5.3 Portfolio Optimization Problem

We are now in a position to move towards our main goal, i.e. to study the *portfolio optimization problem*. We first define a portfolio and other associated terms.

Definition 5.3.1 (Portfolio) *A portfolio is a collection of two or more assets, say, $a_1, \ldots, a_n$, represented by an ordered n-tuple $\Theta = (x_1, \ldots, x_n)$, where $x_i \in \mathbf{R}$ is the number of units of the asset a_i $(i = 1, \ldots, n)$ owned by the investor.*

We consider only a single period model, that is, in between the initial time taken as $t = 0$ and the final transaction time taken as $t = T$, no transaction ever takes place.

Let $V_i(0)$ and $V_i(T)$ be the values of the i-th asset at $t = 0$ and $t = T$, respectively. Let $V_\Theta(0)$ and $V_\Theta(T)$ denote the values of the portfolio $\Theta = (x_1, \ldots, x_n)$ at $t = 0$ and $t = T$, respectively. Then, we have

$$V_\Theta(0) = \sum_{i=1}^{n} x_i V_i(0), \quad \text{and} \quad V_\Theta(T) = \sum_{i=1}^{n} x_i V_i(T).$$

The quantity $r_\Theta(T) = \dfrac{V_\Theta(T) - V_\Theta(0)}{V_\Theta(0)}$ is referred as the *return of the portfolio* Θ.

Definition 5.3.2 (Asset Weights) *The weight w_i of the asset a_i is the proportion of the value of the asset in the portfolio at $t = 0$, i.e.*

$$w_i = \frac{x_i V_i(0)}{\sum_{j=1}^{n} x_j V_j(0)} \qquad (i = 1, \ldots, n).$$

It can be observed that $w_1 + \cdots + w_n = 1$.

In view of Definition 5.3.2, a portfolio can now be represented by the weights w_i, $(i = 1, 2, \ldots, n)$ such that $w_1 + \cdots + w_n = 1$. Thus we can imagine that we are having one unit of money (say *Rs* 1) and if its allocation in the i^{th} asset is w_i, $(i = 1, 2, \ldots, n)$ then the resulting portfolio is $(w_1, w_2, \ldots, w_n)$.

Remark 5.3.1 *In a portfolio, if $w_i < 0$, for some i, it indicates that the investor has taken a short position on the i-th asset a_i.*

Let r_i be the return on the i-th asset. Then

$$r_i = \frac{V_i(T) - V_i(0)}{V_i(0)} \qquad (i = 1, \ldots, n).$$

Definition 5.3.3 (Mean of the Portfolio Return) *Let $(w_1, w_2, \ldots, w_n)$ be a portfolio of n assets $a_1, a_2, \ldots, a_n$. Let r_i $(i = 1, 2, \ldots, n)$, be the return on the i^{th} asset a_i and $E(r_i) = \mu_i$ $(i = 1, 2, \ldots, n)$, be its expected value. Then the mean of the portfolio return is defined as*

$$\mu = E(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i.$$

Definition 5.3.4 (Variance of the Portfolio) *Let $(w_1, w_2, \ldots, w_n)$ be a portfolio of n assets $a_1, a_2, \ldots, a_n$. Let r_i be the return on the i^{th} asset a_i and $E(r_i) = \mu_i$, $(i = 1, 2, \ldots, n)$. Let Cov $(r_i, r_j) = \sigma_{ij}$, for $i, j = 1, 2, \ldots n$. Then the variance of the portfolio return is defined as*

$$\sigma^2 = Var(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}.$$

Here the covariance term σ_{ij} equals $\rho_{ij}\sigma_i\sigma_j$, where $Var(r_i) = \sigma_i^2$, $Var(r_j) = \sigma_j^2$ and ρ_{ij} is the correlation coefficient between r_i and r_j.

In practice, the mean of the portfolio return is simply referred as the *return of the portfolio.* Also the variance or rather the standard deviation of the portfolio return is referred as the *risk of the portfolio.* Therefore given a portfolio A : $(w_1, w_2, \ldots, w_n)$, we can compute its mean μ_A and standard deviation σ_A and therefore get the point $A : (\sigma_A, \mu_A)$ in (σ, μ)-plane. Thus irrespective of the number of assets, a portfolio can always be identified as a point in the (σ, μ)-plane. This representation is called *(σ, μ)-diagram* or (σ, μ)-graph (see Fig. 5.1) and is very convenient for further discussion.

The *portfolio optimization problem* refers to the problem of determining weights w_i $(i = 1, 2, \ldots, n)$, such that the return of the portfolio is maximum and the risk of the portfolio is minimum. Thus we aim to solve the following optimization problem

$$\begin{aligned} &\text{Min } \textstyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ &\text{and} \\ &\text{Max } \textstyle\sum_{i=1}^{n} w_i \mu_i \\ &\textit{subject to} \\ &\qquad w_1 + w_2 + \ldots + w_n = 1. \end{aligned}$$

5.4 Two Assets Portfolio Optimization

Consider a portfolio with two assets, say, a_1, a_2 with weights w_1, w_2, returns r_1, r_2 and standard deviations σ_1, σ_2, respectively. Then the portfolio expected return μ and portfolio variance σ^2 are respectively given by

$$\mu = E(w_1 r_1 + w_2 r_2) = w_1\mu_1 + w_2\mu_2, \tag{5.1}$$
$$\sigma^2 = Var(w_1 r_1 + w_2 r_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2. \tag{5.2}$$

Here ρ is the coefficient of correlation between r_1 and r_2, and the value of ρ lies in $[-1, 1]$. We pause here to analyze the effect of ρ on the risk involved in a portfolio. What we shall be observing is that the value of ρ provides a measure of the extent of diversification of portfolio so as to reduce risk. The more negative the value of ρ, the greater are the benefits of the portfolio diversification.

As w_1 and w_2 are weights representing the proportions of total investment in two assets a_1 and a_2, respectively, we have $w_1 + w_2 = 1$. Moreover, in case of short selling, the weights can be negative. Subsequently, we write $w_1 = 1 - s$, and so, $w_2 = s$, $s \in \mathbf{R}$. Now, it follows from relations (5.1) and (5.2) that

$$\mu = (1 - s)\mu_1 + s\mu_2, \tag{5.3}$$
$$\sigma^2 = (1 - s)^2\sigma_1^2 + s^2\sigma_2^2 + 2\rho(1 - s)s\sigma_1\sigma_2. \tag{5.4}$$

Relation (5.4) can be simplified as

$$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)s^2 - 2\sigma_1(\sigma_1 - \rho\sigma_2)s + \sigma_1^2. \tag{5.5}$$

Without loss of generality we assume that $0 < \sigma_1 \leq \sigma_2$. We discuss the following two independent cases

$$(i) \quad \rho = \pm 1 \qquad\qquad (ii) \quad -1 < \rho < 1.$$

Case (i) $\rho = \pm 1$. From relations (5.3) and (5.5), the portfolio expected return and standard deviation are respectively given by

$$\mu = (1 - s)\mu_1 + s\mu_2$$
$$\sigma = |(1 - s)\sigma_1 \pm s\sigma_2|.$$

For $s \in [0, 1]$ both weights are non-negative and thus the portfolio has no short positions. If $s > 1$ then $w_1 < 0$, and therefore that asset a_1 is held short. If $s < 0$ then $w_2 < 0$ and therefore it indicates that asset a_2 is held short. It may be noted that an investor can not take short position on both the assets simultaneously.

