Q 8) Let $\{W(t), t \geq 0\}$ be a Wiener process. Is $\exp\{\sigma W(t) - \frac{\sigma^2}{2} t\}$ a martingale where σ is a positive constant?

Ans) Let $0 \leq s \leq t$ since $W(t) - W(s)$ is independent of F_s where $W(s)$ be F_s measurable and σ be any +ve constant, we have

$$E[e^{\sigma W(t)} | F_s] = E[e^{\sigma(W(t) - W(s))} e^{W(s)} | F_s]$$

$$= e^{\sigma W(s)} E[e^{\sigma(W(t) - W(s))} | F_s]$$

$$= e^{\sigma W(s)} E[e^{\sigma(W(t) - W(s))}]$$

$$E[e^{\sigma(W(t) - W(s))}] = \sigma \frac{(t-s)}{2}$$

Since $W(t) - W(s)$ has normal distribution

Hence $E[e^{\sigma W(t)} | F_s] = e^{\sigma W(s)} \frac{(t-s)}{2}$

This gives

$$E[e^{\sigma W(t) - \sigma^2 t/2} | F_s] = e^{-\sigma^2 t/2} E[e^{\sigma W(t)} | F_s]$$

$$e^{\sigma^2 (W(s) - s/2)}$$

It follows that $\exp(\sigma W(t) - \frac{t^2}{2} \sigma)$ is a martingale