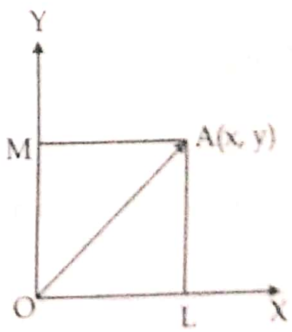
CHAPTER 4

VECTOR ALGEBRA

To apply mathematics in many real-world situations, we need vector. Some of the things we measure are determined by magnitudes. For example; mass, length, time. While describing force, displacement, or velocity, we need more information. For example, to describe a force, we need the direction in which it acts as well as how large it is. Therefore, in this chapter we introduce vectors. Building on what we already know about vectors in xy-plane, we establish vectors in space by adding a third axis that measures distance above and below the xy-plane.

4.1 Vectors in Terms of Coordinates

Let OX and OY be the two mutually perpendicular co-ordinate axes. Consider any point A(x, y) in XY plane. Join O and A.



Then $OL = x =$ projection of OA on x axis

$OM = y =$ projection of OA on y axis

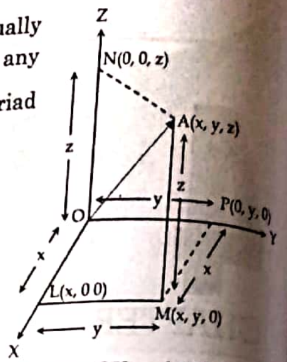
where AL and AM are perpendiculars drawn from A to OX and OY respectively. The directed line segment OA is known as $\overrightarrow{OA}$. Also, the $\overrightarrow{OA}$ is defined by

$$\overrightarrow{OA} = (x, y) = (\text{projection of OA on x axis, projection of OA on y axis}).$$

The vector $\overrightarrow{OA} = (x, y)$ is a vector in the Cartesian plane so it is called a plane vector.

Similarly, if OX, OY, OZ be the three mutually perpendicular co-ordinate axes and A(x, y, z) be any point in the space then $\overrightarrow{OA} = (x, y, z)$ an ordered triad is called a space vector.

The length of the line segment OA measures the magnitude of the vector $\overrightarrow{OA}$ in both two and three dimensional cases.



The Unit Vectors $\vec{i}, \vec{j}$ and $\vec{k}$

The unit vectors along two mutually perpendicular coordinate axes OX and OY denoted by $\vec{i}$ and $\vec{j}$ are defined by $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$ respectively. Similarly the unit vectors along three mutually perpendicular coordinate axes OX, OY and OZ denoted by $\vec{i}, \vec{j}$ and $\vec{k}$ are defined by, $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$ respectively.

Therefore,

$$\overrightarrow{OA} = (x, y) = x\vec{i} + y\vec{j}$$

and, $$\overrightarrow{OA} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$$

Position Vector

Let O be the origin and P(x, y) be any point in the plane. Then the vector $\overrightarrow{OP} = (x, y)$ is called the position vector of P. Similarly if P(x, y, z) be any point in the space, then the vector $\overrightarrow{OP} = (x, y, z)$ is called position vector of point P. Thus position vector of any point P is defined as a vector from the origin O to any point P.

Modulus of a Vector

The length of a line segment representing a vector which is a non-negative number is called the modulus of a vector. The modulus of a vector $\vec{a}$ and $-\vec{a}$ are denoted by

$$|\vec{a}| = a \text{ and } |-\vec{a}| = a$$

respectively.

Consider A(x, y) be any point in the xy-plane. From right angled triangle AOL,

$$OA^2 = OL^2 + AL^2 = x^2 + y^2$$

i.e. $$OA = \sqrt{x^2 + y^2},$$

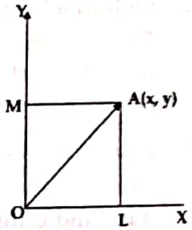
which gives the modulus of vector $\overrightarrow{OA}$.

That is, if

$$\overrightarrow{OA} = (x, y)$$

then modulus of vector $\overrightarrow{OA}$ is $|\overrightarrow{OA}|$ and

$$|\overrightarrow{OA}| = \sqrt{x^2 + y^2}.$$



Similarly, if A(x, y, z) be any point in the space the modulus of vector $\overrightarrow{OA}$ is given by $\sqrt{x^2 + y^2 + z^2}$. i.e. $|\overrightarrow{OA}| = \sqrt{x^2 + y^2 + z^2}$.

Cartesian Coordinates Plane

The Cartesian coordinates (x, y, z) of a point P in space are the values at which the planes through P perpendicular to the axes cut the axes. Cartesian coordinates for space are also called rectangular coordinates because the axes that define them meet at right angles.

The coordinates of the x-axis is (x, 0, 0), of y-axis is (0, y, 0) and of z-axis is (0, 0, z). The planes determined by the coordinates axes are the xy-plane whose standard equation is z = 0; the yz-plane whose standard equation is x = 0; and the xz-plane whose standard equation is y = 0.

4.2 Types of Vectors

1. **Zero Vector (Null Vector)**

A vector whose magnitude is zero is called zero or Null vector. The initial and terminal points of a zero vector are coincident. Its direction is arbitrary.

Note: A non-zero vector is called proper vector.

2. **Unit Vector**

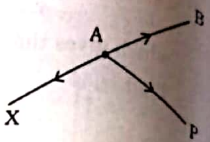
A vector whose magnitude is unity is called unit vector. If $\vec{a}$ is any vector and $\hat{a}$ is a unit vector in the direction of $\vec{a}$, then

$$\vec{a} = |\vec{a}|\,\hat{a} \Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|}.$$

Note: Unit vector is distinguished from other vectors by putting a cap (^) over them.

3. **Co-initial Vectors**

Two or more vectors having the same initial point are called co-initial vectors.



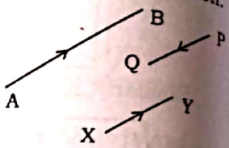
Thus, $\overrightarrow{AB}$, $\overrightarrow{AP}$, $\overrightarrow{AX}$ are co-initial vectors.

4. **Equal Vectors**

Two vectors are said to be equal if they have equal magnitude and same direction.

5. **Like and Unlike Vectors**

Two vectors are said to be like vectors if their directions are same whatever their magnitude may be. If their directions are opposite, they are said to be unlike vectors.



In figure, $\overrightarrow{AB}$ and $\overrightarrow{XY}$ are like vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are unlike vectors.

6. **Negative of a vector**

A vector whose magnitude is the same as that of a given vector $\vec{a}$ but has opposite direction is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$.

Clearly, $\vec{a} = \overrightarrow{PQ}$, $-\vec{a} = \overrightarrow{QP}$.

7. **Free and Localized Vectors**

Vectors having the same magnitude and direction and acting at different points in the same line or parallel lines are called free vectors i.e. free vector is free to have any point as its initial point.

A vector acting at a fixed point and parallel to a given vectors is called localized vector. Thus the initial point of a localized vector is fixed and hence it is represented by a unique directed line segment.

9. **Collinear Vectors**

Vectors along the same or parallel lines irrespective of their magnitudes and sense, are said to be collinear. If $\vec{a}$ and $\vec{b}$ are collinear, then $\vec{a} = k\vec{b}$, for scalar k.

Note that if $\vec{a}$ and $\vec{b}$ be two non-zero and non-collinear vectors and k_1, k_2 two scalars such that $k_1\vec{a} + k_2\vec{b} = 0$ then $k_1 = 0$, $k_2 = 0$.

10. **Coplanar Vectors**

A system of vectors is said to be coplanar if a plane can be drawn parallel to all of them. That is any vector $\vec{r}$ coplanar with two non-collinear vectors $\vec{a}$ and $\vec{b}$ can uniquely be expressed as $\vec{r} = k_1\vec{a} + k_2\vec{b}$, where k_1 and k_2 are scalars.

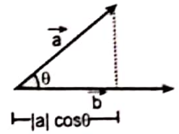
Note that If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-coplanar vectors and k_1, k_2, k_3 are three scalars such that $k_1\vec{a} + k_2\vec{b} + k_3\vec{c} = 0$ then $k_1 = 0$, $k_2 = 0$, $k_3 = 0$.

4.3 Product of Two Vectors

Product of vectors is of two types. A vector has both magnitude and direction and based on this the two product of vectors are, the dot product of two vectors and the cross product of two vectors. The dot product of two vectors is also referred to as scalar product, as the resultant value is a scalar quantity. The cross product is called the vector product as the result is a vector, which is perpendicular to these two vectors.

