DePaul University
College of Computing and Digital Media

CSC 202 NOTES
Spring 2010: Amber Settle

Week 4: Monday, April 19, 2010

Announcements

- The third assignment is due now
- The fourth assignment is due Monday, May 3rd
- The midterm is next week
    - Monday, April 26th for section 901
    - To be arranged in a set window of dates for section 910
    - A study guide has been posted
    - Discussion at the end of class
- Review of the second assignment
    - Other questions?

(Review of) Relations

We write **x R y** or **R(x,y)** to express that x is in relation to y.

A binary relation R is **reflexive** if R(x, x) for all x
A binary relation R is **symmetric** if $R(x,y) \rightarrow R(y, x)$ for all x and y.
A binary relation R is **transitive** if $R(x,y) \wedge R(y,z) \rightarrow R(x,z)$ for all x, y, and z.

A binary relation is **an equivalence relation** if it is reflexive, symmetric, and transitive.

Let R be an equivalence relation and for every element a define **the equivalence class of a under R** as follows: $[a]_R := \{b: R(a,b)\}$. Note that **an equivalence class is never empty**. It always contains at least one element.

Ordering relations

A binary relation R is **anti-symmetric** if $R(x,y) \wedge R(y, x) \rightarrow x = y$

An ordering relation is a reflexive, anti-symmetric, and transitive relation.

Pair exercise: Show that $\subseteq$ is an ordering relation on sets.

The previous example shows us that ordering relations can be quite different than $\leq$. The main difference is that two elements might not be comparable by the relation.

For numbers x and y we always have $x \leq y$ or $y \leq x$.

But it is not true that for an arbitrary ordering $\preccurlyeq$ we always have either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Example: $\{\text{Amber}\} \nsubseteq \{\text{André}\}$ and $\{\text{André}\} \nsubseteq \{\text{Amber}\}$

An ordering relation $\preccurlyeq$ for which $x \preccurlyeq y$ or $y \preccurlyeq x$ for all elements x and y is called a **total ordering**. Otherwise it is a **partial ordering**.

Example: Consider the set W of all possible "words", meaning arbitrary sequences of letters, including "parsimonious" but also including "pdjwejslk".

This set can be totally ordered as follows:

Given two words $x \neq y \in W$, let w be their longest common prefix.

- If $w = x$ or $w = y$, then let $x \leq_{LEX} y$ if $|x| \leq |y|$ where $|x|$ and $|y|$ are the lengths of x and y.
- If neither $w = x$ or $w = y$, then both have a letter following w. This means $x = wl_1\ldots$ and $y = wl_2\ldots$ where l_1 and l_2 are letters of the alphabet. We let $x \leq_{LEX} y$ if l_1 comes before l_2 in the alphabet.

This ordering is called the **lexicographic** or **dictionary ordering** of words. It is a simplified variant of the ordering used by dictionaries to arrange words.

Some **specific examples**:

- x = "hand" and y = "handle" have longest common prefix "hand" which is x. So "hand" $\leq_{LEX}$ "handle" since "hand" is shorter.
- x = "brutal" and y = "brunch" have longest common prefix "bru". In x "bru" is followed by "t" whereas in y "bru" is followed by "n". Since "n" < "t" is the English ordering of the alphabet, "brunch" $\leq_{LEX}$ "brutal"

The lexicographic ordering does **disagree with our standard ordering of the numbers** (if we extend the notion of words to be made from letters as well as numbers.

For example, if we consider 112 and 12 as words "112" and "12", then "112" would be listed before "12". Under standard numeric ordering, 12 comes before 112.

We didn't specify which alphabet we were using (although you probably assumed the English alphabet), and the lexicographic ordering makes sense for any alphabet. We can also use any ordering of the alphabet that we like.

Strict ordering relations

For numbers, we distinguish between $<$ and $\leq$, and the same distinction can be made for arbitrary orders.

