(*vii*) $p = y^2 e^{xy^2} + 4x^3,\ q = 2xy\, e^{xy^2} - 3y^2$ **Ans.** $z = e^{xy^2} + x^4 - y^3 + c$

(*viii*) $p = \sin x \cos y + e^{3x},\ q = \cos x \sin y + \tan y$

Ans. $x = (1/3) \times e^{3x} - \cos x \cos y + \log \sec x + c$

3.7. Charpit's method.* (General method of solving partial differential equations of order one but of any degree. [Agra 2003; Delhi Maths (H) 2000, 05, 06, 08-11; Kanpur 1998; Meeerut 2003, 05; Nagpur 2002, 04, 06, 08; Rohilkhand 2001, 04]

Let the given partial equation differential of first order and non–linear in p and q be

$$f(x, y, z, p, q) = 0. \qquad ...(1)$$

We know that

$$dz = p\, dx + q\, dy. \qquad ...(2)$$

The next step consists in finding another relation $F(x, y, z, p, q) = 0$ ...(3) such that when the values of p and q obtained by solving (1) and (3), are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (3), differentiate partially (1) and (3) with respect to x and y and get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0, \qquad ... (4)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0, \qquad ... (5)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad ... (6)$$

and

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0. \qquad ... (7)$$

Eliminating $\partial p/\partial x$ from (4) and (5), we get

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p} = 0$$

or

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0. \qquad ...(8)$$

Similarly, eliminating $\partial q/\partial y$ from (6) and (7), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right) q + \left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0. \qquad ...(9)$$

Since $\partial q/\partial x = \partial^2 z/\partial x\partial y = \partial p/\partial y$, the last term in (8) is the same as that in (9), except for a minus sign and hence they cancel on adding (8) and (9).

Therefore, adding (8) and (9) and rearranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0. \qquad ...(10)$$

This is a linear equation of the first order to obtain the desired function F. As in Art 2.20 of chapter 2, integral of (10) is obtained by solving the auxiliary equations

$$\frac{dp}{(\partial f/\partial x) + p(\partial f/\partial z)} = \frac{dq}{(\partial f/\partial y) + q(\partial f/\partial z)} + \frac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dF}{0}. \qquad ...(11)$$

---

*This is general method for solving equations with two independent variables. Since the solution by this method is generally more complicated, this method is applied to solve equations which cannot be reduced to any of the standard forms which will be discussed later on. Thus, Charpit's method is used in two situations (*i*) When you are asked to solve a problem by Charpit's method (*ii*) when the given equation is not of any four standard forms given in Articles 3.10, 3.12, 3.14 and 3.17.

Since any of the integrals of (11) will satisfy (10), an integral of (11) which involves p or q (or both) will serve along with the given equation to find p and q. In practice, however, we shall select the simplest integral.

Note. In what follows we shall use the following standard notations:

$$\partial f/\partial x = f_x, \qquad \partial f/\partial y = f_y, \qquad \partial f/\partial z = f_z, \qquad \partial f/\partial p = f_p, \qquad \partial f/\partial q = f_q.$$

Therefore, Charpit's auxiliary equations (11) may be re–written as

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0} \qquad \text{... (11)}'$$

3.8A. WORKING RULE WHILE USING CHARPIT'S METHOD

Step 1. Transfer all terms of the given equation to L.H.S. and denote the entire expression by f.

Step 2. Write down the Charpit's auxiliary equations (11) or (11)′.

Step 3. Using the value of f in step 1 write down the values of $\partial f/\partial x$, $\partial f/\partial y$ ..., *i.e.*, f_x, f_y, ... etc. occuring in step 2 and put these in Charpit's equations (11) or (11)′.

Step 4. After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5. The simplest relation of step 4 is solved along with the given equation to determine p and q. Put these values of p and q in $dz = p\,dx + q\,dy$ which on integration gives the complete integral of the given equation.

The Singular and General integrals may be obtained in the usual manner.

Remark. Sometimes Charpit's equations give rise to $p = a$ and $q = b$, where a and b are constants. In such cases, putting $p = a$ and $q = b$ in the given equation will give the required complete integral.

3.8.B. SOLVED–EXAMPLES BASED ON ART. 3.8A.

Ex. 1. *Find a complete integral of* $z = px + qy + p^2 + q^2$.

[Bilaspur 2000l; Bhopal 1996, I.A.S. 1996; Indore 2000; Jabalpur 2000; K.U. Kurukshetra 2005; Ravishankar 2000; 04; Meerut 2010; Garhwal 2010]

Sol. Let $$f(x, y, z, p, q) \equiv z - px - qy - p^2 - q^2 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \qquad \text{... (2)}$$

From (1), $f_x = -p$, $f_y = -q$, $f_z = 0$, $f_p = -x - 2p$ and $f_q = -y - 2q$ ... (3)

Using (3), (2) reduces to

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p) + q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} \qquad \text{... (4)}$$

Taking the first fraction of (4), $dp = 0$ so that $p = a$ ... (5)

Taking the second fraction of (4), $dq = 0$ so that $q = b$ ... (6)

Putting $p = a$ and $q = b$ in (1), the required complete integral is

$z = ax + by + a^2 + b^2$, a, b being arbitrary constants.

Ex. 2. *Find a complete integral of* $q = 3p^2$. **[Agra 2006]**

Sol. Here given equation is $$f(x, y, z, p, q) \equiv 3p^2 - q = 0. \qquad \text{...(1)}$$

$\therefore$ Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{0+p.0}=\frac{dq}{0+q.0}=\frac{dz}{-6p^2+q}=\frac{dx}{-6p}=\frac{dy}{1}, \text{ using (1)} \qquad \text{...(2)}$$

Taking the first fraction of (1), $dp = 0$ so that $p = a$. ...(3)

Substituting this value of p in (1), we get $q = 3a^2$. ...(4)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = adx + 3a^2dy$ so that $z = ax + 3a^2y + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 3. *Find the complete integral of* $zpq = p + q$ **[Nagpur 2010; Meerut 2006]**

Sol. Let $f(x, y, z, p, q) = zpq - p - q = 0$ ... (1)

Here Charpit's auxiliary equations are $$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q} \qquad \text{... (2)}$$

From (1), $f_x = 0$, $f_y = 0$, $f_z = pq$, $f_p = zq - 1$ and $f_q = zp - 1$ ... (3)

Using (3), (2) reduces to

$$\frac{dp}{p^2q}=\frac{dq}{pq^2}=\text{......} \quad \text{or} \quad \frac{dp}{p}=\frac{dq}{q} \quad \text{so that} \quad p = aq \qquad \text{... (4)}$$

Solving (1) and (2), $p = (1 + a)/z$ and $q = (1 + a)/az$.

$\therefore$ $dz = pdx + qdy = [(1 + a)/z]dx + [(1 + a)/az]dy$ or $2zdz = 2(1 + a)\,[dx + (1/a)dy]$

Integrating, $z^2 = 2(1 + a)\,[x + (1/a)y] + b$, a, b being arbitrary constants

Ex. 4. *Find a complete integral of* $p^2 - y^2q = y^2 - x^2$. **[M.D.U. Rohtak 2006]**

Sol. Here given equation is $f(x, y, z, p, q) = p^2 - y^2q - y^2 + x^2 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{2x}=\frac{dq}{-2qy-2y}=\frac{dz}{-p(2p)-q(-q^2)}=\frac{dx}{-2p}=\frac{dy}{y^2}, \text{ using (1)} \qquad \text{...(2)}$$

Taking the first and fourth fractions, $pdp + xdx = 0$ so that $p^2 + x^2 = a^2$ ... (3)

Solving (1) and (3) for p and q, $p = (a^2 - x^2)^{1/2}$, $q = a^2y^{-2} - 1$.

$\therefore$ $dz = pdx + qdy = (a^2 - x^2)^{1/2}dx + (a^2y^{-2} - 1)dy$.

Integrating, $z = (x/2) \times (a^2 - x^2)^{1/2} + (a^2/2) \times \sin^{-1}(x/a) - (a^2/y) - y + b$.

Ex. 5. *Find a complete integral of* $z^2(p^2z^2 + q^2) = 1$. **[I.A.S. 1997; Meerut 2007]**

Sol. Here given equation is $f(x, y, z, p, q) = p^2z^4 + q^2z^2 - 1 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dq}{-f_q}$$

or $$\frac{dp}{p(4p^2z^3+2zq^2)}=\frac{dq}{q(4p^2z^3+2zq^2)}=\frac{dz}{-2p^2z^4-2q^2z^2}=\frac{dx}{-2pz^4}=\frac{dy}{-2qz^2}, \text{ by (1)} \quad \text{... (2)}$$

Taking the first two fractions, $(1/p)dp = (1/q)dq$ so that $p = aq$.

Solving (1) and (2) for p and q, $$p = \frac{a}{z(a^2z^2+1)^{1/2}}, \qquad q = \frac{1}{z(a^2z^2+1)^{1/2}}.$$

$\therefore$ $dz = pdx + qdy = (a\,dx + dy)/z\,(a^2z^2 + 1)^{1/2}$ or $adx + dy = z(a^2z^2 + 1)^{1/2}dz$.

Integrating, $ax + y = \int (a^2z^2+1)^{1/2}.zdz$. ...(3)

Putting $a^2z^2 + 1 = t^2$ so that $2a^2zdz = 2tdt$, (3) becomes

$ax + y = \int (1/a^2)\, t.t\, dt$ or $ax + y + b = (1/3a^2)t^3$, where $t = (a^2z^2 + 1)^{1/2}$

or $ax + y + b = (1/3a^2) \times (a^2z^2 + 1)^{3/2}$ or $9a^4(ax + y + b)^2 = (a^2z^2 + 1)^3$,

which is a complete integral, a and b being arbitrary constants.

Ex. 6. *Find a complete integral of* $px + qy = pq$. **[Kurukshetra 2006 Rajasthan 2000, 01, Gulbarga 2005; Meerut 2002; Kanpur 2004; Jiwaji 2004; Rewa 2001; Vikram 2000, 03, 04; Bhopal 2010]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv px + qy - pq = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

or $$\frac{dp}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p+p.0} = \frac{dq}{q+q.0}, \text{ by (1)} \qquad ...(2)$$

Taking the last two fractions of (2), $(1/p)dp = (1/q)dq$.

Integrating, $\log p = \log q + \log a$ or $p = aq$. ...(3)

Substituting this value of p in (1), we have

$aqx + qy - aq^2 = 0$ or $aq = ax + y$, as $q \neq 0$ ...(4)

$\therefore$ From (3) and (4), $q = (ax + y)/a$ and $p = ax + y$. ...(5)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (ax + y)dx + [(ax + y)/a]\, dy$ or $adz = (ax + y)\,(adx + dy)$

or $adz = (ax + y)\, d(ax + y) = udu$, where $u = ax + y$.

Integrating, $az = u^2/2 + b = (ax + y)^2/2 + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 7. *Find the complete integrals of following equations:*

(*i*) $q = (z + px)^2$ **[Indore 2004; Ravishanker 2005]**

(*ii*) $p = (z + qy)^2$ **[Meerut 2008, 09; Agra 2001; Delhi B.Sc. (Prog) 2008; Kurukshetra 2005]**

Sol. (*i*). Here given equations is $f(x, y, z, p, q) = (z + px)^2 - q = 0$ ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

or $$\frac{dp}{2p(z+px)+2p(z+px)} = \frac{dq}{2q(z+px)} = \frac{dz}{-2px(z+px)+q} = \frac{dx}{-2x(z+px)} = \frac{dy}{0}, \text{ by (1)}$$

Taking the second and fourth fractions, $(1/q)dq = -(1/x)dx$.

Integrating, $\log q = \log a - \log x$ so that $q = a/x$. ...(2)

Substituting the above value of q in (1), we have

$(z + px)^2 = a/x$ or $px = \sqrt{a}/\sqrt{x} - z$ or $p = \sqrt{a}/x\sqrt{x} - z/x$. ...(3)

$\therefore$ $dz = pdx + qdy = \left(\dfrac{\sqrt{a}}{x\sqrt{x}} - \dfrac{z}{x}\right) dx + \dfrac{a}{x}dy$, by (2) and (3)

or $xdz = \sqrt{a}\, x^{-1/2}dx - zdx + ady$ or $xdz + zdx = \sqrt{a}\, x^{-1/2}dx + ady$

or $d(xz) = \sqrt{a}\, x^{-1/2}dx + ady$.

Integrating, $xz = 2\sqrt{a}\sqrt{x} + ay + b$, a, b being arbitrary constants

(*ii*) **Sol.** Do as in part (1). **Ans.** $yz = ax + \sqrt{ay} + b$.

Ex. 8. *Find a complete integral of* $yzp^2 - q = 0$.

Sol. Here $f(x, y, z, p, q) = yzp^2 - q = 0.$

Charpit's uxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{0 + p(yp^2)} = \frac{dq}{zp^2 + q(yp^2)} = \frac{dz}{-2yzp^2 + q} = \frac{dx}{-2yzp} = \frac{dy}{1}, \text{ by (1)} \qquad ...(2)$$

Taking the first and fifth fractions, $(1/yp^3)\,dp = dy$

or $p^{-3}dp = ydy$ or $-2p^{-3}dp = -2ydy$.

Integrating, $p^{-2} = a^2 - y^2$ so that $p = 1/(a^2 - y^2)^{1/2}$. ...(3)

Using (3), (1) $\Rightarrow$ $q = yzp^2$ $\Rightarrow$ $q = yz/(a^2 - y^2)$. ...(4)

$$\therefore \qquad dz = pdx + qdy = \frac{dx}{(a^2 - y^2)^{1/2}} + \frac{yzdy}{(a^2 - y^2)}$$

or $$(a^2 - y^2)^{1/2}\,dz - \frac{yzdy}{(a^2 - y^2)^{1/2}} = dx \qquad \text{or} \qquad d\,[z\,(a^2 - y^2)^{1/2}] = dx.$$

Integrating, $z\,(a^2 - y^2)^{1/2} = x + b$ or $z^2(a^2 - y^2) = (x + b)^2$, a, b being arbitrary constants.

Ex. 9. *Find a complete integral of* $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$. **[I.A.S. 1994]**

Sol. Given equation is $f(x, y, z, p, q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-p(32p^2z + 18q^2z + 8z)} = \frac{dq}{q(32p^2z + 18q^2z + 8z)} = \frac{dz}{-p(32pz^2) - q(18qz^2)} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2}.$$

Taking the first and second fractions, $(1/p)dp = (1/q)dq$ so that $p = aq$ ... (2)

Solving (1) and (2) for p and q, we have

$$q = \frac{2(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}} \qquad \text{and} \qquad p = \frac{2a(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}}. \qquad ...(3)$$

Hence, $$dz = pdx + qdy = \frac{2(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}}\,(adx + dy), \text{ using (3)}$$

or $$(1/2) \times (16a^2 + 9)^{1/2}\,(1 - z^2)^{-1/2}\,(-2zdz) = -\,2(adx + dy). \qquad ...(4)$$

Putting $1 - z^2 = t$ so that $-2zdz = dt$, (4) becomes

or $$(1/2) \times (16a^2 + 9)^{1/2}\,t^{-1/2}\,dt = -\,2\,(adx + dy).$$

Integrating, $(16a^2 + 9)^{1/2}\,t^{1/2} = -\,2(ax + y) + b$, a, b being arbitrary constants.

or $$(16a^2 + 9)^{1/2}\sqrt{(1 - z^2)} + 2(ax + y) = b, \quad \text{as} \quad t = 1 - z^2.$$

Ex. 10(a). *Find a complete integral of* $(p^2 + q^2)x = pz$.

[Agra 2003; Rajasthan 2005; Ravishankar 2001; Delhi Maths (Hons) 2004, 05]

(b). *Find the complete integral of the partial differential equation* $(p^2 + q^2)x = pz$ *and deduce the solution which passes through the curve* $x = 0$, $z^2 = 4y$. **[Meerut 2007]**

Sol. Let $f(x, y, q, p, q) = (p^2 + q^2)x - pz = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

giving $dp/q^2 = dq/(-pq)$, by (1) or $2pdp + 2qdq = 0$.

Integrating, $p^2 + q^2 = a^2$, where a is an arbitrary constant. ...(2)

Solving (1) and (2), $p = a^2x/q$ and $q = (a/z) \times \sqrt{(z^2 - a^2x^2)}$. ...(3)

$\therefore$ $dz = pdx + qdy = \dfrac{a^2xdx}{z} + \dfrac{a\sqrt{(z^2 - a^2x^2)}\,dy}{z}$ or $\dfrac{zdz - a^2xdx}{\sqrt{(z^2 - a^2x^2)}} = ady.$

Putting $z^2 - a^2x^2 = t$ so that $2(zdz - a^2xdx) = dt$, we get

$(1/2\sqrt{t})dt = ady$ or $(1/2) \times t^{-1/2} = ady.$

Integrating, $t^{1/2} = ay + b$ or $\sqrt{(z^2 - a^2x^2)} = ay + b,$ as $t = \sqrt{z^2 - a^2x^2}$

or $z^2 - a^2x^2 = (ay + b)^2$ or $z^2 = a^2x^2 + (ay + b)^2$. ... (4)

(b) Proceeding as in part (a), (4) is the complete integral.

The parametric equations of the given curve $x = 0$, $z^2 = 4y$ are given by

$x = 0,$ $y = t^2,$ $z = 2t$ ... (5)

Therefore the intersections of (1) and (2) are determined by

$4t^2 = (at^2 + b)^2$ or $a^2t^4 + 2(ab - 2)t^2 + b^2 = 0$ ... (6)

Equation (6) has equal roots if its discriminant = 0, *i.e.*, if

$4(ab - 2)^2 - 4a^2b^2 = 0$ or $a^2b^2 = 1$ so that $b = 1/a$

Hence from (4), the appropriate one paremeter sub–system is given by

$z^2 = a^2x^2 + (ay + 1/a)^2$ or $a^4(x^2 + y^2) + a^2(2y - z^2) + 1 = 0,$

which is a quatratic equation in parameter 'a'. Therefore, this has for its envelope surface

$(2y - z^2)^2 - 4(x^2 + y^2) = 0$ or $(2y - z^2)^2 = 4(x^2 + y^2)$ ... (7)

The desired solution is given by the function z defined by equation (7).

Ex. 10(c). *Find a complete, singular and general integrals of* $(p^2 + q^2)y = qz$.
[Guwahati 2007; Agra 2001; Bilaspur 1998; Delhi Maths (H) 2003, 05; Garhwal 2005; Meerut 2010, 11; K.V. Kurukshetra 2004; Kanpur 2005; Rohilkhand 2001; Pune 2010]

Sol. Here the given equation is $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-pf_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{-pq} = \dfrac{dq}{p^2} = \dfrac{dz}{-2p^2y + qz - 2q^2y} = \dfrac{dx}{-2py} = \dfrac{dy}{-2qy + z}$, by (1) ...(2)

Taking the first two fractions, we get $2pdp + 2qdq = 0$ so that $p^2 + q^2 = a$ ... (3)

Using (3), (1) gives $a^2y = qz$ or $q = a^2y/z.$

Putting this value of q in (3), we get

$$p = \sqrt{(a^2 - q^2)} = \sqrt{a^2 - (a^4y^2/z^2)} = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}\,.$$

Now putting these values of p and q in $dz = pdx + qdy$, we have

$dz = \dfrac{a}{z}\sqrt{(z^2 - a^2y^2)}\,dx + \dfrac{a^2y\,dy}{z}dy$ or $\dfrac{zdz - a^2y\,dy}{\sqrt{(z^2 - a^2y^2)}} = a\,dx.$

Integrating, $(z^2 - a^2y^2)^{1/2} = ax + b$ or $z^2 - a^2y^2 = (ax + b)^2$, ...(4)
which is a required complete integral, a, b being arbitrary constants.

Singular Integral. Differentiating (4) partially w.r.t. a and b, we have

$$0 = 2ay^2 + 2\,(ax + b)x \qquad ...(5)$$

and

$$0 = 2(ax + b). \qquad ...(6)$$

Eliminating a and b between (4), (5) and (6), we get $z = 0$ which clearly satisfies (1) and hence it is the singular integral.

General Integral. Replacing b by $\phi(a)$ in (4), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2. \qquad ...(7)$$

Differentiating (7) partially w.r.t. a, $-2ay^2 = 2[ax + \phi(a)]\,.\,[x + \phi'(a)]$. ...(8)

General integral is obtained by eliminating a from (7) and (8).

Ex. 11. *Find a complete integral of* $p\,(1 + q^2) + (b - z)q = 0$. **[Agra 1996]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv p(1 + q^2) + (b - z)q = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\dfrac{dx}{-\dfrac{\partial f}{\partial p}}=\dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or

$$\frac{dp}{pq}=\frac{dq}{p^2}=\frac{dz}{-p(1+q^2)-(b-z)q}=\frac{dx}{-(q^2+1)}=\frac{dy}{-2pq-(b-z)}, \text{ by (1)}$$

First two fractions give $(1/p)dp = (1/q)dq$ so that $q = pc.$

Putting $q = pc$ in (1), we have $p = \sqrt{[c(z-b)-1]}\,/c$.

$\therefore$ $q = pc$ gives $q = \sqrt{[c(z-b)-1]}$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{[c(z-b)-1]}\,\left(\frac{dx}{c}+dy\right) \quad \text{or} \quad \frac{cdz}{\sqrt{[c(z-b)-1]}} = dx + c\,dy.$$

Integrating, $2\sqrt{[c(z-b)-1]} = x + cy + a$ or $4\{c(z - b) - 1\} = (x + cy + a)^2$
which is a complete integral, a and c being arbitrary constants.

Ex. 12. *Find a complete and singular integrals of* $2xz - px^2 - 2q\,xy + pq = 0$. **[I.A.S. 1991, 93, 2007, 2008; Delhi Hons. 2001, 01, 05; Kanpur 2001, 03; Meerut 2005; Bhopal 2004, 10; Indore 1999; M.D.U. Rohtak 2004, Ravishanker 2004; Rajasthan 2000, 03, 05, 10]**

Sol. Here given equation is $f\,(x, y, z, p, q) = 2xz - px - 2qxy + pq = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\dfrac{dx}{-\dfrac{\partial f}{\partial p}}=\dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or

$$\frac{dp}{2z-2qy}=\frac{dq}{0}=\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2+2xyq-2pq}, \text{ by (1)}$$

The second fraction gives $dq = 0$ so that $q = a$

Putting $q = a$ in (1), we get $p = 2x(z - ay)/(x^2 - a)$

Putting values p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2x(z-ay)}{x^2-a}\,dx + a\,dy \quad \text{or} \quad \frac{dz-ady}{z-ay}=\frac{2xdx}{x^2-a}.$$

Integrating, $\log\,(z - ay) = \log\,(x^2 - a) + \log b$

or $\quad z - ay = b(x^2 - a) \quad$ or $\quad z = ay + b(x^2 - a), \quad$ ...(2)

which is the complete integral, a and b being arbitrary constants.

Differentiating (2) partially with respect to a and b, we get

$$0 = y - b \quad \text{and} \quad 0 = x^2 - a. \qquad \text{...(3)}$$

Solving (3) for a and b, $\quad a = x^2 \quad$ and $\quad b = y. \quad$ ...(4)

Substituting the values of a and b given by (4) in (2), we get $z = x^2 y$, which is the required singular integral.

Ex. 13. *Find a complete integrals of the following partial differential equations:*

(i) $q = px + p^2$. **[Sagar 2003; Meerut 1994]**

(ii) $q = -px + p^2$.

Sol. (i) Here given equation is $\quad f(x, y, z, p, q) \equiv q - px - p^2 = 0. \quad$ ... (1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{-p} = \frac{dq}{0} = \frac{dz}{-p(-x-2p)-q} = \frac{dx}{-(-x-2p)} = \frac{dy}{-1}, \text{ by (1)}$$

The 2nd fraction gives $\quad dq = 0 \quad$ so that $\quad q = a.$

Putting $\quad q = a$ in (1) gives $\; p^2 + px - a = 0 \quad$ so that $\quad p = (1/2) \times \left[-x \pm (x^2 + 4a)^{1/2}\right]$

Putting these values of p and q in $\; dz = pdx + qdy$, we get

$$dz = -(x/2) \times dx \pm (1/2) \times (x^2 + 4a)^{1/2}\, dx + a\, dy.$$

Integrating, the required complete integral is

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}(x^2 + 4a)^{1/2} + 2a \log\{x + (x^2 + 4a)^{1/2}\}\right] + ay + b,$$

Part (ii). Proceed like part (i) yourself. Complete integral is

$$z = \frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}(x^2 + 4a)^{1/2} + 2a \log\{x + (x^2 + 4a)^{1/2}\}\right] + ay + b.$$

Ex. 14. *Find a complete integral of* $pxy + pq + qy = yz$. **[Delhi B.A. (Prog) H 2010]**

[Garhwal 2001; Rohilkhand 1999; Meerut 2001, 02; Kanpur 2005]

Sol. Given $\quad f(x, y, z, p, q) \equiv pxy + pq + qy - yz = 0. \quad$ ...(1)

Charpit's auxiliary equation are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{0} = \frac{dq}{(px+q)+qy} = \frac{dz}{-p(xy+q)-q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}, \text{ by (1)}$$

The first fraction gives $\quad dp = 0 \quad$ so that $\quad p = a.$

Putting $p = a$ in (1) gives $\quad axy + aq + qy = yz \quad$ so that $\quad q = y(z - ax)/(a + y).$

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = adx + \frac{y(z-ax)}{a+y}dy \quad \text{or} \quad \frac{dz - adx}{z - ax} = \frac{y\,dy}{a+y} = \left(1 - \frac{a}{a+y}\right)dy.$$

Integrating, $\quad \log(z - ax) = y - a \log(a + y) + \log b$, a, b, being arbitrary constants.

or $\quad \log(z - ax) + \log(a + y)^a - \log b = y \quad$ or $\quad (z - ax)(y + a)^a = be^y$

Ex. 15. *Find a complete integral* $p^2 + q^2 - 2px - 2qy + 1 = 0$.

[Patna 2003; Meerut 99, 2003; Delhi Maths Hons 91; Ravishanker 2010]

Sol. Given $$f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 1 = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-pf_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-p(2p-2x) - q(2q-2y)} = \frac{dx}{-(2p-2y)} = \frac{dy}{-(2q-2y)}, \text{ by (1)}$$

The first two fractions give $(1/p)dp = (1/q)dq$ so that $p = aq$.

Putting $p = aq$ in (1), $a^2q^2 + q^2 - 2aqx - 2qy + 1 = 0$ or $(a^2 + 1)q^2 - 2(ax - y)q + 1 = 0$.

$$\Rightarrow \; q = \frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}, \; p = aq \; = a\,\frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}.$$

Putting these values of p and q in $dz = p\,dx + y\,dy$, we get

$$dz = \frac{(ax+y) \pm \sqrt{\{(ax+y)^2 - (a^2+1)\}}}{(a^2+1)}(adx + dy). \qquad ...(2)$$

Put $ax + y = v$ so that $a\,dx + dy = dv$. Then (2) gives

$$(a^2 + 1)dz = [v \pm \sqrt{\{v^2 - (a^2+1)\}}\,]dv.$$

Integrating, $$(a^2 + 1)\,z = v^2/2 \pm [\,(v/2) \times \sqrt{\{v^2 - (a^2+1)\}}$$
$$- (1/2) \times (a^2 + 1) \log (v + \sqrt{\{v^2 - (a^2+1)\}}\,)] + b$$

is the complete integral, where $v = ax + b$ and a, b are arbitrary constants.

Ex. 16. *Find a complete integral of* $p^2 + q^2 - 2px - 2qy + 2xy = 0$. **[PCS (U.P.) 2001; Garhwal 1993; Delhi 1997; Kanpur 1996; I.A.S. 1999; Meerut 2003; Rohitkhand 1998]**

Sol. Given equation is $$f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $$\frac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \frac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \frac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \frac{dx}{-\dfrac{\partial f}{\partial p}} = \frac{dy}{-\dfrac{\partial f}{\partial q}}$$

or $$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}, \text{ by (1)}$$

which gives $$\frac{dp+dq}{2(x+y-p-q)} = \frac{dx+dy}{2(x+y-p-q)}$$

or $dp + dq = dx + dy$ *i.e.*, $dp - dx + dq - dy = 0$.

