

Gwinnett School of Math, Science, and Technology

Multivariable Calculus Yearlong Notes

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1 Systems of Linear Equations and Matrices

1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal \iff they have the same dimensions and values

1.1.1 Addition & Subtraction

Two matrices can be added/subtracted \iff they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3 \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

1.1.3 Matrix Multiplication

- We can **only** multiply an $(m \times n)$ by $(n \times p)$ matrix.
- The resulting matrix will be $(m \times p)$

1.1.4 Properties of Matrix Arithmetic

- (a) $A + B = B + A$ (**Commutative law for addition**)
- (b) $A + (B + C) = (A + B) + C$ (**Associative law for addition**)
- (c) $A(BC) = (AB)C$ (**Associative law for multiplication**)
- (d) $A(B + C) = AB + AC$ (**Left distributive law**)
- (e) $(B + C)A = BA + CA$ (**Right distributive law**)
- (f) $A(B - C) = AB - AC$
- (g) $(B - C)A = BA - CA$
- (h) $a(B+C) = aB + aC$
- (i) $a(B-C) = aB - aC$
- (j) $(a+b)C = aC + bC$
- (k) $(a-b)C = aC - bC$
- (l) $a(bC) = (ab)C$
- (m) $a(BC) = (aB)C = B(aC)$

1.1.5 Examples

1.

$$\begin{aligned} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \end{aligned}$$

2.

$$\begin{aligned} & \begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix} \end{aligned}$$

3.

$$\begin{aligned} & [4 \ 5 \ -1] \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} \\ &= [4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2] \\ &= [30] \end{aligned}$$

1.2 Transpose of a Matrix

The transpose of an $(m \times n)$ matrix is the $(n \times m)$ matrix where the rows and columns are swapped.

$$\text{If } B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix} \end{aligned}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix A is equal to the sum of all the elements on the main diagonal: $\text{tr}(A)$

1.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

1.3 Homework — “Matrix Stuff” (08/03/2023)

1.3.1 Suppose that A, B, C, D and E are matrices with the following sizes:

A	B	C	D	E
(3×2)	(2×3)	(3×3)	(3×2)	(2×3)

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

Undefined	Defined; (4×2)	Defined; (5×5)	Defined; (5×2)
BA	$AC + D$	$E(A + B)$	$(A^T + E)D$
$AB + B$			$E(AC)$
$E^T A$			
$AE + B$			

1.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2. $2A^T + C$

$$\begin{aligned} 2A^T + C &= 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2 \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3. $\mathbf{B}^T + 5\mathbf{C}^T$

$$\begin{aligned} B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\ &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\ &= \text{Undefined} \end{aligned}$$

4. $2\mathbf{E}^T - 3\mathbf{D}^T$

$$\begin{aligned} 2E^T - 3D^T &= 2 \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3 \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T \\ &= 2 \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix} \end{aligned}$$

5. $\text{tr}(\mathbf{DE})$

$$\begin{aligned} \text{tr}(DE) &= \text{tr} \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right) \\ &= \text{tr} \left(\begin{bmatrix} 1 \cdot 6 + 5 \cdot (-1) + 2 \cdot 4 & 1 \cdot 1 + 5 \cdot 1 + 2 \cdot 1 & 1 \cdot 3 + 5 \cdot 2 + 2 \cdot 3 \\ (-1) \cdot 6 + 0 \cdot (-1) + 1 \cdot 4 & (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1 & (-1) \cdot 3 + 0 \cdot 2 + 1 \cdot 3 \\ 3 \cdot 6 + 2 \cdot (-1) + 4 \cdot 4 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 & 3 \cdot 3 + 2 \cdot 2 + 4 \cdot 3 \end{bmatrix} \right) \\ &= \text{tr} \left(\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix} \right) \\ &= 34 \end{aligned}$$

