

Sequence

A sequence is a discrete structure used to represent an ordered list. For example, 1, 2, 3, 5, 8 is a sequence with five terms and $1, 3, 9, 27, 81, \dots, 3^n, \dots$ is an infinite sequence.

A *sequence* is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \dots\}$ or the set $\{1, 2, 3, \dots\}$) to a set S . We use the notation a_n to denote the image of the integer n . We call a_n a *term* of the sequence.

EXAMPLE 1 Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}.$$

The list of the terms of this sequence, beginning with a_1 , namely,

$$a_1, a_2, a_3, a_4, \dots,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \dots, ar^n, \dots$$

where the *initial term* a and the *common ratio* r are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

EXAMPLE 2 The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and -1 , 2 and 5, and 6 and $1/3$, respectively, if we start at $n = 0$. The list of terms $b_0, b_1, b_2, b_3, b_4, \dots$ begins with

$$1, -1, 1, -1, 1, \dots;$$

the list of terms $c_0, c_1, c_2, c_3, c_4, \dots$ begins with

$$2, 10, 50, 250, 1250, \dots;$$

and the list of terms $d_0, d_1, d_2, d_3, d_4, \dots$ begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots$$



An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \dots, a + nd, \dots$$

where the *initial term* a and the *common difference* d are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

EXAMPLE 3 The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to -1 and 4 , and 7 and -3 , respectively, if we start at $n = 0$. The list of terms $s_0, s_1, s_2, s_3, \dots$ begins with

$$-1, 3, 7, 11, \dots,$$


and the list of terms $t_0, t_1, t_2, t_3, \dots$ begins with

$$7, 4, 1, -2, \dots$$




A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, \dots, a_{n-1} , for all integers n

EXAMPLE 5 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \dots$, and suppose that $a_0 = 2$. What are a_1 , a_2 , and a_3 ?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$. 

EXAMPLE 6 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3 ?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$. We can find a_4 , a_5 , and each successive term in a similar way. 

The *Fibonacci sequence*, f_0, f_1, f_2, \dots , is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \dots$.

EXAMPLE 7 Find the Fibonacci numbers f_2, f_3, f_4, f_5 , and f_6 .

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$

$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$

$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$



EXAMPLE 11

Extra
Examples

Compound Interest Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution: To solve this problem, let P_n denote the amount in the account after n years. Because the amount in the account after n years equals the amount in the account after $n - 1$ years plus interest for the n th year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10,000$.

We can use an iterative approach to find a formula for P_n . Note that

$$P_1 = (1.11)P_0$$

$$P_2 = (1.11)P_1 = (1.11)^2P_0$$

$$P_3 = (1.11)P_2 = (1.11)^3P_0$$

$$\vdots$$

$$P_n = (1.11)P_{n-1} = (1.11)^nP_0.$$

When we insert the initial condition $P_0 = 10,000$, the formula $P_n = (1.11)^n 10,000$ is obtained.

Inserting $n = 30$ into the formula $P_n = (1.11)^n 10,000$ shows that after 30 years the account contains


$$P_{30} = (1.11)^{30} 10,000 = \$228,922.97.$$

Find formulae for the sequences with the following first five terms: (a) $1, 1/2, 1/4, 1/8, 1/16$
(b) $1, 3, 5, 7, 9$ (c) $1, -1, 1, -1, 1$.

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(b) $1, 3, 5, 7, 9$ (c) $1, -1, 1, -1, 1$.


Solution: (a) We recognize that the denominators are powers of 2. The sequence with $a_n = 1/2^n$, $n = 0, 1, 2, \dots$ is a possible match. This proposed sequence is a geometric progression with $a = 1$ and $r = 1/2$.

(b) We note that each term is obtained by adding 2 to the previous term. The sequence with $a_n = 2n + 1$, $n = 0, 1, 2, \dots$ is a possible match. This proposed sequence is an arithmetic progression with $a = 1$ and $d = 2$.

(c) The terms alternate between 1 and -1 . The sequence with $a_n = (-1)^n$, $n = 0, 1, 2, \dots$ is a possible match. This proposed sequence is a geometric progression with $a = 1$ and $r = -1$. 

How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

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Solution: In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer n appears exactly n times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match. 

How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

TABLE 1 Some Useful Sequences.

<i>n</i> th Term	First 10 Terms
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...

Summation

$$a_m, a_{m+1}, \dots, a_n$$

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^n a_j, \quad \sum_{j=m}^n a_j, \quad \text{or} \quad \sum_{m \leq j \leq n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j) to represent

$$a_m + a_{m+1} + \dots + a_n.$$

Here, the variable j is called the **index of summation**, and the choice of the letter j as the variable is arbitrary; that is, we could have used any other letter, such as i or k . Or, in notation,

$$\sum_{j=m}^n a_j = \sum_{i=m}^n a_i = \sum_{k=m}^n a_k.$$

Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \dots$

Solution: The lower limit for the index of summation is 1, and the upper limit is 100. We write this sum as

$$\sum_{j=1}^{100} \frac{1}{j}.$$



What is the value of $\sum_{j=1}^5 j^2$?

Solution: We have

$$\begin{aligned}\sum_{j=1}^5 j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55.\end{aligned}$$

What is the value of $\sum_{k=4}^8 (-1)^k$?

Solution: We have

$$\begin{aligned}\sum_{k=4}^8 (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1.\end{aligned}$$

Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^4 \sum_{j=1}^3 ij.$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\begin{aligned} \sum_{i=1}^4 \sum_{j=1}^3 ij &= \sum_{i=1}^4 (i + 2i + 3i) \\ &= \sum_{i=1}^4 6i \\ &= 6 + 12 + 18 + 24 = 60. \end{aligned}$$



What is the value of $\sum_{s \in \{0,2,4\}} s$?

Solution: Because $\sum_{s \in \{0,2,4\}} s$ represents the sum of the values of s for all the members of the set $\{0, 2, 4\}$, it follows that

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$



TABLE 2 Some Useful Summation Formulae.

<i>Sum</i>	<i>Closed Form</i>
$\sum_{k=0}^n ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^n k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^n k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^n k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, x < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, x < 1$	$\frac{1}{(1-x)^2}$