Integrating, $$(p - x) + (q - y) = a \qquad ...(2)$$

Re–writing (1), $$(p - x)^2 + (q - y)^2 = (x - y)^2. \qquad ...(3)$$

Putting the value of $(q - y)$ from (2) in (3), we get

$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2$ or $2(p - x)^2 - 2a(p - x) + \{a^2 - (x - y)^2\} = 0$.

$$\therefore \; p - x = \frac{2a \pm \sqrt{[4a^2 - 4.2.\{a^2 - (x-y)^2\}]}}{4} \quad \Rightarrow \quad p = x + \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}],$$

$$\therefore \text{ (2) gives} \quad q = a + y - p + x \quad \text{or} \quad q = y + (1/2) \times [a \mp \sqrt{\{2(x-y)^2 - a^2\}}].$$

Putting these value of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = x\,dx + y\,dy + (a/2) \times (dx + dy) \pm (1/2)\sqrt{\{2(x-y)^2 - a^2\}}(dx - dy)$$

or
$$dz = x\,dx + y\,dy + \frac{a}{2}(dx + dy) \pm \frac{1}{\sqrt{2}}\sqrt{(x-y)^2 - a^2/2}.(dx - dy).$$

Integrating, the desired complete integral is

$$z = \frac{x^2+y^2}{2} + \frac{a(x+y)}{2} \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2 - a^2/2} - \frac{a^2}{4}\log\left[(x-y) + \sqrt{(x-y)^2 - a^2/2}\right]\right)$$

Ex. 17. *Find a complete integral of* $p^2x + q^2y = z$. **[Gujarat 2005; K.U. Kurukshetra 2001; Meerut 2008; Agra 2004; I.A.S. 2004, 06 ; Delhi Maths Hons. 1997; Punjab 2001]**

Sol. Given equation is $f(x, y, z, p, q) = p^2x + q^2y - z = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} - \frac{dy}{-f_q}$$

or
$$\frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}, \text{ by (1)} \qquad ...(2)$$

Now, each fraction in (2) $$= \frac{2px\,dp + p^2dx}{2px(-p+p^2) + p^2(-2px)} = \frac{2qy\,dq + q^2dy}{2qy(-q+q^2) + q^2(-2qy)}$$

or
$$\frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2qy} \qquad i.e., \qquad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}.$$

Integrating it, $\log(p^2x) = \log(q^2y) + \log a$ or $p^2x = q^2ya$. ...(3)

Form (1) and (3), $aq^2y + q^2y = z$ or $q = [z/(1+a)]^{1/2}$. ...(4)

Form (3) and (4),
$$p = q\left(\frac{ya}{x}\right)^{1/2} = \left\{\frac{za}{(1+a)x}\right\}^{1/2}.$$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2}dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2}dy \quad \text{or} \quad (1+a)^{1/2}z^{-1/2}\,dz = \sqrt{a}x^{-1/2}dx + y^{-1/2}dy.$$

Integrating, $(1+a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$, a, b being arbitrary constants.

Ex. 18. *Find a complete integral of* $2z + p^2 + qy + 2y^2 = 0$. **[I.F.S. 2005; Meerut 2000; Rohilkhand 1993; Bilaspur 2004, M.D.U Rohtak 2005; Rawa 1999; Ranchi 2010]**

Sol. Given equation is $f(x, y, z, p, q) = 2z + p^2 + qy^2 + 2y^2 = 0$. ... (1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dp}{-f_p} = \frac{dq}{-f_q}$$

or
$$\frac{dp}{0+2p} = \frac{dq}{(q+4y)+2q} = \frac{dz}{-p\times(2p) - qy} = \frac{dx}{-2p} = \frac{dy}{-y}, \text{ by (1)}$$

Taking the first and fourth fractions, $dp = -\,dx$.

Integrating, $p = a - x$ or $p = -(x - a)$. ...(2)

Using (2), (1) becomes $2z + (a-x)^2 + qy + 2y^2 = 0$

$\therefore$ $q = -[2z + (x-a)^2 + 2y^2]/y$. ...(3)

$\therefore \qquad dz = p\,dx + q\,dy = -(x-a)\,dx - [\{2z + (x-a)^2 + 2y^2\}/y]dy$, by (2) and (3)

Multiplying both sides by $2y^2$ and re–writing, we have

$$2y^2dz = -2\,(x-a)y^2\,dx - 4zydy - 2y(x-a)^2dy - 4y^3dy$$

or
$$2(y^2dz + 2zy\,dy) + [2(x-a)^2y^2dx + 2y(x-a)^2dy] + 4y^3dy = 0$$

or
$$2d(y^2z) + d[y^2(x-a)^2] + 4y^3dy = 0.$$

Integrating, $\quad 2y^2z + y^2(x-a)^2 + y^4 = b$, a, b being arbitrary constants

Ex. 19(a). *Find a complete integral of* $2(z + px + qy) = yp^2$.

[Delhi B.A. (Prog.) II 2007, 10; CDLU 2004; Delhi Maths Hons. 1998, 2008]

Sol. Given equation is $\quad f(x, y, z, p, q) = 2(z + px + qy) - yp^2 = 0 \quad$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-pf_p - q\,f_q} = \dfrac{dp}{-f_p} = \dfrac{dq}{-f_q}$

or
$$\frac{dp}{2p+2p} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p\,(2x-2yp)-q\times 2y} = \frac{dx}{-(2x-2yp)} = \frac{dy}{-2y}, \text{ by (1)}$$

Taking the first and the last fractions, $\quad \dfrac{dp}{4p} = \dfrac{dy}{-2y} \quad$ or $\quad \dfrac{dp}{p} + 2\dfrac{dy}{y} = 0.$

Integrating, $\quad \log p + 2\log y = \log a \quad$ or $\quad py^2 = a. \quad$ ...(2)

Solving (1) and (2) for p and q, $\quad p = \dfrac{a}{y^2} \quad$ and $\quad q = -\dfrac{z}{y} - \dfrac{ax}{y^3} + \dfrac{a^2}{2y^4}.$

$$\therefore \qquad dz = p\,dx + q\,dy = \frac{a}{y^2}dx + \left[-\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}\right]dy$$

Multiplying both sides by y and re–arranging, we get

$$(ydz + zdy) - a\left(\frac{y\,dx - x\,dy}{y^2}\right) - \frac{a^2}{2y^3}\,dy = 0 \qquad \text{or} \qquad d(yz) - ad\left(\frac{x}{y}\right) - \frac{a^2}{2}y^{-3}\,dy = 0.$$

Integrating, $\quad yz - a(x/y) + (a^2/4y^2) = b$, a, b being arbitrary constants. $\quad$ ... (3)

Ex. 19(b). *Find the complete integral, general integral and the singular integral of* $2(z + xp + yq) = yp^2$ **[Delhi B.Sc. (H) 1998, 2008]**

Sol. Proceed as in solved Ex. 19(a) to get the complete integral (3).

General integral. Replacing b by $\phi(a)$ in (3), we get

$$yz - a(x/y) + (a^2/4y^2) = \phi(a) \qquad \text{... (4)}$$

Differentiating (4) partially w.r.t. 'a', $\quad -(x/y) + (a/2y^2) = \phi'(a) \quad$ ... (5)

Then the general integral is obtained by eliminating a from (4) and (5).

Singular integral. Differentiating (3) partially w.r.t. 'a' and 'b' by turn, we get

$-(x/y) + (a/2y^2) = 0 \quad$ ... (6) $\qquad 0 = 1 \quad$ ... (7)

Relation (7) is absurd and hence there is no singular solution of the given equation.

Ex. 20. *Find a complete integral of* $z^2 = pqxy$. **[Delhi B.A. (Prog) II 2010]**

[Delhi Maths (H) 2004 Jabalpur 2004; Meerut 2006; Lucknow 2010]

Sol. The given equation is $\quad f(x, y, z, p, q) = z^2 - pqxy = 0. \quad$ ...(1)

Charpits's auxiliary equations are $\quad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{-pqy+2pz}=\frac{dq}{-pqx+2qz}=\frac{dz}{-p(-qxy)-q(-pxy)}=\frac{dx}{qxy}=\frac{dy}{pxy}, \text{ by (1)} \qquad ...(2)$$

Each fraction of (2) $= \dfrac{x\,dp+p\,dx}{x(-pqy+2pz)+pqxy} = \dfrac{y\,dq+q\,dy}{y(-pqx+2qz)+pqxy}$

or $$\frac{xdp+pdx}{2pxz}=\frac{ydq+qdy}{2qyz} \quad \text{or} \quad \frac{d(xp)}{xp}=\frac{d(yq)}{yq}.$$

Integrating, $\log(xp) = \log(yq) + \log a^2$ or $xp = a^2yq$. ...(3)

Solving (1) and (2) for p and q, $p = (az)/x$ and $q = z/(ay)$.

$\therefore\ dz = p\,dx + q\,dy = (az/x)\,dx + (z/ay)\,dy$ or $(1/z)\,dz = (a/x)\,dx + (1/ay)\,dy$.

Integrating, $\log z = a \log x + (1/a)\log y + \log b$ or $z = x^a y^{1/a}\, b$.

Ex. 21. *Using Charpit's method, find three complete integrals of $pq = px + qy$.*

(Kanpur 2004; Meerut 2002; Rajasthan 2001)

Sol. Here given equation is $f(x, y, z, p, q) = pq - px - qy = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_z}=\frac{dz}{-p\,f_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-p}=\frac{dq}{-q}=\frac{dz}{-p(q-x)-p(p-y)}=\frac{dx}{-(q-x)}=\frac{dy}{-(p-y)}, \text{ by (1)} \qquad ...(2)$$

To find first complete integral. Taking the first two fractions of (2), we get

$(1/p)dp = (1/q)dq$ so that $\log p = \log q + \log a$ or $p = aq$. ...(3)

Using (3), (1)$\Rightarrow$ $aq^2 = q(ax + y)$ $\Rightarrow$ $q = (ax + y)/a$. ...(4)

Hence, from (3), we have $p = ax + y$. ...(5)

$\therefore\ dz = p\,dx + q\,dy = (ax + y)dx + [(ax + y)/a]dy = (1/a)(ax + y)(a\,dx + y)$.

Putting $ax + y = t$ so that $adx + dy = dt$, we get

$dz = (1/a)\times t\,dt$ so that $z = (1/2a) \times t^2 + b$ or $z = (1/2a) \times (ax + y)^2 + b$, as $t = ax + y$.

To find second complete integral. Taking the second and the fourth ratios in (2), we get

$dx/(q - x) = dq/q$ or $q\,dx + x\,dq = q\,dq$.

Integrating, $qx = q^2/2 + a/2$ or $q^2 - 2xq + a = 0$.

$\therefore\ q = [2x \pm 2(x^2-a)^{1/2}]/2$ so that $q = x + (x^2-a)^{1/2}$. ...(6)

Using (6), (1) $\Rightarrow$ $p[x + (x^2-a)^{1/2}] - px - y[x + (x^2-a)^{1/2}] = 0$

so that $$p = \left\{1+x/(x^2-a)^{1/2}\right\}y. \qquad ...(7)$$

$$\therefore \quad dz = p\,dx + q\,dy = \left\{1+x/(x^2-a)^{1/2}\right\} ydx + [x + (x^2-a)^{1/2}]dy$$

or $$dz = (y\,dx + x\,dy) + \left[\frac{xy\,dy}{(x^2-a)^{1/2}} + (x^2-a)^{1/2}dy\right] \quad \text{or} \quad dz = d(xy) + d[y\,(x^2-a)^{1/2}].$$

Integraing, $z = xy + y\,(x^2-a)^{1/2} + b$, a, b being arbitrary constants.

To find third complete integral. Taking the first and the fifth ratios of (2) and proceeding as above third complete integral is $z = xy + x\,(y^2-a)^{1/2} + b$.

Ex. 22. *Find complete integral of $xp + 3yq = 2(z - x^2q^2)$.* **[Delhi B.Sc. (Prog) II 2009; Delhi B.Sc. (Hons) II 2010;]**

Sol. Given equation is $f(x, y, z, p, q) = xp + 3yq - 2z + 2x^2q^2 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-p+4xq^2} = \frac{dq}{q} = \frac{dz}{-p\,x - q(3y+4x^2q)} = \frac{dx}{-x} = \frac{dy}{-3y-4x^2q}, \text{ by (1)} \qquad ...(2)$$

(2) $\Rightarrow$ $\dfrac{dq}{q} = \dfrac{dx}{-x}$ $\Rightarrow$ $\log q = \log a - \log x$ $\Rightarrow$ $qx = a$ $\Rightarrow$ $q = \dfrac{a}{x}$. ...(3)

Using (3), (1) $\Rightarrow$ $xp + 3y(a/x) - 2z + 2x^2(a^2/x^2) = 0$ $\Rightarrow$ $p = \dfrac{2(z-a^2)}{x} - \dfrac{3ay}{x^2}$. ...(4)

$$\therefore \qquad dz = p\,dx + q\,dy = \left\{\frac{2(z-a^2)}{x} - \frac{3ay}{x^2}\right\}dx + \frac{a}{x}dy$$

or $x^2dz = 2x(z - a^2)dx - 3ay\,dx + ax\,dy$ or $x^2dz - 2x(z - a^2)\,dx = -3ay\,dx + ax\,dy$

or $$\frac{x^2dz - 2x(z-a^2)dx}{x^4} = -\frac{3ay\,dx}{x^4} + \frac{a\,dy}{x^3} \quad \text{or} \quad d\left(\frac{z-a^2}{x^2}\right) = d\left(\frac{ay}{x^3}\right)$$

Integrating, $(z - a^2)/x^2 = (ay)/x^3 + b$ or $z = a(a + y/x) + bx^2$.

Ex. 23. *Find complete integrals of the following equations* :

(*i*) $(p^2 + q^2)^n (qx - py) = 1$.

(*ii*) $qx + py = (p^2 - q^2)^n$.

Sol. (*i*) Given equation is $f(x, y, z, p, q) = (p^2 + q^2)^n (qx - py) - 1 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $\dfrac{dp}{q(p^2+q^2)^n} = \dfrac{dq}{-p(p^2+q^2)^n} = \ldots$ or $\dfrac{dp}{q} = \dfrac{dq}{-p}$ or $pdp + qdq = 0$.

Integrating, $p^2 + q^2$ = constant = $(1/a^2)$, say ...(2)

Using (2), (1) $\Rightarrow qx - py = a^{2n}$ or $qx = py + a^{2n}$. ...(3)

Using (3), (2) $\Rightarrow$ $p^2 + (p^2y^2 + a^{4n} + 2a^{2n}yp)/x^2 = 1/a^2$

or $p^2(x^2 + y^2) + 2a^{2n}yp + \{a^{4n} - (x^2/a^2)\} = 0$ so that

$$p = \frac{-ya^{2n} + \sqrt{\left\{a^{4n}y^2 - (x^2+y^2)\left(a^{4n} - x^2/a^2\right)\right\}}}{x^2+y^2} = \frac{-ya^{2n} + x\sqrt{\left[\left\{(x^2+y^2)/a^2\right\} - a^{4n}\right]}}{x^2+y^2} \qquad ...(4)$$

$$\therefore \qquad (3) \Rightarrow \qquad q = \frac{xa^{2n} + y\sqrt{\left[\left\{(x^2+y^2)/a^2\right\} - a^{4n}\right]}}{x^2+y^2}. \qquad ...(5)$$

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = a^{2n}\left(\frac{x\,dy - y\,dx}{x^2+y^2}\right) + \frac{x\,dx + y\,dy}{x^2+y^2}\sqrt{\left\{\left(\frac{x^2+y^2}{a^2}\right) - a^{4n}\right\}}.$$

Integrating, $z + b = a^{2n}\tan^{-1}\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\displaystyle\int \dfrac{1}{u}(ua^{-2} - a^{4n})^{1/2}\,du$, where $u = x^2 + y^2$.

Part (ii). Proceed as in part (i). If $u = x^2 + y^2$, then complete integral is

$$z + b = -\frac{1}{2}a^{2n}\log\frac{x-y}{x+y} - \frac{1}{2}\int\frac{1}{u}\sqrt{(a^{4n}+a^2u)}\,du.$$

Ex. 24. *Find complete integral of* $p^2 + q^2 - 2pq \tanh 2y = \operatorname{sech}^2 2y$.

Sol. Given $\qquad f(x, y, z, p, q) = p^2 + q^2 - 2pq \tanh 2y - \operatorname{sech}^2 2y = 0. \qquad$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\quad \dfrac{dp}{0} = \dfrac{dq}{-4pq\operatorname{sech}^2 2y + 4\operatorname{sech}^2 2y \tanh 2y} = \cdots$, by (1)

Then, $\quad$ first fraction $\Rightarrow \quad dp = 0 \quad \Rightarrow \quad p$ = constant = a, say. $\qquad$ ...(2)

Using (2), $\quad$ (1) $\Rightarrow \quad q^2 - (2a \tanh 2y)q + a^2 - \operatorname{sech}^2 2y = 0$

$\Rightarrow \quad q = [2a \tanh 2y \pm 2\sqrt{(a^2\tanh^2 2y - a^2 + \operatorname{sech}^2 2y)}\,]/2$

$\Rightarrow \quad q = a \tanh 2y + \sqrt{(1-a^2)}$. sech 2y. $\qquad$ ...(3)

[Note that $\operatorname{sech}^2 2y = 1 - \tanh^2 2y$]

Using (2) and (3), $dz = p\,dx + q\,dy$ reduces to

$$dz = a\,dx + \{a \tanh 2y + \sqrt{(1-a^2)}\operatorname{sech} 2y\}dy$$

Integrating, $\quad z + b = ax + \dfrac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\displaystyle\int\frac{2dy}{e^{2y}+e^{-2y}}$

or $\quad z + b = ax + \dfrac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\displaystyle\int\frac{2e^{2y}dy}{1+(e^{2y})^2}$

or $\quad z + b = ax + \dfrac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\tan^{-1}(e^{2y})$,

$\left[\because \text{ on putting } e^{2y} = t \text{ and } 2e^{2y}\,dy = dt, \ \displaystyle\int\frac{2e^{2y}dy}{1+(e^{2y})^2} = \int\frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1} e^{2y}\right]$

Ex. 25. *Find complete integral of the equation* $q = \{(1 + p^2)/(1 + y^2)\}x + yp(z - px)^2$.

Sol. Let $\qquad f(x, y, z, p, q) = \{(1 + p^2)/(1 + y^2)\}x + yp(z - px^2) - q = 0. \qquad$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\quad \dfrac{dp}{\{(1+p^2)/(1+y^2)\} - 2yp^2(z-px) + 2yp^2(z-px)} = \dfrac{dy}{1} = \ldots\ldots\ldots$, by (1)

or $\quad \dfrac{dp}{1+p^2} = \dfrac{dy}{1+y^2} \quad$ so that $\quad \tan^{-1} p - \tan^{-1} y$ = constant = $\tan^{-1} a$

$\Rightarrow \quad (p - y)/(1 + py) = a \quad \Rightarrow \quad p = (y + a)/(1 - ay). \qquad$ ...(2)

Using (2), $\quad$ (1) $\Rightarrow \quad q = \dfrac{1+a^2}{(1-ay)^2}x + \dfrac{y(y+a)}{(1-ay)^3}\{z(1-ay) - x(y+a)\}^2. \qquad$ ...(3)

Using (2) and (3), $dz = p\,dx + q\,dy$ reduces to

$$dz = \frac{y+a}{1-ay}dx + \left[\frac{1+a^2}{(1-ay)^2}x + \frac{y(y+a)}{(1-ay)^3}\{z(1-ay)-x(y+a)\}^2\right]dy$$

or $dz = d\left(\dfrac{y+a}{1-ay}x\right)+\dfrac{y(y+a)}{(1-ay)^3}\{z(1-ay)-x(y+a)\}^2 dy$ or $dz = du + \dfrac{y(y+a)}{1-ay}(z-u)^2 dy$, ...(4)

where $u = x(y+a)/(1-ay)$. ...(5)

(4) $\Rightarrow$ $\dfrac{dz-du}{(z-u)^2} = \dfrac{y(y+a)}{1-ay}\text{dy}.$ or $\dfrac{d(z-u)}{(z-u)^2}=\left\{-1-\dfrac{1}{a^2}(1+ay)+\dfrac{a^2+1}{a^2}\dfrac{1}{1-ay}\right\}dy.$

Integrating, $b-\dfrac{1}{z-u} = -y-\dfrac{1}{a^2}\left(y+\dfrac{ay^2}{2}\right)-\dfrac{a^2+1}{a^3}\log(1-ay)$, where u is given by (5).

Ex. 26. Find complete integral of $xp - yq = xq\,f(z-px-qy)$.

Sol. Let $F(x, y, z, p, q) = xp - yq - xq\,f(z-px-qy) = 0.$ ...(2)

Charpit's auxiliary equations are

$$\frac{dp}{\partial F/\partial x+p(\partial F/\partial z)}=\frac{dq}{\partial F/\partial y+q(\partial F/\partial z)}=\frac{dz}{-p(\partial F/\partial p)-q(\partial F/\partial q)}=\frac{dx}{-(\partial F/\partial p)}=\frac{dy}{-(\partial F/\partial q)}$$

or $$\frac{dp}{p-qf+xqpf'-pqxf'}=\frac{dq}{-q+xq^2f'-xq^2f'}=\cdots\text{, by (2)} \qquad ...(3)$$

Each ratio of (3) $= \dfrac{x\,dp+y\,dq}{xp-yq-qxf}=\dfrac{x\,dp+y\,dq}{0}$, by (2)

$\Rightarrow$ $x\,dp + y\,dq = 0$ $\Rightarrow$ $x\,dp + y\,dq + p\,dx + q\,dy = p\,dx + q\,dy$

$\Rightarrow$ $dz - d(xp) - d(yq) = 0$, as $dz = pdx + qdy$

Integrating, $z - xp - yq =$ constant $= a$, say ...(4)

$\therefore$ $xp + yq = z - a.$ ...(5)

Using (4), (1) becomes $x\,p - y\,q = x\,q\,f(a).$ ...(6)

Subtracting (6) from (5), $2yq = z - a - xqf(a)$ $\Rightarrow$ $q = (z-a)/\{2y + xf(a)\}$ ...(7)

Using (7), (5) $\Rightarrow$ $p = \dfrac{(z-a)\{y+xf(a)\}}{x\{2y+xf(a)\}}.$ ...(8)

Using (7) and (8), $dz = p\,dx + q\,dy$ reduces to

$$dz = (z-a)\left[\frac{\{y+xf(a)\}\,dx}{x\{2y+xf(a)\}}+\frac{dy}{2y+xf(a)}\right]$$

or $$\frac{2dz}{z-a} = \frac{2y\,dx+2xf(a)dx+2x\,dy}{x\{2y+xf(a)\}}=\frac{2d(xy)+2xf(a)\,dx}{2xy+x^2f(a)}.$$

Integrating, $2\log(z-a) = \log\{2xy + x^2f(a)\} + \log b$ or $(z-a)^2 = b\,\{2xy + x^2f(a)\}$.

Ex. 27. *Find a complete integral of* $px + qy = z(1 + pq)^{1/2}$

[Meerut 2001, 02; Kanpur 1995, I.A.S. 1992]

Sol. Given $f(x, y, z, p, q) = px + qy - z(1 + pq)^{1/2} = 0.$ ...(1)

Charpit's auxiliary equation are $\dfrac{dp}{f_x+p\,f_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-pf_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

or $\dfrac{dp}{p-p(1+pq)^{1/2}}=\dfrac{dq}{q-q(1+pq)^{1/2}}=\ldots\ldots\ldots$ so that $\dfrac{dp}{p}=\dfrac{dq}{q}$, by (1)

$\Rightarrow$ $\log p = \log a + \log q$ $\Rightarrow$ $p = aq.$ ...(2)

Using (2), (1) $\Rightarrow$ $q(ax + y) = z\,(1 + aq^2)^{1/2}$ or $q^2\,[(ax + y)^2 - az^2] = z^2.$

$\therefore \quad q = \dfrac{z}{[(ax+y)^2 - az^2]^{1/2}}$ and $p = aq = \dfrac{az}{[(ax+y)^2 - az^2]^{1/2}}$.

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{z(a\,dx+dy)}{\sqrt{\{(ax+y)^2 - az^2\}}} \quad \text{or} \quad \frac{dz}{z} = \frac{a\,dx+dy}{\sqrt{\{(ax+y)^2 - az^2\}}}. \qquad ...(3)$$

Let $ax + y = \sqrt{a}\,u$ so that $a\,dx + dy = \sqrt{a}\,du$.

$$\therefore \quad (3) \Rightarrow \quad \frac{dz}{z} = \frac{\sqrt{a}\,du}{\sqrt{(au^2 - az^2)}} \quad \text{or} \quad \frac{du}{dz} = \frac{\sqrt{(u^2 - z^2)}}{z} = \sqrt{\left\{\left(\frac{u}{z}\right)^2 - 1\right\}}, \qquad ...(4)$$

which is linear homogeneus equation. To solve it, we put

$$\frac{u}{z} = v \quad \text{or} \quad u = vz \quad \text{so that} \quad \frac{du}{dz} = v + z\frac{dv}{dz}.$$

$$\therefore \quad (4) \text{ yields} \quad v + z\frac{dv}{dz} = (v^2 - 1)^{1/2}. \quad \text{or} \quad \frac{dz}{z} = \frac{dv}{(v^2-1)^{1/2} - v}$$

or $\qquad (1/z)dz = -\left[(v^2-1)^{1/2} + v\right]dv$, on rationalization.

Integrating, $\log z = -\left[\dfrac{v}{2}(v^2-1)^{1/2} - \dfrac{1}{2}\log\{v + (v^2-1)^{1/2}\}\right] - \dfrac{v^2}{2} + b$, where, $v = \dfrac{u}{z} = \dfrac{ax+y}{z\sqrt{a}}$

Ex. 28. *Find complete integral of* $(x^2 - y^2)\,pq - xy(p^2 - q^2) = 1$.

Sol. Let $\qquad f(x, y, z, p, q) = (x^2 - y^2)\,pq - xy(p^2 - q^2) - 1 = 0. \qquad ...(1)$

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\quad \dfrac{dp}{2pqx - z(p^2-q^2)} = \dfrac{dq}{-2pqy - x(p^2-q^2)} = \dfrac{dx}{-(x^2-y^2)y + 2pxy} = \dfrac{dy}{-(x^2-y^2)p - 2pxy}$, by (1)

Using x, y, p, q as multipliers, each fraction $= \dfrac{x\,dp + y\,dq + p\,dx + q\,dy}{0} = \dfrac{d(xp) + d(yq)}{0}$

$$\Rightarrow \quad d(xp + qy) = 0 \quad \Rightarrow \quad xp + yq = a \quad \Rightarrow \quad p = (a - qy)/x. \qquad ...(2)$$

Using (2), $\qquad (1) \Rightarrow \quad (x^2 - y^2)\left(\dfrac{a - qx}{x}\right)q - xy\left[\left(\dfrac{a-qy}{x}\right)^2 - q^2\right] - 1 = 0$

or $\quad \dfrac{a-qy}{x}\{(x^2 - y^2)q - (a - qy)y\} + xyq^2 - 1 = 0$ or $\{(a - qy)/x\}\,(x^2q - ay) + xyq^2 - 1 = 0$

or $\quad (a - qy)\,(x^2q - ay) + x^2yq^2 - x = 0$ $\quad$ or $\quad aq(x^2 + y^2) = a^2y + x$

$$\therefore \quad q = \frac{a^2y + x}{a(x^2+y^2)} \quad \text{and} \quad p = \frac{1}{x}\left[a - \frac{(a^2y+x)y}{a(x^2+y^2)}\right] = \frac{a^2x - y}{a(x^2+y^2)}.$$

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{(a^2x - y)dx + (a^2y + x)dy}{a(x^2+y^2)} = a\frac{x\,dx + y\,dy}{x^2+y^2} + \frac{x\,dy - y\,dx}{a(x^2+y^2)}.$$

Integrating, $\qquad z = (a/2) \times \log(x^2 + y^2) + (1/a) \times \tan^{-1}(y/x) + b.$

Ex. 29. *Find a complete integral of* $2(pq + yp + qx) + x^2 + y^2 = 0$. **[Kanpur 1993]**

Sol. Given equation is $\quad f(x, y, z, p, q) = 2(pq + yp + qx) + x^2 + y^2 = 0. \quad$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{2q+2x}=\frac{dq}{2p+2y}=\frac{dz}{-p(2q+2y)-q(2p+2x)}=\frac{dx}{-(2q+2y)}=\frac{dy}{-(2p+2x)}, \text{ by (1)}$$

Each of these above fractions $= \dfrac{dp+dq+dx+dy}{(2q+2x)+(2p+2y)-(2q+2y)-(2p+2x)}$

$= (dp + dq + dx + dy)/0$

This $\Rightarrow \quad dp + dq + dx + dy = 0 \quad$ so that $\quad (p + x) + (q + y) = a. \quad$ ...(2)

Re–writing (1), $2(p + x)(q + y) + (x - y)^2 = 0$ or $(p + x)(q + y) = -(x - y)^2/2. \quad$ ...(3)

Now, $\quad (p + x) - (q + y) = \sqrt{\{(p+x)^2+(q+y)\}^2-4(p+x)(q+y)}$

$\therefore \quad (p + x) - (q + y) = \sqrt{a^2+2(x-y)^2}$, using (2) and (3) $\quad$ ...(4)

Adding (2) and (4), $\quad 2(p + x) = a + \sqrt{a^2+2(x-y)^2}$.

Substracting (4) from (2), $\quad 2(q + y) = a - \sqrt{a^2+2(x-y)^2}$.

These give $\quad p = -x + \dfrac{a}{2}+\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}, \qquad q = -y + \dfrac{a}{2}-\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}$

Substituting the above values of p and q, $dz = p\,dx + q\,dy$ becomes

$$dz = -(x\,dx + y\,dy) + (a/2) \times (dx + dy) + (1/2)\times\sqrt{a^2+2(x-y)^2}\;(dx - dy)$$

or $$dz = -\frac{1}{2}d(x^2 + y^2) + \frac{a}{2}d(x + y) + \sqrt{2}\times\frac{1}{2}\sqrt{\frac{a^2}{2}+(x-y)^2}\;d(x - y) \qquad \text{...(5)}$$

Put $\quad x - y = t \quad$ so that $\quad d(x - y) = dt$. Then (5) becomes

$$dz = -(1/2) \times d(x^2 + y^2) + (a/2) \times d(x + y) + (1/\sqrt{2}) \times \sqrt{(a/\sqrt{2})^2+t^2}\;dt.$$

$$\therefore \quad z = -\frac{x^2+y^2}{2}+a\frac{x+y}{2}+\frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{(a/\sqrt{2})^2+t^2}+\frac{(a/\sqrt{2})^2}{2}\log\left\{t+\sqrt{(a/\sqrt{2})^2+t^2}\right\}\right]+b$$

Putting the value of t, the required complete integral is

$$z=-\frac{x^2+y^2}{2}+\frac{a(x+y)}{2}+\frac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\frac{a^2}{2}+(x-y)^2}+\frac{a^2}{2}\log\left\{x-y+\sqrt{\frac{a^2}{2}+(x-y)^2}\right\}\right]+b.$$

Ex. 30. *Solve* $z=(1/2)\times(p^2+q^2)+(p-x)(q-y)$ **[I.A.S. 2002]**

Sol. Given $\quad z=(1/2)\times(p^2+q^2)+(p-x)(q-y)$

Re–writing (1), $\quad f(x,y,z,p,q)=(1/2)\times(p^2+q^2)+pq-xq-yp+xy-z=0 \quad$ ... (2)

Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_z}=\frac{dz}{-p\,f_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-q+z-p}=\frac{dq}{-p+x-q}=\frac{dz}{-p(p+q-y)-q(p+q-x)}=\frac{dx}{-(p+q-y)}=\frac{dy}{-(p+q-x)}$$

Taking the first and the fourth fractions, we have

$dp = dx$ so that $p = x + a$, a being an arbitratry constant. ... (3)

Taking the second and the fifth fractions, we have

$dq = dy$ so that $q = y + b$, b being an arbitrary constant ... (4)

Putting $p = x + a$ and $q = y + b$ in (1), the required solution is

$z = (1/2) \times \{(x+a)^2 + (y+b)^2\} + ab$, a and b being arbitrary constants.

