PARTIAL DIFFERENTIAL EQUATIONS

MC-406

ASSIGNMENT- 4+5

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Q1) Solve the differential equation $u_t - \alpha^2 u_{xx} = 0$ for the conduction of heat along a rod subject to the following conditions:

a) u is not infinite for $t \to \infty$

Given equation $u_t - \alpha^2 u_{xx} = 0$

On substituting $u = X(x)T(t)$ we get

$$XT' = \alpha^2 X'' T$$

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = -k^2$$

$$\therefore \frac{d^2X}{dx^2} + k^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + k^2\alpha^2 T = 0 \quad \text{—— (1)}$$

The solutions are:-

$$\left.\begin{aligned} X &= C_1 \cos kx + C_2 \sin kx \\ T &= C_3 e^{-k^2\alpha^2 T} \end{aligned}\right\} \quad \text{—— (2)}$$

If k^2 is changed to $-k^2$ solutions are

$$X = C_4 e^{kx} + C_5 e^{-kx} \qquad T = C_6 e^{k^2\alpha^2 t} \quad \text{—— (3)}$$

If $k^2 = 0$ solutions are:-

$$X = C_7 x + C_8 \;,\; T = C_9 \quad \text{—— (4)}$$

In eq. (3) $T \to \infty$ for $t \to \infty$ thus $u \to \infty$ i.e. the given condition (a) is not satisfied so, solution (3) is rejected. While (2) and (4) satisfy this equation

Applying condition (b) to equation (4), we get

$$C_7 = 0$$

$$\therefore u = XT = C_3 C_9 = a_0 \quad (5)$$

From eq. (2) $\frac{du}{dx} = [-C_1 \sin(kx) + C_2 \cos(kx)]\, k C_3 e^{-k^2 \alpha^2 t}$

applying the condition, we get $C_2 = 0$

$$-C_1 \sin(kl) + C_2 \cos(kl) = 0$$

i.e. $C_2 = 0$ and $kl = n\pi$

$$u = C_1 \cos(kx)\, C_3 e^{-k^2 \alpha^2 t}$$

$$= a_n \cos\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2 \pi^2 \alpha^2 t}{l^2}} \quad \text{——— (6)}$$

Thus the general solution is the sum of eq (5) and (6)

$$u = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right) e^{-(n^2\pi^2\alpha^2 t)/l} \quad \text{——— (7)}$$

Now, using condition (c), we get

$$lx - x^2 = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and $a_n = \frac{2}{l}\int_0^l (lx - x^2) \cos\left(\frac{n\pi x}{l}\right) dx$

$$\frac{2}{l}\Big[(lx - x^2)\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{l}\right)\right)\Big]_0^l$$

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$$\frac{2}{l}\left[0 - \frac{l^3}{n^2\pi^2}\left(\cos(n\pi+1)+0\right)\right] = -\frac{4l^2}{n^2\pi^2}, \quad n \in \cancel{2i}\; 2i\cancel{+1},\; i \in \mathbb{Z}$$

$$= 0 \qquad \text{otherwise}$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-(4m^2\pi^2\alpha^2 t)/l^2}$$

Q.2) $u_t = k u_{xx}$

$u_x(0,t) = 0$

$u_x(200,t) = -\frac{h}{k}[u(200,t) - 20]$

$u(x,0) = \sin(\pi x)$

Taking Laplace transform

$$sU(x,s) - \sin(\pi x) = k\frac{\partial^2 U}{\partial x^2} U(x,s)$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} - \frac{s}{k}U = -\frac{\sin(\pi x)}{k}$$

$$U_{CF}(x,s) = c_1 e^{\sqrt{s/k}\,x} + c_2 e^{\sqrt{s/k}\,x}$$

$$U_{PI}(x,s) = -\frac{\sin(\pi x)}{k[D^2 - s/k]} = \frac{\sin(\pi x)}{k(\pi^2 + s/k)} = \frac{\sin(\pi x)}{s + k\pi^2}$$

$$\therefore U(x,s) = c_1 e^{\sqrt{s/k}\,x} + c_2 e^{-\sqrt{s/k}\,x} + \frac{\sin(\pi x)}{s + k\pi^2} \quad \text{—— (1)}$$

