

Chapter 1: Systems of Linear Equations and Matrices

Gwinnett School of Math, Science, and Technology

Multivariable Calculus Yearlong Notes

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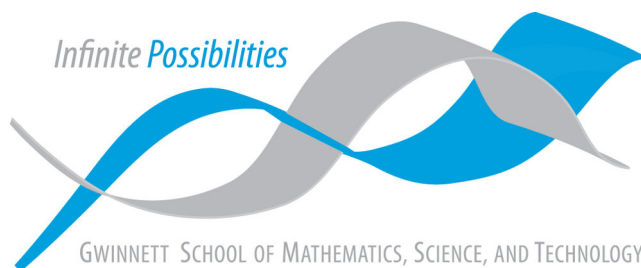


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0.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal \iff they have the same dimensions and values

0.1.1 Addition & Subtraction

Two matrices can be added/subtracted \iff they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

0.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3 \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

0.1.3 Matrix Multiplication

- We can **only** multiply an $(m \times n)$ by $(n \times p)$ matrix.
- The resulting matrix will be $(m \times p)$

0.1.4 Properties of Matrix Arithmetic

- (a) $A + B = B + A$ (**Commutative law for addition**)
- (b) $A + (B + C) = (A + B) + C$ (**Associative law for addition**)
- (c) $A(BC) = (AB)C$ (**Associative law for multiplication**)
- (d) $A(B + C) = AB + AC$ (**Left distributive law**)

$$(e) (B + C)A = BA + CA \text{ (Right distributive law)}$$

$$(f) A(B - C) = AB - AC$$

$$(g) (B - C)A = BA - CA$$

$$(h) a(B+C) = aB + aC$$

$$(i) a(B-C) = aB - aC$$

$$(j) (a+b)C = aC + bC$$

$$(k) (a-b)C = aC - bC$$

$$(l) a(bC) = (ab)C$$

$$(m) a(BC) = (aB)C = B(aC)$$

0.1.5 Examples

1.

$$\begin{aligned} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \end{aligned}$$

2.

$$\begin{aligned} & \begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix} \end{aligned}$$

3.

$$\begin{aligned} & \begin{bmatrix} 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} \\ &= [4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2] \\ &= [30] \end{aligned}$$

0.2 Transpose of a Matrix

The transpose of an $(m \times n)$ matrix is the $(n \times m)$ matrix where the rows and columns are swapped.

$$\text{If } B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix} \end{aligned}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix A is equal to the sum of all the elements on the main diagonal: $\text{tr}(A)$

0.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

0.3 Homework — “Matrix Stuff” (08/03/2023)

0.3.1 Suppose that A, B, C, D and E are matrices with the following sizes:

A	B	C	D	E
(3×2)	(2×3)	(3×3)	(3×2)	(2×3)

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

Undefined	Defined; (4×2)	Defined; (5×5)	Defined; (5×2)
BA	$AC + D$	$E(A + B)$	$(A^T + E)D$
$AB + B$			$E(AC)$
$E^T A$			
$AE + B$			

0.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2. $2A^T + C$

$$\begin{aligned} 2A^T + C &= 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2 \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3. $B^T + 5C^T$

$$\begin{aligned}
 B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\
 &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\
 &= \text{Undefined}
 \end{aligned}$$

4. $2E^T - 3D^T$

$$\begin{aligned}
 2E^T - 3D^T &= 2 \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3 \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T \\
 &= 2 \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}
 \end{aligned}$$

5. $\text{tr}(DE)$

$$\begin{aligned}
 \text{tr}(DE) &= \text{tr} \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right) \\
 &= \text{tr} \left(\begin{bmatrix} 1 \cdot 6 + 5 \cdot (-1) + 2 \cdot 4 & 1 \cdot 1 + 5 \cdot 1 + 2 \cdot 1 & 1 \cdot 3 + 5 \cdot 2 + 2 \cdot 3 \\ (-1) \cdot 6 + 0 \cdot (-1) + 1 \cdot 4 & (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1 & (-1) \cdot 3 + 0 \cdot 2 + 1 \cdot 3 \\ 3 \cdot 6 + 2 \cdot (-1) + 4 \cdot 4 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 & 3 \cdot 3 + 2 \cdot 2 + 4 \cdot 3 \end{bmatrix} \right) \\
 &= \text{tr} \left(\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix} \right) \\
 &= 34
 \end{aligned}$$

1 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

- Infinite solutions
- One solution
- No solutions

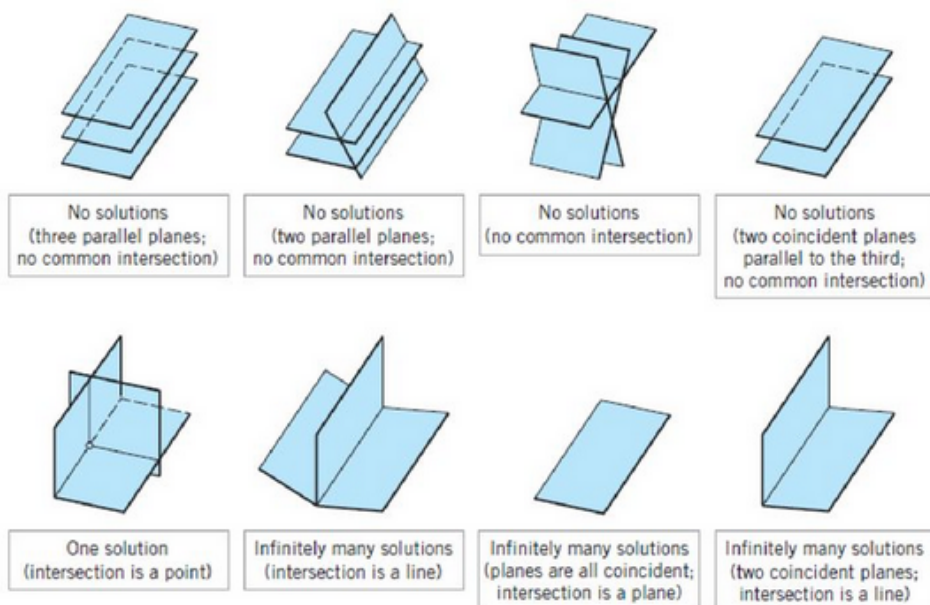
Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear *really* means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.

Linear Systems in Three Unknowns



1.1 Review: Solve the following systems

1.
$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$5x = 15$$

$$x = 3$$

$$2(3) + y = 10$$

$$6 + y = 10$$

$$y = 4$$

2.
$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$y = 10 - 2x$$

$$6x + 3(10 - 2x) = 10$$

$$6x + 30 - 6x = 10$$

$$30 = 10. \therefore \text{no solution}$$

3.
$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$0 = 0$$

$$12 = 12. \therefore \text{infinite solutions}$$

1.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

1.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

1.2 The Augmented Matrix

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array} \right]$$

1.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/subtract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

1.3.1 Example 1... again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\begin{aligned} \left[\begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array} \right] &\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array} \right] \\ &\xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array} \right] \xrightarrow{R_1 - \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array} \right] \end{aligned}$$

And so... $x = 3$ and $y = 4$!

