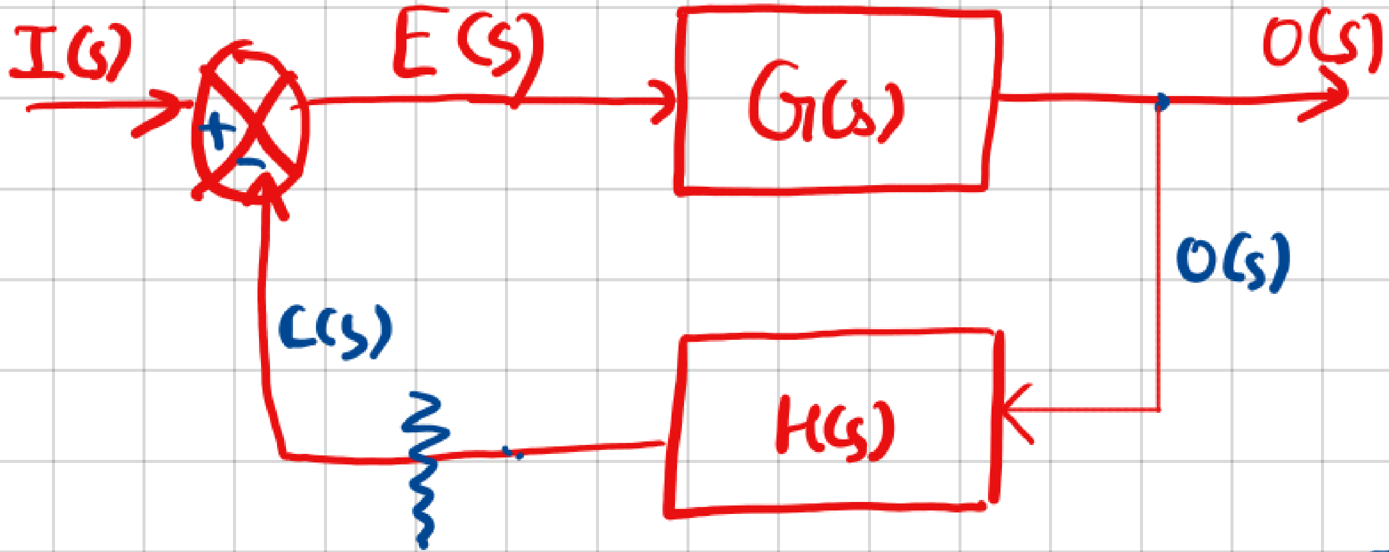
Root Locus Techniques

Learning Outcomes

1. Define the behavior of closed loop poles as proportional gain increases while only knowing the open loop poles and zeros
2. Draw an approximate sketch of the locus of the closed loop poles (Root Locus), and find critical points of interest for controller design
3. Understand the use cases for RL
4. Learn how to plot the RL using Matlab



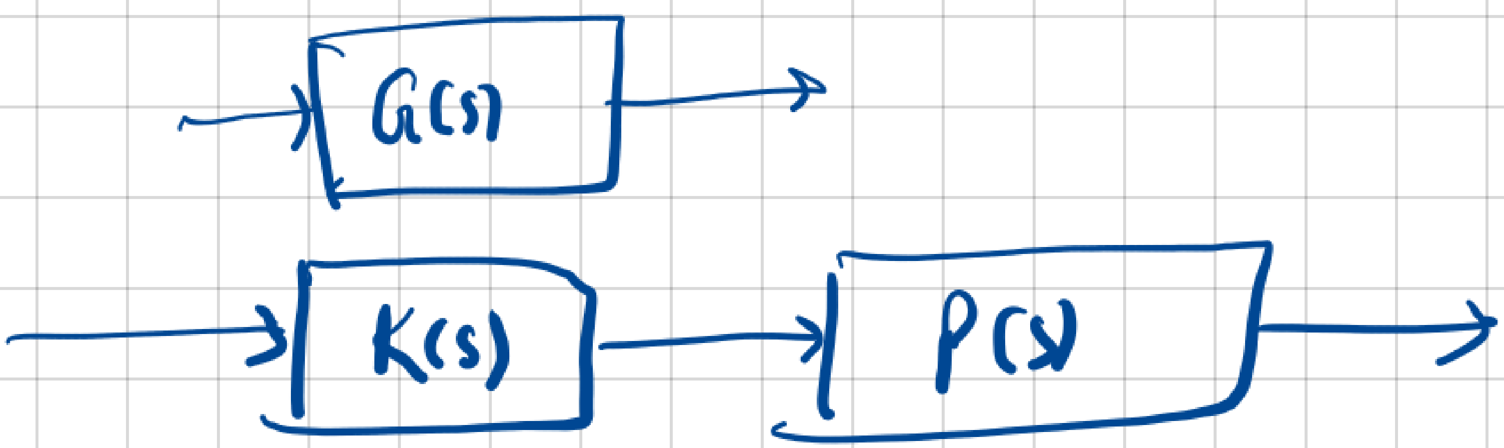
$$E(s)\cdot G(s) = O(s)$$

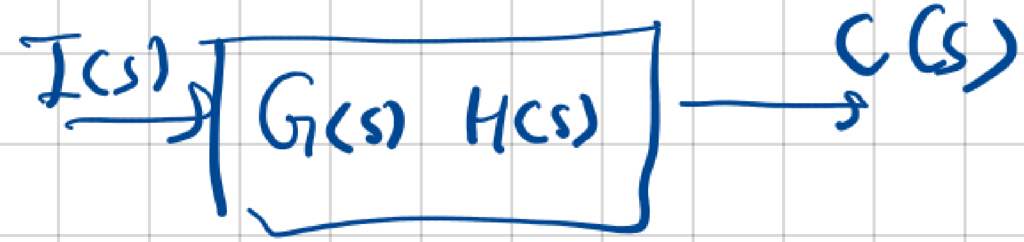
$$E(s) = I(s) - C(s)$$
$$= I(s) - O(s)\cdot H(s)$$

$$\left[I(s) - O(s)\,H(s)\right]\cdot G(s) = O(s)$$

$$I(s)\cdot G(s) - O(s)\,H(s)\,G(s) = O(s)$$

$$Q(s) = \frac{O(s)}{I(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

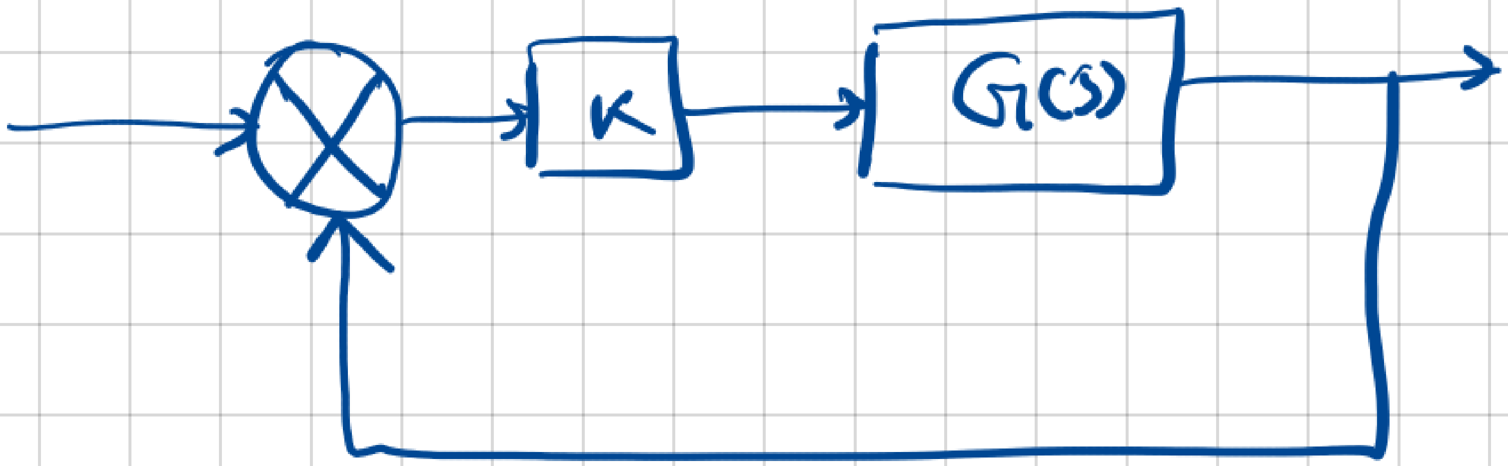




↳ open loop Tf.

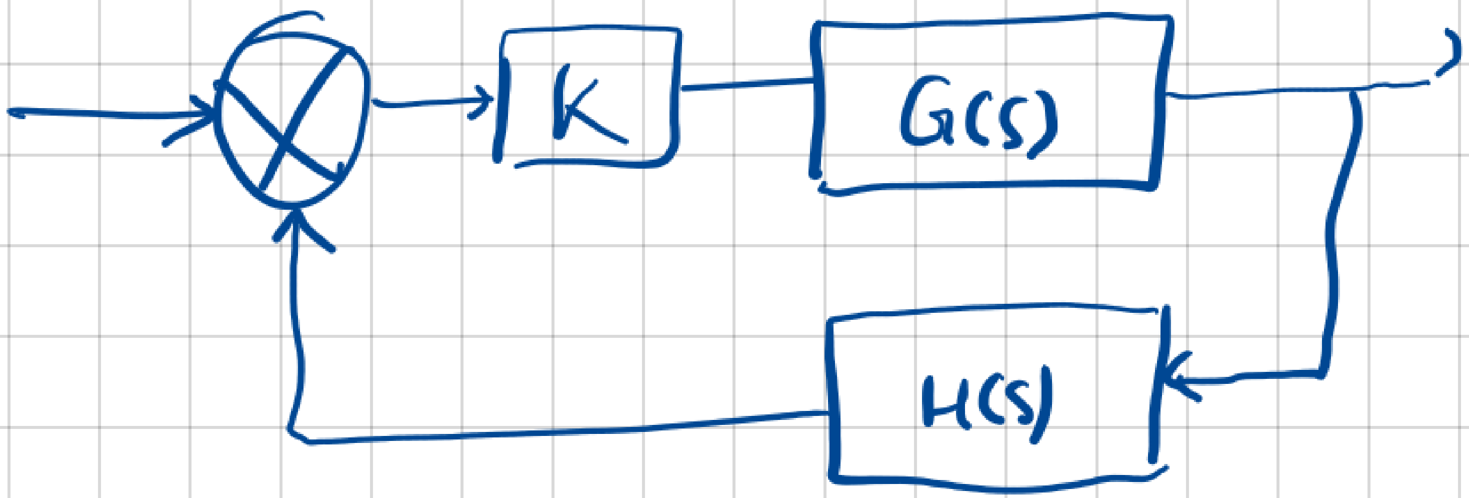
$$G(s) = \frac{(s+a)}{(s+b)(s+c)(s+q)(s+r)}$$

$$Q(s) = \frac{KG(s)}{1+KG(s)}$$



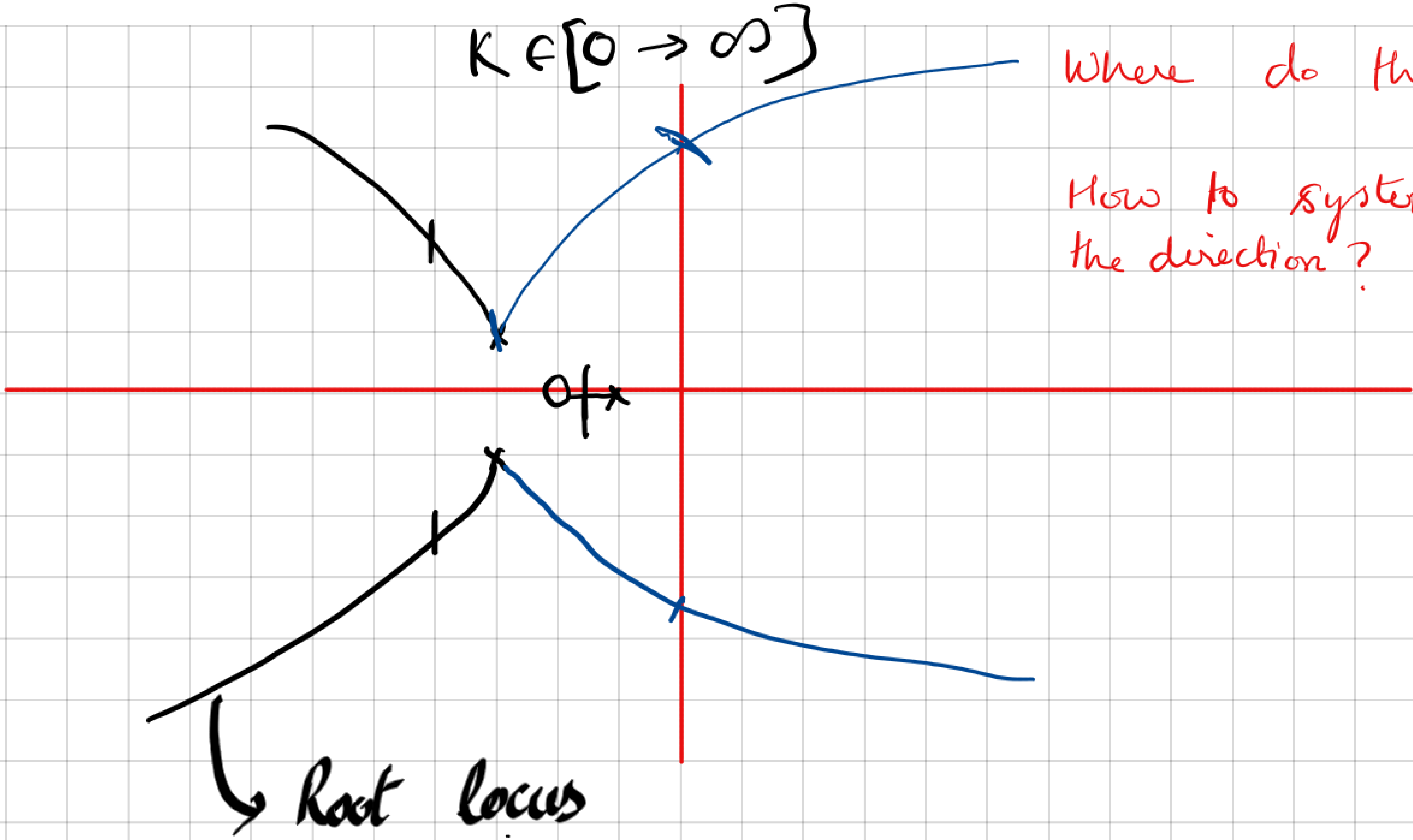
$$G(s) = \frac{N_G}{D_G}$$

$$H(s) = \frac{N_H}{D_H}$$



$$Q(s) = \frac{N_G / D_G}{1 + K\frac{N_G}{D_G}\frac{N_H}{D_H}}$$

$$\boxed{Q(s) = \frac{N_G \cdot D_H}{D_H D_G + K N_G N_H}}$$

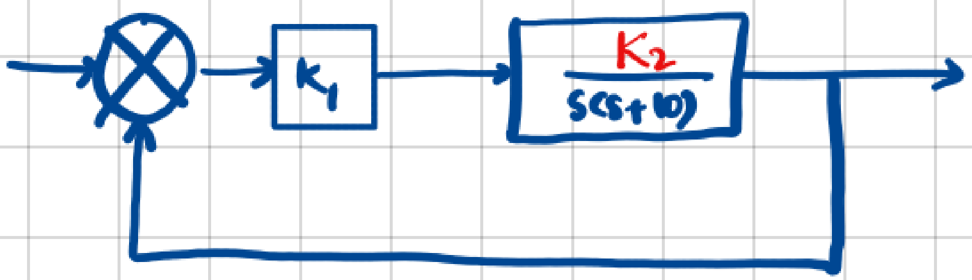


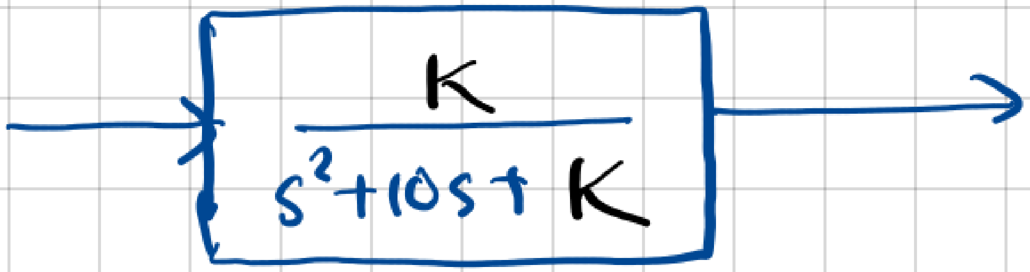
Where do these branches go?

How to systematically identify the direction?

Path taken by poles of the closed loop sys as $K \uparrow$

Defining a R.L.