$$\frac{\partial U}{\partial x}(x,s) = \sqrt{\frac{s}{k}}\left[c_1 e^{x\sqrt{s/k}} - c_2 e^{x\sqrt{s/k}}\right] + \frac{\pi\cos(\pi x)}{s + k\pi^2}$$

$$\mathcal{L}[u_x(0,t)] = \mathcal{L}[0]$$

$$\sqrt{\frac{s}{k}}[c_1 - c_2] + \frac{\pi}{s + k\pi^2} = 0 \quad \text{—— (2)}$$

$$\mathcal{L}[u_x(200,t)] = \mathcal{L}\left[-\frac{h}{k}[u(200,t) - 20]\right]$$

$$\sqrt{\frac{s}{k}}\left[c_1 e^{200\sqrt{s/k}}\right] - c_2 e^{\sqrt{s/k}\,200}\left[\sqrt{s/k} - \frac{h}{k}\right] - \frac{s}{\sqrt{k}}[c_1 - c_2]$$

$$= \frac{20h}{sk}$$

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$$\Rightarrow C_1 e^{\sqrt{s/k}\,200}\frac{h}{k} + C_2 e^{-200\sqrt{s/k}}\frac{h}{k} = \frac{20}{s}\frac{h}{k}$$

$$C_1 e^{200\sqrt{s/k}} + C_2 e^{-200\sqrt{s/k}} = \frac{20}{s} \quad \text{--- (4)}$$

$$\& \ (2) \Rightarrow C_1 - C_2 = \frac{\pi}{s+k\pi^2}\sqrt{\frac{k}{s}}$$

Solving (2) and (4):-

$$C_2 = \frac{1}{2\cosh\left(200\frac{\sqrt{s}}{\sqrt{k}}\right)}\left[\frac{20}{s} + \frac{\pi}{s+k\pi^2}\sqrt{\frac{k}{s}}\,e^{\sqrt{s/k}\,200}\right]$$

$$C_1 = \frac{1}{2\cosh\left(\frac{\sqrt{s}}{\sqrt{k}}200\right)}\left[\frac{20}{s} - \frac{\pi}{s+k\pi^2}\sqrt{\frac{k}{s}}\,e^{-200\sqrt{s/k}}\right]$$

$$\therefore u(x,s) = \frac{1}{2\cosh(200\sqrt{s/k})}\left[\frac{20}{s}\left[e^{\sqrt{\frac{s}{k}}x} + e^{-\sqrt{s/k}\,x}\right] + \frac{\pi}{s+k\pi^2}\sqrt{\frac{k}{s}}\right] + \frac{\sin(\pi x)}{s+k\pi^2}$$

Taking Inverse Laplacian Transform,

$$u(x,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{t}\left[\frac{20}{s\cosh(200\sqrt{s/k})}\cosh\left(\frac{\sqrt{s}}{\sqrt{k}}x\right) + \frac{1}{2(s+\pi^2 k)}\left[\pi\sqrt{\frac{k}{s}} + \sin(\pi x)\right]\right]$$

I. Roots lie at $s=0$, & $200\frac{\sqrt{s}}{\sqrt{k}} = j(2n+1)\frac{\pi}{2} \Rightarrow s = \frac{-(2n+1)^2\pi^2 k}{16\cdot 10^4}$

$$\text{Residue}_{s=0} \ \lim_{s\to 0} \frac{20\cosh\sqrt{s/k}\,x\,e^{st}}{\cosh\sqrt{s/k}} = 20$$

$$\text{Residue}_{s} \ \lim_{s\to s'}\left[s + \frac{(2n+1)^2\pi^2 k}{16\cdot 10^4}\right]\frac{\cosh\left(x\frac{\sqrt{s}}{\sqrt{k}}\right)e^{st}}{\cosh(200\sqrt{s/k})}$$

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$$= -\frac{4j}{(2n+1)\pi} \frac{\cos\left((2n+1)\frac{\pi}{400}\right)}{\sin\left((2n+1)\frac{\pi}{2}\right)} \frac{\sqrt{k}}{200}$$

$$\therefore I = \frac{10}{\pi j} - \frac{2}{(2n+1)\pi^2} \frac{\cos\left[(2n+1)\frac{\pi}{400}\right]}{\sin\left[(2n+1)\frac{\pi}{2}\right]} e^{-\frac{(2n+1)^2 4^2 k t}{16 \cdot 10^4}}$$

