GRAPH THEORY
MC-405

Assignment-I

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DTU/2K16/MC/13

9th September 2020

Q1) Model the following situation as a graph. Draw the graph, and give the corresponding adjacency matrix.

It is well known that in the Netherlands, there is a 2-lane highway from Amsterdam to Breda, another 2-lane highway from Amsterdam to Capelle aan der IJsel, a 3-lane highway from Breda to Dordrecht, a 1-lane road from Breda to Ede and another from Dordrecht to Ede, and a 5-lane superhighway from Capele aan den IJsel to Ede.

In this problem I assume that in both highways and roads there is a 2-way path i.e. cars can go in both directions, hence there is no requirement for a directed graph as any highway/road is equivalent for both interconnecting cities. I assume all places as vertices and denote ~~the~~ interconnecting edges/paths/highways as edges (undirected) weighted edges.

I represent the cities with following labels.

A = Amsterdamn

B = Breda

C = Cappele aan den IJsel

D = Dordrecht

E = Ede

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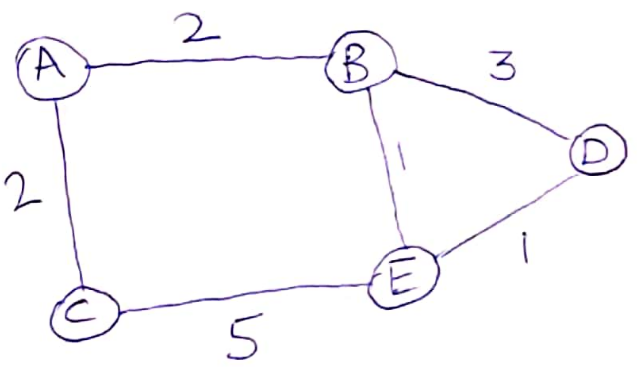
The undirected weighted graph G is denoted as:-



Here the edges denote the capacity of the highway/road with lanes per road. The adjacency matrix is as follows:-

$$G = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 3 & 1 \\ 2 & 0 & 0 & 0 & 5 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 5 & 1 & 0 \end{bmatrix} \end{array}$$

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Q2) Are the sequences given below graphical?

i) $(6, 6, 5, 4, 3, 3, 2)$

This sequence has 3 odd vertices $\{5, 3, 3\}$. This isn't possible in a graph as all graphs ~~Eton~~ must have an even number of odd vertices, hence this sequence isn't graphical.

ii) $(6, 6, 5, 4, 3, 3, 1)$

For a sequence to be graphical some necessary, but insufficient conditions are:-

$$\deg_G(v) \leq |V| - 1$$

$$\sum_{v \in V(G)} \deg_G(v) \equiv_2 0 \pmod 2 \quad (\text{is even})$$

Now, for this sequence there are $7 = n$ vertices and both conditions are satisfied, but that isn't sufficient to call this sequence graphical. We see that there are 2 vertices with degree 6 in this graph. This implies that 2 vertices are connected to every other vertex.

This implies that every vertex must have minimum degree 2. i.e. $\deg_G(v) \geq 2 \ \forall \ v \in V(G)$, but it is also given that one vertex has degree 1 which isn't possible, hence the sequence isn't graphical.

Q3) Give an example of a degree sequence that is realizable as the degree sequence by only a disconnected graph.

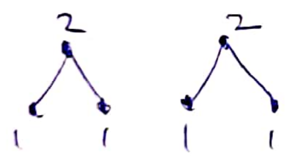
We consider trivial sequences and also forests.

i) $(0,0)$

ii) $(0,0,0,0,0)$

iii) $(1,1,1,1,1,1)$ The disjoint 1-regular components

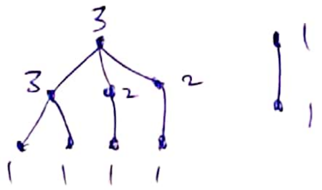
iv) $(2,2,1,1,1,1)$ Forest with P_3, P_3



This has $|V(G)| = n = 6$ and $|E(G)| = m = 4$. If we create connected graph from this forest and connected component has no cycles, then it is a tree and ~~therefore~~ must have $n-1 = 5$ edges > 4, hence not possible. If we take cycle then smallest cycle is C_3 which will result in 3 vertices with degree 2 and hence also not possible.

We can also take other forests such as $\{P_3 P_4\}$, $\{P_4 P_5\}$.....

v) (3, 3, 2, 2, 1, 1, 1, 1, 1, 1) A more complicated forest example



Same case $n = 10$ vertices, $m = 1 + 7 = 8$ edges. When combined forms tree with 10 vertices and hence requires 9 edges which isn't possible.

These are a few examples.

Q4) A graph G has adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

a) Is G a simple graph?

For G to be a simple graph it must not have any loops and it must not have parallel edges. G doesn't have any loops but it does have parallel edges as in a simple graph edge connection is denoted by 1 but in the adjacency matrix above $A(1,4)=2$ hence <u>G is not simple</u>.

b) What is degree sequence of G?

Degree of vertex v is row sum and vertice degree can be used to find the degree sequence.

$S = (4, 2, 3, 3, 2)$ reordering ⇓

$S = (4, 3, 3, 2, 2)$

c) How many edges does G have?

$$\deg(G) = \sum_i \sum_j A_{ij} = 14$$

$$|E(G)| = \frac{1}{2}\deg(G) = \frac{14}{2} = 7 \text{ edges}$$

Q5) If δ and Δ are respectively the minimum and maximum degrees of a graph G, show that $\delta \leq \frac{2m}{n} \leq \Delta$, where G is (n, m) graph.

We know by definition that

$$\delta = \min\{\deg_G(v)\ \forall v \in V(G)\}$$

$$\Delta = \max\{\deg_G(v)\ \forall v \in V(G)\}$$

Hence, this clearly implies :-

$$\delta \leq \deg_G(v) \leq \Delta \quad \forall v \in G$$

$$\sum_{v \in V(G)} \delta \leq \sum_{v \in V(G)} \deg_G(v) \leq \sum_{v \in V(G)} \Delta$$

We know that $\sum_{v \in V(G)} 1 = |V(G)| = n$ (No. of vertices)

$$\delta n \leq \sum_{v \in V(G)} \deg_G(v) \leq \Delta n \quad \text{—— (1)}$$

Now, we also know that $\sum_{v \in V(G)} \deg_G(v) = 2|E(G)| = 2m$

as each edge in graph G is counted twice for both vertices it is incident on.

$$\sum_{v \in V(G)} \deg_G(v) = 2|E(G)| = 2m \quad \text{—— (2)}$$

Using (2) in (1), we get

$$\delta n \leq 2m \leq \Delta n$$

$$\delta \leq \frac{2m}{n} \leq \Delta \quad \text{Dividing both sides by } n$$

Hence proved ∎

Q6) Using techniques from Graph Theory, show that

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Let us consider a fully connected graph with $n+1$ vertices K_{n+1}. ~~For~~ ~~th~~ Let the vertices in this graph be $V(K_{n+1}) = \{v_1 v_2 v_3 \dots v_{n+1}\}$ $|V(K_{n+1})| = n+1$

and as this graph is fully connected, we also know that

$|E(K_{n+1})| = \binom{n+1}{2} = {}^{n+1}C_2$ in a fully connected graph.

We can also count number of edges using another method.

- Method of construction

Let there be an empty graph G with $n+1$ vertices and we need to add edges to convert it into fully connected graph K_{n+1}.

$V(G) = \{v_1 v_2 v_3 \dots v_{n+1}\}$ $|V(G)| = n+1$

Now, every vertex is connected to every other vertex, so adding n edges from v_1 to v_i $\forall$ $i = 2, 3, 4 \dots n+1$

So now we have added n edges in the graph G.

$|E(G)| = n$

We now add edges for v_2, but $v_1 v_2$ is already an edge so we add for v_i $\forall$ $i = 3, 4 \dots n$. We add $n-1$ edges.

