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CHAPTER

LINEAR PROGRAMMING PROBLEM

Introduction

In an **optimization problem,** the objective is to optimize (maximize or minimize) some function f. This function f is called the **objective function**. It is the focal point or goal of our optimization problem. In most optimization problems the objective function f depends on several variables

$$x_1, x_2, \dots, x_n$$

These are called **control variables** because we can "control" them, that is, choose their values. In many problems the choice of values of is not entirely free but is subject to some **constraints**, that is, additional restrictions arising from the nature of the problem and the variables. For example, if its production cost, then and there are many other variables (time, weight, distance traveled by a salesman, etc.) that can take nonnegative values only. Constraints can also have the form of equations (instead of inequalities).

3.1 Linear Programming (LP)

Linear programming or **linear optimization** consists of methods for solving optimization problems **with constraints**, that is, methods for finding a maximum (or a minimum) value of a **linear** objective function satisfying the constraints. Problems of this kind arise frequently, almost daily, for instance, in production, inventory

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management, bond trading, operation of power plants, routing delivery vehicles, airplane scheduling, and so on. Progress in computer technology has made it possible to solve programming problems involving hundreds or thousands or more variables. Let us explain the setting of a linear programming problem and the idea of a "geometric" solution, so that we shall see what is going on.

3.2 Model Formulation

The linear function which is to be optimized is called the objective function and the conditions of the problem expressed as simultaneous linear equations (or inequalities) are referred as constraints.

A general linear programming problem can be stated as follows:

Find $x_1, x_2, \ldots, x_n$, which optimize the linear function

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \ldots (1)$$

Subject to the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n\ (\le = \ge)b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n\ (\le = \ge)b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n\ (\le = \ge)b_m \end{array}\right\} \qquad \ldots (2)$$

and non-negative restrictions

$$x_j \ge 0,\ j = 1, 2, \ldots n \qquad \ldots (3)$$

In the conditions given by (2) there may be any of the three sings $\le, =, \ge$.

The function z given by (1) is called the objective function and the conditions given by (2) are termed as the constraints of the linear programming problems.

The above linear programming problem may also be stated in matrix form as follows:

Optimize $Z = cx$

Subject to

$Ax\ (\le, =, \ge)\ b$

and $x \ge 0$,

where $A = [a_{ij}]$, is the matrix of coefficient of order $m \times n$.

$c = [c_1, c_2, \ldots c_n]$ is a row matrix known as price vector.

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_m \end{bmatrix}$$ is column matrix called the requirement vector.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}$$ is a column matrix of variables.

3.3 Mathematical Formulation of LP

Step 1: Define the decision variables $x_1, x_2, x_3, \ldots, x_n$.

Step 2: Construct the objective function which has to be optimized as a linear equation involving decision variables.

Step 3: Express every condition as a linear inequality involving decision variables

Step 4: State the non-negativity condition and hence express the given problem as a mathematical model.

Example 1: A manufacturer of a line of patent medicines is preparing a production plant on medicine A and B. There are sufficient ingredients available to make 20,000 bottles of A and 40,000 bottles of B but there are only 45,000 bottles into which either of the medicine can be put. Further-more, it takes 3 hours to prepare enough material to full 1,000 bottles of A, it takes one hour prepare enough material to fill 1,000 bottles of B and there are 66 hours available for this operation. The profit is Rs. 8/- per bottle for A and Rs. 7% per bottle of B. Formulate this problem as a linear programming problems.

Solution:

Let the manufacture product x_1 and x_2 bottles of medicine A and B, respectively.

$$\text{Total profit (in Rs.) } Z = 8x_1 + 7x_2 \qquad \ldots (1)$$

The time required to prepare x_1 bottles of medicine A

$$= \frac{3x_1}{1000 \text{ hours}}$$

and the time required to prepare x_2 bottles of medicine B

$$= \frac{x_2}{1000 \text{ hours}}$$

Total time required to prepare x_1 bottles of medicine A and x_2 bottles of medicine B is

$$\frac{3x_1}{1000} + \frac{x_2}{1000} \text{ hours.}$$

Since total time available for this operation is 66 hours.

$$\frac{3x_1}{1000} + \frac{x_2}{1000} \le 66.$$

$$\Rightarrow \quad 3x_1 + x_2 = 66,000$$

Since there are only 45,000 bottles into which the medicines can be put,

So, the linear programming problem of the given problem is as follows;

$$\text{Max. } Z = 8x_1 + 7x_2$$

Subject to the constraints

$$3x_1 + x_2 \le 66,000,\ x_1 + x_2 \le 45,000$$

$$x_1 \le 20,000,\ x_2 \le 40,000$$

and $x_1 \ge 0,\ x_2 \ge 0.$

Example 2: A resourceful home decorator manufactures two types of lamps say A and B. Both lamps go through two technicians, first a cutter, second a finisher. Lamp A requires 2 hours of the cutter's tie and 1 hours of the finisher time. The cutter has 104 hours and finisher 76 hours of time available each month. Profit on one lamp A is Rs. 6.00 and on one lamp B is Rs. 11.00. Formulate this problem as a linear programming problems.

Solution:

Let the decorator manufacture x_1 and x_2 lamps of type A and B respectively.

$$\text{Total profit (in Rs.): } Z = 6x_1 + 11x_2.$$

Total time of the cutter used in preparing x_1 lamps of type A and x_2 of type B is

$$2x_1 + x_2$$

Since cutter has 104 hours only for each month is

$$2x_1 + x_2 \le 104.$$

Similarly, the total time of the finisher used in preparing x_1 lamps of type A and x_2 of type B is

$$x_1 + 2x_2 \le 76$$

Hence, the decorator problem is, to find x_1, x_2 which maximize

$$Z = 6x_1 + 11x_2$$

Subject to the constraints.

$$2x_1 + x_2 \le 104$$

$$x_1 + 2x_2 \le 76$$

and $x_1 \ge 0,\ x_2 \ge 0.$

3.4 Normal Form of a Linear Programming Problem (LPP)

To prepare for general solution methods, we show that constraints can be written more uniformly.

Consider, an inequality,

$$2x_1 + 8x_2 \le 60 \qquad \text{... (i)}$$

and the inequality (i) can now be written as

$$x_3 = 60 - 2x_1 - 8x_2$$

which is non-negative. Clearly, $x_3 \ge 0$. Then (i) can be written as

$$2x_1 + 8x_2 + x_3 = 60 \qquad \text{... (ii)}$$

Such a variable x_3 is called a **slack variable**, because it "takes up the slack" or difference between the two sides of the inequality. Here, the equation (ii) is the normal form of inequality (i).

