Degree Distributions of Scale-Free and Random Networks

Anisha Kadri
Department of Physics, Imperial College London
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Abstract

Three attachment mechanisms for network growth in simple networks are presented; (a) preferential attachment, (b) random attachment and (c) attachment via a random walk mechanism. The resulting degree distributions from these three mechanisms are simulated numerically and compared to theoretically expected values. It was found that preferential attachment lead to a scale-free network with a power-law degree distribution $p(k) \sim k^{-3}$, with the exponent estimated at $\alpha = ?2.83 \pm 0.291$. However the exponent was found to depend on initial conditions and various other parameters unaccounted for. Fitting for the theoretical from for $m = 32$ (the number of new edges added per time increment) and a system size of $N = 10^5$ nodes in the power-law region gave a $\chi^2 \sim 16.3$. Degree distribution for different system sizes and $m = 4$ could be collapsed and was found to obey a scaling relation, implying scale-free properties. In contrast, pure random attachment gives an exponential degree distribution with that could not be collapsed for different system sizes, although cutoff values k_1 were shown to be closer to theoretical expectation than those for preferential attachment. The random walk mechanism was found to scale similar to the random attachment mechanism for small random walks of length L, however the degree distribution tended to a fat-tail for large random walk length.

[Word Count: 2488 (Excluding Captions, Abstract and Title Page)]

1 Introduction

Different processes of network growth (i.e. how a new node attaches to the existing network) can be shown to heavily influence the resultant network dynamics and structural properties in the long time limit.

Three methods of attachment for network growth are explored within this study, growth via preferential attachment, pure random attachment and preferential attachment via random walks.

2 Basic Algorithm for Network Growth

- Initialisation of Network: Create a graph/ network G_0 at $t = t_0$.
- Drive: Increment time by one: $t \to t+1$, $N \to N+1$.
- Add one new vertex, with m edges.
- Attach each of the m edges to pre-existing vertcies with probability Π.
- Iteration: Repeat from 2.

The probability of vertex selection is defined separately for each method of attachment.

3 Theoretical Degree Distributions

3.1 The Master equation

The master equation governs the time evolution of number of nodes of degree k in a network [1];

$$n(k,t+1) = n(k,t) + m\Pi(k-1,t)n(k-1,t) - m\Pi(k,t)n(k,t) + \delta_{k,m} \tag{1}$$

where $n(k,t)$ represents the number of nodes of degree at a certain time, m is the number of new edges added at each time increment and $\Pi(k,t)$ is the probability of selecting a node of degree k at time t, and is defined differently for each attachment method. The equation follows intuitive logic; the number of nodes of degree k at time $t+1$ is equal to the increase in number due to nodes of degree $k-1$ gaining an edge and the decrease in number due to nodes of degree k gaining an edge. Here the $\delta_{k,m}$ is the kronecker delta function, and is equal to 1 only if k=m. This is the increase due the the new node of m edges being added to the network. The time-dependent degree distribution is by definition $p(k,t) = \frac{n(k,t)}{N(t)}$, and the fact that $N(t+1) = N(t)+1$ allows us to express the master equation in terms of degree distribution;

$$p(k,t+1) = m\Pi(k-1,t)p(k-1,t)N(t) - m\Pi(k,t)p(k,t)N(t) + \delta_{k,m} \tag{2}$$

Since we seek a stationary solution for degree distribution in the long time limit $N \to \infty$, such that $p(k,t) \to p_\infty(k)$:

$$p_\infty(k) = mN(t)p_\infty(k-1)\Pi(k-1,t) - mN(t)p_\infty(k)\Pi(k,t) + \delta_{k,m} \tag{3}$$

3.2 The BA Model $p_\infty(k)$

Otherwise known as the Barabasi- Albert Model, this method relies on growth by preferential attachment, such that the probability of attaching a new node to a vertex is proportional to its degree $\Pi_{pa}(k) \propto k$. Normalisation of the probability over all values of k yields the simple form:

$$\Pi_{pa}(k,t) = \frac{k}{\sum_{i=1}^{N(t)} k_i}. \tag{4}$$

However, since $E(t) = \frac{\sum_{i=1}^{N(t)} k_i}{2}$ and assuming no initial edges $E(t) = mN(t)$, a more complete expression for Π_{pa} is:

$$\Pi_{pa}(k,t) = \frac{k}{2E(t)} = \frac{k}{2mN(t)} \tag{5}$$

Substitution for Π into the master equation (3) gives the following expression:

$$p_\infty(k) = \frac{mN(t)}{2mN(t)}(k-1)p_\infty(k-1) - \frac{mN(t)}{2mN(t)}kp_\infty(k) + \delta_{k,m} \tag{6}$$

$$p_\infty(k) = \frac{1}{2}\{(k-1)p_\infty(k-1) - kp_\infty(k)\} + \delta_{k,m} \tag{7}$$