1.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array} \right],$$

then the solution to the system will be the ordered triple (c_1, c_2, c_3) .

1.4.1 Example 2: again

$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{R_2-6R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array} \right]$$

This is inconsistent, so there is no solution.

1.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{R_2-15R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array} \right]$$

Since $0 = 0$, there are infinitely many solutions.

1.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \xrightarrow{R_3+4R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_3+\frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\begin{matrix} R_1+7R_3 \\ R_2+4R_3 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \end{aligned}$$

Therefore the solution to (x_1, x_2, x_3) is $(29, 16, 3)$.

1.4.4 Elementary Row Operations & REF Homework Problem (08/08/2023)

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] \xrightarrow[-R_3]{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array} \right] \\ & \xrightarrow[R_3-10R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array} \right] \xrightarrow{\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow[R_2+5R_3]{R_1-7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

Therefore, the solution to (x, y, z) is $(3, 1, 2)$.

1.5 Gaussian Elimination

Vocabulary: A matrix is in Row Echelon Form (REF) if:

- (a) Any rows of all zeroes are placed at the bottom of the matrix
- (b) All other rows have a leading 1 ("pivot")
- (c) As we move down the matrix, each leading 1 is further to the right than the 1 above it

A matrix is in Row Reduced Echelon Form if the three above conditions are met in addition to:

- (d) Each column with a leading 1 has all other entries in the column as a 0. ("pivot column")

1.5.1 Examples

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF? ✓
RREF? ✓

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF? ✓
RREF? ✗

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

REF? ✗
RREF? ✗

1.6 Gaussian Elimination With Back-Substitution

1.6.1 Goal:

To get the augmented matrix in REF

Solve:
$$\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array} \right] \xrightarrow[R_3+R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

$$x + 9z = 19$$

$$y + 3z = 5$$

$$z = 2$$

$$\therefore z = 2, y = 5 - 3z, x = 19 - 9z$$

$$z = 2, y = 5 - 3(2), x = 19 - 9(2)$$

$$z = 2, y = -1, x = 1$$

Therefore, the solution (x_1, x_2, x_3) is $(1, -1, 2)$.

1.6.2 Gaussian Elimination Homework Problem (08/09/2023)

$$\begin{cases} -2w + y + z = -3 \\ x + 2y - z = 2 \\ -3w + 2x + 4y + z = -2 \\ -w + x - 4y - 7z = -19 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{cccc|c} -2 & 0 & 1 & 1 & -3 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -1 & 1 & -4 & -7 & -19 \end{array} \right] \xrightarrow{R_4} \left[\begin{array}{cccc|c} -1 & 1 & -4 & -7 & -19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array} \right] \xrightarrow{-R_1} \\ & \left[\begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array} \right] \xrightarrow{\begin{matrix} R_3+3R_1 \\ R_4+2R_1 \end{matrix}} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & 16 & 22 & 55 \\ 0 & -2 & 9 & 15 & 35 \end{array} \right] \xrightarrow{\begin{matrix} R_1+R_2 \\ R_3+R_2 \\ R_4+2R_2 \end{matrix}} \\ & \left[\begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 18 & 21 & 57 \\ 0 & 0 & 13 & 13 & 39 \end{array} \right] \xrightarrow{\frac{1}{18}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 13 & 13 & 39 \end{array} \right] \xrightarrow{\begin{matrix} R_1-6R_3 \\ R_2-2R_3 \\ R_4-13R_3 \end{matrix}} \\ & \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & -\frac{13}{6} & -\frac{13}{6} \end{array} \right] \xrightarrow{\frac{-6}{13}R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{matrix} R_1+R_4 \\ R_2+\frac{10}{3}R_4 \\ R_3-\frac{7}{6}R_4 \end{matrix}} \\ & \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \Rightarrow \begin{cases} w = 3 \\ x = -1 \\ y = 2 \\ z = 1 \end{cases} \end{aligned}$$

1.7 Gauss-Jordan Elimination

1.7.1 Goal:

To get the matrix into RREF

$$\text{Solve: } \begin{cases} x_1 - 3x_3 = -2 \\ 3x_1 + x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + x_3 = 4 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array} \right] \xrightarrow[R_3-2R_1]{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array} \right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array} \right] \\ & \xrightarrow{\frac{-1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow[R_2-7R_3]{R_1+3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = 2 \end{cases} \end{aligned}$$

1.8 Matrix Properties, Equations, and Inverses

1.8.1 With Real Numbers

- If $ab = bc$, then $a = c$, if $b \neq 0$
- If $ab = 0$, then $a = 0$ or $b = 0$, or both

1.8.2 With Matrices

- If $AB = AC$, then $B = C$, if A is invertible
- If $AB = [0]$, then $A = [0]$ or $B = [0]$, or both

1.8.2.1 Multiply:

$$\begin{aligned} & \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(5) + 3(-3) & 2(-3) + 3(2) \\ 3(5) + 5(-3) & 3(-3) + 5(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

1.8.3 Matrix Inverses

- If a matrix has an inverse, it is said to be invertible or non-singular.
- If a matrix does not have an inverse, it is said to be singular.
- Every square matrix has a “special number” associated with it called the **determinant**.
- For the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $ad - bc$
- $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- When $\det A = 0$, the matrix is singular and has no inverse (since you cannot divide by zero)

Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}^{-1} &= \frac{1}{\det A} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{(4)(2) - (3)(1)} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix} \end{aligned}$$

2 Chapter 2: Determinants

2.1 Prior Knowledge:

$$\begin{bmatrix} 10 & -4 \\ -3 & -5 \end{bmatrix} = -50 - = -62$$

$$\begin{aligned} & \begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix} \\ & = ((2 \cdot 2 \cdot -2) + (4 \cdot 3 \cdot 3) + (3 \cdot -1 \cdot 0)) - ((3 \cdot 2 \cdot 3) + (0 \cdot 3 \cdot 2) + (-2 \cdot -1 \cdot 4)) \\ & = (-8 + 36 + 0) - (18 + 0 + 8) \\ & = 28 - 26 \\ & = 2 \end{aligned}$$

2.2 Minors & Cofactors

Given a square matrix A, the minor of matrix element a_{ij} , (M_{ij}) is the determinant of the matrix formed by removing the i^{th} row and j^{th} column from matrix A.

The cofactor of matrix element a_{ij} , $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

2.2.1 Example

Let $\det \begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$. What is the cofactor of element (1, 1)?