The general linear optimization problem in normal form is given by

$$\text{Maximize: } f = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\dots \qquad \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$x_i \ge 0\ (i = 1, 2, \dots, n)$$

with all $b_j \ge 0$ $(i = 1, 2, \dots, m)$ (if any $b_j < 0$ then multiply by -1).

Here, the variables $x_1, x_2, \dots, x_n$ include the slack variables (for which the c_j's in f are zero). We assume that the above equations are linearly independent.

Some Basic Terms of LPP

An n-tuple that satisfies all the constraints is called a **feasible point** or **feasible solution**. A feasible solution is called an **optimal solution** if, for it, the objective function f becomes maximum (or minimum), compared with the values of f at all feasible solutions. Finally, by a **basic feasible solution** we mean a feasible solution for which at least $n - m$ of the variables $x_1, x_2, \dots, x_n$ are zero. The basic feasible solutions are the vertices of the basic feasible region.

3.5 Simplex Method

For finding an optimal solution of this problem, we need to consider only the basic feasible solutions, but there are still so many that we have to follow a systematic search procedure. In 1948 G. B. Dantzig published an iterative method, is called the simplex method, for that purpose. In this method, one proceeds stepwise from one basic

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feasible solution to another in such a way that the objective function f always increases its value. Let us explain this method in terms of the example in the last section.

A linear optimization problem (linear programming problem) can be written in normal form:

$$\text{Maximize: } z = f(x) = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots \dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$
$$x_i \geq 0 \ (i = 1, 2, \dots, n)$$

Here, the variables $x_1, x_2, \dots, x_n$ include the slack variables (for which the c_j's in f are zero). We assume that the above equations are linearly independent.

For an example, the maximization of the objective function

$$\text{Maximize: } z = 40x_1 + 88x_2$$

Subject to the constraints

$$2x_1 + 8x_2 \leq 60$$
$$5x_1 + 2x_2 \leq 60$$
$$x_i \geq 0 \ (i = 1, 2)$$

With introducing two slack variables x_3, x_4, we obtain the normal form of the problem is

$$\text{Maximize: } z - 40x_1 - 88x_2 = 0$$

Subject to the constraints

$$2x_1 + 8x_2 + x_3 = 60$$
$$5x_1 + 2x_2 + x_4 = 60$$
$$x_i \geq 0 \ (i = 1, 2, 3, 4)$$

To find an optimal solution of it, we may consider its augmented matrix is

	z	x_1	x_2↓	x_3	x_4	Constant	ratio
R_1	1	−40	−88	0	0	0	0
R_2	0	2	8	1	0	60	← 7.5
R_3	0	5	2	0	1	60	30

This matrix is called a simplex tableau or simplex table (the initial simplex table). Every simplex table contains two kinds of variables – basic variables and non-basic variables. By basic variables we mean those whose columns have only one nonzero entry. Thus, in above table only x_3 and x_4 are basic variables and x_1 and x_2 are non basic variables.

Every simplex table gives a basic feasible solution. It is obtained by setting the non basic variables to zero.

The optimal solution (its location and value) is now obtained stepwise by pivoting, designed to take us to basic feasible solutions with higher and higher values of z until the maximum of z is reached. Here, the choice of the pivot equation and pivot are quite different from that in the Gauss elimination. The reason is that are restricted to nonnegative values.

Now, we have to maximize the function. So, we observe the negative entry in first row i.e. in R_1. The greatest negative entry is −88. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$, R_2 is the pivot row (row if least positive ratio). Therefore, 8 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 11R_2$, $R_3 \to 4R_3 - R_2$ then the above table becomes,

	z	x_1↓	x_2	x_3	x_4	Constant	ratio
R_1	1	−18	0	11	0	660	−ve value
R_2	0	2	8	1	0	60	30
R_3	0	18	0	−1	4	180	← 10

Again, R_1 has negative entry and that is, −20. So, the column of x_1 is pivot column then by ratio, R_3 is the pivot row and pivot point is 18.

Now, applying $R_1 \to R_1 + R_3$, $R_2 \to 9R_2 - R_3$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	Constant	ratio
R_1	1	0	0	10	4	840	
R_2	0	0	72	10	−4	360	
R_3	0	18	0	−1	4	180	

In above table only x_1 and x_2 are basic variables and x_3 and x_4 are non basic variables. So, setting the non basic variables to zero i.e $x_3 = 0$ and $x_4 = 0$ then the above table gives

Then by R_1, $z = 840$

by R_2, $72x_2 = 360 \Rightarrow x_2 = 5$

by R_3, $18x_1 = 180 \Rightarrow x_1 = 10.$

Thus, $\max(z) = 840$ at $(x_1, x_2) = (10, 5)$.

Process to Solve the Linear Programming Problem

Step 1.1: Selection of the Column of the Pivot

Select as the column of the pivot the first column with a negative entry in first row i.e. in R_1. In above table this is Column 2 (because of the −40).

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Step 1.2: **Selection of the Row of the Pivot**

Divide the right sides (i.e. constant) by the corresponding entries of the column just selected in step 1. Take as the pivot equation the equation that gives the smallest quotient (or smallest ratio). In first table, 12 is the smallest non-negative ration and therefore the pivot is 5, the corresponding value.

Step 1.3: **Elimination by Row Operations**

This gives zeros above and below the pivot, we use the row operation method for the work.

Step 2.1: The basic feasible solution given by (Table 2) is not yet optimal because of the greatest negative entry −72 in first row i.e. in R_1. Accordingly, we perform the operations as in step 1 again, choosing a pivot in the column of −72.

Step 2.2: Select Column 3 of in above table (Table 2) as the column of the pivot because − 72 is negative.

Step 2.3: As step 1.2, the pivot is 36.

Step 2.4: Elimination by row operations gives (as by step 1.3), we obtain the new table (Table 3).

Step 3.1: The basic feasible solution given by (Table 3) is optimal because there is no negative entry in first row i.e. in R_1.

Step 4.1: Now, we see the variables are basic and nonbasic. Setting the nonbasic to zero, we obtain the solution from the basic feasible solution.

This gives the solution of our problem by the simplex method of linear programming.

Minimization the Problem

If we want to minimize the function $z = f(x)$ (instead of maximize), we take as the columns of the pivots those whose entry in first row (i.e. in Row 1) is positive (instead of negative). In such a column k, we consider only positive entries t_{jk} and take as pivot t_{jk} for which $\dfrac{\text{constant}}{\text{entries in corresponding pivot column}}$ is smallest (as above).

