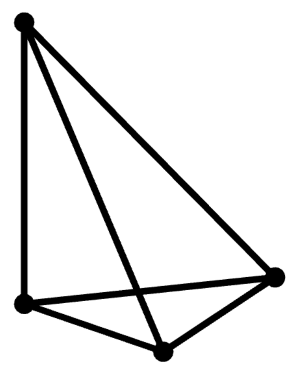
1. Find the volume of the pyramid with vertices at the coordinates

$$(1,0,0),(0,1,0),(0,0,1),(0,0,0).$$



Solution: The formula for the volume of a pyramid is $\frac{1}{3}Bh$. The area of the base is $1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$, and the height is 1. Thus, the volume is $\frac{1}{2} \cdot \frac{1}{3} \cdot 1 = \boxed{\frac{1}{6}}$.

2. Consider the following system of equations:

$$w + x + y = 8$$

$$y + z = 10$$

$$w + x + z = 12.$$

Find $w + x + y + z$.

Solution: Add the three equations together. We find that

$$(w + x + y) + (y + z) + (w + x + z) = 8 + 10 + 12$$

$$2(w + x + y + z) = 30$$

$$w + x + y + z = \boxed{15}.$$

3. What is the number of 5-digit numbers that have strictly decreasing digits from left to right?

Solution: For every possible choice of 5 distinct digits, there is exactly one permutation that works. For example, for the choice of the digits $0, 3, 4, 6, 7$, the only possible valid 5-digit number is 76430. Thus, the solution is the number of choices of 5 distinct digits out of 10, which is simply $\binom{10}{5} = \boxed{252}$.

4. Call a number balanced if it is divisible by $p + 10$ where p is its smallest prime divisor. How many numbers from 1 to 100, inclusive, are balanced?

Solution: We iterate over the primes and see what numbers are multiples of p and $p + 10$.

For 2, any balanced number needs to be divisible by 12. There are 8 such numbers:
$12, 24, 36, 48, 60, 72, 84, 96$.

For 3, the balanced number cannot be divisible by 2 and must be divisible by 3 and 13. The only such number is 39.

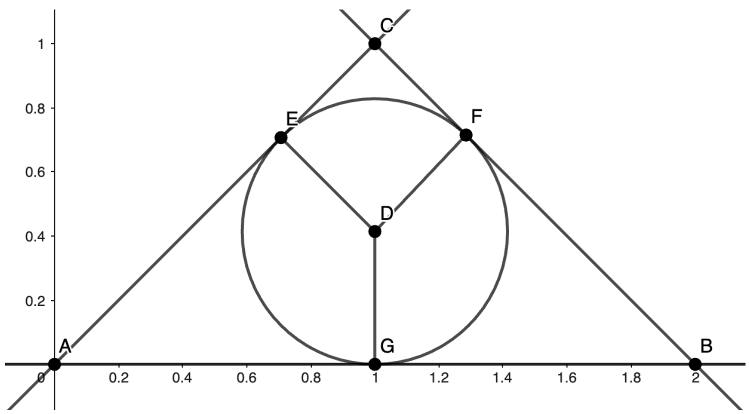
For 5, the number must be divisible by 15. However, 15 is divisible by 3, so there are no balanced numbers with smallest prime divisor 5.

For any prime larger than 6, we would have $p(p+10) > 100$, so there are no more balanced numbers.

Thus, we have a total of $8 + 1 = \boxed{9}$ balanced numbers.

5. Let $\mathcal{A}$ be the region in the xy-plane bounded by $y = 0$, $y = x$, and $y = 2 - x$. $\mathcal{A}$ includes the area enclosed by these boundaries, as well as the boundaries themselves. What is the maximum possible radius of a circle that lies in $\mathcal{A}$?

   Solution:



   Notice that $EDFC$ is a square, and that $AE = AG = 1$. Thus, $CE = \sqrt{2} - 1$, so $ED = \sqrt{2} - 1$, and the radius is $\boxed{\sqrt{2} - 1}$.

6. Find the area enclosed by the relation:

$$|x + y| + |x - y| = 16.$$

   Solution: The two absolute expressions could be positive or negative. Therefore, there are four cases. Plugging these in, we obtain that the boundary is described by the four lines $2x = 16, 2y = 16, -2x = 16$, or $-2y = 16$. Therefore, the relation describes a square with side length 16. Therefore, the area is simply 16 squared, so the answer is $\boxed{256}$.