$K_1 K_2 = K$

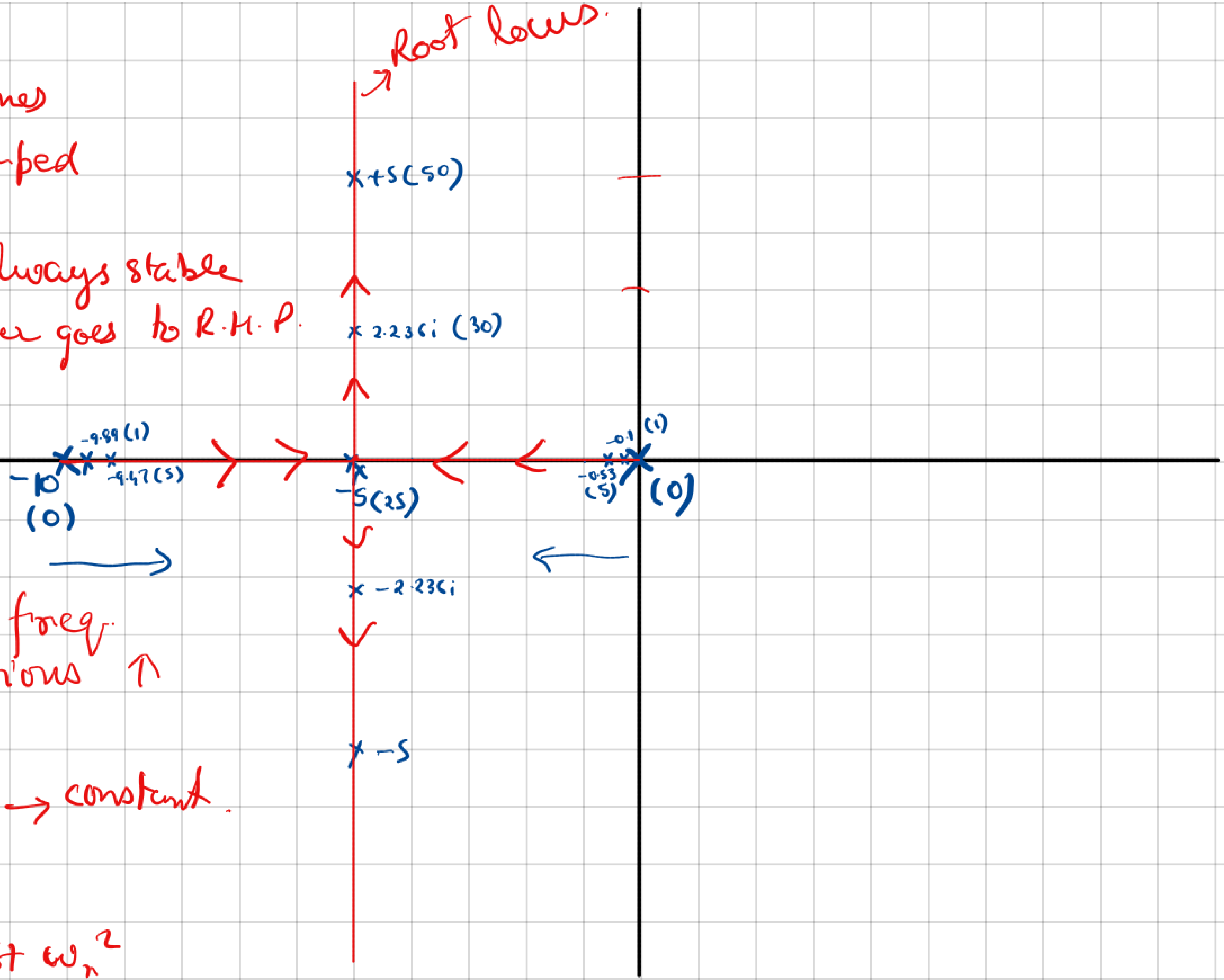
verify if at $K=5$, -9.4721 is a pole of the closed loop system

$$\frac{5}{(-9.4721)(-9.4721+10)} = -0.9999\ldots \simeq -1$$

K	Pole 1	Pole 2
0	−10	0
1	−9.899	−0.1010
5	−9.4721	−0.5279
25	−5	−5
36	−5 + 2.361i	−5 − 2.361i
35	−5 + 3.1623i	−5 − 3.1623i
50	−5 + 5i	−5 − 5i

$K > 25$ → Sys becomes underdamped

→ Sys is always stable as R.L. never goes to R.H.P.



→ Damped freq. of oscillations ↑

98% T_s → $\dfrac{4}{\xi\omega_n}$ → constant.

$s^2 + 2\xi\omega_n s + \omega_n^2$

Vector Representation of Complex Nos.

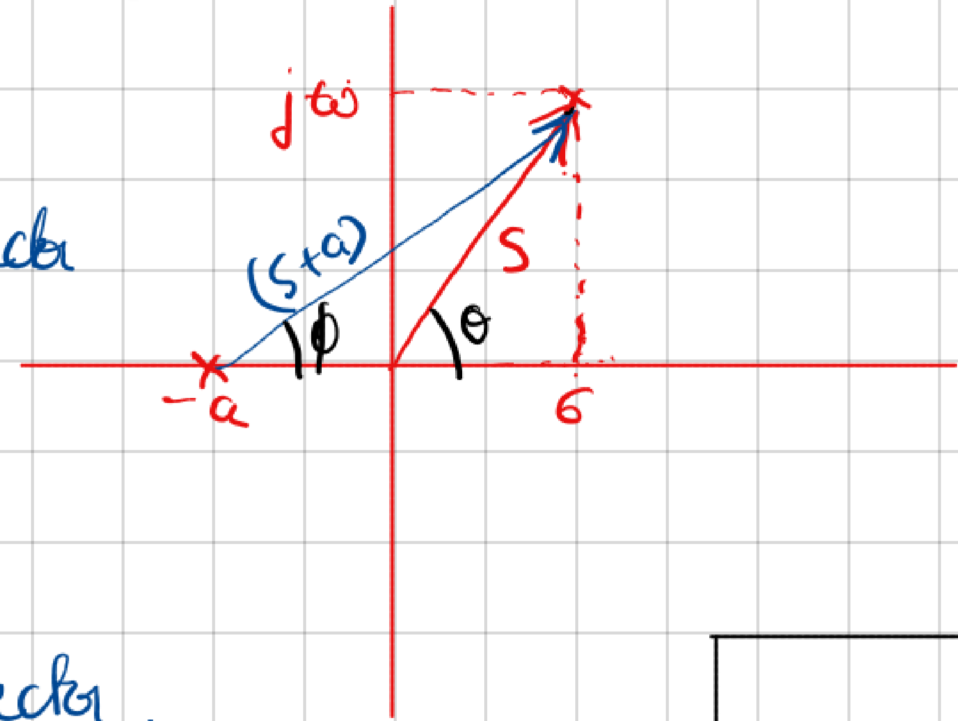
$s = \sigma + j\omega$

can be represented as a vector

$(s+a)$

vector representation is as shown

$s = -a$ is the tail of the vector.



$(s+a)$

$|s+a| \angle (s+a)$

Assume a function

$$F(s) = \frac{\prod_{i=1}^{m} (s+z_i)}{\prod_{j=1}^{n} (s+p_j)}$$

\# poles = n

\# zeros = m

magnitude of $F(s)$ at a given value of s.

$$\boxed{M = \frac{\prod_{i=1}^{m} |s+z_i|}{\prod_{j=1}^{n} |s+p_j|}}$$

$$\angle F(s) = \sum_{1}^{m} \text{Zero } \angle\text{'s} - \sum_{1}^{n} \text{Pole angles}$$

$$= \sum_{1}^{m} \angle (s+z_i) - \sum_{1}^{n} \angle (s+p_j)$$

$$F(s) = \frac{(s+1)}{s(s+2)} \qquad @\, s = -3+4j$$

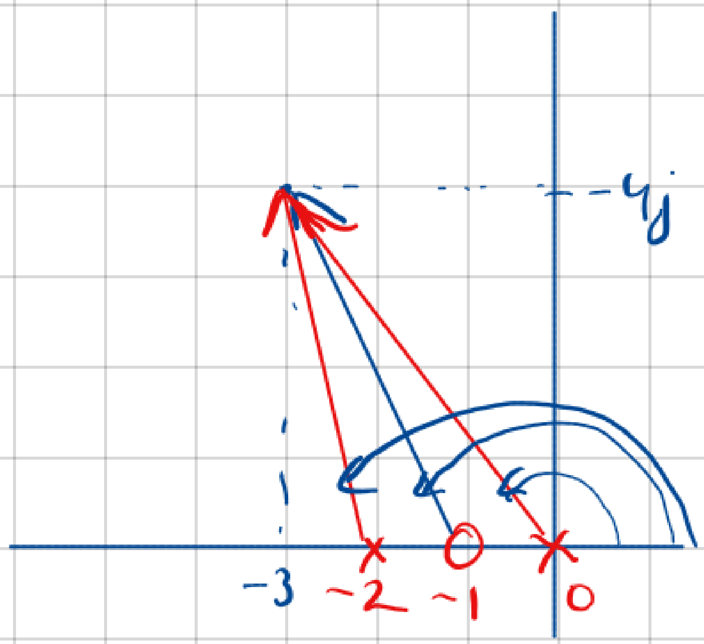
$(s+1) \rightarrow \sqrt{20}\angle 116.56^\circ$

$s \rightarrow 5\angle 126.9^\circ$

$(s+2) \rightarrow \sqrt{17}\angle 104$

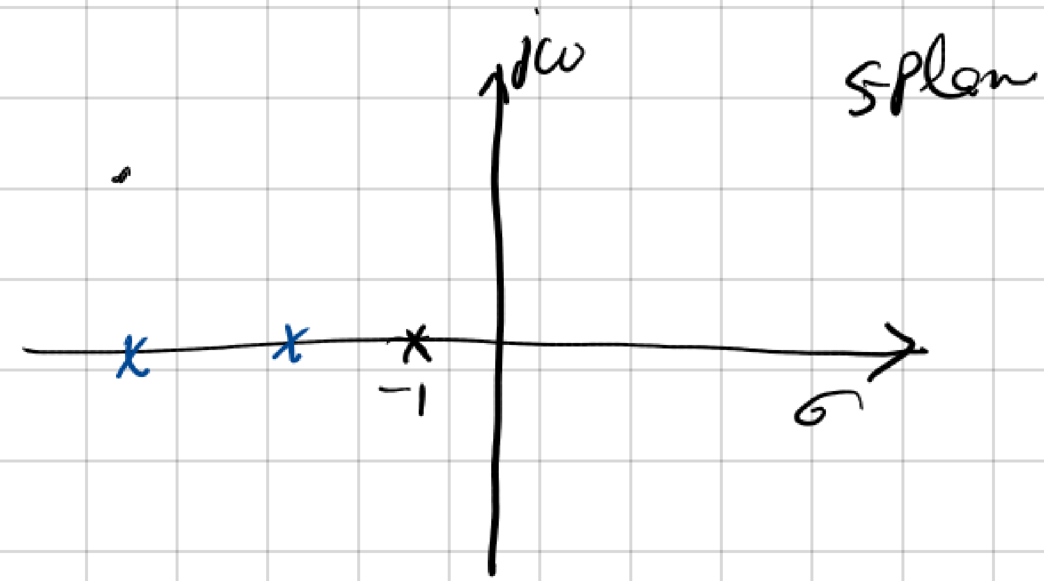
$$F(s) = \frac{\sqrt{20}}{5\sqrt{17}} \angle (116.56 - 126.9 - 104)$$

$$= 0.217 \angle -114.35^\circ$$



Properties of Root locus

$$F(s) = \frac{K\,G(s)}{1 + K\,G(s)\,H(s)}$$



For S to lie on R.L. $\left(1 + K\,G(s)\,H(s)\right)_S = 0$

$$K\,G(s)\,H(s) = -1$$

$|K\,G(s)\,H(s)| = 1$ & $\angle K\,H(s)\,G(s) = 180^\circ$ ⇒ $\boxed{\angle H(s)\,G(s) = (2p+1)\,180^\circ}$

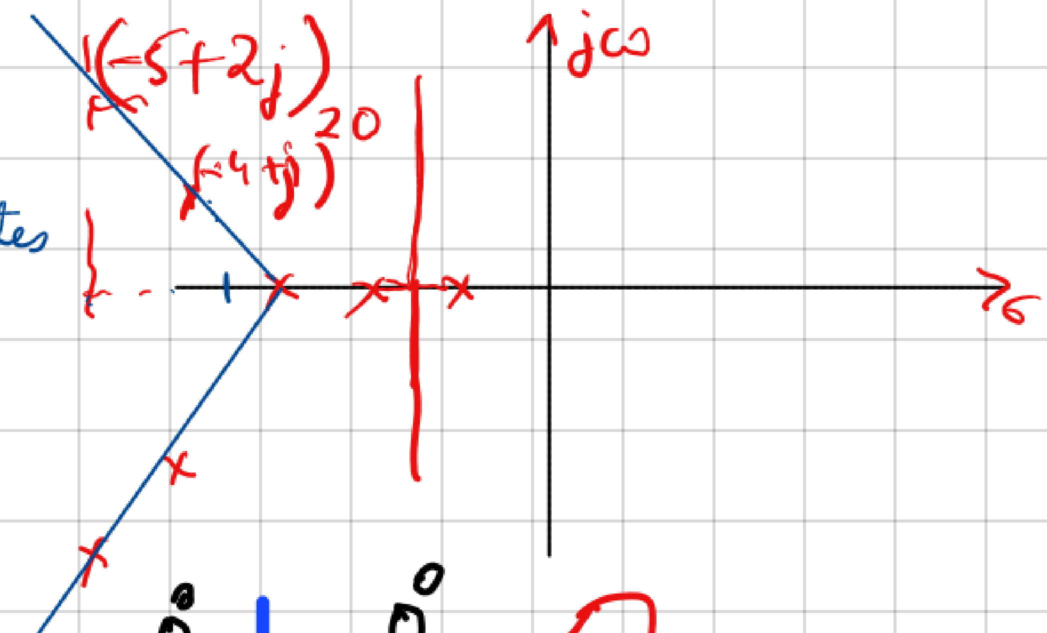
K is positive ⇒ $\boxed{K = \dfrac{1}{|G(s)\,H(s)|}}$

Pole causes an angle of 180° for $K\,G(s)\,H(s)$

Sketch a Root Locus:

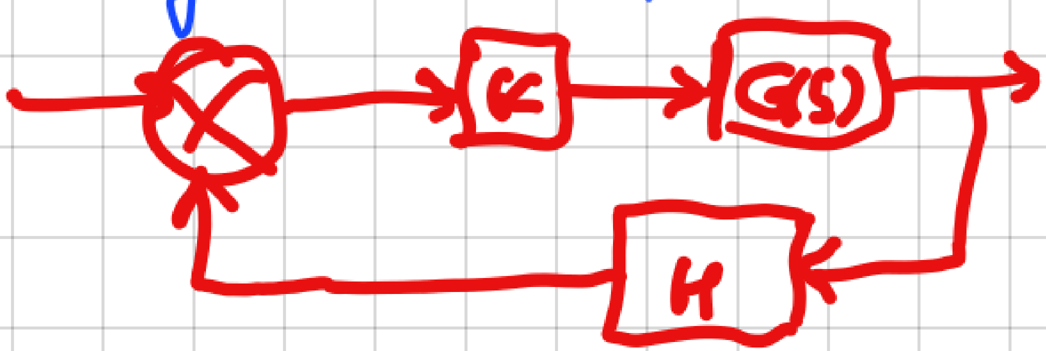
Rule 1 : No. of branches = No. of poles of the closed loop system.

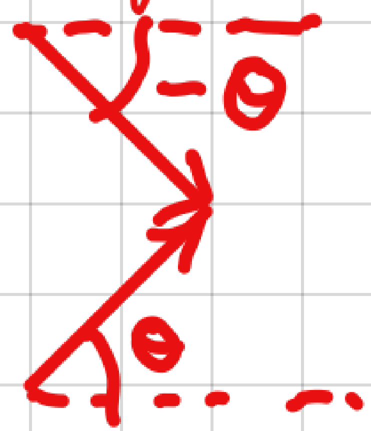
Rule 2 : Root Locus is symmetric about the real axis since complex roots can only exist as complex conjugates

