GSMST

Applications of Linear Algebra in Programming

Chapter 2 Assignment

Submitted By:
Anish Goyal
4th Period

Submitted To:
Mrs. Denise Stiffler
Educator

February 6, 2023

Infinite Possibilities

Gwinnett School of Mathematics, Science, and Technology

Table of contents

2.13 Review Questions

What is vector addition?

Vector addition is the adding of two vectors of the same size into a single vector. Say we have two vectors v and k. The addition of the vectors v and k can be defined as follows:
$[u_1, u_2, ..., u_n] + [v_1, v_2, ..., v_n] = [u_1 + v_1, u_2 + v_2, ..., u_n + v_n]$

What is the geometric interpretation of vector addition?

The geometric interpretation of vector addition is placing the tail of the second vector at the head of the first vector and drawing a new vector from the tail of the first vector to the head of the second vector, which represents the sum of the two vectors.

What is scalar-vector multiplication?

Scalar-vector multiplication is an operation performed between a scalar value and a vector that results in a new vector. The scalar multiplies each component of the vector, which results in a new vector that has the same direction as the original director, but with a length (or magnitude) that is scaled by that scalar value.

What is the distributive property that involves scalar-vector multiplication but not vector addition?

The distributive property that involves scalar-vector multiplication but not vector addition is:
$(\alpha + \beta)u = \alpha u + \beta u$
where α and β are scalars and u is a vector. This property states that when a vector is multiplied by the sum of two scalars, the result is equivalent to the sum of the scalar multiplication of one scalar to that vector and the scalar multiplication of the other scalar to that vector.

What is the distributive property that involves both scalar-vector multiplication and vector addition?

The distributive property that involves both scalar-vector multiplication and vector addition is:
$a \cdot (u + v) = a \cdot u + a \cdot v$
where a is a scalar, and u and v are vectors. This property states that when a scalar is multiplied to the sum of two vectors, the result is equivalent to the sum of the scalar multiplication to each of those vectors. In other words, you can distribute a scalar over the sum of two vectors, and the resulting vector will be the same as adding the scalar multiples to each vector separately.

How is scalar-vector multiplication used to represent the line through a pair of given points?

The line through a pair of given points u and v, known as the u-to-v line segment, consists of the set of convex combinations of u where $\{\alpha v + \beta u : \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0, \alpha + \beta = 1\}$.

What is dot product?

The dot product is defined as the sum of the elements with the same index multiplied together across two vectors with equal length.

What is the ***homogeneity*** property that relates dot-product to scalar-vector multiplication?

The *homogeneity* property says multiplying one of the vectors being dot-producted together by a scalar is equivalent to multiplying that scalar to the actual dot product:
$(\alpha u) \cdot v = \alpha(u \cdot v)$

What is the distributive property property that relates dot-product to vector addition?

Dot product is distributive over vector addition:
$(u + v) \cdot w = u \cdot w + v \cdot w$

What is a linear equation (expressed using dot-product)?

A linear equation expressed using dot product involves finding a scalar product of a vector x and a vector w, and comparing it to a scalar value b. This can be written as:
$a \cdot x = \beta$
where a is a vector, x is a vector variable, and β is a scalar.

What is a linear system?

A linear system is a list of linear equations with the same vector variable expressed using dot-product:
$a_1 \cdot x = \beta_1$
$a_2 \cdot x = \beta_2$
...
$a_m \cdot x = \beta_m$

What is an upper-triangular linear system?

An upper-triangular linear system is a linear system that is in row-echelon form with zeros across the scalar values of the bottom left triangle.

How can one solve an upper-triangular linear system?

You can solve an upper-triangle linear system with Gaussian Elimination with backwards substitution. Once you have solved for a single scalar at the bottommost equation of the linear system, you can plug in that scalar into the following equations and work your way up.

