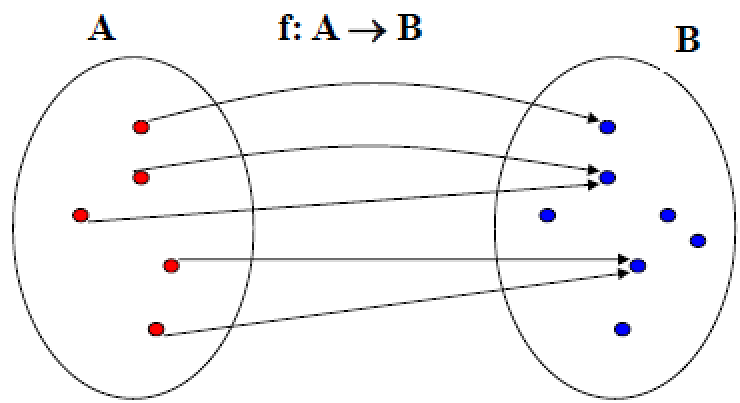
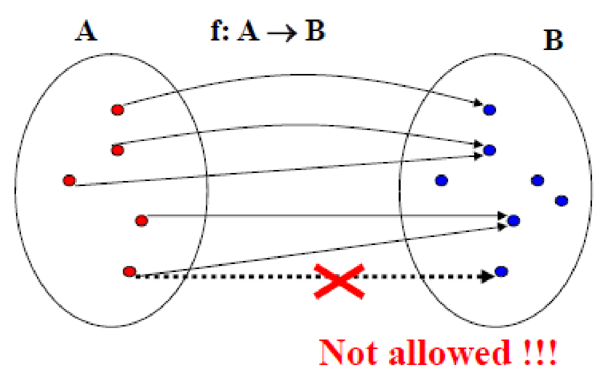
Function

Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \to B$** , is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.





Representing functions

Example1:

- Let A = {1,2,3} and B = {a,b,c}
- Assume f is defined as:
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow c$
- Is f a function ?
- **Yes.** since f(1)=c, f(2)=a, f(3)=c. each element of A is assigned an element from B

Example 2:

- Let A = {1,2,3} and B = {a,b,c}
- Assume g is defined as
    - $1 \rightarrow c$
    - $1 \rightarrow b$
    - $2 \rightarrow a$
    - $3 \rightarrow c$
- Is g a function?
- **No.** g(1) = is assigned both c and b.

Important sets

Definitions: Let f be a function from A to B.

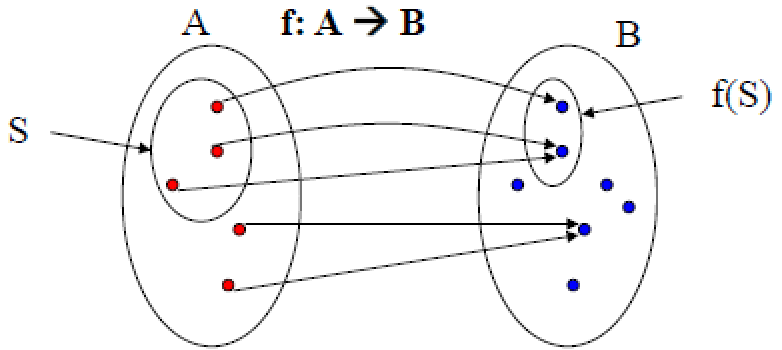
- We say that A is the **domain** of f and B is the **codomain** of f.
- If f(a) = b, **b is the image of a** and **a is a pre-image of b**.
- **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let A = {1,2,3} and B = {a,b,c}

- Assume f is defined as: 1 → c, 2 → a, 3 → c
- What is the image of 1?
- 1 → c   c is the image of 1
- What is the pre-image of a?
- 2 → a   2 is a pre-image of a.
- Domain of f ? {1,2,3}
- Codomain of f ? {a,b,c}
- Range of f ? {a,c}

Image of a subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that f(S) = { f(s) | s ∈ S }.



