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Q7) Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter $\lambda$. Which of the following are martingales.

i) $\{N(t) - t, t \geq 0\}$

$N(t)$ is not a martingale as $E[N(t)] \leq \lambda t < \infty$
for $0 < s < t$

$E[N(t) | F_s] = E[N(t) - N(s) + N(s) | F_s]$

$E[N(t) - N(s) | F_s] + E[N(s) | F_s]$

$\lambda(t-s) + N(s)$

Hence, $E[N(t) | F_s] \neq N(s)$

So, $\{N(t), t \geq 0\}$ is not a martingale.
It is sub martingale.

ii) $\{N^2(t) - t, t \geq 0\}$

A Brownian motion is a continuous process with independent, Gaussian increments. A Brownian motion $W$ is standard if the increments $W(t+s) - W(t)$, $t, s \geq 0$ have mean zero (0) and variance $s$.

$W$ is martingale

$E[W(t+s) | F_t^W] = E[W(t+s) - W(t) | F_t^W] + W(t) = W(t)$

$F_t^W = \sigma(W(s) : s \leq t)$

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$W$ has quadratic variation $t$, that is

$$[W]_t = \lim_{\sup|t_{i+1} - t_i| = 0} \sum_i \left( W(t \wedge t_{i+1}) - W(t \wedge t_i) \right)^2 = t$$

In probability, or more precisely

$$\lim_{\sup|t_{i+1} - t_i| = 0} \sup_{t \le T} \left| \sum_i \left( W(t \wedge t_{i+1}) - W(t \wedge t_i) \right)^2 - t \right| = 0$$

in probability for each $T > 0$

$W(t)^2 - t$ is martingale.

Hence, we can say that $\{ N^2(t) - t, \; t \ge 0 \}$ is a martingale.

iii) $\{ (N(t) - t)^2 - t, \; t \ge 0 \}$

$$W_t = (N(t) - t)^2 - t$$

We need to prove that $E[W_{t+s} \mid F_s] = W_s$ to prove that this is a martingale.

$$E[W_{t+s} \mid F_s] = E[[N(t+s) - (t+s)]^2 - (t+s) \mid F_s]$$

$$E[N^2(t+s) + (t+s)^2 - 2(t+s) N(t+s) - (t+s) \mid F_s]$$

$$E[N^2(t+s)] + (t+s)^2 - 2(t+s) E[N(t+s) \mid F_s] - (t+s)$$

$$E[N(t+s)]^2 + (t+s)^2 - 2(t+s) E[N(t+s) \mid F_s] - (t+s)$$

$$(N(s) + \lambda(t-s))^2 + (t+s)^2 - 2(t+s)[N(s) + \lambda(t-s)] - (t+s)$$

Now, we know that $\lambda = 1$

$$\cancel{N^2(t)} \left(N(s) + (t-s)\right)^2 + (t+s)^2 - 2(t+s)\left[N(s) + (t-s)\right] - (t+s)$$

$$N^2(s) + (t-s)^2 + 2(t-s)N(s) + t^2 + s^2 + 2ts - 2(t+s)N(s) - 2(t^2-s^2) - (t+s)$$

As, we can clearly see this is not equal to $M_s$

or $M_s = (N(s) - t)^2 - t$

and hence this is not a martingale.