2.14 Problems

Vector addition practice

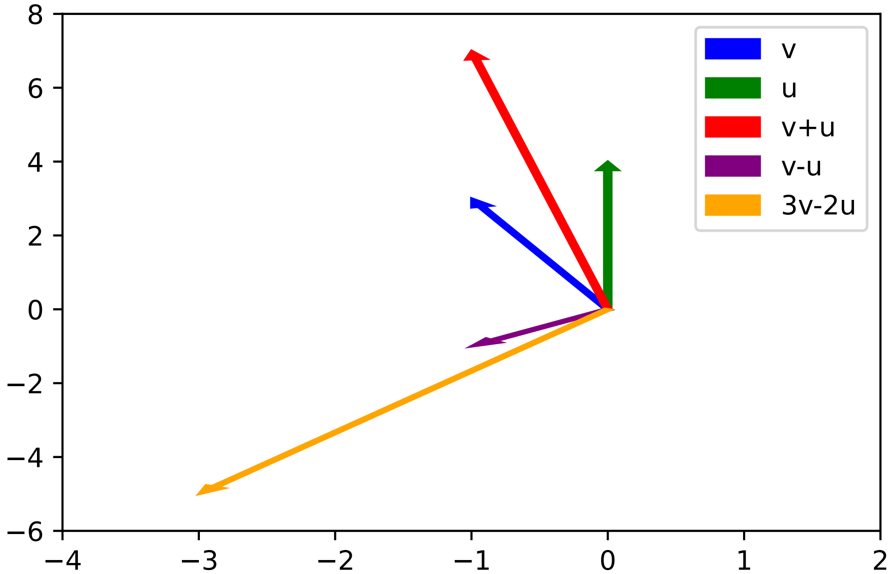
Problem 2.14.1

For vectors $v = [-1, 3]$ and $u = [0, 4]$, find the vectors $v + u$, $v - u$, and $3v - 2u$. Draw these arrows as arrows on the same graph.

$v + u =$ [-1, 7]
$v - u =$ [-1, -1]
$3v - 2u =$ [-3, 1]

```
import matplotlib.pyplot as plt

v = [-1, 3]
u = [0, 4]
v_plus_u = [-1, 7]
v_minus_u = [-1, -1]
three_v_minus_two_u = [-3, -5]

plt.arrow(0, 0, v[0], v[1], color='blue', width=0.05, length_includes_head=True, label='v')
plt.arrow(0, 0, u[0], u[1], color='green', width=0.05, length_includes_head=True, label='u')
plt.arrow(0, 0, v_plus_u[0], v_plus_u[1], color='red', width=0.05, length_includes_head=True, label='v+u')
plt.arrow(0, 0, v_minus_u[0], v_minus_u[1], color='purple', width=0.05, length_includes_head=True, label='v-u')
```

```python
plt.arrow(0, 0, three_v_minus_two_u[0], three_v_minus_two_u[1], color='orange', width=0.05, length_includes_head=True, label='3v-2u')

plt.xlim(-4, 2)
plt.ylim(-6, 8)

plt.legend()
plt.show()
```



Problem 2.14.2

Given the vectors $v = [2, -1, 5]$ and $u = [-1, 1, 1]$, find the vectors $v + u$, $v - u$, $2v - u$, and $v + 2u$.

$v + u = [1, 0, 6]$
$v - u = [3, -2, 4]$ $\quad 2v - u = [5, -3, 9]$
$v + 2u = [0, 1, 7]$

Problem 2.14.3

For the vectors $v = [0, one, one]$ and $u = [one, one, one]$ over $GF(2)$, find $v + u$ and $v + u + u$.

$v + u = [0, one, one] + [one, one, one] = [one, 0, 0]$
$v + u + u = [one, 0, 0] + [one, one, one] = [0, one, one]$

Expressing one $GF(2)$ vector as a sum of others

Problem 2.14.4

Here are six 7-vectors over $GF(2)$:

$\mathbf{a} = 1100000$	$\mathbf{d} = 0001100$
$\mathbf{b} = 0110000$	$\mathbf{e} = 0000110$
$\mathbf{c} = 0011000$	$\mathbf{f} = 0000011$

For each of the following vectors u, find a subset of the above vectors whose sum is u, or report that no such subset exists.