$|E(G)| = n + (n-1)$

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Similarly if we add edges for v_3, but v_1v_3 and v_2v_3 are already present, so we add $n-2$ edges.

$$|E(G)| = n + (n-1) + (n-2).$$

We repeat this process for all vertices $v \in V(G)$ in the graph and for v_2 we add only one edge v_2v, and we add zero (0) edges for v_1.

In general we add edges v_iu where $u = \{v_{i-1}\, v_{i-2} \cdots v_1\}$ $\forall\, v_i, u \in V(G)$

and we get: $|E(G)| = n + (n-1) + (n-2) + (n-3) + \cdots 3 + 2 + 1 + 0$

$$|E(G)| = 1 + 2 + 3 + \cdots + n$$
$$= \sum_{i=1}^{n} i$$

But we know that in fully connected graph K_{n+1} with $n+1$ vertices there are $\binom{n+1}{2}$ edges, hence

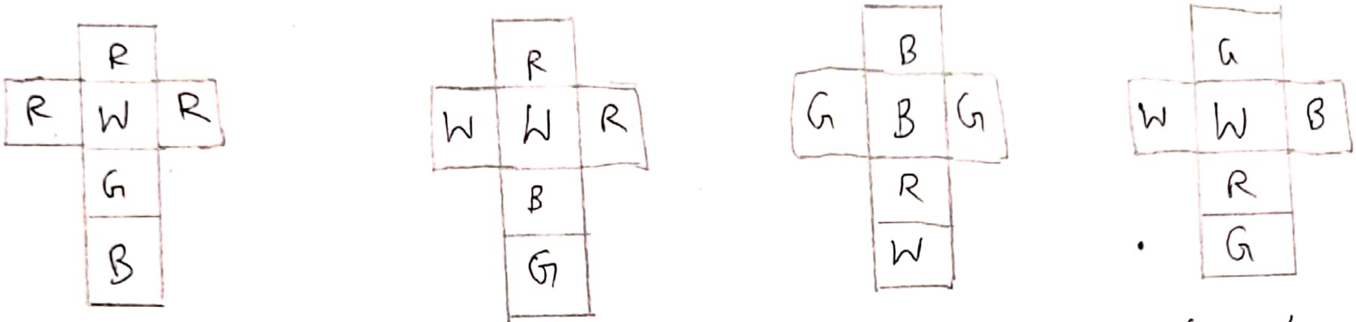
$$\sum_{i=1}^{n} i = \binom{n+1}{2} = {}^{n+1}C_2 = \frac{(n+1)!}{2!\,(n-1)!}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

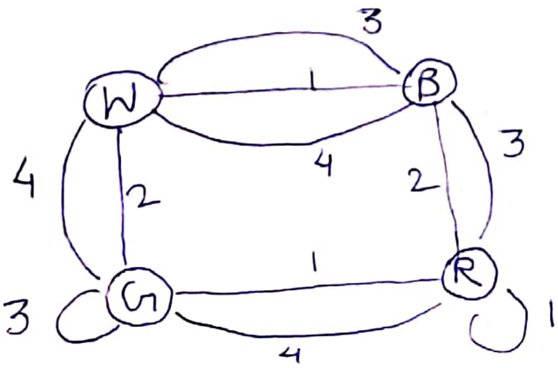
Hence proved ∎

Q7) The figure below shows 4 unwrapped cubes that form the instant Insanity puzzle. The letters R, W, B and G stand for Red, White, Blue, Green. The objective of the puzzle is to stack the blocks in a pile of 4 in such a way that each color appears exactly once on each of the 4 sides of the stack.



Cube 1 Cube 2 Cube 3 Cube 4

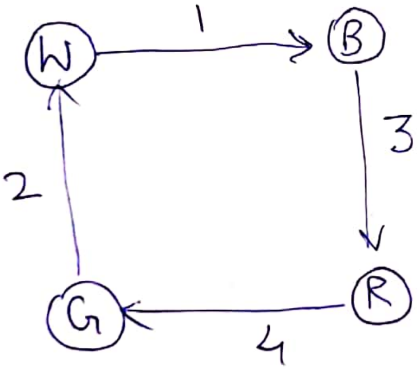
Given the already coloured cubes of the 4 distinct colors, we generate a graph which gives clear picture of all the positions of colors in all the cubes. The resultant graph will contain four vertices; one for each color and we will number each edge from one through four (one number for each cube). If an edge connects 2 vertices (Red and Green) and the number of edge is 3, then it means that in the third cube has red and green faces opposite to each other.



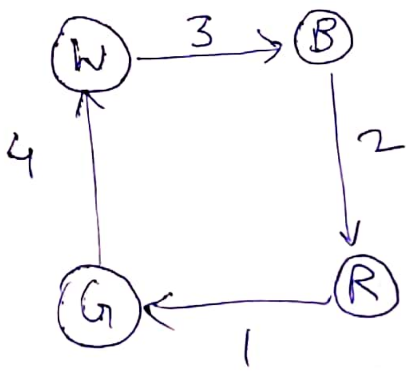
To find a solution, we need the arrangement of four faces of each of the cubes. To represent the information of two opposite faces we need a directed subgraph instead of an undirected one because two directions can only represent two opposite faces, but not whether face should be at front or back.

So, if we have two directed subgraphs, we can actually represent all the four faces (which matter) of all four cubes.

First directed graph represents the front and rear faces and second directed graph will represent the left and right faces.



Front → Back



Left → Right

We can't randomly select any two subgraphs, so what is the criteria for selecting?

We need to choose graphs such that:

i) the two subgraphs have no edges in common, because if there is an edge which is common that means at least one cube has the pair of opposite faces of exactly same color.

ii) A subgraph contains only one edge from each cube, because the subgraph has to account for all the cubes and one edge can completely represent a pair of opposite faces.

iii) A subgraph can contain only vertices of degree 2, because degree 2 implies that color can be present at only faces of 2 cubes. Eight faces / Four colors = 2 faces / color.

From the first subgraph we derive :-

i) First cube will have white in front face and Blue in rear.

ii) Second cube will have green in front face and white in rear

iii) ~~Thir~~ Third cube will have blue in front face and red in the rear.

iv) Fourth cube will have red in front face and green in rear.

From second subgraph we derive the left and right face colors :-

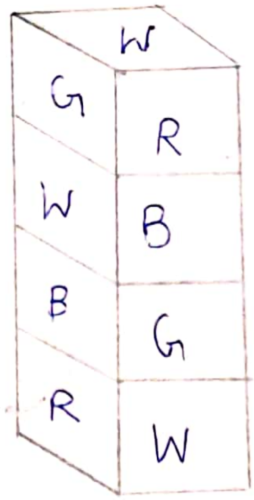
i) 1^{st} cube will have red in left face and green in the right

ii) 2^{nd} cube will have blue in the left face and red in the right

iii) 3^{d} cube will have white in left face and blue in the right

iv) 4^{th} cube will have green in left face and white in the right.

The solution will look as follows:—



Q.8) Prove that a simple graph with n vertices and k components can have atmost $(n-k)(n-k+1)/2$ edges.

<u>Lemma</u>: Let n_i represent the number of vertices ~~with~~ in component with degree k. and let $n_1, n_2, n_3 \ldots\ldots n_k \in \mathbb{N}$. Then

$$\sum_{i=1}^{k} n_i^2 \leq \left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right).$$

<u>Proof</u>:

We have

$$(n_1-1)+(n_2-1)+\cdots(n_k-1) = (n_1+n_2+n_3+\cdots n_k)-k$$

Squaring on both sides we get

$$[(n_1-1)+(n_2-1)+\cdots(n_k-1)]^2 = [(n_1+n_2+\cdots n_k)-k]^2$$

$$\sum_{i=1}^{k}(n_i-1)^2 + \sum_{i=1}^{k}\sum_{j=1, j\neq i}^{k}(n_i-1)(n_j-1) = \left(\sum_{i=1}^{k} n_i\right)^2 - 2k\sum_{i=1}^{k} n_i + k^2$$

$$\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k + \underbrace{\sum_{i=1}^{k}\sum_{j=1, j\neq i}^{k}(n_i-1)(n_j-1)}_{\text{positive}} = \left(\sum_{i=1}^{k} n_i\right)^2 - 2k\sum_{i=1}^{k} n_i + k^2$$

Hence, we have

$$\sum_{i=1}^{k} n_i^2 \leq 2\sum_{i=1}^{k} n_i - k + \left(\sum_{i=1}^{k} n_i\right)^2 - 2k\sum_{i=1}^{k} n_i + k^2$$

or equivalently,

$$\sum_{i=1}^{k} n_i^2 \leq \left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right)$$

Using this lemma, we now prove the theorem

<u>Theorem</u>: A simple graph with n vertices and k components can have at most $\frac{1}{2}(n-k)(n-k+1)$ edges.

