9

Monge's Methods

9.1 INTRODUCTION

The most general form of partial differential equation of order two is

$$f(x, y, z, p, q, r, s, t) = 0. \qquad ...(1)$$

It is only in special cases that (1) can be integrated. Some well known methods of solutions were given by Monge. His methods are applicable to a wide class (but not all) of equations of the form (1). Monge's methods consists in finding one or two first integrals of the form

$$u = \phi(v), \qquad ...(2)$$

where u and v are known functions of x, y, z, p and q and ϕ is an arbitrary function. In other words, Monge's methods consists in obtaining relations of the form (2) such that equation (1) can be derived from (2) by eliminating the arbitrary function. A relation of the form (2) is known as an *intermediate integral* of (1). Every equation of the form (1) need not possess an intermediate integral. However, it has been shown that most general partial differential equations having (2) as an intermediate integral are of the following forms

$$Rr + Ss + Tt = V \qquad \text{and} \qquad Rr + Ss + Tt + U(rt - s^2) = V, \qquad ...(3)$$

where R, S, T, U and V are functions of x, y, z, p and q. Even equations (3) need not always possess an intermediate integral. In what follows we shall assume that an intermediate integral of (3) exists.

9.2. MONGE'S METHOD OF INTERGRATING Rr + Ss + Tt = V. [Agra 2005; Delhi Maths (Hons) 2000, 02, 08, 09, 11; Garhwal 1994; Patna 2003; Kanpur 1997; Meerut 2000]

Given $$Rr + Ss + Tt = V, \qquad ...(1)$$

where R, S, T and V are functions of x, y, z, p and q.

We know that $p = \partial z/\partial x, \qquad q = \partial z/\partial y,$

$$\left.\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}, & t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}, \\ s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x} \quad \text{and} & s &= \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y} \end{aligned}\right\} \quad ...(2)$$

Now, $$dp = (\partial p/\partial x)\,dx + (\partial p/\partial y)dy = rdx + sdy, \text{ using (2)} \qquad ...(3)$$

and $$dq = (\partial q/\partial x)dx + (\partial q/\partial y)dy = sdx + tdy, \text{ using (2)} \qquad ...(4)$$

From (3) and (4), $\quad r = (dp - s\,dy)/dx \quad$ and $\quad t = (dq - s\,dx)/dy \qquad ...(5)$

Substituting the values of r and s given by (5) in (1), we get

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V \quad \text{or} \quad R(dp - sdy)dy + Ss\,dxdy + T(dq - sdx)dx = V\,dxdy$$

or $$(Rdpdy + Tdqdx - Vdxdy) - s\{R(dy)^2 - Sdxdy + T(dx)^2\} = 0. \qquad ...(6)$$

Clearly any relation between x, y, z, p and q which satisfies (6) must also satisfy the following two simultaneous equations

$$Rdpdy + Tdq\,dx - Vdxdy = 0. \qquad ...(7)$$

and $$(dy)^2 - Sdxdy + T(dx)^2 = 0. \qquad ...(8)$$

The equations (7) and (8) are called *Monge's subsidiary equations* and the relations which satisfy these equations are called *intermediate integrals.*

Equation (8) being a quadratic, in general, it can be resolved into two equations, say

$$dy - m_1\, dx = 0 \qquad ...(9)$$

and $$dy - m_2\, dx = 0. \qquad ...(10)$$

Now the following two cases arise :

Case I. When m_1 and m_2 are distinct in (9) and (10).

In this case (7) and (9), if necessary by using well known result $dz = pdx + qdy$, will give two integrals $u_1 = a$ and $v_1 = b$, where a and b are arbitrary constants. These give

$$u_1 = f_1(v_1), \qquad ...(11).$$

where f_1 is an arbitrary function. It is called an *intermediate integral* of (1).

Next, taking (7) and (10) as before, we get another intermediate integral of (1), say

$u_2 = f_2(v_2)$, where f_2 is an arbitrary function. ...(12)

Thus we have in this case two distinct intermediate integrals (11) and (12). Solving (11) and (12), we obtain values of p and q in terms of x, y and z. Now substituting these values of p and q in well known relation $\quad dz = pdx + qdy \quad$ ...(13)

and then integrating (13), we get the required complete integral of (1).

Case II . When $m_1 = m_2$ i.e., (8) is a perfect square.

As before, in this we get only one intermediate integral which is in Lagrange's form

$$Pp + Qq = R. \qquad ...(14)$$

Solving (14) with help of Lagrange's method (refer Art. 2.3, chapter 2), we get the required complete integral of (1).

Remark 1. Usually while dealing with case I, we obtain second intermediate integral directly by using symmetry. However sometimes in absence of any symmetry, we find the complete integral with help of only one indetermediate integral. This is done with help of using Lagrange's method.

Remark 2. While obtaining an intermediate integral, remember to use the relation $dx = pdx + qdy$ as explained below :

(*i*) $pdx + qdy + 2xdx = 0$ can be re–written as $\quad dz + 2xdx = 0 \quad$ so that $\quad z + x^2 = c$.

(*ii*) $xdp + ydq = dx$ can be re–written as $\quad xdp + ydq + pdx + qdy = dx + pdx + qdy$

or $\quad d(xp) + d(yq) = dx + dz \quad$ so that $\quad xp + yq = x + z + c$, on integration

Remark 3. While integrating, we shall use the following types of calculations. In what follows, f and g are arbitrary functions and k and a are a constants.

(*i*) $\int k\, f(t)\,dt = g(t)$ $\qquad$ (*ii*) $\int k \frac{1}{t} f(t)\,dt = g(t)$. $\qquad$ (*iii*) $\int k \frac{1}{t^2} f(t^2)\,d(t^2) = g(t^2)$

(*iv*) $\int k\, f(x+y)\,d(x+y) = g(x+y)$. $\qquad$ (*v*) $\int k t^2 f\left(\frac{1}{t}\right) d\left(\frac{1}{t}\right) = \int \frac{k}{(1/t)^2} f\left(\frac{1}{t}\right) d\left(\frac{1}{t}\right) = g\left(\frac{1}{t}\right)$

(*vi*) $\int \frac{k}{t^2} f(at^2)\,d(t^2) = \int \frac{k}{(at^2)} f(at^2)\,d(at^2) = g(at^2)$

Proof of (vi). Putting $at^2 = u$, and $d(at^2) = du$ we have

$$\int \frac{k}{t^2} f(at^2)\,d(t^2) = \int \frac{k}{u} f(u)\,d(u) = g(u) = g(at^2), \text{ as } u = at^2.$$

Similarly, other results can be proved. In examination we shall not use substitution as explained above. With good practice, the students will be able to write direct results of integration very easily.

Important Note. For sake of convenience, we have divided all questions based on $Rr + Ss + Tt = V$ in four types. We shall now discuss them one by one.

9.3. Type 1. When the given equation Rr + Ss + Tt = V leads to two distict intermediate intergrals and both of them are used to get the desired solution.

Working rule for solving problems of type 1.

Step 1. Write the given equation in the standard form $Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations:

$Rdpdy + Tdqdx - Vdxdy = 0$ ...(1) $\qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ... (2)

Step 3. Factorise (1) into two distinct factors.

Step 4. Using one of the factors obtained in (1), (2) will lead to an intermediate integral. In general, the second intermediate integral can be obtained from the first one by inspection, taking advantage of symmetry. In absence of any symmetry, the second factor obtained in step 3 is used in (2) to arrive at second intermediate integral. You should use remark 2 of Art. 9.2 while finding intermediate integrals.

Step 5. Solve the two intermediate integrals obtained in step 4 and get the values of p and q.

Step 6. Substitute the values of p and q in $dz = pdx + q\,dy$ and integrate to arrive at the required general solution. You should use remark 3 of Art. 9.2 while integrating $dz = pdx + qdy$.

9.4. SOLVED EXAMPLES BASED ON ART. 9.3.

Ex. 1. (a) *Solve* $r = a^2t$. **[Agra 2008; Lucknow 2010; Patna 2003; Meerut 2008]**

(b) $r = t$. **[Agra 2006]**

(c) Solve one-dimensions wave equation by Monge's method: $\partial^2 y/dx^2 = a^2(\partial^2 y/\partial t^2)$. **[Meerut 2003]**

Sol. (*a*) Given equation is $r - a^2t = 0$.

Comparing it with $Rr + Ss + Tt = V$, we have $R = 1$, $S = 0$, $T = -a^2$, $V = 0$.

Hence Monge's subsidiary equations

$Rdpdy + Tdq\,dx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T\,(dx)^2 = 0$

become
$$dpdy - a^2dqdx = 0 \qquad ...(1)$$
and
$$(dy)^2 - a^2(dx)^2 = 0. \qquad ...(2)$$

Equation (2) may be factorised as $(dy - adx)\,(dy + adx) = 0$

Hence two systems of equations to be considered are
$$dpdy - a^2\,dqdx = 0, \qquad dy - adx = 0. \qquad ...(3)$$
and
$$dpdy - a^2\,dqdx = 0, \qquad dy + adx = 0. \qquad ...(4)$$

Integrating the second equation of (3), we get $y - ax = c_1$. ...(5)

Eliminating dy/dx between the equations of (3), we get

$dp - adq = 0$ so that $p - aq = c_2$. ...(6)

Hence the intermediate integral corresponding to (3) is $p - aq = \phi_1(y - ax)$. ...(7)

Similarly another intgermediate integral corresponding to (4) is $p + aq = \phi_2(y + ax)$....(8)

Here ϕ_1 and ϕ_2 are arbitrary functions.

Solving (7) and (8) for p and q, we have
$$p = (1/2) \times \{\phi_2(y + ax) + \phi_1(y - ax)\} \quad \text{and} \quad q = (1/2a) \times \{\phi_2(y + ax) - \phi_1(y - ax)\}.$$

Substituting these values of p and q in $dz = pdx + qdy$, we get
$$dz = (1/2) \times \{\phi_2(y + ax) + \phi_1(y - ax)\}dx + (1/2a) \times \{\phi_2(y + ax) - \phi_1(y - ax)\}dy$$
$$= (1/2a) \times \phi_2(y + ax)\,(dy + adx) - (1/2a) \times \phi_1(y - ax)(dy - adx)$$

Integrating, $z = \psi_2(y + ax) + \psi_1(y - ax)$, ψ_1, ψ_2 being arbitrary functions.

(b) This is a particular case of part (a). Here $a = 1$. **Ans.** $z = \psi_2(y + x) + \psi_1(y - x)$.

(c) Refer part (a). Note that $\partial^2 y/\partial x^2 = r$ and $\partial^2 y/\partial t^2 = t$

Ex. 2. *Solve* $r + (a + b)s + abt = xy$. **[Vikram 2003]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = 1$, $S = a + b$, $T = ab$, $V = xy$. The usual Monge's subsidiary equations

$Rdpdy + Tqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$.

become $$dp\,dy + a\,b\,dq\,dx - xy\,dx\,dy = 0 \qquad ...(1)$$

and $$(dy)^2 - (a + b)\,dxdy + ab\,(dx)^2 = 0. \qquad ...(2)$$

Factorizing, (2) gives $(dy - bdx)\,(dy - adx) = 0$.

Hence two systems to be considered are

$$dp\,dy + ab\,dq\,dx - xy\,dx\,dy = 0, \qquad dy - b\,dx = 0. \qquad ...(3)$$

and $$dp\,dy + ab\,dq\,dx - xy\,dx\,dy = 0, \qquad dy - a\,dx = 0. \qquad ...(4)$$

Integrating the second equation of (3), $y - bx = c_1$. ...(5)

Eliminating dy/dx between the equations of (3), we get

$dp + a\,dq - xy\,dx = 0$ or $dp + a\,dq - x(c_1 + bx)\,dx = 0$, by (5) ...(6)

Integrating (6), $p + aq - (c_1/2)x^2 - (b/3)x^3 = c_2$ or $p + aq - (x^2/2)\,(y - bx) - (b/3)x^3 = c_2$, using (5)

or $$p + aq - (1/2) \times yx^2 + (1/6) \times bx^3 = c_2. \qquad ...(7)$$

Using (5) and (7), the first intermediate integral corresponding to (3) is

$p + aq - (1/2) \times yx^2 + (1/6) \times bx^3 = \phi_1(y - bx)$, ϕ_1 being an arbitrary function ...(8)

Similarly, another intermediate integral corresponding to (4) is

$p + bq - (1/2) \times yx^2 + (1/6) \times ax^3 = \phi_2(y - ax)$, ϕ_2 being an arbitrary functionJ ...(9)

Solving (8) and (9) for p and q, we have

$$p = (1/2) \times x^2y - (1/6) \times (a + b)x^3 + (a - b)^{-1}\,[a\phi_2(y - ax) - b\phi_1(y - ax)]$$

and $$q = (1/6) \times x^3 + (a - b)^{-1}\,[\phi_1(y - bx) - \phi_2(y - ax)].$$

Substituting these values in $dz = pdx + qdy$, we get

$$dz = (1/2) \times x^2ydx - (1/6) \times (a + b)x^3dx + (a - b)^{-1}\,[\phi_2(y - bx)dx - \phi_1(y - ax)dx]$$
$$+ (1/6) \times x^3dy + (a - b)^{-1}\,[\phi_1(y - bx)dy - \phi_2(y - ax)dy]$$

or $$dz = (1/6) \times (3x^2ydx + x^3dy) - (1/6) \times (a + b)\,x^3dx - (b - a)^{-1}\,[\phi_2(y - bx)dx$$
$$- \phi_1(y - ax)dx] - (b - a)^{-1}\,[(\phi_1(y - bx)dy - \phi_2(y - ax)dy]$$

or $$dz = (1/6) \times d(x^3y) - (1/6) \times (a + b)\,x^3dx + (b - a)^{-1}\phi_2(y - ax)\,(dy - adx)$$
$$- (b - a)^{-1}\phi_1(y - bx)\,(dy - bdx)$$

or $$dz = (1/6) \times d(x^3y) - (1/6) \times (a + b)\,x^3dx + (b - a)^{-1}\,\phi_2(y - ax)\,d(y - ax)$$
$$- (b - a)^{-1}\,\phi_1(y - bx)\,d(y - bx)$$

Integrating, $$z = (1/6) \times x^3y - (1/24) \times (a + b)x^4 + \psi_2(y - ax) + \psi_1(y - bx),$$

where ψ_1 and ψ_2 are arbitrary functions.

Ex. 3. *Solve* $r - t\cos^2 x + p\tan x = 0$. **[K.U. Kurukshetra 2005; Meerut 1993]**

Sol. Given $$r - t\cos^2 x = -p\tan x \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = 1$, $S = 0$, $T = -\cos^2 x$ and $V = -p\tan x$. ...(2)

Monge's subsidiary equations are $$Rdp\,dy + Tdq\,dx - V\,dx\,dy = 0 \qquad ...(3)$$

and $$R\,(dy)^2 - S\,dx\,dy + T\,(dx)^2 = 0 \qquad ...(4)$$

Putting the values of R, S, T and V, (3) and (4) become

$$dp\,dy - \cos^2 x\,dq\,dx + p\tan x\,dx\,dy = 0 \qquad ...(5)$$

and $$(dy)^2 - \cos^2 x\,(dx)^2 = 0 \qquad ...(6)$$

Equation (6) may be factoriesed as $(dy - \cos x\, dx)\,(dy + \cos x\, dx) = 0$

$\therefore$ $$dy - \cos x\, dx = 0 \qquad ...(7)$$

or $$dy + \cos x\, dx = 0 \qquad ...(8)$$

Putting the value of dy from (7) in (5), we get

$dp \cos x\, dx - \cos^2 x\, dq\, dx + p \tan x\, dx \cos x\, dx = 0$ or $dp - \cos x\, dq + p \tan x\, dx = 0$

or $\sec x\, dp + p \sec x \tan x\, dx - dq = 0$ or $d\,(p \sec x) - dq = 0.$

Integrating it, $p \sec - q = c_1$, c_1 being an arbitrary constant ....(9)

Integrating (7), $y - \sin x = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one integral of (1) is $p \sec x - q = f(y - \sin x)$. ...(11)

In a similar manner, (8) and (5) give another integral of (1)

$$p \sec x + q = g(y + \sin x). \qquad ...(12)$$

Solving (11) and (12) for p and q, we find

$p = (f + g)/2 \sec x = (1/2) \times (f + g) \cos x$ and $q = (g - f)/2$ ...(13)

Now, $dz = p\, dx + q\, dy$ or $dz = (1/2) \times (f + q) \cos x\, dx + (1/2) \times (g - f)\, dy$, by (13)

or $dz = -(1/2) \times f(y - \sin x)\,(dy - \cos x\, dx) + (1/2) \times g(y + \sin x)\,(dy + \cos x\, dx)$

Integrating, $z = F(y - \sin x) + G(y + \sin x)$, F and G being arbitrary functions.

Ex. 4. *Solve* $t - r \sec^4 y = 2q \tan y$. **[Delhi Maths Hons 1995; Kanpur 1995; Meerut 1995]**

Sol. Given $$t - r \sec^4 y = 2q \tan y. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = -\sec^4 y$, $S = 0$, $T = 1$, $V = 2q \tan y$. ...(2)

Monge's subsidiary equations are $$R dp\, dy + T\, dq\, dx - V\, dx\, dy = 0 \qquad ...(3)$$

and $$R(dy)^2 - S\, dxdy + T\,(dx)^2 = 0 \qquad ...(4)$$

Putting the values of R, S, T and V, (3) and (4) become

$$-\sec^4 y\, dp\, dy + dq\, dx - 2q \tan y\, dx\, dy = 0 \qquad ...(5)$$

and $$-\sec^4 y\,(dy)^2 + (dx)^2 = 0. \qquad ...(6)$$

Equation (6) may be factorised as $(dx - \sec^2 y\, dy)\,(dx + \sec^2 y\, dy) = 0$ so that

$$dx - \sec^2 y\, dy = 0 \qquad ...(7)$$

or $$dx + \sec^2 y\, dy = 0. \qquad ...(8)$$

Putting the value of dx from (7) in (5), we get

$-\sec^4 y\, dp\, dy + dq \sec^2 y\, dy - 2q \tan y\, dy \times \sec^2 y\, dy = 0$ or $-dp + \cos^2 y\, dq - 2q \sin y \cos y\, dy = 0$

or $dp - (\cos^2 x\, dq - q \times 2 \sin y \cos y\, dy) = 0$ or $dp - d(q \cos^2 y) = 0.$

Integrating it, $p - q \cos^2 y = c_1$, c_1 being an arbitrary constant ...(9)

Integrating (7), $x - \tan y = c_2$, being an arbitrary constant ...(10)

From (9) and (10), one integral of (1) is $p - q \cos^2 y = f(x - \tan y)$. ...(11)

Similarly, from (8) and (5) the other integral of (1) is $p + q \cos^2 y = g(x + \tan y)$. ...(12)

Solving (11) and (12) for p and q, we find

$p = (f + g)/2$ and $q = (g - f)/(2 \cos^2 y) = (1/2) \times (g - f) \times \sec^2 y$ ...(13)

Now, we have $$dz = pdx + qdy$$

or $$dz = (1/2) \times (f + g)dx + (1/2) \times (g - f) \times \sec^2 y\, dy, \text{ using (13)}$$

or $$dz = (1/2) \times f(x - \tan y)\,(dx - \sec^2 y\, dy) + (1/2) \times g(x + \tan y)\,(dx + \sec^2 y\, dy)$$

or $$dz = (1/2) \times f(x - \tan y)\, d(x - \tan y) + (1/2) \times g(x + \tan y)\, d(x + \tan y).$$

Integrating, $z = F(x - \tan y) + G(x + \tan y)$, F, G being arbitrary functions.

Ex. 5. *Solve* $q(yq + z)r - p\,(2yq + z)s + yp^2 t + p^2 q = 0$. **[Delhi 2008]**

Sol. As usual, here Monge's subsidiary equations are

$$q(yq + z)dp\, dy + yp^2 dqdx + p^2 qdxdy = 0 \qquad ...\ (1)$$

and $$q(yq + z)(dy)^2 + p(2yq + z)dxdy + yp^2(dx)^2 = 0. \qquad ...\ (2)$$

On factorization, (2) gives $(qdy + pdx)\ \{(yq + z)dy + ypdx\} = 0$.

Hence two systems to be considered are

$$q(yq + z)dpdy + yp^2dqdx + p^2qdxdy = 0, \qquad qdy + pdx = 0 \qquad \ldots (3)$$

and $$q(yq + z)dpdy + yp^2dqdx + p^2q\ dxdy = 0, \qquad (yq + z)dy + ypdx = 0 \ldots (4)$$

Using $dz = pdx + qdy$, the second equation of (3) reduces to

$dz = 0$ so that $z = c_1$. ... (5)

From second equation of (3), $qdy = -pdx$. Hence first equation of (3) reduces to

$(yq + z)dp - ypdq - pqdy = 0$ or $(yq + z)dp - p\ d(yq) = 0$

or $(yq + z)dp - pd(yq+ z) = 0$, as $dz = 0$, by (5)

or $\dfrac{d(yq+z)}{yq+z} - \dfrac{dp}{p} = 0$ so that $\log (yq + z) - \log p = \log c_1$

or $(yq + z)/p = c_2$, c_2 being an arbitrary constant ... (6)

From (5) and (6), the intermediate integral corresponding to (3) is

$(yq + z)/p = \phi_1(z)$ or $yq + z = p\phi_1(z)$, ...(7)

where ϕ_1 is an arbitrary function.

Using $dz = pdx + qdy$, the second equation of (4) becomes

$y(qdy + pdx) + zdy = 0$ or $ydz + zdy = 0$ or $d(yz) = 0$.

Integrating it, $yz = c_3$, c_3 being an arbitrary constant ...(8)

From second equation of (4), $(yq + z)dy = -ypdx$.

Using this fact, first equation of (4) reduces to

$qdp - pdq - (pq/y)dy = 0$ or $-(1/p)dp + (1/q)dq + (1/y)dy = 0$.

Integrating, $-\log p + \log q + \log y = \log c_1$ or $(yq)/p = c_2$ ...(9)

From (8) and (9), another intermediate integral corresponding to (4) is

$(qy)/p = \phi_2(yz)$, where ϕ_2 is an arbitrary function. ...(10)

Solving (7) and (10) for p and q, we have $p = \dfrac{z}{\phi_1(z) - \phi_2(yz)}$, $q = \dfrac{z\phi_2(yz)}{y\{\phi_1(z) - \phi_2(yz)\}}$.

