Heat equation

k - interval of time

HEAT EQUATION

We have, one dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2}{\partial x^2} u(x,t) \quad (1)$$

$$\frac{\partial u(x_i,t_j)}{\partial t} = c^2 \frac{\partial^2 u(x_i,t_j)}{\partial x^2}$$

or, $$\frac{u_{i,j+1} - u_{i,j}}{k} = \left(\frac{c^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2}\right)$$

or, $$u_{i,j+1} = u_{i,j} + \frac{kc^2}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \quad (2)$$

if $\frac{kc^2}{h^2} = \frac{1}{2}$ then (2) is

$$u_{i,j+1} = u_{i,j} + \frac{1}{2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$= \cancel{u_{i,j}} + \frac{1}{2}u_{i+1,j} - \cancel{u_{i,j}} + \frac{1}{2}u_{i-1,j}$$

$$u_{i,j+1} = \frac{1}{2}\left(u_{i+1,j} + u_{i-1,j}\right)$$

which is Bender-schmid formula for solving heat equation.

2017

3 (a)

t	0	0.1	0.2	0.3
$F(t)$	30.13	31.62	32.87	33.95

t global interval 0.1

Find velocity at $t = 0.2, 0, 0.3, 0.15$

Solution:

By using central difference quotient

$$F'(t) = \frac{f(t+h) - f(t-h)}{2h}$$

$$F'(0.2) = \frac{F(0.2+h) - F(0.2-h)}{2h}$$

$$= \frac{F(0.2+0.1) - F(0.2-0.1)}{2 \times 0.1}$$

$$= \frac{F(0.3) - F(0.1)}{0.2}$$

$$= \frac{33.95 - 31.62}{0.2}$$

$$= 11.65$$

For, $t = 0$, using forward difference quotient

$$F'(t) = \frac{F(t+h) - F(t)}{h}$$

$$F'(0) = \frac{F(0+h) - F(0)}{h}$$

$$= \frac{F(0+0.1) - F(0)}{0.1}$$

$$= \frac{31.62 - 30.13}{0.1}$$

$$F'(0) = 14.9$$

$\Delta^2 f_0 = f_2 - 2f_1 + f_0$

$\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$

For $t = 0.3$, using backward difference quotient

$$f'(t) = \frac{f(t) - f(t-h)}{h}$$

$$f'(0.3) = \frac{f(0.3) - f(0.3-0.1)}{h}$$

$$= \frac{33.95 - 32.87}{0.1}$$

$$f'(0.3) = 10.8$$

for $t = 0.15$, we need Newton's forward interpolating polynomial

So, Newton's forward difference table is formed as below:

t	$f(t)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	30.13	1.49	-0.24	0.07
0.1	31.62	1.25	-0.17	
0.2	32.87	1.08		
0.3	33.95			

$h = 0.1$

$$s = \frac{t - t_0}{h} = \frac{t-0}{0.1} \quad \therefore s = 10t$$

~~$\frac{d}{dt}(10t)(10t-1)$~~

~~$(10t-1)10 + 10t\frac{d}{dt}(10t-1)$~~

we have,

$$f(t) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0$$

$$\text{or, } f(t) = 30.13 + 10t \times 1.49 + \frac{1}{2}10t(10t-1)(-0.24) + \frac{10t(10t-1)(10t-2)}{6} \times 0.07$$

$$\text{or, } f(t) = 30.13 + 14.9t - 1.2t(10t-1) + 0.1167(10t-1)(10t-2)$$

To derivative each term

$$f'(t) = 14.9 - 1.2(t \times 10 + 10t - 1) + 0.1167\big((10t-1)(10t-2) + 10t(10t-2) + t(10t-1) \times 10\big)$$

$$f'(0.15) = 12.47$$

(3) If $f(x) = x^2$ at $x = 1$ for $h = 0.1, 0.2, 0.03, 0.01$ find error

Solution:

By central difference quotient, we know that

$$F'(x) = \frac{F(x+h) - F(x-h)}{2h}$$

$$\text{or, } F'(1) = \frac{F(1+h) - F(1-h)}{2h}$$

$$= \frac{(1+h)^2 - (1-h)^2}{2h}$$

$$= \frac{1+2h+h^2 - 1+2h-h^2}{2h}$$

$$F'(1) = \frac{4}{2} = 2$$

For exact derivative,

$$F'(x) = 2x$$

$$\therefore F'(1) = 2$$

By central difference quotient,

exact derivative = numerical derivative for all value of h

By Backward difference quotient,

$$F'(x) = \frac{F(x+h) - F(x)}{h}$$

$$F'(1) = \frac{F(1+h) - F(1)}{h}$$

$$= \frac{(1+h)^2 - 1}{h}$$

$$= \frac{1+2h+h^2 - 1}{h}$$

$$F'(1) = 2+h$$

If $h = 0.01$

$$F'(1) = 2.01$$

and similarly

Q. Evaluate $\int_0^1 \frac{1}{x^2+1}\,dx$

1. By Trapezoid Rule
2. By Simpson's $\frac{1}{3}$ Rule
3. By Simpson's $\frac{3}{8}$ Rule

and compare the result with exact value taking number of interval $n = 6$

$h = \frac{b-a}{n} = \frac{1-0}{6} = 1/6$

let $x_0 = 0$

$x_1 = x_0 + h = 0 + 1/6 = 1/6$

$x_2 = x_1 + h = 1/6 + 1/6 = 2/6$

$x_3 = 3/6$

$x_4 = 4/6$

$x_5 = 5/6$

$x_6 = 6/6 = 1$

where, $f(x) = \frac{1}{x^2+1}$

$f_0 = f(x_0) = f(0) = \frac{1}{0^2+1} = 1$ //

$f_1 = f(x_1) = f(1/6) = \frac{1}{(1/6)^2+1} = 36/37$

$f_2 = f(x_2) = f(2/6) = \frac{1}{(2/6)^2+1} = 9/10$

$f_3 = f(x_3) = f(3/6) = \frac{1}{(3/6)^2+1} = 4/5$

$f_4 = f(x_4) = f(4/6) = \frac{1}{(4/6)^2+1} = 9/13$

$f_5 = f(x_5) = f(5/6) = \frac{1}{(5/6)^2+1} = 36/61$

$f_6 = f(x_6) = f(1) = \frac{1}{1+1^2} = 1/2 = 0.5$

Integrating by Trapezoid rule

$I_T = \frac{h}{2}[f_0 + f_6 + 2(f_1 + f_2 + f_3 + f_4 + f_5)]$

$= \frac{1}{2} \times \frac{1}{6}\left[1 + 0.5 + 2\left(\frac{36}{37} + \frac{9}{10} + \frac{4}{5} + \frac{9}{13} + \frac{36}{61}\right)\right]$

$I_T = \cancel{0.8}\ 0.7842$ //

Integrating by simpson's 1/3 rule

$I_s = \frac{h}{3}[f_0 + f_6 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4)]$

$= \frac{1}{3} \times \frac{1}{6}\left\{1 + 0.5 + 4 \times \left(\frac{36}{37} + \frac{4}{5} + \frac{36}{61}\right) + 2\left(\frac{36}{40} + \frac{36}{52}\right)\right]$

$= \frac{1}{18}\left\{1.5 + 4 \times \left[\frac{36}{37} + \frac{4}{5} + \frac{36}{61}\right] + 2\left(\frac{9}{10} + \frac{9}{13}\right)\right]$

$= 0.7853$

Integrating by simpson's 3/8 rule.

