1(v) = \int \phi(v)\,dv$$

or $\qquad z = \dfrac{1}{y(x+y)}\,\phi_1(y) + \dfrac{1}{y}\phi_2(x+y)$, by (2); ϕ_1, ϕ_2 being arbitrary functions

(*ii*) **Hint.** Given $\qquad xys - x^2r - px - qy + z = -2x^2y. \qquad$ ...(1)

Here, $R = -x^2$, $S = xy$, $T = 0$ and λ–quadratic is $\qquad -x^2\lambda^2 + xy\lambda = 0$

so that $\lambda_1 = y/x$ and $\lambda_2 = 0$. Hence, characterstic equations

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Integrating these, $\quad xy = c_1, \quad y = c_2$. $\quad$ So we take $\quad u = xy \quad$ and $\quad v = y. \qquad$ ...(2)

Then, $\qquad p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u}y = v\dfrac{\partial z}{\partial u}$, by (2) $\qquad$ ...(3)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\,\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{...(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = v\frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right) = v^2\,\frac{\partial^2 z}{\partial u^2}, \text{ by (3)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = v\frac{\partial}{\partial u}\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = v\left(\frac{1}{v}\frac{\partial z}{\partial u} + \frac{u}{v}\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right), \text{ by (3) and (4)}$$

Substituting these values in (1), we have

$$xy\left(\frac{\partial z}{\partial u} + u\frac{\partial^2 z}{\partial u^2} + v\frac{\partial^2 z}{\partial u \partial v}\right) - x^2v^2\frac{\partial^2 z}{\partial u^2} - v\frac{\partial z}{\partial u}x - y\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + z = -2x^2y$$

or $$u\frac{\partial z}{\partial u} + u^2\frac{\partial^2 z}{\partial u^2} + uv\frac{\partial^2 z}{\partial u \partial v} - u^2\frac{\partial^2 z}{\partial u^2} - u\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} + z = -2(u^2/v^2)v, \text{ by (2)}$$

or $$uv\frac{\partial^2 z}{\partial u \partial v} - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} + z = -\frac{2u^2}{v} \qquad \text{or} \qquad \frac{\partial^2 z}{\partial u \partial v} - \frac{1}{v}\frac{\partial z}{\partial u} - \frac{1}{u}\frac{\partial z}{\partial v} + \frac{z}{uv} = -\frac{2u}{v^2}.$$

or $$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} - \frac{z}{v}\right) - \frac{1}{u}\left(\frac{\partial z}{\partial v} - \frac{z}{v}\right) = -\frac{2u}{v^2}. \qquad \text{...(5)}$$

Let $\qquad \partial z/\partial v - z/v = w. \qquad$ ...(6)

Then (5) becomes $\dfrac{\partial w}{\partial u} - \dfrac{1}{u}w = -\dfrac{2u}{v^2}$, which is linear differential equation $\qquad$ ...(7)

I.F. of (7) $= e^{-\int (1/u)\,du} = e^{-\log u} = e^{\log u^{-1}} = (1/u)$ and so its solution is

$$\frac{w}{u} = -\int\left(\frac{2u}{v^2} \times \frac{1}{u}\right)du = -\frac{2u}{v^2} + \phi(v) \qquad \text{or} \qquad w = -\frac{2u^2}{v^2} + u\,\phi(v)$$

Substituting this value of w in (6), we get $\qquad \dfrac{\partial z}{\partial v} - \dfrac{1}{v}z = -\dfrac{2u^2}{v^2} + u\phi(u).$

Its I.F. $= e^{-\int (1/v)\,dv} = e^{-\log v} = e^{\log v^{-1}} = (1/v)$ and so its solution is

$$\frac{z}{v} = \int \frac{1}{v}\left[-\frac{2u^2}{v^2} + u\,\phi(v)\right]dv = \frac{u^2}{v^2} + u\psi(v) + \phi_2(u)$$

or $\quad z = (u^2/v) + uv\psi(v) + v\phi_2(u) = (u^2/v) + u\phi_1(v) + v\phi_2(u)$ or $\quad z = x^2y + xy\phi_1(y) + y\phi_2(xy)$, by (2).

Ex. 12. *Solve* $x^2r - y^2t + px - qy = x^2$. $\qquad$ **[Kurukshetra 2003; Delhi Maths (H) 1998]**

Sol. Given $\qquad x^2r - y^2t + (px - qy - x^2) = 0. \qquad$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we get

$R = x^2$, $\qquad S = 0 \qquad$ and $\qquad T = -y^2$. $\quad$ ...(2)

Now, the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ and (2) give

$x^2\lambda^2 - y^2 = 0$ so that $\lambda = \pm\, y/x$. (real and distinct roots)

Take $\lambda_1 = y/x$ and $\lambda_2 = -y/x$.

Hence characterstic equations $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$

become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

or $(1/x)dx + (1/y)dy = 0$ and $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x + \log y = \log c_1$ and $\log x - \log y = \log c_2$

or $xy = c_1$ and $x/y = c_2$.

To reduce (1) to canonical form, we change the independent variables x, y to new independent variables u, v by taking

$$u = xy \quad \text{and} \quad v = x/y. \qquad \text{...(3)}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}, \text{ using (3).} \qquad \text{...(4)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}, \text{ using (3).} \qquad \text{...(5)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= y\left(\frac{\partial^2 z}{\partial u^2} \times y + \frac{\partial^2 z}{\partial v \partial u} \times \frac{1}{y}\right) + \frac{1}{y}\left(\frac{\partial^2 z}{\partial v \partial u} \times y + \frac{\partial^2 z}{\partial v^2} \times \frac{1}{y}\right), \text{ using (3)}$$

$$\therefore \quad r = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}. \qquad \text{...(6)}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - \left[-\frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x}{y^2}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2} \times x + \frac{\partial^2 z}{\partial v \partial u} \times \left(-\frac{x}{y^2}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial^2 z}{\partial u \partial v} \times x + \frac{\partial^2 z}{\partial v^2} \times \left(-\frac{x}{y^2}\right)\right]$$

$$\therefore \quad t = x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}. \qquad \text{...(7)}$$

Substituting the values of r, t, p and q given by (6), (7) (3) and (4) in (1), we obtain

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial u^2}\right) - y^2\left(x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right)$$

$$+ x\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) - y\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) - x^2 = 0$$

or $4x^2\dfrac{\partial^2 z}{\partial u \partial v} = x^2$ so that $\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial v}\right) = \dfrac{1}{4}$, ...(8)

which is the canonical form of (1).

