1

Origin of Partial Differential Equations

1.1 INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. In the present part of the book, we propose to study various methods to solve partial differential equations.

1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E.) [Delhi Maths (H) 2001]

Definition. An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a *partial differential equation.*

For examples of partial differential equations we list the following:

$$\partial z/\partial x + \partial z/\partial y = z + xy \quad \text{... (1)} \qquad (\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x(\partial z/\partial x) \quad \text{... (2)}$$

$$z(\partial z/\partial x) + \partial z/\partial y = x \quad \text{... (3)} \qquad \partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz \quad \text{... (4)}$$

$$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2} \quad \text{... (5)} \qquad y\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\right\} = z(\partial z/\partial y) \quad \text{... (6)}$$

1.3 ORDER OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

Definition. The *order* of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

In Art. 1.2, equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

1.4 DEGREE OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

In 1.2, equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

1.5 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

In Art. 1.2, equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

1.6 NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$$p = \partial z/\partial x, \qquad q = \partial z/\partial y, \qquad r = \partial^2 z/\partial x^2, \qquad s = \partial^2 z/\partial x \partial y \qquad \text{and} \qquad t = \partial^2 z/\partial y^2$$

In case there are n independent variables, we take them to be $x_1, x_2, \ldots, x_n$ and z is then regarded as the dependent variable. In this case we use the following notations :

$p_1 = \partial z/\partial x_1$, $p_2 = \partial z/\partial x_2$, $p_3 = \partial z/\partial x_3$, and $p_n = \partial z/\partial x_n$.

Sometimes the partial differntiations are also denoted by making use of suffixes. Thus we write $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xy} = \partial^2 u/\partial x \partial y$ and so on.

1.7 Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations with examples. [Delhi Maths (H) 2001; 2004]

Linear equation. *A first order equation* $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$.

For examples, $yx^2 p + xy^2 q = xyz + x^2 y^3$ and $p + q = z + xy$ are both first order linear partial differential equations.

Semi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form* $P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$

For examples, $xyp + x^2 yq = x^2 y^2 z^2$ and $yp + xq = (x^2 z^2/y^2)$ are both first order semi-linear partial differential equations.

Quasi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form*

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

For examples, $x^2 zp + y^2 zp = xy$ and $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$ are first order quasi-linear partial differential equations.

Non-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *which does not come under the above three types, in known as a non-liner equation.*

For examples, $p^2 + q^2 = 1$, $p\,q = z$ and $x^2 p^2 + y^2 q^2 = z^2$ are all non-linear partial differential equations.

1.8 Origin of partial differential equations. We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

1.9 Rule I. Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation
$$F(x, y, z, a, b) = 0, \qquad \ldots(1)$$
where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get
$$\partial F/\partial x + p(\partial F/\partial z) = 0 \quad \text{and} \quad \partial F/\partial y + q(\partial F/\partial z) = 0 \qquad ..(2)$$

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form
$$f(x, y, z, p, q) = 0, \qquad \ldots(3)$$
which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems: For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$$F(x, y, z, a, b) = 0, \qquad \partial F / \partial x = 0 \qquad \text{and} \qquad \partial F / \partial y = 0.$$

The equation free from a and b will be the required partial differential equation.

Three situations may arise :

Situation I. *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

For example, consider $$z = ax + y, \qquad \text{... (1)}$$
where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $$\partial z / \partial x = a \qquad \text{... (2)}$$

Differentiating (1) partially w.r.t. 'y', we get $$\partial z / \partial y = 1 \qquad \text{... (3)}$$

Eliminating a between (1) and (2) yields $$z = x(\partial z / \partial x) + y \qquad \text{... (4)}$$

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Situation II. *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.*

Example: *Eliminate a and b from* $$az + b = a^2 x + y \qquad \text{... (1)}$$

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$a(\partial z / \partial x) = a^2 \quad \text{... (2)} \qquad a(\partial z / \partial y) = 1 \quad \text{... (3)}$$

Eliminating a from (2) and (3), we have $$(\partial z / \partial x)\,(\partial z / \partial y) = 1,$$

which is the unique partial differential equation of order one.

Situation III. *When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.*

Example: *Eliminate a, b and c from* $$z = ax + by + cxy \qquad \text{... (1)}$$

Differentiating (1) partially w.r.t., 'x' and 'y', we have

$$\partial z / \partial x = a + c\,y \quad \text{... (2)} \qquad \partial z / \partial y = b + c\,x \quad \text{... (3)}$$

From (2) and (3), $$\partial^2 z / \partial x^2 = 0, \qquad \partial^2 z / \partial y^2 = 0 \qquad \text{... (4)}$$

and $$\partial^2 z / \partial x \partial y = c \qquad \text{... (5)}$$

Now, (2) and (3) $\Rightarrow$ $$x(\partial z / \partial x) = ax + cxy \qquad \text{and} \qquad y(\partial z / \partial y) = by + cxy$$

$$\therefore \qquad x(\partial z / \partial x) + y(\partial z / \partial y) = ax + by + cxy + cxy$$

or $$x(\partial z / \partial x) + y(\partial z / \partial y) = z + xy(\partial^2 z / \partial x \partial y), \text{ using (1) and (5)} \qquad \text{... (6)}$$

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

1.10 SOLVED EXAMPLES BASED ON RULE I OF ART 1.9

Ex. 1. *Find a partial differential equation by eliminating a and b from* $z = ax + by + a^2 + b^2$.

Sol. Given $$z = ax + by + a^2 + b^2. \qquad \text{...(1)}$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = a \qquad \text{and} \qquad \partial z/\partial y = b.$$

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2,$$

which is the required partial differential equation.

Ex. 2. *Eliminate arbitrary constants a and b from* $z = (x - a)^2 + (y - b)^2$ *to form the partial differential equation.* **[Jiwaji 1999; Banglore 1995]**

Sol. Given $\qquad z = (x - a)^2 + (y - b)^2. \qquad$ ...(1)

Differentiating (1) partially with respect to a and b, we get

$\partial z/\partial x = 2(x - a) \qquad$ and $\qquad \partial z/\partial y = 2(y - b).$

Squatring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x - a)^2 + 4(y - b)^2 = 4\ [(x - a)^2 + (y - b)^2]$$

or $\qquad (\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z,$ using (1).

Ex. 3. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations :*

(a) $z = a(x + y) + b.$ $\qquad$ (b) $z = ax + by + ab.$ **[Bhopal 2010, Rewa 1996]**

(c) $z = ax + a^2y^2 + b.$ **[Agra 2010]** $\qquad$ (d) $z = (x + a)\ (y + b).$ **[Madurai Kamraj 2008]**

Sol. (a) Given $\qquad z = a(x + y) + b \qquad$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a \qquad$ and $\qquad \partial z/\partial y = a.$

Eliminating a between these, we get $\qquad \partial z/\partial x = \partial z/\partial y,$

which is the required partial differential equation.

(b) Given $\qquad z = ax + by + ab. \qquad$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a \qquad$ and $\qquad \partial z/\partial y = b \qquad$ ....(2)

Substituting the values of a and b from (2) in (1), we get

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\mathrm{d}x)(\partial z/\partial y),$$

which is the required partial differential equation.

