Relation

Relations

Relations are a natural way to associate objects of various sets

A **binary relation**,[1] or simply a **relation**, R from a set A into a set B is a subset of $A \times B$.

Let R be a relation from A into B, i.e., $R \subseteq A \times B$. If $(a, b) \in R$, we say that, a is ***R*-related** (or **related**, if the relation under consideration is understood) to b and write $a \, R \, b$ (or, $R(a) = b$). If $(a, b) \notin R$, i.e., if a is not R-related to b, we denote it by $a \not R \, b$.

Representing Relations

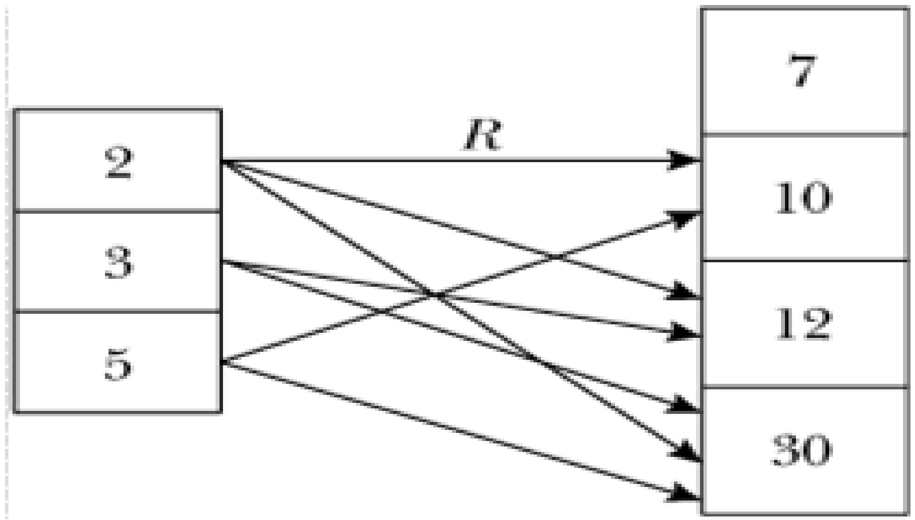
- Set – ordered pairs
- Set definition – membership values
- Arrow Diagram
- Digraph (Directed Graph)

Relations

Arrow Diagram

- Write the elements of A in one column
- Write the elements B in another column
- Draw an arrow from an element, a, of A to an element, b, of B, if $(a, b) \in R$
- Here, $A = \{2,3,5\}$ and $B = \{7,10,12,30\}$ and R from A into B is defined as follows: For all $a \in A$ and $b \in B$, $a\ R\ b$ if and only if a divides b
- The symbol $\rightarrow$ (called an arrow) represents the relation R

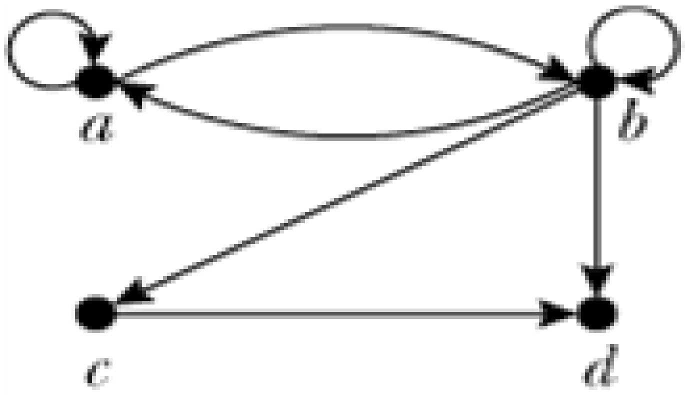
Relations



Relations

- Directed Graph
  - Let R be a relation on a finite set A
  - Describe R pictorially as follows:
    - For each element of A , draw a small or big dot and label the dot by the corresponding element of A
    - Draw an arrow from a dot labeled a , to another dot labeled, b , if $a \, R \, b$.
    - Resulting pictorial representation of R is called the **directed graph representation** of the relation R

Relations



Relations

Let R be a relation from a set A into a set B. Then the **domain** of R, denoted by $\mathcal{D}(R)$, is the set

$$\mathcal{D}(R) = \{a \mid a \in A \text{ and there exists } b \in B \text{ such that } (a, b) \in R\}.$$

The **range**, or **image**, of R, denoted by $\mathcal{I}(R)$, or $\mathrm{Im}(R)$, is the set

$$\mathrm{Im}(R) = \{b \mid b \in B \text{ and there exists } a \in A \text{ such that } (a, b) \in R\}.$$

How do these definitions compare to Defn 12.5 on p732 in your book?