Substituting these in $dz = pdx + qdy$, $dz = \dfrac{z}{\phi_1(z) - \phi_2(yz)}\{dx + (1/y) \times \phi_2(yz)\ dy\}$

or $\phi_1(z)dz = zdx + \phi_2(yz)\dfrac{zdy + ydz}{y}$ or $\dfrac{\phi_1(z)\,dz}{z} = dx + \dfrac{\phi_1(yz)\,d(yz)}{yz}$.

Integrating, $\psi_1(z) = x + \psi_2(yz)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex. 6. *Solve* $(r - t)xy - s(x^2 - y^2) = qx - py$. **[Delhi Maths 2005, Kurukshetra 2005 (H)]**

Sol. Usual Monge's auxiliary equations are

$$xydpdy - xydqdx - (qx - py)dxdy = 0 \qquad \ldots(1)$$

and $$xy(dy)^2 + (x^2 - y^2)\ dxdy - xy\ (dx)^2 = 0. \qquad \ldots(2)$$

On factorizing, (2)gives $(xdy - ydx)\ (ydx + xdy) = 0$.

Hence, two systems to be considered are

$$xydpdy - xydqdx - (qx - py)\ dxdy = 0, \qquad xdy - ydx = 0 \qquad \ldots(3)$$

and $$xydpdy - xydqdx - (qx - py)\ dxdy = 0, \qquad ydx + xdy = 0. \qquad \ldots(4)$$

Second equation of (3) gives $y/z = c_1$, c_1 being an arbitrary constant ...(5)

Using second equation, first equation of (3) reduces to

$ydp - xdq - qdx + pdy = 0$ or $d(yp - xq) = 0$

Integrating, $yp - xq = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), intermediate integral corresponding to (3) is

$yp - xq = \phi_1(y/x)$, where ϕ_1 is an arbitrary function. ...(7)

Second equation of (4) gives $x^2 + y^2 = c_3$, c_3 being arbitrary constant ...(8)

Using second equation, first equation of (4) reduces to

$xdp + ydq + qdy + pdx = 0$ or $d(xp) + d(yq) = 0$

Integrating, $xp + yq = c_4$, c_4 being an arbitrary constant ...(9)

From (8) and (9), another intermediate integral corresponding to (4) is

$xp + yq = \phi_2(x^2 + y^2)$, where ϕ_2 is an arbitrary function. ...(10)

Solving (7) and (10) for p and q, we have

$$p = \frac{1}{x^2+y^2}\left\{y\phi_1\left(\frac{y}{x}\right)+x\,\phi_2(x^2+y^2)\right\} \quad \text{and} \quad q = \frac{1}{x^2+y^2}\left\{y\,\phi_2(x^2+y^2)-x\,\phi_1\left(\frac{y}{x}\right)\right\}.$$

Substituting these values in $dz = pdx + qdy$, we get

$$dz = \frac{1}{x^2+y^2}\left[\left\{y\,\phi_1\left(\frac{y}{x}\right)+x\,\phi_2(x^2+y^2)\right\}dx+\left\{y\,\phi_2(x^2+y^2)-x\,\phi_1\left(\frac{y}{x}\right)dy\right\}\right]$$

or $$dz = \frac{ydx-xdy}{x^2+y^2}\phi_1\left(\frac{y}{x}\right)+\frac{xdx+ydy}{x^2+y^2}\phi_2(x^2+y^2) \quad \text{or} \quad dz = -\frac{\phi_1\,(y/x)}{1+(y/x)^2}d\left(\frac{y}{x}\right)+\frac{1}{2}\frac{\phi_2\,(x^2+y^2)}{x^2+y^2}d(x^2+y^2).$$

Integrating, $z = \psi_1(y/x) + \psi_2(x^2 + y^2)$, ψ_1, ψ_2 being arbitrary functions.

Ex. 7. *Solve* $(r - s)\, x = (t - s)\, y$. **(M.D.U Rohtak 2005)**

Sol. Usual Monge's subsidiary equations are $xdpdy - ydqdx = 0$ ...(1)

and $x(dy)^2 + (x - y)\, dxdy - y(dx)^2 = 0.$ ...(2)

Factorising, (2) $\Rightarrow$ $(xdy - ydx)\,(dy + dx) = 0.$

Hence two systems to be considered are

$xdpdy - ydqdx = 0,$ $xdy - ydx = 0$ ...(3)

and $xdpdy - ydqdx = 0,$ $dy + dx = 0.$ ...(4)

Integrating second equation of (3), $y/x = c_1$, c_1 being an arbitrary constant ...(5)

Eliminating dy/dx between equations of (3), we get

$dp - dq = 0$ so that $p - q = c_2$, c_2 being an arbitrary constant ...(6)

Hence the intermediate integral corresponding to (3) is $p - q = \phi_1(y/x)$. ...(7)

Integrating second equation of (4), $x + y = c_3$, c_3 being an arbitrary constant ...(8)

Eliminating dy/dx between equations of (4), we get

$xdp + ydq = 0$ or $xdp + ydq + pdx + qdy = pdx + qdy$

or $d(xp) + d(yq) - dz = 0,$ as $dz = pdx + qdy.$

Integrating, $xp + yq - z = c_4$, c_4 being an arbitrary constant ...(9)

Hence the intermediate integral corresponding to (4) is

$xp + yq - z = \phi_2(x + y)$ or $xp + yq = z + \phi_2(x + y),$ ...(10)

Solving (7) and (10) for p and q, we have

$$p = \frac{1}{x+y}\left\{z+\phi_2(x+y)+y\phi_1\left(\frac{y}{x}\right)\right\} \quad \text{and} \quad q = \frac{1}{x+y}\left\{z+\phi_2(x+y)-x\,\phi_1\left(\frac{y}{x}\right)\right\}.$$

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \frac{1}{x+y}\left[\left\{z+\phi_2(x+y)+y\,\phi_1\left(\frac{y}{x}\right)\right\}dx+\left\{z+\phi_2(x+y)-x\,\phi_1\left(\frac{y}{x}\right)\right\}dy\right]$$

$$\Rightarrow \quad \frac{(x+y)\,dx-zdx}{(x+y)^2} = \frac{\phi_2(x+y)\,d\,(x+y)}{(x+y)^2}+\frac{(ydx-xdy)\,\phi_1\,(y/x)}{(x+y)^2}$$

$$\Rightarrow \qquad d\left(\frac{z}{x+y}\right) = \frac{\phi_2(x+y)}{(x+y)^2}\,d(x+y) - \frac{\phi_1(y/x)}{1+(y/x)^2}\,d\left(\frac{y}{x}\right).$$

Integrating, $z/(x+y) = \psi_2(x+y) + \psi_1(y/x)$, ψ_1, ψ_2 being arbitrary functions.

Ex. 8. *Solve* $r + ka^2 t - 2as = 0$.

Sol. Given $$r - 2as + ka^2 t = 0. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we have $R = 1$, $S = -2a$, $T = ka^2$, $V = 0$.

Hence the Monge's subsidiary equations

$Rdp\,dy + Tdq\,dx - Vdx\,dy = 0$ and $R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0$

become $$dp\,dy + ka^2\,dq\,dx = 0 \qquad ...(2)$$

and $$(dy)^2 + 2a\,dx\,dy + ka^2\,(dx)^2 = 0. \qquad ...(3)$$

From (3), $dy = [-2a\,dx \pm \{4a^2(dx)^2 - 4ka^2\,(dx)^2\}^{1/2}]/2 = -a\,dx \pm a\sqrt{(1-k)}\,dx$

or $dy + a\,\{1 \pm \sqrt{(1-k)}\}dx = 0$ or $dy + a\,(1 \pm l)\,dx = 0$, where $l = \sqrt{(1-k)}$.

Hence (3) reduces to the following two equations :

$$dy + a(1 + l)dx = 0 \qquad ...(4)$$

and $$dy + a(1 - l)dx = 0. \qquad ...(5)$$

From (2) and (4) , eliminating dy, we have

$dp\{-a\,(1 + l)\,dx\} + ka^2\,dqdx = 0$ or $(1 + l)dp - ka\,dq = 0$.

Integrating it, $(1 + l)p - kaq = c_1$, c_1 being an arbitrary constant ...(6)

Again, integrating (4), $y + a(1 + l)x = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), first intermediate integral is

$(1 + l)p - kaq = f_1\{y + a(1 + l)x\}$, where f_1 is an arbitrary function. ...(8)

Similary, from (2) and (5), second intermediate intgegral is given by (replacing l by $-l$ in (8) since (5) differs from (4) in having $-l$ in place of l)

$(1 - l)p - kaq = f_2\{y + a(1 - l)x\}$, where f_2 is an arbitrary function ...(9)

Solving (8) and (9) for p and q, $p = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]$

and $q = (1/2akl) \times [(1 - l)f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}]$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]dx$$
$$+ (1/2akl) \times [(1 - l)\,f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}]dy$$

or $$dz = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]dx$$

$+ \dfrac{1}{2al(1-l^2)}\,[\,(1 - l)\,f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}]dy$, as $l = (1 - k)^{1/2} \Rightarrow k = 1 - l^2$

or $$dz = (1/2l)\,[dx f_1\,\{y + a(1+l)x\} - dx f_2\{y + a(1-l)x\}] + \frac{1}{2al}\left[\frac{dy}{1+l} f_1\{y + a(1+l)x\} - \frac{dy}{1-l} f_2\{y + (1-l)x\}\right]$$

$$= \frac{1}{2al(l+1)} f_1\{y + a(1 + l)x\}\{dy + a(1 + l)dx\} - \frac{1}{2al(1-l)}\,f_2\{y + a(1 - l)x\}\ \{dy + a(1 - l)dx\}$$

or $$dz = \frac{1}{2al(l+1)}\,f_1\{y + a(1 + l)x\}\ d\{y + a(1 + l)x\} - \frac{1}{2al(1-l)} f_2\{y + a(1 - l)x\}\ d\{y + a\,(1 - l)x\}.$$

Integrating, $z = F_1\{y + a(1 + l)x\} + F_2\{y + a\,(1 - l)x\}$, where F_1 and F_2 are arbitrary functions.

Ex. 9. *Solve* $x^{-2}r - y^{-2}t = x^{-3}p - y^{-3}q$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^{-2}$, $S = 0$, $T = y^{-2}$, $V = x^{-3}p - y^{-3}q$. Then Monge's subsidiary equations

$Rdpdy + Tdqdx + Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^{-2}dpdy + y^{-2}dqdx - (x^{-3}p - y^{-3}q)\,dxdy = 0. \qquad ...(1)$$

and $$x^{-2}(dy)^2 - y^{-2}(dx)^2 = 0. \qquad ...(2)$$

Multiplying both sides of (1) by x^3y^3, we get

$$xy^3dpdy - x^3ydqdx - py^3dxdy + qx^3dxdy = 0. \qquad ...(3)$$

Again, (2) $\Rightarrow$ $x^2y^2(y^2dy^2 - x^2dx^2) = 0$ or $x^2y^2(ydy + xdx)\,(ydy - xdx) = 0$

Hence (2) is equivalent to the equations

$ydy + xdx = 0$ *i.e.,* $ydy = -xdx$ ...(4)

and $$ydy - xdx = 0. \qquad ...(5)$$

Integrating (4), $y^2/2 + x^2/2 = c_1/2$ or $x^2 + y^2 = c_1$. ...(6)

From (3), $xy^2dp(ydy) - x^2ydq(xdx) - py^2dx(ydy) + qx^2dy(xdx) = 0$

or $xy^2dp(-xdx) - x^2ydq(xdx) - py^2dx\,(-xdx) + qx^2dy(xdx) = 0$, using (4)

or $-xy^2dp - x^2ydq + py^2dx + qx^2dy = 0$ or $y^2(xdp - pdx) + x^2(ydq - qdy) = 0$

or $$\frac{xdp - pdx}{x^2} + \frac{ydq - qdy}{y^2} = 0 \quad \text{or} \quad d\left(\frac{p}{x}\right) + d\left(\frac{q}{y}\right) = 0.$$

Integrating, $(p/x) + (q/y) = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), an intermediate integral is

$(1/x)p + (1/y)q = f(x^2 + y^2)$, where f is an arbitrary function. ...(8)

Similarly, from (3) and (5), another intermediate integral is

$(1/x)p - (1/y)q = g(x^2 - y^2)$, where g is an arbitrary function ...(9)

Solving (8) and (9) for p and q, we obtain

$p = (x/2) \times \{f(x^2 + y^2) + g(x^2 - y^2)\}$ and $q = (y/2) \times \{f(x^2 + y^2) - g(x^2 - y^2)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (x/2) \times \{f(x^2 + y^2) + g(x^2 - y^2)\}dx + (y/2) \times \{f(x^2 + y^2) - g(x^2 - y^2)\}dy$$

or $$dz = (1/4) \times f(x^2 + y^2)\,(2xdx + 2ydy) + (1/4) \times g(x^2 - y^2)\,(2xdx - 2ydy) \quad ...(10)$$

Putting $x^2 + y^2 = u$, $x^2 - y^2 = v$ so that $2xdx + 2ydy = du$ and $2xdx - 2ydy = dv$, (10) gives

$$dz = (1/4) \times f(u)\,du + (1/4) \times g(v)dv, \qquad ...(11)$$

Integrating (11), $z = F(u) + G(v) = F(x^2 + y^2) + G(x^2 - y^2)$,

where F and G are arbitrary functions.

Ex. 10. *Solve* $rx^2 - 3s\,xy + 2t\,y^2 + px + 2qy = x + 2y$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^2$, $S = -3xy$, $T = 2y^2$, $V = x + 2y - px - 2qy$.

Hence Monge's subsidiary equations are

$Rdpdy + Tdqdy - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^2\,dpdy + 2y^2dqdx - (x + 2y - px - 2qy)\,dxdy = 0 \qquad ...(1)$$

and $$x^2(dy)^2 + 3xy\,dxdy + 2y^2(dx)^2 = 0. \qquad ...(2)$$

Here (2) $\Rightarrow$ $(xdy + 2ydx)\,(xdy + ydx) = 0$.

Hence (2) resolves into the following two equations

$xdy + 2ydx = 0$ *i.e.,* $2ydx = -xdy$ ...(3)

and $$xdy + ydx = 0. \qquad ...(4)$$

Re–writing (3), $(1/y)dy + 2(1/x)dx = 0$

Integrating, $\log y + 2 \log x = \log c_1$ or $yx^2 = c_1$. ...(5)

Re–writing (1), $(xdp)(xdy) + ydq(2ydx) - dx(xdy) - dy(2ydx) + pdx(xdy) + qdy(2ydx) = 0$

or $(xdp)(xdy) + ydq(-xdy) - dx(xdy) - dy(-xdy) + pdx(xdy) + qdy(-xdy) = 0$, using (3)

or $xdp - ydq - dx + dy + pdx - qdy = 0$

or $(xdp + pdx) - (ydq + qdy) - dx + dy = 0$ or $d(xp) - d(yq) - dx + dy = 0$.

Integrating, $xp - yq - x + y = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), an intermediate integral is

$$xp - yq - x + y = f(x^2y), \text{ where } f \text{ is an arbitrary function.} \qquad ...(7)$$

Similarly from (1) and (4), another intermediate integral is

$$xp - 2yq - x + 2y = g(xy), \text{ where } g \text{ is an arbitrary function.} \qquad ...(8)$$

Solving (7) and (8) for p and q, we have

$p = (1/x) \times \{x + 2f(x^2y) - g(xy)\}$, and $q = (1/y) \times \{y + f(x^2y) - g(xy)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/x) \times \{x + 2f(x^2y) - g(xy)\}dx + (1/y) \times \{y + f(x^2y) - g(xy)\}]dy$$

or
$$dz = dx + dy + f(x^2y)\left(\frac{2}{x}dx + \frac{1}{y}dy\right) - g(xy)\left(\frac{dx}{x} + \frac{dy}{y}\right)$$

or
$$dz = dx + dy + f(x^2y)\, d[\log(x^2y)] - g(xy)\, d[\log(xy)].$$

Integrating, $z = x + y + F(x^2y) + \mathrm{G}(xy)$, G, and F being arbitrary functions.

Ex. 11. *Find the general solution of the equation $r + 4t = 8xy$, by Monge's method. Find also the particular solution for which $z = y^2$ and $p = 0$, when $x = 0$* **[Delhi Maths (Hons) 2006, 09]**

Sol. Given $$r + 4t = 8xy \qquad ... (1)$$

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 1$, $S = 0$, $T = 4$ and $V = 8xy$. Hence Monge's subsidians equations $Rdp\,dy + Tdq\,dx - Vdxdy = 0$ and $R(dy)^2 - Sdx\,dy + T(dx)^2 = 0$ become

$$dpdy + 4\,dqdx - 8xydxdy = 0 \qquad ... (2)$$

and
$$(dy)^2 + 4\,(dx)^2 = 0 \qquad ... (3)$$

Re–writing (3), $dy^2 - 4i^2dx^2 = 0$ or $(dy - 2idx)(dy + 2idx) = 0$

so that $dy - 2idx = 0$ or $dy = 2idx$ ... (4)

and $dy + 2idx = 0$ or $dy = -2idx$ ... (5)

We first consider (4) and (2). Integrating (4), $y - 2ix = C_1$ ... (6)

Using (4) and (6), (2) gives $dp\,(2i\,dx) + 4\,dq\,dx - 8x(C_1 + 2ix)\,(2i\,dx)\,dx = 0$

or $i\,dp + 2dq - 8xi\,(C_1 + 2ix) = 0$, by (6) or $idp + 2dq - 8C_1ix\,dx + 16x^2dx = 0$

Integrating, $ip + 2q - 4C_1ix^2 + (16/3)x^3 = C_2$, C_2 being an arbitrary constant

or $$ip + 2q - 4ix^2(y - 2ix) + (16/3) \times x^3 = C_2, \text{ by (6)} \qquad ... (7)$$

From (6) and (7) first intermediate integral of (1) is $ip + 2q - 4ix^2(y - 2ix) + (16/3)x^3 = f(y - 2ix)$

or $$ip + 2q = (8/3) \times x^3 + 4ix^2y + f(y - 2ix), \; f \text{ being an abritrary function} \qquad ... (8)$$

Similarly considering the pair (5) and (2), the second intermediate integral of (1) is

$$ip - 2q = -(8/3) \times x^3 + 4ix^2 y + g\ (y + 2ix),\ g \text{ being an arbitrary function} \quad \ldots (9)$$

Solving (8) and (9) for p and q, $\quad p = \{8ix^2 y + f(y - 2ix) + g(y + 2ix)\}/2i$

and $\quad q = \{(16/3) \times x^3 + f(y - 2ix) - g(y + 2ix)\}/4$

Putting the above values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2i) \times \{8ix^2 y + f(y - 2ix) + g(y + 2ix)\}dx \ + \ (1/4) \times \{(16/3) \times x^3 + f(y - 2ix) - g(y - 2ix)\}dy$$

$$= (4/3) \times (3x^2 ydx + x^3 dy) + (1/4) \times f(y - 2ix)d(y - 2ix) - (1/4) \times g(y + 2ix)d(y + 2ix)$$

$$\therefore \qquad dz = (4/3) \times d(x^3 y) + (1/4) \times f(y - 2ix)\ d(y - 2ix) - (1/4) \times g(y + 2ix)d(y + 2ix)$$

Integrating, $\quad z = (4/3) \times x^3 y + F(y - 2ix) + G\ (y + 2ix), \quad \ldots (10)$

which is the general solution of (1) containing F and G as arbitrary functions

To find particular solution of (1) Given conditions are

$z = y^2 \quad$ and $\quad p = \partial z/\partial x = 0 \quad$ when $x = 0 \quad \ldots (11)$

From (11), $\quad \partial z/\partial y = 2y \quad$ when $\quad x = 0 \quad \ldots (12)$

Differentiating (10) partially w.r.t. 'x' and 'y', we get

$$\partial z/\partial x = 4x^2 y - 2i\ F'(y - 2ix) + 2i\ G'(y + 2ix) \quad \ldots (13)$$

and $\quad \partial z/\partial y = (4/3) \times x^3 + F'(y - 2ix) + G'(y + 2ix) \quad \ldots (14)$

Using (11) and (12), (10), (13) and (14) reduce to

$$F(y) + G(y) = y^2 \quad \ldots (15)$$

$$F'(y) - G'(y) = 0 \quad \ldots (16)$$

and $\quad F'(y) + G'(y) = 2y \quad \ldots (17)$

From (16) and (17), $\quad F'(y) = y \quad$ and $\quad G'(y) = y$

Integrating these, $\quad F(y) = y^2/2 \quad$ and $\quad G(y) = y^2/2 \quad \ldots (18)$

which also satisfy (15).

From (18), $\quad F(y - 2ix) = (y - 2ix)^2/2 \quad$ and $\quad G(y + 2ix) = (y + 2ix)^2/2$

Putting these values in (10), the required particular solution is

$z = (4/3) \times x^3 y + (y - 2ix)^2/2 + (y + 2ix)^2/2 \quad$ or $\quad z = (4/3) \times x^3 y + y^2 - 4x^2$.

9.5. Type 2. When the given equation *Rr + Ss + Tt = V* leads to two distinct intermediate integrals and only one is employed to get the desired solution.

Working rule for solving problems of type 2.

Step 1. Write the given equation in the standard form $\quad Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0 \quad \ldots (1) \qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0 \quad \ldots (2)$

Step 3. Factorise (1) into two distinct factors.

Step 4. Take one of the factors of step 3 and use (2) to get an intermediate integral. Don't find second intermediate integral as we did in type 1. If required use remark 1 of Art. 9.2.

Step 5. Re–write the intermediate integral of the step 4 in the form of Lagrange equation, namely, $Pp + Qq = R$ (refer chapter 2). Using the well known Lagrange's method we arrive at the desired general solution of the given equation.

9.6 SOLVED EXAMPLES BASED ON ART. 9.5.

Ex. 1. *Solve* $(r - s)y + (s - t)x + q - p = 0$.