Example 3: Solve the problem: Maximize: $z = 5x_1 + 10x_2$ subject to $0 \le x_1 \le 5$, $x_1 + x_2 \le 6$, $0 \le x_2 \le 4$.

Solution:

Given that,

$$\text{max: } z = 5x_1 + 10x_2$$

Subject to

$$x_1 \le 5;\ x_1 + x_2 \le 6;\ x_2 \le 4$$

With introducing new variables x_3, x_4 and x_5 the normal form of the above problem is,

$$\text{Max: } z - 5x_1 - 10x_2 = 0$$

Subject to

$$x_1 + x_3 = 5$$
$$x_1 + x_2 + x_4 = 6$$
$$x_2 + x_5 = 4$$

The tabled form of above problem is,

	z	x_1	x_2↓	x_3	x_4	x_5	Constant	ratio
R_1	1	−5	−10	0	0	0	0	0
R_2	0	1	0	1	0	0	5	undefined
R_3	0	1	1	0	1	0	6	6
R_4	0	0	1	0	0	1	4	← 4

Now, we have to maximize the function. So, we observe the greatest negative entry in R_1. The greatest negative entry is −10. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$, R_4 is the pivot row (row if least positive ratio).

Therefore, 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 10R_4$, $R_3 \to R_3 - R_4$ then the above table becomes,

	z	x_1↓	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	−5	0	0	0	10	40	−ve value
R_2	0	1	0	1	0	0	5	5
R_3	0	1	0	0	1	−1	2	← 2
R_4	0	0	1	0	0	1	4	undefined

Again, R_1 has negative entry and that is, −5. So, the column of x_1 is pivot column then by ratio, R_3 is the pivot row and pivot point is 1.

Now, applying $R_1 \to R_1 + 5R_3$, $R_2 \to R_2 - R_3$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	0	0	0	5	5	50	
R_2	0	0	0	1	−1	1	3	
R_3	0	1	0	0	1	−1	2	
R_4	0	0	1	0	0	1	4	

Here R_1 has no negative entry, so the table gives optimal solution.

Assume the non-basic variables are zero i.e. $x_4 = 0 = x_5$.

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Then by R_1, by $z = 50$

by R_2, $x_1 = 2$

by R_3, $x_2 = 4$

by R_4, $x_3 = 3$

Thus, $\max(z) = 50$ at $(x_1, x_2) = (2, 4)$.

Example 4: Maximize the linear programming problem, $z = 300x_1 + 500x_2$ subject to: $x_1 + 4x_2 \le 30$, $x_1 + x_2 \le 5$, $2x_1 + x_2 \le 30$ by using simplex method.

Solution:

Given that,

$$\max: z = 300x_1 + 500x_2$$

Subject to

$$x_1 + 4x_2 \le 30$$

$$x_1 + x_2 \le 5$$

$$2x_1 + x_2 \le 30$$

Introducing new variables x_3, x_4 and x_5 so that,

Maximize: $z - 300x_1 - 500x_2 = 0$

Subject to

$$x_1 + 4x_2 + x_3 = 30$$

$$x_1 + x_2 + x_4 = 5$$

$$2x_1 + x_2 + x_5 = 30$$

The tabled form of above problem is,

	z	x_1	$x_2\downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	1	–300	–500	0	0	0	0	0
R_2	0	1	4	1	0	0	30	7.5
R_3	0	1	1	0	1	0	5	$\leftarrow$ 5
R_4	0	2	1	0	0	1	30	30

Now, we have to maximize the function. So, we observe the greatest negative entry in R_1. The greatest negative entry is –500. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$, R_3 is the pivot row (row if least positive ratio). Therefore, 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply,

$$R_1 \to R_1 + 500R_3,\ R_2 \to R_2 - 4R_3,\ R_4 \to R_4 - R_3$$

then the above table becomes,

	z	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	200	0	0	500	0	2500	
R_2	0	–3	0	1	–4	0	10	
R_3	0	1	1	0	1	0	5	
R_4	0	1	0	0	–1	1	25	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non-basic variables are zero i.e. $x_1 = 0 = x_4$.

Then by R_1, $z = 2500$

by R_2, $x_3 = 10$

by R_3, $x_2 = 5$

by R_4, $x_5 = 25$

Thus, $\max(z) = 2500$ at $(x_1, x_2) = (0, 5)$.

Example 5: Solve the LPP, minimize: $z = 5x_1 - 20x_2$ subject to $-2x_1 + 10x_2 \le 5$, $2x_1 + 5x_2 \le 10$.

Solution:

Given that,

$$\min: z = 5x_1 - 20x_2$$

Subject to

$$-2x_1 + 10x_2 \le 5$$

$$2x_1 + 5x_2 \le 10$$

Introducing new variables x_3 and x_4 so that,

Minimize: $z - 5x_1 + 20x_2 = 0$

Subject to

$$-2x_1 + 10x_2 + x_3 = 5$$

$$2x_1 + 5x_2 + x_4 = 10$$

The tabled form of above problem is,

	z	x_1	$x_2\downarrow$	x_3	x_4	Constant	ratio
R_1	1	–5	20	0	0	0	0
R_2	0	–2	10	1	0	5	$\leftarrow$ 0.5
R_3	0	2	5	0	1	10	2

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Now, we have to minimize the function. So, we observe the greatest positive entry in R_1 and the greatest positive entry is 20. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_2 is the pivot row (row if least positive ratio).

Therefore, 10 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply,

$$R_1 \to R_1 - 2R_2,\ R_3 \to 2R_3 - R_2$$

then the above table becomes,

	z	x_1	$x_2\downarrow$	x_3	x_4	Constant	ratio
R_1	1	–1	0	–2	0	–10	
R_2	0	–2	10	1	0	5	
R_3	0	6	0	–1	2	15	

Here R_1 has no positive entry and so the table gives optimal solution.

Assume the non-basic variables are zero i.e. $x_1 = 0 = x_3$.

Then by R_1, $z = -10$

by R_2, $10x_2 = 5 \Rightarrow x_2 = 0.5$

by R_3, $2x_4 = 15 \Rightarrow x_4 = 7.5$

Thus, max(z) = – 10 at $(x_1, x_2) = (0, 0.5)$.

Example 6: Solve the linear programming problem, maximize $f = x_1 + x_2 + x_3$ subject to $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, 4x_1 + 5x_2 + 8x_3 \leq 12, 8x_1 + 5x_2 + 4x_3 \leq 12$.

