Meetali Tomer

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→ Book ⇒ Introduction to financial Engineering — Suresh Chandra

4 authors → Suresh

Asset ⇒ anything we have except currency

Asset — Risk free (whoose future value is known) govt bond / FD
— Risky (whoose future value is not known)
eg: stock, gold, foreign currency
↓
it may go up or down (value)

Risk free debunture, FD fixed deposit in private company.

If the money is same for an asset, then we will consider it loss.

Risky ⇒ X (price — p ↑ / 1−p ↓ — at any time the price may go up or down.

$S(t)$ ⇒ price of a Stock.

we start at time $t=0$ — $S(0)$ → known quantity

t only has 2 values $t=0$ Starting time.
$t=1$ when it ends

We are not concerned with what happens in between.

$t=0$ current time $S(0)$ → known quantity.

$t=1$ future time $S(1)$ → unknown quantity.

$S(0)$ → S_{up} $(t=1)$ probability p
$S(0)$ → S_{down} $(t=1)$ probability $1-p$

Rate of return or return } same thing

$$K_S = \frac{S[1]-S[0]}{S[0]} \Bigg\} \text{ rate of return}$$

$$= \frac{\text{Change in price}}{\text{initial price}}$$

for bonds $A(0)$ price of bond at $t=0$

$A(1)$ price of bond at $t=1$

$$K_A = \frac{A(1) - A(0)}{A(0)} \Bigg\} \text{ rate of bond.}$$

Assumptions for mathematical modelling

① Randomness => The future of stock price $S(t)$ is a random variable with atleast two different values,
while $A(t)$ the future bond price is a known number

② Positivity of prices => All stock and bond prices are strictly positivite i.e $A(t) > 0$ and $S(t) > 0$ for $t = 0, 1$

The total wealth of an investor holding x stocks and y bonds at $t = 0$ and 1 will be given as

$$V(0) = x\, S(0) + y\, A(0)$$

$$V(1) = x\, S(1) + y\, A(1)$$

The ordered pair (x, y) is called the portfolio.
order is important, first we write the risky asset.
Rate of return on this portfolio $= \dfrac{V(1) - V(0)}{V(0)}$

$x, y \in$ Real nos.

③ Divisibility, Liquidity and short selling =>
An investor may hold any no x and y of stock share and bond
i.e $x, y \in \mathbb{R}$ — ①
x and y can also be -ve. if we have borrowed from bank
Short selling => we are reducing our portfolio even though we do not own the share or bond, we are selling it.

currency & liquidity should not be associated with it.
~~If~~ worth of an asset in the market can be said liquidity.

Since by (1), x and y are unbounded., they can approach to $\pm\infty$ $x \to \pm\infty$, $y \to \pm\infty$ (they can be as large as possible)
$\Rightarrow$ Any asset can be brought ~~and~~ or sold on demand at market price in arbitarary quantities & This is liquidity.

If the no of securities (some as stocks and bonds) of a particular kind held in a portfolio is +ve, then investor is said to have long position, otherwise said to have taken a short position.

(4) Solvency
wealth of an investor should be non-ve at all times
$V(t) \geq 0$, $t = 0, 1$
A portfolio satisfying this condition is called admissable

(5) Discrete unit price $\Rightarrow$ The future price $S(1)$ of a share or stock is a random variable taking only finitely many values

(6) No Arbitoge principle $\Rightarrow$ withouth risk, no profit
Market does not allow for risk free profit with no initial ~~investme~~ investment.

If $V(0) = 0$ (initial ~~is 0~~) investment is 0)
~~then~~ ~~$V(t)$~~ $V(t) > 0$ not possible

There is no admissable portfolio with inital value $V(0) = 0$ such that $V(t) > 0$ with non zero probability.
If a portfolio violating this principle exists, we can say that ~~arbitragy~~ arbitoge oppurtuning is avaiable.

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arbitrage will be marginally small every time, but investment there will not be fruitfull.
Stock costs should be at parity.

Risk | Return

Since the future price of a stock is a random variable then return is also a random variable
Depending upon many factors, performance of the company, natural calamities, national and international politics

Return r_i

If r_i be the return with the probability p_i with phase or scenario i
$i = 1, 2, 3, \ldots$ (integral values)

Expected return $R = E(r)$

$$R = \sum p_i r_i \quad \text{—} (1)$$

$$\sigma^2 = E(r_0 - E(r))^2$$

$$\sigma^2 = \sum (r_i - R)^2 p_i \quad \text{—} (2)$$

$$\sigma = \sqrt{\sum (r_i - R)^2 p_i} \quad \} \text{ Risk} = \sigma \quad \text{—} (3)$$

p_i depends on the judgement of the investor

Problem) A Company having today's stock price ₹ 261.25 per share, may go through 4 equally likely economic condition; high growth, expansion, Stagnation, and decline. Under these states the price of the stock is expected to be ₹ 305.50, ₹ 285.50, ₹ 261.50, ₹ 243.50 respectively. Decide whether to invest.

$S(0) = ₹\ 261.25$

Scenario	Price ~~rate of return~~	p_i	Rate of return ~~16.9%~~
High Growth	305.50	0.25	16.9 %
Expansion	285.50	0.25	9.3 %
Stagnation	261.50	0.25	0
Decline	243.50	0.25	-6.8 %

$R = \sum p_i r_i$

Return $= \frac{16.9 + 9.3 + 0 + (-6.8)}{4}$

$= \underline{4.85}$

$\sigma^2 = \frac{(16.9 - 4.85)^2 + (9.3 - 4.85)^2 + (-4.85)^2 + (-6.8 - 4.5)^2}{4}$

$= 81.06$

$\sigma \approx \underline{9\%}$ Risk

Risk > Return

=> investment is not feasible

Problem) Let $A(0) = \$100$, $A(1) = \$110$

$S(0) = \$80$, $S(1) = \begin{cases} \$100 & \text{probability } 0.8 \\ \$60 & \text{probability } 0.2 \end{cases}$

Discuss per portfolio (50, 60) (125, 0)

Return and Risk

1 (50, 60) $V(0) = 50 \times 80 + 60 \times 100$

$= 4000 + 6000$

$= 10,000$

$V(1) = 50 \times 100 + 60 \times 110$ probability 0.8

$= 5000 + 6600$

$= 11,600$ probability 0.8

$= 50 \times 60 + 60 \times 110$

$= 60(160) = 9600$ probability 0.2

$~~Return~~ K_S = \frac{11,600 - 10,000}{10,000} \times 100 = \frac{1600}{100} = 16\%$ proba 0.8

$= \frac{9600 - 10,000}{10,000} \times 100 = \frac{-400}{100} = -4\%$ p = 0.2

Average Return = $E(R) = 0.16 \times 0.8 + (-0.04 \times 0.2)$

$= 0.12$

$= 12\%$

Risk $\sigma = 0.08$ or 8%

$\sigma^2 = \frac{0.8(16-12)^2 + 0.2(-4-12)^2}{2} = 0.8(16) + 256(0.2) = \cancel{156}$

$\sigma^2 = 12.8 + 51.2 = 64$

$\Rightarrow \sigma = \sqrt{64} = 8 = 0.08\%$

(ii) (125, 0)

$V(0) = 125 \times 80 = 10,000$

$V(1) = 125 \times 100$ $p = 0.8$

$= 12500$ $p = 0.8$

$= 125 \times 60$ $p = 0.2$

$= 7500$ $p = 0.2$

$K_s = \frac{12,500 - 10,000}{10,000} \times 100 =$

$= \frac{2500}{100} = 25\%$ $p = 0.8$

$= \frac{7500 - 10,000}{10,000} \times 100$

$= \frac{-2500}{100} = -25\%$ $p = 0.2$

Average Return $= 0.25 \times 0.8 - 0.25 \times 0.2$

$= 0.25\,(0.8 - 0.2)$

$= 0.25 \times 0.6 = 0.15$

$= 15\%$

$\sigma = 20\%$

$$\sigma^2 = (25-15)^2 \times 0.8 + (-25-15)^2 \times 0.2$$

$$= 100 \times 0.8 + 1600 \times 0.2$$

$$= 80 + 320 = 400$$

$$\Rightarrow \sigma = \sqrt{400} = 20\%$$

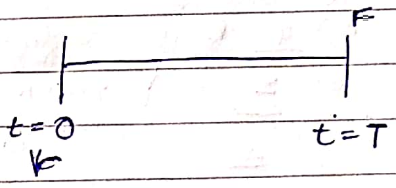
Zero Coupon Bond

F - face value of the bond

T - term of maturity

r - rate of interest

value of the bond $V(0) = (1+r)^{-1} F$



$1+r$ } growth factor

$(1+r)^{-1}$ } discount factor

Types of Compounding

- Simple Interest
- Compound Interest

when more than once in a year $\left(1+\frac{r}{m}\right)^m$

m is the frequency in ~~which~~ a year compounding is done

when compounding is done daily $\left(1+\frac{r}{365}\right)^{365}$

- Continuous compounding } e^{rt}

$\left(1+\frac{r}{n}\right)^n$ ∞ $n \to \infty$ it tends to exponential

$P = 100$ $R = 12\%$ $T = 200$ days.

$$SI = \frac{PRT}{100} = \frac{100 \times 12}{100} \times \frac{200}{365}$$

$=$

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Zero Coupon bond is a risk free asset when the payment is ~~to~~ guaranteed to the customer.

Assumption = Interest rate remains constant, which is not true, because interest rate changes over time.

but we take it constant.

that is why it is called implied interest rate } the current interest rate.

$$\boxed{r = \frac{F}{V} - 1}$$ } implied interest rate } the current rate of interest.

The customer will get a fixed amount of cost at the ~~stop~~ time of maturity. In between no amount has to be paid.

eg: $F = 100$, $r = 12\%$. $T = 1$ year.

$$V(0) = (1+r)^{-1} \times F$$

$$= \frac{100}{(1+0.12)} = \frac{100}{1.12} \approx ₹\,89$$

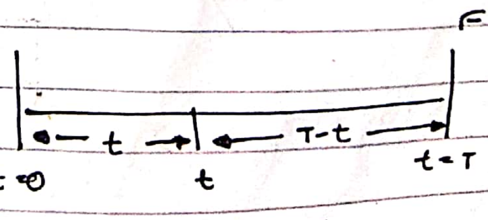
So after 1 year if we need 100 ₹, the agency will charge us ₹89 today.

if face value of the bond is 1 $F=1$ } it becomes currency independent

$$V(0) = (1+r)^{-1}$$

But these bonds are ~~sold~~ sold in the market, because of the fluctuation of interest rate.

we need to find the value of the bond at time t



we will discount the value of F to the period t

$$V(t) = (1+r)^{-(T-t)} F$$

F=1 ↙ Annual compounding

initiated at

$$B(0,T) = \frac{1}{1+r}$$ } value of the bond at time 0 with maturity at time T

$$B(t,T) = (1+r)^{-(T-t)}$$ } value of the bond initiated at time t with maturity at time T

Face value of the bond here =1

If ~~comp~~ periodic compounding is taking place (~~continuous compounding~~) with frequency m

$$V(t) = \left(1 + \frac{r}{m}\right)^{-(T-t)m}$$

If simply rate of interest is given, it is for 1 year, unless explicitly mentioned.

