

Gwinnett School of Math, Science, and Technology

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## Multivariable Calculus Yearlong Notes

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# 1 Chapter 1: Systems of Linear Equations and Matrices

## 1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal  $\iff$  they have the same dimensions and values

### 1.1.1 Addition & Subtraction

Two matrices can be added/subtracted  $\iff$  they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

### 1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3 \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

### 1.1.3 Matrix Multiplication

- We can **only** multiply an  $(m \times n)$  by  $(n \times p)$  matrix.
- The resulting matrix will be  $(m \times p)$

### 1.1.4 Properties of Matrix Arithmetic

- (a)  $A + B = B + A$  (**Commutative law for addition**)
- (b)  $A + (B + C) = (A + B) + C$  (**Associative law for addition**)
- (c)  $A(BC) = (AB)C$  (**Associative law for multiplication**)
- (d)  $A(B + C) = AB + AC$  (**Left distributive law**)
- (e)  $(B + C)A = BA + CA$  (**Right distributive law**)
- (f)  $A(B - C) = AB - AC$
- (g)  $(B - C)A = BA - CA$
- (h)  $a(B+C) = aB + aC$
- (i)  $a(B-C) = aB - aC$
- (j)  $(a+b)C = aC + bC$
- (k)  $(a-b)C = aC - bC$
- (l)  $a(bC) = (ab)C$
- (m)  $a(BC) = (aB)C = B(aC)$

### 1.1.5 Examples

1.

$$\begin{aligned} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \end{aligned}$$

2.

$$\begin{aligned} & \begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix} \end{aligned}$$

3.

$$\begin{aligned} & \begin{bmatrix} 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} \\ &= [4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2] \\ &= [30] \end{aligned}$$

## 1.2 Transpose of a Matrix

The transpose of an  $(m \times n)$  matrix is the  $(n \times m)$  matrix where the rows and columns are swapped.

$$\text{If } B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix} \end{aligned}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix  $A$  is equal to the sum of all the elements on the main diagonal:  $\text{tr}(A)$

### 1.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

### 1.3 Homework — “Matrix Stuff” (08/03/2023)

**1.3.1 Suppose that  $A, B, C, D$  and  $E$  are matrices with the following sizes:**

|                |                |                |                |                |
|----------------|----------------|----------------|----------------|----------------|
| $A$            | $B$            | $C$            | $D$            | $E$            |
| $(3 \times 2)$ | $(2 \times 3)$ | $(3 \times 3)$ | $(3 \times 2)$ | $(2 \times 3)$ |

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

| Undefined | Defined; $(4 \times 2)$ | Defined; $(5 \times 5)$ | Defined; $(5 \times 2)$ |
|-----------|-------------------------|-------------------------|-------------------------|
| $BA$      | $AC + D$                | $E(A + B)$              | $(A^T + E)D$            |
| $AB + B$  |                         |                         | $E(AC)$                 |
| $E^T A$   |                         |                         |                         |
| $AE + B$  |                         |                         |                         |

**1.3.2 Consider the matrices**

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2.  $2A^T + C$

$$\begin{aligned} 2A^T + C &= 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2 \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3.  $B^T + 5C^T$

$$\begin{aligned}
 B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\
 &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\
 &= \text{Undefined}
 \end{aligned}$$

4.  $2E^T - 3D^T$

$$\begin{aligned}
 2E^T - 3D^T &= 2 \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3 \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T \\
 &= 2 \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}
 \end{aligned}$$

5.  $\text{tr}(DE)$

$$\begin{aligned}
 \text{tr}(DE) &= \text{tr} \left( \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right) \\
 &= \text{tr} \left( \begin{bmatrix} 1 \cdot 6 + 5 \cdot (-1) + 2 \cdot 4 & 1 \cdot 1 + 5 \cdot 1 + 2 \cdot 1 & 1 \cdot 3 + 5 \cdot 2 + 2 \cdot 3 \\ (-1) \cdot 6 + 0 \cdot (-1) + 1 \cdot 4 & (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1 & (-1) \cdot 3 + 0 \cdot 2 + 1 \cdot 3 \\ 3 \cdot 6 + 2 \cdot (-1) + 4 \cdot 4 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 & 3 \cdot 3 + 2 \cdot 2 + 4 \cdot 3 \end{bmatrix} \right) \\
 &= \text{tr} \left( \begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix} \right) \\
 &= 34
 \end{aligned}$$

## 2 Intro to Systems

What are we looking for?

Lines: How many possible solutions?

- Infinite solutions
- One solution
- No solutions

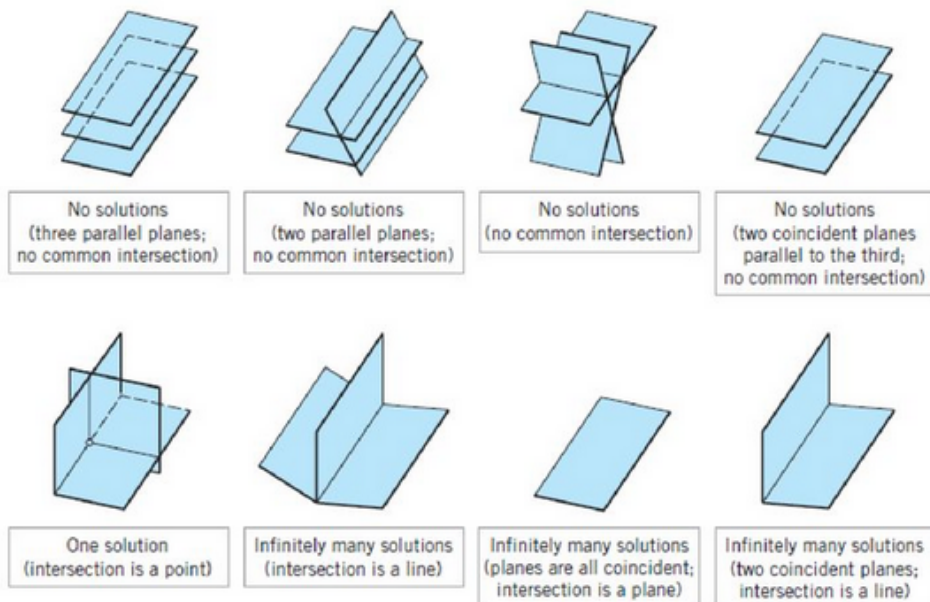
Planes: How many possible solutions?

- Infinite solutions
- No solutions

What does linear actually mean?

- The word linear *really* means that you've got equations with variables and **all** of the variables are degree one.
- This means that there is no limit to the number of dimensions in a linear system.

## Linear Systems in Three Unknowns



## 2.1 Review: Solve the following systems

1. 
$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$5x = 15$$

$$x = 3$$

$$2(3) + y = 10$$

$$6 + y = 10$$

$$y = 4$$

2. 
$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$y = 10 - 2x$$

$$6x + 3(10 - 2x) = 10$$

$$6x + 30 - 6x = 10$$

$$30 = 10. \therefore \text{no solution}$$

3. 
$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$0 = 0$$

$$12 = 12. \therefore \text{infinite solutions}$$

### 2.1.1 Consistent

- A system of equations is **consistent** if it has at least one solution.

### 2.1.2 Inconsistent

- A system of equations is **inconsistent** if it has no solutions.

## 2.2 The Augmented Matrix

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases} \longrightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{array} \right]$$

## 2.3 Elementary Row Operations

1. Interchange 2 rows
2. Multiply a row by a non-zero constant
3. Add/subtract a multiple of one row to/from another row

Doing these things changes the matrix, but it's the same system!

### 2.3.1 Example 1... again

$$\begin{cases} 2x + y = 10 \\ 3x - y = 5 \end{cases}$$

$$\begin{aligned} \left[ \begin{array}{cc|c} 2 & 1 & 10 \\ 3 & -1 & 5 \end{array} \right] &\xrightarrow{\frac{1}{2}R_1} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 3 & -1 & 5 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & -\frac{5}{2} & -10 \end{array} \right] \\ &\xrightarrow{-\frac{2}{5}R_2} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 1 & 4 \end{array} \right] \xrightarrow{R_1 - \frac{1}{2}R_2} \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array} \right] \end{aligned}$$

And so...  $x = 3$  and  $y = 4$ !

## 2.4 Connection to Matrices

If we can make a system's matrix look like

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array} \right],$$

then the solution to the system will be the ordered triple  $(c_1, c_2, c_3)$ .



### 2.4.1 Example 2: again

$$\begin{cases} 2x + y = 10 \\ 6x + 3y = 10 \end{cases}$$

$$\left[ \begin{array}{cc|c} 2 & 1 & 10 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 6 & 3 & 10 \end{array} \right] \xrightarrow{R_2-6R_1} \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 5 \\ 0 & 0 & -20 \end{array} \right]$$

This is inconsistent, so there is no solution.

### 2.4.2 Example 3: again

$$\begin{cases} 5x - 2y = 4 \\ 15x - 6y = 12 \end{cases}$$

$$\left[ \begin{array}{cc|c} 5 & -2 & 4 \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{\frac{1}{5}R_1} \left[ \begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 15 & -6 & 12 \end{array} \right] \xrightarrow{R_2-15R_1} \left[ \begin{array}{cc|c} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{array} \right]$$

Since  $0 = 0$ , there are infinitely many solutions.

### 2.4.3 Example 4: Solve the following system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right] \xrightarrow{R_3+4R_1} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_3+\frac{3}{2}R_2} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -1 & 3 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_2} \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{R_1+2R_2} \left[ \begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\begin{matrix} R_1+7R_3 \\ R_2+4R_3 \end{matrix}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \end{aligned}$$

Therefore the solution to  $(x_1, x_2, x_3)$  is  $(29, 16, 3)$ .