1. $u = 0010010$

2. $u = 0100010$

1) $u = c + d + e$

2) $u = b + c + d + e$

Problem 2.14.5

Here are six 7-vectors over $GF(2)$:

$\mathbf{a} = 1110000$	$\mathbf{d} = 0001110$
$\mathbf{b} = 0111000$	$\mathbf{e} = 0000111$
$\mathbf{c} = 0011100$	$\mathbf{f} = 0000011$

For each of the following vectors u, find a subset of the above vectors whose sum is u, or report that no such subset exists.

1. $u = 0010010$

2. $u = 0100010$

1) $u = c + d$

2) There is no such subset.

Finding a solution to linear equations over $GF(2)$

Problem 2.14.6

Find a vector $x = [x_1, x_2, x_3, x_4]$ over $GF(2)$ satisfying the following linear equations:
$1100 \cdot x = 1$
$1010 \cdot x = 1 \quad 1111 \cdot x = 1$
Show that $x + 1111$ also satisfies the equations.

A vector that satisfies the linear equation is $x = [1, 0, 0, 0]$. $1100 \cdot 1000 \stackrel{\checkmark}{=} 1$
$1010 \cdot 1000 \stackrel{\checkmark}{=} 1$
$1111 \cdot 1000 \stackrel{\checkmark}{=} 1$
$(x = 1000) + 1111 = 0111$ also satisfies the equations:
$1100 \cdot 0111 \stackrel{\checkmark}{=} 0 + 1 + 0 + 0 \stackrel{\checkmark}{=} 1$
$1010 \cdot 0111 \stackrel{\checkmark}{=} 0 + 0 + 1 + 0 \stackrel{\checkmark}{=} 1$
$1111 \cdot 0111 \stackrel{\checkmark}{=} 0 + 1 + 1 + 1 \stackrel{\checkmark}{=} 1$

Formulating equations using dot-product

Problem 2.14.7

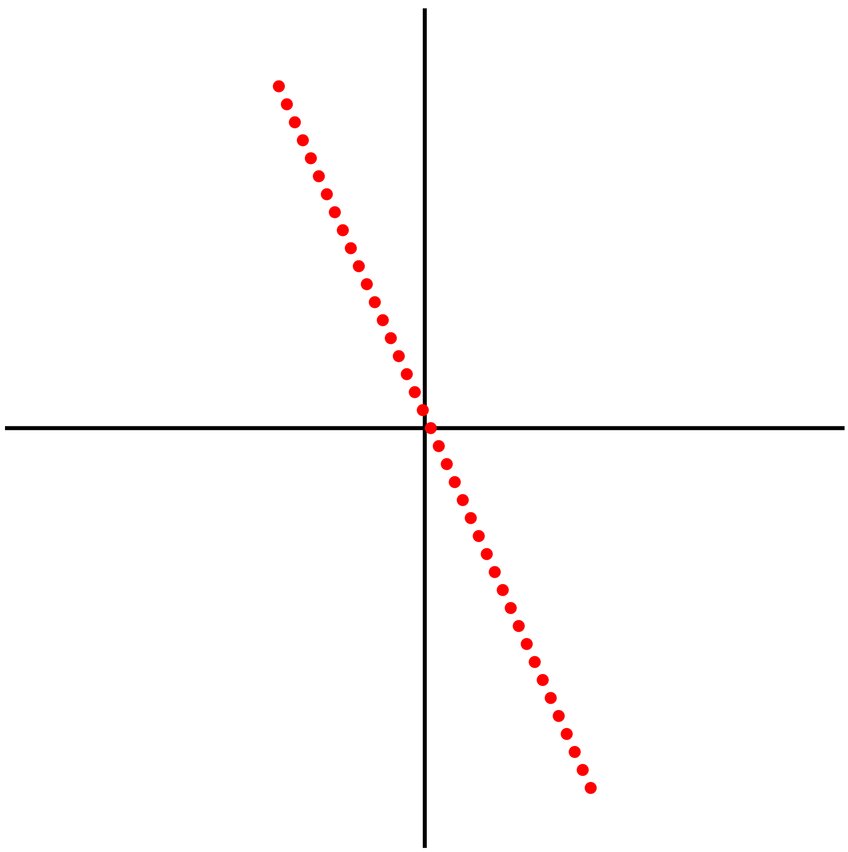
Consider the equations
$2x_0 + 3x_1 - 4x_2 + x_3 = 10$
$x_0 - 5x_1 + 2x_2 + 0x_3 = 35$
$4x_0 + x_1 - x_2 - x_3 = 8$
Your job is not to solve these equations but to formulate them using dot-product. In particular, come up with three vectors v1, v2, and v3 represented as lists so that the above equations are equivalent to
$\text{v1} \cdot x = 10$
$\text{v2} \cdot x = 35$
$\text{v3} \cdot x = 8$
where x is a 4-vector over $\mathbb{R}$.
$v_1 = [2, 3, -4, 1]$
$v_2 = [1, -5, 2, 0]$
$v_3 = [4, 1, -1, -1]$