First we need a definition: A binary relation R is **anti-reflexive** if $\overline{R(x,x)}$ for all x.

We call $\prec$ a **strict ordering relation** if it is anti-reflexive, anti-symmetric, and transitive.

We can also distinguish between partial and total strict orders. A **strict ordering $\prec$ is total** if for any two distinct elements x and y we either have $x \prec y$ or $y \prec x$.

Note that we cannot require this for all pairs of elements since for $x = y$ we do not have either $x \prec y$ or $y \prec x$.

Note: By definition **strict ordering relations are not ordering relations** (since they are not reflexive).

Ordering in databases

SQL allows you to sort your output by any field using ORDER BY.

Example: We can list the students in the Student table in the order in which they started.

```
SELECT *
FROM Student
ORDER BY Started;
```

Example: We can sort students in the Student table within each year by last name and then by first name.

```
SELECT *
FROM Student
ORDER BY Started, LastName, FirstName;
```

Inverse relations

The ordering relations we are familiar with on numbers typically come in pairs. For example, we have < and >.

In general, starting with a binary relation R(x,y) we can always consider the **inverse relation** $R^{-1}(x,y) := R(y,x)$

Note that this works on all binary relations, not just ordering relations.

Examples: What are the inverses of the following relations?

- R(x,y) = "x < y"
- R(x,y) = "x ≥ y"
- R(x,y) = "x is a parent of y"
- R(x,y) = "x is a brother of y"
- R(x,y) = "x is an uncle of y"
- R(x,y) = "x is coprime to y"
- R(x,y) = "x is older than y"

Exercise: Show that R is symmetric if and only if $R^{-1} = R$.

Joining relations

Another natural operation on relations is to combine them.

Let R(x,y) and S(u,v) be two relations. The **join of R and S, written R ∘ S**, is the relation that holds between x and v if there is a y such that R(x,y) and S(y,v).

Examples:

- R(x,y) = "x is a parent of y" and S(u,v) = "u is a parent of v", then R ∘ S(x,v) is true if there is a y such that x is a parent of y and y is a parent of v. In other words, R ∘ S(x,v) is true if x is a grandparent of v.
- R(x,y) = "x is a sibling of y" and S(u,v) = "u is a parent of v". Then R ∘ S(x,v) is true if there is a y such that x is a sibling of y and y is a parent of v. In other words, R ∘ S(x,v) means that x is an uncle or aunt of v (excluding uncles or aunts by marriage).
- Suppose R(x,y) = "x = 2y" and S(u,v) = "u = 3v". Then R ∘ S(x,v) is true if there is a y such that x = 2y and y = 3v. This is true if x = 2(3v) = 6v.

Exercises: What is the join R ∘ S if R(x,y) = "x is married to y" and S(u,v) = "u is a child of v"

The join of a relation is closely related to its transitivity.

For example, “parent of parent” is “grandparent” and is not the same as “parent”. Being a parent is not transitive.

But “taller than someone taller than” is the same as “taller than”. Being taller than is transitive.

Joins and databases

Joins can be defined on arbitrary, not just binary, relations.

In that case we need to specify which argument of R is matched up with which argument of S.

We have **used joins before when connecting multiple tables** in a database. We join them by requiring that the foreign key be the same as the primary key.

Example: This query lists the last names of presidents of student groups.

```
SELECT LastName, SID, Name
FROM Student, StudentGroup
WHERE PresidentID = SID;
```

This is a join of Student(ln, fn, sid, cr, pr, ct, st) and StudentGroup(pid, gn, fd), joined by the condition that sid=pid.

The extensional view of relations

The examples of relations we have seen included many natural relations such as “sibling”, “parent”, <, etc.

These relations differ from the type we have seen in our database examples.

A relation that is described through words or mathematics is called an **intensional** relation, since it is determined by its intension or meaning.

