10

CHAPTER

STRAIGHT LINE

Introduction

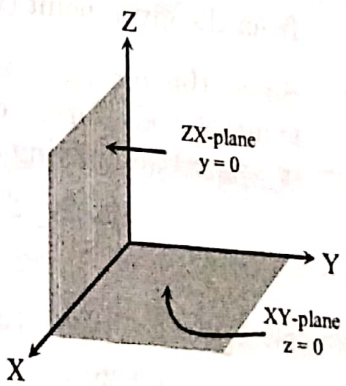
The intersection of two planes P_1 and P_2 is the locus of all the common points on both the planes P_1 and P_2. This locus is a straight line. Any given line can be uniquely determined by any of the two planes containing the line. Thus, a line can be regarded as the locus of the common points of two intersecting planes.

Let us consider the two planes

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

Any set of coordinates (x, y, z) which satisfy these two equations simultaneously will represent a point on the line of intersection of these two planes. Hence, these two equations taken together will represent a straight line. It can be noted that the equation of x-axis are $y = 0$, $z = 0$; the equation of the y–axis is $x = 0$, $z = 0$ and the equation of the z–axis is $x = 0$, $y = 0$. The representation of the straight line by the equations

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is called non–symmetrical form.



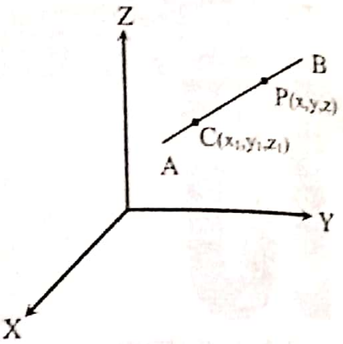
Let us now derive the equations of a straight line in the symmetrical form.

10.1 EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM

Let $C(x_1, y_1, z_1)$ be a point on the straight line AB and P(x, y, z) be any point on the straight line. Then the projections of the line CP on the coordinate axes are $x - x_1, y - y_1, z - z_1$. Then the length of the line is

$$CP = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = r \text{ (say)}$$

Let ℓ, m, n be the direction cosines of the straight line. Then the projections of CP on the coordinate axes are given by ℓr, mr, nr. Therefore,



$$x - x_1 = \ell r;\ y - y_1 = mr;\ z - z_1 = nr$$

i.e. $$\frac{x - x_1}{r} = \ell;\ \frac{y - y_1}{r} = m;\ \frac{z - z_1}{r} = n$$

Therefore,

$$\frac{x - x_1}{\ell} = r;\ \frac{y - y_1}{m} = r;\ \frac{z - z_1}{n} = r$$

This gives,

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad \text{... ... (1)}$$

This is the symmetrical form of the equation of line CP that passes through the point
$C(x_1, y_1, z_1)$ and having the direction cosines ℓ, m, n.

Note:

1. From (1) we have

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$

i.e. $$x = x_1 + \ell r;\ y = y_1 + mr;\ z = z_1 + nr$$

That is, the coordinate of general point (any point) of the line (1) is $(x_1 + \ell r, y_1 + mr, z_1 + nr)$ where r is the actual distance of any point on the line from the given point (x_1, y_1, z_1).

2. Since the direction rations a, b, c is propositional to ℓ, m, n. Therefore, the symmetrical form of the equation of the line which passes through (x_1, y_1, z_1) and having direction rations a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \text{... ... (2)}$$

Example 1: Express the symmetrical form of the equations of the line $x + 2y + z - 3 = 0$, $6x + 8y + 3z - 13 = 0$.

Solution:

Given line is

$$x + 2y + z - 3 = 0 \quad \text{... (i)}$$

$$6x + 8y + 3z - 13 = 0 \quad \text{... (ii)}$$

To express the equations of a line in symmetrical form we have to find (i) the direction ratios of the line and (ii) a point on the line. Let ℓ, m, n be the direction ratios of line. Then

$$\ell + 2m + n = 0$$

and $$6l + 8m + 3n = 0.$$

Then

$$\frac{\ell}{6-8} = \frac{m}{6-3} = \frac{n}{8-12}$$

i.e. $$\frac{\ell}{-2} = \frac{m}{3} = \frac{n}{-4}$$

i.e. $$\frac{\ell}{2} = \frac{m}{-3} = \frac{n}{4}$$

Now, we turn to find the point where the line meets the xy–plane (i.e.) z = 0. Choose z = 0 then (i) and (ii) reduces to

$$x + 2y - 3 = 0$$
$$6x + 8y - 13 = 0$$

Solving we get,

$$\frac{x}{-26+24} = \frac{y}{-18+13} = \frac{1}{8-12}$$

i.e. $$\frac{x}{-2} = \frac{y}{-5} = \frac{1}{-4}$$

This implies

$$x = \frac{1}{2};\ y = \frac{5}{4} \text{ and } z = 0.$$

Therefore, the equation of the line represents by (i) and (ii) is

$$\frac{x-1/2}{2} = \frac{y-5/4}{-3} = \frac{z-0}{4}$$

i.e. $$\frac{2x-1}{1} = \frac{4y-5}{-3} = \frac{z}{1}$$

10.2 Equations of a Straight Line Passing Through the Two Given Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. The direction ratios of the line are $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Therefore, the equations of the straight line are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Example 2: Find the equation of the line joining the points (2, 3, 5) and (–1, 2, –4).

Solution:

Given that the line joining the points (2, 3, 5) and (–1, 2, –4). The direction ratios of the line are

$$2 + 1, 3 - 2, 5 + 4$$

i.e. $$3, 1, 9$$

Therefore, the equation of the line is

$$\frac{x-2}{-1-2}=\frac{y-3}{2-3}=\frac{z-5}{-4-5}$$

i.e. $$\frac{x-2}{3}=\frac{y-3}{1}=\frac{z-5}{9}$$

Example 3: Find the equation of the line passing through the point (3, 2, –6) and is perpendicular to the plane $3x - y - 2z + 2 = 0$.

Solution:

Given plane is

$$3x - y - 2z + 2 = 0 \qquad \text{... (i)}$$

The direction ratios of the line are the direction ratios of the normal to the plane. Therefore, the direction ratios of the line normal to (i), are 3, –1, –2.

Now, the equations of the line which is passing through (3, 2, –6) and is perpendicular to the plane (i) is

$$\frac{x-3}{3}=\frac{y-2}{-1}=\frac{z+6}{-2}$$

Example 4: Show that the equation of the line which is perpendicular from the point (1, 6, 3) to the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$. Also, find the length of perpendicular from (1, 6, 3) to the foot.

Solution:

Given line is,

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}=r \text{ (let)} \qquad \text{...... (i)}$$

So, the general point of the line (i) is,

$$(r, 2r+1, 3r+2)$$

Let M be the foot of the perpendicular from the given point P(1, 6, 3) on the given line (i), then M(r, 2r + 1, 3r +2) for some fixed r.

Therefore the direction ratios of the line joining P and M is,

$$r-1,\ 2r+1-6,\ 3r+2-3$$

i.e. $$r-1,\ 2r-5,\ 3r-1.$$

Since the line PM is perpendicular to line (i) so,

$$\boxed{a_1a_2+b_1b_2+c_1c_2=0}$$

$$(r-1)(1)+(2r-5)(2)+(3r-1)(3)=0$$

$$\Rightarrow\ r-1+4r-10+9r-3=0$$

$$\Rightarrow\ 14r=14$$

$$\Rightarrow\ r=1$$

Then, $x = 1, y = 3, z = 5$. Therefore M = (1, 3, 5).

And, the direction ratio of the perpendicular line is

$$(r-1, 2r-5, 3r-1) = (1-1, 2-5, 3-1) = (0, -3, 2).$$

Then the equation of the line through (1, 6, 3) and having direction ratios 0, –3, 2 be,

$$\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}.$$

Now, distance of perpendicular line is,

$$PM=\sqrt{(1-1)^2+(6-3)^2+(3-5)^2}=\sqrt{13}$$

Thus, the length of perpendicular from (1, 6, 3) to the foot is $\sqrt{13}$.

Example 5: Find the image of the point P (1, 3, 4) in the plane $2x - y + z + 3 = 0$.

Solution:

The equation of line passing through (1, 3, 4) and perpendicular to the plane $2x - y + z + 3 = 0$ is,

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r \text{ (suppose)} \qquad \text{...... (i)}$$

Then the general point of (i) is,

$$(x, y, z) = (2r+1, 3-r, r+4).$$

Since, if the point Q(2r +1, 3 – r, r + 4) is the image of P then the middle point R lies on the plane $2x - y + z + 3 = 0$.

Here, the coordinate of R is,

$$R\left(\frac{2r+1+1}{2}, \frac{3-r+3}{2}, \frac{r+4+4}{2}\right)=R\left((r+1), 3-\frac{r}{2}, 4+\frac{r}{2}\right)$$

Since the point lies on the plane $2x - y + z + 3 = 0$, so,

$$2(r+1)-3+\frac{r}{2}+4+\frac{r}{2}+3=0.$$

$$\Rightarrow\ 2r+2+4+r=0.$$

$$\Rightarrow\ r=-2.$$

Then the coordinate of Q is,

$$(2r+1, 3-r, r+4)=(-4+1, 3+2, -2+4)=(-3, 5, 2)$$

Thus, the image of the point P (1, 3, 4) in the plane $2x - y + z + 3 = 0$ be (–3, 5, 2).

Example 6: Find the distance from the point (3, 4, 5) to the point where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane $x + y + z = 2$.

Solution: The given line is

$$\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2} = r \text{ (say)} \quad \text{............(i)}$$

So, the general point of (i) is

$$(r + 3,\ 2r + 4,\ 2r + 5) \quad \text{.........(ii)}$$

Given that the line (i) meets the plane $x + y + z = 2$. Then the point (ii) is the common point of (i) and (ii). Therefore,

$$r + 3 + 2r + 4 + 2r + 5 = 2$$

$$\Rightarrow \quad 5r + 12 = 2$$

$$\Rightarrow \quad r = -2$$

So the point of intersection of the line and the plane is

$$(-2 + 3, -4 + 4, -4 + 5)$$

i.e $(1, 0, 1)$

Now, length of (3, 4, 5) from (1, 0, 1) is

$$d = \sqrt{(3-1)^2 + (4-0)^2 + (5-1)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6.$$

Example 7: Find the equation of the plane containing the line $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and parallel to the line $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$.

Solution: Given line is

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \text{........ (i)}$$

Clearly, the line (i) passes through he point (x_1, y_1, z_1).

Then the plane containing the line (i) passes through the point (x_1, y_1, z_1).

Now, the equation of plane through the point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \text{........ (ii)}$$

So,

$$al_1 + bm_1 + cn_1 = 0 \quad \text{......... (iii)}$$

By question the plane (ii) is parallel to the line

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad \text{......... (iv)}$$

So,

$$al_2 + bm_2 + cn_2 = 0 \quad \text{.......... (v)}$$

Solving (iii) and (v) we get,

$$\frac{a}{m_1n_2 - m_2n_1} = \frac{b}{n_1l_2 - n_2l_1} = \frac{c}{l_1m_2 - l_2m_1}$$

Thus (2) becomes,

$$(m_1n_2 - m_2n_1)(x - x_1) + (n_1l_2 - n_2l_1)(y - y_1) + (l_1m_2 - l_2m_1)(z - z_1) = 0$$

This is the equation of the plane that contains (i) and is parallel to (ii).

Exercise 10.1

1. Write down the equations of the line through (2, 1, 3) and (4, 2, 4).
2. Find the equation of the line through (1, 2, 3) and normal to the plane $2x + 3y - z = 4$.
3. Find the equation of the line through (1, 5, 3) and normal to the plane $2x + 3y + 7z = 0$.
4. Find the value of k, such that the lines $\frac{x-1}{2} = \frac{y-3}{4k} = \frac{z}{2}$ and $\frac{x-2}{2k} = \frac{y-1}{3} = \frac{2-1}{4}$ are perpendicular.
5. Find the distance of the point (1, – 3, 5) from the plane $3x - 2y + 6z = 15$ along a line with direction cosines proportional to (2, 1, –2).
6. Find the distance to the point (–1, –5, –10) from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.
7. Find the two points on the line $\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+5}{2}$ either side of (2, –3, –5) and at a distance 3 from it.
8. Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.
9. Find the equation to the line passing through (–1, –2, –3) and the perpendicular to each of the lines $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and $\frac{x+2}{4} = \frac{y+3}{5} = \frac{z+4}{6}$.
10. Find the equation to the line through (–1, 3, 2) and perpendicular to the plane $x + 2y + 2z = 3$, the length of perpendicular and the co-ordination of its foot.
11. Find the image of the point P (1, 2, 3) in the plane $2x - y + z + 3 = 0$.
12. Find the image of the point (2, 3, 5) on the plane $2x + y - z + 2 = 0$.
13. Find the image of the line $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-4}{2}$ in the plane $2x - y + z + 3 = 0$.

Answers

1. $\frac{x-2}{2} = y - 1 = z - 3$
2. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{-1}$
3. $\frac{x-1}{2} = \frac{y-5}{3} = \frac{z-3}{7}$
4. $k = \frac{-1}{2}$
5. 9 units
6. 13 units

7. $(3, -1, -3), (1, -5, -7)$
8. 1 unit
9. $\frac{x+1}{1} = \frac{x+2}{-2} = \frac{z+3}{1}$
10. $\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{2}$; length = 2 units; $\left(\frac{-5}{2}, \frac{5}{3}, \frac{2}{3}\right)$
11. $\left(-\frac{11}{3}, \frac{13}{3}, \frac{2}{3}\right)$
12. $\left(-\frac{2}{3}, \frac{5}{3}, \frac{19}{3}\right)$
13. $\frac{x+5}{1} = \frac{y+7}{6} = \frac{z}{1}$

10.3 Equations of a Straight Line in Symmetrical Form

Let the equations of a line in the general form be

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0 \\ \text{and } a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\} \quad \dots\dots (1)$$

and we now express this equation (1) in symmetrical form.

The two equations of (1) separately represent two planes.

Let l, m, n be the direction cosines of the line represented by (1). Clearly, the line lies on the planes and the line is perpendicular to the normal to these two given planes. The direction ratios of the two normal are a_1, b_1, c_1 and a_2, b_2, c_2.

Being the normal are perpendicular to the line we have,

$$a_1l + b_1m + c_1n = 0 \quad \dots\dots (2)$$

$$\text{and} \quad a_2l + b_2m + c_2n = 0 \quad \dots\dots (3)$$

Solving (2) and (3) by cross multiplication,

$$\frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

Therefore, the direction cosines of the line (1) are proportional to

$$b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1$$

Choose a point $(x_1, y_1, 0)$ on the line that intersects the plane $z = 0$, then (1) gives

$$a_1x_1 + b_1y_1 + d_1 = 0 \quad \dots\dots (4)$$

$$a_2x_1 + b_2y_1 + d_2 = 0 \quad \dots\dots (5)$$

Solving (4) and (5), we get

$$x_1 = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1},\ y_1 = \frac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}$$

Therefore, the equation of the line (1) in symmetrical form is

$$\frac{x - \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \frac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

Note: We can also find the point where the line meets the yz-plane i.e. $x = 0$ or zx-plane i.e. $y = 0$.

Example 8: Change the equation $x + y + z + 1 = 0$, $4x + y - 2z + 2 = 0$ in symmetrical form.

Solution:

Given equation of line is,

$$x + y + z + 1 = 0 = 4x + y - 2z + 2 \quad \dots\dots (i)$$

Then we have to change (i) in symmetrical form.

