5

CHAPTER

INFINITE SERIES

Introduction

Since the Newton's idea of representing functions as sum of infinite series, it seen as an important mathematical tool in calculus and in applied mathematics.

5.1 Infinite Sequence

A sequence is an ordered list of things. Such thinks may finite or infinite. In a sequence, the terms are separated by commas (,).

Definition (Infinite Sequence)

A sequence is a list of numbers which are written as a definite order:

$u_1, u_2, u_3, \ldots u_n, \ldots$

Here u_1 is the first term, u_2 is the second term u_3 is the third term and likewise u_n is the n^{th} term (is also known as general term) of the series.

For instance, $\{1, 2, 3, 4, \ldots\}$; $\{\sqrt{n}\}$; $\left\{\frac{(-1)^n}{n}\right\}$ are sequences. Normally, the general term of an infinite sequence is noted by u_n (also known as n^{th} term of the sequence). The following table shows the general term of the sequence.

Sequence	General term
$\{1, 2, 3, 4, \ldots\}$	n
$\{\sqrt{n}\}$	$\sqrt{n}$
$2, 4, 6, 8, \ldots$	$2n$
$1, 8, 27, 64, \ldots$	n^3

Definition (Constant Sequence)

A sequence with every term is same fixed value is called a ***constant sequence.*** *For instance, the sequence {2, 2, 2, ... } is a constant sequence.*

Bounded and Unbounded Sequence

- A sequence $\{u_n\}$ is **bounded** if there are two fixed value k, K $\in$ R such that $k \le u_n \le K$, for all n.
- A sequence $\{u_n\}$ is **bounded above** if there is a real value k $\in$ R such that $u_n \le k$, for all n. In such case, the sequence is called **unbounded below.**
- A sequence $\{u_n\}$ is **bounded below** if there is a real value k $\in$ R such that $k \le u_n$, for all n. In such case, the sequence is called **unbounded above.**
- A sequence $\{u_n\}$ is called **unbounded** if it is no bounded above and bounded below.

Example 1: Consider a sequence $u_n = \frac{1}{n}$. Show that the sequence is bounded.

Solution: Given that,

$$u_n = \frac{1}{n} \qquad \text{for all } n \in N \text{ (set of natural numbers)}$$

Therefore,

$$\{u_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}.$$

This shows that $u_n \le 1$ for all n.

And,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

Thus,

$$0 < u_n \le 1 \qquad \text{for all n.}$$

This shows that $\{u_n\}$ is bounded below by 0 and above by 1. Therefore, the sequence is bounded.

Example 2: Show that the sequence $u_n = 2^n$ is bounded below but is unbounded above.

Solution: Given that,

$$u_n = 2^n \qquad \text{for all n.}$$

Here,

$$\{u_n\} = \{2, 2^2, 2^3, 2^4, \dots\}.$$

This means, the sequence has least value 2 and is increasing.

And,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (2^n) = \infty.$$

This means it has no fixed greatest value.

Therefore, the sequence $\{u_n\}$ is bounded below by 2 but not bounded above.

Convergence and Divergence of Infinite Sequence

In above example–1, we observe that the sequence approaches to a single value 0 as n tends to infinity. Such sequence is known as a **convergent sequence.** But in example–2, the sequence has no fixed value as n tends to infinity, such sequence is a **divergent sequence.**

Definition:

Let $\{u_n\}$ be an infinite sequence. The sequence $\{u_n\}$ converges to a number L if for every positive number $\varepsilon > 0$ there corresponds an integer N such that

$$|u_n - L| < \varepsilon \qquad \text{for } n > N.$$

Mathematically, the sequence $\{u_n\}$ **converges** to a number L if $\forall\ \varepsilon > 0\ \exists$ N such that

$$|u_n - L| < \varepsilon \qquad \forall\ n > N.$$

In such condition, we may observe

$$\lim_{n\to\infty} u_n = L.$$

Here, the number L is called the ***limit*** of the sequence $\{u_n\}$.

Definition:

An infinite sequence $\{u_n\}$ of real numbers is called ***divergent*** *if $\lim_{n\to\infty} u_n$ has no fixed finite value.*

Example 3: Show that the sequence $\left\{1 + \frac{1}{n}\right\}$ converges to 1.

Solution: Let $\varepsilon > 0$ be given choose $N = \frac{1}{e}$.

Now, $\forall\ n > N = \frac{1}{e}$,

$$\left|\left(1 + \frac{1}{n}\right) - 1\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < N = \frac{1}{\varepsilon} \qquad \left[\because n > N. \text{ So } \frac{1}{n} < N\right]$$

This means the sequence $\left\{1 + \frac{1}{n}\right\}$ converges to 1.

Alternatively,

Here,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1+0=1.$$

This means the sequence $\left\{1+\frac{1}{n}\right\}$ converges to 1.

Example 4: Prove that the sequence $\{u_n\}=\{(-1)^n\}$ diverges.

Solution:

Let $\varepsilon>0$ be given choose $N=\varepsilon=1$. Take $L=1$.

Now, $\forall\, n>N=1$ with n is even,

$$|u_n-L|=|(-1)^n-1|=|1-1|=0<\varepsilon$$

But if n is odd,

$$|u_n-L|=|(-1)^n-1|=|-1-1|=|-2|=2>\varepsilon$$

This shows, the sequence $|u_n|$ has no fixed single value L such that $|u_n-L|<\varepsilon$ for $\varepsilon>0$. Therefore, the sequence is divergent.

Theorem: If $\lim_{n\to\infty} u_n=L$ and the function f is continuous at L then $\lim_{n\to\infty} f(u_n)=f(L)$.

Example 5: Find $\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right)$.

Solution:

We know the sine function is continuous on R.

Here,

$$\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right)=\sin\left(\lim_{n\to\infty}\left(\frac{\pi}{n}\right)\right)=\sin 0=0.$$

Definition (Non-Decreasing Sequence)

A sequence $\{u_n\}$ with the property $u_n\le u_{n+1}$ for all n, is called a non-decreasing sequence.

For instance the following sequences are non-decreasing.

(i) $1, 2, 3, \ldots, n, \ldots$ (ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$

Note: Above sequence given in (i) and (ii) both are bounded below but not bounded above.

Example 6: Show that the sequence $\left\{\frac{n}{4n-1}\right\}$ is non-increasing.

Solution:

Let,

$$u_n=\frac{n}{4n-1}$$

Here,

$$u_{n+1}-u_n=\frac{n+1}{4(n+1)-1}-\frac{n}{4n-1}$$
$$=\frac{n+1}{4n+3}-\frac{n}{4n-1}$$
$$=\frac{(n+1)(4n-3)-n(4n+3)}{(4n+3)(4n-1)}$$
$$=\frac{4n^2-n+4n-1-4n^2-3n}{(4n+3)(4n-1)}$$
$$=\frac{-1}{(4n+3)(4n-1)}<0$$
$$\Rightarrow u_{n+1}<u_n \quad \text{for any n.}$$

This means the given sequence is non-increasing.

Example 7: The sequence $\frac{3}{n+5}$ is decreasing because $\frac{3}{n+5}>\frac{3}{(n+1)+5}=\frac{3}{n+6}$.

This means $u_n>u_{n+1}$ for all $n\ge 1$.

Note: (i) A sequence $\{u_n\}$ is called **non-increasing** if

$$u_n\ge u_{n+1} \quad \text{for all n.}$$

(ii) A sequence $\{u_n\}$ is called **decreasing or strictly decreasing** if

$$u_n>u_{n+1} \quad \text{for all n.}$$

Example 8: Show that the sequence $\left\{\frac{2}{n}\right\}$ is bounded above by 2.

Solution:

The given sequence is

$$2, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \ldots, \frac{2}{n}, \ldots$$

Here,

$$u_{n+1}-u_n=\frac{2}{n+1}-\frac{2}{n}$$

$$= \frac{2n - 2n - 2}{n(n+1)}$$

$$= \frac{-2}{n(n+1)} < 0$$

$$\Rightarrow \quad u_{n+1} < u_n \quad \text{for all n.}$$

This means the given sequence is strictly decreasing sequence. Also,

$$\ldots < u_{n+1} < u_n < \ldots < u_2 < u_1 = 2 \qquad \text{(which is finite).}$$

This shows that the greatest value of the sequence is $u_1 = 2$. So, the given sequence is bounded above by 2.

Note: In above sequence, we may write as,

$$u_n \le 2 \qquad \text{for all n.}$$

But no value less than 2 is a upper bound of the sequence. So, 2 is the least upper bound of the sequence.

Definition (Least Upper Bound)

Let $\{u_n\}$ be an infinite sequence. If there is a value M such that

$$u_n \le M \qquad \textit{for all n} \qquad \ldots\ldots(i)$$

and, if there is no value less than M satisfies the relation (i), then M is called ***least upper bound*** *of the sequence $\{u_n\}$.*

Definition (Greatest Lower Bound)

Let $\{u_n\}$ be an infinite sequence. If there is a value Q such that

$$u_n \ge Q \qquad \textit{for all n} \qquad \ldots\ldots(i)$$

and, if there is no value greater than Q satisfies the relation (i), then Q is called ***greatest lower bound*** *of the sequence $\{u_n\}$.*

Sandwich (Squeeze) Theorem

Statement: Let f, g and h be real functions such that $a_n \le u_n \le b_n$ for all x in the common domain of definition. Then,

$$a_n \to \ell \text{ and } b_n \to \ell \text{ then } u_n \to \ell$$

Example 9:

(a) The sequence 1, 2, 3, ... , n, ... has no upper bound but it has 1 as a greatest lower bound.

(b) The sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ has upper bound any value among 1, 2, 3, ... But the least upper bound of the sequence is 1.

