Gwinnett School of Math, Science, and Technology

Multivariable Calculus Yearlong Notes

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1 Chapter 1: Systems of Linear Equations and Matrices

1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal $\iff$ they have the same dimensions and values

1.1.1 Addition & Subtraction

Two matrices can be added/subtracted $\iff$ they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

1.1.3 Matrix Multiplication

- We can **only** multiply an (m x n) by (n x p) matrix.
- The resulting matrix will be (m x p)

1.1.4 Properties of Matrix Arithmetic

(a) $A + B = B + A$ **(Commutative law for addition)**
(b) $A + (B + C) = (A + B) + C$ **(Associative law for addition)**
(c) $A(BC) = (AB)C$ **(Associative law for multiplication)**
(d) $A(B + C) = AB + AC$ **(Left distributive law)**
(e) $(B + C)A = BA + CA$ **(Right distributive law)**
(f) $A(B - C) = AB - AC$
(g) $(B - C)A = BA - CA$
(h) a(B+C) = aB + aC
(i) a(B-C) = aB - aC
(j) (a+b)C = aC + bC
(k) (a-b)C = aC - bC
(l) a(bC) = (ab)C
(m) a(BC) = (aB)C = B(aC)

1.1.5 Examples

1.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix}$$
$$= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix}$$

3.

$$\begin{bmatrix} 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2 \end{bmatrix}$$
$$= \begin{bmatrix} 30 \end{bmatrix}$$

1.2 Transpose of a Matrix

The transpose of an (m x n) matrix is the (n x m) matrix where the rows and columns are swapped.

If $B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$

$$\begin{aligned} B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix} \end{aligned}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix A is equal to the sum of all the elements on the main diagonal: $\mathrm{tr}(A)$

1.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

1.3 Homework — "Matrix Stuff" (08/03/2023)

1.3.1 Suppose that A, B, C, D and E are matrices with the following sizes:

A	B	C	D	E
(3×2)	(2×3)	(3×3)	(3×2)	(2×3)

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

Undefined	Defined; (4×2)	Defined; (5×5)	Defined; (5×2)
BA $AB + B$ $E^T A$ $AE + B$	$AC + D$	$E(A + B)$	$(A^T + E)D$ $E(AC)$

1.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2. $\mathbf{2A^T + C}$

$$\begin{aligned} 2A^T + C &= 2\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3. $\mathbf{B^T} + \mathbf{5C^T}$

$$B^T + 5C^T = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T$$
$$= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix}$$
= Undefined

4. $\mathbf{2E^T} - \mathbf{3D^T}$

$$2E^T - 3D^T = 2\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T$$
$$= 2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$$

5. tr($\mathbf{DE}$)

$$\operatorname{tr}(DE) = \operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\begin{bmatrix} 1\cdot 6 + 5\cdot(-1) + 2\cdot 4 & 1\cdot 1 + 5\cdot 1 + 2\cdot 1 & 1\cdot 3 + 5\cdot 2 + 2\cdot 3 \\ (-1)\cdot 6 + 0\cdot(-1) + 1\cdot 4 & (-1)\cdot 1 + 0\cdot 1 + 1\cdot 1 & (-1)\cdot 3 + 0\cdot 2 + 1\cdot 3 \\ 3\cdot 6 + 2\cdot(-1) + 4\cdot 4 & 3\cdot 1 + 2\cdot 1 + 4\cdot 1 & 3\cdot 3 + 2\cdot 2 + 4\cdot 3 \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}\right)$$
$$= 34$$

2 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

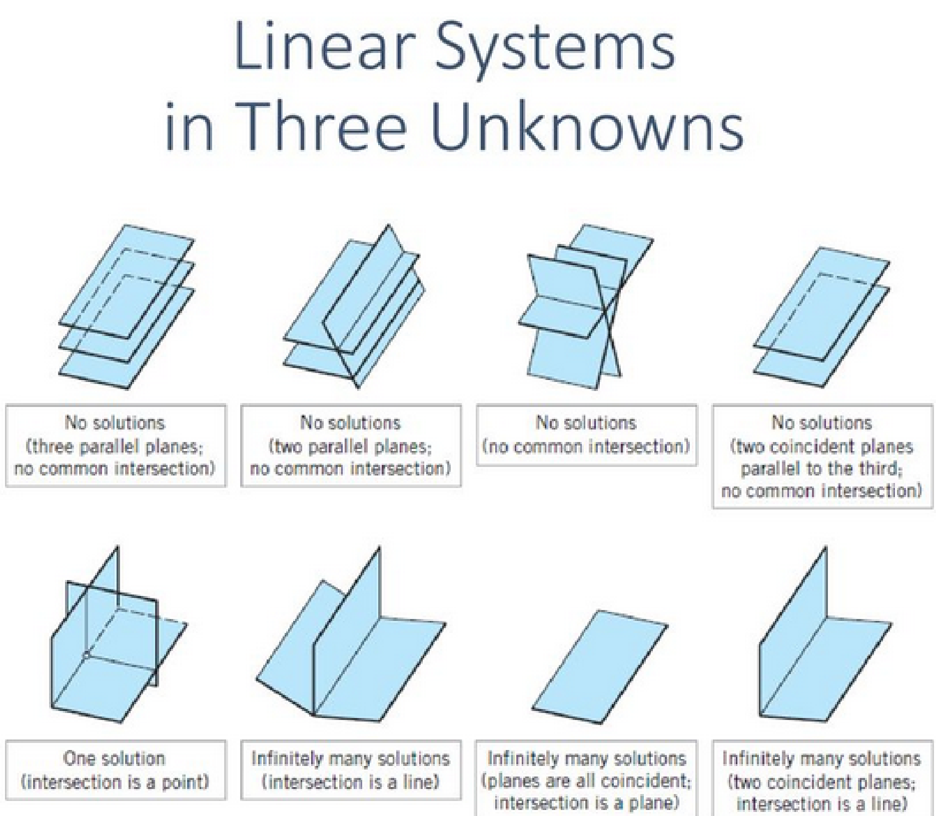
- Infinite solutions
- One solution
- No solutions

Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear really means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.



2.1 Review: Solve the following systems

1. $\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$

$$\begin{aligned} 5x &= 15 \\ x &= 3 \\ 2(3) + y &= 10 \\ 6 + y &= 10 \\ y &= 4 \end{aligned}$$

2. $\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$

$$\begin{aligned} y &= 10 - 2x \\ 6x + 3(10 - 2x) &= 10 \\ 6x + 30 - 6x &= 10 \\ 30 &= 10 \therefore \text{ no solution} \end{aligned}$$

3. $\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$

$$\begin{aligned} 0 &= 0 \\ 12 &= 12 \therefore \text{ infinite solutions} \end{aligned}$$

2.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

2.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

2.2 The Augmented Matrix

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array}\right]$$

2.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/substract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

2.3.1 Example 1… again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array}\right]$$
$$\xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{R1 - \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

And so… $x = 3$ and $y = 4$!

2.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right],$$

then the solution to the system will be the ordered triple (c_1, c_2, c_3).

2.4.1 Example 2: again

$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array}\right] \xrightarrow{\frac{1}{2}R1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array}\right] \xrightarrow{R2-6R1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array}\right]$$

This is inconsistent, so there is no solution.

2.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array}\right] \xrightarrow{\frac{1}{5}R1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array}\right] \xrightarrow{R2-15R1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array}\right]$$

Since 0 = 0, there are infinitely many solutions.

2.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow{R3+4R1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R3+\frac{3}{2}R2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow[R_3+3R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2+4R_3]{R1+7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore the solution to (x_1, x_2, x_3) is $(29, 16, 3)$.

2.4.4 Elementary Row Operations & REF Homework Problem (08/08/2023)

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \xrightarrow[-R_3]{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array}\right]$$

$$\xrightarrow[R_3-10R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array}\right] \xrightarrow{\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2+5R_3]{R_1-7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore, the solution to (x, y, z) is $(3, 1, 2)$.

2.5 Gaussian Elimination

Vocabulary: A matrix is in Row Echelon Form (REF) if:

(a) Any rows of all zeroes are placed at the bottom of the matrix
(b) All other rows have a leading 1 ("pivot")
(c) As we move down the matrix, each leading 1 is further to the right than the 1 above it

A matrix is in Row Reduced Echelon Form if the three above conditions are met in adition to:

(d) Each column with a leading 1 has all other entries in the column as a 0. ("pivot column")

2.5.1 Examples

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF? ✓
RREF? ✓

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF? ✓
RREF? ×

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

REF? ×
RREF? ×

2.6 Gaussian Elimination With Back-Substitution

2.6.1 Goal:

To get the augmented matrix in REF

Solve: $\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow[R_3+R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x + 9z &= 19 \\ y + 3z &= 5 \\ z &= 2 \\ \therefore z = 2, y &= 5 - 3z, x = 19 - 9z \\ z = 2, y &= 5 - 3(2), x = 19 - 9(2) \\ z = 2, y &= -1, x = 1 \end{aligned}$$

Therefore, the solution (x_1, x_2, x_3) is $(1, -1, 2)$.

2.6.2 Gaussian Elimination Homework Problem (08/09/2023)

$$\begin{cases} -2w \qquad\quad + \;\; y + \;\; z = -3 \\ \qquad\quad x + 2y - \;\; z = 2 \\ -3w + 2x + 4y + \;\; z = -2 \\ \;\; -w + \;\; x - 4y - 7z = -19 \end{cases}$$