Ex. 31 *Find a complete integral of* $z = pq$. **[Sagar 2004, Ravishankar 2003, Rewa 2003]**

Sol. Here given equation is $f(x, y, z, p, q) = z - pq = 0$ ... (1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-p} = \frac{dq}{-q} = \frac{dz}{-2pq} = \frac{dx}{-q} = \frac{dy}{-p}, \text{ by (1)} \qquad ... (2)$$

Taking the first and last fractions of (2), $dp = dy$

Integrating $p = y + a$, a being an arbitrary constant ... (3)

Similarly, taking the second and fourth fractions of (2), we get

$dq = dx$ so that $q = x + b$, b being an arbitrary constant. ... (4)

Putting values of p and q given by (3) and (4) in (1), we get

$z = (x+b)(x+a)$, which is the required complete integral.

Ex. 32. *Use Charpit's method to find the complete integral of* $2x\ \{z^2(\partial z/\partial y)^2 + 1\} = z(\partial z/\partial x)$.

[I.A.S. 1998]

Sol. Given $2x(z\partial z/\partial y)^2 + 2x - (z\partial z/\partial x) = 0$ ... (1)

Let $z\,dz = dZ$ so that $z^2 = 2Z$ ... (2)

Then (1) becomes $2x(\partial Z/\partial y)^2 + 2x - (\partial Z/\partial x) = 0$ or $2xQ^2 + 2x - P = 0$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$ ... (3)

Let $f(x, y, Z, P, Q) = 2x\,Q^2 + 2x - P = 0$ ... (4)

Charpit's auxiliary equations are $$\frac{dP}{f_x + Pf_Z} = \frac{dQ}{f_y + Qf_Z} = \frac{dZ}{-Pf_P - Qf_Q} = \frac{dx}{-f_P} = \frac{dy}{-f_Q}$$

giving $\dfrac{dP}{2Q^2+2} = \dfrac{dQ}{O} = ...$, by (4) so that $dQ = 0$.

Integrating, $Q = a$, a being an arbitrars constant ... (5)

Using $Q = a$, (4) gives $P = 2x\,(a^2 + 1)$, $Q = a$ ... (6)

$\therefore$ $dZ = P\,dx + Q\,dy = 2x(a^2+1)dx + a\,dy$, by (5) and (6)

Integrating, $Z = x^2(a^2+1) + ay + b/2$, or $z^2/2 = x^2(a^2+1) + ay + b/2$, using (2) or $$z^2 = 2x^2(a^2+1)+2ay+b$$, which is complete integral of (1)

Ex. 33. *Solve by Charpit's method the partial differential equation.*

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0. \qquad \textbf{[I.A.S. 2000]}$$

Sol. Let $f(x, y, z, p, q) = p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0$ ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-p\,f_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$ ... (2)

From (1), $f_x = p^2(2x-1)+2pqy-2pz$, $f_y = 2pqx+q^2(2y-1)-2qz$,

$f_z=-2px-2qy+2z$, $f_p = 2px(x-1)+2qxy-2xz$; $f_q = 2pxy+2qy(y-1)-2yz$

and so $f_x+pf_z=-p^2$, $f_y+qf_z=-q^2$. Then (2) becomes

$$\frac{dp}{-p^2}=\frac{dq}{-q^2}=\frac{dz}{-p\{2px(x-1)+2qxy-2xz\}-q\{2pxy+2qy(y-1)-2yz\}}$$

$$=\frac{dx}{-(2px^2-2px+2qxy-2xz)}=\frac{dy}{-(2pxy+2qy^2-2qy-2yz)} \qquad \text{... (3)}$$

Each fraction of (3) = $\dfrac{(1/p)\,dp}{-p}=\dfrac{(1/q)\;dq}{-q}=\dfrac{(1/p)\,dp-(1/q)dq}{-p+q}$ ... (4)

Also, each fraction of (3) = $\dfrac{(1/x)\;dx-(1/y)\;dy}{-2px+2p-2qy+2z+2px+2qy-2q-2z}$ ... (5)

$\therefore$ (4) and (5) $\Rightarrow$ $\dfrac{(1/p)\,dp-(1/q)\;dq}{-(p-q)}=\dfrac{(1/x)\;dx-(1/y)\,dy}{2(p-q)}$

or $$(1/2)\times\{(1/x)\;dx-(1/y)dy\}=(1/q)\;dq-(1/p)\;dp$$

Integrating, $(1/2)\times\{\log x-\log y\}=\log q-\log p+\log a$ or $(x/y)^{1/2}=aq/p$

or $p=(ay^{1/2}q)/x^{1/2}$, a being an arbitrary constant. ... (5)

Re–writing (1), $(px+qy-z)^2=p^2x+q^2y$ or $px+qy-z=\pm(p^2x+qy^2)^{1/2}$ ... (6)

Taking + ive sign in (7), $px+qy-z=(p^2x+q^2y)^{1/2}$ ... (7)

[The case of – ve sign in (7) can be discussed similarly]

Substituting the value of p given by (6) in (8), $aqy^{1/2}x^{1/2}+qy-z=(a^2q^2y+q^2y)^{1/2}$

or $q\{y+a(xy)^{1/2}-(1+a^2)^{1/2}y^{1/2}\}=z$ so that $q=z/y^{1/2}\{y^{1/2}+a\;x^{1/2}-(1+a^2)^{1/2}\}$ ... (9)

Then (6) gives $p=az/x^{1/2}\;\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}$ ... (10)

Putting these values of p and q in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{az\;dx}{x^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}+\frac{z\;dy}{y^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}$$

or $$\frac{dz}{z}=\frac{ay^{1/2}dx+x^{1/2}dy}{(xy)^{1/2}\ \{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}$$

Integrating, $$\log z=2\log\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}+\log b$$

or $z=b\ \{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}^2$, a and b being an arbitrary constants.

Ex. 34. *Find the complete integral of* $(p+q)\ (px+qy)=1$.

[Meerut 2007; Delhi Maths (H) 2007, Purvanchal 2007]

Sol. Let $$f(x,y,z,p,q)=(p+q)\ (px+qy)-1=0 \qquad \text{... (1)}$$

Charpit's auxiliary equations $$\frac{dp}{f_x+p\ f_z}=\frac{dq}{f_y+q\ f_z}=\frac{dz}{-p\,f_p-q\ f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

give $$\frac{dp}{p(p+q)}=\frac{dq}{q(p+q)}=\ldots \qquad \text{so that} \qquad \frac{dp}{p}=\frac{dq}{q}, \text{ using (1)}$$

Integrating, $p=aq$, a being an arbitrary constant ... (2)

Putting $p=aq$ in (2) gives $(aq+q)\ (aqx+qy)-1=0$ or $q^2\ (1+a)\ (ax+y)=1$ ... (3)

$\therefore$ From (2) and (3), $q=1/(1+a)^{1/2}\ (ax+y)^{1/2}$, $p=a/(1+a)^{1/2}(ax+y)^{1/2}$

Putting these values of p and q in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{a\ dx}{(1+a)^{1/2}(ax+y)^{1/2}}+\frac{dy}{(1+a)^{1/2}(ax+y)^{1/2}}=\frac{d(ax+y)}{(1+a)^{1/2}(ax+y)^{1/2}}$$

Integrating, $z(1+a)^{1/2}=2(ax+y)^{1/2}+b$, a, b being arbitrary constants.

Ex. 35. *Find the complete integral of the following partial differential equations*

(a) $px^5-4q^2x^2+6x^2z-2=0$. **[Delhi B.Sc. (H) 2002; Delhi B.A. (Proj) II 2011]**

(b) $px^5-4q^3x^2+6x^2z-2=0$

Sol. (a) Let $$f(x,y,z,p,q)=px^5-4q^2x^2+6x^2z-2=0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_y}=\frac{dz}{-p\,f_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{5px^4-8q^2x+12xz+6px^2}=\frac{dq}{6qx^2}=\frac{dz}{-px^5+8q^2x^2}=\frac{dx}{-x^5}=\frac{dy}{8qx^2}, \text{ by (1)}$$

Taking the second and the last fractions, $4\,dq=3dy$

Integrating, $4q=3y+3a$ or $q=3(y+a)/4$ ... (2)

Using (2), (1) gives $$p=\{(9/4)\times(y+a)^2-6x^2z+2\}/x^5 \qquad \text{... (3)}$$

Putting the above values of p and q in $dz=p\,dx+q\,dy$, we get

$$dz=(9/4x^3)\ (y+a)^2\ dx-(6z/x^3)\ dx+(2/x^5)dx+(3/4)\ (y+a)dy$$

or $$(6z/x^3)dx+dz=\{(9/4x^3)\ (y+a)^2dx+(3/4)\ (y+a)\ dy\}+(2/x^5)dx \qquad \text{... (4)}$$

The total differential equation (4) is always integrable. To solve (4), we first proceed to find the integrating factor of the L.H.S. of (4). Comparing L.H.S. of (4) with $M\,dx + N\,dz$ (here on L.H.S. we have variable x, z in place of usual variables x, y), we have $M = 6z/x^3$ and $N = 1$.

$$\frac{1}{N}\left(\frac{\partial M}{\partial z}-\frac{\partial N}{\partial x}\right)=\frac{6}{x^3}$$, which is function x alone and so I.F. $= e^{\int (6/x^3)dx} = e^{-3/x^2}$.

Multiplying both sides of (4) by I.F. e^{-3/x^2}, we get

$$(6z/x^3)\, e^{-3/x^2} dx + e^{-3/x^2} dz = (3/8) \times \{(6/x^3)\,(y+a)^2\, e^{-3/x^2} dx + 2(y+a)\, e^{-3/x^2} dy\} + (2/x^5)\, e^{-3/x^2} dx$$

or $$d(z\, e^{-3/x^2}) = (3/8) \times d\{(y+a)^2\, e^{-3/x^2}\} + (2/x^5) \times e^{-3/x^2} dx$$

Integrating, $$z\, e^{-3/x^2} = (3/8) \times (y+a)^2\, e^{-3/x^2} + 2\int (1/x^2)\, e^{-3/x^2} (1/x^3) dx$$

or $z\, e^{-3/x^2} = (3/8)\times(y+a)^2\, e^{-3/x^2} - (1/9) \times \int u\, e^u du$, putting $(-3/x^2) = u$ so that $(6/x^3)dx = du$

or $$z\, e^{-3/x^2} = (3/8) \times (y+a)^2\, e^{-3/x^2} - (1/9)\times(ue^u - e^u) + b$$

or $$z\, e^{-3/x^2} = (3/8)\times(y+a)^2\, e^{-3/x^2} - (1/9)\times(-3/x^2)e^{-3/x^2} + (1/9)\times e^{-3/x^2} + b$$

or $z = (3/8)\times(y+a)^2 + (1/3x^2) + (1/9) + be^{3/x^2}$, a, b being arbitrary constants.

(*b*) Proceed exacts as in part (*a*) **Ans.** $z = (2/3)\times(y+a)^{3/2} + (1/3x^2) + (1/9) + b\, e^{3/x^2}$

Ex. 36. *Find the complete integral of* $(p+y)^2 + (q+x)^2 = 1$

Sol. Let $$f(x,y,z,p,q) = (p+y)^2 + (q+x)^2 - 1 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-pf_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{2(q+x)} = \frac{dq}{2(p+y)} = \frac{dz}{-2(p^2+q^2+py+qx)} = \frac{dx}{-2(p+y)} = \frac{dy}{-2(q+x)}$$, by (1)

Taking the first and the last fractions, $dp + dy = 0$ so that $p + y = a$ ... (2)

Using (2), (1) gives $a^2 + (q+x)^2 - 1 = 0$ or $q + x = (1-a^2)^{1/2}$ ... (3)

Using (2) and (3) in $dz = p\,dx + q\,dy$, we get

$$dz = (a-y)dx + \{(1-a^2)^{1/2} - x\}dy = a\,dx - (1-a^2)^{1/2} dy - (y\,dx + x\,dy)$$

Integrating, $z = ax - (1-a^2)^{1/2} y - xy + b$, a, b being arbitrary constants.

Ex. 37. *Find the complete integral of* $2\,(y + zq) = q(xp + yq)$ **[Nagpur 2003, 06; Delhi B.Sc. (Prog) II 2011; Delhi B.Sc. (Hons) 2011]**

Sol. Let $$f(x, y, z, p, q) = 2y + 2zq - xpq - yq^2 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_x - q f_y} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \qquad \text{... (2)}$$

$$\frac{dp}{-pq+2pq} = \frac{dq}{2-q^2+2q^2} = \frac{dz}{2pqx+2qy-2qz} = \frac{dx}{qx} = \frac{dy}{xp+2yq-2z}\text{, by (1)}$$

Taking the first and fourth fractions, $(1/pq)dp = (1/qx)dx$ or $(1/p)dp = (1/x)dx$

Integrating, $\log p = \log a + \log x$ or $p = ax$, ... (3)

where a is an arbitrary constant. Substituting the value of p given by (3) in (1), we have

$2y + 2zq - ax^2q - yq^2 = 0$ or $yq^2 + q(ax^2 - 2z) - 2y = 0.$

$$\Rightarrow \qquad q = [-(ax^2-2z) \pm \{(ax^2-2z)^2 + 8y^2\}^{1/2}]/(2y) \qquad \text{... (4)}$$

Substituting the values of p and q given by (3) and (4) in $dz = p\,dx + q\,dy$, we obtain

$$dz = ax\,dx + (1/2y) \times [2z - ax^2 \pm \{(2z-ax^2)^2 + 8y^2\}^{1/2}]dy$$

or
$$\frac{2dz - 2ax\,dx}{(2z-ax^2) \pm \{(2z-ax^2)^2+8y^2\}^{1/2}} = \frac{dy}{y} \qquad \text{... (5)}$$

Putting $2z - ax^2 = u$ and $2dz - 2\,ax\,dx = du$, (5) yields

$$\frac{du}{u \pm (u^2+8y^2)^{1/2}} = \frac{dy}{y} \quad \text{or} \quad \frac{du}{dy} = \frac{u}{y} \pm \left\{\left(\frac{u}{y}\right)^2 + 8\right\}^{1/2}, \qquad \text{... (6)}$$

which is linear homogeneous differential equation. To solve it, we put $u/y = v$, *i.e.*, $u = yv$ so that $du/dy = v + y(dv/dy)$ and so (6) reduces to

$$v + y\frac{dv}{dy} = v \pm (v^2+8)^{1/2} \quad \text{or} \quad \frac{dv}{(v^2+8)^{1/2}} = \frac{dy}{y},$$

taking positive sign. Integrating it, we have

$$\log\{v + (v^2+8)^{1/2}\} = \log y + \log b \quad \text{or} \quad v + (v^2+8)^{1/2} = by$$

or $\quad u/y + \left\{(u/y)^2 + 8\right\}^{1/2} = by \quad$ or $\quad u + (u^2+8y^2)^{1/2} = by^2$

or $\quad 2z - ax^2 + \{(2z-ax^2)^2 + 8y^2\}^{1/2} = by^2$, as $u = 2z - ax^2$; a, b being arbitrary constants

EXERCISE 3(B)

Using Charpit's method, find a complete integral of the following equations :

1. $z = px + qy + pq$. **[Mysore 2004]** **Ans.** $z = ax + by + ab$
2. $pq = xz$. **Ans.** $z = (a + x^2/2)(b + y)$
3. $p^2 + px + q = z$. **Ans.** $z = ax + a^2 + be^y$
4. $(p + q)(z - px - qy) = 1$. **Ans.** $(a + b)(z - ax - by) = 1$
5. $px + qy + pq = 0$ **Ans.** $az = -(1/2) \times (y + ax)^2 + b$
6. $q = px + q^2$ **Ans.** $z = (a - a^2)\log x + ay + b$
7. $p - 3x^2 = q^2 - y$ **Ans.** $z = x^3 - (1/3) \times (a - x)^3 + ay - xy + b$
8. $x^2p^2 + y^2q^2 = 4$ **Ans.** $z = a\log x + (4 - a^2)^{1/2}\log y + b$
9. $xpq + yq^2 = 1$ **Ans.** $(z + b)^2 = 4(ax + b)$
10. $p + q = 3pq$ **Ans.** $az = b - (1/2) \times (y + ax)^2$
11. $pq + x(2y + 1)p + (y^2 + y)q - (2y + 1)z = 0$ **Ans.** $z = ax + b(a + y + y^2)$
12. $z^2(p^2 + q^2) = x^2 + e^{2y}$.

$$\textbf{Ans.}\ \frac{z^2}{2} = \frac{x\sqrt{(x^2+a)}}{2} + \frac{a}{2}\sinh^{-1}\frac{x}{\sqrt{a}} + \sqrt{(e^{2y}-a)} - \sqrt{a}\tan^{-1}\left(\frac{e^{2y}-a}{a}\right) + b$$

13. $p^2 - y^2 q = x^2 - y^2$ **[Madurai Kamraj 2008]**

Ans. $z = \dfrac{x\sqrt{(x^2+b)}}{2} + \dfrac{b}{2}\sinh^{-1}\dfrac{x}{\sqrt{b}} - \dfrac{b}{2y^2} + \log y + c$

14. $p^2 q\,(x^2 + y^2) = p^2 + q$ **Ans.** $z = \log[x + \sqrt{(x^2+a)}] + \dfrac{1}{2\sqrt{a}}\log\dfrac{y-\sqrt{a}}{y+\sqrt{a}} + b$

15. $yp = 2xy + \log q$. **[Lucknow 2010]** **[Ans.** $z = (a+2x)^2/4 + (1/a)\times e^{ay} + b$]

3.9. Special methods of solutions applicable to certain standard forms:

We now consider equations in which p and q occur other than in the first degree, that is non–linear equations. We have already discussed the general method (*i.e.*, Charpit's method — see Art. 3.7). We now discuss four standard forms to which many equations can be reduced, and for which a complete integral can be obtained by inspection or by other shorter methods.

3.10. Standard Form I. Only p and q present. **[Nagpur 2002; Bhopal 2010]**

Under this standard form, we consider equations of the form $f(p, q) = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

giving $\dfrac{dp}{0} = \dfrac{dq}{0}$, by (1)

Taking the first ratio, $dp = 0$ so that p = constant = a, say ...(2)

Substituting in (1), we get $f(a, q) = 0$, giving q = constant = b, say, ... (3)

$z = (a+2x)^2/4 + (1/a)\times e^{ay} + b$

where b is such that $f(a, b) = 0$. ...(4)

Then, $dz = p\,dx + q\,dy = adx + bdy$, using (2) and (3).

Integrating, $z = ax + by + c$, ...(5)

where c is an arbitrary constant. (5) together with (4) give the required solution.

Now solving (4) for b, suppose we obtain $b = F(a)$, say.

Putting this value of b in (5), the *complete integral* of (1) is

$$z = ax + yF(a) + c, \qquad ...(6)$$

which contains two arbitrary constants a and c which are equal to the number of independent variables, namely x and y.

The *singular integral* of (1) is obtained by eliminating a and c between the complete integral (6) and the equations obtained by differentiating (6) partially w.r.t. a and c ; *i.e.*, between

$z = ax + yF(a) + c,$ $\quad 0 = x + yF'(a)$ $\quad$ and $\quad 0 = 1.$ ...(7)

Since the last equation in (7) is meaningless, we conclude that the equations of standard form I have no singular solution.

In order to find the general integral of (1), we first take $c = \phi(a)$ in (6), ϕ being an arbitrary function and obtain $z = ax + yF(a) + \phi(a)$. ...(8)

Now, we differentiate (8) partially with respect to a and get

$$0 = x + yF'(a) + \phi'(a). \qquad ....(9)$$

Eliminating a between (8) and (9), we get the general solution of (1).

Remark. Sometimes change of variables can be employed to transform a given equation to standard form I.

3.11. SOLVED EXAMPLES BASED ON ART. 3.10

Ex. 1. (a) *Solve $pq = k$, where k is a constant.* **[M.S. Univ. T.N. 2007; Meerut 1995]**

(b) Solve pq = 1 by standard from I **[Bhopal 2010]**

Sol. Given that $$pq = k. \qquad ...(1)$$

Since (1) is of the form $f(p, q) = 0$, its solution is $$z = ax + by + c, \qquad ...(2)$$

where $ab = k$ or $b = k/a$, on putting a for p and b for q in (1).

$\therefore$ From (2), the complete integral is $$z = ax + (k/a)y + c, \qquad ...(3)$$

which contains two arbitrary constants a and c.

For singular solution, differentiating (3) partially with respect to a and c, we get $0 = x - (k/a^2)y$ and $0 = 1$. But $0 = 1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get
$$z = ax + (k/a)y + \phi(a). \qquad ...(4)$$

Differentiating (4) partially with respect to 'a', we get $$0 = x - (k/a^2)y + \phi'(a). \qquad ...(5)$$

Eliminating a from (4) and (5), we get the required general solution.

(*b*) Do like part (a) taking $k = 1$

Ex. 2. *Solve* (*a*) $p^2 + q^2 = m^2$, *where m is a constant.* **[Kanpur 1993]**

(*b*) $p^2 + q^2 = 1$ **[Meerut 2011]**

Sol. (*a*) Given that $$p^2 + q^2 = m^2. \qquad ...(1)$$

Since (1) is of the form $f(b, q) = 0$, its solution is $$z = ax + by + c, \qquad ...(2)$$

where $a^2 + b^2 = m^2$ or $b = (m^2 - a^2)^{1/2}$, on putting a for p and b for b in (1).

$\therefore$ From (2), the complete integral is $$z = ax + y\,(m^2 - a^2)^{½} + c, \qquad ...(3)$$

which contains two arbitrary constants a and c.

For singular solution, differentiating (3) partially with respect to a and c, we get $0 = x - ay/(m^2 - a^2)^{1/2}$ and $0 = 1$. But $0 = 1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get
$$z = ax + y(m^2 - a^2)^{1/2} + \phi(a). \qquad ...(4)$$

Differentiating (4) partially wit respect to 'a', we get
$$0 = a - ay/(m^2 - a^2)^{1/2} + \phi'(a). \qquad ...(5)$$

Eliminating a from (4) and (5), we get the required general solution.

(*b*) **Hint.** Do like part (*a*) by taking $m = 1$

EQUATIONS REDUCIBLE TO STANDARD FORM I

Ex. 3. *Find the complete integral of* $z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0$.

Sol. The given equation can be rewritten as
$$z^2y\,(\partial z/\partial x)^2 + 6zxy(\partial z/\partial x) + 2zx^2(\partial z/\partial y) + 4x^2y = 0$$

or $$\left(\frac{z}{x}\frac{\partial z}{\partial x}\right)^2 + 6\left(\frac{z}{x}\frac{\partial z}{\partial x}\right) + 2\left(\frac{z}{y}\frac{\partial z}{\partial y}\right) + 4 = 0, \text{ dividing by } x^2y \qquad ...(1)$$

Put $x\,dx = dX$, $y\,dy = dY$ and $z\,dz = dZ$. ...(2)

so that $x^2/2 = X$, $y^2/2 = Y$ and $z^2/2 = Z$. ...(3)

Using (2), (1) becomes $$(\partial Z/\partial X)^2 + 6\,(\partial Z/\partial X) + 2(\partial Z/dY) + 4 = 0$$

or $P^2 + 6P + 2Q + 4 = 0$, where $P = \partial Z/\partial X$, $Q = \partial Z/\partial Y$. ...(4)

Equation (4) is of the form $f(P, Q) = 0$. Note that now we have P, Q, X, Y, Z in place of p, q, x, y, z in usual equations. Accordingly, solution of (4) is

$$Z = aX + bY + c, \qquad ...(5)$$

where $a^2 + 6a + 2b + 4 = 0$ or $b = -(a^2 + 6a + 4)/2$, on putting a for P and b for Q in (4). So, from (5), the required complete integral is

$$Z = aX - \{(a^2 + 6a + 4)/2\}Y + c, \text{ where } a \text{ and } c \text{ are arbitrary constants.}$$

or $$z^2/2 = a(x^2/2) - (a^2 + 6a + 4) \times (y^2/4) + c, \text{ using (3)}$$

or $$z^2 = ax^2 - (2 + 3a + a^2/2)y^2 + c', \text{ where } c' = 2c.$$

Ex. 4. *Find the complete integral of*

(*i*) $x^2p^2 + y^2q^2 = z$ **[Delhi Maths (H) 2004]**

(*ii*) $p^2x + q^2y = z.$ **[Meerut 1994]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{x^2}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{x\,\partial z}{\sqrt{z}\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{\sqrt{z}\,\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $(1/x)dx = dX$, $(1/y)dy = dY$ and $(1/\sqrt{z})dz = dZ$ ...(2)

so that $\log x = X$, $\log y = Y$ and $2\sqrt{z} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P,Q) = 0$.

$\therefore$ solution of (4) is $Z = aX + bY + c$, ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + Y\sqrt{1-a^2} + c \quad \text{or} \quad 2\sqrt{z} = a\log x + \log y \,.\, \sqrt{1-a^2} + c, \text{ by (3)}$$

or $$\log x^a + \log y^{\sqrt{1-a^2}} - \log c' = 2\sqrt{z}, \text{ taking } c = -\log c'$$

or $$\log\{x^a y^{\sqrt{1-a^2}}/c'\} = 2\sqrt{z} \quad \text{or} \quad x^a\, y^{\sqrt{1-a^2}} = c' e^{2\sqrt{z}}$$

where a and c' are two arbitrary constants.

(*ii*) The given equation can be re–written as

$$\frac{x}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{\sqrt{x}}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\sqrt{y}}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $\left(1/\sqrt{x}\right)dx = dX$, $\left(1/\sqrt{y}\right)dy = dY$ and $\left(1/\sqrt{z}\right)dz = dZ$ ...(2)

so that $2\sqrt{x} = X$, $2\sqrt{y} = Y$ and $2\sqrt{z} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $z = aX + bY + c$, ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + Y\sqrt{1-a^2} + c \quad \text{or} \quad 2\sqrt{z} = 2a\sqrt{x} + 2\sqrt{y}\sqrt{1-a^2} + c, \text{ by (3)}$$

where a and c are two arbitrary constants.

Ex. 5. *Solve* $x^2p^2 + y^2q^2 = z^2$. **[Jabalpur 2000, 03; Gulbarga 2005; Bilaspur 1997; Meerut 2008, Sagar 2004, Vikram 1999; Ravi Shanker 1994, 96; Rohitkhand 2004]**

Sol. The given equation can be rewritten as

$$\frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(\frac{x\,\partial z}{z\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{z\,\partial y}\right)^2 = 1. \qquad \text{...(1)}$$

Put $(1/x)dx = dX$, $(1/y)dy = dY$ and $(1/z) = dZ$ ...(2)

so that $\log x = X$, $\log y = Y$ and $\log z = Z$. ...(3)

Using (2), (1) becomes $(dZ/dX)^2 + (dZ/dY)^2 = 1$ or $P^2 + Q^2 = 1$, ... (4)

where $P = dZ/dX$ and $Q = dZ/dy$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $Z = aX + bY + c$, ... (5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $Z = X \cos\alpha + Y \sin\alpha + \log c'$, taking $a = \cos\alpha$ and $c = \log c'$

or $\log z = \cos\alpha \log x + \sin\alpha \log y + \log c'$ or $z = c'\, x^{\cos\alpha}\, y^{\sin\alpha}$. ...(6)

To determine singular integral. Differentiating (6) partially w.r.t. α and c' successively, we obtain

$$0 = c' \cos\alpha \,.\, x^{\cos\alpha}\, y^{\sin\alpha} \log y - c' \sin\alpha \,.\, x^{\cos\alpha}\, y^{\sin\alpha} \log x \qquad \text{...(7)}$$

and

$$0 = x^{\cos\alpha}\, y^{\sin\alpha}. \qquad \text{...(8)}$$

Eliminating α and c' from (6), (7) and (8), the singlar solution is $z = 0$.

To determine general integral. Putting $c' = \phi(\alpha)$, where ϕ is an arbitrary function, (4) gives

$$z = \phi(\alpha) \sin\alpha\, x^{\cos\alpha}\, y^{\sin\alpha}. \qquad \text{...(9)}$$

Differentiating (9), partially, w.r.t. 'α', we get

$$0 = \phi'(\alpha)\, x^{\cos\alpha}\, y^{\sin\alpha} + \phi(\alpha)\, \{x^{\cos\alpha}\, y^{\sin\alpha} \cos\alpha - y^{\sin\alpha}\, x^{\cos\alpha} \sin\alpha\}. \qquad \text{...(10)}$$

The required general integral is obtained by eliminating α from (9) and (10).