Solution for k=m:
For the case where $k = m (\Rightarrow \delta_{k,m} = 1)$, we have the relation:

$$2p_\infty(m) = (m-1)c - kp_\infty(k) + 2 \tag{8}$$

However, by the growth process is defined so each new vertex is created with a minimum of m edges, implying $kp_\infty(k) = 0$. This gives the solution:

$$p_\infty(m) = \frac{2}{(m+2)} \tag{9}$$

Solution for k > m:
Where $\delta_{k,m} = 0$, rearranging Equation 7 gives a first order difference equation;

$$\frac{p_\infty(k)}{p_\infty(k-1)} = \frac{k-1}{k+2} \tag{10}$$

The solution to the difference equation can be found by considering the properties of the Γ function.
Gamma Function:
The Gamma function [2], defined as $\Gamma(x) = (x-1)!$ for $x \in \mathbb{Z}^+$ has the key property;

$$\Gamma(x+1) = x\Gamma(x) \tag{11}$$

$$\Gamma(1) = 1 \tag{12}$$

It is therefore useful for solving discrete difference equations like equation 10.
General Solution to Equation 10:
A trial solution for the difference equation is proposed.

$$p_\infty(k) = Z\frac{\Gamma(k)}{\Gamma(k+3)} \tag{13}$$

Where Z is some normalisation constant for the distribution. Relabelling gives:

$$p_\infty(k-1) = Z\frac{\Gamma(k-1)}{\Gamma(k+2)} \tag{14}$$

Substituting this into the LHS of Equation 10 gives the expression:

$$\frac{p_\infty(k)}{p_\infty(k-1)} = Z\frac{\Gamma(k)}{\Gamma(k+3)} \cdot \frac{1}{Z}\frac{\Gamma(k+2)}{\Gamma(k-1)} = \frac{\Gamma(k-1)}{\Gamma(k+2)} \cdot \frac{\Gamma(k+2)}{\Gamma(k-1)} \cdot \frac{k-1}{k+2} = \frac{k-1}{k+2} \tag{15}$$

Where the expansion of the Gamma function in Equation 11 has been used. The result is the RHS of Equation 10, thus we can conclude that Equation 13 gives a numerically correct trial solution to the difference equation. Using the property in Equation 11, the solution can also be rewritten as:

$$p_\infty(k) = \frac{Z}{k(k+1)(k+2)} \tag{16}$$

However, as a probability distribution, p_∞ must satisfy normalisation:

$$\sum_{k=1}^{\infty} p_\infty(k) = 1 \tag{17}$$

$$\Rightarrow p_\infty(m) + \sum_{k=m+1}^{\infty} p_\infty(k) = \frac{2}{(m+2)} + \sum_{k=m+1}^{\infty} \frac{Z}{k(k+1)(k+2)} = 1 \tag{18}$$

The infinite series can be deconstructed into partial fractions which sum to give:

$$\frac{2}{(m+2)} + \frac{Z}{2(m+1)} - \frac{Z}{2(m+2)} = 1 \tag{19}$$

$$\Rightarrow Z = 2m(m+1) \tag{20}$$

Specific Solution: The functional form for the long time solution of degree distribution is;

$$p_\infty(k) = \begin{cases} 0 & k < m \\ \frac{2m(m+1)}{k(k+1)(k+2)} & k \geq m \end{cases}$$

Thus in the large degree limit, we expect that $p_\infty(k) \sim k^{-3}$.

3.3 Pure Random Attachment $p_\infty(k)$

In this model, m vertices are chosen with uniform probability for attachment to a new node of degree m. Thus, the probability of choosing a vertex is $\Pi_{rand}(k,t) = \frac{1}{N(t)}$. Substituting this expression into the master equation form given in Equation 3 gives an expression for the stationary distribution $p_\infty(k)$:

$$p_\infty(k) = \frac{1}{(1+m)}[mp_\infty(k-1) + \delta_{m,n}] \tag{21}$$

Since all new vertices have a minimum of m edges, it is assumed that $p_\infty(k) = 0$ for $k < m$. For the case of $k = m$, taking into account that $p_\infty(m-1) = 0$ we have the solution:

$$p_\infty(m) = \frac{1}{(1+m)} \tag{22}$$

For the case where $k > m$, the kronecker delta term vanishes, leaving:

$$p_\infty(k) = \frac{m}{(1+m)} p_\infty(k-1) \tag{23}$$

Since the equation does not have any other terms in k except the distribution function, we can expand the function in the following manner:

$$p_\infty(k) = \frac{m}{(1+m)} p_\infty(k-1) = \left(\frac{m}{(1+m)}\right)^2 p_\infty(k-2) = \left(\frac{m}{(1+m)}\right)^3 p_\infty(k-3)\ldots \tag{24}$$

