Lorenz system and Chaos

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We consider the Lorenz system of equations to show the phenomenon of Chaos and Butterfly effect and look into the characteristics of chaos as a strange attractor. Analysis of bifurcations and change in behaviour of fixed points as we change the parameter r is done.

INTRODUCTION

Chaos was first observed by Lorenz when was simulating weather predictions on a computer. He observed that a slight change in the initial condition leads to a catastrophic change in weather prediction patterns. This phenomena lead to a significant change in our understanding of how prediction models work. The behaviour of the model over a larger time span is highly sensitive to initial conditions. Chaos is a deterministic non-linear phenomena where small changes in initial state may lead to large differences in longer run. This phenomena can be observed in time travel where a change in the past event will lead to insurmountable paradoxes in the future. For example if Hitler would have been killed earlier, there might not have been a World war.[1]

LORENZ EQUATIONS FOR CHAOS

Before Edward Lorenz, transient chaos was discovered (one that would come and go in some time) but in Lorenz's system for the first time we got chaos that remains self-sustained i.e. the chaos goes on forever.

$$\frac{dx}{dt} = \sigma(y - x) \tag{1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{3}$$

$\sigma = PrandtlNumber$
$\beta = AspectRatio$
$\rho = RayleighNumber$
σ, β, ρ should always be positive.

Lorenz always kept σ and *beta* as constants where $\sigma = 10$ and $\beta = 8/3$ thus reducing the number of parameters to 1. i.e. ρ. Lorenz equations represent a simplified model for weather prediction. It is these non-linear terms -xz and xy in the Lorenz equations which are responsible for a chaotic behaviour beyond a certain value of parameter ρ.

Properties of Lorenz Systems

1. Symmetry and Z-axis invariance: The equations are invariant to the transformation of (x, y, z) to (-x, -y, z). Here we simply have z'$=-\beta$. Thus the z-axis is always a part of the stable manifold for the equilibrium at the origin.

2. System is dissipative: Assuming an arbitrary ellipsoidal closed surface, we can show that the rate of change in volume for the given system is $-(\sigma + 1 + \beta)$ V. Hence, volumes in phase space contracts under the flow.

ANALYSIS OF LORENZ EQUATIONS

Fixed Points

$$\tfrac{dx}{dt} = \sigma(y - x) = 0 = \tfrac{dy}{dt} = x(\rho - z) - y = \tfrac{dz}{dt} = xy - \beta z$$

The origin is a fixed point for this system which exists for all values of ρ. $(x^*, y^*, z^*) = (0, 0, 0)$. The system also has two symmetric pair of fixed points which exist for $\rho > 1$
$C^+ = (x^*, y^*, z^*) = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r - 1)$

$C^- = (x^*, y^*, z^*) = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r - 1)$

1. **Parameter $\rho < 1$**
   There is only one fixed point which is at the origin. It behaves as a stable fixed point.

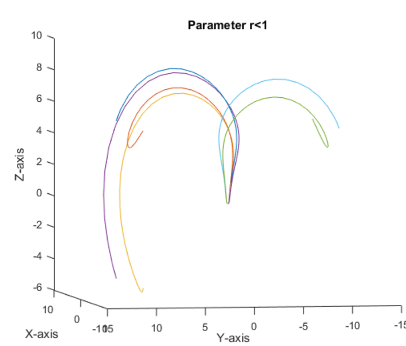

FIG. 1: Trajectories of different initial conditions for ρ=0.75 end up at the origin since

Theoretical Analysis
The following linear stability analysis at the origin shows that it is a stable fixed point. We linearize the system of equations and compute the Jacobian matrix at the origin. This decouples the motion along z-axis which will behave as invariant stable manifold. The Jacobian(A) obtained is as follows:

A= $\begin{bmatrix} \sigma & \sigma \\ \rho & \text{-1} \end{bmatrix}$

Trace= $\tau = -\sigma - 1 < 0$ since $\sigma = 10$
Determinant= $\Delta = \sigma(1-\rho) > 0$ for $0 < \rho < 1$
D=$\tau^2 - 4\Delta = (\sigma-1)^2 + 4\sigma\rho > 0$ since $\sigma, \rho > 0$

These three conditions suggest that fixed point would be a stable node. The tr(A) and det(A) show that both the eigen values are negative while the last one from the above statement shows that the eigen values will be real and hence no spiral.
Also, not only is the origin locally stable but we can conclude it behaves as a global stable fixed point since it is the only fixed point for $\rho < 1$.

