2

CHAPTER

VECTOR SPACE

Introduction

Linear algebra is the study of vector spaces and the function of vector space (linear transformation). One we define a vector space. We will study the properties of vector spaces. Their importance lies in the fact that many mathematical questions can be rephrased as a question about vector spaces. Thus, each fact that we can prove about vector spaces gives us corresponding information about many different mathematical questions.

2.1 VECTORS AND VECTOR SPACES IN R^2 AND R^3

Consider a non-empty set V of vectors where each vector has the same number of components then the set V is said to be vector space if it satisfy the following laws:

(a) Commutative law: $v_1 + v_2 = v_2 + v_1$ for all $v_1. v_2 \in V$

(b) Associative law: $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$, for all $v_1, v_2, v_3 \in V$.

(c) Existence of additive identity: For all $v_1 \in V$, there exists an element $0 \in V$ (called additive identify) such that

$$0 + v_1 = v_1 + 0 = v_1.$$

(d) Existence of additive inverse: For all $v_1 \in V$, there exists $-v_1 \in V$, (called additive inverse of v_1) such that

$$v_1 + (-v_1) = (-v_1) + v_1 = 0 \text{ for all } v_1 \in V.$$

(e) Associative law of vector multiplication by scalar

$$(ab)v_1 = a(bv_1) = b(av_1) \quad \text{for all } a, b \in K, v_1 \in V.$$

(f) Distributive law

(i) $(a + b)v_1 = av_1 + bv_1$

(ii) $a(v_1 + v_2) = av_1 + av_2$, for all $v_1, v_2 \in V$; $a, b \in K$

(g) Existence of multiplicative identity: $1 \in K$ such that

$$1.v_1 = v_1.1 = v_1 \qquad \text{for all } v_1 \in V.$$

The discussion of vectors in plane can now be extended to a discussion of vectors in n-space. A vector in n-space is represented by an ordered n-tuple $(x_1, x_2, \ldots, x_n)$. The set of all ordered n-tuples is called the n-space and is denoted by R^n. So,

1. $R = R^1$ space = set of all real numbers.
2. R^2 = set of all ordered pairs (x_1, x_2) of real numbers.
3. R^3 = set of all ordered triples (x_1, x_2, x_3) of real numbers.

Vector Subspace

Let V be a vector space over the field K. Then a non empty subset W of V is called a subspace of V if W itself is a vector space.

In other word, let V be a vector space over the field K. Then a non-empty subset W of V is called a subspace of V if W satisfies the conditions:

(i) $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$.

(ii) $aw_1 \in W$ for all $w_1 \in W$, $a \in K$.

(iii) $0 \in W$.

Moreover, the single equivalent condition with the above three condition of a vector subspace is $aw_1 + bw_2 \in W$ for all $a, b \in K$ and $w_1, w_2 \in W$. For $a = b = 1$, the first condition holds, if $b = 0$ then the second condition holds. Similarly, if $a = b = 0$, then third condition holds.

Example 1: If $W = \{(x, y, z) : x + y + z = 0\}$ show that it is a subspace of vector space R^3 over the field of real numbers R.

Solution:

Let,

$$W = \{(x, y, z) : x + y + z = 0\}.$$

Then clearly W is subset of R^3 being W is a three dimensional.

Let,

$$u = (x_1, y_1, z_1) \text{ and } v = (x_2, y_2, z_2)$$

Then $u, v \in W$.

Set $x = y = z = 0$ then $0 + 0 + 0 = 0$. So, $(0, 0, 0) \in W$. Therefore $W \neq \varnothing$.

And,

$$x_1 + y_1 + z_1 = 0$$

and $$x_2 + y_2 + z_2 = 0$$

Now,

$$au + bv = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \text{ ...... (i)}$$

Here,

$$(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2) = a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2)$$
$$= a.0 + b.0$$
$$= 0$$

This shows that

$$(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2) \in W.$$

This implies $au + bv \in W$

This means W is a subspace in R^3.

Example 2: The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as $\{0\}$. It is also called trivial subspace of V.

Example 3: Let $V = R^3 = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix}, x, y, z \in R\right\}$ is a vector space over R and the set $W = \left\{\begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \text{ are real}\right\}$ is a subspace of V.

Solution:

(a) Taking, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, is zero element in W. so, $W \neq \varnothing$.

(b) For all $\alpha, \beta \in R$ and $w_1 = \begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix} \in W$ then

$$\alpha w_1 + \beta w_2 = \begin{bmatrix} as_1 + bs_2 \\ at_1 + bt_2 \\ 0 \end{bmatrix} \in W$$

Hence, W is a subspace of V.

Example 4: The vector space R^2 is not a subspace of R^3 because R^2 is not subset of R^3.

Example 5: Let $W = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0\right\}$ prove that W is not subspace of R^2 by showing that it is not closed under scalar multiplication.

Solution:

Since $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in W$ and $c = -1$ then

$$cu = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin W$$

$\Rightarrow$ W is not subspace of R^2.

2.2 Linearly Independent Vectors

Here we with vectors and their combination as well as linearly independence of the vectors. A linear combination of vectors $\{v_1, v_2, \ldots, v_p\}$ in a vector space V is an expression

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p$$

where $c_1, c_2, \ldots, c_p$ are scalars.

An indexed set of vectors $\{v_1, v_2, \ldots, v_p\}$ in V is said to be linearly independent if the vector equation.

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0 \qquad \ldots \text{(i)}$$

has only the trivial solution, i.e. $c_1 = 0, c_2 = 0, \ldots, c_p = 0$.

The set $\{v_1, v_2, \ldots, v_p\}$ is said to be linearly dependent if

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0 \qquad \ldots \text{(ii)}$$

has non-trivial solution. So there are some weights (which we called scalars) $c_1, c_2, \ldots, c_p$ not all zero such that (ii) hold.

Moreover, a set containing a single vector v is linearly independent if $v \neq 0$. Also, a set of two vectors is linearly dependent of one can be expressed as a multiple of other.

