# Modern Control Theory –

→ Uses State space Representation vs transfer function rep

→ Can represent & compute MIMO & nonlinear systems. vs only LTI SISO systems

→ Computationally efficient vs. Closed form solution efficient.

→ Can be easily extended to discrete time domain. i.e. digital systems vs Requires Z-Transform to account for sampling.

★ Time domain rep. (can also do freq.) vs. complex freq. domain rep.

# State Space

What is state?

Smallest set of $\boxed{\text{State variables}}$ to convey full knowledge of sys. at $t = t_1$ if values of s.v. at $t = 0$ is known.

→ Any set of variables that can represent the systems characteristics.

The min # of state variables stays the same.

Completely define the characteristics of the sys.

## State Vector :

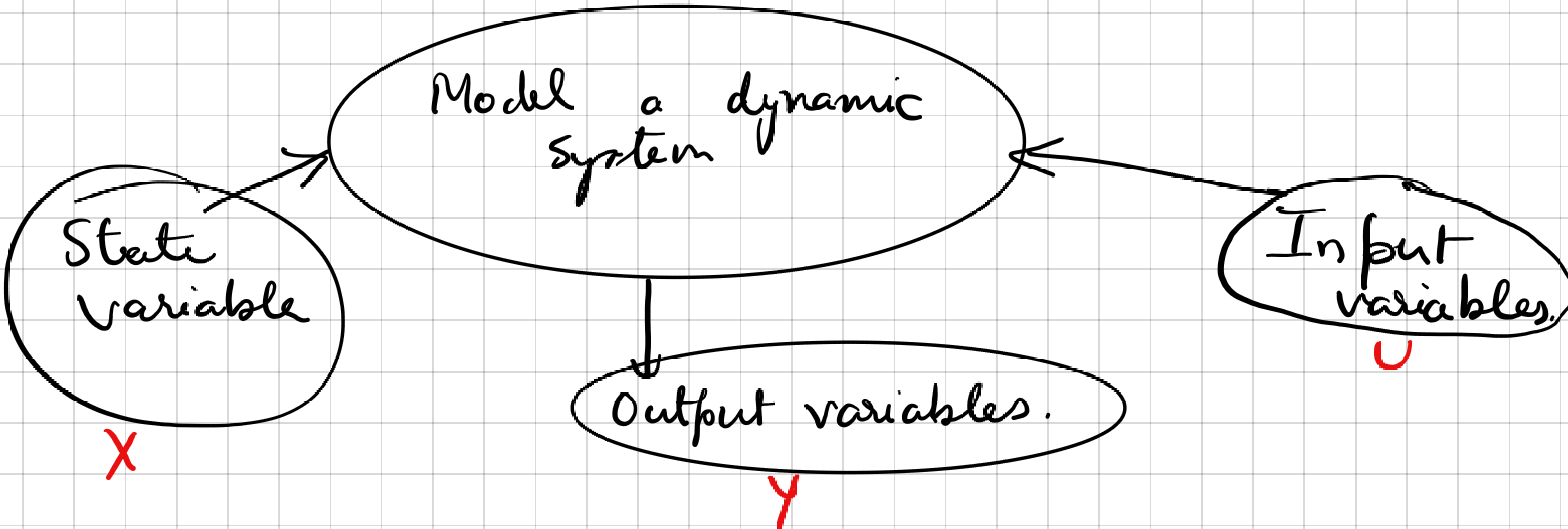
$$\begin{bmatrix} Svar_1 \\ Svar_2 \\ \vdots \\ Svar_n \end{bmatrix} \rightarrow$$

Minimum.
$n$ state variables are required to define some system

## State Space :

$n$ dimensional space whose co-ordinate axes consist of $Svar_1, Svar_2 \ldots Svar_n$ axis defines the state space.