4.3.1 Scalar (dot) Product of Two Vectors

If $\vec{a}$ and $\vec{b}$ are the two non-zero vectors then scalar product of these two vectors is denoted by $\vec{a} \cdot \vec{b}$ and is defined as



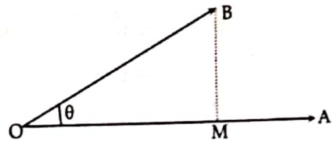
$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

where θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

Geometrical Interpretation of Scalar Product of Two Vectors

Let $\vec{a}$ and $\vec{b}$ are two non-zero vectors are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively.

Then



$$\begin{aligned}
\vec{a} \cdot \vec{b} &= |\vec{a}|\,|\vec{b}| \cos\theta \\
&= ab \cos\theta \\
&= (OA)\,(OB)\,\frac{OM}{OB} \\
&= (OA)\,(OM) \\
&= |\vec{a}| \text{ (Scalar component of } \overrightarrow{OB} \text{ on OA)}
\end{aligned}$$

Derivation of Dot Product of Two Vectors

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are the two vectors then scalar product of these two vectors is given by

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Angle between Two given Vectors

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are two given vectors then by scalar product of these two vectors,

$$|\vec{a}|\,|\vec{b}| \cos\theta = \vec{a} \cdot \vec{b}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

$$\text{i.e. } \theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

Properties of Scalar Product of Two Vectors:

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three non zero vectors and k be any scalars then

a. If $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow |\vec{a}|\,|\vec{b}|\cos\theta = 0$

$\Leftrightarrow \cos\theta = 0$ [being $\vec{a} \neq 0$, $\vec{b} \neq 0$]

$\Leftrightarrow \cos\theta = \cos\frac{\pi}{2}$

$\Leftrightarrow \theta = \frac{\pi}{2}$

Thus, $\vec{a}\cdot\vec{b} = 0$ if and only if (i.e. iff) $\vec{a}$ and $\vec{b}$ are perpendicular (i.e. orthogonal or normal) to each other.

b. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

c. $k(\vec{a}\cdot\vec{b}) = (k\vec{a})\cdot\vec{b} = \vec{a}\cdot(k\vec{b})$.

d. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (left distributive)

e. $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$ (right distributive)

f. $\vec{a}\cdot\vec{a} = |\vec{a}|^2 = |a|^2 = a^2$

Because $\vec{a}\cdot\vec{a} = |\vec{a}|\,|\vec{a}|\cos 0$

$= a\,a\cos 0$ [angle between overlapped lines]

$= a^2$ [$\cos 0 = 1$]

g. $\vec{0}\cdot\vec{a} = 0$

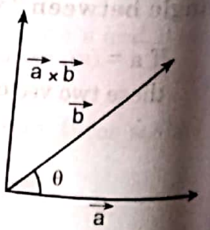
h. If $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the axes then

$\vec{i}\cdot\vec{i} = 1$, $\vec{j}\cdot\vec{j} = 1$, $\vec{k}\cdot\vec{k} = 1$, $\vec{i}\cdot\vec{j} = 0$, $\vec{j}\cdot\vec{k} = 0$, $\vec{k}\cdot\vec{i} = 0$

4.3.2. Vector Product (Cross Product) of Two Vectors

Let $\vec{a}$ and $\vec{b}$ are any two non-zero vectors. The cross product of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ which is defined as

$$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$$



where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector in the direction of $\vec{a}\times\vec{b}$.

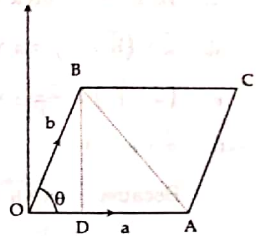
Note that $\vec{a}\times\vec{b}$ is perpendicular to both vectors $\vec{a}$ and $\vec{b}$.

Geometrical Interpretation of Vector Product of Two Vectors

Let $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ are two coinitial vectors. Let θ is the angle between $\vec{a}$ and $\vec{b}$. Construct a parallelogram OACB, whose adjacent sides are $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively. Draw a perpendicular BD on OA.

Then in right angle triangle OBD,

$$\sin\theta = \frac{BD}{OB}$$



By definition we have

$$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$$

$$= (OA)\,OB\,\frac{BD}{OB}\,\hat{n}$$

$$= (OA)\,BD\,\hat{n}$$

Then

$$|\vec{a}\times\vec{b}| = (OA)(BD) \quad [\because |\hat{n}| = 1]$$

= (Base of OACB)(breadth of OACB)

= Area of parallelogram

Thus geometrically, the vector product of two vectors $\vec{a}\times\vec{b}$ represents a vector which is perpendicular to the vectors $\vec{a}$ and $\vec{b}$ and the magnitude $|\vec{a}\times\vec{b}|$ is equal to the area of the parallelogram of whose adjacent sides are represents by the vectors $\vec{a}$ and $\vec{b}$.

Derivation of Cross Product of Two Vectors

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are the two vectors then vector product of these two vectors is given by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2,\ a_1b_3 - a_3b_1,\ a_1b_2 - a_2b_1)$$

Properties of the Cross Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three non zero vectors and m be any scalars then

a. If $\vec{a}\times\vec{b}=0 \Leftrightarrow |\vec{a}||\vec{b}|\sin\theta\,\hat{n}=0$

$\Leftrightarrow \sin\theta=0$ [being $\vec{a}\neq 0$, $\vec{b}\neq 0$]

$\Leftrightarrow \sin\theta=\sin 0$ or $\sin\pi$

$\Leftrightarrow \theta=0$ or π

Thus, $\vec{a}\times\vec{b}=0$ if and only if (i.e. iff) $\vec{a}$ and $\vec{b}$ are parallel (may like or unlike as given vectors) to each other.

b. $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$

c. $(m\vec{a})\times\vec{b}=m(\vec{a}\times\vec{b})=\vec{a}\times(m\vec{b})$

d. $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

e. $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$

f. $\vec{a}\times\vec{a}=0$

Because $\vec{a}\times\vec{a}=|\vec{a}||\vec{a}|\sin\theta\,\hat{n}$

$=a\,a\sin 0\,\hat{n}$ [angle between overlapped lines]

$=0$ [cos0 = 0]

g. If $\vec{i},\vec{j},\vec{k}$ are unit vectors along the axes, them.

$\vec{i}\times\vec{i}=0;\ \vec{j}\times\vec{j}=0;\ \vec{k}\times\vec{k}=0;\ \vec{i}\times\vec{j}=\vec{k};\ \vec{j}\times\vec{k}=\vec{i};\ \vec{k}\times\vec{i}=\vec{j}$.

Example 1: If $\vec{u}=\vec{i}-2\vec{j}+3\vec{k}$ and $\vec{v}=2\vec{i}-\vec{j}-\vec{k}$ then find $\vec{u}\cdot\vec{v}$, $\vec{u}\times\vec{v}$ and magnitude of $\vec{v}$.

Solution:

Given that,

$\vec{u}=\vec{i}-2\vec{j}+3\vec{k}$ and $\vec{v}=2\vec{i}-\vec{j}-\vec{k}$.

Then,

$\vec{u}\cdot\vec{v}=(\vec{i}-2\vec{j}+3\vec{k})\cdot(2\vec{i}-\vec{j}-\vec{k})=2+2-3=1.$

And,

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 3\\ 2 & -1 & -1\end{vmatrix}$$

$=\vec{i}(2+3)-\vec{j}(-1-6)+\vec{k}(-1+4)$

$=5\vec{i}+7\vec{j}+3\vec{k}$

Also, the magnitude of $\vec{v}$ is

$$|\vec{v}|=|2\vec{i}-\vec{j}-\vec{k}|=\sqrt{4+1+1}=\sqrt{6}$$

Example 2: Find the angle between $\vec{u}=4\vec{i}-2\vec{j}-\vec{k}$ and $\vec{v}=4\vec{i}-2\vec{j}+4\vec{k}$.

Solution:

Given that,

$\vec{u}=4\vec{i}-2\vec{j}-\vec{k}$ and $\vec{v}=4\vec{i}-2\vec{j}+4\vec{k}$.

Then,

$\vec{u}\cdot\vec{v}=(4\vec{i}-2\vec{j}-\vec{k})\cdot(4\vec{i}-2\vec{j}+4\vec{k})=16+4-4=16$

Also,

$|\vec{u}|=\sqrt{4^2+(-2)^2+(-1)^2}=\sqrt{16+4+1}=\sqrt{21}$

$|\vec{v}|=\sqrt{4^2+(-2)^2+4^2}=\sqrt{16+4+16}=\sqrt{36}=6$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$ then by definition

$$\theta=\cos^{-1}\left(\frac{\vec{u}.\vec{v}}{|u|\ |v|}\right)$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)$$

Example 3: Show that the three points A, B, C with the position vectors $-2\vec{a}+3\vec{b}+5\vec{c}$, $\vec{a}+2\vec{b}+3\vec{c}$, $7\vec{a}-\vec{c}$ are collinear.