For $\mu = (1-s)\mu_1+s\mu_2$, $\tilde{\sigma} = (1-s)\sigma_1+s\sigma_2$, we plot the points $(\tilde{\sigma}, \mu)$ in $(\tilde{\sigma}, \mu)$-plane, and get the first graph of Fig. 5.2. The second graph in Fig. 5.2 corresponds to the case when $\mu = (1-s)\mu_1+s\mu_2$, $\tilde{\sigma} = (1-s)\sigma_1 - s\sigma_2$. These graphs are essentially straight lines with the bold parts corresponding to $s \in [0, 1]$.

Subsequently, we plot the standard deviation-mean diagram of the portfolio, i.e. the (σ, μ)-graph, for $\rho = \pm 1$. Observe that $\sigma = |\tilde{\sigma}|$ and therefore to get the required graphics we simply need to flip the portion of the line that lies in the left half-plane over the μ axis. The two graphs so obtained are depicted in Fig. 5.3. From Fig. 5.3 we have the following observations to share.

Remark 5.4.1 *When the returns of the two assets are perfectly positively or negatively correlated, i.e. $\rho = \pm 1$, the expected portfolio return increases with the gradual decrease in the overall risk of the portfolio (point A to point P Fig. 5.3) till the risk is completely eliminated (point P). After that the higher return comes with higher risk as weight of riskier asset increases. For the case $\rho = +1$, the point P corresponds to a portfolio in which a short position is taken on asset a_2; where as for the case $\rho = -1$, the point P corresponds to a portfolio in which weights of both assets are non-negative.*

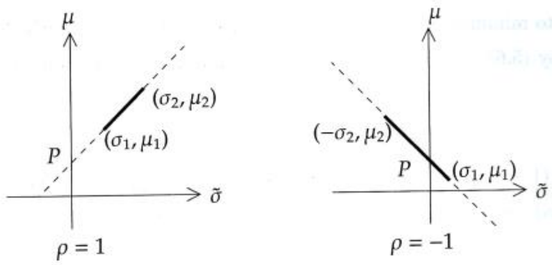

Fig. 5.2. $(\tilde{\sigma}, \mu)$-graph for$\rho = \pm 1$.

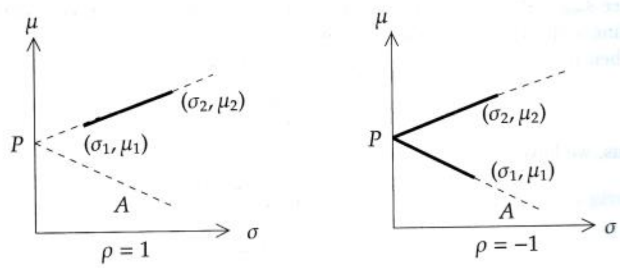

Fig. 5.3. (σ, μ)-graph for$\rho = \pm 1$.

Below we provide a mathematical justification of the above observations. For the case when $\rho = 1$ and $\sigma_1 = \sigma_2$, then $\sigma_{\min} = \sigma_1$ for all $s \in \mathbf{R}$. Therefore we need to consider following two cases only.

(a) $\rho = 1$ and $\sigma_1 < \sigma_2$. In this case we have

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2(1-s)s\sigma_1\sigma_2.$$

Our aim is to minimize σ, or equivalently minimize σ^2. Now

$$\frac{d\sigma^2}{ds} = 2[s(\sigma_1 - \sigma_2)^2 - \sigma_1(\sigma_1 - \sigma_2)], \tag{5.6}$$

and

$$\frac{d^2\sigma^2}{ds^2} = 2(\sigma_1 - \sigma_2)^2 > 0. \tag{5.7}$$

Thus, to minimize the risk σ, we must choose the weight s such that $\frac{d\sigma^2}{ds} = 0$. Thereby (5.6) yields s_{min} (the value of s for which σ^2 is minimum) as

$$s_{min} = \frac{\sigma_1}{\sigma_1 - \sigma_2} < 0.$$

Hence $(1 - s_{min}) = \frac{\sigma_2}{\sigma_2 - \sigma_1} > 0$. Let μ_{min} and σ^2_{min} respectively denote the expected return and variance of the portfolio with $(w_1 = 1 - s_{min}, w_2 = s_{min})$ then

$$\mu_{\min} = \frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sigma_1 - \sigma_2} \quad \text{and} \quad \sigma^2_{min} = 0.$$

Since $s_{min} < 0$ (i.e. $w_2 < 0$), an investor can eliminate risk in the portfolio by taking a short position with asset a_2.

(b) When $\rho = -1$ and $\sigma_1 \leq \sigma_2$. In this case we have

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 - 2(1-s)s\sigma_1\sigma_2.$$

Thus, we have

$$s_{min} = \frac{\sigma_1}{\sigma_1 + \sigma_2} > 0$$

$$1 - s_{min} = \frac{\sigma_2}{\sigma_1 + \sigma_2} > 0$$

$$\mu_{\min} = \frac{\sigma_1\mu_2 + \sigma_2\mu_1}{\sigma_1 + \sigma_2}$$

$$\sigma^2_{min} = 0.$$

Since $s_{min} > 0$ and $1 - s_{min} > 0$ hence the investor can eliminate the risk in the portfolio without restoring to short selling.

Case (ii) We now consider the second case when $-1 < \rho < 1$. Recalling relations (5.3) and (5.5), we have

$$\mu = (1-s)\mu_1 + s\mu_2,$$

and

$$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)s^2 - 2\sigma_1(\sigma_1 - \rho\sigma_2)s + \sigma_1^2,$$

which represents the parametric equation of a parabola in (σ^2, μ)-plane. Now

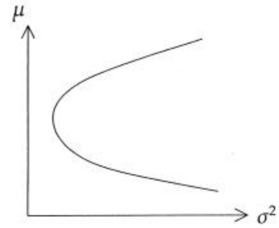

Fig. 5.4. (σ^2, μ)-graph for $-1 < \rho < 1$.

$$\frac{d\sigma^2}{ds} = 0 \Rightarrow s = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Also

$$\frac{d^2\sigma^2}{ds^2} = 2((\sigma_1 - \rho\sigma_2)^2 + \sigma_2^2(1-\rho^2)) > 0.$$

Consequently $s_{min} = \dfrac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$, and the minimum value of σ^2 is given by

$$\sigma_{\min}^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Further, the corresponding expected portfolio return equals

$$\mu_{\min} = (\mu_2 - \mu_1)s_{\min} + \mu_1.$$

Remark 5.4.2 *It is important to take note of the following points.*

(i) The condition $-1 \le \rho < \dfrac{\sigma_1}{\sigma_2}$ *is equivalent to* $0 < s_{\min} < 1$. *Thus the minimum risk can be achieved without short selling. Also, in this case*

$$\sigma_{\min}^2 = 0 \Leftrightarrow \rho = -1.$$

(ii) $\rho = \dfrac{\sigma_1}{\sigma_2} \Leftrightarrow s_{\min} = 0 \Leftrightarrow \sigma_{\min}^2 = \sigma_1^2.$

(iii) The condition $\frac{\sigma_1}{\sigma_2} < \rho \leq 1$ *is equivalent to* $s_{\min} < 0$. *In this case the investor has taken a short position on asset* a_2 *in order to minimize the portfolio risk. Further, in this case*

$$\sigma^2_{\min} = 0 \Leftrightarrow \rho = 1.$$

The entire theory of this section is summarized in Fig. 5.5.

