Real Analysis
Solution Set

Methods of Real Analysis

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Contents

1 Sets and Functions

at12.0pt at12.0pt**1.1 Exercise 1.1**

1. Describe the following sets of real numbers geometrically :

   (a) $A = \{x|\ \ x < 7\}$

   (b) $B = \{x|\ \ |x| \geq 2\}$

   (c) $C = \{x|\ \ |x| = 1\}$

2. Describe the following sets of points in the plane geometrically:

   (a) $A = \left\{\langle x, y\rangle \mid\ x^2 + y^2 = 1\right\}$

   (b) $B = \{\langle x, y\rangle \mid\ x \leq y\}$

   (c) $C = \{\langle x, y\rangle \mid\ x + y = 2\}$

3. Let P be the set of prime Integers, which of the following are true ?

   (a) $7 \in P$

   (b) $9 \in P$

   (c) $11 \notin P$

1.2 Exercise 1.2

1.3 Exercise 1.3

1.4 Exercise 1.4

1.5 Exercise 1.5

1.6 Exercise 1.6

1.7 Exercise 1.7

2 Sequences Of Real Numbers

2.1 Exercise 2.1

1. Let $\{s_n\}_{n=1}^{\infty}$ be the sequence defined by

$$s_1 = 1$$

$$s_2 = 1$$

$$s_{n+1} = s_n + s_{n-1} \; (n = 3, 4, 5, \cdots)$$

Find s_8

$$\begin{aligned} s_3 &= s_2 + s_1 \\ s_3 &= 1 + 1 \\ s_3 &= 2 \\ s_4 &= s_3 + s_2 \\ s_4 &= 2 + 1 \\ s_4 &= 3 \\ s_5 &= s_4 + s_3 \\ s_5 &= 3 + 2 \\ s_5 &= 5 \\ s_6 &= s_5 + s_4 \\ s_6 &= 5 + 3 \\ s_6 &= 8 \\ \text{and so on we get} \cdots \\ s_8 &= 21 \end{aligned}$$

2. Write a formula or formulae for s_n for each of the following sequences.

(a) 1,0,1,0...
$s_n = 1 \,\forall\, \mathrm{N} \in \mathbb{I}$ where $n = 2\mathrm{N} - 1$
$s_n = 0 \,\forall\, \mathrm{N} \in \mathbb{I}$ where $n = 2\mathrm{N}$

(b) 1,3,6,10,15...
$s_n = s_{n-1} + n \,\forall n \in \mathbb{N}$

(c) 1,-4,9,-16,25,-36...
$s_n = (-1)^{n+1} n^2 \,\forall n \in \mathbb{N}$

(d) 1,1,1,2,1,3,14,1,5,1,6...
$s_n = 1 \,\forall\, \mathrm{N} \in \mathbb{I}$ where $n = 2\mathrm{N} - 1$
$s_n = \frac{n}{2} \,\forall\, \mathrm{N} \in \mathbb{I}$ where $n = 2\mathrm{N}$

3. Which of the following sequences (a), (b), (c) and (d) in the previous exercise are subsequences of $\{\text{n}\}_{n=1}^{\infty}$?

   The sequences (a), (b) and (d) are subsequences of $\{\text{n}\}_{n=1}^{\infty}$

4. If $\text{S} = \{s_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty}$ and $\text{N} = \{n_i\}_{n=1}^{\infty} = \left\{i^2\right\}_{i=1}^{\infty}$. Find s_5, s_9, n_2, s_{n_3}. Is N a subsequence of $\{\text{k}\}_{k=1}^{\infty}$?

$$
\begin{aligned}
s_5 &= 2 \cdot 5 - 1 \\
s_5 &= 9 \\
s_9 &= 2 \cdot 9 - 1 \\
s_9 &= 17 \\
n_2 &= 2^2 = 4 \\
s_{n_3} &= 2 \cdot n_3 - 1 \\
\text{We know } n_3 &= 3^2 = 9 \\
s_{n_3} &= s_9 = 17
\end{aligned}
$$

   Now $\{k\}_{k=1}^{\infty} = \{1, 2, 3, 4 \cdots\}$ and the sequence $\text{N} = \left\{i^2\right\}_{i=1}^{\infty} = \{1, 4, 9 \cdots\}$ is clearly a subsequence.

2.2 Exercise 2.2

1. If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, if $s_n \leqslant$ M $(n \in \mathbb{I})$ and if $\lim\limits_{n\to\infty} =$ L. Prove L $\leqslant$ M

$$\begin{aligned} \text{We Know } s_n &\leqslant \text{M} \ \forall \, \text{n} \in \mathbb{I} \\ \text{and} \lim_{n\to\infty} s_n &= \text{L} \\ \text{Now, L } &\leqslant \max \ (s_n) \\ \text{and} \max (s_n) &\leqslant \text{M} \\ \text{So, L} &\leqslant \text{M} \end{aligned}$$

2. If L $\in \mathbb{R}$, M $\in \mathbb{R}$ and L $\leq$ M $+\epsilon$ for every $\epsilon \geq 0$, prove that L $\leq$ M

$$\begin{aligned} \text{We Know}, |s_n| &\leq \text{M} \ \forall n \in N \\ \text{L} &\leq |s_n| \ \ \forall n \in N \\ \text{Now we know that, } L &\ngtr |s_n| \\ \text{Hence, } L &\leq M \end{aligned}$$

3. If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if, for every $\epsilon > 0$,
$|s_n - L| < \epsilon \ (n \geq \text{N})$
where N does not depend on ϵ, prove that all but a finite number of terms of $\{s_n\}_{n=1}^{\infty}$ are equal to L.

4. (a) find N $\in$ I such that

$$\left|\frac{2n}{n+3} - 3\right| < \frac{1}{5} \ (n \geq N)$$

$$\begin{aligned} \left|\frac{2n}{n+3} - 3\right| &< \frac{1}{5} && (1) \\ \left|\frac{2n - 3(n+3)}{n+3}\right| &< \frac{1}{5} \\ \left|\frac{-n-9}{n+3}\right| &< \frac{1}{5} \\ \left|\frac{n+9}{n+3}\right| &< \frac{1}{5} \\ \frac{-1}{5} < \left|\frac{n+9}{n+3}\right| &< \frac{1}{5} && (2) \end{aligned}$$

From (2)

$$\frac{n+9}{n+3} < \frac{1}{5}$$
$$5(n+9) < n+3$$
$$5n+45 < n+3$$
$$4n+42 < 0$$
$$2n+21 < 0$$
$$n < -21/2 \tag{3}$$

Now according to (3), we have to find n $< -21/2$ which isn't possible.
Hence for no N $\in$ I is equation (1) satisfied.

(b) Prove $\lim\limits_{n\to\infty} \frac{2n}{n+3} = 2$
To prove : $\{\frac{2n}{n+3}\}_{n=1}^{\infty} = 2$
Let Limit L = 2

$$\left|\frac{2n}{n+3} - L\right| < \epsilon$$
$$\left|\frac{2n}{n+3} - 2\right| < \epsilon \tag{1}$$
$$\left|\frac{2n-2(n+3)}{n+3}\right| < \epsilon$$
$$\left|\frac{6}{n+3}\right| < \epsilon$$
$$\frac{6}{n+3} < \epsilon$$
$$\frac{n+3}{6} < \frac{1}{\epsilon}$$
$$n < \frac{6}{\epsilon} - 3 \tag{2}$$

From (2) we can see that for any positive N $\in \mathbb{I}$, where N $> \frac{6}{\epsilon} - 3$
$\forall n >$ N the equation (1) is satisfied
Hence, 2 is the limit of the function:
$\{\frac{2n}{n+3}\}_{n=1}^{\infty}$

5. (a) Find $N \in I$ such that $\frac{1}{\sqrt{n+1}} < 0.3$ when $n > N$

$$\frac{1}{\sqrt{n+1}} < 0.3 \tag{1}$$
$$\sqrt{n+1} > \frac{1}{0.3}$$
$$n+1 > \frac{1}{0.3^2}$$
$$n+1 > \frac{1}{0.09}$$
$$n > 11.\overline{11} - 1$$
$$n > 10.\overline{11} \tag{2}$$

So, for any N $\in \mathbb{I}$ where N is positive and N $> 10.\overline{11}$
and n $>$ N, the equation (1) is satisfied

(b) Prove that $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$
To prove: limit L = 0
Let the limit of the function as $n \to \infty$ be L = 0
Now,

$$\left|\frac{1}{\sqrt{n+1}} - L\right| < \epsilon \;\; (\epsilon > 0) \tag{1}$$

$$\left|\frac{1}{\sqrt{n+1}} - 0\right| < \epsilon$$

$$\left|\frac{1}{\sqrt{n+1}}\right| < \epsilon \tag{2}$$

We need to find N $\in \mathbb{I}$ such that N is positive and equation (1) is satisfied $\forall n \geq$ N

$$\frac{1}{\sqrt{n+1}} < \epsilon$$

$$\sqrt{n+1} > \frac{1}{\epsilon}$$

$$n+1 > \frac{1}{\epsilon^2}$$

$$n > \frac{1}{\epsilon^2} - 1 \tag{3}$$

Now let us take a N such that N $> \frac{1}{\epsilon^2} - 1$. So for all n $\geq$ N the equation (1) is satisfied.
Hence the limit of $\{\frac{1}{\sqrt{n+1}}\}_{n=1}^{\infty}$ is 0.

6. If θ is a rational number prove that the sequence $\{\sin n\theta_\pi\}_{n=1}^{\infty}$ has a limit.

7. For each of the following sequences, prove either that the sequence has a limit or that the sequence does not have a limit.

