Portfolio Optimization Using Hybrid Quantum Algorithms

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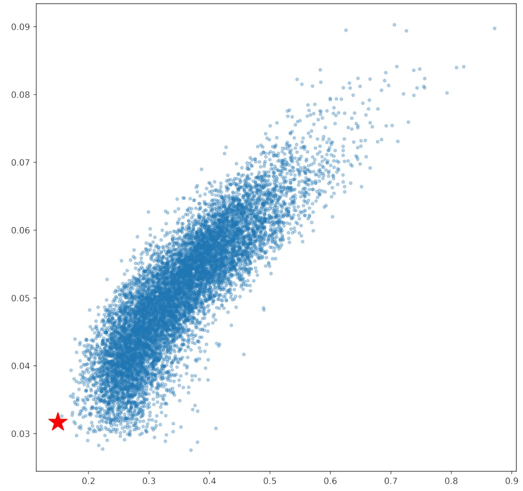
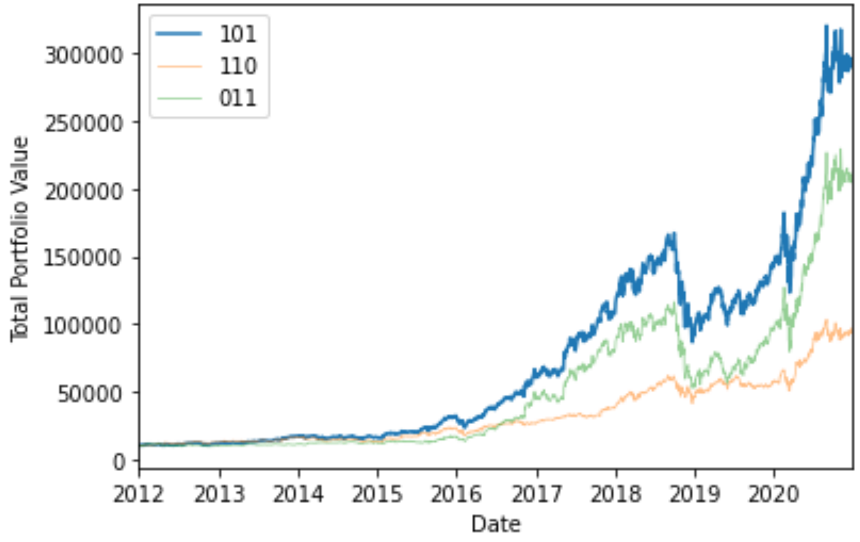
A Refresher – Portfolio Optimization

- Portfolio Optimization is relevant to both retail investors and investment groups.
- Pick the "best" portfolio is an optimization problem.
- There are different ways to set up and perform portfolio optimization – Efficient Frontier, etc.
- Optimal portfolio may look different depending on investor – high return, low volatility, etc.



Efficient Frontier portfolio optimization



Example portfolios over time.

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Introduction – Quantum Computing

- Current quantum computing hardware is limited by the number of qubits, connectivity, and a number of other factors.

- We can use small-scale hardware, using a combination of classical and quantum computing techniques.

- Two relevant hybrid algorithms for portfolio optimization :
  - Variational Quantum Eigensolver (VQE)
  - Quantum Approximate Optimization Algorithm (QAOA)

Introduction – Quantum Portfolio Optimization

We will solve a portfolio optimization problem

- Classically,
- With the Variational Quantum Eigensolver, and
- With the Quantum Approximate Optimization Algorithm,

using qiskit (IBM's quantum SDK).

The Context of Quantum Realism:

- Prepare for the "Quantum Surprise" by understand where quantum computing is ***right now***, and where benefits are to be had as the hardware evolves in ***the future***.

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Portfolio Optimization Problem

Portfolio Optimization Problem

The form of this problem is adapted from [Improving Variational Quantum Optimization using CVaR (Barkoutsos et al. 2019).](), where for n assets that can be chosen from:

$$\max_{x\in\{0,1\}^n} \left(\mu^T x - q x^T \sigma x\right)$$

$$\text{subject to: } 1^T x = B.$$

In this optimization problem:

- $x \in \{0, 1\}^n$ is the vector of binary decision variables, which indicate which assets to pick ($x[i] = 1$) and which not to pick ($x[i] = 0$),
- $\mu \in \mathbb{R}^n$ is the vector of means for the daily percent returns of each asset,
- $\sigma \in \mathbb{R}^{n\times n}$ is the covariances matrix of the daily percent returns of the assets,
- $q > 0$ is the risk factor,
- B is the budget, i.e. the number of assets to be selected out of n.

We assume the following simplifications:

- the full budget B has to be spent, i.e. one has to select exactly B assets.

Considerations and Limitations

There is a desire to forecast results, however this method doesn’t guarantee a prediction of the best portfolio.

The method calculates the “best” portfolio by looking back over a period of time.

- The covariance calculation is for a fixed period.

A need to address these limitations in the real world.

- Can make an estimate on how long the portfolio calculated will be relevant.
- Can recalculate and rebalance as needed.

Variational Quantum Eigensolver

Hamiltonian: From Classical …

Hamiltonian – the sum of kinetic energy and the potential energy.

- Consider a ball of mass m attached to a massless spring with a spring constant k. This system has the Hamiltonian:

$$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- Describes the energy and can be used to derive the system dynamics.

- Equations of motions can be derived from the Hamiltonian by using Hamilton's Equations to solve for the time derivatives of momentum and position.

Hamiltonian: From Classical to Quantum

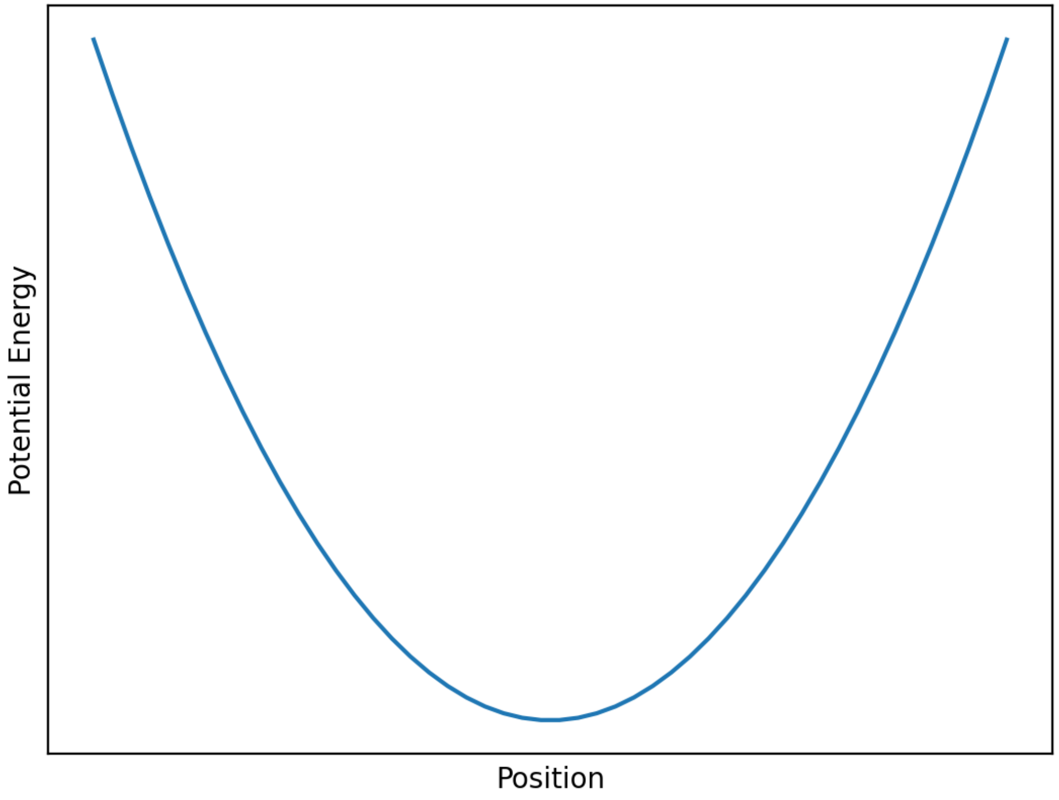
- Formalism also applies to quantum systems.
- A quantum particle of mass m moving in a harmonic potential defined by k (a particle in a laser ion trap) has the Hamiltonian:

$$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

- This quantum system must obey the Schrödinger Equation

$$H\psi = E\psi$$

- The wavefunction, ψ, that describes the dynamics of the system can be solved for directly.



The potential that the quantum particle occupies.

Variational Quantum Eigensolver

- Relies on the Variational Method from Quantum Mechanics:

$$\langle \psi_0 | H | \psi_0 \rangle = E_0$$
$$\langle \psi | H | \psi \rangle = E$$
$$E \geq E_0$$

- VQE has makes use of a parameterized circuit – the “ansantz”.

- Parameters are incrementally changed and optimized classically, then input into the next iteration.

- This is then repeated until convergence and until the minimum energy and optimal parameters are found.

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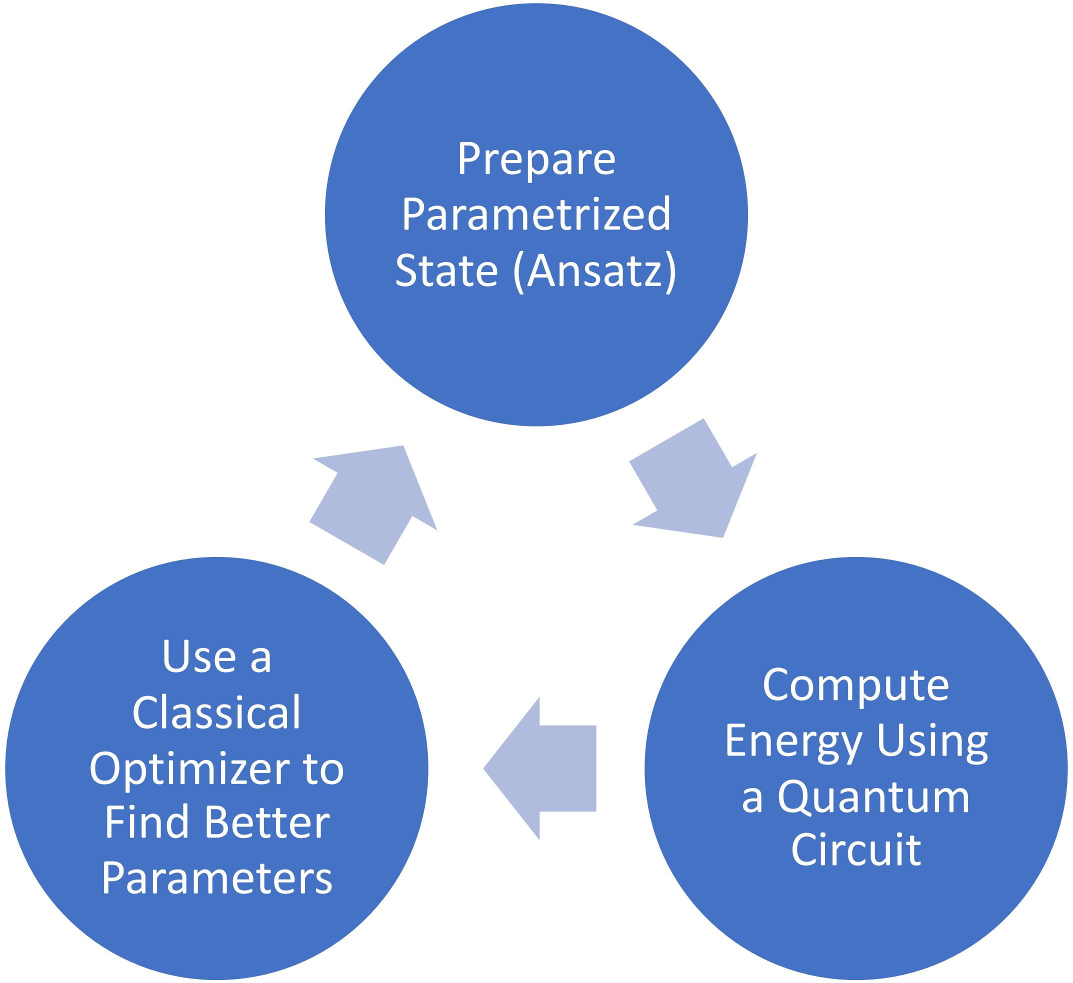
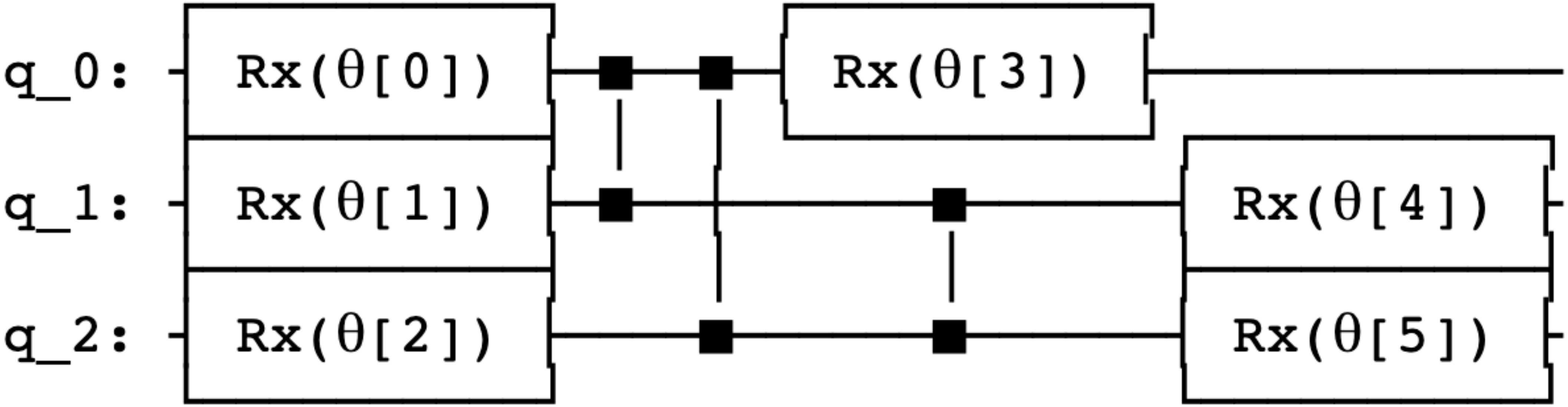
Variational Quantum Eigensolver