Observation: From figure 1, we observe that a particle placed at different initial locations will eventually reach the origin. Thus, it shows that the origin is globally stable.

2. **Parameter $\rho = 1$**
All the three fixed points merge at the origin and ρ closer to 1^+ 3 fixed points emerge out. The two new fixed points emerge out stable and the origin becomes a saddle node hence at $\rho = 1$ **we see supercritical pitchfork bifurcation**

3. **Parameter $\rho > 1$**
The linearization of the Lorenz equations is given by the matrix

A(x,y,z)= $\begin{bmatrix} \sigma & \sigma & 0 \\ \rho - z & \text{-1} & \text{-x} \\ \text{y} & \text{x} & \text{-b} \end{bmatrix}$

Changing the parameter from $\rho = 1$ to ρ close to 1^+, the number of fixed points increase from one to three fixed points.
Out of the three, the origin becomes a saddle fixed point while the other two become stable nodes. The origin becomes a saddle point since
Determinant= $\Delta = \sigma(1-\rho) < 0$ for $\rho > 1$

$(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ are the 2 new symmetric fixed points which are referred to as $C^{\pm}$ respectively.
Based on the symmetric property of the Lorenz equation, we can consider either of these points. Now, their characteristic equation is
$f_{c\pm}(\lambda) = \lambda^3 + (1+b+\sigma)\lambda^2 + \beta(\sigma+\rho)\lambda + 2\beta\sigma(\rho-1)$

The behaviour of the symmetric fixed points change as we vary the parameter ρ's value since it changes the values of the roots-eigen values (λ) of the characteristic equation. The eigen values in turn determines the nature of these fixed points.

- Parameter $1 < \rho < 1.346$
Determinant of the characteristic equation is greater than 0. Hence we get 3 real eigen values all of which are negative. Hence here, $C^{\pm}$ are stable nodes. The nature of the fixed points at $\rho = 1.2$ can be seen in figure 2.

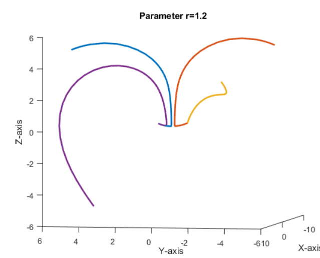

FIG. 2: Trajectories of different initial conditions for ρ=1.2 converge to either of the 2 symmetric stable fixed points

- Parameter $1.346 < \rho < 13.926$
The nature of the two symmetric fixed points C^+ and C^- changes from stable node to a stable spiral at $\rho = 1.346$. The eigen values change from negative real values to complex values as we move from $\rho < 1.346$ to $\rho > 1.346$. The imaginary part introduces oscillatory behaviour, while the negative real part is responsible for the damping and hence stable spiral structure. At $\rho = 1.346$, the oscillations are at a very low frequency and damping is significant, hence, it immediately converges to origin hence the spiral behaviour is not well defined as shown in figure 3 for $rho = 4$. To observe a clear spiral structure, we increased the parameter ρ to 10 as shown in figure 4. As we increase the parameter value ρ, the frequency of the oscillations is much more significant as compared to the damping.

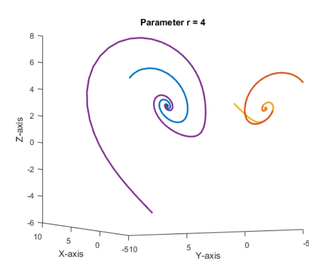

FIG. 3: Trajectories of different initial conditions for ρ=4 spirally converge to either of the 2 symmetric stable spiral fixed points

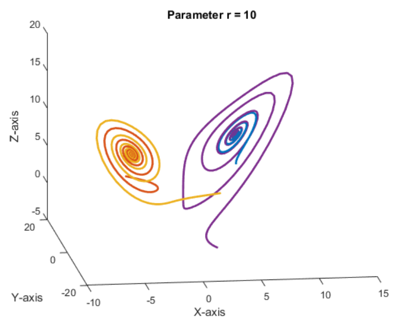

FIG. 4: Trajectories of different initial conditions for ρ=10 spirally converge to either of the 2 symmetric stable spiral fixed points

- Parameter $\rho = 13.926$ As we decrease ρ from ρ_H , the unstable limit cycles increase such that it passes very close to the saddle point at the origin. At approx r=13.92655741, homoclinic bifurcation takes place when the these limit cycles meet the saddle point to form homoclinic orbits. This is also known as "homoclinic explosion". This is the point at which a pair of unstable limit cycles are formed for the 2 fixed points $C^{\pm}$. The figure 5 and 6 suggests homoclinic bifurcation where for the same initial point and ρ changed by 10^{-9} the trajectory completely changes. Quoting Sparrow's discussion "an amazingly complicated invariant is born at $\rho = 13.926$, along with unstable limit cycles."

