5. Fuzzy Relations and Fuzzy Graphs

1. Introduction

The classical notion of 'relation' describes a relationship between two or more objects. For example, 'friend' classifies members in a group being either friends or not (A is friend of B but C is not a friend of B). Such a relation between two objects is called a binary relation. More generally we can also have an n-ary relation – a relation between n elements. For instance, a student X took a course A in semester B in year C. This relation has four arguments (student, course, semester and year). An n-ary relation may be formally defined as a set of ordered list of n objects.

In fact, a binary relation R on the variables X and Y can be regarded as a set of ordered pairs on $X \times Y$. For instance, binary relation 'greater than' between two real numbers x and y may be formally defined as

$$R = \{(x, y) \mid x > y, x, y \in R\}$$

It is easy to see that the relation R is a subset of $X \times Y$. In general, an n-ary relation on $X_1, X_2, \ldots, X_n$ whose domains are $X_1, X_2, \ldots, X_n$ is a subset of $X_1 \times X_2 \times \ldots \times X_n$.

In analogy with crisp relations defined above, we can also define fuzzy relations, which we now proceed to do. These can be defined on crisp as well as fuzzy sets. Application of fuzzy relations are widespread and important.

2. Fuzzy Relations on Crisp Sets

Fuzzy relation between two elements x and y belonging to crisp sets X and Y, respectively is a fuzzy subset of $X \times Y$. In other words, a fuzzy relation between elements of X and Y is a fuzzy mapping from $X \rightarrow Y$.

Definition: Let $X, Y \subseteq R$ be subsets of real numbers R, then

$$\tilde{R} = \{(x, y), \mu_R(x, y) \mid (x, y) \subseteq X \times Y\}$$

is called a fuzzy relation on $X \times Y$.

Example 1: Let $X = Y = R$ and $\tilde{R}$: "considerably larger than". Then the membership function of this fuzzy relation $\tilde{R}$ which is a fuzzy set on $X \times Y$ can be defined as

$$\mu_R(x,y) = \begin{cases} 0, & x \le y \\ \dfrac{x-y}{10y}, & y \le x \le 11y \\ 1, & x \ge 11y \end{cases}$$

An alternative membership function for this relation could be

$$\mu_R(x,y) = \begin{cases} 0 \\ \left(1+(y-x)^{-2}\right)^{-1}, & x > y \end{cases}$$

A fuzzy relation generalizes the classical notion of relation into matter of degree. The fuzzy relation considerably larger than discussed above gives an estimate of the magnitude by which x is greater than y. Obviously, for $x < y$ the membership value is zero and it increases gradually to value 1 as x becomes more and more larger than y. Similarly, the fuzzy relation 'friend' will describe the degree of friendship between two persons.

2.1 Fuzzy Relations as Matrices

For discrete supports, fuzzy relations can also be expressed in matrix form. Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3, y_4)$, then $\tilde{R}$ = 'x considerably larger than y' may be expressed as

$$\tilde{R}: \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} .8 & 1 & .1 & .7 \\ 0 & .8 & 0 & 0 \\ .9 & 1 & .7 & .8 \end{bmatrix} \end{array}$$

and $\tilde{S}$ = 'y very close to x' may be expressed as

$$\tilde{S}: \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} .4 & 0 & .9 & .6 \\ .9 & .4 & .5 & .7 \\ .3 & 0 & .8 & .5 \end{bmatrix} \end{array}$$

Thus, formally we can say that a fuzzy relation $\tilde{R}$ between two variables x and y whose domains are X and Y, respectively, is a function that maps ordered pairs in $X \times Y$ to their degree of relationship which is some number between 0 and 1, i.e. $\tilde{R}: X \times Y \to [0, 1]$.

More generally, a fuzzy n-ary relation $\tilde{R}$ in $x_1 \times x_2 \times ... \times x_n$, whose domain is $X_1 \times X_2 \times ... \times X_n$ is defined by a function that maps an n-tuple $\langle x_1 \times x_2 \times ... \times x_n \rangle$ in $X_1 \times X_2 \times ... \times X_n$ to a real number in the interval [0, 1]. In other words, $\tilde{R}: X_1 \times X_2 \times ... \times X_n \to [0, 1]$.