#### 2.4.4 Elementary Row Operations & REF Homework Problem (08/08/2023)

$$\begin{cases} x + y + 2z = 8 \\ -x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \xrightarrow[\substack{R_2+R_1 \\ R_3-3R_1}]{\substack{R_2+R_1 \\ R_3-3R_1}} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] \xrightarrow[\substack{-R_2 \\ -R_3}]{\substack{-R_2 \\ -R_3}} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array} \right] \\ & \xrightarrow[\substack{R_1-R_2 \\ R_3-10R_2}]{\substack{R_1-R_2 \\ R_3-10R_2}} \left[ \begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array} \right] \xrightarrow[\substack{R_1-7R_3 \\ R_2+5R_3}]{\substack{R_1-7R_3 \\ R_2+5R_3}} \left[ \begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

Therefore, the solution to  $(x, y, z)$  is  $(3, 1, 2)$ .

#### 2.5 Gaussian Elimination

Vocabulary: A matrix is in Row Echelon Form (REF) if:

- (a) Any rows of all zeroes are placed at the bottom of the matrix
- (b) All other rows have a leading 1 ("pivot")
- (c) As we move down the matrix, each leading 1 is further to the right than the 1 above it

A matrix is in Row Reduced Echelon Form if the three above conditions are met in addition to:

- (d) Each column with a leading 1 has all other entries in the column as a 0. ("pivot column")

### 2.5.1 Examples

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF? ✓  
RREF? ✓

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF? ✓  
RREF? ✗

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

REF? ✗  
RREF? ✗

## 2.6 Gaussian Elimination With Back-Substitution

### 2.6.1 Goal:

To get the augmented matrix in REF

Solve: 
$$\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$$

$$\begin{aligned} &\left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array} \right] \xrightarrow[R_3+R_2]{R_1+2R_2} \left[ \begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array} \right] \\ &\xrightarrow{\frac{1}{2}R_3} \left[ \begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

$$x + 9z = 19$$

$$y + 3z = 5$$

$$z = 2$$

$$\therefore z = 2, y = 5 - 3z, x = 19 - 9z$$

$$z = 2, y = 5 - 3(2), x = 19 - 9(2)$$

$$z = 2, y = -1, x = 1$$

Therefore, the solution  $(x_1, x_2, x_3)$  is  $(1, -1, 2)$ .

## 2.6.2 Gaussian Elimination Homework Problem (08/09/2023)

$$\begin{cases} -2w + y + z = -3 \\ x + 2y - z = 2 \\ -3w + 2x + 4y + z = -2 \\ -w + x - 4y - 7z = -19 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{cccc|c} -2 & 0 & 1 & 1 & -3 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -1 & 1 & -4 & -7 & -19 \end{array} \right] \xrightarrow{R_4} \left[ \begin{array}{cccc|c} -1 & 1 & -4 & -7 & -19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array} \right] \xrightarrow{-R_1} \\ & \left[ \begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ -3 & 2 & 4 & 1 & -2 \\ -2 & 0 & 1 & 1 & -3 \end{array} \right] \xrightarrow{\begin{matrix} R_3+3R_1 \\ R_4+2R_1 \end{matrix}} \left[ \begin{array}{cccc|c} 1 & -1 & 4 & 7 & 19 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & 16 & 22 & 55 \\ 0 & -2 & 9 & 15 & 35 \end{array} \right] \xrightarrow{\begin{matrix} R_1+R_2 \\ R_3+R_2 \\ R_4+2R_2 \end{matrix}} \\ & \left[ \begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 18 & 21 & 57 \\ 0 & 0 & 13 & 13 & 39 \end{array} \right] \xrightarrow{\frac{1}{18}R_3} \left[ \begin{array}{cccc|c} 1 & 0 & 6 & 6 & 21 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 13 & 13 & 39 \end{array} \right] \xrightarrow{\begin{matrix} R_1-6R_3 \\ R_2-2R_3 \\ R_4-13R_3 \end{matrix}} \\ & \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & -\frac{13}{6} & -\frac{13}{6} \end{array} \right] \xrightarrow{-\frac{6}{13}R_4} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & \frac{7}{6} & \frac{19}{6} \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{matrix} R_1+R_4 \\ R_2+\frac{10}{3}R_4 \\ R_3-\frac{7}{6}R_4 \end{matrix}} \\ & \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \Rightarrow \begin{cases} w = 3 \\ x = -1 \\ y = 2 \\ z = 1 \end{cases} \end{aligned}$$

## 2.7 Gauss-Jordan Elimination

### 2.7.1 Goal:

To get the matrix into RREF

$$\text{Solve: } \begin{cases} x_1 - 3x_3 = -2 \\ 3x_1 + x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + x_3 = 4 \end{cases}$$

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array} \right] \xrightarrow[R_3-2R_1]{R_2-3R_1} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array} \right] \xrightarrow{R_3-2R_2} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array} \right] \\ & \xrightarrow{\frac{-1}{7}R_3} \left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow[R_2-7R_3]{R_1+3R_3} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right] \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = 2 \end{cases} \end{aligned}$$

## 2.8 Matrix Properties, Equations, and Inverses

### 2.8.1 With Real Numbers

- If  $ab = bc$ , then  $a = c$ , if  $b \neq 0$
- If  $ab = 0$ , then  $a = 0$  or  $b = 0$ , or both

### 2.8.2 With Matrices

- If  $AB = AC$ , then  $B = C$ , if  $A$  is invertible
- If  $AB = [0]$ , then  $A = [0]$  or  $B = [0]$ , or both

#### 2.8.2.1 Multiply:

$$\begin{aligned} & \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(5) + 3(-3) & 2(-3) + 3(2) \\ 3(5) + 5(-3) & 3(-3) + 5(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

### 2.8.3 Matrix Inverses

- If a matrix has an inverse, it is said to be invertible or non-singular.
- If a matrix does not have an inverse, it is said to be singular.
- Every square matrix has a “special number” associated with it called the **determinant**.
- For the  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , the determinant is  $ad - bc$
- $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- When  $\det A = 0$ , the matrix is singular and has no inverse (since you cannot divide by zero)

Find the inverse of  $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}^{-1} &= \frac{1}{\det A} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{(4)(2) - (3)(1)} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix} \end{aligned}$$

## 3 Chapter 2: Determinants

### 3.1 Prior Knowledge:

$$\begin{bmatrix} 10 & -4 \\ -3 & -5 \end{bmatrix} = -50 - = -62$$

$$\begin{aligned} & \begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix} \\ & = ((2 \cdot 2 \cdot -2) + (4 \cdot 3 \cdot 3) + (3 \cdot -1 \cdot 0)) - ((3 \cdot 2 \cdot 3) + (0 \cdot 3 \cdot 2) + (-2 \cdot -1 \cdot 4)) \\ & = (-8 + 36 + 0) - (18 + 0 + 8) \\ & = 28 - 26 \\ & = 2 \end{aligned}$$

### 3.2 Minors & Cofactors

Given a square matrix A, the minor of matrix element  $a_{ij}$ , ( $M_{ij}$ ) is the determinant of the matrix formed by removing the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column from matrix A.

The cofactor of matrix element  $a_{ij}$ ,  $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

#### 3.2.1 Example

Let  $\det \begin{bmatrix} 2 & 4 & 3 \\ -1 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix}$ . What is the cofactor of element (1, 1)?

Cofactor checkerboard:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} = -4$$

$$C_{11} = 1 \cdot -4 = -4$$

Find the minor and cofactor of: \ a)  $a_{21} = -1$

$$M_{21} = \begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} = -8$$

$$C_{21} = 8$$

b)  $a_{33} = -2$

$$M_{33} = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 8$$

$$C_{33} = 8$$

### 3.3 Cofactor Expansion

- 1) Pick a row or column
- 2) Multiply every entry in that row or column by it's corresponding cofactor
- 3) Add those together. That's it

$$A = \begin{bmatrix} 6 & 7 & -1 \\ 0 & 4 & 1 \\ 2 & 5 & -3 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 6 \begin{vmatrix} 4 & 1 \\ 5 & -3 \end{vmatrix} + 7 \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} + -1 \begin{vmatrix} 0 & 4 \\ 2 & 5 \end{vmatrix} \\ &= 6(-17) + 7(2) + (-1(-8)) \\ &= -102 + 14 + 8 \\ &= -80 \end{aligned}$$

#### 3.3.1 Example

$$A = \begin{bmatrix} 6 & 4 & 2 \\ 5 & -6 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

$$\begin{aligned} &6 \begin{vmatrix} -6 & 1 \\ 3 & 0 \end{vmatrix} + 4 \begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix} + 2 \begin{vmatrix} 5 & -6 \\ 0 & 3 \end{vmatrix} \\ &= 6(-3) + 0 + 2(15) \\ &= -18 + 30 \\ &= 12 \end{aligned}$$



### 3.3.2 Does the method generalize to 2×2 matrices?

$$\begin{aligned} & \begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix} \\ &= 3|2| - 5|7| \\ &= 6 - 35 \\ &= -29 \end{aligned}$$

The determinant of a 1×1 matrix is... **itself!**

### 3.3.3 Find the determinant of a 4×4

$$A = \begin{bmatrix} -3 & 2 & 0 & 8 \\ 2 & 1 & 0 & -4 \\ 5 & -2 & 1 & 5 \\ 2 & 3 & 0 & 6 \end{bmatrix}$$

$$\begin{aligned} &= 0 + 0 + \begin{vmatrix} -3 & 2 & 8 \\ 2 & 1 & -4 \\ 2 & 3 & 6 \end{vmatrix} + 0 \\ &= -2 \begin{vmatrix} 2 & 8 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} -3 & 8 \\ 2 & 6 \end{vmatrix} - \left( -4 \begin{vmatrix} -3 & 2 \\ 2 & 3 \end{vmatrix} \right) \\ &= 24 - 34 - 52 \\ &= -62 \end{aligned}$$

### 3.4 Theorem

If  $A$  is an  $n \times n$  matrix, then regardless of which row or column of  $A$  is chosen, the number obtained by multiplying the elements in that row or column by their corresponding cofactors is **always the same** and is called the determinant of  $A$ .