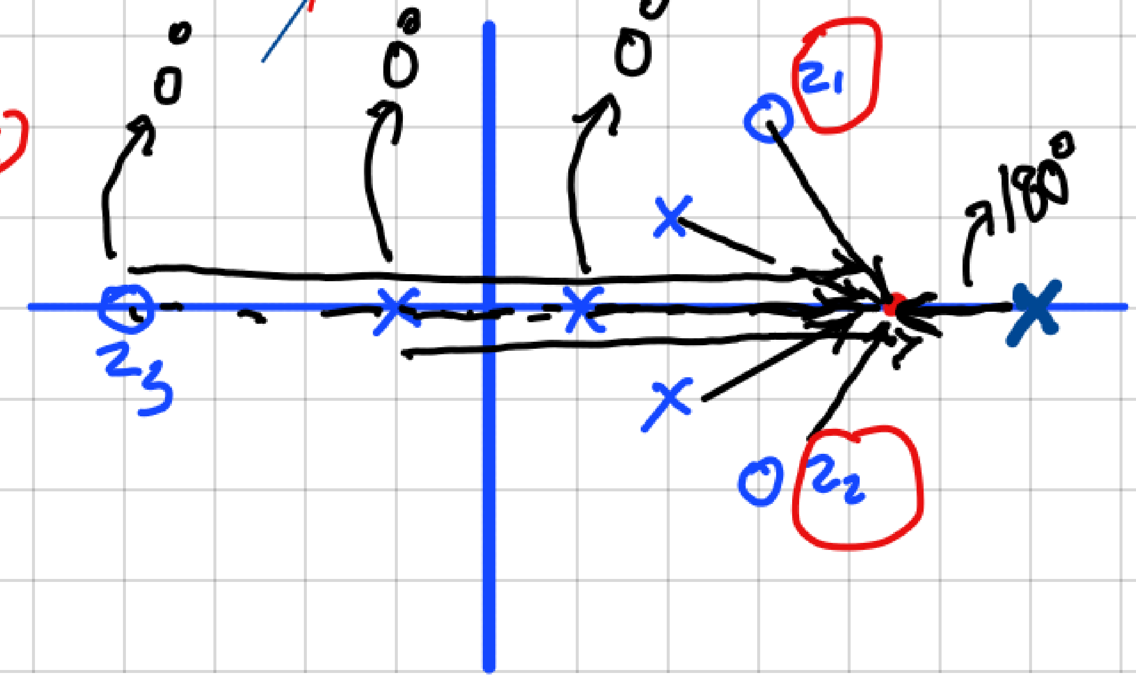


Rule 3: Segments of Real Axis

Say - Poles (x) zeros (o) → of $K\,G(s)\,H(s)$







Angle contribution of complex poles & zeros TO THE REAL AXIS segme

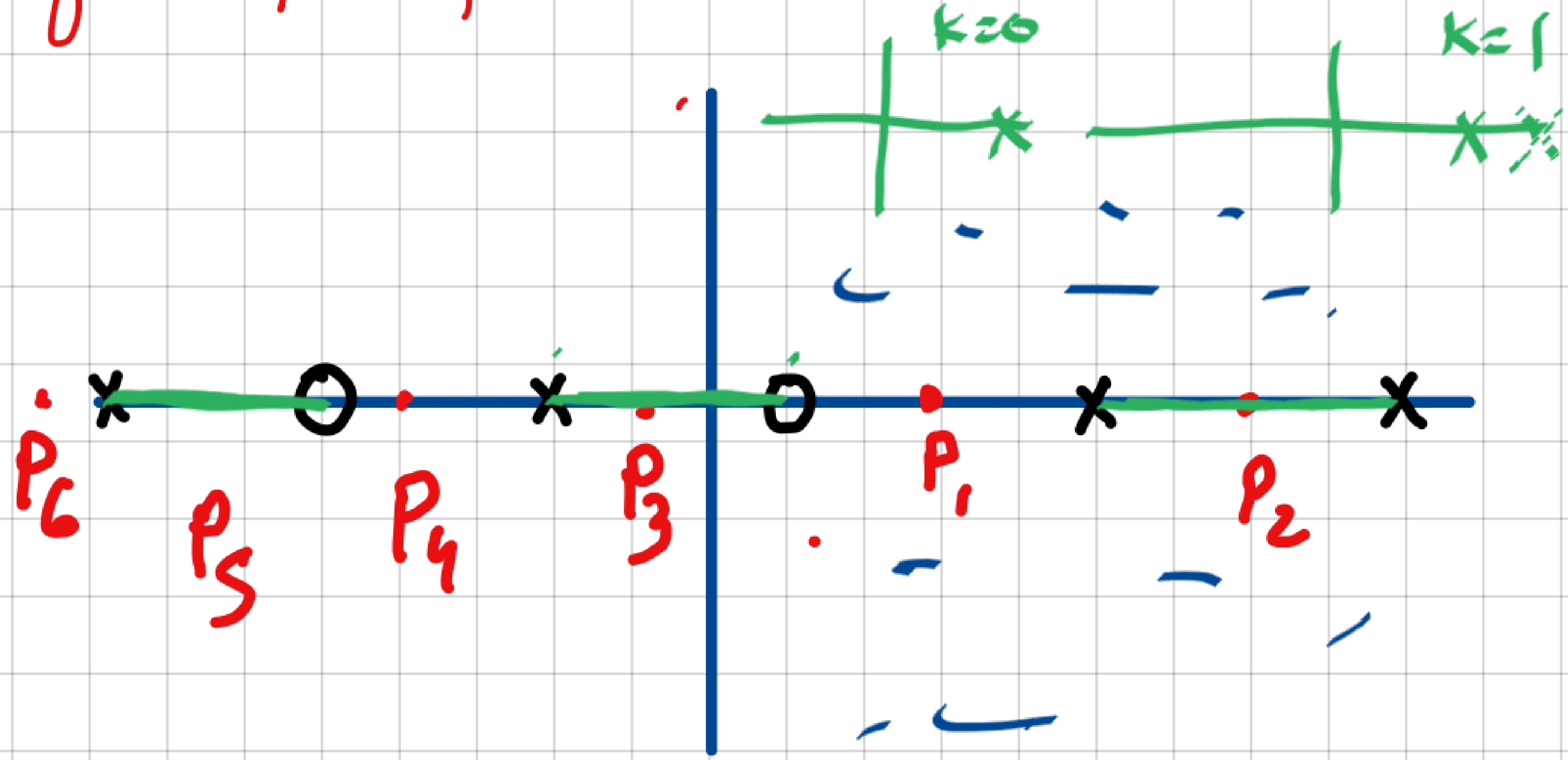
is = 0

Part of R.L.

P_2

P_3

P_5



All points on the LEFT of ODD # poles & zeros lie on the R.L real axis

Start & end of Branches of Root Locus

Extend the R.L. beyond the real axis

$$T(s) = \frac{K\,N_G D_H}{\underline{D_G D_H} + \underline{K N_G N_H}}$$

$K \to 0$
ε

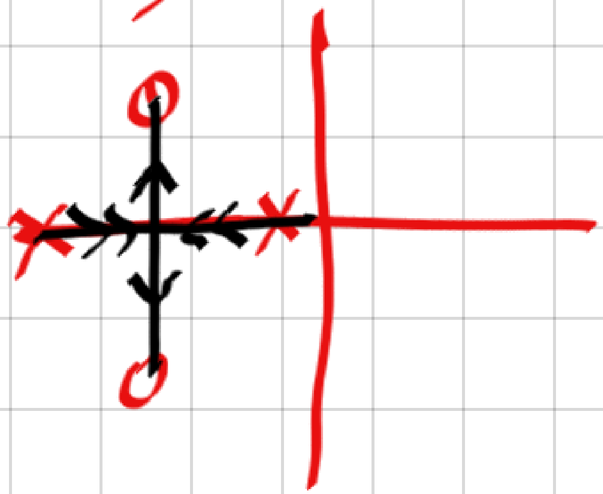
$D_T = D_G D_H + \underset{\to 0}{\varepsilon} \Rightarrow$ poles at $K=0$ = poles of open loop system

$K \to \infty$

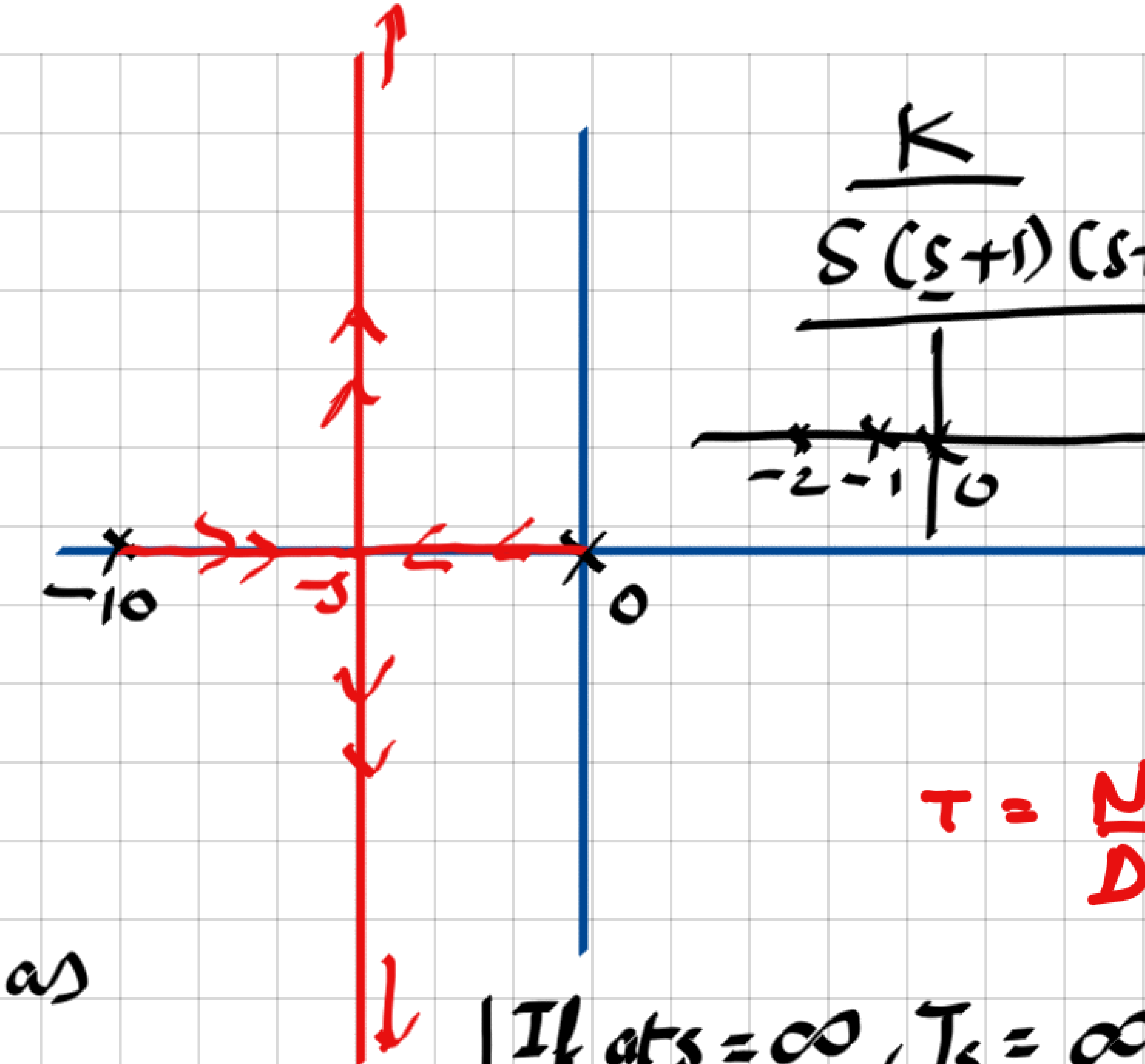
$D_T = \dfrac{K N_G D_H}{\varepsilon + K N_G N_H} \Rightarrow$ poles at $K = \infty$ = zeros of open loop system

Rule 5 Behavior at ∞

$$\frac{(s+a)(s+b)}{(s+c)(s+d)}$$



$$\frac{K}{s(s+1)(s+2)}$$



$$T = \frac{N \to \infty}{D \to 0}$$

There are as many zeros as poles when ∞ is considered as the poles & zeros of the system.

If at $s = \infty$, $T_s = \infty$ as $\boxed{K \to \infty}$ — ∞ is a POLE

If at $s = \infty$, $T_s \to 0$, $\boxed{K \to \infty}$ — $\infty \to$ Zero of the system

$$\boxed{\frac{K}{s(s+1)(s+2)}}$$

$s \to \infty \quad \approx \frac{K}{s\, s\, s}$

$\Rightarrow$ system has 3 zeros at ∞

$$\frac{K}{s(s+1)(s+2)}$$

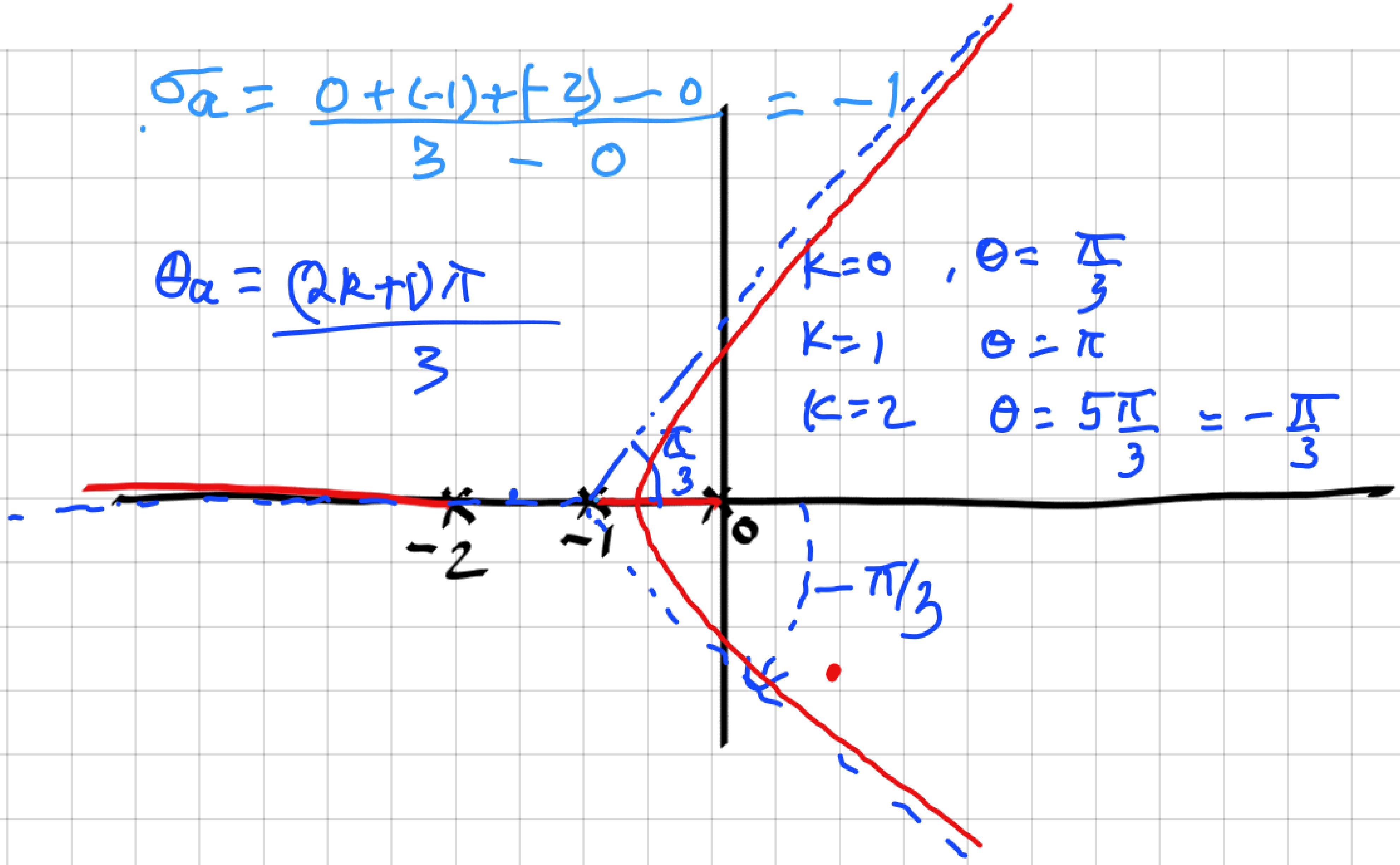
$$\sigma_a = \frac{0 + (-1) + (-2) - 0}{3 - 0} = -1$$

$$\theta_a = \frac{(2k+1)\pi}{3}$$

$k=0, \quad \theta = \frac{\pi}{3}$

$k=1 \quad \theta = \pi$

$k=2 \quad \theta = \frac{5\pi}{3} = -\frac{\pi}{3}$



Root locus approaches straight lines as asymptotes as the locus $\to \infty$.

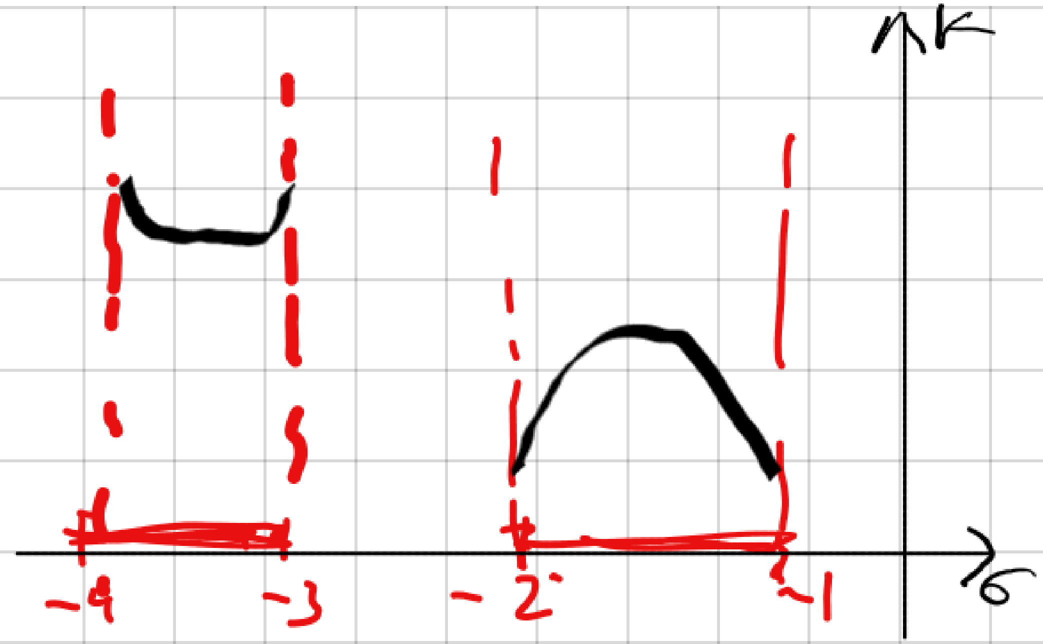
Root locus approaches straight lines as asymptotes as the locus $\rightarrow \infty$,

The intercept of these lines on Real Axis is

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\ \text{finite poles} - \#\ \text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\ \text{finite poles} - \#\ \text{finite zeros}}$$

$$K = -\frac{1}{|G(s)H(s)|}$$

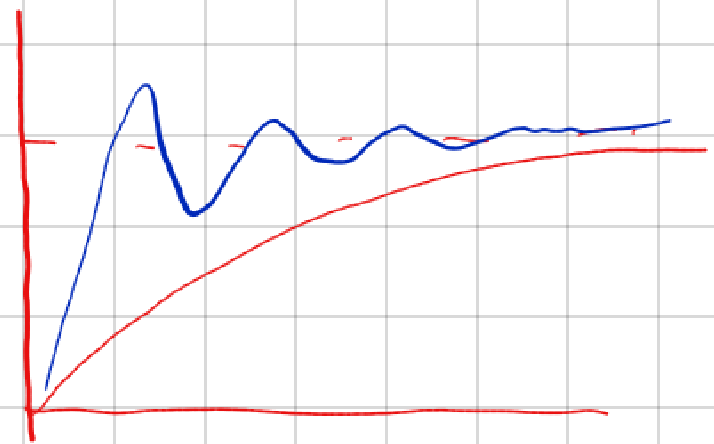


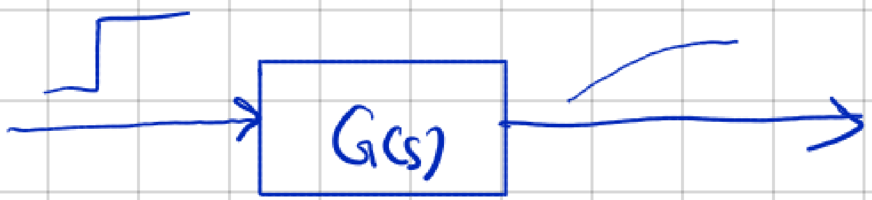
$$G(s)H(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)}$$

$s = \sigma$ i.e. s lies on the real axis.