2.3 Rank of Matrix in Terms of Linearly Independent Column Vectors

A common approach to finding the rank of a matrix is to reduce it to a simpler form, generally row echelon form, by elementary row operations. Row operations do not change the row space. Once in row echelon form, the rank is clearly the same for both row rank and column rank, and equals the number of pivots (or basic columns) and also the number of non-zero rows.

In Matrix form, if the number of variables (vectors) is same as the rank of the matrix developed by the vectors then we called the vectors are linearly independent.

Example 6: Prove that the set of vectors (3, 0, –1), (0, 1, 2), (1, –1, 1) are linearly independent in R^3.

Solution: The matrix form of the given vectors v_1, v_2, v_3 is

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$$

For linearly independent, set $Ax = 0$ then

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -8 & 0 \end{bmatrix} \quad (\text{applying } R_3 \to 3R_3 + R_1)$$

which is the echelon form of the matrix.

Given vectors are 3 and the echelon form has 3 pivots (1 in first column, 1 in second column and –8 in third column), so vectors v_1, v_2, v_3 are linearly independent.

Also, the rank of the matrix is 3.

Example 7: Check whether the following vectors are linearly independent or not: (1, –1, 1), (1, 1, –1), (–1, 1, 1), (0, 1, 0).

Solution:

Given vectors are,

$$(1, -1, 1), (1, 1, -1), (-1, 1, 1), (0, 1, 0)$$

For linear independent set $Ax = 0$ then

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \end{bmatrix} \begin{pmatrix} \text{applying} \\ R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{pmatrix} \text{applying} \\ R_3 \to R_3 + R_2 \end{pmatrix}$$

which is the echelon form of the matrix.

Given vectors are 4 and the echelon form has 3 pivots (1 in first column, 2 in second column and 2 in third column), so vectors are linearly dependent.

Also, the rank of the matrix is 3.

2.4 Basis for a Vector Space

If any vector in V can be expressed as a linear combination of the vectors in the set, we also say that the set of vectors **span** the vector space V. A linearly independent set in V consisting of a maximum possible number of vectors in V is called a **basis for V**.

Definition (Spanning Set of a Subspace)

Let V be a vector space and W be a subspace defined by

$$W = \{a_1v_1 + a_2v_2 + \dots + a_nv_n : a_i \in R\}$$

and $v_1, v_2, \dots, v_n \in V$. Then the set $\{v_1, v_2, \dots, v_n\}$ is called spanning set of W.

Definition (Basis)

Let H be a subspace of a vector space V. An indexed set vectors $B = \{b_1, b_2, \dots, b_p\}$ in V is a **basis for** H. If

(i) the set $\{b_1, b_2, \dots, b_p\}$ is linearly independent,

(ii) $H = \text{Span}\{b_1, b_2, \dots, b_p\}$.

The maximum number of linearly independent vectors in V is called the **dimension** of V and is denoted by dim V. If the number of vectors of a basis for V equals dim V.

In addition, the given vectors are linearly independent and number of vectors is same as the dimV then the vectors form a basis for that vector space.

Example 8: Check the vectors (1, 1, 1), (1, 3, 2), (–1, 0, 1) forms a basis of R^3 or not.

Solution:

Let $V = R^3 = \{(x, y, z): x, y, z \in R\}$ be a vector space over a field F.

Then we wish to show (1, 1, 1), (1, 3, 2), (–1, 0, 1) form a basis for $V = R^3$.

Then,

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{vmatrix}$$

$$\sim \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{vmatrix} \begin{pmatrix} \text{applying} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{vmatrix} \begin{pmatrix} \text{applying} \\ R_3 \to R_3 - 2R_2 \end{pmatrix}$$

which is the echelon form of the matrix.

Given vectors are 3 and the echelon form has 3 pivots (1 in first column, 2 in second column and –3 in third column), so vectors are linearly independent.

So, the dimension of the vector space is 3.

Also, the number of vectors is same as the dimension of the vector space, so the vectors span the vector space. Therefore, the vectors form a basis for V.

Example 9: Check the vectors (1, 1, 0), (1, 0, 1) forms a basis of R^3 or not.

Solution:

Let $V = R^3 = \{(x, y, z): x, y, z \in R\}$ be a vector space over a field F.

Then we wish to show (1, 1, 0), (1, 0, 1) form a basis for $V = R^3$.

Then,

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$\sim \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix} \begin{pmatrix} \text{applying} \\ R_2 \to R_2 - R_1 \end{pmatrix}$$

$$\sim \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{pmatrix} \text{applying} \\ R_3 \to R_3 + R_2 \end{pmatrix}$$

which is the echelon form of the matrix.

Clearly, the R_3 has no pivot value, so the column vectors of the matrix are linearly dependent.

Therefore, the vectors does not form a basis for V.

Exercise 2.1

1. a. Let $V = R^2$ be a vector space. Show that $W = \{(x, y): x + 2y = 0\}$ is a vector subspace of V.

   b. Let $V = R^3$ be a vector space. Show that $W = \{(x, y, z): x + 2y + z = 0\}$ is a vector subspace of V.

c. Let $V = R^3$ be a vector space. Show that $W = \{(x, y, z) : 2x + y + 2z = 0\}$ is a vector subspace of V.

d. Let $V = R^3$ be a vector space. Show that $W = \{(x, 0, z): x, z \in R\}$ is a vector subspace of V.

e. Let $V = R^3$ be a vector space. Show that $W = \{(0, y, z): y, z \in R\}$ is a vector subspace of V.

f. Let V = set of all 2×2 matrices, be a vector space. Let $W = \left\{\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in R\right\}$

Show that W is vector subspace of V.

g. Let V = set of all 2×2 matrices, be a vector space. Let $W = \left\{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} : c, d \in R\right\}$

Show that W is vector subspace of V.

h. If $W = \{(x, y, z) : x + y + z = 0\}$, show that it is a subspace of vector space R^3 over the field of real numbers R.

