Partial Differential Equations

Assignment-3

MC-406

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1. $u_{tt} - c^2 u_{xx} = 0$ —— (1)

$$\begin{cases} u(0,t) = 0, \ u(L,t) = 0 & \text{(boundary condns)} \\ u(x,0) = f(x) \ \& \ u_t(x,0) = g(x) & \text{(initial condn)} \end{cases}$$

Let $u_x(x,t) = T(t)X(x)$ —— (2)

from boundary condns

we get, $X(0) = 0$, $X(l) = 0$

from (2) $u_{tt} = T''(t)X(x)$ & $u_{xx} = T(t)X''(x)$ —— (3)

from (1) & (3)

$T''(t)X(x) = c^2 T(t)X''(x)$

$$\Rightarrow \frac{T''(t)}{c^2 T(t)} = \frac{X''(t)}{X(x)} = -\lambda \text{ (say)}$$

$\Rightarrow$ $T''(t) + \lambda c^2 T(t) = 0$ —— (4)

$$\begin{cases} X''(t) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{cases} \text{ —— (5)}$$

Case 1 $\lambda = -\mu^2 < 0$ where $\mu > 0$

gen sol^n to $X'' - \mu^2 X = 0$

$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$X(0) = 0 \Rightarrow c_1 + c_2 = 0$

& $X(l) = 0 \Rightarrow c_1 e^{\mu l} + c_2 e^{-\mu l} = 0$

$\Rightarrow c_1 = c_2 = 0$

Hence, No non trivial sol^n.

Case 2. ~~$\mu = 0$~~ $\lambda = 0$.

gen sol^n to $X'' = 0$

$X(x) = c_1 x + c_2$

$X(0) = 0 \Rightarrow c_2 = 0$

$X(l) = 0 \Rightarrow c_1 l = 0 \Rightarrow c_1 = 0$

again no non trivial soln.

case 3. $\lambda = \mu^2 \geq 0$, $\mu > 0$.

$- X'' + \mu^2 X = 0$.

gen sol^n $\Rightarrow$ $X(x) = c_1 \sin \mu x + c_2 \cos \mu x$. ——— (6)

$X(0) = 0 \Rightarrow c_2 = 0$

$X(l) = 0 \Rightarrow c_1 \sin \mu l = 0$

$\Rightarrow \mu_k = \frac{k\pi}{l}$; $k = 1, 2, \ldots$

eigenvalue & eigenfunctions will be.

$\lambda_k = \left(\frac{k\pi}{l}\right)^2$, $X_k(x) = \sin \frac{k\pi x}{l}$, $k = 1, 2, \ldots$

sim solving for $T(t)$ with $\lambda = \lambda_k$

we get $T_k(t) = A_k \cos\frac{k\pi ct}{l} + B_k \sin\frac{k\pi ct}{l}$

$$\Rightarrow u_k(x,t) = \left(A_k \cos\frac{k\pi ct}{l} + B_k \sin\frac{k\pi ct}{l}\right)\sin\left(\frac{k\pi x}{l}\right)$$

$$k = 1, 2, \ldots \quad \text{———} \quad (7)$$

u can be written as an ~~ifin~~ infinite series of u_k

i.e

$$u(x,t) = \sum_{k=1}^{\infty}\left(A_k \cos\frac{k\pi ct}{l} + B_k \sin\frac{k\pi ct}{l}\right)\sin\frac{k\pi x}{l}$$

from initial condns.

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi x}{l} = f(x)$$

$$u_t(x,0) = \sum_{k=1} \frac{k\pi c}{l} B_k \sin\frac{k\pi x}{l} = g(x)$$

using the formulae for the fourier coefficients in the sine expansion of f & g in $[0, l]$

we get,

$$A_k = \frac{2}{l}\int_0^l f(y)\sin\frac{k\pi y}{l}\,dy$$

$$B_k = \frac{2}{k\pi c}\int_0^l g(y)\sin\frac{k\pi y}{l}\,dy.$$

Q.
$$\left.\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2} \\ y(x,0)=f(x) \;&\&\; \frac{\partial y}{\partial t}(t,0) = g(x)\end{aligned}\right\} \quad (1)$$

let us make change of variable.

$$\left.\begin{aligned} u &= x+ct \quad \text{---} (i) \\ v &= x-ct \quad \text{---} (2) \end{aligned}\right\} (2)$$

$$\frac{\partial u}{\partial t} = c, \quad \frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial t} = -c \quad \text{---} (3)$$

$$\cancel{\frac{\partial y}{\partial x} = \frac{\partial y}{\partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial t}}$$

$$\cancel{\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + c\frac{\partial y}{\partial u}}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \quad \text{---} (4)$$

diff again wrt x.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} + \frac{2\,\partial^2 y}{\partial u\,\partial v} \quad \text{---} (5)$$

