Q3) Let $\{W(t), t \geq 0\}$ be a Brownian motion. Prove that $\{tW(1/t), t \geq 0\}$ where $tW(1/t)$ is taken to be zero when $t=0$ is a Brownian Motion.

Brownian motion is defined as a stochastic process $\{X(t), t \geq 0\}$ if :-

i) $X(0) = 0$

ii) $\{X(t), t \geq 0\}$ has stationary and independent increments

iii) For every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

Now, we are given a brownian process $W(t)$ and we need to prove that $tW(1/t)$ is also a brownian process (Weiner Process).

Let $\hat{W}_t = tW_{1/t}$ $t > 0$ and $\hat{W}_0 = 0$. This proves the first condition for Brownian process.

Now, $0 \leq s \leq t$ implies $1/t \leq 1/s$

Thus

$$\begin{aligned} \mathrm{Cov}(\hat{W}_s, \hat{W}_t) &= \mathrm{Cov}(sW_{1/s}, tW_{1/t}) \\ &= st\,\mathrm{Cov}(W_{1/s}, W_{1/t}) \\ &= st \cdot \frac{1}{t} = s \end{aligned}$$

Date ________
Page ________

Now, to show the continuity of $\hat{W}_t$ it is sufficient to show the continuity of $\hat{W}_t$ at $t=0$.

But it is already given that $\left[t W(1/t)\right]_{t=0} = 0$

Also,

$$\lim_{t \to} \hat{W}_t = \lim_{t\to 0} t W_{1/t} = \lim_{t \to \infty} \frac{W_t}{t} = 0$$

Hence, from continuity we can state that the new process $\hat{W}_t$ is also a Brownian process. (Wiener Process)