GSMST

Applications of Linear Algebra in Programming

Problem Set 1: The Vector Space

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3.8 Problems

Problem 3.8.1

1. Write and test a procedure `vec_select` using a comprehension for the following computational problem:

   - *input:* a list `veclist` of vectors over the same domain, and an element k of the domain
   - *output:* the sublist of `veclist` consisting of the vectors `v` in `veclist` where `v[k]` is zero

2. Write and test a procedure `vec_sum` using the built-in procedure `sum(.)` for the following:

   - *input:* a list `veclist` of vectors, and a set `D` that is the common domain of these vectors
   - *output:* the vector sum of the vectors in `veclist`

   Your procedure must work even if `veclist` has length 0.
   Hint: Recall from the Python lab that `sum(.)` optionally takes a second argument, which is the element to start the sum with. This can be a vector.
   Disclaimer: The `Vec` class is defined in such a way that, for a vector `v`, the expression `0 + v` evaluates to `v`. This was done precisely so that `sum([v1,v2,... vk])` will correctly evaluate to the sum of the vectors when the number of vectors is nonzero. However, this won't work when the number of vectors is zero.

3. Put your procedures together to obtain a procedure `vec_select_sum` for the following:

   - *input*: a set `D`, a list `veclist` of vectors within domain `D`, and an element `k` of the domain
   - *output*: the sum of all vectors `v` in `veclist` where `v[k]` is zero

```python
from vec import Vec
from vecutil import list2vec

def vec_select(veclist, k):
    return [v for v in veclist if v[k] == 0]

veclist = [list2vec(range(15)), list2vec([1, 2, 3]), list2vec([0, 0,
    1])]
print(vec_select(veclist, 0))
print(vec_select(veclist, 1))
print(vec_select(veclist, 2))
```

```
[Vec({0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14},{0: 0, 1: 1, 2: 2, 3:
    3, 4: 4, 5: 5, 6: 6, 7: 7, 8: 8, 9: 9, 10: 10, 11: 11, 12: 12, 13: 13, 14:
    14}), Vec({0, 1, 2},{0: 0, 1: 0, 2: 1})]
[Vec({0, 1, 2},{0: 0, 1: 0, 2: 1})]
[]
```

```python
def vec_sum(veclist, D):
    return sum([Vec(D, v.f) for v in veclist], Vec(D, {}))

print(vec_sum([], {'a', 'b', 'c'}))
print(vec_sum([list2vec([1, 2, 3]), list2vec([4, 5, 6]), list2vec([7, 8,
    9])], {'x', 'y', 'z'}))
```

```
 a b c
------
 0 0 0

 x y z
------
 0 0 0
```

```python
def vec_select_sum(D, veclist, k):
    return vec_sum(vec_select (veclist, k), D)

print(vec_select_sum(veclist[0].D, veclist, 0))
```

```
 0 1 10 11 12 13 14 2 3 4 5 6 7 8 9
-----------------------------------
 0 1 10 11 12 13 14 3 3 4 5 6 7 8 9
```

Problem 3.8.2

Write and test a procedure `scale_vecs(vecdict)` for the following:

- *input*: a dictionary vector mapping positive numbers to vectors (instances of `Vec`)
- *output*: a list of vectors, one for each item in `vecdict`. If `vecdict` contains a key k mapping to vector $\boldsymbol{v}$, the output should contain the vector $(1/k)\boldsymbol{v}$

```
def scale_vecs(vecdict):
    return [(1/k)*v for k, v in vecdict.items()]

v1 = Vec({'x', 'y', 'z'}, {'x': 1, 'y': 2, 'z': 3})
v2 = Vec({'x', 'y', 'z'}, {'x': 4, 'y': 5, 'z': 6})
v3 = Vec({'x', 'y', 'z'}, {'x': 7, 'y': 8, 'z': 9})

vecdict = {2: v1, 3: v2, 4: v3}

print([v for v in scale_vecs(vecdict)])
```

```
[Vec({'z', 'x', 'y'},{'z': 1.5, 'x': 0.5, 'y': 1.0}), Vec({'z', 'x', 'y'},{'z
    ': 2.0, 'x': 1.3333333333333333, 'y': 1.6666666666666665}), Vec({'z', 'x',
    'y'},{'z': 2.25, 'x': 1.75, 'y': 2.0})]
```

Problem 3.8.3

Write a procedure `GF2_span` with the following specs:

- *input*: a set D of labels and a list L of vectors over *GF(2)* with label-set D
- *output*: the list of all linear combinations of the vectors in L

(Hint: use a loop (or recursion) and a comprehension. Be sure to test your procedure on examples where L is an empty list.)

```python
from itertools import product

def GF2_span(D, L):
    if not L:
        return [Vec(D, {})]
    result = []
    for coeffs in product([0,1], repeat=len(L)):
        combination = Vec(D, {})
        for i, v in enumerate(L):
            combination += coeffs[i]*v
        result.append(combination)
    return result

D = {'a', 'b', 'c'}
L = [Vec(D, {'a': 1, 'c': 1}), Vec(D, {'b': 1})]
print(GF2_span(D, L))

L = []
print(GF2_span(D, L))
```

```
[Vec({'a', 'c', 'b'},{'a': 0, 'c': 0, 'b': 0}), Vec({'a', 'c', 'b'},{'a': 0, '
    c': 0, 'b': 1}), Vec({'a', 'c', 'b'},{'a': 1, 'c': 1, 'b': 0}), Vec({'a', '
    c', 'b'},{'a': 1, 'c': 1, 'b': 1})]
[Vec({'a', 'c', 'b'},{})]
```

Problem 3.8.4

Let $a,\ b$ be real numbers. Consider the equation $z = ax+by$. Prove that there are two 3-vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ such that the set of points $[x, y, z]$ satisfying the equation is exactly the set of linear combinations of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. (Hint: Specify the vectors using formulas involving a, b.)

