Gwinnett School of Math, Science, and Technology

# Multivariable Calculus Yearlong Notes

Anish Goyal
1st Period

Donny Thurston
Educator

2023-2024

# Table of Contents

# 1 Chapter 1: Systems of Linear Equations and Matrices

## 1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal $\iff$ they have the same dimensions and values

### 1.1.1 Addition & Subtraction

Two matrices can be added/subtracted $\iff$ they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

### 1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

### 1.1.3 Matrix Multiplication

- We can **only** multiply an (m x n) by (n x p) matrix.
- The resulting matrix will be (m x p)

### 1.1.4 Properties of Matrix Arithmetic

(a) $A + B = B + A$ **(Commutative law for addition)**
(b) $A + (B + C) = (A + B) + C$ **(Associative law for addition)**
(c) $A(BC) = (AB)C$ **(Associative law for multiplication)**
(d) $A(B + C) = AB + AC$ **(Left distributive law)**
(e) $(B + C)A = BA + CA$ **(Right distributive law)**
(f) $A(B - C) = AB - AC$
(g) $(B - C)A = BA - CA$
(h) a(B+C) = aB + aC
(i) a(B-C) = aB - aC
(j) (a+b)C = aC + bC
(k) (a-b)C = aC - bC
(l) a(bC) = (ab)C
(m) a(BC) = (aB)C = B(aC)

### 1.1.5 Examples

1.

$$\begin{aligned}
&\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix} \\
&= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}
\end{aligned}$$

2.

$$\begin{aligned}
&\begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\
&= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix}
\end{aligned}$$

3.

$$\begin{aligned}&\begin{bmatrix}4 & 5 & -1\end{bmatrix}\begin{bmatrix}8\\0\\2\end{bmatrix}\\&=\begin{bmatrix}4\cdot 8+5\cdot 0+(-1)\cdot 2\end{bmatrix}\\&=\begin{bmatrix}30\end{bmatrix}\end{aligned}$$

## 1.2 Transpose of a Matrix

The transpose of an (m x n) matrix is the (n x m) matrix where the rows and columns are swapped.

If $B=\begin{bmatrix}4 & 2\\-1 & 0\\3 & 5\end{bmatrix}, B^T=\begin{bmatrix}4 & -1 & 3\\2 & 0 & 5\end{bmatrix}$

$$\begin{aligned}B\cdot B^T&=\begin{bmatrix}4 & 2\\-1 & 0\\3 & 5\end{bmatrix}\begin{bmatrix}4 & -1 & 3\\2 & 0 & 5\end{bmatrix}\\&=\begin{bmatrix}4\cdot 4+2\cdot 2 & 4\cdot(-1)+2\cdot 0 & 4\cdot 3+2\cdot 5\\(-1)\cdot 4+0\cdot 2 & (-1)\cdot(-1)+0\cdot 0 & (-1)\cdot 3+0\cdot 5\\3\cdot 4+5\cdot 2 & 3\cdot(-1)+5\cdot 0 & 3\cdot 3+5\cdot 5\end{bmatrix}\\&=\begin{bmatrix}20 & -4 & 22\\-4 & 1 & -3\\22 & -3 & 34\end{bmatrix}\end{aligned}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix $A$ is equal to the sum of all the elements on the main diagonal: $\mathrm{tr}(A)$

### 1.2.1 Transpose Matrix Properties

- $(A^T)^T=A$
- $(A+B)^T=A^T+B^T$
- $(A-B)^T=A^T-B^T$
- $(kA)^T=kA^T$
- $(AB)^T=B^TA^T$

### 1.3 Homework — "Matrix Stuff" (08/03/2023)

#### 1.3.1 Suppose that $A, B, C, D$ and $E$ are matrices with the following sizes:

| $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|
| $(3 \times 2)$ | $(2 \times 3)$ | $(3 \times 3)$ | $(3 \times 2)$ | $(2 \times 3)$ |

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

| Undefined | Defined; $(4 \times 2)$ | Defined; $(5 \times 5)$ | Defined; $(5 \times 2)$ |
|---|---|---|---|
| $BA$<br>$AB + B$<br>$E^T A$<br>$AE + B$ | $AC + D$ | $E(A + B)$ | $(A^T + E)D$<br>$E(AC)$ |

