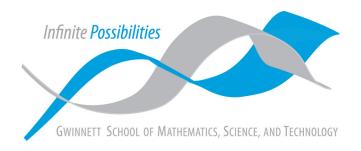
# Gwinnett School of Math, Science, and Technology

# **Multivariable Calculus Yearlong Notes**

Anish Goyal 1st Period Donny Thurston Educator

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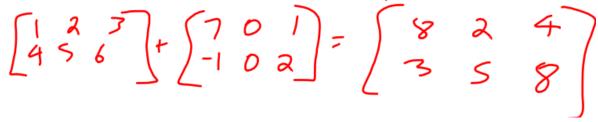
# 1 Systems of Linear Equations and Matrices

### 1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- ullet Two matrices are equal  $\iff$  they have the same dimensions and values

#### 1.1.1 Addition & Subtraction

Two matrices can be added/subtracted  $\iff$  they have the same dimensions.



#### 1.1.2 Scalar Multiplication

• Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

#### 1.1.3 Matrix Multiplication

- We can **only** multiply an (m x n) by (n x p) matrix.
- The resulting matrix will be (m x p)

## 1.1.4 Properties of Matrix Arithmetic

(a) 
$$A + B = B + A$$
 (Commutative law for addition)

(b) 
$$A + (B + C) = (A + B) + C$$
 (Associative law for addition)

(c) 
$$A(BC) = (AB)C$$
 (Associative law for multiplication)

(d) 
$$A(B+C)=AB+AC$$
 (Left distributive law)

(e) 
$$(B+C)A=BA+CA$$
 (Right distributive law)

$$(f) \ A(B-C) = AB - AC$$

(g) 
$$(\hat{B} - C)\hat{A} = BA - CA$$

(h) 
$$a(B+C) = aB + aC$$

(i) 
$$a(B-C) = aB - aC$$

(j) 
$$(a+b)C = aC + bC$$

(k) 
$$(a-b)C = aC - bC$$

(I) 
$$a(bC) = (ab)C$$

(m) 
$$a(BC) = (aB)C = B(aC)$$

#### 1.1.5 Examples

1.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix}$$

3.

$$\begin{bmatrix} 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2 \end{bmatrix}$$
$$= \begin{bmatrix} 30 \end{bmatrix}$$

## 1.2 Transpose of a Matrix

The transpose of an (m x n) matrix is the (n x m) matrix where the rows and columns are swapped.

If 
$$B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}$$
 ,  $B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$ 

$$\begin{split} B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix} \end{split}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix A is equal to the sum of all the elements on the main diagonal: tr(A)

#### 1.2.1 Transpose Matrix Properties

• 
$$(A^T)^T = A$$

$$\bullet \ (A-B)^T = A^T - B^T$$

• 
$$(kA)^T = kA^T$$

$$\bullet \ (AB)^T = B^T A^T$$

#### 1.3 Homework — "Matrix Stuff" (08/03/2023)

#### 1.3.1 Suppose that A, B, C, D and E are matrices with the following sizes:

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

| Undefined                          | Defined; (4 $	imes$ 2) | Defined; ( $5 \times 5$ ) | Defined; (5 $	imes$ 2)                               |
|------------------------------------|------------------------|---------------------------|--|
| $BA \\ AB + B \\ E^{T}A \\ AE + B$ | AC + D                 | E(A+B)                    | $ \begin{array}{c} (A^T + E)D \\ E(AC) \end{array} $ |

#### 1.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2. 
$$2A^{T} + C$$

$$2A^{T} + C = 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^{T} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$
$$= 2 \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

3.  $\mathbf{B^T} + \mathbf{5C^T}$ 

$$\begin{split} B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\ &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\ &= \text{Undefined} \end{split}$$

4.  $2E^T - 3D^T$ 

$$2E^{T} - 3D^{T} = 2 \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^{T} - 3 \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^{T}$$

$$= 2 \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$$

5.  $tr(\mathbf{DE})$ 

$$\begin{split} \operatorname{tr}(DE) &= \operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right) \\ &= \operatorname{tr}\left(\begin{bmatrix} 1 \cdot 6 + 5 \cdot (-1) + 2 \cdot 4 & 1 \cdot 1 + 5 \cdot 1 + 2 \cdot 1 & 1 \cdot 3 + 5 \cdot 2 + 2 \cdot 3 \\ (-1) \cdot 6 + 0 \cdot (-1) + 1 \cdot 4 & (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1 & (-1) \cdot 3 + 0 \cdot 2 + 1 \cdot 3 \\ 3 \cdot 6 + 2 \cdot (-1) + 4 \cdot 4 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 & 3 \cdot 3 + 2 \cdot 2 + 4 \cdot 3 \end{bmatrix}\right) \\ &= \operatorname{tr}\left(\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}\right) \\ &= 34 \end{split}$$