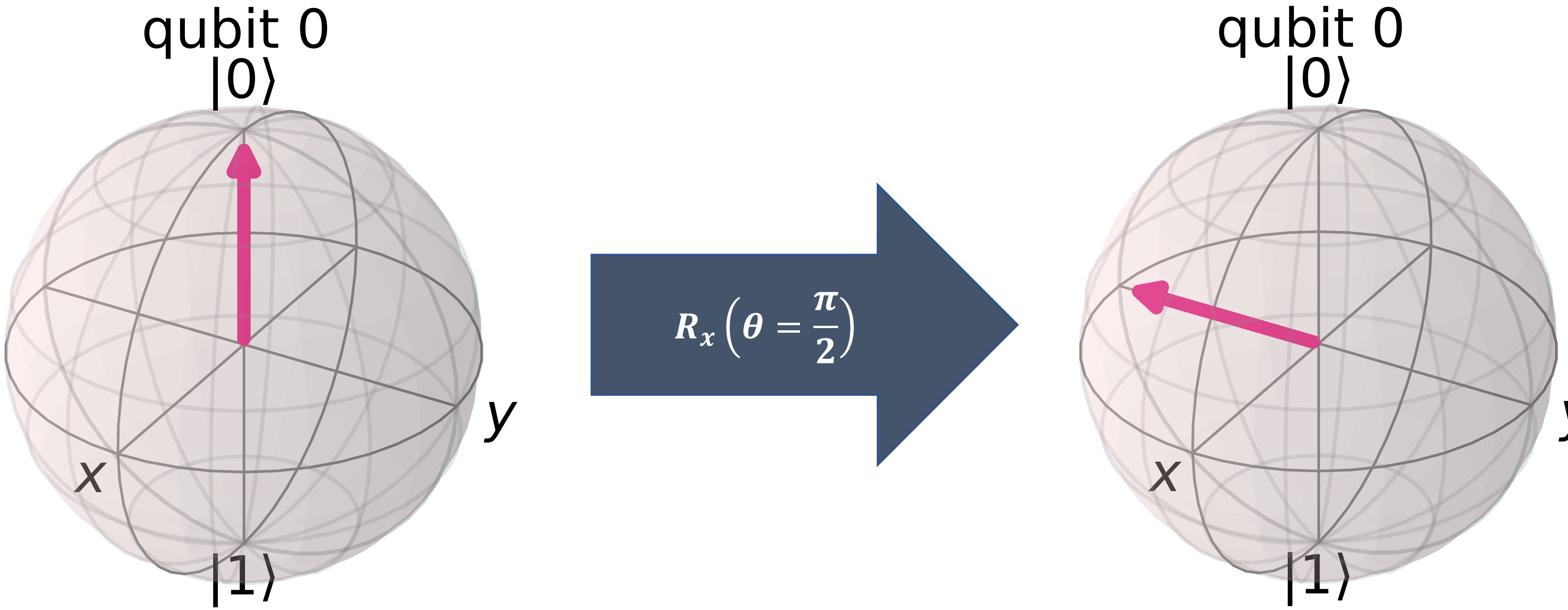
Variational Quantum Eigensolver

- The choice of Ansatz dictates the accuracy of your results, and the efficiency of your algorithm.
- The choice is limited by hardware capabilities.



Rotating Qubits

- The “bloch sphere” is a representation of qubit states.
- Can rotate these states with a rotation operator, by an angle θ.



Quantum Approximate Optimization Algorithm

Quantum Approximate Optimization Algorithm

- A hybrid algorithm that relies on the variational principle, like VQE.
- The Hamiltonian directly informs the Ansatz.

$$|\psi(\beta,\gamma)\rangle = e^{-iH_M\beta_1}e^{-iH_C\gamma_1}\dots e^{-iH_M\beta_p}e^{-iH_C\gamma_p}|0\rangle$$

- H_C is the Hamiltonian that corresponds to the optimization problem.
- H_M is a mixer Hamiltonian, which acts to entangle the qubits.

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Interpreting Results

Dirac Formalism (Bra-Ket Notation)

- Through Dirac formalism the problem is cast into a Linear Algebra problem.

- Wavefunctions, Hamiltonian operators, and other quantum states can be represented as vectors and matrices in a complex Hilbert space.

$$\psi \rightarrow |\psi\rangle = a_1|a_1\rangle + a_2|a_2\rangle + \cdots + a_n|a_n\rangle = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

- The normalized wavefunction, $|\psi\rangle$, can be expressed as a linear combination of basis states, $|a_i\rangle$, weighted by a complex scalar a_i.

- The modulus squared of each scalar, $|a_i|^2$, is the probability of the system occupying the associated state.

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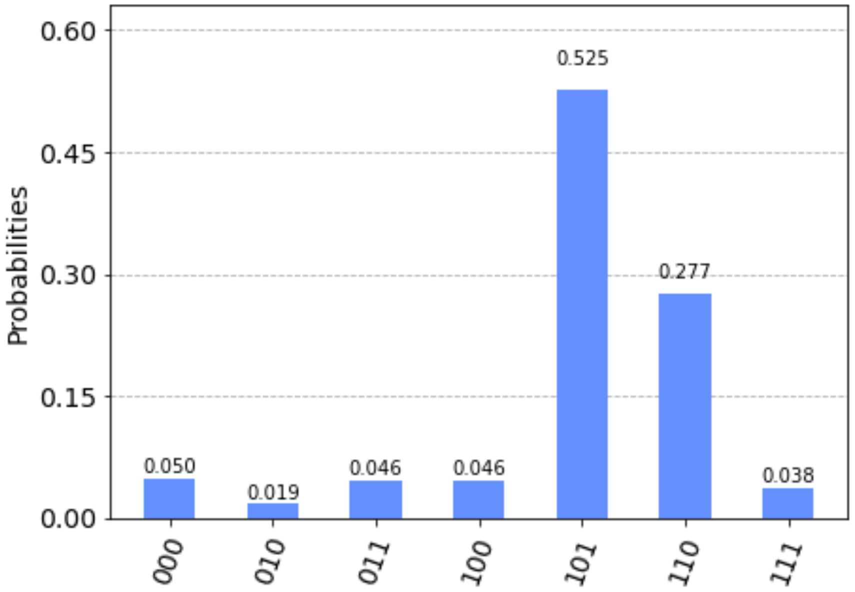
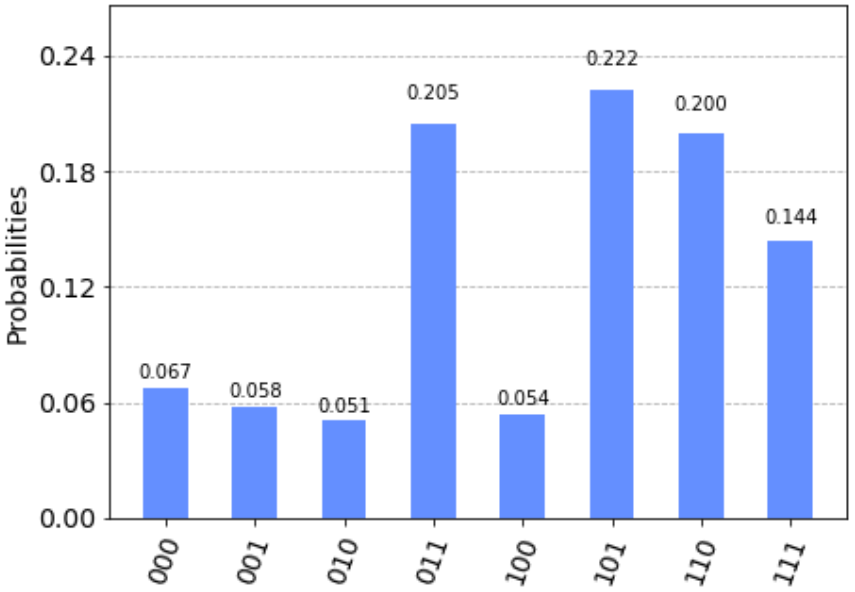
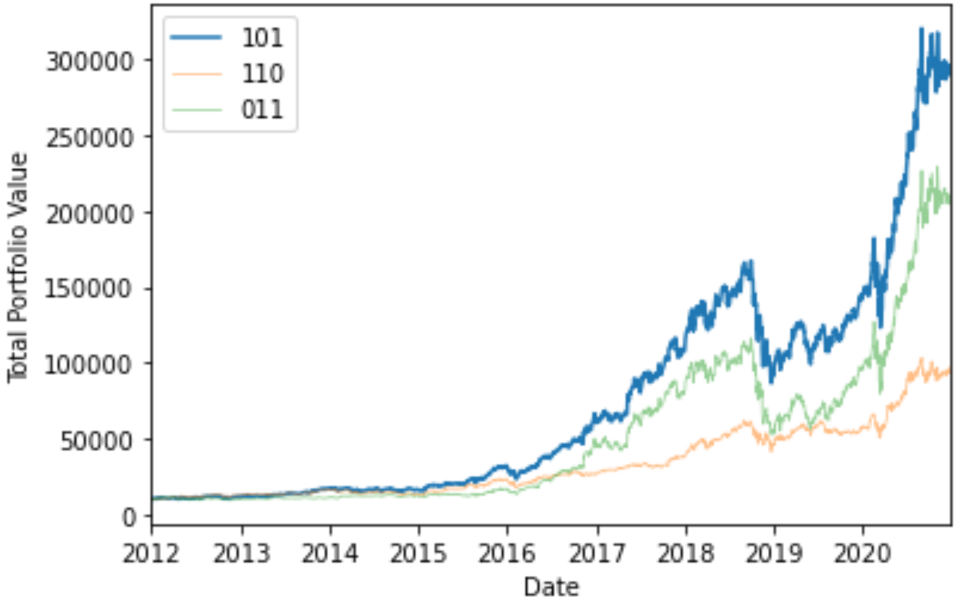
Results