Now, integrating (8) w.r.t. 'u', $\partial z/\partial u = (u/4) + f(v)$. ...(9)

Integrating (9) w.r.t. 'v', $z = (uv)/4 + \int f(v)\,dx + \phi(u)$

or $z = (uv)/4 + \psi(v) + \phi(u)$, where $\psi(v) = \int f(v)\,dv$

or $z = x^2/4 + \psi(x/y) + \phi(xy)$, which is the required solution, ϕ, ψ being arbitrary functions.

Ex. 13. *(a) Reduce* $x^2\,(\partial^2 z/\partial x^2) - y^2\,(\partial^2 z/\partial y^2) = 0$ *to canonical form and hence solve it.*

(b) Reduce $y^2(\partial^2 z/\partial x^2) - x^2(\partial^2 z/\partial y^2) = 0$ *to canonical form.*

Sol. (*a*) Re-writing the given equation, $x^2\,r - y^2\,t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2 y^2 > 0$ for $x \neq 0$, $y \neq 0$ and hence (1) is hyperbolic. The λ-quadrate equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ so that $\lambda = y/x,\ -y/x$ and hence the corresponding characteristic equations become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

Integrating these, $xy = c_1$ and $x/y = c_2$

In order to reduce (1) to its canonical form, we choose $u = xy$ and $v = x/y$ ... (2)

Now, doing exactly as in solved Ex. 12, we get

$$r = y^2\,\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} \quad \text{and} \quad t = x^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$$

Putting these values of r and t in (1), we get

$$x^2\left(y^2\,\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right) - y^2\left(x^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\,\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $$4x^2\,\frac{\partial^2 z}{\partial u \partial v} - \frac{2x}{y}\,\frac{\partial z}{\partial v} = 0 \quad \text{or} \quad 2xy\,\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} = 0$$

or $$2u\,(\partial^2 z/\partial u \partial v) - (\partial z/\partial v) = 0, \text{ using (2).} \quad \text{... (3)}$$

This is the required canonical form of (1).

We now proceed to find solution of (1). Multiplying both sides of (3) by v, we get

$$2uv\,\frac{\partial^2 z}{\partial u\,\partial v} - v\,\frac{\partial z}{\partial v} = 0 \quad \text{or} \quad (2uv\,DD' - vD')z = 0 \quad \text{... (4)}$$

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial v$. We now reduce (4) to a linear equation with constant coefficients by usual method (refer Art. 6.3 of chapter 6).

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $y = \log v$ ... (5)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$. Then (4) reduces to

$$(2D_1 D_1' - D_1')z = 0 \quad \text{or} \quad D_1'\,(2D_1 - 1)\,z = 0$$

Its general solution is given by (use Art. 5.6 of chapter 5)

$$z = e^{X/2}\,\phi_1(Y) + \phi_2(X) = u^{1/2}\phi_1(\log v) + \phi_2(\log u) = u^{1/2}\psi_1(v) + \psi_2(u), \text{ using (5)}$$

$$= (xy)^{1/2}\psi_1(x/y) + \psi_2(xy) = x(y/x)^{1/2}\,\psi_1(x/y) + \psi_2(xy) = xf(x/y) + \psi_2(xy), \text{ using (2)}$$

where f and ψ_2 are arbitrary functions

(*b*) Try yourself. Choose $u = (y^2 - x^2)/2$, $v = (y^2 + x^2)/2$.

Ans. $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{2(u^2 - v^2)}\left(v\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial v}\right).$$

Ex. 14. *Reduce the equation* $x(xy-1)r-(x^2y^2-1)s+y(xy-1)t+(x-1)p+(y-1)q=0$ *to canonical form and hence solve it.*

Sol. Comparing the given equation with $\quad Rr+Ss+Tt+f(x,y,z,p,q)=0,$

here, $\quad R=x(xy-1), \qquad S=-(x^2y^2-1), \qquad T=y(xy-1). \qquad ...(1)$

Now, the λ-quadratic equation $R\lambda^2+S\lambda+T=0$ and (1) give

$x(xy-1)\lambda^2-(x^2y^2-1)\lambda+y(xy-1)=0 \qquad$ or $\qquad x\lambda^2-(xy+1)\lambda+y=0$

or $\quad (x\lambda-1)(\lambda-y)=0 \quad$ so that $\quad \lambda=1/x, \quad y.\quad$ Take $\quad \lambda_1=1/x \quad$ and $\quad \lambda_2=y.$

Hence characterstic equations $\quad (dy/dx)+\lambda_1=0 \quad$ and $\quad (dy/dx)+\lambda_2=0$

become $\quad (dy/dx)+(1/x)=0 \quad$ and $\quad (dy/dx)+y=0$

or $\quad dy+(1/x)dx=0 \quad$ and $\quad (1/y)dy+dx=0. \qquad ...(2)$

Intergrating (2), $\quad y+\log x=\log c_1 \quad$ and $\quad \log y+x=\log c_2$

or $\quad \log e^y+\log x=\log c_1 \quad$ and $\quad \log y+\log e^x=\log c_2$

$\quad x\,e^y=c_1 \quad$ and $\quad y\,e^x=c_2.$

To reduce the given equation to canonical form, we change the independent variables x, y to new independent variables u, v, by taking

$u=x\,e^y \qquad$ and $\qquad v=y\,e^x. \qquad ...(3)$

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^y\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v}, \text{ using (3)} \qquad ...(4)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}, \text{ using (3).} \qquad ...(5)$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(e^y\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+ye^x\frac{\partial z}{\partial v}+y\,e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+y\,e^x\frac{\partial z}{\partial v}+ye^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+y\,e^x\frac{\partial z}{\partial v}+y\,e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$\therefore \quad r=e^{2y}\frac{\partial^2 z}{\partial u^2}+2y\,e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y^2\,e^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{\partial z}{\partial v}.$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\,e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial z}{\partial u}+x\,e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial z}{\partial v}+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\frac{\partial z}{\partial u}+xe^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$=x\,e^{2y}\frac{\partial^2 z}{\partial u^2}+(xy+1)\,e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y\,e^{2x}\frac{\partial^2 z}{\partial v^2}+e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}$$