$$G(\sigma)H(\sigma) = \frac{\sigma^2 + 7\sigma + 12}{\sigma^2 + 3\sigma + 2}$$

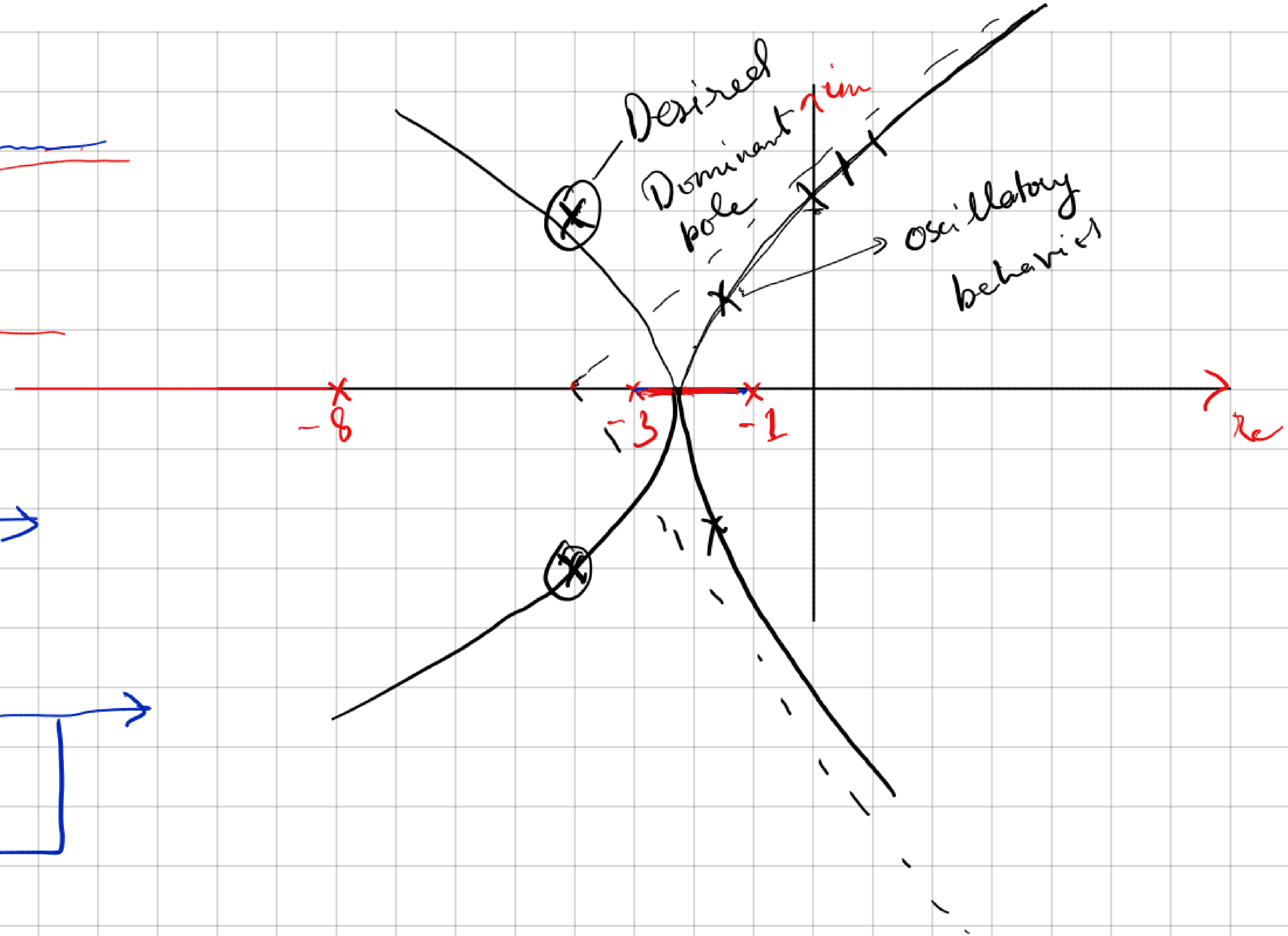
Differentiate the above w.r.t σ and set that $= 0$ to find Maxima & Minima corresponding to breakin & breakaway points respectively.



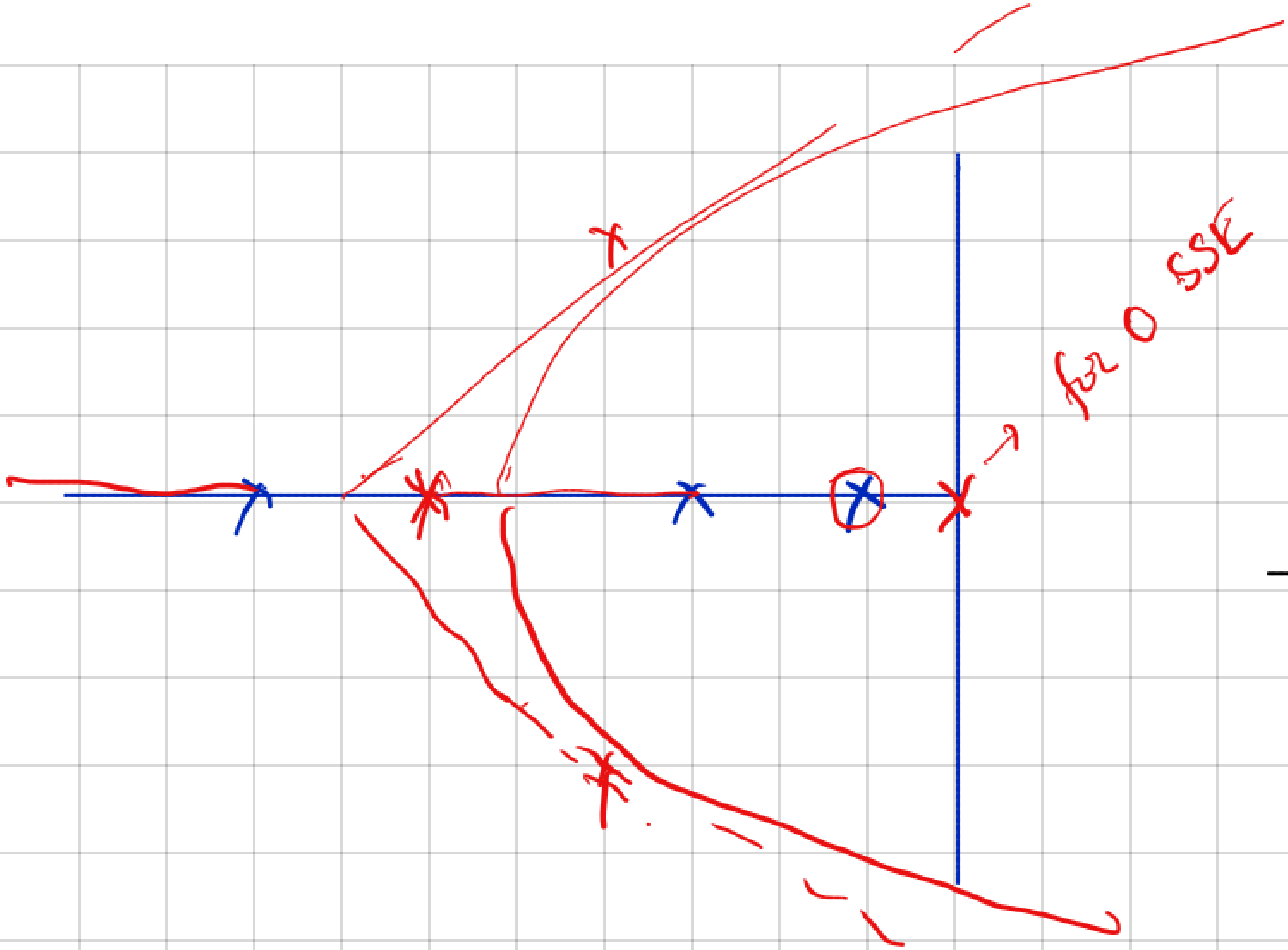


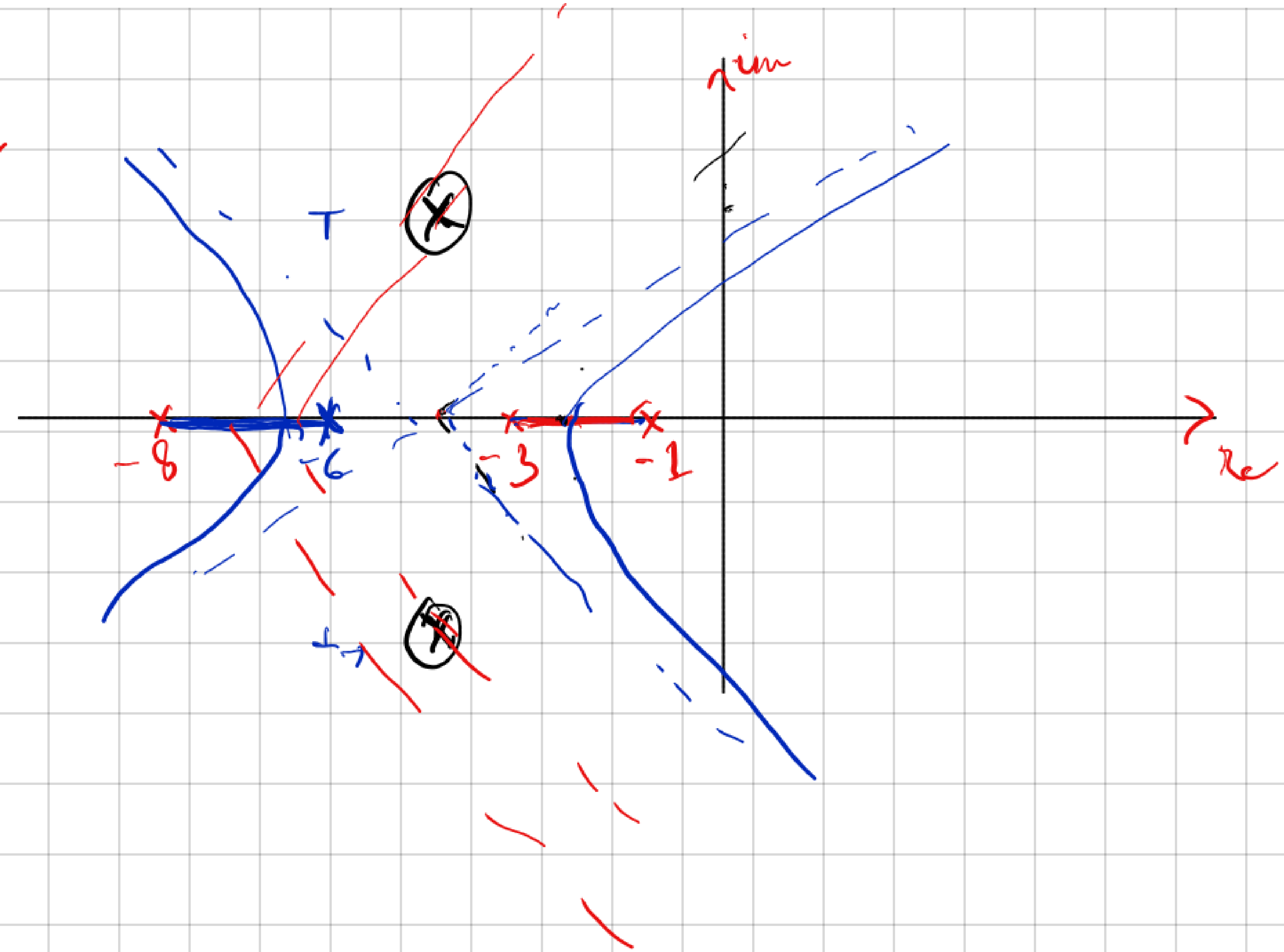
G(s)

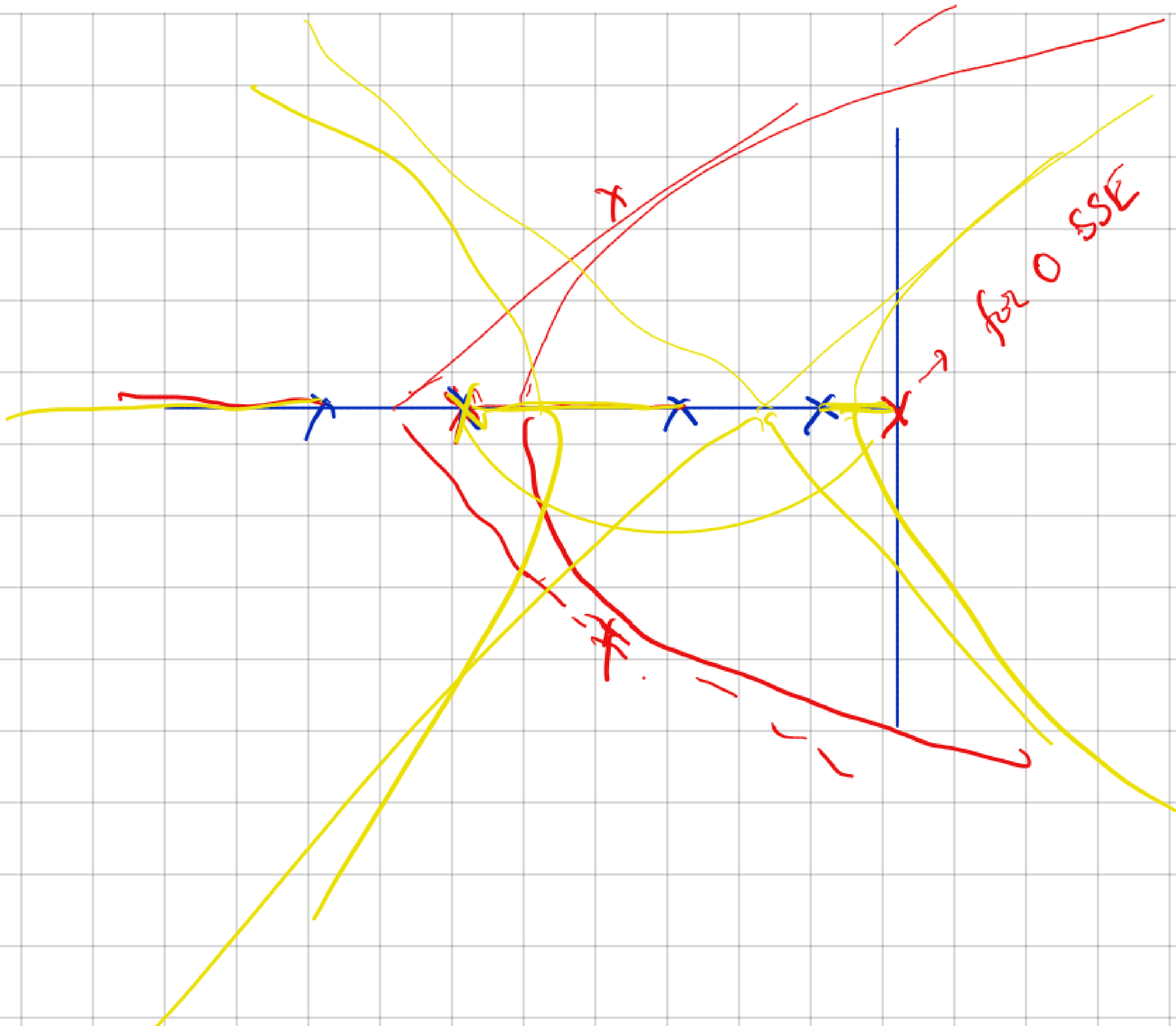
K
G(s)