Ex. 6. *Find a complete integral of* (*i*) $pq = x^m y^n z^{2l}$ **[Delhi B.Sc. (Prog) II 2007]**

(*ii*) $pq = x^m y^n z^l$ **[I.A.S. 1989, 94]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{z^{-l}z^{-l}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}\right) = 1. \qquad \text{...(1)}$$

Put $x^m dx = dX$, $y^n dy = dY$ and $z^{-l}dz = dZ$ ...(2)

so that $\dfrac{x^{m+1}}{m+1} = X$, $\dfrac{y^{n+1}}{n+1} = Y$ and $\dfrac{z^{1-l}}{1-l} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)(\partial Z/\partial Y) = 1$ or $PQ = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c$, ... (5)

where $ab = 1$ or $b = 1/a$, on putting a from P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \qquad \text{or} \qquad \frac{z^{1-l}}{1-l} = a\frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

(*ii*) The given equation can be rewritten as

$$\frac{z^{-l/2}z^{-l/2}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}=1 \qquad \text{or} \qquad \left(\frac{z^{-l/2}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l/2}}{y^n}\frac{\partial z}{\partial y}\right)=1. \qquad ...(1)$$

Put $\quad x^m dx = dX, \qquad y^n dy = dY \qquad$ and $\qquad z^{-l/2}dz = dZ \qquad ...(2)$

so that $\quad \dfrac{x^{m+1}}{m+1}=X, \qquad \dfrac{y^{n+1}}{n+1}=Y \qquad$ and $\qquad \dfrac{z^{1-(l/2)}}{1-(l/2)}=Z. \qquad ...(3)$

Using (2), (1) becomes $\qquad (\partial Z/\partial X)(\partial Z/\partial Y) = 1 \qquad$ or $\qquad PQ = 1, \qquad ...(4)$

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\qquad z = aX + bY + c, \qquad ... (5)$

where $\quad ab = 1 \quad$ or $\quad b = 1/a$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \qquad \text{or} \qquad \frac{z^{1-(l/2)}}{1-(l/2)} = a\frac{x^{m+1}}{m+1}+\frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

Ex. 7. *Find complete integral of* $p^m \sec^{2m}x + z^l q^n \operatorname{cosec}^{2n}y = z^{lm/(m-n)}$

Sol. The given equation can be re–written as

$$\frac{1}{z^{lm/(m-n)}}\left(\frac{1}{\cos^2 x}\frac{\partial z}{dx}\right)^m+\frac{z^l}{z^{lm/(m-n)}}\left(\frac{1}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n = 1 \text{ or } \left(\frac{z^{-l/(m-n)}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^m+\left(\frac{z^{-l/(m-n)}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n = 1. \quad ...(1)$$

Put $\quad \cos^2 x\, dx = dX, \qquad \sin^2 y\, dy = dY \qquad$ and $\qquad z^{-l/(m-n)}dz = dZ \qquad ...(2)$

i.e., $\{(1 + \cos 2x)/2\}dx = dX, \qquad \{(1 - \cos 2y)/2\}dy = dY \qquad$ and $\qquad z^{-l/(m-n)}\, dz = dZ$

so that $\quad \dfrac{1}{2}(x + \dfrac{1}{2}\sin 2x) = X, \quad \dfrac{1}{2}(y - \dfrac{1}{2}\sin 2y) = Y \quad$ and $\quad \dfrac{(m-n)z^{(m-n-l)/(m-n)}}{m-n-l} = Z. \qquad ...(3)$

Using (2), (1) becomes $\quad (\partial Z/\partial X)^m + (\partial Z/\partial Y)^n = 1 \qquad$ or $\qquad P^m + Q^n = 1, \qquad ...(4)$

where $P = \partial Z/dX$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\qquad Z = aX + bY + c, \qquad ... (5)$

where $\quad a^m + b^n = 1 \quad$ or $\quad b = (1 - a^m)^{1/n}$, on putting a for P and b for Q is (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + (1 - a^m)^{1/n}Y + c$, a and c being two arbitrary constants.

or $$\frac{m-n}{m-n-l}z^{(m-n-l)/(m-n)} = \frac{a}{4}(2x + \sin 2x) + \frac{(1-a^m)^{1/n}}{4}(2y - \sin 2y) + c, \text{ by (3).}$$

Ex. 8. *Find the complete integral of* $(1 - x^2)\, yp^2 + x^2 q = 0$.

Sol. The given equation can be rewritten as

$$\frac{1-x^2}{x^2}\left(\frac{\partial z}{\partial x}\right)^2+\frac{1}{y}\frac{\partial z}{\partial y}=0 \qquad \text{or} \qquad \left(\frac{(1-x^2)^{1/2}}{x}\frac{\partial z}{\partial x}\right)^2+\left(\frac{1}{y}\frac{\partial z}{\partial y}\right)=0. \qquad ...(1)$$

Put $\qquad \{x/(1 - x^2)^{1/2}\}dx = dX \qquad$ and $\qquad y\, dy = dY \qquad ...(2)$

so that $\quad X = \displaystyle\int\frac{x\,dx}{(1-x^2)^{1/2}}=-\frac{1}{2}\int(1-x^2)^{-1/2}(-2x)dx = -(1 - x^2)^{1/2} \qquad$ and $\qquad Y = \dfrac{y^2}{2} \qquad ...(3)$

Using (2), (1) becomes $\quad (\partial z/\partial X)^2 + (\partial z/\partial Y) = 0 \qquad$ or $\qquad P^2 + Q = 0, \qquad ...(4)$

where $\quad P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Note carefully that here the old variable z remains unchanged even after transformation (2). Here (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\qquad z = aX + bY + c,$ ... (5)

where $\quad a^2 + b = 0 \quad$ or $\quad b = -a^2$, on putting a for P and b for Q in (4),

$\therefore$ from (5), the required complete integral is

$z = aX - a^2Y + c \qquad$ or $\qquad z = -a(1 - x^2)^{1/2} - (a^2y^2)/2 + c$, by (3).

Ex. 9. *Find the complete integral of* $(y - x)(qy - px) = (p - q)^2$. **[Delhi Maths (H) 2005; Ravishankar 2010; Meerut 1995, 97; Agra 1999; Kanpur 2001, 04, 07, 08]**

Sol. Let X and Y be two new variables such that

$X = x + y \qquad$ and $\qquad Y = xy.$ ...(1)

Given equation is $\qquad (y - x)(qy - px) = (p - q)^2.$ ...(2)

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y} \qquad ...(3)$$

[$\because$ from (1), $\partial X/\partial x = 1$ and $\partial Y/\partial x = y$]

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}. \qquad ...(4)$$

[$\because$ from (1), $\partial X/\partial y = 1$ and $\partial Y/\partial y = x$]

Substituting the above values of p and q in (2), we have

$$(y - x)\left[y\left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right) - x\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right)\right] = \left[\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right) - \left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right)\right]^2$$

or $$(y - x)^2\frac{\partial z}{\partial X} = (y - x)^2\left(\frac{\partial z}{\partial Y}\right)^2 \quad \text{or} \quad \frac{\partial z}{\partial X} = \left(\frac{\partial z}{\partial Y}\right)^2 \quad \text{or} \quad P = Q^2, \qquad ...(5)$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\qquad z = aX + bY + c,$ ...(6)

where $a = b^2$, on putting a for P and b for Q in (5).

$\therefore$ from (6), the required complete integral is

$z = b^2X + bY + c \qquad$ or $\qquad z = b^2(x + y) + bxy + c$, by (1).

Ex. 10. *Find the complete integral of* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$. **[I.A.S. 1991; Kanpur 2006; Meerut 1997]**

Sol. Let X and Y be two new variables such that

$X^2 = x + y \qquad$ and $\qquad Y^2 = x - y.$ ...(1)

Given equation is $\qquad (x + y)(p + q)^2 + (x - y)(p - q)^2 = 1.$ ...(2)

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{1}{2X}\frac{\partial z}{\partial X} + \frac{1}{2Y}\frac{\partial z}{\partial Y} \qquad ...(3)$$

[$\because$ from (1), $\partial X/\partial x = 1/2X$ and $\partial Y/\partial x = 1/2Y$]

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{1}{2X}\frac{\partial z}{\partial X} - \frac{1}{2Y}\frac{\partial z}{\partial Y}. \qquad ...(4)$$

[$\because$ from (1), $\partial X/\partial y = 1/2X$ and $\partial Y/\partial y = -1/2Y$]

(3) and (4) $\Rightarrow$ $$p + q = \frac{1}{X}\frac{\partial z}{\partial X} \quad \text{and} \quad p - q = \frac{1}{Y}\frac{\partial z}{\partial Y}. \qquad ...(5)$$

Using (1) and (5), (2) reduces to

$$X^2 \times \frac{1}{X^2}\left(\frac{\partial z}{\partial X}\right)^2 + Y^2 \times \frac{1}{Y^2}\left(\frac{\partial z}{\partial Y}\right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1, \qquad ...(6)$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $$z = aX + bY + c, \qquad ...(7)$$

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (6).

$\therefore$ from (7), the required complete integral is

$$z = aX + Y\sqrt{1-a^2} + c \qquad \text{or} \qquad z = a\sqrt{x+y} + \sqrt{x-y}\sqrt{1-a^2} + c, \text{ by (1).}$$

Ex. 11. *Find a complete integral of* $(x^2 + y^2)(p^2 + q^2) = 1$.

[Agra 2008; Indore 2004; Vikram 2000; Meerut 1995; Rohitkhand 1994]

Sol. Put $x = r\cos\theta$ and $y = r\sin\theta$. ...(1)

Then, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$. ...(2)

Differentiating (2) partially with respect to x and y, we get

$$2r(\partial r/dx) = 2x \qquad \text{and} \qquad 2r(\partial r/\partial y) = 2y$$

$$\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta \qquad \text{and} \qquad \frac{\partial r}{\partial y} = \frac{r\sin\theta}{r} = \sin\theta. \qquad ...(3)$$

and $$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2} \times \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \qquad ...(4)$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2} \times \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \qquad ...(5)$$

Given equation is $$(x^2 + y^2)(p^2 + q^2) = 1. \qquad ....(6)$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}, \text{ by (3) and (4)}$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{\cos 0}{r}\frac{\partial z}{\partial\theta}, \text{ by (3) and (5).}$$

Hence $$p^2 + q^2 = (\partial z/\partial r)^2 + (1/r^2) \times (\partial z/\partial\theta)^2. \qquad ...(7)$$

$\therefore$ (6) becomes $$r^2[(\partial z/\partial r)^2 + (1/r^2) \times (\partial z/\partial\theta)^2] = 1, \text{ using (2) and (7)}$$

or $$\left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 = 1. \qquad ...(8)$$

Let R be a new variable such that $(1/r)dr = dR$ so that $\log r = R$. ...(9)

Then (8) becomes $(\partial z/\partial R)^2 + (\partial z/\partial\theta)^2 = 1$ or $P^2 + Q^2 = 1$, ...(10)

where $P = \partial z/\partial R$ and $Q = \partial z/\partial\theta$. (10) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $$z = aR + b\theta + c, \qquad ...(11)$$

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (10)

$\therefore$ from (11), the required complete integral is

$$z = aR + \theta\sqrt{1-a^2} + c \qquad \text{or} \qquad z = a\log r + \theta\sqrt{1-a^2} + c,$$

or $$z = a\log(x^2 + y^2)^{1/2} + \tan^{-1}(y/x)\,.\,\sqrt{1-a^2} + c, \text{ by (2)}$$

or $z = (a/2) \times \log(x^2 + y^2) + \sqrt{1-a^2}\tan^{-1}(y/x) + c$, a and c being arbitrary constants.

Ex. 12. *Find the complete integral of* $z^2 = pqxy$ **[Meerut 2007; Punjab 2005]**

Sol. The given equation can be re–written as

$$\frac{xy}{z^2}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)\left(\frac{y}{z}\frac{\partial z}{\partial y}\right) = 1 \qquad ...(1)$$

Put $(1/x)dx = dX$, $(1/y)\, dy = dY$ and $(1/z)dz = dZ$

so that $\log x = X$, $\log y = Y$ and $\log z = Z$ ... (2)

Then (1) becomes $(\partial Z/\partial X)\,(\partial Z/\partial Y) = 1$ or $PQ = 1$ ... (3)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. Then, solution of (3) is

$$Z = aX + bY + C', \quad \text{where} \quad ab = 1 \quad \text{so that} \quad b = 1/a.$$

$\therefore$ $\log z = a\log x + (1/a)\log y + \log C$, taking $C' = \log C$ and using (2)

or $z = x^a\ y^{1/a} C$, a and C being arbitrary constants.

Ex. 13. *Find the complete integral of* $(x/p)^n + (y/q)^n = z^n$.

Sol. The given can be re–written as $(x/zp)^n + (y/zq)^n = 1$ ... (1)

Let $X = x^2/2$, $Y = y^2/2$, $Z = z^2/2$, $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$ ... (2)

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial Z}\dfrac{\partial Z}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{x}{z}P$. Similarly, $q = \dfrac{y}{z}Q$, using (2)

Hence $x/zp = 1/P$ and $y/zq = 1/Q$ and so (1) reduces to $P^{-n} + Q^{-n} = 1$, whose solution is

$Z = aX + bY + C'$, where $a^{-n} + b^{-n} = 1$ so that $b = (1 - a^{-n})^{-1/n}$ ... (3)

$\therefore$ (2) and (3) $\Rightarrow$ $z^2/2 = a(x^2/2) + (1 - a^{-n})^{-1/n}\ (y^2/2) + C/2$, taking $C' = C/2$

or $z^2 = ax^2 + (1 - a^{-n})^{-1/n} y^2 + C$, a and C being arbitrary constants.

Ex. 14. *Find the complete integral of* $p^3 \sec^6 x + z^2 q^2 \operatorname{cosec}^4 y = z^6$

Sol. The given equation can be re–written as

$$\frac{z^{-6}}{\cos^6 x}\left(\frac{\partial z}{\partial x}\right)^3 + \frac{z^{-4}}{\sin^4 y}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{z^{-2}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{z^{-2}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^2 = 1 \qquad ... (1)$$

Let $\cos^2 x\, dx = dX$, $\sin^2 y\, dy = dY$, $z^{-2} dz = dZ$ ... (2)

$\Rightarrow$ $X = (1/2)\displaystyle\int (1 + \cos 2x)dx$, $Y = (1/2)\displaystyle\int (1 - \cos 2y)dy$, $Z = -(1/z)$

$\Rightarrow$ $X = (1/2)\,\{x + (1/2)\sin 2x\}$, $y = (1/2)\,\{y - (1/2)\sin 2y\}$, $Z = -(1/z)$

$\Rightarrow$ $X = (1/2)\ (x + \sin x \cos x)$, $y = (1/2)\ (y - \sin y \cos y)$, $Z = -(1/z)$ ... (3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$ ... (4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. Now, solution of (4) is

$Z = aX + bY + C/2$, where $a^2 + b^2 = 1$ so that $b = (1 - a^2)^{1/2}$ ... (5)

$\therefore$ $-(1/z) = a(1/2)\ (x + \sin x \cos x) + (1 - a^2)^{1/2}(1/2)(y - \sin y \cos y) + C/2$, by (3) and (5)

or $(2/z) + a(x + \sin x \cos x) + (1 - a^2)^{1/2}(y - \sin y \cos y) + C = 0$.

Ex. 15. *Find the complete integral of* $yp + xq = pq$.

Sol. The given equation can be re–written as

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{1}{xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \qquad \text{or} \qquad \frac{\partial z}{x\partial x}+\frac{\partial z}{y\partial y}=\left(\frac{\partial z}{x\partial x}\right)\left(\frac{\partial z}{y\partial y}\right) \qquad ... (1)$$

Put $x\,dx = dX,\quad y\,dy = dY$ so that $x^2/2 = X,\quad y^2/2 = Y$ ... (2)

Then (1) becomes $\partial z/\partial X + \partial z/\partial Y = (\partial z/\partial X)\,(\partial z/\partial Y)$ or $P + Q = PQ$ ... (3)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Then solution of (3) is

$z = aX + bY + c$, where $a + b = ab$ so that $b = a/(a-1)$ ... (4)

or $z = a(x^2/2) + a(a-1)^{-1}\,(y^2/2) + c$, a and c being arbitrary constants, by (2) and (4)

Ex. 16. *Find the complete integral of* $p^2x^2 + px = q$

Sol. The given equation can be re–written as

$$x^2\left(\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x} = q \qquad \text{or} \qquad \left(x\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x} = q \qquad ... (1)$$

Putting $(1/x)dx = dX$ so that $\log x = X$, (1) gives

$(\partial z/\partial X)^2 + \partial z/\partial X = q$ or $P^2 + P = q$, where $P = \partial z/\partial X$.

Its solution is $z = aX + by + c$ where $a^2 + a = b$

or $z = a\log x + (a^2 + a)y + c$, a and c being arbitrary constants. $[\because\ X = \log x]$

Ex. 17. *Find the complete integral, general integral and singular integral of* $pq = 4xy$. *Show that the equation is satisfied by* $z = 2xy + C$, *C being an arbitrary constant. What is the character of this integral.* **[Delhi Maths (H) 2007]**

Sol. The given equation can be re–written as

$$\frac{pq}{4xy} = 1 \qquad \text{or} \qquad \frac{1}{4xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) = 1 \qquad ... (1)$$

Putting $2x\ dx = dX,\ 2y\ dy = dY$ so that $x^2 = X,\ y^2 = Y$, (1) gives

$(\partial z/\partial X)\ (\partial z/\partial Y) = 1$ or $PQ = 1$ whose solution is

$z = aX + bY + d$, where $ab = 1$ so that $b = 1/a$.

$$\therefore \qquad z = ax^2 + (1/a)y^2 + d \qquad ... (2)$$

is complete integral of (1) containing two arbitrary constants a and d.

General integral. Putting $d = \phi(a)$ in (2), we get

$$z = ax^2 + (1/a)\ y^2 + \phi(a) \qquad ... (3)$$

Differenting (3) partially w.r.t. 'a', $0 = x^2 - (1/a^2)\ y^2 + \phi'(a)$ ... (4)

Then general integral is obtained by eliminating a from (3) and (4).

Singular integral. Differenting (2) partially w.r.t. 'a' and 'd' by turn, we get

$$0 = x^2 + (-1/a^2)\, y^2 \qquad \text{... (5)} \qquad\qquad 0 = 1 \qquad \text{... (6)}$$

Since (6) is absurd, so (1) has no singular solution.

Discussion of the character of the given integral

$$z = 2xy + C, \ C \text{ being an arbitrary constant} \qquad \text{... (7)}$$

Differenting (7) partially w.r.t. x and y, we get $\partial z/\partial x = p = 2x$ and $\partial z/\partial y = q = 2y$. These values of p and q satisfy (1). Hence (1) is satisfied by (7).

Now, (7) can be derived from (2), if the values of p and q given by (7) and (2) are same, that is if $2ax = 2y$ and $2y/a = 2x$, *i.e.,* if we choose $a = y/x$. Putting $a = y/x$ and taking $d = C$ in (2), we have

$$z = (y/x)\, x^2 + (x/y)\, y^2 + C \qquad \text{or} \qquad z = 2xy + C,$$

showing that (7) is a particular case of the complete integral (2)

We now show that (7) is a particular case of the general integral. To this end, replace $\phi(a)$ by C in (3) and write

$$z = ax^2 + (1/a)\, y^2 + C \qquad \text{... (8)}$$

Differenting (8) partially w.r.t. 'a', we get

$$0 = x^2 - (1/a^2) y^2 \qquad \text{or} \qquad a = y/x \qquad \text{... (9)}$$

Eliminating a from (8) and (9), we get $\qquad z = 2xy + C$

Ex. 18. *Find the complete integral of* $z = p^2 - q^2$. **[Delhi Maths (G) 2006]**

Sol. Re–writing the given equation, we have

$$\frac{1}{z}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{1}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(z^{-1/2}\frac{\partial z}{\partial x}\right)^2 - \left(z^{-1/2}\frac{\partial z}{\partial y}\right)^{1/2} = 1 \qquad \text{... (1)}$$

Let X, Y and Z be new variables such that

$dX = dx, \quad dY = dy \quad$ and $\quad dZ = z^{-1/2} dz \quad$ so that $\quad X = x, \quad Y = y, \quad Z = 2z^{1/2}$ ... (2)

Let $P = \partial Z/\partial X$ and $Q = \partial Z/\partial y$. Using (2), (1) becomes

$$P^2 - Q^2 = 1, \qquad \text{... (3)}$$

which is of the form $f(P, Q) = 0$. Hence a solution of (3) is $\quad Z = aX + by + c, \quad$ ... (4)

where $a^2 - b^2 = 1$. Then $b = \pm(a^2 - 1)^{1/2}$ and so from (4), we have

$$Z = aX \pm (a^2 - 1)^{1/2}\, Y + c \qquad \text{or} \qquad 2z^{1/2} = ax \pm (a^2 - 1)^{1/2}\, y + c,$$

which is the complete integral, a and c being arbitrary constants and $|a| \geq 1$.

EXERCISE 3(C)

Solve the following partial differential equations (1 – 10)

1. $p^2 - q^2 = 1$ **Ans. C.I.** $z = ax + (a^2 - 1)^{1/2} + c$, a and c are arbitrary constants and $|a| \geq 1$; **S.I.** Does not exist, **G.I.** It is given by $z - ax - (a^2 - 1)^{1/2} y - \psi(a) = 0$, $-x - a(a^2 - 1)^{-1/2}\, y - \psi'(a)$, where ψ is an arbitrary function.

2. $p^2 - q^2 = \lambda$ **Ans. C.I.** $z = ax + (a^2 - \lambda)^{1/2} y + c,$ where a and c are arbitrary constants and $\lambda \le a^2$; **S.I.** Does not exist, **G.I.** It is given by $z - ax - (a^2 - \lambda)^{1/2} y - \psi(a) = 0,$

$-x - a(a^2 - \lambda)^{1/2} y - \psi'(a) = 0$

3. $p + q = pq$ **[Mysore 2004; Gulberga 2005; Kanpur 2011; Pune 2010]**

Ans. C.I. $z = ax + \{a/(a-1)\}y + c$, where a and c are arbitrary constants and $a \ne 1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - \{a/(a-1)\}y - \psi(a) = 0$ and $-x - y(a-1)^{-2} - \psi'(a) = 0,$ where ψ is an arbitrary function

4. $p + q + pq = 0$. **Ans. C.I.** $z = ax - \{a/(a+1)\}y + c$, where a and c are arbitrary constants and $a \ne -1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax + \{a/(a+1)\}y - \psi(a) = 0,$

$-x - \{(2a+1)/2(a^2+a)^{1/2}\}y - \psi'(a) = 0,$ where ψ is an arbitrary function.

5. $p^2 + q^2 = npq$. **(M.S. Univ. T.N. 2007), Ans. C.I.** $z = ax + (a/2) \times \{n + (n^2-4)^{1/2}y + c,$ where a and c are arbitrary constants and $n^2 \ge 4$; **S.S.** Does not exists; **G.S.** It is given by $z - ax - (a/2) \times \{n + (n^2-4)^{1/2}\}y - \psi(a) = 0,\ -x - (1/2) \times \{n + (n^2-4)^{1/2\}} y - \psi'(a) = 0,$ where ψ is an arbitrary function.

6. $p = 2q^2 + 1$. **Ans. C.I.** $z = ax + \{(a-1)/2\}^{1/2} y + c,$ where a and c are arbitrary constants and $a \ge 1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - \{(a-1)/2\}^{1/2} y - \psi(a) = 0,$

$-x - (2\sqrt{2}\sqrt{a-1})^{-1} y - \psi'(a) = 0,$ where ψ is an arbitrary function.

7. $p = e^q$. **Ans. C.I.** $z = ax + y \log a + c$, where a and c are arbitrary constants and $a > 0$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - y\log a - \psi(a) = 0, - x - (y/a) - \psi'(a) = 0,$ where ψ is an arbitrary function.

8. $p^2q^3 = 1$ **Ans. C.I.** $z = ax + a^{-2/3}y + c$, where a and c are arbitrary constants and $a > 0$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - a^{-2/3}y - \psi(a) = 0,\ \ -x + (2/3) \times a^{-5/3} y - \psi(a) = 0,$ where ψ is an arbitrary function.

9. $p^2 + p = q^2$. **Ans. C.I.** $z = ax + (a^2 + a)^{1/2}y + c$, where a and c are arbitrary constants and $a \in \boldsymbol{R} - (-1, 0)$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - (a^2 + a)^{1/2}\ y - \ \psi(a) = 0,$

$-x - \{(2a+1)/2(a^2+a)^{1/2}\}y - \psi'(a) = 0,$ where ψ is an arbitrary function.

10. $p^2 + 6p + 2q + 4 = 0$. **C.I.** $z = ax - (2 + 3a + a^2/2)y + c$, where a and c are arbitrary constants; **S.S.** Does not exist; **G.S.** It is given by $z - ax + (2 + 3a + a^2/2)y - \psi(a) = 0,$

$-x + (a+3)y - \psi'(a) = 0,$ where ψ is an arbitrary function.

Find the complete integral (*solution*) *of the following equations* (*Ex. 11—18*).

11. $zy^2p = x(y^2 + z^2q^2)$. **Ans.** $z^2 = ax^2 \pm y^2(a-1)^{1/2} + c$, where $a \ge 1$

12. $z^2(p^2/x^2 + q^2/y^2) = 1$. **Ans.** $z^2 = ax^2 \pm y^2(1-a^2)^{1/2} + c$, where $-1 \le a \le 1$

13. $yp + x^2q^2 = 2x^2y$. **Ans.** $(3z - ax^3 - b)^2 = 4(2-a)y^2$

14. $(1 - y^2)\,xq^2 - y^2p = 0$. **Ans.** $(2z - ax^2 - b)^2 = a(1 - y^2)$

15. $p^2y(1 + x^2) = qx^2$. **Ans.** $z = a(1 + x^2)^{1/2} + (1/2) \times a^2y^2 + c$

16. $x^4p^2 + y^2zq - z^2 = 0$. **Ans.** $xy \log z = ay + (a^2 - 1)x + bxy$

17. $p^2 + q^2 = z$. **[Bangalore 1995]** **Ans.** $2z^{1/2} = ax \pm (1 - a^2)^{1/2} y + c$, where $-1 \le a \le 1$

18. $x^2p^2 + y^2q^2 = 4z^2$. **Ans.** $\log z = a \log x + (4 - a^2)^{1/2} + c$, $-2 \le a \le 2$

3.12. Standard form II. Clairaut equation. [Meerut 2009; Nagpur 2002]

A first order partial differential equation is said to be of Clariaut form if it can be written in the form

$$z = px + qy + f(p, q). \qquad ...(1)$$

Let

$$F(x, y, z, p, q) \equiv px + qy + f(p, q) - z. \qquad ...(2)$$

Charpit's auxiliary equations are

$$\frac{dp}{\dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z}} = \frac{dq}{\dfrac{\partial F}{\partial y} + q\dfrac{\partial F}{\partial z}} = \frac{dz}{-p\dfrac{\partial F}{\partial p} - q\dfrac{\partial F}{\partial q}} = \frac{dx}{-\dfrac{\partial F}{\partial p}} = \frac{dy}{-\dfrac{\partial F}{\partial q}}$$

or

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px - qy - p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-x - (\partial f/\partial p)} = \frac{dy}{-y - (\partial f/\partial q)}, \text{ by (1)}$$

Then, first and second fractions $\Rightarrow$ $dp = 0$ and $dq = 0$ $\Rightarrow$ $p = a$ and $q = b$.

Substituting these values in (1), the complete integral is $z = ax + by + f(a, b)$

Remark 1. Observe that the complete integral of (1) is obtained by merely replacing p and q by a and b respectively. Singular and general integrals can be obtained by usual methods.

Remark 2. Sometimes change of variables can be employed to transform a given equation to standard form II.

3.13. SOLVED EXAMPLES BASED ON ART. 3.12

Ex. 1. *Solve* $z = px + qy + pq$. **[Ravishanker 1997; Bangalore 2005; Sagar 1995, 96]**

Sol. The complete integral is $z = ax + by + ab$, a, b being arbitrary constants ...(1)

Singular integral. Differentiating (1) partially w.r.t. a and b, we have

$$a = x + b \quad \text{and} \quad 0 = y + a. \qquad ...(2)$$

Eliminating a and b between (1) and (2), we get $z = -xy - xy + xy$ *i.e.*, $z = -xy$, which is the required singular solution, for it satisfies the given equation.

General Integral. Take $b = \phi(a)$, where ϕ denotes an arbitrary function.

Then (1) becomes $z = a\,x + \phi(a)\,y + a\,\phi(a)$. ...(3)

Differentiating (3) partially w.r.t. a, $0 = x + \phi'(a)y + \phi(a) - a\,\phi'(a)$. ...(4)

The general integral is obtained by eliminating a between (3) and (4).