Solution:

As given,

$\overrightarrow{OA}=-2\vec{a}+3\vec{b}+5\vec{c}$, $\overrightarrow{OB}=\vec{a}+2\vec{b}+3\vec{c}$, $\overrightarrow{OC}=7\vec{a}-\vec{c}$

Then,

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{AO}=(\vec{a}+2\vec{b}+3\vec{c})-(-2\vec{a}+3\vec{b}+5\vec{c})$

$=3\vec{a}-\vec{b}-2\vec{c}$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=6\vec{a}-2\vec{b}-4\vec{c}=2(3\vec{a}-\vec{b}-2\vec{c})=2\overrightarrow{AB}$

This shows the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear. This means the points A, B, C are collinear.

Example 4: Find the unit vector perpendicular to both vector $\vec{a} = 2\vec{j} - 3\vec{k}$ and $\vec{b} = 2\vec{i}$.

Solution:

Let,

$$\vec{a} = 2\vec{j} - 3\vec{k} \quad \text{and} \quad \vec{b} = 2\vec{i}.$$

Since $\vec{a} \times \vec{b}$ be the vector perpendicular to $\vec{a}$ and $\vec{b}$.

Here,
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -3 \\ 2 & 0 & 0 \end{vmatrix}$$
$$= \vec{i}\,(0-0) - \vec{j}\,(0+6) + \vec{k}\,(0-4)$$
$$= -6\vec{j} - 4\vec{k}.$$

So,
$$|\vec{a} \times \vec{b}| = \sqrt{(0)^2 + (-6)^2 + (-4)^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}.$$

Let $\hat{n}$ be unit vector along $\vec{a} \times \vec{b}$ then

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{-6\vec{j} - 4\vec{k}}{2\sqrt{13}} = \frac{-(3\vec{j} + 2\vec{k})}{\sqrt{13}}.$$

Thus the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\dfrac{-(3\vec{j} + 2\vec{k})}{\sqrt{13}}$.

Example 5: Find the area of the parallelogram having adjacent sides are $\vec{i} + 2\vec{j} + 3\vec{k}$ and $3\vec{i} - 2\vec{j} + \vec{k}$.

Solution:

Let

Then
$$\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k} \quad \text{and} \quad \vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}.$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$$
$$= \vec{i}\,(2+6) - \vec{j}\,(1-9) + \vec{k}\,(-2-6)$$
$$= 8\vec{i} + 8\vec{j} - 8\vec{k}$$

So,
$$|\vec{a} \times \vec{b}| = \sqrt{8^2 + 8^2 + (-8)^2} = \sqrt{3 \times 8^2} = 8\sqrt{3}.$$

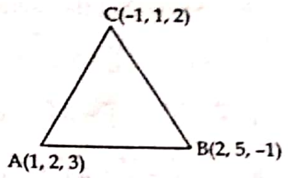
Thus, the area of parallelogram whose adjacent sides represents by the given vectors, is $8\sqrt{3}$ sq. unit.

Example 6: Find the area of the triangle whose vertices are A(1, 2, 3), B(2, 5, –1) and C(–1, 1, 2).

Solution:

Given that,

A(1, 2, 3), B(2, 5, –1) and C(–1, 1, 2).



Then,

$$\vec{AB} = (2, 5, -1) - (1, 2, 3) = (1, 3, -4).$$

$$\vec{AC} = (-1, 1, 2) - (1, 2, 3) = (-2, -1, -1).$$

Now,

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -4 \\ -2 & -1 & -1 \end{vmatrix} = -7\vec{i} + 9\vec{j} + 5\vec{k}.$$

Therefore, the area of triangle ABC is

$$A = \frac{1}{2}\,|\vec{AB} \times \vec{AC}|$$
$$= \frac{1}{2}\sqrt{(-7)^2 + 9^2 + 5^2}$$
$$= \frac{\sqrt{155}}{2} \text{ sq. units.}$$

Exercise 4.1

1. a. If $\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} - \vec{k}$, find $\vec{a} \cdot \vec{b}$ and the angle between them.

   b. Find the cosine of the angle between $\vec{a} = 3\vec{i} + 2\vec{j}$ and $\vec{b} = 5\vec{j} + \vec{k}$. of

   c. Find the angle between two position vectors of the points P(1, 2, 3) and Q(1, –3, 2).

   d. Find the sine of the angle between $\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$.

2. Determine the value of λ, so that $\vec{a} = 2\vec{i} + \lambda\vec{j} + \vec{k}$ and $\vec{b} = 4\vec{i} - 2\vec{j} - 2\vec{k}$ are perpendicular to each other.

3. a. Show that the vectors $\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 4\vec{j} + 2\vec{k}$ and $\vec{c} = -10\vec{j} - 2\vec{j} + 4\vec{k}$ are orthogonal to each other.

   b. Show that the vectors $\vec{a} = \frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k})$, $\vec{b} = \frac{1}{7}(3\vec{i} - 6\vec{j} + 2\vec{k})$ and $\vec{c} = \frac{1}{7}(6\vec{i} + 2\vec{j} - 3\vec{k})$ are orthogonal to each other

4. If $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} - 3\vec{j} + 2\vec{k}$, $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$.

5. a. If $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$, find unit vector along $\vec{a} \times \vec{b}$.

   b. Find a unit vector normal to the plane containing $\vec{a} = 3\vec{i} - 2\vec{j} + 4\vec{k}$, $\vec{b} = \vec{i} + \vec{j} - 2\vec{k}$.

6. a. If $\vec{a} = 5\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = -15\vec{i} + 3\vec{j} - 3\vec{k}$ then show that $\vec{a}$ and $\vec{b}$ are parallel to each other.

   b. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vector such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$; $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$; $\vec{a} \neq \vec{0}$ then show that $\vec{b} = \vec{c}$.

Answers

1. a. 60°

   b. $\cos^{-1}\left(\frac{5\sqrt{2}}{13}\right)$

   c. $\cos^{-1}\left(\frac{1}{2\sqrt{13}}\right)$

   d. $\sin^{-1}\left(\frac{\sqrt{185}}{3\sqrt{26}}\right)$

2. $\lambda = 3$

4. $13\vec{i} + \vec{j} - 5\vec{k}$; $-13\vec{i} - \vec{j} + 5\vec{k}$

5. a. $\frac{\vec{i} - \vec{k}}{\sqrt{2}}$

   b. $\frac{2\vec{j} + \vec{k}}{\sqrt{5}}$

4.4 PRODUCT BETWEEN THREE VECTORS

We already discuss that there are two types of product between vectors – dot or scalar product and cross or vector product.

4.4.1 Scalar or Dot Product of Three Vectors

The scalar product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = (a_1, a_2, a_3)$,

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} = (b_1, b_2, c_3)$,

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k} = (c_1, c_2, c_3)$.

Properties of Scalar Triple Product

1. In a scalar triple product, the position of dot and cross can be interchanged without changing its value. That is, i.e. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Proof: Let

$$\vec{a} = (a_1, a_2, a_3),\ \vec{b} = (b_1, b_2, b_3),\ \vec{c} = (c_1, c_2, c_3)$$

Now,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{c} \cdot (\vec{a} \times \vec{b})$$

Similarly, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

Thus, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$.

2. In a scalar triple product, if three vectors are coplanar then the value of scalar triple product is zero and vice-versa.

Proof: Let $\vec{a}, \vec{b}, \vec{c}$ be three given vectors in which $\vec{a}$ and $\vec{b}$ are coplanar then any one among these vectors should be zero. Then,

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = 0.$$

3. In a scalar triple product, if any two vectors are parallel then the value of scalar triple product is zero.

Proof: Let any two vectors $\vec{b}$ and $\vec{c}$ are parallel. Then, $\vec{b} = k\vec{c}$ for some scalar k.

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ k\vec{c}\ \vec{c}] = k[\vec{a}\ \vec{c}\ \vec{c}] = 0$$

being $\vec{c}\times\vec{c} = 0 \Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0.$

Note: In a scalar triple product, if any two vectors are parallel then the vectors are coplanar.

Geometrical Meaning of Scalar Triple Product

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$ are three given vectors. Then we already discussed that $|\vec{b}\times\vec{c}|$ gives area of parallelogram OBEC.

Construct a parallelepiped OABCEFGH having the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as its adjacent sides.

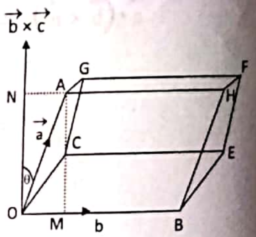
Let θ be the angle between $\vec{a}$ and $\vec{b}\times\vec{c}$.