(a) $\{\frac{n^2}{n+5}\}_{n=1}^{\infty}$
Let the sequence have a limit L such that $\lim_{n\to\infty}\{\frac{n^2}{n+5}\} = L$
Now if the function has a limit, it would also satisfy the equation :

$$\left|\frac{n^2}{n+5} - L\right| < \epsilon \;\; (\forall n \geq N) \text{ and } (\epsilon > 0) \tag{1}$$

$$\left|\frac{n}{1+5/n} - L\right| < \epsilon \tag{2}$$

Dividing Numerator and denominator by n

$$1 + 5/n > 1$$
$$\frac{1}{1+5/n} < 1$$
$$\frac{n}{1+5/n} < n \tag{3}$$

Substituting the value of (3) in (2)

$$|n - L| < \epsilon \tag{4}$$

We can now clearly see that for any value of ϵ $|n - L|$ would be greater than that of ϵ, hence limit does not exist and the function diverges

$\lim\limits_{n\to\infty} \frac{n^2}{n+5} = +\infty$

(b) $\{\frac{3n}{n+7^{1/2}}\}_{n=1}^{\infty}$
Let us suppose that the limit of the funtion exists and it's value be L, that is
$\{\frac{n}{n+7^{1/2}}\}_{n=1}^{\infty} = L$
Now if the limit exists, the function will satisfy the equation

$$\left|\frac{n}{n+7^{1/2}} - L\right| < \epsilon \ \ (\forall n \geq \text{N}) \ \text{ and } \ (\epsilon > 0) \tag{1}$$
$$\left|\frac{1}{1+\frac{7^{1/2}}{n}} - L\right| < \epsilon \tag{2}$$

Dividing both numerator and denominator by n

$$1 + \frac{7^{1/2}}{n} > 1$$
$$\frac{1}{1+\frac{7^{1/2}}{n}} < 1 \tag{3}$$

Substituting the value of equation (3) in (2)

$$|1 - L| < \epsilon \tag{4}$$

We can clearly find $\epsilon > 0$ such that equation (4) is satisfied and hence the equation (1) will also be satisfied.
So there exists N $\in \mathbb{I}$ such that $\forall$ n $\geq$ N eq. (1) is satisfied.
$\lim\limits_{n\to\infty} \frac{n}{n+7^{1/2}} = 3$

(c) $\{\frac{3n}{n+7n^2}\}_{n=1}^{\infty}$
Let the limit of the function exist and let it be L.
If the limit exists then the function will also satisfy the equation

$$\left|\frac{3n}{n+7n^2} - L\right| < \epsilon \ \ (\forall n \geq N) \tag{1}$$

where N $\in \mathbb{I}$ and $\epsilon > 0$

$$\left|\frac{3}{1+7n} - L\right| < \epsilon \tag{2}$$

Dividing numerator and denominator by n

$$\begin{aligned} 1+7n &> 7n \\ \frac{1}{1+7n} &< \frac{1}{7n} \\ \frac{3}{1+7n} &< \frac{3}{7n} \end{aligned} \tag{3}$$

Substituting equation (3) into (2)

$$\left|\frac{3}{7n} - L\right| < \epsilon \tag{4}$$

We can see that this resembles the series $\{\frac{1}{n}\}$ and hence can assume L=0 in equation (4)

$$\left|\frac{3}{7n}\right| < \epsilon \tag{5}$$

$$\frac{3}{7n} < \epsilon$$

$$\frac{7n}{3} > \frac{1}{\epsilon}$$

$$n > \frac{3}{7\epsilon} \tag{6}$$

Now let us take N $\in \mathbb{I}$ where N $> \frac{3}{7\epsilon}$ for any $\epsilon > 0$.
So for n $\geq$ N the equation (6) is satisfied and in conclusion equation (1) is also satisfied. So the limit for the function $\frac{3n}{n+7n^2}$ exists.

8. (a) Prove that the sequence $\{10^7/n\}_{n=1}^{\infty}$ has a limit 0.
To prove that the function has a limit L =0
$\lim\limits_{n\to\infty} \frac{10^7}{n} = 0$
Let the limit of the function exist and let $L = 0$

So the function would satisfy the equation

$$\left|\frac{10^7}{n} - L\right| < \epsilon \;\; (\forall n \geq \mathrm{N}) \tag{1}$$

where $\epsilon > 0$ and $\mathrm{N} \in \mathbb{I}$ and $\mathrm{N} > 0$

$$\left|\frac{10^7}{n}\right| < \epsilon \tag{2}$$

$$\frac{10^7}{n} < \epsilon$$

$$\frac{n}{10^7} > \frac{1}{\epsilon}$$

$$n > \frac{10^7}{\epsilon} \tag{3}$$

If we take $\mathrm{N} > \frac{10^7}{\epsilon}$, we will get $\forall$ n $\geq$ N and the equation (1) will be satisfied
Hence the limit for the function exists
$\lim\limits_{n\to\infty} \frac{10^7}{n} = 0$

(b) Prove that $\{n/10^7\}_{n=1}^{\infty}$ does not have a limit.
To prove: We have to prove that limit for $\{\frac{n}{10^7}\}_{n=1}^{\infty}$ doesn't exist.
Let us assume that the limit of the function exists and it is L.
As teh Limit L exists the function will satisfy the equation :

$$\left|\frac{n}{10^7} - L\right| < \epsilon \;\; \forall n \geq N \tag{1}$$

Here L is the limit and N>0. $\mathrm{N} \in \mathbb{I}$ and ϵ is an arbitrary positive rational number.

Now as the limit exists we would be able to find some N for any arbitrary ϵ where equation (1) is satisfied.
Let $\epsilon = 1$

$$\left|\frac{n}{10^7} - L\right| < 1$$

$$\frac{n}{10^7} \in (L-1, L+1)$$

$$n \in (10^7(L-1), 10^7(L+1)) \tag{2}$$

For any Limit value L that we take, we can find a value for n where equation (2) is not satisfied.
Hence our assumption that this function converges was wrong.
The series $\{\frac{n}{10^7}\}_{n=1}^{\infty}$ clearly diverges

(c) Note that the first 10^7 terms of the sequence in (a) are greater than the corresponding terms in sequence (b. This emphasizes that the existence of a limit for a sequence does not depend on the first few ('few' = any finite number of terms) terms.

9. Prove that $\{n - 1/n\}_{n=1}^{\infty}$ does not have a limit.
   To prove : that series $\{n - 1/n\}_{n=1}^{\infty}$ is divergent
   We know that:

$$n - 1/n > \frac{n}{2} \ \ (\forall n \in \mathbb{N}) \ \text{ and } \ n > 1 \tag{1}$$

   Now from (1) we can infer that if the sequence $\{\frac{n}{2}\}_{n=1}^{\infty}$ is divergent then the given sequence $\{n-1/n\}_{n=1}^{\infty}$ will also diverge.
   We have previously proved that the sequence $\{n\}_{n=1}^{\infty}$ is divergent.
   Hence we can also state that $\{\frac{n}{2}\}_{n=1}^{\infty}$ diverges using the property that

$$\lim_{n\to\infty} \{c \cdot s_n\} = \lim_{n\to\infty} c\,\{s_n\} \tag{2}$$

   Hence we can state from (2) that $\{\frac{n}{2}\}_{n=1}^{\infty}$ is divergent.
   We can now state that sequence $\{n - 1/n\}_{n=1}^{\infty}$ is also divergent
   Hence proved

10. If $s_n = 5^n/n!$ show that $\lim_{n\to\infty} s_n = 0$.
   e can write s_n as:

$$s_n = \frac{5 \cdot 5 \cdot 5 \cdots}{1 \cdot 2 \cdot 3 \cdot 4 \cdots}$$
$$s_n = \left(\frac{5}{1}\right) \cdot \left(\frac{5}{2}\right) \cdot \left(\frac{5}{3}\right) \cdot \left(\frac{5}{4}\right) \cdot \left(\frac{5}{5}\right) \cdot \left(\frac{5}{6}\right) \cdots$$

   We can write this as:

$$s_n = \frac{5^5}{5!}\Pi\frac{5}{n}$$

   So,

$$s_n < \frac{5^5}{5!} \cdot \frac{5}{n} \tag{1}$$

   From equation (1) we can infer that $\{s_n\}_{n=1}^{\infty}$ will be convergent if the series $\{\frac{5^5}{5!} \cdot \frac{5}{n}\}_{n=1}^{\infty}$ is convergent.
   Now we know that $\{1/n\}_{n=1}^{\infty}$ is a convergent series. (Proved above)
   we also know that

$$\lim_{n\to\infty} c \cdot s_n = c \cdot \lim_{n\to\infty} s_n \tag{2}$$

   using (2) we can state that the series $\{\frac{5^5}{5!} \cdot \frac{5}{n}\}_{n=1}^{\infty}$ is convergent
   We can then state that the series $\{s_n\}_{n=1}^{\infty}$
   Hence proved

11. If P is a polynomial function of the third degree

$$P(x) = ax^3 + bx^2 + cx + d \ (a, b, c, d, x \in \mathbb{R})$$

Prove that

$$\lim_{n\to\infty} \frac{P(n+1)}{P(n)} = 1$$

To prove : That limit exists and Limit L = 1.
We know that if the limit for any given function exists then it will satisfy the equation:

$$\left|\frac{P(n+1)}{P(n)} - L\right| < \epsilon \ \ (\forall n \geq N) \tag{1}$$

Here ϵ is an arbitrary constant and L is the value of the limit.
Here Limit, L =1.
N $\in \mathbb{I}$ and N > 0

Now,

$$\left|\frac{P(n+1)}{P(n)} - 1\right| < \epsilon \tag{2}$$

$$\left|\frac{a(n+1)^3 + b(n+1)^2 + c(n+1) + d}{an^3 + bn^2 + cn + d} - 1\right| < \epsilon$$

Substituting the value of the polynomial in the equation

$$\left|\frac{a(\frac{n+1}{n})^3 + b(\frac{n+1}{n})^2\frac{1}{n} + c(\frac{n+1}{n})\frac{1}{n^2} + d\frac{1}{n^3}}{a + b/n + c/n^2 + d/n^3} - 1\right| < \epsilon$$

Dividing Numerator and Denominator by n^3

2.3 Exercise 2.3

1. For any a, b $\in \mathbb{R}$ show that

$$||a| - |b|| \leq |a - b|$$

Then prove that $\{|s_n|\}_{n=1}^{\infty}$ converges to $|L|$ if $\{s_n\}_{n=1}^{\infty}$ converges to L.

2. Give an example of a sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers for which $\{|s_n|\}_{n=1}^{\infty}$ converges but $\{s_n\}_{n=1}^{\infty}$ does not.
The sequence $\{(-1)^n\}_{n=1}^{\infty}$ doesn't converge, but the sequence $\{|(-1)^n|\}_{n=1}^{\infty}$ converges

3. Prove that if $\{|s_n|\}_{n=1}^{\infty}$ converges to 0 then $\{s_n\}_{n=1}^{\infty}$ converges to 0.

We are given that $\{|s_n|\}_{n=1}^{\infty}$ converges to 0 and we have to prove that $\{s_n\}_{n=1}^{\infty}$ should also converge to 0.
To prove: $\lim\limits_{n\to\infty} s_n = 0$

Now let s_n converge to 0 so it will satisfy the equation:

$$|s_n - 0| < \epsilon \ \ (\forall n > N) \tag{1}$$

Here ϵ is an arbitrary positive rational number and 0 is the limit of the sequence.
N $\in \mathbb{I}$ and also N > 0.