#### 1.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2. $\mathbf{2A^T + C}$

$$\begin{aligned} 2A^T + C &= 2\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3. $\mathbf{B^T + 5C^T}$

$$B^T + 5C^T = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix}$$

= Undefined

4. $\mathbf{2E^T - 3D^T}$

$$2E^T - 3D^T = 2\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T$$

$$= 2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$$

5. tr(**DE**)

$$\mathrm{tr}(DE) = \mathrm{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right)$$

$$= \mathrm{tr}\left(\begin{bmatrix} 1\cdot 6 + 5\cdot(-1) + 2\cdot 4 & 1\cdot 1 + 5\cdot 1 + 2\cdot 1 & 1\cdot 3 + 5\cdot 2 + 2\cdot 3 \\ (-1)\cdot 6 + 0\cdot(-1) + 1\cdot 4 & (-1)\cdot 1 + 0\cdot 1 + 1\cdot 1 & (-1)\cdot 3 + 0\cdot 2 + 1\cdot 3 \\ 3\cdot 6 + 2\cdot(-1) + 4\cdot 4 & 3\cdot 1 + 2\cdot 1 + 4\cdot 1 & 3\cdot 3 + 2\cdot 2 + 4\cdot 3 \end{bmatrix}\right)$$

$$= \mathrm{tr}\left(\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}\right)$$

$$= 34$$

## 2 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

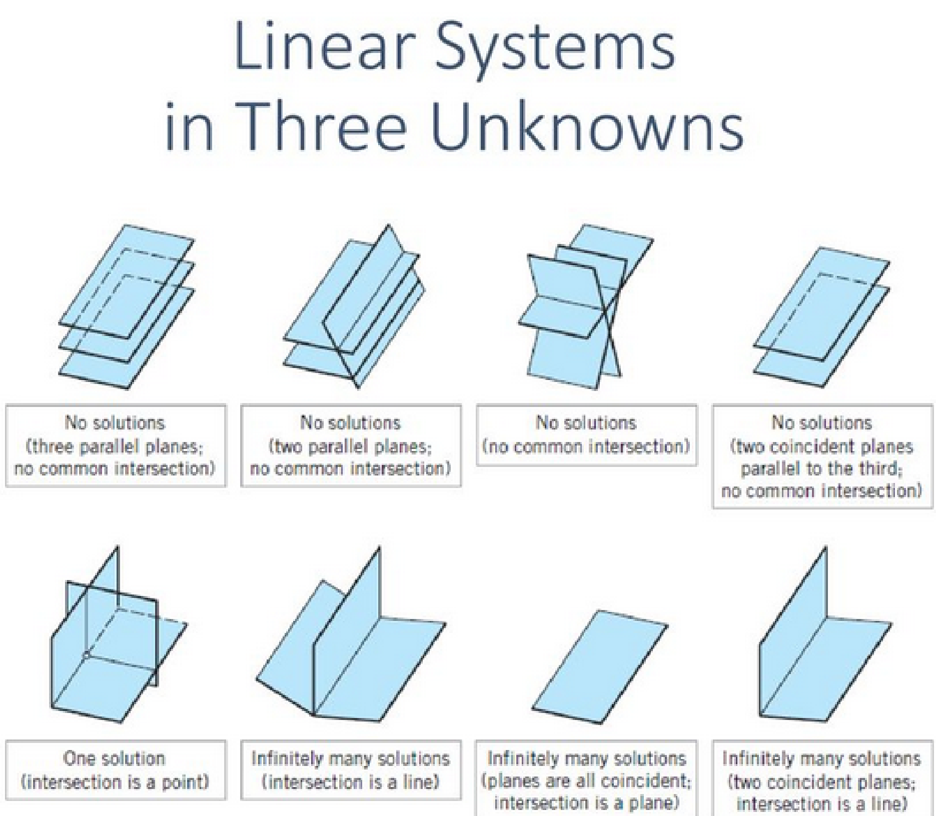
- Infinite solutions
- One solution
- No solutions

Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear *really* means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.