As another example, suppose the height h in inches and weight w in kg of interest are [60, 62, 64, 66, 68, 72, 72] and [50, 55, 60, 65, 70, 75, 80, 85], then we may express the fuzzy relation 'healthy' based on these two parameters as

$$
\begin{array}{r} \\ 60 \\ 62 \\ 64 \\ \tilde{R} = 66 \\ 68 \\ 70 \\ 72 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 50 & 55 & 60 & 65 & 70 & 75 & 80 & 85 \end{array} \\
\begin{bmatrix}
1 & 1 & 0.8 & 0.6 & 0.5 & 0.5 & 0.3 & 0.2 \\
1 & 1 & 1 & 0.9 & 0.8 & 0.8 & 0.5 & 0.2 \\
1 & 1 & 1 & 1 & 0.8 & 0.7 & 0.5 & 0.2 \\
0.8 & 1 & 1 & 1 & 0.8 & 0.7 & 0.6 & 0.3 \\
0.6 & 0.8 & 0.1 & 1 & 1 & 0.8 & 0.6 & 0.3 \\
0.5 & 0.6 & 0.8 & 1 & 1 & 1 & 0.8 & 0.4 \\
0 & 0.2 & 0.4 & 0.6 & 1 & 1 & 1 & 0.6
\end{bmatrix}
\end{array}
$$

Each entry in the matrix indicates the degree to which a person having height and weight indicated in the row and column is considered to be 'healthy'. For instance, entry 0.5 in the row of height 62 and column of weight 80 indicates the degree to which such a person is considered healthy.

Once we define the fuzzy relation 'healthy' in terms of the elements of the matrix as defined above, we can answer the following two types of questions:

(i) What is the degree or extent to which a person with specified height and specified weight can be regarded as healthy?

(ii) What is the possibility that a healthy person has a specified pair of height and weight?

In answering the first question, the fuzzy relation is equivalent to the membership function of a multi-dimensional fuzzy set. In the second case, the fuzzy relationship becomes a possibility distribution to be assigned to a healthy person whose actual weight and height are not known.

3. Fuzzy Relations on Fuzzy Sets

In the above definition of fuzzy relation it has been assumed that μ is a mapping from $X \times Y$ to [0, 1], i.e. it assigns to each pair a degree of membership in the unit interval [0, 1]. In some cases, such as graph theory, it is useful to consider fuzzy relations that map from the fuzzy sets to points in the unit interval [0, 1].

Definition: Let $X, Y \subseteq R$, and

$$\tilde{A} = \{(x, \mu_A(x)) \mid x \in X\}, \tilde{B} = \{(y, \mu_B(y)) \mid y \in Y\}$$

be two fuzzy sets, then

$$\tilde{R} = \{[(x, y), \mu_R(x, y)] \mid (x, y) \in X \times Y\}$$

is a **fuzzy relation** on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_R(x, y) \leq \min \{\mu_A(x), \mu_B(y)\}, \forall (x, y) \in X \times Y$$

This definition is useful in defining graphs. Let the elements of a **fuzzy relation** $\tilde{R}$ be the nodes of a fuzzy graph which is represented by this fuzzy relation. Let the membership elements of the related fuzzy sets define the **flow** in respective edges of the graph while the degrees of membership of the corresponding pairs in the relation denote **capacities** of the graph. The requirement $\mu_{\tilde{R}}(x, y) \leq \min\{\mu_A(x), \mu_B(y)\}$ then ensures that flows in the edges of the graph cannot exceed the flow in the respective nodes (pair of edges).