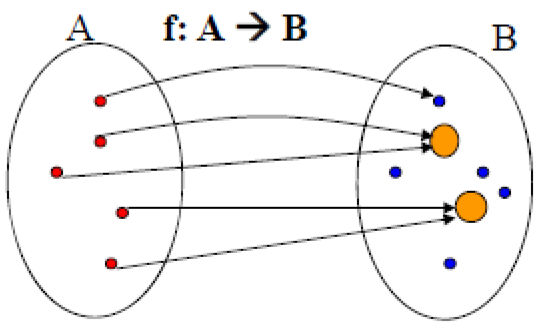
Example:

- Let A = {1,2,3} and B = {a,b,c} and f: 1 → c, 2 → a, 3 → c
- Let S = {1,3} then image f(S) = {c}.

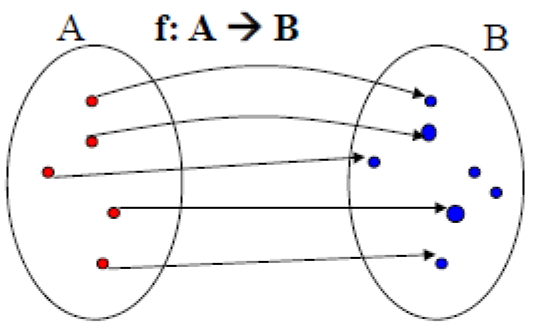
Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternate: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$. This is the contrapositive of the definition.



Not injective



Injective function

Injective functions

Example 1: Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is f one to one? **No**, it is not one-to-one since $f(1) = f(3) = c$, and $1 \neq 3$.

Example 2: Let $g : Z \rightarrow Z$, where $g(x) = 2x - 1$.

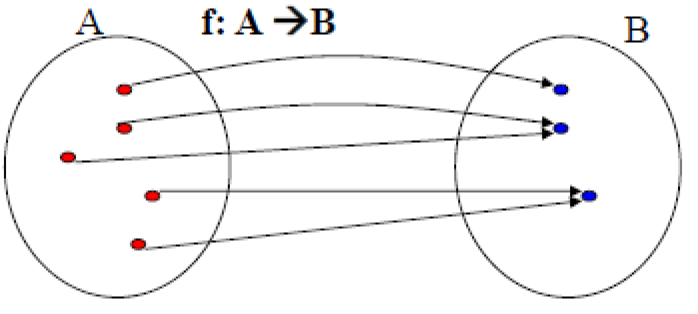
- Is g is one-to-one (why?)
- **Yes.**
- Suppose $g(a) = g(b)$, i.e., $2a - 1 = 2b - 1 \Rightarrow 2a = 2b$

$$\Rightarrow a = b.$$

Surjective function

Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

Alternative: all co-domain elements are covered



Surjective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$

- Is f an onto?
- **No.** f is not onto, since $b \in B$ has no pre-image.

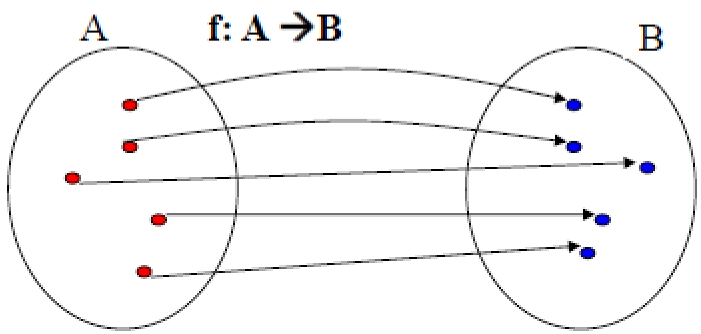
Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

- Define h: $A \rightarrow B$ as h(x) = x mod 3.

- Is h an onto function?
- **Yes.** h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one (injection) and onto (surjection)**.



Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f a bijection?
- **Yes.** It is both one-to-one and onto.

Identity function

Definition: Let A be a set. The **identity function** on A is the function i_A: A $\rightarrow$ A where i_A (x) = x.

