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2K17/MC/055

PDE ASSIGNMENT-2

Ans1

a) $y^2 r - x^2 t = 0 \quad -(1)$

Comparing (1) with

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0$$

$\Rightarrow R = y^2, \; S = 0, \; T = -x^2$

$\Rightarrow S^2 - 4RT = -4y^2(-x^2) = 4x^2y^2$

its > 0 & hence ~~parabolic~~ hyperbolic everywhere except $x = 0$ & $y = 0$

The λ quadratic is

$$R\lambda^2 + S\lambda + T = 0$$

$\Rightarrow y^2\lambda^2 - x^2 = 0$

$\Rightarrow \lambda = \pm \frac{x}{y}$

$\Rightarrow$ Characteristic eqns are:-

$\frac{dy}{dx} = \frac{x}{y}$ & $\frac{dx}{dy} = -\frac{x}{y}$

$\Rightarrow x\,dx - y\,dy = 0 \;, \; y\,dx + x\,dy = 0$

$\Rightarrow x^2 + y^2 = c_1, \; x^2 - y^2 = c_2$ are

req families ie hyperbola & circles

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b) $x^2 r + 2xys + y^2 t = 0$

Comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$\Rightarrow R = x^2, S = 2xy, T = y^2$

$\Rightarrow S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$

$\Rightarrow$ its parabolic everywhere

The λ quadratic is

$$R\lambda^2 + S\lambda + T = 0$$

$$\Rightarrow x^2\lambda^2 + 2xy\lambda + y^2 = 0$$

$$\Rightarrow (x\lambda + y)^2 = 0$$

$$\Rightarrow \lambda = -\frac{y}{x}, -\frac{y}{x}$$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow x\,dy = y\,dx$

$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$

$\Rightarrow y = c_1 x$

ie straight line is req family

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A2

a) $$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

$\Rightarrow \quad r - t = 0$

On compari $Rr + Tt + Ss + f(x, y, z, p, q) = 0$

$\Rightarrow R = 1, S = 0, T = -1$

$\Rightarrow S^2 - 4RT \Rightarrow 4 > 0 \Rightarrow$ its hyperbolic

The λ quadratic eqn

$$R\lambda^2 + S\lambda + T = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

$\Rightarrow \frac{dy}{dx} = \pm 1 \quad \Rightarrow y + x = c_1$
$\quad \& \; y - x = c_2$

To reduce to canonical form

$u = y + x \quad \& \quad v = y - x$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \quad \& \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial u} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Illy $t = \frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$

Using r & t in given eqn

$$\boxed{\frac{\partial^2 z}{\partial u \partial v} = 0}$$

b) $y^2\left(\frac{\partial^2 z}{\partial x^2}\right) + x^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$

$\Rightarrow y^2 r + x^2 t = 0$

On comparing with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

$\Rightarrow R = y^2,\ T = x^2,\ S = 0$

$\Rightarrow S^2 - 4RT = -4x^2y^2 < 0$ & so the given eqn is elliptic

The λ quadratic eqn is

$$R\lambda^2 + S\lambda + T = 0$$

$$\Rightarrow y^2\lambda^2 + x^2 = 0$$

$$\Rightarrow \lambda = \pm \frac{xi}{y}$$

$\Rightarrow \frac{dy}{dx} + \frac{xi}{y} = 0 \qquad \frac{dy}{dx} - \frac{xi}{y} = 0$

$\Rightarrow y^2 + x^2 i = C_1, \quad y^2 - x^2 i = C_2$

In order to reduce to canonical form, we choose

$u = y^2 + x^2 i$ & $v = y^2 - x^2 i$

Let $\alpha = y^2$ & $\beta = x^2$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = 2x\frac{\partial z}{\partial \beta}$

Illy $q = 2y\frac{\partial z}{\partial \alpha}$

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$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{2\,\partial z}{\partial \beta} + 4x^2 \frac{\partial^2 z}{\partial \beta^2}$$

$$\text{Illy } t = \frac{2\,\partial z}{\partial \alpha} + 4y^2 \frac{\partial^2 z}{\partial \alpha^2}$$

Using this in given eqn

$$\boxed{\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}\right) = 0}$$

A3

(1) $y(x+y)r - y(x+y)s - xp - yq - z = 0$

On compari $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$R = y(x+y)$, $S = -y(x+y)$, $T = 0$