Example 2: Let

$$\tilde{A} = \{(a_1, .2), (a_2, .4), (a_3, .6)\}$$

$$\tilde{B} = \{(b_1, .3), (b_2, .4), (b_3, .5), (b_4, .2)\}$$

Then $\tilde{R} = \{((a_1, b_1), .2), ((a_1, b_2), .2), ((a_1, b_3), .2), ((a_1, b_4), .2), ((a_2, b_1), .3), ((a_2, b_2), .4), ((a_2, b_3), .4), ((a_2, b_4), .2), ((a_3, b_1), .3), ((a_3, b_2), .4), ((a_3, b_3), .5), ((a_3, b_4), .2)\}$

is a relation on $\tilde{A}$ and $\tilde{B}$. Such relations are best expressed in matrix form as

$$\mu_R(a, b) : \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} .2 & .2 & .2 & .2 \\ .3 & .4 & .4 & .2 \\ .3 & .4 & .5 & .2 \end{bmatrix} \end{array}$$

Such a matrix is sometimes referred to as a **fuzzy matrix**. A fuzzy arrow graph for relation $\tilde{R}$ on $\tilde{A}$ and $\tilde{B}$ may be drawn as shown in Fig. 1.

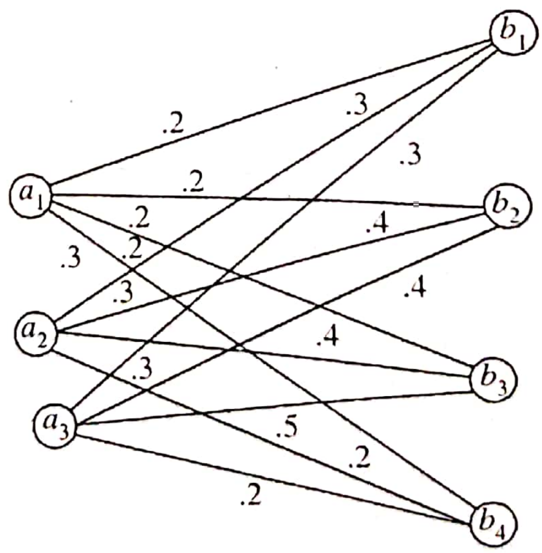

Fig. 1

In addition to defining a binary relation that exists between two different sets, it is also possible to define crisp as well as fuzzy binary relation among the elements of a single set X. A binary relation of this type is usually denoted by $\tilde{R}(X, X)$. For example

$$\tilde{R}: \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & .2 & 0 & .2 \\ x_2 & .4 & .2 & .8 \\ x_3 & 1 & .6 & .2 \end{array}$$

These relations are often referred to as directed graphs or digraphs.

3.1 Domain and Range of a Fuzzy Relation

Definition: Given a fuzzy relation $\tilde{R}(X, Y)$ its **domain** is a fuzzy set on X denoted by dom $(\tilde{R})$ whose membership function is defined by

$$\mu \text{ dom } (\tilde{R}) = \max_{y \in Y} \tilde{R}(x, y), \quad \text{for each } x \in X$$

In other words, each element of the set X belongs to the domain R to the degree equal to the strength of its strongest relation to any member of the set Y.

The range $\tilde{R}(X, Y)$ is a fuzzy relation on Y denoted by range $\tilde{R}(y)$ whose membership function is defined by

$$\mu \text{ range } \tilde{R}(y) = \max_{x \in X} \tilde{R}(x, y), \quad \text{for each } y \in Y$$

That is the strength of the strongest relation that each element of Y has to an element of X is equal to the degree of that element's membership in the range.

In addition **height** $h(\tilde{R})$ of a fuzzy relation $\tilde{R}(X, Y)$ is the number

$$h(\tilde{R}) = \max_{y \in Y} \max_{x \in X} \tilde{R}(x, y)$$

In other words, it is the largest membership grade attained by any pair (x, y) in $\tilde{R}(x, y)$.

For instance, in the case of fuzzy relation of Example 2, dom $(\tilde{R})$ = $[(a_1, .2), (a_2, .4), (a_3, .5)]$ and range $\tilde{R}(y) = [(b_1, .3), (b_2, .4), (b_3, .5), (b_4, .2)]$. Its height is $h(\tilde{R}) = .5$.

4. Union and Intersection of Fuzzy Relations

Fuzzy relations are in fact sets in product spaces. Therefore, the set-theoretic and algebraic operations can be defined for them based on the extension principle.