2. Are the following sets of vectors linearly independent or dependent?

a. (1, 0, 0), (1, 1, 0), (1, 1, 1)
b. (–1, 5, 0), (16, 8, –3), (– 64, 56, 9)
c. (2, – 4), (1, 9), (3, 5)
d. (1, 9, 9, 8), (2, 0, 0, 3), (2, 0, 8)
e. (1, 2), (1, 3)
f. (1, 1, 1), (1, –1, 0), (0, 1, 1)
g. (2, 3, 5), (4, 9, 11)
h. (1, 1, 2), (3, 1, 2), (0, 1, 4)
i. (1, 0, 0), (0, 1, 0), (0, 0, 1)
j. (1, 0, 1), (1, 1, 0), (–1, 0, –1)
k. (1, 2, –1), (2, 3, 0), (0, 0, 0)

3. Check which of the following forms a basis of R^3.

a. (1, 1, 1), (1, 3, 2), (–1, 0, 1)
b. (1, 2, 1), (2, 1, 0), (1, –1, 2)
c. (1, 1, 0), (1, 0, 1)
d. (1, 1, 0), (0, 1, 0), (0, 0, 1), (2, 3, 4)
e. (0, 1, 0), (0, 0, 1), (1, 1, 1)
f. (2, 1, 1), (3, –2, 2), (–1, 2, –1)
g. (1, 2, 0), (0, 3, 1) and (–1, 0, 1)
h. (1, –2, 3, 4), (–2, 4, –1, –3), (–1, 2, 7, 6)

Answers

2. a. linearly independent
b. linearly independent
c. linearly dependent
d. linearly independent
e. linearly independent
f. linearly independent
g. linearly independent
h. linearly independent
i. linearly independent
j. linearly dependent
k. linearly dependent

3. a. basis
b. basis
c. not a basis
d. not a basis
e. basis
f. basis
g. basis
h. not a basis

2.5 Linear Transformation

We have already studied about the equation Ax = b. This shows that every value of x associates with A that gives the value b of particular Ax. Such association is a transformation.

This concept generalizes the following definition.

Definition (Transformation or Function or Mapping)

A transformation T from R^n to R^m (noted as $T: R^n \to R^m$) is a rule that assigns each vector x in R^n to a vector T(x) in R^m. In such condition, R^n is domain of T and R^m is co-domain of T.

Let X and Y be any vector spaces. To each vector x in X we assign a unique vector y in Y. Then we say that a mapping (or transformation or operator) of X into Y is given. Such a mapping is denoted by a capital letter, say T.

Definition (Linear Transformation)

Let T:X→Y be a transformation, is called linear if for all vectors u and v in X and scalars c,

$$T(u + v) = T(u) + T(v)$$
$$T(cu) = cT(u)$$

The single equivalent condition for linearity of T is for all α, β ∈ F, u, v in domain of T, is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

Note: If $T : R^n \to R^m$ is a linear then T(0) = 0, if T(0) ≠ 0 then T is not linear.

Example 10: Show that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is not linear.

Solution:

Let T is a transformation, defined by

$$T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2).$$

Now,

$$\begin{aligned} T(u + v) &= T(u_1 + v_1, u_2 + v_2) \\ &= (2(u_1 + v_1) - 3(u_2 + v_2), (u_1 + v_1) + 4, 5(u_2 + v_2)) \\ &= (2u_1 + 2v_1 - 3u_2 - 3v_2, u_1 + v_1 + 4, 5u_2 + 5v_2) \end{aligned}$$

and,

$$\begin{aligned} T(u) + T(v) &= T(u_1, u_2) + T(v_1, v_2) \\ &= (2u_1 - 3u_2, u_1 + 4, 5u_2) + (2v_1 - 3v_2, v_1 + 4, 5v_2) \\ &= (2u_1 + 2v_1 - 3u_2 - 3v_2, u_1 + v_1 + 8, 5u_2 + 5v_2) \\ &\neq T(u, v) \end{aligned}$$

This implies that T is not a linear transformation.

Contraction and Dilation:

A transformation $T: R^2 \to R^2$ defined by $T(x) = rx$ for some scalar r. Then T is called contraction when $0 \le r \le 1$ and T is called dilation when $r > 1$.

Example 11: Prove that contraction map is linear transformation.

Solution:

We know that map $T : R^2 \to R^2$ defined by $T(x) = rx$, where $0 \le r \le 1$ is called contraction map.

Let $u, v \in R^2$ and c and d are scalar. Then

$$\begin{aligned} T(cu + dv) &= r(cu + dv) \\ &= rcu + rdv \\ &= c(ru) + d(rv) \\ &= cT(u) + dT(v) \end{aligned}$$

This means the transformation T is linear.

2.6 Transpose of a Matrix

The transpose of a matrix A is the matrix obtained by interchanging the rows into columns or the columns into rows. It is denoted by A' or A^T or A^t.

That is,

$$\text{if } A = (a_{ij})_{m\times n} \text{ then } A^T = (a_{ji})_{n\times m}$$

Example 12: If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Some Facts

For two matrices A and B,

1. $(A + B)^T = A^T + B^T$.
2. $(AB)^T = B^T A^T$.

Symmetric Matrix

A square matrix A is called symmetric if $A = A^T$. That means a matrix $A = (a_{ij})_{n\times n}$ is symmetric if $a_{ij} = a_{ji}$ for all i, j.

Example 13: If $A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} = A$. So, A is a symmetric matrix.

Skew-symmetric Matrix

A square matrix A is called skew-symmetric matrix if $A^T = -A$. That is, if $A = (a_{ij})_{n\times n}$ then A is skew-symmetric if $a_{ij} = -a_{ji}$ for all i, j.

Example 14: Let $A = \begin{bmatrix} a & -b & -c \\ b & d & -e \\ c & e & f \end{bmatrix}$. Then $A^T = -A$, so, A is skew-symmetric matrix.

2.7 Matrix of Linear Transformation

The transformation via matrix of an objects can be change in different ways:

Rotation

i. Rotation through 90° about origin: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

ii. Rotation through 180° about origin: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

iii. Rotation through 270° about origin: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

For, if the point(2, – 3) is revolved through the angle 90° about the origin, then the co-ordinates of new position is obtained by, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ Let, the new point is (3, 2)

Example 15: Find the vertices of rectangle obtained by revolving the rectangle with vertices O(0, 0), A (3, 0), B (3, 2), C(0, 2) though 270° about origin.