VQE





QAOA



Summary of Results

- We refreshed ourselves on Portfolio Optimization and cast the problem as a qubo.
- We introduced concepts from quantum mechanics and how they apply to the hybrid quantum computing algorithms – VQE, QAOA.
- We learned how to interpret the probabilistic output from the two algorithms.
- We saw how they map back to the original and familiar problem.
- Now we can apply this to a more realistic scenario.

Applied to Real Data

Motivating Research Questions

1. How many and which equity benchmarks does an endowment portfolio really need?

2. How does this change for different periods?

Data

Picking from $n = 8$ assets:

- Global Equity
- S&P500
- US Midcap
- EAFE (Europe, Australasia, Far East)
- Europe
- Japan
- Asia Ex-Japan
- Emerging Markets

Data

Data has been processed in two ways:

1. Monthly returns (percentage) taken at end of month.
2. Risk-adjusted monthly return data (percentage), perceive
   a. Returns are expressed per one unit of risk, which should emphasize correlations and the relative return per unit risk:

$$\mu_{i,\text{Risk Adj.}} = \frac{\mu_i}{\sigma_i/\sqrt{12}}$$

How VQE/QAOA Can Be Applied

1. From risk-adjusted data, VQE and QAOA algorithms pick assets.

2. A practitioner then weights the picked assets according to perceived risk with the lower risk benchmarks getting a higher weight instead of using equal weights.
   a. IE risk–parity analysis.

$$\text{Portfolio Weight}_i = \frac{\left(\sum \sigma_j{}^{-1}\right)^{-1}}{\sigma_i}$$

Methods

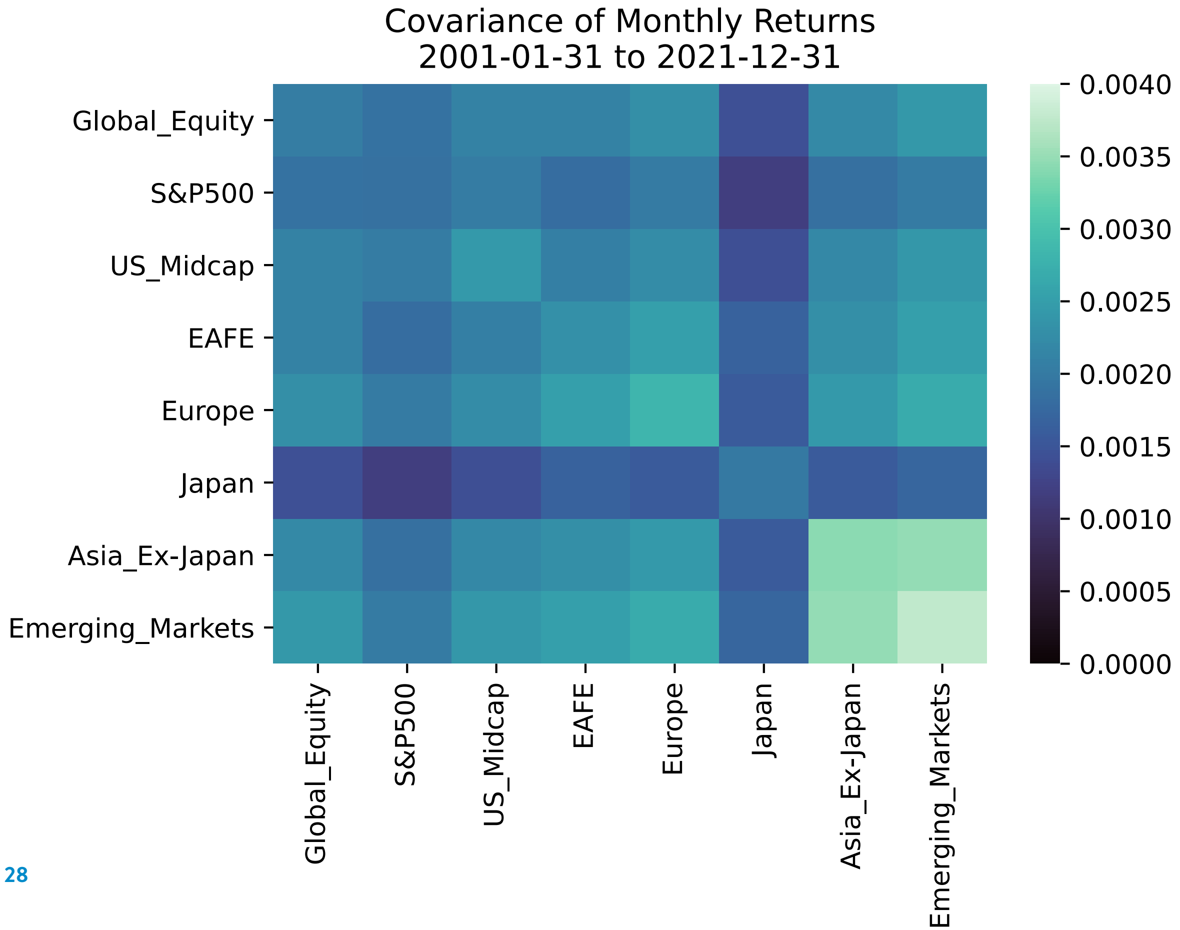
- Used both VQE and QAOA with small ansatz (p=2).
- Data Collected for all B portfolios, for all time periods of interest.
- Periods of interest are:

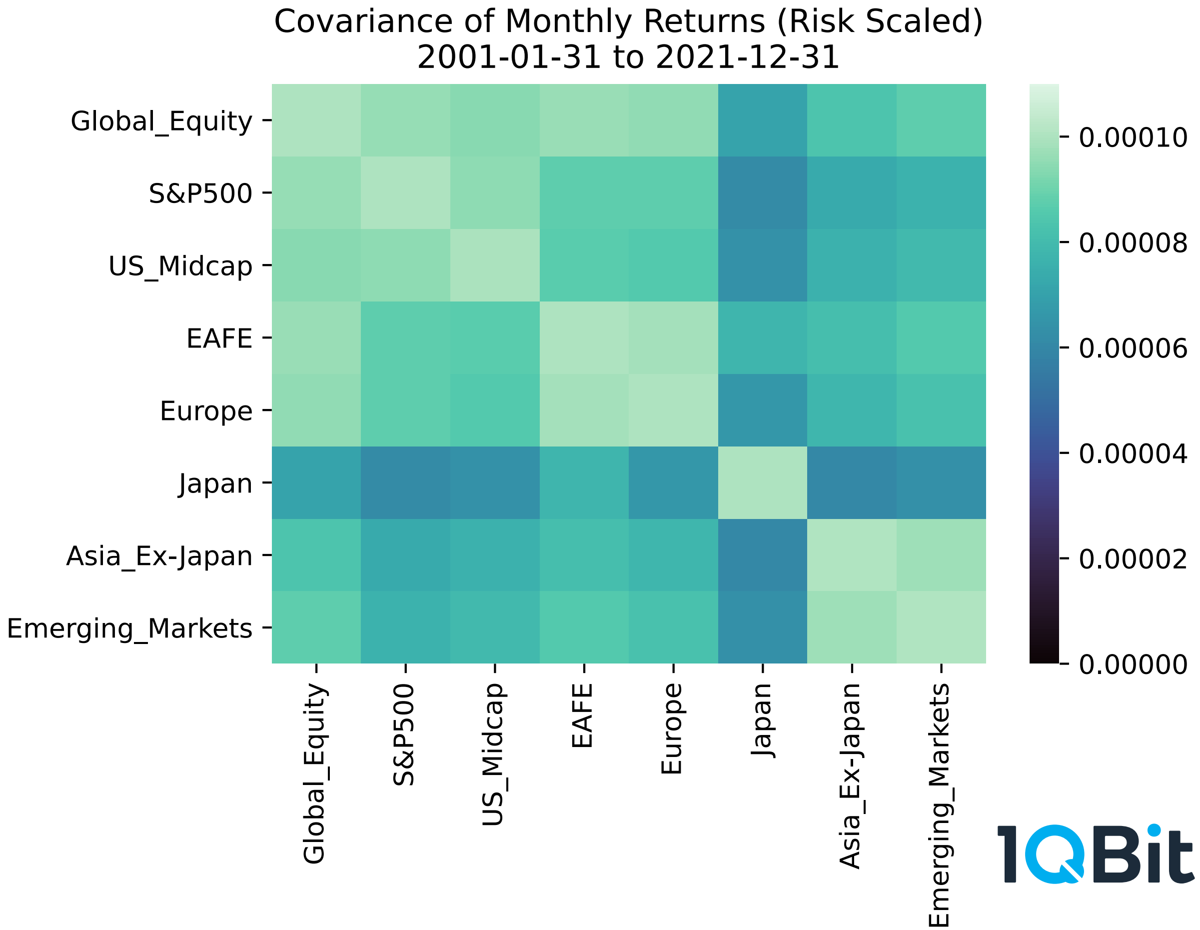
Jan-2001 through Dec-2021	Whole Period
Jan-2001 through Jun-2008	Before the 2008 Market Panic
Jan-2010 through Dec-2019	QE Era before the Pandemic
Jun-2020 through Dec-2021	Pandemic Rebound

- Used two simulators, statevector_simulator and qasm_simulator

Results

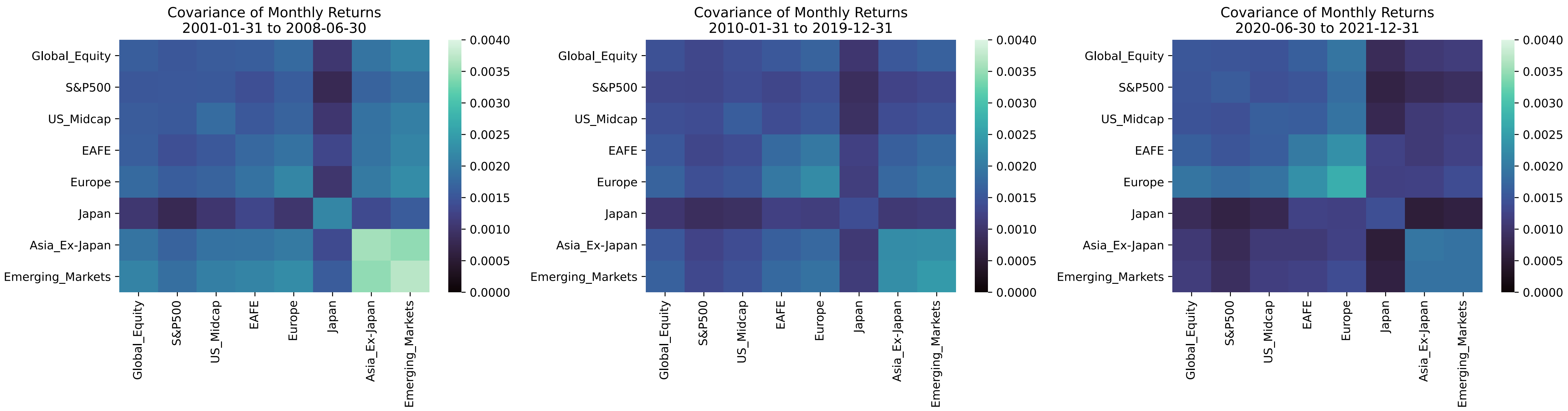
- Left – non-scaled, Right – risk-scaled
- Japan looks to have much lower covariance over all periods





Results

- From period-to-period, there is a shift in the covariances.
- Below are non-scaled.