Cofactor checkerboard:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} = -4$$

$$C_{11} = 1 \cdot -4 = -4$$

Find the minor and cofactor of: \ a) $a_{21} = -1$

$$M_{21} = \begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} = -8$$

$$C_{21} = 8$$

b) $a_{33} = -2$

$$M_{33} = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 8$$

$$C_{33} = 8$$

2.3 Cofactor Expansion

- 1) Pick a row or column
- 2) Multiply every entry in that row or column by it's corresponding cofactor
- 3) Add those together. That's it

$$A = \begin{bmatrix} 6 & 7 & -1 \\ 0 & 4 & 1 \\ 2 & 5 & -3 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 6 \begin{vmatrix} 4 & 1 \\ 5 & -3 \end{vmatrix} + 7 \left(- \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} \right) + -1 \begin{vmatrix} 0 & 4 \\ 2 & 5 \end{vmatrix} \\ &= 6(-17) + 7(2) + (-1(-8)) \\ &= -102 + 14 + 8 \\ &= -80 \end{aligned}$$

2.3.1 Example

$$A = \begin{bmatrix} 6 & 4 & 2 \\ 5 & -6 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

$$\begin{aligned} &6 \begin{vmatrix} -6 & 1 \\ 3 & 0 \end{vmatrix} + 4 \left(- \begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix} \right) + 2 \begin{vmatrix} 5 & -6 \\ 0 & 3 \end{vmatrix} \\ &= 6(-3) + 0 + 2(15) \\ &= -18 + 30 \\ &= 12 \end{aligned}$$

2.3.2 Does the method generalize to 2×2 matrices?

$$\begin{aligned} & \begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix} \\ &= 3|2| - 5|7| \\ &= 6 - 35 \\ &= -29 \end{aligned}$$

The determinant of a 1×1 matrix is... **itself!**

2.3.3 Find the determinant of a 4×4

$$A = \begin{bmatrix} -3 & 2 & 0 & 8 \\ 2 & 1 & 0 & -4 \\ 5 & -2 & 1 & 5 \\ 2 & 3 & 0 & 6 \end{bmatrix}$$

$$\begin{aligned} &= 0 + 0 + \begin{vmatrix} -3 & 2 & 8 \\ 2 & 1 & -4 \\ 2 & 3 & 6 \end{vmatrix} + 0 \\ &= -2 \begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} -3 & 8 \\ 2 & 6 \end{vmatrix} - \left(-4 \begin{vmatrix} -3 & 2 \\ 2 & 3 \end{vmatrix} \right) \\ &= 24 - 34 - 52 \\ &= -62 \end{aligned}$$

2.4 Theorem

If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the elements in that row or column by their corresponding cofactors is **always the same** and is called the determinant of A .

2.4.1 Example

Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} & 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\ &= (-2) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ &= -6 \end{aligned}$$

2.5 Triangular Matrices

Find the determinant of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$\begin{aligned} & \begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} \\ &= 2 \begin{vmatrix} 3 & 3 \\ 0 & 4 \end{vmatrix} \\ &= 2(3 \cdot 4) \\ &= 2 \cdot 12 \\ &= 24 \end{aligned}$$

If A is an $n \times n$ triangular matrix, then $\det(A)$ is equal to the product of the elements along the main diagonal.

2.6 An Important Definition

Elementary Matrix a matrix that can be obtained from the $n \times n$ identity matrix by performing a single row operation. \

Are the following matrices elementary? 1) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} + (R_3 + 5R_1)$ yes 2) $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix} + (R_1 + 5R_2)$...
no

2.7 A Pair of Theorems

2.7.1 Theorem: If a square matrix A has a row of column of zeros, then $\det(A) = 0$

2.7.2 Theorem: If A is a square matrix, then $\det(A) = \det(A^T)$

2.8 Unit 1 & 2 Homework Problems

2.8.1 "Gaussian Elimination" (08/11/2023)

2.8.1.1 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \xrightarrow[\substack{R_2+R_1 \\ R_3-3R_1}]{\substack{R_2+R_1 \\ R_3-3R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array} \right]$$

$$\xrightarrow{R_3+10R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array} \right] \xrightarrow{-\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\therefore \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ x_2 - 5x_3 = -9 \\ x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

2.8.1.2 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ -2x_1 - 3x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + 4x_3 = 0 \end{cases}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 4 & 0 \end{array} \right] \xrightarrow[\substack{R_2+2R_1 \\ R_3-2R_1}]{\substack{R_2+2R_1 \\ R_3-2R_1}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right] \xrightarrow[\substack{-\frac{1}{7}R_2 \\ -\frac{1}{2}R_3}]{\substack{-\frac{1}{7}R_2 \\ -\frac{1}{2}R_3}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & \frac{2}{7} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\therefore \begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ x_2 + \frac{2}{7}x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow 1 \neq 0 \therefore \text{no solution}$$

2.8.2 "Inverses and Determinants" (08/14)

2.8.2.1 Find the determinants of the following:

$$1) \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -3 \\ 4 & 4 \end{vmatrix} = 2(4) - (-3)(4) = 8 + 12 = 20$$

$$2) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2(3) - 0(0) = 6$$

$$3) \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta + \sin^2 \theta = 1$$

2.8.2.2 Find the INVERSES of those matrices:

$$1) \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}^{-1} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

$$2) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$3) \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

2.8.3 Inverses and Determinants (08/15)

2.8.3.1 Use a matrix equation to solve the following problems:

$$1) \begin{cases} 3x_1 - 2x_2 = 1 \\ 4x_1 + 5x_2 = 3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{23} \\ \frac{9}{23} \end{bmatrix}$$

$$2) \begin{cases} 6x_1 + x_2 = 0 \\ 4x_1 - 3x_2 = -2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} -3 & -1 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} \\ \frac{4}{11} \end{bmatrix}$$

2.8.4 Consistent Systems (08/21)

2.8.4.1 Solve the linear systems together by reducing the appropriate augmented matrix.

$$\begin{cases} x_1 - 5x_2 = b_1 \\ 3x_1 + 2x_2 = b_2 \end{cases}$$

1) $b_1 = 1, b_2 = 4$

2) $b_1 = -2, b_2 = 5$

First, let's solve it for the general case:

$$\left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 3 & 2 & b_2 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 17 & b_2 - 3b_1 \end{array} \right] \xrightarrow{\frac{1}{17}R_2} \left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 1 & \frac{b_2 - 3b_1}{17} \end{array} \right] \xrightarrow{R_1 + 5R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{2b_1 + 5b_2}{17} \\ 0 & 1 & \frac{-3b_1 + b_2}{17} \end{array} \right]$$

Therefore, the solution to the general case is $(x_1, x_2) = \left(\frac{2b_1 + 5b_2}{17}, \frac{-3b_1 + b_2}{17} \right)$

And so, for the specific cases:

1) $(x_1, x_2) = \left(\frac{2(1) + 5(4)}{17}, \frac{-3(1) + 4}{17} \right) = \left(\frac{13}{17}, \frac{1}{17} \right)$

2) $(x_1, x_2) = \left(\frac{2(-2) + 5(5)}{17}, \frac{-3(-2) + 5}{17} \right) = \left(\frac{16}{17}, \frac{11}{17} \right)$

2.8.4.2 Determine the conditions on b , if any, in order to guarantee that the linear system is consistent.

$$\begin{cases} x_1 + 3x_2 = b_1 \\ -2x_1 + x_2 = b_2 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array} \right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & b_2+2b_1 \end{array} \right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array} \right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{b_1-3b_2}{7} \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array} \right]$$

There are no conditions. The system is consistent for all values of b_1 and b_2 .