$$\left[\begin{array}{cccc|c} -2 & 0 & 1 & 1 & -3 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -1 & 1 & -4 & -7 & -19 \end{array}\right] \begin{array}{c} \xrightarrow{R_4} \\ \\ \\ \xrightarrow{R_1} \end{array} \left[\begin{array}{cccc|c} -1 & 1 & -4 & -7 & -19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array}\right] \xrightarrow{-R_1}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array}\right] \begin{array}{c} \xrightarrow{R_3+3R_1} \\ \xrightarrow{R_4+2R_1} \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & 16 & 22 & 55 \\ 0 & -2 & 9 & 15 & 35 \end{array}\right] \begin{array}{c} \xrightarrow{R_1+R_2} \\ \xrightarrow{R_3+R_2} \\ \xrightarrow{R_4+2R_2} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 18 & 21 & 57 \\ 0 & 0 & 13 & 13 & 39 \end{array}\right] \xrightarrow{\frac{1}{18}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 13 & 13 & 39 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-6R_3} \\ \xrightarrow{R_2-2R_3} \\ \\ \xrightarrow{R_4-13R_3} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & -\frac{13}{6} & -\frac{13}{6} \end{array}\right] \xrightarrow{\frac{-6}{13}R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_1+R_4} \\ \xrightarrow{R_2+\frac{10}{3}R_4} \\ \xrightarrow{R_3-\frac{7}{6}R_4} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \implies \begin{cases} w = 3 \\ x = -1 \\ y = 2 \\ z = 1 \end{cases}$$

2.7 Gauss-Jordan Elimination

2.7.1 Goal:

To get the matrix into RREF

Solve: $\begin{cases} x_1 \qquad\quad - 3x_3 = -2 \\ 3x_1 + \ \ x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + \ \ x_3 = 4 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2-7R_3]{R_1+3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longrightarrow \begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = 2 \end{cases}$$

2.8 Matrix Properties, Equations, and Inverses

2.8.1 With Real Numbers

- If $ab = bc$, then $a = c$, if $b \neq 0$
- If $ab = 0$, then $a = 0$ or $b = 0$, or both

2.8.2 With Matrices

- If $AB = AC$, then $B = C$, if A is invertible
- If $AB = [0]$, then $A = [0]$ or $B = [0]$, or both

2.8.2.1 Multiply:

$$\begin{aligned} &\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(5) + 3(-3) & 2(-3) + 3(2) \\ 3(5) + 5(-3) & 3(-3) + 5(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

2.8.3 Matrix Inverses

- If a matrix has an inverse, it is said to be invertible or non-singular.
- If a matrix does not have an inverse, it is said to be singular.
- Every square matrix has a "special number" associated with it called the **determinant**.
- For the 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $ad - bc$
- $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- When $\det A = 0$, the matrix is singular and has no inverse (since you cannot divide by zero)

Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{(4)(2) - (3)(1)}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

3 Chapter 2: Determinants

3.1 Prior Knowledge:

$\begin{bmatrix} 10 & -4 \\ -3 & -5 \end{bmatrix} = -50- = -62$

$$\begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$$
$$= ((2 \cdot 2 \cdot -2) + (4 \cdot 3 \cdot 3) + (3 \cdot -1 \cdot 0)) - ((3 \cdot 2 \cdot 3) + (0 \cdot 3 \cdot 2) + (-2 \cdot -1 \cdot 4))$$
$$= (-8 + 36 + 0) - (18 + 0 + 8)$$
$$= 28 - 26$$
$$= 2$$

3.2 Minors & Cofactors

Given a square matrix A, the minor of matrix element a_{ij}, (M_{ij}) is the determinant of the matrix formed by removing the i^{th} row and j^{th} column from matrix A.

The cofactor of matrix element $a_{ij}, C_{ij} = (-1)^{i+j} \cdot M_{ij}$

3.2.1 Example

Let $\det \begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$. What is the cofactor of element (1, 1)?

Cofactor checkerboard:

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$M_{11} = \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} = -4$

$C_{11} = 1 \cdot -4 = -4$

Find the minor and cofactor of: \ a) $a_{21} = -1$

$$M_{21} = \begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} = -8$$
$$C_{21} = 8$$

b) $a_{33} = -2$

$$M_{33} = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 8$$
$$C_{33} = 8$$

3.3 Cofactor Expansion

1) Pick a row or column
2) Multiply every entry in that row or column by it's corresponding cofactor
3) Add those together. That's it

$A = \begin{bmatrix} 6 & 7 & -1 \\ 0 & 4 & 1 \\ 2 & 5 & -3 \end{bmatrix}$

$$\begin{aligned} \det(A) &= 6\left(\begin{vmatrix} 4 & 1 \\ 5 & -3 \end{vmatrix}\right) + 7\left(-\begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix}\right) + -1\left(\begin{vmatrix} 0 & 4 \\ 2 & 5 \end{vmatrix}\right) \\ &= 6(-17) + 7(2) + (-1(-8)) \\ &= -102 + 14 + 8 \\ &= -80 \end{aligned}$$

3.3.1 Example

$A = \begin{bmatrix} 6 & 4 & 2 \\ 5 & -6 & 1 \\ 0 & 3 & 0 \end{bmatrix}$

$$\begin{aligned} &6\begin{vmatrix} -6 & 1 \\ 3 & 0 \end{vmatrix} + 4\left(-\begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix}\right) + 2\begin{vmatrix} 5 & -6 \\ 0 & 3 \end{vmatrix} \\ &= 6(-3) + 0 + 2(15) \\ &= -18 + 30 \\ &= 12 \end{aligned}$$

