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2K19/MSCPHY/16

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Assignment 4 & 5
MC : 406 Partial Differential Eqs.

1. Solve the differential equations $u_t - \alpha^2 u_{xx} = 0$, for the conduction of heat along a rod subject to the following conditions:
(a) u is not infinite for $t \to \infty$
(b) $\frac{\partial u}{\partial x} = 0$, for $x=0$ and $x=l$
(c) $u = lx - x^2$ for $t=0$, between $x=0$ and $x=l$

1. Given equation, $u_t - \alpha^2 u_{xx} = 0$
On substituting, $u = X(x)\,T(t)$, we get
$$XT' = \alpha^2 X'' T \quad \text{i.e.} \quad \frac{X''}{X} = \frac{T'}{\alpha^2 T} = -k^2$$
$$\therefore \quad \frac{d^2X}{dx^2} + k^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + k^2\alpha^2 T = 0 \quad -(1)$$
Solutions are,
$$\left.\begin{aligned} X &= C_1 \cos kx + C_2 \sin kx \\ T &= C_3 e^{-k^2\alpha^2 t} \end{aligned}\right\} \quad -(2)$$

If k^2 is changed to $-k^2$ solutions are
$$X = C_4 e^{kx} + C_5 e^{-kx}, \quad T = C_6 e^{k^2\alpha^2 t} \quad -(3)$$
If $k^2 = 0$, solutions are
$$X = C_7 x + C_8, \quad T = C_9 \quad -(4)$$

In eq^n(3), $T \to \infty$ for $t \to \infty$, thus $u \to \infty$ i.e. the given condition (a) is not satisfied. So, solution (3) is rejected, while (2) & (4) satisfy this condition.

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applying condition, (b) to eq^n (4), we get
$$c_7 = 0$$
$$\therefore \; u = XT = c_3 c_9 = a_0 \quad (5)$$
from eq^n (2), $\frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx) k c_3 e^{-k^2\alpha^2 t}$

applying the condition (b), we get $c_2 = 0$ &
$$-c_1 \sin kl + c_2 \cos kl = 0$$

i.e. $c_2 = 0$ & $kl = n\pi$
$$u = c_1 \cos kx \; c_3 e^{-k^2\alpha^2 t}$$
$$= a_n \cos\left(\frac{n\pi x}{l}\right) \frac{e^{-n^2\pi^2\alpha^2 t}}{l^2} \quad -(6)$$

Thus, the general solution is the sum of eq^n(5) & (6),
$$u = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right) e^{-(n^2\pi^2\alpha^2 t)/l^2} \quad -(7)$$

now, using condition (c), we get
$$lx - x^2 = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and, $a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$
$$= \frac{2}{l}\left((lx - x^2)\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)$$
$$= -(l - 2x)\left(\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) +$$
$$+ (-2)\left(\frac{-l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\Big|_0^l$$

$= \frac{2}{l}\left\{0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right\} = -\frac{4l^2}{n^2\pi^2}$ (when n is even, otherwise it is zero)

Hence, taking $n = 2m$, the required solution is

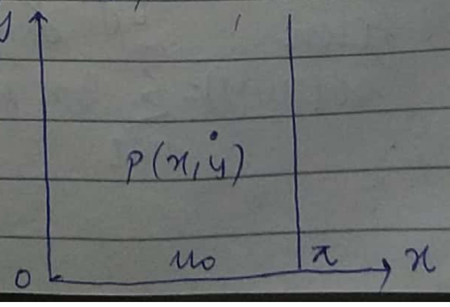
$$\boxed{u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-(4m^2\pi^2\alpha^2 t)/l^2}}$$

6. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angel to them. The breadth is π, this end is maintained at a temperature u_0 at all points and other edges are 0 temperatures. Determine the temperature at any point of the plate in the steady state.