Draw a perpendicular AM to OB and a perpendicular AN to the vector $\vec{b}\times\vec{c}$.

Then, OMAN is a rectangle. Therefore, ON = AM.

In right angle triangle OAN, we have

$$\cos\theta = \frac{ON}{OA}$$



Since, $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

Then by definition,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = |\vec{a}|\,|\vec{b}\times\vec{c}|\cos\theta$$

$$= (OA)\ (\text{area of parallelogram OBEC})\ \frac{ON}{OA}$$

$$= (\text{area of parallelogram OBEC})\ (ON)$$

$$= (\text{area of parallelogram OBEC})\ (AM)$$

$$= (\text{area of parallelogram OBEC})\ (\text{height of parallelepiped})$$

$$= \text{volume of parallelepiped OABCEFGH}$$

This means the scalar triple product gives the volume of parallelepiped whose adjacent sides are $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Linearly Independent Vectors

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three given vectors. If the scalar triple product of these vectors, $\vec{a}\cdot(\vec{b}\times\vec{c})$ is non zero then we called the vectors are linearly independent. That is, if $\vec{a}\cdot(\vec{b}\times\vec{c}) \neq 0$ then $\vec{a}, \vec{b}, \vec{c}$ are linearly independent.

Note: If the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent then they are coplanar.

4.4.2 Vector or Cross Product of Three Vectors

The vector product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$$

Justification:

Let,

$$\vec{a} = (a_1, a_2, a_3),\ \vec{b} = (b_1, b_2, b_3) \text{ and } \vec{c} = (c_1, c_2, c_3).$$

Here,

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_1c_2 - b_2c_1)\vec{i} - (b_1c_3 - c_1b_3)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}$$

Now,

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1c_2 - b_2c_1 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}.$$

$$= \sum_{i\,j\,k} [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\,\vec{i}$$

$$= \sum_{i\,j\,k} [a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3]\,\vec{i}$$

$$= \sum_{i\,j\,k} [a_2b_1c_2 + a_3b_1c_3 + a_1b_1c_1 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1]\,\vec{i}$$

$$= \sum_{i\,j\,k} [(a_1b_1 + a_2c_2 + a_3c_3)\,b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1]\,\vec{i}$$

$$= \sum_{i\,j\,k} [(\vec{a}\cdot\vec{c})b_1 - (\vec{a}\cdot\vec{b})c_1]\,\vec{i}$$

$$= (\vec{a}\cdot\vec{c})\sum_{i\,j\,k} b_1\vec{i} - (\vec{a}\cdot\vec{b})\sum_{i\,j\,k} c_1\vec{i}$$

$$= (\vec{a}\cdot\vec{c})(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) - (\vec{a}\cdot\vec{b})(c_1\vec{i} + c_2\vec{j} + c_3\vec{k})$$
$$= (\vec{a}\cdot\vec{c})(b_1, b_2, c_3) - (\vec{a}\cdot\vec{b})(c_1, c_2, c_3)$$
$$= (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$$

Geometrical Interpretation of Vector Product of Three Vectors

Let $\vec{a}\times(\vec{b}\times\vec{c})$ be the vector product of three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Since $(\vec{b}\times\vec{c})$ be a vector and is perpendicular to both vectors $\vec{b}$ and $\vec{c}$ therefore the vector $(\vec{b}\times\vec{c})$ is normal to the plane determined by $\vec{b}$ and $\vec{c}$. And therefore the vector $\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to the plane containing the vectors $\vec{a}$ and $(\vec{b}\times\vec{c})$. Therefore, the vectors $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}$ and $\vec{c}$ are coplanar.

Example 7: Find the volume of a parallelepiped whose concurrent edges are represented by $\vec{i}+\vec{j}+\vec{k}$, $2\vec{i}+\vec{j}-2\vec{k}$ and $3\vec{i}+2\vec{j}-\vec{k}$.

Solution:

Let the given vectors are

$\vec{a} = \vec{i}+\vec{j}+\vec{k}$, $\vec{b} = 2\vec{i}+\vec{j}-2\vec{k}$, $\vec{c} = 3\vec{i}+2\vec{j}-\vec{k}$.

Now, volume of a parallelepiped whose concurrent edges represented by $\vec{a}$, $\vec{b}$ and $\vec{c}$ is,

$$V = \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 3 & 2 & -1 \end{vmatrix}$$
$$= (1)(3) - (1)(4) + (1)(1)$$
$$= 3 - 4 + 1$$
$$= 0$$

So, the given vectors are coplanar.

Example 8: Find the value of p so that the vectors $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i}+2\vec{j}+3\vec{k}$ and $\vec{c} = 3\vec{i}+p\vec{j}+5\vec{k}$ are coplanar.

Solution:

Given that $\vec{a} = (2, -1, 1)$, $\vec{b} = (1, 2, 3)$ and $\vec{c} = (3, p, 5)$ are coplanar. So,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & p & 5 \end{vmatrix} = 0$$
$$\Rightarrow 2(10-3p) + 1(-4) + 1(P-6) = 0$$
$$\Rightarrow -5p + 10 = 0$$
$$\Rightarrow p = 2$$

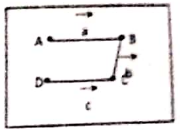
Thus the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar when p = 2.

Example 9: Prove that the four points $2\vec{i}+3\vec{j}-\vec{k}$, $\vec{i}-2\vec{j}+3\vec{k}$, $3\vec{i}+4\vec{j}-2\vec{k}$ and $\vec{i}-6\vec{j}+6\vec{k}$ are coplanar;

Solution:

Let, the given points are A, B, C and D. So, their position vectors are,

$$\overrightarrow{OA} = 2\vec{i}+3\vec{j}-\vec{k},\quad \overrightarrow{OB} = \vec{i}-2\vec{j}+3\vec{k},$$
$$\overrightarrow{OC} = 3\vec{i}+4\vec{j}-2\vec{k},\quad \overrightarrow{OD} = \vec{i}-6\vec{j}+6\vec{k}$$



Then,
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i}-5\vec{j}+4\vec{k}$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\vec{i}+6\vec{j}-5\vec{k}$$
$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -2\vec{i}-10\vec{j}+8\vec{k}$$

Now,
$$[\overrightarrow{AB}\ \ \overrightarrow{BC}\ \ \overrightarrow{CD}] = \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ -2 & -10 & 8 \end{vmatrix}$$
$$= (-1)(48-50) - (-5)(16-10) + (4)(-20+12)$$
$$= 2 + 30 - 32$$
$$= 0.$$

Thus, $[\overrightarrow{AB}\ \ \overrightarrow{BC}\ \ \overrightarrow{CD}] = 0$. This means the given points are coplanar.

Example 10: Show that the vector $\vec{a} = 4\vec{i} - 3\vec{j} + 2\vec{k}$, $\vec{b} = 2\vec{i} - 4\vec{j} - 4\vec{k}$ and $\vec{c} = 3\vec{i}+2\vec{j}-\vec{k}$ are linearly independent.

Solution:

Let,

$$\vec{a} = \vec{i}+2\vec{j}+2\vec{k};\quad \vec{b} = 3\vec{i}+7\vec{j}-4\vec{k};\quad \vec{c} = \vec{i}-5\vec{j}+3\vec{k}$$

Now,

$$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix}$$

$$= (1)(21-20) - (2)(9+4) + (2)(-15-7)$$

$$= 1 - 26 - 44$$

$$= -69$$

$$\neq 0.$$

This shows that the vectors are linearly independent.

Example 11: *Show that* $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0.$

Solution:

By definition we have

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$$

and $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

Here,

$$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$$

$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$$

$= 0$ [being the dot product is commutative]

Thus, $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0.$

Example 12: Show that $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ if and only if the vectors $\vec{a}$ and $\vec{c}$ are collinear.

Solution: Suppose that,

$$(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$$

$$\Leftrightarrow (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$\Leftrightarrow (\vec{c}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\vec{c} = 0$$

$$\Leftrightarrow (\vec{c}\cdot\vec{b})\vec{a} = (\vec{a}\cdot\vec{b})\vec{c}$$

$$\Leftrightarrow \vec{a} = \frac{(\vec{a}\cdot\vec{b})}{(\vec{c}\cdot\vec{b})}\vec{c}$$

$$\Leftrightarrow \vec{a} = \lambda\vec{c} \quad \text{where } \lambda = \frac{(\vec{a}\cdot\vec{b})}{(\vec{c}\cdot\vec{b})} = \text{some scalar quantity}$$

Thus, $\vec{a}$ and $\vec{c}$ are collinear.

Hence, $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ if and only if $\vec{a}$ and $\vec{c}$ are collinear.