## State Space Equations:




$X \rightarrow$ Matrices of functions

$x \rightarrow$ Individual variables or parameters.

$$\dot{x}_1 = f_1(x_1, x_2, x_3 \ldots x_n, u_1 \ldots u_m)$$
$$\dot{x}_2 = f_2( \quad '' \quad )$$
$$\vdots$$
$$\dot{x}_n = f_n( \quad '' \quad )$$

$\dot{x}(t) = f(x,u,t)$

State equation

If $\dot{x}$ is known & I.C. of $x$ at $t=0$ is known

then at $t = t_1$

$$\boxed{x_{t_1} = x_0 + \underbrace{(t_1 - t_0)}_{\to 0}\dot{x}}$$

$\dot{x}$

$\ddot{x}$

$\dddot{x}$

$$y_1 = g_1(x_1, x_2 \ldots x_n, u_1 \ldots u_m)$$
$$y_2 = g_2( \quad '' \quad )$$
$$\vdots$$
$$y_k = g_n( \quad '' \quad )$$

$y(t) = g(x, u, t)$

output equation

Here : We will deal with linear systems using theories of linear algebra.

$$\dot{X}(t) = A(t)\,x(t) + B(t)\,u(t) \rightarrow \text{State eqn.}$$

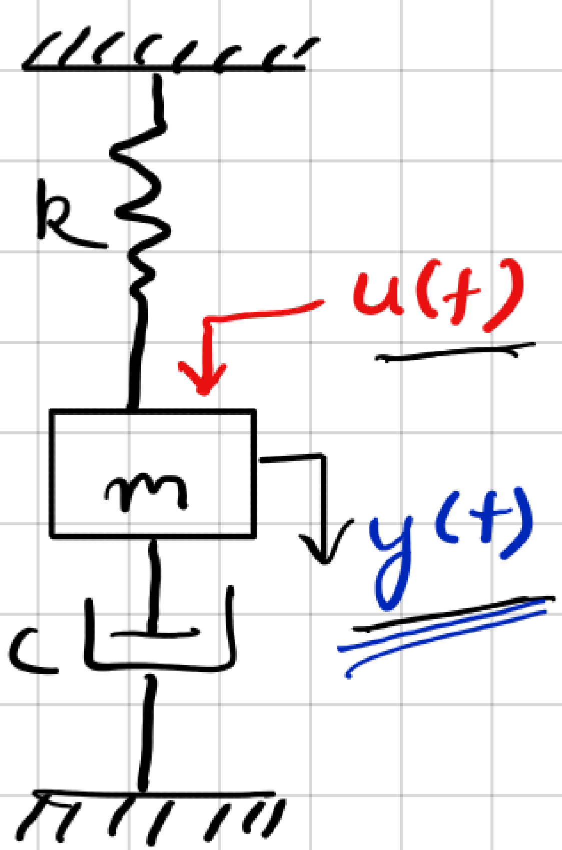
$$Y(t) = C(t)\,x(t) + D(t)\,u(t) \rightarrow \text{Output eqn.}$$

$A(t) \rightarrow$ State Matrix

$B(t) \rightarrow$ Input Matrix

$C(t) \rightarrow$ Output Matrix

$D(t) \rightarrow$ Direct transmission Matrix




$$m\ddot{y} + c\dot{y} + ky = u$$

$$\ddot{y} = \frac{1}{m}\left[-c\underbrace{\dot{y}}_{x_2} - k\underbrace{y}_{x_1} + u\right]$$

