PARTIAL DIFFERENTIAL EQUATIONS (MC-406)

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ASSIGNMENT - II

Q1) Find the characteristics of the following:-

a) $y^2 r - x^2 t = 0$

Using the characteristic equation:-

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

$R = y^2 \quad S = 0 \quad T = -x^2$

$\Rightarrow S^2 - 4rt = 0 + 4y^2x^2 = 4x^2y^2 > 0$

This equation is a hyperbolic curve with $f(x) > 0$ except when $x = 0$ or $y = 0$

λ quadratic is

$$R\lambda^2 + S\lambda + T = 0$$

$$y^2\lambda^2 - x^2 = 0$$

$$\lambda = \pm \frac{x}{y}$$

Characteristic equations are:-

$\frac{dy}{dx} + \frac{x}{y} = 0$

$x\,dx + y\,dy = 0$

$\int x\,dx + \int y\,dy = \int 0\,\cancel{dx}$

$\frac{x^2}{2} + \frac{y^2}{2} = A$

$x^2 + y^2 = 2A = C_1$

$x^2 + y^2 = C_1$

$\frac{dy}{dx} - \frac{x}{y} = 0$

$x\,dx - y\,dy = 0$

$\int x\,dx - \int y\,dy = 0$

$x^2 - y^2 = C_2$

$x^2+y^2=c_1$ & $x^2-y^2=c_2$ are required families of characteristic and they are hyperbolas.

b) $x^2 r + 2yxs + y^2 t = 0$

$R = x^2, \; S = 2xy, \; T = y^2$

$S^2 - 4RT = 0$

This is a parabolic equation, parabolic everywhere

λ quadratic $\rightarrow R\lambda^2 + S\lambda + T = 0$

$$x^2\lambda^2 + 2xy\lambda + y^2 = 0$$

solving, $(x\lambda + y)^2 = 0$

$$\lambda = -\frac{y}{x}, -\frac{y}{x}$$

The characteristic equations are:-

$$\frac{dy}{dx} + \left(-\frac{y}{x}\right) = 0 \Rightarrow \frac{1}{y}dy - \frac{1}{x}dx = 0$$

$$\int \frac{1}{y}dy - \int \frac{1}{x}dx = \int 0$$

$$\ln y - \ln x = A$$

$$\ln\left(\frac{y}{x}\right) = A$$

$$\frac{y}{x} = e^A = c_1 \Rightarrow y = c_1 x$$

characteristic equation represents family of straight lines passing through origin.

Q2) Reduce the following partial differential Equations:-

a) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

$R = 1, S = 0, T = -1$

λ quadratic $\Rightarrow \lambda^2 - 1 = 0$

$\lambda = \pm 1 \quad \lambda_1 = 1, \lambda_2 = -1$ (Real and distinct)

$\frac{dy}{dx} + 1 = 0 \qquad \frac{dy}{dx} - 1 = 0$

$y + x = c_1 \qquad y - x = c_2$

$u = x + y \qquad v = y - x \quad \text{—— (1)}$

We know

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \quad \text{—— (2)}$$

From (1), we have:-

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{—— (3)}$$

$$r = \frac{\partial^2 z}{\partial^2 u} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial u^2} - \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{—— (4)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{———} \quad (5)$$

Using (4) and (5), the canonical form is:-

$$\frac{\partial^2 z}{\partial^2 u} - \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) = 0$$

$$\boxed{\frac{\partial^2 z}{\partial u \partial v} = 0}$$

b) $y^2\left(\frac{\partial^2 z}{\partial x^2}\right) + x^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$

$R = y^2, \; S = 0, \; T = x^2 \Rightarrow S^2 - 4RT = -4x^2y^2 < 0 \; \forall x \neq 0, y \neq 0$