If continuous compounding is taking place.

$B(0,T) = e^{-rt}$ $\quad$ $r=1$ $\quad \Rightarrow \quad = e^{-r}$ } for 1 year.

$B(t,T) = e^{-r(T-t)}$

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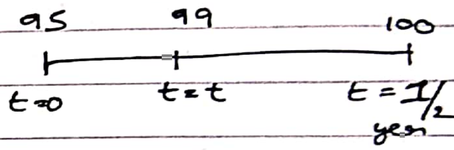
$$V(0) = (1+r)^{-1/2}\, 100$$

$\Rightarrow$ value of r will be calculated from here

$$(1+r)^{-1/2} = \frac{95}{100}$$

$r = 10.8\%$

$$(1+r)^{-1} = \left(\frac{95}{100}\right)^2$$



$\cancel{1+r = 0.902}$

$$100\left[1 + 0.108\right]^{-\left(\frac{1}{2} - t\right)} = 95$$

we have to find this value of t

$$[1.108]^{-\frac{1}{2}+t} = \frac{95}{100}$$

$$[1.108]^{-\frac{1}{2}+t} = 0.95$$

$\cancel{\left(-\frac{1}{2}+t\right)\log(1.108) = \log(0.95)}$

$\cancel{-\frac{1}{2}+t = \frac{\log(0.95)}{\log(1.108)} = -0.022}$

called coupon.

<u>Coupon bond</u> ? Intermediatory payments is made here.

100 of coupons paid and cost of each coupon is ~~paid~~. Fixed last coupon will be paid at time of maturing.

Value of each coupon is same

Face value of the bond = F Time of maturi...

Continuous compounding

$V(0) = ce^{-r} + ce^{-2r} + \cdots + ce^{-nr}(C+F)$ — ①

One coupon is paid after 1 year

$V(1) = ce^{-r} + \cdots ce^{-(n-1)r}(C+F)$

→ Same value if we issue the bond for n-1 years instead of n years.

$\cancel{V(1) =}$

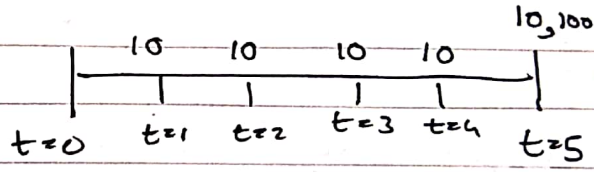
multiply eq ① by e^{r}

$V(0)e^{r} = V(1) + C$

Q) A bond with face value \$100 maturity in 5 years with coupons of \$10 paid annually. If the rate of interest being 12%. What is the price of the bond. Continuous compounding

$r = 12\%$
$= 0.12$

$\cancel{V(0)}$



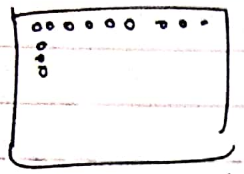
$V(0) = 10e^{-0.12} + 10e^{-(0.12\times2)} + \cdots$
$+ 10\times e^{-(0.12\times4)} + 110e^{-(0.12\times5)}$

→ this is the value of the bond.

making

- Mathematical elements of Computer graphics by Roger's and ADAMS
- Procedural computer graphics

<u>CG</u>



black and white

colour

} in the form of triplet (triad)

using these 3 colours (red, green, blue) we can get all the colours possible.

In colour display, we have 3 electron beam corresponding to each colour. → red, green, blue

The beam should fall on that electron ~~which it should~~ corresponding to the same colour.

The intensity of the picture will depend on the energy with which electron beam strikes.
after some time intensity of the image will decrease.
to keep the intensity constant, we can do continuous striking of electron beam.

<u>~~Image~~ resolution</u> — image pixels per inch (ppi) — no of pixels displayed in one horizontal ~~dis~~ line.

<u>~~Screen~~ resolution</u> — Screen dots per inch (dpi)

<u>Aspect ratio</u> - of an image is the ratio of no of pixel in x direction to that of y direction.
Standard aspect ratio => 4:3 or 5:4

<u>Screen size</u> } is the distance between two opposite ends of screen.

<u>Resolution</u> - no of pixels in x direction or y direction

<u>flickering</u> => when screen quality is not good, Image screen is good or vice versa

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Forward contract } always made on the risky asset.

If forward contract is an agreement, to buy or sell a risky asset, at a specified future time known as delivery date; for a price F fixed at present moment called the forward price.

An investor who agrees to buy the asset is said to enter into a long forward contract or to take a long forward position and the investor agrees to sell the asset is said to enter into a short forward contract or take short forward position.

No money is paid at the time when forward contract is exchanged.

Exercise of contract is mandatory for both the parties.

At time $t=0$ $S(0)$. } cost of the risky asset.

price at maturity $S(1)$ $t=1$

F } price at which parties have agreed to enter the contract. (at $t=0$)

→ pay off long forward position holder $= S(1) - F$

pay off short forward position holder $= F - S(1)$

} one of them will be +ve and the other will be −ve

if $S(1) = F$ then both will be zero

One will have to bear the loss and the other will be cutting profit.

for portfolio Z → represents the no of forward contract.

portfolio will become a triplet (x, y, z) } ordered pair

x = no of risky asset y = no of bonds z = no of forward contract.

For long position holder $t=0$ $V(0) = x\,S(0) + y\,A(0) + 0$

$t=1$ $V(1) = x\,A(1) + y\,A(1) + z(S(1)-F)$

will be different for short position holder.

~~For short position~~

We shall represent the agreed forward price as $F(0,T)$

long position holder

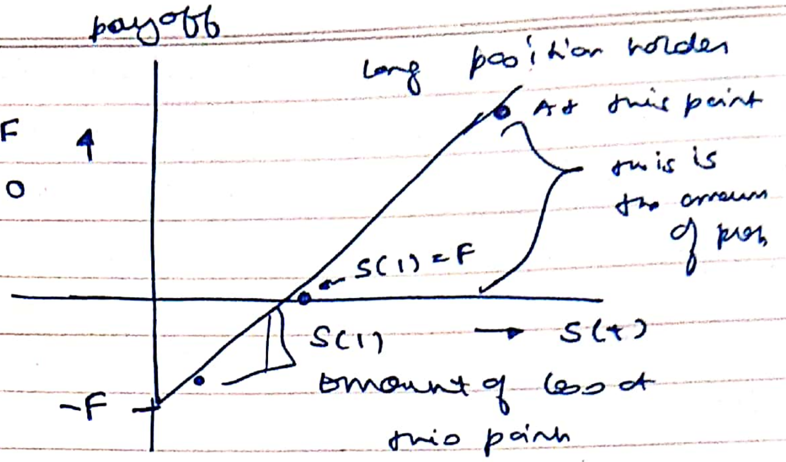
maximum loss he can incur = F

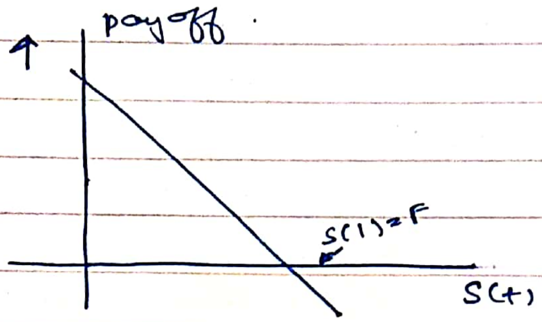
when $S(1) = F$ payoff will be 0

and when $S(1) > F$

it will be a profit.

↳ Case will be opposite for short position holder.





Some times the contract may be initiated at any time $t < T$ instead of $t = 0$, In that case the payoff for long forward position will be

$$\left.\begin{array}{ll} & S(T) - F(t,T) \\ \text{short position holder} & F(t,T) - S(T) \end{array}\right\} \text{payoff}$$

where $F(t,T)$ is the forward contract price at time t

Theorem ⇒ Let the price of an asset at $t=0$ be $S(0)$, then the forward price $F(0,T)$ is given as $F(0,T) = S(0)$

- Borrow the amount $S(0)$ for time T
- Buy one unit of underlined stock at $S(0)$ price
- take a short forward position with delievery time T and $F(0,T)$ cost.

$(1, -1, 1)$ } portfolio

$V(0) = 1 \cdot S(0) + (-1) A(0) + 0$ here $A(0) = S(0)$

$V(0) = 0$ current worth $= 0$

At time $t = T$

- Close the short forward position by selling the price at $F(0,T)$
- Return the borrowed money with interest $\frac{S(0)}{d(0,T)}$

$$V(1) = F(0,T) - \frac{S(0)}{d(0,T)}$$

By virtue of A $F(0,T) - \frac{S(0)}{d(0,T)} > 0$

$\Rightarrow V(1) > 0$ payoff is +ve

No arbitage principle is violeted.

$\Rightarrow F(0,T)$ can not be greater than $\frac{S(0)}{d(0,T)}$

$$F(0,T) \ngtr \frac{S(0)}{d(0,T)} \quad \text{— } A_1$$

Case 2   Assume that $F(0,T) < \frac{S(0)}{d(0,T)}$ — B

Construct a portfolio at time $t = 0$

- Short sell one unit of underlined asset for $S(0)$.
- Invest $S(0)$ in risk free for $t = T$
- Enter into a long forward contract with delievery at T and forward price $F(0,T)$

portfolio $= (-1, 1, 1)$

$V(0) = -1 . S(0) + (1) A(0) + 0$ $A(0) = S(0)$

$= 0$ current worth $= 0$

at $t = T$

- Receive the money from the bank, amount $\frac{S(0)}{d(0,T)}$
- Buy the stock under forward contract by paying $F(0,T)$
- Close the short sell position by returning the one unit of underlined asset.

$$V(T) = \frac{S(0)}{d(0,T)} - F(0,T)$$

By (B) $V(T) > 0$

No arbitrage principle is violated. $F(0,T)$ can not be less then $\frac{S(0)}{d(0,T)}$

$$\Rightarrow F(0,T) \nless \frac{S(0)}{d(0,T)} \quad \text{— } (B_1)$$

By (A_1) and (B_1) $\boxed{F(0,T) = \frac{S(0)}{d(0,T)}}$

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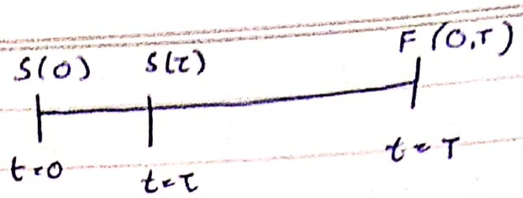
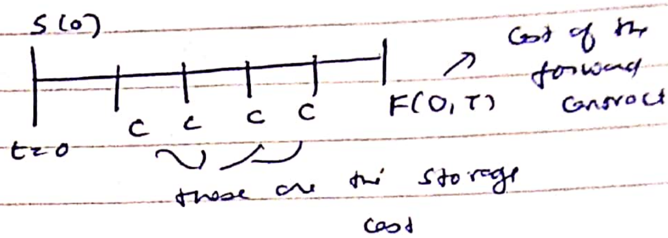
Value of the forward contract



if forward contract is initiated at time $t = \tau$ at $t = T$ it will be of the cost $F(\tau, T)$ at time $t = T$

- Storage Cost →

will be paid by short forward position holder.



the cost c will also grow

Let an asset carry a holding cost $c(i)$ per unit in period $i=0$ to $n-1$

Also, let the price of the asset at $t=0$ be $S(0)$ and short selling is allowed

$$F(0,T) = \frac{S(0)}{d(0,T)} + \sum_{i=0}^{n-1} \frac{c(i)}{d(i,n)} \quad \Big\} \text{ paying in advance}$$

if we have to pay in advance → 1 month will be extra cost unit will be done on $(n-1)$th unit.

