PARTIAL DIFFERENTIAL EQUATIONS (MC-406)

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ASSIGNEMENT - I

1) Form a partial Differential equation by eliminating arbitrary constants.

a) a, b, c from $\cancel{\frac{x^2}{y}}$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{———} \quad (1)$$

Differentiating w.r.t x

$$\frac{2x}{a^2}+0+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$\frac{x}{a^2}+\frac{z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{———} \quad (2)$$

Differentiating (1) w.r.t y partially:-

$$0+\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\frac{y}{b^2}+\frac{z}{c^2}\frac{\partial z}{\partial y}=0 \quad \text{———} \quad (3)$$

Differentiating (2) w.r.t x partially:-

$$\frac{\partial}{\partial x}\left[\frac{x}{a^2}+\frac{z}{c^2}\frac{\partial z}{\partial x}\right]=0$$

$$\frac{1}{a^2}+\frac{1}{c^2}\left[\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}\right]=0 \quad \text{———} \quad (4)$$

Differentiating (3) w.r.t (y) partially

$$\frac{\partial}{\partial y}\left[\frac{y}{b^2}+\frac{z\partial z}{c^2\partial y}\right]=0$$

$$\frac{1}{b^2}+\frac{1}{c^2}\left[\left(\frac{\partial z}{\partial y}\right)^2+z\frac{\partial^2 z}{\partial y^2}\right]=0 \quad \text{—— (5)}$$

From (2), we have:-

$$\frac{x}{a^2}+\frac{z\partial z}{c^2\partial x}=0$$

$$c^2x+a^2z\frac{\partial z}{\partial x}=0$$

$$c^2=-\frac{a^2z\partial z}{x\partial x} \quad \text{—— (6)}$$

Putting (6) in (4) we obtain:-

$$\frac{1}{a^2}+\frac{1}{c^2}\left[\left(\frac{\partial z}{\partial x}\right)^2+z\left(\frac{\partial^2 z}{\partial x^2}\right)\right]=0$$

$$c^2+a^2\left[\left(\frac{\partial z}{\partial x}\right)^2+z\left(\frac{\partial^2 z}{\partial x^2}\right)\right]=0$$

$$-\frac{a^2z\partial z}{x\partial x}+a^2\left[\left(\frac{\partial z}{\partial x}\right)^2+z\left(\frac{\partial^2 z}{\partial x^2}\right)\right]=0$$

$$-\frac{z\partial z}{x\partial x}+\left[\left(\frac{\partial z}{\partial x}\right)^2+z\left(\frac{\partial^2 z}{\partial x^2}\right)\right]=0$$

$$zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0 \quad \text{— (7)}$$

Similarly from (3) and (5) :-

$$2y\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \quad \text{—— (8)}$$

Differentiating (2) w.r.t y partially, we get :-

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + z\left(\frac{\partial^2 z}{\partial x \partial y}\right) = 0 \quad \text{—— (9)}$$

The equations (7), (8) and (9) are all possible required partial differential equations.

b) a and b from $z = (x^2+a)(y^2+b)$

Differentiating Equation (1) w.r.t x partially:-

$$z = (x^2+a)(y^2+b) \quad \text{—— (1)}$$

$$\frac{\partial z}{\partial x} = (y^2+b)\ 2x$$

$$\frac{\partial z}{\partial x} = 2x(y^2+b) \quad \text{—— (2)}$$

Partially Differentiating (1) w.r.t y :-

$$\frac{\partial z}{\partial y} = 2y(x^2+a)$$

$$x^2+a = \frac{1}{2y}\frac{\partial z}{\partial y} \quad \text{—— (3)}$$

Using (2) and (3) in (1):—

$$z = \left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)$$

$$4xyz = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

This is the required reduced form.

2) Form a partial differential equation by eliminating arbitrary functions:-

a) f from $x+y+z = f(x^2+y^2+z^2)$

$$x+y+z = f(x^2+y^2+z^2) \quad — (1)$$

Differentiating partially (1) w.r.t x

$$1+\frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left(2x+2z\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x} = p$$

$$1+p = f'(x^2+y^2+z^2)(2x+2zp) \quad — (2)$$

Differentiating (1) w.r.t y:—

$$1+q = f'(x^2+y^2+z^2)(2y+2zq) \quad — (3)$$ where $q = \frac{\partial z}{\partial y}$

Using (2) and (3) to eliminate $f'(x^2+y^2+z^2)$

$$\frac{1+p}{2x+2zp} = \frac{1+q}{2y+2zq}$$

$$\frac{1+p}{x+zp} = \frac{1+q}{y+zq}$$

$$(1+p)(y+zq) = (1+q)(x+zp)$$

This is the desired partial Differential Equation of first order.