### 3.4.1 Example

Find the determinant of  $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} & 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} \\ &= \left( -2 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \right) \\ &= -6 \end{aligned}$$

### 3.5 Triangular Matrices

Find the determinant of  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$\begin{aligned} & \begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} \\ &= 2 \begin{vmatrix} 3 & 3 \\ 0 & 4 \end{vmatrix} \\ &= 2(3 \cdot 4) \\ &= 2 \cdot 12 \\ &= 24 \end{aligned}$$

If  $A$  is an  $n \times n$  triangular matrix, then  $\det(A)$  is equal to the product of the elements along the main diagonal.

### 3.6 An Important Definition

Elementary Matrix a matrix that can be obtained from the  $n \times n$  identity matrix by performing a single row operation. \

Are the following matrices elementary? 1)  $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} + (R_3 + 5R_1)$  yes 2)  $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix} + (R_1 + 5R_2)$ ...  
no

### **3.7 A Pair of Theorems**

**3.7.1 Theorem: If a square matrix  $A$  has a row of column of zeros, then  $\det(A) = 0$**

**3.7.2 Theorem: If  $A$  is a square matrix, then  $\det(A) = \det(A^T)$**

### 3.8 Unit 1 & 2 Homework Problems

#### 3.8.1 "Gaussian Elimination" (08/11/2023)

##### 3.8.1.1 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$
$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] \xrightarrow{-R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array} \right]$$
$$\xrightarrow{R_3+10R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array} \right] \xrightarrow{-\frac{1}{52}R_3} \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right]$$
$$\therefore \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ x_2 - 5x_3 = -9 \\ x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

##### 3.8.1.2 Solve this system using Gaussian Elimination

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ -2x_1 - 3x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + 4x_3 = 0 \end{cases}$$
$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 4 & 0 \end{array} \right] \xrightarrow[R_3-2R_1]{R_2+2R_1} \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{7}R_2} \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & \frac{2}{7} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$
$$\therefore \begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ x_2 + \frac{2}{7}x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow 1 \neq 0 \therefore \text{no solution}$$

### 3.8.2 "Inverses and Determinants" (08/14)

#### 3.8.2.1 Find the determinants of the following:

$$1) \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -3 \\ 4 & 4 \end{vmatrix} = 2(4) - (-3)(4) = 8 + 12 = 20$$

$$2) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2(3) - 0(0) = 6$$

$$3) \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta + \sin^2 \theta = 1$$

#### 3.8.2.2 Find the INVERSES of those matrices:

$$1) \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}^{-1} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

$$2) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$3) \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

### 3.8.3 Inverses and Determinants (08/15)

#### 3.8.3.1 Use a matrix equation to solve the following problems:

$$1) \begin{cases} 3x_1 - 2x_2 = 1 \\ 4x_1 + 5x_2 = 3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{23} \\ \frac{9}{23} \end{bmatrix}$$

$$2) \begin{cases} 6x_1 + x_2 = 0 \\ 4x_1 - 3x_2 = -2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} -3 & -1 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} \\ \frac{4}{11} \end{bmatrix}$$

### 3.8.4 Consistent Systems (08/21)

**3.8.4.1 Solve the linear systems together by reducing the appropriate augmented matrix.**

$$\begin{cases} x_1 - 5x_2 = b_1 \\ 3x_1 + 2x_2 = b_2 \end{cases}$$

- 1)  $b_1 = 1, b_2 = 4$
- 2)  $b_1 = -2, b_2 = 5$

First, let's solve it for the general case:

$$\left[ \begin{array}{cc|c} 1 & -5 & b_1 \\ 3 & 2 & b_2 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[ \begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 17 & b_2 - 3b_1 \end{array} \right] \xrightarrow{\frac{1}{17}R_2} \left[ \begin{array}{cc|c} 1 & -5 & b_1 \\ 0 & 1 & \frac{b_2 - 3b_1}{17} \end{array} \right] \xrightarrow{R_1 + 5R_2} \left[ \begin{array}{cc|c} 1 & 0 & \frac{2b_1 + 5b_2}{17} \\ 0 & 1 & \frac{-3b_1 + b_2}{17} \end{array} \right]$$

Therefore, the solution to the general case is  $(x_1, x_2) = \left( \frac{2b_1 + 5b_2}{17}, \frac{-3b_1 + b_2}{17} \right)$

And so, for the specific cases:

- 1)  $(x_1, x_2) = \left( \frac{2(1) + 5(4)}{17}, \frac{-3(1) + 4}{17} \right) = \left( \frac{13}{17}, \frac{1}{17} \right)$
- 2)  $(x_1, x_2) = \left( \frac{2(-2) + 5(5)}{17}, \frac{-3(-2) + 5}{17} \right) = \left( \frac{16}{17}, \frac{11}{17} \right)$

**3.8.4.2 Determine the conditions on  $b$ , if any, in order to guarantee that the linear system is consistent.**

$$\begin{cases} x_1 + 3x_2 = b_1 \\ -2x_1 + x_2 = b_2 \end{cases}$$

$$\left[ \begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array} \right] \xrightarrow{R_2+2R_1} \left[ \begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & b_2+2b_1 \end{array} \right] \xrightarrow{\frac{1}{7}R_2} \left[ \begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array} \right] \xrightarrow{R_1-3R_2} \left[ \begin{array}{cc|c} 1 & 0 & \frac{b_1-3b_2}{7} \\ 0 & 1 & \frac{b_2+2b_1}{7} \end{array} \right]$$

There are no conditions. The system is consistent for all values of  $b_1$  and  $b_2$ .

**3.8.5 Another “determining the conditions” problem:**

$$\begin{cases} x_1 - 2x_2 - x_3 = b_1 \\ -4x_1 + 5x_2 + 2x_3 = b_2 \\ -4x_1 + 7x_2 + 4x_3 = b_3 \end{cases}$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array} \right] \xrightarrow[R_3+4R_1]{R_2+4R_1} \left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & b_2+4b_1 \\ 0 & -1 & 0 & b_3+4b_1 \end{array} \right] \xrightarrow{-\frac{1}{3}R_2} \left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{b_3+4b_1}{3} \end{array} \right]$$

$$\xrightarrow{-\frac{3}{2}R_3} \left[ \begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & \frac{2}{3} & \frac{-b_2-4b_1}{3} \\ 0 & 0 & 1 & \frac{-b_3-4b_1}{2} \end{array} \right]$$

Therefore, the system is consistent for all values of  $b_1$ ,  $b_2$ , and  $b_3$ .



### 3.8.6 Triangular and Diagonal Matrices

#### 3.8.6.1 Find $A^2$

$$1) A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 0(0) & 1(0) + 0(-2) \\ 0(1) + (-2)(0) & 0(0) + (-2)(-2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

$$2) A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (-6)(-6) + (0)(0) + (0)(0) & (-6)(0) + (0)(3) + (0)(0) & (-6)(0) + (0)(0) + (0)(5) \\ (0)(-6) + (3)(0) + (0)(0) & (0)(0) + (3)(3) + (0)(0) & (0)(0) + (3)(0) + (0)(5) \\ (0)(-6) + (0)(0) + (5)(0) & (0)(0) + (0)(3) + (5)(0) & (0)(0) + (0)(0) + (5)(5) \end{bmatrix} \\ &= \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} \end{aligned}$$

**3.8.6.2 Find  $A^{-k}$ , such that  $k$  is some nonzero constant**

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} A^{-k} &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-k} \\ &= \begin{bmatrix} 2^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & 2^{-k} \end{bmatrix} \end{aligned}$$

4. Determine whether each matrix is symmetric or not.

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

Not symmetric

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -8 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & -6 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

**3.8.6.3 Find a diagonal matrix A that satisfies the given condition**

$$1) A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{\frac{1}{5}} \\ &= \begin{bmatrix} 1^{\frac{1}{5}} & 0 & 0 \\ 0 & (-1)^{\frac{1}{5}} & 0 \\ 0 & 0 & (-1)^{\frac{1}{5}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

$$2) A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-\frac{1}{2}} \\ &= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 & 0 \\ 0 & 4^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1^{-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

### 3.8.7 Determinants and Triangular Matrices (08/29)

#### 3.8.7.1 What is $C_{32}$

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} C_{32} &= (-1)^{3+2} \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} \\ &= - \left( 2 \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 3 \\ 3 & 0 \end{vmatrix} + 1 \begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} \right) \\ &= - (2(-3) - (-1)(-9) + 1(-3)) \\ &= -(-6 + 9 - 3) \\ &= 0 \end{aligned}$$

#### 3.8.7.2 Find all values of $\lambda$ such that $|A| = 0$

$$A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (\lambda - 2)(\lambda + 4) - (-5)(1) \\ &= \lambda^2 + 2\lambda - 8 + 5 \\ &= \lambda^2 + 2\lambda - 3 \\ &= (\lambda + 3)(\lambda - 1) \\ &= 0 \end{aligned}$$