If we have to pay afterwards; cost payment will be done in the nth unit → cost payment will not grow.

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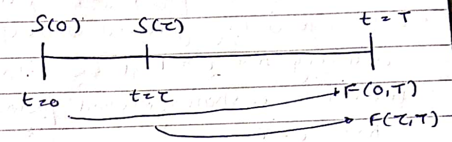
$$F(0,T) = \frac{S(0) - \text{'div'}\, d(0,\tau)}{d(0,T)}$$

→ If ~~Continuous~~ the asset pays dividend continuously at a rate r_{div}

$$F(0,T) = S(0)\, e^{(r - r_{div})T}$$

$$\infty \quad \cancel{F(0,T) = \frac{S(0)}{d(0,T)} - div}$$

<u>value of the forward contract</u>



If a person wants to invest at time $t = \tau$ he has two ~~opt~~ options.

- take the stock initiated at time $t=0$ with cost $F(0,T)$ at time T
- take the stock initiated at time $t = \tau$ with cost $F(0,T)$ at time T

T ~ time of maturity

$f(\tau)$ ~ value of contract at time $t = \tau$ for the contract at time $t=0$ (worth)

Let a forward contract initiated at $t=0$ with delivery at $t = T$ of the price $F(0,T)$. Consider an intermediate time $t = \tau$ $(0 < \tau < T)$ and let $F(\tau, T)$ be the forward price of the contract initiated at $t = \tau$ with delivery $t = T$.

As the time passes by, the value of the forward contract, initiated at $t=0$ will be changing and let $f(\tau)$ be the value at $t = \tau$

The value of the forward contract will be given by the expression.

$$f(\tau) = [F(\tau,T) - F(0,T)]\, d(\tau,T)$$

where $d(\tau,T)$ is the risk free discount factor over the period $t = \tau$ to $t = T$

$\rightarrow \quad f(\tau) = [F(\tau,T) - F(0,T)]\, d(\tau,T)$

Proof:

Case 1) Let $f(\tau) < (F(\tau,T) - F(0,T))\, d(\tau,T)$

Construct a ~~portf~~ portfolio at $t=\tau$

$$F(\tau,T) - F(0,T) - \frac{f(\tau)}{d(t,T)} > 0 \quad \text{— (A)}$$

- Borrow the amount $f(\tau)$ (risk free)
- Enter into a long forward contract with forward price $F(0,T)$ and delivery time $t=T$ (by paying the value $f(\tau)$ at $t=\tau$)
- Enter into a short forward position with price $f(\tau,T)$ with delivery time $t=T$

The worth of the portfolio at time τ is 0 ~~V(0)=0~~ $V_p(\tau)=0$

At time $t=T$

- Close the long forward position by paying $F(0,T)$
- Close the short forward position by receiving the amount $F(\tau,T)$ and Deliver the stock
- Return the risk free appreciated money $\frac{f(\tau)}{d(\tau,T)}$.

$$V(T) = F(T_0,T) - F(0,T) - \frac{f(\tau)}{d(\tau,T)}$$

using (A) $\Rightarrow V(T) > 0$

no arbitrage is ~~not~~ violated.

hence $f(\tau)$ can not be less than $[F(\tau,T) - F(0,T)]\, d(\tau,T)$

$\Rightarrow f(\tau) \nless [F(\tau,T) - F(0,T)]\, d(\tau,T)$ — (A1)

Case 2 Let $f(\tau) > [F(\tau,T) - F(0,T)]\, d(\tau,T)$

Construct a portfolio at time $t=\tau$

$$\frac{f(\tau)}{d(\tau,T)} + F(0,T) - F(\tau,T) > 0 \quad \text{— (B)}$$

time $t = \tau$

- Sell the forward contract which was initiated at $t=0$ (short sell) for the amount $f(\tau)$
- Invest $f(\tau)$ in risk free for $t=\tau$ to $t=T$
- Enter into a long forward contract with delivery time $t=T$ at the price $F(\tau,T)$

Value of portfolio at time $t=\tau$ is 0

$$V(\tau)=0$$

At time $t=\tau$

- Receive the risk free appreciated money $\frac{f(\tau)}{d(\tau,T)}$
- Close the short forward position by receiving $F(0,T)$ and delivering the stock
- Close the long forward position by paying $F(\tau,T)$ and receiving the stock.

$$V(T) = \frac{f(\tau)}{d(\tau,T)} + F(0,T) - F(\tau,T) > 0$$

from (B) $\quad V(T) > 0$

no arbitrage principle is violated

and ~~then~~ Hence $f(\tau)$ can not be greater than $[F(\tau,T) - F(0,T)]d(\tau,T)$

$$f(\tau) \ngtr [F(\tau,T) - F(0,T)]\, d(\tau,T) \quad \text{— (B1)}$$

from (A1) and (B1) $\quad f(\tau) = [F(\tau,T) - F(0,T)]\, d(\tau,T)$

Storage cost.

Q) The current price of sugar is 60/kg and is carrying a cost of 10 paise / kg / month ; storage cost has to be paid in advance (start of the month). The constant interest rate is 9% per annum compounded monthly. Find the forward price of the sugar to be delivered in 5 month.

$S(0) = 60$ per kg $\quad r = 9\%$ per annum $\quad$ monthly compounding

$c = 10$ paise

$c = 0.1$ per kg / month $\quad t = 5$ month

$$\text{growth factor} = 1 + \frac{r}{12} = 1 + \frac{0.09}{12}$$

$$= 1.0075 \quad \text{per month growth factor}$$

$$F(0,5) = \frac{S(0)}{d(0,T)} + \text{storage cost} \qquad \frac{1}{d(0,T)} = \text{growth factor.}$$

$d(0,T)$ = discount factor.

$$F(0,5) = 60 \times (1.0075)^5$$
$$+ 0.1(1.0075)^5$$
$$+ 0.1(1.0075)^4$$
$$+ 0.1(1.0075)^3 +$$
$$+ 0.1(1.0075)^2 + 0.1(1.0075)^1$$

0.1 0.1 0.1 0.1 0.1

t=0 t=1 t=2 t=3 t=4 t=5

• Let the stock be sold at the beginning of the year for ₹45.

$r = 6$ percent continuous compounding.

a) Find the forward price if the stock is to be delivered in 1 year

b) Also find it's value after 9 months, if it is given that, stock price at that time turn out to be ₹49.

a) $S(0) = 45 \quad r = 0.06\% \quad T = 1$

$$F(0,T) = S(0) \times e^{rT}$$
$$= 45 \times e^{0.06} \approx 47.78$$

b) $\tau = 9$ months $= 9/12$ year $\quad S(\tau) = 49$

$$F(\tau, T) = F(9/12, 1) = S(\tau) \times e^{0.06[1 - 9/12]}$$

$$F(\tau, T) = 49 \times e^{0.06 \times \frac{1}{4}}$$
$$= 49.74$$

$$f(\tau) = [F(\tau,T) - F(0,T)] \times d(\tau,T)$$
$$= [49.74 - 47.78] \times e^{-0.06 \times (1 - 9/12)}$$
$$= 1.93$$

<u>Options</u> =>

An option is a contract that gives the holder a right to trade without any obligation.

that is to buy or sell an asset at specified time (on or before) on an agreed price fixed now. It gives us the right to buy or sell an asset at a fixed price. (it is our choice to exercise it or not)

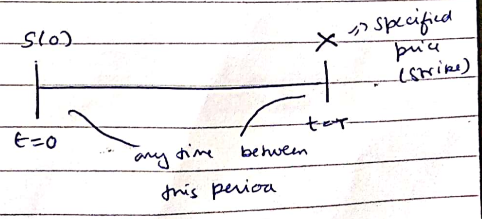
Options are usually traded in a block of 100 shares.

Two types of options are there =>

1) <u>Call option</u> - is the right to buy an asset at a specified price.

2) <u>Put option</u> - is the right to sell an asset at a specified price

The specified price is known as the strike price or exercise price and the asset on which the call or put option is created is referred to as underlying asset.



European option } it puts a restriction that all transaction will take place at time $t = T$ (maturity time)

but American option does not have this restriction.

The person who gives the option does not have to do anything.

The ~~person~~ who sells the option is called the righter of the option.

Replicating the portfolio. (because the portfolio is constructed on the same stock)

One period Binomial Model & pricing of options using

what should be value of the S(t) ?

$t=0$ ——— $t=1$, X

$S(0)$ $S(t)$

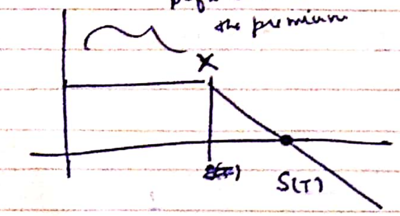
X = strike price

$S(T) > X$ for call option

$S(T) < X$ for put option to be exercised (premium)

If $S(T) < X$ he will not exercise the call option
then Loss = the money he has paid.

If $S(T) > X$ then payoff $= S(T) - X$
and there will be loss for the writer



Writer will try to make his loss less & by charging the premium.

for writer (call option)

$$S(0) \begin{cases} S^u & \text{prob } p \\ S^d & \text{prob } 1-p \end{cases}$$

S(1) is splitted into two parts.

(x, y) is the portfolio of the writer ~~containing~~ using ~~money~~ ~~gained~~ ~~by payoff~~

this portfolio is created in the favour of the writer

$V_p(0) = x\, S(0) + y\, A(0)$

$V_p(t) = x\, S(1) + y\, A(1)$

it will have two expressions

$$V_p(1) = \begin{cases} x S^u + y A(1) & \text{(A) probability } p \\ x S^d + y A(1) & \text{(B) probability } 1-p \end{cases}$$

Call option will be exercised in (A) (when prices go up)

~~then~~ writer will make his loss equal to the payoff
(so that he does not faces any losses) (he will construct a portfolio)

$x S^u + y A(1) = \text{payoff}$

(what is happening in between ⇒ we are not concerned.

