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CHAPTER

MATRIX AND SYSTEM OF LINEAR EQUATIONS

Introduction

Linear algebra is a fairly extensive subject that covers vector and matrix, determinant and their application to various forms. Matrix, which is rectangular arrays of numbers or functions is the main tool of linear algebra. Linear algebra is an active field that has many applications in engineering physics, economics, and others. Matrices are important because they let us express large amounts of data and functions in an organized and concise form. Matrices can hold enormous amounts of data. It also contributes to a deeper understanding of mathematics itself. And, the determinant provides the scalar value of the square matrix that plays a vital role in solving of the system of linear equations.

1.1 Matrix and Determinant

Matrices are used to represent relation between quantities. For example, a plane vector can be represented by two numbers arranged in a single column in which the upper number represents its x-component and the lower number represents its y-component. Matrices can also be used to represent and solve a system of linear equations. Matrix mechanics was the first formulation of quantum mechanics stated by Werner Heisenberg in 1925 and was developed by Heisenberg and Max Born and the German Physicist Pascual Jordan (1902 – 1980).

In fact, a set or block of numbers are arranged in some rows and columns forming a rectangle and enclosed in parentheses is called a matrix. Such blocks of numbers arise while dealing with problems in different fields such as Physics, Chemistry and Social Sciences etc.

Definition of Matrix

A rectangular arrangement of numbers (may real or complex), that are enclosed by a bracket [ ] or ( ), is called a matrix. It is denoted by capital alphabet A, B, C, . . . etc. For example,

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix},\ B = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \text{ etc.}$$

Notation and Terminology

Matrices are associated with some rows and columns. In any matrix, a particular i^{th} row is denoted by R_i and j^{th} column is denoted by C_j. The total number of rows (say m) followed by the total number of columns (say n) of a matrix written as $m \times n$ is called the order or size or dimension of the matrix. The numbers which constitute a matrix are called the elements or entries of the matrix. Matrices are usually denoted by the capital letters A, B, C . . . etc or $A_{m\times n}$, $B_{p\times q}$, $C_{r\times s}$ . . . etc. where the double subscripts $m \times n$, $p \times q$, $r \times s$, . . . etc. in the product form represent the orders of the matrices.

Any element of a matrix A which belongs to the i^{th} row and j^{th} column is called $(i, j)^{th}$ element of the matrix and that particular element is denoted by a small letter a_{ij} with double subscript where the first letter i stands for the row position and the second letter j stands for the column position of the element.

A matrix of order $m \times n$ is written by using small letters with double subscript as follows:

$$A = (a_{ij})_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

or

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

$A = (a_{ij})_{m\times n}$ is also written as $A = (a_{ij})$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$

In particular, if B is a matrix of order 2×3, we write

$$B = (b_{ij})_{2\times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

Definition of Determinant

A unique real number associated with a square matrix A of order m of real numbers is called the determinant of A of order m and is denoted by |A| or det(A).

Example 1: The value of the determinant corresponding to a square matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ of order 2 is given by

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example 2: Show that $\begin{vmatrix} 6 & 1 & 9 \\ 2 & 4 & 7 \\ 18 & 3 & 27 \end{vmatrix} = 0.$

Solution:

Here, $\begin{vmatrix} 6 & 1 & 9 \\ 2 & 4 & 7 \\ 18 & 3 & 27 \end{vmatrix}$

$= 3\begin{vmatrix} 6 & 1 & 9 \\ 2 & 4 & 7 \\ 6 & 1 & 9 \end{vmatrix}$

$= 3 \times 0$ being R_1 and R_3 have same entries.

$= 0$

Singular and Non-Singular Matrices

As the determinant value of a matrix, it classified into two types-singular and non-singular matrix.

Definition

A square matrix A is said to be a singular matrix if |A| = 0. Otherwise, the matrix is called non-singular i.e. the matrix A is non-singular if $|A| \neq 0$.

Example 3:

(i) Let $A = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ then A is a singular matrix being

$$|A| = \begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 3\times4 - 2\times6 = 12 - 12 = 0.$$

(ii) Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ then A is a non-singular matrix being

$$|A| = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 0 + 2 = 2 \neq 0.$$

1.2 Systems of Linear Equations

A linear equation in the variables $x_1, x_2, \dots, x_n$ is an equation in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where the coefficients $a_1, a_2, \dots, a_n$ and the value of b are real or complex numbers.

Definition (System of Linear Equations)

A collection of one or more linear equations, is a **system of linear equations** or a **linear system.**

Example 4: The systems,

(a) $x_1 + 5x_2 = 7$

$2x_1 + 7x_2 = 5.$

(b) $x_2 - 4x_3 = 8$

$2x_1 - 3x_2 + 2x_3 = 1$

$5x_1 - 8x_2 + 7x_3 = 1.$

are examples of linear system.

Definition (Solution of the System of Linear Equations)

A solution of the system is a list of values $x_1, x_2, \ldots, x_n$ of number that satisfies the given system.

For an example, the linear system

$$x_1 + 5x_2 = 7$$
$$2x_1 + 7x_3 = 5$$

is satisfied by a list $(x_1, x_2) = (-8, 3)$. So, $(-8, 3)$ is the solution of the system.

