Contents

Math 123: Notes

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§1 Introduction

Consider $\vec{y}' = \vec{f}(t, \vec{y})$ denoted as (1):

Definition 1.1. If $\vec{f} = A(t) \cdot \vec{y}$ then (1) is linear and homogeneous.

Definition 1.2. If $\vec{f} = A(t) \cdot \vec{y} + g(t)$ then (1) is linear and inhomogeneous.

§2 Solution Techniques

§2.1 Separation of Variables

Consider

$$y' = g(t)h(y)$$

Then:

$$\frac{y'}{h(y)} = g(t) \implies \int_{t_0}^{t} \frac{y'}{\ln(y)} dt = \int_{t_0}^{t} g(s)ds$$

Applying change of variables $w = y$ then $dw = y'dt$:

$$\int_{y(t_0)}^{y(t_1)} \frac{dw}{h(w)} = \int_{t_0}^{t_1} g(s)ds$$

§2.2 Method of Integrating Factors

Consider where $a, b : \mathbb{R} \to \mathbb{R}$ are continuous:

$$y' = a(t)y + b(t)$$

We want to find $I(t)$ such that $(Iy)' = I(y' - a(t)y) = Ib(t)$.

$$I(t)'y + I'(t)y = I(y' + a(t)y)$$

$$\Rightarrow I'y = -Ta(t)y$$

$$\Rightarrow (\ln(I(t)))' = -a(t)I(t)$$

$$\Rightarrow I(t) = e^{-\int_{t_0}^{t_1} a(s)ds}$$

$$\Rightarrow (Iy)' = I(t)b(t)$$

Integrating both sides:

$$\int_{t_0}^{t}(Iy)'ds = \int_{t_0}^{t} I(s)b(s)ds \implies I(t)y(t) - I(t_0)y(t_0) = \int_{t_0}^{t} I(s)b(s)ds$$

$$y(t) = y(t_0)e^{\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{\int_s^t a(w)dw}b(s)ds$$

Consider the following example:

$$y' = a(t)y + b(t)y^n$$

Dividing by y^n:

$$\frac{y'}{y^n} = a(t)y^{1-n} + b(t)$$

$$\Rightarrow (y^{-n+1})' = (-n+1)y^{-n}y' \Rightarrow y^{-n}y' = \frac{1}{1-n}(y^{-n+1})'$$

$$\Rightarrow \frac{1}{1-n}(y^{-n+1})' = a(t)y^{-n+1} + b(t)$$

Set $z = y^{-n+1}$ and proceed using method of integrating factor.

§3 Gronwall's Inequality

Theorem 3.1

Let f, g be continuous functions on $I = [a, b]$ where $g \geq 0$. Suppose f is differentiable on (a, b) and that $f'(t) \leq g(tf(t))$ for all $t \in (a, b)$. Then $f(t) \leq f(a)e^{\int_a^t g(s)ds}$.

Proof. We know that $f'(t) - g(t)f(t) \leq 0$ for all $t \in I$. Let $H = e^G$ where $G = -\int_a^t g(s)ds$ then:

$$H(t)(f'(t) - g(t)f(t)) \leq 0$$

Using $H' = -g(t)H(t)$:

$$\Rightarrow (H(t)f(t))' \leq 0$$

Claim 3.2 — $(f(t)H(t))' \leq 0$

Proof. We know that $f'H + fH' \leq 0$. By defintion $H' = -gH$ therefore $(fH)' = f'H - fgH = H(f' - fg) \leq 0$. Integrating $(fH)'$:

$$\int_a^t (H(s)f(s))'ds \leq 0$$

By the Fundamental Thm of Calculus:

$$Hf - H(a)f(a) \leq 0$$

$$\Rightarrow f(t) \leq H^{-1}f(0)$$

$$H(a) = e^{\int_a^t g(s)ds}f(a)$$

□

Theorem 3.3

Let $K > 0$ where $K \in \mathbb{R}$ and f, g are continuous on I such that $f \leq K + \int_a^t f(s)g(s)ds$ for all $t \in I$ where $g \geq 0$. Then $f(t) \leq Ke^{\int_a^t g(s)ds}$.

Proof. Let $U(t) = K^{-1} \int_a^t f(s)g(s)ds$ then $U' = f(t)g(t)$. Clearly:

$$U' \leq g(t) \left(K + \int_a^t f(s)g(s)ds \right) = g(t)U(t)$$

Using $U(a) = K$ by Gronwall's inequality in Differential Form:

$$U(t) \leq Ke^{\int_a^t g(s)ds}$$

□

□

§4 Well-Posedness

Lemma 4.1

Let $f : R \to R$ where $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$. If f is continuous on R and $|\partial_y f(t, y)| \leq K$ for all $(t, y) \in R$ then f is Lipschitz continuous with respect to y.

Proof. Suppose $y_2 \geq y_1$ then:

$$\begin{aligned} |f(t, y_2) - f(t, y_1)| &\leq \left| \int_{y_1}^{y_2} \partial_y f(t, \bar{y}) d\bar{y} \right| \\ &\leq \int_{y_1}^{y_2} |\partial_y f(t, \bar{y}) d\bar{y}| \\ &\leq \int_{y_1}^{y_2} K d\bar{y} = K|y_2 - y_1| \end{aligned}$$

□

§5 Picard-Lindelof Theorem

Theorem 5.1

Consider the following initial value problem:

$$(**) \begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Let $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$. Assume that:
(1) f is continuous in t and y on R, with $|f(t, y)| \leq M$ for any $(t, y) \in R$.
(2) f is Lipschitz continuous with respect to y on R.
Then there exists an $\alpha > 0$ and a solution $\phi : I \to R$ of $(**)$ where $I = (t_0 - \alpha, t_0 + \alpha)$.

Lemma 5.2

ϕ is a solution to $(**)$ iff ϕ is continuous and satisfies:

$$(1) \qquad \phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

for all $t \in I$

Proof. Suppose ϕ is a solution to $(**)$ then:

$$\int_{t_0}^{t} \phi'(s)ds = \phi(t) - \phi(t_0) = \phi(t) - y_0$$

$$\Rightarrow \phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

Conversely define $\phi(t)$ by (1), then $\phi(t)$ is differentiable and:

$$\begin{aligned}\phi'(t) &= \left(y_0 \int_{t_0}^{t} f(s, \phi(s))ds\right)' \\ &= f(s, \phi(s))\Big|_{t_0}^{t} = f(t, \phi(t))\end{aligned}$$

□

Remark: $\alpha = \min\left\{a, \frac{b}{M}\right\}$

Notice that the variation of $\phi(t)$ in I is at most:

$$|\phi(t_0) - \phi(t)| \leq M\alpha$$

And $M\alpha < b$ which implies that $\alpha < \frac{b}{M}$.

Proof. **Step 1**: It suffices to show that there exists a $\phi : I \to R$, where ϕ is continuous such that:

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

Step 2: Picard Iteration.
Construct a sequence of functions ϕ_j such that for all $j = 0, 1, \cdots$:

$$\begin{cases}\phi_0(t) = y_0 \\ \phi_{j+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_j(s))ds\end{cases}$$

Remark: Spoiler for the Proof

The solution to $(**)$ will be found by $\phi(t) = \lim_{j\to\infty} \phi_j(t)$.