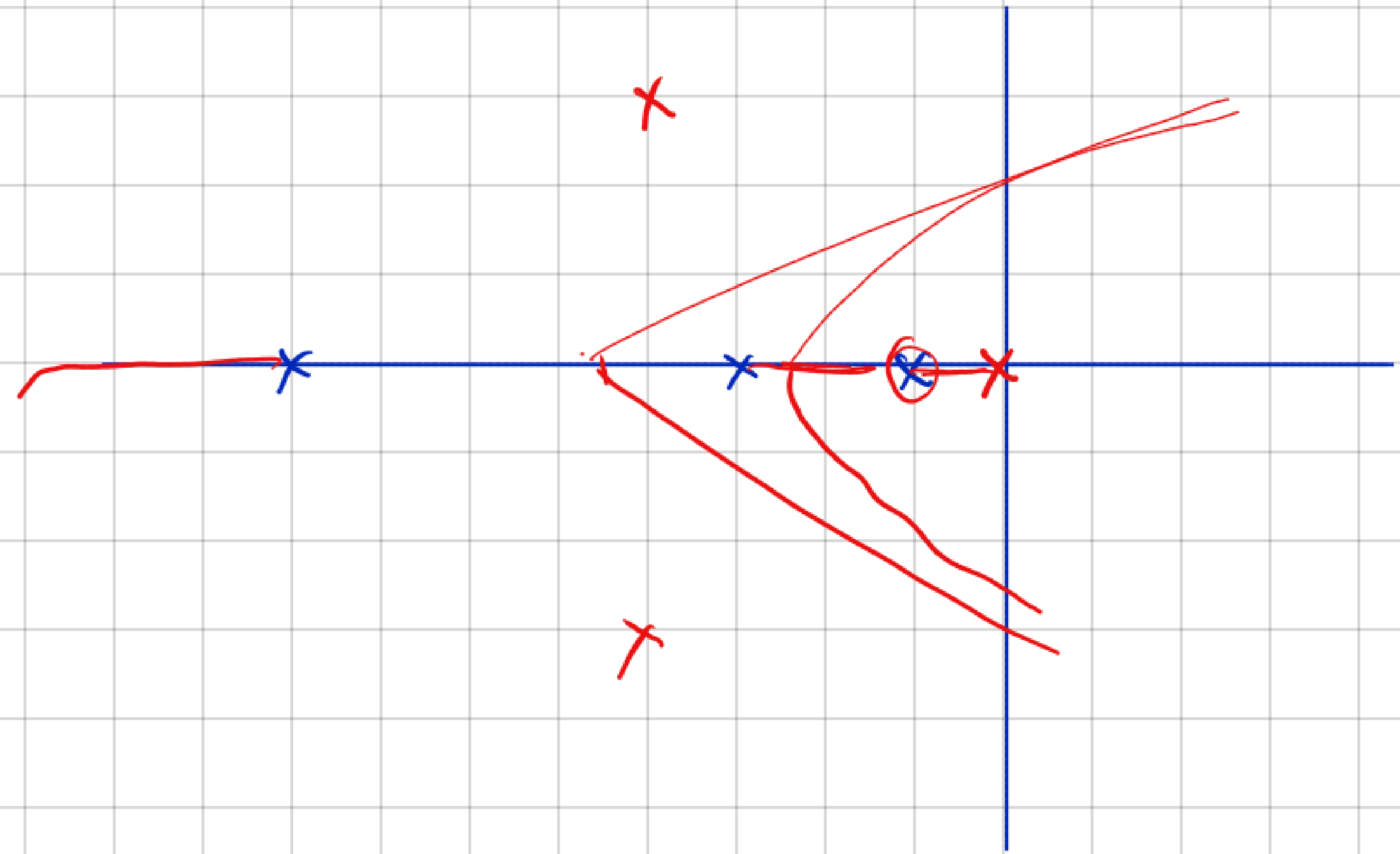


Desired Dominant pole
Im
oscillatory behavior
Re
-8
-3
-1


for 0 SSE


im
Re
T
-8
-6
-3
-1


for 0 SSE

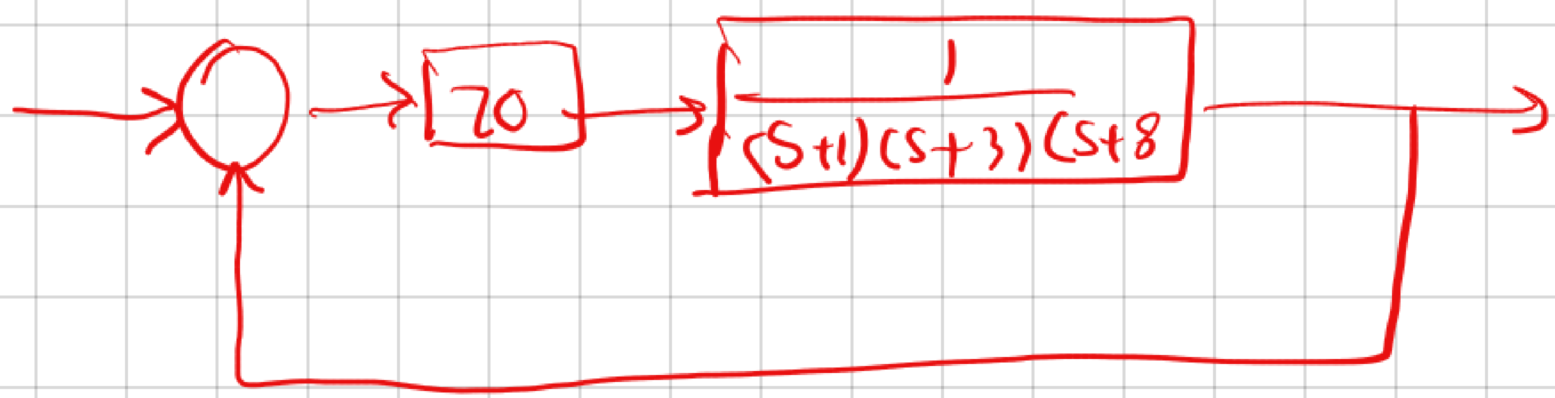


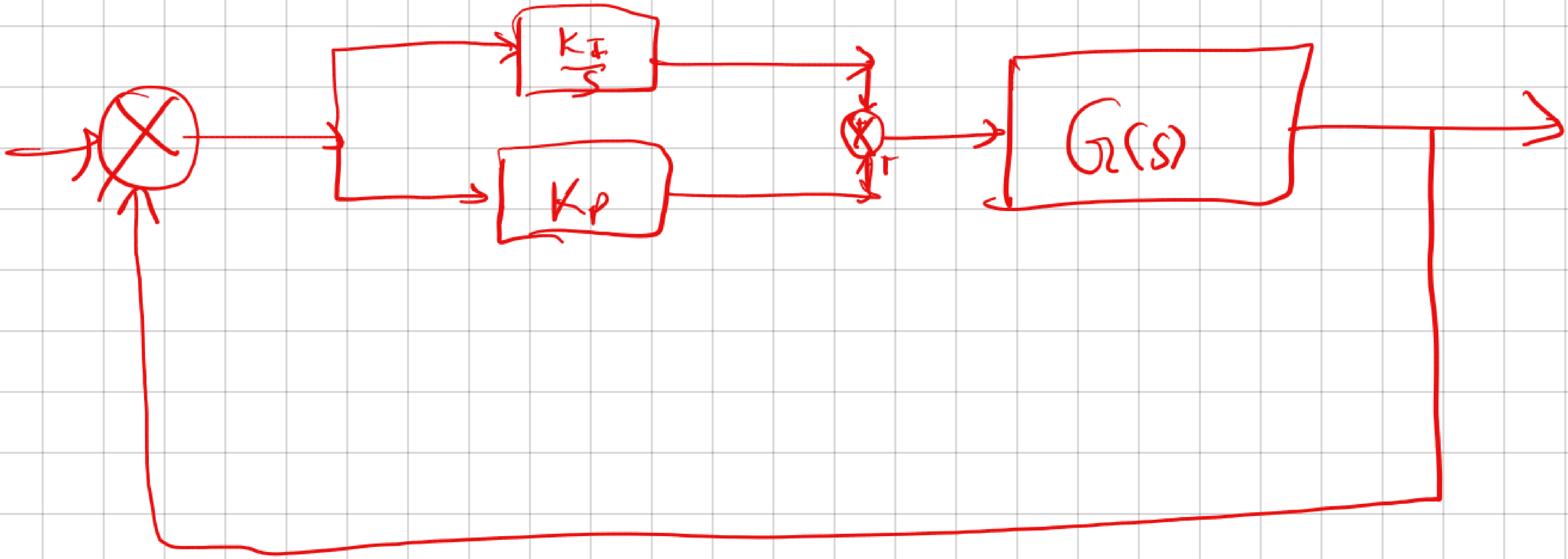
$$G(s) = \frac{K \prod (s - z_m)}{\prod (s - p_n)}$$

$zpk([z_1, z_2, z_3 \ldots], [p_1, p_2 \ldots p_n], K)$

feedback (G, H)

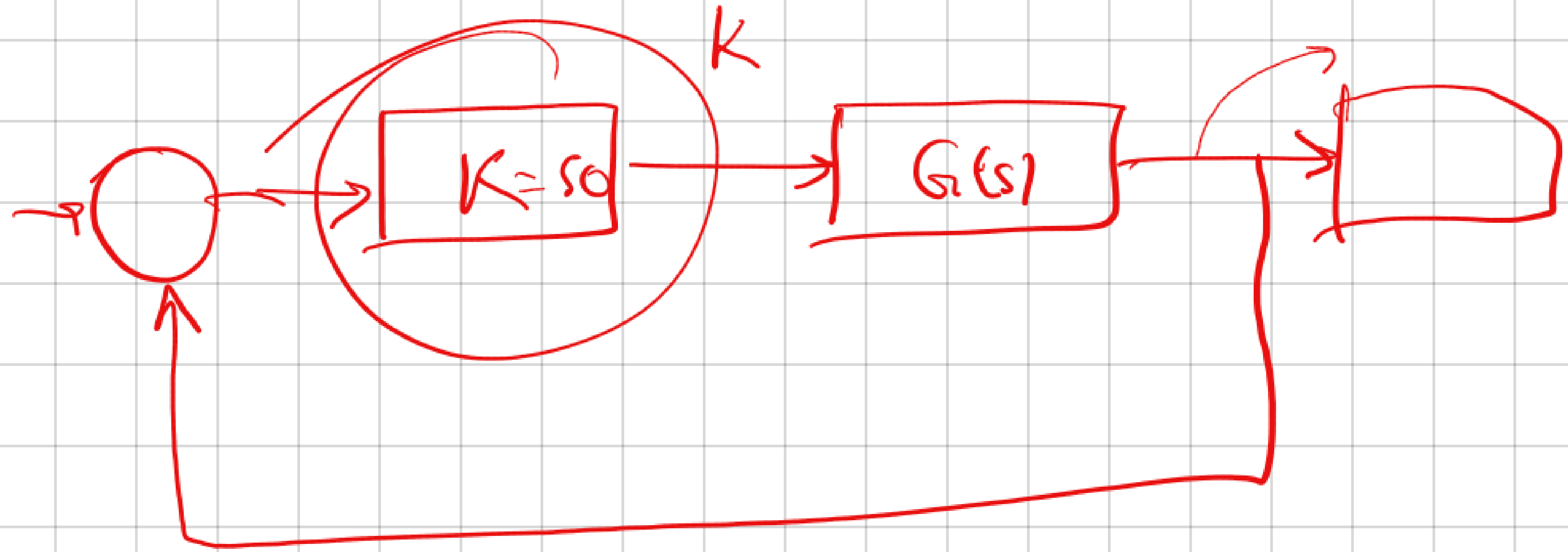
$$= \frac{G}{1 + GH}$$

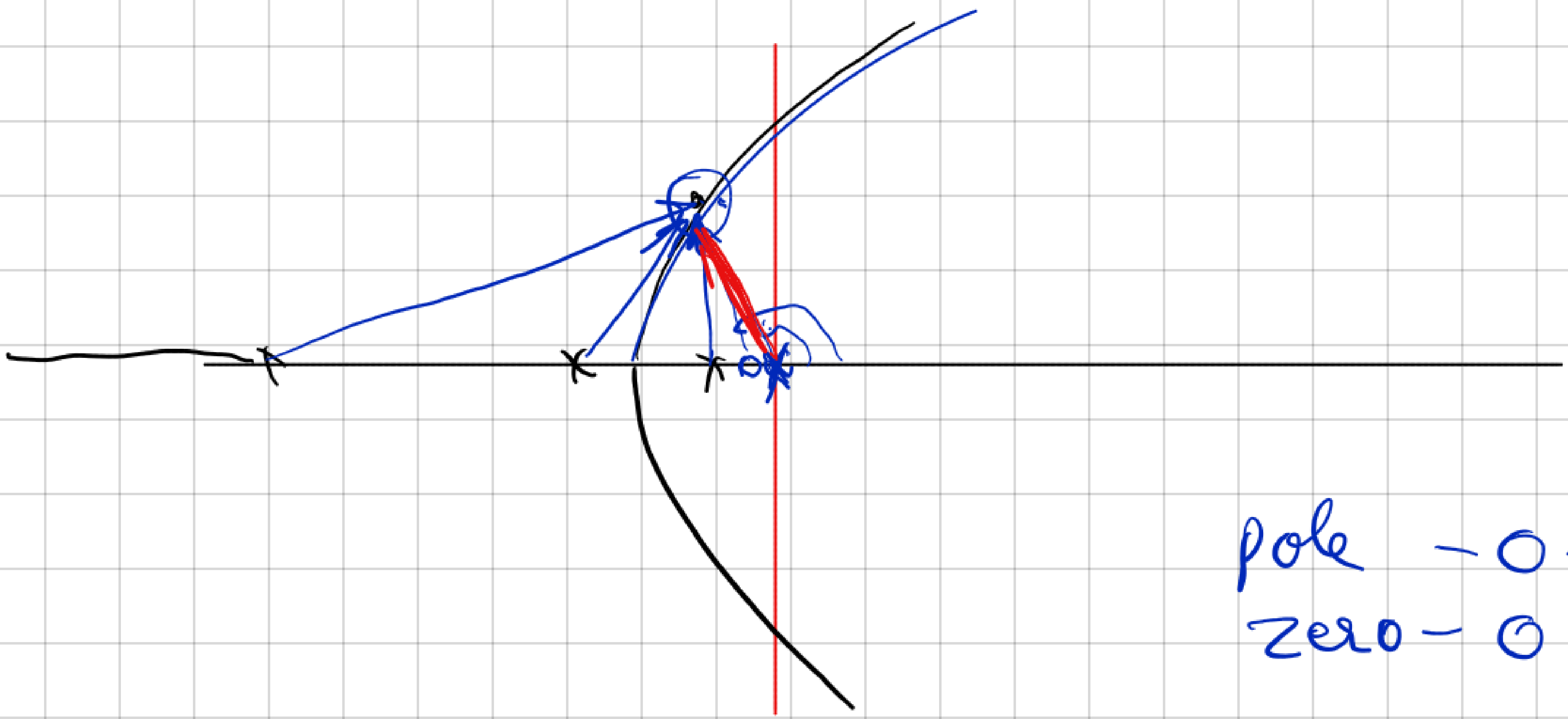




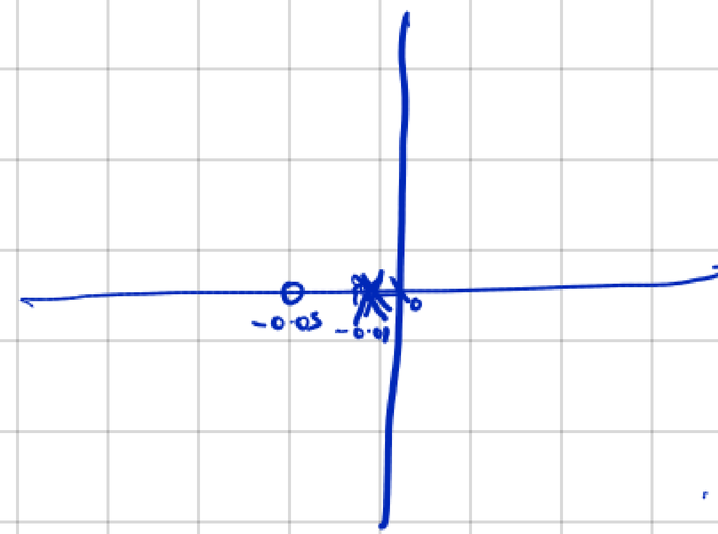
$$\left(K_p + \frac{K_I}{s}\right) \cdot G(s)$$