Note that the solution of a linear system does not necessarily exist. This means, some linear system may not have solution. And, sometimes the system has exactly one solution and sometimes the system has infinitely many solutions.

1.3 Classification of System of Linear Equations

As the solution of the system of linear equations, we classify the system as: consistent and inconsistent.

Definition (Consistent and Inconsistent System)

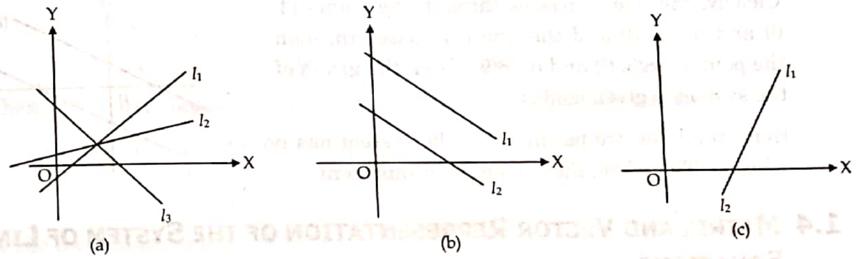
A system of linear equations is called **consistent** if it has solution (that may be one solution or infinitely many solutions) and is called **inconsistent** if it has no solution.

Note that the consistent system are of two different types: has unique solution and has infinite solution.

Graphical Representation of Consistency of Linear Equations

Geometrically, linear equation represents a straight line in two variables. A system of linear equations is a set of straight lines in which the number of lines is exactly equal to number of equations in the system. The system of linear equations is consistent if the equations have a common (at least one) solution. This means, if the lines of the system intersect each other at a point (at least one point) then the system is called consistent and if they all are not intersected at any point then the system is called inconsistent. For instance,

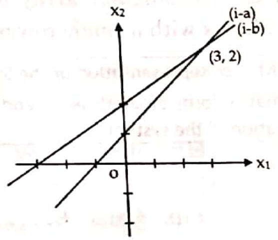
1. The equation of lines l_1, l_2 and l_3 in figure (a) is consistent and has exactly one solution because the lines are intersect at a point.
2. The equations of lines l_1 and l_2 are parallel. So, the system of equations of l_1 and l_2, is inconsistent, in figure (b).
3. The line l_1 is overlap to l_2. This means the lines have infinitely many points of intersect. So, the system of equations of line l_1 and l_2 has infinitely many solutions and is consistence, in figure (c).



Example 5: Consider a system

$$x_1 - 2x_2 = -1$$
$$x_1 - 3x_2 = -3$$

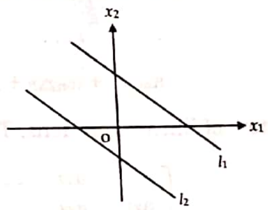
Lines satisfy by the point (3, 2). So, the system is consistent and has exactly one solution because the lines intersect each other only at a point.



Example 6: Consider a system

$$x_1 + 2x_2 = -1$$
$$x_1 + 2x_2 = 2$$

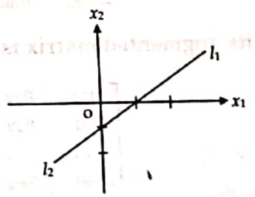
This shows that the lines represented by the system, are parallel. So, the system has no solution. The graphical representation of the system shows the lines are parallel. So, the system is inconsistent.



Example 7: Consider a system

$$x_1 - x_2 = 1$$
$$-3x_1 + 3x_2 = -3$$

This system has the overlapping lines. So, the system has infinite solutions and is consistent.



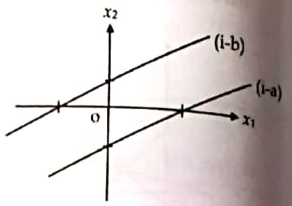
Example 8: Why the system $x_1 - 3x_2 = 4$, $-3x_1 + 9x_2 = 8$ is inconsistent? Give graphical representation.

Solution:

Given system is,

$$x_1 - 3x_2 = 4 \quad \text{...(i)}$$
$$-3x_1 + 9x_2 = 8 \quad \text{... (ii)}$$

Clearly, the line (i) passes through the points (4, 0) and (0, −4/3) and the line (ii) passes through the points (−8/3, 0) and (0, 8/9). Then the graph of the system is given asides.



Here, the lines are parallel. So, the system has no solution. Therefore, the system is inconsistent.

1.4 Matrix and Vector Representation of the System of Linear Equations

A matrix is a rectangular array of numbers or functions which we will enclose in brackets. Matrix with a single row or column are called vectors.

Definition (Matrix Representation of the System)
A matrix form of coefficients and the constant values of a linear system is known as matrix notation of the system.

Let,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots. + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots. + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots. + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \ldots (1)$$

be a system of linear equations. Then the coefficient matrix of (1) is

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \quad \ldots (2)$$

and its augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} : b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} : b_2 \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} : b_m \end{bmatrix} \quad \ldots (3)$$

If the matrix notation involves only the coefficient of variable then the matrix is called **coefficient matrix**. And, if the matrix notation involves the coefficient of linear system as well as constant value then the matrix is called **augmented matrix** of the system.