(c) Try yourself. $\qquad$ **Ans.** $\partial z/\partial y = 2y(\partial z/\partial x)^2.$

(d) Try yourself. $\qquad$ **Ans.** $z = (\partial z/\partial y)\ (\partial z/\partial x).$

Ex. 4. *Eliminate a and b from* $z = axe^y + (1/2) \times a^2e^{2y} + b.$ **[Meerut 2006]**

Sol. Given $\qquad z = axe^y + (1/2) \times a^2e^y + b. \qquad$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = ae^y \qquad \text{...(2)}$$

and $\qquad \partial z/\partial y = axe^y + a^2e^{2y} = x(ae^y) + (ae^y)^2. \qquad$ ...(3)

Substituting the value of ae^y from (2) in (3), we get $\qquad \partial z/\partial y = x(\partial z/\partial x) + (\partial z/\partial x)^2.$

Ex. 5(*a*). *Form the partial differential equation by eliminating h and k from the equation* $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2.$ **[Gulbarga 2005; I.A.S. 1996]**

Sol. Given $\qquad (x - h)^2 + (y - k)^2 + z^2 = \lambda^2. \qquad$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$2(x - h) + 2z(\partial z/\partial x) = 0 \qquad$ or $\qquad (x - h) = -z(\partial z/\partial x) \qquad$ ...(2)

and $\qquad 2(y - k) + 2z(\partial z/\partial y) = 0 \qquad$ or $\qquad (y - k) = -z(\partial z/\partial y). \qquad$ ...(3)

Substituting the values of $(x - h)$ and $(y - k)$ from (2) and (3) in (1) gives

$z^2(\partial z/\partial x)^2 + z^2(\partial z/\partial y)^2 + z^2 = \lambda^2 \qquad$ or $\qquad z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = \lambda^2,$

which is the required partial differential equation.

Ex. 5(*b*). *Find the differential equation of all spheres of radius* λ, *having centre in the xy-plane*. **[M.D.U. Rohtak 2005; I.A.S. 1996, K.U. Kurukshetra 2005]**

Sol. From the coordinate geometry of three-dimensions, the equation of any sphere of radius λ, having centre $(h, k, 0)$ in the xy-plane is given by

$$(x-h)^2 + (y-k)^2 + (z-0)^2 = \lambda^2 \quad \text{or} \quad (x-h)^2 + (y-k)^2 + z^2 = \lambda^2, \qquad ...(1)$$

where h and k are arbitray constants. Now, proceed exactly in the same way as in Ex. 5(a).

Ex. 6. *Form the differential equation by eliminating a and b from* $z = (x^2 + a)(y^2 + b)$.

[Madras 2005; Sagar 1997, I.A.S. 1997]

Sol. Given $$z = (x^2 + a)(y^2 + b). \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2x(y^2 + b) \quad \text{or} \quad (y^2 + b) = (1/2x) \times (\partial z/\partial x) \qquad ...(2)$$

and $$\partial z/\partial y = 2y(x^2 + a) \quad \text{or} \quad (x^2 + a) = (1/2y) \times (\partial z/\partial y). \qquad ...(3)$$

Substituting the values of $(y^2 + b)$ and $(x^2 + a)$ from (2) and (3) in (1) gives

$$z = (1/2y) \times (\partial z/\partial y) \times (1/2x) \times (\partial z/\partial x) \quad \text{or} \quad 4xyz = (\partial z/\partial x)(\partial z/\partial y),$$

which the required partial differential equation.

Ex. 7. *Form differential equation by eliminating constants A and p from* $z = A\, e^{pt} \sin px$.

Sol. Given $$z = A\, e^{pt} \sin px. \qquad ...(1)$$

Differentiating (1) partially with respect to x and t, we get

$$\partial z/\partial x = Ap\, e^{pt} \cos px \qquad ...(2) \qquad\qquad \partial z/\partial t = Ap\, e^{pt} \sin px. \qquad ...(3)$$

Differentiating (2) and (3) partially with respect to x and t respectively gives

$$\partial^2 z/\partial x^2 = -Ap^2 e^{pt} \sin px. \qquad ...(4) \qquad\qquad \partial^2 z/\partial t^2 = Ap^2 e^{pt} \sin px. \qquad ...(5)$$

Adding (4) and (5), $$\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0,$$

which is the required partial differential equation.

Ex. 8. *Find the differential equation of the set of all right circular cones whose axes coincide with z-axix.* **[I.A.S. 1998]**

Sol. The general equation of the set of all right circular cones whose axes coincide with z-axis, having semi-vertical angle α and vertex at (0, 0, c) is given by

$$x^2 + y^2 = (z-c)^2 \tan^2 \alpha, \qquad ...(1)$$

in which both the constants c and α are arbitrary.

Differentiating (1) partially, w.r.t. x and y, we get

$$2x = 2(z-c)\,(\partial z/\partial x)\tan^2\alpha \quad \text{and} \quad 2y = 2(z-c)\,(\partial z/\partial y)\tan^2\alpha$$

$$\Rightarrow \quad y(z-c)\,(\partial z/\partial x)\tan^2\alpha = xy \quad \text{and} \quad x(z-c)\,(\partial z/\partial y)\tan^2\alpha = xy$$

$$\Rightarrow \quad y(z-c)\,(\partial z/\partial x)\tan^2\alpha = x(z-c)\,(\partial z/\partial y)\tan^2\alpha$$

Thus, $y\,(\partial z/\partial x) = x\,(\partial z/\partial y)$, which is the required partial differential equation.

Ex. 9. *Show that the differential equation of all cones which have their vertex at the origin is* $px + qy = z$. *Verify that* $yz + zx + xy = 0$ *is a surface satisfying the above equation.*

[I.A.S. 1979, 2009]

Sol. The equation of any cone with vertex at origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad ...(1)$$

where a, b, c, f, g, h are parameters. Differentiating (1) partially w.r.t. 'x' and 'y' by turn, we have (noting that $p = \partial z/\partial x$ and $q = \partial z/\partial y$)

$$2ax + 2czp + 2fyp + 2g(px + z) + 2hy = 0 \quad \text{or} \quad ax + gz + hy + p(cz + gx + fy) = 0 \qquad ...(2)$$

and $2by + 2czq + 2f(yq + z) + 2gxq + 2hx = 0$ or $by + fz + hx + q(cz + fy + gx) = 0.$ ...(3)

Multiplying (2) by x and (3) by y and adding, we have

$$(ax^2 + by^2 + gzx + fyz + 2hxy) + (cz + fy + gx)(px + qy) = 0.$$

$$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0, \text{ using (1)}$$

or $(cz + fy + gx)(px + qy - z) = 0$ or $px + qy - z = 0,$ ...(4)

which is required partial differential equation.

