#) Fubinis Theorem (first form):

Let, $f(x,y)$ is continuous on the rectangular region, $R : a \leq x \leq b$

$c \leq y \leq d$

$$\therefore \iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Double integrals of bounded by non-rectangle region:

Fubinis Theorem (Stronger form):

Let, $f(x,y)$ be a function, continuous on the region R.

① If R is defined by $a \leq x \leq b$, $f_1(x) \leq y \leq f_2(x)$.

$$\iint_R f(x,y)\, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y) \cdot dy\, dx$$

② If R is defined by $c \leq y \leq d$, $g_1(y) \leq x \leq g_2(y)$

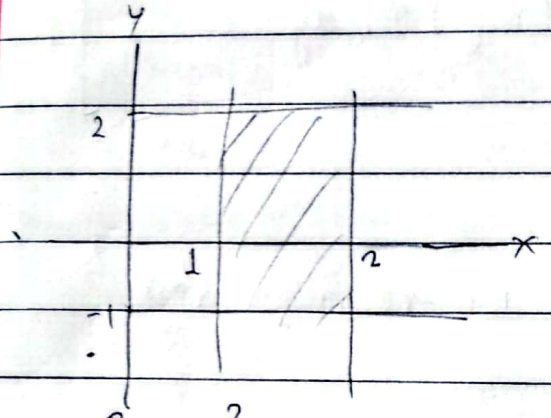
$$\iint_R f(x,y)\, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y) \cdot dx\, dy$$

(iii) $\int_1^2\int_{-1}^{2} (12xy^2 - 8x^3)\, dy\, dx$

Here,

$R: -1 \le y \le 2$, i.e $y = -1$, $y = 2$

$1 \le x \le 2$, i.e $x = 1$, $x = 2$



$$= \int_1^2\int_{-1}^{2} (12xy^2 - 8x^3)\cdot dy\, dx$$

$$= \int_1^2 \left[12x\frac{y^3}{3} - 8x^3y\right]_{-1}^{2} dx$$

$$= \int_1^2 \left[4xy^3 - 8x^3y\right]_{-1}^{2} \cdot dx$$

$$= \int_1^2 \left[(4x\times 2^3 - 8x^3\times 2) - (4x\times(-1)^3 - 8x^3\times(-1))\right] dx$$

$$= \int_1^2 (32x - 16x^3) - (-4x + 8x^3)\, dx$$

$$= \int_1^2 (32x - 24x^3)\, dx$$

$$= \left(32\cdot\frac{x^2}{2} - 24\cdot\frac{x^4}{4}\right)_1^2$$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4$

$$= \left[18x^2 - 6x^4\right]_1^2$$

$$= (18\times 2^2 - 6\times 2^4) - (18\times 1^2 - 6\times 1^4)$$

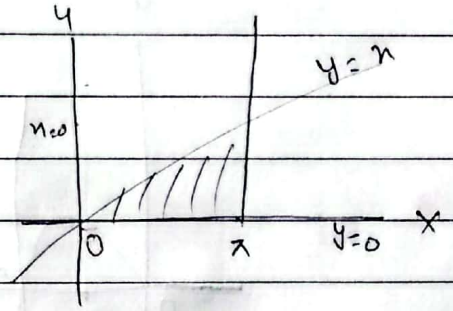
$$= 72 - 96 - 18 + 6$$

$$= -36$$

(#) $$\int_0^{\pi}\int_0^{x} x \sin y \, dy \, dx$$

$R: 0 \le x \le \pi$ i.e $x=0, x=\pi$

$0 \le y \le x$ i.e $y=0, y=x$.



$$\int_0^{\pi}\left[x\cdot(-\cos y)\right]_0^{x} dx$$

$$= -\int_0^{\pi}(x\cdot\cos x - x\cdot\cos 0)\,dx$$

$$= -\int_0^{\pi}(x\cos x - x)\,dx$$

$$= -\left[x\sin x - 1\cdot(-\cos x) - \frac{x^2}{2}\right]_0^{\pi}$$

$$= -\left[\pi\sin\pi + \cos\pi - \frac{\pi^2}{2} - 0 - 1 + 0\right]$$

$$= -\left[0 - 1 - \frac{\pi^2}{2} - 1\right]$$

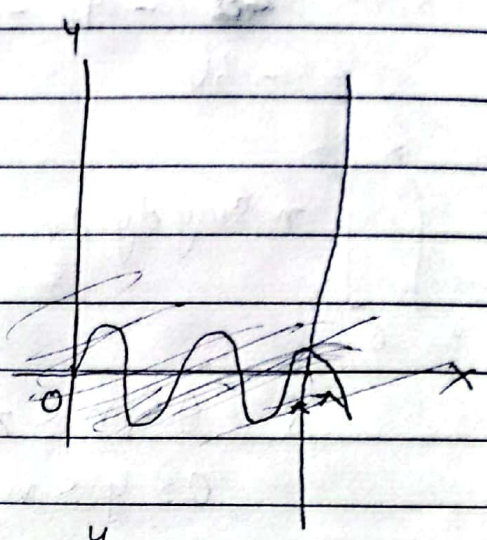
$$= 2 + \frac{\pi^2}{2} = \frac{4+\pi^2}{2}$$

(4) $$\int_0^{\pi}\int_0^{\sin x} y \cdot dy \cdot dx$$

R : $0 \le y \le \sin x$ i.e $y = 0$ & $y = \sin x$

$0 \le x \le \pi$ i.e $x = 0$ & $x = \pi$

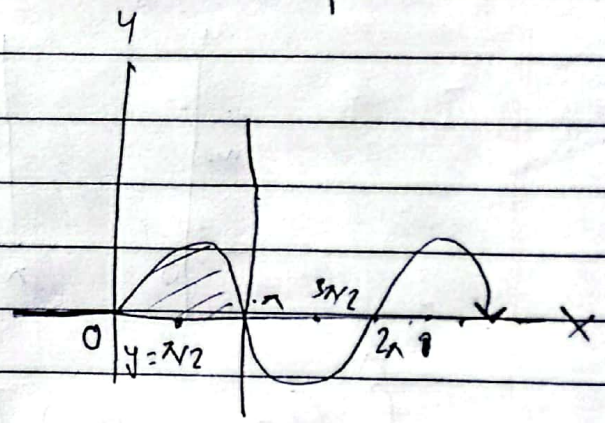
$$\int_0^{\pi} \left[\frac{y^2}{2}\right]_0^{\sin x} \cdot dx$$



$$= \int_0^{\pi} \frac{\sin^2 x}{2} \cdot dx$$

$$= \frac{1}{4}\int_0^{\pi} 2\sin^2 x \cdot dx$$

$$= \frac{1}{4}\int_0^{\pi} (1 - \cos 2x) \cdot dx$$



$$= \frac{1}{4}\left[x - \sin 2x\right]_0^{\pi}$$

$$= \frac{1}{4}\left(\pi - \sin 4\pi - 0 + 0\right)$$

$$= \frac{1}{4}(\pi - 0)$$

$$= \frac{\pi}{4}$$

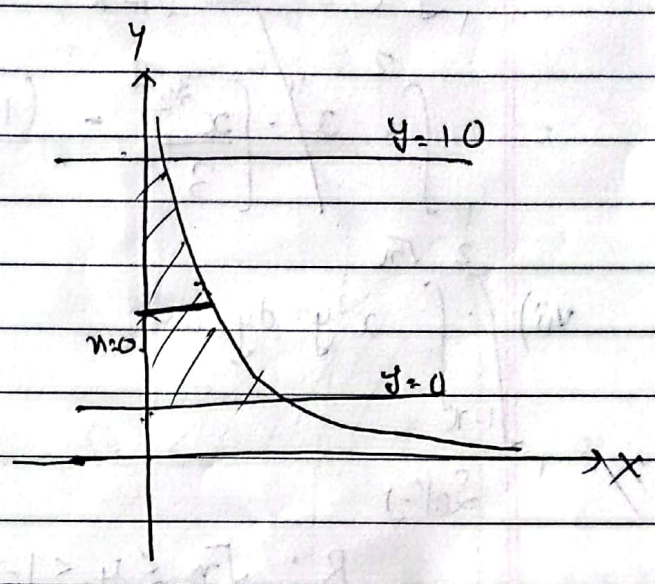
(vi) $$\int_{10}^{1}\int_{0}^{1/y} y e^{xy}\, dx\, dy$$

Sol^n: $R: \ 1/y \le x \le 0$ i.e $x = 1/y$ & $x = 0$

$1 \le y \le 10$ i.e $y = 1$ & $y = 10$

$$= \int_{10}^{1} \left[\cancel{y} \cdot \frac{e^{xy}}{\cancel{y}} \right]_{0}^{1/y} \cdot dy$$

$$= \int_{10}^{1} (e^{1} - e^{0}) \cdot dy$$



$$= (e-1) \int_{10}^{1} dy$$

$$= (e-1) \left[y \right]_{10}^{1}$$

$$= (e-1)(1-10)$$

$$= (e-1) \times (-9)$$

$$= 9 - 9e$$

$$\int_1^2 \int_{1-x}^{\sqrt{x}} x^2y \; x^2y \; dy \cdot dx$$

Sol^n ⇒ $R: \sqrt{x} \le y \le 1-x$ i.e $y=\sqrt{x}$ & $y=1-x$

$2 \le x \le 1$ i.e $y=2$ & $y=1$

$$\int_1^2 \left[x^4 \cdot \frac{y^3}{3} \right]_{1-x}^{\sqrt{x}} \cdot dx$$

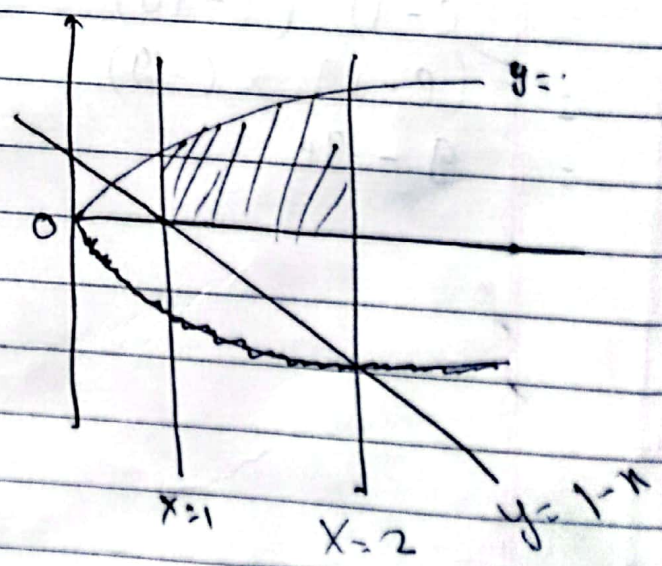
$$\int_1^2 x^4 \cdot \left[\frac{x^{3/2}}{3} - \frac{(1-x)^3}{3} \right] \cdot dx$$

$$\int_1^2 \int_{1-x}^{\sqrt{x}} x^2y \cdot dy \cdot dx$$

Sol^n ⇒

$R: \sqrt{x} \le y \le 1-x$ i.e $y=\sqrt{x}$ & $y=1-x$

$2 \le x \le 1$ $x=2$ & $x=1$



Evaluate the integral by Interchanging the Equivalent integral obtained by reversing the order of Integration if necessary

(i) $\int_0^2 \int_1^{e^x} dy\, dx$

Sol(n):

$R: \quad 1 \le y \le e^x \qquad$ i.e $y = 1$ & $y = e^x$

$0 \le x \le 2 \qquad$ i.e $x = 0$, $x = 2$

$$= \int_0^2 \int_1^{e^x} dy\, dx$$

$$= \int_0^2 \left[y\right]_1^{e^x} \cdot dx$$

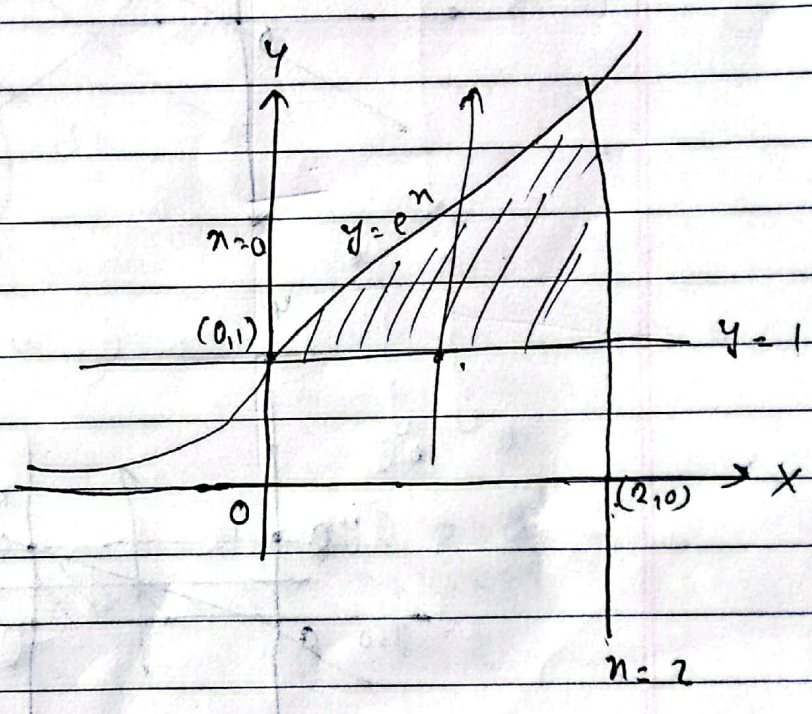
$$= \int_0^2 (e^x - 1) \cdot dx$$

$$= \left[e^x - x\right]_0^2$$

$$= \left[e^2 - 2\right] - \left[e^0 - 0\right]$$

$$= e^2 - 2 - 1$$

$$= e^2 - 3$$



Q. $\int_0^1 \int_{2y}^{2} \cos x^2 \cdot dx\, dy$

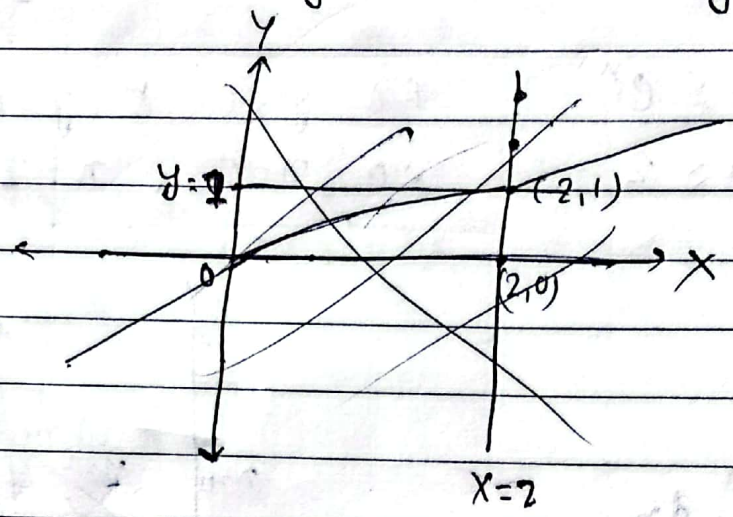
$x = 2y$

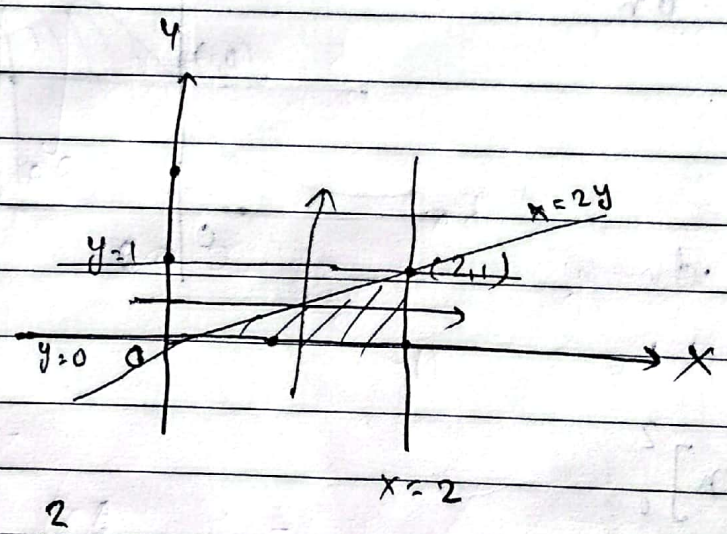
x	0	2	4
y	0	1	2

Sol(?)

R : $2y \leq x \leq 2$ i.e $x = 2y$, $x = 2$

$0 \leq y \leq 1$ i.e $y = 0$, $y = 1$





$$= \int_0^1 \int_{2y}^{2} \cos x^2 \cdot dx\, dy$$

$$= \int_0^1 \int_{2y}^{2} \left(\frac{\cos 2x + 1}{2} \right) \cdot dx$$

Changing the order of integration.