Solution:

Given problem is

$$\text{Max. } f = x_1 + x_2 + x_3$$

$$\text{s.t. } 4x_1 + 5x_2 + 8x_3 \leq 12;\ 8x_1 + 5x_2 + 4x_3 \leq 12,$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Introducing new variables x_4 and x_5 so that,

$$\text{Max. } f - x_1 - x_2 - x_3 = 0$$

subject to

$$4x_1 + 5x_2 + 8x_3 + x_4 = 12$$

$$8x_1 + 5x_2 + 4x_3 + x_5 = 12$$

The tabled form of above problem is,

	f	$x_1\downarrow$	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	–1	–1	–1	0	0	0	0
R_2	0	4	5	8	1	0	12	3
R_3	0	8	5	4	0	1	12	← 1.5

Now, we have to maximize the function. So, we observe the greatest negative entry in R_1. The greatest negative entry is –1. So, the column of x_1 (we choose as first comes) is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_3 is the pivot row (row if least positive ratio). Therefore, 8 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply,

$$R_1 \to 8R_1 + R_3,\ R_2 \to 2R_2 - R_3$$

then the above table becomes,

	f	x_1	x_2	$x_3\downarrow$	x_4	x_5	Constant	ratio
R_1	8	0	–3	–4	0	1	12	–ve value
R_2	0	0	5	12	2	–1	12	← 1
R_3	0	8	5	4	0	1	12	3

Again, R_1 has greatest negative entry and that is, –4. So, the column of x_3 is pivot column then by ratio, R_2 is the pivot row and pivot point is 12.

Now, applying $R_1 \to 3R_1 + R_2$, $R_3 \to 3R_3 - R_2$ then the above table becomes,

	f	x_1	$x_2\downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	24	0	–4	0	2	2	48	–ve value
R_2	0	0	5	12	2	–1	12	← 2.4
R_3	0	24	10	0	–2	4	24	2.4

Again, R_1 has greatest negative entry and that is, –4. So, the column of x_2 is pivot column then by ratio, R_2 is the pivot row and pivot point is 5.

Now, applying $R_1 \to 5R_1 + 4R_2$, $R_3 \to R_3 - 2R_2$ then the above table becomes,

	f	x_1	$x_2\downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	120	0	0	48	18	6	288	
R_2	0	0	5	12	2	–1	12	
R_3	0	24	0	–24	–6	6	0	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $x_3 = 0 = x_4 = x_5$.

Then by R_1, by R_1 $120f = 288 \Rightarrow f = 2.4$

by R_2, $5x_2 = 12 \Rightarrow x_2 = 2.4$

by R_3, $24x_1 = 0 \Rightarrow x_1 = 0$

Thus, max(f) = 3 at $(x_1, x_2, x_3) = (0, 2.4, 0)$.

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6 The Simplex Method: Difficulties

metimes in solving a linear optimization problem by the simplex method, the above ocess does not always smoothly as above. Occasionally, we encounter two kinds of fficulties-the first one is the degeneracy and the second one concern difficulties in arting.

.6.1 The Simplex Method: Degeneracy

degenerate feasible solution is a feasible solution at which more than the usual umber $n - m$ of variables is zero. Here n is the number of variables (slack and others) nd m the number of constraints (not counting the x_j conditions).

n the case of a degenerate feasible solution we do an extra elimination step in which a basic variable that is zero for that solution becomes nonbasic (and a nonbasic variable becomes basic instead).

We explain this in a typical case.

Example 7: (Degenerate Case): Solve the problem, maximize: $z = 4x_1 + x_2 + 2x_3$, subject to $x_1 + x_2 + x_3 \le 1$, $x_1 + x_2 - x_3 \le 0$, $x_1, x_2, x_3 \ge 0$.

Solution:

Given that,

$$\max: z = 4x_1 + x_2 + 2x_3$$

Subject to

$$x_1 + x_2 + x_3 \le 1,$$
$$x_1 + x_2 - x_3 \le 0,$$
$$x_1, x_2, x_3 \ge 0$$

Introducing new variable x_4 and x_5 so that

Maximize: $z - 4x_1 - x_2 - 2x_3 = 0$

Subject to

$$x_1 + x_2 + x_3 + x_4 = 1,$$
$$x_1 + x_2 - x_3 + x_5 = 0$$

The tabled form of above problem is,

	f	$x_1\downarrow$	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	–4	–1	–2	0	0	0	0
R_2	0	1	1	1	1	0	1	1
R_3	0	1	1	–1	0	1	0	← 0

Now, we have to maximize the function. So, we observe the greatest negative entry in R_1 and is –4. So, the column of x_1 is pivot column and there arise degenerate condition on R_3 [Because, by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$, we observe that R_3 has ratio 0 and the basic variable x_5 is valued for this which is not possible], so, R_3 is the pivot row and 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply,

$$R_1 \to R_1 + 4R_3,\ R_2 \to R_2 - R_3$$

then the above table becomes,

	f	x_1	x_2	$x_3\downarrow$	x_4	x_5	Constant	ratio
R_1	1	0	3	–6	0	4	0	0
R_2	0	0	0	2	1	–1	1	← 2
R_3	0	1	1	–1	0	1	0	0

Again, R_1 has negative entry –6. So, the column of x_3 is pivot column then by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$(we observe least positive ratio) [We shift the pivoting after used once], R_2 is the pivot row and pivot point is 2.

Now, apply

$$R_1 \to R_1 + 3R_2,\ R_3 \to 2R_3 + R_2$$

then the above table becomes,

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	0	3	0	3	1	3	
R_2	0	0	0	2	1	–1	1	
R_3	0	2	2	0	1	1	1	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $x_2 = 0 = x_4 = x_5$.

Then by R_1, by R_1, $f = 3$

by R_2, $2x_3 = 1 \Rightarrow x_3 = 0.5$

by R_3, $2x_1 = 1 \Rightarrow x_1 = 0.5$

Thus, $\max(f) = 3$ at $(x_1, x_2, x_3) = (0.5, 0, 0.5)$.

3.6.2 The Simplex Method: Difficulties in Starting i.e. Big M Method

As a second kind of difficulty, it may sometimes be hard to find a basic feasible solution to start from. When a basic feasible solution is not readily apparent, the Big M method or the two phase simplex method may be used to solve the problem. In such a case the idea of an artificial variable (or several such variables) is helpful.

The Big M method is a version of the Simplex Algorithm that first finds a basic feasible solution by adding "artificial" variables to the problem. The objective function of the original linear programming must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

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Process to Solve by Big M method

Step 1: Modify the constraints so that the right side of each constraint is non negative (This requires that each constraint with a negative right side be multiplied by −1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as $\geq$, $\leq$ or = constraint.