2 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

- Infinite solutions
- One solution
- No solutions

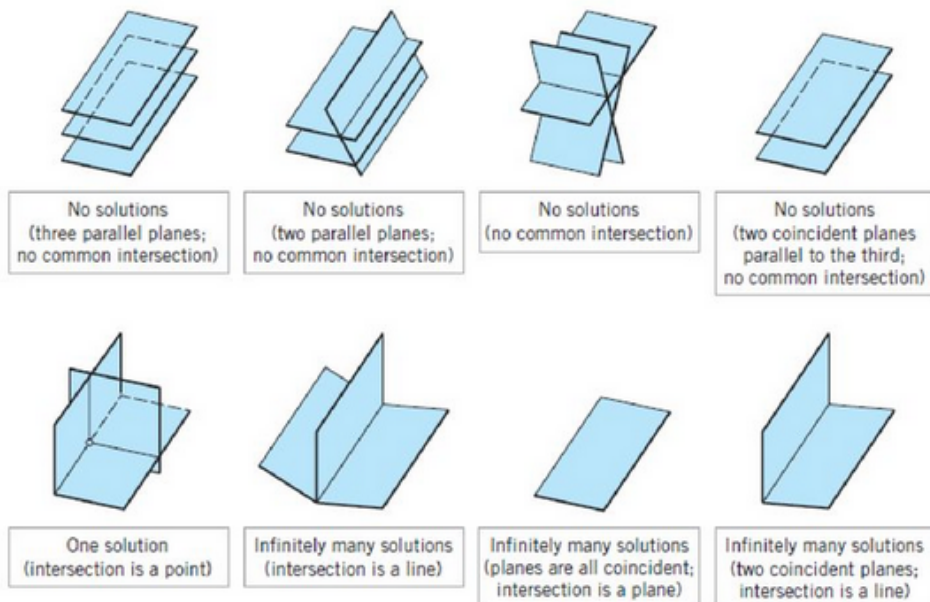
Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear *really* means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.

Linear Systems in Three Unknowns



2.1 Review: Solve the following systems

1.
$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$5x = 15$$

$$x = 3$$

$$2(3) + y = 10$$

$$6 + y = 10$$

$$y = 4$$

2.
$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$y = 10 - 2x$$

$$6x + 3(10 - 2x) = 10$$

$$6x + 30 - 6x = 10$$

$$30 = 10 \therefore \text{no solution}$$

3.
$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$0 = 0$$

$$12 = 12 \therefore \text{infinite solutions}$$

2.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

2.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

2.2 The Augmented Matrix

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array} \right]$$

2.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/subtract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

2.3.1 Example 1... again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\begin{aligned} \left[\begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array} \right] &\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array} \right] &\xrightarrow{R_2-3R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array} \right] \\ &\xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array} \right] &\xrightarrow{R_1-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array} \right] \end{aligned}$$

And so... $x = 3$ and $y = 4$!

2.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array} \right],$$

then the solution to the system will be the ordered triple (c_1, c_2, c_3) .

2.4.1 Example 2: again

$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{R_2-6R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array} \right]$$

This is inconsistent, so there is no solution.

2.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{R_2-15R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array} \right]$$

Since $0 = 0$, there are infinitely many solutions.

2.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \xrightarrow{R_3+4R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_3+\frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_3+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \\ & \xrightarrow{R_1+7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{R_2+4R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \end{aligned}$$

Therefore the solution to (x_1, x_2, x_3) is $(29, 16, 3)$.

2.4.4 Elementary Row Operations & REF Homework Problem (08/08/2023)

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \xrightarrow{\substack{R_2+R_1 \\ R_3-3R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] \xrightarrow{\substack{-R_2 \\ -R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array} \right] \\ & \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array} \right] \xrightarrow{R_3-10R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array} \right] \xrightarrow{\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{\substack{R_1-7R_3 \\ R_2+5R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

Therefore, the solution to (x, y, z) is $(3, 1, 2)$.

2.5 Gaussian Elimination

Vocabulary: A matrix is in Row Echelon Form (REF) if:

- (a) Any rows of all zeroes are placed at the bottom of the matrix
- (b) All other rows have a leading 1 ("pivot")
- (c) As we move down the matrix, each leading 1 is further to the right than the 1 above it

A matrix is in Row Reduced Echelon Form if the three above conditions are met in addition to:

- (d) Each column with a leading 1 has all other entries in the column as a 0. ("pivot column")

2.5.1 Examples (come back)

$\begin{bmatrix} 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 1 & & \end{bmatrix}$	REF? ✓
	RREF? ✓
$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	REF? ✓
	RREF? ✗
	REF? ✗
	RREF? ✗
$[hi]$	

2.6 Gaussian Elimination With Back-Substitution

2.6.1 Goal:

To get the augmented matrix in REF

$$\text{Solve: } \begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$$

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right] & \xrightarrow[\substack{R_2+R_1 \\ R_3-2R_1}]{\substack{R_1+2R_2 \\ R_3+R_2}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

$$x + 9z = 19$$

$$y + 3z = 5$$

$$z = 2$$

$$\therefore z = 2, y = 5 - 3z, x = 19 - 9z$$

$$z = 2, y = 5 - 3(2), x = 19 - 9(2)$$

$$z = 2, y = -1, x = 1$$

Therefore, the solution (x_1, x_2, x_3) is $(1, -1, 2)$.

2.6.2 Gaussian Elimination Homework Problem (08/09/2023)

$$\begin{cases} -2w & + & y & + & z & = & -3 \\ & x & + & 2y & - & z & = & 2 \\ -3w & + & 2x & + & 4y & + & z & = & -2 \\ -w & + & x & - & 4y & - & 7z & = & -19 \end{cases}$$