$0 \le x \le 2$

$0 \le y \le x/2$

$$\int_0^2 \left[\int_0^{x/2} (\cos x^2)\, dy \right] dx$$

$$= \int_0^2 \left[y \cdot \cos x^2 \right]_0^{x/2} \cdot dx$$

$$= \int_0^2 \left(\frac{x}{2} \cdot \cos x^2 - 0 \right) \cdot dx$$

$$= \frac{1}{2} \int_0^2 x \cdot \cos x^2 \cdot dx$$

let, $z = x^2$ when $x = 0$, $z = 0$,

or, $dz = 2x \cdot dx$ when $x = 2$, $z = 4$

or, $\frac{dz}{2} = x \cdot dx$

$$= \frac{1}{4} \int_0^4 \cos z \cdot dz$$

$$= \frac{1}{4} \left[\sin z \right]_0^4$$

$$= \frac{1}{4} (\sin 4 - 0)$$

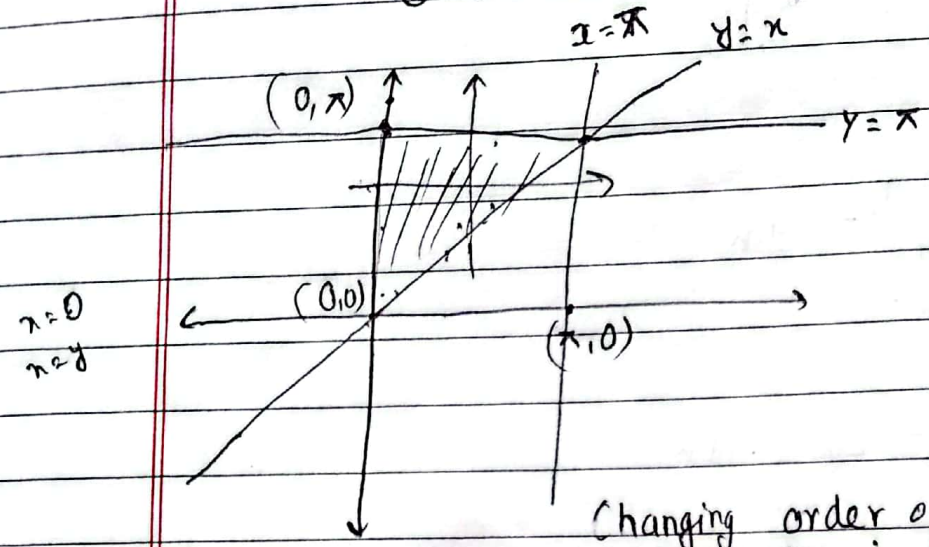
$$= \frac{\sin 4}{4}$$

V.V

Q. $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$

R: $x \le y \le \pi$ When i.e. $y = x$ & $y = \pi$

$0 \le x \le \pi$ i.e. $x = 0$ & $x = \pi$



Changing order of integration : $0 \le y \le \pi$, $0 \le x \le y$

$$= \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy$$

$$= \int_0^{\pi} \left[x \cdot \frac{\sin y}{y}\right]_0^{y} \cdot dy$$

$$= \int_0^{\pi} \left(\pi \cdot \frac{\sin y}{y} - \cancel{y} \cdot \frac{\sin y}{\cancel{y}}\right) \cdot dy$$

$$= \int_0^{\pi} \left(0 - y \cdot \frac{\sin y}{y}\right) \cdot dy$$

$$= \int_0^{\pi} \sin y \cdot dy$$

$$= -\left[\cos y\right]_0^{\pi}$$

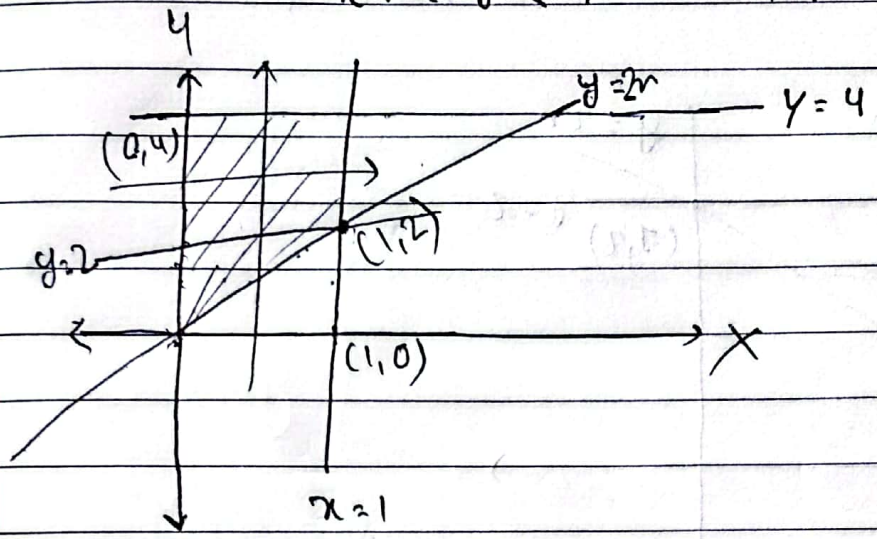
$$= -(\cos\pi - \cos 0)$$

$$= -(-1-1)$$

Q. $\int_0^1 \int_{2x}^{4} e^{y^2} \cdot dy\, dx$

Sol^n: R : $0 \le x \le 1$ i.e $x=0$ & $x=1$

$x \le y \le 4$ i.e $y=2x$ & $y=4$



Changing the order of integration

$0 \le y \le 2$; $0 \le x \le \frac{y}{2}$ } And, $2 \le y \le 4$; $0 \le x \le 1$

$$= \int_0^2 \int_0^{y/2} e^{y^2} \cdot dx \cdot dy \qquad \int_2^4 \int_0^1 e^{y^2} \cdot dx \cdot dy$$

$$= \int_0^2 e^{y^2} \cdot [x]_0^{y/2} \cdot dy \qquad = \int_2^4 [x]_0^1 \cdot e^{y^2} \cdot dy$$

$$= \int_0^2 e^{y^2} \cdot \frac{y}{2} \cdot dy \qquad = \int_2^4 e^{y^2} \cdot dy$$

$$= \frac{1}{4} \int_0^2 (2y) \cdot e^{y^2} \cdot dy$$

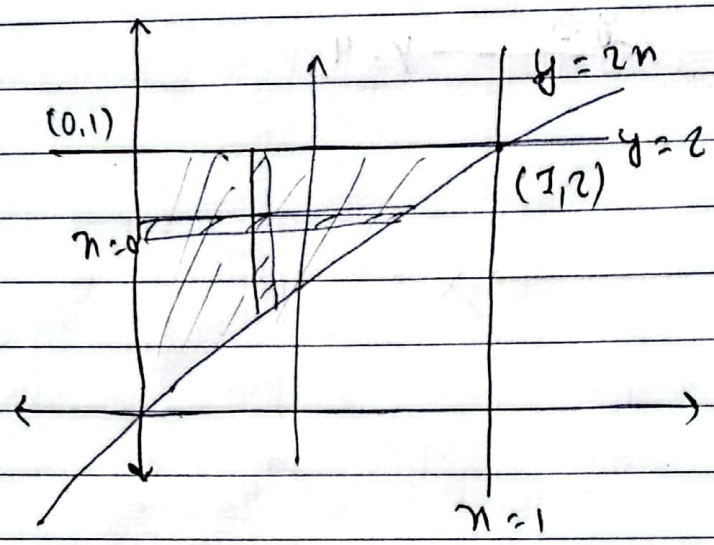
$$= \frac{1}{4} \left[e^{y^2} \right]_0^2$$

$$\therefore \frac{1}{4} e^{4^2} \Rightarrow \frac{e^4}{4} - \frac{1}{4} \Rightarrow \frac{e^4 - 1}{4} \text{ Ans.}$$

Q. $\int_0^1 \int_{2x}^{2} e^{y^2} \cdot dy\, dx$

Sol: $R: 2x \le y \le 2$ i.e. $y = 2x$, $y = 2$

$0 \le x \le 1$ i.e. $x = 0$ & $x = 1$



$$\int_0^1 \int_{2x}^{2} e^{y^2} \cdot dy \cdot dx$$

Changing order of integration

$0 \le y \le 2$

$0 \le x \le y/2$

$$= \int_0^2 \int_0^{y/2} e^{y^2} \cdot dx \cdot dy$$

$$= \int_0^2 e^{y^2} \cdot [x]_0^{y/2}\, dy$$

$$= \int_0^2 e^{y^2} \cdot \frac{y}{2} \cdot dy$$

$$= \frac{1}{4} \int_0^2 (2y) \cdot e^{y^2} \cdot dy$$

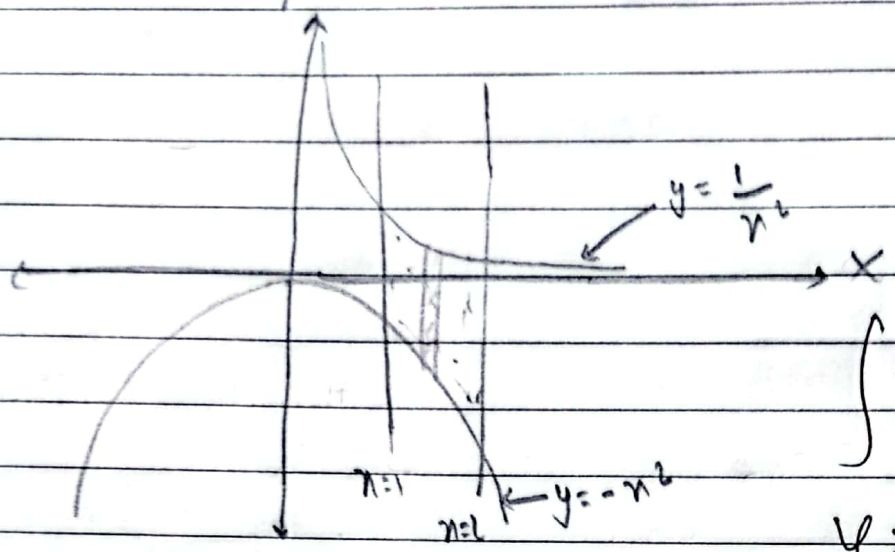
$$= \frac{1}{4} \left[e^{y^2}\right]_0^2 = \frac{1}{4}\left[e^4 - 1\right] \text{ Ans}$$

H.W.

Q. $$\int_0^1 \int_{4y}^{4} e^{x^2} \cdot dx \cdot dy$$

(#) Sketch the region bounded by the graph of Equation and find its Area using one or more double integral.

(i) $y = \frac{1}{x^2}$, $y = -x^2$, $x = 1$, $x = 2$



$$\int y\,dx$$

$$y = f(x)$$

$$\iint_R f(x,y)$$

$$z = f(x,y)$$

$$x^2 + z^2 = 9$$

$$z = \sqrt{9 - x^2}$$

$$\int_1^2 \left[\int_{-x^2}^{1/x^2} dy \right] dx$$

$$= \int_1^2 \left[y \right]_{-x^2}^{1/x^2} \cdot dx$$

$$= \int_1^2 \left[\frac{1}{x^2} + x^2 \right] \cdot dx$$

$$= \left[\frac{x^{-2+1}}{-2+1} + \frac{x^3}{3} \right]_1^2$$

$$= \left[\frac{-1}{x} + \frac{x^3}{3} \right]$$

$$= \frac{-1}{2} + \frac{8}{3} + 1 - \frac{1}{3}$$

$$= \frac{1}{2} + \frac{7}{3}$$

$$= 17/6$$

(#) Sketch the solid in the first octant bounded by the curves and find its volume.

i) $\boxed{x^2+z^2=9,\quad y=2x,\quad y=0,\quad z=0}$

Solⁿ:

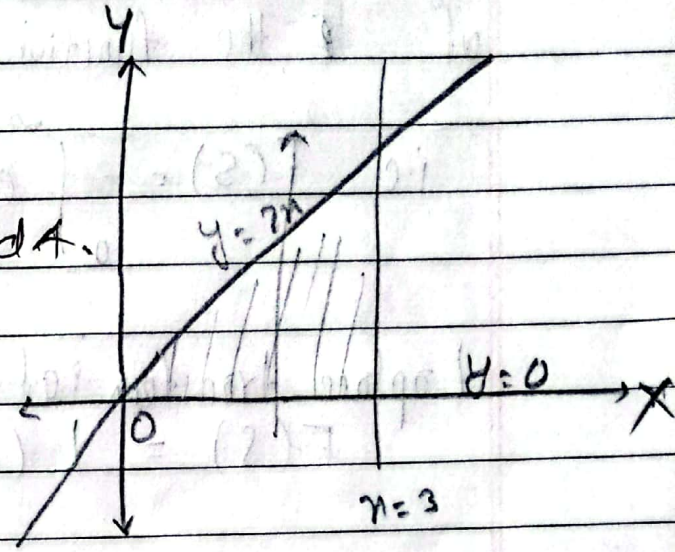
Here $x^2+z^2=9$

If $z=0$, $x^2=9$

$\therefore x=\pm 3$

Now.

$z=\pm\sqrt{9-x^2}$ $\quad V=\iint_R z\,dA.$



$$\text{Volume} = \int_0^3\left[\int_0^{2x}\sqrt{9-x^2}\,dy\right]dx$$

$$= \int_0^3 \sqrt{9-x^2}\,[y]_0^{2x}\,dx$$

$$= \int_0^3 2x\sqrt{9-x^2}\,dx$$

$$= -\int_0^3 (-2x)\cdot(9-x^2)^{1/2}\,dx$$

$$= -\left[\frac{(9-x^2)^{1/2+1}}{1/2+1}\right]_0^3$$

$$= -\left[\frac{0}{3/2}-\frac{(9)^{3/2}}{3/2}\right]$$

$$= \frac{2}{3}\times(3^2)^{3/2}$$

$$= 18$$

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(#) Laplace transformation:-

Let, $f(x)$ be a given function define for all $t>0$. Then, the function $F(s)$ (with $s>0$) obtained by multiplying $f(t)$ by e^{-st} and integrating with respect to 't' from zero to ∞ is called laplace transform of the function $f(t)$, if integral exist.

i.e $F(S) = \int_0^\infty e^{-st} f(t) \cdot dt$ ....

Laplace transform of $f(t)$ is denoted by $L(f(t))$ i.e

$$F(S) = L(f(t)) = \int_0^\infty e^{-st} f(t) \cdot dt$$

Formula, $L(1) = \frac{1}{S}$

Find the laplace transformation of t, t^2, $\sin t$, $\cos t$, e^{at}

(i) t^2.

$$L(t^2) = \int_0^\infty e^{-st} t^2 \cdot dt$$

$$= \int_0^\infty t^2 \cdot e^{-st} \cdot dt$$

$$= t^2 \left(\frac{e^{-st}}{s}\right) - 2t\left(\frac{e^{-st}}{(-s)(-s)}\right) + 2\left(\frac{e^{-st}}{-s \cdot -s \cdot -s}\right)\Bigg|_0^\infty$$

$$= (0 - 0 + 0) - \left[0 - 0 + \frac{2}{-s^3}\right]$$

$$= \frac{2}{s^3}$$

Formula: $L(1) = \frac{1}{s}$ $\qquad L^{-1}\left(\frac{1}{s}\right) = 1$

$L(t) = \frac{1}{s^2}$ $\qquad L^{-1}\left(\frac{1}{s^2}\right) = t$

$L(t^n) = \frac{n!}{s^{n+1}}$, $\qquad L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

$n = 1, 2, 3, 4, \ldots$

Find the laplace transformation of : $2t+7$.

Solⁿ:

Let, $f(t) = 2t + 7$

By defⁿ of Laplace transformation,

$$L(f) = \int_0^\infty e^{-st} \cdot f \, dt$$

$$L(2t+7) = \int_0^\infty e^{-st} (2t+7) \cdot dt$$

$$\text{or, } L(2t+7) = 2\int_0^\infty t \cdot e^{-st} \cdot dt + 7\int_0^\infty e^{-st} \cdot dt$$

$$= 2\left[t \cdot \frac{e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{-s \cdot -s}\right]_0^\infty + 7\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= 2\left(0 - 0 - 0 + \frac{1}{s^2}\right) + 7 \times \frac{1}{s}$$

$$= \frac{2}{s^2} + \frac{7}{s}$$

$$L(2t+7) = L(2t) + L(7)$$

$$= 2L(t) + 7L(1)$$

$$= 2 \times \frac{1}{s^2} + 7 \times \frac{1}{s}$$

$$= \frac{2}{s^2} + \frac{7}{s}$$

$L(\sin at) = \frac{a}{s^2+a^2}$, $L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$

$L(\cos at) = \frac{s}{s^2+a^2}$, $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

$L(\sinh at) = \frac{a}{s^2-a^2}$, $L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$

$L(\cosh at) = \frac{s}{s^2-a^2}$, $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

$L(e^{at}) = \frac{1}{s-a}$, $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$L(t^{\alpha}) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

where α is positive

Q. Find $f(t)$ of $\frac{7}{s^2} + \frac{2}{s}$.

Sol^n:)

Let, $F(s) = \frac{7}{s^2} + \frac{2}{s}$

By def^n of inverse laplace transform

$f(t) = L^{-1}(F(s))$

$= L^{-1}\left(\frac{7}{s^2} + \frac{2}{s}\right)$

$= 7\,L^{-1}\left(\frac{1}{s^2}\right) + 2\,L^{-1}\left(\frac{1}{s}\right)$

$= 7t + 2 \cdot 1$

$= 7t + 2$

Q. Find Laplace transform of e^{at}.

Sol^n) Let, $f(t) = e^{at}$

And,

$$L(f(t)) = L(e^{at})$$

$$= \int_0^\infty e^{-st} \cdot e^{at} \cdot dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= \frac{e^{-(s-a)\infty}}{-(s-a)} - \frac{e^0}{-(s-a)}$$

$$= 0 + \frac{1}{s-a}$$

$$= \frac{1}{s-a}$$

u.v $\int e^{ax} \cdot \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$

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Q. Find the laplace transform of $f(t) = \sin 5t$.

Sol^n :-

$$f(t) = \sin 5t$$

$$L(\sin 5t) = \int_0^\infty \sin 5t \cdot e^{-st} \cdot dt$$

$$\left[= \sin 5t \int_0^\infty e^{-st} \cdot dt - 5\int_0^\infty (-\cos 5t) \cdot \left[\frac{e^{-st}}{-s}\right]_0^\infty \cdot dt \right.$$

$$\left. = \sin 5t \cdot \left[\frac{e^{-st}}{-s}\right]_0^\infty + 5\int \cos 5t \left(0 + \frac{1}{s}\right) \right]$$

$$= \left[\frac{e^{-st}}{s^2+25}\left[-s \cdot \sin 5t - 5 \cdot \cos 5t\right]_0^\infty\right]_0^\infty$$

$$= 0 - \left(\frac{e^{-0}}{s^2+25}(0 - 5 \times 1)\right)$$

$$= \frac{5}{s^2+25}$$

Find the laplace transform of $5t^3 + 7t + e^{-2t} + \cosh 3t$.

Sol^n:

$f(t) = 5t^3 + 7t + e^{-2t} + \cosh 3t$

$L(f(t)) = 5L(t^3) + 7L(t) + L(e^{-2t}) + L(\cosh 3t)$

$= 5L(t^3) + 7L(t) + L(e^{-2t}) + L(\cosh 3t)$

$= 5 \cdot \frac{3!}{s^{3+1}} + 7 \cdot \frac{1}{s^2} + \frac{1}{s-(-2)} + \frac{s}{s^2-3^2}$

$L(f(t)) = \frac{30}{s^4} + \frac{7}{s^2} + \frac{1}{s+2} + \frac{s}{s^2-9}$

Find the $f(t)$ of $F(s) = \frac{30}{s^4} + \frac{7}{s^2} + \frac{1}{s+2} + \frac{s}{s^2-9}$

Sol^n:

By using def^n of inverse laplace transform.