Put $z = 0$ then,

$$x + y + 1 = 0 \text{ and } 4x + y + 2 = 0 \quad \dots\dots (ii)$$

Solving these equations, we get

$$x = \frac{-1}{3} \text{ and } y = \frac{-2}{3}$$

Thus, the point on the line is $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$

Let the directions ratios of the line (i) are a, b, c. Then, using the perpendicularity condition we get

$$a + b + c = 0$$

$$4a + b - 2c = 0$$

Solving we get

$$\frac{a}{-2-1} = \frac{b}{4+2} = \frac{c}{1-4}$$

$$\Rightarrow \frac{a}{-3} = \frac{b}{6} = \frac{c}{-3}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{-2} = \frac{c}{1} = \lambda \text{ (let)}$$

This implies,

$$a = \lambda,\ b = -\lambda,\ c = \lambda$$

Now, the equation of line which is passing through the point $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$ and having the direction ratio $a = \lambda$, $b = -\lambda$, $c = \lambda$ be,

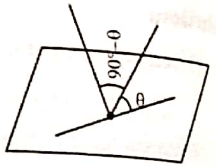
$$\frac{x + (1/3)}{\lambda} = \frac{y + (2/3)}{-2\lambda} = \frac{z - 0}{\lambda}$$

$$\Rightarrow \frac{3x+1}{3} = \frac{3y+2}{-6} = \frac{z}{1}$$

10.4 Angle between a Plane and a Line

Let the equation of the plane be

$$ax + by + cz + d = 0 \quad \dots\dots (1)$$



and a straight line be

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad \dots\dots (2)$$

The direction ratios of the line (1) are ℓ, m, n and the direction ratios of normal to the plane (2) are a, b, c. Let θ be the angle between the line (1) and the plane (2), therefore the angle between the normal to the plane and the line is $\left(\frac{\pi}{2} - \theta\right)$.

Now, the angle between the line and the normal line to the plane is given by

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{a\ell + bm + cn}{\sqrt{a^2 + b^2 + c^2}\sqrt{\ell^2 + b^2 + c^2}}$$

i.e. $$\sin\theta = \frac{a\ell + bm + cn}{\sqrt{a^2 + b^2 + c^2}\sqrt{\ell^2 + b^2 + c^2}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{a\ell + bm + cn}{\sqrt{a^2 + b^2 + c^2}\sqrt{\ell^2 + b^2 + c^2}}\right) \quad \dots\dots (3)$$

Note:

1. The line (1) and the plane (2) are parallel to each other if and only if θ = 0. That is, the line (1) and plane (2) are parallel to each other if and only if

   $$a\ell + bm + cn = 0.$$

   Since (x_1, y_1, z_1) is a point on the line and does not lie on the plane (1). Therefore,

   $$ax_1 + by_1 + cz_1 + d \neq 0.$$

   Hence, the conditions for the line (1) to be parallel to the plane (2) are

   $$a\ell + bm + cn = 0 \text{ and } ax_1 + by_1 + cz_1 + d \neq 0.$$

2. The line (1) and the plane (2) are perpendicular (normal or orthogonal) to each other if and only if $\theta = \frac{\pi}{2}$. That is, the line (1) and plane (2) are perpendicular to each other if and only if

   $$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}.$$

Example 9: Find the value of k such that the line $\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k}$ is parallel to the plane $2x - 3y + z = 3$.

Solution:

Given line and plane are

$$\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k} \quad \dots (i)$$

and $$2x - 3y + z = 3 \quad \dots (ii)$$

As given the line (i) is parallel to the plane (ii). So, the line is perpendicular to the normal of the plane (i.e. to the direction ratios of the plane). Therefore,

$$\boxed{a\ell + bm + cn = 0}$$

i.e. $(2)(2) + (5)(-3) + (k)(1) = 0$

$\Rightarrow \quad 4 - 15 + k = 0$

$\Rightarrow \quad k = 11.$

Thus, for k = 11 the given line and the plane are in the form of parallel.

Conditions for a Line to lie on a Plane

Let the equation of the line be

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad \dots\dots (1)$$

Let the equation of the plane be

$$ax + by + cz + d = 0 \quad \dots\dots (2)$$

Since the line normal to the plane (2) has direction cosines are a, b, c. Being the line (1) lies on the plane (2) therefore the line whose direction cosines are a, b, c is normal to the line (1). Then,

$$a\ell + bm + cn = 0 \quad \dots\dots (3)$$

Since the line lies on the plane, so every point on the line (1) is also a point of the plane (2). That is, the point (x_1, y_1, z_1) is a point on the line and therefore lies on the plane. Hence, $ax_1 + by_1 + cz_1 + d = 0$. Therefore, the conditions for the line (1) to be parallel to the plane (2) be

$$a\ell + bm + cn = 0 \text{ and } ax_1 + by_1 + cz_1 + d = 0.$$

10.5 Length of the Perpendicular from a Point to a Line

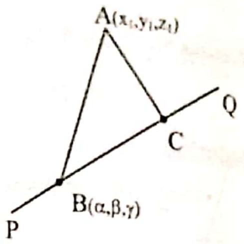
Let $A(x_1, y_1, z_1)$ be the point and the line PQ that passes through the point $B(\alpha, \beta, \gamma)$ and having direction ratios ℓ, m, n is

$$\frac{x - \alpha}{\ell} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \quad \dots\dots (1)$$

Draw AC perpendicular to the line PQ then

BC = projection of AB on the line PQ

$= (x_1 - \alpha)\ell + (y_1 - \beta)m + (z_1 - \gamma)n$



Again,

$$BA^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2$$

Now, from right angled ΔABC,

$$AC^2 = BA^2 - BC^2$$

$$= (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - [(x_1 - \alpha)\ell + (y_1 - \beta)m + (z_1 - \gamma)n]^2$$

From this relation we can get the perpendicular distance AC from A.

Example 10: Find the equation of a plane containing the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$.

Solution:

Given line is,

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4} \qquad \text{...... (i)}$$

Let the equation of plane containing the line is (i) is,

$$a(x-1) + b(y+1) + c(z-3) = 0 \qquad \text{...... (ii)}$$

Clearly, the direction ratios of the normal line to the plane (ii) are a, b, c. Therefore,

$$2a + (-1)b + 4c = 0$$

$$\Rightarrow \quad 2a - b + 4c = 0 \qquad \text{...... (iii)}$$

As given, the plane (ii) is perpendicular to the plane $x + 2y + z = 12$ therefore,

$$(a)(1) + (b)(2) + (1)(c) = 0$$

$$\Rightarrow \quad a + 2b + c = 0 \qquad \text{...... (iv)}$$

Solving equation (iii) and (iv), we get

$$\frac{a}{-1-8} = \frac{b}{4-2} = \frac{c}{4+1}$$

$$\Rightarrow \quad \frac{a}{-9} = \frac{b}{2} = \frac{c}{5} = \lambda \text{ (let)}$$

$$\Rightarrow \quad a = -9\lambda;\ b = 2\lambda;\ c = 5\lambda.$$

Now, the equation (ii) becomes,

$$-9(x-1) + 2(y+1) + 5(z-3) = 0$$

$$\Rightarrow \quad 9(x-1) - 2(y+1) - 5(z-3) = 0$$

$$\Rightarrow \quad 9x - 2y - 5z + 4 = 0.$$

Thus, the equation of plane containing the line (i) and is perpendicular to the plane $x + 2y + z = 12$ is $9x - 2y - 5z + 4 = 0$.

Example 11: Find the equation of the plane which passes through the point (1, 2, −1) and contains the line $\frac{x+1}{2} = \frac{y-1}{3} = \frac{z+2}{-1}$

Solution:

Given straight line is

$$\frac{x+1}{2} = \frac{y-1}{3} = \frac{z+2}{-1} \qquad \text{--- (i)}$$

Any plane through the line (i) is

$$A(x+1) + B(y-1) + C(z+2) = 0 \qquad \text{--- (ii)}$$

Therefore, the normal line to (ii) has the direction ratios A, B, C. So,

$$2A + 3B - C = 0 \qquad \text{--- (iii)}$$

As given the plane (ii) passes through the point (1, 2, −1). So, (ii) reduces to

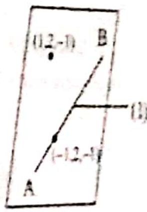
$$A(1+1) + B(2-1) + C(-1+2) = 0$$

i.e. $\quad 2A + B + C = 0 \qquad \text{--- (iv)}$

Solving (iii) and (iv), we get

$$\frac{A}{3+1} = \frac{B}{-2-2} = \frac{C}{2-6}$$

or, $\quad \frac{A}{1} = \frac{B}{-1} = \frac{C}{-1}$



Substituting the values of A, B, C in (ii), we get

$$x + 1 - y + 1 - z - 2 = 0$$

$$\Rightarrow \quad x - y - z = 0$$

This is the equation of required plane.

Example 12: Find the equation of the line through (2, −1, −1) is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $2x + y = 0 = x - z + 5$.

Solution:

The equation of line through the point (2, −1, −1) is

$$\frac{x-2}{a} = \frac{y+1}{b} = \frac{z+1}{c} \qquad \text{---- (i)}$$

where, a, b, c are the direction ratio of the line.

Since the equation (i) is parallel to the plane $4x + y + z + 2 = 0$ so, ---- (ii)

$$4a + b + c = 0$$

Again, given a line,

$$2x + y = 0 = x - z + 5$$

This gives,

$$x = -\frac{y}{2} \text{ and } x = z - 5$$

Therefore,

$$\frac{x}{1} = \frac{-y}{2} = \frac{z-5}{1} \qquad \text{...... (iii)}$$

Given that the line (i) is perpendicular to the line (iii). So,

$$\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$$

$$\Rightarrow \quad (a)(1) + (-2)(b) + (c)(1) = 0$$

$$\Rightarrow \quad a - 2b + c = 0 \qquad \text{....... (iv)}$$

Solving the equations (ii) and (iv), we get

$$\frac{a}{1+2} = \frac{b}{1-4} = \frac{c}{-8-1} = \lambda$$

$$\Rightarrow \quad a = 3\lambda,\ b = -3\lambda \text{ and } c = -7\lambda$$

Substituting the value of a, b and c in equation (i) then

$$\frac{x-2}{3\lambda} = \frac{y+1}{-3\lambda} = \frac{z+1}{-9\lambda}$$

$$\Rightarrow \quad \frac{x-2}{1} = \frac{y+1}{-1} = \frac{z+1}{-3}$$

This is the equation of required line.

Example 13: Find the equation of line through (α, β, δ) parallel, to the plane $lx + my + nz = p$, $l_1x + m_1y + n_1z = p$.

Solution:

The equation of line passes through (α, β, δ) and having the direction ratio (a, b, c) is

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\delta}{c} \qquad \text{......... (i)}$$

Since equation (i) is parallel to the plane $lx + my + nz = p$. Then,

$$al + bm + cn = 0 \qquad \text{........ (ii)}$$

Similarly, equation (i) is parallel to $l_1x + m_1y + n_1z = p$. Then,

$$al_1 + bm_1 + cn_1 = 0 \qquad \text{........ (iii)}$$

Solving the equations (ii) and (iii), we get

$$\frac{a}{mn_1 - m_1n} = \frac{b}{l_1n - ln_1} = \frac{c}{lm_1 - l_1m} = \lambda \text{ (Say)}$$

$$\Rightarrow \quad a = \lambda(mn_1 - m_1n),\ b = \lambda(l_1n - ln_1),\ c = \lambda(lm_1 - l_1m)$$

Substituting the value of a, b and c in equation (i) then

$$\frac{x-\alpha}{(mn_1 - m_1n)} = \frac{y-\beta}{(l_1n - ln_1)} = \frac{z-\delta}{(lm_1 - l_1m)}$$

This is the equation of required line.

Example 14: Find the equations to the planes through the line $2x + 3y - 5z - 4 = 0 = 3x - 4y + 3z - 6$ and is parallel to the coordinate axes.

Solution:

Given line is

$$2x + 3y - 5z - 4 = 0 = 3x - 4y + 3z - 6 \qquad \text{... (i)}$$

The equation of plane passes through the line (i) is

$$2x + 3y - 5z - 4 + k(3x - 4y + 3z - 6) = 0$$

$$\Rightarrow \quad (2 + 3k)x + (3 - 4k)y + (-5 + 3k)z - 4 + 6k = 0 \qquad \text{... (ii)}$$

Let the plane (ii) is parallel to the x–axis and we know the x-axis have direction ratios 1, 0, 0 then,

$$al + bm + cn = 0$$

$$\Rightarrow \quad (2 + 3k)(1) + (3 - 4k)(0) + (-5 + 3k)(0) = 0$$

$$\Rightarrow \quad 2 + 3k = 0$$

$$\Rightarrow \quad k = \frac{-2}{3}$$

Then, the equation (ii) becomes

$$2x + 3y - 5z - 4 + \frac{-2}{3}(3x - 4y + 5z - 6) = 0$$

$$\Rightarrow \quad 17y - 25z = 0$$

Let the plane (ii) is parallel to the y–axis and we know the y-axis have direction ratios 0, 1, 0 then,

$$al + bm + cn = 0$$

$$\Rightarrow \quad (2 + 3k)(0) + (3 - 4k)(1) + (-5 + 3k)(0) = 0$$

$$\Rightarrow \quad 3 - 4k = 0$$

$$\Rightarrow \quad k = \frac{3}{4}$$

Then, the equation (ii) becomes

$$2x + 3y - 5z - 4 + \frac{3}{4}(3x - 4y + 5z - 6) = 0$$

$$\Rightarrow \quad 17x - 5z - 34 = 0$$

Let the plane (ii) is parallel to the z-axis and we know the z-axis have direction ratios 0, 0, 1 then,

$$al + bm + cn = 0$$

$$\Rightarrow \quad (2 + 3k)(0) + (3 - 4k)(0) + (-5 + 3k)(1) = 0$$

$$\Rightarrow \quad -5 + 3k = 0$$

$$\Rightarrow \quad k = \frac{5}{3}$$

Then, the equation (ii) becomes

$$2x + 3y - 5z - 4 + \frac{5}{3}(3x - 4y + 5z - 6) = 0$$

$$\Rightarrow \quad 6x + 9y - 15z - 12 + 15x - 20y + 25z - 30 = 0$$

$$\Rightarrow \quad 21x - 11y + 10z - 42 = 0.$$

Thus, the equations of required planes are $17y - 25z = 0$, $17x - 5z - 34 = 0$, $21x - 11y + 10z - 42 = 0$.

Example 15: Find the equation of the plane through the points (2, 2, 1), (1, −2, 3) and parallel to the line joining the points (2, 1, −3), (−1, 5, −8).

Solution: The plane through the points (2, 2, 1) is,

$$a(x - 2) + b(y - 2) + c(z - 1) = 0 \qquad \text{...... (i)}$$

As given the plane (i) passes through the points (1, −2, 3) then,

$$a(1 - 2) + b(-2 - 2) + c(3 - 1) = 0$$

$$\Rightarrow \quad -a - 4b + 2c = 0$$

$$\Rightarrow \quad a + 4b - 2c = 0 \qquad \text{...... (ii)}$$

The equation of line joining points (2, 1, −3) and (−1, 5, −8) is

$$\frac{x - 2}{2 - (-1)} = \frac{y - 1}{1 - 5} = \frac{z - (-3)}{-3 - (-8)}$$

$$\Rightarrow \quad \frac{x - 2}{3} = \frac{y - 1}{-4} = \frac{z + 3}{5} \qquad \text{...... (iii)}$$

Given that the plane (i) to is parallel to the line (iii). So, by the condition of parallelism,

$$3a - 4b + 5c = 0 \qquad \text{...... (iv)}$$

Solving equation (ii) and (iv), we get

$$\frac{a}{20 - 8} = \frac{b}{-6 - 5} = \frac{c}{-4 - 12} = \lambda$$

$$\Rightarrow \quad a = 12\lambda,\ b = -11\lambda,\ c = -16\lambda.$$

Now, the equation (i) becomes

$$12(x - 2) - 11(y - 2) - 16(z - 1) = 0$$

$$\Rightarrow \quad 12x - 11y - 16z + 14 = 0.$$

Thus, the equation of required plane is $12x - 11y - 16z + 14 = 0$.