3.3.2 Does the method generalize to 2×2 matrices?

$$\begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix}$$
$$= 3|2| - 5|7|$$
$$= 6 - 35$$
$$= -29$$

The determinant of a 1×1 matrix is... **itself!**

3.3.3 Find the determinant of a 4×4

$$A = \begin{bmatrix} -3 & 2 & 0 & 8 \\ 2 & 1 & 0 & -4 \\ 5 & -2 & 1 & 5 \\ 2 & 3 & 0 & 6 \end{bmatrix}$$

$$= 0 + 0 + \begin{vmatrix} -3 & 2 & 8 \\ 2 & 1 & -4 \\ 2 & 3 & 6 \end{vmatrix} + 0$$
$$= -2\begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} -3 & 8 \\ 2 & 6 \end{vmatrix} - \left(-4\begin{vmatrix} -3 & 2 \\ 2 & 3 \end{vmatrix}\right)$$
$$= 24 - 34 - 52$$
$$= -62$$

3.4 Theorem

If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the elements in that row or column by their corresponding cofactors is **always the same** and is called the determinant of A.

3.4.1 Example

Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$
\begin{aligned}
& 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\
& = \left(-2 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\right) \\
& = -6
\end{aligned}
$$

3.5 Triangular Matrices

Find the determinant of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$
\begin{aligned}
& \begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} \\
& = 2 \begin{vmatrix} 3 & 3 \\ 0 & 4 \end{vmatrix} \\
& = 2(3 \cdot 4) \\
& = 2 \cdot 12 \\
& = 24
\end{aligned}
$$

If A is an $n \times n$ triangular matrix, then $\det(A)$ is equal to the product of the elements along the main diagonal.

3.6 An Important Definition

Elementary Matrix a matrix that can be obtanied from the $n \times n$ identity matrix by performing a single row operation. \

Are the following matrices elementary? 1) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} + (R_3 + 5R_1)$ yes 2) $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix} + (R_1 + 5R_2)$... no

3.7 A Pair of Theorems

3.7.1 Theorem: If a square matrix A has a row of column of zeros, then $\det(A) = 0$

3.7.2 Theorem: If A is a square matrix, then $\det(A) = \det(A^T)$

3.8 Unit 1 & 2 Homework Problems

3.8.1 "Gaussian Elimination" (08/11/2023)

3.8.1.1 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$\xrightarrow{R_3+10R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] \xrightarrow{-\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\therefore \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ \phantom{x_1 + {}} x_2 - 5x_3 = -9 \\ x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

3.8.1.2 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ -2x_1 - 3x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + 4x_3 = 0 \end{cases}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 4 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{7}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & \frac{2}{7} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\therefore \begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ \phantom{x_1 - {}} x_2 + \frac{2}{7}x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow 1 \neq 0 \therefore \text{no solution}$$

3.8.2 "Inverses and Determinants" (08/14)

3.8.2.1 Find the determinants of the following:

1) $\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

$$\begin{vmatrix} 2 & -3 \\ 4 & 4 \end{vmatrix} = 2(4) - (-3)(4) = 8 + 12 = 20$$

2) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2(3) - 0(0) = 6$$

3) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

3.8.2.2 Find the INVERSES of those matrices:

1) $\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

$$\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}^{-1} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

2) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

3) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3.8.3 Inverses and Determinants (08/15)

3.8.3.1 Use a matrix equation to solve the following problems:

1) $\begin{cases} 3x_1 - 2x_2 = 1 \\ 4x_1 + 5x_2 = 3 \end{cases}$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{23} \\ \frac{9}{23} \end{bmatrix}$$

2) $\begin{cases} 6x_1 + x_2 = 0 \\ 4x_1 - 3x_2 = -2 \end{cases}$

$$\Rightarrow \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} -3 & -1 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} \\ -\frac{4}{11} \end{bmatrix}$$

3.8.4 Consistent Systems (08/21)

3.8.4.1 Solve the linear systems together by reducing the appropriate augmented matrix.

$$\begin{cases} x_1 - 5x_2 = b_1 \\ 3x_1 + 2x_2 = b_2 \end{cases}$$

1) $b_1 = 1,\ b_2 = 4$
2) $b_1 = -2,\ b_2 = 5$

First, let's solve it for the general case:

$$\left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 3 & 2 & b_2 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 17 & b_2 - 3b_1 \end{array}\right] \xrightarrow{\frac{1}{17}R_2} \left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 1 & \frac{b_2 - 3b_1}{17} \end{array}\right] \xrightarrow{R_1 + 5R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{2b_1 + 5b_2}{17} \\ 0 & 1 & \frac{-3b_1 + b_2}{17} \end{array}\right]$$

