6

CHAPTER

Two Dimensional Geometry

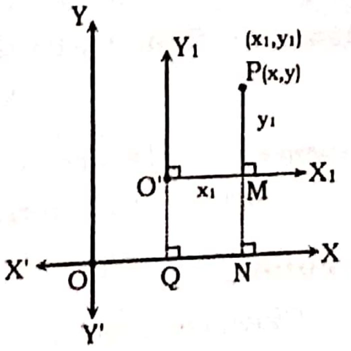
Introduction

It sometimes happens that the choice of axes at the beginning of the solution of a problem does not lead to the simplest form of the equation. By a proper transformation of axes an equation may be simplified. This may be accomplished in two steps, one called translation of axes, the other rotation of axes.

6.1 TRANSFORMATION OF COORDINATES

6.1.1 Translation of Axes

Let OX and OY be the original axes and let $O'X_1$ and $O'Y_1$ be the new axes, parallel respectively to the old ones. Also, let $O'(h, k)$ referred to the origin of new axes. Let $P(x, y)$ be any point in the plane XOY, and let its coordinates referred to the new axes be $P(x_1, y_1)$.



To determine x and y in terms of x_1 and y_1, draw $O'Q \perp OX$ and $PN \perp OX$ which cuts $O'X_1$ at M. Then

$$OQ = h,\ O'Q = k = MN,$$

$$O'M = x_1 = QN,\ PM = y_1$$

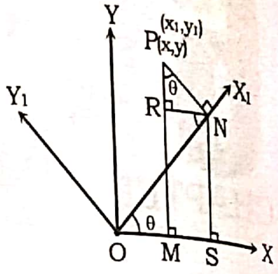
Therefore,

$$x = ON = OQ + QN = h + x_1$$

$$y = PN = PM + MN = k + y_1$$

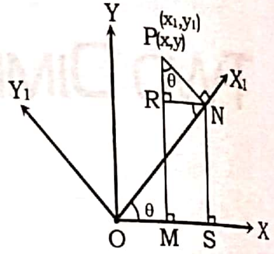
These equations represent the equation of translation and hence, the coefficients of the first degree terms are vanished.

6.1.2 Rotation of Axes



Let OX and OY be the original axes and OX_1 and OY_1 the new axes. Since O is the origin for each set of axes. Let the angle XOX_1 through which the axes have been rotated and represented by θ, therefore $\angle XOY = \theta$.

Let P(x, y) be any point in the plane, and referred to the new axes be $P(x_1, y_1)$. To determine x and y in terms of x_1, y_1 and θ, draw $PM \perp OX$, $PN \perp OX_1$ and $NR \perp PM$, $NS \perp OX$ then



$$\angle NOX = \angle RNO = \angle RPN = \theta$$

From ΔPRN,

$$\cos\theta = \frac{PR}{PN} \Rightarrow PR = PN\cos\theta$$

and $$\sin\theta = \frac{RN}{PN} \Rightarrow RN = PN\sin\theta$$

And, ΔONS gives

$$\cos\theta = \frac{OS}{ON} \Rightarrow OS = ON\cos\theta$$

and $$\sin\theta = \frac{NS}{ON} \Rightarrow NS = ON\sin\theta$$

Now,

$$OM = OS - MS = OS - RN = ON\cos\theta - PN\sin\theta$$

i.e. $x = x_1\cos\theta - y_1\sin\theta$

Again, $$PM = PR + RM = PN\cos\theta + NS = PN\cos\theta + ON\sin\theta$$

i.e. $y = y_1\cos\theta + x_1\sin\theta$

Hence, the formulae for the rotation of the axes through an angle θ are

$$x = x_1\cos\theta - y_1\sin\theta; \quad y = y_1\cos\theta + x_1\sin\theta$$

Example 1: Determine the equation of the curve $2x^2 + 3y^2 - 8x + 6y = 7$ when the origin is translated to the point (2, −1).

Solution:

Given equation is

$$2x^2 + 3y^2 - 8x + 6y = 7 \qquad \text{... (i)}$$

As given we have to translate the origin to the point (2, −1) with the coordinate axes remaining parallel. Therefore, substituting

$$x = x_1 + 2,\ y = y_1 - 1$$

Then the curve equation (i) becomes

$$2(x_1+2)^2 + 3(y_1-1)^2 - 8(x_1+2) + 6(y_1-1) = 7$$

$$\Rightarrow \quad 2(x_1)^2 + 8x_1 + 8 + 3(y_1)^2 - 6y_1 + 3 - 8x_1 - 16 + 6y_1 - 6 = 7$$

$$\Rightarrow \quad 2(x_1)^2 + 3(y_1)^2 = 18$$

Therefore, the locus is

$$2(x)^2 + 3(y)^2 = 18$$

This is the standard equation of the ellipse with its center at the new origin and its major axis on the x–axis, with semi axes $a = 3$, $b = \sqrt{6}$.

Example 2: Determine a translation of axes that will transform the equation $3x^2 - 4y^2 + 6x + 24y = 135$ into one in which the coefficient of the first degree terms are zero.

Solution:

Given equation is

$$3x^2 - 4y^2 + 6x + 24y = 135 \qquad \text{... (i)}$$

As given the equation will transform to new form that has the coefficients of the first degree terms are zero and is possible only when we translate the axes. So, substitute for x and y the values $x_1 + h$ and $y_1 + k$ respectively in (i) then

$$3(x_1+h)^2 - 4(y_1+k)^2 + 6(x_1+h) + 24(y_1+k) = 135$$

$$\Rightarrow \quad 3(x_1)^2 - 4(y_1)^2 + 6x_1(h+1)^2 + y_1(-8k+24) + (3h^2 - 4k^2 + 6h + 24k - 135) = 0 \qquad \text{... (ii)}$$

Being the equation (ii) has first degree terms are zero, we get

$$h = -1 \text{ and } k = 3$$

Then (ii) becomes,

$$3(x_1)^2 - 4(y_1)^2 = 102$$

Its locus is

$$3x^2 - 4y^2 = 102$$

This is the standard form for the hyperbola with its center at the origin.

Example 3: Find the equation of the curve $x^2 + y^2 + 2gx + 2fy + c = 0$, if the origin translated into O′(−g, −f) and the new axes parallel to the old ones.

Solution:

Given equation is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (i)}$$

As given the origin translated into O′(−g, −f) then the variables shifted to $x = x_1 - g$, $y = y_1 - f$. Therefore, (i) reduces to

$$(x_1 - g)^2 + (y_1 - f)^2 + 2g(x_1 - g) + 2f(y_1 - f) + c = 0$$

$$\Rightarrow (x_1)^2 + (y_1)^2 + x_1(-2g + 2g) + y_1(-2f + 2f) + (g^2 + f^2 - 2g^2 - 2f^2 + c) = 0$$

$$\Rightarrow (x_1)^2 + (y_1)^2 = g^2 + f^2 - c$$

The locus of the equation is

$$x^2 + y^2 = g^2 + f^2 - c$$

Example 4: If the axes rotate by angle $\frac{\pi}{4}$, find the next equation in the new coordinates, $5x^2 + 2xy + 5y^2 = 2$.

Solution:

Given equation is

$$5x^2 + 2xy + 5y^2 = 2 \qquad \dots \text{(i)}$$

As given the axes rotate by angle $\frac{\pi}{4}$, therefore

$$x = x_1 \cos\frac{\pi}{4} - y_1 \sin\frac{\pi}{4}; \quad y = y_1 \cos\frac{\pi}{4} + x_1 \sin\frac{\pi}{4}$$

i.e. $$x = x_1 \frac{1}{\sqrt{2}} - y_1 \frac{1}{\sqrt{2}} = \frac{x_1 - y_1}{\sqrt{2}}; \quad y = y_1 \frac{1}{\sqrt{2}} + x_1 \frac{1}{\sqrt{2}} = \frac{x_1 + y_1}{\sqrt{2}}$$

Then, (i) reduces to

$$5\left(\frac{x_1 - y_1}{\sqrt{2}}\right)^2 + 2\left(\frac{x_1 - y_1}{\sqrt{2}}\right)\left(\frac{x_1 + y_1}{\sqrt{2}}\right) + 5\left(\frac{x_1 + y_1}{\sqrt{2}}\right)^2 = 2$$

$$\Rightarrow 5(x_1^2 + y_1^2 - 2x_1y_1) + 2(x_1^2 - y_1^2) + 5(x_1^2 + y_1^2 + 2x_1y_1) = 4$$

$$\Rightarrow 12x_1^2 + 8y_1^2 = 4$$

$$\Rightarrow 3x_1^2 + 2y_1^2 = 1$$

The locus of the equation is

$$3x^2 + 2y^2 = 1$$

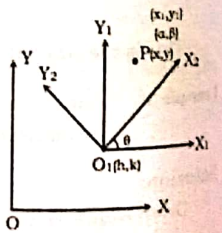
This is the standard form for the ellipse with its center at the origin, and its major axis on the y–axis.

6.1.3 Rotation and Translation of Axes

Let the origin O(0, 0) translate to the point $O_1(h, k)$ without changing direction.

If P(x, y) is any point on the plane and its co-ordinate in new axes be $P(\alpha, \beta)$ then

$$x = \alpha + h, \; y = \beta + k \qquad \dots\dots\dots (1)$$



Rotate the axes about O_1 through an angle θ and coordinates of P be (x_1, y_1) then

$$\alpha = x_1 \cos\theta - y_1 \sin\theta$$

$$\beta = y_1 \cos\theta + x_1 \sin\theta$$

Therefore, (1) reduces to

$$x = x_1 \cos\theta - y_1 \sin\theta + h$$

$$y = y_1 \cos\theta + x_1 \sin\theta + k$$

Example 5: What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ become when referred to rectangular axes through the point (–2, –3), the new axes being inclined at an angle $\frac{\pi}{4}$, with the old?

Solution:

Given equation is

$$2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0 \qquad \dots \text{(i)}$$

When the axes turn by an angle θ without change in origin, then we replace,

$$x = x\cos\theta - y\sin\theta \text{ and } \quad y = x\sin\theta + y\cos\theta.$$

Again when axes are transferred from origin to point (h, k) we replace,

$$x = x + h \quad \text{and} \quad y = y + k.$$

Here, after changing origin and turning the axes,

$$x = (x + h)\cos\theta - (y + k)\sin\theta$$

$$y = (x + h)\sin\theta + (y + k)\cos\theta$$

The equation (i) is transforms the axes parallel with (h, k) = (–2, –3) and turned with an angle $\theta = \frac{\pi}{4}$. Therefore

$$x = (x - 2)\cos\frac{\pi}{4} - (y - 3)\sin\frac{\pi}{4} = \frac{x - 2 - y + 3}{\sqrt{2}} = \frac{x - y + 1}{\sqrt{2}}$$

$$y = (x - 2)\sin\frac{\pi}{4} + (y - 3)\cos\frac{\pi}{4} = \frac{x - 2 + y - 3}{\sqrt{2}} = \frac{x + y - 1}{\sqrt{2}}$$

Then equation (i) becomes,

$$2\left(\frac{x - y + 1}{\sqrt{2}}\right)^2 + 4\left(\frac{x - y + 1}{\sqrt{2}}\right)\left(\frac{x + y - 5}{\sqrt{2}}\right) - 5\left(\frac{x + y - 5}{\sqrt{2}}\right)^2 + 20\left(\frac{x - y + 1}{\sqrt{2}}\right)$$
$$- 22\left(\frac{x + y - 5}{\sqrt{2}}\right) - 14 = 0$$

$$\Rightarrow (x - y + 1)^2 + 2(x - y + 1)(x + y - 5) - \frac{5}{2}(x + y - 5)^2 + 10\sqrt{2}(x - y + 1)$$
$$- 11\sqrt{2}(x + y - 5) - 14 = 0$$

$$\Rightarrow \quad 2(x-y+1)^2 + 4(x-y+1)(x+y-5) - 5(x+y-5)^2 + 20\sqrt{2}(x-y+1) - 22\sqrt{2}(x+y-5) - 28 = 0$$

$$\Rightarrow \quad 2(x^2+y^2+1-2xy+2x-2y) + 4(x^2+xy-5x-xy-y^2+5y+x+y-5) - 5(x^2+y^2+25+xy-5y-5x) + 20\sqrt{2}x - 20\sqrt{2}y + 20\sqrt{2} - 22\sqrt{2}x - 22\sqrt{2}y + 110\sqrt{2} - 28 = 0$$

$$\Rightarrow \quad x^2 - 7y^2 - 9xy + (3-2\sqrt{2})x + (45-42\sqrt{2})y - 171 + 130\sqrt{2} = 0.$$

This is the required equation.

Exercise 6.1

1. Transform the equation $x^2 - y^2 + 2x + 4y = 0$ by transferring the origin to $(-1, 2)$ the coordinate axes remaining parallel.
2. Transform the equation $x^2 - 3y^2 + 4x + 6y = 0$ by transferring the origin to the point $(-2, 1)$ the co–ordinate axes remaining parallel.
3. Translate the axes so as to change the equation $3x^2 - 2xy + 4y^2 + 8x - 10y + 8 = 0$ into an equation with linear terms missing.
4. Transform the equation $x^2 + 2cxy + y^2 = a^2$, by turning the rectangular axes through the angle $\frac{\pi}{4}$.
5. By transforming to parallel axes through a properly chosen point (h, k) prove that the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$ can be reduced to one containing only terms of second degree.
6. Transform the equation $x^2 + 3xy + y^2 = 0$ through the point (2, 3) and turning though 45°.

Answers

1. $x^2 - y^2 + 3 = 0$
2. $x^2 - 3y^2 - 1 = 0$
3. $3x^2 - 2xy + 4y^2 - 1 = 0$
4. $x^2(1 + c) + y^2(1 - c) = a^2$
5. $12x^2 - 10xy + 2y^2 = 0$
6. $(x_1 - y_1)^2 + (x_1 + y_1)^2 + 3(x_1^2 - y_1^2) + 13\sqrt{2}(x_1 - y_1) + 12\sqrt{2}(x_1 + y_1) + 62 = 0$

6.2 Review of Conic Section

Introduction

Conic sections were discovered during the classical Greek period around 600 to 300 B.C. By that time, enough was known of conics for Apollonius (262–190 B.C.). However, it was not until the early seventeenth century that the broad applicability of conics became apparent. These days, conic sections are important to model many physical processes in nature. For example, trajectories of heavenly bodies are conics (circle, ellipse, parabola, and hyperbola).

6.2.1 Conic Section

We begin with the following basic definitions of conic sections.

Conic section: A conic section is a locus of a point that moves so that the ratio of its distance from a fixed point to its perpendicular distance from a fixed line is constant.

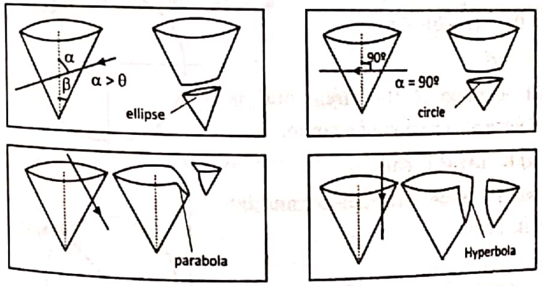
Focus and directrix: The fixed point of a conic is called the focus and the fixed line is called the line of directrix.

Eccentricity: The fixed ratio is the eccentricity of the conic and is denoted by e.