Relations

Let $A = \{4, 5, 6, 11\}$ and $B = \{20, 23, 24, 28, 31\}$. Let us define a relation R from A into B for all $a, b \in R$:

$$a\,R\,b \quad \text{if and only if} \quad a \text{ divides } b.$$

Now $4 \mid 20$, $4 \mid 24$, and $4 \mid 28$. Thus, $4\,R\,20$, $4\,R\,24$, and $4\,R\,28$. In fact, it can be checked that

$$R = \{(4, 20), (4, 24), (4, 28), (5, 20), (6, 24)\}.$$

We can now conclude that

$$\mathcal{D}(R) = \{4, 5, 6\}$$

and

$$\mathrm{Im}(R) = \{20, 24, 28\}.$$

The arrow diagram of R is as shown in Figure 3.3.

Relations

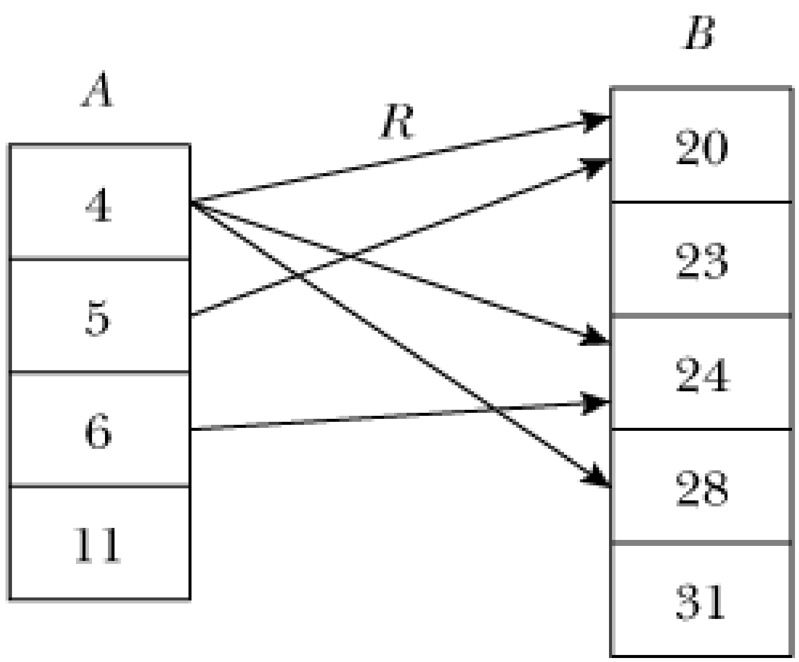


FIGURE 3.3 Arrow diagram of the relation R from A into B

