Ordering of Fuzzy Numbers

Defn :- A fuzzy Number is a convex normalized fuzzy set $\tilde{A}$ on $\mathbb{R}$ such that

(i) $\exists\ x_0 \in \mathbb{R},\ \mu(x_0) = 1$ (x_0 is the mean value of $\tilde{A}$)

(ii) $\mu_A(x)$ is piecewise cts

(Not cts in general)

Note :- Since standard addition and multiplication of fuzzy number is not possible, therefore min and max lattice operations are used to define the ordering of fuzzy numbers.

Defn :- let $\tilde{A}$ and $\tilde{B}$ be any two fuzzy Numbers with continuous membership fn $\mu_A(x)$ and $\mu_B(y)$ respectively. Then, the membership fn of

$\exists$ exist x_0 s.t $\mu_A(x_0) = \mu_B(x_0)$

$$\boxed{\begin{aligned} \min(\tilde{A},\tilde{B})(z) &= \sup_{z=\min(x,y)} \min\left[\mu_A(x), \mu_B(y)\right] \\ \max(\tilde{A},\tilde{B})(z) &= \sup_{z=\max(x,y)} \max\left[\mu_A(x), \mu_B(y)\right] \end{aligned}}$$

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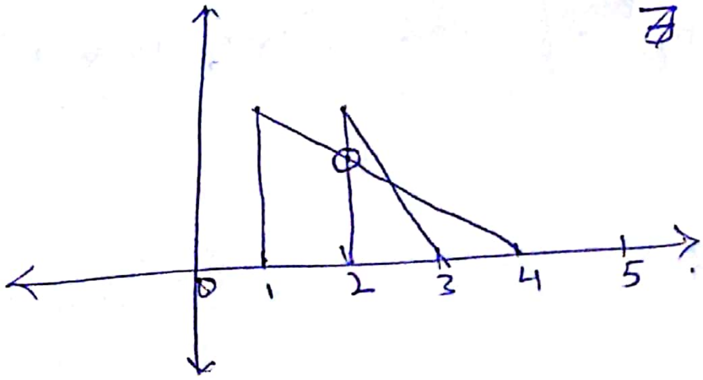
Ex $\tilde{A} = (-2, 1, 4)$ & $\tilde{B} = (1, 2, 3)$

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < -2 \\ \frac{x+2}{3} & -2 \le x \le 1 \\ \frac{4-x}{3} & 1 \le x \le 4 \\ 0 & x > 4 \end{cases} \quad \& \quad \mu_{\tilde{B}}(x) = \begin{cases} 0 & x < 1 \\ x-1 & 1 \le x \le 2 \\ 3-x & 2 \le x \le 3 \\ 0 & x > 3 \end{cases}$$

Then,

$$\text{MIN}(\tilde{A}, \tilde{B}) = \mu(x) \atop \min(\tilde{A}, \tilde{B}) = \begin{cases} 0 & x < -2 \text{ and } x > 3 \\ \frac{x+2}{3} & -2 \le x \le 1 \\ \frac{x-1}{3} & 1 \le x \le 2.5 \\ 3-x & 2.5 \le x \le 3 \end{cases}$$



$\nexists$ To search for a point for which $\underline{\mu_A(x_0) = \mu_B(x_0)}$.

$$\frac{4-x}{3} = 3-x.$$

$$\Rightarrow 4-x = 9-3x$$

$$\Rightarrow 2x = 5$$

$$\Rightarrow \boxed{x = 2.5}$$

$$\text{MAX}(\tilde{A}, \tilde{B}) = \mu(x) = \begin{cases} 0 & x < 1 \;\&\; x > 4 \\ x-1 & 1 \le x \le 2 \\ 3-x & 2 \le x \le 2.5 \\ \frac{4-x}{3} & 2.5 \le x \le 4 \end{cases}$$

Check?