Theorem (Non-Decreasing Sequence Theorem)

A non-decreasing sequence of real numbers converges if and only if it is bounded from above. If a non-decreasing sequence converges, it converges to its least upper bound.

Exercise 5.1

1. List the first five terms of the sequence.

a. $u_n = \frac{2n}{n^2+1}$ b. $u_n = \frac{3^n}{1+2^n}$

c. $\{2, 4, 6, \ldots, (2n)\}$ d. $u_1 = 1, u_{n+1} = 5u_n - 3$

e. $u_1 = 6, u_{n+1} = \frac{u_n}{n}$

2. Find the formula for the general term u_n of the sequence, assuming that the pattern of the first five terms continues.

a. $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right\}$ b. $\left\{1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots\right\}$

c. $\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, \frac{-16}{27}, \ldots\right\}$ d. $\{5, 8, 11, 14, 17, \ldots\ldots\}$

e. $\left\{\frac{1}{2}, \frac{-4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\right\}$

3. Determine whether the sequence converges or diverges. If it converges, find the limit.

a. $u_n = 1 - (0.2)^n$ b. $u_n = \frac{3+5n^2}{n+n^2}$

c. $u_n = \frac{n^3}{n+1}$ d. $u_n = e^{1/n}$ e. $u_n = \frac{3^{n+2}}{5^n}$

f. $u_n = \tan\left(\frac{2np}{1+8n}\right)$ g. $u_n = \sqrt{\frac{n+1}{9n+1}}$

h. $u_n = \frac{n^2}{\sqrt{n^3+4n}}$ i. $u_n = \frac{(2n-1)!}{(2n+1)!}$

j. $u_n = ln(n+1) - ln(n)$

4. Determine if the sequence is non–decreasing and if it is bounded above or below.

(a) $u_n = \dfrac{3n+1}{n+1}$ (b) $u_n = \dfrac{2^n 3^n}{n!}$

5. Determine whether the sequence is increasing, decreasing or not monotonic. Is the sequence bounded?

a. $u_n = (-2)^{n+1}$ b. $u_n = \dfrac{1}{2n+3}$ c. $u_n = \dfrac{2n-3}{3n+4}$

d. $u_n = ne^{-n}$ e. $u_n = n + \dfrac{1}{n}$

Answers

1. (a) $1, \frac{4}{5}, \frac{3}{5}, \frac{8}{17}, \frac{5}{13}$ (b) $1, \frac{9}{5}, 3, \frac{81}{27}, \frac{81}{11}$ (c) 2, 24, 720, 40320, 3628800

(d) 1, 2, 7, 32, 157 (e) 6, 6, 3, 1, $\frac{1}{4}$

2. (a) $u_n = \dfrac{1}{2n-1}$ (b) $u_n = (-1)^{n+1}\dfrac{1}{3^{n-1}}$ (c) $u_n = -3\left(-\dfrac{2}{3}\right)^{n-1}$

(d) $u_n = 2 + 3n$ (e) $u_n = (-1)^{n+1}\dfrac{n^2}{n+1}$

3. (a) 1 (b) 5 (c) diverges

(d) 1 (e) 0 (f) 1

(g) $\frac{1}{3}$ (h) diverges (i) 0

(j) 0

4. (a) non–decreasing, bounded

(b) not non–decreasing, bounded

5. (a) not monotonic, not bounded

(b) decreasing, bounded

(c) increasing, bounded

(d) decreasing, bounded

(e) increasing, not bounded

5.2 INFINITE SERIES

The summing form of infinite sequence as an infinite series.

Definition (Infinite Series)

Given a sequence of numbers $\{u_n\}$, an expression

$$u_1 + u_2 + \dots + u_n + \dots = \sum_{n=1}^{\infty} u_n$$

is called an ***infinite series.***

The number u_n is known as n^{th} term (or general term) of the series.

If we take only the finite number of terms from the infinite series, then series of such finite terms is called **partial sum** of the series.

Definition (Partial Sum)

Given a series with finite k–terms in the form

$$\sum_{n=1}^{k} u_n$$

is called k^{th} ***partial sum of the series*** *$\sum_{n=1}^{\infty} u_n$ and denoted by S_k i.e. $S_k = \sum_{i=1}^{k} u_i$*

Definition (Convergence and Divergence of an Infinite Series)

Let $\sum_{n=1}^{\infty} u_n$ be an infinite series. If there is a finite value L such that

$$\lim_{n\to\infty} \sum_{i=1}^{n} u_i = L \text{ i.e. } \lim_{n\to\infty} S_n = L.$$

then we say the given series is convergent to L.

Otherwise, the series is divergent.

Example 10: Examine the convergency of the series whose sum of first n–terms is $\dfrac{2n}{3n+5}$.

Solution:

Suppose the sum of first n–terms of the infinite series is $\dfrac{2n}{3n+5}$. Therefore,

$$S_n = u_1 + u_2 + \dots + u_n = \frac{2n}{3n+5}.$$

Then

$$\sum_{n=1}^{\infty} u_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{2n}{3n+5}\right) = \lim_{n\to\infty}\left(\frac{2}{3+(5/n)}\right) = \frac{2}{3+0} = \frac{2}{3}.$$

This shows that the series is convergent and its limit is $\frac{2}{3}$.

Telescoping series

A series in which on expansion of n^{th} partial sum, every term except first and last term is cancelled out is called telescoping series.

Example 11: Examine the convergency of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right)$.

Solution: This series is telescoping series because,

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad \text{(by partial decomposition)}$$

$$S_k = \sum_{n=1}^{k}\left(\frac{1}{n(n+1)}\right) = \sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_k = \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \dots + \left(\frac{1}{k}-\frac{1}{k+1}\right).$$

$$= 1 - \frac{1}{k+1}.$$ (First and last term are remaining so, it is telescoping series)

Then,

$$\lim_{k\to\infty} S_k = 1 - \lim_{k\to\infty}\left(\frac{1}{k+1}\right) = 1.$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right) = 1.$$

Hence, the series $\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right)$ is convergent and its sum is 1.

5.2.1 Geometric Series

Definition (Geometric Series)

A series of the form

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

where a is the non-zero first term r is fixed ratio. Such series is known as ***geometric series.***

The following theorem helps to determine the series (if the series is in geometric form) is convergent or divergent.

Theorem (Geometric Ratio Test)

The geometric series $a + ar + ar^2 + \dots$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Proof: Let the geometric series is,

$$a + ar + ar^2 + ar^3 + \dots$$

Let S_n be the partial sum of n-terms of series (1). Then,

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

So,

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n$$

Then,

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow (1-r)S_n = a(1-r^n)$$

$$\Rightarrow S_n = \frac{a(1-r^n)}{1-r} \text{ for } r \neq 1.$$

Case I: If $|r| < 1$ then r^n decreases as n increases.

And finally, $\lim_{n\to\infty} r^n \to 0$. Therefore, $s_n = \frac{a}{1-r}$ for $|r| < 1$

This shows that the series converges to $\frac{a}{1-r}$ for $|r| < 1$.

Case II: If $|r| > 1$ then r^n increases as n increases. So, $\lim_{n\to\infty} r^n \to \infty$ for very large value of n. So, the series diverges for $|r| > 1$.

Case II: If $|r| = 1$ then $S_n = na \to \infty$ as $n \to \infty$.

That means, the series diverges for $|r| = 1$.

Thus the series converges for $|r| < 1$ and diverges for $|r| \geq 1$.

Remember that if the geometric series

$$a + ar + ar^2 + ar^3 + \dots$$

is convergent then its sum is $\frac{a}{1-r}$ where a is the first term and r is the common ratio of the infinite series.

Example 12: Show that the series $\sum_{n=1}^{\infty}\left[\frac{1}{9}\left(\frac{1}{3}\right)^{n-1}\right]$ is convergent.

Solution:

Given series is,

$$\sum_{n=1}^{\infty}\left[\frac{1}{9}\left(\frac{1}{3}\right)^{n-1}\right], \text{ it is geometric series.}$$

Here, common ratio $(r) = \frac{1}{3}$

Clearly $|r| = \frac{1}{3} < 1$. Then, the given series is convergent by geometric ratio test theorem.

Example 13: Show that the series $\sum_{n=0}^{\infty}\left[\frac{2(-1)^n}{5 \times 3^n}\right]$ is convergent to $\frac{3}{10}$.

Solution:

Given series is

$$\sum_{n=0}^{\infty}\left[\frac{2(-1)^n}{5 \times 3^n}\right], \text{ it is geometric series.}$$

Clearly, the series has

the first term $(a) = \frac{2}{5}$

and common ratio $(r) = \frac{-1}{3}$

Here, $|r| = \frac{1}{3} < 1$. So, the given series is convergent by geometric ratio test theorem.

Also,

$$S_\infty = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = \left(\frac{\frac{2}{5}}{1+\frac{1}{3}}\right) = \left(\frac{\frac{2}{5}}{\frac{4}{3}}\right) = \frac{3}{10}.$$

This shows that the given series convergent to $\frac{3}{10}$.

Example 14: Examine the convergency of the series $\sum_{n=0}^{\infty}(\sqrt{2})^n$

Solution:

Given series is,

$$\sum_{n=0}^{\infty}(\sqrt{2})^n, \text{ it is geometric series.}$$

Here, the first term $(a) = 1$

and the common ratio $(r) = \sqrt{2}$.

Since, $|r| = \sqrt{2} > 1$. This means the series is divergent by geometric ratio test.

Example 15: Find the sum of the geometric series: $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{29} + \ldots$

Solution:

Given geometric series is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{29} + \ldots$$

Here, the first term of the series is, $a = 5$ and the common ratio is $r = \frac{-2}{3}$.

Clearly, $|r| = \left|\frac{-2}{3}\right| = \frac{2}{3} < 1$.