Also

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial t}$$

$$\frac{\partial y}{\partial t} = c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \quad \text{—— (6)}$$

or $$\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)$$

diff (6) again wrt t

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)\right)$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)y$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} - \frac{2\partial^2 y}{\partial u \partial v}\right) \quad \text{—— (7)}$$

from (6), (7) & (1)

$$\cancel{c^2}\left(\cancel{\frac{\partial^2 y}{\partial u^2}} + \cancel{\frac{\partial^2 y}{\partial v^2}} - \frac{2\partial^2 y}{\partial u \partial v}\right) = \cancel{c^2}\left(\cancel{\frac{\partial^2 y}{\partial u^2}} + \frac{2\partial^2 y}{\partial u \partial v} + \cancel{\frac{\partial^2 y}{\partial v^2}}\right)$$

$$\Rightarrow \frac{\partial^2 y}{\partial u \partial v} = 0$$

integrating wrt ~~x~~ u

$$\frac{\partial y}{\partial u} = \Phi(u) \quad (8)$$

integrating again wrt u

$$y = \int \Phi(u)\,du + \psi(v)$$

$$y = \phi(u) + \psi(v)$$

or $\boxed{y = \phi(x+ct) + \psi(x-ct)}$

imposing boundary condns

we get, $y(x,0) = f(x)$ (9) , $\frac{\partial y}{\partial t}(x,0) = g(x)$ (10)

(9) $\Rightarrow$ $y(x,0) = f(x) = \phi(x) + \psi(x)$ (11)

$$\frac{\partial y}{\partial t}(x,0) = g(x) = c\phi'(x) - c\psi'(x) \quad (12)$$

Integrating (12) on both sides from x_0 to x

$$c\phi(x) - c\psi(x) = \int_{x_0}^{x} g(\xi)\,d\xi + A \quad (13)$$

from (11) we have $\phi(x) + \psi(x) = f(x)$ (14)

adding (13) & (14)

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi + \frac{A}{2}$$

for $\psi(x)$ substitute value of ϕ from (15) into (11) we get, $\psi(x) = f(x) - \phi(x) = f(x) - \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi - \frac{A}{2}$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi - A/2$$

$$y(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x_0+ct} g(\xi)\,d\xi + \frac{A}{2}]$$

$$+ \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x_0-ct} g(\xi)\,d\xi - \frac{A}{2}]$$

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi)$$

3.

$$\begin{cases} y_{tt} - c^2 y_{xx} = h(x,t), & x \in \mathbb{R} \\ y(x,0) = 0 & \& \; y_t(x,0) = 0 \end{cases} \quad \text{--- (1)}$$

$\Rightarrow f(x) = 0, \; g(x) = 0$

let $y(x,t) = u(x,t) + v(x,t)$

where u solves $\begin{bmatrix} u_{tt} = c^2 u_{xx} & x \in \mathbb{R} \; t > 0 \\ u(x,0) = \cancel{f(x)}\, 0 & u_t(x,0) = \cancel{g(x)}\, 0 \end{bmatrix}$ --- (2)

and v solves $\begin{bmatrix} v_{tt} = c^2 v_{xx} + h(x,t) \\ v(x,0) = 0 \; \& \; v_t(x,0) = 0 \end{bmatrix}$ --- (3)

verification $\frac{\partial^2}{\partial t^2}(u+v) = u_{tt} + v_{tt} = c^2 u_{xx} + c^2 v_{xx} + h(x,t)$
$= c^2 \frac{\partial^2}{\partial x^2}(u+v) + h(x,t)$

$(u+v)(x,0) = u(x,0) + v(x,0) = f(x,0) + 0 = f(x,0) =$
$\frac{\partial}{\partial t}(u+v)(x,0) = u_t(x,0) + v_t(x,0) = g(x) + 0 = 0$

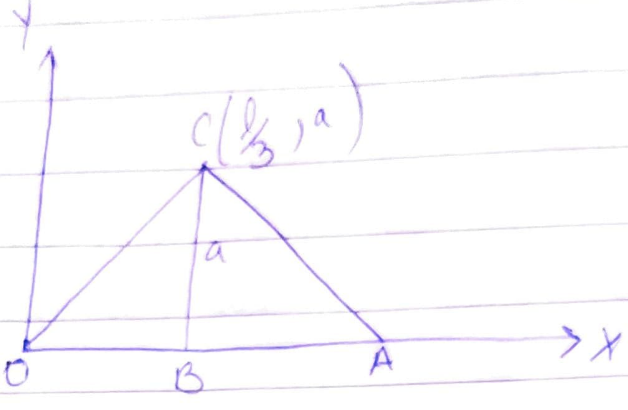
$(u+v)$ satisfies ① if u solves ② & v solves ③

since ② is a homogenous eqn its solⁿ will be.
$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$
$= \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi = 0$

