An Introduction to Linear Algebra & Quantum Mechanics

Welcome!

What do you think a vector is?

Background

A set is can be thought of as a collection of objects.

1.8 **Definition** ***list, length***

Suppose n is a nonnegative integer. A ***list*** of ***length*** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \ldots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

F denotes either the set of complex numbers (C) or real numbers (R)

F is filled with scalars (fancy word for 'numbers')

1.10 **Definition** $\mathbf{F}^n$

$\mathbf{F}^n$ is the set of all lists of length n of elements of $\mathbf{F}$:

$$\mathbf{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \ldots, n\}.$$

For $(x_1, \ldots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the j^{th} ***coordinate*** of $(x_1, \ldots, x_n)$.

Background

1.12 **Definition** ***addition in $\mathbf{F}^n$***

Addition in $\mathbf{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n).$$

1.13 **Commutativity of addition in $\mathbf{F}^n$**

If $x, y \in \mathbf{F}^n$, then $x + y = y + x$.

In math we like to use definitions to prove things…
Definitions → Statements → Proofs

A statement can be either true or false, we use proofs to determine the validity of a statement

Background

1.13 Commutativity of addition in $\mathbf{F}^n$

If $x, y \in \mathbf{F}^n$, then $x + y = y + x$.

Proof Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$
\begin{aligned}
x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\
&= (x_1 + y_1, \ldots, x_n + y_n) \\
&= (y_1 + x_1, \ldots, y_n + x_n) \\
&= (y_1, \ldots, y_n) + (x_1, \ldots, x_n) \\
&= y + x,
\end{aligned}
$$

Background

1.14 **Definition** 0

Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0).$$

1.16 **Definition** ***additive inverse in*** $\mathbf{F}^n$

For $x \in \mathbf{F}^n$, the ***additive inverse*** of x, denoted $-x$, is the vector $-x \in \mathbf{F}^n$ such that

$$x + (-x) = 0.$$

In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$.

1.17 **Definition** ***scalar multiplication in*** $\mathbf{F}^n$

The ***product*** of a number λ and a vector in $\mathbf{F}^n$ is computed by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n);$$

here $\lambda \in \mathbf{F}$ and $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

Vectors and Vector Spaces

1.19 **Definition** ***vector space***

A ***vector space*** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity
$u + v = v + u$ for all $u, v \in V$;

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;

additive identity
there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;

additive inverse
for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;

multiplicative identity
$1v = v$ for all $v \in V$;

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

1.18 **Definition** ***addition, scalar multiplication***

- An ***addition*** on a set V is a function that assigns an element $u+v \in V$ to each pair of elements $u, v \in V$.
- A ***scalar multiplication*** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

This property is commonly known as being ‘closed’ under addition and scalar multiplication.

Subspaces

1.32 Definition ***subspace***

A subset U of V is called a ***subspace*** of V if U is also a vector space (using the same addition and scalar multiplication as on V).

I told you what a set was… what do you think a subset is?

Quantum Mechanics

In quantum mechanics the state of a particle is represented as a vector which lives in a complex Hilbert space (infinite-dimensional vector space)

Vectors in Dirac Notation

In quantum we commonly use Dirac Notation, where vectors such as v and u above are represented as kets $v \to |\psi\rangle$.

A special property of Hilbert spaces is that they are L^2 spaces

L^2 Spaces

L^2 defines the set of all square integrable functions on a specific interval $[a, b]$.

$$L^2(a, b) := \left\{ f(x) : \int_a^b |f(x)|^2 dx < \infty \right\}$$

For physicists Hilbert spaces are L^2 spaces, by defintion. But mathemathicians can refer to them as separate things.

Quantum Mechanics

Why are there complex numbers in quantum mechanics?

Schrodinger's Eqn

H represents the Hamiltonian which represents the total energy of the system

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$$

There is a real-valued counterpart:

$$\hbar^2\frac{\partial^2\psi}{\partial t^2} = -H^2\psi$$

The complex version is much simpler to calculate

- In general there isn't a conclusive answer

Linear Combinations

2.3 Definition ***linear combination***

A ***linear combination*** of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

2.17 Definition ***linearly independent***

- A list $v_1, \ldots, v_m$ of vectors in V is called ***linearly independent*** if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$.
- The empty list ( ) is also declared to be linearly independent.

2.5 Definition ***span***

The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the ***span*** of $v_1, \ldots, v_m$, denoted $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbf{F}\}.$$

The span of the empty list ( ) is defined to be $\{0\}$.

Bases and Dimensionality

2.27 Definition ***basis***

A ***basis*** of V is a list of vectors in V that is linearly independent and spans V.

Before thinking about dimensionality let's consider an important question...

Do you think that all basis of a specific vector space have the same length?

Bases and Dimensionality

2.35 Basis length does not depend on basis

Any two bases of a finite-dimensional vector space have the same length.