Before proceeding we need to check that the ϕ_j's are defined on an interval independent of j. To do so we shall prove the following lemma:

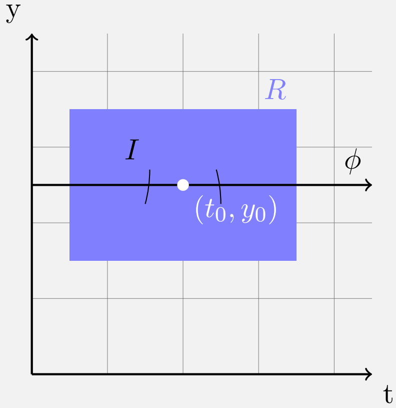
Lemma 5.3

Let $\alpha = \min\left\{a, \frac{b}{M}\right\}$ then $\phi_j : I \to R$ is well defined where $I = (t_0 - \alpha, t_0 + \alpha)$ and satisfies $(t, \phi_j(t)) \in R$ for any $j = 0, 1, \cdots$ and any $t \in I$.

Proof. We shall proceed using induction.
(1) **Base Case**: $\phi_0(t) \in R$ for any $t \in I$. This is true because $\alpha < a$ and $y_0 \in (y_0 - b, y_0 + b)$.

Remark: A Bit More Understanding

Consider the graph of ϕ_0 inside the rectangle of R over the interval I:



Notice that the constant line doesn't escape the rectangle over the interval I.

(2) **Inductive Step:** We want to show that if $(t, \phi_j(t)) \in R$ for any $t \in I$, then $(t, \phi_{j+1}(t)) \in R$ for any $t \in I$.
Compute (assuming $t > t_0$):

$$\begin{aligned}
|\phi_{j+1}(t) - y_0| &= \left| \int_{t_0}^{t} f(s, \phi_j(s)) ds \right| \\
&\leq \int_{t_0}^{t} |f(s, \phi_j(s))| ds && \text{by Minkowski inequality} \\
&\leq \int_{t_0}^{t} M ds = M(t - t_0) && \text{by assumption } f \leq M \\
&< M\alpha = \frac{b}{M} \cdot M = b
\end{aligned}$$

This clearly implies that $(t, \phi_{j+1}(t)) \in R$ for any $t \in I$, completing the proof of this lemma. $\square$

Step 3: Show that the ϕ_j's converge.
We shall use the following method to prove convergence.

Remark: Method for Proof of Convergence

Consider the summation:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n(n+1)}=\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\lim_{N\to\infty}\left(1-\frac{1}{N+1}\right)=1$$

Specifcally $\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\cdots$ is a telescoping series.

Consider that $\phi_j=\phi_j-\phi_{j-1}+\phi_{j-1}-\phi_{j-2}+\phi_{j-2}+\cdots+\phi_0$. We can represent this as a summation:

$$\phi_j=\phi_0+\sum_{j'=1}^{j}(\phi_{j'}-\phi_{j'-1})$$

Our goal now is to study $\phi_{j'}-\phi_{j'-1}$. Consider the following:

$$\phi_{j'}=y_0+\int_{t_0}^{t}f(s,\phi_{j'-1}(s))ds \tag{1}$$

$$\phi_{j'-1}=y_0+\int_{t_0}^{t}f(s,\phi_{j'-2}(s))ds \tag{2}$$

Subtracting (1)-(2):

$$\phi_{j'}-\phi_{j'-1}=\int_{t_0}^{t}(f(s,\phi_{j'-1}(s))-f(s,\phi_{j'-2}(s)))ds$$

Taking absolute values, let us attempt to bound this difference:

$$\begin{aligned}|\phi_{j'}-\phi_{j'-1}|&=\left|\int_{t_0}^{t}(f(s,\phi_{j'-1}(s))-f(s,\phi_{j'-2}(s)))ds\right|\\&\leq\int_{t_0}^{t}|f(s,\phi_{j'-1}(s))-f(s,\phi_{j'-2}(s))|ds\qquad\text{by Minkowski inequality}\\&\leq\int_{t_0}^{t}K|\phi_{j'-1}-\phi_{j'-2}|ds\qquad\text{since }|f(t,y_1)-f(t,y_2)|\leq K|y_1-y_2|\text{ by Lipschitz condition}\end{aligned}$$

Wthout loss of generality assume that $t>t_0$. Here we are a bit stick in how to proceed with the bounds. So, let us try to find a pattern in the difference, which we can represent as a summation, to better study $|\phi_{j'}-\phi_{j'-1}|$.
First, $j'=1$

$$\begin{aligned}|\phi_1-\phi_0|&\leq\int_{t_0}^{t}|f(s,\phi(s))|ds\\&\leq\int_{t_0}^{t}Mds=M(t-t_0)\qquad\text{since }|f(t,y)|\leq M\text{ for any }(t,y)\in R\end{aligned}$$

Let us proceed with $j'=2$:

$$\begin{aligned}|\phi_2-\phi_1|&\leq\int_{t_0}^{t}K|\phi_1(s)-\phi_0(s)|ds\\&\leq\int_{t_0}^{t}MK(s-t_0)ds=\frac{MK(t-t_0)^2}{2}\end{aligned}$$

And $j' = 3$:

$$|\phi_3 - \phi_2| \leq MK^2 \frac{(t-t_0)^3}{2 \cdot 3}$$

Claim 5.4 — $I(j') = |\phi_{j'} - \phi_{j'-1}| \leq M \cdot K^{j'-1} \frac{(t-t_0)^{j'}}{j'!}$

We shall proceed to prove the remainder of this step using induction.
(1) **Base Case:** We have already shown this above for $j' = 1$.
(2) **Inductive Step:** Assume that $I(j')$ holds and we shall prove the case for $I(j'+1)$.

$$\begin{aligned}
|\phi_{j'+1} - \phi_{j'}| &\leq \int_{t_0}^{t} K|\phi_{j'} - \phi_{j'-1}|ds \\
&\leq MK^{j'-1}K \int_{t_0}^{t} \frac{(s-t_0)^{j'}}{j'!} ds && \text{Substituting in } I(j'). \\
&\leq MK^{j'} \frac{1}{j'} \cdot \frac{(t-t_0)^{j'+1}}{j'!} = \frac{MK^{j'}(t-t_0)^{j'+1}}{j'!}
\end{aligned}$$

Now since we wnat a qualitative bound on ϕ_j we shall proceed to finish the proof of convergence by using the Comparison Test (even though we have already shown that the series is Cauchy). Recall that:

Remark: Comparison Test

If $\{a_n\}$ and $\{b_n\}$ are such that $|a_n| \leq |b_n|$ and $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges.

Applying this to ϕ_j:

$$\phi_j = \phi_0 + \sum_{j'=1}^{j} \phi_{j'} - \phi_{j'-1}$$

Taking absolute values:

$$\begin{aligned}
|\phi_j| &= \left| \phi_0 + \sum_{j'=1}^{j} \phi_{j'} - \phi_{j'-1} \right| \\
&\leq |\phi_0| + \left| \sum_{j'=1}^{j} \phi_{j'} - \phi_{j'-1} \right| && \text{by Triangle Inequality} \\
&\leq |y_0| + \sum_{j'=1}^{j} MK^{j'-1} \frac{(t-t_0)^{j'}}{j'!} \\
&= \frac{M}{K}(e^{K(t-t_0)} - 1)^{j'} && \text{since } \sum^{\infty} j' = 0 \frac{B^{j'}}{j'!} = e^B
\end{aligned}$$

The sequence of partial sums defining ϕ_j converges by the Comparison test for series. $\square$

A recap of what has been done so far:
Step 1: Reduction of the problem $(**)$ to an integral form. Look for $\phi : I \to \mathbb{R}$ such that

$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds$.
Step 2: Set up Picard Iteration where:

$$\begin{cases} \phi_0(t_0) = y_0 \\ \phi_j(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds \text{ for all } j = 1, 2, \cdots \end{cases}$$

Step 3: Proved that $\phi_j(t)$ is well-defined for $y \in \mathbb{R}$ and $t \in I$ such that $(t, \phi_j(t)) \in R$ for all $j \in \mathbb{N}$ and $t \in I$.
Step 4: Proved convergence of $\lim_{j\to\infty} \phi_j(t) = \phi(t)$.