So, the given series is convergent by geometric ratio test. And, the sum of the series is,

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{29} + \ldots = \left(\frac{5}{1-(-2/3)}\right) = \left(\frac{5}{5/3}\right) = 3.$$

Example 16: Show that $\sum_{n=0}^{\infty} x^n$ is convergent for $|x| < 1$ and find its sum.

Solution:

Given series is

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

Clearly, the first term of the series is $a = 1$ and the common ratio is $r = x$.

By geometric ratio test, the given series is convergent for $|x| < 1$.

And, the sum of the series is

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

Thus, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$.

Example 17: Show that the harmonic series $\sum_{n=0}^{\infty}\left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ is divergent.

Solution:

Given series is

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$$

Let S_n be the partial sum of the series of first n^{th} terms. So

$$S_1 = 1$$

$$S_2 = \left(1+\frac{1}{2}\right)$$

$$S_4 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 1+\frac{1}{2}+\frac{1}{2} = 1+\frac{2}{2}$$

$$S_8 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$$

$$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$= 1+\frac{3}{2} \text{ and so on.}$$

This shows

$$S_{2^n} > 1+\frac{n}{2}$$

$$\Rightarrow \lim_{n\to\infty} (S_{2^n}) = \infty.$$

This means $\{S_n\}$ is divergent. That is, the harmonic series is divergent.

Theorem: The necessary condition for the convergence of an infinite series Σu_n is $\lim_{n\to\infty} u_n = 0$. But, the condition is not sufficient.

Proof: Let Σu_n be an infinite series.

Let, $S_n = u_1 + u_2 + \ldots + u_n$.

Suppose that the series Σu_n is convergent to a finite value S. Then

$$\lim_{n\to\infty} S_n = S.$$

Also,

$$\lim_{n\to\infty} S_{n-1} = S.$$

Clearly,

$$S_n - S_{n-1} = \sum_{i=1}^{n} u_i - \sum_{i=1}^{n-1} u_i = u_n$$

Therefore,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S$$

$$= 0.$$

Thus, if the series converges then necessarily $\lim_{n\to\infty} u_n = 0$.

But, the condition is not sufficient. That is $\lim_{n\to\infty} u_n = 0$ may not imply that the series converges.

Take a series $\Sigma\left(\frac{1}{n}\right)$. Here,

$$u_n = \frac{1}{n}.$$

which is divergent series.

Thus, the condition $\lim_{n\to\infty} u_n = 0$ is only the necessary condition for convergency of an infinite series but not a sufficient.

5.2.2 n^{th} term test for Divergence

If $\lim_{n\to\infty} u_n$ does not exist or $\lim_{n\to\infty} u_n \neq 0$ then the series Σu_n is divergent.

Example 18: Prove that the given series is divergent: $\frac{1}{3}+\frac{2}{4}+\frac{3}{5}+\frac{4}{6}+\ldots$

Solution:

Given series is,

$$\frac{1}{3}+\frac{2}{4}+\frac{3}{5}+\frac{4}{6}+\ldots+\frac{n}{n+2}+\ldots$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n}{n+2}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{2}{n}}\right) = \frac{1}{1+0} = 1 \neq 0.$$

Therefore, by n^{th} term test for divergence, the given series is divergent.

__Warning:__ If we find $\lim_{n\to\infty} u_n \neq 0$ then the series Σu_n is divergent. On the other hand, if we find $\lim_{n\to\infty} u_n = 0$ then the series Σu_n is not necessarily convergent (see above THEOREM). This means if we find $\lim_{n\to\infty} u_n = 0$ then the series Σu_n might be convergent or might be divergent.

5.2.3 Combining series

If the two series are convergent, then the basic mathematical operations between the series does not destroy the behavior.

Theorem:

If $\sum u_n = A$ and $\sum v_n = B$ are two convergent series. Then,

(i) Sum Rule: $\sum(u_n + v_n) = \sum u_n + \sum v_n = A + B$

(ii) Difference rule: $\sum(u_n - v_n) = \sum u_n - \sum v_n = A - B$

(iii) Constant multiple rule: $\sum(ku_n) = k\sum u_n = kA$ for any constant number k.

Example 19: Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3^{n-1}-1}{6^{n-1}}\right)$

Solution:

Given series is,

$$\sum_{n=1}^{\infty}\left(\frac{3^{n-1}-1}{6^{n-1}}\right) = \sum_{n=1}^{\infty}\left(\frac{3}{6}\right)^{n-1} - \sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1} \quad \ldots \text{(i)}$$

Since, the series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$ is a geometric series with having common ratio $(r) = \frac{1}{2}$ and the first term $(a) = 1$.

So,

$$S_\infty = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{1-\frac{1}{2}}\right) = 2. \qquad \left[\because S_\infty = \frac{a}{1-r}\right]$$

Also, the series $\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1}$ is a geometric series with having common ratio $(r) = \frac{1}{6}$ and the first term $(a) = 1$. Therefore,

$$S_\infty = \sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1} = \left(\frac{1}{1-\frac{1}{6}}\right) = \frac{6}{5}.$$

Then (i) reduces to,

$$\sum_{n=1}^{\infty}\left(\frac{3^{n-1}-1}{6^{n-1}}\right) = \left(2 - \frac{6}{5}\right) = \frac{4}{5}.$$

Thus the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3^{n-1}-1}{6^{n-1}}\right)$ is $\frac{4}{5}$.

Exercise 5.2

Test the convergency of the following infinite series. If convergent, find its sum.

1. Applying geometric series test.

a. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{3^n}\frac{2}{5}\right)$ b. $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$ c. $\sum_{n=2}^{\infty}(\sqrt{3})^n$ d. $\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$

e. $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\left(\frac{-1}{5}\right)^n\right)$ f. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$ g. $\sum_{n=0}^{\infty}\cos(n\pi)$ h. $\sum_{n=0}^{\infty}e^{-2n}$

i. $\sum_{n=1}^{\infty}\frac{2}{10^n}$ j. $\sum_{n=0}^{\infty}\frac{2^n-1}{3^n}$

2. a. $\sum_{n=1}^{\infty}\left(\frac{4}{(4n+1)(4n-3)}\right)$ b. $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ c. $\sum\left(\frac{1}{(n+1)(n+2)}\right)$

(All these infinite series are telescoping series.)

3. Show that following series are divergent

a. $\sum_{n=1}^{\infty}n^2$ b. $\sum_{k=1}^{\infty}\frac{k+1}{k}$ c. $\sum_{n=1}^{\infty}\frac{-n}{2n+5}$

Answers

1. (a) converges, $\frac{3}{10}$ (b) converges, $\frac{\sqrt{2}}{\sqrt{2}-1}$ (c) diverges

(d) converges, $\frac{21}{2}$ (e) converges, $\frac{17}{6}$ (f) converges, -1

(g) diverges (h) converges, $\frac{e^2}{e^2-1}$ (i) converges, $\frac{4}{9}$

(j) converges, $\frac{3}{2}$

2. (a) converges, 1 (b) converges, 1 (c) converges, 1

5.3 Convergence Tests of Infinite Series

In this section, we study about different test to examine the convergency of an infinite series. Also, if the series is convergent, what is its sum? We will observe the answer of the question.

Theorem: A series $\sum_{n=1}^{\infty} u_n$ of non-negative terms converges if and only if its partial sums are bounded above.

5.3.1 Integral Test

Suppose that f is continuous, positive, decreasing function on $[1, \infty)$ and $u_n = f(n)$.

Then the series $\sum_{n=1}^{\infty} u_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words,

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} u_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} u_n$ is divergent.

Note: When we use the integral test, it is not necessary to start the series or the integral at n = 1. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \text{ we use } \int_4^{\infty} \frac{1}{(x-3)^2}\,dx$$

Also, it is not necessary that f be always decreasing.

Example 20: (Divergence of Harmonic Series)

Show that a harmonic series $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots,$ is divergent.

Solution: Here,

$$\int_1^{\infty} \left(\frac{1}{x}\right) dx = \lim_{k\to\infty} \int_1^k \frac{dx}{x}$$

$$= \lim_{k\to\infty} [\ln|x|]_1^k$$

$$= \lim_{k\to\infty} [\ln(k) - \ln(1)]$$

$$= \lim_{k\to\infty} \ln(k) \quad [\because \ln(1) = 0]$$

$$= \ln(\infty)$$

$$= \infty$$

Thus, the integral $\int_1^{\infty} \left(\frac{1}{x}\right) dx$ has no fixed finite value. So, the integral is divergent.

Then, by integral test the series $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$ is divergent.

Example 21: Does the series $\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)$ converge?

Solution: Here,

$$\int_1^{\infty} \left(\frac{1}{x^2}\right) dx = \lim_{k\to\infty} \int_1^k \left(\frac{1}{x^2}\right) dx$$

$$= \lim_{k\to\infty} \left(\frac{-1}{x}\right)_1^k$$

$$= \lim_{k\to\infty} \left(\frac{-1}{k} + 1\right)$$

$$= -0 + 1$$

$$= 1.$$

This shows that the integral is convergent. Therefore, by integral test, the series $\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)$ is convergent.

Example 22: Test the convergency of a p–series (hyper harmonic series) $\sum_{n=1}^{\infty} \left(\frac{1}{n^p}\right)$ (p is real constant) by integral test.

Solution: Since,

$$f(n) = \frac{1}{n^p} \quad \text{(p is real constant).}$$

<u>Case I:</u> For $p > 1$, $f(x) = \frac{1}{x^p}$ is a positive decreasing function of x. Here,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{k\to\infty}\int_1^k \left(\frac{1}{x^p}\right)dx = \lim_{k\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^k \qquad [\because\ p>1 \Rightarrow p-1>0]$$

$$= \frac{1}{1-p}(0-1)$$

$$= \frac{1}{p-1} \quad \text{(a non–zero finite value).}$$

This means the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges. Then, by integral test, the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ converges for $p > 1$.