$L^{-1}(F(s)) = L^{-1}\left(\frac{30}{s^4} + \frac{7}{s^2} + \frac{1}{s+2} + \frac{s}{s^2-9}\right)$

$f(t) = L^{-1}\left(\frac{30}{s^4}\right) + L^{-1}\left(\frac{7}{s^2}\right) + L^{-1}\left(\frac{1}{s+2}\right) + L^{-1}\left(\frac{s}{s^2-9}\right)$

$= 5L^{-1}\left(\frac{3!}{s^{3+1}}\right) + 7L^{-1}\left(\frac{1}{s^2}\right) + e^{-2t} + \cosh 3t$

$= 5t^3 + 7t + e^{-2t} + \cosh 3t$

Q. Find $f(t)$ of $F(S) = \frac{1}{S^7}$

Sol^n:

$$L^{-1}(F(S)) = L^{-1}\left(\frac{1}{S^7}\right)$$

or, $$f(t) = \frac{1}{6!} \cdot L^{-1}\left(\frac{6!}{S^{6+1}}\right)$$

or, $$f(t) = \frac{1}{6!} \cdot t^6$$

$$\therefore f(t) = \frac{t^6}{6!}$$

~~(ii) Linearity of Laplace transform:~~

~~Imp (ii)~~ Linearity of Laplace transform:

Statement: The laplace transform is a linear operation.

Proof: Let, $f(t)$ and $g(t)$ be any two function whose Laplace transform exist and let a and b be any two constant, then we have to show,

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t))$$

Now, by def^n of laplace,

$$L(af(t) + bg(t)) = \int_0^\infty e^{-st}\left[af(t) + bg(t)\right]dt$$

$$= \int_0^\infty e^{-st} af(t)\,dt + \int_0^\infty e^{-st} bg(t)\,dt$$

$$= a\int_0^\infty e^{-st} f(t)\cdot dt + b\int_0^\infty e^{-st} g(t)\cdot dt$$

$$= aL(f(t)) + bL(g(t))$$

Q. Find the Laplace transform of $\cosh at$.

Sol:)

$$L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right)$$

$$= \frac{1}{2} L\left(e^{at} + e^{-at}\right)$$

$$= \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s-(-a)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}s}{s^2-a^2}\right)$$

$$\therefore L(\cosh at) = \frac{s}{s^2-a^2}$$

Q. Find $f(t)$ of $F(s) = \frac{7s}{s^2-3}$

Sol:)

$$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{7s}{s^2-3}\right)$$

$$= \frac{7}{1} L^{-1}\left(\frac{s}{s^2-(\sqrt{3})^2}\right) = 7\cosh\sqrt{3}t$$

$$= \frac{7}{2} L^{-1}\left(\frac{s+\sqrt{3}+s-\sqrt{3}}{(s-\sqrt{3})(s+\sqrt{3})}\right)$$

$$= \frac{7}{2} L^{-1}\left(\frac{1}{s-\sqrt{3}} + \frac{1}{s+\sqrt{3}}\right)$$

$$= \frac{7}{2}\left(L^{-1}\left(\frac{1}{s-\sqrt{}}\right.\right.$$

Q. $F(s) = \dfrac{s}{s^2-5s+6}$, find $f(t)$

Sol^n:

$$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2-5s+6}\right)$$

$$= L^{-1}\left(\frac{s}{s^2-3s-2s+6}\right)$$

$$= L^{-1}\left(\frac{s}{(s-3)(s-2)}\right)$$

$$= \cancel{\frac{1}{2}\, L^{-1}\left(\frac{2s-5+5}{s}\right)}$$

Now, $\dfrac{s}{(s-3)(s-2)} = \dfrac{A}{(s-2)} + \dfrac{B}{(s-3)}$

or, $\dfrac{s}{\cancel{(s-2)(s-3)}} = \dfrac{A(s-3)+B(s-2)}{\cancel{(s-2)(s-3)}}$

or, $s = A(s-3) + B(s-2)$

When, $s=3$, $3 = B$

$\therefore B = 3$

When, $s=2$, $2 = A\cdot(-1)$

$\therefore A = -2$

$\therefore \dfrac{s}{(s-3)(s-2)} = \dfrac{-2}{(s-2)} + \dfrac{3}{(s-3)}$

Ans, $f(t) = L^{-1}\left(\dfrac{-2}{s-2} + \dfrac{3}{s-3}\right)$

$$= -2\, L^{-1}\left(\frac{1}{s-2}\right) + 3\, L^{-1}\left(\frac{1}{s-3}\right)$$

$\therefore f(t) = -2e^{2t} + 3e^{3t}$

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Q. Find $f(t)$ of $F(s) = \left(\frac{3^5}{s^4(s^2+3^2)}\right)$

Sol^n:

$$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{3^5}{s^4(s^2+3^2)}\right)$$

$$= L^{-1}\left(\frac{3^5}{s^4(s^2+3^2)}\right)$$

Now, $\frac{3^5}{s^4(s^2+3^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{Es+F}{s^2+3^2}$

or, $\frac{3^5}{s^4(s^2+3^2)} = \frac{s^3(s^2+3^2)\cdot A + s^2(s^2+3^2)\cdot B + s\cdot(s^2+3^2)\cdot C + (s^2+3^2)\cdot D + s^4(Es+F)}{s^4\cdot(s^2+3^2)}$

or, $1 = s^5A + 9s^3A + s^4B + 9s^2B + s^3C + 9sC + s^2D + 9D + s^5E + s^4F$

or, $3^5 = s^5(A+E) + s^4(B+F) + s^3(9A+C) + s^2(9B+D) + 9sC + 9D$

Equating both sides,

$A + E = 0$

$B + F = 0$

$9A + C = 0$

$9B + D = 0$

$9C = 0 \Rightarrow C = 0$

$9D = 3^5 \Rightarrow D = 27$

$9A + C = 0$	$A + E = 0$	$9B + D = 0$	$B + F = 0$
or, $9A + 0 = 0$	$\therefore E = 0$	or, $9B + 27 = 0$	or, $F = 3$
$\therefore A = 0$		$\therefore B = -3$	

$\therefore A = 0, \quad B = -3, \quad C = 0, \quad D = 27, \quad E = 0, \quad F = 3$

Now,

$$\frac{3^5}{s^4(s^2+3^2)} = \frac{\overset{\to 0}{A}}{s} + \frac{B}{s^2} + \frac{\overset{\to 0}{C}}{s^3} + \frac{D}{s^4} + \frac{Es+F}{(s^2+3^2)}$$

$$\therefore \frac{3^5}{s^4(s^2+3^2)} = \frac{-3}{s^2} + \frac{27}{s^4} + \frac{3}{s^2+3^2}$$

$$f(t) = L^{-1}\left(\frac{3^5}{s^4(s^2+3^2)}\right)$$

$$= L^{-1}\left(\frac{-3}{s^2} + \frac{27}{s^4} + \frac{3}{s^2+3^2}\right)$$

$$= -3\, L^{-1}\left(\frac{1}{s^2}\right) + 27\, L^{-1}\left(\frac{1}{s^4}\right) + 3\, L^{-1}\left(\frac{1}{s^2+3^2}\right)$$

$$= -3 \cdot t + \frac{27}{3!} \cdot L^{-1}\left(\frac{3!}{s^{3+1}}\right) + L^{-1}\left(\frac{3}{s^2+3^2}\right)$$

$$= -3t + \frac{27}{3\times2\times1} \cdot t^3 + \sin 3t$$

$$= -3t + \frac{9}{2} t^3 + \sin 3t$$

First Shifting Theorem [LAPLACE TRANSFORM]:-

Statement:

If $L(f(t)) = F(s)$

Then,

$$L(e^{at} f(t)) = [L(f(t))]_{s \to s-a}$$

$$= [F(s)]_{s \to s-a}$$

$$= F(s-a)$$

Proof:

By defn of laplace transform,

$$L(e^{at} f(t)) = \int_0^\infty e^{-st} e^{at} f(t) \cdot dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) \cdot dt$$

$$= F(s-a)$$

proved

Theorem:

If $L(f(t)) = F(s)$

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} L(f(t))$$

(*) $t^2 e^{3t}$

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} L(f(t))$$

or, $$L(t^2 f(t)) = (-1)^2 \frac{d^2}{ds^2} L(e^{3t})$$

or, $$L(t^2 e^{3t}) = (-1)^2 \frac{d^2}{ds^2} L(e^{3t})$$

$$= \frac{d^2}{ds} (s-3)^{-2}$$

$$= (-1) \frac{d}{ds} (s-3)^{-2}$$

$$= (-1)(-2) \cdot (s-3)^{-3}$$

$$= \frac{2}{(s-3)^3}$$

Q. Find $f(t)$, where $F(s) = \log\left(\frac{s+a}{s+b}\right)$

Sol^n:

Diff. both sides w.r.t. s,

$$\frac{dF(s)}{ds} = \frac{d}{ds}\log\left(\frac{s+a}{s+b}\right)$$

or, $$F'(s) = \frac{(s+b)}{s+a}\left[\frac{b-a}{(s+b)^2}\right]$$

We know that,

$$L(t f(t)) = -F'(s)$$
$$= \frac{a-b}{(s+a)(s+b)}$$

or, $$t f(t) = L^{-1}\left(\frac{a-b}{(s+a)(s+b)}\right) \quad \text{--- (i)}$$

Here, $$\frac{a-b}{(s+a)(s+b)} = \frac{A}{(s+a)} + \frac{B}{(s+b)}$$

or, $$\frac{a-b}{(s+a)(s+b)} = \frac{As + Ab + Bs + Ba}{(s+a)(s+b)}$$

or, $$a-b = (A+B)s + (Ab+Ba)$$

Equating coeff. of like terms, we get

$A+B=0$

$Ab+Ba = a-b$

$\cancel{Ab} + Bb = 0$

$\cancel{Ab} + Ba = a-b$

$B(b-a) = (b-a)$

$\therefore B = 1$

Solving, we get

$\therefore A = -1,\ B = 1.$

Now, Eq^n ① is,

$$t\,f(t) = L^{-1}\left(-\frac{1}{s+a} + \frac{1}{s+b}\right)$$

or, $$t\,f(t) = -L^{-1}\left(\frac{1}{s+a}\right) + L^{-1}\left(\frac{1}{s+b}\right)$$

$$\therefore f(t) = \frac{-e^{-at} + e^{-bt}}{t}$$

Q. Find the Laplace transform of $te^{-t}\cos t$.

Sol^n:

We know that $L(t\,f(t)) = -\frac{d}{ds}L(f(t))$

$$L(t\,\underbrace{e^{-t}\cos t}_{f(t)}) = -\frac{d}{ds}L(e^{-t}\cos t)$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+1^2}\right]_{s \to s+1}$$

$$= -\frac{d}{ds}\left[\frac{s+1}{(s+1)^2+1}\right]$$

We know,

$$L\left(e^{at} f(t)\right) = \left[L(f(t))\right]_{s \to s-a}$$

Here,

$$L\left[e^{-t}\, t \cos t\right] = L\left[t \cos t\right]_{s \to s-(-1)}$$

$$= L\left(t \cos t\right)_{s \to s+1}$$

$$= \left[-\frac{d}{ds} L(\cos t)\right]_{s \to s+1}$$

$$= \left[-\frac{d}{ds} \frac{s}{s^2+1}\right]_{s \to s+1}$$

$$= -\frac{d}{ds} \frac{s+1}{(s+1)^2+1}$$

(#) <u>Integration of Laplace transform:</u>

If $L(f(t)) = F(s)$

Then, $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\bar{s}) \cdot d\bar{s}$

Q Find the laplace transform of $f(t) = \frac{\sin 2t}{t}$

We know that,

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\bar{s}) \cdot d\bar{s}$$

where $L(f(t)) = F(\bar{s})$.

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{\bar{s}^2+4} \cdot d\bar{s}$$

or, $$L\left(\frac{\sin 2t}{t}\right) = 2\left[\frac{1}{2}\tan^{-1}\frac{\bar{s}}{2}\right]_s^\infty$$

$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right]$$

$$\therefore L\left(\frac{\sin 2t}{t}\right) = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

Q. Find laplace transform, $f(t) = \frac{\sinh t}{t}$.

We know that,

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\bar{s}) \cdot d\bar{s}$$

where, $L(f(t)) = F(\bar{s})$

$$L\left(\frac{\sinh t}{t}\right) = \int_s^\infty \frac{1}{\bar{s}^2 - 1^2} \cdot d\bar{s}$$

or, $$L\left(\frac{\sinh t}{t}\right) = \frac{1}{2} \cdot \left[\log\left(\frac{\bar{s}+1}{\bar{s}-1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s+1}{s-1}\right)\right]$$

$$= \frac{1}{2} \log\left(\frac{s+1}{s-1}\right)^{-1}$$

$$\therefore L\left(\frac{\sinh t}{t}\right) = \frac{1}{2} \log\left(\frac{s-1}{s+1}\right)$$

Q. Find $f(t)$ if $F(s) = \frac{2s}{(s^2-4)^2}$

We know that,

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\bar{s}) \cdot d\bar{s}$$

$$= \int_s^\infty \frac{2\bar{s}}{(\bar{s}^2-4)^{-2}} \cdot 2\bar{s} \cdot d\bar{s}$$

$$= \int^\infty (\bar{s}^2-4)^{-2} \cdot 2\bar{s} \cdot d\bar{s}$$

$$= \frac{(s^2-4)^{-2+1}}{-2+1}\Bigg|_s^{\infty}$$

$$= -\left[s^2-4\right]^{-1}\Bigg|_s^{\infty}$$

$$= -\left[\frac{1}{s^2-4}\right]_s^{\infty}$$

$$= -\left[0 - \frac{1}{s^2-4}\right]$$

$$= \frac{1}{s^2-4}$$

Q. Find the laplace transform of,

$$f(t) = e^{-2t}(\sinh t + \cos 3t)$$

Sol^n:)

$$L(f(t)) = L\left(e^{-2t}(\sinh t + \cos 3t)\right)$$

$$= L\left(e^{-2t}\sinh t + e^{-2t}\cos 3t\right)$$

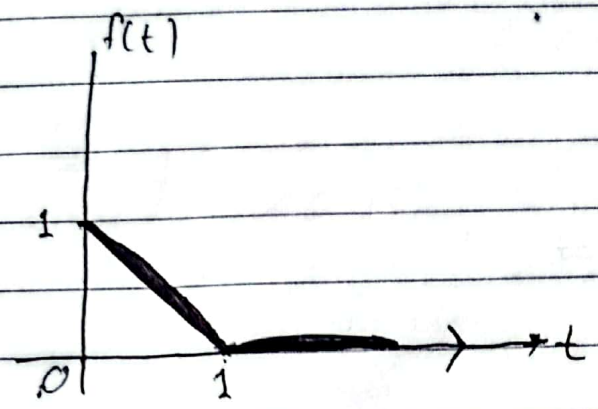
$$= L\left(e^{-2t}\sinh t\right) + L\left(e^{-2t}\cos 3t\right)$$

$$= \left[\frac{1}{s^2-a^2}\right]_{s\to s-(-2)} + \left[\frac{s}{s^2+3^2}\right]_{s\to s-(-2)}$$

$$= \left[\frac{1}{s^2-a^2}\right]_{s\to s+2} + \left[\frac{s}{s^2+9}\right]_{s\to s+2}$$

$$= \frac{1}{(s+2)^2-a^2} + \frac{s+2}{(s+2)^2+9}$$

Q. Find the laplace transform of $f(t)$ where $f(t)$ is defined as.



Here,

$$f(t) = \begin{cases} 1-t & 0 \le t \le 1 \\ 0 & t \geq 1 \end{cases}$$

By def^n of Laplace transform,

$$L(f(t)) = \int_0^\infty e^{-st} f(t) \cdot dt$$

$$= \int_0^1 e^{-st}(1-t)\,dt + \int_1^\infty e^{-st} \cdot 0 \cdot dt$$

$$= \int_0^1 (e^{-st} - e^{-st} \cdot t) \cdot dt$$

$$= \int_0^1 e^{-st} \cdot dt - \int_0^1 e^{-st} \cdot t\, dt$$

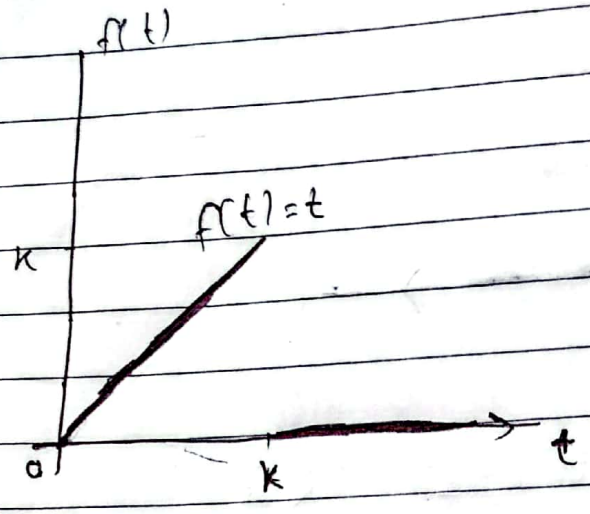
$$= \left[\frac{e^{-st}}{-s}\right]_0^1 - \left[t \cdot \frac{e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{-s \cdot -s}\right]_0^1$$

$$= \left(\frac{e^{-s}}{-s} - \frac{e^0}{-s}\right) - \left[1 \cdot \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} - \left(0 - \frac{e^0}{s^2}\right)\right]$$

$$= -\cancel{\frac{e^{-s}}{s}} + \frac{1}{s} + \cancel{\frac{e^{-s}}{s}} + \frac{e^{-s}}{s^2} - \frac{1}{s^2}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{1}{s^2}$$

Q. Find the laplace transform of $f(t)$, where $f(t)$ defined as,



Here,

$$f(t) = \begin{cases} t & 0 \le t \le k \\ 0 & t \geq 1 \end{cases}$$

By def^n,

$$L(f(t)) = \int_0^\infty e^{-st} f(t) \cdot dt$$

$$= \int_0^k e^{-st}\, t\, dt + \int_k^\infty e^{-st}\, 0 \cdot dt$$

$$= \left[t \frac{e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{s^2} \right]_0^k$$

$$= \left[k \frac{e^{-sk}}{-s} - \frac{e^{-sk}}{s^2} \right] - \left[0 - \frac{1}{s^2} \right]$$

$$= \frac{e^{-sk}}{s} \left(-k - \frac{1}{s} \right)$$

$$= k \frac{e^{-sk}}{-s} + \frac{1}{s^2} - \frac{e^{-sk}}{s^2}$$

Q. Find $f(t)$ of, $F(s) = \cot^{-1}(s+1)$

We know,

$$L(t f(t)) = -\frac{d}{ds}(F(s))$$

$$= -\frac{d}{ds}\cot^{-1}(s+1)$$

$$= -\left(-\frac{1}{1+(s+1)^2}\right)$$

or, $L(t f(t)) = \frac{1}{1+(s+1)^2}$

or, $t \cdot f(t) = L^{-1}\left(\frac{1}{1+(s+1)^2}\right)$

or, $\cancel{t \cdot f(t) = t^{-1}\left(\frac{1}{s^2+2s+2}\right)}$

or, $t \cdot f(t) = e^{-t} L^{-1}\left(\frac{1}{s^2+1^2}\right)$

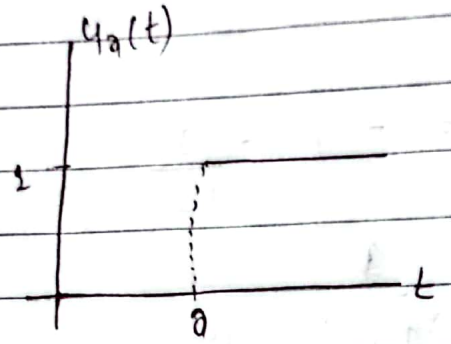
or, $t \cdot f(t) = e^{-t} \cdot \sin t$