Therefore,  $\lambda = -3, 1$

**3.8.7.3 For the matrix  $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$  find the determinant 3 sp.different ways with cofactor expansion. Pick sp.different rows and columns each time.**

$$\begin{aligned} \det(A) &= 3 \begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0 \begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + 0 \begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} \\ &= 3(-1(-4) - 5(9)) - 0(2(-4) - 5(1)) + 0(2(9) - (-1)(1)) \\ &= 3(4 - 45) - 0(-8 - 5) + 0(18 + 1) \\ &= 3(-41) - 0(-13) + 0(19) \\ &= 36 \end{aligned}$$

$$\begin{aligned} \det(A) &= 0 \begin{vmatrix} 2 & 5 \\ 9 & -4 \end{vmatrix} - 3 \begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 0 \begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} \\ &= 0(2(-4) - 5(9)) - 3(3(-4) - 0(1)) + 0(3(5) - 0(2)) \\ &= 0(-8 - 45) - 3(-12 - 0) + 0(15 - 0) \\ &= 0(-53) - 3(-12) \\ &= 36 \end{aligned}$$

$$\begin{aligned} \det(A) &= 0 \begin{vmatrix} 2 & -1 \\ 9 & -4 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 0(2(-4) - (-1)(9)) - 0(3(-4) - 0(1)) + 3(3(-1) - 0(2)) \\ &= 0(-8 + 9) - 0(-12 - 0) + 3(-3 - 0) \\ &= 0(1) - 0(-12) + 3(-3) \\ &= 0 + 0 - 9 \\ &= 36 \end{aligned}$$

**3.8.7.4 Evaluate  $\det(A)$  by a cofactor expansion along a row or column of your choice**

$$A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - k \begin{vmatrix} 1 & k^2 \\ 1 & k^2 \end{vmatrix} + k^2 \begin{vmatrix} 1 & k \\ 1 & k \end{vmatrix} \\ &= 1(k^2 - k^2) - k(1(k^2) - k^2(1)) + k^2(1(k) - k(1)) \\ &= 0 \end{aligned}$$

**3.8.7.5 Evaluate the determinant of the following matrices by just looking at them.**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1(-1)(1) = -1$$

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det(A) = 1(1)(2)(3) = 6$$

**3.8.7.6 Show that the value of the determinant is independent of  $\theta$**

$$A = \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ \sin \theta - \cos \theta & \sin \theta + \cos \theta & 1 \end{vmatrix}$$

$$\begin{aligned} \det(A) &= \sin \theta \begin{vmatrix} \sin \theta & 0 \\ \sin \theta + \cos \theta & 1 \end{vmatrix} - \cos \theta \begin{vmatrix} \cos \theta & 0 \\ \sin \theta + \cos \theta & 1 \end{vmatrix} \\ &\quad + 0 \begin{vmatrix} \cos \theta & \sin \theta \\ \sin \theta + \cos \theta & \sin \theta \end{vmatrix} \\ &= \sin \theta (\sin \theta(1) - 0(\sin \theta + \cos \theta)) - \cos \theta (\cos \theta(1) - 0(\sin \theta + \cos \theta)) \\ &\quad + 0 (\cos \theta(\sin \theta) - \sin \theta(\sin \theta + \cos \theta)) \\ &= \sin^2 \theta - \cos^2 \theta \\ &= 1 \end{aligned}$$

### 3.8.8 Row operations and Determinants (08/31)

3.8.8.1 Find the determinant of  $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$  **WITHOUT** using cofactor expansion

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{28}{2} \end{vmatrix} \\ &= 1(-2)\left(\frac{28}{2}\right) \\ &= -28\end{aligned}$$



**3.8.8.2 Find the determinant of**  $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -5 & -1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -3 & 2 \end{vmatrix} \\ &= 2(-2)(-4)(2) \\ &= 64 \end{aligned}$$

### 3.8.9 Adjoints and Cramer's Rule (09/05)

**3.8.9.1 Find the inverse of  $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$  using the adjoint method**

$$\begin{aligned}\det(A) &= 2 \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} - 5 \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} + 5 \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\ &= 2(-3) - 5(-3) + 5(-2) \\ &= -6 + 15 - 10 \\ &= -1\end{aligned}$$

$$\begin{aligned}\text{adj}(A) &= \begin{bmatrix} (-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} & (-1)^{1+2} \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} & (-1)^{1+3} \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} \\ (-1)^{2+1} \begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix} & (-1)^{2+2} \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} & (-1)^{2+3} \begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} \\ (-1)^{3+1} \begin{vmatrix} 5 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix} & (-1)^{3+3} \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} (-1)(3) & -(-1)(3) & -4 + 2 \\ -(15 - 20) & 6 - 10 & -(8 - 10) \\ 5 & -5 & -2 + 5 \end{bmatrix}^T \\ &= \begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}^T \\ &= \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix} \\ \therefore A^{-1} &= - \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}\end{aligned}$$

**3.8.9.2 Solve the following system of equations using Cramer's Rule**

$$\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases} \rightarrow \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \rightarrow 4 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} = -132$$

$$\begin{aligned} \det(x) &= \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \\ &= 2 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 2(2 - 10) - 5(6 - 2) \\ &= -16 - 20 \\ &= -36 \end{aligned}$$

$$\begin{aligned} \det(y) &= \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \\ &= 4 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 4(6 - 2) - 2(22 - 2) \\ &= 16 - 40 \\ &= -24 \end{aligned}$$

$$\begin{aligned} \det(z) &= \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \\ &= 4 \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 5 \begin{vmatrix} 11 & 3 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 11 & 1 \\ 1 & 5 \end{vmatrix} \\ &= 4(1 - 15) - 5(33 - 3) + 2(55 - 1) \\ &= -56 - 150 + 108 \\ &= -98 \end{aligned}$$

Therefore, the solution  $(x, y, z) = \left(\frac{3}{11}, \frac{2}{11}, -\frac{49}{66}\right)$

## 4 Chapter 5: Eigenvectors and Eigenvalues

### 4.1 Eigenvalues and Eigenvectors (11/06)

If  $A$  is an  $n \times n$  matrix, then a non-zero vector  $\mathbf{x}$ , in  $R^n$ , is called an eigenvector of  $A$  if  $A\mathbf{x}$  is a scalar multiple of  $\mathbf{x}$ ; that is  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ . This scalar  $\lambda$  is called an eigenvalue of  $A$  and  $\mathbf{x}$  is said to be an eigenvector corresponding to  $\lambda$ .

See, normally, multiplying a vector by a square matrix changes both the magnitude and the direction of the vector. Really screws it up.

Some examples:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 23 \\ 4 \end{bmatrix}$$

However, there are some ways to get consistent results.

#### 4.1.1 Examples

**4.1.1.1**  $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is an eigenvector of  $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$  because

$$A\vec{x} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{x} \therefore \lambda = 2$$

**4.1.1.2 Let**  $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ ,  $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ . **Are  $\vec{u}$  and  $\vec{v}$  eigenvectors of  $A$ ?**

$$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 1(6) + 6(-5) \\ 5(6) + 2(-5) \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \therefore \lambda = -4$$

$$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 6(-2) \\ 5(3) + 2(-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \vec{v}$$

## 4.2 Eigenvector Homework Problem (11/06)

**Confirm by multiplication that  $\mathbf{x}$  is an eigenvector of  $A$ , and find the corresponding eigenvalue.**

**4.2.1**  $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$A\mathbf{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1) + 0(2) + 1(1) \\ 2(1) + 3(2) + 2(1) \\ 1(1) + 0(2) + 4(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \therefore \lambda = 5$$

## 4.3 Finding Eigenvalues and Eigenvectors (11/07)

Essential question:

**If we know an  $n \times n$  matrix  $A$ , can we find its  $\lambda$ ?**

If  $A\vec{x} = \lambda\vec{x}$ , then:

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

This equation is familiar. It's the homogeneous system of equations  $A\vec{x} = \vec{0}$ , the solution of which is the nullspace of  $A - \lambda I$ . Therefore,  $\vec{x}$  is an eigenvector of  $A \iff \vec{x}$  is in the nullspace of  $A - \lambda I$ .

In this situation, what do we know about that matrix?

Everything in the equivalent statements is false because  $\vec{x}$  cannot be the zero vector. Therefore, we can see that  $\det(A - \lambda I)$  OR  $\det(\lambda I - A)$  MUST be 0.

Big Idea: If  $A$  is an  $n \times n$  matrix, then  $\lambda$  is an eigenvalue of  $A \iff \det(\lambda I - A) = 0$ . This is called the characteristic equation of  $A$ .

**4.3.1 Find the characteristic equation and the eigenvalues of  $A = \begin{bmatrix} 3 & 0 & 5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$**

$$\begin{aligned} \det(\lambda I - A) &= 0 \\ \begin{vmatrix} \lambda - 3 & 0 & 5 \\ -\frac{1}{5} & \lambda + 1 & 0 \\ -1 & -1 & \lambda + 2 \end{vmatrix} &= 0 \\ 0 &= (\lambda - 3)((\lambda + 1)(\lambda + 2)) + 5\left(\frac{1}{5} + \lambda + 1\right) \\ 0 &= (\lambda - 3)(\lambda^2 + 3\lambda + 2) \\ 0 &= \lambda^3 - 2\lambda \\ 0 &= \lambda(\lambda^2 - 2)\lambda &= 0, \pm\sqrt{2} \end{aligned}$$

**4.3.2 Find the characteristic equation and the eigenvalues of  $A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$**

$$\begin{aligned} \begin{vmatrix} -1 - \lambda & 0 & 1 \\ -1 & 3 - \lambda & 0 \\ -4 & 13 & -\lambda \end{vmatrix} &= 0 \\ (-1 - \lambda)((3 - \lambda)(-\lambda) - 0(13)) + (-1(13) - (3 - \lambda)(-4)) &= 0 \\ (-1 - \lambda)(\lambda^2 - 3\lambda) + (-13 - 4\lambda + 12) &= 0 \\ (-1 - \lambda)(\lambda^2 - 3\lambda) + (-4\lambda - 1) &= 0 \\ -\lambda^3 + 3\lambda^2 + 2 &= 0 \\ (-\lambda + 2)(-\lambda^2 - \lambda - 1) &= 0 \\ (-\lambda + 2)(-\lambda - 1)(-\lambda + 1) &= 0 \\ \lambda &= 2 \end{aligned}$$

**4.3.3 Find the eigenvalues of**  $A = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 2 & 0 & 0 \\ 6 & \lambda - 3 & 0 \\ 1 & 4 & \lambda - 5 \end{vmatrix} = 0$$

$$(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$$

$$\lambda = 2, 3, 5$$

Theorem 1: For a triangular matrix, the eigenvalues are the elements on the main diagonal.