Note: Generally Simpson's 3/8 rule is more accurate than Simpson's 1/3 rule which is also more accurate than trapezoid rule.

$$I = \frac{3h}{8}\left[f_0 + f_6 + 3(f_1 + f_2 + f_4 + f_5) + 2 \times f_3 \right]$$

$$= \frac{3}{8} \cdot \frac{1}{6}\left[1 + 0.5 + 3\left(\frac{36}{37} + \frac{9}{10} + \frac{9}{13} + \frac{36}{61}\right) + 2 \times \frac{4}{5} \right]$$

$= 0.7853$

Exact integral $= \int_0^1 \frac{1}{x^2+1} dx$

$= [\tan^{-1} x]_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4}$

$= 0.7853$

Error in trapezoid integration

$$= \left| \frac{\text{Exact value} - \text{integrate by trapezoid}}{\text{exact value}} \right| \times 100\%$$

$$= \left| \frac{0.7853 - 0.7842}{0.7853} \right| \times 100\%$$

$= 0.14\%$

Since Simpson's 3/8 rule's error is ... $\frac{2\pi}{12}$ ... radian ...

Error in Simpson's 1/3 rule = 0%

Simpson's rule are more accurate than trapezoid rule.

Q. Evaluate: $\int_0^{\pi/2} \sin x \, dx$ by Trapezoid Rule and Simpson's rule and compare the result with exact value.

Solution: $h = \frac{b-a}{n} = \frac{1}{6}\left(\frac{\pi}{2} - 0\right)$

$\therefore h = \frac{\pi}{12}$

Let $x_0 = 0$	$f(x) = \sin x$
$x_1 = \frac{\pi}{12}$	$f(0) = \sin 0 = 0$
$x_2 = \frac{2\pi}{12}$	$f_1 = f(x_1) = f\left(\frac{\pi}{12}\right)$
$x_3 = \frac{3\pi}{12}$	$= \sin\left(\frac{\pi}{12}\right)$
$x_4 = \frac{4\pi}{12}$	$= 0.2588$
$x_5 = \frac{5\pi}{12}$	$f_2 = f(x_2) = f\left(\frac{2\pi}{12}\right)$
$x_6 = \frac{6\pi}{12} = \frac{\pi}{2}$	$= 0.5$

$f_3 = f(x_3) = f\left(\frac{3\pi}{12}\right) = 0.7071$

$f_4 = f(x_4) = f\left(\frac{4\pi}{12}\right) = 0.8660$

$f_5 = f(x_5) = f\left(\frac{5\pi}{12}\right) = 0.9659$

$f_6 = f(x_5) = f\left(\frac{6\pi}{12}\right) = f\left(\frac{\pi}{2}\right) = 1$

Integrate by Trapezoidal Rule:-

$I_T = h/_2 \left[f_0 + f_6 + 2(f_1 + f_2 + f_3 + f_4 + f_5)\right]$

$= \frac{1}{2} \times \frac{\pi}{12} \left[0 + 1 + 2(0.2588 + 0.5 + 0.7071 + 0.8660 + 0.9659)\right]$

$= 0.9942$

Integrate by simpson's $\frac{1}{3}$ rule:

$I_S = h/_3 \left[f_0 + f_6 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4)\right]$

$= \frac{1}{3} \times \pi/_{12} \left[0 + 1 + 4(0.2588 + 0.7071 + 0.9659) + 2(0.5 + 0.8660)\right]$

$= 1.000003848$

Integrate by Simpson's $3/_8$ rule

$I_S = \frac{3h}{8} \left[f_0 + f_6 + 3(f_1 + f_2 + f_4 + f_5) + 2f_3\right]$

$= 3/_8 \times \pi/_{12} \left[0 + 1 + 3 \times (0.2588 + 0.5 + 0.8660 + 0.9659) + 2 \times 0.7071\right]$

$= \underline{1.00003764}$

Q. Evaluate $\int_0^1 \frac{\sin x}{x} dx$

Q. $\int_1^2 x^2 e^{-x} \sqrt{1 + \sin x}\, dx$

Q. $\int_0^{3\pi/2} \sin x\, dx$

By Trapezoid and Simpson's Rule

Q. Evaluate $\int_0^{\pi/2} \sqrt{\sin x}\, dx$ by Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule

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Q. Evaluate $\int_1^2 x^2 \sin(x^3 - 1) \tan_2 x\, dx$

Solution:

Here,

$h = \frac{b-a}{6} = \frac{2-1}{6} = \frac{1}{6}$

Let $x_0 = 1$

$x_1 = 1 + 1/6 = 7/6$

$x_2 = 7/6 + 1/6 = 8/6$

$x_3 = 8/6 + 1/6 = 9/6$

$x_4 = 9/6 + 1/6 = 10/6$

$x_5 = 10/6 + 1/6 = 11/6$

$x_6 = 2$

$f(x) = x^2 \sin(x^3 - 1) \tanh x$

$f_0 = f(1) = 0$

$f_1 = f(7/6) = 0.6214$

$f_2 = f(8/6) = 1.5158$

$f_3 = f(9/6) = 1.4127$

$f_4 = f(10/6) = -1.2127$

$f_5 = f(11/6) = -2.8761$

$f_6 = f(2) = 2.5334$

Integrate by trapezoidal rule

$I_T = h/2 \left[f_0 + f_6 + 2(f_1 + f_2 + f_3 + f_4 + f_5) \right]$

$= \frac{1}{12} [0 + 2.5334 + 2(0.6214 + 1.5158 + 1.4127 - 1.2127 - 2.8761)]$

$= 0.1263$

Integrate by simpson's 1/3 rule

$I = h/3 \left[f_0 + f_6 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) \right]$

$= \frac{1}{6 \times 3} [0 + 2.5334 + 4(0.6214 + 1.4127 - 2.8761) + 2(1.5158 - 1.2127)]$

$= -0.01268$

Q. Evaluate $\int_0^2 \frac{e^x + \sin x}{1 + x^2} dx$ by Romberg's method correct to two d.p.

Solution:

Here,

$f(x) = \frac{e^x + \sin x}{1 + x^2}$

Taking $n = 2$, $h = \frac{b - a}{n} = \frac{2 - 0}{2} = 1$

Let $x_0 = 0$

$x_1 = 0 + 1 = 1$

$F_0 = F(x_0) = F(0) = \frac{e^0 + \sin 0}{1+0} = 1$

$F_1 = F(x_1) = F(1) = \frac{e^1 + \sin 1}{1+1} = 1.7798$

$F_2 = F(x_2) = F(2) = \frac{e^2 + \sin 2}{1+4} = 1.6596$

$I = \frac{h}{2}[F_0 + 2F_1 + F_2]$

$= \frac{1}{2} \times 1 \, [1 + 2 \times 1.7798 + 1.6596]$

$= 3.1096$

Taking number of interval = 4

$\therefore h = \frac{2-0}{4} = 0.5$

let $x_0 = 0$

$x_1 = x_0 + h = 0.5$

$x_2 = x_1 + h = 0.5 + 0.5 = 1$

$x_3 = x_2 + h = 1 + 0.5 = 1.5$

$x_4 = x_3 + h = 1.5 + 0.5 = 2$

$F_0 = F(x_0) = F(0) = \frac{e^0 + \sin 0}{1+0} = 1$

$F_1 = F(x_1) = F(0.5) = \frac{e^{0.5} + \sin 0.5}{1 + 0.5^2} = 1.7025$

$F_2 = F(x_2) = F(1) = \frac{e^1 + \sin 1}{1+1^2} = 1.7798$

$F_3 = F(x_3) = F(1.5) = \frac{e^{1.5} + \sin 1.5}{1 + 1.5^2} = 1.6859$

$F_4 = F(x_4) = F(2) = \frac{e^2 + \sin 2}{1+2^2} = 1.6596$

$I = \frac{h}{2}[F_0 + F_4 + 2(F_1 + F_2 + F_3)]$

$= \frac{0.5}{2}[1 + 1.6596 + 2(1.7025 + 1.7798 + 1.6859)]$

$= 3.249$

Taking number of interval = 8

$h = \frac{2-0}{8} = 0.25$

let $x_0 = 0$	$F_0 = F(x_0) = F(0) = 1$
$x_1 = 0.25$	$F_1 = F(x_1) = F(0.25) = 1.4413$
$x_2 = 0.5$	$F_2 = F(x_2) = F(0.5) = 1.7025$
$x_3 = 0.75$	$F_3 = F(x_3) = F(0.75) = 1.7911$
$x_4 = 1$	$F_4 = F(x_4) = F(1) = \cancel{1.7025}\ 1.7798$
$x_5 = 1.25$	$F_5 = F(x_5) = F(1.25) = 1.7324$
$x_6 = 1.5$	$F_6 = F(x_6) = F(1.5) = 1.6859$
$x_7 = 1.75$	$F_7 = F(x_7) = F(1.75) = 1.6587$
$x_8 = 2$	$F_8 = F(x_8) = F(2) = 1.6596$

$I = \frac{h}{2}[F_0 + F_8 + 2(F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7)]$

$= \frac{0.25}{2}[1 + 1.6596 + 2(1 + 1.4413 + 1.7025 + 1.7911 + 1.7798 + 1.7324 + 1.6859 + 1.6587)]$

$= 3.2803$

In the comparison of 3.1096 and 3.2490

More Accurate (M.A) = 3.2490

Less Accurate (L.A) = 3.1096

$$\text{Best estimate} = M.A + \frac{1}{2^n - 1}(MA - LA)$$

$$= 3.249 + \frac{1}{2^2 - 1}(3.249 - 3.1096)$$

$$= 3.249 + 1/3\,(3.249 - 3.1096)$$

$$= 3.2954$$

In the comparison of 3.2490 and 3.2803

More Accurate (MA) = 3.2803

Less Accurate (LA) = 3.2490

$$\text{Best estimate} = MA + \frac{1}{2^n - 1}(MA - LA)$$

$$= 3.2803 + \frac{1}{2^2 - 1}(3.2803 - 3.2490)$$

$$= 3.2803 + 1/3\,(3.2803 - 3.2490)$$

$$= 3.2907$$

In the comparison of 3.2954 and 3.2907

More Accurate (M.A) = 3.2907

Less Accurate (L.A) = 3.2954

$$\text{Best estimate} = MA + \frac{1}{2^n - 1}(MA - LA)$$

$$= 3.2907 + \frac{1}{2^4 - 1}(3.2907 - 3.2954)$$

$$= 3.2907 + 1/15\,(3.2907 - 3.2954)$$

$$= 3.2903$$

Q. Evaluate $\int_{0.5}^{1.5} e^{-x^2}\,dx$

Solution:

Here, $f(x) = e^{-x^2}$

Taking $n = 1$, $h = \frac{b-a}{n} = \frac{1.5 - 0.5}{1} = 1$

Let, $x_0 = a = 0.5$

$x_1 = x_0 + h = 0.5 + 1 = 1.5$

$f_0 = f(x_0) = f(0.5) = e^{-0.5^2} = 0.7788$

$f_1 = f(x_1) = f(1.5) = e^{-1.5^2} = 0.1053$

$$I = h/2\,[f_0 + f_1] = 1/2\,[0.7788 + 0.1053]$$

$$= 0.44205$$

Taking number of intervals, $n = 2$

$$h = \frac{b-a}{n} = \frac{1.5 - 0.5}{2} = 0.5$$

Let $x_0 = a = 0.5$

$x_1 = x_0 + h = 0.5 + 0.5 = 1$

$x_2 = x_1 + h = 1 + 0.5 = 1.5$

$F_0 = F(x_0) = F(0.5) = 0.7788$

$F_1 = F(x_1) = F(1) = 0.3678$

$F_2 = F(x_2) = F(1.5) = 0.1053$

$I = \frac{h}{2}[F_0 + F_2 + 2F_1] = \frac{0.5}{2}[0.7788 + 0.1053 + 2 \times 0.3678]$

