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Definition 8.4.1 (Filtration in a Discrete Time) *Let Ω be the set of all possible outcomes of a random experiment and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Then a filtration in discrete time is an increasing sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ of σ-fields, one per time instant.*

Example 8.4.1 *Consider $\Omega = \{a, b, c, d\}$. Construct σ-fields $\mathcal{F}_i$, $(i = 0, 1, 2)$, such that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$.*

Solution Obviously $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $\mathcal{F}_1 = \{\emptyset, \{a, b\}, \{c, d\}, \Omega\}$ and $\mathcal{F}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, d\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{c, d, a\}, \{d, a, b\}, \{b, c, d\}, \Omega\}$. Then, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$.

Definition 8.4.2 (Filtration in a Continuous Time) *Let Ω be the set of all possible outcomes of a random experiment. Let T be a fixed positive number and assume that for each $t \in [0, T]$, there is a σ-field $\mathcal{F}_t$. Assume further that, if $s \leq t$, then every set in $\mathcal{F}_s$ is also in $\mathcal{F}_t$. Then, the collection of σ-fields $\{\mathcal{F}_t,\ 0 \leq t \leq T\}$ is called a filtration in continuous time.*

Thus a collection of σ-fields $\{\mathcal{F}_t,\ t \geq 0\}$ is called a filtration in continuous time if $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s \leq t$.

Definition 8.4.4 (Adapted Process) *We say that a discrete time stochastic process $\{X_0, X_1, \ldots\}$ is adapted to a given filtration $\{\mathcal{F}_n, n = 0, 1, \ldots\}$ if the σ-field generated by X_n is a subset of $\mathcal{F}_n$ means $\sigma(X_n) \subset \mathcal{F}_n$, for every n. In a similar manner, a continuous time stochastic process $\{X(t), t \geq 0\}$ is said to be adapted to a given filtration $\mathcal{F}_t, t \geq 0\}$ if $\sigma(X(t)) \subset \mathcal{F}_t$ for all $t \geq 0$.*

Remark 8.4.2 *The natural filtration corresponding to a process is the smallest filtration to which it is adapted. If the process $\{Y_0, Y_1, \ldots\}$ is adapted to the natural filtration of a stochastic process $\{X_0, X_1, \ldots\}$ then for each n the variable Y_n is a function $\sigma(X_0, X_1, \ldots, X_n)$ of the sample path of the process X up till time n.*

Martingales

Definition 8.6.1 (Discrete Time Martingale) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{X_n, n = 0, 1, \ldots\}$ be a stochastic process and $\{\mathcal{F}_n, n = 0, 1, \ldots\}$ be the filtration. The stochastic process $\{X_n, n = 0, 1, \ldots\}$ is said to be a martingale corresponding to the filtration $\{\mathcal{F}_n, n = 0, 1, \ldots\}$ if it satisfies the following conditions*

(i) For every n, $E(X_n)$ exists.
(ii) Each X_n is $\mathcal{F}_n$-measurable.
(iii) For every n, $E(X_{n+1}/\mathcal{F}_n) = X_n$.

Remark 8.6.2 *From the definition of martingale and using the properties of conditional expectation, we observe that if $\{X_n\}$ is a martingale then $E(X_{n+1}) = E(X_n)$ for every n. This implies that $E(X_n) = c$, a constant. Therefore if, for some $n > 0$, $E(X_n) < \infty$ and the increments $X_{n+1} - X_n$ of the martingale $\{X_n\}$ are bounded, then $E(X_n) = E(X_0)$.*

Example 8.6.2 *Let $X_1, X_2, \ldots$ be a sequence of i.i.d random variables each taking two values +1 and -1 with equal probabilities. Let us define $S_0 = 0$ and $S_n = \sum_{j=1}^{n} X_j$, $(n = 1, 2, \ldots)$. This discrete time stochastic process $\{S_n, n = 0, 1, \ldots\}$ is a symmetric random walk. Prove that, $\{S_n, n = 0, 1, \ldots\}$ is a martingale with respect to $\{X_n, n = 1, 2, \ldots\}$.*

Solution We have $E(|S_n|) \leq E(|X_1|) + E(|X_2|) + \cdots + E(|X_n|) < \infty$. Also

$$\begin{aligned} E(S_{n+1}/X_1, X_2, \ldots, X_n) &= E\left((S_n + X_{n+1})/X_1, X_2, \ldots, X_n\right) \\ &= E(S_n/X_1, X_2, \ldots, X_n) + E(X_{n+1}/X_1, X_2, \ldots, X_n) \\ &= S_n + E(X_{n+1}) \ \ \text{(using independent of } X_1, X_2, \ldots, X_n, X_{n+1}) \\ &= S_n + 0 \end{aligned}$$

Hence $\{S_n, n = 0, 1, \ldots\}$ is a martingale with respect to $\{X_n, n = 1, 2, \ldots\}$.