The relations found in a database are explicitly listed (e.g. Abigail Winter took CSC 440 in the Fall 2005), and such relations are called **extensional** since they are determined by their explicit listing.

If we take an extensional view of relations, we can view them as sets. This means that we can take unions, intersections, and differences (complements) of them.

Examples:

- Let R(x,y) = "x < y" and S(x,y) = "x = y". Then $R \cup S(x,y)$ = "$x \leq y$". $R \cap S(x,y) = \emptyset$ since there are no x and y such that x < y and x = y.
- Let R(x,y) = "x is a sister of y" and S(x,y) = "x is older than y". Then $R \cup S(x,y)$ = "x is older than y or a sister of y". $R \cap S(x,y)$ = "x is an older sister of y". R – S(x,y) = "x is a younger (or same-age) sister of y".

Note that the set-theoretic operations correspond to logical operations on the relation.

- $R \cup S$ is the same as $(R \vee S)(x,y) := R(x,y) \vee S(x,y)$
- $R \cap S$ is the same as $(R \wedge S)(x,y) := R(x,y) \wedge S(x,y)$
- $R - S$ is the same as $(R \wedge \bar{S})(x,y) := R(x,y) \wedge \bar{S}(x,y)$

Representing binary relations

For binary relations there is a good alternative representation to what we have seen.

Example: Consider the set S = {Chad, Charlotte, Eva, Noelle} and the relation R(x,y) = "x is a parent of y".

We can represent this as a **table**, where the row corresponds to x and the column corresponds to y.

For example, Chad is a parent of Eva and Charlotte so in the following table the row corresponding to Chad has two true entries, one in the column labeled Eva and one in the column labeled Charlotte:

	Chad	Charlotte	Eva	Noelle
Chad	F	T	T	F
Charlotte	F	F	F	F
Eva	F	F	F	F
Noelle	F	T	T	F

Another way to represent this is to remove the labels on the row and columns (assuming that we know and remember what they mean) and replace the T/F entries with 1/0 entries to get a matrix representation of the set:

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

This matrix encodes all of the information about "is a parent of" for my brother's family.

M[i,j] is the entry in row i and column j
M[i,j] = 1 if and only if the ith Settle is a parent of the jth Settle where we order the Settles as specified in the set S above.

This is called the **matrix view** of binary relations.

Exercises: Construct matrices for the following relations on the Settles ordered as in the set S above:

- "is a child of" (compare to the above – do you see a connection?)
- "has the same sex as"
- "is identical to"
- "is younger than" compared with "is older than" – what is the connection and what happens at the diagonal?

Relation properties and the matrix view

Some of the properties of relations that we covered earlier translate into visually clear properties of the matrices that represent them.

To aid our translations we need a bit of matrix background.

The **transpose** of a matrix M is written M^T and is defined by $M^T[i,j] = M[j,i]$

Example: Find the transpose of the matrix representing the parent relationships in the Settle family set.

Translations:

- A relation is **reflexive** if the diagonal of the associated matrix contains only 1s.
- A relation is **symmetric** if the associated matrix remains unchanged when transposing it, or precisely stated, $M^T = M$. This is the same as saying that $R^{-1} = R$.

Examples: Consider the transposes of the matrices produced for the set S given above under the following relations

- "is identical to"
- "has the same sex as"

Transitive closure

In the previous section we did not characterize transitivity.

Transitivity is closely related to the join of a binary relation with itself.

Example: "parent of parent" is "grandparent" and not the same as "parent" but "taller than someone taller than" is the same as "taller than"

Moving back to the set theoretic view of relations we can characterize this further.

Lemma: $R \circ R \subseteq R$ if and only if R is transitive
Proof: Assume that $R \circ R \subseteq R$. Pick an arbitrary $(x,y) \in R$ and $(y,z) \in R$. By definition of $R \circ R$, $(x,z) \in R \circ R$. But by assumption $R \circ R \subseteq R$ so that $(x,z) \in R$. This means that R is transitive.