Second Part : Given surface is $yz + zx + xy = 0$ ...(5)

Differentiating (5) partially w.r.t. 'x' and 'y' by turn, we get

$yp + px + z + y = 0$ and $z + qy + xq + x = 0.$ ...(6)

Solving (6) for p and q, $p = -(z + y)/(x + y)$ and $q = -(z + x)/(x + y)$.

$$\therefore \quad px + qy - z = -\frac{x(z+y)}{x+y} - \frac{y(z+x)}{x+y} - z = -\frac{2(xy+yz+zx)}{x+y} = 0, \text{ using (5)}$$

Hence (5) is a surface satisfying (4).

Ex. 10. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations:*

(*a*) $2z = x^2/a^2 + y^2/b^2$ **[Nagpur 1995; M.D.U. Rohtak 2006]**

(*b*) $2z = (ax + y)^2 + b$ **[Nagpur 1996; Delhi Maths (G) 2006; Pune 2010]**

Sol. (*a*) Given $2z = x^2/a^2 + y^2/b^2$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$2(\partial z/\partial x) = 2x/a^2$ ... (2) $\qquad 2(\partial z/\partial y) = 2y/b^2$ ... (3)

From (2) and (3), $p = x/a^2$, $q = y/b^2$ $\Rightarrow$ $a^2 = x/p$, $b^2 = y/q$

Substituting these values of a^2 and b^2 in (1), we get

$2z = px + qy$, which is the required partial differential equation

(*b*) Given $2z = (ax + y)^2 + b$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$2p = 2a(ax + y)$ ... (2) $\qquad 2q = 2(ax + y)$ ... (3)

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Dividing (2) by (3) yields $p/q = a$.

Substituting this value of a in (3), we get $q = (p/q)\,x + y$ or $px + qy = q^2$.

Ex. 11. *Eliminate a, b and c from* $z = a(x + y) + b(x - y) + abt + c$ **[I.A.S. 1998]**

Sol. Given $z = a(x + y) + b(x - y) + abt + c$ ... (1)

Differentiating (1) partially w.r.t. 'x', 'y' and 't', we get

$\partial z/\partial x = a + b$ ... (2) $\qquad \partial z/\partial y = a - b$ ... (3) $\qquad \partial z/\partial t = ab$ ... (4)

We have the identity: $(a + b)^2 - (a - b)^2 = 4ab$

$$\therefore \quad (\partial z/\partial x)^2 - (\partial z/\partial y)^2 = 4(\partial z/\partial t), \text{ using (2), (3) and (4)}$$

Ex. 12. *Form the partial differential equation by eliminating the arbitrary constants a and b from log* $(az - 1) = x + ay + b$. **[I.A.S. 2002]**

Sol. (*a*) Given $\log(az - 1) = x + ay + b$ ... (1)

Differentiating (1) partially w.r.t. 'x', we get $\dfrac{a}{az-1}\dfrac{\partial z}{\partial x} = 1$ ... (2)

Differentiating (1) partially w.r.t. 'y', we get $\dfrac{a}{az-1}\dfrac{\partial z}{\partial y} = a$ ... (3)

From (3), $az - 1 = \dfrac{\partial z}{\partial y}$ so that $a = \dfrac{1 + (\partial z / \partial y)}{z}$ ... (4)

Putting the above values of $az - 1$ and a in (2), we have

$$\frac{1 + (\partial z / \partial y)}{z(\partial z / \partial y)} \frac{\partial z}{\partial x} = 1 \quad \text{or} \quad \left(1 + \frac{\partial z}{\partial y}\right) \frac{\partial z}{\partial x} = z \frac{\partial z}{\partial y}.$$

Ex. 13. *Find a partial differential equation by eliminating a, b, c, from* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

[Bhopal 2004; Jabalpur 2000, 03, Jiwaji 2000, Vikram 2002, 04; Ravishanker 2010]

Sol. Given $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} \frac{dz}{dx} = 0 \quad \text{or} \quad c^2 x + a^2 z \frac{dz}{dx} = 0 \qquad \text{... (2)}$$

and

$$\frac{2y}{b^2} + \frac{2x}{c^2} \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2 y + b^2 z \frac{\partial z}{\partial y} = 0. \qquad \text{... (3)}$$

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2 + a^2 \left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \qquad \text{... (4)} \qquad c^2 + b^2 \left(\frac{\partial z}{\partial x}\right)^2 + b^2 z \frac{\partial^2 z}{\partial y^2} = 0. \qquad \text{... (5)}$$

From (2), $c^2 = -(a^2 z / x) \times (\partial z / \partial x)$ ... (6)

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x} \frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad zx \frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial x}\right)^2 - z \frac{\partial z}{\partial x} = 0. \qquad \text{... (7)}$$

Similarly, from (3) and (5), $zy \dfrac{\partial^2 z}{\partial y^2} + y \left(\dfrac{\partial z}{\partial y}\right)^2 - z \dfrac{\partial z}{\partial y} = 0.$ ... (8)

Differentiating (2) partially w.r.t. y, $0 + a^2 \left\{(\partial z / \partial y)\,(\partial z / \partial x) + z(\partial^2 z / \partial x \partial y)\right\} = 0$

or

$$(\partial z / \partial x)\,(\partial z / \partial y) + z(\partial^2 z / \partial x \partial y) = 0 \qquad \text{... (9)}$$

(7), (8) and (9) are three possible forms of the required partial differential equations.

Ex. 14. *Find the partial differential equation of all planes which are at a constant distance 'a' from the origin.*

Sol. Let $lx + my + nz = a$ ... (1)

be the equation of the given plane where l, m, n are direction colines of the normal to the plane so that $l^2 + m^2 + n^2 = 1$, l, m, n being parameters ... (2)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$l + np = 0$ ...(3) $m + nq = 0,$ ... (4)

where $p = \partial z / \partial x$ and $q = \partial z / \partial y$. From (3) and (4), $l = -np$ and $m = -nq$. Substituting these values in (2), we have

$n^2(p^2 + q^2 + 1) = 1$ so that $n = (p^2 + q^2 + 1)^{-1/2}$ ... (5)

$\therefore\ l = -np = -p(p^2 + q^2 + 1)^{-1/2}$ and $m = -nq = -q(p^2 + q^2 + 1)^{-1/2}$ ... (6)

Substituting the values of l, m, n given by (5) and (6) in (1), we get

$$-px\,(p^2+q^2+1)^{-1/2} - qy\,(p^2+q^2+1)^{-1/2} + z(p^2+q^2+1)^{-1/2} = a$$

or $\quad z = px + qy + a\,(p^2+q^2+1)^{1/2}$, which is the required partial differential equation.

Ex. 15. *Show that the partial differential equation obtained by eliminating the arbitrary constants a and c from $z = ax + g(a)\,y + c$, where $g\,(a)$ is an arbitrary function of a, is free of the variables x, y, z.*

Sol. Differentiating $z = ax + g(a)\,y + c$ partially w.r.t. 'x' and 'y' yields $p = a$ and $q = g(a)$. Eliminating a between them leads to $q = g(p)$ or $f(p, q) = 0$, where f is an arbitrary function of p and q. Clearly, the resulting partial differential equation contains p and q but none of the variables x, y, z.