Order of differential eqn = # state variables

i.e. 2 state variables

$x_1(t)$

$x_2(t)$

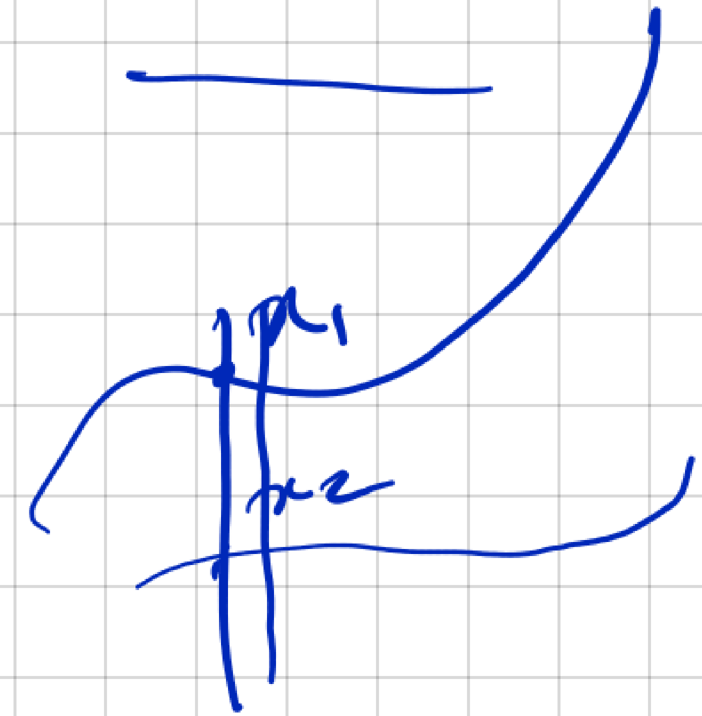
$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{array}{l} \to y \\ \to \dot{y} \end{array} + \begin{bmatrix} 0 \end{bmatrix} u$$

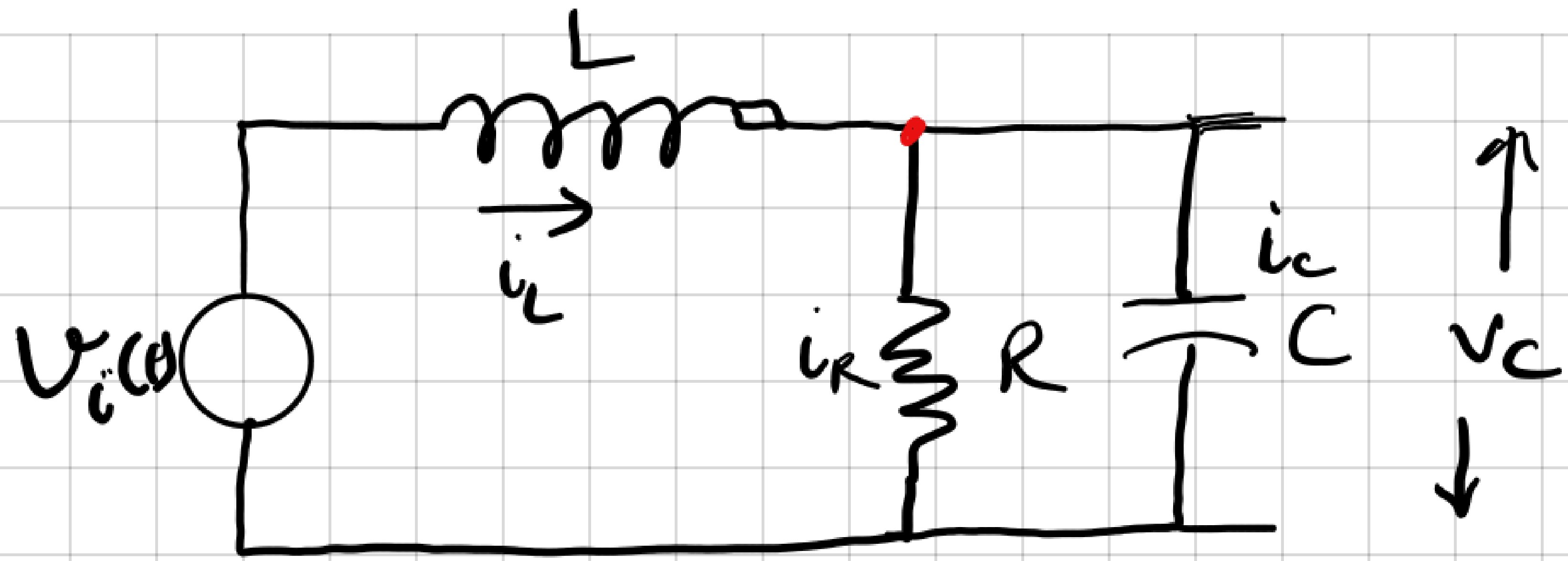
$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}}$$




State space representation of the spring mass damper system.

