Set Theory

Sets

- Definition: Well-defined collection of distinct objects
- Members or Elements: part of the collection
- Roster Method: Description of a set by listing the elements, enclosed with braces
  - Examples:
    - Vowels = {a,e,i,o,u}
    - Primary colors = {red, blue, yellow}
- Membership examples
  - "a belongs to the set of Vowels" is written as:

    $a \in$ Vowels
  - "j does not belong to the set of Vowels:

    $j \notin$ Vowels

Sets

- Set-builder method
  - $A = \{ x \mid x \in S, P(x) \}$ or $A = \{ x \in S \mid P(x) \}$
    - A is the set of all elements x of S, such that x satisfies the property P
    - Example:
      - If $X = \{2,4,6,8,10\}$, then in set-builder notation, X can be described as

        $X = \{n \in Z \mid n \text{ is even and } 2 \leq n \leq 10\}$

Sets

- Standard Symbols which denote sets of numbers
  - N : The set of all natural numbers (i.e.,all positive integers)
  - Z : The set of all integers
  - Z^+ : The set of all positive integers
  - Z* : The set of all nonzero integers
  - E : The set of all even integers
  - Q : The set of all rational numbers
  - Q* : The set of all nonzero rational numbers
  - Q^+ : The set of all positive rational numbers
  - R : The set of all real numbers
  - R* : The set of all nonzero real numbers
  - R^+ : The set of all positive real numbers
  - C : The set of all complex numbers
  - C* : The set of all nonzero complex numbers

Sets

- Subsets
  - "X is a subset of Y" is written as $X \subseteq Y$
  - "X is not a subset of Y" is written as $X \nsubseteq Y$
  - Example:
    - $X = \{a,e,i,o,u\}$, $Y = \{a, i, u\}$ and

      $Z= \{b,c,d,f,g\}$

      – $Y \subseteq X$, since every element of Y is an element of X

      – $Y \nsubseteq Z$, since $a \in Y$, but $a \notin Z$

Sets

- Superset
  - X and Y are sets. If $X \subseteq Y$, then “X is contained in Y” or “Y contains X” or Y is a superset of X, written $Y \supseteq X$
- Proper Subset
  - X and Y are sets. X is a proper subset of Y if $X \subseteq Y$ and there exists at least one element in Y that is not in X. This is written $X \subset Y$.
  - Example:
    - X = {a,e,i,o,u}, Y = {a,e,i,o,u,y}
      - $X \subset Y$, since $y \in Y$, but $y \notin X$

Sets

- Set Equality
  - X and Y are sets. They are said to be equal if every element of X is an element of Y and every element of Y is an element of X, i.e. $X \subseteq Y$ and $Y \subseteq X$
  - Examples:
    - {1,2,3} = {2,3,1}
    - X = {red, blue, yellow} and Y = {c | c is a primary color} Therefore, X=Y
- Empty (Null) Set
  - A Set is Empty (Null) if it contains no elements.
  - The Empty Set is written as $\varnothing$
  - The Empty Set is a subset of every set

Sets

- Finite and Infinite Sets
  - X is a set. If there exists a nonnegative integer n such that X has n elements, then X is called a finite set with n elements.
  - If a set is not finite, then it is an infinite set.
  - Examples:
    - Y = {1,2,3} is a finite set
    - P = {red, blue, yellow} is a finite set
    - E , the set of all even integers, is an infinite set
    - $\emptyset$, the Empty Set, is a finite set with 0 elements

Sets

- Cardinality of Sets
  - Let S be a finite set with n distinct elements, where $n \geq 0$. Then $|S| = n$, where the cardinality (number of elements) of S is n
  - Example:
    - If P = {red, blue, yellow}, then $|P| = 3$
  - Singleton
    - A set with only one element is a singleton
    - Example:
      - H = { 4 }, $|H| = 1$, H is a singleton