Ex. 16. *Show that the partial differential equation obtained by eliminating the arbitrary constants a and b from $z = ax + by + f\,(a, b)$ is given by $z = px + qy + f(p, q)$.*

Sol. Differentiating $\quad z = ax + by + f\,(a, b) \quad$ ... (1)

partially with respect to 'x' and 'y', we get $\quad p = a \quad$ and $\quad q = b \quad$ ... (2)

Eliminating a and b from (1) and (2) yields $\quad z = px + qy + f\,(p, q)$

Ex. 17. *Form a partial differential equation by eliminating a, b and c from the relation $ax^2 + by^2 + cz^2 = 1$.* **[Mysore 2004]**

Sol. Given $\quad ax^2 + by^2 + cz^2 = 1. \quad$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$$2ax + 2cz(\partial z/\partial x) = 0 \quad \ldots (2) \qquad 2by + 2cz\,(\partial z/\partial y) = 0 \quad \ldots (3)$$

Differentiating (2) partially w.r.t. 'y', we get

$$0 + 2c\left\{(\partial z/\partial y)\,(\partial z/\partial x) + z(\partial^2 z/\partial y\partial x\right\} = 0 \quad \text{or} \quad (\partial z/\partial x)\,(\partial z/\partial y) + z(\partial^2 z/\partial x\partial y) = 0\,, \quad \ldots (4)$$

since c is an arbitrary constant. (4) is the desired partial differential equation.

Again, differentiating partially (2) w.r.t. x and (3) w.r.t. y, we get

$$2a + 2c\left\{(\partial z/\partial x)^2 + z(\partial^2 z/\partial x^2)\right\} = 0 \quad \ldots (5) \qquad 2b + 2c\left\{(\partial z/\partial y)^2 + z(\partial^2 z/\partial y^2)\right\} = 0 \quad \ldots (6)$$

From (2), $a = -(cz/x) \times (\partial z/\partial x)$. Putting this in (5), we get

$$-(cz/x) \times (\partial z/\partial x) + c\left\{(\partial z/\partial x)^2 + z(\partial^2 z/\partial x^2\right\} = 0 \quad \text{or} \quad zx(\partial^2 z/\partial x^2) + x(\partial z/\partial x)^2 - z(\partial z/\partial x) = 0 \quad \ldots (7)$$

Similarly, from (3) and (6), we get $\quad zy(\partial^2 z/\partial y^2) + y(\partial z/\partial y)^2 - z(\partial z/\partial y) = 0\,. \quad$ ... (8)

(4), (7) and (8) are three possible forms of the required partial differential equations.

EXERCISE 1 (A)

Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial differential equations.

1. $z = A\,e^{pt} \sin px$, (p and A). **Ans.** $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0$.

2. $z = A\,e^{-p^2 t} \cos px$, ($p$ and A) **(Sagar 1999; Ranchi 2010)** **Ans.** $\partial^2 z/\partial x^2 = dz/\partial t$

3. $z = ax^3 + by^3$; (a, b) **Ans.** $x(\partial z/\partial x) + y(\partial z/\partial y) = 3z$

4. $4z = [ax + (y/a) + b]^2$; (a, b). **(Delhi B.A. (Prog) II 2011)** **Ans.** $z = (dz/\partial x)\,(\partial z/\partial y)$

5. $z = ax^2 + bxy + cy^2$, (a, b, c) **Ans.** $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = 2z$

6. $z^2 = ax^3 + by^3 + ab$, (a, b)

Ans. $9x^2y^2z = 6x^3y^2(\partial z/\partial x) + 6x^2y^3(\partial z/\partial y) + 4z(\partial z/\partial x)\,(\partial z/\partial y)$

7. $e^{1/\{z-(x^2/y)\}} = \dfrac{ax^2}{y^2} + \dfrac{b}{y}, (a, b)$ **Ans.** $x\dfrac{\partial z}{\partial x} + 2y\dfrac{\partial z}{\partial y} = 2(z - x^2/y)^2$

8. Find the differential equation of the family of spheres of radius 4 with centres on the xy-plane. **Ans.** $(x-y)^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = 16(\partial z/\partial x - \partial z/\partial y)^2$

9. Find the P.D.E of planes having equal x and y intercepts. **Ans.** $p - q = 0$

10. Find the partial differential equation of the family of spheres of radius 7 with centres on the plane $x - y = 0$. **Ans.** $(p^2 + q^2 + 1)(x - y)^2 = 49(p - q)$

11. Find the partial differential equation of all spheres whose centres lie on z-axis.

Ans. $xq - yp = 0$

1.11 Rule II. Derivation of partial differential equation by the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z.

[Meerut 1995]

Proof. Given $\phi(u, v) = 0.$ ...(1)

We treat z as dependent variable and x and y as independent variables so that

$\partial z/\partial x = p,$ $\partial z/\partial y = q,$ $\partial y/\partial x = 0$ and $\partial x/\partial y = 0.$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

or
$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

or
$$\frac{\partial \phi}{\partial u}\Big/\frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right). \quad ...(3)$$

Similarly, differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\Big/\frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) \quad ...(4)$$

Eliminating ϕ with the help of (3) and (4), we get

$$\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

or
$$\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

or
$$Pp + Qq = R, \quad ...(5)$$

where $P = \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}$, $Q = \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}$, $R = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}$.

Thus we obtain a linear partial differntial equation of first order and of first degree in p and q.

Note. If the given equation between x, y, z contains two arbitrary functions, then in general, their elimination gives rise to equations of higher order.

1.12 SOLVED EXAMPLES BASED ON RULE II OF ART. 1.11.

Ex. 1. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x + y + z, x^2 + y^2 - z^2) = 0$. *What is the order of this partial differential equation ?*

[Bilaspur 2003; Indore 2003; Jiwaji 2003; Vikram 2001]

Sol. Given $\phi(x + y + z, x^2 + y^2 - z^2) = 0.$ ...(1)

Let $u = x + y + z$ and $v = x^2 + y^2 - z^2.$ ...(2)

Then (1) becomes $\phi(u, v) = 0$. ...(3)

Differentiating (3) w.r.t., 'x' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0. \quad ...(4)$$

From (2), $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial z} = 1$, $\frac{\partial v}{\partial x} = 2x$, $\frac{\partial v}{\partial z} = -2z$, $\frac{\partial u}{\partial y} = 1$, $\frac{\partial v}{\partial y} = 2y$. ..(5)

From (4) and (5), $(\partial\phi/\partial u)(1 + p) + 2(\partial\phi/\partial v)(x - pz) = 0$

or $(\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(x - pz)/(1 + p)$. ...(6)

Again, differentiating (3) w.r.t., 'y' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = 0$$

or $(\partial\phi/\partial u)(1 + q) + 2(\partial\phi/\partial v)(y - \mathrm{z}q) = 0$, by (5)

or $(\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(y - qz)/(1 + q)$. ...(7)

Eliminating ϕ from (6) and (7), we obtain

$(x - pz)/(1 + p) = (y - qz)/(1 + q)$ or $(1 + q)(x - pz) = (1 + p)(y - qz)$

or $(y + z)p - (x + z)q = x - y$, which is the desired partial differential equation of first order.