7. What is the probability of obtaining a sum of 9 by rolling 4 six-sided dice?

   Solution: This is equivalent to asking the number of ways to distribute 9 balls in 4 buckets, such that every bucket has at least one ball, and no bucket has more than 6 balls.

   Because every bucket has at least one ball, there are a total of $9 - 4 = 5$ more balls to distribute. We can distribute these balls in any way, so this is equivalent to a stars and bars problem with 5 stars and 3 bars. We have $\binom{8}{3} = 56$. There are 6^4 possible results from rolling 4 dice, so our answer is $\frac{56}{6^4} = \boxed{\frac{7}{162}}$.

8. For some real constant c, the roots of the quadratic $x^2 + cx - 2024$ are r and s. If the quadratic $x^2 + rx + s$ has one distinct root t (not necessarily real), find t.

   Solution: Since the quadratic $x^2 + rx + s$ has one distinct root, its discriminant $r^2 - 4s$ is zero. Thus, by Vieta's formula on $x^2 + cx - 2024$, we have $rs = r(r^2/4) = -2024$. Since $c = -(r + s) = -(r + r^2/4)$ is real, it follows that $r = -2\sqrt[3]{1012}$. By Vieta's formula on $x^2 + rx + s$, we have $t + t = -r$, and $t = -r/2 = \boxed{\sqrt[3]{1012}}$.

9. Call two positive integers *similar* if their prime factorization have the same number of distinct prime divisors, and when ordered in some way, the exponents match. For example, 250 and 24 are *similar* because $250 = 5^3 \cdot 2$, and $24 = 2^3 \cdot 3$. How many positive integers less than or equal to 200 are *similar* to 18 (including itself)?

   Solution: We have $18 = 2 \cdot 3^2$, so we seek numbers of the form pq^2. Noting that $2 \cdot 11^2$ already exceeds 200, we may assume that $q \leq 7$. In the case $q = 7$, $p = 2$ or $p = 3$. If $q = 5$, then $p = 2, 3, 7$. If $q = 3$, then p is any prime less than or equal to 22 except for 3, i.e. $2, 5, 7, 11, 13, 17, 19$. If $q = 2$, then p is any prime less than or equal to 50, except for 2, i.e. $3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$. Counting, there are in total $2 + 3 + 7 + 14 = \boxed{26}$ such numbers.

10. What is the sum of the possible values of c such that the polynomial $x^2 - 40x + c = 0$ has positive integer roots (possibly equal to each other)?

    Solution: The only possible root pairs are $(1, 39), (2, 38), (3, 37), \ldots, (20, 20)$. Then,

$$1 \cdot 39 + 2 \cdot 38 + \ldots + 20 \cdot 20$$

$$= (20^2 - 19^2) + (20^2 - 18^2) + \ldots + (20^2 - 0^2)$$

$$= 20(20^2) - (1^2 + 2^2 + \ldots + 19^2)$$

$$= 20(20^2) - \frac{19(19+1)(2(19)+1)}{6}$$

$$= 8000 - 2470 = \boxed{5530}.$$

11. Let $\triangle ABC$ be an equilateral triangle with side length 6. Three circles of radius 6 are centered at A, B, and C. Compute the radius of the circle that is centered at the center of $\triangle ABC$, is internally tangent to these three circles, and lies in the interior of the three circles.

    Solution: Denote the circle that we want to compute the radius of as ω. Let the center of $\triangle ABC$ be O and let line AO intersect ω at point D such that O lies between A and D. Note that this is also the point where ω is tangent to the circle centered at A. Using $30 - 60 - 90$ triangles, one can show that the circumradius of an equilateral triangle is $\frac{1}{\sqrt{3}}$ of its side length. The desired radius is thus $OD = AD - OA = 6 - \frac{6}{\sqrt{3}} = \boxed{6 - 2\sqrt{3}}$.