# RLC Circuit

$v_i,\ v_c,\ R,\ C,\ L$




$$C\frac{dv_c}{dt} = i_c$$

$$L\frac{di_L}{dt} = v_L$$

$$i_L = i_R + i_c$$

$$i_c = i_L - i_R$$

$$= i_L - \frac{v_c}{R}$$

Chosen state variables

$$\begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

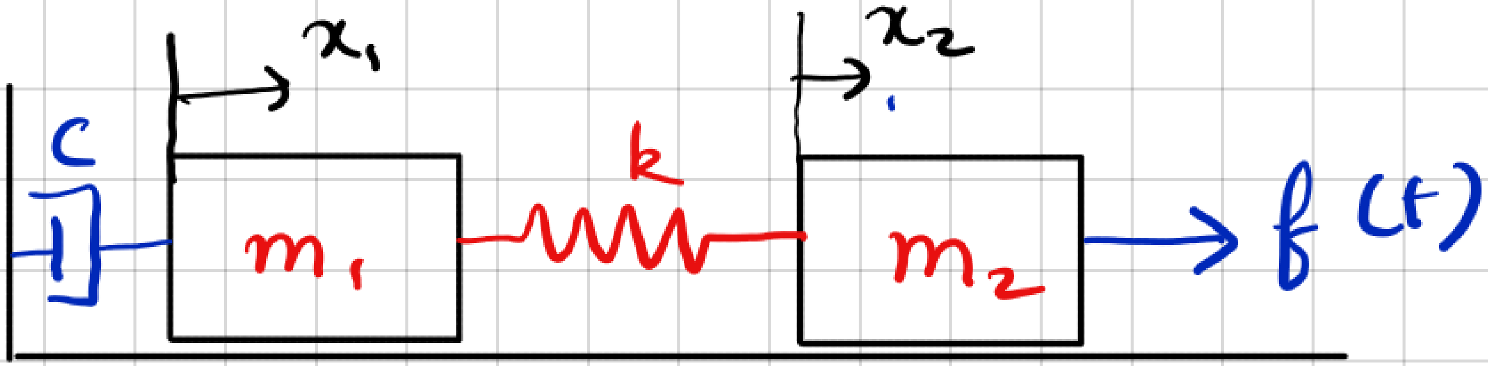
$$\dot{v}_c = \frac{i_c}{C} = \frac{1}{C}\left[i_L - \frac{v_c}{R}\right]$$

$v_L = ? \quad v_i - v_c$

$$\dot{i}_L = \frac{1}{L} v_L = \frac{1}{L}[-v_c] + \frac{1}{L} v_i$$

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad u = v_i$$

$$y = v_c = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + 0\,u$$




$$(m_1, m_2, c, k) \to A$$

$$x_1, x_2 \to y$$

$$f(t) = u$$

$$\bar{X}$$

$$\begin{bmatrix} m_1 \ddot{x}_1 + c\dot{x}_1 + k(x_1 - x_2) = 0 \\ m_2 \ddot{x}_2 + k(x_2 - x_1) = f(t) \end{bmatrix}$$

$$A_2 - \left(-\frac{c}{m_1} A_1\right)$$

$$\bar{X}$$

$$x_1 - (x_1 - x_2)$$

$$t=0 \quad = x_2$$

$$\lambda(A) = 0$$

$$\left.\begin{matrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{matrix}\right\} \text{State}$$

$$\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{c}{m_1} & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} f$$

$x_1, \dot{x}_2, \dot{x}_1$ $\qquad (x_2 - x_1)$

$\dot{x}_1 = \dot{x}_1$

$\dot{x}_2 =$

$\ddot{x}_1 = -\frac{c}{m_1}\dot{x}_1 - \frac{k}{m_2}x_1 + \frac{k}{m_1}x_2$

# LINEAR SYSTEMS

$\dot{x} = Ax$ → state vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ $x \in \mathbb{R}^n$

$\bar{A}\bar{z} = \underset{\text{number}}{\lambda}\, \underset{\text{vector}}{\bar{z}}$

$\lambda \rightarrow$ eigen value

$z \rightarrow$ eigen vector

$(A - \lambda I) z = 0$ ✓ characteristic equation

$|A - \lambda I| = 0$ → Relevant set of eigen values.