Ex. 2. *Form a partial differential equation by eliminating the arbitrary function f from the equation* $x + y + z = f(x^2 + y^2 + z^2)$. **(Kanpur 2011)**

Sol. Given $x + y + z = f(x^2 + y^2 + z^2)$. ...(1)

Differentiating partially w.r.t. 'x' and 'y', (1) gives

$1 + p = f'(x^2 + y^2 + z^2).(2x + 2zp)$. ...(2)

and $1 + q = f'(x^2 + y^2 + z^2).(2y + 2zq)$. ...(3)

Eliminating $f'(x^2 + y^2 + z^2)$ from (2) and (3), we obtain

$(1 + p)/(2x + 2zp) = (1 + q)/(2y + 2zq)$ or $(1 + p)(y + zq) = (1 + q)(x + zp)$

or $(y - z)p + (z - x)q = x - y$, which is the required partial differential equations.

Ex. 3. *Eliminate the arbitrary functions f and F from* $y = f(x - at) + F(x + at)$.

(Sagar 1997; Vikram 1995; Jabalpur 2002)

Sol. Given $y = f(x - at) + F(x + at)$. ...(1)

From (1), $\partial y/\partial x = f'(x - at) + F'(x + at)$

and hence $\partial^2 y/\partial x^2 = f''(x - at) + F''(x + at)$. ...(2)

Also, $\partial y/\partial t = f'(x - at) . (-a) + F'(x + at) . (a)$

and hence $\partial^2 y/\partial t^2 = f''(x - at) . (-a)^2 + F''(x + at) . (a)^2$

or $\partial^2 y/\partial t^2 = a^2[f''(x - at) + F''(x + at)]$. ...(3)

Then, (2) and (3) $\Rightarrow$ $\partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2)$.

Ex. 4. *Eliminate arbitrary function f from*

(*i*) $z = f(x^2 - y^2)$. **[Bilaspur 1996; Sagar 1996; Bangalore 1995]**

(*ii*) $z = f(x^2 + y^2)$. **[Meerut 1995; Pune 2010]**

Sol. (*i*) Given $z = f(x^2 - y^2)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(x^2 - y^2) \times 2x$ so that $f'(x^2 - y^2) = (1/2x) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(x^2 - y^2) \times (-2y)$ so that $f'(x^2 - y^2) = -(1/2y) \times (\partial z/\partial y)$. ...(3)

Eliminating $f'(x^2 - y^2)$ between (2) and (3), we have

$$\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y} \quad \text{or} \quad y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

(*ii*) Proceed as in part (1). **Ans.** $y(\partial z/\partial x) - x(\partial z/\partial y) = 0$

Ex. 5. *Form a partial differential equation by eliminating the function f from*

(*i*) $z = f(y/x)$. **[Sagar 2000]** (*ii*) $z = x^n f(y/x)$.

Sol. Given $z = f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(y/x) \times (-y/x^2)$ or $f'(y/x) = -(x^2/y) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(y/x) \times (1/x)$ or $f'(y/x) = x(\partial z/\partial y)$. ...(3)

Eliminating $f'(y/x)$ between (2) and (3), we have

$$-\frac{x^2}{y}\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y} \quad \text{or} \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

which is the required partial differential equation.

(*ii*) Given $z = x^n f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = n\, x^{n-1} f(y/x) + x^n f'(y/x) \times (-y/x^2)$ ...(2)

and $\partial z/\partial y = x^n f'(y/x) \times (1/x)$. ...(3)

Multiplying both sides of (2) by x, we have $x(\partial z/\partial x) = n\, x^n f(y/x) - yx^{n-1} f'(y/x)$. ...(4)

Multiplying both sides of (3) by y, we have $y(\partial z/\partial y) = y\, x^{n-1} f'(y/x)$. ...(5)

Adding (4) and (5), $x(\partial z/\partial x) + y(\partial z/\partial y) = n\, x^n f(y/x)$

or $x(\partial z/\partial x) + y(\partial z/\partial y) = nz$, by (1)

Ex. 6. *Form a partial differential equation by eliminating the function* ϕ from $lx + my + nz = \phi(x^2 + y^2 + z^2)$. **[Ravishankar 2003; Vikram 2003]**

Sol. Given $lx + my + nz = \phi(x^2 + y^2 + z^2)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$l + n(\partial z/\partial x) = \phi'(x^2 + y^2 + z^2) \times \{2x + 2z(\partial z/\partial x)\}$ ...(2)

and $m + n(\partial z/\partial y) = \phi'(x^2 + y^2 + z^2) \times \{2y + 2z(\partial z/\partial y)\}$ ...(3)

Dividing (2) by (3), we get $$\frac{l+n(\partial z/\partial x)}{m+n(\partial z/\partial y)} = \frac{2\{x+z(\partial z/\partial x)\}}{2\{y+z(\partial z/\partial y)\}}$$

or $(ny - mz)(\partial z/\partial x) + (lz - nx)(\partial z/\partial y) = mx - ly$, which is the required partial differential equation.

Ex. 7. *Form partial differential eqn. by eliminating the function f from* $z = e^{ax+by} f(ax - by)$.

Sol. Given $z = e^{ax+by} f(ax - by)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = e^{ax+by}\, a\, f'(ax - by) + a\, e^{ax+by} f(ax - by)$ ...(2)

and $\partial z/\partial y = e^{ax+by} \{-b\, f'(ax - by)\} + b\, e^{ax+by} f(ax - by)$. ...(3)

Multiplying (2) by b and (3) by a and adding, we get

$b(\partial z/\partial x) + a(\partial z/\partial y) = 2ab\, e^{ax+by} f(ax - by)$ or $b(\partial z/\partial x) + a(\partial z/\partial y) = 2abz$, by (1)

Ex. 8. *Form a partial differential equation by eliminating the arbitrary functions f and F from* $z = f(x + iy) + F(x - iy)$, *where* $i^2 = -1$. **[Bilaspur 2004; Jiwaji 1998; Meerut 2010]**

Sol. Given $z = f(x + iy) + F(x - iy)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(x + iy) + F'(x - iy)$ ...(2)

and $\partial z/\partial y = i f'(x + iy) - iF'(x - iy)$. ...(3)

Differentiating (2) and (3) partial w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = f''(x+iy) + F''(x-iy) \qquad ...(4)$$

and $$\partial^2 z/\partial y^2 = i^2 f''(x+iy) + i^2 F''(x-iy) = -\{f''(x+iy) + F''(x+iy)\}. \qquad ...(5)$$

Adding (4) and (5), $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$, which is the required equation.