Line of symmetry or axis of symmetry: The line that passes through the focus and is perpendicular to the line of directrix, is called the axis of symmetry. Generally, the axis of symmetry gives the mirror figure of the conic.

6.2.2 Classification of Conic Section

A conic section can be classified as:



(a) If the plane cut a cone perpendicularly to the axis of the cone then it gives a circle.

(b) If the plane cut a cone at an angle greater than the semi–vertical angle then it gives an ellipse.

(c) If the plane cut a cone with the plane is parallel to the generator and if it does not pass through the vertex then it gives a parabola.

(d) If the plane cut a cone at an angle less than the semi-vertical angle then it gives an hyperbola.

6.2.3 Eccentricity of Conic Section

The constant ratio between the distance from a fixed point and distance from a fixed straight line of a conic section is called eccentricity of the conic section. It is denoted by e.

Classification of Conic Section as their Eccentricity

As, the value of eccentricity, the conic section is classified as,

If $e = 1$, then the conic section is a parabola.

If $e < 1$, then the conic section is an ellipse.

If $e > 1$ then the conic section is a hyperbola.

If $e = 0$ then the conic is called a circle.

6.3 REVIEW OF CIRCLE

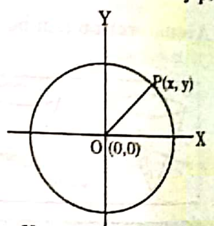
Introduction

A circle is a closed curve traced out by a moving point moving in a plane in such a way that its distance from a fixed point is always constant. The fixed point is called the center of the circle and the constant distance is called its radius.

6.3.1 Equation of a Circle

Centre at the Origin (standard form)

Let O(0, 0) be the centre and r be the radius of a circle and let P(x, y) be any point on it. Then



$$OP = r$$

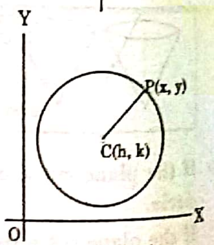
i.e. $\sqrt{(x-0)^2 + (y-0)^2} = r$

$\Rightarrow \quad x^2 + y^2 = r^2$

which is the required equation of the circle and is known as the standard form of equation of a circle.

Centre at any Point (h, k) (central form)

Let C(h, k) be the center of a circle with radius r and let P(x, y) be any point on it. Then



$$CP = r$$

$\Rightarrow \quad \sqrt{(x-h)^2 + (y-k)^2} = r$

$\Rightarrow \quad (x-h)^2 + (y-k)^2 = r^2$

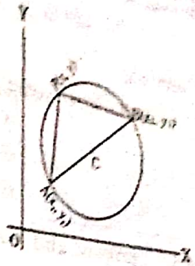
which is the required equation of the circle.

Circle with a given Diameter (diameter form)

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the ends of a diameter of a circle and let P(x, y) be any point on the circle.

Since an angle in a semi-circle is a right angle, so $\angle APB = 90°$.

Now,



Slope of AP $= \dfrac{y - y_1}{x - x_1}$

and slope of BP $= \dfrac{y - y_2}{x - x_2}$

Since AP is perpendicular to BP, so we must have

slope of AP × slope of BP = – 1

i.e. $\dfrac{y - y_1}{x - x_1} \times \dfrac{y - y_2}{x - x_2} = -1$

$\Rightarrow \quad (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

which is the required equation of the circle.

Example 6: Find the equation of the circle with center at (– 4, 5) and radius 6.

Solution:

The equation of the circle with center at (–4, 5) and radius 6 is

$$(x + 4)^2 + (y - 5)^2 = 6^2$$

$\Rightarrow \quad x^2 + 8x + 16 + y^2 - 10y + 25 = 36$

$\Rightarrow \quad x^2 + y^2 + 8x - 10y + 5 = 0$

Example 7: Find the center and radius of the circle $2x^2 + 2y^2 - 12x + 4y = 1$.

Solution:

The equation of the circle is

$$2x^2 + 2y^2 - 12x + 4y = 1$$

$\Rightarrow \quad x^2 + y^2 - 6x + 2y - \dfrac{1}{2} = 0$

Comparing this equation with the general equation of a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

We have,

$$2g = -6,\ 2f = 2,\ c = -\frac{1}{2}$$

i.e. $\quad g = -3,\ f = 1$

Hence, the center of the circle is at (– g, – f) i.e. (3, – 1) and the radius is $= \sqrt{g^2 + f^2 - c} = \sqrt{(-3)^2 + 1^2 - (-1/2)} = \sqrt{9 + 1 + 1/2} = \sqrt{21/2}$.

Example 8: Determine the equation of the circle if the ends of a diameter be at (1, 2) and (3, 5).

Solution:

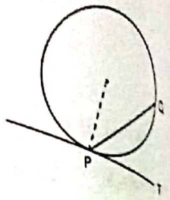
The required equation of the circle is

$$(x-1)(x-3)+(y-2)(y-5)=0$$

$$\Rightarrow \quad x^2-4x+3+y^2-7y+10=0$$

$$\Rightarrow \quad x^2+y^2-4x-7y+13=0$$

6.3.2 Tangent and Normal to a Circle

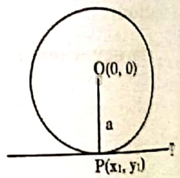


Let P and Q be any two neighboring points on the circle. Now, keeping P fixed, as the point Q moves along the curve towards the point P, the chord through P and Q turns about the point P and ultimately it goes through the point P when Q coincides with P. This limiting position PT of the chord PQ is called the tangent to the circle at the point P.

Thus, tangent to a circle is a line which meets the circle at a unique point. This unique point is called the point of contact. A line drawn perpendicular to the tangent at the point of contact is called the normal to the circle at that point. Normal to a circle at any point on it always passes through the center of the circle.

6.3.3 Equation of the Tangent and Normal to the Circle at a Point (x_1, y_1)

Equation of the Tangent to the Circle $x^2 + y^2 = a^2$ at a Point (x_1, y_1) on the Circle



Let the equation of circle with radius a and center at O(0, 0) is

$x^2 + y^2 = a^2$

Let $P(x_1, y_1)$ be a point on the circle. Then the slope of the line OP is $\frac{y_1}{x_1}$.

Being (x_1, y_1) lies on the circle $x^2 + y^2 = a^2$, so

$x_1^2 + y_1^2 = a^2$

By geometry, OP is perpendicular to the tangent PT. Then OP is normal to PT.

Therefore, the slope of the tangent PT is $m = \frac{-x_1}{y_1}$

Now, the equation of the tangent line PT is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

$$\Rightarrow \quad yy_1-y_1^2=-xx_1+x_1^2$$

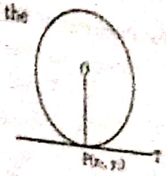
$$\Rightarrow \quad xx_1+yy_1=x_1^2+y_1^2$$

$$\Rightarrow \quad xx_1+yy_1=a^2$$

Equation of the Normal to the Circle $x^2 + y^2 = a^2$ at a Point (x_1, y_1)

The equation of the tangent to the circle at the point $P(x_1, y_1)$ is

$$xx_1+yy_1=a^2$$



Clearly, the tangent has slope is $-\frac{x_1}{y_1}$. Then the slope of the normal line to the tangent line at P is $\frac{y_1}{x_1}$.

Then the equation of the normal at P is

$$y-y_1=\frac{y_1}{x_1}(x-x_1)$$

$$\Rightarrow \quad yx_1-x_1y_1=xy_1-x_1y_1$$

$$\Rightarrow \quad xy_1=yx_1$$

6.3.4 Condition of Tangency to a Circle (condition of tangency of the line $y = mx + c$ to the circle $x^2 + y^2 = a^2$)

The given equation of line and circle are

$$y=mx+c \qquad \text{... (1)}$$

$$x^2+y^2=a^2 \qquad \text{... (2)}$$

Eliminating y from (1) and (2), we get

$$(mx+c)^2=a^2-x^2$$

$$\Rightarrow \quad m^2x^2+2mcx+c^2=a^2-x^2$$

$$\Rightarrow \quad (1+m^2)x^2+(2mc)x+(c^2-a^2)=0 \qquad \text{... (3)}$$

which is quadratic in x.

Since the line (1) is tangent to the curve (2) then the discriminate value of (3) is zero. Then,

$$(2mc)^2-4(1+m^2)(c^2-a^2)=0 \qquad [B^2-4AC=0]$$

$$\Rightarrow \quad m^2c^2-c^2+a^2-m^2c^2+m^2a^2=0$$

$$\Rightarrow \quad -c^2+a^2+m^2a^2=0$$

$$\Rightarrow \quad c^2=a^2(1+m^2)$$

This is the condition that the line (1) is tangent to the curve (2).

For point of contact,

Since line (1) is tangent to the parabola (2), so the discriminate value of (3) is zero i.e. $B^2 - 4AC = 0$. Therefore, (3) gives

$$x=\frac{-B}{2A}$$

$$\text{i.e.} \quad x=\frac{-2mc}{2(1+m^2)}=\frac{-a^2mc}{c^2}=\frac{-a^2m}{c}$$

Then (i) gives

$$y = mx + c = m\left(\frac{-a^2m}{c}\right) = \frac{-a^2m^2}{c}$$

So, the point of contact of (i) and (ii) is $\left(\frac{-a^2m}{c}, \frac{-a^2m^2}{c}\right)$.

The Condition of Tangency of the Line $\ell x + my = n$ to the circle $x^2 + y^2 = a^2$

The equation of the line is

$$\ell x + my = n$$

$$\Rightarrow \quad my = -\ell x + n$$

$$\Rightarrow \quad y = -\frac{\ell}{m}x + \frac{n}{m}$$

The condition of tangency of the line $y = mx + c$ to the circle $x^2 + y^2 = a^2$ is

$$c^2 = a^2(1 + m^2)$$

$$\Rightarrow \quad \left(\frac{n}{m}\right)^2 = a^2\left[1 + \left(-\frac{\ell}{m}\right)^2\right]$$

$$\Rightarrow \quad n^2 = a^2(\ell^2 + m^2)$$

This is the condition of tangency.

Example 9: Find the equation of tangent to the circle $x^2 + y^2 - 2x - 10y + 1 = 0$ at the point (–3, 2).

Solution:

Given equation of circle is

$$x^2 + y^2 - 2x - 10y + 1 = 0$$

$$\Rightarrow \quad (x - 1)^2 + (y - 5)^2 - 1 - 25 + 1 = 0$$

$$\Rightarrow \quad (x - 1)^2 + (y - 5)^2 = 25$$

Clearly, the center of the circle is at (1, 5) and has radius 5. Then the tangent to the given circle at the point (–3, 2) is

$$xx_1 + yy_1 = a^2$$

i.e. $\quad (x - 1)(-3) + (y - 5)(2) = 25$

$$\Rightarrow \quad -3x + 2y = 38$$

Example 10: For what value of c will the line $y = 2x + c$ be a tangent to the circle $x^2 + y^2 = 5$?

Solution:

The condition of tangency of the line $y = mx + c$ to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$. So, the line $y = 2x + c$ will be tangent to $x^2 + y^2 = 5$ when

$$c^2 = 5(1 + 4)$$

$$\Rightarrow \quad c^2 = 25$$

$$\Rightarrow \quad c = \pm 5$$

Exercise 6.2

1. Find the center and radius of the following circles.
   (a) $x^2 + (y - 1)^2 = 2$ (b) $x^2 + y^2 - 2ax - 2ay = 0$
   (c) $x^2 + y^2 - 6x + 4y - 36 = 0$ (d) $4(x^2 + y^2) + 4ax - 6ay - 3a^2 = 0$
2. Find the equations of circles with the following data.
   (a) center at (0, 0) and diameter 10
   (b) center at ($a \cos \alpha$, $a \sin \alpha$) and radius a.
   (c) center at (2, – 1) and through (3, 6).
3. Find the equation to the circle whose diameter is the straight line joining the points as the end of diameter.
   (a) (0, 0) and (– 8, 4) (b) (0, –1) and (2, 3).
4. Find the equations of tangent and normal to the following circles at the point mentioned against each.
   (a) $x^2 + y^2 = 169$ at the point (5, 12). (b) $x^2 + y^2 = 2$ at the point (1, 1).
5. (a) If the line $x = a$ touches the circle $x^2 + y^2 = 25$, what are the possible values of a?
   (b) For what value of k will the line $4x + 3y + k = 0$ touch the circle $x^2 + y^2 = 4$?
6. Find the condition for the line $y = mx + c$ to be a normal to the circle $x^2 + y^2 = a^2$.

Answers

1. (a) $(0, 1), \sqrt{2}$ (b) $(a, a), \sqrt{2}a$ (c) $(3, -4), 7$ (d) $\left(-\frac{a}{2}, \frac{3a}{4}\right), \frac{5a}{4}$
2. (a) $x^2 + y^2 = 25$ (b) $x^2 + y^2 - 2xa \cos \square - 2ya \sin \square = 0$
   (c) $x^2 + y^2 - 4x + 2y - 45 = 0$
3. (a) $x^2 + y^2 + 8x - 4y = 0$ (b) $x^2 + y^2 - 2x - 2y - 3 = 0$
4. (a) $5x + 12y = 169$, $12x - 5y = 0$ (b) $x + y = 2$, $x - y = 0$
5. (a) ± 5 (b) ± 10
6. $c = 0$

6.4 REVIEW OF PARABOLA

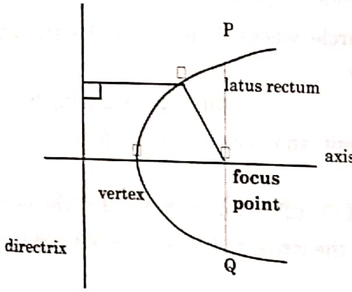
Introduction

A parabola is a conic that has eccentricity 1. That is, a parabola is a plane curve defined a locus of points in which the ratio of distance from a fixed point with distance from a fixed line is 1. In short,

$$\frac{\text{Distance from a fixed point}}{\text{Distance from a fixed line}} = 1$$

6.4.1 Parabola

A parabola is a locus of point in a plane which is equidistant from a given fixed point (or focus) and a given fixed straight line (or directrix).



In the definition of parabola,

(a) Focus ↔ The fixed point of parabola, is called focus.

(b) Directrix ↔ The fixed line of parabola, is called directrix.

(c) Axis ↔ The straight line passing through focus and perpendicular to directrix, is called axis.

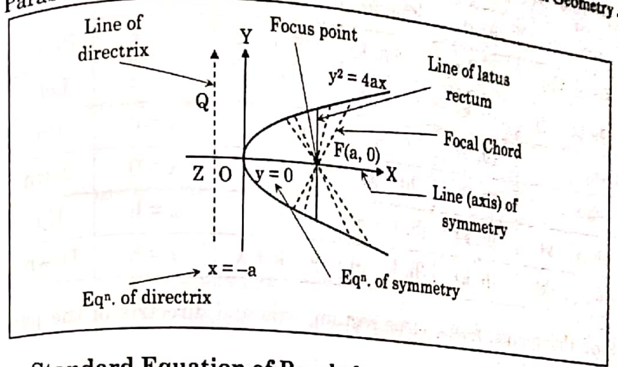
(d) Vertex ↔ The meeting point of axis and parabola, is called vertex.

(e) Chord ↔ An interior straight line of a conic that touches two points of the conic, is called chord.

(f) Focal Chord ↔ The chord of the parabola which is passing through the focus point.

(e) Latus rectum ↔ Among the focal chords, one special chord which is perpendicular to axis, is called latus rectum. The distance between the meeting points of latus rectum to the parabola, is called length of latus rectum. In the figure above, PQ is called length of latus rectum.

