

# Matrices

A *matrix* is a rectangular array of numbers. A matrix with  $m$  rows and  $n$  columns is called an  $m \times n$  matrix. The plural of matrix is *matrices*. A matrix with the same number of rows as columns is called *square*. Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

**EXAMPLE 1** The matrix  $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$  is a  $3 \times 2$  matrix.

Let  $m$  and  $n$  be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The  $i$ th *row* of  $\mathbf{A}$  is the  $1 \times n$  matrix  $[a_{i1}, a_{i2}, \dots, a_{in}]$ . The  $j$ th *column* of  $\mathbf{A}$  is the  $m \times 1$  matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}.$$

The  $(i, j)$ th *element* or *entry* of  $\mathbf{A}$  is the element  $a_{ij}$ , that is, the number in the  $i$ th row and  $j$ th column of  $\mathbf{A}$ . A convenient shorthand notation for expressing the matrix  $\mathbf{A}$  is to write  $\mathbf{A} = [a_{ij}]$ , which indicates that  $\mathbf{A}$  is the matrix with its  $(i, j)$ th element equal to  $a_{ij}$ .

# Matrix Arithmetic

Let  $\mathbf{A} = [a_{ij}]$  and  $\mathbf{B} = [b_{ij}]$  be  $m \times n$  matrices. The *sum* of  $\mathbf{A}$  and  $\mathbf{B}$ , denoted by  $\mathbf{A} + \mathbf{B}$ , is the  $m \times n$  matrix that has  $a_{ij} + b_{ij}$  as its  $(i, j)$ th element. In other words,  $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$ .

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}.$$

Let  $\mathbf{A}$  be an  $m \times k$  matrix and  $\mathbf{B}$  be a  $k \times n$  matrix. The *product* of  $\mathbf{A}$  and  $\mathbf{B}$ , denoted by  $\mathbf{AB}$ , is the  $m \times n$  matrix with its  $(i, j)$ th entry equal to the sum of the products of the corresponding elements from the  $i$ th row of  $\mathbf{A}$  and the  $j$ th column of  $\mathbf{B}$ . In other words, if  $\mathbf{AB} = [c_{ij}]$ , then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

## Cont...

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Find  $\mathbf{AB}$  if it is defined.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

## Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Does  $\mathbf{AB} = \mathbf{BA}$ ?

*Solution:* We find that

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}.$$

Hence,  $\mathbf{AB} \neq \mathbf{BA}$ .

# Identity Matrix

The *identity matrix of order  $n$*  is the  $n \times n$  matrix  $\mathbf{I}_n = [\delta_{ij}]$ , (the *Kronecker delta*) where  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ . Hence,

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

## Transpose of a Matrix

Let  $\mathbf{A} = [a_{ij}]$  be an  $m \times n$  matrix. The *transpose* of  $\mathbf{A}$ , denoted by  $\mathbf{A}^t$ , is the  $n \times m$  matrix obtained by interchanging the rows and columns of  $\mathbf{A}$ . In other words, if  $\mathbf{A}^t = [b_{ij}]$ , then  $b_{ij} = a_{ji}$  for  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, m$ .

The transpose of the matrix  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$  is the matrix  $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ .

## Zero-one Matrix

A matrix all of whose entries are either 0 or 1 is called a **zero-one matrix**.

Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

*Solution:* We find that the join of  $\mathbf{A}$  and  $\mathbf{B}$  is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of  $\mathbf{A}$  and  $\mathbf{B}$  is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

# Boolean product of Matrices

Let  $\mathbf{A} = [a_{ij}]$  be an  $m \times k$  zero-one matrix and  $\mathbf{B} = [b_{ij}]$  be a  $k \times n$  zero-one matrix. Then the *Boolean product* of  $\mathbf{A}$  and  $\mathbf{B}$ , denoted by  $\mathbf{A} \odot \mathbf{B}$ , is the  $m \times n$  matrix with  $(i, j)$ th entry  $c_{ij}$  where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Find the Boolean product of  $\mathbf{A}$  and  $\mathbf{B}$ , where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

*Solution:* The Boolean product  $\mathbf{A} \odot \mathbf{B}$  is given by

$$\begin{aligned} \mathbf{A} \odot \mathbf{B} &= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix} \\ &= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$