

Asymptotic Notation Cheat Sheet

Growth of Functions - Quick Reference (Portrait)

THE 5 NOTATIONS

| Not. | Meaning | Example |
|----------|--------------|------------|
| Θ | Tight bound | $a = b$ |
| O | Upper bound | $a \leq b$ |
| Ω | Lower bound | $a \geq b$ |
| o | Strict upper | $a < b$ |
| ω | Strict lower | $a > b$ |

(Applies to all: O, Ω, o, ω)

Reflexivity: $f(n) = \Theta(f(n))$, $f(n) = O(f(n))$, $f(n) = \Omega(f(n))$

Symmetry: $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$

$\Theta(g(n))$: $\exists c_1, c_2, n_0 > 0 : 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \forall n \geq n_0$

$O(g(n))$: $\exists c, n_0 > 0 : 0 \leq f(n) \leq c \cdot g(n) \forall n \geq n_0$

$\Omega(g(n))$: $\exists c, n_0 > 0 : 0 \leq c \cdot g(n) \leq f(n) \forall n \geq n_0$

$o(g(n))$: $\forall c > 0, \exists n_0 : 0 \leq f(n) < c \cdot g(n) \forall n \geq n_0$

OR: $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$

$\omega(g(n))$: $\forall c > 0, \exists n_0 : 0 \leq c \cdot g(n) < f(n) \forall n \geq n_0$

OR: $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$

KEY THEOREMS

Th 3.1: $f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ AND } f(n) = \Omega(g(n))$

Polynomials: $p(n) = \sum_{i=0}^d a_i n^i, a_d > 0 \implies p(n) = \Theta(n^d)$

Transpose Terminology: $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
 $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$

GROWTH HIERARCHY

$$O(1) < O(\lg n) < O(\sqrt{n}) < O(n)$$

$$< O(n \lg n) < O(n^2) < O(n^3)$$

$$< O(2^n) < O(n!)$$

COMMON COMPLEXITIES

| Class | Example Algorithm |
|--------------|-------------------------------|
| $O(1)$ | Array access, Stack push/pop |
| $O(\lg n)$ | Binary search |
| $O(n)$ | Linear search, Traversal |
| $O(n \lg n)$ | Merge sort, Heap sort |
| $O(n^2)$ | Bubble sort, Insertion sort |
| $O(2^n)$ | Power Set, Fibonacci (naive) |
| $O(n!)$ | Traveling Salesperson (brute) |

PROPERTIES

Transitivity: $f = \Theta(g), g = \Theta(h) \implies f = \Theta(h)$

CRITICAL LIMITS

Exp vs Poly: $\lim_{n \rightarrow \infty} \frac{n^b}{a^n} = 0$ ($a > 1$) $\implies n^b = o(a^n)$

Poly vs Log: $\lim_{n \rightarrow \infty} \frac{\lg^b n}{n^a} = 0$ ($a > 0$) $\implies \lg^b n = o(n^a)$

Factorial vs Exp: $n! = \omega(2^n)$, $n! = o(n^n)$

LOGARITHMS & EXPONENTIALS

$$\lg n = \log_2 n, \ln n = \log_e n$$

Log Identities:

$$a = b^{\log_b a} \quad \log(ab) = \log a + \log b$$

$$\log(a^k) = k \log a \quad \log_b a = \frac{\log a}{\log b}$$

Exp Identities: $a^0 = 1, a^1 = a, a^{-1} = 1/a, (a^m)^n = a^{mn}$

Useful Series: $e^x = 1 + x + \frac{x^2}{2!} + \dots \approx 1 + x$

FACTORIALS & STIRLING

$$n! = 1 \cdot 2 \cdot 3 \cdots n$$

Weak bound: $n! \leq n^n$

Stirling's Approximation: $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$

Log Factorial: $\lg(n!) = \Theta(n \lg n)$

MASTER THEOREM

For $T(n) = aT(n/b) + f(n)$: Let $c_{crit} = \log_b a$.

Case 1: $f(n) = O(n^c)$ where $c < c_{crit}$ $\implies T(n) = \Theta(n^{c_{crit}})$

Case 2: $f(n) = \Theta(n^{c_{crit}}) \implies T(n) = \Theta(n^{c_{crit}} \lg n)$

Case 3: $f(n) = \Omega(n^c)$ where $c > c_{crit}$ (and $af(n/b) \leq kf(n)$ for $k < 1$) $\implies T(n) = \Theta(f(n))$

LOOP ANALYSIS

Single loop: for $i=1$ to $n \rightarrow O(n)$

Nested (independent): for $i=1..n, j=1..m \rightarrow O(nm)$

Nested (dependent): for $i=1..n, j=1..i \rightarrow \sum_{i=1}^n i = \frac{n(n+1)}{2} = \Theta(n^2)$

Halving loop: while $n \geq 1$: $n=n/2 \rightarrow O(\lg n)$

SUMMATION FORMULAS

$$\sum_{i=1}^n 1 = n$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} = \Theta(n^2)$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$$

$$\sum_{i=0}^n a^i = \frac{a^{n+1}-1}{a-1}$$
 (Geometric Series)

$$\text{For } |x| < 1: \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

PRACTICAL LIMITS (1 SEC)

| Complexity | Max Input Size n |
|--------------|--------------------|
| $O(n)$ | 10^8 |
| $O(n \lg n)$ | 10^7 |
| $O(n^2)$ | 10^4 |
| $O(n^3)$ | 500 |
| $O(2^n)$ | 25 |