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Que 1

a) $y^2 r - x^2 t = 0$

$Rs + Ss + Tt + f(x, y, z, p, q) = 0$

$R = y^2,\ S = 0,\ T = -x^2$

$\Rightarrow S^2 - 4RT = 0 + 4y^2(x^2) = 4x^2y^2 > 0 \rightarrow$ hyperbolic everywhere except on cordinate axis $x = 0$, $y = 0$

The λ quadratic is

$R\lambda^2 + S\lambda + T = 0$

OR

$y^2\lambda^2 - x^2 = 0$

$\lambda = \pm x/y$

Corresponding characteristic equation are.

$dy/dx + x/y = 0$ and $dy/dx - x/y = 0$

$\int x\,dx + y\,dy = 0$ and $\int x\,dy - y\,dy = 0$

$x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$

$x^2 + y^2 = c_1$ & $x^2 - y^2 = c_2$ are required. families of characteristic & there are hyperbolas

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b) $x^2 r + 2yxs + y^2 t = 0$

$R = x^2,\ S = 2xy,\ T = y^2 \Rightarrow S^2 - 4RT = 0$ (parabolic everywhere)

λ quadratic $\rightarrow R\lambda^2 + S\lambda + T = 0$

$x^2\lambda^2 + 2xy\lambda + y^2 = 0$

solving $(x\lambda + y)^2 = 0$ we get $\lambda = -y/x, -y/x$

The characteristic equation are -

$dy/dx + (-y/x) = 0$ or $1/y\, dy - 1/x\, dx = 0$

giving $y/x = c_1$ or $\boxed{y = c_1 x}$

Characteric equation represents family of straight line passing through origin

Que 2

a) $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$

$R = 1,\ S = 0,\ T = -1$

λ quadratic $\Rightarrow \lambda^2 - 1 = 0$

hence $\lambda = 1, -1 \rightarrow \lambda_1 = 1, \lambda_2 = -1$ (Real and distinct)

$dy/dx + 1 = 0$ and $dy/dx - 1 = 0$

$y + x = c_1$ $y - x = c_2$

$u = x+y$ and $v = y-x$ $\longrightarrow$ (1)

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

(using 1)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{(using 1)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right).$$

$$= \frac{\partial^2 z}{\partial u} - \frac{2\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \longrightarrow (2)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \longrightarrow (3)$$

using (2) & (3) our canonical form is

$$\frac{\partial^2 z}{\partial u^2} - \frac{2\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial u\partial v} = 0}$$

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b) $y^2(\partial^2 z/\partial x^2) + x^2(\partial^2 z/\partial y^2) = 0$

$R = y^2, S = 0, T = x^2 \Rightarrow S^2 - 4RT = -4x^2y^2 < 0$ for $x \neq 0, y \neq 0$ the curve is elliptic

λ quadratic eqn is

$y^2\lambda^2 + x^2 = 0$ or $\lambda^2 = -x^2/y^2$

$\Rightarrow \lambda = ix/y, -ix/y \longrightarrow (1)$

The corresponding characteristic eqn are

$\frac{dy}{dx} + \frac{ix}{y} = 0$ and $\frac{dy}{dx} - \frac{ix}{y} = 0$

$\Rightarrow y^2 + ix^2 = c_1$ and $y^2 - ix^2 = c_2$

$u = y^2 + ix^2 = \alpha + i\beta$ and $v = y^2 - ix^2 = \alpha - i\beta \longrightarrow (2)$

~~Classification of P.D.E Reduction to canonical or normal form using~~

Where $\alpha = y^2$ & $\beta = x^2 \longrightarrow (3)$

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = 2x\frac{\partial z}{\partial \beta}$ by (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = 2y\frac{\partial z}{\partial \alpha}$ by (3)

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial \beta}\right) = 2\frac{\partial z}{\partial \beta} + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$