<u>Proof</u>. Let X be a graph with k components. Let n_i be the number of vertices in the i^{th} component, where $1 \le i \le k$. Then, the number of edges in the graph is equal to the sum of edges in each of it's components.

$$M = |E(X)| = \sum_{i=1}^{k} m_i \quad (m_i \text{ is number of edges in } i^{th} \text{ component})$$

Thus X has maximum number of edges if each component is a complete graph. Hence, the maximum possible number of edges in the graph X is $\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

$$\max(M) = \sum_{i=1}^{k} \binom{n_i}{2}$$

But from lemma above we have :-

$$\frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \le \frac{1}{2}\left[\left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right)\right] - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$\le \frac{1}{2}(n-k)(n-k+1)$$

Hence proved ■

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Q9) Show that for a simple Bipartite Graph $m \leq \frac{n^2}{4}$, where $G = (n; m)$

Let us consider a simple bipartite graph $K_{v_1 v_2}$ where v_1 and v_2 are the bipartite set sizes.

Such that $|X| = v_1$
$|Y| = v_2$

Now, the edges will be maximum when this is a fully connected bipartite graph. So, number of edges in fully connected bipartite graph.

$$\cancel{E(G)} \quad |E(K_{v_1 v_2})| = |X| \cdot |Y|$$
$$= v_1 v_2 \quad \text{———} \quad (1)$$

As there exists edge between every possible element of X and Y. Now let the ~~total~~ number of vertices in bipartite graph be n:-

$$|V(K_{v_1 v_2})| = n$$

So, $v_1 + v_2 = n$ ——— (2)

Putting (2) in (1):-

$$\max\{|E(K_{v_1 v_2})|\} = v_1 v_2$$
$$= v_1 (n - v_1)$$

To find maxima we differentiate with respect to (w.r.t.) v_1

$$\frac{d}{dv_1} max\{|E(K_{v_1v_2})|\} = \frac{d}{dv_1}(v_1n - v_1^2) = 0$$

$$\Rightarrow n - 2v_1 = 0$$

$$\boxed{v_1 = \frac{n}{2}}$$

So, $max\{|E(K_{v_1,v_2})|\} = v_1v_2$

$$= v_1(n - v_1)$$

$$= \frac{n}{2}\left(n - \frac{n}{2}\right)$$

$$= \frac{n^2}{4} \quad \text{— (3)}$$

So, from (3) we can clearly see that

$$|E(K_{v_1,v_2})| \leq \frac{n^2}{4}$$

$$m \leq \frac{n^2}{4}$$

Hence proved ■

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Q10) Show that every simple graph of order n is isomorphic to a subgraph of the complete graph with n vertices.

Consider the graph G of order n and consider the complete graph K_n of order n. Now, let us create a bijection $\sigma : V(G) \to V(K_n)$ such that

$$\sigma(v_i) = u_i \quad \forall\, i = 1,2,3 \ldots n \quad \exists\, v_i \in V(G) \; u_i \in V(K_n)$$

Now, let us construct a subgraph S from K_n such that edge $xy \in S$ iff $\sigma^{-1}(x)\sigma^{-1}(y) \in E(G)$ otherwise $xy \notin E(S)$.

Now, we show that we have an isomorphism with G and S (A subgraph of K_n). The σ function is bijective, hence $\sigma(v) \in V(S)\ \forall\, v \in V(G)$. Also, when

$$xy \in E(G)$$
$$\sigma(x)\sigma(y) \in E(S) \quad \text{(By construction)}$$

and when $xy \notin E(G)$

$$\sigma(x)\sigma(y) \notin E(S) \quad \text{(Also, by construction)}$$

Hence $G \cong S$ and the theorem is proved. ∎

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Q 11) Show that two graphs need not be isomorphic even when they both have the same order and same size.

Let us consider the graphs $K_{1,3}$ and P_4.

P_4

order: 4

size: 3

$K_{1,3}$

order: 4

size: 3

In the above examples both graphs have samee size and order, yet they are not isomorphic. Let us ~~writh~~ write their respective degree sequences :-

$S(K_{1,3}) = \{3, 1, 1, 1\}$ $\qquad$ $S(P_4) = \{2, 2, 1, 1\}$

We clearly see that there exists an vertex of degree 3 in $K_{1,3}$ wheras the highest degree in P_4 is 2. Hence the 2 graphs despite having same size and order can't be isomorphic.

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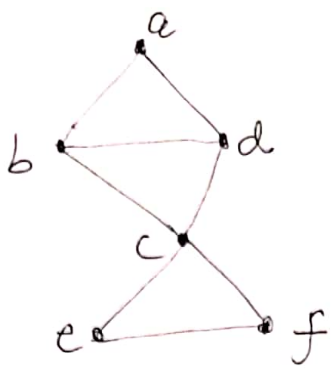
Q12) For each of the following sequences of vertices, state whether or not it represents a walk, Path, closed Walk, circuit or cycle in the illustrated graph.



	Walk	Path	Closed Walk	Circuit	Cycle
a) abcefcbd	✓	✗	✗	✗	✗
b) abcefcd	✓	✗	✗	✗	✗
c) abcefcdba	✓	✗	✓	✗	✗
d) bcefcdb	✓	✗	✓	✓	✗
e) bcdb	✓	✗	✓	✓	✓
f) abefcd	✗	✗	✗	✗	✗

Q13) Show that it is not possible to have a group of 7 people such that each knows exactly 3 persons in the group.

Firstly let us model this problem with a graph G.
Let the 7 people be 7 vertices on this graph G and let the people they know be represented as edges in the graph. If $uv \in E(G)$ then that implies that u knows v and v knows u.

Now, according to the problem every one knows exactly 3 people in the graph. So, $deg_G(v) = 3 \ \forall \ v \in V(G)$.

Now, no. of edges in graph G $|E(G)| = \frac{1}{2} deg(G)$ where

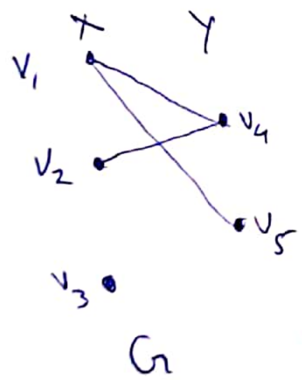
$$deg(G) = \sum_{v \in V(G)} deg_G(v)$$

$$deg(G) = \sum_{v \in V(G)} 3 = 7 \cdot 3 = 21 \Rightarrow\Leftarrow$$

This is a contradiction as we clearly know that the degree of a graph should be even so that edges $|E(G)| \in \mathbb{N}$, but if $deg(G)$ is odd that's not possible, hence the assumption that in a group of 7 people everyone knows exactly 3 people is incorrect.

Q14) Show that the complement of a bipartite graph need not be a bipartite graph.

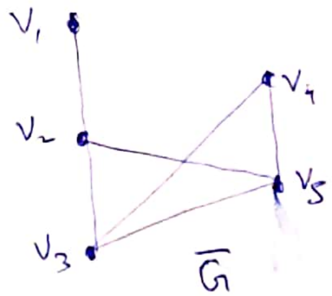
Let us consider the following bipartite graph:-



This graph G is bipartite as for every edge $e \in E(G)$ it can be represented as xy such that $x \in X$ and $y \in Y$.