Step 2: Convert each inequality constraint to standard form. If

constraint $\leq$

then we add slack variable(s). And, if

constraint $\geq$

we subtract surplus variable(s).

Step 3: Add an artificial variable a_i to the constraints identified as $\geq$ or = constraints at the end of Step 1. Also add the sign restriction $a_i \geq 0$.

Step 4: Let M denote a very large positive number. If the linear programming is a min problem, add (for each artificial variable) Ma_i to the objective function (before equal to 0).

If the linear programming is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function (before equal to 0).

Step 5: Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from objective row before beginning the simplex.

Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

Note: If all artificial variables are equal to zero in the optimal solution, we have found the optimal solution to the original problem. If any artificial variables are positive in the optimal solution, the original problem is infeasible.

We explain this method in terms of a typical example.

Example 8: **(Big M Method):** Solve the problem, maximize: $z = 2x_1 - x_2$ subject to $x_1 + x_2 \geq 5, -x_1 + x_2 \leq 1, 5x_1 + 4x_2 \leq 40$.

Solution:

Given that,

$$\max: z = 2x_1 - x_2$$

Subject to

$$x_1 + x_2 \geq 5,$$
$$-x_1 + x_2 \leq 1$$
$$5x_1 + 4x_2 \leq 40$$
$$x_1 \geq 0, x_2 \geq 0$$

Introducing new variable x_3, x_4 and x_5 so that

$$\text{Maximize: } z - 2x_1 + x_2 = 0$$

Subject to

$$x_1 + x_2 - x_3 = 5$$
$$-x_1 + x_2 + x_4 = 1$$
$$5x_1 + 4x_2 + x_5 = 40$$

(Here, x_3 is a surplus variable slack variable, x_4, x_5 are slack variable)

The tabled form of above problem is,

	z	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	−2	1	0	0	0	0	
R_2	0	1	1	−1	0	0	5	
R_3	0	−1	1	0	1	0	1	
R_4	0	5	4	0	0	1	40	

Here the basic variable x_3 is negative, which is not possible. And, the table gives

$$x_3 = x_1 + x_2 - 5$$

Introduce a new artificial variable x_6 such that

$$x_3 = x_1 + x_2 - 5 + x_6$$

Choose a value M (so large) such that, the problem can be written as

$$\text{Maximize: } \bar{z} = z - Mx_6$$

i.e. $\quad \text{Maximize: } \bar{z} = 2x_1 - x_2 - M(x_3 - x_1 - x_2 + 5)$

$\Rightarrow \quad \text{Maximize: } \bar{z} = (2 + M)x_1 + (M - 1)x_2 - (M)x_3 - 5M$

$\Rightarrow \quad \text{Maximize: } \bar{z} - (2 + M)x_1 - (M - 1)x_2 + (M)x_3 = -5M$

Then the augmented table is

	$\bar{z}$	x_1	$x_2\downarrow$	x_3	x_4	x_5	x_6	Constant	ratio
R_1	1	−(2 + M)	−(M − 1)	M	0	0	0	−5M	−
R_2	0	1	1	−1	0	0	0	5	← 5
R_3	0	−1	1	0	1	0	0	1	ve value
R_4	0	5	4	0	0	1	0	40	8
R_5	0	1	1	−1	0	0	1	5	5

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Now, we have to maximize the function. So, we observe the negative entry in R_1 and is $-(2+M)$. So, the column of x_1 is pivot column then by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$ is 5 and therefore R_2 is the pivot row and pivot point is 1.

Now, apply

$$R_1 \to R_1 + (2+M)R_2,\ R_3 \to R_3 + R_2,\ R_4 \to R_4 - 5R_2,\ R_5 \to R_5 - R_2$$

then the above table becomes,

	$\bar{z}$	x_1	x_2	$x_3 \downarrow$	x_4	x_5	x_6	Constant	ratio
R_1	1	0	2	–2	0	0	0	10	– ve value
R_2	0	1	1	–1	0	0	0	5	– ve value
R_3	0	0	2	–1	1	0	0	6	– ve value
R_4	0	0	–1	5	0	1	0	15	$\leftarrow$ 3
R_5	0	0	0	0	0	0	1	0	undefined

Again, R_1 has negative entry and that is, -2. So, the column of x_3 is pivot column and by the ratio R_4 is working row. Apply

$$R_1 \to 5R_1 + 2R_4,\ R_2 \to 5R_2 + R_4,\ R_3 \to 5R_4 + R_4$$

then the above table becomes,

	$\bar{z}$	x_1	x_2	x_3	x_4	x_5	x_6	Constant	ratio
R_1	5	0	8	0	0	2	0	80	
R_2	0	5	4	0	0	1	0	40	
R_3	0	0	9	0	5	1	0	45	
R_4	0	0	–1	5	0	1	0	15	
R_5	0	0	0	0	0	0	1	0	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non-basic variables are zero i.e. $x_2 = 0 = x_5$.

Then by R_1, by R_1, $5\bar{z} = 80 \Rightarrow \bar{z} = 16$

by R_2, $5x_1 = 40 \Rightarrow x_1 = 8$

by R_3, $5x_4 = 45 \Rightarrow x_4 = 9$

by R_4, $5x_3 = 15 \Rightarrow x_3 = 3$

by R_5, $x_6 = 0$

Thus, $\max(\bar{z}) = 16$ at $(x_1, x_2, x_3, x_4, x_5, x_6) = (8, 0, 3, 9, 0, 0)$.

Therefore, $\max(z) = 16$ at $(x_1, x_2) = (8, 0)$.

Example 9: Solve the problem, maximize $f = -10x_1 + 2x_2$ subject to $x_1 \geq 0$, $x_2 \geq 0$; $-x_1 + x_2 \geq -1$, $x_1 + x_2 \leq 6$, $x_2 \leq 5$.

Solution:

Given problem is

$$\text{Max. } f = -10x_1 + 2x_2$$

$$\text{s.t. } -x_1 + x_2 \geq -1;\ x_1 + x_2 \leq 6;\ x_2 \leq 5$$

Introducing new variables x_3, x_4 and x_5 so that,

$$\text{Max. } f + 10x_1 - 2x_2 = 0$$

subject to

$$x_1 - x_2 + x_3 = 1$$

$$x_1 + x_2 + x_4 = 6$$

$$x_2 + x_5 = 5.$$

The tabled form of above problem is, $+ x_4$

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	10	–2	0	0	0	0	0
R_2	0	1	–1	1	0	0	1	–ve value
R_3	0	1	1	0	1	0	6	6
R_4	0	0	1	0	0	1	5	$\leftarrow$ 5

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is -2 in R_1. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$, R_4 is the pivot row (row if least positive ratio).