II: Roots lie at $s=0$, $s=-k\pi^2$

Residue $\lim_{s\to 0} \frac{\pi}{2} \frac{\sqrt{sk}}{s+k\pi^2} = 0$
$s=0$

Residue $\lim_{s\to()} \frac{\pi}{2}\sqrt{\frac{k}{s}} = \frac{\pi}{2}\sqrt{\frac{k}{-k\pi^2}} = \frac{j}{2}$
$s=-k\pi^2$

$$II = \frac{1}{2\pi j} \frac{j}{2} = \frac{1}{4\pi}$$

$$III: \frac{1}{2\pi j} \sin(\pi x)\, \frac{e^{-k\pi^2 t}}{2}$$

$$\therefore u(x,t) = 2\pi j\,(I + II + III)$$

$$= 20 + \frac{j}{2} + \frac{1}{2}\sin(\pi x)\, e^{-k\pi^2 t} + \sum_{n=0}^{\infty} \frac{-4j}{\pi(2n+1)} \frac{\cos\left[(2n+1)\frac{\pi}{400}\right]}{\sin\left[(2n+1)\frac{\pi}{2}\right]} e^{\frac{(2n+1)^2 \pi^2 k}{16\cdot 10^4}}$$

Q3) State and prove a maximum principle for solutions of an initial boundary value problem for $u_t = k\Delta u$ where Δ is the Laplacian in $\mathbb{R}$.

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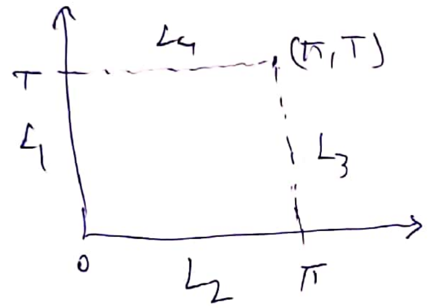
Let $u \in C_1$ be a solution of the heat equation. Then the maximum value of u on R is achieved on the parabolic boundary of R.

Let $M = \max\limits_{R} u$ and $m = \min \max\limits_{\partial p\, u} u$

To prove the max value principle we must show $m < M$ is not possible.

Where R is boundary (rectangular) formed by $L_1 L_2 L_3 L_4$.



The parabolic boundary is $\partial_p R$ defined by $\partial p R = L_1 \cup L_2 \cup L_3$

Let $(x_1 t_1) \in R \cup L_4$ be such that $u(x_1, t_1) = M$

Let $v : \bar{R} \to R$ be defined by

$$v(x,t) = u(x,t) + \frac{M-m}{4\pi^2}(x - x_1)^2 \quad \text{——} \quad (1)$$

For $(x,t) \in \partial p R$, we have

$$u(x,t) \leq m + \frac{M-m}{4\pi^2}\pi^2 = m + \frac{M-m}{4} < M \quad \text{——} \quad (2)$$

Further $v(x,t) = u(x_1, t_1) = M$. Thus the function v consumes it's maximum value namely M on $R \cup L_4$

Let $(x_2 t_2) \in R \cup L_4$ be such that $v(x_2, t_2) = M$

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Note that $0 < x < \pi$

if $(x_2, t_2) \in R$, then $v_t(x_2, t_2) = 0$

if $(x_2, t_2) \in L_4$, then $v_t(x_2, t_2) \geq 0$

Thus, we have $v_t(x_2, t_2) \geq 0$

In view of the relations,

$$v_t(x_2, t_2) = u_t(x_2, t_2)$$

$$= k \Delta u(x_2, t_2)$$

$$k \left[\Delta v(x_2, t_2) - \frac{M-m}{2\pi^2} \right] \quad (\Delta u = \text{Laplacian})$$

We get $0 \leq v_t(x_2, t_2) < k \Delta v(x_2, t_2)$ —— ③

However $\Delta v(x_2, t_2) \leq 0$ since v attains a max. at (x_2, t_2) which contradicts ③ $\Rightarrow\Leftarrow$