Now we know that $||a| - |b|| \leq |a - b|$
It is also given that :

$$||s_n| - 0| < \epsilon \ \ \text{(given)} \tag{2}$$

From equation (2) and the above mentioned identity we can infer that:

$$||s_n| - |0|| \leq |s_n - 0| \tag{3}$$

Plugging equation (3) in (1) we get that :

$$|s_n - 0| < \epsilon \ \ \text{(Standard Form of Limit Equation)} \tag{4}$$

Hence we can say that the series $\{s_n\}_{n=1}^{\infty}$ is convergent and converges to 0.

4. Can you find a sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ which has no convergent subsequence and yet $\{|s_n|\}_{n=1}^{\infty}$ converges?

   If the sequence $\{s_n\}_{n=1}^{\infty}$ has no convergent subsequence, then that implies that $\{s_n\}_{n=1}^{\infty}$ is divergent to either positive or negative Infinity.

   If the sequence $\{s_n\}_{n=1}^{\infty}$ is divergent to positive or negative infinity then $\{|s_n|\}_{n=1}^{\infty}$ will also diverge to positive Infinity.
   Hence such a case is not possible where $\{s_n\}_{n=1}^{\infty}$ has no convergent subsequence but $\{|s_n|\}_{n=1}^{\infty}$ converges.

5. If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if

$$\lim_{m\to\infty} s_{2m} = L$$
$$\lim_{m\to\infty} s_{2m-1} = L$$

   prove that $s_n \to L$ as $n \to \infty$.

   It is given that terms with even subscripts converge to L as $n \to \infty$ and terms with odd terms also converge to L as $n \to \infty$.

   We can say that $\lim_{n\to\infty} s_n$ as $\lim_{n\to\infty} s_{2m}$ where $n = 2m$ for all even numbers.

   Similarly we can say that $\lim_{n\to\infty} s_n$ as $\lim_{n\to\infty} s_{2m-1}$ where $n = 2m - 1$ for all odd numbers.

   We know that:

$$\lim_{m\to\infty} s_{2m} = L$$
$$\lim_{m\to\infty} s_{2m-1} = L$$

   So, we can conclusively say that $\lim_{n\to\infty} s_n = L$
   Hence proved.

2.4 Exercise 2.4

1. Label each of the following sequences either (A) convergent. (B) divergent to infinity, (C) divergent to -Infinity, or (D) Oscillating

   (a) $\{\sin n\pi/2\}_{n=1}^{\infty}$
   (D) Oscillating

   (b) $\{\sin n\pi\}_{n=1}^{\infty}$
   (A) Convergent

   (c) $\{e^n\}_{n=1}^{\infty}$
   (B) Divergent to $+\infty$

   (d) $\{e^{1/n}\}_{n=1}^{\infty}$
   (A) Convergent

   (e) $\{n\sin(\pi/n)\}_{n=1}^{\infty}$
   (A) Convergent

   (f) $\{(-1)^n\tan(\pi/2-1/n)\}_{n=1}^{\infty}$
   (D) Oscillating

   (g) $\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right\}_{n=1}^{\infty}$
   (B) Divergent to $+\infty$

   (h) $\{-n^2\}_{n=1}^{\infty}$
   (C) Divergent to $-\infty$

2. Prove that $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to Infinity.
   To prove that the sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to $+\infty$ we must prove that for any given $M>0$:

$$s_n > M \ \ (n \geq N) \tag{1}$$

   Here N $\in \mathbb{I}$ and N > 0
   Now,

$$\sqrt{n} > M \tag{2}$$
$$n > M^2 \tag{3}$$

   So for any N $> M^2$ will satisfy the equation (3) and hence satisfy (1). So we can conclusively say that the sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to $+\infty$.

3. Prove that $\{\sqrt{n+1}-\sqrt{n}\}_{n=1}^{\infty}$ is convergent.
   First let us simplify the expression

$$\sqrt{n+1}-\sqrt{n}$$
$$\frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} \tag{1}$$

Rationalizing numerator and denominator
Further simplifying

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} \tag{2}$$

We know that :

$$\begin{aligned}\sqrt{n+1}+\sqrt{n} &> \sqrt{n}+\sqrt{n}\\ \sqrt{n+1}+\sqrt{n} &> 2\sqrt{n}\\ \frac{1}{\sqrt{n+1}+\sqrt{n}} &< \frac{1}{2\sqrt{n}}\end{aligned} \tag{3}$$

Now if the equation in (3) is convergent we can say that $\{\sqrt{n}\}_{n=1}^{\infty}$ will also be convergent.
$\{\frac{1}{\sqrt{n}}\}_{n=1}^{\infty}$ is a bounded above series by 0. It is also monotonically decreasing as $\frac{1}{\sqrt{n-1}} > \frac{1}{\sqrt{n}} \forall n \in \mathbb{N}$.
As this series is both monotonically non-increasing as well as bounded above it is also convergent.
We also know that:

$$\lim_{n\to\infty} c \cdot s_n = c \cdot \lim_{n\to\infty} s_n \tag{4}$$

From (4) we can infer that $\lim\limits_{n\to\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2} \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$
Now as $\{\frac{1}{2\sqrt{n}}\}_{n=1}^{\infty}$ is convergent we also conclude that $\{\sqrt{n+1}-\sqrt{n}\}_{n=1}^{\infty}$ is convergent.

4. Prove that if the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ diverges to infinity, then $\{-s_n\}_{n=1}^{\infty}$ diverges to minus infinity.
   It is given that the sequence $\{s_n\}_{n=1}^{\infty}$ diverges to $+\infty$. So we can write it mathematically as:

$$s_n > M \ \ (n \geq N) \tag{1}$$

Here M > 0 is an arbitrary constant and N $\in \mathbb{I}$ where N > 0 is a term subscript where s_n surpasses the value of M. Solving further :-

$$-s_n < -M \ \ (n \geq N) \tag{2}$$

Now let the negative sequence be denoted by $\{-s_n\}_{n=1}^{\infty}$. Let the sequence be represented by S.

$$\begin{aligned}S &= \{-s_n\}_{n=1}^{\infty}\\ S &< -M \ \ (n \geq N)\end{aligned} \tag{3}$$

Equation (3) is the standard form of a divergent series that diverges to $-\infty$. Hence $\{-s_n\}_{n=1}^{\infty}$ diverges to $-\infty$ when $\{s_n\}_{n=1}^{\infty}$ diverges tp $+\infty$.

5. Suppose $\{s_n\}_{n=1}^{\infty}$ converges to 0. Prove that $\{(-1)^n s_n\}_{n=1}^{\infty}$ converges to 0.
   It is given that $\{s_n\}_{n=1}^{\infty}$ converges to 0. We can also then state that $\{-s_n\}_{n=1}^{\infty}$ converges to 0. As

$$\lim_{n\to\infty} c \cdot s_n = c \cdot \lim_{n\to\infty} s_n \tag{1}$$

   Using (1) $\lim_{n\to\infty} -s_n = -\lim_{n\to\infty} s_n = 0$
   Now, we can represent $\{s_n\}_{n=1}^{\infty}$ as

$$s_n = \{s_1, s_2, s_3, s_4 \cdots\} \tag{2}$$

   And we can represent$\{-s_n\}_{n=1}^{\infty}$ as

$$-s_n = \{-s_1, -s_2, -s_3 \cdots\} \tag{3}$$

   Taking sub-sequences from (2) and (3) we get

$$\{s_2, s_4, s_6 \cdots\} \text{ and } \{-s_1, -s_3, -s_5 \cdots\} \tag{4}$$

   respectively

   We know that sub-sequences of a convergent sequence are also convergent and they converge to the same value as their parent sequence.

   So limits for the sub-sequences (4) are 0.

   On combining the sub-sequences that we have created we create the sequence:

$$s_n = \{-s_1, s_2, -s_3 \cdots\} \tag{5}$$

   Hence $\{(-1)^n s_n\}_{n=1}^{\infty} = \{-s_1, s_2, -s_3 \cdots\}$. This sequence has been formed by the combination of 2 convergent sub-sequences that are convergent to the same value 0, and hence $\{(-1)^n s_n\}_{n=1}^{\infty}$ is also convergent.

6. Suppose $\{s_n\}_{n=1}^{\infty}$ converges to L $\neq 0$. Prove that $\{(-1)^n s_n\}_{n=1}^{\infty}$ oscillates.
   Similar to the above example we can state that the series $\{-s_n\}_{n=1}^{\infty}$ will also converge. We know that:

$$\lim_{n\to\infty} c \cdot s_n = c \cdot \lim_{n\to\infty} s_n \tag{1}$$

   From (1) we can deduce that lim)$n \to \infty - s_n = -\lim_{n\to\infty} s_n = -L$

   Both these series can now be represented as :-

$$\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, s_3 \cdots\} \tag{2}$$
$$\{-s_n\}_{n=1}^{\infty} = \{-s_1, -s_2, -s_3 \cdots\} \tag{3}$$

We know that sub-sequences of convergent sequences are convergent hence both the sub-sequences (2) and (3) will be convergent. Sequence (2) will converge to L and sequence (3) will converge to -L.

Now let us create a sequence by the combination of these sub-sequences:

$$\{(-1)^n s_n\}_{n=1}^{\infty} = \{-s_1, s_2, -s_3 \cdots\} \tag{4}$$

Clearly (4) will oscillate as it's sub-sequences are converging to different points L $\neq$ -L.

7. Suppose $\{s_n\}_{n=1}^{\infty}$ diverges to infinity. Prove that $\{(-1)^n s_n\}_{n=1}^{\infty}$ oscillates.
From example 6 above it can be proved that any sequence $\{(-1)^n s_n\}_{n=1}^{\infty}$ oscillates if $\lim\limits_{n\to\infty} = L$ where L$\neq$ 0.

In this question it is given that $\lim\limits_{n=to\infty} s_n$ diverges to positive infinity, or $\lim\limits_{n\to\infty} s_n = L$ where $L \neq 0$.

So it is evident that $\{(-1)^n s_n\}_{n=1}^{\infty}$ will oscillate.

2.5 Exercise 2.5

1. True or false? If a sequence of positive numbers is not bounded then the sequence diverges to infinity.
   False, it isn't necessary that the sequence will diverge to ∞.

2. Give an example of a sequence $\{s_n\}_{n=1}^{\infty}$ which is not bounded but for which $\lim_{n\to\infty} s_n = 0$.
   The sequence $\{e^{-x^2} \tan x\}_{n=1}^{\infty}$ converges to 0 as $n \to \infty$ but it isn't bounded.