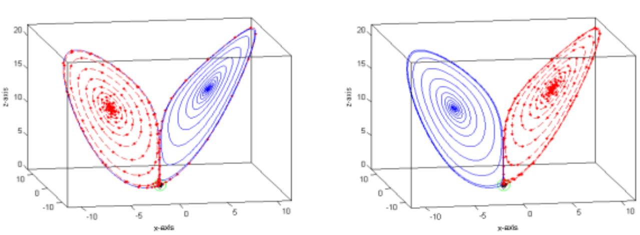

FIG. 5: Red curves with dots: x0=(1.e-16,1.e-16,1.e-16), Blue solid curve: x0=(1.e-16,-1.e-16,1.e-16).for case 1:ρ=13.92655741 case 2:ρ=13.92655741

- Parameter $\rho = 16$ Here, we get eigen values which have imaginary values. Thus here also we g stable spiral structures are formed which converge to the two symmetric fixed points. The convergence to these fixed point is determined based on the initial conditions. For initial conditions within a certain critical region from the given fixed point, the particle remains attracted to that fixed point. Interestingly if the initial condition is outside the critical region of the fixed point, the particle leaves the oscillations around that fixed point and gets attracted to the further fixed point. The size of these critical regions become smaller as we increase the value of parameter ρ. Hence as ρ's value increases for more points the trajectory leaves the fixed point near it and settles to the one further away.This can be seen in figure 5.

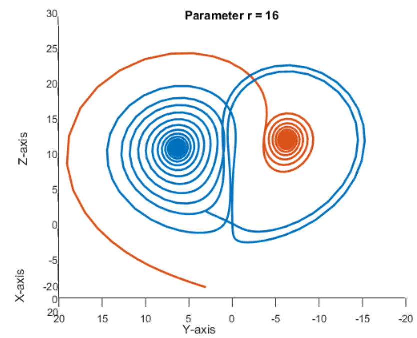

FIG. 6: Trajectories of different initial conditions(I.C.s) for ρ=16 spirally converge to the stable spiral fixed point further from the I.C.

- Parameter $\rho = 24.74$ (=ρ_H)
  Below this value of ρ $C^{\pm}$ were surrounded by unstable limit cycles. At this parameter value, the fixed points lose stability by absorbing an unstable limit cycle in a subcritical Andronov-Hopf Bifurcation. This is an important point after which chaos begin i.e. for ρ greater than 24.74 .
  From after homoclinic explosion till Hopf Bifurcation, the trajectories used to show transient chaos behaviour but used to settle down in either $C^{\pm}$ fixed points.But as ρ's value increases the trajectory spends more and more time in the phase space until ρ_H where this time becomes infinite i.e. chaos happens.

Figure 8 (taken from Strogartz book) explains the behaviour of fixed points with variation in parameter.

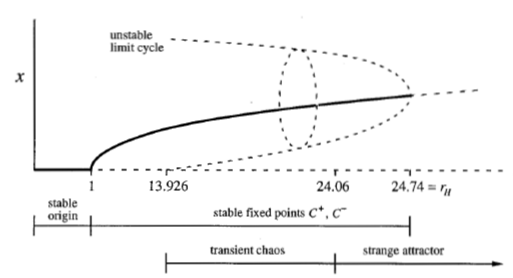

FIG. 7: Parameter Space

CHAOS

Chaos is defined by Strogartz as "aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions".

The whole motion of the trajectory becomes random after chaos. It is unpredictable in terms of how many times the trajectory would circuit around one of the atrractors before shifting to the other one. (Attractor would be explained shortly.) Parameter $\rho = 28$ is a standard value of ρ which is widely used for experimenting and observing chaos. The figure 7 shows a chaos diagram.