$v(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(\xi; s)\,d\xi\,ds$

$y(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(\xi, s)\,d\xi\,ds$

4.



eqn of wave $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— (1)

eqn line OC $\Rightarrow$ $y - 0 = \frac{a-0}{\frac{l}{3} - 0}(x-0)$

$\Rightarrow y = \frac{3ax}{l}$. —— (2)

eq line CA $\Rightarrow$ $y - a = \frac{-a}{\frac{2l}{3}}\left(x - \frac{l}{3}\right)$

$y = \frac{3a}{2}\left(1 - \frac{x}{l}\right)$ —— (3)

Hence boundary condns are.

$$\left.\begin{aligned} y(0,t) &= 0 \\ y(l,t) &= 0 \end{aligned}\right\} \quad (4)$$

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ —— (5)

& $y(x,0) = \begin{cases} \frac{3ax}{l} & 0 < x < \frac{l}{3} \\ \frac{3a}{2}\left(1 - \frac{x}{l}\right), & \frac{l}{3} < x < l \end{cases}$ —— (6)

soln of ① $y(x,t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$

$y(0,t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_3$

$\Rightarrow c_3 = 0$

$y(x,t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px$.

$y(l,t) = 0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin pl$.

$\sin pl = 0 = \sin n\pi$

$p = \frac{n\pi}{l}$.

$y(x,t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$

$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}$.

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$

$\Rightarrow c_2 = 0$

$y(x,t) = c_1 b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$ where $b_n = c_1 c_4$

$\Rightarrow$ general soln will be.

$$y(x,t) = \sum_{1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad (7)$$

$$y(x,0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

$$b_n = \frac{2}{l}\int_0^l y(x,0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{l} \frac{3a}{2}\left(1-\frac{x}{2}\right)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\frac{3a}{l}\int_0^{l/3} \sin\frac{n\pi x}{l}\,dx + \frac{3a}{2}\int_{l/3}^{l}\left(1-\frac{x}{l}\right)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left[\left(x\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}\right]_0^{l/3} - \int_0^{l/3} 1\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$+ \frac{3a}{l}\left[\left(1-\frac{x}{l}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right]_{l/3}^{l}$$

$$- \int_{l/3}^{l}\left(-\frac{1}{l}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\Bigg]$$

$$= \frac{6a}{l^2}\left[\frac{-l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right)$$

$$+ \frac{3a}{l}\left[\frac{2l}{3n\pi}\cos\frac{n\pi}{3} - \frac{l}{n^2\pi^2}\left(0-\sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{9a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$\boxed{y(x,t) = \frac{9a}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}}$$

(5):

$$\begin{cases} u_{tt} - 9u_{xx} = 2\sinh x. & x \in \mathbb{R}\ t>0 \\ u(x,0) = x \ \& \ u_t(x,0) = \sin x \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$
$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(z;s)\,dz\,ds$$

(i) $c^2 = 9 \Rightarrow c = 3$ or $c = -3$ ~~let~~, ~~c=3~~, take $c = 3$.

(ii) $f(x) = x$

$\Rightarrow f(x+ct) = x+3t$

$\& \ f(x-ct) = x-3t$

(iii) $g(z) = \sin z$ · (iv) $h(z;s) = 2\sinh z$

$$u(x,t) = \frac{1}{2}\left[x+3t + (x-3t)\right] + \frac{1}{2(3)} \cdot \int_{x-3t}^{x+3t} 2\sin z\,dz$$
$$+ \frac{1}{2(3)}\int_0^t \int_{x-3(t-s)}^{x+3(t-s)} 2\sinh z\,dz\,ds.$$

$$= x + \left[-\cos(x+3t) + \cos(x-3t)\right]$$
$$+ \frac{1}{3}\int_0^t \left(\cosh(x+3(t-s)) - \cosh(x-3(t-s))\right)ds$$

$$= x + \frac{1}{3}\sin x \sin 3t + \frac{1}{3}\left[-\frac{1}{3}\sinh(x+3(t-s))\right.$$
$$\left. - \frac{1}{3}\sinh(x-3(t-s))\right]_0^t$$

$$= x + \frac{1}{3}\sin x \sin 3t - \frac{2}{9}\sinh x + \frac{2}{9}\sinh x \cosh 3t$$

cos 0