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$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Find eigen values.

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix}$$ solve for $\lambda$

$$(2-\lambda)^2 - 3 = 0$$

$$\lambda = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$$\lambda_1 \bar{z}_1 = \bar{\bar{A}} \bar{z}_1$$

$$\lambda_2 \bar{z}_2 = \bar{\bar{A}} \bar{z}_2$$

$$\underbrace{\bar{\bar{T}} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{\text{columns are eigen vectors}}$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$AT = TD$$

$D$ → Diagonal matrix with eigen values as diagonals.

$$\boxed{T^{-1} A T = D}$$

$$\begin{bmatrix} \vdots & & \\ z_1 & \cdots & \\ \vdots & & \end{bmatrix} \begin{bmatrix} z_1 \\ \\ \end{bmatrix}$$

$$\underline{x} = \underline{Tz}$$

$$\underline{\dot{x}} = Ax$$

$$\dot{x} = T^{-1} T \dot{z} = T^{-1} A T z$$

$$\dot{z} = \underline{T^{-1} A T} z$$

$$\boxed{\dot{z} = Dz}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\dot{z}_1 = \lambda_1 z_1$$

$$\dot{z}_2 = \lambda_2 z_2$$

$$\vdots$$

$$\dot{z}_n = \lambda_n z_n$$

$\Rightarrow$ Dynamics along the eiven vectors are decoupled.

$$z_1 = c_1 e^{\lambda_1 t}$$

$$z_2$$

$$\vdots$$

$$z_n$$

Generalized $\text{sol}^n$ of sys. of Linear Diff. Eqn. $\dot{x} = Ax$

$$x(t) = e^{At}\, x(0)$$

$x(0) \hookrightarrow$ Initial Value.

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots\cdots$$

$$AT = TD$$

$$A = TDT^{-1}$$

$$e^{TDT^{-1}t} = TT^{-1} + TDT^{-1}t + \frac{(TD\overbrace{T^{-1}T}^{I}DT^{-1})t^2}{2!} + TD^3T^{-1}\frac{t^3}{3!} + \cdots$$

$$TDT^{-1}TDT^{-1} = TD^2T^{-1}$$

$$A^n = TD^nT^{-1}$$

$$X = T\underbrace{\left[ I + Dt + D^2\frac{t^2}{2!} + D^3\frac{t^3}{3!} + \cdots \right]}T^{-1}$$

$$X = T\, e^{Dt}\, T^{-1}$$

mapped to eigenvectors.

Desired state vector

$$\begin{bmatrix} e^{\lambda_1 t}c_1 & & & \\ & c_2 e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & c_n e^{\lambda_n t} \end{bmatrix}$$

Transfer Fcⁿ

$m\ddot{x} + c\dot{x} + kx = u$

$$\frac{X(s)}{U(s)} = \frac{1}{\boxed{ms^2 + cs + k}}$$

↳ Roots of this eqn. → poles of the sys.

State space

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & \left(-\frac{c}{m} - \lambda\right) \end{vmatrix}$$

$$= \lambda\left(\frac{c}{m} + \lambda\right) + \frac{k}{m}$$

$$= \underline{\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}}$$

$\lambda =$ eigen values

Pole - zero cancellation → Might miss out some dynamics.

Some information is lost about INTERNAL STATES (values you could measure) in a T.F.