Sets

- Power Set
  - For any set X ,the power set of X ,written P(X),is the set of all subsets of X
  - Example:
    - If X = {red, blue, yellow}, then P(X) = { ∅ , {red}, {blue}, {yellow}, {red,blue}, {red, yellow}, {blue, yellow}, {red, blue, yellow} }
- Universal Set
  - An arbitrarily chosen, but fixed set

Sets

- Venn Diagrams
  - Abstract visualization of a Universal set, U as a rectangle, with all subsets of U shown as circles.
  - Shaded portion represents the corresponding set
  - Example:
    - In Figure 1, Set X, shaded, is a subset of the Universal set, U

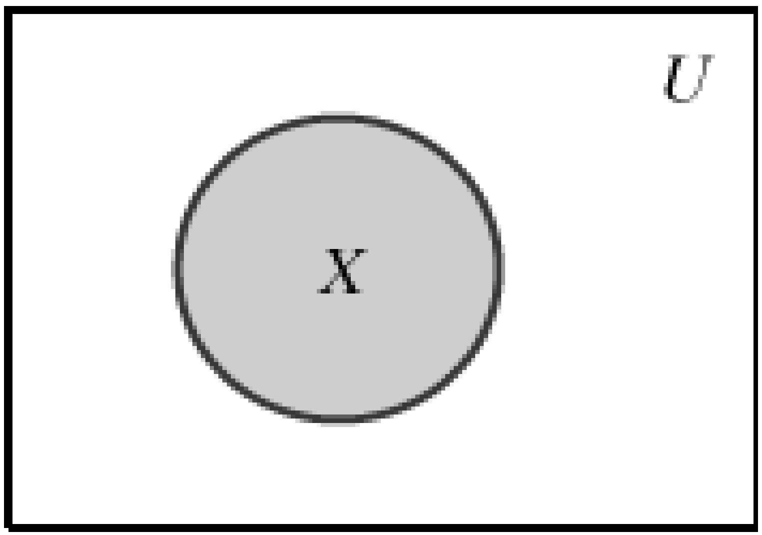

FIGURE 1.1 Set X

Set Operations and Venn Diagrams

- Union of Sets

The **union** of two sets X and Y, denoted by $X \cup Y$, is defined to be the set

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}.$$

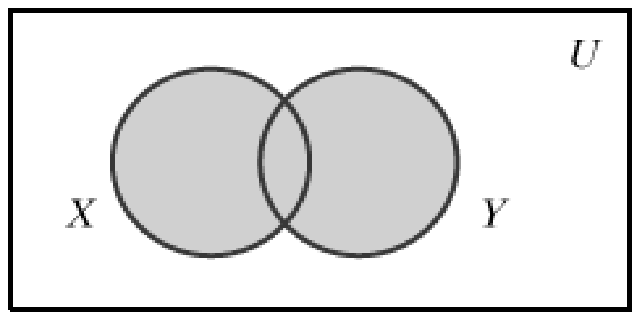

FIGURE 1.2 Venn diagram of $X \cup Y$

Example: If X = {1,2,3,4,5} and Y = {5,6,7,8,9}, then
XUY = {1,2,3,4,5,6,7,8,9}

Sets

- Intersection of Sets

The **intersection** of two sets X and Y, denoted by $X \cap Y$, is defined to be the set

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}.$$

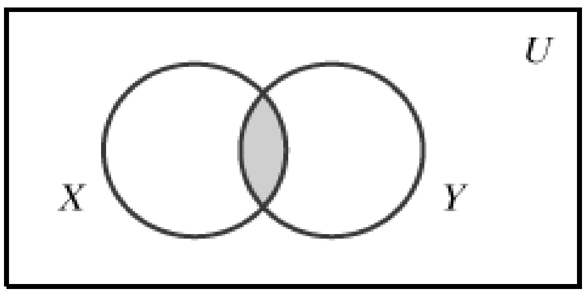

FIGURE 1.3 Venn diagram of $X \cap Y$

Example: If X = {1,2,3,4,5} and Y = {5,6,7,8,9}, then $X \cap Y$ = {5}

Sets

- Disjoint Sets

Two sets X and Y are said to be **disjoint** if $X \cap Y = \emptyset$.