Therefore, the solution to the general case is $(x_1, x_2) = (\frac{2b_1+5b_2}{17}, \frac{-3b_1+b_2}{17})$

And so, for the specific cases:

1) $(x_1, x_2) = \left(\frac{2(1)+5(4)}{17}, \frac{-3(1)+4}{17}\right) = \left(\frac{13}{17}, \frac{1}{17}\right)$
2) $(x_1, x_2) = \left(\frac{2(-2)+5(5)}{17}, \frac{-3(-2)+5}{17}\right) = \left(\frac{16}{17}, \frac{11}{17}\right)$

3.8.4.2 Determine the conditions on b, if any, in order to guarantee that the linear system is consistent.

$$\begin{cases} x_1 + 3x_2 = b_1 \\ -2x_1 + x_2 = b_2 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & b_2 + 2b_1 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{b_1-3b_2}{7} \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array}\right]$$

There are no conditions. The system is consistent for all values of b_1 and b_2.

3.8.5 Another "determining the conditions" problem:

$$\begin{cases} x_1 - 2x_2 - x_3 = b_1 \\ -4x_1 + 5x_2 + 2x_3 = b_2 \\ -4x_1 + 7x_2 + 4x_3 = b_3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right] \xrightarrow[R_3+4R_1]{R_2+4R_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & b_2 + 4b_1 \\ 0 & -1 & 0 & b_3 + 4b_1 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{b_3+4b_1}{3} \end{array}\right]$$

$$\xrightarrow{-\frac{3}{2}R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & 1 & \frac{-b_3-4b_1}{2} \end{array}\right]$$

Therefore, the system is consistent for all values of b_1, b_2, and b_3.

3.8.6 Triangular and Diagonal Matrices

3.8.6.1 Find A^2

1) $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 0(0) & 1(0) + 0(-2) \\ 0(1) + (-2)(0) & 0(0) + (-2)(-2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

2) $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$\begin{aligned} A^2 &= \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (-6)(-6) + (0)(0) + (0)(0) & (-6)(0) + (0)(3) + (0)(0) & (-6)(0) + (0)(0) + (0)(5) \\ (0)(-6) + (3)(0) + (0)(0) & (0)(0) + (3)(3) + (0)(0) & (0)(0) + (3)(0) + (0)(5) \\ (0)(-6) + (0)(0) + (5)(0) & (0)(0) + (0)(3) + (5)(0) & (0)(0) + (0)(0) + (5)(5) \end{bmatrix} \\ &= \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} \end{aligned}$$

3.8.6.2 Find A^{-k}, such that k is some nonzero constant

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} A^{-k} &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-k} \\ &= \begin{bmatrix} 2^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & 2^{-k} \end{bmatrix} \end{aligned}$$

4. Determine whether each matrix is symmetric or not.

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

Not symmetric

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

3.8.6.3 Find a diagonal matrix A that satisfies the given condition

1) $A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{\frac{1}{5}} \\ &= \begin{bmatrix} 1^{\frac{1}{5}} & 0 & 0 \\ 0 & (-1)^{\frac{1}{5}} & 0 \\ 0 & 0 & (-1)^{\frac{1}{5}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

2) $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} A &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-\frac{1}{2}} \\ &= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 & 0 \\ 0 & 4^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1^{-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3.8.7 Determinants and Triangular Matrices (08/29)

3.8.7.1 What is C_{32}

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} C_{32} &= (-1)^{3+2} \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \left(2 \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 3 \\ 3 & 0 \end{vmatrix} + 1 \begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} \right) \\ &= -(2(-3) - (-1)(-9) + 1(-3)) \\ &= -(-6 + 9 - 3) \\ &= 0 \end{aligned}$$

3.8.7.2 Find all values of λ such that $|A| = 0$

$$A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (\lambda - 2)(\lambda + 4) - (-5)(1) \\ &= \lambda^2 + 2\lambda - 8 + 5 \\ &= \lambda^2 + 2\lambda - 3 \\ &= (\lambda + 3)(\lambda - 1) \\ &= 0 \end{aligned}$$

Therefore, $\lambda = -3, 1$

3.8.7.3 For the matrix $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$ find the determinant 3 different ways with cofactor expansion. Pick different rows and columns each time.

$$
\begin{aligned}
\det(A) &= 3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0\begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} \\
&= 3(-1(-4) - 5(9)) - 0(2(-4) - 5(1)) + 0(2(9) - (-1)(1)) \\
&= 3(4 - 45) - 0(-8 - 5) + 0(18 + 1) \\
&= 3(-41) - 0(-13) + 0(19) \\
&= 36
\end{aligned}
$$

$$
\begin{aligned}
\det(A) &= 0\begin{vmatrix} 2 & 5 \\ 9 & -4 \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 0\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} \\
&= 0(2(-4) - 5(9)) - 3(3(-4) - 0(1)) + 0(3(5) - 0(2)) \\
&= 0(-8 - 45) - 3(-12 - 0) + 0(15 - 0) \\
&= 0(-53) - 3(-12) \\
&= 36
\end{aligned}
$$

$$
\begin{aligned}
\det(A) &= 0\begin{vmatrix} 2 & -1 \\ 9 & -4 \end{vmatrix} - 0\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 3\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \\
&= 0(2(-4) - (-1)(9)) - 0(3(-4) - 0(1)) + 3(3(-1) - 0(2)) \\
&= 0(-8 + 9) - 0(-12 - 0) + 3(-3 - 0) \\
&= 0(1) - 0(-12) + 3(-3) \\
&= 0 + 0 - 9 \\
&= 36
\end{aligned}
$$