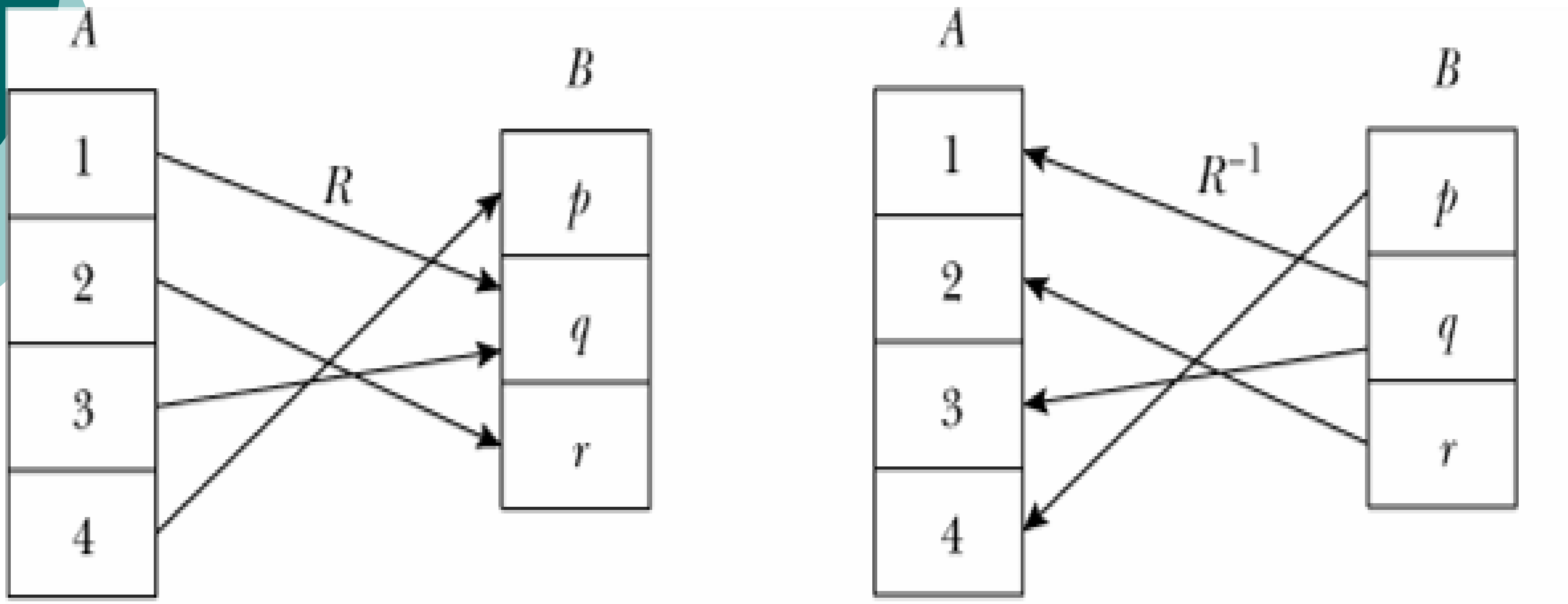
Inverse of Relations

Let R be a relation from a set A into a set B. The **inverse** of R, denoted by R^{-1}, is the relation from B into A, which consists of those ordered pairs that, when reversed, belong to R, i.e.,

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

- Let $A = \{1, 2, 3, 4\}$ and $B = \{p, q, r\}$. Let $R = \{(1, q), (2, r), (3, q), (4, p)\}$. Then $R^{-1} = \{(q, 1), (r, 2), (q, 3), (p, 4)\}$
- To find R^{-1}, just reverse the directions of the arrows
- $D(R) = \{1, 2, 3, 4\} = \mathrm{Im}(R^{-1})$, $\mathrm{Im}(R) = \{p, q, r\} = D(R^{-1})$