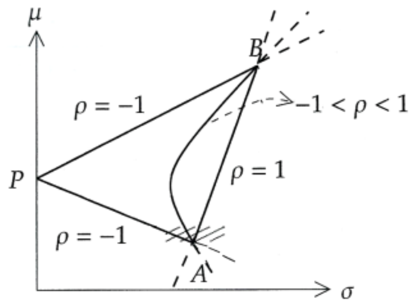

Fig. 5.5. Feasible region for two asset problem.

The risk-return relation of two assets for various values of ρ provides us with a triangle APB. The points A and B signify undiversified portfolios. Since $-1 \leq \rho \leq 1$, $\triangle APB$ specifies the limit of diversification. The risk-return relation for all values of ρ except ± 1 lie within this triangle. Here the bold portions of the graphs represent the case $0 \leq s \leq 1$. We verify the two-assets portfolio theory by considering the below given example.

5.5 Multi Asset Portfolio Optimization

The weights of the various assets $a_1, \ldots, a_n$ in the portfolio are written in the vector form $w^T = (w_1, \ldots, w_n)$. Let $e^T = (1, \ldots, 1) \in \mathbf{R}^n$. Then $w_1 + \cdots + w_n = 1$

can be expressed as $e^T w = 1$. Let $m^T = (\mu_1, \ldots, \mu_n)$ be the expected return vector of the portfolio, where, $\mu_i = E(r_i)$ $(i = 1, \ldots, n)$, and $C = [c_{ij}]$ denotes the $n \times n$ variance-covariance matrix with entries $c_{ij} = Cov(r_i, r_j)$ $(i, j = 1, \ldots, n)$. Note that $c_{ii} = \sigma_i^2$ $(i = 1, \ldots, n)$. Obviously C is a symmetric matrix. Now the expected return μ of the portfolio is given by

$$\mu = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i \mu_i = m^T w,$$

and the risk σ^2 of the portfolio is

$$\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i,j=1}^{n} c_{ij} w_i w_j = w^T C w. \tag{5.8}$$

Here C is certainly positive semidefinite. In practice, it is also assumed to be positive definite (and hence invertible) because the minimum risk of a general n-asset portfolio is rarely zero.

Here C is certainly positive semidefinite. In practice, it is also assumed to be positive definite (and hence invertible) because the minimum risk of a general n-asset portfolio is rarely zero.

During a portfolio selection, every investor is faced with a choice of either minimizing a risk with respect to certain value of return or maximizing a return with respect to certain value of risk. Now, from (5.8), we observe that the portfolio risk σ^2 depends on three factors, viz.,

(i) risk of each individual asset;
(ii) coefficient of correlation between assets returns;
(iii) weights of the assets.

Out of these contributing factors, the only factor that an investor can control is the weights of the assets. Our main aim is to examine the optimal choice of these weights.

The Feasible Region of a Portfolio Problem

Let $W = \{w \in \mathbf{R}^n : e^T w = 1\}$ be the collection of all portfolios. We have earlier observed that every portfolio $w \in W$ corresponds to a point in the (σ, μ)-plane, say $(\sigma^{(w)}, \mu^{(w)})$. The set $\{(\sigma^{(w)}, \mu^{(w)}) \mid w \in W\}$ is called the *feasible region* or *feasible set* of the given portfolio optimization problem. It should be of interest to know the geometry of the feasible region and that is the point of our discussion now.

Consider the n-dimensional hyperplane $e^T w = 1$ in which the weight vector w resides. Let f be the mapping that takes each weight vector in the weight hyperplane to the corresponding portfolio point in the (σ, μ)-plane. We try and find the image of any straight line in the weight hyperplane $e^T w = 1$ under the mapping f. For this we note that the parametric equation of any line in the weight hyperplane is of the form

e.g. $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = \xi$

$$
\begin{aligned}
l(\xi) &= (s_1\xi + b_1, \dots, s_n\xi + b_n)^T \\
&= \xi s + b, \qquad\qquad -\infty < \xi < \infty,
\end{aligned}
$$

$x = l\xi + a$, $y = m\xi + b$, $z = n\xi + c$

where $s = (s_1, \dots, s_n)^T$ and $b = (b_1, \dots, b_n)^T$. Let w be any point on this line. Then

$$
\begin{aligned}
\mu &= m^T w \\
&= m^T(\xi s + b) \\
&= \xi(m^T s) + (m^T b).
\end{aligned}
$$

Let $\alpha = (m^T s)^{-1}$, $\beta = -(m^T b)(m^T s)^{-1}$. Then, $\xi = \alpha\mu + \beta$. Moreover,

$$
\begin{aligned}
\sigma^2 &= w^T C w \\
&= (\xi s + b)^T C(\xi s + b) \\
&= (s^T C s)\xi^2 + (s^T C b + b^T C s)\xi + b^T C b \\
&\equiv \gamma\xi^2 + \delta\xi + \eta.
\end{aligned}
$$

Substituting the value of ξ we get

$$
\sigma^2 = \gamma(\alpha\mu + \beta)^2 + \delta(\alpha\mu + \beta) + \eta. \tag{5.9}
$$

As ξ varies from $-\infty$ to ∞, the ordered pair (σ^2, μ) traces out a parabola given by (5.9) which lies in (σ, μ)-plane with axis parallel to σ-axis and sides open on the right.

We are actually interested in (σ, μ)-graph. Taking the square root of σ^2, the resulting curve is

$$
\sigma = \sqrt{\gamma(\alpha\mu + \beta)^2 + \delta(\alpha\mu + \beta) + \eta}. \tag{5.10}
$$

This curve is called a *Markowitz curve.* Thus, each line in the weight hyperplane is mapped onto a Markowitz curve. This phenomena is depicted in Fig. 5.6.

Remark 5.5.1 *Here it is important to note that the Markowitz curve (5.10) is not a parabola. In fact the main difference between the parabola (5.9) and the Markowitz curve (5.10) in (σ, μ)-graph is that a tangent can be drawn to the parabola (5.9) from any point on the μ-axis, whereas the Markowitz curve behaves almost as a straight line as $\mu \to \infty$, thereby, it is not possible to draw a tangent to*

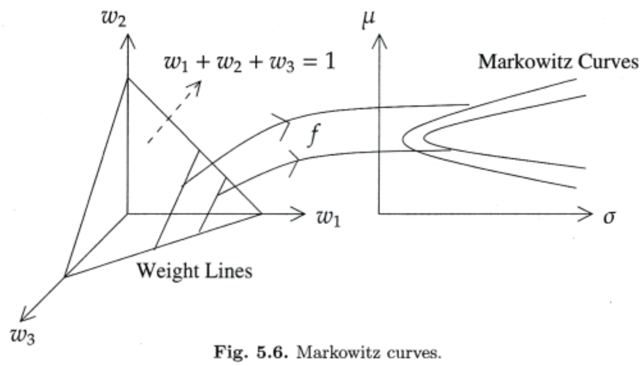

Fig. 5.6. Markowitz curves.

the Markowitz curve as $\mu \to \infty$. This difference may not sound significant right now but it plays a vital role when the portfolio consists of one risk-free asset as well. We shall be addressing to this type of portfolio in the next section. For the current discussion we have assumed that all the assets in the portfolio are risky.

Remark 5.5.2 *As we cover the weight hyperplane by taking weight lines, we trace a family of Markowitz curves in the (σ, μ)-plane. It is not difficult to get convinced that this region in the (σ, μ)-plane is going to be a solid region and its shape will be like a bullet, which is appropriately called the Markowitz bullet.*

5.6 The Minimum Variance Set, The Minimum Variance Point and the Efficient Frontier

Consider the feasible set of a given portfolio optimization problem as depicted in (σ, μ)-diagram (Fig. 5.8). Its left boundary is called the *minimum variance set* because for any level of the mean rate return, the feasible point with the smallest variance (or standard deviation) is the corresponding left boundary point. This situation is illustrated in Fig. 5.8.