$$\frac{K_p s + K_I}{s} \, G(s)$$



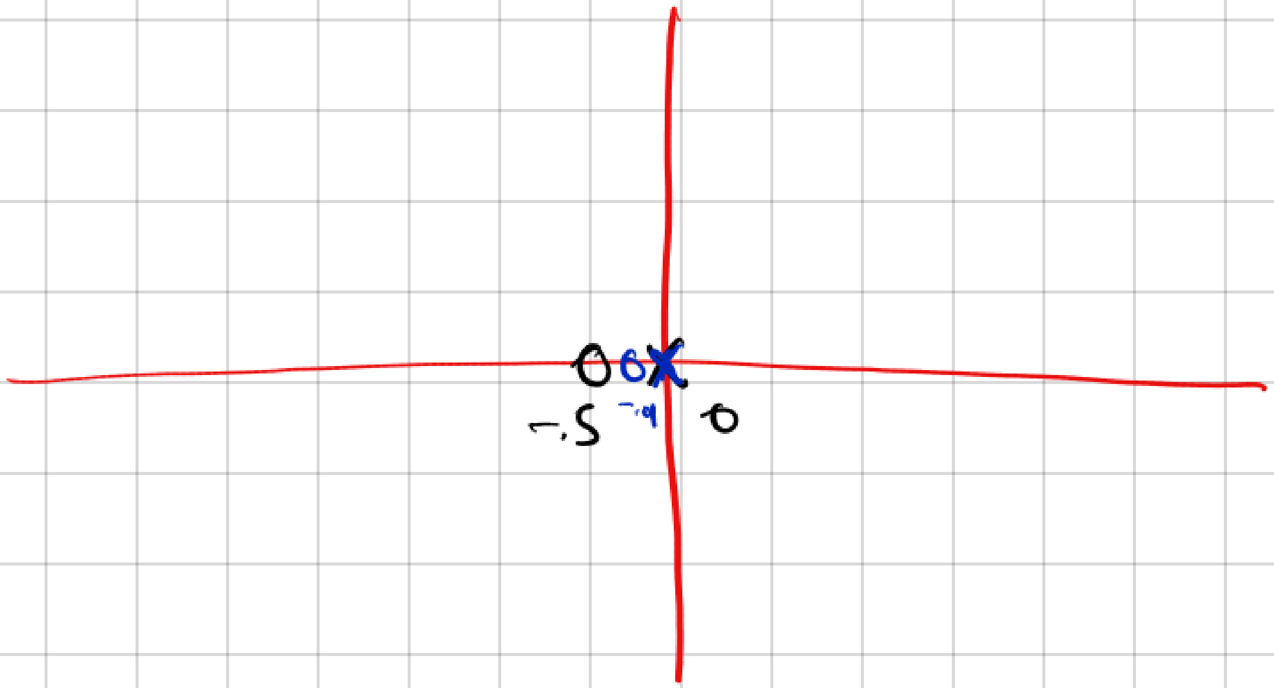


Lag compensator



.33
.09

pole -0.01
zero -0.1



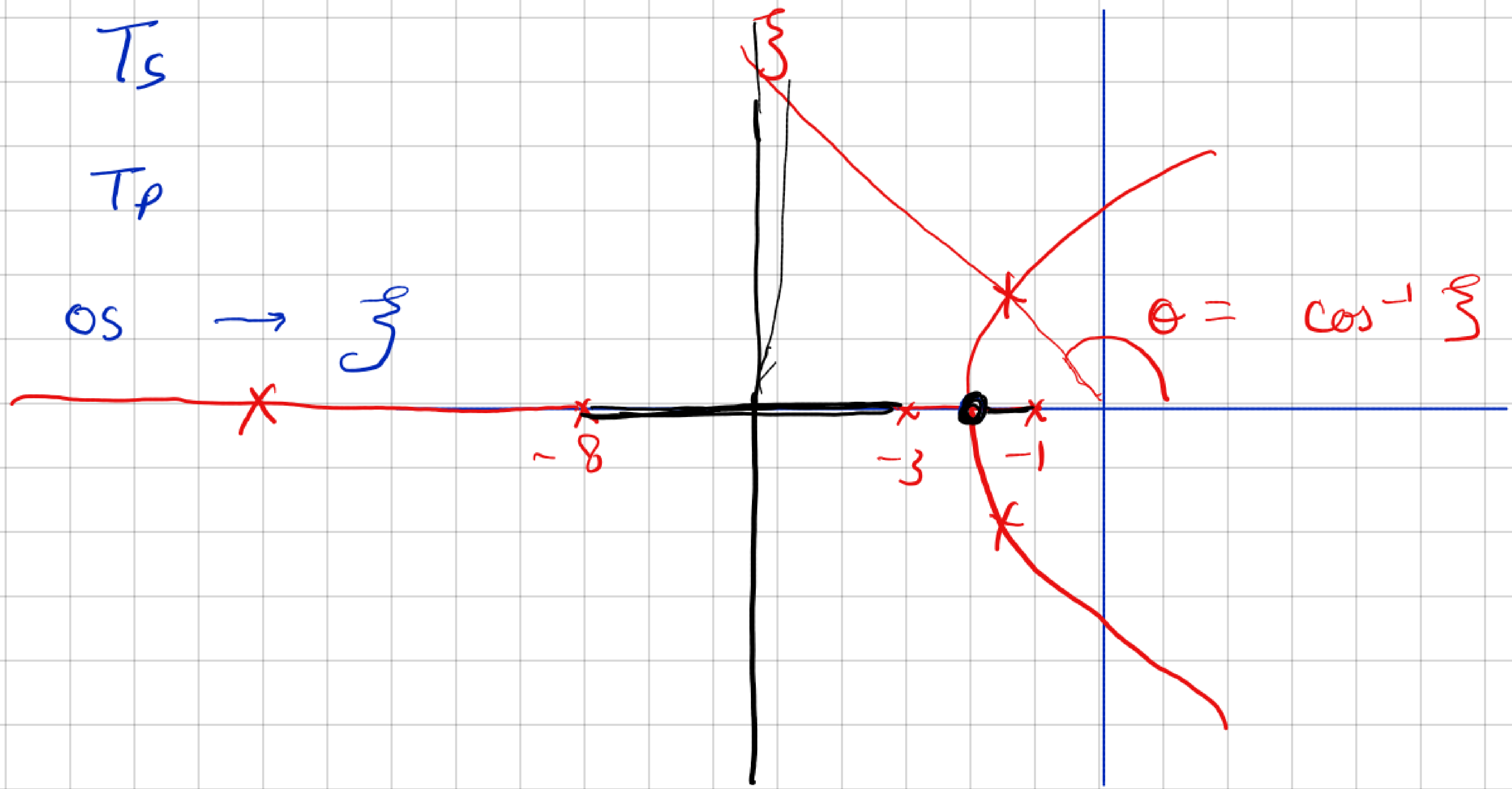
-.5
-.9
0

What if I want to speed up the dynamics of the system

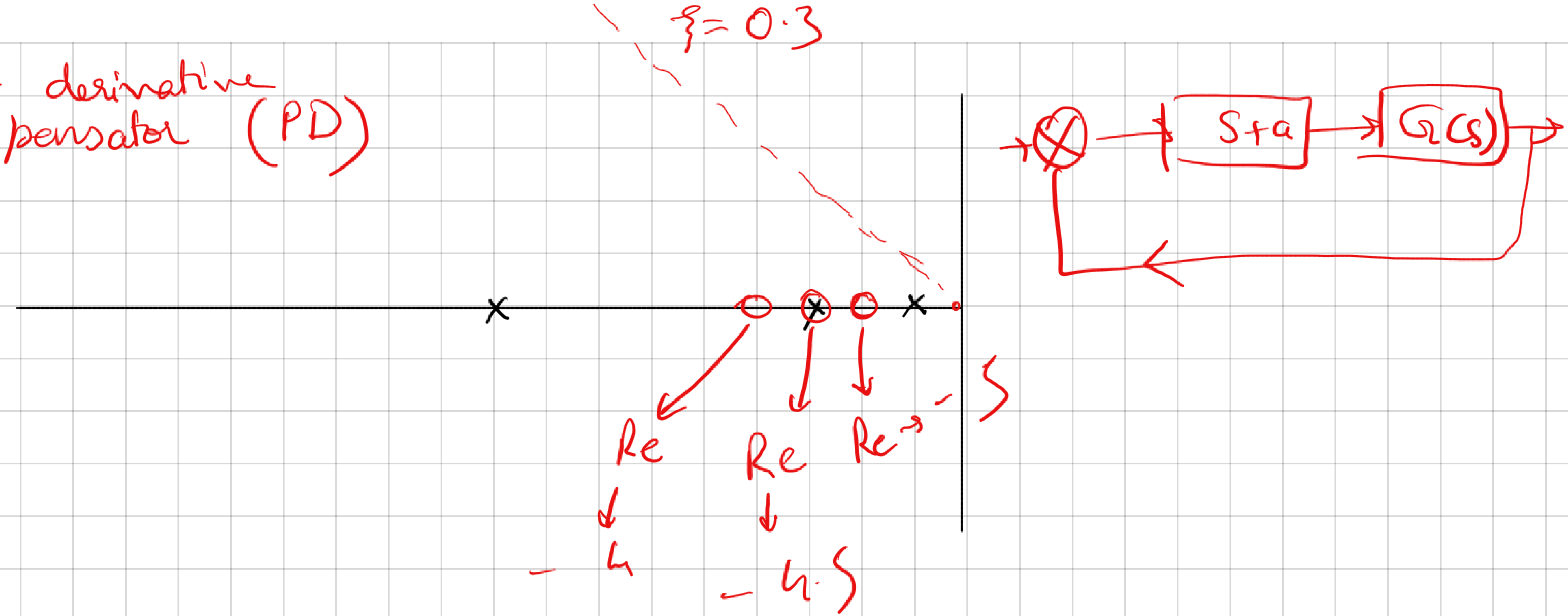
Reduce T_s

T_p

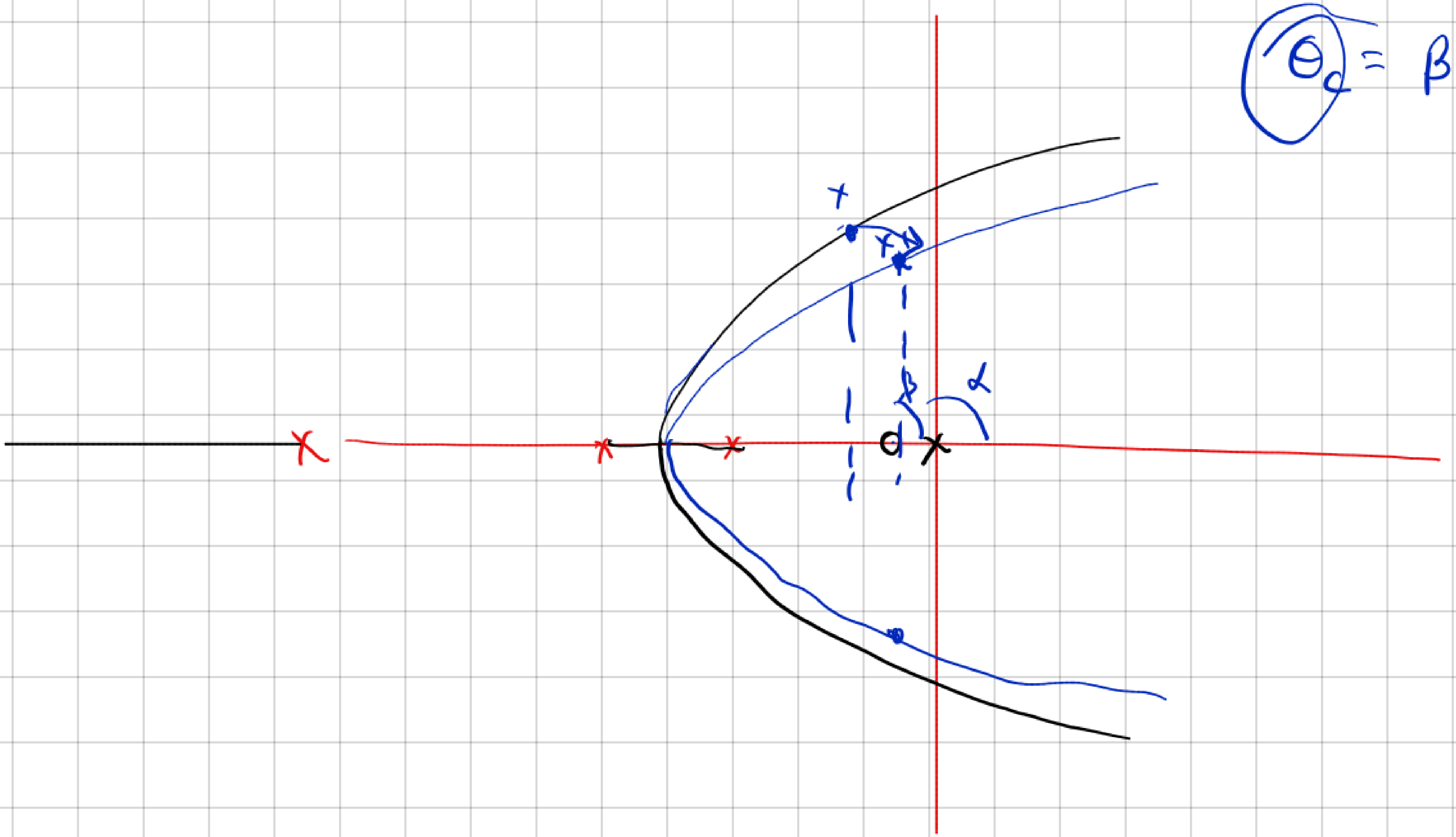
Reduce OS $\rightarrow \xi$



Ideal derivative compensator (PD)



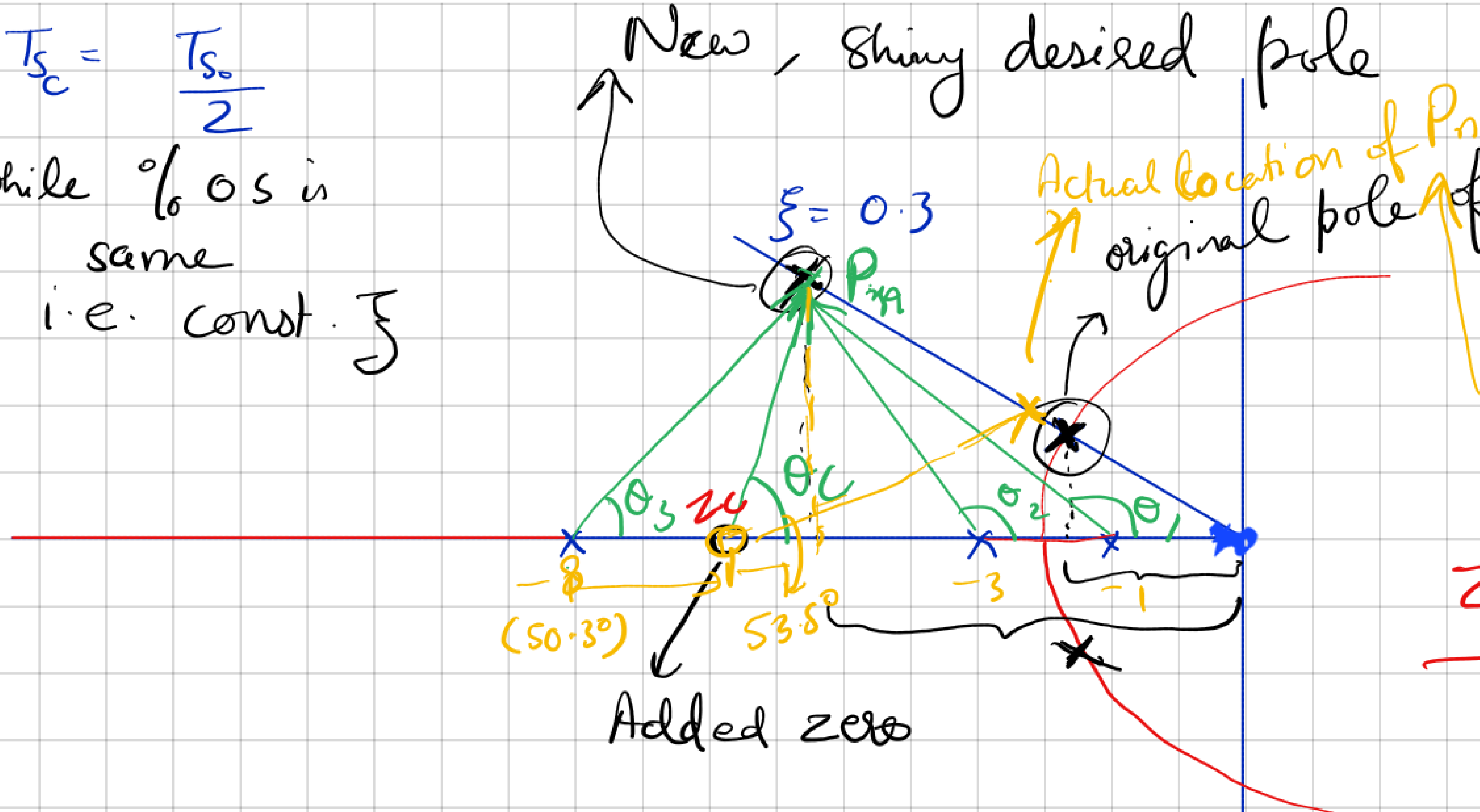
$$\theta_d = \beta - \alpha < 0^\circ$$



Require:- $T_{s_c} = \dfrac{T_{s_0}}{2}$

while %OS is same

i.e. const. ζ



$P_n = -2.2 \pm 6.9956i$

$Z_c = -7.8674$

$$\theta_c - \theta_1 - \theta_2 - \theta_3 = 180(2k+1)$$

$$-\underbrace{(99.7^\circ + 83.5^\circ + 50.3^\circ)}_{} = 233.5^\circ$$

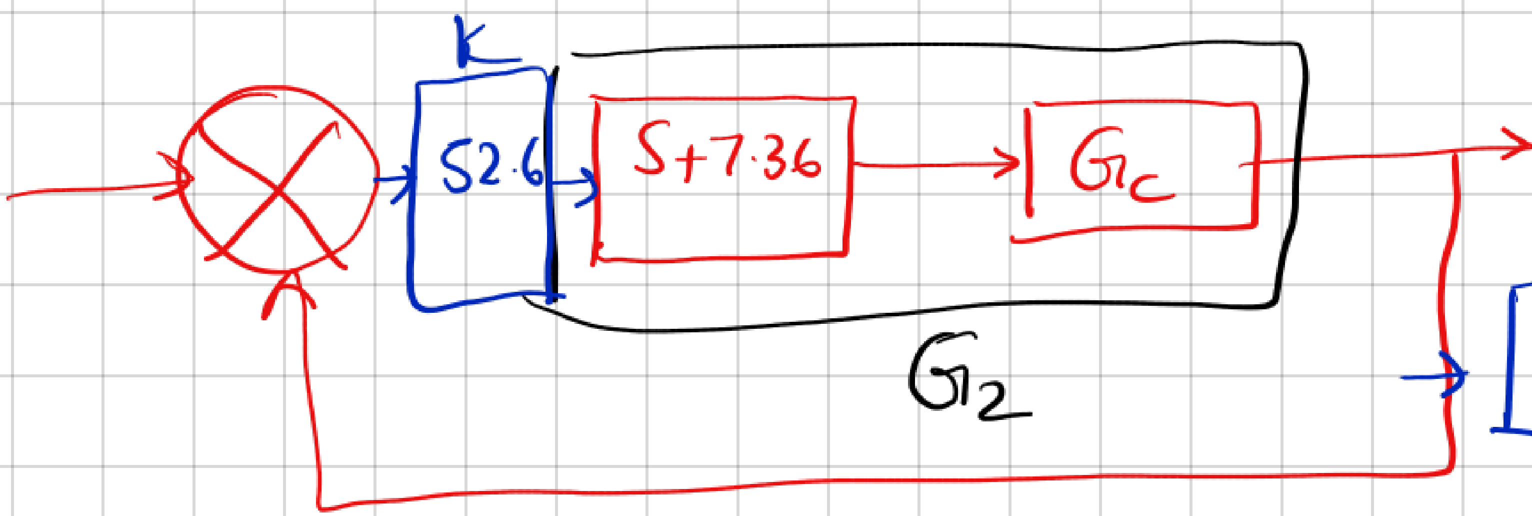
$$233.5 - 180 = 53.5^\circ = \text{Angle deficit}$$