$= 0.4049$

Taking number of intervals, $n = 4$

$h = \frac{b-a}{n}$ or, $h = \frac{1.5-0.5}{4} = 0.25$

Let $x_0 = a = 0.5$

$x_1 = x_0 + h = 0.5 + 0.25 = 0.75$

$x_2 = x_1 + h = 0.75 + 0.25 = 1$

$x_3 = x_2 + h = 1 + 0.25 = 1.25$

$x_4 = x_3 + h = 1.25 + 0.25 = 1.5$

$F_0 = F(x_0) = F(0.5) = 0.7788$

$F_1 = F(x_1) = F(0.75) = 0.5697$

$F_2 = F(x_2) = F(1) = 0.3678$

$F_3 = F(x_3) = F(1.25) = 0.2096$

$F_4 = F(x_4) = F(1.5) = 0.1053$

$I = \frac{h}{2}[F_0 + F_4 + 2(F_1 + F_2 + F_3)]$

$= \frac{0.25}{2}[0.7788 + 0.1053 + 2(0.5697 + 0.3678 + 0.2096)]$

$= 0.3972$

In the comparison of 0.4420 and 0.4049

More accurate (MA) = 0.4049

Less accurate (LA) = 0.4420

Best estimate $= MA + \frac{1}{2^n - 1}(MA - LA)$

$= 0.4049 + \frac{1}{2^2 - 1}(0.4049 - 0.4420)$

$= 0.3925$

In the comparison of 0.4049 and 0.3972

More accurate (MA) = 0.3972

Less accurate (LA) = 0.4049

Best estimate $= MA + \frac{1}{2^n - 1}(MA - LA)$

$= 0.3972 + \frac{1}{3}(0.3972 - 0.4049)$

$= 0.3946$

In the comparison of 0.3925 and 0.3946

More accurate (MA) = 0.3946

Less accurate (LA) = 0.3925

Best estimate $= MA + \frac{1}{2^n - 1}(MA - LA)$

$= 0.3946 + \frac{1}{2^4 - 1}(0.3946 - 0.3925)$

$= 0.3947$ //

Q. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ by Romberg integration. (1, 2, 4) (and (2, 4, 8) Yeslai bakixa)

Solution:

Here, $F(x) = \frac{1}{1+x^2}$

Taking no. of intervals, $n = 1$

$h = \frac{b-a}{n}$

or, $h = \frac{1-0}{1}$ $\therefore h = 1$

Let $x_0 = a = 0$

$x_1 = x_0 + h = 0 + 1 = 1$

$F_0 = F(x_0) = F(0) = 1$

$F_1 = F(x_1) = \frac{1}{1+1} = F(1) = 1/2$

$I = h/2\,[F_0 + F_1] = 1/2 \times 1\,[1 + 1/2]$

$= 0.75$

Taking no. of intervals, $n = 2$

$h = \frac{b-a}{n} = \frac{1-0}{2}$ $\therefore h = 0.5$

Let $x_0 = a = 0$

$x_1 = 0 + 0.5 = 0.5$

$x_2 = 0.5 + 0.5 = 1$

$F_0 = F(x_0) = F(0) = 1$

$F_1 = F(x_1) = F(0.5) = 0.8$

$F_2 = F(x_2) = F(1) = 0.5$

$I = h/2\,[F_0 + F_2 + 2F_1] = 1/2 \times 0.5\,[1 + 0.5 + 2 \times 0.8]$

$= 0.775$

Taking no. of intervals, $n = 4$

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

Let $x_0 = a = 0$

$x_1 = 0 + 0.25 = 0.25$

$x_2 = 0.25 + 0.25 = 0.5$

$x_3 = 0.5 + 0.25 = 0.75$

$x_4 = 0.75 + 0.25 = 1$

$F_0 = F(x_0) = F(0) = 1$

$F_1 = F(x_1) = F(0.25) = 0.9411$

$F_2 = F(x_2) = F(0.5) = 0.8$

$F_3 = F(x_3) = F(0.75) = 0.64$

$F_4 = F(x_4) = F(1) = 0.5$

$I = \frac{h}{2}[f_0 + f_4 + 2(f_1 + f_2 + f_3)]$

$= \frac{0.25}{2}[1 + 0.5 + 2(0.9411 + 0.8 + 0.64)]$

$= 0.7827$

In the comparison of 0.75 and 0.775

More accurate (MA) = 0.775

Less accurate (LA) = 0.75

Best estimate = MA + $\frac{1}{2^n - 1}$ (MA − LA)

$= 0.775 + \frac{1}{2^2 - 1}(0.775 - 0.75)$

$= 0.7833$

In the comparison of 0.775 and 0.7827

More accurate (MA) = 0.7827

Less accurate (LA) = 0.775

Best estimate = MA + $\frac{1}{2^n - 1}$ (MA − LA)

$= 0.7827 + \frac{1}{2^2 - 1}(0.7827 - 0.775)$

$= 0.7852$

In the comparison of 0.7833 and 0.7852

More accurate (MA) = 0.7852

Less accurate (LA) = 0.7833

Best estimate = MA + $\frac{1}{2^n - 1}$ (MA − LA)

$= 0.7852 + \frac{1}{2^n - 1}(0.7852 - 0.7833)$

$= 0.7853$

$\therefore I = 0.785$ correct to 3 dp.

Q. $\int_0^4 \log x \, dx$

Q. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}} dx$

Solution:

$f(x) = \frac{\cos x}{\sqrt{1+\sin x}}$

Taking no. of intervals n = 1

$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{1} = \pi/2$

Let $x_0 = a = 0$

$x_1 = x_0 + h = 0 + \pi/2 = \pi/2$

$f_0 = f(x_0) = f(0) = 1$

$f_1 = f(x_1) = f(\pi/2) = 0$

$\therefore I = \frac{h}{2}[f_0 + f_1] = \frac{\pi/2}{2}(1 + 0) = 0.7853$

$I = \frac{h}{2}\{f_0 + f_4 + 2(f_1 + f_2 + f_3)\}$

$= \frac{0.25}{2}[1 + 0.5 + 2(0.9411 + 0.8 + 0.64)]$

$= 0.7827$

In the comparison of 0.75 and 0.775

More accurate (MA) = 0.775

Less accurate (LA) = 0.75

Best estimate = $MA + \frac{1}{2^n - 1}(MA - LA)$

$= 0.775 + \frac{1}{2^2 - 1}(0.775 - 0.75)$

$= 0.7833$

In the comparison of 0.775 and 0.7827

More accurate (MA) = 0.7827

Less accurate (LA) = 0.775

Best estimate = $MA + \frac{1}{2^n - 1}(MA - LA)$

$= 0.7827 + \frac{1}{2^2 - 1}(0.7827 - 0.775)$

$= 0.7852$

In the comparison of 0.7833 and 0.7852

More accurate (MA) = 0.7852

Less accurate (LA) = 0.7833

Best estimate = $MA + \frac{1}{2^n - 1}(MA - LA)$

$= 0.7852 + \frac{1}{2^4 - 1}(0.7852 - 0.7833)$

$= 0.7853$

$\therefore I = 0.785$ correct to 3 d.p.