2.8.5 Another “determining the conditions” problem:

$$\begin{cases} x_1 - 2x_2 - x_3 = b_1 \\ -4x_1 + 5x_2 + 2x_3 = b_2 \\ -4x_1 + 7x_2 + 4x_3 = b_3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array} \right] \xrightarrow[R_3+4R_1]{R_2+4R_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & b_2+4b_1 \\ 0 & -1 & 0 & b_3+4b_1 \end{array} \right] \xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{b_3+4b_1}{3} \end{array} \right]$$

$$\xrightarrow{-\frac{3}{2}R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & 1 & \frac{-b_3-4b_1}{2} \end{array} \right]$$

Therefore, the system is consistent for all values of b_1 , b_2 , and b_3 .

2.8.6 Triangular and Diagonal Matrices

2.8.6.1 Find A^2

$$1) A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 0(0) & 1(0) + 0(-2) \\ 0(1) + (-2)(0) & 0(0) + (-2)(-2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

$$2) A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (-6)(-6) + (0)(0) + (0)(0) & (-6)(0) + (0)(3) + (0)(0) & (-6)(0) + (0)(0) + (0)(5) \\ (0)(-6) + (3)(0) + (0)(0) & (0)(0) + (3)(3) + (0)(0) & (0)(0) + (3)(0) + (0)(5) \\ (0)(-6) + (0)(0) + (5)(0) & (0)(0) + (0)(3) + (5)(0) & (0)(0) + (0)(0) + (5)(5) \end{bmatrix} \\ &= \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} \end{aligned}$$

2.8.6.2 Find A^{-k} , such that k is some nonzero constant

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} A^{-k} &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-k} \\ &= \begin{bmatrix} 2^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & 2^{-k} \end{bmatrix} \end{aligned}$$

4. Determine whether each matrix is symmetric or not.

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

Not symmetric

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

2.8.6.3 Find a diagonal matrix A that satisfies the given condition

$$1) A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{\frac{1}{5}} \\ &= \begin{bmatrix} 1^{\frac{1}{5}} & 0 & 0 \\ 0 & (-1)^{\frac{1}{5}} & 0 \\ 0 & 0 & (-1)^{\frac{1}{5}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

$$2) A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-\frac{1}{2}} \\ &= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 & 0 \\ 0 & 4^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1^{-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

2.8.7 Determinants and Triangular Matrices (08/29)

2.8.7.1 What is C_{32}

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} C_{32} &= (-1)^{3+2} \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \left(2 \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 3 \\ 3 & 0 \end{vmatrix} + 1 \begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} \right) \\ &= - (2(-3) - (-1)(-9) + 1(-3)) \\ &= - (-6 + 9 - 3) \\ &= 0 \end{aligned}$$

2.8.7.2 Find all values of λ such that $|A| = 0$

$$A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (\lambda - 2)(\lambda + 4) - (-5)(1) \\ &= \lambda^2 + 2\lambda - 8 + 5 \\ &= \lambda^2 + 2\lambda - 3 \\ &= (\lambda + 3)(\lambda - 1) \\ &= 0 \end{aligned}$$

Therefore, $\lambda = -3, 1$

2.8.7.3 For the matrix $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$ find the determinant 3 sp.different ways with cofactor expansion. Pick sp.different rows and columns each time.

$$\begin{aligned} \det(A) &= 3 \begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0 \begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + 0 \begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} \\ &= 3(-1(-4) - 5(9)) - 0(2(-4) - 5(1)) + 0(2(9) - (-1)(1)) \\ &= 3(4 - 45) - 0(-8 - 5) + 0(18 + 1) \\ &= 3(-41) - 0(-13) + 0(19) \\ &= 36 \end{aligned}$$

$$\begin{aligned} \det(A) &= 0 \begin{vmatrix} 2 & 5 \\ 9 & -4 \end{vmatrix} - 3 \begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 0 \begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} \\ &= 0(2(-4) - 5(9)) - 3(3(-4) - 0(1)) + 0(3(5) - 0(2)) \\ &= 0(-8 - 45) - 3(-12 - 0) + 0(15 - 0) \\ &= 0(-53) - 3(-12) \\ &= 36 \end{aligned}$$

$$\begin{aligned} \det(A) &= 0 \begin{vmatrix} 2 & -1 \\ 9 & -4 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 0(2(-4) - (-1)(9)) - 0(3(-4) - 0(1)) + 3(3(-1) - 0(2)) \\ &= 0(-8 + 9) - 0(-12 - 0) + 3(-3 - 0) \\ &= 0(1) - 0(-12) + 3(-3) \\ &= 0 + 0 - 9 \\ &= 36 \end{aligned}$$

2.8.7.4 Evaluate $\det(A)$ by a cofactor expansion along a row or column of your choice

$$A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - k \begin{vmatrix} 1 & k^2 \\ 1 & k^2 \end{vmatrix} + k^2 \begin{vmatrix} 1 & k \\ 1 & k \end{vmatrix} \\ &= 1(k^2 - k^2) - k(1(k^2) - k^2(1)) + k^2(1(k) - k(1)) \\ &= 0 \end{aligned}$$

2.8.7.5 Evaluate the determinant of the following matrices by just looking at them.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1(-1)(1) = -1$$

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det(A) = 1(1)(2)(3) = 6$$

2.8.7.6 Show that the value of the determinant is independent of θ

$$A = \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ \sin \theta - \cos \theta & \sin \theta + \cos \theta & 1 \end{vmatrix}$$

$$\begin{aligned} \det(A) &= \sin \theta \begin{vmatrix} \sin \theta & 0 \\ \sin \theta + \cos \theta & 1 \end{vmatrix} - \cos \theta \begin{vmatrix} \cos \theta & 0 \\ \sin \theta + \cos \theta & 1 \end{vmatrix} \\ &\quad + 0 \begin{vmatrix} \cos \theta & \sin \theta \\ \sin \theta + \cos \theta & \sin \theta \end{vmatrix} \\ &= \sin \theta (\sin \theta(1) - 0(\sin \theta + \cos \theta)) - \cos \theta (\cos \theta(1) - 0(\sin \theta + \cos \theta)) \\ &\quad + 0 (\cos \theta(\sin \theta) - \sin \theta(\sin \theta + \cos \theta)) \\ &= \sin^2 \theta - \cos^2 \theta \\ &= 1 \end{aligned}$$