Assume that R is transitive. Pick an arbitrary $(x,z) \in R \circ R$. By definition of $\in R \circ R$ there has to be a y such that $(x,y) \in R$ and $(y,z) \in R$. But then by transitivity of R, we know that $(x,z) \in R$. Thus we know that $R \circ R \subseteq R$.

Transitivity can be expressed in the matrix view using multiplication. Matrix multiplication is beyond the scope of this class, but you will very likely see it in other computer science classes.

What do we do if a relation isn't transitive, but we would like it to be?

Example: "is a parent of" is not transitive because your grandparents are not your parents. But we could include grandparents. But now your great grandparents are violating transitivity. So we could add them.

This is equivalent to replacing R with $R \circ R$ at each step. Repeating this over and over until we don't add any additional connections, we produce the **transitive closure** of the original relation R.

Exercise: What is the transitive closure of "is a child of"?

Quantifiers

Many of the definitions we made in the section on relations went beyond what we saw in the section on propositional logic.

For example, we said a relation R is reflexive if R(x,x) for all x.

This requires the truth of a proposition for all objects x in the universe, which takes us into what is called **first-order logic**.

It is helpful if we can express this in a more mathematical way to allow us to better reason about the kind of statements we are making.

So a relation R is reflexive if $(\forall x)\ [R(x,x)]$.

The part in square brackets tells us to what the quantifier for all x applies. Here that is unambiguous, so we could write: $\forall x\ R(x,x)$

The quantifier $\forall$ is called the **universal quantifier**.

Example: If P(x) = "x is an American citizen" and Q(x) = "x has a social security number" then $(\forall x)\ [P(x) \rightarrow Q(x)]$ expresses (whether true or not) that every American citizen has a social security number.

Exercise: What do the following formulas express if P(x) and Q(x) are as given above?

- $(\forall x)\ [Q(x) \rightarrow P(x)]$
- $(\forall x)\ [P(x)] \rightarrow (\forall x)\ [Q(x)]$
- $(\forall x)\ [P(x)]$
- $(\forall x)\ [Q(x)]$
- $(\forall x)\ [P(x)] \wedge (\forall x)\ [Q(x)]$
- $(\forall x)\ [Q(x) \leftrightarrow P(x)]$

The universal quantifier distributes over conjunction, meaning that,

$(\forall x)\ [P(x) \wedge Q(x)] \leftrightarrow (\forall x)\ [P(x)] \wedge (\forall x)\ [Q(x)]$

Example: Let P(x) = "x is male" and Q(x) = "x is female". What do the following statements express?

- $(\forall x)\ [P(x) \vee Q(x)]$
- $(\forall x)\ [P(x)] \vee (\forall x)\ [Q(x)]$

The above example shows that **the universal quantifier does not distribute over disjunction**.

Example: A prime number is a natural number larger than 1 whose only divisors are 1 and the number itself. We can easily express this kind of condition using quantifiers.

To express than n is prime we need to say that any number dividing n is either 1 or n.

We saw earlier than $a \mid b$ denotes that a divides b (evenly).

So the following says that a divisor k of n has to be 1 or n:

$$k \mid n \rightarrow (k = 1 \vee k = n)$$

For n to be prime, every divisor of n must be either 1 or n, so we need to require that any number that divides n has to be 1 or n:

$$(\forall k)\ [k \mid n \rightarrow (k = 1 \vee k = n)]$$

Now the only thing missing is the requirement that n be larger than 1:

n is prime if and only if $n > 1 \wedge (\forall k)\ [k \mid n \rightarrow (k = 1 \vee k = n)]$

Multiple quantifiers

A formula can have multiple quantifiers, and we have seen several examples of it implicitly when talking about relations.

