

Gwinnett School of Math, Science, and Technology

Multivariable Calculus Yearlong Notes

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1st Period

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2023-2024



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1 Systems of Linear Equations and Matrices

1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal \Leftrightarrow they have the same dimensions and values

1.1.1 Addition & Subtraction

Two matrices can be added/subtracted \Leftrightarrow they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3 \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

1.1.3 Matrix Multiplication

- We can **only** multiply an $(m \times n)$ by $(n \times p)$ matrix.
- The resulting matrix will be $(m \times p)$

1.1.4 Properties of Matrix Arithmetic

- (a) $A + B = B + A$ (**Commutative law for addition**)
- (b) $A + (B + C) = (A + B) + C$ (**Associative law for addition**)
- (c) $A(BC) = (AB)C$ (**Associative law for multiplication**)
- (d) $A(B + C) = AB + AC$ (**Left distributive law**)
- (e) $(B + C)A = BA + CA$ (**Right distributive law**)
- (f) $A(B - C) = AB - AC$
- (g) $(B - C)A = BA - CA$
- (h) $a(B+C) = aB + aC$
- (i) $a(B-C) = aB - aC$
- (j) $(a+b)C = aC + bC$
- (k) $(a-b)C = aC - bC$
- (l) $a(bC) = (ab)C$
- (m) $a(BC) = (aB)C = B(aC)$

1.1.5 Examples

1.

$$\begin{aligned} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \end{aligned}$$

2.

$$\begin{aligned} & \begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix} \end{aligned}$$

3.

$$\begin{aligned} & \begin{bmatrix} 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} \\ &= [4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2] \\ &= [30] \end{aligned}$$

1.2 Transpose of a Matrix

The transpose of an $(m \times n)$ matrix is the $(n \times m)$ matrix where the rows and columns are swapped.

$$\text{If } B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix} \end{aligned}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix A is equal to the sum of all the elements on the main diagonal: $\text{tr}(A)$

1.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

1.3 Homework — “Matrix Stuff” (08/03/2023)

1.3.1 Suppose that A, B, C, D and E are matrices with the following sizes:

A	B	C	D	E
(3×2)	(2×3)	(3×3)	(3×2)	(2×3)

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

Undefined	Defined; (4×2)	Defined; (5×5)	Defined; (5×2)
BA	$AC + D$	$E(A + B)$	$(A^T + E)D$
$AB + B$			$E(AC)$
$E^T A$			
$AE + B$			

1.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2. $2A^T + C$

$$\begin{aligned} 2A^T + C &= 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2 \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3. $B^T + 5C^T$

$$\begin{aligned}
 B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\
 &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\
 &= \text{Undefined}
 \end{aligned}$$

4. $2E^T - 3D^T$

$$\begin{aligned}
 2E^T - 3D^T &= 2 \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3 \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T \\
 &= 2 \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}
 \end{aligned}$$

5. $\text{tr}(DE)$

$$\begin{aligned}
 \text{tr}(DE) &= \text{tr} \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right) \\
 &= \text{tr} \left(\begin{bmatrix} 1 \cdot 6 + 5 \cdot (-1) + 2 \cdot 4 & 1 \cdot 1 + 5 \cdot 1 + 2 \cdot 1 & 1 \cdot 3 + 5 \cdot 2 + 2 \cdot 3 \\ (-1) \cdot 6 + 0 \cdot (-1) + 1 \cdot 4 & (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1 & (-1) \cdot 3 + 0 \cdot 2 + 1 \cdot 3 \\ 3 \cdot 6 + 2 \cdot (-1) + 4 \cdot 4 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 & 3 \cdot 3 + 2 \cdot 2 + 4 \cdot 3 \end{bmatrix} \right) \\
 &= \text{tr} \left(\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix} \right) \\
 &= 34
 \end{aligned}$$