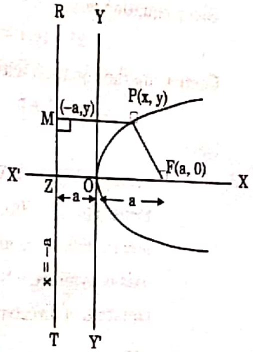
Parabola: $y^2 = 4ax$.



6.4.2 Standard Equation of Parabola

A. When the Vertex is at Origin and the axis of the Parabola is x-axis.

Let F(a, 0) be the focus point and RT be the directrix of a parabola. Let O (taken as origin) be the vertex of the parabola. Then the equation of the directrix RT is $x = -a$. Let P(x, y) be any point on parabola and PM is perpendicular to RT. Then we join PF.



By definition of parabola,

the point P is in equidistance from the fixed point F and the fixed line RT i.e. say at M.

Then the coordinate of M is $(-a, y)$.

Therefore,

$$PF = PM$$

$$\Rightarrow (PF)^2 = (PM)^2$$

$$\Rightarrow (x-a)^2 + (y-0)^2 = (x+a)^2 + (y-y)^2$$

$$\Rightarrow x^2 - 2xa + a^2 + y^2 = x^2 + 2xa + a^2$$

$$\Rightarrow y^2 = 4ax$$

This is equation of parabola in standard form.

B. When the Vertex at any Point V(h, k)

The equation of parabola having vertex at a point V(h, k) is

$$(y-k)^2 = 4a(x-h).$$

or $$(x-h)^2 = 4a(y-k)$$

Some Results of Parabola when $a > 0$

Equation of parabola	Vertex	Focus	Equation of Directrix	Equation of Axis	Opens
$y^2 = 4ax$	(0, 0)	(a, 0)	$x = -a$	$y = 0$	Right

	(0, 0)	(–a, 0)	x = a	y = 0	Left
$y^2 = -4ax$	(0, 0)	(–a, 0)	x = a	y = 0	Left
$(y-k)^2 = 4a(x-h)$	(h, k)	(h + a, k)	x = h – a	y = k	Right
$(y-k)^2 = -4a(x-h)$	(h, k)	(h – a, k)	x = h + a	y = k	Left
$x^2 = 4ay$	(0, 0)	(0, a)	y = –a	x = 0	Up
$x^2 = -4ay$	(0, 0)	(0, –a)	y = a	x = 0	Down
$(x-h)^2 = 4a(y-k)$	(h, k)	(h, k + a)	y = k – a	x = h	Up
$(x-h)^2 = -4a(y-k)$	(h, k)	(h, k – a)	y = k + a	x = h	Down

Example 11: Find the vertex, focus, latus rectum, axis and directrix of the parabola $x^2 - 4y = 0$.

Solution:

Given equation is,

$$x^2 - 4y\ 4 = 0 \Rightarrow x^2 = 4y$$

Comparing the equation with $x^2 = 4ay$, we get

$$a = 1 > 0$$

Now,

vertex is V(h, k) = V(0, 0)

focus, F(0, a) = F(0, 1)

length of latus rectum = 4a = 4

axis of symmetry be, x = 0 i.e. y–axis

equation of directrix is, y = –a i.e. y = –1.

Example 12: Find the vertex, focus, directrix, axis and latus rectum of the parabola $y^2 = 4x + 4y$.

Solution:

The given equation is

$$y^2 = 4x + 4y$$
$$\Rightarrow y^2 - 4y = 4x$$
$$\Rightarrow y^2 - 4y + 4 - 4 = 4x$$
$$\Rightarrow (y-2)^2 = 4x + 4$$
$$\Rightarrow (y-2)^2 = 4(x+1) \quad \dots (i)$$

Now, comparing this equation with the equation of parabola $(y-k)^2 = 4a(x-h)$ we get,

$$a = 1, h = -1, k = 2$$

Now, the parabola has

vertex (h, k) = (–1, 2)

focus (h + a, k) = (–1 + 1, 2) = (0, 2)

equation of directrix, x = h – a ⇒ x = –1 – 1 ⇒ x = –2

equation of axis, y = k ⇒ y = 2

length of latus rectum is, 4a = 4×1 = 4.

Example 13: Find the equation of the parabola having focus at (–3, 0) and equation of directrix is x + 5 = 0.

Solution:

Given that the parabola has,

focus is F(h + a, k) = F(–3, 0)

and equation of directrix is, x + 5 = 0 ⇒ x = –5

Since distance between focus and directrix be,

$$2a = 2 \Rightarrow a = 1.$$

Then, vertex of the parabola is

$$V(h, k) = V(h + a - a, k) = V(-3 - 1, 0) = V(-4, 0).$$

Now, the equation of parabola is

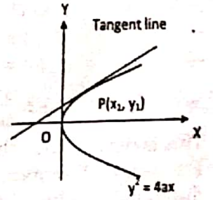
$$(y-k)^2 = 4a(x-h)$$
$$\Rightarrow (y-0)^2 = 4 \times 1\,(x+4)$$
$$\Rightarrow y^2 = 4(x+4)$$

This is the equation of required parabola.

6.4.3 Equation of Tangent at (x_1, y_1) to the Parabola $y^2 = 4ax$.

The equation of parabola in standard form is

$$y^2 = 4ax \quad \dots (1)$$



Differentiating equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2a}{y} \quad \dots (2)$$

At (x_1, y_1),

$$y_1^2 = 4ax_1 \quad \dots (3)$$

and

$$\frac{dy_1}{dx_1} = \frac{2a}{y_1}$$

Since the equation of a tangent line to (1), is passing through the point (x_1, y_1) is,

$$y - y_1 = m(x - x_1) \quad \dots (4)$$

where m be the slope of the line.

Since, the point (x_1, y_1) be the common point to the line (4) and the given parabola (1). So,

$$m = \frac{dy_1}{dx_1} = \frac{2a}{y_1}.$$

Then (4) becomes,

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$\Rightarrow$ $yy_1 - y_1^2 = 2ax - 2ax_1$

$\Rightarrow$ $yy_1 = 2ax + y_1^2 - 2ax_1$

$\Rightarrow$ $yy_1 = 2ax + 2ax_1$ [Using (3)]

$\Rightarrow$ $yy_1 = 2a(x + x_1)$

Thus the equation of tangent at (x_1, y_1) on the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

Note: Equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1).$$

6.4.4 Condition for Tangency that a Line $y = mx + c$ Touches the given Parabola $y^2 = 4ax$.

The given equation of line and parabola are

$y = mx + c$ ... (1)

$y^2 = 4ax$ ... (2)

Eliminating y from (1) and (2), we get

$(mx + c)^2 = 4ax$

$\Rightarrow$ $m^2x^2 + 2mcx + c^2 = 4ax$

$\Rightarrow$ $m^2x^2 + (2mc - 4a)x + c^2 = 0$

which is quadratic in x. ... (3)

Since the line (1) is tangent to the curve (2) then the discriminate value of (3) is zero. Then,

$(2mc - 4a)^2 - 4.m^2c^2 = 0$ $[B^2 - 4AC = 0]$

$\Rightarrow$ $4(mc - 2a)^2 . 4m^2c^2 = 0$

$\Rightarrow$ $(mc - 2a)^2\ m^2c^2 = 0$

$\Rightarrow$ $m^2c^2 - 4mca + 4a^2 - m^2c^2 = 0$

$\Rightarrow$ $-4mca + 4a^2 = 0$

$\Rightarrow$ $4a^2 = 4mca$

$\Rightarrow$ $c = \frac{a}{m}$

This is the condition that the line (1) is tangent to the curve (2).

For point of contact, since line (1) is tangent to the parabola (2), so the discriminate value of (3) is zero i.e. $B^2 - 4AC = 0$

Therefore, (3) gives

$$x = \frac{-B}{2A}$$

i.e. $x = \frac{-(2mc - 4a)}{2m^2} = \frac{2a - mc}{m^2} = \frac{2a - a}{m^2} = \frac{a}{m^2}$ $\left(\because c = \frac{a}{m} \Rightarrow mc = a\right)$

Then (1) gives

$$y = mx + c = m\left(\frac{a}{m^2}\right) + \frac{a}{m} = \frac{2a}{m}$$

So, the print of contact of (1) and (2) is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Example 14: Find the condition that the line $y = mx + c$ may touch the parabola $y^2 = 4a(x + a)$.

Solution:

Given equation of line and parabola is

$y = mx + c$ ... (i)

and $y^2 = 4a(x + a)$ ... (ii)

Since (i) touches (ii) so

$(mx + c)^2 = 4a(x + a)$

$\Rightarrow$ $m^2x^2 + 2mxc + c^2 = 4ax + 4a^2$

$\Rightarrow$ $m^2x^2 + (2mc - 4a)x + (c^2 - 4a^2) = 0$ ... (iii)

which is quadratic in x. Since (i) touches (ii) so the discriminate term of (iii) is equal to zero.

$\boxed{b^2 - 4ac = 0}$

i.e. $(2mc - 4a)^2 - 4m^2(c^2 - 4a^2) = 0$

$\Rightarrow$ $4m^2c^2 - 16mca + 16a^2 - 4m^2c^2 + 16m^2a^2 = 0$

$\Rightarrow$ $16mca = 16a^2 + 16m^2a^2.$

$\Rightarrow$ $c = \frac{-16a^2(1 + m^2)}{16ma}$

$\Rightarrow$ $c = \frac{a(1 + m^2)}{m}$

$\Rightarrow$ $c = am + \frac{a}{m}.$

Thus the line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = am + \frac{a}{m}$.

Example 15: Find the equation of tangent on $y^2 = 25x$ through (4, 10).

Solution:

Given equation of parabola is,

$y^2 = 25x$ ... (i)

which has a tangent at (4, 10).

Comparing equation (i) with $y^2 = 4ax$ we get

$$a = \frac{25}{4}.$$

Now the equation of tangent on $y^2 = 25x$ at (4, 10) is,

$yy_1 = 2a(x + x_1)$

$\Rightarrow$ $y . 10 = 2 \times \frac{25}{4}(x + 4)$

$\Rightarrow$ $40y = 50(x + 4)$

$\Rightarrow$ $4y = 5x + 20.$

Thus, the equation of tangent on $y^2 = 25x$ that passes through the point (4, 10) is $5x - 4y + 20 = 0$.

Example 16: Show that the straight line $7x + 6y - 13 = 0$ is a tangent to the parabola $y^2 - 7x - 8y + 14 = 0$. Find the point of contact.

Solution:

Given equation of tangent is

$$7x + 6y - 13 = 0 \quad \dots (i)$$

and given equation of parabola is

$$y^2 - 7x - 8y + 14 = 0$$

$\Rightarrow$ $y^2 - (-6y + 13) - 8y + 14 = 0$

$\Rightarrow$ $y^2 - 2y + 1 = 0$

$\Rightarrow$ $(y - 1)^2 = 0$

$\Rightarrow$ $y = 1$

Then (i) gives

$$7x + 6 - 13 = 0$$

$\Rightarrow$ $7x = 7$

$\Rightarrow$ $x = 1$

This shows the given line (i) and curve (ii) meet at a single point (1, 1). This means line (i) is tangent to curve (ii) and their point of contact is (1, 1).

Exercise 6.3

1. Find the vertex, focus, directrix and length of latus rectum.
   (a) $y^2 = 8x$ (b) $x^2 = 100y$
   (c) $x^2 - 2x - 8y - 15 = 0$ (d) $y^2 + 6y + 2x + 5 = 0$
2. Find the equation of the following parabola, whose V is vertex and L is directrix are given
   (a) V(1, –2), L is the x–axis. (b) V(–3, 1), L is the line $x = 1$.
3. Find the value of λ, when the line $x - y + 1 = 0$ is a tangent to the parabola $y^2 = \lambda x$.
4. Show that the line $lx + my + n = 0$ touches the parabola $y^2 = 4a(x - b)$ if $am^2 = bl^2 + nl$.
5. Find the equation of tangent at (2, ¼) on the parabola $y^2 = 16x$.
6. For what value of k, the line $2x - y - 1 = 0$ is a tangent to the parabola $y^2 = kx$.
7. Prove that the line $lx + my + n = 0$ touches $y^2 = 4ax$ if $ln = am^2$.

Answers

1. (a) Vertex (0, 0), Focus (2, 0), directrix $x = -2$, latus rectum = 8
   (b) Vertex (0, 0), Focus (0, 25), directrix $y = -25$, latus rectum = 100
   (c) Vertex (1, –2), Focus (1, 0), directrix $y = -4$, latus rectum = 8
   (d) Vertex (2, – 3), Focus (3/2, –3), directrix $2x - 5 = 0$, latus rectum = 2
2. (a) $y^2 = 8(x - 2)$ (b) $(y - 1)^2 = -16(x + 3)$
3. $\lambda = 4$ 5. $y = 32x + 64$ 6. $k = 0, -8$

□□□

7
CHAPTER

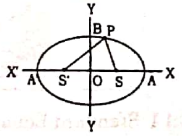
ELLIPSE AND HYPERBOLA

Introduction

As the value of eccentricity of the conic is known as ellipse if the eccentricity is less than 1 and is hyperbola if the eccentricity is greater than 1

7.1 ELLIPSE

An ellipse is the locus of a point on a plane such that the sum of its distance from two fixed points is constant. Two fixed points are called foci (plural form of focus) and the line joining foci is called the major axis of the ellipse. SS' is the major axis of the ellipse.



An ellipse is the locus of point which moves in a plane in such a way that the ratio of its distance from a fixed point (focus) in the same plane to its distance from a fixed straight line (directrix) is always constant and is less than 1. In other words, a conic section in which the eccentricity is less than 1 ($e < 1$) is called ellipse.

Definitions

(a) Foci: The two fixed points of ellipse are called foci.

(b) Major axis: The straight line passing through the foci is called major axis.

(c) Centre of ellipse: The middle point of the join of foci is called center.

(d) Minor axis: The straight line passing through the center and perpendicular to the major axis is called minor axis.

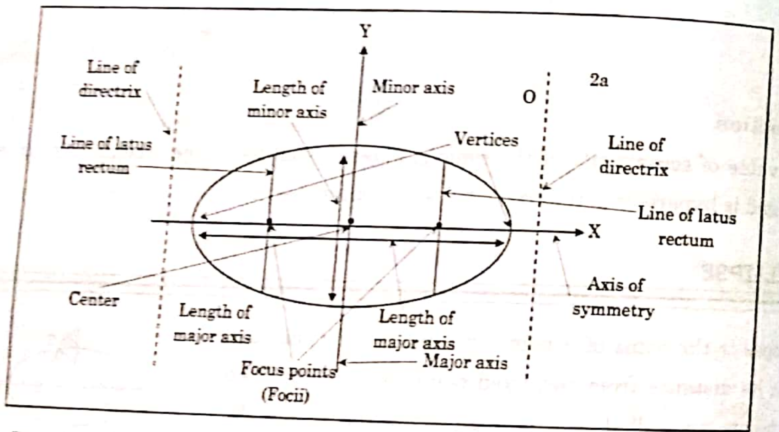
(e) Vertices: The meeting points of the major axis with ellipse are called vertices.

(f) Length of major axis: The distance between the vertices on major axis is called length of major axis.

(g) Length of minor axis: The distance between the meeting points of minor axis with ellipse is called length of minor axis.

