Convergence of a Series

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1 Preamble

I will first begin by summarizing some terms and conclusions that were discussed within the video lecture.

Definition 1.1. Given a real sequence $(a_n)_{n=0}^{\infty}$ we can form a sequence of sums $S_N = a_0 + \cdots + a_N = \sum_{n=0}^{\infty} a_n$.

Definition 1.2. If the sequence $(S_N)_{N=0}^{\infty}$ converges to some $s \in \mathbb{R}$ we say that the series given by the partial sums S_N **converges** and write:

$$s = \sum_{k=0}^{\infty} a_k = \lim_{N\to\infty} S_N$$

Definition 1.3. A series $\sum_{n=0}^{\infty} a_n$ **diverges** if the $\lim_{N\to\infty} S_N$ does not converge.

Example 1.4

Let $0 < r < 1$ and define $a_n = r^n$, we shall prove from the defintion that $\sum_{n=0}^{\infty} a_n$ converges.

Proof. We know that $S_N = a_0 + a_1 + \cdots + a_n = 1 + r + r^2 + \cdots + r^N$. Since $(1-r)(1 + r + r^2 + \cdots + r^N) = (1 - r^{N+1})$ we can conclude that $S_N = \frac{1-r^{N+1}}{1-r}$. Taking limits:

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1-r^{N+1}}{1-r} = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

$\square$

Definition 1.5. Let s_n be a sequence of real numbers and let E be the set of numbers x such that $s_{n_k} \to x$ for some subsequence s_{n_k}. Define $a = \sup E$ then:

$$\lim_{n\to\infty} \sup s_n = a$$

Theorem 1.6

If $|a_n| \leq c_n$ for $n \geq N_0$ where N_0 is some fixed integer and if $\sum c_n$ converges then $\sum a_n$ converges.

The proof for Theorem 1.6 is simply an application for the Cauchy Criterion for series which is discussed in the video and omitted here due to lack of space.

Theorem 1.7

If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

The proof for Theorem 1.7 is a specific case of the Cauchy Criterion for series where $m = n$ that is discussed in the video and omitted here due to lack of space.

Theorem 1.8

Given $\sum a_n$, define $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{a_n}$ then:
(1) If $\alpha < 1$ then $\sum a_n$ converges.
(2) If $\alpha > 1$ then $\sum a_n$ diverges.
(3) If $\alpha = 1$ then the test gives no information

Proof. (1) Since $\alpha > 1$, let $\epsilon > 0$ such that $\alpha < \alpha + \epsilon < 1$. From the defintion of $\lim\sup$ observe that:

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{N\to\infty} \sup \left\{ |a_n|^{\frac{1}{n}} : n > N \right\} = \alpha$$

Now by the limit definition and our chosen ϵ, there must exist some N_1 such that if $N > N_1$ then:

$$\left| \sup \left\{ |a_n|^{\frac{1}{n}} : n > N \right\} - \alpha \right| < \epsilon$$

Since the statement above is true notice that:

$$\sup \left\{ |a_n|^{\frac{1}{n}} : n > N \right\} - \alpha < \epsilon$$

$$\sup \left\{ |a_n|^{\frac{1}{n}} : n > N \right\} < \epsilon + \alpha$$

Now by the defintion of the supremum we know that for all $n > N$:

$$|a_n|^{\frac{1}{n}} < \alpha + \epsilon$$

$$|a_n| < (\alpha + \epsilon)^n$$

Now taking the summation we can conclude that:

$$\sum_{n=N+1}^{\infty} |a_n| \leq \sum_{n=N+1}^{\infty} (\alpha + \epsilon)^n$$

We know that $\alpha + \epsilon < 1$, therefore the series is above is simply a geometric series, which we know converges. Thus:

$$\sum_{n=N+1}^{\infty} |a_n| \leq \sum_{n=N+1}^{\infty} r^n$$

By Theorem 1.6, $\sum a_n$ converges.
(2) We know that for any sequence s_n there exists a subsequence s_{n_k} that converges to