Ex. 2. *Prove that complete integral of the equations* $(px + qy - z)^2 = 1 + p^2 + q^2$ is $ax + by + cz = (a^2 + b^2 + c^2)^{1/2}$. **[I.A.S. 1989]**

Sol. Re–writting the given equation, we have

$$px + qy - z = \pm\sqrt{(1 + p^2 + q^2)} \qquad \text{or} \qquad z = px + qy \pm \sqrt{(1 + p^2 + q^2)}$$

which is of standard form II and so its complete integral is

$$z = Ax + By \pm (1 + A^2 + B^2)^{1/2}. \qquad ...(1)$$

To get the desired form of solution we take +ve sign in (1) and set $A = -a/c$ and $B = -b/c$. Then (1) becomes

$$z = -(ax + by)/c + (c^2 + a^2 + b^2)^{1/2}/c$$

or

$$ax + by + cz = (a^2 + b^2 + c^2)^{1/2}.$$

Ex. 3. *Solve* $z = px + qy + c\sqrt{(1 + p^2 + q^2)}$. **[I.A.S. 1989; Meerut 1998]**

Sol. The complete integral of the given equation is

$$z = ax + by + c\sqrt{(1 + a^2 + b^2)}, \; a, b \text{ being arbitrary constants.} \qquad ...(1)$$

Singular Integral. Differentiating (1) partially w.r.t. a and b, we get

$0 = x + ac/\sqrt{(1+a^2+b^2)}$ ...(2) $\qquad 0 = y + bc/\sqrt{(1+a^2+b^2)}$. ...(3)

$\therefore$ From (2) and (3), $\qquad x^2 + y^2 = (a^2c^2 + b^2c^2)/(1 + a^2 + b^2)$.

$$\therefore \qquad c^2 - x^2 - y^2 = c^2 - \frac{a^2c^2+b^2c^2}{1+a^2+b^2} = \frac{c^2}{1+a^2+b^2}$$

so that $\qquad 1 + a^2 + b^2 = c^2/(c^2 - x^2 - y^2)$. ...(4)

From (2), $\qquad a = -\dfrac{x\sqrt{(1+a^2+b^2)}}{c} = -\dfrac{x}{\sqrt{(c^2-x^2-y^2)}}$, by (4)

Similarly from (3) and (4), we obtain $\qquad b = -y/\sqrt{c^2-x^2-y^2}$.

Putting these values of a and b in (1), the singular solution is

$$z = -\frac{x^2}{\sqrt{(c^2-x^2-y^2)}} - \frac{y^2}{\sqrt{(c^2-x^2-y^2)}} + \frac{c^2}{\sqrt{(c^2-x^2-y^2)}} = \frac{c^2-x^2-y^2}{\sqrt{(c^2-x^2-y^2)}}$$

or $\quad z = (c^2 - x^2 - y^2)^{1/2}$ $\quad$ or $\quad z^2 = c^2 - x^2 - y^2$ $\quad$ or $\quad x^2 + y^2 + z^2 = c^2$. ...(5)

We can easily verify that (1) is satisfied by (5).

General Integral. Take $b = \phi(a)$, where ϕ is an arbitrary function.

Then, (1) yeilds $\qquad z = ax + y\phi(a) + c[1 + a^2 + \{\phi(a)\}^2]^{1/2}$. ...(6)

Differentiating both sides of (6) partially w.r.t. 'a', we get

$$0 = x + y\phi'(a) + (c/2) \times [1 + a^2 + \{\phi(a)\}^2]^{-1/2} \times [2a + 2\phi(a)\,\phi'(a)]. \qquad ...(7)$$

Eliminating a from (6) and (7), we get the general integral.

Ex. 4. *Find the complete and singular integrals of the following equations:*

(*i*) $z = px + qy + \log(pq)$ **[Indore 2004; K.U. Kurukshetra 2006]**

(*ii*) $z = px + qy - 2\sqrt{pq}$. **[Bangalore 1993; Lucknow 2010]**

Sol. (*i*) The complete integral is $\qquad z = ax + by + \log(ab)$

or $\qquad z = ax + by + \log a + \log b$, a, b being arbitrary constants ...(1)

Differentiating (1) partially with respect to a and b, we get

$0 = x + (1/a)$ $\quad$ and $\quad 0 = y + (1/b)$ $\quad$ so that $\quad a = -1/x$ $\quad$ and $\quad b = -1/y$. ...(2)

Eliminating a and b from (1) and (2), the required singular integral is

$$z = -1 - 1 + \log(1/xy) \qquad \text{or} \qquad z = -2 - \log(xy).$$

(*ii*) The complete integral is $\qquad z = ax + by - 2\sqrt{ab}$. ...(1)

Differentiating (1) partially with respect to a and b, we get

$0 = x - \dfrac{2b}{2\sqrt{ab}}$ $\quad$ and $\quad 0 = y - \dfrac{2a}{2\sqrt{ab}}$ $\quad$ so that $\quad x = \sqrt{\dfrac{b}{a}}$ $\quad$ and $\quad y = \sqrt{\dfrac{a}{b}}$. ...(2)

Now, using (1) $\qquad x - z = x - (ax + by - 2\sqrt{ab}) = \sqrt{\dfrac{b}{a}} - a\sqrt{\dfrac{b}{a}} - b\sqrt{\dfrac{a}{b}} + 2\sqrt{ab}$, using (2)

$$\therefore \qquad x - z = \sqrt{(b/a)}. \qquad ...(3)$$

Similarly, using (1) $\quad y - z = y - (ax + by - 2\sqrt{ab})$, $= \sqrt{\dfrac{a}{b}} - a\sqrt{\dfrac{b}{a}} - b\sqrt{\dfrac{a}{b}} + 2\sqrt{ab}$

$$\therefore \qquad y - z = \sqrt{(a/b)}. \qquad ...(4)$$

From (3) and (4), $\qquad (x - z)(y - z) = 1$,

which is singular integral as it satisfies the given equation.

Ex. 5. *Prove that the complete integral of* $z = px + qy - 2p - 3q$ *represents all possible planes through the point* (2, 3, 0). *Also find the envelope of all planes represented by the complete integral* (*i.e.*, *find the singular integral*). **(M.D.U. Rohtak 2006)**

Sol. Given that $z = px + qy - 2p - 3q$, ...(1)

which is of the form $z = px + qy + f(p, q)$ and so its complete integral is

$$z = ax + by - 2a - 3b, \quad a, b \text{ being arbitrary constants} \qquad ...(2)$$

Since (2) is a linear equation in x, y, z, it follows that (2) represents planes for various values of a and b. Again putting $x = 2$, $y = 3$, $z = 0$ in (2), we have

$$0 = 2a + 3b - 2a - 3b \qquad i.e., \qquad 0 = 0,$$

showing that coordinates of the point (2, 3, 0) satisfy (2). Hence the complete integral (2) of (1) represents all possible planes passing through the point (2, 3,0).

Differentiating (2) partially with respect to a and b, we get

$0 = x - 2$ and $0 = y - 3$ so that $x = 2$ and $y = 3$.

Substituting these values in (2), we get $z = 0$ as the required envelope (*i.e.*, singular integral).

Ex. 6. *Prove that the complete integral of* $z = px + qy + [pq/(pq - p - q)]$ *represents all planes such that the algebraic sum of the intercepts on three coordinate axes is unity.*

Sol. Since the given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is

$$z = ax + by + [ab/(ab - a - b)], \quad a \text{ and } b \text{ being arbitrary constants.} \qquad ...(1)$$

Since (2) is a linear equation in x, y, z, it follows that (1) represents planes for various values of a and b. We now rewrite (1) in the intercept form of a plane as follows :

$$ax + by - z = ab/(a + b - ab)$$

or
$$\frac{x}{[b/(a+b-ab)]} + \frac{y}{[a/(a+b-ab)]} + \frac{z}{[-ab/(a+b-ab)]} = 1.$$

$\therefore$ The algebric sum of the intercepts on three coordinate axes

$$= \frac{b}{a+b-ab} + \frac{a}{a+b-ab} + \frac{(-ab)}{a+b-ab} = \frac{b+a-ab}{a+b-ab} = 1, \text{ as required.}$$

Ex. 7. *Show that the complete integral of the equation* $z = px + qy + (p^2 + q^2 + 1)^{1/2}$ *represents all planes at unit distance from the origin.*

Sol. Given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is

$$z = ax + by + (a^2 + b^2 + 1)^{1/2}, \; a, b \text{ being an arbitrary constants.}$$

or
$$ax + by - z + (a^2 + b^2 + 1)^{1/2} = 0. \qquad ...(1)$$

Since (2) is a linear equation in x, y, z, it follows that (1) represents planes for various values of a and b.

The perpendicular distance of (1) from origin (0, 0, 0)

$$= \frac{a.0 + b.0 - 0 + \sqrt{a^2+b^2+1}}{\sqrt{\{a^2+b^2+(-1)^2\}}} = \frac{\sqrt{a^2+b^2+1}}{\sqrt{a^2+b^2+1}} = 1, \text{ as required}$$

Ex. 8. *Find the complete integral of the following equations:*

(*i*) $(p+q)(z-px-qy) = 1$ **[Pune 2010]**

(*ii*) $pqz = p^2(xq+p^2) + q^2(yp+q^2)$ **[Delhi B.A. (Prog) II 2008, 10]**

Sol. (*i*) Re–writing the given equation in the standard form $z = px + qy + f(p,q)$, we get

$$z - px - qy = 1/(p+q) \qquad \text{or} \qquad z = px + qy + 1/(p+q)$$

$\therefore$ Its complete integral is $z = ax + by + 1/(a+b)$, where a and b are arbitrary constants.

(ii) Dividing both sides of the given equation by pq, $\quad z = px + qy + (p^4 + q^4)/pq$,

Its complete integral is $\quad z = ax + by + (a^4 + b^4)/ab$, a, b being arbitrary constants.

Ex. 9. (a) *Find the complete integral the equation* $\quad 2(y + zq) = q(xp + yq)$.

[Delhi Maths (H) 1999]

Sol. Re–writing the given equation, we have

$2zq = xpq + yq^2 - 2y \quad$ or $\quad z = (1/2)px + (1/2)qy - (y/q)$

or
$$z = x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) - \frac{1}{2}\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)^{-1} \qquad \text{... (1)}$$

Putting $\quad 2x\,dx = dX$ and $\quad 2y\,dy = dY \quad$ so that $\quad x^2 = X$ and $y^2 = Y$, $\quad$ (1) gives

$z = X\,(\partial z/\partial X) + Y\,(\partial z/\partial Y) - 1/\{2\,(\partial z/\partial Y)\} \quad$ or $\quad z = PX + QY - (1/2Q)$,

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. The above equation is of the form $z = PX + Qy + f(P,Q)$ and hence its complete integral is

$z = aX + bY - (1/2b) \quad$ or $\quad z = ax^2 + by^2 - (1/2b)$, a and b being arbitrary constants.

Ex. 9. (b) *Find the complete integral of* $2q(z - px - qy) = 1 + q^2$.

Sol. Re–writing the given equation in the form $z = px + qy + f(p,q)$, we have

$z - px - qy = (1 + q^2)/2q \quad$ or $\quad z = px + qy + (1 + q^2)/2q$,

Its complete integral is $z = ax + by + (1 + b^2)/2b$, a and b being arbitrary constants.

Ex. 10. *Find the compelte integral of* $p^2x + q^2y = (z - 2px - 2qy)^2$.

Sol. Taking positive root, the given equation reduces to

$z - 2px - 2qy = (p^2x + q^2y)^{1/2} \quad$ or $\quad z = 2px + 2qy + (p^2x + q^2y)^{1/2}$

or
$$z = \sqrt{x}\,\frac{\partial z}{(1/2\sqrt{x})\partial x} + \sqrt{y}\,\frac{\partial z}{(1/2\sqrt{y})\partial y} + \frac{1}{2}\left[\left(\frac{\partial z}{(1/2\sqrt{x})\partial x}\right)^2 + \left(\frac{\partial z}{(1/2\sqrt{y})\partial y}\right)^2\right]^{1/2} \qquad \text{... (1)}$$

Put $\quad (1/2\sqrt{x})\,dx = dX \quad$ and $\quad (1/2\sqrt{y})dy = dY \quad$ so that $\quad \sqrt{x} = X \quad$ and $\quad \sqrt{y} = Y$ ... (2)

Using (2), (1) gives $\quad z = (\partial z/\partial X)X + (\partial z/\partial Y)Y + (1/2) \times \{(\partial z/\partial X)^2 + (\partial z/\partial Y)^2\}^{1/2}$

or $\quad z = PX + QY + (1/2) \times (P^2 + Q^2)^{1/2}, \quad$ where $\quad P = \partial z/\partial X \quad$ and $\quad Q = \partial z/\partial Y$.

It is of the Clairaut's form $z = Px + Qy + f(P,Q)$ and so its complete integral is given by

$z = aX + bY + (1/2) \times (a^2 + b^2)^{1/2} \quad$ or $\quad z = a\sqrt{x} + b\sqrt{y} + (1/2) \times (a^2 + b^2)^{1/2}$

Ex. 11. *Find a complete and the singular integral of* $4xyz = pq + 2px^2y + 2qxy^2$

Sol. The given equation can be rewritten as

$$z = \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) + x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right). \qquad \text{... (1)}$$

Put $2x\,dx = dX$ and $2y\,dy = dY$ ... (2)

so that $x^2 = X$ and $y^2 = Y.$ ... (3)

Using (2), (1) becomes $z = (\partial z/\partial X)(\partial z/\partial Y) + X(\partial z/\partial X) + Y(\partial z/\partial Y)$

or $$z = XP + YQ + PQ, \qquad \text{... (4)}$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $z = XP + YQ + f(P, Q)$.

$\therefore$ Solution of (4) is $z = aX + bY + ab$, a, b being arbitrary constants.

or $z = ax^2 + by^2 + ab$, which is complete integral. ... (5)

Differentiating (5) partially w.r.t a and b, we have

$0 = x^2 + b$ and $0 = y^2 + b$ so that $b = -x^2$ and $a = -y^2$ .... (6)

Eliminating a and b between (5) and (6), the required singular integral is

$z = -x^2y^2 - x^2y^2 + x^2y^2$ or $z = -x^2y^2$.

Ex. 12. *Find the complete and singular solutions of* $z = px + qy + p^2q^2$. **[Jabalpur 2000; Sagar 1995; Rewa 2003; Ravishankar 2004]**

Sol. Given $z = px + qy + p^2q^2$ ... (1)

Since (1) is in Clairaut's form, its complete solution is

$z = ax + by + a^2b^2$, a, b being arbitrary constants ... (2)

To find singular solution of (1). Differentiating (2) partially w.r.t. 'a' and 'b' successively,

$0 = x + 2ab^2$ and $0 = y + 2a^2b$ ... (3)

From (3), $a = -(y^2/2x)^{1/3}$ and $b = -(x^2/2y)^{1/3}$ ... (4)

Substituting the values of a and b given by (4) in (2), we get

$z = -x(y^2/2x)^{1/3} - y(x^2/2y)^{1/3} + (x^2y^2/16)^{1/3}$ or $z = -(3/4) \times 4^{1/3}\, x^{2/3}\, y^{2/3}$,

which is the required singular solution of (1)

EXERCISE 3 (D)

Solve the following partial differential equations : (1 – 9)

1. $z = px + qy - 2p - 3q$. **[M.D.U. Rohtak 2006]**

Ans. C.I. $z = ax + by - 2a - 3b$; **S.S.** $z = 0$; **G.S.** It is given by $z - ax - \psi(a)y + 2a + 3\psi(a) = 0$,

$$x + (y-3)\psi'(a) - 2 = 0$$

2. $z = px + qy + 5pq$. **Ans. S.I.** $z = ax + by + 5ab$; **S.S.** $5z + xy = 0$

G.S. $z - ax - \psi(a)y - 5a\,\psi(a) = 0,\ \ x + 5\psi(a) + (y + 5a)\,\psi'(a) = 0$

3. $z = px + qy + p^2 - q^2$ **[Purvanchal 2007]** **Ans. S.I.** $z = ax + by + a^2 - b^2$;

S.S. $x^2 - y^2 + 4z = 0$; **G.S.** $z - ax - \psi(a)y - a^2 + \{\psi(a)\}^2 = 0;\ \ x + 2a + \{y - 2\psi(a)\}\psi'(a) = 0;$

4. $z = px + qy + (q/p) - p$. **[Madras 2005]**

Ans. C.I. $z = ax + by + (b/a) - a$; **S.S.** $yz = 1 - x$; **G.S.** It is given by

$$z - ax + \psi(a)y + (1/a) \times \psi(a) - a,\ \ -x + \psi'(a)y - (1/a^2)\,\psi(a) + (1/a) \times \psi'(a) = 0$$

5. $z = px + qy + p/q$ **Ans. C.I.** $= ax + by + a/b$; **S.S.** $xz + 4 = 0$;

G.S. $z - ax - \psi(a)y - a/\psi(a) = 0;\ \ x + \psi'(a)y + 1/\psi(a) - \{a\psi'(a)\}/\{\psi(a)\}^2 = 0$

6. $z = px + qy + 2\sqrt{pq}$ **[Bangalore 1994]**

Ans. C.I. $z = ax + by + 2\sqrt{ab}$; **S.S.** $(x - z)(y - z) = 1$; **G.S.** $z - ax - \psi(a)y - 2\sqrt{a\psi(a)} = 0$,

$$x + \psi'(a) + \{\psi(a) + a\psi'(a)\}/2\sqrt{a\psi(a)} = 0$$

7. $z = px + qy - 2\sqrt{pq}$. **Ans. C.I.** $z = ax + by - 2\sqrt{ab}$; **S.S.** $(x - z)(y - z) = 1$;

G.S. $z - ax - \psi(a)y + 2\sqrt{a\psi(a)} = 0,\ x + \psi'(a)y - \{\psi(a) + a\psi'(a)\}/\sqrt{a\psi(a)} = 0$

8. $z = px + qy + p^2 + pq + q^2$. **[Ranchi 2010]**

Ans. C.I. $z = ax + by + a^2 + ab + b^2$; **S.S.** $x^2 + y^2 - xy + 3z = 0$, **G.S.** $z - ax - \psi(a)y - a^2 - a\psi(a) - \{\psi(a)\}^2 = 0,\quad x + \{y + a + 2\psi(a)\}\psi'(a) + 2a + \psi(a) = 0$

9. $z = px + qy + (\alpha p^2 + \beta q^2 + 1)^{1/2}$. **Ans. C.I.** $z = ax + by + (\alpha a^2 + \beta b^2 + 1)^{1/2}$;

S.S. $x^2/\alpha + y^2/\beta + z^2 = 1$; **G.S.** $z - ax - \psi(a)y - [a\alpha^2 + \beta\{\psi(a)\}^2 + 1]^{1/2} = 0; x + \psi'(a)y$

$$+\{a\alpha + \beta\psi(a)\psi'(a)\}/[a\alpha^2 + \beta(\psi(a)\}^2 + 1]^{1/2} = 0$$

10. Find the complete integral of $z = px + qy - \sin(pq)$ **[GATE 2003]**

Ans. $z = ax + by - \sin(ab)$ a, b being arbitrary constants.

11. Find the complete integral and singular integral of the differential equation $z = px + qy + p^2 - q^2$. Find also a developable surface belonging to the general integral of this differential equation. **[I.A.S 1983]**

Ans. Complete integral is $z = ax + by + a^2 - b^2$; singular integral is $4z = 3(x^2 - y^2)$

3.14. Standard form III. Only p, q and z present. [Nagpur 2003; Delhi Maths (H) 2006]

Under this standard form we consider differential equation of the form

$$f(p, q, z) = 0. \qquad \text{...(1)}$$

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or $$\frac{dp}{p(\partial f/\partial z)} = \frac{dq}{q(\partial f/\partial z)} = \frac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q}, \text{ using (1)}$$

Taking the first two ratios, $(1/p)dp = (1/q)dq$

Integrating, $q = ap$, a being an arbitrary constant. ...(2)

Now, $dz = p\,dx + q\,dy = p\,dx + ap\,dy$, using (2)

or $dz = p(dx + ady) = pd(x + ay) = p\,du$, ...(3)

where $u = x + ay$. ...(4)

Now, (3) $\Rightarrow$ $p = dz/du$ and so by (2) $q = ap = a(dz/du)$.

Substituting these values of p and q in (1), we get $f\left(\dfrac{dz}{du}, a\dfrac{dz}{du}, z\right) = 0$, ...(5)

which is an ordinary differential equation of first order. Solving (5), we get z as a function of u. Complete integral is then obtained by replacing u by $(x + ay)$.

3.15. Working rule for solving equations of the form $f(p,q, z) = 0$. ...(1)

Step I. Let $u = x + ay$, where a is an arbitrary constant. ...(2)

Step II. Replace p and q by dz/du and $a(dz/du)$ respectively in (1) and solve the resulting ordinary differential equation of first order by usual methods.

Step III. Replace u by $x + ay$ in the solution obtained in step II.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form III.

Remark 2. Singular and general integrals are obtined by well known methods.

3.16. SOLVED EXAMPLES BASED ON ART 3.15.

Ex. 1. *Find a complete integral of* $9(p^2z + q^2) = 4$.

[Delhi Maths (H) 2006; Banglore 1995; I.A.S. 1988; Meerut 1996; Rohilkhand 1995]

Sol. Given equation is $$9(p^2z + q^2) = 4, \qquad ...(1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4 \qquad \text{or} \qquad \left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)}.$$

or $\quad du = \pm\, (3/2) \times (z + a^2)^{1/2} dz$, separating variables u and z.

Integrating, $\quad u + b = \pm\, (3/2) \times [(z + a^2)^{3/2}/(3/2)] \quad$ or $\quad u + b = \pm\, (z + a^2)^{3/2}$

or $\quad (u + b)^2 = (z + a^2)^3 \quad$ or $\quad (x + ay + b)^2 = (z + a^2)^3$, as $u = x + ay$

which is a complete integral contaning two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p^2 = qz$. **[Bilaspur 1996; Sagar 2004]**

Sol. Given equation is $$p^2 = qz, \qquad ...(1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$\left(\frac{dz}{du}\right)^2 = \left(a\frac{dz}{du}\right)z \qquad \text{or} \qquad \frac{dz}{du} = az \qquad \text{or} \qquad \frac{dz}{z} = a\,du.$$

Integrating, $\quad \log z - \log b = au \quad$ or $\quad z = be^{au} \quad$ or $\quad z = be^{a(x + ay)}$,

which is a complete integral containing two arbitrary constants a and b.

Ex. 3.(a) *Find a complete integral of* $z = pq$. **[Meerut 1994]**

Sol. Given equation is $$z = pq, \qquad ... (1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$z = a\left(\frac{dz}{du}\right)^2 \qquad \text{or} \qquad \frac{dz}{du} = \pm\frac{\sqrt{z}}{\sqrt{a}} \qquad \text{or} \qquad \pm\sqrt{a}\, z^{-1/2} dz = du.$$

Integrating, $\quad \pm 2\sqrt{az} = u + b \quad$ or $\quad 4(az) = (x + ay + b)^2$, as $u = x + ay$

Ex. 3.(b) *Find a complete integral of* $pq = 4z$.

Sol. Proceed as in Ex. 3.(a). **Ans.** $(x + ay + b)^2 = az$

Ex. 4.(a) *Find a complete integral of* $p(1 + q^2) = q(z - \alpha)$. **[Meerut 1999; Bilaspur 2002; Jiwaji 2003; Ravishanker 2005 Rewa 1998, Vikram 2004]**

(b) *Find a complete integral of* $p(1 + q^2) = q(z - 1)$. **[M.S. Univ. T.N. 2007]**

Sol. (*a*) Given equation is $$p(1 + q^2) = q(z - \alpha), \qquad ...(1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$\frac{dz}{du}\left\{1+\left(a\frac{dz}{du}\right)^2\right\} = a\frac{dz}{du}\,(z - \alpha) \qquad \text{or} \qquad 1 + a^2\left(\frac{dz}{du}\right)^2 = a(z - \alpha)$$

or $$\frac{dz}{du} = \pm\frac{\sqrt{\{a(z-\alpha)-1\}}}{a} \qquad \text{or} \qquad du = \pm\frac{a\,dz}{\sqrt{\{a(z-\alpha)-1\}}}.$$

Integrating, $u + b = \pm 2\sqrt{\{a(z-\alpha)-1\}}$ or $(u + b)^2 = 4\{a(z - \alpha) - 1\}^2$

or $(x + ay + b)^2 = 4\{a(z - \alpha) - 1\}^2$, a and b being arbitrary constants.

(***b***) Proceed as in part (a) by taking $\alpha = 1$.

Ex. 5.(a) *Find a complete integral of* $pz = 1 + q^2$. **[Meerut 1996]**

Sol. Given equation is $$pz = 1 + q^2, \qquad ...(1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$z\frac{dz}{du} = 1 + a^2\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad a^2\left(\frac{dz}{du}\right)^2 - z\frac{dz}{du} + 1 = 0.$$

$$\therefore \frac{dz}{du} = \frac{z \pm (z^2-4a^2)^{1/2}}{2a^2} \quad \text{or} \quad \frac{dz}{z \pm (z^2-4a^2)^{1/2}} = \frac{du}{2a^2}$$

or $$\frac{[z \mp (z^2-4a^2)^{1/2}]dz}{[z \pm (z^2-4a^2)^{1/2}][z \mp (z^2-4a^2)^{1/2}]} = \frac{du}{2a^2} \quad \text{or} \quad \frac{z \mp (z^2-4a^2)^{1/2}}{4a^2} = \frac{du}{2a^2}$$

or $$[z \mp (z^2-4a^2)^{1/2}]\, dz = 2du.$$

Integrating, $$\frac{z^2}{2} \mp \left[\frac{z}{2}(z^2-4a^2)^{1/2} - \frac{4a^2}{2}\log\left\{z + (z^2-4a^2)^{1/2}\right\}\right] = 2u + \frac{b}{2}$$

or $$z^2 \mp \left[z(z^2-4a^2)^{1/2} - 4a^2\log\left\{z + (z^2-4a^2)^{1/2}\right\}\right] = 4(x + ay) + b.$$

Ex. 5. (b) *Find a complete integral of* $1 + p^2 = qz$.

Sol. Proceed as in Ex. 5.(*a*). The requied complete integral is

$$a^2z^2 \mp \left[az\sqrt{(a^2z^2-4)} - 4\log\left\{az + \sqrt{(a^2z^2-4)}\right\}\right] = 4(x + ay) + b.$$

Ex. 6. *Find complete integrals of the following partial differential equations*

(*i*) $p(z + p) + q = 0$

(*ii*) $p(1 + q) = qz$. **[Gulbarga 2005]**

Sol. (*i*) The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we get

$$\frac{dz}{du}\left(z + \frac{dz}{du}\right) + a\frac{dz}{du} = 0 \quad \text{or} \quad \frac{dz}{du} = -(z + a) \quad \text{or} \quad \frac{dz}{z+a} = -du.$$

Integrating, $\log(z + a) - \log b = -u$ or $z + a = be^{-u}$ or $z + a = be^{-(x + ay)}$.

(*ii*) Proceed as in part (*i*). **Ans.** $az - 1 = be^{x + ay}$.

Ex. 7. *Find a complete integral of* $p^3 + q^3 - 3pqz = 0$. **[I.A.S. 1991]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation,

$$\left(\frac{dz}{du}\right)^3 + a^3\left(\frac{dz}{du}\right)^3 - 3az\left(\frac{dz}{du}\right)^2 = 0 \quad \text{or} \quad (1 + a^3)\frac{dz}{du} = 3az \quad \text{or} \quad \frac{1+a^3}{z}dz = 3au.$$

Integrating $(1 + a^3)\log z = 3au + b$ or $(1 + a^3)\log z = 3a(x + ay) + b$.

Ex. 8. *Find a complete integrals of* (*i*) $p + q = z/c$.

(*ii*) $p + q = z$.

Sol. (*i*) The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we get

$$\frac{dz}{du}+a\frac{dz}{du}=\frac{z}{c} \quad \text{or} \quad (1+a)\frac{dz}{du}=\frac{z}{c} \quad \text{or} \quad \frac{c(1+a)}{z}dz = du.$$

Integrating, $c(1 + a)\log z = u + b$ or $c(1 + a)\log z = x + ay + b$.

(*ii*) Proceed as in part (*i*). **Ans.** $(1 + a)\log z = x + ay + b$.

Ex. 9. *Find a complete integral of* $p^2 = z^2(1 - pq)$. **[Jiwaji 1998; Meerut 2001]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(\frac{dz}{du}\right)^2 = z^2\left\{1-a\left(\frac{dz}{du}\right)^2\right\} \quad \text{or} \quad \left(\frac{dz}{du}\right)^2(1+az^2) = z^2$$

or $$\frac{dz}{du} = \pm\frac{z}{\sqrt{(1+az^2)}} \quad \text{or} \quad \pm\, du = \frac{\sqrt{(1+az^2)}}{z}dz = \pm\frac{(1+az^2)dz}{z\sqrt{(1+az^2)}}.$$

or $$\pm\int du \pm b = \int\frac{dz}{z\sqrt{(1+az^2)}}+\frac{1}{2}\int\frac{2az\,dz}{\sqrt{(1+az^2)}}. \qquad ...(1)$$

Now, $$\int\frac{dz}{z\sqrt{(1+az^2)}}=\int\frac{(-1/t^2)\,dt}{(1/t)\times\sqrt{\{(1+(a/t^2)\}}}, \text{ putting } z = 1/t \text{ so that } dz = -(1/t^2)dt$$

$$= -\int\frac{dt}{(t^2+a)^{1/2}} = -\sinh^{-1}\frac{t}{\sqrt{a}} = -\sinh^{-1}\frac{1}{z\sqrt{a}}, \text{ as } t=\frac{1}{z} \qquad ...(2)$$

and $$\frac{1}{2}\int\frac{2az\,dz}{(1+az^2)^{1/2}} = \frac{1}{2}\int\frac{2v\,dv}{v}, \text{ putting } 1+az^2=v^2 \text{ and } 2az\,dz = 2v\,dv$$

$$= v = (1+az^2)^{1/2}.$$

Using (2) and (3), (1) reduces to $\pm(u + b) = -\sinh^{-1}(1/z\sqrt{a}) + (1 + az^2)^{1/2}$

or $$\pm(x + ay + b) = -\sinh^{-1}(1/z\sqrt{a}) + (1 + az^2)^{1/2}.$$

Ex. 10. *Find complete and singular integrals of* $4(1 + z^3) = 9z^4pq$.