**4.3.4 Find the eigenvalues of  $A^3$  if**  $A = \begin{bmatrix} \frac{1}{2} & 4 & 5 & -2 \\ 0 & -1 & 3 & -8 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$\lambda_A = \frac{1}{2}, -1, 2, 4$$

$$\lambda_{A^3} = \frac{1}{8}, -1, 8, 64$$

Theorem 2: The eigenvalues of  $A^k$  are  $\lambda_1^k, \lambda_2^k, \dots$

**4.3.5 Give me a matrix with eigenvalues  $\lambda = 0, 2, 5$**

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 5 \end{bmatrix}$$

Theorem 3: A square matrix  $A$  is invertible  $\iff \lambda \neq 0$  (which also means its determinant is 0).

### 4.3.6 Finding eigenvectors!

Find the nontrivial eigenvectors of:

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$\begin{aligned} \begin{vmatrix} \lambda - 1 & -6 \\ -5 & \lambda - 2 \end{vmatrix} &= 0 \\ (\lambda - 1)(\lambda - 2) - (-6)(-5) &= 0 \\ \lambda^2 - 3\lambda - 28 &= 0 \\ (\lambda - 7)(\lambda + 4) &= 0 \\ \lambda &= 7, -4 \end{aligned}$$

Substitute each  $\lambda$ , one at a time into the  $\lambda I - A$  matrix and find the null space.

For  $\lambda = -4$ :

$$\begin{aligned} \left( \begin{array}{cc|c} -5 & -6 & 0 \\ -5 & -6 & 0 \end{array} \right) \\ \left( \begin{array}{cc|c} -5 & -6 & 0 \\ 0 & 0 & 0 \end{array} \right) \\ \langle -\frac{6}{5}t, t \rangle \\ \vec{x} = \{ \langle -6, 5 \rangle \} \end{aligned}$$

For  $\lambda = 7$ :

$$\begin{aligned} \left( \begin{array}{cc|c} 6 & -6 & 0 \\ -5 & 5 & 0 \end{array} \right) \\ \left( \begin{array}{cc|c} 6 & -6 & 0 \\ 0 & 0 & 0 \end{array} \right) \\ \langle t, t \rangle \\ \vec{x} = \{ \langle 6, 6 \rangle \} \end{aligned}$$

Therefore, the eigen space is:  $\{ \langle -6, 5 \rangle, \langle 6, 6 \rangle \}$

## 4.4 Diagonalization and Similar Triangles

Similar matrices: If  $A$  and  $D$  are square matrices, we say that  $A$  and  $D$  are “similar” if there exists an invertible matrix  $P$  such that:

$$D = P^{-1}AP.$$



#### 4.4.1 Properties of Similar Matrices

- They have the same determinant
- If one is invertible, so is the other
- They have the same trace
- They have the same characteristic polynomial
- They have the same eigenvalues

#### 4.4.2 Procedure

1. Find the eigenvectors for the  $n \times n$  matrix  $A$ .
  - Theorem: If an  $n \times n$  matrix  $A$  has  $n$  distinct eigenvalues, then  $A$  is **for sure** diagonalizable.
2. Make matrix  $P$  out of the eigenvectors ( $P$  is the matrix that diagonalizes  $A$ )
3. Check your work to find matrix  $D$  if reasonable

#### 4.4.3 Example: Find a matrix $P$ that diagonalizes $A$ and compute $P^{-1}AP$

1.  $A = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$

Find the eigenvalues:

$$\begin{aligned} \begin{bmatrix} \lambda - 3 & -7 \\ -5 & \lambda - 5 \end{bmatrix} &= 0 \\ (\lambda - 3)(\lambda - 5) - (-7)(-5) &= 0 \\ \lambda^2 - 5\lambda - 3\lambda + 15 - 35 &= 0 \\ \lambda^2 - 8\lambda - 20 &= 0 \\ \lambda &= -2, 10 \end{aligned}$$

Find the eigenvectors:

$$\begin{aligned} \lambda = -2 : \left[ \begin{array}{cc|c} -5 & -7 & 0 \\ -5 & -7 & 0 \end{array} \right] \vec{x} &= \{ \langle -7, 5 \rangle \} \\ \lambda = 10 : \left[ \begin{array}{cc|c} -7 & -7 & 0 \\ -5 & 5 & 0 \end{array} \right] \vec{x} &= \{ \langle 1, 1 \rangle \} \end{aligned}$$

Create the matrix  $P$ :

$$P = \begin{bmatrix} -7 & 1 \\ 5 & 1 \end{bmatrix}$$

Find matrix  $D$ :

$$\begin{aligned} D &= P^{-1}AP \\ &= \begin{bmatrix} 7 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & 10 \end{bmatrix} \end{aligned}$$

#### 4.4.4 Conclusion

- If  $D$  has the same eigenvalues of  $A$  and if  $D$  must be diagonal, then  $D$  is **THE** diagonal matrix with eigenvalues of  $A$  on the diagonal.

$$2. A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

First, find  $D$ :

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, find  $P$ :

$$\lambda = 2 : \left[ \begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \vec{x} = \langle 1, 0, 0 \rangle$$

$$\lambda = 3 : \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \vec{x} = \langle 0, 1, 0 \rangle$$

$$\lambda = 1 : \left[ \begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \vec{x} = \langle 2, 0, 1 \rangle$$

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 4.5 More on Similar Matrices

There are a few more properties of similar matrices:

- They have the same rank (non-zero eigenvalues)
- They have the same nullity
- They have the same column space
- They have the same row space

### 4.5.1 Example

\*\*Matrix A is similar to the following matrix:

$$D = \begin{bmatrix} 3 & -1 & 1 & 4 & 5 & 2 \\ 0 & -3 & 5 & -10 & -16 & 1 \\ 0 & 0 & 5 & 7 & -8 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A: 4

Nullity of A: 2

Eigenvalues: 3, -3, 5, 2, 0, 0

Characteristic Polynomial:

$$\det(\lambda I - A) = 0$$
$$\begin{vmatrix} \lambda - 3 & 1 & -1 & -4 & -5 & -2 \\ 0 & \lambda + 3 & -5 & 10 & 16 & -1 \\ 0 & 0 & \lambda - 5 & -7 & 8 & -2 \\ 0 & 0 & 0 & \lambda - 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} = 0$$
$$(\lambda - 3)(\lambda + 3)(\lambda - 5)(\lambda - 2)\lambda^2 = 0$$

#### 4.5.2 Some review

- Eigenspace of  $\lambda$ : The nullspace of  $\lambda I - A$ . Each eigenvalue will have its own eigenspace.
- Algebraic multiplicity: The number of times a given  $\lambda$  appears as a root of the characteristic equation.
- Geometric multiplicity: The number of eigenvectors it maps to.

##### 4.5.2.1 Theorem: Geometric and Algebraic Multiplicity

If  $A$  is a square matrix, then: a. For every eigenvalue of  $A$ , the geometric multiplicity is less than or equal to the algebraic multiplicity. b.  $A$  is diagonalizable  $\iff$  the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity.

#### 4.6 Similar Matrices Continued (11/13/2023)

##### 4.6.1 Warm-Up

Can you write a new statement involving eigenvalues to add to the list of equivalent statements?

$\lambda = 0$  is not an eigenvalue of  $A$

##### 4.6.2 Homework Review

$$A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 19 & 9 & 6 \\ -25 & \lambda + 11 & 9 \\ 17 & 9 & \lambda + 4 \end{bmatrix} = (\lambda - 19)((\lambda + 11)(\lambda + 4) - 81) + 25(9\lambda + 36 - 54) - 17(81 - 6\lambda - 66) \\ = (\lambda - 1)^2(\lambda - 2)$$

$\lambda = 1$  has an algebraic multiplicity of 2 and  $\lambda = 2$  has an algebraic multiplicity of 1.

#### 4.6.3 Suppose that a characteristic polynomial of some matrix $A$ is found to be:

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$$

- a. What are the dimensions of  $A$ ?

$$6 \times 6$$

- b. What are the algebraic multiplicities of each eigenvalue?

$$\lambda = 1 : 1, \lambda = 3 : 2, \lambda = 4 : 3$$

- c. What are the possible dimensions of the eigenspace associated with each of the eigenvalues?

$$\lambda = 1 : 1, \lambda = 3 : 1 \text{ or } 2, \lambda = 4 : 1 \text{ or } 2 \text{ or } 3$$

- d. If  $\{v_1, v_2\}$  is a linearly independent set of eigenvectors of  $A$ , all of which correspond to the same eigenvalue of  $A$ , what can you say about the eigenvalue?

The eigenvalue must be 3 or 4.

## 5 Semester II

### 5.1 Introduction to Multivariable functions (01/04)

#### 5.1.1 Definition of a Multivariable Function

Suppose  $D$  is a set of  $n$ -tuples of real numbers  $(x_1, x_2, \dots, x_n)$ . A function  $f$  on  $D$  is a rule that assigns a real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in  $D$ . The set  $D$  is the function's **domain**. The set of  $w$ -values taken on by  $f$  is the function's **range**. The symbol  $w$  is the **dependent variable** of  $f$ , and  $f$  is a function of the  $n$  **independent variables**  $x_1$  to  $x_n$ . The  $x$ 's are the function's **input variables**;  $w$  is the function's **output variable**.