2.8.8 Row operations and Determinants (08/31)

2.8.8.1 Find the determinant of $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$ **WITHOUT** using cofactor expansion

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{28}{2} \end{vmatrix} \\ &= 1(-2)\left(\frac{28}{2}\right) \\ &= -28\end{aligned}$$

2.8.8.2 Find the determinant of $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -3 & 2 \end{vmatrix} \\ &= 2(-2)(-4)(2) \\ &= 64 \end{aligned}$$

2.8.9 Adjoints and Cramer's Rule (09/05)

2.8.9.1 Find the inverse of $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ using the adjoint method

$$\begin{aligned}\det(A) &= 2 \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} - 5 \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} + 5 \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\ &= 2(-3) - 5(-3) + 5(-2) \\ &= -6 + 15 - 10 \\ &= -1\end{aligned}$$

$$\begin{aligned}\text{adj}(A) &= \begin{bmatrix} (-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} & (-1)^{1+2} \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} & (-1)^{1+3} \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\ (-1)^{2+1} \begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix} & (-1)^{2+2} \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} & (-1)^{2+3} \begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} \\ (-1)^{3+1} \begin{vmatrix} 5 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+3} \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} (-1)(3) & -(-1)(3) & -4 + 2 \\ -(15 - 20) & 6 - 10 & -(8 - 10) \\ 5 & -5 & -2 + 5 \end{bmatrix}^T \\ &= \begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}^T \\ &= \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix} \\ \therefore A^{-1} &= - \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}\end{aligned}$$

2.8.9.2 Solve the following system of equations using Cramer's Rule

$$\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases} \rightarrow \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \rightarrow 4 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} = -132$$

$$\begin{aligned} \det(x) &= \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \\ &= 2 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 2(2 - 10) - 5(6 - 2) \\ &= -16 - 20 \\ &= -36 \end{aligned}$$

$$\begin{aligned} \det(y) &= \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \\ &= 4 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 4(6 - 2) - 2(22 - 2) \\ &= 16 - 40 \\ &= -24 \end{aligned}$$

$$\begin{aligned} \det(z) &= \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \\ &= 4 \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 5 \begin{vmatrix} 11 & 3 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 11 & 1 \\ 1 & 5 \end{vmatrix} \\ &= 4(1 - 15) - 5(33 - 3) + 2(55 - 1) \\ &= -56 - 150 + 108 \\ &= -98 \end{aligned}$$

Therefore, the solution $(x, y, z) = \left(\frac{3}{11}, \frac{2}{11}, -\frac{49}{66}\right)$

3 Chapter 5: Eigenvectors and Eigenvalues

3.1 Eigenvalues and Eigenvectors (11/06)

If A is an $n \times n$ matrix, then a non-zero vector \mathbf{x} , in R^n , is called an eigenvector of A if $A\mathbf{x}$ is a scalar multiple of \mathbf{x} ; that is $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ . This scalar λ is called an eigenvalue of A and \mathbf{x} is said to be an eigenvector corresponding to λ .

See, normally, multiplying a vector by a square matrix changes both the magnitude and the direction of the vector. Really screws it up.

Some examples:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 23 \\ 4 \end{bmatrix}$$

However, there are some ways to get consistent results.

3.1.1 Examples

3.1.1.1 $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ because

$$A\vec{x} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{x} \therefore \lambda = 2$$

3.1.1.2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. **Are \vec{u} and \vec{v} eigenvectors of A ?**

$$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 1(6) + 6(-5) \\ 5(6) + 2(-5) \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \therefore \lambda = -4$$

$$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 6(-2) \\ 5(3) + 2(-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \vec{v}$$

3.2 Eigenvector Homework Problem (11/06)

Confirm by multiplication that \mathbf{x} is an eigenvector of A , and find the corresponding eigenvalue.

3.2.1 $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$A\mathbf{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1) + 0(2) + 1(1) \\ 2(1) + 3(2) + 2(1) \\ 1(1) + 0(2) + 4(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \therefore \lambda = 5$$

3.3 Finding Eigenvalues and Eigenvectors (11/07)

Essential question:

If we know an $n \times n$ matrix A , can we find its λ ?

If $A\vec{x} = \lambda\vec{x}$, then:

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

This equation is familiar. It's the homogeneous system of equations $A\vec{x} = \vec{0}$, the solution of which is the nullspace of $A - \lambda I$. Therefore, \vec{x} is an eigenvector of $A \iff \vec{x}$ is in the nullspace of $A - \lambda I$.

In this situation, what do we know about that matrix?

Everything in the equivalent statements is false because \vec{x} cannot be the zero vector. Therefore, we can see that $\det(A - \lambda I)$ OR $\det(\lambda I - A)$ MUST be 0.

Big Idea: If A is an $n \times n$ matrix, then λ is an eigenvalue of $A \iff \det(\lambda I - A) = 0$. This is called the characteristic equation of A .

3.3.1 Find the characteristic equation and the eigenvalues of $A = \begin{bmatrix} 3 & 0 & 5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

$$\begin{aligned} \det(\lambda I - A) &= 0 \\ \begin{vmatrix} \lambda - 3 & 0 & 5 \\ -\frac{1}{5} & \lambda + 1 & 0 \\ -1 & -1 & \lambda + 2 \end{vmatrix} &= 0 \\ 0 &= (\lambda - 3)((\lambda + 1)(\lambda + 2)) + 5\left(\frac{1}{5} + \lambda + 1\right) \\ 0 &= (\lambda - 3)(\lambda^2 + 3\lambda + 2) \\ 0 &= \lambda^3 - 2\lambda \\ 0 &= \lambda(\lambda^2 - 2)\lambda &= 0, \pm\sqrt{2} \end{aligned}$$

3.3.2 Find the characteristic equation and the eigenvalues of $A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

$$\begin{aligned} \begin{vmatrix} -1 - \lambda & 0 & 1 \\ -1 & 3 - \lambda & 0 \\ -4 & 13 & -\lambda \end{vmatrix} &= 0 \\ (-1 - \lambda)((3 - \lambda)(-\lambda) - 0(13)) + (-1)(13) - (3 - \lambda)(-4) &= 0 \\ (-1 - \lambda)(\lambda^2 - 3\lambda) + (-13 - 4\lambda + 12) &= 0 \\ (-1 - \lambda)(\lambda^2 - 3\lambda) + (-4\lambda - 1) &= 0 \\ -\lambda^3 + 3\lambda^2 + 2 &= 0 \\ (-\lambda + 2)(-\lambda^2 - \lambda - 1) &= 0 \\ (-\lambda + 2)(-\lambda - 1)(-\lambda + 1) &= 0 \\ \lambda &= 2 \end{aligned}$$

3.3.3 Find the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 2 & 0 & 0 \\ 6 & \lambda - 3 & 0 \\ 1 & 4 & \lambda - 5 \end{vmatrix} = 0$$

$$(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$$

$$\lambda = 2, 3, 5$$

Theorem 1: For a triangular matrix, the eigenvalues are the elements on the main diagonal.