## 2.1 Review: Solve the following systems

1. $\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$

$$\begin{aligned} 5x &= 15 \\ x &= 3 \\ 2(3) + y &= 10 \\ 6 + y &= 10 \\ y &= 4 \end{aligned}$$

2. $\begin{cases} 2x + \phantom{3}y = 10 \\ 6x + 3y = 10 \end{cases}$

$$\begin{aligned} y &= 10 - 2x \\ 6x + 3(10 - 2x) &= 10 \\ 6x + 30 - 6x &= 10 \\ 30 &= 10 \therefore \text{ no solution} \end{aligned}$$

3. $\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$

$$\begin{aligned} 0 &= 0 \\ 12 &= 12 \therefore \text{ infinite solutions} \end{aligned}$$

### 2.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

### 2.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

## 2.2 The Augmented Matrix

$$\begin{cases} x - \phantom{2}y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array}\right]$$

## 2.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/substract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

### 2.3.1 Example 1... again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array}\right]$$
$$\xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{R1 - \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

And so... $x = 3$ and $y = 4$!

## 2.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right],$$

then the solution to the system will be the ordered triple $(c_1, c_2, c_3)$.

### 2.4.1 Example 2: again

$$\begin{cases} 2x + \phantom{3}y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array}\right] \xrightarrow{\frac{1}{2}R1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array}\right] \xrightarrow{R2-6R1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array}\right]$$

This is inconsistent, so there is no solution.

### 2.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array}\right] \xrightarrow{\frac{1}{5}R1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array}\right] \xrightarrow{R2-15R1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array}\right]$$

Since $0 = 0$, there are infinitely many solutions.

### 2.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + \phantom{9}x_3 = 0 \\ \phantom{-4x_1 +} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow{R3+4R1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R3+\frac{3}{2}R2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow[R_3+3R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2+4R_3]{R1+7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore the solution to $(x_1, x_2, x_3)$ is $(29, 16, 3)$.

### 2.4.4 Elementary Row Operations & REF Homework Problem (08/08/2023)

$$\begin{cases} x + \ \ y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \xrightarrow[-R_3]{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array}\right]$$
$$\xrightarrow[R_3-10R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array}\right] \xrightarrow{\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2+5R_3]{R_1-7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore, the solution to $(x, y, z)$ is $(3, 1, 2)$.

## 2.5 Gaussian Elimination

Vocabulary: A matrix is in Row Echelon Form (REF) if:

(a) Any rows of all zeroes are placed at the bottom of the matrix
(b) All other rows have a leading 1 ("pivot")
(c) As we move down the matrix, each leading 1 is further to the right than the 1 above it

A matrix is in Row Reduced Echelon Form if the three above conditions are met in adition to:

(d) Each column with a leading 1 has all other entries in the column as a 0. ("pivot column")

### 2.5.1 Examples

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF? ✓
RREF? ✓

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF? ✓
RREF? ×

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

REF? ×
RREF? ×

## 2.6 Gaussian Elimination With Back-Substitution

### 2.6.1 Goal:

To get the augmented matrix in REF

Solve: $\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 \qquad = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow[R_3+R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x + 9z &= 19 \\ y + 3z &= 5 \\ z &= 2 \\ \therefore z = 2, y &= 5 - 3z, x = 19 - 9z \\ z = 2, y &= 5 - 3(2), x = 19 - 9(2) \\ z = 2, y &= -1, x = 1 \end{aligned}$$

Therefore, the solution $(x_1, x_2, x_3)$ is $(1, -1, 2)$.

### 2.6.2 Gaussian Elimination Homework Problem (08/09/2023)

$$
\begin{cases}
-2w \phantom{{}+2x} + y + z = -3 \\
\phantom{-2w+{}} x + 2y - z = 2 \\
-3w + 2x + 4y + z = -2 \\
-w + x - 4y - 7z = -19
\end{cases}
$$

$$
\left[\begin{array}{cccc|c}
-2 & 0 & 1 & 1 & -3 \\
0 & 1 & 2 & -1 & 2 \\
-3 & 2 & 4 & 1 & -2 \\
-1 & 1 & -4 & -7 & -19
\end{array}\right]
\begin{array}{c} \xrightarrow{R_4} \\ \\ \xrightarrow{R_1} \end{array}
\left[\begin{array}{cccc|c}
-1 & 1 & -4 & -7 & -19 \\
0 & 1 & 2 & -1 & 2 \\
-3 & 2 & 4 & 1 & -2 \\
-2 & 0 & 1 & 1 & -3
\end{array}\right]
\xrightarrow{-R_1}
$$