$\therefore f(t) = \frac{e^{-t}\sin t}{t}$

(#) Unit step function:-

The unit step function $u_a(t)$ or $u(t-a)$ as defined as,

$$u(t-a) = u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$



V.V.I. (#) Second shifting theorem:-

Statement: If $L(f(t)) = F(s)$

then, $L(f(t-a)\,u(t-a)) = e^{-as}F(s)$

Proof: By def^n,

$$L(f(t-a)\cdot u(t-a)) = \int_0^\infty e^{-st} f(t-a)\cdot u(t-a)\cdot dt$$

$$= \int_0^a e^{-st} f(t-a)\cdot 0\, dt + \int_a^\infty e^{-st} f(t-a)\cdot dt$$

$$= \int_a^\infty e^{-st} f(t-a)\cdot dt$$

Put, $t-a = u$, when $t=a$, $u=0$

$1 = \frac{du}{dt}$ when $t=\infty$, $u=\infty$

$dt = du$

$$= \int_0^\infty e^{-s(u+a)} \cdot f(u) \cdot du$$

$$= \int_0^\infty e^{-su} \cdot e^{-as} f(u) \cdot du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u) \cdot du$$

$$= e^{-as} F(s) \quad \text{where } F(s) = L(f(t))$$

provid.

laplace transform of unit step function.

$$L(u(t-a)) = \int_0^\infty e^{-st} u(t-a) \cdot dt$$

$$= \int_0^a e^{-st} \cdot 0 \, dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt$$

$$= 0 + \int_a^\infty e^{-st} \cdot dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_a^\infty$$

$$= \left[0 - \frac{e^{-as}}{-s} \right]$$

$$\boxed{\therefore L(u(t-a)) = \frac{e^{-as}}{s}}$$

Find the inverse of $F(s) = \frac{e^{-2s}}{s^2}$

Sol^n:)

$$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{e^{-2s}}{s^2}\right)$$

$$= f(t-2) \cdot u(t-2)$$

$$L(u(t-a) f(t-a) = e^{-as} F(s)$$

$$L^{-1}(e^{-as} F(s)) = u(t-a) \cdot f(t-a)$$

$$L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u(t-2) \cdot (t-2)$$

$$L^{-1}\left(\frac{e^{-2s}}{s^3}\right) = u(t-2) \cdot \left(\frac{1}{2}(t-2)^2\right)$$

$$L(t \cdot u(t-3)) = t \cdot u(t-2)$$

Now,

$$L((t-2) \cdot u(t-2)) = e^{-2s} \cdot F(s)$$

$$= \frac{e^{-2s}}{s^2}$$

Note: $L(f(t)u(t-a)) = e^{\quad} L(f(t+a))$

Q. $L(t\,u(t-2)) = L(\{(t-2)+2\}\,u(t-2))$

$= L((t-2)\,u(t-2) + 2u(t-2))$

$= L((t-2)\,u(t-2)) + 2L(u(t-2))$

$= \dfrac{e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s}$

Q. Find laplace transform of, $t^2\,u(t-3)$.

$L(t^2 u(t-3)) = L((t-3+3)^2\,u(t-3))$

$= L(\{(t-3)^2 + 6(t-3) + 9\}\,u(t-3))$

$= L((t-3)^2\,u(t-3) + 6(t-3)\cdot u(t-3) + 9\,u(t-3))$

$= L((t-3)^2 \cdot u(t-3)) + 6L((t-3)u(t-3)) + 9L(u(t-3))$

$= e^{-3s}\cdot\dfrac{2!}{s^{2+1}} + 6\cdot e^{-3s}\cdot\dfrac{1}{s^2} + 9\cdot\dfrac{e^{-3s}}{s}$

$= \dfrac{2e^{-3s}}{s^3} + 6\dfrac{e^{-3s}}{s^2} + 9\cdot\dfrac{e^{-3s}}{s}$

(#) Laplace transform of Derivative :-

Let, $y = f(t)$

$$\frac{dy}{dt} = \frac{d(f(t))}{dt}$$

$$\therefore y' = f'(t)$$

$$L[f'(t)] = sL(f(t)) - f(0)$$

$$\therefore \boxed{L(y') = sL(y) - y(0)}$$

$$L[f''(t)] = s^2 L(f(t)) - sf(0) - f'(0)$$

$$\therefore \boxed{L[y''] = s^2 L(y) - sy(0) - y'(0)}$$

$$L[f'''(t)] = s^3 L(f(t)) - s^2 f(0) - sf'(0) - f''(0)$$

$$\therefore \boxed{L[y'''] = s^3 L(y) - s^2 y(0) - sy'(0) - y''(0)}$$

(*) Solve the initial value problem using Laplace transform.

$y' + 3y = 10 \sin t$, $y(0) = 0$.

Given,

$$y' + 3y = 10 \sin t$$

Taking Laplace transform,

$$L(y' + 3y) = L(10 \sin t)$$

or, $L(y') + 3L(y) = 10 L(\sin t)$

or, $sL(y) - y(0) + 3L(y) = 10 \frac{1}{s^2+1}$

or, $sL(y) - 0 + 3L(y) = \frac{10}{s^2+1}$

or, $L(y)[s+3] = \frac{10}{s^2+1}$

or, $L(y) = \frac{10}{(s^2+1)(s+3)}$

Taking L^{-1},

$$y = L^{-1}\left[\frac{10}{(s^2+1)(s+3)}\right]$$

Here, $\frac{10}{(s^2+1)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s+3}$

or, $10 = (As+B)(s+3) + C(s^2+1)$

or, $10 = As^2 + 3As + Bs + 3B + Cs^2 + C$

or, $10 = (A+C)s^2 + (3A+B)s + (3B+C)$

Equating coeff. of like terms,

$A + C = 0$ — ①

$3A + B = 0$ — ②

$3B + C = 10$ — ③

Solving ① & ②,

$\cancel{3A} + 3C = 0$

$\cancel{3A} + B = 0$

$3C - B = 0$ — ④

Solving ③ & ④,

$\cancel{3B} + C = 10$

$-\cancel{3B} + 9C = 0$

$10C = 10$

$\therefore C = 1$

$\therefore A = -1, \quad B = 3, \quad C = 1$

Now, $y = L^{-1}\left[\frac{-s+3}{s^2+1} + \frac{1}{s+3}\right]$

or, $y = L^{-1}\left[\frac{-s+3}{s^2+1}\right] + L^{-1}\left[\frac{1}{s+3}\right]$

or, $y = -L^{-1}\left[\frac{s}{s^2+1}\right] + 3L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{1}{s+3}\right]$

$\therefore y = -\cos t + 3\sin t + e^{-3t}$

(*) Solve the initial value problem using Laplace transform
$y'' + 2y' + 5y = 0$, $y(0) = 2$, $y'(0) = -4$

Given,

$y'' + 2y' + 5y = 0$

Taking Laplace transform,

$L[y'' + 2y' + 5y] = L(0)$

or, $L(y'') + 2L(y') + 5L(y) = 0$

or, $s^2L(y) - sy(0) - y'(0) + 2[sL(y) - y(0)] + 5L(y) = 0$

or, $s^2L(y) - s\cdot 2 - (-4) + 2sL(y) - 2\cdot 2 + 5L(y) = 0$

or, $s^2L(y) - 2s + \cancel{4} + 2sL(y) - \cancel{4} + 5L(y) = 0$

or, $(s^2 + 2s + 5)\,L(y) = 2s$

or, $L(y) = \frac{2s}{s^2+2s+5}$

or, $\cancel{L(y)} =$

Taking Inverse L^{-1},

$$y = L^{-1}\left(\frac{2s}{s^2+2s+5}\right)$$

or, $$y = L^{-1}\left(\frac{2s-2+2}{s^2+2s+1+4}\right)$$

or, $$y = L^{-1}\left(\frac{2(s+1)-2}{(s+1)^2+2^2}\right)$$

or, $$y = e^{-t}\, L^{-1}\left(\frac{2s-2}{s^2+2^2}\right)$$

or, $$y = e^{-t}\, L^{-1}\left(\frac{2s}{s^2+4} - \frac{2}{s^2+4}\right)$$

or, $$y = e^{-t}\left[2\, L^{-1}\left[\frac{s}{s^2+2^2}\right] - L^{-1}\left[\frac{2}{s^2+2^2}\right]\right]$$

or, $$y = e^{-t}\left(2\cdot\cos 2t - \sin 2t\right)$$

$$\therefore\ y = e^{-t}\left(2\cos 2t - \sin 2t\right)$$

(#) Convolution of two function:

Let, $f(t)$ and $g(t)$ are two function defined for all $t>0$. Then, convolution of these function denoted by $f*g$ and defined as,

$$f*g = \int_0^t f(T) \cdot g(t-T)\, dT$$

or,

$$f*g = \int_0^t f(t-T) \cdot g(T)\, dT$$

Q. $e^t * e^{-t}$

Let, $f(t) = e^t$, $g(t) = e^{-t}$

$$f*g = \int_0^t f(T)\, g(t-T)\, dT$$

$$= \int_0^t e^T \cdot e^{-(t-T)} \cdot dT$$

$$= \int_0^t e^T \cdot e^{-t} e^T \cdot dT$$

$$= e^{-t} \int_0^t e^{2T} \cdot dT$$

$$= e^{-t} \left[\frac{e^{2T}}{2} \right]_0^t$$

$$= e^{-t} \left[\frac{e^{2t} - e^0}{2} \right]$$

$$\therefore f*g = \frac{e^t - e^{-t}}{2}$$

$= \sinh t$

Q. $t * e^{-t}$

Let, $f(t) = t$ & $g(t) = e^{-t}$

$$f * g = \int_0^t f(t-T) \cdot g(T)\, dT$$

$$= \int_0^t (t-T) \cdot e^{-T} \cdot dT$$

$$= \int_0^t t \cdot e^{-T}\, dT - \int_0^t T \cdot e^{-T} \cdot dT$$

$$= t \int_0^t e^{-T} dT - \int_0^t T \cdot e^{-T} \cdot dT$$

$$= t \left[\frac{e^{-T}}{-1} \right]_0^t - \left[T \cdot \frac{e^{-T}}{-1} - 1 \cdot \frac{e^{-T}}{(-1)\cdot(-1)} \right]_0^t$$

$$= t \left[\frac{e^{-t} - e^0}{-1} \right] - \left[\frac{T \cdot e^{-T}}{-1} - e^{-T} \right]_0^t$$

$$= -te^{-t} + t - \left[\frac{t \cdot e^{-t}}{-1} - e^{-t} - (0 - e^0) \right]$$

$$= -\cancel{te^{-t}} + t + \cancel{t \cdot e^{-t}} + e^{-t} - 1$$

$$= e^{-t} + t - 1$$

IMP

(II) Convolution Theorem:

Statement

Let, $f(t)$ and $g(t)$ are two function defined for all $t \geq 0$ and their Laplace exist.

Let, $L(f(t)) = F(S)$ and $L(g(t)) = G(S)$.

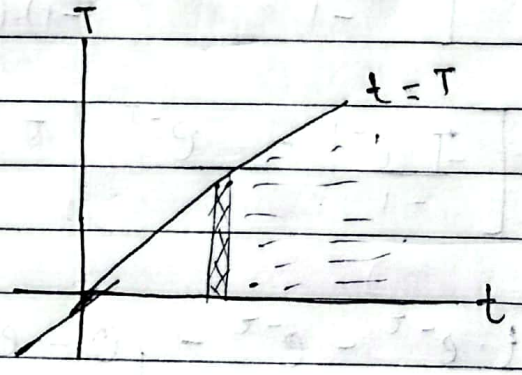
Then, $L(f*g) = F(S) \cdot G(S)$.

Proof: By defination of Laplace transform,

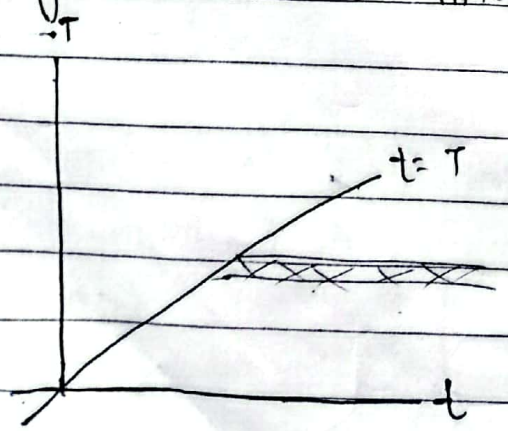
$$L(f*g) = \int_0^\infty e^{-st}(f*g)\,dt$$

$$= \int_0^\infty e^{-st} \int_0^t f(T) \cdot g(t-T)\,dT\,dt$$

$$= \int_0^\infty \int_0^t e^{-st}\, g(T) \cdot g(t-T) \cdot dT\,dt$$



Reversing the order of integration



$$\int_0^\infty \int_T^\infty e^{-st} f(T) \cdot g(t-T)\, dt\, dT$$

put, $t - T = v \Rightarrow t = T + v$

$1 = \frac{dv}{dt}$ when, $t = T$, $v = 0$

$dt = dv$ when $t \to \infty$, $v \to \infty$

$$\int_0^\infty \int_0^\infty e^{-s(T+v)} f(T) \cdot g(v)\, dv\, dT$$

$$= \int_0^\infty \int_0^\infty e^{-sT} e^{-sv} f(T)\, g(v)\, dv\, dT$$

$$= \int_0^\infty e^{-sT} f(T) \cdot dT \int_0^\infty e^{-sv} g(v)\, dv$$

$\therefore L(f * g) = F(S) \cdot G(S)$

Provid

Q. Find $f(t)$ where, $F(S) = \frac{1}{S^3}$ by convolution.

By convolution theorem,

$L(f * g) = F(S) \cdot G(S)$ — ①.

where, $L(f(t)) = F(S)$, $L(g(t)) = G(S)$

$L^{-1}(F(S) \cdot G(S)) = f * g$ — ②

$$L^{-1}\left(\frac{1}{s^3}\right) = L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2}\right)$$

where, $L^{-1}\left(\frac{1}{s}\right) = 1$, $L^{-1}\left(\frac{1}{s^2}\right) = t$

Now, $L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2}\right) = 1 * t = f * g$

$$= \int_0^t 1.(t-T).dT$$

$$= \left[tT - \frac{T^2}{2}\right]_0^t$$

$$= \left(t.t - \frac{t^2}{2}\right) - 0$$

$$= t^2 - \frac{t^2}{2}$$

$$\therefore L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}$$

Q. Find Inverse of $\frac{1}{s^2(s^2+4)}$ by convolution,

By convolution theorem,

$L(f*g) = F(s).G(s)$ — ①

where, $L(f(t)) = F(s)$, $L(g(t)) = G(s)$.

$L^{-1}(F(s).G(s)) = f*g$ — ②

$$L^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = L^{-1}\left(\frac{1}{s^2}\right)\cdot\frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$L^{-1}\left(\frac{1}{s^2}\right) = t$, $\frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right) = \sin 2t$

$$= \frac{1}{2} t * \sin 2t$$

Now,

$$L^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \frac{1}{2} t * \sin 2t$$

$$= \frac{1}{2}\int_0^t T \cdot \sin 2(t-T) \cdot dT$$

$$= \frac{1}{2}\int_0^t T \cdot \sin(2t-2T) \cdot dT$$

$$= \frac{1}{2}\left[T \cdot \frac{-\cos(2t-2T)}{-2} - 1(-)\frac{\sin(2t-2T)}{(-2)(-2)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{T \cdot \cos(2t-2T)}{2} + \frac{\sin(2t-2T)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{t \cdot \cos 0}{2} + \frac{\sin 0}{4}\right) - \left(0 + \frac{\sin 2t}{4}\right)\right]$$

$$\therefore L^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \frac{1}{2}\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)$$

$$= \frac{1}{4}\left(t - \frac{\sin 2t}{2}\right)$$

CW Solve by using Laplace transform,

$x'' - 3x' + 2x = 1 - e^{2t}$, $x(0) = 1$, $x'(0) = 0$

Given,

$$x'' - 3x' + 2x = 1 - e^{2t}$$

Taking laplace transform,

$$L(x'' - 3x' + 2x) = L(1 - e^{2t})$$

or, $L(x'') - 3L(x') + 2L(x) = L(1) - L(e^{2t})$

or, $s^2 L(x) - s\,x(0) - x'(0) - 3[sL(x) - x(0)] + 2L(x)$
$= \frac{1}{s} - \frac{1}{s-2}$

or, $s^2 L(x) - s \cdot 1 - 0 - 3sL(x) + 3 \cdot 1 + 2L(x) = \frac{\cancel{s} - 2 - \cancel{s}}{s(s-2)}$

or, $L(x)[s^2 - 3s + 2] = \frac{-2}{s(s-2)} + (s-3)$

or, $L(x) = \frac{-2}{s(s-2)(s^2-3s+2)} + \frac{(s-3)}{(s^2-3s+2)}$

or, $L(x) = \frac{-2}{s(s-2)(s^2-3s+2)} + \frac{(s-3)}{(s-1)(s-2)}$

or, $L(x) = \frac{-2}{}$

or, $x = L^{-1}\left(\frac{-2}{s(s-1)(s-2)^2}\right) + L^{-1}\left(\frac{(s-3)}{(s-1)(s-2)}\right)$

Here,

$$\frac{-2}{s(s-1)(s-2)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

or, $$\frac{-2}{s(s-1)(s-2)^2} = \frac{A(s-1)(s-2)^2 + B\cdot s(s-2)^2 + C\cdot s\cdot(s-1)(s-2) + D.s(s-1)}{s(s-1)(s-2)^2}$$

or, $$-2 = \cancel{(As - A)(s-2)^2 + Bs(s-2)^2 + Cs(s-2)}$$

or, $$-2 = A(s-1)(s-2)^2 + Bs(s-2)^2 + Cs(s-1)(s-2) + Ds(s-1)$$

Put, $s = 1$,

$-2 = 0 + B\cdot 1(-1)^2 + 0 + 0$

$\therefore B = -2$

& Put, $s = 2$

$-2 = 0 + 0 + 0 + 2D$

$\therefore D = -1$

Put, $s = 0$,

$-2 = A\cdot(-1)\cdot(-2)^2 + 0 + 0 + 0$

or, $-2 = -4A$

$\therefore A = \frac{1}{2}$

Put, $s = -1$,

$-2 = -2\cdot A\cdot 9 - B\cdot 9 + C\cdot 6 + 2D$

or, $-2 = -2\times\frac{1}{2}\cdot 9 + 18 - 6C - 2$

or, $-18 + 9 = -6C$

or, $-9 = -6C$

$\therefore C = \frac{3}{2}$

$$\therefore \frac{-2}{s(s-1)(s-2)^2} = \frac{1}{2}\cdot\frac{1}{s} - \frac{2}{s-1} + \frac{3}{2}\cdot\frac{1}{s-2} - \frac{1}{(s-2)^2}$$

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Again, $\frac{(s-3)}{(s-1)(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-2)}$

or, $\frac{(s-3)}{(s-1)(s-2)} = \frac{A(s-2)+B(s-1)}{(s-1)(s-2)}$

or, $s-3 = As - 2A + Bs - B$

or, $s-3 = (A+B)s - (2A+B)$

Equating, we get

$A+B = 1$ — ③

& $2A+B = 3$ — ④

Solving ③ & ④,

$$\begin{array}{r} 2A + \cancel{B} = 3 \\ A + \cancel{B} = 1 \\ - \quad - \quad - \\ \hline A = 2 \end{array}$$

& $B = -1$

$\therefore \frac{(s-3)}{(s-1)(s-2)} = \frac{2}{(s-1)} - \frac{1}{(s-2)}$

Now, ① is,

$$x = L^{-1}\left(\frac{1}{2}\frac{1}{s} - \frac{2}{s-1} + \frac{3}{2}\frac{1}{s-2} - \frac{1}{(s-2)^2}\right) + L^{-1}\left(\frac{2}{s-1} - \frac{1}{s-2}\right)$$

or, $x = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s-1}\right) + \frac{3}{2}L^{-1}\left(\frac{1}{s-2}\right) - L^{-1}\left(\frac{1}{(s-2)^2}\right)$

$+ 2L^{-1}\left(\frac{1}{(s-1)}\right) - L^{-1}\left(\frac{1}{s-2}\right)$

or, $x = \frac{1}{2} \cancel{1} - \cancel{2e^{t}} + \frac{3}{2} e^{2t} - e^{2t} \cdot t + \cancel{2e^{t}} - e^{2t}$

or, $x = \frac{1}{2} + \frac{1}{2} e^{2t} - e^{2t} \cdot t$

$\therefore x = \frac{1}{2} + e^{2t}\left(\frac{1}{2} - t\right)$.