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$$= \frac{2\partial z}{\partial \beta} + 2x\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

$$= 2\frac{\partial z}{\partial \beta} + 4x^2\frac{\partial^2 z}{\partial \beta^2} \longrightarrow (4)$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial \alpha}\right) = 2\frac{\partial z}{\partial \alpha} + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= 2\frac{\partial z}{\partial \alpha} + 2y\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right\}$$

$$= 2\frac{\partial z}{\partial \alpha} + 4y^2\frac{\partial^2 z}{\partial \alpha^2} \longrightarrow (5)$$

using (4) & (5)

$$2y^2\frac{\partial z}{\partial \beta} + 4x^2y^2\frac{\partial^2 z}{\partial \beta^2} + 2x^2\frac{\partial z}{\partial \alpha} + 4x^2y^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

$$= 2\alpha\beta\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) + \alpha\frac{\partial z}{\partial \beta} + \beta\frac{\partial z}{\partial \alpha} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}\right) = 0}$$

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Ques $q(yq+z)r - p(2yq+z)s + yp^2t + p^2q = 0$ using Monge's method.

Solⁿ Monge's subsidiary equations are

$$q(yq+z)\,dp\,dy + yp^2\,dq\,dx + p^2q\,dx\,dy = 0 \rightarrow (1)$$
$$q(yq+z)(dy)^2 + p(2yq+z)\,dx\,dy + yp^2(dx)^2 = 0 \rightarrow (2)$$

on factorizing (2) gives $(q\,dy + p\,dx)\{(yq+z)\,dy + yp\,dx\} = 0$

Hence 2 systems to be considered are

$$q(yq+z)\,dp\,dy + yp^2\,dq\,dx + p^2q\,dx\,dy = 0, \quad q\,dy + p\,dx = 0 \rightarrow (3)$$
& $$q(yq+z)\,dp\,dy + yp^2\,dq\,dx + p^2q\,dx\,dy = 0, \quad (yq+z)\,dy + yp\,dx = 0 \rightarrow (4)$$

using $dz = p\,dx + q\,dy$, the second equation of (3) reduces to

$dz = 0$ so that $z = c_1 \longrightarrow (5)$

From second equⁿ of (3) $q\,dy = -p\,dx$, Hence 1st equation of 3 reduces to

$(yq+z)\,dp - yp\,dq - pq\,dy = 0$ or $(yq+z)\,dp - p\,d(yq) = 0$

or $(yq+z)\,dp - p\,d(yq+z) = 0$ and $dz = 0$ by (5)

or $\dfrac{d(yq+z)}{yq+z} - \dfrac{dp}{p} = 0$ so that $\log(yq+z) - \log p = \log c_1$

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or $(yq+z)/p = c_2$, c_2 being arbitrary constant

from (5) & (6), the intermediate integral corresponding to (3) is

$(yq+z)/p = \phi_1(z)$ or $yq + z = p\phi_1(z) \rightarrow (7)$

Using $dz = p\,dx + q\,dy$, the second eqn of (4) becomes

$y(q\,dy + p\,dx) + z\,dy = 0$ or $y\,dz + z\,dy = 0$ or $d(yz) = 0$

Integrating it, $yz = c_3$, c_3 being arbitrary const. $\longrightarrow (8)$

~~From this fact~~ Using this fact, first equation of (4) reduces to

$q\,dp - p\,dq - (pq/y)\,dy = 0$ or $-(1/p)\,dp + (1/q)\,dq + 1/y\,dy = 0$

Integrating, $-\log p + \log q + \log y = \log c_4$ or

$$yq/p = c_4 \longrightarrow (9)$$

from (8) & (9) another intermediate integral corresponding to (4) is

$qy/p = \phi_2(yz)$, where ϕ_2 is aribitrary $f^n \rightarrow (10)$

solving (7) & (10) for p & q, we have

$p = z/\{\phi_1(z) - \phi_2(yz)\}$, $q = \dfrac{z\,\phi_2(yz)}{y\{\phi_1(z) - \phi_2(yz)\}}$

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substituting these in $dz = p\,dx + q\,dy$,

$$dz = \frac{z}{\phi_1(z) - \phi_2(yz)}\left\{dx + \frac{z}{y} \times \phi_2(yz)\,dy\right\}$$

or $\phi_1(z)\,dz = z\,dx + \phi_2(yz)\,\dfrac{z\,dy - y\,dz}{y}$

or $\dfrac{\phi_1(z)\,dz}{z} = dx + \dfrac{\phi_1(yz)\,d(yz)}{yz}$

Integrating $\varphi_1(z) = x + \varphi_2(yz)$ where φ_1 & φ_2 are arbitrary f^n.