## 5.2 Limits and Continuity (01/05)

### 5.2.1 Level Curve, Graph, Surface

The set of points in the plane where a function  $f(x, y)$  has a constant value  $k$  is called a **level curve** of  $f$ . The graph of  $f$  is the set of all points  $(x, y, z)$  in space, where  $z = f(x, y)$ . The graph of  $f$  is a surface in space.

### 5.2.2 Limits With Multivariable Functions

*Limits:* Let  $f$  be a function of two variables defined on an open region, except possibly at  $(x_0, y_0)$ .

In 2-D:

$$\lim_{x \rightarrow c} f(x) \text{ exists iff } \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L \in \mathbb{R}$$

In 3-D:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) \text{ exists iff } \lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L \in \mathbb{R} \text{ from all directions.}$$

### 5.2.3 Examples

$$1. \lim_{(x,y) \rightarrow (1,2)} \frac{5x^2y}{x^2 + y^2} = \frac{5 \cdot 1 \cdot 2}{1 + 4} = \frac{10}{5} = 2$$

$$2. \lim_{(x,y) \rightarrow (1,1)} \frac{x - y}{x^2 - y^2} = \frac{0}{0}$$

Since this is indeterminate, you need to switch to traditional limit solving methods. There is no *L'Hopital's Rule* for multivariable functions. Instead, factor the numerator and denominator and cancel out common factors.

$$\lim_{(x,y) \rightarrow (1,1)} \frac{x - y}{(x - y)(x + y)} = \frac{1}{2}$$

$$3. \lim_{(x,y) \rightarrow (4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$$

## 5.2.4 Homework

**5.2.4.1 Evaluate the limit**  $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} \bigg|_{(0,0)} = \frac{5}{2}$$

**5.2.4.2 Evaluate the limit**  $\lim_{(x,y) \rightarrow (1,1)} \cos \sqrt[3]{|xy| - 1}$

$$\lim_{(x,y) \rightarrow (1,1)} \cos \sqrt[3]{|xy| - 1} \bigg|_{(1,1)} = \cos 0 = 1$$

**5.2.4.3 Evaluate the limit**  $\lim_{(x,y) \rightarrow (1,1)} \frac{xy - y - 2x + 2}{x - 1}$

$$\lim_{(x,y) \rightarrow (1,1)} \frac{xy - y - 2x + 2}{x - 1} = \frac{(x-1)(y-2)}{x-1} = y - 2 \big|_{(1,1)} = -1$$

**5.2.4.4 On what interval is the function  $f(x, y) = \sin(x + y)$  continuous?**

The sin function is continuous everywhere, so  $(x, y) \in \mathbb{R}^2$

**5.2.4.5 On what interval is the function  $f(x, y, z) = x^2 + y^2 - 2z^2$  continuous?**

$(x, y, z) \in \mathbb{R}^3$

**5.2.4.6 On what interval is the function  $f(x, y, z) = xy \sin\left(\frac{1}{z}\right)$  continuous?**

Can't divide by zero, so  $(x, y, z) \in \mathbb{R}^3 | z \neq 0$

## 5.3 Limits that DO NOT EXIST in 3-Space (01/08)

**5.3.1 Question: Does the function  $f(x, y) = \frac{2x^2y}{x^4 + y^2}$  have a limit as  $(x, y)$  approaches  $(0, 0)$ ?**

Nope, it approaches the point sp.differently.

**5.3.2 Find  $f(x, y)|_{y=x^2}$  and compute  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  along  $y = x^2$**

$$f(x, y) = \frac{2x^2y}{x^4 + y^2} \bigg|_{y=x^2} = \frac{2x^4}{2x^4} = 1$$

**5.3.3 Find  $f(x, y)|_{y=-x^2}$  and compute  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  along  $y = -x^2$**

$$f(x, y) = \frac{2x^2y}{x^4 + y^2} \bigg|_{y=-x^2} = \frac{-2x^4}{2x^4} = -1$$

**5.3.4 Explain why we can conclude that  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist**

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = x^2 \neq \lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = -x^2$$

**5.3.5 How did we know to choose  $y = x^2$  and  $y = -x^2$  to evaluate the limit?**

You want to try to choose things that, as you plug them in, you get a nice expression that you can simplify. For this problem in particular, we chose paths towards  $(0, 0)$  that would be easy to solve and yield sp.different answers upon simplification. We could've also used the equation of the x-axis ( $y = 0$ ).

**5.3.6 Show that these functions have no limit as  $(x, y)$  approaches  $(0, 0)$  by considering sp.different paths of approach.**

**5.3.6.1**  $f(x, y) = \frac{x^4}{x^4 + y^2}$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = 2x^2 = \frac{x^4}{5x^4} = \frac{1}{5} \neq \lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = x^2 = \frac{x^4}{2x^4} = \frac{1}{2}$$

**5.3.6.2**  $f(x, y) = \frac{x^2 + y}{y}$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = x^2 = \frac{2y}{y} = 2 \neq \lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } y = 2x^2 = \frac{3x^2}{2x^2} = \frac{3}{2}$$



## 5.4 Partial Derivatives (01/10)

### 5.4.1 First Order Partial Derivatives

A **partial derivative** is obtained by holding all but one of the independent variables constant and differentiating with respect to that variable.

#### 5.4.1.1 Notation

$\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x} f(x, y)$  is the partial derivative of  $f$  with respect to  $x$

#### 5.4.2 Examples

**5.4.2.1**  $f(x, y) = 2x^3y - 4x^2y^3 + 5x^4$

$$f_x = 6x^2y - 8xy^3 + 20x^3$$

$$f_y = 2x^3 - 12x^2y^2$$

**5.4.2.2**  $f(x, y) = 4x^2y - 8x^3y^4 + 2xy^7$

$$f_x = 8xy - 24x^2y^4 + 2y^7$$

$$f_y = 4x^2 - 32x^3y^3 + 14xy^6$$

**5.4.2.3**  $f(x, y) = \tan(2x - y)$

$$f_x = 2 \sec^2(2x - y)$$

$$f_y = -\sec^2(2x - y)$$

### 5.4.3 The Second Fundamental Theorem of (Multivariable) Calculus

- $\frac{d}{dx} \int_5^x f(t)dt = f(x)$
- $\frac{d}{dx} \int_5^{x^2} f(t)dt = 2x \cdot f(x^2)$

## 5.4.4 Examples

### 5.4.4.1 $f(x, y) = \int_{3x}^{2y} (t^2 - 1) dt$

```
1 # import libraries
2 import sympy as sp
3 from IPython.display import display, Math, Latex
4 x, y, t, z, r, , p, q = sp.symbols('x y t z r p q', real=True)

1 f = sp.integrate(t**2-1, (t, 3*x, 2*y))
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
```

$$3 - 27x^2$$

$$8y^2 - 2$$

### 5.4.4.2 Find both partial derivatives and evaluate each at the point $(1, \ln 2)$

1.  $f(x, y) = xe^{x^2y}$

```
1 f = x*sp.exp(x**2*y)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
4 display(sp.diff(f, x).subs({x:1, y:sp.ln(2)}))
5 display(sp.diff(f, y).subs({x:1, y:sp.ln(2)}))
```

$$2x^2ye^{x^2y} + e^{x^2y}$$

$$x^3e^{x^2y}$$

$$2 + 4 \log(2)$$

$$2$$

## 5.5 2nd Partial Derivatives (01/11)

**5.5.1 First, let's see this one. If  $f(x, y, z) = x \sin(y + 3z)$ , find  $f_x, f_y, f_z$ .**

```

1 f = x*sp.sin(y+3*z)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
4 display(sp.diff(f, z))

```

$$\sin(y + 3z)$$

$$x \cos(y + 3z)$$

$$3x \cos(y + 3z)$$

## 5.5.2 2nd-Order Partial Derivatives

The **second-order partial derivatives** of  $f$  are the partial derivatives of the partial derivatives of  $f$ .

### 5.5.2.1 Notation

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$$

## 5.5.3 Mixed Partial Derivatives

The **mixed partial derivatives** of  $f$  are the partial derivatives of  $f$  with respect to many other different variables.

### 5.5.3.1 Notation

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$$

To compute a mixed partial derivative, you must evaluate the order from right to left. For example,  $f_{xy}$  is the partial derivative of  $f_x$  with respect to  $y$ , which is notated as  $\frac{\partial^2 f}{\partial x \partial y}$ .

## 5.5.4 Theorem (Clairaut's Theorem)

If  $f_{xy}$  and  $f_{yx}$  are continuous on an open region  $D$ , then  $f_{xy}(a, b) = f_{yx}(a, b)$  on  $D$ .