Q. $\int_2^4 \log x \, dx$

Q. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1 + \sin x}} dx$

Solution:

$f(x) = \frac{\cos x}{\sqrt{1 + \sin x}}$

Taking no. of intervals $n = 1$

$h = \frac{b - a}{n} = \frac{\pi/2 - 0}{1} \therefore h = \pi/2$

Let $x_0 = a = 0$

$x_1 = x_0 + h = 0 + \pi/2 = \pi/2$

$f_0 = f(x_0) = f(0) = 1$

$f_1 = f(x_1) = f(\pi/2) = 0$

$\therefore I = \frac{h}{2}[f_0 + f_1] = \frac{1}{2}\frac{\pi}{2}[1 + 0] = 0.7853$

Taking no. of intervals $n=2$

$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{2} = \pi/4$

let $x_0 = a = 0$

$x_1 = 0 + \pi/4 = \pi/4$

$x_2 = \pi/2$

$F_0 = F(x_0) = F(0) = 1$

$F_1 = F(x_1) = F(\pi/4) = 0.5411$

$F_2 = F(x_2) = F(\pi/2) = 0$

$I = h/2 \, [F_0 + F_2 + 2F_1] = \frac{\pi}{8} [1 + 0 + 2 \times 0.5411]$

$= 0.8176$

Taking no. of intervals $n=4$

$h = \frac{b-a}{n}$

or, $h = \frac{\pi/2 - 0}{4} = \pi/8$

let $x_0 = a = 0$

$x_1 = 0 + \pi/8 = \pi/8$

$x_2 = \pi/8 + \pi/8 = \pi/4$

$x_3 = 3\pi/8$

$x_4 = \pi/2$

$F_0 = F(x_0) = F(0) = 1$

$F_1 = F(x_1) = F(\pi/8) = 0.7856$

$F_2 = F(x_2) = F(\pi/4) = 0.5411$

$F_3 = F(x_3) = F(3\pi/8) = 0.2758$

$F_4 = F(x_4) = F(\pi/2) = 0$

$I = h/2 \, [F_0 + F_4 + 2(F_1 + F_2 + F_3)]$

$= \frac{\pi}{16} [1 + 0 + 2(0.7856 + 0.5411 + 0.2758)]$

$= 0.8256$

In comparison of 0.7853 and 0.8176

More accurate (MA) = 0.8176

Less accurate (LA) = 0.7853

$\therefore$ Best estimate = $MA + \frac{1}{2^n - 1} (MA - LA)$

$= 0.8176 + \frac{1}{3} (0.8176 - 0.7853)$

$= 0.8283$

In comparison of 0.8176 and 0.8256

More accurate (MA) = 0.8256

less accurate (LA) = 0.8176

$\therefore$ Best estimate = $MA + \frac{1}{2^n - 1} (MA - LA)$

$= 0.8256 + \frac{1}{2^2 - 1} (0.8256 - 0.8176)$

$= 0.8282$

In comparison of 0.8283 and 0.8282

More Accurate (M.A.) = 0.8282

Less Accurate (L.A.) = 0.8283

$\therefore$ Best estimate $= MA + \frac{1}{2^n - 1}(MA - LA)$

$= 0.8282 + \frac{1}{15}(0.8282 - 0.8283)$

$= 0.8281$

Gaussian Integration (Gauss quadrature Integration)

(Gauss - Legendre's integration):-

2023 New

3) Evaluate $\int_2^4 (x^4+1)\,dx$ by Gaussian 2 and 3 point formula.

Solution:

$I = \int_2^4 (x^4+1)\,dx$ ---(1)

To change limit putting

$x = \frac{(4-2)t + 4+2}{2}$

or, $x = \frac{2t+6}{2} = t+3$

or, $\frac{dx}{dt} = 1$

$\therefore dx = dt$

when $x=2$,

$2 = t+3$

$\therefore t = -1$

when $x=4$,

$t = 1$

$\therefore I = \int_{-1}^{1} [(t+3)^4+1]\,dt$

By two points formula

$I = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right)$

where, $f(t) = (t+3)^4+1$

$f(1/\sqrt{3}) = \left(\frac{1}{\sqrt{3}}+3\right)^4 + 1 = 164.7743$

$f\left(\frac{-1}{\sqrt{3}}\right) = \left(\frac{-1}{\sqrt{3}}+3\right)^4 + 1 = 35.4478$

$\therefore \int_{-1}^{1} (t+3)^4+1\,dt = 164.7743 + 35.4478$

$= 200.221$

By three points formula

$$I = 0.5556\,F(-0.77460) + 0.88897(0) + 0.5556\,F(0.77460)$$

Here, $F(-0.77460) = (-0.77460+3)^4 + 1$
$= 25.5263$

$F(0) = (0+3)^4 + 1 = 82$

$F(0.77460) = (0.77460+3)^4 + 1$
$= 203.9942$

$\therefore I = 0.5556 \times 25.5263 + 0.8889 \times 82 + 0.5556 \times 203.9942$
$= 200.4173$

Solution of a system of EQUATIONS

(i) Gauss elimination ~~Method~~ Model:-
2015 spring

Solve: $3x + 2y + z = 10$
$2x + 3y + 2z = 14$
$x + 2y + 3z = 14$

solution:

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 2 & 3 & 2 & : & 14 \\ 3 & 2 & 1 & : & 10 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$, $R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -1 & -4 & : & -14 \\ 0 & -4 & -8 & : & -32 \end{bmatrix}$$

$R_2 : R_2/-1$, $R_3 : R_3/-4$

$$\begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & 1 & 4 & : & 14 \\ 0 & 1 & 2 & : & 8 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & 1 & 4 & : & 14 \\ 0 & 0 & -2 & : & -6 \end{bmatrix}$$

From R_3

$-2z = -6$
$\therefore z = 3$

From R_2

$y + 4z = 14$
or, $y + 12 = 14$
$\therefore y = 2$

From R

Q. Solve: $2f_{xx}(x,t) = f_t(x,t)$, $0 \le t \le 1.5$ and $0 \le x \le 4$

given initial conditions $f(x,0) = 50(4-x)$, $0 \le x$

$f(0,t) = 0$ for $0 \le t \le 1.5$

$f(4,t) = 0$ for $0 \le t \le 1.5$

Solution.

$$2f_{xx}(x,t) = f_t(x,t)$$

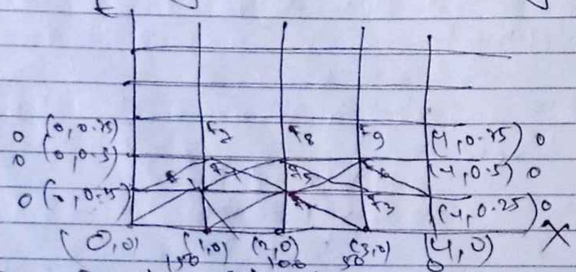
or, $$\frac{\partial}{\partial t} f(x,t) = 2\frac{\partial^2}{\partial x^2} f(x,t) \quad (1)$$