3.3.4 Find the eigenvalues of A^3 **if** $A = \begin{bmatrix} \frac{1}{2} & 4 & 5 & -2 \\ 0 & -1 & 3 & -8 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$\lambda_A = \frac{1}{2}, -1, 2, 4$$

$$\lambda_{A^3} = \frac{1}{8}, -1, 8, 64$$

Theorem 2: The eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \dots$

3.3.5 Give me a matrix with eigenvalues $\lambda = 0, 2, 5$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 5 \end{bmatrix}$$

Theorem 3: A square matrix A is invertible $\iff \lambda \neq 0$ (which also means its determinant is 0).

3.3.6 Finding eigenvectors!

Find the nontrivial eigenvectors of:

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$\begin{aligned} \begin{vmatrix} \lambda - 1 & -6 \\ -5 & \lambda - 2 \end{vmatrix} &= 0 \\ (\lambda - 1)(\lambda - 2) - (-6)(-5) &= 0 \\ \lambda^2 - 3\lambda - 28 &= 0 \\ (\lambda - 7)(\lambda + 4) &= 0 \\ \lambda &= 7, -4 \end{aligned}$$

Substitute each λ , one at a time into the $\lambda I - A$ matrix and find the null space.

For $\lambda = -4$:

$$\begin{aligned} \left(\begin{array}{cc|c} -5 & -6 & 0 \\ -5 & -6 & 0 \end{array} \right) \\ \left(\begin{array}{cc|c} -5 & -6 & 0 \\ 0 & 0 & 0 \end{array} \right) \\ \langle -\frac{6}{5}t, t \rangle \\ \vec{x} = \{ \langle -6, 5 \rangle \} \end{aligned}$$

For $\lambda = 7$:

$$\begin{aligned} \left(\begin{array}{cc|c} 6 & -6 & 0 \\ -5 & 5 & 0 \end{array} \right) \\ \left(\begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array} \right) \\ \langle t, t \rangle \\ \vec{x} = \{ \langle 6, 6 \rangle \} \end{aligned}$$

Therefore, the eigen space is: $\{ \langle -6, 5 \rangle, \langle 6, 6 \rangle \}$

3.4 Diagonalization and Similar Triangles

Similar matrices: If A and D are square matrices, we say that A and D are “similar” if there exists an invertible matrix P such that:

$$D = P^{-1}AP.$$

3.4.1 Properties of Similar Matrices

- They have the same determinant
- If one is invertible, so is the other
- They have the same trace
- They have the same characteristic polynomial
- They have the same eigenvalues

3.4.2 Procedure

1. Find the eigenvectors for the $n \times n$ matrix A .
 - Theorem: If an $n \times n$ matrix A has n distinct eigenvalues, then A is **for sure** diagonalizable.
2. Make matrix P out of the eigenvectors (P is the matrix that diagonalizes A)
3. Check your work to find matrix D if reasonable

3.4.3 Example: Find a matrix P that diagonalizes A and compute $P^{-1}AP$

1. $A = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$

Find the eigenvalues:

$$\begin{aligned} \begin{bmatrix} \lambda - 3 & -7 \\ -5 & \lambda - 5 \end{bmatrix} &= 0 \\ (\lambda - 3)(\lambda - 5) - (-7)(-5) &= 0 \\ \lambda^2 - 5\lambda - 3\lambda + 15 - 35 &= 0 \\ \lambda^2 - 8\lambda - 20 &= 0 \\ \lambda &= -2, 10 \end{aligned}$$

Find the eigenvectors:

$$\begin{aligned} \lambda = -2 : \left[\begin{array}{cc|c} -5 & -7 & 0 \\ -5 & -7 & 0 \end{array} \right] \vec{x} &= \{ \langle -7, 5 \rangle \} \\ \lambda = 10 : \left[\begin{array}{cc|c} -7 & -7 & 0 \\ -5 & 5 & 0 \end{array} \right] \vec{x} &= \{ \langle 1, 1 \rangle \} \end{aligned}$$

Create the matrix P :

$$P = \begin{bmatrix} -7 & 1 \\ 5 & 1 \end{bmatrix}$$

Find matrix D :

$$\begin{aligned} D &= P^{-1}AP \\ &= \begin{bmatrix} 7 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & 10 \end{bmatrix} \end{aligned}$$

3.4.4 Conclusion

- If D has the same eigenvalues of A and if D must be diagonal, then D is **THE** diagonal matrix with eigenvalues of A on the diagonal.

$$2. A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

First, find D :

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, find P :

$$\begin{aligned} \lambda = 2 : & \left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \vec{x} = \langle 1, 0, 0 \rangle \\ \lambda = 3 : & \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \vec{x} = \langle 0, 1, 0 \rangle \\ \lambda = 1 : & \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \vec{x} = \langle 2, 0, 1 \rangle \\ & P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3.5 More on Similar Matrices

There are a few more properties of similar matrices:

- They have the same rank (non-zero eigenvalues)
- They have the same nullity
- They have the same column space
- They have the same row space

3.5.1 Example

**Matrix A is similar to the following matrix:

$$D = \begin{bmatrix} 3 & -1 & 1 & 4 & 5 & 2 \\ 0 & -3 & 5 & -10 & -16 & 1 \\ 0 & 0 & 5 & 7 & -8 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A: 4

Nullity of A: 2

Eigenvalues: 3, -3, 5, 2, 0, 0

Characteristic Polynomial:

$$\det(\lambda I - A) = 0$$
$$\begin{vmatrix} \lambda - 3 & 1 & -1 & -4 & -5 & -2 \\ 0 & \lambda + 3 & -5 & 10 & 16 & -1 \\ 0 & 0 & \lambda - 5 & -7 & 8 & -2 \\ 0 & 0 & 0 & \lambda - 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} = 0$$
$$(\lambda - 3)(\lambda + 3)(\lambda - 5)(\lambda - 2)\lambda^2 = 0$$

3.5.2 Some review

- Eigenspace of λ : The nullspace of $\lambda I - A$. Each eigenvalue will have its own eigenspace.
- Algebraic multiplicity: The number of times a given λ appears as a root of the characteristic equation.
- Geometric multiplicity: The number of eigenvectors it maps to.