## 5.5.5 Examples

### 5.5.5.1 Given that $f(x, y) = x^2y - y^3 + \ln x$ , find all 2nd order partial derivatives

```
f = x**2*y - y**3 + sp.ln(x) display(sp.diff(f, y, x)) display(sp.diff(f, x, y)) display(sp.diff(f, x, x)) display(sp.diff(f, y, y))
```

#### Given that  $w = xy + \frac{e^y}{y^2 + 1}$ , find  $\frac{\partial^2 w}{\partial x \partial y}$

For this problem, you need to "choose wisely." Knowing that the function and it's partial derivatives

### Partial Derivatives Homework (01/11)

### Find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  for  $f(x, y)$

####  $f(x, y) = x(y-1)$

```
```{python}
f = x*(y-1)
display(sp.diff(f, x))
display(sp.diff(f, y))
```

### 5.5.6 $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

```
1 f = 5*x*y - 7*x**2 - y**2 + 3*x - 6*y + 2
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
```

### 5.5.7 $f(x, y) = \sqrt{x^2 + y^2}$

```
1 f = sp.sqrt(x**2 + y**2)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
```

**5.5.8**  $f(x, y) = e^{(x+y+1)}$

```
1 f = sp.exp(x+y+1)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
```

**5.5.9**  $f(x, y) = \int_x^y g(t)dt$

```
1 g = sp.Function('g')
2 f = sp.integrate(g(t), (t, x, y))
3 display(sp.diff(f, x))
4 display(sp.diff(f, y))
```

**5.5.10**  $f(x, y, z) = xy + yz + xz$

```
1 f = x*y + y*z + x*z
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
4 display(sp.diff(f, z))
```

**5.5.11**  $f(x, y, z) = \ln(x + 2y + 3z)$

```
1 f = sp.ln(x + 2*y + 3*z)
2 display(sp.diff(f, x))
3 display(sp.diff(f, y))
4 display(sp.diff(f, z))
```

**5.5.12** If  $f(x, y) = x \cos(y) + ye^x$ , find  $\frac{\partial^2 f}{\partial x^2}$ ,  $\frac{\partial^2 f}{\partial y^2}$ ,  $\frac{\partial^2 f}{\partial x \partial y}$ , and  $\frac{\partial^2 f}{\partial y \partial x}$

```

1 f = x*sp.cos(y) + y*sp.exp(x)
2 display(sp.diff(f, x, x))
3 display(sp.diff(f, y, y))
4 display(sp.diff(f, x, y))
5 display(sp.diff(f, y, x))

```

## 5.6 Definition of Differentiability (01/16)

So, how do we solve the apparent contradiction?

Definition of differentiability in multivariable calculus:

The function  $f(x, y)$  is said to be differentiable if there exists a *linear function* at the point  $(a, b)$ :

$$L(x, y) = f(a, b) + p(x - a) + q(y - b)$$

### 5.6.1 Remember local linearization?

...what it means for a function to be “locally linear?”

Consider the following:

Is this function differentiable at  $x = 0$ ?

$$f(x) = |x| + 1$$

What about this one?

$$g(x) = \sqrt{x^2 + 0.0001} + 0.99$$

The linearization of a function  $f(x, y)$  at a point  $(x_0, y_0)$  where  $f$  is differentiable is given by the function:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad z = z_0 + m_x(x - x_0) + m_y(y - y_0) \quad 0 = m_x(x - x_0) + m_y(y - y_0) + (z - z_0)$$

$$0 = m_x(x - x_0) + m_y(y - y_0) + (z - z_0)$$

$$0 = A(x - x_0) + B(y - y_0) + C(z - z_0) \quad \text{...where } \vec{n} = \langle A, B, C \rangle \text{ and } p = (x_0, y_0, z_0)$$

### 5.6.2 Examples

**5.6.2.1 Find the linearization of  $f(x, y) = y^2 + 2xy - \frac{1}{2}x^2$  at the point  $(2, 3)$ .**

### 5.6.3 Homework

#### 5.6.3.1 Find the linearization of $f(x, y) = x^2 + y^2 + 1$ at $(1, 1)$

```
1 f = x**2 + y**2 + 1
2 x_0 = 1
3 y_0 = 1
4 z_0 = f.subs({x:x_0, y:y_0})
5 f_x = sp.diff(f, x).subs({x:x_0, y:y_0})
6 f_y = sp.diff(f, y).subs({x:x_0, y:y_0})
7 display(z_0 + f_x*(x-x_0) + f_y*(y-y_0))
```

#### 5.6.3.2 Find the linearization of $f(x, y) = 3x - 4y + 5$ at $(1, 1)$

```
1 f = 3*x - 4*y + 5
2 x_0 = 1
3 y_0 = 1
4 z_0 = f.subs({x:x_0, y:y_0})
5 f_x = sp.diff(f, x).subs({x:x_0, y:y_0})
6 f_y = sp.diff(f, y).subs({x:x_0, y:y_0})
7 display(z_0 + f_x*(x-x_0) + f_y*(y-y_0))
```

#### 5.6.3.3 Find the linearization of $f(x, y) = e^x \cos y$ at $(0, \frac{\pi}{2})$

```
1 f = sp.exp(x)*sp.cos(y)
2 x_0 = 0
3 y_0 = sp.pi/2
4 z_0 = f.subs({x:x_0, y:y_0})
5 f_x = sp.diff(f, x).subs({x:x_0, y:y_0})
6 f_y = sp.diff(f, y).subs({x:x_0, y:y_0})
7 display(z_0 + f_x*(x-x_0) + f_y*(y-y_0))
```

## 5.7 Chain Rule (01/18)

### 5.7.1 Warmup: Find the linearization of $f(x, y) = x^3 y^4$ at the point $(1, 1)$

```

1 f = x**3*y**4
2 x_0 = 1
3 y_0 = 1
4 z_0 = f.subs({x:x_0, y:y_0})
5 f_x = sp.diff(f, x).subs({x:x_0, y:y_0})
6 f_y = sp.diff(f, y).subs({x:x_0, y:y_0})
7 display(z_0 + f_x*(x-x_0) + f_y*(y-y_0))

```

### 5.7.2 Chain Rule (1-Variable)

The “normal” chain rule formula:

$$\frac{d}{dt}f(g(t)) = f'(g(t)) \cdot g'(t) = \frac{df}{dt} \cdot \frac{dg}{dt} = \frac{df}{dt}$$

### 5.7.3 Chain Rule (Multi Variable)

Let  $w = f(x, y)$  where  $f$  is a differentiable function of  $x$  and  $y$ . If  $x = g(t)$  and  $y = h(t)$  where  $g$  and  $h$  are differentiable functions of  $t$ , then  $w$  is also a differentiable function of  $t$  and  $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$

### 5.7.4 Examples

**5.7.4.1** If  $w = x^2y - y^2$  where  $x = \sin t$  and  $y = e^t$ , find  $\frac{dw}{dt}$  at  $t = 0$

```

1 w = x**2*y - y**2
2 w_t = w.subs({x: sp.sin(t), y: sp.exp(t)})
3 dw_dt = sp.diff(w_t, t)
4 display(dw_dt.subs(t, 0))

```

**5.7.4.2** If  $w = xy + xz + yz$  and  $x = t - 1, y = t^2 - 1, z = t$ , find  $\frac{dw}{dt}$

```

1 w = x*y + x*z + y*z
2 w_t = w.subs({x: t-1, y: t**2-1, z: t})
3 display(sp.diff(w_t, t))

```



## 5.7.5 Homework

**5.7.5.1**  $w = x^2 + y^2, x = \cos t, y = \sin t$  at  $t = \pi$

```
1 w = x**2 + y**2
2 w_t = w.subs({x: sp.cos(t), y: sp.sin(t)})
3 display(sp.diff(w_t, t))
4 display(sp.diff(w_t, t).subs(t, sp.pi))
```

**5.7.5.2**  $w = 2ye^x - \ln z, x = \ln(t^2 + 1), y = \arctan t, z = e^t$  at  $t = 1$

```
1 w = 2*y*sp.exp(x) - sp.ln(z)
2 part_wx = sp.diff(w, x)
3 part_wy = sp.diff(w, y)
4 part_wz = sp.diff(w, z)
5 dx_dt = sp.diff(sp.ln(t**2 + 1), t)
6 dy_dt = sp.diff(sp.atan(t), t)
7 dz_dt = sp.diff(sp.exp(t), t)
8 dw_dt = part_wx * dx_dt + part_wy * dy_dt + part_wz * dz_dt
9 display(dw_dt)
10 display(dw_dt.subs(t, 1))
```

**5.7.5.3**  $z = 4e^x \ln y, x = \ln(r \cos \theta), y = r \sin \theta; (r, \theta) = \left(2, \frac{\pi}{4}\right)$

```
1 z = 4*sp.exp(x)*sp.ln(y)
2 part_zx = sp.diff(z, x)
3 part_zy = sp.diff(z, y)
4 dx_dr = sp.diff(sp.ln(r*sp.cos()), r)
5 dx_d = sp.diff(sp.ln(r*sp.cos()), )
6 dy_dr = sp.diff(r*sp.sin(), r)
7 dy_d = sp.diff(r*sp.sin(), )
8 dz_dr = part_zx * dx_dr + part_zy * dy_dr
9 dz_d = part_zx * dx_d + part_zy * dy_d
10 display(Latex(r"$\frac{\partial z}{\partial r} = " + sp.latex(dz_dr) + "$"))
11 display(Latex(r"$\frac{\partial z}{\partial \theta} = " + sp.latex(dz_d) +
    "$"))
12 display(dz_dr.subs({r:2, :sp.pi/4}))
13 display(dz_d.subs({r:2, :sp.pi/4}))
```

**5.7.5.4**  $u = \frac{p-q}{q-r}, p = x + y + z, q = x - y + z, r = x + y - z; (x, y, z) = (\sqrt{3}, 2, 1)$

```

1 u = (p-q)/(q-r)
2 part_ux = sp.diff(u.subs({p: x+y+z, q: x-y+z, r: x+y-z}), x)
3 part_uy = sp.diff(u.subs({p: x+y+z, q: x-y+z, r: x+y-z}), y)
4 display(Latex(r"$\frac{\partial u}{\partial x} = " + sp.latex(part_ux) +
   "$"))
5 display(Latex(r"$\frac{\partial u}{\partial y} = " + sp.latex(part_uy) +
   "$"))
6 display(Latex(r"$\frac{\partial u}{\partial z} = \frac{4y}{(-2y+2z)^2}$"))
7 display(part_ux.subs({x:sp.sqrt(3), y:2, z:1}))
8 display(part_uy.subs({x:sp.sqrt(3), y:2, z:1}))
9 display(Latex(r"$-2$"))

```

## 5.8 Related Rates and Implicit Differentiation (01/22)

### 5.8.1 Related rates in Calc I

- 1) Come up with a formula (usually volume or something)
- 2) Pray to the math gods that you can somehow put the formula in terms of one variable
- 3) Take the derivatives, plug in tons of values and rates, and solve for the missing one

In Calc III, you can remove step 2 and just use chain rule for step 3.

### 5.8.2 Examples

**5.8.2.1 A right circular cylinder with a n open top has height  $h$ , radius  $r$ , and surface area  $A$ . If  $\frac{dh}{dt} = 3$  and  $\frac{dr}{dt} = -2$ , find  $\frac{dA}{dt}$  when  $h = 10$  and  $r = 5$ .**

