

Gwinnett School of Math, Science, and Technology

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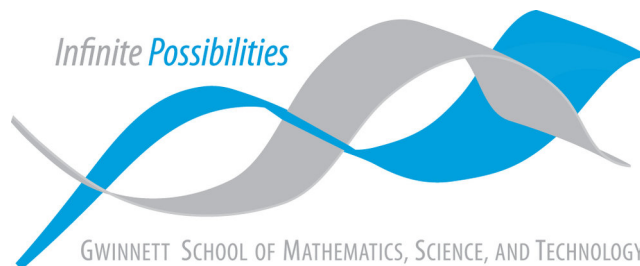
# Multivariable Calculus Yearlong Notes

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# 1 Chapter 1: Systems of Linear Equations and Matrices

## 1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal  $\Leftrightarrow$  they have the same dimensions and values

### 1.1.1 Addition & Subtraction

Two matrices can be added/subtracted  $\Leftrightarrow$  they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

### 1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3 \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

### 1.1.3 Matrix Multiplication

- We can **only** multiply an  $(m \times n)$  by  $(n \times p)$  matrix.
- The resulting matrix will be  $(m \times p)$

### 1.1.4 Properties of Matrix Arithmetic

- (a)  $A + B = B + A$  (**Commutative law for addition**)
- (b)  $A + (B + C) = (A + B) + C$  (**Associative law for addition**)
- (c)  $A(BC) = (AB)C$  (**Associative law for multiplication**)
- (d)  $A(B + C) = AB + AC$  (**Left distributive law**)
- (e)  $(B + C)A = BA + CA$  (**Right distributive law**)
- (f)  $A(B - C) = AB - AC$
- (g)  $(B - C)A = BA - CA$
- (h)  $a(B+C) = aB + aC$
- (i)  $a(B-C) = aB - aC$
- (j)  $(a+b)C = aC + bC$
- (k)  $(a-b)C = aC - bC$
- (l)  $a(bC) = (ab)C$
- (m)  $a(BC) = (aB)C = B(aC)$

### 1.1.5 Examples

1.

$$\begin{aligned} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \end{aligned}$$

2.

$$\begin{aligned} & \begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix} \end{aligned}$$

3.

$$\begin{aligned} & \begin{bmatrix} 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} \\ &= [4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2] \\ &= [30] \end{aligned}$$

## 1.2 Transpose of a Matrix

The transpose of an  $(m \times n)$  matrix is the  $(n \times m)$  matrix where the rows and columns are swapped.

$$\text{If } B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix} \end{aligned}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix  $A$  is equal to the sum of all the elements on the main diagonal:  $\text{tr}(A)$

### 1.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

### 1.3 Homework — “Matrix Stuff” (08/03/2023)

**1.3.1 Suppose that  $A, B, C, D$  and  $E$  are matrices with the following sizes:**

$A$	$B$	$C$	$D$	$E$
$(3 \times 2)$	$(2 \times 3)$	$(3 \times 3)$	$(3 \times 2)$	$(2 \times 3)$

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

Undefined	Defined; $(4 \times 2)$	Defined; $(5 \times 5)$	Defined; $(5 \times 2)$
$BA$	$AC + D$	$E(A + B)$	$(A^T + E)D$
$AB + B$			$E(AC)$
$E^T A$			
$AE + B$			

**1.3.2 Consider the matrices**

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2.  $2A^T + C$

$$\begin{aligned} 2A^T + C &= 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2 \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$



3.  $B^T + 5C^T$

$$\begin{aligned}
 B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\
 &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\
 &= \text{Undefined}
 \end{aligned}$$

4.  $2E^T - 3D^T$

$$\begin{aligned}
 2E^T - 3D^T &= 2 \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3 \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T \\
 &= 2 \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}
 \end{aligned}$$

5.  $\text{tr}(DE)$

$$\begin{aligned}
 \text{tr}(DE) &= \text{tr} \left( \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right) \\
 &= \text{tr} \left( \begin{bmatrix} 1 \cdot 6 + 5 \cdot (-1) + 2 \cdot 4 & 1 \cdot 1 + 5 \cdot 1 + 2 \cdot 1 & 1 \cdot 3 + 5 \cdot 2 + 2 \cdot 3 \\ (-1) \cdot 6 + 0 \cdot (-1) + 1 \cdot 4 & (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1 & (-1) \cdot 3 + 0 \cdot 2 + 1 \cdot 3 \\ 3 \cdot 6 + 2 \cdot (-1) + 4 \cdot 4 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 & 3 \cdot 3 + 2 \cdot 2 + 4 \cdot 3 \end{bmatrix} \right) \\
 &= \text{tr} \left( \begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix} \right) \\
 &= 34
 \end{aligned}$$