4.5 Reciprocal System of Vectors

If $\vec{a}, \vec{b}, \vec{c}$ be a set of non coplanar vectors then the reciprocal vectors of $\vec{a}, \vec{b}, \vec{c}$ are denoted by $\vec{a}', \vec{b}', \vec{c}'$ and are defined by

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]},\ \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}\ \text{ and } \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$$

Properties of Reciprocal System of Vectors

1. The scalar product of a vector of a system with a corresponding vector of other system is always unity. That is, if $\vec{a}', \vec{b}', \vec{c}'$ are the reciprocal system to $\vec{a}, \vec{b}, \vec{c}$ then $\vec{a}\cdot\vec{a}' = 1, \vec{b}\cdot\vec{b}' = 1, \vec{c}\cdot\vec{c}' = 1.$

Proof: Here,

$$\vec{a}\cdot\vec{a}' = \vec{a}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} = \frac{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} = 1.$$

Similarly, $\vec{b}\cdot\vec{b}' = 1$ and $\vec{c}\cdot\vec{c}' = 1.$

2. The scalar product of a vector of a system with a non-corresponding vector of other system is always zero. That is,

$$\vec{a}\cdot\vec{b}' = 0,\ \vec{a}\cdot\vec{c}' = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = \vec{c}\cdot\vec{b}' = 0.$$

Proof: Here,

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Then

$$\vec{a}\cdot\vec{b}' = \vec{a}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} = \frac{[\vec{a}\ \ \vec{c}\ \ \vec{a}]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} = \frac{0}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} = 0$$

Similarly,

$$\vec{a}\cdot\vec{c}' = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = \vec{c}\cdot\vec{b}' = 0.$$

3. The orthogonal vectors $\vec{i}, \vec{j}, \vec{k}$ are itself reciprocal.

Proof:

Let $\vec{i}, \vec{j}, \vec{k}$ are orthogonal vectors. Then

$$[\vec{i} \quad \vec{j} \quad \vec{k}] = \vec{i}\cdot(\vec{j}\times\vec{k}) = \vec{i}\cdot(\vec{i}) = 1$$

Let $\vec{i}', \vec{j}', \vec{k}'$ are reciprocal vectors of $\vec{i}, \vec{j}, \vec{k}$. Then,

$$\vec{i}' = \frac{\vec{j}\times\vec{k}}{[\vec{i}\ \vec{j}\ \vec{k}]} = \frac{\vec{i}}{1} = \vec{i}$$

Similarly,

$$\vec{j}' = \vec{j} \text{ and } \vec{k}' = \vec{k}.$$

Example 13: Find the reciprocal system of the set of vectors. $2\vec{i} + 3\vec{j} - \vec{k}$, $-\vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} - 4\vec{j} + 2\vec{k}$.

Solution:

Let

$$\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k},\ \vec{b} = -\vec{i} + 2\vec{j} - 3\vec{k} \text{ and } \vec{c} = 3\vec{i} - 4\vec{j} + 2\vec{k}$$

Here,

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix} = -35$$

Also,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -1 & 2 & -3 \end{vmatrix} = -7\vec{i} + 7\vec{j} + 7\vec{k}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix} = -8\vec{i} - 7\vec{j} - 2\vec{k}$$

$$\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -4 & 2 \\ 2 & 3 & -1 \end{vmatrix} = -2\vec{i} + 7\vec{j} + 17\vec{k}$$

Let $\vec{a}', \vec{b}', \vec{c}'$ be the reciprocal system of $\vec{a}, \vec{b}, \vec{c}$. Then,

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{1}{35}(8\vec{i} + 7\vec{j} + 2\vec{k})$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{1}{35}(2\vec{i} - 7\vec{j} - 17\vec{k})$$

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{1}{5}(\vec{i} - \vec{j} - \vec{k}).$$

Example 14: If the reciprocal vectors of $\vec{a}$, $\vec{b}$ and $\vec{c}$ are $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ then show that the reciprocal of $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

Solution:

Let $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are reciprocal vectors of $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively then

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]},\ \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]},\ \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$$

Here,

$$\vec{b}'\times\vec{c}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} \times \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$$

$$= \frac{1}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2}\{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\}$$

$$= \frac{1}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2}\{[\vec{c}\ \ \vec{a}\ \ \vec{b}]\vec{a} - [\vec{c}\ \ \vec{a}\ \ \vec{a}]\vec{b}\}$$

$$= \frac{1}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2}[\vec{c}\ \ \vec{a}\ \ \vec{b}]\vec{a} \quad [\because (\vec{c}\ \ \vec{a}\ \ \vec{a}) = 0]$$

$$= \frac{1}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2}[\vec{a}\ \ \vec{b}\ \ \vec{c}]\vec{a}$$

$$= \frac{\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} \quad \text{..... (i)}$$

and

$$[\vec{a}'\ \ \vec{b}'\ \ \vec{c}'] = \vec{a}'\cdot(\vec{b}'\times\vec{c}')$$

$$= \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}\cdot\frac{\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$$

$$= \frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]} \qquad \text{.... (ii)}$$

Now, using (i) and (ii)

$$\frac{\vec{b}' \times \vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']} = \frac{\dfrac{\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}}{\dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}} = \vec{a}$$

Similarly,

$$\vec{b} = \frac{\vec{c}' \times \vec{a}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']} \text{ and } \vec{c} = \frac{\vec{a}' \times \vec{b}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$$

Exercise 4.2

1. a. If $\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{c} = \vec{i} - 3\vec{j} - 2\vec{k}$, then find $\left|\vec{a}\times(\vec{b}\times\vec{c})\right|$. Also, verify the formula for vector triple product.

   b. Verify the cross triple product by the vectors $\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{c} = \vec{i} + 3\vec{j} - 2\vec{k}$.

2. Find the volume of a parallelepiped whose concurrent edges are represented by the vectors.

   a. $\vec{i} + \vec{j} + \vec{k}$; $\vec{i} - \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} - \vec{k}$

   b. $2\vec{i} - 3\vec{j} + 4\vec{k}$, $\vec{i} + 2\vec{j} - \vec{k}$ and $3\vec{i} - \vec{j} + 2\vec{k}$

   c. $\vec{i} + 2\vec{j} + 2\vec{k}$, $3\vec{i} + 7\vec{j} - 4\vec{k}$ and $\vec{i} - 5\vec{j} + 3\vec{k}$

   d. $\vec{i} + \vec{j} + \vec{k}$, $2\vec{i} + \vec{j} + 3\vec{k}$ and $3\vec{i} + 2\vec{j} + 4\vec{k}$

3. Prove that the following vectors are coplanar;

   a. $\vec{i} + \vec{j} + \vec{k}$, $2\vec{i} + \vec{j} - 2\vec{k}$ and $3\vec{i} + 2\vec{j} - \vec{k}$

4. Determine the following four points are coplanar or not;

   a. $-6\vec{i} + 3\vec{j} + 2\vec{k}$, $3\vec{i} - 2\vec{j} + 4\vec{k}$, $5\vec{i} + 7\vec{j} + 3\vec{k}$ and $-13\vec{i} + 17\vec{j} - \vec{k}$

   b. $-\vec{i} + 4\vec{j} - 3\vec{k}$, $3\vec{i} + 2\vec{j} - 5\vec{k}$, $3\vec{i} + 8\vec{j} - 5\vec{k}$ and $-3\vec{i} + 2\vec{j} + \vec{k}$

   c. $4\vec{i} + 5\vec{j} + \vec{k}$, $-\vec{j} - \vec{k}$, $3\vec{i} + 9\vec{j} + 4\vec{k}$ and $-4\vec{i} + 4\vec{j} + 4\vec{k}$

5. The position vectors of the points A, B, C and D are $3\vec{i} - 2\vec{j} - \vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$, $-\vec{i} + \vec{j} + 2\vec{k}$ and $4\vec{i} + 5\vec{j} + p\vec{k}$ respectively. If the points A, B, C and D are coplanar, find the value of p.

6. a. Find the constant p such that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} + p\vec{j} + 5\vec{k}$ are coplanar.

   b. If the vectors $2\vec{i} - \vec{j} + 2\vec{k}$, $5\vec{i} + \lambda\vec{j}$ and $\vec{i} + 6\vec{k}$ are coplanar, find the value of λ.

   c. Find the value of λ, if the volume of parallelepiped whose edges are represented by $-12\vec{i} + \lambda\vec{k}$; $3\vec{j} - \vec{k}$ and $2\vec{i} + \vec{j} - 15\vec{k}$ is 546.

7. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors and $\vec{a} + \vec{b} + \vec{c} = 0$, then show that $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\frac{3}{2}$.