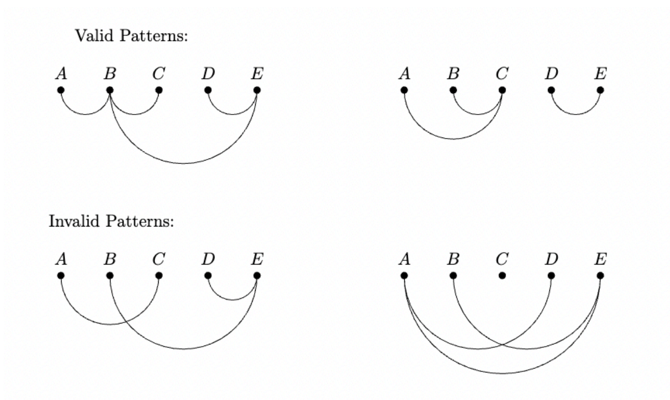
12. Compute the largest positive integer x less than 1000 that satisfies $x^2 \equiv 24 \pmod{1000}$.

    Solution: We need to find x such that $x^2 \equiv 24 \pmod{8}$ and $x^2 \equiv 24 \pmod{125}$. The first is equivalent to $x^2 \equiv 0 \pmod{8}$, which is satisfied when $x \equiv 0 \pmod{4}$. The second is equivalent to

$$x^2 - 24 \equiv 0 \pmod{125}$$
$$x^2 - 1024 \equiv 0 \pmod{125}$$
$$(x-32)(x+32) \equiv 0 \pmod{125}.$$

Since $(x+32)-(x-32)=64$, which is not divisible by 5, we must have $125 \mid (x-32)$ or $125 \mid (x+32)$. Then, $x \equiv 32 \pmod{125}$ or $x \equiv -32 \pmod{125}$. Now, we can see that the largest solution less than 1000 is $1000 - 32 = \boxed{968}$.

13. Jana is decorating her room by hanging zero or more strings of lights. She has 5 collinear attachment points (A, B, C, D, and E), and she can connect any two attachment points with a semicircular string of lights (direction hanging downward), as long as no two strings cross. In how many different patterns can she hang the lights?

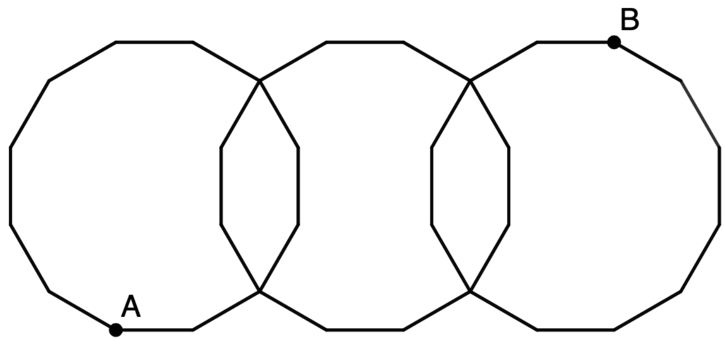


Solution: We can always choose to connect A and E (referred to as A-E) or not, giving us 2 options. We can also choose to connect as many or as few of A-B, B-C, C-D, and D-E connections as we wish, giving us another $2^4 = 16$ options.
Thus, the only connections we need to worry about are the A-C, B-D, and C-E connections, as well as the A-D and B-E connections. A manual inspection of these shows that there are 11 possible combinations of these connections. $2 \cdot 2^4 \cdot 11 = \boxed{352}$.

14. Let $0.\overline{d}_b = 0.ddd..._b$ denote a repeating decimal written in base b. If $0.\overline{4}_a + 0.\overline{7}_b = 1$ for positive integers a and b such that $a \neq b$, what is the minimum possible value of $a+b$?

Solution: We have $0.\overline{4}_a + 0.\overline{7}_b = \frac{4}{a-1} + \frac{7}{b-1} = 1$, which gives us $ab - 8a - 5b = -12$. This can be rewritten as $(a-5)(b-8) = 28$. Let $a' = a-5$ and $b' = b-8$. Then, $a'b' = 28$. We have $a' + \frac{28}{a'} = \left(\sqrt{a'} - \sqrt{\frac{28}{a'}}\right)^2 + 2\sqrt{28}$, so in order to minimize $a' + \frac{28}{a'}$ we want to find the values of a' and b' that are closest to $\sqrt{28}$. The pair of divisors of 28 closest to each other is $(4,7)$. This gives us $a + b = 4 + 7 + 5 + 8 = \boxed{24}$, and we can let $a = 4 + 5 = 9$, $b = 7 + 8 = 15$, so that $a \neq b$.

