Book → Theory of SP by Cox and Miller
PS book - HC Taneja - Ch 12
SP by Sheldon & Ross.

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STOCHASTIC PROCESSES

Random variable — Any variable which has a $\boxed{\text{distribution}}$
Different values have diff probabilities

Random process is a family of random variables
↓
Something should be common.

Any random Process will have a sample space

A stochastic process (Random process) is a family of random variables $\{X(t),\ t \in T\}$ defined on a given probability space indexed by parameter t where $t \in T$.

If a variable is not random it is deterministic.

Index Set (parameter) — Discrete / Continuous

$X \Rightarrow$ outcome of rolling a die.

$X_n \Rightarrow$ outcome of rolling a die n times
State space → possible values that X_n can take
→ Random Process

Nature of stochastic Process depends on (1) Index Set and (2) State space.

$X(t, s)$ $t \in T$ — Index Set
$s \in S$ — State space

STATE SPACE	INDEX SET	
discrete	cont / dis	— STOCHASTIC CHAIN

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Monday Class at 5 - 6:30
~~Tuesday~~ ~~12-2~~

Date.....................

State space	Index Set	
	Discrete	Continuous
Discrete	Discrete parameter stochastic chain	Continuous Parameter Stochastic chain
Continuous	Continuous parameter stochastic process	Continuous parameter stochastic process

Q) Define random variable and random process. Give specific examples to differentiate between the two. Classify a stochastic process on the basis of its index set and state space. Give one specific example of each case and give their graphical representations. Pg 307, 310, 311

Lab ⇒ Simulate a stochastic process with discrete parameters — Exp 1

Simulate a stochastic process with continuous parameters — Exp 2
(Index Set)

BERNOUILLI PROCESS

A random variable X is s.t.b. a binomial variable with parameters n and p if X can take values from 0 to n and the probability of x taking that value is $^nC_r\, p^r q^{n-r}$ ✓

Bernoulli variable → If the variable can take only two values — Success or failure.
Eg → Throwing a ~~d~~ coin

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$x <^{0}_{1}$

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Probability Distribution ⇒ sum of all probabilities is 1 and the individual probabilities are ≥ 0

$P\{x=1\} = p = 1-q$

$P\{x=0\} = q = 1-p$

→ eg- throw of coins

Consider a sequence of independent and identically distributed Bernoulli random variable, each with probability p of success

$P_n\{x_i = 1\} = p$

$P_n\{x_1 = 0\} = 1-p$

$S_n = x_1 + x_2 + \dots + x_n$ be the number of successes in n bernoulli trials

$0 \leq S_n \leq n$

S_1 = # successes in 1 trial

:

Process S_n is defined as a bernoulli process

Index set → All natural numbers

State space → Non negative integers

$S_n = S_{n-1} + X_n$

$P\{S_n = K \mid S_{n-1} = K-1\}$ — conditional prob

$\hookrightarrow = P(X_n = 1) = p$

$P\{S_n = K \mid S_{n-1} = K\} = P(X_n = 0) = 1-p$

→ Now $S_{n-1} = K-1$ or $K-2$ → But this does not affect the conditional probability of S_n

Future depends only on the present and not on the past.

$P\{S_n = K \mid S_{n-1} = K-1,\ S_{n-2} = K-1\} = P\{S_n = K \mid S_{n-1} = K-1,\ S_{n-2} = K-$

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A stochastic process in which a future state depends only on the present and not on the past is called a Markovial process and this property is called memory less probability

Non markovial → future depends on present and past

S_n → binomial distribution with parameter n and p

Expected value of $S_n = np$
Variance of $S_n = npq$] Binomial distribution
Moment generation function = $(q+pe^t)^n$

Starting at particular bernoulle trial
The number of succeeding trials T before the next success occurs

$$P[T=K] = (1-p)^K \cdot p = \underline{q^K p} \quad \text{where} \quad K=0,1,2\ldots$$

$$p + qp + q^2p + q^3p + \ldots = p(1+q+q^2+q^3+\ldots) \rightarrow \text{Geometric series.}$$

$$= p \cdot \frac{1}{1-q} = p \cdot \frac{1}{p} = 1$$

↓

Since it is after K trials, it is at the $K+1$th trial.

