Gwinnett School of Math, Science, and Technology

Multivariable Calculus Yearlong Notes

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Infinite Possibilities

GWINNETT SCHOOL OF MATHEMATICS, SCIENCE, AND TECHNOLOGY

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1 Systems of Linear Equations and Matrices

1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal $\iff$ they have the same dimensions and values

1.1.1 Addition & Subtraction

Two matrices can be added/subtracted $\iff$ they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

1.1.3 Matrix Multiplication

- We can **only** multiply an (m x n) by (n x p) matrix.
- The resulting matrix will be (m x p)

1.1.4 Properties of Matrix Arithmetic

(a) $A + B = B + A$ **(Commutative law for addition)**
(b) $A + (B + C) = (A + B) + C$ **(Associative law for addition)**
(c) $A(BC) = (AB)C$ **(Associative law for multiplication)**
(d) $A(B + C) = AB + AC$ **(Left distributive law)**
(e) $(B + C)A = BA + CA$ **(Right distributive law)**
(f) $A(B - C) = AB - AC$
(g) $(B - C)A = BA - CA$
(h) a(B+C) = aB + aC
(i) a(B-C) = aB - aC
(j) (a+b)C = aC + bC
(k) (a-b)C = aC - bC
(l) a(bC) = (ab)C
(m) a(BC) = (aB)C = B(aC)

1.1.5 Examples

1.

$$\begin{aligned}
&\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix} \\
&= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}
\end{aligned}$$

2.

$$\begin{aligned}
&\begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\
&= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix}
\end{aligned}$$

3.

$$
\begin{aligned}
&\begin{bmatrix} 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} \\
&= [4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2] \\
&= [30]
\end{aligned}
$$

1.2 Transpose of a Matrix

The transpose of an (m x n) matrix is the (n x m) matrix where the rows and columns are swapped.

If $B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}$, $B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$

$$
\begin{aligned}
B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\
&= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix}
\end{aligned}
$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix A is equal to the sum of all the elements on the main diagonal: tr(A)

1.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

1.3 Homework — "Matrix Stuff" (08/03/2023)

1.3.1 Suppose that A, B, C, D and E are matrices with the following sizes:

A	B	C	D	E
(3×2)	(2×3)	(3×3)	(3×2)	(2×3)

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

Undefined	Defined; (4×2)	Defined; (5×5)	Defined; (5×2)
BA	$AC + D$	$E(A + B)$	$(A^T + E)D$
$AB + B$			$E(AC)$
$E^T A$			
$AE + B$			

1.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2. $\mathbf{2A^T + C}$

$$\begin{aligned} 2A^T + C &= 2\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3. $\mathbf{B^T + 5C^T}$

$$\begin{aligned}
B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\
&= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\
&= \text{Undefined}
\end{aligned}$$

4. $\mathbf{2E^T - 3D^T}$

$$\begin{aligned}
2E^T - 3D^T &= 2\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T \\
&= 2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} \\
&= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}
\end{aligned}$$

5. $\text{tr}(\mathbf{DE})$

$$\begin{aligned}
\text{tr}(DE) &= \text{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right) \\
&= \text{tr}\left(\begin{bmatrix} 1\cdot 6 + 5\cdot(-1) + 2\cdot 4 & 1\cdot 1 + 5\cdot 1 + 2\cdot 1 & 1\cdot 3 + 5\cdot 2 + 2\cdot 3 \\ (-1)\cdot 6 + 0\cdot(-1) + 1\cdot 4 & (-1)\cdot 1 + 0\cdot 1 + 1\cdot 1 & (-1)\cdot 3 + 0\cdot 2 + 1\cdot 3 \\ 3\cdot 6 + 2\cdot(-1) + 4\cdot 4 & 3\cdot 1 + 2\cdot 1 + 4\cdot 1 & 3\cdot 3 + 2\cdot 2 + 4\cdot 3 \end{bmatrix}\right) \\
&= \text{tr}\left(\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}\right) \\
&= 34
\end{aligned}$$