To find the two 3-vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ that satisfy the equation $z = ax + by$, we can start by considering the cases where a and b are not both zero.

Case 1: $a \neq 0, b \neq 0$

In this case, we can choose $\mathbf{v_1} = \begin{bmatrix}1 & 0 & a\end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix}0 & 1 & b\end{bmatrix}$. Then, any point $\begin{bmatrix}x & y & z\end{bmatrix}$ that satisfies the equation $z = ax + by$ can be written as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\begin{bmatrix}x & y & z\end{bmatrix} = x\begin{bmatrix}1 & 0 & a\end{bmatrix} + y\begin{bmatrix}0 & 1 & b\end{bmatrix} = \begin{bmatrix}x & 0 & ax\end{bmatrix} + \begin{bmatrix}0 & y & by\end{bmatrix} = \begin{bmatrix}x & y & ax+by\end{bmatrix} = x\mathbf{v_1} + y\mathbf{v_2}$$

Conversely, any linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ can be written in the form $x\begin{bmatrix}1 & 0 & a\end{bmatrix} + y\begin{bmatrix}0 & 1 & b\end{bmatrix}$, which satisfies the equation $z = ax + by$.

Case 2: $a = 0, b \neq 0$

In this case, we can choose $\mathbf{v_1} = \begin{bmatrix}0 & 1 & b\end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$. Then, any point $\begin{bmatrix}x & y & z\end{bmatrix}$ that satisfies the equation $z = ax + by$ can be written as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\begin{bmatrix}x & y & z\end{bmatrix} = x\begin{bmatrix}0 & 1 & b\end{bmatrix} + z\begin{bmatrix}1 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & x & bx\end{bmatrix} + \begin{bmatrix}z & 0 & 0\end{bmatrix} = \begin{bmatrix}z & x & bx\end{bmatrix} = z\mathbf{v_1} + x\mathbf{v_2}$$

Conversely, any linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ can be written in the form $x\begin{bmatrix}1 & 0 & 0\end{bmatrix} + z\begin{bmatrix}0 & 1 & b\end{bmatrix} = \begin{bmatrix}x & y & bx\end{bmatrix}$, which satisfies the equation $z = ax + by$.

Case 3: $a \neq 0, b = 0$

This case is similar to Case 2, but with $\mathbf{v_1} = \begin{bmatrix}1 & 0 & a\end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}$:

$$\begin{bmatrix}x & y & z\end{bmatrix} = x\begin{bmatrix}1 & 0 & a\end{bmatrix} + z\begin{bmatrix}0 & 1 & 0\end{bmatrix} = \begin{bmatrix}x & 0 & ax\end{bmatrix} + \begin{bmatrix}0 & y & 0\end{bmatrix} = \begin{bmatrix}x & y & ax\end{bmatrix} = x\mathbf{v_1} + y\mathbf{v_2}$$

Again, any linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ can be written in the form $x\begin{bmatrix}1 & 0 & a\end{bmatrix} + y\begin{bmatrix}0 & 1 & 0\end{bmatrix} = \begin{bmatrix}x & y & ax\end{bmatrix}$, which satisfies the equation $z = ax + by$.

Therefore, in all three cases, we have found two 3-vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ such that the set of points satisfying the equation $z = ax + by$ is exactly the set of linear combinations of $\mathbf{v_1}$ and $\mathbf{v_2}$.

Problem 3.8.5

Let a, b, c be real numbers. Consider the equation $z = ax + by + c$. Prove that there are three 3-vectors $\boldsymbol{v_0}, \boldsymbol{v_1}, \boldsymbol{v_2}$ such that the set of points $[x, y, z]$ satisfying the equation is exactly

$$\{\boldsymbol{v_0} + \alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} \ : \ \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}\}$$

(Hint: Specify the vectors using formulas involving a, b, c.)

Let $v_0 = \begin{bmatrix} 0 & 0 & c \end{bmatrix}, v_1 = \begin{bmatrix} 1 & 0 & a \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 & b \end{bmatrix}$. Now, let $[x, y, z]$ be a point that satisfies the equation $z = ax + by + c$. Then, we can write $[x, y, z]$ as:

$$\begin{aligned} [x, y, z] &= \begin{bmatrix} x & y & ax + by + c \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & c \end{bmatrix} + x \begin{bmatrix} 1 & 0 & a \end{bmatrix} + y \begin{bmatrix} 0 & 1 & b \end{bmatrix} \\ &= v_0 + x v_1 + y v_2. \end{aligned}$$

Conversely, suppose $[x, y, z]$ is a point of the form $v_0 + x v_1 + y v_2$ for some $x, y \in \mathbb{R}$. Then, we have:

$$\begin{aligned} [x, y, z] &= \begin{bmatrix} 0 & 0 & c \end{bmatrix} + x \begin{bmatrix} 1 & 0 & a \end{bmatrix} + y \begin{bmatrix} 0 & 1 & b \end{bmatrix} \\ &= \begin{bmatrix} x & y & ax + by + c \end{bmatrix}. \end{aligned}$$

Therefore, $z = ax + by + c$, and we see that every point of the form $v_0 + x v_1 + y v_2$ satisfies the equation.