- Estimate $\phi_j(t) - \phi_{j-1}(t)$:

$$|\phi_j - \phi_{j-1}| \leq \frac{M}{K} \frac{K^j (t - t_0)^j}{j!} \text{ where } t - t_0 \leq \alpha$$

$$\phi_j = \phi_0(t) + \sum_{j'=1}^{j} (\phi_{j'}(t) - \phi_{j'-1}(t))$$

- From the first bullet point, it follows that for fixed t, $\sum_{j'=1}^{j} |\phi_{j'}(t) - \phi_{j'-1}(t)| < \infty$ which means that the sum $\sum_{j'=1}^{j} (\phi_{j'}(t) - \phi_{j'-1}(t))$ must converge.

Now we shall proceed to prove that ϕ is continuous.

Proof. First consider the following rewrite:

$$\begin{aligned} |\phi - \phi_j| &= \left| \phi_0 + \sum_{j'=1}^{\infty} r_{j'} - \left(\phi_0 + \sum_{j'=1}^{j} r_{j'} \right) \right| && \text{where } r_{j'} = \phi_{j'} - \phi_{j'-1} \\ &= \left| \sum_{j' \leq j'+1}^{\infty} r_{j'} \right| \leq \sum_{j'=j'+1}^{\infty} |r_{j'}| \leq \sum_{j'=j'+1}^{\infty} \frac{M}{K} \frac{K^{j'} \alpha^{j'}}{j'!} \\ &= \sum_{n=0}^{\infty} \frac{M}{K} \frac{K^{n+j+1} \alpha^{n+j+1}}{(n+j+1)!} && \text{re-index } j' = j + 1 + n \\ &\leq \sum_{n=0}^{\infty} \frac{M}{K} \frac{k^{j+1} \alpha^{j+1}}{(j+1)!} \frac{K^n \alpha^n}{n!} && \text{since } (n + j + 1)! \geq (j + 1)! n! \\ &= \frac{M}{K} \frac{k^{j+1} \alpha^{j+1}}{(j+1)!} \sum_{n=0}^{\infty} \frac{K^n \alpha^n}{n!} \\ &= \frac{M}{K} \frac{k^{j+1} \alpha^{j+1}}{(j+1)!} e^{K\alpha} \end{aligned}$$

Let $\epsilon > 0$, we want to find some $\delta > 0$ such that $|t' - t| < \delta$ where $t, t' \in I$ for which $|\phi(t) - \phi(t')| < \epsilon$. We shall proceed to use $\frac{\epsilon}{3}$ argument. Consider:

$$\begin{aligned} |\phi(t) - \phi(t')| &= |\phi(t) - \phi_j(t) + \phi_j(t) - \phi_j(t') + \phi_j(t') - \phi(t')| \\ &\leq |\phi(t) - \phi_j(t)| + |\phi_j(t) - \phi_j(t')| + |\phi_j(t') - \phi(t')| \end{aligned}$$

From here we use $|\phi - \phi_j| \leq \frac{M}{K}\frac{k^{j+1}\alpha^{j+1}}{(j+1)!}e^{K\alpha}$. We can choose and N_1 such that for $j \geq N_1$ $|\phi(t) - \phi_j(t)| \leq \frac{\epsilon}{3}$ since $\lim_{j\to\infty} \frac{c^j}{j!} = 0$. And similarily we choose and N_2 such that for $j \geq N_2$ $|\phi_j(t) - \phi_j(t')| \leq \frac{\epsilon}{3}$. And by continuity of ϕ_j there exists a $\delta > 0$ such that $|\phi_j(t) - \phi_j(t')| \leq \frac{\epsilon}{3}$ for $|t - t'| < \delta$. ☐

To complete the proof we wnat to show that $\phi(t)$ satisfies $\phi(t) = y_0 + \int_{t_0}^t f(s, \phi(s))ds$.

Proof. We know $\phi_j(t) = y_0 + \int_{t_0}^t f(s, \phi_{j-1}(s))ds$. Taking limits of both sides as $j \to \infty$.
(1) LHS: $\phi_j(t) \to \phi$ by **Step 3** in the proof above.
(2) RHS: Consider the following difference:

$$\begin{aligned}\left|\int_{t_0}^t f(s, \phi_{j-1}(s))ds - \int_{t_0}^t f(s, \phi(s))ds\right| &\leq \int_{t_0}^t |f(s, \phi_{j-1}(s))ds - f(s, \phi(s))|ds \\ &\leq \int_{t_0}^t K|\phi_{j-1}(s) - \phi(s)|ds \\ &\leq K\int_{t_0}^t \frac{M}{K}\frac{K^j\alpha^j e^{K\alpha}}{j!}ds \\ &\leq \frac{MK^j\alpha^j e^{K\alpha}}{j!} \to 0 \text{ as } j \to \infty\end{aligned}$$

Since the LHS=RHS limit it's clear that:

$$\phi(t) = \lim_{j\to\infty} \phi_j(t) = \lim_{j\to\infty}\left(y_0 + \int_{t_0}^t f(s, \phi(s))ds\right) = y_0 + \int_{t_0}^t f(s, \phi(s))ds$$

☐

§6 Peano's Theorem

Theorem 6.1

Suppose that f is continuous on $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$. Suppose that there exists an $M \geq 0$ such that $|f(t, y)| \leq M$ for all $(t, y) \in R$. Then there exists an $I = (t_0 - \alpha, t_0 + \alpha)$ and a solution $\phi : I \to R$ of $(**)$.

Example of Peano's Theorem

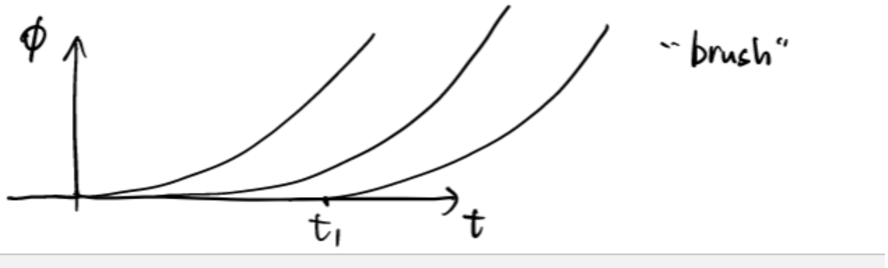
Let $R = [-1, 1] \times [-1, 1]$ and $f : \mathbb{R} \to \mathbb{R}$ such that $f = y^{\frac{1}{3}}$ is continuous but not Lipschitz continuous with respect to y.

$$\begin{cases} y' = y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$$

This does not satisfy the hypothesis of Picard-Linderlof, however it satisfies the hypothesis of Peano's Theorem. We proceed to guess the solution of type $y = ct^a$ then:

$$\begin{cases} \phi(t) = 0 & \text{for all } t \in R \\ \phi(t) = \left(\frac{2}{3}\right)^{\frac{3}{2}} t^{\frac{3}{2}} & \text{for } t \geq 0 \end{cases}$$

These are both solutions to the ODE above. And we end up with a brush of different solutions.