Solution:

Let the vertices O, A, B, C transforms to new vertices O', A', B', C'. As given the object revolves through 270° about origin, so the transformation matrix for given transformation is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

Then,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & -3 & -3 & 0 \end{bmatrix}$$

Hence the co-ordinates of the vertices are O' (0, 0), A' (O, – 3), B' (2, – 3) and C' (2, 0)

Rotation:

(i) Reflection:

i. Reflection in x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

ii. Reflection in y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

iii. Reflection in the line $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

iv. Reflection in the line $y = x$ $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Illustration

If the point $(-5, -1)$ i.e. $\begin{bmatrix} -5 \\ -1 \end{bmatrix}$ is reflected in the line $y = -x$ as a mirror, its image point is obtained by, $\begin{bmatrix} 0 & -1 \\ - & 0 \end{bmatrix}\begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Example 16: Find the vertices of the image if the rectangle with vertices O (0, 0), A(3, 0), B (3, – 2), C (0, – 2) is reflected in x-axis.

Solution:

Let the vertices O, A, B, C reflected to new vertices O', A', B', C'. As given the object reflected about x-axis by the required transformation matrix is $\begin{bmatrix} 0 & -1 \\ - & 0 \end{bmatrix}$

Then,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Hence, the vertices of image are O' (0, 0), A' (3, 0), B' (3, 2), C' (0, 2).

Example 17: Find the matrix of transformation which transforms the point (1, 1) into point (–2, 0) and the point (0, 1) into (–1, 1).

Solution:

We have to transform $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ to $\begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}$ with help of matrix transformation. Let the matrix of transformation be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}$$

or, $$\begin{bmatrix} a+b & d \\ c+d & d \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}$$

This implies,

$$a + b = -2 \qquad b = -1$$
$$c + d = 0 \qquad d = 1$$

This gives

$$a = -1,\ b = -1 = -1,\ d = 1$$

Hence the matrix of transformation is $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$

2.8 Orthogonal Matrix

A square matrix A is said to be orthogonal if $A^TA = I = AA^T$.

Example:

Let

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Then

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

So, A is an orthogonal matrix.

Theorem: If A and B are any two orthogonal matrices of same order then AB is also orthogonal.

Proof:

Let A and B are orthogonal matrices of same order. Then AB is defined.

Being A is orthogonal,

$$AA^T = I = A^TA.$$

Also, being B is orthogonal,

$$BB^T = I = B^TB.$$

Now,

$$(AB)\,(AB)^T = (AB)\,(B^TA^T) = A(BB^T)A^T = AIA^T = AA^T = I$$

Also,

$$(AB)^T\,(AB) = (B^TA^T)\,(AB) = B^T(A^TA)B = B^TIB = B^TB = I$$

Thus,

$$(AB)(AB)^T = I = (AB)^T(AB).$$

This shows that AB is orthogonal.

Note: AB may replace by BA in above statement.

Transformation by Orthogonal Matrix

The linear transformation Y = AX is said to be orthogonal if the transformation matrix A is orthogonal.

Let Y = AX be orthogonal transformation,

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Since A is orthogonal matrix when $A^T A = I$

Given the

$$Y = AX \dots\dots (i)$$

Taking transpose on both side,

$$Y^T = (AX)^T$$

or, $Y^T = X^T X$.

Multiplying (ii) by (i) we get

$$Y^T Y = (X^T X^T)(AX)$$

or, $Y^T Y = X^T(A^T)X$

or, $Y^T Y = X^T X$

or, $[y_1, y_2, \dots y_n] \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} = [x_1, x_2, \dots x_n] \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$

or, $y_1^2 + y_2^2 + \dots + y_2^2 = x_1^2 + x_2^2 + \dots \dots + x_n^2$

Thus the linear transformation Y = AX is orthogonal if it transforms $y_1^2 + y_2^2 + \dots \dots + y_n^2$ into $x_1^2 + x_2^2 + \dots \dots + x_n^2$.

If the transformation Y = AX, since A^{-1} exists, we can determine the inverse transformation, which caries the vector Y back to the vector X as $X = A^{-1}Y$.

Exercise 2.2

1. Check the following transformations are linear or not.
   (a) $T: R^2 \to R$ be defined by $T(x, y) = x + y$.
   (b) $T: R^2 \to R^3$ be defined by $T(x, y) = (x, y, xy)$.
   (c) $T: R^2 \to R^2$ be defined by $T(x, y) = (x + 3, y)$.
   (d) $T: R \to R$ be defined by $T(x) = x + 4$.
   (e) $T: R^2 \to R^2$ be defined by $T(x, y) = (x, -2y)$.
   (f) Let $T: R^2 \to R^2$ be a transformation which is defined by $T(x, y) = (x + y, x - y)$, check the linearity of T.
   (g) Show that the transformation $T: R^2 \to R$ defined by $T(x, y) = |x + y|$ is not linear.

2. (a) If $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$, show that $(A^2)^T = (A^T)^2$.
   (b) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Verify $(AB)^T = B^T A^T$.
   (c) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Verify that $(AB)^T = B^T A^T$.

Answers

1. (a) linear (b) not linear (c) not linear
   (d) not linear (e) linear (f) linear.

2.9 EIGENVALUE AND EIGENVECTOR

The goal of this title is to show how Ax is related to x, where A is $n\times n$ matrix and x is column vector in R^n. For example, if A is 2×2 matrix and if x is non-zero vector in R such that $Ax = \lambda x$ for some scalar λ, then each vector on the line through orig determined by x gets mapped back on to the same line under the multiplication b matrix A.

Definition (Eigenvalue and Eigenvector)

If A is $n \times n$ matrix, then a scalar λ is called an eigenvalue of matrix A if equation $Ax = \lambda x$ has a non trivial solution. Such an x is called eigenvector corresponding to eigenvalue λ and the correspondin vector $x \in R^n$ is called an eigen-vector of matrix A.