Thus $m < M$ is not possible

Q4) Solve the problem:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin(\pi x) \quad 0 \leq x \leq 1, \; 0 \leq t \leq \infty \\ &u(x, 0) = 1, \; 0 \leq x \leq 1 \end{aligned}\right] \text{—— ①}$$

Taking the Laplace transform of ①

$$sU(x, s) - u(x, 0) = \frac{\partial^2 U}{\partial x^2}(x, s) + \frac{\sin(\pi x)}{s}$$

$$\frac{\partial^2 U}{\partial x^2} - sU = \frac{\sin(\pi x)}{s} + 1$$

$$U_{CF} = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} \quad \text{—— ②}$$

$$U_{PI} = \frac{\sin(\pi x)}{s[D^2 - s]} + \frac{1}{D^2 - s} \quad \text{—— ③}$$

$$\frac{\sin(\pi x)}{s(\pi^2+s)} - \frac{1}{s}$$

Taking inverse Laplacian of ②

$$U_{CF} = C_1\left[\frac{-xe^{-x^2/4t}}{2\sqrt{\pi t^3}}\right] + C_2\left[\frac{xe^{-x^2/4t}}{2\sqrt{\pi t^3}}\right] = \frac{Cxe^{-x^2/4t}}{2\sqrt{\pi t^3}}$$

Taking inverse Laplacian of ③

$$U_{PI} = \mathcal{L}^{-1}\left[\frac{\sin(\pi x)}{\pi^2}\left(\frac{1}{s} - \frac{1}{s+\pi^2}\right)\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$\frac{\sin(\pi x)}{\pi^2}\left[e^{-\pi^2 t} - 1\right] + 1$$

$$\therefore u(x,t) = \frac{Cxe^{-x^2/4t}}{2\sqrt{\pi t^3}} + \frac{\sin(\pi x)}{\pi^2}\left[e^{-\pi^2 t} - 1\right] + 1$$

Q5) Using duhamel's principle find the solution

$$\left.\begin{array}{ll} u_t = u_{xx} & 0 \le x \le 1 \quad 0 \le t \le \infty \\ u(1,t) = \sin t & 0 \le t \le \infty \end{array}\right] \text{——} ①$$

Taking Laplace transform of ①

$$sU = \frac{\partial^2 U}{\partial x^2} \qquad \text{[Assuming no initial conditions]}$$

$$U(x,s) = C_1 e^{\sqrt{s}x} + C_2 e^{-x\sqrt{s}}$$

$$\because \mathcal{L}[u(0,t)] = \mathcal{L}[0]$$

$$\Rightarrow C_1 = -C_2$$

$$\mathcal{L}[u(1,t)] = \mathcal{L}[\sin t]$$

$$C_1\left[e^{\sqrt{s}} - e^{-\sqrt{s}}\right] = \frac{1}{s^2+1}$$

$$C_1 = \frac{1}{(s^2+1)\,2\sinh(\sqrt{s})} = \frac{1}{2(s^2+1)\sinh(\sqrt{s})}$$

$$\therefore\ U(x,s) = \frac{1}{s^2+1}\,\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})} \qquad \text{—— (2)}$$

Taking inverse Laplace of (2)

$$u(x,t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} \frac{\sinh(x\sqrt{s})}{(s^2+1)\sinh(x\sqrt{s})}\, e^{st}\, ds$$

Roots lie at $s = \pm j$, $s = -x^2\pi^2$

$$\underset{s=j}{\text{Residue}}\ \lim_{s\to j} \frac{1}{(s+j)}\,\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\, e^{jt} = \frac{1}{2j}\,\frac{\sin(\pi/4\,x)}{\sin(\pi/4)}\, e^{jt} \qquad \text{—— (3)}$$

$$\underset{s=-j}{\text{Residue}}\ \lim_{s\to -j}\ -\frac{1}{2j}\,\frac{\sin\left(\frac{\pi x}{4}\right)}{\sin(\pi/4)}\, e^{-jt} \qquad \text{—— (4)}$$