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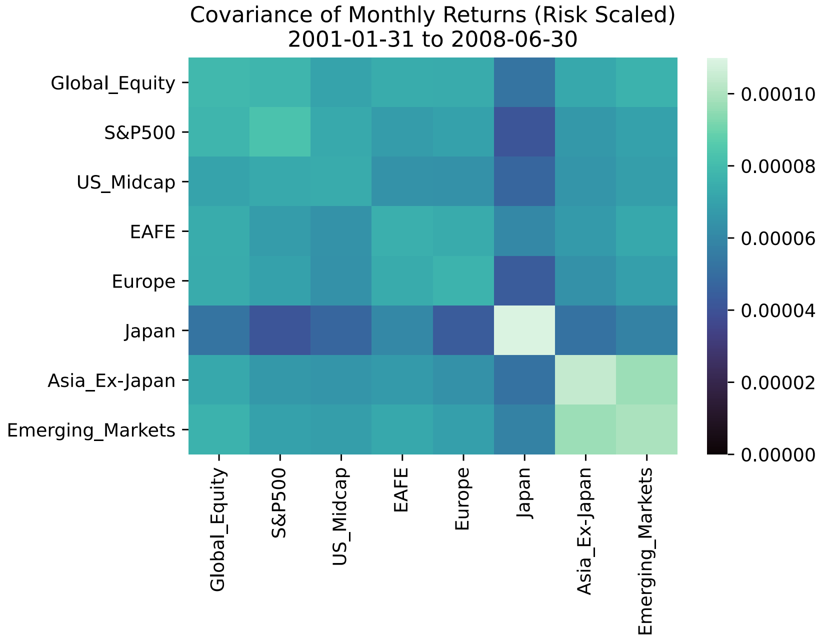
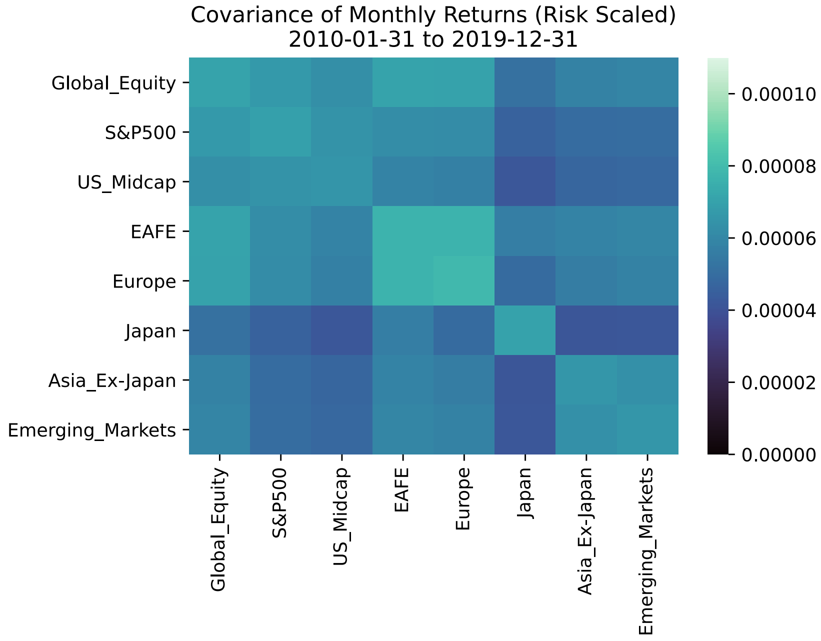
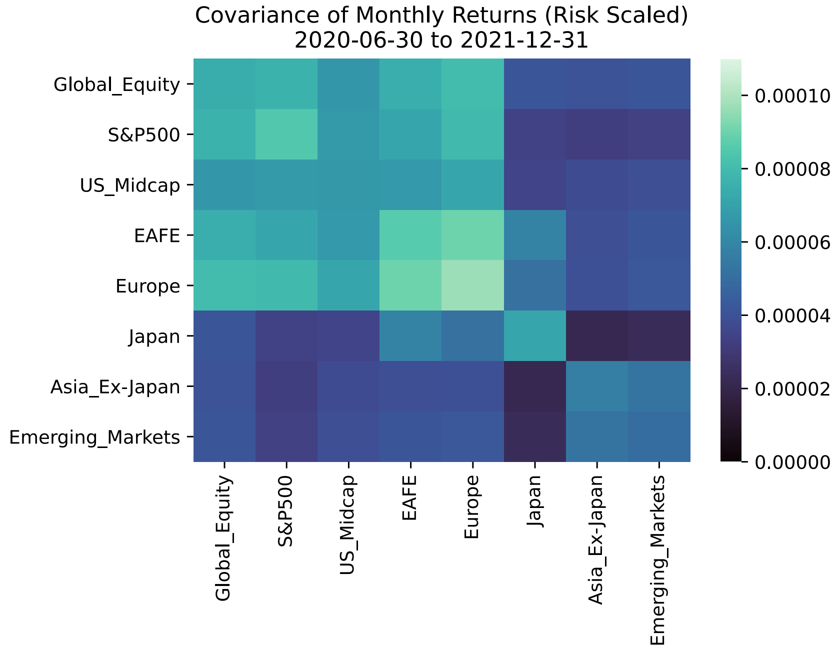


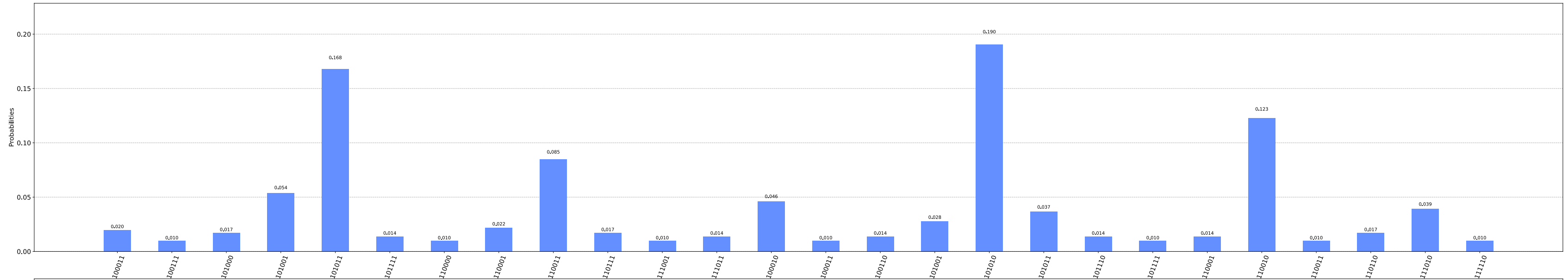




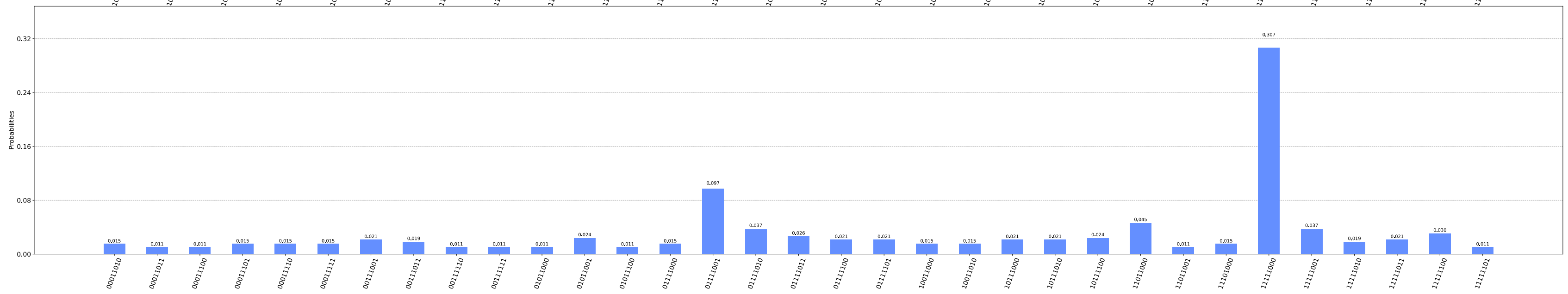
Results

- QAOA tends to be more accurate than VQE, relative to classical.
- *Statevector_simulator* is more consistent than *qasm_simulator*. The latter samples results in shots (similar to a real device)
- Generally, the risk-scaled distributions (bottom) from the results lead to better distributions than non-scaled (top).