Here when the return is at level μ_0, the feasible point with the smallest variance is P_0. Similarly for return levels μ_1 and μ_2 we get the respective points P_1 and P_2. Amongst all such points (σ, μ) lying on the left boundary of the feasible region, there is a point P_{min} which has the least variance. This point is called the *minimum variance point.*

Mathematically, to find the minimum variance point we need to solve the following risk minimization problem

$$\begin{aligned} &\text{Min} \qquad \sigma^2 = w^T C w \\ &\text{subject to} \\ &\qquad\qquad e^T w = 1. \end{aligned}$$

The following theorem gives a closed form solution of the above problem

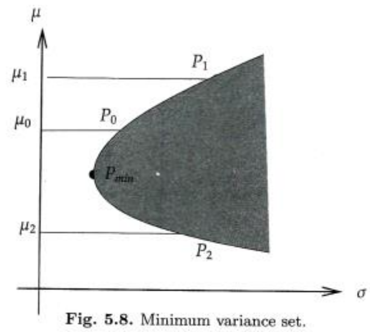

Fig. 5.8. Minimum variance set.

Theorem 5.6.1 *A portfolio with minimum risk has weights given by*

$$w = \frac{C^{-1}e}{e^T C^{-1} e}.$$

Proof. We desire to solve the following optimization problem

$$\begin{aligned} &\text{Min} \qquad \sigma^2 = w^T C w \\ &\text{subject to} \\ &\qquad\qquad e^T w = 1. \end{aligned} \tag{5.11}$$

Using the Lagrange multiplier $\lambda \in \mathbf{R}$, we minimize the Lagrangian

$$L(w, \lambda) = w^T C w + \lambda(1 - e^T w). \tag{5.12}$$

Note that λ is unrestricted in sign because the constraint in the risk minimization problem is an equation $e^T w = 1$. Now, differentiating (5.12) with respect to w, we obtain

$$2w^T C - \lambda e^T = 0 \implies w = \frac{\lambda}{2} C^{-1} e.$$

Using (5.11), we get

$$e^T \left(\frac{\lambda}{2} C^{-1} e \right) = 1 \implies \frac{\lambda}{2} = \frac{1}{e^T C^{-1} e}.$$

Thus the requisite result follows. □

Markowitz Efficient Frontier
Looking at the minimum variance set in the (σ, μ)-diagram, we observe that for a given level of risk, (say σ_0), there are two values of returns ($\mu_0^{(L)}$ and $\mu_0^{(U)}$ with $\mu_0^{(U)} > \mu_0^{(L)}$). Since our aim in portfolio optimization is to maximize return for a given level of risk, we shall obviously choose the larger of these two returns, i.e $\mu_0^{(U)}$. Therefore in the minimum variance set, it is only the upper half which is of importance for investment. This upper half portion of the minimum variance set is called the *Markowitz efficient frontier*. A typical efficient frontier is illustrated in Fig. 5.9.

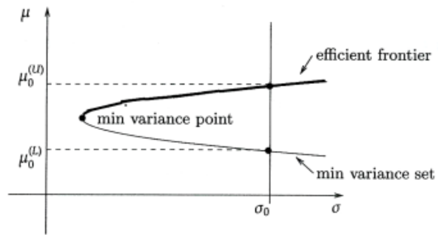

Fig. 5.9. Efficient frontier.

We now formalize the definition of efficient frontier and outline a procedure to determine the same.

Definition 5.6.1 (Dominating Point and Efficient Frontier) *Let $P : (\sigma_1, \mu_1)$ and $Q : (\sigma_2, \mu_2)$ be two points in the feasible region of a given portfolio optimization problem. Then the point P is said to dominate Q if $\sigma_1 \leq \sigma_2$ and $\mu_1 \geq \mu_2$.*

Definition 5.6.2 (Non-dominated Point and Efficient Frontier) *Let A be a point in the feasible region. The point A is said to be a non-dominated point if there is no point P in the feasible region which dominates A. The set of all non-dominated points in the feasible region is called its efficient frontier.*

Geometrically the efficient frontier is the upper portion of the minimum variance set as depicted in Fig. 5.9. Obviously for an investor it is only the efficient frontier which will be of interest and therefore we should explore methodologies to determine the same.

Before we proceed in that direction we notice that in many cases, it is more likely that an investor provides a fixed value of the expected return, say μ, that he/she desires to achieve. Therefore the investor's problem is to decide the right investment strategy to obtain the return μ with the minimum risk. We look at this scenario in the result to follow.

Theorem 5.6.2 *For a given expected return μ, the portfolio with minimum risk has weights given by*

$$w = \frac{det\begin{pmatrix} \mu & m^TC^{-1}e \\ 1 & e^TC^{-1}e \end{pmatrix} C^{-1}m + det\begin{pmatrix} m^TC^{-1}m & \mu \\ e^TC^{-1}m & 1 \end{pmatrix} C^{-1}e}{det\begin{pmatrix} m^TC^{-1}m & m^TC^{-1}e \\ e^TC^{-1}m & e^TC^{-1}e \end{pmatrix}}. \tag{5.13}$$

Proof. We wish to solve the following quadratic programming problem

$$\text{Min} \qquad \sigma^2 = \frac{1}{2}w^TCw$$

subject to

$$\begin{aligned} m^Tw &= \mu \\ e^Tw &= 1. \end{aligned} \tag{5.14}$$

This is a convex quadratic programming problem with unrestricted variable vector w. Therefore we define the Lagrangian

$$L(w, \alpha, \beta) = \frac{1}{2}w^TCw + \alpha(\mu - m^Tw) + \beta(1 - e^Tw)$$

where $\alpha, \beta \in \mathbf{R}$ are Lagrange multipliers. Now solving $\frac{\partial L}{\partial w} = 0$ gives

$$w^TC - \alpha m^T - \beta e^T = 0$$

i.e.

$$w = C^{-1}(\alpha m + \beta e). \tag{5.15}$$

Substituting the value of w in (5.14), we get

$$\begin{aligned} (m^TC^{-1}m)\alpha + (m^TC^{-1}e)\beta &= \mu \\ (e^TC^{-1}m)\alpha + (e^TC^{-1}e)\beta &= 1. \end{aligned}$$

Solving the above two equations for α and β, we obtain

$$\alpha = \frac{det\begin{pmatrix} \mu & m^TC^{-1}e \\ 1 & e^TC^{-1}\cancel{m}e \end{pmatrix}}{det\begin{pmatrix} m^TC^{-1}m & m^TC^{-1}e \\ e^TC^{-1}m & e^TC^{-1}e \end{pmatrix}}, \quad \beta = \frac{det\begin{pmatrix} m^TC^{-1}m & \mu \\ e^TC^{-1}m & 1. \end{pmatrix}}{det\begin{pmatrix} m^TC^{-1}m & m^TC^{-1}e \\ e^TC^{-1}m & e^TC^{-1}e \end{pmatrix}}.$$

Substituting these values in the expression (5.15) for w, we get the required expression (5.13).