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\,e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial u}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial^2 z}{\partial u^2}xe^y+\frac{\partial^2 z}{\partial u\partial v}e^x\right]+e^x\left[\frac{\partial^2 z}{\partial u\partial v}xe^y+\frac{\partial^2 z}{\partial v^2}e^x\right],$$

$$\therefore \qquad t = x^2 e^{2y}\frac{\partial^2 z}{\partial u^2} + 2x\, e^{x+y}\frac{\partial^2 z}{\partial u \partial v} + x\, e^{y}\frac{\partial z}{\partial u} + e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Putting the above values of r, s, t, p, q in the given equation and simplifying, we obtain the required canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) = 0. \qquad ...(6)$$

Integrating (6) w.r.t. 'v', $\partial z/\partial u = f(u)$, f being an arbitrary function ...(7)

Integrating (7) w.r.t. 'u', $z = \int f(u)\,du + \psi(v)$ or $z = \phi(u) + \psi(v)$, where $\phi(u) = \int f(u)\,du$.

Using (3), the required solution is $z = \phi(xe^y) + \psi(ye^x)$, ϕ and ψ being arbitrary functions.

Ex. 15. (*a*) *Reduce the one-dimensional wave equation* $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *(c > 0) to canonical form and hence find its general solution.*

(b) Find the D'Alembert's solution of the Cauchy's problem: $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *(c > 0) satisfying z (x, 0) = f (x) and* $z_t(x,0) = g(x)$ *where f (x) and g (x) are given functions representing the initial displacement and initial velocity, respectively. Also,* $z_t = \partial z/\partial t$

Sol. (*a*) Given $\partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$, $c > 0$. ... (1)

To re-write (1), put $y = ct$, ... (2)

Then, (1) reduces to $\partial^2 z/\partial x^2 - (\partial^2 z/\partial y^2) = 0$ or $r - t = 0$ ... (3)

Proceed now exactly as in solved Ex. 1 to reduce (3) to its canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = 0 \qquad ... (4)$$

where $u = y + x$, $v = y - x$ or $u = ct + x$ and $v = ct - x$. ... (5)

Integrating (4) w.r.t. 'u', $\partial z/\partial v = f(v)$, where f is an arbitrary function ... (6)

Integrating (6) w.r.t. 'v', $z = \int f\ (v)\ dv + \psi(u) = F(v) + \psi(u)$, where $f(v) = \int f(v)\,dv$

or $$z(x,t) = F(ct - x) + \psi\ (ct + x), \text{ using (5)}$$

or $$z(x,t) = \phi(x - ct) + \psi(x + ct), \qquad ... (7)$$

where we take $\phi\ (x - ct) = F\ (ct - x)$ and ϕ, ψ as arbitrary functions.

(7) is the required general solution of (1).

(***b***) We are to solve $\partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$ ... (*i*)

subject to the conditions $z(x,0) = f(x)$ ... (*ii*)

and $(\partial z/\partial t)_{t=0} = g(x)$ ... (*iii*)

Proceed exactly as in part (*a*) and get solution of (*i*) as

$$z(x,t) = \phi(x - ct) + \psi(x + ct) \qquad ... (iv)$$

Differentiating (*iv*) partially w.r.t. 't', we get

$$\partial z/\partial t = -c\ \phi'(x - ct) + c\ \psi'(x + ct) \qquad ... (v)$$

where dash denotes the derivative w.r.t. the argument. Putting $t = 0$ in (*iv*) and (*v*) and using (*ii*) and (*iii*) respectively, we get $\phi(x) + \psi(x) = f(x)$ ... (*vi*)

and $-c\ \phi'(x) + c\,\psi'(x) = g(x)$ ... (*vii*)

Integrating (*vii*), $-c\ \phi(x) + c\ \psi\ (x) = \int_a^x g(u)\ du$, ... (*viii*) where a is an arbitrary constant. Solving (vi) and ($viii$) for $\phi(x)$ and $\psi(x)$, we have

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_a^x g(x)\, du, \qquad \text{and} \qquad \psi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_a^x g(u)\, du$$

so that
$$\phi\,(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_a^{x-ct} g(u)\, du \quad \text{... } (ix)$$

and
$$\psi\,(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_a^{x+ct} g(u)\, du \quad \text{... } (x)$$

Using (ix) and (x) in (iv), we get the required so called D'Alembert's solutiion of the Cauchy problem (which represents the vibrations of an infinite string in the present problem)

$$z(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{a} g(u)\, du + \int_a^{x+ct} g(u)\, dx\right]$$

or
$$z(x,t) = \frac{1}{2}\,\{f\,(x-ct) + f(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du \quad \text{... } (xi)$$

Particular Case I. If in the above problem, we take $g\,(x) = 0$ so that the initial velocity of the string is zero, then (xi) reduces to

$$z(x,t) = \{f(x-ct) + f(x+ct)\}/2\,,$$

where $f\,(x - ct)$ represents a right travelling wave travelling with the speed c (along OX) and $f\,(x + ct)$ represents a left travelling wave travelling with the speed c.

Particular case II. If $f\,(x) = \sin x$ and $g\,(x) = \cos x$ in the above problem, then the corresponding solution (xi) reduces to

$$z(x,t) = \frac{1}{2}\{\sin\,(x-ct) + \sin(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos u\; du$$

or $z(x,t) = \sin x \cos ct + (1/2c) \times \{\sin(x+ct) - \sin(x-ct)\}$ or $z(x,t) = \sin x \cos ct + (1/c) \times \cos x \sin ct$.

Particular case III. If $f(x) = \sin x$ and $g(x) = x^2$, then (xi) gives

$z(x,\ t) = \sin x \cos ct + x^2 t + (c^3 t^3)/3$, on simplification.