Now, we take complement of this graph G, such that the complement G' will contain edges $e \in E(G')$ iff $e \notin E(G)$ and $V(G') = V(G)$

The complement G' :-



We clearly see that the resulting complement contains ~~to~~ 2 3-cycles $\{v_2v_3v_5v_2, v_3v_4v_5v_3\}$ These are C_3 isomorphic sub-graphs.

We also know that any ~~bipa~~ graph which contains an odd length cycle is not bipartite, hence the resulting graph $\overline{G}$ can't be bipartite.

Theorem: A cycle of odd length is not a bipartite graph

Proof: Let us have a cycle C_n of odd length where n is odd. Now, let us create 2 sets X and Y to form the bipartite sets and let us add vertices in them as

$$X = \{v_1 v_3 v_5 v_7 \ldots\ldots v_{n-2}\}$$

$$Y = \{v_2 v_4 v_6 \ldots\ldots v_{n-1}\}$$

We put all odd indexed vertices in X and all even indexed vertices in Y. We have not placed v_n in any set till now.

Now, we know from the definition of a cycle C_n

$$E(C_n) = \{v_iv_{i+1} \mid i=1,2\ldots\ldots n-1\} \cup \{v_nv_1\}$$

Hence ~~exist~~ there exists no edge between vertices whose index difference >1 except v_1 and v_n.

So, no vertex in set X is adjacent to each other

as $|v_i - v_j| \equiv_2 0 \bmod 2 \;\; \forall\, v_i, v_j \in X \;\; i \neq j$

Same applies for set Y, hence sets X and Y form 2 bipartite sets.

But, we have to place v_n also in one of these sets.

v_n is adjacent to both v_1 and v_{n-1}. $v_1 \in X$ and $v_{n-1} \in Y$, hence irrespective of the set we put it in, it will not form a bipartite set.

Hence, cycle of odd length can't form a bipartite set.

Q15) A simple graph that is isomorphic to it's complement is called self complementary graph. Find a self-complementary graph of order 4.

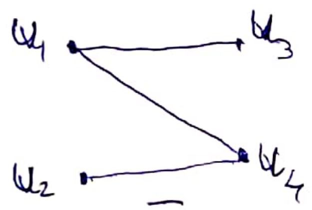
Let us consider the following graph G.

G

Now, let us take the complement of G, $\bar{G}$ such that

$V(\bar{G}) = V(G)$ and $e \in E(\bar{G})$ iff $e \notin E(\bar{G})$ Also

$e \notin E(\bar{G})$ iff $e \in E(\bar{G})$.



$\bar{G}$

The graphs G and $\bar{G}$ are isomorphic and we prove that by defining a bijection $\sigma : V(G) \to V(\bar{G})$.

$\sigma(v_1) = u_3$ $\sigma(v_3) = u_4$

$\sigma(v_2) = u_1$ $\sigma(v_4) = u_2$

We show that if $xy \in E(G)$, then $\sigma(x)\sigma(y) \in E(\bar{G})$ $\forall xy \in V(G)$

$v_1v_2 \in E(G)$

$\sigma(v_1)\sigma(v_2) = u_3u_1 \in E(\bar{G})$

$v_3v_4 \in E(G)$

$\sigma(v_3)\sigma(v_4) = u_4u_2 \in E(\bar{G})$

$v_2v_3 \in V(G)$

$\sigma(v_2)\sigma(v_3) = u_1u_4 \in E(\bar{G})$

Also, $|V(G)| = |V(\bar{G})|$ $|E(G)| = |E(\bar{G})|$

Hence, G and $\bar{G}$ are self complementary isomorphic graphs.

Q16) State and prove the characterization of Eulerian graph. Also give an example of an Eulerian graph with 5 vertices and 8 edges.

The following theorem by Euler characterizes Eulerian graphs. Euler proved the necessity part and the sufficient part was proved by Hierholzer, hence is often called the Euler - Hierholzer theorem:-

<u>Theorem</u>: A connected graph G is an Euler graph iff all vertices of G are of even degree.

We need to prove both sides of this theorem to prove that condition is necessary and sufficient.

Proof 1

→ Suppose that G is connected Eulerian graph, ~~it was~~ then all vertices have even degree.

Let $W: u \xrightarrow{t} u$ be an euler tour in G and let $v \neq u \in G$ such that the vertex v occurs k times in the tour W.

Every time we enter the vertex v using a new edge and leave on a distinct edge. Let us say that we enter the vertex ~~k~~ v k times i.e. the vertex occurs k times on the tour W, then:

$$\deg_G(v) = 2k \text{ which is even}$$

Hence for $\forall v \in V(G)$ $\deg_G(v)$ is even if it is not the start and end vertex.

Now, we consider the start and end vertex $u \in V(G)$ in the Euler tour $W: u \xrightarrow{*} u$.

Let us assume that the vertex u is encountered k times in the tour and each time we use a different ~~e~~ edge to enter and a different edge to leave the vertex u.

Also, we initially use an edge to leave u and finally use an edge to return to u and not leave again.

So, $\deg_G(u) = 2k$ (for k instances in W) $+1$ (for leaving S)
$+1$ (for entering back again)

$= 2k+2$

which is even degree

Hence, every vertex in Eulerian graph G is of even degree.

<u>Proof 2: Sufficiency</u>

If a graph G is even, then it is Eulerian.

Let G be a non-trivial connected graph whose vertices all have even degree. Let us consider the longest trail in the graph

$W: v_0 e_1 v_1 e_2 v_2 e_3 \ldots\ldots\ldots e_{\ell-1} v_{\ell-1} e_\ell v_\ell$

Since we know that all vertices have even degree, if v_ℓ is not the last vertex, that implies we still have some vertices left, but that isn't possible ~~as~~, hence v_ℓ must be the last vertex and also $v_\ell = v_0$ as this is the longest trail and with all ~~ed~~ vertices having even degree $v_\ell = v_0$.

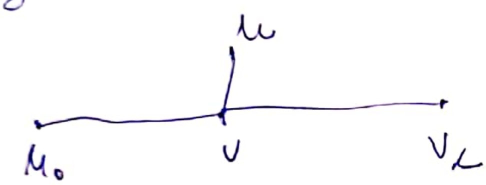
So, W is a closed trail.

Suppose that W is not an Euler tour. Since G is connected, there exists an edge which we have not traversed on.

$$W = u_0 u_1 u_2 \cdots v_i \cdots u \cdots u_n$$

There exists an $f(\text{edge}) = v_i u$ which we haven't traversed.

We can now create a longer trail using this untraversed edge



$$W' : uv \cdots \underbrace{v_\ell v_0}_{\text{same vertex}} \cdots v$$

We know that $v_\ell = v_0$, hence we begin at u, go to v, go to v_ℓ then $v_\ell = v_0$ go from v_0 to v. This is now a longer trail than we have for W.

$$|W'| = |W| + 1$$

$$|W'| > |W| \Rightarrow\Leftarrow$$

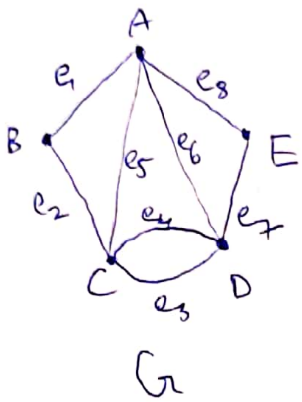
This is a contradiction as we clearly stated when we began our proof that W is the longest trail in the graph G, hence our assumption that G is not Eulerian was incorrect, and G is in fact Eulerian. So, given a non-trivial connected graph of even degree, it contains an Eulerian tour. ∎

Example of Eulerian Graph with 5 vertices and 8 edges :-

The vertices are 5, hence max degree should be 4.