□

Now to generate the entire efficient frontier we need to solve problems of type (5.14) for all values of $\mu \in \mathbf{R}$. This is almost impossible. But then an extremely interesting observation is made here. Recall from (5.14) and (5.15) that, for a given value of return μ, the points of minimum variance must satisfy the following system of $(n+2)$ linear equations in $(n+2)$ unknowns $w \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$

$$\begin{aligned} w^TC - \alpha m^T - \beta e^T &= 0 \\ m^Tw &= \mu \\ e^Tw &= 1. \end{aligned} \tag{5.16}$$

Suppose we solve the system (5.16) for two distinct values of expected return μ, say $\bar{\mu}^{(1)}$ and $\bar{\mu}^{(2)}$. Let the two solutions be $(w^{(1)})^T = \left((w_1^{(1)}, \ldots, w_n^{(1)}), \alpha^{(1)}, \beta^{(1)}\right)^T$ and $(w^{(2)})^T = \left((w_1^{(2)}, \ldots, w_n^{(2)}), \alpha^{(2)}, \beta^{(2)}\right)^T$, respectively. Then it is simple to verify that the combination portfolio, $\lambda(w^{(1)}, \alpha^{(1)}, \beta^{(1)})^T + (1-\lambda)(w^{(2)}, \alpha^{(2)}, \beta^{(2)})^T$, $\lambda \in \mathbf{R}$, is also a solution of the system (5.16) corresponding to the return $\lambda\bar{\mu}^{(1)} + (1-\lambda)\bar{\mu}^{(2)}$. Therefore, in order to solve (5.16) for every value of μ, one is only required to solve it for two distinct values of μ and then form the combination of the two solutions. Thus, the knowledge of two distinct portfolios yielding the minimum variances is sufficient to generate the entire minimum variance set. This result is significant from investor's point of view. Also, it demonstrates a very good application of Karush-Kuhn-Tucker optimality conditions. The result is known as the *two fund theorem*.

Theorem 5.6.3 (Two Fund Theorem) *Two efficient portfolios can be established so that any other efficient portfolio can be duplicated, in terms of mean and variance, as a linear combination of these two assets. In other words, it says that, an investor seeking an efficient portfolio need to invest only in the combination of these two assets.*

The most convenient way to get two solutions of (5.16) is to assign two distinct values to α and β, and then work out the solutions. The most convenient choices are $\alpha = 1,\ \beta = 0$ and $\alpha = 0,\ \beta = 1$. The above discussion is illustrated through the below given example.

Example 5.6.1 *Consider three risky assets with the variance-covariance matrix and expected returns as follows.*

variance - covariance matrix(C)			return(M)
2	*1*	*0*	*0·4*
1	*2*	*1*	*0·8*
0	*1*	*2*	*0·8*

Find two portfolios yielding the minimum variance. Also, determine the expected returns from these two portfolios. Using the two fund theorem, construct the portfolio giving the return of $33\cdot 4\%$ *with minimum risk.*

Solution Taking $\alpha = 0,\ \beta = 1$ in (5.16), we need to solve: $\sum_{j=1}^{3}\sigma_{ij}v_j^{(1)} = 1$ $(i = 1,2,3)$, resulting in the following system of linear equations

$$\begin{aligned} 2v_1^{(1)} + v_2^{(1)} &= 1\\ v_1^{(1)} + 2v_2^{(1)} + v_3^{(1)} &= 1\\ v_2^{(1)} + 2v_3^{(1)} &= 1. \end{aligned}$$

The solution is $V^{(1)} = (0\cdot 5,\ 0,\ 0\cdot 5)^T$. We next take $\alpha = 1,\ \beta = 0$ in (5.16), to solve $\sum_{j=1}^{3}\sigma_{ij}v_j^{(2)} = \mu_i$ $(i = 1,2,3)$, i.e.

$$\begin{aligned} 2v_1^{(2)} + v_2^{(2)} &= 0\cdot 4\\ v_1^{(2)} + 2v_2^{(2)} + v_3^{(2)} &= 0\cdot 8\\ v_2^{(2)} + 2v_3^{(2)} &= 0\cdot 4. \end{aligned}$$

The solution of the above system is $V^{(2)} = (0\cdot 1,\ 0\cdot 2,\ 0\cdot 3)^T$.

Note that $\sum_{j=1}^{3}v_j^{(1)} = 1$, thus we take $w^{(1)} = V^{(1)} = (0.5,\ 0,\ 0.5)$. Normalizing $V^{(2)}$, we get, $w^{(2)} = (1/6,\ 1/3,\ 1/2)^T$ (so that, $\sum_{j=1}^{3}w_j^{(2)} = 1$). The corresponding returns from the two portfolios with weights $w^{(1)}$ and $w^{(2)}$ are $\bar{\mu}^{(1)} = m^T w^{(1)} = 0\cdot 6$ and $\bar{\mu}^{(2)} = m^T w^{(2)} = 0\cdot 733$, respectively.

Next, we consider the case when the investor desired a return of $\mu = 0\cdot 334$ at minimum risk. It is simple to check that for $\lambda = 3$, $\lambda\bar{\mu}^{(1)} + (1-\lambda)\bar{\mu}^{(2)} = 0\cdot 334$.

Thus, by the two fund theorem the requisite portfolio is given by $w = \lambda w^{(1)} + (1 - \lambda)w^{(2)} = (7/6, -2/3, 1/2)$. Observe that the second asset has a short position in this portfolio. The variance corresponding to this portfolio is

$$w^T C w = \begin{pmatrix} 7/6 & -2/3 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 7/6 \\ -2/3 \\ 1/2 \end{pmatrix} = 2/9.$$

□

5.7 Capital Asset Pricing Model (CAPM)

So far we have assumed that all assets in the portfolio are risky assets. So it is natural to query as to what would be the scenario if one risk-free asset is included in the portfolio? In this section we make an attempt to study this aspect of portfolio selection.

Consider a portfolio with n risky assets, $a_1, \ldots, a_n$ with weights $w_1, \ldots, w_n$ and one risk-free asset a_{rf} with weight w_{rf}. Then

$$w_{\mathrm{risky}} + w_{\mathrm{rf}} = \sum_{i=1}^{n} w_i + w_{\mathrm{rf}} = 1. \tag{5.17}$$

Also, the expected return and the variance associated with this portfolio are respectively given by

$$\mu = \sum_{i=1}^{n} w_i \mu_i + w_{\mathrm{rf}} \mu_{\mathrm{rf}} = \mu_{\mathrm{risky}} + w_{\mathrm{rf}} \mu_{\mathrm{rf}},$$

and

$$\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i + w_{\mathrm{rf}} r_{\mathrm{rf}} \right) = Var\left(\sum_{i=1}^{n} w_i r_i \right) = \sigma^2_{\mathrm{risky}}.$$

If we remove the risk-free asset from the portfolio and readjust the weights of the risky assets so that their sum remain 1, the resultant portfolio so obtained is referred to as the *derived risky portfolio.* We use μ_{der} and σ^2_{der} to denote the derived risky portfolio expected return and risk, respectively. Then,

$$\begin{aligned}\mu &= \sum_{i=1}^{n} w_i\mu_i + w_{\text{rf}}\mu_{\text{rf}} \\ &= w_{\text{risky}}\left(\sum_{i=1}^{n}\frac{w_i}{w_{\text{risky}}}\mu_i\right) + w_{\text{rf}}\mu_{\text{rf}} \\ &= w_{\text{risky}}\mu_{\text{der}} + w_{\text{rf}}\mu_{\text{rf}} \\ &= w_{\text{risky}}\mu_{\text{der}} + (1 - w_{\text{risky}})\mu_{\text{rf}} \\ &= w_{\text{risky}}(\mu_{\text{der}} - \mu_{\text{rf}}) + \mu_{\text{rf}}.\end{aligned} \tag{5.18}$$

Also,

$$\begin{aligned}\sigma^2 &= Var\left(\sum_{i=1}^{n} w_i r_i\right) \\ &= w_{\text{risky}}^2\, Var\left(\sum_{i=1}^{n}\frac{w_i}{w_{\text{risky}}} r_i\right) \\ &= w_{\text{risky}}^2\ \sigma_{\text{der}}^2,\end{aligned} \tag{5.19}$$

which gives $w_{\text{risky}} = \dfrac{\sigma}{\sigma_{der}}$. From (5.18) and (5.19) we get

$$\mu = \mu_{\text{rf}} + \left(\frac{\mu_{\text{der}} - \mu_{\text{rf}}}{\sigma_{der}}\right)\sigma, \tag{5.20}$$

which is an equation of the line joining $(0, \mu_{\text{rf}})$ and $(\sigma_{\text{der}}, \mu_{\text{der}})$ in the (σ, μ)-graph.