Sol. The given can be written as $\qquad yr + s(x - y) - tx = p - q. \qquad ...(1)$

Comparing (1) with $Rr + Ss + Tt = V$, $\quad R = y$, $\quad S = x - y$, $\quad T = -x$ $\quad$ and $\quad V = p - q$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ $\qquad$ and $\qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $\qquad ydpdy - xdqdx + (q - p)dxdy = 0 \qquad ...(1)$

and $\qquad y(dy)^2 - (x - y)\, dxdy - x\,(dx)^2 = 0. \qquad ...(2)$

Re–writing (2), $\qquad (dy + dx)\,(ydy - xdx) = 0.$

so that $\qquad dy + dx = 0 \qquad$ or $\qquad dy = - dx \qquad ...(3)$

and $\qquad ydy - xdx = 0. \qquad ...(4)$

Using (3), (1) becomes $\qquad - ydpdx - xdqdx + q\, dx(- dx) - p\, dxdy = 0$

or $\quad ydp + xdq + qdx + pdy = 0 \qquad$ or $\qquad (ydp + pdy) + (xdq + qdx) = 0$

or $\quad d(yp) + d(xq) = 0 \qquad$ so that $\qquad yp + xq = c_1. \qquad ...(5)$

Integrating (3), $\qquad x + y = c_2$, c_2 being an arbitrary constant $\qquad ...(6)$

From (5) and (6), one intermediate integral is $\qquad yp + xq = f(x + y), \qquad ...(7)$

which is of the Lagrange's form and so its subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{f(x+y)}. \qquad ...(8)$$

From first and second fractions of (8), $\qquad 2xdx - 2ydy = 0.$

Integrating, $\qquad x^2 - y^2 = a$, a being an arbitrary constant $\qquad ...(9)$

Taking first and third fractions of (8), we get

$$\frac{dx}{y} = \frac{dz}{f(x+y)} \qquad \text{or} \qquad \frac{dx}{(x^2-a)^{1/2}} = \frac{dz}{f\left[x+(x^2-a)^{1/2}\right]}, \text{ as (9)} \quad \Rightarrow y = (x^2 - a)^{1/2}$$

or $\qquad dz = f[x + (x^2 - a^2)^{1/2}]\,(x^2 - a^2)^{-1/2}dx \qquad ...(10)$

Put $\qquad x + (x^2 - a)^{1/2} = v \qquad$ so that $\qquad \left[1 + x/(x^2 - a)^{1/2}\right]dx = dv \qquad ...(11)$

or $\qquad \dfrac{x+(x^2-a)^{1/2}}{(x^2-a)^{1/2}}\,dx = dv \qquad$ or $\qquad \dfrac{dx}{(x^2-a)^{1/2}} = \dfrac{dv}{v}$, using (11)

Then, (10) reduces to $\qquad dz - (1/v)\, f(v)dv = 0.$

Integrating, $\qquad z - F(v) = b \qquad$ or $\qquad z - F[x + (x^2 - a)^{1/2}] = b$, by (11)

or $\qquad z - F(x + y) = b, \qquad$ as $\qquad y = (x^2 - a)^{1/2}$, by (9) $\qquad ...(12)$

From (9) and (12), the required general solution is $\qquad z - F(x + y) = G(x^2 - y^2)$

or $\qquad z = F(x + y) + G(x^2 - y^2)$, where F and G are arbitrary functions.

Ex. 2. *Solve* : $q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0$. $\qquad$ **[Meerut 1994; I.A.S. 1974]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we find

$R = q(1 + q), \qquad S = - (p + q + 2pq), \qquad T = p(1 + p), \qquad V = 0 \qquad ...(1)$

Monge's subsidiary equations are $\qquad Rdpdy + Tdq\,dx - Vdxdy = 0 \qquad ...(2)$

and $\qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0 \qquad ...(3)$

Using (1), (2) and (3) become $\qquad (q + q^2)dpdy + (p + p^2)dqdx = 0 \qquad ...(4)$

and $\qquad (q + q^2)\,(dy)^2 + (p + q + 2pq)dxdy + (p + p^2)\,(dx)^2 = 0. \qquad ...(5)$

In order to factorise (5), we re–write it as

$$q(1 + q)(dy)^2 + (p + pq)dxdy + (q + pq)dxdy + p(1 + p)(dx)^2 = 0$$

or $$q(1 + q)(dy)^2 + p(1 + q)dxdy + q(1 + p)dxdy + p(1 + p)(dx)^2 = 0$$

or $$(1 + q)dy(qdy + pdx) + (1 + p)dx(qdy + pdx) = 0$$

or $$(qdy + pdx)\,[(1 + q)dy + (1 + p)dx] = 0. \qquad \text{... (6)}$$

Then, from (6), we get $qdy + pdx = 0$ *i.e.*, $qdy = -pdx$ ...(7)

and $$(1 + q)dy + (1 + p)dx = 0. \qquad \text{...(8)}$$

Keeping (7) in view, (4) may be re–written as $(1 + q)dp\ (qdy) - (1 + p)dq\ (-pdx) = 0$

From (7), qdy and $(-pdx)$ are equivalent. Hence dividing each term of the above equation by qdy, or its equivalent $(-pdx)$, we get

$(1 + q)dp - (1 + p)dq = 0$ or $dp/(1 + p) - dq/(1 + q) = 0.$

Integrating it, $\log (1 + p) - \log (1 + q) = \log c_1$ or $(1 + p)/(1 + q) = c_1$. ...(9)

Using $dz = pdx + qdy$, (7) becomes $dz = 0$ so that $z = c_2$. ...(10)

From (9) and (10), one intermediate integral of (1) is given by

$(1 + p)/(1 + q) = f(z)$ or $p - f(z)q = f(z) - 1$, ...(11)

which is of the form $Pp + Qq = R$. Here Lagrange's auxiliary equations for (11) are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{f(z)-1}. \qquad \text{...(12)}$$

Choosing 1, 1, 1 as multipliers, each fraction in (12) = $\dfrac{dx+dy+dz}{1-f(z)+f(z)-1} = \dfrac{dx+dy+dz}{0}$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_2$. ...(13)

From first and third fractions in (12), we get $dx - [f(z) - 1]^{-1}\,dz = 0.$

Integrating it, $x + F(z) = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$$x + F(z) = G(x + y + z),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 3. *Solve* $(x - y)\,(xr - xs - ys + yt) = (x + y)(p - q)$.

[Delhi Maths (H) 97, 2000; Meerut 1999; Garhwal 1996]

Sol. Given $(x - y)xr - (x^2 - y^2)s + (x - y)yt = (x + y)(p - q)$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = x(x - y)$, $S = -(x^2 - y^2)$, $T = y(x - y)$, $V = (x + y)(p - q)$. ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ ...(3)

and $$R(dy)^2 - Sdxdy + T(dx)^2 = 0. \qquad \text{...(4)}$$

Using (2), (3) and (4) become

$$x(x - y)dpdy + y(x - y)dqdx - (x + y)(p - q)dxdy = 0 \qquad \text{...(5)}$$

and $$x(x - y)(dy)^2 + (x^2 - y^2)dxdy + y(x - y)(dx)^2 = 0. \qquad \text{...(6)}$$

Since $x^2 - y^2 = (x - y)(x + y)$, dividing (6) by $(x - y)$ gives

$xdy^2 + (x + y)dxdy + ydx^2 = 0$ or $(xdy + ydx)\,(dx + dy) = 0$

Thus we get $xdy + ydx = 0$ or $xdy = -\,ydx$ ...(7)

and $$dx + dy = 0. \qquad \text{...(8)}$$

Keeping (7) in view, (5) may be rewritten as

$$(x - y)dp\ (xdy) - (x - y)\ dq\ (-ydx) - (p - q)\ dx\ (xdy) + (p - q)dy(-\,ydx) = 0.$$

From (7), $x\,dy$ and $(-y\,dx)$ are equal. So dividing each term of the above equation by $x\,dy$, or its equivalent $(-y\,dx)$, we get

$(x - y)\ dp - (x - y)dq - (p - q)\ dx + (p - q)dy = 0$ or $(x - y)\ (dp - dq) - (p - q)\ (dx - dy) = 0$

or $\quad \dfrac{dp-dq}{p-q} - \dfrac{dx-dy}{x-y} = 0 \quad$ so that $\quad \dfrac{p-q}{x-y} = c_1$ ...(9)

Integrating (7), $\quad xy = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one intermediate integral of (10) is

$(p-q)/(x-y) = f(xy) \quad$ or $\quad p - q = (x-y)f(xy)$ ...(11)

which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x-y)f(xy)}. \quad ...(12)$$

Taking the first two fractions of (12), we get

$dx + dy = 0 \quad$ so that $\quad x + y = c_3$, c_3 being an arbitrary constant ...(13)

Taking $y f(xy)$, $x f(xy)$, 1 as multipliers, each fraction of (12) = $\dfrac{y\,f(xy)\,dx + x\,f(xy)\,dy + dz}{0}$

so that $\quad f(xy) \times (ydx + x\,dy) + dz = 0 \quad$ or $\quad f(xy) \times d(xy) + dz = 0.$

Integrating it, $\quad F(xy) + z = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$F(xy) + z = G(x + y)$, where F and G are arbitrary functions.

Ex. 4. $xy\,(t - r) + (x^2 - y^2)\,(s - 2) = py - qx.$ **[Delhi Maths (H) 2001]**

Sol. Given $\quad - xyr + (x^2 - y^2)\,s + xyt = py - qx + 2(x^2 - y^2).$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = -xy, \quad S = x^2 - y^2, \quad T = xy, \quad V = py - qx + 2(x^2 - y^2).$ ...(2)

Monge's subsidiary equations are $\quad Rdp\,dy + Tdq\,dx - V\,dx\,dy = 0$ ...(3)

and $\quad R(dy)^2 - Sdxdy + T(dx)^2 = 0.$ ...(4)

Using (2), (3) and (4) become

$$- xy\,dp\,dy + xydqdx - [py - qx + 2(x^2 - y^2)]dxdy = 0 \quad ...(5)$$

and $\quad - xy\,(dy)^2 - (x^2 - y^2)dxdy + xy(dx)^2 = 0.$ ...(6)

From (6), $\quad xy(dy)^2 + x^2dxdy - y^2dxdy - xy(dx)^2 = 0$

or $\quad xdy(ydy + xdx) - ydx\,(ydy + xdx) = 0 \quad$ or $\quad (xdy - ydx)\,(ydy + xdx) = 0.$

So, we get $\quad xdx + ydy = 0, \quad$ *i.e.*, $\quad xdx = -ydy$ ...(7)

and $\quad xdy - ydx = 0.$ ...(8)

Keeping (7) in view, (5) may be re–written as

$$xdp(-ydy) + ydq\,(xdx) + pdx\,(-ydy) + qdy(xdx) - 2xdy(xdx) - 2ydx(-ydy) = 0.$$

From (7), xdx and $(-ydy)$ are equivalent. So dividing each term of the above equation by xdx, or its equivalent $(-ydy)$, we get

$xdp + ydq + pdx + qdy - 2xdy - 2ydx = 0 \quad$ or $\quad (xdp + pdx) + (ydq + qdy) - 2(xdy + ydx) = 0.$

Integrating it, $\quad xp + yq - 2xy = c_1$, being an arbitrary constant ...(9)

Integrating (7), $\quad x^2/2 + y^2/2 = c_2/2 \quad$ or $\quad x^2 + y^2 = c_2.$ ...(10)

From (9) and (10), one intgegral of (1) is

$xp + qy - 2xy = f(x^2 + y^2) \quad$ or $\quad xp + yq = 2xy + f(x^2 + y^2),$ ...(11)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations for (11)are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{2xy + f(x^2 + y^2)}. \quad ..(12)$$

Taking the first two fractions in (12), we get

$\log y - \log x = \log c_3 \quad$ or $\quad y/x = c_3 \quad$ or $\quad y = xc_3$ ...(13)

Taking the first and the last fractions in (12) and using $y = xc_3$ in it, we get

$dz = (1/x) \times [2c_3x^2 + f(x^2 + x^2c_3^2)]dx$ or $dx = 2c_3xdx + (1/x) \times f\{(1 + c_3^2)x^2\}dx$

or $$dz = 2c_3xdx + (1/2x^2) \times f\{(1 + c_3^2)x^2\}d\,(x^2).$$

Integrating $z - 2c_3(x^2/2) + F\{(1 + c_3^2)x^2\} = c_4$ or $z - (y/x)x^2 + F\{(1 + y^2/x^2)x^2\} = c_4$, by (13)

or $$z - xy + F(x^2 + y^2) = c_4,\ c_4 \text{ being an arbitrary constant} \qquad ...(14)$$

From (13) and (14), the required general solution is

$$z - xy + F(x^2 + y^2) = G(y/x), \text{ where } F \text{ and } G \text{ are arbitrary functions.}$$

Ex. 5. *Solve* $x^2r - y^2t - 2xp + 2z = 0.$

Sol. Given $$x^2r - y^2t = 2xp - 2z. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = x^2$, $S = 0$, $T = -y^2$, $V = 2xp - 2z.$

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^2dpdy - y^2dqdx - (2xp - 2z)dxdy = 0 \qquad ...(2)$$

and $$x^2(dy)^2 - y^2(dx)^2 = 0. \qquad ...(3)$$

On factorizing, $(3) \Rightarrow (xdy - ydx)\,(xdy + ydx) = 0$

Thus, we have $xdy - ydx = 0$ *i.e.*, $xdy = ydx.$ ...(4)

and $$xdy + ydx = 0. \qquad ...(5)$$

Re–writing (2), $xdp\,(xdy) - ydq(ydx) - 2(xp - z)\,(xdy)(1/x)dx = 0$

or $xdp(xdy) - ydq(xdy) - 2(xp - z)\,(xdy)\,(1/x)dx = 0$, using (4)

or $$xdp - ydq - 2(xp - z)\,(1/x)dx = 0$$

or $xdp - dz + pdx + qdy - ydq - 2(xp - z)\,(1/x)dx = 0$ as $dz = pdx + qdy \Rightarrow -dz + pdx + qdy = 0$

or $$d(xp - z) - d(yq) + 2qdy - 2(xp - z)\,(1/x)dx = 0$$

or $d(xp - yq - z) + 2qy(1/x)dx - 2(xp - z)\,(1/x)dx = 0$, as from (4), $dy = (y/x)dx$

or $d(xp - yq - z) - 2\,(xp - yq - z)\,(1/x)\,dx = 0$ or $$\frac{d(xp-yq-z)}{xp-yq-z} - \frac{2dx}{x} = 0.$$

Integrating, $\log\,(xp - yq - z) - 2\log x = \log c_1$ or $(xp - yq - z)/x^2 = c_1.$ ...(6)

From (4), $(1/y)dy - (1/x)dx = 0$ so that $\log y - \log x = \log c_2$

or $$y/x = c_2,\ c_2 \text{ being an arbitrary constant} \qquad ...(7)$$

From (6) and (7), an intermediate integral is

$(xp - yq - z)\,/x^2 = \phi_1(y/x)$ or $xp - yq = z + x^2\phi_1(y/x).$ ...(8)

Lagrange's auxiliary equations for (8) are $$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{z+x^2\phi_1(y/x)}. \qquad ...(9)$$

From the first two ratios of (9), we get

$(1/x)\,dx + (1/y)\,dy = 0$ so that $xy = c_3.$ ...(10)

Taking the second and third ratios of (9), we get

$$\frac{dz}{dy} + \frac{z}{y} = -\frac{x^2}{y}\phi_1\left(\frac{y}{x}\right) = -\frac{c_3^2}{y^3}\phi_1\left(\frac{y^2}{c_3}\right), \text{ by (10)}$$

Its I.F $= e^{\int(1/y)dy} = y$ and so solution is $$zy = -\int \frac{c_3^2}{y^2}\phi_1\left(\frac{y^2}{c_3}\right)dy + c_4$$

or $$zy + \frac{c_3^{3/2}}{2}\int\left(\frac{c_3}{y^2}\right)\phi_1\left(\frac{y^2}{c_3}\right)\left(\frac{\sqrt{c_3}}{y}\right)d\left(\frac{y^2}{c_3}\right) = c_4 \quad \text{or} \quad zy + c_3^{3/2}\,\psi_1\left(\frac{y^2}{c_3}\right) = c_4$$

or $$zy + (xy)^{3/2}\,\psi_1(y/x) = c_4, \text{ using (10).} \qquad ...(11)$$

From (10) and (11), the required general solution is

$zy + (xy)^{3/2}\,\psi_1(y/x) = \psi_2(xy)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex. 6. *Solve* $(r - t)xy - s(x^2 - y^2) = qx - py$.

Sol. Given $$xyr - (x^2 - y^2)s - xyt = qx - py. \qquad ...(1)$$

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$xy\,dpdy - xy\,dqdx - (qx - py)\,dxdy = 0 \qquad ...(2)$$

and $$xy\,(dy)^2 + (x^2 - y^2)\,dxdy - xy\,(dx)^2 = 0. \qquad ...(3)$$

Now, (3) $\Rightarrow$ $(xdx + ydy)\,(xdy - ydx) = 0$

Hence, $xdx + ydy = 0$ *i.e.*, $xdx = -\,ydy$ ...(4)

and $$xdy - ydx = 0 \qquad ...(5)$$

Re–writing (2), $(xdp)\,(ydy) - ydq(xdx) - qdy(xdx) + pdx\,(ydy) = 0$

or $(xdp)\,(ydy) - ydq(-ydy) - qdy(-ydy) + pdx(ydy) = 0$, using (4)

or $xdp + ydq + qdy + pdx = 0$ or $d(xp) + d(yq) = 0$.

Integrating, $xp + yq = c_1$, c_1 being an arbitrary constant ...(6)

Integrating (4) $x^2/2 + y^2/2 = c_2/2$ or $x^2 + y^2 = c_2$. ...(7)

From (6) and (7), are intremediate integral is

$xp + yq = f(x^2 + y^2)$, f being an arbitrary function. ...(8)

Lagrange's subsidiary equations for (8) are $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{f(x^2+y^2)}. \qquad ...(9)$$

Taking the first and second fractions of (9), $(1/y)dy - (1/x)dx = 0$.

Integrating, $\log y - \log x = \log a$ or $y/x = a$, ...(10)

where a is an arbitrary constant.

Taking the first and third fraction of (9), we get

$$\frac{dx}{x} = \frac{dz}{f(x^2+y^2)} \quad \text{or} \quad \frac{dx}{x} = \frac{dz}{f(x^2+a^2x^2)}, \text{ using (10)}$$

or $$dz = (1/x) \times f[x^2\,(1 + a^2)]\,dx = (1/x^2) \times f[x^2\,(1 + a^2)]\,xdx. \qquad ...(11)$$

Putting $x^2(1 + a^2) = v$ and $2x(1+ a^2)dx = dv$, (11) gives

$$dz = \frac{1+a^2}{v}\,f(v) \times \frac{1}{2(1+a^2)}\,dv = \left(\frac{1}{2v}\right) f(v)dv.$$

Integrating, $z = F(v) + b$ or $z - F[x^2\,(1 + a^2)] = b$

or $z - F(x^2 + x^2a^2) = b$ or $z - F(x^2 + y^2) = b$, using (10). ...(12)

Here b is an arbitrary constant. From (10) and (12), general solution of (1) is

$z - F(x^2 + y^2) = G(y/x)$ or $z = F(x^2 + y^2) + G(y/x)$,

where F and G are arbitrary functions.

Ex. 7. *Solve* $2xr - (x + 2y)s + yt = [(x + 2y)\,(2p - q)]/(x - 2y)$

Sol. Comparing the given equation with $Rr + Ss + Tr = V$, we have

$R = 2x$, $S = -(x + 2y)$, $T = y$, $V = [(x - 2y)\,(2p - q)]/(x - 2y)$.

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$2xdpdy + ydqdx - \frac{x+2y}{x-2y}\,(2p - q)\,dxdy = 0 \qquad ...(1)$$

and $$2x(dy)^2 + (x + 2y)dxdy + y(dx)^2 = 0. \qquad ...(2)$$

The equation (2) can be resolved into the following two equations

$$xdy + ydx = 0 \qquad i.e., \qquad xdy = -\,ydx \qquad ...(3)$$

and $$dx + 2ydy = 0. \qquad ...(4)$$

Re–writing (1), $$2dp(xdy) + dq(ydx) - \frac{2p-q}{x-2y}[(xdy)dx + 2(ydx)dy]$$

or $$2dp(-ydx) + dq(ydx) - \frac{2p-q}{x-2y}\{(-ydx)dx + 2(ydx)dy\} = 0 \text{ using (3)}$$

or $$-2dp + dq - \frac{2p-q}{x-2y}(-dx + 2dy) = 0 \qquad \text{or} \qquad \frac{2dp-dq}{2p-q} - \frac{dx-2dy}{x-2y} = 0.$$

Intergrating, $\log(2p - q) - \log(x - 2y) = \log c_1$ or $(2p - q)/(x - 2y) = c_1$. ...(5)

Re–writing (3), $(1/y)dy + (1/x)dx = 0$ so that $\log x + \log y = \log c_2$

$\therefore$ $xy = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), an intermediate integral is

$(2p - q)/(x - 2y) = f(xy)$ or $2p - q = (x - 2y)\,f(xy)$, ...(7)

where f is an arbitrary function. The equation (7) is of Lagrange's form $Pp + Qq = R$. So Lagrange's, subsidiary equation for (7) are

$$\frac{dx}{2} = \frac{dy}{-1} = \frac{dz}{(x-2y)\,f(xy)}. \qquad ...(8)$$

Taking the first and second fractions of (8), $dx + 2dy = 0$.

Integrating, $x + 2y = a$, a being an arbitrary constant ...(9)

Taking $y\,f(xy)$, $x\,f(xy)$, 1 as multipliers, each fraction of (8)

$$= \frac{y\,f(xy)\,dx + x\,f(xy)\,dy + dz}{2y\,f(xy) - x\,f(xy) + (x-2y)\,f(xy)} = \frac{f(xy)(ydx + x\,dy) + dz}{0}$$

This $\Rightarrow$ $f(xy)\,d(xy) + dz$, as $ydx + xdy = d(xy)$

Integrating, $F(xy) + z = b$, b being an arbitrary constant. ...(10)

From (9) and (10), the required complete integral is

$F(xy) + z = G(x + y)$, F and G being arbitrary functions.