Inverse of Relations



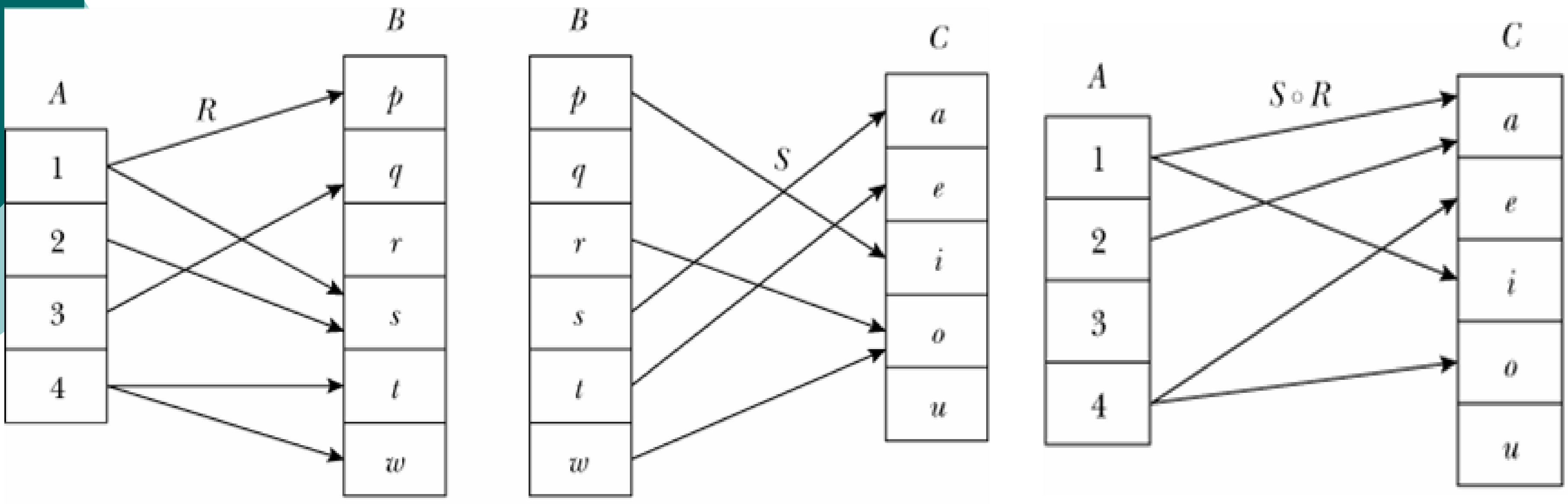
Composition of Relations

Le R be a relations whose domain is A and whose image is B. Let S be a relation whose domain contains B and whose range is C. The composition of S and R is a subset of AXC. It is defined by:

$$S \circ R = \{(a,c) \mid \exists b \in B \, with \, (a,b) \in R \, and \, (b,c) \in S\}$$

Compositive is Associative

Composition of Relations



- Example:
  - Consider the relations *R* and *S* as given above
  - The composition $S \circ R$ is shown on the right

Properties of Relations

Let A be a set and let R be a relation on A. Then R is called

(i) **reflexive**, if for all $a \in A$, $a \, R \, a$;

(ii) **symmetric**, if for all $a, b \in A$, whenever $a \, R \, b$ holds, $b \, R \, a$ must also hold;

(iii) **transitive**, if for all $a, b, c \in A$, whenever $a \, R \, b$ and $b \, R \, c$ hold, $a \, R \, c$ must also hold.

Let A be a set and let R be a relation on A. Observe that

- R is *not reflexive*, if there exists an $a \in A$ such that $(a, a) \notin R$;
- R is *not symmetric*, if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$;
- R is *not transitive*, if there exist $a, b, c \in A$ such that $(a, b) \in R$, $(b, c) \in R$ but $(a, c) \notin R$.

Equivalence Relations

A relation R on a set A is called an **equivalence relation** if R is reflexive, symmetric, and transitive.

Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$. We can show that R is an equivalence relation on A.

Perhaps one of the most natural examples of an equivalence relation is the equality relation on the set of all real numbers. To be specific, let R be the relation on $\mathbb{R}$ defined by $a \; R \; b$ if and only if $a = b$ for all $a, b \in \mathbb{R}$. Then R is an equivalence relation called the **equality relation** on $\mathbb{R}$.

Equivalence Relations

If R is a relation on a finite set A, then we can effectively use the digraph of R to determine if R is reflexive, symmetric, or transitive. In fact, we can show that

1. R is reflexive if and only if there is a loop at each vertex of the digraph (A, R).
2. R is symmetric if and only if in the digraph of R if there is a directed edge from one vertex a to another vertex b, then there must exist a directed edge from vertex b to vertex a.
3. R is a transitive relation if and only if in the digraph of R if there is a directed edge from one vertex a to another vertex b, and if there exists a directed edge from vertex b to vertex c, then there must exist a directed edge from vertex a to vertex c.

Equivalence Relations

Let R be an equivalence relation on a set X. For all $x \in X$, let $[x]$ denote the set

$$[x] = \{y \in X \mid y \, R \, x\}.$$

The subset $[x]$ of X is called the **equivalence class** (***R*-class**, or ***R*-equivalence class**) of the equivalence relation R determined by x.

The relation

$$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,4), (4,1), (2,3), (3,2)\}$$

on the set $A = \{1,2,3,4,5\}$ is an equivalence relation. The equivalence class $[1]$ is the subset of those elements of A that are related to 1. Because only $1 \, R \, 1$ and $4 \, R \, 1$, we have $[1] = \{1,4\}$.