Properties of Eigenvalue and Eigenvector

1. The eigenvalues of a triangular matrix are the entries on its main diagonal.
2. Product of the eigen values is equal to the determinant of the matrix.
3. If λ is an eigenvalue of a matrix A then $\frac{1}{\lambda}$, (for $\lambda \neq 0$) is the eigenvalue of A^{-1}.
4. If λ is an eigenvalue of an orthogonal matrix A, then $\frac{1}{\lambda}$, (for $\lambda \neq 0$) is an eigenvalue A^{-1}.
5. If $v_1, v_2, \ldots\ldots, v_r$ are eigenvectors that correspond to distinct eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_r$ of an $n\times n$ matrix A, then the set $\{v_1, v_2, \ldots\ldots, v_r\}$ is linearly independent.

Characteristic Polynomial and Characteristic Equation

If λ be an eigenvalue of a square matrix A, then det $(A - \lambda I)$ is called characteristic polynomial and $\det(A - \lambda I) = 0$ is called characteristic equation of the matrix A.

Statement of Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation. That is, if the characteristic equation of n^{th} order of square matrix A satisfies $|A - \lambda I| = 0$.

i.e. $(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots + k_n = 0$.

Then $(-1)^n A^n + k_1A^{n-1} + k_2A^{n-2} + \ldots + k_nI = 0$.

Example 18: Show that -2 is eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

Solution:

Given,

$$\lambda = -2 \text{ and } A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}.$$

If

$$Ax = \lambda x \Rightarrow Ax = -2x \Rightarrow (A + 2I)x = 0 \qquad \ldots\ldots(i)$$

has non-trivial solution, then $\lambda = -2$ is eigenvalue of matrix A.

Since,

$$A + 2I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

So, row reduced the augmented matrix is

$$[A + 2I \quad 0]$$

$$\sim \begin{bmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus homogeneous system has free variable (here x_2 is free variable), so equation (i) has non-trivial solution. Thus $\lambda = -2$ is eigenvalue of given matrix A.

Example 19: Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$?

Solution:

Let,

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So,

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3x.$$

Hence, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is eigenvector of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

Note: The collection of all eigenvectors of a vectorspace is called the eigenspace.

Example 20: Find the basis for the eigenspace corresponding to listed eigenvalue, where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ and $\lambda = 3$.

Solution:

Since $\lambda = 3$ is eigenvalue for given matrix A, so $Ax = 3x$ has non-trivial solution.

i.e. $(A - 3I)x = 0 \qquad \ldots\ldots(i)$

has non-trivial solution.

Here,

$$A-3I=\begin{bmatrix}4&2&3\\-1&1&-3\\2&4&9\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&2&3\\-1&-2&-3\\2&4&6\end{bmatrix}.$$

So, reduce augmented matrix is $[A-3I \quad 0]$

$$=\begin{bmatrix}1&2&3&0\\-1&-2&-3&0\\2&4&6&0\end{bmatrix}.$$

$$\sim\begin{bmatrix}1&2&3&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Thus the homogeneous system has non-trivial solution, because x_2 and x_3 are free variable.

Also,

$$x_1+2x_2+3x_3=0 \Rightarrow x_1=-2x_2-3x_3$$

x_2 is free

x_3 is free.

Hence,

$$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2x_2-3x_3\\x_2\\x_3\end{bmatrix}=\left\{x_2\begin{pmatrix}-2\\1\\0\end{pmatrix}+x_3\begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$$

is eigenspace and basis for eigenspace is $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix},\begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$.

Example 21: Find eigenvalue and eigenvector of the matrix: $\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

Solution:

Let

$$A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

Let λ and X are corresponding eigenvalue and eigenvector of A. The characteristics equation of A is

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix}1-\lambda&0\\0&-1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)(-1-\lambda)=0$$

$$\Rightarrow \lambda^2-1=0$$

$$\Rightarrow \lambda=\pm 1$$

Thus, the eigenvalue of A are $\lambda=1, \lambda=-1$.

And for the corresponding eigenvector is

$$AX=\lambda X$$

$$\Rightarrow \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=1\begin{bmatrix}x\\y\end{bmatrix} \quad \text{at } \lambda=1$$

$$\Rightarrow \begin{bmatrix}x\\-y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

This gives,

$$x=x \text{ and } -y=y$$

$$\Rightarrow x=x \text{ (i.e. x is free) and } y=0$$

Thus, the required eigen vector with corresponding to $\lambda=1$ is $(x, 0)$.

And, at $\lambda=-1$,

$$\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=-1\begin{bmatrix}x\\y\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\-y\end{bmatrix}=\begin{bmatrix}-x\\-y\end{bmatrix}$$

This gives,

$$x=-x \text{ and } -y=-y$$

$$\Rightarrow x=0 \text{ and } y=y \quad \text{(i.e. y is free)}$$

Thus the eigenvector with corresponding to $\lambda=-1$ is $(0, y)$.

Hence, the required eigenvector of A be $(x, 0)$ associated with $\lambda=1$ and $(0, y)$ associated with $\lambda=-1$.

Example 22: Find eigenvalue as well as vector of the matrix, $\begin{bmatrix}2&1&2\\0&-1&3\\0&1&1\end{bmatrix}$.

Solution:

Let,

$$A=\begin{bmatrix}2&1&2\\0&-1&3\\0&1&1\end{bmatrix}$$

Let λ and x are eigenvalue and eigenvector of A, respectively.

The characteristic equation of A is,

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\,[(-1-\lambda)(1-\lambda) - 3] = 0$$

$$\Rightarrow (2-\lambda)\,[(1+\lambda)(1-\lambda) + 3] = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda^2+3) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-4) = 0$$

$$\Rightarrow \lambda = 2, -2$$

These are the eigenvalue of A.