(i) Latus rectum: In ellipse, the chord passing through focus (singular of foci) and perpendicular to major axis is called latus rectum and the distance between the meeting points of latus rectum to the ellipse is called length of latus rectum.

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



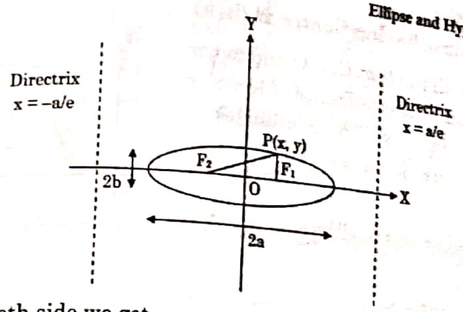
7.1.1 Standard Equation of Ellipse

A. Standard Equation of Ellipse Having Centre at (0, 0)

Let O(0, 0) be the center of the ellipse. Let $F_1(c, 0)$ and $F_2(-c, 0)$ be foci of the ellipse and let P(x, y) be any point of the ellipse. Then by definition of the ellipse, the sum of distance of P from F_1 and F_2, is constant. That is, $PF_1 + PF_2 = 2a$

i.e. $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$

$\Rightarrow \sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$



Squaring both side we get,

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$\Rightarrow x^2 - 2xc + c^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2$$

$$\Rightarrow 4a\sqrt{(x+c)^2 + y^2} = 4(a^2 + xc)$$

Again, squaring both sides, we get

$$a^2[(x+c)^2 + y^2] = a^4 + 2a^2xc + x^2c^2$$

$$\Rightarrow a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2xc + x^2c^2$$

$$\Rightarrow (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad \text{... (i)}$$

In any triangle, we have the sum of two sides is always greater than the third side. So, in figure in ΔPF_1F_2,

$$(PF_1 + PF_2) > F_1F_2$$

$$\Rightarrow 2a > 2c$$

$$\Rightarrow a > c$$

$$\Rightarrow (a^2 - c^2) > 0$$

So, let, $b^2 = a^2 - c^2$. Then the equation (i) reduces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

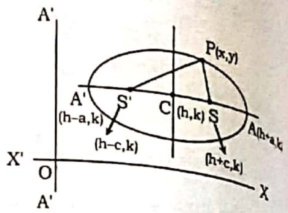
This is the equation of ellipse having center at (0, 0) and foci at (+ c, 0), where $b^2 = a^2 - c^2$.

Note 1: Position of a Point

A point (x_1, y_1) lies inside, on or outside of the ellipse according as $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ is negative, zero or positive.

B. Equation of an Ellipse having Centre at (h, k)

Let the center of the ellipse is at C(h, k) and major axis parallel to X-axis. The coordinates of foci S is (h + c, k) and S' is (h − c, k). The coordinates of vertices A and A' are (h + a, k) and (h − a, k) respectively.



Let P(x, y) be any point on the ellipse. Then by definition, we have

$$SP + S'P = 2a$$

$$\Rightarrow \sqrt{(x-h-c)^2+(y-k)^2}+\sqrt{(x-h+c)^2+(y-k)^2}=2a$$

$$\Rightarrow \left\{\sqrt{(x-h-c)^2+(y-k)^2}\right\}^2=\left\{2a-\sqrt{(x-h+c)^2+(y-k)^2}\right\}^2$$

$$\Rightarrow x^2+h^2+c^2-2hx+2hc-2cx+(y-k)^2=4a^2-4a\sqrt{(x-h+c)^2+(y-k)^2}+x^2+h^2+c^2-2hx-2hc+2cx+(y-k)^2$$

$$\Rightarrow 4cx-4hc-4a^2=-4a\sqrt{(x-h+c)^2+(y-k)^2}$$

$$\Rightarrow (cx-hc-a^2)^2=\left\{-a\sqrt{(x-h+c)^2+(y-k)^2}\right\}^2$$

$$\Rightarrow c^2x^2+h^2c^2+a^4-2hc^2x+2hca^2-2a^2cx=a^2\{x^2+h^2+c^2-2hx-2hc+2cx+(y-k)^2\}$$

$$\Rightarrow x^2(a^2-c^2)-2hx(a^2-c^2)+h^2(a^2-c^2)+a^2(y-k)^2=a^4-c^2a^2$$

$$\Rightarrow (a^2-c^2)(x-h)^2+a^2(y-k)^2=a^2(a^2-c^2)$$

$$\Rightarrow \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{(a^2-c^2)}=1 \qquad \ldots (1)$$

In any triangle, we have the sum of two sides is always greater than the third side. So, in figure in ΔPSS',

$$(PS + PS') > SS'$$

$$\Rightarrow 2a > 2c$$

$$\Rightarrow a^2 > c^2$$

$$\Rightarrow (a^2-c^2) > 0$$

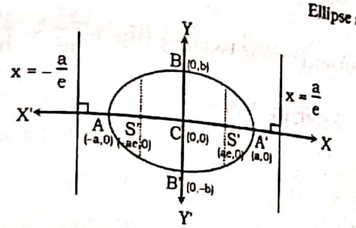
So, let, $b^2 = a^2 - c^2$. Then the equation (1) reduces

$$\Rightarrow \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

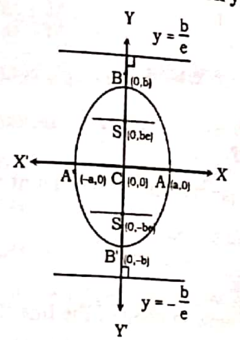
7.1.2 Two Standard Forms of Ellipse

As value of a and b, we categories the ellipse in two different forms as:

(a) If a > b > 0 then the foci of ellipse are on x-axis or parallel to x-axis.



(b) If b > a > 0 then the foci of ellipse are on y-axis or parallel to y-axis.



Some Equations of Terms of Ellipse

The table gives the equation and coordinates of a ellipse as its situation.

Equation of ellipse	Centre	Vertex	Focus	Length of major axis	Length of minor axis	Length of latus rectum	Eccentricity	Equation of direction
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$	(0, 0)	(±a, 0)	(±ae, 0)	2a	2b	$\frac{2b^2}{a}$	$e=\sqrt{1-\frac{b^2}{a^2}}$	$x=\pm\frac{a}{e}$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $b>a$	(0, 0)	(0, ±b)	(0, ±be)	2b	2a	$\frac{2a^2}{b}$	$e=\sqrt{1-\frac{a^2}{b^2}}$	$y=\pm\frac{b}{e}$
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, $a>b$	(h, k)	(h ± a, k)	(h ±ae, k)	2a	2b	$\frac{2b^2}{a}$	$e=\sqrt{1-\frac{b^2}{a^2}}$	$x=h\pm\frac{a}{e}$
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, $b>a$	(h, k)	(h, k ± b)	(h, k ± be)	2b	2a	$\frac{2a^2}{b}$	$e=\sqrt{1-\frac{a^2}{b^2}}$	$y=k\pm\frac{b}{e}$

7.1.3 Equation of Tangent at (x_1, y_1) to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The equation of ellipse in standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{..... (i)}$$

At (x_1, y_1), the ellipse is,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \text{..... (ii)}$$

Differentiating equation (i) w. r. t. x, we get

$$\frac{dy_1}{dx_1} = \frac{-b^2x_1}{a^2y_1}$$

Since the equation of line that passes through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1) \quad \text{... (iii)}$$

where m be the slope of the line.

Since the point (x_1, y_1) be the common point of the line (iii) and the given ellipse (i). This means the line is a tangent to (i) at (x_1, y_1) and so, the slope m of (iii) is same as the slope $\frac{dy_1}{dx_1}$ of (i).

That is,

$$m = \frac{dy_1}{dx_1} = \frac{-b^2x_1}{a^2y_1}$$

Therefore, the line (iii) becomes

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\Rightarrow \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{xx_1}{a^2} + \frac{x_1^2}{a^2}$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1. \quad \text{[Using (ii)]}$$

Thus, the equation of tangent at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

Note: Equation of normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1).$$

7.1.4 Condition for tangency that a line y = mx + c touches the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let the equation of the given line is

$$y = mx + c \quad \text{... (i)}$$

and the equation of the given curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2 \quad \text{... (ii)}$$

Eliminating y from equation (i) and (ii), we get

$$b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

$$\Rightarrow b^2x^2 + a^2(m^2x^2 + 2mxc + c^2) = a^2b^2$$

$$\Rightarrow b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

$$\Rightarrow (b^2 + a^2m^2)x^2 + 2a^2mcx + (a^2c^2 - a^2b^2) = 0 \quad \text{... (iii)}$$

This is quadratic in x.

Since, (iii) be the common value of (i) and (ii). And, (i) is tangent on (ii). So, its discriminate term of (iii) should be equal to zero. So,

$$(2a^2mc)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$$

$$\Rightarrow a^4m^2b^2 + a^2b^4 = a^2b^2c^2$$

$$\Rightarrow a^2m^2 + b^2 = c^2.$$

This is the condition for tangency.

Next, for point of contact, being the discriminate term is zero i.e. $B^2 - 4AC = 0$,

$$x = -\frac{B}{2A} = -\frac{2a^2mc}{2(b^2 + a^2m^2)} = -\frac{2a^2mc}{c^2} = -\frac{a^2m}{c}$$

Then (i) gives,

$$y = mx + c = m\left(-\frac{ma^2}{c}\right) + c = -\frac{m^2a^2 - c^2}{c} = \frac{b^2}{c} \quad \text{[Using (iv)]}$$

Thus, the point of contact of (i) and (ii) is $\left(-\frac{ma^2}{c}, \frac{b^2}{c}\right)$.

Example 1: Find the equation of ellipse which has center at C(0, 2), focus at F(0, 0) and length of major axis is 8. Also, calculate its eccentricity.

Solution:

As given the required the ellipse has center at C(0, 2) and focus at F(0, 0).

Here the x value is fixed and y value is varying in center and focus. So,

$$C(h, k) = C(0, 2) \text{ and } F(h, k + c) = F(0, 0)$$

Then,

$$h = 0,\ k = 2 \text{ and } c = -2 \quad \text{along y–axis.}$$

Being, y is varying we observe b > a.

As given $\qquad b = 4$

Then,

$$a^2 = b^2 - c^2 = 9 - 4 = 5$$

$$\Rightarrow \quad a = \sqrt{5}.$$

Now the equation of ellipse with center (0, 0) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

$$\Rightarrow \quad \frac{(x-0)^2}{5} + \frac{(y-2)^2}{1} = 1$$

$$\Rightarrow \quad \frac{x^2}{5} + \frac{(y-2)^2}{9} = 1.$$

and its eccentricity is, $\quad e = \frac{c}{b} = \frac{2}{3}.$

Example 2: Find center, vertices and foci of the ellipse $x^2 + 5y^2 + 4x = 1$.

Solution:

Given equation is,

$$x^2 + 5y^2 + 4x = 1$$

$$\Rightarrow \quad (x+2)^2 + 5y^2 = 5$$

$$\Rightarrow \quad \frac{(x+2)^2}{5} + \frac{y^2}{1} = 1 \qquad \text{... (i)}$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$$h = -2,\quad k = 0,\quad a^2 = 5 \text{ and } b^2 = 1.$$

Here, a > b. So,

$$c^2 = a^2 - b^2 = 5 - 1$$

$$\Rightarrow \quad c = 2.$$

Therefore, center $(h, k) = C(-2, 0)$.

Since a > b, so the major axis is parallel to x-axis. Therefore, the foci lie on the line that is parallel to x–axis. So, the foci and vertex of (i) are

$$F(h + c, k) = F(-2 + 2, 0).$$

$$V(h + a, k) = V(-2 + \sqrt{5}, 0).$$

Example 3: Find the eccentricity and the co-ordinate of the foci of the ellipse $2x^2 + 3y^2 - 1 = 0$.

Solution:

Given ellipse is

$$2x^2 + 3y^2 = 1$$

$$\Rightarrow \quad \frac{x^2}{1/2} + \frac{y^2}{1/3} = 1 \qquad \text{... (i)}$$

Comparing the equation (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, we get,

$$h = 0,\quad k = 0,\quad a^2 = \frac{1}{2} \text{ and } b^2 = \frac{1}{3}.$$

Here, a > b. So,

$$c = \sqrt{a^2 - b^2} = \sqrt{\frac{1}{2} - \frac{1}{3}} = \frac{1}{\sqrt{6}}.$$

Therefore, eccentricity of (i) is

$$e = \frac{c}{a} = \frac{1/\sqrt{6}}{1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}.$$

Since, a > b, so the foci lie on the line parallel to x-axis. Therefore, the foci of the ellipse (i) is

$$F(h + c, k) = F\left(0 + \frac{1}{\sqrt{6}}, 0\right) = F\left(\frac{1}{\sqrt{6}}, 0\right)$$

Example 4: Find the equation of the ellipse whose focus, directrix and eccentricity are given as: $F(0, 3)$, $x + 7 = 0$ and $e = \frac{1}{3}$

Solution:

Given that the ellipse has focus F(0, 3), equation of directrix is $x + 7 = 0$ and eccentricity is $e = \frac{1}{3}$.

Let P(x, y) be any point on the ellipse. Then by definition of eccentricity,

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix } x + 7 = 0}$$

$$\Rightarrow \frac{1}{3} = \frac{\sqrt{(x-0)^2 + (y-3)^2}}{+\frac{x+7}{1}}$$

$$\Rightarrow (x+7)^2 = 9[x^2 + (y-3)^2]$$

$$\Rightarrow x^2 + 14x + 49 = 9x^2 + 9y^2 - 54y + 81$$

$$\Rightarrow 8x^2 + 9y^2 - 14x - 54y + 32 = 0.$$

This is the equation of the required ellipse.

Example 5: Find the equation of tangent and normal at the point (4, 3) on the ellipse $3x^2 + 4y^2 = 84$.

Solution:

Given, ellipse is

$$3x^2 + 4y^2 = 84 \qquad \text{... (i)}$$

Differentiating (i) w. r. t. x, we get

$$6x + 8y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-6x}{8y} = -\frac{3x}{4y}.$$

At (4, 3)

$$\frac{dy}{dx} = m = -\frac{3 \times 4}{4 \times 3} = -1.$$

Now, equation of tangent to (i) that passes through point (4, 3) is

$$y - 3 = -1(x - 4)$$

$$\Rightarrow x + y = 7.$$

And, equation of normal to (i) at (4, 3) is,

$$y - 3 = -(-1)(x - 4)$$

$$\Rightarrow y - 3 = x - 4$$

$$\Rightarrow x - y = 1$$

Example 6: Find the value of λ, when the straight line $y = x + \lambda$ touches the ellipse $2x^2 + 3y^2 = 6$.

Solution:

Given equation of ellipse is

$$2x^2 + 3y^2 = 6$$

$$\Rightarrow \frac{x^2}{3} + \frac{y^2}{2} = 1 \qquad \text{... (i)}$$

Comparing equation (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$a^2 = 3 \quad \text{and} \quad b^2 = 2$$

Given straight line is

$$y = x + \lambda \qquad \text{... (ii)}$$

Comparing with $y = mx + c$, we get

$$m = 1, \quad c = \lambda$$

For the line to touch the ellipse is,

$$a^2m^2 + b^2 = c^2$$

$$\Rightarrow 3 \times 1 + 2 = \lambda^2$$

$$\Rightarrow 5 = \lambda^2$$

$$\Rightarrow \lambda = \pm\sqrt{5}.$$

Example 7: Condition of tangency of the line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the point of contact.

