Gwinnett School of Math, Science, and Technology

Multivariable Calculus Unit 4 Quiz Project

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1 Determine whether this set is a vector space. Show the work for all ten axioms from 4.1, including closure under addition and scalar multiplication.

The set of all triples of real numbers, (x, y, z) with the normal rules for vector addition, but scalar multiplication defined by:

$$\mathbf{k}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{k}^3\mathbf{x}, \mathbf{k}^3\mathbf{y}, \mathbf{k}^3\mathbf{z})$$

1.1 Closure under addition

Let $V = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ and (x_2, y_2, z_2) be arbitrary elements in V. Their sum under the given conditions is $(x + x_2, y + y_2, z + z_2)$. This operation satisfies the normal rules of vector addition for all x, y, z and $x_2, y_2, z_2 \in \mathbb{R}$. Therefore, V is closed under addition.

1.2 Closure under scalar multiplication

For the previously described set V, let $\mathbf{v} = (x, y, z) \mid x, y, z \in \mathbb{R}$ be an arbitrary element in V, and let k be a scalar. The operation $\mathbf{kv} = (k^3x, k^3y, k^3z)$ is still in V for any $k \in \mathbb{R}$. Therefore, V is closed under scalar multiplication.

1.3 A1. $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$

For any $u = (x_1, y_1, z_1), v = (x_2, y_2, z_2)$, and $w = (x_3, y_3, z_3)$ in V, the left-hand side of the equation is:

$$\begin{aligned} u + (v + w) &= (x_1, y_1, z_1) + ((x_2, y_2, z_2) + (x_3, y_3, z_3)) \\ &= (x_1, y_1, z_1) + (x_2 + x_3, y_2 + y_3, z_2 + z_3) \\ &= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3), z_1 + (z_2 + z_3)) \\ &= (x_1 + x_2 + x_3, y_1 + y_2 + y_3, z_1 + z_2 + z_3) \end{aligned}$$

The right-hand side of the equation is:

$$\begin{aligned} (u + v) + w &= ((x_1, y_1, z_1) + (x_2, y_2, z_2)) + (x_3, y_3, z_3) \\ &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) + (x_3, y_3, z_3) \\ &= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3, (z_1 + z_2) + z_3) \\ &= (x_1 + x_2 + x_3, y_1 + y_2 + y_3, z_1 + z_2 + z_3) \end{aligned}$$

Since the left-hand side of the equation is equal to the right-hand side of the equation, $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.

1.4 A2. $u + v = v + u$ for all $u, v \in V$

For any $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ in V, the left-hand side of the equation is:

$$\begin{aligned} u + v &= (x_1, y_1, z_1) + (x_2, y_2, z_2) \\ &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \end{aligned}$$

The right-hand side of the equation is:

$$\begin{aligned} v + u &= (x_2, y_2, z_2) + (x_1, y_1, z_1) \\ &= (x_2 + x_1, y_2 + y_1, z_2 + z_1) \end{aligned}$$

Since the left-hand side of the equation is equal to the right-hand side of the equation, $u + v = v + u$ for all $u, v \in V$.

1.5 A3. There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$

The additive identity of V is $(0, 0, 0)$. For any $v = (x, y, z) \in V$, we have:

$$v + (0, 0, 0) = (x + 0, y + 0, z + 0) = (x, y, z) = v$$

1.6 A4. For every $v \in V$, there exists an element $w \in V$ such that $v + w = 0$

For any $v = (x, y, z) \in V$, its additive inverse is $(-x, -y, -z)$ because:

$$v + (-x, -y, -z) = (x + (-x), y + (-y), z + (-z)) = (0, 0, 0) = 0$$

1.7 S1. $(xy)\mathbf{v} = x(y\mathbf{v})$ for all $x, y \in \mathbb{R}$, $\mathbf{v} \in V$

For any $x, y \in \mathbb{R}$ and $v = (x_1, y_1, z_1)$ in V, the left-hand side of the equation is:

$$(xy)\mathbf{v} = (xy)^3(x_1, y_1, z_1) = (x^3y^3x_1, x^3y^3y_1, x^3y^3z_1)$$

The right-hand side of the equation is:

$$x(y\mathbf{v}) = x(y^3x_1, y^3y_1, y^3z_1) = (x^3y^3x_1, x^3y^3y_1, x^3y^3z_1)$$

Since both sides are equal, S1 is satisfied.