## 2 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

- Infinite solutions
- One solution
- No solutions

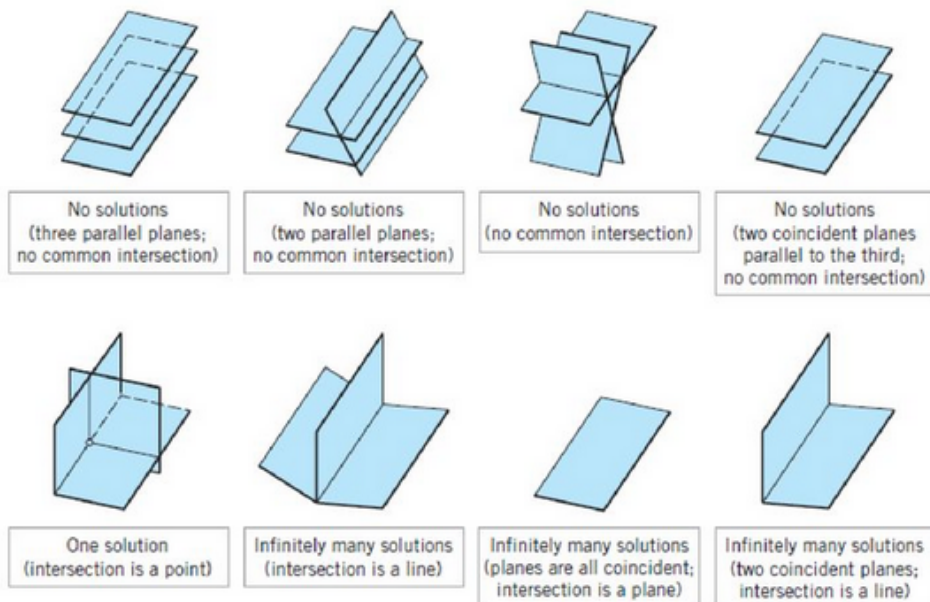
Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear *really* means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.

## Linear Systems in Three Unknowns



## 2.1 Review: Solve the following systems

1. 
$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$5x = 15$$

$$x = 3$$

$$2(3) + y = 10$$

$$6 + y = 10$$

$$y = 4$$

2. 
$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$y = 10 - 2x$$

$$6x + 3(10 - 2x) = 10$$

$$6x + 30 - 6x = 10$$

$$30 = 10. \therefore \text{no solution}$$

3. 
$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$0 = 0$$

$$12 = 12. \therefore \text{infinite solutions}$$

### 2.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

### 2.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

## 2.2 The Augmented Matrix

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array} \right]$$

## 2.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/subtract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

### 2.3.1 Example 1... again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\begin{aligned} \left[ \begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array} \right] &\xrightarrow{\frac{1}{2}R_1} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array} \right] \\ &\xrightarrow{-\frac{2}{5}R_2} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array} \right] \xrightarrow{R_1 - \frac{1}{2}R_2} \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array} \right] \end{aligned}$$

And so...  $x = 3$  and  $y = 4$ !

## 2.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array} \right],$$

then the solution to the system will be the ordered triple  $(c_1, c_2, c_3)$ .

### 2.4.1 Example 2: again

$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[ \begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{R_2-6R_1} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array} \right]$$

This is inconsistent, so there is no solution.

### 2.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[ \begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{\frac{1}{5}R_1} \left[ \begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{R_2-15R_1} \left[ \begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array} \right]$$

Since  $0 = 0$ , there are infinitely many solutions.

### 2.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \xrightarrow{R_3+4R_1} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_3+\frac{3}{2}R_2} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_2} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_1+2R_2} \left[ \begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\begin{matrix} R_1+7R_3 \\ R_2+4R_3 \end{matrix}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \end{aligned}$$

Therefore the solution to  $(x_1, x_2, x_3)$  is  $(29, 16, 3)$ .

#### 2.4.4 Elementary Row Operations & REF Homework Problem (08/08/2023)

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \xrightarrow[\substack{R_2+R_1 \\ R_3-3R_1}]{\substack{R_2+R_1 \\ R_3-3R_1}} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] \xrightarrow[\substack{-R_2 \\ -R_3}]{\substack{-R_2 \\ -R_3}} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array} \right] \\ & \xrightarrow[\substack{R_1-R_2 \\ R_3-10R_2}]{\substack{R_1-R_2 \\ R_3-10R_2}} \left[ \begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array} \right] \xrightarrow[\substack{1/52 R_3}]{\substack{1/52 R_3}} \left[ \begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow[\substack{R_1-7R_3 \\ R_2+5R_3}]{\substack{R_1-7R_3 \\ R_2+5R_3}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

Therefore, the solution to  $(x, y, z)$  is  $(3, 1, 2)$ .