```

1 h, r, t = sp.symbols('h r t')
2 A = sp.pi*r**2 + 2*sp.pi*r*h
3 dhdt = 3
4 drdt = -2
5 dAdt = sp.diff(A, r)*drdt + sp.diff(A, h)*dhdt
6 display(dAdt.subs({h:10, r:5}))

```

### 5.8.3 Implicit Differentiation From Calc I

**5.8.3.1 Find  $\frac{dy}{dx}$  given that  $y^3 + 4x^2 - 2xy + 3x = 19$**

$$y' = \frac{8x+2y-3}{3y^2-2x}$$

**5.8.3.2 Now suppose that  $f(x, y) = y^3 + 4x^2 - 2xy + 3x - 19$ . Find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .**

```
1 display(sp.diff(f, x))
2 display(sp.diff(f, y))
```

If  $f(x, y) = 0$  defines  $y$  implicitly as a differentiable function of  $x$ , then  $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

Similarly, if  $f(x, y, z) = 0$  defines  $z$  implicitly as a differentiable function of  $x$  and  $y$ , then

$$\frac{dz}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \text{ and } \frac{dz}{dy} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

### 5.9 Directional Derivatives and Gradient Vectors (01/25)

Suppose that  $f$  is differentiable at  $(x_0, y_0)$  and  $\vec{u} = \langle u_1, u_2 \rangle$  is any unit vector. Then the directional derivative of  $f$  at  $(x_0, y_0)$  in the direction of  $\vec{u}$  is given by:  $D_{\vec{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$

In other words, this is partial  $x$  times the  $x$  component of the unit vector plus partial  $y$  times the  $y$  component of the unit vector.

The directional derivative is also denoted by:

$(D_u f)_{p_0}$ ; "The derivative of  $f$  at  $P_0$  in the direction of  $u$ "

**5.9.1 Find the directional derivative of  $f(x, y) = x^2 \sin 2y$  at  $(1, \frac{\pi}{2})$  in the direction of  $\vec{v} = 3\vec{i} - 4\vec{j}$ .**

$$\begin{aligned} f_x &= 2x \sin 2y \\ f_y &= 2x^2 \cos 2y \\ f_x(1, \frac{\pi}{2}) &= 0 \\ f_y(1, \frac{\pi}{2}) &= -2 \\ \vec{u}_1 &= \frac{3}{5} \\ \vec{u}_2 &= -\frac{4}{5} \\ D_{\vec{u}} f(1, \frac{\pi}{2}) &= 0 \cdot \frac{3}{5} - 2 \cdot -\frac{4}{5} \\ &= \frac{8}{5} \end{aligned}$$

### 5.9.2 Thinking questions

Not too bad, right?

What if I wanted the directional derivative of  $f(x, y)$  in the direction of  $\vec{v} = \vec{i}$ ? What can you tell me about that derivative?

$$f_x$$

What if I just wanted it in the direction of  $\vec{v} = \vec{j}$ ?

$$f_y$$

So, all of this time, when you've been finding  $f_x$  and  $f_y$  at a point  $(x_0, y_0)$ , you've really just been finding the components to a (not unit) vector that contains both derivatives

### 5.9.3 The Gradient

Let  $z = f(x, y)$  be a function of  $x$  and  $y$  such that  $f_x$  and  $f_y$  exist. Then the gradient of  $f$  denoted by  $\nabla f(x, y)$  is the vector:

$$\nabla f(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}$$

#### 5.9.4 Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point $(e^3, 2)$

$$\begin{aligned}f_x &= \frac{y}{x} + y^2 \\f_y &= \ln x + 2xy \\ \nabla f(x, y) &= \left\langle \frac{2}{e^3} + 4, 3 + 4e^3 \right\rangle\end{aligned}$$

Wait a second... go back to how we calculate a directional derivative...

$$D_{\vec{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

This is the same as the dot product of the gradient and the unit vector!

$$D_{\vec{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

#### 5.9.5 Homework

Find the gradient of the following:

##### 5.9.5.1 $g(x, y) = y - x^2, (-1, 0)$

```
1 g = y - x**2
2 g_x = sp.diff(g, x)
3 g_y = sp.diff(g, y)
4 display(g_x.subs({x:-1, y:0}))
5 display(g_y.subs({x:-1, y:0}))
```

Find the directional derivative of  $f$  at  $P_0$  in the direction of  $A$ .

##### 5.9.5.2 $f(x, y) = 2xy - 3y^2, P_0 = (5, 5), A = \langle 4, 3 \rangle$

```
1 f = 2*x*y - 3*y**2
2 f_x = sp.diff(f, x).subs({x:5, y:5})
3 f_y = sp.diff(f, y).subs({x:5, y:5})
4 A = sp.sqrt(4**2 + 3**2)
5 u_1 = 4/A
6 u_2 = 3/A
7 display(f_x*u_1 + f_y*u_2)
```

**5.9.5.3**  $f(x, y) = x - \left(\frac{y^2}{x}\right) + \sqrt{3} \sec^{-1}(2xy), P_0 = (1, 1), A = \langle 12, 5 \rangle$

```

1 f = x - y**2/x + sp.sqrt(3)*sp.asec(2*x*y)
2 f_x = sp.diff(f, x).subs({x:1, y:1})
3 f_y = sp.diff(f, y).subs({x:1, y:1})
4 A = sp.sqrt(12**2 + 5**2)
5 u_1 = 12/A
6 u_2 = 5/A
7 display(f_x*u_1 + f_y*u_2)

```

**5.9.5.4**  $f(x, y) = xy + yz + zx, P_0 = (1, -1, 2), A = \langle 3, 6, -2 \rangle$

```

1 x, y, z = sp.symbols('x y z', real=True)
2 f = x*y + y*z + z*x
3 f_x = sp.diff(f, x).subs({x:1, y:-1, z:2})
4 f_y = sp.diff(f, y).subs({x:1, y:-1, z:2})
5 f_z = sp.diff(f, z).subs({x:1, y:-1, z:2})
6 A = sp.sqrt(3**2 + 6**2 + (-2)**2)
7 u_1 = 3/A
8 u_2 = 6/A
9 u_3 = -2/A
10 display(f_x*u_1 + f_y*u_2 + f_z*u_3)

```

## 5.10 More Gradient and Tangent Planes (01/29)

### 5.10.1 Properties of Gradients

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = ||\nabla f|| ||\vec{u}|| \cos \theta$$

In order to maximize this value,  $\cos \theta$  must be 1.

Some consequences:

- $f$  increases most rapidly when  $\cos \theta = 1 (\theta = 0)$ , which implies that  $\nabla f$  and  $\vec{u}$  are in the same direction.
- This maximum rate of increase =  $||\nabla f||$
- $f$  decreases most rapidly when  $\cos \theta = -1 (\theta = \pi)$ , which implies that  $\nabla f$  and  $\vec{u}$  are in opposite directions.
- This maximum rate of decrease =  $-||\nabla f||$

Bit more interesting: What if I wanted the directional derivative to be zero?

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = 0$$

When the dot product equals 0.

## 5.10.2 Examples

**5.10.2.1 Find the maximum rate of increase/decrease of the function  $f(x, y) = x^2 + y^2$  at  $(1, 3)$**

```
1 f = x**2 + y**2
2 f_x = sp.diff(f, x)
3 f_y = sp.diff(f, y)
4 display(Latex(r"$\nabla f(x, y) = " + sp.latex(sp.Matrix([f_x, f_y])) + "$"))
5 display(Latex(r"$||\nabla f(x, y)|| = " + sp.latex(sp.sqrt(f_x**2 + f_y**2))
6         + "$"))
6 display(Latex(r"$||\nabla f(1, 3)|| = " + sp.latex(sp.sqrt(f_x.subs({x:1,
        y:3})**2 + f_y.subs({x:1, y:3})**2)) + "$"))
```

**5.10.2.2 The temperature in Celsius on the surface of a metal plate is  $T(x, y) = 20 - 4x^2 - y^2$  where  $x$  and  $y$  are measured in cm. In what direction from  $(2, -3)$  does the temperature**

a. Increase most rapidly?

```
1 T = 20 - 4*x**2 - y**2
2 T_x = sp.diff(T, x)
3 T_y = sp.diff(T, y)
4 # The gradient at (2, -3)
5 gradient = sp.Matrix([T_x.subs({x:2, y:-3}), T_y.subs({x:2, y:-3})])
6 magnitude = sp.sqrt(T_x.subs({x:2, y:-3})**2 + T_y.subs({x:2, y:-3})**2)
7 unit_vector = gradient/magnitude
8 display(unit_vector)
```

b. Decrease most rapidly?

```
1 display(-unit_vector)
```

c. Not change at all?

```
1 display(sp.Matrix([6, 16]), sp.Matrix([-6, -16]))
```

### 5.10.3 More Applications of Gradients

Given a surface  $z = f(x, y)$ ,  $[f(x, y) - z = 0]$ , the plane tangent to  $z$  at the point  $P_0(x_0, y_0, z_0)$  is given by:

The equation of a plane is given by:

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0 \text{ where } f_x, f_y, f_z \in \nabla(f(x, y) - z)$$

The equations of the normal line to the surface are given by:

- $x = x_0 + f_x(x_0, y_0, z_0)t$
- $y = y_0 + f_y(x_0, y_0, z_0)t$
- $z = z_0 + f_z(x_0, y_0, z_0)t$