Therefore, 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 2R_4$, $R_2 \to R_2 + R_4$, $R_3 \to R_3 - R_4$

then the above table becomes,

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	12	0	0	0	2	10	
R_2	0	1	0	1	0	1	6	
R_3	0	1	0	0	1	–1	1	
R_4	0	0	1	0	0	1	5	

Now, we have to maximize the function. So, we observe the negative entry in R_1 and is $-(2 + M)$. So, the column of x_1 is pivot column then by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$ is 5 and therefore R_2 is the pivot row and pivot point is 1.

Now, apply

$$R_1 \to R_1 + (2 + M)R_2,\ R_3 \to R_3 + R_2,\ R_4 \to R_4 - 5R_2,\ R_5 \to R_5 - R_2$$

then the above table becomes,

	$\overline{z}$	x_1	x_2	$x_3 \downarrow$	x_4	x_5	x_6	Constant	ratio
R_1	1	0	2	–2	0	0	0	10	– ve value
R_2	0	1	1	–1	0	0	0	5	– ve value
R_3	0	0	2	–1	1	0	0	6	– ve value
R_4	0	0	–1	5	0	1	0	15	$\leftarrow$ 3
R_5	0	0	0	0	0	0	1	0	undefined

Again, R_1 has negative entry and that is, –2. So, the column of x_3 is pivot column and by the ratio R_4 is working row. Apply

$$R_1 \to 5R_1 + 2R_4,\ R_2 \to 5R_2 + R_4,\ R_3 \to 5R_4 + R_4$$

then the above table becomes,

	$\overline{z}$	x_1	x_2	x_3	x_4	x_5	x_6	Constant	ratio
R_1	5	0	8	0	0	2	0	80	
R_2	0	5	4	0	0	1	0	40	
R_3	0	0	9	0	5	1	0	45	
R_4	0	0	–1	5	0	1	0	15	
R_5	0	0	0	0	0	0	1	0	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $x_2 = 0 = x_5$.

Then by R_1, by R_1, $5\overline{z} = 80 \Rightarrow \overline{z} = 16$

by R_2, $5x_1 = 40 \Rightarrow x_1 = 8$

by R_3, $5x_4 = 45 \Rightarrow x_4 = 9$

by R_4, $5x_3 = 15 \Rightarrow x_3 = 3$

by R_5, $x_6 = 0$

Thus, $\max(\overline{z}) = 16$ at $(x_1, x_2, x_3, x_4, x_5, x_6) = (8, 0, 3, 9, 0, 0)$.

Therefore, $\max(z) = 16$ at $(x_1, x_2) = (8, 0)$.

Example 9: Solve the problem, maximize $f = -10x_1 + 2x_2$ subject to $x_1 \ge 0,\ x_2 \ge 0;$ $-x_1 + x_2 \ge -1,\ x_1 + x_2 \le 6,\ x_2 \le 5.$

Solution:

Given problem is

$$\text{Max. } f = -10x_1 + 2x_2$$

$$\text{s.t. } -x_1 + x_2 \ge -1;\ x_1 + x_2 \le 6;\ x_2 \le 5$$

Introducing new variables x_3, x_4 and x_5 so that,

$$\text{Max. } f + 10x_1 - 2x_2 = 0$$

subject to

$$x_1 - x_2 + x_3 = 1$$

$$x_1 + x_2 + x_4 = 6$$

$$x_2 + x_5 = 5.$$

The tabled form of above problem is, $+ x_4$

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	10	–2	0	0	0	0	0
R_2	0	1	–1	1	0	0	1	–ve value
R_3	0	1	1	0	1	0	6	6
R_4	0	0	1	0	0	1	5	$\leftarrow$ 5

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is –2 in R_1. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$, R_4 is the pivot row (row if least positive ratio).

Therefore, 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 2R_4,\ R_2 \to R_2 + R_4,\ R_3 \to R_3 - R_4$

then the above table becomes,

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	12	0	0	0	2	10	
R_2	0	1	0	1	0	1	6	
R_3	0	1	0	0	1	–1	1	
R_4	0	0	1	0	0	1	5	

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Here R_0 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $x_2 = 0 = x_3$.

Then by R_1, $f = 10$

by R_2, $x_1 = 6$

by R_3, $x_4 = 1$

by R_4, $x_2 = 5$

Thus, $\max(f) = 10$ at $(x_1, x_2) = (0, 5)$.

Exercise 3.1

1. Solve the following problems by simplex method:

a. Maximize: $z = 15x_1 + 10x_2$ subject to $2x_1 + 2x_2 \le 10$; $2x_1 + 3x_2 \le 10$, $x_1, x_2 \ge 0$.

b. Maximize the function $Z = 2x_1 + x_2$ subject to $-x_1 + x_2 \le 4$; $x_1 + x_2 \le 6$; $x_1 \ge 0, x_2 \ge 0$.

c. Maximize the total output $z = 3x_1 + x_2$ subject to input constraints $x_1 + x_2 \le 12$; $x_1 + 3x_2 \le 5$, and $x_1 \ge 0, x_2 \ge 0$.

d. Maximize $z = 6x_1 + 12x_2$, s.t. $0 \le x_1 \le 4$, $0 \le x_2 \le 4$, $6x_1 + 12x_2 \le 72$.

e. Maximize the function $Z = 5x_1 + 4x_2$ subject to the conditions $x_1 + x_2 \le 20$; $2x_1 + x_2 \le 35$; $-3x_1 + x_2 \le 12$; and $x_1 \ge 0, x_2 \ge 0$.

f. Maximize $z = 150x_1 + 300x_2$ subject to $2x_1 + x_2 \le 16$, $x_1 + x_2 \le 8$, $x_2 = 3.5$, $x_1 \ge 0, x_2 \ge 0$.

g. Maximize: $z = 90x_1 + 50x_2$ subject to $x_1 + 3x_2 \le 18$; $x_1 + x_2 \le 10$; $3x_1 + x_2 \le 24$.

h. Maximize: $z = 20x_1 + 20x_2$ subject to $-x_1 + x_2 \le 1$, $x_1 + 3x_2 \le 15$, $3x_1 + x_2 \le 21$, $x_1 \ge 0, x_2 \ge 0$.

i. Maximize $z = 2x_1 + x_2 + 3x_3$ subject to $4x_1 + 3x_2 + 6x_3 \le 12$.

j. Maximize $f = 6x_1 + 6x_2 + 9x_3$ subject to $x_j \ge 0$ {for j = 1, 2, 3, 4, 5} and $x_1 + x_3 + x_4 = 1$, $x_2 + x_3 + x_5 = 1$.

k. Maximize the daily output in producing x_1 glass plates by a process P_1 and x_2 glass plates by a process P_2 subject to the constrains

$2x_1 + 3x_2 \le 130$ (labor hours)

$3x_1 + 8x_2 \le 300$ (machine hours)

$4x_1 + 2x_2 \le 140$ (raw material supply).