A degree sequence could be :-

$S = (4, 4, 4, 2, 2)$



Here, $deg_G(A) = 4$

$deg_G(C) = deg_G(D) = 4$

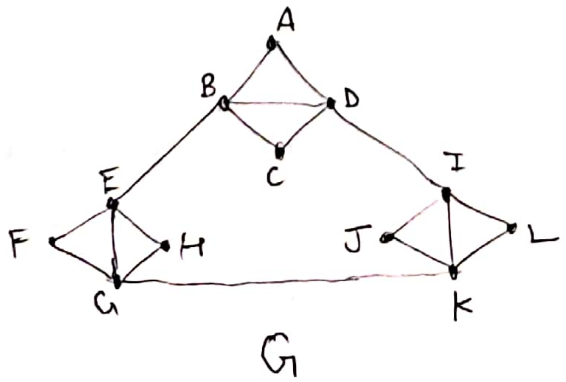
$deg_G(B) = deg_G(E) = 2$

Now, an eulerian cycle in G could be.

$W: Ae_1Be_2Ce_3De_4Ce_5Ae_6De_7Ee_8A$

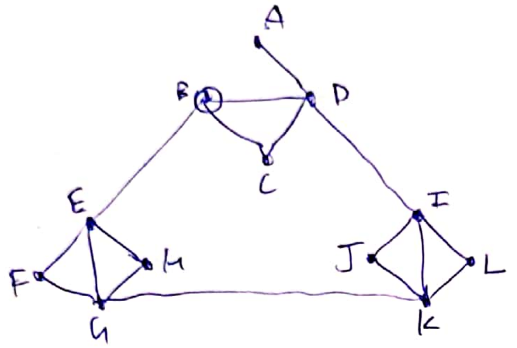
This is a closed Euler trail with no repeated edges.

Q17) Apply Fleury's Algorithm beginning with vertex A, to find an eulerian trail in the following graph. In applying the algorithm at each state choose the edge (from those available) which visits the vertex which comes first in alphabetical order (Lexographical order).



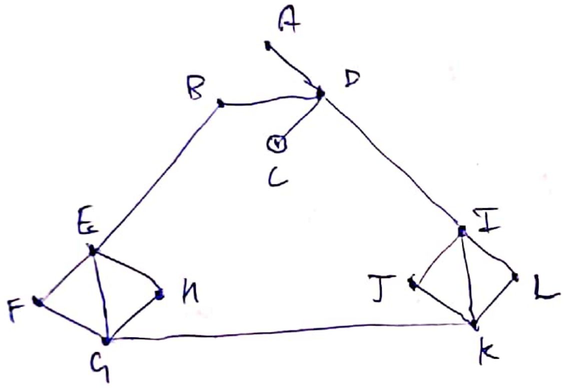
Since $\deg_G(v)$ is even $\forall\, v \in V(G)$ there will exist an Euler path from $A \xrightarrow{*} A$ in the graph G.

Starting with A, we select edge AB as B is lexographically smallest ~~we~~ neighbouring vertex of A.



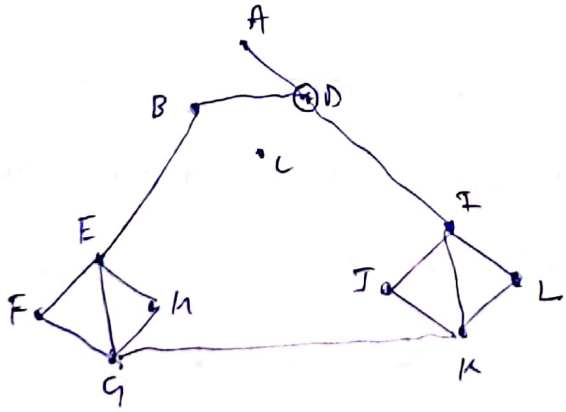
circuit : AB

From {BD, BC, BE} we select lexographically smallest BC



Circuit : ABC

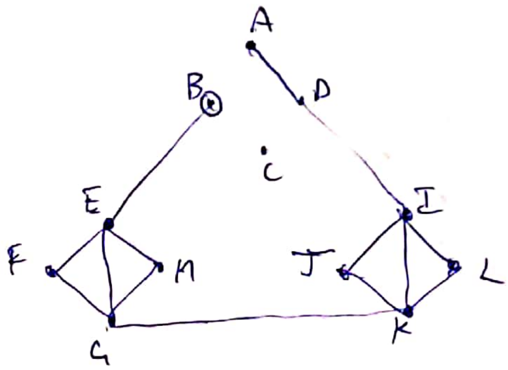
We have only one edge ~~E~~ CD which we can remove now.



circuit : ABCD

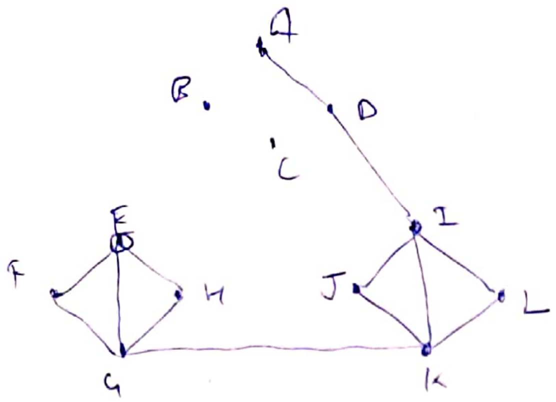
We have edges {DA, DB, DI}

Removing DA will create disconnected components hence not allowed, we select lexographically smallest DB.



circuit : ABCDB

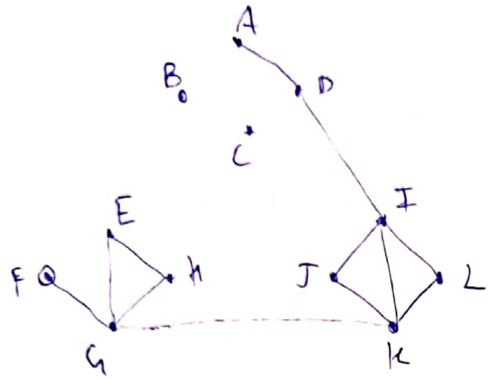
We have only one edge {BE} so we remove that.



Circuit : ABCDBE

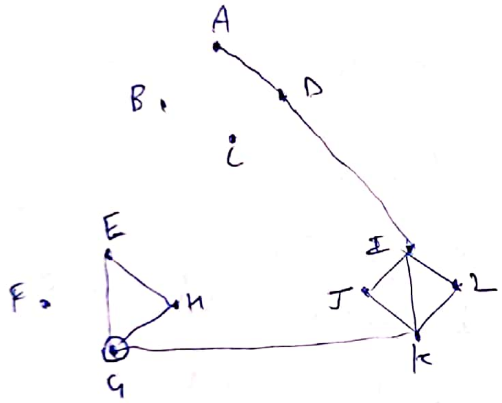
We have edges {EF, EG, EH}

We select lexographically smallest EF.



Circuit ABCDBEF

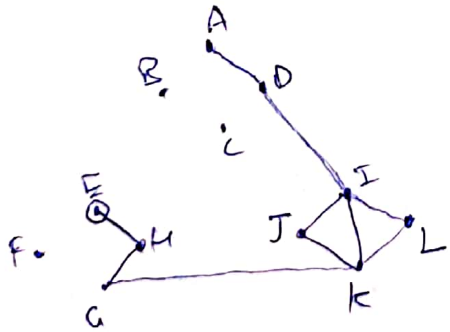
Only one edge we can select {FG}



Circuit : ABCDBEFG

We have 2 edges {GE, GH}

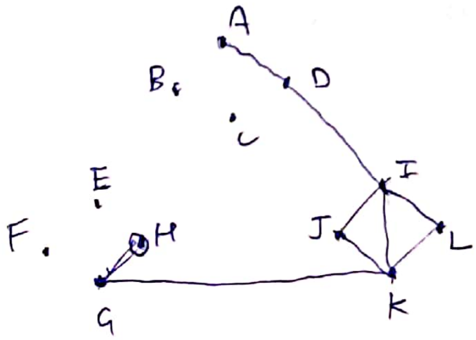
We select lexographically smallest (GE).