To overcome the above limitation, fuzzy Number is determined by α-cut

$$[A]^\alpha = [a^1(\alpha), a^2(\alpha)], \quad 0 \le \alpha \le 1$$

The method to implement the operation of MIN and MAX for fuzzy NOs using α-cut-representation

Thm :- let A_i, $i = 1, 2, \dots, n$ be fuzzy NOs with $[A_i]^\alpha = [a_i^{1}(\alpha), a_i^{2}(\alpha)]$, $a_i^{(1)} \le a_i^{(2)}$

Then, the operation MIN and MAX can be implemented as

$$[MIN(A_1, A_2, \dots, A_n)]^\alpha = [\min_{1 \le i \le n} a_i^{(1)}(\alpha), \min_{1 \le i \le n} a_i^{(2)}(\alpha)]$$

$$[MAX(A_1, A_2, \dots, A_n)]^\alpha = [\max_{1 \le i \le n} a_i^{(1)}(\alpha), \max_{1 \le i \le n} a_i^{(2)}(\alpha)]$$

5. Fuzzy Equations

Fuzzy equations are equations in which coefficients and/or unknowns are fuzzy numbers and these have been constructed by operations of fuzzy arithmetic. Unfortunately their theory is still not sufficiently developed. In this section we discuss simple fuzzy equations of the type $\tilde{A} \oplus \tilde{X} = \tilde{B}$ and $\tilde{A} \odot \tilde{X} = \tilde{B}$.

5.1 Solution of Fuzzy Equations of Type $\tilde{A} \oplus \tilde{X} = \tilde{B}$ and $\tilde{A} \ominus \tilde{X} = \tilde{B}$

Unlike corresponding crisp equation $a + x = b$ whose solution is $x = b - a$, we should not rush to the conclusion that the solution of $\tilde{A} \oplus \tilde{X} = \tilde{B}$ is $\tilde{X} = \tilde{B} \ominus \tilde{A}$. The reason being that unlike the system of real numbers where $a + (-a) = 0$, in the case of fuzzy numbers $\tilde{A} \oplus (-\tilde{A})$ need not be neutral or null corresponding to zero of real numbers. In such cases, therefore, the best way to solve such type of equations is to first perform $\oplus$ operation on the left side and then equate the resulting fuzzy number on the left with the fuzzy number $\tilde{B}$ on the right to obtain the value of unknown fuzzy number $\tilde{X}$.

Example 15: Find $\tilde{X}$ satisfying $\tilde{A} \oplus \tilde{X} = \tilde{B}$, where $\tilde{A} = (-2, -1, 3)$ and $\tilde{B} = (1, 5, 10)$ are triangular fuzzy numbers.

Let $\tilde{X} = (x_1, x_2, x_3)$ then $\tilde{A} \oplus \tilde{X} = \tilde{B}$ implies

$$(-2, -1, 3) \oplus (x_1, x_2, x_3) = (1, 5, 10)$$

or

$$(-2 + x_1, -1 + x_2, 3 + x_3) = (1, 5, 10)$$

or $-2 + x_1 = 1, -1 + x_2 = 5, 3 + x_3 = 10$ yielding $x_1 = 3, x_2 = 6, x_3 = 7$

So that $X = (3, 6, 7)$.

Same approach can be used to solve $\tilde{A} \ominus \tilde{X} = \tilde{B}$ or $\alpha \tilde{A} \oplus \beta \tilde{X} = \tilde{B}$, or $\alpha \tilde{A} \ominus \beta \tilde{X} = \tilde{B}$, α, β real scalars.

In the case of numbers other than triangular and trapezoidal, crisp interval approach based on α-level sets may be used.

It is worth noting that a fuzzy equation may or may not have a solution.

For example if we want to solve $\tilde{A} \oplus \tilde{X} = \tilde{B}$, where $\tilde{A} = (-2, -1, 3)$, $\tilde{B} = (1, 3, 5)$.

Then $\tilde{A} \oplus \tilde{X} = \tilde{B} \Rightarrow (-2 + x_1, -1 + x_2, 3 + x_3) = (1, 3, 5) \Rightarrow -2$, This yields $-2 + x_1 = -1, -1 + x_2 = 3$ and $3 + x_3 = 5$ so that $x_1 = 1, x_2 = 4, x_3 = 2$ resulting in a triangular fuzzy number $(1, 4, 2)$ which does not exist.