Hence, yes

In a bernoulle (process) variable, the number of trials having failures between two successive successes is distributed like a geometric random variable.

$$E(T) = \sum_{K=0}^{\infty} Kq^K p = \frac{q}{p}$$

$$Var(T) = \frac{q}{p^2}$$

$$Moment(T) = qe^t$$

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Q. What is a generating function? Give examples. Specifically, what is a moment generating function in case of a probability distribution

Expectation ⇒ mean = $E(x) = \sum_{i=1}^{n} x_i p_i$

$E(x^2)$ = second moment about origin = $\sum_{i=1}^{n} x_i^2 p_i$

$E(x-a)^r$ = rth moment about the point a.

$E(x-\bar{x})^r$ - Central moment [replacing a with $\bar{x}$]

$$M_x(t) = E[e^{tx}]$$

$$= E\left[1 + tx + \frac{t^2x}{2!} + \frac{t^3x}{3!} + \dots\right]$$

$$= 1 + t u_1' t + u_2' \frac{t^2}{2!} + u_3' \frac{t^3}{3!} + \dots$$

Differentiate this thrice wrt t and put t=0 to get third moment

Moment generating function ⇒ used to find all the moments from a single function.

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$$

↳ differentiating thrice and putting $t=0$ gives $3!$

↳ This generating function generates $n!$

If the Bernoulli trials are non homogeneous, i.e

$P\{x_i = 1\} = p_i$

$P\{x_i = 0\} = 1 - p_i$ for $i = 1, 2, \dots n, \dots$

Then find the expression for the probability that out of n trials, there are k successes. $P_x\{S_n = k\}$

Lab ⇒ Simulation of ~~Bio~~ ~~binomial~~ ~~program~~ bernoulli process

Write a program to find the number of successes in n trials in case of a bernoulli process

i) Homogeneous ii) Non-homogeneous

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Q. Write a program to find the number of failures between two successive successes in case of a bernoulli process (homogeneous)

POISSON PROCESS - basically counting in an interval

continuous parameter - discrete state process.

Number of customers arriving in interval $(0, t]$

Index set $\Rightarrow$ time (continuous)

Event occurs at random times within the interval

Th Let $N(t)$ denote the number of events that occur in the time interval $(0, t]$. These events are said to constitute a Poisson process with rate λ where $\lambda > 0$, if the following conditions are satisfied.

i) $N(0) = 0$ Number of events at time $t = 0$ are 0

ii) Number of events that occur in disjoint time intervals are independent.

no overlap of intervals

iii) The distribution of the number of events that occur in a given interval of time depends only on the length of the interval and not the location

iv) $P[N(\Delta t) = 1] = \lambda \Delta t + \cancel{\alpha \Delta t}\ o(\Delta t)$

$P[N(\Delta t) \geq 2] = o(\Delta t)$

Need to know these 4 properties

$N(\Delta t) =$ number in the interval $(0, \Delta t]$ which is a very small interval

$\Delta t =$ length of interval.

$o(\Delta t) =$ Order of Δt.

small o order

Big / Capital O order

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Small Order $\qquad$ Capital Order

$\lim_{f\to 0} \frac{o(f)}{f} = 0 \qquad \lim_{f\to 0} \frac{O(f)}{f} = \text{constant}$

Review small and capital order.

$P_n(t) \Rightarrow$ Probability $[N(t) = n] \qquad n = 0, 1, 2 \ldots$

$\boxed{n=0}$: $P_0(t+\Delta t) = p_0(t)\left[1 - \lambda\Delta t + o(\Delta t)\right]$?

$$\Rightarrow \frac{P_0(t+\Delta t) - p_0(t)}{\Delta t} = -\lambda p_0(t) + \frac{o(\Delta t)}{\Delta t}$$

$$\frac{dp_0}{dt} = -\lambda p_0(t)$$

$$p_0(0) = 1$$

$$p_0(t) = e^{-\lambda t}$$

$\boxed{n>0}$: $P_n(t+\Delta t) = p_n(t)\left[1 - \lambda\Delta t + o(\Delta t)\right] + p_{n-1}(t)\left[\lambda(\Delta t) + o(\Delta t)\right]$

$$\frac{p_n(t+\Delta t) - p_n(t)}{\Delta t} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(\Delta t)}{\Delta t}$$

