Insertion and Deletion Correction in Polymer-based Data Storage

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Joint work with

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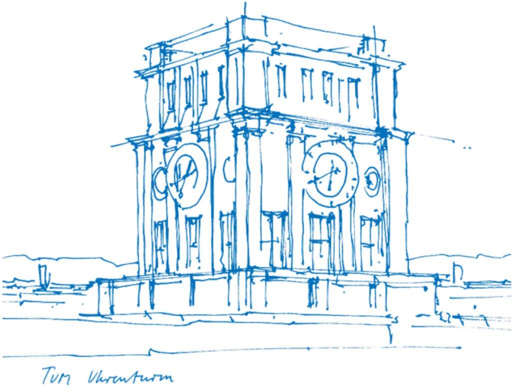

Introduction

Problem Statement & Results

Conclusion

Motivation

- Need for high-density and reliable data storage media.
  - Molecular storage paradigms, e.g. DNA storage

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- Need for high-density and reliable data storage media.
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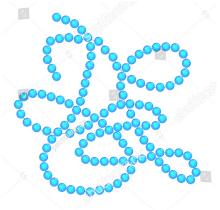
- One potential candidate is polymer-based data storage[OACL17].
  - Low readout latency.
  - Low storage cost.
  - Common analytical equipment usable.



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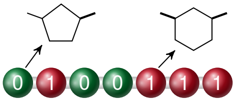

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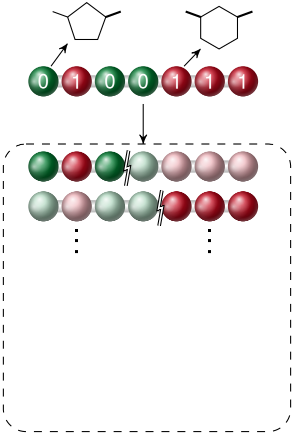

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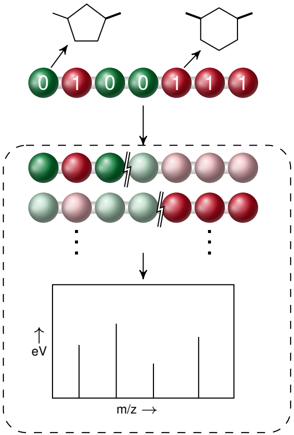

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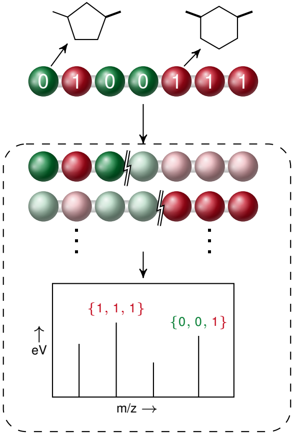

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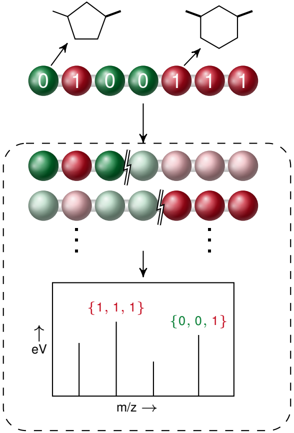

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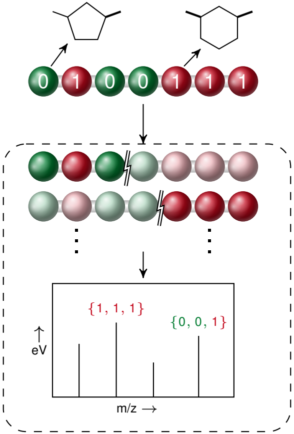

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- Read using tandem mass spectrometer.
  - Breaks links & generates fragments.
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- Assume we get compositions of all contiguous substrings.

Model

- For $\boldsymbol{s} \in \{0,1\}^n$, define **multiset** $C_k(\boldsymbol{s})$ as

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Note: If $\widetilde{\boldsymbol{s}} = (1,0,1,0)$, $C(\widetilde{\boldsymbol{s}}) = C(\boldsymbol{s})$. $\rightarrow$ A string and its reversal have identical composition multisets!

Previous Work

- [ADMOP15] showed that strings of length one less than a prime or twice a prime are uniquely reconstructable, up to reversal.

[ADMOP15] J. Acharya, H. Das, O. Milenkovic, A. Orlitsky, and S. Pan, "String Reconstruction from Substring Compositions", *SIAM Journal on Discrete Mathematics*, vol. 29, no. 3, pp. 1340–1371, Jan. 2015

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- [PGM19] proposed a code of $O(\log n)$ redundancy to allow unique reconstruction for all lengths.

Construction

$$
\begin{aligned}
\mathcal{S}_R(n) = \{\boldsymbol{s} \in \{0,1\}^n, s_1 = 0, s_n = 1, \text{ and } \\
\exists I \subset \{2, \ldots, n-1\} \text{ such that } \\
\text{for all } i \in I, s_i \neq s_{n+1-i}, \text{ for all } i \notin I, s_i = s_{n+1-i}, \\
\boldsymbol{s}_{[n/2] \cap I} \text{ has each prefix with strictly more 0s than 1s.}\}
\end{aligned}
$$

[PGM19] S. Pattabiraman, R. Gabrys, and O. Milenkovic, "Reconstruction and Error-Correction Codes for Polymer-Based Data Storage", in *2019 IEEE Information Theory Workshop (ITW)*, Visby, Sweden: IEEE, Aug. 2019, pp. 1–5

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2. How robust is $\mathcal{S}_R(n)$ to insertions and deletions in composition multisets?

   $\rightarrow$ Can correct deletion of at most one complete multiset.

   $\rightarrow$ Similar construction exists to correct deletions of more multisets.