#### 2.5 Gaussian Elimination

Vocabulary: A matrix is in Row Echelon Form (REF) if:

- (a) Any rows of all zeroes are placed at the bottom of the matrix
- (b) All other rows have a leading 1 ("pivot")
- (c) As we move down the matrix, each leading 1 is further to the right than the 1 above it

A matrix is in Row Reduced Echelon Form if the three above conditions are met in addition to:

- (d) Each column with a leading 1 has all other entries in the column as a 0. ("pivot column")

### 2.5.1 Examples

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF? ✓  
RREF? ✓

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF? ✓  
RREF? ✗

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

REF? ✗  
RREF? ✗

## 2.6 Gaussian Elimination With Back-Substitution

### 2.6.1 Goal:

To get the augmented matrix in REF

Solve: 
$$\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array} \right] \xrightarrow[R_3+R_2]{R_1+2R_2} \left[ \begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_3} \left[ \begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

$$x + 9z = 19$$

$$y + 3z = 5$$

$$z = 2$$

$$\therefore z = 2, y = 5 - 3z, x = 19 - 9z$$

$$z = 2, y = 5 - 3(2), x = 19 - 9(2)$$

$$z = 2, y = -1, x = 1$$

Therefore, the solution  $(x_1, x_2, x_3)$  is  $(1, -1, 2)$ .

## 2.6.2 Gaussian Elimination Homework Problem (08/09/2023)

$$\begin{cases} -2w + y + z = -3 \\ x + 2y - z = 2 \\ -3w + 2x + 4y + z = -2 \\ -w + x - 4y - 7z = -19 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{cccc|c} -2 & 0 & 1 & 1 & -3 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -1 & 1 & -4 & -7 & -19 \end{array} \right] \xrightarrow{R_4} \left[ \begin{array}{cccc|c} -1 & 1 & -4 & -7 & -19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array} \right] \xrightarrow{-R_1} \\ & \left[ \begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array} \right] \xrightarrow{\begin{array}{l} R_3+3R_1 \\ R_4+2R_1 \end{array}} \left[ \begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & 16 & 22 & 55 \\ 0 & -2 & 9 & 15 & 35 \end{array} \right] \xrightarrow{\begin{array}{l} R_1+R_2 \\ R_3+R_2 \\ R_4+2R_2 \end{array}} \\ & \left[ \begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 18 & 21 & 57 \\ 0 & 0 & 13 & 13 & 39 \end{array} \right] \xrightarrow{\frac{1}{18}R_3} \left[ \begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 13 & 13 & 39 \end{array} \right] \xrightarrow{\begin{array}{l} R_1-6R_3 \\ R_2-2R_3 \\ R_4-13R_3 \end{array}} \\ & \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & -\frac{13}{6} & -\frac{13}{6} \end{array} \right] \xrightarrow{-\frac{6}{13}R_4} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} R_1+R_4 \\ R_2+\frac{10}{3}R_4 \\ R_3-\frac{7}{6}R_4 \end{array}} \\ & \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \Rightarrow \begin{cases} w = 3 \\ x = -1 \\ y = 2 \\ z = 1 \end{cases} \end{aligned}$$



## 2.7 Gauss-Jordan Elimination

### 2.7.1 Goal:

To get the matrix into RREF

$$\text{Solve: } \begin{cases} x_1 - 3x_3 = -2 \\ 3x_1 + x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + x_3 = 4 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array} \right] \xrightarrow[\substack{R_2-3R_1 \\ R_3-2R_1}]{R_2-3R_1} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array} \right] \xrightarrow{R_3-2R_2} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array} \right] \\ & \xrightarrow{\frac{-1}{7}R_3} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow[\substack{R_1+3R_3 \\ R_2-7R_3}]{R_1+3R_3} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = 2 \end{cases} \end{aligned}$$

## 2.8 Matrix Properties, Equations, and Inverses

### 2.8.1 With Real Numbers

- If  $ab = bc$ , then  $a = c$ , if  $b \neq 0$
- If  $ab = 0$ , then  $a = 0$  or  $b = 0$ , or both

### 2.8.2 With Matrices

- If  $AB = AC$ , then  $B = C$ , if  $A$  is invertible
- If  $AB = [0]$ , then  $A = [0]$  or  $B = [0]$ , or both

#### 2.8.2.1 Multiply:

$$\begin{aligned} & \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(5) + 3(-3) & 2(-3) + 3(2) \\ 3(5) + 5(-3) & 3(-3) + 5(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

### 2.8.3 Matrix Inverses

- If a matrix has an inverse, it is said to be invertible or non-singular.
- If a matrix does not have an inverse, it is said to be singular.
- Every square matrix has a “special number” associated with it called the **determinant**.
- For the  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , the determinant is  $ad - bc$
- $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- When  $\det A = 0$ , the matrix is singular and has no inverse (since you cannot divide by zero)

Find the inverse of  $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}^{-1} &= \frac{1}{\det A} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{(4)(2) - (3)(1)} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix} \end{aligned}$$

## 3 Chapter 2: Determinants

### 3.1 Prior Knowledge:

$$\begin{bmatrix} 10 & -4 \\ -3 & -5 \end{bmatrix} = -50 - = -62$$

$$\begin{aligned} & \begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix} \\ & = ((2 \cdot 2 \cdot -2) + (4 \cdot 3 \cdot 3) + (3 \cdot -1 \cdot 0)) - ((3 \cdot 2 \cdot 3) + (0 \cdot 3 \cdot 2) + (-2 \cdot -1 \cdot 4)) \\ & = (-8 + 36 + 0) - (18 + 0 + 8) \\ & = 28 - 26 \\ & = 2 \end{aligned}$$

### 3.2 Minors & Cofactors

Given a square matrix A, the minor of matrix element  $a_{ij}$ , ( $M_{ij}$ ) is the determinant of the matrix formed by removing the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column from matrix A.

The cofactor of matrix element  $a_{ij}$ ,  $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

#### 3.2.1 Example

Let  $\det \begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$ . What is the cofactor of element (1, 1)?

Cofactor checkerboard:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} = -4$$

$$C_{11} = 1 \cdot -4 = -4$$

Find the minor and cofactor of: \ a)  $a_{21} = -1$

$$M_{21} = \begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} = -8$$

$$C_{21} = 8$$

b)  $a_{33} = -2$

$$M_{33} = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 8$$

$$C_{33} = 8$$

### 3.3 Cofactor Expansion

- 1) Pick a row or column
- 2) Multiply every entry in that row or column by it's corresponding cofactor
- 3) Add those together. That's it

$$A = \begin{bmatrix} 6 & 7 & -1 \\ 0 & 4 & 1 \\ 2 & 5 & -3 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 6 \begin{vmatrix} 4 & 1 \\ 5 & -3 \end{vmatrix} + 7 \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} + -1 \begin{vmatrix} 0 & 4 \\ 2 & 5 \end{vmatrix} \\ &= 6(-17) + 7(2) + (-1(-8)) \\ &= -102 + 14 + 8 \\ &= -80 \end{aligned}$$

#### 3.3.1 Example

$$A = \begin{bmatrix} 6 & 4 & 2 \\ 5 & -6 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

$$\begin{aligned} &6 \begin{vmatrix} -6 & 1 \\ 3 & 0 \end{vmatrix} + 4 \begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix} + 2 \begin{vmatrix} 5 & -6 \\ 0 & 3 \end{vmatrix} \\ &= 6(-3) + 0 + 2(15) \\ &= -18 + 30 \\ &= 12 \end{aligned}$$

### 3.3.2 Does the method generalize to 2×2 matrices?

$$\begin{aligned} & \begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix} \\ &= 3|2| - 5|7| \\ &= 6 - 35 \\ &= -29 \end{aligned}$$

The determinant of a 1×1 matrix is... **itself!**

### 3.3.3 Find the determinant of a 4×4

$$A = \begin{bmatrix} -3 & 2 & 0 & 8 \\ 2 & 1 & 0 & -4 \\ 5 & -2 & 1 & 5 \\ 2 & 3 & 0 & 6 \end{bmatrix}$$

$$\begin{aligned} &= 0 + 0 + \begin{vmatrix} -3 & 2 & 8 \\ 2 & 1 & -4 \\ 2 & 3 & 6 \end{vmatrix} + 0 \\ &= -2 \begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} -3 & 8 \\ 2 & 6 \end{vmatrix} - \left( -4 \begin{vmatrix} -3 & 2 \\ 2 & 3 \end{vmatrix} \right) \\ &= 24 - 34 - 52 \\ &= -62 \end{aligned}$$

### 3.4 Theorem

If  $A$  is an  $n \times n$  matrix, then regardless of which row or column of  $A$  is chosen, the number obtained by multiplying the elements in that row or column by their corresponding cofactors is **always the same** and is called the determinant of  $A$ .