§7 Existence Theorem for Systems of 1st Order ODEs

Definition 7.1. Let $\vec{x} \in \mathbb{R}^n$. Define

- $|x| = \sum_{j=1}^{n} |x_j|$
- $||x|| = \sqrt{\sum_{j=1}^{n} x_j^2}$

Lemma 7.2

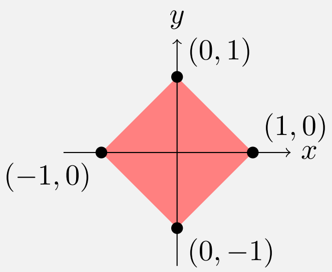
Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ then:

$$|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$$

$$||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$$

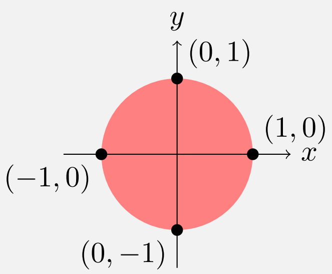
Definition 7.3. Let $\vec{x} \in \mathbb{R}^n$ and let $s \in \mathbb{R}$ such that $s > 0$ then $R_s(\vec{a}) = \{\vec{x} \in \mathbb{R}^n : |\vec{x} - \vec{a}| < s\}$

In $n = 2$ $R_1(0) = \{(x, y); |x_1| + |y_1| = 1\}$



Definition 7.4. Let $\vec{a} \in \mathbb{R}^n$, let $r > 0$ then $B_r(\vec{a}) = \{\vec{x} \in \mathbb{R}^n : ||\vec{x} - \vec{a}|| \leq r\}$

If $n = 2$ $B_1(0)$



Definition 7.5. A set $U \subseteq \mathbb{R}^n$ is said to be **open** if for any $\vec{x} \in U$ there exists an $r > 0$ such that $B_r(\vec{x}) \subseteq U$.

Definition 7.6. A set $U \subseteq \mathbb{R}^n$ is said to be **closed** if $\mathbb{R}^n \backslash U$ is open.

Definition 7.7. Let $D \subseteq \mathbb{R}^n$ be open $\vec{f} : D \to \mathbb{R}^n$ is **continuous** at $\vec{x}_0 \in D$. If for any $\epsilon > 0$ there exists a $\delta > 0$ such that for $|\vec{x} - \vec{x}_0| < \delta$ implies $|f(\vec{x}) - f(\vec{x}_0)| < \epsilon$.

Definition 7.8. Consider the IVP (Initial Value Problem) where $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $\vec{f} = \begin{bmatrix} f_1(t, \vec{y}) \\ \vdots \\ f_n(t, \vec{y}) \end{bmatrix}$:

$$\begin{cases} \vec{y}' = f(t, \vec{y}) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

Let $D \subseteq \mathbb{R}^n$ be an open set. Suppose that $f : D \to \mathbb{R}$ is continuous on D.
We say that f is **Lipschitz continuous** with respect to $\vec{y}$ if there exists a $K > 0$ such that $|f(t, \vec{y}_1) - f(t, \vec{y}_2)| \leq K|\vec{y}_1 - \vec{y}_2|$ for all $(t, \vec{y}_i) \in D$.

Theorem 7.9

Let $R = [t_0 - a, t_0 + a] \times \{y \in \mathbb{R}^n : |\vec{y} - \vec{y}_0| < 0\}$. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is:
(1) Continuous on R.
(2) Lipschitz continuous on R with respect to $\vec{y}$.
(3) There exists an $M > 0$ such that $|\vec{f}(t, \vec{y})| \leq M$ for all $(t, \vec{y}) \in R$.
Then there exists an $\alpha > 0$ and a solution $\vec{\phi}(t)$ of

$$(***) \begin{cases} \vec{y}' = f(t, \vec{y}) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

on $I = (t_0 = \alpha, t_0 + \alpha)$ and $\alpha = \min\{a, \frac{b}{M}\}$

§8 Uniqueness for System of 1st Order ODE's

Theorem 8.1

Let R and f satisfy the assumptions of Thm 3.9. If $\vec{\phi}_1 : J_1 \to \mathbb{R}^n$ and $\vec{\phi}_2 : J_2 \to \mathbb{R}^\ltimes$ (where J_1, J_2 are open intervals) are both solutions for $(***)$ then $\phi_1 = \phi_2$ for all $t \in J_1 \cap J_2$.

Proof. Let $J = J_1 \cap J_2$ then since both ϕ_1 and ϕ_2 are solutions we have that:

$$\begin{cases} \vec{\phi}_1' = \vec{f}(t, \vec{\phi}_1) \\ \vec{\phi}_2' = \vec{f}(t, \vec{\phi}_2) \end{cases}$$

Integrating

$$\begin{cases} \vec{\phi}_1 - \vec{y}_0 = \int_{t_0}^{t} \vec{f}(s, \vec{\phi}_1(s))ds \\ \vec{\phi}_2 - \vec{y}_0 = \int_{t_0}^{t} \vec{f}(s, \vec{\phi}_2(s))ds \end{cases}$$

Subtracting and taking the mod of the two expressions:

$$\begin{aligned}
|\vec{\phi}_2(t) - \vec{\phi}_1(t)| &= \left| \int_{t_0}^{t} \vec{f}(s, \vec{\phi}_1(s))ds - \int_{t_0}^{t} \vec{f}(s, \vec{\phi}_2(s))ds \right| \\
&\leq \int_{t_0}^{t} |\vec{f}(s, \vec{\phi}_1(s))ds - \vec{f}(s, \vec{\phi}_2(s))ds| \\
&\leq \int_{t_0}^{t} K|\vec{\phi}_2(s) - \vec{\phi}_1(s)|ds && \text{by Lipschitz}
\end{aligned}$$

Let $G(t) = |\vec{\phi}_2(t) - \vec{\phi}_1(t)$ then:

$$G(t) \leq K \int_{t_0}^{t} G(s)ds$$

From here we apply Gronwall's inequality in intrgral form.

Gronwall in Integral Form

Let f, g be continuous on $[a, b]$ and let $g \geq 0$ on $[a, b]$ and let $A \geq 0$. Suppose:

$$f(t) \leq A + \int_{t_0}^{t} g(s)f(s)ds$$

Then:

$$f(t) \leq Ae^{\int_{t_0}^{t} g(s)ds}$$

Therefore $G(t) \leq 0$ which is only possible if $\phi_1 = \phi_2$. □

§9 Maximized Interval of Defintion for Solutions

Theorem 9.1

Let D be an open set of $\mathbb{R}^{n+1}$, consider:

$$(*) \begin{cases} \vec{y}' = \vec{f}(t, \vec{y}) \\ \vec{y}(t_0) = 0 \end{cases}$$

Suppose that:
(1) $\vec{f} : D \to \mathbb{R}^n$ is continuous on D.
(2) $\partial_y \vec{f}$ is continuous on D.
(3) There exists some $M > 0$ such that $|\vec{f}(t, \vec{y})| \leq M$ for all $(t, \vec{y}) \in D$.
Then the solution $\phi(t)$ given by Picard-Lindelof can be extended until items graph reaches ∂D (the boundary of D).

Boundary of D

Recall that $\partial D = \bar{D} \backslash D^o$ where $\bar{D} = \{x \in \mathbb{R}^n :$ there exists a sequence $x_j \in \mathbb{R}^n$ such that $x_j \in D$ and $\lim_{j \to \infty} x_j \to x\}$.
Equivalently, since D is open $\partial D = \{\vec{x} \in \mathbb{R}^n : \exists x_j \in D$ which tends to $x \notin D\}$

Lemma 9.2

Consider $(*)$ assume (1) and (3). Then for any solution $\phi : (c, d) \to \mathbb{R}^n$ of (1) the limits $\lim_{t\to d^-} \phi(t)$ and $\lim_{t\to d^+} \phi(t)$ exist.

Remak

By Picard-Lindelof Thm we know that a solution $\phi(t)$ exists on $(t - \alpha, t + \alpha)$. The lemma is proving that $\lim_{t\to(t\pm\alpha)^\pm} \phi(t)$ exists.