$\Delta t \to 0$

$$\frac{dp_n}{dt} = -\lambda p_n(t) + \lambda p_{n-1}(t). \qquad p_n(0) = 0$$

$n=1 \qquad \frac{dp_1}{dt} = -\lambda p_1(t) + \lambda p_0(t). \qquad p_1(0) = 0$

$$p_1(t) = \lambda t e^{-\lambda t}$$

$$p_{n-1}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}$$

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

Let $N(t)$, $t \geq 0$ be a poisson ~~variable~~ process with rate λ. Then the random variable $N(t)$ giving the number of events in any time interval of

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length t has a poisson distribution with para...

The average number of events occurring in average time interval of length t is λt.

$0 = t_0 < t_1 < t_2 < t_3 \dots$

$T_n = t_n - t_{n-1} \quad n = 1, 2, 3 \dots$

$$P[T_n > s] = P[N(t_{n-1} + s) - N(t_{n-1}) = 0]$$

$$= P[N(s) = 0] = e^{-\lambda s}$$

$$P[T_n \leq s] = 1 - e^{-\lambda s}$$

Random variable

In book it is written as λs and that is incorrect

Distribution function of exponential variable

Check distribution & density function.

In case of poisson, some density function is exponential whereas #failures between two successive successes in binomial is a geometric distribution

$E[N(t)] = \lambda t$ — Non stationary system.

Any system that doesnt evolve with time will have a constant expectation value

Rate of arrival per unit time

$E\left[\frac{N(t)}{t}\right] =$ #successes per unit time

$$= \frac{1}{t} E[N(t)] = \frac{\lambda t}{t} = \boxed{\lambda}$$

$$Var\left[\frac{N(t)}{t}\right] = \frac{1}{t^2} Var[N(t)] = \frac{\lambda t}{t^2} = \boxed{\frac{\lambda}{t}}$$

As $t \to \infty$ (as the system has settled), $E\left(\frac{N(t)}{t}\right) = \lambda$ and variance $\frac{N(t)}{t}$ becomes 0.

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If λ is a function of t i.e. $\lambda = \lambda(t)$, then the process is called a non homogeneous Poisson process.

Do Eq 12.1 and 12.2 from Book

BIRTH AND DEATH PROCESS

$\{N(t),\ t > 0\}$ arrival $\Rightarrow$ birth

departure $\Rightarrow$ death

λ_n will depend on E_n which is the current population

u_n will be the ~~mean~~ death rate when population $= n$.

λ_n and u_n may depend on n or may be constant

We consider a continuous parameter is a S.P with a discrete state space.

$\{X(t),\ t \geq 0\}$ $\underset{\text{state}}{E_n}$ $\underset{\text{birth and death}}{\{X(t) = n\}}$ $n = 0, 1, 2 \ldots$

Assumptions for birth & death:

1) State changes from E_n to E_{n+1} or E_n to E_{n-1} if $n \geq 1$ but from E_0 to E_1

2) Probability that $E_n \to E_{n+1} = \lambda_n \Delta t + o(\Delta t)$

Probability that $E_n \to E_{n-1} = u_n \Delta t + o(\Delta t)$.

Probability of more than one transition is $o(\Delta t)$

To find $\Rightarrow$ $P_n(t) = P_r[\cancel{X(t}\ X(t) = n]$

$$P_n(t + \Delta t) = \text{prob size} = n \text{ at time } t + \Delta t$$

$$= P_n(t)[1 - \lambda_n \Delta t - u_n \Delta t] + P_{n-1}(t)\lambda_{n-1}\Delta t + P_{n+1}(t)\, u_{n+1}\Delta t + o(\Delta t)$$

Prob of no birth $= 1 - u_{n-1}\Delta t$

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— Differential Difference ~~Dif~~ Equation

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$$\frac{dp_n}{dt} = -(\lambda_n + u_n)\, p_n(t) + \lambda_{n-1} P_{n-1}(t) + u_{n+1} P_{n+1}(t)$$

for $n \geq 1$

$$\frac{dp_0}{dt} = -\lambda_0 P_0(t) + u_1 P_1(t)$$

Assume process starts when population size $= i$ i.e. $p_i(0) = 1$ and $p_j(0) = 0$ $j \neq i$

This is known as general birth and death differential equation.

