Appendix A

Laplace Transform Tables

Appendix A first presents the complex variable and complex function. Then it presents tables of Laplace transform pairs and properties of Laplace transforms. Finally, it presents frequently used Laplace transform theorems and Laplace transforms of pulse function and impulse function.

Complex Variable. A complex number has a real part and an imaginary part, both of which are constant. If the real part and/or imaginary part are variables, a complex quantity is called a *complex variable*. In the Laplace transformation we use the notation s as a complex variable; that is,

$$s = \sigma + j\omega$$

where σ is the real part and ω is the imaginary part.

Complex Function. A complex function $G(s)$, a function of s, has a real part and an imaginary part or

$$G(s) = G_x + jG_y$$

where G_x and G_y are real quantities. The magnitude of $G(s)$ is $\sqrt{G_x^2 + G_y^2}$, and the angle θ of $G(s)$ is $\tan^{-1}(G_y/G_x)$. The angle is measured counterclockwise from the positive real axis. The complex conjugate of $G(s)$ is $\bar{G}(s) = G_x - jG_y$.

Complex functions commonly encountered in linear control systems analysis are single-valued functions of s and are uniquely determined for a given value of s.

A complex function $G(s)$ is said to be *analytic* in a region if $G(s)$ and all its derivatives exist in that region. The derivative of an analytic function $G(s)$ is given by

$$\frac{d}{ds}G(s) = \lim_{\Delta s \to 0} \frac{G(s + \Delta s) - G(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta G}{\Delta s}$$

Since $\Delta s = \Delta\sigma + j\Delta\omega$, Δs can approach zero along an infinite number of different paths. It can be shown, but is stated without a proof here, that if the derivatives taken along two particular paths, that is, $\Delta s = \Delta\sigma$ and $\Delta s = j\Delta\omega$, are equal, then the derivative is unique for any other path $\Delta s = \Delta\sigma + j\Delta\omega$ and so the derivative exists.

For a particular path $\Delta s = \Delta\sigma$ (which means that the path is parallel to the real axis),

$$\frac{d}{ds}G(s) = \lim_{\Delta\sigma \to 0}\left(\frac{\Delta G_x}{\Delta\sigma} + j\frac{\Delta G_y}{\Delta\sigma}\right) = \frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma}$$

For another particular path $\Delta s = j\Delta\omega$ (which means that the path is parallel to the imaginary axis),

$$\frac{d}{ds}G(s) = \lim_{j\Delta\omega \to 0}\left(\frac{\Delta G_x}{j\Delta\omega} + j\frac{\Delta G_y}{j\Delta\omega}\right) = -j\frac{\partial G_x}{\partial\omega} + \frac{\Delta G_y}{\partial\omega}$$

If these two values of the derivative are equal,

$$\frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} - j\frac{\partial G_x}{\partial\omega}$$

or if the following two conditions

$$\frac{\partial G_x}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} \qquad \text{and} \qquad \frac{\partial G_y}{\partial\sigma} = -\frac{\partial G_x}{\partial\omega}$$

are satisfied, then the derivative $dG(s)/ds$ is uniquely determined. These two conditions are known as the Cauchy–Riemann conditions. If these conditions are satisfied, the function $G(s)$ is analytic.

As an example, consider the following $G(s)$:

$$G(s) = \frac{1}{s + 1}$$

Then

$$G(\sigma + j\omega) = \frac{1}{\sigma + j\omega + 1} = G_x + jG_y$$

where

$$G_x = \frac{\sigma + 1}{(\sigma + 1)^2 + \omega^2} \qquad \text{and} \qquad G_y = \frac{-\omega}{(\sigma + 1)^2 + \omega^2}$$

It can be seen that, except at $s = -1$ (that is, $\sigma = -1$, $\omega = 0$), $G(s)$ satisfies the Cauchy–Riemann conditions:

$$\frac{\partial G_x}{\partial \sigma} = \frac{\partial G_y}{\partial \omega} = \frac{\omega^2 - (\sigma + 1)^2}{\left[(\sigma + 1)^2 + \omega^2\right]^2}$$

$$\frac{\partial G_y}{\partial \sigma} = -\frac{\partial G_x}{\partial \omega} = \frac{2\omega(\sigma + 1)}{\left[(\sigma + 1)^2 + \omega^2\right]^2}$$

Hence $G(s) = 1/(s + 1)$ is analytic in the entire s plane except at $s = -1$. The derivative $dG(s)/ds$, except at $s = 1$, is found to be

$$\frac{d}{ds}G(s) = \frac{\partial G_x}{\partial \sigma} + j\frac{\partial G_y}{\partial \sigma} = \frac{\partial G_y}{\partial \omega} - j\frac{\partial G_x}{d\omega}$$

$$= -\frac{1}{(\sigma + j\omega + 1)^2} = -\frac{1}{(s + 1)^2}$$

Note that the derivative of an analytic function can be obtained simply by differentiating $G(s)$ with respect to s. In this example,

$$\frac{d}{ds}\left(\frac{1}{s + 1}\right) = -\frac{1}{(s + 1)^2}$$

Points in the s plane at which the function $G(s)$ is analytic are called *ordinary* points, while points in the s plane at which the function $G(s)$ is not analytic are called *singular* points. Singular points at which the function $G(s)$ or its derivatives approach infinity are called *poles*. Singular points at which the function $G(s)$ equals zero are called *zeros*.