3. Prove that if $\lim_{n\to\infty} s_n/n = L \neq 0$ then $\{s_n\}_{n=1}^{\infty}$ is not bounded.
   The sequence $\{s_n/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n/n = L \neq 0$. Now,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} n \cdot \left(\lim_{n\to\infty} s_n/n\right) \tag{1}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} n \cdot L \tag{2}$$

   From (2) we can clearly see that $\{s_n\}_{n=1}^{\infty}$ is divergent and is diverging to positive ∞. As the sequence $\{s_n\}_{n=1}^{\infty}$ is diverging hence it will not be bounded from theorem 2.5B

4. If $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, and $\{t_n\}_{n=1}^{\infty}$ converges to 0, prove that $\{s_n t_n\}_{n=1}^{\infty}$ converges to 0.
   Let us take some arbitrary $\epsilon > 0$.

   Now we know that the sequence $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence so:

$$|s_n| \leq M \;\; (M > 0) \tag{1}$$

   We also know that $\{t_n\}_{n=1}^{\infty}$ converges to 0, so

$$|t_n| < \epsilon/M \;\; (n \geq \mathrm{N}) \tag{2}$$

   Here $\mathrm{N} \in \mathbb{I}$ and $\mathrm{N} > 0$. Equation (2) will be true for some value of N.

   Now we can achieve

$$|s_n \cdot t_n| < \epsilon \;\; (n \geq \mathrm{N})$$

$$|s_n \cdot t_n - 0| < \epsilon \;\; (n \geq \mathrm{N}) \tag{3}$$

   Equation (3) is the standard equation for limit and we can clearly see that the series $\{s_n \cdot t_n\}_{n=1}^{\infty}$ converges to 0.

5. If the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded, prove that for any $\epsilon > 0$ there is a closed interval $J \subset R$ of length ϵ such that $s_n \subset J$ for infinitely many values of n.

   It is given that the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded, so

$$|s_n| \leq M \;\; (\text{for some}\, M > 0) \tag{1}$$

   Now the sequence is bounded between $[-M.M]$ and the terms of the sequence can be expresses as

$$\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, s_3 \cdots\} \tag{2}$$

   The number of terms inside $[-M, M]$ are countably infinite. Our region is of length $2M$. Let us divide that into 2 parts of length $2M - \rho$ and ρ such that each part has respectively N_1 and N_2 terms where total terms are N.

$$N = N_1 + N_2 \tag{3}$$

   We know that N is countably infinite, so from (3) we can infer that both N_1 and N_2 are countably infinite or one of them is. If either of them is countably infinite, we will obtain a set J $\in \mathbb{R}$ of finite arbitrary length ϵ, such that it contains infinite elements.

2.6 Exercise 2.6

1. Which of the following sequences are Monotone?

   (a) $\{\sin n\}_{n=1}^{\infty}$

   (b) $\{\tan n\}_{n=1}^{\infty}$

   (c) $\{\frac{1}{1+n^2}\}_{n=1}^{\infty}$

   (d) $\{2n + (-1)^n\}_{n=1}^{\infty}$

   The sequences (c) and (d) are monotonic.

2. If $\{s_n\}_{n=1}^{\infty}$ is nondecreasing and bounded above and L = $\lim\limits_{n\to\infty} s_n$, prove that $s_n \leq L$ (n $\in I$).

   It is given that the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded above , hence there will be an M > 0 such that

$$M = l.u.b\,\{s_1, s_2, s_3 \cdots\} \tag{1}$$

Now we will prove that this upper bound M is also the limit of the sequence $\{s_n\}_{n=1}^{\infty}$. Now for any arbitrary $\epsilon > 0$, we know that $M - \epsilon$ is not an upper bound so for some N $\in \mathbb{I}$

$$s_n > M - \epsilon \;\; (n \geq N) \tag{2}$$

For some value of n. Now we know that M is the upper bound of $\{s_n\}_{n=1}^{\infty}$. So

$$s_n \leq M \;\; (\forall n \in \mathbb{N}) \tag{3}$$

Now combining (2) and (3), we get:-

$$|s_n - M| < \epsilon \;\; (n \geq N) \tag{4}$$

This is the standard for of the limit equation and this indicates that $\lim\limits_{n\to\infty} s_n = M$. Now we are also given that $\lim\limits_{n\to\infty} s_n = L$.

We know that there cannot be multiple limits for the same sequence that is convergent and hence $M = L$. This shows that the limit of the sequence was the upper bound and hence

$$s_n \leq L$$

3. If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$are non-decreasing bounded sequences, and if $s_n \leq t_n$ (n $\in I$), prove that $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$.

   Let the l.u.b for the sequence $\{s_n\}_{n=1}^{\infty}$ be M_1 and the l.u.b for the sequence $\{t_n\}_{n=1}^{\infty}$ be M_2.

   Now we know that l.u.b of any sequence $\{s_n\}_{n=1}^{\infty} = \max(s_1, s_2, s_3 \cdots)$.

   So, $M_1 = \max(s_1, s_2, s_3 \cdots)$ and
   $M_2 = \max(t_1, t_2, t_3 \cdots)$

   We also know that $s_n < t_n \ \forall n \in \mathbb{I}$
   So, $\max(s_n) < \max(t_n)$ and hence $M_1 < M_2$

   Now we also know that if a sequence is bounded above and monotonically non-decreasing, then the l.u.b of the sequence is the limit of that sequence. That is

$$\lim_{n\to\infty} s_n = M_1 \tag{1}$$
$$\lim_{n\to\infty} t_n = M_2 \tag{2}$$

   We know that $M_1 < M_2$, hence $\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n$.
   Hence proved.

4. Find the limit of $\{n^{-n-1}(n+1)^n\}_{n=1}^{\infty}$.

   The sequence $\{n^{-n-1}(n+1)^n\}_{n=1}^{\infty}$ can be further simplified to:

$$s_n = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n} \tag{1}$$
$$s_n = \left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{n} \tag{2}$$

Expanding the Binomial of the terms

$$s_n = \left(1 + \frac{C_1}{n} + \frac{C_2}{n^2} + \frac{C_3}{n^3} \cdots \right) \cdot \frac{1}{n} \tag{3}$$
$$s_n = \left(1 + 1 + \frac{n(n-1)}{2} + \cdots + \frac{n(n-1)\cdots 1}{1 \cdot 2 \cdot 3 \cdots n} \cdot \frac{1}{n^n}\right) \cdot \frac{1}{n} \tag{4}$$

   For $k = 1 \cdots n$, the k^{th} terms on the right are

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{1 \cdot 2 \cdots k} \cdot \frac{1}{n^k} \cdot \frac{1}{n} \tag{5}$$

   which equals

$$\frac{1}{1\cdot 2\cdots k}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{1}{n} \tag{6}$$

If we expand s_{n+1} we obtain n+2 terms (One more for s_n) and, for $k = 1\cdots n$, the $(k+1)$st term is

$$\frac{1}{1\cdot 2\cdots k}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)\frac{1}{n}$$

which is greater than or equal to the quantity (1). This shows that $s_{n+1} \leq s_n$. That is $\{s_n\}_{n=1}^{\infty}$ is monotonically non-increasing. And also :

$$s_n < \left(1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots\right)\cdot\frac{1}{n} \tag{7}$$

s_n is bounded below by 0 and that is it's lower bound. As the sequence $\{s_n\}_{n=1}^{\infty}$ is monotonically non-increasing a well as bound below by it's g.l.b 0. It has a limit and
$\lim\limits_{n\to\infty} n^{-n-1}(n+1)^n = 0$

5. If $s_n = 10/n!$ find $N \in I$ such that

$$s_{n+1} < s_n \quad (n \geq N)$$

We have to find s_{n+1} such that $s_{n+1} < s_n$.

$$\begin{aligned}
s_{n+1} &< s_n \\
\frac{10}{(n+1)!} &< \frac{10}{n!} \\
\frac{1}{(n+1)!} &< \frac{1}{n!} \\
(n+1)! &> n! \\
n+1 &> 1 \\
n &> 0
\end{aligned}$$

So for all values of $n \in \mathbb{I}$ the equation is satisfied and $s_{n+1} < s_n$. This sequence is strictly monotonically non-increasing. We can take value of N as 1.

6. For $n \in I$, let

$$s_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$$

Prove that $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} s_n \leq 1/2$

To prove that the series $\{s_n\}_{n=1}^{\infty}$ where

$$\{s_n\}_{n=1}^{\infty} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \tag{1}$$

is convergent, we must prove that $\{s_n\}_{n=1}^{\infty}$ is monotonic and that it is bounded.
We know that the series $\{s_n\}_{n=1}^{\infty}$ consists of positive real quantities, hence

$$s_n \geq 0 \tag{2}$$

From (2), we can say that the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded below. Now we also know that

$$\begin{aligned} 2 &> 1 \\ 4 &> 3 \\ 6 &> 5 \\ &\;. \\ &\;. \\ &\;. \\ 2n &> 2n-1 & (3) \\ \prod_{k=1}^{n} 2n &> \prod_{k=1}^{n} 2n-1 & (4) \end{aligned}$$

From (4) we can further say that

$$\begin{aligned} \frac{\prod 2n-1}{\prod 2n} &< 1 & (5) \\ s_n &< 1 & (6) \end{aligned}$$

Hence from (6) we can say that $\{s_n\}_{n=1}^{\infty}$ is also bounded above. As $\{s_n\}_{n=1}^{\infty}$ is bounded below and bounded above $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence.

Now we have to prove that $\{s_n\}_{n=1}^{\infty}$ is a monotonic sequence. We know that

$$\begin{aligned} s_n &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} & (7) \\ s_n &= \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)}\right) \cdot \frac{2n-1}{2n} & (8) \\ s_n &= s_{n-1} \cdot \frac{2n-1}{2n} & (9) \end{aligned}$$

Now, we also know that $\forall n \in \mathbb{I}$

$$\begin{aligned} 2n-1 &< 2n \\ \frac{2n-1}{2n} &< 1 \\ s_{n-1}\cdot\frac{2n-1}{2n} &< s_{n-1} \end{aligned} \tag{10}$$

From (9) and (10), we can say that

$$s_n < s_{n-1} \tag{11}$$

Hence the series $\{s_n\}_{n=1}^{\infty}$ is strictly monotonically non-increasing and as this series is also bounded, from the monotonic Theorem we can say that the series $\{s_n\}_{n=1}^{\infty}$ is convergent.

