GSMST

Applications of Linear Algebra in Programming

Inverse Matrices

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1. Find the inverse of each matrix. You must use each method from class (Augmented/Cofactoring and linear row reduction with identity matrix):

Gauss-Jordan Elimination Method

$$A = \begin{bmatrix} -2 & -6 & 1 \\ -3 & -2 & -5 \\ -5 & -4 & -1 \end{bmatrix}$$

$$[A \quad I] = \left[\begin{array}{ccc|ccc} -2 & -6 & 1 & 1 & 0 & 0 \\ -3 & -2 & -5 & 0 & 1 & 0 \\ -5 & -4 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -3 & -2 & -5 & 0 & 1 & 0 \\ -5 & -4 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{3R_1+2R_2} \xrightarrow{5R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 7 & -\frac{13}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 11 & -\frac{7}{2} & -\frac{5}{2} & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{13}{14} & -\frac{3}{14} & \frac{1}{7} & 0 \\ 0 & 11 & -\frac{7}{2} & -\frac{5}{2} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{16}{7} & \frac{1}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{13}{14} & -\frac{3}{14} & \frac{1}{7} & 0 \\ 0 & 11 & -\frac{7}{2} & -\frac{5}{2} & 0 & 1 \end{array}\right] \xrightarrow{R_3-11R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{16}{7} & \frac{1}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{13}{14} & -\frac{3}{14} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{47}{7} & -\frac{1}{7} & -\frac{11}{7} & 1 \end{array}\right]$$

$$\xrightarrow{\frac{4}{47}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{16}{7} & \frac{1}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{13}{14} & -\frac{3}{14} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & -\frac{1}{47} & -\frac{11}{47} & \frac{7}{47} \end{array}\right] \xrightarrow{R_1-\frac{16}{7}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{47} & \frac{5}{47} & -\frac{16}{47} \\ 0 & 1 & -\frac{13}{14} & -\frac{3}{14} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & -\frac{1}{47} & -\frac{11}{47} & \frac{7}{47} \end{array}\right]$$

$$\xrightarrow{\frac{13}{14}R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{47} & \frac{5}{47} & -\frac{16}{47} \\ 0 & 1 & 0 & -\frac{11}{47} & -\frac{7}{94} & \frac{13}{94} \\ 0 & 0 & 1 & -\frac{1}{47} & -\frac{11}{47} & \frac{7}{47} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{9}{47} & \frac{5}{47} & -\frac{16}{47} \\ -\frac{11}{47} & -\frac{7}{94} & \frac{13}{94} \\ -\frac{1}{47} & -\frac{11}{47} & \frac{7}{47} \end{bmatrix}$$

Adjoint Method

$$B = \begin{bmatrix} 3 & 4 & 5 \\ -2 & 3 & 3 \\ -1 & 2 & -5 \end{bmatrix}$$

$$\begin{aligned}
\det B &= \begin{vmatrix} 3 & 4 & 5 \\ -2 & 3 & 3 \\ -1 & 2 & -5 \end{vmatrix} \\
&= 3\begin{vmatrix} 3 & 3 \\ 2 & -5 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 1 & -5 \end{vmatrix} + 5\begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix} \\
&= 3([3 \cdot -5] - [3 \cdot 2]) + 4([2 \cdot -5] - [3 \cdot 1]) + 5([-2 \cdot 2] - [3 \cdot -1]) \\
&= 3(-15 - 6) + 4(-10 - 3) + 5(-4 + 3) \\
&= 3(-21) + 4(-13) + 5(-1) \\
&= -63 - 52 - 5 \\
&= -120
\end{aligned}$$

$$\begin{aligned}
&\operatorname{cof}(B_{11}) = \begin{vmatrix} 3 & 3 \\ 2 & -5 \end{vmatrix} = -21 \qquad && \operatorname{cof}(B_{12}) = -\begin{vmatrix} -2 & 3 \\ -1 & -5 \end{vmatrix} = -13 \qquad && \operatorname{cof}(B_{13}) = \begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix} = -1 \\
&\operatorname{cof}(B_{21}) = -\begin{vmatrix} 4 & 5 \\ 2 & -5 \end{vmatrix} = 30 && \operatorname{cof}(B_{22}) = \begin{vmatrix} 3 & 5 \\ -1 & -5 \end{vmatrix} = -10 && \operatorname{cof}(B_{23}) = -\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = -10 \\
&\operatorname{cof}(B_{31}) = \begin{vmatrix} 4 & 5 \\ 3 & 3 \end{vmatrix} = -3 && \operatorname{cof}(B_{32}) = -\begin{vmatrix} 3 & 5 \\ -2 & 3 \end{vmatrix} = -19 && \operatorname{cof}(B_{33}) = \begin{vmatrix} 3 & 4 \\ -2 & 3 \end{vmatrix} = 17
\end{aligned}$$