Q. Find $f(t)$ of $F(s) = \frac{s}{(s+1/2)^2 + 1}$

Soln)

$$L^{-1}\left(\frac{s+1/2-1/2}{(s+1/2)^2+1}\right) = e^{-1/2 t} \cdot L^{-1}\left(\frac{s-1/2}{s^2+1}\right)$$

$$= e^{-1/2 t}\left[L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1/2}{s^2+1}\right)\right]$$

or, $f(t) = -\frac{1}{2} e^{-t/2} \cdot \sin t + e^{-1/2 t} \cos t$

or, $f(t) = -\frac{1}{2} \cdot e^{-t/2} \cdot \sin t + e^{-t/2} \cos t$

$\therefore f(t) = e^{-t/2}\left(\cos t - \frac{\sin t}{2}\right)$

Fourier Series

(#) Periodic function:-

A function $f(x)$ is said to be periodic function of there exist non-zero positive number p such that,

$$f(x+p) = f(x)$$

or, $$f(x) = f(x+p)$$

This number p is said to be period of function $f(x)$. The smallest value of p is called the fundamental period or primitive period or basic period or prime period.

Example $\sin x, \cos x$ are periodic with period 2π

Q. Find the period of $f(x) = \sin nx$

By def^n of periodicity,

$f(x) = f(x+p)$ where p is period of $f(x)$.

$$\sin nx = \sin(n(x+p))$$

or, $$\sin nx = \sin(nx + np)$$

or, $$\sin nx = \sin nx \cdot \cos np + \sin np + \cos nx$$

Equating the coeff. of like terms,

$$1 = \cos np$$

or, $$\cos 2\pi = \cos np$$

or, $$2\pi = np$$

$$\therefore \quad p = \frac{2\pi}{n}$$

(#) Even and odd function:

A function $f(x)$ is said to be Even function, if $f(-x) = f(x)$.

Ex: $f(x) = x^2$

x replace by $-x$,

$f(-x) = (-x)^2 = x^2 = f(x)$

A function $f(x)$ is said to be odd function, if $f(-x) = -f(x)$,

Example: $f(x) = x^3$

x replace by $-x$

$\therefore f(-x) = (-x)^3 = -x^3 = -f(x)$

Q. Check $f(x) = x^3$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

is even or odd function or neither.

Sol^n:

Replace x by $-x$,

$f(-x) = (-x)^3$ for $-\frac{\pi}{2} < -x < \frac{\pi}{2}$

or, $f(-x) = -x^3$

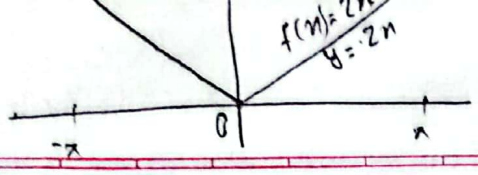
~~for, $x < 0$ & $x > \frac{\pi}{2}$~~

~~$f(-x) = -x^3 \times (-)$~~

or, $f(-x) = -x^3$ for $\frac{\pi}{2} > x > -\frac{\pi}{2}$

$\therefore f(-x) = -x^3$ for, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore f(x)$ is the odd function.



Check odd or even function of

$$f(x) = \begin{cases} -2x & \text{if } -\pi < x < 0 \\ 2x & \text{if } 0 < x < \pi \end{cases}$$

Replace x with $-x$,

$$f(-x) = \begin{cases} -2\cdot(-x) & \text{if } -\pi < -x < 0 \\ 2\cdot(-x) & \text{if } 0 < -x < \pi \end{cases}$$

or, $$f(-x) = \begin{cases} 2x & \text{if } \pi > x > 0 \\ -2x & \text{if } 0 > x > -\pi \end{cases}$$

or, $$f(-x) = \begin{cases} 2x & \text{if } 0 < x < \pi \\ -2x & \text{if } -\pi < x < 0 \end{cases}$$

or, $$f(-x) = \begin{cases} -2x & \text{if } -\pi < x < 0 \\ 2x & \text{if } 0 < x < \pi \end{cases}$$

$\therefore f(-x) = f(x)$

$\therefore f(x)$ is even function

<u>Properties of odd and Even function:</u>

Product of two odd function is Even.

Let $f(x)$ and $g(x)$ are two odd function for all x

$f(-x) = -f(x)$, $g(-x) = -g(x)$

Now, $f(-x)\cdot g(-x) = -f(-x)\cdot -g(x)$

$= f(x)\cdot g(x)$

Product of odd an Even function is odd function

③ Product of two even function is even function

④ If $f(x)$ is an Even function Then, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

⑤ If $f(x)$ is an odd function. Then, $\int_{-a}^{a} f(x)\cdot dx = 0$

(#) Trigonometric Series :-

An infinite series of the form,

$$a_0 + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \dots$$

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0, a_1, a_2, \dots\dots$ $b_1, b_2, b_3 \dots\dots$ are constant is called a trigonometric series, where a_n and b_n are coefficient of series.

Fourier series with ~~2π~~ period 2π :-

If $f(x)$ be a periodic function with 2π and is Integrable over the period. Then, the trigonometric series of the form,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{—①}$$

is said to be fourier series of $f(x)$ if the series ① converges to $f(x)$, ~~where~~

where a_0, a_n, b_n are the Euler coefficient,

where, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx\, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx\, dx$, $n = 1, 2, 3, \dots$

Find the fourier series of,

$f(x) = x^2, \quad -\pi < x < \pi$

Sol^n:

Period of $f(x)$ is,

$P = (\pi - (-\pi)) = 2\pi$

We know, the fourier series of $f(x)$ with period 2π.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

where, $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \cdot dx$$

Here, $f(-x) = (-x)^2 = x^2$

$\therefore f(x)$ is the even function.

for a_0,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \cdot dx$$

$$= \frac{1}{2\pi} \cdot 2 \int_{0}^{\pi} x^2 \cdot dx$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \cdot \frac{\pi^3}{3}$$

$$= \frac{\pi^2}{3}$$

for a_n, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx$

$$= \frac{1}{\pi} \cdot 2 \int_{0}^{\pi} x^2 \cdot \cos nx \cdot dx$$

$$= \frac{2}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - 2x \cdot \frac{(-\cos nx)}{n \cdot n} + \frac{2(-\sin nx)}{n \cdot n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi^2 \cdot \frac{\sin n\pi}{n} + 2\pi \frac{(\cos n\pi)}{n^2} - \frac{2 \sin n\pi}{n^3}\right) - (0-0+0)\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi \cdot (-1)^n}{n^2} - 0\right]$$

$$= \frac{4\pi}{\pi} \cdot \frac{(-1)^n}{n^2}$$

$$\therefore a_n = \frac{4(-1)^n}{n^2}$$

for b_n, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot \sin nx \cdot dx$

$$= 0$$

$$\therefore x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2} \cos nx + 0 \cdot \sin nx\right]$$

$$\therefore x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

Ans

Q. Find the fourier series of,

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

Here,

period $f(x)$,

$$p = \pi - (-\pi) = 2\pi$$

We know that,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

for $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} 0 \cdot dx + \int_{0}^{\pi} 1 \cdot dx \right]$$

$$= \frac{1}{2\pi} \left[0 + (x)_0^{\pi} \right]$$

$$= \frac{1}{2\pi} \cdot (\pi - 0)$$

$$= \frac{1}{2}$$

for a_n, $a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx$

or $a_n = \frac{1}{\pi} \left[0 + \int_{0}^{\pi} 1 \cdot \cos nx \cdot dx \right]$

$$= \frac{1}{\pi} \cdot \int_{0}^{\pi} \cos nx \cdot dx$$

$$= \frac{1}{\pi} \left[\frac{\sin nx}{n} - 0 \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\sin n\pi}{n} - 0 \right]$$

for b_n, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx\cdot dx$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi} 1\cdot\sin nx\cdot dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{\cos 0}{n}\right]$$

$$= -\frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n} - \frac{\cos n\pi}{n}\right]$$

$$\therefore b_n = \frac{1}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]$$

The required fourier series of $f(x)$ is,

$$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]\sin nx$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[\sum_{n=1}^{\infty}\frac{\sin nx}{n} - \frac{(-1)^n\cdot\sin nx}{n}\right]$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[\frac{\sin x}{1} - \frac{(-1)\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{(-1)\sin 3x}{3}\right.$$
$$\left. + \frac{\sin 4x}{4} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - \frac{(-1)\sin 5x}{5} + \dots\right]$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[2\sin x + 2\frac{\sin 3x}{3} + 2\frac{\sin 5x}{5} + \dots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right]$$

(#) Fourier series of $f(x)$ having period 2π in the interval $[\alpha, 2\pi+\alpha]$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

where,

$$a_0 = \frac{1}{2\pi} \int_{\alpha}^{2\pi+\alpha} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{2\pi+\alpha} f(x) \cdot \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{2\pi+\alpha} f(x) \cdot \sin nx \cdot dx$$

(#) Fourier series of a function with Arbitrary period $2L$:

Let, the fourier series of $f(x)$ having period $2L$ in the interval $[-L, L]$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \cdot dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos \frac{n\pi x}{L} \cdot dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \sin \frac{n\pi x}{L} \cdot dx$$

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Q. Find the fourier series of $f(x) = x - x^2$, $-1 < x < 1$

Sol^n:

Here period of $f(x)$, $2L = (L + (-L))$

$= 1 - (-1)$

or, $2L = 2$

$\therefore L = 1$

Suppose, the fourier series of $f(x)$ having period $2L$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

(i) For a_0, $a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \cdot dx$

$$= \frac{1}{2} \int_{-1}^{1} (x - x^2) \cdot dx$$

$$= \frac{1}{2} \left[\int_{-1}^{1} x \cdot dx - \int_{-1}^{1} x^2 \cdot dx \right]$$

$$= \frac{1}{2} \left[0 - 2 \int_{0}^{1} x^2 \cdot dx \right]$$

$$= \frac{1}{2} \times (-2) \cdot \left[\frac{x^3}{3} \right]_0^1$$

$$= -1 \left[\frac{1}{3} - 0 \right]$$

$$= -\frac{1}{3}$$

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(ii) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$= \frac{1}{1}\int_{-1}^{1}(x-x^2)\cdot\cos\frac{n\pi x}{1}\,dx$

$= \int_{-1}^{1} x\cdot\cos n\pi x\cdot dx - \int_{-1}^{1} x^2\cdot\cos n\pi x\cdot dx$

$= 0 - 2\int_{0}^{1} x^2\cdot\cos n\pi x\cdot dx$

$= -2\left[x^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2x\cdot\left(\frac{-\cos n\pi x}{n\pi\cdot n\pi}\right) + 2\left(\frac{-\sin n\pi x}{n\pi\cdot n\pi\cdot n\pi}\right)\right]$

$= -2\left[\frac{\sin n\pi}{n\pi} + \frac{2\cos n\pi}{n^2\pi^2} - \frac{2\sin n\pi}{n^3\pi^3} - (0-0-0)\right]$

$= -2\left[0 + \frac{2\cdot 1\cdot(-1)^n}{n^2\pi^2} - 0 + 0\right]$

$\therefore a_n = \frac{-4\cdot(-1)^n}{n^2\pi^2}$

(iii) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\sin\frac{n\pi x}{L}\,dx$

$= \frac{1}{1}\int_{-1}^{1}(x-x^2)\cdot\sin n\pi x\cdot dx$

$= \int_{-1}^{1} x\cdot\sin n\pi x\cdot dx - \int_{-1}^{1} x^2\cdot\sin n\pi x\cdot dx$

$= 2\int_{0}^{1} x\cdot\sin n\pi x - 0$

$$= 2\left[x\cdot\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\cdot\left(-\frac{\sin n\pi x}{n\pi}\right)\right]_0^1$$

$$= 2\left[-\frac{1}{n\pi}(-1)^n + 0 - (0+0)\right]$$

$$\therefore b_n = -2\frac{(-1)^n}{n\pi}$$

Now: The required fourier series is,

$$x - x^2 = -\frac{1}{3} + \sum_{n=1}^{\infty}\left(-\frac{4(-1)^n}{n^2\pi^2}\cos n\pi x - \frac{2(-1)^n}{n\pi}\sin n\pi x\right)$$

Q. Find the fourier series of,
$f(x) = \frac{\pi - x}{2}$ for $0 < x < 2$

Sol^n:
Period of $f(x)$, $2L = (L - (-L))$
$= (2 - 0)$
or, $2L = 2$
$\therefore L = 1$

The fourier series of given function is,
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right).$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\cdot dx$$

$$= \frac{1}{2}\int_{0}^{2}\left(\frac{\pi - x}{2}\right)\cdot dx$$

$$= \frac{1}{2}\left[\frac{\pi x}{2} - \frac{x^2}{4}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{\pi \times 2}{2} - \frac{2^2}{4}\right]$$

$$= \frac{1}{2} \times 2\left(\frac{\pi}{2} - \frac{1}{2}\right)$$

$$\therefore \quad a_0 = \left(\frac{\pi - 1}{2}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot \cos\frac{n\pi x}{L}\cdot dx$$

$$= \int_{0}^{2}\left(\frac{\pi - x}{2}\right)\cdot \cos\frac{n\pi x}{1}\cdot dx$$

$$= \frac{\pi}{2}\int_{0}^{2}\cos n\pi x\cdot dx - \frac{1}{2}\int_{0}^{2} x\cdot \cos n\pi x\cdot dx$$

$$= \frac{\pi}{2}\left[\frac{\sin n\pi x}{n\pi}\right]_0^2 - \frac{1}{2}\left[x\cdot\frac{\sin n\pi x}{n\pi} - 1\left(\frac{-\cos n\pi x}{n\pi\cdot n\pi}\right)\right]_0^2$$

$$= \frac{\pi}{2}\left[\frac{\sin n\pi\cdot 2}{n\pi} - 0\right] - \frac{1}{2}\left[2\cdot\frac{\sin n\pi 2}{n\pi} + \frac{\cos n\pi 2}{n^2\pi^2} - 0 + \frac{\cos 0}{n^2\pi^2}\right]$$

$$= 0 - \frac{1}{2}\left[0 + \right.$$

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(ii) Fourier sine and cosine series (Fourier series of Even & odd function):

Fourier sine and Cosine series of $f(x)$ having period 2π in $(-\pi, \pi)$:-

We know that fourier series of $f(x)$ having period 2π,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx \cdot dx$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \cdot dx$$

Case I: If $f(x)$ is even function.

$a_0 = \frac{1}{2\pi} \cdot 2 \int_0^{\pi} f(x) \cdot dx$ $\quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos nx \cdot dx$ $\quad b_n = 0$

$$\therefore a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \cdot dx$$

Then (1) becomes,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \cdot dx$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos nx \cdot dx$

which is known as the fourier cosine series in $(0, \pi)$

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Case II : If $f(x)$ is odd function

$a_0 = 0$, $a_n = 0$ $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin nx \cdot dx$

~~which is known~~ .

Then, ① becomes,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin nx \, dx$

which is known as fourier sine series in $(0, \pi)$.

(*) Fourier series sine and cosine series in any interval having period 2L :

We know the fourier series of $f(x)$ having period 2L in $[-L, L]$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad \text{—①}$$

where, $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\cdot dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot \cos\frac{n\pi x}{L}\cdot dx$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot \sin\frac{n\pi x}{L}\cdot dx$$

Case I: If $f(x)$ is a Even function,

$$a_0 = \frac{2}{2L}\int_0^L f(x) \quad , \quad a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi x}{L}\,dx$$

$b_n = 0$

Then, ① becomes,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where, $a_0 = \frac{1}{L}\int_0^L f(x)\cdot dx \quad , \quad a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi x}{L}\,dx$

which is known as the fourier cosine series of $f(x)$ in $(0, L)$.

Case II: If $f(x)$ is a Odd function.