8.8 Working rule for reducing a parabolic equation to its canonical form.

Step 1. Let the given equation $\quad Rr + Ss + Tt + f(x,y,z,p,q) = 0 \quad$ ... (1)

be parabolic so that $\quad S^2 - 4RT = 0$.

Step 2. Write λ-quadratic equation $\quad R\lambda^2 + S\lambda + T = 0 \quad$ ... (2)

Let λ_1, λ_1 be two equal roots of (2)

Step 3. Write the characteristic equation corresponding to $\lambda = \lambda_1$, *i.e.*, $(dy/dx) + \lambda_1 = 0$

Solving it, we get $\quad f_1(x,y) = C_1$, C_1 being an arbitrary constant ... (3)

Step 4. Choose $\quad u = f_1(x,y) \quad$ and $\quad v = f_2(x,y) \quad$ ... (4)

where $f_2\,(x, y)$ is an arbitrary function of x and y and is independent of $f_1\,(x, y)$. For this verify that Jacobian J of u and v given by (4) is non-zero,

i.e.
$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0 \quad \text{... } (5)$$

Step 5. Using relations (4), find p, q, r, s and t in terms of u and v as shown in Art. 8.5.

Step 6. Substituting the values of p, q, r, s and t obtained in step (1) and simplifying we get the following canonical forms of (1)

$$\partial^2 z/\partial u^2 = \phi\,(u, v, z, \partial z/\partial u,\ \partial z/\partial v) \qquad \text{or} \qquad \partial^2 z/\partial v^2 = \phi(u, v, z, \partial z/\partial u,\ \partial z/\partial v)$$

8.9 SOLVED EXAMPLES BASED ON ART. 8.8

Ex. 1. *Reduce the equation* $\partial^2 z/\partial x^2 + 2(\partial^2 z/\partial x\partial y) + \partial^2 z/\partial y^2 = 0$ *to canonical form and hence solve it.* **[Delhi Maths (H) 2000, 06; 08; Jabalpur 2004; Delhi Maths (Prog) II 2008; Delhi B.Sc. (Prog) II 2008, 11; Himanchal 2001; 05 Rajasthan 2003; Lucknow 2010]**

Sol. Re-writing the given equation, we get $\qquad r + 2s + t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2$, $T = 1$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadrate equation reduces to $\quad \lambda^2 + 2\lambda + 1 = 0 \quad$ so that $\quad \lambda = -1, -1$ (equal roots).

The corresponding characteristic equation is $\quad (dy/dx) - 1 = 0 \quad$ or $\quad dx - dy = 0$

Integrating, $\quad x - y = c$, c being an arbitrary constant.

Choose $\quad u = x - y \quad$ and $\quad v = x + y, \quad$ ... (2)

where we have chosen $v = x + y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1\cdot 1 + 1\cdot 1 = 2 \neq 0\,.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

From (3) and (4), $$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \text{... (5)}$$

$\therefore$ $$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4) and (5)}$$

$$= -\frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 v}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

and $$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4) and (5)}$$

$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (8)}$$

Using (6), (7) and (8) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial v^2} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0 \qquad \text{... (9)}$$

To find the required solution. Integrating (9) partially w.r.t. 'v', we get

$$\partial z / \partial v = \phi(u), \ \phi \text{ being an arbitrary function.} \qquad \text{... (10)}$$

Integrating (10) partially w.r.t 'v', $\quad z = \int \phi(u)\, dv + \psi(u) = v\phi(u) + \psi(u)$

or $\quad z = (x+y)\,\phi(x-y) + \psi(x-y)$, which is the desired solution, ϕ, ψ being arbitrary functions.

Ex. 2. *Reduce the equation* $y^2(\partial^2 z/\partial x^2) - 2xy(\partial^2 z/\partial x\,\partial y) + x^2(\partial^2 z/\partial x^2) = (y^2/x)(\partial z/\partial x) + (x^2/y)(\partial z/\partial y)$ *to canonical form and hence solve it.* [**Nagpur 2005; Delhi Maths (H) 2001, 05, 09; Avadh 2001, Himanchal 2009; Delhi B.Sc. (Prog) II 2007; Meerut 2005, 06, 11; G.N.D.U. Amritsar 2005**]

Sol. Re-writing the given equation, $\quad y^2 r - 2xys + x^2 t - (y^2/x)\,p - (x^2/y)q = 0 \quad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = -2xy$, $T = x^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$y^2\lambda^2 - 2xy\,\lambda + x^2 = 0 \qquad$ or $\qquad (y\lambda - x)^2 = 0 \qquad$ so that $\qquad \lambda = x/y,\ x/y$.

The corresponding characteristic equation is $\qquad dy/dx + x/y = 0$

or $\qquad x\,dx + y\,dy = 0 \qquad$ so that $\qquad x^2/2 + y^2/2 = C_1$

Choose $\qquad u = x^2/2 + y^2/2 \qquad$ and $\qquad v = x^2/2 - y^2/2, \quad$ ... (2)

where we have chosen $v = x^2/2 - y^2/2$ in such a manner that u and v are independent functions as verified below

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -2xy \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right), \text{ using (2)} \qquad \text{... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right] = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ by (4)}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{... (6)}$$

and $$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left\{y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

or $$s = xy\,(\partial^2 z/\partial u^2 - \partial^2 z/\partial v^2) \quad \text{... (7)}$$

Using (3), (4), (5), (6) and (7) in (1) and simplifying, we get

$4x^2y^2\ (\partial^2 z/\partial v^2) = 0$ so that $\partial^2 z/\partial v^2 = 0$, ... (8)

which is the required canonical form.

Integrating (8) partially w.r.t. 'v', $\partial z/\partial v = \phi(u)$, ϕ being arbitrary function. ... (9)

Integrating (9) partially w.r.t. 'v', $z = v\ \phi(u) + \psi(u)$, ψ being arbitrary function.

or $$z = [(x^2 - y^2)/2]\ \phi\{(x^2 + y^2)/2\} + \psi\ \{(x^2 + y^2)/2\}, \text{ using (2)}$$

or $$z = (x^2 - y^2)\ F(x^2 + y^2) + G\ (x^2 + y^2), \text{ F, G being arbitrary functions}$$

Ex. 3. *(a) Reduce* $r + 2xs + x^2 t = 0$ *to canonical form*

(b) Reduce $r - 6s + 9t + 2p + 3q - z = 0$ *to canonical form*

(c) Reduce $r - 2s + t + p - q = 0$ *to canonical form and hence solve it.*

Sol. (*a*) Given $$r + 2xs + x^2 t = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2x$ and $T = x^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + 2\lambda x + x^2 = 0$ or $(\lambda + x^2) = 0$ so that $\lambda = -x, -x$.