Example 8.6.3 *Consider a symmetric random walk $\{S_n, n = 0, 1, \ldots\}$ which is a martingale with respect to the filtration $\{\mathcal{F}_n, n = 0, 1, \ldots\}$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n), (n \geq 1)$, is the σ-field of information corresponding to the first n random variables X_n. Verify if $\{S_n^2, n = 0, 1, \ldots\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n, n = 0, 1, \ldots\}$.*

Solution For each $n = 1, 2, \ldots$, S_n^2 is $\mathcal{F}_n$-measurable. Also

$$E(S_n^2) = \sum_{i=1}^{n} E(X_i^2) < \infty .$$

Now,

$$\begin{aligned} E(S_{n+1}^2/\mathcal{F}_n) &= E\left[(S_{n+1} - S_n + S_n)^2/\mathcal{F}_n\right] \\ &= E\left[(S_{n+1} - S_n)^2/\mathcal{F}_n\right] - 2E\left[S_n(S_{n+1} - S_n)/\mathcal{F}_n\right] + E(S_n^2/\mathcal{F}_n) \\ &= E(X_{n+1}^2/\mathcal{F}_n) - 2E(X_{n+1}\dot{S}_n/\mathcal{F}_n) + E(S_n^2/\mathcal{F}_n). \end{aligned}$$

Since X_{n+1} is independent of $\mathcal{F}_n$ and since S_n^2 is $\mathcal{F}_n$-measurable, we have

$$\begin{aligned} E(S_{n+1}^2/\mathcal{F}_n) &= E(X_{n+1}^2) - 2S_n E(X_{n+1}) + S_n^2 \\ &= 1 - 0 + S_n^2 \\ &= 1 + S_n^2 . \end{aligned}$$

Hence, $E(S_{n+1}^2/\mathcal{F}_n) \neq S_n^2$. Therefore, $\{S_n^2, n = 0, 1, \ldots\}$ is not a martingale. Since

Example 8.6.5 *Let a person start with Rs 1. A fair coin is tossed infinitely many times. For nth toss, if it turns up 'head', the person gets Rs 2, but if turn up 'tail', the person does not get any amount. Let Y_n be his/her fortune at the end of nth toss. Prove that Y_n is a martingale.*

Solution Let $X_1, X_2, \ldots$ be a sequence of i.i.d random variables each defined by

$$X_i = \begin{cases} 2, & \text{with probability } 0.5 \\ 0, & \text{with probability } 0.5 . \end{cases}$$

Since the game is double or nothing, his/her fortune at the end of nth toss is given by

$$Y_n = X_1 X_2 \cdots X_n \ (n = 1, 2, \ldots) .$$

Let $\mathcal{F}_n$ be the σ-field generated by $X_1, X_2, \ldots, X_n$. We note that $0 \leq Y_n \leq 2^n$ and $E[X_{n+1}] = 1$. Now,

$$\begin{aligned} E\left[Y_{n+1}/\mathcal{F}_n\right] &= E\left[Y_n X_{n+1}/\mathcal{F}_n\right] \\ &= Y_n E\left[X_{n+1}/\mathcal{F}_n\right] \\ &= Y_n E[X_{n+1}] \\ &= Y_n . \end{aligned}$$

Hence, $\{Y_n, n = 1, 2, \ldots\}$ is a martingale.