15. Perry bakes a pineapple upside-down cake with 3 slices of pineapple on top, each in the shape of regular dodecagons of side length 1. The pineapple slices overlap each other as shown in the figure. Compute the length of the cut AB that Perry makes to slice the cake in half.



Solution: Let the third vertex counterclockwise to A in the left dodecagon be C, and let the third vertex counterclockwise to B in the right dodecagon be D. Note that AC makes a 120° angle with the right-most side of the left dodecagon, BD makes a 120° angle with the left-most side of the right dodecagon, and CD forms 60° angles with these sides of the dodecagon. This means that points A, C, D, B are collinear. We see that the length of AC is the sum of the heights of two equilateral triangles of side length 1 and the side length of a unit square, so $AC = 1 + \sqrt{3}$. CD is the hypotenuse of a $30 - 60 - 90$ triangle with a short leg that has length 1, so $CD = 1$. Then,

$$\begin{aligned} AB &= AC + CD + DB \\ &= 1 + \sqrt{3} + 2 + 1 + \sqrt{3} \\ &= \boxed{4 + 2\sqrt{3}}. \end{aligned}$$

16. Let $P(n)$ represent the number of real roots x for the equation

$$x^n + x^{n-1} + ... + x^1 + 1 = 0.$$

Compute

$$P(1) + P(2) + P(3) + ... + P(2024).$$

Solution: Clearly, there are no solutions at $x = 1$, since then every term is strictly positive.

Since the polynomial is the sum of a geometric series with n terms, we consider when $\frac{x^{n+1}-1}{x-1} = 0$ where $x \neq 1$. If n is odd and x is real, then the numerator is 0 precisely at $x = -1$ and nowhere else. If n is even, then the only real solution for the numerator to be 0 is where $x = 1$, but we already forbade this case.

Therefore, all odd n have one solution, and all even n have no solutions. Then the answer is simply the number of odd natural numbers less than or equal to 2024, which is $\boxed{1012}$.

17. Nacho is building a sandcastle. Each time he adds a scoop of sand, he has a $\frac{5}{6}$ chance that the sandcastle will increase by 1 inch in height. Nacho is a clumsy engineer, so each time the height doesn't increase, the sandcastle topples and loses $\frac{1}{3}$ of its current height. Suppose Nacho starts his sandcastle at height H. What H should he choose so that after any number of scoops, the expected height of his sandcastle is still H?

Solution: It must be the case that $H = \frac{5}{6}(H+1) + \frac{1}{6} \cdot \frac{2}{3} \cdot H$. Rearranging, this implies that $H = \boxed{15}$.

Intuitive solution: For every 5 times he succeeds in adding 1 to the height, he fails once causing $\frac{1}{3}$ of the sand castle to topple. Until the height reaches $\frac{5}{\frac{1}{3}} = 15$, his net progress will be positive. Now, once he reaches 15, he is going to stay there.

18. Consider the horizontal line that intersects the ellipse $\frac{x^2}{9} + y^2 = 1$ at points A and B above the x-axis such that $\angle AOB = 120°$, where point O is the origin. Compute the area of the region of the ellipse that lies above this line.

Solution: Let the midpoint of AB be M and WLOG let A be to the left of the y-axis. Since $\triangle AOM$ is a $30-60-90$ triangle, we know that $A = (-k, k/\sqrt{3})$ for some positive k. Solving for k in $\frac{k^2}{9} + \frac{k^2}{3} = 1$ gives us $k = \frac{3}{2}$.

Consider the circle $x^2 + y^2 = 1$, which can be obtained by compressing the x-coordinates on the ellipse by a factor of 3. Then, $A'M' = \frac{1}{2}$ while we still have $M'O' = \frac{\sqrt{3}}{2}$, so we deduce that $\triangle A'O'M'$ is a $30-60-90$ triangle with $\angle A'O'M' = 30°$. We can then compute the area of the sector subtended by $A'B'$ as $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$. To get the area for the ellipse, we multiply by 3 to get $\boxed{\frac{\pi}{2} - \frac{3\sqrt{3}}{4}}$.

