

Asymptotic Notation Cheat Sheet

Growth of Functions - Quick Reference (Portrait Minimal)

THE 5 NOTATIONS

Not.	Meaning	Condition
Θ	Tight bound	$c_1g \leq f \leq c_2g$
O	Upper bound	$f \leq cg$
Ω	Lower bound	$cg \leq f$
o	Strict upper	$f < cg$
ω	Strict lower	$cg < f$

$\Theta(g(n))$: $\exists c_1, c_2, n_0 > 0 : 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \forall n \geq n_0$

$O(g(n))$: $\exists c, n_0 > 0 : 0 \leq f(n) \leq c \cdot g(n) \forall n \geq n_0$

$\Omega(g(n))$: $\exists c, n_0 > 0 : 0 \leq c \cdot g(n) \leq f(n) \forall n \geq n_0$

$o(g(n))$: $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$

$\omega(g(n))$: $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$

KEY THEOREMS

Th 3.1: $f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ AND } f(n) = \Omega(g(n))$

Polynomials: $p(n) = \sum_{i=0}^d a_i n^i \implies \Theta(n^d)$

Transpose Terminology: $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$

GROWTH HIERARCHY

$$\begin{aligned} O(1) &< O(\lg n) < O(\sqrt{n}) < O(n) \\ &< O(n \lg n) < O(n^2) < O(n^3) \\ &< O(2^n) < O(n!) \end{aligned}$$

COMMON COMPLEXITIES

Class	Algorithm
$O(1)$	Stack op, Hash map
$O(\lg n)$	Binary search
$O(n)$	Linear scan
$O(n \lg n)$	Optimal sort
$O(n^2)$	Nested loops
$O(2^n)$	Subsets
$O(n!)$	Permutations

PROPERTIES

Transitivity: $f = \Theta(g), g = \Theta(h) \implies f = \Theta(h)$

Reflexivity: $f(n) = \Theta(f(n))$

Symmetry: $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$

CRITICAL LIMITS

Exp vs Poly: $\lim_{n \rightarrow \infty} \frac{n^b}{a^n} = 0 \quad (a > 1) \implies n^b = o(a^n)$

Poly vs Log: $\lim_{n \rightarrow \infty} \frac{\lg^b n}{n^a} = 0 \quad (a > 0) \implies \lg^b n = o(n^a)$

Factorial vs Exp: $n! = \omega(2^n)$

LOGARITHMS & EXPONENTIALS

$\lg n = \log_2 n, \ln n = \log_e n$

Log Identities:

$$\begin{aligned} a &= b^{\log_b a} & \log(ab) &= \log a + \log b \\ \log(a^k) &= k \log a & \log_b a &= \frac{\log a}{\log b} \end{aligned}$$

Exp Identities: $(a^m)^n = a^{mn}, \quad e^x \approx 1 + x$

FACTORIALS & STIRLING

$n! = 1 \cdot 2 \cdot 3 \cdots n$

Stirling's Approximation (Weak): $n! \approx (n/e)^n$

Log Factorial: $\lg(n!) = \Theta(n \lg n)$

MASTER THEOREM

For $T(n) = aT(n/b) + f(n)$: Let $c_{crit} = \log_b a$.

- $f(n) = O(n^c) \quad (c < c_{crit}) \implies \Theta(n^{c_{crit}})$
- $f(n) = \Theta(n^{c_{crit}}) \implies \Theta(n^{c_{crit}} \lg n)$
- $f(n) = \Omega(n^c) \quad (c > c_{crit}) \implies \Theta(f(n))$

LOOP ANALYSIS

Single loop: $O(n)$

Nested (independent): $O(nm)$

Nested (dependent): $\sum_{i=1}^n i = \Theta(n^2)$

Halving loop: $O(\lg n)$

SUMMATION FORMULAS

$$\begin{aligned} \sum_{i=1}^n 1 &= n \\ \sum_{i=1}^n i &= \Theta(n^2) \\ \sum_{i=1}^n i^2 &= \Theta(n^3) \\ \sum_{i=0}^n a^i &= \frac{a^{n+1}-1}{a-1} \end{aligned}$$