Case II: If $p < 1$ then $(1-p) > 0$. Here,

$$\int_1^\infty \frac{1}{x^p}\,dx = \left(\frac{1}{1-p}\right)\lim_{k\to\infty}\left[x^{-p+1}\right]_1^k$$

$$= \left(\frac{1}{1-p}\right)\lim_{k\to\infty}\left[x^{1-p}\right]_1^k$$

$$= \left(\frac{1}{1-p}\right)\lim_{k\to\infty}\left(k^{1-p}-1\right)$$

$$= \left(\frac{1}{1-p}\right)(\infty - 1).$$

$$= \infty$$

This means the integral is divergent for $p < 1$. Then, by integral test the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ divergent for $p < 1$.

Case III: If $p = 1$ then the series reduces to $\sum_{n=1}^{\infty}\frac{1}{n}$, which is a harmonic series. Clearly, this series is divergent by Example 15.

Thus, the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ converges for $p > 1$ and diverges for $p \le 1$.

Example 23: Test the convergency of $\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\right)$ by integral test.

Solution:

Since, $f(n) = \frac{1}{n^2+1}$ which is continuous, positive and decreasing for $n \ge 1$.

Here,

$$\int_1^\infty \left(\frac{1}{x^2+1}\right)dx = \lim_{k\to\infty}\int_1^k \frac{dx}{x^2+1}$$

$$= \lim_{k\to\infty}\left[\tan^{-1}x\right]_1^k$$

$$= \lim_{k\to\infty}\left[\tan^{-1}(k) - \tan^{-1}(1)\right]$$

$$= \tan^{-1}(\infty) - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}.$$

This means the integral converges. Therefore, by integral test, the given series is convergent.

Remarks: To use the integral test, it is not necessary to start the limit of integral (as the series) from $n = 1$. For example, to test the convergence of $\sum_{n=5}^{\infty}\left(\frac{1}{n-4}\right)$, we use the integral $\int_5^\infty \left(\frac{1}{x-4}\right)dx$.

Note: We should not infer from the Integral Test that the sum of the series is equal to the value of the integral. In fact,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) = \frac{\pi^2}{6} \text{ whereas } \int_1^\infty \frac{dx}{x^2} = 1.$$

That is

$$\sum_{n=1}^{\infty} u_n \neq \int_1^\infty f(x)\,dx.$$

Example 24: Determine whether the series $\sum_{i=1}^{\infty}\left(\frac{ln(n)}{n}\right)$ converges or diverges.

Solution:

Let

$$f(x) = \left(\frac{ln(n)}{n}\right) > 0 \text{ for } x > 1.$$

Here,

$$\int_1^{\infty}\left(\frac{ln(n)}{n}\right)dx = \int_0^{\infty}(y)\,dy \qquad \text{Put } \ln(x) = y.$$

$$= \lim_{k\to\infty}\int_0^{k}(y)\,dy$$

$$= \lim_{k\to\infty}\left[\frac{y^2}{2}\right]_0^k$$

$$= \lim_{k\to\infty}\left(\frac{k^2}{2}\right)$$

$$= \infty$$

This means the integral is divergent, so the series $\sum_{i=1}^{\infty}\left(\frac{ln(n)}{n}\right)$ is also divergent by Integral Test.

5.3.2 Theorem (p-series Test)

A series of the form $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ (p is real constant) is called p–series (hyper harmonic series).

The series $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ (p is real constant) is convergent for $p > 1$ and is diverges for $p \le 1$.

Example 25: (Divergence of Harmonic Series)

Show that a harmonic series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$, is divergent.

Solution:

Clearly the given series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$ is p-series with $p = 1$ and is divergent by p-test.

Example 26: Does the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ converge?

Solution: Clearly the given series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ is p-series with $p = 2 > 1$ and is convergent by p-test.

5.3.3 Theorem: (Comparison Test)

Let, $\sum u_n$ be a series of non-negative terms.

(a) If there is a convergent series $\sum v_n$ with $u_n \le v_n$ for all $n \ge N$ (some integer N) then $\sum u_n$ is also convergent.

(b) If there is a divergent series $\sum v_n$ with $u_n \ge v_n$ for all $n \ge N$ (some integer N) then $\sum u_n$ is also divergent.

Example 27: Apply the Comparison Test for $\sum_{n=1}^{\infty}\left(\frac{7}{7n-2}\right)$.

Solution: Here,

$$u_n = \left(\frac{7}{7n-2}\right) = \left(\frac{1}{n-\frac{2}{7}}\right)$$

Suppose $v_n = \frac{1}{n}$

We know that

$$\left(n - \frac{2}{7}\right) < n \quad \text{for all n}$$

$$\left(\frac{1}{n-\frac{2}{7}}\right) > \frac{1}{n}$$

$$u_n > v_n \quad \text{for all n}$$

Also, $\sum v_n = \sum\left(\frac{1}{n}\right)$ is divergent by p–series test. Thus, $\sum u_n$ also divergent by comparison test.

5.3.4 Theorem (The Limit Comparison Test)

Suppose $\sum u_n$ and $\sum v_n$ are series of positive terms. If,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = l$$

(a) with l is a finite number and $l > 0$ then either both $\sum u_n$ and $\sum v_n$ converge or both diverge.

(b) with $l = 0$ or $l = \infty$ then the series $\sum u_n$ diverges.

Example 28: Examine the convergency of the following series

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \dots + \frac{2n+1}{(n+1)^2} + \dots$

(b) $1+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\dots+\frac{1}{2^n-1}+\dots$

(c) $\frac{1+2\ln(2)}{9}+\frac{1+3\ln(3)}{14}+\frac{1+4\ln(4)}{21}+\dots+\frac{1+n\ln(n)}{n^2+5}+\dots$

Solution:

(a) Let,

$$u_n=\frac{2n+1}{(n+1)^2}.$$

Here,

$$\frac{2n+1}{(n+1)^2}=\left[\frac{n\left(2+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}\right]=\frac{1}{n}\left[\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2}\right].$$

Choose, $v_n=\frac{1}{n}$.

Clearly, the series $\sum v_n$ is diverges by p–test with p = 1.

And,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{\frac{(2n+1)}{(n+1)^2}}{\frac{1}{n}}\right)=\lim_{n\to\infty}\left(\frac{2+\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2}\right)=\frac{2+0}{(1+0)^2}=2>0.$$

Then by limit comparison test, the given series $\sum u_n$ is divergent.

(b) Let,

$$u_n=\frac{1}{2^n-1}=\frac{1}{2^n\left(1-\frac{1}{2^n}\right)}$$

Choose $v_n=\frac{1}{2^n}$.

Since, $\sum v_n$ is a geometric series with ratio $|r|=\frac{1}{2}<1$, so by geometric series ratio test, the series is convergent.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\frac{1}{\left(1-\frac{1}{2^n}\right)}=\frac{1}{1-0}=1>0.$$

This means the series Σu_n converges by limit comparison test.

(c) Here,

$$\sum_{n=1}^{\infty}\left(\frac{1+n\ln(n)}{n^2+5}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n^2+5}\right)+\sum_{n=1}^{\infty}\left(\frac{n\ln(n)}{n^2+5}\right).$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{n^2\left(1+\frac{5}{n^2}\right)}\right)+\sum_{n=1}^{\infty}\left(\frac{\ln(n)}{n^2+5}\right)$$

Let,

$$u_1=\frac{1}{n^2\left(1+\frac{5}{n^2}\right)}\quad\text{and}\quad u_2=\frac{\ln(n)}{n\left(1+\frac{5}{n^2}\right)}.$$

Choose $v_1=\frac{1}{n^2}$ and $v_2=\frac{1}{n}$.

Clearly $\sum v_1$ converges by p–test with p = 2 and $\sum v_2$ diverges by p–test with p = 1.

And,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{1}{1+\frac{5}{n^2}}\right)=\frac{1}{1+0}=1>0.$$

This means $\sum u_1$ converges by limit comparison test.

Also,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{\ln(n)}{1+\frac{5}{n^2}}\right)=\frac{\infty}{1+0}=\infty.$$

This means $\sum u_2$ diverges by limit comparison test.

Thus, $\sum u_1$ converges but $\sum u_2$ diverges. So, $\sum(u_1+u_2)$ diverges, therefore, the given series diverges.

Example 29: Examine the convergency of the series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^3+2}}\right)$.

Solution:

Given series is,

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^3+2}}\right)$$

The general term of the series is,

$$u_n=\left(\frac{1}{\sqrt{n^3+2}}\right)=\left(\frac{1}{n^{3/2}\sqrt{1+2/n^3}}\right)$$

Take,

$$v_n = \frac{1}{n^{3/2}}.$$

Clearly, this series is convergent by p–test with $p = \frac{3}{2} > 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+2/n^3}}\right) = 1 \text{ (which is a finite non–zero value.)}$$

Thus, Σu_n is convergent by limit comparison test.

Example 30: Examine the convergency of the series $\sum_{n=1}^{\infty}\left(\frac{n}{n^2+1}\right)$.

Solution: Given series is,

$$\sum_{n=1}^{\infty}\left(\frac{n}{n^2+1}\right)$$

The general term of the series is,

$$u_n = \left(\frac{n}{n^2+1}\right) = \frac{1}{n(1+1/n^2)}$$

Choose, $v_n = \frac{1}{n}$. Clearly this series Σv_n is divergent by p–test with $p = 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1+1/n^2}\right) = 1 \text{ (which is a finite non–zero value.)}$$

Since Σv_n is divergent so Σu_n is also divergent by limit comparison test.