$S^2 - 4RT = -4x^2y^2 < 0 \; \forall x, y \neq 0$

This is an elliptical curve

Quadratic equation is

$$y^2\lambda^2 + x^2 = 0 \quad \text{or} \quad \lambda^2 = -x^2/y^2$$

$$\Rightarrow \lambda = ix/y, \; -ix/y \quad \text{———} \quad (1)$$

The characteristic equations are:-

$$\frac{dy}{dx} + \frac{ix}{y} = 0 \qquad\qquad \frac{dy}{dx} - \frac{ix}{y} = 0$$

$$\int y\,dy + \int ix\,dx = 0 \qquad\qquad \int y\,dy - \int ix\,dx = 0$$

$$y^2 + ix^2 = c_1 \qquad\qquad y^2 - ix^2 = c_2 \quad \text{———} \quad (2)$$

where $\alpha = y^2$ and $\beta = x^2$ ——— (3)

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = 2x\frac{\partial z}{\partial \beta} \quad \text{Using (3)} \quad \text{———— (4)}$$

Using (3), we get:-

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = 2y\frac{\partial z}{\partial \alpha} \quad \text{———— (5)}$$

$$r = \frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(2x\frac{\partial z}{\partial \beta}\right) = 2\frac{\partial z}{\partial \beta} + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= 2\frac{\partial z}{\partial \beta} + 2x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]$$

$$= 2\frac{\partial z}{\partial \beta} + 4x^2\frac{\partial^2 z}{\partial \beta^2} \quad \text{———— (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial \alpha}\right) = 2\frac{\partial z}{\partial \alpha} + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= 2\frac{\partial z}{\partial \alpha} + 2y\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right]$$

$$= 2\frac{\partial z}{\partial \alpha} + 4y^2\frac{\partial^2 z}{\partial \alpha^2} \quad \text{———— (7)}$$

Using (6) and (7):-

$$2y^2\frac{\partial z}{\partial \beta} + 4x^2y^2\frac{\partial^2 z}{\partial \beta^2} + 2x^2\frac{\partial z}{\partial \alpha} + 4x^2y^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

$$2\alpha\beta\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) + \alpha\frac{\partial z}{\partial \beta} + \beta\frac{\partial z}{\partial \alpha} = 0$$

$$\boxed{\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}\right) = 0}$$

Q3) Solve the following partial Differential Equations

a) $y(x+y)(r-s) - xp - yq - z = 0$

Comparing with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

we get $R = y(x+y)$ $S = -y(x+y)$, $T = 0$

The λ quadratic equation

$$R\lambda^2 + S\lambda + T = 0$$

$$y(x+y)\lambda^2 - y(x+y)\lambda = 0$$

$$\lambda y(x+y)[\lambda - 1] = 0$$

$$\lambda = 0, 1 \quad \text{——— (1)}$$

Characteristic equations are :-

$\frac{dy}{dx} = 0$

$y = c_1$

$\frac{dy}{dx} + 1 = 0$

$dy + dx = 0$

$\int dy + \int dx = \int 0$

$y + x = c_2$ ——— (2)

$$u = x + y, \quad v = y \quad \text{——— (3)}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial u}.1 + \frac{\partial z}{\partial v}.(0) = \frac{\partial z}{\partial u} \quad \text{— (4)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1)$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{——— (5)}$$

$$r = \frac{\partial^2 z}{\partial u^2}$$

$$s = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}$$

$$t = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting this in the given equation we get:-

$$y(x+y)\left(-\frac{\partial^2 z}{\partial u \partial v}\right) - x\frac{\partial z}{\partial u} - y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0$$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0$$

let $\frac{\partial z}{\partial v} = w - \frac{z}{v}$

then $\frac{\partial w}{\partial u} + \frac{w}{u} = 0$

$$wu = \phi(v)$$

$$w = \frac{\phi(v)}{u} \qquad \text{———} \qquad (6)$$

Substituting (6) :-

$$\frac{\partial z}{\partial v} + \frac{z}{v} = \frac{1}{u}\phi(v)$$

$$IF = e^{\int \frac{1}{v} dv} = v$$

$$zv = \frac{1}{u}\int \phi(v)\, dv + \phi_2(u)$$

$$z = \frac{1}{uv}\phi_1(v) + \frac{1}{v}\phi_2(u) \quad \text{where } \phi_1(v) = \int \phi(v)$$