Problem)

A call

Find th

p

$$x S^d + y A(1) = 0$$

Problem) Let $A(0) = \$100$, $A(1) = \$110$

$S(0) = \$100$, $S(1) = \$120$ prob p

$\$80$ prob $1-p$

A call option with strike price $100 and exercise time 1

Find the premium of the call option.

Portfolio = (x, y) $\quad V_p(0) = x(100) + y(100)$

$V_p(1) = x(120) + y(110)$ prob p

$= x(80) + y(110)$ prob $1-p$

$x(120) + y(110) =$ ~~premium~~ payoff.

$x(80) + y(110) = 0$

$S(T) - X$

writer of the option should buy $1/2$ share in his portfolio (of the same stock)

if we are using put option only the payoff will change.

$$C(1) = \begin{cases} 0 & \text{prob } p \\ 20 & \text{prob } 1-p \end{cases}$$

$$\Rightarrow \quad 120x + 110y = 0$$

$$80x + 110y = 20$$

$$-40x = 20 \qquad x = -1/2$$

$$+120 \times \left(-\frac{1}{2}\right) + 110y = 0$$

$$110y = 60 \qquad y = \frac{6}{11}$$

$\left(-\frac{1}{2}, \frac{6}{11}\right)$ } value of the writer for the put option.

$\Rightarrow$ Short sell $1/2$ share (of the same stock)

$$V_P(0) = \frac{-1}{2} \times 100 + \frac{6}{11} \times 100$$

price of the european put option p^E

$$= -50 + \frac{600}{11} = \frac{50}{11} = \$4.5$$

because they are usually sold in bunch of 100's } find the answer up to 2 decimal points

$$x^+ = \begin{cases} x & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

function

payoff call option option holder =

$$\max(S(T) - X, 0)$$

for writer $\min(S(T) - X; 0)$

$\Rightarrow$ payoff for call option (option holder) $= (S(T) - X)^+$

Put-call parity formula

$C^E \to$ European call option price

$P^E \to$ European put option price

For the stock paying no dividends, the following relation holds between the price of European call and put options, both with exercise price X and exercise time T.

$$C^E - P^E = S(0) - X e^{-rT}$$

$r =$ risk free interest rate.

Proof

Case (i) Let $C^E - P^E > S(0) - X e^{-rT}$ $\Rightarrow [C^E - P^E - S(0)] e^{rT} + X > 0$ — (A)

Construct a portfolio at time $t = 0$

- Buy one share for $S(0)$
- Buy one put option for P^E
- Write and sell one call option, ~~invest the sum £~~.
- Invest the sum $[C^E - P^E - S(0)]$ (is +ve and borrow if -ve) ~~and~~ in money market at interest rate r (risk free)

$$V_p(0) = 0.$$ (asset - liability = 0)

Value of the portfolio at time $t = 0$ is 0

At time $t = T$

- Close the market position, by collecting (or paying if borrowed) the sum $[C^E - P^E - S(0)] e^{rT}$
- Sell the share for X to close either by exercise of put, if $S(T) \leq X$ or settling the short position in call if $S(T) > X$

The worth at time $t = T$ will be $[C^E - P^E - S(0)] e^{rT} + X$

$$V_p(T) = [C^E - P^E - S(0)] e^{rT} + X$$

from (A) $V_p(T) > 0$

no arbitrage principle is violated

$\Rightarrow c^E - p^E \not> S(0) - Xe^{-rT}$ — (A1)

Case (2) Let $c^E - p^E < S(0) - Xe^{-rt}$

$\Rightarrow (p^E - c^E + S(0))e^{rT} - X > 0$ — (B)

At time $t=0$ Construct a portfolio

- Sell short one share for $S(0)$
- Write and sell one put option for p^E
- Buy one call option for c^E.
- Invest $[p^E + S(0) - c^E)$ (borrow if $-ve$) in risk free

wealth at $t=0$ is 0 $V_p(0) = 0$

At time $t = T$

- Close the money market position by collecting or paying the sum $(p^E - c^E + S(0))$
- Buy one share for X if $S(T) > X$ by exercising call or settling put option if $S(T) \le X$ and close the short position of the stock

The balance will be $[p^E - c^E + S(0)]e^{rT} - X$

$$V_p(T) = [p^E - c^E + S(0)]e^{rT} - X$$

by (B) $V_p(T) > 0$

no arbitrage principle is violated

$\Rightarrow c^E - p^E \not< S(0) - Xe^{-rT}$ — (B1)

By (A1) and (B1) $c^E - p^E = S(0) - Xe^{-rT}$

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Bounds on ~~C~~ the option price

$C^E \le S(0)$ —— ① upper bound.   money market = same as risk free

X if $C^E \geq S(0)$ $-S(0)$

X $\therefore C^E - S(0) \geq 0$

X initially at time $t=0$ C^E $(C^E - S(0))e^{rT}$

X initial value of portfolio is 0

→ ~~Let~~ If possible let $C^E \geq S(0)$

Construct a portfolio at $t=0$ with

(1) Write and sell ~~an~~ the call option for a strike price X ~~and~~ ~~mo~~ at maturity at T

(2) ~~Buy~~ Buy a stock for S(0)

(3) Invest $C^E - S(0)$ (if +ve) in the money market (risk-free)

initial wealth = 0

At exercise time T, ~~...~~

(1) Receive the risk free money i.e $[C^E - S(0)]e^{rT}$

(2) Settle the option position by collecting minimum of S(T), X

if $S(T) > X$ ⟹ we will get X

if $S(T) < X$ ⟹ we will get S(T)

~~(=)~~ Total profit will be. $[C^E - S(0)]e^{rT} + \min(S(T), X)$

⟹ $C^E \ngtr S(0)$

∴ ~~to~~ no arbitrage principle is violated

⟹ $C^E < S(0)$

•

$C^E - P^E = S(0) - X e^{-rT}$ (already proved.)

⟹ $C^E > S(0) - X e^{-rT}$ (lower bound)

from this $C^E - S(0) = P^E - X e^{-rT}$ ∴ $C^E - S(0) < 0$

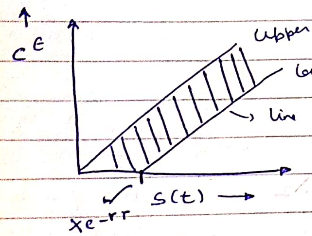
⟹ $P^E - X e^{-rT} < 0$

2) $p^E < Xe^{-rT}$ } upper bound

$-S(0) + ~~Xe^{-rT}~~ = ~~p^E/c^E~~$

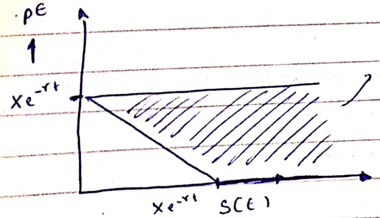
$~~p^E~~ X.$

$-S(0) + Xe^{-rT} \leq p^E$ } lower bound



The point of intersection is the place where call option should take place.

$c^E(t) < S(t)$ (the value of the ~~stock~~ call option at any time should be less than the cost of the stock at that price.



bounding region.

~~max [0,~~

$\boxed{\max\ [0, S(0) - Xe^{-rT}] \leq c^E < S(0)}$

$\boxed{\max\ (0, -S(0) + Xe^{-rT} \leq p^E < Xe^{-rT}}$

these are the bounds

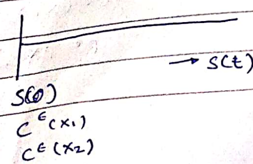
Variables determining the option prices

~~Consider options on the same underlying asset and with the same exercise time T but with~~

Dependency on the strike price

Consider options on the same underlying asset and with the same exercise time T but varying strike prices, same rate of interest, same time of maturity. and the underlying asset price $S(0)$ will be kept fixed ~~for time~~

two strike prices X_1 and X_2 $(X_1 < X_2)$



~~the~~ $c^E(X_1) > c^E(X_2)$

∞ chances of X_1 being exercised is high.

~~loss~~ and if call option is exercised, loss incurred by the writer for case X_1 will be more.

in case of put option $p^E(X_1) < p^E(X_2)$

NOTE ⇒ This means that $c^E(X)$ is a strictly decreasing function and $p^E(X)$ is increasing function

• Let $c^E(X_1) \leq c^E(X_2)$

$c^E(X_2) - c^E(X_1) \geq 0$

At time $t=0$

write and sell a call option for price X_2, receive $c^E(X_2)$

and buy a call option of price X_1 → pay $c^E(X_1)$ } time of maturity should be same

$[c^E(X_2) - c^E(X_1)]$ invest in risk free

initial wealth $=0$ } of the portfolio

At time $t=T$ settle the call options.

• if $S(T) > X_2 > X_1$ both of the call options will be exercised.

payoff in case of X_2 ⇒ $X_2 - S(T)$

in case of X_1 ⇒ $S(T) - \cancel{S(T)}\ X_1$

Total payoff $= X_2 - S(T) + S(T) - X_1$

$= X_2 - X_1$

$X_2 > 1$

⇒ payoff >0

+ risk free value $[c^E(X_2) - c^E(X_1)]e^{RT}$ will also be receive

⇒ no arbitrage principle will be violated. ⇒ not possible.

If $X_1 < X_2$

$$c^E(X_1) - c^E(X_2) < e^{-rT}(X_2 - X_1)$$

$$p^E(X_2) - p^E(X_1) < e^{-rT}(X_2 - X_1)$$

Proof Using the put call parity formula.

$$C^E - P^E = S(0) - Xe^{-rT}$$

$X_1 \Rightarrow \quad C^E(X_1) - P^E(X_1) = S(0) - X_1 e^{-rT}$

$X_2 \Rightarrow \quad C^E(X_2) - P^E(X_2) = S(0) - X_2 e^{-rT}$

Subtract $\quad [C^E(X_1) - C^E(X_2)] + [P^E(X_2) - P^E(X_1)]$

$$= (X_2 - X_1) e^{-rT}$$

both the brackets of LHS is +ve

as $C^E(X_1) > C^E(X_2)$ and $P^E(X_1) < P^E(X_2)$

$(X_2 - X_1)$ is also +ve.

sum of 2 +ve quantities = 3rd quantity

$a + b = c \qquad a, b, c > 0$

$\Rightarrow a \leq c \quad , b \leq c$

$\Rightarrow \quad C^E(X_1) - C^E(X_2) \leq (X_2 - X_1) e^{-rT}$

$P^E(X_2) - P^E(X_1) < (X_2 - X_1) e^{-rT}$

Dependency on the underlying asset price $S(0)$

$p^e(S_1) \geq p^e(S_2)$ → decreasing function

Proof let $c^e(S_1) \geq c^e(S_2)$

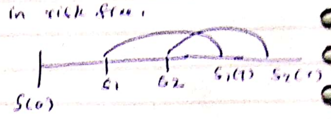
$\Rightarrow c^e(S_1) - c^e(S_2) \geq 0$ for $S_1 < S_2$

time $t = 0$

Write and sell a call option for S_1

Buy a call option for S_2

$c^e(S_1) - c^e(S_2)$ & invest in risk free.



time $t = T$ (maturing)

If $X > S_2 > S_1$

both call will be exercised.

payoff for S_1 $X - x_1 S(T)$

payoff for S_2 $x_2 S(T) - X$

total payoff $= (x_2 - x_1) S(T)$

$x_2 > x_1$

$\Rightarrow$ payoff > 0

& risk free grown value is also received.