3.5.2.1 Theorem: Geometric and Algebraic Multiplicity

If A is a square matrix, then: a. For every eigenvalue of A , the geometric multiplicity is less than or equal to the algebraic multiplicity. b. A is diagonalizable \iff the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity.

3.6 Similar Matrices Continued (11/13/2023)

3.6.1 Warm-Up

Can you write a new statement involving eigenvalues to add to the list of equivalent statements?

$\lambda = 0$ is not an eigenvalue of A

3.6.2 Homework Review

$$A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 19 & 9 & 6 \\ -25 & \lambda + 11 & 9 \\ 17 & 9 & \lambda + 4 \end{bmatrix} = (\lambda - 19)((\lambda + 11)(\lambda + 4) - 81) + 25(9\lambda + 36 - 54) - 17(81 - 6\lambda - 66) \\ = (\lambda - 1)^2(\lambda - 2)$$

$\lambda = 1$ has an algebraic multiplicity of 2 and $\lambda = 2$ has an algebraic multiplicity of 1.

3.6.3 Suppose that a characteristic polynomial of some matrix A is found to be:

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$$

a. What are the dimensions of A ?

$$6 \times 6$$

b. What are the algebraic multiplicities of each eigenvalue?

$$\lambda = 1 : 1, \lambda = 3 : 2, \lambda = 4 : 3$$

c. What are the possible dimensions of the eigenspace associated with each of the eigenvalues?

$$\lambda = 1 : 1, \lambda = 3 : 1 \text{ or } 2, \lambda = 4 : 1 \text{ or } 2 \text{ or } 3$$

d. If $\{v_1, v_2\}$ is a linearly independent set of eigenvectors of A , all of which correspond to the same eigenvalue of A , what can you say about the eigenvalue?

The eigenvalue must be 3 or 4.

4 Semester II

4.1 Introduction to Multivariable functions (01/04)

4.1.1 Definition of a Multivariable Function

Suppose D is a set of n -tuples of real numbers (x_1, x_2, \dots, x_n) . A function f on D is a rule that assigns a real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D . The set D is the function's **domain**. The set of w -values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f , and f is a function of the n **independent variables** x_1 to x_n . The x 's are the function's **input variables**; w is the function's **output variable**.

4.2 Limits and Continuity (01/05)

4.2.1 Level Curve, Graph, Surface

The set of points in the plane where a function $f(x, y)$ has a constant value k is called a **level curve** of f . The graph of f is the set of all points (x, y, z) in space, where $z = f(x, y)$. The graph of f is a surface in space.

4.2.2 Limits With Multivariable Functions

Limits: Let f be a function of two variables defined on an open region, except possibly at (x_0, y_0) .

In 2-D:

$$\lim_{x \rightarrow c} f(x) \text{ exists iff } \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L \in \mathbb{R}$$

In 3-D:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) \text{ exists iff } \lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L \in \mathbb{R} \text{ from all directions.}$$

4.2.3 Examples

$$1. \lim_{(x,y) \rightarrow (1,2)} \frac{5x^2y}{x^2 + y^2} = \frac{5 \cdot 1 \cdot 2}{1 + 4} = \frac{10}{5} = 2$$

$$2. \lim_{(x,y) \rightarrow (1,1)} \frac{x - y}{x^2 - y^2} = \frac{0}{0}$$

Since this is indeterminate, you need to switch to traditional limit solving methods. There is no *L'Hopital's Rule* for multivariable functions. Instead, factor the numerator and denominator and cancel out common factors.

$$\lim_{(x,y) \rightarrow (1,1)} \frac{x - y}{(x - y)(x + y)} = \frac{1}{2}$$

$$3. \lim_{(x,y) \rightarrow (4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$$

4.2.4 Homework

4.2.4.1 Evaluate the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} \bigg|_{(0,0)} = \frac{5}{2}$$

4.2.4.2 Evaluate the limit $\lim_{(x,y) \rightarrow (1,1)} \cos \sqrt[3]{|xy| - 1}$

$$\lim_{(x,y) \rightarrow (1,1)} \cos \sqrt[3]{|xy| - 1} \bigg|_{(1,1)} = \cos 0 = 1$$

4.2.4.3 Evaluate the limit $\lim_{(x,y) \rightarrow (1,1)} \frac{xy - y - 2x + 2}{x - 1}$

$$\lim_{(x,y) \rightarrow (1,1)} \frac{xy - y - 2x + 2}{x - 1} = \frac{(x-1)(y-2)}{x-1} = y - 2 \big|_{(1,1)} = -1$$

4.2.4.4 On what interval is the function $f(x, y) = \sin(x + y)$ continuous?

The sin function is continuous everywhere, so $(x, y) \in \mathbb{R}^2$

4.2.4.5 On what interval is the function $f(x, y, z) = x^2 + y^2 - 2z^2$ continuous?

$(x, y, z) \in \mathbb{R}^3$

4.2.4.6 On what interval is the function $f(x, y, z) = xy \sin\left(\frac{1}{z}\right)$ continuous?

Can't divide by zero, so $(x, y, z) \in \mathbb{R}^3 | z \neq 0$

4.3 Limits that DO NOT EXIST in 3-Space (01/08)

4.3.1 Question: Does the function $f(x, y) = \frac{2x^2y}{x^4 + y^2}$ have a limit as (x, y) approaches $(0, 0)$?

Nope, it approaches the point sp.differently.

4.3.2 Find $f(x, y)|_{y=x^2}$ and compute $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ along $y = x^2$

$$f(x, y) = \frac{2x^2y}{x^4 + y^2} \bigg|_{y=x^2} = \frac{2x^4}{2x^4} = 1$$

4.3.3 Find $f(x, y)|_{y=-x^2}$ and compute $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ along $y = -x^2$

$$f(x, y) = \frac{2x^2y}{x^4 + y^2} \bigg|_{y=-x^2} = \frac{-2x^4}{2x^4} = -1$$

4.3.4 Explain why we can conclude that $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = x^2 \neq \lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = -x^2$$

4.3.5 How did we know to choose $y = x^2$ and $y = -x^2$ to evaluate the limit?

You want to try to choose things that, as you plug them in, you get a nice expression that you can simplify. For this problem in particular, we chose paths towards $(0, 0)$ that would be easy to solve and yield sp.different answers upon simplification. We could've also used the equation of the x-axis ($y = 0$).

4.3.6 Show that these functions have no limit as (x, y) approaches $(0, 0)$ by considering sp.different paths of approach.

4.3.6.1 $f(x, y) = \frac{x^4}{x^4 + y^2}$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = 2x^2 = \frac{x^4}{5x^4} = \frac{1}{5} \neq \lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = x^2 = \frac{x^4}{2x^4} = \frac{1}{2}$$

4.3.6.2 $f(x, y) = \frac{x^2 + y}{y}$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = x^2 = \frac{2y}{y} = 2 \neq \lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = 2x^2 = \frac{3x^2}{2x^2} = \frac{3}{2}$$

4.4 Partial Derivatives (01/10)

4.4.1 First Order Partial Derivatives

A **partial derivative** is obtained by holding all but one of the independent variables constant and differentiating with respect to that variable.