6. In steady state, the temperature $u(x,y)$ at any point $P(x,y)$ satisfies the equation, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —(1)

The boundary conditions are,

$u(0,y) = 0$, for all values of y —(2)
$u(\pi,y) = 0$ for all values of y —(3)
$u(x,\infty) = 0$, in $0 < x < \pi$ —(4)
$u(x,0) = u_0$, in $0 < x < \pi$ —(5)



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The possible solutions for eq^n (1) are,

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad -(6)$$
$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad -(7)$$
$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad -(8)$$

now, we have to choose a suitable solution. Solution (6) cannot satisfy condition (2) for $u \neq 0$ for $x=0$ for all values of y. Solution (8) can not satisfy condition (4). Thus, the only possible solution is (7) i.e.,

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad -(9)$$

From (2), $u(0,y) = c_1(c_3 e^{py} + c_4 e^{-py}) = 0$ for all y.

Hence, $c_1 = 0$ and (9) reduces to
$$u(x,y) = c_2 \sin px \, (c_3 e^{py} + c_4 e^{-py}) \quad -(10)$$
By (3), $u(\pi,y) = c_2 \sin p\pi (c_3 e^{py} + c_4 e^{-py}) = 0$ for all y

This requires $\sin p\pi = 0$, i.e. $p\pi = n\pi$ $(c_2 \neq 0)$
$p = n$ (integer)

also, to satisfy condition (4)
i.e. $u=0$ as $y \to \infty$, $c_3 = 0$

Hence, u takes the form
$$u(x,y) = b_n \sin nx \cdot e^{-ny}$$
(where, $b_n = c_2 c_4$)

$\therefore$ The solution satisfying (2), (3) & (4) is of the form
$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx \cdot e^{-ny} \quad -(11)$$

putting $y=0$,

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx \quad -(12)$$

In order that the condition (5) may be satisfied, (5) and (12) must be same. This requires the expansion of u as a half-range Fourier sine series in $(0, \pi)$. Thus,

$$u = \sum_{n=1}^{\infty} b_n \sin nx \quad \left(\text{where, } b_n = \frac{2}{\pi}\int_0^{\pi} u \sin nx \, dx = \frac{2u_0}{n\pi}\left[1-(-1)^n\right]\right)$$

i.e., $b_n = 0$ if n is even

$(b_n = 4u_0/n\pi$ if n is odd)

Hence, eq^n (11) becomes,

$$u(x,y) = \frac{4u_0}{\pi}\left[e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \dots\right]$$

8. Solve the Laplace equation

$$u_{xx} + u_{yy} = 0$$

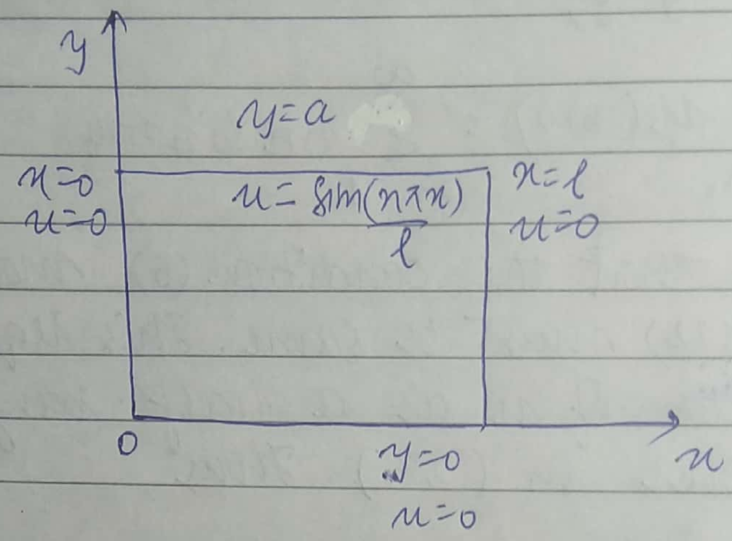
subject to the conditions $u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin\frac{n\pi x}{l}$.