$\Rightarrow$ no arbitrage principle is violated.

If $S_1(T) < X < S_2(T)$

S_1 will not be exercised

S_2 will be exercised. payoff $= x_2 S(T) - X$

> 0

+ grown value.

$\Rightarrow$ no arbitrage principle is violated.

If $X > S_1(T)$ and $X > S_2(T)$

no call option will be received.

but risk free grown value will be received

$\Rightarrow$ no arbitrage principle is violated.

- If $A(0) = S(0)$, show that $S^d < A(1) < S^u$
(in one period binomial)

Case 1) $A(1) \leq S^d$
Case 2) $A(1) > S^u$
} no arbitrage principle

DYNAMICS OF STOCK PRICE

$$S(0) \begin{cases} (1+u)\,S(0) & \text{with probability } p \\ (1+d)\,S(0) & \text{with probability } 1-p \end{cases}$$

Wer value of p lies between 0 and 1 (excluding 0 and 1)

$$-1 < d < u$$

$$E(R(1)) = p \cdot u + (1-p) d$$

Now if for some probability p^*, the average return on a risky asset is equal to risk free return, then such a probability is known as risk neutral probability.

Return under p^* =

$$\Rightarrow p^* u + (1-p^*) d = r$$

$$p^* \times (u-d) = r-d$$

$$\left. p^* = \frac{r-d}{u-d} \right\}$$ risk neutral probability

in terms of return u and d.

in terms of growth factor

$$p^* = \frac{R-d}{u-d}$$

S_1 is a random variable.

~~$E(S_1)$~~ $E(S(1)) = p \cdot (1+u) S(0) + (1-p)(1+d) S(0)$

in terms of growth factor

$$= S(0) [\cancel{p} + pu + 1 - \cancel{p} + d - dp]$$

$$= S(0) [1 + pu + (1-p) d]$$

w.r.t. to return

$$= S(0) \{1 + E(R(1))\}$$

Since the step returns $k(1), k(2)$ are independent, so are the random variables $1+k(1)$, $1+k(2)$ and so on.

$$E(S(n)) = E\left[S(0)\,(1+k(1))\,(1+k(2)) \cdots (1+k(n))\right]$$
$$= S(0)\,E\left[(1+k(1))\,(1+k(2)) \cdots (1+k(n))\right]$$

$$E(XY) = E(X)\,E(Y)$$

$$E(S(n)) = S(0)\,E[1+k(1)]\,E[1+k(2)] \cdots E[1+k(n)]$$
$$= S(0)\,[1+E(k(1))]\,[1+E(k(2))] \cdots [1+E(k(n))]$$

k_n are identically and independently distributed random variables.

$$\Rightarrow E(k(1)) = E(k(2)) = \cdots = E(k(n))$$

$$\Rightarrow E[S(n)] = S(0)\,[1+E(k_1)]\,[1+E(k_1)] \cdots [1+E(k(1))]$$
$$E[S(n)] = S(0)\,[1+E(k_1)]^n$$

In case of probability being risk neutral probability p^*

$$E^*[S(n)] = S(0)\,[1+E^*(k_1)]^n$$
$$E^*(k_1) = r$$

$$E^*[S(n)] = S(0)\,[1+r]^n$$

The risk neutral expected price expectation.

<u>Problem</u> Consider a two step binomial tree model such that $S(0) = \$100$, $u = 0.2$, $d = -0.1$, $r = 0.1$. Find the risk neutral expected price after second step i.e. $E^*[S(2)]$

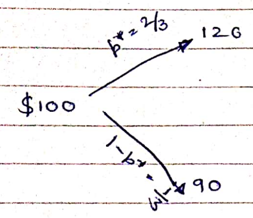
return

$$E^*[S(n)] = S(0)\,[1+r]^n$$
$$E^*[S(2)] = 100\,[1+0.1]^2 = 100 \times 1.1^2 = \$121$$

$\Rightarrow$ growth factor $\quad 1+u = 1.2$

$\quad\quad\quad\quad\quad\quad\quad\quad 1+d = 0.9$

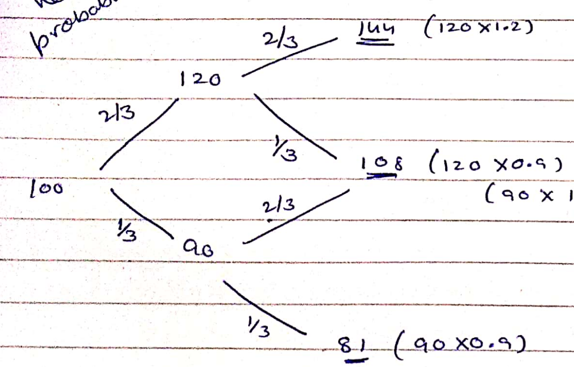
~~S(0)~~



Period 1

$$p^* = \frac{r-d}{u-d} = \frac{0.1-(-0.1)}{0.2-(-0.1)} = \frac{0.2}{0.3} = \frac{2}{3}$$

(risk neutral probability)



$E^*[S(2)] = ?$

Case 1) (price goes up)

$$E^*[S(2)] = \frac{2}{\cancel{3}} \times \overset{48}{\cancel{144}} + \frac{1}{\cancel{3}} \times \overset{36}{\cancel{108}}$$

$$= \$132$$

Case 2) price goes down

$$E^*[S(2)] = \frac{2}{\cancel{3}} \times \overset{36}{\cancel{108}} + \frac{1}{\cancel{3}} \times \cancel{81}\ 27$$

$$= 72 + \overset{27}{\cancel{9}} = \cancel{81}\ \$99$$

$\Rightarrow$ final $E^*[S(2)] = \overset{44}{\cancel{132}} \times \frac{2}{\cancel{3}_1} + \overset{33}{\cancel{99}} \times \frac{1}{\cancel{3}} = 88 + 33$

$= \$121$

36
×2
72

88
33

We introduce a function with stock S as the underlying asset

$f \rightarrow$ payoff

$D(T)$ is a random variable, then $D(T) = f(S(T))$

$D(T) = S(T) - X$ X is constant

In particular for call option $f(S) = (S-X)^+$

and for put option $f(S) = (X-S)^+$

Now the stock with current price $S(0)$ will be $S(1)$ after time 1.

and may take 2 values

$$S(1) \begin{cases} (1+u)\, S(0) = S^u & \text{with prob } p \\ (1+d)\, S(0) = S^d & \text{with prob } 1-p \end{cases}$$

To replicate a general derivative security, with payoff f, we have

solve the linear equations

$$x\, S(1) + y\,(1+r) = f(S^u)$$

$$x\, S(1) + y\,(1+r) = f(S^d)$$

$$\Rightarrow \quad x S^u + y\,(1+r) = f(S^u) \quad \text{---}(1)$$

$$x S^d + y\,(1+r) = f(S^d) \quad \text{---}(2)$$

$$\cancel{x(1)}\ x(1) = \frac{f(S^u) - f(S^d)}{S^u - S^d}$$

~~This~~ which is the replicating position in the stock is called the delta of the option.

$$y(1) = -\frac{(1+d)\, f(S^u) - (1+u)\, f(S^d)}{(u-d)\,(1+r)}$$

multiply ~~eq~~ eq (1) by $(1+d)$ and eq (2) by $(1+u)$

$\therefore$ Initial values of the replicating portfolio $= x\, S(0) + y(1)$

$$D(0) = x\, S(0) + y(1)$$

$$D(0) = \frac{f(S^u) - f(S^d)}{u-d} - \frac{(1+d)\, f(S^u) - (1+u)\, f(S^d)}{(u-d)\,(1+r)}$$

$S(0)$ is the payoff at time 0, which is the price of the option.

Payoff at time 0 = option price

Theorem The expectation of discounted payoff computed with respect to risk neutral probability is equal to the present value of european derivative security. (~~…~~ or Contingent Claim)

derivative, because they are derived from assets. stocks

$$D(0) = E^*\left[\frac{1}{1+r} f(S(1))\right]$$

Proof $D(0) = \dfrac{f(S^u) - f(S^d)}{u-d} + \dfrac{(1+u) f(S^d) - (1+d) f(S^u)}{(u-d)(1+r)}$

$$D(0) = \frac{1}{1+r}\left[\frac{(1+r)[f(S^u) - f(S^d)]}{u-d} + \frac{(1+u) f(S^d) - (1+d) f(S^u)}{u-d}\right]$$

$$= \frac{1}{1+r}\left[\frac{(r-d)}{u-d} f(S^u) + \frac{(u-r)}{(u-d)} f(S^d)\right]$$

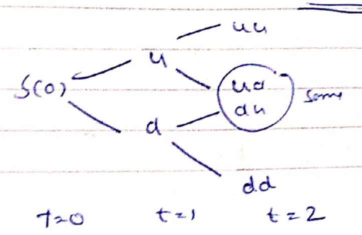
$$p^* = \left(\frac{r-d}{u-d}\right) \qquad 1-p^* = \frac{u-\cancel{d}-r+\cancel{d}}{u-d} = \frac{u-r}{u-d}$$

$$\Rightarrow D(0) = \frac{1}{1+r}\left[p^* f(S^u) + (1-p^*) f(S^d)\right]$$

$$= \frac{1}{1+r}\left[E^* f(S(1))\right]$$

$$= E^*\left[\frac{1}{1+r}(f(S(1)))\right]$$

Now we consider a two step binomial model



Two possibilities:

$S^u = (1+u)\, S(0)$

$S^d = (1+d)\, S(0)$

$$S^{uu} = (1+u)(1+u)\,S(0)$$
$$S^{ud} = S^{du} = (1+u)(1+d)\,S(0)$$
$$S^{dd} = (1+d)(1+d)\,S(0)$$

For each sub-tree, we can use one-step replication ~~pro~~ procedure.