Ex. 8. *Solve $xr + (x + y)s + yt + p + q = 0$ by Monge's method.*

Sol. Given $$xr + (x + y)s + yt = -(p + q) \qquad ...(1)$$

Comparing (1) with $Rr + Sr + Tt = V$, here $R = x$, $S = x + y$, $T = y$ and $V = -(p + q)$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$xdpdy + ydqdx + (p + q)dxdy = 0 \qquad ...(2)$$

and $$x(dy)^2 - (x + y)dxdy + y(dx)^2 = 0 \qquad ...(3)$$

Re–writing (3), $$(xdy - ydx)(dy - dx) = 0$$

so that $$xdy - ydx = 0 \qquad ...(4)$$

and $$dy - dx = 0 \qquad i.e., \qquad dy = dx \qquad ...(5)$$

For the required solution, we consider relation (5) only.

Integrating (5), $x - y = c_1$, being an arbitrary constant ... (6)

Using (5), (2) becomes $xdpdx + ydqdx + (p + q)(dx)^2 = 0$

or $xdp + ydq + pdx + qdx = 0$, on dividing by dx (as $dx \neq 0$)

or $(xdp + pdx) + (ydq + qdx) = 0$ or $(xdp + pdx) + (ydq + qdy) = 0$ by(5)

or $d(xp) + d(yq) = 0$ so that $xp + yq = c_2$. ...(7)

From (6) and (7), one intermediate integral of (1) is

$$xp + yq = f(x - y), \ f \text{ being an arbitrary function} \qquad ...(8)$$

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{f(x-y)} \qquad ...(9)$$

Taking the first two fractions of (9), $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x - \log y = \log c_3$ or $x/y = c_3$ ...(10)

Now, each fraction of (9) $= \dfrac{dx - dy}{x - y} = \dfrac{d(x-y)}{x-y}$ ...(11)

Combining this fraction with last fraction of (9), we get

$$\frac{dz}{f(x-y)} = \frac{d(x-y)}{x-y} \quad \text{or} \quad dz = \frac{f(x-y)}{x-y}d(x-y) = \frac{f(u)du}{u}, \text{ if } \quad u = x - y$$

Integrating, $z = F(u) + c_4 = F(x-y) + c_4$, where $F(u) = \int \frac{1}{u} f(u)\, du$

or $z - F(x - y) = c_4$, c_4 being an arbitrary constant ...(12)

From (10) and (12), the required solution is

$z - F(x-y) = G(x/y)$ or $z = G(x/y) + F(x-y)$,

where F and G are arbitrary functions.

Ex. 9. *Solve $rq^2 - 2pqs + p^2t = pt - qs$ by Monge's method.* **[Delhi Maths (Hons) 2002]**

Sol. Given $q^2r - q(2p - 1)s + p(p - 1)t = 0$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = q^2$, $S = -q(2p - 1)$, $T = p(p - 1)$, $V = 0$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$ become

$$q^2dpdy + p(p - 1)dqdx = 0 \qquad ...(2)$$

and $$q^2(dy)^2 + q(2p - 1)dxdy + p(p - 1)(dx)^2 = 0 \qquad ...(3)$$

Re–writing (3), $(qdy + pdx)\{qdy + (p - 1)dx\} = 0$

so that $qdy + pdx = 0$ *i.e.*, $qdy = -pdx$ ...(4)

and $qdy + (p - 1)dx = 0$ ...(5)

For the required solution, we consider relation (4) only.

Since $dz = pdx + qdy$, (4) reduces to $dz = 0$ and so $z = c_1$ ...(6)

Re–writing (2), $(qdp)(qdy) + (p - 1)dq(pdx) = 0$

or $(qdp)(-pdx) + (p - 1)dq(pdx) = 0$, since from (4), $qdy = -pdx$

or $-qdp + (p - 1)dq = 0$ or $\{1/(p - 1)\}dp - \{1/q\}dq = 0$

Integrating, $\log(p-1) - \log q = \log c_2$ or $(p-1)/q = c_2$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$(p - 1)/q = f(z)$ or $p - qf(z) = 1$, ...(8)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{1} \qquad ...(9)$$

From the first and the last fractions of (9), $dx - dz = 0$ so that $x - z = c_3$ ...(10)

From the last two fractions of (9), $dy - f(z)dz = 0$

Integrating, $y - F(z) = c_4$, where $F(z) = \int f(z)dz$ ...(11)

From (10) and (11), the required solution is $y - F(z) = G(x - z)$

or $y = F(z) + G(x - z)$, where F, G are arbitrary functions.

Ex. 10. *Solve $e^{2y}(r - p) = e^{2x}(t - q)$ by Monge's method.*

Sol. Given $e^{2y}r - e^{2x}t = pe^{2y} - qe^{2x}$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = e^{2y}$, $S = 0$, $T = -e^{2x}$ and $V = pe^{2y} - qe^{2x}$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $e^{2y}dxdy - e^{2x}dqdx - (pe^{2y} - qe^{2x})dxdy = 0$ ...(2)

and $e^{2y}(dy)^2 - e^{2x}(dx)^2 = 0$ ...(3)

From (3), $(e^y dy - e^x dx)(e^y dy + e^x dx) = 0$

so that $e^y dy - e^x dx = 0$, that is, $e^x dx = e^y dy$ ...(4)

and $e^y dy + e^x dx = 0$ ...(5)

For the required solution, we consider relation (4) only.

Integrating (4), $e^x - e^y = c_1$, c_1 being arbitrary constant ...(6)

Rewriting (2), $(e^y dp)(e^y dy) - (e^x dq)(e^x dx) - p(e^y dy)(e^y dx) + q(e^x dx)(e^x dy) = 0$

or $(e^y dp)(e^x dx) - (e^x dq)(e^x dx) - p(e^x dx)(e^y dx) + q(x^x dx)(e^y dy) = 0$, by (4)

or $e^y dp - e^x dq - pe^y dx + qe^x dy = 0$ or $\{d(e^y p) - pe^y dy\} - \{d(e^x q) - qe^x dx\} = pe^y dx - qe^x dy$

or $d(e^y p) - d(e^x q) = pe^y(dx + dy) - qe^x(dx + dy)$ or $d(e^y p - e^x q) = (e^y p - e^x q)(dx + dy)$

or $$\frac{d(e^y p - e^x q)}{e^y p - e^x q} = d(x+y)$$

Integrating, $\log(e^y p - e^x q) - \log c_2 = x + y$ or $(e^y p - e^x q)/c_2 = e^{x+y}$

or $(e^y p - e^x q)/e^{x+y} = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), one intermediate integral of (1) is

$(e^y p - e^x q)/e^{x+y} = f(e^x - e^y)$ or $e^y p - e^x q = e^{x+y} f(e^x - e^y)$

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{e^y} = \frac{dy}{-e^x} = \frac{dz}{e^{x+y} f(e^x - e^y)} \qquad ...(8)$$

From the first two fractions of (8), $e^x dx + e^y dy = 0$ so that $e^x + e^y = c_3$ ...(9)

Taking the first and third fraction of (8) and noting that $e^y = c_3 - e^x$ from (9), we get

$$\frac{dx}{e^y} = \frac{dz}{e^x e^y f(e^x - c_3 + e^x)} \quad \text{or} \quad dz = e^x f(2e^x - c_3)dx$$

or $dz - (1/2) \times f(2e^x - c_3)d(2e^x - c_3) = 0$ or $dz - (1/2) \times f(u)du = 0$, taking $u = 2e^x - c_3$

Integrating, $z - F(u) = c_3$, where $F(u) = \int (1/2) \times f(u)du$

or $z - F(2e^x - c_3) = c_4$ or $z - F(e^x - e^y) = c_4$, by (9) ...(10)

From (9) and (10), the required solution is $z - F(e^x - e^y) = G(e^x + e^y)$

or $z = F(e^x - e^y) + G(e^x + e^y)$, where F, G are arbitrary functions.

Ex. 11. *Solve $x^2r - y^2t = xp - yq$ by Monge's method.*

Sol. Given $$x^2r - y^2t = xp - yq \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, here $R = x^2$, $S = 0$, $T = -y^2$ and $V = xp - yq$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$x^2dpdy - y^2dqdx - (xp - yq)dxdy = 0 \qquad ...(2)$$

and $$x^2(dy)^2 - y^2(dx)^2 = 0 \qquad ...(3)$$

Re–writing (3), $$(xdy - ydx)\,(xdy + ydx) = 0$$

so that $xdy - ydx = 0$ that is, $xdy = ydx$ ...(4)

and $$xdy + ydx = 0 \qquad ...(5)$$

From (4), $(1/y)dy - (1/x)dx = 0$ so that $y/x = c_1$ ...(6)

For the required solution, we consider relation (4) only.

Re–writing (2), $$(xdp)(xdy) - (ydq)(ydx) - (pdx)(xdy) + (qdy)(ydx) = 0$$

or $$(xdp)(ydx) - (ydq)(ydx) - (pdx)(ydx) + (qdy)(ydx) = 0, \text{ by (4)}$$

or $xdp - ydq - pdx + qdy = 0$ or $\{d(xp) - pdx\} - \{d(yq) - qdy\} - pdx + qdy = 0$

or $d(xp - yq) - 2pdx + 2qdy = 0$ or $d(xp - yq) - 2p\,dx + 2(y/x)dx = 0$, by (4)

or $d(xp - yq) - (2/x)(xp - yq)dx = 0$ or $\dfrac{d(xp - yq)}{xp - yq} - \dfrac{2dx}{x} = 0$

Integrating, $\log(xp - yq) - 2\log x = c_2$ or $(xp - yq)/x^2 = c_2$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$(xp - yq)/x^2 = f(y/x)$ or $xp - yq = x^2 f(x/y)$ ...(8)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{x^2 f(y/x)} \qquad ...(9)$$

Taking the first two ratios of (9), $(1/x)\,dx + (1/y)\,dy = 0$ so that $\log x + \log y = c_3$

or $$xy = c_3,\ c_3 \text{ being an arbitrary constant} \qquad ...(10)$$

Taking the first and last fractions of (9), we get

$dz = x\,f(y/x)dx$ or $dz = x\,f(c_3/x^2)$, since by (10), $y = c_3/x$

$$\therefore\quad z = \int\left(-\frac{x^4}{2c_3}\right) f\left(\frac{c_3}{x^2}\right)\left(-\frac{2c_3}{x^3}\right)dx = \int\left(-\frac{c_3^2}{2c_3t^2}\right)f(t)dt, \text{ putting } \frac{c_3}{x^2} = t \text{ and } -\frac{2c_3}{x^3}dx = dt$$

or $z = -\dfrac{c_3}{2}\displaystyle\int \frac{f(t)}{t^2}dt + c_4 = c_3F(t) + c_4,$ where $F(t) = -\dfrac{1}{2}\displaystyle\int \frac{f(t)}{t^2}dt$

or $z - c_3F(c_3/x^2) = c_4$ or $z - xy\,F(y/x) = c_4$, by (10) ... (11)

From (10) and (11), the required solution is

$z - xy\ F(y/x) = G(xy)$ or $z = x^2(y/x)\ F\ (y/x) + G(xy)$

or $z = x^2H(y/x) + G(xy)$ where $H(y/x) = (y/x)\ F(y/x)$ and H, G are arbitrary functions.

Ex. 12. *Solve* $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$.
and hence find the surface satisfying the above equation and touching the hyperbolic paraboloid $z = x^2 - y^2$ *along its section by the plane* $y = 1$. **[Meerut 2001, I.A.S. 1978, Ranchi 2010]**

Sol. Given $\quad 2x^2r - 5xys + 2y^2t = -2(px + qy)$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = 2x^2$, $S = -5xy$, $T = 2y^2$, $V = -2(px + qy)$

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$.

become $\quad 2x^2dpdy + 2y^2dqdx + 2(px + qy)dxdy = 0$. ...(2)

and $\quad 2x^2(dy)^2 + 5xydxdy + 2y^2(dx)^2 = 0$. ...(3)

Re–writing (3), $\quad (xdy + 2ydy)\,(2xdy + ydx) = 0$.

so that $\quad xdy + 2ydx = 0$, *i.e.*, $xdy = -2ydx$ ...(4)

and $\quad 2xdy + ydx = 0$. ...(5)

Keeping (4) in view, (2) may be re–written as

$$2xdp(xdy) - ydq\,(-2ydx) + 2pdx\,(xdy) - qdy\,(-2ydx) = 0.$$

or $\quad 2xdp(xdy) - ydq\,(xdy) + 2pdx\,(xdy) - qdy\,(xdy) = 0$, using (4)

or $\quad 2xdp - ydq + 2pdx - qdy = 0$ or $2(xdp + pdx) - (ydq + qdy) = 0$

or $\quad 2d(xp) - d(yq) = 0$ so that $2xp - yq = c_1$. ...(6)

From (4), $(1/y)dy + 2(1/x)dx = 0$ so that $\log y + 2\log x = \log c_2$

or $\quad \log y + \log x^2 = \log c_2$ or $x^2y = c_2$. ...(7)

From (6) and (7), one intermediate integral is

$2xp - yq = f(x^2y)$, f being an arbitrary function. ...(8)

which is of Lagrange's form. Hence Lagrange's subsidiary equations are

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{f(x^2y)}. \qquad ...(9)$$

Taking the first two fractions of (9), $\quad 2(1/y)dy + (1/x)dx = 0$.

Integrating, $2\log y + \log x = \log a$ or $y^2x = a$ or $x = a/y^2$. ...(10)

Taking the second and third fractions of (9) and using (10), we get

$$\frac{dy}{-y} = \frac{dz}{f(a^2/y^3)} \qquad \text{or} \qquad dz + \frac{1}{y} f\left(\frac{a^2}{y^3}\right) dy = 0. \qquad ...(11)$$

Putting $(a^2/y^3) = v$ so that $-(3a^2/y^4)\,dy = dv$, (11) gives

$$dz + \frac{1}{y} f(v) \times \left(-\frac{y^4}{3a^2}\right) dv = 0 \qquad \text{or} \qquad dz - \frac{f(v)}{3(a^2/y^3)}\,dv = 0$$

or $\quad dz - (1/3v) \times f(v)\,dv = 0$, as $v = a^2/y^3$.

Integrating, $z - F(v) = b$ or $z - F(a^2/y^3) = b$, b being an arbitrary constant.

or $\quad z - F(x^2y) = b$, as $y^2x = a$. ...(12)

From (10) and (12), the required complete solution is

$z - F(x^2y) = G(xy^2)$, F and G being arbitrary fucntions.

or $\quad z = F(x^2y) + G(xy^2)$. ...(13)

Second Part. The given surface is $\quad z = x^2 - y^2$. ...(14)

(13) $\Rightarrow$ $p = \partial z/\partial x = 2xy\,F'(x^2y) + y^2G'(xy^2)$ and $q = \partial z/\partial y = x^2F'(x^2y) + 2xyG'(xy^2)$. ...(15)

From (14), $\quad p = \partial z/\partial x = 2x$ and $q = \partial x/\partial y = -2y$. ...(16)

Since (13) and (14) touch each other along their section by the plane $y = 1$, the values of p and q given by (15) and (16) at any point on $y = 1$ must be equal

Thus, $$2xyF'(x^2y) + y^2G'(xy^2) = 2x, \text{ where } y = 1 \qquad ...(17)$$

and $$x^2F'(x^2y) + 2xy\, G'(xy^2) = -2y, \text{ where } y = 1. \qquad ...(18)$$

From (17), $$2xF'(x^2) + G'(x) = 2x. \qquad ...(19)$$

From (18), $$x^2F'(x^2) + 2xG'(x) = -2. \qquad ...(20)$$

Solving (19) and (20) for $F'(x^2)$ and $G'(x)$, we have

$$F'(x^2) = (4/3) + (2/3) \times (1/x^2). \qquad ...(21)$$

and $$G'(x) = -(2/3)\times x - (4/3) \times (1/x). \qquad ...(22)$$

(21) $\Rightarrow$ $\quad F'(u) = (4/3) + (2/3) \times (1/u)$, on putting $x^2 = u$

Integrating, $\quad F(u) = (4/3) \times u + (2/3) \times \log u + c_1$, c_1 being an arbitrary constant

This $\Rightarrow$ $$F(x^2y) = (4/3) \times x^2y + (2/3) \times \log (x^2y) + c_1. \qquad ...(23)$$

Integrating (22), $\quad G(x) = -(2/3)(x^2/2) - (4/3) \log x + c_2$, being an arbitrary constant

This $\Rightarrow$ $$G(xy^2) = -(1/3) \times x^2y^4 - (4/3) \times \log (xy^2) + c_2. \qquad ...(24)$$

Putting values of $F(x^2y)$ and $G(xy^2)$ given by (23) and (24) in (13), we get

$$z = (4/3) \times x^2y + (2/3) \times \log (x^2y) + c_1 - (1/3) \times x^2y^4 - (4/3) \times \log (xy^2) + c_2$$

or $$z = (4/3) \times x^2y - (1/3)\times x^2y^4 + (2/3) \times [\log (x^2y) - 2 \log (xy^2)] + c, \text{ taking } c_1 + c_2 = c$$

or $$z = (4/3)\times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log (x^2y) - \log (xy^2)^2]$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log \{(x^2y)/(x^2y^4)\} + c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times \log y^{-3} + c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y + c. \qquad ...(25)$$

Now at the point of contact of (14) and (25), the values of z must be the same and hence

$$x^2 - y^2 = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y + c, \text{ where } y = 1$$

$$\Rightarrow \quad x^2 - 1 = (4/3) \times x^2 - (1/3) \times x^2 + c, \text{ putting } y = 1$$

$$\Rightarrow \quad x^2 - 1 = x^2 + c \quad \Rightarrow \quad c = -1.$$

Putting $c = -1$ in (25), the required surface is

$$z = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y - 1 \quad \text{or} \quad 3z = 4x^2y - x^2y^4 - 6 \log y - 3.$$

9.7. Type 3. When the given equation Rr + Ss + Tt = V leads to two identical intermediate intergrals.

Working rule for solving problems of type 3

Step 1. Write the given equation in the standard form $\quad Rr + Ss + Tt = V.$

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rpdy + Tdqdx - Vdx\, dy = 0 \quad ...(1) \qquad R(dy)^2 - S\, dxdy + T(dx)^2 = 0 \quad ...(2)$

Step 3. R.H.S. of (2) reduces to a perfect square and hence it gives only one distinct factor in place of two as in type 1 and type 2.

Step 4. Start with the only one factor of step 3 and use (2) to get an intermediate integral.

Step 5. Re–write the intermediate integral of the step 4 in the form of $Pp + Qq = R$ and use Lagrange's method to obtain the required general solution of the given equation.

9.8. Solved examples based on Art 9.7

Ex. 1. *Solve* : $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$

[Meerut 2002, Delhi Maths (H) 1999 2007, 10; Rohailkhand 1997; Kanpur 1994]

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, ...(1)

$R = (1 + q)^2$, $S = -2(1 + p + q + pq)$, $T = (1 + p)^2$, $V = 0$. ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$. ...(4)

Using (2), (3) and (4) become

$$(1 + q)^2dpdy + (1 + p)^2dqdx = 0 \qquad ...(5)$$

and $$(1 + q)^2(dy)^2 + 2(1 + p + q + pq)dxdy + (1 + p)^2 (dx)^2 = 0. \qquad ...(6)$$

Since $1 + p + q + pq = (1 + p)(1 + q)$, (6) becomes $[(1 + q)dy + (1 + p)dx]^2 = 0$

so that $(1 + q)dy + (1 + p)dx = 0$ or $(1 + q)dy = -(1 + p)dx$. ...(7)

Keeping (7) in view, (5) may be re–written as

$$(1 + q)dp\ \{(1 + q)dy\} - (1 + p)dq\ \{-(1 + p)dx\} = 0. \qquad ...(8)$$

Dividing each term of (8) by $(1 + q)dy$, or its equivalent $-(1 + p)dx$, we get

$(1 + q)dp - (1 + p)dq = 0$ or $dp/(1 + p) - dq/(1+ q) = 0$.

Integrating it, $(1+ p)/(1 + q) = c_1$, c_1 being an arbitrary constant ...(9)

From (7), $dx + dy + pdx + qdy = 0$ or $dx + dy + dz = 0$, as $dz = pdx + qdy$

Integrating it, $x + y + z = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one intermediate integral of (1) is

$(1 + p)/(1 + q) = F(x + y + z)$ or $1 + p = (1 + q)\ F(x + y + z)$

or $p - q\ F(x + y + z) = F(x + y + z) - 1$, ...(11)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-F(x+y+z)} = \frac{dz}{F(x+y+z)-1} \qquad ...(12)$$

Choosing 1, 1, 1 as multipliers, each fraction of (12) $= (dx + dy + dz)/0$

so that $dx + dy + dz = 0$ giving $x + y + z = c_2$ ... (13)

Using (13) and taking the first two fractions of (12), we have

$dx = -dy/F(c_2)$ or $dy + F(c_2)dx = 0$.

Integrating it, $y + xF(c_2) = c_3$ or $y + x\ F(x + y + z) = c_3$ ...(14)

From (13) and (14), the required general solution is

$y + x\ F(x + y + z) = G(x + y + z)$, F, G being arbitrary functions.

Ex. 2. *Solve* $y^2r + 2xys + x^2t + px + qy = 0$. **[Bilaspur 2004]**

Sol. Given $y^2r + 2xys + x^2t = -(px + qy)$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = y^2$, $S = 2xy$, $T = x^2$, $V = -(px + qy)$. ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx + Vdxdy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$. ...(4)

Using (2), (3) and (4) become

$$y^2dpdy + x^2dqdx + (px + qy)\ dxdy = 0 \qquad ...(5)$$

and $$y^2(dy)^2 - 2xydxdy + x^2(dx)^2 = 0. \qquad ...(6)$$

From (6), $(xdx - ydy)^2 = 0$ so that $xdx - ydy = 0$ or $xdx = ydy$. ...(7)

Keeping (7) in view, (5) may be re–written as

$$ydp\ (ydy) + xdq\ (xdx) + pdy\ (xdx) + qdx\ (ydy) = 0. \qquad ...(8)$$

Dividing each term of (8) by xdx, or its equivalent ydy, we get

$ydp + xdq + pdy + qdx = 0$ or $(ydp + pdy) + (xdq + qdx) = 0$

Integrating it, $yp + xq = c_1$, being an arbitrary constant ...(9)

Integrating (7), $x^2/2 - y^2/2 = c_2/2$ or $x^2 - y^2 = c_2$. ...(10)

From (9) and (10), one intermediate integral of (1) is $yp + xq = F(x^2 - y^2)$, ...(11)
which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{F(x^2 - y^2)}. \qquad ...(12)$$

From the first two fractions of (2), $xdx - ydy = 0$ so that $x^2 - y^2 = c_2$. ...(13)

Taking the last two fractions and using (13), we get

$$\frac{dy}{(y^2 + c_2)^{1/2}} = \frac{dz}{F(c_2)} \qquad \text{or} \qquad dz - F(c_2)\frac{dy}{(y^2 + c_2)^{1/2}} = 0.$$

Integrating, $$z - F(c_2) \log [y + (y^2 + c_2)^{1/2}] = c_3$$

or $$z - F(x^2 - y^2) \log [y + \sqrt{(y^2 + x^2 - y^2)}\,] = c_3, \text{ using } (13)$$

or $z - F(x^2 - y^2) \log (x + y) = c_3$, c_3 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$z - F(x^2 - y^2) \log (x + y) = G(x^2 - y^2)$, F, G being arbitrary functions.