The corresponding characteristic equation is $(dy/dx) - x = 0$ or $dy - x\,dx = 0$

Integrating, $y - x^2/2 = c_1$, c_1 being an arbitrary constant. ... (2)

Choose $u = y - x^2/2$ and $v = x$ ... (2)

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial\,(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -1 \neq 0$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}, \text{ using (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= -\frac{\partial z}{\partial u} - x\left(-x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = x^2\frac{\partial^2 z}{\partial u^2} - 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} \quad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} = -x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}, \text{ by (4)} \quad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} = \frac{\partial^2 z}{\partial u^2}, \text{ by (4)} \quad \text{... (7)}$$

Using (5), (6) and (7) in (1), we finally obtain $\partial^2 z/\partial v^2 = \partial z/\partial u$, which is required canonical form.

3. (*b*) Hint. Here $\lambda = 3$, $u = y + 3x$. Choose $v = y$. The canonical form will be

$$\partial^2 z/\partial v^2 = z/9 - (\partial z/\partial u) + (1/3) \times (\partial z/\partial v).$$

3. (c) Hints. Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = \partial z/\partial v$.

Solution is $\qquad z = \phi\,(x + y) + e^y \psi\,(x + y)$, ϕ, ψ being arbitrary functions

Ex. 4. *Reduce the following to canonical form and hence solve*

(*a*) $x^2 r + 2xy s + y^2 t = 0$

(*b*) $r - 4s + 4t = 0$

(*c*) $x^2 r + 2xys + y^2 t + xyp + y^2 q = 0$

(*d*) $2r - 4s + 2t + 3z = 0$.

Sol. (a) Given $\qquad x^2 r + 2xys + y^2 t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$x^2\lambda^2 + 2xy\lambda + y^2 = 0 \qquad$ or $\qquad (x\lambda + y)^2 = 0 \qquad$ giving $\qquad \lambda = -y/x,\ -y/x$.

The corresponding characteristic equation is $\qquad dy/dx - y/x = 0$

or $\quad (1/y)dy - (1/x)dx = 0 \quad$ so that $\quad \log y - \log x = c_1 \quad$ or $\quad y/x = c_1$

Choose $\qquad u = y/x \qquad$ and $\qquad v = y, \qquad$ ... (2)

where we have chosen $v = y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -\frac{y}{x^2} \neq 0.$$

Now,
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\frac{y}{x^2}\frac{\partial z}{\partial u}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial z}{\partial u}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial u} - \frac{y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right), \text{ by (3)}$$

$$= \frac{2y}{x^3}\frac{\partial z}{\partial u} - \frac{y}{x^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{2y}{x^3}\frac{\partial z}{\partial u} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial u^2} \quad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}.$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} - \frac{y}{x^3}\frac{\partial^2 z}{\partial u^2} - \frac{y}{x^2}\frac{\partial^2 z}{\partial u \partial v} \quad \text{... (6)}$$

$$t = \frac{\partial^2 y}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \text{ using (4)}$$

$$= \frac{1}{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + \frac{2}{x}\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial v^2} \quad \text{... (7)}$$

Using (5), (6) and (7) in (1), we finally get as the canonical form $\quad \partial^2 z/\partial v^2 = 0$ ... (8)

Integrating (8) partially w.r.t. 'v', $\quad \partial z/\partial v = \phi(u)$ ... (9)

Integrating (9) partially w.r.t 'v', $\quad z = v\phi(u) + \psi(u)$

or $\quad z = y\phi(y/x) + \psi(y/x), \phi, \psi$ being arbitrary functions.

(*b*) **Hint.** Here $\lambda = 2$, $u = y + 2x$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = 0$ and solution is $z = y\phi(y+2x) + \psi(y+2x)$.

(*c*) **Hint.** Here $\lambda = -y/x$, $u = y/x$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = -(\partial z/\partial v)$ and solution is $z = \phi(y/x) + e^{-y}\psi(y/x)$

(*d*) **Hint.** Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = -(3z/2)$ and solution is $z = e^{(i\sqrt{3}/2)y}\phi(y+x) + e^{-(i\sqrt{3}/2)}\psi(y+x)$,

Ex. 5. *Reduce the following in canonical form and solve them*

(*a*) $r - 2s + t + p - q = e^x(2y-3) - e^y$

(*b*) $r - 2s + t + p - q = e^{x+y}$

Sol. (*a*) Given $\quad r - 2s + t + p - q - e^x(2y-3) + e^y = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$, here $R = 1$, $S = -2$ and $T = 1$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 - 2\lambda + 1 = 0 \quad$ or $\quad (\lambda-1)^2 = 0 \quad$ so that $\quad \lambda = 1, 1$ (equal roots)

So the corresponding characteristic equation is $\quad dy/dx + 1 = 0 \quad$ or $\quad dx + dy = 0$

Integrating it, $\quad x + y = c_1$, c_1 being an arbitrary constant.

Choose $\quad u = x + y \quad$ and $\quad v = y$ ... (2)

where we have chosen $v = y$ in such a manner that u and v are independent functions as verified below

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1 \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2} \text{ by (3)} \qquad \text{... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (4)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (6)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}, \text{ using (4)} \qquad \text{... (7)}$$

Using (2) (3), (4), (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial u^2} - 2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u} - \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = e^{u-v}(2v-3) - e^v$$

or $$\partial^2 z/\partial v^2 - \partial z/\partial v = e^{u-v}(2v-3) - e^v \qquad \text{... (8)}$$

which is the required canonical form of (1) Let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$.