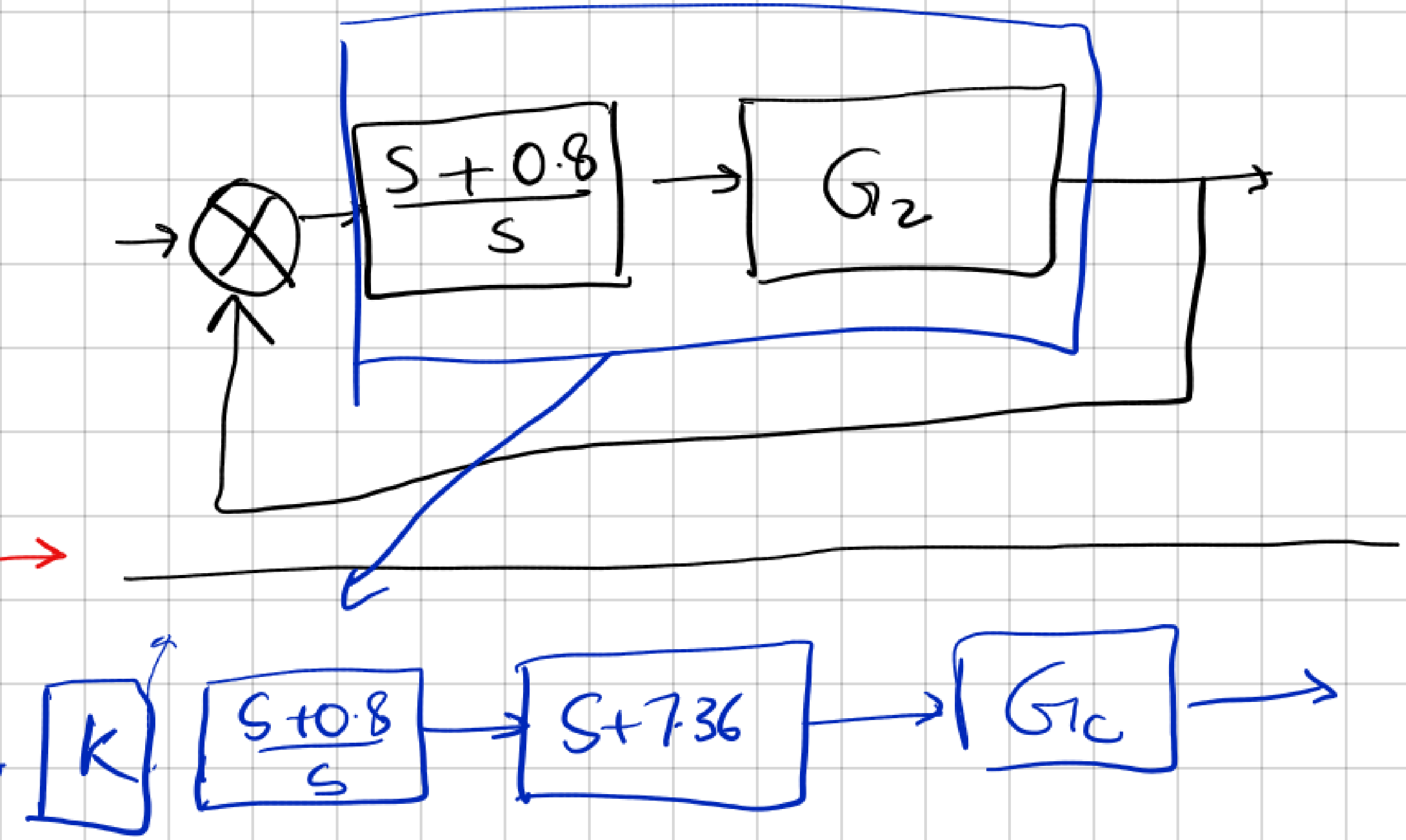
$$G_c = \frac{1}{(s+1)(s+3)(s+8)}$$

now remove the e_{ss}

$\xi = 0.3$

Design a controller to improve T_s by 2 times



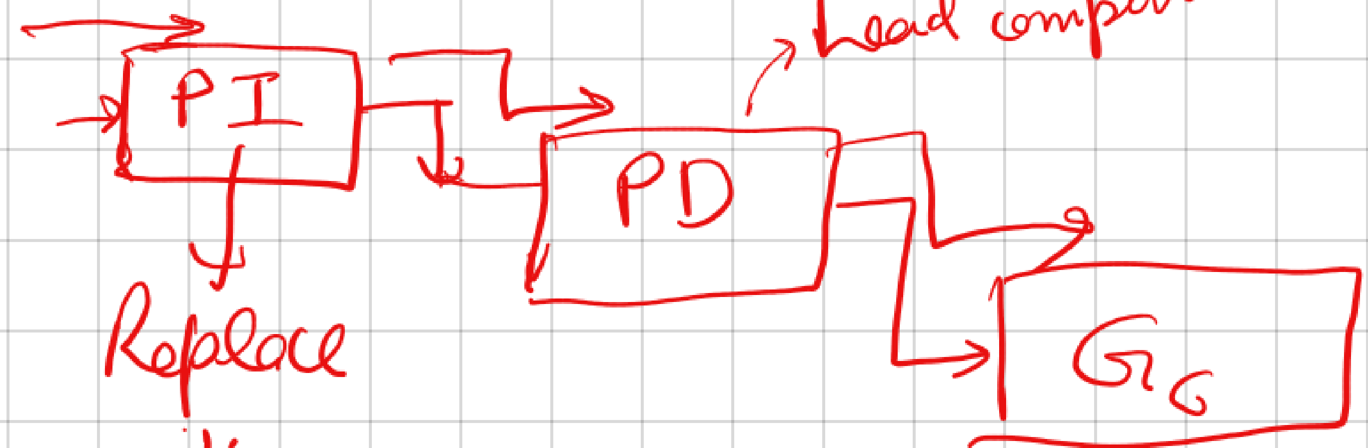


$$52.6\frac{(s+0.8)}{s}(s+7.36) = 52.6\frac{(s^2+8.16s+5.88)}{s}$$

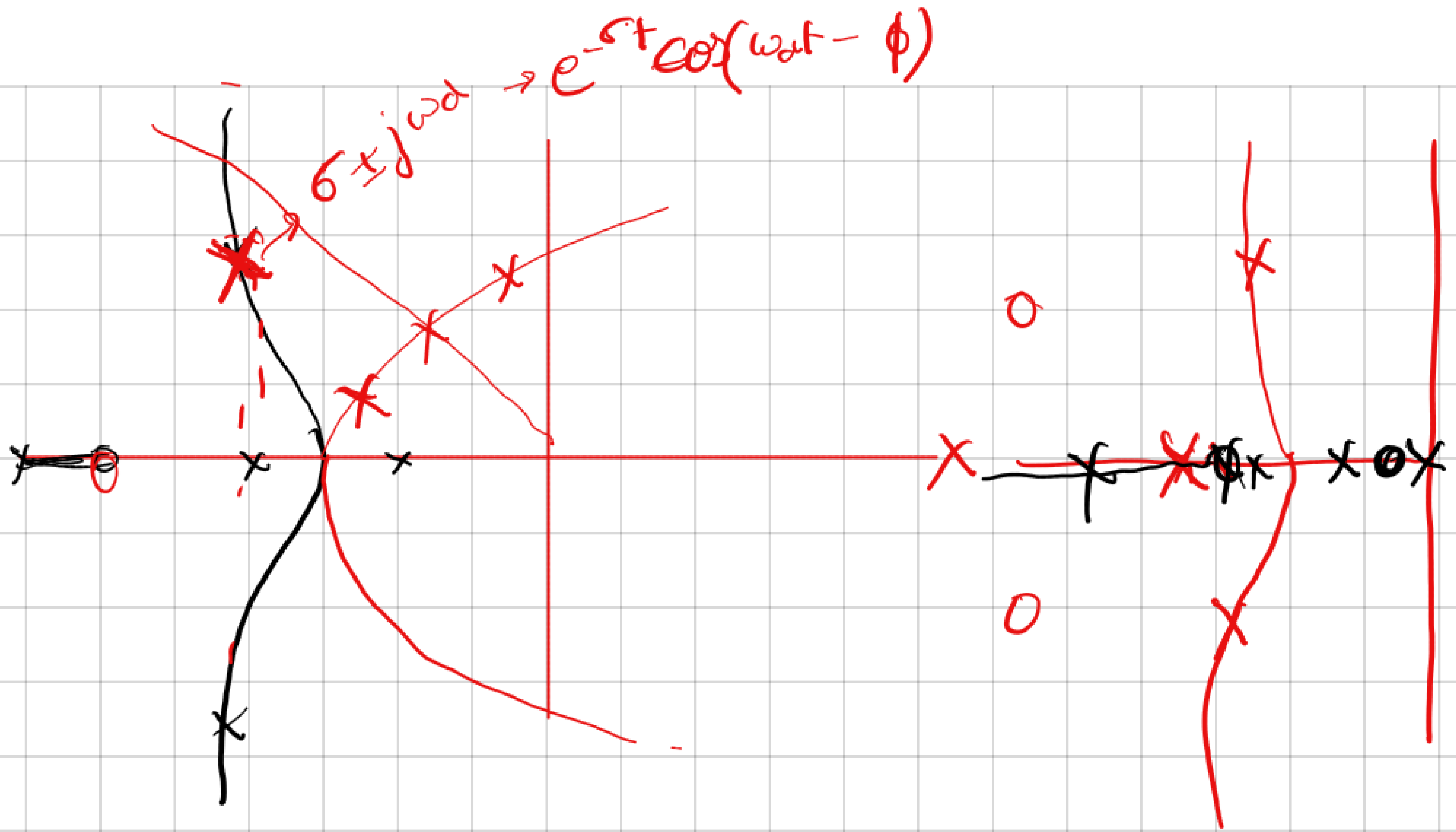
$$= \boxed{\underbrace{52.6s}_{K_D} + \underbrace{429.2}_{K_P} + \overbrace{\frac{309.7}{s}}^{K_I}}$$

Lead compensator.

Cascade controllers →



Replace with a Lag compensator


$\sigma \pm j\omega_d \rightarrow e^{-\sigma t}\cos(\omega_d t - \phi)$

Steps to systematically develop a controller to change dynamics & Steady State behavior

1.) Evaluate the performance of the uncompensated system.
(Simulate)
→ To determine the correction in performance.
→ convert these into desired pole location

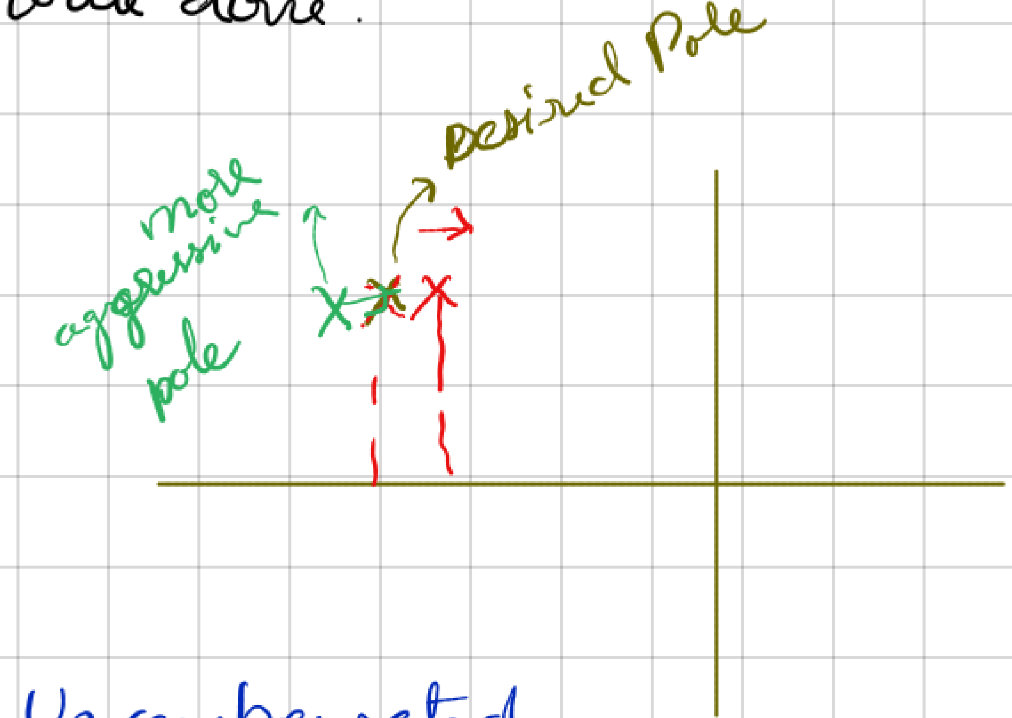
2.) Design PD / Lead-Compensator, to force / reshape the R.L. to pass through the desired pole.

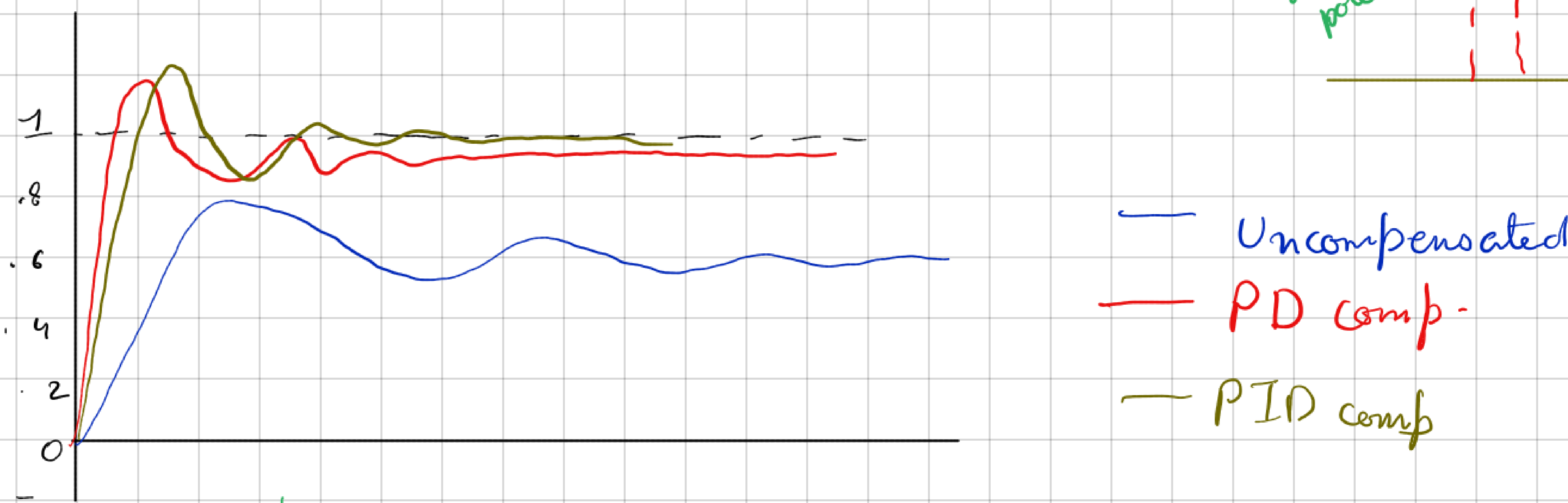
Simulate the system

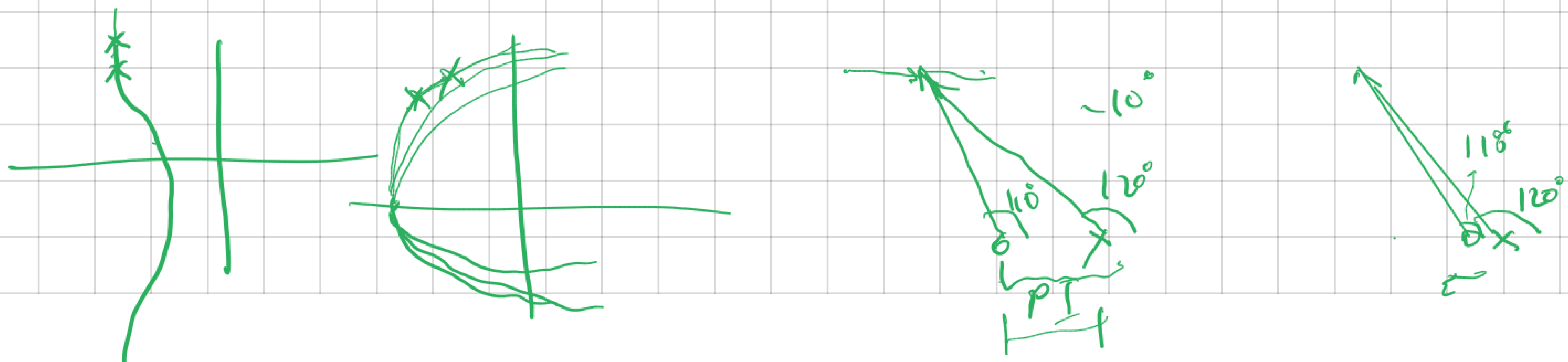
3.) Design a PI / Lag-compensator to eliminate steady state error.
Simulate

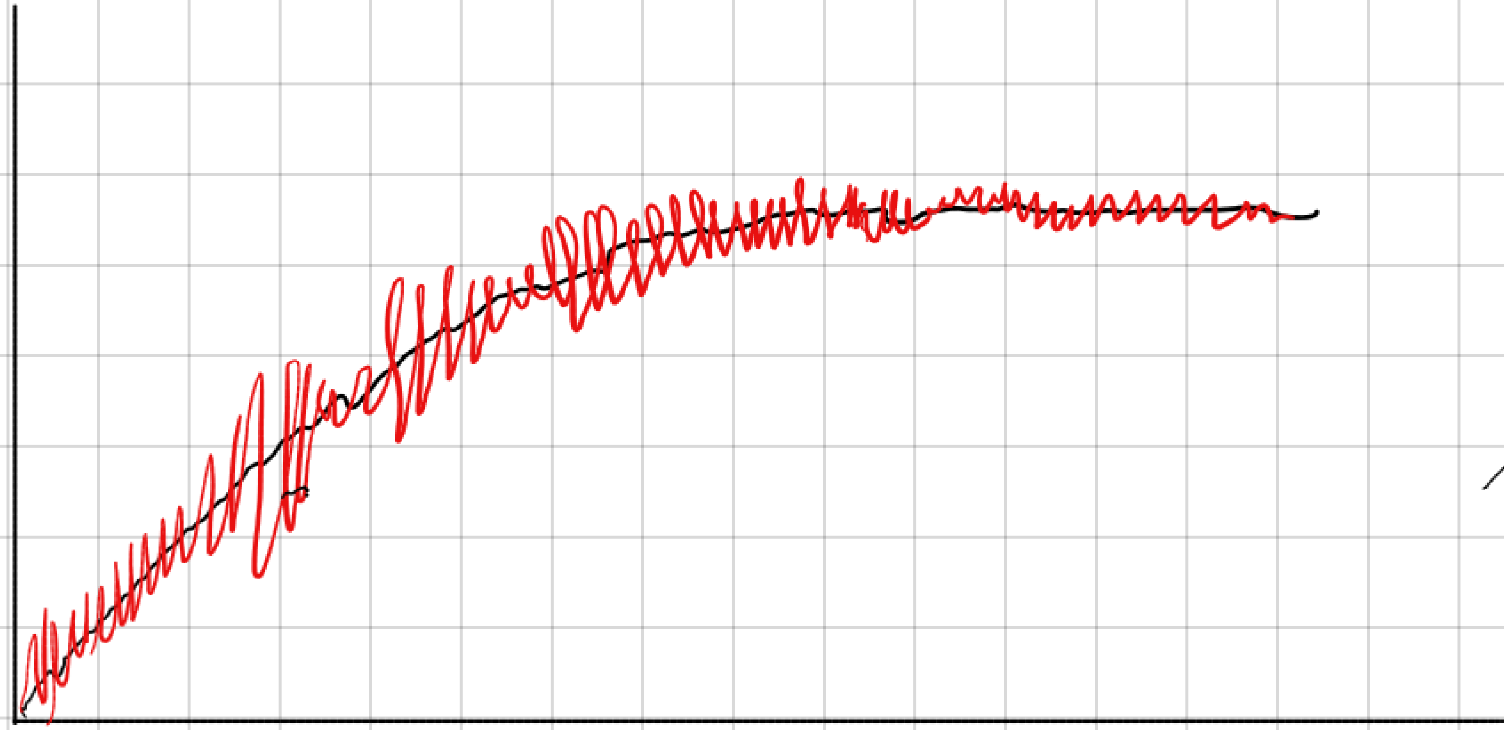
If new dynamics are agreeable, you're done.