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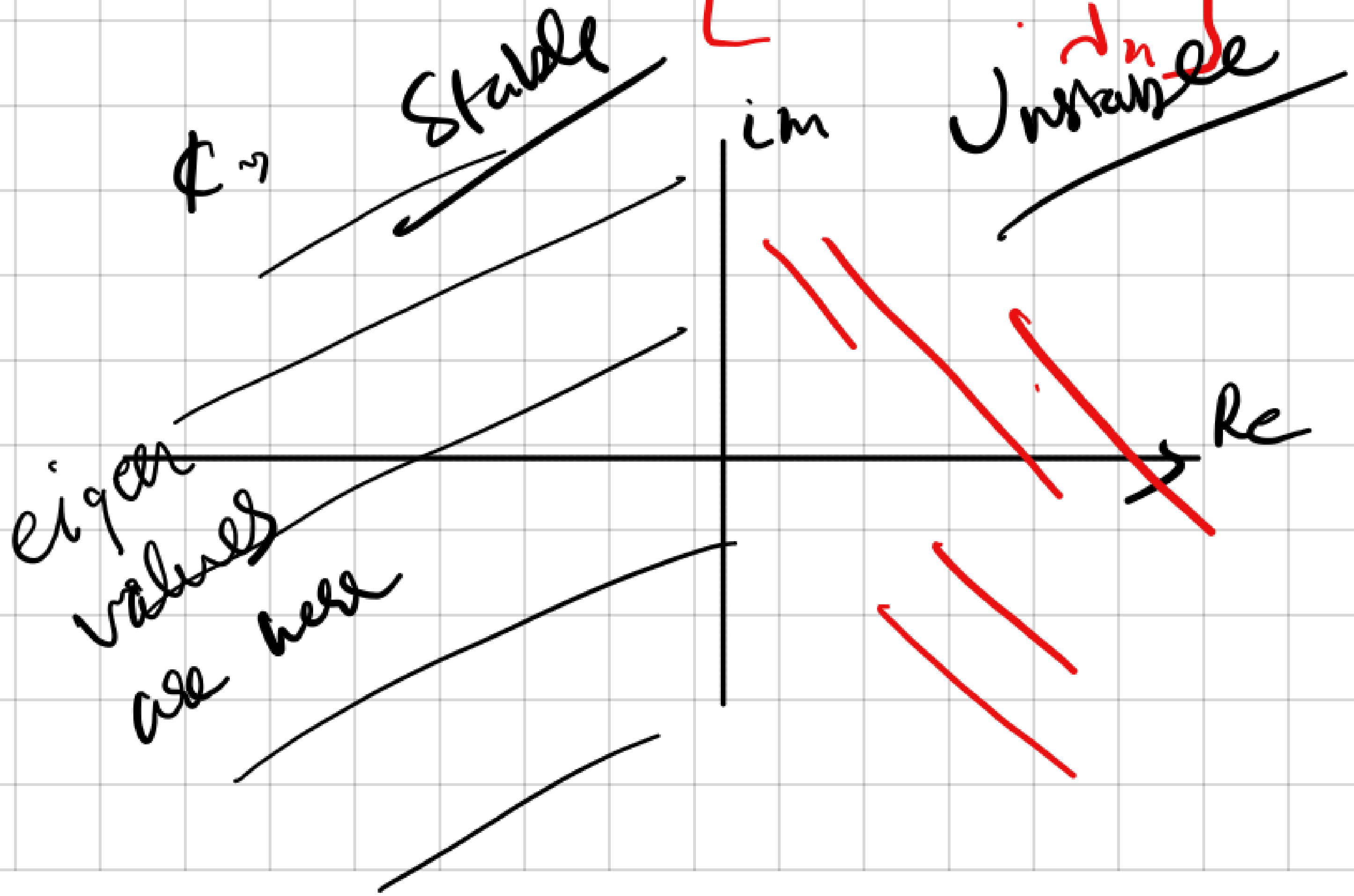
$$\boxed{\dot{x} = Ax} + Bu \rightarrow \text{Actuator}$$

# STABILITY

$\dot{x} = Ax \qquad x \in \mathbb{R}^n$

$AT = TD \qquad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$




$$x = e^{At} x(0)$$

$$= T e^{Dt} T^{-1} x(0)$$

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

if $Re(\lambda) < 0$, system is asymptotically stable.