Example 16: Find the distance from the point (3, 4, 5) to the point where the line $\frac{x - 3}{1} = \frac{y - 4}{2} = \frac{z - 5}{2}$ meets the plane $x + y + z = 2$.

Solution:

The given line is

$$\frac{x - 3}{1} = \frac{y - 4}{2} = \frac{z - 5}{2} = r \text{ (say)} \qquad \text{...... (i)}$$

The general point of (i) is

$$(r + 3,\ 2r + 4,\ 2r + 5) \qquad \text{...... (ii)}$$

Given that the line (i) meets the plane

$$x + y + z = 2 \qquad \text{...... (iii)}$$

Then the point (ii) is the common point of (i) and (iii). Therefore,

$$r + 3 + 2r + 4 + 2r + 5 = 2$$

$$\Rightarrow \quad 5r + 12 = 2$$

$$\Rightarrow \quad r = -2$$

So, the point of intersection of the line and the plane is

$$(-2 + 3,\ -4 + 4,\ -4 + 5)$$

i.e. $(1, 0, 1)$

Now, the length from (3, 4, 5) to (1, 0, 1) is

$$d = \sqrt{(3 - 1)^2 + (4 - 0)^2 + (5 - 1)^2}$$

$$= \sqrt{4 + 16 + 16} = \sqrt{36} = 6.$$

Exercise 10.2

1. Find the equation of the plane through (−1, 1, −1) and perpendicular to the line $x - 2y + z = 4$, $4x + 3y - z + 4 = 0$.
2. Find the equation of the line passing through (2, 3, 4) parallel to the line $x - 2y + z = 4$, $4x + 3y - z + 4 = 0$.
3. Find the angle between the lines in which the plane $x - y + z = 5$ is cut by the planes $2x + y - z = 3$ and $2x + y + 3z - 1 = 0$.

4. Prove that the lines $x = -2y + 7,\ z = 3y + 10$ and $x = 5y - 1,\ z = 3y - 6$ are perpendicular to each other.
5. Find the co-ordinate of the foot of the line which is the perpendicular from the origin on the straight line given by $x + 2y + 3z + 4 = 0,\ x + y + z + 1 = 0$.
6. Find the equation of the plane parallel to the line $x - 2 = \frac{y-1}{3} = \frac{z-3}{2}$ containing $(0, 0, 0)$ and $(-3, 1, 2)$.
7. Find the equation of the plane through $(2, -3, 1)$ normal to the line joining $(3, 4, -1)$ and $(2, -1, 5)$.
8. Find the equation of plane through the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ parallel to y-axis.
9. Find the equation of plane through (α, β, γ) and the line $x = py + q = rz + s$.

Answers

1. $x - 5y - 11z = 5$
2. $\frac{x-2}{-1} = \frac{y-3}{5} = \frac{z-4}{11}$
3. $\cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$
5. $\left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$
6. $2x - 4y + 5z = 0$
7. $x + 5y - 6z + 19 = 0$
8. $2x - z + 1 = 0$
9. $x(q + \beta p - s - rq) - yp(\alpha - s - r\delta) + zr(\alpha - q - \beta p) - (\alpha q + \alpha\beta p + \alpha s + \alpha rq + \alpha\beta q - \beta sq - \alpha\beta\delta q - \alpha\delta r + \delta rq + \beta\delta rp) = 0$

10.6 Coplanar Lines

Two lines are said to be coplanar if both the lines lie in a plane. Two lines will lie in a plane if they are either parallel or intersecting. Following are the conditions for the lines to be co-planar.

A. When the two lines are given in symmetrical form

Let, the two lines in symmetrical form are

$$\frac{x - x_1}{\ell_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \quad \text{--- (1)}$$

and

$$\frac{x - x_2}{\ell_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \quad \text{--- (2)}$$

These two straight lines will be coplanar if the plane through the line (1) and also contain the line (2).

The equation of the plane containing the line (1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \text{---(3)}$$

satisfies the relation

$$a\ell_1 + bm_1 + cn_1 = 0 \quad \text{--- (4)}$$

Let the plane (3) contains the line (2), then

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \quad \text{--- (5)}$$

satisfies the relation

$$a\ell_2 + bm_2 + cn_2 = 0 \quad \text{--- (6)}$$

Now, eliminating a, b, c from (4), (5) and (6), we get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 \quad \text{--- (7)}$$

This is the condition that the lines (1) and (2) are coplanar.

And, eliminating a, b, c from (3), (4) and (6), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

This is the equation of plane containing the lines (1) and (2).

B. **If the line is in general form then change it into symmetrical form and then use above process.**

Note: If the lines are intersect to each other then the lines are coplanar.

Example 17: Examine, whether the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar or not. If so, find the equation of plane containing them.

Solution:

Given line is,

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad \text{...... (i)}$$

and

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5} \quad \text{...... (ii)}$$

Clearly, the line (i) passes through the point $(x_1, y_1, z_1) = (1, 2, 3)$ and having the direction ratios $(l_1, m_1, n_1) = (2, 3, 4)$. Also, the line (ii) passes through the point $(x_2, y_2, z_2) = (2, 3, 4)$ and having the direction ratio $(l_2, m_2, n_2) = (3, 4, 5)$.

We know, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now,

$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 1(15-16) - 1(10-12) + 1(8-9)$$

$$= -1+2-1$$

$$= 0$$

This shows that the lines are coplanar.

And, the equation of plane containing the lines (i) and (ii) is,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(15-16) - (y-2)(10-12) + (z-3)(8-9) = 0$$

$$\Rightarrow -x+1+2y-4-z+3 = 0$$

$$\Rightarrow x-2y+z=0.$$

Thus, the equation of the plane containing the lines be $x-2y+z=0$.

Example 18: Prove that the lines $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ and $x+2y+3z-8=0 = 2x+3y+4z-11$ are coplanar and find the point of intersection and the equation of the plane containing them.

Solution:

Given lines are

$$\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3} \quad \text{---- (i)}$$

And,

$$x+2y+3z-8=0=2x+3y+4z-11 \quad \text{---- (ii)}$$

First we set (ii) in symmetrical form. For this set $z=0$, then

$$x+2y-8=0 \quad \text{---- (*)}$$

$$2x+3y-11=0 \quad \text{---- (**)}$$

Solving (*) and (**) we get,

$$x=-2 \quad \text{and} \quad y=5.$$

Therefore the line (ii) passes through the point $(-2, 5, 0)$.

Let direction ratio of lines is (a, b, c) of (ii) then

$$a+2b+3c=0 \quad \text{...... (iii)}$$

$$2a+3b+4c=0 \quad \text{...... (iv)}$$

Solving equation (iii) and (iv), we get

$$\frac{a}{8-9}=\frac{b}{6-4}=\frac{c}{3-4}=\lambda$$

$$\Rightarrow a=-\lambda,\ b=2\lambda,\ c=-\lambda$$

Now, the equation of the line passes through the point $(-2, 5, 0)$ and have direction ratio $(-\lambda, 2\lambda, -\lambda)$ is

$$\frac{x+2}{-\lambda}=\frac{y-5}{2\lambda}=\frac{z-0}{-\lambda}$$

$$\Rightarrow \frac{x+2}{-1}=\frac{y-5}{2}=\frac{z-0}{-1} \quad \text{...... (v)}$$

Since the line (i) passes through the point $(x_1, y_1, z_1) = (-1, -1, -1)$ and having the direction ratio $(l_1, m_1, n_1) = (1, 2, 3)$.

And, Then the line (ii) passes through the point $(x_2, y_2, z_2) = (-2, 5, 0)$ and having the direction ratio $(l_2, m_2, n_2) = (-1, 2, -1)$.

Here,

$$\begin{vmatrix} -2-(-1) & 5-(-1) & 0-(-1) \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 6 & 1 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= -1(-2-6) - 6(-1+3) + 1(2+2)$$

$$= 8-12+4$$

$$= 0$$

This means the given lines are coplanar.

For point of intersection, let,

$$\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3} = r \text{ (let)} \qquad \text{...... (vi)}$$

Then the general point of (vi) is,

$$(r-1, 2r-1, 3r-1).$$

And,

$$\frac{x+2}{-1} = \frac{y-5}{2} = \frac{z-0}{-1} = r_1 \qquad \text{...... (vii)}$$

Then the general point of (v) is, $(-r_1-2, 2r_1+5, -r_1)$.

For point of intersection, at least one point of the line (vi) and (vii) is same.

So,

$$x_1 = x_2,\ y_1 = y_2 \text{ and } z_1 = z_2$$

That is,

$$r - 1 = -r_1 - 2$$

$$2r - 1 = 2r_1 + 5 \qquad \text{...... (viii)}$$

$$3r - 1 = -r_1 \qquad \text{...... (ix)} \qquad \text{........ (x)}$$

Solving the equation (viii) and (ix) we get,

$$r = 1.$$

Therefore the point of intersection is $(r-1, 2r-1, 3r-1) = (0, 1, 2)$.

And, the equation of plane containing the given lines be,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x+1 & y+1 & z+1 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x+1)(-8) - (y+1)(2) + (z+1)(4) = 0$$

$$\Rightarrow -8x - 8 - 2y - 2 + 4z + 4 = 0$$

$$\Rightarrow -8x - 2y + 4z - 6 = 0$$

$$\Rightarrow 4x + y - 2z + 3 = 0$$

This is the equation of plane that contains the given lines (i) and (ii).

Example 19: Show that the lines $x + y + z - 3 = 0 = 2x + 3y + 4z - 3$ and $4x - y + 5z - 7 = 0 = 2x - 5y - z - 3$ are co-planar and find the equation of the plane in which they lie.

Solution:

The given two lines are

$$\left.\begin{array}{l} x + y + z - 3 = 0 \\ 2x + 3y + 4z - 5 = 0 \end{array}\right\} \qquad \text{... ... (i)}$$

and

$$\left.\begin{array}{l} 4x - y + 5z - 7 = 0 \\ 2x - 5y - z - 3 = 0 \end{array}\right\} \qquad \text{--- (ii)}$$

To reduce one line (i), to the symmetrical form, let l_1, m_1, n_1 be the direction cosines of the line (i), then

$$\ell_1 + m_1 + n_1 = 0$$

and

$$2\ell_1 + 3m_1 + 4n_1 = 0$$

Solving we get

$$\frac{\ell_1}{4-3} = \frac{m_1}{2-4} = \frac{n_1}{3-2}$$

$$\text{i.e. } \frac{\ell_1}{1} = \frac{m_1}{-2} = \frac{n_1}{1}$$

This shows the direction ratios of the line (i) are 1, −2, 1.

For the passes through of the line (i), set $z = 0$ then (i) gives

$$x + y - 3 = 0$$

and

$$2x + 3y - 5 = 0$$

Solving we get

$$\frac{x}{-5+9} = \frac{y}{-6+5} = \frac{1}{3-2}$$

$$\text{i.e. } x = 4,\ y = -1.$$

Thus, the line (i) meets xy plane at (4, −1, 0).

Thus, the symmetrical form of the line (i) is

$$\frac{x-4}{1} = \frac{y+1}{-2} = \frac{z}{1} \qquad \text{--- (iii)}$$

Now, we are to prove that the lines (ii) and (iii) are co-planar.

Next, the plane that passes through line (ii) is

$$(4x - y + 5z - 7) + k(2x - 5y - z - 3) = 0 \qquad \text{--- (iv)}$$

Clearly, the plane (iv) contains the line (iii) if

a. The point (4, −1, 0) on the line (iii) lies on this plane.

b. The line (iii) is perpendicular to the normal to this plane.

Let, the point (4, −1, 0) lies on the line (iv) then

$$(16 + 1 - 7) + k(8 + 5 - 3) = 0$$

$$\Rightarrow k = -1$$

Then (4) becomes

$$4x - y + 5z - 7 - 2x + 5y + z + 3 = 0$$

$$\Rightarrow 2x + 4y + 6z - 4 = 0 \qquad \text{--- (v)}$$

$$\Rightarrow x + 2y + 3z - 2 = 0$$

The direction cosines of line (iii) are proportional to 1, −2, 1 and the direction cosines of the normal to the plane (v) are proportional to 1, 2, 3.

Since the line (iii) will be perpendicular to the normal to plane, if

$$1\times1 + 2\times(-2) + 3\times1 = 0$$

i.e. $$0 = 0$$

This means the lines (i) and (ii) are coplanar.

And, the equation of plane in which the lines (i) and (ii) lie is $x + 2y + 3z - 2 = 0$.

Exercise 10.3

1. Show that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = y-4 = \frac{z-5}{3}$ are coplanar. Find their common point and equation of plane in which they lie.
2. Show that the lines $x - 1 = 2y - 4 = 3z$ and $3x - 5 = 4y - 9 = 3z$ meet in a point and the equation of the plane is which they lie is $3x - 8y + 3z + 13 = 0$.
3. Show that the lines $\frac{x+4}{3} = \frac{y+6}{5} = \frac{1-z}{2}$ and $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ are coplanar. Find their point of intersection and the plane in which they lie.
4. Find the equation of the plane containing $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-4}{5}$.
5. Prove that the lines $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ and $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ intersect. Find also their point of intersection and plane through them.
6. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$ are coplanar. Also find their point of contact.

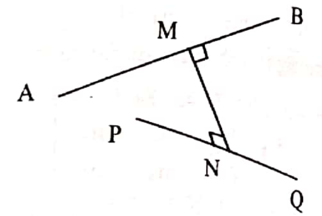
Answers

1. $(1, 3, 2)$; $17x - 47y - 24z + 172 = 0$
3. $(2, 4, -3)$; $45x - 17y + 25z + 53 = 0$
4. $2x + y - 2z + 3 = 0$
5. $(5, -7, 6)$; $11x - 6y - 5z - 67 = 0$
6. $(-1, -1, -1)$

10.7 The Shortest Distance Between Two Lines

The two non-intersecting and non-parallel lines are called skew lines. There also exists a shortest distance between the skew lines and the line of the shortest distance which is common perpendicular to both of these. The shortest distance is denoted by S.D.

In the figure, AB and PQ are two skew lines and MN is the line of shortest distance.



10.7.1 Length and Equations of the Line of the Shortest Distance

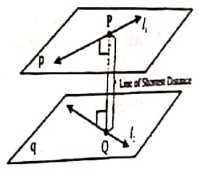
Let the equation of the skew lines be

$$\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \dots\dots (1)$$

and

$$\frac{x-x_2}{\ell_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad \dots\dots (2)$$

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the points on the two skew lines, respectively, as shown in figure. Let PQ is the shortest distance (S.D.) between these two lines and let its direction cosines are ℓ, m, n then,



$$\ell\ell_1 + mm_1 + nn_1 = 0 \quad \dots\dots (3)$$

$$\ell\ell_2 + mm_2 + nn_2 = 0 \quad \dots\dots (4)$$

Solving (3) and (4) we get,

$$\frac{\ell}{m_1n_2 - n_1m_2} = \frac{m}{n_1\ell_2 - \ell_1n_2} = \frac{n}{\ell_1m_2 - m_1\ell_2}$$

Therefore, the direction ratios of PQ are

$$m_1n_2 - n_1m_2,\ n_1\ell_2 - \ell_1n_2,\ \ell_1m_2 - m_1\ell_2$$

Now, dividing each by $\sqrt{\Sigma(m_1n_2 - n_1m_2)^2}$ then

$$\frac{m_1n_2 - n_1m_2}{\sqrt{\Sigma(m_1n_2 - n_1m_2)^2}},\ \frac{n_1\ell_2 - \ell_1n_2}{\sqrt{\Sigma(m_1n_2 - n_1m_2)^2}},\ \frac{\ell_1m_2 - m_1\ell_2}{\sqrt{\Sigma(m_1n_2 - n_1m_2)^2}}$$

which are directions ratios.