If $G(s)$ approaches infinity as s approaches $-p$ and if the function

$$G(s)(s + p)^n, \qquad \text{for } n = 1, 2, 3, \ldots$$

has a finite, nonzero value at $s = -p$, then $s = -p$ is called a pole of order n. If $n = 1$, the pole is called a simple pole. If $n = 2, 3, \ldots$, the pole is called a second-order pole, a third-order pole, and so on.

To illustrate, consider the complex function

$$G(s) = \frac{K(s + 2)(s + 10)}{s(s + 1)(s + 5)(s + 15)^2}$$

$G(s)$ has zeros at $s = -2$, $s = -10$, simple poles at $s = 0$, $s = -1$, $s = -5$, and a double pole (multiple pole of order 2) at $s = -15$. Note that $G(s)$ becomes zero at $s = \infty$. Since for large values of s

$$G(s) \doteqdot \frac{K}{s^3}$$

$G(s)$ possesses a triple zero (multiple zero of order 3) at $s = \infty$. If points at infinity are included, $G(s)$ has the same number of poles as zeros. To summarize, $G(s)$ has five zeros ($s = -2$, $s = -10$, $s = \infty$, $s = \infty$, $s = \infty$) and five poles ($s = 0$, $s = -1$, $s = -5$, $s = -15$, $s = -15$).

Laplace Transformation. Let us define

$f(t)$ = a function of time t such that $f(t) = 0$ for $t < 0$

s = a complex variable

$\mathscr{L}$ = an operational symbol indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^\infty e^{-st}\,dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called the *inverse Laplace transformation*. The notation for the inverse Laplace transformation is $\mathscr{L}^{-1}$, and the inverse Laplace transform can be found from $F(s)$ by the following inversion integral:

$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds, \qquad \text{for } t > 0$$

where c, the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount c from it. This path of integration is to the right of all singular points.

Evaluating the inversion integral appears complicated. In practice, we seldom use this integral for finding $f(t)$. We frequently use the partial-fraction expansion method given in Appendix B.

In what follows we give Table A–1, which presents Laplace transform pairs of commonly encountered functions, and Table A–2, which presents properties of Laplace transforms.

Table A–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \qquad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \qquad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!}t^{n-1}e^{-at} \qquad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \qquad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}\left(1 - e^{-at}\right)$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}\left(be^{-bt} - ae^{-at}\right)$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}\left(be^{-at} - ae^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$

(*continues on next page*)

Table A–1 *(continued)*

18	$\frac{1}{a^2}\left(1 - e^{-at} - ate^{-at}\right)$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}\left(at - 1 + e^{-at}\right)$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t \quad (0 < \zeta < 1)$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t - \phi\right)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi\right)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$
25	$1 - \cos\omega t$	$\frac{\omega^2}{s\left(s^2 + \omega^2\right)}$
26	$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2\left(s^2 + \omega^2\right)}$
27	$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{\left(s^2 + \omega^2\right)^2}$
28	$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{\left(s^2 + \omega^2\right)^2}$
29	$t\cos\omega t$	$\frac{s^2 - \omega^2}{\left(s^2 + \omega^2\right)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2}\left(\cos\omega_1 t - \cos\omega_2 t\right) \qquad \left(\omega_1^2 \neq \omega_2^2\right)$	$\frac{s}{\left(s^2 + \omega_1^2\right)\left(s^2 + \omega_2^2\right)}$
31	$\frac{1}{2\omega}\left(\sin\omega t + \omega t\cos\omega t\right)$	$\frac{s^2}{\left(s^2 + \omega^2\right)^2}$

Table A–2 Properties of Laplace Transforms

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_{\pm}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_{\pm}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_{\pm}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_{\pm}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0\pm}$
7	$\mathscr{L}_{\pm}\left[\int\cdots\int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left[\int\cdots\int f(t)(dt)^k\right]_{t=0\pm}$
8	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
10	$\mathscr{L}[e^{-\alpha t}f(t)] = F(s + a)$
11	$\mathscr{L}[f(t - \alpha)1(t - \alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
12	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad (n = 1, 2, 3, \dots)$
15	$\mathscr{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \qquad \text{if } \lim_{t\to 0}\frac{1}{t}f(t) \text{ exists}$
16	$\mathscr{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$
18	$\mathscr{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(p)G(s-p)\,dp$

Finally, we present two frequently used theorems, together with Laplace transforms of the pulse function and impulse function.

Initial value theorem	$f(0+) = \lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s)$
Final value theorem	$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$
Pulse function $f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$	$\mathscr{L}[f(t)] = \frac{A}{t_0 s} - \frac{A}{t_0 s} e^{-st_0}$
Impulse function $g(t) = \lim_{t_0 \to 0} \frac{A}{t_0}, \quad \text{for } 0 < t < t_0$ $= 0, \quad \text{for } t < 0, t_0 < t$	$\mathscr{L}[g(t)] = \lim_{t_0 \to 0} \left[\frac{A}{t_0 s} (1 - e^{-st_0}) \right]$ $= \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}[A(1 - e^{-st_0})]}{\frac{d}{dt_0}(t_0 s)}$ $= \frac{As}{s} = A$