Solution:

Given line is

$$lx + my + n = 0$$

$$\Rightarrow y = -\left(\frac{l}{m}\right)x - \frac{n}{m} \qquad \text{... (i)}$$

And, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2 \qquad \text{... (ii)}$$

Eliminating y from given equations (i) and (ii)

$$b^2x^2 + a^2\left(-\frac{lx + n}{m}\right)^2 = a^2b^2$$

$$\Rightarrow m^2b^2x^2 + a^2(l^2x^2 + 2lxn + n^2) = a^2b^2m^2$$

$$\Rightarrow (m^2b^2 + a^2l^2)x^2 + 2a^2lnx + (a^2n^2 - a^2b^2m^2) = 0 \qquad \text{... (iii)}$$

Since given line is tangent on the ellipse, so the discriminate term of (iii) is zero.

i.e. $(2a^2 l n)^2 - 4(m^2b^2 + a^2l^2)(a^2n^2 - a^2b^2m^2) = 0$

$\Rightarrow \quad 4a^4l^2n^2 - 4\{m^2b^2(a^2n^2 - a^2b^2m^2) + a^2l^2(a^2n^2 - a^2b^2m^2)\} = 0$

$\Rightarrow \quad m^4b^4 + a^2b^2l^2m^2 = m^2n^2b^2$

$\Rightarrow \quad a^2l^2 + m^2b^2 = n^2.$

This is the required condition.

And, for point of contact, being the discriminate term is zero i.e. $B^2 - 4AC = 0$,

$$x = -\frac{B}{2A} = -\frac{2a^2ln}{2(m^2b^2 + a^2l^2)} = -\frac{2a^2ln}{2n^2} = -\frac{a^2l}{n}$$

Then (i) gives,

$$y = -\frac{lx - n}{m} = -\frac{l\left(-\frac{a^2l}{n}\right) - n}{m}$$

$$\Rightarrow \quad y = \frac{a^2l^2 - n^2}{mn} = \frac{-b^2m^2}{mn} = \frac{-b^2m}{n}.$$

Thus the point of contact is $\left(-\frac{a^2l}{n}, -\frac{b^2m}{n}\right)$.

Exercise 7.1

1. Find the equation of ellipse which has centre C, focus F and semi–major axis a or b and calculate eccentricity.
   (a) C(0, 0), F(0, 2), b = 4
   (b) C(2, 2), F(–1, 2), $a = \sqrt{10}$.
2. Find center, vertices and foci of the ellipse
   (a) $x^2 + 2y^2 - x - 4y + 1 = 0$
   (b) $25(x - 3)^2 + 4(y - 1)^2 = 100$
   (c) $x^2 + 10x + 25y^2 = 0$
   (d) $9x^2 + 16y^2 + 18x - 96y + 4 = 0$
   (e) $16(x - 2)^2 + 9(y + 3)^2 = 144$
3. Find the center, vertices, foci, eccentricity and length of latus rectum of the ellipse $9x^2 + 16y^2 + 18x - 96y + 9 = 0$.
4. Show that $25x^2 + 9y^2 - 100x + 54y - 44 = 0$ represents an ellipse. Then, find its center, and foci.
5. Obtain the vertices, center, coordinates of foci, eccentricity of the ellipse $9x^2 + 4y^2 + 36x - 8y + 4 = 0$.

6. Show that $2x^2 + y^2 = 3x$ represents an ellipse; find its eccentricity and coordinates of foci.
7. Find the equation of ellipse whose foci are at (–2, 4) and (4, 4); length of major axis is 10. Also, find the eccentricity.
8. Show that the line $y = x + \frac{\sqrt{5}}{\sqrt{6}}$ touches the ellipse $2x^2 + 3y^2 = 1$. Find the point of contact.
9. Find the equation of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the line $y = 3x + 7$.
10. Show that the line $3x + 4y + \sqrt{7} = 0$ touches the ellipse $3x^2 + 4y^2 = 1$. Also find the point of contact.

Answers

1. (a) $\frac{x^2}{12} + \frac{y^2}{16} = 1;\ e = \frac{c}{b} = \frac{2}{4} = \frac{1}{2}$
   (b) $\frac{(x-2)^2}{10} + \frac{(y-2)^2}{1} = 1;\ e = \frac{c}{a} = \frac{3}{\sqrt{10}}$
2. (a) C(–1, –4); F$(1/2, 1 + \sqrt{5/4})$; V$(1/2, 1 + \sqrt{5/2})$
   (b) C(3, 1); F$(3, 1 + \sqrt{21})$; V(3, 1 + 5)
   (c) C(–5, 0); F$(-5 + 2\sqrt{6}, 0)$; F(0, 0)
   (d) C(–1, 3); F$(-1 + \sqrt{7}, 3)$; V(3, 3)
   (e) C(2, –3); F$(2, -3 + \sqrt{7})$; V(2, 1)
3. C(–1, 3); V(–1 + 4, 3); F$(-1 + \sqrt{7}, 3)$; $\frac{\sqrt{7}}{4}, \frac{9}{2}$
4. C(2, –3); F(2, –3 + 4)
5. C(–2, 1); V(–2, 1 + 3); F$(-2, 1 + \sqrt{5})$; $e = \frac{\sqrt{5}}{3}$
6. $F\left(\frac{3}{4}, \frac{3}{4}\right)$; $e = \frac{1}{2}$
7. $16x^2 + 25y^2 - 32x - 200y + 16 = 0$; $e = \frac{3}{5}$
8. $\left(\sqrt{\frac{3}{10}}, \sqrt{\frac{2}{15}}\right)$
9. $3x - y + \sqrt{\frac{155}{12}} = 0$
10. $\left(-\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$

7.2 HYPERBOLA

Introduction

A hyperbola is defined as the locus of a point that moves in a plane such that its distance from a fixed point is always e times (e > 1) its distance from a fixed line. The fixed point is called the focus of the hyperbola. The fixed straight line is called the directrix and the constant e is called the eccentricity of the hyperbola.

A conic section in which the eccentricity is always greater than 1 is called a hyperbola. In other words, the locus of a point whose distance from a fixed point (focus) to its distance from a fixed straight line (directrix) bears a constant (greater than 1), is called a hyperbola. Alternatively, the locus of point in a plane such that the difference of its distances from two fixed points is constant, is called a hyperbola.

Definitions

(a) **Foci:** The two fixed points in hyperbola are called foci.

(b) **Directrix:** The fixed straight line is called directrix corresponding to two foci.

(c) **Transverse axis:** The straight line passing through foci is called transverse axis.

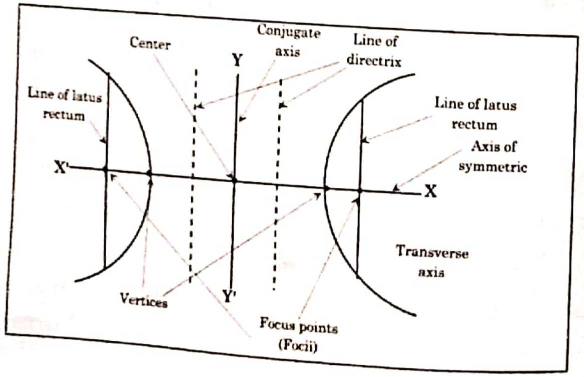
(d) **Centre:** In hyperbola, the middle point of the join of foci is called center.

(e) **Conjugate axis:** The straight line passing through the center and perpendicular to the transverse axis is called conjugate axis.

(f) **Vertices:** In hyperbola, the meeting points of transverse axis with hyperbola are called vertices.

(g) **Latus rectum:** In hyperbola, the straight line segment passing through focus and perpendicular to transverse axis is called latus rectum. There are two latus rectum of a hyperbola.

Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$



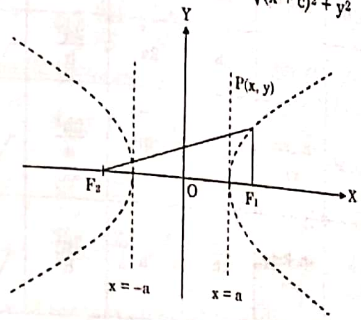
7.2.1 Standard Equation of Hyperbola having center at O(0, 0)

Let O(0, 0) be the center of the hyperbola. Let $F_1(c, 0)$ and $F_2(-c, 0)$ be foci of the hyperbola and let P(x, y) be any point of the hyperbola. The difference of distance of foci form P is constant. That is,

$$PF_1 - PF_2 = 2a$$

i.e. $\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a$

$\Rightarrow \sqrt{(x-c)^2 + y^2} = 2a + \sqrt{(x+c)^2 + y^2}$



Squaring both side we get,

$(x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$

$\Rightarrow x^2 - 2xc + c^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2$

$\Rightarrow 4a\sqrt{(x+c)^2 + y^2} = -4(a^2 + xc)$

$\Rightarrow a^2[(x+c)^2 + y^2] = a^4 + 2a^2xc + x^2c^2$

$\Rightarrow a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2xc + x^2c^2$

$\Rightarrow (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$ ... (i)

In any triangle, we have the sum of two sides is always greater than the third side. So, in figure in ΔPF_1F_2,

$(PF_1 - PF_2) < F_1F_2$

$\Rightarrow 2a < 2c$

$\Rightarrow a^2 < c^2$

$\Rightarrow (c^2 - a^2) > 0.$

So, let, $b^2 = c^2 - a^2$. Thus equation (i) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is the equation of hyperbola having center (0, 0) and foci at (± c, 0).

Some Equations of Terms of Hyperbola

The table gives the equation and coordinates of a hyperbola as its situation,

Equation of hyperbola	Centre	Vertex	Focus	Length of transverse axis	Length of conjugate axis	Length of latus rectum	Eccentricity	Equation of direction
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	(0, 0)	$(\pm a, 0)$	$(\pm ae, 0)$	2a	2b	$\frac{2b^2}{a}$	$\sqrt{1+\frac{b^2}{a^2}}$	$x=\pm\frac{a}{e}$
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$	(0, 0)	$(0, \pm b)$	$(0, \pm be)$	2b	2a	$\frac{2a^2}{b}$	$\sqrt{1+\frac{a^2}{b^2}}$	$y=\pm\frac{b}{e}$
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	(h, k)	$(h\pm a, k)$	$(h \pm ae, k)$	2a	2b	$\frac{2b^2}{a}$	$\sqrt{1+\frac{b^2}{a^2}}$	$x=h\pm\frac{a}{e}$
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1$	(h, k)	$(h, k \pm b)$	$(h, k \pm be)$	2b	2a	$\frac{2a^2}{b}$	$\sqrt{1+\frac{a^2}{b^2}}$	$y=k\pm\frac{b}{e}$

7.2.2 Rectangular Hyperbola

A hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is called a rectangular hyperbola if a = b. That is the equation of hyperbola $x^2-y^2=a^2$ is the rectangular hyperbola.

7.2.3 Conjugate Hyperbola

Let the equation of a hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then the hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ or $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1\right)$ is called a conjugate hyperbola.

7.2.4 Equation of tangent at (x_1, y_1) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The equation of hyperbola in standard form is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{..... (1)}$$

At (x_1, y_1), the hyperbola is,

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \qquad \text{..... (2)}$$

Differentiating equation (1) w. r. t. x, we get

$$\frac{dy_1}{dx_1}=\frac{b^2x_1}{a^2y_1}$$

Since the equation of line that passes through the point (x_1, y_1) is

$$y-y_1=m(x-x_1) \qquad \text{... (3)}$$

where m be the slope of the line.

Since the point (x_1, y_1) be the common point of the line (3) and the given ellipse (1). This means the line is a tangent to (1) at (x_1, y_1) and so, the slope m of (3) is same as the slope $\frac{dy_1}{dx_1}$ of (1).

That is,

$$m=\frac{dy_1}{dx_1}=\frac{b^2x_1}{a^2y_1}$$

Therefore, the line (3) becomes

$$y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\Rightarrow \frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=\frac{xx_1}{a^2}-\frac{x_1^2}{a^2}$$

$$\Rightarrow \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1. \qquad \text{[Using (ii)]}$$

Thus, the equation of tangent at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

Note: Equation of normal at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1).$$

7.2.5 Equation of Normal to the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at point (x_1, y_1)

The equation of tangent at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \qquad \text{... ... (1)}$$

The slope of tangent (1) is

$$m_1=\frac{b^2x_1}{a^2y_1}$$

And, the slope of normal is

$$m_2=-\frac{1}{m_1}=-\frac{a^2y_1}{b^2x_1}$$

The equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$$\Rightarrow \quad \frac{x - x_1}{x_1/a^2} = -\frac{y - y_1}{y_1/b^2}$$

This is the equation of required normal.

7.2.6 Condition for Tangency that a Line y = mx + c Touches the given Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let the equation of the given line is

$$y = mx + c \qquad \text{... (1)}$$

and the equation of the given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad b^2x^2 - a^2y^2 = a^2b^2 \qquad \text{... (2)}$$

Now, eliminating y from equation (1) and (2), we get

$$b^2x^2 - a^2(mx + c)^2 = a^2b^2$$

$$\Rightarrow \quad b^2x^2 - a^2(m^2x^2 + 2mxc + c^2) = a^2b^2$$

$$\Rightarrow \quad b^2x^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0$$

$$\Rightarrow \quad (b^2 - a^2m^2)\,x^2 - 2a^2mcx - (a^2c^2 + a^2b^2) = 0 \qquad \text{... (3)}$$

This is quadratic in x.

Since, (3) be the common value of (1) and (2). And, (1) is tangent on (2). So, its discriminate term of (3) should be equal to zero. So,

$$(-2a^2mc)^2 - 4(b^2 - a^2m^2)(-a^2c^2 - a^2b^2) = 0$$

$$\Rightarrow \quad a^4m^2b^2 - a^2b^4 = a^2b^2c^2$$

$$\Rightarrow \quad a^2m^2 - b^2 = c^2.$$

This is the required condition for tangency.

For point of contact.

Since the tangent to a curve touches the curve at a single point. So, for the point of contact of (1) and (2), we observe the discriminate value of (3), is zero.

Therefore, from (3)

$$x = -\frac{B}{2A} = -\frac{2a^2mc}{2(b^2 + a^2m^2)} = -\frac{2a^2mc}{c^2} = -\frac{a^2m}{c}$$

Then (1) gives,

$$y = mx + c = m\left(-\frac{ma^2}{c}\right) + c = -\frac{m^2a^2 - c^2}{c} = \frac{-b^2}{c}$$

Thus, the point of contact of (1) and (2) is $\left(-\frac{ma^2}{c}, \frac{-b^2}{c}\right)$.

Example 8: Find the length of transverse axis, conjugate axis, coordinates of foci, vertices, eccentricity and latus rectum of the hyperbola $3x^2 - 4y^2 = 36$.