From (3) and (4)

$$\underset{s=\pm j}{\text{Residue}} \left[\frac{\sin(\pi x/4)}{\sin(\pi/4)}\right] \sin(t) \qquad \text{—— (5)}$$

$$\underset{s=-n^2\pi^2}{\text{Residue}}\ \lim_{s\to n^2\pi^2} \frac{s+n^2\pi^2}{s^2+1}\,\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\, e^{st}$$

$$\frac{(-1)^{n+1}}{n^4\pi^4+1}\, 2n\pi j\, \sin(n\pi x)\, e^{-n^2\pi^2 t} \qquad \text{—— (6)}$$

From (5) and (6)

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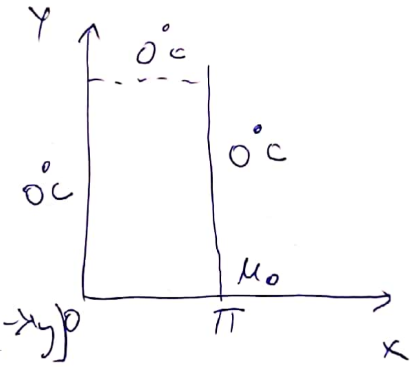
$$u(x,t) = \left[\frac{\sin(\pi x/4)}{\sin(\pi/4)}\right]\sin(t) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4\pi^4+1}(n)\sin(n\pi x)\, e^{-n^2\pi^2 t}$$

Q-6) An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angle to them. The breadth is π. This end is maintained at a temperature u_0 at all points and other edges are at 0 temperature. Determine the temperature at any point of the plate in the steady state.

$$u_{xx} + u_{yy} = 0 \quad \text{—— ①}$$



$u = X(x)\,Y(y)$

Let the solution of ① be :-

$u(x,y) = [A\cos\lambda x + B\sin\lambda x][Ce^{\lambda y} + De^{-\lambda y}]$

$\because u(0,y) = 0$

$A[Ce^{\lambda y} + De^{-\lambda y}] = 0$

$\Rightarrow A = 0$

$\because u(\pi, y) = 0$

$B\sin\lambda\pi\,[Ce^{\lambda y} + De^{-\lambda y}] = 0$

$B\sin\lambda\pi = 0$

$\sin\lambda\pi = 0$

$\lambda\pi = n\pi \quad , \quad n \in \mathbb{Z}$

$\lambda = n$

$\therefore u(x,y) = B\sin(nx)[Ce^{ny} + De^{-ny}] \quad \text{—— ②}$

$\because u(x,\infty) = 0$

$$[\beta \sin(\alpha(x))][ce^{\infty} + 0] = 0 \quad \text{——— (3)}$$

$C = 0$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} C_n \sin(nx)\, e^{-ny} \quad \text{——— (4)}$$

$$\because u(x,0) = u_0 = \sum_{n=1}^{\infty} C_n \sin(nx)$$

$$C_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin(nx)\, dx$$

$$\frac{2}{\pi} u_0 \frac{1}{n}[1-(-1)^{n+1}] = \begin{cases} \frac{4u_0}{\pi n} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

$$\therefore u(x,y) = \sum_{n=1,3,5,7\ldots} \frac{2u_0}{\pi} \frac{1}{n}[2] - \sin(nx)$$

$$= \sum_{n=1,3,5\ldots} \frac{4u_0}{n\pi} \sin(nx)$$

Q 7) Solve $u_{xx} + u_{yy} = 0 \quad -\infty \le x \le \infty, \; y \ge 0$ ——— (1)

$u_y(x,0) = g(x) \quad -\infty \le x \le \infty$ ——— (2)

with the condition that u is bounded as $y \to \infty$ and u and u_x vanish as $|x| \to \infty$ and

$$\int_{-\infty}^{\infty} g(x)\, dx = 0$$

$u(x,\infty) = 0$ ——— (3)

$u(\pm\infty, y) = 0$ ——— (4)

$u_x(\pm\infty, y) = 0$ ——— (5)

Let $u(x,y) = X(x)Y(y)$

$$X''(x)Y(y) + Y''(y)X(x) = 0$$

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -k^2 \text{ const}$$

$$X''(x) + k^2 X(x) = 0$$

If $k = 0$

$$X(x) = C_0 x + C_1$$

violates (3) and (4)