5.2 Solution of Fuzzy Equations of the type $\tilde{A} \odot \tilde{X} = \tilde{B}$, $\tilde{A} \oslash \tilde{X} = \tilde{B}$

Same approach, as recommended for $\tilde{A} \oplus \tilde{X} = \tilde{B}$ and $\tilde{A} \ominus \tilde{X} = \tilde{B}$, can be adopted in this case also. In other words, first perform the specified operation on the left hand side to obtain a fuzzy number and then equate this to $\tilde{B}$. However, the solution in the present case is all the more difficult since $\odot$ and $\oslash$ operations even when performed on triangular fuzzy numbers do not yield triangular fuzzy numbers as in the case of $\oplus$ and $\ominus$ operations. In such cases it is generally preferable to write the equation in terms of crisp intervals using α-level sets and then proceed.

Example 16: Find $\tilde{X}$ such that $\tilde{A} \odot \tilde{X} = \tilde{B}$, where $\tilde{A} = (1, 3, 4)$ and $\tilde{B} = (2, 12, 48)$.

Writing in terms of α-level sets we obtain:

$\tilde{A}_\alpha \cdot \tilde{B}_\alpha = [1 + 2\alpha, 4 - \alpha] \cdot [x_1 + \alpha(x_2 - x_1), x_3 + \alpha(x_2 - x_3)] = [2 + 10\alpha, 48 - 36\alpha]$
$\Rightarrow \min[(1 + 2\alpha) \cdot (x_1 + \alpha(x_2 - x_1)), (1 + 2\alpha) \cdot (x_3 + \alpha(x_2 - x_3)), (4 - \alpha) \cdot (x_1 + \alpha(x_2 - x_1)), (4 - \alpha)(x_3 + \alpha(x_2 - x_3)] = 2 + 10\alpha, 0 \le \alpha \le 1.$

and $\max[(1 + 2\alpha) \cdot (x_1 + \alpha(x_2 - x_1)), (1 + 2\alpha) \cdot (x_3 + \alpha(x_2 - x_3)), (4 - \alpha) \cdot (x_1 + \alpha(x_2 - x_1)), (4 - \alpha) \cdot (x_3 + \alpha(x_2 - x_3)] = 48 - 36\alpha, 0 \le \alpha \le 1$

These yield

for $\alpha = 0$, $\min\{x_1, x_3, 4x_1, 4x_3) = 2 \Rightarrow x_1 = 2$
and $\max\{x_1, x_3, 4x_1, 4x_3) = 48 \Rightarrow 4x_3 = 48$ or $x_3 = 12$
for $\alpha = 1$ $\min 3x_2 = 12 \Rightarrow x_2 = 4$

If required spreads for other values of a between 0 and 1 can be found. Or solution may be approximated to triangular fuzzy number $\tilde{X} = (2, 4, 12)$.

Example 17: Solve $\tilde{A} \oslash \tilde{X} = \tilde{B}$, given $\tilde{A} = (4, 6, 10)$ and $\tilde{B} = (2, 4, 20)$.

$\tilde{A} \oslash \tilde{X} = \tilde{B} \Rightarrow (4, 6, 10) \oslash (x_1, x_2, x_3) = (2, 4, 20)$

$$\text{or } [4+2\alpha, 10-4\alpha] \cdot \left[\frac{1}{x_3} + \alpha\left(\frac{1}{x_2} - \frac{1}{x_3}\right), \frac{1}{x_1} + \alpha\left(\frac{1}{x_2} - \frac{1}{x_1}\right)\right] = [2+2\alpha, 20-4\alpha]$$

For $\alpha = 0$

$$[4, 10] \cdot \left[\frac{1}{x_3}, \frac{1}{x_1}\right] = [2, 10]$$

$$\Rightarrow \quad \left[\text{Min}\left(\frac{4}{x_3}, \frac{4}{x_1}, \frac{10}{x_3}, \frac{10}{x_4}\right), \text{Max}\left(\frac{4}{x_3}, \frac{4}{x_4}, \frac{10}{x_3}, \frac{10}{x}\right)\right] = [2, 20]$$

For $x_3 > x_1$, this implies

$$\left[\frac{4}{x_3}, \frac{10}{x_1}\right] = [2, 20] \Rightarrow \frac{4}{x_3} = 2 \quad \text{and} \quad \frac{10}{x_1} = 20$$

Thus, $x_3 = 2$, $x_1 = ½$

$$\text{For } \alpha = 1, \; 6.\frac{1}{x_2} = 4 \Rightarrow x_2 = \frac{3}{2}$$

Thus, $\tilde{X} \approx (1/2, 3/2, 2)$.