Example 31: Examine the convergency of the series $\Sigma[\sqrt{n^2+1} - n]$.

Solution: Given series is,

$$\Sigma[\sqrt{n^2+1} - n]$$

The general term of the series is,

$$u_n = \sqrt{n^2+1} - n$$

which is in $\infty - \infty$ form as $n \to \infty$. So, multiply numinator and denominator by its conjugate. Then,

$$u_n = (\sqrt{n^2+1} - n) \times \frac{(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)}$$

$$= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$= \frac{1}{n}\left(\frac{1}{\sqrt{1+\frac{1}{n^2}}+1}\right)$$

Choose $v_n = \frac{1}{n}$. Clearly this series Σv_n is divergent by p–test with $p = 1$.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{(\sqrt{1+1/n^2}+1)}$$

$$= \frac{1}{\sqrt{1+0}+1}$$

$$= \frac{1}{2} \text{ (which is a finite non–zero value.)}$$

Since Σv_n is divergent so Σu_n is also divergent by limit comparison test.

5.3.5 Theorem (D'Alembert Ratio Test)

Let $\sum u_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = l.$$

Then,

a. If $l < 1$ then the series converges.

b. If $l > 1$ then the series diverges.

c. If $l = 1$ the test is inconclusive and further test is needed.

Example 32: Examine the convergency of the series $\sum_{n=1}^{\infty}\left(\frac{n^2}{3^n}\right)$.

Solution: Given series is,

$$\sum_{n=1}^{\infty}\left(\frac{n^2}{3^n}\right)$$

The general term of the series is,

$$u_n = \frac{n^2}{3^n}$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1)^2}{3^{n+1}} \times \frac{3^n}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n^2(1+1/n)^2}{3^n.\,3} \times \frac{3^n}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{(1+1/n)^2}{3}\right) \sim \frac{(1+0)^2}{3} = \frac{1}{3} < 1.$$

This shows that the given series converges, by D'Alembert ratio test.

Example 33: Examine the convergency of the series, $\Sigma\left(\frac{2^{n-1}}{3^{n+1}}\right)$

Solution: Given series is,

$$\Sigma\left(\frac{2^{n-1}}{3^{n+1}}\right)$$

The general term of the series is,

$$u_n = \frac{2^{n-1}}{3^{n+1}}$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{(n+1)}-1}{3^{(n+1)+1}} \times \frac{3^{n+1}}{2^{n-1}}$$

$$= \lim_{n\to\infty} \frac{2^n}{3^{n+2}} \times \frac{3^{n+1}}{2^{n-1}}$$

$$= \lim_{n\to\infty} \frac{2^n}{3.3^{n+1}} \times \frac{3^{n+1}}{2^n.\,2^{-1}}$$

$$= \lim_{n\to\infty} \frac{1}{3.2^{-1}}$$

$$= \frac{2}{3} < 1$$

Thus, by D'Alembert ratio test, the given series is convergent.

Example 34: Examine the convergency of the series $1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \ldots$

Solution: Given series is,

$$1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \ldots$$

The general term of the series is,

$$u_n = \frac{n!}{n^2}$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^2} \times \frac{n^2}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\times n!}{(n+1)^2} \times \frac{n^2}{n!}$$

$$= \lim_{n\to\infty} \frac{n^2}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{1+1/n}$$

$$= \infty > 1$$

That is, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$. So, by D'Alembert ratio test the given series diverges.

Example 35: Test the convergency of the series $\Sigma\left(\frac{n^2(n+1)^2}{n!}\right)$.

Solution: Given series is,

$$\Sigma \frac{n^2(n+1)^2}{n!}$$

The general term of the series is,

$$u_n = \frac{n^2(n+1)^2}{n!}$$

Here,

$$\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \left(\frac{(n+1)^2\,(n+1+1)^2}{(n+1)!}\right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^2\,(n+2)^2}{(n+1)\,n(n-1)\,(n-2)\,(n-3)!}\right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)\,n^2\,(1+2/n)^2}{n.n^2\,(1-1/n)\,(1-2/n)\,(n-3)!}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n(1+1/1n)\,(1+2/n)^2}{n(1-1/n)\,(1-2/n)\,(n-3)!}\right)$$

$$= \frac{(1+0)\,(1+0)^2}{(1-0)\,(1-0)\,\infty}$$

$$= \frac{1}{\infty}$$

$$= 0 < 1.$$

Thus, $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = 0 < 1$. So, by D'Alembert ratio test the given series is convergent.

Example 36: Test, whether the series $\sum_{n=1}^{\infty} \left(\frac{n^n}{n!}\right)$ is convergent.

Solution:

Given series is

$$\sum_{n=1}^{\infty} \left(\frac{n^n}{n!}\right)$$

The general term of the series is

$$u_n = \frac{n^n}{n!}$$

Here

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(n+1)!}\times\frac{n!}{n^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)\,(n+1)^n}{(n+1)\,n!}\times\frac{n!}{n^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e$$

$$\approx 2.71 > 1.$$

So, the given series is divergent by D'Alembert ratio test.

Example 37: Test the convergency of the series, $\Sigma\left(\frac{3^n-2}{3^n+1}\right)$.

Solution:

Given series is,

$$\Sigma\left(\frac{3^n-2}{3^n+1}\right)$$

The general term of the series is,

$$u_n = \frac{3^n-2}{3^n+1}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{3^n-2}{3^n+1} = \lim_{n\to\infty}\frac{1-2/3^n}{1+1/3^n} = \frac{1-0}{1+0} = 1 \neq 0.$$

Thus, the given series is divergent by n^{th} term test for divergency.

Example 37: Test the convergency of the series $\Sigma\left(\frac{1}{\ln(n+1)}\right)$.

Solution:

Given series is,

$$\Sigma\left(\frac{1}{\ln(n+1)}\right)$$

The general term is,

$$u_n = \frac{1}{\ln(n+1)}.$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{1}{\ln(n+2)}\times\frac{\ln(n+1)}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\ln(n+1)}{\ln(n+2)}\right)$$

which in $\frac{\infty}{\infty}$ form, so using l'Hospital rule,

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n+1}}{\frac{1}{n+2}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)$$

$$= 1.$$

Therefore by D'Alembert's ratio test, further test is needed.

Since, $\left(\frac{1}{\ln(n+1)}\right) > \left(\frac{1}{n+1}\right)$

$$\Rightarrow \frac{1}{\ln(n)} > \frac{1}{n}.$$

By p–test, the series $\sum\left(\frac{1}{n}\right)$ diverges and then by direct comparison test the series $\Sigma\left(\frac{1}{\ln(n+1)}\right)$ is divergent.

Note: For the series given above, it is better to choose integral test to determine the convergency.

5.3.6 Theorem (The n^{th} Root Test) (Cauchy's Radical Test)

Let $\sum u_n$ be a series with $u_n \geq 0$ for $n \leq N$ and suppose that

$$\lim_{n\to\infty}(u_n)^{1/n} = r$$

Then,

a. If $r < 1$ then the series converges.

b. If $r > 1$ then the series diverges.

c. If $r = 1$ then test is inconclusive and further test is needed.

Example 38: Examine the convergency of the $\Sigma\left(\frac{n-ln(n)}{2n}\right)^n$.

Solution: Given series is,

$$\Sigma\left(\frac{n-ln(n)}{2n}\right)^n.$$

The general term of the series is,

$$u_n = \left(\frac{n-ln(n)}{2n}\right)^n.$$

Then,

$$u_n^{1/n} = \frac{n-ln(n)}{2n} = \frac{1}{2}\left(1-\frac{ln(n)}{n}\right).$$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\frac{1}{2}\left(1-\frac{ln(n)}{n}\right)\right] = \frac{1}{2}(1-0) = \frac{1}{2} < 1.$$

Therefore, by the Cauchy's radical test the given series is convergent.

Example 39: Examine the convergency of the $\Sigma\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

Solution:

Given series is,

$$\Sigma\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

The general term of the series is,

$$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}.$$

Then,

$$u_n^{1/n} = \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right]^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}\right)$$

$$= \frac{1}{e} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$= \frac{1}{2.71} < 1.$$

Therefore, the given series is convergent by Cauchy's root test.

Example 40: Examine the convergency of the $\Sigma\left(1+\frac{1}{n}\right)^n$

Solution: Given series is,

$$\Sigma\left(1+\frac{1}{n}\right)^n$$

The general term of the series is,

$$u_n = \left(1+\frac{1}{n}\right)^n$$

Here,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1+0 = 1.$$

So, the test fail.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}(e) = e \neq 0.$$

Therefore, the given series is divergent.