Plotting lines and segments

Problem 2.14.8

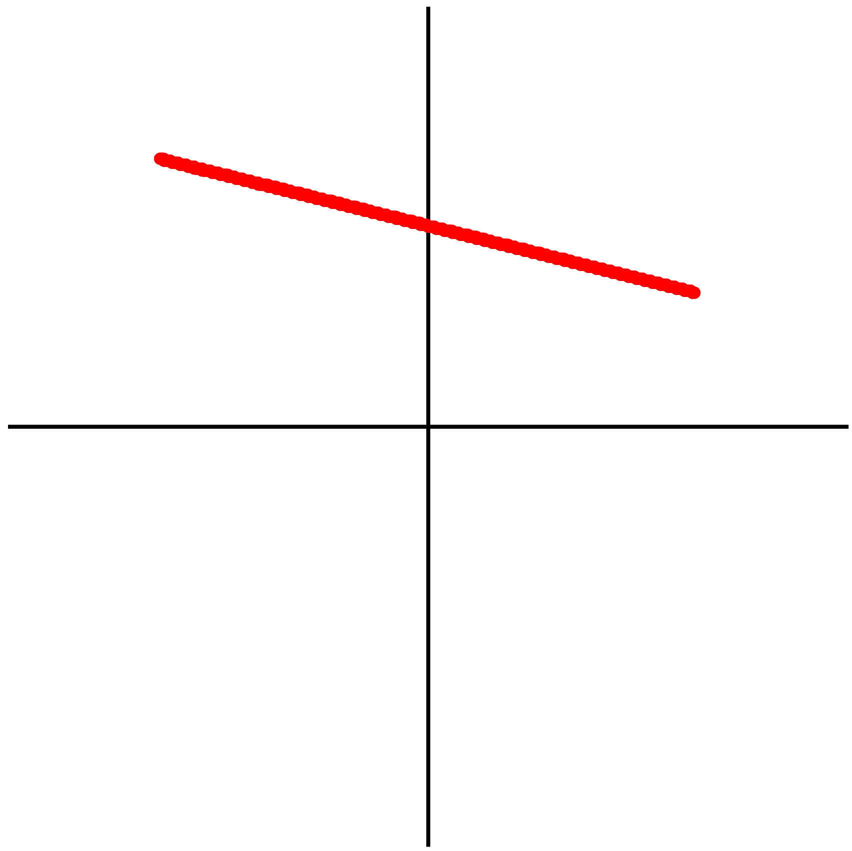
Use the `plot` module to plot
(a) a substantial portion of the line through [-1.5, 2] and [3, 0], and
(b) the line segment between [2, 1] and [-2, 2].
For each, provide the Python statements you used and the plot obtained.

```
from plotting import *
from IPython.display import SVG, display
L=[(3 + i*(-4), i*9) for i in range(-20, 20)]
display(SVG(plot(L, 200)))
```



```
import numpy as np
from plotting import *
```

```
from IPython.display import SVG, display
L=[(2-i, 1+0.25*i) for i in np.arange(0, 4, 0.001)]
display(SVG(plot(L, 3)))
```