Example 9: Consider a linear system

$$x_2 + 4x_3 = -5$$
$$x_1 + 3x_2 + 5x_3 = -2$$
$$3x_1 + 7x_2 + 7x_3 = 6$$

Then, the matrix notation with coefficients of each variable in the form,

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 3 & 5 \\ 3 & 7 & 7 \end{bmatrix}$$

is the coefficient matrix.

And, the matrix notation of the system

$$\begin{bmatrix} 0 & 1 & 4 & : & -5 \\ 1 & 3 & 5 & : & -2 \\ 3 & 7 & 7 & : & 6 \end{bmatrix}$$

is the augmented matrix.

Elementary Row Operations on Matrix

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Example 10: Solve the systems of linear equation by elementary row operation.

$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2.$$

Solution:

Given system is,

$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2.$$

The matrix notation of the system is,

$$\begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 2 & 2 & 9 & : & 7 \\ 0 & 1 & 5 & : & -2 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 2 & 2 & 9 & : & 7 \\ 0 & 1 & 5 & : & -2 \end{bmatrix}$$

Apply $R_3 \to 2R_3 - R_2$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 0 & 2 & 15 & : & -9 \\ 0 & 0 & -5 & : & 5 \end{bmatrix} \text{ is triangular form.}$$

The equation form of the matrix notation is,

$$x_1 - 3x_3 = 8 \qquad \ldots \text{(i)}$$

$$2x_2 + 15x_3 = -9 \qquad \ldots \text{(ii)}$$

$$-5x_3 = 5 \qquad \ldots \text{(iii)}$$

From (iii), we get $x_3 = -1$.

Substituting the value $x_3 = -1$ in (ii) then it gives,

$$2x_2 - 15 = -9.$$

$$\Rightarrow \quad x_2 = 3.$$

And, substituting the value $x_3 = -1$ in (i) then it gives,

$$x_1 + 3 = 8.$$

$$\Rightarrow \quad x_1 = 5.$$

Thus, the solution of the given linear system is $(x_1, x_2, x_3) = (5, 3, -1)$.

1.5 Row Reduction and Echelon Forms

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

The following matrices are in echelon form:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

In which the second matrix is the row reduced echelon form.

The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the started entries (*) may have any value (including zero).

$$\begin{bmatrix} ■ & * & * & * \\ 0 & ■ & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & ■ & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & ■ & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & ■ & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & ■ & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ■ & * \end{bmatrix}$$

The following matrices are in row reduced echelon form because the leading entries are 1's, and there are 0's below and above each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Note:

1. Different sequences (operations or formulae) of row operation gives different echelon matrix of any non-zero matrix. This means, a matrix may have more than one echelon matrix as the operated row operations.
2. Remember that a non-zero matrix have a unique (i.e. one and only one or exactly one) reduced echelon matrix.

This note leads the concept of following theorem.

Theorem 1 (Uniqueness of the Row Reduced Form)

Each matrix is row equivalent to one and only one row reduced echelon matrix.

Here, the row reduces the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible-namely, in the second column. Choose the 2 in this position as the next pivot.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad \ldots(1)$$

Next pivot column

Add $\frac{-5}{2}$ times row 2 to row 3, and add 3/2 times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \quad \ldots(2)$$

There is no way to create a leading entry in column 3! (We can't use row 1 or 2 because doing so would destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{General form} \quad \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

Pivot positions

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \quad \ldots(3)$$

Pivot columns

A pivot, as in above, is a nonzero number in a pivot position that is used as needed to create zeros via row operations. The pivots as above were 1, 2, and –5. Notice that these numbers are not the same as the actual elements of A in the highlighted pivot positions shown in (3).

Definition (Pivot Position)

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Row Reduction Algorithm

The algorithm helps to obtain a matrix in echelon form and in reduced echelon form. The process of the algorithm is as follows:

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the sub matrix that remains. Repeat the process until there are no more nonzero rows to modify.
5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Existence and Uniqueness Questions

In our study of linear system, we observed that sometimes, the linear system consists free variable (or free variables) and sometimes there is no one such free variable exist. If, the system has free variable then the system will satisfied by many different solutions (as different value of the free variable). This means the solution have infinitely many solutions. On the other hand, if a system has no one free variable then the solution will unique.

The following theorem leads the concept:

Theorem 2 (Existence and Uniqueness Theorem)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. (That is, a linear system has solution if and only if an echelon form of the augmented matrix has no row of the form [0 0 ... 0 b] with $b \neq 0$). If the system is consistent then the solution set contains either unique solution when there is no free variable or infinitely many solutions when there is at least one free variable.

Example 11: Consider a matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 5 \\ 3 & 1 & 1 & 8 \end{bmatrix}$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - 3R_1$ then

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -1 \\ 0 & -2 & -2 & -10 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ then

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & -9 \end{bmatrix}$$

which is the echelon form of the given matrix A.