5.3.7 Strategy for Testing Series

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its form.

a. If the series is of the form $\Sigma\left(\frac{1}{n^p}\right)$, it is a p–series, which we know to be convergent if $p > 1$ and divergent if $p \le 1$.

b. If the series has the form Σar^{n-1} or Σar^n, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \ge 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

c. If the series has a form that is similar to a p–series or a geometric series, then one of the comparison tests should be considered. In particular, if u_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p–series. (The value of p should be chosen as by keeping only the highest powers of n in the numerator and denominator). The comparison tests apply only to series with positive terms, but if Σu_n has some negative terms, then we can apply the Comparison Test to $\Sigma|u_n|$ and test for absolute convergence.

d. If you can see at a glance that $\lim_{n\to\infty} u_n \neq 0$, then the test for divergence should be used.

e. Series that involve factorials or other products (including a constant raised to the n^{th} power) are often conveniently tested using the ratio test. Bear in mind that $\left|\frac{u_{n+1}}{u_n}\right| \to$ $n \to \infty$ for all p–series and therefore all rational or algebraic functions of n. Thus the ratio test should not be used for such series.

f. If u_n is of the form $(b_n)^n$, then the Root Test may be useful.

g. If $u_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Exercise 5.3

1. Test the convergence of series by integral test

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (b) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ (c) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

(d) $\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$ (e) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ (f) $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

(g) $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$ (h) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

2. Explain why the integral test can not be used to determine whether the series is divergent.

(a) $\sum_{n=1}^{\infty} \frac{\cos(np)}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{1+n^2}$

3. Test the convergence of series by comparison test

(a) $\sum_{n=1}^{\infty} \frac{3}{3\sqrt{n}-2}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ (c) $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

(d) $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^2}}$ (e) $\sum_{n=1}^{\infty} \left(\frac{1}{n^3+1}\right)$ (f) $\Sigma[\sqrt{n+1}-\sqrt{n}]$

(g) $\Sigma\left[\sqrt{n^2+1}-n\right]$ (h) $\Sigma\left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]$ (i) $\Sigma[(n^3+1)^{1/3}-n]$

4. Test the convergence of series

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ (b) $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ldots$ (c) $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots$

(d) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$ (e) $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\ldots$ (f) $\frac{2}{3^2}+\frac{3}{4^2}+\frac{4}{5^2}+\ldots$

(g) $\frac{14}{1^3}+\frac{24}{2^3}+\frac{34}{3^3}+\ldots\ldots\ldots+\frac{4+10n}{n^3}+\ldots\ldots$

5. Investigate the convergence of the following series

(a) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ (b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ (c) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ (d) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$

(e) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (f) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ (g) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ (h) $\sum_{n=1}^{\infty} \sqrt{\frac{2^n-1}{3^n-1}}$

(i) $\sum_{n=1}^{\infty} \left(\frac{4}{3^{2n}-1}\right)$ (j) $\sum_{n=1}^{\infty} \left(\frac{9^n}{3+10^n}\right)$ (k) $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ (l) $\sum_{n=1}^{\infty} \left(\frac{1+4^n}{1+3^n}\right)$

(m) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ (n) $\sum_{n=1}^{\infty} e^{-2n}$

6. Investigate the convergence of the following series:

(a) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$ (b) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ (c) $\Sigma\left(\frac{(n!)^n}{(n^n)^2}\right)$

(d) $\sum_{n=1}^{\infty} \left(\frac{n^n}{2^{n^2}}\right)$ (e) $\Sigma \frac{n^n}{(2^n)^2}$ (f) $\Sigma \frac{n}{(n^2+2)^n}$

(g) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ (h) $\sum_{n=1}^{\infty} \left(\frac{2^{n+1}}{5^{n+1}}\right)^{1/n}$

Answers

1. (a) converges (b) diverges (c) converges (d) converges
   (e) Diverges (f) converges (g) diverges (h) converges
2. (a) f is neither positive nor negative
   (b) f is positive but not monotonically decreasing
3. (a) diverges (b) converges (c) converges (d) diverges
   (e) Converges (f) diverges (g) converges (h) converges
   (i) Converges
4. (a) diverges (b) diverges (c) converges (d) converges
   (e) diverges (f) diverges (g) convergent
5. (a) converges (b) diverges (c) diverges (d) converges
   (e) converges (f) diverges (g) converges (h) converges
   (i) converges (j) converges (k) diverges (l) diverges
   (m) diverges (n) converges
6. (a) converges (b) converges (c) diverges (d) diverges
   (e) diverges (f) converges (g) converges (h) converges

5.4 Absolute and Conditional Convergence

Definition (Alternative Series)

An infinite series $\sum(-1)^n u_n$ is known as an alternative series.

For an example,

(i) $1-2+3-4+\ldots+(-1)^{n+1}n+\ldots$

(ii) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{(-1)^{n+1}}{n}+\ldots$

are alternative series.

Definition (Alternative Harmonic Series)

A series given in above example (ii) is known as ***alternative harmonic series.***

5.4.1 Leibnitz's Test (The Alternative Series Test):

The series,

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1-u_2+u_3-u_4+\ldots$$

converges if it satisfies the following conditions:

(a) $u_n > 0$, for all n.

(b) $u_n \geq u_{n+1}$, for all $n \geq N$

(c) $\lim_{n\to\infty} u_n = 0$.

Example 41: (Convergence of an Alternative Harmonic Series):

Show that the alternative harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots \text{ converges.}$$

Solution: Comparing the given series with $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ then we get,

$$u_n = \frac{1}{n}$$

Here, (a) $u_n \geq 0$ for all $n \geq 1$.

(b) $u_{n+1}-u_n = \frac{1}{n+1}-\frac{1}{n} = \frac{-1}{n(n+1)} < 0$.

$\Rightarrow u_{n+1} < u_n$.

(c) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$.

Therefore, by Leibnitz's test the given series is convergent.

Example 42: Test the alternative series $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\ldots$ converges.

Solution: Given series is,

$$1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\ldots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n\sqrt{n}}$$

Comparing the given series with $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ then we get,

$$u_n = \frac{1}{n\sqrt{n}}$$

Here, (a) $u_n = \frac{1}{n\sqrt{n}} \geq 0$, for all $n \geq 1$.

(b) $\frac{u_{n+1}}{u_n} = \left(\frac{1}{(n+1)\sqrt{n+1}}\times\frac{n\sqrt{n}}{1}\right) = \left(\frac{1}{(1+1/n)\sqrt{1+1/n}}\right) < 1$.

$\Rightarrow u_{n+1} < u_n$.

(c) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n\sqrt{n}} = 0$.

Therefore, by Leibnitz's test the given series is convergent.

Example 43: Test the convergency of the series $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n!}\right)$.

Solution: Given series is,

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n!}\right)$$

Comparing the given series with $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ then we get,

$$u_n = \frac{1}{n!}$$

Here, (a) $u_n = \frac{1}{n!} \geq 0$, for all $n \geq 1$.

(b) $\frac{u_{n+1}}{u_n} = \left(\frac{n!}{(n+1)!}\right) = \left(\frac{1}{n+1}\right) < 1$.

$\Rightarrow u_{n+1} < u_n$.

(c) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n!} = 0$.

Therefore, by Leibnitz's test the given series is convergent.

Example 44: Test the convergency $\sum_{n=1}^{\infty}\left(\frac{(-1)^n 3n}{4n-1}\right)$

Solution: Given series is

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^n 3n}{4n-1}\right)$$

Comparing it with $\sum_{n=1}^{\infty}(-1)^n u_n$ then

$$u_n = \frac{3n}{4n-1}$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{3n}{4n-1}\right) = \lim_{n\to\infty}\left(\frac{3}{4-\frac{1}{n}}\right) = \frac{3}{4-0} = \frac{3}{4} \neq 0.$$

This means the given series is divergent by Leibnitz test.

5.5 Absolute and Conditional Convergence

We consider a series $\sum u_n$ which may have both negative and positive terms. An alternative series is of such type of series. But if we reform the series as $\sum |u_n|$ then the series takes only positive terms.

Definition (Absolute Convergence)

A series $\sum_{n=1}^{\infty} u_n$ converges absolutely if the corresponding series $\sum_{n=1}^{\infty}|u_n|$ converges.

Theorem (The Absolute Convergence Test)

If $\sum_{n=1}^{\infty}|u_n|$ converges then $\sum_{n=1}^{\infty} u_n$ converges.

Proof: Since,

$$-|u_n| \le u_n \le |u_n| \quad \text{for all n.}$$

$$\Rightarrow \quad 0 \le u_n + |u_n| \le 2|u_n|.$$

Let the series $\sum_{n=1}^{\infty}|u_n|$ converges. Then $2\sum_{n=1}^{\infty}|u_n|$ converges. By direct comparison test, the series $\sum_{n=1}^{\infty}(u_n + |u_n|)$ also converges.

Since,

Then,

$$u_n = (u_n + |u_n|) - |u_n|.$$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}(u_n + |u_n|) - \sum_{n=1}^{\infty}|u_n|$$

Since both series $\sum_{n=1}^{\infty}(u_n + |u_n|)$ and $\sum_{n=1}^{\infty}|u_n|$ are convergent. Therefore, the difference of those series $\sum_{n=1}^{\infty} u_n$ is also convergent.

Theorem (Re-arrangement Theorem for absolutely convergent series)

If $\sum_{n=1}^{\infty} u_n$ converges absolutely and $\sum_{n=1}^{\infty} v_n$ be any arrangement of the sequence $\{u_n\}$ then $\sum v_n$ converges absolutely and

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} v_n.$$

Definition (Conditional Convergence)

If a series $\sum u_n$ converges but $\sum |u_n|$ does not converge, then the series is called **convergent conditionally.**

Normally, absolute convergence of a series implies the general convergence.

Example 45: Is the series $\sum_{n=1}^{\infty}\left(\frac{n(-1)^{n-1}}{5^n}\right)$ converges absolutely?

Solution: Here,

$$\sum_{n=1}^{\infty}\left|\frac{n(-1)^{n-1}}{5^n}\right| = \sum_{n=1}^{\infty}\left(\frac{n}{5^n}\right).$$

Comparing the given series with $\sum_{n=1}^{\infty} u_n$ then we get,

$$u_n = \left(\frac{n}{5^n}\right).$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{n+1}{5^{n+1}} \times \frac{5^n}{n}\right) = \lim_{n\to\infty}\frac{1+(1/n)}{5} = \frac{1}{5} < 1$$

So, the series $\sum |u_n| = \sum_{n=1}^{\infty}\left(\frac{n}{5^n}\right)$ is convergent by D'Alembert's ratio test.

This means the given series is convergent absolutely.

Example 46: Test the absolute convergency of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{n^3+1}\right)$.