Now, for a given risk σ, if we choose various weight combinations of risk-free asset and risky assets satisfying (5.17), we generate different lines represented by (5.20) in (σ, μ)-graph. Obviously, among all such lines, the line that produces the point with highest expected return for a given risk is tangent to the upper portion of the Markowitz bullet. This is illustrated in Fig. 5.10.

Definition 5.7.1 (Capital Market Line) *Among all the lines (5.20) for various weight combinations of risk-free asset and risky assets, the line giving the highest return for a given risk is called the capital market line.*

Definition 5.7.2 (Market Portfolio) *The point on the Markowitz bullet where the capital market line is tangential is said to represent the market portfolio.*

Theoretically, the market portfolio must contain all risky assets, for if some asset is not in it then it will wither and die. Since the market portfolio contains all risky assets, it is a completely diversified portfolio with no unsystematic risk.

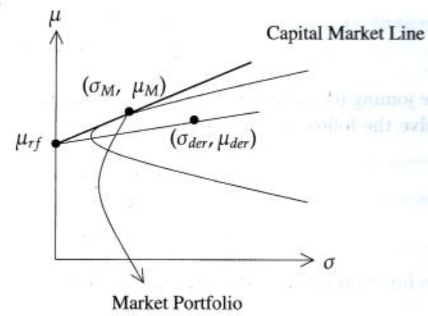

Fig. 5.10. Capital market line.

The basic idea of the capital asset pricing model (CAPM) is that an investor can improve the risk-expected return balance by investing partially in a portfolio of risky assets and partially in a risk-free asset. All investors will end up with portfolios along the capital market line as all *efficient portfolios* lie along this line while any other combination of risk-free asset and risky assets, except those which are efficient, lies below the capital market line. It is thus important to observe that all investors will hold combinations of only two assets, viz. the market portfolio M and a risk-free asset. This fund scenario is summarized in the following theorem.

Theorem 5.7.1 (One Fund Theorem) *There exists a single portfolio, namely the market portfolio M, of risky assets such that any efficient portfolio can be constructed as a linear combination of the market portfolio M and the risk-free asset.*

Unlike with the two fund theorem where any two efficient portfolios are sufficient, in this case, the tangent portfolio is a specific portfolio.

Theorem 5.7.2 *For any expected risk-free return μ_{rf}, the weight vector w_M of the market portfolio is given by*

$$w_M = \frac{C^{-1}(m - \mu_{\mathrm{rf}}\, e)}{e^T C^{-1}(m - \mu_{\mathrm{rf}}\, e)}\,.$$

Proof. From Fig. 5.10, we observe that for any point (σ, μ) in the Markowitz bullet, the slope of the line joining $(0, \mu_{\mathrm{rf}})$ and (σ, μ) is

$$s = \frac{\mu - \mu_{\text{rf}}}{\sigma} = \frac{\sum_{i=1}^{n} \mu_i w_i - \mu_{\text{rf}}}{(\sum_{i,j=1}^{n} c_{ij} w_i w_j)^{\frac{1}{2}}}.$$

For the line joining $(0, \mu_{\text{rf}})$ to (σ, μ) to be a tangent line to the Markowitz bullet, we need to solve the following optimization problem

$$\begin{aligned} &\text{Max} \qquad \frac{m^T w - \mu_{\text{rf}}}{(w^T C w)^{1/2}} \\ &\text{subject to} \quad e^T w = 1. \end{aligned} \tag{5.21}$$

The Lagrange function $L : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ for the problem (5.21) is described as

$$L(w, \lambda) = \frac{m^T w - \mu_{\text{rf}}}{(w^T C w)^{1/2}} + \lambda(1 - e^T w).$$

Now, solving (5.21) is same as maximizing $L(w, \lambda)$. So, $\nabla_w L(w, \lambda) = 0$, giving

$$\frac{1}{w^T C w} \Big((w^T C w)^{1/2} m - (m^T w - \mu_{\text{rf}})(w^T C w)^{-1/2} C w \Big) = \lambda e.$$

The above expression can be rewritten as

$$\sigma m - (\mu - \mu_{\text{rf}}) \frac{C w}{\sigma} = \lambda \sigma^2 e.$$

Multiplying by σ, we obtain

$$\sigma^2 m - (\mu - \mu_{\text{rf}}) C w = \lambda \sigma^3 e, \tag{5.22}$$

which in turn yields

$$\sigma^2 w^T m - (\mu - \mu_{\text{rf}}) w^T C w = \lambda \sigma^3 w^T e.$$

Since $e^T w = 1,\ \mu = w^T m$, and $\sigma^2 = w^T C w$, we get

$$\lambda = \frac{\mu_{\text{rf}}}{\sigma}. \tag{5.23}$$

The requisite value of weight vector w_M now follows from (5.22) and (5.23).

□

Remark 5.7.1 *Suppose the market portfolio* (σ_M, μ_M) *is known. Then, from (5.20), the equation of the capital market line is given by*

$$\mu = \mu_{rf} + \left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)\sigma.$$

If the investor is willing to take a positive risk σ, *he/she can earn an additional return* $\left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)\sigma$ *over and above the risk-free return* μ_{rf} *to compensate the risk taken by him/her. Therefore sometimes the quantity* $\left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)$ *is called the price of risk.*

Example 5.7.1 *Suppose a portfolio comprises of one risk-free asset with return* 0.5, *and three mutually independent risky assets with expected returns* 1, 2, 3 *and variances* 1, 1, 1, *respectively. Determine the equation of the capital market line.*

Solution The given information gives, $m^T = (\mu_1, \mu_2, \mu_3) = (1, 2, 3)$, $\mu_{rf} = 0.5$, $C = [\sigma_{ij}] = I_{3\times3}$, $e^T = (1, 1, 1)$. Therefore, the weight vector of the market portfolio is given by

$$w_M = \frac{C^{-1}(m - \mu_{rf}e)}{e^T C^{-1}(m - \mu_{rf}e)} = \begin{pmatrix} 1/9 \\ 1/3 \\ 5/9 \end{pmatrix}.$$

Consequently, the expected return and variance of the market portfolio are

$$\mu_M = m^T w_M = \frac{22}{9}, \quad \sigma_M = ((w_M)^T C w_M)^{1/2} = \frac{\sqrt{35}}{9}.$$

Thus, the equation of the capital market line is

$$\begin{aligned} \mu &= \mu_{rf} + \left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)\sigma \\ &= \frac{1}{2} + \frac{\sqrt{35}}{2}\sigma. \end{aligned}$$