Then (8) can be re-written as $$D'(D'-1)\, z = e^{u-v}(2v-3) - e^v, \qquad \text{... (9)}$$

which is non-honmogeneous linear partial differential equation with constant coefficients. To solve it, we shall use results of chapter 5. Accordingly, we have

$$\text{C.F.} = \phi(u) + e^v\ \psi(u) = \phi(x+y) + e^y \psi(x+y), \text{ by (2)}$$

P.I. corresponding to $e^{u-v}(2v-3)$

$$= \frac{1}{D'(D'-1)} e^{u+(-1)v}(2v-3) = e^{u+(-1)v}\frac{1}{(D'-1)\,(D'-1-1)}(2v-3)$$

$$= (1/2) \times e^{u-v}\ (1-D')^{-1}(1-D'/2)^{-1}(2v-3) = (1/2) \times e^{u-v}\ (1+D'+\ldots)(1+D'/2+\ldots)\ (2v-3)$$

$$= (1/2) \times e^{u-v}(1+3D'/2+\ldots)\ (2v-3) = (1/2) \times e^{u-v}\ (2v-3+3) = v\ e^{u-v} = y\ e^{x+y-y} = ye^x, \text{ using(2)}$$

P.I. Corresponding to $(-\,e^v)$

$$= \frac{1}{D'(D'-1)}(-e^v) = -\frac{1}{D'-1}\frac{1}{D'}e^v = -\frac{1}{D'-1}\ (e^v \times 1) = -e^v\frac{1}{D'+1-1}1 = -e^y\frac{1}{D'}1$$

$$= -e^v v = -e^y y, \text{ using (2)}$$

Hence the required general solution is given by $y = \phi\ (x+y) + e^y\ \psi\ (x+y) + y\ e^x - y\ e^y$

or $$y = \phi\ (x+y) + e^y\ \psi\ (x+y) + ye^x - (x+y)\ e^y + x\ e^y$$

or $$y = \phi\ (x+y) + e^y\ \{\phi\ (x+y) + (x+y)\} + y\ e^x + x\ e^y$$

or $$y = \phi\ (x + y) + e^y F\ (x + y) + y\ e^x + x\ e^y,$$

where ϕ and F are arbitrary functions and $F\ (x + y) = \phi\ (x + y) + x + y$

(*b*) Hint. Here $\lambda = 1$, $u = x + y$, choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = \partial z/\partial v + e^u$ and solution is $z = \phi\ (x+y) + e^y \psi(x+y) - y\ e^{x+y}$, ϕ, ψ being arbitrary functions.

Ex. 6. *Reduce the equation $x^2 r - 2xys + y^2 t - xp + 3yq = 8y/x$ to canonical form.*

[Delhi B.Sc. (H) 1999]

Sol. Given $$x^2 r - 2xy\ s + y^2 t - xp + 3yq - 8y/x = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f\,(x, y, y, p, q) = 0$, here $R = x^2$, $S = -2xy$, $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$x^2\lambda^2 - 2xy\lambda + y^2 = 0 \qquad \text{or} \qquad (x\lambda - y)^2 = 0 \qquad \text{so that} \qquad \lambda = y/x,\ y/x.$$

The corresponding characteristic equation is

$$dy/dx + y/x = 0 \qquad \text{or} \qquad (1/y)\ dy + (1/x)\ dx = 0 \qquad \text{so that} \qquad xy = C_1$$

Choose $$u = xy \qquad \text{and} \qquad v = x \qquad \text{... (2)}$$

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -x \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u}, \text{ by (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= y\left(y\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right) + y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}, \qquad \text{.... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial z}{\partial u} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{\partial z}{\partial u} + xy\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial u \partial v} \qquad \text{... (6)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right), \text{ by (4)}$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] = x^2\frac{\partial^2 z}{\partial u^2}, \text{ by (2)} \qquad \text{... (7)}$$

Using (2), (3), (4), (5), (6) and (7) in (1), we have

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-2xy\left(\frac{\partial z}{\partial u}+xy\frac{\partial^2 z}{\partial u^2}+x\frac{\partial^2 z}{\partial u\partial v}\right)+y^2x^2\frac{\partial^2 z}{\partial u^2}-x\left(y\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+3y\,x\frac{\partial z}{\partial u}-\frac{8y}{x}=0$$

or $\quad x^2\dfrac{\partial^2 z}{\partial v^2}--x\dfrac{\partial z}{\partial v}=\dfrac{8y}{x}\quad$ or $\quad v^2\dfrac{\partial^2 z}{\partial v^2}-v\dfrac{\partial z}{\partial v}=\dfrac{8u}{v^2}$, by (2)

or $\quad (v^2D'^2-vD')z=8u/v^2,\quad$ where $\quad D\equiv\partial/\partial u,\quad D'\equiv\partial/\partial v\quad$ ... (8)

As explained in chapter 6, we shall reduce (8) to linear partial differential equation with constant coefficients and then use methods of chapter 5 to solve the resulting equation.

To solve (8), let $\quad u=e^X\quad$ and $\quad v=e^Y\quad$ so that $\quad X=\log u,\quad Y=\log v\quad$ ... (9)

Then (8) becomes $\quad \{D'\,(D'-1)-D'\}z=8e^{X-2y}\quad$ or $\quad D'(D'-2)z=8e^{X-2y}$.

C.F. $=\ \phi(X)+e^{2Y}\psi\,(X)=\phi(\log u)+v^2\psi(\log u)$, using (9)

$\quad = F(u)+v^2G(u)=F(xy)+x^2G(xy)$, using (2)

P.I. $=\ \dfrac{1}{D'(D'-1)}\,8\,e^{X-2Y}\ =\ 8\,e^{X-2Y}\dfrac{1}{(D'-2)\,(D'-2-2)}\cdot 1$

$$=\frac{8e^X}{(e^Y)^2}\times\frac{1}{8}\left(1-\frac{D'}{2}\right)^{-1}\left(1-\frac{D'}{4}\right)^{-1}\cdot 1=\frac{u}{v^2}\left(1+\frac{D'}{2}+...\right)\left(1+\frac{D'}{4}+...\right)\cdot 1=\frac{u}{v^2}\times 1=\frac{xy}{x^2}=\frac{y}{x},\text{ by (2)}$$

$\therefore$ Required solution is $\qquad z=F(xy)+x^2G(xy)+y/x$, F, G being arbitrary functions.