Ex. 9. *Form partial differential equation by eliminating arbitrary functions f and g from* $z = f(x^2 - y) + g(x^2 + y)$. **[Nagpur 1996 ; I.A.S. 1996; Kanpur 2011]**

Sol. Given $$z = f(x^2 - y) + g(x^2 + y). \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2xf'(x^2 - y) + 2xg'(x^2 + y) = 2x\{f'(x^2 - y) + g'(x^2 + y)\}. \qquad ...(2)$$

and $$\partial z/\partial y = -f'(x^2 - y) + g'(x^2 + y). \qquad ...(3)$$

Differentiating (2) and (3) w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = 2\{f'(x^2 - y) + g'(x^2 + y)\} + 4x^2\{f''(x^2 - y) + g''(x^2 + y)\} \qquad ...(4)$$

and $$\partial^2 z/\partial y^2 = f''(x^2 - y) + g''(x^2 + y). \qquad ...(5)$$

Again, (2) $\Rightarrow$ $f'(x^2 - y) + g'(x^2 + y) = (1/2x) \times (\partial z/\partial x)$. ...(6)

Substituting the values of $f''(x^2 - y) + g''(x^2 + y)$ and $f'(x^2 - y) + g'(x^2 + y)$ from (5) and (6) in (4), we have

$$\frac{\partial^2 z}{\partial x^2} = 2 \times \left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2 \frac{\partial^2 z}{\partial y^2} \qquad \text{or} \qquad x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3 \frac{\partial^2 z}{\partial y^2},$$

which is the required partial differential equation.

Ex. 10. *Find the differential equation of all surfaces of revolution having z-axis as the axis of rotation.* **[I.A.S. 1997]**

Sol. From coordinate geometry of three dimensions, equation of any surface of revolution having z-axis as the axis of rotation may be taken as

$$z = \phi[(x^2 + y^2)^{1/2}], \text{ where } \phi \text{ is an arbitrary function.} \qquad ...(1)$$

Differentating (1) partially with respect to x and y, we get

$$\partial z/\partial x = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2x \qquad ...(2)$$

and $$\partial z/\partial y = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2y. \qquad ...(3)$$

Dividing (2) by (3), $$\frac{\partial z/\partial x}{\partial z/\partial y} = \frac{x}{y} \qquad \text{or} \qquad y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}.$$

Ex. 11. *Form a partial differential equation by eliminating the arbitrary functions f and g from* $z = y\ f(x) + x\ g(y)$. (*Guwahati 2007*)

Sol. Given $$z = y\, f(x) + x\, g(y). \qquad ...(1)$$

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\partial z/\partial x = y\, f'(x) + g(y)$ ...(2) $\qquad \partial z/\partial y = f(x) + x\, g'(y)$. ...(3)

Differentiating (3) with respect to x, $$\partial^2 z/\partial x\partial y = f'(x) + g'(y). \qquad ...(4)$$

From (2) and (3), $$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] \qquad \text{and} \qquad g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right].$$

Substituting these values in (4), we have

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

or $$xy\frac{\partial^2 z}{\partial x\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \{x\, g(y) + y\, f(x)\} \qquad \text{or} \qquad xy\frac{\partial^2 z}{\partial x\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z, \text{ by (2)}$$

Ex. 12. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. **[Nagpur 1996; 2002]**

Sol. Given $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. ...(1)

Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy$. ...(2)

Then, (1) becomes $\phi(u, v) = 0$. ...(3)

Differentiating (3) partially w.r.t. 'x', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0, \qquad ...(4)$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Now, from (2), we have

$\partial u/\partial x = 2x$, $\partial u/\partial y = 2y$, $\partial u/\partial z = 2z$, $\partial v/\partial x = -2y$, $\partial v/\partial y = -2x$, $\partial v/\partial z = 2z$. ...(5)

Using (5), (4) reduces to $(\partial\phi/\partial u)(2x + 2pz) + (\partial\phi/\partial v)(-2y + 2pz) = 0$

or $(x + pz)(\partial\phi/\partial u) = (y - pz)(\partial\phi/\partial v)$. ...(6)

Again, differentiating (3) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial y}\right) = 0$$

or $(\partial\phi/\partial u)(2y + 2qz) + (\partial\phi/\partial v)(-2x + 2qz) = 0$, by (5)

or $(y + qz)(\partial\phi/\partial u) = (x - qz)(\partial\phi/\partial v)$. ...(7)

Dividing (6) by (7), $(x + pz)/(y + qz) = (y - pz)/(x - qz)$

or $pz(y + x) - qz(y + x) = y^2 - x^2$ or $(p - q)z = y - x$.

Ex. 13. *Eliminate the arbitrary function f and obtain the partial differential equation from* $z = e^y f(x + y)$ **[Madras 2005]**

Sol. Given $z = e^y f(x + y)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$\partial z/\partial x = e^y f'(x+y)$ and $\partial z/\partial y = e^y f(x+y) + e^y f'(x+y)$ ... (2)

From (1) and (2), we have $\partial z/\partial y = z + \partial z/\partial x$

Ex. 14. *If* $z = f(x+ay) + \phi(x-ay)$, *prove that* $\partial^2 z/\partial y^2 = a^2(\partial^2 z/\partial x^2)$

Hint. Refer solved Ex. 3. **[Madurai Kamraj 2008; Jabalpur 2002]**

Ex. 15. *Equation of any cone with vertex at P* (a, b, c) *is of the form* $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$. *Find the differential equation of the cone.*

Sol. Let $(x - a)/(z - c) = u$ and $(y - b)/(z - c) = v$ ... (1)

Then, the equation of the given cone becomes $f(u, v) = 0$ ... (2)

Differentiating (2) partially with respect to 'x', we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}\left(\frac{1-0}{z-c} - \frac{x-a}{(z-c)^2}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial x}\right) = 0, \text{ using (1)}$$

or
$$\frac{\partial f}{\partial u}\left(\frac{1}{z-c} - p\frac{x-a}{(z-c)^2}\right) - \frac{\partial f}{\partial v}\left(p\frac{y-b}{(z-c)^2}\right) = 0, \quad \text{where} \quad p = \frac{\partial z}{\partial x} \qquad ... (3)$$

Differentiating (2) partially with respect to 'y', we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}\left(-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{1-0}{z-c} - \frac{y-b}{(z-c)^2}\frac{\partial z}{\partial y}\right) = 0, \text{ using (1)}$$

or $$-\frac{\partial f}{\partial u}\left(q\frac{x-a}{(z-c)^2}\right)+\frac{\partial f}{\partial v}\left(\frac{1}{z-c}-q\frac{y-b}{(z-c)^2}\right)=0, \quad \text{where} \quad q=\frac{\partial z}{\partial y} \qquad \text{... (4)}$$

Eliminating $\partial f/\partial u$ and $\partial f/\partial v$ from (3) and (4), we have

$$\begin{vmatrix} \frac{1}{z-c}-p\frac{x-a}{(z-c)^2} & -p\frac{y-b}{(z-c)^2} \\ -q\frac{x-a}{(z-c)^2} & \frac{1}{z-c}-q\frac{y-b}{(z-c)^2} \end{vmatrix}=0$$

or $$\begin{vmatrix} z-c-p(x-a) & -p(y-b) \\ -q(x-a) & z-c-q(y-b) \end{vmatrix}=0$$

or $$\{z-c-p(x-a)\}\ \{z-c-q(y-b)\}-pq(x-a)(y-b)=0$$

or $$(z-c)^2-p(x-a)(z-c)-q(y-b)(z-c)=0 \quad \text{or} \quad (x-a)p+(y-b)q=z-c.$$

which in the required partial differential equation of the given cone.