Ex. 3(a). *Obtain the integral of* $q^2r - 2pqs + p^2t = 0$ *in the form* $y + xf(z) = F(z)$.

[Delhi Maths Hons. 1999, 2007; Meerut 1994, 95; Nagpur 2005]

(b) *Show also that this solution represents a surface generated by straight lines that are parallel to a fixed plane.*

Sol. (*a*) Given $q^2r - 2pqs + p^2t = 0$. ...(1)

As ususal Monge's subsidiary equations are $q^2dpdy + p^2dp\ dx = 0$ ...(2)

and $q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0$ or $(qdy + pdx)^2 = 0$. ...(3)

From (3), we have $qdy + pdx = 0$ or $qdy = -pdx$. ...(4)

In view of (4), (2) may be re–written as $qdp\ (qdy) - pdq\ (-pdx) = 0$. ...(5)

Dividing each term of (5) by qdy, or its equivalent ($-pdx$), we find

$qdp - pdq = 0$ or $(1/p)dp - (1/q)\ dp = 0$.

Integrating it, $p/q = c_1$, c_1 being an arbitrary constant ...(6)

From (4), $dz = 0$, (as $dz = pdx + qdy$) so that $z = c_2$. ...(7)

From (6) and (7), one integral of (1) is $p/q = f(z)$ or $p - f(z)q = 0$, ...(8)
which is of the form $Pp + Qq = R$. Here f is an arbitrary function. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{0}. \qquad ...(9)$$

The last fraction in (9) gives $dz = 0$ so that $z = c_2$ ...(10)

From the first two fractions in (9) and (10), we find

$$\frac{dx}{1} = \frac{dy}{-f(c_2)} \qquad \text{or} \qquad dy + f(c_2)dx = 0.$$

Integrating, $y + xf(c_2) = c_3$ or $y + xf(z) = c_3$, by (10). ...(11)

From (10) and (11), the required integral is $y + xf(z) = \mathrm{F}(z)$. ...(12)

Part (*b*). Let $z = k$, k being an arbitrary constant. Then (12) is the locus of the straight lines given by the intersection of the planes

$$z = k \qquad \text{and} \qquad y + xf(k) - F(k) = 0. \qquad ...(13)$$

Clearly the lines are parallel to the plane $z = 0$ (which is a fixed plane) because these lie on the plane $z = k$ for different values of k.

Ex. 4. *Solve* $y^2r - 2ys + t = p + 6y$. **[Agra 1993; Bhopal 2004; Vikram 2004; Meerut 2009; Delhi Maths Hons 1994, 98, 2006, 09, 10]**

Sol. As usual Monge's subsidiary equations are

$$y^2 dpdy + dqdx - (p + 6y)dxdy = 0 \qquad ...(1)$$

and $\quad y^2(dy)^2 + 2ydydx + (dx)^2 = 0 \quad$ or $\quad (ydy + dx)^2 = 0. \qquad ...(2)$

From (2), $\quad ydy + dx = 0 \quad$ or $\quad dx = -ydy. \qquad ...(3)$

Putting the value of dx from (3) in (1), we find

$$y^2 dpdy + dq(-ydy) - (p + 6y)\, dy\, (-ydy) = 0$$

or $\quad ydp - dq + (p + 6y)\, dy = 0 \quad$ or $\quad (ydp + pdy) - dq + 6ydy = 0.$

Integrating it, $\quad yp - q + 3y^2 = c_1$, c_1 being an arbitrary constant $\qquad ...(4)$

Integrating (4), $\quad y^2/2 + x = c_2/2 \quad$ or $\quad y^2 + 2x = c_2. \qquad ...(6)$

From (5) and (6), one integral of (1) is

$yp - q + 3y^2 = F(y^2 + 2x) \quad$ or $\quad yp - q = F(y^2 + 2x) - 3y^2, \qquad ...(7)$

which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{F(y^2+2x)-3y^2}. \qquad ...(8)$$

From the first two fractions of (8), $\quad 2ydy + 2dx = 0 \quad$ so that $\quad y^2 + 2x = c_2. \qquad ...(9)$

Taking the last two fractions of (8) and using (9), $\quad dz + [F(c_2) - 3y^2]dy = 0.$

Integrating, $\quad z + yF(c_2) - y^3 = c_2 \quad$ or $\quad z + yF(y^2 + 2x) - y^3 = c_3. \qquad ...(10)$

From (9) and (10), the required general solution is

$$z + yF(y^2 + 2x) - y^3 = G(y^2 + 2x),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 5. *Solve* $(b + cq)^2 r - 2(b + cq)(a + cp)s + (a + cp)^2 t = 0$

Sol. Usual Monge's subsidiary equations are $\quad (b + cq)^2\, dpdy + (a + cp)^2\, dqdx = 0. \qquad ...(1)$

and $\quad (b + cq)^2\,(dy)^2 + 2(b + cq)(a + cp)\, dxdy + (a + cp)^2\,(dx)^2 = 0. \qquad ...(2)$

(2) $\Rightarrow \quad \{(b + cq)dy + (a + cp)dx\}^2 = 0 \qquad ...(3)$

or $\quad (b + cq)dy + (a + cp)dx = 0 \quad$ or $\quad adx + bdy + c(pdx + qdy) = 0$

or $\quad adx + bdy + cdz = 0, \quad$ as $\quad dz = pdx + qdy.$

Integrating, $\quad ax + by + cz = c_1$, c_3 being an arbitrary constant $\qquad ...(4)$

From (3), $(b + cq)dy = -(a + cp)dx$. So (1) reduces to $\quad (b + cq)dp - (a + cp)dq = 0$

or $\quad \dfrac{dp}{a+cp} - \dfrac{dq}{b+cq} = 0 \quad$ so that $\quad \dfrac{a+cp}{b+cq} = c_2 \qquad ...(5)$

So the intermediate integral of the given equation is $\quad (a + cp)/(b + cq) = \phi_1(ax + by + cz)$

or $\quad cp - c\phi_1(ax + by + cz)q = -a + b\,\phi_1(ax + by + cz). \qquad ...(6)$

Lagrange's auxiliary equations are

$$\frac{dx}{c} = \frac{dy}{-c\,\phi_1(ax+by+cz)} = \frac{dz}{-a+b\,\phi_1(ax+by+cz)}. \qquad ...(7)$$

Using a, b, c as multipliers, each fraction of (7) $= (adx + bdy + cdz)/0$

$\therefore \quad adx + bdy + cdz = 0 \quad$ so that $\quad ax + by + cz = c_3. \qquad ...(8)$

Using (8) and taking the first two ratios of (7), we get

$dx = -\,dy/\phi_1(c_3) \quad$ or $\quad dy + \phi_1(c_3)dx = 0.$

Integrating, $\quad y + x\phi_1(c_3) = c_4 \quad$ or $\quad y + x\phi_1(ax + by + cz) = c_4. \qquad ...(9)$

From (8) and (9), the required solution is

$$y + x\phi_1(ax + by + cz) = \phi_2(ax + by + cz),\ \phi_1,\ \phi_2 \text{ being arbitrary functions.}$$

Ex. 6. *Solve* $x^2 r - 2xs + t + q = 0.$ **[K.U. Kurukshetra 2004; Ravishankar 2005]**

Sol. Usual Monge's subsidiary equations are $\quad x^2 dpdy + dqdx + qdxdy = 0 \qquad ...(1)$

and $\quad x^2(dy)^2 + 2xdxdy + (dx)^2 = 0. \qquad ...(2)$

Now, $(2) \Rightarrow \quad (xdy + dx)^2 = 0 \quad \Rightarrow \quad xdy + dx = 0$ ...(3)

$(3) \Rightarrow \quad (dx)/x + dy = 0 \quad \Rightarrow \quad y + \log x = c_1.$ ...(4)

Using (3), (1) reduces to $\quad x^2dpdy + dq\,(-x\,dy) + q(-x\,dy)dy = 0$

or $\quad dp - \left(\dfrac{dq}{x} - \dfrac{qdx}{x^2}\right) = 0 \quad$ or $\quad d\left(p - \dfrac{q}{x}\right) = 0.$

Integrating, $\quad p - (q/x) = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), the intermediate integral of the given equation is

$p - (q/x) - \phi_1(y + \log x) \quad$ or $\quad xp - q = x\phi_1(y + \log x).$ ...(6)

Lagrange's auxiliary equations for (6) are $\quad \dfrac{dx}{x} = \dfrac{dy}{-1} = \dfrac{dz}{x\,\phi_1(y+\log x)}.$ ...(7)

Taking the first two fractions of (7), $\quad (1/x)dx + dy = 0 \Rightarrow y + \log x = c_3.$ ...(8)

Using (8), first and third fractions of (7) give $\quad \dfrac{dx}{x} = \dfrac{dz}{x\,\phi_1(c_3)} = \Rightarrow z - x\phi_1(c_3) = c_4$

or $\quad z - x\phi_1(y + \log x) = c_4$, c_4 being an arbitrary constant ...(9)

From (8) and (9) the required solution is

$z - x\phi_1(y + \log x) = \phi_2(y + \log x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 7. *Solve* $(y - x)\,(q^2r - 2pqs + p^2t) = (p + q)^2\,(p - q)$.

Sol. The usual Monge's subsidiary equations are

$$(y - x)\,(q^2dpdy + p^2dqdx) - (p + q)^2\,(p - q)dxdy = 0 \quad \text{...(1)}$$

and
$$q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0. \quad \text{...(2)}$$

$(2) \Rightarrow \quad (qdy + pdx)^2 = 0 \quad$ or $\quad qdy + pdx = 0.$ ...(3)

$dz = pdx + qdy$ and (3) $\quad \Rightarrow \quad dz = 0 \quad \Rightarrow \quad z = c_1.$ ...(4)

Using (3), (1) reduces to $\quad (y - x)\,(qdp - pdq) - (p^2 - q^2)\,(dx - dy) = 0$

or $\quad q^2d\left(\dfrac{p}{q}\right) - (p^2 - q^2)\,\dfrac{d(x-y)}{y-x} = 0 \quad$ or $\quad \dfrac{d(x-y)}{x-y} + \dfrac{d(p/q)}{(p/q)^2-1} = 0$

Integrating, $\quad \log(x - y) + \dfrac{1}{2}\log\dfrac{(p/q)-1}{(p/q)+1} = \dfrac{1}{2}\log c_2 \quad$ or $\quad (x - y)^2\dfrac{p-q}{p+q} = c_2.$ ...(5)

From (4) and (5), the intermediate integral of the given equation is

$(x - y)^2\dfrac{p-q}{p+q} = \phi_1(z) \quad$ or $\quad (x - y)^2(p - q) = (p + q)\phi_1(z)$

or $\quad p\{(x - y)^2 - \phi_1(z)\} - q\{(x - y)^2 + \phi_1(z)\} = 0.$ ...(6)

Here Lagrange's subsidiary equation for (6) are

$$\frac{dx}{(x-y)^2 - \phi_1(z)} = \frac{dy}{-\{(x-y)^2 + \phi_1(z)\}} = \frac{dz}{0}. \quad \text{...(7)}$$

Now, the third fraction of (7) $\Rightarrow \quad dz = 0 \quad$ so that $\quad z = a$, ...(8)

where 'a' is an arbitrary constant.

Now, each fraction of (7) $= \dfrac{dx+dy}{-2\phi_1(z)} = \dfrac{dx-dy}{2(x-y)^2} \quad \Rightarrow \quad d(x + y) = -\,\phi_1(a)\dfrac{d(x-y)}{(x-y)^2}$, by (8).

Integrating it, $\; x + y - \phi_1(a)\,(x - y)^{-1} = b \;$ or $\; x + y - \phi_1(z)\,(x - y)^{-1} = b$, using (8). ...(9)

From (8) and (9), the required general solution is

$x + y - (x - y)^{-1}\,\phi_1(z) = \phi_2(z)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 8. *Solve* $x^2r + 2xys + y^2t = 0$. **[Meerut 2003, Garhwal 1993; Delhi Maths (H) 2001]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^2$, $S = 2xy$, $T = y^2$. Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$

become $$x^2dpdy + y^2dqdx = 0 \qquad \text{...(1)}$$

and $$x^2(dy)^2 - 2xydxdy + y^2(dx)^2 = 0. \qquad \text{...(2)}$$

Now, (2) gives $(xdy - ydx)^2 = 0$ so that $xdy - ydx = 0$. ...(3)

Re–writing (1), $$(xdp)\,(xdy) + (ydx)\,(ydq) = 0$$

or $(xdp)(xdy) + (xdy)\,(ydq) = 0$ [$\because$ from (3), $ydx = xdy$]

or $xdp + ydq = 0$ or $xdp + ydq + pdx + qdy = pdx + qdy$

or $d(xp) + d(yq) - dz = 0$, as $dz = pdx + qdy$.

Integrating (1) $xp + yq - z = c_1$, c_1 being an arbitrary constant ...(4)

Now (3) gives $$(1/y)dy - (1/x)dx = 0.$$

Integrating, $\log y - \log x = \log c_2$ or $y/x = c_2$. ...(5)

From (4) and (5), the intermediate integral of the given equation is

$xp + yq - z = f(y/x)$ or $xp + yq = z + f(y/x)$, ...(6)

where f is an arbitrary function. Lagrange's subsidiary equation for (6) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + f(y/x)}. \qquad \text{...(7)}$$

Taking the first two fractions of (7), $(1/y)dy - (1/x)dx = 0$.

Integrating, $\log y - \log x = \log a$ so that $y/x = a$. ...(8)

Taking the last two fractions of (7) and using (8), we get $\dfrac{dz}{z + f(a)} - \dfrac{dy}{y} = 0.$

Integrating it, $\log\,[z + f(a)] - \log y = \log b$, b being an arbitrary constant

so that $[z + f(a)]/y = b$ or $[z + f(y/x)]/y = b$, using (8) ...(9)

From (8) and (9), the required solution is

$[z + f(y/x)]/y = g(y/x)$ or $z = yg(y/x) - f(y/x)$, where f and g are arbitrary functions.

Ex. 9. *Solve* $r - 2s + t = \sin\,(2x + 3y)$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = 1$, $S = -2$, $T = 1$, $V = \sin\,(2x + 3y)$. So Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$dpdy + dqdx - \sin\,(2x + 3y)dxdy = 0. \qquad \text{...(1)}$$

and $$(dy)^2 + 2\,dxdy + (dx)^2 = 0. \qquad \text{...(2)}$$

Now, (2) gives $(dy + dx)^2 = 0$ so that $dy + dx = 0$. ...(3)

From (3), $dy = -\,dx$. Then, (1) becomes $-\,dpdx + dqdx + \sin\,(2x + 3y)dxdy = 0$

or $$dp - dq + \sin\,(2x + 3y)dy = 0, \text{ as } dx \neq 0. \qquad \text{...(4)}$$

Now, integrating (3), $x + y = c_1$, c_1 being an arbitrary constant ...(5)

From (4), $dp - dq + \sin\,[2(x + y) + y]dy = 0$ or $dp - dq + \sin\,(2c_1 + y)dy = 0$, using (5).

Integrating, $$p - q - \cos\,(2c_1 + y) = c_2$$

or $$p - q - \cos(2x + 3y) = c_2, \text{ as } c_1 = x + y \qquad \text{...(6)}$$

From (5) and (6), an intermediate integral is

$p - q - \cos\,(2x + 3y) = f(x + y)$ or $p - q = \cos\,(2x + 3y) + f(x + y)$, ...(7)

where f is an arbitrary function. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\cos(2x+3y)+f(x+y)}. \qquad ...(8)$$

Taking the first two fractions of (8), $dx + dy = 0$ so that $x + y = a$ ... (9)
Taking the last two fractions of (8) and using (9), we get

$$\frac{dy}{-1} = \frac{dz}{\cos(2a+y)+f(a)} \quad \text{or} \quad dz + [\cos(2a+y) + f(a)]dy = 0.$$

Integrating it, $z + \sin(2a + y) + y f(a) = b$, b being an arbitrary constant

or $z + \sin(2x + 3y) + y f(x + y) = b$, using (9). ...(10)

From (9) and (10) the required complete integral is

$z + \sin(2x + 3y) + y f(x + y) = g(x + y)$, f and g being an arbitrary functions.

Ex. 10. *Solve* $q^2r - 2pqs + p^2t = pq^2$. **[I.A.S. 1986]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = q^2$, $S = -2pq$, $T = p^2$, $V = pq^2$. The Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $q^2dpdy + p^2dqdx - pq^2dxdy = 0$ ...(1)

and $q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0.$ ...(2)

Re–writing (2), $(qdy + pdx)^2 = 0$ so that $pdx + qdy = 0.$ ...(3)

Since $dz = pdx + qdy$, (3) $\Rightarrow$ $dz = 0$ so that $z = c_1$. ...(4)

Re–writing (1), $(qdy)(qdp) + (pdx)(pdq) - (qdy)(pqdx) = 0$

or $(qdy)(qdp) - (qdy)(pdq) - (qdy)(pqdx) = 0$, as from (3), $pdx = -qdy$

or $qdp - pdq - pqdx = 0$ or $(1/p)dp - (1/q)dq = dx.$

Integrating, $\log p - \log q - \log c_2 = x$ or $p/(c_2q) = e^x$

or $(p/q)e^{-x} = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), the intermediate integral of the given equation is

$(p/q)e^{-x} = f(z)$ or $px^{-x} - f(z)q = 0.$ ...(6)

Lagrange's auxiliary equations for (6) are $\dfrac{dx}{e^{-x}} = \dfrac{dy}{-f(z)} = \dfrac{dz}{0}.$ ...(7)

The last fraction of (7) $\Rightarrow$ $dz = 0$ so that $z = a.$ ...(8)

Taking the first fractions of (7) and using (8), we get

$$\frac{dx}{e^{-x}} = \frac{dy}{-f(a)} \quad \text{or} \quad e^x f(a)dx + dy = 0.$$

Integrating, $e^x f(a) + y = b$ or $e^x f(z) + y = b$, as from (8), $a = z$ ...(9)

From (8) and (9), the required complete integral is

$e^x f(z) + y = g(z)$, where f and g are arbitrary functions.

Ex. 11. *Solve* $q^2r - 2q(1+p)s + (1+p)^2t = 0$ *by Monge's method.*

Sol. Given $q^2r - 2q(1+p)s + (1+p)^2t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = q^2$, $S = -2q(1+p)$ and $T = (1+p)^2$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdx\,dy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$q^2 dpdy + (1+p)^2 \; dqdx = 0 \qquad \text{... (2)}$$

and $$q^2 (dy)^2 + 2q(1+p)dxdy + (1+p)^2 \; (dx)^2 = 0 \qquad \text{... (3)}$$

Rewriting (3), $\{qdy + (1+p)dx\}^2 = 0$ or $qdy + (1+p)\,dx = 0$ ... (4)

From (4), $dx + (pdx + qdy) = 0$ or $dx + dz = 0$, as $dz = pdx + qdy$

Integrating, $x + z = C_1$, C_1 being an arbitrary constant ... (5)

Re–writing (2), $(qdy)\;(qdp) + [(1+p)\;dx\} \times \{(1+p)dq\} = 0$

or $(qdy)\;(qdp) + (-qdy)\;[(1+p)\;dq] = 0$, using (4)

or $qdp - (1+p)\,dq = 0$ or $\{1/(1+p)\}dp - (1/q)dq = 0$

Integrating, $\log(1+p) - \log q = \log C_2$ or $(1+p)/q = C_2$ ... (6)

From (5) and (6), the intermediate integral of (1) is

$(1+p)/q = f(x+z)$ or $p - q\,f(x+z) = -1$ ... (7)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(x+z)} = \frac{dz}{-1} \qquad \text{... (8)}$$

Taking the first and last ratios, $dx + dz = 0$ $\Rightarrow$ $x + z = C_3$ ... (9)

Using (9) and taking the first two ratios of (8), we get

$dy + f(C_3)dx = 0$ so that $y + xF(C_3) = C_4$

or $$y + x\;f(x+z) = C_4, \text{ using (9)} \qquad \text{... (10)}$$

From (9) and (10), the required general solution is

$$y + xf(x+z) = g(x+z), \; f, g \text{ are arbitrary functions}$$

Ex. 12. *Solve* $(x-y)\;(x^2 - 2xys + y^2 t) = 2xy\;(p-q)$. **[Delhi B.Sc. (Hons) 2011]**

Sol. Given $$x^2(x-y)r - 2xy(x-y)s + y^2(x-y)t = 2xy(p-q) \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt = V$, here $R = x^2(x-y)$, $S = -2xy\;(x-y)$, $T = y^2\;(x-y)$ and $V = 2xy\;(p-q)$. Hence Monge's subsidiars equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dyxy + T(dx)^2 = 0$ become

$$x^2(x-y)\;dpdy - 2xy\;(p-q)\;dxdy + y^2(x-y)dqdx = 0 \qquad \text{... (2)}$$

and $$(x-y)\{x^2(dy)^2 + 2xy\;dx\;dy + y^2(dy)^2\} = 0 \qquad \text{... (3)}$$

Since $x \neq y$, (3) gives $(xdy + ydx)^2 = 0$ so that $ydx = -xdy$ ... (4)

From (4), $(1/x)dx + (1/y)dy = 0$ so that $xy = C_1$ ... (5)

Re–writing (2), $x(x-y)\;dp\;(xdy) - 2(p-q)(xdy)\;(ydx) + y(x-y)\,dq(ydx) = 0$

or $$x(x-y)\,dp(xdy) - 2(p-q)\;(xdy)\;(ydx) + y(x-y)dq(-xdy), by (4)$$

or $$x(x-y)dp - 2\;(p-q)\;(ydx) - y(x-y)\;dq = 0$$

or $(x-y)\ (xdp-ydq)=2y\ (p-q)dx$ or $xdp-ydq=\{2y(p-q)dx\}/(x-y)$

or $(xdp+pdx)-(ydq+qdy)=\{2y(p-q)\ dx\}/(x-y)+pdx-qdy$

or $d(xp)-d(yq)=\{2(p-q)\ ydx+(x-y)p\,dx-(x-y)\ qdy\}/(x-y)$

or $(x-y)d(xp-yq)=2pydx-2qydx+xpdx-ypdx-xqdy+yqdy$

$= pydx-2qydx+xpdx+qydx+yqdy\ =\ -pxdy-qydx+xpdx+yqdy,\ by(4)$

$\therefore$ $(x-y)\ d(xp-yq)=xp(dx-dy)-yq(dx-dy)=(xp-yq)(dx-dy)$

or $\dfrac{d(xp-yq)}{xp-yq}=\dfrac{dx-dy}{x-y}$ or $\dfrac{d(xp-yq)}{xp-yq}-\dfrac{d(x-y)}{x-y}=0$.