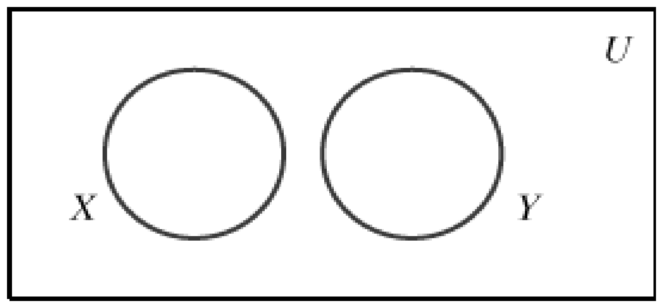

FIGURE 1.4 $X \cap Y = \emptyset$

Example: If X = {1,2,3,4,} and Y = {6,7,8,9}, then $X \cap Y = \emptyset$

Sets

- Difference

Let X and Y be sets. The **difference** of X and Y (or the **relative complement** of Y in X), written $X - Y$, is the set

$$X - Y = \{x \mid x \in X \text{ but } x \notin Y\}.$$

- Example: If X = {a,b,c,d} and Y = {c,d,e,f}, then X – Y = {a,b} and Y – X = {e,f}

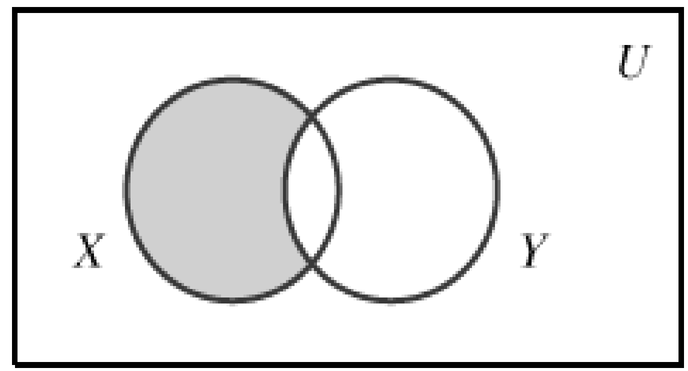

FIGURE 1.6 Venn diagram of $X - Y$

Sets

- Complement

The complement of a set X with respect to a universal set U, denoted by $\overline{X}$, is defined to be $\overline{X}$ = {x | x $\in$ U, but x $\notin$ X}