2. Solve the following problems by simplex method:

a. Minimize: $x = 30x_1 + 20x_2$ subject to $-x_1 + x_2 \le 5$; $2x_1 + x_2 \le 10$.

b. Minimize: $z = 2x_1 - 10x_2$ subject to, $x_1 - x_2 \le 4$, $2x_1 + x_2 \le 14$, $x_1 + x_2 \le 9$, $-x_1 + 3x_2 \le 15$, $x_1 \ge 0, x_2 \ge 0$.

3. Solve the following problems by simplex method:

a. Maximize: $z = 2x_1 + x_2$ subject to $2x_1 + x_2 \le 2$; $x_1 + 2x_2 \ge 6$; $x_1 + x_2 \le 4$; and $x_1 \ge 0$; $x_2 \ge 0$.

b. Maximize $z = 2x_1 + x_2$ subject to $x_1 - \frac{x_2}{2} \ge 1$, $x_1 - x_2 \le 2$, $x_1 - x_2 \le 4$, $x_1 \ge 0$, $x_2 \ge 0$, by using simplex methods.

c. Maximize $Z = 3x + 2y$ Subject to $2x + y \le 9$, $x + 2y \ge 9$, $x, y \ge 0$.

Answers

1. a. Max(z) = 75 at (5, 0) b. Max(z) = 12 at (6, 0)

c. Max(z) = 36 at (12, 0) d. Max(z) = 72 at (4, 4)

e. Max(z) = 140 at (0, 35) f. Max(z) = 1462.5 at (6.25, 1.75)

g. Max(z) = 1620 at (18, 0) h. Max(z) = 180 at (6, 3)

i. Max(z) = 6 on line from (3, 0, 0) to (0, 0, 2)

j. Max(f) = 12 at (1, 1, 0) k. Max value = 50 at (20, 30)

2. a. Min (x) = $\frac{550}{3}$ at $\left(\frac{5}{3}, \frac{20}{3}\right)$ b. Min(z) = – 54 at (3, 6)

3. a. Max(z) = 2 at (0, 2) b. Max(z) = 7 at (3, 1)

c. Max(Z) = 18 at (0, 9)

3.7 Duality in Linear Programming

There is always a corresponding Linear Programming Problem (LPP) associated to every LPP, which is called as dual problem of the original LPP (Primal Problem). The original linear programming problem is called **Primal,** while the derived linear problem is called **Dual.** Here, we will develop an understanding of the dual linear program. This understanding translates to important insights about many optimization problems and algorithms.

Before solving for the duality, the original linear programming problem is to be formulated in its standard form. Standard form means, all the variables in the problem should be non-negative and "≥," "≤" sign is used in the minimization case and the maximization case respectively.

Symmetrical Primal – Dual Form

A. Maximize Problem

Consider a Linear Programming Problem (LPP),

$$\text{Maximize: } z = f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$$
$$\ldots \ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$$
$$x_i \ge 0 \ (i = 1, 2, \ldots, n)$$

The matrix form of the problem is

$$\text{Maximize: } z = CX$$
$$\text{Subject to } AX \le B$$
$$\text{with } x_i \ge 0$$

Then its dual form is

$$\text{Minimize: } \bar{z} = B^TY$$
$$\text{Subject to } A^TY \ge C^T$$
$$\text{with } y_i \ge 0$$

B. Minimize Problem

Consider a Linear Programming Problem (LPP),

$$\text{Minimize: } z = f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ge b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ge b_2$$
$$\ldots \ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ge b_m$$
$$x_i \ge 0 \ (i = 1, 2, \ldots, n)$$

The matrix form of the problem is

$$\text{Maximize: } z = CX$$
$$\text{Subject to } AX \ge B$$
$$\text{with } x_i \ge 0$$

Then its dual form is

$$\text{Maximize: } \bar{z} = B^TY$$
$$\text{Subject to } A^TY \le C^T$$
$$\text{with } y_i \ge 0$$

Some Characteristics of Duality

1. Dual of dual is primal
2. If either the primal or dual problem has a solution then the other also has a solution and their optimum values are equal.
3. If any of the two problems has an infeasible solution, then the value of the objective function of the other is unbounded.
4. The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.
5. If either the primal or dual has an unbounded solution, then the solution to the other problem is infeasible.
6. If the primal has a feasible solution, but the dual does not have then the primal will not have a finite optimum solution and vice versa.

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Advantages and Applications of Duality

1. Sometimes dual problem solution may be easier than primal solution, particularly when the number of decision variables is considerably less than slack / surplus variables.
2. In the areas like economics, it is highly helpful in obtaining future decision in the activities being programmed.
3. In physics, it is used in parallel circuit and series circuit theory.
4. In game theory, dual is employed by column player who wishes to minimize his maximum loss while his opponent i.e. Row player applies primal to maximize his minimum gains. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.
5. When a problem does not yield any solution in primal, it can be verified with dual.
6. Economic interpretations can be made and shadow prices can be determined enabling the managers to take further decisions.

Steps for a Standard Primal Form

Step 1: Change the objective function to Maximization form

Step 2: If the constraints have an inequality sign '≥' then multiply both sides by –1 and convert the inequality sign to '≤'.

Step 3: If the constraint has an '=' sign then replace it by two constraints involving the inequalities going in opposite directions.

For example

$$x_1 + 2x_2 = 4$$

is written as

$$x_1 + 2x_2 \le 4,$$

and $x_1+2x_2 \ge 4$ (using step2) $\rightarrow - x_1 - 2x_2 \le - 4$

Step 4: Every unrestricted variable is replaced by the difference of two non-negative variables.

Step 5: We get the standard primal form of the given LPP in which,

- All constraints have '≤' sign, where the objective function is of maximization form.
- All constraints have '≥' sign, where the objective function is of minimization from.

Rules for Converting any Primal into its Dual

1. Transpose the rows and columns of the constraint co-efficient.
2. Transpose the co-efficient $(c_1, c_2, \dots, c_n)$ of the objective function and the right side constants $(b_1, b_2, \dots, b_n)$
3. Change the inequalities from '≤' to '≥' sign.
4. Minimize the objective function instead of maximizing it.