3.8.7.4 Evaluate $\det(A)$ **by a cofactor expansion along a row or column of your choice**

$$A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - k\begin{vmatrix} 1 & k^2 \\ 1 & k^2 \end{vmatrix} + k^2\begin{vmatrix} 1 & k \\ 1 & k \end{vmatrix} \\ &= 1(k^2 - k^2) - k(1(k^2) - k^2(1)) + k^2(1(k) - k(1)) \\ &= 0 \end{aligned}$$

3.8.7.5 Evaluate the determinant of the following matrices by just looking at them.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1(-1)(1) = -1$$

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det(A) = 1(1)(2)(3) = 6$$

3.8.7.6 Show that the value of the determinant is independent of θ

$$A = \begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

$$\det(A) = \sin\theta\begin{vmatrix} \sin\theta & 0 \\ \sin\theta + \cos\theta & 1 \end{vmatrix} - \cos\theta\begin{vmatrix} \cos\theta & 0 \\ \sin\theta + \cos\theta & 1 \end{vmatrix}$$

$$+0\begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta + \cos\theta & \sin\theta \end{vmatrix}$$

$$= \sin\theta\,(\sin\theta(1) - 0(\sin\theta + \cos\theta)) - \cos\theta\,(\cos\theta(1) - 0(\sin\theta + \cos\theta))$$

$$+0\,(\cos\theta(\sin\theta) - \sin\theta(\sin\theta + \cos\theta))$$

$$= \sin^2\theta - \cos^2\theta$$

$$= 1$$

3.8.8 Row operations and Determinants (08/31)

3.8.8.1 Find the determinant of $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$ **WITHOUT using cofactor expansion**

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{28}{2} \end{vmatrix} \\ &= 1(-2)\left(\frac{28}{2}\right) \\ &= -28 \end{aligned}$$

3.8.8.2 Find the determinant of $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -3 & 2 \end{vmatrix} \\
&= 2(-2)(-4)(2) \\
&= 64
\end{aligned}$$

3.8.9 Adjoints and Cramer's Rule (09/05)

3.8.9.1 Find the inverse of $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ **using the adjoint method**

$$\begin{aligned}
\det(A) &= 2\begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} - 5\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} + 5\begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\
&= 2(-3) - 5(-3) + 5(-2) \\
&= -6 + 15 - 10 \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\operatorname{adj}(A) &= \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 5 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} \end{bmatrix} \\
&= \begin{bmatrix} (-1)(3) & -(-1)(3) & -4+2 \\ -(15-20) & 6-10 & -(8-10) \\ 5 & -5 & -2+5 \end{bmatrix}^T \\
&= \begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}^T \\
&= \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore A^{-1} &= -\begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}
\end{aligned}$$

3.8.9.2 Solve the following system of equations using Cramer's Rule

$$\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases} \longrightarrow \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \longrightarrow 4\begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5\begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} = -132$$

$$\begin{aligned}
\det(x) &= \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \\
&= 2\begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\
&= 2(2 - 10) - 5(6 - 2) \\
&= -16 - 20 \\
&= -36 \\
\det(y) &= \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \\
&= 4\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} \\
&= 4(6 - 2) - 2(22 - 2) \\
&= 16 - 40 \\
&= -24 \\
\det(z) &= \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \\
&= 4\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 5\begin{vmatrix} 11 & 3 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 11 & 1 \\ 1 & 5 \end{vmatrix} \\
&= 4(1 - 15) - 5(33 - 3) + 2(55 - 1) \\
&= -56 - 150 + 108 \\
&= -98
\end{aligned}$$

Therefore, the solution $(x, y, z) = (\frac{3}{11}, \frac{2}{11}, -\frac{49}{66})$

4 Chapter 5: Eigenvectors and Eigenvalues

4.1 Eigenvalues and Eigenvectors (11/06)

If A is an $n \times n$ matrix, then a non-zero vector **x**, in R^n, is called an eigenvector of A if $A\mathbf{x}$ is a scalar multiple of **x**; that is $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. This scalar λ is called an eigenvalue of A and **x** is said to be an eigenvector corresponding to λ.

See, normally, multiplying a vector by a square matrix changes both the magnitude and the direction of the vector. Really screws it up.

Some examples:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$$

$$\begin{bmatrix}5 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$$

$$\begin{bmatrix}7 & 8\\-2 & 3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}23\\4\end{bmatrix}$$

However, there are some ways to get consistent results.