### 3.4.1 Example

Find the determinant of  $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} & 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\ &= \left( -2 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \right) \\ &= -6 \end{aligned}$$

### 3.5 Triangular Matrices

Find the determinant of  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$\begin{aligned} & \begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} \\ &= 2 \begin{vmatrix} 3 & 3 \\ 0 & 4 \end{vmatrix} \\ &= 2(3 \cdot 4) \\ &= 2 \cdot 12 \\ &= 24 \end{aligned}$$

If  $A$  is an  $n \times n$  triangular matrix, then  $\det(A)$  is equal to the product of the elements along the main diagonal.

### 3.6 An Important Definition

Elementary Matrix a matrix that can be obtained from the  $n \times n$  identity matrix by performing a single row operation. \

Are the following matrices elementary? 1)  $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} + (R_3 + 5R_1)$  yes 2)  $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix} + (R_1 + 5R_2)$ ...  
no

### **3.7 A Pair of Theorems**

**3.7.1 Theorem:** If a square matrix  $A$  has a row of column of zeros, then  $\det(A) = 0$

**3.7.2 Theorem:** If  $A$  is a square matrix, then  $\det(A) = \det(A^T)$

### 3.8 Homework Makeups

#### 3.8.1 "Gaussian Elimination" (08/11/2023)

##### 3.8.1.1 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$
$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \xrightarrow[\substack{R_2+R_1 \\ R_3-3R_1}]{\substack{R_2+R_1 \\ R_3-3R_1}} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] \xrightarrow{-R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array} \right]$$
$$\xrightarrow{R_3+10R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array} \right] \xrightarrow{-\frac{1}{52}R_3} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right]$$
$$\therefore \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ x_2 - 5x_3 = -9 \\ x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

##### 3.8.1.2 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ -2x_1 - 3x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + 4x_3 = 0 \end{cases}$$
$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 4 & 0 \end{array} \right] \xrightarrow[\substack{R_2+2R_1 \\ R_3-2R_1}]{\substack{R_2+2R_1 \\ R_3-2R_1}} \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right] \xrightarrow[\substack{-\frac{1}{7}R_2 \\ -\frac{1}{2}R_3}]{\substack{-\frac{1}{7}R_2 \\ -\frac{1}{2}R_3}} \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & \frac{2}{7} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$
$$\therefore \begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ x_2 + \frac{2}{7}x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow 1 \neq 0 \therefore \text{no solution}$$



### 3.8.2 "Inverses and Determinants" (08/14)

#### 3.8.2.1 Find the determinants of the following:

$$1) \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -3 \\ 4 & 4 \end{vmatrix} = 2(4) - (-3)(4) = 8 + 12 = 20$$

$$2) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2(3) - 0(0) = 6$$

$$3) \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta + \sin^2 \theta = 1$$

#### 3.8.2.2 Find the INVERSES of those matrices:

$$1) \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}^{-1} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

$$2) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$3) \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

### 3.8.3 Inverses and Determinants (08/15)

#### 3.8.3.1 Use a matrix equation to solve the following problems:

$$1) \begin{cases} 3x_1 - 2x_2 = 1 \\ 4x_1 + 5x_2 = 3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{23} \\ \frac{9}{23} \end{bmatrix}$$

$$2) \begin{cases} 6x_1 + x_2 = 0 \\ 4x_1 - 3x_2 = -2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} -3 & -1 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} \\ \frac{4}{11} \end{bmatrix}$$

### 3.8.4 Consistent Systems (08/21)

**3.8.4.1 Solve the linear systems together by reducing the appropriate augmented matrix.**

$$\begin{cases} x_1 - 5x_2 = b_1 \\ 3x_1 + 2x_2 = b_2 \end{cases}$$

- 1)  $b_1 = 1, b_2 = 4$
- 2)  $b_1 = -2, b_2 = 5$

First, let's solve it for the general case:

$$\left[ \begin{array}{cc|c} 1 & -5 & b_1 \\ 3 & 2 & b_2 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[ \begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 17 & b_2 - 3b_1 \end{array} \right] \xrightarrow{\frac{1}{17}R_2} \left[ \begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 1 & \frac{b_2 - 3b_1}{17} \end{array} \right] \xrightarrow{R_1 + 5R_2} \left[ \begin{array}{cc|c} 1 & 0 & \frac{2b_1 + 5b_2}{17} \\ 0 & 1 & \frac{-3b_1 + b_2}{17} \end{array} \right]$$

Therefore, the solution to the general case is  $(x_1, x_2) = \left( \frac{2b_1 + 5b_2}{17}, \frac{-3b_1 + b_2}{17} \right)$

And so, for the specific cases:

- 1)  $(x_1, x_2) = \left( \frac{2(1) + 5(4)}{17}, \frac{-3(1) + 4}{17} \right) = \left( \frac{13}{17}, \frac{1}{17} \right)$
- 2)  $(x_1, x_2) = \left( \frac{2(-2) + 5(5)}{17}, \frac{-3(-2) + 5}{17} \right) = \left( \frac{16}{17}, \frac{11}{17} \right)$

**3.8.4.2 Determine the conditions on  $b$ , if any, in order to guarantee that the linear system is consistent.**

$$\begin{cases} x_1 + 3x_2 = b_1 \\ -2x_1 + x_2 = b_2 \end{cases}$$

$$\left[ \begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array} \right] \xrightarrow{R_2+2R_1} \left[ \begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & b_2+2b_1 \end{array} \right] \xrightarrow{\frac{1}{7}R_2} \left[ \begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array} \right] \xrightarrow{R_1-3R_2} \left[ \begin{array}{cc|c} 1 & 0 & \frac{b_1-3b_2}{7} \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array} \right]$$

There are no conditions. The system is consistent for all values of  $b_1$  and  $b_2$ .

**3.8.5 Another “determining the conditions” problem:**

$$\begin{cases} x_1 - 2x_2 - x_3 = b_1 \\ -4x_1 + 5x_2 + 2x_3 = b_2 \\ -4x_1 + 7x_2 + 4x_3 = b_3 \end{cases}$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array} \right] \xrightarrow[R_3+4R_1]{R_2+4R_1} \left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & b_2+4b_1 \\ 0 & -1 & 0 & b_3+4b_1 \end{array} \right] \xrightarrow{-\frac{1}{3}R_2} \left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{b_3+4b_1}{3} \end{array} \right]$$

$$\xrightarrow{-\frac{3}{2}R_3} \left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & 1 & \frac{-b_3-4b_1}{2} \end{array} \right]$$

Therefore, the system is consistent for all values of  $b_1$ ,  $b_2$ , and  $b_3$ .

### 3.8.6 Triangular and Diagonal Matrices

#### 3.8.6.1 Find $A^2$

$$1) A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 0(0) & 1(0) + 0(-2) \\ 0(1) + (-2)(0) & 0(0) + (-2)(-2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

$$2) A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (-6)(-6) + (0)(0) + (0)(0) & (-6)(0) + (0)(3) + (0)(0) & (-6)(0) + (0)(0) + (0)(5) \\ (0)(-6) + (3)(0) + (0)(0) & (0)(0) + (3)(3) + (0)(0) & (0)(0) + (3)(0) + (0)(5) \\ (0)(-6) + (0)(0) + (5)(0) & (0)(0) + (0)(3) + (5)(0) & (0)(0) + (0)(0) + (5)(5) \end{bmatrix} \\ &= \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} \end{aligned}$$

**3.8.6.2 Find  $A^{-k}$ , such that  $k$  is some nonzero constant**

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} A^{-k} &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-k} \\ &= \begin{bmatrix} 2^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & 2^{-k} \end{bmatrix} \end{aligned}$$

4. Determine whether each matrix is symmetric or not.

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

Not symmetric

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

**3.8.6.3 Find a diagonal matrix A that satisfies the given condition**

$$1) A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{\frac{1}{5}} \\ &= \begin{bmatrix} 1^{\frac{1}{5}} & 0 & 0 \\ 0 & (-1)^{\frac{1}{5}} & 0 \\ 0 & 0 & (-1)^{\frac{1}{5}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

$$2) A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-\frac{1}{2}} \\ &= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 & 0 \\ 0 & 4^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1^{-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$



### 3.8.7 Determinants and Triangular Matrices (08/29)

#### 3.8.7.1 What is $C_{32}$

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} C_{32} &= (-1)^{3+2} \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \left( 2 \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 3 \\ 3 & 0 \end{vmatrix} + 1 \begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} \right) \\ &= - (2(-3) - (-1)(-9) + 1(-3)) \\ &= -(-6 + 9 - 3) \\ &= 0 \end{aligned}$$

#### 3.8.7.2 Find all values of $\lambda$ such that $|A| = 0$

$$A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (\lambda - 2)(\lambda + 4) - (-5)(1) \\ &= \lambda^2 + 2\lambda - 8 + 5 \\ &= \lambda^2 + 2\lambda - 3 \\ &= (\lambda + 3)(\lambda - 1) \\ &= 0 \end{aligned}$$