19. Let $a_1, a_2, \ldots$ be a strictly increasing sequence of positive integers such that a_{3k-2} is divisible by 8 and a_{3k} is divisible by 9 for all positive integers k. Find the largest possible positive integer i such that $a_i > 2024$ and $a_{i-1} \le 2024$.

Solution: Clearly every term should be the smallest it possibly can be to maximize i. Therefore, we will have a sequence of 3 types of terms: the smallest multiple of 8 larger than the last term, a term 1 larger than the previous term, and the smallest multiple of 9 larger than the last term.

If we compare the terms a_{3k}, they will be consecutive multiples of 9 except when the term preceding some a_{3k} is a multiple of 9. This happens exactly when $a_{3k-3} = 72c$, $a_{3k-2} = 72c + 8$, and $a_{3k-1} = 72c + 9$. So, this occurs after every common multiple of 9 and 8.

We have $\left\lfloor \frac{2024}{72} \right\rfloor = 28$ and $28 \cdot 72 = 2016$. In each cycle between multiples of 72 we have 21 terms. This is because $a_{3k} = 72c + 18$, as shown previously, and every multiple of 9 afterward appears until the next multiple of 72, so we have $\frac{72}{9} - 1 = 7$ sets of 3 terms. After the term 2016 we follow with $2016 + 8 = 2024$ and $2024 + 1 = 2025$ which is the term we are looking for.

In total, we used $28 \cdot 21 + 2 = \boxed{590}$ terms.

20. I have 3 red balls, 3 blue balls, and 3 yellow balls. These 9 balls are randomly arranged on a 3×3 grid. Let a *3-in-a-row* denote when 3 balls of the same color are aligned in a line in the grid (a row, column, or diagonal consisting of 3 balls of the same color). What is the expected number of *3-in-a-rows* that will show up in the grid?

Solution: For any ball that is in the center, the probability that there is a *3-in-a-row* through that point is $\frac{1}{7}$. This is because there are $\binom{8}{2} = 28$ ways to choose where we can place the other two balls and of those, there are 4 valid placings of the other two balls to get *3-in-a-row* through the middle (middle row, middle column, upper right diagonal, lower right diagonal). For any ball in a corner, the probability of a *3-in-a-row* through that point is $\frac{3}{28}$. This is because there are again 28 ways to choose where we can place the other two balls and only 3 valid placings of the other two balls to get *3-in-a-row* through that point. Finally, for any ball on an edge of the grid, the probability of a *3-in-a-row* through that point is $\frac{2}{28}$. This is because there are again 28 ways to choose where we can place the other two balls and only 2 valid placings of the other two balls to get *3-in-a-row* through that point. By linearity of expectation, the expected number of *3-in-a-rows* that pass through all points of the grid is $\frac{1}{7} \cdot 1 + \frac{3}{28} \cdot 4 + \frac{2}{28} \cdot 4 = \frac{6}{7}$ as on the grid there are 4 corner points, 4 edge points, and 1 center point. This number would overcount the number of *3-in-a-rows* in the grid by a factor of 3 as each *3-in-a-row* is counted once per point in a *3-in-a-row* in the grid. Thus, the expected number of *3-in-a-rows* in the entire graph is $\frac{\frac{6}{7}}{3} = \boxed{\frac{2}{7}}$.

Alternate solution: Consider the case where we only have 3 balls to place in a 3×3 grid. There are $\frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ ways to do so. This is because there are $9 \cdot 8 \cdot 7$ ways to choose the squares on the grid and we divide by 3! because order does not matter. Thus, the probability that 3 particular balls end up in a given set of 3 grid cells is $\frac{1}{84}$. Notice that there are 8 possible ways to form a *3-in-a-row* (all 3 rows, all 3 columns, and 2 diagonals) and there are 3 colors to choose from. Thus, our answer is $\frac{8 \cdot 3}{84} = \boxed{\frac{2}{7}}$.