Definition: Let $\tilde{R}$ and $\tilde{S}$ be two fuzzy relations defined in the same product space. Then we define the **union** of $\tilde{R}$ and $\tilde{S}$ as a fuzzy relation

$$\tilde{R} \cup \tilde{S} = [\{(x, y), \mu_{R \cup S}(x, y)\} \mid (x, y) \in X \times Y]$$

where $\mu_{R \cup S}(x, y) = \max\{\mu_R(x, y), \mu_S(x, y)\}, \forall (x, y) \in X \times Y$

Similarly, the intersection of $\tilde{R}$ and $\tilde{S}$ is a fuzzy relation

$$\tilde{R} \cap \tilde{S} = [\{(x, y), \mu_{R \cap S}(x, y)\} \mid (x, y) \in X \times Y]$$

where $\mu_{R \cap S}(x, y) = \min\{\mu_R(x, y), \mu_S(x, y)\}, \forall (x, y) \in X \times Y$

Example 3: Let $\tilde{R}$ and $\tilde{S}$ be relations from $A = [a_1, a_2]$ to $B = [b_1, b_2, b_3]$ defined by the matrices

$$\tilde{R}: \begin{array}{c} \\ a_1 \\ a_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.5 & 0 \\ 0.2 & 0.8 & 0.2 \end{bmatrix} \end{array} \text{ and } \tilde{S}: \begin{array}{c} \\ a_1 \\ a_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 1.0 \\ 0.7 & 0.8 & 0 \end{bmatrix} \end{array}$$

then

$$\tilde{R} \cup \tilde{S} = \begin{array}{c} \\ a_1 \\ a_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.5 & 1.0 \\ 0.7 & 0.8 & 0.2 \end{bmatrix} \end{array} \text{ and } \tilde{R} \cap \tilde{S} = \begin{array}{c} \\ a_1 \\ a_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0 \\ 0.2 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 4: Let $\tilde{R}$ and $\tilde{S}$ be two fuzzy relations 'x considerably larger than y' and 'y very close to x' as defined in section 2.1.

Then, $\tilde{R} \cup \tilde{S}$ which can be interpreted as implying 'x considerably larger than or very close to y' is given by

$$\tilde{R} \cup \tilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 1 & 0.9 & 0.7 \\ 0.9 & 0.8 & 0.5 & 0.7 \\ 0.9 & 1 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

Similarly, $\tilde{R} \cap \tilde{S}$ which can be interpreted to imply 'x is considerably larger than as well as very close to y' is given by

$$\tilde{R} \cap \tilde{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.4 & 0 & 0.1 & 0.6 \\ 0 & 0.4 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

6. Composition of Fuzzy Relations

Fuzzy relations in different product spaces can be combined with each other by the operation 'composition'. Composition of two fuzzy relations is the result of three operations: (i) cylindrically extending each relation so that their dimensions are identical, (ii) intersecting the two extended relations and (iii) projecting the intersection to the dimensions not shared by the two original relations. This is formally stated below for the composition of binary fuzzy relations.

Definition: Let $\tilde{R}$ and $\tilde{S}$ be two fuzzy relations in $U_1 \times U_2$ and $U_2 \times U_3$, respectively. The **composition** of these two fuzzy relations denoted by $\tilde{R} \circ \tilde{S}$ is

$$\tilde{R} \circ \tilde{S} = \underset{U_1 \cdot U_3}{\mathrm{Proj}}(\tilde{R}^* \cap \tilde{S}^*)$$

where $\tilde{R}^*$ and $\tilde{S}^*$ are cylindrical extensions of $\tilde{R}$ and $\tilde{S}$ in $U_1 \times U_2 \times U_3$.

Different versions of 'composition' have been suggested which differ in their results and also with respect to their mathematical properties. The 'max-min' composition is the best known and most frequently used. However, max-product and max-average compositions yield results that are sometimes more appealing.