$\Rightarrow$ The λ quadratic eqn

$R\lambda^2 + S\lambda + T = 0$

$\Rightarrow y(x+y)\lambda^2 - y(x+y)\lambda = 0$

$\Rightarrow \lambda = 0, 1$

$\Rightarrow \frac{dy}{dx} = 0 \quad \& \quad \frac{dy}{dx} + 1 = 0$

$\Rightarrow x+y = c_1 \quad \& \quad y = c_2$

$\Rightarrow u = x+y \quad \& \quad y = v$

$p = \frac{\partial z}{\partial x} = \cdots \frac{\partial z}{\partial u} \qquad q = \frac{\partial z}{\partial v}, \quad r = \frac{\partial^2 z}{\partial u^2}$

$$s = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}$$

$$t = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

On substituting in given eqn we have

$$y(x+y)\left(\frac{-\partial^2 z}{\partial u \partial v}\right) - x\frac{\partial z}{\partial u} - y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0$$

$$\Rightarrow \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0$$

let $\frac{\partial z}{\partial v} = w - \frac{z}{v}$

then $\frac{\partial w}{\partial u} + \frac{w}{u} = 0$

$\Rightarrow wu = \phi(v)$

$\Rightarrow w = \frac{\phi(v)}{u}$

$$\Rightarrow \frac{\partial z}{\partial v} + \frac{z}{v} = \frac{1}{u}\phi(v)$$

$IF = e^{+\int \frac{1}{v} dv} = v$

$$\Rightarrow zv = \frac{1}{u}\int \phi(v)\, dv + \phi_2(u)$$

$$\Rightarrow z = \frac{1}{uv}\phi_1(v) + \frac{1}{v}\phi_2(u)$$

where $\phi_1(v) = \int \phi(v)$

$$\Rightarrow \boxed{z = \frac{1}{y(x+y)}\phi_1(v) + \frac{1}{y}\phi_2(x+y)}$$

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(11) $x^2r - y^2t + px - qy = x^2$

Compare with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

$R = x^2, \quad S = 0, \quad T = -y^2$

λ quad eqn $\Rightarrow$ $R\lambda^2 + S\lambda + T = 0$

$\Rightarrow x^2\lambda^2 - y^2 = 0$

$\Rightarrow \lambda = \pm \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = 0 \qquad \frac{dy}{dx} - \frac{y}{x} = 0$

$\Rightarrow \cancel{y + x = c_1} \qquad \cancel{y - x = c_2}$

$xy = c_1 \qquad x/y = c_2$

$\Rightarrow$ We can take $xy = u$, $\frac{x}{y} = v$

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}$

||ly $q = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}$

$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = y^2\left(\frac{\partial^2 z}{\partial u^2}\right) + \frac{2\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}$

$t = x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$

On substituting in given eqn:—

$4x^2\frac{\partial^2 z}{\partial u \partial v} = x^2 \Rightarrow \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{4}$

Integ wrt u $\Rightarrow \frac{\partial z}{\partial v} = \frac{u}{4} + f(v)$

In wrt v $\Rightarrow z = \frac{uv}{4} + \int f(v)\,dv + \phi(u)$

$$\Rightarrow \boxed{z = \frac{x^2}{4} + \psi(x,y) + \phi(x,y)}$$

(iii) $r - 4s + 4t = 0$

On compairs wrt $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

$\Rightarrow R = 1,\ S = -4,\ T = +4$

λ quad eqn $\Rightarrow R\lambda^2 + S\lambda + T = 0$

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow (\lambda - 2)^2 \Rightarrow \lambda = 2, 2$

$\Rightarrow \frac{dy}{dx} = -2 \Rightarrow y = -2x + c_1$

$\Rightarrow y + 2x = c_1$

We can take $u = y + 2x$

let $v = y$

then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2\frac{\partial z}{\partial u} + 0 = 2\frac{\partial z}{\partial u}$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2\partial z}{\partial u}\right) = 2\left[\frac{\partial\left(\frac{\partial z}{\partial u}\right)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\left(\frac{\partial z}{\partial u}\right)}{\partial v}\frac{\partial v}{\partial x}\right]$

$= 4\frac{\partial^2 z}{\partial u^2}$

$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

$= 2\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} \times 0$

$+ \frac{\partial^2 z}{\partial u \partial v} \times 2 + \frac{\partial^2 z}{\partial v^2} \times 0$

$= 2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right)$

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$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial y^2} + \frac{2\partial^2 z}{\partial u \partial y}$$

On substituting in given eqn.