What does the following express?

$$(\forall x)\ (\forall y)\ [R(x,y) \rightarrow R(y,x)]$$

What does the following express?

$$(\forall x)\ (\forall y)\ (\forall z)\ [R(x,y) \wedge R(y,z) \rightarrow R(x,z)]$$

Exercises:

- Express the condition that the binary relation R is anti-reflexive.
- Express the condition that the binary relation R is anti-symmetric
- Express the condition that the binary relation R is not reflexive
- Let P(x,y) = "x is taller than y". Express that if x is taller than y, then y cannot be taller than x.

Example: Let P(x,y) = "x has a social security number y". Using this relation, let's express that everybody has at most one social security number.

This means that no one can have two social security numbers. Suppose someone did have two social security numbers.

That means P(x,y) and P(x,z) for some x, y, and z.
To rule this out we would have to require that y = z.

So the final form is $(\forall x)\ (\forall y)\ (\forall z)\ [P(x,y) \wedge P(x,z) \rightarrow y = z]$

Another quantifier

How could we express that somebody does have a social security number?

Let R(x,y) = "x has social security number y".
Suppose that x is a particular American.
We want to say that there is a y such that R(x,y).

We write this using the **existential quantifier ∃**

$(\exists y)[R(x,y)]$ where R(x,y) is given above

This is itself a formula with a variable x, so we can quantify again, for example to say that every American has a social security number: $(\forall x)(\exists y)[R(x,y)]$

Order of quantifiers

When you have multiple, different quantifiers then the order of quantifiers matters.

Examples: How are the following statements different if R(x,y) = "x has a social security number y"?

- $(\forall x)\ (\exists y)[R(x,y)]$
- $(\exists y)\ (\forall x)[R(x,y)]$
- $(\exists x)\ (\forall y)[R(x,y)]$

When reading quantifiers you should read from left to right. It also helps to digest the inner-most quantifier first and then consider what the additional quantifiers add to the situation.

Exercises: For each of the following formulas find a binary relation R that makes the formula true and a relation R that makes the formula false.

- $(\exists x)\ (\forall y)[R(x,y)]$
- $(\forall x)\ (\exists y)[R(x,y)]$
- $(\exists y)\ (\forall x)[R(x,y)]$
- $(\forall y)\ (\exists x)[R(x,y)]$

Exercises: State in plain English what the following formulas express. Also, decide whether it is true or not interpreted in the universe of the natural numbers ($\mathbb{N}$), the integers ($\mathbb{Z}$), or the real numbers ($\mathbb{R}$).

- $(\exists x)\ (\forall y)[x \le y]$
- $(\forall x)\ (\exists y)[x \le y]$
- $(\exists y)\ (\forall x)[x \le y]$

Negations and quantifiers

Earlier we saw how to express primality using a universal quanitifer.

$$n \text{ is prime if and only if } n > 1 \wedge (\forall k)\,[k \mid n \rightarrow (k = 1 \vee k = n)]$$

But you will often find a prime number defined as a number larger than 1 that has no divisors other than 1 and itself.

Using an existential quantifier, this could be directly expressed as:

$$n \text{ is prime if and only if } n > 1 \wedge \overline{(\exists k)[k|n \wedge k\neq 1 \wedge k\neq n]}$$

The two definitions are identical: Every divisor or n is either 1 or n is the same as saying that there is no divisor of n which is different from 1 and n.

The previous example is a special case of a powerful observation:

$$(\forall x)[P(x)] \leftrightarrow \overline{(\exists x)[\overline{P(x)}]}$$

This says that a property P is always true if and only if there is no counterexample, that is no x that makes it false.

There is another way of expressing the relationship between universal and existential quantification:

$$(\exists x)[P(x)] \leftrightarrow \overline{(\forall x)[\overline{P(x)}]}$$

This says that there is an x that makes P true if and only if P is not false for all x.

These are an example of DeMorgan's Laws in a new context.

They tell us that we don't need both universal and existential quantifiers: we could get away with just using one.

For ease of understanding it is helpful to use both.