8.10 Working rule for reducing an elliptic equation to its canonical form.

Step 1. Let the given equation $\qquad Rr+Ss+Tt+f(x,y,z,p,q)=0\qquad$ ... (1)

be ellipitic so that $\qquad S^2-4RT<0\cdot\qquad$ ... (2)

Step 2. Write λ quadratic equation $\qquad R\lambda^2+S\lambda+T=0\qquad$ ... (2)

Let roots λ_1,λ_2 of (2) be complex conjugates.

Step 3. Then corresponding charactersitic equations are

$(dy/dx)+\lambda_1=0\qquad$ and $\qquad dy/dx+\lambda_2=0$

Solving these, we shall obtain solutions of the form

$f_1(x,y)+i\ f_2(x,y)=c_1\qquad$ and $\qquad f_1(x,y)-i\ f_2(x,y)=c_2\qquad$ ... (3)

Step 4. Choose $\qquad u=f_1(x,y)+i\,f_2(x,y),\qquad v=f_1(x,y)-i\ f_2(x,y)$

Let α and β be two new real independent variables such that $u=\alpha+i\beta$ and $v=\alpha-i\beta$,

so that $\qquad \alpha=f_1(x,y)\qquad$ and $\qquad \beta=f_2(x,y)\qquad$ ... (4)

Step 5. Using relations (4), find *p*, *q*, *r*, *s* and *t* in terms of α and β (in place of *u* and *v* as we did in Art 8.6 and 8.8 corresponding to the cases of hyperbolic and parabolic equations).

Step 6. Substituting the values of *p*, *q*, *r*, *s* and *t* and relations (4) in (1) and simplifying we shall get the following canonical form of (1)

$$\partial^2 z/\partial\alpha^2+\partial^2 z/\partial\beta^2\ =\ \phi\,(\alpha,\beta,z,\partial z/\partial\alpha,\ \partial z/\partial\beta)\,.$$

8.11 SOLVED EXAMPLES ON ART 8.10

Ex. 1. *Reduce the following partial differential equations to canonical forms:*

$(a)\,\partial^2 z/\partial x^2+x^2(\partial^2 z/\partial y^2)=0$ *or* $\qquad r+x^2t=0$

[Delhi B.Sc. (Prog) II 2010; Delhi B.Sc. (Hons) III 2011; Kanpur 2011]

(*b*) $y^2(\partial^2 z/\partial y^2)+\partial^2 z/\partial x^2 = 0$ **[Delhi Math (Hons.) 1995, 98, 2005]**

Sol. (*a*) Re-writing the given equations, we get $r + x^2 t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$, $T = x^2$ so that $S^2 - 4RT = -4x^2 < 0$, $x \neq 0$, showing that (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + x^2 = 0$ giving $\lambda = ix, -ix$. The corresponding characteristic equations are given by

$$dy/dx + ix = 0 \quad \text{and} \quad dy/dx - ix = 0$$

Integrating, $y + i\,(x^2/2) = c_1$ and $y - i\,(x^2/2) = c_2$.

Choose $u = y + i(x^2/2) = \alpha + i\beta$ and $v = y - i(x^2/2) = \alpha - i\beta$,

where $\alpha = y$ and $\beta = x^2/2$ ... (2)

are now two new independent variables.

Now,
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}, \text{ by (2)} \qquad \text{.... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}, \text{ by (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right), \text{ by (3)}$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right] = \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} \qquad \text{... (5)}$$

and
$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}, \text{ by (4)} \qquad \text{... (6)}$$

Using (5) and (6) in (1) the required canonical form is

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0 \quad \text{or} \quad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta}, \text{ as } \beta = \frac{x^2}{2}.$$

(***b***) Do as in part (a). **Ans.** $\partial^2 z/\partial \alpha^2 + \partial^2 z/\partial \beta^2 = -(1/2\alpha)\times(\partial z/\partial \alpha)$, where $\alpha = y^2/2$, $\beta = x$.

Ex. 2. *Reduce* $y^2(\partial^2 z/\partial x^2) + x^2(\partial^2 z/\partial y^2) = 0$ *to canonical form*

Sol. Re-writing the given equation, we get $y^2 r + x^2 t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = 0$, $T = x^2$ so that $S^2 - 4RT = -4x^2 y^2 < 0$ for $x \neq 0$, $y \neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$y^2\lambda^2 + x^2 = 0$ or $\lambda^2 = -x^2/y^2$ so that $\lambda = ix/y, \quad -ix/y$

The corresponding characteristic equations are

$$dy/dx + ix/y = 0 \quad \text{and} \quad dy/dx - ix/y = 0$$

Integrating, $y^2 + ix^2 = C_1$ and $y^2 - ix^2 = C_2$

Choose $u = y^2 + ix^2 = \alpha + i\beta$ and $v = y^2 - ix^2 = \alpha - i\beta$,

where $\alpha = y^2$ and $\beta = x^2$ ... (2)
are now two new independent variables

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = 2x\frac{\partial z}{\partial \beta}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = 2y\frac{\partial z}{\partial \alpha}, \text{ by (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial \beta}\right) = 2\frac{\partial z}{\partial \beta} + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right), \text{ by (3)}$$

$$= 2\frac{\partial z}{\partial \beta} + 2x\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\} = 2\frac{\partial z}{\partial \beta} + 4x^2\frac{\partial^2 z}{\partial \beta^2} \qquad \text{... (5)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial \alpha}\right) = 2\frac{\partial z}{\partial \alpha} + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= 2\frac{\partial z}{\partial \alpha} + 2y\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right\} = 2\frac{\partial z}{\partial \alpha} + 4y^2\frac{\partial^2 z}{\partial \alpha^2} \qquad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$2y^2\frac{\partial z}{\partial \beta} + 4x^2y^2\frac{\partial^2 z}{\partial \beta^2} + 2x^2\frac{\partial z}{\partial \alpha} + 4x^2y^2\frac{\partial^2 z}{\partial \alpha^2} = 0 \quad \text{or} \quad 2\alpha\beta\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) + \alpha\frac{\partial z}{\partial \beta} + \beta\frac{\partial z}{\partial \alpha} = 0$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}\right) = 0$$

Ex. 3. *Reudce* $\partial^2 z/\partial x^2 + y^2(\partial^2 z/\partial y^2) = y$ *to canonical form.*

Sol. Re-writing the given equation, we get $r + y^2 t - y = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here R = 1, S = 0 and $T = y^2$ so that $S^2 - 4RT = -4y^2 < 0$ for $y \neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + y^2 = 0 \Rightarrow \lambda = iy, -iy$.