b) f and g from $y = f(x-ct) + g(x+ct)$

Given $y = f(x-ct) + g(x+ct)$ — (1)

$$\frac{\partial y}{\partial x} = f'(x-ct) + g'(x+ct)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x-ct) + g''(x+ct) \quad \text{— (2)}$$

Partially Differentiating (1) w.r.t t

$$\frac{\partial y}{\partial t} = (-c) f'(x-ct) + c g'(x+ct)$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 f''(x-ct) + c^2 g''(x+ct)$$

$$= c^2 [f''(x-ct) + g''(x+ct)]$$

$$= c^2 \left[\frac{\partial^2 y}{\partial x^2}\right]$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

3) Solve the following partial Differential Equations

a) $y^2 p - xyq = x(z-2y)$

Writing this in Lagrange's Form :-

$Pp + Qq = R$ and comparing we get

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

comparing first 2 fractions:-

$$\frac{dx}{y^2} = -\frac{dy}{xy}$$

$$x\,dx = -y\,dy$$

$$x\,dx + y\,dy = 0$$

$$\int x\,dx + \int y\,dy = \int 0$$

$x^2 + y^2 = c_1$ —— (1) where c_1 is arbitrary constant

Taking the last 2 fractions :-

$$-\frac{dy}{xy} = \frac{dz}{x(z-2y)}$$

$$-\frac{dy}{y} = \frac{dz}{z-2y}$$

$$-\frac{dy}{y} = \frac{dz}{z-2y}$$

$$\frac{dy}{dz} = \frac{y}{2y-z}$$

$$\frac{dz}{dy} = \frac{2y-z}{y}$$

$$= 2 - \frac{z}{y}$$

$$\frac{dz}{dy} + \frac{z}{y} = 2$$ This is a linear integral equation whose solution is:-

$$I.F = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$$

$$z\,(IF) = \int 2\,(IF)\, dy$$

$$zy = \int 2y\, dy$$

$$zy = y^2 + c_2$$

$$zy - y^2 = c_2$$

The solution of P.D.E (1) will be

$$\phi\,(x^2+y^2,\ zy-y^2) = 0$$ where ϕ is arbitrary function.

b) $xzp + yzq = xy$

Comparing this with Lagrange's Equation $Pp + Qq = R$, we get

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

Comparing First 2 fractions

$$\frac{dx}{xz} = \frac{dy}{yz}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\frac{dx}{x} - \frac{dy}{y} = 0$$, Integrating both sides

$$\ln x - \ln y = A_1$$

$$\ln\left(\frac{x}{y}\right) = A_1$$

$$\frac{x}{y} = e^{A_1} = c_1 \Rightarrow \frac{x}{y} = c_1 \quad \text{—— (1)}$$

Taking second and third fractions:-

$$\frac{dy}{yz} = \frac{dz}{xy} \Rightarrow \frac{dy}{z} = \frac{dz}{x}$$

$$x\,dy = z\,dz$$

$$yc_1\,dy - z\,dz = 0$$

Integrating both sides:-

$$c_1\frac{y^2}{2} - \frac{z^2}{2} = A$$

$$c_1y^2 - z^2 = 2A$$

$$(c_1y)y - z^2 = 2A \Rightarrow xy - z^2 = c_2 \quad \text{—— (2)}$$

Combining (1) and (2), we get :-

$\phi(xy - z^2, x/y) = 0$, ϕ being an arbitrary function.

c) $z(x+y)p + z(x-y)q = x^2 + y^2$

Given $z(x+y)p + z(x-y)q = x^2 + y^2$ —— (1)

The Lagrange's subsidiary equations are

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} \quad \text{—— (2)}$$

Choosing $x, -y, -z$ as multipliers, each fraction

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

$\therefore x\,dx - y\,dy - z\,dz = 0$

Integrating, we get

$x^2 - y^2 - z^2 = c_1$ —— (3)

Choosing $y, x, -z$ as multipliers for fractions in (2) :-

$$\frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}$$

$\therefore y\,dx + x\,dy - z\,dz = 0$

$d(xy) - z\,dz = 0 \Rightarrow 2xy - z^2 = c_2$ —— (4)

From (3) and (4), we get

$$\phi(x^2-y^2-z^2,\ 2xy-z^2)=0 \quad (\phi \text{ being an arbitrary function}).$$

4) Find the integral surface of the linear P.D.E

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains straight line $x+y=0$, $z=1$