2 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

- Infinite solutions
- One solution
- No solutions

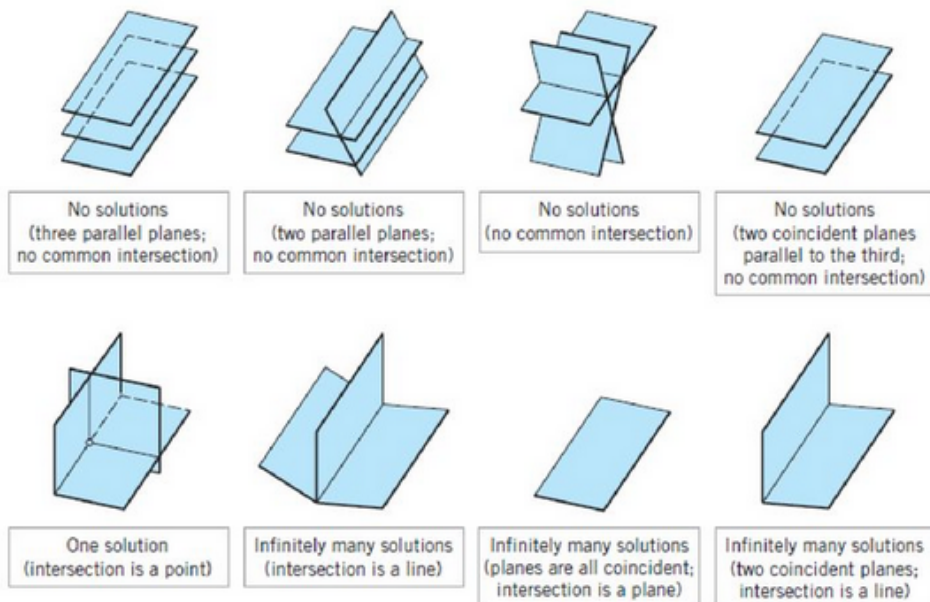
Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear *really* means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.

Linear Systems in Three Unknowns



2.1 Review: Solve the following systems

1.
$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$5x = 15$$

$$x = 3$$

$$2(3) + y = 10$$

$$6 + y = 10$$

$$y = 4$$

2.
$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$y = 10 - 2x$$

$$6x + 3(10 - 2x) = 10$$

$$6x + 30 - 6x = 10$$

$$30 = 10. \therefore \text{no solution}$$

3.
$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$0 = 0$$

$$12 = 12. \therefore \text{infinite solutions}$$

2.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

2.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

2.2 The Augmented Matrix

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array} \right]$$

2.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/subtract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

2.3.1 Example 1... again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\begin{aligned} \left[\begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array} \right] &\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array} \right] \\ &\xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array} \right] \xrightarrow{R_1 - \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array} \right] \end{aligned}$$

And so... $x = 3$ and $y = 4$!

2.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array} \right],$$

then the solution to the system will be the ordered triple (c_1, c_2, c_3) .

2.4.1 Example 2: again

$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{R_2-6R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array} \right]$$

This is inconsistent, so there is no solution.

2.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{R_2-15R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array} \right]$$

Since $0 = 0$, there are infinitely many solutions.

2.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \xrightarrow{R_3+4R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_3+\frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\begin{matrix} R_1+7R_3 \\ R_2+4R_3 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \end{aligned}$$

Therefore the solution to (x_1, x_2, x_3) is $(29, 16, 3)$.

2.4.4 Elementary Row Operations & REF Homework Problem (08/08/2023)

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] \xrightarrow[-R_3]{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array} \right] \\ & \xrightarrow[R_3-10R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array} \right] \xrightarrow{\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow[R_2+5R_3]{R_1-7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

Therefore, the solution to (x, y, z) is $(3, 1, 2)$.

2.5 Gaussian Elimination

Vocabulary: A matrix is in Row Echelon Form (REF) if:

- (a) Any rows of all zeroes are placed at the bottom of the matrix
- (b) All other rows have a leading 1 ("pivot")
- (c) As we move down the matrix, each leading 1 is further to the right than the 1 above it

A matrix is in Row Reduced Echelon Form if the three above conditions are met in addition to:

- (d) Each column with a leading 1 has all other entries in the column as a 0. ("pivot column")

2.5.1 Examples

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF? ✓
RREF? ✓

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF? ✓
RREF? ✗

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

REF? ✗
RREF? ✗

2.6 Gaussian Elimination With Back-Substitution

2.6.1 Goal:

To get the augmented matrix in REF

Solve:
$$\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$$

$$\begin{aligned} &\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array} \right] \xrightarrow[R_3+R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array} \right] \\ &\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

$$x + 9z = 19$$

$$y + 3z = 5$$

$$z = 2$$

$$\therefore z = 2, y = 5 - 3z, x = 19 - 9z$$

$$z = 2, y = 5 - 3(2), x = 19 - 9(2)$$

$$z = 2, y = -1, x = 1$$

Therefore, the solution (x_1, x_2, x_3) is $(1, -1, 2)$.

2.6.2 Gaussian Elimination Homework Problem (08/09/2023)

$$\begin{cases} -2w + y + z = -3 \\ x + 2y - z = 2 \\ -3w + 2x + 4y + z = -2 \\ -w + x - 4y - 7z = -19 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{cccc|c} -2 & 0 & 1 & 1 & -3 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -1 & 1 & -4 & -7 & -19 \end{array} \right] \xrightarrow{R_4} \left[\begin{array}{cccc|c} -1 & 1 & -4 & -7 & -19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array} \right] \xrightarrow{-R_1} \\ & \left[\begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array} \right] \xrightarrow{\begin{matrix} R_3+3R_1 \\ R_4+2R_1 \end{matrix}} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & 16 & 22 & 55 \\ 0 & -2 & 9 & 15 & 35 \end{array} \right] \xrightarrow{\begin{matrix} R_1+R_2 \\ R_3+R_2 \\ R_4+2R_2 \end{matrix}} \\ & \left[\begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 18 & 21 & 57 \\ 0 & 0 & 13 & 13 & 39 \end{array} \right] \xrightarrow{\frac{1}{18}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 13 & 13 & 39 \end{array} \right] \xrightarrow{\begin{matrix} R_1-6R_3 \\ R_2-2R_3 \\ R_4-13R_3 \end{matrix}} \\ & \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & -\frac{13}{6} & -\frac{13}{6} \end{array} \right] \xrightarrow{-\frac{6}{13}R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{matrix} R_1+R_4 \\ R_2+\frac{10}{3}R_4 \\ R_3-\frac{7}{6}R_4 \end{matrix}} \\ & \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \Rightarrow \begin{cases} w = 3 \\ x = -1 \\ y = 2 \\ z = 1 \end{cases} \end{aligned}$$

2.7 Gauss-Jordan Elimination

2.7.1 Goal:

To get the matrix into RREF

$$\text{Solve: } \begin{cases} x_1 - 3x_3 = -2 \\ 3x_1 + x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + x_3 = 4 \end{cases}$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array} \right] \xrightarrow[R_3-2R_1]{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array} \right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array} \right] \\ & \xrightarrow{\frac{-1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow[R_2-7R_3]{R_1+3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = 2 \end{cases} \end{aligned}$$

2.8 Matrix Properties, Equations, and Inverses

2.8.1 With Real Numbers

- If $ab = bc$, then $a = c$, if $b \neq 0$
- If $ab = 0$, then $a = 0$ or $b = 0$, or both

2.8.2 With Matrices

- If $AB = AC$, then $B = C$, if A is invertible
- If $AB = [0]$, then $A = [0]$ or $B = [0]$, or both

2.8.2.1 Multiply:

$$\begin{aligned} & \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(5) + 3(-3) & 2(-3) + 3(2) \\ 3(5) + 5(-3) & 3(-3) + 5(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

2.8.3 Matrix Inverses

- If a matrix has an inverse, it is said to be invertible or non-singular.
- If a matrix does not have an inverse, it is said to be singular.
- Every square matrix has a “special number” associated with it called the **determinant**.
- For the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $ad - bc$
- $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- When $\det A = 0$, the matrix is singular and has no inverse (since you cannot divide by zero)

Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}^{-1} &= \frac{1}{\det A} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{(4)(2) - (3)(1)} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix} \end{aligned}$$