

Gwinnett School of Math, Science, and Technology

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# Multivariable Calculus Yearlong Notes

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# 1 Systems of Linear Equations and Matrices

## 1.1 Matrix Operations

- Matrix operations are given as: rows x columns
- Two matrices are equal  $\iff$  they have the same dimensions and values

### 1.1.1 Addition & Subtraction

Two matrices can be added/subtracted  $\iff$  they have the same dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 5 & 8 \end{bmatrix}$$

### 1.1.2 Scalar Multiplication

- Scalar multiplication is defined as multiplying each element of a matrix by a number

$$3 \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 15 & 6 \end{bmatrix}$$

### 1.1.3 Matrix Multiplication

- We can **only** multiply an  $(m \times n)$  by  $(n \times p)$  matrix.
- The resulting matrix will be  $(m \times p)$

### 1.1.4 Properties of Matrix Arithmetic

- (a)  $A + B = B + A$  (**Commutative law for addition**)
- (b)  $A + (B + C) = (A + B) + C$  (**Associative law for addition**)
- (c)  $A(BC) = (AB)C$  (**Associative law for multiplication**)
- (d)  $A(B + C) = AB + AC$  (**Left distributive law**)
- (e)  $(B + C)A = BA + CA$  (**Right distributive law**)
- (f)  $A(B - C) = AB - AC$
- (g)  $(B - C)A = BA - CA$
- (h)  $a(B+C) = aB + aC$
- (i)  $a(B-C) = aB - aC$
- (j)  $(a+b)C = aC + bC$
- (k)  $(a-b)C = aC - bC$
- (l)  $a(bC) = (ab)C$
- (m)  $a(BC) = (aB)C = B(aC)$

### 1.1.5 Examples

1.

$$\begin{aligned} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \end{aligned}$$

2.

$$\begin{aligned} & \begin{bmatrix} 2 & -3 \\ 5 & 0 \\ -2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 3 \\ 5 \cdot (-1) + 0 \cdot 3 \\ -2 \cdot (-1) + 4 \cdot 3 \\ 1 \cdot (-1) + 2 \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} -11 \\ -5 \\ 14 \\ 5 \end{bmatrix} \end{aligned}$$

3.

$$\begin{aligned} & [4 \ 5 \ -1] \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} \\ &= [4 \cdot 8 + 5 \cdot 0 + (-1) \cdot 2] \\ &= [30] \end{aligned}$$

## 1.2 Transpose of a Matrix

The transpose of an  $(m \times n)$  matrix is the  $(n \times m)$  matrix where the rows and columns are swapped.

$$\text{If } B = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, B^T = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\begin{aligned} B \cdot B^T &= \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 4 + 2 \cdot 2 & 4 \cdot (-1) + 2 \cdot 0 & 4 \cdot 3 + 2 \cdot 5 \\ (-1) \cdot 4 + 0 \cdot 2 & (-1) \cdot (-1) + 0 \cdot 0 & (-1) \cdot 3 + 0 \cdot 5 \\ 3 \cdot 4 + 5 \cdot 2 & 3 \cdot (-1) + 5 \cdot 0 & 3 \cdot 3 + 5 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -4 & 22 \\ -4 & 1 & -3 \\ 22 & -3 & 34 \end{bmatrix} \end{aligned}$$

- The transpose of a matrix is **always** multiplicative with the original.
- There is also a **main diagonal** that is the diagonal from the top left to the bottom right, but only square matrices have these.
- The **trace** of a square matrix  $A$  is equal to the sum of all the elements on the main diagonal:  $\text{tr}(A)$

### 1.2.1 Transpose Matrix Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$

### 1.3 Homework — “Matrix Stuff” (08/03/2023)

**1.3.1 Suppose that  $A, B, C, D$  and  $E$  are matrices with the following sizes:**

$A$	$B$	$C$	$D$	$E$
$(3 \times 2)$	$(2 \times 3)$	$(3 \times 3)$	$(3 \times 2)$	$(2 \times 3)$

For each matrix operation, sort them into undefined if the operation can't be done, or defined if it can along with the correct dimensions of the outcome.

Undefined	Defined; $(4 \times 2)$	Defined; $(5 \times 5)$	Defined; $(5 \times 2)$
$BA$	$AC + D$	$E(A + B)$	$(A^T + E)D$
$AB + B$			$E(AC)$
$E^T A$			
$AE + B$			

#### 1.3.2 Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

2.  $2A^T + C$

$$\begin{aligned} 2A^T + C &= 2 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= 2 \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \end{aligned}$$

3.  $\mathbf{B}^T + 5\mathbf{C}^T$

$$\begin{aligned} B^T + 5C^T &= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^T + 5 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \\ &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 20 & 5 \\ 10 & 25 \end{bmatrix} \\ &= \text{Undefined} \end{aligned}$$

4.  $2\mathbf{E}^T - 3\mathbf{D}^T$

$$\begin{aligned} 2E^T - 3D^T &= 2 \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - 3 \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T \\ &= 2 \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 9 & -5 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix} \end{aligned}$$

5.  $\text{tr}(\mathbf{DE})$

$$\begin{aligned} \text{tr}(DE) &= \text{tr} \left( \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right) \\ &= \text{tr} \left( \begin{bmatrix} 1 \cdot 6 + 5 \cdot (-1) + 2 \cdot 4 & 1 \cdot 1 + 5 \cdot 1 + 2 \cdot 1 & 1 \cdot 3 + 5 \cdot 2 + 2 \cdot 3 \\ (-1) \cdot 6 + 0 \cdot (-1) + 1 \cdot 4 & (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1 & (-1) \cdot 3 + 0 \cdot 2 + 1 \cdot 3 \\ 3 \cdot 6 + 2 \cdot (-1) + 4 \cdot 4 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 & 3 \cdot 3 + 2 \cdot 2 + 4 \cdot 3 \end{bmatrix} \right) \\ &= \text{tr} \left( \begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix} \right) \\ &= 34 \end{aligned}$$