user@machine: $ asymptotic_notation

--verbose --cheat-sheet

> The 5 Notations

Not.	Meaning	Like
Θ	Tight bound	$a = b$
O	Upper bound	$a \le b$
Ω	Lower bound	$a \ge b$
o	Strict upper	$a < b$
ω	Strict lower	$a > b$

> Definitions

>> $\Theta(g(n))$

$\exists c_1, c_2, n_0 > 0:$

$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$

$\forall n \ge n_0$

>> $O(g(n))$

$\exists c, n_0 > 0:$

$0 \le f(n) \le c \cdot g(n)$

$\forall n \ge n_0$

>> $\Omega(g(n))$

$\exists c, n_0 > 0:$

$0 \le c \cdot g(n) \le f(n)$

$\forall n \ge n_0$

>> $o(g(n))$

$\forall c > 0, \exists n_0:$

$0 \le f(n) < c \cdot g(n)$

OR: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

>> $\omega(g(n))$

$\forall c > 0, \exists n_0:$

$0 \le c \cdot g(n) < f(n)$

OR: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

> Key Theorems

[Th 3.1] $f = \Theta(g) \iff f = O(g)$ AND $f = \Omega(g)$

[Poly] $p(n) = \sum_{i=0}^{d} a_i n^i \implies p(n) = \Theta(n^d)$

[Transpose] $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$

> Growth Hierarchy

$O(1) < O(\lg n) < O(\sqrt{n}) < O(n)$

$< O(n \lg n) < O(n^2) < O(n^3)$

$< O(2^n) < O(n!)$

> Complexity Reference

Class	Example
$O(1)$	Array access
$O(\lg n)$	Binary search
$O(n)$	Linear search
$O(n \lg n)$	Merge/Heap sort
$O(n^2)$	Bubble/Insert sort
$O(2^n)$	Subset gen
$O(n!)$	Permutations

> Properties

[Transitivity]

$f = \Theta(g), g = \Theta(h) \implies f = \Theta(h)$

[Reflexivity]

$f(n) = \Theta(f(n))$

[Symmetry]

$f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$

> Critical Limits

[Exp vs Poly]

$\lim_{n\to\infty} \frac{n^b}{a^n} = 0 \;\; (a > 1)$

$\implies n^b = o(a^n)$

[Poly vs Log]

$\lim_{n\to\infty} \frac{\lg^b n}{n^a} = 0 \;\; (a > 0)$

$\implies \lg^b n = o(n^a)$

[Factorial]

$n! = \omega(2^n)$, $n! = o(n^n)$

> Logarithms

$\lg n = \log_2 n$, $\ln n = \log_e n$

Identities:

$$a = b^{\log_b a}$$
$$\log(ab) = \log a + \log b$$
$$\log(a^k) = k \log a$$
$$\log_b a = \frac{\log a}{\log b}$$
$$\log(1/a) = -\log a$$

> Exponentials

$a^0 = 1$, $a^1 = a$, $a^{-1} = 1/a$

$(a^m)^n = a^{mn}$, $a^m a^n = a^{m+n}$

[Natural exp]

$e^x \ge 1 + x$

$e^x = 1 + x + \Theta(x^2)$ as $x \to 0$

> Factorials

$n! = 1 \cdot 2 \cdot 3 \cdots n$

[Stirling]

$n! \approx \sqrt{2\pi n}(n/e)^n$

[Log factorial]

$\lg(n!) = \Theta(n \lg n)$

> Floor/Ceiling

$\lfloor x \rfloor$ :: max int $\leq x$

$\lceil x \rceil$ :: min int $\geq x$

$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

> Modular Arithmetic

$a \bmod n = a - n\lfloor a/n \rfloor$

$0 \leq a \bmod n < n$

$a \equiv b \pmod{n} \iff n|(b-a)$

> Sequences

[Fibonacci]

$F_0 = 0, F_1 = 1, F_i = F_{i-1} + F_{i-2}$

$F_i \approx \phi^i/\sqrt{5}$ where $\phi \approx 1.618$

[Iterated Log]

$\lg^* n$: grows very slowly.

$2, 4, 16, 65536 \to 1, 2, 3, 4$

> Loop Analysis

Single loop: $O(n)$

Nested (both n): $O(n^2)$

Dependent: $\sum i \to O(n^2)$

Halving: $n = n/2 \to O(\lg n)$

Tree recursion: $O(n \lg n)$

> Summation

$\sum_{i=1}^{n} 1 = n$

$\sum_{i=1}^{n} i = \Theta(n^2)$

$\sum_{i=1}^{n} i^2 = \Theta(n^3)$

$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$

> Practical Limits

Max n for 1 sec:

$O(n) \to 10^8$

$O(n \lg n) \to 10^7$

$O(n^2) \to 10^4$

$O(n^3) \to 500$

$O(2^n) \to 25$

$O(n!) \to 11$

> Quick Rules

Polynomials: Drop lower terms

Nested loops: Multiply

Sequential: Add (Take Max)

Recursive: Master Thm

> Common Mistakes

[ERR] "at least O(n)"

[ERR] 2^n vs n^2

[ERR] Counting input in space

> Space Complexity

Count only aux space.

Recursion: $O(depth)$

> Master Theorem

$T(n) = aT(n/b) + f(n)$

$c_{crit} = \log_b a$

[Case 1] $f(n)$ small

$\implies T(n) = \Theta(n^{c_{crit}})$

[Case 2] $f(n)$ equal

$\implies T(n) = \Theta(n^{c_{crit}} \lg n)$

[Case 3] $f(n)$ large

$\implies T(n) = \Theta(f(n))$

> Proving Bounds

[Theta] Find c_1, c_2, n_0

[Big-O] Find c, n_0 OR limit $< \infty$

[Omega] Find c, n_0 OR limit > 0