Proof. Recall that the left limit $\lim_{t\to d^-} \phi(t)$ exists iff for any $\epsilon > 0$ there exists a $\delta > 0$ such that for $t_1, t_2 \in (d - \delta, d)$ then $|\vec{\phi}(t_1) - \vec{\phi}(t_2)| < \epsilon$. And similarily for the RHS. Using $\vec{\phi}(t) = \vec{y}_0 + \int_{t_0}^{t} \vec{f}(t, \vec{\phi}(s))ds$ then:

$$\begin{aligned}
|\vec{\phi}(t_1) - \vec{\phi}(t_2)| &= \left| \int_{t_0}^{t_1} \vec{f}(s, \vec{\phi}(s))ds - \int_{t_0}^{t_2} \vec{f}(s, \vec{\phi}(s))ds \right| \\
&= \left| \int_{t_0}^{t_1} \vec{f}(s, \vec{\phi}(s))ds \right| \leq \int_{t_2}^{t_1} |f(s, \vec{\phi}(s))|ds \\
&\leq M(t_1 - t_2) && \text{since } |f| \leq M \\
&\leq M\delta
\end{aligned}$$

Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{M}$ then we obtain that:

$$|\vec{\phi}(t_1) - \vec{\phi}(t_2)| \leq \epsilon$$

$\square$

Use posted notes here for proof of the thm.

Definition 9.3. Let $\vec{\phi} : I \to \mathbb{R}^n$ be a solution to (*). We say that $\vec{\phi}$ is **maximal** if it does not admit a non-trivial extension of $\vec{\psi}$ which is also a solution to (*).
Let $\phi : (T, T^*) \to \mathbb{R}^n$. If $\vec{\psi} : (S, S^*)$ is an extension of $\vec{\phi}$ and $\vec{\psi}$ is a solution to (*) then $S = T$, $S^* = T^*$ and $\phi = \psi$.

§10 Dependence of Initial Conditions

Theorem 10.1

Consider $(*)$ from before. Let D is an open set of $\mathbb{R}^n$ and suppose that:
(1) $\vec{f} : D \to \mathbb{R}^n$ is continuous on D.
(2) $\vec{f}$ is Lipschitz with respect to $\vec{y}$.
(3) There exists some $M > 0$ such that $|\vec{f}(t, \vec{y})| \leq M$ for all $(t, \vec{y}) \in D$.
Then if ϕ_1 and ϕ_2 are solitions to $(*)$ defined on a common interval $[\alpha, \beta]$ then:

$$|\phi_1 - \phi_2| \leq |\vec{y}_1 - \vec{y}_2| e^{K|t-t_0|} \quad \text{where } \vec{y}_i = \vec{\phi}_i(t_0)$$

Proof. Integrating and subtracting as previously done:

$$\begin{aligned}
|\vec{\phi}_1 - \vec{\phi}_2| &= \left| \vec{y}_1 - \vec{y}_2 + \int_{t_0}^{t} f(s, \vec{\phi}_1) - f(s, \vec{\phi}_2(s)) \right| \\
&\leq |\vec{y}_1 - \vec{y}_2| + \int -{t_0}^{t} |f(s, \vec{\phi}_1) - f(s, \vec{\phi}_2(s))| ds \\
&\leq |\vec{y}_1 - \vec{y}_2| + \int_{t_0}^{t} K|\phi_1 - \phi_2| ds && \text{by Lipschitz}
\end{aligned}$$

Define $F(t) = |\phi_1 - \phi_2|$ then $F(t) \leq |\vec{y}_1 - \vec{y}_2| + K\int_{t_0}^{t} F(s)ds$. Now applying Gronwall we obtain the desired result. $\square$

Definition 10.2. We say that solutions to $(*)$ depend continuously on initial data if the map $y : H \to \mathcal{C}(I)$ is continuous where $H \subseteq \mathbb{R}^n$. H is open and $\mathcal{C}(I)$ is the space of continuous functions defined on I.

More precisely, let $a_0 \in H$, we say that y is continuous at a_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\vec{y}_0 - \vec{a}_0| < \delta$ then $d(\phi(y_0), \phi(a_0)) < \epsilon$. Here $d(\phi(y_0), \phi(y_f)) = \sup |\phi(y_0) - \phi(y_f)|$

§11 Systems of Linear 1st Order ODEs

Theorem 11.1

Consider:

$$(1) \begin{cases} \vec{y}' = A(t) \cdot \vec{y} + g(t) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

Suppose $A(t)$ and $g(t)$ are continuous on I then there exists a unique solution to (1) on I

Proof. Proof sketch:
(1) Show that $f = Ay + g$ is continuous and (2) Lipschitz in terms of $\vec{y}$. Proceed to apply Picard Lindelof (continued in the book). $\square$

§11.1 Homogeneous Case

Theorem 11.2

The set of all solutions to:

$$(2) \begin{cases} \vec{y}' = A(t) \cdot \vec{y} \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

is a n-dimensional vector space denoted as V.

Proof. Refer to notebook Thm 2.2 $\square$

Definition 11.3. An n-tuple of solutions to (2) is a **fundamental set of solutions** of it is a basis for V.

Definition 11.4. A **solutions matrix** is a matrix whose columns are all the solutions of (2). Denoted as $\Phi = (\phi_1 | \cdots | \phi_n)$ where ϕ_j are the solutions to (2).

Definition 11.5. A **fundamental matrix** is a solution matrix whose columns are linearly independent.

We want to be able to detect whether a certain set of solutions is a fundamental set of solutions.

Theorem 11.6

Abel's Formula: Let $\Phi = (\phi_1 | \cdots | \phi_n)$ with ϕ_j solving (2) for all j. Let $t_0, t \in (a, b)$ then $\det \Phi(t) = \det \Phi(t_0) e^{\int_{t_0}^{t} \operatorname{Tr}(A(s)) ds}$

Claim 11.7 — (Jacob's Formula): Let $\Phi(t) : [a, b] \to \mathbb{R}^{n\times n}$. Assume $\Phi(t)$ is differentiable then $\det(\Phi(t))' = \det(\Phi(t)) \operatorname{Tr}(\Phi'(t)\Phi^{-1}(t))$

Proof. Proof. (**Special Case**) Assume further that $\Phi(t)$ has distinct (non-zero) eigenvalues equivalent for all $t \in [a, b]$. Then $\Phi(t) = L(t)D(t)L^{-1}(t)$ where $L(t), D(t)$ and differentiable (need to prove $D(t)$ is a differentiable matrix and $L(t)$ is invertible).

In this case $D(t) = \begin{bmatrix} \lambda_1(t) & 0 & \cdots \\ & \vdots & \\ 0 & \cdots & \lambda_n(t) \end{bmatrix}$ where $\lambda_i(t) \in \mathbb{C}$. Then $\det(D(t)) = \lambda_1(t)\cdots\lambda_n(t)$ which means:

$$\begin{aligned}
(\det(D(t)))' &= \det(L(t)D(t)L^{-1}(T))' \\
&= \lambda_1'\lambda_2\cdots\lambda_n + \lambda_1\lambda_2'\cdots\lambda_n + \cdots + \lambda_1\cdots\lambda_n' \\
&= \frac{\lambda_1'}{\lambda_1}\cdot\lambda_1\cdots\lambda_n + \cdots + \frac{\lambda_n'}{\lambda_n}\lambda_1\cdots\lambda_n \\
&= \det(D(t))\cdot\left(\frac{\lambda_1'}{\lambda_1} + \cdots + \frac{\lambda_n'}{\lambda_n}\right) \\
&= \det(\Phi(t)) \operatorname{Tr}(D'(t)D^{-1}(t))
\end{aligned}$$