1.8 S2. $(x+y)\mathbf{v} = x\mathbf{v} + y\mathbf{v}$ for all $x, y \in \mathbb{R}$, $\mathbf{v} \in V$

For any $x, y \in \mathbb{R}$ and $v = (x_1, y_1, z_1) \in V$, the left-hand side of the equation is:

$$\begin{aligned}(x+y)\mathbf{v} &= (x+y)^3(x_1, y_1, z_1) = (x^3 + 3x^2y + 3xy^2 + y^3)(x_1, y_1, z_1) \\ &= (x^3x_1 + 3x^2yx_1 + 3xy^2x_1 + y^3x_1, x^3y_1 + 3x^2yy_1 + 3xy^2y_1 + y^3y_1, x^3z_1 + 3x^2yz_1 + 3xy^2z_1 + y^3z_1)\end{aligned}$$

The right-hand side of the equation is:

$$\begin{aligned}x\mathbf{v} + y\mathbf{v} &= x^3(x_1, y_1, z_1) + y^3(x_1, y_1, z_1) \\ &= (x^3x_1, x^3y_1, x^3z_1) + (y^3x_1, y^3y_1, y^3z_1) \\ &= (x^3x_1 + y^3x_1, x^3y_1 + y^3y_1, x^3z_1 + y^3z_1)\end{aligned}$$

Both sides are **not** equal, so S2 is **not** satisfied. Therefore, V is **not** a vector space.

1.9 S3. $x(\mathbf{v} + \mathbf{w}) = x\mathbf{v} + x\mathbf{w}$ for all $x \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in V$

For any $x \in \mathbb{R}$ and $v = (x_1, y_1, z_1), w = (x_2, y_2, z_2) \in V$, the left-hand side of the equation is:

$$\begin{aligned}x(\mathbf{v} + \mathbf{w}) &= x((x_1, y_1, z_1) + (x_2, y_2, z_2)) \\ &= x(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= x(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (xx_1 + xx_2, xy_1 + xy_2, xz_1 + xz_2)\end{aligned}$$

The right-hand side of the equation is:

$$\begin{aligned}x\mathbf{v} + x\mathbf{w} &= x(x_1, y_1, z_1) + x(x_2, y_2, z_2) \\ &= (xx_1, xy_1, xz_1) + (xx_2, xy_2, xz_2) \\ &= (xx_1 + xx_2, xy_1 + xy_2, xz_1 + xz_2)\end{aligned}$$

Since both sides are equal, S3 is satisfied.

1.10 S4. $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$

For any $v = (x, y, z) \in V$, the left-hand side of the equation is:

$$1\mathbf{v} = 1^3(x, y, z) = (x, y, z) = v$$

Since scalar multiplication by 1 does not change the vector, S4 is satisfied.

2 Let V be the set of all ordered triples of real numbers, and define vector addition of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ as

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 7, u_2 + v_2, u_3 + v_3 - 16)$$

2.1 What is the zero vector, 0 of V?

To find the zero vector of V, we need to find an ordered triple (x, y, z) that, when added to any vector $(u_1, u_2, u_3) \in V$, the result is still (u_1, u_2, u_3). In other words, we need to find x, y, z such that the following equations hold for any $u_1, u_2, u_3 \in \mathbb{R}$ by substituting (x, y, z) into (v_1, v_2, v_3), respectively:

$$\begin{aligned} u_1 + x + 7 &= u_1 \\ u_2 + y &= u_2 \\ u_3 + z - 16 &= u_3 \end{aligned}$$

Although we could use an augmented matrix to solve this system of equations, it is trivial to see that $x = -7, y = 0$, and $z = 16$ is the solution to the system. Therefore, the zero vector of V is $(-7, 0, 16)$

2.2 What is the additive inverse of u in V?

The additive inverse of a vector $\mathbf{u} = (u_1, u_2, u_3) \in V$ is a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = 0$. Because we know that the zero vector of V is $(-7, 0, 16)$, we can rearrange for $-\mathbf{u}$ as follows:

$$\begin{aligned} \mathbf{u} + (-\mathbf{u}) &= (-7, 0, 16) \\ (-\mathbf{u}) &= (-7, 0, 16) - \mathbf{u} \\ (-\mathbf{u}) &= (-7 - u_1, 0 - u_2, 16 - u_3) \\ (-\mathbf{u}) &= (-u_1 - 7, -u_2, -u_3 + 16) \end{aligned}$$

Therefore, the additive inverse of $\mathbf{u} = (u_1, u_2, u_3)$ in V is $(-u_1 - 7, -u_2, -u_3 + 16)$.