# Discrete time domain

$$x_{k+1} = \tilde{A} x_k \longrightarrow \neq \quad \dot{x} = Ax$$

$x_k = x(k\Delta t)$ where $\Delta t$ is sampling time
& $k$ is the sampled instance

Start at $x_0$

$$x_1 = \tilde{A} x_0$$

$$x_2 = \tilde{A} \cdot x_1 = \tilde{A}^2 x_0$$

$$x_3 = \tilde{A}^3 x_0$$

$$\vdots$$

$$x_m = \tilde{A}^m x_0$$

Let eigen values of $\tilde{A}$
$\lambda_1, \lambda_2 \ldots \lambda_n$

$\tilde{A} = \tilde{T} \tilde{D} \tilde{T}^{-1}$ (Recall)

$$x_m = \tilde{T} \tilde{D}^m \tilde{T}^{-1} x_0$$

$$\tilde{D}^m = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}^m$$

$$= \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_n^m \end{bmatrix}$$

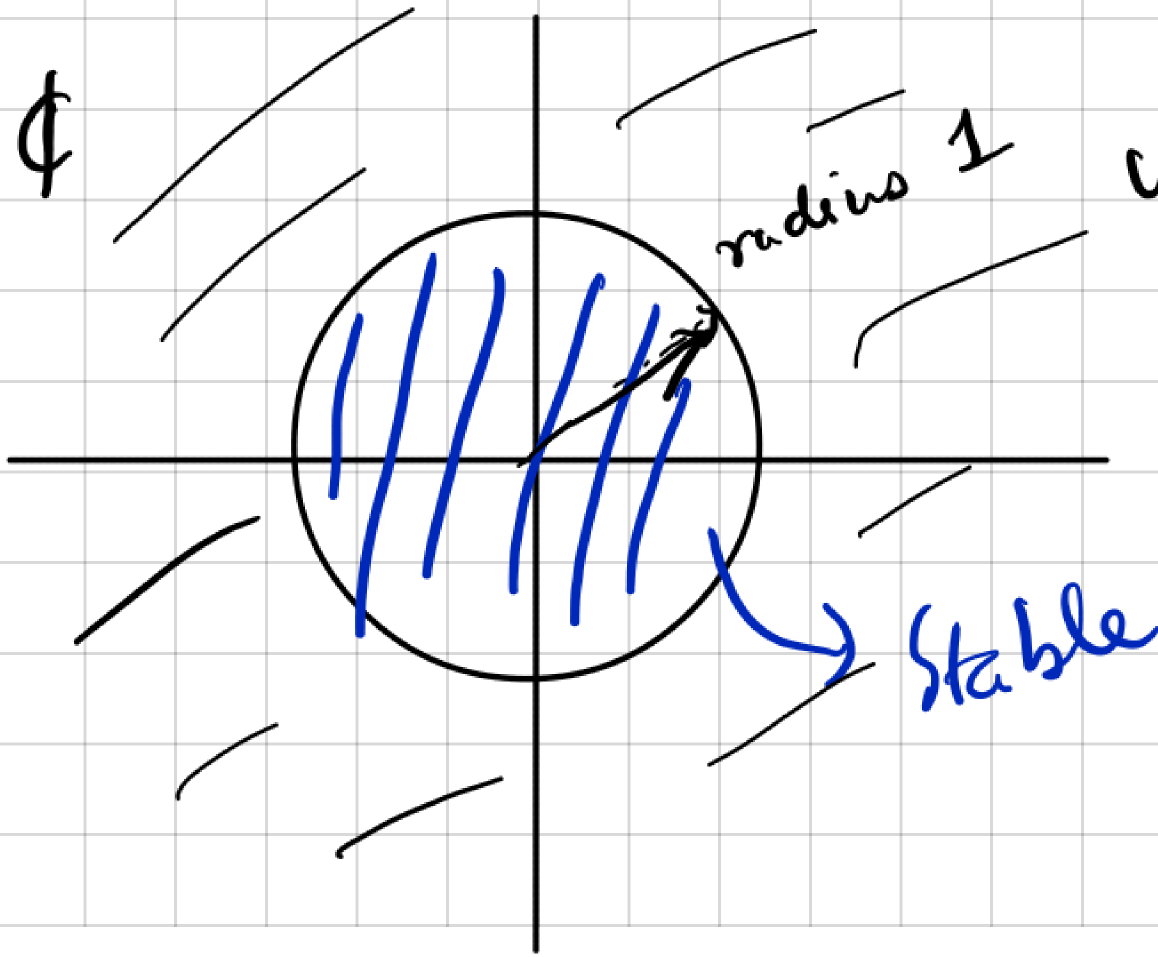
if you track any $\lambda_i$ through time,

then at $t = t_m$

$$x_m = \tilde{A}^m x_0 = \tilde{T}\tilde{D}^m\tilde{T}^{-1} x_0$$

$$\lambda_i^m$$

If $\lambda_i > 1$, system response will blow up.