The last step because $D'(t) = \begin{bmatrix} \lambda_1'(t) & 0 & \cdots \\ & \vdots & \\ 0 & \cdots & \lambda_n'(t) \end{bmatrix}$ and $(D(t)^{-1}) = \begin{bmatrix} \frac{1}{\lambda_1(t)} & 0 & \cdots \\ & \vdots & \\ 0 & \cdots & \frac{1}{\lambda_n(t)} \end{bmatrix}$. So $D'(t)D^{-1}(t) = \begin{bmatrix} \frac{\lambda_1'(t)}{\lambda_1(t)} & 0 & \cdots \\ & \vdots & \\ 0 & \cdots & \frac{\lambda_n(t)'}{\lambda_n(t)} \end{bmatrix}$. Finally we proceed to prove the claim:

$$\begin{aligned}
\Phi'(t)\Phi^{-1}(t) &= (L(t)D(t)L^{-1}(t))'(L(t)D(t)L^{-1}(t)) \\
&= L'DL^{-1}LD^{-1}L^{-1} + LD'L^{-1}LD^{-1}L^{-1} + LD(L^{-1})'LD^{-1}L^{-1} \\
&= L'L^{-1} + LD'D^{-1}L^{-1} + LD(L^{-1})'LD^{-1}L^{-1}
\end{aligned}$$

This implies:

$$\begin{aligned}
\operatorname{Tr}(\Phi'(t)\Phi^{-1}(t)) &= \operatorname{Tr}(L'L^{-1} + LD'D^{-1}L^{-1} + LD(L^{-1})'LD^{-1}L^{-1}) \\
&= \operatorname{Tr}(L'L^{-1}) + \operatorname{Tr}(L^{-1}L') + Tr(D'D^{-1}) \qquad \text{since } \operatorname{Tr}(ABCD) = \operatorname{Tr}(DABC) \\
&= \operatorname{Tr}(D'D^{-1})
\end{aligned}$$

For the final step $(LL^{-1})' = L'L^{-1} + L(L^{-1})' = 0$ $\square$

Using this claim we proceed to prove Abel's Formula. Assume that $\Phi'(t) = A(t)\Phi(t)$. From the Jacob's Formula $\det(\Phi'(t)) = \det(\Phi(t)) \operatorname{Tr}(\Phi'(t)\Phi^{-1}(t))$. Take $d(t) = \det(\Phi(t))$ then:

$$\Rightarrow d'(t) = d(t) \operatorname{Tr}(A(t)\Phi(t)\Phi^{-1}(t)) = d(t) \operatorname{Tr}(A(t))$$

$$\Rightarrow \frac{d}{dt}(\ln d(t)) = \operatorname{Tr}(A(t))$$

Integrating from t_0 to t:

$$\ln(d(t)) - \ln(d(t_0)) = \int_{t_0}^{t} \operatorname{Tr} A(s) ds$$

$$\Rightarrow d(t) = d(t_0) e^{\int_{t_0}^{t} \operatorname{Tr} A(s) ds}$$

And we obtain the desired result. □

Theorem 11.8

$\phi_1, \cdots, \phi_n$ is a fundamental set of solutions iff $\det \Phi(t) \neq 0$ at some $t \in [a, b]$.

Proof. Suppose $\det \Phi(t_0) \neq 0$ then $(\Phi(t_0))^{-1}$ exists. Consider $\Psi \in V$. Then:

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = (\Phi(t_0))^{-1}\Psi(t_0)$$

Define $\Psi_* = \Phi(t) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Compute:

$$\Psi_*(t_0) = \Phi(t_0) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Phi(t_0) \cdot (\Phi(t_0))^{-1}\Psi(t_0) = \mathbb{I} \cdot \Psi(t_0) \cdot \Psi(t)$$

Therefore $\Psi(t) = \Psi_*(t)$ for all $t \in [a, b]$ therefore Ψ is a linear combination of $\phi_1, \cdots, \phi_n$. □

Consider again the problem:

$$\begin{cases} \vec{y}'(t) = A(t)\vec{y}(t) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

(1) Find the fundamental matrix $\Phi(t)$. Make sure to use the criterion to check that it's a fundamental matrix.
(2) Compute $\det(\Phi(t_0))$. If $\det(\Phi) \neq 0$ then Φ a fundamental matrix by the criterion and Φ^{-1} exists.
(3) The solution to the problem above is $\phi(t) = \Phi(t)\Phi^{-1}(t_0) \cdot \vec{y}_0$ by the uniqueness of solution.

§11.2 Non-Homogeneous Case

Theorem 11.9

Consider systems of the form:

$$(*) \begin{cases} \vec{y}' = A(t)\vec{y} + \vec{g}(t) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

Let $\vec{\phi}_p$ be a particular solution to $(*)$. Then, any solution to $(*)$ satisfies:

$$\vec{\phi} = \vec{\phi}_p + \vec{\phi}_h$$

where $\vec{\phi}_h$ solves the homogeneous equation:

$$\vec{\phi}_h' = A(t)\vec{\phi}_h$$

Proof. Let $\vec{\phi}$ be a solution to $(*)$. Consider $\phi_h = \phi - \phi_p$ then:

$$\phi_h' = (\phi - \phi_p)' = \phi' - \phi_p' = A(t)\phi + g(t) - A(t)\phi_p - g(t) = A(t)(\phi - \phi_p) = A(t)\phi_h$$

Consider the 1D proof for the expression:

$$y' = a(t)y + g(t)$$

We know that:

$$y_h = Ce^{\int_{t_0}^t a(s)ds}$$

We guess that the solution is given by $\phi(t) = c(t)y_h(t)$ such that $\phi' = a(t)\phi + g(t)$. Thus:

$$\text{(LHS:)} \quad c'(t)y_h(t) + c(t)y_h'(t) = c'(t)y_h + c(t)a(t)y_h$$

$$\text{(RHS:)} \quad a(t)s(t)y_h + g(t) \implies c'(t)y_h = g(t) \implies c(t) = \int_{t_0}^t \frac{g(s)}{y_h(s)}ds$$

□

Theorem 11.10

Let $\Phi(t)$ be a fundamental matrix for the system:

$$\vec{y}' = A(t)\vec{y}$$

Then,

$$\vec{\phi}_p = \Phi(t)\int_{t_0}^t \Phi^{-1}(s)g(s)ds$$

is a particular solution to $\vec{y}' = A(t)\vec{y} + g(t)$.

§12 Linear Systems with Constant Coefficients

§12.1 Matrix Exponential

Consider the linear homogeneous system

$$(1) \quad \vec{y}' = A\vec{y} \quad \text{where } A \in \mathbb{R}^{n\times n}$$

Theorem 12.1

Let $\Phi(t) = e^{tA}$, then $\Phi(t)$ is a fundamental matrix for (1)

§12.2 Jordan Canonical Form

Definition 12.2. Jordan block of size n with eigenvalue λ is the $n \times n$ matrix:

$$\mathcal{J}_{n,\lambda} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$$

Suppose $A = \text{blockdiag}(M_1, \cdots, M_n)$ then $e^{tA} = \text{blockdiag}(e^{tM_1}, \cdots, e^{tM_n})$.

Lemma 12.3

$$e^{t\mathcal{J}_{n,\lambda}} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t \cdots & \frac{t^{n-2}}{(n-2)!} & \\ & \ddots & \ddots & \ddots & \vdots \\ & & & & \frac{t^2}{2!} \\ & & & & t \\ & & & & 1 \end{bmatrix}$$

Proof. Prove using special cases. □

If P is invertible and $A = P^{-1}JP$ for some matrix J, then $e^{tA} = P^{-1}e^{tJ}P$.

Theorem 12.4

(Jordan Canonical Form): For each $A \in \mathbb{C}^{n\times n}$ there exists k numbers where $k \le n$ and $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$ and $n_1, \cdots, n_k \in \mathbb{C}$ such that $A = P^{-1}JP$ where:

$$J = \begin{bmatrix} \mathcal{J}_{n_1,\lambda_1} & & \\ & \ddots & \\ & & \mathcal{J}_{n_k,\lambda_k} \end{bmatrix}$$

Definition 12.5. Let $A \in \mathbb{C}^{n\times n}$. λ is an eigenvalue of A if there exists a $\vec{v} \neq 0$ where $\vec{v} \in \mathbb{C}^n$ such that $A\vec{v} = \lambda\vec{v}$.