Now we also know that

$$\begin{aligned} s_1 &= \frac{1}{2} && (12) \\ \text{and } \ s_2 &< s_1 \ \text{ From (11)} && (13) \\ \text{similarly } \ s_3 &< s_2 < s_1 \\ &. \\ &. \\ &. \\ s_n < s_{n-1} < \cdots &< s_1 \ \text{ From (11) and (13)} && (14) \\ s_n &\le \frac{1}{2} \ \text{ From (14) and (12)} && (15) \end{aligned}$$

From (15) we can say that

$$s_n \le \frac{1}{2}$$

7. For $n \in I$, let

$$s_n = \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\cdot\frac{1}{n^2}$$

Verify that $s_1 > s_2 > s_3$. Prove that $\{s_n\}_{n=1}^{\infty}$ is non-increasing.

We have to prove that $\{s_n\}_{n=1}^{\infty}$ where

$$s_n = \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\cdot\frac{1}{n^2} \tag{1}$$

is strictly non-increasing or monotonically strictly non-increasing. To do that we can rewrite $\{s_n\}_{n=1}^{\infty}$ as

$$s_n = \frac{\prod 2n}{\prod 2n-1} \cdot \frac{1}{n^2} \tag{2}$$

Now

$$s_{n+1} = \frac{\prod^{n+1} 2n}{\prod^{n+1} 2n-1} \cdot \frac{1}{(n+1)^2} \tag{3}$$

$$s_{n+1} = \frac{\prod^{n} 2n}{\prod^{n} 2n-1} \cdot \frac{1}{n^2} \cdot \frac{2n+2}{2n+1} \cdot \left(\frac{n}{n+1}\right)^2 \tag{4}$$

From (2) we can rewrite this as

$$s_{n+1} = s_n \cdot \frac{2n+2}{2n+1} \cdot \left(\frac{n}{n+1}\right)^2 \tag{5}$$

We know that that $\forall n \in \mathbb{I}$

$$2n^3 + 2n^2 < 2n^3 + 5n^2 + 4n + 1 \tag{6}$$

$$\frac{2n^3 + 2n^2}{2n^3 + 5n^2 + 4n + 1} < 1 \tag{7}$$

$$\text{On simplifying } \frac{2n+2}{2n+1} \cdot \left(\frac{n}{n+1}\right)^2 < 1 \tag{8}$$

Putting (8) in (5)

$$s_{n+1} < s_n \tag{9}$$

From (9), we can clearly se that the sequence $\{s_n\}_{n=1}^{\infty}$ is strictly monotonically non-increasing.

Now we can also show that

$$s_1 = 2$$

$$s_2 = \frac{2 \cdot 4}{1 \cdot 3} \cdot \frac{1}{2^2} = \frac{8}{3} \cdot \frac{1}{4} = \frac{2}{3} \approx 0.667$$

Clearly $s_2 < s_1$

$$s_3 = \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5} \cdot \frac{1}{3^2} = \frac{16}{5} \cdot \frac{1}{9} \approx 0.355$$

Clearly $s_3 < s_2 < s_1$

Hence Proved

8. For $n \in I$, let

$$t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

   (a) Prove that $\{t_n\}_{n=1}^{\infty}$ is non-decreasing

   (b) Using only facts established in the proof 2.6C, prove that $\{t_n\}_{n=1}^{\infty}$ is bounded above and then prove $\lim_{n\to\infty} t_n \geq \lim_{n\to\infty} (1 + 1/n)^n$.

Firstly we have to prove that $\{t_n\}_{n=1}^{\infty}$ is non-decreasing. We know that

$$t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \tag{1}$$

$$t_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} \tag{2}$$

Substituting (1) into (2)

$$t_{n+1} = t_n + \frac{1}{n!} \tag{3}$$

We know that

$$\frac{1}{n!} > 0 \;\; \forall n \in \mathbb{I} \tag{4}$$

$$t_n + \frac{1}{n!} > t_n \;\; \forall n \in \mathbb{I} \tag{5}$$

From (3) we get

$$t_{n+1} > t_n \forall n \in \mathbb{I} \tag{6}$$

Hence, from (6) we can say that the sequence $\{t_n\}_{n=1}^{\infty}$ is strictly non-decreasing

9. Let ζ denote the class of all sequences of real numbers. Let γ denote the class of all convergent sequences and ξ the class of all divergent sequences. Further let ξ_P and ξ_M denote the classes of sequences that diverge to plus infinity and minus infinity, respectively. Let ϱ denote the class of oscillating sequences. Finally. let β denote the class of all bounded sequences and let ϖ denote the class of all monotone sequences.
   By citing the proper definitions and theorums, verify the following statements.

   (a) $\zeta = \gamma \cup \xi$

   (b) $\xi = \xi_P \cup \xi_M \cup \varrho$

   (c) $\gamma \subset \beta$

   (d) $\varpi \cap \beta \subset \gamma$

   (e) $\varpi \cap \beta' \subset \xi_P \cup \xi_M$

   (f) $\beta \cap \xi_P = \phi$

2.7 Exercise 2.7

1. Prove

   (a) $\lim_{n\to\infty} \frac{2n^3+5n}{4n^3+n^2} = \frac{1}{2}$

   Dividing numerator and denominator by n^3

$$\frac{2+5/n^2}{4+1/n} \tag{1}$$

   We have proved previously that $\lim_{n\to\infty} 1/n = 0$, and we also know that

$$\lim_{n\to\infty} s_n \cdot t_n = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} t_n \tag{2}$$

So,

$$\lim_{n\to\infty} 1/n^2 = \lim_{n\to\infty} 1/n \cdot \lim_{n\to\infty} 1/n = 0 \tag{3}$$

We know that

$$\lim_{n\to\infty} c \cdot s_n = c \cdot \lim_{n\to\infty} s_n \tag{4}$$

Now we can also that from (4)

$$\lim_{n\to\infty} 5/n^2 = 5 \cdot \lim_{n\to\infty} 1/n^2 = 0 \tag{5}$$

We know that

$$\lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n \tag{6}$$

From (6)

$$\lim_{n\to\infty} 2 + 5/n^2 = \lim_{n\to\infty} 2 + \lim_{n\to\infty} 5/n^2 \tag{7}$$

We know that

$$\lim_{n\to\infty} 1 = 1 \tag{8}$$

So, $\lim_{n\to\infty} 2 = 2 \cdot 1 = 2$

$$\lim_{n\to\infty} 2 + 5/n^2 = 2 \tag{9}$$

Similarly,

$$\lim_{n\to\infty} 4 + 1/n = 4 \tag{10}$$

We know that

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} \tag{11}$$

From (11)

$$\lim_{n\to\infty} \frac{2+5/n^2}{4+1/n} = \frac{\lim_{n\to\infty} 2+5/n^2}{\lim_{n\to\infty} 4+1/n} \tag{12}$$

$$\lim_{n\to\infty} \frac{2+5/n^2}{4+1/n} = \frac{2}{4} = 1/2$$

   Hence Proved

(b) $\lim_{n\to\infty} \frac{n^2}{(n-7)^2-6} = 1$

Simplifying the equation

$$\lim_{n\to\infty} \frac{n^2}{n^2-14n+49-6} \tag{1}$$

$$\lim_{n\to\infty} \frac{n^2}{n^2-14n+43} \tag{2}$$

$$\text{Dividing Numerator and denominator by} n^2$$

$$\lim_{n\to\infty} \frac{1}{1-14/n+43/n^2} \tag{3}$$

$$\text{We know that}$$

$$\lim_{n\to\infty} 1/n = 0 \text{ and} \tag{4}$$

$$\lim_{n\to\infty} c\cdot s_n = c\cdot \lim_{n\to\infty} s_n \tag{5}$$

$$\text{Hence}$$

$$\lim_{n\to\infty} -14/n = -14 \lim_{n\to\infty} 1/n = 0 \tag{6}$$

$$\text{We know that}$$

$$\lim_{n\to\infty} s_n\cdot t_n = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} t_n \tag{7}$$

$$\lim_{n\to\infty} 1/n^2 = \left(\lim_{n\to\infty} 1/n\right)^2 \tag{8}$$

$$\text{Hence}$$

$$\lim_{n\to\infty} 1/n^2 = 0 \tag{9}$$

$$\text{Now, From (5)}$$

$$\lim_{n\to\infty} 43/n^2 = 43 \lim_{n\to\infty} 1/n^2 = 0 \tag{10}$$

$$\text{We also know that}$$

$$\lim_{n\to\infty} 1 = 1 \tag{11}$$

$$\text{Now, From (6), (10) and (11)}$$

$$\lim_{n\to\infty} 1-14/n+43/n^2 = 1-0+0 = 1 \tag{12}$$

$$\text{We know that}$$

$$\lim_{n\to\infty} s_n/t_n = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} \tag{13}$$

$$\text{From (13)}$$

$$\lim_{n\to\infty} \frac{1}{1-14/n+43/n^2} = 1$$

Hence Proved

2. Prove that if $\{s_n\}_{n=1}^{\infty}$ converges to 1 then $\{s_n^{1/2}\}_{n=1}^{\infty}$ converges to 1.

The question asks us to prove that is the series $\{s_n\}_{n=1}^{\infty}$ is converging to one then $\{s_n^{1/2}\}_{n=1}^{\infty}$ must also converge to 1, but I shall prove a much more general statement, that is :
If a sequence $\{s_n\}_{n=1}^{\infty}$ is convergent and $\{s_n\}_{n=1}^{\infty}$ converges to L, then the series $\{s_n^{1/2}\}_{n=1}^{\infty}$ must also be convergent and will converge to $\sqrt{L}$.