Example 12: Consider a matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & -3 & 3 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Interchanging R_1 and R_2 to make non-zero pivot then,

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Apply $R_4 \to R_4 - 3R_1$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_2$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

Apply $R_4 \to R_4 + 3R_3$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix} \text{ is a echelon form}$$

Now, for row reduced echelon form, apply $R_3 \to \frac{R_3}{3}$ and $R_4 \to \frac{R_4}{-5}$ then the above matrix reduces to,

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -4/3 & 7/3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Apply $R_2 \to R_2 + 3R_4$ and $R_3 \to R_3 + \frac{3}{4}R_4$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Apply $R_1 \to R_1 - 3R_3$ then the matrix reduces to,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

is a row reduced echelon form or canonical form of given matrix A.

Example 13: Illustrate by an example that a system of linear equation has either no solution or exactly one solution.

Solution:

Consider linear equations $x + y = 5$ and $x + y = 3$.

The augmented matrix of the system and the echelon form is

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

This shows the system is inconsistent (because last row is a form of $[0 \quad 0 \quad b]$, where $b \neq 0$) so has no solution.

Consider the linear equations

$$x + y = 3$$
$$x - y = 1$$

The augmented matrix and its echelon form is

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \end{bmatrix}$$

This shows the system is consistent (because last row is not a form of $[0 \quad 0 \quad b]$, where $b \neq 0$) and x_1 and x_2 are basic variable but there is no free variable so the system has exactly one solution.

Example 14: Solve the following system of linear equations, if consistent.

$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$2x + 19y - 47z = 32$$

Solution:

The augmented matrix of given system is,

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$$

Applying $R_2 \to R_2 - \frac{3}{2}R_1$ and $R_3 \to R_3 - R_1$ then

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & 11/2 & -27/2 & 11/2 \\ 0 & 22 & -54 & 27 \end{bmatrix}$$

Applying $R_2 \to 2R_2$ then

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 22 & -54 & 27 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$ then

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Applying $R_1 \to R_1/2$ and $R_2 \to R_2/11$ then

$$\begin{bmatrix} 1 & -3/2 & 7/2 & 5/2 \\ 0 & 1 & -27/11 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Applying $R_1 \to R_1 + \frac{3}{2} R_2$ then

$$\begin{bmatrix} 1 & 0 & 75/11 & 4 \\ 0 & 1 & -27/11 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Here, the last row is in form of $[0 \quad 0 \quad 0 \quad b]$, $b \neq 0$. So, system is inconsistent.

Example 15: Determine the value of h and k so that system of linear equation

$$x_1 + 3x_2 = 2$$
$$3x_1 + hk_2 = k$$

has infinite many solution.

Solution:

The augmented matrix of given system is,

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & 3k-2 \end{bmatrix}$$

is an echelon form.

For infinite solution in above echelon form must has at least one free variable, and is possible if

$$h - 9 = 0 \text{ and } 3k - 2 = 0$$

i.e. $h = 9$ and $k = 2/3$

Example 16: If one row in an echelon form of an augmented matrix is $[0 \quad 0 \quad 0 \quad 5]$ then the associated linear system is inconsistent.

Solution:

Given that an augmented matrix has a row of the form $[0 \quad 0 \quad 0 \quad 5]$ so, the associated equation of the row is,

$$0.x_1 + 0.x_2 + 0.x_3 = 5.$$
$$\Rightarrow 0 + 0 + 0 = 5.$$
$$\Rightarrow 0 = 5.$$

This is not possible. This means, the system has no solution. So, the linear system is inconsistent.

Exercise 1.1

1. Determine the values of h such that the matrix is the augmented matrix of a consistent linear system

   a. $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -2 & 3 \\ 3 & h & -2 \end{bmatrix}$

2. Find the value of h and k so that the system has (i) no solution (ii) a unique solution and (iii) many solutions for the system of equation.

   c. $x_1 + hx_2 = 2$, $4x_1 + 8x_2 = k$ d. $x_1 + 3x_2 = 2$, $3x_1 + hx_2 = k$

3. Find the general solutions of the system whose augmented matrix is:

   a. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix}$ c. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$

4. Solve the following systems:

   a. $2x + 3y + 4z = 20$, $3x + 4y + 5z = 26$, $3x + 5y + 6z = 31$

   b. $x_1 - 3x_3 = 8$, $2x_1 + 2x_2 + 9x_3 = 7$, $x_2 + 5x_3 = -2$.

5. Determine if the system is consistent.

   a. $x_1 + 3x_3 = 2$, $x_2 - 3x_4 = 3$, $-2x_2 + 3x_3 + 2x_4 = 1$, $3x_1 + 7x_4 = -5$

   b. $x_2 - 8x_3 = 8$, $2x_1 - 3x_2 + 2x_3 = 1$, $5x_1 - 8x_2 + 7x_3 = 1$

Answers

1. (a) h = 7/2 (b) h ≠ –6
2. (a) (i) h = 2 and k □ 8 (ii) h ≠ 2 (iii) h = 2 and k = 8