$$\cancel{\frac{4\partial^2 z}{\partial u^2}} - 4\left(2\left(\cancel{\frac{\partial^2 z}{\partial u^2}} + \cancel{\frac{\partial^2 z}{\partial u \partial v}}\right)\right) + 4\left(\cancel{\frac{\partial^2 z}{\partial u^2}} + \frac{\partial^2 z}{\partial v^2} + \cancel{\frac{2\partial^2 z}{\partial u \partial y}}\right) = 0$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial v^2} = 0 \quad \text{is cannonical form}$$

Integr wrt v $\Rightarrow$ $\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0$

Integr wrt v $\Rightarrow$ $\frac{\partial z}{\partial v} = \phi(u)$

$$\Rightarrow \quad z = v\,\phi_1(u) + \phi_2(u)$$

$$\Rightarrow \quad \boxed{z = y\,\phi_1(y+2x) + \phi_2(y+2x)}$$

Ay $\quad q(yq+z)r - p(2yq+z)s + yp^2 t + p^2 q = 0$

Here Monge's subsidiary eqns are

$$q(yq+z)\,dp\,dy + yp^2\,dq\,dx + p^2 q\,dx\,dy = 0 \quad (1)$$

$$q(yq+z)(dy)^2 + p(2yq+z)\,dx\,dy + yp^2(dx)^2 = 0 \quad (2)$$

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(3) gives,

$$(q\,dy + p\,dx)[(yq+z)dy + yp\,dx] = 0$$

Hence (1), (2) can be written as

$$q\,dy + p\,dx = 0 \quad -(3)$$

$$(yq+z)dy + yp\,dx = 0 \quad -(4)$$

Using $dz = p\,dx + q\,dy$

(3) $\Rightarrow$ $dz = 0 \Rightarrow z = c_1$ — (5)

$\cancel{(4)}$ $\Rightarrow$ $q\,dy = -p\,dx$

$$(yq+z)dp - yp\,dq - pq\,dy = 0$$

$$\Rightarrow (yq+z)\,dp - p\,d(yq) = 0$$

$$\Rightarrow (yq+z)dp - p\,d(yq+z) = 0 \quad \text{as } dz = 0$$

$$\Rightarrow \frac{d(yq+z)}{yq+z} - \frac{dp}{p} = 0$$

$$\Rightarrow \frac{yq+z}{p} = c_2 \quad -(6)$$

From (5) & (6)

$$\frac{yq+z}{p} = \phi_1(z) \Rightarrow yq+z = p\phi_1(z) - (7)$$

Using $dz = p\,dx + q\,dy$

(4) $\Rightarrow$ $q\,dp - p\,dq = \frac{pq}{y}\,dy = 0 \Rightarrow \frac{yq}{p} = c_2$ — (8)

$$\Rightarrow \frac{qy}{p} = \phi_2(yz) \quad (8)$$

From (7) & (8), $p = z/(\phi_1(z) - \phi_2(yz))$

$$q = z\phi_2(yz)/y(\phi_1(z) - \phi_2(yz))$$

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Substituting in $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{z}{\phi_1(z) - \phi_2(yz)}\left(dx + \frac{1}{y}\phi_2(yz)\,dy\right)$$

$$\Rightarrow \frac{\phi_1(z)\,dz}{z} = dx + \phi_1(yz)\frac{d(yz)}{yz}$$

On Integrating

$$\boxed{\psi_1(x) = x + \psi_2(yz)}$$

A5

$$(q+1)s = (p+1)t$$

On Comparing with $Rr + Ss + Tt + \cancel{f(x,y,z,p,q)=0} = V$

$\Rightarrow R = 0$

$S = (q+1)$

$T = -(p+1)$

$V = 0$

Acc Monge's subsidiary eqns:-

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0$$

& $R(dy)^2 + T(dx)^2 - S\,dx\,dy = 0$

$\Rightarrow -(p+1)\,dq\,dx = 0$ & $-(q+1)\,dx\,dy - (p+1)(dx)^2 = 0$

$\Rightarrow dq = 0$

Integrate

$q = C_2$

$\hookrightarrow (1)$

$(q+1)dy + (p+1)dx = 0$

$\Rightarrow dz = -(dy + dx)$

Integrat

$\Rightarrow z = -x - y + C_1$

$\hookrightarrow (2)$

from (1) & (2), integral of given eqn is

$q = f(x + y + z) \rightarrow \frac{\partial z}{\partial y}$

Integrate partially w.r.t y

$$\boxed{z = F(x+y+z) + G(x)}$$

where F & G are arbitrary eqns.