All eigenvalues are solutions to $P(\lambda) = 0$ where $P(\lambda) = \det(A - \lambda\mathbb{I})$.
If the eigenvalues are pairwise different, then A is diagonalizable where $A = P^{-1}DP$ ($P = (v_1|\cdots|v_n)$) and

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Definition 12.6. Let λ be an eigenvalue of A. Then it's algebraic multiplicity is the multiplicity of λ as a root of $P(\lambda)$.

Definition 12.7. Let λ be an eigenvalue of A. The geometric multiplicity of λ is $g(\lambda) = \dim(\ker(\lambda\mathbb{I} - A))$

§12.3 Linear Systems with Complex Eigenvalues

If $\vec{\phi}' = A\vec{\phi}$ where $\vec{\phi} : I \to \mathbb{C}^n$ then $\Re(\phi)$ and $\Im(\phi)$ are also solutions. Then

$$\vec{\phi}_1 = \Re(e^{\lambda_1 t}\vec{v}_1) \quad \text{where } \lambda_1 = \alpha + i\beta \text{ and } \vec{v}_1 = \Re\vec{v}_1 + i\Im\vec{v}_1$$

Hence $\vec{\phi}_1(0) = \Re(\vec{v_1})$ and $\vec{\phi}_2 = \Im\vec{v}_1$:

$$\begin{aligned} \vec{\phi}_1 &= \Re(e^{t(\alpha+i\beta)}(\Re\vec{v}_1 + i\Im\vec{v}_1)) \\ &= \Re(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\Re\vec{v}_1 + i\Im\vec{v}_1)) \\ &= e^{\alpha t}(\cos(\beta t)\Re\vec{v}_1 - \sin(\beta t)\Im\vec{v}_1) \end{aligned}$$

$$\begin{aligned} \vec{\phi}_2 &= \Im(e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\Re\vec{v}_1 + i\Im\vec{v}_1)) \\ &= e^{\alpha t}(\cos(\beta t)\Im\vec{v}_1 - \sin(\beta t)\Re\vec{v}_1) \end{aligned}$$

Thus the fundamental matrix is $\Phi(t) = (\phi_1|\phi_2)$

§12.4 Asymptotic Behavior of Solutions

Theorem 12.8

Suppose that A has $\lambda_1, \cdots, \lambda_n$ distinct eigenvalues and suppose that $P > \Re(\lambda_j)$ for all $j = 1, \cdots, k$. Then there exists $K > 0$ such that $|e^{tA}| \leq Ke^{\rho t}$ for all $t \geq 0$.

Proof. Write $A = PJP^{-1}$ where $J =$blockdaig$(J_{n_1,\lambda_1}, \cdots, J_{n_k,\lambda_k})$. Thus:

$$e^{tA} = e^{tP^{-1}JP} = P^{-1}e^{tJ}P$$

$$\Rightarrow |e^{tA}| = |Pe^{tJ}P^{-1}| \leq |P||e^{tJ}||P^{-1}| \leq K_1|e^{tJ}| \quad \text{where } K_1 = |P||P^{-1}|$$

We know that:

$$|e^{tJ_{n_m,\lambda_m}}| = \left| e^{\lambda_m} \begin{bmatrix} 1 & t & \cdots & \frac{t^{n_m-1}}{(n_m-1)!} \\ & \ddots & \ddots & \vdots \\ & & & t \\ & & & 1 \end{bmatrix} \right| \leq |e^{\lambda_m t}| \left(n_m + t(n_m - 1) + \cdots + \frac{t^{n_m-1}}{(n_m-1)!} \right)$$

Notice that $s^k \leq k!e^s$ because $e^t = \sum_n^\infty \frac{t^n}{n!}$ therefore if $s = \epsilon t$ then:

$$\epsilon^k t^k \leq k!e^{st} \quad (*)$$

Using $(*)$. We have that:

$$\begin{aligned}
|e^{tJ_{n_m,\lambda_m}}| &= |e^{\lambda_m t}| \left(n_m + t(n_m - 1) + \cdots + \frac{t^{n_m - 1}}{(n_m - 1)!} \right) \\
&\leq e^{t\Re(\lambda_m)} \left(n_m e^{\epsilon t} + \frac{(n_m - 1)}{\epsilon} e^{\epsilon t} + \cdots + \frac{(n_m - 1)!}{(n_m - 1)!} \frac{e^{\epsilon t}}{\epsilon^{n_m - 1}} \right) \\
&= e^{t\Re(\lambda_m)} e^{\epsilon t} \left(n_m + \frac{(n_m - 1)}{\epsilon} + \cdots + \frac{1}{\epsilon^{n_m - 1}} \right) \\
&= e^{t(\Re\lambda_m + \epsilon)} K_n
\end{aligned}$$

Therefore:

$$\begin{aligned}
|e^{tA}| \leq |P||P^{-1}||e^{tJ}| &= |P||P^{-1}| \sum_{m=1}^{k} |e^{tJ_{n_m,\lambda_m}}| \\
&\leq |P||P^{-1}| \sum_{m=1}^{k} K_n e^{t\Re(\lambda_m) + \epsilon}
\end{aligned}$$

Let $P > \Re(\lambda_m)$ for all $m = 1, \cdots, k$ then there exists a $\epsilon > 0$ such that $\rho > \Re(\lambda_m) + \epsilon$ thus $e^{t(\Re(\lambda_m))} \leq e^{t\rho}$. Finally:

$$|e^{tA}| \leq |P||P^{-1}| \sum_{m=1}^{k} K_m e^{t\rho} = K e^{t\rho}$$

□

§12.5 Autonomous First Order Systems

We shall be discussing systems of the form:

$$\vec{y}' = \vec{g}(\vec{y}) \quad \text{where } \vec{g} : \mathbb{R}^n \to \mathbb{R}^n \text{ does not depend on } t$$

Definition 12.9. $\vec{y}_0$ is a **critical point** if $\vec{g}(\vec{y}_0) = 0$ therefore $\phi(t) = \vec{y}_0$ is an equilibrium solution.

Consider the system $y'' = \sin(y)$ then take $y_1 = y$ and $y_2 = y'$:

$$\vec{y}' = \begin{bmatrix} y_2 \\ -\sin(y_1) \end{bmatrix} \vec{y}$$

The critical points of this system are $y_2 = 0$ and $y_1 = n\pi$ where $n \in \mathbb{Z}$. Suppose we want to study this sytem at the point $(0, 0)$. Then we can proceed to linearize the system. Notice that:

$$\sin(y_1) \approx y_1 - \frac{y_1^3}{3!} + \cdots$$

Therefore:

$$\begin{bmatrix} y_2 \\ \sin(y_1) \end{bmatrix} \approx \begin{bmatrix} y_2 \\ y_1 \end{bmatrix}$$

Therefore the system close to zero is the following solvable system:

$$\vec{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{y}$$

Consider:

$$\vec{y}' = A\vec{y} \text{ where } A = PJP^{-1}$$

Notice that if ϕ is a solition to $\phi' = J\phi$ then $\phi_1 = e^{tJ}\vec{y}_0$ and $\phi_2 = e^{At}\vec{y}_0 = Pe^{Jt}P^{-1}\vec{y}_0$. Hence the phase portraits of $\vec{y}' = A\vec{y}$ is equivalent to $\vec{y}' = J\vec{y}$. Hence all of the possible phase portraits can be redued to 6 cases (refer to notebook for drawings of these).