2 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

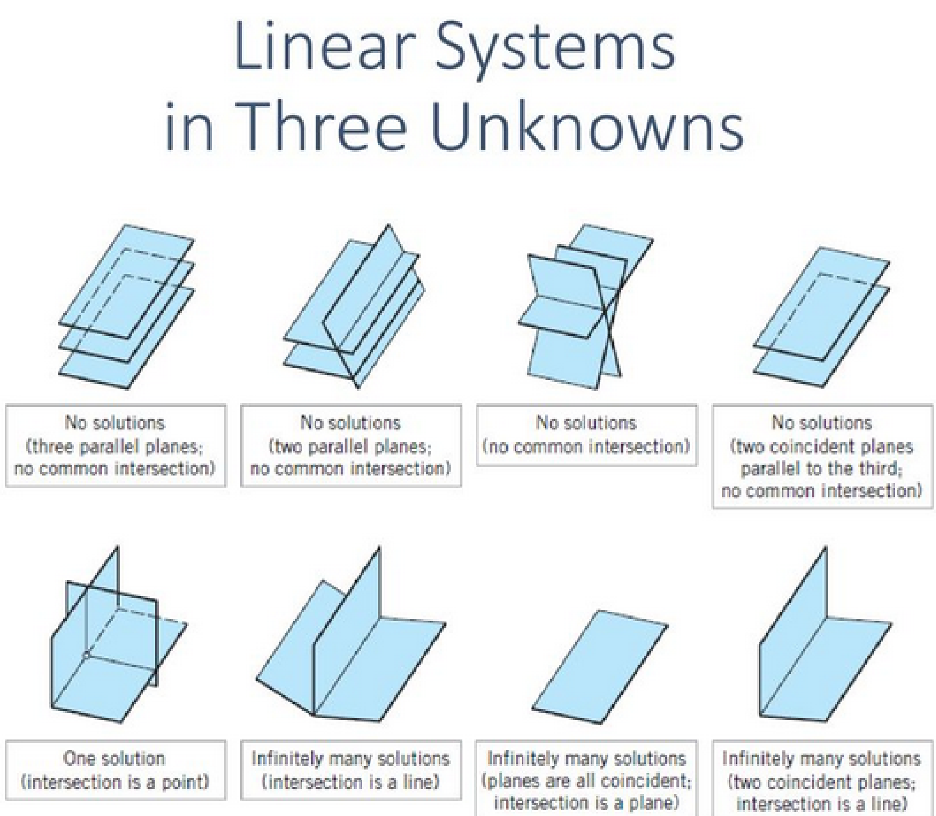
- Infinite solutions
- One solution
- No solutions

Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear *really* means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.



2.1 Review: Solve the following systems

1. $\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$

$$\begin{aligned} 5x &= 15 \\ x &= 3 \\ 2(3) + y &= 10 \\ 6 + y &= 10 \\ y &= 4 \end{aligned}$$

2. $\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$

$$\begin{aligned} y &= 10 - 2x \\ 6x + 3(10 - 2x) &= 10 \\ 6x + 30 - 6x &= 10 \\ 30 &= 10 \therefore \text{ no solution} \end{aligned}$$

3. $\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$

$$\begin{aligned} 0 &= 0 \\ 12 &= 12 \therefore \text{ infinite solutions} \end{aligned}$$

2.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

2.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

2.2 The Augmented Matrix

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array}\right]$$

2.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/substract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

2.3.1 Example 1... again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array}\right]$$
$$\xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{R1 - \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

And so... $x = 3$ and $y = 4$!

2.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right],$$

then the solution to the system will be the ordered triple (c_1, c_2, c_3).

2.4.1 Example 2: again

$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array}\right] \xrightarrow{\frac{1}{2}R1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array}\right] \xrightarrow{R2-6R1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array}\right]$$

This is inconsistent, so there is no solution.

2.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array}\right] \xrightarrow{\frac{1}{5}R1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array}\right] \xrightarrow{R2-15R1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array}\right]$$

Since $0 = 0$, there are infinitely many solutions.

2.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow{R3+4R1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R3+\frac{3}{2}R2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \begin{array}{c} \xrightarrow{R_1+2R_2} \\ \xrightarrow{R_3+3R_2} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} \xrightarrow{R1+7R_3} \\ \xrightarrow{R_2+4R_3} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore the solution to (x_1, x_2, x_3) is $(29, 16, 3)$.

2.4.4 Homework

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \xrightarrow[-R_3]{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \xrightarrow{R_3-10R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array}\right] \xrightarrow{\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2+5R_3]{R_1-7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore, the solution to (x, y, z) is $(3, 1, 2)$.