$$z = \frac{1}{y(x+y)}\phi_1(v) + \frac{1}{y}\phi_2(x+y)$$

b) $x^2 r - y^2 t + px - qy = x^2$

Comparing with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

$$R = x^2,\ S = 0,\ T = -y^2$$

λ quadratic equation

$$R\lambda^2 + S\lambda + T = 0$$

$$x^2\lambda^2 - y^2 = 0$$

$$\lambda = y/x,\ -y/x \quad \text{——} \quad (1)$$

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

$$\cancel{\int y\,dy + \int x\,dx = 0}$$

$$\cancel{y^2 + x^2 = c_1}$$

$$\int \frac{dy}{y} + \int \frac{dx}{x} = 0$$

$$\ln y + \ln x = \ln c_1$$

$$\ln(yx) = \ln c_1$$

$$xy = c_1$$

$$\frac{dy}{dx} - \frac{y}{x} = 0$$

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

$$\int \frac{dy}{y} - \int \frac{dx}{x} = 0$$

$$\ln y - \ln x = \ln c_2$$

$$\frac{y}{x} = c_2$$

We can take $u = xy,\ v = \dfrac{x}{y}$ —— (2)

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = y^2\left(\frac{\partial^2 z}{\partial u^2}\right) + \frac{2\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u\partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}$$

$$t = x^2 \frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y}\frac{\partial^2 z}{\partial v^2}$$

substituting in the given equation:—

$$4x^2 \frac{\partial^2 z}{\partial u \partial v} = x^2 \Rightarrow \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{4}$$

Integrating w.r.t u

$$z = \frac{u}{4} + f(v)$$

Integrating w.r.t $v \Rightarrow$

$$z = \frac{uv}{4} + \int f(v)\,dv + \phi(u)$$

$$z = \frac{x^2}{4} + \phi(xy) + \psi\left(\frac{x}{y}\right)$$

6) $r - 4s + 4t = 0$

Comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$R = 1, S = -4, T = 4$

λ quadratic $\lambda^2 R + \lambda S + T = 0$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2, 2$$

$$\frac{dy}{dx} = -2$$

$$y = -2x + c_1$$

Let us take $u = y + 2x, v = y$

then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2\frac{\partial z}{\partial u} + 0 = 2\frac{\partial z}{\partial u}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2\frac{\partial z}{\partial u}\right) = 2\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial^2 z}{\partial u^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= 2\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}(0) + \frac{\partial^2 z}{\partial u \partial v}(2) + \frac{\partial^2 z}{\partial v^2}(0)$$

$$= 2\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right]$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial y^2} + \frac{2\partial^2 z}{\partial u \partial y}$$

On substituting in given equation, we get:-

$\frac{\partial^2 z}{\partial v^2} = 0$ is the cannonical form

Integrating w.r.t u

$$z = \frac{\partial z}{\partial u}\left(\frac{\partial z}{\partial v}\right) = 0$$

Integrating w.r.t v

$$\frac{\partial z}{\partial v} = \phi(u)$$

$$z = v\,\phi_1(u) + \phi_2(u)$$

$$z = y\,\phi_1(y+2x) + \phi_2(y+2x)$$

Q4) Solve $q(yq+z)r - p(2yq+z)s + yp^2t + p^2q = 0$ using Monge's method

Monge's subsidiary equations are :-

$$q(yq+z)\,dp\,dy + yr^2\,dp\,dx + p^2q\,dx\,dy = 0 \quad \text{—— (1)}$$

$$q(yq+z)(dy)^2 + p(2yq+z)\,dx\,dy + yp^2(dx)^2 = 0 \quad \text{—— (2)}$$

On factorizing (2), it gives us:-

$$(q\,dy + p\,dx)[4(yq+z)\,dy + yp\,dx] = 0$$

The 2 systems to be considered are :-

$$q(yq+z)\,dp\,dy + yp^2\,dq\,dx + p^2q\,dx\,dy = 0$$

$$q(yq+z)\,dp\,dy + yp^2\,dq\,dx + p^2q\,dx\,dy = 0$$

$$q\,dy + p\,dx = 0 \quad \text{—— (3)}$$

$$(yq+z)\,dy + yp\,dx = 0 \quad \text{—— (4)}$$

Using $dz = p\,dx + q\,dy$ the second equation of (3) reduces to:-

$$dz = 0 \implies z = c_1 \quad \text{—— (5)}$$

From second equation of (3) $q\,dy = -p\,dx$, Hence 1st equation of 3 reduces to

$(yq+z)\,dp - yp\,dq - pq\,dy = 0$ OR $(yp+z)(dp) - p\,d(yq) = 0$

$(yq+z)\,dp - p\,d(yq+z) = 0$ as $dz = 0$ ~~to~~ by (5)