And, the corresponding eigenvector of A is,

$$Ax = \lambda x$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+y+2z \\ -y+3z \\ y+z \end{bmatrix} = \begin{bmatrix} lx \\ ly \\ lz \end{bmatrix}$$

$$\Rightarrow 2x+y+2z = \lambda x,\quad -y+3z = \lambda y,\quad y+z = \lambda z$$

At $\lambda = 2$,

$$2x+y+2z = 2x,\ -y+3z = 2y,\ y+z = 2z$$

$$\Rightarrow 2x+y+2z = 2x,\ 3y-3z = 0,\ y-z = 0$$

$$\Rightarrow 2x+y+2z = 2x \quad \ldots \text{(i)}$$

$$y - z = 0 \quad \ldots \text{(ii)}$$

$$y - z = 0 \quad \ldots \text{(iii)}$$

From (ii) and (iii) we have, $y = z$

And from (i), it is possible only if $y = z = 0$ then $2x = 2x \Rightarrow 0 = 0$.

Therefore, x is free and $y = z = 0$. Thus, the corresponding vector is (x, 0, 0).

Next, at $\lambda = -2$,

$$2x+y+2z = -2x,\quad -y+3z = -2y,\quad y+z = -2z$$

$$\Rightarrow 4x+y+2z = 0,\ y+3z = 0,\quad y+3z = 0$$

$$\Rightarrow 4x+y+2z = 0 \quad \ldots \text{(iv)}$$

$$y+3z = 0 \quad \ldots \text{(v)}$$

$$y+3z = 0 \quad \ldots \text{(vi)}$$

From (v) $y = -3z$ then (iv) gives us,

$$4x - 3z + 2z = 0$$

$$\Rightarrow 4x = z$$

Thus, for x = 1, z = 4 and y = −12. Therefore, the corresponding eigenvector is, (1, −12, 4).

Hence, the eigenvalue of A are $\lambda = 2$, and $\lambda = -2$ and their corresponding eigenvector are (x, 0, 0) and (x, −12y, 4z).

Example 23: Find eigenvalue as well as vector of the matrix, $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution:

Let,

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Let λ and x is eigenvalue and eigenvector, respectively to corresponding matrix A.

The characteristic equation A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-1 & 1 & 0 \\ 1 & 3-1 & 0 \\ 0 & 0 & 2-1 \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)^2(2-\lambda) - (2-\lambda) = 0$$

$$\Rightarrow (2-\lambda)\{(3-\lambda)^2 - 1\} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda-1)(3-\lambda+1) = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda)(4-\lambda) = 0$$

$$\Rightarrow \lambda = 2, 2, 4$$

Thus, the eigenvalue are $\lambda = 2$ and $\lambda = 4$.

And, the corresponding eigenvector is,

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$3x + y = \lambda x;\ x + 3y = \lambda y;\quad 2z = \lambda z$$

At, $\lambda = 2$,
$$x + y = 0,\ x + y = 0,\ 2z = 2z$$

This implies
$$x = -y,\ z = z$$

At $\lambda = 4$
$$-x + y = 0,\ x - y = 0,\ 2z = 0$$

This implies
$$x = y,\ z = 0$$

Thus, the eigenvector are $(x, -x, z)$ at $\lambda = 2$ and $(x, x, 0)$ at $\lambda = 4$.

Example 24: Find the characteristic polynomial of matrix, $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$. Also, find its eigenvalue.

Solution:

Let
$$A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

Characteristic polynomial is $|A - \lambda I|$, that is
$$A - \lambda I = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix}.$$

Therefore, characteristic polynomial is,
$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) + 1$$
$$= \lambda^2 - 6\lambda + 9$$

So, characteristic equation is
$$|A - \lambda I| = 0$$
$$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$$
$$\Rightarrow \lambda^2 - 3\lambda - 3\lambda + 9 = 0$$
$$\Rightarrow \lambda(\lambda - 3) - 3(\lambda - 3) = 0$$
$$\Rightarrow (\lambda - 3)(\lambda - 3) = 0$$
$$\Rightarrow \lambda = 3$$

Therefore, $\lambda = 3$ is eigenvalue of matrix $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

Example 25: Verify the Cayley Hamilton theorem by matrix A, and using it, find the inverse of the matrix, $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$.

Solution:

The given matrix is
$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

Now,
$$A - \lambda I = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} - \lambda I \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}$$

Its characteristics equation is
$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (3-\lambda)(2-\lambda) + 1 = 0$$
$$\Rightarrow 6 - 5\lambda + \lambda^2 + 1 = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 7 = 0$$

To verify Cayley-Hamilton theorem, we have to show that

$A^2 - 5A + 7I = 0$...... (i)

Now,
$$A^2 - 5A + 7I = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} - 5\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 9-1 & -3-2 \\ 3+2 & -1+4 \end{bmatrix} - \begin{bmatrix} 15 & -5 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & -5 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 8 & -5 \\ 5 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= 0$$
$$\Rightarrow A^2 - 5A + 7I = 0$$

This verifies Cayley-Hamilton theorem.

To find the inverse, multiplying both sides of (i) by A^{-1}, we get
$$A^{-1}(A^2 - 5A + 7I) = A^{-1}(0)$$
$$\Rightarrow A - 5I + 7A^{-1} = 0$$
$$\Rightarrow 7A^{-1} = 5I - A$$

$$= 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Example 26: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 1 & 13 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ and hence find its inverse.

Solution:

Let

$$A = \begin{bmatrix} 1 & 1 & 13 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

The characteristic equation of the matrix A is

$$\begin{vmatrix} 1-\lambda & 1 & 13 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)\begin{vmatrix} 3-\lambda & -3 \\ -4 & -4-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ -2 & -4-\lambda \end{vmatrix} + 13\begin{vmatrix} 1 & 3-\lambda \\ -2 & -4 \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)\,[(3-\lambda)\,(4+\lambda) - 12] - [-(4+\lambda) - 6] + 3(-4 + 6 - 2\lambda) = 0$

$\Rightarrow \quad (1-\lambda)\,(-12 + \lambda + \lambda^2 - 12) - (-4 - \lambda - 6) + 3(2 - 2\lambda) = 0$

$\Rightarrow \quad (1-\lambda)\,(\lambda^2 + \lambda - 24) + \lambda + 10 + 6 - 6\lambda = 0$

$\Rightarrow \quad \lambda^2 + \lambda - 24 - \lambda^3 - \lambda^2 + 24\lambda + 16 - 5\lambda = 0$

$\Rightarrow \quad \lambda^3 - 20\lambda + 8 = 0$

By Cayley-Hamilton theorem, it satisfies the equation

$A^3 - 20A + 8I = 0$

Multiplying both sides by $A^{-1} = 0$, we get

$A^{-1}(A^3 - 20A + 8I) = A^{-1}(0)$

$$\Rightarrow \quad A^{-1} = \frac{5}{2}I - \frac{1}{8}A^2$$

$$= \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

2.10 Similarity Transformation

Let A and B are two n×n matrices. The matrix A is similar to matrix B if there exists an invertible matrix P such that $P^{-1}AP = B$, or equivalently $A = PBP^{-1}$. The changing A into $P^{-1}AP$ is called similarity transformation. It is noted as $B \sim A$. In such condition P is called model matrix.