Thus, the solution for $p_\infty(k)$ can be found in terms of $p_\infty(m)$, which is known:

$$p_\infty(k) = \left[\frac{m}{(1+m)}\right]^{(k-m)} \cdot p_\infty(m) = \frac{m^{(k-m)}}{(1+m)^{(k-m+1)}} \tag{25}$$

Which is the stationary long time solution for all values of $k \geq m$. Since it is derived form a uniform vertex attachment probability, it trivially fulfils the normalisation requirement. The theoretical degree distribution for random attachment indicates that the distribution is not a power-law, and that growth is not sufficient for scale-free behaviour.

3.4 Largest Expected Degree (k_1)

Mathematically the largest expected degree k_1 in a system of final size N, is defined as the threshold above which the expected number of observations drops to 1:

$$\sum_{k=k_1}^{\infty} N p_\infty(k) = 1 \tag{26}$$

For the BA model, substituting the probability gives the expression:

$$\sum_{k=k_1}^{\infty} p_\infty(k) = \frac{2m(m+1)}{2} \sum_{k=k_1}^{\infty} \left[\frac{1}{(k+2)} - \frac{2}{(k+1)} + \frac{1}{k} \right] = \frac{1}{N} \tag{27}$$

In this case the infinite sum can be calculated, giving the equality:

$$\frac{m(m+1)}{k_1(k_1+1)} = \frac{1}{N} \Rightarrow k_1^2 + k_1 - Nm(m+1) = 0 \tag{28}$$

The positive solution to this quadratic is:

$$k_1^{BA} = \frac{-1 + \sqrt{1 + 4Nm(m+1)}}{2} \tag{29}$$

Thus the cut-off degree scales with network size as $k_1 \sim N^{0.5}$. The same principle can be applied for the random attachment model, but substituting a different $p_\infty(k)$ into Equation 26;

$$\sum_{k=k_1}^{\infty} \frac{m^{(k-m)}}{(1+m)^{(k-m+1)}} = \frac{1}{N} \tag{30}$$

This can be rearranged as:

$$\sum_{k=k_1}^{\infty} \frac{m^k}{(1+m)^k} \cdot \frac{(1+m)^{(m-1)}}{m^m} \tag{31}$$

$$\Rightarrow \sum_{k=k_1}^{\infty} \left[\frac{m}{(1+m)} \right]^k = \frac{m^m}{N(1+m)^{(m-1)}} \tag{32}$$

Relabelling the index k as $k - k_1$ gives a numerically equal sum from the limits $0 \to \infty$:

$$\left(\frac{m}{1+m} \right)^{k_1} \sum_{k=0}^{\infty} \left[\frac{m}{(1+m)} \right]^k = \frac{m^m}{N(1+m)^{(m-1)}} \tag{33}$$

The sum of the geometric series to infinity is $1 + m$, so:

$$\left(\frac{m}{m+1} \right)^{k_1} = \frac{m^m}{N(1+m)^m} \tag{34}$$

with the solution for k_1:

$$k_1^{rand} = m - \frac{\ln(N)}{\ln(m) - \ln(m+1)} \tag{35}$$

This gives the characteristic scaling with network size as $k_1 \sim \ln(N)$. This indicates that random graphs give exponential cutoffs.

4 Pure Preferential Attachment

4.1 Algorithm

As outlined in section 2, the algorithm for preferential attachment grows a network by introducing a new node of m edges. The new node attaches to pre-exisitng nodes by selecting them randomly from a list. Fore preferential attachment, the list (named `node_list`) contains each pre-existing node i with frequency equal to its degree k_i in the list. In order to maintain this list, every time an edge is created between nodes, the node indices are added to the list. The details of the algorithm are described in greater detail in the flowchart shown in Figure 1.

Initially a complete graph of m nodes with degree $k = 0$ are created. Thus, the node added at $t = 1$ will connect to all of the existing nodes. So at t=1 the network should consist of m nodes of degree $m + 1$, and 1 node of degree m for every run (i.e. graphs of specified m are isomorphic at $t = 1$). This was checked visually by network mapping the first few time increments[1].

4.2 Numerical Results for Preferential Attachment

The degree distribution resulting from the algorithm above is shown in Figure 1, and the log-binned distribution in Figure 2. Although all the curves exhibit power law behaviour for low k values, there is significant deviation from power-law behaviour at the fat-tails. The graph shows a clear offset on the x-axis for different values of m. The computational model has several limiting features and approximations made that cause differences in the observed degree distribution from the theoretically expected value. The main challenges of numerical modelling are outlined below.