$a_0 = 0 \quad , \quad a_n = 0 \quad , \quad b_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\frac{n\pi x}{L}\cdot dx$

Then, ① becomes,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where, $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

which is known as the fourier sine series of $f(x)$ in $(0, L)$

Q. Find the fourier cosine series of $f(x) = x \sin x$ in $(0, \pi)$ and show that,

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \dots = \frac{\pi - 2}{4}$$

Sol^n:

We know that fourier cosine series of $f(x)$ in $(0, \pi)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \cdot dx$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos nx \cdot dx$

① Now, $a_0 = \frac{1}{\pi} \int_0^{\pi} x \cdot \sin x \cdot dx$

$$= \frac{1}{\pi} \left[x(-\cos x) - 1 \cdot (-\sin x) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\pi(-\cos \pi) + \sin \pi - (0 + 0) \right]$$

$$= 1$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cdot \sin x \cdot \cos nx \cdot dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cdot (2 \sin x \cdot \cos nx) \cdot dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left[\sin(nx + x) - \sin(nx - x) \right] \cdot dx$$

$$= \int^{\pi} x \cdot \sin(n+1)x - x \sin(n-1)x \cdot dx$$

② To find a_n,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos nx\cdot dx$$

or, $$a_n = \frac{2}{\pi}\int_0^{\pi} x\cdot\sin x\cdot\cos nx\cdot dx$$

or, $$a_n = \frac{1}{\pi}\int_0^{\pi} x\cdot(2\sin x\cdot\cos nx)\cdot dx$$

or, $$a_n = \frac{1}{\pi}\int_0^{\pi} x\{\sin(x+nx)+\sin(x-nx)\}\cdot dx$$

or, $$a_n = \frac{1}{\pi}\int_0^{\pi} x\cdot\sin(n+1)x\cdot dx - \frac{1}{\pi}\int_0^{\pi} x\cdot\sin(n-1)x\cdot dx$$

or, $$a_n = \frac{1}{\pi}\left[x\cdot\left(\frac{-\cos(n+1)x}{n+1}\right) - 1\left(\frac{-\sin(n+1)x}{(n+1)(n+1)}\right)\right]_0^{\pi}$$

$$-\frac{1}{\pi}\left[x\cdot\left(\frac{-\cos(n-1)x}{n-1}\right) - 1\left(\frac{-\sin(n-1)x}{(n-1)(n-1)}\right)\right]_0^{\pi}$$

or. for $n \geq 2$

or, $$a_n = \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{(n+1)} + \frac{\sin(n+1)\pi}{(n+1)^2} - 0\right]$$

$$-\frac{1}{\pi}\left[-\pi\frac{\cos(n-1)\pi}{(n-1)} + \frac{\sin(n-1)\pi}{(n-1)^2} - 0\right]$$

or, $$a_n = \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{(n+1)} + 0\right] - \frac{1}{\pi}\left[-\pi\frac{\cos(n-1)\pi}{(n-1)} + 0\right]$$

or, $a_n = -\frac{1}{\pi} \cdot \pi \frac{(-1)^{n+1}}{n+1} + \frac{1}{\pi} \cdot \pi \cdot \frac{(-1)^{n-1}}{(n-1)}$

or, $a_n = -\frac{(-1)^n \cdot (-1)^1}{n+1} + \frac{(-1)^n \cdot (-1)^{-1}}{n+1}$

or, $a_n = (-1)^n \left[\frac{1}{n+1} - \frac{1}{n-1} \right]$

or, $a_n = (-1)^n \left[\frac{\cancel{n}-1 - \cancel{n} - 1}{n^2-1} \right]$

$\therefore a_n = \frac{-2(-1)^n}{n^2-1}$

Now, For $n=1$,

$a_1 = \frac{1}{\pi} \int_0^{\pi} x \cdot \sin 2x \cdot dx - 0$

$= \frac{1}{\pi} \left[x \cdot \frac{(-\cos 2x)}{2} - 1 \cdot \left(-\frac{\sin 2x}{2 \cdot 2} \right) \right]_0^{\pi}$

$= \frac{1}{\pi} \left[-\pi \cdot \frac{\cos 2\pi}{2} + \frac{\sin 2\pi}{4} - 0 \right]$

$= \frac{1}{\cancel{\pi}} \times \cancel{\pi}(-1) \cdot \frac{(-1)^3}{2} + 0$

$= \frac{-1}{2}$

$\cos 3\pi = (-1)$

$\cos (n+1)\pi = (-1)^{n+1}$

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Then the required fourier cosine series ① becomes,

$$x \cdot \sin x = 1 + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

or, $$x \cdot \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{(-2)\cdot(-1)^n}{n^2-1} \cdot \cos nx$$

or, $$x \cdot \sin x = 1 - \frac{1}{2}\cos x + \frac{(-2)}{3} \cdot \cos 2x + \frac{2}{8} \cdot \cos 3x - \frac{2}{15} \cos 4x + \ldots$$

or, At $x = \frac{\pi}{2}$,

$$\frac{\pi}{2} \cdot \sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - \frac{2}{3} \cdot \cos 2 \cdot \frac{\pi}{2} + \frac{2}{8} \cos 3 \cdot \frac{\pi}{2}$$
$$- \frac{2}{15} \cos 4 \cdot \frac{\pi}{2} + \ldots$$

or, $$\frac{\pi}{2} = 1 - 0 - \frac{2}{3} \times (-1) + \frac{2}{8} \times 0 - \frac{2}{15} \times (-1)^2$$

or, $$\frac{\pi}{2} - 1 = \frac{2}{3} - \frac{2}{15} + \ldots$$

or, $$\frac{\pi - 2}{2} = \frac{2}{3} - \frac{2}{15} + \ldots$$

or, $$\frac{\pi - 2}{4} = \frac{1}{3} - \frac{1}{15} + \ldots$$

$$\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \ldots = \frac{\pi - 2}{4}$$

Q. Find the ~~sine~~ fourier sine series,
$f(x) = \pi \cdot \sin \pi x$ for $0 < x < 1$ $(0, 1)$.

Sol^n:

We know that fourier sine of $f(x)$ in $(0, L)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad \text{—— (1)}$$

where, $b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} \cdot dx$

Here, $L = 1$,

For b_n,

$$b_n = \frac{2}{1} \int_0^1 \pi \sin \pi x \cdot \sin n\pi x \cdot dx$$

$$= \pi \int_0^1 (2 \sin \pi x \cdot \sin n\pi x) dx$$

$$= \pi \int_0^1 \cos(\pi x - n\pi x) - \cos(\pi x + n\pi x) \cdot dx$$

$$= \pi \int_0^1 [\cos(1-n)\pi x - \cos(1+n)\pi x] \cdot dx$$

$$= \pi \int_0^1 [\cos(n-1)\pi x - \cos(n+1)\pi x] \cdot dx$$

$$= \pi \left[\frac{\sin(n-1)\pi x}{(n-1)\pi} - \frac{\sin(n+1)\pi x}{(n+1)\pi} \right]_0^1$$

$$= \pi \left[\frac{\sin(n-1)\pi}{(n-1)\pi} - \frac{\sin(n+1)\pi}{(n+1)\pi} \right]$$

$$= \pi \times 0$$

$$\therefore b_n = 0$$

Vector Calculus :

Find the angle between the tangents to the curve at the point $t = \pm 1$.

$$\vec{r} = t^2 \vec{i} + 2t \vec{j} - t^3 \vec{k}$$

Diff. w.r. to 't',

$$\frac{d\vec{r}}{dt} = 2t \vec{i} + 2 \vec{j} - 3t^2 \vec{k}$$

at $t = 1$,

Let, $T_1 = \frac{d\vec{r}}{dt} = 2\vec{i} + 2\vec{j} - 3\vec{k}$

Let, $T_2 = \frac{d\vec{r}}{dt} = -2\vec{i} + 2\vec{j} - 3\vec{k}$

Dot product of two vector :-

Let, $\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

Then, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

where θ = be the angle betn 2 vector $\vec{a}$ & $\vec{b}$

$\therefore \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

or, $\vec{a} \cdot \vec{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$

$= (a_1 b_1 + a_2 b_2 + a_3 b_3)$

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Let, θ be the angle between T_1 & T_2.

$$\cos\theta = \frac{\vec{T_1}\cdot\vec{T_2}}{|\vec{T_1}||\vec{T_2}|}$$

or, $$\cos\theta = \frac{(2\vec{i}+2\vec{j}-3\vec{k})(-2\vec{i}+2\vec{j}-3\vec{k})}{|2\vec{i}+2\vec{j}-3\vec{k}|\,|-2\vec{i}+2\vec{j}-3\vec{k}|}$$

or, $$\cos\theta = \frac{2\times(-2)+2\times2+(-3)(-3)}{\sqrt{2^2+2^2+3^2}\cdot\sqrt{2^2+2^2+3^2}}$$

or, $$\cos\theta = \frac{\cancel{-4}+\cancel{4}+9}{\sqrt{17}\cdot\sqrt{17}}$$

or, $$\cos\theta = \frac{9}{17}$$

$$\therefore \theta = \cos^{-1}\left(\frac{9}{17}\right)$$

Q. A particle moves along the curve $x=4\cos t$, $y=4\sin t$ and $z=6t$. Find the velocity and acceleration at $t=0$, and $t=\frac{\pi}{2}$.

Soln:

Let, $\vec{r}$ be the position vector on the curve.

$$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\vec{r} = 4\cos t\,\hat{i}+4\sin t\,\hat{j}+6t\,\hat{k}$$

Let, $$\vec{v} = \frac{d\vec{r}}{dt} = -4\sin t\,\vec{i}+4\cos t\,\hat{j}+6\vec{k}$$

& $$\vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = -4\cos t\,\vec{i}-4\sin t\,\vec{j}+0\vec{k}$$

$$\therefore \vec{a} = \frac{d^2\vec{r}}{dt^2} = -4\cos t\,\vec{i}-4\sin t\,\vec{j}+0\cdot\vec{k}$$

At, $t=0$,

velocity $\vec{v} = -4.0\vec{i} + 4.1\vec{j} + 6\vec{k}$

$= 0\vec{i} + 4\vec{j} + 6\vec{k}$

acceleration, $\vec{a} = -4\vec{i} - 0.\vec{j} + 0.\vec{k}$

At $t = \frac{\pi}{2}$,

velocity $\vec{v} = -4\times1\,\vec{i} + 4\times0\,\vec{j} + 6\vec{k}$

$= -4\vec{i} + 0.\vec{j} + 6\vec{k}$

acceleration $\vec{a} = -4\times0\,\vec{i} - 4\times1\,\vec{j} + 0.\vec{k}$

$= \cancel{0\vec{i}} - 4,$

$= 0\vec{i} - 4\vec{j} + 0.\vec{k}$

Q. Find all partial Derivatives of the vector function.

$\vec{v} = e^x.\cos y\,\vec{i} + e^x \sin y\,\vec{j}$

Diff. partially w.r.to 'x',

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[e^x.\cos y\,\vec{i} + e^x \sin y\,\vec{j}\right]$$

or, $$V_x = \cos y\,\vec{i}.\frac{\partial e^x}{\partial x} + \sin y\,\vec{j}\,\frac{\partial e^x}{\partial x}$$

$$\therefore V_x = \cos y\,\vec{i}\,e^x + \sin y\,\vec{j}\,e^x$$

Diff. both sides w.r.to y,

$$V_y = e^x(-\sin y)\,\vec{i} + e^x \cos y\,\vec{j}$$

(#) Scalar point function :-

A scalar point function is denoted by,

$$\phi = \phi(x, y, z)$$

This means ϕ is a scalar function of x, y, z.

Vector point function:

A vector point function is denoted by,

$$\vec{v} = \vec{v}(x, y, z)$$

This means $\vec{v}$ is a vector function.

(#) Vector Differential operator :-

A differential operator is defined by,

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

∇ is read as nabla or Del.

(*) Gradient of a scalar point function :-

Let, $\phi = \phi(x, y, z)$ be a scalar point function.

Then, its gradient is defined by,

$$\text{grad}\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$\therefore$ The gradient of scalar function is a vector function.

Q. $\phi = x^3 + y^3 + z^3 - 3xyz$

Find $\nabla \phi$.

Sol^n:

$$\nabla \phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$= (3x^2 - 3yz)\vec{i} + (3y^2 - 3xz)\vec{j} + (3z^2 - 3xy)\vec{k}$$

(#) <u>Angle between two surfaces</u>:

Let, $\phi_1(x, y, z) = 0$ and,

$\phi_2(x, y, z) = 0$ are

two surfaces.

Let, $\vec{n_1} = \nabla \phi_1$ and $\vec{n_2} = \nabla \phi_2$.

Let, θ be the angle between surface.

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}| \cdot |\vec{n_2}|}$$

Q. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$ and $z = 3t - 5$

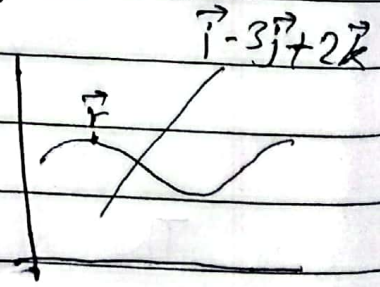
Find the component of its velocity and acceleration at $t = 1$ sec in the direction $\vec{i} - 3\vec{j} + 2\vec{k}$.

Sol^n:

Let, $\vec{r}$ be the position vector,

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r} = 2t^2\vec{i} + (t^2 - 4t)\vec{j} + (3t - 5)\vec{k}$



$$\text{Velocity} = \vec{v} = \frac{d\vec{r}}{dt} = 4t\vec{i} + (2t - 4)\vec{j} + 3\vec{k}$$

$$\text{acceleration} = \vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\vec{i} + 2\vec{j} + 0 \cdot \vec{k}$$

at $t=1$,

$\vec{v} = 4\vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{a} = 4\vec{i} + 2\vec{j} + 0\vec{k}$

Here, Let, $\vec{n} = \vec{i} - 3\vec{j} + 2\vec{k}$

Thus, the component of velocity $\vec{v}$ in the direction

$$\vec{n} = \frac{\vec{n}}{|\vec{n}|} \cdot \vec{v}$$

$$= \frac{(\vec{i} - 3\vec{j} + 2\vec{k})}{\sqrt{1+9+4}} \cdot (4\vec{i} - 2\vec{j} + 3\vec{k})$$

$$= \frac{4+6+6}{\sqrt{14}}$$

$$= \frac{16}{\sqrt{14}}$$

The component of accleration in direction $\vec{n} = \hat{n} \cdot \vec{a}$

$$\Rightarrow \frac{(\vec{i} - 3\vec{j} + 2\vec{k})}{\sqrt{14}} \cdot (4\vec{i} - 2\vec{j} + 0\vec{k})$$

$$= \frac{4+6+0}{\sqrt{14}}$$

$$= \frac{10}{\sqrt{14}}$$

(#) <u>Directional Derivative</u>:

The directional Derivative of scalar function ϕ in the direction $\vec{a}$ is denoted by $D_{\vec{a}}(\phi)$ is defined as,

$$D_{\vec{a}}(\phi) = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

Find the directional derivative of given function at given point in the given direction.

Q $f = xyz$ at $P(-1,1,3)$ along $\vec{a} = \vec{i} - 2\vec{j} + 2\vec{k}$

Sol^n:

By def^n Directional Derivative,

$$D_{\vec{a}} f = \text{grad} f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla f = \left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} \right)$$

$$= \vec{i}\frac{\partial xyz}{\partial x} + \vec{j}\frac{\partial xyz}{\partial y} + \vec{k}\frac{\partial xyz}{\partial z}$$

$$\therefore \nabla f = yz\vec{i} + xz\vec{j} + xy\vec{k}$$

at $(-1,1,3)$

$$\nabla f = 3\vec{i} - 3\vec{j} - \vec{k}$$

$$\therefore D_{\vec{a}} f = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (3\vec{i} - 3\vec{j} - \vec{k}) \cdot \frac{(\vec{i} - 2\vec{j} + 2\vec{k})}{\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{3 + 6 - 2}{3}$$

$$= \frac{7}{3}$$

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(b) $f = 4xz^3 - 3x^2yz^2$ at $(2,-1,2)$ along z-axis $(0,0,1)$

$0\cdot\vec{i} + 0\cdot\vec{j} + \vec{k}$

Sol^n:

By def^n of Directional Derivative,

$$D_{\vec{a}} f = \text{grad } f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla f = \left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right)$$

$$\therefore \nabla f = (4z^3 - 6xyz^2)\vec{i} - 3x^2y^2\vec{j} + (12xz^2 - 6x^2yz)\vec{k}$$

at $(2,-1,2)$

$$\therefore \nabla f = [4\times 2^3 - 6\times 2\times(-1)\times 2^2]\vec{i} - 3\times 2^2\times(-1)^2\cdot\vec{j} + (12\times 2\times 2^2 - 6\times 2^2\times(-1)\times 2)\vec{k}$$

$$= (32+48)\vec{i} - 48\vec{j} + (96+48)\vec{k}$$

$$= 80\vec{i} - 48\vec{j} + 144\vec{k}$$

$$\therefore D_{\vec{a}} f = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (80\vec{i} - 48\vec{j} + 144\vec{k}) \cdot \times \frac{\vec{k}}{\sqrt{1^2}}$$

$$= 144$$

(#) Divergence of vector field:

Let, $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$

be a vector. Then, Divergence of $\vec{F}$ is denoted by div $\vec{F}$ or $\nabla \cdot \vec{F}$ and defined as,

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right)$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)$$

Q. find divergence:

$\vec{V} = x\vec{i} + y^2\vec{j} + z^3\vec{k}$

$$\text{div } \vec{V} = \nabla \cdot \vec{V}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x\vec{i} + y^2\vec{j} + z^3\vec{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^3}{\partial z}$$

$$= 1 + 2y + 3z^2$$

(#) Curl of vector field:-

Let, $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$

be a vector. Then Curl of $\vec{F}$ is denoted by Curl $\vec{F}$ or $\nabla \times \vec{F}$ and defined as,

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

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Find the curl of $\vec{F} = (x^2\vec{i} + y^3\vec{j} + z^4\vec{k})$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^3 & z^4 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial z^4}{\partial y} - \frac{\partial y^3}{\partial z}\right) - \vec{j}\left(\frac{\partial z^4}{\partial x} - \frac{\partial x^2}{\partial z}\right) + \vec{k}\left(\frac{\partial y^3}{\partial x} - \frac{\partial x^2}{\partial y}\right)$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0\cdot\vec{i} - 0\cdot\vec{j} + 0\cdot\vec{k}$$

If $\text{div } \vec{v} = 0$ Then the vector $\vec{v}$ is said to be solenoidal.

If $\text{curl } \vec{v} = 0$, Then the vector $\vec{v}$ is said to be irrotational.

Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is both solenoidal and Irrotational.