3 Let Q be the vector space of polynomials in the form: $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$. Let P be a subset of Q, and be polynomials of the form: $b_0 + b_1x^2 + b_2x^4$. Show that P is a subspace of Q and explain your reasoning.

3.1 The zero vector of Q is in P:

The zero vector of Q is the polynomial $0 + 0x + 0x^2 + 0x^3 + 0x^4$, which can be written as $0 + 0x^2 + 0x^4$. This polynomial is in the form required for P, so the zero vector of Q is in P.

3.2 P is closed under addition:

Let $p_1(x) = b_0^{(1)} + b_1^{(1)}x^2 + b_2^{(1)}x^4$ and $p_2(x) = b_0^{(2)} + b_1^{(2)}x^2 + b_2^{(2)}x^4$ be two polynomials in P. Their sum is:

$$p_1(x) + p_2(x) = (b_0^{(1)} + b_0^{(2)}) + (b_1^{(1)} + b_1^{(2)})x^2 + (b_2^{(1)} + b_2^{(2)})x^4$$

This is also in the form required for P, so P is closed under addition.

3.3 P is closed under scalar multiplication:

Let c be a scalar, and let $p(x) = b_0 + b_1x^2 + b_2x^4$ be a polynomial in P. When we multiply $p(x)$ by c, we get:

$$cp(x) = cb_0 + cb_1x^2 + cb_2x^4$$

This is also in the form required for P, so P is closed under scalar multiplication.

Since P satisfies all three properties, it is a subspace of Q.

4 Let $v_1 = (2, 1, 0, 3)$, $v_2 = (3, -1, 5, 2)$, and $v_3 = (-1, 0, 2, 1)$.

4.1 Is the vector $(0, 0, 0, 0)$ in $\text{span}\{v_1, v_2, v_3\}$?

The zero vector is always in the span of any vector combination by letting the combination coefficients all be 0.

4.2 Is the vector $(1, 1, 1, 1)$ in $\text{span}\{v_1, v_2, v_3\}$?

There must exist scalars a, b, c such that:

$$a\begin{bmatrix}2\\1\\0\\3\end{bmatrix} + b\begin{bmatrix}3\\-1\\5\\2\end{bmatrix} + c\begin{bmatrix}-1\\0\\2\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

From this, we can create the following system of equations:

$$\begin{cases} 2a + 3b - c = 1 \\ \quad a - b = 1 \\ \quad 5b + 2c = 1 \\ 3a + 2b + c = 1 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow[R_4-3R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 5 & 2 & 1 \\ 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{-\frac{2}{5}R_2} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 5 & 2 & 1 \\ 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{1}{2} \end{array}\right] \xrightarrow[R_4+\frac{5}{2}R_2]{R_3-5R_2} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 5 & 1 \end{array}\right] \Rightarrow \begin{cases} a + \frac{3}{2}b - \frac{1}{2}c = \frac{1}{2} \\ \quad b - \frac{1}{5}c = -\frac{1}{5} \\ \quad 5c = 1 \end{cases}$$

Since this is a consistent system, the vector $(1, 1, 1, 1)$ is in $\text{span}\{v_1, v_2, v_3\}$.

5 Is the following set of 2×2 matrices linearly independent? Explain why or why not.

We need to check if the only solution to the equation $aA + bB + cC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is $a = b = c = 0$. Let's set up the equation:

$$a\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} -2 & 5 \\ 6 & 8 \end{bmatrix} + c\begin{bmatrix} 5 & 4 \\ 7 & 7 \end{bmatrix}$$

This leads to the following system:

$$\begin{aligned} a - 2b + 5c &= 0 \quad \text{(for the first row, first column)} \\ 2a + 5b + 4c &= 0 \quad \text{(for the first row, second column)} \\ a + 6b + 7c &= 0 \quad \text{(for the second row, first column)} \\ 2a + 8b + 7c &= 0 \quad \text{(for the second row, second column)} \end{aligned}$$

Now, we can write the system in matrix form. For the coefficient matrix A, the system only has a non-trivial solution $\iff \det A \neq 0$. Let's find the determinant of A:

$$\det(A) = \begin{vmatrix} 1 & -2 & 5 \\ 2 & 5 & 4 \\ 1 & 6 & 7 \end{vmatrix}$$

Expanding the determinant along the first row:

$$\det(A) = 1 \times \begin{vmatrix} 5 & 4 \\ 6 & 7 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} + 5 \times \begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix}$$

Evalulating the determinants inside:

$$\begin{aligned} \det(A) &= (1 \times (5 \times 7 - 4 \times 6)) - (-2 \times (2 \times 7 - 4 \times 1)) + (5 \times (2 \times 6 - 5 \times 1)) \\ &= (1 \times 3) - (-2 \times 10) + (5 \times 7) \\ &= 11 + 20 + 35 \\ &= 66 \end{aligned}$$

Therefore, the set is **linearly independent**.