(a)Statistical Noise

Since preferential attachment is a probabilistic process, there is some level of statistical variance in data sets in each run. Increasing the number of data sets vastly improves the statistics of the numerically obtained degree distribution. Amalgamating degree values from different runs of the same m and N minimises statistical deviations, but the number of runs is limited by computational power and code efficiency. The 'noisy' fat-tail observed in the log-log plot of the degree distribution is also problematic for fitting purposes, but this can be resolved by employing log-binning solutions.

(b)The Effect of Initial Attractiveness

Because of finite timescales, the initial configuration of the network was found to have significant impact on the resulting probability distribution. Two initial sparse network configurations were employed, an initial network of m nodes and 0 edges and a complete graph of m nodes. It was found that the first variant produced a degree distribution with 2 humps before cutoff, corresponding to the hubs created from the initial nodes which existed earlier, but with only degree 1 at $t = 1$[4]. The second hump corresponds to nodes added with degree m at an earlier timescale. The slope of the graph for this initial configuration was also closer to 2, than 3 with the linear fit in logspace for $m = 4, N = 10^5$ giving a slope of 2.53 ± 0.03. The 2000 study of preferential linking by Dorogovtsev, S.N., Mendes, J.F.F. and Samukhin proposes that the exponent of the power law distribution is in fact dependent on the initial attractiveness of the nodes (or the degree of inital nodes), and is only equal to 3 if the initial attractiveness = m [3]. In comparison, the complete graph configuration is the most consistent with theoretical constraints ($p(k < m) = 0$) and has all nodes with equal degree at $t = 1$, so only one hump emerges in the degree distribution.

[1] For testing purposes, feeding a seed to the random number generator ensures the same network can be recreated multiple times