$$
\left[\begin{array}{cccc|c}
1 & -1 & 4 & 7 & 19 \\
0 & 1 & 2 & -1 & 2 \\
-3 & 2 & 4 & 1 & -2 \\
-2 & 0 & 1 & 1 & -3
\end{array}\right]
\begin{array}{c} \xrightarrow{R_3+3R_1} \\ \xrightarrow{R_4+2R_1} \end{array}
\left[\begin{array}{cccc|c}
1 & -1 & 4 & 7 & 19 \\
0 & 1 & 2 & -1 & 2 \\
0 & -1 & 16 & 22 & 55 \\
0 & -2 & 9 & 15 & 35
\end{array}\right]
\begin{array}{c} \xrightarrow{R_1+R_2} \\ \xrightarrow{R_3+R_2} \\ \xrightarrow{R_4+2R_2} \end{array}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 6 & 6 & 21 \\
0 & 1 & 2 & -1 & 2 \\
0 & 0 & 18 & 21 & 57 \\
0 & 0 & 13 & 13 & 39
\end{array}\right]
\xrightarrow{\frac{1}{18}R_3}
\left[\begin{array}{cccc|c}
1 & 0 & 6 & 6 & 21 \\
0 & 1 & 2 & -1 & 2 \\
0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\
0 & 0 & 13 & 13 & 39
\end{array}\right]
\begin{array}{c} \xrightarrow{R_1-6R_3} \\ \xrightarrow{R_2-2R_3} \\ \xrightarrow{R_4-13R_3} \end{array}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 0 & -1 & 2 \\
0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\
0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\
0 & 0 & 0 & -\frac{13}{6} & -\frac{13}{6}
\end{array}\right]
\xrightarrow{\frac{-6}{13}R_4}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & -1 & 2 \\
0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\
0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\
0 & 0 & 0 & 1 & 1
\end{array}\right]
\begin{array}{c} \xrightarrow{R_1+R_4} \\ \xrightarrow{R_2+\frac{10}{3}R_4} \\ \xrightarrow{R_3-\frac{7}{6}R_4} \end{array}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 1
\end{array}\right]
\Longrightarrow
\begin{cases}
w = 3 \\
x = -1 \\
y = 2 \\
z = 1
\end{cases}
$$

## 2.7 Gauss-Jordan Elimination

### 2.7.1 Goal:

To get the matrix into RREF

Solve: $\begin{cases} x_1 \qquad\quad - 3x_3 = -2 \\ 3x_1 + \;\; x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + \;\; x_3 = 4 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2-7R_3]{R_1+3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longrightarrow \begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = 2 \end{cases}$$

## 2.8 Matrix Properties, Equations, and Inverses

### 2.8.1 With Real Numbers

- If $ab = bc$, then $a = c$, if $b \neq 0$
- If $ab = 0$, then $a = 0$ or $b = 0$, or both

### 2.8.2 With Matrices

- If $AB = AC$, then $B = C$, if $A$ is invertible
- If $AB = [0]$, then $A = [0]$ or $B = [0]$, or both

#### 2.8.2.1 Multiply:

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2(5) + 3(-3) & 2(-3) + 3(2) \\ 3(5) + 5(-3) & 3(-3) + 5(2) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### 2.8.3 Matrix Inverses

- If a matrix has an inverse, it is said to be invertible or non-singular.
- If a matrix does not have an inverse, it is said to be singular.
- Every square matrix has a “special number” associated with it called the **determinant**.
- For the 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $ad - bc$
- $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- When $\det A = 0$, the matrix is singular and has no inverse (since you cannot divide by zero)

Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$
\begin{aligned}
\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}^{-1} &= \frac{1}{\det A} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\
&= \frac{1}{(4)(2) - (3)(1)} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}
\end{aligned}
$$

# 3 Chapter 2: Determinants

## 3.1 Prior Knowledge:

$\begin{bmatrix} 10 & -4 \\ -3 & -5 \end{bmatrix} = -50- = -62$

$$\begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$$
$$= ((2 \cdot 2 \cdot -2) + (4 \cdot 3 \cdot 3) + (3 \cdot -1 \cdot 0)) - ((3 \cdot 2 \cdot 3) + (0 \cdot 3 \cdot 2) + (-2 \cdot -1 \cdot 4))$$
$$= (-8 + 36 + 0) - (18 + 0 + 8)$$
$$= 28 - 26$$
$$= 2$$

## 3.2 Minors & Cofactors

Given a square matrix A, the minor of matrix element $a_{ij}$, ($M_{ij}$) is the determinant of the matrix formed by removing the $i^{\text{th}}$ row and $j^{\text{th}}$ column from matrix A.