   (b) (i) h = 9, k ≠ 6 (ii) h ≠ 9 (iii) h = 9 and k = 6
3. (a) $\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$ (b) $\begin{cases} x_1 = 4 + 5x_3 \\ x_2 = 5 + 6x_3 \\ x_3 \text{ is free} \end{cases}$ (c) $\begin{cases} x_1 = 4/3\, x_2 - 2/3\, x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$
4. (a) (x, y, z) = (1, 2, 3) (b) (5, 3, –1)
5. (a) Consistent (b) Consistent

1.6 CRAMER'S RULE

Determinant is originally introduced for solving linear system. Although **impractical in computation**, it has important engineering applications in eigenvalue problems, differential equations, vector algebra, and in other areas. Here, our definition is particularly for dealing with linear system and particularly with Cramer's Rule. A **determinant of order** n is a scalar associated with a (hence **square**) matrix and is denoted by

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}$$

In linear algebra, **Cramer's rule** is a specific formula used for solving a system of linear equations containing as many equations as unknowns, efficient whenever the system of equations has a unique solution. This rule is named after Gabriel Cramer (1704–1752), who published the rule for an arbitrary number of unknowns in 1750. The solution obtained using Cramer's rule will be in terms of the determinants of the coefficient matrix and matrices obtained from it by replacing one column with the column vector of the right-hand sides of the equations.

Cramer's Rule Formula

Consider a system of linear equations with n variables $x_1, x_2, x_3, \dots, x_n$ written in the matrix form

$$AX = B.$$

where A = Coefficient matrix (must be a square matrix)

X = Column matrix with variables

B = Column matrix with the constants (which are on the right side of the equations)

Now, we have to find the determinants as:

$$D = |A|, Dx_1, Dx_2, Dx_3, \dots, Dx_n$$

Here, Dx_i for i = 1, 2, 3,..., n is the same determinant as D such that the column is replaced with B.

Therefore, the solution of the system of linear equations

$$x_1 = \frac{Dx_1}{D};\ x_2 = \frac{Dx_2}{D};\ x_3 = \frac{Dx_3}{D};\ \dots;\ x_n = \frac{Dx_n}{D}$$

$x_1 = \frac{Dx_1}{D}$; $x_2 = \frac{Dx_2}{D}$; $x_3 = Dx_3/D$; ....; $x_n = Dx_n/D$ where D is not equal to 0.

Note: If D = 0 then the system has no unique solution.

Example 17: Solve the system, if the solution exists;

$$5x - 3y = 37$$
$$-2x + 7y = -38$$

Solution:

Given system of linear equations be,

$$5x - 3y = 37$$
$$-2x + 7y = -38$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 5 & -3 \\ -2 & 7 \end{vmatrix} = 35 - 6 = 29 \neq 0$$

So, the solution of the system has unique solution.

Here,

$$D_1 = \begin{vmatrix} 37 & -3 \\ -38 & 7 \end{vmatrix} = 259 - 114 = 145$$

And,

$$D_2 = \begin{vmatrix} 5 & 37 \\ -2 & -38 \end{vmatrix} = -190 + 74 = -116$$

Now, by Cramer's rule,

$$x = \frac{D_1}{D} = \frac{145}{29} = 5 \quad \text{and} \quad y = \frac{D_2}{D} = \frac{-116}{29} = -4$$

Thus, x = 5, y = – 4 be solution of given system of equations.

Example 18: Solve the system, if the solution exists;

$$x + y + z = 0$$
$$2x + 5y + 3z = 1$$
$$-x + 2y + z = 2$$

Solution:

Given system of linear equations

$$x + y + z = 0$$
$$2x + 5y + 3z = 1$$
$$-x + 2y + z = 2$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 3 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= 1(5-6) - 1(2+3) + 1(4+5) = -1 - 5 + 9 = 3 \neq 0$$

So, the solution of the system is possible by the method and has unique solution.

Here,

$$D_1 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 5 & 3 \\ 2 & 2 & 1 \end{vmatrix} = 0 - 1(1-6) + 1(2-10) \quad = 0 + 5 - 8 \ = -3$$

and,

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = 1(1-6) - 0 + 1(4+1) \ = -5 - 0 + 5 \ = 0.$$

also,

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ -1 & 2 & 2 \end{vmatrix} = 1(10-2) - 1(4+1) + 0 \quad = 8 - 5 + 0 = 3$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{-3}{3} = -1, \quad y = \frac{D_2}{D} = \frac{0}{3} = 0 \quad \text{and,} \quad z = \frac{D_3}{D} = \frac{3}{3} = 1$$

Thus, $x = -1$, $y = 0$ and $z = 1$ be the solution of the given system of equations.

1.7 Rank of a Matrix

The order of highest order non-zero minor is said to be the rank of a matrix. This means, the rank of a matrix is 'r' if all the minors of order (r + 1) and more if exists, are should be zero and if there exists at least one non-zero minor of order 'r'.

Alternative Definition

The number of non-zero rows present in the matrix echelon form is also known as rank of a matrix.

The rank of a matrix is noted by ρ.