8. a. Show that $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ iff $(\vec{a}\times\vec{c})\times\vec{b} = 0$.

   b. Prove that $2\vec{a} = \vec{i}\times(\vec{a}\times\vec{i}) + \vec{j}\times(\vec{a}\times\vec{j}) + \vec{k}\times(\vec{a}\times\vec{k})$

   c. Show that $\left\{(\vec{a} + \vec{b} + \vec{c})\times(\vec{b}+\vec{c})\right\}\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$.

9. Find a set of reciprocal vector of

   a. $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = \vec{i} - \vec{j} - 2\vec{k}$ and $\vec{c} = -\vec{i} + 2\vec{j} + 2\vec{k}$.

   b. $-\vec{i} + \vec{j} + \vec{k}$, $\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + \vec{j} - \vec{k}$

10. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ are reciprocal system of each other then show that $[\vec{a}'\ \vec{b}'\ \vec{c}'] = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

Answers

1. a. $\sqrt{378}$
2. a. 4 b. 7 c. 69 d. 2
4. a. Coplanar b. Non-coplanar c. Non coplanar
5. $p = \dfrac{-146}{17}$
6. a. $p = -4$ b. $\lambda = -3$ c. $\lambda = -3$
9. a. $\vec{a}' = \dfrac{2\vec{i} + \vec{k}}{3}$; $\vec{b}' = \dfrac{-8\vec{i} + 3\vec{j} - 7\vec{k}}{3}$; $\vec{c}' = \dfrac{-7\vec{i} + 3\vec{j} - 5\vec{k}}{3}$

   b. $\vec{a}' = \dfrac{\vec{j} + \vec{k}}{2}$; $\vec{b}' = \dfrac{\vec{i} + \vec{k}}{2}$; $\vec{c}' = \dfrac{\vec{i} + \vec{j}}{2}$

4.6 Product Between Four Vectors

We already discuss that there are two types of product between vectors – dot or scalar product and cross or vector product.

4.6.1 Scalar or Dot Product of Four Vectors

The scalar product of two vectors each of which is again vector product of two vectors gives the scalar product of four vectors. That is,

$$\vec{u}\cdot\vec{v} = (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$$

where $\vec{u} = \vec{a}\times\vec{b}$ and $\vec{v} = \vec{c}\times\vec{d}$.

Definition

The scalar product of four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ is denoted given by

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

4.6.2 Vector or Cross Product of Four Vectors

The vector product of two vectors each of which is again vector product of two vectors gives the vector product of four vectors. That is,

$$\vec{u}\times\vec{v} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$$

where $\vec{u} = \vec{a}\times\vec{b}$ and $\vec{v} = \vec{c}\times\vec{d}$.

Definition

The vector product of four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ is denoted given by

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \ \vec{b}\ \ \vec{d}]\,\vec{c} - [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{d}$$

or equally

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \ \vec{c}\ \ \vec{d}]\,\vec{b} - [\vec{b}\ \ \vec{c}\ \ \vec{d}]\,\vec{a}$$

Example 15: Show that $\left[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}\right] = \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]^2$.

Solution:

Here,

$$\left[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}\right]$$

$$= (\vec{b}\times\vec{c})\cdot\left\{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\right\} \qquad [\because \text{By definition}]$$

$$= (\vec{b}\times\vec{c})\cdot\left\{[\vec{c}\ \ \vec{a}\ \ \vec{b}]\,\vec{a} - [\vec{c}\ \ \vec{a}\ \ \vec{a}]\,\vec{b}\right\}$$

$$= (\vec{b}\times\vec{c})\cdot\left\{[\vec{c}\ \ \vec{a}\ \ \vec{b}]\,\vec{a} - 0\right\} \qquad [\because [\vec{c}\ \ \vec{a}\ \ \vec{a}] = 0]$$

$$= (\vec{b}\times\vec{c})\cdot[\vec{c}\ \ \vec{a}\ \ \vec{b}]\,\vec{a}$$

$$= \left\{(\vec{b}\times\vec{c})\cdot\vec{a}\right\}\cdot[\vec{c}\ \ \vec{a}\ \ \vec{b}]$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]\cdot[\vec{a}\ \ \vec{b}\ \ \vec{c}]$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]^2$$

Thus, $\left[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}\right] = \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]^2$.

Example 16: Show that $(\vec{b}\times\vec{c})\cdot(\vec{b}\times\vec{a}) + (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c}) = b^2(\vec{a}\cdot\vec{c})$ and $(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c}) = [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{a}$.

Solution: Here,

$$(\vec{b}\times\vec{c})\cdot(\vec{b}\times\vec{a}) + (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$$

$$= \begin{vmatrix} \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{a} \\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a} \end{vmatrix} + (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$$

$$= (\vec{b}\cdot\vec{b})(\vec{c}\cdot\vec{a}) - (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{b}) + (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$$

$$= b^2(\vec{a}\cdot\vec{c}).$$

And,

$$(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c}) = \vec{u}\times(\vec{a}\times\vec{c}) \quad \text{where } \vec{u}\times\vec{a}\times\vec{b}$$

$$= (\vec{u}\cdot\vec{c})\,\vec{a} - (\vec{u}\cdot\vec{a})\,\vec{c}$$

$$= (\vec{a}\times\vec{b}\cdot\vec{c})\,\vec{a} - (\vec{a}\times\vec{b}\cdot\vec{a})\,\vec{c}$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{a} - 0$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{a}.$$

Example 17: Show that the vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to the vector $\vec{a}$.

Solution:

Here we have to show the vector

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) \quad \ldots \text{(i)}$$

is parallel to the vector $\vec{a}$. For this it sufficient to show the vector (i) is equal to scalar times $\vec{a}$.

Since we have,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

$$\Rightarrow (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{c}\ \vec{d}\ \vec{a}]\vec{b}-[\vec{c}\ \vec{d}\ \vec{b}]\vec{a}$$

Here,

$$\vec{r}=(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$

$$=[\vec{c}\ \vec{d}\ \vec{a}]\vec{b}-[\vec{c}\ \vec{d}\ \vec{b}]\vec{a}+[\vec{d}\ \vec{b}\ \vec{a}]\vec{c}-[\vec{d}\ \vec{b}\ \vec{c}]\vec{a}+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$

$$=[\vec{c}\ \vec{d}\ \vec{a}]\vec{b}-[\vec{c}\ \vec{d}\ \vec{b}]\vec{a}+[\vec{d}\ \vec{b}\ \vec{a}]\vec{c}-[\vec{d}\ \vec{b}\ \vec{c}]\vec{a}+[\vec{a}\ \vec{d}\ \vec{c}]\vec{b}-[\vec{a}\ \vec{d}\ \vec{b}]\vec{c}$$

$$=[\vec{c}\ \vec{d}\ \vec{a}]\vec{b}-[\vec{c}\ \vec{d}\ \vec{b}]\vec{a}+[\vec{d}\ \vec{b}\ \vec{a}]\vec{c}-[\vec{c}\ \vec{d}\ \vec{b}]\vec{a}-[\vec{c}\ \vec{d}\ \vec{a}]\vec{b}-[\vec{d}\ \vec{b}\ \vec{a}]\vec{c}$$

$$=-2[\vec{c}\ \vec{d}\ \vec{b}]\vec{a}$$

$\Rightarrow \vec{r}=\lambda\vec{a}$ where $\lambda=-2[\vec{c}\ \vec{d}\ \vec{b}]$ = scalar.

Hence $\vec{r}$ is parallel to $\vec{a}$.

Exercise 4.3

1. Show that $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})=[\vec{a}\ \vec{b}\ \vec{c}]\vec{c}$.
2. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar then show that $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})=0$.
3. Show that $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$
4. Show that $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0$
5. Show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})=-2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$.
6. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar then show that $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ are also coplanar.
7. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar then show that $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, and $\vec{a}\times\vec{b}$ are also non coplanar.
8. Show that the vectors $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar.

4.7 Application of Product of Two Vectors

Vector is useful tool in derivation and solving various problem related to physics, engineering and applied mechanics etc. As application in this section we use the vector to finding projection, equation of line and equation of plane.

4.7.1 Projection of a Vector on a Vector

Let $\vec{a}$ and $\vec{b}$ are two vectors. Then the projection of $\vec{a}$ onto $\vec{b}$ is given by,

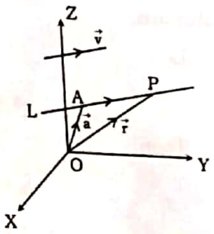
$$\text{Proj}_{\vec{b}}(\vec{a})=\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

And, the scalar projection of $\vec{a}$ onto $\vec{b}$ is,

$$\left|\text{Proj}_{\vec{b}}(\vec{a})\right|=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

4.7.2 Equation of a Line through a Given Point and Parallel to Given Vector

Suppose a line L in space is passes through $A=(a_1, a_2, a_3)$ and parallel to the given non-zero vector $\vec{v}=(b_1, b_2, b_3)$.