Practice with dot-product

Problem 2.14.9

For each of the following pairs of vectors u and v over $\mathbb{R}$, evaluate the expression $u \cdot v$:
(a) $u = [1, 0], v = [5, 4321]$
(b) $u = [0, 1], v = [12345, 6]$
(c) $u = [-1, 3], v = [5, 7]$
(d) $u = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right], v = \left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right]$

(a) $[1, 0] \cdot [5, 4321] = 5 + 0 = 5$

(b) $[0,1]\cdot[12345,6] = 0+6 = 6$

(c) $[-1,3]\cdot[5,7] = -5+21 = 16$

(d) $\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]\cdot\left[\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right] = -\frac{1}{2}-\frac{1}{2} = -1$

Writing procedures for the `Vec` class

Problem 2.14.10

Download the file `vec.py` to your computer, and edit it. The file defines procedures using the Python statement `pass`, which does nothing. You can import the `vec` module and create instances of `Vec` but the operations such as * and + currently do nothing. Your job is to replace each occurrence of the `pass` statement with appropriate code. Your code for a procedure can include calls to others of the seven. You should make no changes to the class definition.

Docstrings

At the beginning of each procedure body is a multi-line string (deliminated by triple quotation marks). This is called a documentation string (*docstring*). It specifies what the procedure should do.

Doctests

The documentation string we provide for a procedure also includes examples of the functionality that procedure is supposed to provide to `Vec`s. The examples show an interaction with Python: statements and expressions are evaluated by Python, and Python's responses are shown. These examples are provided to you as tests (called *doctests*). You should make sure that your procedure is written in such a way that the behavior of your Vec implementation matches that in the examples. If not, your implementation is incorrect.

Download the file `vec.py` to your computer, and edit it. Fill in the procedure definitions and test the doctests with
`python3 -m doctest vec.py`.

```
def getitem(v,k):
    """
    Return the value of entry k in v.
    Be sure getitem(v,k) returns 0 if k is not represented in v.f.

    >>> v = Vec({'a','b','c', 'd'},{'a':2,'c':1,'d':3})
```

```
    >>> v['d']
    3
    >>> v['b']
    0
    """
    assert k in v.D
    return v.f[k] if k in v.f else 0

def setitem(v,k,val):
    """
    Set the element of v with label d to be val.
    setitem(v,d,val) should set the value for key d even if d
    is not previously represented in v.f, and even if val is 0.

    >>> v = Vec({'a', 'b', 'c'}, {'b':0})
    >>> v['b'] = 5
    >>> v['b']
    5
    >>> v['a'] = 1
    >>> v['a']
    1
    >>> v['a'] = 0
    >>> v['a']
    0
    """
    assert k in v.D
    v.f[k] = val
    return

def equal(u,v):
    """
    Return true iff u is equal to v.
    Because of sparse representation, it is not enough to compare dictionaries

    Consider using brackets notation u[...] and v[...] in your procedure
    to access entries of the input vectors.  This avoids some sparsity bugs.

    >>> Vec({'a', 'b', 'c'}, {'a':0}) == Vec({'a', 'b', 'c'}, {'b':0})
    True
```

```
    >>> Vec({'a', 'b', 'c'}, {'a': 0}) == Vec({'a', 'b', 'c'}, {})
    True
    >>> Vec({'a', 'b', 'c'}, {}) == Vec({'a', 'b', 'c'}, {'a': 0})
    True

    Be sure that equal(u, v) checks equalities for all keys from u.f and v.f even if
    some keys in u.f do not exist in v.f (or vice versa)

    >>> Vec({'x','y','z'},{'y':1,'x':2}) == Vec({'x','y','z'},{'y':1,'z':0})
    False
    >>> Vec({'a','b','c'}, {'a':0,'c':1}) == Vec({'a','b','c'}, {'a':0,'c':1,'b':4})
    False
    >>> Vec({'a','b','c'}, {'a':0,'c':1,'b':4}) == Vec({'a','b','c'}, {'a':0,'c':1})
    False

    The keys matter:
    >>> Vec({'a','b'},{'a':1}) == Vec({'a','b'},{'b':1})
    False

    The values matter:
    >>> Vec({'a','b'},{'a':1}) == Vec({'a','b'},{'a':2})
    False
    """
    assert u.D == v.D
    first = []
    second = []
    for k in u.D:
        if k in u.f:
            first.append(u.f[k])
        else:
            first.append(0)
        if k in v.f:
            second.append(v.f[k])
        else:
            second.append(0)
    return first == second