Therefore,  $\lambda = -3, 1$

**3.8.7.3 For the matrix  $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$  find the determinant 3 different ways with cofactor expansion. Pick different rows and columns each time.**

$$\begin{aligned} \det(A) &= 3 \begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0 \begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + 0 \begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} \\ &= 3(-1(-4) - 5(9)) - 0(2(-4) - 5(1)) + 0(2(9) - (-1)(1)) \\ &= 3(4 - 45) - 0(-8 - 5) + 0(18 + 1) \\ &= 3(-41) - 0(-13) + 0(19) \\ &= 36 \end{aligned}$$

$$\begin{aligned} \det(A) &= 0 \begin{vmatrix} 2 & 5 \\ 9 & -4 \end{vmatrix} - 3 \begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 0 \begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} \\ &= 0(2(-4) - 5(9)) - 3(3(-4) - 0(1)) + 0(3(5) - 0(2)) \\ &= 0(-8 - 45) - 3(-12 - 0) + 0(15 - 0) \\ &= 0(-53) - 3(-12) \\ &= 36 \end{aligned}$$

$$\begin{aligned} \det(A) &= 0 \begin{vmatrix} 2 & -1 \\ 9 & -4 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 0(2(-4) - (-1)(9)) - 0(3(-4) - 0(1)) + 3(3(-1) - 0(2)) \\ &= 0(-8 + 9) - 0(-12 - 0) + 3(-3 - 0) \\ &= 0(1) - 0(-12) + 3(-3) \\ &= 0 + 0 - 9 \\ &= 36 \end{aligned}$$

**3.8.7.4 Evaluate  $\det(A)$  by a cofactor expansion along a row or column of your choice**

$$A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - k \begin{vmatrix} 1 & k^2 \\ 1 & k^2 \end{vmatrix} + k^2 \begin{vmatrix} 1 & k \\ 1 & k \end{vmatrix} \\ &= 1(k^2 - k^2) - k(1(k^2) - k^2(1)) + k^2(1(k) - k(1)) \\ &= 0 \end{aligned}$$

**3.8.7.5 Evaluate the determinant of the following matrices by just looking at them.**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1(-1)(1) = -1$$

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det(A) = 1(1)(2)(3) = 6$$

**3.8.7.6 Show that the value of the determinant is independent of  $\theta$**

$$A = \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ \sin \theta - \cos \theta & \sin \theta + \cos \theta & 1 \end{vmatrix}$$

$$\begin{aligned} \det(A) &= \sin \theta \begin{vmatrix} \sin \theta & 0 \\ \sin \theta + \cos \theta & 1 \end{vmatrix} - \cos \theta \begin{vmatrix} \cos \theta & 0 \\ \sin \theta + \cos \theta & 1 \end{vmatrix} \\ &\quad + 0 \begin{vmatrix} \cos \theta & \sin \theta \\ \sin \theta + \cos \theta & \sin \theta \end{vmatrix} \\ &= \sin \theta (\sin \theta(1) - 0(\sin \theta + \cos \theta)) - \cos \theta (\cos \theta(1) - 0(\sin \theta + \cos \theta)) \\ &\quad + 0 (\cos \theta(\sin \theta) - \sin \theta(\sin \theta + \cos \theta)) \\ &= \sin^2 \theta - \cos^2 \theta \\ &= 1 \end{aligned}$$

### 3.8.8 Row operations and Determinants (08/31)

3.8.8.1 Find the determinant of  $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$  **WITHOUT** using cofactor expansion

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{28}{2} \end{vmatrix} \\ &= 1(-2)\left(\frac{28}{2}\right) \\ &= -28\end{aligned}$$

**3.8.8.2 Find the determinant of**  $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -3 & 2 \end{vmatrix} \\ &= 2(-2)(-4)(2) \\ &= 64 \end{aligned}$$

### 3.8.9 Adjoints and Cramer's Rule (09/05)

**3.8.9.1 Find the inverse of**  $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$  **using the adjoint method**

$$\begin{aligned}
 \det(A) &= 2 \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} - 5 \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} + 5 \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\
 &= 2(-3) - 5(-3) + 5(-6) \\
 &= -6 + 15 - 30 \\
 &= -21
 \end{aligned}$$

$$\begin{aligned}
 \text{adj}(A) &= \begin{bmatrix} (-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} & (-1)^{1+2} \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} & (-1)^{1+3} \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\ (-1)^{2+1} \begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix} & (-1)^{2+2} \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} & (-1)^{2+3} \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} \\ (-1)^{3+1} \begin{vmatrix} 5 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+3} \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} (-1)(-3) & (-1)(-3) & (-1)(-6) \\ (-1)(-5) & (-1)(-1) & (-1)(-1) \\ (-1)(-5) & (-1)(-2) & (-1)(-3) \end{bmatrix} \\
 &= \begin{bmatrix} 3 & 3 & 6 \\ 5 & 1 & 1 \\ 5 & 2 & 3 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \therefore A^{-1} &= -\frac{1}{21} \begin{bmatrix} 3 & 3 & 6 \\ 5 & 1 & 1 \\ 5 & 2 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{21} & -\frac{1}{21} & -\frac{1}{21} \\ -\frac{5}{21} & -\frac{2}{21} & -\frac{1}{7} \end{bmatrix}
 \end{aligned}$$