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$4(1+z^3) = 9z^4a\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad \pm\frac{3\sqrt{a}z^2}{(1+z^3)^{1/2}}dz = 2\,du$$

or $$\pm\left(\sqrt{a}/t\right)\times 2t\,dt = 2\,du, \text{ putting } 1+z^3=t^2 \text{ so that } 3z^2dz = 2t\,dt$$

Integrating, $\pm\sqrt{a}\,t = u + b$ or $\pm\sqrt{a}\,(1 + z^3)^{1/2} = x + ay + b$

or $$a(1 + z^3) = (x + ay + b)^2, \qquad ...(1)$$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially w.r.t. a and b by turn, we get

$$1 + z^3 = 2y(x + ay + b) \qquad ...(2)$$

and $$0 = 2(x + ay + b). \qquad ...(3)$$

Eliminating a and b from (1), (2) and (3), the singular integral is $1 + z^3 = 0$. ...(4)

From (4), $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Thus these values of p and q together with $1 + z^3 = 0$ satisfy the given equation. Hence $1 + z^3 = 0$ is the required singular integral.

Ex. 11. *Find complete and singular integrals of* $q^2 = z^2p^2(1 - p^2)$.

[Madurai Kamraj 2008; CDLU 2004]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(a\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left[1-\left(\frac{dz}{du}\right)^2\right] \qquad \text{or} \qquad a^2 = z^2\left[1-\left(\frac{dz}{du}\right)^2\right]$$

or
$$\left(\frac{dz}{du}\right)^2 = \frac{z^2-a^2}{z^2} \qquad \text{or} \qquad du = \pm\frac{zdz}{(z^2-a^2)^{1/2}}.$$

Integrating, $\quad u + b = \pm (z^2 - a^2)^{1/2} \quad$ or $\quad (x + ay + b)^2 = z^2 - a^2, \quad ...(1)$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$-2a = 2y(x + ay + b) \qquad ...(2)$$

and
$$0 = 2(x + ay + b). \qquad ...(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ satisfy the given equation and hence the required singular integral is $z = 0$.

Ex. 12. *Find complete, singular and general integral of* $p^3 + q^3 = 27z$. **[Ravishankar 2005]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(\frac{dz}{du}\right)^3 + \left(a\frac{dz}{du}\right)^3 = 27z \qquad \text{or} \qquad \frac{dz}{du}(1 + a^3)^{1/3} = 3z^{1/3}$$

or
$$du = (1/3) \times (1 + a^3)^{1/3}\, z^{-1/3}\, dz.$$

Integrating, $\quad u + b = (1/3) \times (1 + a^3)^{1/3} \times [z^{2/3}/(2/3)] \quad$ or $\quad 2(u + b) = (1 + a^3)^{1/3}z^{2/3}$

or $\quad 8(u + b)^3 = (1 + a^3)z^2 \quad$ or $\quad 8(x + ay + b)^3 = (1 + a^3)z^2, \quad ...(1)$

which is a complete integral containing two arbitrary contants a and b.

Singular Integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$24y(x + ay + b)^2 = 3a^2z^2 \qquad ...(2)$$

and
$$24(x + ay + b) = 0. \qquad ...(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ satisfy the given equation and hence the required singular integral is $z = 0$.

General integral. Let $b = \phi(a)$, where ϕ is an arbitrary function. Then (1) becomes

$$8[x + ay + \phi(a)]^3 = z^2(1 + a^3). \qquad ...(4)$$

Differentiating (4) partially w.r.t. 'a', $\quad 24[x + ay + \phi(a)]^2\, [y + \phi'(a)] = 3a^2z^2. \quad ...(5)$

General integral is obtained by eliminating a from (4) and (5).

Ex. 13. *Find complete and singular integrals of* $z^2(p^2z^2 + q^2) = 1$.

[Delhi Maths Hons 2005; Meerut 2003]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$z^2\left[z^2\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 1 \qquad \text{or} \qquad z^2(z^2 + a^2)\left(\frac{dz}{du}\right)^2 = 1$$

or
$$du = \pm\, z(z^2 + a^2)^{1/2}\, dz = \pm\, (1/2) \times (z^2 + a^2)^{1/2}\, (2zdz)$$

Integrating, $\quad u + b = \pm\, (1/2) \times [(z^2 + a^2)^{3/2}/(3/2)]$

or $\quad 9(u + b)^2 = (z^2 + a^2)^3 \quad$ or $\quad 9(x + ay + b)^2 = (z^2 + a^2)^3, \quad ...(1)$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially, w.r.t. 'a' and 'b', we get

$$18(x + ay + b)y = 3(z^2 + a^2) \times 2a \qquad ...(2)$$

and
$$18(x + ay + b) = 0. \qquad ...(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ do not satisfy the given equation. Hence $z = 0$ is not a singular solution of the given equation.

Ex. 14. (*i*) *Find a complete integral of* $z^2(p^2 + q^2 + 1) = k^2$.

[Jabalpur 2004; Banglore 1993; I.A.S. 1996; Meerut 1997]

(*ii*) *Find a complete and singular integral of* $z^2(p^2 + q^2 + 1) = 1$. **[I.A.S. 1979]**

Sol. (*i*) The given equation is of the form $f(p, q\, z) = 0$. Let $u = x + ay$ where a is an arbitrary constant. Replacing p by (dz/du) and q by $a(dz/du)$ in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = k^2 \qquad \text{or} \qquad (1 + a^2)\left(\frac{dz}{du}\right)^2 = \frac{k^2 - z^2}{z^2}$$

or
$$\pm (1 + a^2)^{1/2} \frac{z}{(k^2 - z^2)^{1/2}} dz = du \qquad \text{or} \qquad \pm \frac{1}{2}(1 + a^2)(k^2 - z^2)^{-1/2}(-2zdz) = du.$$

Integrating, $\pm (1 + a^2)^{1/2} (k^2 - z^2)^{1/2} = u + b$ or $(1 + a^2)(k^2 - z^2) = (u + b)^2$

or
$$(1 + a^2)(k^2 - z^2) = (x + ay + b)^2.$$

(*ii*) Here $k = 1$. Proceed as in part (*i*) and get complete integral

$$(1 + a^2)(1 - z^2) = (x + ay + b)^2. \qquad ...(1)$$

Differentiating (1) partially w.r.t. a and b, we get

$$2a(1 - z^2) = 2(x + ay + b) \times y \qquad ...(2)$$

and
$$0 = 2(x + ay + b). \qquad ...(3)$$

From (2) and (3), we get $x + ay + b = 0$ and $a = 0$. With these values (1) reduces to $z^2 = 1$, which is free from a and b. Again, from $z^2 = 1$, $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Now, $p = 0$, $q = 0$ and $z^2 = 1$, satisfy the given equation and hence singular integral of the given equation is $z^2 = 1$.

Ex. 15. *Find a complete integral of* (*i*) $q^2y^2 = z(z - px)$ **[Meerut 1997]**

(*ii*) $p^2x^2 = z(z - qy)$.

Sol. (*i*) Given equation can be rewritten as
$$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right). \qquad ...(1)$$

We choose new variables X and Y such that $(1/x)dx = dX$ and $(1/y)dy = dY$. ...(2)

so that $\log x = X$ and $\log y = Y$. ...(3)

Using (2), (1) becomes
$$\left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right) \qquad \text{or} \qquad Q^2 = z\,(z - P), \qquad ...(4)$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q, z) = 0$. Let $u = X + aY$, where a is an arbitrary constant. Replacing P by dz/du and Q by $a(dz/du)$ in (4), we get

$$a^2\left(\frac{dz}{du}\right)^2 = z\left(z - \frac{dz}{du}\right) \qquad \text{or} \qquad a^2\left(\frac{dz}{du}\right)^2 + z\frac{dz}{du} - z^2 = 0.$$

$$\therefore \qquad \frac{dz}{du} = \frac{-z \pm (z^2 + 4a^2z^2)^{1/2}}{2a^2} = \frac{-1 \pm (1 + 4a^2)^{1/2}}{2a^2} z = kz, \qquad ...(5)$$

where $$k = [-1 \pm (1 + 4a^2)^{1/2}]/2a^2. \qquad ...(6)$$

From (5), $(1/kz)dz = du$ so that $(1/k) \log z = u + \log b$

or $\log z^{1/k} = X + aY + \log b = \log x + a \log y + \log b = \log (xby^a)$.

$\therefore$ $z^{1/k} = xby^a$ is complete integral containing two arbitrary constants a and b and an absolute constant k given by (6).

(*ii*) Proceed as in part (*i*). **Ans.** $xby^a = z^{1/k}$, where $k = [-1 \pm (a^2 + 4)^{1/2}]/2$

Ex. 16. *Solve* $p^2 + q^2 = z$.

Sol. Given equation is $$p^2 + q^2 = z, \qquad ... (1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we have
$$(du/dz)^2 + a^2(du/dz)^2 = z \quad \text{or} \quad (dz/du)^2 = z/(1 + a^2)$$
or $$\frac{dz}{du} = \pm\frac{z^{1/2}}{(1+a^2)^{1/2}} \quad \text{or} \quad \pm z^{-1/2}(1+a^2)^{1/2}dz = du$$

Integrating, $\pm 2z^{1/2}(1+a^2)^{1/2} = u + b$ or $\pm 2z^{1/2}(1+a^2)^{1/2} = x + ay + b$

Thus, $4z(1 + a^2) = (x + ay + b)^2$, a, b being arbitrary constants ... (2)

(2) is the complete integral of the given equation (1).

Differentiating (2) partially w.r.t. 'a' and 'b', we get

$8az = 2y(x + ay + b)$ or $4az = y(x + ay + b)$ ... (3)

$0 = 2(x + ay + b)$ or $x + ay + b = 0$ ... (4)

Substituting the value of $x + ay + b$ from (4) in (3), we have

$4az = 0$ or $z = 0$, which is the singular solution.

In order to get the general solution, put $b = \psi(a)$ in (2) and get
$$4z(1+a^2) - \{x + ay + \psi(a)\}^2 = 0 \qquad ... (5)$$

Differentiating (5) partially w.r.t. 'a', $8az - 2\{x + ay + \psi(a)\} \times \{y + \psi'(a)\} = 0$ ... (6)

The required general solution is given by (5) and (6)

Ex. 17. *Find the complete integral of* $16p^2z^2 + 9q^2z^2 + 4(z^2 - 1) = 0$

Sol. Given equation is of the form $f(p, q, z) = 0$

Let $u = x + ay$, a being an arbitrary constant. ... (1)

Now replacing p and q by dz/du and $a(dz/du)$ respectively in the given equation, we have
$$16z^2\ (dz/du)^2 + 9a^2z^2\ (dz/du)^2 + 4(z^2 - 1) = 0$$
or $$(16 + 9a^2)z^2\left(\frac{dz}{du}\right)^2 = 4(1 - z^2) \quad \text{or} \quad \frac{dz}{du} = \frac{2(1-z^2)^{1/2}}{z(16+9a^2)^{1/2}}$$
or $$(-1/2) \times (16 + 9a^2)^{1/2}\ (1 - z^2)^{-1/2}(-2z)dz = du$$

Integrating, $-(16 + 9a^2)^{1/2}\ (1 - z^2)^{1/2} = u + b = x + ay + b$, by (1)

or $(16 + 9a^2)\ (1 - z^2) = (x + ay + b)^2$ is the complete integral, a, b being arbitrary constants

Ex. 18. *Find the complete integral of* $pq = x^4y^3z^2$

Sol. Re–writing the given equation, we get $(\partial z / x^4\partial x)\ (\partial z / y^3\partial y) = z^2$ ... (1)

Putting $x^4 dx = dX$, $y^3 dy = dY$ so that $x^5/5 = X$, $y^4/4 = Y$, (1) gives

$$(\partial z/\partial X)\ (\partial z/\partial Y) = z^2 \quad \text{or} \quad PQ = z^2 \qquad \text{... (2)}$$

which is of the form $f(P, Q, z) = 0$. Let $u = X + aY$, a being an arbitrary constant. Replacing P and Q be dz/du and $a(dz/du)$ respectively in (2), we get

$$a\,(dz/du)^2 = z^2 \quad \text{giving} \quad (\sqrt{a}/z)dz = du$$

Integrating $\sqrt{a}\log z = u + b = X + aY + b$, as $u = X + aY$

or $\sqrt{a}\ \log z = x^5/5 + (ay^4)/4 + b,\ a,\ b$ being arbitrars constants

EXERCISE 3(E)

Find the complete integral of the following equations (1–7)

1. $z = p^2 - q^2$. **Ans.** $x + ay + b = 4(1 - a^2)z$

2. $zpq = p + q$. **(Delhi 2007)** **Ans.** $x + ay + b = (4az)/(1 - a)$

3. $z^2p^2 + q^2 = p^2q$. **Ans.** $z = a\tan(x + ay + b)$

4. $p^2z^2 + q^2 = 1$. **[Delhi B.A. (Prog) II 2010, 11; M.S. Univ. T.N. 2007, Nagpur 2001; Meerut 2008]** **Ans.** $x + ay + b = \pm[(z/2)\times(z^2 + a^2)^{1/2} + (a^2/2)\times \sinh^{-1}(z/a)]$

5. $p^3 = qz$. **(Delhi B.Sc. (Prog) II 2007)** **Ans.** $4z = (x + ay + b)^2$

6. $16z^2p^2 + 25z^2a^2 + 9z^2 - 81 = 0$. **Ans.** $(16 + 25a^2)\,(9 - z^2) = 9(x + ay + b)^2$

7. $z^2 = 1 + p^2 + q^2$. **Ans.** $z = \cosh\{x + ay + b)/(1 + a^2)^{1/2}\}$

8. Using Charpit's method, discuss how to solve equations of the form $f(z, p, q) = 0$. Hence find complete integral of the equation $9(p^2z + q^2) = 4$. **[Delhi Maths (H) 2006]**

Hint : Refer Art. 3.14 and Ex. 1 of Art. 3.15

Solve the following partial differential equations (9 – 14)

9. $z^2(p^2 + q^2 + 2) = 1$ **Ans. C.I.** $(1 + a^2)\,(1 - 2z^2) = 4(x + ay + b)^2$; **S.S.** $2z^2 - 1 = 0$;

G.S. It is given by $(1 + a^2)\,(1 - 2z^2) - \{x + ay + \psi(a)\}^2 = 0,\quad a(1 - 2z^2)$
$-4\{x + ay + \psi(a)| \times \{y + \psi'(a)\} = 0$

10. $z = pq$. **Ans. C.I.** $4az = (x + ay + b)^2$; **S.S.** $z = 0$; **G.S.** It is given by

$4az - \{x + ay + \psi(a)\}^2,\quad 2z - \{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$

11. $p(1 - q^2) = q(1 - z)$. **Ans. C.I.** $4(1 - a + az) = (x + ay + b)^2$; **S.S.** Does not exist; **G.S.** It is given by $4(1 - a + az) - \{x + ay + \psi(a)\}^2 = 0,\quad 2z - 2 - \{x + ay + \psi(a)\}\{y + \psi'(a)\} = 0$

12. $p^2 + pq = 4z$. **Ans. C.I.** $(1 + a)z = (x + ay + b)^2$; **S.S.** $z = 0$; **G.S.** $(1 + a)z - \{x + ay + \psi(a)\}^2 = 0$,

$z - 2\{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$

13. $p^3 + q^3 = 3pqz,\ z > 0$. **Ans. C.S.** $(1 + a^3)\log z = 3a(x + ay) + b$; **S.S.** Does not exist. **G.S.** It is given by $(1 + a^3)\log z - 3a(x + ay) - \psi(a) = 0,\quad 3a^2\log z - 3x - 6ay - \psi'(a) = 0$

14. $p^2 + q^2 = 4z$. **Ans. C.I.** $4(1+a^2)z - (x + ay + b)^2 = 0$; **S.S.** $z = 0$; **G.S.** It is given by

$(1 + a^2)z - \{x + ay + \psi(a)\}^2 = 0,\quad az - \{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0.$

3.17. Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$. *i.e.*, a form in which z does not appear and the terms containing x and p are on one side and those containing y and q on the other side. **[Bhopal 2010; Ravishankar 1999]**

Let $$F(x, y, z, p, q) = f_1(x, p) - f_2(y, q) = 0. \qquad ...(1)$$

Then Charpit's auxiliary equations are

$$\frac{dp}{\dfrac{\partial F}{\partial x}+p\dfrac{\partial F}{\partial z}} = \frac{dq}{\dfrac{\partial F}{\partial y}+q\dfrac{\partial F}{\partial z}} = \frac{dz}{-p\dfrac{\partial F}{\partial p}-q\dfrac{\partial F}{\partial q}} = \frac{dx}{-\dfrac{\partial F}{\partial p}} = \frac{dy}{-\dfrac{\partial F}{\partial q}}$$

or $$\frac{dp}{\partial f_1/\partial x} = \frac{dq}{-\partial f_2/\partial y} = \frac{dz}{-p(\partial f_1/\partial p)+q(\partial f_2/\partial q)} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_2/\partial q}, \text{ by (1)}$$

Taking the first and the fourth ratios, we have

$(\partial f_1 / \partial p)\, dp + (\partial f_1 / \partial x) dx = 0$ or $df_1 = 0.$

Integrating, $f_1 = a$, a being an arbitrary constant.

$\therefore$ (1) $\Rightarrow$ $f_1(x, p) = f_2(y, q) = a.$ ...(2)

Now, (2) $\Rightarrow$ $f_1(x, p) = a$ and $f_2(y, q) = a.$ ...(3)

From (3), on solving for p and q respectively, we get

$p = F_1(x, a)$, say and $q = F_2(y, a)$, say ...(4)

Substituting these values in $dz = p\, dx + q\, dy$, we get $dz = F_1(x, a)\, dx + F_2(y, a)\, dy.$

Integrating, $$z = \int F_1(x,a)\, dx + \int F_2(y,a) dy + b,$$

which is a complete integral containing two arbitrary constants a and b.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form IV.

Remark 2. Singular and general integral are obtained by well known methods.

3.18. SOLVED EXAMPLES BASED ON ART 3.17

Ex. 1. *Find a complete integral of* $x(1 + y)p = y(1 + x)q$. **[Agra 1991]**

Sol. Separating p and x from q and y, the given equation reduces to

$$(xp)/(1 + x) = (yq)/(1 + y)$$

Equating each side to an arbitrary constant a, we have

$$\frac{xp}{1+x} = a \quad \text{and} \quad \frac{yq}{1+y} = a \quad \text{so that} \quad p = a\left(\frac{1+x}{x}\right) \quad \text{and} \quad q = a\left(\frac{1+y}{y}\right).$$

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = \frac{a(1+x)}{x}dx + \frac{a(1+y)}{y}dy \quad \text{or} \quad dz = a\left(\frac{1}{x}+1\right)dx + a\left(\frac{1}{y}+1\right)dy.$$

Integrating, $z = a(\log x + x) + a(\log y + y) + b = a(\log xy + x + y) + b,$

which is a complete integral containing two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p - 3x^2 = q^2 - y$. **[Meerut 1996]**

Sol. Equating each side to an arbitrary constant a, we get

$p - 3x^2 = a$ and $q^2 - y = a$ so that $p = a + 3x^2$ and $q = (a + y)^{1/2}.$

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (a + 3x^2)dx + (a + y)^{1/2}\, dy$ so that $z = ax + x^3 + (2/3) \times (a + y)^{3/2} + b.$

Ex. 3. *Find a complete integral of* $yp = 2yx + \log q$. **[Ravishankar 2005]**

Sol. Rewritting the given equation, $p = 2x + (1/y) \log q$ or $p - 2x = (1/y) \log q$.

Equating each side to an arbitrary constant a, we get

$p - 2x = a$ and $(1/y) \log q = a$ so that $p = a + 2x$ and $q = e^{ay}$.

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$dz = (a + 2x)dx + e^{ay}dy$ so that $z = (ax + x^2) + (1/a) \times e^{ay} + b$.

Ex. 4. *Find a complete integral of* $q = px + p^2$.

[Agra 1995; Meerut 1994; Bilaspur 2004; Jabalpur 1998]

Sol. Equating each side of the given equation to an arbitrary constant a, we have

$q = a$ and $px + p^2 = a$ or $q = a$ and $p^2 + px - a = 0$.

$\therefore$ $q = a$ and $p = [-x \pm (x^2 + 4a)^{1/2}]/2$.

Putting thse values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = (1/2) \times [-x \pm (x^2 + 4a)^{1/2}]dx + a\, dy.$$

Integrating, $$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{(x^2 + 4a)} + 2a \log\left\{x + \sqrt{(x^2 + 4a)}\right\}\right] + ay + b.$$

Ex. 5. *Solve* $py + qx + pq = 0$. **[Kurukshetra 2004; I.A.S 1990]**

Sol. Given $py + q(x + p) = 0$. or $p/(p + x) = -q/y$.

Equating each side to an arbitrary constant a, we get

$p/(p + x) = a$ and $-q/y = a$ $\Rightarrow$ $p = (xa)/(1 - a)$ and $q = -ay$.

Putting thse values of p and q in $dz = p\, dx + q\, dy$, we get

$dz = \{a/(1 - a)\}\, x\, dx - ay\, dy$ so that $z = \{a/(1 - a)\} \times (x^2/2) - a \times (y^2/2) + b/2$

or $2z = \{a/(1 - a)\}x^2 - ay^2 + b$, a, b being arbitrary constants.

Ex. 6. *Find a complete integral of* $z^2(p^2 + q^2) = x^2 + y^2$, *i.e.*, $z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2] = x^2 + y^2$.

[Agra 2006; Jabalpur 2004; Rewa 2002 Sagar 1999; Vikram 1996 Delhi Maths Hons 1990; I.A.S. 1989; Kanpur 1994; Meerut 2003]

Sol. Given $$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 \quad \text{or} \quad \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \quad ...(1)$$

Let $z\, dz = dz$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$,

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side of the above equation to an arbitrary constant a^2, we get

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$.

Putting these values of P and Q in $dZ = P\, dx + Q\, dy$, we have

$$dZ = (a^2 + x^2)^{1/2}\, dx + (y^2 - a^2)^{1/2}\, dy.$$

Integrating, $$Z = (x/2) \times (a^2 + x^2)^{1/2} + (a^2/2) \times \log\{x + (a^2 + x^2)^{1/2}\}$$
$$+ (y/2) \times (y^2 - a^2)^{1/2} - (a^2/2) \times \log\{y + (y^2 - a^2)^{1/2}\} + (b/2)$$

or $$z^2 = x^2(a^2 + x^2)^{1/2} + a^2 \log[x + (a^2 + x^2)^{1/2}\} + y(y^2 - a^2)^{1/2} - a^2 \log\{y + (y^2 - a^2)^{1/2}\} + b$$

[$\because$ From (2), $Z = z^2/2$]

Ex. 7. *Find a complete integral of* $z(p^2 - q^2) = x - y$.

[Bilaspur 2003; Indore 2002, 02; Jiwaji 2000; Bangalore 1995; I.A.S 1989]

Sol. Re–writting the given equation, $(\sqrt{z}\partial z/\partial x)^2 - (\sqrt{z}\partial z/\partial y)^2 = x - y.$ ...(1)

Let $\sqrt{z}\, dz = dZ$ so that $(2/3) \times z^{3/2} = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 - (\partial Z/\partial y)^2 = x - y$ or $P^2 - Q^2 = x - y,$
where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = Q^2 - y. \qquad ...(3)$$

Equating each side to an arbitrary constant a, we get

$P^2 - x = a$ and $Q^2 - y = a$ so that $P = (x + a)^{1/2}$ and $Q = (y + a)^{1/2}$

Putting these values of P and Q in $dZ = P\, dx + Q\, dy$, $dZ = (x + a)^{1/2} dx + (y + a)^{1/2} dy.$

Integrating, $Z = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$

or $(2/3) \times z^{3/2} = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$, as $Z = (2/3) \times z^{3/2}$

or $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$, a, b being arbitrary constants.

Ex. 8. *Find a complete integral of* $z(xp - yq) = y^2 - x^2$.

Sol. Re–writting the given equation, we have

$$xz\frac{\partial z}{\partial x} - yz\frac{\partial z}{\partial y} = y^2 - x^2 \quad \text{or} \quad x\left(z\frac{\partial z}{\partial x}\right) - y\left(z\frac{\partial z}{\partial y}\right) = y^2 - x^2. \qquad ...(1)$$

Let $z\, dz = dZ$ so that $z^2/2 = Z.$ ...(2)

Using (2), (1) becomes $x(\partial Z/\partial x) - y(\partial Z/\partial y) = y^2 - x^2$ or $xP - yQ = y^2 - x^2,$
where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$xP + x^2 = yQ + y^2.$$

Equating each side to an arbitrary constant a, we have

$xP + x^2 = a$ and $yQ + y^2 = a$ so that $P = a/x - x$ and $Q = a/y - y.$

Putting these values of P and Q in $dZ = P\, dx + Q\, dy$, $dZ = (a/x - x)dx + (a/y - y)dy.$

Integrating, $Z = a \log x - (x^2/2) + a \log y - (y^2/2) + b/2$

or $z^2/2 = a(\log x + \log y) - (x^2 + y^2 - b)/2$ or $z^2 = 2a \log (xy) - x^2 - y^2 + b.$

Ex. 9. *Find a complete integral of* $p^2 + q^2 = z^2(x + y)$. **[Agra 2010; M.S. Univ. T.N. 2007]**

Sol. Given $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z^2(x + y)$ or $\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{z}\frac{\partial z}{\partial y}\right)^2 = x + y$ ....(1)

Let $(1/z)dz = dZ$ so that $\log z = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x + y$ or $P^2 + Q^2 = x + y,$
where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = y - Q^2.$$

Equating each side to an arbitrary constant a, we have

$P^2 - x = a$ and $y - Q^2 = a$ so that $P = (a + x)^{1/2}$ and $Q = (y - a)^{1/2}.$

Putting these values of P and Q in $dZ = P\, dx + Q\, dy$, $dZ = (a + x)^{1/2} dx + (y - a)^{1/2} dy.$

Integrating, $Z = (2/3) \times [(a + x)^{3/2} + (y - a)^{3/2}] + (2/3) \times b$

$\therefore$ $\log z = (2/3) \times [(a + x)^{3/2} + (y - a)^{3/2} + b]$ is a complete integral, using $Z = \log z$

Ex. 10. *Find a complete integral of* $p^2 + q^2 = (x^2 + y^2)z$. **[Delhi Maths Hons. 1995]**

Sol. The given equation can be rewritten as

$$\frac{1}{z}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2 + y^2 \quad \text{or} \quad \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \qquad ...(1)$$

Let $\left(1/\sqrt{z}\right)dz = dZ$ *i.e.,* $z^{-1/2}dz = dZ$ so that $2\sqrt{z} = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$,

where $P = \partial Z/\partial x$ *and* $Q = \partial Z/\partial y$. Separting P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side to an arbitrary constant a^2, we have

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a^2 + x^2)^{1/2}dx + (y^2 - a^2)^{1/2}dy$.

Integrating, $$Z = \frac{x}{2}(x^2 + a^2)^{1/2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}(y^2 - a^2)^{1/2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + \frac{b}{2}$$

or $4z^{1/2} = x(x^2 + a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2 - a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$, as $Z = 2\sqrt{z}$

Ex. 11. *Find a complete integral of* $(p^2/x) - (q^2/y) = (1/z) \times [(1/x) + (1/y)]$.

[Delhi B.Sc. Hons. 1996]

Sol. The given equation can be re–written as

$$\frac{z}{x}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{z}{y}\left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{or} \quad \frac{1}{x}\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \frac{1}{y}\left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad ....(1)$$

Let $\sqrt{z}\,dz = dZ$ so that $(2/3) \times z^{3/2} = Z$. ...(2)

Using (2), (1) becomes

$$\frac{1}{x}\left(\frac{\partial Z}{\partial x}\right)^2 - \frac{1}{y}\left(\frac{\partial Z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{or} \quad \frac{P^2}{x} - \frac{Q^2}{y} = \frac{1}{x} + \frac{1}{y},$$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$(P^2 - 1)/x = (Q^2 + 1)/y.$$

Equating each side to an arbitrary constant a, we have

$(P^2 - 1)/x = a$ and $(Q^2 + 1)/y = a$ so that $P = (1 + ax)^{1/2}$ and $Q = (ay - 1)^{1/2}$.

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (1 + ax)^{1/2}dx + (ay - 1)^{1/2}dy$.

Integrating, $Z = (2/3a) \times (1 + ax)^{3/2} + (2/3a) \times (ay - 1)^{3/2} + (2/3a) \times b$

or $az^{3/2} = (1 + ax)^{3/2} + (ay - 1)^{3/2} + b$, as $Z = (2/3) \times z^{3/2}$.

Ex. 12. *Find a complete integral of* $yzp^2 = q$. **[M.S. Univ. T.N. 2007]**

Sol. Given $$yz^2\left(\frac{\partial z}{\partial x}\right)^2 = z\frac{\partial z}{\partial y} \quad \text{or} \quad y\left(z\frac{\partial z}{\partial x}\right)^2 = \left(z\frac{\partial z}{\partial y}\right). \quad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $y(\partial Z/\partial x)^2 = \partial Z/\partial y$ or $yP^2 = Q$, ...(3)

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P from y and Q, we get

$P^2 = Q/y = a^2$, (say) ; a being an arbitrary constant. Hence $P = a$ and $Q = ya^2$.