If $k^2 = +k^2$

$$X(x) = C_4 \sin(kx) + C_5 \cos(kx)$$

If $k^2 = -k^2$

$$X(x) = C_8 e^{kx} + C_9 e^{-kx}$$

From (7), (4)

$$u(\infty, y) = 0 \Rightarrow C_8 = 0$$

$$u(-\infty, y) = 0 \Rightarrow C_9 = 0$$

$$Y''(y) - k^2 Y(y) = 0$$

$$Y(y) = C_2 y + C_3$$

$$Y(y) = C_6 e^{ky} + C_7 e^{-ky} \quad \text{— (6)}$$

$$Y(y) = C_{10} \sin(ky) + C_{11} \cos(ky) \quad \text{— (7)}$$

$$\therefore u(x,y) = [C_4 \sin(kx) + C_5 \cos(kx)][C_6 e^{ky} + C_7 e^{-ky}]$$

From (13):

$$u(x, \infty) = 0$$

$$\Rightarrow C_6 = 0$$

From (1): $u(\infty, y) = \lim_{k \to \infty} [A_1 \sin(ky) + A_2 \cos(kx)] = 0$ — (8)

From ⑤

$u_x(\infty, y) = k[A_1 \cos(kx_1) + A_2 \sin(kx_1)]$ —— ⑨

$\because \int_{-\infty}^{\infty} g(x) = 0 \Rightarrow$ odd function on +ve/-ve function

$\Rightarrow u(x, y)$ must be a function of one

$\therefore$ From ⑧ and ⑨ $\Rightarrow A_2 = 0$

From ⑧ $\lim_{l \to \infty} \sin(kl) = 0$

$$k = \lim_{l \to \infty} \frac{n\pi}{l}$$

$\therefore u(x, y) = A_1 \sin(kx) e^{-ky}$

$u_y(x, 0) = -kA_1 \sin(kx) = g(x)$

$\therefore A_1 = \lim_{l \to \infty} \frac{2}{l} \int_0^l \frac{-g(x)\, l}{n\pi} \sin\left(\frac{n\pi}{l}\right) x \, dx$

$\Rightarrow A_1 = \lim_{l \to \infty} \frac{-2}{n\pi} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$

Q8) Solve the Laplace Equation

$$u_{xx} + u_{yy} = 0 \quad \text{——} \quad ①$$

subject to the conditions $u(0, y) = u(l, y) = u(x, 0) = 0$

and $u(x, a) = \sin(n\pi x / l)$

$u(0, y) = 0$

$u(l, y) = 0$

$u(x, 0) = 0$

$u(x, 0) = \sin\left(\frac{n\pi x}{l}\right)$

Let $u(x,y) = X(x)Y(y)$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -k^2$$

$X''(x) + k^2 X(x) = 0$

$X(x) = C_1 \sin(kx) + C_2 \cos(kx)$

If $k = 0$

$X(x) = C_5 x + C_6$

If $k = -k^2$

$X(x) = C_9 e^{kx} + C_{10} e^{-kx}$

$Y''(y) - k^2 Y(y) = 0$

$Y(y) = C_3 e^{ky} + C_4 e^{-ky}$ — (2)

$Y(y) = C_7 y + C_8$ — (3)

$Y(y) = C_{11} \sin(ky) + C_{12} \cos(ky)$ — (4)

From (3)

$u(0,y) = C_6 [C_7 y + C_8] = 0$

$u(x,0) = 0 \Rightarrow C_8 = 0$

$u(l,y) = 0 \Rightarrow C_5 = 0$

$\Rightarrow u = 0$

From (4) $u(0,y) = 0$

$C_9 + C_{10} = 0$

$C_9 e^{kl} + C_{10} e^{-kl} = 0$

$\Rightarrow C_9 = 0 \quad \Rightarrow C_{10} = 0$

$\Rightarrow u = 0$

From (2):