6.1 The max-min, max-product and max-average compositions

Let $\tilde{R}_1(x, y)$, $(x, y) \in X \times Y$ and $\tilde{R}_2(y, z)$, $(y, z) \in Y \times Z$ be two fuzzy relations. The max-min composition ($\tilde{R}_1$ max-min $\tilde{R}_2$) is then the fuzzy set

$$\tilde{R}_1 \circ \tilde{R}_2 = \left\{ \left[(x,y), \max_y \left\{ \min\left(\mu_{R_1}(x,y), \mu_{R_2}(y,z) \right) \right\} \right] \mid x \in X, y \in Y, z \in Z \right\}$$

μ_{R1oR2} is used to denote the membership function of the fuzzy composition of $\tilde{R}_1 \circ \tilde{R}_2$.

A more general definition of the composition, which can also take care of max-product and max-average compositions is as follows:

Definition: Let $\tilde{R}_1\ (x, y)$, $(x, y) \in X \times Y$ and $\tilde{R}_2\ (y, z)$, $(y, z) \in Y \times Z$ be two fuzzy relations. The max-composition of $\tilde{R}_1 \circ \tilde{R}_2$ is then defined as the fuzzy set

$$\tilde{R}_1 \circ \tilde{R}_2 = \left\{ \left[(x,y), \max_y \{ \mu_R(x,y) * \mu(y,z) \} \right] \mid x \in X, y \in Y, z \in Z \right\}$$

If o is an associative operation, which is monotonically non-decreasing in each argument, then the max-*composition essentially corresponds to the max-min composition.

Based on this general definition max-product composition $\tilde{R}_1 \cdot \tilde{R}_2$ and max-average composition $\tilde{R}_1 \therefore \tilde{R}_2$ are defined as

$$\tilde{R}_1 \cdot \tilde{R}_2(x,z) = \left\{\left[(x,z), \max_y \left\{\mu_{R_1}(x,y) \cdot \mu_{R_2}(y,z)\right\}\right] \mid x \in X, y \in Y, z \in Z\right\}$$

$$\tilde{R}_1 \therefore \tilde{R}_2(x,z) = \left\{\left[(x,z), \frac{1}{2}\max_{x,z} \left\{\mu_{R_1}(x,y) + \mu_{R_2}(y,z)\right\}\right] \mid x \in X, y \in Y, z \in Z\right\}$$

Example 7: Let $\tilde{R}_1(x, y)$ and $\tilde{R}_2(y, z)$ be defined by the following relational matrices:

$$\tilde{R}_1 : \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} .1 & .2 & 0 & .1 & .7 \\ .3 & .5 & 0 & .2 & 1 \\ .8 & 0 & 1 & .4 & .3 \end{bmatrix} \end{array} \qquad \tilde{R}_2 : \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} .9 & 0 & .3 & .4 \\ .2 & 1 & .8 & 0 \\ .8 & 0 & .7 & 1 \\ .4 & .2 & .7 & 0 \\ 0 & 1 & 0 & .8 \end{bmatrix} \end{array}$$

We first compute the max-min composition $\tilde{R}_1$ o $\tilde{R}_2$ (x, z). We show in detail the determination for $x = x_1$, $z = z_1$. The remaining results shown at the end can be similarly verified.

For $x = x_1$, $z = z_1$, $y = y_i$, $i = 1, 2, \ldots, 5$ we have
$\min\{\mu_{R1}(x_1, y_1), \mu_{R2}(y_1, z_1)\} = \min(.1, .9) = .1$
$\min\{\mu_{R1}(x_1, y_2), \mu_{R2}(y_2, z_1)\} = \min(.2, .2) = .2$
$\min\{\mu_{R1}(x_1, y_3), \mu_{R2}(y_3, z_1)\} = \min(0, .8) = 0$
$\min\{\mu_{R1}(x_1, y_4), \mu_{R2}(y_4, z_1)\} = \min(1, .4) = .4$
$\min\{\mu_{R1}(x_1, y_5), \mu_{R2}(y_5, z_1)\} = \min(.7, 0) = 0$
Hence $\mu_{R_1 \cdot R_2}(x_1, z_1) = \max\{.1, .2, 0, .4, 0\} = .4$

Similarly, we find $\mu_{R_1 \cdot R_2}$ for other combinations of (x_i, z_i) and arrive at

$$\tilde{R}_1 \cdot \tilde{R}_2 : \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} .4 & .7 & .7 & .7 \\ .3 & 1 & .5 & .8 \\ .8 & .3 & .7 & 1 \end{bmatrix} \end{array}$$