At time $t=2$, derivative security will be represented by it's payoff $D(2) = f[S(2)]$

$f[S(2)]$ has 3 possibilities

The derivative security price $D(1)$ has two values

$$\frac{1}{1+r}\left[p^* f(S^{uu}) + (1-p^*) f(S^{ud})\right]$$

when we discount S^{uu}

and $$\frac{1}{1+r}\left[p^* f(S^{ud}) + (1-p^*) f(S^{dd})\right]$$

This gives $$D(1) = \frac{1}{1+r}\left[p^* f(S(1)(1+u)) + (1-p^*) f(S(1)(1+d))\right]$$

now $S(1)$ will be splitted into two

which we can write as $g(S(1))$ when ~~$g(x)$~~.

$$g(x) = \frac{1}{1+r}\left[p^* f(x(1+u)) + (1-p^*) f(x(1+d))\right]$$

As a result, $D(1)$ can be regarded as a derivative security expiring at time 1 with payoff g. This means that one step procedure can be applied once again to the single sub tree at the root of the tree.

$$D(0) = \frac{1}{1+r} E^*[g(S(1))]$$
$$= \frac{1}{1+r}\left[p^* g[S(0)(1+u)] + (1-p^*) g[S(0)(1+d)]\right]$$

$$= \frac{1}{(1+r)^2}\Big[p^* \big[p^* f(S(0)(1+u)^2) + (1-p^*) f(S(0)(1+u)(1+d))\big]$$
$$+ (1-p^*)\big[p^* f(S(0)(1+d)(1+u)) + (1-p^*) f(S(0)(1+u)^2)\big]\Big]$$

$$= \frac{1}{(1+r)^2}\Big[p^* f(S^{uu}) + 2p^*(1-p^*) f(S^{ud}) + (1-p^*)^2 f(S^{dd})\Big]$$

** For 2 period binomial distribution calculation, do not do calculation at time $t=1$, do calculations at time $t=2$ because actual payoff will not be equal to the discounted payoff

→ The expectation of the discounted payoff is equal to the present value of derivative security

$\cancel{D(1) \neq \frac{1}{(1+r)^2}}$ $\quad D(0) = E^*\left[\frac{1}{(1+r)^2} f(S(2))\right]$

<u>Multiperiod N-Step ~~Binomial~~ Binomial Model</u>

$$D(0) = \frac{1}{(1+r)^N} \sum_{k=1}^{N} \left\{ {}^N C_k \, p^{*k} (1-p^*)^{N-k} f\left(S(0)(1+u)^k (1+d)^{N-k}\right)\right\}$$

The value of a European derivative security with payoff $f(S(n))$ in n-step binomial model is the expectation of discounted payoff under the risk neutral probability.

$$D(0) = E^*\left[\frac{1}{(1+r)^n} f(S(N))\right]$$

$$D(0) = \frac{1}{(1+0.1)^2}\left[\left(\frac{2}{3}\right)^2 \times 52.8 + 2\cdot\frac{2}{3}\times\frac{1}{3}\times 9.6 + \left(\frac{1}{3}\right)^2 \times 0\right]$$

$$= \$\ 22.92$$

Replicating strategy. ⇒ If the bond value is 1 at time t = 0 ⇒ it's value will be 1+r after 1 time period.

Let (x, y) be the portfolio

$$x\,S(1) + y\,(1+r) = f(S(1))$$

f breaks into 2

$$x\,S^u + y\,(1+r) = f(S^u)$$
$$x\,S^d + y\,(1+r) = f(S^d)$$

$S^u = 120 \times 1.2 = \$\ 144$

$S^d = 120 \times 0.9 = \$\ 108$

$$x \cdot 144 + y\cdot(1.1) = f(S^u)$$
$$x \cdot 108 + y\cdot(1.1) = f(S^d)$$

do not calculate $f(S^u)$ and $f(S^d)$ directly, discount the payoff at time T = 2.

~~f(S^u)~~ =

$$f(S(1)) \begin{cases} f(S^u) = \frac{1}{1+0.1}\left[\frac{2}{3}\times 52.8 + \frac{1}{3}\times 9.6\right] \\ f(S^d) = \frac{1}{1+0.1}\left[\frac{2}{3}\times 9.6 + \frac{1}{3}\times 0\right] \end{cases}$$

$$f(S^u) = \frac{384}{11} \qquad f(S^d) = \frac{64}{11}$$

now equations are

$$144x + 1.1y = \frac{384}{11}$$
$$108x + 1.1y = \frac{64}{11}$$

Solve these equation

$$(144-108)x = \frac{(384-64)}{11}$$

$$\cancel{36}^{\,99}\,x = \frac{\cancel{320}^{\,80}}{11} \qquad x = \frac{80}{99} = 0.8081$$

$\overset{12}{\cancel{144}}^{\,14}$
108
3.

384
64
320

$$1.1\,y + \cancel{108}^{\cancel{36}\;12} \times \left(\cancel{}\ \frac{80}{\cancel{99}\,\cancel{33}\,11}\right) = \frac{64}{11}$$

$$\Rightarrow y = -74.05$$

The portfolio ~~will~~ is to be constructed with 0.8081 units of the underlined stock by borrowing 74.05 unit of the bond.

CRR Model Cox - Ross - Rubinston Model

The assumptions on the financial market mode for single period binomial model are carried forward as following ⇒

1) the underlying stock on which the option is written is perfectly divisible

2) the underlying stock pays no dividend.

3) There is no transaction cost in buying or selling the option and no taxes.

4) Short selling is allowed

5) The risk free interest rate is known and constant till time of expiration of option.

6) No arbitage principle holds

We divide the time ~~origin~~ horizon 0 to T $[0,T]$ into n sub intervals and assume that in each subinterval the stock price changes like one period binomial case. Thus in each case, stock price either moves up by a constant factor u or moves down by constant factor d with the probability.

We define
$$\xi_k = \begin{cases} u & \text{prob } p \\ d & \text{prob } 1-p \end{cases} \qquad k = 1, 2, \ldots n \qquad 0 < p < 1$$

$$\delta t = \frac{T}{n}$$

now ξ_k is a Bernoulli random variable

Matching of CRR model with multiperiod Binomial Model

$$\mu \delta t = E(\ln E_k) \qquad E_k = \begin{cases} u & \text{prob } p \\ d & \text{prob } 1-p \end{cases} \text{ (growth factor)}$$

$$\sigma^2 \delta t = Var(\ln E_k)$$

$$S(T) = S(0)\exp(\mu T + \sigma\sqrt{T}\, Y)$$

To determine three parameters u, d and p, using eq (8)
Assuming that σ is known from the past data of the stock since we need to solve, three variables with two equations uniquely.

Let us take $U = \ln u$ and $D = \ln d$ and assume $D = -U$

$$\Rightarrow d = \frac{1}{u}$$

That means, we are matching the CRR model to a particular multiperiod binomial model where $d = 1/u$.

Under this assumption eq (4) becomes $\Rightarrow$

$$(2p-1)U = \mu \Delta t \quad (10A)$$

$$4p(1-p)U^2 = \sigma^2 \Delta t \quad (10B)$$

Squaring (10A) and adding (10B)

$$(2p-1)^2 U^2 + 4p(1-p)U^2 = \mu^2 \Delta t^2 + \sigma^2 \Delta t$$

$$U^2 [\cancel{4p^2} + 1 - \cancel{4p} + \cancel{4p} - \cancel{4p^2}] = \mu^2 \Delta t^2 + \sigma^2 \Delta t$$

$$U^2 = \mu^2 \Delta t^2 + \sigma^2 \Delta t$$

$$\left.\begin{aligned} U &= \sqrt{\mu^2 \Delta t^2 + \sigma^2 \Delta t} \\ D &= -U \end{aligned}\right\} \quad (11)$$

From (10A) and (11)

$$p = \frac{1}{2} + \frac{1}{2}\frac{\mu \Delta t}{\sqrt{\mu^2 \Delta t^2 + \sigma^2 \Delta t}} \quad \text{— (11A)}$$

Let us take n sufficiently large. $\Delta t = \frac{T}{n}$

as n increases Δt increases as $n \to \infty$ $\Delta t \to 0$

~~[illegible]~~ and so Δt^2 can be neglected. (as n is sufficiently large)

$$p = \frac{1}{2} + \frac{1}{2}\frac{\mu \Delta t}{\sqrt{\sigma^2 \Delta t}} = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\Delta t}$$

$$U = \ln u = \sqrt{\sigma^2 \Delta t} = \sigma\sqrt{\Delta t} \qquad (12)$$

$$u = e^{\sigma\sqrt{\Delta t}} \Rightarrow d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$$

- A non dividend paying stock is currently selling at ₹100, annual volatility 20%. Assuming continuously compounded risk free interest rate 5%, use 2 period CRR model to find the price of the european call option on this stock with strike price ₹80 and time to expiration 4 years.

$S(0) = ₹100/-$ $\quad \sigma = 20\% = 0.2$ $\quad r = 5\% = 0.05$

$X = ₹80/-$ $\quad T = 4$ years $\quad n = 2$

$$D(0) = \frac{1}{(1+r)^2}\left[p^{*2} f(suu) + 2p^*(1-p^*) f(sud) + (1-p^*)^2 f(sdd)\right]$$

in case of continuous compounding $(1+r) \rightarrow e^{r\Delta t}$

$\Delta t = \frac{4}{2} = 2$ $\quad R = e^{r\Delta t} = e^{0.05\times 2} = e^{+0.1}$

discount factor $= e^{-0.1}$

$$D(0) = \frac{1}{(e^{0.1})^2} E^*\left[f(s(2))\right]$$

$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\times\sqrt{2}} = 1.3269$

$d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.2\times\sqrt{2}} = \frac{1}{u} = 0.7536$

these are the value of growth factor.

$$p^* = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05\times 2} - 0.7536}{1.3769 - 0.7536} = 0.6132$$

$S^{uu} = S(0)\,u^2 = 100 \times (1.3269)^2$

$= ₹176.0664$

$S^{ud} = S(0)\,ud = 100 \times 1.3269 \times 0.7536$ (as $u \times d = 1$)

$= ₹100$

$S^{dd} = S(0)\,d^2 = 100 \times (0.7536)^2$

$= ₹56.7913$

payoff · $f(S^{uu}) = 176.0664 - 80$

$= ₹96.0664$

$f(S^{ud}) = 100 - 80$

$= ₹20$

Call option will be exercised.