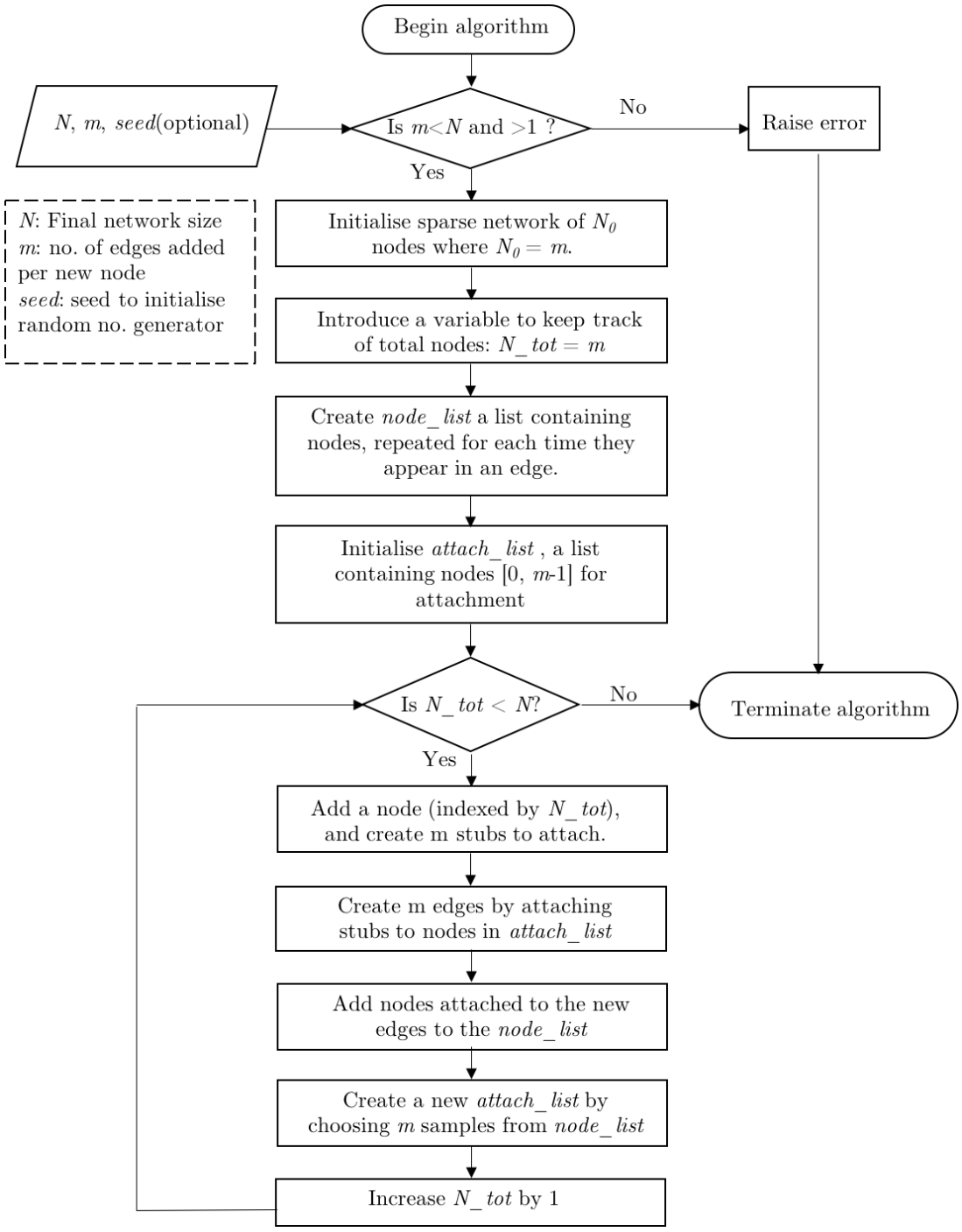

Figure 1: *A flowchart describing the algorithm for preferential attachment.*

(c)Finite System Size

The theoretical forms described in Section 2 are stationary solutions for the long time limit $N \to \infty$. However, the degree distributions plotted in Figure 2 are for finite network sizes, with visible finite size effects- most notably the humps at the tail end of the distribution. These effects decrease for cases where the network size is much larger than the degree of new nodes, $N >> m$.

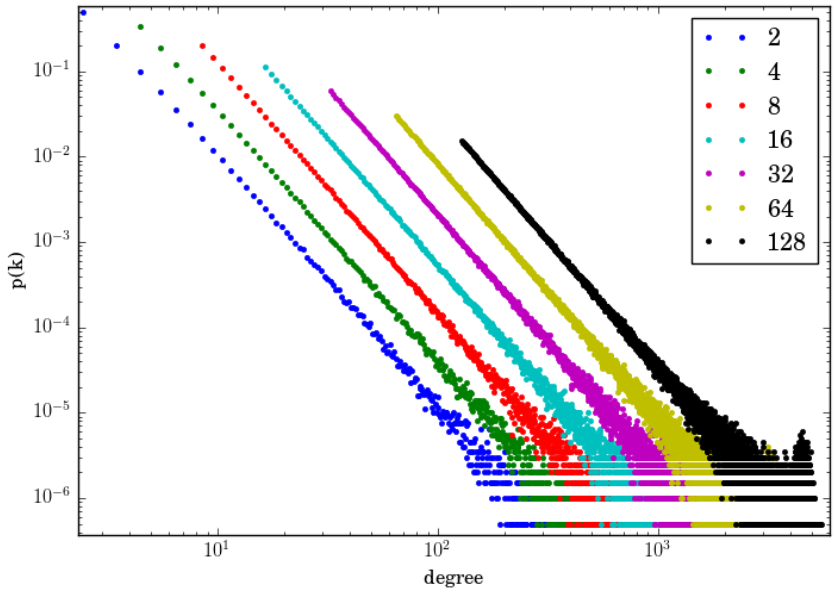

Figure 2: *Numerical results for the frequency of nodes $n(k)$ with degree k, for system size of $N = 10^5$ nodes and m values as indicated in the graph. For low k values, a clear linear relationship can be seen for $n(k)$ and k, implying a power law distribution. The data was fitted to a straight line in log space with some success, with an average gradient of $\alpha = -2.83 \pm 0.291$ (consistent with the theoretically expected value of the exponent). However low k values have outliers and in general, higher m-value data sets had very high residuals from fitting. This can be attributed to the fat-tail, where statistical variations allow a significant number of large k hubs to emerge creating a wider tail.*

Correspondence with Theory: Goodness of Fit

The data in Figure 3 was fitted to the theoretical degree distribution: $p_{inf}(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$. The goodness of fit was determined by the Pearson's Chi-Squared test, as the observed degree values can be treated as a normalised random variable. The χ^2 statistic was calculated via the following method:

$$\sum_{i=1}^{N} \frac{(O_i - E_i)^2}{E_i} \tag{36}$$

Where O_i represents the observed frequency for the degree k_i and E_i the theoretically expected observations. The goodness of fit test was conducted for regions of the graph of $m = 32$ in Figure 3, where the finite size effects are negligible and the numerical results are expected to follow theory (i.e. $k << k_1$). The χ^2 value obtained was 16.3, for 10 bins (9 degrees of freedom), giving a p-value between 0.05 and 0.1 significance interval. The p-value corresponds to the probability of a null hypothesis creating the same set of results, so the observed degree distribution is strong evidence for the theoretical model.