Similarly, for the max-product operation we have

For $x = x_1$, $z = z_1$, $y = y_i$, $i = 1, 2, \ldots, 5$
$\mu_{R1}(x_1, y_1) \cdot \mu_{R2}(y_1, z_1) = .1 \times .9 = .09$
$\mu_{R1}(x_1, y_2) \cdot \mu_{R2}(y_2, z_1) = .2 \times .2 = .04$
$\mu_{R1}(x_1, y_3) \cdot \mu_{R2}(y_3, z_1) = 0 \times .08 = 0$

$\mu_{R1}(x_4, y_4) \cdot \mu_{R2}(y_4, z_1) = 1 \times .4 = .4$
$\mu_{R1}(x_1, y_5) \cdot \mu_{R2}(y_5, z_1) = .7 \times 0 = 0$
Hence $\mu_{R_1 \cdot R_2}(x_1, z_1) = \max(.09, .04, 0, .4, 0) = .4$

After similarly performing the other operations we finally obtain

$$\begin{array}{r c} & \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \tilde{R}_1 \cdot \tilde{R}_2 : \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} .4 & .7 & .7 & 56 \\ .27 & 1 & .4 & .8 \\ .8 & .3 & .7 & 1 \end{bmatrix} \end{array}$$

For max-av composition

i	$=$	1	2	3	4	5
$\mu_{R_1}(x_1, y_i) + \mu_{R_2}(y_i, z_1)$	$=$	$(.1 + .9)$	$(.2 + .2)$	$(0 + .8)$	$(1 + .4)$	$(.7 + 0)$
	$=$	1	.4	.8	1.4	.7

Hence $$\frac{1}{2}\max_{y}\{\mu_{R_1}(x_1, y_i) + \mu_{R_2}(y_i + z_1)\} = \frac{1}{2} \times (1.4) = .7$$

After performing similar operations on remaining terms, we finally obtain

$$\begin{array}{r c} & \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \tilde{R}_1 \therefore \tilde{R}_2 : \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} .7 & .85 & .85 & .75 \\ .6 & 1 & .65 & .9 \\ .9 & .65 & .85 & 1 \end{bmatrix} \end{array}$$

6.2 Properties of max-min Composition

Since max-min composition is most widely used we enlist here some of its properties.

(i) **Associativity:** The max-min composition is associative, i.e. $(\tilde{R}_3 \cdot \tilde{R}_2) \cdot \tilde{R}_1 = \tilde{R}_3 \cdot (\tilde{R}_2 \cdot \tilde{R}_1)$

Hence, $\tilde{R}_1 \cdot \tilde{R}_1 \cdot \tilde{R}_1 = \tilde{R}_1^3$ defines the third power of a fuzzy relation.

(ii) **Reflexivity:** Let $\tilde{R}$ be a fuzzy relation in $X \times X$, then
$\tilde{R}$ is called '**reflexive**' if $\mu_R(x, x) = 1, \forall x \in X$
$\tilde{R}$ is called '**ε-reflexive**' if $\mu_R(x, x) \geq \in, \forall x \in X$
$\tilde{R}$ is called **weakly reflexive** if

$$\{\mu_R(x, y) \leq \mu_R(x, x) \text{ and } \mu_R(y, x) \leq \mu_R(x, x)\}, \quad \forall x, y \in X$$

7. Symmetric and Anti Symmetric Relations

A fuzzy relation $\tilde{R}$ is called **symmetric** if $\tilde{R}(x, y) = \tilde{R}(y, x)$.
It is called **'anti symmetric'** if

for $x \neq y$ either $\mu_R(x, y) \neq \mu_R(y, x)$ or $\mu_R(x, y) = \mu_R(y, x) = 0$, $\forall\, x, y \in X$

A relation is called **'perfectly anti symmetric'** if for $x \neq y$ whenever $\mu_R(x, y) > 0$, then $\mu_R(y, x) = 0$, $\forall\, x, y \in X$.