4.1.1 Examples

4.1.1.1 $\vec{x} = \begin{bmatrix}2\\1\end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix}3 & -2\\1 & 0\end{bmatrix}$ because

$$A\vec{x} = \begin{bmatrix}3 & -2\\1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix} = 2\begin{bmatrix}2\\1\end{bmatrix} = 2\vec{x} \therefore \lambda = 2$$

4.1.1.2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$**. Are** $\vec{u}$ **and** $\vec{v}$ **eigenvectors of** A**?**

$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 1(6) + 6(-5) \\ 5(6) + 2(-5) \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} \therefore \lambda = -4$

$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 6(-2) \\ 5(3) + 2(-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\vec{v}$

4.2 Eigenvector Homework Problem (11/06)

Confirm by multiplication that x is an eigenvector of A, and find the corresponding eigenvalue.

4.2.1 $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$A\mathbf{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1) + 0(2) + 1(1) \\ 2(1) + 3(2) + 2(1) \\ 1(1) + 0(2) + 4(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \therefore \lambda = 5$

4.3 Finding Eigenvalues and Eigenvectors (11/07)

Essential question:

If we know an n × n matrix A, can we find its $\boldsymbol{\lambda}$**?**

If $A\vec{x} = \lambda\vec{x}$, then:

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A\vec{x} - \lambda\vec{x} &= \vec{0} \\ (A - \lambda I)\vec{x} &= \vec{0} \end{aligned}$$

This equation is familiar. It's the homogeneous system of equations $A\vec{x} = \vec{0}$, the solution of which is the nullspace of $A - \lambda I$. Therefore, $\vec{x}$ is an eigenvector of $A \iff \vec{x}$ is in the nullspace of $A - \lambda I$.

In this situation, what do we know about that matrix?

Everything in the equivalent statements is false because $\vec{x}$ cannot be the zero vector. Therefore, we can see that $\det(A - \lambda I)$ OR $\det(\lambda I - A)$ MUST be 0.

Big Idea: If A is an $n \times n$ matrix, then λ is an eigenvalue of $A \iff \det(\lambda I - A) = 0$. This is called the characteristic equation of A.

4.3.1 Find the characteristic equation and the eigenvalues of $A = \begin{bmatrix} 3 & 0 & 5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 3 & 0 & 5 \\ -\frac{1}{5} & \lambda + 1 & 0 \\ -1 & -1 & \lambda + 2 \end{vmatrix} = 0$$

$$0 = (\lambda - 3)((\lambda + 1)(\lambda + 2)) + 5(\frac{1}{5} + \lambda + 1)$$

$$0 = (\lambda - 3)(\lambda^2 + 3\lambda + 2)$$

$$0 = \lambda^3 - 2\lambda$$

$$0 = \lambda(\lambda^2 - 2)\lambda \qquad = 0, \pm\sqrt{2}$$

4.3.2 Find the characteristic equation and the eigenvalues of $A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

$$\begin{vmatrix} -1 - \lambda & 0 & 1 \\ -1 & 3 - \lambda & 0 \\ -4 & 13 & -\lambda \end{vmatrix} = 0$$

$$(-1 - \lambda)((3 - \lambda)(-\lambda) - 0(13)) + (-1(13) - (3 - \lambda)(-4)) = 0$$

$$(-1 - \lambda)(\lambda^2 - 3\lambda) + (-13 - 4\lambda + 12) = 0$$

$$(-1 - \lambda)(\lambda^2 - 3\lambda) + (-4\lambda - 1) = 0$$

$$-\lambda^3 + 3\lambda^2 + 2 = 0$$

$$(-\lambda + 2)(-\lambda^2 - \lambda - 1) = 0$$

$$(-\lambda + 2)(-\lambda - 1)(-\lambda + 1) = 0$$

$$\lambda = 2$$

4.3.3 Find the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 2 & 0 & 0 \\ 6 & \lambda - 3 & 0 \\ 1 & 4 & \lambda - 5 \end{vmatrix} = 0$$
$$(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$$
$$\lambda = 2, 3, 5$$

Theorem 1: For a triangular matrix, the eigenvalues are the elements on the main diagonal.

4.3.4 Find the eigenvalues of A^3 if $A = \begin{bmatrix} \frac{1}{2} & 4 & 5 & -2 \\ 0 & -1 & 3 & -8 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$\lambda_A = \frac{1}{2}, -1, 2, 4$$
$$\lambda_{A^3} = \frac{1}{8}, -1, 8, 64$$

Theorem 2: The eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots$

4.3.5 Give me a matrix with eigenvalues $\lambda = 0, 2, 5$

$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 5 \end{bmatrix}$

Theorem 3: A square matrix A is invertible $\iff$ $\lambda \neq 0$ (which also means its determinant is 0).