Example:

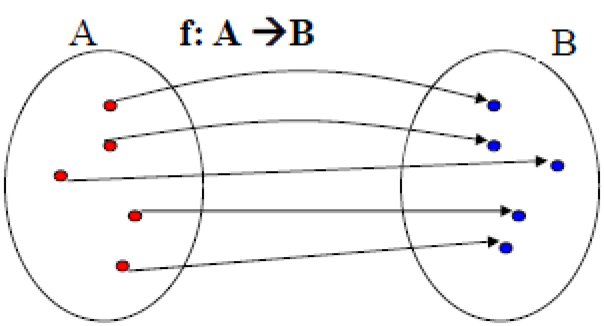
- Let A = {1,2,3}

Then:

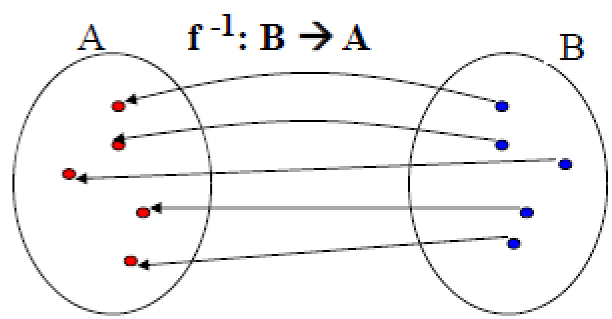
- i_A (1) = 1
- i_A (2) = 2
- i_A (3) = 3.

Inverse functions

Definition: Let f be a **bijection** from set A to set B. The **inverse function of f** is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by $\mathbf{f^{-1}}$. Hence, $f^{-1}(b) = a$, when f(a) = b. If the inverse function of f exists, f is called **invertible.**



f is bijective



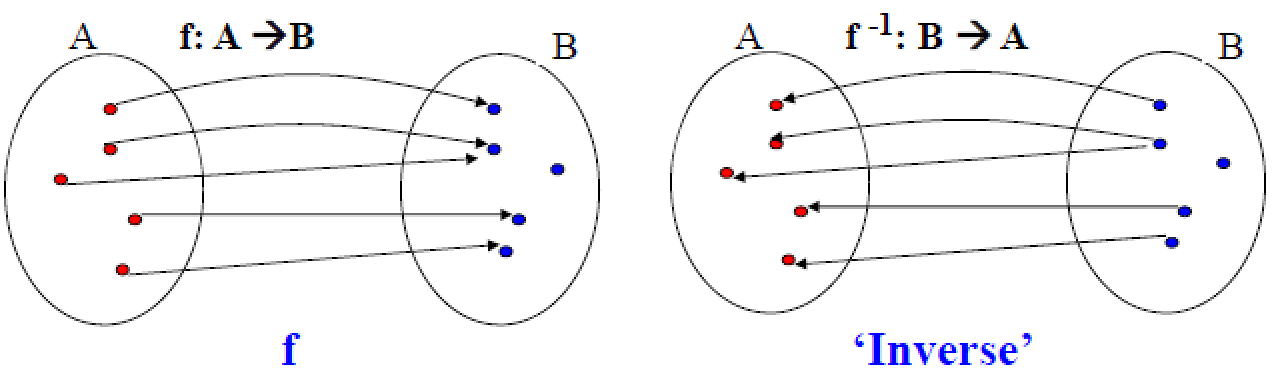
Inverse of f

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

Inverse is not a function. One element of B is mapped to two different elements.

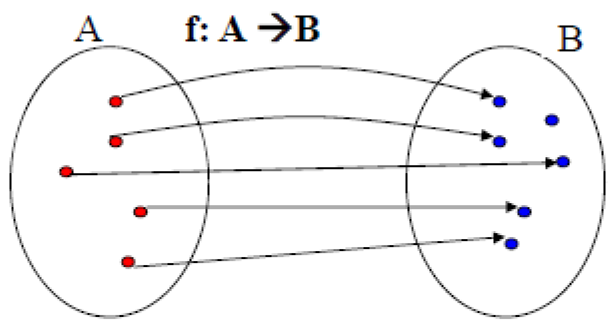


Inverse functions

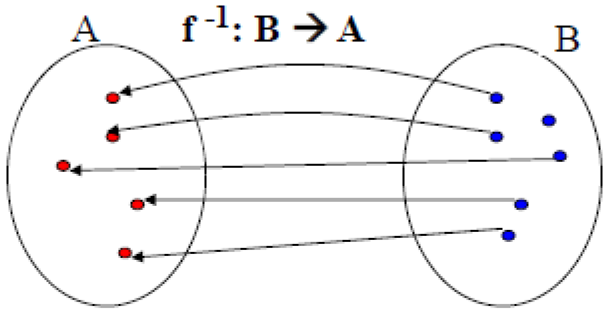
Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not onto:

Inverse is not a function. One element of B is not assigned any value in B.



f



'Inverse'

Inverse functions

Example 1:

- Let A = {1,2,3} and i_A be the identity function

- $i_A(1) = 1$ $\quad\quad$ $i_A^{-1}(1) = 1$
- $i_A(2) = 2$ $\quad\quad$ $i_A^{-1}(2) = 2$
- $i_A(3) = 3$ $\quad\quad$ $i_A^{-1}(3) = 3$

- Therefore, the inverse function of i_A is i_A.

Example 2:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$y = 2x - 1 \;\Rightarrow\; y + 1 = 2x$

$\Rightarrow (y+1)/2 = x$

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

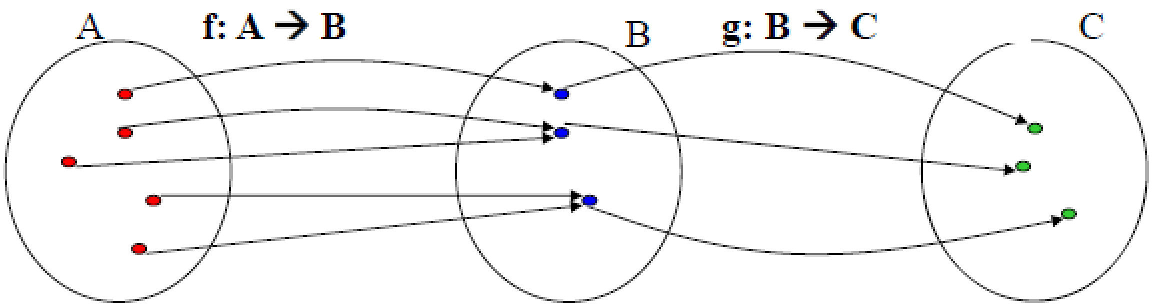
Test the correctness of inverse:

- $g(3) = 2*3 - 1 = 5$
- $g^{-1}(5) = (5+1)/2 = 3$
- $g(10) = 2*10 - 1 = 19$
- $g^{-1}(19) = (19+1)/2 = 10$.

Composition of functions

Definition: Let f be a function from set A to set B and let g be a function from set B to set C. The **composition of the functions g and f**, denoted by $g \circ f$ is defined by

- $(g \circ f)(a) = g(f(a))$.



Example 1:

- Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$

$g : A \rightarrow A,$	$f: A \rightarrow B$
$1 \rightarrow 3$	$1 \rightarrow b$
$2 \rightarrow 1$	$2 \rightarrow a$
$3 \rightarrow 2$	$3 \rightarrow d$

$f \circ g : A \rightarrow B$:

- $1 \rightarrow d$
- $2 \rightarrow b$
- $3 \rightarrow a$

Example 2:

- Let f and g be two functions from Z to Z, where
- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \rightarrow Z$
- $(f \circ g)(x) \;=\; f(g(x)) \;=\; f(x^2) \;=\; 2(x^2)$
- $g \circ f : Z \rightarrow Z$
- $(g \circ f)(x) \;=\; g(f(x)) \;=\; g(2x) \;=\; (2x)^2 \;=\; 4x^2$

Note that the order of the function composition matters

Example 3:

- **$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.**
- Let $f : \mathbf{R} \rightarrow \mathbf{R}$, where $f(x) = 2x – 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f((x+1)/2) = 2((x+1)/2) - 1 = (x+1) - 1 = x$
- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(2x - 1) = (2x)/2 = x$