$$\frac{d(yq+z)}{yq+z} - \frac{dp}{p} = 0$$

$$\int \frac{d(yq+z)}{d(yq+z)} - \int \frac{dp}{p} = 0$$

$$\ln(yq+z) - \ln p = A$$

$$\ln\left|\frac{yq+z}{p}\right| = A$$

$$\frac{yq+z}{p} = e^A = c_2,$$ where c_2 is arbitrary constant.

From (3), (5) and (6) we obtain the intermediate integral corresponding to $4(yq+z) = \phi_1(z)$

$$yp + z = p\,\phi_1(z) \quad \text{—— (7)}$$

Using $dz = p\,dx + q\,dy$, the second equation of (4) becomes

$y(q\,dy + p\,dx) + z\,dy = 0$

$y\,dz + z\,dy = 0$

$\int d(yz) = \int 0 \Rightarrow yz = c_3$ —— (8)

where c_3 is arbitrary constant.

Now, the first equation of (4) reduces to

$$q\,dp - p\,dq - \left(\frac{pq}{y}\right) dy = 0$$

$$-\frac{1}{p}dp + \frac{1}{q}dq + \frac{1}{y}dy = 0$$

Integrating, $-\log p + \log q + \log y = \log c$

$$\frac{yp}{q} = c_2 \quad \text{———} \quad (9)$$

From (8) and (9) another intermediate integral corresponding to (4) is :-

$$\frac{qy}{p} = \phi_2(yz) \quad \text{where } \phi_2 \text{ is arbitrary } f^n \quad \text{——} \quad (10)$$

solving (7) and (10) for p and q we get :-

$$p = \frac{z}{\phi_1(z) - \phi_2(zy)} \qquad q = \frac{z\,\phi_2(yz)}{y\{\phi_1(z) - \phi_2(yz)\}}$$

Substituting in $z = p\,dx + q\,dy$

$$dz = \frac{z}{\phi_1(z) - \phi_2(y)}\left[dx + \frac{1}{y}\phi_2(y_2)\,dy\right]$$

$$\phi_1(z)\,dz = z\,dx + \phi_2(yz) \cdot \frac{z\,dy - y\,dz}{y}$$

$$\frac{\phi_1(z)\,dz}{z} = dx + \frac{\phi_1(yz)\,d(yz)}{yz}$$

Integrating, we get :-

$\phi_1(z) = x + \phi_2(yz)$ where ϕ_1 and ϕ_2 are arbitrary functions.

Q5) Solve $(q+1)s = (p+1)t$

~~Comparing with Rr + Ss + Tt + f(x, y, z, p, q) = 0~~

Comparing with $Rr + Ss + Tt = V$

$R = 0 \quad S = q+1 \quad T = p+1 \quad V = 0$

As per monge's subsidiary equations :-

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0$$

$$R(dy)^2 + T(dx)^2 - S\,dx\,dy = 0$$

$-(p+1)\,dq\,dx = 0$

$\Rightarrow dq = 0$

$q = c_1$ —— (1)

$-(q+1)\,dx\,dy - (p+1)(dx)^2 = 0$

$(q+1)\,dy + (p+1)\,dx = 0$

$dz = -(dy + dx)$

$z = -x - y + c_1$

$z + x + y = c_1$ —— (2)

From (1) and (2) integral of given equation will be:-

$\phi = f(x+y+z) - \frac{\partial z}{\partial y}$

Integrating partially w.r.t y

$$\boxed{z = F(x+y+z) + G(x)}$$

Here G and F are arbitrary functions.