Now, the length of shortest distance between the lines is

$$PQ = \text{projection of line joining the lines on } PQ$$

$$= \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

$$= \frac{1}{\sqrt{\Sigma(m_1n_2 - n_1m_2)^2}}\left[(x_2 - x_1)(m_1n_2 - n_1m_2) + (y_2 - y_1)(n_1\ell_2 - \ell_1n_2) + (z_2 - z_1)(\ell_1m_2 - m_1\ell_2)\right]$$

$$= \frac{1}{\sqrt{\Sigma(m_1n_2 - n_1m_2)^2}}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}$$

And, the equation of plane containing the shortest distance is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ \ell_2 & m_2 & n_2 \\ \ell_1 & m_1 & n_1 \end{vmatrix}$$

Example 20: Find the shortest distance between the lines, $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5}$.

Solution:

Given that,

$$\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4} \qquad \text{...... (i)}$$

Clearly the line (i) passes through the point $(x, y, z) = (3, 4, 5)$ and it has the direction ratio $(\ell_1, m_1, n_1) = (2, 3, 4)$.

Also, given line is,

$$\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5} \qquad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (4, 5, 7)$ and it has the direction ratio $(\ell_2, m_2, n_2) = (3, 4, 5)$

Let ℓ, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$2\ell + 3m + 4n = 0$$

$$3\ell + 4m + 5n = 0$$

Solving by cross multiplication, we get

$$\frac{\ell}{15 - 16} = \frac{m}{12 - 10} = \frac{n}{8 - 9}$$

$$\Rightarrow \quad \frac{\ell}{-1} = \frac{m}{2} = \frac{n}{-1} = k$$

$$\Rightarrow \quad \ell = -k,\ m = 2k,\ n = -k$$

We know that,

$$\ell^2 + m^2 + n^2 = 1$$

$$\Rightarrow \quad (-k)^2 + (2k)^2 + (-k)^2 = 1$$

$$\Rightarrow \quad 6k^2 = 1$$

$$\Rightarrow \quad k = \frac{1}{\sqrt{6}}, \text{ taking +ve sign only.}$$

So that,

$$\ell = \frac{1}{\sqrt{6}},\quad m = -2\frac{1}{\sqrt{6}},\quad n = \frac{1}{\sqrt{6}}$$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\,\ell + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$

$$= (4 - 3)\left(\frac{1}{\sqrt{6}}\right) + (5 - 4)\left(\frac{-2}{\sqrt{6}}\right) + (7 - 5)\left(\frac{1}{\sqrt{6}}\right)$$

$$= \frac{1 - 2 + 2}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}.$$

Thus, the shortest distance between the lines is $\frac{1}{\sqrt{6}}$ units.

Example 21: Find the shortest distance between the z-axis and $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$.

Solution:

We have the equation of the y-axis is

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{1} \qquad \text{...... (i)}$$

Another given line is,

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \qquad \text{... (ii)}$$

The plane containing (ii) is

$$ax + by + cz + d + k(a'x + b'y + c'z + d') = 0 \qquad \text{... (iii)}$$

Let the line (i) is parallel to (iii) then the normal to (iii) is perpendicular to (i). Therefore,

$$0(a + ka') + 0(b + kb') + 1(c + c') = 0$$

$$\Rightarrow \quad k = \frac{-c}{c'}$$

Therefore (iii) becomes

$$ax + by + cz + d - \frac{c}{c'}(a'x + b'y + c'z + d') = 0$$

$$\ldots + (dc' - d'c) = 0 \qquad \text{... (iv)}$$

Thus (iv) contains (ii). Clearly, the origin contained in y-axis. Therefore the shortest distance from y-axis to (ii) is the perpendicular form origin to (ii). So,

$$SD = \pm \frac{c'd - d'c}{\sqrt{(ca' - c'a)^2 + (bc' - b'c)^2}}$$

Example 22: Find the magnitude and equation of the shortest distance between the lines $\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7}$ and $\frac{x-9}{3} = \frac{y-13}{8} = \frac{15-z}{5}$.

Solution:

Given that,

$$\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7} \qquad \text{...... (i)}$$

Clearly the line (i) passes through the point $(x_1, y_1, z_1) = (5, 7, 3)$ and it has the direction ratio $(\ell_1, m_1, n_1) = (3, -16, 7)$.

Also, given line is,

$$\frac{x-9}{3} = \frac{y-13}{8} = \frac{15-z}{5} \qquad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (9, 13, 15)$ and it has the direction ratio $(\ell_2, m_2, n_2) = (3, 3, -5)$.

Let ℓ, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$3\ell - 16m + 7n = 0$$

$$3\ell + 8m - 5n = 0$$

Solving by cross multiplication, we get

$$\frac{\ell}{80-56} = \frac{m}{21+15} = \frac{n}{24+48}$$

$$\Rightarrow \frac{\ell}{24} = \frac{m}{36} = \frac{n}{72} = k \text{ (let)}$$

$$\Rightarrow \ell = 24k,\ m = 36k,\ n = 72k$$

We know that,

$$\ell^2 + m^2 + n^2 = 1$$

$$\Rightarrow (24k)^2 + (36k)^2 + (72k)^2 = 1$$

$$\Rightarrow 576\,k^2 + 1296\,k^2 + 5184\,k^2 = 1$$

$$\Rightarrow 7056\,k^2 = 1$$

$$\Rightarrow k = \frac{1}{84}, \text{ taking +ve sign only.}$$

So that,

$$\ell = \frac{2}{7},\ m = \frac{3}{7},\ n = \frac{6}{7}$$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\ell + (y_2 - y_1)m + (z_2 - z_1)n$$

$$= (9-5) \times \frac{2}{7} + (13-7) \times \frac{3}{7} + (15-3) \times \frac{6}{7}$$

$$= \frac{8 + 18 + 72}{7} = 14.$$

Thus, the shortest distance between the lines is 14 units.

Next, for equation of shortest distance,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-5 & y-7 & z-3 \\ 3 & -16 & 7 \\ 2/7 & 3/7 & 6/7 \end{vmatrix} = 0 = \begin{vmatrix} x-9 & y-13 & z-15 \\ 3 & 8 & -5 \\ 2/7 & 3/7 & 6/7 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-5 & y-7 & z-3 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix} = 0 = \begin{vmatrix} x-9 & y-13 & z-15 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix}$$

$$\Rightarrow (x-5)(-96-21) - (y-7)(18-14) + (z-3)(9+32) = 0$$

$$= (x-9)(48+15) - (y-13)(18+10) + (z-15)(9-16)$$

$$\Rightarrow (x-5)(-117) - (y-7)4 + (z-3)41 = 0 = (x-9)9 - (y-13)4 + (z-15)(-1)$$

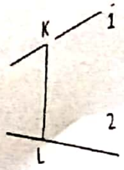
$$\Rightarrow -117(x-5) - 4(y-7) + 41(z-3) = 0 = (x-9)9 - (y-13)4 - (z-15).$$

$$\Rightarrow -117x - 4y + 41z + 490 = 0 = 9x - 4y - z - 14.$$

Example 23: Find the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{x-3}{1}$ and $\frac{x+3}{-3} = \frac{y-7}{2} = \frac{z-6}{4}$. Find also its equations and the points in which it meets the given lines.

Solution:

(By alternative method)



The given lines are

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \qquad \text{. . . (1)}$$

and

$$x + 3 \quad y + 7 \quad z - 6 \qquad (2)$$

Let the S.D., meet the lines (1) and (2) in K and L.

Any point on the line (1) is

$(3r_1 + 3, -r_1 + 8, r_1 + 3)$. Let it be K. . . . (3)

Again, any point on the line (2) is

$(-3r_2 - 3, 2r_2 - 7, 4r_2 + 6)$ Let it be L. . . . (4)

Then the d.r.'s of LK are

$3r_1 + 3 + 3r^2 + 3, -r_1 + 8 - 2r_2 + 7, r_1 + 3 - 4r_2 - 6$

$\Rightarrow \quad 3r_1 + 3r_2 + 6, -r_1 - 2r_2 + 15, r_1 - 4r_2 - 3.$

Since KL is normal to both (1) and (2),

$3(3r_1 + 3r_2 + 6) + (-1)(-r_1 - 2r_2 + 15) + 1(r_1 - 4r_3 - 3) = 0$

$\Rightarrow \quad 11r_1 + 7r_2 = 0$ . . . (5)

and

$-3(3r_1 + 3r_2 + 6) + 2(-r_1 - 2r_2 + 15) + 4(r_1 - 4r_2 - 3) = 0$

$\Rightarrow \quad 7r_1 + 29r_2 = 0$ . . . (6)

Solving (5) and (6) for r_1, r_2, we get

$$r_1 = 0, r_2 = 0$$

Putting these values of r_1, r_2 in (3) and (4), we get K(3, 8, 3) and L (– 3, – 7, 6), which are the required points of intersection of the S.D. with the given lines,

Length of S.D. = KL $= \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2}$

$$= 3\sqrt{30}$$

The equations of KL, the S.D. are

$$\frac{x-3}{-3-3} = \frac{y-8}{-7-8} = \frac{z-3}{6-3}$$

or $$\frac{x-3}{2} = \frac{y-8}{1} = \frac{z-3}{-1}$$

Exercise 10.4

1. Find the shortest distance between the lines, $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5}$.

2. Find the shortest distance between the lines $x = y + 4 = \frac{z}{3}$, $\frac{x-1}{3} = \frac{y}{2} = z$

3. Find the magnitude and equation of the line of shortest distance between the line of shortest distance the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

4. Find the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$. Also, find the equation of the line of shortest distance.

5. Define shortest distance between two skew lines in space. Find the length and equation of shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$.

6. Find the magnitude and equation of the shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$.

7. Find shortest distance between the lines $\frac{x-1}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-6}{5}$. Also, find the equation of the shortest distance.

8. Find the length of the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $2x - 3y + 27 = 0, 2y - z + 20 = 0$.

9. Find the length and equation of the shortest distance between the lines $x - y + z = 0 = 2x - 3y + 4z$ and $x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4$.

Answers

1. $\frac{1}{\sqrt{6}}$ unit

2. $\frac{9}{\sqrt{10}}$ unit

3. 14 unit; $117x + 4y - 41z - 490 = 0 = 9x - 4y - z - 14$

4. $3\sqrt{30}$ unit; $4x - 5y - 17z + 79 = 0 = 22x - 3y + 19z - 83$.

5. $\frac{1}{\sqrt{3}}$ unit; $4x + y - 5z = 0 = 7x + y - 8z - 31$

6. $\frac{1}{\sqrt{3}}$ unit; $4x + y - 5z = 0 = 7x + y - 8z - 31$

7. $\frac{5}{\sqrt{6}}$ unit; $11x + 2y - 7z + 16 = 0 = 14x + 2y - 10z - 6$

8. $\frac{108}{\sqrt{22}}$ unit

9. $\frac{13}{\sqrt{66}}$ unit; $3x - y - z = 0 = x + 2y + z - 1$

❑❑❑

11

CHAPTER

SPHERE

Introduction

A **sphere** is a three-dimensional object that is round in shape. The sphere is defined in three axes, i.e., x-axis, y-axis and z-axis. This is the main difference between circle and sphere. A sphere does not have any edges or vertices, like other 3D shapes.

The points on the surface of the sphere are equidistant from the center. Hence, the distance between the center and the surface of the sphere are equal at any point. This distance is called the radius of the sphere. Examples of spheres are a ball, a globe, the planets, etc.

11.1 DEFINITION OF A SPHERE

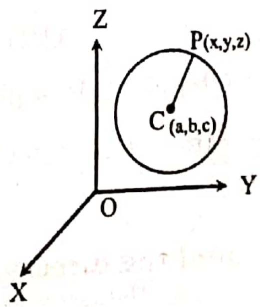
A round in shape in a three-dimensional object is a sphere which is the points on the surface with equidistant from the center and is perfectly symmetrical.

Definition (Sphere)

A sphere is the locus of a point which moves so that its distance from a fixed point always remains constant. The fixed point is called the center and constant distance is called the radius of sphere.

11.2 EQUATION OF A SPHERE

Let C(a, b, c) be the center of the sphere and r be its radius. If P(x, y, z) be any point on the sphere, then



$$CP = r = \text{radius}$$

$$\Rightarrow \quad CP^2 = r^2$$

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \quad \ldots\ldots (1)$$

which is the required equation of sphere.

Note: The equation of sphere whose center is (0, 0, 0) and radius r is $x^2 + y^2 + z^2 = r^2$.

11.3 GENERAL EQUATION OF A SPHERE

The second degree symmetrical equation in three dimensional can be written in the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \dots\dots (1)$$

The equation (1) is known as the general form of the equation of sphere.

The equation can be re-written as

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = -d$$

$$\Rightarrow (x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$$

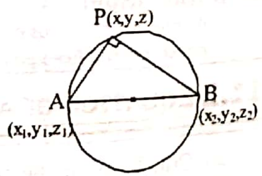
$$\Rightarrow [x - (-u)]^2 + [y - (-v)]^2 + [z - (-w)]^2 = \left[\sqrt{u^2 + v^2 + w^2 - d}\right]^2$$

which clearly represents a sphere with center $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$

Note:

1. If $u^2 + v^2 + w^2 - d > 0$, the radius is real hence the sphere is real.
   If $u^2 + v^2 + w^2 - d = 0$, the sphere in this case is a point sphere.
   If $u^2 + v^2 + w^2 - d < 0$, the radius is imaginary.
2. The general equation of second degree
   $$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$
   represents a sphere, if $a = b = c$ and does not contain terms involving the products xy, yz, zx.
3. If the sphere passes through the origin then $x^2 + y^2 + z^2 = r^2$.

11.4 EQUATION OF A SPHERE IN A DIAMETER FORM



Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two end points of the diameter of the given sphere.

Let $P(x, y, z)$ be any point on the sphere. Join AP and BP then there we find a right angle triangle form ΔABP in which

$$\angle APB = \text{angle in semi-circle} = 90°.$$

Clearly, AP is perpendicular to BP.

Since, the direction ratios of AP are

$$x - x_1, y - y_1, z - z_1$$

and the direction ratios of BP are

$$x - x_2, y - y_2, z - z_2.$$

As $AP \perp BP$, we get

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

i.e. $(x - x_1)(x - x_1) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$

Example 1: Find the equation of the sphere whose center $(3, -4, 5)$ and radius 7.

Solution:

Given that, the sphere has,

Centre $(a, b, c) = (3, -4, 5)$

and Radius $(r) = 7$.

Now the equation of sphere is,

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

i.e. $(x - 3)^2 + (y + 4)^2 + (z - 5)^2 = 7^2$

$$\Rightarrow x^2 + y^2 + z^2 - 6x + 8y - 10z + 1 = 0.$$

Example 2: Find the center and radius of sphere $x^2 + y^2 + z^2 + 2x - 3y - 4z - 12 = 0$.

Solution:

The given equation is,

$$x^2 + y^2 + z^2 + 2x - 3y - 4z - 12 = 0 \quad \dots (i)$$

Comparing it with equation (i) then we get

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

then we get

$$u = 1, v = -\frac{3}{2}, w = -2 \text{ and } d = -12.$$

Then the center of (i) is

$$C(-u, -v, -w) = C\left(-1, \frac{3}{2}, 2\right)$$

and radius of (i) is

$$r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{1 + \frac{9}{4} + 4 + 12} = \frac{\sqrt{77}}{2}$$

Thus the center is $\left(-1, \frac{3}{2}, 2\right)$ and radius is $\left(\frac{\sqrt{77}}{2}\right)$.

Example 3: Find the equation of the sphere passing through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 0, 0)$.