Comparing (1) with

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

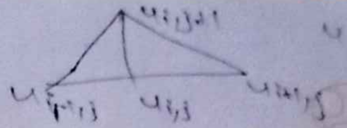
$$c^2 = 2$$

According to question on figure is formed



For Bender Schmidt Formula

$\frac{kc^2}{h^2} = 1/2$ | $\frac{2k}{h^2} = 1/2$ if $h=1$, | $k = \frac{h^2}{4}$, $k = 0.25$



$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$

$f(x,0) = 50(4-x)$ is given

$f(1,0) = 50(4-1) = 150$

$f(2,0) = 100$

$f(0,t) = 0$

$f(0,0) = 0$

$f(0,0.25) = 0$

$f(0,0.5) = 0$

$f(0,0.75) = 0$

$f(4,t) = 0$

$f(4,0.25) = 0$

$f(4,0.5) = 0$

$f(4,0.75) = 0$

By using Bender-Schmidt formula,

$f_1 = \frac{1}{2}(0+100) = 50$

$f_2 = \frac{1}{2}(150+50) = 100$

$f_3 = \frac{1}{2}(100+0) = 50$

$f_4 = \frac{1}{2}(0+f_2) = \frac{1}{2} \times 100 = 50$

$f_5 = \frac{1}{2}(f_1+f_3) = \frac{1}{2}(50+50) = 50$

$f_6 = \frac{1}{2}(f_2+0) = \frac{1}{2}(100) = 50$

$f_7 = \frac{1}{2}(0+f_5) = \frac{1}{2} \times 50 = 25$

$f_8 = \frac{1}{2}(f_4+f_6) = \frac{1}{2} \times 100 = 50$

$f_9 = \frac{1}{2}(f_5 + 0) = \frac{1}{2} \times 50 = 25$

Similarly, we can find temperature distributions at other points which is required solution.

Second order partial differential equation

The general second order partial differential equation is of the form.

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = F\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \quad (1)$$

(1) is classified on the basis of values of $b^2 - 4ac$

(a) Elliptical type of partial differential equation -

The differential equation is called elliptical type of p.d.e. if $b^2 - 4ac < 0$

For example,

$$4f_{xx} + f_{xy} + 4f_{yy} + 2 = 0 \quad (2)$$

$a = 4$, $c = 4$
$b = 1$

Here, $b^2 - 4ac = (1)^2 - 4 \times 4 \times 4$
$= -15 < 0$

$\therefore$ (2) is elliptical type of p.d.e.

(b) Parabolic type of partial differential equation

The partial differential eq^n is called parabolic type of p.d.e if
$b^2 - 4ac = 0$ for example,

$$f_{xx} + 2f_{xy} + f_{yy} + f = 0 \quad (3)$$

$a = 1$
$b = 2$
$c = 1$

$b^2 - 4ac = 4 - 4 \times 1 \times 1 = 0$

$\therefore$ (3) is a parabolic type of p.d.e.

(c) Hyperbolic type of partial differential equation

The second partial differential equation is called hyperbolic type of p.d.e if
$b^2 - 4ac > 0$

For example,

$$f_{xx} + 10f_{xy} + f_{yy} + f_x + f_y + f = 7 \quad (4)$$

$a = 1$, $c = 1$
$b = 10$

$b^2 - 4ac = 100 - 4 \times 1 \times 1$
$= 96 > 0$

$\therefore$ (4) is hyperbolic type of p.d.e.

2021 fall

(c) $u_{xx} + 2u_{xy} + u_{yy} = 0$

Solution:

$u_{xx} + 2u_{xy} + u_{yy} = 0$ (1)

Comparing (1) with

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = F\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad (2)$$

$\therefore a = 1$
$b = 2$
$c = 1$

Here, $b^2 - 4ac = 4 - 4 \times 1 \times 1$
$= 0$

Try Questions

Q Solve:

$x + y + z + w = 2$
$x + y + 3z - 2w = -6$
$2x + 3y - z + 2w = 7$
$x + 2y + z - w = -4$

by Gauss elimination method

Ill condition system of linear equation:

Rounding of coefficient of some equation gives very different in solution such a system of equation is called ILL condition system of linear equations.

For example; In solving the equations;

$$\begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad (1)$$

$x = 1$
$y = 1$

If we modify R.H.S of (1) is

$$\begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.98 \\ 2.02 \end{bmatrix}$$

$\therefore x = 0$
$y = 2$

If we have another modification of RHS as

$$\begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.02 \\ 1.98 \end{bmatrix} \quad \therefore x = 2, \; y = 0$$

* ILL condition system of equation in the view of geometry

If the difference of slopes of two lines are very small, at that condition instead of intersecting two straight lines at a single point, they overlap of each other in an interval. At that condition ILL condition is occured.

* ILL condition system of equation in the view of determinant.

If the determinant of coefficient matter is very small or 0, at that condition, ILL condition system is occured.

* Factorization of matrices

Let, A be a matrix. Conversion of A in the form of $A = LU$
where, L is lower triangular and U is upper triangular matrix.
is called factorization of matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \text{---(i)}$$

we have following type of matrix factorization.

(1) Dolittle's method:

In this method in (i), we set $l_{11} = l_{22} = l_{33} = 1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

(2) Crout's method:

In crout's method (i) is set as $u_{11} = u_{22} = u_{33} = 1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

(3) Cholesky's decomposition:

In cholesky's decomposition, we set

$A = LL^t$

OR

$A = UU^t$

i.e.
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Q. Factorize:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$ By Dollitle's method.

solution:

For Dolittle's method setting

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

or, $$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

By $u_{11} = 3$, $u_{12} = 2$, $u_{13} = 1$

$l_{21}u_{11} = 2$

$l_{21} \times 3 = 2$

$\therefore l_{21} = 2/3$

$l_{31}u_{11} = 1$

$3 l_{31} = 1$

$\therefore l_{31} = 1/3$

$l_{21}u_{12} + u_{22} = 3$

$l_{21} \cdot 2 + u_{22} = 3$

$\frac{2}{3} \cdot 2 + u_{22} = 3$

$u_{22} = 3 - \frac{4}{3}$

$u_{22} = 5/3$

$l_{21}u_{13} + u_{23} = 2$

or, $\frac{2}{3} \cdot 1 + u_{23} = 2$

or, $u_{23} = 2 - \frac{2}{3}$

$\therefore u_{23} = 4/3$

$l_{31}u_{12} + l_{32}u_{22} = 2$

$\frac{1}{3} \times 2 + l_{32} \times \frac{5}{3} = 2$

$\frac{5}{3} l_{32} = 2 - \frac{2}{3}$

$\frac{5}{3} l_{32} = 4/3$

$\therefore l_{32} = 4/5$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3$

$\frac{1}{3} \times 1 + \frac{4}{5} \times \frac{4}{3} + u_{33} = 3$

$\frac{1}{3} + \frac{16}{15} + u_{33} = 3$

$\frac{16}{15} + u_{33} = 3 - \frac{1}{3}$

$u_{33} = \frac{8}{3} - \frac{16}{15}$

$\therefore u_{33} = 8/5$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix}$$

which is required factorization

Q. Factorize:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

By Crout's and Cholesky's method.