The equation is

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z \qquad \text{--- (1)}$$

Lagrange's Auxiliary Equations of (1) are :-

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

choosing $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{x\,dx+y\,dy-dz}{0}$$

$$\Rightarrow 2x\,dx + 2y\,dy - 2\,dz = 0 \quad \text{such that}$$

$$x^2+y^2-2z = c_1 \qquad \text{--- (2)}$$

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Choosing $1/x$, $1/y$, $1/z$ as multipliers of each fraction in (1)

$$\frac{(1/x)\,dx + (1/y)\,dy + (1/z)\,dz}{y^2+z-(x^2+z)+x^2-y^2} = \frac{(1/x)\,dx + (1/y)\,dy + (1/z)\,dz}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating both sides

$$\ln x + \ln y + \ln z = c$$

$$\ln(xyz) = c$$

$$xyz = e^c = c_2 \quad \text{——} \quad (3)$$

From (2) and (3) we get the following solution

$\phi(x^2+y^2-2z,\ xyz) = 0$ where ϕ is an arbitrary function

Now, taking t as a parameter and given equation of straight line $x+y=0$, $z=1$

let $x = t$, $y = -t$, $z = 1$

Using (3)

$$xyz = c_2$$

$$t(-t) = c_2$$

$$-t^2 = c_2 \quad \text{——} \quad (4)$$

Using (2):

$$x^2+y^2-2z = c_1$$

$$t^2+t^2-2 = c_1 \Rightarrow 2t^2-2 = c_1 \quad \text{——} \quad (5)$$

Eliminating t from the equations of (4) and (5)

$$2(-c_2) - 2 = c_1$$

$$-2c_2 - 2 = c_1$$

$$2c_2 + c_1 + 2 = 0 \quad \text{——} \quad (6)$$

Putting values of c_1 and c_2 in (6) :-

$$2(xyz) + x^2 + y^2 - 2z + 2 = 0$$

$$x^2 + y^2 + 2xyz - 2z + 2 = 0$$

This is the desired integral Surface.

5) Use Charpit's method to find three complete integrals of $pq = px + qy$

The given equation is

$$f(x, y, z, p, q) \equiv px + qy - pq = 0 \quad \text{——} \quad (1)$$

Charpit's Auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dp}{p+p(0)} = \frac{dq}{q+q(0)} \quad \text{——} (2)$$

Taking Last 2 fractions in (2) :-

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating both sides

$$\ln p = \ln q + \ln a$$

$$\ln p = \ln(qa)$$

$$p = qa \quad \text{———} \quad (3)$$

Substituting (3) value of p in (1):-

$$aqx + qy - aq^2 = 0$$

$$q(ax + y - aq) = 0$$

$q \neq 0$, hence $ax + y - aq = 0$

$$q = \frac{ax+y}{a} \quad \text{———} \quad (4)$$

$$p = ax + y \quad \text{———} \quad (5)$$

Putting values of p and q in

$$dz = p\,dx + q\,dy$$

$$dz = (ax+y)\,dx + \left(\frac{ax+y}{a}\right) dy$$

$$a\,dz = (ax+y)(a\,dx + dy)$$

$$\int a\,dz = \int (ax+y)(a\,dx+dy)$$

$$az = \frac{(ax+y)^2}{2} + b$$

This is the complete integral with a and b being arbitrary constants.

6) Find the complete integral of

a) $(p+q)(px+qy)=1$

$$f(x,y,z,p,q) = (p+q)(px+qy) - 1 = 0 \quad \text{——— (1)}$$

$$f_x = (p+q)p \quad f_y = (p+q)q \quad f_z = 0$$

$$f_p = px+qy + x(p+q)$$

$$f_q = (px+qy) + y(p+q)$$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{-dy}{f_q}$$

$$\frac{dp}{(p+q)p} = \frac{dq}{(p+q)q} = \frac{dz}{-(p+q)(px+qy)-(p+q)(px+qy)} = \frac{dx}{-(px+qy)-x(p+q)}$$

$$= \frac{dy}{-(px+qy)-y(p+q)}$$

——— (2)