Solution:

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \dots\dots (i)$$

As given, the sphere (i) passes through the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and (0, 0, 0). Then,

$$1 + 2u + d = 0 \qquad \text{...... (ii)}$$

$$1 + 2v + d = 0 \qquad \text{...... (iii)}$$

$$1 + 2w + d = 0 \qquad \text{...... (iv)}$$

$$\text{and} \quad d = 0 \qquad \text{...... (v)}$$

Therefore, solving (ii), (iii), (iv) and (v), we get

$$u = v = w = \frac{-1}{2} \text{ and } d = 0$$

Then (i) becomes

$$x^2 + y^2 + z^2 - x - y - z = 0$$

The equation of the sphere is $x^2 + y^2 + z^2 - x - y - z = 0$.

Example 4: Obtain the equation of sphere through (1, 0, 0), (0, 1, 0), (0, 0, 1) and its radius as small as possible.

Solution:

Let the equation of sphere in general form is,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{...... (i)}$$

having center at $(-u, -v, -w)$ and radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

Since, it passes through the point (1, 0, 0) then (i) gives

$$1 + 2u + d = 0 \Rightarrow u = -\frac{(d+1)}{2} \qquad \text{...... (ii)}$$

Similarly (i) passes through (0, 1, 0) then

$$v = -\frac{(d+1)}{2} \qquad \text{...... (iii)}$$

Also, (i) passes through (0, 0, 1) then

$$w = -\frac{(d+1)}{2} \qquad \text{...... (iv)}$$

This shows,

$$u = v = w = -\frac{(d+1)}{2}$$

So, the radius of the sphere is,

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

Therefore,

$$r^2 = u^2 + v^2 + w^2 - d = 3\left(\frac{d+1}{2}\right)^2 - d = \frac{3d^2 + 2d + 3}{4}$$

Then,

$$\frac{d(r^2)}{dd} = \frac{6d+2}{4} \quad \text{and} \quad \frac{d^2(r^2)}{dd^2} = \frac{6}{4} > 0.$$

This means r^2 is minimum. For the point, set

$$\frac{d(r^2)}{dd} = 0 \Rightarrow \frac{6d+2}{4} = 0 \Rightarrow d = \frac{-1}{3}$$

That is the radius is minimum at $d = \frac{-1}{3}$.

Then,

$$u = v = w = -\left(\frac{d+1}{2}\right) = -\left(\frac{(-1/3)+1}{2}\right) = -\left(\frac{-1+3}{6}\right) = \frac{-1}{3}.$$

Now, putting the value of u, v, w, d in equation (i) then it becomes,

$$x^2 + y^2 + z^2 + 2\left(\frac{-1}{3}\right)x + 2\left(\frac{-1}{3}\right)y + 2\left(\frac{-1}{3}\right)z + \left(\frac{-1}{3}\right) = 0$$

$$\Rightarrow \quad 3(x^2 + y^2 + z^2) - 2x - 2y - 2z - 1 = 0$$

$$\Rightarrow \quad 3(x^2 + y^2 + z^2) - 2(x - y - z) - 1 = 0$$

This is the equation of sphere.

Example 5: Find the equation of sphere described in the join of (4, 5, 6) and (2, 3, 4) as a diameter.

Solution:

The equation of sphere joining the point (x_1, y_1, z_1) and (x_2, y_2, z_2) as the ends of its diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Given that,

$$(x_1, y_1, z_1) = (4, 5, 6) \quad \text{and} \quad (x_2, y_2, z_2) = (2, 3, 4)$$

Then the equation sphere be,

$$(x - 4)(x - 2) + (y - 5)(y - 3) + (z - 6)(z - 4) = 0$$

$$\Rightarrow \quad x^2 - 4x - 2x + 8 + y^2 - 3y - 5y + 15 + z^2 - 4z - 6z + 24 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - 6x - 8y - 10z + 47 = 0$$

Example 6: A plane passes through a fixed (a, b, c) and cuts the axes in A, B, C. Prove that the locus of the center of the sphere OABC is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$.

Given that the plane cut the axes at A, B, C.

Let the coordinate, $A = (l, 0, 0)$, $B = (0, m, 0)$ and $C = (0, 0, n)$

Let the sphere contains the origin so its general equation be,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \qquad \text{......... (i)}$$

it passes through A, B, C

$$l^2 + 2ul = 0 \Rightarrow u = \frac{-l}{2}$$

Similarly,

$$v = \frac{-m}{2} \quad \text{and} \quad w = \frac{-n}{2}$$

Now, the center of (i) is

$$(-u, -v, -w) = \left(\frac{l}{2}, \frac{m}{2}, \frac{n}{2}\right).$$

Since the equation of plane ABC in intercept form is,

$$\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 1$$

where l, m, n be the intercept made by plane.

Given that the plane passes through the point (a, b, c). So,

$$\frac{a}{l} + \frac{b}{m} + \frac{c}{n} = 1 \qquad \text{........... (ii)}$$

Let $(-u = x_1, -v = y_1, -w = z_1)$ be center of sphere OABC then,

$$x_1 = \frac{l}{2}, \quad y_1 = \frac{m}{2}, \quad z_1 = \frac{n}{2}$$

$$\Rightarrow \quad l = 2x_1, \quad m = 2y_1, \quad n = 2z_1$$

Then the equation (ii) becomes,

$$\frac{a}{2x_1} + \frac{b}{2y_1} + \frac{c}{2z_1} = 1$$

$$\Rightarrow \quad \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 2$$

Hence, locus of center is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$.

Exercise 11.1

1. Find the equation of the following spheres with:
   a. center (1, −2, 3) and radius 3.
   b. center (2, −3, 1) and radius 5.
2. Find the center and the radius of the following spheres:
   a. $x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$.
   b. $x^2 + y^2 + z^2 - 2x + 4y - 4z - 7 = 0$.
   c. $x^2 + y^2 + z^2 + 4x - 6y + 8z = 10$
3. Find the equation of the sphere described on the join of following points as diameter:
   a. (3, 4, 5) and (1, 2, 3).
   b. (2, −1, 4) and (−2, 2, −2).
4. Find the equation of sphere passing the points (0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c). Find it center and radius.
5. Find the equation of the sphere through the points (0, 0, 0), (−a, b, c), (a, −b, c) and (a, b, −c).
6. Find the equation of the sphere through the four points (1, 2, 3), (0, −2, 4), (4, −4, 2) and (3, 1, 4).
7. Find the equation of the sphere through (1, 0, 0), (0, 1, 0), (0, 0, 1) and its center lies on the plane $x + y + z = 6$.
8. Find the equation of the sphere through the points (1, −3, 4), (1, −5, 2) and (1, −3, 0) and whose center lies in the plane $x + y + z = 0$.
9. Obtain the equation of the sphere passing through the points (3, 0, 2), (−1, 1, 1), (2, −5, 4) and having its center on the plane $2x + 3y + 4z = 6$.
10. Find the equation of sphere which passes through the points (1, 2, 3) and (2, 3, 4) and has its center on the line $x = y = z$.
11. A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

Answers

1. a. $x^2 + y^2 + z^2 - 2x + 4y - 6z + 6 = 0$ b. $x^2 + y^2 + z^2 - 4x + 6y - 2z - 11 = 0$
2. a. $(-1, 2, 3); 3$ b. $(1, -2, 2); 4$ c. $(-2, 3, -4); \sqrt{19}$
3. a. $x^2 + y^2 + z^2 - 4x - 6y - 6z + 26 = 0$ b. $x^2 + y^2 + z^2 - y - 2z - 14 = 0$
4. $x^2 + y^2 + z^2 - ax - by - cz = 0$; center $= \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$; radius $= \frac{1}{2}\sqrt{a^2 + b^2 + c^2}$
5. $\frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0$
6. $x^2 + y^2 + z^2 - 8x + 2y - 2z - 8 = 0$
7. $x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0$
8. $x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0$
9. $x^2 + y^2 + z^2 + 4y - 6z - 1 = 0$
10. $x^2 + y^2 + z^2 - 5(x + y + z) + 16 = 0$

11.5 Circle as Intersection of a Plane and a Sphere

The section of a sphere by a plane is a circle. Suppose the equation of the sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{---- (1)}$$

and the plane section is

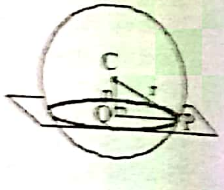
$$ax + by + cz + k = 0 \quad \text{---- (2)}$$

Then any point on the circle also lie on the sphere (1) as well as the plane section (2). Hence, the equations of the circle of the sphere are given by

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,\ ax + by + cz + k = 0.$$

11.5.1 Section of a Sphere by a Plane

Let C be the center of the sphere and P be any point on the section of the sphere by the plane.



Draw CQ perpendicular to the plane. Then N is the foot of the perpendicular from P on the plane section.

Join CP. Since CQ is perpendicular to PQ, then CPQ is a right angled triangle. Therefore,

$$CP^2 = CQ^2 + PQ^2$$

i.e. $PQ^2 = CQ^2 - CP^2$

$\Rightarrow$ $PQ = \sqrt{CQ^2 - CP^2}$ = a constant

Since CP and CQ are constants, so PQ is constant. This shows that the locus of P is a circle with center at Q and radius equal to PQ.

Note1: If the radius of the circle is less than the radius of the sphere then the circle is called a small circle. In other words, a circle of the sphere not passing through the center of the sphere is called a small circle.

Note 2: If the radius of the circle is equal to the radius of the sphere then the circle is called a great circle of the sphere. In other words, a circle of the sphere passing through the center of the sphere is called a great circle.

Note 3: If a sphere together with a plane always determines a circle i.e. the two equations of the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,\ \ell x + my + nz = p$$

always represents a circle.

Note 4: The center of a circle is the foot of perpendicular from the center of sphere on the plane.

11.6 Intersection of Two Spheres

The curve given by the intersection of two spheres, is a circle. Let the two spheres are

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

Then

$$S - S_1 = 2(u - u_1)x + 2(v - v_1)y + 2(w - w_1)z + d - d_1 = 0 \quad \text{--- (1)}$$

Clearly, the equation (1) is a linear equation in x, y, z and therefore represents a plane and this plane passes through the point of intersection of the given two spheres.

In addition, we know that section of the sphere by a plane is a circle. So, the curve of intersection of the spheres is given by $S_1 - S_2 = 0$

11.7 Sphere Through a Given Circle

Let the equation of a circle be

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

and $$P = \ell x + my + nz - p = 0$$

Then the equation of sphere through the given circle is

$$S + kP = 0$$

where k is some constant value represents a sphere as the coefficients of x^2, y^2, z^2 are equal and there are no terms containing xy, yz, zx.

Also, this equation is satisfied by all the points satisfying S = 0 and P = 0. Thus it follows that S + kP = 0 represents a sphere passing through the circle in which the sphere S = 0 cut by a plane P = 0.

Example 7: Find the equation of sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and (1, 2, 3) find its center and radius.

Solution:

The equation of sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ is

$$x^2 + y^2 + z^2 - 9 + k(2x + 3y + 4z - 5) = 0 \quad \text{---- (i)}$$

Since the equation (i) passes through (1, 2, 3), so,

$$1^2 + 2^2 + 3^2 - 9 + k(2 + 6 + 12 - 5) = 0$$

$\Rightarrow$ $1 + 4 + 9 - 9 + k(15) = 0$

$\Rightarrow$ $k = \frac{-1}{3}$

Then equation (i) becomes,

$$x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow \quad 3(x^2 + y^2 + z^2) - (2x + 3y + 4z) - 22 = 0 \quad \text{...... (ii)}$$

Since, the center and radius of (ii) is,

$$\text{Centre} = \left(\frac{\text{coeff. of x}}{-2}, \frac{\text{coeff. of y}}{-2}, \frac{\text{coeff. of z}}{-2}\right)$$

$$= \left(\frac{(-2/3)}{-2}, \frac{-1}{-2}, \frac{(-4/3)}{-2}\right)$$

$$= \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$$

And,

$$\text{Radius} = \sqrt{u^2 + v^2 + w^2 - d}$$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^2 - \left(\frac{-22}{3}\right)}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{2} + \frac{2}{3} + \frac{22}{3}}$$

$$= \sqrt{\frac{293}{36}}$$

Thus center is $\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ and radius is $\sqrt{\frac{293}{36}}$.

Example 8: Find the equation of sphere through the circle $x^2 + y^2 = 4$, $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3.

Solution:

Given circle is

$$x^2 + y^2 + z^2 = 4, \ z = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - 4 + kz = 0 \quad \text{... ... (i)}$$

its center is,

$$P\left(\frac{0}{-2}, \frac{0}{-2}, \frac{-k}{2}\right) = P\left(0, 0, \frac{-k}{2}\right)$$

and, its radius is,

$$r = \sqrt{0^2 + 0^2 + \left(\frac{-k}{2}\right)^2 - (-4)} = \sqrt{\frac{k^2}{4} + 4}$$

Let M be the center of the circle which is the cut off part plane $x + 2y + 2z = 0$ and sphere (i). Let N be the point on the surface of the sphere. Then ΔMNP is a right angle. From the right angled triangle in the figure,

$$ON^2 = OM^2 + MN^2$$

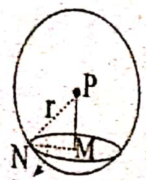
$$\Rightarrow \quad \left(\sqrt{\frac{k^2}{4} + 4}\right)^2 = \left(\frac{0 + (2)(0) + (2)(-k/2)}{\sqrt{1^2 + 2^2 + 2^2}}\right)^2 + 3^2$$

$$\Rightarrow \quad \frac{k^2}{4} + 4 = \frac{k^2}{3^2} + 9$$

$$\Rightarrow \quad \frac{9k^2 - 4k^2}{36} = 9 - 4$$

$$\Rightarrow \quad 5k^2 = 5(36)$$

$$\Rightarrow \quad k = \pm 6$$



Then equation (i) becomes

$$x^2 + y^2 + z^2 - 4 + 6z = 0$$

$$\text{and} \quad x^2 + y^2 + z^2 - 4 - 6z = 0.$$

Example 9: Find the equation of the sphere, its radius and center which has the circle, $x^2 + y^2 + z^2 = 9$, $x - 2y + 2z = 5$ as a great circle.

Solution:

The given circles is determined by

$$x^2 + y^2 + z^2 = 9 \quad \text{...... (i)}$$

$$x - 2y + 2z = 5 \quad \text{...... (ii)}$$

The equation of sphere through given circle is

$$x^2 + y^2 + z^2 - 9 + k(x - 2y + 2z - 5) = 0 \quad \text{...... (iii)}$$

As given, let the circle is a great circle of (iii), then the center of (iii) lies on the plane (ii).

Clearly, the center of (iii) is

$$C\left(-\frac{k}{2}, k, -k\right)$$

Since the center lies on the plane (ii), so

$$-\frac{k}{2} - 2k - 2k = 5$$

$$\Rightarrow \quad k = \frac{-10}{9}$$

Now equation (iii) becomes,

$$x^2 + y^2 + z^2 - 9 - \frac{10}{9}(x - 2y + 2z - 5) = 0 \quad \text{...... (iv)}$$

Then comparing (iv) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ then we get,

(a) Centre of (iv) is,

$$(-u, -v, -w) = \left(\frac{5}{9}, -\frac{10}{9}, \frac{10}{9}\right).$$

(b) Radius of (iv) is,

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

$$= \sqrt{\frac{25}{81}[1+4+4] - \left(-9+\frac{50}{9}\right)}$$

$$= \sqrt{\frac{25}{9}+\frac{31}{9}}$$

$$= \sqrt{\frac{56}{9}}$$

Thus the equation of sphere is (iv) and its center and radius are $\left(\frac{5}{9}, -\frac{10}{9}, \frac{10}{9}\right)$ and $\sqrt{\frac{56}{9}}$, respectively.

Example 10: A sphere S has points (0, 1, 0), (3, –5, 2) as opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x - 2y + 4z + 7 = 0$ as a great circle.