$f(S^{dd}) = 0$ (as call option will not be exercised)

$$D(0) = C(0) = \left[e^{-0.05 \times 2}\right]^2 \left[p^{*2} f(S^{uu}) + 2p^*(1-p^*) f(S^{ud}) + (1-p^*) f(S^{dd})\right]$$

$$= \left(e^{-0.05 \times 2}\right)^2 \left[(0.6132)^2 \times 96.0664 + 2 \times (0.6132)(0.3868)\,20 + 0\right]$$

$= ₹37.34$

if σ is given only, u and d are absent ∴ we have to use CRR model find u and d from σ

Black Scholes Formula

$u = e^{\sigma\sqrt{\Delta t}}$ $\quad d = \frac{1}{u}$

$S(T) = S(0)\exp\left[\mu T + \sigma\sqrt{\Delta T}\,Y\right]$

Now, we define a counter, an uptick and downtick movements of stock price at time $k\Delta t$ $(k = 1, 2, \ldots, n)$ as a Bernoulli Random variable.

$$Y_k = \begin{cases} 1 & \text{with prob } p \quad \text{if stock goes up} \\ 0 & \text{with prob } 1-p \quad \text{if stock goes down} \end{cases}$$

$$S(T) = S(n\Delta T) = S(0)\, u^{\left(\sum_{k=1}^{n} y_k\right)} d^{\left(n - \sum_{k=1}^{n} y_k\right)}$$

$$S(T) = S(0) \left(\frac{u}{d}\right)^{\sum_{k=1}^{n} y_k} d^n$$

$$\frac{S(T)}{S(0)} = d^{T/\Delta t} \left(\frac{u}{d}\right)^{\sum_{k=1}^{n} y_k} \quad \text{— (13)}$$

$Y = \sum_{k=1}^{n} y_k$ is a simple random walk

$$E(Y) = np = \frac{pT}{\Delta t}$$

$$Var(Y) = np(1-p) = \frac{p(1-p)T}{\Delta t}$$

From (13), we have $\ln \frac{S(T)}{S(0)} = -\frac{\sigma T}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t} \sum_{k=1}^{n} y_k$ — (14)

Hence $E\left[\ln \frac{S(T)}{S(0)}\right] = E\left[-\frac{\sigma T}{\sqrt{\Delta T}} + 2\sigma\sqrt{\Delta T} \sum_{k=1}^{n} Y_k\right]$

$$= -\frac{\sigma T}{\sqrt{\Delta T}} + 2\sigma \cancel{\sqrt{\Delta T}} \frac{T}{\cancel{\sqrt{\Delta T}}} \frac{p}{\sqrt{\Delta t}}$$

$$= \frac{T}{\sqrt{\Delta T}} \sigma \left[-1 + 2p\right]$$

$$= (2p-1) \frac{\sigma T}{\sqrt{\Delta T}}$$

using (12) $E\left\{\ln \frac{S(T)}{S(0)}\right] = (2p-1) \frac{\sigma T}{\sqrt{\Delta T}}$

$$Var\left[\ln \frac{S(T)}{S(0)}\right] = Var\left[-\frac{\sigma T}{\sqrt{\Delta T}} + 2\sigma\sqrt{\Delta T} \sum_{k=1}^{n} y_k\right]$$

$$= 4\sigma^2 \Delta t \sum_{k=1}^{n} Var\, Y_k$$

$$= 4\sigma^2 \cancel{\Delta t} \cdot p(1-p) \frac{T}{\cancel{\Delta t}}$$

$$= 4\sigma^2 p(1-p) T$$

as $n \to \infty$ $\quad p \to 1/2$

$$\Rightarrow \operatorname{Var}\left[\ln\left(\frac{S(T)}{S(0)}\right)\right] \to \cancel{4}\sigma^2 \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} T$$

$$\to \sigma^2 T$$

By application of central limit theorem, Y_k follows normal distribution when time step approaches to zero.

$$\ln \frac{S(T)}{S(0)} \sim N(\mu T, \sigma^2 T)$$

Since $u = e^{\sigma\sqrt{\Delta t}} = e^{\sigma\sqrt{\frac{T}{n}}} = 1 + \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2}{2}\frac{T}{n}$ — (15)

neglecting $\Delta T^{3/2}$ and higher ~~terms~~ powers

$d = e^{-\sigma\sqrt{\Delta T}} = e^{-\sigma\sqrt{\frac{T}{n}}} = 1 - \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2}{2}\frac{T}{n}$ — (16)

So the risk neutral probability measure (RNPM), p^*

$$p^* = \frac{R-d}{u-d}$$

$$R = e^{r\Delta t} \approx 1 + r\Delta t = 1 + \frac{rT}{n}$$

$$p^* = \frac{\cancel{1} + \frac{rT}{n} - \left(\cancel{1} - \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2}{2}\frac{T}{n}\right)}{\left[\cancel{1} + \sigma\sqrt{\frac{T}{n}} + \cancel{\frac{\sigma^2}{2}\frac{T}{n}}\right] - \left[\cancel{1} - \sigma\sqrt{\frac{T}{n}} + \cancel{\frac{\sigma^2}{2}\frac{T}{n}}\right]}$$

$$p^* = \frac{1}{2} + \frac{(2r - \sigma^2)\sqrt{\frac{T}{n}}}{4\sigma}$$ — (17)

The European call option price for n period binomial model is described as

$$D(0) = \frac{1}{\left(1+\frac{rT}{n}\right)^n} E^*[S(N)]$$

$$= \frac{1}{\left(1+\frac{rT}{n}\right)^n} E^*[S(T)-X]^+$$

$$= \frac{1}{\left(1+\frac{rT}{n}\right)^n} E^*\left[S(0)\left(\frac{u}{d}\right)^{\Sigma y_k} d^n - X\right]^+$$

using (15) and (16) for u and d =>

$$C(0) = D(0) = \left[1+\frac{rT}{n}\right]^{-n} E^*\left[S(0)e^{W} - X\right]^+$$

where $W = 2\sigma\sqrt{\frac{T}{n}}\,Y - \sigma\sqrt{nT}$ — (18)

↳ random quantity.

$$E(W) = 2\sigma\sqrt{\frac{T}{n}}\,E(Y) - \sigma\sqrt{nT}$$

$$= 2\sigma\sqrt{\frac{T}{n}}\,p \times \cancel{\frac{T}{\Delta t}}^{\,n} - \sigma\sqrt{nT}$$

$$= 2\sigma\sqrt{\frac{T}{n}}\,np - \sigma\sqrt{nT}$$

$$= \sigma\sqrt{nT}\,[2p-1]$$

$$= \sigma\,[2p-1]\sqrt{nT}$$

$$= \mu\sqrt{\Delta T}\sqrt{nT} = \mu T$$

$$Var(W) = \cancel{4}\,4\sigma^2\frac{T}{\cancel{n}}\,\cancel{n}\,p(1-p)$$

$$= 4\sigma^2 T\,p(1-p)$$

as $n \to \infty$ $\quad p \to 1/2$

$Var(\omega) \to \cancel{4}\,\sigma^2 \times T \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}}$

$Var(\omega) \to \sigma^2 T$

using $E^*(\omega) = 2\sigma\sqrt{\frac{T}{n}}\,E^*(Y) - \sigma\sqrt{nT}$

$= 2\sigma\sqrt{\frac{T}{n}}\, n p^* - \sigma\sqrt{nT}$ using (17)

$= 2\sigma\sqrt{\frac{T}{n}} \times n\left[\frac{1}{2} + \frac{(2r-\sigma^2)}{4\sigma}\sqrt{\frac{T}{n}}\right] - \sigma\sqrt{nT}$

$= \sigma\sqrt{nT}\left[\left\{\cancel{2}\times\frac{1}{\cancel{2}} + \cancel{2}\times\frac{(2r-\sigma^2)}{2\cancel{4}\sigma}\sqrt{\frac{T}{n}}\right\} - 1\right]$

$= \sigma\sqrt{nT}$

$= \left(r - \frac{\sigma^2}{2}\right)T$

$Var^*(\omega) = 4\sigma^2\frac{T}{n}\,Var^*(Y)$

$= 4\sigma^2\frac{T}{n}\,p^*(1-p^*)$

$\to \sigma^2 T \quad , \infty\; n \to \infty,\; p^* \to 1/2$

$\to \sigma^2 T$

using above relations of (18) gives

$C(0) = D(0) = e^{-rT}\int_{+\infty}^{-\infty} [S(T) - x]^+$

$\times \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{S(t) - \mu T}{\sigma\sqrt{T}}\right]^2} d(S(t))$

$C(0) = e^{-rT}\int_{-\infty}^{\infty} [S(0)e^{\omega} - x]^T \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{\omega - \mu T}{\sigma\sqrt{T}}\right]^2} d\omega$

$\left(r - \frac{\sigma^2}{2}\right)$

$\left[S(0)\, e^{w} - X\right]^{+}$ will be non zero if and only if $S(0)\, e^{w} > X$

$$\Rightarrow w > \log\left[\frac{X}{S(0)}\right] = w_1 \text{ (say)}$$

$$C(0) = \left(1 + \frac{rt}{n}\right)^{-n} E^{*}\left[S(0)\, e^{w} - K\right]^{+}$$

$$= e^{-rT} \int_{-\infty}^{\infty} \left[S(0)\, e^{w} - K\right]^{+} \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{w - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right]^2}$$

$= \left[S(0)\, e^{w} - K\right]^{+}$ will be non-zero iff

$S(0)\, e^{w} > K \Rightarrow w > \ln\left[\frac{K}{S(0)}\right] = w_1$ (say)

$$= e^{-rT} \int_{w>w_1}^{\infty} \left[S(0)\, e^{w} - K\right] \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{w - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right]^2} dw$$

Let $y = \dfrac{w - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

$dw = \sigma\sqrt{T}\, dy$

$\Rightarrow w = y\sigma\sqrt{T} + \left(r - \frac{\sigma^2}{2}\right)T$

For $w > w_1 \Rightarrow y > y_1$ (say) $= \dfrac{w_1 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

$$C(0) = e^{-rT} \int_{y>y_1}^{\infty} \left[S(0)\, e^{y\sigma\sqrt{T} + \left(r - \frac{\sigma^2}{2}\right)T} - K\right] \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dw$$

$$= e^{-rT} \int_{y>y_1}^{\infty} S(0)\, e^{y\sigma\sqrt{T} + \left(r - \frac{\sigma^2}{2}\right)T} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy$$

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$$= I - K e^{-rT} \phi(-y_1) \quad — (19)$$

where $\phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-x^2/2}\, dx$

is the distribution function (CDF) of a standard ~~[illegible]~~ normal variable.

Consider $I = \dfrac{e^{-\cancel{r}T}}{\sqrt{2\pi}} S(0) \displaystyle\int_{y>y_1}^{\infty} e^{y\sigma\sqrt{T} + (\cancel{r} - \frac{\sigma^2}{2})T}\, e^{-y^2/2}\, dy$

$$= \frac{S(0)}{\sqrt{2\pi}} \int_{y>y_1}^{\infty} e^{-\frac{1}{2}[y - \sigma\sqrt{T}]^2}\, dy$$

Let $y - \sigma\sqrt{T} = s \Rightarrow y = s + \sigma\sqrt{T}$

for $y > y_1 \Rightarrow s > s_1$ (say)

$s_1 = y_1 - \sigma\sqrt{T}$

$\Rightarrow dy = ds$

substituting

$$I = \frac{S(0)}{\sqrt{2\pi}} \int_{s>s_1}^{\infty} e^{-\frac{1}{2}s^2}\, ds$$

$$= S(0)\, \phi(-s_1)$$

$$= S(0)\, \phi(\sigma\sqrt{T} - y_1)$$

$$y_1 = \frac{w_1 - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \qquad w_1 = \ln\frac{K}{S(0)}$$

$$-s_1 = \sigma\sqrt{T} - y_1$$

$$= \sigma\sqrt{T} - \left[\frac{\ln\frac{K}{S(0)} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right]$$

$$= \frac{\ln \frac{S(0)}{K} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

$$I = S(0) \; \phi(d_1)$$

$$\text{where} \quad d_1 = \frac{\ln \frac{S(0)}{K} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \quad \text{—} \; (20)$$

$$C(0) = S(0)\, \phi(d_1) - K e^{-rT} \phi(d_2)$$

where $d_2 = -y_1$

$= -(s_1 + \sigma\sqrt{T})$

$d_2 = d_1 - \sigma\sqrt{T}$ — (21) Black Scholes formula for European call option price.

d_1 is ~~the~~ given by (20)

Objection of Black Scholes formula ⇒ σ is constant here, which is not the case.

