Q) Gamma Distribution (Mean, Variance, Mode)

The gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Dividing $\Gamma(\alpha)$ we get :-

$$1 = \int \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} dy$$

where we made a change of variables $x = \beta y$.
Therefore if we define

$$f(x|\alpha,\beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Then $f(x|\alpha,\beta)$ is a P.D.F as it integrates to 1 and is non-negative.

Definition : let us compute the k^{th} moment of gamma distribution

$$E[X^k] = \int_0^\infty x^k \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{(\alpha+k)-1} e^{-\beta x} dx$$

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} \underbrace{\int_0^{\infty} \frac{\beta^{\alpha+k}}{\Gamma(\alpha+k)} x^{\alpha+k-1} e^{-\beta x} dx}_{\text{Integrate to } 1}$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\beta^k}$$

$$= \frac{(\alpha+k-1)\Gamma(\alpha+k-1)}{\Gamma(\alpha)\beta^k}$$

$$\frac{(\alpha+k-1)(\alpha+k-2)\ldots\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)\beta^k} = \frac{(\alpha+k-1)\ldots\alpha}{\beta^k}$$

The mean of the gamma distribution is

$$\boxed{E[X] = \frac{\alpha}{\beta}}$$

We compute the second moment :-

$$E[X^2] = \frac{(\alpha+1)\alpha}{\beta}$$

The variance will be :-

$$Var[X] = E[X^2] - (E[X])^2$$

$$\frac{(\alpha+1)\alpha}{\beta} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha^2}{\beta^2}$$

$$\boxed{Var[X] = \frac{\alpha}{\beta^2}}$$

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To compute the mode we need to maximize

$x^{\alpha-1}e^{-\beta x}$ over $x \in (0, \infty)$

$$\frac{d}{dx}\left[x^{\alpha-1}e^{-\beta x}\right] = e^{-\beta x}\left[(\alpha-1)x^{\alpha-2} - \beta x^{\alpha-1}\right]$$

$$= x^{\alpha-2}e^{-\beta x}(\alpha-1-\beta x)$$

Equating $\frac{d}{dx}\left[x^{\alpha-1}e^{-\beta x}\right] = 0$

$$\boxed{\text{Mode} = \frac{\alpha-1}{\beta}}$$

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