Solution:

The given equation of the hyperbola is

$$3x^2 - 4y^2 = 36$$

$$\Rightarrow \quad \frac{x^2}{12} - \frac{y^2}{9} = 1 \qquad \text{... (i)}$$

Comparing (i) with the equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$$a^2 = 12,\ b^2 = 9$$

i.e. $a = 2\sqrt{3},\ b = 3$

Now,

the length of transverse axis of (i) is, $2a = 4\sqrt{3}$

the length of conjugate axis of (i) is, $2b = 6$

the vertices of (i) is, $V(\pm a, 0) = V(\pm 2\sqrt{3}, 0)$

the eccentricity of (i) is, $e = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{12 + 9}{12}} = \sqrt{\frac{21}{12}} = \frac{\sqrt{7}}{2}$

the foci of (i) is, $F(\pm ae, 0) = F\left(\pm 2\sqrt{3}\frac{\sqrt{7}}{2}, 0\right) = F(\pm\sqrt{21}, 0)$

the length of the latus rectum of (i) is, $\frac{2b^2}{a} = \frac{2 \times 9}{2\sqrt{3}} = 3\sqrt{3}$

Example 9: Find the center, eccentricity, foci and the directrix of the hyperbola: $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

Solution:

The given equation to the hyperbola is

$$9x^2 - 16y^2 - 72x + 96y - 144 = 0$$

$$\Rightarrow \quad 9(x^2 - 8x) - 16(y^2 - 6y) - 144 = 0$$

$$\Rightarrow \quad 9(x^2 - 8x + 16 - 16) - 16(y^2 - 6y + 9 - 9) - 144 = 0$$

$$\Rightarrow \quad 9(x - 4)^2 - 144 - 16(y - 3)^2 + 144 - 144 = 0$$

$$\Rightarrow \quad 9(x - 4)^2 - 16\,(y - 3)^2 = 144$$

$$\Rightarrow \quad \frac{9\,(x - 4)^2}{144} - \frac{16\,(y - 3)^2}{144} = 1$$

$$\Rightarrow \quad \frac{(x - 4)^2}{16} - \frac{(y - 3)^2}{9} = 1 \qquad \text{.... (i)}$$

Comparing (i) with the equation of hyperbola $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$, we get

$$a^2 = 16,\ b^2 = 9,\ h = 4,\ k = 3$$

Now,

The coordinate of center of (i) is, C(h, k) = C(4, 3)

The eccentricity of (i) is, $e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{16+9}{16}} = \frac{5}{4}$

The coordinate of foci of (i) is,

$$F(h \pm ae, k) = \left(4 \pm 4 \times \frac{5}{4}, 3\right) = (4 \pm 5, 3) = (9, 3), (-1, 3)$$

The equation of directrix of (i) is,

$$x = h \pm \frac{a}{e} = 4 \pm \frac{4}{(5/4)} = 4 \pm \frac{16}{5}$$

$\Rightarrow$ $5x = 20 \pm 16$

$\Rightarrow$ $5x = 36$ and $5x = 4$

Example 10: Find the equation of hyperbola whose foci is at (± 5, 0) and vertex at (± 2, 0).

Solution:

Given that a hyperbola has

Foci, F(± ae, 0) = F(± 5, 0)

and vertex, V(± a, 0) = V(± 2, 0)

These implies us,

$a = 2$ and $ae = 5$

And the equation of the hyperbola having foci at (± 5, 0) and vertex at (± 2, 0) is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

i.e. $\frac{x^2}{4} - \frac{y^2}{b^2} = 1$ ... (i)

From foci and vertex, we observe that both coordinate has fixed y-value. So,

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$\Rightarrow$ $ae = \sqrt{a^2 + b^2}$

$\Rightarrow$ $5 = \sqrt{4 + b^2}$

$\Rightarrow$ $25 = 4 + b^2$

$\Rightarrow$ $b^2 = 21$

Now, the equation (i) becomes,

$$\frac{x^2}{4} - \frac{y^2}{21} = 1.$$

Example 11: Find the equation of hyperbola having foci at (± 7, 0) and eccentricity is $\frac{7}{4}$

Solution:

Given that the hyperbola has

Foci, F(± ae, 0) = F(± 7, 0) and eccentricity, $e = \frac{7}{4}$

These implies us,

$e = \frac{7}{4}$ and $ae = 7 \Rightarrow e = \frac{7}{a}$

Solving we get $a = 4$

And, the equation of the hyperbola having foci at (± 7, 0) and eccentricity $\frac{7}{4}$ is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

i.e. $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$ ... (i)

From foci, we observe that the coordinate has fixed y-value and dual x-value. So,

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$\Rightarrow$ $\frac{7}{4} = \sqrt{1 + \frac{b^2}{16}}$

$\Rightarrow$ $\frac{7}{4} = \frac{\sqrt{16 + b^2}}{4}$

$\Rightarrow$ $7 = \sqrt{16 + b^2}$

$\Rightarrow$ $49 = 16 + b^2$

$\Rightarrow$ $b^2 = 33$

Now, the equation (i) becomes,

$$\frac{x^2}{16} - \frac{y^2}{33} = 1.$$

Example 12: Show that the line $x + y - 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$. What is the point of contact?

Solution:

The given line and the hyperbola are

$$x + y - 1 = 0 \quad \text{......... (i)}$$

and

$$3x^2 - 4y^2 = 12 \quad \text{......... (ii)}$$

Solving (i) and (ii), we get

$$3x^2 - 4(1 - x)^2 = 12$$

$$\Rightarrow \quad 3x^2 - 4(1 - 2x + x^2) = 12$$

$$\Rightarrow \quad 3x^2 - 4 + 8x - 4x^2 = 12$$

$$\Rightarrow \quad x^2 - 8x + 16 = 0$$

$$\Rightarrow \quad (x - 4)^2 = 0$$

$$\Rightarrow \quad x = 4$$

Using this to (i) then we get

$$y = -3.$$

Thus, the point of contact of (i) and (ii) is (4, –3). This means the line (i) is tangent to the hyperbola (ii).

Example 13: Find the equation of the tangent to the hyperbola $2x^2 - 3y^2 = 6$ which passes through the point (–2, –1).

Solution:

The equation of hyperbola is

$$2x^2 - 3y^2 = 6$$

$$\Rightarrow \quad \frac{x^2}{3} - \frac{y^2}{2} = 1 \quad \text{... (i)}$$

Comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$$a^2 = 3 \text{ and } b^2 = 2.$$

Since the equation of line passes through the point (–2, –1) and having slope m is

$$y + 1 = m(x + 2)$$

$$\Rightarrow \quad y = mx + (2m - 1) \quad \text{... (ii)}$$

As given the line (ii) is tangent to (i), so using the condition of tangency $c = \pm\sqrt{a^2m^2 - b^2}$, we get

$$2m - 1 = \pm\sqrt{3m^2 - 2}$$

$$\Rightarrow \quad 4m^2 - 4m + 1 = 3m^2 - 2$$

$$\Rightarrow \quad m^2 - 4m + 3 = 0$$

$$\Rightarrow \quad (m - 3)(m - 1) = 0$$

$$\Rightarrow \quad m = 1, 3.$$

Therefore, the tangent (i) is

$y + 1 = 1(x + 2)$ i.e. $x - y + 1 = 0$

and $\quad y + 1 = 3(x + 2)$ i.e. $3x - y + 5 = 0.$

Example 14: Find the equation of tangent to the hyperbola $3x^2 - 4y^2 - 12 = 0$, which are parallel to the line $y = x + 2$, find the point of contact.

Solution:

The given equation of hyperbola is

$$3x^2 - 4y^2 = 12$$

$$\Rightarrow \quad \frac{x^2}{4} - \frac{y^2}{3} = 1 \quad \text{... (i)}$$

Comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$$a^2 = 4 \text{ and } b^2 = 3.$$

Given line is

$$y = x + 2 \quad \text{... (ii)}$$

Clearly, the slope of (ii) is

$$m = 1$$

As given the tangent line to (i) is parallel to (ii) therefore the slope of the tangent line is m = 1.

Therefore, the equation of tangent to (i) is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

i.e. $\quad y = x \pm \sqrt{4 - 3}$

$$\Rightarrow \quad y = x \pm 1$$

Example 15: Find the condition that the line $lx + my + n = 0$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solution:

Given equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \cdots (i)$$

Given line is

$$lx + my + n = 0$$

$$\Rightarrow \quad y = -\frac{(lx + n)}{m} \qquad \cdots (ii)$$

Eliminating y from equation (i) and (ii), then

$$\frac{x^2}{a^2} - \frac{\left(\frac{-lx + n}{m}\right)^2}{b^2} = 1$$

$$\Rightarrow \quad b^2m^2x^2 - a^2l^2x^2 - 2a^2lxn - a^2n^2 = a^2m^2b^2$$

$$\Rightarrow \quad (b^2m^2 - a^2l^2)\, x^2 - 2a^2lnx - a^2n^2 - a^2m^2b^2 = 0 \qquad \cdots (iii)$$

If equation (ii) is tangent on (i), then the discriminate term of (iii) should be zero.

i.e. $$(-2a^2ln)^2 + 4(b^2m^2 - a^2l^2)\,(a^2n^2 + a^2m^2b^2) = 0$$

$$\Rightarrow \quad a^4l^2n^2 + a^2b^2m^2n^2 + a^2b^4m^4 - a^4n^2l^2 - a^4m^2l^2b^2 = 0$$

$$\Rightarrow \quad n^2 + b^2m^2 - a^2l^2 = 0$$

$$\Rightarrow \quad -b^2m^2 + a^2l^2 = n^2 \qquad [\because a^2b^2m^2 \neq 0]$$

This shows that the line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, when $a^2l^2 - b^2m^2 = n^2$.

And, for point of contact,

$$x = -\frac{B}{2A} = -\frac{2a^2ln}{2(m^2b^2 - a^2l^2)} = -\frac{2a^2ln}{-2n^2} = \frac{a^2l}{n}.$$

Then (i) gives,

$$y = -\frac{lx - n}{m} = -\frac{l\left(\frac{a^2l}{n}\right) - n}{m}$$

$$\Rightarrow \quad y = -\frac{a^2l^2 - n^2}{mn} = \frac{b^2m^2}{mn} = \frac{b^2m}{n}.$$

Thus the point of contact is $\left(\frac{a^2l}{n}, \frac{b^2m}{n}\right)$.

Exercise 7.2

1. Find the coordinate of vertices, eccentricity and foci of the hyperbola:
   (a) $3x^2 - 2y^2 = 1$
   (b) $\frac{x^2}{9} - \frac{y^2}{16} = 1$
   (c) $9x^2 - 16y^2 - 18x - 64y - 199 = 0$
   (d) $3x^2 - 6y^2 = -18$
2. Find the center, vertices, foci, eccentricity of the following hyperbola
   (a) $4(x - 2)^2 - 9(y + 3)^2 = 36$
   (b) $9(x - 2)^2 - 4(y + 3)^2 = 36$
   (c) $5x^2 - 4y^2 + 20x + 8y = 4$
   (d) $4y^2 = x^2 - 4x$
   (e) $x^2 - y^2 - 2x + 4y = 4$
   (f) $4x^2 - 5y^2 - 16x + 10y + 31 = 0$
3. Find the equation of the hyperbola having:
   (a) focus at F(0, 5) and vertex at V(0, –3).
   (b) foci are at F(8, 3) and F(0, 3) and eccentricity, $e = \frac{4}{3}$.
4. Find the equation of the straight lines which are tangents both to the parabola $y^2 = 8x$ and the hyperbola $3x^2 - y^2 = 3$.
5. Show that the line $y = x + 2$ touches to the hyperbola $5x^2 - 9y^2 = 45$. Find the point of contact.
6. Find the value of λ, when the line $y = 2x + \lambda$ is tangent to the hyperbola $3x^2 - y^2 = 3$.
7. Find the equation of the tangents to the hyperbola $3x^2 - 4y^2 = 12$, which are perpendicular to the line $y = x + 2$. Also, find the point of contact.
8. Find the equation of a conic section with focus at (2, 0), directrix $x = 4$ and eccentricity $e = 1$.

Answers

1. (a) $V\left(\pm\frac{1}{\sqrt{3}}, 0\right)$; $e = \sqrt{\frac{5}{3}}$; $F\left(\pm\sqrt{\frac{5}{6}}, 0\right)$

   (b) $V(\pm 3, 0)$; $e = \frac{5}{3}$; $F(\pm 5, 0)$

   (c) $V(1 \pm 4, -2)$; $e = \frac{5}{4}$; $F(1 \pm 5, -2)$

   (d) $V(0, \pm\sqrt{3})$; $e = \sqrt{3}$; $F(0, \pm 3)$

2. (a) $C(2, -3)$; $V(2 \pm 3, -3)$; $F(2 \pm \sqrt{13}, -3)$; $e = \frac{\sqrt{13}}{3}$.

   (b) $C(2, 3)$; $V(2 \pm 2, -3)$; $F(2 \pm \sqrt{13}, -3)$; $e = \frac{\sqrt{13}}{2}$

   (c) $C(-2, 1)$; $V(-2 \pm 2, 1)$; $F(-2 \pm 3, 1)$; $e = \frac{3}{2}$

   (d) $C(2, 0)$; $V(2 \pm 2, 0)$; $F(2 \pm \sqrt{5}, 0)$; $e = 1$

   (e) $C(1, 2)$; $F(1 \pm \sqrt{2}, 2)$; $V(1 \pm 1, 2)$; $e = \sqrt{2}$

   (f) $C(2, 1)$; $V(2, 1 \pm 2)$; $F(2, 1 \pm \sqrt{21})$; $e = 5$

3. (a) $16y^2 - 9x^2 = 144$

   (b) $7x^2 - 9y^2 - 56x + 54y = 81$

4. $y = 2x + 2$ and $y + 2x + 1 = 0$

5. $\left(-\frac{9}{2}, -\frac{5}{2}\right)$

6. $\lambda = \pm 1$

7. $x + y = \pm 1$; $(4, -3)$, $(-4, 3)$

8. $4x + y^2 - 12 = 0$

8

CHAPTER

General Equation of Conic Section

Introduction

A coordinate system represents a point in a plane by an ordered pair of numbers called coordinates. Earlier we used Cartesian coordinates which are directed distances from two perpendicular axes. Now, we describe another coordinate system introduced by Newton called polar coordinates which is more convenient for some special purposes.

8.1 The General Equation of Conic Section in Cartesian Form

Let

$$ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0 \quad \text{... (i)}$$

be a general equation of second degree in x and y.

By rotating the axes with an angle θ then x and y becomes $x\cos\theta - y\sin\theta$ and $x\cos\theta - y\sin\theta$ in new axes so the equation (i) reduces to

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2$$
$$+ 2f(x\cos\theta - y\sin\theta) + 2g(x\sin\theta + y\cos\theta) + c = 0 \quad \text{... (ii)}$$

which is in the form

$$Ax^2 + Hxy + By^2 + Fx + Gy + C = 0 \quad \text{... (iii)}$$

Then clearly,

$$H^2 - 4AB = h^2 - 4ab$$

Our goal is to eliminate the xy cross-term which makes classifying different conics difficult. Thus, we wish to find θ we make the coefficient of xy in (ii) is zero. That is

$$H = 0$$

i.e. $-2a \sin\theta \cos\theta + 2h(\cos^2\theta - \sin^2\theta) + 2b \sin\theta \cos\theta = 0$

$\Rightarrow \quad 2(b-a) \sin\theta \cos\theta + 2h \cos 2\theta = 0$

$\Rightarrow \quad (b-a) \sin 2\theta + 2h \cos 2\theta = 0$

$$\Rightarrow \quad \tan 2\theta = \frac{2h}{b-a} \qquad \text{... (iv)}$$

Whatever be the value of a, b, h, there is always a value $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

Therefore, the equation (iii) becomes

$$Ax^2 + By^2 + Fx + Gy + C = 0 \qquad \text{... (v)}$$

Now, we obtain different conic as value of constants.