4.3 Finite Size Effect

To explore the finite size effect, the degree distribution was computed for $m = 4$ and network sizes of $N = \{10^3, 10^4, 10^5, 10^6, 10^7\}$, with data averaged over $\{10^2\}$ runs. In comparison, for large values of m,

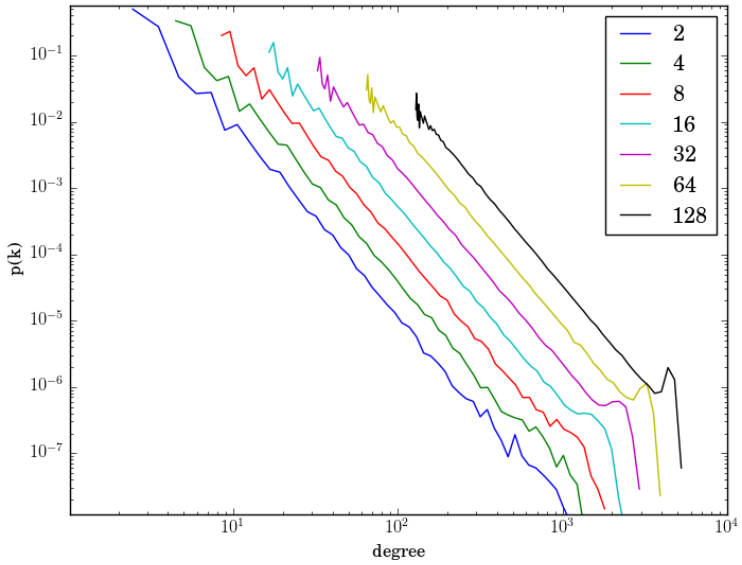

Figure 3: *Statistics were improved by conducting* 10^2 *repeats per graph, and Log-binning the degree values allowed a smother curve to emerge from the fat-tail. The resulting curves in log space exhibit 2 clear regions, the linear regression of the scaling region where the power-law is a good approximation of* $p(k)$ *and an exponential cutoff region. The cutoff increases with* m *for the same system size as higher* m *allow for a higher largest expected degree. The 'humps' indicate a tendency of older, larger hubs to accrue more edges than expected by a power law, at the expense of younger nodes. This effect is expected to disappear in the long-time limit, which is why it is less noticeable for smaller* m*, where* N *is large in comparison.*

the number of possible network structures emerging, and the range of k-values diminishes significantly (as seen in Figure 2). The lower bound for system size was chosen as 10^2 because the corrections to scaling become very significant for network sizes below this value.

Finite-Size Scaling and Data Collapse

The degree distribution is expected theoretically to be scale-free, thus a FSS ansatz can be proposed for degree distribution[5]. This would imply a possible data collapse similar to the mathematical model for self-organised complex systems. Theoretical expressions derived in Section 2 motivate the scaling relation:

$$p(k) \propto p_\infty(k) \cdot G\left(\frac{k}{k_1}\right) \tag{37}$$

Where G is a FSS scaling function. Thus, plotting $p(k)/p_\infty(k)$ against k/k_1 should give a data collapse onto G. For the graph in Figure 5, $N = 10^3$ was omitted due to significant scaling effects, and $N = 10^7$ due to computational constraints. Plotting the rescaled degree distribution against degree gives a successful data collapse for these systems.

Largest Expected Degree (k_1)

The k_1 values used for the data collapse were numerically computed from the smoothed cutoff for data from each network size, these can be seen in Figure 6, with the theoretically expected values. The uncertainty of the largest expected degree was found by evaluating the standard deviation of the data set over multiple runs.

