Introduction to proof

- A theorem is a statement that can be shown to be true.
- A proof is a valid argument that establishes the truth of a mathematical statement.
- The statements used in a proof can include axioms (or postulates), which are statements we assume to be true.
- A less important theorem that is helpful in the proof of other results is called a lemma.
- A corollary is a theorem that can be established directly from a theorem that has been proved.
- A conjecture is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert.

Understanding How Theorems Are Stated

- Many theorems assert that a property holds for all elements in a domain, such as the integers or the real numbers.
- The precise statement of such theorems needs to include a universal quantifier, the standard convention in mathematics is to omit it.

Example:

"If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$"

really means

"For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$."

Methods of Proving Theorems

Direct Proofs:

- A direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.

Definition 1:

The integer n is *even* if there exists an integer k such that $n = 2k$, and n is *odd* if there exists an integer k such that $n = 2k + 1$. (Note that every integer is either even or odd, and no integer is both even and odd.) Two integers have the *same parity* when both are even or both are odd; they have *opposite parity* when one is even and the other is odd.

EXAMPLE 1:

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution:

- The theorem states $\forall nP((n) \to Q(n))$, where *P(n)* is "n is an odd integer" and *Q(n)* is "n^2 is odd."
- To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd.
- By the definition of an odd integer, it follows that n = 2k + 1, where k is some integer. We want to show that n^2 is also odd.
- Square both sides of the equation n = 2k + 1 to obtain a new equation that expresses n^2.
- $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k2 + 2k) + 1$.
- By the definition of an odd integer, we can conclude that n^2 is an odd integer (it is one more than twice an integer). Consequently, we have proved that if n is an odd integer, then n2 is an odd integer.

Proof by Contraposition:

- Direct proofs lead from the premises of a theorem to the conclusion. They begin with the premises, continue with a sequence of deductions, and end with the conclusion.
- Proofs of theorems of this type that are not direct proofs, that is, that do not start with the premises and end with the conclusion, are called indirect proofs.
- An extremely useful type of indirect proof is known as proof by contraposition. Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$. This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true.
- In a proof by contraposition of $p \rightarrow q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.

EXAMPLE 2:

Prove that if n is an integer and 3n + 2 is odd, then n is odd.

Solution:

- The first step in a proof by contraposition is to assume that the conclusion of the conditional statement “If 3n + 2 is odd, then n is odd” is false.
- Namely, assume that n is even.
- Then, by the definition of an even integer, n = 2k for some integer k.
- Substituting 2k for n, we find that 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1).
- This tells us that 3n + 2 is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the theorem.
- Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem “If 3n + 2 is odd, then n is odd.”