```

```python
def add(u,v):
    """
    Returns the sum of the two vectors.

    Consider using brackets notation u[...] and v[...] in your procedure
    to access entries of the input vectors.  This avoids some sparsity bugs.

    Do not seek to create more sparsity than exists in the two input vectors.
    Doing so will unnecessarily complicate your code and will hurt performance.

    Make sure to add together values for all keys from u.f and v.f even
    if some keys in u.f do not exist in v.f (or vice versa)

    >>> a = Vec({'a','e','i','o','u'}, {'a':0,'e':1,'i':2})
    >>> b = Vec({'a','e','i','o','u'}, {'o':4,'u':7})
    >>> c = Vec({'a','e','i','o','u'}, {'a':0,'e':1,'i':2,'o':4,'u':7})
    >>> a + b == c
    True
    >>> a == Vec({'a','e','i','o','u'}, {'a':0,'e':1,'i':2})
    True
    >>> b == Vec({'a','e','i','o','u'}, {'o':4,'u':7})
    True
    >>> d = Vec({'x','y','z'}, {'x':2,'y':1})
    >>> e = Vec({'x','y','z'}, {'z':4,'y':-1})
    >>> f = Vec({'x','y','z'}, {'x':2,'y':0,'z':4})
    >>> d + e == f
    True
    >>> d == Vec({'x','y','z'}, {'x':2,'y':1})
    True
    >>> e == Vec({'x','y','z'}, {'z':4,'y':-1})
    True
    >>> b + Vec({'a','e','i','o','u'}, {}) == b
    True
    """
    assert u.D == v.D
    return Vec(u.D, {d:getitem(u,d)+getitem(v,d) for d in u.D})

def dot(u,v):
```

```
    """
    Returns the dot product of the two vectors.

    Consider using brackets notation u[...] and v[...] in your procedure
    to access entries of the input vectors.  This avoids some sparsity bugs.

    >>> u1 = Vec({'a','b'}, {'a':1, 'b':2})
    >>> u2 = Vec({'a','b'}, {'b':2, 'a':1})
    >>> u1*u2
    5
    >>> u1 == Vec({'a','b'}, {'a':1, 'b':2})
    True
    >>> u2 == Vec({'a','b'}, {'b':2, 'a':1})
    True
    >>> v1 = Vec({'p','q','r','s'}, {'p':2,'s':3,'q':-1,'r':0})
    >>> v2 = Vec({'p','q','r','s'}, {'p':-2,'r':5})
    >>> v1*v2
    -4
    >>> w1 = Vec({'a','b','c'}, {'a':2,'b':3,'c':4})
    >>> w2 = Vec({'a','b','c'}, {'a':12,'b':8,'c':6})
    >>> w1*w2
    72

    The pairwise products should not be collected in a set before summing
    because a set eliminates duplicates
    >>> v1 = Vec({1, 2}, {1 : 3, 2 : 6})
    >>> v2 = Vec({1, 2}, {1 : 2, 2 : 1})
    >>> v1 * v2
    12
    """
    assert u.D == v.D
    sum = 0
    for k in u.D:
        if k in u.f and k in v.f:
            sum += u.f[k]*v.f[k]
    return sum

def scalar_mul(v, alpha):
    """
```

```
    Returns the scalar-vector product alpha times v.

    Consider using brackets notation v[...] in your procedure
    to access entries of the input vector.  This avoids some sparsity bugs.

    >>> zero = Vec({'x','y','z','w'}, {})
    >>> u = Vec({'x','y','z','w'},{'x':1,'y':2,'z':3,'w':4})
    >>> 0*u == zero
    True
    >>> 1*u == u
    True
    >>> 0.5*u == Vec({'x','y','z','w'},{'x':0.5,'y':1,'z':1.5,'w':2})
    True
    >>> u == Vec({'x','y','z','w'},{'x':1,'y':2,'z':3,'w':4})
    True
    """
    return Vec(v.D, {d:alpha*getitem(v, d) for d in v.D})

def neg(v):
    """
    Returns the negation of a vector.

    Consider using brackets notation v[...] in your procedure
    to access entries of the input vector.  This avoids some sparsity bugs.

    >>> u = Vec({1,3,5,7},{1:1,3:2,5:3,7:4})
    >>> -u
    Vec({1, 3, 5, 7},{1: -1, 3: -2, 5: -3, 7: -4})
    >>> u == Vec({1,3,5,7},{1:1,3:2,5:3,7:4})
    True
    >>> -Vec({'a','b','c'}, {'a':1}) == Vec({'a','b','c'}, {'a':-1})
    True
    """
    return scalar_mul(v, -1)

###############################################################################

class Vec:
    """
```

```
    A vector has two fields:
    D - the domain (a set)
    f - a dictionary mapping (some) domain elements to field elements
        elements of D not appearing in f are implicitly mapped to zero
    """
    def __init__(self, labels, function):
        assert isinstance(labels, set)
        assert isinstance(function, dict)
        self.D = labels
        self.f = function

    __getitem__ = getitem
    __setitem__ = setitem
    __neg__ = neg
    __rmul__ = scalar_mul #if left arg of * is primitive, assume it's a scalar

    def __mul__(self,other):
        #If other is a vector, returns the dot product of self and other
        if isinstance(other, Vec):
            return dot(self,other)
        else:
            return NotImplemented  #  Will cause other.__rmul__(self) to be invoked

    def __truediv__(self,other):  # Scalar division
        return (1/other)*self

    __add__ = add

    def __radd__(self, other):
        "Hack to allow sum(...) to work with vectors"
        if other == 0:
            return self

    def __sub__(a,b):
        "Returns a vector which is the difference of a and b."
        return a+(-b)

    __eq__ = equal

