Linear Approximations

This approximation is crucial to many known numerical techniques such as Euler's Method to approximate solutions to ordinary differential equations. The idea to use linear approximations rests in the closeness of the tangent line to the graph of the function around a point.

Let x_0 be in the domain of the function $f(x)$. The equation of the tangent line to the graph of $f(x)$ at the point (x_0,y_0), where $y_0 = f(x_0)$, is

$$y - y_0 = f'(x_0)(x - x_0).$$

If x_1 is close to x_0, we will write $x_1 = x_0 + \Delta x$, and we will approximate $f(x_0 + \Delta x)$ by the point (x_1,y_1) on the tangent line given by

$$y_1 = y_0 + \Delta x\, f'(x_0).$$

If we write $\Delta y = y_1 - y_0$, we have

$$\Delta y = \Delta x f'(x_0).$$

In fact, one way to remember this formula is to write $f'(x)$ as $\frac{dy}{dx}$ and then replace d by Δ. Recall that, when x is close to x_0, we have

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)\,.$$

Example. Estimate $\sqrt{9.2}$.

Let $f(x) = \sqrt{3 + x}$. We have $f(6) = \sqrt{9} = 3$. Using the above approximation, we get

$$f(6.2) \approx f(6) + f'(6)(6.2 - 6)$$

We have

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x+3}}.$$

So $f'(6) = \frac{1}{6}$. Hence

$$f(6.2) \approx f(6) + f'(6)(6.2-6) = 3.033$$

or $\sqrt{9.2} \approx 3.033$. Check with your calculator and you'll see that this is a pretty good approximation for $\sqrt{9.2}$.**Remark.** For a function *f*(*x*), we define the **differential** *df* of *f*(*x*) by

$$df = f'(x)\,dx\,.$$

Example. Consider the function $y = f(x) = 5x^2$. Let Δx be an increment of *x*. Then, if Δy is the resulting increment of *y*, we have

$$\begin{aligned} \Delta y &= f(x+\Delta x) - f(x) \\ &= 5(x+\Delta x)^2 - 5x^2 \\ &= 10x(\Delta x) + 5(\Delta x)^2\,. \end{aligned}$$

On the other hand, we obtain for the differential *dy*:

$$dy = f'(x)\,dx = 10x\,dx\,.$$

In this example we are lucky in that we are able to compute Δy exactly, but in general this might be impossible. The error in the approximation, the difference between *dy*(replacing *dx* by Δx) and Δy, is $5(\Delta x)^2$, which is small compared to Δx.