2.23 Length of linearly independent list $\leq$ length of spanning list

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof Suppose V is finite-dimensional. Let B_1 and B_2 be two bases of V. Then B_1 is linearly independent in V and B_2 spans V, so the length of B_1 is at most the length of B_2 (by 2.23). Interchanging the roles of B_1 and B_2, we also see that the length of B_2 is at most the length of B_1. Thus the length of B_1 equals the length of B_2, as desired. ■

Bases and Dimensionality

2.36 **Definition** ***dimension*, dim V**

- The ***dimension*** of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by dim V.

Basis in Dirac Notation

In a n-dimensional Hilbert space, we can write any ket $|\psi\rangle$ in terms of $|n\rangle$ other kets (the basis vectors) multiplied by specific scalars.

$$|\psi\rangle = a_1|1\rangle + a_2|2\rangle + a_3|3\rangle + \cdots + a_n|n\rangle$$

ASIDE: We can also add different kets. Assuming $|\psi_1\rangle = a_1|1\rangle + \cdots + a_n|n\rangle$ and $|\psi_2\rangle = b_1|1\rangle + \cdots + b_n|n\rangle$ then:

$$|\psi_1\rangle + |\psi_2\rangle = (a_1 + b_1)|1\rangle + \cdots + (a_n + b_n)|n\rangle$$

Linear Maps & Matrices

3.2 Definition ***linear map***

A ***linear map*** from V to W is a function $T: V \to W$ with the following properties:

additivity

$$T(u + v) = Tu + Tv \text{ for all } u, v \in V;$$

homogeneity

$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbf{F} \text{ and all } v \in V.$$

Linear Maps & Matrices

Vectors and states can also be represented as matrices…

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_n \end{bmatrix}$$

What do you think a matrix is?

Linear Maps and Matrices

3.32 **Definition** ***matrix of a linear map, $\mathcal{M}(T)$***

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The ***matrix of*** T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases are not clear from the context, then the notation $\mathcal{M}\big(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m)\big)$ is used.

3.62 **Definition** ***matrix of a vector, $\mathcal{M}(v)$***

Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. The ***matrix of*** v with respect to this basis is the n-by-1 matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \ldots, c_n$ are the scalars such that

$$v = c_1v_1 + \cdots + c_nv_n.$$

Inner Product Spaces

Hilbert spaces are inner product spaces.

6.3 Definition ***inner product***

An ***inner product*** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbf{F}$ and has the following properties:

positivity

$\langle v, v\rangle \geq 0$ for all $v \in V$;

definiteness

$\langle v, v\rangle = 0$ if and only if $v = 0$;

additivity in first slot

$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;

homogeneity in first slot

$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$;

conjugate symmetry

$\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

6.5 Definition ***inner product space***

An ***inner product space*** is a vector space V along with an inner product on V.

Inner Product Spaces

We can define inner products in any way we wish as long as the conditions previously are satisfied.

- The Euclidean Inner Product (Dot Product).

$$\langle (v_1, \cdots, v_n), (u_1, \cdots, u_n) \rangle = v_1\bar{u}_1 + \cdots + v_n\bar{u}_n$$

- Continuous Real Valued Funtions.

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

- On $\mathcal{P}(\mathbb{R})$ (the set of all polynomails with real coefficients).

$$\langle p, q \rangle = \int_0^\infty p(x)q(x)e^{-x}dx$$

Observables

In QM the things that you can measure (position, momentum, velocity, etc) are denoted as observables.

Observables are represented as Hermitian operators!

Defintion: Operator

A linear map from a vector space to itself is called an **operator**.

Defintion: Adjoint

Suppose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is a function $T^\dagger : W \to V$ such that:

$$\langle Tv, w \rangle = \langle v, T^\dagger w \rangle$$

for every $v \in V$ and $w \in W$.

Defintion: Hermitian

An operator is called **hermitian** if $T = T^\dagger$.

Eigenvalues and Eigenvectors

Defintion: Eigenvalues/Eigenvectors

Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists a $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. And v is the corresponding **eigenvector**.

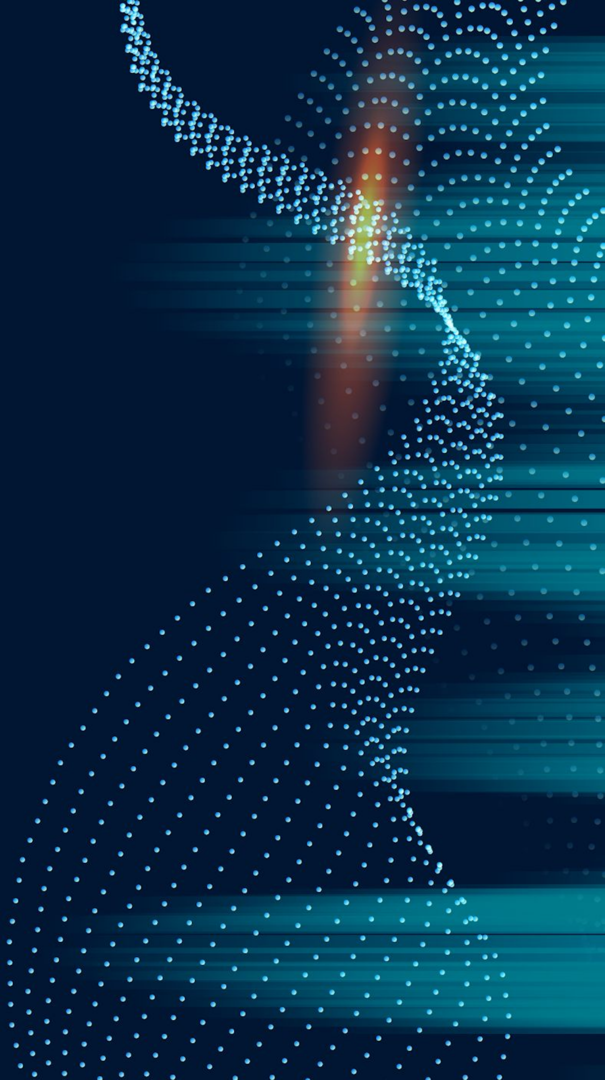
One of the postulates of QM….