Then, $dZ = P\,dx + Q\,dy$ reduces to $dZ = a\,dx + ya^2dy$ so that $Z = ax + (a^2/y^2)/2 + b/2$

or $z^2/2 = ax + (a^2y^2)/2 + b/2$ or $z^2 = 2ax + a^2y + b$.

Ex. 13. *Find a complete integral of* $zpy^2 = x(y^2 + z^2q^2)$.

Sol. Given $$y^2\left(z\frac{\partial z}{\partial x}\right) = x\,y^2 + x\left(z\frac{\partial z}{\partial y}\right)^2 \quad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $y^2(\partial Z/\partial x) = xy^2 + x\,(\partial Z/\partial y)^2$ or $y^2P = x(y^2 + Q^2)$,

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from, Q and y, we get

$$P/x = (y^2 + Q^2)/y^2.$$

Equating each side to an arbitrary constant a, we get

$P/x = a$ and $1 + (Q^2/y^2) = a$ so that $P = ax$ and $Q = \pm (a-1)^{1/2} y$.

$\therefore$ $$dZ = P\,dx + Q\,dy = ax\,dx \pm (a-1)^{1/2} y\,dy$$

Integrating, $Z = (ax^2/2) \pm (a-1)^{1/2}(y^2/2) + b/2$ or $z^2 = ax^2 \pm (a-1)^{1/2} y^2 + b$, as $Z = z^2/2$.

Ex. 14. *Find the complete integral of the partial differential equation*

$$2p^2q^2 + 3x^2y^2 = 8x^2q^2(x^2+y^2)$$ **[I.A.S. 2001]**

Sol. Re–writing the given equation, we have

$2q^2(p^2 - 4x^4) = x^2y^2(8q^2 - 3)$ or $(p^2 - 4x^4)/x^2 = y^2\,(8q^2-3)/2q^2 = 4a^2$, say

where a is an arbitrary constant. Then, $p^2 = 4x^2(a^2 + x^2)$ and $8q^2(y^2 - a^2) = 3y^2$

so that $p = 2x(a^2 + x^2)^{1/2}$ and $q = (3/2)^{1/2} \times (y/2) \times (y^2 - a^2)^{-1/2}$

Substituting these values in $dz = p\,dx + q\,dy$, we get

$$dz = 2x(a^2+x^2)^{1/2}dx + (3/2)^{1/2} \times (y/2) \times (y^2-a^2)^{-1/2}dy$$

Integrating, $z = 2\int x(a^2+x^2)^{1/2}dx + (3/2)^{1/2} \times (1/2) \times \int y(y^2-a^2)^{-1/2}dy + b$ ... (1)

Put $x^2 + a^2 = u$ and $y^2 - a^2 = v$ so that $2xdx = du$ and $2ydy = dv$ ... (2)

i.e., $xdx = (1/2) \times du$ and $ydy = (1/2) \times dv$. Then (1) reduces to

$$z = \int u^{1/2}du + (3/2)^{1/2} \times (1/4) \times \int v^{-1/2}dv + b$$

or $$z = (2/3) \times u^{3/2} + (3/2)^{1/2} \times (1/4) \times 2v^{1/2} + b$$

or $$z = (2/3) \times (x^2+a^2)^{3/2} + (3/2)^{1/2} \times (1/2) \times (y^2-a^2)^{1/2} + b,$$

which is the required complete integral containing a and b as arbitrary constants.

Ex. 15. *Find the complete integral of the partial differential equation* $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$ **[Delhi Maths (H) 2002; Agra 2005]**

Sol. Re–writing, $p^2/x^2 + y^2/q^2 = x^2 + y^2$ or $(p^2/x^2) - x^2 = y^2 - (y^2/q^2) = a^2$, say

$\Rightarrow$ $p = x(x^2+a^2)^{1/2}$, and $q = y/(y^2 - a^2)^{1/2}$

$\therefore$ $dz = p\,dx + q\,dy$ becomes $dz = x(x^2+a^2)^{1/2}dx + y(y^2-a^2)^{-1/2}dy$

Integrating, $$z = (1/3) \times (x^2+a^2)^{1/2} + (y^2-a^2)^{1/2} + b,$$

which is complete integral with a and b as arbitrary constants.

Ex. 16. *Find the complete integral of* $(1-x^2)yp^2 + x^2q = 0$

Sol. Re–writing, we have $(x^2-1)p^2/x^2 = q/y = a^2$, say

$\therefore$ $p = ax/(x^2-1)^{1/2}$ and $q = a^2y$. Hence $dz = p\,dx + q\,dy$ becomes

$dz = ax(x^2-1)^{-1/2}dx + a^2ydy$ so that $z = a(x^2-1)^{1/2} + (a^2y^2)/2 + b$.

Ex. 17. *Find the the complete integral of* $p + q - 2px - 2qy + 1 = 0$.

Sol. Re–writing, $p - 2px = 2qy - q - 1 = a$, say

$\therefore$ $p = \dfrac{a}{1-2x}$, $q = \dfrac{a+1}{2y-1}$ and so $dz = p\,dx + q\,dy = \dfrac{a\,dx}{1-2x} + \dfrac{(a+1)dy}{2y-1}$

Integrating, $z = -(a/2) \times \log|1-2x| + (1/2) \times (a+1) \log|2y+1| + b$.

Ex. 18. *Find the complete integral of* $2x(z^2q^2+1) = pz$

Sol. Re–writing the given equation, we have $2x\ \{(z\partial z/\partial y)^2 + 1\} = (z\partial z/\partial x)$ ... (1)

Putting $z\,dz = dZ$ so that $z^2/2 = Z$, (1) reduces to

$$2x\{(\partial Z/\partial y)^2 + 1\} = \partial Z/\partial x \quad \text{or} \quad 2x(Q^2+1) = P, \qquad \text{... (2)}$$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Re–writing (2), we have

$(P/2x) - 1 = Q^2 = a^2$, say so that $P = 2x(1+a^2)$, $Q = a$

$\therefore\ d\,Z = P\,dx + Q\,dy$ becomes $dZ = 2x(1+a^2)dx + a\,dy$

Integrating, $Z = (1+a^2)x^2 + ay + b$ or $z^2/2 = (1+a^2)x + ay + b$.

EXERCISE 3 (F)

Find a complete integral of the following equations (1 – 9)

1(a). $p^2 = q + x$. **Ans.** $z = (2/3) \times (a + x)^{3/2} + ay + b$.

(b). $p^2y\,(1 + x^2) = qx^2$. **[Delhi B.A (Prog) II 2011]** **Ans.** $z = a(1 + x^2)^{1/2} + (a^2y/2) + b$.

2. $p^2 + q^2 = x + y$. **[Agra 2009; Meerut 2007]** **Ans.** $3z = 2(x + a)^{3/2} + 2(y - a)^{3/2} + b$.

3. $p^2 + q^2 = x^2 + y^2$. **[Jiwaji 1999; Ravishankar 2003]**

Ans. $2z = x(x^2 + a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2 - a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$.

4. $pe^y = qe^x$. **[Jiwaji 1996]** **Ans.** $z = ae^x + ae^y + b$.

5. $p^{1/3} - q^{1/3} = 3x - 3y$. **Ans.** $z = 3x^3 - 3ax^2 + a^2x + 2y^4 - 4ay^3 + 3a^2y^2 - a^3y + b$.

6. $q = 2yp^2$. **Ans.** $z = ax + a^2y^2 + b$.

7. $p^2 - y^3q = x^2 - y^2$. **Ans.** $2z = x(x^2 + a^2)^{1/2} + a^2\sinh^{-1}(x/a) - (a^2/2) + \log y^2 + b$.

8. $z^2(p^2 + q^2) = x^2 + e^{2y}$. **[Delhi Maths (H) 2005]**

Ans. $z^2 = x(x^2 + a)^{1/2} + a\sinh^{-1}(x/\sqrt{a}) + 2(e^{2y} - a)^{1/2} - \sqrt{a}\ \tan^{-1}\{(e^{2y} - a)/a\}^{1/2} + b$

9. $p + q = px + qy$. **[Bangalore 1996]** **Ans.** $z = -a\log(1 - x) + a\log(y - 1) + b$.

Solve the following partial differential equations: (10 – 17)

10. $pq = xy$ **Ans. C.I.** $2z = ax^2 + y^2/a + b$; **S.S.** Does not exist **G.S.** $2z - ax^2 - y^2/a - \psi(a) = 0$,

$$x^2 - y^2/a^2 + \psi'(a) = 0$$

11. $\sqrt{p} + \sqrt{q} = 2x$. **Ans. C.I.** $z = (2x - a)^3/6 + a^2y + b$; **S.S.** Does not exit; **G.S.**

$$z - (2x-a)^3/6 - a^2y - \psi(a) = 0,\ \ (2x-a)^2/2 - 2ay - \psi'(a) = 0$$

12. $q\,(p - \cos x) = \cos y$. **Ans.** $z = ax + \sin x + (1/a) \times \sin y + b$; **S.S.** Does not exist

G.S. $z - ax - (1/a) \times \sin y - \psi(a) = 0,\ \ -x - (1/a^2) \times \sin y + \psi'(a) = 0$

13. $q = xyp^2$ **Ans. C.I.** $2z = 4\sqrt{ax} + ay^2 + b$; **S.S.** Does not exist; **G.S.** $2z - 4\sqrt{ax} - ay^2 - \psi(a) = 0$,

$$2\sqrt{(x/a)} + y^2 + \psi'(a) = 0$$

14. $x^2p^2 = q^2y$. **Ans. C.I.** $z = \sqrt{a}\log x + 2\sqrt{ay} + b$; **S.S.** Does not exist.

G.S. $z - \sqrt{a}\log x - 2\sqrt{ay} - \psi(a) = 0,\ \ \log x + 2\sqrt{y} + 2\sqrt{a}\,\psi'(a) = 0$

15. $p - q = x^2 + y^2$. **Ans. C.I.** $z = (x^3 - y^3)/3 + a(x + y) + b$; **S.S.** Does not exit;

G.S. $z - (x^3 - y^3)/3 - a(x+y) - \psi(a) = 0,\ \ x + y + \psi'(a) = 0$

16. $p^2 - x = q^2 - y$ **Ans. C.I.** $3z = 2(x + a)^{3/2} + 2(y + a)^{3/2} + b$; **S.S.** Does not exist

G.S. $3z-2(x+a)^{3/2}-2(y+a)^{3/2}-\psi(a)=0, 3(x+a)^{1/2}+3(y+a)^{1/2}+\psi'(a)=0$

17. $px+q=p^2$. **Ans. C.I.** $z=(1/4)\times\{x^2+x(x^2+4a)^{1/2}\}+a\log\{x+(x^2+4a^2)^{1/2}\}\ ay$ $+b$; **S.S.** Does not exist **G.S.** $z-(1/4)\times\{x^2+x(x^2+4a)^{1/2}\}-a\log\{x+(x^2+4a)^{1/2}\}-ay$ $-\psi(a)=0$, $(x/2)\times(x^2+4a)^{-1/2}+\log\{x+(x^2+4a)^{1/2}\}+(2a)/[\{x+(x^2+4a)^{1/2}\}\times(x^2+4a)]$ $+y+\psi'(a)=0$

3.19. JACOBI'S METHOD [Himachel 2005; Meerut 2005, 06, 08; Pune 2010]

This method is used for solving partial differential equations involving three or more independent variables. The central idea of Jacobi's method is almost the same as that of Charpit's method for two independent variables. We begin with the case of three independent variables. The results arrived at are, however, general and will be used with suitable modification for the case of four independent variables and so on.

Let $p_1=\partial z/\partial x_1$, $p_2=\partial z/\partial x_2$ and $p_3=\partial z/\partial x_3$.

Consider a partial differential equation $f(x_1, x_2, x_3, p_1, p_2, p_3)=0$, ...(1)

where the dependent variable z does not occur except by its partial differential coefficients with respect to the three independent variables x_1, x_2, x_3.

The main idea in Jacobi's method is to get two additional partial differential equations of the first order $F_1(x_1, x_2, x_3, p_1, p_2, p_3)=a_1$ ...(2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3)=a_2$, ...(3)

where a_1 and a_2 are two arbitrary constants such that (1), (2) and (3) can be solved for p_1, p_2, p_3 in terms of x_1, x_2, x_3 which when substituted in

$$dz=p_1dx_1+p_2dx_2+p_3dx_3, \qquad \text{...(4)}$$

makes it integrable, for which the conditions are

$$\partial p_2/\partial x_1=\partial p_1/\partial x_2, \qquad \partial p_3/\partial x_2=\partial p_2/\partial x_3, \qquad \text{and} \qquad \partial p_1/\partial x_3=\partial p_3/\partial x_1 \qquad \text{...(5)}$$

Differentiating (1) and (2) partially, w.r.t. x_1, we have

$$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0 \qquad \text{...(6)}$$

and

$$\frac{\partial F_1}{\partial x_1}+\frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0. \qquad \text{...(7)}$$

Eliminating $\partial p_1/\partial x_1$ from (6) and (7), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_1}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_1}=0. \quad \text{...(8)}$$

Similarly, differentiating (1) and (2) partially w.r.t. x_2 and then eliminating $\partial p_2/\partial x_2$ from the resulting equations, we have

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_2}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_2}=0. \quad \text{...(9)}$$

Again, differentiating (1) and (2) partially w.r.t. x_3 and then eliminating $\partial p_3/\partial x_3$ from the resulting equation, we have

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_3}+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_3}=0. \quad \text{...(10)}$$

Adding (8), (9) and (10) and using the relations (5), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)=0. \quad ...(11)$$

The L.H.S. of (11) is generally denoted by (f, F_1). Then, (11) becomes

$$(f, F_1)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right)=0. \quad ...(11)'$$

Starting with (1) and (3) in place of (1) and (2) and proceeding as above, we have a similar relation

$$(f, F_2)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(12)$$

Again, starting with (2) and (3) in place of (1) and (2) and proceeding as above, we again have a similar relation

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(13)$$

(11) [or (11)′] and (12) are linear equtions of first order with x_1, x_2, x_3, p_1, p_2, p_3 as independent variables and F_1, F_2 as dependent variables respectively. For both of these equations, Lagrange's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}, \quad ...(14)$$

which are known as *Jacobi's auxiliary equations*.

We try to find two independent integrals $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$ with help of (14). If these relations satisfy (13), these are the required two additional relations (2) and (3).

We now solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Substituting these values in (4) and then integrating the resulting equation, we shall obtain a complete integral of the given equation containing three arbitrary constants of integration.

3.20. Working rules for solving partial differential equations with three or more independent variable. Jacobi's method

Step 1 : Suppose the given equation with three independent variables is

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0. \quad ...(1)$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}.$$

Solving these equation we obtain two additional equations

$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ ...(2) $\qquad$ $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$. ...(3)

where a_1 and a_2 are arbitrary constants.

While obtaining (2) and (3), try to select simple equations so that later on solutions of (1), (2) and (3) may be as easy as possible.

Step III. Verify that relations (2) and (3) satisfy the condition

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(4)$$

If (4) is satisfied then solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Their substitution in

$$dz = p_1dx_1 + p_2dx_2 + p_3dx_3$$

and subsequent integration leads to a complete integral of the given equation.

Remark 1. Sometime, change of variables can be employed to reduce the given equation in a form solvable by Jacobian method.

Remark 2. While solving a partial differential equation with four independent variables, we modify the above working rule as follows :

Step I. Suppose the given equation with four independent variables is

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0. \qquad ...(1)$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_4}{\partial f/\partial x_4} = \frac{dx_4}{-\partial f/\partial p_4}$$

Solving these equations we obtain three additional equations

$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1$, ...(2) $\qquad F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2$, ...(3)

and $\qquad F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3$, ...(4)

where a_1, a_2 and a_3 are arbitrary constants.

Step IV. Verify that relations (2), (3) and (4) satisfy following three conditions:

$$(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0, \ ...(4) \qquad (F_2, F_3) = \sum_{r=1}^{4}\left(\frac{\partial F_2}{\partial x_r}\frac{\partial F_3}{\partial p_r} - \frac{\partial F_2}{\partial p_r}\frac{\partial F_3}{\partial x_r}\right) = 0 \ ...(5)$$

and $$(F_3, F_1) = \sum_{r=1}^{4}\left(\frac{\partial F_3}{\partial x_r}\frac{\partial F_1}{\partial p_r} - \frac{\partial F_3}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right) = 0. \qquad ...(6)$$

If (4), (5) and (6) are satisfied, then solve (1), (2), (3) and (4) for p_1, p_2, p_3 and p_4 in terms of x_1, x_2, x_3 and x_4. Their substitution in $\quad dz = p_1dx_1 + p_2dx_2 + p_3dx_3 + p_4dx_4$
and subsequent integration leads to a complete integral of the given equation.

3.21 SOLVED EXAMPLES BASED ON ART 3.20.

Ex. 1. *Find a complete integral of* $p_1^3 + p_2^2 + p_3 = 1$. **[I.A.S. 1997; Meerut 2006]**

Sol. Let the given equation be rewritten as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0. \qquad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1}, \text{ using (1)}$$

From first and third fractions, $dp_1 = 0$ and $dp_2 = 0$ so that $p_1 = a_1$ and $p_2 = a_2$.

$\therefore$ Here $\qquad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1$. ...(2)

and $\qquad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$. ...(3)

Now, $$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

or $$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

or $\quad (F_1, F_2) = (0)(0) - (1)(0) + (0)(1) - (0)(0) + (0)(0) - (0)(0) = 0$, by (3) and (4).

Thus, we have verified that for relations (2) and (3), $(F_1, F_2) = 0$. Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, $\quad p_1 = a_1, \quad p_2 = a_2, \quad p_3 = 1 - a_1^3 - a_2^2$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we have

$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3.$$

Integrating, $\quad z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2)\, x_3 + a_3$,

which is a complete integral of given equation containing three arbitrary constants a_1, a_2, and a_3.

Ex. 2. *Find a complete integral of* $x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0$. **[Delhi Maths (H) 2006]**

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{0} = \frac{dx_1}{-(2p_1 x_3^2 p_2^2 p_3^2 + 2p_1 p_2^2)} = \frac{dp_2}{0} = \frac{dx_2}{-(2p_2 x_3^2 p_1^2 p_3^2 + 2p_2 p_1^2)} = \ldots\ldots, \text{ by (1)}$$

From first and third fractions, $\quad dp_1 = 0$ and $dp_2 = 0 \quad$ so that $\quad p_1 = a_1$ and $\quad p_2 = a_2$.

$\therefore$ Here $\quad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1, \quad$ ...(2)

and $\quad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2. \quad$ ...(3)

As in Ex. 1, verify that for relations (2) and (3), $\quad (F_1, F_2) = 0$.

Hence (2) and (3) may be taken as the additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, we have $p_1 = a_1,\ p_2 = a_2,\ \ p_3 = \pm a_1 a_2 \big/ \sqrt{(1 - a_1^2 a_2^2 x_3^2)}$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$dz = a_1 dx_1 + a_2 dx_2 \pm \left\{ a_1 a_2 \big/ \sqrt{(1 - a_1^2 a_2^2 x_3^2)} \right\} dx_3$, whose integration gives

$z = a_1 x_1 + a_2 x_2 \pm\ \sin^{-1}(a_1 a_2 x_3) + a_3$, a_1, a_2, a_3 being arbitrary constants.

Ex. 3. *Find a complete integral of* $p_1 x_1 + p_2 x_2 = p_3^2$. **[Meerut 2007]**

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 x_1 + p_2 x_2 - p_3^2 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{p_1} = \frac{dx_1}{-x_1} = \frac{dp_2}{p_2} = \frac{dx_2}{-x_2} = \frac{dp_3}{0} = \frac{dx_3}{2p_3}, \text{ using (1)} \qquad \text{...(2)}$$

Taking the first two fractions of (2), $(1/x_1)dx + (1/p_1)dp_1 = 0. \Rightarrow \log x_1 + \log p_1 = \log a_1$.

$\therefore x_1 p_1 = a_1 \quad$ and $\quad$ let $\quad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 p_1 = a_1$. ...(3)

Taking the third and fourth fractions of (2), $\quad (1/x_2)dx_2 + (1/p_2)dp_2 = 0$.

$\therefore x_2 p_2 = a_2 \quad$ and $\quad$ let $\quad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = x_2 p_2 = a_2$. ...(4)

As in Ex. 1, verify that for relations (3) and (4), $(F_1, F_2) = 0$.

Solving (1), (3) and (4) for p_1, p_2, p_3, $\quad p_1 = a_1/x_1, \quad p_2 = a_2/x_2 \quad$ and $\quad p_3 = (a_1 + a_2)^{1/2}$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we have

$$dz = (a_1/x_1)dx_1 + (a_2/x_2)dx_2 + (a_1 + a_2)^{1/2}\, dx_3.$$

Integrating, $\quad z = a_1 \log x_1 + a_2 \log x_2 + x_3 (a_1 + a_2)^{1/2} + a_3$.

Ex. 4. *Find complete integral of* $2p_1 x_1 x_3 + 3p_2 x_3^2 + p_2^2 p_3 = 0$.

[I.A.S. 1998, Meerut 1999]

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1 x_1 x_3 + 3p_2 x_3^2 + p_2^2 p_3 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{2p_1x_3}=\frac{dx_1}{-2x_1x_3}=\frac{dp_2}{0}=\frac{dx_2}{-3x_3^2-2p_2p_3}=\frac{dp_3}{2p_1x_1+6p_2x_3}=\frac{dx_3}{-p_2^2}\text{, by (1)} \qquad ...(2)$$

Taking the first two fractions of (2), $(1/p_1)dp_1 + (1/x_1)dx_1 = 0.$ so $p_1x_1 = a_1$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1x_1 = a_1.$ ...(3)

From the third fraction of (2), $dp_2 = 0$ so that $p_2 = a_2.$

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2.$ ...(4)

As in Ex. 1, verify that for relations (3) and (4), $(F_1, F_2) = 0.$

Solving (1), (3) and (4) for p_1, p_2, p_3, $p_1 = a_1/x_1$, $p_2 = a_2$, $p_3 = -(2a_1x_3 + 3a_2x_3^2)/a_2^2.$

Putting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_2$, we have

$dz = (a_1/x_1)dx_1 + a_2dx_2 - \{(2a_1x_3 + 3a_2x_3^2)/a_2^2\}dx_3$, whose integration gives

$z = a_1 \log x_1 + a_2x_2 - (a_1x_3^2 + a_2x_3^3)/a_2^2 + a_3.$ which is required complete integral

Ex. 5. *Find a complete integral of* $p_3x_3 (p_1 + p_2) + x_1 + x_2 = 0.$

Sol. Given $f(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3(p_1 + p_2) + x_1 + x_2 = 0.$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{1}=\frac{dx_1}{p_3x_3}=\frac{dp_2}{1}=\frac{dx_2}{-p_3x_3}=\frac{dp_3}{p_3(p_2+p_3)}=\frac{dx_3}{-x_3(p_1+p_2)}\text{, by (1)} \qquad ...(2)$$

Taking the two fractions of (2), $dp_1 - dp_2 = 0$ so $p_1 - p_2 = a_1$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 - p_2 = a_1.$ ...(3)

Taking the fifth and sixth fractions of (2), $(1/p_3)dp_3 + (1/x_3)dx_3 = 0$ giving $p_3x_3 = a_3$

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3 = a_2.$ ...(4)

Now, $$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

$$=\left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1}-\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right)+\left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2}-\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)+\left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3}-\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)$$

$= (0)(0) - (1)(0) + (0)(0) - (-1)(0) + (0)(x_3) - (0)(p_3) = 0$ by (3) and (4)

Thus, we have verified that for the relations (3) and (4), $(F_1, F_2) = 0.$

From (1) and (4), $a_2(p_1 + p_2) + x_1 + x_2 = 0$ or $p_1 + p_2 = -(x_1 + x_2)/a_2.$ ...(5)

Solving (3) and (5), $p_1 = \dfrac{a_1}{2}-\dfrac{x_1+x_2}{2a_2}$ and $p_2 = -\dfrac{a_1}{2}-\dfrac{x_1+x_2}{2a_2}.$ ...(6)

Again, from (4), $p_3 = a_2/x_3.$ ...(7)

Putting the values of p_1, p_2, p_3 given by (6) and (7) in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we have

$$dz = \frac{a_1}{2}(dx_1 - dx_2) - \frac{(x_1+x_2)}{2a_2}(dx_1 + dx_2) + \frac{a_2}{x_3}dx_3.$$

Integrating, $z = (a_1/2) \times (x_1 - x_2) - (1/4a_2) \times (x_1 + x_2)^2 + a_2 \log x_3 + a_3.$

Ex. 6. *Find a complete integral of* $(p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 = 3(x_1 + x_2 + x_3)$.

Sol. Let the given partial differential equation be re–written as

$$f(x_1, x_3, x_3, p_1, p_2, p_3) = (p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 - 3(x_1 + x_2 + x_3) = 0. \quad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}, \text{ giving}$$

$$\frac{dp_1}{2(p_1+x_1)-3} = \frac{dx_1}{-2(p_1+x_1)} = \frac{dp_2}{2(p_2+x_2)-3} = \frac{dx_2}{-2(p_2+x_2)} = \frac{dp_3}{2(p_3+x_3)-3} = \frac{dx_3}{-2(p_3+x_3)}. \quad ...(2)$$

$$\text{Each fraction of (2)} = \frac{dp_1+dx_1}{-3} = \frac{dp_2+dx_2}{-3} = \frac{dp_3+dx_3}{-3} \quad ...(3)$$

Then (3) $\Rightarrow$ $dp_1 + dx_1 = dp_2 + dx_2$ and $dp_3 + dx_3 = dp_2 + dx_2$

Integrating, $p_1 + x_1 = p_2 + x_2 + a_1$ and $p_3 + x_3 = p_2 + x_2 + a_2$,

where a_1 and a_2 are arbitrary constants

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 + p_1 - x_2 - p_2 = a_1$. ...(4)

and $F_2(x_1, x_2, x_2, p_1, p_2, p_3) = x_3 + p_3 - x_2 - p_2 = a_2$. ...(5)

As in Ex. 1, verify that for relations (4) and (5), the condition $(F_1, F_2) = 0$ is satisfied. Hence (4) and (5) may be taken as two additional equations.

With help of (4) and (5), (1) reduces to

$$(x_2 + p_2 + a_1)^2 + (x_2 + p_2)^2 + (x_2 + p_2 + a_2)^2 = 3(x_1 + x_2 + x_3)$$

or $$3(p_2 + x_2)^2 + 2(p_2 + x_2)(a_1 + a_2) + a_1^2 + a_2^2 - 3(x_1 + x_2 + x_3) = 0.$$

$$\therefore \quad p_2 + x_2 = \left[-2(a_1+a_2) \pm \sqrt{[4(a_1+a_2)^2 - 12\{a_1^2 + a_2^2 - 3(x_1+x_2+x_3)\}}\right] \Big/ 6$$

$$\Rightarrow \quad p_2 = -x_2 + \left[-(a_1+a_2) \pm \sqrt{\{9(x_1+x_2+x_3) - 2a_1^2 - 2a_2^2 + 2a_1a_2\}}\right] \Big/ 3$$

For sake of simplification, we take $a_1 = 3c_1$ and $a_2 = 3c_2$. Then, we get

$$p_2 = -x_2 - (c_1 + c_2) \pm \sqrt{\{(x_1+x_2+x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}. \quad ...(6)$$

$\therefore$ From (4), $p_1 = x_2 + p_2 + 3c_1 - x_1$

$$\Rightarrow \quad p_1 = -x_1 + 2c_1 - c_2 \pm \sqrt{\{(x_1+x_2+x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}, \text{by (6)}$$

Again, from (5), $p_3 = x_2 + p_2 + 3c_2 - x_3$

$$\Rightarrow \quad p_3 = -x_3 + 2c_2 - c_1 \pm \sqrt{\{(x_1+x_2+x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}, \text{ by (6)}$$

Substituting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we get

$$dz = -(x_1dx + x_2dx_2 + x_3dx_3) + [(2c_1 - c_2)dx_1 - (c_1 + c_2)dx_2 + (2c_2 - c_1)dx_3]$$
$$\pm (x_1 + x_2 + x_3 - 2c_1^2 - 2c_2^2 + 2c_1c_2)^{1/2} \ (dx_1 + dx_2 + dx_3).$$

Integrating, $z = -(1/2) \times (x_1^2 + x_2^2 + x_3^2) + (2c_1 - c_2)x_1 - (c_1 + c_2)x_2 + (2c_2 - c_1)x_3$

$$\pm (2/3) \times (x_1 + x_2 + x_3 - 2c_1^2 - 2c_2^2 + 2c_1c_2)^{3/2} + c_3,$$

which is a complete integral containing c_1, c_2, c_3 as arbitrary constants.

Ex. 7. *Find a complete integral of* $(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0$. **[Delhi B.Sc. (Hons) III 2011]**

Sol. Given $(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0$. ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Re–writting (1), we have

$$(x_2 + x_3)\left(\frac{1}{z}\frac{\partial z}{\partial x_2} + \frac{1}{z}\frac{\partial z}{\partial x_3}\right)^2 + \frac{1}{z}\frac{\partial z}{\partial x_1} = 0. \qquad ...(2)$$

Let $(1/z)dz = dZ$ so that $Z = \log z$. ...(3)

Then, (2) $\Rightarrow$ $(x_2 + x_3)\,(\partial Z/\partial x_2 + \partial Z/\partial x_3)^2 + \partial Z/\partial x_1 = 0.$ ...(4)

Let $P_1 = \partial Z/\partial x_1$, $P_2 = \partial Z/\partial x_2$, $P_3 = \partial Z/\partial x_3$. Then (4) becomes

$$(x_2 + x_3)(P_2 + P_3)^2 + P_1 = 0.$$

So here $f(x_1, x_2, x_3, P_1, P_2, P_3) \equiv (x_2 + x_3)(P_2 + P_3)^2 + P_1 = 0.$ ...(5)

Jacobi's auxiliary equations take the form

$$\frac{dP_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial P_3}$$

or
$$\frac{dx_1}{-1} = \frac{dP_1}{0} = \frac{dx_2}{-2(x_2+x_3)(P_2+P_3)} = \frac{dP_2}{(P_2+P_3)^2} = \frac{dx_3}{-2(x_2+x_3)(P_2+P_3)} = \frac{dP_3}{(P_2+P_3)^2}. \qquad ...(6)$$

Taking second ratio of (6), we have $dP_1 = 0$ $\Rightarrow$ $P_1 = -a_1$.