For Solenoidal,

Curl div $\vec{F} = 0$

$$\text{div}\,\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \vec{F}$$

$$= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy)$$

$$+ \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2$$

$$\text{div}\,\vec{F} = 0$$

$\therefore$ $\vec{F}$ is the solenoidal vector.

For Irrotational,

Curl $\vec{F} = 0$

$$\text{Curl}\,\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \vec{i}(\cancel{3x} - \cancel{3x}) - \vec{j}(3y - \cancel{2z} + \cancel{2z} - 3y) + \vec{k}(\cancel{3z} + \cancel{2y} - \cancel{2y} - \cancel{3z})$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$= 0$$

$\therefore$ This show that $\vec{F}$ is Irrotational.

Q. Find Directional Derivative of $\vec{F} = x^2y^2z^2$ at $(1,1,-1)$ in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.

Sol^n:

$F = x^2y^2z^2$ at $(1,1,-1)$

Let, $\vec{r}$ be posn vector

$\vec{r} = e^t\,\vec{i} + (\sin 2t + 1)\,\vec{j} + (1 - \cos t)\,\vec{k}$

at $t = 0$,

$\therefore \vec{a} = \vec{i} + \vec{j} + 0\vec{k}$

$$D_{\vec{a}} F = \text{grad } F \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla F = \left(\vec{i}\,\frac{\partial F}{\partial x} + \vec{j}\,\frac{\partial F}{\partial y} + \vec{k}\,\frac{\partial F}{\partial z}\right)$$

$$= 2xy^2z^2\,\vec{i} + 2x^2yz^2\,\vec{j} + 2x^2y^2z\,\vec{k}$$

at $(1,1,-1)$,

$$\nabla F = 2\times 1\times(1)^2\times(-1)^2\,\vec{i} + 2\times 1\times 1\times(-1)^2\,\vec{j} + 2\times 1\times 1\times(-1)\,\vec{k}$$

$$\nabla F = 2\vec{i} + 2\vec{j} - 2\vec{k}$$

$$D_{\vec{a}} F = \nabla F \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= 2\vec{i} + 2\vec{j} - 2\vec{k} \cdot \frac{(\vec{i} + \vec{j} + 0\vec{k})}{\sqrt{1^2 + 1^2 + 0^2}}$$

$$= \frac{2 + 2 - 2\times 0}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}$$

$$= \frac{2\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}}$$

$$D_{\vec{a}} F = 2\sqrt{2}$$

$$\frac{d\vec{r}}{dt} = e^t \cdot \vec{i} + \cos 2t \cdot 2\vec{j} + \sin t\, \vec{k}$$

At $t=0$,

$$\text{Tangent} = \frac{d\vec{r}}{dt} = \vec{i} + 2\vec{j} + 0\vec{k}$$

$\therefore$ Let, $\vec{a} = \vec{i} + 2\vec{j} + 0 \cdot \vec{k}$

Then, Directional Derivative is,

$$D_{\vec{a}} F = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (2\vec{i} + 2\vec{j} - 2\vec{k}) \cdot \frac{\vec{i} + 2\vec{j} + 0\vec{k}}{\sqrt{1^2+2^2+0^2}}$$

$$= \frac{2+4-0}{\sqrt{5}}$$

$$\therefore D_{\vec{a}} f = \frac{6}{\sqrt{5}}$$

Q. Find the angle between the surface $xy^2z - 3x - z^2 = 0$ and $3x^2 + 2z = 1 + y^2$ at $(1,-2,1)$.

Sol^n:

Let, $\phi_1 = xy^2z - 3x - z^2$ — (1), $\phi_2 = 3x^2 + 2z - 1 - y^2$ — (2)

Then, normal vector to (1) is,

$$\nabla \phi_1 = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (xy^2z - 3x - z^2)$$

$$= (y^2z - 3)\vec{i} + 2xyz\,\vec{j} + (xy^2 - 2z) \cdot \vec{k}$$

at $(1,-2,1)$

$$\nabla \phi_1 = \vec{i} - 4\vec{j} + 2\vec{k}$$

$\therefore$ Let, $\vec{n}_1 = \nabla \phi_1 = \vec{i} - 4\vec{j} + 2\vec{k}$

The normal to ② is,

$$\nabla \phi_2 = 6x\,\vec{i} - 2y\,\vec{j} + 2\vec{k}$$

at $(1,-2,1)$

$$\therefore \nabla \phi_2 = 6\vec{i} + 4\vec{j} + 2\vec{k}$$

$$\therefore \vec{n_2} = \nabla \phi_2 = 6\vec{i} + 4\vec{j} + 2\vec{k}$$

Let, θ be the angle between surfaces

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

$$= \frac{(\vec{i} - 4\vec{j} + 2\vec{k}) \cdot (6\vec{i} + 4\vec{j} + 2\vec{k})}{\sqrt{1^2+4^2+2^2} \cdot \sqrt{6^2+4^2+2^2}}$$

$$= \frac{6 - 16 + 4}{\sqrt{21} \cdot \sqrt{56}}$$

$$= \frac{-6}{14\sqrt{6}}$$

or, $\cos\theta = \dfrac{-3}{7\sqrt{6}}$

or, $\theta = \cos^{-1}\left(\dfrac{-3}{7\sqrt{6}}\right)$

$\therefore \theta$

(#) Vector Integral Calculus :

Line integral : $\int_C \vec{F} \cdot d\vec{r}$

Ex. 4.3

① Evaluate $\int_C \vec{F} \cdot d\vec{r}$

ⓐ $\vec{F} = (xy, x^2y^2)$, C : AB where, coordinates A(2,0) & B(0,2)

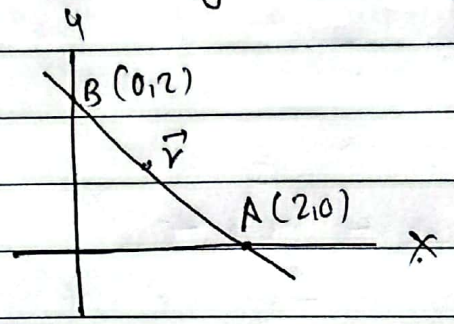
Sol^n :

Eq^n of line joining A(2,0) & B(0,2),

$$y - 0 = \frac{2-0}{0-2}(x-2)$$

or, $y = -x + 2$

$\therefore C : x + y = 2$



Let, $\vec{r}$ be the position vector on C.

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$\vec{F} = xy\,\vec{i} + x^2y^2\,\vec{j}$

$\vec{F} \cdot d\vec{r} = xy \cdot dx + x^2y^2\,dy$

Here, $y = 2 - x$

or, $\frac{dy}{dx} = -1$

$\therefore dy = -dx$

Thus, $\vec{F}\cdot d\vec{r} = x(2-x)\,dx + x^2(2-x)^2(-dx)$

$$\int_C \vec{F}\cdot d\vec{r} = \int_2^0 x(2-x)\cdot dx - x^2(4-4x+x^2)\cdot dx$$

$$= \int_2^0 (2x-x^2)\cdot dx - \int_2^0 (4x^2-4x^3+x^4)\cdot dx$$

$$= \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_2^0 - \left[\frac{4x^3}{3} - \frac{4x^4}{4} + \frac{x^5}{5}\right]_2^0$$

$$= \left[(0-0) - \left(4 - \frac{8}{3}\right)\right] - \left[(0-0+0) - \left(\frac{4\cdot 8}{3} - 16 + \frac{32}{5}\right)\right]$$

$$= \frac{8}{3} - 4 + \frac{32}{3} - 16 + \frac{32}{5}$$

$$= \frac{40}{3} - 20 + \frac{32}{5}$$

$$= \frac{40-60}{3} + \frac{32}{5}$$

$$= \frac{32}{5} - \frac{20}{3}$$

$$= \frac{96-100}{15}$$

$$= -\frac{4}{15}$$

(d) $\vec{F} = (2z, x, -y)$ $C: \vec{r} = (\cos t, \sin t, 2t)$ from $(0,0,0)$ to $(1, 0, 4\pi)$

Sol^n:

Let, $\vec{r}$ be the position vector

$$\vec{r} = \cos t\, \vec{i} + \sin t\, \vec{j} + 2t\, \vec{k}$$

Here, $x = \cos t$, $y = \sin t$, $z = 2t$

$$\frac{d\vec{r}}{dt} = -\sin t\, \vec{i} + \cos t\, \vec{j} + 2\, \vec{k}$$

$$\therefore d\vec{r} = (-\sin t\, \vec{i} + \cos t\, \vec{j} + 2\vec{k})\, dt$$

$$\therefore \vec{F} = 2 \cdot 2t\, \vec{i} + \cos t\, \vec{j} - \sin t\, \vec{k}$$

$$\vec{F} \cdot d\vec{r} = (-4t \sin t + \cos^2 t - 2\sin t)\, dt$$

for the line integral of t, $z = 2t$

When $z = 0$, $t = 0$
When $z = 4\pi$, $t = 2\pi$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-4t \sin t + \cos^2 t - 2\sin t)\, dt$$

$$= -4 \int_0^{2\pi} t \sin t\, dt + \int_0^{2\pi} \frac{1 + \cos 2t}{2}\, dt - 2 \int_0^{2\pi} \sin t\, dt$$

$$= -4 \left[t(-\cos t) - 1(-\sin t) \right]_0^{2\pi} + \frac{1}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$- 2 \left[-\cos t \right]_0^{2\pi}$$

$$= 4 \left[t \cos t - \sin t \right]_0^{2\pi} + \frac{1}{4} \left[2t + \sin 2t \right]_0^{2\pi} + 2 \left[\cos t \right]_0^{2\pi}$$

$$= 4\left[2\pi \cdot 1 - 0 - (0-0)\right] + \frac{1}{4}\left[4\pi + 0 - 0 - 0\right] + 2\left[1 - 0\right]$$

$$= 8\pi + \pi + 0$$

$$= 9\pi$$

(2) $\vec{F} = (2xy - z)\vec{i} + yz\vec{j} + x\vec{k}$, $C: (t, 2t, t^2 - 1)$ from $t=0$ to $t=1$

Soln)

Here, $C : (t, 2t, t^2 - 1)$

i.e $x = t$, $y = 2t$, $z = t^2 - 1$

Let, $\vec{r}$ be the posn vector of :

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

or, $\vec{r} = t\vec{i} + 2t\vec{j} + (t^2 - 1)\vec{k}$

or, $\frac{d\vec{r}}{dt} = \vec{i} + 2\vec{j} + 2t\vec{k}$

$\therefore d\vec{r} = (\vec{i} + 2\vec{j} + 2t\vec{k})\,dt$

$$\vec{F} = (2 \cdot t \cdot 2t - t^2 + 1)\vec{i} + (2t \cdot (t^2 - 1))\vec{j} + t\vec{k}$$

$$= (4t^2 - t^2 + 1)\vec{i} + (2t^3 - 2t)\vec{j} + t\vec{k}$$

$$\therefore \vec{F} = (3t^2 + 1)\vec{i} + (2t^3 - 2t)\vec{j} + t\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (3t^2 + 1) + 4t^3 - 4t + 2t^2)\,dt$$

$$\therefore \vec{F} \cdot d\vec{r} = (5t^2 + 4t^3 - 4t + 1)\,dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (5t^2 + 4t^3 - 4t + 1) \cdot dt$$

$$= \left[\frac{5t^3}{3} + \frac{\cancel{4}t^4}{\cancel{4}} - \frac{\cancel{4}^{2}t^2}{\cancel{2}} + t\right]_0^1$$

$$= \frac{5}{3} + 1 - 2 + 1$$

$$= \frac{5}{3}$$

(h) $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ $C: x^2 = 4y$, $3x^3 = 8z$

from $0 \le x \le 2$

Soln:

Let, $\vec{r}$ be the position vector

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = 3x^2\cdot dx + (2xz - y)\,dy + z\cdot dz \quad \text{--- (1)}$$

Here, $x^2 = 4y$

or, $2x = 4\cdot\dfrac{dy}{dx}$

$\therefore dy = \dfrac{1}{2}x\cdot dx$

Also, $3x^3 = 8z$

$3\times 3x^2 = 8\dfrac{dz}{dx}$

$\therefore dz = \dfrac{9x^2}{8}\cdot dx$

from (1),

$$\vec{F}\cdot d\vec{r} = 3x^2\cdot dx + \left(2x\cdot\frac{3}{8}x^3 - \frac{x^2}{4}\right)\cdot\frac{1}{2}x\cdot dx + \frac{3}{8}x^3\cdot\frac{9x^2}{8}\cdot dx$$

$$= \left[3x^2 + \left(\frac{3}{4}x^4 - \frac{x^2}{4}\right)\cdot\frac{1}{2}x + \frac{27}{64}\cdot x^5\right]\cdot dx$$

$$= \left[3x^2 + \frac{3}{8}x^5 - \frac{x^3}{8} + \frac{27}{64}x^5\right]dx$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^2\left(3x^2 + \frac{51}{64}x^5 - \frac{x^3}{8}\right)\cdot dx$$

$$= \left[\frac{\cancel{3}x^3}{\cancel{3}} + \frac{51}{64}\cdot\frac{x^6}{6} - \frac{x^4}{8\times 4}\right]_0^2$$

$$= \left[8 + \frac{51}{\cancel{64}}\times\frac{\cancel{64}}{6} - \frac{16}{8\times 4} - 0\right]$$

$$= 8 + 51 - 1$$

$$= \frac{15}{2} + \frac{51}{6}$$

$$= \frac{1}{2}\left(15 + \frac{51}{3}\right)$$

$$\cancel{= \frac{1}{2}\left(\frac{45+51}{3}\right)}$$

$$= \frac{1}{2}\left(15 + 17\right)$$

$$= \frac{1}{2} \times 32$$

$$= 16$$

<u>Example 26</u>: Calculate $\int_C \left[(x^2+y^2)\vec{i} - 2x\vec{j}\right] \cdot d\vec{r}$, where

C is a rectangle on xy-plane bounded by $y=0$, $y=b$, $x=0$, $x=a$.

Q. Calculate $\int_C f\,ds$ for

(a) $f = x^2 y$ $\quad C: \vec{r} = (2\cos t, 2\sin t), (0 \le t \le \frac{\pi}{2})$

Soln:

$\vec{r} = (2\cos t, 2\sin t)$

Here, $x = 2\cos t$, $y = 2\sin t$

$\vec{r} = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$

$\frac{d\vec{r}}{dt} = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$

Now,

$$\int_C f\,ds = \int_0 f \frac{ds}{dt}\,dt \quad \text{--- (1)}$$

where, $\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}$

or, $\frac{ds}{dt} = \sqrt{(-2\sin t\,\vec{i} + 2\cos t\,\vec{j}) \cdot (-2\sin t\,\vec{i} + 2\cos t\,\vec{j})}$

$= \sqrt{4\sin^2 t + 4\cos^2 t}$

$\therefore \frac{ds}{dt} = 2$

from (1),

$$\int_C f\,ds = \int_C x^2 y \cdot 2\,dt$$

$$= \int_0^{\pi/2} 4\cos^2 t \cdot 2\sin t \cdot 2\,dt$$

$$= 16 \int_0^{\pi/2} \cos^2 t\,(-\sin t)\,dt$$

$$= -16 \left[\frac{\cos^3 t}{3}\right]_0^{\pi/2}$$

$$= -16\left[0 - \frac{1}{3}\right] \Rightarrow \frac{16}{3}$$

Q. Calculate $\int_C f\,ds$ for

(a) $f = x^2 y$ $C: \vec{r} = (2\cos t, 2\sin t)$, $(0 \le t \le \frac{\pi}{2})$

Sol:

$\vec{r} = (2\cos t, 2\sin t)$

Here, $x = 2\cos t$, $y = 2\sin t$

$\vec{r} = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$

$\frac{d\vec{r}}{dt} = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$

Now,

$$\int_C f\,ds = \int_0 f \frac{ds}{dt}\,dt \quad \text{--- (1)}$$

where, $\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}$

or, $\frac{ds}{dt} = \sqrt{(-2\sin t\,\vec{i} + 2\cos t\,\vec{j}) \cdot (-2\sin t\,\vec{i} + 2\cos t\,\vec{j})}$

$= \sqrt{4\sin^2 t + 4\cos^2 t}$

$\therefore \frac{ds}{dt} = 2$

from (1),

$$\int_C f\,ds = \int_C x^2 y \cdot 2\,dt$$

$$= \int_0^{\pi/2} 4\cos^2 t \cdot 2\sin t \cdot 2\,dt$$

$$= 16 \int_0^{\pi/2} \cos^2 t\,(-\sin t)\,dt$$

$$= -16 \left[\frac{\cos^3 t}{3}\right]_0^{\pi/2}$$

$$= -16\left[0 - \frac{1}{3}\right] \Rightarrow \underline{16}$$

1. Prove that the given vector are conservative (Irrotational) and find scalar function ϕ such $\vec{F} = \nabla\phi$

(a) $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

Sol^n:-

If $\text{Curl}\,\vec{F} = 0$. Then $\vec{F}$ is said to be conservative.

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z)$$

$$= 0$$

$\therefore \vec{F}$ is conservative.

$$\vec{F} = \nabla\phi$$

or, $$yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Equating

$$\frac{\partial\phi}{\partial x} = yz, \quad \frac{\partial\phi}{\partial y} = xz, \quad \frac{\partial\phi}{\partial z} = xy$$

We know that,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\therefore d\phi = yz\,dx + xz\,dy + xy\,dz$$

Integrating

$$\int d\phi = \int(yz\,dx + xz\,dy + xy\,dz)$$

or, $$\phi = \int d(xyz)$$

Exactness Condition:

Let an Integral $\int_C F_1 dx + F_2 dy + F_3 dz$

is Integrable. Then, we called the value under integral sign is Exact if,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

If the Integrand Exact. Then its solution,

$$= \int F_1 dx + \int (\text{term of } F_2 \text{ free from } x)\, dy$$

$$+ \int (\text{term of } F_3 \text{ free from } x \,\&\, y)\, dz$$

Ex 4.4

Show that the value under the integral sign in Exact and Evaluate.

(a) $$\int_{(0,1)}^{(2,5)} (2x + y^3)\, dx + (3xy^2 + 4)\, dy$$

Comparing ~~Integrant~~ Integrand with,

$F_1 dx + F_2 dy + F_3 dz$

$\therefore F_1 = 2x + y^3 \qquad F_2 = 3xy^2 + 4$

$\therefore$

$$\frac{\partial F_1}{\partial y} = 3y^2 \qquad \frac{\partial F_2}{\partial x} = y^2$$

Here, $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

$\therefore$ The given integrand is Exact.