Non-scaled



Risk-scaled

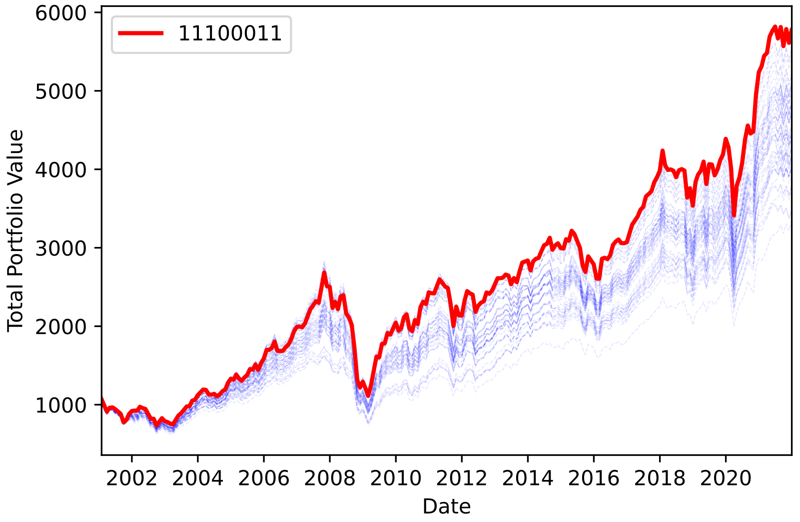


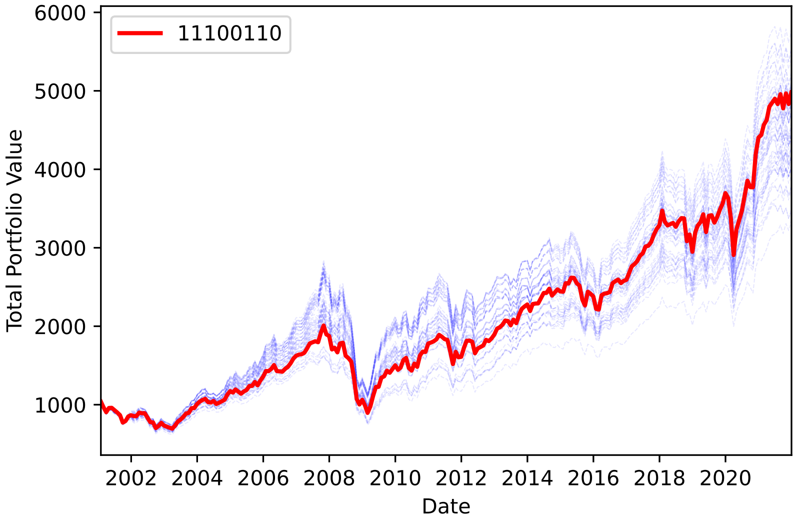
Results

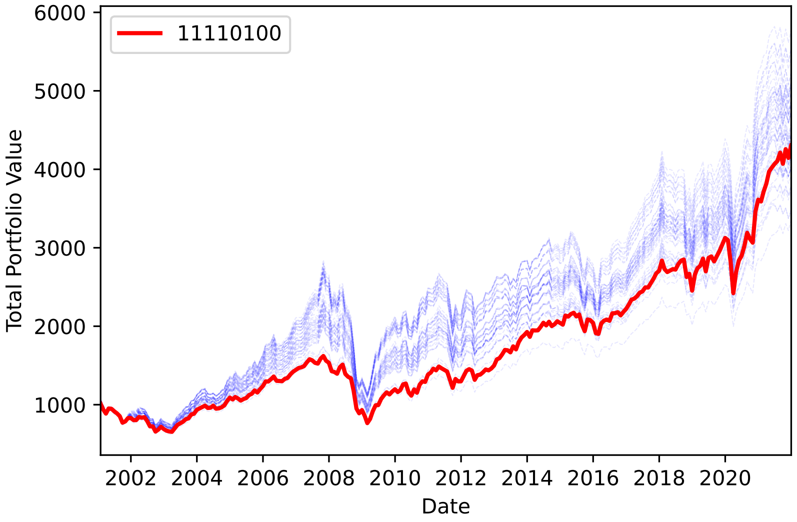
- Non-scaled example with $B = 5$, $q = 0.0, 0.5, 20$.
- Sanity check to show that our qubo is valid.