Solution:

Given that,

$$(x_1, y_1, z_1) = (0, 1, 0) \text{ and } (x_2, y_2, z_2) = (3, -5, 2)$$

The equation sphere opposite ends of a diameter are (0, 1, 0), (3, –5, 2) is,

$$(x-0)(x-3) + (y-3)(y+5) + (z-0)(z-2) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - 3x + 2y - 2z - 15 = 0 \qquad \text{..... (i)}$$

The equation of sphere passing through the circle $x^2 + y^2 + z^2 - 3x + 2y - 2z - 15 = 0$, $5x - 2y + 4z + 7 = 0$ is,

$$x^2 + y^2 + z^2 - 3x + 2y - 2z - 15 + k(5x - 2y + 4z + 7) = 0 \qquad \text{.... (ii)}$$

Clearly, the center of (ii) is,

$$\left(\frac{-3+5k}{-2}, \frac{2-2k}{-2}, \frac{-2+4k}{-2}\right)$$

As given, the sphere (ii) gives great circle while intersecting with the plane

$$5x - 2y + 4z + 7 = 0 \qquad \text{... ... (iii)}$$

So, the center of (ii) lies on the plane (iii) then

$$\frac{-3+5k}{-2} + \frac{2-2k}{-2} + \frac{-2+4k}{-2} + 7 = 0.$$

$$\Rightarrow \quad 7k - 3 = 14$$

$$\Rightarrow \quad k = \frac{17}{7}$$

Then the equation (ii) becomes,

$$x^2 + y^2 + z^2 - 3x + 2y - 2z - 15 + \frac{17}{7}(5x - 2y + 4z + 7) = 0$$

$$\Rightarrow \quad 7(x^2 + y^2 + z^2) - 64x - 20y - 54z + 34 = 0$$

Example 11: A sphere passes through the circle $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 - 2x + 4y - 6z - 11 = 0$ and its center lies on the plane $x + 2y - 3z = 12$ find its equation.

Solution:

Given circle is,

$$x^2 + y^2 + z^2 = 4, x^2 + y^2 + z^2 - 2x + 4y - 6z - 11 = 0 \qquad \text{... (i)}$$

Subtracting second equation from first of (i) then,

$$2x - 4y + 6z + 7 = 0 \qquad \text{... (ii)}$$

The equation of sphere through given circle is

$$x^2 + y^2 + z^2 - 4 + k(2x - 4y + 6z + 7) = 0 \qquad \text{... (iii)}$$

Clearly, the center of (iii) is,

$$\left(\frac{2k}{-2}, \frac{-4k}{-2}, \frac{6k}{-2}\right)$$

i.e. $\quad (-k, 2k, -3k)$

As given, the center $(-k, 2k, -3k)$ lies on the plane $x + 2y - 3z = 12$. So,

$$-k + 4k + 9k = 12$$

$$\Rightarrow \quad k = 1$$

Therefore, the equation (iii) becomes,

$$x^2 + y^2 + z^2 - 4 + 1(2x - 4y + 6z + 7) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 2x - 4y + 6z + 3 = 0.$$

Example 12: Show that the two circles $x^2 + y^2 + z^2 - y + 2z = 0 = x - y - z - 2$ and $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0 = 2x - y + 4z - 1$ lie on the same sphere and find its equation.

Solution:

The given circles are

$$x^2 + y^2 + z^2 - y + 2z = 0 = x - y + z - 2 \qquad \text{..... (i)}$$

$$x^2 + y^2 + z^2 - x - 3y + z - 5 = 0 = 2x - y + 4z - 1 \qquad \text{..... (ii)}$$

Then the equation of circle passing through circle (i) and (ii) be,

$$x^2 + y^2 + z^2 - y + 2z + k_1(x - y + z - 2) = 0$$

$$x^2 + y^2 + z^2 - y + 2z + k_1(x - y + z - 2) = 0 \qquad \text{...... (iii)}$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + k_1x + (-1 - k_1)y + (2 + k_1)z - 2k_1 = 0$$

And,

$$x^2 + y^2 + z^2 - x - 3y + z - 5 + k_2(2x - y + 4z - 1) = 0$$

$$x^2 + y^2 + z^2 + (-1 + 2k_2)x + (-3 - k_2)y + (1 + 4k_2)z + (-5 - k_2) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + (-1 + 2k_2)x + (-3 - k_2)y + (1 + 4k_2)z + (-5 - k_2) = 0 \qquad \text{...(iv)}$$

The two circle (i) and (ii) lie on same sphere. So, for some values of k_1 and k_2 the spheres (iii) and (iv) are identical. Therefore, comparing the terms on (iii) and (iv) we get,

$$k_1 = -1 + 2k_2; \quad -1 - k_1 = -3 - k_2; \quad 2 + k_1 = 1 + 4k_2$$

Solving first two equations we get,

$$k_1 = 3 \text{ and } k_2 = 1$$

These value also satisfies the first two equation hence sphere (iii) and (iv) are identical. So, from (iii) we get the equation of sphere is

$$x^2 + y^2 + z^2 - y + 2z + 3(x - y + z - 2) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - y + 2z + 3x - 3y + 3z - 6 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0.$$

This is the equation of required sphere.

Exercise 11.2

1. Find the equation of the sphere for which the circle, $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle. Determine its center and radius.
2. Find the equation to a sphere which passes through the point $(1, -2, 3)$ and the circle $z = 0$, $x^2 + y^2 = 9$.
3. A sphere has a points $(0, 0, 0)$, $(4, 5, 6)$ at opposite ends of a diameter. Find the equation of the sphere having the intersection of the circle with the plane $x + y + z = 3$ as great circle.
4. Find the center and radius of the circle in which the sphere $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$ is cut by plane $x - 2y + 2z = 3$.
5. Find the equation of the sphere one of whose great circles is $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$.
6. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $x + 3y + 4z - 2 = 0$ and the point $(2, 2, 3)$. Also, determine the radius and center of the sphere.
7. Show that the two circles $x^2 + y^2 + z^2 - y + 2z = 0 = x - y - z - 2$ and $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0 = 2x - y + 4z - 1$ lie on the same sphere and find its equation.

Answers

1. $x^2 + y^2 + z^2 + 7y - 2x + 4y - 6z + 10 = 0$; $(1, -2, 3)$; 2
2. $3(x^2 + y^2 + z^2) - 5z - 27 = 0$
3. $x^2 + y^2 + z^2 - x - 2y - 3z - 9 = 0$
4. $\left(\frac{13}{3}, \frac{-8}{3}, \frac{-10}{3}\right)$; $4\sqrt{5}$
5. $4(x^2 + y^2 + z^2) + 2x$ [illegible] $y + 8z - 9 = 0$
6. $9(x^2 + y^2 + z^2) - 4x - 12y - 16z - 73 = 0$; $\left(\frac{2}{9}, \frac{6}{9}, \frac{8}{9}\right)$; $\sqrt{\frac{293}{36}}$
7. $x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0$

11.8 Tangent Plane to a Sphere

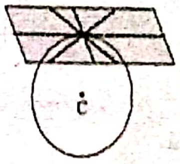
A plane is tangent to a spherical surface if the distance from the center of the sphere to the plane is equal to the radius of the sphere. That is, the tangent plane at a point on the surface of the sphere is the locus of lines through that point.

11.9 Equation of Tangent Plane

Let the given circle is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \dots\dots (1)$$

and (x_1, y_1, z_1) be a point on the sphere. To find the equation of tangent plane at (x_1, y_1, z_1), any line through (x_1, y_1, z_1) is



$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (let)} \qquad \dots\dots (2)$$

Any point on the line (2) is $(\ell r + x_1, mr + y_1, nr + z_1)$

If it lies on the sphere, (1), we have

$$(\ell r + x_1)^2 + (mr + y_1)^2 + (nr + z_1)^2 + 2u(\ell r + x_1) + 2v(mr + y_1) + 2w(nr + z_1) + d = 0$$

$$\Rightarrow \quad r^2[\ell^2 + m^2 + n^2] + 2r[\ell(u + x_1) + m(v + y_1) + n(w + z_1)] + [x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d] = 0 \qquad \dots\dots (3)$$

Since the point (x_1, y_1, z_1) is also lie on the sphere (1), so

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

Then the equation (3) reduces to,

$$r^2(\ell^2 + m^2 + n^2) + 2r[\ell(u + x_1) + m(v + y_1) + n(w + z_1)] = 0 \qquad \dots (4)$$

Since the equation (4) is quadratic in r, and clearly one root of (4) is zero. Since the line (2) touches the sphere, so both the values of r must be equal. Therefore,

$$\ell(u + x_1) + m(v + y_1) + n(w + z_1) = 0 \qquad \dots\dots (5)$$

The equation of the tangent plane at (x_1, y_1, z_1) is the locus of all such tangent lines and its equation is obtained by eliminating ℓ, m, n from (2) and (5), we get,

$$(x - x_1)(u + x_1) + (y - y_1)(v + y_1) + (z - z_1)(w + z_1) = 0$$

$$(x - x_1)(u + x_1) + (y - y_1)(v + y_1) + wz + zz_1 - wz_1 - z_1^2 = 0$$

$$\Rightarrow \quad ux + xx_1 - ux_1 - x_1^2 + vy + yy_1 - vy_1 - y_1^2 + wz + zz_1 - wz_1 - z_1^2 = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + zz_1 + ux + vy + wz = x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1$$

$$\Rightarrow \quad xx_1 + yy_1 + zz_1 + ux + vy + wz + ux_1 + vy_1 + wz_1 + d = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

$$\Rightarrow \quad xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0 \qquad \text{[Using (1)]}$$

This is the required equation of the tangent plane at (x_1, y_1, z_1) to the sphere (1).

Alternative Method

Let the given circle is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \dots\dots (1)$$

and $P(x_1, y_1, z_1)$ be a point on the surface of sphere.

Clearly, the center of the sphere (1) is $C(-u, -v, -w)$. Then the direction ratios of the line CP joining $P(x_1, y_1, z_1)$ and $C(-u, -v, -w)$ are $(x_1 + u, y_1 + v, z_1 + w)$.

Since the line CP is normal to the sphere at P, therefore

$$(x - x_1)(u + x_1) + (y - y_1)(v + y_1) + (z - z_1)(w + z_1) = 0$$

$$\Rightarrow\ ux + xx_1 - ux_1 - x_1^2 + vy + yy_1 - vy_1 - y_1^2 + wz + zz_1 - wz_1 - z_1^2 = 0$$

$$\Rightarrow\ xx_1 + yy_1 + zz_1 + ux + vy + wz = x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1$$

$$\Rightarrow\ xx_1 + yy_1 + zz_1 + ux + vy + wz + ux_1 + vy_1 + wz_1 + d = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

$$\Rightarrow\ xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

[Using (1)]

This is the required equation of the tangent plane at (x_1, y_1, z_1) to the sphere (1).

Note:

1. The equation of the tangent plane at (x_1, y_1, z_1) to the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$.
2. The radius of the sphere through the point of contact of the tangent plane is perpendicular to the tangent plane.
3. The tangent plane cut the sphere at a circle of radius 0 (this circle we called point circle).

11.10 Condition of Tangency

Let the plane and sphere be

$$lx + my + nz = p \qquad \dots\dots (1)$$

and

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \dots\dots (2)$$

Clearly, the center of (2) is $(-u, -v, -w)$ and radius of (2) is $\sqrt{u^2 + v^2 + w^2 - d}$ respectively.

Now, the length of perpendicular from the center of the sphere to the plane is

$$\text{length} = \pm\frac{l(-u) + m(-v) + n(-w) - p}{\sqrt{l^2 + m^2 + n^2}}$$

The plane will be tangent to the sphere if the length of the perpendicular from the center of the sphere to the plane is equal to the radius of the sphere. So,

$$\pm\frac{l(-u) + m(-v) + n(-w) - p}{\sqrt{l^2 + m^2 + n^2}} = \sqrt{u^2 + v^2 + w^2 - d}$$

$$\Rightarrow\ (lu + mv + nw + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$$

This is the condition for tangency of (1) to (2).

Example 13: Find the equation tangent plane of sphere $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ at $(1, 2, -2)$.

Solution:

Given sphere is,

$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0 \qquad \dots (i)$$

Comparing the equation (i) with equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then we get,

$$u = 1, \quad v = -3, \quad w = 0, \quad d = 1.$$

Then the equation of the tangent plane to $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ at $(1, 2, -2)$ is,

$$x + 2y - 2z + (1)(x + 1) + (-3)(y + 2) + (0)(z + 2) + 1 = 0.$$

$$\Rightarrow\ 2x - y - 2z - 4 = 0.$$

This is the equation of required tangent plane.

Example 14: For what value of k, the plane $x - 2y - 2z = k$ touches the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$.

Solution:

Given sphere is,

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0. \qquad \dots\dots (i)$$

Comparing (i) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then the center of (i) is,

$$(-u, -v, -w) = (1, -2, 3)$$

and the radius of (i) is,

$$r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{1^2 + (-2)^2 + 3^2 - 5} = \sqrt{1 + 4 + 9 - 5} = 3.$$

Since the length of perpendicular from center to the plane $x - 2y - 2z = k$ is equal to the radius of the sphere. So,

$$\text{length} = \pm\frac{1(1) - 2(-2) - 2(3) - k}{\sqrt{1^2 + (-2)^2 + 2^2}} = 3$$

$$\Rightarrow\ \pm(-1 - k) = 9$$

For $\quad (-1 - k) = 9 \Rightarrow k = -10.$

And, $\quad -(-1 - k) = 9 \Rightarrow k = 8.$

Thus, the value of k is 8 or −10.

Example 15: Show that the plane $2x - y + 2z = 14$ touches the sphere $x^2 + y^2 + z^2 - 4x + 2y - 4 = 0$. Find the point of contact.

Solution:

Given sphere is

$$x^2 + y^2 + z^2 - 4x + 2y - 4 = 0 \qquad \dots (i)$$

and given plane is

$$2x - y + 2z = 14 \quad \text{... (ii)}$$

Clearly, the center and the radius of the sphere (i) are

center = (2, −1, 0) and radius $= \sqrt{4+1+0+4} = 3$

Now, the length of the perpendicular from the center of the sphere to the given plane (ii) is

$$\left|\frac{2\times2 - 1(-1) + 2\times0 - 14}{\sqrt{4+1+4}}\right| = 3$$

This means the radius of the sphere is same as the perpendicular distance from the center of the sphere to the given plane (ii). So, the plane (ii) touches the sphere.

Again, the direction ratios of the line perpendicular to the plane (ii) are 2, − 1, 2. And, we know the line normal to (ii) is passing through the center of the sphere (i). Therefore, the equation of line through (2, −1, 0) and having direction ratios 2, − 1, 2 is

$$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-0}{2} = r \text{ (say)}$$

The general point of the line is $(2r + 2, -r - 1, 2r)$.

Clearly, the line meets the plane (ii), so the point lies on the plane (ii) therefore

$$2(2r+2) - (-r-1) + 2.2r = 14$$

$$\Rightarrow \quad r = 1$$

Thus, the point of contact is (4, −2, 2).

Example 16: Find the equation of the sphere which passes through the circle $x^2 + y^2 + z^2 - 2x + 2z = 2$, $y = 0$ and touches the plane $y - z = 7$.