Likely, if $Q = P^{-1}$, then $A = Q^{-1}BQ$, so B is similar to A and we say A and B are similar. The changing A into $P^{-1}AP$ is called similarity transformation.

Note: If n × n matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Example 27: If n × n matrices A and B are similar, then show that det(A) = det(B).

Solution:

Since A and B are similar, so there exist invertible matrix P such that

$$B = P^{-1}AP.$$

So,

$$|B| = |P^{-1}AP|$$
$$= |P^{-1}|\,|A|\,|P|$$
$$= |P^{-1}|\,|P|\,|A|$$
$$= |P^{-1}P|\,|A|$$
$$= |I|\,|A|$$
$$= |A|.$$

Thus, $|B| = |A|$.

2.10 DIAGONALIZABLE MATRIX

A square matrix A is called diagonalizable if there exist an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

Note 1: An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Note 2: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Procedure for Diagonalizing a Matrix

Step 1: Find n linearly independent eigenvectors of A say $v_1, v_2, \ldots, v_n$.

Step 2: For matrix P having $v_1, v_2, \ldots, v_n$ as its column vectors.

Step 3: The matrix D will be the diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to v_i for $i = 1, 2, \ldots, n$.

Note: Here, $A = PDP^{-1}$ or $AP = PD$, if so our P and D really work and A is diagonalizable.

Example 28: Diagonalize $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution:

Here, $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

The characteristics, values of A are given by

$$|A - \lambda I| = 0$$

or, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)(2-l) - 4 = 0$

or, $\lambda^2 - 7\lambda + 6 = 0$

or, $\lambda^2 = 6, 1$

These are characteristics values of A.

Let x be the characteristics vector corresponding to A. So,

$Ax = \lambda x$

$\Rightarrow (A - \lambda) = 0$

At $\lambda = 6$ is

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduce to the equations

$$-x_1 + 4x_2 = 0, x_1 - 4x^2 = 0$$

Solving we get,

$$x_1 = 4, x_2 = 1$$

So, the characteristics vector is

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Again at $\lambda = 1$

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduce to the equations

$$4x_1 + 4x_2 = 0, x_1 + x_2 = 0$$

Solving we get

$$x_1 = 1, x_2 = -1$$

So the characteristics vector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence the model matrix is

$$\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$$

We know the diagonal of the given matrix A is

$$C^{-1}AC = \frac{1}{-5}\begin{bmatrix} -1 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} -1 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 20+4 & 5-4 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} -1 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 24 & 1 \\ 6 & -1 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} -24-6 & -1+1 \\ -24+24 & -1+4 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -30 & 0 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 29: Let $A = PDP^{-1}$, compute A^4; if $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution:

We know that

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^4\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2^4 & 0 \\ 0 & 1^4 \end{bmatrix}\left(\frac{1}{15-14}\right)\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 7 \\ 32 & 3 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}.$$

Example 30: Compute A^8, where $A=\begin{bmatrix}4&-3\\2&-1\end{bmatrix}$.

Solution:

If A is diagonalizable, then $A = PDP^{-1}$, so

$$A^8 = PD^8P^{-1} \quad .....(i)$$

For P and D, we find eigenvalue of A.

Its characteristic equation is,

$$|A-\lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix}4-l & -3\\ 2 & -1-l\end{vmatrix} = 0.$$

$$\Rightarrow \lambda^2-3\lambda+2=0$$

This gives, $\lambda = 1, 2$.

For $\lambda = 1$, the augmented matrix of A is,

$$[A-\lambda I \quad 0]$$

i.e. $[A-I \quad 0]$ [Being $\lambda = 1$]

$$= \begin{bmatrix}3&-3&0\\2&-2&0\end{bmatrix}$$

$$\sim \begin{bmatrix}1&-1&0\\1&-1&0\end{bmatrix}$$

$$\sim \begin{bmatrix}1&-1&0\\0&0&0\end{bmatrix}.$$

From the last matrix,

x_2 is free and $x_1 - x_2 = 0$

This implies, x_2 = free, $x_1 = x_2$.

Therefore,

$$x = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\x_2\end{bmatrix} = \left\{x_2\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

Thus, the basis for eigenspace (corresponding $\lambda = 1$) is $v_1 = \begin{bmatrix}1\\1\end{bmatrix}$.

Similarly, for $\lambda = 2$, basis for eigenspace is $v_2 = \begin{bmatrix}3\\2\end{bmatrix}$.