SPECIAL CASES

1) Pure Birth Process

$\lambda_n = \lambda$ and $u_n = 0$

then

$$\frac{dp_n}{dt} = -\lambda p_n(t) + \lambda p_{n-1}(t) \text{ where } n \geq 1$$

$$\frac{dp_0}{dt} = -\lambda p_0(t) \qquad n = 0$$

If condition is $P_0(0) = 1$, then this is a Poisson process.

Write all this for description.

2) Pure death process

We assume that the system starts with N inventories at $t = 0$. $P_N(0) = 1$ and $P_k(0) = 0$ $0 \leq k < N$

Departure occurs at constant rate $u_n = u$

Birth rate $\lambda_n = 0$

Differential difference equation:

$$\frac{dp_n}{dt} = -uP_N(t) \qquad n = N$$

$$\frac{dp_N}{dt} = -uP_n(t) + uP_{n+1}(t) \qquad 0 < n < N$$

$$\frac{dp_0}{dt} = uP_1(t)$$

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$P_N(0) = 1$

$p_n(0) = 0 \quad , \quad 0 < n < N$

After solving we get that $P_n(t) = \dfrac{(ut)^{N-n}}{(N-n)!} e^{-ut}$ $\quad n = 1, 2, \ldots$

$$P_0(t) = 1 - \sum_{n=1}^{N} p_n(t)$$

This is called truncated poisson distribution.

↳ cut short

Q) What are truncated probability distributions? ~~Speciffi~~ Specifically truncated poisson distribution and their applications.

STEADY STATE ⇒ independent of time

$$\frac{dp_n}{dt} = 0$$

Steady state solution for the general birth and death process. In case of steady state, the general birth and death equations become:

$$\begin{cases} 0 = -(\lambda_n + u_n) p_n + \lambda_{n-1} P_{n-1} + u_{n+1} P_{n+1} & \boxed{n \geq 1} \\ 0 = -\lambda_0 p_0 + u_1 p_1 & \boxed{n = 0} \end{cases}$$

Pure difference equations.

$$\lambda_n p_n - u_{n+1} P_{n+1} = \lambda_{n-1} P_{n-1} - u_n p_n$$

$$= \lambda_{n-2} P_{n-2} - u_{n-1} P_{n-1}$$

$$\vdots$$

$$= \lambda_0 P_0 - u_1 p_1 = 0$$

Book says n-1 which is wrong

$$\frac{\lambda_n}{u_{n+1}} P_n \; ; \quad n \geq 0$$

$$p_n = \frac{\lambda_{n-1}}{u_n} P_{n-1}$$

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$\Sigma \to$ sum

$\Pi \to$ multiplication

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$$\Rightarrow p_n = \frac{\lambda_{n-1}}{u_n} \cdot \frac{\lambda_{n-2}}{u_{n-1}} \cdots \frac{\lambda_0}{u_1} p_0$$

$$p_n = p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{u_{i+1}} \qquad n \geq 1$$

$$\sum_{n=0}^{\infty} p_n = 1$$

Taking summation when n goes from 1 to ∞, we get

$$1 - p_0 = p_0 \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{u_{i+1}}$$

$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{u_{i+1}}} \qquad 1 + \frac{\lambda_0}{u_1} + \frac{\lambda_0}{u_1} \cdot \frac{\lambda_1}{u_2} + \cdots +$$

The non zero value of p_0 exists if the denominator is convergent (finite)

Steady state Processes of Birth & Death exist under the condition that series is convergent.

<u>Eg 12.4</u> $\to$ Highkey Imp

If there is a process with inter arrival time distributed exponentially then it is a poisson process. Not exponential then not a poisson process.

<u>Q</u> Retail store

\# customers = 1-3   \# customers = 4-6 ≥ 7

\# operations = 1   2   3

Customers arrive according to poisson distribution with a mean = 10 per hour

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$X(t) = n$

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Average checkout time per customer is exponential with mean 12 minutes.] This is also poisson.