4.3.6 Finding eigenvectors!

Find the nontrivial eigenvectors of:

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 1 & -6 \\ -5 & \lambda - 2 \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda - 2) - (-6)(-5) = 0$$

$$\lambda^2 - 3\lambda - 28 = 0$$

$$(\lambda - 7)(\lambda + 4) = 0$$

$$\lambda = 7, -4$$

Substitute each λ, one at a time into the $\lambda I - A$ matrix and find the null space.

For $\lambda = -4$:

$$\left(\begin{array}{cc|c} -5 & -6 & 0 \\ -5 & -6 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} -5 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\langle -\frac{6}{5}t, t \rangle$$

$$\vec{x} = \{\langle -6, 5 \rangle\}$$

For $\lambda = 7$:

$$\left(\begin{array}{cc|c} 6 & -6 & 0 \\ -5 & 5 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\langle t, t \rangle$$

$$\vec{x} = \{\langle 6, 6 \rangle\}$$

Therefore, the eigen space is: $\{\langle -6, 5 \rangle, \langle 6, 6 \rangle\}$

4.4 Diagonalization and Similar Triangles

Similar matrices: If A and D are square matrices, we say that A and D are “similar” if there exists an invertible matrix P such that:

$D = P^{-1}AP$.

4.4.1 Properties of Similar Matricces

- They have the same determinant
- If one is invertible, so is the other
- They have the same trace
- They have the same characteristic polynomial
- They have the same eigenvalues

4.4.2 Procedure

1. Find the eigenvectors for the $n \times n$ matrix A.

- Theorem: If an $n \times n$ matrix A has n distinct eigenvalues, then A is **for sure** diagonalizable.

2. Make matrix P out of the eigenveectors (P is the matrix that diagonalizes A)
3. Check your work to find matrix D if reasonable

4.4.3 Example: Find a matrix P that diagonalizes A and compute $P^{-1}AP$

1. $A = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$

Find the eigenvalues:

$$\begin{aligned} \begin{bmatrix} \lambda - 3 & -7 \\ -5 & \lambda - 5 \end{bmatrix} &= 0 \\ (\lambda - 3)(\lambda - 5) - (-7)(-5) &= 0 \\ \lambda^2 - 5\lambda - 3\lambda + 15 - 35 &= 0 \\ \lambda^2 - 8\lambda - 20 &= 0 \\ \lambda &= -2, 10 \end{aligned}$$

Find the eigenvectors:

$$\lambda = -2 : \left[\begin{array}{cc|c} -5 & -7 & 0 \\ -5 & -7 & 0 \end{array}\right] \vec{x} = \{\langle -7, 5 \rangle\}$$

$$\lambda = 10 : \left[\begin{array}{cc|c} -7 & -7 & 0 \\ -5 & 5 & 0 \end{array}\right] \vec{x} = \{\langle 1, 1 \rangle\}$$

Create the matrix P:

$P = \begin{bmatrix} -7 & 1 \\ 5 & 1 \end{bmatrix}$

Find matrix D:

$$\begin{aligned} D &= P^{-1}AP \\ &= \begin{bmatrix} 7 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & 10 \end{bmatrix} \end{aligned}$$

4.4.4 Conclusion

- If D has the same eigenvalues of A and if D must be diagonal, then D is **THE** diagonal matrix with eigenvalues of A on the diagonal.

2. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

First, find D:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, find P:

$$\lambda = 2 : \left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \vec{x} = \langle 1, 0, 0 \rangle$$

$$\lambda = 3 : \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \vec{x} = \langle 0, 1, 0 \rangle$$

$$\lambda = 1 : \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \vec{x} = \langle 2, 0, 1 \rangle$$

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.5 More on Similar Matrices

There are a few more properties of similar matrices:

- They have the same rank (non-zero eigenvalues)
- They have the same nullity
- They have the same column space
- They have the same row space

4.5.1 Example

**Matrix A is similar to the following matrix:

$$D = \begin{bmatrix} 3 & -1 & 1 & 4 & 5 & 2 \\ 0 & -3 & 5 & -10 & -16 & 1 \\ 0 & 0 & 5 & 7 & -8 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A: 4

Nullity of A: 2

Eigenvalues: 3, -3, 5, 2, 0, 0

Characteristic Polynomial:

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 3 & 1 & -1 & -4 & -5 & -2 \\ 0 & \lambda + 3 & -5 & 10 & 16 & -1 \\ 0 & 0 & \lambda - 5 & -7 & 8 & -2 \\ 0 & 0 & 0 & \lambda - 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} = 0$$

$$(\lambda - 3)(\lambda + 3)(\lambda - 5)(\lambda - 2)\lambda^2 = 0$$

4.5.2 Some review

- Eigenspace of λ: The nullspace of $\lambda I - A$. Each eigenvalue will have its own eigenspace.
- Algebraic multiplicty: The number of times a given λ appears as a root of the characteristic equation.
- Geometric multiplicity: The number of eigenvectors it maps to.

4.5.2.1 Theorem: Geometric and Algebraic Multiplicity

If A is a square matrix, then: a. For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity. b. A is diagonalizable $\iff$ the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity.