Gwinnett School of Math, Science, and Technology

Multivariable Calculus Yearlong Notes

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1 Chapter 1: Systems of Linear Equations and Matrices

1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal $\iff$ they have the same dimensions and values

1.1.1 Addition & Subtraction

Two matrices can be added/subtracted $\iff$ they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

1.1.3 Matrix Multiplication

- We can **only** multiply an (m x n) by (n x p) matrix.
- The resulting matrix will be (m x p)

1.1.4 Properties of Matrix Arithmetic

(a) $A + B = B + A$ **(Commutative law for addition)**
(b) $A + (B + C) = (A + B) + C$ **(Associative law for addition)**
(c) $A(BC) = (AB)C$ **(Associative law for multiplication)**
(d) $A(B + C) = AB + AC$ **(Left distributive law)**
(e) $(B + C)A = BA + CA$ **(Right distributive law)**
(f) $A(B - C) = AB - AC$
(g) $(B - C)A = BA - CA$
(h) a(B+C) = aB + aC
(i) a(B-C) = aB - aC
(j) (a+b)C = aC + bC
(k) (a-b)C = aC - bC
(l) a(bC) = (ab)C
(m) a(BC) = (aB)C = B(aC)

1.1.5 Examples

1.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix}$$
$$= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix}$$

3.

$$\begin{bmatrix} 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2 \end{bmatrix}$$
$$= \begin{bmatrix} 30 \end{bmatrix}$$

1.2 Transpose of a Matrix

The transpose of an (m x n) matrix is the (n x m) matrix where the rows and columns are swapped.

If $B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$

$$B \cdot B^T = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix}$$
$$= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix A is equal to the sum of all the elements on the main diagonal: tr(A)

1.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

1.3 Homework — “Matrix Stuff” (08/03/2023)

1.3.1 Suppose that A, B, C, D and E are matrices with the following sizes:

A	B	C	D	E
(3×2)	(2×3)	(3×3)	(3×2)	(2×3)

For each matrix operation, sort them into undefined if the operation can’t be done, or defined if it can along with the correct dimensions of the outcome.

Undefined	Defined; (4×2)	Defined; (5×5)	Defined; (5×2)
BA $AB + B$ $E^T A$ $AE + B$	$AC + D$	$E(A + B)$	$(A^T + E)D$ $E(AC)$

1.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2. $\mathbf{2A^T + C}$

$$\begin{aligned} 2A^T + C &= 2\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3. $\mathbf{B^T + 5C^T}$

$$
\begin{aligned}
B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\
&= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\
&= \text{Undefined}
\end{aligned}
$$

4. $\mathbf{2E^T - 3D^T}$

$$
\begin{aligned}
2E^T - 3D^T &= 2\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T \\
&= 2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} \\
&= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}
\end{aligned}
$$

5. tr(**DE**)

$$
\begin{aligned}
\operatorname{tr}(DE) &= \operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right) \\
&= \operatorname{tr}\left(\begin{bmatrix} 1\cdot 6 + 5\cdot(-1) + 2\cdot 4 & 1\cdot 1 + 5\cdot 1 + 2\cdot 1 & 1\cdot 3 + 5\cdot 2 + 2\cdot 3 \\ (-1)\cdot 6 + 0\cdot(-1) + 1\cdot 4 & (-1)\cdot 1 + 0\cdot 1 + 1\cdot 1 & (-1)\cdot 3 + 0\cdot 2 + 1\cdot 3 \\ 3\cdot 6 + 2\cdot(-1) + 4\cdot 4 & 3\cdot 1 + 2\cdot 1 + 4\cdot 1 & 3\cdot 3 + 2\cdot 2 + 4\cdot 3 \end{bmatrix}\right) \\
&= \operatorname{tr}\left(\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}\right) \\
&= 34
\end{aligned}
$$

2 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

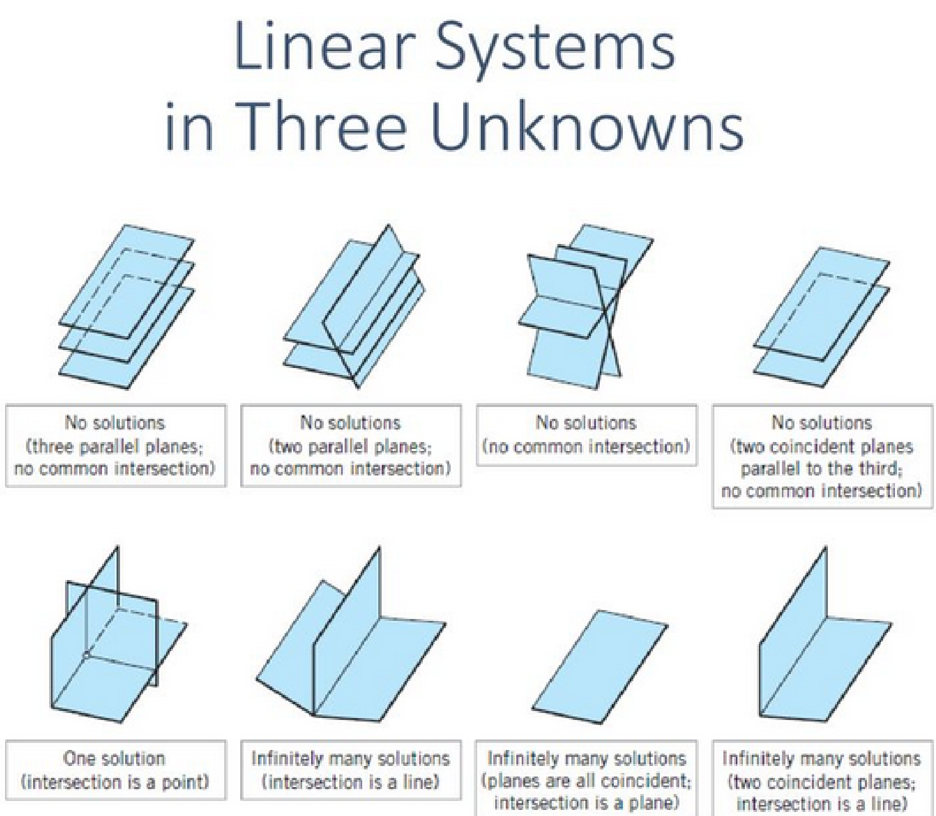
- Infinite solutions
- One solution
- No solutions

Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear *really* means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.



2.1 Review: Solve the following systems

1. $\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$

$$\begin{aligned} 5x &= 15 \\ x &= 3 \\ 2(3) + y &= 10 \\ 6 + y &= 10 \\ y &= 4 \end{aligned}$$

2. $\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$

$$\begin{aligned} y &= 10 - 2x \\ 6x + 3(10 - 2x) &= 10 \\ 6x + 30 - 6x &= 10 \\ 30 &= 10 \therefore \text{no solution} \end{aligned}$$

3. $\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$

$$\begin{aligned} 0 &= 0 \\ 12 &= 12 \therefore \text{infinite solutions} \end{aligned}$$

2.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

2.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

2.2 The Augmented Matrix

$$\begin{cases} x - \quad y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array}\right]$$

2.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/substract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

2.3.1 Example 1... again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array}\right]$$
$$\xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{R1-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

And so... $x = 3$ and $y = 4$!

2.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right],$$

then the solution to the system will be the ordered triple (c_1, c_2, c_3).

2.4.1 Example 2: again

$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array}\right] \xrightarrow{\frac{1}{2}R1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array}\right] \xrightarrow{R2-6R1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array}\right]$$

This is inconsistent, so there is no solution.

2.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array}\right] \xrightarrow{\frac{1}{5}R1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array}\right] \xrightarrow{R2-15R1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array}\right]$$

Since $0 = 0$, there are infinitely many solutions.

2.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow{R3+4R1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R3+\frac{3}{2}R2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array}\right]$$
$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \begin{array}{c} \xrightarrow{R_1+2R_2} \\ \xrightarrow{R_3+3R_2} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} \xrightarrow{R1+7R_3} \\ \xrightarrow{R_2+4R_3} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore the solution to (x_1, x_2, x_3) is $(29, 16, 3)$.

2.4.4 Elementary Row Operations & REF Homework Problem (08/08/2023)

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \xrightarrow[-R_3]{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array}\right]$$

$$\xrightarrow[R_3-10R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array}\right] \xrightarrow{\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2+5R_3]{R_1-7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore, the solution to (x, y, z) is $(3, 1, 2)$.

2.5 Gaussian Elimination

Vocabulary: A matrix is in Row Echelon Form (REF) if:

(a) Any rows of all zeroes are placed at the bottom of the matrix
(b) All other rows have a leading 1 ("pivot")
(c) As we move down the matrix, each leading 1 is further to the right than the 1 above it

A matrix is in Row Reduced Echelon Form if the three above conditions are met in adition to:

(d) Each column with a leading 1 has all other entries in the column as a 0. ("pivot column")

2.5.1 Examples

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF? ✓
RREF? ✓

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF? ✓
RREF? ×

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

REF? ×
RREF? ×

2.6 Gaussian Elimination With Back-Substitution

2.6.1 Goal:

To get the augmented matrix in REF

Solve: $\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 \qquad = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow[R_3+R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x + 9z &= 19 \\ y + 3z &= 5 \\ z &= 2 \\ \therefore z &= 2, y = 5 - 3z, x = 19 - 9z \\ z &= 2, y = 5 - 3(2), x = 19 - 9(2) \\ z &= 2, y = -1, x = 1 \end{aligned}$$

Therefore, the solution (x_1, x_2, x_3) is $(1, -1, 2)$.

2.6.2 Gaussian Elimination Homework Problem (08/09/2023)

$$\begin{cases} -2w \phantom{{}+2x} + y + z = -3 \\ \phantom{-3w+{}} x + 2y - z = 2 \\ -3w + 2x + 4y + z = -2 \\ -w + x - 4y - 7z = -19 \end{cases}$$

$$\left[\begin{array}{cccc|c} -2 & 0 & 1 & 1 & -3 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -1 & 1 & -4 & -7 & -19 \end{array}\right] \begin{array}{c} \xrightarrow{R_4} \\ \\ \\ \xrightarrow{R_1} \end{array} \left[\begin{array}{cccc|c} -1 & 1 & -4 & -7 & -19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array}\right] \xrightarrow{-R_1}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array}\right] \begin{array}{c} \\ \\ \xrightarrow{R_3+3R_1} \\ \xrightarrow{R_4+2R_1} \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & 16 & 22 & 55 \\ 0 & -2 & 9 & 15 & 35 \end{array}\right] \begin{array}{c} \xrightarrow{R_1+R_2} \\ \\ \xrightarrow{R_3+R_2} \\ \xrightarrow{R_4+2R_2} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 18 & 21 & 57 \\ 0 & 0 & 13 & 13 & 39 \end{array}\right] \xrightarrow{\frac{1}{18}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 13 & 13 & 39 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-6R_3} \\ \xrightarrow{R_2-2R_3} \\ \\ \xrightarrow{R_4-13R_3} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & -\frac{13}{6} & -\frac{13}{6} \end{array}\right] \xrightarrow{\frac{-6}{13}R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_1+R_4} \\ \xrightarrow{R_2+\frac{10}{3}R_4} \\ \xrightarrow{R_3-\frac{7}{6}R_4} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Longrightarrow \begin{cases} w = 3 \\ x = -1 \\ y = 2 \\ z = 1 \end{cases}$$

2.7 Gauss-Jordan Elimination

2.7.1 Goal:

To get the matrix into RREF

Solve: $\begin{cases} x_1 \qquad - 3x_3 = -2 \\ 3x_1 + \ \ x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + \ \ x_3 = 4 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2-7R_3]{R_1+3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \Longrightarrow \begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = 2 \end{cases}$$

2.8 Matrix Properties, Equations, and Inverses

2.8.1 With Real Numbers

- If $ab = bc$, then $a = c$, if $b \neq 0$
- If $ab = 0$, then $a = 0$ or $b = 0$, or both

2.8.2 With Matrices

- If $AB = AC$, then $B = C$, if A is invertible
- If $AB = [0]$, then $A = [0]$ or $B = [0]$, or both

2.8.2.1 Multiply:

$$\begin{aligned} &\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(5) + 3(-3) & 2(-3) + 3(2) \\ 3(5) + 5(-3) & 3(-3) + 5(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

2.8.3 Matrix Inverses

- If a matrix has an inverse, it is said to be invertible or non-singular.
- If a matrix does not have an inverse, it is said to be singular.
- Every square matrix has a “special number” associated with it called the **determinant**.
- For the 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $ad - bc$
- $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- When $\det A = 0$, the matrix is singular and has no inverse (since you cannot divide by zero)

Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$
\begin{aligned}
\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}^{-1} &= \frac{1}{\det A} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\
&= \frac{1}{(4)(2) - (3)(1)} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}
\end{aligned}
$$

3 Chapter 2: Determinants

3.1 Prior Knowledge:

$\begin{bmatrix} 10 & -4 \\ -3 & -5 \end{bmatrix} = -50- = -62$

$$\begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$$
$$= ((2 \cdot 2 \cdot -2) + (4 \cdot 3 \cdot 3) + (3 \cdot -1 \cdot 0)) - ((3 \cdot 2 \cdot 3) + (0 \cdot 3 \cdot 2) + (-2 \cdot -1 \cdot 4))$$
$$= (-8 + 36 + 0) - (18 + 0 + 8)$$
$$= 28 - 26$$
$$= 2$$

3.2 Minors & Cofactors

Given a square matrix A, the minor of matrix element a_{ij}, (M_{ij}) is the determinant of the matrix formed by removing the i^{th} row and j^{th} column from matrix A.

The cofactor of matrix element $a_{ij}, C_{ij} = (-1)^{i+j} \cdot M_{ij}$

3.2.1 Example

Let det $\begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$. What is the cofactor of element (1, 1)?

Cofactor checkerboard:

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$M_{11} = \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} = -4$

$C_{11} = 1 \cdot -4 = -4$

Find the minor and cofactor of: \ a) $a_{21} = -1$

$$M_{21} = \begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} = -8$$
$$C_{21} = 8$$

b) $a_{33} = -2$

$$M_{33} = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 8$$
$$C_{33} = 8$$

3.3 Cofactor Expansion

1) Pick a row or column
2) Multiply every entry in that row or column by it's corresponding cofactor
3) Add those together. That's it

$$A = \begin{bmatrix} 6 & 7 & -1 \\ 0 & 4 & 1 \\ 2 & 5 & -3 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 6\left(\begin{vmatrix} 4 & 1 \\ 5 & -3 \end{vmatrix}\right) + 7\left(-\begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix}\right) + -1\left(\begin{vmatrix} 0 & 4 \\ 2 & 5 \end{vmatrix}\right) \\ &= 6(-17) + 7(2) + (-1(-8)) \\ &= -102 + 14 + 8 \\ &= -80 \end{aligned}$$

3.3.1 Example

$$A = \begin{bmatrix} 6 & 4 & 2 \\ 5 & -6 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

$$\begin{aligned} &6\begin{vmatrix} -6 & 1 \\ 3 & 0 \end{vmatrix} + 4\left(-\begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix}\right) + 2\begin{vmatrix} 5 & -6 \\ 0 & 3 \end{vmatrix} \\ &= 6(-3) + 0 + 2(15) \\ &= -18 + 30 \\ &= 12 \end{aligned}$$

3.3.2 Does the method generalize to 2×2 matrices?

$$\begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix}$$
$$= 3|2| - 5|7|$$
$$= 6 - 35$$
$$= -29$$

The determinant of a 1×1 matrix is... **itself!**

3.3.3 Find the determinant of a 4×4

$$A = \begin{bmatrix} -3 & 2 & 0 & 8 \\ 2 & 1 & 0 & -4 \\ 5 & -2 & 1 & 5 \\ 2 & 3 & 0 & 6 \end{bmatrix}$$

$$= 0 + 0 + \begin{vmatrix} -3 & 2 & 8 \\ 2 & 1 & -4 \\ 2 & 3 & 6 \end{vmatrix} + 0$$
$$= -2\begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} -3 & 8 \\ 2 & 6 \end{vmatrix} - \left(-4\begin{vmatrix} -3 & 2 \\ 2 & 3 \end{vmatrix}\right)$$
$$= 24 - 34 - 52$$
$$= -62$$

3.4 Theorem

If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the elements in that row or column by their corresponding cofactors is **always the same** and is called the determinant of A.

3.4.1 Example

Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$
\begin{aligned}
&1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\
&= \left(-2 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\right) \\
&= -6
\end{aligned}
$$

3.5 Triangular Matrices

Find the determinant of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$
\begin{aligned}
&\begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} \\
&= 2 \begin{vmatrix} 3 & 3 \\ 0 & 4 \end{vmatrix} \\
&= 2(3 \cdot 4) \\
&= 2 \cdot 12 \\
&= 24
\end{aligned}
$$

If A is an $n \times n$ triangular matrix, then $\det(A)$ is equal to the product of the elements along the main diagonal.

3.6 An Important Definition

Elementary Matrix a matrix that can be obtanied from the $n \times n$ identity matrix by performing a single row operation. \

Are the following matrices elementary? 1) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} + (R_3 + 5R_1)$ yes 2) $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix} + (R_1 + 5R_2)$... no

3.7 A Pair of Theorems

3.7.1 Theorem: If a square matrix A has a row of column of zeros, then $\det(A) = 0$

3.7.2 Theorem: If A is a square matrix, then $\det(A) = \det(A^T)$

3.8 Unit 1 & 2 Homework Problems

3.8.1 “Gaussian Elimination” (08/11/2023)

3.8.1.1 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$\xrightarrow{R_3+10R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] \xrightarrow{-\frac{1}{52}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\therefore \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ \phantom{x_1 + {}} x_2 - 5x_3 = -9 \\ x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

3.8.1.2 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ -2x_1 - 3x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + 4x_3 = 0 \end{cases}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 4 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{7}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & \frac{2}{7} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\therefore \begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ \phantom{x_1 - 2{}} x_2 + \frac{2}{7}x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow 1 \neq 0 \therefore \text{ no solution}$$

3.8.2 "Inverses and Determinants" (08/14)

3.8.2.1 Find the determinants of the following:

1) $\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

$$\begin{vmatrix} 2 & -3 \\ 4 & 4 \end{vmatrix} = 2(4) - (-3)(4) = 8 + 12 = 20$$

2) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2(3) - 0(0) = 6$$

3) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

3.8.2.2 Find the INVERSES of those matrices:

1) $\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

$$\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}^{-1} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

2) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

3) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3.8.3 Inverses and Determinants (08/15)

3.8.3.1 Use a matrix equation to solve the following problems:

1) $\begin{cases} 3x_1 - 2x_2 = 1 \\ 4x_1 + 5x_2 = 3 \end{cases}$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{23} \\ \frac{9}{23} \end{bmatrix}$$

2) $\begin{cases} 6x_1 + x_2 = 0 \\ 4x_1 - 3x_2 = -2 \end{cases}$

$$\Rightarrow \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} -3 & -1 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} \\ -\frac{4}{11} \end{bmatrix}$$

3.8.4 Consistent Systems (08/21)

3.8.4.1 Solve the linear systems together by reducing the appropriate augmented matrix.

$$\begin{cases} x_1 - 5x_2 = b_1 \\ 3x_1 + 2x_2 = b_2 \end{cases}$$

1) $b_1 = 1,\ b_2 = 4$
2) $b_1 = -2,\ b_2 = 5$

First, let's solve it for the general case:

$$\left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 3 & 2 & b_2 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 17 & b_2 - 3b_1 \end{array}\right] \xrightarrow{\frac{1}{17}R_2} \left[\begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 1 & \frac{b_2-3b_1}{17} \end{array}\right] \xrightarrow{R_1+5R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{2b_1+5b_2}{17} \\ 0 & 1 & \frac{-3b_1+b_2}{17} \end{array}\right]$$

Therefore, the solution to the general case is $(x_1, x_2) = (\frac{2b_1+5b_2}{17}, \frac{-3b_1+b_2}{17})$

And so, for the specific cases:

1) $(x_1, x_2) = \left(\frac{2(1)+5(4)}{17}, \frac{-3(1)+4}{17}\right) = \left(\frac{13}{17}, \frac{1}{17}\right)$
2) $(x_1, x_2) = \left(\frac{2(-2)+5(5)}{17}, \frac{-3(-2)+5}{17}\right) = \left(\frac{16}{17}, \frac{11}{17}\right)$

3.8.4.2 Determine the conditions on b, if any, in order to guarantee that the linear system is consistent.

$$\begin{cases} x_1 + 3x_2 = b_1 \\ -2x_1 + \ \ x_2 = b_2 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & b_2 + 2b_1 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{b_1-3b_2}{7} \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array}\right]$$

There are no conditions. The system is consistent for all values of b_1 and b_2.

3.8.5 Another "determining the conditions" problem:

$$\begin{cases} x_1 - 2x_2 - \ \ x_3 = b_1 \\ -4x_1 + 5x_2 + 2x_3 = b_2 \\ -4x_1 + 7x_2 + 4x_3 = b_3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right] \xrightarrow[R_3+4R_1]{R_2+4R_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & b_2 + 4b_1 \\ 0 & -1 & 0 & b_3 + 4b_1 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{b_3+4b_1}{3} \end{array}\right]$$

$$\xrightarrow{-\frac{3}{2}R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & 1 & \frac{-b_3-4b_1}{2} \end{array}\right]$$

Therefore, the system is consistent for all values of b_1, b_2, and b_3.

3.8.6 Triangular and Diagonal Matrices

3.8.6.1 Find A^2

1) $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 0(0) & 1(0) + 0(-2) \\ 0(1) + (-2)(0) & 0(0) + (-2)(-2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

2) $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$\begin{aligned} A^2 &= \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (-6)(-6) + (0)(0) + (0)(0) & (-6)(0) + (0)(3) + (0)(0) & (-6)(0) + (0)(0) + (0)(5) \\ (0)(-6) + (3)(0) + (0)(0) & (0)(0) + (3)(3) + (0)(0) & (0)(0) + (3)(0) + (0)(5) \\ (0)(-6) + (0)(0) + (5)(0) & (0)(0) + (0)(3) + (5)(0) & (0)(0) + (0)(0) + (5)(5) \end{bmatrix} \\ &= \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} \end{aligned}$$

3.8.6.2 Find A^{-k}, such that k is some nonzero constant

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} A^{-k} &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-k} \\ &= \begin{bmatrix} 2^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & 2^{-k} \end{bmatrix} \end{aligned}$$

4. Determine whether each matrix is symmetric or not.

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

Not symmetric

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

3.8.6.3 Find a diagonal matrix *A* that satisfies the given condition

1) $A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{\frac{1}{5}} \\ &= \begin{bmatrix} 1^{\frac{1}{5}} & 0 & 0 \\ 0 & (-1)^{\frac{1}{5}} & 0 \\ 0 & 0 & (-1)^{\frac{1}{5}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

2) $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} A &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-\frac{1}{2}} \\ &= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 & 0 \\ 0 & 4^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1^{-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3.8.7 Determinants and Triangular Matrices (08/29)

3.8.7.1 What is C_{32}

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} C_{32} &= (-1)^{3+2} \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \left(2 \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 3 \\ 3 & 0 \end{vmatrix} + 1 \begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} \right) \\ &= - (2(-3) - (-1)(-9) + 1(-3)) \\ &= -(-6 + 9 - 3) \\ &= 0 \end{aligned}$$

3.8.7.2 Find all values of λ such that $|A| = 0$

$$A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (\lambda - 2)(\lambda + 4) - (-5)(1) \\ &= \lambda^2 + 2\lambda - 8 + 5 \\ &= \lambda^2 + 2\lambda - 3 \\ &= (\lambda + 3)(\lambda - 1) \\ &= 0 \end{aligned}$$

Therefore, $\lambda = -3, 1$

3.8.7.3 For the matrix $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$ find the determinant 3 sp.different ways with co-factor expansion. Pick sp.different rows and columns each time.

$$\begin{aligned}
\det(A) &= 3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0\begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} \\
&= 3(-1(-4) - 5(9)) - 0(2(-4) - 5(1)) + 0(2(9) - (-1)(1)) \\
&= 3(4 - 45) - 0(-8 - 5) + 0(18 + 1) \\
&= 3(-41) - 0(-13) + 0(19) \\
&= 36
\end{aligned}$$

$$\begin{aligned}
\det(A) &= 0\begin{vmatrix} 2 & 5 \\ 9 & -4 \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 0\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} \\
&= 0(2(-4) - 5(9)) - 3(3(-4) - 0(1)) + 0(3(5) - 0(2)) \\
&= 0(-8 - 45) - 3(-12 - 0) + 0(15 - 0) \\
&= 0(-53) - 3(-12) \\
&= 36
\end{aligned}$$

$$\begin{aligned}
\det(A) &= 0\begin{vmatrix} 2 & -1 \\ 9 & -4 \end{vmatrix} - 0\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 3\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \\
&= 0(2(-4) - (-1)(9)) - 0(3(-4) - 0(1)) + 3(3(-1) - 0(2)) \\
&= 0(-8 + 9) - 0(-12 - 0) + 3(-3 - 0) \\
&= 0(1) - 0(-12) + 3(-3) \\
&= 0 + 0 - 9 \\
&= 36
\end{aligned}$$

3.8.7.4 Evaluate $\det(A)$ **by a cofactor expansion along a row or column of your choice**

$$A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - k\begin{vmatrix} 1 & k^2 \\ 1 & k^2 \end{vmatrix} + k^2\begin{vmatrix} 1 & k \\ 1 & k \end{vmatrix} \\ &= 1(k^2 - k^2) - k(1(k^2) - k^2(1)) + k^2(1(k) - k(1)) \\ &= 0 \end{aligned}$$

3.8.7.5 Evaluate the determinant of the following matrices by just looking at them.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1(-1)(1) = -1$$

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det(A) = 1(1)(2)(3) = 6$$

3.8.7.6 Show that the value of the determinant is independent of θ

$$A = \begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

$$\det(A) = \sin\theta\begin{vmatrix} \sin\theta & 0 \\ \sin\theta + \cos\theta & 1 \end{vmatrix} - \cos\theta\begin{vmatrix} \cos\theta & 0 \\ \sin\theta + \cos\theta & 1 \end{vmatrix}$$

$$+0\begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta + \cos\theta & \sin\theta \end{vmatrix}$$

$$= \sin\theta\,(\sin\theta(1) - 0(\sin\theta + \cos\theta)) - \cos\theta\,(\cos\theta(1) - 0(\sin\theta + \cos\theta))$$

$$+0\,(\cos\theta(\sin\theta) - \sin\theta(\sin\theta + \cos\theta))$$

$$= \sin^2\theta - \cos^2\theta$$

$$= 1$$

3.8.8 Row operations and Determinants (08/31)

3.8.8.1 Find the determinant of $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$ **WITHOUT using cofactor expansion**

$$
\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{28}{2} \end{vmatrix} \\
&= 1(-2)\left(\frac{28}{2}\right) \\
&= -28
\end{aligned}
$$

3.8.8.2 Find the determinant of $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$
\begin{aligned}
\det(A) &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -3 & 2 \end{vmatrix} \\
&= 2(-2)(-4)(2) \\
&= 64
\end{aligned}
$$

3.8.9 Adjoints and Cramer's Rule (09/05)

3.8.9.1 Find the inverse of $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ **using the adjoint method**

$$
\begin{aligned}
\det(A) &= 2\begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} - 5\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} + 5\begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\
&= 2(-3) - 5(-3) + 5(-2) \\
&= -6 + 15 - 10 \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{adj}(A) &= \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 5 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} \end{bmatrix} \\
&= \begin{bmatrix} (-1)(3) & -(-1)(3) & -4+2 \\ -(15-20) & 6-10 & -(8-10) \\ 5 & -5 & -2+5 \end{bmatrix}^T \\
&= \begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}^T \\
&= \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\therefore A^{-1} &= -\begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}
\end{aligned}
$$

3.8.9.2 Solve the following system of equations using Cramer's Rule

$$\begin{cases} 4x + 5y \qquad\; = 2 \\ 11x + \;\; y + 2z = 3 \\ \;\; x + 5y + 2z = 1 \end{cases} \longrightarrow \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \longrightarrow 4\begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5\begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} = -132$$

$$\begin{aligned} \det(x) &= \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 2(2 - 10) - 5(6 - 2) \\ &= -16 - 20 \\ &= -36 \\ \det(y) &= \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \\ &= 4\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 4(6 - 2) - 2(22 - 2) \\ &= 16 - 40 \\ &= -24 \\ \det(z) &= \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \\ &= 4\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 5\begin{vmatrix} 11 & 3 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 11 & 1 \\ 1 & 5 \end{vmatrix} \\ &= 4(1 - 15) - 5(33 - 3) + 2(55 - 1) \\ &= -56 - 150 + 108 \\ &= -98 \end{aligned}$$

Therefore, the solution $(x, y, z) = (\frac{3}{11}, \frac{2}{11}, -\frac{49}{66})$

4 Chapter 5: Eigenvectors and Eigenvalues

4.1 Eigenvalues and Eigenvectors (11/06)

If A is an $n \times n$ matrix, then a non-zero vector $\mathbf{x}$, in R^n, is called an eigenvector of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. This scalar λ is called an eigenvalue of A and $\mathbf{x}$ is said to be an eigenvector corresponding to λ.

See, normally, multiplying a vector by a square matrix changes both the magnitude and the direction of the vector. Really screws it up.

Some examples:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 7 & 8 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 23 \\ 4 \end{bmatrix}$

However, there are some ways to get consistent results.

4.1.1 Examples

4.1.1.1 $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ because

$A\vec{x} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{x} \therefore \lambda = 2$

4.1.1.2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?

$$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 1(6) + 6(-5) \\ 5(6) + 2(-5) \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} \therefore \lambda = -4$$

$$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 6(-2) \\ 5(3) + 2(-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\vec{v}$$

4.2 Eigenvector Homework Problem (11/06)

Confirm by multiplication that x is an eigenvector of A, and find the corresponding eigenvalue.

4.2.1 $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$A\mathbf{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1) + 0(2) + 1(1) \\ 2(1) + 3(2) + 2(1) \\ 1(1) + 0(2) + 4(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \therefore \lambda = 5$$

4.3 Finding Eigenvalues and Eigenvectors (11/07)

Essential question:

If we know an n × n matrix A, can we find its λ?

If $A\vec{x} = \lambda\vec{x}$, then:

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A\vec{x} - \lambda\vec{x} &= \vec{0} \\ (A - \lambda I)\vec{x} &= \vec{0} \end{aligned}$$

This equation is familiar. It's the homogeneous system of equations $A\vec{x} = \vec{0}$, the solution of which is the nullspace of $A - \lambda I$. Therefore, $\vec{x}$ is an eigenvector of $A \iff \vec{x}$ is in the nullspace of $A - \lambda I$.

In this situation, what do we know about that matrix?

Everything in the equivalent statements is false because $\vec{x}$ cannot be the zero vector. Therefore, we can see that $\det(A - \lambda I)$ OR $\det(\lambda I - A)$ MUST be 0.

Big Idea: If A is an $n \times n$ matrix, then λ is an eigenvalue of $A \iff \det(\lambda I - A) = 0$. This is called the characteristic equation of A.

4.3.1 Find the characteristic equation and the eigenvalues of $A = \begin{bmatrix} 3 & 0 & 5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 3 & 0 & 5 \\ -\frac{1}{5} & \lambda + 1 & 0 \\ -1 & -1 & \lambda + 2 \end{vmatrix} = 0$$

$$0 = (\lambda - 3)((\lambda + 1)(\lambda + 2)) + 5(\frac{1}{5} + \lambda + 1)$$

$$0 = (\lambda - 3)(\lambda^2 + 3\lambda + 2)$$

$$0 = \lambda^3 - 2\lambda$$

$$0 = \lambda(\lambda^2 - 2)\lambda \qquad = 0, \pm\sqrt{2}$$

4.3.2 Find the characteristic equation and the eigenvalues of $A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

$$\begin{vmatrix} -1 - \lambda & 0 & 1 \\ -1 & 3 - \lambda & 0 \\ -4 & 13 & -\lambda \end{vmatrix} = 0$$

$$(-1 - \lambda)((3 - \lambda)(-\lambda) - 0(13)) + (-1(13) - (3 - \lambda)(-4)) = 0$$

$$(-1 - \lambda)(\lambda^2 - 3\lambda) + (-13 - 4\lambda + 12) = 0$$

$$(-1 - \lambda)(\lambda^2 - 3\lambda) + (-4\lambda - 1) = 0$$

$$-\lambda^3 + 3\lambda^2 + 2 = 0$$

$$(-\lambda + 2)(-\lambda^2 - \lambda - 1) = 0$$

$$(-\lambda + 2)(-\lambda - 1)(-\lambda + 1) = 0$$

$$\lambda = 2$$

4.3.3 Find the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 2 & 0 & 0 \\ 6 & \lambda - 3 & 0 \\ 1 & 4 & \lambda - 5 \end{vmatrix} = 0$$
$$(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$$
$$\lambda = 2, 3, 5$$

Theorem 1: For a triangular matrix, the eigenvalues are the elements on the main diagonal.

4.3.4 Find the eigenvalues of A^3 if $A = \begin{bmatrix} \frac{1}{2} & 4 & 5 & -2 \\ 0 & -1 & 3 & -8 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$\lambda_A = \frac{1}{2}, -1, 2, 4$$
$$\lambda_{A^3} = \frac{1}{8}, -1, 8, 64$$

Theorem 2: The eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots$

4.3.5 Give me a matrix with eigenvalues $\lambda = 0, 2, 5$

$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 5 \end{bmatrix}$

Theorem 3: A square matrix A is invertible $\iff$ $\lambda \neq 0$ (which also means its determinant is 0).

4.3.6 Finding eigenvectors!

Find the nontrivial eigenvectors of:

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\begin{aligned}
\begin{vmatrix} \lambda - 1 & -6 \\ -5 & \lambda - 2 \end{vmatrix} &= 0 \\
(\lambda - 1)(\lambda - 2) - (-6)(-5) &= 0 \\
\lambda^2 - 3\lambda - 28 &= 0 \\
(\lambda - 7)(\lambda + 4) &= 0 \\
\lambda &= 7, -4
\end{aligned}$$

Substitute each λ, one at a time into the $\lambda I - A$ matrix and find the null space.

For $\lambda = -4$:

$$\begin{aligned}
&\left(\begin{array}{cc|c} -5 & -6 & 0 \\ -5 & -6 & 0 \end{array}\right) \\
&\left(\begin{array}{cc|c} -5 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right) \\
&\langle -\frac{6}{5}t, t\rangle \\
\vec{x} &= \{\langle -6, 5\rangle\}
\end{aligned}$$

For $\lambda = 7$:

$$\begin{aligned}
&\left(\begin{array}{cc|c} 6 & -6 & 0 \\ -5 & 5 & 0 \end{array}\right) \\
&\left(\begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right) \\
&\langle t, t\rangle \\
\vec{x} &= \{\langle 6, 6\rangle\}
\end{aligned}$$

Therefore, the eigen space is: $\{\langle -6, 5\rangle, \langle 6, 6\rangle\}$

4.4 Diagonalization and Similar Triangles

Similar matrices: If A and D are square matrices, we say that A and D are “similar” if there exists an invertible matrix P such that:

$D = P^{-1}AP$.

4.4.1 Properties of Similar Matricces

- They have the same determinant
- If one is invertible, so is the other
- They have the same trace
- They have the same characteristic polynomial
- They have the same eigenvalues

4.4.2 Procedure

1. Find the eigenvectors for the $n \times n$ matrix A.

- Theorem: If an $n \times n$ matrix A has n distinct eigenvalues, then A is **for sure** diagonalizable.

2. Make matrix P out of the eigenveectors (P is the matrix that diagonalizes A)
3. Check your work to find matrix D if reasonable

4.4.3 Example: Find a matrix P that diagonalizes A and compute $P^{-1}AP$

1. $A = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$

Find the eigenvalues:

$$\begin{aligned} \begin{bmatrix} \lambda - 3 & -7 \\ -5 & \lambda - 5 \end{bmatrix} &= 0 \\ (\lambda - 3)(\lambda - 5) - (-7)(-5) &= 0 \\ \lambda^2 - 5\lambda - 3\lambda + 15 - 35 &= 0 \\ \lambda^2 - 8\lambda - 20 &= 0 \\ \lambda &= -2, 10 \end{aligned}$$

Find the eigenvectors:

$$\lambda = -2 : \left[\begin{array}{cc|c} -5 & -7 & 0 \\ -5 & -7 & 0 \end{array}\right] \vec{x} = \{\langle -7, 5 \rangle\}$$
$$\lambda = 10 : \left[\begin{array}{cc|c} -7 & -7 & 0 \\ -5 & 5 & 0 \end{array}\right] \vec{x} = \{\langle 1, 1 \rangle\}$$

Create the matrix P:

$P = \begin{bmatrix} -7 & 1 \\ 5 & 1 \end{bmatrix}$

Find matrix D:

$$\begin{aligned} D &= P^{-1}AP \\ &= \begin{bmatrix} 7 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & 10 \end{bmatrix} \end{aligned}$$

4.4.4 Conclusion

- If D has the same eigenvalues of A and if D must be diagonal, then D is **THE** diagonal matrix with eigenvalues of A on the diagonal.

2. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

First, find D:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, find P:

$$\lambda = 2 : \left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \vec{x} = \langle 1, 0, 0 \rangle$$

$$\lambda = 3 : \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \vec{x} = \langle 0, 1, 0 \rangle$$

$$\lambda = 1 : \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \vec{x} = \langle 2, 0, 1 \rangle$$

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.5 More on Similar Matrices

There are a few more properties of similar matrices:

- They have the same rank (non-zero eigenvalues)
- They have the same nullity
- They have the same column space
- They have the same row space

4.5.1 Example

**Matrix A is similar to the following matrix:

$$D = \begin{bmatrix} 3 & -1 & 1 & 4 & 5 & 2 \\ 0 & -3 & 5 & -10 & -16 & 1 \\ 0 & 0 & 5 & 7 & -8 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A: 4

Nullity of A: 2

Eigenvalues: 3, -3, 5, 2, 0, 0

Characteristic Polynomial:

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 3 & 1 & -1 & -4 & -5 & -2 \\ 0 & \lambda + 3 & -5 & 10 & 16 & -1 \\ 0 & 0 & \lambda - 5 & -7 & 8 & -2 \\ 0 & 0 & 0 & \lambda - 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} = 0$$

$$(\lambda - 3)(\lambda + 3)(\lambda - 5)(\lambda - 2)\lambda^2 = 0$$

4.5.2 Some review

- Eigenspace of λ: The nullspace of $\lambda I - A$. Each eigenvalue will have its own eigenspace.
- Algebraic multiplicty: The number of times a given λ appears as a root of the characteristic equation.
- Geometric multiplicity: The number of eigenvectors it maps to.

4.5.2.1 Theorem: Geometric and Algebraic Multiplicity

If A is a square matrix, then: a. For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity. b. A is diagonalizable $\iff$ the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity.

4.6 Similar Matrices Continued (11/13/2023)

4.6.1 Warm-Up

Can you write a new statement involving eigenvalues to add to the list of equivalent statements?

$\lambda = 0$ is not an eigenvalue of A

4.6.2 Homework Review

$$A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} \lambda - 19 & 9 & 6 \\ -25 & \lambda + 11 & 9 \\ 17 & 9 & \lambda + 4 \end{bmatrix} &= (\lambda - 19)((\lambda + 11)(\lambda + 4) - 81) + 25(9\lambda + 36 - 54) - 17(81 - 6\lambda - 66) \\ &= (\lambda - 1)^2(\lambda - 2) \end{aligned}$$

$\lambda = 1$ has an algebraic multiplicity of 2 and $\lambda = 2$ has an algebraic multiplicity of 1.

4.6.3 Suppose that a characteristic polynomial of some matrix A is found to be:

$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda = 4)^3$

a. What are the dimensions of A?

6×6

b. What are the algebraic multiplicities of each eigenvalue?

$\lambda = 1 : 1, \lambda = 3 : 2, \lambda = 4 : 3$

c. What are the possible dimensions of the eigenspace associated with each of the eigenvalues?

$\lambda = 1 : 1, \lambda = 3 :$ 1 or 2, $\lambda = 4 :$ 1 or 2 or 3

d. If $\{v_1, v_2\}$ is a linearly independent set of eigenvectors of A, all of which correspond to the same eigenvalue of A, what can you say about the eigenvalue?

The eigenvalue must be 3 or 4.

5 Semester II

5.1 Introduction to Multivariable functions (01/04)

5.1.1 Definition of a Multivariable Function

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A function f on D is a rule that assigns a real number

$w = f(x_1, x_2, \ldots, x_n)$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is a function of the n **independent variables** x_1 to x_n. The x's are the function's **input variables**; w is the function's **output variable**.

5.2 Limits and Continuity (01/05)

5.2.1 Level Curve, Graph, Surface

The set of points in the plane where a function $f(x, y)$ has a constant value k is called a **level curve** of f. The graph of f is the set of all points (x, y, z) in space, where $z = f(x, y)$. The graph of f is a surface in space.

5.2.2 Limits With Multivariable Functions

Limits: Let f be a function of two variables defined on an open region, except possibly at $(x_0, y_0).

In 2-D:

$\lim_{x\to c} f(x)$ exists iff $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L \in \mathbb{R}$

In 3-D:

$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists iff $\lim_{(x,y)\to(x_0,y_0)^-} f(x, y) = L \in \mathbb{R}$ from <u>all</u> directions.

5.2.3 Examples

1. $\lim_{(x,y)\to(1,2)} \dfrac{5x^2y}{x^2 + y^2} = \dfrac{5 \cdot 1 \cdot 2}{1 + 4} = \dfrac{10}{5} = 2$
2. $\lim_{(x,y)\to(1,1)} \dfrac{x - y}{x^2 - y^2} = \dfrac{0}{0}$

Since this is indeterminate, you need to switch to traditional limit solving methods. There is no *L'Hopital's Rule* for multivariable functions. Instead, factor the numerator and denominator and cancel out common factors.

$\lim_{(x,y)\to(1,1)} \dfrac{x - y}{(x - y)(x + y)} = \dfrac{1}{2}$

3. $\lim_{(x,y)\to(4,3)} \dfrac{\sqrt{x} - \sqrt{y + 1}}{x - y - 1}$

5.2.4 Homework

5.2.4.1 Evaluate the limit $\lim_{(x,y)\to(0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

$$\lim_{(x,y)\to(0,0)} \left.\frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}\right|_{(0,0)} = \frac{5}{2}$$

5.2.4.2 Evaluate the limit $\lim_{(x,y)\to(1,1)} \cos \sqrt[3]{|xy| - 1}$

$$\lim_{(x,y)\to(1,1)} \cos \left.\sqrt[3]{|xy| - 1}\right|_{(1,1} = \cos 0 = 1$$

5.2.4.3 Evaluate the limit $\lim_{(x,y)\to(1,1)} \dfrac{xy - y - 2x + 2}{x - 1}$

$$\lim_{(x,y)\to(1,1)} \frac{xy - y - 2x + 2}{x - 1} = \frac{(x-1)(y-2)}{x - 1} = y - 2|_{(1,1)} = -1$$

5.2.4.4 On what interval is the function $f(x, y) = \sin(x + y)$ **continuous?**

The sin function is continuous everywhere, so $(x, y) \in \mathbb{R}^2$

5.2.4.5 On what interval is the function $f(x, y, z) = x^2 + y^2 - 2z^2$ **continuous?**

$(x, y, z) \in \mathbb{R}^3$

5.2.4.6 On what interval is the function $f(x, y, z) = xy \sin\left(\frac{1}{z}\right)$ **continuous?**

Can't divide by zero, so $(x, y, z) \in \mathbb{R}^3 | z \neq 0$

5.3 Limits that DO NOT EXIST in 3-Space (01/08)

5.3.1 Question: Does the function $f(x, y) = \dfrac{2x^2y}{x^4 + y^2}$ have a limit as (x,y) approaches (0,0)?

Nope, it approaches the point sp.differently.

5.3.2 Find $f(x,y)|_{y=x^2}$ and compute $\lim_{(x,y)\to(0,0)} f(x,y)$ along $y = x^2$

$$f(x,y) = \left.\frac{2x^2y}{x^4+y^2}\right|_{y=x^2} = \frac{2x^4}{2x^4} = 1$$

5.3.3 Find $f(x,y)|_{y=-x^2}$ and compute $\lim_{(x,y)\to(0,0)} f(x,y)$ along $y = -x^2$

$$f(x,y) = \left.\frac{2x^2y}{x^4+y^2}\right|_{y=x^2} = \frac{-2x^4}{2x^4} = 1$$

5.3.4 Explain why we can conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

$\lim_{(x,y)\to(0,0)} f(x,y)$ along $y = x^2 \neq \lim_{(x,y)\to(0,0)} f(x,y)$ along $y = -x^2$

5.3.5 How did we know to choose $y = x^2$ and $y = -x^2$ to evaluate the limit?

You want to try to choose things that, as you plug them in, you get a nice expression that you can simplify. For this problem in particular, we chose paths towards $(0,0)$ that would be easy to solve and yield sp.different answers upon simplification. We could've also used the equation of the x-axis ($y = 0$).

5.3.6 Show that these functions have no limit as (x,y) approaches $(0,0)$ by considering sp.different paths of approach.

5.3.6.1 $f(x,y) = \dfrac{x^4}{x^4+y^2}$

$\lim_{(x,y)\to(0,0)} f(x,y)$ along $y = 2x^2 = \dfrac{x^4}{5x^4} = \dfrac{1}{5} \neq \lim_{(x,y)\to(0,0)} f(x,y)$ along $y = x^2 = \dfrac{x^4}{2x^4} = \dfrac{1}{2}$

5.3.6.2 $f(x,y) = \dfrac{x^2+y}{y}$

$\lim_{(x,y)\to(0,0)} f(x,y)$ along $y = x^2 = \dfrac{2y}{y} = 2 \neq \lim_{(x,y)\to(0,0)} f(x,y)$ along $y = 2x^2 = \dfrac{3x^2}{2x^2} = \dfrac{3}{2}$

5.4 Partial Derivatives (01/10)

5.4.1 First Order Partial Derivatives

A **partial derivative** is obtained by holding all but one of the independent variables constant and differentiating with respect to that variable.

5.4.1.1 Notation

$\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x} f(x, y)$ is the partial derivative of f with respect to x

5.4.2 Examples

5.4.2.1 $f(x, y) = 2x^3y - 4x^2y^3 + 5x^4$

$f_x = 6x^2y - 8xy^3 + 20x^3$
$f_y = 2x^3 - 12x^2y^2$

5.4.2.2 $f(x, y) = 4x^2y - 8x^3y^4 + 2xy^7$

$f_x = 8xy - 24x^2y^4 + 2y^7$
$f_y = 4x^2 - 32x^3y^3 + 14xy^6$

5.4.2.3 $f(x, y) = \tan(2x - y)$

$f_x = 2\sec^2(2x - y)$
$f_y = -\sec^2(2x - y)$

5.4.3 The Second Fundamental Theorem of (Multivariable) Calculus

- $\frac{\mathrm{d}}{\mathrm{d}x} \int_5^x f(t)\mathrm{d}t = f(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x} \int_5^{x^2} f(t)\mathrm{d}t = 2x \cdot f(x^2)$

5.4.4 Examples

5.4.4.1 $f(x, y) = \int_{3x}^{2y}(t^2 - 1)\text{d}t$

```
# import libraries
import sympy as sp
from IPython.display import display, Math, Latex
x, y, t, z, r,  , p, q = sp.symbols('x y t z r  p q', real=True)
```

```
f = sp.integrate(t**2-1, (t, 3*x, 2*y))
display(sp.diff(f, x))
display(sp.diff(f, y))
```

$3 - 27x^2$

$8y^2 - 2$

5.4.4.2 Find both partial derivatives and evaluate each at the point $(1, \ln 2)$

1. $f(x, y) = xe^{x^2y}$

```
f = x*sp.exp(x**2*y)
display(sp.diff(f, x))
display(sp.diff(f, y))
display(sp.diff(f, x).subs({x:1, y:sp.ln(2)}))
display(sp.diff(f, y).subs({x:1, y:sp.ln(2)}))
```

$2x^2ye^{x^2y} + e^{x^2y}$

$x^3e^{x^2y}$

$2 + 4\log(2)$

2

5.5 2nd Partial Derivatives (01/11)

5.5.1 First, let's see this one. If $f(x, y, z) = x\sin(y + 3z)$, find f_x, f_y, f_z.

```
f = x*sp.sin(y+3*z)
display(sp.diff(f, x))
display(sp.diff(f, y))
display(sp.diff(f, z))
```

$\sin(y + 3z)$

$x\cos(y + 3z)$

$3x\cos(y + 3z)$

5.5.2 2nd-Order Partial Derivatives

The **second-order partial derivatives** of f are the partial derivatives of the partial derivatives of f.

5.5.2.1 Notation

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

5.5.3 Mixed Partial Derivatives

The **mixed partial derivatives** of f are the partial derivatives of f with respect to many other different variables.

5.5.3.1 Notation

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

To compute a mixed partial derivative, you must evaluate the order from right to left. For example, f_{xy} is the partial derivative of f_x with respect to y, which is notated as $\frac{\partial^2 f}{\partial x \partial y}$.

5.5.4 Theorem (Clairaut's Theorem)

If f_{xy} and f_{yx} are continuous on an open region D, then $f_{xy}(a, b) = f_{yx}(a, b)$ on D.

5.5.5 Examples

5.5.5.1 Given that $f(x, y) = x^2y - y^3 + \ln x$, find all 2nd order partial derivatives

f = x**2*y - y**3 + sp.ln(x) display(sp.diff(f, y, x)) display(sp.diff(f, x, y)) display(sp.diff(f, x, x)) display(sp.diff(f, y, y))

```
#### Given that $w = xy+\frac{e^y}{y^2+1}$, find $\frac{\partial^2 w}{\partial x \partial y}$

For this problem, you need to "choose wisely." Knowing that the function and it's partial de

### Partial Derivatives Homework (01/11)

### Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y)$

#### $f(x, y) = x(y-1)$

```{python}
f = x*(y-1)
display(sp.diff(f, x))
display(sp.diff(f, y))
```

5.5.6 $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

```
f = 5*x*y - 7*x**2 - y**2 + 3*x - 6*y + 2
display(sp.diff(f, x))
display(sp.diff(f, y))
```

5.5.7 $f(x, y) = \sqrt{x^2 + y^2}$

```
f = sp.sqrt(x**2 + y**2)
display(sp.diff(f, x))
display(sp.diff(f, y))
```
```

5.5.8 $f(x, y) = e^{(x+y+1)}$

```
f = sp.exp(x+y+1)
display(sp.diff(f, x))
display(sp.diff(f, y))
```

5.5.9 $f(x, y) = \int_x^y g(t)\mathrm{d}t$

```
g = sp.Function('g')
f = sp.integrate(g(t), (t, x, y))
display(sp.diff(f, x))
display(sp.diff(f, y))
```

5.5.10 $f(x, y, z) = xy + yz + xz$

```
f = x*y + y*z + x*z
display(sp.diff(f, x))
display(sp.diff(f, y))
display(sp.diff(f, z))
```

5.5.11 $f(x, y, z) = \ln(x + 2y + 3z)$

```
f = sp.ln(x + 2*y + 3*z)
display(sp.diff(f, x))
display(sp.diff(f, y))
display(sp.diff(f, z))
```

5.5.12 If $f(x, y) = x\cos(y) + ye^x$, find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$

```
f = x*sp.cos(y) + y*sp.exp(x)
display(sp.diff(f, x, x))
display(sp.diff(f, y, y))
display(sp.diff(f, x, y))
display(sp.diff(f, y, x))
```

5.6 Definition of Differentiability (01/16)

So, how do we solve the apparent contradiction?

Defintion of differentiability in multivaraible calculus:

The function $f(x, y)$ is said to be differentiable if there exists a *linear function* at the point (a, b):
$L(x, y) = f(a, b) + p(x - a) + q(y - b)$

5.6.1 Remember local linearization?

...what it means for a function to be "locally linear?"

Consider the following:

Is this function differentiable at $x = 0$?

$f(x) = |x| + 1$

What about this one?

$g(x) = \sqrt{x^2 + 0.0001} + 0.99$

The linearization of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is given by the function:

$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ $z = z_0 + m_x(x - x_0) + m_y(y - y_0)$ $0 = m_x(x - x_0) + m_y(y -$

x(x-x_0)+m_y(y-y_0)   0 = m_x(x-x_0)+m_y(y-y_0)+(z-z_0)
0 = A(x-x_0) + B(y-y) + C(z-z_0) \end{align*} ...where $\vec{n} = \langle A, B, C \rangle$ and $p = (x_0, y_0, z_0)$

5.6.2 Examples

5.6.2.1 Find the linearization of $f(x, y) = y^2 + 2xy - \frac{1}{2}x^2$ at the point $(2, 3)$.

5.6.3 Homework

5.6.3.1 Find the linearization of $f(x, y) = x^2 + y^2 + 1$ at $(1, 1)$

```
f = x**2 + y**2 + 1
x_0 = 1
y_0 = 1
z_0 = f.subs({x:x_0, y:y_0})
f_x = sp.diff(f, x).subs({x:x_0, y:y_0})
f_y = sp.diff(f, y).subs({x:x_0, y:y_0})
display(z_0 + f_x*(x-x_0) + f_y*(y-y_0))
```

5.6.3.2 Find the linearization of $f(x, y) = 3x - 4y + 5$ at $(1, 1)$

```
f = 3*x - 4*y + 5
x_0 = 1
y_0 = 1
z_0 = f.subs({x:x_0, y:y_0})
f_x = sp.diff(f, x).subs({x:x_0, y:y_0})
f_y = sp.diff(f, y).subs({x:x_0, y:y_0})
display(z_0 + f_x*(x-x_0) + f_y*(y-y_0))
```

5.6.3.3 Find the linearization of $f(x, y) = e^x \cos y$ at $(0, \frac{\pi}{2})$

```
f = sp.exp(x)*sp.cos(y)
x_0 = 0
y_0 = sp.pi/2
z_0 = f.subs({x:x_0, y:y_0})
f_x = sp.diff(f, x).subs({x:x_0, y:y_0})
f_y = sp.diff(f, y).subs({x:x_0, y:y_0})
display(z_0 + f_x*(x-x_0) + f_y*(y-y_0))
```

5.7 Chain Rule (01/18)

5.7.1 Warmup: Find the linearization of $f(x, y) = x^3 y^4$ at the point $(1, 1)

```
f = x**3*y**4
x_0 = 1
y_0 = 1
z_0 = f.subs({x:x_0, y:y_0})
f_x = sp.diff(f, x).subs({x:x_0, y:y_0})
f_y = sp.diff(f, y).subs({x:x_0, y:y_0})
display(z_0 + f_x*(x-x_0) + f_y*(y-y_0))
```

5.7.2 Chain Rule (1-Variable)

The "normal" chain rule formula:

$$\frac{d}{dt}f(g(t)) = f'(g(t)) \cdot g'(t) = \frac{df}{dt} \cdot \frac{dg}{dt} = \frac{df}{dt}$$

5.7.3 Chain Rule (Multi Variable)

Let $w = f(x, y)$ where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$ where g and h are differentiable functions of t, then w is also a differentiable function of t and $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$

5.7.4 Examples

5.7.4.1 If $w = x^2y - y^2$ where $x \sin t$ and $y = e^t$, find $\frac{dw}{dt}$ at $t = 0$

```
w = x**2*y - y**2
w_t = w.subs({x: sp.sin(t), y: sp.exp(t)})
dw_dt = sp.diff(w_t, t)
display(dw_dt.subs(t, 0))
```

5.7.4.2 If $w = xy + xz + yz$ and $x = t - 1, y = t^2 - 1, z = t$, find $\frac{dw}{dt}$

```
w = x*y + x*z + y*z
w_t = w.subs({x: t-1, y: t**2-1, z: t})
display(sp.diff(w_t, t))
```

5.7.5 Homework

5.7.5.1 $w = x^2 + y^2, x = \cos t, y = \sin t$ **at** $t = \pi$

```
w = x**2 + y**2
w_t = w.subs({x: sp.cos(t), y: sp.sin(t)})
display(sp.diff(w_t, t))
display(sp.diff(w_t, t).subs(t, sp.pi))
```

5.7.5.2 $w = 2ye^x - \ln z, x = \ln(t^2 + 1), y = \arctan t, z = e^t$ **at** $t = 1$

```
w = 2*y*sp.exp(x) - sp.ln(z)
part_wx = sp.diff(w, x)
part_wy = sp.diff(w, y)
part_wz = sp.diff(w, z)
dx_dt = sp.diff(sp.ln(t**2 + 1), t)
dy_dt = sp.diff(sp.atan(t), t)
dz_dt = sp.diff(sp.exp(t), t)
dw_dt = part_wx * dx_dt + part_wy * dy_dt + part_wz * dz_dt
display(dw_dt)
display(dw_dt.subs(t, 1))
```

5.7.5.3 $z = 4e^x \ln y, x = \ln(r \cos \theta), y = r \sin \theta; (r, \theta) = \left(2, \frac{\pi}{4}\right)$

```
z = 4*sp.exp(x)*sp.ln(y)
part_zx = sp.diff(z, x)
part_zy = sp.diff(z, y)
dx_dr = sp.diff(sp.ln(r*sp.cos( )), r)
dx_d  = sp.diff(sp.ln(r*sp.cos( )),  )
dy_dr = sp.diff(r*sp.sin( ), r)
dy_d  = sp.diff(r*sp.sin( ),  )
dz_dr = part_zx * dx_dr + part_zy * dy_dr
dz_d  = part_zx * dx_d  + part_zy * dy_d
display(Latex(r"$\frac{\partial z}{\partial r} = " + sp.latex(dz_dr) + "$"))
display(Latex(r"$\frac{\partial z}{\partial\theta} = " + sp.latex(dz_d ) +
    "$"))
display(dz_dr.subs({r:2,  :sp.pi/4}))
display(dz_d .subs({r:2,  :sp.pi/4}))
```

5.7.5.4 $u = \frac{p-q}{q-r}, p = x + y + z, q = x - y + z, r = x + y - z; (x, y, z) = (\sqrt{3}, 2, 1)$

```
u = (p-q)/(q-r)
part_ux = sp.diff(u.subs({p: x+y+z, q: x-y+z, r: x+y-z}), x)
part_uy = sp.diff(u.subs({p: x+y+z, q: x-y+z, r: x+y-z}), y)
display(Latex(r"$\frac{\partial u}{\partial x} = " + sp.latex(part_ux) +
    "$"))
display(Latex(r"$\frac{\partial u}{\partial y} = " + sp.latex(part_uy) +
    "$"))
display(Latex(r"$\frac{\partial u}{\partial z} = \frac{4y}{(-2y+2z)^2}$"))
display(part_ux.subs({x:sp.sqrt(3), y:2, z:1}))
display(part_uy.subs({x:sp.sqrt(3), y:2, z:1}))
display(Latex(r"$-2$"))
```

5.8 Related Rates and Implicit Differentiation (01/22)

5.8.1 Related rates in Calc I

1) Come up with a formula (usually volume or something)
2) Pray to the math gods that you can somehow put the formula in terms of one variable
3) Take the derivatives, plug in tons of values and rates, and solve for the missing one

In Calc III, you can remove step 2 and just use chain rule for step 3.

5.8.2 Examples

5.8.2.1 A right circular cylinder with a n open top has height h, radius r, and surface area A. If $\frac{dh}{dt} = 3$ and $\frac{dr}{dt} = -2$, find $\frac{dA}{dt}$ when $h = 10$ and $r = 5$.

```
h, r, t = sp.symbols('h r t')
A = sp.pi*r**2 + 2*sp.pi*r*h
dhdt = 3
drdt = -2
dAdt = sp.diff(A, r)*drdt + sp.diff(A, h)*dhdt
display(dAdt.subs({h:10, r:5}))
```

5.8.3 Implicit Differentiation From Calc I

5.8.3.1 Find $\frac{\mathrm{dy}}{\mathrm{d}x}$ given that $y^3 + 4x^2 - 2xy + 3x = 19$

$y' = \frac{8x+2y-3}{3y^2-2x}$

5.8.3.2 Now suppose that $f(x, y) = y^3 + 4x^2 - 2xy + 3x - 19$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

```
display(sp.diff(f, x))
display(sp.diff(f, y))
```

If $f(x, y) = 0$ defines y implicitly as a differentiable fnuction of x, then $\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

Similarly, if $f(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\mathrm{dz}}{\mathrm{dx}} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \text{ and } \frac{\mathrm{dz}}{\mathrm{dy}} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

5.9 Directional Derivatives and Gradient Vectors (01/25)

Suppose that f is differentiable at (x_0, y_0) and $\vec{u} = \langle u_1, u_2 \rangle$ is any unit vector. Then the directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is given by: $D_{\vec{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$

In other words, this is partial x times the x component of the unit vector plus partial y times the y component of the unit vector.

The directional derivative is also denoted by:

$(D_u f)_{p_0}$; "The derivative of f at P_0 in the direction of u"

5.9.1 Find the directional derivative of $f(x,y) = x^2 \sin 2y$ at $\left(1, \frac{\pi}{2}\right)$ in the direction of $\vec{v} = 3\vec{i} - 4\vec{j}$.

$$
\begin{aligned}
f_x &= 2x \sin 2y \\
f_y &= 2x^2 \cos 2y \\
f_x(1, \frac{\pi}{2}) &= 0 \\
f_y(1, \frac{\pi}{2}) &= -2 \\
\vec{u}_1 &= \frac{3}{5} \\
\vec{u}_2 &= -\frac{4}{5} \\
D_{\vec{u}} f(1, \frac{\pi}{2}) &= 0 \cdot \frac{3}{5} - 2 \cdot -\frac{4}{5} \\
&= \frac{8}{5}
\end{aligned}
$$

5.9.2 Thinking questions

Not too bad, right?

What if I wanted the directional derivative of $f(x,y)$ in the direction of $v = i$? What can you tell me about that derivative?

f_x

What if I just wanted it in the direction of $v = j $?

f_u

So, all of this time, when you've been findign f_x and f_y at a point (x_0, y_0), you've really just been finding the components to a (not unit) vector that contains both derivatives

5.9.3 The Gradient

Let $z = f(x,y)$ be a function of x and y such that f_x and f_y exist. Then the gradient of f denoted by $\nabla f(x,y)$ is the vector:

$\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$

5.9.4 Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point $(e^3, 2)$

$$f_x = \frac{y}{x} + y^2$$
$$f_y = \ln x + 2xy$$
$$\nabla f(x, y) = \langle \frac{2}{e^3} + 4, 3 + 4e^3 \rangle$$

Wait a second... go back to how we calculate a directional derivative...

$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$

This is the same as the dot product of the gradient and the unit vector!

$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

5.9.5 Homework

Find the gradient of the following:

5.9.5.1 $g(x, y) = y - x^2, (-1, 0)$

```
g = y - x**2
g_x = sp.diff(g, x)
g_y = sp.diff(g, y)
display(g_x.subs({x:-1, y:0}))
display(g_y.subs({x:-1, y:0}))
```

Find the directional derivative of f at P_0 in the direction of A.

5.9.5.2 $f(x, y) = 2xy - 3y^2, P_0 = (5, 5), A = \langle 4, 3 \rangle$

```
f = 2*x*y - 3*y**2
f_x = sp.diff(f, x).subs({x:5, y:5})
f_y = sp.diff(f, y).subs({x:5, y:5})
A = sp.sqrt(4**2 + 3**2)
u_1 = 4/A
u_2 = 3/A
display(f_x*u_1 + f_y*u_2)
```

5.9.5.3 $f(x, y) = x - \left(\frac{y^2}{x}\right) + \sqrt{3}\sec^{-1}(2xy), P_0 = (1, 1), A = \langle 12, 5\rangle$

```
f = x - y**2/x + sp.sqrt(3)*sp.asec(2*x*y)
f_x = sp.diff(f, x).subs({x:1, y:1})
f_y = sp.diff(f, y).subs({x:1, y:1})
A = sp.sqrt(12**2 + 5**2)
u_1 = 12/A
u_2 = 5/A
display(f_x*u_1 + f_y*u_2)
```

5.9.5.4 $f(x, y) = xy + yz + zx, P_0 = (1, -1, 2), A = \angle 3, 6, -2\rangle$

```
x, y, z = sp.symbols('x y z', real=True)
f = x*y + y*z + z*x
f_x = sp.diff(f, x).subs({x:1, y:-1, z:2})
f_y = sp.diff(f, y).subs({x:1, y:-1, z:2})
f_z = sp.diff(f, z).subs({x:1, y:-1, z:2})
A = sp.sqrt(3**2 + 6**2 + (-2)**2)
u_1 = 3/A
u_2 = 6/A
u_3 = -2/A
display(f_x*u_1 + f_y*u_2 + f_z*u_3)
```

5.10 More Gradient and Tangent Planes (01/29)

5.10.1 Properties of Gradients

$D_{\vec{u}}f = \nabla f \cdot \vec{u} = ||\nabla f||||u|| \cos\theta$

In order to maximize this value, $\cos\theta$ must be 1.

Some consequences:

- f increases most rapidly when $\cos\theta = 1(\theta = 0)$, which implies that ∇f and $\vec{u}$ are in the same direction.
- This maximum rate of increase = $||\nabla f||$
- f decreases most rapidly when $\cos\theta = -1(\theta = \pi)$, which implies that ∇f and $\vec{u}$ are in opposite directions.
- This maximum rate of decrease = $-||\nabla f||$

Bit more interesting: What if I wanted the directional derivative to be zero?

$D_{\vec{u}}f = \nabla f \cdot \vec{u} = 0$

When the dot product equals 0.

5.10.2 Examples

5.10.2.1 Find the maximum rate of increase/decrease of the function $f(x, y) = x^2 + y^2$ at $(1, 3)$

```
f = x**2 + y**2
f_x = sp.diff(f, x)
f_y = sp.diff(f, y)
display(Latex(r"$\nabla f(x, y) = " + sp.latex(sp.Matrix([f_x, f_y])) + "$"))
display(Latex(r"$||\nabla f(x, y)|| = " + sp.latex(sp.sqrt(f_x**2 + f_y**2))
    + "$"))
display(Latex(r"$||\nabla f(1, 3)|| = " + sp.latex(sp.sqrt(f_x.subs({x:1,
    y:3})**2 + f_y.subs({x:1, y:3})**2)) + "$"))
```

5.10.2.2 The temperature in Celsius on the surface of a metal plate is $T(x, y) = 20 - 4x^2 - y^2$ where x and y are measured in cm. In what direction from $(2, -3)$ does the temperature

a. Increase most rapidly?

```
T = 20 - 4*x**2 - y**2
T_x = sp.diff(T, x)
T_y = sp.diff(T, y)
# The gradient at (2, -3)
gradient = sp.Matrix([T_x.subs({x:2, y:-3}), T_y.subs({x:2, y:-3})])
magnitude = sp.sqrt(T_x.subs({x:2, y:-3})**2 + T_y.subs({x:2, y:-3})**2)
unit_vector = gradient/magnitude
display(unit_vector)
```

b. Decrease most rapidly?

```
display(-unit_vector)
```

c. Not change at all?

```
display(sp.Matrix([6, 16]), sp.Matrix([-6, -16]))
```

5.10.3 More Applications of Gradients

Given a surface $z = f(x, y), [f(x, y) - z = 0]$, the plane tangent to z at the point $P_0(x_0, y_0, z_0)$ is given by:

The equation of a plane is given by:

$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$ where $f_x, f_x, f_z \in \nabla(f(x, y) - z)$

The equations of the normal line to the surface are given by:

- $x = x_0 + f_x(x_0, y_0, z_0)t$
- $y = y_0 + f_y(x_0, y_0, z_0)t$
- $z = z_0 + f_z(x_0, y_0, z_0)t$

5.11 Local Extrema and Saddle Points (02/02)

In multivariable calculus, mins/maxes on a closed region occur in the following conditions:

- An interior point where **both** first partial derivatives equal 0 (Critical point!)
- An interior point where one **or** both first partial derivatives DNE (Critical point?)
- Any boundary point of the region (End point)

What would a tangent plane look like at one of those interior critical points?

$0(x\ldots) + 0(y\ldots) + f_z(z\ldots) = 0$

5.11.1 Types of Extrema

- $f(a, b)$ is a relative max of f if $f(a, b) \geq f(x, y)\forall$ domain points in an open disk centered at (a, b)
- $f(a, b)$ is a relative min of f if $f(a, b) \leq f(x, y)\forall$ domain points in an open disk centered at (a, b)
- If (a, b) is a critical point ($f_x = f_y = 0$) and there are domain points where $f(x, y) > f(a, b)$ and where $f(x, y) < f(a, b)$, then $(a, b, f(a, b))$ is a saddle point.

5.11.2 The Second Partials Test

Suppose $f(x, y)$ and its first and second partial derivatives are continuous on an open region centered at (a, b) and $f_x(a, b) = f_y(a, b) = 0$. Then:

a) f has a rel <u>max</u> at (a, b) if $f_{xx} < 0$ AND $f_{xx}f_{yy} - (f_{xy})^2 > 0$ at (a, b)

- (if f is concave down in the x and y direction, then it's a relative max)

b) f has a rel <u>min</u> at (a, b) if $f_{xx} > 0$ AND $f_{xx}f_{yy} - (f_{xy})^2 > 0$ at (a, b)

- (if f is concave up in the x and y direction, then it's a relative min)

c) f has a saddle point if $f_{xx}f_{yy} - (f_{xy})^2 < 0$ at (a, b)

- (if the concavity of f disagree **or** f_{xy} is too large, then it's a saddle point)

d) The test is inconclusive if $f_{xx}f_{yy} - (f_{xy})^2 = 0$ at (a, b)

5.12 Absolute Extrema (02/07)

Recap of the extreme value theorem (EVT):
If $f(x)$ is continuous over $[a, b]$, then $f(x)$ *must* have an absolute max *and* min.

5.12.1 Calc I Approach

1) Create list of candidates (critical points, end points)
2) Brute force which one is largest/smallest. Plug and chug

5.12.2 Problems with Calc I Approach in 3D

It's not that different in multi! Although... we don't have endpoints because we don't have *ends*. In 2D, boundaries to possible values of x are 1-dimensional ([little x, big x]).

In 3D, both **x** and **y** can have boundaries, so we *could* have [little x, big x] and [little y, big y]. But that boundary system describes a rectangle...

What if I wanted boundaries that looked like a triangle? A circle? Anything else...?

5.12.3 Multi Approach

To find absolute extrema of a continuous function $f(x, y)$ on a closed bounded region R: 1) Graph region R 2) Find the critical points of f 3) Find the boundary points of R which are also critical points 4) Find the coordinates of corners of R 5) Evaluate f *at all these points* and answer the question

5.13 Lagrange Multipliers (02/08)

When looking for a max/min on a surface constrained by another function, the two gradients of those functions at the maximum/minimum point are multiples of each other. That multiplier is the **Lagrange Multiplier**.

5.13.1 More formal definition

Suppose $f(x, y, z)$ and $g(x, y, z)$ are functions with continuous first partial derivatives and $\nabla g(x, y, z) \neq 0$ on the surface $g(x, y, z) = 0$. Suppose also that the minimum/maximum of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at (x_0, y_0, z_0). Then $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ for some non-zero constant λ called the Lagrange multiplier.

Essentially, it's saying the two functions have a proportional set of partial derivatives.

5.13.2 Steps

5.13.2.1 Two variables

1) Set up the equation $\nabla f = \lambda \nabla g$
2) Solve each equation for λ and set them equal to create an equation with just x and y
3) Solve for a variable and plug it into the constraint equation
4) Solve the constraint equation and back substitute to find the other variable
5) Evaluate the function at that point to find the max/min

5.13.2.2 Three variables

1) Set up the equation $\nabla f = \lambda \nabla g$
2) Solve each equation for a non-λ variable so that each is a function of λ
3) Plug all lambda equations into constraint to create equations in terms of one variable
4) Solve for lambda and back substitute into each variable
5) If you have multiple points, evaluate the function at each point to find the max/min

5.13.3 More than one multiplier

When optimizing with two constraints, g and h, we introduce a second multiplier, μ and our system becomes: - $f_x = \lambda g_x + \mu h_x$ - $f_y = \lambda g_y + \mu h_y$ - $f_z = \lambda g_z + \mu h_z$