Find the steady state probability P_n of n customers in the checkout area. Also find the probability that all the three counters will be open.

$\lambda_n = \lambda = 10 \quad n = 0, 1, 2, \cdots$ → Homogeneous poisson process

No zero because no dept at zero

$$u_n = \begin{cases} \frac{60}{12} = 5 & n = 1, 2, 3 \\ 5 \times 2 = 10 & n = 4, 5, 6 \\ 5 \times 3 = 15 & n = 7, 8, 9 \cdots \end{cases}$$

→ Non-homogeneous

$\cancel{\frac{\lambda_{n-1}}{u_n}}$ $\cancel{P_n}$

$$P_n = \frac{\lambda_{n-1}}{u_n} \cdot P_{n-1} \quad n \geq 1$$

$$P_1 = \frac{\lambda_0}{u_1} \cdot P_0 = 2P_0$$

$$P_2 = \frac{\lambda_1}{u_2} \cdot P_1 = 4P_0$$

$$P_3 = 8P_0$$

$$P_4 = \frac{\lambda_3}{u_4} P_3 = \frac{\lambda_3}{u_3} \cdot \frac{\lambda_2}{u_3} \cdot \frac{\lambda_1}{u_2} = 8P_0$$

$$P_5 = 8P_0$$

$$P_6 = 8P_0$$

$$P_7 = \frac{\lambda_6}{u_7} \cdot 8P_0 = \left(\frac{2}{3}\right) 8P_0$$

$$P_8 = \left(\frac{2}{3}\right)^2 8P_0$$

$$P_9 = \left(\frac{2}{3}\right)^3 P_0$$

$\vdots$

$$P_0 + P_1 + P_2 + P_3 + \cdots = 1$$

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$$p_0 + 2p_0 + 4p_0 + 8p_0 + 8p_0 + 8p_0 + 8p_0\left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right] = 1$$

From this calculate p_0

$$\boxed{p_0 = \frac{1}{55}}$$

Prob(all counters are open) = $P(n \geq 7)$
= Take sum from P_7 onwards

Renewal Process

In case of poisson process ⇒ inter arrival time is exponential then the process is poisson.
Renewal process is a generalisation of poisson.
Inter-arrival time is not necessarily exponential.

Exponential distribution is memory less
Renewal process is also known as Recurrent process

Renewal process ⇒ Discrete parameter independent process $\{X_n, n \geq 1\}$ where x_is are independent and identically distributed, non negative random variable.
Here X_i may be considered as the time between the $(i-1)$th and ith renewal; removing the restriction of exponential distribution

$\{X_n ; n \geq 1\}$

$$F(x) = P_r[X_n \leq x] \qquad n = 1, 2, \ldots \quad \underbrace{u}_{\text{common mean}}, \ \underbrace{\sigma^2}_{\text{variance}}$$

$W_n = X_1 + X_2 + \ldots X_n$

$n \geq 1$ for $W_0 = 0$

W_n - waiting time until the nth renewal

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The process $\{N(t), t \geq 0\}$ is the renewable countable process then $N(t)$ is called a renewal random variable.

$M(t) = E[N(t)] \rightarrow$ Renewal function

↳ expected number of renewals in the interval $[0,t]$

$m(t) = \frac{dM}{dt}$ = Renewal density.

= Prob of occurance of renewal in the interval $(t, t+\Delta t)$

FUNDAMENTAL RENEWAL EQUATIONS.

$\{X_i, n \geq 1\}$ $F(x) \rightarrow$ common distribution for X_is

$W_n = \sum_{i=1}^{n} x_n$ $F^{(n)}(t)$ is the ~~related~~ distribution function of W_n.

↳ Waiting Time till the nth renewal

Convolution theorem:

f and g are 2 functions.

$$f \circ g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

X < f density function, F – distribution
Y < g, G
} Independent

Distribution function of $X+Y$ is the convolution of their distributions. Convolution of two distribution functions is defined as the convolution of distribution function of one and the density function of the other.

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Truncated Poisson ⇒ will come in exam

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$F^{(n)}$ is the n-fold convolution of F with itself.