Circuit : ABCDBEFGE

We have only one edge {EH}

which we select.



Circuit : ABCDBEFGEH

We have only one edge {HG} which we select.

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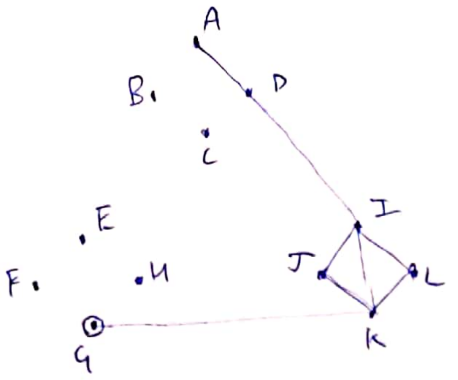
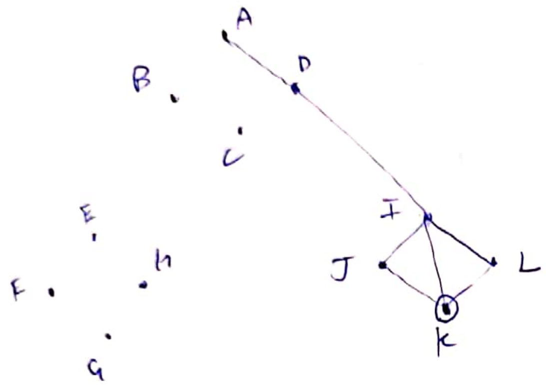


circuit : ABCDBEFGEHG

We have only one edge {GK}

which we will now remove.

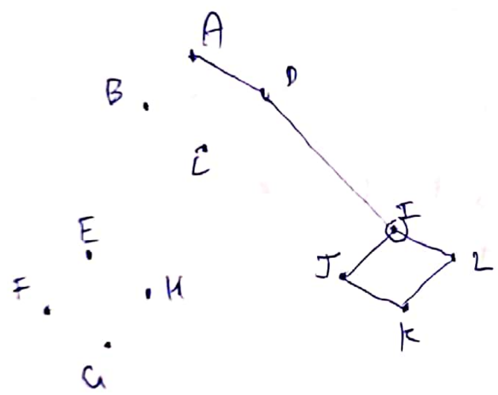


circuit: ABCDBEFGEHGK

We have edges {KI, KJ, KL}

We remove lexographically

smallest KI.



circuit : ABCDBEFGEHGKI

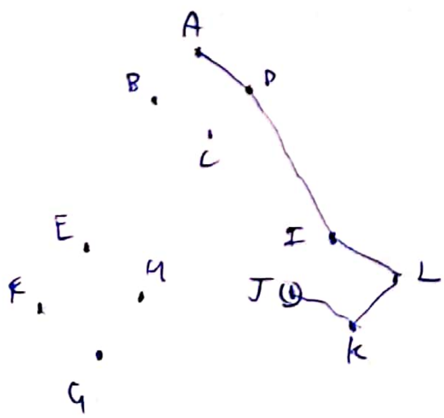
We have edges {ED, EJ, EL}

Removing ED forms disconnected

component, hence not allowed.

We remove lexographically smallest

IJ.



circuit : ABCDBEFGEHGKIJ

We have only one edge, hence

we remove {JK}.

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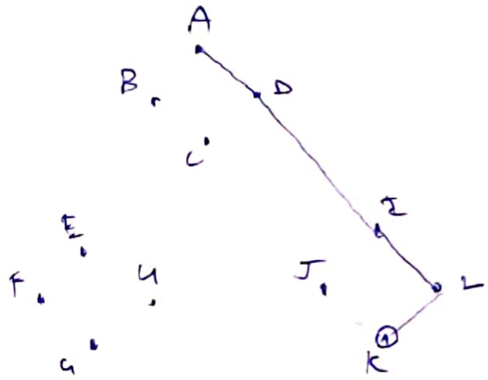
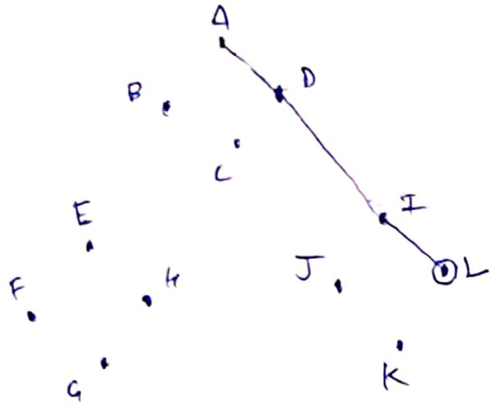


Circuit: ABCDBEFGEHGKIJK

We have only one edge {K L}

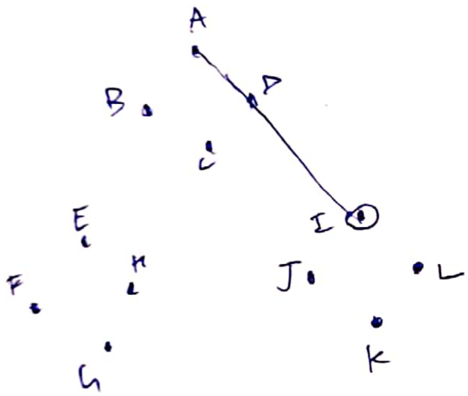
which we remove.



Circuit: ABCDBEFGEHGKIJKL

We have only one edge {L I} which

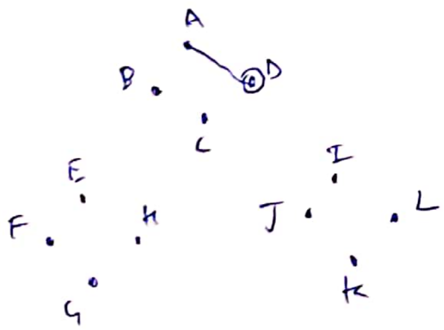
we remove.



Circuit: ABCDBEFGEHGKIJKLI

We have only one edge {I D}

which we will remove.



Circuit: ABCDBEFGEHGKIJKLID

We have only one edge {D A}

which we will remove.

After applying Fleury's algorithm

we obtain the Eulerian Tour in

graph G as follows:-

T: ABCDBEFGEHGKIJKLIDA

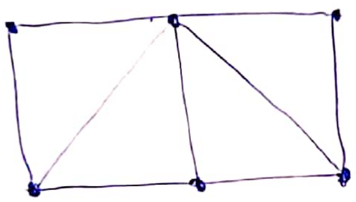
Q18) Draw a graph which is :-

a) Hamiltonian & Eulerian

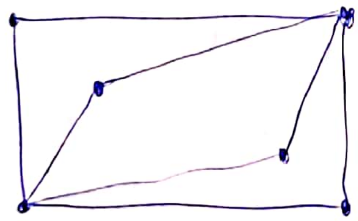
All cycles are both Eulerian and Hamiltonian, hence the following are valid examples.

C_3 C_4 C_5

b) Hamiltonian & Non-Eulerian



c) Non-Hamiltonian & Eulerian



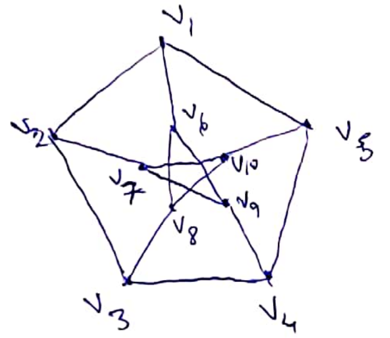
d) Non-Hamiltonian & Non-Eulerian

$K_{1,3}$ $K_{1,4}$

Q 19) Define & Draw the Peterson Graph. Is peterson graph Hamiltonian? If yes, then write down the path.