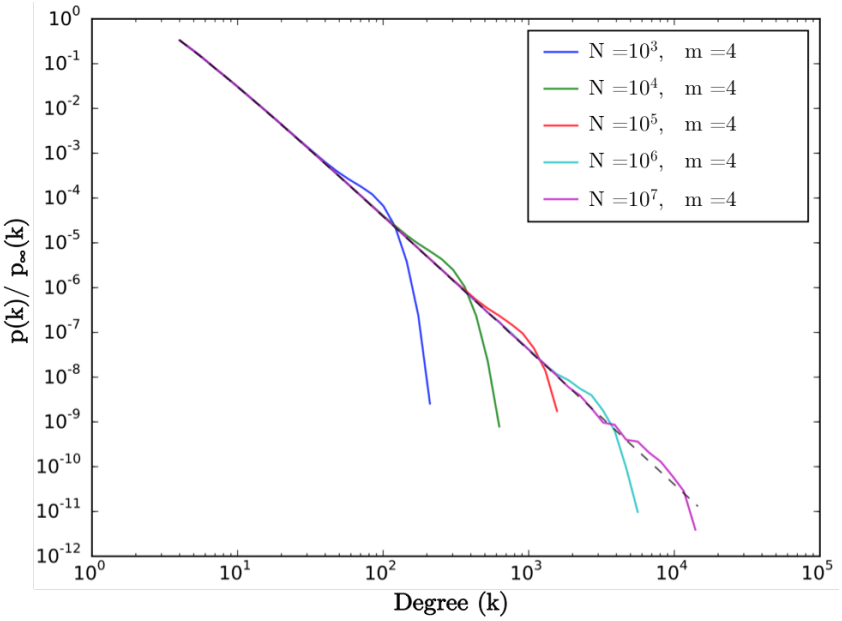

Figure 4: *A graph of $p(k)/p_\infty(k)$ is plotted against degree value. The data collapses onto the dashed line (the expected power-law), and the humps occur at different characteristic cutoff points (k_1).*

5 Pure Random Attachment

5.1 Algorithm

The algorithm for random attachment was implemented in a similar manner as the Preferential Attachment algorithm described in Figure 1. Both algorithms maintain a list of nodes to choose m attachees from. The key difference is that whereas the preferential attachment algorithm maintains a list of nodes with node multiplicity corresponding to its degree, the random attachment model maintains a list of nodes with each node index occurring once. This ensures that each vertex is equally probable for selection.

5.2 Numerical Results for Random Attachment

The degree distribution seen in Figure 7 for random attachment is of the exponential form, giving the characteristic parabolic cutoff seen in log-log graphs for random graphs. As for the BA model, the graphs for logarithmically increasing values of m are also evenly spaced in log-space. However, there does not appear to be any scale free behaviour as with the power law in the preferential attachment degree distribution. Unlike the preferential attachment results, the initial configuration of the network was found to have little to no effect on the overall degree distribution.

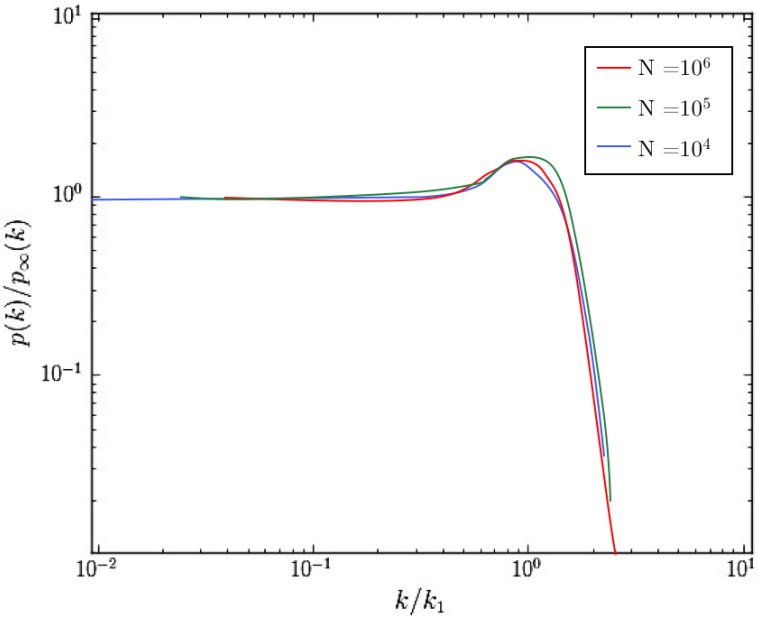

Figure 5: *The data collapsed degree distribution for system sizes* $N = \{10^4, 10^5, 10^6\}$ *onto the scaling function for preferential attachment. The data has been averaged over* $10^4, 10^3, 10^2$ *runs respectively to improve statistics further. Networks created by this mechanism exhibit scale-free properties.*