$N(t)$ is the number of renewals till time t.

$$P_r[N(t) = n] = P[W_n \leq t < W_{n+1}]$$

$$= P[W_n \leq t] - P[W_{n+1} \leq t]$$

$$= F^{(n)}(t) - F^{(n+1)}(t)$$

Lab

Exp 3: Bernoulli Process (binomial)

Write a program to find the prob of k successes in a bernoulli process with parameters n & p

Write a program to find the number of failures preceding the first success in case of a bernoulli process with parameter n and p (geometric).

Exp 4: Poisson Process & Birth & Death Process

Write a program to find the probability for the specific number of arrivals in a time interval of length t in case of poisson process with rate $\lambda > 0$

Write a program to find the probability for n deaths in an interval $[0, t]$ in case of a pure death process with rate $\mu > 0$. (Truncated Poisson)

Write a program to find the probability that in steady state the system is in state n in case of birth and death process when the rates depend on the state of the system.

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Renewal Process

$P(N(t)=n) = P[W_n \le t \le W_{n+1}]$

$= P[W_n \le t] - P[W_{n+1} \le t] \rightarrow$ Here $B \Rightarrow A$ so yes this formula is correct

$A = W_n \le t$

$B = W_{n+1} \le t \Rightarrow B^c = W_{n+1} > t$

$t < W_{n+1}$

$P(A \cap B^c) = P(A) - P(B)$ if $B \subset A$

This means if B happens, then A will def happen

This means $B \Rightarrow A$ (converse may not be true)

$B \subset A$

$B \Rightarrow$ Getting 2 on a dice

$A \Rightarrow$ Getting 2, 4, 6 on dice

$P(N(t)=n) = F^{(n)}(t) - F^{(n+1)}(t)$ — Imp

$M(t) = E(N(t))$

$= \sum_{n=0}^{\infty} n \, P[N(t)=n]$

$M(t) = \sum_{n=0}^{\infty} n F^{(n)}(t) - \sum_{n=0}^{\infty} n F^{(n+1)}(t)$

$= \sum_{n=1}^{\infty} F^{(n)}(t)$

$= F(t) + \sum_{n=1}^{\infty} F^{(n+1)}(t)$

$F^{(n+1)}(t) =$ convolution $\Rightarrow \int_0^t F^{(n)}(t-x) \, f(x) \, dx$

$M(t) = F(t) + \sum_{n=1}^{\infty} \left[\int_0^t F^{(n)}(t-x) \, f(x) \, dx \right]$

$= F(t) + \int_0^t \sum_{n=1}^{\infty} F^{(n)}(t-x) \, f(x) \, dx$

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$\boxed{M(t) = F(t) + \int_0^t M(t-x)\, f(x)\, dx}$ — Fundamental Renewal Equation.

↳ This is an ~~differential~~ integral equation.

Differentiating it w.r.t t:

Differentiation under integral sign.

$M'(t) = F'(t) + \cancel{M(t-x) f(x)} \int_0^t m(t-x)\, f(x)\, dx$

↳ Another form of renewal equation is the density function.

To solve this we need Laplace transforms

$L[m(t)] = \mathcal{L}[f(t)] + \mathcal{L}\left[\int_0^t m(t-x)\, f(x)\, dx\right]$

convolution theorem of Laplace

Laplace of convolution of two function is the product of the laplace of the two functions

$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(x)\, dx$

$L[m(t)] = \mathcal{L}[f(t)] + \mathcal{L}_m(s)\, \mathcal{L}_f(s)$ — Algebraic Equation

Laplace transforms change differential / integral equations to simple algebraic equations.

$$\mathcal{L}_m(s) = \frac{\mathcal{L}_f(s)}{1 - \mathcal{L}_f(s)}$$

$$\mathcal{L}_f(s) = \frac{\mathcal{L}_m(s)}{1 + \mathcal{L}_f(s)}$$

In case of Poisson Process $f(t) = \lambda e^{-\lambda t}$

$$\mathcal{L}_f(s) = \int_0^\infty f(t)\, e^{-\lambda t} = \frac{\lambda}{\lambda + s}$$

$$\mathcal{L}_m(s) = \frac{\lambda}{s}$$

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$m(t) = \mathcal{L}^{-1}\left(\frac{\lambda}{s}\right)$

