1. Rectangle $ABCD$ has side lengths $AB = 10$ and $BC = 12$. Let the midpoint of CD be point M. Compute the area of the overlap between $\triangle AMB$ and $\triangle ADC$.

   Answer: 20

   Solution: We have that the area of $\triangle ADC$ is $\frac{12\cdot10}{2} = 60$. The area of $\triangle ADM$ is $\frac{12\cdot5}{2} = 30$. Let the intersection of AC and BM be N. Then, we can see that $\triangle ABN$ and $\triangle CMN$ are similar, with ratio $\frac{AB}{CM} = \frac{10}{5} = 2$, so the height of $\triangle CMN$ is $\frac{12}{3} = 4$ and the area of $\triangle CMN$ is $\frac{5\cdot4}{2} = 10$. Then, the area of the overlap between $\triangle AMB$ and $\triangle ADC$ is $[ADC] - [ADM] - [CMN] = 60 - 30 - 10 = \boxed{20}$.

2. Let ω_1 be the incircle of $\triangle ABC$ with side lengths $AB = AC = 13$ and $BC = 10$, and let ω_2 be the circle inside $\triangle ABC$ that is externally tangent to ω_1 and tangent to segments AB and AC. Compute the radius of the circle inside $\triangle ABC$ that is externally tangent to ω_1 and ω_2 and tangent to segment AB.

   Answer: $\frac{8}{15}$

   Solution: The radius r_1 of ω_1 is $\frac{[ABC]}{(AB+BC+CA)/2} = \frac{(10\cdot12)/2}{(13+13+10)/2} = \frac{10}{3}$. Then, we can note that ω_2 is a dilation of ω_1 centered at A, so the radius r_2 of ω_2 is $\frac{12-20/3}{12}\cdot\frac{10}{3} = \frac{40}{27}$. Let ω_1 and ω_2 be tangent to AB at P_1 and P_2, respectively. Let the centers of ω_1 and ω_2 be O_1 and O_2, respectively, and let the center of the circle we are finding the radius of be O_3. Let the line passing through O_3 parallel to AB meet O_1P_1 and O_2P_2 at Q_1 and Q_2, respectively. Then, $Q_1Q_2 = P_1P_2$. We can find that $AP_1 = 12\cdot\frac{10/3}{5} = 8$ and $AP_2 = 12\cdot\frac{40/27}{5} = \frac{32}{9}$, so $P_1P_2 = 8 - \frac{32}{9} = \frac{40}{9}$. Denote the radius that we want to compute as r. From right triangle $O_1Q_1O_3$, we have $O_1Q_1 = \frac{10}{3} - r$ and $O_1O_3 = \frac{10}{3} + r$, so $Q_1O_3 = 2\sqrt{10r/3}$. Similarly, $Q_2O_3 = 2\cdot\sqrt{40r/27}$. Finally, we have

$$Q_1O_3 + Q_2O_3 = Q_1Q_2$$

$$2\cdot\sqrt{10r/3} + 2\cdot\sqrt{40r/27} = \frac{40}{9}$$

$$\sqrt{r} = \frac{2\sqrt{30}}{15}$$

$$r = \boxed{\frac{8}{15}}.$$

3. Let circles ω_1 and ω_2 be circles with radii 6 and 13, respectively, such that the distance between their centers is 25. A common external tangent touches ω_1 at point P and ω_2 at point Q. A common internal tangent touches ω_1 at point R and ω_2 at point S, and intersects line PQ at point T such that $TP < TQ$. Compute the length of segment TR.

   Answer: $12 - \sqrt{66}$

   Solution: Denote the intersection of the common external tangent with the other internal tangent as U, and let the centers of ω_1 and ω_2 be O_1 and O_2, respectively. Let the midpoint of segment O_1O_2 be M. Note that since M is the midpoint of O_1O_2, and O_1P, O_2Q are perpendicular to PQ, then the line passing through M perpendicular to PQ passes through the midpoint of segment of PQ, which we denote N. Thus, M lies on the perpendicular bisector of segment PQ, so M is equidistant from P and Q.

Next, note that TO_1 bisects $\angle PTR$ and TO_2 bisects $\angle QTS$ by considering that TP, TR are tangent to ω_1 and TQ, TS are tangent to ω_2. This gives us $O_1TO_2 = 90^\circ$. We can similarly argue that $O_1UO_2 = 90^\circ$. Then, O_1TUO_2 is a cyclic quadrilateral with center M. Since P and Q are equidistant from M, we know that the powers of P and Q with respect to (O_1TUO_2) are equal, which means $(PT)(PT+TU) = (QU)(QU+UT)$, so we conclude that $PT = QU$. Denote $PT = x$. We have $MN = \frac{O_1P+O_2Q}{2} = \frac{19}{2}$ and $PQ = \sqrt{O_1O_2^2 - (O_2Q - O_1P)^2} = \sqrt{25^2 - 7^2} = 24$ so $PN = 12$. Then, $MP = \sqrt{\left(\frac{19}{2}\right)^2 + 12^2}$ and the power of P with respect to (O_1TUO_2) is $\left(\frac{19}{2}\right)^2 + 12^2 - \left(\frac{25}{2}\right)^2 = 78$. Then, $(PT)(PU) = x(24-x) = 78$ which gives us $x = 12 \pm \sqrt{66}$. Since we see that $2x < PQ = 24$, the valid solution is $x = 12 - \sqrt{66}$, so $TR = TP = \boxed{12 - \sqrt{66}}$.