Example 10: Consider the fuzzy relations

$$\tilde{R}_1 : \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & .4 & 0 & .1 & .8 \\ x_2 & .8 & 1 & 0 & 0 \\ x_3 & 0 & .6 & .7 & 0 \\ x_4 & 0 & .2 & 0 & 0 \end{array} \qquad \tilde{R}_2 : \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & .4 & 0 & .7 & 0 \\ x_2 & 0 & 1 & .9 & .6 \\ x_3 & .8 & .4 & .7 & .4 \\ x_4 & 0 & .1 & 0 & 0 \end{array} \qquad \tilde{R}_3 : \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & .4 & .8 & .1 & .8 \\ x_2 & .8 & 1 & 0 & .2 \\ x_3 & .1 & .6 & .7 & .1 \\ x_4 & 0 & .2 & 0 & 0 \end{array}$$

Of these relations $\tilde{R}_1$ is perfectly anti symmetric while $\tilde{R}_2$ is anti-symmetric but not perfectly anti symmetric (why?). $\tilde{R}_3$ is a non symmetric relation, that is there exist $x, y \in X$ with $\mu_R(x, y) \neq \mu_R(y, x)$, and hence is not anti symmetric and therefore also not perfectly anti-symmetric.

Again let $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$, then the relation:

$$\tilde{R}(x, y) : \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0 & .1 & 0 & .1 \\ x_2 & .1 & 1 & .2 & .3 \\ x_3 & 0 & .2 & .8 & .8 \\ x_4 & .1 & .3 & .8 & 1 \end{array}$$

is a symmetric relation.

7.1 Transitivity

A fuzzy relation $\tilde{R}$ is called **(max-min) transitive** if $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$. ??

uuck.

Example 11: Let a fuzzy relation $\tilde{R}$ be defined as

$$\tilde{R} : \begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & .2 & 1 & .4 & .4 \\ x_2 & 0 & .6 & .3 & 0 \\ x_3 & 0 & 1 & .3 & 0 \\ x_4 & .1 & 1 & 1 & .1 \end{array}$$

Then

$$\tilde{R} \circ \tilde{R} : \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} .1 & .6 & .4 & .2 \\ 0 & .6 & .3 & 0 \\ 0 & .6 & .3 & 0 \\ .1 & 1 & .3 & .1 \end{bmatrix} \end{array}$$

It can be easily seen that $\mu_{R \circ R}(x, y) \leq \mu_R(x, y)$ holds for all $x, y \in X$. Hence, $\tilde{R}$ is max-min transitive.

It can easily be verified that for the max-min composition, the following properties hold:

(i) If $\tilde{R}_1$ is a reflexive and $\tilde{R}_2$ an arbitrary fuzzy relation then $\tilde{R}_1 \circ \tilde{R}_2 \supseteq \tilde{R}_2$ and $\tilde{R}_2 \circ \tilde{R}_1 \supseteq \tilde{R}_2$.

(ii) If $\tilde{R}$ is reflexive then $\tilde{R}_1 \circ \tilde{R} \subseteq \tilde{R} \circ \tilde{R}$.

(iii) If $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive relations then so is $\tilde{R}_1 \circ \tilde{R}_2$.

(iv) If $\tilde{R}_1$ and $\tilde{R}_2$ are symmetric then $\tilde{R}_1 \circ \tilde{R}_2$ is also symmetric i.e., $\tilde{R}_1 \cdot \tilde{R}_2 = \tilde{R}_1 \cdot \tilde{R}_2$.

(v) If $\tilde{R}$ is symmetric then so is each power of $\tilde{R}$.

(vi) If $\tilde{R}$ is symmetric and transitive then $\mu_R(x, y) \leq \mu_R(x, x)$ for all $x, y \in X$.

(vii) If $\tilde{R}$ is reflexive and transitive, then $\tilde{R} \circ \tilde{R} = \tilde{R}$.

(viii) If $\tilde{R}_1$ and $\tilde{R}_2$ are transitive and $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$ then $\tilde{R}_1 \circ \tilde{R}_2$ is transitive.

It may be noted that for the max-prod and max-average compositions properties (i) and (iii) hold. Property (iv) is true for any commutative operator.