5.1 If you can, write one of them as a linear combination of the other two. Show all work, and if you can't, explain why.

Let's set up the equations $aB + bC = A$:

$$a\begin{bmatrix} -2 & 5 \\ 6 & 8 \end{bmatrix} + b\begin{bmatrix} 5 & 4 \\ 7 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

This leads to the following system:

$$
\begin{aligned}
-2a + 5b &= 1 \quad \text{(for the first row, first column)} \\
6a + 4b &= 2 \quad \text{(for the first row, second column)} \\
5a + 7b &= 1 \quad \text{(for the second row, first column)} \\
7a + 7b &= 2 \quad \text{(for the second row, second column)}
\end{aligned}
$$

We can subtract the first equation from the second equation to get $7a+2b = 0$. However, this equation contradicts the second row of the given matrix A. Therefore, there are **no constants** a, b, and c such that $aB + bC = A$.

6 Justify *why* each of the following is true or false:

6.1 A finite set with at least two vectors and contains 0 can be linearly independent.

False: If the set contains the zero vector, it is linearly dependent by default.

6.2 If two sets span the same subspace of a vector space V, then those sets must be the same set.

False: Two sets can span the same subspace without being the same set. Consider the vector space $V = \mathbb{R}^2$. The sets $\{(1,0),(0,1)\}$ and $\{(1,0),(1,1)\}$ both span $\mathbb{R}^2$, but the sets are different.

6.3 The polynomials $x - 1, (x - 1)^2$, and $(x - 1)^3$ span the set of all polynomials of degree 3.

True: They form a basis for the set of all polynomials of degree 3. Any polynomial of degree 3 can be expressed as $a(x - 1)^3 + b(x - 1)^2 + c(x - 1)$, where a, b, and c are constants.

6.4 A set with exactly two vectors is linearly independent $\iff$ the vectors are not scalar multiples of each other

True: A set with two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ is linearly independent if and only if one vector is not a scalar multiple of the other. The only solution to $c_1\mathbf{v_1} + c_2\mathbf{v_2} = \mathbf{0}$ is $c_1 = c_2 = 0$. If the vectors are scalar multiples, then one can be expressed as $k\mathbf{v_1} = \mathbf{v_2}$ where k is a scalar, making the set is linearly dependent.

6.5 If the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent, then $\{kv_1, kv_2, kv_3\}$ is also independent for every nonzero scalar k.

True: If the original set $\{v_1, v_2, v_3\}$ is linearly independent, scaling the vectors by a nonzero scalar k won't change their linear independence because the only solution to $c_1(kv_1) + c_2(kv_2) + c_3(kv_3) = \mathbf{0}$ is $c_1 = c_2 = c_3 = 0$.

7 Prove the following:

Let the vectors u, v, and w be in the vector space V. The vectors $u - v$, $v - w$, and $w - u$ form a linearly dependent set.

We need to show that there exist constants a, b, and c, not all equal to zero, such that:

$$a(u - v) + b(v - w) + c(w - u) = 0$$

We can start by expanding the left side of the equation and simplifying it:

$$(au - av) + (bv + bw) + (cw - cu) = 0$$

Distribute the constants and regroup the terms:

$$au - av + bv - bw + cw - cu = 0$$
$$(au - cu) + (-av + bv) + (-bw) = 0$$

Next, factor out common terms:

$$u(a - c) + v(-a + b) + w(-b) = 0$$

Since we want to show that this equation holds for constants a, b, and c, we need to find a solution for a, b, and c that ensures they're not all equal to zero:

$$\begin{aligned} a - c &= 0 \quad (1) \\ -a + b &= 0 \quad (2) \\ -b &= 0 \quad (3) \end{aligned}$$

...However, according to equation (3), we can see that b MUST equal zero. Substituting this into equation (2), we get $-a = 0$, which implies $a = 0$. Finally, using the value of a in equation (1), we find that $-c = 0$, which implies $c = 0$.

Since we've shown that the only solution to the system of equations is $a = b = c = 0$, we can conclude that the vectors $u - v$, $v - w$, and $w - u$ are linearly independent.