To prove: that $\{s_n^{1/2}\}_{n=1}^{\infty}$ converges to $\sqrt{L}$. If a sequence is convergent then:
For $\exists\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ for which $\exists N \in \mathbb{I}$, such that $\forall n \in \mathbb{R}$ where $n \geq N$. If we can find such N, then we can call $\{s_n^{1/2}\}_{n=1}^{\infty}$ convergent.

$$\left|\sqrt{s_n} - \sqrt{L}\right| < \epsilon \quad \forall n \geq N \tag{1}$$

is satisified.

$$\text{Rationalizing}$$

$$\left|\frac{(\sqrt{s_n} - \sqrt{L})(\sqrt{s_n} + \sqrt{L})}{\sqrt{s_n} + \sqrt{L}}\right| < \epsilon \tag{2}$$

$$\left|\frac{s_n - L}{\sqrt{s_n} + \sqrt{L}}\right| < \epsilon \tag{3}$$

Now, if we can find N such that (3) is satisfied, we can prove the equality (1), and then prove convergence for given series.
We know that the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded as $\{s_n\}_{n=1}^{\infty}$ is convergent. Hence $\exists \mathrm{M} \in \mathbb{R}$, where $\mathrm{M} > 0$

$$|s_n| \leq \mathrm{M} \quad \forall n \in \mathbb{R} \tag{4}$$

$$|\sqrt{s_n}| \leq \mathrm{M} \tag{5}$$

$$\text{We also know that}$$

$$\left|\sqrt{s_n} + \sqrt{L}\right| \leq |\sqrt{s_n}| + \left|\sqrt{L}\right| \leq \mathrm{M} + \left|\sqrt{L}\right| \tag{6}$$

We are given that the sequence $\{s_n\}_{n=1}^{\infty}$ is convergent and converges to L, hence there must exist a $\mathrm{N}_1 \in \mathbb{I}$ such that the equation

$$|s_n - L| < \epsilon(\mathrm{M} + \left|\sqrt{L}\right|) \quad (\forall n \geq \mathrm{N}_1) \quad (\epsilon > 0) \tag{7}$$

$$\left|\sqrt{s_n} - \sqrt{L}\right| \left|\sqrt{s_n} + \sqrt{L}\right| < \epsilon(\mathrm{M} + \left|\sqrt{L}\right|) \tag{8}$$

$$\mathrm{Re-writing}$$

$$\left|\sqrt{s_n} - \sqrt{L}\right| < \frac{\epsilon(\mathrm{M} + \left|\sqrt{L}\right|)}{\left|\sqrt{s_n} + \sqrt{L}\right|} \tag{9}$$

$$\left|\sqrt{s_n} - \sqrt{L}\right| < \epsilon \tag{10}$$

Hence from (10), we can clearly see that $\{s_n^{1/2}\}_{n=1}^{\infty}$ is convergent and converges to $\sqrt{L}$. Now for the given question we can see that if $\lim_{n\to\infty} s_n = 1$, then $\lim_{n\to\infty} s_n^{1/2} = \sqrt{1} = 1$. Hence proved.

3. Evaluate $\lim\limits_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n})$.

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n})$$

$$\text{Rationalizing}$$

$$\lim_{n\to\infty} \sqrt{n}\cdot\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \tag{1}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \tag{2}$$

$$\text{Dividing both sides by } \sqrt{n}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{\frac{n+1}{n}}+\sqrt{\frac{n}{n+1}}} \tag{3}$$

Now if we can find the limit of $\lim\limits_{n\to\infty} \frac{n}{n+1}$, we will be able to compute the given limit. Solving

$$\lim_{n\to\infty} \frac{n}{n+1} \tag{4}$$

$$\text{Dividing numerator and denominator by } n$$

$$\lim_{n\to\infty} \frac{1}{1+1/n} \tag{5}$$

$$\text{We know that } \lim_{n\to\infty} 1/n = 0 \tag{6}$$

$$\text{We also know that}$$

$$\lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n \tag{7}$$

$$\text{Hence} \quad \lim_{n\to\infty} 1+1/n = \lim_{n\to\infty} 1 + \lim_{n\to\infty} 1/n \tag{8}$$

$$\lim_{n\to\infty} 1+1/n = 1 \quad \text{From (6) and (7)} \tag{9}$$

$$\text{We know that}$$

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{\lim\limits_{n\to\infty} s_n}{\lim\limits_{n\to\infty} t_n} \tag{10}$$

$$\text{Using (10)}$$

$$\lim_{n\to\infty} \frac{1}{1+1/n} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1+1/n} \tag{11}$$

$$\text{Using (9) and (11)}$$

$$\lim_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1} = 1$$

$$\text{Hence} \quad \lim_{n\to\infty} \frac{n}{n+1} = 1 \tag{12}$$

We know that if $\lim\limits_{n\to\infty} s_n = \mathrm{L}$, then $\lim\limits_{n\to\infty} s_n^{1/2} = \sqrt{\mathrm{L}}$. So

$$\lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \sqrt{1} = 1 \tag{13}$$

We also know that if $\lim\limits_{n\to\infty} s_n = \mathrm{L}$, then $\lim\limits_{n\to\infty} \frac{1}{s_n} = \frac{1}{\mathrm{L}}$. So,

$$\lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}$$
$$\lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = 1 \tag{14}$$

Plugging the values obtained in equations (13) and (14) into the equation (3), we get

$$\lim_{n\to\infty}\frac{1}{\sqrt{\frac{n+1}{n}}+\sqrt{\frac{n}{n+1}}} = \frac{1}{1+1}$$
$$\lim_{n\to\infty}\frac{1}{\sqrt{\frac{n+1}{n}}+\sqrt{\frac{n}{n+1}}} = \frac{1}{2}$$
$$\text{Hence,}\quad \lim_{n\to\infty}\sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \frac{1}{2}$$

4. Suppose $\{s_n\}_{n=1}^{\infty}$ is a sequence of positive numbers and $0 < x < 1$. If $s_{n+1} < xs_n$ (n $\in \mathbb{I}$), prove that $\lim_{n\to\infty} s_n = 0$

Let us assume another sequence t_n such that $t_n \geq s_n$ and t_n consist of positive real terms such that $t_{n+1} = xt_n$, where $0 < x < 1$. Now we know that

$$x < 1 \tag{1}$$
$$s_n \cdot x < s_n \tag{2}$$
$$\text{We also know that}$$
$$s_{n+1} < x \cdot s_n \tag{3}$$
$$\text{So,}\quad s_{n+1} < s_n \tag{4}$$

From (4), we can conclude that s_n is a strictly monotonically non-increasing sequence. We also know that s_n consists only of positive real numbers, so it is also bounded below by 0 and hence $\{s_n\}_{n=1}^{\infty}$ must converge to a limit L such that

$$\text{L} \geq 0 \tag{5}$$

Now let us inspect the sequence $\{t_n\}_{n=1}^{\infty}$. We know

$$x < 1$$
$$x \cdot t_n < t_n \tag{6}$$
$$t_{n+1} = x \cdot t_n \quad \text{(Assumed Series)}$$
$$\text{From (6)}$$
$$t_{n+1} < t_n \tag{7}$$

Hence from (7) we can see that the sequence $\{t_n\}_{n=1}^{\infty}$ is strictly monotonically non-increasing and $\{t_n\}_{n=1}^{\infty}$ is a sequence consisting of positive terms hence $t_n \geq 0 \ \ \forall n \in \mathbb{I}$. So it is also bounded below by 0. hence by the monotonic theorem, the sequence $\{t_n\}_{n=1}^{\infty}$ must be convergent.
Let the sequence $\{t_n\}_{n=1}^{\infty}$ converge to a positive quantity M. So it will hold that

$$\lim_{n\to\infty} t_n = \mathrm{M} \tag{8}$$

$$\lim_{n\to\infty} t_{n+1} = \mathrm{M} \tag{9}$$

and

$$\lim_{n\to\infty} t_{n+1} = x \cdot \lim_{n\to\infty} t_n \tag{10}$$

$$\mathrm{M} = x \cdot \mathrm{M} \tag{11}$$

$$\mathrm{M} - \mathrm{M} \cdot x = 0$$

$$\mathrm{M}(x-1) = 0$$

This is only possible when either $x = 0$ or M$= 0$, but we know that $x \neq 0$, hence M$= 0$.

Now we know that $s_n \leq t_n \ \ \forall n \in \mathbb{I}$. So according to theorem

$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n \tag{12}$$

$$\lim_{n\to\infty} s_n \leq 0 \tag{13}$$

We know that $\lim_{n\to\infty} s_n = \mathrm{L}$

From (5) We also know that

$$\mathrm{L} \geq 0 \tag{14}$$

From (13) and (14)

$$\mathrm{L} = 0 \tag{15}$$

$$\text{Hence, } \lim_{n\to\infty} s_n = 0$$

5. Suppose

$$\lim_{n\to\infty} \frac{s_n - 1}{s_n + 1} = 0$$

Prove $\lim_{n\to\infty} s_n = 1$

$$\lim_{n\to\infty}\frac{s_n-1}{s_n+1}=0 \tag{1}$$

We know that

$$\lim_{n\to\infty}\frac{s_n}{t_n}=\frac{\lim\limits_{n\to\infty} s_n}{\lim\limits_{n\to\infty} t_n} \tag{2}$$

Applying (2) in (1)

$$\frac{\lim\limits_{n\to\infty} s_n-1}{\lim\limits_{n\to\infty} s_n+1}=0$$

$$\lim_{n\to\infty} s_n-1=0 \tag{3}$$

We also know that

$$\lim_{n\to\infty} s_n-t_n=\lim_{n\to\infty} s_n-\lim_{n\to\infty} t_n \tag{4}$$

Applying (4) in (3)

$$\lim_{n\to\infty} s_n-\lim_{n\to\infty} 1=0 \tag{5}$$

$$\lim_{n\to\infty} s_n=\lim_{n\to\infty} 1 \tag{6}$$

$$\text{We know that } \lim_{n\to\infty} 1=1 \tag{7}$$

From (6) and (7)

$$\lim_{n\to\infty} s_n=1$$

Hence Proved

6. Prove that $\lim\limits_{n\to\infty}(1+1/n)^{n+1}=e$. Also, prove that

$$\lim_{n\to\infty}\left[1+\frac{1}{1+n}\right]^n=e$$

7. Using the identity

$$1+\frac{2}{n}=\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n}\right)$$

Prove that

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n=e^2$$

We know that

$$\lim_{n\to\infty} s_n \cdot t_n = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} t_n \tag{1}$$

we are given

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{n+1}\right)^n\left(1+\frac{1}{n}\right)^n\right] \tag{2}$$

From (1) and (2)

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{3}$$

We know that

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = e \text{ and} \tag{4}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{5}$$

Substituting (4) and (5) in (3)

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e\cdot e$$

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2$$

Hence Proved

8. If $c > 0$, prove that $\lim_{n\to\infty} c^{1/n} = 1$

9. Let $s_1 = \sqrt{2}$ and let $s_{n+1} = \sqrt{2}\cdot\sqrt{s_n}$ for n ≥ 2.

   (a) Prove, by induction that $s_n \leq 2$ for all n.

   (b) Prove that $s_{n+1} \geq s_n$ for all n

   (c) Prove that $\{s_n\}_{n=1}^{\infty}$ is convergent.

   (d) Prove that $\lim_{n\to\infty} s_n = 2$.