Case I: When $A \neq 0$, $B \neq 0$. Then (v) will be in complete square form, so that

$$A\left(x + \frac{F}{A}\right)^2 + B\left(y + \frac{G}{B}\right)^2 = \frac{F^2}{A} + \frac{G^2}{B} - C$$

Shifting the origin to the point $\left(-\frac{F}{A}, -\frac{G}{B}\right)$, the equation becomes

$$Ax^2 + By^2 = D \qquad \text{... (vi)}$$

where $D = \frac{F^2}{A} + \frac{G^2}{B} - C$.

If $D = 0$, then the equation (vi) represents a pair of straight lines.

If $D \neq 0$, the the equation (vi) can be written as

$$\frac{x^2}{D/A} + \frac{y^2}{D/B} = 1.$$

which is an ellipse.

Case II: When $A = 0$ and $B \neq 0$. Then (v) becomes

$$By^2 + 2Fx + 2Gy + C = 0$$

$$\Rightarrow \quad B\left(y + \frac{G}{B}\right)^2 + 2F\left(x - \frac{G^2}{2BF} + \frac{G}{2F}\right) = 0$$

which represents a parabola.

Similarly, if $A \neq 0$, $B = 0$, then we also get the equation (v) represents a parabola.

Case III: When $h^2 - 4ab = -4AB > 0$. If $A = 0$ or $B = 0$ then $h^2 - 4ab = 0$ so that the equation (i) represents a parabola.

When $A \neq 0$ or $B \neq 0$ then $h^2 - 4ab \neq 0$. If $A > 0$, $B > 0$ then (i) represent an ellipse. If A and B have opposite sign then (i) represents a hyperbola.

Summary

The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{... (*)}$$

where,

$$(\Delta = abc + 2fgh - af^2 - bg^2 - ch^2)$$

i. represents a pair of strength lines, if $\Delta = 0$

ii. If $\Delta \neq 0$ then

$ab - h^2 = 0$ then (*) represents a parabola.

$ab - h^2 > 0$ then (*) represents an ellipse.

$ab - h^2 < 0$ then (*) represents a hyperbola.

$a = b \neq 0$, $h = 0$ then (*) represents a circle.

Example 1: Identify the curve represented by the equation,

$$2x^2 + 5xy + 3y^2 - 5x - 7y + 2 = 0.$$

Solution:

Given equation is

$$2x^2 + 5xy + 3y^2 - 5x - 7y + 2 = 0 \qquad \text{... (i)}$$

Comparing (i) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$a = 2,\ b = 5,\ h = 5,\ g = \frac{-5}{2},\ f = \frac{-7}{2} \text{ and } c = 2$$

Now,

$$\begin{aligned}\Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= 20 + \frac{175}{2} - \frac{49}{2} - \frac{125}{4} - 50 \\ &= \frac{7}{4} \neq 0\end{aligned}$$

and $\quad ab - h^2 = 10 - 25 = -15 < 0.$

Since $\Delta \neq 0$ and $ab - h^2 < 0$, so the equation (i) represents a hyperbola.

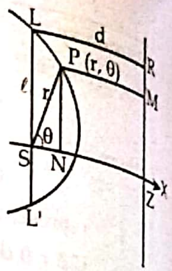
8.2 The Conic Section in Polar Form

We have already defined the conic section in previous sub-topic, which is the locus of a point moving in a plane so that the ratio of its distance from fixed point is to the distance from a fixed line is always constant. The fixed point is called focus and fixed line is directrix. The constant ratio is called eccentricity and denoted by e. The segment of a line through focus and right angle to axis included within the conic section is called latus rectum.

Let S be the focus, ZM be the directrix, ℓ be semi-latus rectum (SL), e be the eccentricity.

Draw SZ⊥ZM and take as initial line, taking S as pole. P (r, θ) be an y point on the conic section.

Draw PN⊥SZ, PM⊥ZM and LR⊥ZM where, LL' = 2ℓ be the latus rectum of the conic section.



Then,

$$SP = e\,PM$$

$$\Rightarrow \quad r = e\,NZ = e(SZ - SN) = e(LR - SN) = e(d - SN)$$

$$\Rightarrow \quad r = e(d - SP\cos\theta) = e(d - r\cos\theta)$$

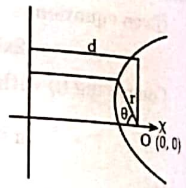
$$\Rightarrow \quad r = \frac{ed}{1 + e\cos\theta}$$

which is the conic with moves horizontally and moves towards left.

Other Forms of Conic

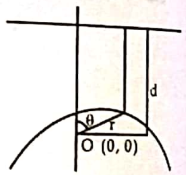
A. If the focus is at right of directrix, then the equation is

$$r = \frac{ed}{1 - e\cos\theta}$$



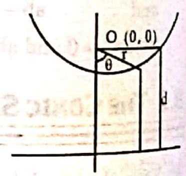
B. If the axis is vertical, then

$$r = \frac{ed}{1 + e\sin\theta}$$



C. If the axis is vertical, then

$$r = \frac{ed}{1 - e\sin\theta}$$



Note: In above equations, if e = 1 then the conic is parabola, if e < 1 then the conic is ellipse and if e > 1 then the conic is hyperbola. And the directrix in each conic is – d.

Example 2: Identify the conic $r = \dfrac{6}{3 + 2\sin\theta}$ with focus at the origin, the line of directrix, and the eccentricity.

Solution:

Given conic is

$$r = \frac{6}{3 + 2\sin\theta} \quad \text{... (i)}$$

$$r = \frac{6}{3 + 2\sin\theta} = \frac{6}{3[1 + (2/3)\sin\theta]} = \frac{2}{1 + (2/3)\sin\theta} \quad \text{... (ii)}$$

Comparing (ii) with $r = \dfrac{ed}{1 + e\sin\theta}$ then we get

$$e = \frac{2}{3} < 1 \text{ and } ed = 2 \Rightarrow d = \frac{2}{e} = 3$$

Here, e < 1 so the conic is ellipse. And, line of directrix is y = –3.

Example 3: Identify the conic $r = \dfrac{15}{4 - 4\cos\theta}$ with focus at the origin, the line of directrix, and the eccentricity.

Solution:

Given equation is

$$r = \frac{15}{4 - 4\cos\theta} = \frac{15}{4(1 - \cos\theta)} = \frac{15/4}{1 - \cos\theta} \quad \text{... (i)}$$

Comparing (i) with $r = \dfrac{ed}{1 - e\cos\theta}$ then we get

$$e = 1 \text{ and } ed = \frac{15}{4} \Rightarrow d = \frac{15}{4}$$

Here e = 1, so the conic is parabola. And the line of directrix is $x = \dfrac{-15}{4}$.

Example 4: Find the polar equation of conic section with focus at pole and given information's: eccentricity, $e = \dfrac{4}{3}$, equation of directrix, x = –3

Solution:

Given that

$$e = \frac{4}{3} > 1 \text{ and directrix is } x = -3, \text{ so } d = 3$$

So the conic is hyperbola.

Now, the equation of the conic section is

$$r = \frac{ed}{1 - e\cos\theta} = \frac{\frac{4}{3}\cdot 3}{1 - \frac{4}{3}\cos\theta} = \frac{12}{3 - 4\cos\theta}$$

Exercise 8.1

1. Identify the given conic with focus at the origin, and find the line of directrix, and the eccentricity.

(a) $r = \dfrac{12}{2 - 6\cos\theta}$ (b) $r = \dfrac{12}{6 + 2\sin\theta}$ (c) $r = \dfrac{3}{2 + 2\cos\theta}$

(d) $r = \dfrac{12}{2 + 6\cos\theta}$ (e) $r = \dfrac{3}{6 - 2\sin\theta}$ (f) $r = \dfrac{6}{2 + 3\sin\theta}$

(g) $r = \dfrac{4}{2 - \cos\theta}$ (h) $r = \dfrac{4\sec\theta}{\sec\theta + 3}$

2. Find the polar equation of conic with focus a pole and having following eccentricity and directrix.

(a) $e = \dfrac{2}{3}, x = 3$ (b) $e = 1, y = 2$ (c) $e = \dfrac{1}{2}, y = 4$

Answers

1. (a) hyperbola; $x = -2$; $e = 3$
(b) ellipse ; $y = 6$; $e = \frac{1}{3}$
(c) Parabola; $x = \frac{3}{2}$; $e = 1$
(d) hyperbola; $x = 2$; $e = 3$
(e) ellipse; $y = \frac{-3}{2}$; $e = \frac{1}{3}$
(f) hyperbola; $y = 2$; $e = \frac{2}{3}$
(g) ellipse; $x = -4$; $e = \frac{1}{2}$
(h) hyperbola; $x = \frac{4}{3}$; $e = 3$

2. (a) $r = \dfrac{6}{3 + 2\cos\theta}$
(b) $r = \dfrac{2}{1 + \sin\theta}$
(c) $r = \dfrac{16}{2 - \sin\theta}$

9 CHAPTER

Review of Coordinate in Space and Plane

Introduction

The coordinate geometry of three dimensions is an extension of the coordinate geometry of the plane. In three dimensional coordinate geometry, the position of a point in space is determined by an ordered triad (x, y, z) with respect to mutually perpendicular straight lines.

9.1 Coordinate in Space

9.1.1 Coordinate Planes

The plane containing Y- and Z-axis is called YZ-plane. Therefore the plane YOZ is YZ-plane, or $x = 0$ plane. Similarly the planes containing Z- and X-axes is called ZX-plane or $y = 0$ plane and the plane containing X- and Y-axes is called XY-plane or $z = 0$ plane.

9.1.2 Distance between Two Points

Let O be the origin and $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the given two points. Then the distance of the point from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Note: The distance of the point P(x, y, z) from the origin O(0, 0, 0) is

$$OP = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = \sqrt{x^2 + y^2 + z^2}$$

Example 1: Show that the points A(0, 7, 10), B(−1, 6, 6) and C(−4, 9, 6) form an isosceles right-angled triangle.

Solution:

We have,

$$AB = \sqrt{(-1-0)^2 + (6-7)^2 + (6-10)^2} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(-4+1)^2 + (9-6)^2 + (6-6)^2} = \sqrt{18} = 3\sqrt{2}$$

and $$CA = \sqrt{(-4-0)^2 + (9-7)^2 + (6-10)^2} = \sqrt{36} = 6$$

Clearly, $AB = BC$ and $AB^2 + B^2C = 18 + 18 = 36 = AC^2$

Hence, triangle ABC is an isosceles right-angled triangle.

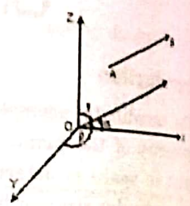
9.1.3 Direction Cosines of a Line

Let α, β, γ be the angles which a given line makes with the positive directions of coordinate axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the given line and are generally denoted by ℓ, m, n respectively.

Thus, $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

Note: Clearly, the direction cosines of x-axis are 1, 0, 0; of y-axis are 0, 1, 0 and of z axis are 0, 0, 1.

9.1.4 Relation between Direction Cosines of a Line



Let α, β and γ are the angles made by AB with axes of coordinates then $\ell = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$ are the direction cosines of line AB. Let OP is a line parallel to AB drawn through origin and OP = r. Hence OP makes same angles with axes of co-ordinates as the line AB.

Draw PM⊥OY, then from right angled triangle OPM,

$$\cos\beta = \frac{OM}{OP} = \frac{y}{r} \quad \text{i.e. } y = rm$$

Similarly, $x = r\cos\alpha = r\ell$ and $z = r\cos\gamma = rn$

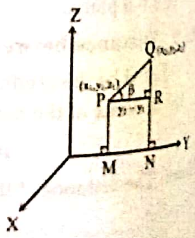
Now,

$$OP^2 = x^2 + y^2 + z^2$$

or, $$r^2 = r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$\Rightarrow \ell^2 + m^2 + n^2 = 1$$

9.1.5 Direction Ratios of a Line



Three numbers a, b, c which are proportional to the direction cosines ℓ, m, n respectively of a line are called ***direction ratios*** of the line.

Then,

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ (say)}$$

$$\Rightarrow \ell = ak,\ m = bk,\ n = ck$$

Since,

$$\ell^2 + m^2 + n^2 = 1$$

i.e. $$k^2(a^2 + b^2 + c^2) = 1$$

$$\Rightarrow k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

Hence,

$$\ell = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

If ℓ, m, n be the direction cosines of a line PQ joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. If α, β, γ be the angles made by PQ with the axes, then, $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. Draw PM and QN perpendicular to y-axis and draw PR perpendicular QN, so that $\angle RPQ = \beta$.

Then

$$PR = MN = ON - OM = y_2 - y_1$$

From right angled triangle PQR,

$$m = \cos\beta = \frac{PR}{PQ} = \frac{y_2 - y_1}{PQ}$$

Similarly,

$$\ell = \frac{x_2 - x_1}{PQ} \text{ and } n = \frac{z_2 - z_1}{PQ}$$

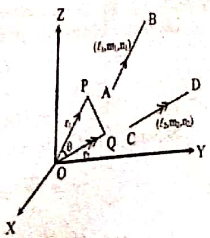
Hence, the direction cosines of PQ are

$$\frac{x_2 - x_1}{PQ},\ \frac{y_2 - y_1}{PQ},\ \frac{z_2 - z_1}{PQ}$$

As the direction cosines of PQ are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ the direction ratios of the line are

$$x_2 - x_1,\ y_2 - y_1 \text{ and } z_2 - z_1.$$

9.1.6 Angle between the Straight Lines



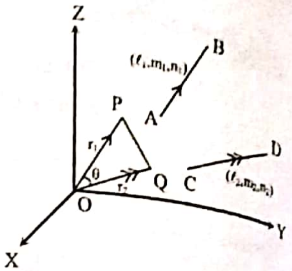
Let AB and CD are two given straight lines with direction cosines, ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 and θ be the angle between them. Through O, draw OP and OQ parallel to AB and CD respectively then $\angle POQ = \theta$. If the co-ordinates of P and Q are $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and $OP = r_1$, and $OQ = r_2$, then,

$$x_1 = r_1\ell_1,\ y_1 = r_1m_1,\ z_1 = r_1n_1$$

$$x_2 = r_2\ell_2,\ y_2 = r_2m_2,\ z_2 = r_2n_2$$

Then

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$
$$= x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2)$$
$$= r_1^2 + r_2^2 - 2r_1r_2(\ell_1\ell_2 + m_1m_2 + n_1n_2)$$



From ΔPOQ,

$$\cos\theta = \frac{OP^2 + OQ^2 - PQ^2}{2.OP.OQ}$$

i.e. $$\cos\theta = \frac{r_1^2 + r_2^2 - r_1^2 - r_2^2 + 2r_1r_2(\ell_1\ell_2 + m_1m_2 + n_1n_2)}{2r_1r_2}$$

$$\Rightarrow \quad \cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$$

Note:

1. Condition of Perpendicularity:

   The given two lines are perpendicular to each other. Then,

   $$\ell_1\ell_2 + n_1n_2 + n_1n_2 = 0$$

   or, $$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

2. Condition of Parallelism:

   The given two straight lines are parallel to each other. Then,

   $$\ell_1 = \ell_2,\ m_1 = m_2 \text{ and } n_1 = n_2$$

   or, $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

3. If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two straight lines and θ be the angle between them

   $$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}$$

Example 2: Find the direction cosines of the line which is perpendicular to the lines whose direction cosines are proportional to 1, –1, 2 and 2, 1, –1.