21. How many positive integers n are there such that the powers of 2024 $\pmod{n}$ repeat in a cycle of length 2? In other words, how many positive integers n are there such that

$$2024^0 \pmod{n} \equiv 2024^2 \pmod{n} \equiv 2024^4 \pmod{n}...$$

and

$$2024^1 \pmod{n} \equiv 2024^3 \pmod{n} \equiv 2024^5 \pmod{n}...$$

but

$$2024^0 \pmod{n} \not\equiv 2024^1 \pmod{n}?$$

Solution: If the powers of 2024 $\pmod{n}$ repeat in a cycle of length 2, then

$$2024^2 - 2024^0 = 2024^2 - 1 \equiv 0 \pmod{n}$$

and

$$2024^1 - 2024^0 = 2024 - 1 \not\equiv 0 \pmod{n}.$$

Every factor of $2024^2 - 1$ that is not also a factor of $2024 - 1$ will have this property.

We have $2024^2 - 1 = 2023 \cdot 2025 = 3^4 \cdot 5^2 \cdot 7 \cdot 17^2$, which gives $5 \cdot 3 \cdot 2 \cdot 3 = 90$ factors.

$2024 - 1 = 2023 = 7 \cdot 17^2$. This is a total of $2 \cdot 3 = 6$ factors.

Thus, the final answer is $90 - 6 = \boxed{84}$.

22. Consider a quadratic function $P(x) = ax^2 + bx + c$ with distinct positive roots r_1 and r_2, and a second polynomial $Q(x) = cx^2 + bx + a$ with roots r_3 and r_4. John writes four numbers on the whiteboard: r_1, r_2, $4r_3$ and $4r_4$. What is the smallest possible integer value of the sum of the numbers John wrote down?

Solution: We first apply Vieta's formula to $P(x)$, which gives that $r_1 r_2 = \frac{c}{a}$ and $r_1 + r_2 = -\frac{b}{a}$. Rearranging, we have $c = ar_1 r_2$, and $b = -a(r_1 + r_2)$. Next, we apply Vieta's to $Q(x)$, which gives $r_3 r_4 = \frac{a}{c}$ and $r_3 + r_4 = -\frac{b}{c}$. Substituting, we have

$$r_3 r_4 = \frac{a}{c} = \frac{a}{ar_1 r_2} = \frac{1}{r_1 r_2}$$

and

$$r_3 + r_4 = -\frac{b}{c} = \frac{a(r_1 + r_2)}{c} = \frac{a}{c}(r_1 + r_2) = \frac{r_1 + r_2}{r_1 r_2} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Since we now know that $r_3 + r_4 = \frac{1}{r_1} + \frac{1}{r_2}$, we can express the sum S of the numbers John wrote down as

$$S = r_1 + r_2 + \frac{4}{r_1} + \frac{4}{r_2}$$

We can re-order this as $\left(r_1 + \frac{4}{r_1}\right) + \left(r_2 + \frac{4}{r_2}\right)$. Observe that for positive k, any expression of the form $k + \frac{4}{k}$ is always at least 4, and is minimised when $k = 2$. Thus,

$$S = \left(r_1 + \frac{4}{r_1}\right) + \left(r_2 + \frac{4}{r_2}\right) \geq 8.$$

However, $S = 8$ is only achieved when $r_1 = r_2 = 2$, which violates our requirement that the roots of $P(x)$ be distinct. The total sum therefore must be greater than 8, yielding a minimum integer value of $\boxed{9}$ (note that the roots themselves don't necessarily have to be integer values). This can achieved by setting $r_1 = 1$ and $r_2 = 2$.

23. Let $ABCDE$ be a regular pentagon with side length 1. Circles ω_B, ω_C, and ω_D are centered at B, C, and D respectively, each with radius 1. ω_B intersects ω_C inside $ABCDE$ at point F, and ω_C intersects ω_D inside $ABCDE$ at point G. Compute the ratio of the measure of $\angle AFB$ to the measure of $\angle AGB$.

    Solution: Note that $\triangle CDG$ is an equilateral triangle, so $\angle BCG = 108^\circ - 60^\circ = 48^\circ$. Next, we see that $\triangle CBG$ is isosceles with $CG = CB = 1$. Then, $\angle CBG = (180^\circ - 48^\circ)/2 = 66^\circ$. We get $\angle GBA = 108^\circ - 66^\circ = 42^\circ$. Next, we note that AG is the angle bisector of $\angle BAE$ since G is symmetric with respect to points C and D, so $\angle BAG = 54^\circ$. Now we have $\angle AGB = 180^\circ - \angle GBA - \angle BAG = 180^\circ - 42^\circ - 54^\circ = 84^\circ$. Finally, note that $\triangle BAF \cong \triangle CBG$, so $\angle BFA = \angle CGB = 66^\circ$. The ratio we want is then $\frac{66^\circ}{84^\circ} = \boxed{\frac{11}{14}}$.