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 = -a_1$. ...(7)

Taking the fourth and sixth ratios in (6), we get $dP_2 = dP_3$ $\Rightarrow$ $P_2 - P_3 = a_2$.

Let $F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 - P_3 = a_2$. ...(8)

Using (7), (5) $\Rightarrow P_2 + P_3 = \pm\,\{a_1/(x_2 + x_3)\}^{1/2}$. ...(9)

Solving (8) and (9) for P_2 and P_3, we have

$$P_2 = \frac{1}{2}\left[a_2 \pm \left(\frac{a_1}{x_2+x_3}\right)^{1/2}\right] \quad \text{and} \quad P_3 = \frac{1}{2}\left[\pm\left(\frac{a_1}{x_2+x_3}\right)^{1/2} - a_2\right]. \qquad ...(10)$$

Using (7) and (10), $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$ becomes

$$dZ = -a_1dx_1 + \frac{1}{2}\left[a_2 \pm \frac{\sqrt{a_1}}{(x_2+x_3)^{1/2}}\right]dx_2 + \frac{1}{2}\left[\pm\frac{\sqrt{a_1}}{(x_2+x_3)^{1/2}} - a_2\right]dx_3$$

or
$$dZ = -\,a_1dx_1 + (1/2)\times a_2dx_2 - (1/2)\times a_2dx_3 \pm (1/2)\times\sqrt{a_1}\,(x_2 + x_3)^{-1/2}(dx_2 + dx_3).$$

Integrating and noting that $dZ = (1/z)dz$, complete integral is given by

$$\log z = -\,a_1x_1 + (a_2/2)\times(x_2 - x_3) \pm \sqrt{a_1}\,(x_2 + x_3)^{1/2} + a_3.$$

Ex. 8. *Find a complete integral of* $p_1p_2p_3 = z^3x_1x_2x_3$. **[Meerut 1998]**

i.e., $(\partial z/\partial x_1)(\partial z/\partial x_2)\,(\partial z/\partial x_3) = z^3x_1x_2x_3$. **[Delhi Maths (H) 2000, 10; I.A.S. 1995]**

Sol. Given $p_1p_2p_3 = z^3x_1x_2x_3$ or $(\partial z/\partial x_1)(\partial z/\partial x_2)\,(\partial z/\partial x_3) = z^3x_1x_2x_3$. ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Re–writting (1) we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = x_1x_2x_3. \qquad ...(2)$$

Let $(1/z)dz = dZ$ so that $\log z = Z$. Then (2) becomes

$(\partial Z/\partial x_1)(\partial Z/\partial x_2)\,(\partial Z/\partial x_3) = x_1x_2x_3$ or $P_1P_2P_3 = x_1x_2x_3$.

$\therefore$ Here $f(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_1P_2P_3 - x_1x_2x_3 = 0.$ ...(3)

$\therefore$ Jacobi's auxilliary equations are

$$\frac{dP_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dP_1}{-x_2x_3}=\frac{dx_1}{-P_2P_3}=\frac{dP_2}{-x_1x_3}=\frac{dx_2}{-P_1P_3}=\frac{dP_3}{-x_1x_2}=\frac{dx_3}{-P_1P_2}, \text{ by (3)}$$

Since from (3), $P_2P_3 = (x_1\, x_2\, x_3)/P_1$, hence first and second fractions give

$$\frac{dP_1}{-x_2x_3}=\frac{dx_1}{-(x_1x_2x_3/P_1)} \quad \text{or} \quad \frac{dP_1}{P_1}=\frac{dx_1}{x_1}.$$

Integrating, $\log P_1 = \log x_1 + \log a_1$ or $P_1 = a_1x_1$.

Thus, here we have $F_1(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_1 - a_1x_1 = 0.$ ...(4)

Similarly, $F_2(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_2 - a_2x_2 = 0.$ ...(5)

As in Ex. 1, verify that for (4) and (5) the condition $(F_1, F_2) = 0$ is satisfied. Hence (4) and (5) can be taken as two additional equations. Solving (3), (4) and (5) for P_1, P_2, P_3, we have

$P_1 = a_1x_1$, $P_2 = a_2x_2$ and $P_3 = x_3/(a_1a_2)$.

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$, we have

$$dZ = a_1x_1dx + a_2x_2dx_2 + \{x_3/(a_1a_2)\}dx_3.$$

Integrating, $Z = (1/2) \times a_1x_1^2 + (1/2) \times a_2x_2^2 + \{1/(2a_1a_2)\}x_3^2 + a_3/2$

or $$2 \log z = a_1x_1^2 + a_2x_2^2 + \{1/(a_1a_2)\}x_3^2 + a_3, \text{ as } Z = \log z$$

Ex. 9. *Find a complete integral of* $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0.$ **[Delhi Maths (H) 2009]**

Sol. Given equation is $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Dividing each term by z^2, (1) can be re–written as

$$\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)^2+\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right)-\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)-\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right)=0. \qquad ...(2)$$

Let $(1/z)dz = dZ$ so that $\log z = Z.$ ...(3)

Using (3), (2) becomes $P_1^2 + P_2P_3 - P_2 - P_3 = 0,$ ...(4)

Let us write $f(x_1, x_2, x_3, P_1, P_2, P_3) = P_1^2 + P_2P_3 - P_2 - P_3 = 0.$ ...(5)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dP_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial P_1}=\frac{dP_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial P_2}=\frac{dP_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dP_1}{0}=\frac{dx_1}{-2P_1}=\frac{dP_2}{0}=\frac{dx_2}{-P_3+1}=\frac{dP_3}{0}=\frac{dx_3}{-P_2+1}, \text{ by (5)}$$

Taking the third and fifth fractions, $dP_2 = 0$ and $dP_3 = 0$ so that $P_2 = a_1$ and $P_3 = a_2$.

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 = a_1.$ ...(6)

and $F_3(x_1, x_2, x_3, P_1, P_2, P_3) = P_3 = a_2.$ ...(7)

As in Ex. 1, verify that for (6) and (7), the condition $(F_1, F_2) = 0$ is satisfied. Hence (6) and (7) can be taken as two additional equations. Solving (4), (6) and (7) for P_1, P_2, P_3, we have

$P_2 = a_1$, $P_3 = a_2$, $P_1 = (a_1 + a_2 - a_1a_2)^{1/2}$.

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$, we have

$$dZ = (a_1 + a_2 - a_1a_2)^{1/2}dx_1 + a_1dx_2 + a_2dx_3.$$

Integrating, $Z = (a_1 + a_2 - a_1a_2)^{1/2}x_1 + a_1x_2 + a_2x_3 + a_3$. Then, the complete integral is

$$\log z = (a_1 + a_2 - a_1a_2)^{1/2}x_1 + a_1x_2 + a_2x_3 + a_3, \text{ using (3).}$$

Ex. 10. *Find a complete integral of* $2x_1x_3zp_1p_3 + x_2p_2 = 0.$

Sol. Given equation is $2x_1x_3zp_1p_3 + x_2p_2 = 0.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Multiplying each term by z, (1) can be re–written as

$$2x_1x_3\left(z\frac{\partial z}{\partial x_1}\right)\left(z\frac{\partial z}{\partial x_3}\right)+x_2\left(z\frac{\partial z}{\partial x_2}\right)=0. \qquad ...(2)$$

Let $zdz = dZ$ so that $z^2/2 = Z$. ...(3)

Using (3), (2) becomes $2x_1x_3P_1P_3 + x_2P_2 = 0$, ...(4)

where $P_1 = \partial Z/\partial x_1$, $P_2 = \partial Z/\partial x_2$ and $P_3 = \partial Z/\partial x_3$. We re–write (4) as

$$f(x_1, x_2, x_3, P_1, P_2, P_3) = 2x_1x_3P_1P_3 + x_2P_2 = 0. \qquad ...(5)$$

$\therefore$ Jacobi's auxilairy equations are

$$\frac{dP_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial P_1}=\frac{dP_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial P_2}=\frac{dP_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dP_1}{2x_3P_1P_3}=\frac{dx_1}{-2x_1x_3P_3}=\frac{dP_2}{P_2}=\frac{dx_2}{-x_2}=\frac{dP_3}{2x_1P_1P_3}=\frac{dx_3}{-2x_1x_3P_2}, \text{ by (5)} \qquad ...(6)$$

Taking the first and second fractions of (6) and simplifying, we get

$(1/P_1)dP_1 + (1/x_1)dx_1 = 0$ so that $\log P_1 + \log x_1 = \log a_1$ or $P_1x_1 = a_1$

So here $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1x_1 = a_1$. ...(7)

Taking the fifth and sixth fractions of (6) and simplifying, we get

$(1/P_3)dP_3 + (1/x_3)dx_3 = 0$ so that $\log P_3 + \log x_3 = \log a_3$ or $P_3x_3 = a_2$

So here $F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_3x_3 = a_2$. ...(8)

As in Ex. 1, verify that for (7) and (8), the condition $(F_1, F_2) = 0$ is satisfied. Hence (7) and (8) can be taken as additional equations. Solving (5), (7) and (8) for P_1, P_2, P_3, we have

$$P_1 = a_1/x, \qquad P_3 = a_2/x_3, \qquad P_2 = -(2a, a_2)/x_2.$$

Putting these values in $dZ = P_1dx + P_2dx_2 + P_3dx_3$, we have

$$dZ = (a_1/x_1)dx_1 - \{(2a_1a_2)/x_2\}dx_2 + (a_2/x_3)dx_3.$$

Integrating, $Z = a_1\log x_1 - 2a_1a_2 \log x_2 + a_2 \log x_3 + a_3$

or $z^2/2 = a_1 \log x_1 - 2a_1a_2 \log x_2 + a_2 \log x_3 + a_3$, by (3).

Ex. 11. *Find a complete integral of* $p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0$.

Sol. [In the present problem we have four independent variables in places of three. According we shall use modified working as explained in remark 2 of Art 3.20]

The given equation can be written as

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0. \qquad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}=\frac{dp_4}{\partial f/\partial x_4}=\frac{dx_4}{-\partial f/\partial p_4}, \text{ giving}$$

$$\frac{dp_1}{p_4^3x_2x_3x_4^3}=\frac{dx_1}{-p_2p_3}=\frac{dp_2}{p_4^3x_1x_3x_4^3}=\frac{dx_2}{-p_1p_3}=\frac{dp_3}{p_4^3x_1x_2x_4^3}=\frac{dx_3}{-p_1p_2}=\frac{dp_4}{3p_4^3x_1x_2x_3x_4^2}=\frac{dx_4}{-3p_4^3x_1x_2x_3x_4^3}$$

Since from (1), $p_4^3x_2x_3x_4^3 = -p_1p_2p_3/x_1$, the first two fractions give

$$\frac{dp_1}{-(p_1p_2p_3/x_1)}=\frac{dx_1}{-p_2p_3} \qquad \text{or} \qquad \frac{dp_1}{p_1}=\frac{dx_1}{x_1}.$$

Integrating, $\log p_1 = \log x_1 + \log a_1$ or $p_1 = a_1 x_1$.

Let $F_1(x_1, x_2, x_3, x_4\, p_1, p_2, p_3, p_4) = p_1 - a_1 x_1 = 0.$ ...(2)

Similarly, $F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_2 - a_2 x_2 = 0$ ...(3)

and $F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_3 - a_3 x_3 = 0.$ ...(4)

With these values of F_1, F_2 and F_3, we can verify that

$$(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0.$$

Similarly, we see that $(F_2, F_3) = 0$ and $(F_3, F_1) = 0$. Hence (2), (3) and (4) can be taken as the three desired additional equations. Now solving (1), (2) (3) and (4) for p_1, p_2, p_3 and p_4, we get

$$p_1 = a_1 x_1, \qquad p_2 = a_2 x_2, \qquad p_3 = a_3 x_3 \qquad \text{and} \qquad p_4 = (a_1 a_2 a_3)^{1/3}/x_4.$$

Putting these in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$ and integrating the desired complete integral is $z = (1/2) \times (a_1 x_1{}^2 + a_2 x_2{}^2 + a_3 x_3{}^2) - (a_1 a_2 a_3)^{1/2} \log x_4 + a_4/2$

or $2z = a_1 x_1{}^2 + a_2 x_2{}^2 + a_3 x_3{}^2 - 2(a_1 a_2 a_3)^{1/2} \log x_4 + a_4,$

Ex. 12. *Find a complete integral by Jacobi's method of the equation* $2x^2 y(\partial u/\partial x)^2 (\partial u/\partial z) = x^2(\partial u/\partial y) + 2y(\partial u/\partial x)^2$. **[Delhi Maths (H) 2001]**

Sol. Let $x = x_1$, $y = x_2$, $z = x_3$, $\partial u/\partial x = p_1$, $\partial u/\partial y = p_2$, and $\partial u/\partial z = p_3$

Then given equation becomes $2x_1^2 x_2 p_1^2 p_3 = x_1^2 p_2 + 2x_2 p_1^2$

Dividing by $x_1^2\ x_2$, $2p_1^2 p_3 = (p_2/x_2) + (2p_1^2/x_1^2)$, which can be written as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1^2(p_3 - 1/x_1^2) - p_2/x_2 = 0 \qquad \text{... (1)}$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{4p_1^2/x_1^3} = \frac{dx_1}{-4p_1(p_3 - 1/x_1^2)} = \frac{dp_2}{p_2/x_2^2} = \frac{dx_2}{1/x_2} = \frac{dp_3}{0} = \frac{dx_3}{-2p_1^2}, \text{ by (1)}$$

Taking the fifth fraction, $dp_3 = 0$ so that $p_3 = a_1$

Taking the second and fourth fractions, $(1/p_2)\, dp_2 = (1/x_2)\, dx_2$

Integrating, $\log p_2 = \log x_2 + \log(2a_2^2)$ or $p_2/x_2 = 2a_2^2$

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_3 = a_1$ ... (2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2/x_2 = 2a_2^2$ ... (3)

Now, $$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

or
$$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

$$= (0)\,(0) - (0)\,(0) + (0)\,(0) - (0)\,(0) + (0)\,(0) - (1)\,(0) = 0$$

Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, $\quad p_1 = a_2 x_1/(a_1 x_1^2 - 1)^{1/2}$, $\quad p_2 = 2a_2^2 x_2$, $\quad p_3 = a_1$

Putting these in $du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 = a_2 x_1 (a_1 x_1^2 - 1)^{-1/2} dx_1 + 2a_2^2 x_2 dx_2 + a_1 dx_3$.

Integrating, $$u = (a_2/a_1) \times (a_1 x_1^2 - 1)^{1/2} + a_2^2 x_2^2 + a_1 x_3 + a_3,$$

which is the complete integral with a_1, a_2, a_3 as arbitrary constants.

Ex. 13. *Show that a complete integral of the equation* $f(\partial u/\partial x, \partial u/\partial y, \partial u/\partial z) = 0$ *is* $u = ax + by + \theta(a, b)z + c$, *where a, b and c are arbitrary constants and* $f(a, b, \theta) = 0$

(*b*)*Find a complete integral of the equation* $\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = (\partial u/\partial x)\,(\partial u/\partial y)\,(\partial u/\partial z)$. **[Allahabad 2004, 06; Meerut 2004, 06; Purvanchal 2003]**

Sol. (*a*) Let $\quad \partial u/\partial x = p_1, \quad \partial u/\partial y = p_2 \quad$ and $\quad \partial u/\partial z = p_3$.

Then gievn equation becomes $$f(p_1, p_2, p_3) = 0 \qquad \text{... (1)}$$

We shall now proceed as in Ex. 1, Art. 3.21. Here Jacobi's auxiliary equations are given by

$$\frac{dp_1}{\partial f/\partial x} = \frac{dx}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial y} = \frac{dy}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial z} = \frac{dz}{-\partial f/\partial p_3}$$

$$\Rightarrow \quad \frac{dp_1}{0} = \frac{dp_2}{0}, \text{ using (1)} \quad \Rightarrow \quad dp_1 = 0 \quad \text{and} \quad dp_2 = 0$$

Integrating, $\quad p_1 = a, \quad p_2 = b, \quad$ *a* and *b* being arbitrary constants ... (2)

Putting $p_1 = a$ and $p_2 = b$ in (1), $f(a, b, p_3) = 0$ so that

$$p_3 = \text{a function of } a, b = \theta(a, b), \text{ say} \qquad \text{... (3)}$$

Now, we have $$du = (\partial u/\partial x)\, dx + (\partial u/\partial y) dy + (\partial u/\partial z) dz = p_1 dx + p_2 dy + p_3 dz$$

or $$du = a\, dx + b\, dy + \theta(a, b)\, dz, \text{ by (2) and (3)}$$

Integrating, $$u = ax + by + \theta\,(a,b)\, z + c, \qquad \text{... (4)}$$

where c is an arbitrary constant and a, b, θ are connected by relation

$$f(a, b, \theta\,(a, b)) = 0, \text{ by (1), (2) and (3)} \qquad \text{... (5)}$$

(***b***) Given $$\partial u/\partial x + \partial u/\partial y + \partial u/\partial z - (\partial u/\partial x)\,(\partial u/\partial y)\,(\partial u/\partial z) = 0 \qquad \text{... }(i)$$

Let $\quad p_1 = \partial u/\partial x, \quad p_2 = \partial u/\partial y \quad$ and $\quad p_3 = \partial u/\partial z$. Then, (*i*) gives

$$p_1 + p_2 + p_3 - p_1 p_2 p_3 = 0 \qquad \text{... }(ii)$$

Comparing (*ii*) with (1) of part (*a*), here

$$f(p_1, p_2, p_3) = p_1 + p_2 + p_3 - p_1 p_2 p_3 \qquad \text{... }(iii)$$

Hence required complete integral is given by (4) and (5) of part (*a*) *i.e.*,

$$u = a\,x + b\,y + \theta\,(a,b)\, z + c, \qquad \text{... }(iv)$$

where $$a + b + \theta\,(a,b) - ab\;\theta(a, b) = 0 \qquad \text{... }(v)$$

From (*v*), $$\theta(a, b) = (a + b)/(ab - 1) \qquad \text{... }(vi)$$

From (*iv*) and (*vi*), $\quad u = ax + by + \{(a + b)/(ab - 1)\} + c,$

which is the required complete integral of (*i*), *a*, *b*, *c* being arbitrary constants.

EXERCISE 3(G)

Find the complete integral of the following equation: (1 – 5)

1. $f(p_1, p_2, p_3) = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$, where $f(a_1, a_2, a_3) = 0$

2. $p_1 + p_2 + p_3 - p_1p_2p_3 = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$, where $a_1 + a_2 + a_3 - a_1a_2a_3 = 0$

3. $p_1x_1^2 - p_2^2 - ap_3^2 = 0$ **Ans.** $z = -(a_1^2 + a a_2^2)x_1^{-1} + a_1x_2 + a_2a_3x_3 + a_3$

4. $x_3(x_3 + p_3) = p_1^2 + p_2^2$ **Ans.** $z = a_1x_1 + a_2x_2 + (a_1^2 + a_2^2)\log x_3 - x_3^2/2 + a_3$

5. $x_3 + 2p_3 - (p_1 + p_3^2) = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + (a_1 + a_2)^2 \times (x_3/2) - (x_3^2/4) + a_3$

6. $x_1 + p_1^2 + x_2 + p_2^2 - x_3p_3^2 = 0$ **Ans.** $z = 2\,(a_1x_1)^{1/2} + 2\,(a_2x_2)^{1/2} + 2\,\{(a_1 + a_2)\,x_3\}^{1/2} + a_3$

7. *Show how to solve, by Jacobi method, a partial differential equation of the type* $f(x, \partial u/\partial x, \partial u/\partial z) = g(y, \partial u/\partial y, \partial u/\partial z)$ *and illustrate the method by finding a complete integral of equation* $2x^2y(\partial u/\partial x)^2(\partial u/\partial z) = x^2(\partial u/\partial y) + 2y(\partial u/\partial x)^2$. **[Meerut 2005]**

Sol. Try yourself **Ans.** $u = (ax^2 - b)^{1/2} + ay^2 + (z/b) + c$

8. *Prove that an equation of the "Clairaut" form* $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z)$ $= f(\partial u/\partial x, \partial u/\partial y, \partial u/\partial z)$ *is always solvable by Jacobi's method. Hence solve*

$$(\partial u/\partial x + \partial u/\partial y + \partial u/\partial z)\,\{x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z)\} = 1$$

3.22. Jacobi's method for solving a non-linear first order partial differential equation in two independent variables.

[Delhi Maths (H) 1997; Amaravati 2001; Himanchal 2003, 05]

Let $$F(x, y, z, p, q) = 0 \qquad \text{... (1)}$$

be the non-linear first order equation in two independent variables x, y.

Then we know that a solution of (1) is of the form $$u(x, y, z) = 0 \qquad \text{... (2)}$$

showing that u can be treated as a dependent variable and x, y, z as three independent variables.

Differentiating (2) partially w.r.t. 'x' and 'y', respectively, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$$

or $$p_1 + p_3 p = 0 \qquad \text{and} \qquad p_2 + p_3\ q = 0 \qquad \text{... (3)}$$

where $p = \partial z/\partial x,\ q = \partial z/\partial y,\ p_1 = \partial u/\partial x = \partial u/\partial x_1,\ p_2 = \partial u/\partial y = \partial u/\partial x_2,\ p_3 = \partial u/\partial z = \partial u/\partial x_3$

by taking $$x = x_1, \qquad y = y_2 \qquad \text{and} \qquad z = x_3 \qquad \text{... (4)}$$

From (3), $$p = -(p_1/p_3) \qquad \text{and} \qquad q = -(p_2/p_3) \qquad \text{... (5)}$$

Using (4) and (5), (1) reduces to $$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \qquad \text{... (6)}$$

We now solve (6) by Jacobi's method as usual (refer Art. 3.20) to get the complete integral of (6). Finally, putting $x_1 = x,\ x_2 = y,\ x_3 = z$, we obtain solution of (6) containing original variables x, y, z and new dependent variable u. The solution so obtained will contain three arbitrary constants a_1, a_2, a_3 (say). However, for the given equation in the form (1), we need only two arbitrary constants in the final solution. The required solution $u = 0$ of (1) is obtained by making different choices of our third arbitrary constant.

Ex. 1. *Solve* $p^2x+q^2y=z$ *by Jacobi's method.* **[Nagpur 2002; Himanchal 2003, 05]**

Sol. Given $p^2x+q^2y=z$.

Let a solution of (1) be of the form $u(x,y,z)=0$ ... (2)

So treating u as dependent variable and x, y, z as three independent variables, differentiation of (2) partially w.r.t 'x' and 'y' respectively gives

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{and} \quad \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}=0 \quad i.e. \quad p_1+p_3p=0 \quad \text{and} \quad p_2+p_3\,q=0$$

so that $p=-p_1/p_3$ and $q=-p_2/p_3$ .... (3)

where $p_1=\partial u/\partial x=\partial u/\partial x_1$, $p_2=\partial u/\partial y=\partial u/\partial x_2$, $p_3=\partial u/\partial z=\partial u/\partial x_3$, $p=\partial z/\partial x, q=\partial z/\partial y$

by taking $x=x_1$, $y=x_2$ and $z=x_3$ ... (4)

Using (3) and (4), (1) $\Rightarrow$ $x_1(p_1/p_3)^2+x_2(p_2/p_3)^2=x_3$. $\Rightarrow$ $x_1p_1^2+x_2p_2^2-x_3p_3^2=0$

Let $f(x_1, x_2, x_3, p_1, p_2, p_3)=x_1p_1^2+x_2p_2^2-x_3\;p_3^2=0$ ... (5)

Now, the Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{p_1^2}=\frac{dx_1}{-2p_1x_1}=\frac{dp_2}{p_2^2}=\frac{dx_2}{-2p_2x_2}=\frac{dp_3}{-p_3^2}=\frac{dx_3}{2p_3x_3}, \text{ by (5)}$$

Taking the first two fractions, $(2/p_1)\;dp_1+(1/x)\,dx=0$.

Integrating, $2\log p_1+\log x_1=\log a_1$ so that $x_1p_1^2=a_1$ or $p_1=(a_1/x_1)^{1/2}$

Similarly, the third and fourth fractions give $p_2=(a_2/x_2)^{1/2}$

Substituting these values of p_1 and p_2 in (5), we get $p_3=\{(a_1+a_2)/x_3\}^{1/2}$.

Putting the above values of p_1, p_2 and p_3 in $du=p_1dx_1+p_2dx_2+p_3dx_3$, we get

$$du=a_1^{1/2}\;x_1^{-1/2}dx_1+a_2x_2^{-1/2}dx_2+(a_1+a_2)^{1/2}x_3^{-(1/2)}dx_3.$$

Integrating, $u=2(a_1x_1)^{1/2}+(a_2x_2)^{1/2}+2(a_1+a_2)^{1/2}x_3^{1/2}+a_3$ ... (6)

Taking $a_2=1$ and using (4), the required solution $u=0$ is given by

$$2(a_1x)^{1/2}+2y^{1/2}+2(a_1+1)^{1/2}z^{1/2}+a_3=0,$$

which is the complete integral containing two arbitrary constants a_1 and a_3.

Ex. 2. *Solve* $p^2+q^2=k^2$ by Jacobi's method **[Delhi B.A./B.Sc. (Prog) Maths 2007]**

Sol. Given $p^2+q^2=k^2$ ... (1)

Let a solution of (1) be of the form $u(x, y, z)=0$ ... (2)

So treating u as dependent variable and x, y, z as three independent variable, differentiation of (2) partially w.r.t. 'x' and 'y' respectively gives

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{and} \quad \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}=0 \quad i.e., \quad p_1+p_3p=0 \quad \text{and} \quad p_2+p_3q=0$$

so that $\quad p=-(p_1/p_3) \quad$ and $\quad q=-(p_2/p_3) \quad$ ... (3)

where $p_1=\partial u/\partial x=\partial u/\partial x_1,\ p_2=\partial u/\partial y=\partial u/\partial x_2,\ p_3=\partial u/\partial z=\partial u/\partial x_3,\ p=\partial z/\partial x,\ q=\partial z/\partial y$

by taking $\quad x=x_1, \quad y=x_2 \quad$ and $\quad z=x_3 \quad$ ... (4)

Using (3) and (4), (1) reduces to $\quad p_1^2/p_3^2+p_2^2/p_3^2=k^2 \quad$ or $\quad p_1^2+p_2^2=k^2p_3^2$

Let $\quad f(x_1,x_2,x_3,p_1,p_2,p_3)=p_1^2+p_2^2-k^2p_3^2=0 \quad$ ... (5)

Now, the Jacobi auxilliary equations are given by

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{0}=\frac{dx_1}{-2p_1}=\frac{dp_2}{0}=\frac{dx_2}{-2p_2}=\frac{dp_3}{0}=\frac{dx_3}{2k^2p_3}, \text{ using (5)}$$

From the first and third fractions of (5), $\quad dp_1=0 \quad$ and $\quad dp_2=0$

Integrating, $\quad p_1=a_1$ and $p_2=a_2$, a_1 and a_2 being arbitrary constants

With $\quad p_1=a_1 \quad$ and $\quad p_2=a_2, \quad$ (5) gives $\ p_3=(a_1^2+a_2^2)^{1/2}/k$

Putting the above values of p_1, p_2 and p_3 in $du=p_1dx_1+p_2dx_2+p_3dx_3$, we get

$$du=a_1dx_1+a_2dx_2+\{(a_1^2+a_2^2)^{1/2}/k\}dx_3$$

Integrating, $\quad u=a_1x+a_2x_2+\{(a_1^2+a_2^2)^{1/2}/k\}x_3+a_3 \quad$ ... (6)

Taking $a_2=1$ and using (4), the required solution $u=0$ is given by

$$a_1x+x_2+\{(a_1^2+1)^{1/2}/k\}x_3+a_3=0,$$

which is the complete integral of (1) containing two arbitrary constants a_1 and a_3.

Ex. 3. *Solve the following partial differential equations by Jacobi's method:*

(*i*) $p=(z+qy)^2 \qquad$ (*ii*) $(p^2+q^2)x=pz \qquad$ (*iii*) $xpq+yq^2=1 \qquad$ **[Nagpur 2005]**

Hint. Proceed as in the above solved Ex. 1

3.23 Cauchy's method of characteristics for solving non-linear partial differential equation $\quad f(x,y,z,\ \partial z/\partial x,\ \partial z/\partial y)=0 \quad i.e., \quad f(x,y,z,p,q)=0 \quad$ ... (1)

We know that the plane passing through the point $P(x_0,y_0z_0)$ with its normal parallel to the direction **n** whose direction ratios are $p_0,q_0,-1$ is uniquely given by the set of five numbers $D(x_0,y_0,z_0,p_0,q_0)$ and conversely any such set of five numbers defines a plane in three dimensional space. In view of this fact a set of five numbers $D(x,y,z,p,q)$ is known as a *plane element* of a three dimensional space. As a special case a plane element (x_0,y_0,z_0,p_0,q_0) whose components satisfy (1) is known as an *integral element* of (1) at P. Solving (1) for q, suppose we get