$$\lambda_i = R e^{i\theta}$$

$$\lambda_i^m = R^m e^{im\theta}$$

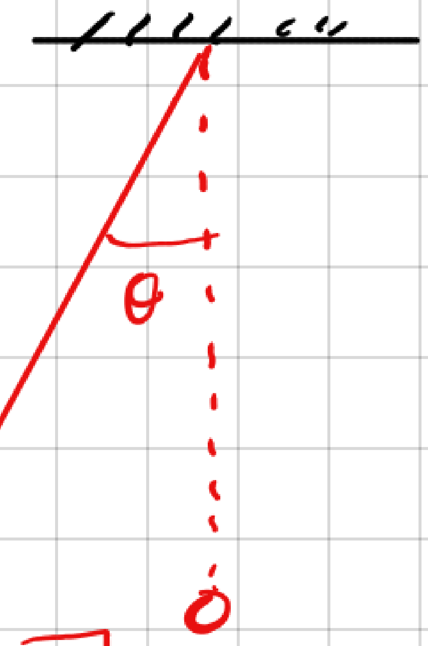
in cont. time domain

$$e^{\sigma t}[\cos(\omega t + \phi)]$$

# Linearization

$$\dot{x} = f(x) \Rightarrow \dot{x} = Ax \quad x \in \mathbb{R}^n$$

$$\dot{x}_1 = f_1(x_1)$$
$$\dot{x}_2 = f_2(x_2)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_n)$$




fixed point $\bar{x}$ $\quad f(\bar{x}) = 0$

Evaluate Jacobian of the sys. at $\bar{x}$ $\rightarrow$

$$J = \frac{Df}{Dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\bar{x}}$$

Please study this from any engineering text book

Example:

$$\dot{x}_1 = f_1(x_1, x_2) = x_1^2 + x_2^3$$

$$\dot{x}_2 = f_2(x_1, x_2) = x_1^2 x_2$$

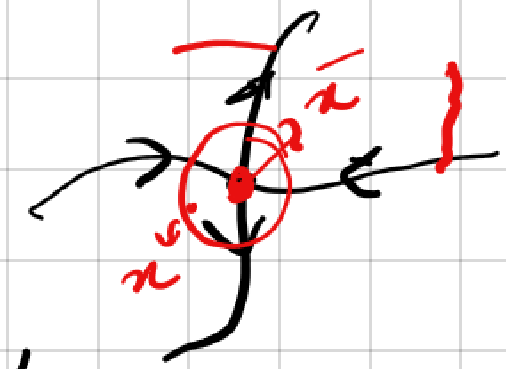
find fixed points of $f_1$ & $f_2$

$$J = \begin{bmatrix} 2x_1 & 3x_2^2 \\ 2x_1 x_2 & x_1^2 \end{bmatrix}_{\bar{x}}$$

$$\bar{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$J = \begin{bmatrix} 2a & 3b^2 \\ 2ab & a^2 \end{bmatrix}$$

$\dot{x} = f(x) \rightarrow$ about $\bar{x}$ Taylor series.



$$= f(\bar{x}) + \overbrace{\frac{Df}{Dx}}^{J}\Bigg|_{\bar{x}} (x-\bar{x})$$

$$+ \frac{D^2 f}{Dx^2}\Bigg|_{\bar{x}} (x-\bar{x})^2 \ldots\ldots$$

$$\boxed{(x-\bar{x}) \rightarrow 0}$$

higher order terms can be ignored.

$$\dot{x} = \overset{\nearrow 0}{f(\bar{x})} + J(x-\bar{x})$$

$$\boxed{\Delta\dot{x} = J\,\Delta x + \epsilon} \rightarrow \text{linear system.}$$