Thus, there are two basis vectors in total, which are linearly independent and are

$$v_1 = \begin{bmatrix}1\\1\end{bmatrix} \text{ and } v_2 = \begin{bmatrix}3\\2\end{bmatrix}.$$

So,

$$P = \begin{bmatrix}1&3\\1&2\end{bmatrix} \text{ and } D = \begin{bmatrix}1&0\\0&2\end{bmatrix}.$$

From (i) we get,

$$A^8 = PD^8P^{-1} = \begin{bmatrix}1&3\\1&2\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}^8\left(\frac{1}{-1}\right)\begin{bmatrix}2&-3\\-1&1\end{bmatrix}$$

$$= \begin{bmatrix}1&3\\1&2\end{bmatrix}\begin{bmatrix}1^8&0\\0&2^8\end{bmatrix}\begin{bmatrix}-2&3\\1&-1\end{bmatrix}$$

$$= \begin{bmatrix}1&3\\1&2\end{bmatrix}\begin{bmatrix}1&0\\0&256\end{bmatrix}\begin{bmatrix}-2&3\\1&-1\end{bmatrix}$$

$$= \begin{bmatrix}1&768\\1&512\end{bmatrix}\begin{bmatrix}-2&3\\1&-1\end{bmatrix}$$

$$= \begin{bmatrix}766&-765\\510&-509\end{bmatrix}.$$

Exercise 2.3

1. Find the characteristics polynomial and eigenvalue of

(a) $\begin{bmatrix}2&7\\7&2\end{bmatrix}$ (b) $\begin{bmatrix}5&-3\\-4&3\end{bmatrix}$ (c) $\begin{bmatrix}2&3\\3&-6\end{bmatrix}$

(d) $\begin{bmatrix}5&3\\-4&4\end{bmatrix}$ (e) $\begin{bmatrix}2&2&-1\\1&3&-1\\-1&-2&2\end{bmatrix}$ (f) $\begin{bmatrix}-1&4&-2\\-3&4&0\\-3&1&3\end{bmatrix}$

(g) $\begin{bmatrix}1&0&0\\1&2&0\\-3&5&2\end{bmatrix}$ (h) $\begin{bmatrix}-1&0&1\\-3&4&1\\0&0&2\end{bmatrix}$

2. Find the characteristics equation of the matrix $A = \begin{bmatrix}1&2&3\\2&-1&4\\3&1&-1\end{bmatrix}$. Show that th equation is satisfied by A.

3. Find eigenvalue and eigenvector of the following matrices;

(a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ (b) $\begin{bmatrix}2&1\\1&2\end{bmatrix}$ (c) $\begin{bmatrix}5&4\\1&2\end{bmatrix}$

4. Find eigenvalue as well as vector of the following:

(a) $\begin{bmatrix}3&0&0\\0&-8&0\\0&0&4\end{bmatrix}$ (b) $\begin{bmatrix}2&1&2\\0&-1&3\\0&1&1\end{bmatrix}$ (c) $\begin{bmatrix}3&1&0\\1&3&0\\0&0&2\end{bmatrix}$

(d) $\begin{bmatrix}3&1&4\\0&2&0\\0&0&5\end{bmatrix}$ (e) $\begin{pmatrix}0&7&0\\0&0&0\\0&0&-2\end{pmatrix}$ (f) $\begin{bmatrix}1&2&0\\2&1&0\\0&0&-1\end{bmatrix}$

5. Using Cayley Hamiltan theorem find the inverse of

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

6. Find the characteristics equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that th equation is satisfied by A and hence obtain the inverse of the given matrix.

7. Show that if $A = QR$ with Q invertible then A is similar to $A_1 = RQ$.

   Hint: Here, $Q^{-1}AQ = Q^{-1}(QR)Q = (Q^{-1}Q)RQ = RQ = A_1$, so A and A_1 are similar.

8. Diagonalize the following matrix, if possible

   (a) $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

9. Let $A = PDP^{-1}$ compute A^5, where $P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

10. If $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ compute A^4.

Answers

1. (a) $\lambda^2 - 4\lambda - 45, \lambda = 9, -5$
   (b) $\lambda^2 - 8\lambda + 3, 4 \pm \sqrt{13}$
   (c) $\lambda^2 + 4\lambda - 21, \lambda = 3, -7$
   (d) $\lambda^2 - 9\lambda + 32$, no real eigen value
   (e) $(\lambda - 1)^2 (\lambda - 5), \lambda = 1, 5$
   (f) $(3 - \lambda)(\lambda - 1)(\lambda - 2), \lambda = 1, 2, 3$
   (g) $(1 - \lambda)(2 - \lambda)^2, \lambda = 1, 2$
   (h) $(2 - \lambda)(-1 - \lambda)(4 - \lambda), \lambda = -1, 2, 4$
2. $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$
3. (a) $\lambda = 1, -1$; $(x, 0)$ at $\lambda = 1$; $(0, y)$ at $\lambda = -1$
   (b) $\lambda = 1, 3$; $(x, -x)$ at $\lambda = 1$; (x, x) at $\lambda = 3$
   (c) $\lambda = 1, 6$; $(x, \lambda x)$ at $\lambda = 1$; $(4x, x)$ at $\lambda = 6$
4. (a) $\lambda = 3, 4, -8$; $(x, 0, 0)$ at $\lambda = 3$; $(0, y, 0)$ at $\lambda = 4$; $(0, 0, z)$ at $\lambda = -8$
   (b) $\lambda = 2, -2$; $(x, 0, 0)$ at $\lambda = 2$; $(x, -12y, 4z)$ at $\lambda = -2$
   (c) $\lambda = 2, 2, 4$; $(x, -x, z)$ at $\lambda = 2$ and $(x, x, 0)$ at $\lambda = 4$
   (d) $\lambda = 2, 3, 5$; $(x, -x, 0)$ at $\lambda = 2$; $(x, 0, 0)$ at $\lambda = 3$; $(2x, 0, x)$ at $\lambda = 5$
   (e) $\lambda = 0, 0, -2$; $(0, y, 0)$ at $\lambda = 0$; $(0, 0, z)$ at $\lambda = -2$
   (f) $\lambda = -1, -1, 3$; $(x, -x, z)$ at $\lambda = -1$; $(x, x, 0)$ at $\lambda = 3$
5. (a) $A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ (b) $\frac{1}{130}\begin{bmatrix} 4 & 20 & -9 \\ -42 & 50 & -3 \\ 20 & -30 & 20 \end{bmatrix}$
6. $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$; $A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$
8. (a) $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$
   (c) Not diagonalizable
9. $A^5 = \begin{bmatrix} -92 & 372 \\ -31 & 125 \end{bmatrix}$
10. $A^4 = \begin{bmatrix} 1169 & 544 \\ -1088 & -463 \end{bmatrix}$

□□□