Integrating, $\log(xp-yq)-\log(x-y)=\log C_2$ or $(xp-yq)/(x-y)=C_2$ ... (6)

From (5) and (6), the intermediate integral of the given equation is

$(xp-yq)/(x-y)=f(xy)$ or $xp-yq=(x-y)f(xy)$, ... (7)

which is of Lagrange's form. Its auxiliary equations are $\dfrac{dx}{x}=\dfrac{dy}{-y}=\dfrac{dz}{(x-y)f(xy)}$ ... (8)

Taking the first two fractions, $(1/x)dx+(1/y)dy=0$ so that $xy=C_3$ ... (9)

Now, each fraction of (8) = $\dfrac{dx+dy}{x-y}=\dfrac{dz}{(x-y)f(xy)}$

or $dz=f(xy)\ d(x+y)$ or $dz=f(C_3)\ d(x+y)$, by (9)

Integrating, $z-(x+y)\ f(C_3)=C_4$ or $z-(x+y)\ f(xy)=C_4$ ... (10)

From (9) and (10), the required solution is $z-(x+y)f(xy)=g(xy)$

or $z=(x+y)f(xy)+g(xy)$, f and g being arbitrary functions.

9.9 Type 4. When the given equation Rr + Ss + Tt = V fails to yield an intermediate integral as in cases 1, 2 and 3.

Working rule for solving problems of type 4.

Suppose the R.H.S. of $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ neither gives two factors nor a perfect square (as in Types 1, 2 and 3 above). In such cases factors dx, dy, p, $1 + p$ etc. are cancelled as the case may be and an integral of given equation is obtained as usual. This integral is then integrated by methods explained in chapter 7.

9.10 SOLVED EXAMPLES BASED ON ART 9.9

Ex. 1. *Solve* $(q + 1)s = (p + 1)t$. **[Agra 2009]**

Sol. Given $(q + 1)s - (p + 1)t = 0$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find $R = 0$, $S = (q + 1)$. $T = -(p + 1)$, $V = 0$....(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$. ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$. ...(4)

Using (2), (3) and (4) become $-(p + 1)dqdx = 0$ ...(5)

and $-(q + 1)dxdy - (p + 1)(dx)^2 = 0$. ...(6)

Dividing (5) by $-(p + 1)dx$, we obtain $dq = 0$. ...(7)

and dividing (6) by $-dx$ we get $(q + 1) + (p + 1)dx = 0$. ...(8)

From (8), $dx + dy + pdx + qdy = 0$ or $dx + dy + dz = 0$, as $dz = pdx + qdy$

Integrating it, $x + y + z = c_1$, being an arbitrary constant ...(9)

Integrating (7), $q = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), an integral of (1) is

$q = f(x + y + z)$ or $\partial z/\partial y = f(x + y + z)$ ...(11)

Integrating (11) partially w.r.t. y (treating x as constant), we find

$$z = F(x + y + z) + G(x),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 2. *Solve* $pq = x(ps - qr)$. **[Delhi. Maths (H) 2002, 08]**

Sol. Given $xqr - xps + 0.t = -pq$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = xq$, $S = xp$, $T = o$ and $V = -pq$

Monge's subsidiary equations $Rdp\,dy + T\,dq\,dx - V\,dx\,dy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $xqdpdy + pqdxdy = 0$. ...(2)

and $xq(dy)^2 + xpdxdy = 0$. ...(3)

Dividing (2) by qdy we get $xdp + pdx = 0$ ...(4)

and dividing (3) by xdy, we get $qdy + pdx = 0$. ...(5)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = c_1$ ...(6)

Integrating (4), $xp = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), one integral of (1) is

$xp = f(z)$ or $x\dfrac{\partial z}{\partial x} = f(z)$ or $\dfrac{1}{f(z)}\dfrac{\partial z}{\partial x} = \dfrac{1}{x}$.

Integrating it partially w.r.t. x, $F(z) = \log x + G(y)$, F, G being arbitrary functions.

Ex. 3. *Solve* $pt - sqs = q^3$ **[MDU Rohtak 2004; Ravishankar 2004; Delhi Maths (H) 2005; Meerut 2005; 06 ; Rohilkhand 1994]**

Sol. Given $pt - qs = q^3$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 0$, $S = -q$, $T = p$, $V = q^3$.

$\therefore$ Monge's subsidiary equations $Rdpdy + Tdqdx - Vdxdy = 0$, $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $pdqdx - q^3dxdy = 0$ ...(2)

and $qdxdy + p(dx)^2 = 0$. ...(3)

Dividing (2) by dx, we get $pdq - q^3dy = 0$ ...(4)

and dividing (3) by dx, we get $pdx + qdy = 0$. ...(5)

From (5), $dy = -(pdx)/q$. Putting this value of dy into (4) gives

$pdq - q^3(pdx/q) = 0$ or $(1/q^2)dq + dx = 0$.

Integrating it, $-1/q + x = C_1$, C_1 being an arbitrary constant ...(6)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = C_2$. ...(7)

From (6) and (7), one integral of (1) is

$-\dfrac{1}{q} + x = f(z)$ or $\dfrac{\partial y}{\partial z} = x - f(z)$, as $q = \dfrac{\partial z}{\partial y}$,

Integrating with respect to z partially (treat x as constant), we obtain

$$y = xz - F(z) + G(x),\ F,\ G \text{ being arbitrary functions, where } F(z) = \int f(z)\,dz.$$

Ex. 4. *Solve* $z(qs - pt) = pq^2$. **[Delhi Maths (H) 1998; 2004, 11]**

Sol. Given $zqs - zpt = pq^2$. ...(1)

The usual Monge's subsidiary equations are $-zpdqdx - pq^2dxdy = 0$ ...(2)

and $-zqdxdy - zp(dx)^2 = 0$. ...(3)

Dividing (2) by $-pdx$, we get $zdq + q^2dy = 0$ ...(4)

and dividing (3) by $-z\,dx$ we get $\qquad qdy + pdx = 0.$ ...(5)

Using $dz = pdx + qdy$, (5) gives $\quad dz = 0 \quad$ so that $\quad z = C_1.$ ...(6)

Using (6) in (4), $\quad C_1 dq + q^2 dy = 0 \quad$ or $\quad (1/q^2)dq + (1/C_1)dy = 0.$

Integrating it, $\quad -1/q + y/C_1 = C_2 \quad$ or $\quad -1/q + y/z = C_2$, by (6) ...(7)

From (6) and (7), one integral of (1) is

$$-\frac{1}{q} + \frac{y}{z} = f(z) \qquad \text{or} \qquad \frac{\partial y}{\partial z} - \frac{1}{z}y = -f(z), \qquad \text{as } q = \frac{\partial y}{\partial z}$$

which is linear in variables y and z (treating x as constant).

Its integrating factor (I.F.) $= e^{-\int(1/z)dz} = e^{-\log z} = z^{-1}$ and so its solution is

$$yz^{-1} = -\int z^{-1} f(z)\, dz + G(x) \qquad \text{or} \qquad yz^{-1} = F(z) + G(x), \qquad \text{where} \quad F(z) = \int f(z)\, dz$$

or $\quad y = zF(z) + zG(x) \quad$ or $\quad y = H(z) + zG(x),$

where $H(z)[= zF(z)]$ and $G(x)$ are arbitrary functions.

Ex. 5. *Solve* $2yq + y^2 t = 1.$

Sol. Given equation is $\qquad 0.r + 0.s + y^2.t = 1 - 2yq.$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 0, \quad S = 0, \quad T = y^2, \quad V = 1 - 2yq.$

Hence the usual subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0 \qquad$ and $\qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $\qquad y^2 dqdx - (1 - 2yq)dxdy = 0$ ...(2)

and $\qquad y^2(dx)^2 = 0.$ ...(3)

From (3), $\quad dx = 0 \quad$ so that $\quad x = c_1.$ ...(4)

From (2), $\quad y^2 dq + 2yq\, dy - dy = 0 \quad$ or $\quad d(y^2 q) - dy = 0.$

Integrating it, $\quad y^2 q - y = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), an intermediate integral is

$y^2 q - y = f(x) \qquad$ or $\qquad y^2(\partial z/\partial y) - y = f(x)$

or $\qquad \partial z/\partial y = 1/y + (1/y^2) \times f(x)$ ...(6)

Integrating (6) w.r. t. y, treating x as constant, we get

$z = \log y - (1/y) f(x) + g(x) \qquad$ or $\qquad yz = y \log y - f(x) + y\, g(x),$

where f and g being arbitrary functions.

Ex. 6. *Solve* $(e^x - 1)(qr - ps) = pqe^x.$

Sol. Given $\qquad q(e^x - 1)r - p(e^x - 1)s = pqe^x.$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $\quad R = q(e^x - 1), \quad S = -p(e^x - 1), \quad T = 0, \quad V = pqe^x.$

Then the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0 \qquad$ and $\qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $\qquad q(e^x - 1)dpdy - pqe^x dxdy = 0$ ...(2)

and $\qquad q(e^x - 1)(dy)^2 + p(e^x - 1)dxdy = 0.$ ...(3)

Now, (3) $\Rightarrow \quad qdy + pdx = 0 \quad \Rightarrow \quad dz = 0, \quad$ as $\quad dz = pdx + qdy.$

Integrating, $\qquad z = c_1$, c_1 being an arbitrary constant ...(4)

Again, from (2), $\quad (e^x - 1)dp - pe^x dx = 0 \quad$ or $\quad \dfrac{dp}{p} - \dfrac{e^x}{e^x - 1}dx = 0$

Integrating, $\quad \log p - \log(e^x - 1) = \log c_2 \quad$ or $\quad p/(e^x - 1) = c_2.$ ...(5)

From (4) and (5), an intermediate integral is $p/(e^x - 1) = f(z)$, $\ f$ being an arbitrary function or $\dfrac{\partial z}{\partial x} = (e^x - 1)f(z)$, or $\dfrac{1}{f(z)}\dfrac{\partial z}{\partial x} = e^x - 1.$

Integrating w.r.t. 'x', treating y as constant, we get

$F(z) = e^x - x + G(y)$ or $x = e^x + G(y) - F(z)$,

F and G being arbitrary functions, where $\int (1/f(z))dz = F(z)$.

Miscellaneous problems based on types 1, 2, 3 and 4

Solve the following partial differential equations by using Monge's method:

1. $x^2r - y^2t = xy$. **Ans.** $z = xy \log x + x\, F(y/x) + G(xy)$

2. $(1 + pq + q^2)r + s(q^2 - p^2) - (1 + pq + p^2)t = 0$ **Ans.** $z\{2 + (x + y)\}^{1/2} = F(x + y) + G(x - y)$

3. $q(1 + q)r - (1 + 2q)(1 + p)s + (1 + p)^2t = 0$ **Ans.** $x = F(x + y + z) + G(x + z)$

4. $x^2r - y^2t - xp + yq = xy$. **Ans.** $z = (xy/4) \times \{(\log x)^2 - (\log y)^2\} + xyF(x/y) + G(xy)$

9.11. Monge's Method of integrating the equation Rr + Ss + Tt + U(rt – s²) = V, where r, s, t have their usual meaning and R, S, T, U, V are functions of x, y, z.

Given $$Rr + Ss + Tt + U(rt - s^2) = V. \qquad ...(1)$$

We have $$dp = (\partial p/\partial x)\,dx + (\partial p/\partial y)\,dy = rdx + sdy$$

and $$dq = (\partial q/\partial x)\,dx + (\partial q/\partial y)\,dy = sdx + tdy$$

which give $r = (dp - sdy)/dx$ and $t = (dq - sdx)/dy$.

Putting these values in (1) and simplifying, we get

$$(Rdpdy + Tdqdx - Udpdq - Vdxdy) - s\{R(dy)^2 - Sdxdy + T(dx)^2 + Udpdx + Udqdy\} = 0.$$

Hence the usual Monge's subsidiary equations are

$$L \equiv Rdpdy + Tdqdx + Udpdq - Vdxdy = 0 \qquad ...(2)$$

and $$M \equiv R(dy)^2 - Sdxdy + T(dx)^2 + Udpdx + Udqdy = 0. \qquad ...(3)$$

We cannot factorise M as we did before (see Art 9.1), on account of the presence of the additional terms, $Udpdx + Udqdy$. Hence let us factorise $M + \lambda L$, where λ is some multiplier to be determined later. Now, we have

$$M + \lambda L \equiv R(dy)^2 + T(dx)^2 - (S + \lambda V)dxdy + Udpdx + Udqdy + \lambda Rdpdy + \lambda Tdqdx + \lambda Udpdq = 0. \qquad ...(4)$$

Factorising L.H.S. of (4), let k and m be constants such that

$$M + \lambda L \equiv (Rdy + mTdx + kUdp)\left(dy + \frac{1}{m}dx + \frac{\lambda}{k}dq\right) = 0. \qquad ...(5)$$

Comparing coefficients in (4) and (5), we get $R/m + mT = -(S + \lambda V)$, ...(6)

$k = m$ and $R\lambda/k = U$. ...(7)

Now, the two relations of (7) give $m = R\lambda u$

Putting this valus of m in (6) and simplifying, we get $\lambda^2(UV + RT) + \lambda US + U^2 = 0$, ...(8)

which is quadratic in λ. Let λ_1 and λ_2 be its roots.

When $\lambda = \lambda_1$, (7) $\Rightarrow$ $R\lambda_1/k = U$ $\Rightarrow$ $k = R\lambda_1/U$ $\Rightarrow$ $m = R\lambda_1/U$

Hence (5) gives $$\left(Rdy + \frac{R\lambda_1}{U}Tdx + R\lambda_1 dp\right)\left(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq\right) = 0$$

or $$(Udy + \lambda_1 Tdx + \lambda_1 Udp)(Udx + \lambda_1 Rdy + \lambda_1 Udq) = 0. \qquad ...(9)$$

Similarly for $\lambda = \lambda_2$, (5) gives

$$(Udy + \lambda_2 Tdx + \lambda_2 Udp)(Udx + \lambda_2 Rdy + \lambda_2 Udq) = 0. \qquad ...(10)$$

Now one factor of (9) is combined with one factor of (10) to give an intermediate integral. Exactly similarly, the other pair will give rise to another intermediate integral. In this connection remember that we must combine first factor of (9) with the second factor of (10) and similarly the second factor of (9) with the first factor of (10). Thus for the desired solution the proper method is to combine the factors in the following manner :

$$U dy + \lambda_1 T dx + \lambda_1 U dp = 0, \qquad U dx + \lambda_2 R dy + \lambda_2 U dq = 0 \qquad ...(11)$$

$$U dy + \lambda_2 T dx + \lambda_2 U dp = 0, \qquad U dx + \lambda_1 R dy + \lambda_1 U dq = 0 \qquad ...(12)$$

Let equations (11) give two integrals $u_1 = c$ and $v_1 = d_1$ so that one intermediate integral is

$$u_1 = f_1(v_1), f_1 \text{ being an arbitrary function} \qquad ...(13)$$

Similarly, (12) gives second intermediate integral $\qquad u_2 = f_2(v_2), \qquad ...(14)$

where f_2 is an arbitrary function

We now solve (13) and (14) for p and q and substitute in $dz = pdx + qdy$, which after integration gives the desired general solution.

Remark 1. There are in all four ways of combining factors of (9) and (10). By combining the first factors in these equations, we would get $u\, dy = 0$ on substraction (after dividing equations by λ_1 and λ_2 respectively) and this would not produce any solution. Similarly, combining the second factors in these equations would give $u\, dx = 0$ and hence would produce no solution. Hence for getting integrals of the given equation we must proceed as explained in (11) and (12).

Remark 2. In what follows we shall use the following two results of equation $a\lambda^2 + b\lambda + c = 0$

(*i*) $a = b = 0$, *i.e.*, the coefficients of λ^2 and λ both equal to zero imply that both roots of the equatin are equal to ∞

(*ii*) $a = 0$ but $b \neq 0$, *i.e.*, the coefficient of λ^2 is zero but that of λ is non–zero imply that one root of the equation is ∞ and the other is $-c/b$.

Remark 3. When the two values of λ are equal, we shall have only one intermediate integral $u_1 = f(v_1)$ and proceed as explained in solved examples of type 1 based on $Rr + Ss + Tt + U(rt - s^2) = V$ given below.

An integral of *a* more general form can be obtained by taking the arbitrary function occuring in the intermediate integral to be linear.

Let $u_1 = mv_1 + n$, where m and n are some constants. Then integrating it by Lagrange's method we find the solution of the given equation.

9.12. Type 1: When the roots of λ –quadratic (8) of Art 9.11 are identical.

Solved examples of type 1 based on Rr + Ss + Tt + U(rt – s²) = V

Ex. 1. *Solve* $5r + 6s + 3t + 2(rt - s^2) + 3 = 0.$ **[I.A.S. 1973 ; Meerut 1998]**

Sol. Given equation $\qquad 5r + 6s + 3t + 2(rt - s^2) = -3. \qquad ...(1)$

Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have $R = 5$, $S = 6$, $T = 3$, $U = 2$ and $V = -3$. Hence the λ–quadratic $\qquad \lambda^2(UV + RT) + \lambda SU + U^2 = 0$

becomes $\qquad 9\lambda^2 + 12\lambda + 4 = 0 \qquad$ or $\qquad (3\lambda + 2)^2 = 0 \qquad$ so that $\qquad \lambda_1 = \lambda_2 = -2/3.$

There is only one intermediate integral given by the equations

$U dy + \lambda_1 T dx + \lambda_1 U dp = 0 \qquad$ and $\qquad U dx + \lambda_2 R dy + \lambda_2 U dq = 0$

or $\quad 2dy + (-2/3) \times 3dx + (-2/3) \times 2dp = 0 \qquad$ and $\qquad 2dx + (-2/3) \times 5dy + (-2/3) \times 2dq = 0$

or $\qquad 3dy - 3dx - 2dp = 0 \qquad$ and $\qquad 3dx - 5dy - 2dq = 0.$

Integrating, $\qquad 3y - 3x - 2p = c_1 \qquad$ and $\qquad 3x - 5y - 2q = c_2. \qquad ...(2)$

Hence here the only intermediate integral is

$$3y - 3x - 2p = f(3x - 5y - 2q), \text{ where } f \text{ is an arbitrary function.} \qquad ...(3)$$

Solving the two equations of (2) for p and q, we have

$p = (1/2) \times (3y - 3x - c_1) \qquad$ and $\qquad q = (1/2) \times (3x - 5y - c_2).$

Putting these values of p and q in $dz = pdx + qdy$, we have

$$dz = (1/2) \times (3y - 3x - c_1)dx + (1/2) \times (3x - 5y - c_2)dy$$

or $$2dz = 3(ydx + xdy) - 3xdx - 5ydy - c_1dx - c_2dy.$$

Integrating, $$2z = 3xy - (3x^2/2) - (5y^2/2) - c_1x - c_2y + c_3,$$

which is the required complete integral, c_1, c_2 and c_3 being arbitrary constants.

Alternative solution. An integral of a more general form can be obtained by supposing the arbitrary function f occuring in the intermediate integral (3) to be linear, giving

$$3y - 3x - 2p = m(3x - 5y - 2q) + n, \text{ where } m \text{ and } n \text{ are arbitrary constants.} \quad ...(4)$$

Re–writing (4), $$2p - 2mq = 3y - 3x + 5my - 3mx - n. \quad ...(5)$$

Lagrange's auxiliary equations for (5) are $$\frac{dx}{2} = \frac{dy}{-2m} = \frac{dz}{3y-3x+5my-3mx-n}. \quad ...(6)$$

Taking the first two fractions of (6), we have

$$dy + mdx = 0 \qquad \text{so that} \qquad y + mx = a. \quad ...(7)$$

Now, $$\text{each fraction of (6)} = \frac{3xdx+5ydy+2dz}{6x-10my+6y-6x+10my-6mx-2n} \quad ...(8)$$

Hence taking first fraction of (6) and fraction (8), we have

$$\frac{dx}{2} = \frac{3xdx+5ydy+2dz}{6y-6mx-2n} \qquad \text{or} \qquad dx = \frac{3xdx+5ydy+2dz}{3y-3mx-n}$$

or $$3xdx + 5ydy + 2dz = (3y - 3mx - n)dx$$

or $$2dz + 3xdx + 5ydy = \{3(a - mx) - 3mx - n\}dx, \text{ using (7)}$$

or $$2dz + 3xdx + 5ydy = (3a - 6mx - n)dx\ .$$

Integrating, $$2z + (3x^2/2) + (5y^2/2) = 3ax - 3mx^2 - nx + b/2$$

or $$4z + 3x^2 + 5y^2 = 6x(y + mx) - 6mx^2 - 2xn + b, \text{ using (7)}$$

or $$4z - 6xy + 3x^2 + 5y^2 + 2nx = b. \quad ...(9)$$

From (7) and (9), the required general solution is $4z - 6xy + 3x^2 + 2nx = \phi(y + mx)$, where ϕ is an arbitrary function and m and n are arbitrary constants.