□

In practice there are certain assets which are listed in the stock called *index stocks*. These limited assets are significant ones that can capture the pulse of the whole market. The most regularly quoted market indices are broad-base indices comprising of the stocks of large companies listed on a nation's largest stock exchanges, such as the American Dow Jones Industrial Average and S&P 500 Index, the British FTSE 100, the French CAC 40, the Japanese Nikkei 225. The

Bombay Stock Exchange is the largest in India, with over 6000 stocks listed and it accounts for over two thirds of the total trading volume in the country. The index stocks finally help us to compute the market portfolio (σ_M, μ_M). The knowledge of the market portfolio yields the equation of capital market line, see Remark 5.7.1. Now suppose an investor P is willing to take risk σ_P. Then for this risk, the expected return μ_P is maximum if the point (σ_P, μ_P) lies on the capital market line. Thus,

$$\mu_P = \mu_{rf} + \left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)\sigma_P.$$

If we let $w_P = \dfrac{\sigma_P}{\sigma_M}$ then

$$\mu_P = w_P\mu_M + (1 - w_P)\mu_{rf}.$$

Remark 5.7.2 *The above relation suggests that if an investor is willing to take a risk σ_P, then he/she should invest $w_P = \dfrac{\sigma_P}{\sigma_M}$ proportion of investment in index fund and $(1 - w_P)$ proportion of investment in the risk-free investment schemes.*

We now aim to examine how an individual asset behaves with respect to the market portfolio. For this, we attempt to build a relationship between the expected return along with the risk of an individual asset with the market portfolio. This gives the CAPM formula (5.24).

Theorem 5.7.3 *Suppose the market portfolio is (σ_M, μ_M). The expected return of an asset a_i is given by*

$$\mu_i = \mu_{rf} + \beta_i(\mu_M - \mu_{rf}), \quad \text{where} \quad \beta_i = \frac{\mathrm{Cov}(r_i, r_M)}{\sigma_M^2}. \tag{5.24}$$

Proof. Suppose an investor portfolio comprises of asset a_i with weight w and the market portfolio M with weight $1 - w$. Then the expected return and risk of the investor portfolio are respectively given by

$$\begin{aligned} \mu &= w\mu_i + (1 - w)\mu_M \\ \sigma^2 &= w^2\sigma_i^2 + (1 - w)^2\sigma_M^2 + 2\rho w(1 - w)\sigma_i\sigma_M \end{aligned} \tag{5.25}$$

where ρ is the coefficient of correlation between the returns of asset a_i and the market portfolio M.

As w varies, these values trace out a curve in the (σ, μ)-graph. It can be observed from Fig. 5.11 that as w passes through zero, the capital market line becomes tangent to the curve at M. This tangency condition can be translated into the

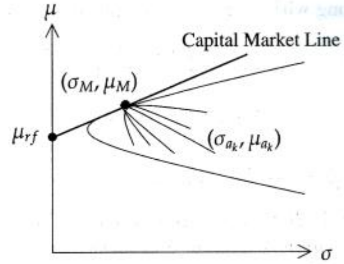

Fig. 5.11. Market portfolio.

condition that the slope of the curve is equal to the slope of the capital market line at M (corresponding to $w = 0$).

Now the slope of the curve at M is given by

$$\left.\frac{d\mu}{d\sigma}\right|_{(w=0)} = \left.\frac{d\mu}{dw}\frac{dw}{d\sigma}\right|_{(w=0)} = (\mu_i - \mu_{\mathrm{M}})\left.\frac{dw}{d\sigma}\right|_{(w=0)}.$$

Differentiating (5.25) with respect to w and computing its value at $w = 0$, we get

$$\left.\frac{d\sigma}{dw}\right|_{(w=0)} = \left.\frac{w\sigma_i^2 - (1-w)\sigma_{\mathrm{M}}^2 + \rho\sigma_i\sigma_{\mathrm{M}}(1-2w)}{\sigma}\right|_{(w=0)} = \frac{\sigma_{\mathrm{iM}} - \sigma_{\mathrm{M}}^2}{\sigma_{\mathrm{M}}}, \qquad \sigma_{\mathrm{iM}} = \rho\sigma_i\sigma_{\mathrm{M}}.$$

Consequently,

$$\left.\frac{d\mu}{d\sigma}\right|_{(w=0)} = \frac{(\mu_i - \mu_{\mathrm{M}})\sigma_{\mathrm{M}}}{\sigma_{\mathrm{iM}} - \sigma_{\mathrm{M}}^2}. \tag{5.26}$$

As discussed above, the slope of the curve needs to be equal to the slope of the capital market line at M, thereby yielding that

$$\frac{\mu_{\mathrm{M}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{M}}} = \frac{d\mu}{d\sigma}\Big|_{(w=0)}.$$

The above relation along with (5.26), on simplification, yields

$\beta_i = \frac{\rho \sigma_i \sigma_M}{\sigma_M^2}$

$\sigma_i = \frac{\beta_i \sigma_M}{\rho}$

$$
\begin{aligned}
\mu_i &= \mu_{\text{rf}} + \left(\frac{\mu_{\text{M}} - \mu_{\text{rf}}}{\sigma_{\text{M}}^2}\right)\sigma_{\text{iM}} \\
&= \mu_{\text{rf}} + \beta_i\,(\mu_{\text{M}} - \mu_{\text{rf}}).
\end{aligned}
$$

□

Remark 5.7.3 *Here,* $\beta_i = \dfrac{\sigma_{\text{iM}}}{\sigma_{\text{M}}^2}$ *is called the beta of an asset. Note that, for the market portfolio,* $\beta_M = 1$. *Beta is generally calculated for individual assets using regression analysis. As can be observed, beta measures an asset's volatility or risk in relation to the rest of the market. It is thus appropriately referred to as financial elasticity or correlated relative volatility, and it is all what is required to be known about the asset's risk characteristics in CAPM formula. In other words, an investor ready to bear some systematic risk gets rewarded for it. For instance, if* $\beta_i = 2$, *it indicates that the* i^{th} *asset return is expected to increase (decrease) by* 2% *when the market increases (decreases) by* 1%. *Equivalently, if the market return fluctuates over a specific range of values, the asset returns will fluctuate over a larger range of values. Thus, the market risk is magnified in the asset risk.*

(σ_i, σ_{iM})

$= \frac{\beta_i}{\rho}(\sigma_M, \sigma_M)$

Remark 5.7.4 *If we take* μ_{rf} *as a base, then the above theorem states that the expected return of a particular asset over the risk-free asset* (*namely* $(\mu_i - \mu_{\text{rf}})$) *is proportional to that of the market* (*namely* $(\mu_{\text{M}} - \mu_{\text{rf}})$) *and the constant of proportionality is* β_i. *The CAPM essentially tells that the expected excess rate of return of an assets is directly proportional to its covariance with the market. So it is important to understand this covariance or equivalently the beta of the asset.*

If $\beta_i = 0$, *the given asset is completely uncorrelated with the market and therefore CAPM gives* $\mu_i = \mu_{\text{rf}}$. *At first this looks surprising because it tells that no matter how risky the given asset is (even if* σ_i *is very large) the expected rate of return of this asset cannot be improved over the base point* μ_{rf} *and therefore there is no premium for risk. The main reason being that as the given asset is uncorrelated with the market its risk can be diversified away by having a small amount of this asset in the portfolio.*

If $\beta_i < 0$ *then CAPM gives* $\mu_i < \mu_{\text{rf}}$. *This again looks odd because it says that even though asset risk* σ_i *may be large, its expected rate of return is less than the base* μ_{rf}. *But then if, this asset is combined with the market it reduces the overall portfolio risk. Therefore the investor may be willing to have this asset in the portfolio as it has the capability of reducing its overall risk. Such an asset is expected to do well even when other assets are not doing that well and therefore it is called an insurance.*

The above discussion suggests that though for a portfolio an appropriate measure of risk is σ but for an individual asset the proper measure of risk is its beta. Thus there is a paradigm shift in understanding the risk of an asset.