□

Example 8.6.6 *Consider a binomial lattice model. Let S_n be the stock price at period n and*

$$S_{n+1} = \begin{cases} uS_n, & \text{with probability } p \\ dS_n, & \text{with probability } 1-p \end{cases}.$$

Define a related process R_n as

$$R_n = ln(S_n) - n\ [p\ ln(u) + (1-p)\ ln(d)]\ .$$

Prove that $\{ln(S_n), n = 1,2,\ldots\}$ is not a martingale whereas $\{R_n, n = 1,2,\ldots\}$ is a martingale with respect to $\{S_n, n = 1,2,\ldots\}$. Also, prove that the discounted stock process $\{S_0, e^{-r}S_1, e^{-2r}S_2, \ldots\}$ is a martingale only if

$$p = \frac{e^r - d}{u-d}\ ,$$

where r is the nominal interest rate.

Solution In this binomial lattice model $\{S_0, S_1, \ldots\}$ with the natural filtration $\{\mathcal{F}_0, \mathcal{F}_1, \ldots\}$, we have

$$P(S_{n+1} = uS_n/\mathcal{F}_n) = 1 - P(S_{n+1} = dS_n/\mathcal{F}_n) = p\ .$$

Hence,

$$E(S_{n+1}/\mathcal{F}_n) = p\ uS_n + (1-p)\ dS_n = S_n[p\ u + (1-p)\ d]\ .$$

We consider the variable $ln(S_n)$ and observe that

$$E\left(ln\left(\frac{S_n}{S_{n-1}}\right)/S_{n-1}, S_{n-2}, \ldots, S_0\right) = p\ ln(u) + (1-p)\ ln(d)\ .$$

Therefore,

$$E\left(ln(S_n)/S_{n-1}, S_{n-2}, \ldots, S_0\right) = ln(S_{n-1}) + p\ ln(u) + (1-p)\ ln(d)\ . \qquad (8.3)$$

Here $\{ln(S_n), n = 1,2,\ldots\}$ is not a martingale and depending upon the values of p, u and d it may be either a submartingale or a supermartingale. Next, we consider the process R_n.

$$E(R_n/R_{n-1}, R_{n-2}, \ldots, R_0) = E\left(ln(S_n) - n\left[p\ ln(u) + (1-p)\ ln(d)\right]/R_{n-1}, R_{n-2}, \ldots, R_0\right)$$

Using equation (8.3), and noting that the history of $S_{n-1}, S_{n-2}, \ldots, S_0$ yields the history of $R_{n-1}, R_{n-2}, \ldots, R_0$ and vice-versa, we get

$$\begin{aligned} E(R_n/R_{n-1}, R_{n-2}, \ldots, R_0) &= ln(S_{n-1}) - (n-1)\left[p\ ln(u) + (1-p)\ ln(d)\right] \\ &= R_{n-1}\ . \end{aligned}$$

Therefore $\{R_n, n = 1,2,\ldots\}$ is martingale.

Now, consider the discounted process $\{S_0, e^{-r}S_1, e^{-2r}S_2, \ldots\}$ where r is the interest rate. We have

$$E(e^{-(n+1)r}S_{n+1}/\mathcal{F}_n) = p\ ue^{-(n+1)r}S_n + (1-p)\ de^{-(n+1)r}S_n\ .$$

The discounted process is a martingale only if the right hand side of the above equation is equal to $e^{-nr}S_n$. That is,

$$e^{-nr}S_n = p\ ue^{-(n+1)r}S_n + (1-p)\ de^{-(n+1)r}S_n$$

or

$$e^r = p\ u + (1-p)\ d\ .$$

Thus, the discounted process is a martingale only if

$$p = \frac{e^r - d}{u-d}.$$

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Wealth Process

Let Δ_k be the number of shares of a stock held between time k and $k+1$. We assume that Δ_k is $\mathcal{F}_k$-measurable and X_0 is the amount of money we have started with time $t = 0$. If we have Δ_k shares between time k and $k+1$, then at time $k+1$ those shares will be worth $\Delta_k S_{k+1}$, where S_{k+1} is the share price at time $k+1$. The amount of cash we hold between time k and $k+1$ is X_k minus the amount held in stock, that is $X_k - \Delta_k S_k$. Hence, the worth of this amount at time $k+1$ is $(1+r)[X_k - \Delta_k S_k]$. Therefore, the amount of money we have at time $k+1$ is

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)\left[X_k - \Delta_k S_k\right] .$$

When $r = 0$, this reduces to

$$X_{k+1} - X_k = \Delta_k (S_{k+1} - S_k) .$$

Thus,

$$X_{k+1} = X_0 + \sum_{i=0}^{k} \Delta_i (S_{i+1} - S_i) .$$

The stochastic process $\{X_k, k = 0, 1, \ldots\}$ is called the *wealth process*.