If you measure an observable $\hat{Q}$ on a particle in some state $|\Psi\rangle$ you will get one of the eigenvalues of $\hat{Q}$.

Observables as Hermitian Operators

A complicated question that is worth asking…

Why are observables represented as Hermitian operators?



Observables as Hermitian Operators

Let $\mathcal{H}$ be a hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be an Hermitian operator. Then all the eigenvalues of T are real.

When you measure something you want to get a real number!

Observables as Hermitian

Eigenvalues of Hermitian Operators are Real

Let $\mathcal{H}$ be a hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be an Hermitian operator. Then all the eigenvalues of T are real.

Proof. Let λ be an eigenvalue of T corresponding with eigenvector $v \in \mathcal{H}$. Then $Tv = \lambda v$. Consider the following:

$$
\begin{aligned}
\lambda\langle v, v\rangle &= \langle \lambda v, v\rangle \\
&= \langle Tv, v\rangle \\
&= \langle v, Tv\rangle \\
&= \overline{\langle Tv, v\rangle} \\
&= \overline{\langle \lambda v, v\rangle} \\
&= \bar{\lambda}\langle v, v\rangle
\end{aligned}
$$

The only way $\lambda = \bar{\lambda}$ is if $\lambda \in \mathbb{R}$. $\square$

Observables

The term ‘complete’ means that any wavefunction can be represented in terms of this basis.

Defintion: Orthogonal

Two vectors $u, v \in V$ are called **orthogonal** if $\langle u, v \rangle = 0$.

Defintion: Norm

For $v \in V$ the **norm** of v, denoted as $||v||$, is defined by:

$$||v|| = \sqrt{\langle v, v \rangle}$$

Defintion: Orthonormal

A list of vectors is called **orthonormal** if each vector in the list has norm 1 and is orthogonal to all other vectors in the list.

Defintion: Orthonormal Basis

An **orthonormal basis** of V is an orthonormal list of vectors in V that is also a basis of V.

Probabilities

In quantum things exist in spectrum of probability…

The probability of measuring a specific value for an observable is given by the Born Rule.

Born Rule

If a system is in a state $|\Psi\rangle$ (assuming pure state) then the probability $\mathbb{P}$ that an eigenvalue λ_i of q_i is found when $\hat{Q}$ is measured is:

$$\mathbb{P}(\lambda_i) = |(q_i, \Psi)|^2$$

Born Rule

Born Rule

If a system is in a state $|\Psi\rangle$ (assuming pure state) then the probability $\mathbb{P}$ that an eigenvalue λ_i of q_i is found when $\hat{Q}$ is measured is:

$$\mathbb{P}(\lambda_i) = |(q_i, \Psi)|^2$$

Let's proceed to break this down. Consider again our previous representation of $|\Psi\rangle$ in the $\hat{Q}$ basis:

$$|\Psi\rangle = c_1|q_1\rangle + \cdots + c_n|q_n\rangle$$

Assume that this state is properly normalized such that $\sum_i |c_i|^2 = 1$. Then the Born rule is basically saying that:

$$\begin{aligned}
\mathbb{P}(\lambda_i) &= |(q_i, \Psi)|^2 \\
&= \left|\left\langle q_i \middle| c_1|q_1\rangle + \cdots + c_n|q_n\rangle \right\rangle\right|^2 \\
&= |c_1\langle q_i|q_1\rangle + \cdots + c_i\langle q_i|q_i\rangle + \cdots + c_n\langle q_i|q_n\rangle|^2 \\
&= |c_i\langle q_i|q_i\rangle|^2 \quad \text{all } \langle q_i|q_j\rangle = 0 \text{ if } i \neq j \text{ by orthogonality} \\
&= |c_i|^2 \quad \text{by orthonormality } \langle q_1|q_i\rangle = 1
\end{aligned}$$

Measurement Collapse

A common somewhat unexplained phenomena in QM is the measurement collapse of every quantum system.

$$|\Psi_{\text{before}}\rangle = c_1|q_1\rangle + \cdots + c_n|q_n\rangle \implies |\Psi_{\text{after}}\rangle = |q_j\rangle$$

So if you were to measure Q again you'd obtain the eigenvalue of q_j with 100% certainty.