Properties of Rank of a Matrix

The following are the properties of the rank of matrix;

a. Only rank of null matrix is zero.

b. Rank of an identity matrix of order n is n i.e. $\rho(I_n) = n$ where I_n = unit matrix of order n.

c. Rank of a matrix $A_{m\times n}$ is

$$\rho(A_{m\times n}) \le \min\{m, n\}$$

$$\rho(A_{n\times n}) = n \text{ if } |A| \ne 0 < n$$

and $$\rho(A_{n\times n}) < n \text{ if } |A| = 0$$

d. If $\rho(A) = m$ and $\rho(B) = n$ then

$$\rho(AB) \le \min\{m, n\}$$

e. If A and B are square matrices of same order 'n' then

$$\rho(AB) \ge \rho(A) + \rho(B) - n$$

f. If $A_{m\times 1}$ is a non-zero column matrix and $B_{1\times n}$ is a non-zero row matrix then $\rho(AB) = 1$.

g. If A^{-1} is exists and B is a matrix of any order then ρ(AB) does not dependent on the matrix A i. e. $\rho(AB) = \rho(B)$.

(we will study the linearly dependent and independent in chapter 2)

h. If A and B are two matrices then

$$\rho(A+B) \le \rho(A) + \rho(B), \ \rho(A-B) \ge \rho(A) - \rho(B), \ \rho(A^T) = \rho(A).$$

Example 19: Find the rank of the matrices, $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \\ 0 & 8 & 7 \end{bmatrix}$.

Solution:

Let,

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \\ 0 & 8 & 7 \end{bmatrix}$$

Applying $R_2 = R_2 - 2R_1$ then

$$A \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -8 & -7 \\ 0 & 8 & 7 \end{bmatrix}$$

Again applying $R_3 = R_3 + R_2$ then

$$A \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -8 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that A has $\{R_1, R_2\}$ as linearly independent rows. So, rank of A = 2.

Example 20: Find the rank of the matrix, $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

Solution:

Let,

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying $R_2 = R_2 - 2R_1$ then

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ then

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that A has $\{R_1, R_2\}$ as independent rows. So, rank of A = 2.

Example 21: Find rank of the matrices, $A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$

Solution:

Let,

$$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Applying $R_2 = 3R_2 - R_1$, $R_3 = 9R_3 - R_1$ then,

$$A \sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 6 & 8 & 9 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Again applying $R_3 = R_3 + 2R_2$, $R_4 = R_4 - 2R_2$ then

$$A \sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & -3 & 45 \end{bmatrix}$$

Again applying $R_4 = 4R_4 + R_3$ then

$$A \sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & 0 & 153 \end{bmatrix}$$

This shows that A has $\{R_1, R_2, R_3, R_4\}$ as linearly independent rows. So, rank of $A = 4$.

Example 22: Find the rank of the matrix via echelon form and its canonical form,

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 3 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Solution:

Let

$$A = \begin{bmatrix} 0 & 1 & 0 & -3 & 3 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Interchanging R_1 and R_2 to make non-zero pivot then,

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Apply $R_4 \to R_4 - 3R_1$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_2$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

Apply $R_4 \to R_4 + 3R_3$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$$

This is an echelon form of A. Clearly, it has 4 non-zero row, so the rank of A is 4.
Now, for canonical or normal or row reduced echelon form, apply

$$R_3 \to \frac{R_3}{3} \text{ and } R_4 \to \frac{R_4}{-5}$$

then the above matrix reduces to,

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -4/3 & 7/3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Apply $R_2 \to R_2 + 3R_4$ and $R_3 \to R_3 + \frac{3}{4}R_4$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Apply $R_1 \to R_1 - 3R_3$ then the matrix reduces to,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

is a row reduced echelon form or canonical form of given matrix A.
Clearly, the matrix has 4 non-zero rows so, the matrix has rank 4.

Note: The rank of a matrix is same as the rank of the echelon form or canonical form of the matrix.

1.8 Row Equivalent Matrices

Sometimes the given two matrices may have same size and shape. There are certain conditions that must be met for matrices to be equivalent (or equal) to each other.

Definition

Given two matrices are said to be **row equivalent** if they have same rank.

Example 23: Let,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

Clearly, both matrices have rank 2, so they are row equivalent matrices.

Note: For any non-zero matrix, the matrix and its echelon form are row equivalent.

Exercise 1.2

1. Solve by using Cramer's rule of the following system of linear equations:

a. $3x + 7y + 8z = -13$
$2x + 9z = -5$
$-4x + y - 26z = 2$

b. $3x + y + 2z = 3$
$2x - 3y - z = -3$
$x + 2y + z = 4$

c. $x + y + z = 1$
$2x + 3y + 2z = 2$
$3x + 3y + 4z = 1$

d. $x + 2y + 3z = 6$
$2x + 4y + z = 7$
$3x + 3y + 9z = 14$

e. $x + 3y + 6z = 2$
$3x - y + 4z = 9$
$x - 4y + 2z = 7$

2. Find rank of the following matrices.

a. $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ c. $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

d. $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ e. $\begin{bmatrix} 8 & -3 & 7 \\ -20 & -17 & -15 \\ 11 & 2 & 9 \end{bmatrix}$ f. $\begin{bmatrix} 3 & -1 & 5 \\ 2 & -4 & 6 \\ 10 & 0 & 14 \end{bmatrix}$

g. $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$ h. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ i. $\begin{bmatrix} 1 & 2 & 9 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

j. $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

Answers

1. a. $x = -7, y = 0, z = 1$ b. $x = 1, y = 2, z = -1$
c. $x = 3, y = 0, z = -2$ d. $x = 1, y = 1, z = 1$
e. $x = 2, y = -1, z = ½$