We shall now show that under risk neutral probability measure Q the discounted wealth process is a martingale.

$$\begin{aligned}
E_Q\left[X_{k+1} - X_k / \mathcal{F}_k\right] &= E_Q\left[\Delta_k (S_{k+1} - S_k)/\mathcal{F}_k\right] \\
&= \Delta_k E_Q\left[(S_{k+1} - S_k)/\mathcal{F}_k\right] \quad (\Delta_k \text{ is } \mathcal{F}_k - \text{ measurable}) \\
&= 0 \quad (S_k \text{ is a martingale}) .
\end{aligned}$$

Now writing X_{k+1} as $X_k + \Delta_k (S_{k+1} - S_k)$ and noting that $r > 0$, we have

$$\begin{aligned}
E_Q\left[(1+r)^{-(k+1)} X_{k+1} - (1+r)^{-k} X_k)/\mathcal{F}_k\right] &= E_Q\left[\Delta_k \left[(1+r)^{-(k+1)} S_{k+1} - (1+r)^{-k} S_k\right]/\mathcal{F}_k\right] \\
&= \Delta_k E_Q\left[(1+r)^{-(k+1)} S_{k+1} - (1+r)^{-k} S_k / \mathcal{F}_k\right] \\
&= 0 \quad (S_k \text{ is a martingale under } Q) .
\end{aligned}$$

Hence, discounted wealth process $\{(1+r)^{-k} X_k, k = 1, 2, \ldots\}$ is a martingale.

Definition 8.6.2 (Continuous Time Martingale) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{X(t), t \geq 0\}$ be a stochastic process and $\{\mathcal{F}_t, t \geq 0\}$ be a filtration. The stochastic process $\{X(t), t \geq 0\}$ is said to be a martingale corresponding to the filtration $\{\mathcal{F}_t, t \geq 0\}$ if it satisfies the following conditions*

(i) For every t, $E(X(t))$ exists.
(ii) Each $X(t)$ is $\mathcal{F}_t$-measurable.
(iii) For every $0 < s < t$,

$$E(X(t)/\mathcal{F}_s) = X(s) . \tag{8.4}$$

Example 8.6.8 *Prove that $\{W(t), t \geq 0\}$ is a martingale.*

Solution For $0 < s < t$,

$$\begin{aligned}
E(W(t)/\mathcal{F}_s) &= E(W(t) - W(s) + W(s)/\mathcal{F}_s) \\
&= E(W(t) - W(s)/\mathcal{F}_s) + E(W(s)/\mathcal{F}_s) \\
&= 0 + W(s) \quad \text{(from the property of Brownian motion)} .
\end{aligned}$$

Therefore, $\{W(t), t \geq 0\}$ is a martingale.

Example 8.6.10 *Show that* $\exp\left(W(t) - \frac{t}{2}\right)$ *is a martingale.*

Solution Let $0 \leq s < t$. Since $W(t) - W(s)$ is independent of $\mathcal{F}_s$ and $W(s)$ is $\mathcal{F}_s$-measurable, we have

$$\begin{aligned}
E\left(e^{W(t)}/\mathcal{F}_s\right) &= E\left(e^{W(t)-W(s)} e^{W(s)}/\mathcal{F}_s\right) \\
&= e^{W(s)} E\left(e^{W(t)-W(s)}/\mathcal{F}_s\right) \\
&= e^{W(s)} E\left(e^{W(t)-W(s)}\right) .
\end{aligned}$$

Since $W(t)-W(s)$ has normal distribution with mean zero and variance $(t-s)$, we have

$$E\left(e^{W(t)-W(s)}\right)=e^{\frac{t-s}{2}}.$$

Hence,

$$E\left(e^{W(t)}/\mathcal{F}_s\right)=e^{W(s)}e^{\frac{t-s}{2}}.$$

This gives, for $0\le s<t$,

$$E\left(e^{W(t)-\frac{t}{2}}/\mathcal{F}_s\right)=e^{\frac{-t}{2}}E\left(e^{W(t)}/\mathcal{F}_s\right)=e^{W(s)-\frac{s}{2}}.$$

It follows that $\exp\left(W(t)-\frac{t}{2}\right)$ is a martingale.