If not, repeat from step 2

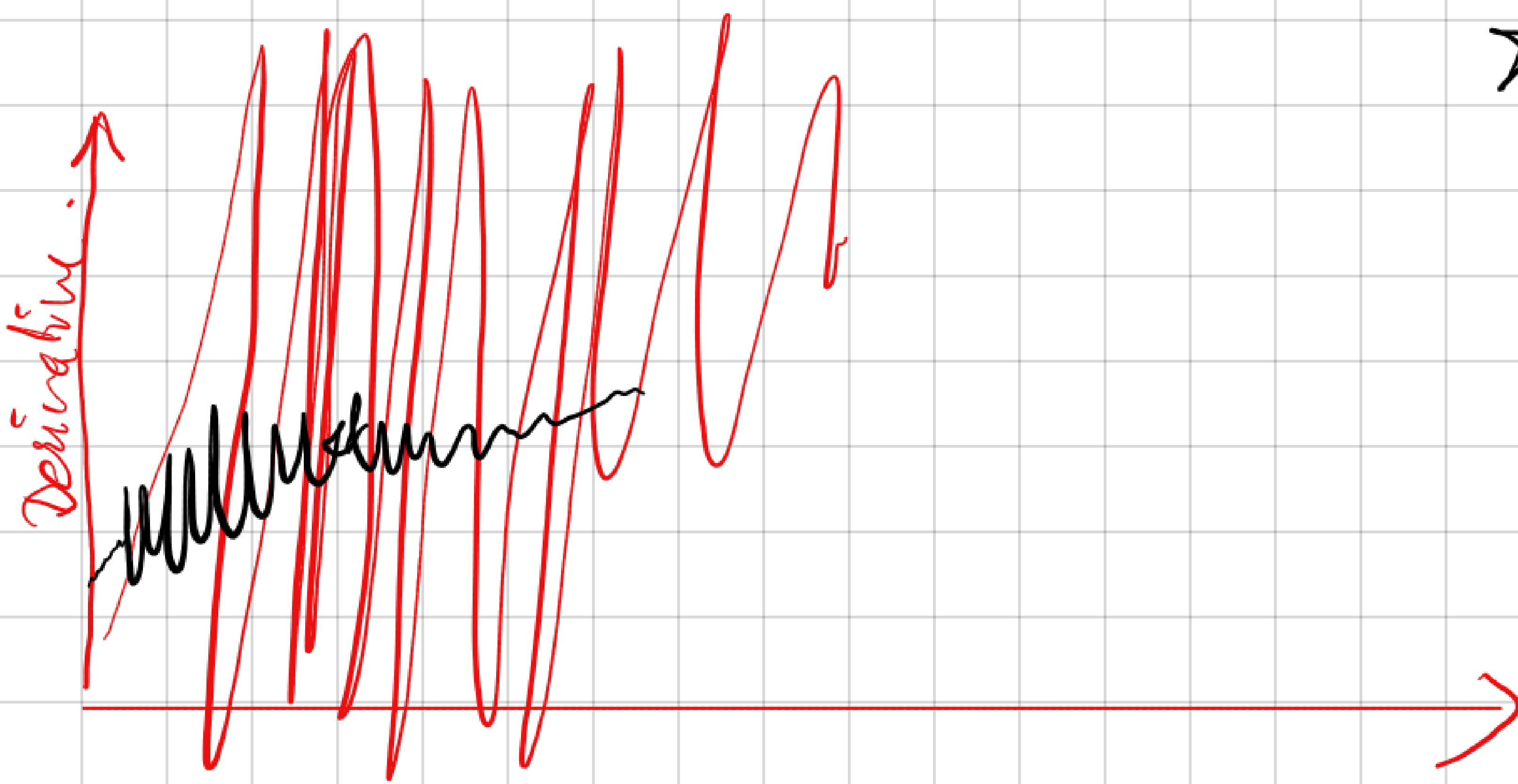


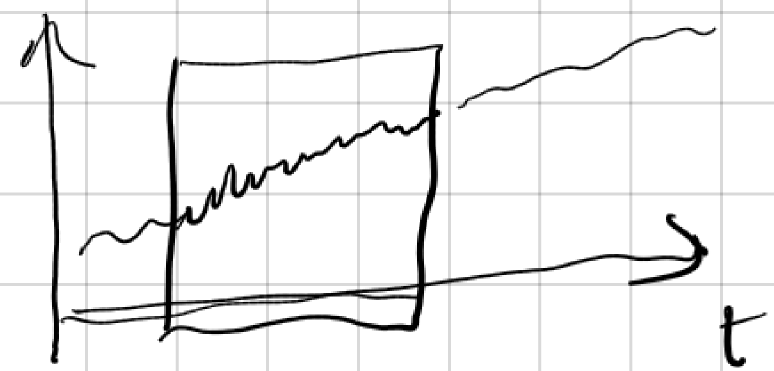




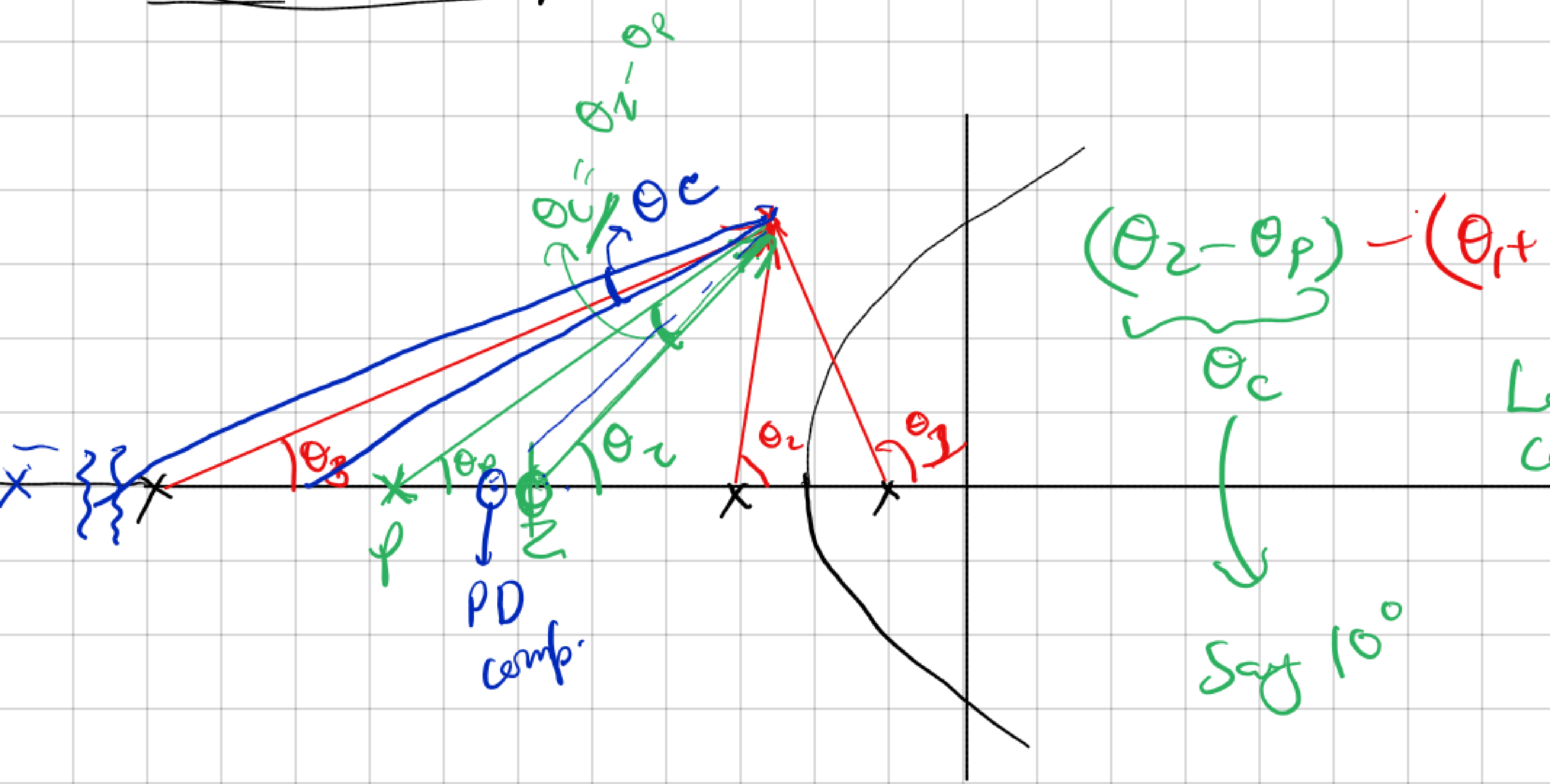


☆ Derivative operation is inherently noisy.





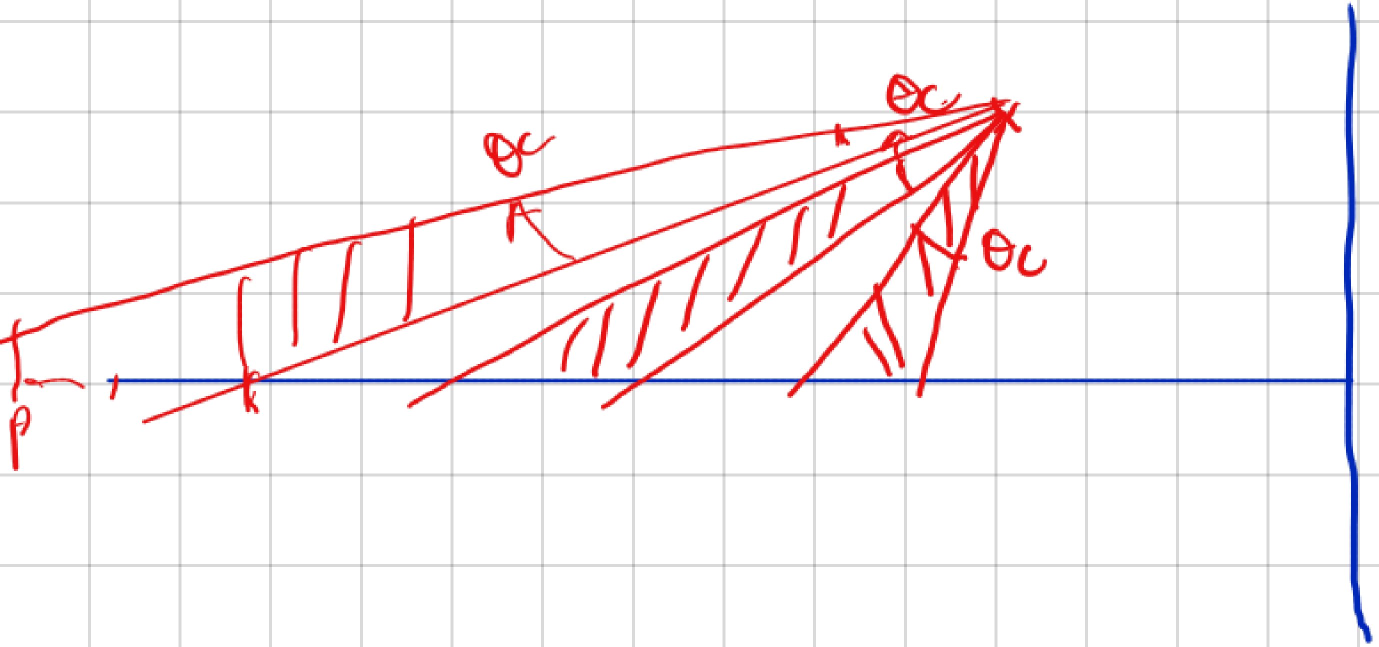
Lead Compensator → Passive implementation & approx. of PD controller.



$$\underbrace{(\theta_z - \theta_p)}_{\theta_c} - (\theta_1 + \theta_2 + \theta_3) = 180^\circ(2k+1)$$

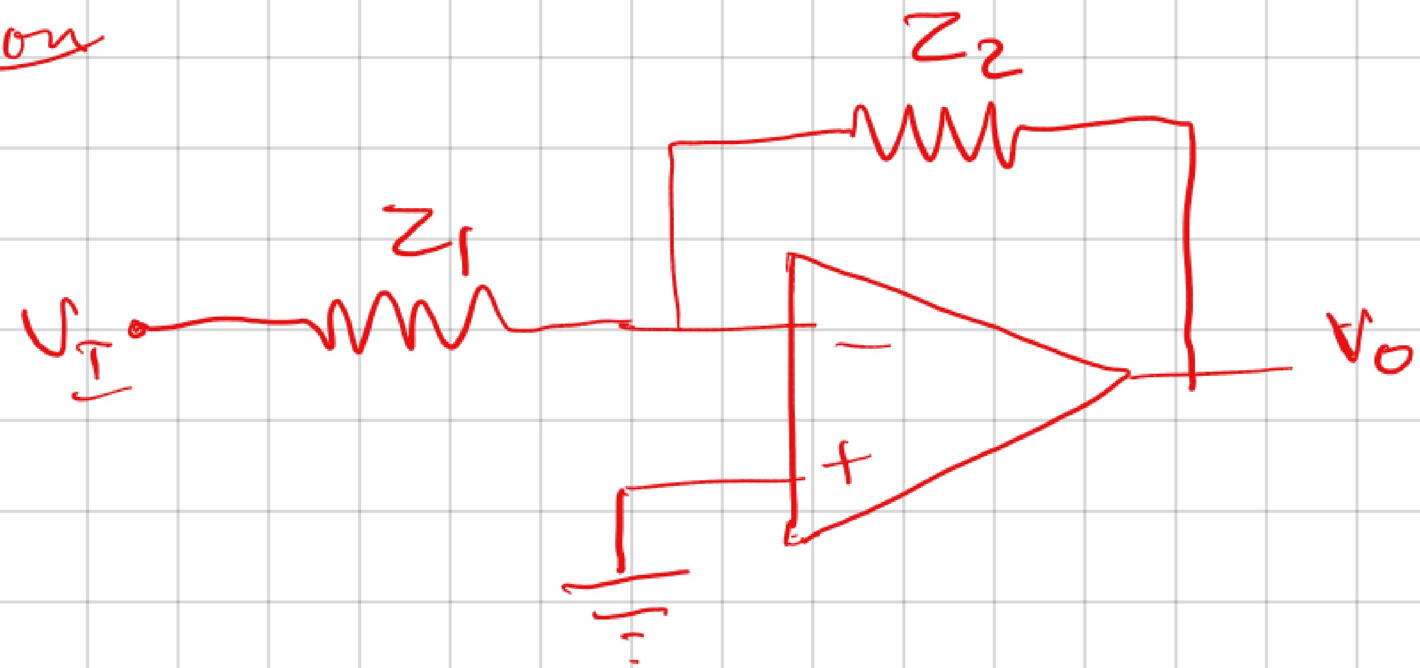
θ_c → Say 10°

Lead compensator $\frac{S+Z}{S+P}$



θc
θc
θc

Implementation

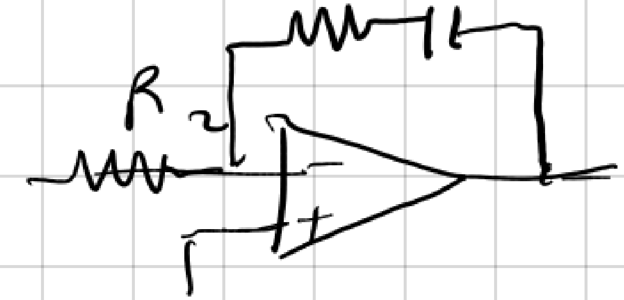


$$\frac{V_o}{V_I} = - \frac{Z_2}{Z_1}$$

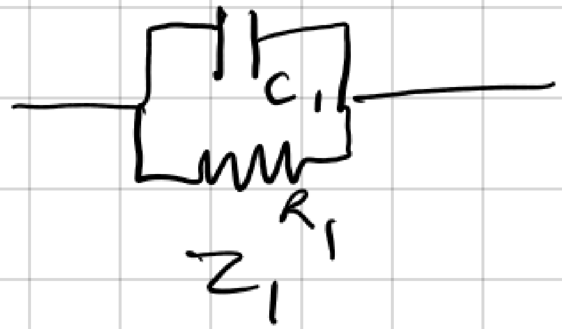
$Z \rightarrow$ Impedance

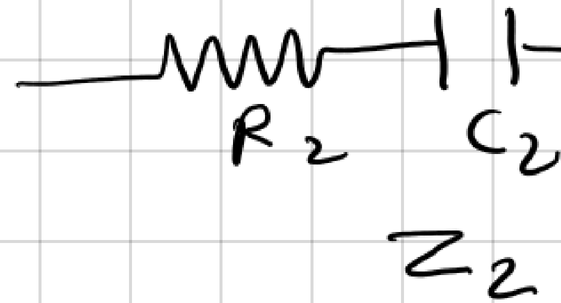
Z_2	Z_1
R_2	R_1

	Z_1	Z_2	$\frac{V_o}{V_I}$
	R	C	$-\frac{1/RC}{s}$
			$-RCs$
PI	R_1	R_2 C	$-\frac{R_2}{R_1}\left(\frac{s+\frac{1}{R_2C}}{s}\right)$
PD	C R_1	R_2	$-R_2C\left(s+\frac{1}{R_1C}\right) \to (s+a)$



PID





$$-\left[\underbrace{\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right)} + R_2 C_1 s + \frac{\frac{1}{R_1 C_2}}{s}\right]$$