Example: If U = {a,b,c,d,e,f} and X = {c,d,e,f}, then $\overline{X}$ = {a,b}

Sets

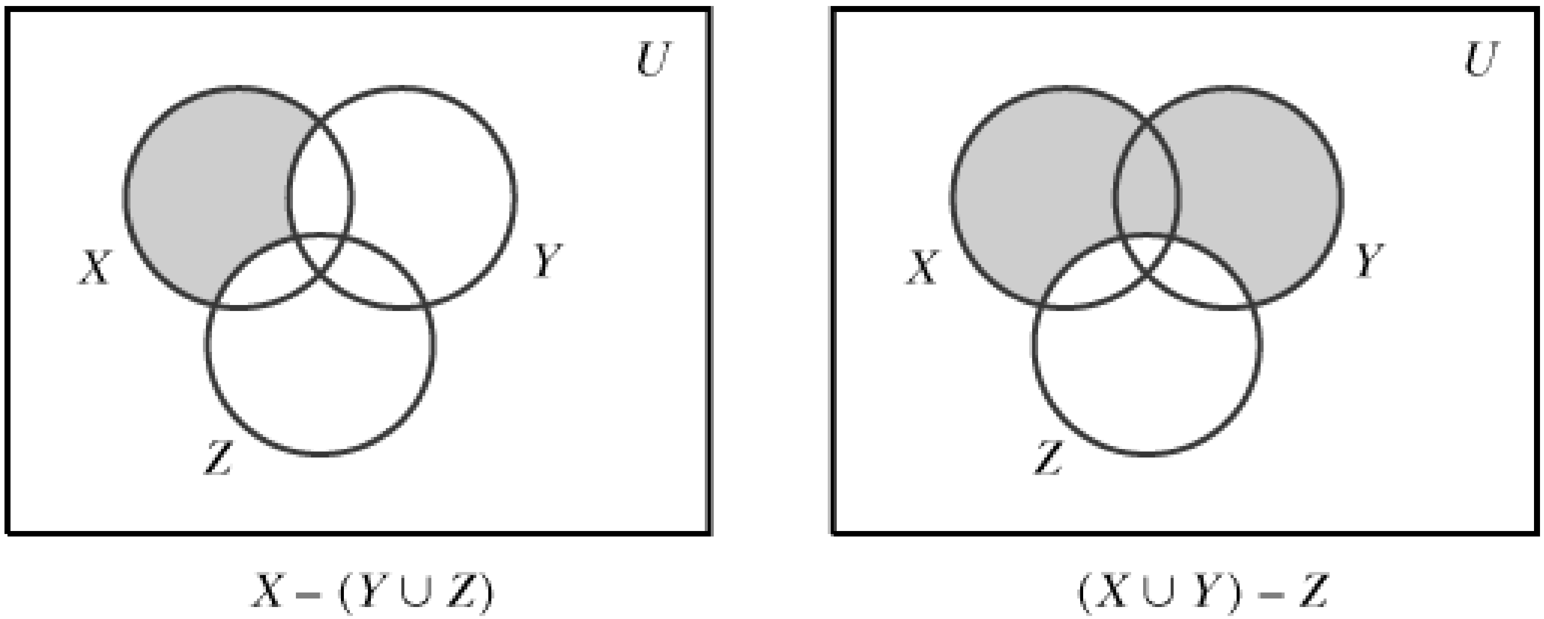

FIGURE 1.8 Venn diagrams of the sets $X - (Y \cup Z)$ and $(X \cup Y) - Z$

Sets

- Ordered Pair
  - X and Y are sets. If $x \in X$ and $y \in Y$, then an ordered pair is written (x,y)
  - Order of elements is important. (x,y) is not necessarily equal to (y,x)
- Cartesian Product
  - The **Cartesian product** of two sets X and Y ,written $X \times Y$,is the set
  - $X \times Y = \{(x,y) | x \in X , y \in Y\}$
    - For any set X, $X \times \varnothing = \varnothing = \varnothing \times X$
  - Example:
    - X = {a,b}, Y = {c,d}
      - $X \times Y = \{(a,c), (a,d), (b,c), (b,d)\}$
      - $Y \times X = \{(c,a), (d,a), (c,b), (d,b)\}$

Fundamental Set Properties

Idempotence

$A \cup A = A$

$A \cap A = A$

Associativity

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Commutativity

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Distributivity (∩ over ∪)

$[A \cap (B \cup C)] = [(A \cap B) \cup (A \cap C)]$

$[(A \cup B) \cap C] = [(A \cap C) \cup (B \cap C)]$

Complement

$A \cup \overline{A} = U$

$A \cap \overline{A} = \varnothing$

Involution

$\overline{(\overline{A})} = A$

Domination

$A \cup U = U$

$A \cap \varnothing = \varnothing$

Identity

$A \cup \varnothing = A$

$A \cap U = A$

De Morgan's Laws

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Distributivity (∪ over ∩)

$[A \cup (B \cap C)] = [(A \cup B) \cap (A \cup C)]$

$[(A \cap B) \cup C] = [(A \cup C) \cap (B \cup C)]$

Complement (continued)

$\overline{\varnothing} = U$

$\overline{U} = \varnothing$