The corresponding characteristic equations are given by

$dy/dx + iy = 0$ and $dy/dx - iy = 0$

Integrating these, $\log y + ix = c_1$ and $\log y - ix = c_2$

Choose $u = \log y + ix = \alpha + i\beta$ and $v = \log y - ix = \alpha - i\beta$,

where $\alpha = \log y$ and $\beta = x$ ... (2)
are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial z}{\partial \beta}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial \alpha}, \text{ using (2)} \qquad \text{.... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) = \frac{\partial^2 z}{\partial \beta^2}, \text{ by (3)} \qquad \text{... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right) = -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial \beta}{\partial y}\right)\right\}$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left(\frac{\partial^2 z}{\partial \alpha^2}\frac{1}{y}\right) = \frac{1}{y^2}\left(\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha}\right) \qquad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} - y = 0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\partial z}{\partial \alpha} + e^{\alpha}, \quad \text{using (2)}$$

Ex. 4. *Reduce* $x\,(\partial^2 z/\partial x^2) + \partial^2 z/\partial y^2 = x^2$ *(x > 0) to canonical form.* **[Delhi Maths(H) 2007, 11]**

Sol. Re-writing the given equation, we get $\qquad xr + t - x^2 = 0, \ (x > 0)$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x$, $S = 0$ and $T = 1$ so that $S^2 - 4RT = -4x < 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$x\lambda^2 + 1 = 0 \qquad \text{or} \qquad \lambda^2 = -(1/x^2) \qquad \text{so that} \qquad \lambda = i/x^{1/2}, \ -i/x^{1/2}$$

The corresponding characteristic equations are given by

$$dy/dx + i\,x^{-1/2} = 0 \qquad \text{and} \qquad dy/dx - i\,x^{-1/2} = 0.$$

Integrating these, $\qquad y + 2i\,x^{1/2} = C_1 \qquad$ and $\qquad y - 2i\,x^{1/2} = C_2$

Choose $\qquad u = y + 2i\,x^{1/2} = \alpha + i\beta \qquad$ and $\qquad v = y - 2i\,x^{1/2} = \alpha - i\beta$,

where $\qquad \alpha = y \qquad$ and $\qquad \beta = 2x^{1/2}$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x^{-1/2}\frac{\partial z}{\partial \beta}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}, \text{ by (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x^{-1/2}\frac{\partial z}{\partial \beta}\right) = -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

or $$r = -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left(x^{-1/2}\frac{\partial^2 z}{\partial \beta^2}\right) = -\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \qquad \text{... (5)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}, \text{ using (4)} \qquad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$x\left(-\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2}\right) + \frac{\partial^2 z}{\partial \alpha^2} = x^2 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = x^2 + \frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta}$$

or $\quad \partial^2 z/\partial\alpha^2 + \partial^2 z/\partial\beta^2 = (\beta^2/4) + (1/\beta)\times(\partial z/\partial\beta)$, as $\beta = 2\,x^{1/2}$.

Ex. 5. *Reduce* $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + 5\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} - 3z = 0$ *to canonical form.*

Sol. Re-writing the given equation, we get $\quad r + 2s + 5t + p - 2q - 3z = 0 \quad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2$ and $T = 5$ so that $S^2 - 4RT = -16 < 0$, showing (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + 2\lambda + 5 = 0 \quad$ so that $\quad \lambda = \{-2 \pm (4-20)^{1/2}\}/2 = -1 \pm 2i$

The corresponding characterisitc equations are given by

$dy/dx + (-1+2i) = 0 \quad$ and $\quad dy/dx + (-1-2i)x = 0.$

Integrating these, $\quad y + (-1+2i)x = C_1 \quad$ and $\quad y + (-1-2i)x = C_2$

Let $\quad u = y - x + 2ix = \alpha + i\beta \quad$ and $\quad v = y - x - 2ix = \alpha - i\beta$,

where $\quad \alpha = y - x \quad$ and $\quad \beta = 2x \quad$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = -\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}, \text{ using (2)} \quad \text{... (4)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}, \text{ using (3)} \quad \text{... (5)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) + 2\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= -\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}\right\} + 2\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

$$= -\left(-\frac{\partial^2 z}{\partial \alpha^2} + 2\frac{\partial^2 z}{\partial \beta \partial \alpha}\right) + 2\left(-\frac{\partial^2 z}{\partial \alpha \partial \beta} + 2\frac{\partial^2 a}{\partial \beta^2}\right), \text{ by (2)}$$

$$\therefore \quad r = \partial^2 z/\partial\alpha^2 + 4(\partial^2 z/\partial\beta^2) - 4(\partial^2 z/\partial\alpha\partial\beta) \quad \text{... (6)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}$$

or $$s = -(\partial^2 z/\partial\alpha^2) + 2(\partial^2 z/\partial\alpha\,\partial\beta), \text{ using (2)} \quad \text{... (7)}$$

Using (3), (4), (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial \alpha^2} + 4\frac{\partial^2 z}{\partial \beta^2} - 4\frac{\partial^2 z}{\partial \alpha \partial \beta} + 2\left(-\frac{\partial^2 z}{\partial \alpha^2} + 2\frac{\partial^2 z}{\partial \alpha \partial \beta}\right) + 5\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta} - 2\frac{\partial z}{\partial \alpha} - 3z = 0$$

or $$4\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) = 3z + 3\frac{\partial z}{\partial \alpha} - 2\frac{\partial z}{\partial \beta}$$ or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{3z}{4} + \frac{3}{4}\frac{\partial z}{\partial \alpha} - \frac{1}{2}\frac{\partial z}{\partial \beta},$$

which is the required canonical form of given equation (1).