Solution:

Let ℓ, m, n be the direction cosines of the required line, then

$$\ell.1 + m.(-1) + n.2 = 0 \text{ and } \ell.2 + m.1 + n.(-1) = 0$$

i.e. $$\ell - m + 2n = 0 \quad \text{...... (1)}$$

and $$2\ell + m - n = 0 \quad \text{...... (2)}$$

Solving (1) and (2) by cross multiplication method, we get

$$\frac{\ell}{-1} = \frac{m}{5} = \frac{n}{3} = \frac{\sqrt{\ell^2 + m^2 + n^2}}{\sqrt{(-1)^2 + 5^2 + 3^2}} = \frac{1}{\sqrt{35}}$$

$$\Rightarrow \quad \ell = \frac{-1}{\sqrt{35}},\ m = \frac{5}{\sqrt{35}},\ n = \frac{3}{\sqrt{35}}$$

Example 3: If A(6, –6, 0), B(–1, –7, 6), C(3, –4, 4) and D(2, –9, 2) be four points in space, then show that the line AB perpendicular to the line CD.

Solution:

Given points are

$$A(6, -6, 0),\ B(-1, -7, 6),\ C(3, -4, 4) \text{ and } D(2, -9, 2)$$

Then the direction ratios of line AB are

$$(a_1, b_1, c_1) = (-1 - 6, -7 + 6, 6 - 0) = (-7, -1, 6)$$

And, the direction ratios of line CD are

$$(a_2, b_2, c_2) = (2 - 3, -9 + 4, 2 - 4) = (-1, -5, -2)$$

If θ be the angle between the lines AB and CD is

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here,

$$a_1a_2 + b_1b_2 + c_1c_2 = (-7)(-1) + (-1)(-5) + 6(-2) = 0$$

$$\Rightarrow \quad \cos\theta = \cos 90°$$

$$\Rightarrow \quad \theta = 90°$$

This means the line AB is perpendicular to CD.

9.2 REVIEW OF PLANE

Introduction

A plane is defined by a surface such that the line joining any two points on the surface lies wholly in the surface.

The general equation of first degree in x, y and z is

$$ax + by + cz = d \quad \text{...... (1)}$$

where a, b, c and d are constants and are not simultaneously zero. Then (1) is the plane.

Thus, the linear equation in first degree of three dimensional variables represents a plane.

9.2.1 Equation of Plane through One Point

Let $A(x_1, y_1, z_1)$ be a given point. Let the plane containing this point be

$$ax + by + cz + d = 0 \quad \text{...... (1)}$$

Since $A(x_1, y_1, z_1)$ lies on it, therefore

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{...... (2)}$$

Thus from (1) and (2), we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

This is the equation of plane passing through a given point (x_1, y_1, z_1) and having direction ratios a, b, c.

Example 4: Find the equation to the plane through the point $(-1, 3, 2)$ and perpendicular to the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$.

Solution:

Any plane through $(-1, 3, 2)$ is

$$a(x+1) + b(y-3) + c(z-2) = 0 \quad \text{... ... (1)}$$

Since (1) is perpendicular to $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$ therefore

$$a + 2b + 2c = 0 \quad \text{... ... (2)}$$

$$3a + 3b + 2c = 0 \quad \text{... ... (3)}$$

Solving (2) and (3), we get

$$\frac{a}{4-6} = \frac{b}{6-2} = \frac{c}{3-6}$$

$$\Rightarrow \quad \frac{a}{2} = \frac{b}{-4} = \frac{c}{3}$$

Putting the values of a, b, c in (1) then

$$2(x+1) - 4(y-3) + 3(z-2) = 0$$

$$\Rightarrow \quad 2x - 4y + 3z + 8 = 0$$

Example 5: Find the equation of plane through (1, 2, 3) and parallel to $4x + 5y - 3z = 7$.

Solution:

Any plane passing through (1, 2, 3) is

$$a(x-1) + b(y-2) + c(z-3) = 0 \quad \text{... ... (1)}$$

As given, the plane (1) is parallel to $4x + 5y - 3z = 7$, then

$$\frac{a}{4} = \frac{b}{5} = \frac{c}{-3} = k \text{ (say)}$$

Substituting the values of a, b, c in (1), we get

$$4k(x-1) + 5k(y-2) - 3k(z-3) = 0$$

$$\Rightarrow \quad 4(x-1) + 5(y-2) - 3(z-3) = 0$$

$$\Rightarrow \quad 4x + 5y - 3z - 5 = 0$$

This is the equation of plane.

Example 6: Find the equation of the plane containing the lines through the origin with direction cosines proportional to (2, 1, −2) and (5, 2, −3).

Solution:

Let ℓ, m, n be the direction cosines of the normal to the required plane. Then the equation of the plane passing through the origin is

$$\ell x + my + nz = 0 \quad \text{... ... (1)}$$

Since the normal to the plane is perpendicular to the lines in the plane, therefore

$$2\ell + m - 2n = 0 \quad \text{... ... (2)}$$

$$5\ell + 2m - 3n = 0 \quad \text{... ... (3)}$$

Solving (2) and (3), we get

$$\frac{\ell}{-3+4} = \frac{m}{-10+6} = \frac{n}{4-5} = k$$

$$\Rightarrow \quad \ell = k,\ m = -4k,\ n = -k$$

Then the equation (1) gives

$$kx - 4ky - kz = 0$$

$$\Rightarrow \quad x - 4y - z = 0$$

This is the equation of the required plane.

Example 7: Find the direction cosines of the line normal to the plane $6x - 3y + 2z = 14$.

Solution:

Given plane is,

$$6x - 3y + 2z = 14 \quad \text{...... (1)}$$

Comparing (1) with $ax + by + cz = d$ then we get

$$a = 6,\ b = -3,\ c = 2 \text{ and } d = 14.$$

Then the direction cosines of the line normal to (1) are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} \Rightarrow l = \frac{6}{\sqrt{36+9+4}} = \frac{6}{7}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} \Rightarrow m = \frac{-3}{\sqrt{36+9+4}} = -\frac{3}{7}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} \Rightarrow n = \frac{2}{\sqrt{36+9+4}} = \frac{2}{7}$$

9.2.2 Equation of Plane through given three non-coplanar Points

Suppose a plane passes through three non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$.

Let the equation of the plane is

$$ax + by + cz + d = 0 \qquad \ldots\ldots (1)$$

Since the plane (1) passes through the points A, B and C, therefore

$$ax_1 + by_1 + cz_1 + d = 0 \qquad \ldots\ldots (2)$$

$$ax_2 + by_2 + cz_2 + d = 0 \qquad \ldots\ldots (3)$$

$$ax_3 + by_3 + cz_3 + d = 0 \qquad \ldots\ldots (4)$$

Eliminating a, b, c and d from (1), (2), (3) and (4), we get

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

This is the required equation of the plane.

Note: The four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) will be co-planer if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & x_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Example 8: Find the equation of plane through the points (4, 5, 1), (3, 9, 4), (–4, 4, 4).

Solution:

The equation of plane through (4, 5, 1), (3, 9, 4), (–4, 4, 4) is

$$\begin{vmatrix} x & y & z & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-4 & y-5 & z-1 & 0 \\ 4 & 5 & 1 & 1 \\ -1 & 4 & -3 & 0 \\ -8 & -1 & -3 & 0 \end{vmatrix} = 0 \qquad \left[\begin{array}{c} \text{Applying} \\ R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \\ R_1 \to R_4 - R_2 \end{array}\right]$$

$$\Rightarrow (-1)^{2+4}(1)\begin{vmatrix} x-4 & y-5 & z-1 \\ -1 & 4 & -3 \\ -8 & -1 & -3 \end{vmatrix} = 0$$

$$\Rightarrow (x-4)(-12-3) - (y-5)(3-24) + (z-1)(1+32) = 0$$

$$\Rightarrow (x-4)(-15) - (y-5)(-21) + (z-1)(33) = 0$$

$$\Rightarrow -15x + 21y + 33z + 60 - 105 - 33 = 0$$

$$\Rightarrow 5x - 7y + 11z + 4 = 0$$

This is the equation of the required plane.

Example 9: Show that the points (1, 3, –1), (1, 1, 0), (2, 5, 4) and (2, 7, 3) are coplanar.

Solution:

Given points are (1, 3, – 1), (1, 1, 0), (2, 5, 4) and (2, 7, 3).

Here,

$$\begin{vmatrix} 1 & 3 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 5 & 4 & 1 \\ 2 & 7 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & -2 & 1 & 0 \\ 1 & 2 & 5 & 0 \\ 1 & 4 & 4 & 0 \end{vmatrix} \qquad \left[\because \begin{array}{c} \text{Applying} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}\right]$$

$$= (-1)^{4+1}(1)\begin{vmatrix} 0 & -2 & 1 \\ 1 & 2 & 5 \\ 1 & 4 & 4 \end{vmatrix}$$

$$= (-1)\,[0 - (-2)(4-5) + 1(4-2)]$$

$$= (-1)\,[-2+2]$$

$$= 0$$

This shows that the given points are coplanar.

9.2.3 Angle between Two Planes

Let the equation of the two planes are

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad \ldots\ldots (1)$$

$$a_2x + b_2y + c_2z + d_2 = 0 \qquad \ldots\ldots (2)$$

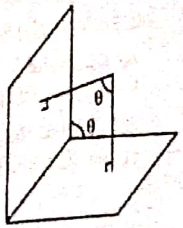
The direction cosines of the line normal to the plane (1) are proportional to a_1, b_1, c_1 and its direction cosines are

$$\frac{a_1}{\sqrt{a_1^2}}, \frac{b_1}{\sqrt{a_1^2}}, \frac{c_1}{\sqrt{a_1^2}}$$

Similarly, the direction cosines of the line normal to the plane (2) are proportional to a_2, b_2, c_2 and its direction cosines are

$$\frac{a_2}{\sqrt{a_2^2}}, \frac{b_2}{\sqrt{a_2^2}}, \frac{c_2}{\sqrt{\Sigma a_2^2}}$$

If θ be the angle between two planes (1) and (2), then the angle between the normal to plane (1) and (2) is also θ, so



$$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$$

$$\text{i.e.}\quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2}\sqrt{a_2^2}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left[\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2}\sqrt{a_2^2}}\right]$$

Note:

1. Two planes (1) and (2) are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

2. Two planes (1) and (2) are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Example 10: Find the angle between the pair of planes $x + 3y + 5z = 0$ and $x - 2y + z = 20$.

Solution:

Given planes are

$$x + 3y + 5z = 0 \quad \text{...... (1)}$$

and

$$x - 2y + z = 20 \quad \text{...... (2)}$$

Clearly, the direction ratios of normal of (1) and (2) are respectively,

$$a_1 = 1, b_1 = 3, c_1 = 5$$

and

$$a_2 = 1, b_2 = -2, c_2 = 1.$$

Let θ be the angle between two planes (1) and (2) then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

i.e.
$$\cos\theta = \frac{1 - 6 + 5}{\sqrt{1 + 9 + 25}\sqrt{1 + 4 + 1}} = \frac{0}{\sqrt{35}\sqrt{6}} = 0 = \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

Thus the angle between (1) and (2) is $\theta = \frac{\pi}{2}$.

9.2.4 Plane passing through the Intersection of Two given Planes

Let the two given planes be

$$a_1x + b_1y + c_1z = d_1 \quad \text{... (1)}$$

$$a_2x + b_2y + c_2z = d_2 \quad \text{... (2)}$$

Then consider the equation

$$(a_1x + b_1y + c_1z - d_1) + k(a_2x + b_2y + c_2z - d_2) = 0 \quad \text{... (3)}$$

for some value k.

This being the first degree equation in x, y and z, represents a plane. Let (x_1, y_1, z_1) be the point on the line of the intersection of planes given by equations (1) and (2). Then (x_1, y_1, z_1) lies on the two given planes,

$$a_1x_1 + b_1y_1 + c_1z_1 = d_1$$

$$a_2x_1 + b_2y_1 + c_2z_1 = d_2$$

From this equation, we infer that the point (x_1, y_1, z_1) lies on the plane given. Similarly, every point in the line of intersection of the planes (1) and (2) lie on the planes (1) and (2). Hence, equation (3) is the plane passing through the intersection of the two given planes.

Example 11: Find the equation of plane through the intersection of $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and is passing through (0, 0, 0).

Solution:

The equation of plane through the intersection of $x + 2y + 3y + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ is,

$$(x + 2y + 3y + 4) + k(4x + 3y + 2z + 1) = 0 \quad \text{...... (1)}$$

where k is some scalar value.

Given that the plane (1) passes through (0, 0, 0). So,

$$4 + k = 0$$

$$\Rightarrow \quad k = -4$$

Then (1) becomes,

$$(x + 2y + 3y + 4) - 4(4x + 3y + 2z + 1) = 0$$

$$\Rightarrow \quad -15x - 10y - 5z = 0$$

$$\Rightarrow \quad 3x + 2y + z = 0 \quad [\because -5 \neq 0]$$

This is the equation of required plane.

Exercise 9

1. Find direction cosines of the normal to the plane $2x - y + 2z = 6$.
2. Find the direction cosines of the perpendicular to the plane $3x - 4y + 5z = 7$.
3. Find the equation of the plane passing through the origin and containing the line joining the points (1, 1, 1) and (3, 4, –5).
4. Find the equation of the plane containing the point (1, –1, 2) and is perpendicular to the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.
5. Find the equation of the plane through P(1, 4, –2) at right angle to OP.
6. Obtain the equation of the plane, which passes through the point (–1, 3, 2) and is perpendicular to each of the two planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$.
7. Find the equation of the plane through three points (1, 1, 1), (1, –1, 1), (–7, –3, –5).
8. Find the equation of the plane through the three points (1, 1, 0), (1, 2, 1) and (–2, 2, –1).

9. Find the equation of the plane through the three points (1, 1, 1), (1, –1, 1), (–7, 3, –5) and show that it is perpendicular to xz–plane.
10. Show that the points (1, 3, –1), (1, 1, 0), (2, 5, 4) and (2, 7, 3) are coplanar.
11. Show that the points (–1, 2, 5), (1, 2, 3) and (3, 2, 1) are collinear.
12. Find the equation of the plane through the points (–1, 1, –1), (6, 2, 1) and normal to the plane $2x + y + z = 5$.
13. Find the angle between the following pair of planes:
    a. $x - 2y - z = 5$ and $2x - y + z = 2$.
    b. $2x + 3y + 5z = 0$ and $x - 2y + z = 20$.
14. Find the angle between the planes $2x + 3y - z = 12$ and $x + 2y - z = 14$.
15. Find the equation of the plane through the intersection of the planes $x + 2y - 3z = 5$ and $5x + 7y + 3z = 10$ through (–1, 2, –3).
16. Find the equation of plane through the intersection of $2x + 3y + 10z = 8$, and $2x - 3y + 7z = 2$; and normal to the plane $3x - 2y + 4z = 5$.

Answers

1. $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$
2. $\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$
3. $9x - 8y - z = 0$
4. $5x - 4y - z = 7$
5. $x + 4y - 2z = 21$
6. $2x - 4y + 3z + 8 = 0$
7. $3x - 4z + 1 = 0$
8. $2x + 3y - 3z = 5$
9. $3x - 4z + 1 = 0$
12. $x + 3y - 5z = 7$
13. a. $\frac{\pi}{3}$ b. $\frac{\pi}{2}$
14. $\cos^{-1}\left(\frac{5}{\sqrt{84}}\right)$
15. $3x - 4y + 1 = 0$
16. $2y + z = 2$