Solution:

Since the equation of the sphere which passing through the given circle $x^2 + y^2 + z^2 - 2x + 2z = 2$, $y = 0$ is,

$$x^2 + y^2 + z^2 - 2x + 2z - 2 + ky = 0 \quad \text{...... (i)}$$

Comparing (i) with the equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then the center of (i) is,

$$(-u, -v, -w) = \left(1, \frac{k}{2}, -1\right).$$

And, the radius is,

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

$$= \sqrt{1^2 + \left(\frac{k}{-2}\right)^2 + (-1)^2 - (-2)}$$

$$= \sqrt{4 + \frac{k^2}{4}}$$

Since the length of perpendicular from center to the plane $y - z - 7 = 0$ is equal to the radius of the sphere. So,

$$\left|\frac{(0)(1) + (1)(k/-2) - (1)(-1) - 7}{\sqrt{1^2 + (-1)^2}}\right| = \sqrt{4 + \frac{k^2}{4}}$$

$$\Rightarrow \quad \left(\frac{k+12}{2\sqrt{2}}\right) = \frac{\sqrt{16+k^2}}{2}$$

$$\Rightarrow \quad k^2 + 24k + 144 = 2(16 + k^2)$$

$$\Rightarrow \quad k^2 - 24k - 112 = 0$$

$$\Rightarrow \quad (k+4)(k-28) = 0$$

$$\Rightarrow \quad k = -4, 28$$

Now, equation (i) becomes

$$x^2 + y^2 + z^2 - 2x + 28y + 2z - 2 = 0$$

or $\quad x^2 + y^2 + z^2 - 2x - 4y + 2z - 2 = 0.$

Example 17: Find the equations of the sphere through the circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ and touches the plane $z = 0$.

Solution:

The equation of sphere through the given circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ be

$$(x^2 + y^2 + z^2 - 1) + k(2x + 4y + 5z - 6) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 2kx + 4ky + 5kz - (1 + 6k) = 0 \quad \text{...... (i)}$$

Comparing (i) with the general equation of sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then we get

$$v = k, \quad v = 2k, \quad w = \frac{5k}{2} \text{ and } \quad d = -(1 + 6k).$$

Then the center of (i) be

$$(-u, -v, -w) = (-k, -2k, -5k/2)$$

and radius of (i) be

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

$$= \sqrt{k^2\left(1 + 4 + \frac{25}{4}\right) + 1 + 6k}$$

$$= \frac{1}{2}\sqrt{45k^2 + 24k + 4}$$

Since the sphere (i) touches the plane $z = 0$. That means the length of z-coordinate is equal to the radius of the sphere.

i.e. $\quad \frac{5k}{2} = \frac{1}{2}\sqrt{45k^2 + 24k + 4}$

$$\Rightarrow \quad 25k^2 = 45k^2 + 24k + 4$$

$$\Rightarrow \quad 20k^2 + 24k^2 + 4 = 0$$

$$\Rightarrow \quad 5k^2 + 6k + 1 = 0$$

$$\Rightarrow \quad (5k+1)(k+1) = 0$$

$$\Rightarrow \quad k = -1, -\frac{1}{5}$$

Therefore (i) become,

$$x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0$$

and $5(x^2 + y^2 + z^2) - 2x - 4y - 5z + 1 = 0.$

These are equation of required spheres.

Exercise 11.3

1. Find the Find the equation of the tangent plane to the sphere, $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$ at the point (1, 2, 3).
2. Find the equations of the tangent planes to the spheres, $x^2 + y^2 + z^2 + 4x + 7y + 9 = 0$ at the point (1, 3, 7).
3. Find the center and radius of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$ and find the point at which the plane $3y + 4z - 31 = 0$ touches the sphere.
4. Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$ and find the point of contact.
5. Find the equations to the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at the point (1, 1, −1) and passes through the origin.
6. Find the equation of tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ which are parallel to the plane $2x + y - z = 0$.
7. Find the equation of the sphere which passes through the circle $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0$, $y = 0$ and touch the plane $3y + 4z + 5 = 0$.

Answers

1. $4x + 9y + 14z - 64 = 0$
2. $6x + 12y + 14z + 43 = 0$
3. C(1, −2, 3); r = 5; (1, 1, 7)
4. (−1, 4, −2)
5. $2(x^2 + y^2 + z^2) - 3x + y + 4z = 0$
6. $2x + y - z \pm 3\sqrt{6} = 0$
7. $x^2 + y^2 + z^2 - 6x - 2z + 5 - 4y = 0$ and $4(x^2 + y^2 + z^2) - 24x - 8z + 20 - 11y = 0$

□□□

12

CHAPTER

Cone and Cylinder

Introduction

A cone is a three-dimensional shape in geometry that narrows smoothly from a flat base (usually circular base) to a point (which forms an axis to the centre of base) called the apex or vertex. We can also define the cone as a pyramid which has a circular cross-section, unlike pyramid which has a triangular cross-section. These cones are also stated as a circular cone.

12.1 Cone

A cone is a shape formed by using a set of line segments or the lines which connects a common point.

Definition (Cone)

A cone is a surface generated by line (the lines are called generators) that passes through a fixed point and touches the given surface. The fixed point is called vertex of the cone and the lines are called generator of the cone.

12.2 Equation of a Cone

A. Equation of a Cone Vertex at Origin

Suppose the second degree equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \quad \ldots\ldots (1)$$

represents a cone with vertex at origin and let $P(x_1, y_1, z_1)$ be the any point on the cone (1).

Since (1) meets origin i.e. (0, 0, 0) so (1) gives

$$d = 0$$

Then (1) becomes

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz = 0 \quad \text{... ... (2)}$$

Being $P(x_1, y_1, z_1)$ be the any point on the cone (1) and O(0, 0, 0) be vertex of the cone (1), the line OP lies on (1). The equation of OP is

$$\frac{x-0}{x_1} = \frac{y-0}{y_1} = \frac{z-0}{z_1}$$

i.e. $$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} = r \text{ (let)}$$

Then the general point of OP is (rx_1, ry_1, rz_1). So, the point lies on (2). Therefore,

$$r^2(ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1) + 2r(ux_1 + vy_1 + wz_1) = 0$$

must be an identity for

$$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0 \quad \text{... ... (3)}$$

and $$ux_1 + vy_1 + wz_1 = 0 \quad \text{... ... (4)}$$

Since the point P lies on cone, but the locus of (4) implies for plane so the locus of P must be,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{... ... (5)}$$

which is homogeneous in x, y, z.

This means the homogeneous second degree equation represents a cone with vertex at O(0, 0, 0).

Note 1: If the cone also passes through the axes then a = b = c = 0, therefore (5) gives

$$2fyz + 2gzx + 2hxy = 0.$$

Note 2: If the cone has three mutual perpendicular generators then

$$a + b + c = 0.$$

B. Equation of a Cone Vertex at any Point

Suppose the general equation of second degree

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \quad \text{.... ....(1)}$$

represents a cone with vertex at (α, β, γ).

Shifting the origin to (α, β, γ), then equation (1) reduces to

$$a(x+\alpha)^2 + b(y+\beta)^2 + c(z+\gamma)^2 + 2f(y+\beta)(z+\gamma) + 2g(z+\gamma)(x+\alpha) + 2h(x+\alpha)(y+\beta) + 2u(x+\alpha) + 2v(y+\beta) + 2w(z+\gamma) + d = 0$$

$$\Rightarrow (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + 2x(a\alpha + h\beta + g\gamma + u) + 2y(h\alpha + b\beta + f\gamma + v) + 2z(g\alpha + f\beta + c\gamma + w) + (a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \quad \text{... ... (2)}$$

Since (2) represents a cone with vertex at origin, it must be homogeneous. Therefore,

$$a\alpha + h\beta + g\gamma + u = 0 \quad \text{... ... (3)}$$

$$h\alpha + b\beta + f\gamma + v = 0 \quad \text{... ... (4)}$$

and $$g\alpha + f\beta + c\gamma + w = 0 \quad \text{... ... (5)}$$

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2u\alpha + 2v\beta + 2w\gamma + d = 0$$

$$\Rightarrow [a\alpha^2 + 2h\alpha\beta + 2g\gamma\alpha + 2u\alpha] + [b\beta^2 + 2f\beta\gamma + 2v\beta] + [c\gamma^2 + 2w\gamma] + d = 0$$

$$\Rightarrow [a\alpha^2 + h\alpha\beta + g\gamma\alpha + u\alpha] + [h\alpha\beta + b\beta^2 + f\beta\gamma + v\beta] + [g\gamma\alpha + f\beta\gamma + c\gamma^2 + w\gamma] + [u\alpha + v\beta + w\gamma + d] = 0$$

$$\Rightarrow \alpha(a\alpha + h\beta + g\gamma + u) + \beta(h\alpha + b\beta + f\gamma + v) + \gamma(g\alpha + f\beta + c\gamma + w) + (u\alpha + v\beta + w\gamma + d) = 0 \quad \text{... (6)}$$

Using the result from (3), (4) and (5) to (6) then (6) reduces to

$$u\alpha + v\beta + w\gamma + d = 0 \quad \text{... ... (7)}$$

Eliminating α, β, γ from equations (3), (4), (5), (7) we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

This represents a cone with vertex at (α, β, γ).

Note: If we eliminate α, β, γ from the equations (3), (4), (5), we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \end{vmatrix} = 0$$

12.3 Equation of the Cone with a Given Vertex and a Given Conic

Let (α, β, γ) be the given vertex and

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0 \quad \text{... (1)}$$

be a given conic for base.

We have the equation of line through the point (α, β, γ) and having direction ratios ℓ, m, n is

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{... ... (2)}$$

Clearly, the line (2) meets the plane z = 0 at the point $\left(\alpha - \frac{\ell\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

This point will lie on the given conic (1) if

$$a\left(\alpha - \frac{\ell\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{\ell\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right)^2 + 2g\left(\alpha - \frac{\ell\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0 \quad \text{... ... (3)}$$

Eliminating ℓ, m, n from (2) and (3), we get

$$a\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)^2 + 2h\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right) + b\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)^2 + 2g\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right) + 2f\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right) + c = 0$$

$$\Rightarrow a(\alpha z - \gamma x)^2 + 2h(\alpha z - \gamma x)(\beta z - \gamma y) + b(\beta z - \gamma y)^2 + 2g(\alpha z - \gamma x)(z - \gamma) + 2f(\beta z - \gamma y)(z - \gamma) + c(z - \gamma)^2 = 0$$

This is the required equation of the cone.

12.4 Right Circular Cone

A right circular cone is a surface generated by a straight line which passes through a fixed point, and makes a constant angle with a fixed straight line through the fixed point. The fixed point is called the vertex of the cone and the constant angle is called the semi-vertical angle and fixed straight line is called the axis of the cone.

The section of right circular cone by any plane perpendicular to its axis is a circle.

12.4.1 Equation of a Right Circular Cone with Vertex V(α, β, γ)

Let $V(\alpha, \beta, \gamma)$ be the vertex of a right circular cone and let $P(x, y, z)$ be any point on the surface of the cone. Then the direction ratios of VP are

$$x - a,\ y - b,\ z - g.$$

The direction ratios of the perpendicular VL are ℓ, m, n.

Let θ be the semi vertical angle then

$$\cos\theta = \frac{(x-\alpha)\ell + (y-\beta)m + (z-\gamma)n}{\sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}\ \sqrt{\ell^2 + m^2 + n^2}}$$

This is the required equation of the cone.

Note: If the vertex is at the origin i.e. $(\alpha, \beta, \gamma) = (0, 0, 0)$ then the equation of the cone becomes

$$(x\ell + ym + zn)^2 = (x^2 + y^2 + z^2)(\ell^2 + m^2 + n^2)\cos^2\theta$$

Example 1: Find the equation of the cone of the second degree which passes through the axes.

Solution:

As given the cone passes through the axes so, the vertex of the cone is the origin.

The equations of the cone is a homogeneous equation of second degree in x, y and z.

$$\text{i.e.}\quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \text{... (i)}$$

Since one generator of the cone (i) is x-axis. And, on x-axis we have $y = 0, z = 0$ then (i) gives

$$a = 0.$$

Similarly, being y-axis and z-axis are also generator of (i) we get

$$b = 0 \text{ and } c = 0.$$

Therefore, the equation of the cone of the second degree which passes through the axes is

$$fyz + gzx + hxy = 0$$

Example 2: Show that the cone $(b-c)x^2 + (c-a)y^2 + (a-b)z^2 + 2fyz + 2gzx + 2hxy = 0$ has set three mutually perpendicular generators.

Solution:

Given cone is

$$(b-c)x^2 + (c-a)y^2 + (a-b)z^2 + 2fygz + 2gzx + 2hxy = 0 \qquad \text{..... (1)}$$

Comparing (1) with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ we get

$$a = (b-c),\ b = (c-a),\ c = (a-b),\ f = 1,\ g = 1,\ h = 1$$

Now,

$$a + b + c = b - c + c - a + a - b = 0$$

This means the cone (i) has three mutually perpendicular generators.

Example 3: Find the equation of the cone with vertex as origin and passing through the circle given by $x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0,\ x - y + z = 2$.

Solution:

Given circle is

$$x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0,\ x - y + z = 2 \qquad \text{..... (1)}$$

From (1),

$$x^2 + y^2 + z^2 + (x - 2y + 3z)\left(\frac{2}{2}\right) - (2)^2 = 0$$

$$\Rightarrow \quad (x^2 + y^2 + z^2) + (x - 2y + 3z)\left(\frac{x - y + z}{2}\right) - (x - y + z)^2 = 0$$

$$\Rightarrow \quad 2(x^2 + y^2 + z^2) + (x - 2y + 3z)(x - y + z) - 2(x^2 + y^2 + z^2 - 2xy - 2yz + 2zx) = 0$$

$$\Rightarrow \quad (x^2 - 3xy + 2y^2 + 4xz - 5yz + 3z^2) + 4xy + 4yz - 4zx) = 0$$

$$\Rightarrow \quad x^2 + 2y^2 + 3z^2 + xy - yz = 0$$

Example 4: Find the equation of the cone whose vertex is the origin and which passes through the curve $x^2 + y^2 + z^2 + x - 2y + 3z = 4$, $x^2 + y^2 + z^2 + 2x - 3y + 4z = 5$.

Solution:

Given curves are

$$x^2 + y^2 + z^2 + x - 2y + 3z = 4 \qquad \text{..... (1)}$$

$$x^2 + y^2 + z^2 + 2x - 3y + 4z = 5 \qquad \text{..... (2)}$$

From (1) and (2), we get

$$x - y + z = 1 \qquad \text{..... (3)}$$

Now making (1) homogeneous with the help of (3), the required equation of the cone is given by

$$(x^2 + y^2 + z^2) + (x - 2y + 3z)(1) = 4(1)^2$$

$$(x^2 + y^2 + z^2) + (x - 2y + 3z)(x - y + z) = 4(x - y + z)^2$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + (x - 2y + 3z)(x - y + z) = 4(x - y + z)^2$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + x^2 - xy + xz - 2yx + 2y^2 - 2yz + 3zx - 3yz + 3z^2$$

$$= 4x^2 + 4y^2 + 4z^2 - 8xy - 8yz + 8zx$$

$$\Rightarrow \quad 2x^2 + y^2 - 5xy - 3yz + 4zx = 0$$

Example 5: Find equation to the cone with vertex at (0, 0, 3) and guiding curve $x^2 + y^2 = 4$ and $z = 0$. (i.e. guiding curve is circle lies on xy plane).

Solution:

Given conic is

$$x^2 + y^2 = 4 \text{ and } z = 0 \qquad \text{...... (i)}$$

Let the cone (i) has the vertex at (0, 0, 3) and P be any point on the conic. Let the direction ratios of AP are ℓ, m, n. Then the equation of the line AP is

$$\frac{x-0}{\ell} = \frac{y-0}{m} = \frac{z-3}{n}$$

$$\text{i.e.} \quad \frac{x}{\ell} = \frac{y}{m} = \frac{z-3}{n} \qquad \text{...... (ii)}$$

As given, the line (ii) intersects xy-plane at z = 0. Therefore,

$$\frac{x}{\ell} = \frac{y}{m} = \frac{0-3}{n}$$

This gives

$$x = \frac{-3\ell}{n} \text{ and } y = \frac{-3m}{n}$$

That is, the coordinate of P is $\left(\frac{-3\ell}{n}, \frac{-3m}{n}, 0\right)$.