8. The possible solutions of $u_{xx} + u_{yy} = 0 \quad -(1)$

are, $u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad -(2)$

$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad -(3)$

$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad -(4)$



we have to solve eqⁿ (1) satisfying the following boundary conditions,

$u(0,y)=0$ -(5) $u(l,y)=0$ (6)
$u(x,0)=$ -(7) $u(x,0)=\sin\left(\frac{n\pi x}{l}\right)$ -(8)

Using (5) and (6) in (2), we get

$$c_1+c_2=0 \;\&\; c_1e^{pl}+c_2e^{-pl}=0$$

on solving these, we get $c_1=c_2=0$
(trivial solution)

using (5) and (6) in (4) gives a trivial solution

Hence, suitable solution is (3)

using (5) in (3) we have,

$$c_5\left(c_7e^{py}+c_8e^{-py}\right)=0, \text{ i.e. } c_5=0$$

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$\therefore$ Eq^n (3) becomes $u = C_6 \sin px \,(C_7 e^{py} + C_8 e^{-py})$ — (9)

Using (6), we have $C_6 \sin pl \,(C_7 e^{py} + C_3 e^{-py}) = 0$

$\therefore$ either $C_6 = 0$ or $\sin pl = 0$

if $C_6 = 0$, we get a trivial solution

Thus, $\sin pl = 0$, $p = \frac{n\pi}{l}$, where $n = 0, 1, 2 \ldots$

Eq. (9) becomes,

$$u = C_6 \sin\left(\frac{n\pi x}{l}\right)\left(C_7 e^{n\pi y/l} + C_8 e^{-n\pi y/l}\right) \quad \text{— (10)}$$

(8) Using (7), we have,

$$0 = C_6 \sin\frac{n\pi x}{l} \cdot (C_7 + C_8) \quad \text{i.e. } C_8 = C_7$$

Thus, solution is,

$$u(x,y) = b_n \sin\frac{n\pi x}{l}\left(e^{n\pi y/l} - e^{-n\pi y/l}\right) \text{ where } b_n = C_6 C_7$$

Using condition (8), we have,

$$u(x,a) = \sin\frac{n\pi x}{l} = b_n \sin$$

we get, $b_n = \dfrac{1}{\left(e^{n\pi a/l} - e^{-n\pi a/l}\right)}$

Hence, the required solution is,

$$u(x,y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}} \sin\frac{n\pi x}{l}$$

Or,

$$\boxed{u(x,y) = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)} \sin\frac{n\pi x}{l}}$$

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3. State and prove a maximum principle for solutions of an initial boundary value problem for $u_t = K\Delta u$, where Δ is the laplacian in $\mathbb{R}$.

3. We know heat equation,

$$u_t - Ku_{xx} = 0 \ , \ K > 0$$

This equation is also known as diffusion equation.

Let D be region in R^n.

Let $x = [x_1, \dots, x_n]^T$ be a vector in R^n.

Let $u(x,t)$ be the temperature at point x, time t, and let $H(t)$ be the total amount of heat contained in D. Let c be the specific heat of the material and ρ its density. Then

$$H(t) = \int_D c\rho u(x,t)\, dx$$

Therefore, the change in heat is given by

$$\frac{dH}{dt} = \int_D c\rho u_t(x,t)\, dx$$

Fourier's Law says that heat flows from hot to cold regions at rate $K>0$ proportional to the temperature gradient. The only way heat will leave D is through the boundary. That is,

$$\frac{dH}{dt} = \int_{\partial D} K \nabla u \cdot n\, dS$$

where ∂D is the boundary of D, n is the outward unit normal vector to ∂D and

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ds is the surface measure over ∂D. Therefore, we have

$$\int_D c\rho u_t(x,t)\,dx = \int_{\partial D} \kappa \nabla u . n\,ds$$

Recall that for a vector field F, the Divergence Theorem says

$$\int_{\partial D} F\cdot n\,ds = \int_D \nabla . F\,dx$$

Therefore, we have

$$\int_D c\rho u_t(x,t) = \int_D \nabla\cdot(\kappa\nabla u)\,dx$$

This leads us to the partial differential equation

$$c\rho u_t = \nabla.(\kappa\nabla u)$$

If c, ρ and κ are constants, we are led to the heat equation

$$u_t = K\Delta u$$

where, $K = \kappa/c\rho > 0$ and $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$

Hence, Proved.

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Q. Derive Poisson integral formula of Laplace equation.