   First we will prove that $s_n \leq 2$ for all n.

$$s_1 = \sqrt{2} \tag{1}$$

$$s_{n+1} = \sqrt{2}\cdot\sqrt{s_n} \tag{2}$$

$$s_2 = \sqrt{2}\cdot\sqrt{\sqrt{2}} < 2 \tag{3}$$

$$\text{Now,}\quad s_3 = \sqrt{2}\cdot\sqrt{\sqrt{\sqrt{2}}} < 2 \tag{4}$$

Hence, we have seen that for the first few terms the condition $s_n \leq 2$ holds. Now let us assume that it holds for upto the k^{th} term of the series. That is

$$s_k \leq 2$$
$$\sqrt{s_k} \leq \sqrt{2} \tag{5}$$
$$s_{k+1} = \sqrt{2} \cdot \sqrt{s_k} \tag{6}$$
$$\text{From (5)}$$
$$\sqrt{2} \cdot \sqrt{s_k} \leq 2 \tag{7}$$
$$\text{From (6)}$$
$$s_{k+1} \leq 2$$
$$\text{Hence Proved}$$

We now need to find out for what values of n is the following condition true:

$$s_{n+1} \geq s_n$$

$$\text{We know that}$$
$$s_{n+1} = \sqrt{2 \cdot s_n}$$
$$s_{n+1} \geq s_n$$
$$\sqrt{2 \cdot s_n} \geq s_n$$

$$\text{Squaring both sides}$$

$$2 \cdot s_n \geq {s_n}^2$$
$${s_n}^2 - 2 \cdot s_n \leq 0$$
$$s_n(s_n - 2) \leq 0$$

For this condition to be satisfied $s_n \in [0, 2]$. we have already proved earlier that $s_n \leq 2$ and it is given that s_n are positive real numbers hence we can say that $s_n \in [0, 2]$. So $s_{n+1} \geq s_n \ \forall n \in \mathbb{I}$.

We can now say that the sequence $\{s_n\}_{n=1}^{\infty}$ is monotonically non-decreasing and we have also proved that $s_n \leq 2$ and that $s_n \in [0, 2]$. So we can also say the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded. By using the monotonicity theorem that whenever a sequence is bounded and monotonic, it must also converge. Hence we can also say that the sequence $\{s_n\}_{n=1}^{\infty}$ converges.

10. Suppose that $s_1 > s_2 > 0$, and let $s_{n+1} = \frac{1}{2}(s_n + s_{n-1})$ $(n \geq 2)$. Prove that

    (a) s_1, s_3, s_5 is non-increasing

    (b) s_2, s_4, s_6 is non-decreasing

    (c) $\{s_n\}_{n=1}^{\infty}$ is convergent

We can reduce the concurrent terms of the series as :

$$s_3 = \frac{1}{2}(s_2 + s_1)$$

We know that

$$s_1 > s_2 \ \text{, (given)} \tag{1}$$

$$\text{From (1)} \ : \ s_1 + s_2 > s_2 + s_2$$

$$\frac{1}{2}(s_1 + s_2) > s_2 \tag{2}$$

$$\text{From (2)} \ : \ s_3 > s_2 \tag{3}$$

$$\text{From (1)} \ : \ s_1 + s_2 < s_1 + s_1$$

$$\frac{1}{2}(s_1 + s_2) < s_1 \tag{4}$$

$$\text{From (4)} \ : \ s_3 < s_1 \tag{5}$$

We now know that

$$s_1 > s_3 > s_2$$

Similarly,

$$s_4 = \frac{1}{2}(s_3 + s_2)$$

$$\text{From (3)} : \ s_3 + s_2 > 2s_2$$

$$\frac{1}{2}(s_3 + s_2) > s_2$$

$$s_4 > s_2 \tag{6}$$

$$\text{From (3)} : \ s_3 + s_2 < 2s_3$$

$$\frac{1}{2}(s_2 + s_3) < s_3$$

$$s_4 < s_3 \tag{7}$$

We now know that

$$s_1 > s_3 > s_4 > s_2$$

and we can continue to create these series which will result in

$$s_1 > s_3 > s_5 > \cdots \ \text{and}$$

$$s_2 < s_4 < s_6 < \cdots$$

Now, Let the sequence $s_1, s_3, s_5 \cdots$ be denoted by $\{s_{2n-1}\}_{n=1}^{\infty}$ or the Odd series and $s_2, s_4, s_6 \cdots$ be denoted by $\{s_{2n}\}_{n=1}^{\infty}$ or the even sequence.

We can now see that all terms of the odd and even series are between s_1 and s_2, hence the odd and even sequences as well as $\{s_n\}_{n=1}^{\infty}$ are bounded between $[s_2, s_1]$.

Now the sequence $\{s_{2n-1}\}_{n=1}^{\infty}$ is bounded and it is monotonic, hence it must be convergent. Let the limit of the sequence be L. That is

$$\lim_{n\to\infty} s_{2n-1} = L \tag{8}$$

We also know that the sequence $\{s_{2n}\}_{n=1}^{\infty}$ is bounded and monotonic, hence it must also converge. Let the limit of this sequence be M. That is

$$\lim_{n\to\infty} s_{2n} = M \tag{9}$$

But we also know that $\lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty} s_{2n}$

$$\begin{aligned} \text{From (9) and (10)}: \lim_{n\to\infty} s_{2n-1} &= \lim_{n\to\infty} s_{2n} \\ L &= M \end{aligned}$$

Hence the sequence $\{s_n\}_{n=1}^{\infty}$ converges.

2.8 Exercise 2.8

1. Give an example of a sequence $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ for which, as $n \to \infty$

   (a) $s_n \to \infty, \quad t_n \to \infty, \quad s_n + t_n \to \infty$

   (b) $s_n \to \infty, \quad t_n \to \infty, \quad s_n - t_n \to 7$

No.	s_n	t_n	$s_n + t_n$	$\lim_{n\to\infty} s_n + t_n$	$s_n - t_n$	$\lim_{n\to\infty} s_n - t_n$
(a)	n	n	$2n$	∞	0	0
(b)	$n+7$	n	$2n+7$	∞	7	7

2. Suppose that $\{s_n\}_{n=1}^{\infty}$ is a divergent sequence of real numbers and $c \in \mathbb{R}$, $c \neq 0$. Prove that $\{c \cdot s_n\}_{n=1}^{\infty}$ diverges.

   It is given that the sequence is divergent, but there can be 3 possible cases how the sequence diverges which isn't mentioned.

   <u>Case I</u>

   Here we are considering the case that the sequence $\{s_n\}_{n=1}^{\infty}$ is diverging to $+\infty$. That is $\lim_{n\to\infty} s_n \to \infty$.

   Now as the sequence is diverging to ∞, then it cannot be bounded and hence there would exist an $M \in \mathbb{R}$ such that $M > 0$ where

$$|s_n| \geq M \quad (\forall n \geq N) \tag{1}$$

   where $N \in \mathbb{I}$ and $N > 0$

$$c \cdot |s_n| \geq c \cdot M \quad (\forall n \geq N) \tag{2}$$

$$\text{let } \; c \cdot M = M^{'}$$

$$\text{So, } \; c \cdot |s_n| \geq M^{'} \tag{3}$$

   here $M^{'}$ is some positive real number which clearly shows that the sequence $\{c \cdot s_n\}_{n=1}^{\infty}$ is unbounded and diverges to $+\infty$.

   <u>Case II</u>

   Here we can consider the case where $\{s_n\}_{n=1}^{\infty}$ diverges to $-\infty$ which can be solved similarly as the above example.

   <u>Case III</u>

   This is the case where the sequence is divergent because it is oscillating, but may or may not be bounded. As we have already shown the case of an unbounded sequence in (Case I), we will consider a bounded oscillating sequence.

   As this sequence is oscillating, it will not be monotonic and for some indexes n_1, n_2 and n_3 where $n_1 < n_2 < n_3$, it will be the case that (considering for some oscillating sequence)

$$s_{n_1} < s_{n_2} \tag{4}$$

$$\text{But } \; s_{n_2} > s_{n_3} \tag{5}$$

Now let us see the case for $\{c \cdot s_n\}_{n=1}^{\infty}$

If $c > 0$

$$\text{From } (4): \; c \cdot s_{n_1} < c \cdot s_{n_2} \tag{6}$$

$$\text{From } (5): \; c \cdot s_{n_2} > s_{n_3} \tag{7}$$

Hence $\{c \cdot s_n\}_{n=1}^{\infty}$ is also not monotonic, hence divergent. Similarly we can prove that $\{c \cdot s_n\}_{n=1}^{\infty}$ will diverge for the case $c < 0$.

3. True or False? If $\{s_n\}_{n=1}^{\infty}$ is oscillating and not bounded,and $\{t_n\}_{n=1}^{\infty}$ is bounded, then $\{s_n + t_n\}_{n=1}^{\infty}$ is oscillating and not bounded.
   True: The sequence $\{s_n + t_n\}_{n=1}^{\infty}$ will also be unbounded and oscillating.

2.9 Exercise 2.9

1. Find the limit superior and limit inferior to the following sequences:

   (a) 1,2,3,1,2,3,1,2,3

   (b) $\{\sin(n\pi/2)\}_{n=1}^{\infty}$

   (c) $\{(1+1/n)\cos n\pi\}_{n=1}^{\infty}$

   (d) $\{(1+1/n)^n\}_{n=1}^{\infty}$

No.	$\limsup\limits_{n\to\infty} s_n$	$\liminf\limits_{n\to\inf} s_n$
(a)	3	1
(b)	1	-1
(c)	1	-1
(d)	e	e

2. If the lim sup of the sequence $\{s_n\}_{n=1}^{\infty} = M$, prove that lim sup of any subsequence of $\{s_n\}_{n=1}^{\infty}$ is $\leq M$.

   Now let

$$P_n = \text{l.u.b}\,\{s_{n+1}, s_{n+2}, s_{n+3}\cdots\} \tag{1}$$

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} P_n \tag{2}$$

$$\text{We know that} \limsup_{n\to\infty} s_n = M \tag{3}$$

$$\text{From (3)}: \lim_{n\to\infty} P_n = M \tag{4}$$

   Now let us take any subsequence of $\{s_n\}_{n=1}^{\infty}$ from the indexes $n_1, n_2, n_3\cdots$ where $n_1 < n_2 < n_3 < \cdots$

$$\{s_{n_j}\}_{j=1}^{\infty} = \{s_{n_1}, s_{n_2}, s_{n_3}\cdots\} \tag{5}$$

Now, let

$$T_j = \text{l.u.b}\left\{s_{n_j}, s_{n_{j+1}}, s_{n_{j+2}}\cdots\right\} \tag{6}$$

$$\text{We know that } \limsup_{j\to\infty} s_{n_j} = \lim_{j\to\infty} T_j \tag{7}$$

Now, We also know that

$$\text{l.u.b}\,\{s_{n_1}, s_{n_2}, s_{n_3}\cdots\} \leq \text{l.u.b}\,\{s_1, s_2, s_3\cdots\} \tag{8}$$

So from (8)

$$\lim_{j\to\infty} \text{l.u.b}\left\{s_{n_j}, s_{n_{j+1}}, s_{n_{j+2}}\cdots\right\} \leq \lim_{n\to\infty}\{s_n, s_{n+1}, s_{n+2}\cdots\} \tag{9}$$

From (9)

$$\limsup_{n\to\infty}\{s_{n_j}\}_{n=1}^{\infty} \leq M \quad \text{Or}$$

   The limit superior of a subsequence of a sequence $\{s_n\}_{n=1}^{\infty}$ where $\limsup\limits_{n\to\infty} s_n = M$ is $\leq M$. Hence Proved.