24. Let F be a set of subsets of $\{1, 2, 3\}$. F is called *distinguishing* if each of 1, 2, and 3 are distinguishable from each other—that is, 1, 2, and 3 are each in a distinct set of subsets from each other. For example $F = \{\{1, 3\}, \{2, 3\}\}$ is *distinguishing* because 1 is in $\{1, 3\}$, 2 is in $\{2, 3\}$, and 3 is in $\{1, 3\}$ and $\{2, 3\}$. $F = \{\{1, 2\}, \{2\}\}$ is also *distinguishing*: 1 is in $\{1, 2\}$, 2 is in $\{1, 2\}$ and $\{2\}$, and 3 is in none of the subsets.

    On the other hand, $F = \{\{1\}, \{2, 3\}\}$ is not *distinguishing*. Both 2 and 3 are only in $\{2, 3\}$, so they cannot be distinguished from each other.

    How many *distinguishing* sets of subsets of $\{1, 2, 3\}$ are there?

Solution: There are a total of 8 possible subsets of $\{1, 2, 3\}$, so there are $2^8 = 256$ possible collections of these subsets. We will solve this problem by complementary counting—determining how many sets are not distinguishable.

There are two cases: either none of the values $\{1, 2, 3\}$ can be distinguished, or only one of the values $\{1, 2, 3\}$ can be distinguished.

If none of the values can be distinguished, then the only possible subsets are $\{\}$ and $\{1, 2, 3\}$. Each of these subsets can be either included or excluded from F, so we have a total of $2^2 = 4$ such collections F.

If only one of the values can be distinguished, say without loss of generality that this value is 1. Then, we are restricted to the subsets $\{\}$, $\{1, 2, 3\}$, $\{1\}$, $\{2, 3\}$ (since 2 and 3 cannot be distinguished from each other). Each of these subsets can be either included or excluded in F, so we have a total of $2^4 = 16$ such collections. However, 4 of these collections (the ones with only $\{\}$ and $\{1, 2, 3\}$ have already been counted in the first case. Thus, we have $16 - 4 = 12$ possible collections. Either 1, 2, and 3 can be distingushed, so we have a total of $12 \cdot 3 = 36$ total collections.

Our final answer is $256 - 4 - 36 = \boxed{216}$.

25. For the numbers $4^4, 4^{44}, 4^{444}, \ldots, 4^{44\ldots4}$ where the last exponent is 2024 digits long, Quincy writes down the remainders when they are divided by 2024. Compute the sum of the distinct numbers that Quincy writes.

Solution: We first find the prime factorization $2024 = 2^3 \cdot 11 \cdot 23$. Note that all of the numbers listed are divisible by 2^3. We have $\phi(253) = 220$, where $\phi(n)$ denotes the number of positive integers up to n that are relatively prime to n. Next, we see that $44\ldots4 \equiv 4 \pmod{220}$ when the number of digits is odd and $44\ldots4 \equiv 44 \pmod{220}$ when the number of digits is even. Using Euler's totient theorem we know that $4^{220} \equiv 1 \pmod{253}$, so we have $4^{44\ldots4} \equiv 4^4 \equiv 3 \pmod{253}$ or $4^{44\ldots4} \equiv 4^{44} \equiv 3^{11} \equiv 3 \cdot (-10)^2 \equiv 47 \pmod{253}$. Thus, there are only 2 distinct remainders that Quincy writes. For the first we have $x_1 \equiv 3 \pmod{253}, x_1 \equiv 0 \pmod{8}$ and for the second we have $x_2 \equiv 47 \pmod{253}, x_2 \equiv 0 \pmod{8}$. We can use $253 \equiv 5 \pmod{8}$ to quickly solve these systems of equivalences to find that $x_1 \equiv 256 \pmod{2024}$ and $x_2 \equiv 1312 \pmod{2024}$. Our answer is then $256 + 1312 = \boxed{1568}$.