Ex. 2. *Solve* $3r + 4s + t + (rt - s^2) = 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get $R = 3$, $S = 4$, $T = 1$, $U = 1$, $V = 1$. Then, λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes $4\lambda^2 + 4\lambda + 1 = 0$ or $(2\lambda + 1)^2 = 0$ so that $\lambda_1 = \lambda_2 = -1/2$.

There is only one intermediate integral given by the equations

$$Udy + \lambda_1Tdx + \lambda_1Udp = 0 \qquad \text{and} \qquad Udx + \lambda_2Rdy + \lambda_2Udq = 0$$

or $$dy + (-1/2) \times dx + (-1/2) \times dp = 0 \qquad \text{and} \qquad dx + (-1/2) \times 3dy + (-1/2) \times dq = 0$$

or $$-2dy + dx + dp = 0 \qquad \text{and} \qquad 3dy - 2dx + dq = 0. \quad ...(1)$$

Integrating, $$-2y + x + p = c_1 \qquad \text{and} \qquad 3y - 2x + q = c_2. \quad ...(2)$$

Hence the only intermediate integral is

$$-2y + x + p = f(3y - 2x + q), \text{ where } f \text{ is an arbitrary function.} \quad ...(3)$$

Solving (2) for p and q, $p = 2y - x + c_1$ and $q = -3y + 2x + c_2$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (2y - x + c_1)dx + (-3y + 2x + c_2)dy$$

or $$dz = 2(ydx + xdy) - xdx - 3ydy + c_1dx + c_2dy.$$

Integrating, $$z = 2xy - (x^2/2) - (3y^2/2) + c_1x + c_2y + c_3,$$

which is the required complete integral, c_1, c_2, c_3 being arbitrary constants.

Alternative solution. In order to get the more general solution, we assume the arbitrary function ϕ in (3) to be linear. Thus, we take

$$-2y + x + p = m(3y - 2x + q) + n, \; m, n \text{ being arbitrary constants}$$

or $$p - mq = 2y - x + 3my - 2mx + n. \quad \text{...(4)}$$

Lagrange's auxiliary equations for (4) are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{2y-x+3my-2mx+n}$. ...(5)

Taking the first two fractions of (5), $dy + mdx = 0$ so that $y + mx = a$. ...(6)

Now, each fraction of (5) $= \dfrac{xdx + 3ydy + dz}{x - 3my + 2y - x + 3my - 2mx + n}$ ...(7)

Taking the first fraction of (5) and the fraction (7), we have $\dfrac{dx}{1} = \dfrac{xdx + 3ydy + dz}{2y - 2mx + n}$

or $$xdx + 3ydy + dz = (2y - 2mx + n)dx$$

or $$xdx + 3ydy + dz = 2(a - mx)dx - 2mxdx + ndx, \text{ using (6)}$$

Integrating, $$(x^2/2) + (3y^2/2) + z = 2ax - mx^2 - mx^2 + nx + b/2$$

or $$x^2 + 3y^2 + 2z - 2x(y + mx) + 2mx^2 - nx = b, \text{ using (6)} \quad \text{...(8)}$$

From (6) and (8), the required general solution is $x^2 + 3y^2 + 2z - 2xy - nx = \phi(y + mx)$, where ϕ is an arbitrary function and m and n are arbitrary constants.

Ex. 3. *Solve* $(q^2 - 1)zr - 2pqzs + (p^2 - 1)zt + z^2(rt - s^2) = p^2 + q^2 - 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have $R = z(q^2 - 1)$, $S = -2pqz$, $T = z(p^2 - 1)$, $U = z^2$ and $V = p^2 + q^2 - 1$.

Hence the λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ becomes

$p^2q^2\lambda^2 - 2pqz + z^2 = 0$ or $(pq\lambda - z)^2 = 0$ so that $\lambda_1 = \lambda_2 = z/pq$.

There is only one intermediate integral given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $z^2dy + \dfrac{z^2(p^2-1)}{pq}dx + \dfrac{z^3}{pq}dp = 0$ and $z^2dx + \dfrac{z^2(q^2-1)}{pq}dy + \dfrac{z^3}{pq}dq = 0$

or $pqdy + (p^2 - 1)dx + zdp = 0$ and $pqdx + (q^2 - 1)dy + zdq = 0$

or $p(qdy + pdx) - dx + zdp = 0$ and $q(pdx + qdy) - dy + zdq = 0$

or $pdz + zdp - dx = 0$ and $qdz + zdq - dy = 0$, as $dz = pdx + qdy$

or $d(pz) - dx = 0$ and $d(qz) - dy = 0$.

Integrating, $pz - x = c_1$ and $qz - y = c_2$. ...(1)

Hence the only intermediate integral is $pz - x = f(qz - y)$, f being an arbitrary function. ...(2)

Solving (1) for p and q, $p = (c_1 + x)/z$ and $q = (c_2 + y)/z$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (1/z) \times (c_1 + x)dx + (1/z) \times (c_2 + y)dy$ or $zdz = (c_1 + x)dx + (c_2 + y)dy$.

Integrating, $$(1/2) \times z^2 = (1/2) \times (c_1 + x)^2 + (1/2) \times (c_2 + y)^2 + (1/2) \times c_3'.$$

or $$z^2 = x^2 + y^2 + 2c_1x + zc_2y + c_3, \text{ where } c_3 = c_1{}^2 + c_2{}^2 + c_3'$$

which is the complete integral, c_1, c_2, c_3 being arbitrary constants.

Alternative solution. To find the more general solution, we take the arbitrary function f in (2) to be linear. So, let $pz - x = m(qz - y) + n$, m, n being arbitrary constants.

or $$pz - mqz = x - my + n. \quad \text{...(3)}$$

Lagrange's auxiliary equation for (3) are $\dfrac{dx}{z} = \dfrac{dy}{-mz} = \dfrac{dz}{x - my + n}$. ...(4)

Taking the first two fractions of (4), $dy + mdx = 0$ so that $y + mx = a$. ...(5)

Now, each fraction of (4) $= \dfrac{(-x/z)dx-(y/z)dy+dz}{z\times(-x/z)-mz\times(-y/z)+x-my+n}$. ...(6)

Taking the first fraction of (4) and fraction (6), $\dfrac{dx}{z} = \dfrac{-(x/z)dx-(y/z)dy+dz}{n}$

or $-xdx - ydy + zdz = ndx$ or $-2zdz + 2xdx + 2ydy + 2ndx = 0.$

Integrating, $-z^2 + x^2 + y^2 + 2nx = b$, b being an arbitrary constant ...(7)

From (5) and (7), the required general solution is $-z^2 + x^2 + y^2 + 2nx = \phi(y + mx)$, where ϕ is an arbitrary function and m, n are arbitrary constants.

Ex. 4. *Solve* $2s + (rt - s^2) = 1$. **[Garwhal 1995; Meerut 2000]**

Sol. Comparing the given equation with the equation $Rr + Ss + Tt + U(rt - s^2) = V$, we get $R = 0$, $S = 2$, $T = 0$, $U = 1$, $V = 1$, so λ–quardratic $\lambda^2 (UV + RT) + \lambda SU + U^2 = 0$ becomes $\lambda^2 + 2\lambda + 1 = 0$ so that $\lambda_1 = \lambda_2 = -1.$

Since we have equal values of l, there would be only one intermediate integral given by

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $dy - dp = 0$ and $dx - dq = 0$, using (1)

which give $y - p = c_1$, and $x - q = c_2$.

Solving these for p and q, $p = y - c_1$ and $q = x - c_2$.

$\therefore$ $dz = pdx + qdy = (y - c_1)dx + (x - c_2)dy = (ydx + xdy) - c_1dx - c_2dy,$

or $dz = d(xy) - c_1dx - c_2dy.$

Integrating, $z = xy - c_1x - c_2y + c_3$, which is solution, c_1, c_2, c_3 being arbitrary constants.

Ex. 5. $z(1 + q^2)r - 2pqzs + z(1 + p^2)t + z^2(s^2 - rt) + 1 + p^2 + q^2 = 0.$

Sol. Comparing the give equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = z(1 + q^2)$, $S = -2pqz$, $T = z(1 + p^2)$, $U = z^2$ and $V = -(1 + p^2 + q^2)$. ... (1)

Hence λ–quadratic *i.e.* $\lambda^2(RT + UV) + \lambda US + U^2 = 0$ gives

$\lambda^2(p^2q^2) - 2\lambda zpq + z^2 = 0$ or $(\lambda pq - z)^2 = 0.$

Thus here we obtain $\lambda_1 = \lambda_2 = z/pq$. Hence there would be only one intermediate integral which is given by

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0. \qquad ...(2)$$

and

$$Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \qquad ...(3)$$

Using (1), (2) becomes $pq\,dy + (1 + p^2)dx + zdp = 0$ ...(4)

Using (1), (3) becomes $pqdx + (1 + q^2)dy + zdq = 0$ ...(5)

Now from (4), $p(pdx + qdy) + dx + zdp = 0$ or $pdz + dx + zdp = 0$, as $dz = pdx + qdy$

or $d(zp) + dx = 0$ so that $zp + x = c_1$. ...(6)

Similarly (5) gives $zq + y = c_2$, c_2 being an arbitrary constant ...(7)

Solving (6) and (7), we get $p = (c_1 - x)/z$ and $q = (c_2 - y)/z$.

$\therefore$ $dz = pdx + qdy = \{(c_1 - x)/z\}dx + \{(c_2 - y)/z\}dy$ or $zdz = c_1dx + c_2dy - (xdx + ydy)$.

Integrating, $(1/2) \times z^2 = c_1x + c_2y - (x_2 + y^2)/2 + c_3/2$ or $z^2 = 2c_1x + 2c_2y - x^2 - y^2 + c_3$, which is complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 6. *Solve* $2r + te^x - (rt - s^2) = 2e^x$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = 2$, $S = 0$, $T = e^x$, $U = -1$ and $V = 2e^x$. ... (1)

Hence the λ–quadratic $\lambda^2(UV+RT)+\lambda SU+U^2=0$ gives $\lambda^2(2e^x-2e^x)+(\lambda\times 0)+1=0.$

Since the coefficient of λ^2 and λ in the above quadratic vanish, it follows from the theory of equations that its both the roots must be infinite. Thus $\lambda_1=\lambda_2=\infty$. Since the two roots are equal there would be only one intermediate integral which is given by

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0 \quad \text{and} \quad Udx+\lambda_2 Rdy+\lambda_2 Udq=0,$$

i.e., by $(U/\lambda_1)dy+Tdx+Udq=0$ and $(U/\lambda_2)dx+Rdy+Udq=0,$

i.e., by $e^x dx-dp=0$ using (1) and $2dy-dq=0$, using (1)

Integrating these $e^x-p=c_1$ and $2y-q=c_2$.

Solving these, $p=e^x-c_1$ and $q=2y-c_2$.

Now, $dz=pdx+qdy=(e^x-c_1)dx+(2y-c_2)dy.$

Integrating, $z=e^x-c_1x+y^2-c_2y+c_3,$

which is complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 7. *Solve* $r+t-(rt-s^2)=1$.

Sol. Comparing the given equation with $Rr+Ss+Tt+U(rt-s^2)=V,$

$$R=1, \quad S=0, \quad T=1, \quad U=-1, \quad V=1. \qquad \text{...(1)}$$

So λ–quadratic $\lambda^2(UV+RT)+\lambda US+U^2=0$ becomes $(0\times\lambda^2)+(0\times\lambda)+1=0$. Since the coeffieicnts of both λ^2 and λ are zero, so both roots of this quadratic are equal to ∞. So $\lambda_1=\lambda_2=\infty$

Now, the only one intermediate integral is given by equations

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0 \quad \text{and} \quad \lambda_1 Rdy+Udx+\lambda_1 Udq=0$$

On dividing each term by λ_1 as λ_1 is infinite, the above equations become

or $(1/\lambda_1)\times Udy+Tdx+Udp=0$ and $Rdy+(1/\lambda_1)\times Udx+Udq=0$

or $Tdx+Udp=0$, as $\lambda_1=\infty$ and $Rdy+Udq=0$, as $\lambda_1=\infty$

or $dx-dp=0$ and $dy-dq=0$, using (1)

Integrating, $p-x=c_1$ and $q-y=c_2$. ...(2)

Solving (2) for p and q, $p=x+c_1$ and $q=y+c_2$.

Putting these values of p and q in $dz=pdx+qdy$, we get $dz=(x+c_1)dx+(y+c_2)dy$

Integrating, $z=x^2/2+c_1x+y^2/2+c_2y+c_3,$

which is the required integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 8. *Solve* $2pr+2qt-4pq\,(rt-s^2)=1$.

Sol. Comparing the given equation with $Rr+Ss+Tt+U(rt-s^2)=V$, we have

$$R=2p, \quad S=0, \quad T=2q, \quad U=-4pq, \quad V=1. \qquad \text{...(1)}$$

Then the λ–quadratic $\lambda^2(UV+RT)+\lambda SU+U^2=0$ becomes $(0\times\lambda^2)+(0\times\lambda)+4p^2q^2=0$. Since the coefficients of both λ^2 and λ are zero, so both roots of the λ– quadratic are equal to ∞. So $\lambda_1=\lambda_2=\infty$.

Now the only intermediate integral is given by the equation

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0 \quad \text{and} \quad \lambda_1 Rdy+Udx+\lambda_1 Udq=0$$

On dividing each term by λ_1 as λ_1 is infinite, the above equations become

$(1/\lambda_1)\times Udy+Tdx+Udp=0$ and $Rdy+(1/\lambda_1)\times Udx+Udq=0$

or $2qdx-4pqdp=0$ and $2pdy-4pqdq=0$, using (1)

or $2pdp-dx=0$ and $2qdq-dy=0.$

Integrating, $p^2 - x = c_1$ and $q^2 - y = c_2$.

Hence $p = \pm (c_1 + x)^{1/2}$ and $q = \pm (c_2 + y)^{1/2}$

Putting values of p and q in $dz = pdx + qdy$ gives $dz = \pm (c_1 + x)^{1/2} dx \pm (c_2 + y)^{1/2} dy$.

Integrating, $z = \pm (2/3) \times (c_1 + x)^{3/2} \pm (2/3) \times (c_2 + y)^{3/2} + c_3/2$

or $3z = \pm 2(c_1 + x)^{3/2} \pm 2(c_2 + y)^{3/2} + c_3$,

which is the complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 9. *Solve* $(1 + q^2)r - 2pqs + (1 + p^2)t + (1 + p^2 + q^2)^{-1/2}(rt - s^2) = -(1 + p^2 + q^2)^{3/2}$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = 1 + q^2$, $S = -2pq$, $T = 1 + p^2$, $U = (1 + p^2 + q^2)^{-1/2}$, $V = -(1 + p^2 + q^2)^{3/2}$ ...(1)

Now, the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes

$$\lambda^2 \{-(1 + p^2 + q^2) + (1 + q^2)(1 + p^2)\} - 2pq(1 + p^2 + q^2)^{-1/2}\lambda + (1 + p^2 + q^2)^{-1} = 0$$

or $$p^2q^2(1 + p^2 + q^2)\lambda^2 - 2pq(1 + p^2 + q^2)^{1/2}\lambda + 1 = 0$$

or $\{pq(1 + p^2 + q^2)^{1/2}\lambda - 1\}^2 = 0$ so that $\lambda_1 = \lambda_2 = 1/pq(1 + p^2 + q^2)^{1/2}$.

Here there is only intermediate integral given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $$\frac{1}{(1+p^2+q^2)^{1/2}}dy + \frac{1+p^2}{pq(1+p^2+q^2)^{1/2}}dx + \frac{dp}{pq(1+p^2+q^2)} = 0, \text{ by (1)}$$

and $$\frac{1}{(1+p^2+q^2)^{1/2}}dx + \frac{1+q^2}{pq(1+p^2+q^2)^{1/2}}dy + \frac{dq}{pq(1+p^2+q^2)} = 0, \text{ by (1)}$$

or $$pqdy + (1 + p^2)dx + [1/(1 + p^2 + q^2)^{1/2}]dp = 0 \qquad ...(2)$$

and $$pqdx + (1 + q^2)dy + \{1/(1 + p^2 + q^2)^{1/2}\}dq = 0. \qquad ...(3)$$

Eliminating dy between (2) and (3), $\{(1 + p^2)(1 + q^2) - p^2q^2\}dx + \dfrac{(1+q^2)dp - pqdq}{(1+p^2+q^2)^{1/2}} = 0$

or $$(1 + p^2 + q^2)dx + \frac{(1 + p^2 + q^2)dp - (p^2dp + pqdq)}{(1 + p^2 + q^2)^{1/2}} = 0$$

or $$dx + \frac{dp}{(1+p^2+q^2)^{1/2}} - \frac{p}{2}\frac{2pdp+2qdq}{(1+p^2+q^2)^{3/2}} = 0 \quad \text{or} \quad dx + d\left\{\frac{p}{(1+p^2+q^2)^{1/2}}\right\} = 0$$

Integrating, $x + p(1 + p^2 + q^2)^{-1/2} = a$, where a is an arbitrary constant. ...(4)

Similarly, eliminating dx between (2) and (3), we have

$y + q(1 + p^2 + q^2)^{-1/2} = b$, where b in an arbitrary constant. ...(5)

From (4) and (5), $x - a = -p(1 + p^2 + q^2)^{-1/2}$, $y - b = -q(1 + p^2 + q^2)^{-1/2}$.

$\therefore$ $\dfrac{x-a}{y-b} = \dfrac{p}{q}$ so that $p = \dfrac{x-a}{y-b}q$. ...(6)

Putting the above value of p in (4), we have

$$x + q\frac{x-a}{y-b}\left\{1+q^2\frac{(x-a)^2}{(y-b)^2}+q^2\right\}^{-1/2} = a \quad \text{or} \quad (x-a) + \frac{x-a}{y-b}q\left[1+\frac{(x-a)^2+(y-b)^2}{(y-b)^2}q^2\right]^{-1/2} = 0$$

or $$1 + \frac{(x-a)^2+(y-b)^2}{(y-b)^2}q^2 = \frac{q^2}{(y-b)^2} \quad \text{or} \quad (y-b)^2 = q^2[1 - \{(x-a)^2 + (y-b)^2\}].$$

Thus, $$q = (y-b)/[1-\{(x-a)^2+(y-b)^2\}]^{1/2}. \qquad \text{... (7)}$$

Now, (6) and (7) $\Rightarrow$ $$p = \frac{x-a}{y-b}q = \frac{x-a}{[1-\{(x-a)^2+(y-b)^2\}]^{1/2}}. \qquad \text{... (8)}$$

$$\therefore\ dz = pdx + qdy = \frac{(x-a)\,dx+(y-b)\,dy}{[1-\{(x-a)^2+(y-b)^2\}]^{1/2}}, \text{ by (7) and (8)}$$

Integrating, $z = [1 - \{(x-a)^2 + (y-b)^2\}]^{1/2} + c$ or $(z-c)^2 = 1 - \{(x-a)^2 + (y-b)^2\}$

$\therefore$ $(x-a)^2 + (y-b)^2 + (z-c)^2 = 1$ is the complete integral, a, b, c being arbitrary constants.

9.13 Type 2. When the roots of λ –quadratic (8) of Art 9.11 are distinct.

Solved Examples of Type –2 based on Rr + Ss + Tt + U(rt – s²) = V

Ex. 1. *Solve* $3s + rt - s^2 = 2$.

Sol. Given $$3s + (rt - s^2) = 2. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0, S = 3, V = 0, U = 1, V = 2$. ...(2)

λ–quadratic is $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad \text{...(3)}$$

Using (2), (3) reduces to $2\lambda^2 + 3\lambda + 1 = 0$ so $\lambda_1 = -1,\ \lambda_2 = -(1/2)$. ... (4)

Two integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp &= 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq &= 0. \end{aligned}\right\} \qquad \text{... (5)}$$

and

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq &= 0. \end{aligned}\right\} \qquad \text{... (6)}$$

Using (2) and (4), (5) and (6) respectively gives

$$\begin{aligned} dy - dp &= 0 \quad \text{or} \quad && \left.\begin{aligned} dp - dy &= 0 \\ dq - 2dx &= 0 \end{aligned}\right\} \qquad \text{... (5A)} \\ dx - (1/2)dq &= 0 \quad \text{or} \quad && \end{aligned}$$

and

$$\begin{aligned} dy - (1/2)dp &= 0 \quad \text{or} \quad && \left.\begin{aligned} dp - 2dy &= 0 \\ dq - dx &= 0. \end{aligned}\right\} \qquad \text{... (6A)} \\ dx - dq &= 0 \quad \text{or} \quad && \end{aligned}$$

Integration of (5A) and (6A) respectively gives

$$p - y = c_1, \qquad q - 2x = c_2 \qquad \text{...(5B)}$$

and $$p - 2y = c_3, \qquad q - x = c_4, \qquad \text{...(6B)}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

From (5B) and (6B), two intermediate integrals of (1) are given by

$$p - y = f(q - 2x) \quad \text{and} \quad p - 2y = F(q - x), \qquad \text{...(7)}$$

where f and F are arbitrary functions.

Let $$q - 2x = \alpha, \qquad \text{...(8)}$$

and $$q - x = \beta. \qquad \text{...(9)}$$

Then from (7) $$p - y = f(\alpha), \qquad \text{...(10)}$$

and $$p - 2y = F(\beta). \qquad \text{...(11)}$$

[If we treat α and β as constants, then solution of four simultaneous equation (8), (9), (10) and (11) would show that x, y, p and q are all constants which is absurd. Hence α and β will be regarded as variables (parameters) and we will get the general solution in parametric form involving α and β as parameters].

Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = \beta - \alpha \qquad \text{...(12)}$$

and $$y = f(\alpha) - F(\beta). \qquad \text{...(13)}$$

From (10) $$p = y + f(\alpha). \qquad \text{...(14)}$$

From (9) $q = x + \beta$. ...(15)

From (12) and (13), $dx = d\beta - d\alpha$, and $dy = f'(\alpha)d\alpha - F'(\beta)d\beta$. ...(16)

$\therefore$ $dz = pdx + qdy = [y + f(\alpha)]dx + (x + \beta)dy$, using (14) and (15)

or $dz = ydx + xdy + f(\alpha)dx + \beta dy = d(xy) + f(\alpha)(d\beta - d\alpha) + \beta[f'(\alpha)d\alpha - F'(\beta)d\beta]$, by (16)

Thus, $dz = d(xy) + [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] - f(\alpha)d\alpha - \beta F'(\beta)d\beta$

or $dz = d(xy) + d[\beta f(\alpha)] - f(\alpha)d\alpha - \beta F'(\beta)d\beta$.