Example 10: Construct the dual problem of the linear programming problems:

Minimize $z = 6x_1 + 4x_2$

s.t. $2x_1 + x_2 \ge 1$; $6x_1 + 8x_2 \ge 3$; $x_1 \ge 0$, $x_2 \ge 0$.

Solution:

Given that

minimize $Z = 6x_1 + 4x_2$

s.t. $2x_1 + x_2 \ge 1$;

$6x_1 + 8x_2 \ge 3$;

$x_1, x_2 \ge 0$.

The given problem is standard minimization problem with all constraint ≥ type.

x_1	x_2	Constant
2	1	1 (y_1)
6	8	3 (y_2)

Let y_1 and y_2 be the dual variable then

Max. $W = y_1 + 3y_2$

s.t. $2y_1 + 6y_2 \le 6$

$y_1 + 8y_2 \le 4$

$y_1, y_2 \ge 0$

This is the dual of given primal.

Example 11: Solve the following linear programming problems by using simplex method:

Minimize $z = 2x_1 - 3x_2$

s.t. $2x_1 - x_2 - x_3 \ge 3$; $x_1 - x_2 + x_3 \ge 2$; $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

Solution:

The given problem is standard form of minimization problems. So

x_1	x_2	x_3	Constant
2	– 1	–1	3 (y_1)
2	–1	1	2 (y_2)
2	–3	0	

Let y_1 and y_2 be dual variable, so

$$\text{Max.: } W = 3y_1 + 2y_2$$

$$\begin{aligned} \text{s.t.: } \quad & 2y_1 + y_2 \le 2 \\ & -y_1 - y_2 \le -3 \\ & -y_1 + y_2 \le 0 \\ & y_1, y_2 \ge 0 \end{aligned}$$

Now, introducing new variables y_3, y_4 and y_5 so that,

$$\text{Max. } W - 3y_1 - 2y_2 = 0$$

subject to

$$\begin{aligned} 2y_1 + y_2 + y_3 &= 2 \\ -y_1 - y_2 + y_4 &= 3 \\ -y_1 + y_2 + y_5 &= 0. \end{aligned}$$

The tabled form of above problem is,

	W	$y_1\downarrow$	y_2	y_3	y_4	y_5	Constant	ratio
R_1	1	–3	–2	0	0	0	0	0
R_2	0	2	1	1	0	0	2	1 ←
R_3	0	–1	–1	0	1	0	3	–ve value
R_4	0	–1	1	0	0	1	0	undefined

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is –3 in R_1. So, the column of y_1 is pivot column and by ratio by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$ (we observe least positive ratio), R_2 is the pivot row and pivot point is 2.

To eliminate the values of the pivot column rather than the pivot, apply,

$$R_1 \to 2R_1 + 3R_2,\ R_3 \to 2R_3 + R_2,\ R_4 \to 2R_4 + R_2$$

then the above table becomes,

	W	y_1	$y_2\downarrow$	y_3	y_4	y_5	Constant	ratio
R_1	2	0	–1	3	0	0	6	–ve value
R_2	0	2	1	1	0	0	2	2
R_3	0	0	–1	1	2	0	8	–ve value
R_4	0	0	3	1	0	2	2	← 1.5

Again, R_1 has negative entry and that is, –1. So, the column of y_2 is pivot column and by ratio R_4 is the pivot row and pivot point is 3.

Now, applying

$$R_1 \to 3R_1 + R_4,\ R_2 \to 3R_2 - R_4,\ R_3 \to 3R_3 + R_4$$

then the above table becomes,

	W	y_1	y_2	y_3	y_4	y_5	Constant	ratio
R_1	6	0	0	10	0	2	20	
R_2	0	6	0	2	0	–2	4	
R_3	0	0	0	4	6	2	26	
R_4	0	0	3	1	0	2	2	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $y_3 = 0 = y_5$. Then

by R_1, $\quad W = 10/3;$

by R_2, $\quad 6y_1 = 4$

by R_3, $\quad 6y_4 = 26$

by R_4, $\quad 3y_2 = 2$

Thus, max(W) = 10/3 at $(y_1, y_2) = (0.66, 0.66)$.

Therefore, min(z) = 10/3 at $(x_1, x_2, x_3) = (5/3, 0, 1/3)$.

Exercise 3.2

Solve the following linear programming problems by using simplex method:

1. Minimize $z = 4x_1 + 3x_2$ subject to $2x_1 + 3x_2 \ge 1$, $3x_1 + x_2 \ge 4$; $x_1 \ge 0$, $x_2 \ge 0$.
2. Minimize $z = 10x_1 + 15x_2$ subject to $x_1 + x_2 \ge 8$, $10x_1 + 6x_2 \ge 60$, $x_1 \ge 0$, $x_2 \ge 0$.
3. Minimize $z = 8x_1 + 9x_2$ subject to $x_1 + x_2 \ge 5$, $3x_1 + x_2 \ge 21$, $x_1 \ge 0$, $x_2 = 0$
4. Minimize: $z = 21x_1 + 50x_2$ subject to $2x_1 + 5x_2 \ge 12$; $3x_1 + 7x_2 \ge 17$; $x_1, x_2 \ge 0$.
5. Minimize $z = 3x_1 + 2x_2$ subject to $2x_1 + 4x_2 \ge 10$; $4x_1 + 2x_2 \ge 10$, $x_2 \ge 4$, $x_1 \ge 0$, $x_2 \ge 0$.
6. Minimize $z = 20x_1 + 30x_2$ subject to $x_1 + 4x_2 \ge 8$, $x_1 + x_2 \ge 5$; $2x_1 + x_2 \ge 7$, $x_1 \ge 0$, $x_2 \ge 0$.
7. Minimize $z = 2x_1 + 9x_2 + x_3$ subject to $x_1 + 4x_2 + 2x_3 \ge 5$; $3x_1 + x_2 + 2x_3 \ge 4$; $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

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Answers

1. $\text{Min}(z) = \frac{16}{3}$ at $\left(\frac{4}{3}, 0\right)$
2. $\text{Min}(z) = 80$ at $(8, 0)$
3. $\text{Min}(z) = 56$ at $(7, 0)$
4. $\text{Min}(z) = 121$ at $(1, 2)$
5. $\text{Min}(z) = 19/2$ at $(x_1, x_2) = (1/2, 4)$
6. $\text{Min}(z) = \frac{670}{7}$ at $\left(\frac{20}{7}, \frac{9}{7}\right)$
7. $\text{Min}(z) = 5/2$ at $(x_1, x_2, x_3) = (0, 0, 5/2)$

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