**3.8.9.2 Solve the following system of equations using Cramer's Rule**

$$\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$$

$$\begin{aligned} x &= \frac{\begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}} \\ &= \frac{2 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix}}{4 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 11 & 1 \\ 1 & 5 \end{vmatrix}} \\ &= \frac{2(2 - 10) - 5(6 - 2) + 0(15 - 1)}{4(2 - 10) - 5(22 - 2) + 0(55 - 1)} \\ &= \frac{-16 - 20}{-32 - 100} \\ &= \frac{-36}{-132} \\ &= \frac{3}{11} \end{aligned}$$

$$\begin{aligned}
 y &= \frac{\begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}} \\
 &= \frac{4 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 11 & 3 \\ 1 & 1 \end{vmatrix}}{4 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 11 & 1 \\ 1 & 5 \end{vmatrix}} \\
 &= \frac{4(6 - 2) - 2(22 - 2) + 0(11 - 3)}{4(2 - 10) - 5(22 - 2) + 0(55 - 1)} \\
 &= \frac{16 - 40}{-32 - 100} \\
 &= \frac{-24}{-132} \\
 &= \frac{2}{11}
 \end{aligned}$$

$$\begin{aligned}
 z &= \frac{\begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}} \\
 &= \frac{4 \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 5 \begin{vmatrix} 11 & 3 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 11 & 1 \\ 1 & 5 \end{vmatrix}}{4 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 11 & 1 \\ 1 & 5 \end{vmatrix}} \\
 &= \frac{4(1 - 15) - 5(11 - 3) + 2(55 - 1)}{4(2 - 10) - 5(22 - 2) + 0(55 - 1)} \\
 &= \frac{-56 - 40 + 108}{-32 - 100} \\
 &= \frac{12}{-132} \\
 &= -\frac{1}{11}
 \end{aligned}$$

Therefore, the solution is  $(x, y, z) = (\frac{3}{11}, \frac{2}{11}, -\frac{1}{11})$



**3.8.9.3 Use Cramer's rule to solve for  $y$  without solving for the unknowns  $x, z$ , and  $w$**

$$\begin{cases} w + 4x + y + z = 6 \\ w + 3x + 7y - z = 1 \\ 8w + 7x + 3y - 5z = -3 \\ 2w + x + y + z = 3 \end{cases}$$

$$\begin{aligned} y &= \frac{\begin{vmatrix} 4 & 6 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 7 & -3 & -5 & -3 \\ 1 & 3 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 2 & 1 \end{vmatrix}} \\ &= \frac{4 \begin{vmatrix} 7 & -3 & -3 \\ 1 & 1 & 3 \\ 3 & -5 & -3 \end{vmatrix} - 6 \begin{vmatrix} 3 & 1 & 3 \\ 7 & -1 & 3 \\ 3 & -5 & -3 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 & 3 \\ 7 & -3 & 3 \\ 3 & -5 & -3 \end{vmatrix} - 3 \begin{vmatrix} 3 & 1 & 3 \\ 7 & -3 & -1 \\ 3 & -5 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 2 & 1 \end{vmatrix}} \\ &= \frac{4 \begin{vmatrix} 7 & -1 & 1 \\ 1 & 1 & 3 \\ 3 & -5 & -3 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 & 3 \\ 7 & -1 & 3 \\ 3 & -5 & -3 \end{vmatrix} - 1 \begin{vmatrix} 3 & 7 & 3 \\ 7 & 1 & 3 \\ 3 & 3 & -3 \end{vmatrix} + 1 \begin{vmatrix} 3 & 7 & -1 \\ 7 & 1 & 3 \\ 3 & 3 & -5 \end{vmatrix}}{\begin{vmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 2 & 1 \end{vmatrix}} \\ &= \frac{4(-3 - 9 + 9) - 6(-9 + 21 - 15) + 1(-9 + 21 - 15) - 3(-9 + 21 + 3)}{4(-7 + 3 + 15) - 1(-9 + 21 - 15) - 1(-9 + 21 - 15) + 1(-9 + 21 + 15)} \\ &= \frac{4(-3) - 6(-3) + 1(-3) - 3(15)}{4(11) - 1(-3) - 1(-3) + 1(27)} \\ &= \frac{-12 + 18 - 3 - 45}{44 + 3 + 3 + 27} \\ &= \frac{-42}{77} \\ &= -\frac{6}{11} \end{aligned}$$