Solution:

Yesari game

$$\begin{bmatrix} l_{11}\times 1+0\times 0+0\times 0 & l_{11}\times u_{12}+0\times 1 \\ l_{21}\times 1+l_{22}\times 0+l\times 0 \\ l_{31}\times 1+l_{32}\times 0+l_{33}\times 0 \end{bmatrix}$$

For Crout's method;

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

or, $$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$l_{11}=3$

$l_{11}u_{12}=2$
or, $3\times u_{12}=2$
$\therefore u_{12}=2/3$

$l_{11}u_{13}=1$
$3u_{13}=1$
$\therefore u_{13}=1/3$

$l_{21}=2$

$l_{21}u_{12}+l_{22}=3$
$2\times \frac{2}{3}+l_{22}=3$
$\therefore l_{22}=5/3$

$l_{21}u_{13}+l_{22}u_{23}=2$
$\frac{2}{3}+\frac{5}{3}u_{23}=2$
$\frac{5}{3}u_{23}=4/3$
$\therefore u_{23}=4/5$

$l_{31}=1$

$l_{31}u_{12}+l_{32}=2$
$2/3+l_{32}=2$
$\therefore l_{32}=4/$

$l_{31}u_{13}+l_{32}u_{23}+l_{33}=3$
$1/3+16/15+l_{33}=3$

$$L=\begin{bmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 8/5 \end{bmatrix}$$

$$U=\begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix}$$

For Cholesky's method;

$A=LL^t$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

or, $$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} (l_{11})^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & (l_{21})^2+(l_{22})^2 & l_{21}l_{31}+l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & (l_{31})^2+(l_{32})^2+(l_{33})^2 \end{bmatrix}$$

$(l_{11})^2=3$
$l_{11}=\sqrt{3}$

$l_{11}l_{21}=2$
$l_{21}=\frac{2}{\sqrt{3}}$

$l_{11}l_{31}=1$
$\sqrt{3}l_{31}=1$
$\therefore l_{31}=1/\sqrt{3}$

$l_{21}l_{11}=2$
$l_{21}=\frac{2}{\sqrt{3}}$

$(l_{21})^2+(l_{22})^2=3$
$4/3+(l_{22})^2=3$
$l_{22}^2=5/3$

$l_{21} u_{31} + l_{22} u_{32} = 2$

$\frac{2}{\sqrt{3}} \frac{1}{\sqrt{3}} + \frac{\sqrt{5}}{\sqrt{3}} l_{32} = 2$

$\frac{\sqrt{5}}{\sqrt{3}} l_{32} = 2 - \frac{2}{3}$

$l_{23} = \frac{4}{3} \times \frac{\sqrt{3}}{\sqrt{5}}$

$l_{23} = \frac{4}{\sqrt{15}}$

$l_{31} \cdot l_{11} = 1$

$l_{31} = \frac{1}{\sqrt{3}}$

$l_{31} l_{21} + l_{32} l_{22} = 2$

$\frac{1}{\sqrt{3}} \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{15}} l_{22} = 2$

$\frac{4}{\sqrt{15}} l_{22} = 2 - \frac{2}{3}$

$l_{22} = \frac{4}{3} \times \frac{\sqrt{15}}{4}$

$l_{31} l_{21} + l_{32} l_{22} = 2$

$\frac{1}{\sqrt{3}} \frac{2}{\sqrt{3}} + l_{32} \frac{\sqrt{5}}{\sqrt{3}} = 2$

$l_{32} \frac{\sqrt{5}}{\sqrt{3}} = \frac{4}{3}$

$\therefore l_{32} = 4/\sqrt{15}$

$(l_{31})^2 + (l_{32})^2 + (l_{33})^2 = 3$

$\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{4}{\sqrt{15}}\right)^2 + (l_{33})^2 = 3$

$\frac{1}{3} + \frac{16}{15} + (l_{33})^2 = 3$

$l_{33} = \sqrt{8}/\sqrt{5}$

$$L = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2/\sqrt{3} & \sqrt{5}/\sqrt{3} & 0 \\ 1/\sqrt{3} & 4/\sqrt{15} & \sqrt{8}/\sqrt{5} \end{bmatrix}$$

$$L^t = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{5}/\sqrt{3} & 4/\sqrt{15} \\ 0 & 0 & \sqrt{8}/\sqrt{5} \end{bmatrix}$$

* Solution to a system of linear equation by iterative method

(i) Jacobi (Jacobian Method):-

It is similar as fixed point iterative method. For Jacobian method, equation should be diagonally dominant (largest)

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Q. Solve:-

$9x_1 - x_2 + 2x_3 = 9$ — (1)

$x_1 + 10x_2 - 2x_3 = 15$ — (2)

$2x_1 - 2x_2 - 13x_3 = -17$ — (3)

solution:

From (1)

$9x_1 = 9 + x_2 - 2x_3$

or, $x_1 = 1 + \frac{1}{9}x_2 - \frac{2}{9}x_3$

$\therefore x_1 = 1 + 0.1111x_2 - 0.2222x_3$ — (4)

From (2)

$10x_2 = 15 - x_1 + 2x_3$

$x_2 = 1.5 - 0.1x_1 + 0.2x_3$ — (5)

From (3)

$13x_3 = 17 + 2x_1 - 2x_2$

$x_3 = \frac{17}{13} + \frac{2}{13}x_1 - \frac{2}{13}x_2$

$x_3 = 1.3076 + 0.1538x_1 - 0.1538x_2$ — (6)

Setting (4), (5) and (6) in Jacobian iteration as (n+1)

$x_1^{n+1} = 1 + 0.1111x_2^{(n)} - 0.2222x_3^{(n)}$ — (7)

$x_2^{n+1} = 1.5 - 0.1x_1^{n} + 0.2x_3^{n}$ — (8)

$x_3^{n+1} = 1.3076 + 0.1538x_1 - 0.1538x_2^{(n)}$ — (9)

let $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$

From (7)

$x_1^{(1)} = 1 + 0.1111x_2^{(0)} - 0.2222x_3^{(0)}$
$= 1$

From (8)

$x_2^{(1)} = 1.5 - 0.1x_1^{(0)} + 0.2x_3^{(0)}$
$= 1.5$

From (9)

$x_3^{(1)} = 1.3076 + 0.1538x_1^{(0)} - 0.1538x_2^{(0)}$
$= 1.3076$

$x_1^{(2)} = 1 + 0.1111x_2^{(1)} - 0.2222x_3^{(1)}$
$= 1 + 0.1111 \times 1.5 - 0.2222 \times 1.3076$
$= 0.8761$

The table of iteration is formed as below.