EXERCISE 1 (B)

Eliminate the arbitrary functions and hence obtain the partial differential equations:

1. $z=e^{mx}\phi(x+y)$. **Ans.** $p-q=mz$

2. $z=f(x+ay)$ **[Bilaspur 1997; Jabalpur 1999]** **Ans.** $q=ap$

3. $z=xy+f(x^2+y^2)$ **[Delhi B.A./B.Sc. (Maths) (Prog.) 2007]** **Ans.** $py-qx=y^2-x^2$

4. $z=x+y+f(xy)$ **[Delhi B.A. (Prog) II 2010]** **Ans.** $px-qy=x-y$

5. $z=f(xy/z)$ **[Nagpur 1995 KU Kurukshetra 2004]** **Ans.** $px-qy=0$

6. $z=f(x-y)$ **[Delhi B.A. (Prog.) II 2011]** **Ans.** $p+q=0$

7. $z=(x-y)\,\phi(x^2+y^2)$ **Ans.** $(x-y)yp-(x-y)xq=(x+y)z$

8. $z=f(x^2+2y^2)$ **Ans.** $xq-yp=x^2-y^2$

9. $x=f(z)+g(y)$ **Ans.** $ps-qr=0$

10. $z=f(y+ax)+g(y+bx)$, $a\neq b$. **Ans.** $r-(a+b)s+abt=0$

11. $f(x+y+z)=xyz$ **Ans.** $x(y-z)p+y(z-x)q=z(x-y)$

12. $z=(x+y)f(x^2-y^2)$ **Ans.** $yp+xq=z$

13. $z=f(x)+e^y g(x)$ **Ans.** $t-q=0$

14. $f(x+y+z, x^2+y^2-z^2)=0$ **(CDLU 2004)** **Ans.** $p(y+z)-(x+z)q=x-y$

15. $z=f(xy)+g(x/y)$ **Ans.** $x^2(\partial^2 z/\partial x^2)-y^2(\partial^2 z/\partial y^2)+x(\partial z/\partial x)-y(\partial z/\partial y)=0$

16. $z=f(x-z)+g(x+y)$ **Ans.** $\dfrac{\partial z}{\partial y}\dfrac{\partial^2 z}{\partial x^2}+\left(1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)\dfrac{\partial^2 z}{\partial x\partial y}-\left(1-\dfrac{\partial z}{\partial x}\right)\dfrac{\partial^2 z}{\partial y^2}=0$

17. $z=f(x\cos\alpha+y\sin\alpha-at)+\phi(x\cos\alpha+y\sin\alpha+at)$.

Ans. $\partial^2 z/\partial x^2+\partial^2 z/\partial y^2=(1/a^2)\times(\partial^2 z/\partial t^2)$

18. $y=f(x+at)+xg(x+at)$ **Ans.** $a^2(\partial^2 z/\partial x^2)-2a(\partial^2 z/\partial x\partial t)+(\partial^2 z/\partial t^2)=0$

19. $y=f(x-at)+xg(x-at)+x^2h(x-at)$. **(Jabalpur 1994)**

Ans. $\partial^3 y/\partial t^3+3a(\partial^3 y/\partial x\partial t^2)+3a^2(\partial^3 y/\partial x^2\partial t)+a^3(\partial^3 y/\partial x^3)=0$

20. $z=f(xy)+g(x+y)$ **Ans.** $x(y-x)r-(y^2-x^2)s+y(y-x)t+(p-q)(x+y)=0$

1.13 CAUCHY'S PROBLEM FOR FIRST ORDER EQUATIONS

The aim of an existence theorem is to establish conditions under which we can decide whether or not a given partial differential equation has a solution at all; the next step of proving that the solution, when it exists, is unique requires a uniqueness theorem. The conditions to be satisfied in the case of a first order partial differential equation are easily contained in the classic problem of Cauchy, which for the two independent variables can be stated as follows:

Cauchy's problem for first order partial differential equation

If (a) $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ *are functions which, together with their first derivatives, are continuous in the interval I defined by* $\mu_1 < \mu < \mu_2$.

(b) And if f (x, y, z, p, q) is a continuous function of x, y, z, p and q in a certain region U of the xyzpq space, then it is required to establish the existence of a function $\Phi(x,y)$ *with the following properties :*

(i) $\Phi(x,y)$ *and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.*

(ii) For all values of x and y lying in R, the point {x, y, $\Phi(x,y)$, $\Phi_x(x,y)$, $\Phi_y(x,y)$*} lies in* $\dfrac{U}{x}$ *and* $f[x,y,\Phi(x,y),\, \Phi_x(x,y),\, \Phi_y(x,y)] = 0$.

(iii) For all μ *belonging to the interval I, the point* $\{x_0(\mu),\, y_0(\mu)\}$ *belongs to the region R, and* $\Phi\{x_0(\mu),\, y_0(\mu)\} = z_0$

Stated geometrically, what we wish to prove is that there exists a surface $z = \Phi(x,y)$ *which passes through the curve C whose parametric equations are given by* $x = x_0(\mu),\ y = y_0(\mu), z = z_0(\mu)$ *and at every point of which the *direction (p, q, – 1) of the normal is such that* $f(x,y,z,p,q) = 0$

Problem 1. *State the properties of* $\Phi(x,y)$ *if there exists a surface* $z = \Phi(x,y)$ *which passes through the curve C with parametric equations* $x = x_0(\mu),\ y = y_0(\mu),\ z = z_0(\mu)$ *and at every point of which the direction* $(p,q,-1)$ *of the normal is such that* $f(x,y,z,p,z) = 0$. **(Delhi B.Sc. (H) 2002)**

Sol. Hint. Refer conditions (*i*), (*ii*) and (*iii*) of the above Art. 1.13

Problem 2. *Solve the Cauchy's problem for* $zp + q = 1$, *when the initial data curve is* $x_0 = \mu,\ y_0 = \mu,\ z_0 = \mu/2,\ 0 \le \mu \le 1$. **[Bangalore 2003; I.A.S. 2004]**

Sol. Given $$f(x,y,z,p,q) = zp + q - 1 = 0 \qquad \text{... (1)}$$

Given inital data curve $$x_0 = \mu, \qquad y_0 = \mu, \qquad z_0 = \mu/2, \qquad 0 \le \mu \le 1 \qquad \text{... (2)}$$

From (1), $$\partial f/\partial p = z, \qquad \partial f/\partial q = 1,$$

and $$\frac{\partial f}{\partial q}\frac{dx_0}{d\mu} - \frac{\partial f}{\partial p}\frac{dy_0}{d\mu} = 1 \times 1 - z \times 1 = 1 - \frac{1}{2}\mu \ne 0, \text{ for } 0 \le \mu \le 1.$$

Now, we have the following ordinary differential equations :

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*Let $$z = \Phi(x,y) \qquad \text{... (1)}$$
be the equation of the given surface