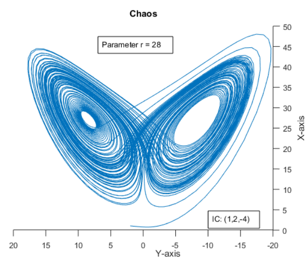

FIG. 8: Chaos

1. Aperiodic and Long term behaviour
   Its a characteristic property of chaos where the trajectory travels forever in the phase space without intersecting with any previous path hence aperiodic and a long term behaviour.Lorenz system is a self-sustaining chaos.

2. Chaos remains in finite space The loops and spirals in chaos although were infinitely dense and never intersecting, yet they stayed inside a finite space.[2] This is because of the Volume Contracting property of the Lorenz system as mentioned previously.

3. Butterfly effect: Sensitive dependence on Initial Conditions
   Chaos shows exponential increase in the distance between nearby trajectories. This is essentially the butterfly effect. Butterfly effect means any minute localised change in a complex system can have catastrophic effects somewhere else. The chaos figure itself above looks like a butterfly.
   $\delta(t)$= Distance between two trajectories.
   $\delta(t) = \delta_0 e^{\lambda t}$ where λ is the Liapunov exponent. This property is illustrated in the figures below.

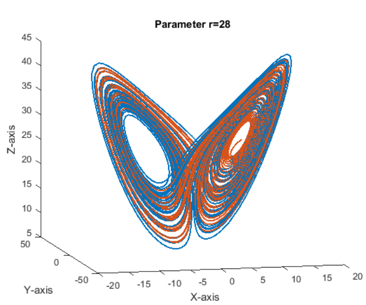

FIG. 9: Chaos using 2 initial conditions very close to each other where only y-component differs by 10^{-4}

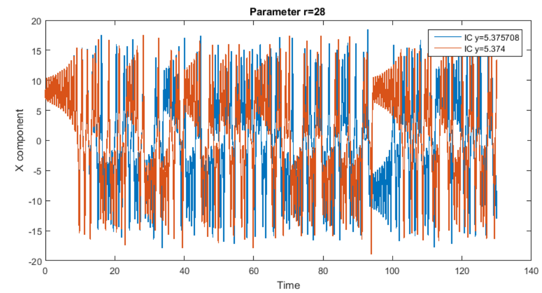

FIG. 10: Chaos using 2 initial conditions very close to each other where only y-component differs by 10^{-4}

Due to this property Edward Lorenz realized why weather prediction was not possible after a certain timespan -Chaos was the reason.A certain error in the measured initial condition from the actual one would lead to exponential increase in the error with time such that at some point the error exceeds the tolerance level and the prediction can no longer be considered. This limited time(t) is called predictability horizon can be measured by,
$t = 1/\lambda \ln(a/\delta_0)$
Where λ is Liapunov exponent, δ_0 is initial measurement error.To increase the timespan by 10 times the increase in accuracy of measurement has to be 10^{10} times. Hence prediction becomes so difficult.

Observation

Changing the parameter ρ to greater than 24.76, analytically the real part of two complex eigen values corresponding to both the symmetric fixed points changes from negative to positive. Hence, the trajectories around the fixed points changes from a stable spiral to an unstable spiral. Interestingly for these values of ρ, the imaginary part of the eigen values i.e. the frequency of the oscillations is always much greater than the real part which is responsible for growing oscillations. Hence beyond a critical distance from the given fixed point, it gets attracted towards another fixed point due to higher frequency. Due to symmetrical behaviour about the fixed points such a mathematical formulation cause formation of trajectory having the shape of the wings of a butterfly.

STRANGE ATTRACTOR

Attractor has the following 3 properties:
(1) Invariant set. (For eg: start in A stay in A forever)
(2) Attracts an open set of Initial Conditions.
(3) Set A should be minimal i.e. No proper subset of A satisfies the above 2 conditions. Our 2 fixed points in the chaotic space fulfills the above 3 conditions. Since any initial condition or point will get attracted to the neighbourhood of the fixed points (2) condition is fulfilled.Lorenz attractor is a strange attractor since it

shows sensitive dependence on initial conditions. Thus a dynamic system with a chaotic attractor is locally unstable yet globally stable: once some sequences have entered the attractor, nearby points diverge from one another but never depart from the attractor.

[1] Strogatz, Steven H. Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. Westview press, 2014.

[2] Gleick, James. Chaos: Making a New Science (Enhanced Edition). Open Road Media, 2011.