No. of iteration	x_1	x_2	x_3
0	0	0	0
1	1	1.5	1.3076
2	0.8761	1.6152	1.2307
3	0.9111	1.6585	1.1868
4	0.9205	1.6462	1.1926
5	0.9178	1.6464	1.1959
6	0.9171	1.6474	1.1955

$\therefore$ Required solution correct to 2 d.p is

$x_1 = 0.91$, $x_2 = 1.64$, $x_3 = 1.19$

Gauss-Siedal Method

It is similar as Jacobian method but only the difference is in iteration. The working process of gauss-siedal method is as below:

Solve: $9x_1 - x_2 + 2x_3 = 9$ — (1)

$x_1 + 10x_2 - 2x_3 = 15$ — (2)

$2x_1 - 2x_2 - 13x_3 = -17$ — (3)

Solution:

From (1)

$9x_1 = 9 + x_2 - 2x_3$

$x_1 = 1 + 0.1111x_2 - 0.2222x_3$ — (4)

From (2)

$10x_2 = 15 - x_1 + 2x_3$

$x_2 = 1.5 - 0.1x_1 + 0.2x_3$ — (5)

From (3)

$13x_3 = 17 + 2x_1 - 2x_2$

$x_3 = 1.3076 + 0.1538x_1 - 0.1538x_2$ — (6)

Setting (4), (5) and (6) in Gauss-siedal iteration as:

$x_1^{(n+1)} = 1 + 0.1111x_2^{(n)} - 0.2222x_3^{(n)}$ — (7)

$x_2^{(n+1)} = 1.5 - 0.1x_1^{(n+1)} + 0.2x_3^{(n)}$ — (8)

$x_3^{(n+1)} = 1.3076 + 0.1538x_1^{(n+1)} - 0.1538x_2^{(n+1)}$ — (9)

$x_1^{(1)} = 1 + 0.1111x_2^{(0)} - 0.2222x_3^{(0)}$

$= 1$

$x_2^{(1)} = 1.5 - 0.1x_1^{(1)} + 0.2x_3^{(0)}$

$= 1.5 - 0.1 \times 1 + 0.2 \times 0$

$= 1.4$

$x_3^{(1)} = 1.3076 + 0.1538 \times 1 - 0.1538 \times 1.4$

$= 1.2460$

The table of iteration is formed as below:

No. of iterations	x_1	x_2	x_3
0	0	0	0
1	1	1.4	1.2460
2	0.8786	1.6613	1.1871
3	0.9208	1.6433	1.1961
4	0.9170	1.6473	1.1952
5	0.9174	1.6473	1.1953

$x_1 = 0.9174$

$x_2 = 1.6473$

$x_3 = 1.1953$

Eigen Value and Eigen vector of Matrix

Let A be a matrix. A scalar number λ is called eigen value of A if $AX = \lambda X$ where X is called eigen vector of A.

$$AX = \lambda X$$

Power method:-

We can determine largest eigen value and corresponding eigen vector by power method. For power method, we set,

$$A X_n = Y_{n+1} \quad (1) \text{ and}$$

$$X_{n+1} = \frac{1}{k} Y_{n+1} \quad (2)$$

where k is magnitudely largest element of Y_{n+1}, where k is called normalizing factor. We repeat the above process (1) and (2) until the change in normalizing factor is negligible. Generally, we choose

$$X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Q. Find the largest eigen value and corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$

Solution:

Let, $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$AX_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-3+2 \\ 4+4-1 \\ 6+3+5 \end{bmatrix}$$

$$\therefore AX_0 = \begin{bmatrix} 0 \\ 7 \\ 14 \end{bmatrix}, k_1 = 14$$

$$\therefore X_1 = \frac{1}{k_1} \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}$$

$$\therefore AX_1 = \begin{bmatrix} 0.5 \\ 1 \\ 6.5 \end{bmatrix}$$

$$X_2 = \frac{1}{6.5} \begin{bmatrix} 0.5 \\ 1 \\ 6.5 \end{bmatrix}, k_2 = 6.5$$

$$X_2 = \begin{bmatrix} 0.0769 \\ 0.1538 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.0769 \\ 0.1538 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.6155 \\ -0.077 \\ 5.9228 \end{bmatrix}$$

$$X_3 = \frac{1}{5.9228} \begin{bmatrix} 1.6155 \\ -0.077 \\ 5.9228 \end{bmatrix} \quad \therefore X_3 = \begin{bmatrix} 0.2727 \\ -0.0130 \\ 1 \end{bmatrix}$$

(Neat calculations while practising)

$$AX_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.2727 \\ -0.0130 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.3177 \\ 0.0388 \\ 6.5972 \end{bmatrix}$$

$$X_4 = \frac{1}{6.5972} \begin{bmatrix} 2.3177 \\ 0.0388 \\ 6.5972 \end{bmatrix}, \quad K_4 = 6.5972$$

$$\therefore X_4 = \begin{bmatrix} 0.3504 \\ 0.0058 \\ 1 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3504 \\ 0.0058 \\ 1 \end{bmatrix}$$

$$\therefore AX_4 = \begin{bmatrix} 2.333 \\ 0.4248 \\ 7.1198 \end{bmatrix}$$

$$X_5 = \frac{1}{7.1198} \begin{bmatrix} 2.333 \\ 0.4248 \\ 7.1198 \end{bmatrix}, \quad K_5 = 7.1198$$

$$X_5 = \begin{bmatrix} 0.3276 \\ 0.0596 \\ 1 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3276 \\ 0.0596 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.1488 \\ 0.5488 \\ 7.1444 \end{bmatrix}$$

$$X_6 = \frac{1}{7.1444} \begin{bmatrix} 2.1488 \\ 0.5488 \\ 7.1444 \end{bmatrix}, \quad K_6 = 7.1444$$

$$= \begin{bmatrix} 0.3007 \\ 0.0768 \\ 1 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3007 \\ 0.0768 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.0703 \\ 6.5100 \\ 7.0346 \end{bmatrix}$$

$$X_7 = \frac{1}{7.0346} \begin{bmatrix} 2.4703 \\ 0.5100 \\ 7.0346 \end{bmatrix} \quad K_7 = 7.0346$$

$$= \begin{bmatrix} 0.2943 \\ 0.0724 \\ 1 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.2943 \\ 0.0724 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.0771 \\ 0.4668 \\ 6.983 \end{bmatrix}$$

$$X_8 = \frac{1}{6.983} \begin{bmatrix} 2.0771 \\ 0.4668 \\ 6.983 \end{bmatrix} = \begin{bmatrix} 0.2974 \\ 0.0668 \\ 1 \end{bmatrix}$$