5.3 Conditions for Existence of Solutions

It may be kept in view that in the majority of situations, equations $\tilde{A} \odot \tilde{X} = \tilde{B}$ and $\tilde{A} \oplus \tilde{X} = \tilde{B}$ may not have realistic solutions as the requirement $x_1 \le x_2 \le x_3$ is more likely be violated in most of the obtained solutions.

In certain cases even equations such as $\tilde{A} \oplus \tilde{X} = \tilde{B}$ and $\tilde{A} \ominus \tilde{X} = \tilde{B}$ may also not admit solution. In this section we consider conditions for the existence of the solution of these fuzzy algebraic equations.

Given $\tilde{A}$ (a_1, a_2, a_3), $\tilde{B} = (b_1, b_2, b_3)$, as triangular fuzzy numbers

(i) $\tilde{A} \oplus \tilde{X} = \tilde{B} \Rightarrow (a_1 + x_1, a_2 + x_2, a_3 + x_3) = (b_1, b_2, b_3)$ i.e. $x_1 = b_1 - a_1$, $x_2 = b_2 - a_2$, $x_3 = b_3 - a_3$. So that solution will exist only if $b_1 - a_1 \le b_2 - a_2 \le b_3 - a_3$.

Similarly

(ii) $\tilde{A} \ominus \tilde{X} = \tilde{B} \Rightarrow (a_1 - x_3, a_2 - x_2, a_3 - x_1) = (b_1, b_2, b_3)$, i.e. $x_3 = a_1 - b_1$, $x_2 = a_2 - b_2$, $x_1 = a_3 - b_3$.

Hence the solution will exist only if $a_3 - b_3 \le a_2 - b_2 \le a_1 - b_1$.

(iii) $\tilde{A} \odot \tilde{X} = \tilde{B} \Rightarrow$ for $\alpha = 0$: $[a_1, a_3] \cdot [x_1, x_3] = [b_1, b_3]$ and for $\alpha = 1$, $a_2 \cdot x_2 = b_2$, $\Rightarrow x_1 = b_1/a_1$, $x_2 = b_2/a_2$, $x_3 = b_3/a_3$. So the solution will exist only if $b_1/a_1 \le b_2/a_2 \le b_3/a_3$.

(iv) $\tilde{A} \odot \tilde{X} = \tilde{B} \Rightarrow$ for $\alpha = 0$: $[a_1, a_3] \cdot [1/x_3, 1/x_1] = [b_1, b_3]$ and for $\alpha = 1$, $a_2/x_2 = b_2$.

Thus $x_3 = a_1/b_1$, $x_2 = a_2/b_2$, $x_1 = a_3/b_3$.

Hence the solution will exist only if $a_3/b_3 \le a_2/b_2 \le a_1/b_1$

Solution of $\tilde{A} \oplus \tilde{X} \subset \tilde{B}$ or $\tilde{A} \oplus \tilde{X} \supset \tilde{B}$ or $\tilde{A} \odot \tilde{X} \subset \tilde{B}$ or $\tilde{A} \odot \tilde{X} \supset \tilde{B}$ can also be similarly found. However, as shown above, it is only in very specific cases that solution to such fuzzy equations exist and can be found. Even in cases where the conditions stated above are satisfied, there is no guarantee that in the case of solution to the equation $\tilde{A} \odot \tilde{X} = \tilde{B}$ and $\tilde{A} \odot \tilde{X} = \tilde{B}$, the membership functions will satisfy convexity conditions although the approximated triangular fuzzy number will satisfy this requirement.

For more details interested reader may refer to Kaufmann and Gupta (Fuzzy Mathematical Models in Engineering and Management Science, Chapters 4 and 7).