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Some Preleminary Concepts.

normal vector: → any vector of length 1. (unit vector).

Consider the vector $\vec{v} = [2,4;1,2]$

we have $|\vec{v}| = \sqrt{2^2+4^2+1^2+2^2} = \sqrt{4+16+1+4} = \sqrt{25} = 5$

Thus $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \left[\frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5}\right]$

check

$|\vec{u}| = \sqrt{\left(\frac{2}{5}\right)^2+\left(\frac{4}{5}\right)^2+\left(\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2} = \sqrt{\frac{25}{25}} = 1$

orthonormal vector.

Vectors of unit length that are orthogonal to each other

$\vec{u} = [2/5, 1/5, -2/5, 4/5]$

$\vec{v} = [3/\sqrt{65}, -6/\sqrt{65}, 4/\sqrt{65}, 2/\sqrt{65}]$

orthonormal

as $|\vec{u}| = \sqrt{(2/5)^2+(1/5)^2+(2/5)^2+(4/5)^2} = 1$

$|\vec{v}| = \sqrt{\frac{9}{65}+\frac{36}{65}+\frac{16}{65}+\frac{4}{65}} = 1$

and $\vec{u}\cdot\vec{v} = \dfrac{6}{5\sqrt{65}} - \dfrac{6}{5\sqrt{65}} - \dfrac{8}{5\sqrt{65}} + \dfrac{8}{5\sqrt{65}} = 0$

SVD] 3 mutually compatible points of view.

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a) Method to transform correlated variables to uncorrelated ones.
b) Identifying and ordering dimensions along which data points exhibit most variation.
c) Find best approx of original data points using fewer dimensions.
   hence SVD → can be seen as a method of data reduction.

Fundamental concepts:

Take a high dimensional, highly variable set of data points and reducing it to a lower dimensional space that exposes the substructure of the original data more clearly, and orders it from most variation to the least.

The SVD Decomposition:

$$A_{mn} = \underbrace{U_{mm}}_{\text{ortho}} \underbrace{S_{mn}}_{\text{diag}} \underbrace{V_{nn}^T}_{\text{ortho}}$$

where $U^T U = I$
$V^T V = I$

→ The columns of U are orthonormal eigen vectors of AA^T.

→ The columns of V are orthonormal eigen vectors of $A^T A$.

→ S is the diag matrix containing square root of eigen values from U or V in descending order.

$$\underset{\substack{[2\times3]\\ m\times n}}{A} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \qquad \underset{[3\times2]}{A^T} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\underset{[3\times2][2\times3]}{A^TA} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

$$\underset{[m\times n][n\times m]}{AA^T} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \underset{\substack{m\times m\\ [2\times2]}}{\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}}$$

$$\therefore \quad AA^T = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

Find Eig. vec. of AA^T

$$AX = \lambda X \qquad \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$

$$11x_1 + x_2 = \lambda x_1$$
$$x_1 + 11x_2 = \lambda x_2. \Rightarrow (11-\lambda)x_1 + x_2 = 0$$
$$x_1 + (11-\lambda)x_2 = 0$$

Char eqn:

$$\begin{vmatrix} (11-\lambda) & 1 \\ 1 & (11-\lambda) \end{vmatrix} = 0$$

$\Rightarrow$

$$(\lambda-10)(\lambda-12) = 0$$

For $\lambda = 10$ ..

$$\lambda = 10, \ \lambda = 12.$$

$$(11-10)x_1 + x_2 = 0 \qquad x_1 = -x_2$$

Choose $x_1 = 1$, $x_2 = -1$.

$$V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Sim For $\lambda = 12$ $\quad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now by descending order of Eigen value. $\lambda = 12$
we get the following matrix of col vectors. $\lambda_{12} > \lambda_{10}$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

[Next step] Convert to ortho Normal Form using Gram Schmidt Process.

Consider the ortho normal form $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$.

Similarly we have $A^TA = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$

we get the Eigen values as. $\lambda = 0$, $\lambda = 10$, $\lambda = 12$

$$\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Setting in descending order we have the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & -5 \end{bmatrix}.$$

ortho normal form $\rightarrow$ $V = \begin{bmatrix} 1/\sqrt{6} & \frac{2}{\sqrt{5}} & 1/\sqrt{30} \\ \frac{2}{\sqrt{6}} & -1/\sqrt{5} & \frac{2}{\sqrt{30}} \\ 1/\sqrt{6} & 0 & -\frac{5}{\sqrt{30}} \end{bmatrix}$.

Thus we have the following matrices at hand.

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}_{2\times2}$$

and we have

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{bmatrix}_{3\times3}$$

$S_{2\times3}$ as. $\begin{bmatrix} S_{11} & 0 & 0 \\ 0 & S_{12} & 0 \end{bmatrix}$

$S_{11} = \sqrt{12}$, $S_{22} = \sqrt{10}$

(only non zero Eigen values)

Thus Finally we have.

$$A_{mn} = U_{mm} S_{mn} V^T_{nn} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \end{bmatrix}$$

Why SVD?

-> Basically we want to solve $Ax = b$
↳ Possible with direct inverse if A is $n \times n$.
and A^{-1} exists.

-> If A is under constrained -> we want entire set of solution

-> if A is over constrained. we ask for least sq. solution.

-> We can find a Matrix B of lower rank than A which approximates A nicely.

Gram Schmidt - Orthonormalization Process.

Process to convert a set of vectors to a set of orthonormal vectors.

eg. Consider A to be having 3 col vectors.

$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

Step 1.

normalize the first col. vector.

$$\vec{u_1} = \frac{\vec{v_1}}{|\vec{v_1}|}$$

Step 2 Rewrite each of remaining vectors iteratively in terms of themselves minus a multiplication of already norm. vectors

Thus **Step 2.1**

Consider. $\vec{w_2} = \vec{v_2} - \vec{u_1} \cdot \vec{v_2} * \vec{u_1}$.

For 2nd col

Step 2.2

Now normalize $\vec{w_2}$

$$\vec{u_2} = \frac{\vec{w_2}}{|\vec{w_2}|}.$$

Now do it for Col 3. **Step 2.3**

Consider $\vec{w_3} = \vec{v_3} - \vec{u_1} \cdot \vec{v_3} * \vec{u_1} - \vec{u_2} \cdot \vec{v_3} * \vec{u_2}$

Step 2.4

Now normalize $\vec{w_3}$

$$\vec{u_3} = \frac{\vec{w_3}}{|\vec{w_3}|}$$