4.4.1.1 Notation

$\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x} f(x, y)$ is the partial derivative of f with respect to x

4.4.2 Examples

4.4.2.1 $f(x, y) = 2x^3y - 4x^2y^3 + 5x^4$

$$f_x = 6x^2y - 8xy^3 + 20x^3$$

$$f_y = 2x^3 - 12x^2y^2$$

4.4.2.2 $f(x, y) = 4x^2y - 8x^3y^4 + 2xy^7$

$$f_x = 8xy - 24x^2y^4 + 2y^7$$

$$f_y = 4x^2 - 32x^3y^3 + 14xy^6$$

4.4.2.3 $f(x, y) = \tan(2x - y)$

$$f_x = 2 \sec^2(2x - y)$$

$$f_y = -\sec^2(2x - y)$$

4.4.3 The Second Fundamental Theorem of (Multivariable) Calculus

- $\frac{d}{dx} \int_5^x f(t)dt = f(x)$
- $\frac{d}{dx} \int_5^{x^2} f(t)dt = 2x \cdot f(x^2)$

4.4.4 Examples

4.4.4.1 $f(x, y) = \int_{3x}^{2y} (t^2 - 1)dt$

```

1 # import libraries
2 import sympy as sp
3 from IPython.display import display

1 x, y = sp.symbols('x y')
2 t = sp.symbols('t')
3 f = sp.integrate(t**2-1, (t, 3*x, 2*y))
4 display(sp.diff(f, x))
5 display(sp.diff(f, y))

```

$$3 - 27x^2$$

$$8y^2 - 2$$

4.4.4.2 Find both partial derivatives and evaluate each at the point $(1, \ln 2)$

1. $f(x, y) = xe^{x^2y}$

```

1 f = x*sp.exp(x**2*y)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
4 display(sp.diff(f, x).subs({x:1, y:sp.ln(2)}))
5 display(sp.diff(f, y).subs({x:1, y:sp.ln(2)}))

```

$$2x^2ye^{x^2y} + e^{x^2y}$$

$$x^3e^{x^2y}$$

$$2 + 4 \log(2)$$

$$2$$

4.5 2nd Partial Derivatives (01/11)

4.5.1 First, let's see this one. If $f(x, y, z) = x \sin(y + 3z)$, find f_x, f_y, f_z .

```
1 x, y, z = sp.symbols('x y z')
2 f = x*sp.sin(y+3*z)
3 display(sp.diff(f, x))
4 display(sp.diff(f, y))
5 display(sp.diff(f, z))
```

$$\sin(y + 3z)$$

$$x \cos(y + 3z)$$

$$3x \cos(y + 3z)$$

4.5.2 2nd-Order Partial Derivatives

The **second-order partial derivatives** of f are the partial derivatives of the partial derivatives of f .

4.5.2.1 Notation

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$

4.5.3 Mixed Partial Derivatives

The **mixed partial derivatives** of f are the partial derivatives of f with respect to many other different variables.

4.5.3.1 Notation

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

To compute a mixed partial derivative, you must evaluate the order from right to left. For example, f_{xy} is the partial derivative of f_x with respect to y , which is notated as $\frac{\partial^2 f}{\partial x \partial y}$.

4.5.4 Theorem (Clairaut's Theorem)

If f_{xy} and f_{yx} are continuous on an open region D , then $f_{xy}(a, b) = f_{yx}(a, b)$ on D .

4.5.5 Examples

4.5.5.1 Given that $f(x, y) = x^2y - y^3 + \ln x$, find all 2nd order partial derivatives

```
f = x**2*y - y**3 + sp.ln(x) display(sp.diff(f, y, x)) display(sp.diff(f, x, y)) display(sp.diff(f, x, x)) display(sp.diff(f, y, y))
```

```
#### Given that $w = xy + \frac{e^y}{y^2+1}$, find $\frac{\partial^2 w}{\partial x \partial y}$
```

For this problem, you need to "choose wisely." Knowing that the function and it's partial derivatives

```
### Partial Derivatives Homework (01/11)
```

```
### Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y)$
```

```
#### $f(x, y) = x(y-1)$
```

```
::: {.cell execution_count=5}
``` {.python .cell-code}
x, y, z = sp.symbols('x y z')
f = x*(y-1)
display(sp.diff(f, x))
display(sp.diff(f, y))
```

$y - 1$

$x$

$\vdots$

##### 4.5.6 $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

```
1 f = 5*x*y - 7*x**2 - y**2 + 3*x - 6*y + 2
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
```

$$-14x + 5y + 3$$

$$5x - 2y - 6$$

**4.5.7**  $f(x, y) = \sqrt{x^2 + y^2}$

```
1 f = sp.sqrt(x**2 + y**2)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
```

$$\frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{y}{\sqrt{x^2 + y^2}}$$

**4.5.8**  $f(x, y) = e^{(x+y+1)}$

```
1 f = sp.exp(x+y+1)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
```

$$e^{x+y+1}$$

$$e^{x+y+1}$$

**4.5.9**  $f(x, y) = \int_x^y g(t)dt$

```
1 g = sp.Function('g')
2 t = sp.symbols('t')
3 f = sp.integrate(g(t), (t, x, y))
4 display(sp.diff(f, x))
5 display(sp.diff(f, y))
```

$$-g(x)$$

$$g(y)$$

**4.5.10**  $f(x, y, z) = xy + yz + xz$ 

```

1 f = x*y + y*z + x*z
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
4 display(sp.diff(f, z))

```

$$y + z$$

$$x + z$$

$$x + y$$

**4.5.11**  $f(x, y, z) = \ln(x + 2y + 3z)$ 

```

1 f = sp.ln(x + 2*y + 3*z)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
4 display(sp.diff(f, z))

```

$$\frac{1}{x + 2y + 3z}$$

$$\frac{2}{x + 2y + 3z}$$

$$\frac{3}{x + 2y + 3z}$$

**4.5.12** If  $f(x, y) = x \cos(y) + ye^x$ , find  $\frac{\partial^2 f}{\partial x^2}$ ,  $\frac{\partial^2 f}{\partial y^2}$ ,  $\frac{\partial^2 f}{\partial x \partial y}$ , and  $\frac{\partial^2 f}{\partial y \partial x}$ 

```

1 f = x*sp.cos(y) + y*sp.exp(x)
2 display(sp.diff(f, x, x))
3 display(sp.diff(f, y, y))
4 display(sp.diff(f, x, y))
5 display(sp.diff(f, y, x))

```

$$ye^x$$

$$-x \cos(y)$$

$$e^x - \sin(y)$$

$$e^x - \sin(y)$$