§13 Stability of Critical Points

§13.0.1 Autonomous Systems

Consider systems of the form:

$$(*) \quad \vec{y}' = \vec{g}(\vec{y})$$

Definition 13.1. Suppose $\vec{y}_0$ is a critical point such that $\vec{g}(y_0) = 0$. We say that $\vec{y}_0$ is **orbitally stable** if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\eta_0 - y_0| < \delta$ then the solution $\vec{\phi}(t)$ to $(*)$ with $\vec{\phi}(0) = \eta_0$ satisfies:

$$|\vec{\phi}(t) - \vec{y}_0| < \delta \qquad \text{for all } t \geq 0$$

Consider the following system

$$\begin{cases} y' = -y \\ y(0) = a \end{cases}$$

Proof. We shall proceed to show that the critical point $y_0 = 0$ is stable. We know that the solution to this ODE is $\phi(t) = ae^{-t}$. Let $\epsilon > 0$. Choose $\delta = \epsilon$ then $|a - y_0| < \delta = \epsilon$. Now consider the following:

$$|\phi(t) - 0| = |ae^{-t}| = |a||e^{-t}| \leq |a| < \delta = \epsilon$$

□

Definition 13.2. We say that $\vec{y}_0$ is **asymptotically stable** if there exists $\delta > 0$ such that if $|\vec{\eta}_0 - \vec{y}_0| < \delta$ then $\phi(t) \to \vec{y}_0$ as $t \to \infty$.

Definition 13.3. A point is **unstable** if it's not stable. For any $\delta > 0$ there exists a η_0 such that $|\vec{\eta}_0 - \vec{y}_0| < \delta$ and $\phi(t)$ is a solution with $\phi(0) = \eta_0$ such that $|\phi(t) - \vec{y}_0| > \epsilon$ for all $t \geq T_\delta$

Theorem 13.4

Consider the following linear system:

$$(1) \quad \vec{y}' = A\vec{y} \qquad \text{where } A \in \mathbb{R}^{n\times n}$$

where $\vec{y}_0 = 0$ is a critical point. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A.
(1) $\Re(\lambda_q) \leq 0$ for all $q = 1, \cdots, n$ and the eigenvalues with $\Re(\lambda_q) = 0$ are simple (geometric multiplicity equivalent to algebraic multiplicity) then, $\vec{y}_0 = 0$ is a stable critical point.
(2) If $\Re(\lambda_q) < 0$ for all $q = 1, \cdots, m$ then $\vec{y}_0$ is asymptotically stable.
(3) If there exists λ_h such that $\lambda_h > 0$ then $\vec{y}_0$ is unstable.

Proof. (3) Suppose that $\lambda_h \in \mathbb{R}$ since λ_h is an eigenvalue there exists some $\vec{v} \neq 0$ such that $A\vec{v} = \lambda_h \vec{v}$. Then $\phi(t) = e^{\lambda_h t}\vec{v}$ is a solution to (1). But $\lim_{t\to\infty} \phi(t) \to \infty$ since $\vec{v} \neq 0$ and $\lambda_h > 0$ then $\vec{y}_0 = 0$ is unstable. □

§13.0.2 Non-Autonomous System

Now consider systems of the form:

$$(2) \quad \vec{y}' = \vec{f}(t, \vec{y}) \quad \vec{f} : D \to \mathbb{R}^n \text{ where } D \subseteq \mathbb{R}_t \times \mathbb{R}^n_y$$

We make the following:
(1) $\vec{f}$ is continuous on D.
(2) $\vec{f}$ is Lipschitz continuous in terms of $\vec{y}$ on D.
Assume that $\vec{\phi} : I \to \mathbb{R}^n$ is a solution (of $\vec{y}_0$).

Definition 13.5. $\vec{\phi}$ is **stable** is for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $\vec{\eta}$ with $|\phi(0) - \vec{\eta}| < \delta$ the solution $\vec{\psi}(t, \vec{\eta})$ to (2) with $\psi(0, \vec{\eta}) = \vec{\eta}$ exists for all $t \in [0, \infty)$ and $|\psi(t, \vec{\eta}) - \phi(t)| < \epsilon$ for all $t \in [0, \infty)$.

Definition 13.6. $\vec{\phi}$ is **asymptotically stable** if ϕ is stable and there exists a $\delta_0 > 0$ such that if $|\vec{\eta} - \vec{\phi}(0)| < \delta_0$ then:

$$\lim_{t\to\infty} |\vec{\psi}(t, \vec{\eta}) - \vec{\phi}(0)| = 0$$

where $\vec{\psi}(t, \eta)$ is the solution to (2) with initial condition η.

Consider the following:

$$\vec{y}' = \left(-1 + \frac{1}{t+1}\right) y$$

Then $\phi_0(t) = 0$ is a solution. We take $y(0) = \eta$ and solve the equation to find:

$$y(t) = \eta e^{-t}(1 + t)$$

Hence ϕ_0 is asymptotically stable.

Theorem 13.7
Consider a system of the form:

$$\vec{y}' = (A + B(t))\vec{y} \quad \text{ where } A \in \mathbb{R}^{n\times n}$$

where $B(t) \in \mathbb{R}^{n\times n}$ with continuous entries. And assume:
(1) $\Re(\lambda_g) < 0$ for all λ_g eigenvalues of A.
(2) $\lim_{t\to\infty} |B(t)| = 0$.
Then $\vec{\phi}_0(t) = 0$ for all t is asymptotically stable.

Theorem 13.8

Consider a system of the type:

$$\vec{y}' = A\vec{y} + \vec{f}(t, \vec{y})$$

Assume the following:
(1) All eigenvalues of A have negative real part.
(2) $\vec{f}$ is continuous and Lipschitz with respect to $\vec{y}$ on $D = [0, \infty) \times \{|\vec{y}| \leq K\}$ for $K > 0$.
(3) $\lim_{\vec{y} \to 0} \sup_{t \in [0,\infty)} \frac{|\vec{f}(t,\vec{y})|}{|\vec{y}|} = 0$.
Then $\vec{y}_0 = 0$ is asymptotically stable.

§13.1 Lyapunov's Theorem

Consider two quantities $y_j' = p_j$ where $j = 1, \cdots, d$ then:

$$\begin{cases} y_j' = p_j \\ p_j' = \partial_{y_j} U \end{cases}$$

where $U : \mathbb{R}^d \to \mathbb{R}$.
The conserved quantity here is defined as the Hamiltonian:

$$H = \frac{1}{2}||\vec{p}||^2 + U(\vec{y})$$

Proof.

$$H^* = \sum_{j=1}^{d} p_j \partial_{y_j} H + \sum_{j=1}^{d} (-\partial_{y_j} U) \partial_{p_j} H = \sum_{j=1}^{d} p_j \partial_{y_j} U + \sum_{j=1}^{d} (-\partial_{y_j} U) p_j = 0$$

□

Theorem 13.9

If $\vec{0}$ is a minimum of $H(\vec{y}, \vec{p})$ is a stable critical point for the system:

$$\begin{cases} y_j' = p_j \\ p_j' = \partial_{y_j} U \end{cases}$$

Theorem 13.10

If there exists a scalar function $V(\vec{y})$, $V(0) = 0$ such that V^* us either positive definite or negative definite on some region Ω containing the origin and if there exists in every neighborhood N of the origin, $N \subset \Omega$ at least one point $\vec{a} \neq 0$ such that $V(\vec{a})$ has the same sign as V^*, then the zero solution of $\vec{y}' = f(\vec{y})$ is unstable.

Theorem 13.11

If ther exists a scalar function V such that in a region Ω containing the origin $V^* = \lambda V + W$ where $\lambda > 0$ is a constant and W is either identically zero or W is a nonnegative or nonpositive function such that in ever neighborhood N of the origin, $N \subset \Omega$, there is at least one point $\vec{a}$ such that $V(\vec{a}) \cdot W(\vec{a}) > 0$, then the zero solution of $y' = f(y)$ is unstable.