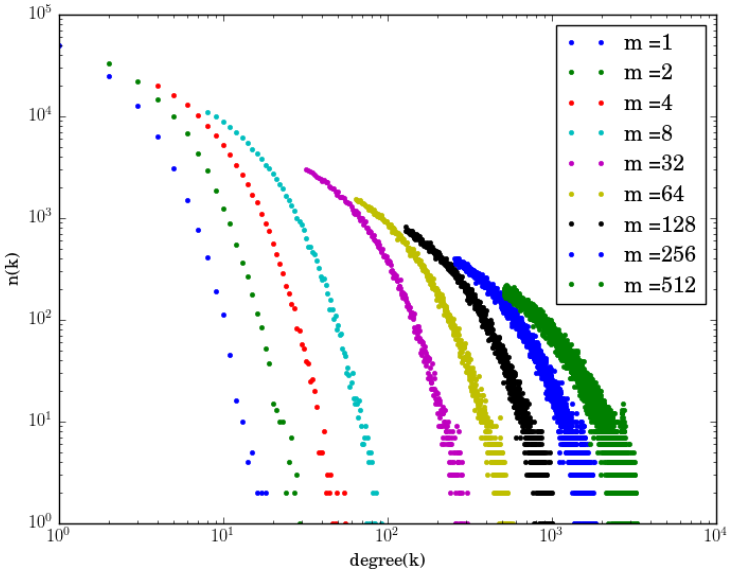

Figure 7: *Degree distribution for Random Attachment Model. The graph shows results for system size* $N = 10^5$ *and* $m = \{1, 2, 4, 8, 32, 64, 128, 256, 512\}$*. There is no fat-tail characteristic of real networks, but instead an exponential cutoff commonly seen in random graphs. The exponential distribution shown is not scale-free and does not facilitate data collapse.*

System size (N)	Observed k_1	Theoretical Expectation
10^2	32.0 ± 1.5	44.22
10^3	131.0 ± 2.4	140.92
10^4	421.0 ± 3.6	446.71
10^5	1390.0 ± 9.4	1413.71
10^6	4490 ± 12.5	4471.64

Figure 6: *Table of theoretical and observed k_1 values for preferential attachment. The data collected is for $m = 4$ and $N = 10^2, 10^3, 10^4, 10^5, 10^6$ and associated uncertainties over 100 runs. The statistics improve for larger system size, but are systematically lower than expected theoretically.*

Largest Expected Degree (k_1)

The table in Figure 8 shows the expected and observed largest expected degree for different N and $m = 4$, from an average of 100 runs. The observed largest degree in random networks is closer to theoretical prediction that the values for preferential attachment found in Figure 6. This indicates that the theoretical model for random attachment is closer to the numerical model, perhaps because there are less variables affecting the outcome.

System size (N)	Observed k_1	Theoretical Expectation
10^2	16 ± 2	24.63
10^3	31 ± 2	34.96
10^4	42 ± 2	45.28
10^5	54 ± 1.9	55.59
10^6	62 ± 1.9	65.91

Figure 8: *Table of theoretical and observed k_1 values for values for random attachment. The data collected is for $m = 4$ and $N = 10^2, 10^3, 10^4, 10^5, 10^6$ and associated uncertainties over 100 runs. Notably, there is very small deviation from theory in comparison to the preferential attachment model.*

6 Random Walk Attachment

This algorithm initialises the network in the same way as the preferential attachment algorithm. However, to attach a new node to existing nodes, m nodes are uniformly chosen from a list. Then, for each node, a node is chosen at random from its list of neighbours and the process is repeated L times, specifying the length of random walk. For the limiting case of a random walk length $L = 0$, the model gives the exponential distribution seen in the random attachment model. However, for larger L, it appears as though the exponential gradually tends to a fat-tailed distribution. Numerical results could not be produced with sufficient quantity for this model due to computational limitations. However, in real networks with fat-tailed distributions, a fractal network structure arises where are a significant amount of hubs, which are connected to a larger number of smaller hubs, which are connected to individual nodes and so on. This random walk approach to attachment in the limit of long L is more likely to choose a larger hub than a node with smaller degree because the hub has more neighbours (i.e. $\pi(k, t) \propto k$).

7 Conclusion

The theoretical and numerical degree distributions arising from networks with (a)preferential attachment, (b) random attachment and (c) random walk attachment were studied. Preferential Attachment was

shown to produce networks where degree distributions follow a power law behaviour, with exponent between $\alpha = 2, \alpha = 3$ dependent on initial conditions. For system size $N = 10^5$ and varying m, an average gradient of -2.83 ± 0.291 was measured. The distributions were also scale-free, and shown to accommodate data collapse by plotting $p(k)/p_\infty(k)$ against k/k_1.

The exponential degree distribution arising from the random attachment mechanism did not exhibit scale-free properties, and had the sharp parabolic cutoff characteristic of random graphs. Differences in degree distribution between vertices is amplified over the process, so initial attractiveness of degrees plays a central role in how preferentially attached networks evolve. However, for the random attachment scenario, initial configuration does not have great impact on the final degree distribution. The preferential attachment model is overall found to be more responsive to parametric changes, which is perhaps the reason for the observed maximum cutoff degrees deviating significantly from theory (Figure 6). In comparison, results of k_1 for the randomly attached networks gave results fairly consistent with theory.

Radom walk attachment was a method that contained some element of randomness, and preferential attachment. For low path lengths, the random walk was equivalent to choosing a vertex uniformly. However, for large L, the distribution began to resemble the fat-tail of the BA model.

Real networks are much more complex in behaviour than simply using one method of attachment, furthermore the models must be extended to include many other possible properties. For example, allowing multigraphs instead of simple graphs, self loops and allowing a new node to randomly choose m, the number of new edges created at each time increment.

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