```

```
    def is_almost_zero(self):
        s = 0
        for x in self.f.values():
            if isinstance(x, int) or isinstance(x, float):
                s += x*x
            elif isinstance(x, complex):
                y = abs(x)
                s += y*y
            else: return False
        return s < 1e-20

    def __str__(v):
        "pretty-printing"
        D_list = sorted(v.D, key=repr)
        numdec = 3
        wd = dict([(k,(1+max(len(str(k)), len('{0:.{1}G}'.format(v[k], numdec))))) if isinstance(v[k], int) or isinstance(v[k], float) else (k,(1+max(len(str(k)), len(str(v[k]))))) for k in D_list])
        s1 = ''.join(['{0:>{1}}'.format(str(k),wd[k]) for k in D_list])
        s2 = ''.join(['{0:>{1}.{2}G}'.format(v[k],wd[k],numdec) if isinstance(v[k], int) or isinstance(v[k], float) else '{0:>{1}}'.format(v[k], wd[k]) for k in D_list])
        return "\n" + s1 + "\n" + '-'*sum(wd.values()) +"\n" + s2

    def __hash__(self):
        "Here we pretend Vecs are immutable so we can form sets of them"
        h = hash(frozenset(self.D))
        for k,v in sorted(self.f.items(), key = lambda x:repr(x[0])):
            if v != 0:
                h = hash((h, hash(v)))
        return h

    def __repr__(self):
        return "Vec(" + str(self.D) + "," + str(self.f) + ")"

    def copy(self):
        "Don't make a new copy of the domain D"
        return Vec(self.D, self.f.copy())

    def __iter__(self):
```

```
            raise TypeError('%r object is not iterable' %
            ↪  self.__class__.__name__)
```

Testing `vec.py`

```
import subprocess
subprocess.run(["python", "-m", "doctest", "vec.py"], check=True)
```

```
CompletedProcess(args=['python', '-m', 'doctest', 'vec.py'], returncode=0)
```

Note that a `returncode` of 0 means that all of the testcases executed successfully.