Using first 2 fractions from (2):-

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\int \frac{dp}{p} = \int \frac{dq}{q}$$

$$\ln p = \ln q + \ln a$$

$$\ln p = \ln(qa)$$

$$p = qa \quad \text{——— (3)}$$

Putting (3) in (1):-

$$(qa+q)(qax+qy)=1$$

$$q^2(a+1)(ax+y)=1$$

$$q=\sqrt{\frac{1}{(a+1)(ax+y)}} \quad \text{——— (4)}$$

$$p=\frac{a}{\sqrt{(a+1)(ax+y)}}$$

We know $dz = p\,dx + q\,dy$

$$dz=\frac{a\,dx+dy}{\sqrt{(a+1)(ax+y)}}$$

$$\frac{1}{\sqrt{a+1}}\ \frac{d(ax+y)}{\sqrt{ax+y}}$$

$$u=ax+y$$

$$dz=\frac{du}{\sqrt{u}\sqrt{a+1}}$$

$$\int dz=\frac{1}{\sqrt{a+1}}\int\frac{du}{\sqrt{u}}$$

$$z=\frac{1}{\sqrt{a+1}}\,2\sqrt{u}+b$$

$$z=\frac{2\sqrt{ax+y}+b}{\sqrt{a+1}}$$

$$z\sqrt{a+1}=2\sqrt{ax+y}+b$$

b) $(y-x)(qy - px) = (p-q)^2$ —— (1)

Let $X = x+y$ & $Y = xy$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x}$$

$$= \frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y} \quad \text{—— (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial z}{\partial X}\frac{\partial X}{\partial y}$$

$$x\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \quad \text{—— (3)}$$

Let $P = \frac{\partial z}{\partial X}$, $Q = \frac{\partial z}{\partial Y}$

From (2) and (3) we have :-

$p = P + Qy$, $q = P + Qx$

From (1)

$$(y-x)((P+Qx)y - (P+Qy)x) = (Q(x-y))^2$$

$$P(y-x)^2 = Q^2(x-y)^2$$

$$P = Q^2 \quad \text{—— (4)}$$

$$f(P,Q) = P - Q^2 = 0 \quad \text{—— (5)}$$

$P = a$, $Q = b$

From (4) : $a = b^2$

$z = aX + bY + c$

$= b^2X + bY + c$

$= b^2(x+y) + bxy + c$

c) $pz = 1 + q^2$

$f(p, q, z) = pz - 1 - q^2 = 0$

Let $u = x + ay$

$\Rightarrow p = \frac{dz}{du}$, $q = a\frac{dz}{du}$ where a is an arbitrary equation

$$z\left(\frac{dz}{du}\right) = 1 + a^2\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow a^2\left(\frac{dz}{du}\right)^2 - z\left(\frac{dz}{du}\right) + 1 = 0$$

$$\frac{dz}{du} = \frac{z \pm \sqrt{z^2 - 4a^2}}{2a^2}$$

$$\Rightarrow \frac{dz}{z \pm \sqrt{z^2 - 4a^2}} = \frac{du}{2a^2}$$

$$\frac{z \mp \sqrt{z^2 - 4a^2}}{4a^2}\, dz = \frac{du}{2a^2}$$

$$\int (z \mp \sqrt{z^2 - 4a^2})\, dz = \int 2\, du$$

$$\frac{z^2}{2} \mp \frac{z}{2}\sqrt{z^2 - 4a^2} - 2a^2 \ln(\sqrt{z^2 - 4a^2} + z) = 2ax + 2y + c$$

7) Use Jacobi's method to find complete integrals of
$p_1^3 + p_2^2 + p_3 = 1$

$p_1^3 + p_2^2 + p_3 = 1$

$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0$ —— (1)

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0 \qquad \frac{\partial f}{\partial p_1} = 3p_1^2 \qquad \frac{\partial f}{\partial p_2} = 2p_2 \qquad \frac{\partial f}{\partial p_3} = 1$$

$$\frac{dp_1}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1} \qquad \text{——— (2)}$$

$$\frac{dp_1}{0} = \frac{dp_2}{0} \Rightarrow dp_1 = 0 \ \& \ dp_2 = 0$$

$$\Rightarrow p_1 = a_1, \ p_2 = a_2$$

$$F_1(x_1 x_2 x_3\, p_1 p_2 p_3) = p_1 = a_1 \qquad \text{——— (3)}$$

$$F_2(x_1 x_2 x_3\, p_1 p_2 p_3) = p_2 = a_2 \qquad \text{——— (4)}$$

We see that $\frac{\partial f_1}{\partial x_1} = \frac{\partial f_1}{\partial x_2} = \frac{\partial f_1}{\partial x_3} = \frac{\partial f_2}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_2}{\partial x_3} = 0$

$$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)^2 = 0$$

$$(F_1, F_2) = 0 \Rightarrow p_3 = 1 - a_1^3 - a_2^2$$

$$dz = p_1\,dx_1 + p_2\,dx_2 + p_3\,dx_3$$

$$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2)x_3 + c$$

This is the complete integral of (1).