Let P(x, y, z) be any point on the line. Then $\overrightarrow{AP}$ will parallel to $\vec{v}$

i.e. $\overrightarrow{AP}=t\vec{v}$

$\Rightarrow \vec{r}-\vec{a}=t\vec{v}$

$\Rightarrow \vec{r}=\vec{a}+t\vec{v}$ .... (i)

$\Rightarrow x\vec{i}+y\vec{j}+z\vec{k}=(a_1+tb_1)\vec{i}+(a_2+tb_2)\vec{j}+(a_3+tb_3)\vec{k}$.

Equating we get

$x=a_1+tb_1$, $y=a_2+tb_2$, $z=a_3+tb_3$

$\Rightarrow \frac{x-a_1}{b_1}=t$, $\Rightarrow z\frac{y-a_2}{b_2}=t$, $\Rightarrow \frac{z-a_3}{b_3}=t$

Therefore,

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3} \quad .... \text{(iii)}$$

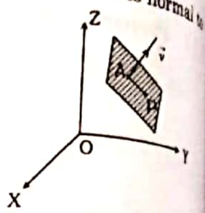
Here, the direction cosines proportional to b_1, b_2, b_3.

4.7.3 Vector Equation of Plane Normal to Given Vector and throug[h] Given Point

Let $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ be a vector normal to a plane which passes through the poi[nt] $A(x_1, y_1, z_1)$. Let $P(x, y, z)$ be any other point on the plane. Then

$$\overrightarrow{AP} = (x - x_1, y - y_1, z - z_1).$$

As our assumption, the vector $\vec{v}$ is normal to the plane. This means $\vec{v}$ is normal t[o] $\overrightarrow{AP}$. So,



$$\vec{v} \cdot \overrightarrow{AP} = 0$$

$$\Rightarrow (a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = 0$$

$$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \ldots\ldots\ldots (i)$$

Example 18: Find the vector projection of $\vec{a}$ onto $\vec{b}$ if $\vec{a} = 3\vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$.

Solution:

Let,

$$\vec{a} = 3\vec{i} - \vec{j} + \vec{k} \quad \text{and} \quad \vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}.$$

Then,

$$\vec{a} \cdot \vec{b} = (3)(2) + (-1)(1) + (1)(-2) = 6 - 1 - 2 = 3.$$

And, $\quad |\vec{b}| = \sqrt{4 + 1 + 4} = 3.$

Now, vector projection of $\vec{a}$ onto $\vec{b}$ is,

$$\text{Proj}_{\vec{b}}(\vec{a}) = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b} = \left(\frac{3}{3^2}\right)\left(2\vec{i} + \vec{j} - 2\vec{k}\right)$$

$$= \left(\frac{2}{3}\right)\vec{i} + \left(\frac{1}{3}\right)\vec{j} - \left(\frac{2}{3}\right)\vec{k}$$

And, the scalar projection of $\vec{a}$ onto $\vec{b}$ is,

$$\left|\text{Proj}_{\vec{b}}(\vec{a})\right| = \left|\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}\right| = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)|\vec{b}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{3}{3} = 1.$$

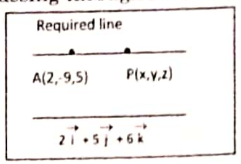
Example 19: Find the equation of line through (2, –9, 5) and is parallel to $2\vec{i} + 5\vec{j} + 6\vec{k}$.

Solution:

Given point is A(2, –9, 5) and the given vector is

$$\vec{v} = 2\vec{i} + 5\vec{j} + 6\vec{k}.$$

Let P(x, y, z) be any point on line and A(2, –9, 5) which is passing through A. Then,



$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x - 2)\vec{i} + (y + 9)\vec{j} + (2 - 5)\vec{k}.$$

Given that $\vec{v}$ and $\overrightarrow{AP}$ are parallel. So,

$$\overrightarrow{AP} = k\vec{v} \quad \text{for some scalar k}$$

$$\Rightarrow (x - 2)\vec{i} + (y + 9)\vec{j} + (2 - 5)\vec{k} = k(2\vec{i} + 5\vec{j} + 6\vec{k}).$$

Equating the coefficient of $\vec{i}, \vec{j}, \vec{k}$ on both sides then we get,

$$x - 2 = 2k, \qquad y + 9 = 5k, \qquad z - 5 = 6k$$

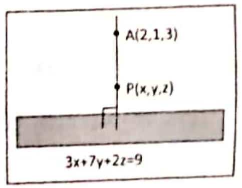
$$\Rightarrow \frac{x - 2}{2} = k, \qquad \Rightarrow \frac{y + 9}{5} = k, \qquad \Rightarrow \frac{z - 5}{6} = k$$

Therefore,

$$\frac{x - 2}{2} = \frac{y + 9}{5} = \frac{z - 5}{6}$$

This is the equation of required line.

Example 20: Find the equation of line through (2, 1, 3) and is perpendicular to $3x + 7y + 2z = 9$.



Solution:

Given equation of plane is

$$3x + 7y + 2z = 9 \quad \ldots (i)$$

Then the vector normal to plane (i) is

$$\vec{n} = 3\vec{i} + 7\vec{j} + 2\vec{k}$$

Again, the given point is A(2, 1, 3) and let P(x, y, z) be the general point on the line which is passing through A. Then the position vector of P and A are,

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + 2\vec{k} \quad \text{and} \quad \overrightarrow{OA} = 2\vec{i} + \vec{j} + 3\vec{k}.$$

Then,

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x - 2)\vec{i} + (y - 1)\vec{j} + (z - 3)\vec{k}.$$

As given condition, $\overrightarrow{AP}$ is perpendicular to the plane (i) and $\vec{n}$ is perpendicular to (i). Therefore, $\overrightarrow{AP}$ and $\vec{n}$ both are perpendicular to same plane.

This means the vectors $\overrightarrow{AP}$ is parallel to $\vec{n}$. So,

$$\overrightarrow{AP} = \lambda \vec{n} \text{ where } \lambda \text{ be some scalar.}$$

$$\Rightarrow \quad (x-2)\vec{i} + (y-1)\vec{j} + (z-3)\vec{k} = \lambda\left(3\vec{i} + 7\vec{j} + 2\vec{k}\right).$$

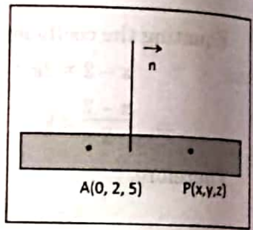
$$\Rightarrow \quad \frac{x-2}{3} = \frac{y-1}{7} = \frac{z-3}{2} = \lambda$$

$$\Rightarrow \quad \frac{x-2}{3} = \frac{y-1}{7} = \frac{z-3}{2}$$

This is the equation of required line.

Example 21: Find the equation of plane through (0, 2, 5) and is normal to $2\vec{i} + 4\vec{j} + \vec{k}$.

Solution:



Given point is A(0, 2, 5), so $\vec{a} = \overrightarrow{OA} = 0.\vec{i} + \vec{j} + 5\vec{k}$.

Let P(x, y, z) be general point of the line which passing through A. Then,

$$\vec{r} = \overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Therefore,

$$\vec{r} - \vec{a} = (x-0)\vec{i} + (y-2)\vec{j} + (z-5)\vec{k}.$$

Also, given vector is, $\vec{n} = 2\vec{i} + 4\vec{j} + \vec{k}$.

Now, the equation of plane which is passing through $\vec{a}$ and is normal to $\vec{n}$ is,

$$\left(\vec{r} - \vec{a}\right) \cdot \vec{n} = 0.$$

$$\Rightarrow \quad \{(x-0)\vec{i} + (y-2)\vec{j} + (z-5)\vec{k}\} \cdot (2\vec{i} + 4\vec{j} + \vec{k}) = 0.$$

$$\Rightarrow \quad (x-0)(2) + (y-2)(4) + (2-5)(1) = 0$$

$$\Rightarrow \quad 2x + 4y - 8 + z - 5 = 0$$

$$\Rightarrow \quad 2x + 4y + z - 13 = 0$$

$$\Rightarrow \quad 2x + 4y + z = 13$$

This is the equation of required plane.

Example 21: Find the equation of the plane through the points (2, 4, 5), (1, 5, 7), (−1, 6, 8).

Solution:

The given points A(2, 4, 5), B(1, 5, 7), C(−1, 6, 8).