The petterson Graph is an undirected graph with 10 vertices and 15 edges. It is a small graph that serves as a useful example and counterexample for many problems in graph Theory. The Petersen Graph is named after Julius Petersen who in 1898 constructed it to be the smallest bridgeless cubic graph with no 3-edge coloring.

It is usually drawn as a pentagram within a Pentagon corresponding vertices attatched to each other.



The Peterson graph has no Hamiltonian cycle, but does have a Hamilton path. One such path P is:-

P: $v_1 v_2 v_3 v_8 v_6 v_9 v_7 v_{10} v_5 v_4$

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Q20) Prove that a simple connected graph with n vertices and m edges is Hamiltonian if $m \geq \frac{n^2-3n+6}{2}$

Proof: Let G be a simple Graph with n vertices and m edges. Let u, v be 2 non-adjacent vertices of G. Consider the subgraph H of G induced by the vertices $v_1 v_2 \ldots v_{n-2}$. This is the subgraph containing all edges of G with both end points from the set $\{v_1, v_2, v_3 \ldots v_{n-2}\}$. Since the number of edges in H is at most $\binom{n-2}{2} = \frac{1}{2}(n-2)(n-3)$

There are at least $m - \binom{n-2}{2}$ edges in G that have endpoints in u or v.

This implies $\Rightarrow$ $\deg(u) + \deg(v) \geq m - \binom{n-2}{2}$

$$\geq \frac{n^2-3n+6}{2} - \frac{(n-2)(n-3)}{2}$$

$$\geq \frac{n^2-3n+6}{2} - \frac{n^2-5n+6}{2}$$

$$\geq \frac{2n}{2}.$$

$$\deg(v) + \deg(u) \geq n \quad \text{--- (1)}$$

We also have Ore's Theorem which states that sum of every pair of vertices in the graph if greater than or equal to the number of vertices is a Hamiltonian graph.

Hence from (1), we can clearly state that using Ore's theorem the graph G with $m \geq \frac{n^2-3n+6}{2}$ is a Hamiltonian Graph.

Q2 b) Prove that a simple connected graph with n vertices $n \geq 3$ is Hamiltonian if $\deg(v) \geq \frac{n}{2}$ $\forall$ $v \in V(G)$

Proof: First we show that the graph is connected. Suppose G is not connected so that G has at least 2 components. Then we could partition $V = V_0 \cup V_1$ into two non-empty pieces so there might be more than 2 components. Instead V_0 and V_1 are unions of components; ~~instead~~ since $n = |V| = |V_0| + |V_1|$, we must have either $|V_0| \leq n/2$ or $|V_1| \leq n/2$. Say $|V_0| \leq n/2$ and pick any $v_0 \in V_0$. Then $\deg(v_0) \geq n/2$, which isn't possible

$\Rightarrow\Leftarrow$

This is a contradiction, hence the graph G with $\deg(v) \geq n/2$ is a connected graph, which is also provided to us.

We prove there exists a Hamilton circuit by induction. Let P_m be the statement "As long as $m+1 \leq n$, there exists a path visiting $m+1$ distinct vertices with no repetitions".

P_0 is trivial – just like a single vertex.

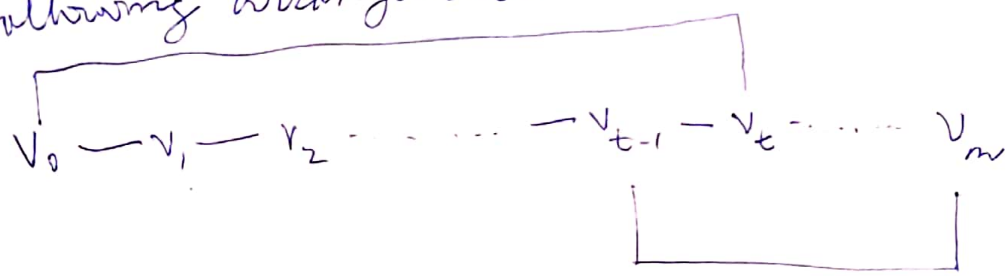
Suppose P_m is true, so we have a path

$$v_0 - v_1 - \cdots\cdots - v_m$$

We want to show that we can extend this to a vertex with one more element. If v_0 is adjacent to any vertex not already in the path, we could just add it before v_0 and be done. Similarly if v_m were adjacent to any vertex not already in the path, we could add it after v_m were adjacent to any vertex not already in the path, we could add it after v_m and be done.

So, we have to consider the hard case. In this case we have an important additional fact. All neighbours of v_0 and v_m are somewhere in the path.

We want to turn our path into a cycle. If v_0 is adjacent to v_m then we already have a circuit. Suppose not, we find the following arrangement:-



Because then we could break the link between v_{t-1} and v_t to obtain the circuit

$$v_t - \cdots\cdots - v_m - v_{t-1} - \cdots\cdots - v_1 - v_0 - v_t$$

We know that v_0 has $n/2$ neighbours, all of them are in this path, and none are v_m. Let A be the vertices adjacent to v_0, so $|A| \geq n/2$. Let B be all the vertices which are adjacent to v_m, so $|B| \geq n/2$. Every vertex in B belongs on the path, so we can ask about the vertex immediately after it on the path. Let C be the set of vertices which are immediately after some vertex B in the path. Then $|C| = |B| \geq n/2$. If $A \cap C = \phi$ if A and C are disjoint - then $|A \cup C| \geq n/2 + n/2 \geq n$, so ~~it~~ $A \cup C$ would have to include all the vertices. But v_0 is neither A nor C, so $A \cup C$ isn't all the vertices, so there is some vertex $v_t \in A \cap C$ and so $v_t \in A$ while $v_{t-1} \in B$.

Therefore we have turned our path into the circuit:-

$$v_t - \cdots - v_m - v_{t-1} - \cdots\cdots - v_1 - v_0 - v_t$$

If $m = n-1$, we have included all the vertices, so we have Hamilton circuit and we're done. If $m+1 < n$, there must be some vertex not included in our circuit, and since G is connected, there must be some vertex w which isn't in our circuit but is adjacent to something in our circuit, say v_u. So, we can rotate our circuit so v_u is the first vertex and then tack on w before that, say:-

$$w - v_u - v_{u+1} - \cdots\cdots - v_m - v_{t-1} - \cdots\cdots - v_1 - v_0 - v_t - \cdots\cdots - v_{u-1}$$

This is a path with $m+2$ elements so we have shown P_{m+1}. By induction we know that for every m, P_{m+1} is true, so in particular there is a path of length $n+1$. In particular we have a path of length n, and by the argument just given, we can turn this path into a circuit.

Hence Proved ∎

Q.22) Let G be a connected graph with n vertices, $n>2$ and no loops or multiple edges. Prove that G has Hamiltonian circuit if $deg(u) + deg(v) \geq n$, where u, v are non-adjacent to each other.

Proof: Suppose for a contradiction, that G does not have a closed Hamiltonian path.

1. Pick any 2 vertices of G which aren't already joined by an edge, and add a new edge between them. Keep on doing this until we reach a graph G_{last} which does have a closed Hamiltonian path.

2. Let $\bar{G}$ be the graph obtained immediately before G_{last} and suppose that $\{x, y\}$ is the edge added to $\bar{G}$ to obtain G_{last}.

Let $(z_1 z_2 \cdots z_n z_1)$ be a closed Hamilton Path in G_{last}. This must use the edge $\{x, y\}$ then $(z_1 z_2 \cdots z_n)$ is a non-closed Hamiltonian path in $\bar{G}$. Otherwise there is some r such that $1 \leq r < n$ and $z_r = x$ and $z_{r+1} = y$; now

$$(z_{r+1}, z_{r+2} \cdots z_n, z_1, \cdots z_r)$$

is a non-closed Hamiltonian path in $\bar{G}$. Note, that either way, all the edges used in this path appear in $\bar{G}$. Relabel the vertices so that this path is $(x_1, x_2 \cdots x_n)$

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3. Suppose we could find a vertex x_i such that x is adjacent to x_i, and y is adjacent to x_{i-1}. Then

$$(x, x_i, \ldots\ldots x_{n-1}, y, x_{i-1} \ldots x)$$

would be a closed Hamiltonian path in $\bar{G}$.