Thus, $\int_{(0,1)}^{(2,3)} (2x+y^3)\,dx + \int_{(0,1)}^{(2,3)} 4\,dy$

$$= \left[\frac{2x^2}{2} + y^3 x\right]_{(0,1)}^{(2,3)} + \left[4y\right]_{(0,1)}^{(2,3)}$$

$$= \left[(4 + 3^3 \cdot 2) - 0\right] + (4 \cdot 3 - 4)$$

$$= 4 + 54 + 8$$

$$= 66$$

Q Show that value under Integral sign is exact and Evaluate. $(0, \pi, 1)$

$$\int_{(\pi, \frac{\pi}{2}, 2)} \left[-2\sin xz\,dx + \cos y\,dy - x\sin xz\,dz\right]$$

Comparing the integrand with,

$F_1\,dx + F_2\,dy + F_3\,dz$

$F_1 = -2\sin xz \qquad F_2 = \cos y \qquad F_3 = -x\sin xz$

$\frac{\partial F_1}{\partial y} = 0 \qquad \frac{\partial F_1}{\partial z} = -2\cos xz \cdot x - \sin xz$

$\frac{\partial F_2}{\partial x} = 0 \qquad \frac{\partial F_2}{\partial z} = 0$

$\frac{\partial F_3}{\partial x} = -x\cos xz \cdot z - \sin xz \qquad \frac{\partial F_3}{\partial y} = 0$

Here, $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

Thus, the integrand under integral sign is exact.

$$= \int_{(\pi,\frac{\pi}{2},2)}^{(0,\pi,1)} -z\sin xz\,dx + \int_{(\pi,\frac{\pi}{2},2)}^{(0,\pi,1)} \cos y\cdot dy + \int_{(\pi,\frac{\pi}{2},2)}^{(0,\pi,1)} 0\cdot dz$$

$$= \left[(-z)\left(-\frac{\cos xz}{z}\right) \cancel{-(1)\left(\frac{-\sin xz}{x^2}\right)}\right]_{(\pi,\frac{\pi}{2},2)}^{(0,\pi,1)} + \left[\sin y\right]_{(\pi,\frac{\pi}{2},2)}^{(0,\pi,1)}$$

$$= \left[\frac{z\cdot\cos xz}{x} - \frac{\sin xz}{x^2}\right]_{(\pi,\frac{\pi}{2},2)}^{(0,\pi,1)} + \left(\sin\pi - \sin\frac{\pi}{2}\right)$$

$$= (\cos 0 - \cos 2\pi) + (0 - 1)$$

$$= 1 - 1 - 1$$

$$= -1$$ Ans

(#) <u>Green's Theorem (Transformation between double integrals & line integral)</u> :-

Let, R be a closed bounded region in the xy-plane bounded by curve C. Let, $F_1(x,y)$ and $F_2(x,y)$ be two function that are continuous and their first order partial derivative are also continuous on R. Then,

$$\oint (F_1\,dx + F_2\,dy) = \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Date
Page

(*) Using Green's Theorem, Evaluate the integral:

(a) $\oint 2xy\,dx + (e^x + x^2)\,dy$

C : Triangle joining (0,0), (1,0) and (1,1)

Soln:

By Green's theorem,

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Here, $F_1\,dx + F_2\,dy = 2xy\,dx + (e^x + x^2)\,dy$

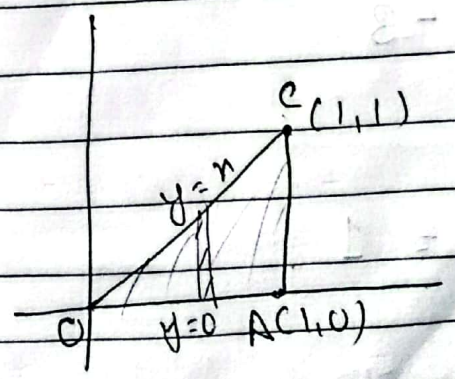
$F_1 = 2xy$ $\qquad$ $F_2 = e^x + x^2$

$\frac{\partial F_1}{\partial y} = 2x$ $\qquad$ $\frac{\partial F_2}{\partial x} = e^x + 2x$

$$\therefore \oint_C 2xy\,dx + (e^x + x^2)\cdot dy = \iint_R (e^x + \cancel{2x} - \cancel{2x})\,dx\,dy$$

$$= \iint_R e^x \cdot dx\,dy$$

C: is the triangle joining the points (0,0), (1,0) & (1,1).



$$\therefore \oint_C 2xy\,dx + (e^x + x^2)\,dy = \int_0^1 \int_0^x e^x \cdot dy\,dx$$

$$= \int_0^1 e^x \cdot \left[y\right]_0^x \cdot dx$$

$$= \int_0^1 e^x \cdot x \cdot dx$$

$$= \left[x \cdot e^x - 1 \cdot e^x\right]_0^1$$

$$= (1 \cdot e^1 - e^1) - (0 - e^0)$$

$$= 1$$

Q. Evaluate $\oint_C (x^3 - 3y)\,dx + (x + \sin y)\,dy$, where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, $(0,2)$

Sol^n:)

By Green's theorem

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Here, $F_1\,dx + F_2\,dy = (x^3 - 3y)\,dx + (x + \sin y)\,dy$

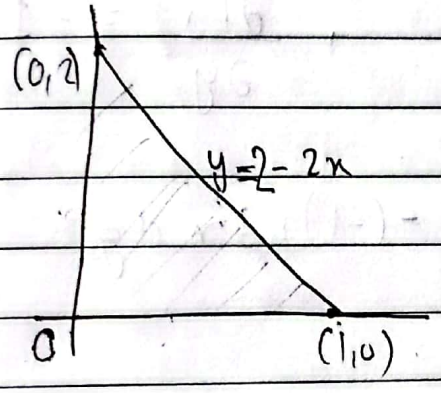
$$\frac{\partial F_1}{\partial y} = \frac{\partial (x^3 - 3y)}{\partial y} = -3$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial y}(x + \sin y) = 1$$

$$\oint_C (x^3-3y)\,dx + (x+\sin y)\,dy = \iint_R (1-(-3))\,dx\,dy$$

$$= \iint_R 4\,dx\,dy$$

C : is the Δ joining $(0,0)$, $(1,0)$, $(0,2)$



$x = \frac{2-y}{2}$

$$\oint_C (x^3-3y)\,dx + (x+\sin y)\,dy = \int_0^1 \int_0^{2-2x} 4 \cdot dy\,dx$$

$$= \int_0^1 \left[4y\right]_0^{2-2x} \cdot dx$$

$$= 4\int_0^1 (2-2x) \cdot dx$$

$$= 4\left[2x - \frac{2x^2}{2}\right]_0^1$$

$$= 4\left((2-1) - (0-0)\right)$$

$$= 4 \times 1$$

$$= 4$$

(#) Application of Green Theorem :-

Let, R be the closed region with boundary C.

Then, by Green's theorem,

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

In particular, we choose $F_2 = x$, $F_1 = -y$ Then,

$$\frac{\partial F_2}{\partial x} = 1 \quad , \quad \frac{\partial F_1}{\partial y} = -1$$

$$\oint_C -y\,dx + x\,dy = \iint_R (1-(-1))\,dx\,dy$$

$$= \iint_R 2\,dx\,dy$$

$$\therefore 2\iint_R dx\,dy = \oint_C x\,dy - y\,dx$$

$$\iint_R dx\,dy = \frac{1}{2}\oint x\,dy - y\,dx$$

$$\boxed{\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

This formula expressed Area of closed region R in term of the line Integral.

We note that the choice of F_1 & F_2 ~~are~~ is not unique.

Again, $x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$\frac{dx}{d\theta} = -r\sin\theta$ $\qquad$ or, $\frac{dy}{d\theta} = r\cos\theta$

$\therefore dx = -r\sin\theta \cdot d\theta$ $\qquad$ $\therefore dy = r\cos\theta \cdot d\theta$

$$x\,dy - y\,dx = r\cos\theta \cdot r\cos\theta\, d\theta + r\sin\theta\, r\sin\theta \cdot d\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta)\,d\theta$$
$$= r^2\, d\theta$$

So, Area formula in polar coordinate becomes,

$$A = \frac{1}{2}\oint_C r^2 \cdot d\theta$$

② $\oint \vec{F} \cdot d\vec{r}$

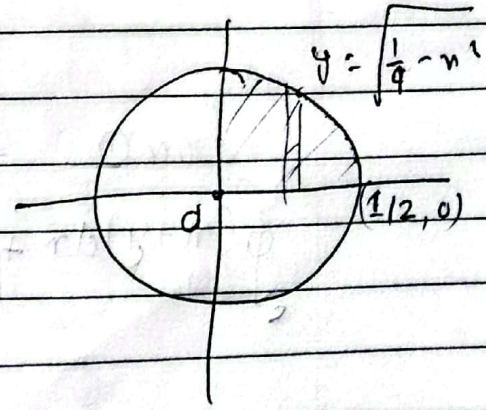
$\vec{F} = (y, -x)$ $\qquad$ $C: 4x^2 + 4y^2 = 1$

$x^2 + y^2 = \left(\frac{1}{2}\right)^2$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$\vec{F} \cdot d\vec{r} = y\,dx - x\,dy$

$$\oint_C y\,dx - x\,dy = \iint_R (-1-1)\,dx\,dy$$



$$= 4 \times -2 \iint_R dx\,dy$$

$$= -2 \int_0^{1/2} \int_0^{\sqrt{1/4 - x^2}} dy\,dx$$

(#) ~~Surface Integral~~ :-

Using Green's theorem:

(e) $\oint (x+y)\,dx + (y+x^2)\,dy$. Area between the circle $x^2+y^2=1$ & $x^2+y^2=4$.

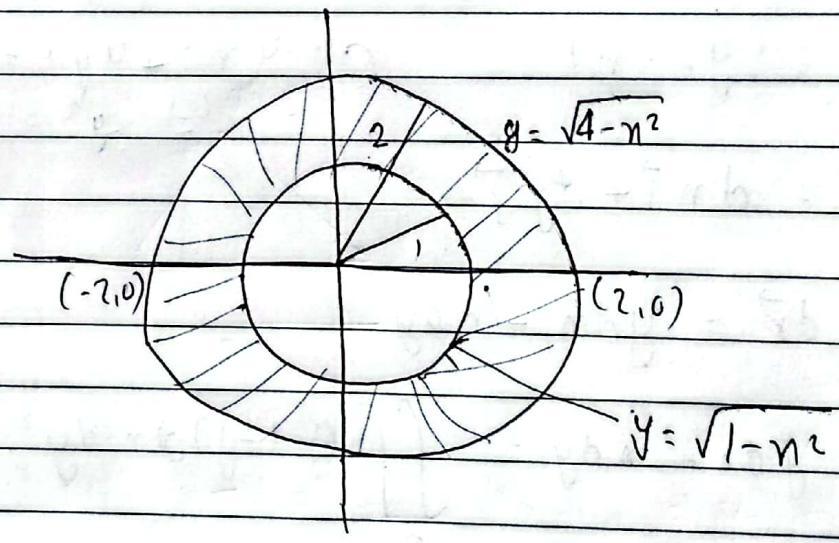
Sol^n ⇒

By Green's theorem,

$$\oint F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \text{— (1)}$$

where, $F_1 = x+y$ & $F_2 = y+x^2$

$\frac{\partial F_1}{\partial y} = 1$ $\qquad$ $\frac{\partial F_2}{\partial x} = 2x$



∴ from (1),

$$\oint_C (x+y)\,dx + (y+x^2)\,dx = \iint_R (2x-1)\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-1)\,dy\,dx - \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x-1)\,dy\,dx$$

Page

Changing polar coordinate

$x = r\cos\theta$, $y = r\sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{2} (2r\cos\theta - 1)\, r\, dr\, d\theta - \int_0^{2\pi}\int_0^{1} (2r\cos\theta - 1)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{2r^3}{3}\cos\theta - \frac{r^2}{2}\right]_0^2 d\theta - \int_0^{2\pi} \left[\frac{2r^3}{3}\cos\theta - \frac{r^2}{2}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{16}{3}\cos\theta - 2 - (0-0)\right) d\theta - \int_0^{2\pi} \left(\frac{2}{3}\cos\theta - \frac{1}{2} - (0-0)\right) d\theta$$

$$= \left[\frac{16}{3}\sin\theta - 2\theta\right]_0^{2\pi} - \left[\frac{2}{3}\sin\theta - \frac{\theta}{2}\right]_0^{2\pi}$$

$$= \left(0 - 4\pi - (0-0)\right) - \left(0 - \pi - (0-0)\right)$$

$$= -4\pi + \pi$$

$$= -3\pi$$

(h) $\int_C \sqrt{y}\,dx + \sqrt{x}\,dy$ C : triangle of vertices $(1,1)$, $(3,1)$ and $(2,2)$.

By Green theorem,

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \text{—(1)}$$

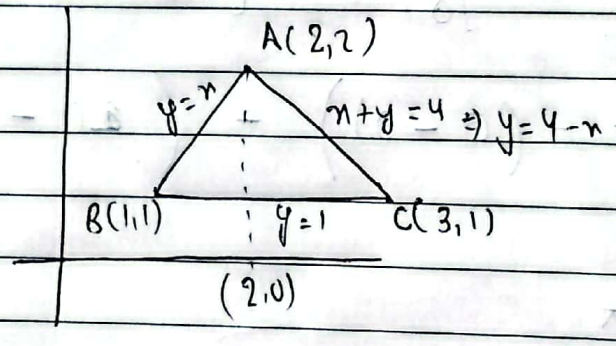
Here,

$F_1 = \sqrt{y}$ $\qquad$ $F_2 = \sqrt{x}$

$\frac{\partial F_1}{\partial y} = \frac{1}{2\sqrt{y}}$ $\qquad$ $\frac{\partial F_2}{\partial x} = \frac{1}{2\sqrt{x}}$

From, $\int_C \sqrt{y}\,dx + \sqrt{x}\,dy = \iint_R \left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{y}}\right) dx\,dy$

Here, R : is the triangle having vertices $(1,1)$, $(3,1)$ & $(2,2)$



Eqⁿ of line AB : $y-1 = \frac{2-1}{(2-1)}(x-1) \Rightarrow y-1 = x-1 \Rightarrow y = x$

Eqⁿ of line AC : $y-2 = \frac{1-2}{3-2}(x-2) \Rightarrow y-2 = -x+2 \Rightarrow x+y=4$

Eqⁿ of line BC : $y-1 = \frac{1-1}{3-1}(x-1) \Rightarrow y-1 = 0 \Rightarrow y = 1$

$$\int_C \sqrt{y}\,dx + \sqrt{x}\,dy = \int_1^2 \int_1^x \left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{y}}\right) dy\,dx + \int_2^3 \int_1^{4-x} \left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{y}}\right) dy\,dx$$

$$= \int_1^2 \left[\frac{y}{2\sqrt{x}} - \frac{\sqrt{y}}{2\times 1/2}\right]_1^x dx + \int_2^3 \left[\frac{y}{2\sqrt{x}} - \frac{\sqrt{y}}{2\times \frac{1}{2}}\right]_1^{4-x} dx$$

$$= \int_1^2 \left[\left(\frac{x}{2\sqrt{x}} - \sqrt{x}\right) - \left(\frac{1}{2\sqrt{x}} - 1\right)\right] dx + \int_2^3 \left[\left(\frac{4-x}{2\sqrt{x}} - \sqrt{4-x}\right) - \left(\frac{1}{2\sqrt{x}} - 1\right)\right] dx$$

$$= \int_1^2 \left[\frac{x - 2x}{2\sqrt{x}} - \left(\frac{1 - 2\sqrt{x}}{2\sqrt{x}}\right)\right] dx + \int_2^3 \left(\frac{4 - x - 2\sqrt{x}(4-x)}{2\sqrt{x}} - \left(\frac{1-2\sqrt{x}}{2\sqrt{x}}\right)\right) dx$$

$$= \int_1^2 \left(\frac{\sqrt{x}}{2} - \sqrt{x} - \frac{1}{2\sqrt{x}} + 1\right) dx + \int_2^3 \left(\frac{4}{2\sqrt{x}} - \frac{\sqrt{x}}{2} - \sqrt{4-x}\right) dx$$

$$= \left[\frac{x^{3/2}}{3} - \frac{x^{3/2}}{3/2} - \frac{\sqrt{x}}{1/2 \times 2} + x\right]_1^2 + \left[\frac{2\sqrt{x}}{1/2} - \frac{x^{3/2}}{3} - \frac{(4-x)^{3/2}}{3/2}\right]_2^3$$

$$= \frac{2^{3/2}}{3} - \frac{2^{3/2}}{3/2} - \sqrt{2} + 2 - \frac{1}{3} + \frac{2}{3} + 1 - 1 + \left[4\sqrt{3} - \frac{3^{3/2}}{3} -\right.$$

$$\Rightarrow \frac{2^{3/2}}{3} - \frac{2 \cdot 2^{3/2}}{3} - \sqrt{2} + 2 + \frac{1}{3}$$

$$\Rightarrow 2^{3/2}\left(\frac{1}{3} - \frac{2}{3}\right) - \sqrt{2} + 2 + \frac{1}{3}$$

$$\Rightarrow$$

View level
Logical
Physical

Level :- 1) Physical
2) Logical
3) View

Independence
1) Logical level data in
2) Physical " "

Database User : who interact with database
1) Naive - simple user/ wr
2) Sophisticate : who
3) Application programme
4) Specialized user

⊛ Data Base Administrator / DBA / Database Manager :-

Person who have central control over database.

Role of DBA :-

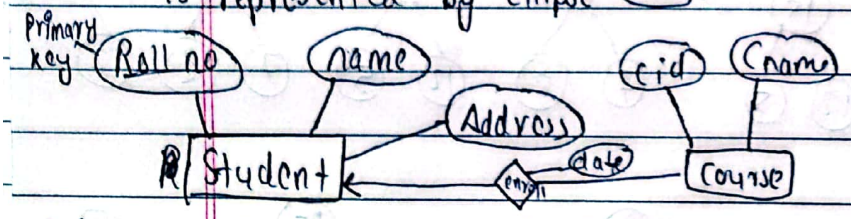
i) Schema Define / Alter
ii) Data Manipulation
iii) Authorization Grant
iv) Routine Maintenance

⊛ E-R Diagram (Entity-Relationship Diagram):

It is a graphical representation of data through entity & its relationship.

Entity: Any object or thing which have physical or logical existence is called entity.
Eg: Student, course - It is represented by Rectangle ▭

Attribute: Property which describe particular entity.
- It is represented by ellipse ⬭.



Relationship: It shows association. It is represented by diamond shape ◇

Cardinality mapping:

1) One - to - one
2) One - to - many
3) Many - to - one
4) Many - to - many