$u(0,y) = 0 \Rightarrow C_2 = 0$

$u(l, y) = 0$

$\sin(kl) = 0$

$k = \frac{n\pi}{l}$

$u(x, 0) = 0$

$\Rightarrow c_3 + c_4 = 0$

$c_3 = -c_4$

$$u(x,y) = c_4 c_3 \sin\left(\frac{n\pi x}{l}\right)\left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right]$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \sinh\left(\frac{n\pi y}{l}\right)$$

$$\because u(x,a) = \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \sinh\left(\frac{n\pi a}{l}\right)$$

$$A_n = \frac{2}{l}\int_0^l \frac{1}{\sinh(n\pi a/l)} \sin^2\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l \sinh(n\pi a/l)} \cdot l$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} \frac{1}{\sinh\left[\frac{n\pi a}{l}\right]} \sin\left[\frac{n\pi x}{l}\right] \sinh\left[\frac{n\pi y}{l}\right]$$

Q4) Derive Poisson integral formula of the Laplace Equation.

$\Delta u = 0$

$u[\text{boundary}] = f(\theta)$ [spherical coordinates]

Let the solution be

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta} \quad \text{—— (1)}$$

In such a case

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) e^{-in\alpha} d\alpha \quad \text{—— (2)}$$

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, e^{-in\alpha}\, d\alpha \cdot r^{|n|} e^{in\theta}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-\alpha)}\, d\alpha$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\left[\sum_{n=0}^{\infty} r^n e^{in(\theta-\alpha)} + \sum_{n=0}^{\infty} r^n e^{-in(\theta-\alpha)} - 1\right] d\alpha$$

$$\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\left[\frac{1}{1-re^{i(\theta-\alpha)}} + \frac{1}{1-re^{-i(\theta-\alpha)}} - 1\right] d\alpha$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\left[\frac{2-2r\cos(\theta-\alpha)}{1-2r\cos(\theta-\alpha)+r^2} - 1\right] d\alpha$$

$$\therefore u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\alpha)(1-r^2)}{1-2r\cos(\theta-\alpha)+r^2}\, d\alpha$$

At centre $r=0$

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, d\alpha$$

$$= \text{Avg}\,[f(\alpha)]$$

Q10) Solve the following Laplace Equation

$$\nabla^2 = 0, \quad 0 \le r \le 1$$

$$u(1,\theta) = 1 + \sin\theta + \frac{1}{2}\sin(3\theta) + \cos(4\theta)$$

$\nabla^2 u = 0$

$$u(1,\theta) = \underset{\mathrm{I}}{1} + \underset{\mathrm{II}}{\sin\theta} + \underset{\mathrm{III}}{\frac{1}{2}\sin 3\theta} + \underset{\mathrm{IV}}{\cos 4\theta} = f(\alpha)$$

From Poisson's formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,\frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}\,d\alpha$$

$$\because \frac{1}{2} + \sum_{n=1}^{\infty} a^n \cos nx = \frac{1-a^2}{2(1+a^2-2a\cos x)}$$

$$u(r,\theta) = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(\alpha)\left[1+2\sum_{n=1}^{\infty} r^n \cos n(\theta-\alpha)\right] d\alpha\right]$$

$$u(1,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\left[1+\sum_{n=1}^{\infty} \cos n(\theta-\alpha)\right] d\alpha$$

I: $u_I = \frac{1}{2\pi}\cdot 2\pi = 1$

II: $u_{II} = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi}\left[1+\sum_{n=1}^{\infty}\cos(n(\theta-\alpha))\right]d\alpha\right]$

I $\sim$ $\frac{1}{2\pi}\frac{2}{n-1}\sin[n\theta]\sin\left[(n-1)\frac{\pi}{2}\right]$

$$\frac{1}{2}\sin(n\theta)\,\frac{\sin[(n-1)\pi/2]}{(n-1)\pi/2}$$

III: $u_{III} = \frac{1}{2}\sin(n\theta)\,\frac{\sin[(n-3)(\pi/2)]}{(n-3)\pi/2}$

IV: $u_{IV} = 0$

$$\therefore u(r,\theta) = 1 + \frac{1}{2\pi}\sum_{n=1}^{\infty} r^n \sin(n\theta)\left[\frac{\sin(n-1)\pi/2}{(n-1)\pi/2} + \frac{\sin(n-3)\pi/2}{(n-3)\pi/2}\right]$$