This is a contradiction $\Rightarrow\Leftarrow$

Aside: It is at this point that we need $n \geq 3$; if $n=2$ then the first step is (x, y), and the second is (y, x), which means we have used an edge twice. Paths are, in particular, trails, so they aren't allowed to repeat edges. As long as $n \geq 3$ this problem doesn't arrise.

4. It remains to be shown there exists such a vertex x_i. This is where we need the hypothesis on degrees. Since $\bar{G}$ is obtained from G by adding edges, it still satisfies this hypothesis. Let

$$A = \{ i : 2 \leq i \leq n \text{ and } x_i \text{ is adjacent to } x \}$$
$$B = \{ i : 2 \leq i \leq n \text{ and } x_{i-1} \text{ is adjacent to } y \}$$

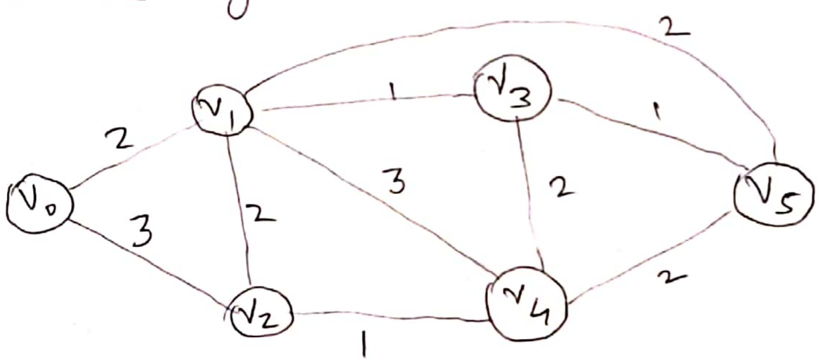
As our graphs have no loops $\delta(x) = |A|$ and $|B| = \delta(y)$. As x and y are not adjacent in $\bar{G}$, our hypothesis tells us that $\delta(x) + \delta(y) \geq n$. Hence A and B are subsets of $\{2, \ldots\ldots n\}$ containing at least n elements between them. It follows that they must intersect non-trivially. If $i \in A \cap B$

then x_i is a suitable vertex for step 3.

Hence, we prove Ore's theorem that for all simple graphs G where $n \geq 3$ and for all pairwise vertices in the graph $x, y \in V(G)$ such that

$\deg_G(x) + \deg_G(y) \geq n$ there exists a Hamiltonian cycle in G. Hence proved.

Q23) Consider the following weighted graph G. Apply Dijkstra's algorithm to vertex v_0.



We maintain 3 structures, a parent map which denotes the path of which vertex we came from?
We maintain a min distance table which will store final result and a temporary distances table to compute min-distance from source to target vertex.

Parent Map

$V_0 \to$ Null

Minimum Distance

$V_0 \to 0$

Distance Table

V_1	∞
V_2	∞
V_3	∞
V_4	∞
V_5	∞

Now, $d(V_0V_1) = \min\{d(V_0V_1),\ d(V_0V_0) + e(V_0V_1)\}$
$= \min\{\infty, 0+2\} = 2$

$d(V_0V_2) = \min\{d(V_0V_2),\ d(V_0V_0) + e(V_0V_2)\}$
$\min\{\infty,\ 0+3\}$
$= 3$

Parent Map

$V_0 \to$ Null
$V_1 \to V_0$
$V_2 \to V_0$

Minimum Distance

$V_0 \to 0$

Distance Table

V_1	2
V_2	3
V_3	∞
V_4	∞
V_5	∞

Extracting minimum distance from Distance table and adding to Minimum Distance table.

Parent Map

$V_0 \to$ Null
$V_1 \to V_0$
$V_2 \to V_0$

Minimum Distance

$V_0 \to 0$
$V_1 \to 2$

Distance Table

$V_2 \to 3$
$V_3 \to \infty$
$V_4 \to \infty$
$V_5 \to \infty$

$d(v_1 v_3) = \min\{d(v_1 v_3), d(v_0 v_1) + e(v_1 v_3)\}$
$= \min\{\infty, 2+1\}$
$= 3$

$d(v_1 v_4) = \min\{d(v_0 v_4), d(v_0 v_1) + e(v_1 v_4)\}$
$= \min\{\infty, 2+3\}$
$= 5$

$d(v_1 v_2) = \min\{d(v_0 v_2), d(v_0 v_1) + e(v_1 v_2)\}$
$= \min\{3, 2+2\}$
$= 3$

$d(v_1 v_5) = \min\{d(v_0 v_5), d(v_0 v_1) + e(v_1 v_5)\} = 4$

Parent Map	Distances (Minimum)	Distances table
$v_0 \to$ null	$v_0 \to 0$	$v_2 \to 3$
$v_1 \to v_0$	$v_1 \to 2$	$v_3 \to 3$
$v_2 \to v_0$		$v_4 \to 5$
$v_3 \to v_1$		$v_5 \to 4$
$v_4 \to v_1$		
$v_5 \to v_1$		

Selecting minimum from Distances table $v_2 \to 3$

Parent Map	Minimum Distances	Distances Table
$v_0 \to$ Null	$v_0 \to 0$	$v_3 \to 3$
$v_1 \to v_0$	$v_1 \to 2$	$v_4 \to 5$
$v_2 \to v_0$	$v_2 \to 3$	$v_5 \to 4$
$v_3 \to v_1$		
$v_4 \to v_1$		
$v_5 \to v_1$		

$d(v_2 v_4) = \min\{d(v_0 v_4), d(v_0 v_2) + e(v_2 v_4)\}$

$= \min\{5, 3+1\}$

$= 4 \quad (v_4 \rightarrow v_2)$

Parent Map

$v_0 \rightarrow$ Null

$v_1 \rightarrow v_0$

$v_2 \rightarrow v_0$

$v_3 \rightarrow v_1$

$v_4 \rightarrow v_2$

$v_5 \rightarrow v_1$

Minimum Distances

$v_0 \rightarrow 0$

$v_1 \rightarrow 2$

$v_2 \rightarrow 3$

Distances Table

$v_3 \rightarrow 3$

$v_4 \rightarrow 4$

$v_5 \rightarrow 4$

Selecting $v_3 \rightarrow 3$ from Distances table.

Parent Map

$v_0 \rightarrow$ Null

$v_1 \rightarrow v_0$

$v_2 \rightarrow v_0$

$v_3 \rightarrow v_1$

$v_4 \rightarrow v_2$

$v_5 \rightarrow v_1$

Minimum Distances

$v_0 \rightarrow 0$

$v_1 \rightarrow 2$

$v_2 \rightarrow 3$

$v_3 \rightarrow 3$

Distances Table

$v_4 \rightarrow 4$

$v_5 \rightarrow 4$

$d(v_0 v_4) = \min\{d(v_0 v_4), d(v_0 v_3) + e(v_3 v_4)\}$

$= \min\{4, 3+2\}$

$= 4$

$d(v_0 v_5) = \min\{d(v_0 5), d(v_0 v_3) + e(v_3 v_5)\}$

$= \min\{4, 3+1\}$

$= 4$

In this iteration nothing changed and as we have solved for ~~V~~ n-1 vertices, we have run the Dijkstra's algorithm. The results are:-

Parent Map	Minimum Distances
$V_0 \rightarrow$ Null	$V_0 \rightarrow 0$
$V_1 \rightarrow V_0$	$V_1 \rightarrow 2$
$V_2 \rightarrow V_0$	$V_2 \rightarrow 3$
$V_3 \rightarrow V_1$	$V_3 \rightarrow 3$
$V_4 \rightarrow V_2$	$V_4 \rightarrow 4$
$V_5 \rightarrow V_1$	$V_5 \rightarrow 4$