Integrating and using integration by parts in the last term on R.H.S. of the above equation, we get $z = xy + \beta f(\alpha) - \int f(\alpha)d\alpha - [\beta F(\beta) - \int 1 \cdot F(\beta)d\beta]$

or $z = xy + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)d\alpha + \int F(\beta)d\beta$. ...(17)

Let $\int f(\alpha)d\alpha = \phi(\alpha)$ and $\int F(\beta)d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$ ...(19)

Using (18) and (19), (12), (13) and (17) give

$x = \beta - \alpha$, $y = \phi'(\alpha) - \psi'(\beta)$ $z = xy + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$

which is the required solution in parametric form, ϕ and ψ being arbitrary functions and α and β being parameters.

Ex. 2. *Solve* $r + 4s + t + rt - s^2 = 2$. **[I.A.S. 1979]**

Sol. Given $r + 4s + t + (rt - s^2) = 2$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 1$, $S = 4$, $T = 1$, $U = 1$, $V = 2$. ...(2)

λ–quadratic is $\lambda^2(UV + RT) + \lambda US + U^2 = 0$. ...(3)

Using (2), (3) reduces to $3\lambda^2 + 4\lambda + 1 = 0$ so $\lambda_1 = -1$, $\lambda_2 = -(1/3)$.

Two integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad \text{... (5)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \quad \text{... (6)}$$

Using (2) and (4), (5) and (6) respectively gives

$$\left.\begin{aligned} dy - dx - dp = 0 \quad &\text{or} \quad dp + dx - dy = 0 \\ dx - (1/3) \times dy - (1/3) \times dq = 0 \quad &\text{or} \quad dq + dy - 3dx = 0 \end{aligned}\right\} \quad \text{... (5A)}$$

$$\left.\begin{aligned} dy - (1/3) \times dx - (1/3) \times dp = 0 \quad &\text{or} \quad dp + dx - 3dy = 0 \\ dx - dy - dq = 0 \quad &\text{or} \quad dq + dy - dx = 0 \end{aligned}\right\} \quad \text{... (6A)}$$

Integration of (5A) and (6A) respectively gives

$p + x - y = c_1$, $q + y - 3x = c_2$ ...(5*B*)

and $p + x - 3y = c_3$, $q + y - x = c_4$, ...(6*B*)

where c_1, c_2, c_3 and c_4 are arbitrary constants.

From (5B) and (6B), two intermediate integrals of (1) are given by

$p + x - y = f(q + y - 3x)$ and $p + x - 3y = F(q + y - x)$. ...(7)

Let $q + y - 3x = \alpha$, ...(8)

and $q + y - x = \beta$. ...(9)

Then from (7), $p + x - y = f(\alpha)$, ...(10)

and $p + x - 3y = F(\beta)$. ...(11)

Here α and β are treated as parameters. Solving (8) and (9) for x and (10) and (11) for y gives

$$x = (\beta - \alpha)/2 \qquad ...(12)$$

and $$y = [f(\alpha) - F(\beta)]/2 \qquad ...(13)$$

From (10), $$p = y - x + f(\alpha) \qquad ...(14)$$

From (9), $$q = x - y + \beta \qquad ...(15)$$

From (12) and (13), $dx = (1/2) \times (d\beta - d\alpha)$, $\quad dy = (1/2) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]$. ...(16)

$\therefore \quad dz = pdx + qdy = [y - x + f(\alpha)]dx + (x - y + \beta)dy$, by (14) and (15)

$$= ydx + xdy - xdx - ydy + f(\alpha)dx + \beta dy$$

$$= d(xy) - xdx - ydy + f(\alpha) \times (1/2)\times(d\beta - d\alpha) + \beta \times (1/2)\times[f'(\alpha)d\alpha - F'(\beta)d\beta], \text{ by (16)}$$

$$= d(xy) - xdx - ydy + (1/2) \times [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] - (1/2) \times f(\alpha)d\alpha - (1/2) \times \beta F'(\beta)d\beta$$

or $$2dz = 2d(xy) - 2xdx - 2ydy + d[\beta f(\alpha)] - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating and using integration by parts in the last term on R.H.S. of the above equation,

we get $$2z = 2xy - x^2 - y^2 + \beta f(\alpha) - \int f(\alpha)d\alpha - [\beta F(\beta) - \int 1 \cdot F(\beta)\, d\beta]$$

or $$2z = 2xy - x^2 - y^2 + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)d\alpha + \int F(\beta)\, d\beta. \qquad ...(17)$$

Let $\int f(\alpha)d\alpha = \phi(\alpha)$ and $\int F(\beta)\, d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$. ...(19)

Using (18) and (19), (12), (13) and (17) give

$$2x = \beta - \alpha, \qquad 2y = \phi'(\alpha) - \psi'(\beta), \qquad 2z = 2xy - x^2 - y^2 + \beta[\phi'(a) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$$

which is the required solution in parametric form, α and β being parameters and ϕ and ψ being arbitrary functions.

Ex. 3. *Solve* $rt - s^2 + 1 = 0$

Sol. Given that $$0\cdot r + 0\cdot s + 0\cdot t + (rt - s^2) = -1. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0$, $S = 0$, $T = 0$, $U = 1$ and $V = -1$. ...(2)

Here λ–quadratic $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad ...(3)$$

becomes $\lambda^2 - 1 = 0$ so that $\lambda_1 = -1$ and $\lambda_2 = 1$. ...(4)

Since the two values of λ are distinct, we shall get two intermediate integrals which are given by the following sets of equations

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp &= 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq &= 0 \end{aligned}\right\} \qquad ...\text{(5A)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq &= 0 \end{aligned}\right\} \qquad ...\text{(5B)}$$

Using (2) and (4), equations (5) and (6) reduces to

$$\left.\begin{aligned} dy - dp = 0 \quad & i.e., \quad dp - dy = 0 \\ dx + dq = 0 \quad & i.e., \quad dq + dx = 0 \end{aligned}\right\} \qquad ...(5A)$$

$$\left.\begin{aligned} dy + dp = 0 \quad & i.e., \quad dp + dy = 0 \\ dx - dq = 0 \quad & i.e., \quad dq - dx = 0 \end{aligned}\right\} \qquad ...(6A)$$

Integrating of (5A) and (6A) respectively gives

$$p - y = c_1, \qquad q + x = c_2. \qquad ...(5B)$$

and $$p + y = c_3, \qquad q - x = c_4, \qquad ...(6B)$$

where c_1, c_2, c_3 are c_4 are arbitrary constants

From (5B) and (6B), two intermediate integrals are given by

$$p - y = f(q + x) \qquad \text{and} \qquad p + y = F(q - x), \qquad ...(7)$$

where f and F are arbitrary functions.

Let $$q + x = \alpha \qquad ...(8)$$

and $$q - x = \beta. \qquad ...(9)$$

Then, from (7), $$p - y = f(\alpha) \qquad ...(10)$$

and $$p + y = F(\beta). \qquad ...(11)$$

In what follows α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (\alpha - \beta)/2 \qquad ...(12)$$

and $$y = [F(\beta) - f(\alpha)]/2 \qquad ...(13)$$

From (10), $$p = y + f(\alpha) \qquad ...(14)$$

From (9), $$q = x + \beta. \qquad ...(15)$$

From (12) and (13), $dx = (1/2) \times (d\alpha - d\beta)$, $dy = (1/2) \times [F'(\beta)d\beta - f'(\alpha)d\alpha]$. ...(16)

$$\begin{aligned}\therefore \qquad dz &= pdx + qdy = [y + f(\alpha)]dx + (x + \beta)dy, \text{ using (14) and (15)}\\ &= (ydx + xdy) + f(\alpha)dx + \beta dy\\ &= d(xy) + f(\alpha) \times (1/2) \times (d\alpha - d\beta) + \beta \times (1/2) \times [F'(\beta)d\beta - f'(\alpha)d\alpha], \text{ by (16)}\\ &= d(xy) + (1/2) \times f(\alpha)d\alpha - (1/2) \times [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] + (1/2) \times \beta F'(\beta)d\beta\end{aligned}$$

or $$2dz = 2d(xy) + f(\alpha)d\alpha - d[\beta f(\alpha)] + \beta F'(\beta)d\beta.$$

Integrating both sides and using integration by parts in the last term on the R.H.S., we obtain

$$2z = 2xy + \int f(\alpha)d\alpha + \beta f(\alpha) + \beta F(\beta) - \int F(\beta)\,d\beta. \qquad ...(17)$$

Let $\int f(\alpha)\,d\alpha = \phi(\alpha)$ and $\int F(\beta)\,d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$. ...(19)

Using (18) and (19), (12), (13) and (17) may be re–written as

$$2x = (\alpha - \beta), \qquad 2y = \psi'(\beta) - \phi'(\alpha), \qquad 2z = 2xy - \phi(\alpha) + \beta\{\phi'(\alpha) + \psi'(\beta)\} - \psi(\beta)$$

which is the required solution in parametric form, α and β being parameters and ϕ and ψ being arbitrary functions.

Ex. 4. *Solve* $r + 3s + t + (rt - s^2) = 1$. **[Rohilkhand 1995]**

Sol. Given $$r + 3s + t + (rt + s^2) = 1 \qquad ... (1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 1, S = 3, T = 1, U = 1, V = 1$. ...(2)

Now, λ–quadratic is $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad ...(3)$$

or $2\lambda^2 + 3\lambda + 1 = 0$ so that $\lambda = -1, \ -1/2$. Here $\lambda_1 = -1$, $\lambda_2 = -1/2$. ...(4)

Two intermediate integrals of (1) are giving by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0\\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0\end{aligned}\right\} \qquad ... (5)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0\\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0\end{aligned}\right\} \qquad ... (6)$$

Using (2) and (4), equations (5) and (6) reduces to

$$\begin{aligned} &dy - dx - dp = 0 && \text{i.e.,} && \left.\begin{aligned} dp + dx - dy = 0\\ dq - 2dx + dy = 0\end{aligned}\right\} \ ... 5(A)\\ &dx - (1/2) \times dy - (1/2) \times dq = 0 && \text{i.e.,} && \end{aligned}$$

and
$$\begin{aligned} &dy - (1/2) \times dx - (1/2) \times dp = 0 && \text{i.e.,} && \left.\begin{aligned} dp + dx - 2dy = 0\\ dq - dx + dy = 0\end{aligned}\right\} \ ...(6A)\\ &dx - dy - dq = 0 && \text{i.e.,} && \end{aligned}$$

Integrating of ($5A$) and ($6A$) respectively gives

$$p + x - y = c_1, \qquad q - 2x + y = c_2 \qquad \text{...(5B)}$$

and $$p + x - 2y = c_3, \qquad q - x + y = c_4, \qquad \text{...(6B)}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

From ($5B$) and ($6B$), two intermediate integrals are given by

$$p + x - y = f(q - 2x + y) \quad \text{and} \quad p + x - 2y = F(q - x + y), \qquad \text{...(7)}$$

where f and F are arbitrary functions

Let $$q - 2x + y = \alpha \qquad \text{...(8)}$$

and $$q - x + y = \beta. \qquad \text{...(9)}$$

Then, from (7) $$p + x - y = f(\alpha) \qquad \text{...(10)}$$

and $$p + x - 2y = F(\beta). \qquad \text{...(11)}$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = \beta - \alpha \qquad \text{...(12)}$$

and $$y = f(\alpha) - F(\beta). \qquad \text{...(13)}$$

From (10), $$p = y - x + f(\alpha) \qquad \text{...(14)}$$

From (9), $$q = x - y + \beta. \qquad \text{...(15)}$$

From (12) and (13), $$dx = d\beta - d\alpha, \qquad dy = f'(\alpha)d\alpha - F'(\beta)d\beta. \qquad \text{...(16)}$$

$\therefore$ $dz = pdx + qdy = [y - x + f(\alpha)]dx + [x - y + \beta]dy$, using (14) and (15)

$= -(x - y)\,(dx - dy) + f(\alpha)dx + \beta dy$

$= -(x - y)\,d(x - y) + f(\alpha)\,(d\beta - d\alpha) + \beta[f'(\alpha)dx - F'(\beta)d\beta]$, by (16)

$= -(x - y)\,d(x - y) - f(\alpha)\,d\alpha + \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - \beta F'(\beta)d\beta$

or $$dz = -(x - y)\,d(x - y) - f(\alpha)\,d\alpha + d[\beta f(\alpha)] - \beta F'(\beta)d\beta.$$

Integrating both sides and using integration by parts in the last term on the R.H.S., we obtain

$$z = -(1/2) \times (x - y)^2 - \int f(\alpha)\,d\alpha + \beta f(\alpha) - \left[\beta F(\beta) - \int F(\beta)\,d\beta\right]. \qquad \text{...(17)}$$

Let $$\int f(\alpha)\,d\alpha = \phi(\alpha) \quad \text{and} \quad \int F(\beta)\,d\beta = \psi(\beta) \qquad \text{...(18)}$$

so that $$f(\alpha) = \phi'(\alpha) \quad \text{and} \quad F(\beta) = \psi'(\beta). \qquad \text{...(19)}$$

Using (18) and (19), (12), (13) and (17) may be written as

$$x = \beta - \alpha, \qquad y = \phi'(\alpha) - \psi'(\beta), \qquad z = -(1/2) \times (x - y)^2 - \phi(\alpha) + \psi(\beta) + \beta[\phi'(\alpha) - \psi'(\beta)]$$

which is the required solution in parametric form, α and β being parameters, and ϕ and ψ being arbitrary functions.

Ex. 5. *Solve* $rt - s^2 + a^2 = 0$. **[Rohilkhand 1993]**

Sol. Given that $$0 \cdot r + 0 \cdot s + 0 \cdot t + (rt - s^2) = -a^2. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0$, $S = 0$, $T = 0$, $U = 1$, $V = -a^2$. ...(2)

Then, the λ–quadratic $$\lambda^2(UV + RT) + \lambda SU + U^2 = 0 \qquad \text{...(3)}$$

becomes $-\lambda^2 a^2 + 1 = 0$ or $\lambda = \pm\, 1/a$. So $\lambda_1 = 1/a$, $\lambda_2 = -1/a$. ...(4)

Two intermediate integrals of (1) are given by the following two sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad \text{... (5)}$$

and $$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad \text{... (6)}$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} dy + (1/a) \times dp = 0 \quad & i.e., \quad dp + ady = 0 \\ dx - (1/a) \times dq = 0 \quad & i.e., \quad dq - adx = 0 \end{aligned}\right\} \quad \text{... (5}A\text{)}$$

and

$$\left.\begin{aligned} dy - (1/a) \times dp = 0 \quad & i.e., \quad dp - ady = 0 \\ dx + (1/a) \times dq = 0 \quad & i.e., \quad dq + adx = 0. \end{aligned}\right\} \quad \text{... (6}A\text{)}$$

Integration of (5A) and (6A) respectively gives

$$p + ay = c_1, \qquad q - ax = c_2 \qquad \text{...(5}B\text{)}$$

and $$p - ay = c_3, \qquad q + ax = c_4. \qquad \text{...(6}B\text{)}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediate integrals are given by

$$p + ay = f(q - ax) \qquad \text{and} \qquad p - ay = F(q + ax). \qquad \text{...(7)}$$

where f and F are arbitrary functions

Let $$q - ax = \alpha \qquad \text{...(8)}$$

and $$q + ax = \beta. \qquad \text{...(9)}$$

Then, from (7) $$p + ay = f(\alpha) \qquad \text{...(10)}$$

and $$p - ay = F(\beta). \qquad \text{...(11)}$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (1/2a) \times (\beta - \alpha) \qquad \text{...(12)}$$

and $$y = (1/2a) \times [f(\alpha) - F(\beta)]. \qquad \text{...(13)}$$

From (10), $$p = f(\alpha) - ay. \qquad \text{...(14)}$$

From (9), $$q = \beta - ax. \qquad \text{...(15)}$$

From (12) and (13), $dx = (1/2a) \times (d\beta - d\alpha)$, $dy = (1/2a) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]$ ...(16)

$$\therefore\ dz = pdx + qdy = [f(\alpha) - ay]dx + (\beta - ay)dy, \text{ using (14) and (15)}$$

$$= f(\alpha)dx + \beta dy - a(ydx + xdy)$$

$$= f(\alpha) \times (1/2a) \times (d\beta - d\alpha) + \beta \times (1/2a) \times [f'(\alpha)d\alpha - F'(\beta)d\beta] - ad(xy), \text{ by (16)}$$

or $$2adz = \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - f(\alpha)d\alpha - 2a^2d(xy) - \beta F'(\beta)d\beta.$$

Integrating both sides and using the formula for integration by parts in the last term on R.H.S., we have

$$2az = \beta f(\alpha) - \int f(\alpha)d\alpha - 2a^2xy - \left[\beta F(\beta) - \int F(\beta)\,d\beta\right]. \qquad \text{...(17)}$$

Let $$\int f(\alpha)d\alpha = \phi(\alpha) \qquad \text{and} \qquad \int F(\beta)\,d\beta = \psi(\beta) \quad \text{...(18)}$$

so that $$f(\alpha) = \phi'(\alpha) \qquad \text{and} \qquad F(\beta) = \psi'(\beta). \quad \text{...(19)}$$

Using (18) and (19), (12), (13) and (17) reduces to

$$2ax = \beta - \alpha, \qquad 2ay = \phi'(\alpha) - \psi'(\beta), \qquad 2az = \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) - 2a^2xy + \psi(\beta).$$

which is the required solution in parametric form, α, β, being parameters and $\phi(\alpha)$ and $\psi(\beta)$ being arbitrary functions.

Ex. 6. *Solve* $7r - 8s - 3t + (rt - s^2) = 36$.

Sol. Given that $$7r - 8s - 3t + (rt - s^2) = 36. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 7$, $S = -8$, $T = -3$, $U = 1$, $V = 36$. ...(2)

The λ–quadratic $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad \text{...(3)}$$

becomes $15\lambda^2 - 18\lambda + 1 = 0$ or $(5\lambda - 1)(3\lambda - 1) = 0$. So $\lambda_1 = 1/5$, $\lambda_2 = 1/3$. ...(4)

Two intermediate integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad \ldots (5)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \quad \ldots (6)$$

Using (2) and (4), equations (5) and (6) reduce to

$$dy + (1/5) \times (-3)dx + (1/5) \times dp = 0 \quad i.e., \quad dp - 3dx + 5dy = 0$$
$$dx + (1/3) \times 7\, dy + (1/3) \times dq = 0 \quad i.e., \quad dq + 7dy + 3dx = 0 \quad \ldots (5A)$$
$$dy + (1/3) \times (-3)dx + (1/3) \times dp = 0 \quad i.e., \quad dp - 3dx + 3dy = 0$$
$$dx + (1/5) \times 7dy + (1/5) \times dq = 0 \quad i.e., \quad dq + 7dy + 5dx = 0 \quad \ldots (6A)$$

Integrating of (5*A*) and (6*A*) respectively, gives

$$p - 3x + 5y = c_1, \qquad q + 7y + 3x = c_2 \qquad \ldots(5B)$$

and $$p - 3x + 3y = c_3, \qquad q + 7y + 5x = c_4, \qquad \ldots(6B)$$

where c_1, c_2 c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediatre integrals are given by

$$p - 3x + 5y = f(q + 7y + 3x) \quad \text{and} \quad p - 3x + 3y = F(q + 7y + 5x) \quad \ldots(7)$$

where f and F are arbitrary functions

Let $$q + 7y + 3x = \alpha \qquad \ldots(8)$$

and $$q + 7y + 5x = \beta. \qquad \ldots(9)$$

Then, from (7) $$p - 3x + 5y = f(\alpha) \qquad \ldots(10)$$

and $$p - 3x + 3y = F(\beta). \qquad \ldots(11)$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (\beta - \alpha)/2 \qquad \ldots(12)$$

and $$y = [f(\alpha) - F(\beta)]/2 \qquad \ldots(13)$$

From (10), $$p = f(\alpha) + 3x - 5y. \qquad \ldots(14)$$

From (9), $$q = \beta - 7y - 5x. \qquad \ldots(15)$$

From (12) and (13), $$dx = (1/2) \times (d\beta - d\alpha), \; dy = (1/2) \times \{f'(\alpha)d\alpha - F'(\beta)\, d\beta\}. \qquad \ldots(16)$$

$\therefore$ $dz = pdx + qdy = \{f(\alpha) + 3x - 5y\}dx + \{\beta - 7y - 5x)dy$, using (14) and (15)

$$= 3xdx - 7ydy - 5(ydx + xdy) + f(\alpha)dx + \beta dy$$
$$= 3xdx - 7ydy - 5d(xy) + f(\alpha) \times (1/2) \times (d\beta - d\alpha) + \beta \times (1/2) \times \{f'(\alpha)d\alpha - F'(\beta)d\beta\}$$

or $$2dz = 6xdx - 14ydy - 10d(xy) + \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - f(\alpha)d\alpha - \beta F'(\beta)d\beta$$

or $$2dz = 6xdx - 14ydy - 10d(xy) + d\{\beta f(\alpha)\} - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating both sides and using the formula for integrating by parts in the last term on R.H.S., we have

$$2z = 3x^2 - 7y^2 - 10xy + \beta f(\alpha) - \int f(\alpha)\, d\alpha - [\beta F(\beta) - \int F(\beta)\, d\beta]$$

or $$2z = 3x^2 - 7y^2 - 10xy + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)\, d\alpha + \int F(\beta)\, d\beta. \qquad \ldots(17)$$

Let $$\int f(\alpha)\, d\alpha = \phi(\alpha) \quad \text{and} \quad \int F(\beta)\, d\beta = \psi(\beta) \qquad \ldots(18)$$

so that $$f(\alpha) = \phi'(\alpha) \quad \text{and} \quad F(\beta) = \psi'(\beta) \qquad \ldots(19)$$

Using (18) and (19), relation (12), (13) and (17) become

$x = (1/2) \times (\beta - \alpha)$, $y = (1/2) \times [\phi'(\alpha) - \psi'(\beta)]$, $2z = 3x^2 - 7y^2 - 10xy + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$.
which is required solution in parametric form, α and β being parameters and $\phi(\alpha)$ and $\psi(\beta)$ being