Let $$F(x,y,z) = \Phi(x,y) - z. \qquad \text{... (2)}$$

From (1) and (2), $$\frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} = \frac{\partial z}{\partial x} = p, \qquad \frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{\partial z}{\partial y} = q, \qquad \frac{\partial F}{\partial z} = -1$$

Since ∇F is normal to the surface $F(x,y,z) = 0,\ \partial F/\partial x,\ \partial F/\partial y,\ \partial F/\partial z$ i.e., $p, q - 1$ are direction ratios of the normal to $F(x, y, z) = 0$ or $z = \Phi(x,y)$.

$$\frac{dx}{dt}=\frac{\partial f}{\partial p}, \qquad \frac{dy}{dt}=\frac{\partial f}{\partial q} \qquad \text{and} \qquad \frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

or $\quad dx/dt = z, \qquad dy/dt = 1$ ... (3)

and $\quad dz/dt = p(\partial f/\partial p)+q\,(\partial f/\partial q) = pz+q = 1$, *by* (1) ... (4)

Integrating (3) and (4), $\quad y = t + C_1 \quad$ and $\quad z = t + C_2$ ... (5)

From (2), at $t = 0$, $\quad x(\mu, 0) = \mu, \quad y(\mu, 0) = \mu \quad$ and $\quad z(\mu, 0) = \mu/2$ ... (6)

Using (6), (5) reduces to $\quad y = t + \mu \quad$ and $\quad z = t + \mu/2$ ... (7)

Then, from (3) and (7), $dx/dt = t + \mu/2$ so that $x = (1/2)\times t^2 + (1/2)\times \mu t + C_3$ ... (8)

Using (6), (8) reduces to $\quad x = (1/2) \times t^2 + (1/2) \times \mu t + \mu$ ... (9)

Solving $y = t + \mu$ with (9) for μ and t in terms of x and y, we get

$$t = \frac{y-x}{1-(y/2)} \qquad \text{and} \qquad \mu = \frac{x-(y^2/2)}{1-(y/2)}$$

Putting these values in $z = t + \mu/2$, the required solution passing through the initial data curve is $\quad z = \{2(y-x)+x-y^2/2\}/(2-y)$.

OBJECTIVE PROBLEMS ON CHAPTER 1

Indicate the correct answer by writing (*a*), (*b*), (*c*) *or* (*d*)

1. Equation $p \tan y + q \tan x = \sec^2 z$ is of order
   (*a*) 1 (*b*) 2 (*c*) 0 (*d*) none of these **[Agra 2005, 2008]**
2. Equation $\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x\partial y) + (\partial z/\partial y)^2 = 0$ is of order
   (*a*) 1 (*b*) 2 (*c*) 3 (*d*) none of these **[Agra 2005, 2006]**
3. The equation $(2x + 3y)p + 4xq - 8pq = x + y$ is
   (*a*) linear (*b*) non-linear (*c*) quasi-linear (*d*) semi-linear **[Agra 2005, 06]**
4. $(x+y-z)\,(\partial z/\partial x)+(3x+2y)\,(\partial z/\partial y)+2z = x+y$ is
   (*a*) linear (*b*) quasi-linear (*c*) semi-linear (*d*) non-linear

Answers **1.** (*a*) **2.** (*b*) **3.** (*b*) **4.** (*b*)

MISCELLANEOUS EXAMPLES ON CHAPTER 1

Ex.1. *Formulate a partial differential equation by eliminating arbitrary constants a and b from the equation* $(x+a)^2+(y+b)^2+z^2=1$. *Examine whether the partial differential equation is linear or non-linear. Also,* find its order and degreee. **[Delhi Maths (H) 2008]**

Hint. Proceed as in Ex. 5(a), page 1.6 with $\lambda = 1$. *Thus we get the partial differential equation* $z^2\left\{(\partial z/\partial x)^2+(\partial z/\partial y)^2+1\right\}=1$, *which is non-linear partial differential equation of order one and degree two.*

Ex. 2. *Eliminate arbitrary constants a and b from the following equations :*

(*i*) $ax^2 + by^2 + z^2 = 1$ **(Delhi B.A. (Prog.) II 2010)**

(*ii*) $z = ax + (1 - a)\,y + b$ **(Lucknow 2010)**

Ans. (*i*) $z\,(z - px - qy) = 1$ (*ii*) $p + q = 1$, where $p = \partial z/\partial x, q = \partial z/\partial y$

Ex. 3. (*i*) *Eliminate the arbitrary function* ϕ *from* $p + x - y = \phi\,(q - x + y)$ **(Ranchi 2010)**

(*ii*) *State true or false with justification. Eliminating arbitrary function f from* $z = f(x^2 + y^2)$, *we get first order non-linear partial differential equation.* **(Pune 2010)**

Ans. (*i*) $(1+\partial^2 z/\partial x^2)\,(1+\partial^2 z/\partial y^2) = (\partial^2 z/\partial x\partial y - 1)^2$ (*ii*) False. see Ex. 4 (ii), page 1.21.

Ex. 4. (*i*) *Obtain the partial differential equation by eliminating arbitrary function of f and g from the equation* $v = \{f(r-at) + g(r+at)\}/r$ **(Nagpur 2010)**

Ans. Given $v = (1/r) \times \{f(r-at) + g(r+at)\}$ ...(1)

$(1) \Rightarrow \partial v/\partial t = (1/r) \times \{-af'(r-at) + ag'(r+at)\} = -(a/r) \times \{f'(r-at) - g'(r+at)\}$ ...(2)

$(2) \Rightarrow \partial^2 v/\partial t^2 = -(a/r) \times \{-af''(r-at) - ag''(r+at)\} = (a^2/r) \times \{f''(r-at) + g''(r+at)\}$ ...(3)

$(1) \Rightarrow \partial v/\partial r = (1/r) \times \{(f'(r-at) + g'(r+at)\} - (1/r^2) \times \{f(r-at) + g(r+at)\}$ ...(4)

$(4) \Rightarrow \partial^2 v/\partial r^2 = (1/r) \times \{f''(r-at) + g''(r+at)\} - (1/r^2) \times \{f'(r-at) + g'(r+at)\}$

$= -(1/r^2) \times \{f'(r-at) + g'(r+at)\} + (2/r^3) \times \{f(r-at) + g(r+at)\}$

$= (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r^2) \times \{f'(r-at)\} + g'(r+at)\} + (2/r^2) \times v$, using (1) and (3)

$= (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r) \times [\partial v/\partial r + (1/r^2) \times \{f(r-at)\} + g(r+at)\} + (2/r^2) \times v$

[Since from (4), $(1/r) \times \{f'(r-at) + g'(r+at)\} = \partial v/\partial r + (1/r^2) \times \{f(r-at) + g(r+at)\}$]

Thus, $\partial^2 v/\partial r^2 = (1/a^2) \times (\partial^2 v/\partial r^2) - (2/r) \times \{\partial v/\partial r + (1/r) \times v\} + (2/r^2) \times v$, using (1)

or $\partial^2 v/\partial r^2 = (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r) \times (\partial^2/\partial r)$, which is the required equation