3. If $\{s_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers and $\liminf_{n\to\infty} s_n = m$, prove that there is a subsequence of $\{s_n\}_{n=1}^{\infty}$ which converges to m.
Also prove that no subsequence of $\{s_n\}_{n=1}^{\infty}$ can converge to a limit less than m.

We are given that $\liminf_{n\to\infty} s_n = m$. So we can state that

$$\lim_{n\to\infty} \text{g.l.b}\,\{s_n, s_{n+1}, s_{n+2}\cdots\} = m \tag{1}$$

$$\text{and} \quad \text{g.l.b}\,\{s_n, s_{n+1}, s_{n+2}\cdots\} \geq m \quad (\forall n \geq N) \tag{2}$$

For some value of $N \in \mathbb{I}$.
Now let us consider some subsequence of $\{s_n\}_{n=1}^{\infty}$ with indexes $n_1, n_2, n_3, \cdots$ such that

$$\{s_j\}_{j=1}^{\infty} = \{s_{n_1}, s_{n_2}\cdots\} \tag{3}$$

$$\text{We know that } \forall j > N' \text{ where } N' > N \text{ the}$$

$$\text{g.l.b}\,\{s_j, s_{j+1}, s_{j+2}\cdots\} \geq m \tag{4}$$

$$\text{From (4)}: \quad \lim_{j\to\infty} s_j \geq m \tag{5}$$

Hence, we cam conclude that all sub-sequences of $\{s_n\}_{n=1}^{\infty}$ will converge to a value $\geq m$.

Now let us again create a subsequence such of indexes $n_1, n_2, n_3\cdots$, such that

$$s_{n_1} \geq m \quad \text{and} \tag{6}$$

$$s_{n_2} \leq s_{n_1} \text{ and } s_{n_2} \geq m$$

$$\text{Similarly we will take } s_{n_3} \leq s_{n_2} \text{ and } s_{n_3} \geq m$$

We can continue this trend to reflect

$$s_{n_{j+1}} \leq s_{n_j} \ \forall j \in \mathbb{I} \text{ and}$$

$$s_{n_j} \geq m$$

$$\text{Now let } T_j = \text{g.l.b}\,\{s_j, s_{j+1}, s_{j+2}\cdots\} \tag{7}$$

$$\text{We know that } \forall j \in \mathbb{I} \ \text{g.l.b}\,\{s_j, s_{j+1}, s_{j+2}\cdots\} \geq m \tag{8}$$

$$\text{Hence, } T_n \geq m \tag{9}$$

We can then create this sequence such that $\liminf_{n\to\infty} s_j = m$. Hence proved.

2.10 Exercise 2.10

2.11 Exercise 2.11

2.12 Exercise 2.12

3 Limits and Metric Spaces

3.1 Exercise 4.1

1. (a) If $|x-2|<1$, prove that $|x^2-4|<5$
   (b) If $|x-3|<\frac{1}{10}$, prove that $|x^2-x-6|<0.51$
   (c) If $|x+1|<\frac{1}{10}$. prove that $|x^3+1|<0.331$

(a)

$$\begin{aligned}
\text{We know that } |x-2| &< 1 && (1)\\
\text{So, } x &\in (1,3) && (2)\\
\text{From (2) } x+2 &\in (3,5) && (3)\\
\text{and } |x+2| &\in (3,5) &&\\
\text{So, } |x+2| &< 5 && (4)\\
\text{So, } |x-2|\cdot|x+2| &< 1\cdot 5 &&\\
|x+2|\cdot|x-2| &< 5 &&\\
|x^2-4| &< 5 && (5)\\
&\text{Hence Proved} &&
\end{aligned}$$

(b)

$$\begin{aligned}
\text{We know that } |x-3| &< \frac{1}{10} && (1)\\
\text{So, } x &\in (3-\frac{1}{10}, 3+\frac{1}{10}) && (2)\\
x &\in (2.9, 3.1) &&\\
\text{We also know that } x+2 &\in (4.9, 5.9) && (3)\\
\text{Hence } |x+2| &< 5.9 &&\\
|x-3|\cdot|x+2| &< \frac{1}{10}\cdot(5.9) && (4)\\
|x^2-x-6| &< 0.51 && (5)\\
&\text{Hence Proved} &&
\end{aligned}$$

(c)

$$\begin{aligned}
\text{We know that } |x+1| &< \frac{1}{10} && (1)\\
\text{So, } x &\in (-1-\frac{1}{10}, -1+\frac{1}{10}) &&\\
x &\in (-1.1, -0.9) && (2)\\
x &< -0.9 && (3)\\
x^3 &< (-0.9)^3 = -0.729 &&\\
x^3+1 &< 0.271 &&\\
|x^3+1| &< 0.331 && (4)\\
&\text{Hence Proved} &&
\end{aligned}$$

2. Let δ be any number such that $0 < \delta < 1$.

   (a) If $|x-2| < \delta$, prove that $|x^2-4| < 5\delta$

   (b) If $|x-3| < \delta$, prove that $|x^2-x-6| < 6\delta$

   (c) If $|x+1| < \delta$, prove that $|x^3+1| < 7\delta$

   (d) If $|x-2| < \delta$, prove that $|\frac{x-2}{x+3}| < \frac{\delta}{4}$

   (a)

$$
\begin{aligned}
\text{We know that } |x-2| &< \delta \\
\text{We also know that } \delta &< 1 \\
\text{So, } |x-2| &< 1 && (1) \\
x &\in (1,3) && (2) \\
x+2 &\in (4,5) && (3) \\
|x+2| &< 5 && (4) \\
|x-2|\cdot|x+2| &< 1\cdot 5 \\
|x^2-4| &< 5\delta \\
\text{Hence Proved}
\end{aligned}
$$

   (b)

$$
\begin{aligned}
\text{We know that } |x-3| &< \delta \\
\text{We also know that } \delta &< 1 \\
\text{So, } |x-3| &< 1 && (1) \\
x &\in (2,4) && (2) \\
x+2 &\in (4,6) && (3) \\
|x+2| &< 6 && (4) \\
\text{From (4)} \\
|x-3|\cdot|x+2| &< 6\delta \\
|x^2-x-1| &< 6\delta && (5) \\
\text{Hence Proved}
\end{aligned}
$$

(c)

$$\begin{aligned}
\text{We know that } |x+1| &< \delta && (1)\\
\text{We also know that } \delta &< 1\\
\text{So, } |x+1| &< 1 && (2)\\
\text{So, } x &\in (-2,0) && (3)\\
x &< 0\\
x^3 &< 0 && (4)\\
\text{From } (4)\\
|x^3+1| &< 1 && (5)\\
\text{That is} |x^3+1| &< \delta\\
\text{So, } |x^3+1| &< 7\delta && (6)\\
\text{Hence Proved}
\end{aligned}$$

(d)

$$\begin{aligned}
\text{We know that } |x-2| &< \delta && (1)\\
\text{We also know that } \delta &< 1 && (2)\\
\text{So, } |x-2| &< 1\\
x &\in (1,3)\\
x+3 &\in (4,6) && (3)\\
\frac{1}{x+3} &\in (1/6, 1/4) && (4)\\
\text{From } (4)\\
\frac{1}{x+3} &< 1/4 && (5)\\
\text{So, } \left|\frac{x-2}{x+3}\right| &< \delta/4\\
\text{Hence Proved}
\end{aligned}$$

3. Let $f(x) = x^2 + 4x$. Find $\delta > 0$ such that

$$|f(x) - 5| < \frac{1}{10} \quad (0 < |x-1| < \delta)$$

This statement implies that the function $f(x)$ will be continuous at the point $x = 1$ and $\lim_{x\to 1} f(x) = 5$.

$$\begin{aligned}
\text{We know that} |f(x) - 5| &< \frac{1}{10} && (1)\\
\text{Putting the value of f(x)}\\
|x^2 + 4x - 5| &< \frac{1}{10} && (2)\\
\text{So, } |x+5| \cdot |x-1| &< \frac{1}{10} && (3)\\
\text{Hence From } (3)\\
|x-1| &< \frac{1}{10 \cdot |x+5|} && (4)
\end{aligned}$$

Now we have to find a value for δ such that $0 < |x-1| < \delta$. So, If we take

$$
\begin{aligned}
\delta &= \frac{1}{10 \cdot |x+5|} && (5)\\
&\text{From } (4)\\
\text{We know that } |x-1| &< \frac{1}{10 \cdot |x+5|} \forall x \in \mathbb{R} && (6)\\
\text{So, } |f(x)-5| &< \frac{1}{10} \quad (0 < |x-1| < \delta) && (7)\\
&\text{Hence Proved}
\end{aligned}
$$

4. Prove directly from definition 4.1A that $\lim_{x \to 1} x^2 + 4x = 5$.

We need to prove that for the function $f(x) = x^2 + 4x$, the limit exists at $x = 1$ and that $\lim_{x \to 1} x^2 + 4x = 5$.

The definition for the limit bat function $f(x)$ states that

$$|f(x) - L| < \epsilon \quad (0 < |x-a| < \delta) \tag{1}$$

where $\epsilon > 0$ and for some $\delta > 0$. Here $a = 1$ and $L = 5$.

$$
\begin{aligned}
|x^2 + 4x - 5| &= |x+5| \cdot |x-1| && (2)\\
\text{From } (1) &\text{ We know that}\\
&|x-1| < \delta\\
&\text{So, let } \delta < 1 && (3)\\
&from(4)\\
&|x-1| < 1\\
&\text{Now, } x \in (0,2) && (4)\\
&\text{So, } |x+5| \in (5,7) && (5)\\
&|x+5| < 7 && (6)\\
&\text{From } (6)\\
|x-1| \cdot |x+5| &< 7\delta && (7)\\
\text{Now, let } \delta &= \min(1, \epsilon/7) && (8)\\
\text{So, } |x^2 + 4x - 5| &< \epsilon
\end{aligned}
$$

So, we have found a δ, namely $\delta = \min(1, \epsilon/7)$, for which (1) holds, hence this function $f(x)$ is continuous at $x = 1$.