Example 5.7.2 *Let the risk-free rate μ_{rf} be 8% and the market has $\mu_M = 12\%$ and $\sigma_M = 15\%$. Let an asset a be given which has covariance of 0.045 with the market. Determine the expected rate of return of the given asset.*

Solution From the given data we have $\beta_a = \dfrac{0 \cdot 045}{(0 \cdot 015)^2} = 2$. Then CAPM gives

$$\mu_a = \mu_{rf} + \beta(\mu_M - \mu_{rf}) = 0.08 + 2(0.12 - 0.08) = 0 \cdot 16.$$

Therefore the expected rate of return of the given asset is 16%.

□

Definition 5.7.3 (Beta of the Portfolio) *The overall β of the portfolio is the weighted average of the betas of the individual assets in the portfolio, with the weights being those that define the portfolio, i.e.* $\beta = \sum_{i=1}^{n} w_i \beta_i$.

The above formula is immediate because for the portfolio $w = (w_1, w_2, \ldots, w_n)$, the rate of return is $\sum_{i=1}^{n} w_i r_i$ and $Cov(r, r_M) = \sum_{i=1}^{n} w_i Cov(r_i, r_M) = \sum_{i=1}^{n} w_i \sigma_{iM} = \sum_{i=1}^{n} w_i \rho \sigma_i \sigma_M$.

Definition 5.7.4 (Security Market Line) *A linear equation*

$$\mu = \mu_{rf} + \beta\,(\mu_M - \mu_{rf}), \quad \text{where} \quad \beta = \frac{\mathrm{Cov}(r, r_M)}{\sigma_M^2}$$

that describes the expected return for all assets in the market is called the security market line.

The security market line highlights the essence of CAPM formula. It says that under the equilibrium conditions assumed by CAPM, all portfolio investments lie along the security market line in the beta-return space. It emphasizes that the risk of an asset is a function of its covariance with the market, or equivalently a function of its beta. The security market line is depicted in bold in Fig. 5.12

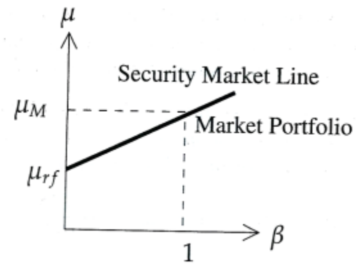

Fig. 5.12. Security market line.

CAPM as a Pricing Formula

We now give another interpretation of CAPM as a pricing formula. Let an asset be purchased at price P and later sold at price Q. Then the rate of return is $r = \frac{(Q-P)}{P}$. Here P is known but Q is random. If we write $E(Q) = \bar{Q}$, then the CAPM formula gives

$$\frac{\bar{Q}-P}{P} = \mu_{r_f} + \beta(\mu_M - \mu_{r_f})$$

i.e.

$$P = \frac{\bar{Q}}{1 + \mu_{r_f} + \beta(\mu_M - \mu_{r_f})}. \tag{5.27}$$

Here β is the beta of the given asset.

The formula (5.27) can be viewed as a pricing formula. Here for the random pay-off Q, $\bar{Q}$ is known and the aim is to determine the price P.

For the deterministic scenario, it is well known that $P = \frac{Q}{1+\mu_{r_f}}$. Therefore, the quantity $\mu_{r_f} + \beta(\mu_M - \mu_{r_f})$ can be interpreted as risk adjusted interest rate. This gives another interpretation of formula (5.27).

Example 5.7.3 *Consider a mutual fund that invests 10% it funds at a risk free rate of 7%. The remaining 90% is invested in a widely diversified portfolio (resembling the market portfolio) which is expected to give a return of 15%. Further it is known that the beta of the fund is 0.90, and one share of the mutual fund costs Rs 100. Is this price of a share of the mutual fund fair? Justify your answer.*

Solution: The value of a share after one year will be $(10\times1.07)+(90\times1.15) = 114.20$. Thus $\bar{Q} = 114.2$. Therefore

$$P = \frac{114.20}{(1.07) + (0.90)(0.15 - 0.07)} = Rs\ 100.$$

This shows that the price of the share, namely *Rs* 100, represents **Rs 100 of assets** in the fund, and therefore CAPM tells that the *price is right*.

□

The CAPM as a Factor Model

The CAPM can be derived as a special case of a single factor model. Let us assume that the asset return r_i and market return r_M (taken as a factor) are related as follows

$$(r_i - \mu_{rf}) = \alpha_i + \beta_i (r_M - \mu_{rf}) + \epsilon_i. \tag{5.28}$$

Here μ_{rf} is the risk-free interest rate and $E(\epsilon_i) = 0$. Also ϵ_i is uncorrelated with the market return r_M and also with other ϵ_j's. Further α_i and β_i are the usual coefficients appearing in a single factor model.

Taking expectation in (5.28) gives

$$(\mu_i - \mu_{rf}) = \alpha_i + \beta_i (\mu_M - \mu_{rf}). \tag{5.29}$$

Here we note that (5.29) is identical with CAPM except that in CAPM, $\alpha_i = 0$. If we further take the covariance of both sides in (5.29) we get

$$\sigma_{iM} = \beta_i \sigma_M^2 .$$

Hence,

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2},$$

which is the same expression as used in CAPM. The equation (5.29) represents a line between the quantities $(\mu_M - \mu_{rf})$ and $(\mu_i - \mu_{rf})$. This line is called the *characteristic equation or characteristic line.*

The characteristic line in a sense is more general than CAPM because here α_i need not be zero. In fact α_i can have a very nice economic interpretation. A stock with non zero α_i can be regarded as mispriced. If $\alpha_i > 0$ then in view of CAPM, the asset is performing better than it should. Similarly if $\alpha_i < 0$, then it is performing worse than it should.

Though we have tried to explain CAPM as a single factor model, we must note that the two are not equivalent. In CAPM we assume that the market is efficient, but in a single factor model we have taken arbitrary covariance matrix σ_{iM} and made no assumption on market efficiency.

The CAPM and β_i can be understood from a different angle if we take the following model

$$r_i = r_{rf} + \beta_i(r_M - r_{rf}) + \epsilon_i,$$

where ϵ_i is a random variable. Taking expectation in the above equation and using CAPM we get $E(\epsilon_i) = 0$. Further taking the correlation with r_M in the above equation we get $Cov(\epsilon_i, r_M) = 0$. Therefore we have

$$\sigma_i^2 = \beta_i^2\sigma_M^2 + Var(\epsilon_i).$$

This equation tells that σ_i^2 is the sum of two expressions. The first expression $\beta_i^2\sigma_M^2$ is called the *systematic risk*. This is the risk associated with the market as a whole. There is no chance of reducing this risk by diversification because all assets with nonzero beta have this risk. The second expression $Var(\epsilon_i)$ is uncorrelated with the market and therefore can be reduced by diversification. The quantity $Var(\epsilon_i)$ is called *unsystematic* risk of the asset. Therefore the systematic risk measured by β becomes more important because it directly combines with the systematic risk of other assets.