Solution: Here,

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\left(\frac{n}{n^3+1}\right)\right| = \sum_{n=1}^{\infty}\left(\frac{n}{n^3+1}\right).$$

Comparing the given series with $\sum_{n=1}^{\infty} u_n$ then we get,

$$u_n = \left(\frac{n}{n^3+1}\right) = \frac{1}{n^2\left(1+\frac{1}{n^3}\right)}.$$

Set, $v_n = \frac{1}{n^2}$ then the series $\Sigma v_n = \Sigma\left(\frac{1}{n^2}\right)$ is convergent by p–test with $p = 2$.

Now,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^3}\right)} = \frac{1}{1+0} = 1 > 0$$

This shows Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = 1 > 0$. So, the series Σu_n is convergent by limit comparison test.

Therefore, the given series is convergent absolutely.

Example 47: Show that $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n^3}\right)$ converges absolutely.

Solution: Here,

$$\Sigma|u_n| = \Sigma\left(\frac{1}{n^3}\right).$$

Clearly this series converges by p–test with $p = 3$. so, the given series converges absolutely.

Example 48: Determine whether the series $\Sigma(-1)^n\left(\frac{1}{n}\right)$ convergent absolutely or not.

Solution:

Given series is

$$\Sigma(-1)^n\left(\frac{1}{n}\right) \qquad \text{.......(i)}$$

Here,

$$\Sigma\left|(-1)^n\left(\frac{1}{n}\right)\right| = \Sigma\left(\frac{1}{n}\right)$$

Clearly $\Sigma\left(\frac{1}{n}\right)$ is a harmonic series, is divergent by p–test.

But comparing (i) with $\Sigma(-1)^n u_n$ then

$$u_n = \left(\frac{1}{n}\right)$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

and

$$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} = \frac{1}{1+(1/n)} < 1 \quad \text{for all } n \geq 1.$$

$$\Rightarrow \quad u_{n+1} < u_n \quad \text{for all } n \geq 1.$$

This implies the given series (i) is convergent by Leibnitz test.

Thus, the series $\Sigma(-1)^n\left(\frac{1}{n}\right)$ converge but $\Sigma\left(\frac{1}{n}\right)$ diverges. That is, the given series converges conditionally.

Examine the absolute convergence of the infinite series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$.

Solution:

Given series is,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1} \qquad \text{... (i)}$$

Here,

$$\Sigma|u_n| = \Sigma\left|\frac{n}{n^2+1}\right| = \Sigma\left(\frac{n}{n^2+1}\right) = \Sigma\left(\frac{n}{n^2(1+1/n^2)}\right) = \Sigma\left(\frac{1}{n(1+1/n^2)}\right)$$

Set, $a_n = \frac{1}{n(1+1/n^2)}$. And, set $v_n = \frac{1}{n}$.

Clearly, the series Σv_n is p–series with $p = 1$. So, by p–test, the series Σv_n diverges.

Here,

$$\lim_{n\to\infty}\left(\frac{a_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{n(1+1/n^2)}\times\frac{n}{1}\right) = \lim_{n\to\infty}\left(\frac{1}{1+1/n^2}\right) = 1$$

This shows that the series Σa_n diverges. That is, $\Sigma|u_n|$ diverges.

Since the series (i) is an alternative series whose general term is,

$$a_n = \frac{n}{n^2+1}$$

Here,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n^2+1}\right) = \lim_{n\to\infty}\frac{n}{n^2(1+1/n^2)} = \lim_{n\to\infty}\frac{1}{n(1+1/n^2)} = 0.$$

And,
$$\frac{a_{n+1}}{a_n} = \frac{n+1}{(n+1)^2+1} \times \frac{n^2+1}{n} = \frac{n^3+n^2+n+1}{n^3+2n^2+2n}$$
$$= 1 - \frac{n^2+n-1}{n^3+2n^2+2n}$$

Since the term $\frac{n^2+n-1}{n^3+2n^2+2n}$ is positive being $n \geq 1$, so
$$\frac{a_{n+1}}{a_n} \leq 1$$
$$\Rightarrow \quad a_{n+1} \leq a_n.$$

This shows that the given series is Σu_n is convergent, by Leibnitz's test.

Thus, the series Σu_n is convergent but the positive term series $\Sigma|u_n|$ is divergent.

So, the given series is conditionally convergent.

Exercise 5.4

1. Test the convergency of the following alternative series.

 (a) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$ (c) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^{3/2}}$

 (d) $\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{\ln(n^2)}$ (e) $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{\ln(n)}$ (f) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$.

2. Test the absolute or conditional convergence of the following series.

 (a) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$

 (d) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{a+nb}$ (e) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\ln(n+1)}$ (f) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

 (g) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^{n-2}}$ (h) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ (i) $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{\sqrt{n}+1}{n+1}\right)$

 (j) $\sum_{n=1}^{\infty} (-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$

Answers

1. (a) Converges (b) Converges (c) Converges (d) Diverges
 (e) Converges (f) Converges.
2. (a) absolutely convergent (b) conditionally convergent
 (c) absolutely convergent (d) conditionally convergent
 (e) absolutely convergent (f) absolutely convergent
 (g) absolutely convergent (h) absolutely convergent
 (i) absolutely convergent (j) conditionally convergent

5.6 POWER SERIES

Definition (Power series)

A power series about $x = 0$ is a series of the form
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n + \ldots$$

A power series about $x = a$ is a series of the form
$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

in which a is centre and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ all are constants.

Interval, Center and Radius of Convergence of a Power Series

Consider a power series
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n + \ldots$$

If there exists a positive number r such that the series converges for $|x| < r$ and diverges for $|x| > r$, then $(-r, r)$ is called the **interval of convergence**, $\frac{r+(-r)}{2} = 0$ is **center of convergence** and r is called the **radius of convergence.**

Note: If there exists a positive number r such that the above series converges for $|x| \leq r$ and diverges for $|x| > r$, then $[-r, r]$ is called the interval, $\frac{r+(-r)}{2} = 0$ is center and r is called the radius of convergence.

Remark: If r is the radius of convergence of the above power series, the interval of convergence (or region of convergence) is any one of the following intervals:

(a) $(-r, r)$ (b) $(-r, r]$ (c) $[-r, r)$ (d) $[-r, r]$.

Example 49: For what value of x does the following series convergence?
$$\sum_{n=0}^{\infty} (-1)^{n-1}\frac{x^n}{n}$$

Solution: Let,
$$u_n = (-1)^{n-1}\frac{x^n}{n}$$

Then,
$$\left|\frac{u_{n+1}}{u_n}\right| = \left|(-1)^n\frac{x^{n+1}}{n+1} \times \frac{n}{(-1)^{n-1}x^n}\right| = \left|\frac{x}{1+1/n}\right|$$

So,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x}{1+1/n}\right| = |x|$$

By ratio test, the given series converges for $|x| < 1$ i.e. $-1 < x < 1$.

At $x = -1$, the given series becomes,

$$\sum_{n=0}^{\infty}\frac{(-1)^{2n-1}}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$$

which is divergent by p–test with p = 1.

At $x = 1$, the given series becomes,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

which is convergent being an alternative harmonic series.

Thus, the given series is convergent for any value of x in $-1 < x \le 1$.

Example 50: Find the interval of convergence of the series

$$1+\frac{2x}{5}+\frac{6x^2}{9}+\frac{14x^3}{17}+\ \dots\ +\frac{(2^n-2)x^{n-1}}{(2^n+1)}$$

Solution: Given series is,

$$1+\frac{2x}{5}+\frac{6x^2}{9}+\frac{14x^3}{17}+\ \dots\ +\frac{(2^n-2)x^{n-1}}{(2^n+1)}$$

This is a power series whose general term is,

$$u_n = \left(\frac{2^n-2}{2^n+1}\right)x^{n-1}$$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)x^n \times \left(\frac{2^n+1}{2^n-2}\right)\cdot\frac{1}{x^{n-1}}$$

$$= \lim_{n\to\infty}\left[\frac{2^n(2-2/2^n)}{2^n(2+1/2^n)}\times\frac{2^n(1+1/2^n)}{2^n(1-2/2^n)}\times\frac{x^n}{x^{n-1}}\right]$$

$$= \lim_{n\to\infty}\left(\frac{2-2/2^n}{2+1/2^n}\times\frac{1+1/2^n}{1-2/2^n}\right)\cdot x$$

$$= \left(\frac{2-0}{2+0}\times\frac{1+0}{1-0}\right)\cdot x$$

$$= x$$

Then by D'Alembert ratio test, the given series is convergent for $|x| < 1$, is divergent for $|x| > 1$ and further test is needed at $|x| = 1$.

At $x = 1$,

$$u_n = \frac{2^n-2}{2^n+1} = \frac{2^n(1-2/2^n)}{2^n(1+1/2^n)} = \frac{1-2/2^n}{1+1/2^n}$$

Here,

$$\lim_{n\to\infty} u_n = \sum_{n=1}^{\infty}\left(\frac{1-2/2^n}{1+1/2^n}\right) = \frac{1-0}{1+0} = 1 \ne 0.$$

This means, the given series is divergent at $x = 1$.

And at $x = -1$,

$$u_n = (-1)^{n-1}\left(\frac{2^n-2}{2^n+1}\right) = (-1)^{n-1}\left(\frac{2-2/2^n}{1+1/2^n}\right)$$

This is an alternative series, whose general term is,

$$v_n = \left(\frac{2^n-2}{2^n+1}\right).$$

Here,

$$\lim_{n\to\infty} v_n = \sum_{n=1}^{\infty}\left(\frac{2^n-2}{2^n+1}\right) = \sum_{n=1}^{\infty}\left(\frac{2-2/2^n}{1+1/2^n}\right) = \frac{2-0}{1+0} = 2 \ne 0.$$

This shows that the given series is divergent by Leibnitz test.

Therefore, the given series converges for $|x| < 1$ and diverges for $|x| \ge 1$.