Then the position vectors of the points are

$$\overrightarrow{OA} = 2\vec{i} + 4\vec{j} + 5\vec{k}, \quad \overrightarrow{OB} = \vec{i} + 5\vec{j} + 7\vec{k}, \quad \text{and, } \overrightarrow{OC} = -\vec{i} + 6\vec{j} + 8\vec{k}.$$

Then,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} + \vec{j} + 2\vec{k}.$$

And,

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -2\vec{i} + \vec{j} + \vec{k}.$$

Since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lies on the same plane. So, $\overrightarrow{AB} \times \overrightarrow{BC}$ is normal to the plane.

Here,

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ -2 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}\,(1-2) - \vec{j}\,(-1+4) + \vec{k}\,(-1+2)$$

$$= -\vec{i} - 3\vec{j} + \vec{k}.$$

Let P(x, y, z) be any point on the plane whose position vector is,

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$$

Then,

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x-2)\vec{i} + (y-4)\vec{j} + (2-5)\vec{k}.$$

Since both A and P are two points on the plane, so $\overrightarrow{AP}$ lies on the plane and $\vec{n}$ is normal to the plane. Therefore, $\overrightarrow{AP}$ is normal to $\vec{n}$.

Therefore,

$$\overrightarrow{AP} \cdot \vec{n} = 0$$

$$\Rightarrow \quad \left\{(x-2)\,\vec{i} + (y-4)\,\vec{j} + (2-5)\vec{k}\right\} \cdot \left(-\vec{i} - 3\vec{j} + \vec{k}\right) = 0.$$

$$\Rightarrow \quad (x-2)(-1) + (y-4)(-3) + (2-5)(1) = 0$$

$$\Rightarrow \quad -x + 2 - 3y + 12 + 2 - 5 = 0$$

$$\Rightarrow \quad -x - 3y + z + 9 = 0$$

$$\Rightarrow \quad x + 3y - z = 9.$$

Therefore, $x - 3y - z = 9$ be the equation of required plane.

Example 22: Find the equation for the plane through (2, 4, 5) and is perpendicular to the line $\frac{x-5}{1}=\frac{y-1}{3}=\frac{z}{4}$.

Solution:

Given equation of line is

$$\frac{x-5}{1}=\frac{y-1}{3}=\frac{z}{4}=\lambda(\text{say}) \qquad \text{... (i)}$$

$$\Rightarrow \quad (x-5)\vec{i}+(y-1)\vec{j}+z\vec{k} = \lambda(\vec{i}+3\vec{j}+4\vec{k})$$
$$= \lambda\vec{n}$$

where $\vec{n}=\vec{i}+3\vec{j}+4\vec{k}$; which is parallel to given line.

Given that the plane passes through the point A(2, 4, 5) and let P(x, y, z) be a any point on the plane. So,

$$\overrightarrow{OP}=x\vec{i}+y\vec{j}+z\vec{k} \text{ and } \overrightarrow{OA}=2\vec{i}+4\vec{j}+5\vec{k}.$$

Then,

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=(x-2)\vec{i}+(y-4)\vec{j}+(2-5)\vec{k}.$$

lies on the plane.

Given that the line (i) is perpendicular to the plane that is through (2, 4, 5). So, $\vec{n}$ is perpendicular to $\overrightarrow{AP}$. Therefore,

$$\overrightarrow{AP}\cdot\vec{n}=0.$$

$$\Rightarrow \quad \{(x-2)\vec{i}+(y-4)\vec{j}+(2-5)\vec{k}\}\cdot(\vec{i}+3\vec{j}+4\vec{k})=0.$$

$$\Rightarrow \quad (x-2)(1)+(y-4)(3)+(2-5)(4)=0$$

$$\Rightarrow \quad x-2+3y-12+4z-20=0.$$

$$\Rightarrow \quad x+3y+4z=34.$$

This is the equation of required plane.

Example 23: Find plane through the points (1, 2, 3) and (3, 2, 1) which is perpendicular to the plane $4x-y+2z=7$.

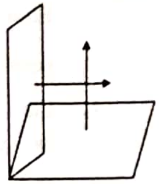
Solution:

Given that the required plane passes through A(1, 2, 3), B(3, 2, 1).

So, $\overrightarrow{AB}=(2, 0, -2)$

lies on the plane.

Let $\vec{n}=(a, b, c)$ be a vector perpendicular to the required plane. Then $\vec{n}$ is perpendicular to $\overrightarrow{AB}$. So,



$$\overrightarrow{AB}\cdot\vec{n}=0$$

$$\Rightarrow \quad 2a-2c=0 \qquad \text{... (i)}$$

Also, given that the plane $4x-y+2z=7$ is perpendicular to the required plane. So, the vector (4, –1, 2) is parallel to the required plane. Therefore,

(4, –1, 2) is normal to $\vec{n}$. So,

$$(4, -1, 2)\cdot\vec{n}=0$$

$$\Rightarrow \quad 4a-b+2c=0 \qquad \text{... (ii)}$$

Solving equation (i) and (ii), we get

$$a=c=\frac{b}{6}=k\ (\text{say})$$

$$\Rightarrow \quad a=k,\ b=6k,\ c=k$$

Thus, $\vec{n}=(k, 6k, k)$ is normal to the plane.

Hence, the equation of plane is passing through $\vec{a}$ and normal to $\vec{n}$ be

$$(\vec{r}-\vec{a})\cdot\vec{n}=0 \qquad \text{for } \vec{a}=\overrightarrow{OA}$$

$$\Rightarrow \quad (x-1, y-2, z-3)\cdot(1, 6, 1)=0 \qquad \text{for } k\neq 0$$

$$\Rightarrow \quad x+6y+z=16$$

This is the equation of required plane.

Exercise 4.4

1. a. Find the projection of the vector $3\vec{i}-\vec{j}+\vec{k}$ on the vector $2\vec{i}+\vec{j}-\vec{k}$.

   b. If $\vec{a}=2\vec{i}-2\vec{j}+\vec{k}$, $\vec{b}=\vec{i}+\vec{j}+\vec{k}$ find the projection of $\vec{b}$ onto $\vec{a}$.

   c. Find the scalar projection of $\vec{a}=\vec{i}-2\vec{j}+\vec{k}$ and $\vec{b}=\vec{i}+2\vec{j}-\vec{k}$.

2. a. Find the equation of the line through (1, 2, 3) and perpendicular to the plane $4x+3y+z=4$.

   b. Find the parametric and Cartesian equations for the line through (1, 2, 3) and parallel to the vector $3\vec{i}-\vec{j}+2\vec{k}$.

   c. Find the equation of line through (1, 2, 0) and (1, 1, –1).

   d. Find the equation of the line passing through (1, 2, 3) and (3, 2, 7); by vector method.

3. a. Find the equation of the plane passing through the point $\vec{i} + \vec{j} + \vec{k}$ and perpendicular to the vector $2\vec{i} + 3\vec{j} - 4\vec{k}$.

b. Find the equation of the plane through (1, –1, 3), parallel to the plane $3x + y + z = 7$.

c. Find the equation for the plane through the points (1, 0, 0), (0, 1, 0), (0, 0, 1).

d. Find by vector method the equation of the plane which passes through the points (1, 1, –1), (2, 0, 2) and (0, –2, 1).

e. Find the equation of a plane through (3, 2, 1), (–1, 1, –2) and (3, –4, 1).

f. By vector method, find the equation of the plane through the origin and that contains the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{4}$.

g. Find the plane through A(1, –2, 1) perpendicular to the vector from the origin to A.

h. Using vector method, obtain the equation of the plane through (2, 1, –1) which is perpendicular to the line of the intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.

Answers

1. a. $\left(\frac{12}{\sqrt{6}}\right)\vec{i} - \left(\frac{4}{\sqrt{6}}\right)\vec{j} + \left(\frac{4}{\sqrt{6}}\right)\vec{k}$

b. $\left(\frac{2}{3}\right)\vec{i} - \left(\frac{2}{3}\right)\vec{j} + \left(\frac{1}{3}\right)\vec{k}$

c. $\left(\frac{-4}{\sqrt{6}}\right)$

2. a. $\frac{x-1}{4} = \frac{y-2}{3} = \frac{z-3}{1}$

b. $x = 1 + 3t,\ y = 2 - t,\ z = 3 + 2t;\ \frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{2}$

c. $\frac{x-1}{0} = \frac{y-2}{-1} = \frac{z}{-1}$

d. $\frac{x-1}{2} = \frac{y-2}{0} = \frac{z-3}{4}$

3. a. $2x + 3y - 4z = 1$

b. $3x + y + z = 5$

c. $x + y + z = 1$

d. $7x - 5y - 4z - 6 = 0$

e. $3x - 4z + 5 = 0$

f. $4x + 4y - 5z = 0$

g. $x - 2y + z = 6$

h. $x - y + z = 0$