In case the European call option price is to be computed at any time t lying between 0 and T, then Black Scholes formula will become.

$$d_1 \; \cancel{c(t)} = \frac{\ln \frac{S(t)}{K} + \left(r + \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}} \quad \text{—} \; (22)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$C(t) = S(t)\, \phi(d_1) - K e^{-r(T-t)} \phi(d_2)$$

where $S(t)$ is the price of stock at time t.

Strike price $X = K$

Derivation of Black Scholes formula

Let a dividend be paid by the underlying stock at time $t = t$, then the stock price will be adjusted at time $t = 0$ as

$S'(0)$ = adjusted price

$S'(0) = S(0) - div \cdot e^{-rt_{div}}$

Problem : nondividend paying stock $S(0) = ₹100$

stock volatility = 24 $\Rightarrow \sigma = 0.24$ $r = 0.05$

European call option $T = 3$ months $= 1/4$ year

$K = ₹125$ price of block of 100 options in black scholes formula?

T will be substituted in terms of years

$$d_1 = \ln\frac{S(0)}{K} + \left(r + \frac{\sigma^2}{2}\right)T$$

$$C(0) = S(0)\,\phi(d_1) - Ke^{-rT}\,\phi(d_2)$$

$$= 100 \times 0.0446 - 125 \times e^{-0.05 \times 0.25} \times 0.0344$$

$$= ₹0.2134$$

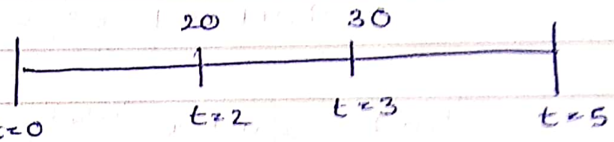
$\Rightarrow$ Cost for 100 options = ₹21.34

<u>Problem</u>: $S(0) = ₹100$ $\quad r = 0.05$ $\quad \sigma = 0.3$

$K = ₹80$ $\quad T = 5$ years

pays dividend ₹20 in 2 years, and another dividend ₹30 in 3 years

↓ adjusted stock price



$$S'(0) = S(0) - 20e^{-r2} - 30e^{-r3}$$

$$= 100 - 20e^{-0.05 \times 2} - 30e^{-0.05 \times 3} = 100 - 18.097 - 25.821$$

$$S'(0) = ₹56.08$$

$$d_1 = \frac{\ln \frac{S'(0)}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{}$$

06/02

↗ 4.5

Problem for european call and put option on a stock, having same expiry ~~with same~~ and same strike price

$S(0) = ₹ 85$ $K = ₹$ ~~125~~ 90

Continuously compounded risk free rate $r = 4\%$

Continuously compounded dividend rate $r_{div} = 2\%$

Call option has premium ₹ 9.91 (c^E) and put option has premium ₹ 12.63 ($= p^E$)

determine the expiry time.

Solution => Using put call parity formule

$$c^E - p^E = S'(0) - K e^{-rT}$$

$$9.91 - 12.63 = S'(0) - 90 \times e^{-0.05 \times T}$$

$$S'(0) = S(0)\, e^{-r_{div} T}$$

$$= 85 \times e^{-0.02 \times T}$$

$$\Rightarrow 9.91 - 12.63 = 85\, e^{-0.02 \times T} - 90\, e^{-0.05 \times T}.$$

Solving $T = 1$ year.

$$\rightarrow \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\, dx$$

$$\frac{d\phi}{dx} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\frac{\partial d_1}{\partial t} = \frac{1}{2\sigma\sqrt{T-t}} \left[\ln\frac{S(T)}{K} - r - \frac{\sigma^2}{2} \right]$$

$$\frac{\partial \phi_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}} \qquad \frac{\partial d_1}{\partial r} = \frac{\sqrt{T-t}}{\sigma}$$

$$\frac{\partial d_1}{d\sigma} = \sqrt{T-t} - \frac{d_1}{\sigma}$$

The delta of the European call option is defined as.

$$\Delta = \frac{\partial C}{\partial S} = \phi(d_1) + \left[S\, \phi'(d_1) - K e^{-r(T-t)} \phi(d_2) \right] \frac{\partial d_1}{\partial S}$$

↓ gives the no of shares of the replicating strategy

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$$= \phi(d_1) + \left\{ \frac{S e^{-d_1^2/2}}{\sqrt{2\pi}} - K e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \right] \frac{1}{S\sigma\sqrt{T-t}}$$

$$= \phi(d_1) + \frac{1}{\sqrt{2\pi}\, S\sigma\sqrt{T-t}} \left[S e^{-d_1^2/2} - \cancel{K e^{-r(T-t)}} \right.$$

$$\left. K \cdot e^{-r(T-t) - \frac{(d_1^2 + \sigma^2(T-t) - 2\sigma d_1\sqrt{T-t})}{2}} \right]$$

$$= \phi(d_1) + \frac{e^{-d_1^2/2}}{\sqrt{2\pi}\, S\sigma\sqrt{T-t}} \left[S - K e^{-(r+\frac{\sigma^2}{2})(T-t) + d_1\sigma\sqrt{T-t}} \right]$$

$$= \phi(d_1) + \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} \underbrace{\left[S - K e^{\ln \frac{S(t)}{K}} \right]}_{0}$$

$$= \phi(d_1)$$

$$\cancel{= \Delta} \qquad \Rightarrow \quad \Delta = \frac{\partial C}{\partial S} = \phi(d_1)$$

Here from what we see $0 < \Delta < 1$

Result $\Rightarrow$ Using call put parity, we have for european put option

$$\Delta = \phi(d_1) - 1$$

Problem Consider a 1 year European call option not paying any dividend.

current price = ₹40 Strike price = ₹45

Continuously compounded risk free rate = 5%

If stock price increases by 0.5, price of the option changes by 0.25

determine the implied volatility. ~~change in C = ₹~~

Solution $S(0) = ₹40$ $K = ₹45$ $r = 0.05$ $T = 1$ year

$\Delta S = 0.5$ $\Delta C = 0.25$

$$\frac{\Delta C}{\Delta S} = \phi(d_1) = \frac{0.25}{0.5} = 0.5$$

$$\Rightarrow d_1 = 0$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0 - \sigma\sqrt{T} = -\sigma\sqrt{T} = -\sigma$$

$$\Rightarrow \ln \frac{S(0)}{K} = -\left(r + \frac{\sigma^2}{2}\right)T$$

$$\ln\left(\frac{40}{45}\right) = -\left[0.05 + \frac{\sigma^2}{2}\right]1$$

$$0.05 + \frac{\sigma^2}{2} = -\ln\left(\frac{40}{45}\right)$$

$$\cancel{0.05} + \frac{\sigma^2}{2} = +0.1177 - 0.05$$

Problem $u = 0.2$ $d = -0.1$ $r = 0.1$

$T = 2$ $n = 2$ $K = ₹120$

If stock price became 130, what will be the

$\Delta S = 10$

$D~~(0) = 22.92~~ \Rightarrow$

$\Delta C = 32.18 - 22.92$

~~S/B~~ FE

~~Random~~ Brownian Motion (Wiener Process)

A stochastic process $\{W(t), t \geq 0\}$ is said to be a Brownian motion if it satisfies the following properties =>

(i) $W(0) = 0$ it starts with 0

(ii) for $t > 0$, the sample path of $W(t)$ is continuous

(iii) the stochastic process $\{W(t), t \geq 0\}$ has independent and stationary increment.

(iv) for $\{0 \leq s \leq t < \infty\}$ $W(t) - W(s)$ is normally distributed random variable with mean 0 and variance $t-s$

$$W(t) - W(s) \sim N(0, t-s)$$

→ The path is always continuous but it is no where differentiable.

$$\frac{dy}{dx} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta t \to 0} \frac{Y(t+\Delta t) - Y(t)}{\Delta t}$$

$$\frac{dW(t)}{dt} = \lim_{\Delta t \to 0} \frac{W(t+\Delta t) - W(t)}{\Delta t}$$

$$\Rightarrow \lim_{\Delta t \to 0} \operatorname{var}\left[\frac{W(t+\Delta t) - W(t)}{\Delta t}\right]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t^2} \operatorname{var}(W(t+\Delta t) - W(t)]$$

$$= \lim_{\Delta t \to 0} \frac{\Delta t}{\Delta t^2} = \text{does not exist.}$$

→ Wiener process is not wide sense stationary

for $0 < s < t$ ~~Cov(W(s), W(t))~~

$$Cov(W(s), W(t)) = E\{(W(t) - E[W(t)])(W(s) - E(W(s))]\}$$

$$= E[W(t) W(s)]$$

Standard Brownian Motion ?

$= E\{(w(t) - w(s) + w(s)) w(s)\}$

$= E\{w(s)(w(t) - w(s))\} + E(w^2(s))$

$= E(w(s)) \, E(w(t) - w(s)) + E[w^2(s)]$

$= 0 + Var(w(s))$

$= s$

$= \min\{s,t\}$

- Given $w(t)$, future $w(t+h)$ having $h>0$ only depend on the increment $w(t+h) - w(t)$ which is independent of the past. Hence, Brownian Motion $\{w(t), t \geq 0\}$ is a markov process

- Brownian motion with drift μ and volatality σ

A stochastic process $\{x(t), t \geq 0\}$ is said to be a Brownian motion with drift μ and volatility σ if $x(t) = \mu t + \sigma w(t)$

When (i) $w(t)$ is a standard Brownian motion

(ii) $-\infty < \mu < +\infty$ (μ is finite) and is constant

(iii) $\sigma > 0$ a constant

$$E[x(t)] = E[\mu t + \sigma w(t)]$$
$$= \mu t + \sigma E(w(t))$$
~~$= \mu t + \sigma^2 Cov$~~

$$Cov(x(t), x(s)) = \sigma^2 Cov(w(t), w(s))$$
$$= \sigma^2 \min\{s,t\}, \quad s,t \geq 0$$

Geometric Brownian Motion

A stochastic process $\{X(t), t \geq 0\}$ is said to be a geometric brownian motion if $x(t) = X(0) e^{w(t)}$ when $w(t)$ is the standard brownian motion.

NOTE:- for any $h>0$ $x(t+h) = X(0) e^{w(t+h)}$

$= X(0) e^{w(t+h) - w(t) + w(t)}$

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$$= X(0)\, e^{W(t)}\, e^{W(t+h) - W(t)}$$

$$X(t+h) = X(t)\, e^{W(t+h) - W(t)}$$

We know that brownian motion has independent increment, Hence the given $X(T)$, the future $X(t+h)$ only depends upon the future increment of the brownian motion. Thus the future is independent of the past, and therefore the Markov property is satisfied, and hence, Consequently the Stochastic process $\{X(t), t \geq 0\}$ is a Markov process