$= \lambda$

$M(t) = \lambda t$

Let's suppose in a renewal process, X_is are distributed like a uniform variate with parameter λ. Then find the expected number of arrivals till time t.
? Find $M(t)$

$f(t) = \frac{1}{\lambda} \qquad 0 \leq t \leq \lambda$

↓

Find Laplace transform

$\mathcal{L}m(s)$

↳ Then inverse laplace transform

↳ The integration

$F^{(n)}(t)$ is the convolution of $F^{(n-1)}(t)$ with $f(t)$.
Prove that in case of convolution of n identical exponential distributions is given by

$$F^{(n)}(t) = 1 - \sum_{k=0}^{n-1} \left(\frac{(\lambda t)^k}{k!}\right) e^{-\lambda t}$$

$$P[N(t) = n] = F^{(N)}(t) - F^{(N+1)}(t)$$

$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

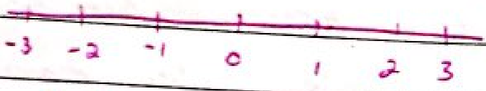
Markov Chain

$X_n \rightarrow$ discrete parameter / discrete state process.

X_n $\quad n = 0, 1, 2 \ldots$

$X_n = i$

the process is in state i at time n.



$Pr\{\text{jump is } 1\} = p$ Random walk.

$Pr\{\text{jump is } -1\} = q$

$Pr\{\text{jump is } 0\} = 1-p-q$

Whenever the process is in state i, there is probability P_{ij} that it will be in state j the next time

⇓

conditional prob.
↓
prob being in j subject to condition that previously it was in i

Such a stochastic process is said to be Markov's chain if $P[X_{n+1}=j / X_n = i, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2} \ldots] = Pr[X_{n+1} = j / X_n = i]$

$= 1$.

Markov Chain ⇒ a process which depends only on the present and not on the past

Eg valve — Satisfactory 0
Unsatisfactory 1
Failed 2

3 State Markov Chain.

$$
\begin{array}{c|c}
 & n+1 \\
n & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ 0 & P_{11} & P_{12} \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

The situation on the next day depends totally on the present day.

$P_{00} + P_{01} + P_{02} = 1$

$P_{11} + P_{12} = 1$ (cannot go higher)

Spiral

Date....................

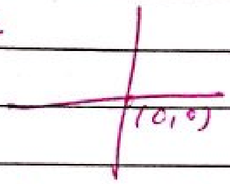
Modifying → If it comes in unsatisfactory position, then it will stay in unsatisfactory stage for 2 days and then it will fail.

$$P[X_{n+1} = 1 \mid X_n = 1, X_{n-1} = 1] = 0$$

$$P[\underset{\text{Future}}{X_{n+1}} = 1 \mid \underset{\text{Present}}{X_n = 1} \quad \underset{\text{Past}}{X_{n-1} = 0}] = 1$$

Here, future depends on the past also. Hence, this is non-markov Chain as it has some memory

Eg 2



Random point (i,j)

Jump ⇒ 1

Next position ⇒ (i, j+1)
(i, j-1)
(i+1, j)
(i-1, j)

Rule ⇒ cannot revisit points and probability for remaining points is equal

Game ends when all points are revisited

Clearly, future does not ONLY depend on the present

Again, an example of Non Markov Chain

Q) What is a Markovian process? Give examples of both markovian & non-markovian and how a non markovian can be changed to a markovian process

Spiral

If P_{ij} depends on n then it called a non homogeneous markov chain
If it is independent of n then the chain is called homogeneous markov chain

We will study ⇒ Homogeneous markov chain
"homogeneous in time"

Complete markov chain is given by **Transition Probability Matrix** [TPM]

$$P = \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0n} \\ P_{10} & P_{11} & \cdots & P_{1n} \\ \vdots & & & \\ P_{n0} & P_{n1} & \cdots & P_{nn} \end{bmatrix}$$ → Represents a process

Total # states = $n+1$
Complete chain is known if all the probabilities are known.