Thus, the interval of convergence of the given series is $(-1, 1)$.

Example 51: Find the interval of convergence of the series

$$\Sigma\left(\frac{x^n}{3^n n^2}\right) \quad \text{for } x > 0.$$

Solution:

Given series is,

$$\Sigma\left(\frac{x^n}{3^n n^2}\right) \quad \text{for } x > 0.$$

This is a power series whose general term is,

$$u_n = \frac{x^n}{3^n n^2}$$

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{x^{n+1}}{3^{n+1}(n+1)^2}\times\frac{3^n x^2}{x^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{x}{3(1+1/n)^2}\right)$$

$$= \frac{x}{3(1+0)^2}$$

$$= \frac{x}{3}$$

By D'Alembert ratio test, the given series is convergent for $\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$ and is divergent for $\left|\frac{x}{3}\right| > 1 \Rightarrow |x| > 3$. And, at $\left|\frac{x}{3}\right| = 1 \Rightarrow |x| = 3$, the further test is needed.

Given that, $x > 0$. So, by the D'Alembert ratio test, the given series is convergent for $0 < x < 3$ and diverges for $x > 3$ and, further test is needed at $x = 3$.

At $x = 3$,

$$u_n = \frac{3^n}{3^n \cdot n^2} = \frac{1}{n^2}$$

This shows that Σu_n is a p-series with $p = 2 > 1$. Then the series is convergent at $x = 3$, by p-test.

Thus, the given series is convergent for $0 < x \le 3$ and is divergent for $x > 3$. Therefore, the interval of convergence of given series is $(0, 3]$.

Example 52: Test the convergence and find the interval, radius and center of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\,5^n}$.

Solution: The general term of given infinite series is

$$u_n = \frac{(x-5)^n}{n5^n}$$

So,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(n-5)^n}$$

$$= \lim_{n\to\infty} \frac{(x-5)}{\left(1+\frac{1}{n}\right)5}$$

$$= \frac{x-5}{5}$$

Then by ratio test the series is convergent for $\left|\frac{x-5}{5}\right| < 1$ and divergent for $\left|\frac{x-5}{5}\right| > 1$. Further test is necessary for $\left|\frac{x-5}{5}\right| = 1$.

At $\left|\frac{x-5}{5}\right| = 1$ implies $\frac{x-5}{5} = 1$ and $\frac{x-5}{5} = -1$.

At $\frac{x-5}{5} = 1$, given series reduces to $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ with general term $u_n = \frac{1}{n}$, which is divergent by p-test.

At $\frac{x-5}{5} = -1$, given series reduced to $-1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \ldots\ldots$ which is an alternating series and each term is less than preceding term and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

So it is convergent at $\frac{x-5}{5} = -1$.

Thus we get given series is convergent for

$$-1 \le \frac{x-5}{5} < 1$$

$$\Rightarrow \quad -5 \le x - 5 < 5$$

$$\Rightarrow \quad 0 \le x < 10$$

Required interval of convergence is $[0, 10)$.

And, the radius of the series is, $r = \frac{10-0}{2} = \frac{10}{2} = 5$

Also, center of the convergence is,

$$C = \frac{0+10}{2} = 5.$$

Example 53: Find the interval and radius of convergence of the series

$$\frac{x}{1.3} + \frac{x^2}{2.5} + \frac{x^3}{3.7} + \ldots + \frac{x^n}{n(2n+1)} + \ldots\ldots$$

Solution: The general term of given series

$$\frac{x}{1.3} + \frac{x^2}{2.5} + \frac{x^3}{3.7} + \ldots\ldots + \frac{x^n}{n(2n+1)} + \ldots\ldots$$

is,

$$u_n = \frac{x^n}{n(2n+1)}$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)(2n+3)} \times \frac{n(2n+1)}{x^n}$$

$$= \lim_{n\to\infty} \frac{x \cdot n(2+1/n)}{n(1+1/n)\,n(2+3/n)}$$

$$= \frac{2x}{1\times 2}$$

$$= x.$$

By D'Alembert ratio test the given series is convergent for $|x| < 1$ and divergent for $|x| > 1$. And further test needed for $|x| = 1$.

At $x = 1$,

$$u_n = \frac{1}{n(2n+1)} = \frac{1}{n^2(2+1/n)}$$

Choose $u_n = \frac{1}{n^2}$ then the series Σv_n converges by p-test with $p = 2 > 1$. Also,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left[\frac{1/n^2(2+1)n}{1/n^2}\right] = \lim_{n\to\infty} \left(\frac{1}{2+1/4}\right) \sim \frac{1}{2} \neq 0$$

Then by comparison test the series $\Sigma u_n = \Sigma \frac{1}{n(2n+1)}$ converges being Σv_n converges.

And at $x = -1$,

$$u_n = \frac{(-1)^n}{n(2n+1)}$$

This is an alternative series. Set,

$$v_n = \frac{1}{n(2n+1)}$$

Here,

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n(2n+1)} \sim \frac{1}{\infty} \sim 0$$

This shows that the series $\Sigma u_n = \Sigma \frac{(-1)^n}{n(2n+1)}$ converges, by Leibnitz theorem.

Thus the given series converges for $|x| \le 1$ and diverges for $|x| > 1$.

So, the interval of convergence of the given series is $[-1, 1]$.

And the radius of convergence is

$$\text{radius} = \frac{1-(-1)}{2} = \frac{2}{2} = 1.$$

Example 54: Find the interval, center and radius of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n+4}$.

Solution: Given series is,

$$\sum_{n=1}^{\infty} \frac{x^n}{n+4}$$

This is a power series whose general term is,

$$u_n = \frac{x^n}{n+4}$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left[\frac{x^{n+1}}{(n+1)+4} \times \frac{n+4}{x^n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{x^n \cdot x}{n(1+5/n)} \times \frac{n(1+4/n)}{x^n}\right]$$

$$= \lim_{n\to\infty} \left(\frac{1+4/n}{1+5/n}\right) x$$

$$= \left(\frac{1+0}{1+0}\right) x$$

$$= x$$

By D'Alembert ratio test, the given series is convergent for $|x| < 1$ and is divergent for $|x| > 1$. And, further test is needed at $|x| = 1$.

At $x = 1$,

$$u_n = \frac{1}{n+4} = \frac{1}{n(1+4/n)}$$

Choose $v_n = \frac{1}{n}$ then Σv_n diverges by p-test with $p = 1$. And,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{1+4/n}\right) = 1 \text{ (a finite value.)}$$

This means the series Σu_n is also divergent by limit comparison test.

And at $x = -1$,

$$u_n = \frac{(-1)^n}{n+4}$$

which is an alternative series, whose general term is,

$$b_n = \frac{1}{n+4}$$

Here,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(\frac{1}{n+4}\right) = \frac{1}{\infty} = 0.$$

And,

$$\frac{b_{n+1}}{b_n} = \frac{1}{n+5} \times \frac{n+4}{1} = \frac{n+5-1}{n+5} = 1 - \frac{1}{n+4} < 1.$$

$$\Rightarrow \quad b_{n+1} < b_n.$$

Thus, each term of the alternative series is numerically less than the preceding term. So, the given series is convergent by Leibnitz theorem.

Thus, the given series is convergent for $-1 \le x < 1$ and diverges for otherwise.

That is, the interval of convergence of the given series is $[-1, 1)$.

And, the centre of convergence is, $\frac{1+(-1)}{2} = 0$.

Also, the radius of convergence is, $\frac{1-(-1)}{2} = 1$.

Exercise 5.5

1. Find for what value of x, the following series converges

(a) $\sum_{n=1}^{\infty} n^2 x^{n-1}$ (b) $\sum_{n=1}^{\infty} \left(\frac{2^n - 2}{2^n + 1}\right) x^{n-1}$ (c) $\sum_{n=1}^{\infty} \left(\frac{x^n}{3^n n^2}\right)$ for $x > 0$

(d) $\sum_{n=1}^{\infty} \left(\frac{x^n}{n}\right)$ for $x > 0$

2. Find the interval, centre and radius of convergence of the following series

(a) $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ (b) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ (c) $\sum_{n=1}^{\infty} \left(\frac{n+1}{10^n}\right)(x-4)^n$

(d) $\sum_{n=0}^{\infty} \frac{3^{2n}}{(n+1)}(x-2)^n$ (e) $\sum_{n=1}^{\infty} \left(\frac{n+1}{10^n}\right)(x-4)^n$ (f) $\sum_{n=0}^{\infty} \frac{n^2}{2^{3n}}(x+4)^n$

(g) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ (h) $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots$ (i) $\sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^{2n}$

(j) $\frac{x}{1.3} + \frac{x^2}{2.5} + \frac{x^3}{3.7} + \ldots. + \frac{x^n}{n(2n+1)} + \ldots\ldots$

Answers

1. (a) $|x| < 1$ (b) $|x| < 1$ (c) $x \le 3$
   (d) $x < 1$
2. (a) Interval = $(-\infty, \infty)$, centre = 0, radius = ∞
   (b) Interval = $[-1, 1]$, centre = 0, radius = 1.
   (c) Interval = $(-6, 14)$, centre = 4, radius = 10.
   (d) Interval = $\left(\frac{17}{9}, \frac{19}{4}\right)$, centre = 2, radius = 1.
   (e) Interval = $(-6, 14)$, centre = 4, radius = 10.
   (f) Interval = $(-2, 2)$, centre = 0, radius = 2.
   (g) Interval = $(-8, 12)$, center = 2, radius = 10.
   (h) Interval = $(-1, 1)$, center = 0, radius = 1.
   (i) Interval = $[-2, 2]$, center = 0, radius = 2
   (j) Interval = $[-1, 1]$, center = 0, radius = 1.