2. a. 1 b. 2 c. 1 d. 2 e. 2
f. 2 g. 2 h. 2 i. 2 j. 2

1.9 Consistency of System of Linear Equations

We already learned that a system of linear equations is **consistent** if it has either one solution or infinitely many solutions. On the other hand, a system of linear equations is said to be **inconsistent** if it has no solution. In matrix form, the system of linear equations is called **consistent** if the rank of coefficient matrix and rank of the augmented matrix of the system of linear equations are same.

Let,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \dots (1)$$

be a system of linear equations. Then the coefficient matrix of (1) is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \dots (2)$$

and its augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} : b_1 \\ a_{21} & a_{22} & \dots & a_{2n} : b_2 \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} : b_m \end{bmatrix} \quad \dots (3)$$

Then the system (1) is called consistent if

rank of matrix (2) = rank of matrix (3)

i.e. rank of coefficient matrix = rank of augmented matrix

Otherwise, the system is called inconsistent.

Note:

1. The system (1) is consistent and has unique solution if

rank of coefficient matrix = rank of augmented matrix = number of independent variables

2. The system (1) is consistent and has infinite solution if

rank of coefficient matrix = rank of augmented matrix ≠ number of independent variables

Solution of a System of Linear Equations by Gauss Elimination

Among the three possible cases of a system, the Gauss elimination method can solve the linear systems that have either unique solution or infinite solution. And the method leaves to a system one with no solution.

Process to Finding the Solution of the System

1. Convert the system to coefficient and augmented matrix form.
2. Complete the coefficient and augmented matrix in echelon form.
3. If rank of coefficient matrix is same as the rank of augmented matrix of the system then the system is consistent.
4. If the system is consistent, convert the echelon matrix to equation (This method we know as Gauss elimination) form and solve the equations that gives the solution either unique or infinite. If the system is inconsistent then the system has no solution.

Example 24: Solve the system by Gauss elimination method (via equation solving method)

$$3x - 4y = 2$$
$$-x + 3y = 1$$

Solution:

Given system of linear equation is

$$\left.\begin{aligned} 3x - 4y &= 2 \\ -x + 3y &= 1 \end{aligned}\right\} \quad \ldots (i)$$

Multiplying second equation of (i) by 3 and then adding with first, we get

$$\left.\begin{aligned} 3x - 4y &= 2 \\ 5y &= 5 \end{aligned}\right\} \quad \ldots (ii)$$

From second equation of (ii) we get

$$y = 1$$

Substituting this value in first equation of (ii), we get

$$x = 2$$

Thus, $x = 2$, $y = 1$ be solution of given system of linear equation.

Solution of above example by matrix method:

Given system of linear equation is

$$3x - 4y = 2$$
$$-x + 3y = 1$$

The augmented matrix of the given equation is

$$[A : B] = \begin{bmatrix} 3 & -4 & : & 2 \\ -1 & 3 & : & 1 \end{bmatrix}$$

[Here, the matrix A is coefficient matrix and [A : B] is augmented matrix]

Applying row operation $R_2 = 3R_2 + R_1$ then

$$[A : B] \sim \begin{bmatrix} 3 & -4 & : & 2 \\ 0 & 5 & : & 5 \end{bmatrix}$$

Here,

rank of A = 2 = rank of [A : B]

[Because A has 2 non-zero rows, so rank of A is 2 and also [A : B] has 2 non-zero, so the rank of [A : B]]

This shows that the rank of coefficient matrix A is equal to the rank of augment matrix [A : B]. So, the given system is linear equations is consistent.

And, from last matrix we have,

$$3x - 4y = 2 \quad \ldots (iii)$$
$$5y = 5 \quad \ldots (iv)$$

From (iv),

$$y = 1$$

From (iii),

$$3x - 4 \times 1 = 2 \Rightarrow 3x = 6$$
$$\Rightarrow x = 2$$

Thus, $x = 2$, $y = 1$ be solution of given equations.

Example 25: Check following system of linear equations is consistent or not, if it is consistent then solve by Gauss elimination method,

$$x - y + 2z = 4$$
$$3x + y + 4z = 6$$
$$x + y + z = 1$$

Solution:

Given system of linear equations is

$$x - y + 2z = 4$$
$$3x + y + 4z = 6$$
$$x + y + z = 1$$

The augmented matrix of given system of equations is

$$[A : B] = \begin{bmatrix} 1 & -1 & 2 & : & 4 \\ 3 & 1 & 4 & : & 6 \\ 1 & 1 & 1 & : & 1 \end{bmatrix}$$

Applying $R_2 = R_2 - 3R_1$ and $R_3 = R_3 - R_1$ then

$$[A:B] \sim \begin{bmatrix} 1 & -1 & 2 & : & 4 \\ 0 & 4 & -2 & : & -6 \\ 0 & 2 & -1 & : & -3 \end{bmatrix}$$

