Graph Theory
Surprise Quiz - 1

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Q1) $x, 1, 2, 3, 5, 5$ – 6 vertices, hence maximum degree can be 5 and also sum must be even so x must be even.

So x can possibly be $\{2, 4, 0\}$

$0, 1, 2, 3, 5, 5$ isn't possible.

ii) $\{2, 1, 2, 3, 5, 5\} \rightarrow \{1, 2, 2, 3, 5, 5\}$

This isn't possible. Also there are 2 vertices with degree 5. That means two vertices are fully connected, which implies ~~that~~ no vertex can have degree less than 2, hence as this sequence has a vertex with degree 1 for no value of x is this a graphical sequence.

Hence no value of x has graphical sequence.

Q2) Let the graph have n vertices, now we are given that this graph is 4-regular that means each vertex has degree 4. We also know that there are 10 edges in the graph is $|E(G)| = 10$ edges.

Now, we know that $\sum_{v \in V(G)} deg_v(G) = 2|E(G)|$

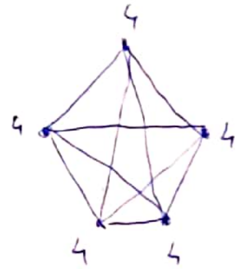
Which implies:-

$$\Rightarrow \sum_{v \in V(G)} \deg_v(G) = 2|E(G)|$$
$$= 2(10)$$
$$= 20$$

$$\underbrace{4+4+\dots 4}_{n\text{-times}} = 20$$

$$4n = 20$$
$$n = \frac{20}{4} = 5$$

Hence, the graph has 5-vertices and is the K_5 regular graph with $\binom{5}{2}$ edges.



Q3) a) We have G_1 and G_2 such that $|V(G_1)| = |V(G_2)|$, they have same order and we also have $|E(G_1)| = |E(G_2)|$ have same size. Now, we are given 2 adjacency matrices A_1 and A_2 for graphs G_1 and G_2 respectively.

Now, if $A_1 = A_2$ then $G_1 \cong G_2$. We prove this by creating a bijection from $\sigma : V(G_1) \to V(G_2)$ and show they are isomorphic.

Let $v_i \in V(G_1)$ and $u_i \in V(G_2)$ Now. $\sigma(v_i) = u_i \ \forall i \leq |V(G)|$

Now, let there be edge $e \in v_i v_j$ in $v_i, v_j \in V(G_1)$ and hence from A_1, we have $A_1[v_i][v_j] = 1$. Now, $A_1 = A_2$, so

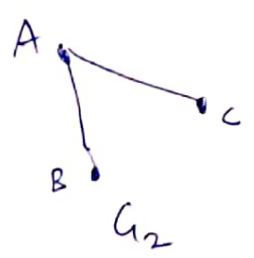
$A_2[u_i][u_j] = 1$ also, but we also know that $\sigma(v_i)\sigma(v_j) = u_i u_j$, hence $\sigma(e) \in E(G_2)$, hence $e \in E(G_2)$ when $e \in E(G_1)$

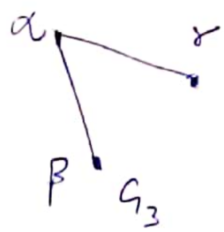
Now, let us take some edge $e \notin E(G_1)$ such that $e = xy$; $x, y \in V(G_1)$, $xy \notin E(G_1)$ ~~for~~ so, $A_1[x][y] = 0$.
Now, similarly $\sigma(xy) = \sigma(x)\sigma(y) = u_x u_y$. We also have $A_2[u_x][u_y] = 0$ as $A_1 = A_2$ and hence for all $e \notin E(G_1)$, $e \notin E(G_2)$ and $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| = |E(G_2)|$, hence the graphs G_1 and G_2 are isomorphic.

$$G_1 \cong G_2 \quad \blacksquare$$

b) If $A_2 \neq A_3$ then $G_2 \not\cong G_3$. This statement is incorrect and we can prove this by showing a counter-example.

Let the 2 graphs G_2 and G_3 be:-





We can very clearly see that $G_2 \cong G_3$ and now, we create adjacency matrices

$$A_2 = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$A_3 = \begin{array}{c c} & \begin{array}{ccc} \beta & \alpha & \gamma \end{array} \\ \begin{array}{c} \beta \\ \alpha \\ \gamma \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

We can clearly see that $A_2 \neq A_3$, but $G_2 \cong G_3$, hence the statement was incorrect and we disprove it using an example.

Q4) Base Case (Base conditions

Let there be a graph G with $|V(G)|$ vertices, we denote by n and $|E(G)|$ edges denoted by m. Now, let us take $m = 0$, so we have a completely disconnected graph ~~with~~ with n components. Hence when we have 0 edges $|V(G)| - |E|$ components exist and the condition holds.

Now, let us take a strong inductive Hypothesis that in graph G with $|V|$ vertices and $|E|$ edges there are at least $|V| - |E|$ components.

Induction Step

Now, let there be an ~~xy~~ edge $xy \in E(G)$ and let us create a graph G' by removing the edge xy. Now, in the new graph after removing the edge xy we have same components as we had in G or we have one more component. So, if initially in G we had at least $|V| - |e|$ components, we now have

$|V| - |e-1| = |V| - |e| + 1$ components.

But from this we also see that components is now at least

$$|V| - |e| + 1 > |V| - |e|.$$

Hence our graph G will have at least $|V| - |e|$ components and our inductive hypothesis is proved.

So, in any graph G, there exists at least $|V| - |E|$ connected components.