$$\max_{x\in\{0,1\}^n} \left(\mu^T x - q x^T \sigma x\right)$$

$$\text{subject to: } 1^T x = B.$$





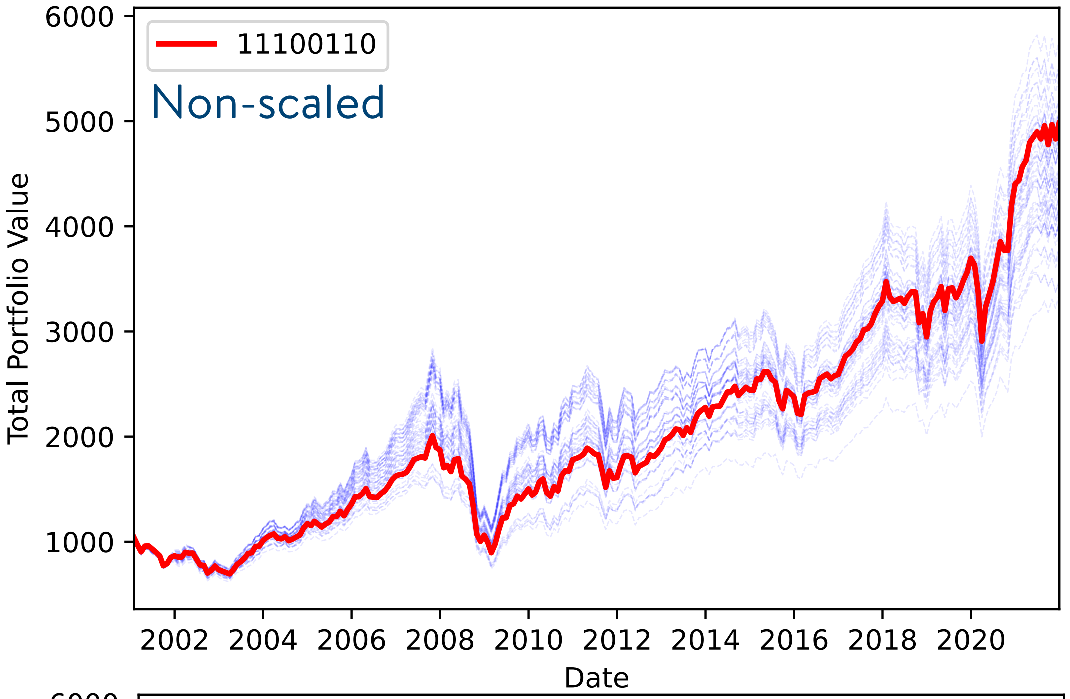


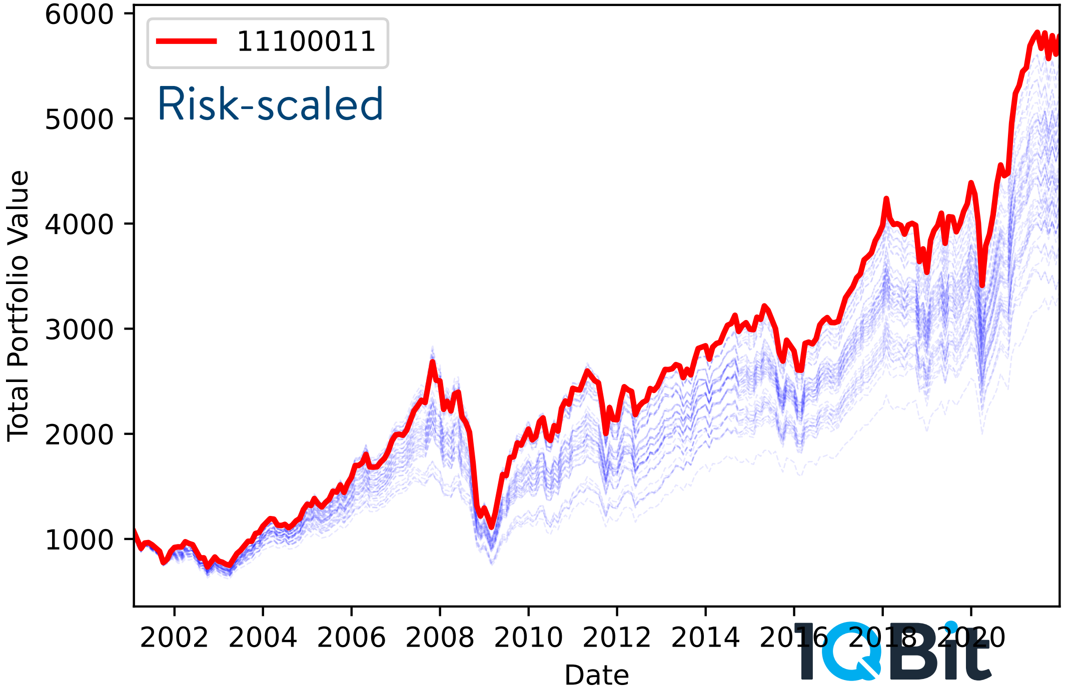
Results

For $B = 5, q = 0.5.$

- Top – non-scaled
  - Global Equity
  - S&P 500
  - US Midcap
  - Asia Ex-Japan
  - Japan
- Bottom – risk-scaled
  - Global Equity
  - S&P 500
  - US Midcap
  - Asia Ex-Japan
  - Emerging Markets

Risk-scaled generally gives a higher return over the periods.





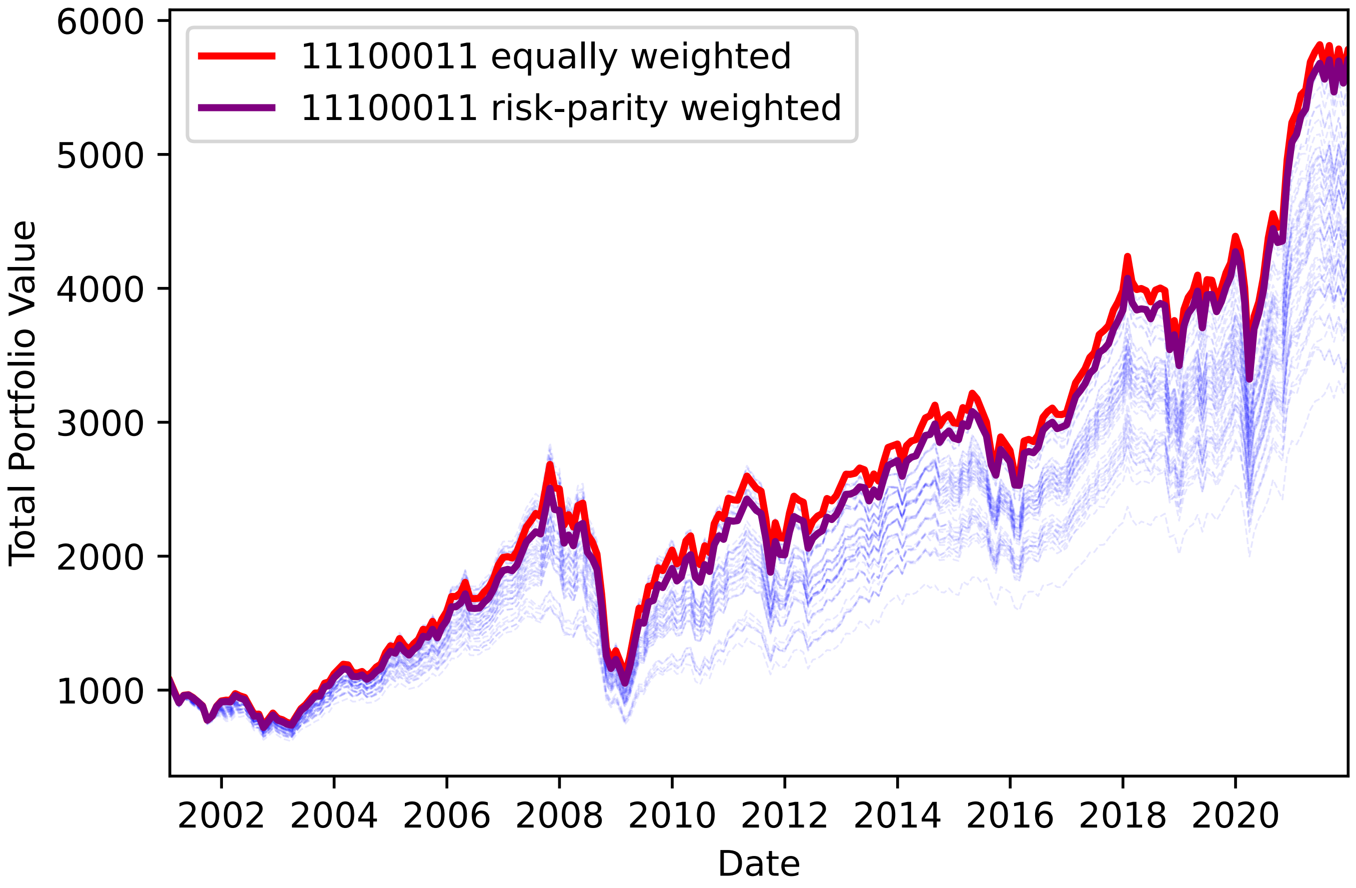
Results

- Pre-2008 Period had different portfolios for Risk-Scaled and Non-Scaled for $B = 5, 6$
- QE Period had different portfolios for Risk-Scaled and Non-Scaled for $B = 5$
- Pandemic Period had different portfolios for Risk-Scaled and Non-Scaled for $B = 1, 4, 5, 6, 7$
- Whole Period had different portfolios for Risk-Scaled and Non-Scaled for $B = 4, 5, 6$

Results

The Risk-Parity weight calculations give a portfolio of:

- Global Equity – 22.3%
- S&P 500 – 23.3%
- US Midcap – 20.4%
- Asia Ex-Japan – 17.3%
- Emerging Markets – 16.5%



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Summary

- We mapped a common problem of portfolio optimization to something that could be calculated using a gate model quantum computer.
- We applied VQE and QAOA to pick the best subset of assets in an endowment portfolio, over a variety of periods.
- We applied this in combination with risk-parity analysis to obtain the weights for the assets within the winning portfolio.

The Academic Collaboration in Finance Program

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