$$\begin{aligned}
[\operatorname{cof}(B_{ij})] &= \begin{bmatrix} -21 & -13 & -1 \\ 30 & -10 & -10 \\ -3 & -19 & 17 \end{bmatrix} \\
\text{Adj } B = [\operatorname{cof}(B_{ij})]^{\intercal} &= \begin{bmatrix} -21 & 30 & -3 \\ -13 & -10 & -19 \\ -1 & -10 & 17 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
B^{-1} &= \frac{\text{Adj } B}{\det B} \\
&= \left(-\frac{1}{120}\right) \begin{bmatrix} -21 & 30 & -3 \\ -13 & -10 & -19 \\ -1 & -10 & 17 \end{bmatrix} \\
&= \begin{bmatrix} -21 \cdot \left(-\frac{1}{120}\right) & 30 \cdot \left(-\frac{1}{120}\right) & -3 \cdot \left(-\frac{1}{120}\right) \\ -13 \cdot \left(-\frac{1}{120}\right) & -10 \cdot \left(-\frac{1}{120}\right) & -19 \cdot \left(-\frac{1}{120}\right) \\ -1 \cdot \left(-\frac{1}{120}\right) & -10 \cdot \left(-\frac{1}{120}\right) & 17 \cdot \left(-\frac{1}{120}\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{-21}{-120} & \frac{30}{-120} & \frac{-3}{-120} \\ \frac{-13}{-120} & \frac{-10}{-120} & \frac{-19}{-120} \\ \frac{-1}{-120} & \frac{-10}{-120} & \frac{17}{-120} \end{bmatrix} \\
&= \begin{bmatrix} \frac{7}{40} & -\frac{1}{4} & \frac{1}{40} \\ \frac{13}{120} & \frac{1}{12} & \frac{19}{120} \\ \frac{1}{120} & \frac{1}{12} & -\frac{17}{120} \end{bmatrix}
\end{aligned}
$$

2. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

The three numbers of c for which A is not invertible are **0, 7, and 2**, as those values of c would make A linearly dependent.

If c was 0, then A would be linearly dependent because it contained a column/row of zeros.

If c was 7, then A would be linearly dependent because it would contain a column of 7s.

If c was 2, then A would be linearly dependent because it would contain a row of 2s.

3. Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

Let's perform a row reduction of the given matrix A:

$$\begin{pmatrix} a & b & b \\ a & a & b \\ a & a & a \end{pmatrix} \sim \begin{pmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{pmatrix} \sim \begin{pmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{pmatrix}$$

Therefore, A is invertible since $a \neq 0$ and $a \neq b \therefore a - b \neq 0$, which means all the pivots are non-zero in fully reduced row-echelon form.

4. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \;\; I]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$\text{Invert } A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and solve } Ax = (1,1,1,1).$$

$$[A \;\; I] = \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_4+R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, let's extend A to be a 5x5 alternating matrix, which we'll denote as B:

$$B = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If we wanted to calculate B^{-1}, we would have to reduce it just like we did to A above... but the pattern here is pretty obvious, since A^{-1} follows a "snake" pattern downward:

$$B^{-1} \text{ (predicted) } = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let's actually calculate it:

$$[B \quad I] = \left[\begin{array}{ccccc|ccccc} 1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R1} \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3+R2} \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_4+R_3} \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_5+R4} \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, let's confirm $B^{-1} \cdot B$ yields the identity matrix:

$$B^{-1} \cdot B = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{15} \\ C_{21} & C_{22} & \cdots & C_{25} \\ \vdots & \vdots & \ddots & \vdots \\ C_{51} & C_{52} & \cdots & C_{55} \end{bmatrix}$$

$$C_{ij} = B_{i1}(B^{-1})_{1j} + B_{i2}(B^{-1})_{2j} + \cdots + B_{in} + (B^{-1})_{nj} = \sum_{k=1}^{n} B_{ik}(B^{-1})_{kj}$$

$$\stackrel{\checkmark}{=} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. A **is a 4x4 matrix with 1's on the diagonal and** $-a, -b, -c$ **on the diagonal above. Find the inverse of this bidiagonal matrix.**

$$[A \;\; I] = \left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{aR_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & -ab & 0 & 1 & a & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{abR_3+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -abc & 1 & a & ab & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{bR_3+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -abc & 1 & a & ab & 0 \\ 0 & 1 & 0 & -bc & 0 & 1 & b & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{abcR_4+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & -bc & 0 & 1 & b & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{bcR_4+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{cR_4+R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$