$\lim_{n\to\infty} \sup s_n$. So there must exist a subsequence $|a_{n_k}|^{\frac{1}{n_k}}$ for $|a_n|^{\frac{1}{n}}$ that converges to $\limsup |a_n|^{\frac{1}{n}} = \alpha > 1$. Let $\epsilon = \alpha - 1$ then since $|a_{n_k}| \to \alpha$ there exists some K such that if $k > K$:

$$\left| |a_{n_k}|^{\frac{1}{n_k}} - \alpha \right| < \epsilon = \alpha - 1$$

Therefore:

$$|a_{n_k}|^{\frac{1}{n_k}} - \alpha > -(\alpha - 1) = 1 - \alpha$$

$$|a_{n_k}|^{\frac{1}{n_k}} > 1$$

$$|a_{n_k}| > 1$$

Now according to Theorem 1.7 if $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$. However this is impossible because the subsequence $a_{n_k} > 1$ for every k. So, $a_n \nrightarrow 0$, and $\sum a_n$ diverges.
(3) Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. Observe that:

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{\frac{1}{n}} = 1$$

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{\frac{1}{n^2}} = 1$$

However $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges, so the test tells us nothing. □

2 Background Information

Now I will discuss some additional topics as they relate to series and conclude with their placement in the boader context of the course.

Definition 2.1. Suppose that f_n is a **sequence of functions** defined on a set E and suppose that the sequence of numbers $f_n(x)$ converges for every $x \in E$. We can define a function f by:

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Definition 2.2. A sequence of functions $(f_n)_n^\infty$ **pointwise converges** to f if for any $\epsilon > 0$ and every $p \in E$ there exists some $N \in \mathbb{N}$ such that $|f_n(p) - f(p)| < \epsilon$ for $n \geq N$.

Definition 2.3. A sequence of functions $(f_n)_n^\infty$ **uniformly converges** to f if for any $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|f_n(p) - f(p)| < \epsilon$ for $n \geq N$ and $p \in E$.

Definition 2.4. Given a real sequence $(a_n)_{n=0}^\infty$ we define the **power series** centered at $x_0 \in \mathbb{R}$ by $\sum_{n=0}^\infty a_n(x - x_0)^n$.

Definition 2.5. A power series **converges** at $r \in \mathbb{R}$ if $\sum_{n=0}^\infty a_n(x - x_0)^n$ converges with x fixed and diverges at x otherwise.

Definition 2.6. Given a subset $E \subseteq \mathbb{R}$ we say that the power series **converges** (pointwise or uniformly) on E if $S_N(x) = a_0 + a_1(x - x_0) + \cdots + a_N(x - x_0)^N$ converges (pointwise or uniformly) as a sequence of functions to some $f(x)$ on E.

3 Context in the Course

Now, the most important attribute of a series and power series is their ability to define specific functions such as e^x or $\sin(x)$ as such functions cannot be studied otherwise. To show this, let us define e^x as a power series:
To make this task easier let us assume that we know the following limit.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Form this, it becomes clear that $e^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$ where $f_n(x) = \left(1+\frac{1}{n}\right)^{nx}$ is a sequence of functions.
Let us rewrite $f_n(x)$ such that $y = nx$ and $\frac{1}{n} = \frac{x}{y}$. Therefore:

$$e^x = \lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y$$

Using the Binomial Expansion Theorem[1] where $y \in \mathbb{Z}$ we obtain the power series expansion of e^x:

$$e^x = \lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y = \lim_{y\to\infty}\left(1+y\left(\frac{x}{y}\right)+\frac{y(y-1)}{2!}\left(\frac{x}{y}\right)^2+\frac{y(y-1)(y-2)}{3!}\left(\frac{x}{y}\right)^3+\cdots\right)$$

We only care about the limits so $\lim_{y\to\infty}\frac{y(y-1)\cdots(y-(n-1))}{y^n} = 1$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Now that we have a defintion of e^x we can also study some of its properties. Specifcially, we were told in Calculus that $\frac{d}{dx}e^x = e^x$. To better understand why let us differentiate the power series of e^x.

$$\frac{d}{dx}(e^x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} = \frac{d}{dx}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

As expected our Calculus knowledge is proven to be correct!
Notice how there would not be a way to study the differentiation of e^x without defining it as a power series. In fact, all of its properties can only be studied by defining e^x as a power series. And this idea is true for a multitude of functions used continuously throughout Calculus, thereby making the idea of a series integral to the study of Real Analysis and beyond.

[1] The Binomial Expansion Theorem only relevent here if the exponent is a whole number, and that is not true for general values of x. However, we are only interested in the limit of a very large n. Since the function is continuous in y, if x is a irrational we can approximate using a sequence of rational numbers to obtain the same result.