Insertion-Deletion Equivalence

Lemma

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Proof idea:

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$$D_t(\boldsymbol{s}) = \{C'(\boldsymbol{s}) = \bigcup_{i \in \{1,\ldots,n\} \setminus \mathcal{I}} C_i(\boldsymbol{s}) : \mathcal{I} \subseteq \{1,\ldots,n\}, |\mathcal{I}| \leq t\}$$

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- Let $\boxed{C'_k(\boldsymbol{s})} = C_k(\boldsymbol{s}) \cup C_k(\boldsymbol{v}) = C'_k(\boldsymbol{v})$.

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  $\rightarrow$ Equal by definition of $D_t(\mathbf{s})$.

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$$\bigcup_{k \in \mathcal{I}} C'_k(\mathbf{s}) \cup \bigcup_{j \in \{1,\dots,n\} \setminus \mathcal{I}} C_j(\mathbf{s}) = \bigcup_{j \in \{1,\dots,n\} \setminus \mathcal{I}} C_j(\mathbf{v}) \cup \bigcup_{k \in \mathcal{I}} C'_k(\mathbf{v}).$$

$\rightarrow$ Equal since $C'_k(\mathbf{s}) = C'_k(\mathbf{v})$.

Insertion-Deletion Equivalence

- Let $\boxed{C'_k(\mathbf{s})} = C_k(\mathbf{s}) \cup C_k(\mathbf{v}) = C'_k(\mathbf{v})$.

  $C_k(\mathbf{s})$ corrupted by insertion of compositions $C_k(\mathbf{v}) \setminus C_k(\mathbf{s})$.

- Define t-multiset insertion ball as

$$I_t(\mathbf{s}) = \{C'(\mathbf{s}) = \bigcup_{i \in \{1,\dots,n\}} C'_i(\mathbf{s}) : \mathcal{I} \subseteq \{1,\dots,n\},\ |\mathcal{I}| \leq t,$$
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- $D_t(\boldsymbol{s}) \cap D_t(\boldsymbol{v}) \neq \emptyset \iff I_t(\boldsymbol{s}) \cap I_t(\boldsymbol{v}) \neq \emptyset$

Multiset Deletions

Definition

An **asymmetric multiset deletion** occurs in the composition multiset $C(\boldsymbol{s})$ of a string $\boldsymbol{s} \in \{0,1\}^n$, if for some $i \in [n]$, the multiset $C_i(\boldsymbol{s})$ is entirely missing, while $C_{n-i+1}(\boldsymbol{s})$ is not corrupted.

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Lemma

$\mathcal{S}_R(n)$ can correct a single asymmetric multiset deletion.

Asymmetric Multiset Deletions

Construction

$$\begin{aligned}\mathcal{S}_{DA}^{(t)}(n) = \{ \boldsymbol{s} \in \{0,1\}^n : s_1 = 0, s_n = 1, \text{ and } \\ \exists I \subset \{2, \ldots, \frac{n}{2}\},\ |I| \geq t, \text{ such that } \\ \forall\, i \in I, s_i \neq s_{n+1-i}, \text{ and } \forall i \notin I, s_i = s_{n+1-i}, \\ \boldsymbol{s}_{[n/2] \cap I} \text{ is a string wherein each } \\ \text{prefix has at least } t \text{ more 0s than 1s} \}.\end{aligned}$$

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- Is an extension of $\mathcal{S}_R(n)$ [PGM19] .
- Involves at most $0.5 \log(n - 2t) + 2t + 3$ redundant bits.

[PGM19] S. Pattabiraman, R. Gabrys, and O. Milenkovic, “Reconstruction and Error-Correction Codes for Polymer-Based Data Storage”, in *2019 IEEE Information Theory Workshop (ITW)*, Visby, Sweden: IEEE, Aug. 2019, pp. 1–5

Asymmetric Multiset Deletions

Theorem

$\mathcal{S}_{DA}^{(t)}(n)$ can correct up to t-asymmetric multiset deletions.

- Proof idea [PGM21]:

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  -Includes all $\boldsymbol{v}$ such that if $(\boldsymbol{s}_1^k, \boldsymbol{s}_{n-k+1}^n) = (\boldsymbol{v}_1^k, \boldsymbol{v}_{n-k+1}^n)$ and $s_{k+1} \neq v_{k+1}$, then for $i \in \{1, \ldots, t+1\}$
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  $$|C_{n-k-i}(\boldsymbol{s}) \setminus C_{n-k-i}(\boldsymbol{v})| \leq 2.$$
  2. We infer that no $\boldsymbol{v}$ simultaneously satisfies
  $$C_{n-k-t-1}(\boldsymbol{s}) = C_{n-k-t-1}(\boldsymbol{v}).$$

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Asymmetric Multiset Deletions

- Thus, if $C_{n-k-1}(\boldsymbol{s}), \ldots, C_{n-k-t}(\boldsymbol{s})$ are deleted, each $\boldsymbol{v} \in \mathcal{S}_{DA}^{(t)}(n)$ upholds

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- When deleted multisets are nonconsecutive, proof follows similarly.

Introduction

Problem Statement & Results

Conclusion

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Summary of Results

- A code that can correct t multiset deletions, iff correct composition insertions in t multisets.
- $\mathcal{S}_R(n)$ can correct the deletion of a single multiset.
- To correct the deletion of t asymmetric multisets, $0.5\log(n-2t)+2t+3$ redundant bits suffice.
- For any $1 \leq i \leq n$, $\mathcal{S}_R(n)$ can correct the deletion of multisets $C_i(\boldsymbol{s})$ and $C_{n-i+1}(\boldsymbol{s})$.

Future Work

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- Extension to **circular polymers**, i.e. bits on a ring.

Thank you!

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