Applying $R_3 = 2R_3 - R_2$ then

$$[A:B] \sim \begin{bmatrix} 1 & -1 & 2 & : & 4 \\ 0 & 4 & -2 & : & -6 \\ 0 & 0 & -4 & : & 0 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B])

So, the system is consistent. And, from the last matrix we have,

$$x - y + 2z = 4 \quad \text{... (i)}$$

$$4y - 2z = -6 \quad \text{... (ii)}$$

$$-4z = 0 \quad \text{... (iii)}$$

From (iii), we get $z = 0$

From (iii), we get $4y - 0 = -6 \Rightarrow y = \frac{-3}{2}$

From (iii), we get $x + \frac{3}{2} + 0 = 4 \Rightarrow x = \frac{5}{2}$

Thus, $x = \frac{5}{2}$, $y = \frac{-3}{2}$ and $z = 0$ is the solution set of given system of linear equations.

Example 26: Check following system of linear equations is consistent or not, if it is consistent then solve by Gauss elimination method,

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

Solution:

Given system of linear equations is,

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

The augmented matrix of given system is

$$[A:B] = \begin{bmatrix} 4 & -2 & 6 & : & 8 \\ 1 & 1 & -3 & : & -1 \end{bmatrix}$$

Applying $R_2 = 4R_2 - R_1$ then

$$[A:B] = \begin{bmatrix} 4 & -2 & 6 & : & 8 \\ 0 & 6 & -18 & : & -12 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, the system is consistent. Now, from the last matrix we have,

$$4x - 2y + 6z = 8 \quad \text{... (i)}$$

and $$6y - 18z = -12 \quad \text{... (ii)}$$

From (ii) we get

$$y - 3z = -2$$

$$\Rightarrow y = 3z - 2$$

From (i) we get

$$2x - y + 3z = 4$$

$$\Rightarrow 2x = 4 + y - 3z$$

$$\Rightarrow 2x = 4 + 3z - 2 - 3z$$

$$\Rightarrow x = 1$$

Thus, $x = 1$, $y = 3z - 2$, $z = z$ is solution set of given system of linear equations.

Example 27: Check following system of linear equations is consistent or not, if it is consistent then solve by Gauss elimination method

$$2x - 3y + 7z = 5$$

$$3x + y - 3z = 13$$

$$2x + 19y - 47z = 32$$

Solution:

Given system of linear equations is

$$2x - 3y + 7z = 5$$

$$3x + y - 3z = 13$$

$$2x + 19y - 47z = 32$$

The augmented matrix of given system is

$$[A:B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{bmatrix}$$

Applying $R_2 = 2R_2 - 3R_1$ and $R_3 = R_3 - R_1$ then

$$[A:B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 0 & 11 & -27 & : & 11 \\ 0 & 22 & -54 & : & 27 \end{bmatrix}$$

Again applying $R_3 = R_3 - 2R_2$ then

$$[A:B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 0 & 11 & -27 & : & 11 \\ 0 & 0 & 0 & : & 5 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) ≠ (rank of augmented matrix [A : B]).

So, the given system is inconsistent.

Exercise 1.3

Check following system of linear equations is consistent or not, if it is consistent then solve by Gauss elimination method:

1. $6x + 4y = 2$
   $3x - 5y = -34$

2. $3x - 0.5y = 0.6$
   $1.5x + 4.5y = 6$

3. $x + y - z = 9$
   $8y + 6z = -6$
   $-2x + 4y - 6z = 40$

4. $13x + 12y = -6$
   $-4x + 7y = -73$
   $11x - 13y = 157$

5. $4y + 3z = 8$
   $2x - z = 2$
   $3x + 2y = 5$

6. $7x - 4y - 2z = -6$
   $16x + 2y + z = 3$

7. $x - 2y + 3z = 11$
   $3x + y - z = 2$
   $5x + 3y + 2z = 3$

8. $2x + 3y + 4z = 20$
   $3x + 4y + 5z = 26$
   $3x + 5y + 6z = 31$

9. $x + y + z = -3$
   $3x + y - 2z = -2$
   $2x + 4y + 7z = 7$

10. $x + y + z = 3$
    $x + 2y + 3z = 4$
    $2x + 3y + 4z = 9$

11. $2x + 5y + 6z = 13$
    $3x + y - 4z = 0$
    $x - 3y - 8z = -10$

12. $5x + 3y + 7z = 4$
    $3x + 26y + 2z = 9$
    $7x + 2y + 10z = 5$

Answers

1. $x = -3, y = 5$
2. $x = 0.4, y = 1.2$
3. $x = 1, y = 3, z = -5$
4. $x = 6, y = -7$
5. inconstant
6. $x = 0, z = 3 - 2y$
7. $x = 2, y = -3, z = 1$
8. $x = 1, y = 2, z = 3$
9. inconsistent
10. inconsistent
11. $x = 2z - 1, y = 3 - 2z, z = z$
12. $5x + 80y = 25, 11y - z = 3, z = z$

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