Conditions:
1) All entries must be non-negative
2) Sum of each row must be 1

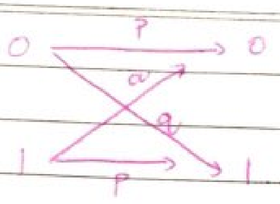
P_{ij} = ith row and jth column

$$\sum_{j=0}^{n} P_{ij} = 1 \quad \forall i$$

If in a particular case, the sum of all columns is also 1 then it is known as doubley stochastic process

Date.....................

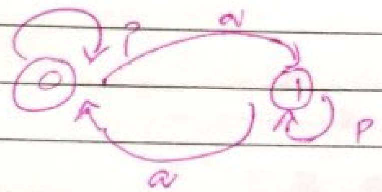
Eg



$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & p & 1-p \\ 1 & 1-p & p \end{array}$$

This is a doubly stochastic process.

<u>Diagramatic</u> Representation:



(Draw nicely)

<u>Q1</u> Find the prob that 0 passed initially is 0 at the 10th state.

Eg $P_r[X_n = i_n \,/\, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2} \dots X_1 = i_1, X_0 = i_0]$

This is a give Markov chain

Joint probability $\rightarrow$ n+1 dimensions

$\hookrightarrow P(AB) = P(A) \cdot P(B|A)$

$\hookrightarrow = P_r[X_n = i_n, X_{n-1} = i_{n-1}, \dots X_1 = i_1] \cdot P[X_{n-1} = i_{n-1}, \dots X_1 = i_1]$

$= P_r[X_n = i_n \mid X_{n-1} = i_{n-1}] \cdot P_r[X_{n-1} = i_{n-1} \dots X_1 = i_1]$

$\vdots \qquad \vdots$ continue breaking down like this

$P_r[X_n = i_n / X_{n-1} = i_{n-1}] \cdot P[X_{n-1} = i_{n-1} \mid X_{n-2} = i_{n-2}] \dots P[X_1 = i_1 / X_0 = i_0]$

$P[X_0 = i_0]$

Reduced to one step conditional probability

Date......................

Bernoulli Trials $\rightarrow$ only 2 outcomes

$p \rightarrow$ probability of success

$X_n \rightarrow$ outcome of the nth trial

$X_n = k$ where $k = 0, 1, 2 \ldots$ where k is uninterrupted length of successes.

<u>Run</u>

$$P_{jk} = P_r\left[X_n = k \mid X_{n-1} = j\right]$$

k can be $j+1$ or 0.

$$P_{jk} = \begin{cases} p & k = j+1 \\ q & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$n+1 \rightarrow$

$$\begin{array}{c} \\ n \downarrow \end{array} \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & \ldots & \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{bmatrix} q & p & 0 & \cdots & \\ q & 0 & p & 0 & \cdots \\ q & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \end{array}$$

This is a Markov chain

$P_{ij} \Rightarrow$ one step

$\hookrightarrow$ diff n

<u>N-step Transition Probability</u>

$$P_{ij}^{(n)} = P\left[X_{n+m} = j \mid X_m = i\right]$$

$$P_{ij}^{(1)} = P_{ij}$$

$$P_{ij}^{(0)} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

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$P_{00}^{(2)}$ → 0 past at initial state is 0 at 2nd state.

$$\boxed{P_{ij}^{(n+m)} = \sum_{K} P_{ik}^{(n)} \cdot P_{kj}^{(m)}}$$ → Chepman Kolmogoro Equation

✓✓ → This is correct.

State & prove Chepman Kolmogoro Equation in case of n-step transition process of Markov chain

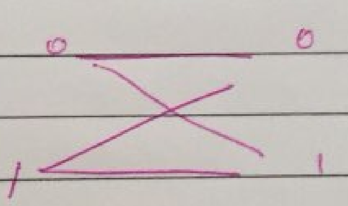
Typo in book: Pg 328 ⇒ P_{ij} → This is wrong
Check proof.

1) Put m=1

$$P_{ij}^{(n+1)} = \sum_{K} P_{ik}^{(n)} \cdot P_{kj}$$

$$\boxed{P^{n} = P^{n-1} P^{m}}$$ idk

Eg



P_{00} $\quad P_{00}^{(5)}$ Example 12.9 ⇒ ~~boo~~ book
P_{01} $\quad$ 12.8 → Remark
P_{10} $\quad$ Super Imp.
P_{11} $\quad$ Eg 12.11