The cofactor of matrix element $a_{ij}, C_{ij} = (-1)^{i+j} \cdot M_{ij}$

### 3.2.1 Example

Let det $\begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$. What is the cofactor of element (1, 1)?

Cofactor checkerboard:
$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$M_{11} = \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} = -4$
$C_{11} = 1 \cdot -4 = -4$

Find the minor and cofactor of: \ a) $a_{21} = -1$

$$M_{21} = \begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} = -8$$
$$C_{21} = 8$$

b) $a_{33} = -2$

$$M_{33} = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 8$$
$$C_{33} = 8$$

## 3.3 Cofactor Expansion

1) Pick a row or column
2) Multiply every entry in that row or column by it's corresponding cofactor
3) Add those together. That's it

$A = \begin{bmatrix} 6 & 7 & -1 \\ 0 & 4 & 1 \\ 2 & 5 & -3 \end{bmatrix}$

$$\begin{aligned} \det(A) &= 6\left(\begin{vmatrix} 4 & 1 \\ 5 & -3 \end{vmatrix}\right) + 7\left(-\begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix}\right) + -1\left(\begin{vmatrix} 0 & 4 \\ 2 & 5 \end{vmatrix}\right) \\ &= 6(-17) + 7(2) + (-1(-8)) \\ &= -102 + 14 + 8 \\ &= -80 \end{aligned}$$

### 3.3.1 Example

$A = \begin{bmatrix} 6 & 4 & 2 \\ 5 & -6 & 1 \\ 0 & 3 & 0 \end{bmatrix}$

$$\begin{aligned} &6\begin{vmatrix} -6 & 1 \\ 3 & 0 \end{vmatrix} + 4\left(-\begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix}\right) + 2\begin{vmatrix} 5 & -6 \\ 0 & 3 \end{vmatrix} \\ &= 6(-3) + 0 + 2(15) \\ &= -18 + 30 \\ &= 12 \end{aligned}$$

### 3.3.2 Does the method generalize to 2×2 matrices?

$$\begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix}$$
$$= 3|2| - 5|7|$$
$$= 6 - 35$$
$$= -29$$

The determinant of a 1×1 matrix is... **itself!**

### 3.3.3 Find the determinant of a 4×4

$$A = \begin{bmatrix} -3 & 2 & 0 & 8 \\ 2 & 1 & 0 & -4 \\ 5 & -2 & 1 & 5 \\ 2 & 3 & 0 & 6 \end{bmatrix}$$

$$= 0 + 0 + \begin{vmatrix} -3 & 2 & 8 \\ 2 & 1 & -4 \\ 2 & 3 & 6 \end{vmatrix} + 0$$
$$= -2\begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} -3 & 8 \\ 2 & 6 \end{vmatrix} - \left(-4\begin{vmatrix} -3 & 2 \\ 2 & 3 \end{vmatrix}\right)$$
$$= 24 - 34 - 52$$
$$= -62$$

## 3.4 Theorem

If $A$ is an $n \times n$ matrix, then regardless of which row or column of $A$ is chosen, the number obtained by multiplying the elements in that row or column by their corresponding cofactors is **always the same** and is called the determinant of $A$.

### 3.4.1 Example

Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$
\begin{aligned}
&1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\
&= \left(-2 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\right) \\
&= -6
\end{aligned}
$$

## 3.5 Triangular Matrices

Find the determinant of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$
\begin{aligned}
&\begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} \\
&= 2 \begin{vmatrix} 3 & 3 \\ 0 & 4 \end{vmatrix} \\
&= 2(3 \cdot 4) \\
&= 2 \cdot 12 \\
&= 24
\end{aligned}
$$

If $A$ is an $n \times n$ triangular matrix, then $\det(A)$ is equal to the product of the elements along the main diagonal.

### 3.5.1 An important definitions

Elementary Matrix a matrix that can be obtanied from the $n \times n$ identity matrix by performing a single row operation. \

Are the following matrices elementary? 1) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \div (R_3 + 5R_1)$ yes 2) $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix} \div (R_1 + 5R_2)$... no