9. Dirichlet problem for a circle of a radius a. The problem is to solve for u from

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad r < a \quad -(i)$$

subject to boundary condition,

$$u(a, \theta) = f(\theta)$$

Eq. (i) is linear and homogenous, we assume a variable separable solution of the form

$$u(r, \theta) = R(r) H(\theta)$$

The equation (i) yields

$$\frac{r^2 R''}{R} + r\frac{R'}{R} = -\frac{H''}{H} = \lambda,$$

where, λ is a constant. This implies,

$$r^2 R'' + r R' - \lambda R = 0$$
$$H'' + \lambda H = 0$$

$\lambda < 0$ doesn't give any acceptable solution, since $H(\theta + 2\pi) = H(\theta)$ is not met in this case,

$\lambda = 0$ gives the solution in the form,

$$u(r, \theta) = (A + B \log r)(C\theta + D)$$

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The periodicity of R gives $c=0$. $r=0$ is a point in the domain and since u must be bounded there, we must have $B=0$. Therefore, in this case, u = constant

let $\lambda>0$. Assume $\lambda=\alpha^2$. Then

$$R(\theta) = A\cos\alpha\theta + B\sin\alpha\theta$$

The periodicity condition implies $\alpha = 1,2,3\ldots$ Then

$$R(r) = Cr^{\alpha} + Dr^{-\alpha}$$

Since, $r^{-\alpha}\to\infty$ as $r\to 0$, D must be zero. Thus, the boundary conditions $u(a,\theta)=f(\theta)$ gives

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \quad n=0,1,2,\ldots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta, \quad n=1,2,\ldots$$

By the very definition, a_n and b_n are bounded. choose M such that $|a_n|<M$ and $|b_n|<M$, $n=1,2,\ldots$ Since

$$u_n(\rho,\theta) = \rho^n(a_n\cos n\theta + b_n\sin n\theta), \quad \rho = r/a$$

we have

$$|u_n| < 2\rho_0^n M, \quad 0\le\rho\le\rho_0<1.$$

Hence, in any closed circular region inside the open unit disc, the series converges uniformly.

also observe,

$$\left|\frac{\partial u_n}{\partial r}\right| = \left|\frac{n}{a}\rho^{n-1}(a_n\cos n\theta + b_n\sin\theta)\right|$$
$$< \frac{2n}{a}\rho_0^{n-1}M$$

consequently,

$$\nabla^2 u = u_{rr} + \frac{1}{\rho}u_r + \frac{1}{\rho^2}u_{\theta\theta}$$
$$= \sum_{n=2}^{\infty}\frac{\rho^{n-2}}{a^2}(a_n\cos n\theta + b_n\sin\theta)[n(n-1)+n-n^2]$$
$$= 0 \quad , \quad 0 \leq \rho \leq \rho_0 < 1$$

Thus u is harmonic in the region $0 \leq \rho < 1$
consider

$$u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau + \frac{1}{\pi}\sum_{n=1}^{\infty}\rho^n\int_0^{2\pi} f(\tau)[\cos n\tau\cos n\theta + \sin n\tau\sin n\theta]\,d\tau,$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 2\sum_{n=1}^{\infty}\rho^n\cos n(\theta-\tau)\right]f(\tau)\,d\tau$$

Hence, due to the uniform convergence of the series, the interchange of summation & integration is allowed.

For $0 \leq \rho < 1$,

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$$\left[1+2\sum_{n=1}^{\infty} \rho^n \cos n(\theta-t)\right] = 1 + \sum_{n=1}^{\infty}\left[\rho^n e^{in(\theta-t)} + \rho^n e^{-in(\theta-t)}\right]$$

$$= 1 + \frac{\rho e^{i(\theta-t)}}{1-\rho e^{i(\theta-t)}} + \frac{\rho e^{-i(\theta-t)}}{1-\rho e^{-i(\theta-t)}}$$

$$= \frac{1-\rho^2}{1-\rho e^{i(\theta-t)} - \rho e^{-i(\theta-t)} + \rho^2}$$

$$= \frac{1-\rho^2}{1-2\rho\cos(\theta-t)+\rho^2}$$

Hence,

$$\boxed{u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho^2}{1-2\rho\cos(\theta-t)+\rho^2} f(t)\,dt}$$

which is called Poisson integral formula.