Using result (ii) in $x^2 + y^2 = 4$ then

$$9\left(\left(\frac{\ell}{n}\right)^2 + \left(\frac{m}{n}\right)^2\right) = 4$$

$$\text{i.e.} \quad \left(\frac{x}{z-3}\right)^2 + \left(\frac{y}{z-3}\right)^2 = \frac{4}{9}$$

$$\Rightarrow \quad 9(x^2 + y^2) = 4(z-3)^2$$

Example 6: Planes through OX and OY include an angle α. Show that their line of intersection lies on the cone $z^2 (x^2 + y^2 + z^2) = x^2y^2 \tan^2 \alpha$.

Solution:

Any plane through OX, i.e., y = 0, z = 0, is

$$y + \lambda z = 0 \qquad \text{...... (1)}$$

Also, a plane through OY, i.e., x = 0, z = 0, is

$$x + \mu z = 0 \qquad \text{...... (2)}$$

The angle between the two planes is α. Therefore,

$$\cos \alpha = \frac{0.1 + 1.0 + \lambda\mu}{\sqrt{1+\lambda^2}\sqrt{1+\mu^2}} = \frac{\lambda\mu}{\sqrt{1+\lambda^2+\mu^2+\lambda^2\mu^2}}$$

So that $\quad \tan^2 \alpha - 1 = \dfrac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{\lambda^2\mu^2}$

$$\Rightarrow \quad \tan^2 \alpha = \frac{1+\lambda^2+\mu^2}{\lambda^2\mu^2}$$

The surface generated by the line of intersection of planes (1) and (2) can be obtained by eliminating λ and μ from (1), (2) and (3). Thus,

$$\tan^2 \alpha = \frac{1 + y^2/z^2 + x^2/z^2}{(y^2/z^2)\,(x^2/z^2)} = \frac{z^2(x^2+y^2+z^2)}{x^2\,y^2}$$

$$\Rightarrow \quad z^2(x^2 + y^2 + z^2) = x^2y^2 \tan^2 \alpha$$

This is the equation of the cone.

Example 7: Find the vertex of the cone $2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2x + 2y + 26z - 9 = 0$.

Solution:

Given equation is

$$2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2x + 2y + 26z - 17 = 0 \qquad \text{... (i)}$$

Let

$$f(x, y, z) = 2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2x + 2y + 26z - 17 = 0$$

To make it homogeneous of second degree by introducing t such that

$$f(x, y, z, t) = 2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2xt + 2yt + 26zt - 17t^2 = 0$$

Differentiate partially w.r. to x, y, z, and t, we get

$$\frac{\partial f}{\partial x} = 4x - 10z + 2t = 0$$

$$\frac{\partial f}{\partial y} = 4y - 10z + 2t = 0$$

$$\frac{\partial f}{\partial y} = 14z - 10y - 10x + 26t = 0$$

At t = 1,

$$4x - 10z + 2 = 0;\ 4y - 10z + 2 = 0;\ 14z - 10y - 10x + 26 = 0$$

Solving we get

$$x = 2,\ y = 2,\ z = 1$$

Thus, the vertex of the cone is (2, 2, 1).

Example 8: Find the equation of the right circular cone with vertex (1, –2, –1) and semi vertical angle θ is equation of the axis as $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$ and semi vertical angle θ is 60°.

Solution:

Given axis of the cone is

$$\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$$

Let P(x, y, z) be any point on the surface of the cone and V(1, –2, –1) be vertex of the cone then the direction ratios of VP are x – 1, y + 2, z + 1 and direction ratios of the axis are 3, –4, 5 and given that the semi vertical angle θ is 60°.

Therefore,

$$\cos 60^\circ = \frac{(x-1)3 + (y+2)(-4) + (z+1)5}{\sqrt{(x-1)^2 + (y+2)^2 + (z+1)^2}\sqrt{3^2 + (-4)^2 + 5^2}}$$

i.e. $$\frac{1}{2} = \frac{3(x-1) - 4(y+2) + 5(z+1)}{\sqrt{(x-1)^2 + (y+2)^2 + (z+1)^2}\sqrt{50}}$$

$$\Rightarrow \quad 50[(x-1)^2 + (y+2)^2 + (z+1)^2] = 4[3(x-1) - 4(y+2) + 5(z+1)]^2$$

$$\Rightarrow \quad 25[x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 + 2z + 1] = 2[3x - 4y + 5z - 6]^2$$

$$\Rightarrow \quad 25[x^2 + y^2 + z^2 - 2x + 4y + 2z + 6] = 2[3x - 4y + 5z - 6]^2$$

Exercise 12.1

1. Find the equation of the cone whose vertex is the origin and guiding curve is the circle $x^2 + y^2 + z^2 + 4x + 2y - 6z + 5 = 0$ and $2x + y + 2z + 5 = 0$.
2. Find the equation of the cone with vertex at the origin and which passes through the curve of intersection of
   a. $x^2 + y^2 + 2z^2 = 1$ and $x - y + 2z = 7$.
   b. $ax^2 + by^2 + cz^2 = 1$ and $\ell x + my + nz = p$.
   c. $x^2 + y^2 + z^2 + 3x - 3y + 3z = 5$ and $x^2 + y^2 + z^2 + 3x - 3y + 4z = 6$.
3. Show that the equation to the cone whose vertex is (α, β, γ) and base the parabola $z^2 = 4ax,\ y = 0$ is $(\beta z - \gamma y)^2 = 4a(\beta - y)(\beta x - \alpha y)$.
4. Find the equation of the cone with vertex (α, β, γ) and base $y^2 = 4ax,\ z = 0$.
5. Find eqn. to the cone with vertex at $(1, 1, 0)$ and guiding curve $x^2 + z^2 = 4$ and $y = 0$.
6. Find the vertex of the cone $2x^2 - 8xy - 4yz - 4x - 2y + 6z + 35 = 0$.
7. Find the equation of the right circular cone with vertex $(3, 2, 1)$, equation of the axis as $\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$ and semi vertical angle θ is 30°.
8. Find the equation of the right circular cone with vertex as $(2, -3, 5)$, axis making equal angle with coordinate axis and semi vertical angle $\theta = 30^\circ$.

Answers

1. $x^2 + 4y^2 + 21z^2 - 4xy + 6yz + 12zx = 0$
2. a. $49(x^2 + y^2 + z^2) = (x - y + 2z)^2$ b. $p^2(ax^2 + by^2 + cz^2) = (\ell x + my + nz)^2$
   c. $x^2 + y^2 - z^2 + 3xz - 3yz = 0$
4. $z^2(\beta^2 - 4a\alpha) - 2z\gamma\{\beta y - 2a(x + \alpha)\} + \gamma^2(y^2 - 4ax) = 0$
5. $x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0$ 6. $\left(4, \frac{3}{2}, \frac{-17}{2}\right)$
7. $39[x^2 + y^2 + z^2 - 6x - 4y - 2z + 14] = [4x - y - 3z - 13]^2$
8. $5[x^2 + y^2 + z^2 - 4x + 6y - 10z + 38] = 4[x + y + z - 4]^2$

12.5 Cylinder

Introduction

A cylinder is a surface generated by a straight line which moves parallel to a fixed straight line and either intersects the given curve or touches a given surface. The fixed line is called **axis**, the given curve is called **guiding curve** and any straight line on the cylinder is known as its **generator.**

If the generating line remains at a constant distance from the fixed line, the surface so generated is called a **right circular cylinder** whose axis is fixed line and radius, the fixed distance.

Section of a right circular cylinder by any plane perpendicular to its axis is a circle. In this case guiding curve is a circle.

A cylinder is a surface generated by a straight line which moves parallel to a fixed straight line and either intersects the given curve or touches a given surface.

Definition:

A cylinder is a locus of lines which remains parallel to a fixed line and intersects a given curve. The lines are called generator of cylinder.

12.6 Equation of Cylinder through a Given Conic whose Generators are Parallel to the Given Line

Let the generator of the cylinder be parallel to the line

$$\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} \qquad \text{..... (1)}$$

and the equation of the conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,\ z = 0 \qquad \text{..... (2)}$$

Let (α, β, γ) be any point on the cylinder, then the equations of generators are

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{..... (3)}$$

This meets the plane $z = 0$ at the point given by $\left(\alpha - \frac{\ell}{n}\gamma, \beta - \frac{m}{n}\gamma, 0\right)$. This point will lie on the conic if

$$a\left(\alpha - \frac{\ell\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{\ell\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + 2g\left(\alpha - \frac{\ell\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0$$

$$\Rightarrow \quad a(n\alpha - \ell\gamma)^2 + b(n\beta - m\gamma)^2 + 2h(n\alpha - \ell\gamma)(n\beta - m\gamma) + 2gn(n\alpha - \ell\gamma) + 2fn(n\beta - m\gamma) + cn^2 = 0$$

Hence, locus of the point (α, β, γ) is

$$a(nx - \ell z)^2 + b(ny - mz)^2 + 2h(nx - \ell z)(ny - mz) + 2gn(nx - \ell z) + 2fn(ny - mz) + cn^2 = 0$$

which is the required equation of the cylinder.

Note:

1. If the axis of the cylinder is along z-axis that is $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$, then the equations of the cylinder reduces to $ax^2+2hxy+by^2+2gx+2fy+c=0$.
2. If the axis is along z-axis and guiding curve is $x^2+y^2=a^2, z=0$, then the equation of the cylinder reduces to the form $x^2+y^2=a^2$.

Example 9: Find the equation of cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and guiding curve is the ellipse $x^2+2y^2=1, z=3$.

Solution:

Given line is

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{3} \qquad \ldots \text{(i)}$$

and the conic is

$$x^2+2y^2=1,\ z=0 \qquad \ldots \text{(ii)}$$

Therefore, the general point of (i) is

$$(r+x_1, -2r+y_1, 3r+z_1)$$

Suppose $P(x_1, y_1, z_1)$ be any point on the cylinder whose generators are parallel to line (i) and formed by (ii).

The generator meets the plane $z=0$ at the point given by

$$(x_1-0, y_1+0, 0)$$

If this point lies on the given ellipse,

$$(x_1)^2+2(y_1)^2=1$$

Hence, the locus of point $P(x_1, y_1, z_1)$ is given by

$$x^2+2y^2=1$$

This is the equation of the cylinder.

Example 10: The axis of a right circular cylinder of radius 2 is $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$. Show that its equation is $10x^2+5y^2+13z^2-12xy-6yz-4zx-8x-30y-74z+59=0$.

Solution:

Given line (axis) is

$$\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1} \qquad \ldots \text{(i)}$$

The direction cosines of the axis are proportional to 2, 3, 1, so its direction cosines are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$.

Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then the square of the perpendicular distance of P from the axis of the cylinder is

$$(x_1-1)^2+(y_1-0)^2+(z_1-3)^2-\left((x_1-1)\frac{2}{\sqrt{14}}+(y_1-0)\frac{3}{\sqrt{14}}+(z_1-3)\frac{1}{\sqrt{14}}\right)^2$$

$$=\frac{1}{14}(10x_1^2+5y_1^2+13z_1^2-12x_1y_1-6y_1z_1-4z_1x_1-8x_1+30y_1-74z_1+115)$$

Equating this to the square of the radius, we have

$$10x_1^2+5y_1^2+13z_1^2-12x_1y_1-6y_1z_1-4x_1z_1-8x_1+30y_1-74z_1+115=56$$

$$\Rightarrow 10x_1^2+5y_1^2+13z_1^2-12x_1y_1-6y_1z_1-4z_1x_1-8x_1+30y_1-74z_1+59=0$$

The required locus of P is

$$10x^2+5y^2+13z^2-12xy-6yz-4zx-8x+30y-74z+59=0$$

This is the equation of the cylinder.

12.7 Right Circular Cylinder

A right circular cylinder is a surface generated by a straight line which remains parallel to a fixed straight line at a constant distance from it. The fixed straight line is called the axis of the cylinder and the constant distance is called the radius of the cylinder.

12.8 Equation of Right Circular Cylinder

Let

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots \text{(i)}$$

be the equation of axis of the right circular cylinder where ℓ, m, n are the actual direction cosines and radius of the cylinder is r.

Let P(x, y, z) be any point on the cylinder. Draw PB normal to axis such that

$$PB = r.$$

Let $A(\alpha, \beta, \gamma)$ be a point on the axis of the cylinder then

$$PA^2=(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$$

Then the projection of PA on axis is AB. Then

$$AB=\frac{(x-\alpha)\ell+(y-\beta)m+(z-\gamma)n}{\sqrt{\ell^2+m^2+n^2}}$$

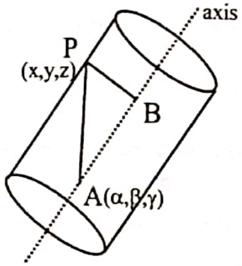
Now,

$$AB^2=PA^2-PB^2$$

i.e. $$\frac{(x-\alpha)\ell+(y-\beta)m+(z-\gamma)n}{\sqrt{\ell^2+m^2+n^2}}=(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2$$

$$\Rightarrow (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-[(x-\alpha)\ell+(y-\beta)m+(z-\gamma)n]^2=r^2$$

being $\ell^2+m^2+n^2=1$.



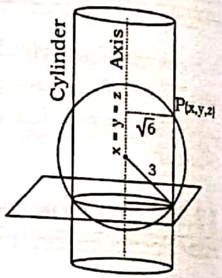
Example 11: Find the equation of a right circular cylinder whose guiding curve is $x^2+y^2+z^2=9$, $x-y+z=3$.

Solution:

Given guiding curve is

$$x^2+y^2+z^2=9,\ x-y+z=3 \quad \ldots (i)$$

Clearly, the curve (i) is a circle.



From (i), the center of sphere is C(0, 0, 0) and radius r = 3.

We know that the radius of a right circular cylinder is equal to the radius of the guiding curve.

So, the center of cylinder is C(0, 0, 0) and radius r = 3.

Then the Length of the perpendicular from (0, 0, 0) to the plane $x-y+z=3$ is

$$\text{length}=\left|\frac{0+0+0-3}{\sqrt{1^2+1^2+1^2}}\right|=\sqrt{3}$$

Now, the radius of the circle (i) is

$$\text{radius}=\sqrt{3^2-(\sqrt{3})^2}=\sqrt{6}$$

The axis of the cylinder passing through (0, 0, 0) and perpendicular to the given plane $x-y+z=3$ is

$$\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$$

i.e. $$\frac{x}{1/\sqrt{3}}=\frac{y}{-1/\sqrt{3}}=\frac{z}{1/\sqrt{3}}$$

Now, the equation of the right circular cylinder is

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-[(x-\alpha)\ell+(y-\beta)m+(z-\gamma)n]^2=r^2$$

i.e. $$(x-0)^2+(y-0)^2+(z-0)^2-\left((x-0)\frac{1}{\sqrt{3}}+(y-0)\frac{-1}{\sqrt{3}}+(z-0)\frac{1}{\sqrt{3}}\right)^2=(\sqrt{6})^2.$$

$$\Rightarrow 3(x^2+y^2+z^2)-(x-y+z)^2=18.$$

Exercise 12.2

1. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and passing through the curve $x^2+2y^2=1$, $z=0$.
2. Find the equation of the cylinder, whose generators are parallel to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and passes through the curve $x^2+y^2=16$, $z=0$.
3. Obtain the equation of the cylinder which passes through $y^2=4ax$, $z=0$ and whose generators are parallel to the line $x=y=z$.
4. Find the equation of the cylinder whose generators have direction cosines ℓ, m, n and which passes through the circle $x^2+z^2=a^2$, $y=0$.

Answers

1. $3x^2+6y^2+3z^2+8yz-2zx-3=0$
2. $9x^2+9y^2+5z^2-12yz-6zx-144=0$
3. $(y-z)^2=4a(x-z)$
4. $(mx-ly)^2+(mz-ny)^2=m^2a^2$

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